Fuchsian DPW potentials for Lawson surfaces

Lynn Heller · Sebastian Heller

Received: 28 June 2022 / Accepted: 2 September 2023 / Published online: 14 September 2023
© The Author(s), under exclusive licence to Springer Nature B.V. 2023

Abstract

The Lawson surface $\xi_{1,g}$ of genus $g$ is constructed by rotating and reflecting the Plateau solution $f_t$ with respect to a particular geodesic 4-gon $\Gamma_t$ across its boundary, where $(t = \frac{1}{2g+2}, \frac{\pi}{2}, t, \frac{\pi}{2})$ are the angles of $\Gamma_t$. Recent progress in integrable surface theory allows for a more explicit construction of these surfaces and for better understanding of their geometric properties using so-called Fuchsian DPW potentials for $t \sim 0$. In this paper we combine the existence and regularity of the Plateau solution $f_t$ in $t \in (0, \frac{1}{4})$ with a detailed investigation of the moduli space of Fuchsian systems on the 4-punctured sphere to obtain the existence of a Fuchsian DPW potential $\eta_t$ for every $f_t$ with $t \in (0, \frac{1}{4}]$. Moreover, the coefficients of $\eta_t$ are shown to depend real analytically on $t$. This implies that the Taylor expansions of the DPW potential $\eta_t$ and of the area in Heller et al. (Complete families of embedded high genus CMC surfaces in the 3-sphere. arXiv:2108.10214) computed at $t = 0$ already determine these quantities for all $\xi_{1,g}$. In particular, this leads to an algorithm to conformally parametrize all Lawson surfaces $\xi_{1,g}$.

Keywords Minimal surfaces · DPW method · Parabolic structures · Fuchsian systems

Mathematics Subject Classification 53A10 · 53C42 · 53C43 · 14H60 · 14H70

Introduction

In 1970 Lawson [26] constructed embedded minimal surfaces of every genus in the round 3-sphere. A Lawson surface is obtained from the Plateau solution of a geodesic polygon $\Gamma$ which is then reflected and rotated across its boundary. Using a similar philosophy, Karcher–Pinkall–Sterling [22] found minimal surfaces in $S^3$ with platonic symmetries. Later Kapouleas and Yang [19, 20] constructed high genus minimal surfaces by doubling a totally geodesic 2-sphere.

Kusner [24] conjectured that the simplest Lawson surfaces $\xi_{1,g}$ minimize the Willmore energy among immersions from a genus $g$ surface in generalization of the famous Willmore
conjecture solved by Marques and Neves [28]. Though few examples of compact minimal or Willmore surfaces of genus $g \geq 2$ are known, this conjecture is supported by computer experiments [24] where an arbitrary compact surface in the 3-sphere is deformed via an energy decreasing flow and converges to a shape resembling a Lawson surface. The Willmore energy of the Lawson surface $\xi_{1,g}$ is strictly below $8\pi$ but converges to $8\pi$ in the large genus limit (see [14, 24]) which coincides with the energy limit of abstract genus $g$ Willmore minimizers [25] for $g \to \infty$. Moreover, the stability properties of these candidates, viewed as minimal surfaces rather than as Willmore surfaces, were studied in [21]. Due to the implicit way the Lawson surfaces are constructed, determining further geometric properties, such as computing their area (or Willmore energy), is very difficult though in view of the Kusner conjecture very desirable.

In an effort to obtain more explicitness, a completely different approach to constructing minimal and constant mean curvature (CMC) surfaces in the 3-sphere is taken in [13–15]. In particular, an alternative existence proof of the Lawson surfaces with large genus is obtained via an implicit function theorem argument using methods from integrable systems. The method itself can be interpreted as a global version of the generalized Weierstraß representation [11]. For tori the approach was pioneered by Hitchin [18] and Pinkall-Sterling [31] around 1990, and Bobenko [6] gave an explicit parametrization of all CMC tori in 3-dimensional space forms.

Consider a conformally parametrized minimal immersion $f$ from a compact genus $g$ Riemann surface $M_g$ into the round 3-sphere. Then, $f$ being harmonic gives rise to a symmetry of the Gauss-Codazzi equations inducing an associated $S^1$-family of (isometric) minimal surfaces on the universal covering of $M_g$ with rotated Hopf differential. The gauge theoretic counterpart of this symmetry is manifested in an associated $\mathbb{C}^*$-family of flat $\text{SL}(2, \mathbb{C})$-connections $\nabla^\lambda$ [18] on the trivial $\mathbb{C}^2$-bundle over $M_g$ solving the following Monodromy Problem

(i) conformality: $\nabla^\lambda = \lambda^{-1} \Phi + \nabla - \lambda \Psi$ for a nilpotent $\Phi \in \Omega^{1,0}(M_g, \mathfrak{sl}(2, \mathbb{C}));$
(ii) intrinsic closing: $\nabla^\lambda$ is unitary for all $\lambda \in S^1$, i.e., $\nabla$ is unitary and $\Psi = \Phi^* \in \Omega^{0,1}(M_g, \mathfrak{sl}(2, \mathbb{C}))$ with respect to the standard hermitian metric on $\mathbb{C}^2$;
(iii) extrinsic closing (also known as Sym-point condition): $\nabla^\lambda$ is trivial for $\lambda = \pm 1$.

The minimal surface can be reconstructed from the associated family of flat connections as the gauge between $\nabla^{-1}$ and $\nabla^1$. Constructing minimal surfaces is thus equivalent to writing down appropriate families of flat connections. The DPW method [11] is a way to generate such families of flat connections on a Riemann surface from so-called DPW potentials, denoted by $\eta = \eta^\lambda$, $\lambda \in \mathbb{C}^*$, using loop group factorization. In fact, $\eta^\lambda$ fixes the gauge class of the connections $\nabla^\lambda$ as

$$d + \eta^\lambda \in [\nabla^\lambda].$$

On simply connected domains $\mathbb{U}$, all DPW potentials give rise to minimal surfaces from $\mathbb{U}$. Whenever the domain has non-trivial topology, finding DPW potentials satisfying conditions equivalent to (i)-(iii) becomes difficult.

Though successful in the case of tori, the first embedded and closed minimal surfaces of genus $g > 1$ using integrable system methods were only recently constructed in [14]. This is due to the fact that, in contrast to tori, the fundamental group of a higher genus surface is non-abelian. A global version of DPW has been developed in [16, 17] under certain symmetry assumptions. The main challenge to actually obtain higher genus minimal and CMC surfaces is to determine the infinitely many parameters in the holomorphic “Weierstraß-data”.

Springer
For large genus, these parameters are determined via the implicit function theorem in [14, 15]. More explicitly, families of minimal and CMC surfaces $f_{\varphi}^t$ for $t \sim 0$ and $\varphi \in (0, \frac{\pi}{2})$ are constructed by starting at two geodesic spheres intersecting at angle $2\varphi$ and by deforming the corresponding DPW potential in the direction of a Scherk surface in $\text{su}(2) \cong \mathbb{R}^3$ with wing angle $2\varphi$. For $\varphi = \frac{\pi}{4}$ we have that

$$f_{\frac{\pi}{4}}^t = \xi_{1,g}$$

at $t = \frac{1}{2(g+1)}$ is the Lawson surface. Consequently, an iterative algorithm to compute the Taylor expansion of the DPW-potentials $\eta_{\varphi}^t$ and the area $\text{Area}_{\varphi}(t)$ of $f_{\varphi}^t$ at $t = 0$ was given in [15]. In particular, we obtain for $\varphi = \frac{\pi}{4}$

$$\text{Area}(f_{\frac{\pi}{4}}^t) \sim 8\pi \left(1 - \log(2)t - \frac{9}{4}\zeta(3)t^3 + O(t^5)\right),$$

(1)

where $\zeta$ is the Riemann $\zeta$-function. By the regularity statement of the implicit function theorem, the family of DPW-potentials $\eta_{\varphi}^t$ as well as the area $\text{Area}_{\varphi}(t)$ depends real analytically on the parameter $t \sim 0$.

This paper is about quantitative results concerning the existence interval of the solutions $\eta_t := \eta_{\frac{\pi}{4}}^t$. The idea is to use the properties of the Plateau solutions for all $t \in (0, \frac{1}{4}]$ to prove existence of the $t$-family of DPW-potentials $\eta_t$ found in [15] on the same time interval. This covers all Lawson surfaces $\xi_{1,g}, g \in \mathbb{N}^>0$. As a byproduct we obtain the real analyticity of $\eta_t$ and $\text{Area}_{\frac{\pi}{4}}(t)$ for all $t \in (0, \frac{1}{4}]$. Together with [15] this leads to an algorithm to computing the area and an explicit conformal parameterization of $\xi_{1,g}$ for every genus by computing their Taylor expansions at $t = 0$. For every coefficient this involves solving a finite dimensional linear system with coefficients given in terms of multi-polylogarithms. In contrast to [13–15] the existence result for the DPW potential in this paper relies on the existence and regularity of the Plateau solutions.

1 Lawson surfaces revisited

Consider $S^3 \subset \mathbb{C}^2$ and the four points in $S^3$ given by

$$P_1 = (1, 0), \quad Q_1 = (0, 1), \quad P_2 = (i, 0), \quad Q_2 = (0, e^{2\pi i t}).$$

(2)

Let $f = f_t: \mathbb{D} \to S^3$ be the Plateau solution with respect to the closed geodesic 4-gon

$$\Gamma = \Gamma_t = P_1 Q_1 P_2 Q_2,$$

where $\mathbb{D}$ is the closed disc of radius 1 in $\mathbb{C}$ and $f(\partial \mathbb{D}) = \Gamma$. As $f$ is immersed except at the points $P_1, P_2$ and $Q_1, Q_2$, we can consider the induced Riemann surface structure on

$$\hat{\mathbb{D}} := \mathbb{D} \setminus f^{-1}\{P_1, Q_1, P_2, Q_2\}.$$

Note that the boundary of $\hat{\mathbb{D}}$ consists of four connected components. Moreover, let $G$ denote the group of automorphisms generated by rotations by $e^{2\pi i t}$ in the $0 \oplus \mathbb{C}$-plane. The Plateau solution extends by reflections across the boundaries to a minimal surface

$$f^t: M_t \to S^3$$

without boundaries. The group $G$ can be considered as symmetries of the minimal surface by uniqueness of the Plateau solutions. The constructed minimal surface $f^t$ is compact (after
adding $P_1, Q_1, P_2, Q_2$ again) if and only if $t$ is rational. The surface $f_t$ is immersed (and embedded of genus $g$) if and only if $t = \frac{1}{2g+2}$ for some $g \in \mathbb{N}$. Many geometric properties, e.g., the structure of singularities, can be derived for rational $t$ from the compactness of $M_t$ using methods of [26]. The following theorem allows us to carry these properties over to all $t \in (0, \frac{1}{4}]$ using continuity arguments.

**Theorem 1** The Plateau solution $f_t$ depends (up to reparametrization) real analytically on the angle $t$ for $t \in (0, \frac{1}{4}]$.

**Proof** For $t \sim 0$ and $t \sim \frac{1}{4}$ real analyticity of the Plateau solution in $t$ follows from the implicit function theorem and the real analyticity of the monodromy problem solved in [13, 14], respectively.

For $t \in (0, \frac{1}{4})$ consider $W^{2,2}_t(\mathbb{D}, S^3) \subset W^{2,2}(\mathbb{D}, \mathbb{R}^4)$ (the completion of) the space of maps from the closed unit disc $\mathbb{D}$ to $S^3$ such that the boundary $\partial \mathbb{D}$ is mapped to $\Gamma_t$. We want to show that the family of Plateau solutions $f_t$ is real analytic in $t$ at every $t_0 \in (0, \frac{1}{4})$.

Therefore, we fix a $t_0 \in (0, \frac{1}{4})$ in the following and consider the linear map

$$\Psi_t: S^3 \subset \mathbb{R}^4 \rightarrow S^3 \subset \mathbb{R}^4; \quad \Psi_t = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -\frac{1}{\tan(2\pi t_0)} + \frac{\cos(2\pi t)}{\sin(2\pi t_0)} \\
0 & 0 & 0 & \frac{\sin(2\pi t_0)}{\sin(2\pi t)}
\end{pmatrix},$$

which is real analytic in $t$, but not isometric unless $t = t_0$. By construction, $\Psi_t(\Gamma_{t_0}) = \Gamma_t$ and this induces a bijection between $W^{2,2}_t(\mathbb{D}, S^3)$ and $W^{2,2}(\mathbb{D}) := W^{2,2}_0(\mathbb{D}, S^3)$ depending real analytically on $t$.

On $W^{2,2}_t(\mathbb{D}, S^3)$ consider the usual area functional. This gives a family of functionals on $W^{2,2}_t(\mathbb{D}, S^3)$ given by

$$A: (-\epsilon + t_0, t_0 + \epsilon) \times W^{2,2}_{t_0}(\mathbb{D}, S^3) \rightarrow \mathbb{R}_+, \quad (t, f) \mapsto A_t(f) = Area(\Psi_t(f)).$$

For $t = t_0$, this is the ordinary area functional on $W^{2,2}_{t_0}(\mathbb{D}, S^3)$. For $t \sim t_0$ consider the family of Euler-Lagrange equations $\delta A_t = 0$. The solutions give rise to the critical points of the area functional $Area$ with boundary $\Gamma_t$. We want to classify all solutions of $\delta A_t = 0$ for $t \sim t_0$ which are $W^{2,2}(\mathbb{D}, S^3)$-close to $f_{t_0}$ using the implicit function theorem. To do so, we restrict to the space of normal variations of $f_{t_0}$. We use the normal $N$ of the minimal surface $f_{t_0}$ to identify normal variations and functions via

$$V = \nu N.$$

Then, (the completion of) the space of normal variations is identified

$$W^{2,2,\perp}(\mathbb{D}) \cong W^{2,2}_{0}(\mathbb{D}, \mathbb{R})$$

with the completion with respect to the $W^{2,2}$-norm of the space of smooth functions $\nu: \mathbb{D} \rightarrow \mathbb{R}$ satisfying

$$\text{supp}(\nu) \subset \mathbb{D}$$

is compact.

