Exact solutions of magnetohydrodynamic flow of PTT fluid

Naeem Faraz\textsuperscript{1, 2, 5}, Zhiming Lu\textsuperscript{1}, Lei Hou\textsuperscript{2}, Yasir Khan\textsuperscript{3} and Ahmed Faisal Siddiqi\textsuperscript{4}

\textsuperscript{1}Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, 200072
\textsuperscript{2}Department of Mathematics, Shanghai University, Shanghai 200444, China
\textsuperscript{3}Department of Mathematics, University of Hafr Al-Batin, Saudia Arabia
\textsuperscript{4}Business School, University of Central Punjab, Lahore 54600, Pakistan
\textsuperscript{5}Email: nfaraz_math@yahoo.com

Abstract. An attempt is made for the first time to investigate the magnetohydrodynamic (MHD) flow of Phan-Thien Tanner fluid by finding travelling wave solution. By using non-dimensional variables, partial differential equation has been converted into ordinary differential equation. The resulted nonlinear ordinary differential equation is solved by using well known \textit{G'/G} expansion method and He’s semi inverse method. We applied He’s semi-inverse method to construct a variational theory for MHD flow of PTT fluid, which helped to get the solitary solution by using Ritz method. The \textit{G'/G}-expansion method is used to seek more general exact solutions. In these methods, it is not important to have boundary conditions as we can get many useful solutions by simulations on the base of the variables appearing in the solution.

1. Introduction

The field of Magnetohydrodynamic (MHD) was initiated by Hannes Alfven, in 1970, for which he was awarded Nobel Prize in Physics. MHD is very import to read the motion of conducting fluids. A broad range of domains such as intensely heated and ionized fluids in an electromagnetic field in astrophysics, plasma physics, liquid metals, salt water, aerodynamics deals with MHD flow\textsuperscript{1}.

However, currently all technological advancements acknowledge the importance of non-Newtonian fluids\textsuperscript{2}. All branches of fluids mainly rate, differential and integral types of fluids explore the nonlinear relationships between rate of strain and stress. Due to this non-linear relation, study of non-Newtonian fluid’s motion became much perplexed and knotty. One of among them is PTT fluid model\textsuperscript{3}. Unlike other non-Newtonian fluids, a classifiable advantage of PTT fluid is the elongation parameter, which reproduces many of the characteristics of the rheology of polymer solutions and other non-Newtonian fluids. The PTT model is one of the most widely used rheological models and can properly describe the common characteristics of viscoelastic non-Newtonian fluids because it is well regarded and derivable from molecular considerations. For this reason, many researchers are interested to study PTT flow from different angles and in different geometries like expansion and contraction entry.

Hou Lei did a remarkable job in PTT fluid and references therein. He developed and solved the nonlinear problem of pressure and stress induced non-Newtonian fluid. Hou Lei studied the boundary layer approach in the contact interface, by use of standard mean variables in material sciences \textsuperscript{3}, \textsuperscript{4}. 

\textsuperscript{1}Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, 200072
\textsuperscript{2}Department of Mathematics, Shanghai University, Shanghai 200444, China
\textsuperscript{3}Department of Mathematics, University of Hafr Al-Batin, Saudia Arabia
\textsuperscript{4}Business School, University of Central Punjab, Lahore 54600, Pakistan
\textsuperscript{5}Email: nfaraz_math@yahoo.com

Abstract. An attempt is made for the first time to investigate the magnetohydrodynamic (MHD) flow of Phan-Thien Tanner fluid by finding travelling wave solution. By using non-dimensional variables, partial differential equation has been converted into ordinary differential equation. The resulted nonlinear ordinary differential equation is solved by using well known \textit{G'/G} expansion method and He’s semi inverse method. We applied He’s semi-inverse method to construct a variational theory for MHD flow of PTT fluid, which helped to get the solitary solution by using Ritz method. The \textit{G'/G}-expansion method is used to seek more general exact solutions. In these methods, it is not important to have boundary conditions as we can get many useful solutions by simulations on the base of the variables appearing in the solution.

1. Introduction

The field of Magnetohydrodynamic (MHD) was initiated by Hannes Alfven, in 1970, for which he was awarded Nobel Prize in Physics. MHD is very import to read the motion of conducting fluids. A broad range of domains such as intensely heated and ionized fluids in an electromagnetic field in astrophysics, plasma physics, liquid metals, salt water, aerodynamics deals with MHD flow\textsuperscript{1}.

