How to reason with inconsistent probabilistic information?

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Abstract

A recent line of research has developed around logics of belief based on information confirmed by a reliable source. In this paper, we provide a finer analysis and extension of this framework, where the confirmation comes from multiple possibly conflicting sources and is of a probabilistic nature. We combine Belnap-Dunn logic and non-standard probabilities to account for potentially contradictory information within a two-layer modal logical framework to account for belief. The bottom layer is to be that of evidence represented by probabilistic information provided by sources available to an agent. The modalities connecting the bottom layer to the top layer, are that of belief of the agent based on the information from the sources in terms of (various kinds of) aggregation. The top layer is to be the logic of thus formed beliefs.

Keywords: epistemic logics, non-standard probabilities, Belnap-Dunn logic, two-layer modal logic.

1 Introduction

As we collect and process more and more data to form beliefs about the world, we are faced with information from multiple sources of different origins, that is mostly incomplete and often conflicting concerning the issues we wish to resolve. In this context, we wish to formalize how an agent can build its belief from the available collected data. The information typically is of a probabilistic nature however, simply confusing (degrees of) belief of a rational agent with a subjective probability, rather than analysing how beliefs are formed based on such probabilistic information coming from multiple sources, would not do.

Context and state of the art. A recent line of research has developed around epistemic logics based on information or evidence [4,6]. In particular, Bílková et al. [6] understand belief as based on information confirmed by a reliable source. However, the nature of information and of sources on which such belief is founded, their reliability, and the way belief is formed, remained somewhat undeveloped. In this paper we follow up on these ideas and present a framework where the information about potentially complex and not fully

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determined facts comes from multiple possibly conflicting sources and is of a probabilistic nature.

**Case studies.** We introduce two case studies that we use throughout the article to illustrate the different concepts at stake.

**Case 1 (Aggregating heterogeneous data)** A company has to launch a new car and needs to decide its selling price and its advertising strategy. Hence, its data analysts must study the reports on the sells of the previous products launched by the company and the success or failure of the advertisement campaigns. This study relies on factual information such as “during the year 2015, the company sold n items of product Y”, but also on statement based on statistical analyses such as “the advertisement broadcasted in June 2016 increased the sells of product Y among the 20-30 years old of 30%”. The second statement is based on aggregated information about the buyers that might be partial and partly false. Plus, the company has access to statistical studies about the population on increase or decrease in expenses for cars.

**Case 2 (How to lead an investigation)** An investigator needs to know if one of the suspects was present at the crime scene. She collects information from various sources (CCTV camera recordings, ATM logs, witnesses’ statements etc.). No information of this kind is absolutely precise and typically different sources of information contradict each other. Sources provide information of a probabilistic nature (e.g. witnesses are not absolutely sure what they have seen). Moreover, the pieces of evidence confirming investigator’s hypothesis (the suspect was present at the place of crime) are different from, and somewhat independent of, those rejecting it (there is a CCTV camera closed to the crime scene vs. ATM in a supermarket in a different city). A lack of evidence supporting the suspect was present at the crime scene does not yield a proof she was not there. In the end, the investigator has to aggregate the available information and form some beliefs about what likely happened.

In Case 1, the agent trying to build an opinion about the launching strategy for the new car has access to a huge amount of heterogeneous, more or less reliable information. However, this agent (the team of data analysts and marketing) has computational abilities one can assume to be unlimited. In Case 2, even though the investigator can be seen as an expert, she can only analyse and combine a limited amount of information in a (computationally) simple manner.

**Belnap-Dunn logic.** There are several proposals of logical frameworks in the literature allowing for (non-trivial) inconsistencies. Belnap-Dunn logic BD (also referred as First Degree Entailment) was specifically designed to deal with (possibly) incomplete and inconsistent information one can find in databases (see Section 2.2 and [5]). This task is accomplished by extending the set of truth values. In BD a formula might not only be true (t) or false (f) as in the classical case, but can also be neither (n) or both (b).2

2 The reading of truth values is not metaphysical, they are interpreted from the point of view of available information.
Non-standard probabilities. One of the background assumptions behind the BD approach is that positive and negative information are distinct. Having information about truth of a proposition $\varphi$ is distinct from the information about its falsity. This assumption is naturally applicable in the probabilistic framework. We will distinguish positive probability $p^+$ of a claim $\varphi$ representing (probabilistic) information supporting $\varphi$ and negative probability of $\varphi$ representing (probabilistic) information rejecting $\varphi$. Non-standard probabilities of this kind have been discussed in the literature, we introduce them in Section 2.3 following the presentation given in [16].

Two-layer modal logics. In order to take into account all such considerations, we will combine BD logic and non-standard probabilities to account for potentially contradictory information about events, within a two-layer modal logical framework (see [3] and Section 4) to account for belief. The bottom layer is to be that of events or facts representing probabilistic information provided by sources available to an agent. The modalities connecting the bottom layer to the top layer encode the belief of the agent (e.g. about an event taking place) based on the information from the sources in terms of (various kinds of) aggregation. The top layer is the logic of thus formed beliefs.

Structure of the paper. First, in Section 2, we give technical preliminaries on the logic BD and non-standard probabilities. Then, in Section 3, we discuss the possible aggregation strategies depending on the type of agent one models. In Section 4, we present two-layer modal logics and provide examples. Finally, in Section 5, we discuss further lines of research.

2 Preliminaries

This section contains the necessary technicalities on algebraic structures (namely certain product bilattices, residuated bilattices and MV algebras) and logics (namely BD logic and two logics derived from Lukasiewicz logic) used in the framework, and finally non-standard probabilities. For more details, the reader can refer to [18,15] for bilattices, [5,11] and [17] for BD logic and de Morgan algebras, [10] for Lukasiewicz logic and MV algebras, and [16] for non-standard probabilities.

2.1 Some bilattices and MV algebras

Semantically, many-valued logics mostly use algebras of truth values which are lattices, where the lattice order is a truth order. An underlying idea of Belnap-Dunn four-valued logic [5] is that not only amount of truth, but also amount of information that each of the values carries matters. The idea was generalized by Ginsberg [13], where he introduced the notion of bilattices, which are algebraic structures that contain two partial orders simultaneously: a truth order, and a knowledge (or an information) order.

Definition 2.1 A bilattice is an algebra $B = (B, \land, \lor, \land, \cup, \neg)$ such that the reducts $(B, \land, \lor)$ and $(B, \land, \lor)$ are both lattices and the negation $\neg$ is a unary
operation satisfying that for every \( a, b \in B \),

\[
\text{if } a \leq_t b \text{ then } \neg b \leq_t \neg a, \quad \text{if } a \leq_k b \text{ then } \neg a \leq_k \neg b, \quad a = \neg \neg a,
\]

with \( \leq_t \) (resp. \( \leq_k \)) the order on \( (B, \land, \lor) \) (resp. \( (B, \cap, \cup) \)) called the truth (resp. knowledge or information) order. A bilattice is interlaced if each one of the four operations \( \land, \lor, \cap, \cup \) is monotone w.r.t. both orders \( \leq_t \) and \( \leq_k \).

