Turbulent thermal diffusion: a way to concentrate dust in protoplanetary discs

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ABSTRACT

Turbulence acting on mixes of gas and particles generally diffuses the latter evenly through the former. However, in the presence of background gas temperature gradients, a phenomenon known as turbulent thermal diffusion appears as a particle drift velocity (rather than a diffusive term). This process moves particles from hot regions to cold ones. We re-derive turbulent thermal diffusion using astrophysical language and demonstrate that it could play a major role in protoplanetary discs by concentrating particles by factors of tens. Such a concentration would set the stage for collective behaviour such as the streaming instability and hence planetesimal formation.

Key words: hydrodynamics – turbulence – planets and satellites: atmospheres – planets and satellites: formation.

1 INTRODUCTION

In this article, we examine a phenomenon in which turbulence transports particles down temperature gradients: turbulent thermal diffusion (TTD), which was originally recognized by Elperin, Kleeorin & Rogachevskii (1996) and subsequently verified in laboratory experiments (Eidelman et al. 2004, 2006). TTD has astrophysical consequences. For example, planetary atmospheres have vertically varying temperature profiles, which play a major role in the formation of hazes and clouds via condensation (Ackerman & Marley 2001). TTD is known to trap particles in such temperature bands in the Earth’s atmosphere (Sofiev et al. 2009) and has been hypothesized to do so elsewhere in the Solar system (Elperin et al. 1997).

Protoplanetary discs also have temperatures that vary strongly on both global and local scales, as has been seen in observations of protoplanetary discs and as has been demonstrated by laboratory examinations of meteorites (Hewins & Radomsky 1990; D’Alessio, Calvet & Woolum 2005). While models of the background disc temperature, including both heating by irradiation from the central star and heating from accretion power, predict temperatures that scale as the square-root of the orbital position, numerical simulations of turbulent accretion discs that focused on gas thermal behaviour have seen order-of-unity temperature fluctuations on far smaller, scaleheight length-scales (McNally et al. 2014). Indeed, large local temperature gradients have many possible sources, including shadowing (Dullemond 2000) and transitional regions such as the boundary between magnetically active and magnetically dead regions (Zhu et al. 2009), evaporation fronts (Dullemond & Monnier 2010) or the edges of planet-opened gaps (Turner et al. 2012). A natural question that arises is to what extent these global and local temperature variations influence the nature and dynamics of solids in such discs and thereby also influence planet formation. In particular, TTD could play a significant role in the transport and concentration of particles in discs, similarly to how it acts in planetary atmospheres.

The smaller spatial scale temperature fluctuations that occurred in our own Solar Nebula and presumably occur in other protoplanetary discs have consequences beyond TTD, i.e. the thermal processing of solid material (e.g. chondrule formation; Hewins 1997). Sufficient heating will cause fluffy particles to compactify, reducing the effective surface area and thereby reducing (although not eliminating) interaction with the gas. This accelerates the rate at which the compactified particles settle or drift radially (Hubbard & Ebel 2014). In addition, temperature fluctuations can also move particles more directly, e.g. through photophoresis, which has been invoked to explain Solar Nebula phenomena such as the outwards radial transport of calcium-aluminium-rich inclusions (Ehrenhaft & Ebel 2000). In an optically thick disc, a particle exchanges thermal information radiatively with a shell an optical depth in radius. Different sides of a particle can therefore see gas at different temperatures, which sets up temperature gradients through the particle. The difference in temperature across the particle’s surface cause gas molecules to recoil more violently from the hotter side, pushing the particles from hot regions to cold, with consequences for protoplanetary discs (McNally & Hubbard 2015). However, photophoresis depends on the radiative flux and hence requires a high background temperature, in addition to a strong temperature gradient.

In this article, we focus only on TTD, where, as the name suggests, it is turbulence rather than the radiation field that plays the central role. Our purpose is twofold: first, we introduce TTD to the astrophysics community and develop it in a more familiar language.
In doing so, we write the conditions for TTD to apply in terms sufficiently general that they can be easily adapted to systems where the transport of solids by gas is important, including exoplanetary and brown dwarf atmospheres. Secondly, we explore TTD’s consequences for particle transport in protoplanetary discs in particular and show that, while background power-law temperature gradients are too weak to support TTD, cold annuli about a local scaleheight wide are expected to concentrate millimetre- to centimetre-scale particles significantly in protoplanetary discs.

2 TURBULENT THERMAL DIFFUSION

As we will quantify, in general, TTD acts as a first-order effect when all of the following are satisfied: (1) the temperature gradient is steeper than the pressure gradient, (2) particles are well, but not perfectly coupled to the turbulence and (3) the turbulence is subsonic. The latter two conditions are linked: the better particles are coupled to the turbulence, the more subsonic the turbulence needs to be. Finally, while TTD can act on very well-coupled particles if the turbulence is sufficiently weak, the time-scale for it to do so can be prohibitive.

Inertial particles, with finite mass $m_p \gg m$, the mean molecular mass of the gas they are embedded in, drift through the gas even if they are frictionally well coupled. We write the equations for $\mathbf{w}$, the particle’s drift speed through the gas (equations A6 and A7), as

\[ \mathbf{w} \equiv \mathbf{v} - \mathbf{u}, \]  
\[ \partial_t \mathbf{w} = -\frac{\tau_s}{\rho} \nabla p, \]  
where $\mathbf{v}$ is the particle’s velocity, $\mathbf{u}$ the gas velocity at the particle’s position and $\tau_s$ the particle’s frictional stopping time.

Because even well-coupled particles embedded in the gas are too large to feel pressure forces, they drift through the gas up pressure gradients according to

\[ \mathbf{w} = \frac{\tau_s}{\rho} \nabla p, \]  
where $\rho$ is the gas density. Taking the divergence of equation (3), we can write that pressure maxima concentrate particles and indeed

\[ n' \simeq \tau_p \partial_t n' \simeq \tau_p \nabla \cdot n \mathbf{w} \simeq n \frac{\tau_s}{\rho} \nabla^2 p \propto n \frac{\tau_s}{\rho} \frac{p'}{\rho} \frac{\nabla T}{\nabla p}, \]  
where $n$ is the background particle number density, $n'$ the fluctuating particle number density, $p'$ a pressure fluctuation, $\tau_p$ its length-scale and $\tau_s$ its lifetime. In the presence of turbulence, there is a fluctuating velocity field $\mathbf{u}'$, which creates a fluctuating pressure field $p'$, which in turn generates a fluctuating particle number density field $n'$. This $n'$ is the signature of preferential concentration (Maxey 1987; Cuzzi et al. 2001), but, in the absence of large-scale gas gradients, symmetry means that there is no preferred direction for averaged vector quantities to be aligned with. As a result,

\[ \langle n' \mathbf{u}' \rangle \propto \langle p' \mathbf{u}' \rangle = 0 \]  
and there is no large-scale particle transport.

The presence of a large-scale gas temperature gradient alters the situation significantly. The key insight of Elperin et al. (1996) was that, for gas in hydrostatic equilibrium, the fluctuating gas density and velocity fields are at most weakly correlated because there is no net turbulent transport of gas mass even in the presence of a gas density gradient; in the presence of a gas temperature gradient, however, there is net turbulent transport of heat down the temperature gradient. Therefore, the correlation of $\mathbf{u}'$ and $p'$ can be approximated

\[ \langle n' \mathbf{u}' \rangle \propto \langle p' \mathbf{u}' \rangle = 0, \]  
where $\langle \cdot \rangle$ is the fluctuating particle number density, $n'$ the fluctuating particle number density, $p'$ the turbulent pressure field, $\tau_p$ its length-scale and $\tau_s$ its lifetime.

Figure 1. Diagram of TTD. Turbulent motions moving down (up) a background temperature gradient drag gas to colder (warmer) regions. The warmer (colder) turbulently advected gas has higher (lower) pressure than the ambient gas. The higher (lower) pressure turbulently advected gas concentrates (disperses) particles, resulting in net particle transport.

\[ \langle p' \mathbf{u}' \rangle \propto \langle p' \mathbf{u}' \rangle = 0, \]  
where $\tau_p$ is the turbulent correlation time-scale. Invoking equation (4), we find our expected scaling:

\[ \langle n' \mathbf{u}' \rangle \propto \frac{\tau_p}{\rho} \langle p' \mathbf{u}' \rangle \propto \frac{\tau_p}{\rho} \frac{\nabla T}{\nabla p}, \]  
where we have used $\tau_p = t_s / t_p$ for turbulent fluctuations. Accordingly, equation (6) implies that turbulence pumps particles from hot regions to cold ones, as sketched in Fig. 1. In essence, TTD acts by having turbulent, small-scale pressure gradients generate local clumps of particles and then having global-scale temperature gradients lead to those clumps moving in an ordered fashion.

