Fast Polynomial Approximation of Heat Diffusion on Manifolds and Its Application to Brain Sulcal and Gyral Graph Pattern Analysis

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Abstract—Heat diffusion has been widely used in brain imaging for surface fairing, mesh regularization and noisy cortical data smoothing. In the previous spectral decomposition of graph Laplacian, Chebyshev polynomials were only used. In this paper, we present a new general spectral theory for the Laplace-Beltrami operator on a manifold that works for an arbitrary orthogonal polynomial with a recurrence relation. Besides the Chebyshev polynomials that was previous used in diffusion wavelets and convolutional neural networks, we provide three other polynomials to show the generality of the method. We also derive the closed-form solutions to the expansion coefficients of the spectral decomposition of the Laplace-Beltrami operator and use it to solve heat diffusion on a manifold for the first time. The proposed fast polynomial approximation scheme avoids solving for the eigenfunctions of the Laplace-Beltrami operator, which are computationally costly for large mesh size, and the numerical instability associated with the finite element method based diffusion solvers. The proposed method is applied in localizing the male and female differences in cortical sulcal and gyral graph patterns obtained from MRI.

Index Terms—Heat diffusion, Laplace-Beltrami operator, cortical surface, sulcal graph pattern, multiscale analysis.

I. INTRODUCTION

Heat diffusion has been widely used in brain image processing as a form of smoothing and noise reduction starting with Perona and Malik’s groundbreaking study [1]. Many techniques have been developed for surface mesh fairing, regularization [2], [3] and surface data smoothing [4]–[7]. The diffusion equation has been solved by various numerical techniques [4], [6], [8]–[11]. In [4], [5], the isotropic heat equation was solved by the least squares estimation of the Laplace-Beltrami (LB) operator and the finite difference method (FDM). In [9], [11], the heat diffusion was solved iteratively by the discrete estimate of the LB-operator using the finite element method (FEM) and an FDM scheme. However, FDM schemes are known to suffer numerical instability if the sufficiently small step size is not chosen in the forward Euler scheme. In [10]–[13], diffusion was solved by expanding the heat kernel as a series expansion of the LB-eigenfunctions. Although the LB-eigenfunction approach avoids the numerical instability associated with the FEM based diffusion solver [9], the computational complexity of computing eigenfunctions is very high for large-scale surface meshes.

In this paper, motivated by the diffusion wavelet transform [14]–[17] and convolutional neural networks [18] on graphs using Chebyshev polynomials, we propose a new spectral method to solve the heat diffusion on manifolds by approximating the heat kernel by orthogonal polynomials. The previous works did the spectral decomposition of mostly graph Laplacian exclusively using Chebyshev polynomials. The LB-operator with other polynomials were not considered before. We present a new general theory for the LB-operator on an arbitrary manifold that works for an arbitrary orthogonal polynomial. Besides the Chebyshev polynomials, we provide three other polynomials to show the generality of the proposed method. We further derive the closed-form expression of the spectral decomposition of the LB-operator and use it to solve heat diffusion on a manifold for the first time. Taking advantage of recurrence relations of orthogonal polynomials [19]–[21], the computational run time of the proposed method is reduced. The proposed method is faster than the LB-eigenfunction approach and FEM based diffusion solvers [11]. We further applied the fast polynomial approximation method to iterative convolution to obtain multiscale features, which is shown to be as good as the diffusion wavelet in detecting localized surface signals [14]–[18].

The proposed method is applied in quantifying brain sulcal and gyral patterns. The sulcal and gyral features such as gyrification index, sulcal depth, curvature, sulcal length and sulcal area [22] were widely used in revealing significant differences between populations. In [23], the difference of the superior temporal sulcus length between males and females was analyzed. [24] computed the lengths of sulcal curves in six landmarks such as central sulcus, sylvian fissure and calcarine sulcus were used in differentiating between the Alzheimer’s disease (AD) and normal control subjects. [25] measured the sulcal depth and average mean curvature along the sulcal lines to reveal group difference between normal controls and AD subjects. The sulcal and gyral graphs, as one of such anatomical features, have also played key roles in cortical surface registration [26]–[28] and brain volume registration [29], [30]. In [31], six distinct metrics were proposed to measure the difference between sulcal graph features, including sulcal pits, sulcal basins and ridge points. [32] created shape-adaptive kernels by
performing wavefront propagation [33] with sulcal and gyral curves as the source, and then computed the local gyration index using the kernels. In [34], [35], a similarity measure between two sulcal graphs was proposed for determining the similarity between twins.

As a demonstration of the method, we analyze the sulcal and gyral graph patterns of the whole brain. We use the proposed fast polynomial scheme to perform heat diffusion on cortical brain surfaces by taking the sulcal and gyral graph patterns as the initial condition. The male and female differences are then localized using both mass univariate and multivariate statistics. The proposed method is then compared against the mean curvature and shape index.

The main contributions of the paper are as follows. 1) The development of a general polynomial approximation based spectral method and its application to solving diffusion equations fast. The derivation of the closed-form solutions of the expansion that enables faster computation of heat diffusion than before. 2) New multiscale shape analysis framework on manifolds that utilizes the iterative kernel convolution property. 3) Application of the faster solver in analyzing the sulcal and gyral patterns on the brain surface meshes with 370,000 vertices for 444 subjects obtained from 3T MRI. The dataset is large enough to demonstrate the effectiveness of our faster solver. Our fast solver can perform diffusion in 40 minutes for the whole dataset.

II. METHODS

We present a new general spectral theory for the Laplace-Beltrami (LB) operator on an arbitrary manifold based on expanding the exponential weight function in the heat kernel using orthonormal polynomials. The new theory works for an arbitrary orthogonal polynomial with the second order recurrence. We present four examples of the proposed methods based on the Jacobi, Chebyshev, Hermite and Laguerre polynomials to show the generality of the proposed method. The analytic closed-form solutions to the expansion coefficients are derived and used to solve the heat diffusion on a manifold.

