Kernel of Trace Operator of Sobolev Spaces on
Lipschitz Domain

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Abstract

We are going to show that on bounded Lipschitz domain $D$: both
$C^\infty_c(D)$, the set of smooth functions on $D$ with compact support, and
$C^\infty_0(D)$, the set of smooth functions on $D$ with (extension) zero boundary,
are dense in $W^{1,p}(D)$, $p \in [1, \infty)$. A proof can be found in Nečas's
monograph [2], Theorem 4.10, §2.4.3.

Our main result in this note is that: we find another proof by showing
that both closures is the same as kernel of trace operator

$T : W^{1,p}(D) \to L^p(\partial D)$.

via some change of variables formulas from Evans and Gariepy’s text-
book [4] for Lipschitz coordinate transformation, to exten-
d the proof of Theorem 2 in §5.5 of Evans’ widespread PDE textbook [3], from $C^1$ to
Lipschitz domain.

Firstly we review some analysis on Lipschitz domain. Let $C^\infty_c(D)$ be the set of
smooth functions on $D$ with compact support, and $C^\infty_0(D)$ be the set of smooth
functions on $D$ with (extension) zero boundary. It is worth to note that in
Grisvard’s classic [1], Corollary 1.5.1.6 in §1.5.1, states without proof a much
more general result which covers above.

1 Preliminary: Lipschitz Domain

1.1 Lipschitz homeomorphism

Let $D \subset \mathbb{R}^d$ be a Lipschitz domain. First we state the usual coordinate transfor-
mation from Appendix C.1 in [3]. Fix $a \in \partial D$, there exist $r > 0$ and Lipschitz
map $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that (upon relabeling and reorienting coordinates axes
if necessary)

$D \cap B(a, r) = \{ x \in B(a, r) \mid x_d > \gamma(x_1, \ldots, x_{d-1}) \}$.

Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz coordinate transformation

$F(x_1, \ldots, x_{d-1}, x_d) := (x_1, \ldots, x_{d-1}, x_d - \gamma(x_1, \ldots, x_{d-1}))$.
and obviously it has inverse $F^{-1}$ as

$$F^{-1}(y_1, \ldots, y_{d-1}, y_d) = (y_1, \ldots, y_{d-1}, y_d + \gamma(y_1, \ldots, y_{d-1})),$$

which is also continuous. Hence $D$ is homeomorphic to an open subset of $\mathbb{R}^d_+ := \{(y_1, \ldots, y_{d-1}, y_d) \in \mathbb{R}^d \mid y_d > 0\}$, and obviously $\partial D$ is homeomorphic to a closed subset of $\{(y_1, \ldots, y_{d-1}, y_d) \in \mathbb{R}^d \mid y_d = 0\}$. To calculate functions on (an open subset of) $\mathbb{R}^d_+$ instead of $D$, we let $\{f_i : U_i \subset D \to \mathbb{R}^d_+\}$ be an atlas of Lipschitz coordinate maps on $D$ as above with $U_i$ compact, and write $\{\rho_i : D \to [0, 1]\}_i$ for a partition of unity associated to $\{U_i\}_i$. Then $\forall \phi \in W^{1,p}(D)$, we can map $\phi$ as

$$\phi \mapsto \sum_i \rho_i \phi \circ f_i \in W^{1,p}(\mathbb{R}^d_+).$$

It is not difficult to show that this form is invariant under partitions of unity. For any $a \in \partial D$ and $r > 0$, let $\{B(a, r) \mid a \in \partial D\}_{r>0}$ be an open cover, then there exists finite sub-cover $\{B_i\}_{i=1}^\nu$. Then let $D_i := D \cap B_i$, $D_0 := D - \left( \bigcup_{i=1}^\nu D_i \right)$, $\varphi_i \in C_\infty^\infty(D_i)$ be a partition of unity on $D$, i.e.,

$$1_D = \sum_{i=0}^\nu \varphi_i.$$

Since our estimate is local (on $D_i$), we are going to show local estimate can be extended to global (on $D$). Let $f \in C^\infty_0(D)$, $h_i \in C^\infty_0(D_i)$, then

$$\|f - h\|_{W^{1,2}(D)}^2 \leq c_1 \sum_{i=0}^\nu \varphi_i^2 \|f - h_i\|^2 + \|Df - Dh_i\|^2 + c_2 \sum_{i=0}^\nu \int |f - h_i|^2 (D\varphi_i)^2.$$

The only remaining obstacle to prove

$$C_0^\infty(D) = C_\infty^\infty(D)$$

in $W^{1,p}(D)$, is to integrate under change of variables. Thereby we need some auxiliary change of variables formulas below. By Rademacher theorem (see §3.1.2 of [4]), Jacobi matrix $DF$ exists a.e. and Jacobian $JF = 1$ a.e.

### 1.2 Change of variables formulas

We state without proofs two auxiliary change of variables formulas. They are based on area or co-area formula.

**Theorem 1 (Theorem 3.9 in [4]).** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \leq m$. Then for each $g \in L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} g(x) Jf(x) \, dx = \int_{\mathbb{R}^m} \left[ \sum_{x \in f^{-1}(w)} g(x) \right] d\mathcal{H}^n(w).$$
Theorem 2 (Theorem 3.11 in [4]). Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be Lipschitz, \( n \geq m \).