The temperature fluctuations $T'$ invoked above are not due to local sonic compression but rather to turbulence advecting gas of different temperatures ($T' = -t_s \mathbf{u}' \cdot \nabla T$) for an appropriate turbulent correlation time $t_s$). This leads to a key aspect of TTD: $\langle p' \mathbf{u}' \rangle \propto \langle \mathbf{u}'^2 \rangle$, so $p'$ depends on the turbulent speed linearly, rather than quadratically as is generally the case (e.g. dynamic pressure and Bernoulli’s principle). Because the presence of a gas temperature gradient enhances the magnitude of $p'$ greatly, it also leads to enhanced preferential concentration beyond the scope of this article (Eidelman et al. 2010). While we do not know of any simple calculation showing why the gas avoids relaxation to more modest pressure fluctuations, in Section A9 we explore further the consequences of assuming

\[ \frac{p'}{\rho} \propto \frac{m_a}{\rho}, \]  
where $Ma$ is the turbulent Mach number. We find that, in that regime, only minor particle transport occurs. Laboratory experiments and atmospheric observations have found strong effects, implying that the approximations in equation (6) are indeed appropriate and that equation (8) should not be used (Eidelman et al. 2004; Sofiev et al. 2009).

2.1 Equations of TTD

We derive TTD beyond the scaling of equation (7) in Appendix A. The derivation is highly involved (note the equation numbers referenced) and is not needed to explore its astrophysical consequences,
so we limit ourselves to using those results. We therefore quote equations (A44), (A61), (A62) and (A68) to encapsulate TTD (up to the approximations made explicit in Appendix A):

$$\partial_t n + \nabla \cdot F_n \simeq 0,$$

(9)

$$F_n = n (D \nabla \left[ \ln \rho - \ln n \right] + w + \hat{V}_{TTD}),$$

(10)

$$V_{TTD} = \frac{4}{3} \frac{C \tau_s k_B T}{m} \ln S r^{-1} \nabla \ln T,$$

(11)

$$w = \frac{\tau_s}{\rho} \hat{\nabla} \rho,$$

(12)

where $n$ and $F_n$ are the particle number densities and fluxes (not normalized to the gas density); $\tau_s$ and $S$ are the particle stopping time and Stokes number normalized to the integral scale of the turbulence (equation A59); $\rho$, $p$, $T$ and $m$ are the gas density, pressure, temperature and mean molecular mass; $D$ is the turbulent diffusion coefficient; and $C$ is a coefficient of order unity. The main difference between equations (7) and (11), the factor of $\ln S r^{-1}$, is due to the fact that turbulence has a power spectrum rather than specific values for its velocity, length-scale and time-scale.

Note that our analysis has been performed in the $S r \ll 1$ limit (i.e. particle frictional stopping times much shorter than turbulent correlation times) to allow for scale separation between the dust stopping time $\tau_s$ and the turbulent correlation time-scales. We have also assumed an approximately adiabatic equation of state, with thermal relaxation negligible on turbulent advective time-scales. Thermal relaxation on time-scales comparable with the turbulent advective time-scale would add a prefactor below unity to equation (A50), reducing our estimate for turbulent thermal fluctuations, while thermal relaxation on a time-scale far shorter than the turbulent time-scale would eliminate the effect.

$V_{TTD}$, the particle turbulent thermal diffusion velocity, is the focus of this article. Note that both $w$ and $V_{TTD}$ are proportional to $\tau_s$, but $w \neq 0$ requires the existence of a large-scale pressure gradient, while $V_{TTD} \neq 0$ relies on the existence of a large-scale temperature gradient. For an ideal gas, those gradients often are, but need not be, related; even when both exist, they need not be aligned or anti-aligned, so the two velocities can act in concert or in opposition. Note also that the form for $V_{TTD}$ in equation (11) does not make reference to the turbulent velocity scales, which have cancelled out. As discussed in Section A7, our analysis is performed in the limit $n'/n \ll 1$, which implies that $V_{TTD} \ll u_0$, the turbulent velocity at the integral scale. When that constraint is violated, equation (11) cannot be used, but the fluctuating particle number density field $n'/n$ is not small and must therefore none the less be treated with care.

In general, we can have both large-scale temperature and pressure gradients and so we need to consider both $V_{TTD}$ and $w$. For TTD to be a first-order effect, we need $|V_{TTD}| \gtrsim |w|$. We therefore determine the conditions for TTD to be important in Section 2.2 and derive the TTD in the special case of $w \sim 0$ in Section 3.3.

2.2 General case

From equations (11) and (12), we have

$$V_{TTD} = \frac{4}{3} \frac{C \tau_s k_B T}{m} \ln S r^{-1} \frac{\hat{\nabla} T}{L_T},$$

(13)

$$w = \frac{\tau_s}{\rho} \frac{\hat{\nabla} \rho}{L_p} = \frac{\tau_s k_B T}{m} \frac{\hat{\nabla} \rho}{L_p},$$

(14)

where $\hat{\nabla}_n$ and $\hat{\nabla}_T$ are the directions of the pressure and temperature gradients, while $L_p$ and $L_T$ are the length-scales over which the pressure and temperature vary:

$$L_p = |\nabla \ln p|^{-1},$$

(15)

$$L_T = |\nabla \ln T|^{-1}.$$
Experiments have found values up to about $\alpha \approx 2.7$ (Eidelman et al. 2006), while observations of aerosols in the troposphere indicate that $\alpha \approx 20$ occurs in nature (Sofiev et al. 2009). Note that our analysis assumes that the dust fluid density is always much less than that of the gas, so that any drag the dust exerts on the gas can be neglected. Large values of $\alpha$ could imply local concentrations of dust large enough to exceed that limit. Our analysis can no longer be applied once such conditions occur, but those conditions would also allow other processes to dominate (e.g. Johansen et al. 2007). In situations where the background dust fluid density is too low to backreact on the gas but TTD would generate concentrations of dust sufficiently dense to backreact on the gas, TTD can be invoked as a trigger for processes such as streaming instability.

3.2 Conditions for relevance

While the parameter $C$ has not yet been determined and likely varies depending on the nature of the turbulence, we expect $C < 1$ and Zilitinkevich et al. (2007) suggest $C \sim 0.3$. In that case, equation (19) becomes

$$St \gtrsim e^{-2.5 L_T/L_p},$$

(27)

The shorter the temperature length-scale is compared to the pressure length-scale, the larger the particles can be and still be transported by TTD. This is important because TTD transports larger particles more quickly and larger particles are more likely to engage in collection behaviour such as streaming instability. In astrophysical cases of interest, such as planetary atmospheres in hydrostatic equilibrium or protoplanetary disc midplanes, temperature, density and pressure generally vary in the same direction and on comparable length-scales in the absence of specific phenomena driving more localized fluctuations. From equation (27), we can see that the limits on $St$ depend very strongly on $L_T/L_p$ and so, in general, TTD can overcome pressure-based particle drift only in the limit of very small $St$ or in the presence of local phenomena that force $L_T \ll L_p$ (i.e. the isobaric limit).

However, the limit of very small $St$ implies particles that are extremely well coupled to the gas and so generally well mixed by turbulence. We define the Mach number of the turbulence as

$$Ma = \frac{u_0^2}{c_s^2} = \frac{\rho u_0}{\gamma p} = \frac{mu_0^2}{\gamma k_B T},$$

(28)

and use that along with equation (A56) to rewrite equation (23) as

$$\alpha - 1 = \frac{4C}{u_0^2} \frac{T_k}{T_0} \frac{\ln St^{-1}}{m} = \frac{8C}{\gamma} \frac{St}{Ma^2} \ln St^{-1}.$$

(29)

If equation (27) is to be satisfied by decreasing $St$, then TTD will have an effect only for weak turbulence ($Ma^2 < St$).