However, currently all technological advancements acknowledge the importance of non-Newtonian fluids\textsuperscript{2}. All branches of fluids mainly rate, differential and integral types of fluids explore the nonlinear relationships between rate of strain and stress. Due to this non-linear relation, study of non-Newtonian fluid’s motion became much perplexed and knotty. One of among them is PTT fluid model\textsuperscript{3}. Unlike other non-Newtonian fluids, a classifiable advantage of PTT fluid is the elongation parameter, which reproduces many of the characteristics of the rheology of polymer solutions and other non-Newtonian fluids. The PTT model is one of the most widely used rheological models and can properly describe the common characteristics of viscoelastic non-Newtonian fluids because it is well regarded and derivable from molecular considerations. For this reason, many researchers are interested to study PTT flow from different angles and in different geometries like expansion and contraction entry.

Hou Lei did a remarkable job in PTT fluid and references therein. He developed and solved the nonlinear problem of pressure and stress induced non-Newtonian fluid. Hou Lei studied the boundary layer approach in the contact interface, by use of standard mean variables in material sciences \textsuperscript{3}, \textsuperscript{4}.
However, like other viscoelastic non-Newtonian fluids the governing equations of the flow of PTT fluids are nonlinear and their exact solutions are rare. A lot of research has been done related to PTT fluid. Recently a series of articles has appeared which present analytical solutions of PTT fluids. For instance, Siddiqi investigated the stationary points and uniform thickness of PTT fluid film on a vertically upward moving belt[5], Ferras studied the channel flow and found the analytical solution[6], Vajravelu investigated peristaltic transport of Phan-Thien-Tanner fluid in an asymmetric channel induced by sinusoidal peristaltic waves travelling down the flexible walls of the channel[7], Dhadwal figured out the effect of viscoelastic relaxation modes on stability of extrusion film casting process modeled using multi mode Phan-Thien Tanner fluid[8].

Above mentioned literature is related to analytical analysis. Analytical solutions are always important to study various fluid flows, which plays a vital role to investigate the role of physical parameters appears in the flow. However, it becomes difficult to compute the series solution of highly nonlinear problems. Although with the speedy advancement of technology, a lot of work has been done to obtain the numerical solution with the help of super computers. The accuracy of all the numerical approximations can only be verified by comparing with the exact solutions. Therefore, it is quite useful to figure out exact solutions not only for verification of numerical methods, but also they are able to describe the detailed behavior of the problem under consideration.

There are various techniques to find the exact solution of nonlinear differential equations. Among them various are soliton solution for solving differential equations [9], Hirota’s bilinear approach[10], Backland transformation[11], and many others to find analytical solutions[12]–[17].

In this investigation, we applied $G'/G$-expansion method[18] and He’s semi inverse method [19]. $G'/G$-expansion method was developed for reliable treatment of nonlinear equations. It is straightforward, and have capacity of producing new applications without incorporating boundary conditions. Furthermore, the solutions obtained by this method are of general nature and a number of specific solutions can be deduced by substituting values to arbitrary constants. He’s semi inverse method is used to obtain Solitary solution of the viscoelastic flow.

2. Basic equations

Akyildiz[20] obtained the following set of ordinary differential equations describing the MHD flow of PTT fluid

\[
\frac{\partial p}{\partial x} = -\sigma B^2 u + \frac{\partial \tau_{xy}}{\partial y} \tag{2.1}
\]

\[
\frac{\partial p}{\partial y} = \frac{\partial \tau_{xy}}{\partial y} \tag{2.2}
\]

\[
f \left( tr(\tau) \right) \tau_{xx} - 2\lambda \tau_{xy} \frac{\partial u}{\partial y} = 0 \tag{2.3}
\]

\[
f \left( tr(\tau) \right) \tau_{yy} - \lambda \tau_{xy} \frac{\partial u}{\partial y} = \mu \frac{\partial u}{\partial y} \tag{2.4}
\]

upon solving above two equation we get the expression

\[
\tau_{xx} = 2 \frac{\lambda}{\mu} \tau_{xy}^2 \tag{2.5}
\]

