The K-essence Emergent f(\bar{R}, L(X))—Gravity with Power-law Expansion and Energy Conditions

Arijit Panda\(^a\) Goutam Manna\(^b\) and Aninda Karmakar\(^c\)
Department of Physics, Prabhat Kumar College, Contai, Purba Medinipur-721404, India

Saibal Ray\(^d\)
Department of Physics, Government College of Engineering & Ceramic Technology, 73, Abinash Chandra Banerjee Lane, Kolkata-700 010, India

Md. Rabiul Islam\(^e\)

\(^a\)Department of Physics, Raiganj University, Raiganj, Uttar Dinajpur-733 134, West Bengal, India.

In this work we reveal a strong and general formalism of f(\bar{R}, L(X))—gravity under the framework of the K-essence emergent geometry, where \(\bar{R}\) is the well known Ricci scalar of this geometry, \(L(X)\) is the Dirac-Born-Infeld (DBI) type non-canonical Lagrangian having the expression \(X = \frac{1}{2}g^{\mu\nu}\nabla_\mu \phi \nabla_\nu \phi\) where \(\phi\) being the scalar field of the K-essence geometry. The emergent gravity metric \(G_{\mu\nu}\) is not conformally equivalent to the well known gravitational metric \(g_{\mu\nu}\). We have set up a modified field equation by metric formalism in \(f(\bar{R}, L(X))\)-gravity and also the corresponding Friedmann equations in the framework of the background gravitational metric considered as Friedmann-Lemaître-Robertson-Walker (FLRW) type. The modified Friedmann equations are solved for the Starobinsky type specific choice of \(f(\bar{R}, L(X))\) by using power law expansion method. It is shown that the model is consistent with the accelerating phase of the Universe when we restrict ourselves to the value of the dark energy density, i.e., \(\delta^2 = \frac{\dot{\rho}}{\rho} = 0.888 < 1\) which indicates that the present Universe is dark energy dominated. Graphical plots for equation of state \(\omega\) w.r.t. time \(t\) based on parametric values are interestingly indicative for this accelerating features. We also put some light on the corresponding energy conditions and constraints of the \(f(\bar{R}, L(X))\) theory with one basic example, which also the fascinating outcome of our work.

PACS numbers: 04.20.-q, 04.50.-h, 04.50.Kd

I. INTRODUCTION

Weyl in 1919 \(^1\) and Eddington in 1923 \(^2\) reshaped Einstein’s theory by incorporating higher-order invariants in its action. The non-renormalizability of General relativity (GR) instructs us to quantize it conventionally. Thanks to Utiyama and De Witt \(^3\) who showed that renormalization at one loop requires the Einstein-Hilbert action to be supplemented by higher-order curvature terms and made renormalizable in 1962. Moreover when quantum corrections or string theory are considered, the effective low energy gravitational action admits higher-order curvature invariants \(^4\). This consideration has inspired the scientific community to modify the Einstein-Hilbert action in applied higher order theories of gravity.

Therefore it becomes obvious to include higher-order curvature invariants concerning Ricci scalar. The corrections applied to GR are considered to be important only at scales close to Plank scale and, consequently, in the early universe or near the black hole singularities such as curvature-driven inflation scenario \(^5\). In the works \(^6\)\(^7\), the authors made a significant contribution in modifying the theory of gravity which is actually the well known \(f(\bar{R})\) theory of gravity Dunsby et al. \(^8\) have reconstructed the \(f(\bar{R})\) gravity from the background Friedman-Lemaître-Robertson-Walker (FLRW) expansion history. Particularly, they have found that the only real-valued Lagrangian for which \(f(\bar{R})\) can mimic an exact \(\Lambda \text{CDM} \) expansion history for a universe, filled with dust-like matter is the Einstein-Hilbert Lagrangian with positive cosmological constant. Mukherjee and Banerjee \(^9\) have introduced a general formalism to investigate the late time dynamics of the universe for any analytic \(f(\bar{R})\) gravity model, along with a cold dark matter. The consequences of energy conditions in \(f(\bar{R})\) gravity in diverse scenario of cosmology has also been investigated in \(^10\)\(^11\). Also, following articles \(^12\)\(^13\) are described the energy conditions using the Raychaudhuri equations in expanding universe.

The generalization of the \(f(\bar{R})\) gravity has been framed by Harko et al. \(^14\) assuming the gravitational Lagrangian as an arbitrary function of the Ricci scalar (\(\bar{R}\)) and the matter Lagrangian (\(L_m\)). They have obtained the gravitational field equations of \(f(\bar{R}, L_m)\) gravity in the metric formalism, as well as the equations of motion for test particles. Also, later on Wang et. al. \(^15\) have developed the general energy conditions of the \(f(\bar{R}, L_m)\)
gravity. In [21,23], they have described the power-law cosmic expansion in higher derivative gravity.

Based on the Dirac-Born-Infeld (DBI) model [24,26], Manna et al. [27–29] have developed the simplest form of K-essence emergent gravity metric $g_{\mu\nu}$ which is not obviously conformally equivalent to the usual gravitational metric $g_{\mu\nu}$. The K-essence model [30,31] is a scalar field model where the kinetic energy of the field dominates over the potential energy of the field. The theoretical form of the K-essence field Lagrangian is non-canonical and it ensures the non-dependency of the Lagrangian on the field explicitly. The differences between the K-essence theory with non-canonical kinetic terms and the relativistic field theories with canonical kinetic terms lie in the non-trivial dynamical solutions of the K-essence equation of motion, which not only spontaneously break Lorentz invariance but change the metric also for the perturbations around these solutions. Thus the perturbations propagate in the so-called emergent or analogue curved space-time with the metric. The general form of the Lagrangian for K-essence model is:

$$L = -V(\phi) F(X) \quad \text{where} \quad \phi \text{ is the K-essence scalar field and} \quad X = \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi.$$  

Nojiri et al. [37] have studied a modified version of $f(R)$ gravity in which higher order kinetic terms of a scalar field has been added in the action of vacuum $f(R)$ gravity. Basically, in the form of action, they added a general class of the K-essence Lagrangian $G(X)$ with the vacuum $f(R)$ gravity Lagrangian. They have investigated the inflationary aspects of their theory in the context of slow-roll approximation. Odintsov et al. [38] have studied the effects of K-essence terms in the late-time phenomenology of $f(R)$ gravity in the presence of cold dark matter and radiation. Perfect fluids with the same model was considered in [37]. They also have studied several cosmological quantities of cosmological interest, such as the dark energy equation of state parameter, the dark energy density parameter, and several state finder quantities. Recently, Oikonomou et al. [39] have discussed the phase space of a simple K-essence $f(R)$ gravity theory. They have performed with both canonical and phantom scalar fields and the canonical scalar K-essence theory is structurally more fascinating in comparison with the phantom theory.

In this paper, which is based on the DBI type non-canonical Lagrangian $L(X)$, we have made the generalization of the $f(R, L(X))$ theory by the metric formalism in the context of the K-essence emergent gravity, where $R$ is the Ricci scalar of the K-essence geometry. Also, we have calculated the energy conditions and modified Friedman equations in $f(R, L(X))$ gravity, where we have considered the flat FLRW type metric as the background gravitational metric. The modified field equation, Friedman equations and energy conditions for the new $f(R, L(X))$ gravity theory are different from the usual $f(R, L_m)$ [19] and $f(R)$ gravity theories. Also, we have solved the modified Friedmann equations using power law cosmic expansion method.

The paper is organized as follows: In Sec. II, the energy conditions [10,14,20,50] in General Relativity with $f(R)$ and $f(R, L_m)$ gravity have been discussed briefly. In Sec. III, we have briefly discussed about the K-essence emergent geometry with the help of [27–34]. In Sec. IV, we have formulated the $f(R, L(X))$ gravity in the context of the K-essence emergent geometry. We also have derived the modified field equations and the condition of the requirement of the conservation of the energy-momentum tensor in the $f(R, L(X))$ gravity. The modified Friedmann equations were brought into lime-light in Sec. V, considering the background gravitational metric as flat FLRW and the K-essence scalar field as a function of time only. The solution of Friedmann equations have been solved for specific choice of $f(R, L(X))$ using power law method in Sec. VI whereas in Sec. VII, we develop the energy conditions and constraints of the $f(R, L(X))$ gravity with example. The last section is the discussion and conclusion of our work. Also, we have briefly discussed the $f(R)$-gravity and $f(R, L_m)$-gravity in Appendix.

II. BRIEF REVIEW OF ENERGY CONDITIONS IN GENERAL RELATIVITY

Following most of the techniques of [10,14,20,50], we will derive the energy conditions for modified ($f(R)$, $f(R, L_m)$, etc.) gravities. From these theories we can approach to the Null Energy Condition (NEC) and Strong Energy Condition (SEC) in the context of GR. The origin of these energy conditions comes from the Raychaudhuri equations. Let $u^\mu$ be the tangent vector field to a congruence of time-like geodesics in a space-time manifold endowed with a metric $g_{\mu\nu}$. Therefore, the Raychaudhuri equation [45,49] is

$$\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} u^\mu u^\nu, \quad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor corresponding to the metric $g_{\mu\nu}$, and $\theta$, $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ are the expansion, shear, and rotation associated with the congruence, respectively. While in the case of a congruence of null geodesics defined by the vector field $k^\mu$, the Raychaudhuri equation [48] is given by

$$\frac{d\theta}{d\tau} = -\frac{1}{2} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} k^\mu k^\nu. \quad (2)$$

These equations are purely based on geometric statements, and as such it makes no reference to any gravitational field equations. In other words, the Raychaudhuri equation can be thought of as geometrical identities which do not depend on any gravitational theory. These equations are provide the evolution of the expansion of a geodesic congruence. However, since the GR field equations relate $R_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$, the combination of Einstein and Raychaudhuri equations can be used to restrict energy-momentum tensors on physical
ground. Indeed, the shear is a “spatial” tensor, given by $\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$.