TTD therefore generally applies to astrophysical systems with temperature fluctuations on length-scales that are simultaneously much larger than the turbulent length-scales and much smaller than the length-scales associated with pressure ($L_0 \ll L_T \ll L_p$). When TTD applies, in practice it concentrates and disperses particles with $1 \gg St > Ma^2$.

4 TEMPERATURE GRADIENTS IN PROTOPLANETARY DISCS

4.1 Global temperature gradients

Turbulent thermal diffusion has been invoked to explain aerosol concentrations in the Earth’s tropopause and could similarly be applied to exoplanetary and brown dwarf atmospheres, including both vertical and horizontal temperature stratification. Here we explore TTD’s consequences for protoplanetary disc midplanes, noting that protoplanetary discs are slightly awkward because anticipated parameters satisfy the assumptions built into our analytical analysis only marginally. We assume the scalings of a Hayashi minimum-mass solar nebula (MMSN: Hayashi 1981). Note that discs are generally assumed to be statistically symmetric about the midplane and are azimuthally periodic, so near the midplane the vertical/azimuthal plane is homogeneous and can be averaged over, leaving only radial variations.

Background disc temperature gradients are shallow power laws, occurring on orbital length-scales, and, as we calculate next, are not expected to support significant TTD. In a Hayashi MMSN, the midplane temperature and pressure scale with orbital position $R$ as

$$T \propto R^{-1/2},$$

(30)

$$p \propto R^{-13/4},$$

(31)

from which we find

$$L_T = 2 R,$$

(32)

$$L_p = \frac{4}{13} R.$$

(33)

Note that, in this case, $\nu$ points radially inwards (headwind-induced infall: Weidenschilling 1977) while $V_{TTD}$ points outwards. With those values for the global gradients, equation (27) is satisfied for $St \lesssim 10^{-7}$,

(34)

i.e. for particles sufficiently coupled to the gas that no meaningful drift occurs regardless. However, for $C = 2/3$ we would have $St \lesssim 7 \times 10^{-4}$, which begins to be relevant. Even for $C = 0.3$, TTD can slow headwind-induced infall by a few tens of per cent: if $St = 0.01$ then $V_{TTD} \sim -0.3 \nu$.

4.2 Local, quasi-isobaric temperature gradients

While background protoplanetary disc global temperature gradients are expected to be too weak relative to global pressure gradients for TTD to be a first-order effect, protoplanetary disc temperature gradients are not expected to follow perfect power laws but rather are also expected to show strong, localized temperature variations on scales much smaller than the orbital length-scale. For example, if the dominant source of heating in a disc is irradiation by the central object, then shadowing can occur: when an annulus of the disc heats, it puffs up, shadowing the disc outside it, which then cools and contracts (Dullemond 2000; Siebenmorgen & Heymann 2012). Similarly, turbulence can generate long-lived, quasi-isobaric order-of-unity temperature fluctuations on length-scales only a few per cent of the orbital position (McNally et al. 2014); boundaries between regions with different chemistry provide highly localized, extremely long-lived sharp temperature gradients (Zhu et al. 2009; Dullemond & Monnier 2010). We can check the requirements for such local temperature variations to allow the TTD to operate.

For quasi-isobaric approximations to apply, we need to satisfy equation (27) for non-negligible values of $St$. As noted above, this requires some form of temperature perturbation on top of the expected background power law of equation (30). One possible source is turbulent dissipation. Magnetohydrodynamical turbulence dissipates its energy in quasi-2D structures known as current sheets; in protoplanetary discs, these structures can be very thin, hot and
in pressure balance with the exterior. McNally et al. (2014) found order-of-unity temperature variation on length-scales only a few per cent of the orbital position with negligible associated pressure structures. These structures would clearly satisfy equation (27) for particles with large enough $St$ to slip through the gas on short time-scales.

Another possible way to generate temperature annuli is shadowing (Dullemond 2000; Siebenmorgen & Heymann 2012). In this case, we note that the gas midplane density in a vertically isothermal disc is

$$\rho_0 = \frac{\Sigma}{\sqrt{2\pi H}} = \frac{\Sigma \Omega_K}{\sqrt{2\pi c_s}}$$

(35)

where $\Sigma$ is the gas surface density, $H$ the pressure scaleheight and $\Omega_K$ the local Keplerian frequency. We also have

$$c_s^2 = \frac{g k_B T}{m}.$$  

(36)

It follows that the midplane pressure is given by

$$p_0 = \frac{\rho_0 c_s^2}{\gamma} = \frac{\sqrt{g k_B T}}{2\pi m} \Sigma \Omega_K.$$  

(37)

While the onset of shadowing will have a major impact on the temperature profile, strong Coriolis forces imply that shadowing is not expected to redistribute mass radially to a significant extent. As long as the temperature gradient caused by shadowing obeys $L_T^{-1} \gg \partial \ln \Sigma / \partial R \sim R^{-1}$, we can therefore use equations (15), (16) and (37) to approximate

$$L_p \sim 2 L_T.$$  

(38)

In that case, equation (27) would require $St \lesssim 0.24$. That upper limit is safely above values associated with fragmentation or bouncing barriers and is safely large enough for streaming instability to act (Zsom et al. 2010; Carrera, Johansen & Davies 2015).

5 QUASI-ISOBARIC PROTOPLANETARY TEST CASE

The potential range of parameters for protoplanetary discs is too large to analyse fully here. Instead, we derive a test case that shows that there are plausible regions of parameter space in which TTD would significantly concentrate dust in protoplanetary discs. The constraints we use to choose the test-case parameters can be used to check other models for the importance of TTD.

5.1 Quasi-isobaric particle concentration

In the common $\alpha$-disc model, protoplanetary discs are assumed to accrete due to turbulent stresses, with a corresponding viscosity $\alpha_{SS} c_s H$, where $\alpha_{SS}$ represents the Shakura–Sunyaev $\alpha$ (Shakura & Sunyaev 1973), not to be confused with the $\alpha$ of equation (23). We assume here that the turbulent viscosity $\alpha_{SS} c_s H$ is approximately equal to the turbulent diffusivity $D$, i.e. a Schmidt number $Sc \sim 1$ (Johansen, Klahr & Mee 2006). The time-scale of the turbulence driven by orbital shear is also generally assumed to lie around $\Omega_K^{-1}$ (Fromang & Papaloizou 2006). It follows that

$$u_0 \sim \sqrt{\alpha_{SS} c_s},$$  

(39)

and hence

$$Ma^2 \sim \alpha_{SS}.$$  

(40)

Accordingly, equation (29) becomes

$$\alpha \sim 1 \sim 1.7 \frac{St}{\alpha_{SS}} \ln \frac{St}{\alpha_{SS}},$$  

(41)

where we have assumed $C = 0.3$ and $\gamma = 1.4$. We can see that significant TTD effects are expected only for $St > \alpha_{SS}$ and that Jacquet, Gounelle & Fromang (2012)'s parameter $S = St/\alpha_{SS}$, which measures dust transport through gas, is very relevant here as well.

Our current understanding of planet formation has difficulties in growing grains with $St \sim 10^{-3}$. Turbulently driven collisions are expected to be growth-neutral at best and often destructive for such grains (Zsom et al. 2010). While the precise size at which collisional growth fails is uncertain, our current picture is that, once grains grow large enough ($St \sim 10^{-2}$), collective dust–gas instabilities such as the streaming instability (SI; Johansen et al. 2007) take hold, collecting dust into gravitationaly unstable clumps which collapse, forming planetesimals directly. However, this leaves us with a clear dust size gap between the end collisional growth and the triggering of SI. One further difficulty for SI is that it requires background dust concentrations that are significantly supersolar and the degree of metallicity enhancement over solar needed is strongly $St$-dependent. Indeed, Carrera et al. (2015) found that the smallest grains that can trigger SI have $St \sim 3 \times 10^{-3}$; this requires a background supersolar dust concentration of about a factor of 5 outside the water-ice line and about a factor of 15 inside it. TTD provides a possible route to generating these supersolar dust concentrations.

