ELLiptic fibrations and the singularities of their Weierstrass models

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Abstract. The aim of this paper is to investigate the structure of non-Kodaira fibres in elliptic threefolds. I will show that for minimal elliptic threefolds these fibres are contractions of Kodaira fibres. I will give a necessary condition for a Weierstrass fibration to be the Weierstrass model of an elliptic threefold and I will present several examples.

1. Introduction

The study of elliptic fibrations, i.e. morphisms $\pi: X \to B$ whose generic fibre is an elliptic curve, dates back to Kodaira’s paper [9], where one can find a detailed description of elliptic surfaces. Since then, the case of elliptic surfaces has been a benchmark for generalizations to higher dimension. Much of what is true for surfaces generalizes to the higher dimensional case, but many of these results are weaker: for example, what is true punitally on the base curve in the case of surfaces becomes true only generically in codimension 1 for higher dimensional varieties.

An important result of Kodaira is the classification of the singular fibres which can occur in a smooth elliptic surface. In this case there is a generalization to the higher dimensional case, and Kodaira’s classification works over the generic point of each irreducible component of the discriminant locus of the fibration, but there appear also new non-Kodaira fibres: see e.g. [13] or the more recent [5, 6].

Despite the great abundance of examples of non-Kodaira fibres, less is known on their structure and a complete classification is still far away. In this paper I will show that for the elliptic fibrations whose total space is a threefold (the first case after the case of surfaces), there is still a link with the Kodaira fibres: the non-Kodaira fibres in an elliptic threefold are contraction of Kodaira fibres.

The structure of the paper is as follows: in Section 2 I will recall the facts of duality theory over Gorenstein varieties which are needed to treat the canonical singularities, in Section 3 These sections are expository, and form the technical heart of the paper. In particular, I will recall an important theorem of Reid on the resolution of threefolds with cDV singularities. In section 4 I will discuss the elliptic fibrations, and in particular the elliptic fibrations with section. I will define what a minimal elliptic fibration is (Definition 4.2) and in the case of three dimensional fibrations (Section 4.3) I will prove two results. The first is a necessary condition for a Weierstrass fibration to be the Weierstrass model of some smooth elliptic threefold (Theorem 4.14), and the second is a result on the structure of non-Kodaira fibres:

Theorem (Theorem 4.15). Let $\pi: X \to B$ be a smooth minimal elliptic threefold with section. If $b \in B$ is a point such that the fibre $X_b$ is of non-Kodaira type, then $X_b$ is a contraction of the Kodaira fibre over $b$ of the elliptic surface obtained restricting $X$ to a generic smooth curve through $b$. 

Finally, in Section 5 I will give some examples, some of which were the original motivation for this paper. Through all the paper, all the varieties are defined over \( C \).

2. Duality

In this section I will recall some facts about duality theory on (singular) varieties. The aim of this section is to keep the paper self-contained, all the results exposed here are well known and can be found in the literature (the main reference is [1]). For this reason, proofs will be omitted.

2.1. The dualizing sheaf. One of the main tools for dealing with smooth varieties is their canonical bundle. Recall that the canonical bundle \( \bigwedge^n \Omega^1_X \) of a smooth variety \( X \) of dimension \( n \) is, by definition, the \( n \)-th exterior power of its cotangent bundle \( \Omega^1_X \). There are many reasons why it is so important, here I want to recall two of them:

1. Adjunction. Let \( X \) be a smooth \( n \)-dimensional variety and \( Y \) a smooth \( \frac{n}{2} \)-dimensional hypersurface of \( X \). Then

   \[
   \bigwedge^{n-1} \Omega^1_Y = i^* \left( \bigwedge^n \Omega^1_X \otimes \mathcal{O}_X(Y) \right)
   \]

   where \( i : Y \hookrightarrow X \) is the inclusion map.

2. Serre duality. Let \( X \) be a smooth \( n \)-dimensional variety and \( L \) a line bundle on \( X \). Then for each \( i = 0, \ldots, n \) there are natural isomorphisms

   \[
   H^i(X, L) \simeq H^{n-i}(X, L^\vee \otimes \bigwedge^n \Omega^1_X),
   \]

   where \( L^\vee \) is the dual bundle of \( L \).

To deal with singular varieties one has to be more general, in the following two senses: first of all, it’s convenient to use sheaves instead of bundles, and second we need to give a new definition of canonical sheaf which extends the usual one for smooth varieties to possibly singular ones.

Definition 2.1 ([8, §III, Prop. 5.7]). Let \( X \subseteq \mathbb{P}^N \) be an irreducible \( n \)-dimensional variety. The dualizing sheaf of \( X \) is

\[
\omega_X = \mathcal{E}xt^{N-n}_{\mathbb{P}^N} \left( \mathcal{O}_X, \bigwedge^N \Omega^1_{\mathbb{P}^N} \right).
\]

Remark 2.1. The dualizing sheaf of a projective variety \( X \) is intrinsic to \( X \), and does not depend on the embedding in \( \mathbb{P}^n \) (see [8] §III, Prop. 7.2).

Although this definition may be strange at first sight, it is satisfactory since it really enjoys the main features of the canonical bundle of a smooth variety. In particular, for smooth varieties the canonical sheaf and the dualizing sheaf coincide.

Proposition 2.2 ([8] §III, Cor. 7.12], [11] §I, Thm. 4.6). Let \( X \) be a projective \( n \)-dimensional smooth variety over an algebraically closed field. Then \( \bigwedge^n \Omega^1_X \simeq \omega_X \).

Notation. From now on, I will denote the canonical sheaf of a smooth variety \( X \) by \( \omega_X \). In case \( X \) is singular, \( \omega_X \) will denote its dualizing sheaf, but with a slight abuse of notation I will still call it the canonical sheaf of \( X \).

I will now explain the reason why this abuse makes sense. It is in fact known that for a projective variety \( X \), its dualizing sheaf is torsion-free of rank 1 (see e.g. [11] §I, Prop. 2.8] or [15] App. to §I, Thm. 7), and satisfies the following property
Whenever $\omega_X \hookrightarrow \mathcal{F}$ with $\mathcal{F}$ a torsion-free sheaf of $\mathcal{O}_X$-modules and $\text{codim} \supp(\mathcal{F}/\omega_X) \geq 2$, then $\mathcal{F} = \omega_X$. In turn, this implies the following proposition.

**Proposition 2.3** ([15, App. to §I, Cor. 8]). Let $X$ be a normal projective variety of dimension $n$. Then

1. $\omega_X$ is the double dual of the sheaf $\wedge^n \Omega^1_X$.
2. $\omega_X = j_*(\wedge^n \Omega^1_{X_{\text{sm}}})$, where $j : X_{\text{sm}} \hookrightarrow X$ is the inclusion of the smooth locus of $X$.
3. There exists a Weil divisor $K_X$ on $X$ such that $\omega_X = \mathcal{O}_X(K_X)$.

In particular, $\omega_X$ is the sheaf of differentials regular in codimension 1.

For this reason, $K_X$ will be called a canonical divisor of $X$.

### 2.2. Cohen–Macaulay and Gorenstein varieties

In this section I will recall the concepts of Cohen–Macaulay and Gorenstein varieties since these are the classes of varieties for which the dualizing sheaf $\omega_X$ behaves closer to the canonical sheaf of a smooth variety.

