NUMERICAL APPROXIMATION FOR NON-COLLIDING PARTICLE SYSTEMS

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Abstract. We apply the semi-discrete method, c.f. N. Halidias and I.S. Stamatiou (2016), On the numerical solution of some non-linear stochastic differential equations using the semi-discrete method, Computational Methods in Applied Mathematics, 16(1), to a class of non-colliding particle systems. The proposed numerical scheme preserves the non-colliding property and strongly converges to the exact solution.

1. Introduction

We are interested in the following system of stochastic differential equations (SDEs),

(1.1) \( X_t^{(i)} = X_0^{(i)} + \int_0^t \left( \sum_{i \neq j} \frac{\gamma_{i,j}}{X_s^{(i)} - X_s^{(j)}} + b^i(X_s^{(i)}) \right) ds + \sum_{j=1}^d \int_0^t \sigma_{i,j} dW_s^{(j)}, \quad i = 1, \ldots, d, \)

where \( X_0 = (X_0^{(1)}, \ldots, X_0^{(d)})^T \in \Delta_d = \{ x = (x^{(1)}, x^{(2)}, \ldots, x^{(d)})^T \in \mathbb{R}^d : x^{(1)} < x^{(2)} < \ldots < x^{(d)} \} \) almost surely (a.s.) and \( \{W_t\}_{t \geq 0} \) is a d-dimensional Wiener process adapted to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). The constants \( \gamma_{i,j} \) satisfy \( \gamma_{i,j} = \gamma_{j,i} \geq 0 \) with \( \gamma_{i,i+1} > 0 \) for \( i = 1, \ldots, d-1 \) and the functions \( b^i(\cdot) \) are globally Lipschitz continuous or non-increasing with the property \( b^i(z) \leq b^{i+1}(z) \) for all \( z \in \mathbb{R} \) and \( \sigma_{i,j} \) are finite constants such that \( \sigma^2 := \sup_{1 \leq i \leq d} \sum_{k=1}^d \sigma_{i,k}^2 \leq 2\gamma_{i,j} \). Under the above assumptions the system (1.1) has a unique strong solution in \( \Delta_d \), c.f. [1], [2]. We want to reproduce the non-colliding property of (1.1). We use a fixed-time step explicit numerical method \( (Y_{n+1}^{i+1})_{n \in \mathbb{N}} \), namely the semi-discrete method, to approximate the difference \( X_{n+1}^{i+1} := X_{t_{n+1}}^{(i+1)} - X_{t_n}^{(i)} \), which reads

(1.2) \( Y_{n+1}^{i+1} = \sqrt{(\tilde{Y}_{n+1}^{i+1})^2 + 4\gamma_{i+1,i}\Delta}, \quad n \in \mathbb{N}, \)

where

\[
\tilde{Y}_{n+1}^{i+1} = e^{\alpha^i(\tilde{Y}_n)}\Delta \left( \tilde{Y}_n^{i+1,i} - \frac{\beta^i(\tilde{Y}_n)}{\alpha^i(\tilde{Y}_n)} (1 - e^{-\alpha^i(\tilde{Y}_n)\Delta}) + \sqrt{\sum_{k=1}^d (\sigma_{i+1,k} - \sigma_{i,k})^2 e^{-\alpha^i(\tilde{Y}_n)\Delta} \Delta D_{n,k}^{i+1}} \right)
\]

and

\[
\alpha^i(Y_n) := \frac{b^{i+1}(Y_n^{(i+1)}) - b^i(Y_n^{(i)})}{Y_n^{(i+1)} - Y_n^{(i)}} = \sum_{k \neq i,i+1} \frac{\gamma_{i+1,i+1}}{Y_{n+1,k}^{i+1,k} Y_{n}^{i+1,k,i}}, \quad \beta^i(Y_n) := \sum_{k \neq i,i+1} \frac{\gamma_{i+1,k} - \gamma_{i+1,i}}{Y_{n+1,k}^{i+1,k} Y_{n}^{i+1,k,i}},
\]

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with $Y_0 = X_0; \Delta = t_{n+1} - t_n$ is the time step-size and $\Delta B_{i+1}^n := B^{i+1}_{t_{n+1}} - B^{i+1}_{t_n}$ are the increments of a Wiener process. For the derivation of (1.2) see Section 3.

Our main goal is to provide an explicit numerical method for the approximation of the solution of system (1.1), which preserves the non-colliding property. The class of systems of SDEs of type (1.1) has been studied thoroughly, see among others in [3] [4], due to its practical importance: in the context of particle systems the term $\sum_{i\neq j} \frac{\gamma_{i,j}}{X^{(i)} - X^{(j)}}$ describes the repulsive force with which each of the $j$-th particle (located at $X^{(j)}$) acts on the $i$-th particle (located at $X^{(i)}$); still few results on their numerical approximations are available: the explicit tamed Euler scheme proposed in [5] for the case of the Dyson Brownian motion, c.f. [6] ($\gamma_{i,j} = \gamma, \sigma_{i,j} = \delta_{i,j} \sigma_i$ where $\delta_{i,j}$ is the Dirac delta function) fails to preserve the non-colliding property; the only method we know of which preserves the non-colliding property is a semi-implicit EM method recently proposed by [2].

The non-colliding property of (1.1) is rephrased in the positivity property of the process $(X^{i+1,n})$; our method preserves this property by construction.

Moreover, we study the strong rate of convergence in $L^p$-norm of the proposed method.

The case of numerical approximations of scalar SDEs with boundaries (like CIR process, mean-reverting CEV process, Wright-Fisher model) has been studied by many researchers, c.f. [7], [8], [9], [10], [11], [12].

The proposed fixed-step method is explicit, strongly convergent, non-explosive and preserves the non-colliding property. The semi-discrete method was originally proposed in [13] and further investigated in [14], [15] for one-dimensional SDEs and [16], [17] in the multivariate case. The basic ingredient of the semi-discrete method is the following: we freeze on each subinterval appropriate parts of the drift and diffusion coefficients of the solution at the beginning of the subinterval in order to obtain explicitly solved SDEs. Apparently the way of freezing (discretization) is not unique. Here, we freeze the nonlinear parts obtaining a linear SDE with explicit solution.

The outline of the article is the following. In Section 2 we present our main results, that is Theorem 2.2 and Corollary 2.3, the proof of which are deferred to Section 3.

2. MAIN RESULTS

Let us restate the assumptions of the model (1.1).

Assumption 2.1

i) $X_0 \in \Delta_d$ a.s.

ii) $\gamma_{i,j} = \gamma_{j,i} \geq 0$ with $\gamma_{i,i+1} > 0$ for $i = 1, \ldots, d - 1$.

iii) $b(\cdot)$ are globally Lipschitz continuous with the property $b'(z) \leq b^{i+1}(z)$ for all $z \in \mathbb{R}$.

iv) $\sigma_{i,j}$ are finite constants such that $\sigma^2 := \sup_{1 \leq i \leq d} \sum_{k=1}^d \sigma_{i,k}^2 \leq 2\gamma_{i,j}$.

Consider the process $X_{i+1}^{i+1,i} := X_{i+1}^{(i+1)} - X_i^{(i)}, i = 0, \ldots, d - 1$, where $X_0^{(i)} \equiv 0$, which satisfies the following SDE

\[
X_{i+1}^{i+1,i} = X_0^{i+1,i} + \int_0^t \left( \frac{2\gamma_{i+1,i}}{X_{i+1}^{i+1,i}} - \sum_{k \neq i,i+1} \frac{\gamma_{i,k} X_{i+1}^{i+1,k}}{X_{i+1}^{i+1,i} X_{i}^{i+1,k}} + b^{i+1}(X_{i}^{(i+1)}) - b'(X_{i}^{(i)}) \right) ds
\]

\[+ \sum_{j=1}^d \int_0^t (\sigma_{i+1,j} - \sigma_{i,j}) dW_s^{(j)}. \tag{2.3} \]
The non-colliding property implies that \((X_t^{i+1, j})\) is positive. In order to find the solution process, we use a splitting technique to get first, see also [12],

\[
\tilde{X}_t^{i+1, j} = \tilde{X}_0^{i+1, j} + \int_0^t \left( - \sum_{k \neq i, i+1} \frac{\gamma_{i,k} \tilde{X}_s^{i+1, k} - \gamma_{i+1,k} \tilde{X}_s^{i,k}}{\tilde{X}_s^{i+1, k} \tilde{X}_s^{i,k}} + b^{i+1} (\tilde{X}_s^{i+1}) - b^i (\tilde{X}_s^i) \right) ds + \sum_{j=1}^d \int_0^t (\sigma_{i+1,j} - \sigma_{i,j}) dW_s^{(j)},
\]