Defining non-dimensional variables[20]

\[
y^* = \frac{y}{H}, u^* = \frac{u}{U}, \tau^* = \frac{\tau H}{\mu U}, p^* = \frac{p H}{\mu U}, N^* = \frac{NH^2}{\mu} \tag{2.6}
\]

and substituting (2.6) in (2.1) by using the non-dimensional variables, we obtain (after dropping the asterisks)
0 = \frac{dp}{dx} + \frac{d\tau_{xy}}{dy} - Nu, \quad (2.7)

By using eq. (2.5) and (2.1) we obtain
\[ (1 + 2\varepsilon D_e^2\tau_{xy}^2)\tau_{xy} = \frac{du}{dy} \tag{2.8} \]

where \( D_e = \lambda U/L \) is the Deborah number and \( N = \sigma B_0^2 > 0 \) is the magnetic parameter and \( \delta_0 = 1 \).

By using eq. (2.1) we get
\[ \frac{du}{dy} = \frac{1}{N} \frac{d^2\tau_{xy}}{dy^2} \tag{2.9} \]

Substituting eq. (2.9) into (2.8) after simple calculations we get
\[ N\tau_{xy}(1 + 2\varepsilon D_e^2\tau_{xy}^2) = \frac{d^2\tau_{xy}}{dy^2} \tag{2.10} \]

we can write eq. (2.10) in more appropriate form as follows:
\[ \frac{d^2\psi}{dy^2} = N\psi + \beta\psi^3 \tag{2.11} \]

where \( \beta = 2\varepsilon D_e, \psi = \tau_{xy} \) and appropriate no-slip boundary conditions for the problem are
\[ u(0) = 0, u(1) = 0 \tag{2.12} \]

the above boundary conditions can be written as
\[ \frac{d}{dy}\tau_{xy}(0) = P, \frac{d}{dy}\tau_{xy}(1) = P, \tag{2.13} \]

3. The \( G'/G \) expansion method
To see the detail procedure of the method one can refer to [20] and reference therein. Assume that it is possible to express solution of (2.11) by a polynomial in \( G'/G \) as
\[ \psi = \sum_{i=0}^{m} k_i \left( G'/G \right)^i \tag{3.1} \]

or
\[ \psi = \sum_{i=-m}^{m} k_i \left( G'/G \right)^i \tag{3.2} \]

where \( \psi = \psi(y) \) satisfies the second order linear ordinary differential equation
\[ G''/ + \lambda G'/ + \mu G = 0 \tag{3.3} \]

from eq. (2.11) we balance \( \psi''/ \) with \( \psi^3 \), which gives us \( m = 1 \). As a result eq. (3.1) and (3.2) takes the following form respectively:
\[ \psi(y) = k_0 + k_1 \left( G'/G \right)^1 \tag{3.4} \]
\[ \psi(y) = k_{-1} \left( G'/G \right)^{-1} + k_0 + k_1 \left( G'/G \right)^1 \tag{3.5} \]

By using eq. (3.4) and (3.5) in (2.11) we get the following system of algebraic equations respectively
\[2k_1 - \beta k_1^3 = 0\]
\[3k_1\lambda - 3\beta k_1^2 k_0 = 0\]
\[k_1\lambda^2 + 2k_1\mu - Nk_1 - 3\beta k_1^2 k_0 = 0\]
\[k_1\lambda\mu - Nk_0 + k_0^3\beta = 0\]  \hspace{1cm} (3.6)

\[2k_{-1}\mu - \beta k_{-1}^3 = 0\]
\[2k_{-1}\lambda - 3\beta k_{-1}^2 k_0 - k_{-1}\lambda\mu = 0\]
\[-2k_{-1} - k_{-1}\lambda^2 - 2k_{-1}\mu - Nk_{-1} - 3\beta k_{-1}^2 k_{-1} - 3\beta k_{-1}^3 k_{-1} = 0\]
\[-3k_{-1}\lambda + k_{-1}\lambda\mu - Nk_{-1} k_0 - k_0^3\beta - 6k_{-1} k_0 k_{1} = 0\]
\[-2k_{-1} + k_{1}\lambda^2 + 2k_{1}\mu - Nk_{1} - 3\beta k_{1}^2 k_{1} - 3\beta k_{1}^3 k_{-1} = 0\]
\[3k_{1}\lambda - 3\beta k_{1} k_{-1} = 0\]
\[2k_{1} - \beta k_{1}^3 = 0\]