A bouncing barrier is expected to occur for particles colliding at approximately $0.1–1$ cm s$^{-1}$ and a fragmentation barrier for particles colliding at around $100$ cm s$^{-1}$ (Güttler et al. 2010). Turbulently induced collisions have a range of possible speeds, however, so particles with characteristic collision speeds that lie between the bouncing and fragmentation barriers are expected to grow slowly through rare collisions at the low end of the speed distribution (Windmark et al. 2012). This leads to a pile-up in grain size as growth becomes ever less efficient. We assume particles with $St = 3 \times 10^{-3}$ (the smallest that can trigger SI; see Carrera et al. 2015) and $\alpha_{SS} = 10^{-3}$. Disc thermal speeds in regions of terrestrial planet formation are of the order of $c_s \sim 10^3$ cm s$^{-1}$ and expected turbulent dust–dust collision speeds are of order (Hubbard 2013)

$$v_{\text{collision}} \sim 0.25 \sqrt{St \alpha_{SS} c_s} \sim 40 \text{ cm s}^{-1}.$$  

(42)

This is large enough to lead to bouncing but likely not fragmentation and hence it lies in the regime associated with a size pile-up.

For $St = 3 \times 10^{-3}$ and $\alpha_{SS} = 10^{-3}$, equation (41) estimates $\alpha \sim 30.6$, which through equation (25) would imply extreme concentrations of particles in cold regions through TTD. Indeed, a temperature perturbation with amplitude $f_T T$ over a length $\ell_T$ has a corresponding $L_T = \ell_T / f_T$, so $f_T = 0.2$, $\ell_T = 0.6H$ satisfies $L_T = 3H \sim 0.1 R \ll R$ and stronger temperature gradients, at quasi-constant pressure, have been seen in simulations of MHD turbulence in protoplanetary discs (McNally et al. 2014). With those values, equation (26) implies a particle concentration by a factor of

$$T^{u-1} \sim (1 + f_T) y^{-1} \sim (1 + 0.2)^{30.6-1} \sim 221.$$  

(43)

This is easily large enough to have strong effects on the behaviour of dust in protoplanetary discs and so TTD could act as a trigger for SI. However, these values of $St$, $\alpha_{SS}$, $f_T$ and $\ell_T$ are at the limits of our analysis.
5.2 Test-case limitations

Note that equation (39) also implies
\[ \ell_0 \simeq \sqrt{\alpha_{SS} H}. \]  
(44)
We require \( \ell_0 \ll \ell_T \). In this test case, this constraint becomes
\[ \ell_0 = 0.03 H \ll \ell_T = 0.6 H, \]  
(45)
which is satisfied, but stronger turbulence (more than an order of magnitude larger \( \alpha_{SS} \)) has length-scales long enough that for small values of \( \ell_T \) we can only marginally linearize the background temperature field, as was done in Appendix A.

More problematically, from equation (24) we have
\[ |V_{TTD}| = (\alpha - 1)\alpha_{SS} \frac{H}{L_T} > \frac{1}{3}(\alpha - 1)\alpha_{SS} c_s. \]  
(46)
For \( St = 3 \times 10^{-3} \) and \( \alpha_{SS} = 10^{-3} \), this becomes
\[ |V_{TTD}| \sim 0.01 c_s \simeq 0.3 \mu m/s, \]  
(47)
and the requirement \( |V_{TTD}| \ll \mu m/s \) is only weakly fulfilled. This means that equation (24) is likely moderately overestimating \( V_{TTD} \), as discussed in Section A7. Because our analysis has \( V_{TTD} \) increasing with particle \( St, St \sim 3 \times 10^{-3} \) lies near the upper limit for which our analysis applies when \( \alpha_{SS} \simeq 10^{-3} \).

If we are overestimating \( V_{TTD} \) by a factor of 2, then we still have
\[ |V_{TTD}| > |\omega|, \]  
but the concentration factor (equation 43) drops from a factor of 221 to a factor of
\[ 1.2^{30-6/2} \simeq 1.2^{14.8} \simeq 15, \]  
(48)
which is on the edge of triggering SI inside the water-ice line. The time-scale for concentration under these conditions is moderate, but not negligible. Halving \( V_{TTD} \), we find
\[ \frac{\ell_T}{0.5 V_{TTD}} \sim 20 \text{ orbits}. \]  
(49)
However, smaller particles, with lower values of \( St \), will feel TTD only in the presence of even weaker turbulence, with a correspondingly larger concentration time-scale.

5.3 Test-case thermal relaxation

A further constraint on this test case is thermal relaxation, in particular any radiative cooling or heating of turbulent parcels of gas. As noted in Section 2.1, if the relaxation time is shorter than the turbulent time then we expect TTD to be weakened. As long as we have small temperature fluctuations and are in the optically thick, dense limit, we can approximate radiative cooling as a diffusive process. For an MMSN midplane at \( R = 1 \) au, we have an approximate thermal diffusion coefficient (McNally et al. 2014)
\[ \mu \simeq 4 \times 10^{12} \left( \frac{T}{270 \text{ K}} \right)^3 \text{ cm}^2 \text{ s}^{-1}. \]  
(50)
The corresponding thermal relaxation time is
\[ \frac{\ell_0^2}{\mu} \simeq 6.25 \times 10^7 \left( \frac{T}{270 \text{ K}} \right)^3 \simeq 13 \left( \frac{T}{270 \text{ K}} \right)^3 \Omega^{-1}. \]  
(51)
As long as the temperature is modest, the time-scale is much longer than the turbulent time-scale of \( \Omega^{-1} \), so the largest scale turbulence is insensitive to radiative thermal relaxation. Thermal relaxation will effect smaller eddies farther down the turbulent cascade. However, this appears in the equation for \( V_{TTD} \) only logarithmically (equation A57) and so is relatively unimportant.

On the other hand, radiative thermal diffusion is a strong function of temperature and equation (51) would seem to limit the TTD to cool regions of the disc. However, the opacity of the gas–dust mixture depends mostly on the abundance of approximately micron-sized dust grains and at high temperatures those evaporate, dramatically lowering the opacity and moving the system into the optically thin limit. Further, in regions with low gas density the gas and dust temperatures can decouple, making radiative cooling even less efficient. Those effects would make equation (51) a significant underestimate for the thermal relaxation time at much higher temperatures. We therefore expect TTD to be effective in cool regions (unlike photophoresis; see McNally & Hubbard 2015) and possibly at very high temperatures (e.g. chondrule formation) in low density regions, but not in warm, high density and opacity regions.

5.4 Other dust and disc parameters

As disc parameters diverge from the ones in Section 5, the situation gets less clear. Larger values of \( \alpha_{SS} \) at constant \( St \) both reduce the strength of TTD (reducing the ratio \( St/\alpha_{SS} \) in equation 41) and increase the length-scale associated with turbulence. This latter would begin to weaken the condition that turbulent length-scales should be much shorter than temperature length-scales. Weaker values of \( \alpha_{SS} \) would reduce the turbulent speed, resulting in ever more severe overestimates of \( V_{TTD} \) (equation 46 and subsequent discussion) and longer concentration time-scales. Thermal relaxation also becomes more pronounced for smaller values of \( \alpha_{SS} \) and their corresponding smaller length-scales.

Weaker turbulence than our test case, or larger particles (higher \( St \) values), constrains the upper limit on \( V_{TTD} \) and the concentration time-scale, while stronger turbulence threatens the scale separation between background and turbulent fields. Nonetheless, it is clear that the presence of scaleheight-scaled temperature fluctuations in turbulent protoplanetary discs should generally result in significant concentration of particles with \( St \gtrsim \alpha_{SS} \) on moderate time-scales. Laboratory or numerical studies will be needed to make precise estimations of the effectiveness of TTD in protoplanetary discs in practice.

6 CONCLUSIONS

We have adapted analysis of turbulent thermal diffusion (TTD) to a form more useful for astronomy. TTD is a process where the combination of turbulent and background pressure gradients act to pump inertial particles from hot regions to cold ones and it can lead to large particle concentrations in the latter (Elperin et al. 1996). As such, it acts similarly to the well-known concentration of particles in high-pressure regions, with the significant difference that temperature, unlike pressure, is not an inherently dynamical parameter (although it is usually strongly correlated with dynamical parameters).