A. Diffusion on Manifolds

Suppose functional data \( f \) on surface \( M \in \mathbb{R}^3 \). We assume \( f \) to be in \( L^2(M) \), the space of square integrable functions on \( M \) with inner product

\[
\langle f, h \rangle = \int_M f(p)h(p)d\mu(p),
\]

where \( \mu(p) \) is the Lebesgue measure such that \( \mu(M) \) is the total area of \( M \). Let \( \Delta \) denote the LB-operator defined on \( M \). The isotropic heat diffusion equation on \( M \) with \( f \) as the initially observed data is given by

\[
\frac{\partial g(p, \sigma)}{\partial \sigma} + \Delta g = 0, \quad g(p, \sigma = 0) = f(p),
\]

where \( \sigma \) is the diffusion time. It has been shown that the convolution of \( f \) with heat kernel \( K_\sigma \) is the unique solution of (1) [8], [11],

\[
g(p, \sigma) = K_\sigma * f(p) = \int_M K_\sigma(p, q)f(q)d\mu(q).
\]

Let \( \psi_j \) be the eigenfunctions of the LB-operator with eigenvalues \( \lambda_j \), i.e., \( \Delta \psi_j = \lambda_j \psi_j \). If we order the eigenvalues as \( 0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \), the heat kernel can be expanded in terms of \( \psi_j \) with exponential weight \( e^{-\lambda_j \sigma} \) [11]:

\[
K_\sigma(p, q) = \sum_{j=0}^{\infty} e^{-\lambda_j \sigma} \psi_j(p)\psi_j(q).
\]

Then, the heat kernel convolution can be expressed as

\[
g(p, \sigma) = K_\sigma * f(p) = \sum_{j=0}^{\infty} e^{-\lambda_j \sigma} f_j \psi_j(p)
\]

with coefficients \( f_j \) computed as

\[
f_j = \int_M f(p)\psi_j(p)d\mu(p).
\]

B. Basic Idea in 1D Diffusion

We start with the simplest case, 1D heat diffusion, to explain the core idea of the proposed method. Consider 1D time series data \( f \) on unit interval \([0, 1]\). It has been shown that the solution of 1D version of heat diffusion (1) is given by the weighted cosine series representation [36]

\[
g = K_\sigma * f = \sum_{l=0}^{\infty} e^{-l^2 \pi^2 \sigma} c_{fl} \psi_l,
\]

where \( \psi_0(x) = 1 \) and \( \psi_l(x) = \sqrt{2} \cos(l\pi x) \) are the eigenfunctions of 1D Laplace operator \( \Delta = -\frac{\partial^2}{\partial x^2} \), and \( c_{fl} = \int_0^1 f(x)\psi_l(x)dx \) are expansion coefficients. Expanding the exponential weight in (4) as the Taylor expansion \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \), the heat diffusion becomes

\[
K_\sigma * f = \sum_{n=0}^{\infty} \frac{(-\sigma)^n}{n!} \sum_{l=0}^{\infty} c_{fl}(l^2 \pi^2)^n \psi_l.
\]

Since \( \Delta \psi_l = -l^2 \pi^2 \psi_l = (l^2 \pi^2)^n \psi_l \), it follows that \( \Delta^n \psi_l = (l^2 \pi^2)^n \psi_l \). Subsequently, we have

\[
K_\sigma * f = \sum_{n=0}^{\infty} \frac{(-\sigma)^n}{n!} \Delta^n f.
\]

Thus, heat diffusion can be computed simply by using the power of Laplacian to the original data. If we can further compute the power of Laplacian quickly using some sort of iterations, the computation can be done without difficulty.

C. Fast Polynomial Approximation

Here we present a general new theory for an arbitrary manifold that works in any type of image domains including surface and volumetric meshes. Consider an orthogonal polynomial \( P_n \), which is often defined using the second order recurrence [21],

\[
P_{n+1}(\lambda) = (A_n \lambda + B_n)P_n(\lambda) + C_n P_{n-1}(\lambda)
\]

with initial conditions \( P_{-1}(\lambda) = 0 \) and \( P_0(\lambda) = 1 \). Assume \( P_n \) are orthogonal over interval \([a, b]\) with inner product

\[
\int_a^b P_n(\lambda)P_k(\lambda)d\mu(\lambda) = \delta_{nk},
\]

the Dirac delta.
To avoid solving for the eigenfunctions \( \psi_j \), which is computationally costly, we expand the exponential weight of the heat kernel by polynomials \( P_n \):

\[
e^{-\lambda \sigma} = \sum_{n=0}^{\infty} c_{\sigma,n} P_n(\lambda), \quad c_{\sigma,n} = \int_{a}^{b} e^{-\lambda \sigma} P_n(\lambda) d\mu(\lambda). \tag{7}
\]

Then, substituting (7) into (3), the solution of heat diffusion can be expressed in terms of the polynomials:

\[
sK_{\sigma} \ast f = \sum_{n=0}^{\infty} c_{\sigma,n} \sum_{j=0}^{\infty} P_n(\lambda_j) f_j \psi_j. \tag{8}
\]

Since \( \Delta \psi_j = \lambda_j \psi_j \), we have \( \Delta^l \psi_j = \lambda_j^l \psi_j \). Assuming \( P_n(\lambda) = \sum_{l=0}^{n} d_l \lambda^l \) for some coefficients \( d_l \), it follows that

\[
P_n(\lambda_j) \psi_j = \sum_{l=0}^{n} d_l \lambda_j^l \psi_j = \sum_{l=0}^{n} d_l \Delta^l \psi_j = P_n(\Delta) \psi_j. \tag{9}
\]

By substituting (9) into (8), the heat diffusion equation is solved by polynomial expansion involving LB-operators but without the LB-eigenfunctions,

\[
sK_{\sigma} \ast f = \sum_{n=0}^{\infty} c_{\sigma,n} P_n(\Delta) f.
\]

Since \( P_n \) is a polynomial of degree \( n \), the direct computation of \( P_n(\Delta) f \) requires the costly computation of \( \Delta f, \Delta^2 f, \ldots, \Delta^nf \). Thus, we compute them Instead, we compute \( P_n(\Delta) f \) by the recurrence

\[
P_{n+1}(\Delta) f = (A_n \Delta + B_n) P_n(\Delta) f + C_n P_{n-1}(\Delta) f
\]

with initial conditions \( P_{-1}(\Delta) f = 0 \) and \( P_0(\Delta) f = f \).

In practice, the expansion is truncated at degree \( m \), which is empirically determined. The expansion coefficients \( c_{\sigma,n} \) can be computed from the closed-form solution to (7). In the following subsections, we present three examples of the fast polynomial approximation methods based on the Jacobi, Hermite and Laguerre polynomials.