(2.4)

and afterwards

\[
X_t^{i+1, j} = \tilde{X}_t^{i+1, j} + \int_0^t \frac{2\gamma_{i+1,i}}{\tilde{X}_s^{i+1,i}} ds = \sqrt{(X_t^{i+1, j})^2 + 4\gamma_{i+1,i}t}.
\]

(2.5)

Therefore instead of approximating directly the solution of (2.3) we work with SDE (2.4). Let \(\tilde{Y}_t^{i+1, j}\) be a fixed-step numerical approximation of \(\tilde{X}_t^{i+1, j}\) with time step size \(\Delta\) and \(\tilde{Y}_0 = \tilde{X}_0\) such that \(\mathbb{E} \sup_{0 \leq t \leq T} |\tilde{Y}_t^{i+1, j} - \tilde{X}_t^{i+1, j}|^2 \leq C\Delta^{2p}\) where \(p\) is the order of strong convergence. Then

\[
|Y_t^{i+1, j} - X_t^{i+1, j}|^2 = \left| \sqrt{(Y_t^{i+1, j})^2 + 4\gamma_{i+1,i}t} - \sqrt{(X_t^{i+1, j})^2 + 4\gamma_{i+1,i}t} \right|^2
= \left| (Y_t^{i+1, j})^2 - (X_t^{i+1, j})^2 \right|
\leq (|Y_t^{i+1, j}| + |X_t^{i+1, j}|)|Y_t^{i+1, j} - X_t^{i+1, j}|,
\]

(2.6)

which implies \(\mathbb{E} \sup_{0 \leq t \leq T} |Y_t^{i+1, j} - X_t^{i+1, j}|^2 \leq C\Delta^p\), provided that the first moments of \((\tilde{Y}_t^{i+1, j})\) and \((\tilde{X}_t^{i+1, j})\) are bounded. Now, we discuss about the approximation \((\tilde{Y}_t^{i+1, j})\) of \((\tilde{X}_t^{i+1, j})\). We propose an application of the semi-discrete method in the following way.

Let the equidistant partition \(0 = t_0 < t_1 < \ldots < t_N = T\) with step size \(\Delta = T/N\) and consider the following process

\[
\tilde{Y}_t^{i+1, j} = \tilde{Y}_0^{i+1, j} + \int_0^t \left( b^{i+1}(\tilde{Y}_s^{i+1}) - b^i(\tilde{Y}_s^i) - \sum_{k \neq i, i+1} \frac{\gamma_{i,k} \tilde{Y}_s^{i+1,k} - \gamma_{i+1,k} \tilde{Y}_s^{i,k}}{\tilde{Y}_s^{i+1,k} \tilde{Y}_s^{i,k}} \right) \tilde{Y}_s^{i+1, j} ds + \int_0^t \left( - \sum_{k \neq i, i+1} \frac{\gamma_{i,k} - \gamma_{i+1,k}}{\tilde{Y}_s^{i+1,k}} \right) ds + \sum_{j=1}^d \int_0^t (\sigma_{i+1,j} - \sigma_{i,j}) dW_s^{(j)}
\]

(2.7)

\[
= \tilde{Y}_0^{i+1, j} + \int_0^t \left( \alpha^i(\tilde{Y}_s^i) \tilde{Y}_s^{i+1, j} - \beta^i(\tilde{Y}_s^i) \right) ds + c^i \int_0^t dB_s^{(i+1)},
\]

\[
\tilde{Y}_t^{i+1, j} = \tilde{Y}_0^{i+1, j} + \int_0^t \left( \alpha^i(\tilde{Y}_s^i) \tilde{Y}_s^{i+1, j} - \beta^i(\tilde{Y}_s^i) \right) ds + c^i \int_0^t dB_s^{(i+1)},
\]
where $B_t^{(i+1)} := \sum_{j=1}^{d} \int_0^t \frac{(\sigma_{i+1,j} - \gamma_{i,j})}{\sqrt{\sum_{k=1}^{d}(\sigma_{i+1,k} - \gamma_{i,k})^2}} dW_s^{(j)}$ is a new Wiener process,

\begin{align}
\alpha^i(Y) &= \frac{b^{i+1}(Y^{(i+1)}) - b^i(Y^{(i)})}{Y_{i+1,i}} - \sum_{k \neq i,i+1} \frac{\gamma_{i,k}}{Y_{i+1,k}}, \\
\beta^i(Y) &= \sum_{k \neq i,i+1} \frac{\gamma_{i,k} - \gamma_{i+1,k}}{Y_{i+1,k}}, \\
\nu^i &= \sqrt{\sum_{k=1}^{d}(\sigma_{i+1,k} - \gamma_{i,k})^2}
\end{align}

and $\tilde{Y}_0^{i+1,i} = \tilde{X}_0^{i+1,i}$ a.s. Here $\tilde{s} = t_n$ when $s \in [t_n, t_{n+1})$ and denotes the freezing times; in particular we keep the diffusion part of (2.4) the same and freeze the drift in such a way that the produced SDE (2.7) is linear in the narrow sense with additive noise and unique strong solution given by (c.f. [18, Sec 4.4.])

\begin{align}
\tilde{Y}_t^{i+1,i} = e^{\alpha^i(\tilde{Y}_s)}t \left( \tilde{Y}_0^{i+1,i} - \frac{\beta^i(\tilde{Y}_s)}{\alpha^i(\tilde{Y}_s)}(1 - e^{-\alpha^i(\tilde{Y}_s)t}) + \nu^i \int_0^t e^{-\alpha^i(\tilde{Y}_s)s} dB_s^{(i+1)} \right).
\end{align}

Note that the drift of (2.7) equals the drift of (2.4) for $\tilde{s} = s$. Our main result is the following.

**Theorem 2.2** [Strong convergence of $\tilde{Y}_t^{i+1,i}$ to $\tilde{X}_t^{i+1,i}$] Let Assumption 2.1 hold, $(\nu^i)^2 \geq (d - 1) \sup_{i \neq j} \gamma_{i,j}$ and $\mathbb{E} \sup_{i \leq d-1} \left( |\tilde{X}_0^{i+1,i}k| \vee |\tilde{Y}_0^{i+1,i}k| \vee |\tilde{Y}_0^{i+1,i}k| \vee |\tilde{Y}_0^{i+1,i}k| \vee |\tilde{Y}_0^{i+1,i}k| \right) < A$, for some $k \geq 6$ and let $\mathbb{E}e^{C(\tilde{X}_0^{i+1,i})} < A_{X_0}$ for any $C > 0$ where $A_{X_0}$ is a finite constant. The semi-discrete scheme (2.7) converges strongly in the mean-square sense to the true solution of (2.4) with order of convergence 1/2, that is

\[ \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{Y}_t^{i+1,i} - \tilde{X}_t^{i+1,i}|^2 \leq C\Delta, \]

where $C$ is a constant independent of $\Delta$. \hfill \Box

The semi-discrete numerical scheme for the approximation of (1.1) is the $d$-dimensional vector $Y_t$ where $Y_t^{(i)} = \sum_{j=0}^{i} Y_t^{(j,1)}$ with $Y_t^{(1)} = Y_t^{(1)}$,

\begin{equation}
Y_t^{i+1,i} = \sqrt{(\tilde{Y}_t^{i+1,i})^2 + 4\gamma_{i+1,i}t}, \quad n \in \mathbb{N},
\end{equation}

where $\tilde{Y}_t^{i+1,i}$ is given in (2.11).

**Corollary 2.3** [Strong convergence of $Y_t$ to $X_t$] Let the assumption of Theorem 2.2 hold. The semi-discrete scheme described by (2.12) converges strongly in the mean-square sense to the true solution of (1.1) with order of convergence 1/4, that is

\[ \mathbb{E} \sup_{0 \leq t \leq T} \|Y_t - X_t\|^2 \leq C\Delta^{1/2}. \]

\hfill \Box
3. Proofs

3.1. Proof of Theorem 2.2. Denote $E_{t}^{i+1,i} := \tilde{Y}_{t}^{i+1,i} - \tilde{X}_{t}^{i+1,i}$. Our goal is to bound $\mathbb{E}\sup_{0 \leq t \leq T} (E_{t}^{i+1,i})^{2}$. We begin with moment bounds for $|\tilde{Y}_{t}^{i+1,i}|^{p}$ and $|\tilde{X}_{t}^{i+1,i}|^{p}$ and later we estimate the local error of the proposed semi-discrete method.