While TTD has already been considered in the context of planetary atmospheres (Sofiev et al. 2009), we also show that it is expected to act in protoplanetary discs that have \( \sim \) scaleheight wide radial temperature banding, concentrating \( St \sim 10^{-3} \) particles in the cold bands by factors of ten. While such a degree of concentration would push our analysis outside its strict region of applicability, it would be a sufficient metallicity enhancement to allow the streaming instability to proceed. This suggests that if cold bands are common in protoplanetary discs, they would allow TTD to act to trigger SI and hence would be natural regions of planetesimal formation. Possible sources for such cold annuli include disc self-shadowing
and localized turbulent dissipation (Dullemond 2000; Siebenmorgen & Heymann 2012; McNally et al. 2014). Shadowing and turbulent dissipation are not expected to be stationary for disc evolution time-scales, but chemical boundaries, including ionization and evaporation fronts, could be and would also drive steep temperature gradients (Zhu et al. 2009; Dullemond & Monnier 2010). This adds to the already significant interest in such regions as possible places for planet formation (Lyra & Mac Low 2012).

The effects of TTD can be implemented in numerical simulations and theory as an ad hoc velocity $V_{\text{TDD}}$, but theoretical estimates are only approximate and factors of a few in $n$ have a large impact on the concentration of particles (equations 25, 43 and 48). Implementing TTD in numerical simulations directly will be difficult and efforts to date have barely been able to detect the effect (Haugen et al. 2012). Capturing TTD requires a large dynamical range between the scale of the temperature variation and the scale of the turbulence and a further large dynamical range between the integral scale of the turbulence and the scale of the turbulence that has the same correlation time as the particle’s frictional stopping time. Further, strong effects are only expected in low Mach number turbulence but depend on gas compressibility, forcing sound waves to be captured fully.

Laboratory experiments can and have constrained $V_{\text{TDD}}$ in the Stokes drag regime appropriate for planetary atmospheres (Eidelberg et al. 2004, 2006); probing the Epstein drag regime, where particle sizes are smaller than the gas molecular mean free path, appropriate for protoplanetary discs would require extremely fine particles in a very dilute gas. Nonetheless, this is the most promising avenue for constraining the numerical factors such as $C$.

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REFERENCES

Ackerman A. S., Marley M. S., 2001, ApJ, 556, 872
Carrera D., Johansen A., Davies M. B., 2015, A&A, 579, A43
Cuzzi J. N., Hogan R. C., Paquette J. M., Dobrovolskis A. R., 2001, ApJ, 546, 496
D’Alessio P., Calvet N., Woolfum D. S., 2005, in Krot A. N., Scott E. R., Reipurth B., eds, ASP Conf. Ser. Vol. 341, Chondrites and the Protoplanetary Disk. Astron. Soc. Pac., San Francisco, p. 353
Dullemond C. P., 2000, A&A, 361, L17
Dullemond C. P., Monnier J. D., 2010, A&A&A, 48, 205
Ehrenhaft F., 1918, Annalen der Physik, 361, 81
Eidelberg A., Elperin T., Kleeorin N., Rogachevskii I., Buchholz J., Grünfeld G., 2004, Nonlinear Processes in Geophysics, 11, 343
Eidelberg A., Elperin T., Kleeorin N., Rogachevskii I., Sipari-Katiraie I., 2006, Exp. Fluids, 40, 744
Eidelberg A. Elperin T., Kleeorin N., Melnik B., Rogachevskii I., 2010, Phys. Rev. E, 81, 056313
Elperin T., Kleeorin N., Rogachevskii I., 1996, Phys. Rev. Lett., 76, 224
Elperin T., Kleeorin N., Podolak M., Rogachevskii I., 1997, Planet. Space Sci., 45, 923
Fromang S., Papaloizou J., 2006, A&A, 452, 751
Güttler C., Blum J., Zsom A., Ormel C. W., Dullemond C. P., 2010, A&A, 513, A56
Haugen N. E. L., Kleeorin N., Rogachevskii I., Brandenburg A., 2012, Phys. Fluids, 24, 075106
Hayashi C., 1981, Progr. Theor. Phys. Suppl., 70, 35
Hewins R. H., 1997, Ann. Rev. Earth Planet. Sci., 25, 61
Hewins R. H., Radomsky P. M., 1990, Meteoritics, 25, 309
Hubbard A., 2013, MNRAS, 432, 1274
Hubbard A., Ebel D. S., 2014, Icarus, 237, 84
Jacquet E., Gouinelle M., Fromang S., 2012, Icarus, 220, 162
Johansen A., Klahr H., Mee A. J., 2002, MNRAS, 370, L71
Johansen A., Oishi J. S., Mac Low M.-M., Klahr H., Henning T., Youdin A., 2007, Nat, 448, 1022
Lyra W., Mac Low M.-M., 2012, ApJ, 756, 62
Maxey M. R., 1987, J. Fluid Mech., 174, 441
McNally C. P., Hubbard A., 2015, ApJ, 814, 37
McNally C. P., Hubbard A., Yang C.-C., Mac Low M.-M., 2014, ApJ, 791, 62
Shakura N. I., Sunyaev R. A., 1973, A&A, 24, 337
Siebenmorgen R., Heymann F., 2012, A&A, 539, A20
Sojiev M., Sofieva V., Elperin T., Kleeorin N., Rogachevskii I., Zilitinkevich S. S., 2009, J. Geophys. Res. (Atmos.), 114, 18209
Turner N. J., Choukroun M., Castillo-Rogez J., Bryden G., 2012, ApJ, 748, 92
Weidenschilling S. J., 1977, MNRAS, 180, 57
Windmark F., Birnstiel T., Ormel C. W., Dullemond C. P., 2012, A&A, 544, L16
Wurm G., Haack H., 2009, Meteoritics and Planetary Science, 44, 689
Zhu Z., Hartmann L., Gammie C., McKinney J. C., 2009, ApJ, 701, 620
Zilitinkevich S. S., Elperin T., Kleeorin N., Rogachevskii I., 2007, Boundary-Layer Meteorol., 125, 167
Zsom A., Ormel C. W., Güttler C., Blum J., Dullemond C. P., 2010, A&A, 513, A57

APPENDIX A: DERIVATION OF TURBULENT THERMAL DIFFUSION

A1 Equations for gas and solids

Here, we derive in exhaustive detail the equations for turbulent particle transport in general and turbulent thermal diffusion more specifically. We start with the continuity and velocity equations for gas and particle fluids. The gas density and velocity are denoted $\rho$ and $u$, while the particle number density and velocity are denoted $n$ and $v$. In this article we assume that the particle fluid has a well-defined, single-valued differentiable velocity field. This single-valued velocity approximation breaks down at small scales, which is what allows particle–particle collisions to occur, but the deviation from a well-defined velocity field is small.

We assume Kolmogorov turbulence with largest (integral) length and velocity scales $L_0$ and $u_0$. The length-scale $L_0$ is assumed to be much smaller than the length-scale associated with global variations in parameters of relevance (such as density), so that those fields can be linearized. We also define the wavenumber $k_0 \equiv L_0^{-1}$ and the time-scale $t_0 \equiv L_0/u_0$ and assume that $t_0$ is much shorter than global system evolution time-scales, so that changes in the background fields can be neglected on turbulent time-scales.

For simplicity, we assume that all global quantities vary along the same direction in a linearizable fashion and we adopt a local cylindrical coordinate system with the $z$-axis aligned with that direction. We also assume that the $xy$ plane can be meaningfully averaged over (i.e. is periodic, closed or sufficiently large in extent that boundary terms can be neglected on the time-scale of TTD). In a planetary atmosphere, the general case would have $z$ aligned with altitude, with the temperature varying with height on length-scales far shorter than the local pressure scaleheight and also far shorter than those associated with latitude or longitude. In a protoplanetary disc, we would generally place ourselves at the midplane, with $z$ aligned with the radial direction, noting that the system is periodic.

MNRS 456, 3079–3089 (2016)
in azimuth and symmetric about the midplane. In this case, for us to be able to average over the radial–azimuthal plane, we need the turbulent length-scale to be short, compared with not only the radial gradients but also the local vertical pressure scaleheight.

