Rate of Convergence of the Empirical Spectral Distribution Function to the Semi-Circular Law

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Abstract
Let \( X = (X_{jk})_{j,k=1}^n \) denote a Hermitian random matrix with entries \( X_{jk} \), which are independent for \( 1 \leq j \leq k \leq n \). We consider the rate of convergence of the empirical spectral distribution function of the matrix \( W = \frac{1}{\sqrt{n}}X \) to the semi-circular law assuming that \( \mathbb{E}X_{jk} = 0 \), \( \mathbb{E}X_{jk}^2 = 1 \) and that
\[
\sup_{n \geq 1} \sup_{1 \leq j, k \leq n} \mathbb{E}|X_{jk}|^4 =: \mu_4 < \infty \quad \text{and} \quad \sup_{1 \leq j, k \leq n} |X_{jk}| \leq Dn^{\frac{1}{4}}. \tag{0.1}
\]

By means of a recursion argument it is shown that the Kolmogorov distance between the empirical spectral distribution of the Wigner matrix \( W \) and the semi-circular law is of order \( O(n^{-1} \log^5 n) \) with high probability.

1 Introduction

The present paper is a continuation of the paper \([14]\), where we proved under the assumptions of Theorem 1.1 below a non improvable bound for the Kolmogorov distance between the expected spectral distribution function of Wigner matrices and the semicircular distribution function. In this paper

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we estimate the $L_p$-norm of the Kolmogorov distance between the empirical spectral distribution function of Wigner matrices and the semicircular distribution function, for $1 \leq p \leq C \log n$.

Consider a family $X = \{X_{jk}\}$, $1 \leq j \leq k \leq n$, of independent real random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any $n \geq 1$. Assume that $X_{jk} = X_{kj}$, for $1 \leq k < j \leq n$, and introduce the symmetric matrices

$$ W = \frac{1}{\sqrt{n}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}. $$

The matrix $W$ has a random spectrum $\{\lambda_1, \ldots, \lambda_n\}$ and an associated spectral empirical distribution function $F_n(x) = \frac{1}{n} \text{card} \{j \leq n : \lambda_j \leq x\}$, $x \in \mathbb{R}$. Averaging over the random values $X_{ij}(\omega)$, define the expected (non-random) empirical distribution functions $F_n(x) = \mathbb{E} F_n(x)$. Let $G(x)$ denote the semi-circular distribution function with density $g(x) = G'(x) = \frac{1}{2\pi} \sqrt{4 - x^2} I_{[-2,2]}(x)$, where $I_{[a,b]}(x)$ denotes the indicator–function of the interval $[a,b]$. The rate of convergence to the semi-circular law has been studied by several authors. For a detailed discussion of previous results see [14].

We shall estimate the Kolmogorov distance between $F_n(x)$ and the distribution function $G(x)$, $\Delta^*_n := \sup_x |F_n(x) - G(x)|$.

The main result of this paper is the following

**Theorem 1.1.** Let $\mathbb{E} X_{jk} = 0$, $\mathbb{E} X_{jk}^2 = 1$. Assume that there exists a constant $\mu_4 > 0$ such that

$$ \sup_{n \geq 1} \sup_{1 \leq j,k \leq n} \mathbb{E}|X_{jk}|^4 =: \mu_4 < \infty. \quad (1.1) $$

Furthermore, assume that there exists a constant $D$ such that for all $n$

$$ \sup_{1 \leq j,k \leq n} |X_{jk}| \leq Dn^\frac{1}{2}. \quad (1.2) $$

Then, there exist positive constants $C = C(D, \mu_4)$ and $C' = C'(D, \mu_4)$ depending on $D$ and $\mu_4$ only such that, for $p \leq C' \log n$

$$ \mathbb{E} \Delta_n^* p \leq Cn^{-1} \log^4 n. \quad (1.3) $$
Corollary 1.1. Let $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X^2_{jk} = 1$. Assume that
\begin{equation}
\sup_{n \geq 1} \sup_{1 \leq j \leq k \leq n} \mathbf{E}|X_{jk}|^8 =: \mu_8 < \infty. \tag{1.4}
\end{equation}
Then, there exists positive constants $C = C(\mu_8)$ and $C' = C'(\mu_8)$ depending on $\mu_8$ only such that, for $p \leq C' \log n$
\begin{equation}
\mathbf{E}^{\frac{1}{p}} \Delta^*_n \leq Cn^{-1} \log^4 n. \tag{1.5}
\end{equation}

Corollary 1.2. Assume that conditions (1.1) and (1.2) or (1.4) hold. Then there exist positive constants $C, c$ depending on $D, \mu_4$ or $\mu_8$ only such that
\begin{equation}
\Pr\{\Delta^*_n \geq Cn^{-1} \log^5 n\} \leq n^{-c \log \log n}. \tag{1.3}
\end{equation}

Proof. The result follows immediately from Theorem 1.1 or Corollary 1.1 and Chebyshev’s inequality.

Corollary 1.3. Inequality (1.3) implies that
\begin{equation}
\Pr\{ \exists j \in [c_1 \log^5 n, n - c_1 \log^5 n] : |\lambda_j - \gamma_{nj}| \geq C \log^5 n \min\{j, n - j + 1\}^{-\frac{4}{3}} n^{-\frac{2}{3}} \} \leq n^{-c \log \log n}. \tag{1.6}
\end{equation}

Proof. For a proof of this Corollary see Subsection 9.1, Appendix in [11]. This shows the localization rigidity of eigenvalues except for a neighborhood of the edges given by $k \leq C \log^5 n$ or $k \geq n - C \log^5 n$.

We denote the Stieltjes transform of $\mathcal{F}_n(x)$ by $m_n(z)$ and the Stieltjes transform of the semi-circular law by $s(z)$. Let $\mathbf{R} = \mathbf{R}(z)$ be the resolvent matrix of $\mathbf{W}$ given by $\mathbf{R} = (\mathbf{W} - z \mathbf{I}_n)^{-1}$, for all $z = u + iv$ with $v \neq 0$. Here and in what follows $\mathbf{I}_n$ denotes the identity matrix of dimension $n$. Sometimes we shall omit the sub index in the notation of an identity matrix. It is well-known that the Stieltjes transform of the semi-circular distribution satisfies the equation
\begin{equation}
s^2(z) + zs(z) + 1 = 0 \tag{1.6}
\end{equation}
(see, for example, equality (4.20) in [13]). Let
\begin{equation}
v_0 := A_0 n^{-1} \log^4 n \tag{1.7}
\end{equation}
and $\gamma(z) := |2 - |u||$, for $z = u + iv$. Introduce the region $\mathbb{G} = \mathbb{G}(A_0, n, \varepsilon) \subset \mathbb{C}_+$
\begin{equation}
\mathbb{G} := \{z = u + iv \in \mathbb{C}_+: -2 + \varepsilon \leq u \leq 2 - \varepsilon, v \geq v_0/\sqrt{\gamma(z)}\}. \tag{1.8}
\end{equation}
Let $a > 0$ be positive number such that
\[
\frac{1}{\pi} \int_{|u| \leq a} \frac{1}{u^2 + 1} du = \frac{3}{4}.
\] 
(1.8)

We prove the following result.

**Theorem 1.2.** Let $\frac{1}{2} > \varepsilon > 0$ be positive numbers such that
\[
\varepsilon \frac{3}{2} := 2v_0 a.
\] 
(1.9)

Assuming the conditions of Theorem 1.1, for any $A_1 > 0$ there exist positive constants $C = C(D, \mu_4, A_1)$ and $A_0 = A_0(\mu_4, D, A_1)$ depending on $D$, $A_1$ and $\mu_4$ only, such that, for $z \in \mathbb{G}$ and for $1 \leq p \leq A_1(nv)^\frac{3}{4}$
\[
\mathbb{E}|m_n(z) - s(z)|^p \leq (Cp)^p(nv)^{-p}.
\]

**Corollary 1.4.** Let $\frac{1}{2} > \varepsilon > 0$ be positive numbers such that the condition (1.9) holds. Let $\mathbb{E}X_{jk} = 0$, $\mathbb{E}X_{jk}^2 = 1$. Assume that there exists a constant $\mu_8 > 0$ such that for any $1 \leq j \leq k \leq n$
\[
\sup_{j,k} \mathbb{E}|X_{jk}|^8 := \mu_8 < \infty.
\]
Then for any $A_1 > 0$ there exist positive constants $C = C(\mu_8, A_1)$ and $A_0 = A_0(\mu_8, A_1)$ depending on $\mu_8$ and $A_1$ only, such that, for $z \in \mathbb{G}$ and $1 \leq p \leq A_1(nv)^\frac{3}{4}$,
\[
\mathbb{E}|m_n(z) - s(z)|^p \leq (Cp)^p(nv)^{-p}.
\]

Similar results were obtained recently in [18], Theorems 1, 2, assuming sub-Gaussian tails for the distribution of the matrix entries.

### 1.1 Sketch of the Proof

1. As in [14] we start with an estimate of the Kolmogorov-distance to the Wigner distribution via an integral over the difference of the corresponding Stieltjes transforms along a contour in the upper half-plane using a smoothing inequality (2.1). This inequality is adapted to the $L_p$-norm of the corresponding Kolmogorov distance. The resulting bound (2.1) involves an integral over a segment, say $V = 4$, at a fixed distance from the real axis and a segment $u + iA_0n^{-1}(2 - |u|)^{-\frac{1}{2}}$, $u \leq x$ at a distance of order $n^{-1}\log^4 n$ but avoiding to come close to the endpoints $\pm 2$ of the support.
These segments are part of the boundary of an $n$-dependent region $G$ where bounds of Stieltjes transforms are needed. Since the Stieltjes-transform and the diagonal elements $R_{jj}(z)$ of the resolvent of the Wigner-matrix $W$ are uniformly bounded on the segment with $\text{Im } z = V$ by $1/V$ (see Section 3.1) proving a bound of order $O(n^{-1} \log n)$ for the latter segment near the $x$-axis is the essential problem.

2. In order to investigate this crucial part of the error we start with the 2nd resolvent or self-consistency equation for the Stieltjes transform resp. the quantities $R_{jj}(z)$ of $W$ (see (5.3) below) based on the difference of the resolvent of $W^{(j)}$ ($j$th row and column removed) and $W$. The necessary bounds of $E|R_{jj}|^p$ for large $p = O(\log n)$ were proved in [14].

3. In Section 6 we prove a bound for the error $E|\Lambda_n|^p := E|m_n(z) - s(z)|^p$ of the form $C_2 p^p (nv)^{-p}$ for $p \leq C_1'(nv)^{1/4}$ which suffices to prove the rate $O(n^{-1} \log^4 n)$ in Theorem 1.1. Here we use a series of martingale-type decompositions to evaluate $E|\Lambda_n|^p$.

4. The necessary auxiliary bounds for all these steps are collected in the Appendix.

2 Bounds for the Kolmogorov Distance Between Distribution Functions via Stieltjes Transforms

To bound the error $\Delta^*_n$ we shall use an approach developed in previous work of the authors, see [13].

We modify the bound of the Kolmogorov distance between an arbitrary distribution function and the semi-circular distribution function via their Stieltjes transforms obtained in [13] Lemma 2.1. For $x \in [-2, 2]$ define $\gamma(x) := 2 - |x|$. Given $\frac{1}{2} > \varepsilon > 0$ introduce the interval $J_\varepsilon = \{x \in [-2, 2] : \gamma(x) \geq \varepsilon\}$ and $J'_{\varepsilon} = J_{\varepsilon/2}$. For a distribution function $F$ denote by $S_F(z)$ its Stieltjes transform,

$$S_F(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dF(x).$$

**Proposition 2.1.** Under the conditions of Proposition 2.1 the following inequality holds

$$\Delta(F, G) \leq 2 \int_{-\infty}^{\infty} |S_F(u + iV) - S_G(u + iV)| du + C_1 v_0 + C_2 \varepsilon^{1/2}$$

$$+ 2 \sup_{x \in J'_\varepsilon} \left| \int_{V'} (S_F(x + iu) - S_G(x + iu)) du \right|.$$
where \( v' = \frac{v_0}{\sqrt{\gamma}} \) with \( \gamma = 2 - |x| \) and \( C_1, C_2 > 0 \) denote absolute constants.

**Remark 2.2.** For any \( x \in \mathbb{J}_\varepsilon \) we have \( \gamma(x) \geq \varepsilon \) and according to condition (1.9), \( \frac{\gamma}{v_0} \leq \frac{\varepsilon}{2} \).

For a proof of this Proposition see [11], Proposition 2.1.

**Corollary 2.1.** Under the conditions of Proposition 2.1 the following inequality holds

\[
\mathbb{E} \int_{-\infty}^{\infty} |m_n(u + iV) - s(u + iV)|^p du \leq C_1 v_0 + C_2 \varepsilon^2.
\]

**Proof.** To prove this Corollary we observe that by Hölder’s inequality

\[
\mathbb{E} \left[ \int_{-\infty}^{\infty} |m_n(u + iV) - s(u + iV)|^p du \right] \leq \prod_{l=1}^{p} \mathbb{E} \left[ \int_{-\infty}^{\infty} |m_n(u_l + iV) - s(u_l + iV)|^p du_l \right]
\]

**3 The proof of Theorem 1.1**

We shall apply the Corollary 2.1 to prove the Theorem 1.1. We choose \( V = 4 \) and \( v_0 \) as defined in (1.7) and introduce the quantity \( \varepsilon = 2v_0 \). We shall denote in what follows by \( C \) a generic constant depending on \( \mu_4 \) and \( D \) only.
3.1 Estimation of the First Integral in (2.1) for \( V = 4 \)

Denote by \( \mathbb{T} = \{1, \ldots, n\} \). In the following we shall systematically use for any \( n \times n \) matrix \( X \) together with its resolvent \( R \), its Stieltjes transform \( m_n \) etc. the corresponding quantities \( X^{(k)}, R^{(k)} \) and \( m^{(k)}_n \) for the corresponding sub matrix with entries \( X_{jk}, j, k \notin \mathbb{A}, \mathbb{A} \subset \mathbb{T} = \{1, \ldots, n\} \). Observe that

\[
m^{(k)}_n(z) = \frac{1}{n} \sum_{j \in \mathbb{T}_k} \frac{1}{\lambda^{(k)} - z}.
\]

(3.1)

By \( \mathfrak{F}^{(j)} \) we denote the \( \sigma \)-algebra generated by \( X_{lk} \) with \( l, k \in \mathbb{T}_j \). If \( \mathbb{A} = \emptyset \) we shall omit the set \( \mathbb{A} \) as exponent index.