**Jacobi polynomials.** The Jacobi polynomials \( P_n^{(\alpha,\beta)}(\lambda) \), which are orthogonal in \([-1,1] \), for \( \alpha, \beta > -1 \), are defined by the recurrence (6) with parameters given by [21]

\[
A_n = \frac{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)}{2(n+1)(n+\alpha+\beta+1)},
\]

\[
B_n = \frac{(\alpha^2 - \beta^2)(2n + \alpha + \beta + 1)}{2(n+1)(n+\alpha+\beta+1)(2n + \alpha + \beta)},
\]

\[
C_n = \frac{(n+\alpha)(n+\alpha+\beta)(2n + \alpha + \beta + 2)}{(n+1)(n+\alpha+\beta+1)(2n + \alpha + \beta)}.
\]

Many polynomials such as Chebyshev, Legendre and Gegenbauer polynomials defined in \([-1,1] \) are all special cases of the Jacobi polynomials [21].

The eigenvalue \( \lambda \) of the LB-operator ranges over \([0,\infty) \). Expanding the exponential weight \( e^{-\lambda \sigma} \) by the Jacobi polynomial may not be able to provide a good fit outside the interval \([-1,1] \). Hence, we shift and scale Jacobi polynomials with parameter \( b > 0 \)

\[
P_n^{(\alpha,\beta)}(\lambda) = P_n^{(\alpha,\beta)}\left( \frac{2\lambda}{b} - 1 \right), \tag{11}
\]

which are orthogonal over \([0,b] \). Then \( e^{-\lambda \sigma} \) is expanded in terms of \( T_n^{(\alpha,\beta)} \).

**Theorem 1.** The Jacobi polynomial expansion of the solution to heat diffusion (1) is given by

\[
sK_{\sigma} \ast f = \sum_{n=0}^{\infty} c_{\sigma,n} T_n^{(\alpha,\beta)}(\lambda) f,
\]

where the coefficients \( c_{\sigma,n} \) have the closed-form solution

\[
c_{\sigma,n} = \frac{\Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} (-\sigma)^n F_1\left( \frac{\beta + n + 1}{\alpha + \beta + 2n + 2};-b\sigma \right),
\]

and \( pF_q \) is the generalized hypergeometric function [21].

**Proof.** We first derive the expansion of \( e^{-\lambda \sigma} \) using the Jacobi polynomials \( P_n^{(\alpha,\beta)} \). The Jacobi polynomials are orthogonal over interval \([-1,1] \) with inner product [21],

\[
\int_{-1}^{1} P_n^{(\alpha,\beta)}(\lambda) P_k^{(\alpha,\beta)}(\lambda)(1 - \lambda)^{\alpha}(1 + \lambda)^{\beta} d\lambda = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)!\Gamma(n+\alpha+\beta+1)\delta_{nk}}.
\]

The algebraic derivation will show that the expansion of \( e^{-\lambda \sigma} \) is given by

\[
e^{-\lambda \sigma} = \sum_{n=0}^{\infty} \gamma_n (-2\sigma)^n e^{\alpha} F_1\left( \frac{\beta + n + 1}{\alpha + \beta + 2n + 2};-\sigma \right) P_n^{(\alpha,\beta)}(\lambda),
\]

where \( \gamma_n = \frac{\Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} \) and \( pF_q \) is the generalized hypergeometric function [21], [37]. The expansion is only valid in interval \([-1,1] \). To obtain the expansion of \( e^{-\lambda \sigma} \) terms of the the shifted and scaled Jacobi polynomial (11), we replace \( \lambda \) by \( \frac{2\lambda}{b} - 1 \) and \( \sigma \) by \( \frac{2\sigma}{b} \) and obtain

\[
e^{-\lambda \sigma + \frac{2\sigma}{b}} = \sum_{n=0}^{\infty} \gamma_n (-\sigma)^n e^{\frac{2\alpha}{b}} F_1\left( \frac{\beta + n + 1}{\alpha + \beta + 2n + 2};-\sigma \right) T_n^{(\alpha,\beta)}(\lambda).
\]

We divide the both sides of the equation by \( e^{\frac{2\alpha}{b}} \) and the expansion of \( e^{-\lambda \sigma} \) follows.

**Chebyshev polynomials.** The Chebyshev polynomials \( T_n(\lambda) = \cos(n \cdot \cos^{-1} \lambda) \) defined in interval \([-1,1] \) are the special cases of the Jacobi polynomials [21]

\[
T_n(\lambda) = \frac{4^n (n!)^2}{(2n)!} P_n^{(-\frac{1}{2},-\frac{1}{2})}(\lambda). \tag{12}
\]

The Chebyshev polynomial satisfies the recurrence relation (6) with parameters \( A_n = 2 - \delta_{n0}, B_n = 0 \) and \( C_n = -1 \). Similar to using the shifted and scaled Jacobi polynomials in Theorem 1, we shift and scale the Chebyshev polynomials to

\[
T_n(\lambda) = T_n\left( \frac{2\lambda}{b} - 1 \right)
\]

for the expansion of exponential weight over interval \([0,b] \).
Theorem 2. The Chebyshev polynomial expansion of the solution to heat diffusion (1) is given by

$$K_{\sigma} \ast f = \sum_{n=0}^{\infty} c_{\sigma,n} T_n(\Delta) f,$$

where the coefficients $c_{\sigma,n}$ have the closed-form solution

$$c_{\sigma,n} = (2 - \delta_{n0})(-1)^n e^{-\frac{b\sigma}{2}} I_n\left(\frac{b\sigma}{2}\right),$$

and $I_n$ is the modified Bessel function of the first kind [21].

Proof. We provide two different proofs. The first proof is based on Theorem 1. The Chebyshev polynomial is a special case of the Jacobi polynomial (12), and thus their shifted and scaled versions have the relation $T_n(\lambda) = 2^n(n!)^2 T_n^\frac{1}{2}-\frac{1}{2}(\lambda)$. Identifying $\alpha = \beta = -\frac{1}{2}$ in Theorem 1 and noting $\Gamma(\alpha+\beta+n+1) = 1$ when $n = 0$, we have

$$c_{\sigma,n} = \frac{2 - \delta_{n0}}{2^{n+1} n!} (-b\sigma)^n F_1\left(n+1/2;2n+1; -b\sigma\right). \quad (13)$$

The modified Bessel function is closely related to the confluent hypergeometric function [21],

$$I_n(z) = \frac{z^n e^{\pm z}}{2^n n!} F_1\left(n+1/2;2n+1; \mp 2z\right). \quad (14)$$

Substitute $I_n(z)$ in (14) with $z = \frac{b\sigma}{2}$ for the term $F_1$ in (13) and the result follows.