Thus it is clear from Raychaudhuri equation that for any hypersurface orthogonal ($\omega_{\mu\nu} \equiv 0$) the condition for attractive gravity (convergence of timelike geodesics or geodesic focusing) reduces to $(R_{\mu\nu}u^\mu u^\nu \geq 0)$, which by virtue of Einstein’s equation implies

$$R_{\mu\nu}u^\mu u^\nu = (T_{\mu\nu} - \frac{T}{2}g_{\mu\nu})u^\mu u^\nu \geq 0,$$  \hspace{1cm} (3)

where $T$ is trace of the energy momentum tensor $T_{\mu\nu}$ ($\kappa = 1$). Here Eq. (3) is nothing but the SEC stated in a coordinate-invariant way in terms of $T_{\mu\nu}$ and vector fields of fixed (time-like) character. Thus, in the context of GR, the SEC ensures the fact that the gravity is attractive. In particular, for a perfect fluid of density $\rho$ and pressure $p$

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}$$ \hspace{1cm} (4)

and the restriction given by Eq. (3) takes the familiar form for the SEC, i.e., $\rho + 3p \geq 0$.

On the other hand, the condition for the convergence (geodesic focusing) of hypersurface orthogonal ($\omega_{\mu\nu} \equiv 0$) congruences of null geodesics along with Einstein’s equation implies

$$R_{\mu\nu}k^\mu k^\nu = T_{\mu\nu}k^\mu k^\nu \geq 0$$ \hspace{1cm} (5)

which is the condition for NEC written in a coordinate-invariant way.

Thus, in GR the NEC ultimately encodes the null geodesic focusing due to the gravitational attraction. For the energy-momentum tensor of a perfect fluid (4) the above condition (5) reduces to the well-known form of the NEC, i.e., $\rho + p \geq 0$.

The Weak Energy Condition (WEC) states that $T_{\mu\nu}u^\mu u^\nu \geq 0$ for all time-like vectors $u^\mu$, or equivalently for perfect fluid it is $\rho > 0$ and $\rho + p > 0$. The Dominant Energy Condition (DEC) includes the WEC as well as the additional requirement that $T_{\mu\nu}u^\mu$ is a non space-like vector i.e., $T_{\mu\nu}T^\nu_\lambda u^\mu u^\lambda \leq 0$. For a perfect fluid, these conditions together are equivalent to the simple requirement that $\rho \geq |p|$, the energy density must be nonnegative, and greater than or equal to the magnitude of the pressure.

**In $f(R)$-gravity** \cite{14, 11}, the energy conditions for perfect fluid are given by

$$\text{SEC : } \rho + 3p - f + Rf' + 3(\bar{R} + \bar{R}H)f'' + 3\bar{R}^2 f''' \geq 0,$$

$$\text{NEC : } \rho + p + (\bar{R} - \bar{R}H)f'' + \bar{R}^2 f''' \geq 0,$$

$$\text{WEC : } \rho + \frac{1}{2}(f - Rf') - 3\bar{R}f'' \geq 0,$$

$$\text{DEC : } \rho - p + f - Rf' - (\bar{R} + 5\bar{R}H)f'' - \bar{R}^2 f''' \geq 0,$$ \hspace{1cm} (6)

where $f' = \frac{df(R)}{dR}$.

In $f(R, L_m)$-gravity \cite{20}, the energy conditions are

$$\text{SEC : } \rho + 3p - \frac{2}{f_{L_m}}[f - Rf'] + \frac{6}{f_{L_m}}[\bar{R}^2 f'' + \bar{R}f'' + H\bar{R}f''] - 2L_m \geq 0.$$

$$\text{NEC : } \rho + p + \frac{2}{f_{L_m}}[f - Rf'] \geq 0,$$

$$\text{WEC : } \rho + \frac{1}{f_{L_m}}[f - Rf'] - \frac{6}{f_{L_m}}H\bar{R}f' + L_m \geq 0,$$

$$\text{DEC : } \rho - p + \frac{2}{f_{L_m}}[f - Rf'] - \frac{2}{f_{L_m}}[\bar{R}^2 f'' + \bar{R}f'' + 6H\bar{R}f''] + 2L_m \geq 0,$$ \hspace{1cm} (7)

where $f' = \frac{df(R, L_m)}{dR}$.

### III. K-ESSSENCE THEORY: BACKGROUND AND DEVELOPMENT

In this section, we will discuss about the development of construction of the effective metric for the emergent spacetime corresponding to a general background geometry and a very general K-essence scalar field vector. The K-essence scalar field $\phi$, minimally coupled with the background space-time metric $g_{\mu\nu}$ has action \cite{30–34}.

$$S_k[\phi, g_{\mu\nu}] = \int d^4x\sqrt{-g}L(X, \phi),$$ \hspace{1cm} (8)

where $X = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ is the canonical kinetic term.

The energy-momentum tensor is

$$T_{\mu\nu} = -2\frac{\delta S_k}{\sqrt{-g}\frac{\delta g_{\mu\nu}}{\delta g_{\mu\nu}}} = -2\frac{\partial L}{\delta g_{\mu\mu}} + g_{\mu\nu}L = -L_X\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}L,$$ \hspace{1cm} (9)

$L_X = \frac{dL}{dX}$, $L_{XX} = \frac{d^2L}{dX^2}$, $L_\phi = \frac{dL}{d\phi}$ and $\nabla_\mu$ is the covariant derivative defined with respect to the gravitational metric $g_{\mu\nu}$.

The equation of motion of the scalar field is

$$-\frac{1}{\sqrt{-g}}\frac{\delta S_k}{\delta \phi} = G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 2XLX\phi - L_\phi = 0,$$ \hspace{1cm} (10)

where

$$G^{\mu\nu} = \frac{c_s^2}{L_X}[L_Xg^{\mu\nu} + L_{XX}\nabla_\mu\phi\nabla_\nu\phi],$$ \hspace{1cm} (11)

with $1 + \frac{2XLXX}{L_X} > 0$ and $c_s^2(X, \phi) \equiv (1 + 2XLXX)^{-1}$.

The inverse metric of $G^{\mu\nu}$ takes the form

$$G_{\mu\nu} = \frac{L_X}{c_s^2}[g_{\mu\nu} - \frac{L_{XX}}{L_X}\nabla_\mu\phi\nabla_\nu\phi].$$ \hspace{1cm} (12)

A further conformal transformation \cite{27, 28} $\tilde{G}_{\mu\nu} \equiv \frac{c_s^2}{L_X}G_{\mu\nu}$ gives

$$\tilde{G}_{\mu\nu} = g_{\mu\nu} - \frac{L_{XX}}{L_X + 2XLXX}\nabla_\mu\phi\nabla_\nu\phi.$$ \hspace{1cm} (13)
Using Eq. (9), the effective emergent metrics (13) can be written as (32, 33):

\[ \bar{G}_{\mu\nu} = \left(1 - \frac{LLXX}{L_X(X + 2XLXX)}\right) g_{\mu\nu} + \frac{LX}{L_X(X + 2XLXX)} T_{\mu\nu}. \] (14)

Here one must always have \( L_X \neq 0 \) for \( \varepsilon^2 \) to be positive definite and only then Eqs. (6) – (11) will be physically meaningful.

It is clear that, for non-trivial space-time configurations of \( \phi \), the emergent metric \( \bar{G}_{\mu\nu} \) is in general, not conformally equivalent to \( g_{\mu\nu} \). So \( \phi \) has different properties as of the canonical scalar fields with the local causal structure. It is distinct from those defined with \( g_{\mu\nu} \). Further, if \( L \) is not an explicit function of \( \phi \), the equation of motion (11) reduces to:

\[-\frac{1}{\sqrt{-g}} \frac{\delta S_k}{\delta \phi} = \bar{G}^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \phi = 0. \] (15)

We shall take the Dirac-Born-Infeld (DBI) type Lagrangian (24, 26, 33, 34) as

\[ L(X, \phi) = 1 - \sqrt{1 - 2X}. \] (16)

It should be noted that here we ignore the any type of potential part of the Lagrangian, since in the K-essence theory, the kinetic energy dominates over the potential energy of the K-essence scalar field. Then \( c^2(X, \phi) = 1 - 2X \) and thus the effective emergent metric (13) becomes

\[ \bar{G}_{\mu\nu} = g_{\mu\nu} - \nabla_{\mu} \phi \nabla_{\nu} \phi \equiv g_{\mu\nu} - \partial_{\mu} \phi \partial_{\nu} \phi, \] (17)

since \( \phi \) is a scalar.

Using Eq. (13), this can also be written in terms of \( T_{\mu\nu} \) and \( \nabla_{\mu} \phi \), (11) as

\[ \bar{G}_{\mu\nu} L = L_X \nabla_{\mu} \phi \nabla_{\nu} \phi + T_{\mu\nu} - L \nabla_{\mu} \phi \nabla_{\nu} \phi. \] (18)

Following (27, 40) the new Christoffel symbols can be related to the old ones by

\[ \Gamma_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha + (1 - 2X)^{-1/2} \bar{G}_{\alpha\gamma} \left[ \bar{G}_{\mu\gamma} \partial_\nu (1 - 2X) \right]^{1/2} + \bar{G}_{\nu\gamma} \partial_\mu (1 - 2X)^{1/2} - \bar{G}_{\mu\nu} \partial_\gamma (1 - 2X)^{1/2} \]

\[-\frac{1}{2 (1 - 2X)} \left[ \delta_{\mu}^\alpha \partial_{\nu} X + \delta_{\nu}^\alpha \partial_{\mu} X \right]. \] (19)

The geodesic equation for the K-essence theory in terms of the new Christoffel connections \( \bar{\Gamma} \) now becomes

\[ \frac{d^2x^\alpha}{d\lambda^2} + \bar{\Gamma}_{\mu\nu}^\alpha \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0, \] (20)

where \( \lambda \) is an affine parameter.