A2 Particle drift velocity

Our equations are

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (A1) \]

\[ \partial_t n + \nabla \cdot (n \mathbf{v}) = 0, \quad (A2) \]

\[ \partial_t u + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + g, \quad (A3) \]

\[ \partial_t v + v \cdot \nabla v = -\frac{v - u}{\tau_s} + g, \quad (A4) \]

where \( p \) is the gas pressure, \( g \) the acceleration due to gravity (and any other accelerations that affect the gas and particles equivalently) and \( \tau_s \) is the frictional stopping time for the particles. In this analysis, we neglect the possibility of any other forces that act on the gas and particles differently and we neglect the back reaction of the particle drag on the gas. This last requires that the particle fluid mass density be much less than the gas density, i.e.

\[ m_p \ll \rho, \quad (A5) \]

where \( m_p \) is the mass of a particle.

Defining the drift of the particle fluid through the gas as

\[ \mathbf{w} \equiv \mathbf{v} - \mathbf{u}, \quad (A6) \]

we can combine equations (A3) and (A4) to write

\[ \partial_t \mathbf{w} + \mathbf{u} \cdot \nabla (\mathbf{w} + \mathbf{u}) + \mathbf{v} \cdot \nabla \mathbf{w} = -\frac{\mathbf{w}}{\tau_s} + \frac{1}{\rho} \nabla p. \quad (A7) \]

In the case of \( \tau_s \ll t_0 \), the particles are well coupled to at least the largest scale turbulence. In that case, the particles will rapidly reach their terminal velocity and we can approximate both

\[ \partial_t \mathbf{w} \simeq 0, \quad (A8) \]

\[ \mathbf{w} \propto \mathbf{u}, \quad (A9) \]

so that equation (A7) becomes

\[ \mathbf{w} \simeq \tau_s \left[ \frac{1}{\rho} \nabla p - \mathbf{w} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{w} \right], \quad (A10) \]

\[ = \tau_s \frac{1}{\rho} \nabla p + [\text{terms in } \tau_s^2 \text{ and higher}]. \quad (A11) \]

Note that turbulence involves motions on a range of size and time-scales and equations (A9) and (A11) neglect the effect of turbulent motions with wavenumbers \( k \) high enough that their corresponding time-scales \( t(k) \ll \tau_s \). This is reasonable, because turbulent structures on those size scales do not live long enough to have a significant effect on the particle trajectories, but this places limits on which wavenumbers can be integrated over, as will be noted in Section A6.

A3 Mean-field decomposition

As mentioned above, for simplicity we assume that all large-scale spatial variations in the background fields are aligned along the \( r \)-axis. We perform a mean-field decomposition, with \( xy \) averages denoted by overbars and fluctuations denoted by primes:

\[ \overline{\mathbf{\rho}} \equiv A^{-1} \int_{xy} \mathrm{d}x \, \mathrm{d}y \, \rho, \quad (A12) \]

\[ \rho' \equiv \rho - \overline{\mathbf{\rho}}, \quad (A13) \]

where

\[ A \equiv \int_{xy} \mathrm{d}x \, \mathrm{d}y. \quad (A14) \]

Note that the average of a fluctuating quantity is zero:

\[ \overline{\rho'} = \overline{[\rho - \overline{\rho}]} = \overline{\rho} - \overline{\rho} = 0. \quad (A15) \]

Using this averaging scheme, only \( t \) and \( z \) derivatives of averaged quantities survive. Note that this averaging scheme obeys the Reynolds averaging rules, so derivatives commute with averaging.

In the case of a protoplanetary disc annulus at the midplane with the temperature gradient pointing radially, we can adopt a spherical coordinate system with the pole (\( \theta = 0 \)) aligned with the rotation axis of the disc. In that case, at an orbital position \( R \), equation (A12) becomes

\[ \overline{\mathbf{\rho}} \equiv \frac{1}{2 \pi R \times 2 \Delta \theta R} \int_{\theta = \pi/2 - \Delta \theta}^{\pi/2 + \Delta \theta} \int_0^{2\pi} R \sin \theta \, \mathrm{d}\phi \, \mathrm{d}\rho. \quad (A16) \]

In equation (A16), \( \phi \) is the azimuthal angle and we require simultaneously that vertical extent \( R \Delta \theta \) be large enough compared with turbulent length-scales to average over and small enough compared with \( R \) that curvature can be neglected. In protoplanetary discs, we expect turbulent length-scales to be much smaller than the local scaleheight, which in turn is expected to be much smaller than the orbital radius, so those conditions can be met.

By horizontally averaging equation (A1), we find

\[ \partial_t \overline{\mathbf{\rho}} = -\partial_r \left( \overline{\rho \mathbf{u}} + \rho' \overline{\mathbf{u}} \right), \quad (A17) \]

noting that the net gas mass flux is

\[ A^{-1} \int_{xy} \mathrm{d}x \, \mathrm{d}y \, \mathbf{u} = \overline{\rho \mathbf{u}} + \rho' \overline{\mathbf{u}}. \quad (A18) \]

A gas quasi-steady state with \( \partial_t \overline{\mathbf{\rho}} \approx 0 \) and no net mass flux in \( z \) then satisfies

\[ \overline{\rho \mathbf{u}} + \rho' \overline{\mathbf{u}} = 0, \quad (A19) \]

which means that

\[ \overline{\mathbf{u}} = -\frac{\rho' \overline{\mathbf{u}}}{\rho} \neq 0 \quad (A20) \]

under this averaging scheme, but also that \( \overline{\mathbf{u}} \) is at most second-order in the fluctuations.

A4 Gas and particle continuity equations

Splitting equations (A1) and (A2) into their mean and fluctuating components using equation (A6), we find

\[ \partial_t \overline{\mathbf{\rho}} + \nabla \cdot (\overline{\rho \mathbf{u}}) + \nabla \cdot (\rho' \overline{\mathbf{u}}) = 0, \quad (A21) \]

\[ \partial_t \rho' + \nabla \cdot (\overline{\rho \mathbf{u}}' + \rho' \overline{\mathbf{u}}) + \nabla \cdot (\rho' \mathbf{u}' - \rho' \overline{\mathbf{u}}) = 0, \quad (A22) \]

\[ \partial_t \overline{n} + \nabla \cdot (\overline{n \mathbf{u}} + \overline{\mathbf{u}}) + \nabla \cdot (n' \overline{\mathbf{u}} + \overline{n' \mathbf{u}}) = 0, \quad (A23) \]

\[ \partial_t n' + \nabla \cdot (\overline{n' \mathbf{u}} + \mathbf{u}') + \nabla \cdot (n' \overline{\mathbf{u}} + \overline{n' \mathbf{u}}) = 0. \quad (A24) \]
To proceed, we need to use equations (A22) and (A24) to estimate \( \rho' \) and \( n' \) for use in equations (A21) and (A23). We assume that the fluctuations are weak enough that we only need to track equations (A22) and (A24) to first order in fluctuating quantities, noting from equation (A20) that \( \overline{u} \) is at most second-order in the fluctuations. We further assume that background gradients are weak enough that \( \overline{w} \) can be neglected when estimating \( n' \).

Under those conditions, we can use the first-order smoothing approximation (FOSA) to close our system, writing

\[
\rho'(t) \simeq \rho'(0) - \int_0^t dt' \nabla \cdot \left[ \rho(t') \overline{u}'(t') \right],
\]

(A25)

\[
n'(t) \simeq n'(0) - \int_0^t dt' \nabla \cdot \left[ n(t') \left( \overline{w}(t') + u'(t') \right) \right].
\]

(A26)

As long as mean quantities vary slowly compared with turbulent time-scales and noting that the fluctuating quantities vary around zero, we need to track the time integrals in equations (A25) and (A26) only for the turbulent correlation times, finding

\[
\rho' \simeq -n_0 \nabla \cdot (\overline{w} u'),
\]

(A27)

\[
n' \simeq -n_0 \nabla \cdot (\overline{w} u) - n_0 \nabla \cdot (n u')
\]

(A28)

for characteristic turbulent time-scales \( n_0 \) and \( t_n \). In what follows, we will assume that at every turbulent length-scale \( t_n \) is equal.