The second proof is based on the generating function of the modified Bessel functions [21]:

$$e^z \cos \theta = I_0(z) + 2 \sum_{n=1}^{\infty} I_n(z) \cos(n\theta). \quad (15)$$

We use the generating function to develop the relation between exponential function and the Chebyshev polynomials. Let $\theta = \cos^{-1} \lambda$, and then (15) can be rewritten in terms of the Chebyshev polynomials $T_n(\lambda) = \cos(n\cos^{-1} \lambda)$,

$$e^z\lambda = I_0(z) T_0(\lambda) + 2 \sum_{n=1}^{\infty} I_n(z) T_n(\lambda), \quad (16)$$

where $T_0(\lambda) = 1$. Replacing $\lambda$ by $\frac{2\lambda}{b\sigma} - 1$ and identifying $z = -\frac{b\sigma}{2}$ in (16) give the expansion of $e^{-\lambda\sigma + \frac{b\sigma}{2}}$ in terms of the shifted and scaled Chebyshev polynomials $T_n$:

$$e^{-\lambda\sigma + \frac{b\sigma}{2}} = I_0\left(-\frac{b\sigma}{2}\right) T_0(\lambda) + 2 \sum_{n=1}^{\infty} I_n\left(-\frac{b\sigma}{2}\right) T_n(\lambda).$$

We divide the both sides of the equation by $e^{\frac{b\sigma}{2}}$ and the expansion of $e^{-\lambda\sigma}$ follows. Note that $I_n\left(-\frac{b\sigma}{2}\right) = (-1)^n I_n\left(\frac{b\sigma}{2}\right)$. \hfill \Box

In numerical implementation, given the maximum eigenvalue $\lambda_{max}$ of the discrete LB-operator, we set $b = \lambda_{max}$ such that the Chebyshev polynomials provide good approximation of the exponential weight over $[0, \lambda_{max}]$ [14].

Hermite polynomials. The Hermite polynomials

$$H_n(\lambda) = (-1)^n e^{\lambda^2} \frac{d^n}{d\lambda^n} e^{-\lambda^2}$$

with $H_{-1}(\lambda) = 0$ and $H_0(\lambda) = 1$ in $(-\infty, \infty)$ satisfies the recurrence relation (6) with parameters [21]

$$A_n = 2, B_n = 0, C_n = -2n.$$

Theorem 3. The Hermite polynomial expansion of the solution to heat diffusion (1) is given by

$$K_{\sigma} \ast f = \sum_{n=0}^{\infty} c_{\sigma,n} H_n(\Delta) f,$$

where the coefficients $c_{\sigma,n}$ have the closed-form solution

$$c_{\sigma,n} = \frac{1}{n!} \left(-\frac{\sigma}{2}\right)^n e^{\frac{\sigma^2}{4}}.$$

Proof. The orthogonal condition of the Hermite polynomials [21] is given by

$$\int_{-\infty}^{\infty} H_n(\lambda) H_m(\lambda) e^{-\lambda^2} d\lambda = \sqrt{\pi} 2^n n! \delta_{nm}.$$ 

It follows that the expansion of $e^{-\lambda\sigma}$ in terms of the Hermite polynomials has coefficients

$$c_{\sigma,n} = \frac{1}{\sqrt{\pi} 2^n n!} \int_{-\infty}^{\infty} e^{-\lambda\sigma} H_n(\lambda) e^{-\lambda^2} d\lambda.$$

The closed-form solution of the expansion coefficients can be derived through the integral property involving the Hermite polynomials $\int_{-\infty}^{\infty} e^{-(\lambda-y)^2} H_n(\lambda) d\lambda = \sqrt{\pi} 2^n y^n$ [38] with $y = -\frac{\sigma}{2}$.

The statement can be also proved using the exponential generating function [21],

$$e^{2\lambda z - z^2} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(\lambda).$$

Here, it is used to derive the expansion coefficients by dividing the both sides of the equation by $e^{-z^2}$ and then identifying $z = -\frac{\sigma}{2}$. \hfill \Box

Laguerre polynomials. The Laguerre polynomials satisfies the recurrence relation (6) with parameters

$$A_n = \frac{1}{n+1}, B_n = \frac{n+1}{n+1}, C_n = -\frac{n}{n+1}$$

and $L_{-1}(\lambda) = 0$ and $L_0(\lambda) = 1$ in $[0, \infty)$ [21].

Theorem 4. The Laguerre polynomial expansion of the solution to heat diffusion (1) is given by

$$K_{\sigma} \ast f(p) = \sum_{n=0}^{\infty} c_{\sigma,n} L_n(\Delta) f,$$

where the coefficients $c_{\sigma,n}$ have the closed-form solution

$$c_{\sigma,n} = \frac{\sigma^n}{(\sigma + 1)^{n+1}}.$$
Proof. From the orthogonal condition of the Laguerre polynomials [21],
\[
\int_0^\infty L_n(\lambda) L_k(\lambda) e^{-\lambda} d\lambda = \delta_{nk},
\]
the expansion of \(e^{-\lambda \sigma}\) in terms of the Laguerre polynomials has coefficients given as the inner product of \(e^{-\lambda \sigma}\) and \(L_n\):
\[
e_{\sigma, n} = \int_0^\infty e^{-\lambda \sigma} L_n(\lambda) e^{-\lambda} d\lambda.
\]
The closed-form solution of the expansion coefficients can be derived through the integral property \(\int_0^\infty e^{-\lambda y} L_n(\lambda) d\lambda = (y-1)^n y^{n-1}\) with \(y = \sigma + 1\).

Alternately, we can prove the theorem using the exponential generating function of the Laguerre polynomials [21],
\[
\frac{1}{1-z} e^{-\lambda z} = \sum_{n=0}^{\infty} z^n L_n(\lambda).
\]

Multiply the both sides of the equation by \(1 - z\). Let \(\frac{1}{1-z} = \sigma\), i.e., \(z = \sigma + 1\) and then the expansion of \(e^{-\lambda \sigma}\) follows. \(\square\)

In numerical implementation, the expansion parameter \(m\) is empirically determined that gives the sufficiently small MSE.