Now we introduces the covariant derivative \( D_\mu \) (32, 33) associated with the emergent metric \( \bar{G}_{\mu\nu} \) \( (D_\alpha \bar{G}^{\alpha\beta} = 0) \) as

\[ D_\mu A_\nu = \partial_\mu A_\nu - \bar{\Gamma}_{\mu\nu}^\lambda A_\lambda \] (21)

and the inverse emergent metric is \( \bar{G}^{\mu\nu} \) such as \( G_{\mu\lambda} \bar{G}^{\lambda\nu} = \delta_\nu^\nu \).

Therefore, the “emergent” Einstein’s equation is

\[ \bar{E}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2} \bar{G}_{\mu\nu} \bar{R} = \kappa T_{\mu\nu}, \] (22)

where \( \kappa = 8\pi G \) is constant, \( \bar{R}_{\mu\nu} \) is Ricci tensor and \( \bar{R} (= R_{\mu\nu} \bar{G}^{\mu\nu}) \) is the Ricci scalar of the emergent space-time.

**IV. \( f(\bar{R}, L(X)) \) GRAVITY IN THE CONTEXT OF K-ESSENCE EMERGENT SPACE-TIME**

We are now considering the action of the modified gravity in the context of the K-essence emergent space-time, which takes the following form \( (\kappa = 1) \)

\[ S = \int d^4x \sqrt{-G} \ f(\bar{R}, L(X)), \] (23)

where \( f(\bar{R}, L(X)) \) is an arbitrary function of the Ricci scalar \( \bar{R} \) and the non-canonical Lagrangian density \( L(X) \) corresponding to the K-essence theory, and

\[ \sqrt{-G} = \sqrt{-\det(\bar{G}_{\mu\nu})}. \]

Based on (19), varying the action \( S \) with respect to the K-essence emergent gravity metric \( \bar{G}^{\mu\nu} \) we obtain

\[ \delta S = \int \left[ f_\bar{R}(\bar{R}, L) \delta \bar{R} + f_L(\bar{R}, L) \frac{\delta L}{\delta \bar{G}^{\mu\nu}} \delta \bar{G}^{\mu\nu} \right] \times \sqrt{-G} \ d^4x, \] (24)

where we have denoted \( f_\bar{R}(\bar{R}, L) = \frac{\partial f(\bar{R}, L)}{\partial \bar{R}} \) and \( f_L(\bar{R}, L) = \frac{\partial f(\bar{R}, L)}{\partial L} \).

Now we obtain the variation of the Ricci scalar for the K-essence emergent gravity metric

\[ \delta \bar{R} = \delta (\bar{R}_{\mu\nu} \bar{G}^{\mu\nu}) = \delta \bar{R}_{\mu\nu} \bar{G}^{\mu\nu} + \bar{R}_{\mu\nu} \delta \bar{G}^{\mu\nu} \]

\[ = \bar{R}_{\mu\nu} \delta \bar{G}^{\mu\nu} + \bar{G}^{\mu\nu} (D_\lambda \delta \bar{G}^{\lambda\mu} - D_\nu \delta \bar{G}^{\lambda\nu}), \] (25)

where

\[ \bar{R}_{\mu\nu} = \partial_\mu \bar{G}^{\alpha\nu} - \partial_\nu \bar{G}^{\alpha\mu} + \bar{G}^{\alpha\beta} \partial_\mu \bar{G}_{\beta\nu} - \bar{G}^{\alpha\beta} \partial_\nu \bar{G}_{\alpha\beta}, \] (26)

\[ \bar{G}^{\mu\nu} [\partial_\mu \bar{G}_{\beta\nu} - \partial_\nu \bar{G}_{\beta\mu} - \bar{G}_{\beta\mu}] = \delta \bar{G}^{\mu\nu} \]

(27)

and the variation of \( \delta \bar{G}^{\lambda\mu} \) is

\[ \delta \bar{G}^{\lambda\mu} = \frac{1}{2} \bar{G}^{\lambda\alpha} \left[ D_\mu \delta \bar{G}_{\rho\alpha} + D_\alpha \delta \bar{G}_{\mu\rho} - D_\rho \delta \bar{G}_{\mu\alpha} \right]. \] (28)

Thus, the expression for the variation of Ricci scalar \( \delta \bar{R} \) is

\[ \delta \bar{R} = \bar{R}_{\mu\nu} \delta \bar{G}^{\mu\nu} + \bar{G}_{\mu\nu} D_\alpha D^\alpha \delta \bar{G}^{\mu\alpha} - D_\mu D_\nu \delta \bar{G}^{\mu\nu}. \] (29)
Therefore, variation of the action [24] is
\[
\delta S = \int [f_R(R,\bar{R},L)\bar{R}_{\mu\nu}\delta \bar{G}^{\mu\nu} + f_R(R,\bar{R},L)\bar{G}_{\mu\nu}D_\alpha D^\alpha \delta \bar{G}^{\mu\nu} - f_R(R,\bar{R},L)D_\mu D_\nu \delta \bar{G}^{\mu\nu} + f_L(R,\bar{R})\frac{\delta L}{\delta \bar{G}^{\mu\nu}}\delta \bar{G}^{\mu\nu} - \frac{1}{2} \bar{G}_{\mu\nu}f(R,\bar{R})\delta \bar{G}^{\mu\nu}] \sqrt{-G}d^4x. \tag{30}
\]

After partially integrating second and third terms of the above Eq. (30), we get
\[
\delta S = \int \left[ f_R(R,\bar{R},L)\bar{R}_{\mu\nu} + \bar{G}_{\mu\nu}D_\mu D^\alpha f_R(R,\bar{R},L) \\
- D_\mu D_\nu f_R(R,\bar{R},L) + f_L(R,\bar{R})\frac{\delta L}{\delta \bar{G}^{\mu\nu}} - \frac{1}{2} \bar{G}_{\mu\nu}f(R,\bar{R})\right] \sqrt{-G}d^4x. \tag{31}
\]

Therefore, using principle of least action, i.e. \( \delta S = 0 \), we have the modified field equation for \( f(R, L(X)) \) theory
\[
f_R(R,\bar{R},L)\bar{R}_{\mu\nu} + \bar{G}_{\mu\nu}D_\alpha D^\alpha f_R(R,\bar{R},L) - D_\mu D_\nu f_R(R,\bar{R},L) \\
- \frac{1}{2} \bar{G}_{\mu\nu}f(R,\bar{R}) + f_L(R,\bar{R})\frac{\delta L}{\delta \bar{G}^{\mu\nu}} = 0. \tag{32}
\]

Now we evaluate the term \( \frac{\delta L}{\delta \bar{G}^{\mu\nu}} \) as
\[
\frac{\delta L}{\delta \bar{G}^{\mu\nu}} = \frac{\delta L}{\delta \bar{X}} \frac{\delta \bar{X}}{\delta \bar{G}^{\mu\nu}} = \frac{1}{2} L X D_\mu D_\nu \phi(1 + D_\alpha \phi D^\alpha \phi), \tag{33}
\]

since for the scalar field: \( \nabla_\mu \phi \equiv D_\mu \phi \equiv D_\mu \phi \).

Using Eqs. (18), (33) and (32), we obtain the expression for modified field equation for the \( f(R, L(X)) \) theory in terms of \( T_{\mu\nu} \) as
\[
f_R(R,\bar{R},L)\bar{R}_{\mu\nu} + (\bar{G}_{\mu\nu}\square - D_\mu D_\nu) f_R(R,\bar{R},L) \\
- \frac{1}{2} \left[ f(R,\bar{R},L) - L f_L(R,\bar{R})(1 + D_\alpha \phi D^\alpha \phi) \right] \bar{G}_{\mu\nu} \\
+ \frac{1}{2} L f_L(R,\bar{R})D_\mu D_\nu \phi + f_L(R,\bar{R}) \left[ \bar{G}_{\mu\nu} \right] \bar{G}_{\mu\nu} + \frac{1}{2} \bar{f}_L(R,\bar{R})T_{\mu\nu}[1 + D_\alpha \phi D^\alpha \phi] = \frac{1}{2} f_L(R,\bar{R})\bar{T}_{\mu\nu}, \tag{34}
\]

where \( \square = D_\mu D^\mu \) and \( \bar{T}_{\mu\nu} = T_{\mu\nu}[1 + D_\alpha \phi D^\alpha \phi] \).

The above Eq. (34) is different from the usual Eq. (114) of \( f(R, L_m) \) theory in the presence of the \( K \)-essence scalar field (vide Appendix). If we consider the emergent gravity metric \( \bar{G}_{\mu\nu} \) is conformally equivalent to the gravitational metric \( g_{\mu\nu} \) and \( L \) can be matter Lagrangian then we get back to the usual \( f(R, L_m) \) theory in the absence of the \( \bar{K} \)-essence scalar field. Also, if we consider \( f(R, L(X)) \equiv f(R, L_m) \equiv \frac{1}{2} R + L_m \), i.e., the Hilbert-Einstein Lagrangian form, then from (34), we lead to the standard Einstein field equation \( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \).