1. For a.e. \( y \in \mathbb{R}^m \) in Lebesgue measure, \( g \in L^1(f^{-1}\{y\}) \) in \((n-m)\) dimensional Hausdorff measure \( H^{n-m} \), and

\[
\int_{\mathbb{R}^n} g \, Jf \, dx = \int_{\mathbb{R}^m} \left[ \int_{f^{-1}\{w\}} g \, dH^{n-m} \right] \, dw.
\]

2. Trace Operator

Remark that a proof of trace Theorem on Lipschitz domain can be found in §4.3 of [4]. Adopting from §5.5 of [3], we can show the existence of trace operator on Lipschitz domain via Theorem 1.

2.1 Trace operator on Lipschitz domain

Theorem 3 (Trace Theorem on Lipschitz domain). Let \( D \subset \mathbb{R}^d \) be a Lipschitz domain. Then for each \( p \in [1, \infty) \), there exists a bounded operator,

\[
T : W^{1,p}(D) \rightarrow L^p(\partial D)
\]

such that

1. \( Tu = u|_{\partial D} \) if \( u \in W^{1,p}(D) \cap C(\overline{D}) \); and
2. \( \|Tu\|_{L^p(\partial D)} \leq C\|u\|_{W^{1,p}(D)} \), where \( C \) is independent of \( u \).

Proof: Fix \( a \in \partial D \). Then there exist \( r > 0 \) and Lipschitz map \( \gamma \) as in §1.1. Let \( f(x) = (x, \gamma(x)) \), \( x \in \mathbb{R}^{d-1} \) and

\[
g(x) = 1_{f^{-1}(D \cap B(a, \frac{r}{2}))}|u(f(x))|^p.
\]

Remark that we need \( p < \infty \). Obviously \( f \) is injective and Jacobian (cf. [4], §3.3.4) \( Jf = 1 + |D\gamma|^2 \leq c \), where \( c \) equals to one plus square of the Lipschitz constant of \( \gamma \). Then

\[
\int_{B(a, \frac{r}{2}) \cap \partial D} |u|^p \, dH^{d-1} = \int_{\mathbb{R}^d} \left[ \sum_{x \in f^{-1}\{w\}} g(x) \right] \, dH^{d-1}(w).
\]

By Theorem 1,

\[
\int_{\mathbb{R}^{d-1}} \left[ \sum_{x \in f^{-1}\{w\}} g(x) \right] \, dH^{d-1}(w) = \int_{\mathbb{R}^{d-1}} g(x) \, Jf(x) \, dx \leq c \int_{\mathbb{R}^{d-1}} g(x) \, dx.
\]

3
Let $\zeta \in C^\infty_c(B(a, r))$ with $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B(a, \frac{r}{2})$. Since $\mathbb{R}^{d-1}$ can be seen as $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_d = 0\}$,

$$
\int_{\mathbb{R}^{d-1}} g(x) dx \leq - \int_{B(a, r)\cap \{x_d \geq 0\}} (\zeta|u|^p)_{x_d} dx.
$$

Expand

$$
- \int_{B(a, r)\cap \{x_d \geq 0\}} (\zeta|u|^p)_{x_d} dx = - \int_{B(a, r)\cap \{x_d \geq 0\}} |u|^p \zeta_{x_d} + p|u|^{p-1} sgn(u)u_{x_d} \zeta dx.
$$

Since $\partial D$ is compact, for any open cover as above, there exists a finite sub-cover. For such finite sub-cover, $\sup |D\zeta|$ is uniformly bounded on this finite sub-cover. Hence we have

$$
\int_{\partial D} |u|^p d\mathcal{H}^{d-1} \leq C \int_D |u|^p + |Du|^p dx,
$$

where constant $C$ does not depend on $u$. Write $T : W^{1,p}(D) \to L^p(\partial D)$ as

$$
Tu := u|_{\partial D},
$$

and this is well-defined since it is a continuous linear operator between Banach spaces.

### 2.2 Kernel of trace operator on Lipschitz domain

Now we are going to complete the proof by using change of variables formulas into the proof of Theorem 2 in §5.5 of [3].

**Theorem 4 (Theorem 4.10, §2.4.3 in [2]).** Let $D \subset \mathbb{R}^d$ be a Lipschitz domain and $u \in W^{1,p}(D)$, $p \in [1, \infty)$. Then $u \in C^\infty_c(D)$ if and only if $Tu = 0$ on $\partial D$.