Lemma 3.4 [Moment bounds] It holds that
\[
\mathbb{E}\sup_{0 \leq t \leq T} (|\tilde{Y}_{t}^{i+1,i}|^{p} \vee |\tilde{X}_{t}^{i+1,i}|^{p}) \leq A,
\]
for any $p \in \mathbb{R}$ where $A$ is a constant. \hfill \Box

Proof of Lemma 3.4. Set the stopping time $\tau_{R} := \inf\{t \in [0,T] : \tilde{Y}_{t}^{i+1,i} > R\}$, for $R > 0$ with the convention $\inf \emptyset = \infty$. Application of Itô’s formula on $(\tilde{Y}_{t/\tau_{R}}^{i+1,i})^{p}$, see (2.7) implies
\[
(\tilde{Y}_{t/\tau_{R}}^{i+1,i})^{p} = (\tilde{Y}_{0}^{i+1,i})^{p} + \int_{0}^{t/\tau_{R}} (p\alpha(\tilde{Y}_{s}) - p\beta(\tilde{Y}_{s}))^{p-1} + \frac{p(p-1)}{2} (\tilde{Y}_{s}^{i+1,i})^{p-2} ds + p\hat{c} \int_{0}^{t/\tau_{R}} \tilde{Y}_{s}^{i+1,i})^{p-1} dB_{t}^{(i+1)}.
\]

Let us denote $b_{Lip}^{i+1}$ the Lipschitz constant of $b^{i+1}$ and $b_{Lip} = \sup_{1 \leq i \leq d} b_{Lip}^{i}$. \hfill (3.1)

\[
|b^{i+1}(v^{(i+1)}) - b^{i}(v^{(i)})| = b^{i+1}(v^{(i+1)}) - b^{i+1}(v^{(i)}) + b^{i+1}(v^{(i)}) - b^{i}(v^{(i)}) \leq b_{Lip}^{i+1,v^{(i+1)}} + (b^{i+1}(0) + b^{i}(0)) + (b_{Lip}^{i+1} + b_{Lip}^{i})|v^{(i)}| \leq b_{Lip}^{i+1,v^{(i+1)}} + (b^{i+1}(0) + b^{i}(0)) + 2b_{Lip}^{i}|v^{(i)}|,
\]

which implies
\[
|b^{i+1}(v^{(i+1)}) - b^{i}(v^{(i)})| \leq b_{Lip}^{i+1} + 3|b^{i+1}(0) + b^{i}(0)| + 2b_{Lip}^{i}|v^{(i)}|.
\]

Moreover
\[
\sum_{k \neq i, i+1} \frac{1}{(\tilde{Y}_{s}^{i,k})(\tilde{Y}_{s}^{i+1,k})} \leq Q^{2} \sup_{1 \leq i \leq d-1} (\tilde{Y}_{s}^{i+1,i})^{-2},
\]

where
\[
Q := 1 + 2 \sum_{k=2}^{i} \frac{1}{k} + 2 \sum_{k=2}^{d-i} \frac{1}{k}.
\]

Recall the definitions (2.8), (2.9) of the functions $\alpha^{i}$ and $\beta^{i}$. \hfill (3.15)

\[
|\alpha^{i}(\tilde{Y}_{s})| \leq b_{Lip}^{i+1} + 3\frac{|b^{i+1}(0) + b^{i}(0)|}{(\tilde{Y}_{s}^{i+1,i})} + 2b_{Lip}^{i}|\tilde{Y}_{s}^{i+1,i}| + \gamma Q^{2} \sup_{1 \leq i \leq d-1} (\tilde{Y}_{s}^{i+1,i})^{-2},
\]

where $\gamma := \sup_{i \neq j} \gamma_{i,j}$, and
\[
|\beta^{i}(\tilde{Y}_{s})| \leq 2\gamma Q \sup_{1 \leq i \leq d-1} (\tilde{Y}_{s}^{i+1,i})^{-1}.
\]

Using the inequality $x^{p-k}y \leq \epsilon^{k}x^{p}y^{p} + \frac{k}{p\epsilon^{p-k}}y^{p/k}$, valid for $x \land y \geq 0$ and $p > k$ with $\epsilon = \frac{1}{2}$, and $k = 1,2$ and the fact that $\tilde{Y}_{s}^{i(i)} = \sum_{j=0}^{i-1} \tilde{Y}_{s}^{i+1,j}$ with the convection $\tilde{Y}_{s}^{1,0} \equiv \tilde{Y}_{s}^{(1)}$ we reach...
\[(\tilde{Y}_{t\wedge \tau_R}^{i+1,i})^p \leq (\tilde{Y}_0^{i+1,i})^p + \int_0^{t\wedge \tau_R} \left( C_1 + C_2(\tilde{Y}_s^{i+1,i}) + C_3 \left( \sum_{j=0}^{i-1} \tilde{Y}_s^{j+1,j} \right) \right) \, ds + M_t, \]

where
\[M_t := p\epsilon \int_0^{t\wedge \tau_R} (\tilde{Y}_s^{i+1,i})^{p-1} dB_s^{(i+1)}, \]

\[C_1 := pb_{Lip} + \frac{3(p-1)|b^{i+1}(0) + b^{i}(0)|}{2} + (p-1)b_{Lip} + \left( \frac{(p-1)}{2} (\epsilon^i)^2 + \gamma (Q^2 + 2Q) \right) \frac{p-2}{4} \]

\[C_2 := 3|b^{i+1}(0) + b^{i}(0)|2^{p-1} + \left( \frac{(p-1)}{2} (\epsilon^i)^2 + \gamma (Q^2 + 2Q) \right) 2^{p-1}, \quad C_3 := 2^p b_{Lip}. \]

Furthermore
\[\left( \sum_{j=0}^{i-1} \tilde{Y}_s^{j+1,j} \right)^p \leq p^{p-1} \sum_{j=0}^{i-1} \left( \tilde{Y}_s^{j+1,j} \right)^p \]
\[\leq p^{p-1} i(i+1) \frac{1}{2} \sup_{0 \leq i \leq d-1} \left( \tilde{Y}_s^{i+1,i} \right)^p \]
\[\leq d^{p-1} \frac{d(d+1)}{2} \sup_{0 \leq i \leq d-1} \left( \tilde{Y}_s^{i+1,i} \right)^p. \]

Thus
\[\sup_{0 \leq i \leq d-1} (\tilde{Y}_{t\wedge \tau_R}^{i+1,i})^p \leq \sup_{0 \leq i \leq d-1} (\tilde{Y}_0^{i+1,i})^p + \int_0^{t\wedge \tau_R} \left( C_1 + C_4 \sup_{0 \leq i \leq d-1} (\tilde{Y}_s^{i+1,i})^p \right) \, ds + M_t \]

where
\[C_4 := C_2 + C_3 d^{p-1} \frac{d(d+1)}{2}. \]

Taking expectations in the above inequality and using that \(M_t\) is a local martingale vanishing at 0, we get
\[\mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{t\wedge \tau_R}^{i+1,i})^p \leq \mathbb{E} (\tilde{Y}_0^{i+1,i})^p + \int_0^{t\wedge \tau_R} \left( C_1 + C_4 \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_s^{i+1,i})^p \right) \, ds \leq \left( \mathbb{E} (\tilde{Y}_0^{i+1,i})^p + C_1 T \right) e^{C_4 T}, \]

where we have applied the Gronwall inequality. Taking the limit as \(R \to \infty\) and applying the monotone convergence theorem leads to
\[\mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_t^{i+1,i})^p \leq \left( \mathbb{E} (\tilde{Y}_0^{i+1,i})^p + C_1 T \right) e^{C_4 T}. \]

