Towards Computing Victory Margins in STV Elections

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Abstract
The Single Transferable Vote (STV) is a system of preferential voting employed in multi-seat elections. Each vote cast by a voter is a (potentially partial) ranking over a set of candidates. No techniques currently exist for computing the margin of victory (MOV) in STV elections. The MOV is the smallest number of vote manipulations (changes, additions, and deletions) required to bring about a change in the set of elected candidates. Knowledge of the MOV of an election gives greater insight into both how much time and money should be spent on the auditing of the election, and whether uncovered mistakes (such as ballot box losses) throw the election result into doubt—requiring a costly repeat election—or can be safely ignored. In this paper, we present algorithms for computing lower and upper bounds on the MOV in STV elections. In small instances, these algorithms are able to compute exact margins.

1 Introduction
The Single Transferable Vote (STV) is a system of preferential voting employed in multi-seat elections. It is used to elect candidates to the Australian Senate, in all elections in Malta, and in most elections in the Republic of Ireland [Farrell and McAllister, 2005]. No techniques currently exist for computing the smallest number of vote manipulations (changes, additions, and deletions) required to bring about a change in the set of elected candidates—the margin of victory (MOV). The ability to compute this margin has significant value. In the 2013 election of six candidates to Western Australia’s Senate a discrepancy of 1,375 initially verified votes was discovered during a recount. The election result was overturned, and a repeat election held in 2014. If the MOV for the original election was known, the question of whether the loss of these votes may have altered the resulting outcome could have been answered, potentially avoiding a repeat election.

In an STV election, each vote is a (potentially partial) ranking over a set of candidates. For example, in an election with candidates $c_1$, $c_2$, $c_3$, and $c_4$, a vote [$c_2$, $c_1$, $c_4$] expresses a first preference for candidate $c_2$, a second for $c_1$, and a third for $c_4$. At the start of the counting process, each vote is initially awarded to its highest ranked candidate. In the above vote, $c_2$ is the highest ranked candidate. The votes awarded to each candidate forms their tally. Candidates whose tallies exceed (or reach) a quota, defined in terms of the number of seats to be filled and votes cast in the election, are elected to a seat. As each candidate is elected, their surplus (the number of votes by which their tally exceeds the quota) is computed, and a subset of their votes (with a combined value equal to the surplus) is distributed to their next preferred candidate (in the above vote, the next preferred candidate after $c_2$ is $c_1$). Where multiple candidates have a quota’s worth of votes in their tally, the candidate with the largest surplus is elected first, and their surplus distributed. Then, if there are still seats to fill, the candidate with the next largest surplus is elected, and their surplus distributed (and so on). If no remaining candidate has a quota’s worth of votes, and one or more seats remain empty, the candidate with the fewest votes is eliminated and their votes distributed to their next preferred candidates. If the number of candidates remaining (unelected and not yet eliminated) equals the number of seats left to be filled, these candidates are elected and the STV counting process terminates.

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Several STV variants exist, differing in the way that surpluses are distributed [Weeks, 2011]. Consider a candidate with a tally of 100 votes and a surplus of 40 votes. The Inclusive Gregory Method redistributes all 100 votes, each with an assigned a value of 0.4 (each vote is worth 0.4 votes), to their next highest ranked candidate that is ‘still standing’ (has not yet been elected or eliminated) [Miragliotta, 2004]. In the STV variant we consider in this paper, candidates whose tallies have reached or exceeded the quota (but have not yet been awarded a seat) receive no further votes from the surplus distributions of other candidates.

We develop, in this paper, an algorithm for computing exact margins of victory in STV elections that use the Inclusive Gregory Method of surplus distribution—arguably the simplest and most straightforward of the existing variants. In Section 2 we step through the counting process that takes place in STV elections, under the Inclusive Gregory Method, in two example STV instances. The algorithm we present in this paper (labelled margin-stv) is an adaptation of existing work for computing margins in Instant Runoff Voting (IRV) elections [Blom et al., 2016]. IRV is a single-seat variant of STV and is employed in lower house elections across Australia, in which a single candidate is elected to a single seat. In an IRV election, candidates with the fewest votes are eliminated, and their votes redistributed to later ranked candidates, until a candidate attains a majority of the available votes, and is declared the winner. The computation of exact IRV and STV victory margins is known to be NP-hard [Bartholdi III and Orlin, 1991, Conitzer et al., 2003, 2007, Xia, 2012, Rothe and Schend, 2013, Narodytska and Walsh, 2014].