In this Section we shall consider \( z = u + iV \) with \( V = 4 \). We shall use the representation

\[
R_{jj} = \frac{1}{-z + \frac{1}{\sqrt{n}} X_{jj} - \frac{1}{n} \sum_{k,l \in \mathbb{T}_j} X_{jk} X_{jl} R^{(j)}_{kl}},
\]

(see, for example, equality (4.6) in [13]). We may rewrite it as follows

\[
R_{jj} = -\frac{1}{z + m_n(z)} + \frac{1}{z + m_n(z)} \varepsilon_j R_{jj},
\]

(3.2)

where

\[
\varepsilon_j := \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3} + \varepsilon_{j4}
\]

with

\[
\varepsilon_{j1} := \frac{1}{\sqrt{n}} X_{jj}, \quad \varepsilon_{j2} := -\frac{1}{n} \sum_{k \neq l \in \mathbb{T}_j} X_{jk} X_{jl} R^{(j)}_{kl}, \quad \varepsilon_{j3} := -\frac{1}{n} \sum_{k \in \mathbb{T}_j} (X^2_{jk} - 1) R^{(j)}_{kk},
\]

\[
\varepsilon_{j4} := \frac{1}{n} (\text{Tr} R - \text{Tr} R^{(j)}).
\]

(3.3)

Let

\[
\Lambda_n := \Lambda_n(z) := m_n(z) - s(z) = \frac{1}{n} \text{Tr} R - s(z).
\]

As follows from (1.6), for the semi-circular law we have

\[
s(z) = -\frac{1}{z + s(z)} \text{ and } |s(z)| \leq 1.
\]

(3.4)

See, for instance [2], p. 632, relations (3.2), (3.3). Summing equality (3.2) in \( j = 1, \ldots, n \) and solving with respect \( \Lambda_n \), we get

\[
\Lambda_n = m_n(z) - s(z) = \frac{T_n}{z + m_n(z) + s(z)},
\]

(3.5)
where
\[ T_n = \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j R_{jj}. \]

Note that for \( V = 4 \)
\[ |m_n(z)| \leq \frac{1}{4} \leq \frac{1}{2} |z + s(z)|, \quad |s(z) - m_n(z)| \leq \frac{1}{2} \text{ a.s.} \quad (3.6) \]

This implies
\[ |z + m_n(z) + s(z)| \geq \frac{1}{2} |z + s(z)|, \quad |z + m_n(z)| \geq \frac{1}{2} |s(z) + z|. \quad (3.7) \]

The last inequality and inequality (3.5) imply as well that, for \( V = 4 \),
\[ |m_n(z)| \leq |s(z)|(1 + 2|T_n(z)|). \quad (3.8) \]

Let
\[ \varphi(z) = \sqrt[2]{|z|^{p-2}}. \]

Using equality (3.5), we may write, for \( p \geq 2 \),
\[ \mathbb{E}|\Lambda_n|^p = \sum_{\nu=1}^{4} \mathbb{E} \frac{T_{\nu\nu}}{z + m_n(z) + s(z)} \varphi(\Lambda_n), \]
where
\[ T_{\nu\nu} = \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j\nu} R_{jj}. \]

We consider first the term with \( \nu = 4 \). Using the relation
\[ \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j4} R_{jj} = -\frac{1}{n} \text{Tr} R^2, \quad (3.9) \]
(see Lemma 5.5 in \[11\] and (3.7)), we get, using Lemma 7.10, inequality (7.16) in the Appendix, and inequalities (3.7) and (3.8)
\[ \left| \frac{T_{44}}{z + m_n(z) + s(z)} \right| \leq \frac{1}{n} \sum_{j=1}^{n} \frac{\varepsilon_{j4} R_{jj}}{z + m_n(z) + s(z)} \leq \frac{C|s(z)|}{n} |m_n(z)| \leq \frac{C}{n} |s(z)|^2 (1 + |T_n|). \quad (3.10) \]
Therefore, by Hölder’s inequality,

\[ |E \frac{T_n^4 \varphi(\Lambda_n)}{z + m_n(z) + s(z)}| \leq C|s(z)|^2 \frac{n}{n} (1 + E_1^\frac{1}{p} |T_n|^p) E_\varphi^\frac{p-1}{p} |\varphi(\Lambda_n)|^\frac{p}{p-1}. \]

It is straightforward to check that, by the Cauchy – Schwartz inequality and \( |R_{jj}| \leq V^{-1} \),

\[ E|T_n|^p \leq E \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{j\nu}|^2 \right)^{\frac{p}{2}} \left( \frac{1}{n} \sum_{j=1}^{n} |R_{jj}|^2 \right)^{\frac{p}{2}} \leq \frac{1}{2^p} E \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{j}|^2 \right)^{\frac{p}{2}}. \]

Applying Corollary 7.5 and Lemmas 7.12 and 7.13 in the Appendix, we get that there exists an absolute constant \( C'' > 0 \) such that for \( p \leq C' \log n \),

\[ E \frac{1}{p} |T_n|^p \leq Cpn^{-\frac{1}{2}} \leq C''CC' \leq C. \]

Therefore,

\[ |E \frac{T_n^4 \varphi(\Lambda_n)}{z + m_n(z) + s(z)}| \leq C|s(z)|^2 \frac{n}{n} E_\varphi^\frac{p-1}{p} |\varphi(\Lambda_n)|^\frac{p}{p-1}. \]

Furthermore, we represent, for \( \nu = 1, 2, 3 \)

\[ \frac{1}{n} E \sum_{j=1}^{n} \varepsilon_{j\nu} R_{jj} \varphi(\Lambda_n) \begin{array}{c} z + m_n(z) + s(z) \\ = H_1 + H_2, \end{array} \]

where

\[ H_1 := \frac{1}{n} E \sum_{j=1}^{n} \varepsilon_{j\nu} s(z) \varphi(\Lambda_n) \]

\[ H_2 := \frac{1}{n} E \sum_{j=1}^{n} \varepsilon_{j\nu} (R_{jj} - s(z)) \varphi(\Lambda_n) \begin{array}{c} z + m_n(z) + s(z) \end{array}, \]

(3.11)

First we bound \( H_2 \). Applying first the Cauchy – Schwartz inequality followed by Hölder’s inequality, we get

\[ |H_2| \leq C|s(z)|E_\varphi^\frac{p}{2} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{j\nu}|^2 \right)^{\frac{p}{2}} E_\varphi^\frac{p}{2} \left( \frac{1}{n} \sum_{j=1}^{n} |R_{jj} - s(z)|^2 \right)^{\frac{p}{2}} E_\varphi^\frac{p-1}{p} |\varphi(\Lambda_n)|^\frac{p}{p-1}. \]

(3.12)

Using the representation (3.12), we may write

\[ R_{jj} = s(z) - s(z) \varepsilon_{j} R_{jj} - s(z) \Lambda_n R_{jj}. \]

(3.13)
Applying the representations (3.13) and (3.5), we obtain
\[
\frac{1}{n} \sum_{j=1}^{n} |R_{jj}(z) - s(z)|^2 \leq C|s(z)|^2 \left( \frac{1}{n} \sum_{l=1}^{n} |\varepsilon_l|^2 \right).
\]
(3.14)

Combining inequalities (3.12) and (3.14), we get
\[
|H_2| \leq C|s(z)|^2 \mathbb{E}^\frac{p-1}{p} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_j|^2 \right)^p \mathbb{E}^\frac{p-1}{p} |\varphi(\Lambda_n)|^\frac{p}{p-1}.
\]

Applying now Corollary 7.5 and Lemmas 7.12, 7.13 in the Appendix, we get
\[
|H_2| \leq \frac{C|s(z)|^2}{n} \mathbb{E}^\frac{p-1}{p} |\varphi(\Lambda_n)|^\frac{p}{p-1}.
\]
(3.15)

We continue with \(H_1\) and represent it in the form
\[
H_1 := H_{11} + H_{12} + H_{13},
\]
(3.16)

where
\[
H_{11} := \frac{1}{n} \mathbb{E} \sum_{j=1}^{n} \varepsilon_{j\nu} s(z) \varphi(\tilde{\Lambda}_n^{(j)}) \left( \frac{z}{z + m_n^{(j)}(z) + s(z)} \right),
\]
\[
H_{12} := \frac{1}{n} \mathbb{E} \sum_{j=1}^{n} \varepsilon_{j\nu} s(z) \left( \varphi(\Lambda_n) - \varphi(\tilde{\Lambda}_n^{(j)}) \right) \left( \frac{z}{z + m_n^{(j)}(z) + s(z)} \right),
\]
\[
H_{13} := -\frac{1}{n} \mathbb{E} \frac{\sum_{j=1}^{n} \varepsilon_{j\nu} s(z) \varepsilon_{j4} \varphi(\Lambda_n)}{(z + m_n(z) + s(z))(z + m_n^{(j)}(z) + s(z))},
\]

where
\[
\tilde{\Lambda}_n^{(j)} = \Lambda_n^{(j)} + \frac{s(z)}{n}.
\]

It is straightforward to check that, by conditional independence of \(\varepsilon_{j\nu}\) and \(\tilde{\Lambda}_n^{(j)}\),
\[
H_{11} = 0.
\]
(3.17)

Using Lemma 7.15 in the Appendix and applying the Cauchy–Schwartz inequality, Hölder’s inequality and inequality (3.7), we get
\[
|H_{13}| \leq \frac{C|s(z)|^2}{n} \mathbb{E} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{j\nu}|^2 \right)^p \mathbb{E}^\frac{p-1}{p} |\varphi(\Lambda_n)|^\frac{p}{p-1} \leq \frac{C|s(z)|^2}{n} \mathbb{E}^\frac{p-1}{p} |\varphi(\Lambda_n)|^\frac{p}{p-1}.
\]
(3.18)
Let $\eta_j = \frac{1}{n} \sum_{k,l \in T_j} X_{jk} X_{jl} [(R^{(j)})^2]_{kl}$. We use that (see Lemma 5.5 in [11])

$$\varepsilon_j = \frac{1}{n} (1 + \eta_j) R_{jj}.$$ 

Note that

$$\delta_{nj} := \Lambda_n - \tilde{\Lambda}_n = \varepsilon_j - \frac{s(z)}{n} = \frac{1}{n} (R_{jj} - s(z)) + \frac{1}{n} \eta_j R_{jj}.$$ 

This yields

$$|\delta_{nj}| \leq \frac{1}{n} |R_{jj} - s(z)| + \frac{1}{n} |\eta_j| (|s(z)| + |R_{jj} - s(z)|).$$  \hspace{1cm} (3.19)$$

By Taylor’s formula $\varphi(x) - \varphi(y) = (x - y) E \varphi'(x - \tau(x - y))$, we may write

$$H_{12} = \frac{s(z)}{n} E \sum_{j=1}^{n} \frac{\varepsilon_{j\nu} \delta_{nj} \varphi' (\Lambda_n - \tau \delta_{nj})}{z + m^{(j)}_n(z) + s(z)},$$

where $\tau$ denotes a uniformly distributed random variable on the unit interval which is independent of all other random variables. Note that, according to Lemma 7.9 in the Appendix with $\zeta = \delta_{nj}$,

$$|\varphi'(\Lambda_n - \tau \delta_{nj})| \leq p|\Lambda_n - \tau \delta_{nj}|^{p-2} \leq C p |\Lambda_n|^{p-2} + p^{p-1} |\tau \delta_{nj}|^{p-2}. \hspace{1cm} (3.20)$$

Therefore, applying inequality (3.7), the Cauchy – Schwartz inequality, Hölder’s inequality and finally inequality (3.20), we get

$$|H_{12}| \leq C p |s(z)|^2 E \left[ \frac{1}{p} \left( \frac{1}{n} \sum_{j=1}^{n} (|\varepsilon_{j\nu}|^2)^{p-2} \right)^{\frac{p}{p-2}} \right] E \left[ \left( \frac{1}{n} \sum_{j=1}^{n} |\delta_{nj}|^2 \right)^{\frac{p}{2}} \right]^{\frac{2}{p}} |\Lambda_n|^p$$

$$+ p^{p-1} |s(z)|^2 \frac{1}{n} \sum_{j=1}^{n} E |\varepsilon_{j\nu}| |\delta_{nj}|^{p-1}. \hspace{1cm} (3.21)$$

Applying inequality (3.19), we get

$$|H_{12}| \leq K_1 + K_2 + K_3 + K_4,$$
where

\[ K_1 := \frac{Cp|s(z)|^2}{n} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{ju}|^2 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |R_{jj} - s(z)|^2 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{p-2}{p}} |\Lambda_n|^p, \]

\[ K_2 := \frac{Cp|s(z)|^3}{n} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{ju}|^2 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\eta_j|^2 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{p-2}{p}} |\Lambda_n|^p, \]

\[ K_3 := \frac{Cp|s(z)|^3}{n} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{ju}|^2 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\eta_j|^4 \right)^{\frac{p}{2}} \times \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |R_{jj} - s(z)|^4 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{p-2}{p}} |\Lambda_n|^p, \]

\[ K_4 := \frac{p^{p-1}|s(z)|^2}{n} \sum_{j=1}^{n} \mathbb{E} |\varepsilon_{ju}| |\delta_{nj}|^{p-1}. \]

Using equality (3.13) and $|\Lambda_n| \leq \frac{1}{2}$ a. s., we get, for $z = u + iv$,

\[ |R_{jj}(z) - s(z)| \leq C|s(z)|^2 \left( |\varepsilon_j| + |\varepsilon_j|^2 + |\Lambda_n| + |\Lambda_n|^2 \right) \leq C|s(z)|^2 \left( |\varepsilon_j| + |\varepsilon_j|^2 + |\Lambda_n| \right). \]  

(3.22)

Therefore,

\[ \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |R_{jj} - s(z)|^2 \right)^{\frac{p}{2}} \leq C|s(z)|^4 \left( \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_j|^2 \right)^{\frac{p}{2}} + \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_j|^4 \right)^{\frac{p}{2}} + \mathbb{E}^{\frac{1}{p}} |\Lambda_n|^p \right). \]  

(3.23)

Applying the last inequality, we get

\[ |K_1| \leq \frac{Cp|s(z)|^4}{n} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{ju}|^2 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_j|^2 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{p-2}{p}} |\Lambda_n|^p \]

\[ + \frac{Cp|s(z)|^4}{n} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{ju}|^2 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_j|^4 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{p-2}{p}} |\Lambda_n|^p \]

\[ + \frac{Cp|s(z)|^4}{n} \mathbb{E}^{\frac{1}{p}} \left( \frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{ju}|^2 \right)^{\frac{p}{2}} \mathbb{E}^{\frac{p-2}{p}} |\Lambda_n|^p. \]
Using Corollary 7.5 and Lemmas 7.12, 7.13 in the Appendix, we get

\[ |K_1| \leq \frac{Cp^2|s(z)|^4}{n^2} E \frac{p-2}{p} |\Lambda_n|^p + \frac{Cp^2|s(z)|^2}{n} E \frac{p-1}{p} |\Lambda_n|^p. \]

According to Corollary 7.5 and Lemmas 7.12, 7.13 inequality (7.14), in the Appendix we have

\[ K_2 \leq \frac{Cp|s(z)|^3}{n^2} E \frac{p-2}{p} |\Lambda_n|^p, \]

and

\[ K_3 \leq \frac{Cp|s(z)|^3}{n^2} E \frac{p-2}{p} |\Lambda_n|^p. \]

To bound \( K_4 \) we use inequalities (3.19) and (3.22) and obtain

\[ K_4 \leq \frac{p^{p-1}|s(z)|^2}{n} \sum_{j=1}^{n} E \left| \varepsilon_{j\nu} \right| \left( \frac{1}{n} |R_{jj} - s(z)| (1 + |\eta_j|) + \frac{1}{n} |\eta_j||s(z)| \right)^{p-1}. \]

(3.24)

Without loss of generality, we may assume that \( p \geq 3 \) and use the inequality

\[ |\varepsilon_{j\nu}| \leq \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} |\varepsilon_{i\nu}|^2 \right)^{\frac{1}{2}}. \]

(3.25)

We rewrite now inequality (3.24) in the form

\[ K_4 \leq \frac{p^{p-1}|s(z)|^2}{n^{p-1}} \sqrt{n} E \left( \frac{1}{n} \sum_{j=1}^{n} \left| \varepsilon_{j\nu} \right|^2 \right)^{\frac{3}{2}} \frac{1}{n} \sum_{j=1}^{n} \left( |R_{jj} - s(z)| (1 + |\eta_j|) + |\eta_j||s(z)| \right)^{p-1}. \]

(3.26)

Applying Hölder’s inequality, we obtain, for \( p \geq 3 \)

\[ K_4 \leq \frac{p^{p-1}|s(z)|^2}{n^{p-1}} \sqrt{n} E^{\frac{3}{2}} \left( \frac{1}{n} \sum_{j=1}^{n} \left| \varepsilon_{j\nu} \right|^2 \right)^{\frac{3}{2}} \frac{1}{n} \sum_{j=1}^{n} E^{\frac{p-1}{p}} \left( |R_{jj} - s(z)| (1 + |\eta_j|) + |\eta_j||s(z)| \right)^{p}. \]

(3.27)

For \( p \leq 3 \) we may apply Hölder’s inequality directly and obtain

\[ K_4 \leq \frac{p^{p-1}|s(z)|^2}{n^p} \sum_{j=1}^{n} E \left| \varepsilon_{j\nu} \right|^p \left( E^{\frac{p-1}{p}} |R_{jj} - s(z)|^p (1 + |\eta_j|)^p + E^{\frac{p-1}{p}} |\eta_j|^p |s(z)|^p \right). \]

(3.28)
Inequalities (3.27) and (3.28) together imply
\[ K_4 \leq \frac{C|s(z)|^{p+1}p^p}{n^p}. \] (3.29)