Using again Itô’s formula on \((\tilde{Y}_t^{i+1,i})^p\), taking the supremum and then using Doob’s martingale inequality on the diffusion term we bound \(\mathbb{E} \sup_{0 \leq t \leq T} |\tilde{X}_t^{i+1,i}|^p\). The same techniques may be applied to show the result for negative \(p\); the moment bounds for \(|\tilde{X}_t^{i+1,i}|^p\) follow by similar arguments. \(\Box\)
Lemma 3.5 [Local error of SD method] Let $s \in [t_{n_s}, t_{n_{s+1}}]$ where $n_s$ is an integer. Then

\begin{equation}
\mathbb{E} \sup_{0 \leq i \leq d-1} |\tilde{Y}_{s}^{i+1,i} - \tilde{Y}_{s}^{i+1,i}|^p \leq C\Delta^{p/2},
\end{equation}

for any $p > 0$ where the constant $C$ does not depend on $\Delta$. \hfill \Box

Proof of Lemma 3.5. Take a $p > 2$. Relation (2.7) implies

\begin{align*}
|\tilde{Y}_{s}^{i+1,i} - \tilde{Y}_{s}^{i+1,i}|^p &= \left| \int_{t_{n_s}}^{s} \left( \alpha^i(\tilde{Y}_{u}) \tilde{Y}_{u}^{i+1,i} - \beta^i(\tilde{Y}_{u}) \right) du + c^i \int_{t_{n_s}}^{s} dB_{s}^{(i+1)} \right|^p \\
&\leq 2^{p-1} \left( \int_{t_{n_s}}^{s} \left( \alpha^i(\tilde{Y}_{u}) \tilde{Y}_{u}^{i+1,i} + \beta^i(\tilde{Y}_{u}) \right) du \right)^p + \left( c^i \right)^p \int_{t_{n_s}}^{s} |dB_{s}^{(i+1)}|^p \\
&\leq 2^{p-1} \left( \int_{t_{n_s}}^{s} \left( \alpha^i(\tilde{Y}_{u}) \tilde{Y}_{u}^{i+1,i} - \beta^i(\tilde{Y}_{u}) \right) du \right)^p + \left( c^i \right)^p \int_{t_{n_s}}^{s} |dB_{s}^{(i+1)}|^p \\
&\leq 4^{p-1} \Delta^{p-1} |\alpha^i(\tilde{Y}_{t_{n_s}})|^p \int_{t_{n_s}}^{s} |\tilde{Y}_{u}^{i+1,i}|^p du + 4^{p-1} \Delta^p |\beta^i(\tilde{Y}_{t_{n_s}})|^p + 2^{p-1} \left( c^i \right)^p \int_{t_{n_s}}^{s} |dB_{s}^{(i+1)}|^p,
\end{align*}

where we have used the Cauchy-Schwarz inequality. Taking expectations in the above inequality and using the uniform moment bounds of $|\tilde{Y}_{t_{n_s}}^{i+1,i}|^p$ described in Lemma 3.4 and Doob’s martingale inequality on the diffusion term we conclude (3.16). The case $0 < p < 2$ follows by Jensen’s inequality for the concave function $\phi(x) = x^{p/2}$ since for a random variable $Z$ it holds $\mathbb{E}|Z|^p \leq (\mathbb{E}|Z|^2)^{p/2}$. \hfill \Box

Relations (2.4) and (2.7) imply

\begin{equation}
\mathcal{E}_{i+1,i}^t = \int_0^t \left( \alpha^i(\tilde{Y}_{s}) \tilde{Y}_{s}^{i+1,i} - \beta^i(\tilde{Y}_{s}) + \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{X}_{s}^{i+1,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,k}}{\tilde{Y}_{s}^{i+1,i} \tilde{X}_{s}^{i+1,k}} - b^{i+1}(\tilde{X}_{s}^{i+1,i}) + b^i(\tilde{X}_{s}^{i,i}) \right) ds,
\end{equation}

where we used $\tilde{Y}_{0}^{i+1,i} = \tilde{X}_{0}^{i+1,i}$. We decompose the above integrand, $I$, in the following way,

\begin{align*}
I &= \alpha^i(\tilde{Y}_{s})(\tilde{Y}_{s}^{i+1,i} - \tilde{X}_{s}^{i+1,i}) + (\alpha^i(\tilde{Y}_{s}) - \alpha^i(\tilde{X}_{s})) \tilde{X}_{s}^{i+1,i} + \alpha^i(\tilde{X}_{s}) \tilde{X}_{s}^{i+1,i} - b^{i+1}(\tilde{X}_{s}^{i+1,i}) + b^i(\tilde{X}_{s}^{i,i}) \\
&+ \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{Y}_{s}^{i+1,i,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,k}}{\tilde{Y}_{s}^{i+1,i} \tilde{X}_{s}^{i+1,k}} - \beta^i(\tilde{X}_{s}) + b^i(\tilde{X}_{s}) - \beta^i(\tilde{Y}_{s}) \\
&= \alpha^i(\tilde{Y}_{s})(\mathcal{E}_{s}^{i+1,i}) + \left( \frac{b^{i+1}(\tilde{Y}_{s}^{i+1,i}) - b^i(\tilde{Y}_{s}^{i,i})}{\tilde{Y}_{s}^{i+1,i} \tilde{X}_{s}^{i+1,i}} - \frac{b^{i+1}(\tilde{X}_{s}^{i+1,i}) - b^i(\tilde{X}_{s}^{i,i})}{\tilde{X}_{s}^{i+1,i} \tilde{X}_{s}^{i+1,k}} \right) \tilde{X}_{s}^{i+1,i} \\
&+ \left( \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{Y}_{s}^{i+1,i,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,k}}{\tilde{Y}_{s}^{i+1,i} \tilde{X}_{s}^{i+1,k}} - \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{X}_{s}^{i+1,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,i}}{\tilde{X}_{s}^{i+1,i} \tilde{X}_{s}^{i+1,k}} \right) \tilde{X}_{s}^{i+1,i} - \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{X}_{s}^{i+1,i,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,k}}{\tilde{X}_{s}^{i+1,i} \tilde{X}_{s}^{i+1,k}} - \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{Y}_{s}^{i+1,i,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,i}}{\tilde{Y}_{s}^{i+1,i} \tilde{X}_{s}^{i+1,k}} \\
&+ \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{X}_{s}^{i+1,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,i}}{\tilde{X}_{s}^{i+1,k} \tilde{X}_{s}^{i+1,k}} - \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{X}_{s}^{i+1,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,i}}{\tilde{X}_{s}^{i+1,k} \tilde{X}_{s}^{i+1,k}} + \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{X}_{s}^{i+1,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,i}}{\tilde{X}_{s}^{i+1,k} \tilde{X}_{s}^{i+1,k}} - \sum_{k \neq i,i+1} \frac{\gamma_{i,k}\tilde{Y}_{s}^{i+1,i,k} - \gamma_{i+1,k}\tilde{X}_{s}^{i,i}}{\tilde{Y}_{s}^{i+1,i} \tilde{X}_{s}^{i+1,k}}.
\end{align*}
where we used (2.8) and (2.9). Moreover,

\[
\left( b^{i+1} \left( \tilde{Y}_s^{(i+1)} \right) - b^i \left( \tilde{Y}_s^{(i)} \right) \right) \frac{\tilde{X}_s^{i+1,i}}{\tilde{Y}_s^{i+1,i}} - b^{i+1} \left( \tilde{X}_s^{(i+1)} \right) - b^i \left( \tilde{X}_s^{(i)} \right)
\]

\[
= \left( b^{i+1} \left( \tilde{Y}_s^{(i+1)} \right) - b^i \left( \tilde{Y}_s^{(i)} \right) \right) \frac{\tilde{X}_s^{i+1,i}}{\tilde{Y}_s^{i+1,i}} + b^{i+1} \left( \tilde{X}_s^{(i+1)} \right) - b^i \left( \tilde{X}_s^{(i)} \right)
\]

\[
= \frac{b^{i+1} \left( \tilde{Y}_s^{(i+1)} \right) - b^i \left( \tilde{Y}_s^{(i)} \right)}{\tilde{Y}_s^{i+1,i}} \left( \tilde{X}_s^{i+1,i} - \tilde{Y}_s^{i+1,i} \right) + b^{i+1} \left( \tilde{X}_s^{(i+1)} \right) - b^i \left( \tilde{X}_s^{(i)} \right)
\]

and

\[
\sum_{k \neq i, i+1} \frac{\gamma_{i,k} \tilde{X}_s^{i+1,k} - \gamma_{i+1,k} \tilde{X}_s^{i+1,i}}{X_s^{i+1,k} X_s^{i+1,i}} - \sum_{k \neq i, i+1} \frac{\gamma_{i,k} \gamma_{i+1,k} \tilde{X}_s^{i+1,i}}{X_s^{i+1,k} X_s^{i+1,i}} - \sum_{k \neq i, i+1} \frac{\gamma_{i,k} \tilde{X}_s^{i+1,i}}{X_s^{i+1,k} X_s^{i+1,i}}
\]