Figure 1 displays the heat diffusion of the left hippocampus surface mesh with 2338 vertices and 4672 triangles, with diffusion time \(\sigma = 1.5\) and expansion degree \(m = 100\). The reconstruction error is measured by the mean squared error (MSE) between the polynomial approximation method and the original surface mesh. Although the methods converged with less than degree \(m = 100\), the Chebyshev approximation method converges the fastest. In many different brain surface meshes we tried, Chebyshev polynomials converged fastest. Thus, they will be mainly used through the paper but other polynomials can be similarly applied.

D. Discretization of the Laplace-Beltrami operator

Let \(p = X(u^1, u^2) \in \mathcal{M}\) be the parametric representation of \(\mathcal{M}\). Then, the inner products \(g_{ij} = \langle \frac{\partial X}{\partial u^i}, \frac{\partial X}{\partial u^j} \rangle\) are the Riemannian metric tensors measuring the amount of deviation from the Cartesian coordinate system [39], [40]. The LB-operator corresponding to the surface parameterization \(X\) is given by
\[
\Delta = (\Delta_{ij}) = \frac{1}{\det g^{1/2}} \sum_{i,j} \frac{\partial}{\partial u^i} \left( \det g^{1/2} g^{ij} \frac{\partial}{\partial u^j} \right),
\]

where \(g = (g_{ij})\) and \(g^{-1} = (g^{ij})\). The LB-operator is discretized in a triangle mesh via the cotan formulation [11], [12], [17], [41] as
\[
\Delta_{ij} = \frac{C_{ij}}{A_i},
\]
where \(A_i\) is the area at vertex \(p_i\), and \(C = (C_{ij})\) is the global coefficient matrix.

The construction of \(C\) is as follows. Let \(T_{ij}^+\) and \(T_{ij}^-\) be the two triangles sharing the vertex \(p_i\) and its neighboring vertex \(p_j\). Let the two angles opposite to the edge connecting \(p_i\) and \(p_j\) be \(\phi_{ij}\) and \(\theta_{ij}\) respectively for \(T_{ij}^+\) and \(T_{ij}^-\) (Figure 2-left).

The off-diagonal entries of the global coefficient matrix are \(C_{ij} = -(\cot \theta_{ij} + \cot \phi_{ij})/2\) if \(p_i\) and \(p_j\) are adjacent and \(C_{ij} = 0\) otherwise. The diagonal entries are \(C_{ii} = -\sum_j C_{ij}\).

For the area \(A_i\), we adopt the computation in [17], [42]. For each vertex \(p_i\), the neighboring triangles are separated into three sets: \(O_i\) is the set of nonobtuse triangles, \(O_i\) is the set of obtuse triangles with obtuse angle at \(p_i\), and \(O_i\) is the set of obtuse triangles with nonobtuse angle at \(p_i\) (Figure 2-right).

Then \(A_i\) is computed as
\[
A_i = \sum_{T \in O_i} V(T) + \sum_{T \in O_i} \frac{A(T)}{2} + \sum_{T \in O_i} \frac{A(T)}{4}
\]
where \(V(T)\) is the Voronoi region (gray area) computed following [42]. The computation of \(A(T)\) is done using the Heron’s formula.

E. Iterative convolution

One can also obtain diffusion related multiscale features obtained at different time points by iteratively performing heat kernel smoothing. Instead of applying the polynomial approximation separately for each \(\sigma\), the computation can also be realized in an iterative fashion. The solution to heat diffusion with larger diffusion time can be broken into iterative heat kernel convolution with smaller diffusion time [11],
\[
K_\sigma * f = K_{\sigma/m} * \cdots * K_{\sigma/m} * f.
\]

Thus, if we computed \(K_{0.25} * f\), then \(K_{0.5} * f\) can be simply computed as two repeated kernel convolution \(K_{0.25} * (K_{0.25} * f)\). Heat diffusion with much larger diffusion time can be
SPHARM representation with degree 0. We used the spherical meshes with 256^2, 1024^2, 4096^2 and other circular region, and all other vertices were assigned value 0. As $\sigma$ increases, we are smoothing the surface more and MSE increases.

done similarly. Figure 3 displays heat diffusion with $\sigma = 0.25$, 0.5, 0.75 and 1 realized by iteratively applying the Chebyshev approximation method with $\sigma = 0.25$ sequentially four times. As $\sigma$ increases, we are smoothing the surface more smoothly and MSE increases.

F. Validation

We compared the performance of the Chebyshev approximation method against the existing FEM based diffusion solver [9] and the LB-eigenfunction approach [10], [11] on the unit sphere $S^2$, where the ground truth can be analytically obtained by the spherical harmonics (SPHARM). The SPHARM, denoted by $Y_{lm}$, are the eigenfunctions of the LB-operator on $S^2$ with eigenvalues $-l(l+1)$. Assume that surface data $f$ has the SPHARM representation given by

$$f(p) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \beta_{lm} Y_{lm}(p), \quad p \in S^2. \quad (17)$$

The heat kernel after diffusion time $\tau$ on the sphere can be analytically constructed by the SPHARM [36] as

$$g(p) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-\tau(l+1)^2} \beta_{lm} Y_{lm}(p) \quad (18)$$

where SPHARM coefficients

$$\beta_{lm} = \int_{S^2} f(p) Y_{lm}(p) d\mu(p).$$

Suppose that the original signal has value 1 at surface mesh vertices within one circular region, $-1$ at vertices within another circular region, and all other vertices were assigned value 0. We used the spherical meshes with 2562, 10242, 40962 and 163842 vertices (Figure 4). We fitted the original signal by the SPHARM representation with degree $l = 100$, which consists of more than 10000 SPHARM basis functions [36]. 100 degree expansion is high enough to provide numerical accuracy up to 4 decimal places in terms of MSE between the original signal and the reconstructed signal. We applied the three methods with different $\sigma$ values (0.005, 0.01, 0.02 and 0.05). Figure 4 displays an example of the reconstructed signal and ground truth with $\sigma = 0.01$ on the sphere with 163842 mesh vertices. The LB-eigenfunction approach with 210 eigenfunctions, FEM based diffusion solver with 405 iterations, and Chebyshev approximation method with degree 45 achieve the similar reconstruction error.