**Definition 2.2.** Let $A$ be a noetherian local ring of (Krull) dimension $r$. Then $A$ is Cohen–Macaulay if there exists a sequence of elements $x_1, \ldots, x_r$ in the maximal ideal of $A$ such that $x_{i+1}$ is not a zero-divisor in $A/(x_1, \ldots, x_i)$ for $i = 0, \ldots, r - 1$.

**Definition 2.3.** Let $X$ be a variety. We say that $X$ is Cohen–Macaulay if $\mathcal{O}_{X,x}$ is a Cohen–Macaulay ring for any $x \in X$.

**Example 2.1.** Since any regular local noetherian ring is Cohen–Macaulay ([1, §III, Cor. 4.12]), we have that smooth varieties are Cohen–Macaulay. More generally, if $Y$ is a local complete intersection in a smooth variety $X$, i.e. if the ideal sheaf $\mathcal{I}_Y$ of $Y$ in $X$ is locally generated by codim $Y$ elements, then

1. $Y$ is Cohen–Macaulay (see [1, §III, Cor. 4.5]);
2. $Y$ is normal if and only if it is regular in codimension 1 (see [8, §II, Prop. 8.23]).

A reason why Cohen–Macaulay varieties are important is that Serre duality holds.

**Proposition 2.4** ([8, §III, Cor. 7.7]). Let $X$ be a projective Cohen–Macaulay variety of pure dimension $n$. Then for any line bundle $\mathcal{L}$ on $X$ there are natural isomorphisms

$$H^i(X, \mathcal{L}) \cong H^{n-i}(X, \mathcal{L}^\vee \otimes \omega_X), \quad i = 0, \ldots, n.$$  

The subclass of the class of Cohen–Macaulay varieties which looks closer to the smooth varieties is the class of Gorenstein varieties.

**Definition 2.4.** Let $X$ be a projective variety. We say that $X$ is Gorenstein if it is Cohen–Macaulay, and its dualizing sheaf $\omega_X$ is locally free, i.e. it’s a line bundle.

In terms of the canonical Weil divisor $K_X$ for $X$, the condition that $\omega_X$ is locally free is equivalent to the condition that $K_X$ is a Cartier divisor.

**Remark 2.5** (Algebraic digression). It’s possible to develop a duality theory for general rings (see [4, §21]) and define what Gorenstein rings are. Then it would be natural to define a Gorenstein variety as a variety $X$ whose local rings $\mathcal{O}_{X,x}$
are all Gorenstein rings, in analogy with the definition of Cohen–Macaulay variety. Anyway, these two ways of defining Gorenstein varieties are equivalent, especially in view of [4, Thm. 21.15].

**Example 2.2.** Let $X$ be a smooth variety, and $Y \subseteq X$ a complete intersection. Then $Y$ is Gorenstein, since it is Cohen–Macaulay by Example 2.1 and $\omega_Y$ is a line bundle by [8, §III, Thm. 7.11].

### 2.3. Gorenstein singularities.

Now I want to address my attention to the singularities of Gorenstein varieties: the idea is that since they are so close to smooth varieties, their singularities should be mild.

Let $X$ be any variety, then we say that a birational morphism $f : Y \to X$ is a **resolution of the singularities** of $X$, or a **resolution** of $X$ for short, if $Y$ is smooth and $f$ is an isomorphism between $f^{-1}(X^{sm})$ and $X^{sm}$. A resolution $f : Y \to X$ is called **minimal** if any other resolution $f' : Y' \to X$ of $X$ factors through $f$.

**Definition 2.5.** Let $X$ be a Gorenstein variety, and $P \in X$ a singular point defined by the ideal sheaf $I_P$. We say that $P$ is a **rational Gorenstein singularity** if there exists a neighbourhood $U$ of $P$ in $X$ and a resolution $f : Y \to U$ of $U$ such that $f_*\omega_Y = \omega_U$. We say that $P$ is an **elliptic Gorenstein singularity** if there exists a neighbourhood $U$ of $P$ in $X$ and a resolution $f : Y \to U$ of $U$ such that $f_*\omega_Y = I_P \cdot \omega_U$.

**Definition 2.6.** We will say that a Gorenstein variety $X$ has rational Gorenstein singularities if all its singular points are rational Gorenstein singularities.

Such singularities were studied in [15, §2], where the following proposition is proved.

**Proposition 2.6 ([15, Thm. 2.6]).** Let $X$ be an $n$-dimensional Gorenstein variety, with $n \geq 2$, and $P \in X$. Then:

1. If $P$ is a rational Gorenstein singularity, then for a generic hyperplane section $H$ through $P$ we have that $P \in H$ is a rational or an elliptic Gorenstein singularity.
2. If there exists a hyperplane section $H$ through $P$ such that $P \in H$ is rational Gorenstein, then $P \in X$ is rational Gorenstein.

In the rest of this section I want to give some examples of such singularities, in particular when $X$ is a surface or a threefold.

**Example 2.3** (Surface rational Gorenstein singularities). Let $X$ be a projective surface and $P \in X$ be a point. Then the following are equivalent ([3]):

1. $P$ is a rational Gorenstein singularity.
2. There exists a resolution of singularities $f : Y \to X$ such that $K_Y = f^*K_X$.
3. In a neighbourhood of $P$ we have that $X$ is analytically isomorphic to a Du Val singularity, i.e. one of the hypersurface singularities of $\mathbb{A}^3$ defined by $f(x,y,z) = 0$, where $f$ is an equation from Table 1.

| Name | Equation |
|------|----------|
|      |          |
The names of Du Val singularities refer to the Dynkin diagrams, in fact in the minimal resolution of a Du Val singularity, the exceptional divisors have the corresponding Dynkin diagram as incidence graph. The minimal resolution of a Du Val singularity can be obtained by a sequence of blow-ups in the singular points, which ends as soon as the blown-up surface becomes smooth. The exceptional curves introduced are all \((-2)\)-curves.

**Example 2.4** (Surface elliptic Gorenstein singularities). Let now \(P\) be an elliptic singularity of \(X\), and \(f : Y \to X\) its minimal resolution. Then \(f^{-1}(P) = \bigcup_{i=1}^{m} A_i\), and we can associate to \(f\) a (unique) cycle \(Z\) in \(Y\) which is effective, satisfies \(Z \cdot A_i \leq 0\) for all \(i = 1, \ldots, m\) and which is minimal with respect to these two properties. Such a cycle is called the fundamental cycle of \(f\), and can be computed as follows (compare [11, p. 1259]): start with \(Z_1 = A_{i_1}\) arbitrary, and then define inductively \(Z_j = Z_{j-1} + A_{i_j}\) such that \(A_{i_j} \cdot Z_{j-1} \geq 0\) until we end with \(Z = Z_l\). The integer \(k = -Z_2\) is a useful invariant of the elliptic Gorenstein surface singularity (see [15, §2, Prop. 2.9]). Not all the exceptional curves introduced are \((-2)\)-curves.

**Example 2.5** (Rational Gorenstein threefold singularities). Let \(X\) be a projective Gorenstein threefold, and \(P \in X\) a rational Gorenstein singularity. By Proposition 2.6, the generic surface through \(P\) has either a rational or an elliptic surface singularity.