\[
= \sum_{k \neq i, i+1} \left( \frac{\gamma_{i,k}}{X_s^{i+1,k}} - \frac{\gamma_{i+1,k}}{X_s^{i+1,k}} - \frac{\gamma_{i,k}}{X_s^{i+1,k}} + \frac{\gamma_{i+1,k}}{X_s^{i+1,k}} \right) - \sum_{k \neq i, i+1} \frac{\gamma_{i,k} \tilde{X}_s^{i+1,i}}{X_s^{i+1,k} X_s^{i+1,i}}
\]

\[
= \sum_{k \neq i, i+1} \gamma_{i,k} \left( \frac{1}{X_s^{i+1,k}} - \frac{1}{Y_s^{i+1,k}} \right) + \sum_{k \neq i, i+1} \gamma_{i,k} \left( \frac{1}{Y_s^{i+1,k}} - \frac{1}{X_s^{i+1,k}} \right)
\]

since \( \tilde{X}_s^{i+1,k} - \tilde{X}_s^{i+1,i} = \tilde{X}_s^{i+1,i} \) and

\[
\left( \sum_{k \neq i, i+1} \frac{\gamma_{i,k}}{X_s^{i+1,k} X_s^{i+1,i}} \right) \tilde{X}_s^{i+1,i} + \sum_{k \neq i, i+1} \frac{\gamma_{i,k}}{X_s^{i+1,k}} - \sum_{k \neq i, i+1} \frac{\gamma_{i,k} \gamma_{i+1,k} \tilde{X}_s^{i+1,i}}{X_s^{i+1,k} X_s^{i+1,i}} - \sum_{k \neq i, i+1} \frac{\gamma_{i,k} \tilde{X}_s^{i+1,i}}{X_s^{i+1,k} X_s^{i+1,i}}
\]

\[
= \sum_{k \neq i, i+1} \gamma_{i,k} \left( \frac{1}{X_s^{i+1,k}} - \frac{1}{Y_s^{i+1,k}} \right) + \sum_{k \neq i, i+1} \gamma_{i,k} \left( \frac{1}{Y_s^{i+1,k}} - \frac{1}{X_s^{i+1,k}} \right)
\]

\[
= \sum_{k \neq i, i+1} \gamma_{i,k} \left( \frac{1}{X_s^{i+1,k}} - \frac{1}{Y_s^{i+1,k}} \right)
\]

\[
= \sum_{k \neq i, i+1} \gamma_{i,k} \left( \frac{1}{X_s^{i+1,k}} - \frac{1}{Y_s^{i+1,k}} \right) - \gamma_{i+1,k} \left( \frac{1}{X_s^{i+1,k}} - \frac{1}{Y_s^{i+1,k}} \right)
\]

\[
= \sum_{k \neq i, i+1} \gamma_{i,k} \left( \frac{1}{X_s^{i+1,k}} - \frac{1}{Y_s^{i+1,k}} \right) - \gamma_{i+1,k} \left( \frac{1}{X_s^{i+1,k}} - \frac{1}{Y_s^{i+1,k}} \right)
\]

\[
+ \frac{\gamma_{i+1,k}}{Y_s^{i+1,k}} \sum_{k \neq i, i+1} \gamma_{i,k} \frac{\tilde{Y}_s^{i+1,k} - \tilde{X}_s^{i+1,k}}{Y_s^{i+1,k}}
\]
The integrand $I$ becomes

\[
I = \alpha^i(\tilde{Y}^i) (\mathcal{E}^{i+1,i} + b^i(\tilde{Y}^{i+1,i}) - b^i(\tilde{Y}^i)) \left( Y_{i+1,i} - \tilde{Y}_{i+1,i} - \mathcal{E}^{i+1,i} \right)
\]

\[
+ b^i(\tilde{Y}^{i+1,i}) - b^i(\tilde{X}^{i+1}) - b^i(\tilde{Y}^i) + b^i(\tilde{X}^i)
\]

\[
+ \sum_{k \neq i,i+1} \gamma_{i,k} \left( \frac{1}{X_{i,k}} - \frac{1}{Y_{i,k}} \right) - \gamma_{i+1,k} \left( \frac{1}{X_{i+1,k}} - \frac{1}{Y_{i+1,k}} \right)
\]

\[
+ \frac{\mathcal{E}^{i+1,i} + (\tilde{Y}_{i+1,i} - \tilde{Y}_{i+1,i})}{\tilde{Y}_{i+1,i}} \sum_{k \neq i,i+1} \gamma_{i,k} \frac{\tilde{Y}_{i+1,k}}{\tilde{Y}_{i+1,k}}
\]

where we used once more (2.8).

\[
(\mathcal{E}^{i+1,i})^2 \leq 5t \int_0^t \left( \frac{b^i(Y^{i+1,i}) - b^i(Y^i)}{Y_{i+1,i}} \right)^2 (Y_{i+1,i} - \tilde{Y}_{i+1,i})^2 ds
\]

\[
+ 5t \int_0^t \left( b^i(Y^{i+1,i}) - b^i(X^{i+1,i}) - b^i(Y^i) + b^i(X^i) \right)^2 ds
\]

\[
+ 5t \int_0^t \left( \sum_{k \neq i,i+1} \gamma_{i,k} \left( \frac{1}{X_{i,k}} - \frac{1}{Y_{i,k}} \right) \right)^2 + \left( \sum_{k \neq i,i+1} \gamma_{i+1,k} \left( \frac{1}{X_{i+1,k}} - \frac{1}{Y_{i+1,k}} \right) \right)^2 ds
\]

(3.17) \[ + 5t \int_0^t (\tilde{Y}_{i+1,i} - \tilde{Y}_{i+1,i})^2 \left( \sum_{k \neq i,i+1} \gamma_{i,k} \frac{\tilde{Y}_{i+1,k}}{\tilde{Y}_{i+1,k}} \right)^2 ds. \]

Squaring both sides of inequality (3.13) yields

\[
(\frac{b^i(Y^{i+1,i}) - b^i(Y^i)}{Y_{i+1,i}})^2 \leq 3b_{Lip}^2 + 3 \left( \frac{b^i(0) + b^i(0)}{(Y_{i+1,i})^2} \right) + 4b_{Lip}^2 \left( \frac{(Y^i)^2}{(Y_{i+1,i})^2} \right).
\]

Furthermore,

\[
\left( \sum_{k \neq i,i+1} \gamma_{i,k} \left( \frac{1}{X_{i,k}} - \frac{1}{Y_{i,k}} \right) \right)^2 \leq 2 \left( \sum_{k \neq i,i+1} \gamma_{i,k} \frac{\tilde{Y}_{i+1,k} - \tilde{Y}_{i,k}}{X_{i,k}Y_{i,k}} \right)^2 + 2 \left( \sum_{k \neq i,i+1} \gamma_{i,k} \frac{\mathcal{E}_{i,k}}{X_{i,k}Y_{i,k}} \right)^2
\]

\[
\leq 2\tilde{Q}^2 \sup_{0 \leq i \leq d-1} (\tilde{Y}_{i+1,i} - \tilde{Y}_{i+1,i})^2 \left( \sum_{k \neq i,i+1} \frac{\gamma_{i,k}}{X_{i,k}Y_{i,k}} \right)^2 + 2\tilde{Q}^2 \sup_{0 \leq i \leq d-1} (\mathcal{E}_{i+1,i})^2 \left( \sum_{k \neq i,i+1} \frac{\gamma_{i,k}}{X_{i,k}Y_{i,k}} \right)^2
\]
and as a consequence
\[
\left( \sum_{k \neq i,i+1} \gamma_{i+1,k} \left( \frac{1}{\tilde{X}^{i+1,k}_s} - \frac{1}{\tilde{Y}^{i+1,k}_s} \right) \right)^2 \leq \hat{Q}^2 \sup_{0 \leq i \leq d-1} (\tilde{Y}^{i+1,i}_s - \tilde{Y}^{i+1,i}_s)^2 \left( \sum_{k \neq i,i+1} \gamma_{i+1,k} \tilde{X}^{i+1,k}_s \right)^2,
\]
where
\[
(3.19) \quad \hat{Q} := 1 + 2 \sum_{k=2}^i k + 2 \sum_{k=2}^{d-i} k.
\]

