Hairiness of $\omega$-bounded surfaces with non-zero Euler characteristic

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Abstract. A little complement concerning the dynamics of non-metric manifolds is provided, by showing that any flow on an $\omega$-bounded surface with non-zero Euler character has a fixed point.

1 Introduction

In a previous paper [2] we adventured timidly into the dynamics of flows on non-metric manifolds. Several missing links foiled the sharpness of the conclusions, leaving great zones of swampy darkness. The present note is supposed to generalise the 2-dimensional $\omega$-bounded hairy ball theorem, which in [2 Thm 4.5] was confined to the simply-connected case. By a hairy ball one understands commonly a topological space such that any flow (i.e., continuous $\mathbb{R}$-action) has a fixed point. In [2 Section 4] we speculated that the hairy ball theorem for $\omega$-bounded surfaces with Euler character $\chi \neq 0$ might be plagued by the existence of wild pipes. The latter jargon (to which we shall not attempt to assign a precise meaning) refers to the phenomenology discovered in Nyikos [7, §6, p. 669–670] effecting that weird continua can be transfinitely amalgamated to construct (“wild”) long pipes lacking a canonical $\omega_1$-exhaustion, whose levels closures are compact annuli. In contradistinction, Theorem 3.1 below indicates rather that the “tameness” of 2-dimensional topology (Schoenflies) conjointly with the Poincaré-Bendixson theory (which applies in the dichotomic pipes, where Jordan separation holds true) seem to unite into a stronger force supplanting the “wildness” of pipes, at least as far as the hairy ball paradigm is concerned. Thus, Conjecture 4.14 in [2] (saying that $\chi(M) \neq 0$ is a sufficient condition in the $\omega$-bounded context for the existence of an equilibrium point) holds true for $n = \dim M = 2$, whereas the Poincaré-Bendixson method used below does not adapt to dimensions $n \geq 3$, leaving the conjecture wide unsettled.

In fact, a third more hidden force decides for the vacillation toward tameness, namely Whitney’s flows, i.e. the creation of a motion compatible with a given oriented one-dimensional foliation. A noteworthy feature of this result of Whitney is that—albeit specifically metrical (as amply discussed in [2])—it proves oft useful in non-metric investigations (cf. optionally [2 Sec. 2.2], where its relevance to the classification of foliations on the long plane $\mathbb{L}^2$ is recalled).

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1 A (topological) space is said to be $\omega$-bounded if the closure of any countable subset is compact. In the case of manifolds it is equivalent to ask that any Lindelöf subset has a compact closure. [For Lindelöf’s locally second countable $\Rightarrow$ second countable $\Rightarrow$ separable.]

2 This means primarily that $M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha$ for any limit ordinal $\lambda$; we shall not use this, but the reader may wish to compare Nyikos [7 Def. 4.3, p. 656].

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2 Homological finiteness of $\omega$-bounded low-dimensional manifolds (after Nyikos)

To state properly Theorem 3.1 below, we need an a priori finiteness of the (singular) homology of $\omega$-bounded surfaces, especially of its Euler characteristic. This section provides an elementary argument, yet it can be noticed that the bagpipe theorem of Nyikos (which we shall anyway use later) also implies the desired finiteness (via Lemma 4.4). Thus the economical reader may prefer to skip completely this section, and move forward to Section 3.

The following argument of Nyikos gives simultaneously the 3-dimensional case, for it depends on the issue that metric manifolds in those low dimensions ($\leq 3$) admit PL-structures (piecewise linear). (Below singular homology is understood, and coefficients may be chosen in the fields $\mathbb{Q}$ or $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.)

Lemma 2.1

(i) An $\omega$-bounded $n$-manifold, $M$, of dimension $n \leq 3$ has finite-dimensional homology in each degree.

(ii) Besides, the groups $H_i(M)$ vanish for $i > n$ (and also for $i = n$ if $M$ is open). Thus, $M$ has a well-defined Euler characteristic, $\chi(M) = \sum_{i=0}^{\infty} (-1)^i b_i(M)$, where the Betti numbers $b_i(M)$ are the dimensions of the $H_i(M)$.

Proof. (We recall briefly the argument of Nyikos [7, Cor. 5.11, p. 665].) The assumption of low-dimensionality ($n \leq 3$) ensures that metric $n$-manifolds are “triangulable”, or better have PL-structures (Radó [8], Moishe for $n = 3$).

Regular neighbourhood theory can be employed to engulf any compact subset in a compact bordered (polyhedral) $n$-submanifold. (Bordered manifold means manifold-with-boundary.) Thus starting with any chart $M_0$, take its closure $\overline{M}_0$ which is compact (by $\omega$-boundedness); cover it by charts to get a metric enlargement $N_0$ (neighbourhood), in which one engulfs the compactum $\overline{M}_0$ into a bordered compact manifold $W_0$, and let $M_1 := \text{int} W_0$ be its interior. By transfinite induction of this clever routine (letting $M_{\lambda} := \bigcup_{\alpha < \lambda} M_\alpha$ for any limit ordinal), one constructs an $\omega_1$-exhaustion of $M$ by open metric manifolds $M = \bigcup_{\alpha < \omega_1} M_\alpha$, where for each non-limit ordinal $\alpha$ the closure $\overline{M}_\alpha = W_{\alpha-1}$ is a compact bordered $n$-manifold. (Unfortunately we cannot claim this at limit ordinals!)