For details about the involved functional spaces see for example [8, Chapter 9]. Then we want to solve the equation

$$\delta A_t(\exp_{f_{t_0}}(V)) = 0$$

for details about the involved functional spaces see for example [8, Chapter 9]. Then we want to solve the equation
for \( t \sim t_0 \) and \( V \sim V_0 \in W^{2,2,1}(\mathbb{D}) \). Differentiating with respect to \( V \) we obtain
\[
\partial_V \delta A_{t_0}(f_{t_0}) \cdot Z = \delta^2 Area(f_{t_0})(Z, \cdot).
\]

Due to the strict stability of Plateau solutions of \( \Gamma_{t_0} \), see for example [21, p. 20, proof Lemma 6.7 (ii)] we have
\[
\delta^2 Area(f_{t_0})(Z, Z) > 0
\]
for all normal variations \( Z \in W^{2,2,1}(\mathbb{D}) \). Therefore, \( \partial_V \delta A_{t_0}(f_{t_0}) \) is invertible and there exist by implicit function theorem a unique real analytic map \( t \mapsto V(t) \in W^{2,2,1}(\mathbb{D}) \) for \( t \sim t_0 \) solving \( \delta A_t(\text{Exp}_{f_{t_0}}(V(t))) = 0 \). This gives rise to a unique real analytic family
\[
(f_t)_{t \sim t_0} = \text{Exp}_{f_{t_0}}(V(t))
\]
of minimal surfaces of disc type with boundary \( \Gamma_t \) and \( f_{t_0} = f_0 \) which are \( W^{2,2}(\mathbb{D}) \)-close to \( f_0 \). These solutions \( f_t \) must then coincide (up to reparametrization) with the Plateau solutions \( f_t \) for all \( t \sim t_0 \) due to the uniqueness of minimal discs with boundary \( \Gamma_t \), see [21, Theorem 4.1] or [26].

\[\square\]

1.1 Riemann surface structures

Let \( \overline{\mathbb{D}} \) denote the complex conjugate Riemann surface of \( \mathbb{D} = \mathbb{D}\setminus \{P_1, Q_1, P_2, Q_2\} \). Define the Riemann surface
\[
\Sigma = \overline{\mathbb{D}} \cup \overline{\mathbb{D}}
\]
glued together along the four boundary components via reflections across the four open (geodesic) edges
\[
P_1Q_1, \ Q_1P_2, \ P_2Q_2, \ Q_2P_1.
\]
Note that the minimal surface into \( S^3 \) is not well-defined as a map from \( \Sigma \). On the other hand, the first and second fundamental forms and the associated family of flat connections are well-defined on \( \Sigma \), as no extrinsic closing conditions is involved to define it.

**Proposition 1** The Riemann surface \( \Sigma \) is (biholomorphic to)
\[
\mathbb{C}P^1 \setminus \{-1, 0, 1, \infty\}.
\]

**Proof** By construction \( \Sigma \) has the topology of the sphere with 4 points removed
\[
\Sigma = S^2 \setminus \{Q_1, Q_2, P_1, P_2\}.
\]
The aim is to show that the Riemann surface structure induced by the minimal immersion \( f_t \) extends through the punctures for all \( t \in (0, \frac{1}{4}] \). Since the angle at the vertices \( Q_1 \) and \( Q_2 \) is \( \frac{\pi}{2} \), the Riemann surface structure around \( Q_1 \) and \( Q_2 \) is by [26, Theorem 1] the Riemann surface structure obtained as the quotient of a non-trivial holomorphic \( \mathbb{Z}_2 \)-action. In particular, the Riemann surface structure of the quotient \( \Sigma \) extends though the points \( Q_1 \) and \( Q_2 \). The same conclusion holds for \( P_1 \) and \( P_2 \) if \( t \) is rational.

In fact, the argument of [26, Theorem 1] works analogously for all rational \( t \) and we obtain a well-defined compact, in general branched, minimal surface \( f_t : M_t \to S^3 \), which induces a Riemann surface structure on the compact surface \( M_t \) for rational \( t \). The Riemann surface \( \Sigma \) is then realized as the 4-punctured sphere obtained by taking a finite quotient of \( M_t \) with
the branch points removed. This shows that the Riemann surface structure also extends into $P_1$ and $P_2$.

Next we determine (for rational $t$) the conformal type of $\Sigma$ by using the additional symmetries of the Plateau solution inherited from the symmetries of its geodesic boundary $\Gamma_t$. The arguments are similar to those in [16, Section 3.3] for the Lawson surface of genus 2.

We have shown that $\Sigma$ is $\mathbb{CP}^1 \setminus \{P_1, P_2, Q_1, Q_2\}$. Note that $\Sigma$ has a real involution $i_S$ which extends to $\mathbb{CP}^1$ with $P_1, P_2, Q_1, Q_2$ contained in its fix point set. Therefore we can assume without loss of generality that the points $P_1, P_2, Q_1, Q_2 \in \mathbb{CP}^1$ lie on the unit circle $S$ of $\mathbb{CP}^1$. Consider the real 3-dimensional hyperplanes $A$ and $B$ in $\mathbb{C}^2$ determined by their normals

$$(0, ie^{\pi i}) \quad \text{and} \quad (e^{\frac{i\pi}{4}}, 0),$$

respectively. By construction and (2), the plane $A$ contains $P_1$ and $P_2$ while the plane $B$ contains $Q_1$ and $Q_2$. Moreover, the reflections across $A$ and $B$ map $\Gamma_t$ to itself, and interchanges the points $Q_1, Q_2$ and $P_1, P_2$, respectively. The uniqueness of the Plateau solution then gives that these reflections also map the Plateau solution to itself. Therefore these reflections induce anti-holomorphic involutions of $\Sigma$ and $\mathbb{CP}^1$, denoted by $i_A$ and $i_B$.

The fix-point sets of $i_A$ and $i_B$ in $\Sigma$ are circles intersecting each other at exactly 2 points, one in each connected component of $\mathbb{CP}^1 \setminus S$. The fix-point sets of $i_A$ and $i_B$ also intersect the unit circle $S$ perpendicularly as $i_S$ commutes with $i_A$ and with $i_B$ by uniqueness of the Plateau solution. Therefore, after a suitable Möbius transformation fixing $S$, these two intersection points can be chosen to be 0 and $\infty$. Then the fix-point sets of $i_A$ and $i_B$ become perpendicularly intersecting lines. Note that $P_1$ and $P_2$ are the two intersection points of the unit circle $S$ with the fix-point line of $i_A$, and $Q_1$ and $Q_2$ are the two intersection points of intersection of $S$ with the fix point line of $i_B$. Therefore, possibly after a further rotation, the 4 singular points must be $P_1=1$, $Q_1=i$, $P_2=-1$, and $Q_2=-i$. Applying another Möbius transformation mapping $Q_1$ to 0 and fix $P_1=1$ and $P_2=-1$, we obtain that $Q_2$ is mapped to $\infty$ as claimed.

Using the real analytic dependency of the Plateau solution $f_t$ on the angle $t$, we show next that the topological 4-punctured sphere $\Sigma$ is of the same conformal type for all $t$. In particular, we have to show that the conformal structure around $P_1$ and $P_2$ is that of a punctured disc. In order to do so, we first claim that for all rational $t \in (0, \frac{1}{4}) \cap \mathbb{Q}$, the (induced) Hopf differential $Q$ is a non-zero real multiple of $\frac{dz^2}{(z^{-1})(z+1)}$. This is due to the fact that the Hopf differential has at most first order poles as the pull-back to $M_t$ is holomorphic. Moreover, $Q$ is not identically zero, as the surface is not totally umbilic for $t \neq 0$. By Riemann-Roch this gives that

$$Q = c \frac{(dz)^2}{z(z-1)(z+1)}$$

for some $c \in \mathbb{C}^\ast$. The segments between the marked points $\{-1, 0, 1, \infty\}$ on the real line $z = \tilde{z}$ on $\mathbb{CP}^1$ are geodesics of the minimal surface, i.e., asymptotic lines. Therefore, the Hopf differential must be purely imaginary along the real line, and thus $c \in i\mathbb{R}$ as claimed.

As a consequence we obtain that

$$p \mapsto \left( \int_{P_k}^p \sqrt{-iQ} \right)^2$$

is a well-defined local holomorphic coordinate $w = w_t$ on $\mathbb{CP}^1$ centered at $P_k$ for rational $t$.  

\[ Springer \]
For irrational $t_0 \in (0, \frac{1}{4})$, consider a sequence of rational times $(t_l)_{l \in \mathbb{N}}$ with $\lim_{l \to \infty} t_l = t_0$, and the corresponding family of holomorphic coordinates $w_{t_l}$. Since the Plateau solutions are never totally umbilic for $t \in (0, \frac{1}{4})$ and the geometric data depend real analytically on $t$, the limit Hopf differential $Q_{t_0}$ is non-vanishing, and cannot have accumulation of zeros at $P_k$. Therefore, the holomorphic coordinates $w_{t_l}$ have a limit $w_0$. By construction it extends continuously to $P_k$. Moreover, $w_0$ is a holomorphic coordinate around $P_k$ and we can identify $\Sigma \cup \{P_1, P_2, Q_1, Q_2\}$ biholomorphically with $\mathbb{C}P^1$. The conformal type of $\Sigma$ is then given by $\mathbb{C}P^1 \setminus \{-1, 0, 1, \infty\}$ by continuity in $t$. \hfill $\Box$

Instead of $\Sigma$ we will be working on a $2$-fold covering

$$\pi : \Sigma \to \Sigma$$

branched over $0, \infty$ which removes the conical singularities of the induced metric at $Q_1$ and $Q_2$. The surface $\Sigma = \Sigma$ is a Riemann sphere with $4$ marked/singular points $p_1, p_3$ and $p_2, p_4$ given by the preimages of $P_1$ and $P_2$ under $\pi$, respectively. Applying a Moebius transformation we can assume without loss of generality that

$$p_1 = -1, \ p_2 = 0, \ p_3 = 1, \ p_4 = \infty.$$  

For $t = \frac{1}{2(\log + 1)}$, the surface is well-defined away from two branch cuts between two pairs of marked points, $p_1$ and $p_2$, and $p_3$ and $p_4$, and gives one handle of the whole genus $g$ minimal surface $\xi_{1,g}$.

Though both Riemann surfaces $\Sigma$ and $\Sigma$ are identified with $\mathbb{C}P^1$ with the same four marked points $-1, 0, 1, \infty$, their singular points have different behavior and geometric meaning, see Fig. 1. Consequently, the (well-defined) associated families of flat connections on $\Sigma$ and on $\Sigma$ are different (but of course related by the pull-back via $\pi$). As the associated family is more symmetric on $\Sigma$, we will work on the Riemann surface $\Sigma$ in the following.

2 Fuchsian systems and parabolic structures

In this section we study the moduli space of logarithmic connections on a $4$-punctured sphere with prescribed conjugacy classes of the local monodromies. Apart from giving the relevant definitions we show that
• there are only two types of logarithmic connections, namely, Fuchsian systems defined on the trivial holomorphic bundle, and exceptional logarithmic connections on $O(-1) \oplus O(1)$, see Lemma 17;
• the space of Fuchsian systems can be parametrized by a parameter $u \in \mathbb{CP}^1$ and a parameter $s \in \mathbb{C}$ via (11) in Sect. 2.4;
• the moduli space of logarithmic connections on $O(-1) \oplus O(1)$ is a complex line lying over $u = -1$. Moreover, the moduli space of all logarithmic connections around the exceptional line can be parametrized using coordinates $(E, c_0)$, see (15), such that the exceptional line is given by $E = 0$, see Sect. 2.5;
• the two coordinates systems $(u, s)$ and $(E, c_0)$ around the exceptional logarithmic connections are related by (16);
• unitarizable logarithmic connections $\nabla$ must be Fuchsian systems by Lemma 21;
• a Fuchsian system which is compatible with the involutions $\sigma$ and $\tau$ can be conjugated into a symmetric normal form by Lemma 16.

In other words, the moduli space of logarithmic connections consists of semi-stable connections, namely Fuchsian systems, which is a complex 2-dimensional space parametrized by $(u, s)$, and unstable connections, which is a complex line that is attached to Fuchsian systems at $u = -1$ via (16). The main aim of this section is to provide tools to track the exceptional connections along deformations. Moreover, we want to provide a normal form into which we can gauge symmetric families of connections, see Lemma 16.

2.1 Logarithmic connections on Riemann surfaces

A holomorphic structure $\bar{\partial}$ on a complex vector bundle $V$ over a Riemann surface $M$ is a linear first order differential operator

$$\bar{\partial} : \Gamma(M, V) \to \Gamma(M, K_M \otimes V)$$

satisfying the $\bar{\partial}$-Leibniz rule

$$\bar{\partial}(f \cdot s) = (df)^{(0,1)} \otimes s + f \bar{\partial}s$$

for all smooth functions $f : M \to \mathbb{C}$ and all smooth sections $s \in \Gamma(M, V)$. One important example of a holomorphic structure is the $(0,1)$-part

$$\bar{\partial} \nabla := \frac{1}{2}(\nabla + i \ast \nabla)$$

of a connection $\nabla$. Another example is the induced holomorphic structure from a holomorphic vector bundle $V$ over $M$, i.e., a complex vector bundle with holomorphic transition functions. The holomorphic structure $\bar{\partial}_V$ is uniquely determined using the Leibniz rule by demanding locally holomorphic sections constituting the kernel of $\bar{\partial}_V$, i.e.,

$$H^0(V|_U) = \{s \in \Gamma(U, V) \mid \bar{\partial}_Vs = 0\}.$$ 

In fact, in the case of complex vector bundles over Riemann surfaces, all holomorphic structures $\bar{\partial}$ are induced from holomorphic vector bundles (see for example [2, Section 5]). Therefore, we will use holomorphic structures and holomorphic vector bundles as synonyms in this paper.

Let $D = p_1 + \ldots + p_n$ be a divisor with pairwise distinct points $p_k \in M$. We call the set $\text{supp}(D) := \{p_1, \ldots, p_n\}$ the support of $D$. The points $p_k$ are called the punctures or singular points of $M$. 

\(\copyright\) Springer
Definition 2 A logarithmic connection $\nabla$ on a holomorphic vector bundle $V$ with singular part contained in the divisor $D$ is a connection on $V$ over $M \setminus \text{supp}(D)$ such that the connection 1-form $\omega$ with respect to any local holomorphic frame of $V$ over an open subset $U \subset M$ is meromorphic with first order poles at $U \cap \text{supp}(D)$ only.

Note that the connection $\nabla$ is automatically flat over $M \setminus \text{supp}(D)$ and the $(0, 1)$-part of the connection satisfies

$$i\partial = i\partial V,$$

and therefore naturally extends to the singular points $p_k \in \text{supp}(D)$.

In the following, we assume $V$ to be an $\text{SL}(2, \mathbb{C})$ vector bundle, i.e., $V$ is of rank 2, has holomorphically trivial determinant line bundle $\Lambda^2 V$, and the holomorphic trivialization of $\Lambda^2 V$ is fixed. A logarithmic connection $\nabla$ on a holomorphic $\text{SL}(2, \mathbb{C})$ vector bundle is called a logarithmic $\text{SL}(2, \mathbb{C})$–connection if the induced connection on the trivial bundle $\Lambda^2 V$ is the trivial connection $d$.