Finally
\[
\left( b^{i+1}(\tilde{Y}^{(i+1)}_s) - b^{i+1}(\tilde{X}^{(i+1)}_s) - b^i(\tilde{Y}^{(i)}_s) + b^i(\tilde{X}^{(i)}_s) \right)^2 \leq 2 \left( b^{i+1}(\tilde{Y}^{(i+1)}_s) - b^{i+1}(\tilde{X}^{(i+1)}_s) \right)^2 + 2 \left( b^i(\tilde{Y}^{(i)}_s) - b^i(\tilde{X}^{(i)}_s) \right)^2 
\leq 2(b_{Lip}^i)^2(\tilde{Y}^{(i+1)}_s - \tilde{X}^{(i+1)}_s)^2 + 2(b_{Lip}^i)^2(\tilde{Y}^{(i)}_s - \tilde{X}^{(i)}_s)^2 
\leq 2b_{Lip}^2(\mathcal{E}_s^{i+1})^2 + 2b_{Lip}^2(\mathcal{E}_s^{i})^2 
\leq 4b_{Lip}^2(\mathcal{E}_s^{i+1,i})^2 + 6b_{Lip}^2 \sum_{j=0}^{i-1} (\mathcal{E}_s^{i+1,j})^2 \leq 4b_{Lip}^2(\mathcal{E}_s^{i+1,i})^2 + 6b_{Lip}^2 \sup_{0 \leq i \leq d-1} (\mathcal{E}_s^{i+1,i})^2.
\]

Plugging all the above estimates and (3.18) into (3.17) yields
\[
(\mathcal{E}_s^{i+1,i})^2 \leq 5t \int_0^t \left( 3b_{Lip}^2 + \frac{(b^{i+1}(0) + b^i(0))^2}{(\tilde{Y}^{i+1+1}_s)^2} + 4b_{Lip}^2 (\tilde{Y}^{(i)}_s)^2 \right) (\tilde{Y}^{i+1,i}_s - \tilde{Y}^{i+1,i}_s)^2 ds 
+ 5t \int_0^t \left( 4b_{Lip}^2(\mathcal{E}_s^{i+1,i})^2 + 6b_{Lip}^2 \sup_{0 \leq i \leq d-1} (\mathcal{E}_s^{i+1,i})^2 \right) ds 
+ 10t \int_0^t \hat{Q}^2 \sup_{0 \leq i \leq d-1} (\mathcal{E}_s^{i+1,i})^2 \left[ \left( \sum_{k \neq i, i+1} \frac{\gamma_{i,k}}{\tilde{X}^{i+k}_s} \right)^2 + \left( \sum_{k \neq i, i+1} \frac{\gamma_{i,k}}{\tilde{X}^{i+k}_s} \tilde{Y}^{i+k}_s \right)^2 \right] ds 
+ 10t \int_0^t \hat{Q}^2 \sup_{0 \leq i \leq d-1} (\tilde{Y}^{i+1,i}_s - \tilde{Y}^{i+1,i}_s)^2 \left[ \left( \sum_{k \neq i, i+1} \frac{\gamma_{i,k}}{\tilde{X}^{i+k}_s} \tilde{Y}^{i+k}_s \right)^2 + \left( \sum_{k \neq i, i+1} \frac{\gamma_{i,k}}{\tilde{X}^{i+k}_s} \tilde{Y}^{i+k}_s \right)^2 \right] ds 
+ 10t \int_0^t (\tilde{Y}^{i+1,i}_s - \tilde{Y}^{i+1,i}_s)^2 \sum_{k \neq i, i+1} \frac{\gamma_{i,k}^2}{(\tilde{Y}^{i,k}_s)^2 (\tilde{Y}^{i+1,k}_s)^2} ds.
\]

We define the process
\[
\zeta(t) := \int_0^t 10T \hat{Q}^2 \gamma^2 \left[ \left( \sum_{k \neq i, i+1} \frac{1}{\tilde{X}^{i+k}_s \tilde{Y}^{i+k}_s} \right)^2 + \left( \sum_{k \neq i, i+1} \frac{1}{\tilde{X}^{i+k}_s \tilde{Y}^{i+1,k}_s} \right)^2 \right] ds.
\]
and the stopping time

$$\tau_l := \inf\{s \in [0, T] : (20 + 30d)Tb_{\text{Lip}}^2s + \zeta(s) \geq l\}.$$ 

Relation (3.20) becomes

$$\mathbb{E} \sup_{0 \leq i \leq \tau} \sup_{0 \leq s \leq d-1} (\mathcal{E}_{l}^{i+1,i})^2 \leq 15Tb_{\text{Lip}}^2 \int_0^\tau \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s}^{i+1,i} - \tilde{Y}_{s}^{i+1,i})^2 ds$$

$$+ 3(b^i(0) + b^i(0))^2 \int_0^\tau \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s}^{i+1,i} - \tilde{Y}_{s}^{i+1,i})^2 ds$$

$$+ 4b_{\text{Lip}}^2 \int_0^\tau \mathbb{E} \sup_{0 \leq i \leq d-1} \frac{(\tilde{Y}_{s}^{i+1,i})^2}{(\tilde{Y}_{s}^{i+1,i})^2} (\tilde{Y}_{s}^{i+1,i} - \tilde{Y}_{s}^{i+1,i})^2 ds$$

$$+ (20 + 30d)Tb_{\text{Lip}}^2 \int_0^\tau \mathbb{E} \sup_{0 \leq s \leq d-1} (\mathcal{E}_{l}^{i+1,i})^2 ds + \int_0^\tau (\zeta(s))^p \mathbb{E} \sup_{0 \leq i \leq d-1} (\mathcal{E}_{l}^{i+1,i})^2 ds$$

$$+ 20Q^2 \int_0^\tau \mathbb{E} \sup_{0 \leq s \leq d-1} (\tilde{Y}_{s}^{i+1,i} - \tilde{Y}_{s}^{i+1,i})^2 \sum_{k \neq i, i+1} \frac{1}{(\tilde{X}_{s}^{i+1,k}l_{\text{Lip}})^2} ds,$$

(3.21)

where $\tau$ is a stopping time. The local error of the semi-discrete method implies

(3.22)

$$\mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s}^{i+1,i} - \tilde{Y}_{s}^{i+1,i})^p \leq \Delta^p/2$$

and

$$\sum_{k \neq i, i+1} \frac{1}{(\tilde{X}_{s}^{i+1,k})^2} + \sum_{k \neq i, i+1} \frac{1}{(\tilde{X}_{s}^{i+1,k})^2} \leq 2Q^2 \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s}^{i+1,i})^2 \sup_{0 \leq i \leq d-1} (\tilde{X}_{s}^{i+1,i})^{-2}$$

and

$$\sum_{k \neq i, i+1} \frac{1}{(\tilde{X}_{s}^{i+1,k})^2} \leq Q^2 \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s}^{i+1,i})^{-4}.$$
We insert these bounds and (3.22) to (3.21) and get

\[
\mathbb{E} \sup_{0 \leq t \leq \tau} \mathbb{E} \sup_{0 \leq i \leq d-1} (\mathcal{E}_{t,i}^{i+1,i})^2 \leq C \Delta + C \int_0^\tau \sqrt{\mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s,i}^{i+1,i})^{-4}} \sqrt{\mathbb{E} (\tilde{Y}_{s,i}^{i+1,i} - \tilde{Y}_{s,i}^{i+1,i})^4} ds \\
+C \int_0^\tau \left( \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s,i}^{i+1,i})^6 \right)^{1/3} \left( \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{X}_{s,i}^{i+1,i})^{-6} \right)^{1/3} \left( \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s,i}^{i+1,i} - \tilde{Y}_{s,i}^{i+1,i})^6 \right)^{1/3} ds \\
+(20 + 30d) T b_{\text{Lip}}^2 \int_0^\tau \mathbb{E} \sup_{0 \leq i \leq d-1} (\mathcal{E}_{t,i}^{i+1,i})^2 ds + \int_0^\tau \left( \zeta_s \right)^{\text{'}E} \sup_{0 \leq s \leq 0 \leq i \leq d-1} (\mathcal{E}_{t,s}^{i+1,i})^2 ds \\
+C \int_0^\tau \left( \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s,i}^{i+1,i})^{-6} \right)^{1/3} \left( \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{X}_{s,i}^{i+1,i})^{-6} \right)^{1/3} \left( \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s,i}^{i+1,i} - \tilde{Y}_{s,i}^{i+1,i})^6 \right)^{1/3} ds \\
+C \int_0^\tau \left( \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s,i}^{i+1,i})^{-6} \right)^{2/3} \left( \mathbb{E} \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s,i}^{i+1,i} - \tilde{Y}_{s,i}^{i+1,i})^6 \right)^{1/3} ds \\
\leq C \Delta + (20 + 30d) T b_{\text{Lip}}^2 \int_0^\tau \mathbb{E} \sup_{0 \leq i \leq d-1} (\mathcal{E}_{t,i}^{i+1,i})^2 ds + \int_0^\tau \left( \zeta_s \right)^{\text{'}E} \sup_{0 \leq s \leq 0 \leq i \leq d-1} (\mathcal{E}_{t,s}^{i+1,i})^2 ds \\
\leq C \Delta + \int_0^\tau \left( (20 + 30d) T b_{\text{Lip}}^2 s + \zeta_s \right)^{\text{'}E} \sup_{0 \leq s \leq 0 \leq i \leq d-1} (\mathcal{E}_{t,s}^{i+1,i})^2 ds.
\]