Our margin-stv algorithm represents the outcome of an STV election as a sequence of candidate elections and eliminations (e.g., \(c_4\) elected, \(c_3\) eliminated, \(c_2\) eliminated, \(c_1\) elected). We present a mixed-integer non-linear program (MINLP) that, given such an outcome, and the set of votes cast in the election, computes the smallest number of vote manipulations required to realise the outcome. A vote manipulation replaces the ranking of a vote (e.g., \([c_2, c_1, c_4]\) with an alternate ranking (e.g., \([c_4, c_3]\)). Consider an election over candidates \(C\), in which candidates \(E \subset C\) are elected to a seat. Our margin-stv algorithm applies branch-and-bound to search the space of alternate election outcomes (in which the set of winning candidates \(E' \neq E\)) for one that requires the least amount of vote manipulation to realise. We show that margin-stv is able to compute exact margins in some small STV election instances. We develop a relaxation of this algorithm capable of computing lower bounds on the margin of victory in larger, more realistic STV election instances.

The remainder of this paper is structured as follows. Section 2 describes the STV counting algorithm. Preliminary definitions and concepts required in the explanation of our margin-stv algorithm are presented in Section 3. Related work is discussed in Section 4. Mechanisms for computing simple upper bounds on the degree of manipulation required to alter the outcome of an STV election are presented in Section 5. Section 6 presents our MINLP for computing the smallest degree of manipulation required to realise a specific election outcome (a specific sequence of elections and eliminations). A rule for computing a lower bound on this degree of manipulation is presented in Section 7. Using these upper and lower bounding techniques as building blocks, we present our margin-stv algorithm, and a relaxed variant of this algorithm, in Section 8. We evaluate these algorithms on a range of both small and large STV election instances in Section 9.

## 2 The Single Transferable Vote (STV)

This section describes the STV vote counting algorithm that we consider in this paper, outlined in Figure 1. We illustrate this algorithm in the example election shown in Table 1. In Step 1 of the counting algorithm the quota of the election is calculated, according to Equation 1. This is known as the Droop quota, and represents a threshold that each candidate must reach before they are elected to a seat. In the election of Table 1 there are 2 seats to be filled and 60 cast votes. The quota in this election is 21 votes.

\[
\text{Quota} = \frac{\text{Total number of votes cast}}{\text{Number of seats} + 1} + 1
\]

(1)

Each vote cast in the election starts with a value of 1. The total value of the votes a candidate has in their tally is computed as shown in Equation 2. As the STV algorithm proceeds, votes will move from the tally of one candidate to that of others. The value of these votes—the extent to which they contribute to a candidate’s tally value—will change over the course of the algorithm.
current value

(Step 7). If the contribution to an eligible next preferred candidate (a candidate that is still standing) are given to that candidate redistributed to candidate $c$ that its next preferred candidate's tally is also $v$ Each of $c$ candidate in this list, $c$, ranking $\tau = \min \left( 1, \frac{\text{Surplus of candidate } c}{\text{Total value of } c\text{'s transferable votes}} \right)$

Consider our example in Table 1. Candidate $c_1$ is elected to a seat, and has 6 transferable votes with ranking $[c_1, c_3]$. The remaining votes in $c_1$’s tally have a ranking of $[c_1]$. These votes have no eligible next preference and are exhausted (not redistributed). The transfer value assigned to $c_1$’s transferable votes is 0.83. All 6 votes with ranking $[c_1, c_3]$ are given to candidate $c_3$, but they now have a combined value of 5.

If, in Step 4, no candidate has a tally value that equals or exceeds the quota, the candidate $c$ with the smallest tally value is eliminated (Step 6). The votes in their tally are redistributed to later preferences. The votes in $c$’s tally that have $c$ as a first preference will have a value of 1 (as a consequence of Step 2). Votes that $c$ has received after prior surplus distributions will have a reduced value. All votes in $c$’s tally that have an eligible next preferred candidate (a candidate that is still standing) are given to that candidate at their current value (Step 7). If the contribution to $c$’s tally value of a vote is $v$, then that votes contribution to its next preferred candidate's tally is also $v$. Consider again the example of Table 1. After the election of candidate $c_1$ and the redistribution of their votes, candidates $c_2$ to $c_4$ have tally values of 10, 14, and 15. No candidate has a quota of votes in their tally. Candidate $c_2$ has the smallest tally value, and is eliminated. Each of $c_2$’s votes—four with ranking $[c_2, c_3]$ and 6 with ranking $[c_2, c_3, c_4]$—still have a value of 1 and are redistributed to candidate $c_3$. Candidates $c_3$ and $c_4$ now have tally values of 24 and 15.
1 Compute the Quota for the election (Equation 1).
2 Assign votes to their first ranked candidates (each vote $b$ has a value of $v_b = 1$)