Let $p_k \in \text{supp}(D)$. Consider a local holomorphic frame $\Phi$ of $V$ on an open neighbourhood of $p_k$, and write $\nabla = d + \omega$ with respect to this holomorphic frame. Changing the holomorphic frame $\Phi$ to another $\tilde{\Phi} = \Phi g$ via a (locally) holomorphic gauge $g$ conjugates the residue $\text{Res}_{p_k}(\omega)$ at a singular point $p_k \in D$ by $g$, as $g$ extends holomorphically to the puncture $p_k$. Thus

$$\text{Res}_{p_k}(\nabla) \in \text{End}(V_{p_k})$$

is independent of the choice of the local frame $\Phi$, i.e., it is a well-defined endomorphism of the fiber $V_{p_k}$. Since $\nabla$ induces the trivial connection on $\Lambda^2 V$, all residues $\text{Res}_{p_k}(\nabla)$ must be trace free. Let $\pm t_k$ denote the eigenvalues of $\text{Res}_{p_k}(\nabla)$ which are also called the weights of $\nabla$. Then a logarithmic $\text{SL}(2, \mathbb{C})$–connection is called non-resonant, if $2 t_k \notin \mathbb{Z}$ for all $k = 1, \ldots, n$. In this case the local monodromy of $\nabla$ along a simple closed curve around $p_k$ is conjugate to the diagonal matrix with entries $\exp(\pm 2\pi i t_k)$, see for example [9]. The following useful lemma is well-known. We include its short proof for completeness and convenience to the reader.

Lemma 3 (Generalized residue formula) Let $\nabla$ be a logarithmic connection with singular part contained in $D$ on a holomorphic line bundle $L$ over a compact Riemann surface $M$. Then,

$$\deg(L) + \sum_{p \in M} \text{Res}_p(\nabla) = 0.$$

Proof Let $s$ be a non-zero meromorphic section of $L$ such that non of its zeros or poles lie in $\text{supp}(D)$. The existence of $s$ is guaranteed by the Riemann-Roch theorem and the Abel-Jacobi Theorem. The degree formula gives

$$\deg(L) = \sum_{p \in M} \text{ord}_p s.$$

Let $\omega$ be the (holomorphic) connection 1-form $\omega$ of $\nabla$ with respect to $s$, i.e.,

$$\nabla s = \omega \otimes s.$$

Then at $p \in M$ with $\text{Res}_p \omega \neq 0$ we have either $p$ is a singular point of $\nabla$ or $p$ is a zero or pole of $s$. In the first case, we have by definition $\text{Res}_p \omega = \text{Res}_p \nabla$ and in the second case we
get $\text{Res}_p \omega = \text{ord}_p \omega$ as a consequence of the Leibniz rule. Since $\omega$ is a meromorphic $1$-form on the compact Riemann surface $M$ the residue theorem gives

$$0 = \sum_{p \in M} \text{Res}_p \omega = \sum_{p \in M} \text{ord}_p \omega + \sum_{p \in M} \text{Res}_p \nabla = \deg(L) + \sum_{p \in M} \text{Res}_p(\nabla)$$

as claimed.

### 2.2 Parabolic structures

Parabolic structures provide a useful tool for investigating logarithmic connections as they describe natural boundary value conditions, see for example [29] or [4].

**Definition 4** Let $M$ be a compact Riemann surface and $V$ be a holomorphic SL$(2, \mathbb{C})$ vector bundle over $M$. A parabolic structure $\mathcal{P}$ on $V$ over the divisor $D = p_1 + \cdots + p_n$ is defined by a collection of complex lines $L_k \subset V_{p_k}$ together with a collection of numbers $t_k \in (0, \frac{1}{2}) \subset \mathbb{R}$ for all $k = 1, \ldots, n$. The divisor $D$ is then called the parabolic divisor, the lines $\{L_k\}_{k=1}^n$ are called the quasiparabolic lines, and the numbers $t_k$ are the parabolic weights at $p_k$.

The parabolic degree of a holomorphic line subbundle $W \subset V$ is defined to be

$$\text{par-deg}(W) := \deg(W) + \sum_{k=1}^n t_k^W,$$

where $t_k^W = t_k$ if $W_{p_k} = L_k$, and $t_k^W = -t_k$ if $W_{p_k} \neq L_k$. The parabolic degree of a parabolic SL$(2, \mathbb{C})$-vector bundle $V$ is always 0 in our setup.

**Remark 5** There are different possible definitions for parabolic structures, compare [23, 29, 30, 34]. There are also different conventions for the range of the weights. We use the ‘trace free’ convention, see [12, 32].

**Definition 6** [29, 32] A parabolic structure $\mathcal{P}$ on the SL$(2, \mathbb{C})$ vector bundle $V$ is called stable (respectively, semistable) if $\text{par-deg}(W) < 0$ (respectively, $\text{par-deg}(W) \leq 0$) for every holomorphic line subbundle $W \subset V$. A semistable parabolic bundle that is not stable is called strictly semistable. A parabolic bundle which is not semistable is called unstable.

An isomorphism $\Psi$ between parabolic structures $\mathcal{P}$ and $\tilde{\mathcal{P}}$ on holomorphic bundles $V$ and $\tilde{V}$ over the same Riemann surface with the same singular divisor and with the same parabolic weights is given by a holomorphic bundle isomorphism $\Psi : V \to \tilde{V}$ such that the parabolic lines are mapped to each other.

Let $\nabla$ be a non-resonant logarithmic SL$(2, \mathbb{C})$–connection on a holomorphic bundle $V$ over a compact Riemann surface. If the eigenvalues of all residues of $\nabla$ are contained in the interval $(-\frac{1}{2}, \frac{1}{2})$ then $\nabla$ naturally induces a parabolic structure $\mathcal{P}$ on $V$. Indeed, the parabolic divisor of $\mathcal{P}$ is the singular locus $D = p_1 + \cdots + p_n$ of $\nabla$, the parabolic weight at $p_k$ is the (unique) positive eigenvalue $t_k$ of $\text{Res}_{p_k}(\nabla)$ and the quasiparabolic line at $p_k$ is the eigenline of $\text{Res}_{p_k}(\nabla)$ with respect to the positive eigenvalue $t_k$.

A strongly parabolic Higgs field on a parabolic SL$(2, \mathbb{C})$ vector bundle $(V, \mathcal{P})$ is a meromorphic endomorphism-valued $1$-form

$$\Psi \in H^0(M, \text{End}(V) \otimes K_M \otimes \mathcal{O}_M(D))$$
such that \( \text{tr}(\Psi) = 0 \) and such that the residues \( \text{Res}_{\Psi_k} \Psi \) are nilpotent with kernels given by the quasiparabolic lines

\[
L_k \subset \ker(\text{Res}_{\Psi_k} \Psi)
\]

for all \( k = 1, \ldots, n \). Note that \( \Psi \) has at most first order poles at the singular points \( p_k \).

Two logarithmic \( SL(2, \mathbb{C}) \)-connections \( \nabla_1 \) and \( \nabla_2 \) on \( V \) with singular part contained in \( D = p_1 + \ldots + p_n \) induce the same parabolic structure on \( V \) if and only if \( \nabla_1 - \nabla_2 \) is a strongly parabolic Higgs field for the parabolic structure induced by \( \nabla_1 \) (or equivalently, for the parabolic structure induced by \( \nabla_2 \)).

The following theorem is a special instance of the famous Mehta-Seshadri theorem ([29, p. 226, Theorem 4.1(2)], see also Biquard [4, p. 246, Théorème 2.5]) for \( SU(2) \)-connections and our sign conventions. For further details see [32, Theorem 3.2.2].

**Theorem 2** Let \( \nabla \) be an irreducible logarithmic \( SL(2, \mathbb{C}) \)-connection with monodromy representation conjugated to an \( SU(2) \)-representation. Then the induced parabolic structure is stable. Conversely, every stable parabolic structure gives rise to a unique irreducible logarithmic \( SL(2, \mathbb{C}) \)-connection with monodromy representation conjugated to an \( SU(2) \)-representation.

Two logarithmic connections which are gauge equivalent via a gauge transformation that is well-defined on the compact surface \( M \), induce isomorphic parabolic structures. Together with the uniqueness part in Theorem 2 this gives a bijection between the space of isomorphism classes of irreducible flat \( SU(2) \)-connections on \( M \setminus \text{supp}(D) \) and the isomorphism classes of stable parabolic \( SL(2, \mathbb{C}) \) vector bundles on \((M, D)\). Another direct consequence of Theorem 2 is that every logarithmic \( SL(2, \mathbb{C}) \)-connection \( \nabla \) on \( V \) with stable parabolic structure \( \mathcal{P} \) admits a unique strongly parabolic Higgs field \( \Psi \) on \((V, \mathcal{P})\) such that the monodromy representation of \( \nabla + \Psi \) is unitary up to conjugation. Since the generalize residue formula given in Lemma 3 states that the parabolic degree of a holomorphic line bundle \( W \) with logarithmic connection has parabolic degree 0, reducible unitary logarithmic connections induce strictly semi-stable parabolic structures. If the underlying parabolic structure of a logarithmic connection \( \nabla \) is unstable then \( \nabla \) cannot be unitary with respect to any hermitian metric. In particular, a logarithmic connection \( \nabla \) with unstable parabolic structure cannot have monodromy which is unitary up to conjugation.

### 2.3 Fuchsian systems on the 4-punctured sphere

A particular class of logarithmic connections is provided by Fuchsian systems. Let \( p_1, \ldots, p_4 \in \mathbb{C} \subset \mathbb{C}P^1 \) be pairwise distinct points and fix the singular part \( D = p_1 + \ldots + p_4 \). An \( SL(2, \mathbb{C}) \)-Fuchsian system on the 4-punctured sphere is a holomorphic connection on the trivial \( \mathbb{C}^2 \)-bundle over \( \mathbb{C}P^1 \setminus \{p_1, \ldots, p_4\} \) of the form \( \nabla = d + \xi \) with

\[
\xi = \sum_{k=1}^{4} A_k \frac{dz}{z - p_k} \quad \text{such that} \quad \sum_{k=1}^{4} A_k = 0. \tag{3}
\]

The vanishing of the sum of the matrices ensures that there is no further singularity at \( z = \infty \). If \( p_4 = \infty \) (as it is the case in following) then

\[
\xi = \sum_{k=1}^{3} A_k \frac{dz}{z - p_k} \quad \text{with} \quad A_4 = -\sum_{k=1}^{3} A_k.
\]
As before, we assume that the residues $A_k \in \text{sl}(2, \mathbb{C})$ have eigenvalues $\pm t_k$ with $t_k \in (0, \frac{1}{4})$ for $k = 1, \ldots, 4$, i.e., $\nabla$ is non-resonant. Therefore, the conjugacy class of the local monodromy along a simple closed curve around a singular point $p_k$ is determined by the residue $A_k$, and this conjugacy class contains
\[
\begin{pmatrix}
e^{2\pi i t_k} & 0 \\
0 & e^{-2\pi i t_k}
\end{pmatrix}.
\]

**Remark 7** In particular, there is always a local holomorphic frame of $V$ such that the connection 1-form of the logarithmic connection with respect to this frame is diagonal, see also [9].

Two Fuchsian systems with the same singular part $D$ and with residues $A_k$ and $\tilde{A}_k$, respectively, are equivalent if and only if there exists an invertible matrix $G$ such that
\[
\tilde{A}_k = G^{-1}A_k G
\]
for $k = 1, \ldots, 4$. This means that two Fuchsian systems with the same weights are equivalent if and only if the connections are gauge equivalent.

**Definition 8** An SL$(2, \mathbb{C})$-Fuchsian system $\nabla$ is called reducible, if there exist a $\nabla$-invariant holomorphic line subbundle. Otherwise, the Fuchsian system is called irreducible.

Given two irreducible and equivalent Fuchsian systems $\nabla$ and $\tilde{\nabla}$, the SL$(2, \mathbb{C})$-gauge matrix $G$ between the two connections is uniquely determined up to sign. Due to the intrinsic closing condition in the monodromy problem, Fuchsian systems admitting a unitary monodromy representation are of particular interest for the construction of minimal surfaces in the 3-sphere.

**Definition 9** A logarithmic SL$(2, \mathbb{C})$-connection $\nabla$ on a holomorphic bundle $V$ over $M$ is called unitarizable if there exist a hermitian metric $h$ on the bundle $V$ restricted to $M \setminus \text{supp}(D)$ such that $\nabla$ is unitary with respect to $h$.

Clearly, a logarithmic SL$(2, \mathbb{C})$-connection $\nabla$ is unitarizable if and only if the monodromy representation of the flat connection $\nabla$ on $M \setminus \text{supp}(D)$ is unitary up to conjugation.

**Convention**

In the following we will only consider SL$(2, \mathbb{C})$-Fuchsian systems on $\mathbb{C}P^1$ with four singular points $D = p_1 + \cdots + p_4$ such that all eigenvalues are $\pm t$ with $t \in (0, \frac{1}{4})$. We call them Fuchsian systems on the 4-punctured sphere with weight $t$ for short. We call a Fuchsian system stable or semi-stable depending on the stability of the induced parabolic structure. To simplify computations further we restrict to the special case $p_1 = -1, p_2 = 0, p_3 = 1, p_4 = \infty \in \mathbb{C}P^1$ as in Sect. 1.1. We want to emphasize that we restrict to $t \in (0, \frac{1}{4})$ instead of $t \in (0, \frac{1}{2})$ which is crucial for many of the following results.

2.4 The parabolic modulus and coordinates

Before defining coordinates for the moduli space of Fuchsian systems, we first collect and prove some folklore facts about them. Though these facts are well-known to experts and can easily be deduced from [27], we include the proofs here to make the paper more self-contained and comprehensible for differential geometers.
Lemma 10 Let $\nabla$ be an $\text{SL}(2, \mathbb{C})$-Fuchsian system on the 4-punctured sphere with parabolic weights $\pm t$ and $t \in (0, \frac{1}{4})$. Then the induced parabolic structure is semi-stable. Moreover, the induced parabolic structure is stable if and only if all quasiparabolic lines are pairwise distinct, and strictly semi-stable if and only if two of the four quasiparabolic lines coincide.

Proof To prove semi-stability we need to show that every holomorphic line subbundle $W$ of $V = \mathcal{O} \oplus \mathcal{O}$ has non-positive parabolic degree. For any holomorphic subbundle $W$ of $V$, the projection to the first (or second) summand of $V = \mathcal{O} \oplus \mathcal{O}$ yields a non-trivial holomorphic section of $W^*$. Therefore, $\deg W \leq 0$. Moreover, if $\deg W < 0$ then also its parabolic degree is negative, as $4t < 1$ by assumption.

Thus let $W$ be a degree 0 holomorphic subbundle $V$. In this case $W$ is constant, i.e., it is parallel with respect to the trivial connection $d$. Choose a complementary holomorphic line bundle $\widetilde{W}$ of $W$ in $V$ and write $\nabla$ with respect to the splitting $V = W \oplus \widetilde{W}$. Then the lower-left entry $\beta^W$ of $\nabla$ is a meromorphic 1-form with at most first order poles at the four singular points. If $W = L_k$ at $p_k$ we obtain $\text{Res}_{p_k} \beta^W = 0$, since $L_k$ is an eigenline of the residue of $\nabla$ at $p_k$ by definition. In this case $\beta^W$ is holomorphic at $p_k$.