The uniform moment bound (3.23) for \( \tau = \tau_1 \) reads

\[
\mathbb{E} \sup_{0 \leq t \leq \tau_1} \mathbb{E} \sup_{0 \leq i \leq d-1} (\mathcal{E}_{t,i}^{i+1,i})^2 \leq C \Delta + \int_0^{\tau_1} \left( (20 + 30d) T b_{\text{Lip}}^2 s + \zeta_s \right)^{\text{'}E} \sup_{0 \leq s \leq 0 \leq i \leq d-1} (\mathcal{E}_{t,s}^{i+1,i})^2 ds \\
\leq C \Delta + \int_0^{\tau_1} \mathbb{E} \sup_{0 \leq j \leq u \leq 0 \leq i \leq d-1} (\mathcal{E}_{t_i}^{i+1,i})^2 du
\]

(3.24)

where in the final step we have used Gronwall’s inequality. Under the change of variables

\( u = (20 + 30d) T b_{\text{Lip}}^2 s + \zeta_s \) relation (3.23) for \( \tau = T \) becomes

\[
\mathbb{E} \sup_{0 \leq t \leq T} \mathbb{E} \sup_{0 \leq i \leq d-1} (\mathcal{E}_{t,i}^{i+1,i})^2 \leq C \Delta + \int_0^T (20 + 30d) T b_{\text{Lip}}^2 T^2 + \zeta_T \mathbb{E} \sup_{0 \leq j \leq u \leq 0 \leq i \leq d-1} (\mathcal{E}_{t_j}^{i+1,i})^2 du \\
\leq C \Delta + \int_0^\infty \mathbb{E} \left( \sup_{0 \leq j \leq u \leq 0 \leq i \leq d-1} (\mathbb{I}_{(20 + 30d) T b_{\text{Lip}}^2 T^2 + \zeta_T \geq u}) (\mathcal{E}_{t_j}^{i+1,i})^2 \right) du \\
+ \int_0^\infty \mathbb{P} ((20 + 30d) T b_{\text{Lip}}^2 T^2 + \zeta_T \geq u) \times \mathbb{E} \left( \sup_{0 \leq j \leq u \leq 0 \leq i \leq d-1} (\mathcal{E}_{t_j}^{i+1,i})^2 \left( (20 + 30d) T b_{\text{Lip}}^2 T^2 + \zeta_T \geq u) \right) \right) du \\
\leq C \Delta + C e^{(20 + 30d) T b_{\text{Lip}}^2 T^2} \Delta + \int_0^\infty \mathbb{P} (\zeta_T \geq u) \mathbb{E} \sup_{0 \leq j \leq u \leq 0 \leq i \leq d-1} (\mathcal{E}_{t_j}^{i+1,i})^2 du \\
(3.25) \leq C \Delta + C \Delta \int_0^\infty \mathbb{P} (\zeta_T \geq u) e^u du,
where in the last steps we have used (3.24). The next step is to show \( u \to \mathbb{P}(\zeta_T \geq u)e^u \in \mathcal{L}^1(\mathbb{R}_+) \). Markov’s inequality implies

\[
\mathbb{P}(\zeta_T \geq u) \leq e^{-\epsilon u} \mathbb{E}(e^{\epsilon \zeta_T}),
\]

for any \( \epsilon > 0 \). The following bound holds

\[
\zeta_T \leq \int_0^T 10TQ^2 \gamma^2 \left[ \left( \sum_{k \neq i, i+1} \frac{1}{\tilde{Y}_{s,k}^{i+1,k} \tilde{Y}_{s,k}^{i,k}} \right)^2 + \left( \sum_{k \neq i, i+1} \frac{1}{\tilde{X}_{s,k}^{i+1,k} \tilde{Y}_{s,k}^{i+1,k}} \right)^2 \right] ds
\]

\[
\leq 20TQ^2 \gamma^2 \int_0^T \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s,k}^{i+1,i})^{-2} \sup_{0 \leq i \leq d-1} (\tilde{X}_{s,k}^{i+1,i})^{-2} ds,
\]

therefore

\[
(3.26) \quad \mathbb{E}(e^{\epsilon \zeta_T}) \leq \mathbb{E} \left( e^{20TQ^2 \gamma^2 \int_0^T \sup_{0 \leq i \leq d-1} (\tilde{Y}_{s,k}^{i+1,i})^{-2} \sup_{0 \leq i \leq d-1} (\tilde{X}_{s,k}^{i+1,i})^{-2} ds} \right).
\]

Applying the Itô formula to (2.4)

\[
(\tilde{X}_{t,i}^{i+1,i})^2 = (\tilde{X}_{0,i}^{i+1,i})^2 + \int_0^t 2\tilde{X}_{s,i}^{i+1,i} \left( - \sum_{k \neq i, i+1} \frac{\gamma_{s,k} \tilde{X}_{s,k}^{i+1,k} - \gamma_{s,i} \tilde{X}_{s,i}^{i,k}}{\tilde{X}_{s,k}^{i+1,k} \tilde{X}_{s,i}^{i,k}} \right) ds
\]

\[
+ \int_0^t 2\tilde{X}_{s,i}^{i+1,i} \left[ b^{i+1}(\tilde{X}_{s,i}^{i+1}) - b^i(\tilde{X}_{s,i}^{i}) + (\epsilon^i)^2 \right] ds + 2(\epsilon^i) \int_0^t \tilde{X}_{s,i}^{i+1,i} dB_{s,i}^{(i+1)}
\]

\[
\geq (\tilde{X}_{0,i}^{i+1,i})^2 + \int_0^t 2\tilde{X}_{s,i}^{i+1,i} \left( - \sum_{k \neq i, i+1} \frac{\gamma_{s,k}}{\tilde{X}_{s,k}^{i+1,k}} + \sum_{k \neq i, i+1} \frac{\gamma_{s,i} \tilde{X}_{s,i}^{i,k}}{\tilde{X}_{s,i}^{i,k}} \right) ds
\]

\[
+ \int_0^t 2\tilde{X}_{s,i}^{i+1,i} \left[ b^i(\tilde{X}_{s,i}^{i}) - b^i(\tilde{X}_{s,i}^{i}) + (\epsilon^i)^2 \right] ds + 2(\epsilon^i) \int_0^t \tilde{X}_{s,i}^{i+1,i} dB_{s,i}^{(i+1)}
\]

\[
\geq (\tilde{X}_{0,i}^{i+1,i})^2 + \int_0^t (-Q\gamma - 2b_{Lip}(\tilde{X}_{s,i}^{i+1,i})^2 + (\epsilon^i)^2) ds + 2(\epsilon^i) \int_0^t \tilde{X}_{s,i}^{i+1,i} dB_{s,i}^{(i+1)}.
\]

In the event \((\tilde{X}_{t,i}^{i+1,i})^2 \geq 1\) we have \((\tilde{X}_{t,i}^{i+1,i})^2 \geq 1\) whereas when \((\tilde{X}_{t,i}^{i+1,i})^2 \leq 1\) and

\[
(3.27) \quad (\epsilon^i)^2 \geq Q\gamma,
\]

we get

\[
(\tilde{X}_{t,i}^{i+1,i})^2 \geq (\tilde{X}_{0,i}^{i+1,i})^2 + \int_0^t -2b_{Lip}(\tilde{X}_{s,i}^{i+1,i})^2 ds + 2(\epsilon^i) \int_0^t \tilde{X}_{s,i}^{i+1,i} dB_{s,i}^{(i+1)}
\]

\[
\geq (\tilde{X}_{0,i}^{i+1,i})^2 \exp \left\{ \int_0^t (-2b_{Lip} - 2(\epsilon^i)^2) ds + 2(\epsilon^i) \int_0^t dB_{s,i}^{(i+1)} \right\}.
\]