Collecting the relations (3.10), (3.11), (3.16), (3.17), (3.18), and (3.29) we get
\[ \mathbb{E}|\Lambda_n|^p \leq \frac{C|s(z)|^{2p}E \frac{p-2}{p} |\Lambda_n|^p + \frac{C|s(z)|^{2p}E \frac{p-1}{p} |\varphi(\Lambda_n)|^{\frac{1}{p-1}} + \frac{C|s(z)|^{p+1}p^p}{n^p}}{n^p}. \] (3.30)

Applying Corollary 7.7 in the Appendix, we get,
\[ \mathbb{E}^{\frac{1}{p}}|\Lambda_n|^p \leq \frac{Cp|s(z)|^{1+\frac{1}{p}}}{n}. \] (3.31)

Consider now the integral
\[ \text{Int}(V) = \int_{-\infty}^{\infty} \mathbb{E}^{\frac{1}{p}}|m_n(u + iV) - s(u + iV)|^p du \]
for \( V = 4 \). Using inequality (3.31), we have
\[ |\text{Int}(V)| \leq \frac{C}{n} \int_{-\infty}^{\infty} |s(u + iV)|^{1+\frac{1}{p}} du. \]

Finally, we note that
\[ \int_{-\infty}^{\infty} |s(z)|^{1+\frac{1}{p}} dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{((x-u)^2 + V^2)^{\frac{p+1}{2p}}} dud\mathcal{G}(x) \leq C(p + 1). \] (3.32)

Inequalities (3.11) – (3.32) together imply, that for \( p \leq C \log n \),
\[ \int_{-\infty}^{\infty} \mathbb{E}^{\frac{1}{p}}|\Lambda_n(u + iV)|^p du \leq \frac{C \log n}{n}. \] (3.33)

3.2 The bound of the second integral in (2.1)

To finish the proof of Theorem 1.1 we need to bound the second integral in (2.1) for \( z \in \mathbb{G}, v_0 = C_7 n^{-1} \log^4(n + 1) \) and \( \varepsilon = C_8 v_0^{\frac{2}{p}} \), where the constant \( C_8 \) is chosen such that so that condition (1.9) holds. We shall use the results of Theorem 1.2. According to these results we have, for \( z \in \mathbb{G}, \)
\[ J_p(z) := \mathbb{E}^{\frac{1}{p}}|m_n(z) - s(z)|^p \leq C p(nv)^{-1}. \] (3.34)
Partition the interval \( J_\varepsilon \) into \( k_n = n^4 \) subintervals of equal length, that is \(-2 + \varepsilon = x_0 < \cdots < x_{k_n} = 2 - \varepsilon\). Note that

\[
\sup_{x \in J_\varepsilon} \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x + iv) - s(x + iv))dv \right| \leq \max_{1 \leq k \leq k_n} \sup_{x_{k-1} \leq x \leq x_k} \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x + iv) - s(x + iv))dv \right|.
\]

Furthermore,

\[
\sup_{x_{k-1} \leq x \leq x_k} \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x + iv) - s(x + iv))dv \right| \leq \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x_{k-1} + iv) - s(x + iv))dv \right| + \int_{x_{k-1}}^{x_k} \int_{v_0/\sqrt{\gamma}}^{V} |m'_n(x + iv) - s'(x + iv)|dvdx.
\]

Note that, for \( z \in \mathbb{G} \),

\[
|m'_n(x + iv) - s'(x + iv)| \leq Cv^{-2} \leq Cn^2.
\]

This yields that

\[
\sup_{x_{k-1} \leq x \leq x_k} \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x + iv) - s(x + iv))dv \right| \leq \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x_{k-1} + iv) - s(x + iv))dv \right| + Cn^{-1},
\]

and

\[
\sup_{x \in J_\varepsilon} \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x + iv) - s(x + iv))dv \right| \leq \max_{0 \leq k \leq k_n - 1} \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x_k + iv) - s(x_k + iv))dv \right| + Cn^{-1}.
\]

Using this inequality, we get

\[
\mathbb{E} \sup_{x \in J_\varepsilon} \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x + iv) - s(x + iv))dv \right|^p \leq 2^{p-1} \left( \sum_{k=0}^{k_n-1} \mathbb{E} \left| \int_{v_0/\sqrt{\gamma}}^{V} (m_n(x_k + iv) - s(x_k + iv))dv \right|^p \right) + \frac{(2C)^p}{n^p}.
\]
Applying Hölder’s inequality, we get similar to (2.2)

\[
E \left[ \int_{\gamma}^{V} (m(x + iv) - s(x + iv)) dv \right]^p \\
\leq \left( \int_{\gamma}^{V} E_{\gamma}^p |m(x + iv) - s(x + iv)|^p dv \right)^p.
\]

Applying now inequality (3.34) for \( p = [4 \log n] \), we obtain

\[
E_{\gamma}^p \sup_{x \in J_\varepsilon} \left| \int_{\gamma}^{V} (m(x + iv) - s(x + iv)) dv \right|^p \\
\leq Ck n^{-1} \log^2 n \leq C n^{-1} \log^2 n.
\]  \( (3.35) \)

Inequalities (3.33) and (3.35) complete the proof of Theorem 1.1.

**Remark 3.1.** To prove the Corollary 1.1 it is enough to use the results of Corollary 1.4, which imply inequality (3.34). Thus Corollary 1.1 is proved.

### 4 Proof of Corollary 1.4

We consider the truncated random variables \( \hat{X}_{jl} \) defined by

\[
\hat{X}_{jl} := X_{jl} I \{|X_{jl}| \leq cn^{\frac{1}{4}} \}.
\]  \( (4.1) \)

Let \( \hat{F}_n(x) \) denote the empirical spectral distribution function of the matrix \( \hat{W} = \frac{1}{\sqrt{n}} (\hat{X}_{jl}) \).

**Lemma 4.1.** Assuming the conditions of Theorem 1.1 there exist constants \( C, c > 0 \) such that, for any \( p \geq 1 \)

\[
E_{\gamma}^p |m_n(z) - s(z)|^p \leq \frac{Cp}{nv}.
\]

**Proof.** We use rank the inequality of Bai. See \[3\], Theorem A.43, p. 503. According to this inequality

\[
|m_n(z) - s(z)| \leq \frac{1}{nv} \text{rank}(X - \hat{X}).
\]
Using the obvious fact that the rank of a matrix is not larger than the number of its non-zero entries, we may write
\[
\mathbb{E}|m_n(z) - s(z)|^p \leq \frac{1}{(nv)^p} \mathbb{E}\left( \sum_{j,k=1}^{n} \mathbb{I}\{|X_{jk}| \geq Cn^{\frac{1}{2}}\}\right)^p \\
\leq \frac{2^p}{(nv)^p} \left( \left( \sum_{j,k=1}^{n} \mathbb{E}\mathbb{I}\{|X_{jk}| \geq Cn^{\frac{1}{2}}\}\right)^p + \mathbb{E} \left| \sum_{j,k=1}^{n} (\mathbb{I}\{|X_{jk}| \geq Cn^{\frac{1}{2}}\} - \mathbb{E}\mathbb{I}\{|X_{jk}| \geq Cn^{\frac{1}{2}}\})\right|^p \right).
\]

Applying Chebyshev’s and Rosenthal’s inequalities, we get
\[
\mathbb{E}|m_n(z) - s(z)|^p \leq \frac{2^p}{(nv)^p} \left( \left( \frac{1}{n^2} \sum_{j,k=1}^{n} \mathbb{E}|X_{jk}|^p\right)^p + C_n^p \left( \frac{1}{n^2} \sum_{j,k=1}^{n} \mathbb{E}|X_{jk}|^p + \frac{1}{n^2} \sum_{j,k=1}^{n} \mathbb{E}|X_{jk}|^p\right)^p \right) \leq (Cp)^p/(nv)^p.
\]

Thus, the Lemma is proved. \qed

Introduce now \( \tilde{X}_{jk} = \tilde{X}_{jk} - \mathbb{E}\tilde{X}_{jk} \) and \( \tilde{W} = \frac{1}{\sqrt{n}}(\tilde{X}_{jk})_{j,k=1}^{n} \). Denote by \( \tilde{m}_n(z) \) the Stieltjes transform of empirical distribution function of the matrix \( \tilde{W} \) and let \( \tilde{m}_n(z) \) denote the Stieltjes transform of the matrix \( \tilde{W} \). Furthermore, we re-normalize the matrix \( \tilde{W} \). Let \( \sigma_{jk}^2 = \mathbb{E}|\tilde{X}_{jk}|^2 \). We introduce the random variables \( \tilde{X}_{jk} = \tilde{X}_{jk} - \tilde{X}_{jk} \). And let \( \tilde{m}_n(z) \) denote the Stieltjes transform of the empirical spectral distribution function of the matrix \( \tilde{W} = \frac{1}{\sqrt{n}}(\tilde{X}_{jk})_{j,k=1}^{n} \).

**Lemma 4.2.**

\[
\mathbb{E}|\tilde{m}_n(z) - \tilde{m}_n(z)|^p \leq \frac{Cp}{(nv)^p}.
\]

**Proof.** Using the resolvent equality (7.7), we get
\[
\tilde{m}_n(z) - \tilde{m}_n(z) = \frac{1}{n} \text{Tr} \tilde{R}(\tilde{W} - \tilde{W})\tilde{R}.
\]

Using the obvious inequalities \( |\text{Tr} AB| \leq \|A\|_2\|B\|_2 \) and \( \|AB\|_2 \leq \|A\|\|B\|_2 \), we obtain
\[
|\tilde{m}_n(z) - \tilde{m}_n(z)| \leq \frac{1}{n} \|\tilde{R}\|\|\tilde{R}\|_2\|\tilde{W} - \tilde{W}\|_2 \quad (4.2)
\]
Rate of Convergence to the Semi-Circular Law

Note that

\[ \| \tilde{W} - \hat{W} \|_2^2 = \frac{1}{n} \sum_{j,k=1}^{n} (1 - \sigma_{jk})^2 \hat{X}_{jk}^2. \]  

(4.3)

Furthermore, we observe that

\[ (1 - \sigma_{jk})^2 \leq (1 - \sigma_{jk}^2)^2 \leq (E \hat{X}_{jk}^2 \mathbb{1}\{|X_{jk}| \geq cn^{\frac{1}{2}}\})^2 \leq C \mu_{\hat{X}}^2 n^{-3}. \]  

(4.4)

Relations (4.3) and (4.4) together imply

\[ \| \tilde{W} - \hat{W} \|_2^2 \leq \frac{C}{n^3} \| \hat{W} \|_2^2. \]

Note that the \( \hat{X}_{jl} \)satisfy the condition \( |\hat{X}_{jl}| \leq Dn^{\frac{1}{4}}, \quad E \hat{X}_{jl} = 0 \) and \( E \hat{X}_{jk}^2 = 1 \),

(4.5)

for some absolute constant \( D \). We may apply Theorem 1.2. According to this theorem we have, for \( q \leq C \log n \),

\[ E|\hat{m}_n(z)|^q \leq C^q. \]  

(4.6)

Furthermore, we note that, by Lemma 7.10 in the Appendix

\[ \frac{1}{n} \| \hat{R} \|_2^2 = v^{-1} \text{Im} \hat{m}_n(z) \leq v^{-1} |\hat{m}_n(z)|, \]

(4.7)

Inequality (4.2) yields

\[ E|\hat{m}_n(z) - \hat{m}_n(z)|^p \leq v^{-p} n^{-\frac{3p}{2}} E^{\frac{1}{2}} \left( n^{-\frac{1}{2}} \| \hat{R} \|_2 \right)^{2p} E^{\frac{1}{2}} \left( n^{-\frac{1}{2}} \| \hat{W} \|_2 \right)^{2p}. \]

Applying inequalities (4.6) and (4.7), we get

\[ E|\hat{m}_n(z) - \hat{m}_n(z)|^p \leq C^p n^{-\frac{3p}{2}} v^{-\frac{p}{2}} E^{\frac{1}{2}} \left( n^{-\frac{1}{2}} \| \hat{W} \|_2 \right)^{2p}. \]  

(4.8)

To bound the last factor in the r.h.s. of (4.8) we use standard arguments based on Rosenthal’s inequality. We may write

\[ E(n^{-1} \| \hat{W} \|_2^2)^p = \frac{1}{n^{2p}} E \left( \sum_{j,k} \hat{X}_{jk}^2 \right)^p \leq \frac{2^p}{n^{2p}} \left( \sum_{j,k} E \hat{X}_{jk}^2 \right)^p \]

\[ + \frac{C^p p^p}{n^{2p}} \left( \left( \sum_{j,k} E(\hat{X}_{jk}^2 - 1)^2 \right)^\frac{p}{2} + \sum_{j,k=1}^n E|\hat{X}_{jk}^2 - 1|^p \right) \leq C^{p} p^p \]

(4.9)

Using now inequalities (4.9) and (4.8), we get the claim. Thus, Lemma 4.2 is proved.
Lemma 4.3. 
\[ \mathbb{E}|\tilde{m}_n(z) - \hat{m}_n(z)| \leq \frac{C\mu_8}{n^{7/2} v^{3/2}}. \]

Proof. According to the resolvent equality (7.7), we have
\[ \tilde{m}_n(z) - \hat{m}_n(z) = \frac{1}{n} \text{Tr} (\bar{R} - \hat{R}) = \frac{1}{n} \text{Tr} (\bar{W} - \hat{W}) \hat{R} \bar{R}. \]

Similar to (4.2) we get
\[ |\tilde{m}_n(z) - \hat{m}_n(z)| \leq n^{-1} \|\bar{R}\| \|\hat{R}\|_2 \|\mathbb{E}\hat{W}\|_2. \] (4.10)

Furthermore, we note that
\[ |\mathbb{E}\hat{X}_{jk}| \leq C n^{-7/4} \mu_8. \]

This yields
\[ n^{-\frac{1}{2}} \|\mathbb{E}\hat{W}\|_2 \leq C n^{-\frac{7}{4}} \text{ a. s.} \]

By Lemma 4.2 we have
\[ \mathbb{E}|\tilde{m}_n(z)|^p \leq C^p. \] (4.11)

This implies that
\[ \mathbb{E}\left(\frac{1}{\sqrt{n}} \|\bar{R}\|_2\right)^p \leq C^p v^{-\frac{p}{2}} \] (4.12)

Combining now inequalities (4.10), (4.11) and (4.12), we get
\[ \mathbb{E}|\tilde{m}_n(z) - \hat{m}_n(z)|^p \leq \frac{C\mu_8}{n^{7/2} v^{3/2}} \leq \frac{C^p}{n^{3/2} v^{3/2}}. \] (4.13)

Thus Lemma 4.3 is proved. 

Lemmas 4.1, 4.2, 4.3 together imply the result of Corollary 1.4. Thus Corollary 1.4 is proved.

5 Proof of Theorem 1.2

The main problem in proving Theorem 1.2 is the the derivation of the following bound
\[ \mathbb{E}|R_{jj}|^p \leq C^p, \]
for \( j = 1, \ldots, n \) and any \( z \in G \). This bound was shown in [14]. To prove this bound we used an approach similar to that of Lemma 3.4 in [18]. We succeeded in the case of finite moments only developing new bounds of quadratic forms of the following type

\[
E \left| \frac{1}{n} \sum_{l \neq k} X_{jl} X_{jk} R_{kl}^{(j)} \right|^p \leq \left( \frac{C_p}{\sqrt{n v}} \right)^p.
\]

These estimates are based on a recursive scheme of using Rosenthal’s and Burkholder’s inequalities.

### 5.1 The Key Lemma

In this Section we state auxiliary lemmas needed for the proof of Theorem 1.2 which have been proved in [14]. Recall that the Stieltjes transform of an empirical spectral distribution function \( F_n(x) \), say \( m_n(z) \), is given by

\[
m_n(z) = \frac{1}{n} \sum_{j=1}^{n} R_{jj} = \frac{1}{n} \text{Tr} R.
\]

(see, for instance, equality (4.3) in [13]).