Hence, if more than two quasiparabolic lines coincide with $W$, $\beta^W \equiv 0$, as there are no non-zero meromorphic 1-forms with at most one pole of order 1 on $\mathbb{C}P^1$. But this would imply that $W$ is a parallel line subbundle of $V$. Then the generalized residue formula gives

$$0 = \deg(W) + \sum_{k=1}^{4} t_k^W \geq 2t$$

contradicting $t > 0$. Therefore, at most two of the quasiparabolic lines can coincide and

$\text{par-deg } W \leq -2t + 2t = 0$,

with equality if and only if exactly two of the quasiparabolic lines $L_1, \ldots, L_4$ are the same and $W$ coincides with this line. \hfill \Box

The next lemma characterizes all unitarizable, reducible Fuchsian systems.

Lemma 11 Let $\nabla$ be a reducible and unitarizable Fuchsian system on the 4-punctured sphere with weight $t$ for $t \in (0, \frac{1}{4})$. Then, up to equivalence, $\nabla$ is given by

$$d + \sigma_1 \left( \begin{array}{cc} t & 0 \\ 0 & -t \end{array} \right) \frac{dz}{z+1} + \left( \begin{array}{cc} t & 0 \\ 0 & -t \end{array} \right) \frac{dz}{z} + \sigma_1 \left( \begin{array}{cc} t & 0 \\ 0 & -t \end{array} \right) \frac{dz}{z-1}, \tag{4}$$

where $(\sigma_1, \sigma_1) \in \{(-1, -1), (-1, 1), (1, -1)\}$.

Proof A reducible Fuchsian system $\nabla$ possesses an invariant holomorphic line subbundle $W$. As before the generalized residue formula shows that

$$0 = \deg(W) + \sum_{k=1}^{4} t_k^W = \text{par-deg } W.$$

Since $t \in (0, \frac{1}{4})$, this implies that the degree of $W$ is 0 and the induced parabolic weights $t_k^W$ on $W$ sum up to 0. Since $\nabla$ is unitarizable there exists a hermitian metric $h$ for which $\nabla$ is unitary. If $W^\perp$ denotes the orthogonal complement of $W$ with respect to $h$, then $W^\perp$ is parallel as well. Moreover, by Remark 7, the holomorphic structure of $W \oplus W^\perp$ extends through the punctures as $\nabla$ is non-resonant. Therefore, $\nabla$ is of the stated form (4) with respect to the splitting $V = W \oplus W^\perp$. \hfill \Box
Consider on $\mathbb{C}P^1$ the involutions
\[ \delta(z) = -\frac{1}{z} \quad \text{and} \quad \tau(z) = \frac{1-z}{z+1} \]
which interchanges the singular points $p_1, ..., p_4$. We call a Fuchsian system symmetric if there exist $\tilde{D}, \tilde{C} \in \text{SL}(2, \mathbb{C})$ such that
\[ \delta^* \nabla = \nabla \tilde{D} \quad \text{and} \quad \tau^* \nabla = \nabla \tilde{C}. \] (5)

A strongly parabolic Higgs field $\Phi$ for a symmetric Fuchsian system $\nabla$ is called symmetric if and only if
\[ \delta^* \Phi = \tilde{D}^{-1} \Phi \tilde{D} \quad \text{and} \quad \tau^* \Phi = \tilde{C}^{-1} \Phi \tilde{C} \]
for the same $\tilde{D}, \tilde{C} \in \text{SL}(2, \mathbb{C})$ as in (5).

**Remark 12** The symmetries $\delta, \tau$ differ from those in [15] only by a Moebius transformation interchanging the different choices of singular points. We will show in Lemma 16 below that up to conjugation the matrices $\tilde{C}$ and $\tilde{D}$ can be chosen to be
\[ D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \] (6)

We characterize symmetric and reducible Fuchsian systems and strictly semi-stable parabolic structures in the following lemma.

**Lemma 13** Let $\nabla$ be a symmetric $\text{SL}(2, \mathbb{C})$-Fuchsian system on the 4-punctured sphere $\Sigma$ with parabolic structure $\mathcal{P}$. If $\mathcal{P}$ is strictly semi-stable, then $\mathcal{P}$ is (up to conjugation) the parabolic structure of one of the three reducible and unitary Fuchsian systems given by Lemma 11. Moreover, if $\nabla$ is reducible, then it coincides with (4) and is automatically unitarizable.

**Proof** Let $\nabla$ be a Fuchsian system that induces a strictly semi-stable parabolic structure $\mathcal{P}$. By Lemma 10 there exists a constant line subbundle $W$ of $V$ which coincides with exactly two of the four quasiparabolic lines $L_k$. Denote the corresponding singular points by $a \neq b \in \{p_1, \ldots, p_4\}$, and the remaining two singular points by $c \neq d \in \{p_1, \ldots, p_4\} \setminus \{a, b\}$.

Consider the group of automorphisms generated by the involutions $\delta$ and $\tau$. This group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and there exist a unique group element $\mu$ interchanging the two pairs $\{a, b\}$ and $\{c, d\}$ of singular points, i.e., $\{c, d\} = \mu(\{a, b\})$. Since $\nabla$ is symmetric we have $\mu^* \nabla = \nabla M$ for some $M \in \text{SL}(2, \mathbb{C})$. Moreover,
\[ MW \neq W, \]
otherwise $W$ would coincide with all four quasiparabolic lines contradicting Lemma 10. Hence,
\[ V = \mathcal{O} \oplus \mathcal{O} = W \oplus MW, \]
and the quasiparabolic lines at $a, b$ are $W$ and the quasiparabolic lines at $c, d$ are $MW$. In particular, since the Fuchsian system is non-resonant, its parabolic structure is gauge equivalent to the one obtained from a reducible unitary Fuchsian system in (4). Therefore, $\nabla$ differs from one of the three connections of Lemma 11 by a strongly parabolic Higgs field $\Psi$ which is necessarily off-diagonal with respect to $W \oplus MW$.

Finally let $\nabla$ be reducible. Since all reducible symmetric connections on the parabolic bundle $\mathcal{P}$ are diagonal with respect to $W \oplus MW$, the strongly parabolic Higgs field $\Psi$ must...
be zero implying that $\nabla$ is of the form (4) for suitable $\sigma_{\pm 1}$, and is therefore automatically unitarizable.

Two Fuchsian systems inducing the same parabolic structure differ by a strong parabolic Higgs field. The next lemma shows that the space of these Higgs fields is complex 1-dimensional for stable Fuchsian systems.

**Lemma 14.** Let $\nabla$ be a stable Fuchsian system. Then the space $\mathcal{H}$ of strongly parabolic Higgs fields is a complex 1-dimensional vector space.

**Proof.** Assume that $\mathcal{H}$ is at least 2-dimensional, i.e., there exist two linear independent strongly parabolic Higgs fields $\Psi_1$ and $\Psi_2$. Recall that the residues of a strongly parabolic Higgs field are nilpotent and their kernel are given by the respective quasiparabolic line. Therefore, we can take a non-zero linear combination of $\Psi_1$ and $\Psi_2$ such that the residue at $p_1$ vanishes. Up to conjugation and scaling, the residues at $p_2$, $p_3$ and $p_4$ are

$$
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
x & y \\
z & -x
\end{pmatrix}, \text{ and } \begin{pmatrix}
-x & -y - 1 \\
-z & x
\end{pmatrix},
$$

respectively, as the sum of all residues must vanish. Since the residues at $p_3$ and at $p_4$ are both nilpotent, we have

$$
-x^2 - yz = 0 \quad \text{and} \quad -x^2 - (y + 1)z = 0
$$

which implies $x = 0$ and $z = 0$. In this case the kernel of the residue at $p_3$ and $p_4$ is the same, which contradicts the fact that the quasiparabolic lines are pairwise distinct for a stable parabolic structure by Lemma 10. A non-trivial parabolic Higgs field for every stable parabolic structure is given in (10) below. Therefore, the space $\mathcal{H}$ is complex 1-dimensional for stable Fuchsian systems.

For strictly semi-stable parabolic structures the space of all strongly parabolic Higgs fields has dimension bigger than 1. Therefore, we restrict to symmetric Higgs fields in this special case.

**Lemma 15.** Let $\nabla$ be symmetric with strictly semi-stable parabolic structure. Then, the space of symmetric strongly parabolic Higgs fields is complex 1-dimensional.

**Proof.** By Lemma 13 the parabolic structure of a symmetric and strictly semi-stable Fuchsian system $\nabla$ is the one of a reducible unitary Fuchsian system, i.e., by Lemma 11 it is diagonal with respect to the splitting $V = W \oplus MW$, and the strongly parabolic Higgs field is off-diagonal. Moreover, the upper right entry of the strongly parabolic Higgs field is

$$
x \left( \frac{dz}{z - a} - \frac{dz}{z - b} \right)
$$

for some $x \in \mathbb{C}$ by the residue theorem, and then the lower left entry is

$$
x \mu^* \left( \frac{dz}{z - a} - \frac{dz}{z - b} \right)
$$

by symmetry. Hence the dimension is 1.
By the proof of Lemma 10 a stable Fuchsian system $\nabla$ induces a stable parabolic structure $P$ with four pairwise distinct quasiparabolic lines, i.e., the four eigenlines with respect to the positive eigenvalues of the four residues are pairwise distinct lines in $\mathbb{C}^2$. Hence, their cross-ratio, denoted by $u$, is well-defined after the choice of ordering. Without loss of generality we can conjugate $\nabla$ by a suitable $g \in \text{SL}(2, \mathbb{C})$ such that the eigenlines of the residues with respect to the positive weights at $p_2 = 0$, $p_3 = 1$ and $p_4 = \infty$ are 

\[
\mathbb{C}\left(\begin{array}{c}
0 \\
1
\end{array}\right), \quad \mathbb{C}\left(\begin{array}{c}
1 \\
0
\end{array}\right), \quad \text{and} \quad \mathbb{C}\left(\begin{array}{c}
1 \\
0
\end{array}\right),
\]

respectively.

Then, there exists a unique $u := u(\nabla) \in \mathbb{C}P^1 \setminus \{0, 1, \infty\}$ such that the eigenline of the residue of $\nabla$ at $p_1 = -1$ is 

\[
\mathbb{C}\left(\begin{array}{c}
u \\
1
\end{array}\right).
\]

Note that the gauge $g$ is unique up to sign, and hence we have fixed the gauge freedom. In particular, if $\nabla$ and $\tilde{\nabla}$ have different $u(\nabla) \neq u(\tilde{\nabla})$ then $\nabla$ and $\tilde{\nabla}$ cannot be gauge-equivalent. This gives rise to a well-defined holomorphic map

\[
u: \{\text{Fuchsian systems with stable parabolic structure}\} \to \mathbb{C}P^1,
\]

which we refer to as modulus map. By construction this map is invariant under conjugation and under gauge transformations. In particular, a holomorphic family of Fuchsian systems $\lambda \in U \subset \mathbb{C} \mapsto \xi(\lambda)$ with stable underlying parabolic structures gives rise to a holomorphic function $u: U \to \mathbb{C}$. Moreover, if $\tilde{\nabla} = \nabla + \Phi$ for a strongly parabolic Higgs field $\Phi$ for a stable Fuchsian system $\nabla$, then

\[u(\tilde{\nabla}) = u(\nabla).
\]

Applying Cauchy’s removable singularity theorem, $u$ extends to the Fuchsian systems with strictly semi stable parabolic structure. More precisely, every holomorphic family of Fuchsian systems $\lambda \in U \subset \mathbb{C} \mapsto \xi(\lambda)$ gives rise to a holomorphic modulus function $u: U \to \mathbb{C}$. In particular, the images of the reducible Fuchsian systems of Lemma 11 (parametrized by $(\sigma_{-1}, \sigma_1) \in \{(-1, -1), (-1, 1), (1, -1)\}$) under $u$ are given by

\[(-1, -1) \mapsto u 1; \quad (-1, 1) \mapsto u \infty; \quad (1, -1) \mapsto u 0.
\]

### 2.4.1 Coordinates

Now we can introduce global coordinates of the moduli space of stable, trace free Fuchsian systems on a 4-punctured sphere. To our best knowledge these have been first introduced by Loray-Saito [27]. A further reference is [12]. Let $t \in (0, \frac{1}{2})$. For $u \in \mathbb{C}\setminus\{0, 1\}$ set

\[
A_1^u = \left(\begin{array}{cc}
-t & 2tu \\
0 & t
\end{array}\right), \quad A_2^u = \left(\begin{array}{cc}
-t & 0 \\
-2t & t
\end{array}\right), \quad A_3^u = \left(\begin{array}{cc}
t & 0 \\
2t & -t
\end{array}\right), \quad A_4^u = \left(\begin{array}{cc}
t & -2tu \\
0 & -t
\end{array}\right).
\]

Then the connection

\[
\nabla^u := d + \sum_{k=1}^{3} A_k^u \frac{dz}{z - p_k}
\]
is a Fuchsian system with poles at $p_k$ for $k = 1, \ldots, 4$ and parabolic weights $\pm t$ and modulus $u$. Moreover, with

$$\Psi_1 = \begin{pmatrix} -u & u^2 \\ -1 & u \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Psi_3 = \begin{pmatrix} u - u \\ u - u \end{pmatrix}, \quad \Psi_4 = \begin{pmatrix} 0 & u - u^2 \\ 0 & 0 \end{pmatrix},$$

(9)

the meromorphic endomorphism-valued 1-form

$$\Psi^u = \Psi := \sum_{k=1}^{4} \Psi_k \frac{dz}{z - p_k}$$

(10)

is a strongly parabolic Higgs field with respect to the parabolic structure of $\nabla^u$ with modulus $u \in \mathbb{C} \setminus \{0, 1\}$. By Lemma 14 the strongly parabolic Higgs field $\Psi^u$ for the stable Fuchsian system $\nabla^u$ is unique up to scaling. Therefore, we can write every stable Fuchsian system up to a unique gauge as

$$\nabla^{u,s} := \nabla^u + s\Psi, \quad s \in \mathbb{C}$$

(11)

for some unique $u \in \mathbb{C} \setminus \{0, 1\}$ and unique $s \in \mathbb{C}$. This gives us a global coordinate system $(u, s)$ on the moduli space of stable Fuchsian systems. We remark that $u = -1$ is special in the sense that it gives the unique stable parabolic structure which admits a nilpotent strongly parabolic Higgs field, as

$$\det(\Psi^u) = -\frac{u - u^3}{z - z^3}. $$

The two other exceptional cases $u = 0, 1$ of

$$u - u^3 = 0$$

correspond to strictly semi-stable parabolic structures, which are not symmetric with respect to the involutions $\delta$ and $\tau$, since they have three distinct quasiparabolic lines. But the strictly semi-stable parabolic structures of the Fuchsian systems $\nabla^u$ for $u = 0, 1$ are in the closure of the gauge orbits of the parabolic structures induced by the symmetric Fuchsian systems in Lemma 11 for $(\sigma_{-1}, \sigma_1) = (1, -1)$ and $(\sigma_{-1}, \sigma_1) = (-1, -1)$, respectively. Similarly, the case $u = \infty$ corresponds to $(\sigma_{-1}, \sigma_1) = (-1, 1)$.