Computational run time over mesh sizes. For the fair comparison, we used the same discretization of the LB-operator [11], [41]. The MSE against the ground truth was used as an error measure (Figure 5). To achieve the similar reconstruction error, the FEM based diffusion solver and Chebyshev approximation method need more iterations and higher degree for larger meshes, while the LB-eigenfunction approach is nearly unaffected by the mesh size. Figure 5-right displays the computational time of the three methods at the similar accuracy (MSE about $10^{-5}$).

Computational run time over diffusion times. The computational run time for different $\sigma$ (0.005, 0.01, 0.02, 0.05) with fixed spherical mesh resolution (40962 vertices) is also investigated. Figure 6-left displays the MSE of the three methods against the ground truth. For the LB-Eigenfunction method, the MSE is plotted over the number of eigenfunctions. For the FEM method, the MSE is plotted over the number of iterations. For the polynomial approximation, the MSE
is plotted over the expansion degree. More eigenfunctions, iterations and expansion degrees are used, we have smaller MSE. Figure 6-right displays the computational run time over $\sigma$ at the same MSE of about $10^{-7}$.

From Figures 5 and 6, in terms of the reconstruction error, the LB-eigenfunction method is the slowest. The polynomial approximation method is up to twelve times faster than the FEM method and took 0.06 seconds for $\sigma = 0.01$ on the sphere with 163842 vertices.

III. APPLICATION

A. Dataset

We used the T1-weighted MRI dataset consisting of 268 females and 176 males collected as the subset of the Human Connectome Project (HCP) [43]. MRI were obtained using a Siemens 3T Connectome Skyra scanner with a 32-channel head coil. T1-weighted images were acquired with the 3D magnetization-prepared rapid gradient echo (3D-MPRAGE) sequence: TE = 2.14 ms, TR = 2400 ms, TI = 1000 ms, flip angle = 8°, pixel bandwidth = 210 Hz, field of view (FOV) = 224 × 224 mm$^2$, resolution 0.7 mm isotropic, and 256 slices per slab.

The details on image and surface registration in HCP processing [44], [45] are as follows. First, T1-weighted MRI were undergone the gradient nonlinearity distortion correction. Then, images were aligned with a rigid body transformation using the FSL’s FLIRT [46] and then averaged. The average images were aligned to the MNI space template using a rigid transform, which aligns the AC, the AC-PC line and the inter-hemispheric plane. A robust initial brain extraction was performed using an initial linear (FLIRT) and non-linear (FNIRT) registration of the image to the MNI template. After removing readout distortion [47], an undistorted native structural volume space was produced for each subject. Finally, a bias field correction was performed, and the T1-weighted image was registered to MNI space with a FLIRT affine and then a FNIRT nonlinear registration. The distortion- and bias-corrected T1w image was then undergone the FreeSurfer’s recon-all pipeline [48], [49] that includes the segmentation of volume into predefined structures, reconstruction of white and pial cortical surfaces, and FreeSurfer’s standard folding-based surface registration to their surface atlas. Then, the white, pial and spherical surfaces of the left and right hemispheres were produced.

B. Sulcal and gyral curve extraction

The automatic sulcal curve extraction method was used to detect concave regions (sulcal fundi) along which sulcal curves are traced [50], [51]. The method consists of two main steps: (1) sulcal point detection and (2) curve delineation by tracing the detected sulcal points. For sulcal point detection, concave points are initially obtained from the vertices of the input surface mesh by thresholding mean curvatures. The concave points are further filtered by employing the line simplification method [52] that simplifies the sulcal regions without significant loss of their morphological details. Specifically, each of the detected concave point is encoded as a part of polyline on the plane of the principal curvature that guarantees the concave point is most distinguishable with maximum curvature with respect to its surface normal. The concave point is then tested if it is still a key part of the simplified curve. For such a test, the extremal point $\hat{p}$ with the maximum distance from the intersecting curve connected from $p_0$ to $p_1$ is selected as

$$\hat{p} = \arg\max_p \|p_1 - p_0\| \times \|p_1 - p_0\|^{-1}.$$  

This procedure is recursively performed until the maximum deviation is smaller than $\delta$. The concave points being selected by this procedure are determined as sulcal points that tightly capture sulcal fundic regions. For curve delineation, the selected sulcal points are connected to form a graph, and the curves are delineated by tracing shortest paths on the graph. Specifically, an undirected graph is constructed by connecting the sulcal points within a given geodesic distance $r$, and the edge weights $w$ are assigned based on geodesic distances $d_u(s)$ from a point $u$ to its neighborhoods $s$:

$$w(u, s) = d_u(s) \cdot e^{(\ln \gamma \cdot \|s - u\| \times \|T(u)\|) \cdot \|s - u\|^{-1}},$$

where $\ln \gamma$ is a regularization of the curve smoothness, and $T(u)$ is the estimated tangent vector at $u$ as the principal direction from $u$ to $s$. The neighboring point of $u$ is penalized if the point is away from the tangent direction $T(u)$. Curve smoothness can be thus controlled by $\gamma$. Finally, the sulcal curves are traced over the graph by the Dijkstra’s algorithm [53]. Similarly, we can extend this idea to gyral curve extraction by finding convex regions.

Since there is no ground truth of sulcal/gyral curves, in the previous studies [50], [51], the curve extraction algorithm was validated by the reproducibility, which is measured by the distance error at each sulcal point between two corresponding surfaces (scan and re-scan sessions), and the robustness to noise by adding synthetic noisy (bumpy) surfaces. We used the available software package TRACE (https://github.com/ilwoolyu/CurveExtraction) in extracting the curve data. The algorithm was validated to show the comparable extraction results to the manually labeled primary curves [54]. In this study, we used the same parameters as in [51]: the smallest deviation for line simplification $\delta = 2.5$ mm, kernel size of geodesic distance $r = 4.0$ mm and curve smoothness $\gamma = e^1$.