Contracting the above field equation (34) with \( \bar{G}^{\mu\nu} \), we have the modified trace equation for the \( f(R, L(X)) \)
\[
\frac{\delta L}{\delta G^{\mu\nu}} \right] \delta G^{\mu\nu} + D^\mu [L \phi D_\nu \phi(1 + D_\alpha \phi D^\alpha \phi) + L\bar{G}_{\mu\nu}D_\alpha \phi D^\alpha \phi] \\
+ 2D^\mu \ln[f_L(R,\bar{R})] \frac{\delta L}{\delta G^{\mu\nu}} + D^\mu [L \phi D_\nu \phi(1 + D_\alpha \phi D^\alpha \phi) + L\bar{G}_{\mu\nu}D_\alpha \phi D^\alpha \phi] = 0. \tag{39}
\]

Thus, the requirement of the conservation of the energy-momentum tensor (\( D^\mu \bar{T}_{\mu\nu} = 0 \)) for the \( K \)-essence Lagrangian, gives an effective functional relation as
\[
2D^\mu \ln[f_L(R,\bar{R})] \frac{\delta L}{\delta G^{\mu\nu}} + D^\mu [L \phi D_\nu \phi(1 + D_\alpha \phi D^\alpha \phi) + L\bar{G}_{\mu\nu}D_\alpha \phi D^\alpha \phi] = 0. \tag{39}
\]
V. MODIFICATION OF THE FRIEDMANN EQUATIONS

We consider the gravitational metric $g_{\mu\nu}$ to be a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the line element is

$$ds^2 = dt^2 - a^2(t) \sum_{i=1}^{3} (dx^i)^2,$$

(40)

with $a(t)$, the scale factor as usual.

From Eq. (17), we have the components of the emergent gravity metric are

$$\bar{G}_{00} = (1 - \dot{\phi}^2) ; \quad \bar{G}_{ii} = -(a^2(t) + (\dot{\phi})^2) ;$$

$$\bar{G}_{0i} = -\dot{\phi} \dot{a} = \bar{G}_{di},$$

(41)

where we consider $\phi \equiv \phi(t, x^i)$, $\dot{\phi} \equiv \partial \phi / \partial t$ and $\dot{a} \equiv \partial a / \partial t$. So the line element of the FLRW emergent gravity metric is

$$dS^2 = (1 - \dot{\phi}^2)dt^2 - [a^2(t) + (\dot{\phi})^2] \sum_{i=1}^{3} (dx^i)^2$$

$$- 2 \dot{\phi} a \dot{a} dt dx^i.$$

(42)

Now from the emergent gravity equation of motion (15), we have

$$G_{00} \left( \partial_0 \partial_0 \phi - \Gamma^0_0 \partial_0 \phi - \Gamma^0_0 \partial_i \phi \right) + \bar{G}_{0i} \left( \partial_0 \partial_i \phi \right) - \Gamma_{i0}^0 \partial_0 \phi - \Gamma_{i0}^0 \partial_0 \phi + \bar{G}_{ii} \left( \partial_i \partial_0 \phi - \Gamma^0_0 \partial_0 \phi - \Gamma^0_i \partial_i \phi \right) = 0.$$  

(43)

For simplification, we consider the homogeneous $K$-essence scalar field $\phi$, i.e., $\phi(t, x^i) \equiv \phi(t)$ then $\bar{G}_{00} = (1 - \dot{\phi}^2)$, $\bar{G}_{0i} = \bar{G}_{i0} = 0 = \partial_i \phi$, $\bar{G}_{ii} = -a^2(t)$ and $X = \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi = \frac{1}{2} \dot{\phi}^2$. This consideration is possible in this case since the dynamical solutions of the $K$-essence scalar fields spontaneously break Lorentz symmetry. Therefore, the flat FLRW emergent gravity line element (12) and the equation of motion (13) become

$$dS^2 = (1 - \dot{\phi}^2)dt^2 - a^2(t) \sum_{i=1}^{3} (dx^i)^2$$

(44)

and

$$\frac{\dot{a}}{a} = H(t) = -\frac{\dot{\phi}}{\phi(1 - \dot{\phi}^2)},$$

(45)

where $H(t) = \frac{\dot{a}}{a}$ is the usual Hubble parameter (always $\dot{a} \neq 0$). The Eq. (15) gives the relation between Hubble parameter and the time derivatives of the $K$-essence scalar field. Note that in the above space-time (13) always $\dot{\phi}^2 < 1$. If $\dot{\phi}^2 > 1$, the signature of this space-time will be ill-defined. Also $\dot{\phi}^2 \neq 0$ condition holds good instead of $\dot{\phi}^2 = 0$, which leads to non-applicability of the $K$-essence theory.

The Ricci tensors and Ricci scalar of the emergent gravity space-time are

$$\bar{R}_{ii} = -\frac{a^2}{1 - \phi^2} \left[ \frac{\dot{a}}{a} + \frac{2}{a} \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dot{a} \ddot{\phi}}{a} \left( 1 - \frac{1}{a^2} \phi^2 \right) \right]$$

$$= -\frac{a^2}{1 - \phi^2} \left[ \frac{\dot{a}}{a} + \left( \frac{a}{a} \right)^2 (2 - \phi^2) \right]$$

$$= -\frac{a^2}{1 - \phi^2} \left[ \dot{H} + H^2 (3 - \phi^2) \right],$$

(46)

$$\bar{R}_{00} = 3 \frac{\ddot{a}}{a} + \frac{3}{a} \frac{\dot{a} \ddot{\phi}}{a} = 3 \frac{\ddot{a}}{a} + 3 \left( \frac{\dot{a}}{a} \right)^2 \phi^2$$

$$= 3 \left[ \dot{H} + H^2 (1 - \phi^2) \right]$$

(47)

and

$$\bar{R} = \frac{6}{1 - \phi^2} \left[ \frac{\ddot{a}}{a} + \left( \frac{a}{a} \right)^2 + \frac{\dot{a} \ddot{\phi}}{a} \left( 1 - \phi^2 \right) \right]$$

$$= \frac{6}{1 - \phi^2} \left[ \dot{H} + H^2 (2 - \phi^2) \right],$$

(48)

where we have used the relation (45) and $H \equiv \frac{\partial H}{\partial t} = \frac{\ddot{a} - \dot{\phi}^2}{a}$.

Combining Eqs. (46) and (47) with (48), we get

$$\bar{R}_{00} = \frac{1}{2} (1 - \dot{\phi}^2) \dot{H} - 3H^2,$$

(49)

$$\bar{R}_{ii} = -\frac{a^2}{1 - \phi^2} \left[ \frac{1}{6} \dot{H} (1 - \dot{\phi}^2) + H^2 \right].$$

(50)

We assume that the energy-momentum tensor is an ideal fluid type which is

$$T^\mu_\nu = \text{diag}(\rho, -p, -p, -p) = (\rho + p) u_\mu u_\nu - \delta^\mu_\nu \rho$$

$$T_{\mu\nu} = \bar{G}_{\mu\nu} T^\nu_\nu,$$

(51)

where $p$ is pressure and $\rho$ is the matter density of the cosmic fluid. In the co-moving frame we have $u^0 = 1$ and $u^\alpha = 0$ : $\alpha = 1, 2, 3$ in the $K$-essence emergent gravity space-time. Now the question is whether this type of energy-momentum tensor is valid or not in case of a perfect fluid model when the kinetic energy $(\dot{\phi}^2)$ of the $K$-essence scalar filed is present. The answer is yes since our Lagrangian is $L(X) = 1 - \sqrt{1 - 2X}$. This class of models is equivalent to perfect fluid models with zero vorticity and the pressure (Lagrangian) can be expressed through the energy density only [33, 34].
Now we substitute Eqs. (49) and (54) in the above $F\bar{R}_{ii} + (\bar{G}_{ii}\square - D_iD_i)\bar{F} - \frac{1}{2}[f - LF_L(1 + \dot{\phi}^2)]\bar{G}_{ii} = \frac{1}{2}L_f\tilde{a}^2(t)\tilde{p}$ \hspace{1cm} (52)

and

$$F\bar{R}_{00} + (\bar{G}_{00}\square - D_0D_0)\bar{F} - \frac{1}{2}[f - LF_L(1 + \dot{\phi}^2)]\bar{G}_{00} + \frac{1}{2}L_f\dot{\phi}^2(1 + \dot{\phi}^2) = \frac{1}{2}f_L(1 - \dot{\phi}^2)\tilde{p}, \hspace{1cm} (53)$$

with $F = f_R(\bar{R}, L) \equiv \frac{\partial f(R, L)}{\partial R}, \tilde{p} = p(1 + \dot{\phi}^2)$ and $\tilde{p} = \rho(1 + \dot{\phi}^2)$.

Now we calculate the terms $\bar{G}_{00}\square F$ and $\bar{G}_{ii}\square F$, using the determinant of the flat FLRW emergent gravity metric $\sqrt{-G} = a^3\sqrt{1 - \dot{\phi}^2}$ and the relation (55):

$$\bar{G}_{00}\square F = \bar{F} + 3\tilde{a}\tilde{\phi} + \bar{F} - \frac{\ddot{\phi}\phi}{(1 - \dot{\phi}^2)} = \bar{F} + H\bar{F}(3 - \dot{\phi}^2) \hspace{1cm} (54)$$

and

$$\bar{G}_{ii}\square F = D_iD_iF - \frac{\tilde{a}^2}{(1 - \dot{\phi}^2)}[\bar{F} + 2H\bar{F}(1 - \dot{\phi}^2)], \hspace{1cm} (55)$$

where we have used $(\partial, t)^2 = \frac{\tilde{a}^2}{1 - \dot{\phi}^2}$ for flat FLRW emergent gravity metric.