Using the normal form provided by Lemma 16 we obtain coordinates on the moduli space of Fuchsian system on some neighborhood of the strictly semistable Fuchsian systems. Note that the normal form in Lemma 16 depends on finitely many choices, and therefore these coordinates are not single valued and do not provide global coordinates on the moduli space of all symmetric Fuchsian systems.

**Lemma 16** Let $\nabla$ be an $SL(2, \mathbb{C})$ Fuchsian system on $\mathbb{C}P^1$ with four singular points $p_1, \ldots, p_4$ and with parabolic weight $t \in (0, \frac{1}{4})$. If $\nabla$ is symmetric with respect to the symmetries $\delta$ and $\tau$, then it is equivalent to

$$d + \sum_{k=1}^{3} A_k \frac{dz}{z - p_k},$$

where the residues $A_k \in \mathfrak{sl}(2, \mathbb{C})$, $k = 1, \ldots, 4$, satisfy

$$A_4 := -A_1 - A_2 - A_3 = D^{-1} A_2 D, \quad A_3 = D^{-1} A_1 D$$

(12)
and

\[ A_4 = C^{-1} A_1 C \quad \text{and} \quad A_3 = C^{-1} A_2 C, \quad (13) \]

for \( D \) and \( C \) defined in (6). This form is unique up to conjugation with elements of the finite group generated by \( D \) and \( C \).

**Proof** To show that every symmetric Fuchsian system \( \nabla \) with parabolic weight \( t \) satisfies the desired symmetries (12) and (13), we distinguish between irreducible and reducible \( \nabla \).

In the reducible case, we can apply Lemma 11 to obtain that \( \nabla \) is given by

\[
d + \sigma_{-1} \left\{ \begin{array}{c} t \\ 0 \\ -t \end{array} \right\} \frac{dz}{z+1} + \left\{ \begin{array}{c} t \\ 0 \\ -t \end{array} \right\} \frac{dz}{z} + \sigma_1 \left\{ \begin{array}{c} t \\ 0 \\ -t \end{array} \right\} \frac{dz}{z-1},
\]

where \( (\sigma_{-1}, \sigma_1) \in \{(-1, -1), (-1, 1), (1, -1)\} \) up to conjugation. From here it is a straightforward computation to see that \( \nabla \) has the desired symmetries after a suitable conjugation.

In the second case, \( \nabla \) is irreducible and \( \nabla \) has no parallel line bundles. Because \( \nabla \) is symmetric there exist \( \tilde{C}, \tilde{D} \in \text{SL}(2, \mathbb{C}) \) such that

\[
\delta^* \nabla = \nabla \tilde{D} = \tilde{D}^{-1} \nabla \tilde{D}
\]

and

\[
\tau^* \nabla = \nabla \tilde{C} = \tilde{C}^{-1} \nabla \tilde{C}.
\]

By irreducibility of \( \nabla \), \( \tilde{C} \) and \( \tilde{D} \) are unique up to sign. Since \( \delta^2 = \tau^2 = \text{Id} \) this implies

\[
\tilde{D}^2 = \pm \text{Id} \quad \text{and} \quad \tilde{C}^2 = \pm \text{Id}.
\]

If \( \tilde{D}^2 = \text{Id} \), we would have \( \tilde{D} = \pm \text{Id} \) and by writing

\[
\nabla = d + \sum_{k=1}^{3} B_k \frac{dz}{z-p_k}
\]

we obtain

\[
B_3 = B_1, \quad B_4 = B_2 \quad \text{and} \quad B_1 + B_2 + B_3 + B_4 = 0.
\]

Therefore, \( \nabla \) would be reducible which is a contradiction to the assumption. Thus \( \tilde{D}^2 = -\text{Id} \), and hence \( \tilde{D} \) is conjugate to \( D \), i.e., there is \( g \in \text{SL}(2, \mathbb{C}) \) with

\[
\tilde{D} = g D g^{-1}.
\]

The connection \( \hat{\nabla} = \nabla \cdot g \) then satisfies

\[
\delta^* \hat{\nabla} = \delta^* \nabla \cdot g = \nabla \cdot g \cdot D = \hat{\nabla} \cdot \tilde{D}
\]

implying the \( \delta \)-symmetry for \( \hat{\nabla} \cdot g \).

We can now assume without loss of generality that \( \nabla \) satisfies the \( \delta \)-symmetry (12). To show the \( \tau \)-symmetry (13) we first recall

\[
\delta \circ \tau = \tau \circ \delta.
\]

Therefore, we obtain from irreducibility of \( \nabla \)

\[
D \tilde{C} = \pm \tilde{C} D.
\]

\( \square \) Springer
This implies $\tilde{C}$ being either diagonal or off-diagonal. If $\tilde{C}$ is diagonal, $\tilde{C}^2 = \pm \text{Id}$ yields again reducibility of $\nabla$. Thus $\tilde{C}$ must be off-diagonal and $\tilde{C}^2 = -\text{Id}$. This gives

$$\tilde{C} = \pm C \quad \text{or} \quad \tilde{C} = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

In the first case $\tilde{C}$ is of the desired form. In the latter case, $\tilde{C}$ and $\pm C$ differ by the conjugation with

$$S^{-1} = \begin{pmatrix} e^{-\frac{\pi i}{4}} & 0 \\ 0 & e^{\frac{\pi i}{4}} \end{pmatrix}. $$

As $S^{-1}$ commutes with $D$, (12) and (13) holds after conjugation.

Uniqueness of the symmetric normal form up to conjugation with $C$ and $D$ follows again from irreducibility: Note that $\pm \text{Id}$ are the only $\text{SL}(2, \mathbb{C})$ matrices commuting with both $C$ and $D$. Moreover, the space of matrices which either commute or anti-commute with $C$ and $D$ is generated by $\text{Id}, C, D$ and $CD$, which proves the claim. $\square$

### 2.5 Exceptional logarithmic connections

Every representation of the first fundamental group of the 4-punctured sphere with prescribed conjugacy classes of the local monodromies is induced by a logarithmic connection, see for example [9]. In fact, the proof of Theorem 3 below is a $\lambda$-family variant of Deligne’s extension method [9]. On the other hand, not all logarithmic connections are Fuchsian, i.e., they are not always defined on the trivial holomorphic bundle $\mathcal{O} \oplus \mathcal{O}$.

**Lemma 17** Let $t \in (0, \frac{1}{4})$. Let $\nabla$ be a logarithmic $\text{SL}(2, \mathbb{C})$-connection on the 4-punctured sphere such that the eigenvalues of all of its residues are $\pm t$. Then, the underlying holomorphic vector bundle $V$ is of the type

$$V = \mathcal{O}(l) \oplus \mathcal{O}(-l)$$

for $l = 0$ or $l = 1$.

**Proof** By Grothendieck splitting the underlying holomorphic vector bundle $V$ of a logarithmic $\text{SL}(2, \mathbb{C})$-connection over $\mathbb{C}P^1$ is given by $V = \mathcal{O}(l) \oplus \mathcal{O}(-l)$ for some $l \in \mathbb{N}$. We have to prove that $l \leq 1$ due to degree considerations. In fact, for $l > 1$, the second fundamental form of $\mathcal{O}(l)$ necessarily vanishes, as it is a meromorphic section with at most 4 simple poles of the holomorphic line bundle $K \mathcal{O}(-2l)$ of degree less than $-4$. Hence, $\mathcal{O}(l)$ would be parallel, contradicting the generalized residue formula (Lemma 3) for $t \in (0, \frac{1}{4})$. $\square$

We have already studied the case of logarithmic connections on $V = \mathcal{O} \oplus \mathcal{O}$ in the last section. In that case the parabolic structure is always semi-stable, and the complex dimension of the moduli space of these Fuchsian systems with prescribed weight is 2. In contrast, for $t \in (0, \frac{1}{4})$, logarithmic connections on $V = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ turn out to be unstable and the moduli space of unstable logarithmic connections is only complex 1-dimensional.

**Proposition 18** Let $t \in (0, \frac{1}{4})$ and consider the holomorphic bundle $V = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ over the Riemann sphere $\Sigma = \mathbb{C}P^1$. Then, the moduli space of trace free logarithmic connections on $V$ with four singular points with parabolic weight $t \in (0, \frac{1}{4})$ is a complex line. The underlying parabolic structure is unstable.
Moreover, the parabolic modulus \( u \) as a holomorphic function from the moduli space of logarithmic connections to \( \mathbb{C}P^1 \) extends holomorphically to this complex line, mapping the whole line to \( u = -1 \).

**Remark 19** The moduli space of parabolic structures admitting logarithmic connections is therefore not Hausdorff, but has a double point over \( u = -1 \).

**Proof** The parabolic structure is unstable, as the line bundle \( O(1) \) is a holomorphic subbundle of \( V \) with positive parabolic degree on the 4-punctured sphere for \( t \in (0, \frac{1}{2}) \).

To determine the moduli space of logarithmic connections on \( V \), we decompose a trace free logarithmic connection \( \nabla \) with respect to \( V = O(-1) \oplus O(1) \) as

\[
\nabla = \begin{pmatrix} \nabla^{-1} & \alpha \\ \beta & \nabla^1 \end{pmatrix},
\]

where \( \nabla^{-1} \) and \( \nabla^1 \) are the induced logarithmic connections on \( O(-1) \) and \( O(1) \), respectively, \( \alpha \) is a meromorphic \( O(-2) \)-valued 1-form with at most 4 simple poles and \( \beta \) is a meromorphic \( O(2) \)-valued 1-form with at most 4 simple poles. Hence \( \alpha \) has exactly 4 poles, i.e., up to scaling it is the unique meromorphic section of \( O(-4) = O(-2) \otimes K_{CP^1} \) with simple poles at \( p_1, \ldots, p_4 \). Note that \( \alpha \) is non-zero since otherwise \( O(1) \) would be parallel contradicting the generalized residue formula

\[
1 = \deg(O(1)) = -\text{Res}(\nabla^1) = -\sum_{k=1}^{4} (-1)^{v_k} t \neq 1
\]

for \( t \in (0, \frac{1}{2}) \) and for suitable \( v_k \in \{0, 1\} \). With respect to the given splitting \( V = O(-1) \oplus O(1) \) a holomorphic \( \text{SL}(2, \mathbb{C}) \) gauge transformation of \( V \) is of the form

\[
g = \begin{pmatrix} a & 0 \\ P & \frac{1}{a} \end{pmatrix},
\]

where \( a \in \mathbb{C}^\ast \) and \( P \in H^0(\mathbb{C}P^1, O(2)) \). Note that the upper right entry vanishes because it is a holomorphic section of \( O(-2) \). The gauged connection is then given by

\[
\nabla.g = \begin{pmatrix} \nabla^{-1} + a^{-1}aP \\ a^2\beta - a\nabla P - \alpha P^2 \nabla^1 - a^{-1}\alpha a P \end{pmatrix}.
\]

Recall that \( \alpha \) has non-trivial first order poles at the singular points \( p_1, \ldots, p_4 \). Therefore, to fix \( P \in H^0(\mathbb{C}P^1, O(2)) \), we can choose its values at 3 of the four singular points, e.g., at \( p_1, \ldots, p_3 \), such that

\[
\text{Res}_{p_k}(\nabla^{-1} + a^{-1}aP) = t
\]

for \( k = 1, \ldots, 3 \). By construction the upper left diagonal entry then has first order poles at \( p_1, \ldots, p_4 \) with residue \( t \) at \( p_1, \ldots, p_3 \). The upper right entry \( a^{-2}\alpha \) has non-trivial poles of order 1 at \( p_1, \ldots, p_4 \). Therefore, the lower left entry cannot have poles at \( p_1, \ldots, p_3 \) since the eigenvalues of the residues of \( \nabla.g \) at \( p_k \) are \( \pm t \). By the generalized residue formula, we further obtain

\[
\text{Res}_{\infty}(\nabla^{-1} + a^{-1}aP) = 1 - 3t \neq t.
\]

Hence, the lower left entry must have a first order pole at \( p_4 = \infty \) since the eigenvalues of the residues of \( \nabla.g \) at \( z = \infty \) is also \( \pm t \).
Next we write the connection 1-form \( \omega \) of \( \nabla, g \) with respect to a particular frame of the bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(1) \). Consider the meromorphic frame \((s_1, s_2)\) of \( \mathcal{O}(-1) \oplus \mathcal{O}(1) \), where \( s_1 \in \Gamma(\mathbb{C}, \mathcal{O}(-1)) \) has a simple pole at \( z = \infty \) and the section \( s_2 \in \Gamma(\mathbb{C}P^1, \mathcal{O}(1)) \) has a simple zero at \( z = \infty \). This leads to the standard trivialization of the bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(1) \) over \( \mathbb{C} \). Another trivialization of \( \mathcal{O}(-1) \oplus \mathcal{O}(1) \) over \( \mathbb{C}P^1 \setminus \{0\} \) is given by the meromorphic frame \((\tilde{s}_1, \tilde{s}_2)\) where \( \tilde{s}_1 \) has a simple pole at \( z = 0 \) and \( \tilde{s}_2 \) has a simple zero at \( z = 0 \). The cocycle for the bundle \( V \) with respect to these frames is

\[
\newcommand{\m}{\widetilde{\omega}}
\m = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix},
\]

i.e.,

\[
(s_1, s_2) = (\tilde{s}_1, \tilde{s}_2) h.
\]

The connection 1-form \( \m \) with respect to the frame \((\tilde{s}_1, \tilde{s}_2)\) over \( \mathbb{C}P^1 \setminus \{0\} \) is therefore given by

\[
\m = h d h^{-1} + h w h^{-1}.
\]

As \( \nabla \) is a logarithmic connection and \((\tilde{s}_1, \tilde{s}_2)\) is a holomorphic frame around \( z = \infty \), \( \m \) has a first order pole at \( z = \infty \). Therefore, with respect to the frame \((s_1, s_2)\), the lower left entry of \( \m \) has a pole of order 3 in \( p_4 \). Thus, after a suitable choice of \( a \in \mathbb{C}^* \) we can assume that the lower left entry of the connection 1-form \( \omega \) of \( \nabla, g \) is of the form

\[
(c_0 + z) dz,
\]

for some \( c_0 \in \mathbb{C} \). Recall that the residues of \( \nabla, g \) at \( z = -1, 0, 1 \) are upper triangular with diagonal entries \( t \) and \(-t\). Moreover, the eigenvalues of the residue of \( \nabla \) at \( z = \infty \) are also \( \pm t \). Therefore, a short computation shows that the connection 1-form with respect to the standard frame \((s_1, s_2)\) over \( \mathbb{C} \) is given by

\[
\omega = \begin{pmatrix} t(1 - 3z^2) & 8z^2 - 6t + 1 \\ z - z^3 & c_0 + z - t(1 - 3z^2) \end{pmatrix} dz.
\]  

(14)

The affine line of logarithmic connections on \( \mathcal{O}(-1) \oplus \mathcal{O}(1) \) (modulo gauge transformations) is thus parametrized by \( c_0 \in \mathbb{C} \).