The solution of these algebraic equations given in (3.6) and (37) can be done by Maple which gives us same solution

\[k_0 = \pm \sqrt{2\lambda - N\over \beta}, \quad k_1 = m \sqrt{2\over \beta}, \quad k_{-1} = 0\]  \hspace{1cm} (3.8)

On substituting the general solution of eq. (3.3) and eq. (3.8) in (3.4) or (3.5) we get the exact solution of eq. (2.11) as follows:

when \(\lambda^2 - 4\mu > 0\)

\[\psi_1(y) = \pm \sqrt{2\lambda - N\over \beta} m \sqrt{2\over \beta} \left[ -\lambda \over 2 + \sqrt{\lambda^2 - 4\mu} \left( C_1 Cosh 1 \over 2 \sqrt{\lambda^2 - 4\mu} y + C_2 Sinh 1 \over 2 \sqrt{\lambda^2 - 4\mu} \right) \right] \]  \hspace{1cm} (3.9)

when \(\lambda^2 - 4\mu < 0\)

\[\psi_1(y) = \pm \sqrt{2\lambda - N\over \beta} m \sqrt{2\over \beta} \left[ -\lambda \over 2 + \sqrt{-\lambda^2 + 4\mu} \left( C_1 Cos 1 \over 2 \sqrt{-\lambda^2 + 4\mu} y - C_2 Cosh 1 \over 2 \sqrt{-\lambda^2 + 4\mu} \right) \right] \]  \hspace{1cm} (3.10)

When \(\lambda^2 - 4\mu = 0\)

\[\psi_1(y) = \pm \sqrt{2\lambda - N\over \beta} m \sqrt{2\over \beta} \left[ -\lambda \over 2 + C_2 \over C_1 + C_2 y \right] \]  \hspace{1cm} (3.11)

If \(C_1 \neq 0, C_2 = 0, \lambda > 0\) and \(\mu = 0\) then

\[\psi_1(y) = \pm \sqrt{2\lambda - N\over \beta} m \sqrt{2\over \beta} \left[ -\lambda \over 2 + \lambda \over 2 Cot \lambda \over 2 y \right] \]  \hspace{1cm} (3.12)

which is the exact solution of magnetohydrodynamic flow of viscoelastic flow of PTT fluid.
4. He’s Semi-Inverse method

In the past few decades, many analytical and numerical techniques have been introduce to handle the nonlinear problems. Also different method has been proposed to find the exact solution of nonlinear partial differential equation. Among them the semi-inverse method, established by Chinese Mathematician in late 90’s[19] is a powerful and effective method for searching for variational principal for physical problems and provides physical insight into the nature of solitary solutions.

By the semi-inverse method, the following variational formulation is constructed

\[ J = \int_{0}^{\infty} \left[ \left( \frac{1}{2} \right) \left( \psi' \right)^2 - N \left( \psi \right)^2 - \left( \frac{\beta}{4} \right) \left( \psi' \right)^4 \right] dy \]  

(4.1)

By using Ritz method, we look for different solitary wave solutions in the form

\[ \psi = p \sec h(qy), \psi = p\cosech(qy), \psi = p \tan h(qy), \psi = p \cot h(qy) \]  

(4.2)

for instant we are using \( \psi = p \sec h(qy) \) for solitary wave solution. Whereas \( p \) and \( q \) are constant to be further determined. On substituting, \( \psi = p \sec h(qy) \) into (4.1), we get:

\[ J = \int_{0}^{\infty} \left[ \left( \frac{1}{2} \right) \left( pq \sec h^2(qy) \tanh^2(qy) \right) - N \left( p \sec h(qy) \right)^2 - \left( \frac{\beta}{4} \right) \left( p \sec h(qy) \right)^4 \right] dy \]

\[ = \left( \frac{pq}{\sqrt{2}} \right)^2 \int_{0}^{\infty} \left( \sec h(qy) \tanh^2(qy) \right) dy - Np^2 \int_{0}^{\infty} \left( \sec h(qy) \right)^2 dy - \left( \frac{\beta}{4} \right) p^4 \int_{0}^{\infty} \left( \sec h(qy) \right)^4 dy \]  