**Proof:** One side is trivial. To show the converse, firstly we establish a priori estimate.

Let $Tu = 0$ on $\partial D$ and $a \in \partial D$. Then there exist $r > 0$, Lipschitz map $\gamma$, $F : \mathbb{R}^d \to \mathbb{R}^d$, $F(x) = y$ be Lipschitz coordinate transformation as in §1.1, and $u_m \in C^1(D)$ such that $u_m \to u$ in $W^{1,p}(D)$ and $Tu_m \to 0$ in $L^p(\partial D)$ as $m \to \infty$. If $y' \in \mathbb{R}^{d-1}$, $y_d > 0$, and $(y', y_d) \in F(B(a, r) \cap \partial D)$, then

$$
|u_m(y', y_d)| \leq |u_m(y', 0)| + \int_0^{y_d} |\partial_{y_d} u_m(y', s)| ds. \quad (1)
$$

We use Theorem 2 (or just ordinary change variables) by taking $m = n = d - 1$,

$$
g(y', y_d) = |u_m(y', y_d)|^p 1_{F(B(a, r) \cap \partial D)}(y'),
$$
and \( f(x') = y' \), thus \( Jf = 1 \). Then

\[
\int_{B(a,r) \cap \partial D} |u_{m}(x', x_d)|^p \, dx' = \int_{F(B(a,r) \cap \partial D)} |u_{m}(y', 0)|^p \, dH^{d-1}(y').
\]

Taking \( p \) power (so we need \( p < \infty \)) on equation (1), we have

\[
|u_{m}(y', y_d)|^p \leq C(|u_{m}(y', 0)|^p + (\int_0^{y_d} |\partial_{y_d} u_{m}(y', s)| \, ds)^p),
\]

and then

\[
(\int_0^{y_d} |\partial_{y_d} u_{m}(y', s)| \, ds)^p \leq y_d^{p-1} (\int_0^{y_d} |\partial_{y_d} u_{m}(y', s)|^p \, ds)
\]

by Jensen’s inequality. Then integrate with respect to \( y' \), on \( B := \{y' \in \mathbb{R}^{d-1} \mid (y', \cdot) \in F(B(a, r)) \} \)

\[
\int_{B} |u_{m}(y', y_d)|^p \, dy' \leq C(\int_{B} |u_{m}(y', 0)|^p \, dy' + y_d^{p-1} (\int_{B} |Du_{m}(y', s)|^p \, dy') \, ds).
\]

Let \( m \to \infty \), and then we have a priori estimate

\[
\int_{B} |u(y', y_d)|^p \, dy' \leq Cy_d^{p-1} (\int_{B} |Du(y', s)|^p \, dy') \, ds,
\]  

(2)

for a.e. \( y_d > 0 \).

Now we are going to approximate \( u \) under Lipschitz coordinate transformation. Let \( \zeta \in C^\infty_0(\mathbb{R}_+) \) such that \( 0 \leq \zeta \leq 1 \), \( \zeta|_{[r, 1]} = 0 \), \( \zeta|_{[0, 1]} = 1 \), \( \zeta_k(y) := \zeta(ky_d) \), \( \forall y \in \mathbb{R}_+^d \), and \( w_k := u(1 - \zeta_k) - u \) in \( L^p(\mathbb{R}_+^d) \) as \( k \to \infty \). Note that \( \sup |\zeta'| < \infty \).

The remainder is to estimate

\[
|Du_k - Du|^p = |\zeta_k Du + ku \zeta'|^p \leq C(|\zeta_k|^p|Du|^p + k^p|\zeta'(ky_d)|^p|u|^p).
\]

Since \( \text{supp} \zeta_k \subset [0, \frac{2}{k}] \), \( \int |\zeta_k|^p|Du|^p \to 0 \) as \( k \to \infty \) by Lebesgue’s dominated convergence Theorem. Since \( \text{supp} \zeta' \subset [0, 2] \), to integrate the last term on \( F^{-1}(B \times [0, 2]) \), we use Theorem 2 again by taking \( m = n = d \),

\[
g(y', y_d) = |u(y', y_d)|^p 1_{B(y')(1, 2/k)}(y_d),
\]

and \( F(x', x_d) = (y', y_d) \), thus \( JF = 1 \) a.e. And then the priori estimate (2) shows

\[
Ck^p (\int_{B} |u(y', y_d)|^p \, dy')^{2/k} dy_d \leq Ck^p (\int_0^{y_d} dy_d)^{2/k} (\int_{B} |Du(y', s)|^p \, dy') \, ds,
\]

(3)
therefore
\[
(3) \leq \int_0^{2/k} \int_B |Du(y', s)|^p dy'ds \to 0
\]
as \( k \to \infty \). By partition of unity as in §1.1 above, we deduce \( w_k \to u \) in \( W^{1,p}(\mathbb{R}^d_+) \), and \( w_k = 0 \) if \( 0 < y_d < 1/k \). To conclude, we can mollify \( w_k \) to produce \( u_k \in C_\infty_c(\mathbb{R}^d_+) \) such that \( u_k \to u \) in \( W^{1,p}(\mathbb{R}^d_+) \), i.e., \( u \in C_\infty_c(\mathbb{R}^d_+) \). This completes the proof.

3 Concluding Remark

The only difference between the Theorem 2 in §5.5 of [3], and this Theorem 4, is that we use change of variables formulas for Lipschitz coordinate transformation. Recall that we need a topological condition: boundary of Lipschitz domain \( \partial D \) is compact, to make sure that the trace operator is bounded. However we have no idea what would happen, if \( D \) is not bounded, or \( p = \infty \), even in smooth domain.

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