For the second part of the Proposition, recall that the moduli space of logarithmic connections with prescribed local monodromies is a smooth complex manifold of dimension 2 away from the reducible connections. Therefore, we introduce a second parameter \( E \) and consider the holomorphic rank 2 bundle \( V(E) \) over \( \mathbb{C}P^1 \) defined by the cocycle

\[
\newcommand{\hE}{h^E}
\hE = \begin{pmatrix} z - E \\ 0 \end{pmatrix}.
\]

on \( \mathbb{C}^* = U_0 \cap U_\infty \). For \( E \neq 0 \) and with respect to the standard frame over \( U_0 \) the sections

\[
s_1 = \begin{pmatrix} 1 \\ E \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

extend to global holomorphic sections of \( V(E) \) on \( \mathbb{C}P^1 \) without zeros, and of determinant \( s_1 \wedge s_2 = 1 \). Therefore, the corresponding holomorphic bundle is trivial for \( E \neq 0 \), while for \( E = 0 \) we have \( V(E) = \mathcal{O}(-1) \oplus \mathcal{O}(1) \).
Let $\nabla^{E,c_0}$ be the meromorphic connection on $V(E)$ with connection 1-form over $U_0$

$$\omega^{E,c_0} = \begin{pmatrix} \frac{t(1-3z^2)}{z-\lambda} + E & -\frac{c_0E^2 + 6t(c_0E-\lambda z) + c_0E(\lambda z - 2) - 8t^2 - E^2z^2 + Ez - 1}{c_0 + z} \\ \frac{t(1-3z^2)}{z-\lambda} & \frac{-t(1-3z^2) - E z^2 + Ez}{c_0 + z} \end{pmatrix}. \quad (15)$$

Using the cocycle $h^E$ one can directly show that $\nabla^{E,c_0}$ is a logarithmic connection for all $c_0, E \in \mathbb{C}$, i.e., that the singularity at $\infty$ is a first order pole with respect to any local holomorphic frame of $V(E)$. Note that (15) coincides with (14) for $E = 0$.

We claim that

$$(E, c_0) \mapsto \nabla^{E,c_0}$$

parametrizes an open neighborhood of the unstable line inside the moduli space of irreducible logarithmic connections, i.e., $(E, c_0)$ are coordinates on the moduli space of logarithmic connections in an open neighbourhood of the exceptional line $E = 0$. This can be proven by using the natural holomorphic symplectic form $\Omega$ on the moduli space of logarithmic connections, see for example [1, 3], or [12] for the particular case of the 4-punctured sphere.

In fact, a direct but lengthy computation shows that

$$\Omega = dc_0 \wedge dE,$$

i.e., $(c_0, E)$ are Darboux coordinates and in particular coordinates.

To relate $(c_0, E)$ with the coordinates $(u, s)$ consider the $z$-independent matrix

$$C = \begin{pmatrix} \frac{c_0E - 2t + 1}{E(c_0E - 2t + E + 1)} & 0 \\ -\frac{c_0E - 2t + E + 1}{c_0E - 2t + E + 1} & 1 \end{pmatrix},$$

and the gauge

$$t^E = \begin{pmatrix} 0 & 1 \\ -1 & \frac{z}{E} \end{pmatrix}.$$

Then a direct computation gives

$$(\nabla)(t^EC) = \nabla^{u_0,s_0},$$

where $\nabla^{u_0,s_0}$ is the Fuchsian system defined in (11) with parameters given by

$$u_0 = -\frac{(c_0E - 2t + 1)(E + 1 - 2t)}{(c_0 - 1)E + 1 - 2t},$$

$$s_0 = -\frac{((1 - c_0)(E + 2t - 1)((1 - c_0)E + 4t - 1))}{2E}. \quad (16)$$

In particular, the function $u = u_0(E, c_0)$ is holomorphic in a neighbourhood of $E = 0$, and maps the exceptional line $\{E = 0, c_0 \in \mathbb{C}\}$ of logarithmic connections on $O(-1) \oplus O(1)$ to $u_0 = -1$.

**Remark 20** An important observation is that for the family of logarithmic connections $\nabla^{E,c_0}$ defined in the proof, the product $(u + 1) \cdot s$ has the simple form

$$(u(E, c_0) + 1) \cdot s(E, c_0) = 1 - 4t + (c_0 - 1)E \quad (17)$$

which extends holomorphically to $E = 0$. In particular, an unstable logarithmic connection appears as the limit of a family of stable Fuchsian systems $\lambda \mapsto \nabla^{u_0,s_0}$ for $\lambda \to \lambda_0 \in \mathbb{C}^*$ if...
and only if
\[
\lim_{\lambda \to \lambda_0} (u_\lambda + 1)s_\lambda = 1 - 4t.
\]

**Lemma 21** Let \( \nabla \) be an \( SL(2, \mathbb{C}) \) logarithmic connection on \( \mathbb{C}P^1 \) with four singular points \( p_1, ..., p_4 \) and with parabolic weight \( t \in (0, \frac{1}{4}) \). Let \( \nabla \) be symmetric with respect to the symmetries \( \delta \) and \( \tau \). If \( \nabla \) has unitarizable monodromy, then \( \nabla \) is a Fuchsian system, i.e, the underlying holomorphic bundle is \( V = \mathcal{O} \oplus \mathcal{O} \).

**Proof** Since a unitarizable logarithmic connection \( \nabla \) has a semi-stable parabolic structure by the Mehta-Seshadri Theorem 2, the underlying holomorphic bundle \( V \) is \( \mathcal{O} \oplus \mathcal{O} \) as a consequence of Lemma 17 and Proposition 18. Therefore, \( \nabla \) is a Fuchsian system. \( \square \)

### 3 Existence of Fuchsian DPW potentials

In this section we want to bring together the analytic properties of the Plateau solutions discussed in Sect. 1 with the properties of the moduli space of Fuchsian systems studied in Sect. 2 to show the existence of a Fuchsian DPW potential for the Lawson surfaces \( \xi_{1, g} \) for every \( g \geq 1 \).

Be aware of the different normalizations here and in [15], where the four singular points are chosen to be
\[
z_1 = e^{i\pi/4}, \quad z_2 = -e^{i\pi/4}, \quad z_3 = e^{i3\pi/4}, \quad z_4 = -e^{i3\pi/4}
\]
with adjusted symmetries \( \delta \) and \( \tau \). These two setups differ only by a Möbius transformation mapping \( z_1, ..., z_k \) to \( p_1, ..., p_k \) and do not affect any properties we have shown.

**Definition 22** A DPW potential on a Riemann surface \( M \) is a closed (i.e., holomorphic) complex linear 1-form
\[
\eta \in \Omega^{1,0}(M, \Lambda \mathfrak{s}(2, \mathbb{C}))
\]
with values in the loop algebra
\[
\Lambda \mathfrak{s}(2, \mathbb{C}) := \{ \xi : S^1 \to \mathfrak{s}(2, \mathbb{C}) \mid \xi \text{ is real analytic} \}
\]
such that \( \lambda \mapsto \lambda \eta(\lambda) \) extends holomorphically to the entire unit disc \( D \subset \mathbb{C} \). Moreover, its residue at \( \lambda = 0 \)
\[
\eta_{-1} := \text{Res}_{\lambda=0}(\eta)
\]
must be a nowhere vanishing and nilpotent 1-form.

To obtain a well-defined surface from a DPW potential, the connections \( d + \eta(\lambda) \) must be unitarizable for all \( \lambda \in S^1 \). The gauge class of an irreducible connection is uniquely determined by the traces of its monodromies along certain closed curves. For unitarizable \( SL(2, \mathbb{C}) \)-connections, these traces are all real and lie in the interval \([-2,2]\). Via the Schwarz reflection principle the DPW potential \( \eta \) restricted to the punctured \( \lambda - \text{disc } D_{1+\varepsilon} \) determines the traces of the monodromies of \( \eta(\lambda) \) for all \( \lambda \in \mathbb{C}^* \) through which the gauge classes of \( d + \eta(\lambda) \) for all \( \lambda \in \mathbb{C}^* \) are determined. This is true even for the possible reducible connections. When \( M = \Sigma \) is the 4-punctured sphere, there exist a particular simple class of DPW potentials for which \( d + \eta(\lambda) \) is a Fuchsian system for all \( \lambda \in D_{1+\varepsilon}^* \subset \mathbb{C}^* \) for
some small $\varepsilon > 0$. These potentials are called \textit{Fuchsian potentials} in the following. Our main Theorem 4 shows the existence of Fuchsian DPW potentials $\eta$ for all Lawson surfaces $\xi_{1,g}$. The map $\lambda \mapsto \eta(\lambda)$ is only well-defined for $\lambda \in \mathbb{D}^*_{1+\varepsilon}$. We do not expect it to extend to a global map from $\mathbb{C}^*$ into the space of Fuchsian systems holomorphically, as unstable logarithmic connections defined on $O(-1) \oplus O(1)$ should accumulate at $\lambda = \infty$.

To obtain the associated family of flat connections of the minimal surface from a DPW potential $\eta$ consider the solution $\Phi_1$ of

$$d\Phi_1 = \Phi_1 \eta$$

with initial condition $\Phi(0) = \Id$. Take the loop group Iwasawa decomposition (see for example [33] or [11])

$$\Phi = FB,$$

where $F$ is unitary for all $\lambda \in \mathbb{S}^1$ and $B$ extends holomorphically to $\lambda = 0$ with upper triangular form and real positive entries on the diagonal. It follows from [11] that $F$ is the extended frame of $f$, i.e.,

$$\nabla^\lambda = d + F^{-1}dF$$

is the associated family of flat connections of $f$. Compare also with [10, Appendix A] for the analogous\footnote{CMC surfaces in $\mathbb{R}^3$ and minimal surfaces in $\mathbb{S}^3$ are related by the Lawson correspondence, and have the same associated family of flat connections. Therefore, all results about normalized potentials also apply for minimal surfaces in $\mathbb{R}^3$.} case of constant mean curvature surfaces in $\mathbb{R}^3$.

### 3.1 The local structure

The following results are a generalization of [7, Proposition 4.2] to arbitrary angles $t \in (0, \frac{1}{4})$. Recall that by Theorem 1 all Gauss-Codazzi data depend real analytically on $t$.

**Lemma 23** The first fundamental form of $f_t: \Sigma \to \mathbb{S}^3$ has a conical singularity with cone angle $4\pi t$ at $p_1, \ldots, p_4$. The Hopf differential $Q$ is given by a meromorphic quadratic differential with first order poles at the four singular points.

**Proof** For rational angles $t \in (0, \frac{1}{4}] \cap \mathbb{Q}$, the claim follows from the discussion in [26, Section 4] by going to the quotient $\Sigma$. For irrational $t$, the claim then follows from the real analytic dependence of the Plateau solutions in $t$, see Theorem 1. \hfill $\Box$

Let $f: U \to \mathbb{S}^3$ be an immersed minimal surface, $w: U \to \mathbb{C}$ be a local holomorphic coordinate, and $Q = Q(dw)^2$ be the Hopf differential, for a holomorphic function $Q: U \to \mathbb{C}$. Note that the induced metric of a minimal surface is real analytic. Hence there is a holomorphic function $G$ in two complex variables such that

$$G(w, \bar{w})dw d\bar{w}$$

is the first fundamental form, i.e., the induced metric, of $f_t$ with respect to the coordinate $w$. Then there is a canonical local choice of a (holomorphic) DPW potential

$$\eta = \begin{pmatrix} 0 & \lambda^{-1}G(w, 0) \\ \frac{Q(w)}{G(w)} & 0 \end{pmatrix} dw$$

(20)

\footnote{CMC surfaces in $\mathbb{R}^3$ and minimal surfaces in $\mathbb{S}^3$ are related by the Lawson correspondence, and have the same associated family of flat connections. Therefore, all results about normalized potentials also apply for minimal surfaces in $\mathbb{R}^3$.}
called the normalized potential, which is gauge equivalent by a positive gauge to its associated family of flat connections $\nabla^\lambda$. By definition, a positive gauge is a holomorphic family $\lambda \mapsto g_\lambda$ of gauge transformations on the Riemann surface which extends homomorphically to $\lambda = 0$.

Note that $G(0, 0) \neq 0$ as we have assumed that $f$ is an immersion. If $f$ is a branched minimal surface with branch point at $w = 0$, then (20) still defines a normalized potential of $f$ as long as (20) is holomorphic at $w = 0$. This last condition is equivalent to the branch order at $w = 0$, i.e., the order of the zero of $w \mapsto G(w, 0)$ at $w = 0$, being less or equal to the umbilic order at $w = 0$, i.e., the order of the zero of $w \mapsto Q(w)$ at $w = 0$.

The Plateau solution $f_t$ on the closed unit disc $\mathbb{D}$ defines a family of flat connections on the trivial rank two bundle over $\mathbb{D}$. Using the Schwarz reflection principle and the symmetries of the surface this gives rise to a family of flat connections $D^\lambda$ over $\Sigma = \mathbb{C}P^1 \setminus \{-1, 0, 1, \infty\}$. Furthermore, $D^\lambda$ can be pulled back by the two-fold covering $\Sigma \to \Sigma$ to give a well-defined family of flat connections

$$D^\lambda = D_t^\lambda$$

over $\Sigma = \mathbb{C}P^1 \setminus \{-1, 0, 1, \infty\}$. The following proposition shows how $D^\lambda$ behaves at the punctures, namely that $\lambda \mapsto D^\lambda$ is locally gauge equivalent to a family of logarithmic connections with weights given by $\pm t$.

**Proposition 24** For each singular point $p_k \in \{-1, 0, 1, \infty\} \subset \mathbb{C}P^1$ there exists an open neighbourhood $U_k \subset \mathbb{C}P^1$ of $p_k$ and a DPW potential $d + \xi_t$ with logarithmic singularity at $p_k$ which is gauge equivalent to $D^\lambda$ on $U_k \setminus \{p_k\}$ by a positive gauge. Moreover, the eigenvalues of the residue of $\xi_t$ at $p_k$ are independent of $\lambda$ and given by $\pm t$.