For any \( J \subset T \) denote \( T_J = T \setminus J \). For any \( J \subset T \) and \( j \in T_J \) define the quadratic form,

\[
Q^{(J,j)} := \frac{1}{n} \sum_{l \in T_J} \left| \sum_{r \in T_J \cap \{1, \ldots, l-1\}} X_{jl} R_{kl}^{(j)} \right|^2.
\]

**Lemma 5.1.** Assuming the conditions of Theorem 1.1 there exist constants \( A_1, C, C_3 \) depending on \( \mu_4 \) and \( D \) only such that we have for \( v \geq v_0 \) and \( p \leq A_1 (nv)^{1/4} \) and for any \( J \subset T \) such that \( |J| \leq C \log n \),

\[
E(Q^{(J,j)})^p \leq (C_3 p)^2 v^{-p}.
\]

**Corollary 5.2.** Assuming the conditions of Theorem 1.1 and for \( z = u + i V \) with \( V = 4 \), we have

\[
E(Q^{(J,j)})^p \leq C p^2 v^{-p}.
\]

**Proof.** The result immediately follows from Lemma 5.1.

**Proof of Lemma 5.1.** For the proof of Lemma 5.1 see [14], Lemma 5.4, Section 5.
5.2 Diagonal Entries of the Resolvent Matrix

Recall that

\[
R_{jj} = -\frac{1}{z + m_n(z)} + \frac{1}{z + m_n(z)} \varepsilon_j R_{jj},
\]

(5.3)

or

\[
R_{jj} = -\frac{1}{z + s(z)} + \frac{\Lambda_n R_{jj}}{(z + s(z))} + \frac{1}{z + s(z)} \varepsilon_j R_{jj},
\]

(5.4)

where \( \varepsilon_j := \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3} + \varepsilon_{j4} \) with

\[
\varepsilon_{j1} := \frac{1}{\sqrt{n}}X_{jj}, \quad \varepsilon_{j2} := -\frac{1}{n} \sum_{k \neq l \in T_j} X_{jk} X_{jl} R_{kl}^{(j)}, \quad \varepsilon_{j3} := -\frac{1}{n} \sum_{k \in T_j} (X_{jk}^2 - 1) R_{kk}^{(j)},
\]

\[
\varepsilon_{j4} := \frac{1}{n}(\text{Tr} R - \text{Tr} R^{(j)}), \quad \Lambda_n := m_n(z) - s(z) = \frac{1}{n} \text{Tr} R - s(z),
\]

\[
\varepsilon_{j4} = \Lambda_n - \Lambda_n^{(j)}.
\]

(5.5)

**Corollary 5.3.** Assuming the conditions of Theorem 1.1, for all \( A_1 > 0 \) there exists a positive constant \( A_0 = A_0(A_1, \mu_4, D) \) depending on \( \mu_4, D \) and \( A_1 \) such that, for \( p \leq A_1 (nv)^{1/2} \) and \( v \geq v_0 = A_0 n^{-1} \log^4 n \) there exist an absolute constants \( C_0 > 0 \) such that

\[
E| R_{jj} |^p \leq C_0^p,
\]

(5.6)

and

\[
E \frac{1}{|z + m_n(z)|^p} \leq C_0^p
\]

(5.7)

**Proof.** For the proof of this Corollary see [14], Section 6, Corollar y 6.10. □

6 Estimation of \( E|m_n(z) - s(z)|^p \).

We return now to the representation (5.4) which may be rewritten as

\[
m_n(z) = \frac{1}{n} \sum_{j=1}^{n} R_{jj} = -\frac{1}{z + m_n(z)} + \frac{T_n(z)}{z + s(z) + m_n(z)}.
\]

(6.1)

We develop the last equality as follows

\[
m_n(z) = s(z) + \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j R_{jj} + \frac{\varepsilon_{j4} R_{jj}}{z + s(z) + m_n(z)} + \frac{\hat{T}_n(z)}{z + s(z) + m_n(z)}.
\]

(6.2)
where
\[ \hat{T}_n = \sum_{\nu=1}^{3} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j\nu} R_{jj}. \]

Note that, by equality (3.9),
\[ \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j4} R_{jj} = - \frac{1}{n} \frac{dm_n(z)}{dz}. \]

Furthermore, we write
\[ \Lambda_n = - \frac{1}{n} \frac{m_n'(z)}{z + s(z) + m_n(z)} + \frac{\hat{T}_n(z)}{z + s(z) + m_n(z)}. \] (6.3)

We shall investigate the quantity
\[ J_p := \mathbb{E}|m_n(z) - s(z)|^p = \mathbb{E}|\Lambda_n|^p, \quad \text{for } p \geq 2. \]

We introduce the notation
\[ \varphi_p(z) = |z|^{p-2}. \]

In these terms we may represent \( J_p \) as
\[ J_p = \mathbb{E}\Lambda_n \varphi_p(\Lambda_n) \]
and expand this equality using the representation (6.3) arriving at
\[ J_p = - \frac{1}{n} \mathbb{E} \frac{m_n'(z) \varphi_p(\Lambda_n)}{z + s(z) + m_n(z)} + \frac{\hat{T}_n(z) \varphi_p(\Lambda_n)}{z + s(z) + m_n(z)}. \]

Denote by
\[ \mathcal{T}_1 = - \frac{1}{n} \mathbb{E} \frac{m_n'(z) \varphi_p(\Lambda_n)}{z + s(z) + m_n(z)}, \]
\[ \mathcal{T}_2 = \frac{\hat{T}_n \varphi_p(\Lambda_n)}{z + s(z) + m_n(z)}. \]

This is an approach similar to that used by us in the proof of Lemma 6.1 [10].

**6.1 Estimation of \( \mathcal{T}_1 \)**

Using Lemma 7.10 in the Appendix, we get
\[ |T_1| \leq \frac{C}{nv} \mathbb{E}|\varphi_p(\Lambda_n)| \leq \frac{1}{nv} \frac{p-1}{p} J_p^{p-1}. \] (6.4)
6.2 Estimation of $\mathfrak{F}_2$

The quantity $\mathfrak{F}_2$ we represent in the form

$$\mathfrak{F}_2 = \mathfrak{F}_{21} + \mathfrak{F}_{22} + \mathfrak{F}_{23},$$

where, for $\nu = 1, 2, 3$,

$$\mathfrak{F}_{2\nu} = -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j,\nu} R_{jj} \varphi(\Lambda)}{z + m_n(z) + s(z)}.$$

6.2.1 Estimation of $\mathfrak{F}_{21}$

We represent $\mathfrak{F}_{21}$ in the form

$$\mathfrak{F}_{21} = L_1 + L_2,$$

where

$$L_1 = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j,1} (1 + nz)^{-1} \varphi(\Lambda_n)}{z + m_n(z) + s(z)}$$

and

$$L_2 = -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j,1} (R_{jj} + 1 + n(z)) \varphi(\Lambda_n)}{z + m_n(z) + s(z)}.$$

We first consider the term $L_1$. Applying H"older’s inequality and Lemma 7.18 in the Appendix, we get

$$|L_1| \leq \frac{1}{\sqrt{|z^2 - 4|}} \frac{n}{E_z} \left| \frac{1}{n \sqrt{n}} \sum_{j=1}^{n} X_{jj} \right| \frac{1}{|z + m_n(z) + s(z)|} \mathbb{E}^{\frac{p-1}{p}} |\varphi(\Lambda_n)|^{\frac{p}{p-1}}$$

Using Corollary 5.3 and the inequality $|\varphi(\Lambda_n)|^{\frac{p}{p-1}} \leq |\Lambda_n|^p$, we get

$$|L_1| \leq \frac{C_0}{\sqrt{|z^2 - 4|}} E_z \frac{1}{n \sqrt{n}} \sum_{j=1}^{n} X_{jj} \left| \frac{1}{|z + m_n(z) + s(z)|} \right| \frac{1}{|z + m_n(z) + s(z)|} \mathbb{E}^{\frac{p-1}{p}} |\varphi(\Lambda_n)|^{\frac{p}{p-1}}.$$

Applying Rosenthal’s inequality to the sum $\frac{1}{n \sqrt{n}} \sum_{j=1}^{n} X_{jj}$, we obtain, for $z \in \mathbb{G}$

$$|L_1| \leq \frac{C_0 p}{n \sqrt{|z^2 - 4|}} \mathbb{E}^{\frac{p-1}{p}} |\Lambda_n|^p \leq \frac{C_0 p}{n \nu v} \mathbb{E}^{\frac{p-1}{p}} |\Lambda_n|^p. \quad (6.5)$$
Using the representation (5.4), we get
\[
L_2 = -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_j \varepsilon_j \varphi(\Lambda_n) R_{jj}}{(z + m_n(z))(z + m_n(z) + s(z))}. 
\]
This representation yields using \(\varepsilon_j \varepsilon_{j^\nu} \leq (\varepsilon_j^2 + \varepsilon_{j^\nu}^2)/2\),
\[
|L_2| \leq \frac{4}{n^2} \sum_{\mu=1}^{n} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\mu}|^2 |R_{jj}| |\varphi(\Lambda_n)|}{|z + m_n(z)||z + m_n(z) + s(z)|} =: \sum_{\mu=1}^{n} L_{2\mu}. 
\]
First we bound \(L_{2\mu}\) for \(\mu = 1\). By definition of \(\varepsilon_j\), we may write
\[
|L_{21}| \leq \frac{2}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|X_{jj}|^2 |R_{jj}| |\varphi(\Lambda_n)|}{|z + m_n(z)||z + m_n(z) + s(z)|}. 
\]
Applying Hölder’s inequality and Lemma 7.18 in the Appendix, we get
\[
|L_{21}| \leq \frac{2}{n^2 \sqrt{|z^2 - 4|}} \sum_{j=1}^{n} \mathbb{E} |X_{jj}|^{2p} |R_{jj}|^{2p} \mathbb{E}^{1/p} |\varphi(\Lambda_n)|^{p-1} \times \mathbb{E}^{1/p} |X_{jj}|^{p-1} |\varphi(\Lambda_n)|^{p-1}. 
\]
Furthermore, we observe that, \(\frac{2p}{p-1} \leq 4\) for \(p \geq 2\) and by Lemma 7.9 with \(\zeta = \varepsilon_{j4}\), we get
\[
\mathbb{E}^{p-1} |X_{jj}|^{p-1} |\varphi(\Lambda_n)|^{p-1} \leq \mathbb{E}^{p-1} |X_{jj}|^{p-1} |\Lambda_n|^p 
\leq \left( \mathbb{E}^{p-1} |X_{jj}|^{p-1} (e|\Lambda_n^{(j)}|^p + \frac{(p + 1)^p}{(nv)^p}) \right) 
\leq e \mu_2^{1/p} \mathbb{E}^{p-1} |\Lambda_n^{(j)}|^p + \frac{(p + 1)^{p-1}}{(nv)^{p-1}}. 
\]
Applying the inequality \((a + b)^p \leq ca^p + (p + 1)^p b^p\), we get
\[
\mathbb{E}|\varphi(\Lambda_n^{(j)})|^p \leq (e \mathbb{E}|\varphi(\Lambda_n)|^p + (p + 1)^p \mathbb{E}|\varphi(\Lambda_n) - \varphi(\Lambda_n^{(j)})|^p). 
\]
Using Corollary 5.3 and Lemmas 7.8 and 7.9 in the Appendix, we obtain
\[
|L_{21}| \leq \frac{Cp}{\sqrt{|z^2 - 4|}} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}^{p-1} |\Lambda_n^{(j)}|^p + \frac{(Cp)^{p-1}}{n^{p/p-1}}. 
\]
Applying that $\sqrt{|z^2 - 4|} \geq Cv$ for $z \in \mathbb{G}$, we get

$$|L_{21}| \leq \frac{Cp}{nv} \left( \frac{n}{p} \right)^{\frac{p}{p-1}} + \frac{(Cp)^{p-1}}{n^p v^{p-1}}. \quad (6.11)$$

Consider now $L_{2\mu}$ for $\mu = 2, 3$. Recall that

$$L_{2\mu} = \frac{2}{n} \sum_{j=1}^{n} \mathbf{E}_{\bar{Z}} |R_{jj}|^p \mathbf{E}^{\frac{1}{p-1}} \left( \frac{1}{|z + m_n(z)|^{\frac{p}{p-1}}} \right) |A_n|^p. \quad (6.12)$$

Using Hölder’s inequality, we may obtain, for $z \in \mathbb{G}$,

$$|L_{2\mu}| \leq \frac{2}{n} \sum_{j=1}^{n} \mathbf{E}_{\bar{Z}} |R_{jj}|^{2p} \mathbf{E}^{\frac{1}{p-1}} \left( \frac{1}{|z + m_n(z)|^{\frac{p}{p-1}}} \right) \mathbf{E}^{\frac{p-1}{p}} \left( \frac{|\varepsilon_{jj}|^{2p}}{|z + m_n(z) + s(z)|^{\frac{p}{p-1}}} \right) |A_n|^p. \quad (6.13)$$

Using Corollary 5.3, Lemma 7.9 with $\zeta = \varepsilon_{j4}$, and Lemma 7.23, we get

$$|L_{2\mu}| \leq \frac{C}{n} \sum_{j=1}^{n} \mathbf{E}_{\bar{Z}} \left( \frac{|\varepsilon_{jj}|^{2p}}{|z + m_n(z) + s(z)|^{\frac{p}{p-1}}} \right) |A_n|^p$$

$$+ \frac{(cp)^{p-1}}{(nv)^{p-1}n} \sum_{j=1}^{n} \mathbf{E}^{\frac{p-1}{p}} \left( \frac{|\varepsilon_{jj}|^{2p}}{|z + m_n(z) + s(z)|^{\frac{p}{p-1}}} \right). \quad (6.14)$$

Furthermore, we use inequality (7.73) in the Appendix. We get, for $z \in \mathbb{G}$,

$$|L_{2\mu}| \leq \frac{C}{n} \sum_{j=1}^{n} \mathbf{E}_{\bar{Z}} \left( \frac{|\varepsilon_{jj}|^{2p}}{|z + m_n^{(j)}(z) + s(z)|^{\frac{p}{p-1}}} \right) |A_n|^p$$

$$+ \frac{(cp)^{p-1}}{(nv)^{p-1}n} \sum_{j=1}^{n} \mathbf{E}^{\frac{p-1}{p}} \left( \frac{|\varepsilon_{jj}|^{2p}}{|z + m_n^{(j)}(z) + s(z)|^{\frac{p}{p-1}}} \right). \quad (6.15)$$

Conditioning on $\mathcal{M}^{(j)}$ and applying Hölder’s inequality, we get

$$|L_{2\mu}| \leq \frac{2}{n} \sum_{j=1}^{n} \mathbf{E}^{\frac{p}{p-1}} \left( \frac{1}{|z + m_n^{(j)}(z) + s(z)|^{\frac{p}{p-1}}} \right) |\mathcal{M}^{(j)}|^p$$

$$+ \frac{(cp)^{p-1}}{(nv)^{p-1}n} \sum_{j=1}^{n} \mathbf{E}^{\frac{p}{p-1}} \left( \frac{|\varepsilon_{jj}|^{4}}{|z + m_n^{(j)}(z) + s(z)|^2} \right). \quad (6.16)$$
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Inequality (6.16), and Corollary 7.22 together imply that, for \( z \in \mathbb{G} \) and for \( \mu = 2, 3 \),

\[
|L_{2\mu}| \leq \frac{Cp}{nv} J_p^{p-1} + \frac{(cp)^{p-1}}{(nv)^p}.
\]

(6.17)

Finally, we observe that

\[
|L_{24}| \leq \frac{Cp}{n^2p^2\sqrt{|z^2 - 4|}} J_p^{p-1} \leq \frac{Cp}{nv} J_p^{p-1}.
\]

Combining inequalities (6.6), (6.11), (6.17), we obtain

\[
|L_2| \leq \frac{Cp}{nv} J_p^{p-1} + \frac{(cp)^{p-1}}{(nv)^p}.
\]

(6.18)

Inequalities (6.18) and (6.5) together imply

\[
|\mathcal{T}_{21}| \leq \frac{Cp}{nv} J_p^{p-1} + \frac{Cp^{p-1}}{(nv)^p}.
\]

(6.19)

6.3 Estimation of \( \mathcal{T}_{2\nu} \), for \( \nu = 2, 3 \)

Recall that

\[
\mathcal{T}_{2\nu} = -\frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu} R_{jj} \varphi(\Lambda_n)}{z + m_n(z) + s(z)}.
\]

