Quantum Computation and Quadratically Signed Weight Enumerators

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We prove that quantum computation is polynomially equivalent to classical probabilistic computation with an oracle for estimating the value of simple sums, \textit{quadratically signed weight enumerators}. The problem of estimating these sums can be cast in terms of promise problems and has two interesting variants. An oracle for the unconstrained variant may be more powerful than quantum computation, while an oracle for a more constrained variant is efficiently solvable in the one-bit model of quantum computation. Thus, problems involving estimation of quadratically signed weight enumerators yield problems in BQP (bounded error quantum polynomial time) that are distinct from the ones studied so far, include a canonical BQP complete problem, and can be used to define and study complexity classes and their relationships to quantum computation.

I. INTRODUCTION

It is widely believed that quantum computers are more efficient than classical deterministic or probabilistic computers. For example, there is an efficient algorithm for factoring integers on a quantum computer, while no such algorithm is known for classical computers \cite{10,12}. Unlike numerous other models more efficient than traditional computation, quantum computation appears to be robustly implementable using reasonable physical devices \cite{11,2,5,7,8}.

To better understand the power of quantum computers, it is desirable to find specific problems to which the problem of simulating a quantum computer on a classical computer can be reduced. In principle, such problems can be extracted from the representation of the amplitudes of the desired answer of a quantum algorithm as a sum over paths of transition amplitudes. (The sum is over all possible evolutions of the computational states consistent with the steps of the quantum algorithm.) This representation can be used to prove that quantum computers can be simulated on classical computers with exponential overhead in time and polynomial overhead in space \cite{3,1,4}. The resulting problems can be simplified by using the fact that transition amplitudes can be restricted to a small set of rational numbers \cite{3}. However, these path sums are still too general for use as canonical problems whose solutions suffice for efficient simulation of quantum computers. Furthermore, it is not clear how to modify the path sums to represent related computational models such as the one-bit model of quantum computation \cite{6}. This model differs from standard quantum computation in that the initial state is random except for one quantum bit, and measurement is destructive. The goal of this paper is to remedy this situation by relating both the standard and the one-bit model of quantum computation to problems of estimating certain sums related to weight generating functions for binary codes. Some of these estimation problems can be cast as promise problems with the property that oracles for these problems can be used to efficiently predict the answers of quantum algorithms. Conversely, since there are efficient quantum algorithms and one-bit quantum algorithms for solving such promise problems, they define a new class of problems in BQP that are apparently hard for classical computation.

\textbf{Quadratically Signed Weight Enumerators.} A general quadratically signed weight enumerator is of the form

\begin{equation}
S(A,B,x,y) = \sum_{b:Ab=0} (-1)^{b^T B b} x^{[b]} y^{n-[b]},
\end{equation}

where \( A \) and \( B \) are \( 0\)-\( 1 \)-matrices with \( B \) of dimension \( n \) by \( n \) and \( A \) of dimension \( m \) by \( n \). The variable \( b \) in the summand ranges over \( 0\)-\( 1 \)-column vectors of dimension \( n \), \( b^T \) denotes the transpose of \( b \), \( [b] \) is the weight of \( b \) (the number of ones in the vector \( b \)), and all calculations involving \( A \), \( B \) and \( b \) are modulo 2. The absolute value of \( S(A,B,x,y) \) is bounded by \((|x|+|y|)^n\)\". In general, one can consider the computational problem of evaluating these sums. Here we consider the following cases, which will be related to quantum computation:

\textbf{Problem 1} Given that \( k \) and \( l \) are positive integers, evaluate \( S(A,B,k,l) \).

Problem 1 is in the class \#P \cite{9}.

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Problem 2 Given that $k$ and $l$ are positive integers and the promise $|S(A, B; k, l)| \geq (k^2 + l^2)^{n/2}/2$, determine the sign of $S(A, B; k, l)$.

The next two problems require that $A$ is square. Let $\text{lwtr}(A)$ denote lower triangular part of $A$, which is the matrix obtained from $A$ by setting to zero all the entries on or above the diagonal. Let $\text{diag}(A)$ denote the diagonal matrix whose diagonal is the same as that of $A$. $I$ denotes the identity matrix. For matrices $C$ and $D$ with the same number of columns, $[C; D]$ denotes the matrix obtained by placing $C$ above $D$.