**Proof** For rational $t = \frac{p}{2q}$, with coprime $p$ and $q$, we proceed analogous to [7, Section 4]: The analytic continuation of the Plateau solution $f_t$ gives rise to a compact branched minimal surface $\tilde{f}_t : M^g \to \mathbb{S}^3$, where $M^g$ is a $q$-fold covering of $\Sigma$ branched over $p_1, \ldots, p_4$. Note that the minimal surface is immersed at $p_k$ if and only if $p = 1$. Let $U \subset M^g$ be a small open subset around $(\text{the preimage of})$ $p_k$ with centered coordinate $w$, and let

$$w \mapsto y = w^q$$

be the $q$-fold covering over $U_k \subset \mathbb{C}P^1$ with centered coordinate $y$. As $t = \frac{p}{2q} \in (0, \frac{1}{4})$, the branch order $p$ at $w = 0$ of the branched minimal surface $\tilde{f}_t$ is less than its umbilical order $q$. Around the branch point $w = 0$ the associated family of flat connection $\nabla^\lambda$ is gauge equivalent to the normalized potential of the form (20) through a positive gauge.

The surface $\tilde{f}_t$ has a rotational $\mathbb{Z}_q$ symmetry at $p_k$. In particular, the first fundamental form and the Hopf differential $Q_t$ are invariant under this $\mathbb{Z}_q$ action and the holomorphic function $z \mapsto G_t(z, 0)$ has a zero of order $p - 1$ at $w = 0$. Let $\mu$ be the map $\omega \mapsto \exp\left(\frac{2\pi i}{q}\right)w$ corresponding to the $\mathbb{Z}_q$ symmetry on $M^g$. Then the normalized potential $\eta_t$ (20) satisfies

$$\mu^* \eta_t = \begin{pmatrix} \exp\left(\frac{\pi ip}{q}\right) & 0 \\ 0 & \exp\left(-\frac{\pi ip}{q}\right) \end{pmatrix} \eta_t \begin{pmatrix} \exp\left(-\frac{\pi ip}{q}\right) & 0 \\ 0 & \exp\left(\frac{\pi ip}{q}\right) \end{pmatrix}.$$

Thus gauging $\eta$ with the gauge

$$g = \text{diag}\left(w^\frac{p}{2}, w^{-\frac{p}{2}}\right)$$

gives $\tilde{\eta}_t = g^{-1} dg + g^{-1} \eta_t g$ which is $\mathbb{Z}_q$-invariant. Therefore, $\tilde{\eta}_t$ is given by the pull-back of an appropriate potential $\xi_t$ on $U_k$ via the branched covering $\pi : w \mapsto w^q = y$. Moreover, the
potentials $\eta_t$ and $\tilde{\eta}_t$ are gauge equivalent to the associated family of flat connections $\nabla_t^\lambda$ by a positive gauge, and therefore the same holds for the potential $\eta_t$ and the associated family $D_t^\lambda$ on the quotient $\Sigma = \mathbb{C}P^1 \setminus \{p_1, \ldots, p_4\}$. As the eigenvalues of the residue of $\tilde{\eta}_t$ at $w = 0$ are $\pm \frac{p}{2q}$ by construction, the eigenvalues of $\xi_t$ are $\pm \frac{p}{2q} = \pm t$.

For irrational $t$ we use the real analytic dependency of the geometric data and of the associated family of connections $D_t^\lambda$ on $\Sigma$, to obtain that $\xi_t$ is a well-defined logarithmic DPW potential, and that its residues have eigenvalues $\pm t$. In fact, for rational $t = \frac{p}{2q}$ we have

$$G(w, 0) = w^q \tilde{G}(w^q) \quad \text{and} \quad \frac{Q(w)}{G(w, 0)} = w^{q-2-p+1} \tilde{Q}(w^q)$$

for holomorphic functions $\tilde{G}$ and $\tilde{Q}$ with $\tilde{G}(0) \neq 0 \neq \tilde{Q}(0)$.

By construction, the holomorphic functions $y \mapsto \tilde{G}(y)$ and $y \mapsto \tilde{Q}(y)$ are determined by the induced metric and the (meromorphic) Hopf differential of the equivariant surface $f_t$ on the 4-punctured sphere $\Sigma = \mathbb{C}P^1 \setminus \{-1, 0, 1, \infty\}$. From (20) and from the definition of the gauge $g$ we obtain on the quotient with local coordinate $y = w^q$, that

$$\xi_t = \frac{p}{2q} = \left( \frac{p}{\tilde{Q}(y)} \lambda^{-1} \tilde{G}(y) - \frac{p}{2q} \right) \frac{dy}{y}. \quad (21)$$

Therefore, the potential (21) extends to all $t \in (0, \frac{1}{2})$ and gives a local DPW potential around the singular point $p_k$ corresponding to $y = 0$ by real analytic dependency of first fundamental form and of the Hopf differential with respect to the parameter $t$.

3.2 The global structure

Analogous to the argumentation in [7, Section 4], there are two families of flat connections for the minimal surface $f_t$, restricted to the local neighbourhoods $U_k$ of a singular point $p_k$. The first is given by the local DPW potential $d + \xi_t$ provided by Proposition 24. The second is given by the associated $\mathbb{C}^*$-family of flat connections $D_t^\lambda$ on $\Sigma = \mathbb{C}P^1 \setminus \{p_1, \ldots, p_4\}$ obtained from the equivariant harmonic map $f_t$ to $\mathbb{S}^3$. By construction of the DPW potential, the two families of flat connections are gauge equivalent on $U_k \setminus \{p_k\}$, $k = 1, \ldots, 4$, by a positive gauge in $\lambda$, i.e., by holomorphic family of gauge transformations which extends holomorphically to $\lambda = 0$. Through these local gauges, we can gauge $D_t^\lambda$ on $U_k$ to the Fuchsian potential $d + \xi_t$ to obtain the following result.

**Theorem 3** On $\Sigma$ there exist a family of flat connections $\nabla_t^\lambda$ with the following properties:

- $\nabla_t^\lambda$ is a $\mathbb{C}^*$-family of logarithmic $\text{SL}(2, \mathbb{C})$-connections with singular part $p_1 + \ldots + p_4$;
- the parabolic weights of $\nabla_t^\lambda$ are $\pm t$;
- $\nabla_t^\lambda$ is unitarizable for every $\lambda \in \mathbb{S}^1$;
- $\nabla_t^\lambda$ determines the complete minimal surface $f_t$ through the DPW recipe, see (19), or [13, Theorem 1.2] and [14, Section 1.4].

The theorem shows that for every $\lambda$ fixed the connection $\nabla_t^\lambda$ is a logarithmic connection on the 4-punctured sphere $\Sigma$. By construction of the minimal surfaces, $\Sigma$ and the induced fundamental forms have the intrinsic symmetries

$$\delta(z) = -\frac{1}{z}, \quad \tau(z) = \frac{1 - z}{z + 1}, \quad \text{and} \quad \sigma(z) = \bar{z}. $$
The symmetries $\delta$ and $\tau$ are same as the ones used in Sect. 2. As shown in [15] the $\sigma$-symmetry relates the connections $\nabla^\lambda_i$ for different (i.e., complex conjugate) $\lambda$-values while the other two symmetries preserve $\lambda$. Therefore, we only obtain that $\nabla^\lambda_i$ is equivariant with respect to $\delta$ and $\tau$, i.e., $\nabla^\lambda_i$ is gauge equivalent to $\delta^*\nabla^\lambda_i$ and $\tau^*\nabla^\lambda_i$.

Moreover, the $\mathbb{C}^*$-family of logarithmic $\text{SL}(2, \mathbb{C})$-connections $\nabla^\lambda_i$ induces a $\mathbb{C}$-family of holomorphic structures $\tilde{\mathcal{F}}^\lambda_i$ on the topologically trivial complex rank 2 bundle over $\mathbb{C}P^1$, since the underlying (0, 1)-parts of the connections extend to $\lambda = 0$ (see also Proposition 4.5. in [7]). Let $V^\lambda_i = (\mathcal{F}, \tilde{\mathcal{F}}^\lambda_i)$ denote the corresponding family of holomorphic bundles. To obtain a Fuchsian DPW potential from $\nabla^\lambda_i$, we first need to show that $V^\lambda_i$ is holomorphically trivial for all $\lambda \in \mathbb{D}_{1+\epsilon}$.

As we have shown in Sect. 2 (Lemmas 10, 17 and Proposition 18), the holomorphic structure of $V^\lambda_i$ is either $\mathcal{O} \oplus \mathcal{O}$, if the parabolic structure is stable, or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$, if the parabolic structure is unstable. If the underlying holomorphic bundle $V^\lambda_i$ is trivial, then the gauge class of $\nabla^\lambda_i$ can be represented by a Fuchsian system

$$\eta^\lambda_i = d + \sum_{k=1}^3 A^i_k(\lambda) \frac{dz}{z - p_k},$$

where $A^i_k(\lambda) \in \mathfrak{sl}(2, \mathbb{C})$ with singular points $p_1 = -1, p_2 = 0, p_3 = 1$ and $p_4 = \infty$. Moreover, being equivariant with respect to $\delta$ and $\tau$ translates into

- $\delta$ symmetry:

$$\delta^* \eta^\lambda_i = D^{-1} \eta^\lambda_i D \quad \text{with} \quad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (22)$$

- $\tau$ symmetry:

$$\tau^* \eta^\lambda_i = C^{-1} \eta^\lambda_i C \quad \text{with} \quad C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (23)$$

by Lemma 16. As in [7, Theorem 4.7] the induced holomorphic structure at $\lambda = 0$ is trivial, i.e., $V_0 \cong \mathcal{O} \oplus \mathcal{O}$. In fact, we obtain modulus $u = -1$ at $\lambda = 0$, but the unstable parabolic structure can be excluded, since for all rational times $t \in (0, \frac{1}{4})$ the zero order of the $(1, 0)$ part of the differential of $f_t$ on $\mathcal{M}_t$ is smaller than the zero order of the Hopf differential on $\mathcal{M}_t$ at the singular points.

We can now state and prove our main theorem.

**Theorem 4** For all $t \in (0, \frac{1}{4})$ there is a Fuchsian DPW potential $\eta_t$, real analytic in $t$,

$$\eta_t = d + \sum_{k=1}^4 A^i_k \frac{dz}{z - p_k}$$

defined on the 4-punctured sphere $\Sigma = \mathbb{C}P^1 \setminus \{-1, 0, 1, \infty\}$ with the following properties:

- there exists $\epsilon > 0$ such that $\eta_t$ is well-defined for all $\lambda \in \mathbb{D}_{1+\epsilon} \setminus \{0\}$;
- the eigenvalues at each of the residues $A^i_k$ are $\pm i$;
- the monodromy representation of $\eta_t$ is unitarizable for all $\lambda \in \mathbb{S}^1$;
- the potential is symmetric, i.e., $\delta^* \eta = \eta.D, \quad \tau^* \eta = \eta.C$.

Moreover, the minimal surface $f^t$ corresponding to the potential $\eta_t$ is the unique analytic continuation of the Plateau solution $f_t$ with respect to the geodesic polygon $\Gamma_t$. In particular, for $t = \frac{1}{2(g+1)}$, $g \in \mathbb{N}_{\geq 1}$, the analytic continuation of $f^t$ is the Lawson surface $\xi_{1,g}$ of genus $g$.
Proof We start with the Plateau solution for $\Gamma_t$ and consider the family of flat logarithmic connections $\widetilde{V}_t^\lambda$ on $\Sigma$ as constructed in Theorem 3. The proof consists of two steps. We first show that the induced holomorphic bundle is $V^\lambda_t = \mathcal{O} \oplus \mathcal{O}$ for all $\lambda \in \mathbb{D}_{1+\varepsilon} \setminus \{0\}$ for an appropriate $\varepsilon > 0$, and for every $t \in (0, \frac{1}{4})$ using continuity arguments. In the second step, we then show that we can gauge $\widetilde{V}_t^\lambda$ into the symmetric normal form of Lemma 16 with holomorphic coefficients in $\lambda$ on $\mathbb{D}_{1+\varepsilon} \setminus \{0\}$, i.e., such that there are no apparent singularities in $\lambda$, and the family has no ambiguities in $\lambda$.

Step 1: For $t \sim 0$ and $t \sim \frac{1}{4}$ the bundle type $V^\lambda_t$ is shown to be $\mathcal{O} \oplus \mathcal{O}$ for all $\lambda \in \mathbb{D}_{1+\varepsilon}$ in [13, 15], respectively. In fact, for $t \sim 0$, Fuchsian DPW potentials on the 4-punctured sphere solving the intrinsic and extrinsic closing conditions via the implicit function theorem are constructed in [15] (for the precise statement see also Sect. 3.3 below). Moreover, it is shown that the corresponding surfaces are the Lawson surfaces [15, Theorem 3 (5)]. For $t \sim \frac{1}{4}$, it is shown in [13, Example 4.1] that the associated family of flat connections of the surface obtained from the Plateau solution for $\Gamma_t$ can be obtained from parametrized families of line bundle connections on the square torus by deforming the spectral data of the Clifford torus. The spectral data for the Clifford torus are well-known (see [18, Section 6] or [13, Section 2]). Recall that the bundle type of a logarithmic connection is not $\mathcal{O} \oplus \mathcal{O}$ if and only if the underlying parabolic structure is unstable. But unstable parabolic structures can only occur at line bundles which are holomorphically trivial (see for example [13, Remark 3.1]). But for the spectral data of the Clifford torus used in [13, Section 2], the trivial holomorphic line bundle only occurs at $\lambda = 0$ inside the unit disc (but there are infinitely many outside the unit disc). At $\lambda = 0$, the parabolic structure is always stable, and the result follows from continuity of the spectral data [13, Theorem 4.1].

We want to show that the bundle type remains $\mathcal{O} \oplus \mathcal{O}$ for all $\lambda \in \mathbb{D}_{1}$ and all $t \in (0, \frac{1}{4})$. Moreover, the bundle type at $\lambda = 0$ is $V^\lambda_{\lambda=0} = \mathcal{O} \oplus \mathcal{O}$ for every $t \in (0, \frac{1}{4}]$. Recall from Sect. 2.4 and Proposition 18 that, for every $t$ fixed, the function $u_t : \mathbb{C}^* \to \mathbb{C}P^1$,

$$\lambda \in \mathbb{C}^* \mapsto u_t(\lambda) \equiv u(\widetilde{V}_t^\lambda) \in \mathbb{C}P^1$$

is holomorphic, well-defined and extends holomorphically to $\lambda = 0$ with value $u_t(0) = -1$, as the underlying parabolic structure extends to $\lambda = 0$ with a nilpotent strongly parabolic Higgs field. The stability of the parabolic structure extends to $\lambda = 0$ with the uniqueness part of the Mehta-Seshadri theorem. Then $\lambda \mapsto \widetilde{V}_t^\lambda$ would be a constant map which contradicts, among others things, the extrinsic closing condition at the Sym points $\lambda = \pm 1$. In fact, for all rational $t$ the trivial monodromy of $\nabla^\lambda_{\pm 1}$ on the compact genus surface corresponds to a totally reducible monodromy on the 4-punctured sphere, as the quotient Riemann surface is obtained by the abelian group $\mathbb{Z}_q$, where $t = \frac{p}{q}$ with coprime $p, q \in \mathbb{N}$. Hence, the corresponding parabolic structure on the 4-punctured sphere is one of the three strictly semi-stable parabolic structures while the parabolic structure is stable at $\lambda = 0$ and given by $u = -1$. By continuity, and because the set of strictly semi-stable parabolic structures is finite, it follows for all $t \in (0, \frac{1}{4})$ that the value $u_t(\pm 1) \neq u_t(0)$, so that $u_t$ is not constant. This implies that, for each $t$, the values of $\lambda \in \mathbb{C}$ for which $u_t(\lambda) = -1$ are discrete. Due to the real analyticity of the Plateau solutions in $t$, the map $t \mapsto \widetilde{V}_t^\lambda$ is also real analytic in $t$.