**Definition 2.7.** If the general surface through \(P\) has a rational singularity in \(P\), i.e. a Du Val singularity, then we say that \(P \in X\) is a compound Du Val singularity, or cDV singularity for short.

In this case, in a suitable neighbourhood of \(P\), we can see \(X\) as a deformation of a Du Val singularity: the definition is equivalent ([15, Def. 2.1]) to ask that around \(P\) the variety \(X\) is locally analytically isomorphic to the hypersurface singularity in \(\mathbb{A}^4\) given by

\[
f(x, y, z) + t \cdot g(x, y, z, t) = 0,
\]

where \(f(x, y, z) = 0\) defines a Du Val singularity (see Table 1). Observe also that while Du Val singularities are isolated, cDV singularities can be isolated or not.

If instead the general surface through \(P\) has an elliptic Gorenstein surface singularity, we can use the invariant \(k\) introduced in Example 2.4 to classify them. We can summarize the classification in the following proposition:

**Proposition 2.7** ([15, §2, Cor. 2.10]). To a rational Gorenstein threefold singularity \(P \in X\) one can attach an integer \(k \geq 0\) such that

1. \(k = 0\) if \(P\) is a cDV singularity.
2. \(k \geq 1\) if the general surface \(H\) through \(P\) has an elliptic Gorenstein singularity whose fundamental cycle has self-intersection \(-k\) (cf. Example 2.4).
In particular, if \( k = 1 \) then \( P \in X \) is locally analytically isomorphic to the singularity in \( \mathbb{A}^4 \) defined by
\[
y^2 = x^3 + f_1(s,t)x + f_2(s,t)
\]
where \( f_1 \) is a sum of monomials of degree at least 4 and \( f_2 \) is a sum of monomials of degree at least 6.

## 3. Canonical singularities

In this section I will recall the definition of canonical singularities (cf. \cite{15} §1, Def. 1.1) and some of their properties. Then I will focus on the link between canonical and \( cDV \) singularities in the case of threefolds.

**Definition 3.1.** Let \( X \) be a (quasi-)projective variety. We say that \( X \) has canonical singularities if \( X \) is normal and

1. There exists an index \( r \geq 1 \) such that \( \omega^r_X \) is locally free.
2. There exists a resolution \( f: Y \to X \) such that \( f_*\omega^r_Y = \omega^r_X \), where \( r \) is as in point (1).

If \( P \in X \) is a singular point, then the smallest \( r \) for which point (1) holds in a neighbourhood of \( P \) is called the index of \( P \) in \( X \).

**Proposition 3.1.** On a threefold, the singularities of \( cDV \) type are canonical.

**Proof.** In fact it follows from the definitions that on any variety, a singular point \( P \) is Cohen–Macaulay and canonical of index 1 if and only if \( P \) is a rational Gorenstein singularity. In particular, \( cDV \) singularities are threefold rational Gorenstein singularities. \( \square \)

Let now \( X \) be a variety (of dimension at least 2) with canonical singularities of index \( r \), and \( f: Y \to X \) a resolution of \( X \). Then we can compare a canonical divisor \( K_Y \) on \( Y \) with \( f^*K_X \): we have
\[
rK_Y = rf^*K_X + \Delta, \quad \Delta = \sum a_i D_i \geq 0
\]
with \( a_i \in \mathbb{Z} \) and \( \{D_i\} \) is the set of effective divisors which are contracted by \( f \), that is \( \text{codim} f(D_i) \geq 2 \). The divisor \( \Delta \) is called the discrepancy of \( f \): a divisor \( D_i \) for which \( a_i = 0 \) is called a crepant divisor, the others are called discrepant.

**Definition 3.2.** Let \( X \) be a singular variety, and \( f: Y \to X \) a resolution of \( X \). We say that \( f \) is a crepant resolution if \( \Delta = 0 \), or equivalently if \( K_Y = f^*K_X \).

**Example 3.1.** Let \( f: Y \to X \) be a resolution of the singularities of the \( n \)-dimensional variety \( X \). We say that \( f \) is small if \( \dim f^{-1}(P) \leq n - 2 \) for all \( P \in X \), or equivalently if the fibres of \( f \) don’t contain divisors. So if \( X \) has isolated singularities and \( f: Y \to X \) is a small resolution, then \( f \) is crepant.

**Remark 3.2.** A small resolution \( f: Y \to X \) can introduce divisors in \( Y \), what is important is that such divisors are not contained in fibres of \( f \).

Small resolutions are important when studying elliptic fibrations and their Weierstrass models, as we will see in Section 4. As one can imagine, the singularities admitting a small resolution are of mild type.

**Proposition 3.3.** Let \( X \) be a Gorenstein threefold with a small resolution \( f: Y \to X \). Then the singularities of \( X \) are of \( cDV \) type.
Proof. Since $X$ is Gorenstein and has a small resolution of the singularities, then $X$ has canonical singularities of index 1. In [10, §5, Thm. 5.35] it’s shown that if $X$ is a variety with canonical singularities of index 1 and $P \in X$ is a singular point, then the following are equivalent:

(1) The general hypersurface section $P \in H \subseteq X$ is an elliptic singularity.
(2) If $g : X' \to X$ is any resolution of singularities then there is a crepant divisor $E \subseteq g^{-1}(P)$.

Let $P \in X$ be a singular point. Since $f$ is small, there is no divisor in $f^{-1}(P)$ and so the general hypersurface section through $P$ is not an elliptic singularity. By Proposition 2.6, we have that the general hypersurface section through $P$ must have a rational (i.e. Du Val) singularity, which proves that $P$ is cDV. □

3.1. Resolution of threefolds with cDV singularities. Let $X$ be a threefold with only cDV singularities. If the singular locus of $X$, $\text{Sing} X$, has dimension 1, then with the exception of possibly a finite number of points (these points are called dissident points in [10]), each singular point has a neighbourhood in which $X$ is analytically equivalent to the product of a Du Val surface singularity with $\mathbb{A}^1$.

In [10, §2], it’s shown that by repeatedly blowing-up the singular locus, we get a partial resolution of $X$, i.e. a proper birational morphism $f : Y \to X$ with $Y$ normal, satisfying the following (natural) properties:

(1) $f$ coincides with the minimal resolution of the Du Val singularity times the identity on $\mathbb{A}^1$ over the Du Val locus.
(2) $f$ is small, i.e. we don’t introduce divisors over the dissident points.

To conclude, I recall here one of the main results in [10], which is a characterization of the crepant resolutions for threefolds with cDV singularities. We will need this proposition in Section 4, to study the non-Kodaira fibres of an elliptic fibration.

Proposition 3.4 ([10, §1, Thm. 1.14]). Let $X$ be a threefold with cDV singularities, and let $f : Y \to X$ be a partial resolution. Then the following are equivalent:

(1) $f$ is crepant;
(2) $f$ is small, and crepant above the generic point of any 1-dimensional component of $\text{Sing} X$;
(3) for every $x$ in $X$ and every hypersurface $H$ through $x$ for which $x \in H$ is a Du Val singularity, $H' = f^{-1}(H)$ is normal and $f_{|H'} : H' \to H$ is crepant. Thus the minimal resolution of $x \in H$ factors through $H'$.