Problem 3 Given that $\text{diag}(A) = I$, $k$ and $l$ are positive integers, and the promise $|S(A, \text{lwtr}(A), k, l)| \geq (k^2 + l^2)^{n/2}/2$, determine the sign of $S(A, \text{lwtr}(A), k, l)$.

Problem 4 Given that $\text{diag}(A) = I$, $k$ and $l$ are positive integers and the promise $|S([A; A^T], \text{lwtr}(A), k, l)| \geq (k^2 + l^2)^{n/2}/2$, determine the sign of $S([A; A^T], \text{lwtr}(A), k, l)$.

We will show that Problem 3 is BQP complete, so that classical probabilistic computation with an oracle for this problem is polynomially equivalent to quantum computation. Problem 4 is solvable efficiently using a one-bit quantum algorithm. In the last two problems, the integers $k$ and $l$ can be restricted to 4 and 3, respectively, without affecting their hardness with respect to polynomial reductions (using classical deterministic algorithms).

II. MODELS OF QUANTUM COMPUTATION

An easy-to-use model of quantum computation consists of a classical random access machine (RAM) with access to any number of addressable quantum bits (qubits) that are initially in the state $|0\rangle$. The qubits can be manipulated by one of a finite set of quantum gates and by measurement. This model is called the quantum random access machine (QRAM). For introductions to the basic notions of quantum computing, see [4].

The basic states of qubit $A$ are denoted by $|0\rangle_A$ and $|1\rangle_A$. These are elementary ket symbols. The basic states of a collection of qubits are obtained by formally multiplying the basic states of each qubit. For example, $|0\rangle_A|1\rangle_B|0\rangle_C$ is a basic state of qubits $A$, $B$ and $C$. We use the convention $|010\rangle_{ABC} = |0\rangle_A|1\rangle_B|0\rangle_C$. Qubit labels are omitted when they can be inferred from the context. The (pure) state space of a collection of qubits consists of the unit complex linear combinations (called superpositions) of their basic states.

Quantum gates act on qubits by applying a unitary operator to the current state. An example is the NOT gate, which in matrix form is given by the Pauli matrix $\sigma_x$. The NOT gate applied to qubit $A$ is denoted by $\sigma_x^{(A)}$ and has the effect of flipping the binary label associated with $A$ in the basic states. The effect on superpositions is obtained by linear extension.

To describe gates and their effects we can use the bra-ket conventions. In addition to the ket symbols already introduced, we introduce bra symbols $\langle b|$ for qubit $X$ with $b = 0$ or $b = 1$. Formal linear combinations of bra and ket symbols can be multiplied using distributivity and associativity rules together with the following:

1. Bras and kets with different labels commute.
2. $\chi\langle a|b\rangle_A = \delta_{a,b}$.
3. Expressions involving two kets or two bras with the same label next to each other are illegal.

If $\phi$ is a bra-ket expression, then so is $\phi^\dagger$, which is obtained by conjugating the complex coefficients, reversing the order of elementary products and changing kets into bras and vice-versa. For example $|0\rangle^\dagger_B = \langle 0|_B$.

With these conventions, we can write the NOT gate acting on qubit $A$ as

$$\sigma_x^A = |0\rangle_A\langle 1| + |1\rangle_A\langle 0|,$$

where $\sigma_x^A$ is intended to be applied to a state by multiplication on the left. The elementary gates available to a QRAM are unitary operators acting on one or two qubits. The operator $U$ is unitary if $U^\dagger U$ acts as the identity. Note that in the bra-ket notation, there are many ways of writing the identity operator. Examples include

$$1 = |0\rangle_A\langle 0| + |1\rangle_A\langle 1| = \sum_b |b\rangle_{AB\ldots AB}\langle b|.$$


The elementary gates to be used here are based on exponentials of products of the Pauli operators $\sigma_x$ and

$$\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|.$$  \hfill (5)

(6)

For qubits labeled by $1, \ldots, n$, a general product of Pauli operators is denoted by $\sigma_b$, where $b$ consists of $n$ pairs of bits and is defined by

$$\sigma_b = \prod_{i=1}^{n} \sigma_{b_i}^{(i)},$$ \hfill (7)

with the conventions $\sigma_{00} = I$, $\sigma_{01} = \sigma_x$, $\sigma_{11} = \sigma_y$ and $\sigma_{10} = \sigma_z$. The weight of $\sigma_b$ is the number of non-zero pairs of bits in $b$. A rotation by $\theta$ around $\sigma_b$ is the operator

$$e^{-i\sigma_b \theta/2} = \cos(\theta/2) - i \sin(\theta/2)\sigma_b.$$ \hfill (8)

A complete set of one and two qubit gates can be obtained from the set of rotations by $\pm 2 \arccos(4/5)$ around operators of weight at most two $\{ \sigma_x, \sigma_y, \sigma_z \}$. A polynomially equivalent model is obtained by allowing such rotations around any product of Pauli operators. We adopt this model.