By Lemma 21 logarithmic connections which are not Fuchsian are characterized by the property that their underlying parabolic structure is unstable. For $t \sim 0$ and $t \sim \frac{1}{4}$, we
have seen that there are no unstable parabolic structures inside the unit disc. By Mehta-Seshadri, logarithmic connections inducing unstable parabolic structures are not unitarizable. Thus, by deforming $t$, unstable parabolic structures cannot cross the unit $\lambda$-circle (where all connections must be unitarizable) in a continuous deformation.

Proposition 18 shows that the modulus for an unstable connection $\tilde{\nabla}_t^\lambda$ must be $u = -1$. Thus it remains to show that unstable structures cannot arise as limits of stable structures with $u = -1$ in our setup. It is important to recall that the family of logarithmic connections $\tilde{\nabla}_t^\lambda$ exist and is well-defined for all $\lambda \in \mathbb{C}^*$ and $t \in (0, \frac{1}{4})$. Consider therefore a continuous sub family of $\tilde{\nabla}_t^\lambda$ given by $t \in (a, b] \subset (0, \frac{1}{4}) \mapsto \lambda_t \in \mathbb{C}$ such that

$$u_t(\lambda_t) = -1.$$ We want to show that if the parabolic structure of $\tilde{\nabla}_t^\lambda$ is stable for all $t < b$, it is also stable at $t = b$.

Since $u_t$ is holomorphic in $\lambda$ (by (7), and Proposition 18) and non-constant, there exist a punctured disc around $\lambda_{t=b}$ such that $s_t = s(\tilde{\nabla}_t^\lambda_t)$ is well-defined and holomorphic, see (11). Therefore, the family of holomorphic functions

$$r(t) := (u_t + 1) \cdot s_t$$

is continuous in $t$. Since the parabolic structure of $\tilde{\nabla}_t^\lambda$ is stable for $t < b$, $s_t \in C$ for each $t < b$ and we obtain

$$((u_t + 1) \cdot s_t)(\lambda_t) = 0.$$ By continuity this also holds at $t = b$, but if the parabolic structure at $t = b$ would be unstable, then (17) would give

$$0 = ((u_b + 1) \cdot s_b)(\lambda_b) = 1 - 4b$$

contradicting $b < \frac{1}{4}$.

Remark 25 This conclusion heavily depends on the strict inequality $t < \frac{1}{4}$. In fact, for the 2-lobed Delaunay tori at $t = \frac{1}{4}$, there exists $\lambda_s$ in the punctured unit disc with $u(\lambda_s) = -1$. (The spectral parameter $\lambda_s$ is the unique branch value of the spectral curve of the Delaunay torus inside the punctured unit disc). Then, as shown in [13, Theorem 4.2] in the setup of spectral curves, one can choose to deform the corresponding spectral data in direction of higher genus CMC surfaces in different ways by specifying to have either an unstable or a stable parabolic structure at $\lambda_s$.

Step 2: To show that the Fuchsian connections $\tilde{\nabla}_t^\lambda$ can be parametrized with holomorphic coefficients in $\lambda$ on $\mathbb{D}_{1+\varepsilon} \setminus \{0\}$ with a simple pole at $\lambda = 0$ only, we proceed as in the proof of [17, Theorem 8]:

We assume for a moment that for every $\lambda_0 \in \mathbb{D}_1$ we can find an open neighborhood $U_{\lambda_0} \subset \mathbb{C}$ of the $\lambda-$plane such that $\tilde{\nabla}_t^\lambda$ is, locally around $\lambda_0$, gauge equivalent to a family of connections $\nabla_{U_{\lambda_0}}^\lambda$ with the desired symmetries (and holomorphic coefficients in $\lambda$) as in Lemma 16. Since the closed disc $\mathbb{D}_1$ is compact, there exist finitely many points $\lambda_l \in \mathbb{D}_1$ and finitely many families $\nabla_{U_{\lambda_l}}^\lambda$ in the symmetric normal form such that $\mathbb{D}_1 \subset \bigcup_l U_{\lambda_l}$. On the intersection $U_{\lambda_l} \cap U_{\lambda_k}$ the connections $\nabla_{U_{\lambda_l}}^\lambda$ and $\nabla_{U_{\lambda_k}}^\lambda$ are by construction gauge equivalent. This gives rise to transition functions

$$g_{lk} : U_{\lambda_l} \cap U_{\lambda_k} \rightarrow SL(2, \mathbb{C}).$$
Due to the uniqueness in Lemma 16 the image of $g_{lk}$ lies in fact the subgroup generated by the matrices $C$ and $D$. Moreover, these transition functions \{ $g_{lk}$ \} define a cocycle on $D_{1+\epsilon}$. This cocycle is integrable, as $D_{1+\epsilon}$ is simply connected, which gives rise to the desired DPW potential on all of $D_{1+\epsilon}^\ast$.

It remains to show the assumption, i.e, the existence of local families $\nabla^{\lambda}_{U_{\lambda_0}}$ for every $\lambda_0 \in D_1$. If $\nabla^{\lambda_0}$ is irreducible, this follows from the uniqueness part of Lemma 16 and the fact that the stabiliser of an irreducible connection is $\pm \text{Id}$. At $\lambda = 0$ the existence of $\nabla^{\lambda_0}_{U_0}$ follows from the fact that the induced parabolic structure is stable and the arguments for irreducible connections in the proof of Lemma 16 carry over verbatim.

For the (finitely many) $\lambda_0$ for which $\nabla^{\lambda_0}_{\tilde{U}_{\lambda_0}}$ is reducible, i.e., where the connection is given by the direct sum of line bundle connections, we proceed as follows: consider the 4-fold covering $\pi : \Sigma \to \mathbb{CP}^1$ by taking the quotient with respect to $\delta$ and $\tau$. Using the push-forward construction (see [5]) one obtains logarithmic connections on a 4-punctured sphere with local eigenvalues $\pm \frac{1}{4}$ at 3 of the 4 singular points, and $\pm t$ at fourth singular point (which is the image of $p_1, \ldots, p_4$ under $\pi$). On the quotient, as $t \in (0, \frac{1}{4})$, there are no reducible connections, see [12, Biswas conditions]. Therefore, we can proceed as in the irreducible case on the quotient to obtain $\hat{\nabla}^{\lambda}_{\lambda_0}$ of the form of Lemma 16, well-defined in a family locally around $\lambda_0$.

3.3 The form of the symmetric potential near $t = 0$

We end the paper by specifying the symmetric DPW potential $\eta$ for the Lawson surfaces $\xi_{1,g}$. Following [15, Section 2.1], $\eta$ is of the form

$$\eta = \left( \begin{array}{c}
-\frac{4a z}{z^4 + 1} \\
\frac{2\sqrt{2}(b(z^2-1)-c(z^2+1))}{z^4 + 1} \\
\frac{4a z}{z^4 + 1}
\end{array} \right) dz$$

for holomorphic functions $a, b, c : D_{1+\epsilon}^\ast \to \mathbb{C}$ with first order poles at $\lambda = 0$, and $a^2 - b^2 - c^2 = -t^2$ with $t = \frac{1}{2g+2}$. Moreover, the residue of the DPW potential at $\lambda = 0$ is nilpotent (and non-zero if $t > 0$) which is equivalent to

$$\text{Res}_{\lambda=0} \left( -\frac{b^2}{c^2} \right) = -1.$$ 

At $t = 0$ we have by [15]

$$a = 0 = b = c.$$ 

In [15, Section 5] we have given an iterative algorithm to compute the Taylor expansion of $a_t(\lambda), b_t(\lambda)$ and $c_t(\lambda)$ in $t$ at $t = 0$. It turns out that the $n$-th $t$-derivative of $a_t, b_t$ and $c_t$ at $t = 0$ are Laurent polynomials in $\lambda$ obtained from solving (non-degenerate) finite-dimensional linear systems with coefficients given by multi-polylogarithms. For example,
the first order derivatives are given by

\[
\begin{align*}
\dot{a} &= \frac{1}{2} (\lambda^{-1} - \lambda) \\
\dot{b} &= -\frac{1}{2 \sqrt{2}} (\lambda^{-1} + \lambda).
\end{align*}
\]  

At \( t = \frac{1}{4} \) the potential can be computed explicitly in terms of elliptic functions.

**Acknowledgements** We thank Reiner Schätzle for providing the ideas to prove Theorem 1. Moreover, we thank Martin Traizet for various fruitful discussions. We would like to thank the anonymous referees for their time, thorough comments and excellent advice. The authors have been supported by the *Deutsche Forschungsgemeinschaft* within the priority program *Geometry at Infinity*.

**Data availability** The manuscript has no associated data.

**References**

1. Alekseev, A., Malkin, A.: Symplectic structure of the Moduli space of flat connections on a Riemann surface. Commun. Math. Phys. **169**, 99–119 (1995)

2. Atiyah, M.F., Bott, R.: The Yang–Mills equations over Riemann surfaces. Philos. Trans. R. Soc. Lond. A **308**, 523–615 (1983)

3. Audin, M.: Lectures on gauge theory and integrable systems. In: Hurtubise, J., Lalonde, F., Sabidussi, G. (eds.) Gauge Theory and Symplectic Geometry. NATO ASI Series, vol. 488. Springer, Dordrecht (1997). [https://doi.org/10.1007/978-94-017-1667-3-1](https://doi.org/10.1007/978-94-017-1667-3-1)

4. Biquard, O.: Fibrés paraboliques stables et connexions singulières plates. Bull. Soc. Math. Fr. **119**, 231–257 (1991)

5. Biswas, I.: Parabolic bundles as orbifold bundles. Duke Math. J. **88**(2), 305–326 (1997)

6. Bobenko, A.I.: All constant mean curvature tori in \( \mathbb{R}^3, S^3, H^3 \) in terms of theta-functions. Math. Ann. **290**(2), 209–245 (1991)

7. Bobenko, A.I., Heller, S., Schmitt, N.: Constant mean curvature surfaces based on fundamental quadrilaterals. Math. Phys. Anal. Geom. **24**, 37 (2021)

8. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, vol. 223. Springer, Berlin (2011)

9. Deligne, P.: Équations différentielles à points singuliers réguliers. Lecture Notes in Mathematics, vol. 163. Springer, Berlin (1970)

10. Dorfmeister, J., Haak, G.: Meromorphic potentials and smooth surfaces of constant mean curvature. Math. Z. **224**(4), 603–640 (1997)

11. Dorfmeister, J., Pedit, F., Wu, H.: Weierstrass type representation of harmonic maps into symmetric spaces. Commun. Anal. Geom. **6**(4), 633–668 (1998)

12. Heller, L., Heller, S.: Abelianisation of Fuchsian systems and applications. J. Symplectic Geom. **14**(4), 1059–1088 (2016)

13. Heller, L., Heller, S., Schmitt, N.: Navigating the space of symmetric CMC surfaces. J. Differ. Geom. **110**(3), 413–455 (2018)

14. Heller, L., Heller, S., Traizet, M.: Area estimates for high genus Lawson surfaces. J. Differ. Geom. (to appear). [arXiv:1907.07139](https://arxiv.org/abs/1907.07139)

15. Heller, L., Heller, S., Traizet, M.: Complete families of embedded high genus CMC surfaces in the 3-sphere. arXiv:2108.10214

16. Heller, S.: Lawson’s genus two minimal surface and meromorphic connections. Math. Z. **274**, 745–760 (2013)

17. Heller, S.: A spectral curve approach to Lawson symmetric CMC surfaces of genus 2. Math. Annalen **360**(3), 607–652 (2014)

18. Hitchin, N.: Harmonic maps from a 2-torus to the 3-sphere. J. Differ. Geom. **31**(3), 627–710 (1990)

19. Kapouleas, N.: Minimal surfaces in the round three-sphere by doubling the equatorial two-sphere. I. J. Differ. Geom. **106**(3), 393–449 (2017)

20. Kapouleas, N., Yang, S.D.: Minimal surfaces in the three-sphere by doubling the Clifford torus. Am. J. Math. **132**(2), 257–295 (2010)

21. Kapouleas, N., Wiygul, D.: The index and nullity of the Lawson surfaces \( \xi_{R,1} \). Camb. J. Math. **8**(2), 363–405 (2020)

22. Karcher, H., Pinkall, U., Sterling, I.: New minimal surfaces in \( S^3 \). J. Differ. Geom. **28**(2), 169–185 (1988)
23. Kim, S., Wilkin, G.: Analytic convergence of harmonic metrics for parabolic Higgs bundles. J. Geom. Phys. 127, 55–67 (2018)
24. Kusner, R.: Comparison surfaces for the Willmore problem. Pac. J. Math. 138(2), 317–345 (1989)
25. Kuwert, E., Li, Y., Schätzle, R.: The large genus limit of the infimum of the Willmore energy. Am. J. Math. 132(1), 37–51 (2010)
26. Lawson, H.B.: Complete minimal surfaces in $S^3$. Ann. Math. (2) 92, 335–374 (1970)
27. Loray, F., Saito, M.-H.: Lagrangian fibrations in duality on Moduli spaces of rank 2 logarithmic connections over the projective line. IMRN 2015(4), 995–1043 (2015)
28. Marques, F.C., Neves, A.: Morse index of multiplicity one min-max minimal hypersurfaces. Adv. Math. 378, Paper No. 107527 (2021)
29. Mehta, V.B., Seshadri, C.S.: Moduli of vector bundles on curves with parabolic structures. Math. Ann. 248, 205–239 (1980)
30. Meneses, C.: Remarks on groups of bundle automorphisms over the Riemann sphere. Geom. Dedicata 196(1), 63–90 (2018)
31. Pinkall, U., Sterling, I.: On the classification of constant mean curvature tori. Ann. Math. (2) 130(2), 407–451 (1989)
32. Pirola, G.: Monodromy of constant mean curvature surface in hyperbolic space. Asian J. Math. 11(4), 651–669 (2007)
33. Pressley, A., Segal, G.: Loop Groups. Oxford Mathematical Monographs. The Clarendon Press, New York (1986)
34. Simpson, C.: Harmonic bundles on noncompact curves. J. Am. Math. Soc. 3(3), 713–770 (1990)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.