Bilattices form a variety, the class of interlaced bilattices is also a variety. Given an arbitrary lattice \( L = (L, \land_L, \lor_L) \), we can construct the product bilattice \( L \circ L = (L \times L, \land, \lor, \cap, \cup, \neg) \) as follows: for all \( (a_1, a_2), (b_1, b_2) \in L \times L \),

\[
\begin{align*}
(a_1, a_2) \land (b_1, b_2) &:= (a_1 \land_L b_1, a_2 \lor_L b_2) \\
(a_1, a_2) \lor (b_1, b_2) &:= (a_1 \lor_L b_1, a_2 \land_L b_2) \\
(a_1, a_2) \cap (b_1, b_2) &:= (a_1 \land_L b_1, a_2 \land_L b_2) \\
(a_1, a_2) \cup (b_1, b_2) &:= (a_1 \lor_L b_1, a_2 \lor_L b_2) \\
\neg (a_1, a_2) &:= (a_2, a_1)
\end{align*}
\]

\( L \circ L \) is always an interlaced bilattice, and any interlaced bilattice can be represented as a product bilattice: a bilattice \( B \) is interlaced if and only if there is a lattice \( L \) such that \( B \cong L \circ L \) [2].

**Example 2.2** The smallest interlaced bilattice is the product bilattice of the two-element lattice \( 2 \circ 2 \) (Figure 1 left). It is isomorphic to Dunn-Belnap square \( 4 \) used as an algebra of truth values for Belnap-Dunn logic (Figure 1 middle). Moreover it is a de Morgan algebra, and it generates the class of de Morgan algebras as a quasivariety.

**Example 2.3** A probabilistic extension of Dunn-Belnap square can be seen as based on the product bilattice \( L_{[0,1]} \circ L_{[0,1]} \) with \( L_{[0,1]} = ([0,1], \min, \max) \).

**Definition 2.4** A residuated lattice is an algebra \( \mathbb{L} = (\mathbb{L}, \land, \lor, \cdot, \top, \bot) \) such that the reduct \( (\mathbb{L}, \land, \lor, \cdot) \) is a lattice, \( (\mathbb{L}, \cdot) \) is a semi-group (i.e. the operation \( \cdot \) is associative and the following residuation properties hold: for all \( a, b, c \in \mathbb{L} \),

\[
\begin{align*}
a \cdot b &\leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c / b.
\end{align*}
\]

**Example 2.5** The standard algebra of Łukasiewicz logic \( [0,1]_L = ([0,1], \land, \lor, \land_L, \lor_L) \) is a residuated lattice, an MV algebra, and it generates the variety of MV algebras. As \( \land_L \) is commutative, the two implications coincide.
For all $a, b \in [0, 1]$, we define a negation $\neg_L a := a \rightarrow_L 0 := 1 - a$, and the standard operations

$$
\begin{align*}
a \wedge b &:= \min\{a, b\}, & a \land b &:= \max\{0, a + b - 1\}, \\
a \lor b &:= \max\{a, b\}, & a \rightarrow_L b &:= \min\{1, 1 - a + b\}.
\end{align*}
$$

$[0, 1]_{\text{op}} = ([0, 1]^\text{op}, \lor, \land, \oplus, \ominus)$ arises turning the standard algebra upside down, and is isomorphic to the original one. Here, we have $\neg_L a := 1 \ominus_L a$.

Finally, we will consider the MV algebra $[0, 1]_L \times [0, 1]_{\text{op}} = ([0, 1] \times [0, 1]^\text{op}, \land, \lor, \&, \rightarrow)$ with operations defined pointwise, in particular $\land$ and $\lor$ are defined as in bilattices, $\neg(a_1, a_2) := a \rightarrow (0, 1) = (\neg a_1, \neg a_2)$, and:

$$(a_1, a_2) \lor (b_1, b_2) := (a_1 \land b_1, a_2 \lor L b_2) \quad (a_1, a_2) \rightarrow (b_1, b_2) := (a_1 \rightarrow_L b_1, a_2 \rightarrow_L b_2) = (a_1 \rightarrow_L b_1, a_2 \rightarrow_L b_2).$$

As both the projections are surjective homomorphisms of MV algebras, this algebra also generates the variety of MV algebras.

**Definition 2.6** [15] Given a residuated lattice $L = (L, \land_L, \lor_L, \cdot, \lor, \land)$ the product residuated bilattice $L \otimes L = (L \times L, \land, \lor, \cap, \cup, \cdot, \lor, \land)$ is defined as follows: the reduct $(L \times L, \land, \lor, \cap, \cup)$ is the product bilattice $(L, \land_L, \lor_L) \otimes (L, \land_L, \lor_L)$ and, for all $(a_1, a_2), (b_1, b_2) \in L \times L$,

$$(a_1, a_2) \lor (b_1, b_2) := (a_1 \land b_1, a_2 \lor b_2) \quad (a_1, a_2) \rightarrow (b_1, b_2) := (a_1 \rightarrow_L b_1, a_2 \rightarrow_L b_2).$$

Given a product residuated bilattice, one can define the following operations: for all $a, b \in L$

$$a \rightarrow b := (a \lor b) \land (\neg a \land b), \quad a \leftarrow b := \neg a \rightarrow \neg b, \quad \text{and} \quad a \ast b := \neg (b \rightarrow \neg a).$$

For any product residuated bilattice, the structure $(L \times L, \land, \lor, \ast, \rightarrow, \leftarrow, \neg)$ is a residuated bilattice endowed with an involutive negation. If $\ast$ is commutative (associative), so is $\ast$.