While a seat remains to be filled do

4 Let surpluses denote the candidates whose tally equals or exceeds the quota (in order of decreasing surplus size)

5 If the surpluses set is empty then

6 Eliminate a candidate $c$, of those remaining, with the smallest tally

7 Redistribute each vote $b$ in $c$'s tally to its next preferred candidate at its current value $v_b$

Else

9 Elect the first candidate in surpluses, $c$, to a seat

10 if all seats are filled then stop

11 Compute the number of transferable votes in $c$'s tally

12 Compute the transfer value $\tau$ of these transferable votes

13 Redistribute each transferable vote $b$ in $c$'s tally with a value of $\tau v_b$, where $v_b$ is its current value, to its next preferred candidate (skipping over candidates that have been elected, eliminated, or are in the surpluses set)

14 If the number of unfilled seats and remaining candidates are equal then

15 Elect all remaining candidates to a seat

Figure 1: The STV vote counting algorithm (under the Inclusive Gregory Method).

After a candidate has been elected or eliminated, we check whether the number of unfilled seats and the number of remaining candidates (that have not yet been elected or eliminated) are equal (Step 14). If so, all remaining candidates are elected to a seat irrespective of their tally value (Step 15). If not, the total value of the votes in each remaining candidate’s tally is recomputed and we return to Step 4. In the example of Table 1, the algorithm recomputes the tally values of candidates $c_3$ and $c_4$ in Step 4, and places $c_3$ in the surpluses list. Candidate $c_3$ is elected to the final seat, and the algorithm terminates in Step 10. The STV algorithm proceeds in rounds that consist of: computing the total value of each candidate’s tally; electing the candidate with the largest surplus (if such a candidate exists) and redistributing their votes; or eliminating the candidate with the smallest tally (if no candidate has a quota) and redistributing their votes.

Let us consider a second example STV election, shown in Table 2. Candidates $c_1$, $c_2$, $c_3$, and $c_4$ have initial tallies of 31, 17, 5, and 10 votes. The quota for the election is 22, and $c_1$ is placed into the list of candidates with a surplus in Step 4. Candidate $c_1$ is elected to the first of two available seats in Step 9, and has 13 transferable votes in their tally (Step 11). The transfer value to be applied to those votes is 0.69 (its surplus of 9 divided by the number of transferable votes 13). In Step 13, candidate $c_2$ is given 5 votes of ranking $[c_1, c_3]$, with the total value of these votes equal to 3.46. Candidate $c_4$ is given 8 votes of ranking $[c_1, c_4, c_2, c_3]$, with the total value of these votes equal to 5.54. In the second round of counting, $c_2$ now has a total tally of 20.46 votes and $c_4$ a total tally value of 15.54. No candidate has a quota, and candidate $c_3$, with the smallest tally, is eliminated (in Step 6). Candidate $c_2$ is given 5 votes with ranking $[c_3, c_2, c_4]$. In the next round of counting, candidate $c_2$ has exceeded a quota and is elected to the last seat (Step 9).

In many STV variants, the last bundle of votes received by a candidate, at any point in the counting process, is known as their last parcel. In the Original Gregory Method, votes in an elected candidate’s last parcel (and no others) are transferred (at a fraction of their value) during surplus distribution. The total value of the votes transferred is equal to the candidate’s surplus. Some jurisdictions do not assign fractional values to distributed votes, but transfer a random selection of votes from a candidate’s last parcel at their full value, the total of which equals the candidate’s surplus. Much of the complexities involved in vote
of an STV election is defined in Definition 3. The primary vote of a candidate—in a sequence of candidate elections and eliminations—is outlined in Definition 2. The margin of victory (MOV) is formally defined as the smallest number of votes required to ensure that a set of candidates is elected to a seat, as defined in Equation 1), (e.g., the vote \(c_1\) must not appear in \(\pi\)’s ranking with the alternate ranking \(\tau\)). For example, consider a vote \(\pi\) that \(c_1\) is elected to a seat in the first round of counting, followed by the elimination of candidates \(c_3\) and \(c_2\), and the election of \(c_4\).