(6.20)

Introduce the quantity

\[
\eta_j = \frac{1}{n} \sum_{i,k \in \mathbb{T}_j} X_{ji} X_{jk} [(R^{(j)})^2]_{ki}.
\]

We have

\[
\varepsilon_{j4} = \frac{1}{n} (\text{Tr} R - \text{Tr} R^{(j)}) = \frac{1}{n} (1 + \eta_j) R_{jj}.
\]

See [11], Lemma 3.3. Denote by

\[
\tilde{\Lambda}_n^{(j)} = \frac{1}{n} \text{Tr} R^{(j)} + \frac{s(z)}{n}.
\]

Similarly as in the previous Section we represent \( \mathcal{T}_{2\nu} \) in the form

\[
\mathcal{T}_{2\nu} = M_1 + M_2 + M_3 + M_4,
\]
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where

\[ M_1 = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu}}{z + m_n^{(j)}(z)} \varphi(\Lambda_n^{(j)}) \]

\[ M_2 = -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu}}{z + m_n(z) + s(z)} (R_{jj} + \frac{1}{z + m_n^{(j)}(z)}) \varphi(\Lambda_n) \]

\[ M_3 = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu}}{z + m_n(z) + s(z)} (\varphi(\Lambda_n) - \varphi(\Lambda_n^{(j)})) \]

\[ M_4 = -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu}}{z + m_n^{(j)}(z) + s(z)} (\varphi(\Lambda_n^{(j)}) - \varphi(\Lambda_n)) \varepsilon_{j4} \]

(6.21)

Note that, by independence of \( X_{jk}, k \in \mathbb{T} \) and \( \mathfrak{M}^{(j)} \),

\[ M_1 = 0. \]  

(6.22)

Furthermore, we represent

\[ M_2 = M_{21} + M_{22} + M_{23}, \]

where, for \( \mu = 1, 2, 3 \)

\[ M_{2\mu} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu} \varepsilon_{j\mu}}{z + m_n^{(j)}(z)} R_{jj} \varphi(\Lambda_n) \]

Using the inequality \( |ab| \leq \frac{1}{2} (a^2 + b^2) \), we obtain, for \( \nu = 2, 3 \), and \( \mu = 1, 2, 3 \)

\[ |M_{2\mu}| \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{(|\varepsilon_{j\nu}|^2 + |\varepsilon_{j\mu}|^2) R_{jj} \varphi(\Lambda_n)}{(z + m_n^{(j)}(z))(z + s(z) + m_n(z))} \]

Similar to inequalities (6.15), (6.16), we get

\[ |M_2| \leq \frac{C_p j^{p-1}}{nv} + \frac{C_p p^{p-1}}{(nv)^p}. \]  

(6.23)
6.4 Estimation of $M_3$

Recall that

$$M_3 = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu} \frac{1}{z + m_n(j)(z)} (\varphi(\Lambda_n) - \varphi(\tilde{\Lambda}_n^{(j)}))}{z + m_n(z) + s(z)}.$$  

Let

$$\delta_j = \Lambda_n - \tilde{\Lambda}_n^{(j)} = \frac{1}{n} (R_{jj} - s(z)) + \frac{1}{n^2} \eta_{jj} R_{jj}. \quad (6.24)$$

Applying Taylor’s formula, we represent it in the form

$$M_3 = M_{31} + M_{32}, \quad (6.25)$$

where

$$M_{31} = \frac{1}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu} (R_{jj} - s(z)) \varphi'(\tilde{\Lambda}_n^{(j)} + \tau \delta_j)}{(z + m_n^{(j)}(z))(z + m_n(z) + s(z))},$$

$$M_{32} = \frac{1}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu} \eta_{jj} R_{jj} \varphi'(\tilde{\Lambda}_n^{(j)} + \tau \delta_j)}{(z + m_n(z) + s(z))(z + m_n^{(j)}(z))}. \quad (6.26)$$

6.4.1 Estimation of $M_{31}$

First we note that

$$R_{jj} - s(z) = \frac{\Lambda_n^{(j)} s(z)}{z + m_n^{(j)}(z)} + \sum_{\mu=1}^{3} \frac{\varepsilon_{j\mu}}{z + m_n^{(j)}(z)} R_{jj}. \quad (6.27)$$

We represent now $M_{31}$ in the form

$$M_{31} = G_1 + G_2 + G_3 + G_4, \quad (6.28)$$

where

$$G_1 = \frac{1}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{s(z) \varepsilon_{j\nu} \Lambda_n^{(j)} \varphi'\tilde{\Lambda}_n^{(j)} + \tau \delta_j}{(z + m_n^{(j)}(z))^2(z + m_n(z) + s(z))},$$

$$G_{\mu+1} = \frac{1}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu} \varepsilon_{j\mu} R_{jj} \varphi'\tilde{\Lambda}_n^{(j)} + \tau \delta_j}{(z + m_n^{(j)}(z))^2(z + m_n(z) + s(z))}, \quad \text{for } \mu = 1, 2, 3.$$
We continue with $G_1$, applying Lemma 7.9. We get

$$|G_1| \leq \frac{C_p}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n^{(j)}|\hat{\Lambda}_n^{(j)} + \tau \delta_j|^{p-2}}{|z + m_n^{(j)}(z)|^2|z + m_n(z) + s(z)|}.$$ 

Furthermore, we use inequality (7.73) and Lemma 7.8 in the Appendix. We get, for $z \in \mathbb{C}$,

$$|G_1| \leq \frac{C_p}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n^{(j)}|\hat{\Lambda}_n^{(j)}|^{p-2}}{|z + m_n^{(j)}(z)|^2|z + m_n^{(j)}(z) + s(z)|} + \frac{(C_p)^{p-2}}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n^{(j)}|\delta_j|^{p-2}}{|z + m_n^{(j)}(z)|^2|z + m_n^{(j)}(z) + s(z)|}.$$ 

This inequality and Lemma 7.9 together imply

$$|G_1| \leq \frac{C_p}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n^{(j)}|^{p-1}}{|z + m_n^{(j)}(z)|^2|z + m_n^{(j)}(z) + s(z)|} + \frac{C_p^{p-2}}{n^p} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n^{(j)}|}{|z + m_n^{(j)}(z)|^2|z + m_n^{(j)}(z) + s(z)|} + \frac{(C_p)^{p-2}}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n^{(j)}|\delta_j|^{p-2}}{|z + m_n^{(j)}(z)|^2|z + m_n^{(j)}(z) + s(z)|}. \tag{6.29}$$

Conditioning on $\mathbb{M}^{(j)}$ and using Lemmas 7.20, 7.21 and Lemma 7.24 in the Appendix, we obtain, for $z \in \mathbb{C}$ and for $\nu = 2, 3$,

$$|G_1| \leq \frac{C_p}{n\sqrt{nu}|z|^2} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n^{(j)}|^{p-1}}{|z + m_n^{(j)}(z)|^2|z + m_n^{(j)}(z) + s(z)|} + \frac{(C_p)^{p-2}}{(nu)^p} + \frac{(C_p)^{p-2}}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n^{(j)}|\delta_j|^{p-2}}{|z + m_n^{(j)}(z)|^2|z + m_n^{(j)}(z) + s(z)|}.$$ 

Without loss of generality we may assume that $p \geq 3$. Applying Lemmas 7.9 and 7.8 we get

$$|G_1| \leq \frac{C_p}{n\sqrt{nu}|z|^2} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n|^{p-1}}{|z + m_n(z)|^2|z + m_n(z) + s(z)|} + \frac{(C_p)^{p-2}}{(nu)^p} + \frac{(C_p)^{p-2}}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|\Lambda_n|\delta_j|^{p-2}}{|z + m_n(z)|^2|z + m_n(z) + s(z)|}. \tag{6.30}$$
According to definition (6.24)
\[ |\delta_j| \leq \frac{1}{n} |R_{jj} - s(z)| + \frac{1}{n} |\eta_j||R_{jj} - s(z)| + \frac{|s(z)|}{n}. \quad (6.31) \]

Inequality (6.31) and equality (6.27) together imply
\[ |\delta_j| \leq \frac{C|\Lambda_n|}{n|z + m_n(z)|} + \sum_{\mu=1}^{4} \frac{C(1 + |\eta_j|)|\varepsilon_{j\mu}|}{n|z + m_n(z)|}|R_{jj}| \]
\[ \leq \frac{C}{nv}|\Lambda^{(j)}_n| + \frac{C(1 + |\eta_j|)|\varepsilon_{j\mu}|}{(nv)^2} \sum_{\mu=1}^{4} \frac{|R_{jj}|}{n|z + m_n(z)|} \]
\[ \leq \frac{C}{nv} |\Lambda^{(j)}_n| + \frac{C(1 + |\eta_j|)|\varepsilon_{j\mu}|}{n^2|z + m_n(z)|}|R_{jj}|. \quad (6.32) \]

Inequalities (6.30), (6.27) and Lemma 7.9 and the inequality
\[ |\Lambda^{(j)}_n|^p \leq e|\Lambda^{(j)}_n|^p + \frac{C_p p^p}{n^p}, \]
yield, for \( z \in \mathbb{G} \),
\[ |G_1| \leq \frac{C_p}{nv n} E^{\frac{p-1}{p}} |\Lambda_n|^p + \frac{(Cp)^{p-2}}{(nv)^p} \]
\[ + \frac{1}{n^2} \sum_{j=1}^{n} E \frac{|\varepsilon_{j\mu}|^2 |\Lambda^{(j)}_n|^{p-1}}{|z + m_n^{(j)}(z)|^2 |z + m_n^{(j)}(z) + s(z)|(nv)^{p-2}} \]
\[ + \frac{4}{n^2} \sum_{\mu=1}^{4} \frac{C}{nv n} E \frac{(Cp)^{p-2} |\varepsilon_{j\mu}| (1 + |\eta_j|)|\varepsilon_{j\mu}|^{p-2} |R_{jj}|^{p-2}}{n^{p-2}|z + m_n(z)|^{p-2}}. \]

Conditioning on \( \mathcal{M}^{(j)} \) and using Lemmas 7.19, 7.20, 7.21 and 7.23 Corollary 7.22 we get, for \( z \in \mathbb{G} \),
\[ E \frac{|\varepsilon_{j\mu}| |\Lambda^{(j)}_n|^{p-1}}{|z + m_n^{(j)}(z)|^2 |z + m_n^{(j)}(z) + s(z)|(nv)^{p-2}} \]
\[ \leq \frac{1}{(nv)^{p-2} |z^2 - 4|^{\frac{3}{2}}} E \left( E^{\frac{4}{p}} \left\{ \left| \varepsilon_{j\mu} \right|^2 \right\} \right) \frac{|\Lambda^{(j)}_n|^p}{|\Lambda^{(j)}_n|^{p-1}} \]
\[ \leq \frac{1}{(nv)^{p-2} |z^2 - 4|^{\frac{3}{2}}} E^{\frac{p-1}{p}} |\Lambda_n|^p + \frac{(Cp)^p}{(nv)^{2p-\frac{3}{2}} |z^2 - 4|^{\frac{3}{4}}}. \quad (6.33) \]

Similarly, applying Hölder’s inequality, we get
\[ E \frac{1 + |\eta_j|)^{p-2} |\varepsilon_{j\mu}|^{p-1}}{n^{p-2}|z + m_n(z)|^{p-2}} |R_{jj}|^{p-2} \]
\[ \leq E^{\frac{1}{2}} |\varepsilon_{j\mu}|^{4(p-1)} E^{\frac{1}{2}} |R_{jj}|^{4(p-2)} E^{\frac{1}{2}} \frac{1}{|z + m_n^{(j)}|^{4(p-2)}} E^{\frac{1}{2}} |1 + \eta_j|^{4(p-2)}. \quad (6.34) \]
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Using condition (1.2), Lemmas 5.1, 7.23, Rosenthal’s and Burkholder’s inequalities and Corollary 5.3, we conclude that for $p \geq 3$

\[ \mathbb{E}^\frac{1}{4} |\varepsilon_{j\mu}|^{4(p-1)} \leq \frac{(Cp)^p}{n} \]  

(6.35)

and

\[ \mathbb{E}^\frac{1}{4} |1 + \eta_j|^{4(p-2)} \leq Cp. \]  

(6.36)

These inequalities and Corollary 5.3 together imply

\[ |G_1| \leq \frac{Cp}{nu} \mathbb{E}^\frac{1}{p} |\Lambda_n|^p + \frac{(Cp)^{p-2}}{(nu)^p}. \]  

(6.37)

To bound $G_{1+\mu}$, for $\mu = 1, 2, 3$, we use Lemma 7.8. We get

\[ |G_{1+\mu}| \leq \frac{Cp}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_{j\mu}| |\tilde{\Lambda}_n^{(j)}|^{p-2}}{z + m_n^{(j)}(z) |z + m_n^{(j)}(z) + s(z)|}. \]

Furthermore, we use inequality (7.73) and Lemma 7.8 in the Appendix. We get, for $z \in \mathbb{G}$,

\[ |G_{1+\mu}| \leq \frac{Cp}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_{j\mu}| |\tilde{\Lambda}_n^{(j)}|^{p-2}}{z + m_n^{(j)}(z) |z + m_n^{(j)}(z) + s(z)|} + \frac{(Cp)^{p-2}}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_{j\mu}| |\delta_j|^{p-2}}{z + m_n^{(j)}(z) |z + m_n^{(j)}(z) + s(z)|}. \]

This inequality and Lemma 7.9 together imply

\[ |G_{1+\mu}| \leq \frac{Cp}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_{j\mu}| |\tilde{\Lambda}_n^{(j)}|^{p-2}}{z + m_n^{(j)}(z) |z + m_n^{(j)}(z) + s(z)|} + \frac{(Cp)^{p-2}}{n^p} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_{j\mu}| |\delta_j|^{p-2}}{z + m_n^{(j)}(z) |z + m_n^{(j)}(z) + s(z)|} + \frac{(Cp)^{p-2}}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_{j\mu}| |\delta_j|^{p-2}}{z + m_n^{(j)}(z) |z + m_n^{(j)}(z) + s(z)|}. \]  

(6.38)

To estimate the first sum on the right hand side of (6.38), we used conditioning on $\mathfrak{M}^{(j)}$, then Corollaries 7.22 and 5.3 and Lemma 7.9. The estimations of the second and third sums on the right hand side of (6.38)
are similar to (6.34). Similarly to (6.37) we get from the last inequality, for \( \mu = 1, 2, 3 \),
\[
|G_{1+\mu}| \leq \frac{C_p}{nv} E^{\frac{p-1}{p}} |\Lambda_n|^p + \frac{(C_p)^{p-2}}{(nv)^p}.
\]
(6.39)

Combining inequalities (6.37) and (6.39), we get
\[
|M_{31}| \leq \frac{C_p}{nv} E^{\frac{p-1}{p}} |\Lambda_n|^p + \frac{(C_p)^{p-2}}{(nv)^p}.
\]
(6.40)

6.4.2 Estimation of \( M_{32} \)

Recall that
\[
M_{32} = \frac{1}{n^2} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu} \varepsilon_{j\mu} R_{jj} \varphi'(\tilde{\Lambda}_n^{(j)}) + \tau \delta_j}{(z + m_n(z) + s(z))(z + m_n^{(j)}(z))}.
\]

Using inequality (7.73) and the definition of \( \varphi \), we get
\[
|M_{32}| \leq \frac{p}{n^2} \sum_{j=1}^{n} E \frac{||\varepsilon_{j\nu}|| ||\varepsilon_{j\mu}|| ||\tilde{\Lambda}_n^{(j)}||^{p-2}}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|}.
\]

Applying now Lemma 7.9, we obtain
\[
|M_{32}| \leq \frac{p}{n^2} \sum_{j=1}^{n} E \frac{||\varepsilon_{j\nu}|| ||\varepsilon_{j\mu}|| ||\tilde{\Lambda}_n^{(j)}||^{p-2}}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|} + \frac{(C_p)^{p-1}}{n^2} \sum_{j=1}^{n} E \frac{||\varepsilon_{j\nu}|| ||\varepsilon_{j\mu}|| ||\delta_j||^{p-2}}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|}.
\]