**Example 2.7** The product residuated bilattice $[0, 1]_L \otimes [0, 1]_{\text{op}} = ([0, 1] \times [0, 1], \land, \lor, \cap, \cup, \subseteq, \neg, 0, 0)$ with $[0, 1]_L = ([0, 1], \land, \lor, \cdot, \lor, \land, 0)$ being the standard MV algebra of Lukasiewicz logic amounts to the following:

$$(a_1, a_2) \ast (b_1, b_2) := (a_1 \land b_1, (a_1 \rightarrow_L b_2) \land (b_1 \rightarrow_L a_2))$$

$$(a_1, a_2) \rightarrow (b_1, b_2) := (a_1 \rightarrow_L b_1) \land (b_2 \rightarrow_L a_2), a_1 \land b_2),$$

and $(1, 1)$ acts as the unit of the $\ast$: $(1, 1) \ast a = a \ast (1, 1) = (1, 1)^3$. We define

$$\neg a := (a \lor (0, 0)) \cup \neg (\neg a \lor (0, 0)) = (\neg a L a_1, \neg a L a_2)$$

$$a \oplus b := (\neg a \lor b) \cup (\neg a \lor \neg b) = (a_1 \lor L b_1, a_2 \lor L b_2)$$

$$a \ominus b := \neg (a \lor b) \cap \neg (\neg a \lor \neg b) = (a_1 \lor L b_1, a_2 \lor L b_2).$$

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$^3$ Definitions of $\rightarrow, \ast$ match those used in [9] for interval based fuzzy logics, under a transformation given by $(x_1, x_2) \mapsto (x_1, 1 - x_2)$ (symmetry across the $(0, 0.5)(1, 0.5)$ line).
From [15], we know that the (isomorphic copies of) product residuated bilattices obtained from MV algebras form a variety, and its axiomatization can be obtained by translating the one of MV algebras (in the language of residuated lattices)\(^4\).

### 2.2 Logics

**Belnap-Dunn four-valued logic BD.** This logic evaluates formulas to Belnap-Dunn square – a lattice built over an extended set of truth values \(\{t, f, b, n\}\) (Figure 1, middle). Alternatively (cf. Example 2.2) we can see BD square as the bilattice 4 (Figure 1, left). We take the language of \(\mathcal{L}_{BD}\) to be built from a (countable) set of propositional variables \(\text{Prop}\) using connectives \(\{\wedge, \vee, \neg\}\), an evaluation \(e : \text{Prop} \times W \to 4\) is extended to all formulas in the standard way. The consequence relation of logic BD is given, based on the logical matrix \((4,F)\) with \(F = \{t, b\}\) being the designated values, as \(\Gamma \models \varphi\) if \(e[\Gamma] \subseteq F \to e(\varphi) \in F\).

A frame semantics can also be given for BD, in two ways. Belnap-Dunn four-valued frame is a couple \((W, 4)\) and the corresponding model as \((W, 4, e)\) where \(W\) is a set of states and \(e\) is a valuation of atomic formulas \(e : \text{Prop} \times W \to 4\). The valuation is extended to formulas of \(\mathcal{L}_{BD}\) using the algebraic operations on 4 in the expected way. Alternatively, following Dunns’ approach [11], we can define a Belnap-Dunn double valuation model \(M = (W, v^+, v^-)\), with two atomic valuations (a positive and a negative one) \(v^+, v^- : W \times \text{Prop} \to \{0, 1\}\) which we extend to two satisfaction relations \(\models^+, \models^-\) on formulas of \(\mathcal{L}_{BD}\):

\[
\begin{align*}
    s \models^+ \neg \varphi & \iff s \models^- \varphi, \\
    s \models^+ \varphi \wedge \psi & \iff s \models^+ \varphi \text{ and } s \models^+ \psi, \\
    s \models^+ \neg \varphi & \iff s \models^- \varphi, \\
    s \models^+ \varphi \wedge \psi & \iff s \models^- \varphi \text{ or } s \models^+ \psi.
\end{align*}
\]

Where \(s \in W\) and \(\varphi, \psi\) are formulas of \(\mathcal{L}_{BD}\) containing only conjunction and negation. Disjunction can be defined as: \(\varphi \vee \psi := \neg(\neg \varphi \wedge \neg \psi)\). The two approaches yield equivalent semantics for BD (see Appendix 6.1). We can speak about a positive (resp. negative) extension of a formula \(\varphi\) which is the set of states which positively (resp. negatively) support it: \(||\varphi||^+ = \{s \in W | s \models^+ \varphi\}\) (resp. \(||\varphi||^- = \{s \in W | s \models^- \varphi\}\). The frame semantics is dual to the matrix one, and yields the same logic when satisfaction and entailment in the double valuation models are defined in the usual way using the positive satisfaction relation, yielding \(\Gamma \models_{BD} \varphi\) if \(\forall M, s (s \models^+ \Gamma \to s \models^+ \varphi)\). BD can be completely axiomatized using the following axioms and rules:

\[
\begin{align*}
    \varphi \wedge \psi & \vdash \varphi \\
    \varphi \wedge \psi & \vdash \psi \\
    \varphi \vdash \varphi \vee \psi \\
    \varphi \vdash \varphi \vee \psi \\
    \varphi \vdash \neg \neg \varphi \\
    \neg \neg \varphi \vdash \varphi \\
    \varphi \wedge (\psi \vee \chi) & \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \\
    \varphi \vdash \psi \vee \chi \\
    \varphi \vdash \psi \wedge \chi \\
    \varphi \vdash \psi \\
    \varphi \vdash \psi \vee \chi \\
    \varphi \vdash \psi \wedge \chi \\
    \neg \psi \vdash \neg \varphi.
\end{align*}
\]

BD logic is known to be (strongly) complete w.r.t. frame semantics mentioned

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\(^4\) [15] hints at the correspondence between subvarieties of residuated lattices and residuated bilattices being categorical. This would mean that the mentioned variety is in fact generated by the product bilattice of the standard MV algebra. One could then use the translation from [15] to obtain axiomatizations of the logic in Example 2.11.
above, and w.r.t. \((4, \{t, b\})\), and therefore also w.r.t. \((\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}, \{\{1, a\} | a \in [0,1]\})\) because it has \((\{1, (0, (1, 1))\}) as a sub-matrix\(^5\). \(\mathbf{BD}\) is also known to be locally finite\(^6\).

\(\mathbf{BD}\) was specifically introduced to process possibly incomplete or inconsistent information one can find in databases. Hence, one understands why we use it to model Case 1. However, \(\mathbf{BD}\) appears to be useful in many other situations. Indeed, computers are not the only ones having to deal with this kind of situation. For instance, the investigator in Case 2 is interviewing eye-witnesses to know if they saw the suspect at the crime scene. She ends up with four kind of answers: “yes”, “no”, “I don’t know”, “maybe, I think so, but it could have been someone else”. We use \(\mathbf{BD}\) mainly to capture the underlying information states on which sources of probabilistic information are based in Section 3.

**Two logics derived from Lukasiewicz logic.** We aim at a logic to reason about beliefs of an agent, some of which preserve the properties of non standard probabilities. We need enough expressivity to capture the probabilistic axioms explained in the next section — this motivates choosing Lukasiewicz logic as a starting point, as it naturally expresses plus and minus.