\begin{align*}
\text{Definition 1. STV Election (}\mathcal{E}\text{)} & \quad \text{An STV election is defined as a tuple } \mathcal{E} = (\mathcal{C}, \mathcal{B}, \mathcal{Q}, N, E) \text{ where } \mathcal{C} \text{ is a set of candidates, } \mathcal{B} \text{ the set of votes cast in the election, } \mathcal{Q} \text{ the election quota (the number of votes a candidate must attain to be elected to a seat, as defined in Equation 1), } N \text{ the number of seats to be filled, and } E \text{ the set of candidates elected to a seat (according to the counting algorithm outlined in Figure 1). Each vote } b \in \mathcal{B} \text{ is a partial or complete ranking over } \mathcal{C} \text{ (e.g., the vote } [c_1, c_3, c_2], \text{ in an election with candidates } \mathcal{C} = \{c_1, c_2, c_3, c_4\}, \text{ expresses a first preference for candidate } c_1, \text{ a second for } c_3, \text{ and a third for } c_2).}
\end{align*}

\begin{align*}
\text{Definition 2. Election Order (}\pi\text{)} & \quad \text{Given an STV election } \mathcal{E} = (\mathcal{C}, \mathcal{B}, \mathcal{Q}, N, E), \text{ we represent the outcome of the election as an election order } \pi—\text{a sequence of tuples } (c, a) \text{ where } c \in \mathcal{C} \text{ and } a \in \{0, 1\}. \text{ The tuple } (c, 1) \text{ indicates that candidate } c \text{ is elected to a seat, and } (c, 0) \text{ that } c \text{ is eliminated. An election order } \pi \text{ defines the sequence of elections and eliminations that arise as the STV counting algorithm (Figure 1) is executed. The order } \pi = [(c_1, 1), (c_3, 0), (c_2, 0), (c_4, 1)], \text{ for example, indicates that candidate } c_1 \text{ is elected to a seat in the first round of counting, followed by the elimination of candidates } c_3 \text{ and } c_2, \text{ and the election of } c_4.}
\end{align*}

\begin{align*}
\text{Definition 3. Margin of Victory (MOV)} & \quad \text{The margin of victory for an STV election } \mathcal{E} = (\mathcal{C}, \mathcal{B}, \mathcal{Q}, N, E) \text{ is defined as the smallest number of vote manipulations required to ensure that a set of candidates } \mathcal{E}' \neq E \text{ are elected to a seat (i.e., at least one candidate in } \mathcal{E}' \text{ must not appear in } E). \text{ A single manipulation changes the ranking on a single vote } b \text{ to an alternate ranking. For example, consider a vote } b \text{ with ranking } [c_1, c_3, c_2]. \text{ Replacing } b\text{'s ranking with the alternate ranking } [c_4, c_1] \text{ represents a single manipulation.}
\end{align*}

\begin{align*}
\text{Definition 4. Primary Vote } f_p(c) & \quad \text{The primary vote of a candidate } c \in \mathcal{C} \text{ in an STV election } \mathcal{E} = (\mathcal{C}, \mathcal{B}, \mathcal{Q}, N, E) \text{ is defined as the total number of votes in } \mathcal{B} \text{ in which } c \text{ is ranked highest (i.e., } c \text{ is ranked first). For example, the vote } [c_1, c_3, c_4] \text{ contributes to the primary vote of candidate } c_1.
\end{align*}

## 4 Related Work

The computation of victory margins in both STV and Instant Runoff Voting (IRV) elections is NP-hard. \cite{Bartholdi, Conitzer, Xia, Rothe, Narodytska} provide a good summary of the range of STV variants used in practice.

### Table 2: Example 2: An STV election profile, stating (a) the number of votes cast with each listed ranking over candidates \(c_1\) to \(c_4\), and (b) the tallies after each round of counting, election, and elimination.