4. Elliptic fibrations

In this section I want to deal with elliptic fibrations, and in particular with elliptic fibrations over a surface. The theory of elliptic fibrations over a curve is well understood, and so fibrations over surfaces are the first step towards higher dimensional fibrations. I will use the results from the previous sections to state a result on the structure of the singular fibres of an elliptic threefold.

Definition 4.1. We say that $\pi : X \to B$ is a smooth elliptic fibration over $B$ with section if

(1) $X$ and $B$ are smooth projective varieties of dimension $n$ and $n - 1$ respectively;
(2) $\pi$ is a surjective morphism;
the fibres of $\pi$ are connected curves, and the generic fibre of $\pi$ is a smooth connected curve of genus 1;

(4) a section $\sigma : B \to X$ of $\pi$ is given.

We denote $S = \sigma(B)$ the image of the section, and still call it a section. When $\pi : X \to B$ satisfies only the first three requirements above, we say that it’s a genus one fibration.

Given two elliptic fibrations $\pi : X \to B$ and $\pi' : X' \to B$, a morphism of elliptic fibrations is a morphism $f : X \to X'$ which is compatible with the fibrations, i.e. such that $\pi = \pi' \circ f$ or equivalently such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
B & & 
\end{array}
$$

is commutative.

Exploiting the presence of the section it’s possible to find in a canonical way a good birational model for an elliptic fibration $\pi : X \to B$: this is the Weierstrass model of the fibration.

Proposition 4.1 ([14, §2, Thm. 2.1]). Let $\pi : X \to B$ be an elliptic fibration with section $S$, and let $i : S \hookrightarrow X$ be the inclusion. Let $L = (\pi_* i_* N_{S|X})^{-1}$. Then the line bundle $O_X(3S)$ defines a birational morphism $f : X \to W$, where $W$ is a (possibly singular) variety defined in $P(L^2 \oplus L^3 \oplus O_X)$ by a Weierstrass equation

$$
y^2 z = x^3 + a_4 x z^2 + a_6 z^3.
$$

Remark 4.2. If the section does not meet all the irreducible components in the fibres of $\pi$, then $f$ is the contraction in the reducible fibres of the irreducible components which don’t meet the section.

Remark 4.3. Let $\pi : X \to B$ be an elliptic fibration, with Weierstrass model $p : W \to B$. Since any fibre of $p$ and of $\pi$ is a curve, we have $\dim f^{-1}(P) \leq 1$ for all $P \in W$.

Remark 4.4. As any hypersurface in a smooth ambient space, the Weierstrass model $W$ of an elliptic fibration is a Gorenstein variety (compare with Example 2.2). We can find this fact also in [14, §1, p. 409], where there is also a formula for the dualizing sheaf of $p : W \to B$:

$$
\omega_W = p^*(\omega_B \otimes L).
$$

The discriminant locus of an elliptic fibration $\pi : X \to B$ is the subset of $B$ which parametrizes the singular fibres of $\pi$:

$$
\Delta = \Delta(\pi) = \{b \in B | X_b \text{ is a singular curve}\}.
$$

Let $p : W \to B$ be a Weierstrass fibration. Its discriminant is the usual discriminant of the Weierstrass cubic polynomial ([14, §1]):

$$
\Delta(p) : 4a_4^3 + 27a_6^2 = 0,
$$

and so $\Delta$ is not only a subset of $B$, but also a subscheme.
Remark 4.5. Let \( \pi : X \to B \) be an elliptic fibration, with Weierstrass model \( p : W \to B \). If \( X_p \) is a smooth elliptic curve then \( W_p \) is smooth, in fact it follows from Remark 4.1 that in this case \( W_p \) is the Weierstrass model of \( X_p \). This implies that \( B \setminus \Delta(\pi) \subseteq B \setminus \Delta(p) \), or equivalently that \( \Delta(p) \subseteq \Delta(\pi) \). So \( f : X \to W \) is a resolution of the singularities of \( W \) if and only if \( \Delta(p) = \Delta(\pi) \), and in this case \( f \) is a small resolution of \( W \) by Remark 4.3.

Example 4.1. Let \( E = \{ y^2w = u^3 + \alpha uw^2 + \beta w^3 \} \subseteq \mathbb{P}^2(u:v:w) \) be an elliptic curve in Weierstrass form, with zero \( O = (0 : 1 : 0) \), and let \( B \) be a smooth surface. Define \( W = B \times E \subseteq B \times \mathbb{P}^2 \), the constant fibration with structure map \( p : W \to B \) and section \( S = B \times \{ O \} \). Choose a smooth curve \( C \subseteq B \) and a point \( Q \in E \setminus \{ O \} \), and let \( f : X \to W \) be the blow-up of \( W \) in \( C \times \{ Q \} \). Then \( \pi = p \circ f \) defines on \( X \) an elliptic fibration over \( B \), whose section is the strict transform of \( S \). The fibre over \( P \in C \) is singular, as it is reducible: its irreducible components are the strict transform of the curve \( W_P \) and the rational curve introduced by the blow up. Since the section meets the strict transform of \( W_P \), by Remark 4.2 the Weierstrass model of \( X \) is \( W \). In this example \( \Delta(\pi) \supseteq \Delta(p) \), in fact \( \Delta(p) \) is empty while \( \Delta(\pi) \) is the curve \( C \).

Example 4.2. Let \( p : W \to B \) be the constant fibration defined in Example 4.1 with section \( S \). Let \( C \subseteq B \) be a smooth curve and \( g \in H^0(B, \mathcal{O}_B(C)) \) an equation for \( C \). Let \( f' : X' \to W \) be the blow-up of \( W \) in \( C \times \{ O \} \): then \( \pi' = p \circ f' \) defines on \( X' \) an elliptic fibration over \( B \), whose section is given by the strict transform of \( S \). The fibre of \( \pi' \) over \( P \in C \) is reducible: its irreducible components are the strict transform of \( W_P \) and the rational curve introduced by the blow up. Since the section meets the rational curve, by Remark 4.2 \( W \) is not the Weierstrass model of \( X' \). In fact this last is the hypersurface \( W' \) in \( Z = \mathbb{P}(\mathcal{O}_B(2C) \oplus \mathcal{O}_B(3C) \oplus \mathcal{O}_B) \) defined by \( y^2z = x^3 + \alpha q^6xz^2 + \beta q^6z^3 \), which is easily seen to be birational to \( W \) over \( B \). Observe that in this example \( \Delta(\pi') = \Delta(p') \), where \( p' : W' \to B \) is the restriction of the structure map of \( Z \).

I want then to give a brief description of how we can define \( X' \) as a resolution of \( W' \). As \( C \) is smooth, around a singular point, \( W' \) is locally isomorphic to the hypersurface of \( \mathbb{A}^4_{(s,t,x,y)} \) described by

\[
y^2 = x^3 + \alpha t^4x + \beta t^6.
\]

Consider then in the product \( \mathbb{A}^4 \times \mathbb{P}^{(1,2,3)} \) the variety \( V \) which is the closure of

\[
\left\{ \left( (s, t, x, y), (T : X : Y) \right) \mid (t, x, y) \neq 0, \quad (t : x : y) = (T : X : Y) \right\},
\]

and the natural projection \( \rho : V \to \mathbb{A}^4 \). Then \( V \) is a sort of weighted blow up of \( \mathbb{A}^4 \) along \( t = x = y = 0 \), and the strict transform of \( W' \) in \( V \) is a resolution of the singularities, which is isomorphic as elliptic fibration over \( B \) to our threefold \( X' \).