**Example 2.8 [An extension of Lukasiewicz logic.]** We first consider the MV algebra \([0, 1]_L \times [0, 1]_L\) from Example 2.5. We consider only \((1, 0)\) to be the designated value. Its logic (understood as the set of theorems - formulas always evaluated at \((1, 0)\), or as a consequence relation - preserving the value \((1, 0)\)) - is Lukasiewicz logic \(L\). It can be axiomatized in the (complete) language \(\{\to, \neg\}\) by axioms of weakening, suffling, commutativity of disjunction, and contraposition, and the rule of Modus Ponens. We will extend the signature of the algebra with the bilattice negation \(\neg(a_1, a_2) = (a_2, a_1)\), and extend the language to \(\{\neg, \neg\}\) (in particular, \(\oplus\) and \(\ominus\) can be defined similarly as in Example 2.5). Together we obtain the following axioms and rules, denoting the resulting consequence relation \(\vdash_{L(\neg)}\) (we will call the three axioms on the right \(\neg\)-axioms):

\[
\begin{align*}
\alpha \to (\beta \to \alpha) & \quad \neg\neg\alpha \leftrightarrow \alpha \\
(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)) & \quad \neg\neg\alpha \leftrightarrow \neg\neg\neg\neg\alpha \\
((\alpha \to \beta) \to \beta) & \quad (\neg\neg\alpha \to \neg\neg\beta) \leftrightarrow \neg\neg(\alpha \to \beta) \\
(\neg\beta \to \neg\alpha) & \quad \alpha, \alpha \to \beta/\beta \quad \alpha/\neg\alpha
\end{align*}
\]

Using the \(\neg\) axioms, one can prove that the \(\neg\) negations can be pushed to the atomic formulas, and consider formulas up to provable equivalence in a *negation normal form (nnf)*, i.e. formulas built using \(\{\to, \neg\}\) from literals of the form \(p, \neg p\). We denote \(\square\Gamma := \neg\neg\Gamma \cup \Gamma\).

\(^5\) The obvious embedding is a strict homomorphism of de Morgan matrices - it preserves and reflects the filters

\(^6\) The inter-derivability relation of the axiom system is a Frege congruence. Local finiteness means there are only finitely many (up to inter-derivability) formulas in a fixed finite set of propositional variables.
Lemma 2.9 For any finite set of formulas $\Gamma, \alpha$ in a nff, 
$$\Gamma \vdash_{L(\neg)} \alpha \iff \text{ for some finite } \Delta, \square \Gamma, \Delta \vdash_{L} \alpha,$$
where $\Delta$ contains instances of $\neg$-axioms.

Lemma 2.9, proved in Appendix 6.3, provides a translation to provability in $L$ and allows us to see that the extension of $L$ by $\neg$ is conservative. Now, using finite completeness of $L$, we can see (Appendix 6.4) that $L(\neg)$ is complete w.r.t. $[0, 1]_{L} \times [0, 1]^p_{L}$.

Lemma 2.10 (Finite strong completeness) For a finite set of formulas $\Gamma$,
$$\Gamma \vdash_{L(\neg)} \alpha \iff \forall e : L \rightarrow [0, 1]_{L} \times [0, 1]^p_{L} (e[\Gamma] \subseteq \{(1, 0)\} \rightarrow e(\alpha) = (1, 0)).$$

Example 2.11 [A bilattice logic of $[0, 1]_{L} \otimes [0, 1]_{L}$] Last, we look at the residuated bilattice $[0, 1]_{L} \otimes [0, 1]_{L} = ([0, 1] \times [0, 1], \wedge, \vee, \sqcap, \sqcup, \neg, (0, 0))$ of Example 2.7. We consider it as a matrix with $F = \{(1, a) \mid a \in [0, 1]\}$ being the designated values. We consider formulas in the language consisting of $\{\wedge, \vee, \sqcap, \sqcup, \neg, 0\}$, and evaluations in $[0, 1]_{L} \otimes [0, 1]_{L}$ that in particular send $0$ to $(0, 0)$, $\top, \bot, \neg, \neg, \oplus, \odot$ are definable (see Example 2.7 and Appendix 6.6).

The consequence relation is defined as $\Gamma \vDash_{L \otimes L} \alpha \iff \forall e[\Gamma] \subseteq F \rightarrow e(\alpha) \in F$.

2.3 Non-standard probabilities

The idea of independence of positive and negative information naturally generalizes to probabilistic extensions of BD logic. We start with the notion of a probabilistic Belnap-Dunn (BD) model, which is a double valuation BD model extended with a classical probability measure on the power set of states $P(W)$ generated by a mass function on the set of states $W$.

Definition 2.12 A probabilistic BD model is a tuple $M = \langle W, v^+, v^-, m \rangle$, such that $\langle W, v^+, v^- \rangle$ is a BD model and $m : W \rightarrow [0, 1], \sum_{s \in W} m(s) = 1$ is a mass function on the set of states.

We define a non-standard (positive) probability of a formula as a (classical) measure of its positive extension: $p^+(\varphi) := \sum_{s \in [\varphi]_m^+} m(s)$. We can define a non-standard negative probability in a similar way. It is shown in [16, Lemma 1], that non-standard probability satisfies the following axioms.

Theorem 2.13 Let $M = \langle W, v^+, v^-, m \rangle$ be a probabilistic BD frame. Then the non-standard probability function $p^+$ induced by $m$ satisfies:

(A1) normalization \quad $0 \leq p^+(\varphi) \leq 1$

(A2) monotonicity \quad if $\varphi \vDash_{BD} \psi$ then $p^+(\varphi) \leq p^+(\psi)$

(A3) import-export \quad $p^+(\varphi \land \psi) + p^+(\varphi \lor \psi) = p^+(\varphi) + p^+(\psi)$.

where $\vDash_{BD}$ is the entailment of BD logic and $\varphi, \psi \in L_{BD}$.

Since positive and negative non-standard probabilities are mutually definable via negation: $p^-(\varphi) = p^+(\neg \varphi)$, it suffices to use just $p^+$. These axioms are weaker than classical Kolmogorovian ones. In particular additivity does not

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7 The probability of a set $X \subseteq W$ is defined as the sum of masses of its elements.
hold and is replaced by A3. As a consequence, $p^+(\neg \varphi) \neq 1 - p^+(\varphi)$ in general, but only $p^+(\varphi \land \neg \varphi) + p^+(\varphi \lor \neg \varphi) = p^+(\varphi) + p^+(\neg \varphi)$ holds. This admits positive non-standard probability of (classical) contradictions ($0 < p^+(\varphi \land \neg \varphi)$) and hence allows for a non-trivial treatment of inconsistent information.

We can diagrammatically represent non-standard probabilities on a continuous extension of Belnap-Dunn square (Figure 1, right), which we can see as a product bilattice $L_{[0,1]} \otimes L_{[0,1]}$ (cf. Example 2.3). In this diagram each point corresponds to a pair $(p^+(\varphi), p^-(\varphi))$ representing positive and negative probabilistic support of a formula $\varphi$. So the point $(0, 0)$ corresponds to the situation when no information concerning $\varphi$ is available, while $(1, 1)$ is the point of maximally conflicting information. The vertical dashed line corresponds to the "classical" case when positive and negative support sum up to 1.