A general QRAM may at any time measure a qubit and act according to the measurement outcome. Suppose the QRAM measures qubit $A$. In bra-ket notation we can expand $\psi = |0\rangle_\psi \psi_0 + |1\rangle_\psi \psi_1$, with $\psi_0$ and $\psi_1$ not containing any kets labeled $A$. Let $p = \psi_0^\dagger \psi_0$ and $q = \psi_1^\dagger \psi_1$. Then $p$ and $q$ are positive reals with $p + q = 1$. The effect of the measurement projects the qubits into the state $|0\rangle_p (1/p) \psi_0$ with probability $p$ and into the state $|1\rangle_q (1/q) \psi_1$ with probability $q$. The answer of the measurement is 0 in the former case, and 1 in the latter, and the answer is placed into a (classical) bit register. We simplify this model by permitting only measurements of qubit 1 and assuming that all the qubits used so far are lost after the measurement. This simplified model is polynomially equivalent to the general one with respect to bounded error algorithms for promise problems.

A version of the one-bit model of quantum computation is given by the Q1RAM, which differs from the (simplified) QRAM only in that the initial state of the qubits has qubit 1 in state $|0\rangle$ and all the other qubits in a state picked uniformly at random from the basic states. A measurement of qubit 1 also re-initializes the qubits. Surprisingly, there are problems for which no efficient classical algorithm is known and that can be solved efficiently using a Q1RAM, while Q1RAMs are not as powerful as QRAMs with respect to oracles.

III. SIMULATING QUANTUM COMPUTERS

As described above, both models of quantum computation can be thought of as being based on classical deterministic RAMs with access to certain oracles. The input to the oracles is a sequence of quantum gates and the answer is 0 or 1 with the appropriate probability distribution. A fundamental question is whether a probabilistic RAM can efficiently implement these oracles. Note that the output probability distribution in such an implementation can deviate from the correct one by $O(\epsilon/N)$, where $N$ is the total number of oracle calls, without significantly affecting the output of an algorithm.

The problems solved by the oracles can be cast in terms of promise problems. In particular, the following promise problems can be solved efficiently by quantum computers and one bit quantum computers, respectively:

Problem 5 Given a quantum network and the promise that after applying the quantum network to the initial state $|00\ldots\rangle$, the probability $p$ that the first qubit is in state $|1\rangle$ satisfies $|2p - 1| \geq 1/2$, determine the sign of $2p - 1$.

Problem 6 Given a quantum network and the promise that after applying the quantum network to the initial state with the first qubit in state $|0\rangle$ and the others random, the probability $p$ that the first qubit is in state $|1\rangle$ satisfies $|2p - 1| \geq 1/2$, determine the sign of $2p - 1$.

Theorem 7 A probabilistic RAM with access to an oracle for Problem 5 can efficiently simulate a quantum computer.

We do not know whether a similar theorem holds for the one-bit model of quantum computation with respect to Problem 5.
Proof. Suppose that we are given a quantum network \( G \). The goal is to produce a random bit with probability distribution close to the output qubit’s distribution for \( G \). The first step is to use an oracle for Problem 5 to estimate the probability that the output qubit is in state \( |1\rangle \). To do so we design new quantum networks \( G_{x,N} \). \( G_{x,N} \) applies \( G \) to \( (N/\epsilon)^2 \) independent sets of qubits, then uses ancillas to (reversibly) determine whether the fraction of \( |1\rangle \)'s in the \( (N/\epsilon)^2 \) output qubits is greater than \( x \) or not, placing the answer into its output qubit. The oracle is queried for \( G_{x,N} \). By using binary search on \( x \), the desired probability can be determined to within \( O(\epsilon/N) \) in \( O(\log(N/\epsilon)) \) queries. The probabilistic RAM then simulates the output of the quantum network by producing a random bit with this estimated bias.