Now we substitute Eqs. (51) and (54) in the above Eq. (55), so that we have the first modified Friedmann equation as

$$3H^2 = \frac{1}{F}[-\frac{1}{2}\tilde{p}f_L(1 - \dot{\phi}^2) + 3H\bar{F} + (1 - \dot{\phi}^2)\frac{1}{2}(F\bar{R} - 2R) + \frac{1}{2}L_f\dot{\phi}^2(1 + \dot{\phi}^2)]$$

$$= \frac{1}{F}[-\frac{1}{2}\tilde{p}f_L(1 - \dot{\phi}^2) + 3H\bar{F} + 3HF_LX\dot{X} + (1 - \dot{\phi}^2)\frac{1}{2}(F\bar{R} - 2R) + \frac{1}{2}L_f\dot{\phi}^2(1 + \dot{\phi}^2)]. \hspace{1cm} (56)$$

We also substitute Eqs. (50), (52) and (56) in the $ii$-components of Eq. (52) and thereafter rearranging we get the second modified Friedmann equation for the flat FLRW $K$-essence emergent gravity space-time under $f(\bar{R}, L(X))$ theory. Hence

$$2\ddot{\phi} + H^2(3 - 2\dot{\phi}^2) = \frac{1}{F}\left[\frac{1}{2}\tilde{p}f_L(1 - \dot{\phi}^2) + \bar{F} + 2HF(1 - \dot{\phi}^2) - \frac{1}{2}(1 - \dot{\phi}^2)(f - RF) + \frac{1}{2}L_f\dot{\phi}^2(1 + \dot{\phi}^2)\right]$$

$$+ 2H\ddot{R}F_R(1 - \dot{\phi}^2) - \frac{1}{2}(1 - \dot{\phi}^2)(f - RF) + \frac{1}{2}L_f\dot{\phi}^2(1 + \dot{\phi}^2)]$$

$$+ \frac{1}{F}[2H(1 - \dot{\phi}^2)F_LX\dot{X} + F_{LL}(LX\dot{X})^2 + F_LXLX(\dot{X})^2 + F_LXL\ddot{X}]. \hspace{1cm} (57)$$

Since the Lagrangian $(L)$ of the $K$-essence theory is function of $X(= \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu\phi)$, therefore, we can write

$$\ddot{\phi} + F = F_R\ddot{\phi} + F_LXL\dot{X} \text{ and } \ddot{\phi} = F_R\ddot{\phi} + (\ddot{R})^2F_R\ddot{R} + F_{LL}(LX\dot{X})^2 + F_LXLX(\dot{X})^2 + F_LXL\ddot{X}. \hspace{1cm} (58)$$

The above Friedmann equations in the presence of the kinetic energy of the $K$-essence scalar field are different from the usual $f(\bar{R})$ gravity model. Note that if we consider $f(\bar{R}, L(\phi)) = f(\bar{R})$ and $G_{\mu\nu} = g_{\mu\nu}$ then the above modified Friedmann equations (56) and (57) reduces to the usual Friedmann equations of $f(\bar{R})$ gravity with $\kappa = 1$ and $T_{\mu\nu}$ replaced by $\frac{1}{2}T_{\mu\nu}$ \hspace{1cm} (56) as

$$3H^2 = \frac{1}{F}\left[-\frac{\ddot{\rho}}{2} + \frac{RF - f}{2} - 3H\ddot{R}F_R\right] \hspace{1cm} (59)$$

and

$$2\ddot{\phi} + 3H^2 = \frac{1}{F}\left[\frac{\ddot{\rho}}{2} + (\ddot{R})^2F_{RR} + 2HF\ddot{R} - \ddot{R}F_R - \frac{f - RF}{2}\right]. \hspace{1cm} (60)$$

VI. SOLUTION TO THE FRIEDMANN EQUATION USING POWER LAW

Now we take the Starobinsky model [3, 4] and write $f(\bar{R}, L)$ as

$$f(\bar{R}, L) = \bar{R} + \alpha\bar{R}^2 + L \hspace{1cm} (61)$$

and $L$ can be written for homogeneous $K$-essence scalar field as

$$L = 1 - \sqrt{1 - 2X} = 1 - \sqrt{1 - \dot{\phi}^2}. \hspace{1cm} (62)$$

Therefore, we get

$$\frac{\partial f}{\partial L} = 1, \quad F = \frac{\partial f}{\partial R} = 1 + 2\alpha\bar{R}, \quad F_R = 2\alpha, \quad F_L = 0. \hspace{1cm} (63)$$
Using these values and after some algebraic calculations we can write Friedmann equation (56) as

\[
(1 + 2\alpha R)3H^2 = -\frac{1}{2} \bar{\rho}(1 - \dot{\phi}^2) + 6\alpha H \dot{R}
+ \frac{1}{2}(1 - \dot{\phi}^2)(\alpha \dot{R}^2 - 1 + \sqrt{1 - \dot{\phi}^2}).
\]  

(64)

Analogous to [21–23], let us now assume there exists an exact power-law solution to the field equations, i.e., the scale factor behaves as

\[
a(t) = a_0 t^m,
\]  

where \( m > 0 \) is a fixed real number.

Then it can be easily calculated that

\[
\dot{a}(t) = a_0 m t^{m-1}, \quad \ddot{a}(t) = a_0 m (m - 1) t^{m-2}.
\]  

(66)

Also, the definition of \( H \) gives us

\[
H = \frac{\dot{a}}{a}, \quad \dot{H} = \frac{m}{t^2}, \quad \ddot{H} = -\frac{6m}{t^4}.
\]  

(67)

Now, taking Eq. (68) into consideration we can evaluate the value of Ricci scalar as

\[
\bar{R} = \frac{6}{(1 - \dot{\phi}^2)t^2} \left[ -m + m^2(2 - \dot{\phi}^2) \right]
\]  

(68)

and then using Eq. (63) we get

\[
\dot{\bar{R}} = \frac{6}{1 - \dot{\phi}^2} \left[ \ddot{H} + 4H \dot{H}(1 - \dot{\phi}^2) - 2H^2 \dot{\phi}^2 \right]
= \frac{12}{t^3(1 - \dot{\phi}^2)} \left[ m - m^3 \dot{\phi}^2 - 2m^2(1 - \dot{\phi}^2) \right].
\]  

(69)

Now putting the values of Eqs. (65) – (69) in (61), we simply get

\[
\frac{1}{2} \bar{\rho}(1 - \dot{\phi}^2) = -\frac{m}{t^2(1 - \dot{\phi}^2)} \left( 2(1 - \dot{\phi}^2) - 3\alpha \dot{\phi}^2 \right)
+ \frac{m^2}{t^2}(3 - 2\dot{\phi}^2) - \frac{3\alpha m^2}{t^4(1 - \dot{\phi}^2)} \left( 56 - 2t^2 \dot{\phi}^2 + \dot{\phi}^4 t^2 \right)
+ \frac{4\alpha m^3}{t^4(1 - \dot{\phi}^2)} \left( 3 - 28\dot{\phi}^2 - 2\dot{\phi}^4 \right)
+ \frac{4\alpha m^4}{t^4(1 - \dot{\phi}^2)} \left( 18 - 9\dot{\phi}^2 + 7\dot{\phi}^4 \right).
\]  

(72)

For our case, the energy-momentum conservation relation is

\[
D^\mu \bar{T}_{\mu\nu} = 0,
\]  

(73)

with \( \bar{T}_{\mu\nu} = T_{\mu\nu}[1 + \alpha_\phi D^\nu \phi] \).

Now, using Eqs. (61) and (73), we have the conserving equation as

\[
\dot{\rho} = 3\alpha \bar{\rho}(\bar{\rho} + \bar{p}),
\]  

(74)

where \( \bar{\rho} \) and \( \bar{p} \) already have been defined. It is essential to mention here that \( \bar{\rho} \) and \( \bar{p} \) are not same as the normal \( \rho \) and \( p \).

Now considering the power law, we get from Eq. (74)

\[
\bar{\rho} = \rho(1 + \dot{\phi}^2) = \rho_0 t^{-3m(1 + \omega)},
\]  

(75)

where \( \omega = \frac{\bar{\rho}}{\bar{p}} = \bar{p} \).

Now put the value of \( \bar{\rho} \) in Friedmann equation (70) we have

\[
\frac{1}{2} \rho_0 t^{-3m(1 + \omega)} = \frac{90\alpha m^2}{t^4(1 - \dot{\phi}^2)^2} - \frac{72\alpha m^4 \dot{\phi}^2}{t^4(1 - \dot{\phi}^2)^2}
- \frac{36\alpha m^3}{t^4} \dot{\phi}^2 - \frac{3m^2}{t^2}.
\]  

(76)

where we have neglected the higher order terms of \( \dot{\phi}^2 \) since the value of \( \dot{\phi}^2 \) belongs to \( 0 < \dot{\phi}^2 < 1 \).

On the other hand, to maintain the energy-momentum conservation, the relation \( \bar{\rho} = \bar{p} \) must be satisfied. So, the effective functional relation (69) for homogeneous K-essence scalar field reduces to

\[
3\dot{\phi}^2 - 2 = 2\sqrt{1 - \dot{\phi}^2},
\]  

(77)

where we have used Eqs. (62) and (53).