**Example 2.14 [Case 2. The eye-witnesses]** Our investigator interviewed four witnesses. If witness 2 affirms with absolute certainty that she did not see the suspect, then it is making the statement that $p(\varphi) = (0, 1)$. If witness 2 says "I am confident with 80% that I saw this man, but I might be wrong with 20%", then it is making the statement that $p(\varphi) = (0.8, 0.2)$. One can observe that this probability is a point on the vertical line (representing classical probabilities) of the probabilistic extension of the Belnap-Dunn square (Figure 1).

Now, if our investigator interviews a witness to know the character of the suspect (here the husband) and his relationship with the victim. He asks “Did they have a happy marriage?” and the witness answers “I think so, because they really seemed to love each other and their kids, but at the same time he was working long hours and travelling a lot for work and she was feeling lonely”. The investigator may interpret this both as a piece of information supporting the fact the husband did not kill his wife and as a piece of information suggesting he might have a motive: $p(\varphi) = (0.8, 0.4)$.

**Example 2.15 [Case 1. Consulting a panel]** The company assembles a panel to which they ask whether a word describes the car well or not. That is, they ask how much they agree with the statements: “the car has property $\varphi$ (e.g. being a family car)?” and “the car does not have property $\varphi$?” If humans were classical agents, every person would answer with a probability $p$ that belongs to the vertical line of the probabilistic extension of the Belnap-Dunn square. However, experience has shown that often people don’t reason classically [1]. If a person answers $(p^+(\varphi), p^-(\varphi))$ with $p^+(\varphi) + p^-(\varphi) < 1$, then she is not conflicted about whether the property $\varphi$ describes the car, however there might be some uncertainty on how to judge whether the car has property $\varphi$. If a person answers $(p^+(\varphi), p^-(\varphi))$ with $p^+(\varphi) + p^-(\varphi) > 1$, then she is conflicted about whether the property $\varphi$ describes the car.

### 3 Aggregating inconsistent probabilistic information

In this section, we consider an agent that (i) considers a set of topics listed by the atomic variables in $\text{Prop}$, (ii) has access to sources giving information within the framework of non-standard probabilities and (iii) builds beliefs based on
these sources using a so-called aggregation strategy (Definition 3.1). We focus on cases where the agent has no prior beliefs about the topics at stake. For instance, in Case 2, the investigator is called on a new crime scene, collects evidence and, only after he has collected enough evidence, tries to build a theory about what happened.

**Definition 3.1** A source $s$ is a non-standard probability assignment over the set of formulas $s : L_{BD} \rightarrow [0, 1] \times [0, 1]$. An aggregation strategy $\text{Agg}$ is a function that takes in input a set of sources $S = \{s_i\}_{i \in I}$ and returns a map $\text{Agg}_{S} : L_{BD} \rightarrow [0, 1] \times [0, 1]$. For every $\varphi \in L_{BD}$, we denote $\text{Agg}_{S}(\varphi)^{+}$ (resp. $\text{Agg}_{S}(\varphi)^{-}$) the positive (resp. negative) support assigned to $\varphi$.

Depending on the context, the aggregation strategy should satisfy different properties.

**Definition 3.2** Let $\text{Agg}$ be an aggregation strategy and $S$ be a set of sources. $\text{Agg}$ is monotone if $\varphi \vdash \psi$ implies $\text{Agg}_{S}(\varphi) \leq \text{Agg}_{S}(\psi)$ for all $\varphi, \psi \in L_{BD}$ and for every $S$. $\text{Agg}$ is $\neg$-compatible if $\text{Agg}_{S}(\varphi)^{-} = \text{Agg}_{S}(\neg \varphi)^{+}$ for every $\varphi \in L_{BD}$ and for every $S$. $\text{Agg}$ preserves non-standard probabilities if $\text{Agg}_{S}$ is a non-standard probability for every $S$.

Many aggregation strategies have been introduced on classical probabilities. Some strategies such as the (weighted) average can be straightforwardly generalized to non-standard probabilities. On belief functions, Dempster-Shafer combination rule [19] allows to combine possibly conflicting beliefs proceeding from multiple independent sources. It has been generalized from Boolean algebras to distributive lattices [20] and arbitrary finite lattices [12]. Intuitively, the combined belief highlights the converging portions of evidence, and downplays the conflicting ones. However, in situations of high conflicts between the sources, it provides counterintuitive results. In the following, we present aggregation strategies for our two case studies.

**Example 3.3** [Case 1. Weighted average] A company has access to a huge amount of heterogeneous data from various sources and to software to analyse these data. Hence, it makes sense to consider potentially “computationally difficult” aggregation methods that preserve non-standard probabilities. A very natural proposal is to grade every source $s_i$ with respect to its reliability $w_i$ and to take the weighted average of the probabilities. The aggregation is then the map $WA : L_{BD} \rightarrow [0, 1] \times [0, 1]$ such that, for every $\varphi \in L_{BD}$,

$$WA^{+}(\varphi) := \frac{\sum_{1 \leq i \leq n} w_i \cdot p_i^{+}(\varphi)}{\sum_{1 \leq i \leq n} w_i} \quad \text{and} \quad WA^{-}(\varphi) := \frac{\sum_{1 \leq i \leq n} w_i \cdot p_i^{-}(\varphi)}{\sum_{1 \leq i \leq n} w_i}.$$ 

One can easily prove that $WA$ is a non-standard probability. This aggregation strategy is not feasible when modelling human reasoning. In what follows, we propose more realistic aggregations for that situation.

**Example 3.4** [Case 2. A very cautious investigator] The investigator builds her opinion on the suspect based on different sources. We assume all
the sources are equally reliable and the investigator does not want to draw conclusions hastily. Hence, she relies only on statements all her sources agree on. The aggregation is then the map \( \text{Min} : \mathcal{L}_{BD} \rightarrow [0, 1] \times [0, 1] \) such that

\[
\text{Min}(\varphi) := \bigcap_{1 \leq i \leq n} p_i(\varphi) = \left( \min_{1 \leq i \leq n} p_i^+(\varphi), \min_{1 \leq i \leq n} p_i^-(\varphi) \right).
\]

Example 3.5 [Case 2. Reasoning with trusted sources] Here, we assume all the sources are perfectly reliable. Hence, the investigator builds her belief on every statement supported by at least one source. The aggregation is then the map \( \text{Max} : \mathcal{L}_{BD} \rightarrow [0, 1] \times [0, 1] \) such that

\[
\text{Max}(\varphi) := \bigcup_{1 \leq i \leq n} p_i(\varphi) = \left( \max_{1 \leq i \leq n} p_i^+(\varphi), \max_{1 \leq i \leq n} p_i^-(\varphi) \right).
\]

Here, one has high chances of reaching contradiction. In a scientific analysis, if one gets experiments or information that is contradictory, there are two options. Either the information is incorrect or there is a mistake in the interpretation of the data. Here, if the sources are 100% reliable, reaching a contradiction state will simply indicate to our investigator that there is a flaw in her analysis of the problem and she needs to change perspective to resolve the conflict.