IV. REDUCTION TO QUADRATICALLY SIGNED WEIGHT ENUMERATORS

For a quantum network \( G \), let \( U(G) \) be the unitary operator defined by \( G \). Observe that without loss of generality, we can restrict \( G \) to have only real gates [3]. (Other networks can be simulated by real networks using one ancilla qubit to keep track of phases, see Appendix A.) These are gates involving rotations around \( \sigma_b \)'s with an odd number of factors of the form \( \sigma_y \). The gate set is still complete if we assume also that the orientation of the rotation is positive if the number of \( \sigma_y \) is 1 mod(4) and negative otherwise.

The results and arguments in [6] show that Problems 5 and 6 are equivalent to problems of estimating specific coefficients of an operator representation of \( U(G) \). In particular, for networks with real gates only, they correspond to the following two problems:

**Problem 8** Promise: \(|\langle 00\ldots|U(G)|00\ldots\rangle| \geq 1/2 \). Determine the sign of \( \langle 00\ldots|U(G)|00\ldots\rangle \).

**Problem 9** Let \( n \) be the number of qubits used by \( G \). Promise: \(|1^{2n}trU(G)| \geq 1/2 \). Determine the sign of \( trU(G) \).

Let \( G \) be determined by the sequence of gates \( G_1,\ldots,G_N \), so that \( U(G) = G_N G_{N-1} \ldots G_1 \). Each gate is of the form

\[
G_k = \frac{4}{5} \pm \frac{3i}{5} \sigma_{b_k},
\]

where \( b \) contains an odd number of pairs of the form 11 and the sign (±) depends on the number of \( \sigma_y \) in \( \sigma_{b_k} \). Let \( |b|_y \) be the number of \( \sigma_y \) occurring in \( \sigma_b \) and define \( \bar{\sigma}_b = (\pm i)^{|b|_y} \sigma_b \). Then, because of the condition on the signs of the rotations,

\[
G_k = \frac{4}{5} + \frac{3i}{5} \bar{\sigma}_{b_k}.
\]

To expand the product of the \( G_k \), we need to determine the multiplication rules for the \( \bar{\sigma}_b \). The property that \( b \) has an odd number of pairs of the form 11 is defined by \( b^T B b = 1 \), where \( B \) is block diagonal with two-by-two blocks given by

\[
B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Direct verification shows that the multiplication rules are given by

\[
\bar{\sigma}_{b_1} \bar{\sigma}_{b_2} = (-1)^{|b_1|_y B_{b_2} \bar{\sigma}_{b_1} + b_2},
\]

where the sum in the subscript is bit-by-bit, modulo two. \( U(G) \) can now be expanded as follows:

\[
U(G) = \prod_{k=N}^1 G_k
\]

\[
= \prod_{k=N}^1 (4 + 3\bar{\sigma}_{b_k})/5
\]

\[
= \frac{1}{5^N} \sum_a (-1)^a \text{tr}(H^T BH)^a 4^{|a|} 3^{N-|a|} \bar{\sigma}_{H_a}.
\]
The last step requires distributing the product over the sum and using the multiplication rules for the $\delta_k$ operators. $H$ is the matrix whose columns are the $b_k$. The sum is over all 0-1 column vectors $a$ of dimension $N$. The bits of the vector $a$ correspond to which of the two terms of each sum in the product are chosen to get a summand of the expansion. The first bit of $a$ determines the term of the factor $G_1$, and so on. Note that every matrix $H$ of dimension $2n$ by $N$ with the property that $\text{diag}(H^T BH) = I$ can occur in this expression. The coefficients of $U(G)$ to be estimated in Problems 8 and 9 are

$$\langle 00\ldots|U(G)|00\ldots \rangle = \frac{1}{5^N} \sum_{a:BHa=0} (-)^a \text{tr}(H^T BH)^a 4^{|a|} 3^{N-|a|}$$  \hspace{1cm} (16)

$$\frac{1}{2^n} \text{tr}U(H) = \frac{1}{5^N} \sum_{a:H_a=0} (-)^a \text{tr}(H^T BH)^a 4^{|a|} 3^{N-|a|}.$$  \hspace{1cm} (17)

Here the condition $BHa = 0$ means that $\sigma_{Ha}$ has no $\sigma_x$ or $\sigma_y$ factors.