(4.3)

\[ = - \left( \frac{\beta}{6q} \right) p^4 - Np^2 \frac{p^2}{2q} + \frac{p^2q}{6} \]

By making \( J \) stationary with respect to \( p \) and \( q \), we get

\[ \frac{\partial J}{\partial p} = - \frac{2\beta}{3q} p^3 - N \frac{p}{q} + \frac{pq}{3} = 0 \]

(4.4)

\[ \frac{\partial J}{\partial q} = \frac{\beta}{6q^2} p^4 + N \frac{p^2}{2q^2} + \frac{p^2}{6} = 0 \]

Or

\[ -2\beta p^3 - 3Np + pq = 0 \]

(4.5)

\[ \beta p^4 - 3Np^2 + q^2 p^2 = 0 \]

By using Maple, we solve equation (4.5) for \( p \) and \( q \), which are as follows:

Case-I

\[ q = \frac{1}{4} \left\{ 1 - \sqrt{\frac{\beta^2 - 24N\beta^2}{\beta}} \right\}, p = \pm \sqrt{\frac{1}{\beta} - \frac{12N}{\beta} - \frac{\sqrt{\beta^2 - 24N\beta^2}}{\beta^2}} \]  

(4.6)

Case-II

\[ q = \frac{1}{4} \left\{ -1 + \sqrt{\frac{\beta^2 - 24N\beta^2}{4\beta}} \right\}, p = \pm \sqrt{\frac{1}{8\beta} - \frac{3N}{2\beta} - \frac{\sqrt{\beta^2 - 24N\beta^2}}{\beta^2}} \]  

(4.7)

So the solitary wave solutions can be obtained by solving equation (4.5), however, we have mentioned two cases i.e. case-I and case-II which gives us a real solution by choosing suitable values for the constant appearing in both cases. Hence final solution can be written in the following form respectively on the bases of aforesaid two cases
\[
\psi = \pm \sqrt{\frac{1}{\beta} - \frac{12N}{\beta^2} - \sqrt{\frac{2}{2} - \frac{24N\beta^2}{\beta^2}}} \cos h\left(\frac{1}{4} - \frac{1}{\beta} - \frac{24N\beta^2}{\beta^2}\right) y \tag{4.8}
\]

\[
\psi = \pm \sqrt{\frac{1}{\beta} - \frac{3N}{2\beta} - \sqrt{\frac{2}{2} - \frac{24N\beta^2}{\beta^2}}} \cos h\left(\frac{1}{4} + \frac{1}{\beta} - \frac{24N\beta^2}{\beta^2}\right) y \tag{4.9}
\]

Similarly, if we substitute both cases one by one in the rest of the solution forms mentioned in eq. (4.2) we can get other forms of the solutions such as:

\[
\psi = \pm \sqrt{\frac{1}{\beta} - \frac{12N}{\beta^2} - \sqrt{\frac{2}{2} - \frac{24N\beta^2}{\beta^2}}} \sec h\left(\frac{1}{4} - \frac{1}{\beta} - \frac{24N\beta^2}{\beta^2}\right) y \tag{4.10}
\]

\[
\psi = \pm \sqrt{\frac{1}{\beta} - \frac{3N}{2\beta} - \sqrt{\frac{2}{2} - \frac{24N\beta^2}{\beta^2}}} \sec h\left(\frac{1}{4} + \frac{1}{\beta} - \frac{24N\beta^2}{\beta^2}\right) y \tag{4.11}
\]

5. Conclusions

Seeking the exact solutions of nonlinear partial differential equations plays an significant role, when we want to understand the physical mechanism of the phenomena such as the wave phenomena observed in fluid dynamics. An exact solution for viscoelastic model has been presented by using well known techniques called \(G'/G\)-expension and He’s semi inverse method. The solution procedure is different then the traditional approach available in the literature. In this study, system has been converted into ordinary differential equation by introducing dimensionless variables rather than by considering the wave transformation. It is found that \(G'/G\)-expension method and He’semi inverse method are relaiable techniques for the solution of non-Newtonian fluid problems. The obtained solutions with free parameters are important to explain some physical phenomena. Obtained results can reveal the solution obtained by Faraz (2015) with the suitable choice of constants. It is proven that the performance of these method is productive, effective and well-built mathematical tool for solving non-linear equations. To summarize, based on the wave transformation, we have obtained many new exact solutions of the nonlinear equation. We have constructed a variety of new solutions of the mentioned equation. Since the technique is direct and powerful it can be used to handle a variety of equations which appears in fluid mechanics.