4 Two layer logics

To make a clear distinction between the logical level of information on which the agent bases her beliefs, and the level of reasoning about her beliefs, we use a two layer logical framework. The formalism originated with Hájek [14], and was further developed in [8] and [3] into an abstract framework with a general theory of syntax, semantics and completeness.

Syntax \((\mathcal{L}_e, \mathcal{L}_u, M)\) of a two layer logic \(\mathcal{L}\) consists of a lower language \(\mathcal{L}_e\), an upper language \(\mathcal{L}_u\), and a set of modalities \(M\). We concentrate on unary modalities in this paper: we propose logics of belief of a single agent. Non-modal formulas of the lower language are defined over a (fixed countable) set \(\text{Prop}\) of propositional variables using connectives of \(\mathcal{L}_e\) in a standard way. Whenever \(\triangledown \in M\) is a modality and \(\varphi\) is a non-modal formula of \(\mathcal{L}_e\), then \(\triangledown(\varphi)\) is a modal atomic formula of \(\mathcal{L}_u\). Given modal atomic formulas, modal formulas of \(\mathcal{L}_u\) are formed using connectives of \(\mathcal{L}_u\) in a standard way.

Semantics of a two layer logic \(\mathcal{L}\) is based on frames of the form

\[
F = (W, (E_s)_{s \in W}, U, (\mu^{\triangledown})_{\triangledown \in M}), \quad \mu^{\triangledown} : \prod_{s \in W} E_s \rightarrow U,
\]

where \(W\) is a set of states, \(E_s\) are local algebras of evaluation in the states, \(U\) is an upper-level algebra, and for each modality \(\triangledown \in M\), its semantics is given by the map \(\mu^{\triangledown}\). A model \(M\) is given by a frame \(F\) together with a collection of valuations \(\{e_s\}_{s \in W}\) of the non-modal formulas in algebras \(E_s\). Thus, for a non-modal formula, we have \(\varphi^M = \{e_s(\varphi)\}_{s \in W} \in \prod_{s \in W} E_s\). For an atomic modal

\[\text{For this paper, we always consider the lower algebras be all the same. But different algebras can be later used when modelling heterogeneous information. We write algebras, but often we use matrices, i.e. algebras with a set of designated values.}\]
formula, we have \( \Diamond (\varphi)^M = \mu^\Diamond (\varphi^M) \in U \). The upper level is then interpreted in the algebra \( U \) in the usual way: for a connective \( c \in L_u \) and modal formulas \( \alpha_1, c(\alpha_1, \ldots, \alpha_m)^M = e^U(\alpha_1^M, \ldots, \alpha_m^M) \). A non-modal formula is said to be valid in a model if its value in all the states \( s \in W \) is designated in \( E_s \). A modal formula is said to be valid if its value in \( U \) is designated. Validity in a frame is defined in the usual way as validity in all the models over the frame.

The resulting logic as an axiomatic system \( L = (L_e, M, L_u) \) consists of an axiomatics of \( L_e \), modal axioms and rules \( M \), and an axiomatics of \( L_u \). We say that \( E \) is: \( K_e \)-based if algebras \( E_s \) are in the class of algebras \( K_e, K_u \)-measured if \( U \) is in the class \( K_u \), and a frame for \( L \) if it validates each axiom of \( M \) and for each rule from \( M \), we have that the frame validates its conclusion whenever it validates all the assumptions. From [8, Theorems 1 and 2] we can extract the following completeness results:

- If \( L_e \) is (finitely) strongly complete w.r.t. \( K_e \) and \( L_u \) is strongly complete w.r.t. \( K_u \), then \( L \) is (finitely) strongly complete w.r.t. \( K_e \)-based \( K_u \)-measured frames for \( L_e \).
- If \( L_e \) is finitely strongly complete w.r.t. \( K_e \) and locally finite, \( M \) finite, and \( L_u \) is finitely strongly complete w.r.t. \( K_u \), then \( L \) is finitely strongly complete w.r.t. \( K_e \)-based \( K_u \)-measured frames for \( L_u \).

**Example 4.1 [Probabilistic belief]** Consider a two layer logic with \( L_e = \{\&, \lor, \neg\} \) language of \( BD \), \( M = \{B\} \) a belief modality, \( L_u = \{\rightarrow, \sim, \neg\} \) of Example 2.8. The intended frames are \( F = (W, 4, [0, 1]_L \times [0, 1]_L^{op}, \mu^B) \) where \( \mu^B \) is computed as follows. A source is given by a mass function on the states \( m : W \rightarrow [0, 1] \). Given \( e \in \prod_{v \in W} 4, \mu^B \) computes the sums of weights over states with positive/negative support: \( \mu^B(e) = (\sum_{v \in \{t,b\}} m(v), \sum_{v \in \{f,b\}} m(v)) \).

Thus, for a non-modal formula \( \varphi \), we obtain its non-standard positive and negative probabilities:

\[
B(\varphi)^M = (\sum_{v \in \mathcal{V}_+} m(v), \sum_{v \in \mathcal{V}_-} m(v)) = (p^+(\varphi), p^-(\varphi)).
\]

The bottom layer is that of events or facts, represented by \( BD \)-information states, and \( L_e \) is \( BD \). The source gives probabilistic information resulting from a mass function on the states. The modality is that of non-standard probabilistic belief, the top layer \( L_u = L(\sim) \) is to be the logic of thus formed beliefs. The modal part \( M \) consists of axioms and a rule reflecting the axioms of non-standard probabilities:

- \( B(\varphi \lor \psi) \leftrightarrow (B\varphi \circ B(\varphi \land \psi)) \circ B\psi \quad B\neg \varphi \leftrightarrow \neg B\varphi \quad B \vdash_B \psi \quad \Gamma(L(\sim)) \vdash_B \varphi \rightarrow B\psi \)

The resulting logic is \( (BD, M, L(\sim)) \). As \( BD \) is locally finite and strongly complete w.r.t. \( [0, 1]_L \times [0, 1]_L^{op} \) by Lemma 2.10, we conclude that

---

9. In a recent survey paper this is contained in [3, Theorems 3 and 4].
10. We can see the upper layer of the semantics as providing a belief state of a single agent.
Lemma 4.2 \((BD, M, L(\neg))\) is finitely strongly complete w.r.t. \(4\) based, \([0, 1]_L \times [0, 1]_{op}^{\text{op}}\)-measured frames validating \(M\).