The point in Examples 4.1 and 4.2 is that the final elliptic fibrations we defined have “useless divisors” introduced by the blow ups. We are then led to the following definition, which wants to capture the concept that for an elliptic fibration being simple means being as close as possible to its Weierstrass model.

In case \( \pi : X \to B \) is an elliptic surface, we say that the fibration is minimal if there are no \((-1\))-curves in the fibres of \( \pi \). In case \( \pi : X \to B \) is an elliptic threefold, Miranda suggested in [12, Point (0.3)] that it should be minimal
if no contraction is compatible with the fibration. The following Definition is a generalization of them.

**Definition 4.2.** An elliptic fibration \( \pi : X \to B \) is *minimal* if for any morphism \( f : X \to X' \) of varieties over \( B \) such that \( f \) contracts at least one divisor, then \( \pi' : X' \to B \) is not an elliptic fibration.

**Proposition 4.6.** Let \( \pi : X \to B \) be a minimal elliptic fibration. Then the morphism on the Weierstrass model \( f : X \to W \) is a resolution of the singularities of \( W \).

**Proof.** In view of Remark 4.5 we only need to show that \( \Delta(\pi) = \Delta(p) \), where \( p : W \to B \) is the Weierstrass fibration. Assume \( C \) is a reduced component of \( \Delta(\pi) \) which is not in \( \Delta(p) \), and let \( P \in C \) be a smooth point. Then by Bertini’s theorem ([8, §III, Cor. III.10.9]) we can find a curve \( Z \) in \( B \) smooth at \( P \) and intersecting \( C \) transversally in \( P \), such that the restriction \( X|_Z \) is a smooth elliptic surface. Since \( P \in \Delta(\pi|_Z) \setminus \Delta(p|_Z) \), then \( X|_Z \) is not a minimal elliptic surface and so there is some \((-1)\)-curve in the fibre of \( \pi|_Z \) over \( P \). In particular, \( X_P \) is a reducible fibre. The morphism to the Weierstrass model thus contracts a divisor in \( \pi^{-1}(C) \), and the result of this contraction is a smooth elliptic curve over the generic point of \( C \). But then we have

\[
\begin{array}{ccc}
  X & \xrightarrow{\text{Contraction}} & X' \\
  \downarrow \pi & & \downarrow \pi' \\
  \downarrow & & \downarrow \\
  B & & \text{Weierstrass model}
\end{array}
\]

and \( \pi' : X' \to B \) is an elliptic fibration. Since \( \pi \) is assumed to be minimal, this is a contradiction. \( \square \)

**Remark 4.7.** The converse of Proposition 4.6 is false. Let \( \pi : X \to B \) be a minimal elliptic fibration, whose discriminant locus \( \Delta \) decomposes in smooth curves. Choose one of these curves, say \( C' \), and let \( C \) be a smooth curve in \( X \) such that \( \pi(C) = C' \). Finally, the blow up \( X' \) of \( X \) in \( C \) is a smooth non-minimal elliptic fibration over \( B \), but it is still a resolution of its Weierstrass model (which is the same as the Weierstrass model of \( X \)).

**4.1. Further discussions on minimality.** I want to give some reasons why the definition of minimality in Definition 4.2 is satisfactory.

It’s known ([13, §II, Cor. II.1.3]) that for each elliptic surface there exists a unique (up to isomorphism) minimal elliptic surface which is birational to the given one: this fact reduces the problem of studying all the elliptic surfaces to the problem of studying only the minimal ones.

We can go further: since any elliptic surface has a unique (up to isomorphism) Weierstrass model, we want to know if it’s possible to study the fibration only from the knowledge of its Weierstrass model. The first step in this direction is to detect, among all the fibrations defined by a Weierstrass-type equation, which are the ones arising from smooth elliptic surfaces.

**Remark 4.8.** In fact, it’s possible to show that the function

\[
\{ \text{Smooth minimal elliptic surfaces} \} \to \{ \text{Weierstrass fibrations} \}
\]
which associates to each minimal elliptic surface its Weierstrass model is injective (see e.g. [13 §II.3]).

**Proposition 4.9** ([13 §III, Prop. III.3.2]). Let $W$ be a surface, and $p : W \to B$ be a Weierstrass fibration, defined by the equation $y^2z = x^3 + a_4xz^2 + a_6z^3$. Then the following are equivalent:

1. $p$ is the Weierstrass model of some minimal elliptic surface.
2. $W$ has only Du Val singularities.
3. There is no point $b \in B$ satisfying $\text{mult}_b a_4 \geq 4$ and $\text{mult}_b a_6 \geq 6$.

In Section 4.3 I will show that in the case of a minimal elliptic threefold $\pi : X \to B$, some of the implications in Proposition 4.9 generalize. In particular, the implication (1) $\Rightarrow$ (2 with cDV singularities) is in Proposition 4.11, while the implication (2 with cDV singularities) $\Rightarrow$ (3) is Theorem 4.14. Unfortunately, the implication (2 with cDV singularities) $\Rightarrow$ (1) is false, since there are examples of cDV singularities which don’t admit small resolutions (for example [16 §1, Cor. 1.16], the singularity in $\mathbb{A}^3$ defined by $x^2 + y^2 + z^2 + w^n = 0$ with odd $n$).

### 4.2. Singular fibres and Tate’s algorithm.

Let $\pi : X \to B$ be a minimal elliptic fibration: now we want to describe the singular fibres of the fibration. In the case of surfaces, the situation is clear and well understood: the possible singular fibres were first listed by Kodaira, who also named them. In the following, I will refer to the singular fibres in the list as Kodaira fibres.

**Proposition 4.10** ([9 Thm. 6.2]). Given a smooth minimal elliptic surface $\pi : X \to B$, the only possible singular fibres of $\pi$ are the ones listed in Table 2.

| Name | Description |
|------|-------------|
| $I_1$ | Nodal rational curve |
| $I_2$ | Two smooth rational curves meeting transversally at two points |
| $I_n$ with $n \geq 3$ | $n$ smooth rational curves meeting with dual graph $\tilde{A}_n$ |
| $I_n^*$ with $n \geq 0$ | $n + 5$ smooth rational curves meeting with dual graph $\tilde{D}_{n+4}$ |
| $II$ | Cuspidal rational curve |
| $III$ | Two smooth rational curves meeting at a point of order two |
| $IV$ | Three smooth rational curves all meeting at a point |
| $IV^*$ | 7 smooth rational curves meeting with dual graph $\tilde{E}_6$ |
| $III^*$ | 8 smooth rational curves meeting with dual graph $\tilde{E}_7$ |
| $II^*$ | 9 smooth rational curves meeting with dual graph $\tilde{E}_8$ |
Moreover, in the case of elliptic surfaces, there is an algorithm which allows one to determine the type of the singular fibre over a point $b$ of the discriminant locus: first of all we put $X$ in Weierstrass form, finding an equation of the form

$$W: y^2z = x^3 + a_4xz^2 + a_6z^3,$$

and then we compute the multiplicities $\text{mult}_b a_4$, $\text{mult}_b a_6$ and $\text{mult}_b \Delta$. The type of the singular fibre over $b$ is then given by Table 3 (compare with [17, Table p. 46])

Table 3: Tate’s algorithm

| Name | mult$_b a_4$ | mult$_b a_6$ | mult$_b \Delta$ |
|------|--------------|--------------|-----------------|
| $I_1$ | 0            | 0            | 1               |
| $I_n$ | 0            | 0            | $n$             |
| $I_0^*$ | $\geq 3$ | 3            | 6               |
|       | 2            | $\geq 4$    | 6               |
| $I_n^*$ | 2            | 3            | $n + 6$         |
| $II$ | $\geq 1$ | 1            | 2               |
| $III$ | 1            | $\geq 2$    | 3               |
| $IV$ | $\geq 2$ | 2            | 4               |
| $IV^*$ | $\geq 3$ | 4            | 8               |
| $III^*$ | 3            | $\geq 5$    | 9               |
| $II^*$ | $\geq 4$ | 5            | 10              |

This procedure to determine the type of a singular fibre is known as Tate’s algorithm.