Combining equations (A21) and (A27) and combining equations (A23) and (A28), using the fact that mean quantities can only vary in the \( z \) direction, we find

\[
\partial_t \overline{\rho} + \partial_z (\overline{\rho} \overline{u}) - \partial_z \left[ \overline{t}_n u' \left( \overline{w} \cdot u' + u' \frac{\partial \overline{w}}{\partial z} \right) \right] = 0,
\]

(A29)

\[
\partial_t \overline{u} + \partial_z (\overline{u} \overline{v}) - \partial_z \left( \overline{t}_n u' \left( \overline{w} \cdot u' + u' \frac{\partial \overline{w}}{\partial z} \right) \right) = 0,
\]

(A30)

As shown in equation (A20), \( \overline{\rho} \) need not be zero, especially in the presence of background density gradients. However, \( \overline{\rho} \) can be considered an artefact of choosing an averaging scheme that obeys the Reynolds averaging rules rather than a density-weighted average and, fortunately, it can be eliminated from the equations for the density in favour of turbulent terms: assuming a gas density statistical steady state, we have \( \partial_t \overline{\rho} = 0 \) and equation (A29) implies that

\[
\rho (\overline{u}_z - \frac{u'_n}{t_n} \overline{w} \cdot u') = \frac{u'_n}{t_n} \overline{w} \cdot \overline{u}.
\]

(A31)

Equation (A31) captures the density evolution of turbulent parcels of gas moving across a background gas density gradient.

Combining and rearranging terms in equation (A30), we arrive at

\[
\partial_t \overline{\rho} + \partial_z (\overline{\rho} \overline{u}) - \partial_z \left( \left[ \overline{t}_n u' + \frac{n_0}{\overline{\rho}} \overline{w} \cdot \overline{u} \right] \frac{\partial \overline{\rho}}{\partial z} \right)
\]

\[
\partial_t \left( \left[ \overline{t}_n u' + \frac{n_0}{\overline{\rho}} \overline{w} \cdot \overline{u} \right] \frac{\partial \overline{\rho}}{\partial z} \right) = 0.
\]

(A32)

### A5 Ordering

We recall from equation (A20) that \( \overline{u} \) is at most quadratic in fluctuating quantities. Averaging equation (A11), we find

\[
\overline{w} \simeq \frac{\overline{\rho}}{\overline{\rho}} \overline{\rho} \overline{w} + \left[ \text{terms quadratic and higher in fluctuations} \right].
\]

(A33)

Because we have assumed that background quantities vary on long length-scales compared with the turbulence, we can approximate

\[
\overline{w} \simeq \left( \frac{\overline{t}_n}{\overline{\rho}} - \frac{\overline{\rho} \overline{w} \overline{u}'}{\overline{\rho}^2} \right) \overline{\rho} \overline{w} + \frac{\overline{\rho}}{\overline{\rho}} \overline{\rho} \overline{w} \overline{p}' \simeq \frac{\overline{\rho}}{\overline{\rho}} \overline{\rho} \overline{w} \overline{p}'.
\]

(A34)

To help identify terms, we define

\[
\overline{D} = \overline{t}_n u'_z,
\]

(A35)

\[
\tilde{\overline{D}} = \overline{t}_n u'_z + 2 \overline{t}_n u'_z u'_z,
\]

(A36)

\[
\bar{w} = -\overline{t}_n u'_z \overline{w} \cdot \overline{u},
\]

(A37)

\[
\tilde{\overline{u}} = -\overline{t}_n u'_z \overline{w} \cdot \overline{u},
\]

(A38)

where \( D \) is the traditional turbulent diffusion coefficient, \( \tilde{D} \) a correction for inertial particles and \( \bar{w} \) and \( \tilde{\overline{u}} \) are correction terms deriving from our use of a volume- (rather than gas density) weighted averaging scheme.

We can now rewrite equation (A32) as

\[
\partial_t \overline{\rho} + \partial_z (\overline{\rho} \overline{u} + \bar{u} + \tilde{\overline{u}}) - \partial_z (\overline{D} \partial_z \overline{\rho})
\]

\[
\partial_z \left( \overline{\rho} \overline{\rho} \overline{w} \cdot \overline{u} \right) \simeq 0.
\]

(A39)

Finally, we can use equation (A31) to write

\[
\overline{\rho} = \bar{u} + \overline{t}_n u'_z \partial_z \overline{\rho} = \overline{D} \partial_z \overline{\rho},
\]

(A40)

reducing equation (A39) to

\[
\partial_t \overline{\rho} + \partial_z \left( \overline{D} \partial_z \overline{\rho} \right) \simeq 0.
\]

(A41)

In equation (A41), the first two terms are the continuity equation for a passive scalar and the remaining terms are the corrections for particle inertia. Note that, as expected, in the absence of particle inertia the steady-state solution obeys \( \overline{\rho} \propto \overline{\rho} \), i.e. a uniform particle concentration.

Turbulence with a wavenumber \( k \) has an associated velocity \( u(k) \). As long as the turbulent time-scale \( t_k \approx \tau_z \), the corresponding relative velocity \( w(k) \ll u(k) \) and we approximate \( w \ll u \). However, the same need not hold when comparing \( \overline{\rho} \overline{w} \cdot \overline{u} \). Because the latter is resisted by pressure forces. We therefore drop \( \bar{w} \) and \( \tilde{D} \) in favour of \( \bar{u} \) and \( D \), reducing equation (A41) to

\[
\partial_t \overline{\rho} + \partial_z \left( \overline{D} \partial_z \overline{\rho} \right) \simeq 0.
\]

(A42)

Combining equations (A34) and (A42), repeating the approximation that derivatives on fluctuating quantities dominate over derivatives on background quantities, we come at long last to

\[
\partial_t \overline{\rho} + \partial_z \left( \overline{D} \partial_z \overline{\rho} \right) \simeq 0.
\]

(A43)

Equation (A43) is the general low Mach number turbulent averaged particle fluid continuity equation under standard mean-field decompositions and for common scale-separation assumptions. We next explore its final term and derive the TTD.
A6 Pressure fluctuations in the presence of a temperature gradient

Using equation (A43), we can define the particle flux \( F_n \) in the \( \hat{e}_z \) direction:

\[
\partial_t \vec{\rho} + \partial_i \left( F_n \hat{e}_z \right) = 0, \tag{A44}
\]

\[
F_n = \pi \left( D \partial_z \left[ \ln \bar{p} - \ln \bar{p}_\tau \right] + \bar{\omega}_z - \frac{\tau}{\rho_T} n u'_z \nabla^2 p \right). \tag{A45}
\]

In equation (A45), the first term on the right-hand side is the diffusive flux, the second the large-scale particle drift due to a background pressure gradient. The third term is the source of the so-called turbulent thermal diffusion.

We can use the ideal gas equation of state to write, to first order in fluctuating quantities,

\[
\overline{p} = \frac{k_b}{m} \bar{T}, \tag{A46}
\]

\[
p' = \frac{k_b}{m} \left( \bar{T}' + \rho \bar{T} \right), \tag{A47}
\]

\[
\overline{\tau u'_z \nabla^2 p'} = \frac{\tau k_b}{m \bar{p}} \overline{u'_z \nabla^2 \left( \rho \bar{T}' + \bar{p} \bar{T}' \right)}, \tag{A48}
\]

where \( m \) is the mean molecular mass of the gas. The key insight of Elperin et al. (1996) was that the correlation of \( u'_z \nabla^2 p' \) in equation (A48) is small because it represents turbulent transport of mass, but the correlation of \( u'_z \bar{T}' \) is large in the presence of a large-scale temperature gradient, because it represents the turbulent transport of temperature down a temperature gradient.

We therefore neglect the \( \rho' \) component of equation (A47) and write the equation for turbulent thermal diffusion as

\[
\overline{\tau u'_z \nabla^2 p'} \simeq \overline{\tau k_b} \overline{u'_z \nabla^2 \bar{T}'}, \tag{A49}
\]

where we have neglected \( \nabla^2 \overline{p} \) because turbulent length-scales are assumed to be shorter than turbulent ones. In the absence of strong cooling terms, temperature is approximately advected:

\[
\bar{T}' = -C u'_z \partial_z \bar{T} \tag{A50}
\]

for a turbulent transport coefficient \( C \) of order unity, which depends on the nature of the turbulence (Zilitinkevich et al. 2007). Combining equations (A49) and (A50), we find

\[
V_{TDD} \simeq \frac{C \tau k_b}{m} \bar{T} \overline{u'_z \nabla^2 u'_z}, \tag{A51}
\]

where we have again used scale separation to neglect derivatives on non-turbulent gradients.