In such frames, \(\mu^B\) interprets \(B\) as a non-standard probability (see Appendix 6.6). From [16, Theorem 4], we know that it is the induced non-standard probability function of exactly one mass function on the \(BD\) states, which in fact yields completeness w.r.t. the intended frames described above. Since the (weighted) average aggregation yields a non-standard probability (see Example 3.3), \((BD, M, L(\neg))\) is also adequate to capture frames with multiple sources and such that \(\mu^B : P(\prod_{s \in W} 4) \to [0, 1]_L \times [0, 1]_{op}^{\text{op}}\) computes the (weighted) average of the probabilities given by the individual sources\(^{11}\).

Alternatively, we can take \(L_u = \{\wedge, \vee, \cap, \cup, \cup, \wedge, 0\}\) as the language of the residuated bilattice \([0, 1]_L \circ [0, 1]_L\) of Examples 2.7 and 2.11. The intended frames are \(F = (W, 4, [0, 1]_L \circ [0, 1]_L, \mu^B)\), with \(\mu^B\) defined as above and interpreting belief as a non-standard probability. We can define \(\ominus, \odot\) as described in Example 2.11, and use literally the same modal axioms \(M\) as above.

Example 4.3 [Monotone coherent belief] The intended frames are \(F = (W, 4, L_{[0, 1]} \circ L_{[0, 1]}, \mu^B)\) where \(L_{[0, 1]} \circ L_{[0, 1]}\) is the bilattice of Example 2.3, we have multiple sources and \(\mu^B : P(\prod_{s \in W} 4) \to L_{[0, 1]} \circ L_{[0, 1]}\) computes the min (max) aggregation of the probabilities given by the individual sources, see Examples 3.4 and 3.5. In general this does not yield a non-standard probability, only the interdefinability of positive and negative support via negation is preserved. This motivates considering logic \((BD, M, BD)\), where the modal part \(M\) consists of the axiom and rule\(^{12}\)

\[B \neg \phi \dashv \vdash_{BD_u} \neg B \phi \quad \phi \vdash_{BD_u} \psi / B \phi \vdash_{BD_u} B \psi.\]

As \(BD\) is strongly complete w.r.t. both \(4\) and \(L_{[0, 1]} \circ L_{[0, 1]}\), we obtain that \((BD, M, BD)\) is strongly complete w.r.t. \(4\)-based \(L_{[0, 1]} \circ L_{[0, 1]}\)-measured frames validating \(M\).\(^{13}\)

5 Conclusion and future research

We lay the foundations for a modular framework to model reasoning with inconsistent probabilistic heterogeneous information. We build on the two layer logical framework presented in [3]. The lower language models the sources and the upper language models the beliefs of the agent. The sources’ inconsistent probabilistic information is formalized via non-standard probabilities and the agent belief is interpreted on the product algebra \([0, 1]_L \times [0, 1]_{op}^{\text{op}}\), or a product bilattice \([0, 1] \circ [0, 1]\]. This framework can be used to model case studies both

\(^{11}\)In case we weight the sources in \([0, 1], \mu^B : \prod_{s \in W} 4 \to [0, 1]_L \times [0, 1]_{op}^{\text{op}}\).

\(^{12}\)We wish to stress the distinction between the lower logic \(BD_u\) and the upper logic \(BD_u\) even though they are the same, only operating in different languages.

\(^{13}\)This kind of generic completeness however provides counterexamples with a single 4-valued state (and a single belief state), which are not quite the intended semantics involving multiple states and sources.
in data analyses and in human reasoning. This article is the first step in a long term research project, future research directions include:

**Aggregation strategies.** In Section 3, we present three aggregation strategies: the weighted average WA, a cautious aggregation Min using only the information all sources agree upon, and a trustful aggregation Max using all the available information. While WA relies on the agent having strong computational abilities and evaluating the reliability of the sources, Min and Max rely on the epistemic attitude of the agent. Another natural aggregation strategy would be to adapt Dempster-Shafer (DS) combination rule. We wish to investigate the link between belief functions and non-standard probabilities. In addition, DS combination rule is problematic in cases of high conflict and provides counter intuitive results. Generalizing DS combination rule to BD should provide a mathematical environment where high conflict can be handled in a more intuitive manner.

**Modelling the sources.** Very often, a source does not give an opinion about each formula of the language. Hence, one needs a framework able to account for sources giving partial probability maps. In addition, the sources’ reasoning might be based on different logical principles. One source might reason classically, while another one might use a very weak logic and refuse (or not be able) to give an opinion about complex statements. This raises further questions of which logic(s) one should use to model sources. Another quest is to capture dynamics of belief given by updates on the level of sources (e.g. adding a source).

**Modelling the agent.** As for the sources, one needs to pick the appropriate logic to model the agent. In Case 1, one assumes the agent has strong computational abilities, hence a logic such as relevant logic might be adequate. In Case 2, we might want to use a weaker logic that takes into account the human limitation of the investigator. In addition, the bilattice framework provides us with many options to define alternative notions of conjunction (for instance, the bilattice meet $\sqcap$ for $\leq_k$ or a lifting of the fusion on $[0,1]$ to the bilattice), implication and negation. What would be the significance of these connectives in that framework?

**From single to multiple agents.** Once we understand the single agent case, it is only natural to generalize the framework to the multi agent setting, involving group modalities and dynamics of belief. Specifically, forming group belief, including common and distributed belief, will involve communication and/or sharing and pooling of sources. It might also call for a use of modalities inside the upper logic to account for reflected beliefs, in contrast to the beliefs grounded directly in the sources.

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6 Appendix

6.1 Two frame semantics for BD.
The two kind of frames for BD mentioned in Subsection 2.2 yield equivalent semantics for BD. They are related as follows:

\[ e(\varphi, s) = t \iff s \models^+ \varphi \text{ and } s \not\models^- \varphi, \]
\[ e(\varphi, s) = b \iff s \models^+ \varphi \text{ and } s \models^- \varphi, \]
\[ e(\varphi, s) = n \iff s \not\models^+ \varphi \text{ and } s \not\models^- \varphi, \]
\[ e(\varphi, s) = f \iff s \not\models^+ \varphi \text{ and } s \models^- \varphi. \]

6.2 Negative normal form in L(\neg)
Each formula of the language of L(\neg) is equivalent to a formula in a \neg\neg-normal form, i.e. \neg\neg occurs only (at most once) in front of atoms. It is easy to see, because we have \neg\neg\neg\alpha \leftrightarrow \neg\neg\neg\neg\neg\neg\alpha and \neg\neg(\alpha \rightarrow \beta) \leftrightarrow \neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\nega...
are valid and the rules sound. We only do some cases:

First the ∼¬-rule: assume that \( e \) is given and \( e(\alpha) = (1, 0) \). Then \( e(\sim \alpha) = \sim \neg (1, 0) = \sim (0, 1) = (1, 0) \).