For minimal elliptic fibrations we can still run it: we put the fibration in Weierstrass form with respect to the given section, and then we consider the irreducible components $\Delta_i$'s of $\Delta$. Since the local rings $\mathcal{O}_{X,\Delta_i}$ are discrete valuation rings, it makes sense to compute the multiplicities $\text{mult}_\Delta a_4$, $\text{mult}_\Delta a_6$ and $\text{mult}_\Delta \Delta$. From the previous table we can then deduce the type of the singular fibre over the generic point of $\Delta_i$.

Observe that in the case of surfaces we have a precise description of any singular fibre, while in the case of higher dimensional elliptic fibrations what happens in codimension 2 (and greater) on the base is not yet well understood.

4.3. Elliptic threefolds. In this section I want to focus on the case of elliptic threefolds, and in particular to describe the singular fibres, even of non-Kodaira type, of the minimal elliptic threefolds.

For sake of simplicity, I summarize the main properties of the Weierstrass model of a minimal elliptic threefolds and of the morphism to the Weierstrass model in the following Proposition.

**Proposition 4.11.** Let $\pi : X \to B$ be a smooth minimal elliptic threefold, with Weierstrass model $p : W \to B$. Then $W$ is normal and Gorenstein, all the singular points are of cDV type and the morphism $f : X \to W$ on the Weierstrass model is a small and crepant resolution.
Proof. We know that $W$ is normal by Point (2) in Example 2.1, that it is Gorenstein by Remark 4.4 and that the morphism $f$ on the Weierstrass model is a small resolution by Remark 4.5. By Proposition 3.3, this means that $W$ has $cDV$ singularities. Finally, by minimality of $\pi : X \rightarrow B$, the resolution $f : X \rightarrow W$ (outside a finite number of points in $\text{Sing} W$) coincides with the minimal resolution of a Du Val singularity (see Section 3.1), which is crepant. By Proposition 3.4, this means that $f : X \rightarrow W$ is crepant. □

As a consequence, we have a way to compute the canonical bundle of the total space of the elliptic fibration.

Corollary 4.12. Let $\pi : X \rightarrow B$ be a smooth minimal elliptic threefold with section $S$, and let $i : S \hookrightarrow X$ be the inclusion. Then

\begin{equation}
\omega_X = \pi^* (\omega_B \otimes \mathcal{L})
\end{equation}

where $\mathcal{L}$ is the line bundle $\mathcal{L} = (\pi_* i_\ast \mathcal{N}_{S|X})^{-1}$.

Proof. Let $p : W \rightarrow B$ be the Weierstrass model of $X$, and $f : X \rightarrow W$ the morphism to the Weierstrass model. Since $f$ is crepant, it suffices to apply $f^*$ to formula (4.2). □

Remark 4.13. The dualizing sheaves of elliptic fibrations have been studied also in more general settings. Under the hypothesis that the discriminant $\Delta$ has only normal crossings, it’s known ([7, §0, Thm. 0.2] or [14, §0, Thm. 0.2]) that

\begin{equation}
\omega_X = \pi^* (\omega_B \otimes \mathcal{L}) \otimes O_X(D),
\end{equation}

where $D$ is a divisor depending on

1. the divisors in $B$ over which $\pi$ has multiple fibres,
2. the divisors $G$ in $X$ such that $\text{codim} \pi(G) \geq 2$,
3. the divisors $E$ in $X$ such that $E \cap X_{l_{i_2}}$ is a vertical $(-1)$-curve for the elliptic surface $X_{l_{i_2}} \rightarrow Z$.

In our case, the presence of the section prevents the presence of multiple fibres, and our assumption that the fibres of $\pi$ are equidimensional implies that $G = 0$. Finally, by Proposition 4.11, minimal elliptic fibrations satisfy $E = 0$. So formula (4.5) agrees with (4.6).

The results on singularities exposed up to now, in particular the condition (3) in Proposition 4.4, make the proof of the following propositions straightforward. Observe that Theorem 4.15 gives a partial answer to the problem of classifying the non-Kodaira fibres.

Theorem 4.14. Let $\pi : X \rightarrow B$ be a smooth minimal elliptic threefold with section, with Weierstrass model $W$ defined by the equation $y^2z = x^3 + a_4xz^2 + a_6z^3$. Then there is no point $b \in B$ such that $\text{mult}_b a_4 \geq 4$ and $\text{mult}_b a_6 \geq 6$.

Proof. Assume that $b \in B$ is a point such that $\text{mult}_b a_4 \geq 4$ and $\text{mult}_b a_6 \geq 6$. From Proposition 2.7, we see that the singular point $(x : y : z) = (0 : 0 : 1)$ in the fibre over $b$ is a rational Gorenstein singularity which is not $cDV$. So $W$ can’t have small resolutions by Proposition 3.3, and in particular it can’t be the Weierstrass model of a smooth minimal elliptic fibration $\pi : X \rightarrow B$ by Proposition 4.11. □
**Theorem 4.15.** Let \( \pi : X \rightarrow B \) be a smooth minimal elliptic threefold with section. If \( b \in B \) is a point such that the fibre \( X_b \) is of non-Kodaira type, then \( X_b \) is a contraction of the Kodaira fibre over \( b \) of the elliptic surface obtained restricting \( X \) to a generic smooth curve through \( b \).

**Proof.** Thanks to Theorem 4.14 we must have \( \text{mult}_b a_4 \leq 3 \) or \( \text{mult}_b a_6 \leq 5 \). By Proposition 4.9, the restriction of \( W \) to the generic smooth curve \( C \) through \( b \) is then an elliptic surface \( W_C \) with only Du Val singularities and finally, by Proposition 3.4, the fibre \( X_b \) is a contraction of the fibre predicted by the smooth minimal elliptic surface corresponding to \( W_C \). \( \Box \)

5. **Examples and non-examples**

In this section I give some examples of elliptic threefolds with non-Kodaira fibres, and show that they are contractions of Kodaira fibres, as stated in Theorem 4.15.

One of the first papers where non-Kodaira fibres appeared is [12], that article is in fact devoted to an explicit desingularization of Weierstrass threefolds. The method described allows one to blow up both the base of the fibration and the fibration itself, in order to deal with a simpler variety (e.g. one can blow up the base surface until the discriminant curve has only normal crossing). With this freedom, the classification in the paper is complete, and in [12, §14] there is the observation that all the non-Kodaira fibres he found are contraction of “the right” Kodaira fibre, in agreement with Theorem 4.15.

