Classification of six-dimensional Leibniz algebras $\mathcal{E}_3$

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Abstract

Leibniz algebras $\mathcal{E}_n$ were introduced as algebraic structure underlying U-duality. Algebras $\mathcal{E}_3$ derived from Bianchi three-dimensional Lie algebras are classified here. Two types of algebras are obtained: Six-dimensional Lie algebras that can be considered extension of semi-Abelian four-dimensional Drinfel’d double and extensions of non-Abelian unimodular Bianchi algebras.

1 Introduction

Algebraic structures underlying U-duality were suggested in [1] as Leibniz algebras $\mathcal{E}_n$ obtained as extensions of $n$-dimensional Lie algebra defining non-symmetric product $\circ$ in $[n+n(n-1)/2]$-dimensional vector space that satisfies Leibniz identity

$$X \circ (Y \circ Z) = (X \circ Y) \circ Z + Y \circ (X \circ Z).$$

(1)

In that paper examples of these Leibniz algebras derived from two-dimensional and Abelian four-dimensional Lie algebras are given. Goal of the present note

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is to write down all algebras that can be derived from three dimensional Lie algebras whose classification given by Bianchi is well known.

Namely, let \((T^a, T^{a_1 a_2})\), \(a, a_1, a_2 \in 1, \ldots, n\), \(T^{a_2 a_1} = -T^{a_1 a_2}\) is a basis of \([n + n(n - 1)/2]\)-dimensional vector space. The algebra product given in [1] is

\[
T_a \circ T_b = f_{ab}^\ c T_c, \\
T_a \circ T^{b_1 b_2} = f_a^{b_1 b_2 c} T_c + 2 f_{ac}^{\ [b_1} T^{b_2]c}, \\
T^{a_1 a_2} \circ T_b = -f_b^{a_1 a_2 c} T_c + 3 f_{c_1 c_2}^{\ [a_1} \delta^{a_2]}_b T^{c_1 c_2}, \\
T^{a_1 a_2} \circ T^{b_1 b_2} = -2 f_d^{a_1 a_2 [b_1} T^{b_2]d},
\]

where \(f_{ab}^\ c\) are structure coefficients of \(n\)-dimensional Lie algebra and \(f_a^{b_1 b_2 b_3} = f_a^{[b_1 b_2 b_3]}\). Moreover, bilinear forms on \(E_n\) are defined

\[
\langle T_a, T^{b_1 b_2} \rangle_c = 2! \delta^{[b_1}_a \delta^{b_2]}_c, \\
\langle T^{a_1 a_2}, T^{b_1 b_2} \rangle_{c_1 \ldots c_4} = 4! \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \delta^{b_1}_{c_3} \delta^{b_2}_{c_4}.
\]

2 Bianchi-Leibniz algebras

We are going to classify Leibniz algebras \(E_3\) derived from three dimensional Lie algebras. In this case \(f_a^{b_1 b_2 b_3} = f_a \varepsilon^{b_1 b_2 b_3}\) where \(\varepsilon\) is totally antisymmetric Levi-Civita tensor. Non-vanishing bilinear forms are

\[
\langle T_1, T^{12} \rangle_2 = \langle T_1, T^{13} \rangle_3 = \langle T_2, T^{23} \rangle_3 = 1, \\
\langle T_2, T^{12} \rangle_1 = \langle T_3, T^{13} \rangle_1 = \langle T_3, T^{23} \rangle_2 = -1.
\]

First of all we shall show that for dimension three the Leibniz identities are satisfied only for unimodular Lie Algebras\(^1\) i.e. \(f_a^b = 0\). Indeed, Leibniz identity

\[
T^{23} \circ (T_1 \circ T_1) = (T^{23} \circ T_1) \circ T_1 + T_1 \circ (T^{23} \circ T_1)
\]

and definitions [2] give

\[
0 = 2 (f_{12}^2 + f_{13}^3)^2 T^{23} = 2 (f_{1b}^b)^2 T^{23}
\]

and similarly for cyclic permutation of \((1, 2, 3)\).

\(^1\)This is not true in general as can be shown explicitely for dimension four or by two-dimensional example in [1].
Next point in our computations is the well known classification of 3–dimensional real Lie algebras. Non–isomorphic Lie algebras can be divided into eleven classes, traditionally known as Bianchi algebras. Their Lie algebra products are (see e.g. [2])

\[
\begin{align*}
[X_1, X_2] &= -a X_2 + n_3 X_3, \\
[X_2, X_3] &= n_1 X_1, \\
[X_3, X_1] &= n_2 X_2 + a X_3,
\end{align*}
\]

(4)

where the parameters \(a, n_1, n_2, n_3\) have the values given in the Table 1. Unimodular Bianchi algebras are those with \(a = 0\), i.e. B1 (Abelian), B2 (Heisenberg), B6 (Euclidean), B7 (Poincare), B8 (sl(2,\(\mathbb{R}\))), and B9 (so(3)).