The algorithm only identified the major gyral and sulcal curves. Minor gyrus and sulcus in almost flat regions like
Fig. 8. The sulcal/gyral curves (left), mean curvature (middle) and shape index (right). 1st and 2nd rows: original data displayed on the white matter surfaces and the enlarged magenta regions. The gyral and sulcal curves are marked by solid and dashed black lines respectively and are assigned heat values 1 and -1 when smoothing. The mean curvature is positive for sulci and negative for gyri. The shape index is positive for gyri and negative for sulci. In the enlarged magenta regions, the noisy mean curvature and shape index showed sulcal patterns in the middle of the gyral region, which is not shown in the sulcal/gyral curve extraction method. Smoothing is done with diffusion time \( \sigma = 0.001 \).

plateau or with very low curvature, shallow depth or short length were not extracted. Figure 7 displays the sulcal/gyral curves and the smoothed mean curvature of four subjects. In the enlarged regions, the first three subjects have no sulcal curves between two gyral curves due to very low mean curvature, while the fourth subject has sulcal curve in the same region because of higher mean curvature.

C. Diffusion maps on sulcal and gyral curves

The junctions between sulci are highly variable [55]. A sulcus corresponding to a long elementary fold in one subject may be made up of several small elementary folds in another subject [56]. Each subject has different number of vertices and edges in sulcal and gyral graphs, and they don’t exactly match across subjects. Thus, it is difficult to directly compare such graphs at the vertex level across subjects. In this paper, the diffusion equation in the spectral domain was mainly used to smooth out the extracted sulcal and gyral curves and obtain the smooth representation of curves.

In this paper, the diffusion equation in the spectral domain was mainly used to smooth out the extracted sulcal and gyral curves and obtain the smooth representation of curves. The extracted gyral curves were assigned heat value 1, and sulcal curves were assigned heat value -1. All other parts of surface mesh vertices were assigned value 0. Then heat diffusion was performed on these values. The diffusion map values range from -1 to 1. The close to the value of 1 indicates the likelihood of the gyral curves while the close to the value of -1 indicates the likelihood of the sulcal curves. The proposed method is motivated by the voxel-based morphometry (VBM) [57], [58], where the segmented white or gray matter regions are compared in 3D volume. Due to the difficulty of exactly aligning the white or gray matter regions separately, Gaussian kernel smoothing with large bandwidth was used to mask the the shape variations across subjects and approximately align the segmented regions. Also a similar approach was used in the tract-based spatial statistics (TBSS) [59], [60] in analyzing white matter regions in diffusion tensor imaging that does not exactly align across subjects.

In numerical implementation, a sufficient large expansion degree \( m = 1000 \) were used. In a desktop with 4.2 GHz Intel Core i7 processor, the construction of the discrete LB-operator took 5.76 seconds, the computation of Chebyshev coefficients took \( 2.6 \times 10^{-4} \) seconds, and heat diffusion by the Chebyshev polynomials took 3.19 seconds for the both hemispheres in average. The total computation took 8.95 seconds per subject in average. One example of diffusion map with diffusion time \( \sigma = 0.001 \) is displayed in Figure 8-left. The diffusion map is then subsequently used in localizing the male and female differences in the sulcal and gyral patterns.

D. Univariate two-sample \( t \)-test

The diffusion maps with diffusion time \( \sigma = 0.001 \) were split into females (268 subjects) and males (176 subjects). The average diffusion maps of females and males (Figure 9-left) show major differences in the temporal lobe, which is responsible for processing sensory input into derived meanings for the appropriate retention of visual memory, language comprehension, and emotion association [61]. The two-sample \( t \)-statistic map shows the statistical significance of the diffusion map differences. The \( t \)-statistics are in the range of \([−6.5, 7.02]\). For multiple comparisons, any \( t \)-statistic with absolute value above 2.75 (red and blue regions) is considered as significant using the false discovery rate (FDR) 0.05 via Benjamini-Hochberg procedure [62]. If the \( t \)-statistic map shows high \( t \)-statistic value at particular vertex, it indicates that one group has consistently more gyral curves than sulcal curves at the vertex. If we use slightly different diffusion time \( \sigma \), we still obtain similar results.

Our results are highly consistent with literature. We mainly found that the differences are mainly in the temporal lobe, especially in the superior temporal gyrus and sulcus, which is highly consistent with literature. [63] reported that females have proportionally larger language areas compared to males, such as superior temporal cortex and Broca’s area. [23] reported statistically difference between males and females in the right superior temporal sulcus and the most posterior point and center of the left superior temporal sulcus. The significant gender differences in sulcal width and depth were reported in the superior temporal, collateral, and cingulate sulci in [64]. Also there was significant gender difference in cortical area of the left frontal lobe and in the gyrification index of the right temporal lobe [65], [66] detected higher...
gray matter concentrations in females in the left posterior superior temporal gyrus and left inferior frontal gyrus. [67] found significant differences between males and females in the sulcal curvature index of the temporal and occipital lobes. [32] reported that females showed higher gyriﬁcation in superior temporal, right inferior frontal, and parieto-occipital sulcal regions. In [68], the mean curvature of the left superior temporal sulcus was identiﬁed as a highly discriminative feature of sex classification.

We did an additional analysis using the standard surface metrics (mean curvature and shape index) and compared against the proposed sulcal/gyral curve analysis. Curvatures and shape index, which is the function of curvatures, are often estimated using the ﬁrst neighbor vertices. We estimated the curvature by ﬁtting the local quadratic surface in the ﬁrst estimated using the ﬁrst neighboring vertices. We estimated the curvature by projecting the local surface neighboring vertices of onto the tangent plane. The mean curvature and shape index are computed as the functions of \( \beta_j \) [57].

Curvature and shape indices are expected to be noisy (Figure 8). Such metrics often require surface smoothing to increase statistical sensitivity and the signal-to-noise ratio [41], [70], [71]. Smoothing surface data before statistical analysis is often done in various cortical surface features: cortical thickness [36], cortical surface coordinates [72], mean curvature [71], shape index and curvedness [70], shape complexity index [73], local gyriﬁcation index [74] and sulcal pits [75]. Even the FreeSurfer pipeline that computes the mean and Gaussian curvatures smooth curvature maps [76], [77]. Figure 8 displays the result of smoothed mean curvature and SI map using the proposed method.

Alternately, curvatures and surface shape features can be estimated by solving partial differential equations (PDE), which provides smooth maps [7], [78]–[80]. However solving a different kind of PDE is beyond the scope of this paper.

The proposed method also solves a PDE (heat equation) in the spectral domain.

We performed the two-sample \( t \)-test on the smoothed mean curvature and shape index maps (Figure 9-middle and right). The results show signiﬁcant gender difference mainly in the temporal lobe, consistent to the ﬁndings in the proposed sulcal/gyral curve analysis (Figure 9-left).