For Kolmogorov turbulence, we have

\[
t_u = 2 \tau_0 \left( \frac{k}{k_0} \right)^{-2/3}, \tag{A52}
\]

\[
|u'| = u_0 \left( \frac{k}{k_0} \right)^{-1/3}. \tag{A53}
\]

Accordingly, we can calculate \( D \) by integrating over the turbulent cascade:

\[
D = \frac{1}{2} \int_{k_0}^{k} B \tau_0 \left( \frac{k}{k_0} \right)^{-2/3} u_0 \left( \frac{k}{k_0} \right)^{-2/3} \frac{dk}{k}. \tag{A54}
\]

For isotropic turbulence Kolmogorov turbulence, we have \( B = 2/9 \) and the dissipation wavenumber

\[
k_\eta = Re^{2/3} k_0, \tag{A55}
\]

where \( Re \) is the Reynolds number of the turbulence. For astrophysical turbulence, we generally have \( Re \gg 1 \) and so

\[
D = \frac{t_u u_0^2}{3}. \tag{A56}
\]

Replacing \( \nabla^2 \rightarrow k^2 \), we can also calculate

\[
\int_{k_0}^{k_1} B \left[ \frac{k}{k_0} \right]^{-4/3} \left[ u_0 \left( \frac{k}{k_0} \right)^{-2/3} \right] \frac{dk}{k} = -\frac{8}{9} \int_{k_0}^{k_1} \frac{dk}{k} = -\frac{8}{9} \ln (k_1/k_0), \tag{A57}
\]

where \( k_1 \) is the limiting wavenumber. We have assumed that \( \tau_s < t_u \), which imposes

\[
k_1 = k_n = Re^{2/3} k_0 \tag{A58}
\]

set by the dissipation scale of the turbulence. Further, \( V_{TDD} \) depends linearly on \( S_t \), so its effects will be negligible unless \( S_t \) is large enough that equation (A58) is indeed the controlling limit.

In much of the existing literature, equation (A55) is the limit used and in the cases where equation (A58) is invoked it is done so in the Stokes drag regime. Many cases of astrophysical interest are in the Epstein drag regime, so we proceed using the more general formulation in equation (A58), resulting in the equations for the particle flux:

\[
F_n = \pi \left( D \partial_z \left[ \ln \bar{p} - \ln \bar{p}_\tau \right] + \bar{\omega}_z + V_{TDD} \right), \tag{A61}
\]

\[
V_{TDD} = -\frac{4}{3} \frac{C \tau k_b \bar{T}}{m} \ln S_t^{-1} \partial_z \ln \bar{T}. \tag{A62}
\]

A7 Velocity limits

An interesting feature of equation (A62) is that the turbulent velocity scale has cancelled out and for an arbitrarily large \( \partial_z \ln \bar{T} \) we would have \( V_{TDD} > u_0 \), which would be absurd. Returning to equation (A51), note that

\[
\frac{n_T}{\bar{n}} = C \frac{k_b \bar{T}}{m} \partial_z \ln \bar{T} \overline{u'_z \nabla^2 u'_z} \tag{A63}
\]

is the fractional particle number density fluctuation due to the temperature fluctuations. We have assumed that the turbulent fluctuations are small enough that they can be treated to first order, so we require \( n_T \ll \bar{n} \), which in turn constrains \( V_{TDD} \ll u_0 \). For Kolmogorov turbulence, the right-hand side of equation (A63) is proportional to

\[
\overline{u'_z \nabla^2 u'_z} \propto k^{1/3} \tag{A64}
\]
and so the requirement that
\[ \frac{n_{T,r}}{\rho} = C \frac{\kappa_b T}{m} \partial \ln \bar{T} \ln^2 \tau, \quad \alpha \ll 1 \]  
(A65)
is a requirement on both \( \partial \ln \bar{T} \) and the upper limit \( k_1/k_0 \) for the integral in equation (A57). Throughout this article, we assume that \( \partial \ln \bar{T} \) is sufficiently small that equation (A62) can be used. When that is not the case, equation (A62) overestimates \( V_{TTD} \), but in that case the concentration of particles invoked by TTD is large enough to be non-negligible on its own terms.

A8 Quasi-isobaric case

In the limit where \( \partial \ln \bar{T} = 0 \), we have
\[ \partial \ln \bar{T} = 0 = \partial \ln \bar{T} \]  
(A66)

and we can write equation (A45) as
\[ F_s = \bar{T} \left( -D \partial \ln \bar{T} \ln \sum \bar{T} + \ln \bar{p} \right) - C \tau \kappa_b \frac{T}{m} \ln \tilde{S} \partial \ln \bar{T} \]
\[ = -D \bar{T} \left( 1 + \frac{4C \tau \kappa_b T}{3D} \ln \tilde{S} \right) \partial \ln \bar{T} \ln \bar{T}. \]  
(A67)

Defining
\[ \alpha = 1 + \frac{4C \tau \kappa_b T}{3D} \ln \tilde{S}, \]
\[ \bar{p} \propto T^{-\alpha}, \]  
(A68)

we see that the steady-state solution is
\[ V_{TTD} = -\left( \alpha - 1 \right) D \partial \ln \bar{T}. \]  
(A70)

where \( \bar{p} \) and \( T \) are observables in both laboratory experiments and the Earth’s atmosphere, allowing \( \alpha \) to be evaluated. Values of \( \alpha > 2 \) have been found in experiments and \( \alpha \geq 20 \) in measurements of aerosols in the Earth’s tropopause. Under these definitions, \( V_{TTD} = -(\alpha - 1)D \partial \ln \bar{T} \).

A9 Falsifying rapid pressure equilibration

It might seem natural to assume that low Mach number turbulence experiences rapid pressure equilibration and therefore that \( \rho' \) is set by the constraint that vertically travelling parcels of gas expand or contract rapidly to maintain pressure equilibration. However, as we show here, it that were the case then particle transport would be negligible, which is incompatible with the results of laboratory investigations of TTD. In the isobaric case, rapid equilibration would imply \( \rho' = 0 \) in the absence of thermal equilibration, which has been ruled out by experiments, which found \( \alpha > 1 \).

Assuming perfect thermal equilibration instead results in the following approximations for equations (A1) and (A3):
\[ \partial \rho \simeq -u'_i \partial \ln \bar{p} - \bar{p} \nabla \cdot u', \]
\[ \rho' = \frac{1}{\bar{p}} \nabla^2 \bar{p}' \]  
(A71)

Under the assumption of small turbulent correlation times, we can also estimate
\[ \nabla \cdot u' \simeq t u' \partial \ln \bar{p}' \]  
(A73)

Inserting equation (A72) into equation (A73), we find
\[ \nabla \cdot u' \simeq - \frac{t}{\bar{p}} \nabla^2 \bar{p}'. \]  
(A74)

Combining equations (A71) and (A74), we arrive at
\[ \nabla^2 \bar{p}' \simeq \frac{t}{\bar{p}} \rho \partial \ln \bar{p}, \]  
(A75)

which, when inserted into equation (A45), would imply
\[ V_{TTD} \simeq -\frac{\tau}{\bar{p}} t u' \partial \ln \bar{p} = -\frac{\tau}{\bar{p}} u' \partial \ln \bar{p} \]  
(A76)

and in the case of Kolmogorov turbulence with zero large-scale pressure gradients this reduces to
\[ V_{TTD} \simeq -2StD \partial \ln \bar{p} = 2StD \partial \ln \bar{T}, \]  
(A77)

which has the opposite sign to the prediction of turbulent thermal diffusion. Thus this approximation would predict \( \alpha < 0 \), which has been ruled out by observation and experiment.

We could also assume that \( \rho' \) is controlled by the turbulent ram pressure \( \bar{p}u'_i \). In the absence of large-scale gradients in the gas, symmetry implies that ram pressure fluctuations cannot correlate with \( u'_i \) in equation (A45). However, in the presence of a temperature gradient, we can postulate
\[ |\rho'| \simeq \bar{p} u'_i k^{-1} \partial \ln \bar{T}, \]  
(A78)

where the \( k^{-1} \ln \bar{T} \) term provides the required isotropy breaking for a correlation to exist. In the case of Kolmogorov turbulence, this reduces to
\[ |V_{TTD}| = |4StD \partial \ln \bar{T}| \ll |D \partial \ln \bar{T}| \]  
(A79)

and so would predict
\[ \alpha = 1 \ll 1, \]  
(A80)

which is also ruled out by experiment and observation.