(i) If $M$ has infinite-dimensional homology, there is a countably infinite sequence $\gamma_n$ of homology classes linearly independent in $H_1(M)$. The union of the supports of representing cycles $c_\alpha \in \gamma_\alpha$ is Lindelöf, thus contained in some $M_\alpha$, $\alpha < \omega_1$. Therefore the $c_\alpha$’s define homology classes in $H_1(M_{\alpha+1})$ which are still linearly independent, violating the finite dimensionality of $H_1(M_{\alpha+1})$.

(ii) This is a standard fact for which we may refer to Samelson [9, Lemma B] (compare also Milnor-Stasheff [5, p. 270–275]).

Remark 2.2 (Very optional reading.) We are not aware of a corruption of the lemma in case $\dim M \geq 4$. Perhaps Nyikos’ argument can be given more ampleness if instead of engulfing by polyhedrons one tries to engulf compacta by topological compact bordered manifolds (perhaps somewhat akin to M. H. A. Newman 1966, The engulfing theorem for topological manifolds, Ann. of Math. (2) 84). Such manifolds or more generally compact metric ANR’s are dominated by finite polyhedra, thus “finitary” homologically. As we shall not use this presently, we prefer to skip this delicate question.

3Of course nothing similar holds in dimension 4 (Rohlin 1952 [3], Freedman 1982).
Non simply-connected $\omega$-bounded hairy ball

Theorem 3.1. Any flow on an $\omega$-bounded surface with non-zero Euler characteristic $\chi(M) \neq 0$ has a stationary point.

Proof. Let $f$ be a flow on such a surface $M$. By Nyikos [7, Thm 5.14, p. 666] the surface admits a bagpipe decomposition $M = B \cup \bigsqcup_{i=1}^n P_i$, where the bag $B$ is a compact connected bordered surface with $n$ contours (=circular boundary components) and the $P_i$ are long pipes. In slight departure from Nyikos [7, Def. 5.2, p. 662] our pipes are supposed to have a boundary circle which seems in better accordance with the combinatorial “cut-and-paste” philosophy. It is easy to check that $\chi(B) = \chi(M)$, and that this equality holds for any bagpipe decomposition of $M$. (For this numerology cf. Lemma 4.4 below, and for the (modified) definition of a long pipe compare eventually the discussion in Section 4 below. Of course Nyikos bagpipe theorem is not jeopardized by this minor change of viewpoint.)

Let propagate the bag $B$ under the dynamics $f: \mathbb{R} \times M \to M$, to obtain $\mathbb{R}B := f(\mathbb{R} \times B)$ which is Lindelöf. By $\omega$-boundedness the closure $\overline{\mathbb{R}B}$ is compact, and flow-invariant (yet, unlikely to be a respectable bag; a priori only a weird compactum stemming from a complicated diffusion process). At any rate, the residual surface $S := M - \overline{\mathbb{R}B}$ is invariant and contained in $M - B$. Clearly, the set $M - B$ (consisting of the residual open pipes) is dichotomic, i.e., divided by any Jordan curve (cf. Lemma 4.2 below), hence by heredity $S$ is likewise dichotomic (compare [2, Lemma 5.3]). This will allow us to apply the Poincaré-Bendixson theory in each pipe to surger out a new flow-invariant bag.

For each $i = 1, \ldots, n$, choose a “remote” point $x_i \in P_i \cap S$ on the pipe $P_i$ and a chart $V_i \subset S$ about it. (Such points $x_i$ exist, because the compact set $\overline{\mathbb{R}B}$ cannot cover completely any of the non-metric pipes $P_i$.) By $\omega$-boundedness the orbit-closure $\overline{\mathbb{R}x_i}$ (in $M$) is compact, yet a priori not contained in $\mathbb{R}V_i$. To arrange the situation we need a little trick, based mostly on Whitney 1933 [10]. We may assume that $f$ has no stationary point, otherwise we are finished. Take $C_i$ a little open collar of the circle $\partial P_i$ such that $C_i \approx S^1 \times [0,1]$ and pairwise disjoint ($C_i \cap C_j = \emptyset$ if $i \neq j$). Aggregate $C_i$ to $\mathbb{R}V_i$. This set $\mathbb{R}V_i \cup C_i =: W_i$

4By a bordered manifold we mean a manifold-with-boundary.
is not flow-invariant a priori, but we may look at the foliation $\mathcal{F}$ on $M$ induced by the non-singular flow $f$ (Whitney+Hausdorff, as discussed in [2, Thm 2.2]) and restrict $\mathcal{F}$ to the open set $W_i$, which is Lindelöf, hence metric (Urysohn). The theorem of (Kerékjártó-)Whitney [2, Thm 2.5] creates a flow-motion $f_i$, compatible with the restricted foliation $\mathcal{F}|W_i$.