Solving the above Eq. (77) we have either \( \dot{\phi}^2 = 0 \), which is not acceptable for our case or

\[
\dot{\phi}^2 = \frac{8}{9} = 0.888 = \text{constant}.
\]  

(78)

The exact solution of field equations are already obtained by the assumption of power law form of the scale factor using Starobinsky Model in [23]. The results of that case is

\[
\rho_\phi = \frac{3n^2}{t^2} - \frac{\rho_0}{t^{3n(1+\omega)}} + \frac{54\alpha n^2(2n - 1)}{t^4}.
\]  

(79)
and 
\[ p_{\phi} = \frac{n(2 - 3n)}{t^2} + \frac{18an(2n - 1)(4 - 3n)}{t^4} - \frac{\omega \rho_{m0}}{t^{3n(1 + \omega)}}, \]  
(80)

where \( \rho_{\phi} \) and \( p_{\phi} \) is the energy density and pressure of the scalar field and \( n \) is synonymous to \( m \) for our case.

Rearranging Eq. (79) we get 
\[ \rho_0 t^{-3n(1+\omega)} = \frac{3n^2}{t^2} + \frac{108an^3}{t^4} - \frac{54an^2}{t^4} - \frac{1}{2} \phi^2 - V(\phi), \]  
(81)

where they have defined \( \rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V(\phi) \), \( p_{\phi} = \frac{1}{2} \phi^2 - V(\phi) \), \( V(\phi) \) is the scalar potential.

Singh et al. [23] in their paper have used canonical Lagrangian but in our case we have used non-canonical Lagrangian. This is the basic difference between these two articles. Also, it should be clearly mentioned that the scalar field of them is not identical with the K-essence scalar field.

Now, putting the value of \( \dot{\phi}^2 \) as 0.888 and using Eqs. (70) and (72), we extract the time evolution of the K-essence scalar field equation of state (EOS) parameter (taking \( m = 1 \)) as 
\[ \omega = \frac{-16180a - 71t^2 + 24\alpha t^2}{1782a - 27t^2 - 2t^4}. \]  
(82)

The variation of \( \omega \) with time for different values of \( \alpha \) has been depicted in Fig. 1. This figure shows that \( \omega \) is negative as \( t \) increases which means that the Universe is dark energy dominated and is in good agreement with the present day observations on the cosmic acceleration [51] [57].

![FIG. 1. (Color online) Variation of \( \omega \) with \( t \) for different value of \( \alpha = 100 - 400 \)](image)

VII. ENERGY CONDITIONS IN \( f(\bar{R}, L(X)) \)-GRAVITY

With the help of the modified field equation [54] for \( f(\bar{R}, L(X)) \) theory, the emergent Einstein’s equation [22] can be written as (\( \kappa = 1 \))
\[ \bar{R}_{\mu\nu} - \frac{1}{2} \bar{G}_{\mu\nu} \bar{R} = T_{\mu\nu}^{\text{eff}}, \]  
(83)

where 
\[ T_{\mu\nu}^{\text{eff}} = \frac{1}{F}[\frac{1}{2} f_L T_{\mu\nu} - \frac{1}{2} \bar{R} F \bar{G}_{\mu\nu} - (\bar{G}_{\mu\nu} \bar{\Box} - D_{\mu} D_{\nu}) F - \frac{1}{2} \bar{G}_{\mu\nu}(f - Lf_L(1 + D_{\alpha} \phi D_{\alpha}^\phi)) - \frac{1}{2} Lf_L D_{\mu} \phi D_{\nu} \phi(1 + D_{\alpha} \phi D_{\alpha}^\phi)] \]  
(84)

with \( F = f_R = \frac{\partial f(R, L(X))}{\partial R} \).

The trace of the effective energy momentum tensor [32] is 
\[ T^{\text{eff}} = \frac{1}{F} \left[ \frac{1}{2} f_L T + 2(f - F \bar{R}) - 3 \bar{\Box} F - 2Lf_L(1 + D_{\alpha} \phi D_{\alpha}^\phi)^2 \right]. \]  
(85)

Now from Eq. (83), we have the emergent Ricci tensor in terms of the effective energy momentum tensor as 
\[ \bar{R}_{\mu\nu} = T_{\mu\nu}^{\text{eff}} - \frac{1}{2} \bar{G}_{\mu\nu} T^{\text{eff}}. \]  
(86)

Let \( \bar{\mu} \) be the tangent vector field to a congruence of time-like geodesics in the K-essence emergent space-time manifold endowed with the metric \( \bar{G}_{\mu\nu} \). Then the SEC [3] in \( f(\bar{R}, L(X)) \) modified gravity can be expressed as 
\[ \bar{R}_{\mu\nu} \bar{\mu}^\mu \bar{\nu}^\nu = (T_{\mu\nu}^{\text{eff}} \bar{\mu}^\mu \bar{\nu}^\nu - \frac{1}{2} T^{\text{eff}}) \geq 0. \]  
(87)

On the other hand, if we consider \( \bar{k}^\mu \) be the tangent vector along the null geodesic congruence (\( \bar{G}_{\mu\nu} \bar{k}^\mu \bar{k}^\nu = 0 \)), then the NEC [3] in \( f(\bar{R}, L(X)) \) gravity is 
\[ \bar{R}_{\mu\nu} \bar{k}^\mu \bar{k}^\nu = T_{\mu\nu}^{\text{eff}} \bar{k}^\mu \bar{k}^\nu \geq 0. \]  
(88)

So, considering an additional condition [22] \( \frac{f_L(R, L)}{f_R(R, L)} > 0 \), and the K-essence scalar field to be homogeneous, i.e., \( \phi(x^i, t) = \phi(t) \) and using the perfect fluid energy momentum tensor [51], we have the SEC and NEC in the \( f(\bar{R}, L(X)) \) gravity are 
\[ \text{SEC} : \quad \bar{\rho} + 3\bar{p} - \frac{2}{f_L} (f - F \bar{R}) + 2L(1 + \dot{\phi}^2)^2 + \frac{6}{f_L(1 - \dot{\phi}^2)} [\bar{\Box}(1 - \frac{2}{3} \dot{\phi}^2) + H \bar{F}(1 - \dot{\phi}^2 + \frac{2}{3} \dot{\phi}^4)] \geq 0, \]  
(89)
\[ \text{NEC} : \quad \bar{\rho} + \bar{p} + \frac{2}{f_L} (\bar{F} - H \bar{F} \dot{\phi}^2) \geq 0, \]  
(90)
where $\bar{\rho} = \rho(1 + \dot{\phi}^2)$ and $\bar{p} = p(1 + \dot{\phi}^2)$.

To evaluate the effective density $\bar{\rho}^{eff}$ and effective pressure $\bar{p}^{eff}$ in the K-essence emergent $f(\bar{R}, L(X))$ gravity, we consider two following equations

$$T^{\mu \nu}_{\mu \nu} - \frac{1}{2} \bar{G}_{\mu \nu} T^{\mu \nu} = \bar{\rho}^{eff} + 3\bar{p}^{eff}$$

and

$$T^{\mu \nu}_{\mu \nu} \kappa^\nu = \bar{\rho}^{eff} + \bar{p}^{eff}. \tag{92}$$

Solving these Eqs. (93) and (94), we get

$$\bar{\rho}^{eff} = \bar{\rho} + \frac{1}{f}(f - F\bar{R}) - \frac{6}{fL(1 - \phi^2)} \frac{1}{3} \bar{F}\dot{\phi}^2$$

$$+ H\bar{\dot{F}}(1 - \frac{1}{3}\dot{\phi}^4) - L(1 + \phi^2)^2, \tag{93}$$

$$\bar{p}^{eff} = \bar{p} - \frac{1}{f}(f - F\bar{R}) + \frac{3}{fL(1 - \phi^2)} \frac{1}{3} \bar{F}(2 - \dot{\phi}^2)$$

$$+ H\bar{\dot{F}}(1 - \frac{2}{3}\dot{\phi}^2 + \frac{1}{3}\dot{\phi}^4)] + L(1 + \phi^2)^2. \tag{94}$$

From the above Eqs. (93) and (94), we have WEC and DEC respectively for the K-essence emergent $f(\bar{R}, L(X))$ gravity as

$$WEC: \quad \bar{\rho} + \frac{1}{f}(f - F\bar{R}) - \frac{6}{fL(1 - \phi^2)} \frac{1}{3} \bar{F}\dot{\phi}^2$$

$$+ H\bar{\dot{F}}(1 - \frac{1}{3}\dot{\phi}^4) - L(1 + \phi^2)^2 \geq 0, \tag{95}$$

$$DEC: \quad \bar{p} - \bar{\rho} + \frac{2}{fL}(f - F\bar{R}) - \frac{2}{fL(1 - \phi^2)}[\bar{F}(1 + \frac{1}{2}\dot{\phi}^2)$$

$$+ 3H\bar{\dot{F}}(\frac{3}{2} - \frac{1}{3}\dot{\phi}^2 - \frac{1}{6}\dot{\phi}^4)] - 2L(1 + \phi^2)^2 \geq 0. \tag{96}$$

These energy conditions (95), (96), (97) and (98) of the K-essence emergent $f(\bar{R}, L(X))$ gravity are different from the usual $f(\bar{R}, L_m)$-gravity (7) and $f(R)$-gravity (8) in the presence of the K-essence scalar field $\phi$. Also note that if we consider $f(\bar{R}, L(X)) \equiv R$ and $\bar{G}_{\mu \nu} \equiv g_{\mu \nu}$, then we can get back to the usual energy conditions of GR i.e.,

- SEC: $\rho + 3p \geq 0$;
- NEC: $\rho + p \geq 0$;
- WEC: $\rho \geq 0$ and
- DEC: $\rho \geq |p|$.