By Lemmas 7.19, 7.20, 7.21, we have
\[
E\{||\varepsilon_{j\nu}|| ||\varepsilon_{j\mu}|| ||\tilde{\Lambda}_n^{(j)}||^{p-2}\} \leq \frac{C \text{Im} m_n^{(j)}}{(nv)}.
\]

We get, for \( z \in \mathbb{C} \),
\[
|M_{32}| \leq \frac{C_p}{nv} \frac{1}{n} \sum_{j=1}^{n} E^{\frac{p-2}{p}} |\tilde{\Lambda}_n^{(j)}|^p

+ \frac{(C_p)^{p-1}}{n^2} \sum_{j=1}^{n} E \frac{||\varepsilon_{j\nu}|| ||\varepsilon_{j\mu}|| ||\delta_j||^{p-2}}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|}.
\]
(6.41)
Inequalities (6.41) and (6.32) yield

\[ |M_{32}| \leq \frac{C_p}{n(nv)} \frac{1}{n} \sum_{j=1}^{n} E^{p-2} |\Lambda_n^{(j)}|^p \]

\[ + \frac{(C_p)^{p-1}}{n^p} \sum_{j=1}^{n} E |\varepsilon_{j\nu}||\varepsilon_{j\mu}||\Lambda_n| \]

\[ + \frac{4}{n^p} \sum_{j=1}^{n} E \frac{|\varepsilon_{j\nu}||\varepsilon_{j\mu}||\varepsilon_{j\varphi}||\Lambda_n|}{|z + m_n^{(j)}(z) + s(z)(z + m_n^{(j)}(z))(z + m_n(z))|^{p-2}} \]

(6.42)

Applying Lemma 7.9 to estimate the first sum on the right hand side of (6.42), Corollary 7.22 to estimate the second one and inequalities (6.35) and (6.36), we get

\[ |M_{32}| \leq \frac{C_p}{n(nv)} \frac{1}{n} \sum_{j=1}^{n} E^{p-1} |\Lambda_n| \]

\[ + \frac{(C_p)^{p-1}}{n^p} \sum_{j=1}^{n} E |\varepsilon_{j\nu}||\varepsilon_{j\mu}||\varepsilon_{j\varphi}| R_{jj}^{p-2} \]

(6.43)

The representation (6.25) and inequalities (6.40) and (6.43) together imply

\[ |M_3| \leq \frac{C_p}{n(nv)} \frac{1}{n} \sum_{j=1}^{n} E^{p-1} |\Lambda_n| \]

\[ + \frac{(C_p)^{p-1}}{n^p} \sum_{j=1}^{n} E |\varepsilon_{j\nu}||\varepsilon_{j\mu}||\varepsilon_{j\varphi}| R_{jj}^{p-2} \]

(6.43)

6.5 Estimation of \( M_4 \)

Recall that

\[ M_4 = -\frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu} \frac{1}{z + m_n^{(j)}(z)} \varphi(\Lambda_n^{(j)}) \varepsilon_{j4}}{(z + m_n^{(j)}(z) + s(z))(z + m_n(z) + s(z))} \]

Consider the following moments

\[ \gamma_n := E \{ |\varepsilon_{j\nu}||\varepsilon_{j\mu}| \mathcal{M}^{(j)} \} \]

By Cauchy – Schwartz inequality we have

\[ |\gamma_n| \leq E^{\frac{1}{2}} \{ |\varepsilon_{j\nu}|^2 \mathcal{M}^{(j)} \} E^{\frac{1}{2}} \{ |\varepsilon_{j4}|^2 \mathcal{M}^{(j)} \} \]

By Lemmas 7.20 and 7.21, we obtain

\[ E^{\frac{1}{2}} \{ |\varepsilon_{j\nu}|^2 \mathcal{M}^{(j)} \} \leq C(nv)^{-\frac{1}{2}} \text{Im} \frac{1}{2} m_n^{(j)}(z). \]
Furthermore,
\[ \mathbb{E}\{|\varepsilon_{j4}|^2|\mathcal{M}^{(j)}\} = \frac{1}{n^2} \mathbb{E}\{(1 + |\eta_j|^2)|R_{jj}|^2|\mathcal{M}^{(j)}\}. \]

It is straightforward to check that
\[ \mathbb{E}\{|\varepsilon_{j4}|^2|\mathcal{M}^{(j)}\} \leq \left( \frac{1}{n^2} + \frac{C}{n^3v^3} \text{Im} m_n^{(j)}(z) \right) \mathbb{E}\left\{ |R_{jj}|^2|\mathcal{M}^{(j)}\right\}. \tag{6.45} \]

The inequalities (6.44) and (6.45) together imply
\[ |\gamma_n| \leq \frac{C \text{Im} \frac{1}{2} m_n^{(j)}(z)}{n^2 \sqrt{nv}} + \frac{C \text{Im} m_n^{(j)}(z)}{(nv)^{3/2}} \mathbb{E}\left\{ |R_{jj}|^2|\mathcal{M}^{(j)}\right\}. \tag{6.46} \]

Using inequality (7.73) and conditioning on \( M^{(j)} \), we may write, for \( z \in \mathbb{G} \)
\[ |M_4| \leq \frac{C}{n} \sum_{j=1}^{n} \mathbb{E}\{|\gamma_n| \sum_{j=1}^{n} \frac{1}{|z + m_n^{(j)}(z)|} |\varphi(\tilde{\Lambda}_n^{(j)})|}. \]

Applying now inequality (6.46), we get
\[ |M_4| \leq \frac{1}{n \sqrt{nv}|z^2 - 4|^2} \sum_{j=1}^{n} \mathbb{E}\{|\gamma_n| \sum_{j=1}^{n} \frac{1}{|z + m_n^{(j)}(z)|} |\varphi(\tilde{\Lambda}_n^{(j)})|}. \tag{6.47} \]

Applying Hölder’s inequality and Corollary 5.3, we obtain
\[ |M_4| \leq \frac{C p}{nv} J_p^{p - 1} + \frac{(C p)^{p - 1}}{(nv)^p}. \tag{6.48} \]

Combining now inequalities (6.23), (6.22), (6.23), (6.25), (6.48), we get
\[ |\Theta_2| \leq \frac{C p}{nv} \mathbb{E}\left\{ \frac{1}{p - 1} |\Lambda_n|^{p - 1} \right\}. \tag{6.48} \]

Together with (6.4) we get
\[ |J_p| \leq \frac{C p}{nv} J_p^{p - 1} + \frac{(C p)^{p - 1}}{(nv)^p} + \frac{C p}{nv} J_p^{\frac{p-1}{p}}. \tag{6.49} \]
Since \( \frac{C_p}{nv} < c < 1 \), we conclude
\[
|J_p| \leq \frac{C_p}{nv} J_p^{-1} + \frac{(C_p)^{p-1}}{(nv)^p}.
\]
Using Lemma 7.6 in the Appendix, we get, for \( z \in G \),
\[
|J_p| \leq \frac{(C_p)^{p-1}}{(nv)^p}.
\]
Thus Theorem 1.2 is proved.

7 Appendix

7.1 Rosenthal’s and Burkholder’s Inequalities

In this subsection we state the Rosenthal and Burkholder inequalities starting with Rosenthal’s inequality. Let \( \xi_1, \ldots, \xi_n \) be independent random variables with \( E \xi_j = 0 \), \( E \xi_j^2 = 1 \) and for \( p \geq 1 \) \( E|\xi_j|^p \leq \mu_p \) for \( j = 1, \ldots, n \).

Lemma 7.1. (Rosenthal’s inequality) There exists an absolute constant \( C_1 \) such that
\[
E|\sum_{j=1}^n a_j \xi_j|^p \leq C_1^p \left( \sum_{j=1}^p |a_j|^2 \right)^{\frac{p}{2}} + \mu_p \sum_{j=1}^p |a_j|^p.
\]
Proof. For the proof of this inequality see [20] and [17].

Let \( \xi_1, \ldots, \xi_n \) be martingale-difference with respect to \( \sigma \)-algebras \( M_j = \sigma(\xi_1, \ldots, \xi_{j-1}) \). Assume that \( E \xi_j^2 = 1 \) and \( E|\xi_j|^p \leq \infty \).

Lemma 7.2. (Burkholder’s inequality) There exist an absolute constant \( C_2 \) such that
\[
E|\sum_{j=1}^n \xi_j|^p \leq C_2^p \left( E\left( \sum_{k=1}^n E\xi_k^2 |M_{k-1}\right) \right)^{\frac{p}{2}} + \sum_{k=1}^n E|\xi_k|^p.
\]
Proof. For the proof of this inequality see [5] and [16].

We rewrite the Burkholder inequality for quadratic forms in independent random variables. Let \( \zeta_1, \ldots, \zeta_n \) be independent random variables such that \( E \zeta_j = 0 \), \( E|\zeta_j|^2 = 1 \) and \( E|\zeta_j|^p \leq \mu_p \). Let \( a_{ij} = a_{ji} \) for all \( i, j = 1, \ldots, n \). Consider the quadratic form
\[
Q = \sum_{1 \leq j \neq k \leq n} a_{jk} \zeta_j \zeta_k.
\]
Lemma 7.3. There exists an absolute constant \( C_2 \) such that
\[
E|Q|^p \leq C_2^p \left( E\left( \sum_{j=2}^{n} \left( \sum_{k=1}^{j-1} a_{jk} \zeta_k \right)^2 \right)^{\frac{p}{2}} + \mu_p \sum_{j=2}^{n} E|\sum_{k=1}^{j-1} a_{jk} \zeta_k|^p \right). \tag{7.1}
\]

Proof. Introduce the random variables
\[
\xi_j = \zeta_j \sum_{k=1}^{j-1} a_{jk} \zeta_k, \quad j = 2, \ldots, n.
\]
It is straightforward to check that
\[
E\{\xi_j | \mathcal{M}_{j-1}\} = 0,
\]
and that \( \xi_j \) are \( \mathcal{M}_j \) measurable. Hence \( \xi_1, \ldots, \xi_n \) are martingale-differences. We may write
\[
Q = 2 \sum_{j=2}^{n} \xi_j
\]
Applying now Lemma 7.2 and using
\[
E\{|\xi_j|^2 | \mathcal{M}_{j-1}\} = (\sum_{k=1}^{j-1} a_{jk} \eta_k)^2 E\zeta_j^2,
\]
\[
E|\xi_j|^p = E|\eta_j|^p E|\sum_{k=1}^{j-1} a_{jk} \zeta_j|^p,
\]
we get the claim. Thus, Lemma 7.3 is proved.


Lemma 7.4. Assuming the conditions of Theorem 1.1 there exists a positive constant \( C = C(\mu_4, D) \), depending on \( \mu_4 \) and \( D \) such that, for any \( 1 \leq q \leq C \log n \),
\[
E\left( \frac{1}{n} \sum_{j=1}^{n} X_{jj}^2 \right)^q \leq C^q.
\]

Proof. Applying the triangle inequality, we get
\[
E\left( \frac{1}{n} \sum_{j=1}^{n} X_{jj}^2 \right)^q \leq 2^q \left( 1 + \frac{1}{n^q} E\left| \sum_{j=1}^{n} (X_{jj}^2 - 1) \right|^q \right).
\]
Using now Rosenthal’s inequality, we get
\[
E\left(\frac{1}{n}\sum_{j=1}^{n} X_{jj}^2\right)^q \leq 2^q(1 + \frac{1}{n^q}(C_1^q q^q n^{\frac{q}{2}} + n \max_{jj} E|X_{jj}|^{2q})).
\]

According to condition (1.2), we have
\[
E\left(\frac{1}{n}\sum_{j=1}^{n} X_{jj}^2\right)^q \leq 2^q(1 + (C_1^q q^q n^{-\frac{q}{2}} + D^{2q-4} n^{-\frac{q}{2}} \mu_4)).
\]}

\[\square\]

**Corollary 7.5.** Under the condition of Theorem 1.1 there exists a positive constant \( C = C(\mu_4, D) \), depending on \( \mu_4 \) and \( D \) such that, for any \( 1 \leq q \leq C \log n \),
\[
E\left(\frac{1}{n}\sum_{j=1}^{n} |\varepsilon_{j1}|^2\right)^q \leq \frac{C^q}{n^q}.
\]

**Proof.** The result immediately follows from the definition
\[
\varepsilon_{j1} = \frac{1}{\sqrt{n}} X_{jj},
\]
and Lemma 7.4. \[\square\]

The next Lemma describes the behavior of the moments of \( \varphi(\Lambda_n) \). Recall that
\[
\Lambda_n = m_n(z) - s(z), \quad \Lambda_n^{(j)} = m_n^{(j)} - s(z), \quad \varepsilon_{j4} = \Lambda_n - \Lambda_n^{(j)},
\]
and
\[
\varphi(z) = \pi |z|^{p-2}.
\]

First we prove

**Lemma 7.6.** Let \( t > r \geq 1 \) and \( a, b > 0 \). Any \( x > 0 \) satisfying the inequality
\[
x^t \leq a + bx^r \quad (7.2)
\]
is explicitly bounded as follows
\[
x^t \leq ea + \left(\frac{2t - r}{t - r}\right)^{\frac{t}{r}} b^{\frac{1}{t-r}}. \quad (7.3)
\]
Proof. First assume that \( x \leq a^\frac{1}{t} \). Then inequality (7.3) holds. If \( x \geq a^\frac{1}{t} \), then according to inequality (7.2)
\[
x^{t-r} \leq a^\frac{t-r}{t} + b,
\]
(7.4)
or
\[
x^t \leq (a \frac{t-r}{t} + b)^{\frac{1}{t-r}}.
\]
(7.5)
Using that for any \( \alpha > 0 \) and \( a > 0, b > 0 \)
\[
(a + b)^\alpha \leq (a + \frac{a}{\alpha})^\alpha + (b + ab)^\alpha \leq ea^\alpha + (1 + \alpha)^\alpha b^\alpha,
\]
(7.6)
we get the claim. \qed

**Corollary 7.7.** Assume that for \( a, b, c, x > 0 \) the following inequality holds
\[
x^t \leq a + bx^{t-1} + cx^{t-2}.
\]
Then
\[
x^t \leq e^2a + e\left(1 + \frac{t}{2}\right)\frac{x}{c^2} + t^e b' .
\]

*Proof.* We apply Lemma 7.6 with \( a' = a + bx^{p-1} \), \( b' = c \) and \( r = t - 2 \) and obtain
\[
x^t \leq ea + ebx^{t-1} + \left(1 + \frac{t}{2}\right)\frac{x}{c^2} .
\]
Using Lemma 7.6 again with \( a'' = ea + (1 + \frac{t}{2})\frac{x}{c^2} \), \( b'' = eb \) and \( r = t - 1 \), we get
\[
x^t \leq e^2a + e\left(1 + \frac{t}{2}\right)\frac{x}{c^2} + t^e b' .
\]
\qed

**Lemma 7.8.** Recall that \( \varepsilon_{j4} = \Lambda_n - \Lambda^{(j)}_n \). Then
\[
|\varphi(\Lambda_n) - \varphi(\Lambda^{(j)}_n)| \leq p|\varepsilon_{j4}|E_r|\Lambda_n - \tau\varepsilon_{j4}|^{p-2} ,
\]
where \( \tau \) denotes a random variable which is uniformly distributed on \([0,1]\) and independent of all \( X_{jk} \), for \( j, k = 1, \ldots, n \).
Proof. For \( x \in [0, 1] \) define the function,

\[
\tilde{\varphi}(x) = \varphi(\Lambda_n - x\varepsilon j).
\]

It is easy to see that \( \tilde{\varphi}(0) = \varphi(\Lambda_n) \), \( \tilde{\varphi}(1) = \varphi(\Lambda_n^{(j)}) \). By Taylor’s formula we have

\[
\varphi(\Lambda_n) - \varphi(\Lambda_n^{(j)}) = -\varepsilon j E_r \tilde{\varphi}'(\Lambda - \tau \varepsilon j).
\]

It is straightforward to check that

\[
|\tilde{\varphi}'(x)| \leq p|\Lambda_n - x\varepsilon j|^{p-2}.
\]

Lemma 7.9. With the notations of Lemma 7.8 we have for any \( q \geq 1 \) and for all \( \zeta \in \mathbb{C} \)

\[
|\Lambda_n - \tau \zeta|^q \leq (q + 1)^q|\zeta|^q + e|\Lambda_n|^q.
\]

Proof. We observe that

\[
|\Lambda_n - \tau \zeta|^q \leq |\Lambda_n - \tau \zeta|^q\{|\Lambda| \leq q|\zeta|\} + |\Lambda_n - \tau \varepsilon j|^q\{|\Lambda_n| \leq q|\zeta|\}.
\]

From here we conclude

\[
|\Lambda_n - \tau \zeta|^q \leq (q + 1)^q|\zeta|^q + (1 + \frac{1}{q})^q|\Lambda_n|^q \leq (q + 1)^q|\zeta|^q + e|\Lambda_n|^q.
\]

Thus Lemma 7.9 is proved.