Consequently,

\[
(\tilde{X}_{t,i}^{i+1,i})^{-2} \leq (\tilde{X}_{0,i}^{i+1,i})^{-2} \exp \left\{ \int_0^t (2b_{Lip} + 2(\epsilon^i)^2) ds - 2(\epsilon^i) \int_0^t dB_{s,i}^{(i+1)} \right\}
\]

\[
\leq (\tilde{X}_{0,i}^{i+1,i})^{-2} e^{(2b_{Lip} + 4(\epsilon^i)^2)T} \exp \left\{ \int_0^t -2(\epsilon^i)^2 ds - 2(\epsilon^i) \int_0^t dB_{s,i}^{(i+1)} \right\}
\]

\[
(3.28) \quad \leq (\tilde{X}_{0,i}^{i+1,i})^{-2} e^{(2b_{Lip} + 4(\epsilon^i)^2)T} \xi_t,
\]
where $\xi_t$ is the exponential martingale

\begin{equation}
\xi_t := \exp \left\{ \int_0^t -2(c)^2 ds - 2(c) \int_0^t dB_s^{(i+1)} \right\}.
\end{equation}

Note that (3.27) holds when

\begin{equation}
(c)^2 \geq (d - 1)\gamma,
\end{equation}

since by the definition of $Q$ (3.15) we get

\[ Q \leq 1 + 2\frac{i - 1}{2} + 2\frac{d - i + 1}{2} \leq d - 1. \]

In the same spirit we bound $(\tilde{Y}_{t}^{i+1,i})^{-2}$. Once again we apply Itô’s formula to (2.7)

\[(\tilde{Y}_{t}^{i+1,i})^2 = (\tilde{Y}_{0}^{i+1,i})^2 + \int_0^t \left(2\alpha^i(\tilde{Y}_s)(\tilde{Y}_{s}^{i+1,i})^2 - 2\tilde{Y}_s^{i+1,i} \beta^i(\tilde{Y}_s) + (c)^2\right) ds + 2c^i \int_0^t \tilde{Y}_s^{i+1,i} dB_s^{(i+1)} \]

Moreover,

\[ 2\alpha^i(\tilde{Y}_s)(\tilde{Y}_{s}^{i+1,i})^2 - 2\tilde{Y}_s^{i+1,i} \beta^i(\tilde{Y}_s) \geq 2\frac{b^i(\tilde{Y}_s^{(i+1)}) - b^i(\tilde{Y}_s^{(i)})}{\tilde{Y}_s^{i+1,i}}(\tilde{Y}_{s}^{i+1,i})^2 \]

\[ \geq -2b_{Lip}(\tilde{Y}_{s}^{i+1,i})^2 - 2\tilde{Y}_s^{i+1,i}\left(\sum_{k\neq i,i+1} \frac{\gamma_{i,k} - \gamma_{i+1,k}}{\tilde{Y}_{s}^{i+1,k}}\right) \]

\[ \geq -2b_{Lip}(\tilde{Y}_{s}^{i+1,i})^2 - Q\gamma, \]

therefore by a comparison theorem we have

\[(\tilde{Y}_{t}^{i+1,i})^2 \geq (\tilde{Y}_{0}^{i+1,i})^2 + \int_0^t \left[(c)^2 - 3\gamma d(d - 1) - 2b_{Lip}(\tilde{Y}_{s}^{i+1,i})^2\right] ds + 2c^i \int_0^t \tilde{Y}_s^{i+1,i} dB_s^{(i+1)} \]

In the event $(\tilde{Y}_{t}^{i+1,i})^2 \geq 1$ we have $(\tilde{Y}_{t}^{i+1,i})^{-2} \leq 1$ whereas when $(\tilde{Y}_{t}^{i+1,i})^2 \leq 1$

\[(\tilde{Y}_{t}^{i+1,i})^2 \geq (\tilde{Y}_{0}^{i+1,i})^2 + \int_0^t \left[(c)^2 - 3\gamma d(d - 1) - 2b_{Lip}(\tilde{Y}_{s}^{i+1,i})^2\right] ds + 2c^i \int_0^t \tilde{Y}_s^{i+1,i} dB_s^{(i+1)} \]

\[ \geq (\tilde{Y}_{0}^{i+1,i})^2 + \int_0^t -2b_{Lip}(\tilde{Y}_{s}^{i+1,i}) ds + 2c^i \int_0^t \tilde{Y}_s^{i+1,i} dB_s^{(i+1)} \]

\[ \geq (\tilde{Y}_{0}^{i+1,i})^2 \exp \left\{ \int_0^t (-2b_{Lip} - 2(c)^2) ds + 2(c) \int_0^t dB_s^{(i+1)} \right\}, \]

when condition (3.27) holds.

Therefore,

\begin{equation}
(\tilde{Y}_{t}^{i+1,i})^{-2} \leq (\tilde{Y}_{0}^{i+1,i})^{-2} e^{(2b_{Lip} + 4(c)^2)t} \xi_t, \end{equation}

where $\xi_t$ is the exponential martingale of (3.29). Now, we plug estimates (3.28) and (3.31) into (3.26) to get
3.2. Proof of Corollary 2.3. It holds that

\[ ||Y_t - X_t||^2 = \sum_{i=1}^{d} |Y_t^{(i)} - X_t^{(i)}|^2 = \sum_{i=1}^{d} \left| \sum_{j=0}^{i-1} Y_t^{j+1,j} - \sum_{j=0}^{d} X_t^{j+1,j} \right|^2 \]

\[ \leq \sum_{i=1}^{d} \sum_{j=0}^{i-1} |Y_t^{j+1,j} - X_t^{j+1,j}|^2, \]

Relation (2.6) implies
\[ \|Y_t - X_t\|_2^2 \leq \sum_{i=1}^{d} \sum_{j=0}^{i-1} \left( (|\tilde{Y}_{t,j}| + |\tilde{X}_{t,j}|) |\tilde{Y}_{t,j} - \tilde{X}_{t,j}| \right). \]

Therefore,

\[ \mathbb{E} \sup_{0 \leq t \leq T} \|Y_t - X_t\|_2^2 \leq \sum_{i=1}^{d} \sum_{j=0}^{i-1} \sqrt{2 \left( \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{Y}_{t,j}|^2 + \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{X}_{t,j}|^2 \right)} \sqrt{ \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{Y}_{t,j} - \tilde{X}_{t,j}|^2 } \]

\[ \leq 4 \sqrt{A} \sum_{i=1}^{d} \sum_{j=0}^{i-1} \sqrt{2 \left( \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{Y}_{t,j}|^2 + \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{X}_{t,j}|^2 \right)} \sqrt{ \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{Y}_{t,j} - \tilde{X}_{t,j}|^2 } \leq C \Delta^{1/2}, \]

where we applied Theorem 2.2 and \( A = \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{Y}_{t,j}|^2 \vee \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{X}_{t,j}|^2 \) which is finite by Lemma 3.4.

**Final comments and future directions.** Here, we investigated the numerical approximation of systems of SDEs which possess the non-colliding property. The proposed numerical scheme preserves this property. There are however more complicated systems to study; the interest in this paper was the constant diffusion case. A natural generalization is the following system of SDEs

\[ X_t^{(i)} = X_0^{(i)} + \int_0^t \left( \sum_{i \neq j} \gamma_{i,j} \frac{X_s^{(i)} - X_s^{(j)}}{X_s^{(i)} - X_s^{(j)}} + b^i(X_s^{(i)}) \right) ds + \sum_{j=1}^{d} \int_0^t \sigma_{i,j}(X_s^{(i)}) dW_s^{(j)}, \quad i = 1, \ldots, d, \]

which arises in various applications such as those in [1, Section 6], [19]. We conjecture that one may use the semi discrete method appropriately in a different way. One other possible solution is to use the Lamperti-type transformation as in [9] to remove the nonlinearity from the diffusion to the drift part of the SDE and then follow the same recipe as the one presented here.

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