### A. Constraints on K-essence emergent $f(\bar{R}, L(X))$-gravity

The inequalities of the energy conditions (95), (96), (97) and (98) can also be expressed in terms of the deceleration ($q$), jerk ($j$), and snap ($s$) parameters such that the Ricci scalar and its derivatives for a spatially flat K-essence emergent FLRW geometry (11) are

$$\bar{R} = \frac{6}{1 - \phi^2} \left[ \dot{H} + H^2(2 - \dot{\phi}^2) \right] = \frac{6H^2}{1 - \phi^2} \left[ 1 - q - \dot{\phi}^2 \right] \tag{97}$$

$$\bar{R} = \frac{6}{1 - \phi^2} \left[ \ddot{H} + 4H\dot{H}(1 - \dot{\phi}^2) - 2H^3\dot{\phi}^2 \right] = \frac{6H^3}{1 - \phi^2} \left[ (j - q - 2) + 2\dot{\phi}^2(1 - 2q) \right] \tag{98}$$

$$\bar{R} = \frac{6}{1 - \phi^2} \left[ (\dddot{H} + 4H\ddddot{H} + 4H\dddot{H}) - 2\dot{\phi}^2(2H^2 + 3H\dot{H} - 2H^4 + 3H^2\dddot{H}) \right] = \frac{6H^4}{1 - \phi^2} \left[ (s + q^2 + 8q + 6) - 2\dot{\phi}^2(3 + 3j + 10q + 2q^2) \right], \tag{99}$$

where $\dot{\phi}^2 \equiv \frac{1}{H^2}\ddot{a}$; $j = \frac{1}{H^2}\dddot{a}$; $q = \frac{1}{H^2}\dddot{a}$; $s = \frac{1}{H^2}\dddot{a}$. \(100\)

Now from Eq. (98), we evaluate the values of $\dot{\phi}$ and $\dddot{\phi}$ in the energy conditions (95), (96), (97) and (98) we have the energy conditions in terms of $q, j$ and $s$. We can easily check that these energy conditions in terms of $q, j$ and $s$ are also different from the $f(\bar{R}, L_m)$-gravity (20) in the presence of the K-essence scalar field $\phi$.

### B. An example of the energy condition

We consider one specific type of model in $f(\bar{R}, L(X))$-gravity as

$$f(\bar{R}, L(X)) = \bar{R} + \alpha\bar{R}^n + L(X), \tag{103}$$

with $\alpha > 0$ and $n > 0$. 

---

**Note:** The above text is a transcription of the content from the given image, formatted for readability. The equations and expressions have been carefully aligned to maintain the logical flow of the text.
\[ f_L = 1; \quad F = 1 + \alpha n \bar{R}^{n-1}; \quad \dot{F} = \alpha n(n-1) \bar{R}^{n-2} \dot{\bar{R}}; \]
\[ \ddot{F} = \alpha n(n-1) \bar{R}^{n-2} \ddot{\bar{R}} + \alpha n(n-1)(n-2) \bar{R}^{-3} \dot{\bar{R}}^2. \]

\[ (104) \]

Applying above expressions in Eqs. \((89), (90), (93)\) and \((96)\), we have the energy conditions for this specific model \((103)\) are

- **SEC**: \[ \ddot{\rho} + 3 \ddot{\rho} - 2[f - (1 + \alpha n \bar{R}^{n-1})] \ddot{\bar{R}} \]
  \[ + 2L(1 + \phi^2)^2 + \frac{6}{1 - \phi^2} \left[ \alpha n(n-1)(\bar{R}^{n-2} \ddot{\bar{R}} \right] \]
  \[ + (n-2) \bar{R}^{n-3}(\dot{\bar{R}})^2 \left[ 1 - \frac{2}{3} \phi^2 \right] \]
  \[ + \frac{6}{1 - \phi^2} \left[ H \alpha n(n-1) \bar{R}^{n-2} \dot{\bar{R}} \right] \left[ 1 - \phi^2 + \frac{2}{3} \phi \dot{\phi} \right] \]
  \[ \geq 0 \]  \[ (105) \]

- **NEC**: \[ \ddot{\rho} + \ddot{\rho} + 2[f(n-1)] \bar{R}^{n-2} \ddot{\bar{R}} \]
  \[ + (n-2) \bar{R}^{n-3}(\dot{\bar{R}})^2 - H \phi^2 \alpha n(n-1) \bar{R}^{n-2} \dot{\bar{R}} \]
  \[ \geq 0 \]  \[ (106) \]

- **WEC**: \[ \ddot{\rho} + [f - \bar{R}(\alpha n \bar{R}^{n-1})] - L(1 + \phi^2)^2 - \frac{6}{1 - \phi^2} \left[ \alpha n(n-1)(\bar{R}^{n-2} \ddot{\bar{R}} \right] \]
  \[ + (n-2) \bar{R}^{n-3}(\dot{\bar{R}})^2 \]
  \[ + \frac{6}{1 - \phi^2} \left[ H \alpha n(n-1) \bar{R}^{n-2} \dot{\bar{R}} \right] \]
  \[ \geq 0 \]  \[ (107) \]

- **DEC**: \[ 2 - \frac{2}{(1 - \phi^2)^2} [(1 + \frac{1}{2} \phi^2) \alpha n(n-1) \bar{R}^{n-2} \dot{\bar{R}} \]
  \[ + (n-2) \bar{R}^{n-3}(\dot{\bar{R}})^2] + \frac{2}{(1 - \phi^2)} [3H \left( \frac{3}{2} \right) \]
  \[ - \frac{1}{3} \phi^2 - \frac{1}{6} \phi \dot{\phi} \alpha n(n-1) \bar{R}^{n-2} \dot{\bar{R}} \geq 0. \]  \[ (108) \]

Again, if we put the values of \( \bar{R}, \dot{\bar{R}} \) and \( \ddot{\bar{R}} \) from \((107), (38) \) and \((109)\) in the above equations \((105), (106), (107)\) and \((108)\), we easily reconstruct the energy conditions in terms of the deceleration \((q)\), jerk \((j)\), and snap \((s)\) parameters.

If we consider \( n = 2 \), i.e., \( f(\bar{R}, L(X)) = f(\bar{R}) + L(X) \) with \( f(\bar{R}) = \bar{R} + \alpha \bar{R}^2 \) being the Starobinsky Model \([5, 6]\) in the K-essence geometry, we get \( f_L = 1; \quad F = 1 + \alpha n \bar{R}; \quad \dot{F} = 0; \quad \ddot{F} = \alpha n(n-1)(n-2) \bar{R}^{-3} \dot{\bar{R}}^2. \) Put these values in the basic energy conditions equations \((89), (90), (93)\) and \((96)\) in the \( f(\bar{R}, L(X)) \) gravity, then we have the energy conditions for the K-essence emergent Starobinsky Model.

**VIII. DISCUSSION AND CONCLUSION**

In this work, we present a new type of modified theory, viz. \( f(\bar{R}, L(X)) \)-gravity, with robust formalism in the context of the K-essence emergent geometry where \( \bar{R} \) is the Ricci scalar of this geometry, \( L(X) \) is the DBI type non-canonical Lagrangian with \( X = \frac{1}{2} g^\mu\nu \partial_\mu \phi \partial_\nu \phi \), \( \phi \) is the K-essence scalar field. The K-essence emergent metric \( G_{\mu\nu} \) is not conformally equivalent to the gravitational metric \( g_{\mu\nu} \). This new type of modified theory is a general mixing between \( f(R) \) gravity and K-essence emergent gravity based on the DBI model.

Let us discuss some salient features of the present study which are as follows:

1. It is to be noted that the modified field equation \((34)\) is different from the usual \( f(R) \) and \( f(R, L_m) \) gravities. If we consider \( f(\bar{R}, L(X)) \equiv R \) and \( G_{\mu\nu} \equiv g_{\mu\nu} \) then we can easily get back to the standard Einstein field equations. The effective functional relation for the requirement of the conservation of the energy-momentum tensor is also different from the \( f(\bar{R}, L_m) \)-gravity. We derive the modified Friedmann equations for the \( f(\bar{R}, L(X)) \)-gravity considering the background gravitational metric as flat FLRW and the K-essence scalar field \( \phi \) being simply a function of time only, which are quite different from the Friedmann equations of the standard \( f(R) \) gravity.

2. From the particular choice, Eq. \((41)\) of \( f(\bar{R}, L) \), the kinetic energy of the K-essence scalar field is a constant. This value of \( \dot{\phi}^2 (= 0.888) \) is less than unity which is comparable with the range of \( \dot{\phi}^2 \). It is also to be noted that the K-essence theory can be used to investigate the effects of the presence of dark energy on cosmological scenarios. In this context, if we consider \( \dot{\phi}^2 \) be dark energy density in unit of critical density as \( \frac{2}{3} \), the value of dark energy density, i.e., \( \dot{\phi}^2 = \frac{2}{3} \) indicates that the present Universe is dark energy dominated.

It is well known that the present observational value \((56, 57)\) of dark energy density is approximately 0.75. Therefore, we note that in the context of dark energy regime our result is in good agreement with the observational data. Now-a-days people believe that the dark energy is one of the reasons for the accelerating Universe. So, our value of dark energy density may indicate that the Universe is more accelerating. From Fig. 1 we also observe the EOS parameter \( (\omega) \) is more negative as \( \dot{\phi} \) increases which is also in good agreement with the observational results, i.e., the Universe is dark energy dominated.