By Poincaré-Bendixson applied to $W_i$ acted upon by the flow $f_i$ (cf. e.g., [2, Lemmas 4.2 and 4.4]), $\bar{\mathcal{F}}_{x_i}$ contains either a fixed point or a periodic orbit, say $K_i \approx S^1$. The first option cannot occur by construction. The circle $K_i$ is certainly contained in the pipe $P_i$, yet may touch its boundary, so we regard it in the slightly enlarged pipe $P_i \cup \overline{C_i} =: P_i^\ast$. Let $B_i = B - \bigcup_{i=1}^n \text{int}P_i^\ast$ be the corresponding smaller “retracted” bag. Filling the pipe $P_i^\ast$ by a disc $D_i^\ast$ yields a simply-connected surface $F_i := P_i^\ast \cup D_i^\ast$ (cf. Definition [1,4]). By Schoenflies [1, Prop. 2] $K_i$ bounds a disc $D_i \subset F_i$ containing $D_i^\ast$ in its interior. Thus $D_i - \text{int}D_i^\ast$ is an annulus $A_i \approx S^1 \times [0,1]$ (again by Schoenflies, or more precisely by its corollary known as the 2-dimensional annulus theorem, e.g., Moise [6, p. 91]). Therefore the non-metric component of $P_i^\ast$ cut along $K_i$, that is $\Pi_i := F_i - \text{int}D_i$, is again a long pipe (fill it by the 2-disc $D_i$). Thus, surgering $M$ along the (disjoint) union $\bigsqcup_{i=1}^n K_i$ yields a new bagpipe decomposition, whose bag is $B^\ast := B_i \cup \bigsqcup_{i=1}^n A_i$ (annular expansion of the retracted bag) and with pipes $\Pi_i$. The new bag $B^\ast$ is flow-invariant (under the original flow $f$), thus a fixed point is created by Lefschetz [4], since $\chi(B^\ast) = \chi(M)$ (by Lemma [4,4] below), which is non-zero by assumption. (As usual one applies Lefschetz to the dyadic times $t_n = 1/2^n$ of the flow, to get a nested sequence of non-void fixed-point sets $K_n = \text{Fix}(f_{t_n})$ (where $f_t(x) = f(t,x)$), whose infinite intersection $\bigcap_{n=0}^\infty K_n$ is non-empty by compactness of the bag $B^\ast$, and a point in this intersection is fixed under all dyadic times, hence under all real times.)

4 Nyikos’s bagpipe decompositions

To put the bagpipe philosophy of Nyikos in closer connection to the classical combinatorial topology, it seems convenient to alter slightly the original definition of a pipe (given in Nyikos [4, Def. 5.2, p. 662]). First, amending a boundary to the pipe gives some material substrate for a sewing procedure along the bag boundaries, and second we may wish to express the “pipe” condition intrinsically without the artifact of an exhaustion (as already implicit in Nyikos [7, p. 668, §6 and p. 644]).

Definition 4.1 A long pipe is a non-metric $\omega$-bounded 2-manifold $P$ with one boundary component $\partial P \approx S^1$ homeomorphic to the circle, which capped-off by a 2-disc $D$ yields a simply-connected $P \cup D =: P_{\text{filled}}$ (called the filled pipe).

Lemma 4.2 The interior of any long pipe is dichotomic, i.e., divided by any embedded circle (alias Jordan curve).

Proof. The dichotomy of the filled pipe $P_{\text{filled}}$ follows at once from the dichotomy of any simply-connected surface (cf. [1, Prop. 6]). Since the interior of the pipe $\text{int}P \subset P_{\text{filled}}$, its dichotomy follows by heredity [2, Lemma 5.3].

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5Compare Kerékjártó 1925 [3], and Whitney 1933 [10] as the original sources.
6Or Poincaré to be historically sharper, yet confined to the vector fields case.
Lemma 4.3 The (singular) Euler characteristic of any long pipe is 0.

Proof. By definition the filled pipe $P \cup D =: P_{\text{filled}}$ is simply-connected, i.e. its fundamental group, hence a fortiori its abelianisation $H_1$, is trivial. Thus, $\chi(P_{\text{filled}}) = 1 - 0 + 0 = 1$, for the second Betti number $b_2(P_{\text{filled}}) = 0$ via the classical vanishing of the top-dimensional homology of an open manifold (cf. Samelson [9, Lemma D]). Via Mayer-Vietoris one has $\chi(P \cup D) = \chi(P) + \chi(D)$ (like in the combinatorial setting). Thus, $\chi(P) = 1 - 1 = 0$.

Lemma 4.4 If a surface $M$ has a bagpipe decomposition $M = B \cup \bigsqcup_{i=1}^n P_i$. Then $\chi(M) = \chi(B)$.

Proof. Set $P = \bigsqcup_{i=1}^n P_i$, thus $M = B \cup P$. The Mayer-Vietoris sequence shows that $\chi(M) = \chi(B) + \chi(P)$. By the obvious additivity of homology, $\chi(P) = \sum_{i=1}^n \chi(P_i)$, where each individual pipe has $\chi(P_i) = 0$ by (4.3).

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