Next, for any \( e \), \( e(\sim (\alpha \rightarrow \beta)) = \sim (e(\alpha) \rightarrow e(\beta)) = \sim (e(\alpha_1 \rightarrow L e(\beta_1)), \sim L (e(\beta_2) \sim L e(\alpha_2)) = ((e(\beta_2) \sim L e(\alpha_2)), \sim L (e(\alpha_1) \rightarrow L e(\beta_1))), \)

and, \( e(\sim \alpha \rightarrow \sim \beta) = \sim e(\alpha) \rightarrow \sim e(\beta) = (\sim L (e(\alpha_2) \sim L e(\alpha_1)) \rightarrow \sim L e(\beta_2), \sim L e(\beta_1)) = (\sim L e(\alpha_2) \rightarrow L ~L e(\beta_1)) = ((e(\beta_2) \rightarrow L \sim L e(\alpha_1), e(\alpha_2)), \sim L (e(\alpha_1) \rightarrow L e(\beta_1))). \)

Last, for any \( e \), \( e(\sim \alpha \sim \alpha) = \sim e(\alpha) = (\sim L e(\alpha_2) \sim L e(\alpha_1)) = \sim e(\alpha) = e(\sim \alpha) \).

For the other direction, let us assume that \( \Gamma \not\models L \alpha \). Then for some finite \( \Delta \) containing instances of \( \sim \)-axioms (in particular those for subformulas of \( \Gamma, \alpha \)), we have \( \not\models \Gamma, \Delta \not\models \alpha \). Because \( L \) is finitely standard complete, there is an evaluation \( e : \Delta \rightarrow [0, 1] \), sending all formulas in \( \Gamma, \Delta \) to 1, while \( e(\alpha) < 1 \). Here, \( \Delta \) contains literals from \( \Gamma, \alpha \) of the form \( p, \neg p \), and atoms and formulas of the form \( \sim \neg \) from \( \Gamma, \Delta \). We define \( e' : \text{Prop} \rightarrow [0, 1] \times [0, 1] \) by \( e'(p) = (e(p), e(\neg p)) \). We can then prove, by routine induction, that for each formula \( \sim e'(\beta) = (e(\beta), e(\beta')) \). We use the fact that \( e(\Delta) \subseteq \{1\} \), and \( \Delta \) contains all instances of \( \sim \)-axioms for all subformulas of \( \Gamma, \alpha \).

We now immediately see that \( e'(\alpha) < (1, 0) \), because \( e(\alpha) < 1 \). To prove that indeed \( e'[\Gamma] \subseteq \{1, 0\} \), we use the fact that \( e[\Gamma] \subseteq \{1\} \): as for all \( \gamma \in \Gamma \), \( e(\sim \neg \gamma) = 1, e(\neg \gamma) = e(0) = 0 \). For the latter, we again need to use the fact that \( e(\Delta) \subseteq \{1\} \), and \( \Delta \) contains all instances of \( \sim \)-axioms for all subformulas of \( \Gamma, \alpha \). As they prove, by means of \( L \), that \( \sim \neg \gamma \equiv \neg \gamma \), and \( e \) has to respect that. Now we conclude, that for all \( \gamma \in \Gamma \), \( e'(\gamma) = (e(\gamma), e(\neg \gamma)) = (1, 0) \).

6.5 Logic of Example 2.11

We consider formulas in the language consisting of \( \{\land, \lor, \land, \lor, \sim, 0\} \), and evaluations in the matrix \( [0, 1]_L \times [0, 1]_L \) with \( F = \{1, a \} \mid a \in [0, 1] \} \), that in particular send 0 to (0, 0). The constants and connectives \( \top, \bot, 1, *, \rightarrow \), \( \sim, \land, \lor \) are definable as follows:

\[
\sim \alpha := (\alpha \lor 0) \land (\neg \neg \sim \alpha \lor 0) \quad \top := 0 \lor 0 \quad \bot := \neg \top \quad 1 := \sim 0 \\
\alpha \rightarrow \beta := (\alpha \land \beta) \lor (\land \sim \beta \lor \alpha) \quad \alpha \lor \beta := (\neg \neg \sim \alpha \lor \beta) \lor (\sim \neg \alpha \lor \beta) \\
\alpha \land \beta := (\sim (\sim \beta \rightarrow \sim \alpha)) \quad \alpha \land \beta := (\sim (\sim \alpha \lor \beta) \lor (\neg \sim \neg \neg \alpha \lor \beta))
\]

For an evaluation \( e \), it holds that \( e(\alpha \rightarrow \beta) \in F \) if \( e(\alpha \rightarrow \beta) \geq_\epsilon (1, 1) \) if \( e(\alpha) \leq_\epsilon e(\beta) \).

6.6 Example 4.1

We have proved that \( (BD, M, L(\neg)) \) is finitely strongly complete w.r.t. 4 based, \([0, 1]_L \times [0, 1]_L \text{op}-measured frames validating \( M \). In such frames, \( \mu^B \) interprets \( B \) as a non-standard probability: recall the axioms and rule in \( M \) are

\[
B(\varphi \lor \psi) \leftrightarrow (B\varphi \lor B(\varphi \land \psi)) \lor B\psi \quad B\neg \varphi \leftrightarrow \neg B\varphi \\
\varphi, \Gamma \models \neg \psi \theta \models \varphi \rightarrow B\psi
\]
For a frame to validate the axioms means they are sent to \((1,0)\), by an evaluation in \([0,1]_L \times [0,1]_{L^op}\) induced by \(\mu^B\) over the lower state valuations (which determines values of modal atomic formulas). An equivalence \(\alpha \leftrightarrow \beta\) is evaluated at \((1,0)\) iff the values of \(\alpha\) and \(\beta\) are equal. \(B(\varphi)^M = \mu^B(\varphi^M) = (p^+(\varphi), p^-(\varphi))\). Therefore, the first two axioms say that

\[
\begin{align*}
p^+(\varphi \lor \psi) &= (p^+(\varphi) - p^+(\varphi \land \psi)) + p^+(\psi) \quad \text{and} \quad p^+(\neg \varphi) = -p^-(\varphi) \\
p^-(\varphi \lor \psi) &= (p^-(\varphi) - p^-(\varphi \land \psi)) + p^-(\psi) \quad \text{and} \quad p^- (\neg \varphi) = -p^+(\varphi).
\end{align*}
\]

Similarly, the fact that the frame validates the rule say that \(p^+(p^-)\) are monotone (antitone) w.r.t. \(\varphi \vdash_{BD} \psi\). Analogous observation holds for the case the upper logic is the bilattice one of Example 2.11.