7.2 Auxiliary Inequalities for Resolvent Matrices

We shall use the following relation between resolvent matrices. Let \( \mathbb{A} \) and \( \mathbb{B} \) be two Hermitian matrices and let \( R_A = (\mathbb{A} - zI)^{-1} \) and \( R_B = (\mathbb{B} - zI)^{-1} \) denote their resolvent matrices. Recall the resolvent equality

\[
R_A - R_B = R_A(B - A)R_B = -R_B(B - A)R_A. \tag{7.7}
\]

Recall the equation, for \( j \in \mathbb{T}_J \), and \( J \subset \mathbb{T} \) (compare with (5.3))

\[
R^{(j)}_{jj} = -\frac{1}{z + m^{(j)}_n(z)} + \frac{1}{z + m^{(j)}_n(z)} \varepsilon^{(j)}_j R^{(j)}_{jj}, \tag{7.8}
\]
where
\begin{align*}
\varepsilon^{(j)}_{j1} &= \frac{X_{jj}}{\sqrt{n}}, \quad \varepsilon^{(j)}_{j2} = \frac{1}{n} \sum_{l \neq k \in T_{j,j}} X_{jl} X_{jk} R^{(j,j)}_{kl}, \\
\varepsilon^{(j)}_j &= \frac{1}{n} \sum_{l \in T_{j,j}} (X_{jl}^2 - 1) R^{(j,j)}_{ll}, \quad \varepsilon^{(j)}_{j4} = m_n^{(j)}(z) - m_n^{(j,j)}(z). \quad (7.9)
\end{align*}

Summing these equations for \( j \in T_j \), we get
\begin{equation}
m_n^{(j)}(z) = -\frac{n - |J|}{n(z + m_n^{(j)}(z))} + \frac{T_n^{(j)}}{z + m_n^{(j)}(z)}, \quad (7.10)
\end{equation}

where
\begin{equation}
T_n^{(j)} = \frac{1}{n} \sum_{j=1}^n \varepsilon^{(j)}_j R^{(j)}_{jj}. \quad (7.11)
\end{equation}

Note that
\begin{equation}
\frac{1}{z + m_n^{(j)}(z)} = \frac{1}{z + s(z)} - \frac{m_n^{(j)}(z) - s(z)}{(s(z) + z)(z + m_n^{(j)}(z))} = -s(z) + \frac{s(z) \Lambda_n^{(j)}(z)}{z + m_n^{(j)}(z)}, \quad (7.12)
\end{equation}

where
\begin{equation}
\Lambda_n^{(j)} = m_n^{(j)}(z) - s(z). \quad (7.13)
\end{equation}

Equalities (7.10) and (7.12) together imply
\begin{equation}
\Lambda_n^{(j)} = -\frac{s(z) \Lambda_n^{(j)}}{z + m_n^{(j)}(z)} + \frac{T_n^{(j)}}{z + m_n^{(j)}(z)} + \frac{|J|}{n(z + m_n^{(j)}(z))}. \quad (7.14)
\end{equation}

Solving this with respect to \( \Lambda_n^{(j)} \), we get
\begin{equation}
\Lambda_n^{(j)} = \frac{T_n^{(j)}}{z + m_n^{(j)}(z) + s(z)} + \frac{|J|}{n(z + m_n^{(j)}(z) + s(z))}. \quad (7.15)
\end{equation}

Lemma 7.10. For any \( z = u + iv \) with \( v > 0 \) and for any \( J \subset T \), we have
\begin{equation}
\frac{1}{n} \sum_{l,k \in T_J} |R_{kl}^{(j)}|^2 \leq v^{-1} \text{Im} m_n^{(j)}(z). \quad (7.16)
\end{equation}

For any \( l \in T_j \)
\begin{equation}
\sum_{k \in T_j} |R_{kl}^{(j)}|^2 \leq v^{-1} \text{Im} R_{ll}^{(j)}. \quad (7.17)\]
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and

\[ \sum_{k \in T_J} |[(R^{(J)})^2]_{kl}|^2 \leq v^{-3} \text{Im} R^{(J)}_{ll}. \] (7.18)

Moreover, for any \( J \subset T \) and for any \( l \in T_J \) we have

\[ \frac{1}{n} \sum_{l \in T_J} |[(R^{(J)})^2]_{ll}|^2 \leq v^{-3} \text{Im} n^{(J)}_n(z), \] (7.19)

and, for any \( p \geq 1 \)

\[ \frac{1}{n} \sum_{l \in T_J} |[(R^{(J)})^2]_{ll}|^p \leq v^{-p} \frac{1}{n} \sum_{l \in T_J} \text{Im}^p R^{(J)}_{ll}. \] (7.20)

Finally,

\[ \frac{1}{n} \sum_{l, k \in T_J} |[(R^{(J)})^2]_{lk}|^2 \leq v^{-3} \text{Im} m_n^{(J)}(z), \] (7.21)

and

\[ \frac{1}{n} \sum_{l, k \in T_J} |[(R^{(J)})^2]_{lk}|^p \leq v^{-3p} \frac{1}{n} \sum_{l \in T_J} \text{Im}^p R^{(J)}_{ll}. \] (7.22)

We have as well

\[ \frac{1}{n^2} \sum_{l, k \in T_J} |[(R^{(J)})^2]_{lk}|^p \leq v^{-2p} (\frac{1}{n} \sum_{l \in T_J} \text{Im}^p R^{(J)}_{ll})^2. \] (7.23)

**Proof.** For \( l \in T_J \) let us denote by \( \lambda_l^{(J)} \) for \( l \in T_J \) the eigenvalues of the matrix \( W^{(J)} \). Then we may write (compare (7.28))

\[ \frac{1}{n} \sum_{l, k \in T_J} |R_{kl}^{(J)}|^2 \leq \frac{1}{n} \sum_{l \in T_J} \frac{1}{|\lambda_l^{(J)} - z|^2}. \] (7.24)

Note that, for any \( x \in \mathbb{R}^1 \)

\[ \text{Im} \frac{1}{x - z} = \frac{v}{|x - z|^2}. \] (7.25)

We may write

\[ \frac{1}{|\lambda_l^{(J)} - z|^2} = v^{-1} \text{Im} \frac{1}{\lambda_l^{(J)} - z}. \] (7.26)
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and

\[
\frac{1}{n} \sum_{l,k \in T_J} |R_{kl}^{(J)}|^2 \leq v^{-1} \text{Im} \left( \frac{1}{n} \sum_{l \in T_J} \frac{1}{\lambda^{(J)}_l - z} \right) = v^{-1} \text{Im} m_n^{(J)}(z). \tag{7.27}
\]

Thus inequality (7.16) is proved. Let denote now by \( u^{(J)}_l = (u^{(J)}_{lk})_{k \in T_J} \) the eigenvector of the matrix \( W^{(J)} \) corresponding to the eigenvalue \( \lambda^{(J)}_l \). Using this notation we may write

\[
R_{kl}^{(J)} = \sum_{q \in T_J} \frac{1}{\lambda^{(J)}_q - z} u^{(J)}_{lq} u^{(J)}_{kq}. \tag{7.28}
\]

It is straightforward to check that the following inequality holds

\[
\sum_{k \in T_J} |R_{kl}^{(J)}|^2 \leq \sum_{q \in T_J} \frac{1}{|\lambda^{(J)}_q - z|^2} |u^{(J)}_{lq}|^2 = v^{-1} \text{Im} \left( \sum_{q \in T_J} \frac{1}{\lambda^{(J)}_q - z} |u^{(J)}_{lq}|^2 \right) = v^{-1} \text{Im} R_{ll}^{(J)}. \tag{7.29}
\]

Thus, inequality (7.17) is proved. Similarly we get

\[
\sum_{k \in T_J} |(R^{(J)})^2_{kl}| \leq \sum_{q \in T_J} \frac{1}{|\lambda^{(J)}_q - z|^4} |u^{(J)}_{lq}|^2 \leq v^{-3} \text{Im} R_{ll}^{(J)}. \tag{7.30}
\]

This proves inequality (7.18). To prove inequality (7.19) we observe that

\[
||(R^{(J)})^2||_2 \leq \sum_{k \in T_J} |R_{ik}^{(J)}|^2. \tag{7.31}
\]

This inequality implies

\[
\frac{1}{n} \sum_{l \in T_J} ||(R^{(J)})^2||_2 \leq \frac{1}{n} \sum_{l \in T_J} \left( \sum_{k \in T_J} |R_{lk}^{(J)}|^2 \right)^2. \tag{7.32}
\]

Applying now inequality (7.17), we get

\[
\frac{1}{n} \sum_{l \in T_J} ||(R^{(J)})^2||_2 \leq v^{-2} \frac{1}{n} \sum_{l \in T_J} \text{Im}^2 R_{ll}^{(J)}. \tag{7.33}
\]

Using \( |R_{ll}^{(J)}| \leq v^{-1} \) this leads to the following bound

\[
\frac{1}{n} \sum_{l \in T_J} ||(R^{(J)})^2||_2 \leq v^{-3} \frac{1}{n} \sum_{l \in T_J} \text{Im} R_{ll}^{(J)} = v^{-3} \text{Im} m_n^{(J)}(z). \tag{7.34}
\]
Thus inequality (7.19) is proved. Furthermore, applying inequality (7.31), we may write
\[ \frac{1}{n} \sum_{l \in T_J} \left| \left| (R(J)^2)_{ll} \right| \right|^4 \leq \frac{1}{n} \sum_{l \in T_J} (\sum_{k \in T_J} |R_{lk}^{(j)}|^2)^4. \]  
(7.35)

Applying (7.17), this inequality yields
\[ \frac{1}{n} \sum_{l \in T_J} \left| \left| (R(J)^2)_{ll} \right| \right|^4 \leq v^{-4} \frac{1}{n} \sum_{l \in T_J} \text{Im}^4 R_{ll}^{(j)}. \]  
(7.36)

The last inequality proves inequality (7.20). Note that
\[ \frac{1}{n} \sum_{l,k \in T_J} \left| \left| (R(J)^2)_{lk} \right| \right|^4 \leq \frac{1}{n} \text{Tr} \left| R(J) \right|^4 = \frac{1}{n} \sum_{l \in T_J} \frac{1}{|\lambda_l^{(j)}| - z}|^4 \leq v^{-3} \text{Im} \frac{1}{n} \sum_{l \in T_J} \frac{1}{|\lambda_l^{(j)}| - z} = v^{-3} \text{Im} m_n^{(j)}(z). \]  
(7.37)

Thus, inequality (7.21) is proved. To finish we note that
\[ \frac{1}{n} \sum_{l,k \in T_J} \left| \left| (R(J)^2)_{lk} \right| \right|^2 \leq \frac{1}{n} \sum_{l \in T_J} (\sum_{q \in T_J} |R_{lq}^{(j)}|^2)^2. \]  
(7.38)

Applying inequality (7.18), we get
\[ \frac{1}{n} \sum_{l,k \in T_J} \left| \left| (R(J)^2)_{lk} \right| \right|^4 \leq v^{-6} \frac{1}{n} \sum_{l \in T_J} (\text{Im} R_{ll}^{(j)})^2. \]  
(7.39)

To prove inequality (7.23), we note
\[ \left| \left| (R(J)^2)_{lk} \right| \right|^2 \leq (\sum_{q \in T_J} |R_{lq}^{(j)}|^2)(\sum_{q \in T_J} |R_{kq}^{(j)}|^2). \]  
(7.40)

This inequality implies
\[ \frac{1}{n^2} \sum_{l,k \in T_J} \left| \left| (R(J)^2)_{lk} \right| \right|^{2p} \leq \left( \frac{1}{n} \sum_{l \in T_J} (\sum_{q \in T_J} |R_{lq}^{(j)}|^2)^2 \right)^p (\text{Im} R_{ll}^{(j)})^2. \]  
(7.41)

Applying inequality (7.16), we get the claim. Thus, Lemma 7.10 is proved.

**Lemma 7.11.** Assuming the conditions of Theorem 1.1, we get
\[ E|\varepsilon_j|^2 \leq \frac{C}{n}. \]

**Proof.** The proof follows immediately from the definition of \( \varepsilon_j \) and the conditions of Theorem 1.1. \( \Box \)
Some Auxiliary Bounds for Resolvent Matrices for $z = u + iV$ with $V = 4$

We shall use the bound for the $\varepsilon_{j\nu}$, and $\eta_j$ for $V = 4$.

**Lemma 7.12.** Assuming the conditions of Theorem 1.1, we get

$$E|\varepsilon_j|^q \leq \frac{C^q n}{n^2}.$$ 

**Proof.** Conditioning on $\mathcal{M}(j)$ and applying Burkholder’s inequality (see Lemma 7.3), we get

$$E|\varepsilon_j|^q \leq C^q n^q \sum_{k \in T_j} \left( \sum_{l=1}^{k-1} R_{kl}^j X_{jl} \right)^2 + \mu q \sum_{k \in T_j} E \left( \sum_{l=1}^{k-1} R_{kl}^j X_{jl} \right)^q.$$ 

Applying now Corollary 5.2 and Rosenthal’s inequality, we get

$$E|\varepsilon_j|^q \leq C^q n^q \sum_{k \in T_j} \left( \sum_{l \in T_j} R_{kl}^j \right)^q + \mu^q \sum_{k, l \in T_j} |R_{kl}^j|^q.$$ 

Using that $|R_{kl}^j| \leq \frac{1}{4}$ and $\sum_{l \in T_j} |R_{kl}^j|^2 \leq \frac{1}{16}$ and $\mu q \leq D \frac{n}{2} \mu_4$, we get

$$E|\varepsilon_j|^q \leq C^q n^q n^{-\frac{q}{2}}.$$ 

Thus Lemma 7.12 is proved.

**Lemma 7.13.** Assuming the conditions of Theorem 1.1, we get

$$E|\varepsilon_j|^q \leq \frac{C^q n}{n^2}.$$ 

**Proof.** Conditioning and applying Rosenthal’s inequality, we obtain

$$E|\varepsilon_j|^q \leq C^q n^q \sum_{l \in T_j} \left( \sum_{k \in T_j} |R_{kl}^j|^2 \right)^{\frac{q}{2}} + \mu^q \sum_{l \in T_j} |R_{kl}^j|^q.$$ 

Using that $|R_{kl}^j| \leq \frac{1}{4}$ and $\mu^q \leq D \frac{2q}{4} n^{-1} \mu_4$, we get

$$E|\varepsilon_j|^q \leq C^q n^q n^{-\frac{q}{2}}.$$ 

Thus Lemma 7.13 is proved.
Lemma 7.14. Assuming the conditions of Theorem 1.1, we get
\[ \mathbb{E}|\eta_j|^q \leq \frac{C^q n^q}{n^2}. \]

Proof. The proof is similar to proof of Lemma 7.12. We need to use that
\[ ||(R^{(j)})^2|_{kl}|| \leq V^{-2} = \frac{1}{16} \text{ and } \sum_{l \in T_j} ||(R^{(j)})^2|_{kl}||^2 \leq V^{-4}. \]

Lemma 7.15. Assuming the conditions of Theorem 1.1, we get, for any \( q \geq 1 \),
\[ |\epsilon_j|^q \leq \frac{C^q}{n^q}. \]

Proof. The result follows immediately from the bound
\[ |\epsilon_j| \leq \frac{1}{nv}, \text{ a. s.} \]

See for instance [10], Lemma 3.3. \( \square \)

Now we investigate the behavior of \( R_{jj} - s(z) \) for \( z = u + iV \) with \( V = 4 \).