We already saw a partial result on the classification of the non-Kodaira fibers of an elliptic threefold: they are all contraction of Kodaira fibers (Theorem 4.15). In Example 5.1 and 5.2 we construct explicitly some of these fibers, which are the contraction of fibers of Kodaira type \( I_0^* \) and \( I_1^* \) respectively. In Example 5.3 we show that some configurations of curves cannot appear as fiber of an elliptic threefold. The last Example is devoted to the construction of a crepant resolution of a Weierstrass fibration which does not satisfy the necessary condition of Theorem 4.14. As we will see, the resolution will not be an elliptic fibration.

**Example 5.1 ([2, §2]).** Let \( Z \) be the projective bundle \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}) \) over \( \mathbb{P}^2 \) and consider the hypersurface

\[
(5.1) \quad X : x^2y + fy^3 + g(y^2z + z^3) = 0,
\]

where \((x : y : z)\) are coordinates in the fibres and \( f, g \) are homogeneous sextic polynomials defining smooth plane sextics intersecting transversally in 36 distinct points. Then \( X \) is a smooth variety, and restricting to \( X \) the bundle projection we get an elliptic fibration over \( \mathbb{P}^2 \), with section \( \sigma \) given by \( P \mapsto (1 : 0 : 0) \in X_P \).

The discriminant of the family is the curve

\[
\Delta : g^4(27f^2 + 4g^2) = g^4(2g + 3\sqrt{-3}f)(2g - 3\sqrt{-3}f) = 0,
\]

over the curve \( g = 0 \) the fibre is of Kodaira type \( IV \), while over each of the curves \( 2g \pm 3\sqrt{-3}f = 0 \) we have nodal cubics. Over the 36 points where \( f = g = 0 \), the fibre has equation \( x^2y = 0 \) and so consists of two concurrent lines, one of which with multiplicity 2 and so this fibre is not in Kodaira’s list (Table 2). A picture of the singular fibres of this fibration is in Figure 1.
Around one of these points there is a suitable neighbourhood where $f$ and $g$ give local centered coordinates. In such a neighbourhood, we choose a generic line through the origin, $g = \lambda f$, and define $X_\lambda$ to be the restriction to this line of the fibration. By Tate’s algorithm, the fibre we expect over $f = 0$ is of Kodaira type $I^*_0$: the fibre of the threefold is a contraction of such fibre, in agreement to Theorem 4.15. In Figure 2 it’s possible to see which components have been contracted.

**Remark 5.1.** Observe that for different values of the parameter $\lambda$, the three singular points in the central fibre of $X_\lambda$ change coordinates. As a consequence these points, which are singular only for some values of $\lambda$, are smooth points for the threefold.
In Example [5.1] the points of the multiplicity 2 component of the non-Kodaira fibres are smooth points also for the elliptic surface obtained by restriction to the generic curve through the points where \( f = g = 0 \). As observed in Remark 5.1, this is enough to ensure that the threefold is smooth at those points. The next example will show that this condition is not necessary.

**Example 5.2.** Consider in \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(4) \oplus \mathcal{O}_{\mathbb{P}^2}(6) \oplus \mathcal{O}_{\mathbb{P}^2}) \) the fibration in Weierstrass form

\[
W : y^2 z = x^3 - \frac{1}{3} t^2 x z^2 + \left( s^4 + \frac{2}{27} t^3 \right) z^3,
\]

where \( s = 0 \) and \( t = 0 \) define a smooth cubic and a smooth quartic respectively, with transverse intersection in 12 distinct points. The discriminant of this fibration is

\[
\Delta = s^4 (27 s^4 + 4 t^3)
\]

and so by Tate’s algorithm we expect that a resolution \( \pi : X \to \mathbb{P}^2 \) of \( W \) has

1. \( I_4 \) fibres over the line \( s = 0 \).
2. \( I_1 \) fibres over \( 27 s^4 + 4 t^3 = 0 \).

The curve \( s = x = y = 0 \) is singular for the threefold, hence we blow it up. The effect is that over \( s = 0 \) instead of nodal cubics now we have triangles, with one of the vertices which is still singular for the whole variety. After a second blow-up of this curve of singular points we have a smooth threefold with \( I_4 \) fibres over \( s = 0 \), as expected. The fibre over one of the points where \( s = 0 \) and \( t = 0 \) meet is of non-Kodaira type, and we have a picture of this fibre in Figure 3.

![Figure 3. The singular fibres of the resolution of (5.2).](image-url)
I want now to give a local description of the fibration around a point over which we have the new fibres, and in a suitable neighbourhood of such points local coordinates are given by $s$ and $t$. Let $X_\lambda$ be the restriction of the fibration to the generic line through the origin $t = \lambda s$. By Tate’s algorithm on this elliptic surface $X_\lambda$ we should have over the origin a fibre of Kodaira type $I^*_1$: what we see is not the whole fibre but a contraction of it (Figure 4), according to Theorem 4.15.

![Figure 4](image-url)

**Figure 4.** On the left a $I^*_1$ fibre, and on the right the non-Kodaira fibre on the resolution of (5.2). The circled components are contracted.

This example is interesting also for the following reason. The surface $X_\lambda$ is singular over $s = 0$, since we don’t have a Kodaira fibre, and so there are singular points whose coordinates depend on $\lambda$. In fact there are always two singular points: one of them is on the multiplicity two component of the non-Kodaira fibre and its coordinates depend on $\lambda$, while the second is at the point of the fibre where the multiplicity 2 component of the $I^*_1$ fibre was blown down. This last has coordinates independent of $\lambda$, but nevertheless the threefold is smooth at this point.

**Example 5.3.** Example 5.2 fits in a more general class of examples, which also show that Theorem 4.14 gives a necessary, but not sufficient, condition. Consider the local elliptic fibration

$$y^2z = x^3 + t^m x^2 z + s^n z^3, \quad m, n \geq 1,$$

in $U \times \mathbb{P}^2$, where $U$ is a neighbourhood of the origin in $\mathbb{C}^2_{(s,t)}$, while $(x : y : z)$ are coordinates in the fibre $\mathbb{P}^2$.

We can put the equation in Weierstrass form, obtaining the standard equation

$$y^2z = x^3 + a_4xz^2 + a_6z^3$$

with

$$a_4 = -\frac{1}{4} t^{2m};$$
$$a_6 = s^n + \frac{1}{27} t^{3m};$$

and so the discriminant locus of the family is

$$\Delta : s^n (27s^n + 4t^{3m}) = 0.$$
By Theorem 4.14 not for every pair \((m, n)\) the corresponding variety can be the Weierstrass model of a smooth elliptic fibration. We have to exclude all the cases with

\[
\begin{align*}
2m & \geq 4 \\
\min(n, 3m) & \geq 6 \\
m & \geq 2 \\
n & \geq 6.
\end{align*}
\]

Remark 5.2. The fibration described in Example 5.2 corresponds to \((m, n) = (1, 4)\).

Remark 5.3. All the cases with \((m, n) = (m, 1)\) are smooth, and the origin is a point of the discriminant where the two components meet with arbitrarily large multiplicity.