It remains to obtain the simpler forms of Problems 8 and 9. Let $H_0$ and $H_1$ be the two $n$ by $N$ matrices obtained from the even and the odd rows of $H$, respectively (starting the count at zero, so that the first row is considered even). The above sums are then equivalent to

$$\langle 00\ldots|U(G)|00\ldots \rangle = \frac{1}{5^N} \sum_{a:H_1a=0} (-)^a \text{tr}(H_1^T H_1)^a 4^{|a|} 3^{N-|a|}$$  \hspace{1cm} (18)

$$\frac{1}{2^n} \text{tr}U(G) = \frac{1}{5^N} \sum_{a:H_1a=0, H_0a=0} (-)^a \text{tr}(H_0^T H_1)^a 4^{|a|} 3^{N-|a|}.$$  \hspace{1cm} (19)

Any pair of matrices $H_0$ and $H_1$ with the property that $\text{diag}(H_0^T H_1) = I$ is possible in these sums. To show that the sums of Problems 8 and 9 are of this form, consider first the case $k = 4$ and $l = 3$. The two sums can then be written as

$$\sum_{a:C_0a=0} (-)^a \text{tr}(C)^a 4^{|a|} 3^{N-|a|}$$  \hspace{1cm} (20)

$$\sum_{a:C_0a=0, C_1a=0} (-)^a \text{tr}(C)^a 4^{|a|} 3^{N-|a|}.$$  \hspace{1cm} (21)

In the former case, let $H_0 = I$ and $H_1 = C$, to see that it is an instance of Sum 18. (The factor of $5^N$ is properly taken care of by the conditions in the promise.) In the latter case, observe that one can write $C = XYT$ with $X$ and $Y$ rectangular matrices with independent columns. This can be done by first using Gaussian elimination to write $UCV^T = I_k$, where $U$ and $V$ are invertible and $I_k$ is a partial identity matrix with $k$ ones, then using such a decomposition for $I_k$. To see that this sum is in fact an instance of Sum 18, let $H_0 = XT$ and $H_1 = YT$ and observe that $H_0^T H_1a = 0$ iff $H_1a = 0$ and similarly for $H_1^T H_0$.

For other $k$ and $l$, use the above reductions to get sums like those of 18 and 19, but with $k$ and $l$ substituted for the numbers 4 and 3, respectively, and $\sqrt{k^2 + l^2}$ substituted for the divisor 5. These sums correspond to sums involving gates with different rotation angles. By universality, these gates can be approximated to within $O(\epsilon/N)$ using the standard ones with $\text{polylog}(N/\epsilon)$ overhead in gates 18. There is a classical algorithm that computes such approximations efficiently. The resulting gate network can be turned back into a sum of the desired form.

To see that Sum 18 can be cast in the form required by Problem 8 requires more work. Let $H_0$ and $H_1$ be as in Sum 18. If $H_0$ has independent rows, then the constraint $H_1 a = 0$ is equivalent to $H_0^T H_1 a = 0$, so the sum is of the desired form. If not, it is necessary to modify $H_0$ so that it has full rank without changing the value of the sum.

**Lemma 10** There exists a full rank $H_2$ such that $\text{tr}(H_2^T H_1) = \text{tr}(H_0^T H_1)$ and $\text{diag}(H_2^T H_1) = I$.

**Proof.** Consider the first $n$ columns of $H_0$ and $H_1$, labeled $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ respectively. To obtain $H_2$, the $c_i$ are replaced by independent $c_i'$. In order for the desired equality to hold, we need $d_i^T c_j' = d_i^T c_j$ for $i \leq j$. The desired $c_j'$ can be constructed starting with $c_n'$. Let $c_n'$ be any solution to $d_n^T c_n' = d_n^T c_n$ for all $i \leq n$. Such a solution exists and is non-zero because $d_n^T c_n = 1$. Suppose $c_n', c_{n-1}', \ldots, c_{k+1}'$ have been constructed. The set of solutions to $d_i^T x = d_i^T c_k$ for $i \leq k$ is an affine subspace not containing 0 of dimension at least $n - k$. Its intersection with the complement of the span of the $c_n', c_{n-1}', \ldots, c_{k+1}'$ is therefore not empty. Let $c_k'$ be an element of this intersection. Proceed until $c_1'$ has been obtained. The vectors constructed by this method satisfy the desired conditions. ■