Inserting (4) and (2) into Leibniz identities (1) we get

\[
\begin{align*}
n_j f_k &= 0, \quad j, k = 1, 2, 3. 
\end{align*}
\]

(5)

This can be shown inspecting e.g. identities (1) for

\[
X = T_1, \quad Y = T^{23}, \quad Z = T^{12},
\]

and

\[
X = T_1, \quad Y = T^{23}, \quad Z = T^{13}.
\]

We get

\[
\begin{align*}
n_2 f_1 T^{13} + n_2 f_2 T^{23} &= 0, \\
n_3 f_1 T^{12} + n_3 f_3 T^{23} &= 0,
\end{align*}
\]

Table 1: Bianchi algebras

| Class | \(a\) | \(n_1\) | \(n_2\) | \(n_3\) |
|-------|------|------|------|------|
| B1    | 0    | 0    | 0    | 0    |
| B2    | 0    | 1    | 0    | 0    |
| B3    | 1    | 0    | 1    | -1   |
| B4    | 1    | 0    | 0    | 1    |
| B5    | 1    | 0    | 0    | 0    |
| B6    | 0    | 1    | -1   | 0    |
| B6\(_a\) \((a > 0, a \neq 1)\) | a    | 0    | 1    | -1   |
| B7\(_0\) | 0    | 1    | 1    | 0    |
| B7\(_a\) \((a > 0)\) | a    | 0    | 1    | 1    |
| B8    | 0    | 1    | 1    | -1   |
| B9    | 0    | 1    | 1    | 1    |

so that
\[ n_2 f_1 = 0, \quad n_2 f_2 = 0, \quad n_3 f_1 = 0, \quad n_3 f_3 = 0. \]  
(6)

By cyclic permutation of \((1, 2, 3)\) we get \((5)\) and it is easy to check that these conditions are sufficient for satisfaction of all Leibniz identities \((1)\). Solution of conditions \((5)\) is either \(n_j = 0, \ j = 1, 2, 3\) or \(f_k = 0, \ k = 1, 2, 3\). It means that we get two types of Bianchi-Leibniz algebras.

The first type are algebras depending only on \(f_k\) with products
\[
T_a \circ T_b = 0, \\
T_a \circ T^{b_1 b_2} = f_a \varepsilon^{b_1 b_2 c} T_c, \\
T^{a_1 a_2} \circ T_b = -f_b \varepsilon^{a_1 a_2 c} T_c, \\
T^{a_1 a_2} \circ T^{b_1 b_2} = -2 f_d \varepsilon^{a_1 a_2 [b_1} T^{b_2]} d. 
\]  
(7)

It is rather easy to check that this product is antisymmetric so that it is a six-dimensional Lie algebra. By linear transformation we can achieve \(f_a \in \{0, 1\}\) and we get four distinct cases. Only one of them is Drinfel’d double, namely if \(f_1 = f_2 = f_3 = 0\).

The Bianchi-Leibniz algebras of the second type depend only on \(n_j\) whose values are given in the Table \(\|\). It means that they are in one to one correspondence with the unimodular Bianchi algebras. Their products are
\[
T_a \circ T_b = [T_a, T_b], \\
T_a \circ T^{b_1 b_2} = \delta_a^{b_1} \varepsilon_{abc} n_{b_2} T^{b_1 c} - \delta_a^{b_2} \varepsilon_{abc} n_{b_1} T^{b_2 c}, \\
T^{a_1 a_2} \circ T_b = 0, \\
T^{a_1 a_2} \circ T^{b_1 b_2} = 0. 
\]  
(8)

Explicit forms of products \(T_a \circ T^{b_1 b_2}\) are
\[
T_1 \circ T^{12} = -T_3 \circ T^{23} = n_2 T^{13}, \\
T_1 \circ T^{13} = T_2 \circ T^{23} = -n_3 T^{12}, \\
T_2 \circ T^{12} = -T_3 \circ T^{13} = n_1 T^{13}. 
\]

Maximal isotropic algebras in both types of algebras are generated by \(\{T_1, T_2, T_3\}\), \(\{T^{12}, T^{13}, T^{23}\}\) and \(\{T_1, T^{23}\}, \{T_2, T^{13}\}, \{T_3, T^{12}\}\).

As mentioned in \(\|\), under some conditions we can choose a subalgebra of dimension \(2(n - 1)\) of the Leibnitz algebra \(\mathcal{E}_n\) that is Lie algebra of Drinfel’d
double. Leibniz algebra then can be considered as an extension of Drinfel’d double of dimension $2(n-1)$. Namely, if we can decompose the generators $\{T_a\}$ as $\{\dot{T}_a, T_z\}$ and $\{T^{ab}, T^{\dot{a}z}\}$ ($\dot{a} = 1, \ldots, n-1$) so that

$$f_{ab}^z = 0, \quad f_{az}^b = 0, \quad f_z^{b_1b_2b_3} = 0, \quad f_{\dot{a}b_1b_2b_3} = 0,$$

then the subalgebra spanned by

$$\langle \dot{T}_a, T^{\dot{a}} \rangle \equiv \langle \dot{T}_a, T_{\dot{a}} \rangle \quad (T_{\dot{a}} \equiv T^{\dot{a}z})$$

becomes Lie algebra of Drinfel’d double with the bilinear form

$$\langle \dot{T}_a, T^{\dot{b}} \rangle := \langle \dot{T}_a, T^{\dot{b}} \rangle_z = \delta_{\dot{a}}^{\dot{b}}.$$

However for $n = 3$, the conditions (9) and unimodularity are satisfied only for Abelian Bianchi algebra $B_1$. It means that Leibniz algebras (7) can be considered extensions of Lie algebras of four-dimensional Drinfel’d doubles generated by $T_1, T_2, T^{13}, T^{23}$ and

$$[T_1, T_2] = 0, \quad [T^{13}, T^{23}] = -f_1 T^{13} - f_2 T^{23}, \quad \langle T_1, T^{13} \rangle = \langle T_2, T^{23} \rangle = 1.$$

3 Conclusions

We have classified six-dimensional Leibniz algebras starting from classification of three-dimensional Lie algebras. Up to linear transformations we have obtained nine inequivalent algebras. Four of them, obtained from Bianchi algebra $B_1$, are six-dimensional Lie algebras that can be considered extension of semi-Abelian four-dimensional Drinfel’d double. The other five are unique Leibniz extensions of unimodular Bianchi algebras $B_2, B_6, B_7, B_8, B_9$.

References

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