As final comments, we would like to put here the following two aspects which emerges from of the present investigation:

1. This model can open up an alternative window to explore the current cosmic acceleration without a stringent condition of invoking an exotic component as the dark energy. In other words, this theory seems interesting from a purely gravitational theory standpoint, rather than the cosmological context of dark energy whose very
existence is still an issue of doubt \[55\] in the context of the latest analysis of data from the Planck consortium \[53, 57\]. However, the arbitrariness in the choice of different functional forms of \(f(R, L(X))\) based on DBI Lagrangian gives rise to the problem of how to constrain the many possible \(f(R, L(X))\) gravity theories on physical grounds. In this context, we have shed some light on this issue by discussing some constraints on general \(f(R, L(X))\) gravity from the so-called energy conditions.

(ii) Starting from the Raychaudhuri equations along with the requirement that the gravity is attractive, we have derived the null, strong, weak and dominant energy conditions in the framework of \(f(R, L(X))\) gravity. These energy conditions are different from the usual \(f(R)\) and \(f(R, L_m)\) theories. With the help of the specific form of \(f(R, L(X))\), we also have derived these energy conditions in the \(f(R, L(X))\)-gravity. We intend to report on this interesting theory in the near future encompassing the multifarious aspects of Cosmology.

Appendix: The modified field equation in \(f(R, L_m)\)-Gravity

Let us consider here the well known \(f(R)\) and \(f(R, L_m)\) gravity, where \(R\) is the Ricci scalar with respect to the gravitational metric \(g_{\mu\nu}\) and \(L_m\) is the matter Lagrangian. The total action for the \(f(R)\) gravity is \[6, 7\]

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M(g_{\mu\nu}, \psi), \tag{109}
\]

where \(S_M\) is matter term, \(\psi\) denotes the matter fields, \(\kappa = 8\pi G\), \(G\) is the gravitational constant, \(g\) is the determinant of the gravitational metric and \(R = g^{\mu\nu} R_{\mu\nu}\) is the Ricci scalar.

Varying with respect to the gravitational metric we achieve the modified field equation as

\[
f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box] f'(R) = \kappa T_{\mu\nu}, \tag{110}
\]

with

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{111}
\]

where \(f'(R) = \frac{\partial f(R)}{\partial R}\), \(\nabla_\mu\) is covariant derivative with respect to the gravitational metric and \(\Box = \nabla^\mu \nabla_\mu\).

On the other hand, the action for the \(f(R, L_m)\) gravity is \[19, 20\]

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R, L_m), \tag{112}
\]

where \(f(R, L_m)\) is an arbitrary function of the Ricci scalar \(R\), and the Lagrangian density corresponding to matter \(L_m\). The energy-momentum tensor is

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(-g L_m)}{\delta g^{\mu\nu}} = -2 \frac{\partial L_m}{\partial g^{\mu\nu}} + g_{\mu\nu} L_m, \tag{113}
\]

where the Lagrangian density \(L_m\) is only matter dependent on the metric tensor components \(g_{\mu\nu}\).

The modified field equations of the \(f(R, L_m)\)-gravity model is

\[
\frac{f(R, L_m)}{R_{\mu\nu}} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f(R, L_m) = \frac{1}{2} f(R, L_m) - L_m f_{L_m} R_{\mu\nu}, \tag{114}
\]

where \(f(R, L_m) = \partial f(R, L_m)/\partial R\) and \(L_m f_{L_m} (R, L_m) = \partial f(R, L_m)/\partial L_m\). However, if \(f(R, L_m) = R/2 + L_m\), then the above Eq. (114) reduces to the usual field equation \(R_{\mu\nu} - (1/2) g_{\mu\nu} R = \kappa T_{\mu\nu}\).

Acknowledgement

G.M. acknowledges the DSTB, Government of West Bengal, India for financial support through the Grants No.: 322(Sanc.)/ST/P/S&T/16G-3/2018 dated 06.03.2019. The authors also thanks to Dr. Pradipta Panchadhayee, Associate Professor, Department of Physics, Prabhat Kumar College, Contai, for his valuable suggestions.

[1] H. Weyl, Ann. Phys. 59, 101 (1919).
[2] A.S. Eddington, Mathematical Theory of Relativity (Cambridge University Press, Cambridge, 1923).
[3] R. Utiyama, B.S. DeWitt, J. Math. Phys. 3, 608 (1962).
[4] N.D. Birrell, P.C.W. Davies, Quantum Fields in Curved Spacetime (Cambridge University Press, Cambridge, 1982).
[5] A.A. Starobinsky, Phys. Lett. B 91, 9 (1980).
[6] T.P. Sotiriou, V. Faraoni, Rev. of Mod. Phys. 82, 451 (2010).
[7] A. De Felice, S. Tsujikawa, Living Rev. Relativity 13, 3 (2010).
[8] P.K.S. Dunsby et. al., Phys. Rev. D 82, 023519 (2010).
[9] A. Mukherjee, N. Banerjee, Astrophys. Space Sci. 352, 893 (2014).
[10] K. Atazadeh et. al., Int. J. Mod. Phys. D 18, 1101 (2009).
[11] J. Santos et. al., Phys. Rev. D 76, 083513 (2007).
[12] S. Capozziello et. al., Phys. Lett. B 781, 99 (2018).
[13] J. Wang et. al., Phys. Lett. B 689, 133 (2010).
[14] S.E. Perez Bergliaffa, Phys. Lett. B 642, 311 (2006).
[15] F.D. Albareti et. al., JCAP 12, 020 (2012).
[16] F.D. Albareti et. al., JCAP, 07, 009 (2013).
[17] F.D. Albareti et. al., JCAP 03, 012 (2014).
[18] K.D. Krori et. al., Ind. J. Phys., 82(5), 531 (2008).
[19] T. Harko, F.S.N. Lobo, Eur. Phys. J. C 70, 373 (2010).
[20] J. Wang, K. Liao, Class. Quantum Gravit. 29, 215016 (2012).
[21] N. Goheer, R. Goswami P. Dunsby, K. Ananda, Phys. Rev. D 79, 121301(R) (2009).
[22] N. Goheer, J. Larena, P.K.S. Dunsby, Phys. Rev. D 80, 061301(R), (2009).
[23] C.P. Singh, V. Singh, Int. J. Theor. Phys. 51, 1889 (2012).
[24] M. Born, L. Infeld, Proc. Roy. Soc. Lond A 144, 425 (1934).
[25] W. Heisenberg, Zeit. Phys. 113 61 (1939).
[26] P.A.M. Dirac, R. Soc. Lond. Proc. Series A 268, 57 (1962).
[27] D. Gangopadhyay, Goutam Manna, Eur. Phys. Lett. 100 49001 (2012).
[28] G. Manna, D. Gangopadhyay, Eur. Phys. J. C 74 2811 (2014).
[29] G. Manna, B. Majumder, Eur. Phys. J. C 79, 553 (2019).
[30] M. Visser, C. Barcelo, S. Liberati, Gen. Relativ. Gravit. 34 1719 (2002).
[31] E. Babichev, V. Mukhanov, A. Vikman, JHEP 0609, 061 (2006).
[32] E. Babichev, V. Mukhanov, A. Vikman, JHEP 0802 101 (2008).
[33] A. Vikman, K-essence: Cosmology, causality and Emergent Geometry, Dissertation an der Fakultat fur Physik, Arnold Sommerfeld Center for Theoretical Physics, der Ludwig-Maximilians-Universitat Munchen, Munchen, den 29.08.2007.
[34] E. Babichev, V. Mukhanov, A. Vikman, Looking beyond the Horizon (WSPC-Proceedings, October 23, 2008).
[35] R.J. Scherrer, Phys. Rev. Lett. 93 011301 (2004).
[36] L.P. Chimento, Phys. Rev. D 69 123517 (2004).
[37] S. Nojiri et. al., Nucl. Phys. B 941, 11 (2019).
[38] S.D. Odintsov et. al., arXiv:2004.08884 (2020).
[39] V.K. Oikonomou, N. Chatzarakis, Nucl. Phys. B 956, 115023 (2020).
[40] R.M. Wald, General Relativity (Univ. Chicago Press, 1984).
[41] C. Misner, K.S. Thorne, J. Wheeler, Gravitation (W.H. Freeman and Company, 1970).
[42] T. Koivisto, Class. Quantum Gravit. 23 4289 (2006).
[43] T. Harko, Phys. Lett. B 669, 376 (2008).
[44] D. Gangopadhyay, G. Manna, arXiv:gr-qc/1502.06255 (2015).
[45] A. Raychaudhuri, Phys. Rev. 98, 1123 (1955).
[46] A. Raychaudhuri, Z. Astrophysik 43, 161 (1957).
[47] A. Raychaudhuri, Phys. Rev. 106, 172 (1957).
[48] M. Blau, Lecture Notes on General Relativity, http://www.blau.itp.unibe.ch/GRLecturenotes.html (2020).
[49] I. Bhattacharyya, S. Ray, Int. J. Mod. Phys. D, 2150092 (2021) [DOI: 10.1142/S02182718215000929].
[50] S. Carroll, Spacetime and Geometry: An Introduction to General Relativity (Addison Wesley, New York, 2004).
[51] E.R. Harrison, Nature (London) 260, 591 (1976).
[52] F. Landsberg, Nature (London) 263, 217 (1976).
[53] M. Blau, Lecture Notes on General Relativity, http://www.blau.itp.unibe.ch/GRLecturenotes.html (2020).
[54] J.T. Nielsen, A Guffanti, S. Sarkar, Sci. Rep. 6 3559 (2016).
[55] P.R. Ade et. al., Planck 2015 results. XIII. Cosmological parameters, Astron. Astrophys. 594, A13 (2016).
[56] N. Aghanim et. al., Planck 2018 results. VI. Cosmological parameters, Planck Collaboration, Astron. Astrophys. 641, A6 (2020).