For the matrix shown to exist by this lemma,
Thus, the constraint in the sum can be replaced by $H_2^T H_1 a = 0$ to obtain a sum of the desired form. We have proved the following:

**Theorem 11** Problem 3 is polynomially equivalent to Problem 5.

**Corollary 12** Probabilistic RAMs with an oracle for Problem 3 are polynomially equivalent to quantum computers.

**Corollary 13** Problem 3 is complete for BQP.

**Theorem 14** Problem 4 can be solved efficiently by one-bit quantum computers.

It is an open problem to determine whether the converse of Theorem 14 holds and to determine the relationships between the various promise problems suggested in the Introduction. Note that it is possible to simulate one-bit quantum computers given access to oracles for Problem 4, if the coefficient $1/2$ in the bound in the promise is replaced by $1/X$, with $X$ given as an input. ($X$ should be given as a unary number to maintain canonical size/complexity relationships.)

**V. CONCLUSION**

We have shown that the problem of simulating a quantum algorithm on a classical computer is equivalent to the difficulty of estimating certain combinatorial sums given by the quadratically signed weight enumerators. The problem of approximating these sum includes a new set of apparently difficult problems solvable efficiently by quantum computers. The class of known problems of this type is still sparse. Except for the ones proposed here, they are generally related to finding periodicities in functions or inferring properties of eigenvalues of unitary operators. Shor’s factoring and discrete logarithm algorithms are of this type [12]. The factoring and discrete logarithm problems have the advantage of not requiring a potentially difficult to verify promise. On the other hand, promise problems are a natural framework to use for both probabilistic and quantum computation and abstract the much more economically significant statistical inference problems underlying many practical applications. Our work demonstrates that quadratically signed weight enumerator problems are both simple to state and have sufficient flexibility to represent the capabilities of both quantum computers and one-bit quantum computers. There are variants that appear to be hard, perhaps even for quantum computers, and others that may be easier than one-bit quantum computation but hard for classical computation. As a result, the investigation of this class of problems will contribute toward a better understanding of classical and quantum complexity classes.

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[1] D. Aharonov. Quantum computation. quant-ph/9812037, 1998.
[2] D. Aharonov and M. Ben-Or. Fault-tolerant quantum computation with constant error. In Proceedings of the 29th Annual ACM Symposium on the Theory of Computing, pages 176–188, New York, New York, 1996. ACM Press. quant-ph/9611025.
[3] E. Bernstein and U. Vazirani. Quantum complexity theory. SIAM J. Comput., 26:1411–1473, 1997.
[4] R. Cleve. An introduction to quantum complexity theory. quant-ph/9906111, 1999.
[5] A. Yu. Kitaev. Quantum computations: algorithms and error correction. Uspekhi Mat. Nauk, 52:53–112, 1997.
[6] E. Knill and R. Laflamme. On the power of one bit of quantum information. Physical Review Letters, 81:5672–5675, 1998. quant-ph/9802037 and LA-UR-98-1567.
[7] E. Knill, R. Laflamme, and W. Zurek. Resilient quantum computation: Error models and thresholds. Proceedings of the Royal Society of London A, 454:365–384, 1998. quant-ph/9702058.
[8] E. Knill, R. Laflamme, and W. H. Zurek. Resilient quantum computation. Science, 279:342–345, 1998.
[9] C. H. Papadimitriou. Computational Complexity. Addison-Wesley, Reading, Mass, 1994.
APPENDIX A: REAL GATES ARE EQUIVALENT TO COMPLEX GATES

Let $G$ be a gate network consisting of the gates $G_N, \ldots, G_1$. Introduce a new qubit, labeled 0, to represent the complex phase by the real orthogonal map

$$R : (\alpha|0\rangle_0 + \beta|1\rangle_0)|b\rangle \rightarrow (\alpha + i\beta)|b\rangle.$$  \hspace{1cm} (A1)

Define $G'_k = \Re(G_k) - i\sigma_y^{(0)}\Im(G_k)$. Then the $G'_k$ are real orthogonal and define a new gate network $G'$. Note that each $G'_k$ can be efficiently approximated using the elementary real gates. The unitary operator defined by $G'$ satisfies

$$U(G') = \Re(U(G)) - i\sigma_y^{(0)}\Im(U(G)),$$  \hspace{1cm} (A2)

and $U(G) = RU(G')R^{-1}$. These relationships can be used to simulate any network by a real network.