E. Multivariate two-sample \( t \)-test

The proposed iterative application of Chebyshev polynomial method was used to compute the diffusion at different time points quickly. The values of diffusion at different time points were then used in constructing the multiscale diffusion features. In this study, we adopted 10 time points \( \sigma = 0.0005, 0.001, \cdots, 0.0045, 0.005 \). The heat diffusion was computed by the proposed iterative convolution. Figure 10 shows the ﬂattened diffusion maps of one representative subject. At each vertex, the multiscale diffusion features are used determining the signiﬁcance of 268 females and 176 males. We used the the two-sample Hotelling’s \( T^2 \) statistic, which is the multivariate generalization of the two-sample \( t \)-statistic [16], [81]. Figure 13 shows the Hotelling’s \( T^2 \) statistics and the corresponding \( p \)-values in the log-scale. The heat diffusion has \( T^2 \) statistics in the range of \([0.13, 8.2]\) with minimum \( p \)-value \( 3.4 \times 10^{-12} \). For multiple comparisons, any \( T^2 \) statistic above 2.28 (yellow and red regions) is considered as signiﬁcant using the FDR 0.05.

In comparison, we used the diffusion wavelet features [14]–[17] at ten different scales and showed that the proposed method can achieve similar performance in localizing signal regions as the wavelet features. The diffusion wavelet [14], [15], [17] has the similar algebraic form as the heat kernel:

\[
\psi_t(p, q) = \sum_{j=0}^{\infty} g(\lambda_j t) \psi_j(p) \psi_j(q).
\]

The difference between the heat kernel and diffusion wavelet transform is the weight function \( g \), which determines the
spectral distribution. In the heat kernel \(k\) (2), the exponential weight \(e^{-\lambda_j \sigma}\) having higher weight value in low frequencies (small \(\lambda_j\)) makes the heat kernel work as a low-pass filter. The diffusion time \(\tau\) controls the bandwidth of the filter (Figure 11).

Compared to the heat kernel, the weight function \(g\) attenuating all low and high frequencies outside the passband makes the diffusion wavelet work as a band-pass filter. The wavelet transform of \(f\) is then given by

\[
\psi_* f(p) = \sum_{j=0}^{\infty} g(\lambda_j \tau)f_j \psi_j(p), \quad f_j = \int_{\mathcal{M}} f(p)\psi_j(p)d\mu(p).
\]

The proposed polynomial approximation scheme can be applied to the diffusion wavelet transform through expanding \(g(\lambda \tau)\) by orthogonal polynomials. In this paper, we used the following cubic spine as \(g(\lambda \tau)\) [14]

\[
g(x) = \frac{x^{-\alpha}x^\alpha}{x^{-\beta}x^\beta}, \quad x < x_1, \quad -5 + 11x - 6x^2 + x^3, \quad x_1 \leq x \leq x_2, \quad x > x_2, \quad (19)
\]

where \(\alpha = \beta = 2, x_1 = 1\) and \(x_2 = 2\). The scaling parameter \(t\) controls the passband of the diffusion wavelet (Figure 11).

Diffusion at different diffusion time \(\tau\) and diffusion wavelets at different scaling parameter \(t\) contain different spectral information of input data \(f\) (Figure 11). Thus, the heat diffusion with a varying \(\tau\) and diffusion wavelet with varying \(t\) provide multiscale features of \(f\). All the heat diffusion features contain low-frequency components. If the initial surface data suffer from significant low-frequency noise, the diffusion wavelet transform would be more suitable. On the other hand, if most noises are in high frequencies, performance of the both methods would be similar and we do not really needs diffusion wavelet features [82].

In this study, we adopted 10 different values of \(t = 0.002, 0.003, \ldots , 0.011\). Figure 12 shows the flattened diffusion wavelet maps of one representative subject. The parameters \(t\) are chosen empirically to match the amount of smoothing (FWHM) in the wavelet to the amount of smoothing in heat diffusion. Using the two-sample Hotelling’s \(T^2\) statistic on the multiscale diffusion wavelet features, we also contrasted 268 females and and 176 males. Figure 13 shows the Hotelling’s \(T^2\) statistics and the corresponding \(p\)-values in the log-scale. The diffusion wavelet transform has \(T^2\) statistics in the range of \([0.09, 7.6]\) with minimum \(p\)-value \(3.4 \times 10^{-11}\). For multiple comparisons, any \(T^2\) statistic above 2.37 (yellow and red regions) is considered as significant using the FDR 0.05. Although there are slight differences, the both methods showed the similar localization of sulcal and gyral graph patterns, mainly in the temporal lobe.

The exponential weight in the heat diffusion has only one parameter, i.e., the diffusion time \(\tau\), and leads to the analytic closed-form solutions to the expansion coefficients. The weight function in the diffusion wavelet transform is more complicated and it may not possible to derive the closed-form expression for the expansion coefficients. The simpler weight function in heat kernel and the iterative convolution scheme lead to faster computational run time compared to the diffusion wavelets. In heat diffusion, we only need to compute the expansion coefficients for \(\tau = 0.0005\) and reuse these coefficients in the iterative convolution to obtain the other nine features. The computation of the 1000 degree expansion coefficients by the proposed closed-form solution costed only \(2.6 \times 10^{-4}\) seconds. In the diffusion wavelet transform, due to the more complicated weight function, the 1000 degree...
expansion coefficients were computed numerically, which took 1.26 seconds [14], [16].

IV. CONCLUSION

The proposed general polynomial approximation of the Laplace-Beltrami (LB) operator works for an arbitrary orthogonal polynomial with the second order recurrence, and the closed-form solutions to the expansion coefficients allow us to compute the expansion analytically. The proposed polynomial expansion method speeds up the computation compared to existing numerical schemes for diffusion equations. Our method avoids various numerical issues associated with the LB-eigenfunction method and FEM-based diffusion solvers. The proposed fast and accurate scheme can be further extended to any arbitrary domain without much computational or memory bottlenecks. Thus, the method can be easily applicable to large-scale images where the existing methods may not be applicable without additional computational resources. Beyond the sulcal and gyral graph analysis on 2D surface meshes, the proposed method can be applied to 3D volumetric meshes for fast diffusion in image volume data [83]. This is left as a future study.

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