Remark 5.4. Consider the cases with \((m, n) = (m, 3)\) and \(m \geq 2\). After one blow up in the singular locus, we obtain a variety which is still singular in three points, where each singularity is locally isomorphic to

\[y^2 = x^2 + s^2 + t^m.\]

As pointed out in [16, §1, Cor. 1.16], these singularities are cDV and admit a small resolution if and only if \(m\) is even. This means that not every Weierstrass fibration satisfying the necessary condition of Theorem 4.14 is actually the Weierstrass model of an elliptic fibration.

Example 5.4. I now want to show that not every singular curve of arithmetic genus 1 can be a fibre in a smooth elliptic threefold.

Let \(F\) be the curve defined in \(\mathbb{P}^3\) by the equations

\[
\begin{align*}
x_0x_3 &= 0 \\
x_1x_3 &= 0 \\
x_1^2x_2 &= x_0^3.
\end{align*}
\]

This curve has two irreducible components: a plane cuspidal cubic and a non coplanar line passing through the cusp (Figure 5).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The fibre \(F\).}
\end{figure}

Such a curve is not in Kodaira’s list of singular fibres (Table 2) and so there is no elliptic surface having it as fibre. We could deduce this fact also from the following observation: the tangent space to the singular point of \(F\) has dimension 3, so we need at least an elliptic threefold to have a smooth total space having \(F\) among its fibres. However, we can not obtain \(F\) as a contraction of a Kodaira fibre, and so no smooth elliptic threefold can have such a fibre.
Example 5.5. In this example I want to show an example of how we can resolve an elliptic fibration if we are not in the case of Theorem 4.14. A simple equation to consider is

\[(5.4) \quad W : y^2 = x^3 + s^4x + t^6,\]

defining a threefold in \(\mathbb{C}^4_{(s,t,x,y)}\) with an isolated singularity at the origin. With the projection on the first two coordinates, by Theorem 4.14 we know that this can not be the Weierstrass model of an elliptic fibration.

Consider in \(\mathbb{C}^4_{(s,t,x,y)} \times \mathbb{P}^{(1,1,2,3)}_{(S:T:X:Y)}\) the subvariety \(V\) of dimension 4 defined by

\[(5.5) \quad V = \left\{ (s, t, x, y), (S : T : X : Y) \mid (s, t, x, y) \neq (0, 0, 0, 0), (s : t : x : y) = (S : T : X : Y) \right\},\]

and let \(\beta\) be the restriction to \(V\) of the projection on \(\mathbb{C}^4\). Then

1. The fibre over a point \((s, t, x, y) \neq (0, 0, 0, 0)\) is the point \((s, t, x, y)\).
2. The fibre over \((0, 0, 0, 0)\) is \(\mathbb{P}^{(1,1,2,3)}\).

Then \(V\) is a sort of weighted blow-up of \(\mathbb{C}^4\) at the origin (see [15, §4]), where it is called the \(\alpha\)-blow up of \(\mathbb{C}^4\), and I claim that the strict transform \(W\) of \(W\) in \(V\) is a crepant resolution of the singularities of \(W\), which introduces a divisor over the singular point. This in fact follows from a more general result, see [15, §2, Thm. 2.11].

There is a nice description of \(V\) using toric geometry. The affine space \(\mathbb{C}^4\) is the toric variety associated to the fan \(\Sigma\), and \(\Sigma\) gives the projection \(\mathbb{C}^4 \twoheadrightarrow \mathbb{C}^4\). From this toric picture, we can also describe \(V\) as a quotient space: we have that

\[V = \left( \mathbb{C}^5_{(S,T,X,Y,w)} \setminus \{ S = T = X = Y = 0 \} \right) / \sim,\]

where \(\sim\) is the equivalence relation induced by the action of \(\mathbb{C}^*\) on \(\mathbb{C}^5 \setminus \{ S = T = X = Y = 0 \}\)

\[\lambda(S, T, X, Y, w) \mapsto (\lambda S, \lambda T, \lambda^2 X, \lambda^3 Y, \lambda^{-1} w).\]

Using the global homogeneous toric coordinates \((S : T : X : Y : w)\) on \(V\), the projection on \(\mathbb{C}^4\) is

\[(S : T : X : Y : w) \mapsto (s, t, x, y) = (Sw, Tw, Xw^2, Yw^3)\]

and so we see that over a point \((s, t, x, y) \neq 0\) there is only the point \((s : t : x : y : 1)\), while over \((0, 0, 0, 0)\) we have the divisor \(w = 0\) in \(V\), which is isomorphic to the weighted projective space \(\mathbb{P}^{(1,1,2,3)}\).

I want now to give a description of \(\beta : \overline{W} \twoheadrightarrow W\) in local coordinates on \(V\): from the quotient description we have that \(V\) is covered by four local charts, corresponding to the open subsets where \(S\), respectively \(T\), \(X\), \(Y\), are non-zero.
Chart $S \neq 0$ This affine chart is smooth, and the projection to $\mathbb{C}^4$ is

$$(T, X, Y, w) \mapsto (s, t, x, y) = (w, Tw, Xw^2, Yw^3):$$

the strict transform $\tilde{W}$ of $W$ via (5.6) is then

$$(\tilde{W} : Y^2 = X^3 + X + T^6),$$

which is smooth. Observe that we have

$$\begin{align*}
ds &= dw \\
dt &= wdT + Tdw \\
dx &= w^2dX + 2wxdw \\
dy &= w^3dY + 3w^2Ydw
\end{align*}$$

and so the pull back of the residue

$$-\frac{1}{2y}ds \wedge dt \wedge dx,$$

which generates $\omega_W$, is the 3-form

$$-\frac{1}{2y}dT \wedge dX \wedge dw.$$

This last is a generator for $\omega_{\tilde{W}}$, and so (5.6) is crepant. To conclude that $\beta : \tilde{W} \rightarrow W$ is crepant we only need to show that $\tilde{W}$ is smooth, since the part of $W$ which is not described in this chart is of codimension 2.

Chart $T \neq 0$ This affine chart is completely analogous to the previous one.

Chart $X \neq 0$ This affine chart is singular, in fact it is isomorphic to the quotient of $\mathbb{C}^4$ by the action

$$(S, T, Y, w) \mapsto (\zeta S, \zeta T, \zeta Y, \zeta w), \quad \zeta \neq 1, \zeta^3 = 1.$$ 

The projection to $\mathbb{C}^4$ is

$$(S, T, Y, w) \mapsto (Sw, Tw, w^2Y, Yw^3)$$

and so $\tilde{W}$ is defined in this chart by

$$\tilde{W} : Y^2 = 1 + S^4 + T^6,$$

which does not pass through the singular point of the chart, and is in fact smooth.

Chart $Y \neq 0$ Also this affine chart is singular, and it is isomorphic to the quotient of $\mathbb{C}^4$ by the action

$$(S, T, X, w) \mapsto (\zeta^2S, \zeta^2T, \zeta_3X, \zeta_3w), \quad \zeta_3^3 = 1, \zeta_3 \neq 1.$$ 

The projection to $\mathbb{C}^4$ is

$$(S, T, X, w) \mapsto (Sw, Tw, w^2X, w^3)$$

and so $\tilde{W}$ is defined in this chart by

$$\tilde{W} : 1 = X^3 + S^4X + T^6,$$

which does not pass through the singular point of the chart, and is in fact smooth.
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