Lemma 7.16. Assuming the conditions of Theorem 1.1, we get,
\[ \mathbb{E}|R_{jj} - s(z)|^4 \leq Cn^{-2}. \]

Proof. By equality [5.3] we have
\[ \mathbb{E}|R_{jj} - s(z)|^4 \leq C(\mathbb{E}|\Lambda_n|^4 + \sum_{\nu=1}^4 \mathbb{E}|\epsilon_{j\nu}|^4). \]

By equation (7.15), for \( V = 4 \),
\[ \mathbb{E}|\Lambda_n|^4 \leq C\mathbb{E}|T_n|^4 \leq \frac{C}{n} \sum_{l=1}^n \mathbb{E}|\varepsilon_l|^4. \]

Direct calculations show that
\[ \mathbb{E}|\varepsilon_{j2}|^4 \leq C\mu_4 n^{-2} \mathbb{E}(\frac{1}{n} \sum_{l,k=1}^n |R^{(j)}|_{lk}|^2)^2 \leq \frac{C}{n^2}. \]

Similarly we get
\[ \mathbb{E}|\varepsilon_{j3}|^4 \leq C\mu_4 n^{-2} \mathbb{E}(\frac{1}{n} \sum_{l \in T_j} |R^{(j)}|_{ll})^2 + C\mu_4 n^{-2} \frac{1}{n} \sum_{l \in T_j} \mathbb{E}|R^{(j)}|_{ll}^4 \leq Cn^{-2}. \]

Finally, by Lemma 7.15 we have
\[ \mathbb{E}|\varepsilon_{j4}|^4 \leq Cn^{-4}. \]

Combining these inequalities we get the claim. Thus Lemma 7.16 is proved. \( \square \)
7.3 Some Auxiliary Bounds for Resolvent Matrices for $z \in \mathbb{G}$

Introduce now the region

$$\mathbb{G} := \{z = u + iv \in \mathbb{C}^+: u \in J_\varepsilon, v \geq v_0/\sqrt{\gamma}\}, \quad \text{where } v_0 = A_0 n^{-1},$$

$$J_\varepsilon = [-2 + \varepsilon, 2 - \varepsilon], \quad \varepsilon := c_1 n^{-\frac{4}{3}}, \quad \gamma = \gamma(u) = \min\{2 - u, 2 + u\}.$$  

(7.42)

In the next lemma we some simple inequalities for the region $\mathbb{G}$

**Lemma 7.17.** For any $z \in \mathbb{G}$ we have

$$|z^2 - 4| \geq 2 \max\{\gamma, v\}, \quad nv\sqrt{|z^2 - 4|} \geq 2A_0.$$  

(7.43)

**Proof.** We observe that

$$|z^2 - 4| = |z - 2||z + 2| \geq 2\sqrt{\gamma^2 + v^2}.$$  

(7.44)

This inequality proves the Lemma.

**Lemma 7.18.** Assuming the conditions of Theorem 1.1, there exists an absolute constant $c_0 > 0$ such that for any $J \subset \mathbb{T}$,

$$|z + m_n^{(J)}(z) + s(z)| \geq \text{Im}\, m_n^{(J)}(z),$$

(7.45)

moreover, for $z \in \mathbb{G}$,

$$|z + m_n^{(J)}(z) + s(z)| \geq c_0 \sqrt{|z^2 - 4|}.$$  

(7.46)

**Proof.** First we note

$$|z + m_n^{(J)}(z) + s(z)| \geq \text{Im} \, (z + s(z)) \geq \frac{1}{2} \text{Im} \, \sqrt{z^2 - 4}.$$  

(7.47)

Furthermore, it is simple to check that, for $z = u + iv$ with $v > 0$

$$\text{Im} \, \sqrt{z^2 - 4} \geq \frac{\sqrt{2}}{2} \sqrt{|z^2 - 4|}.$$  

(7.48)

Thus Lemma 7.18 is proved.

**Lemma 7.19.** Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$ such that for any $j = 1, \ldots, n$,

$$\mathbb{E}\{|z_j|^{-1} |\text{Re}^{(j)}|\} \leq \frac{C \mu_4}{n^2}.$$  

(7.49)
Proof. The result follows immediately from the definition of \( \varepsilon_{j1} \).

Lemma 7.20. Assuming the conditions of Theorem 7.1, there exists an absolute constant \( C > 0 \) such that for any \( j = 1, \ldots, n \),

\[
E\{|\varepsilon_{j2}|^2|\mathcal{M}^{(j)}\} \leq \frac{C}{n^2} \text{Im}m_n^{(j)}(z), \tag{7.50}
\]

and

\[
E\{|\varepsilon_{j2}|^4|\mathcal{M}^{(j)}\} \leq \frac{C\mu_4^2}{n^2v^2} \text{Im}^2m_n^{(j)}(z). \tag{7.51}
\]

Proof. Note that r.v.’s \( X_{jl} \), for \( l \in T_j \) are independent of \( M^{(j)} \) and that for \( l, k \in T_j \) \( R_{lk}^{(j)} \) are measurable with respect to \( \mathcal{M}^{(j)} \). This implies that \( \varepsilon_{j2} \) is a quadratic form with coefficients \( R_{lk}^{(j)} \) independent of \( X_{jl} \). Thus its variance and fourth moment are easily available.

\[
E\{|\varepsilon_{j2}|^2|\mathcal{M}^{(j)}\} = \frac{1}{n^2} \sum_{l \neq k \in T_j} |R_{lk}^{(j)}|^2 \leq \frac{1}{n^2} \text{Tr}|R^{(j)}|^2, \tag{7.52}
\]

Here we use the notation \( |A|^2 = AA^* \) for any matrix \( A \). Applying Lemma 7.10, inequality (7.16), we get equality (7.50).

Furthermore, direct calculations show that

\[
E\{|\varepsilon_{j2}|^4|\mathcal{M}^{(j)}\} \leq \frac{C}{n^2} \left( \frac{1}{n} \sum_{l \neq k \in T_j} |R_{lk}^{(j)}|^2 \right)^2 + \frac{C\mu_4^2}{n^2} \frac{1}{n^2} \sum_{l \in T_j} |R_{ll}^{(j)}|^4
\]

\[
\leq \frac{C\mu_4^2}{n^2} \left( \frac{1}{n} \sum_{l \neq k \in T_j} |R_{lk}^{(j)}|^2 \right)^2 \leq \frac{C\mu_4^2}{n^2v^2} (\text{Im}m_n^{(j)}(z))^2. \tag{7.53}
\]

Here again we used Lemma 7.10 inequality (7.16). Thus Lemma 7.20 is proved.

Lemma 7.21. Assuming the conditions of Theorem 7.1, there exists an absolute constant \( C > 0 \) such that for any \( j = 1, \ldots, n \),

\[
E\{|\varepsilon_{j3}|^2|\mathcal{M}^{(j)}\} \leq \frac{C\mu_4}{n} \sum_{l \in T_j} |R_{ll}^{(j)}|^2, \tag{7.54}
\]

and

\[
E\{|\varepsilon_{j3}|^4|\mathcal{M}^{(j)}\} \leq \frac{C\mu_4}{n^2} \left( \frac{1}{n} \sum_{l \in T_j} |R_{ll}^{(j)}|^2 \right)^2 + \frac{C\mu_4}{n^2} \frac{1}{n} \sum_{l \in T_j} |R_{ll}^{(j)}|^4. \tag{7.55}
\]
Proof. The first inequality is obvious. To prove the second inequality, we apply Rosenthal’s inequality. We obtain

\[
E(|\varepsilon_j|^4 |\mathcal{M}^{(j)}|) \leq \frac{C\mu_4}{n^2} \left( \frac{1}{n} \sum_{l \in \mathcal{S}_j} |R^{(j)}_{il}|^2 \right)^2 + \frac{C\mu_8}{n^3} \frac{1}{n} \sum_{l \in \mathcal{T}_j} |R^{(j)}_{il}|^4. 
\]  

(7.56)

Using \(|X_{jl}| \leq Cn^{1/2}\) we get \(\mu_8 \leq Cn\mu_4\) and the claim. Thus Lemma 7.21 is proved.

Corollary 7.22. Assuming the conditions of Theorem 1.1, there exists an absolute constant \(C > 0\), depending on \(\mu_4\) and \(D\) only, such that for any \(j = 1, \ldots, n\), \(\nu = 1, 2, 3\), \(z \in \mathbb{G}\), and \(1 \leq \alpha \leq \frac{1}{2}A_1(n\nu)^{1/4}\)

\[
E \frac{|\varepsilon_{j\nu}|^2}{|z + m^{(j)}_n(z) + s(z)||z + m^{(j)}_n(z)|^\alpha} \leq \frac{C}{nv}.
\]  

(7.57)

and

\[
E \frac{|\varepsilon_{j\nu}|^4}{|z + m^{(j)}_n(z) + s(z)||z + m^{(j)}_n(z)||z + m^{(j)}_n(z)|^\alpha} \leq \frac{C}{n^2v^2}.
\]  

(7.58)

Proof. For \(\nu = 1\), by Lemma 7.18 we have

\[
E \frac{|\varepsilon_{j\nu}|^2}{|z + m^{(j)}_n(z) + s(z)||z + m^{(j)}_n(z)|^\alpha} \leq \frac{1}{n} \frac{E|X_{jj}|^4E|e_{\nu}|^4}{E|e_{\nu}|^4 E|X_{jj}|^4}. 
\]  

(7.59)

Applying now Corollary 5.3, we get the claim. The proof of the second inequality for \(\nu = 1\) is similar. For \(\nu = 2\) we apply Lemma 7.20 inequality (7.50) and obtain, using that \(\text{Im} m^{(j)}_n(z) \leq |z + m^{(j)}_n(z) + s(z)|\), (see (7.45)),

\[
E \frac{|\varepsilon_{j2}|^2}{|z + m^{(j)}_n(z) + s(z)||z + m^{(j)}_n(z)|^\alpha} \leq \frac{1}{nv} \frac{E|\text{Im} m^{(j)}_n(z)|}{|z + m^{(j)}_n(z) + s(z)||z + m^{(j)}_n(z)|^\alpha} \leq \frac{C}{nv} \frac{1}{|z + m^{(j)}_n(z)|^\alpha}. 
\]  

(7.60)

Similarly, using Lemma 7.20 inequality (7.51), we get

\[
E \frac{|\varepsilon_{j2}|^4}{|z + m^{(j)}_n(z) + s(z)||z + m^{(j)}_n(z)||z + m^{(j)}_n(z)|^\alpha} \leq \frac{C}{n^2v^2} \frac{E|\text{Im}^2 m^{(j)}_n(z)|}{|z + m^{(j)}_n(z) + s(z)||z + m^{(j)}_n(z)|^\alpha} \leq \frac{C}{n^2v^2} \frac{1}{|z + m^{(j)}_n(z)|^\alpha}. 
\]  

(7.61)
Applying Corollary 5.3, we get the claim. For \( \nu = 3 \), we apply Lemma 7.21, inequalities (7.54) and (7.55) and Lemma 7.18. We get

\[
E \left| \varepsilon_{j3} \right|^2 \left| z + m_n^{(j)}(z) + s(z) \right|^2 \left| z + m_n^{(j)}(z) \right|^\alpha \leq \frac{C}{n \sqrt{|z^2 - 4|}} E \frac{1}{|z + m_n^{(j)}(z)|^\alpha} \left( \frac{1}{n} \sum_{l \in \mathcal{T}_j} |R_{ll}^{(j)}|^2 \right),
\]  
(7.62)

and

\[
E \left| \varepsilon_{j3} \right|^4 \left| z + m_n^{(j)}(z) + s(z) \right|^2 \left| z + m_n^{(j)}(z) \right|^\alpha \leq \frac{C}{n^2 |z^2 - 4|} E \frac{1}{|z + m_n^{(j)}(z)|^\alpha} \left( \frac{1}{n} \sum_{l \in \mathcal{T}_j} |R_{ll}^{(j)}|^4 \right).
\]  
(7.63)

Using now the Cauchy – Schwartz inequality and Corollary 5.3, we get the claim.

**Lemma 7.23.** Assuming the conditions of Theorem 1.1, there exists an absolute constant \( C > 0 \) such that for any \( j = 1, \ldots, n \),

\[
\left| \varepsilon_{j4} \right| \leq \frac{C}{nv} \quad \text{a.s.}
\]  
(7.64)

**Proof.** This inequality follows from

\[
\text{Tr} \mathbf{R} - \text{Tr} \mathbf{R}^{(j)} = (1 + \frac{1}{n} \sum_{l,k \in \mathcal{T}_j} X_{jl}X_{jk}[(R^{(j)})^2]_{kl})R_{jj} = R_{jj}^{-1} \frac{dR_{jj}}{dz},
\]  
(7.65)

which may be obtained using the Schur complement formula. For details see, for instance [10], Lemma 3.3.

**Lemma 7.24.** Assuming the conditions of Theorem 1.1, we have, for \( z \in \mathcal{G} \),

\[
E \left| \Lambda_n \right|^2 \leq \frac{C}{nv|z^2 - 4|^2}.
\]  
(7.66)
Proof. We write
\[ E|\Lambda_n|^2 = E\Lambda_n \overline{\Lambda}_n = \frac{T_n}{z + m_n(z) + s(z)} \overline{\Lambda}_n = \sum_{\nu=1}^4 E\frac{T_{n\nu}}{z + m_n(z) + s(z)} \overline{\Lambda}_n, \]
where
\[ T_{n\nu} := \frac{1}{n} \sum_{j=1}^n \varepsilon_{j\nu} R_{jj}, \text{ for } \nu = 1, \ldots, 4. \]

Applying the Cauchy – Schwartz inequality, we get
\[ E\frac{|\Lambda_n|^2}{|z + m_n(z) + s(z)|^2} \leq \sum_{\nu=1}^4 E\frac{|T_{n\nu}|^2}{|z + m_n(z) + s(z)|^2}. \]

First we observe that by (7.65)
\[ |T_{n4}| = \frac{1}{n} |m'_n(z)| \leq \frac{1}{nv} \text{Im} m_n(z). \]

Hence \(|z + m_n(z) + s(z)| \geq \text{Im} m_n(z)\) and Jensen’s inequality yields
\[ E\frac{|T_{n4}|^2}{|z + m_n(z) + s(z)|^2} \leq \frac{1}{n^2 v^2}. \]

Furthermore, we observe that,
\[ \frac{1}{|z + s(z) + m_n(z)|} \leq \frac{1}{|z + s(z) + m_n^{(j)}(z)|} \left(1 + \frac{|\varepsilon_{j4}|}{|z + s(z) + m_n(z)|}\right). \]

Therefore, by Lemmas 7.23 and 7.17 for \(z \in \mathbb{G},\)
\[ \frac{1}{|z + s(z) + m_n(z)|} \leq \frac{C}{|z + s(z) + m_n^{(j)}(z)|}. \]

Applying inequality (7.73), we may write
\[ E\frac{|T_{n\nu}|^2}{|z + m_n(z) + s(z)|^2} \leq \frac{1}{n} \sum_{j=1}^n E\frac{|\varepsilon_{j\nu}|^2 |R_{jj}|^2}{|z + s(z) + m_n^{(j)}(z)|^2}. \]
Applying Cauchy – Schwartz inequality and Lemma 7.18 we get
\[ E \frac{|T_{\nu}|^2}{|z + m_n(z) + s(z)|^2} \leq C \frac{\sum_{j=1}^{n} E_{\frac{1}{2}} \frac{|\varepsilon_{j\nu}|^4}{|z + m_n^{(j)}(z) + s(z)|^2}}{n|z^2 - 4|^2} E_{\frac{1}{2}} |R_{jj}|^4. \]  
(7.75)

Using now Corollary 7.22 inequality (7.58) and Corollary 5.3, we get for \( \nu = 1, 2, 3 \)
\[ E \frac{|T_{\nu}|^2}{|z + m_n(z) + s(z)|^2} \leq C \frac{\sum_{j=1}^{n} E_{\frac{1}{2}} \frac{|\varepsilon_{j\nu}|^4}{|z + m_n^{(j)}(z) + s(z)|^2}}{n|z^2 - 4|^2}. \]  
(7.76)

Inequalities (7.69), (7.71) and (7.76) together complete the proof. Thus Lemma 7.24 is proved.

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