Acknowledgements

We are grateful to the second and third autor for their valuable suggestions and feedback. We are also grateful to the Shanghai University for providing us financial support.

References

[1] Nadeem S, Hussain M and Naz M 2010 MHD stagnation flow of a micropolar fluid through a porous medium Meccanica 45(6) 869-880

[2] Faraz N 2011 Study of the effects of the Reynolds number on circular porous slider via variational iteration algorithm-II Comput. Math. with Appl. 61(8) 1991-1994

[3] Faraz N, Hou L and Khan Y 2015 Analytical Solution of Linear, Quadratic and Cubic Model of PTT Fluid J. Appl. Comput. Mech. 1(4) 220-228

[4] Hou L, Faraz N, Sun X Y, Zhao J J and Qiu L 2014 The convergence of coupled boundary-layer equation in the contact interface 654

[5] Siddiqui A M, Walait A, Haroon T and Ashraf H 2016 On the study of stationary points and
uniform thickness of PTT fluid film on a vertically upward moving belt Can. J. Phys. 94(10)

[6] Ferrás L L, Nóbrega J M and Pinho F T 2012 Analytical solutions for channel flows of Phan-Thien-Tanner and Giesekus fluids under slip J. Nonnewton. Fluid Mech. 171-172 97-105

[7] Vajravelu K, Sreenadh S, Lakshminarayana P, Sucharitha G and Rashidi M M 2016 Peristaltic flow of phan-thien-tanner fluid in an asymmetric channel with porous medium J. Appl. Fluid Mech. 9(4) 1615-1625

[8] Dhadwal R, Banik S, Doshi P and Pol H 2017 Effect of viscoelastic relaxation modes on stability of extrusion film casting process modeled using multi-mode Phan-Thien-Tanner constitutive equation Appl. Math. Model 47 487-500

[9] Eskandari M and Safian R 2010 Inverse scattering method based on contour deformations using a fast marching method Inverse Probl. 26(9)

[10] Tang Y, Zai W, Tao S and Guan Q 2017 Binary Bell polynomials, Hirota bilinear approach to Levi equation Appl. Math. Comput. 293 565-574

[11] Gordoa P R, Pickering A and Zhu Z N 2013 Backlund transformation of matrix equations and a discrete matrix first Painlevé equation Phys. Lett. A 377(19-20) 1345-1349

[12] Faraz N, Hou L and Khan Y 2015 Analytical solution of linear, quadratic and cubic model of PTT fluid J. Appl. Comput. Mech. 1(4)

[13] Faraz N, Khan Y and Austin F 2010 An alternative approach to differential-difference equations using the variational iteration method J. Appl. Math. Comput. Mech. 1(4)

[14] Faraz N and Khan Y 2012 Study of the rate type fluid with temperature dependent viscosity Zeitschrift fur Naturforsch.-Sect. A J. Phys. Sci. 65(12)

[15] Nadeem S and Faraz N 2010 Thin film flow of a second grade fluid over a stretching/shrinking sheet with variable temperature-dependent viscosity Chinese Phys. Lett. 27(3)

[16] Faraz N and Khan Y 2011 Analytical solution of electrically conducted rotating flow of a second grade fluid over a shrinking surface Ain Shams Eng. J. 2(3-4)

[17] Faraz N 2011 Study of the effects of the Reynolds number on circular porous slider via variational iteration algorithm-II Comput. Math. with Appl. 61(8)

[18] Ozkan G and Ahmet B 2016 A novel method for nonlinear fractional differential equations using symbolic computation WAVES IN RANDOM AND COMPLEX MEDIA 2 1-8

[19] He J H 1997 Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics Int. J. Turbo Jet Engines 14(1) 23-28

[20] Akyildiz F T and Vajravelu K 2008 Magnetohydrodynamic flow of a viscoelastic fluid Phys. Lett. A 372(19) 3380-3384