| Ranking  | Count |
|----------|-------|
| \([c_1, c_2, c_3]\) | 5     |
| \([c_1]\)          | 18    |
| \([c_4, c_3]\)     | 10    |
| \([c_3, c_2, c_4]\)| 5     |
| \([c_2, c_4, c_3]\)| 17    |
| \([c_1, c_4, c_2, c_3]\)| 8     |

| Candidate | Round 1 | Round 2 | Round 3 |
|-----------|---------|---------|---------|
| \(c_1\)   | elected |         |         |
| \(c_3\)   | eliminated |       |         |
| \(c_2\)   |         |         |         |
| \(c_4\)   |         | 15.54   | 15.54   |

| Seats: 2 |
|----------|
| Quota: 22 |

| Candidate | Round 1 | Round 2 | Round 3 |
|-----------|---------|---------|---------|
| \(c_1\)   | 31      | —       | —       |
| \(c_2\)   | 17      | 20.46   | 25.46   |
| \(c_3\)   | 5       | 5       | —       |
| \(c_4\)   | 10      | 15.54   | 15.54   |
To the best of our knowledge, the algorithms presented in this paper form the first attempt to compute margins in STV elections. Blom et al. [2016] present a branch-and-bound algorithm for computing victory margins in IRV elections, itself an adaptation of earlier work by Magrino et al. [2011]. An IRV election elects a single winner $w$ from a field of candidates $C$ on the basis of the votes $B$ cast in the election. As in an STV election, each vote $b \in B$ is a (possibly partial) ordering over the candidates in $C$. An IRV election proceeds in rounds of candidate elimination. All votes that preference a candidate first are placed in that candidate’s tally. In the first round, the candidate with the least votes in their tally is eliminated. Each of these votes is placed in the tally of its next preferred candidate. Votes with no next preferred candidate become exhausted (are not redistributed). This process of candidate elimination is repeated until only a single candidate remains—this candidate is the winner of the election. Unlike IRV, STV elects multiple candidates with the counting algorithm alternating between rounds of candidate election and candidate elimination. The votes in the tally of elected candidates are redistributed at a reduced value.

Magrino et al. [2011] represent the outcome of an IRV election as a candidate sequence $\pi$ with candidates listed in the order in which they are eliminated (with the last candidate being the winner). Given such a sequence $\pi$, and a collection of votes, a linear program (LP) is presented that computes the smallest number of vote manipulations required to realise $\pi$. This linear program is labelled $\text{DISTANCETo}$. In an election with winner $w$, Magrino et al. [2011] search the space of alternate elimination sequences (in which a candidate other than $w$ is elected) for one requiring the least number of vote manipulations to realise. A key observation made by Magrino et al. [2011] is that, given a partial sequence of candidates $\pi'$, the $\text{DISTANCETo}$ LP computes a lower bound on the number of vote manipulations required to realise any elimination order that ends in $\pi'$. Magrino et al. [2011] progressively explore and build partial candidate elimination orders in a branch-and-bound algorithm. The last round margin (LRM) of the election (defined as the difference in the tallies of the winning candidate and runner-up, divided by two and rounded up) is used as an upper bound on the MOV. Consider an IRV election with candidates $c_1$, $c_2$, $c_3$, and $c_4$, with outcome $[c_4, c_3, c_2, c_1]$, where $c_1$ is the winning candidate. Partial orders containing a single candidate (not including the original winner $c_1$) are added to a tree. $\text{DISTANCETo}$ is applied to each partial order $\pi'$ in this tree to compute a lower bound on the number of vote manipulations required to realise an elimination sequence ending in $\pi'$. The partial order $\pi'$ with the smallest $\text{DISTANCETo}$ evaluation is expanded by adding a candidate (not already in $\pi'$) to the start of the sequence. Partial orders with evaluations equal to or larger than the current upper bound are pruned. When a complete elimination order (involving all candidates) is formed, its $\text{DISTANCETo}$ evaluation is used to revise the current recorded upper bound. The algorithm terminates once all partial orders have either been expanded or pruned, with the revised upper bound returned as the MOV.

Blom et al. [2016] improve the efficiency of the branch-and-bound algorithm of Magrino et al. [2011] by introducing new rules for computing lower bounds on the manipulation required to realise each partial order in the search tree. These rules typically result in tighter (i.e., higher) lower bounds for each partial order than supplied by solving the $\text{DISTANCETo}$ LP. Consequently, Blom et al. [2016] are able to: prune larger portions of the space of partial elimination sequences; reduce the number of calls to the $\text{DISTANCETo}$ LP; and quickly compute margins in elections for the which the algorithm of Magrino et al. [2011] times out after 72 hours of computation. Our margin-stv algorithm shares a similar structure to that of Blom et al. [2016] and Magrino et al. [2011], in that it searches the space of alternate election and elimination sequences using branch-and-bound. The margin-stv algorithm differs in several key aspects: each node is a partial sequence of candidate elections and eliminations (in place of a sequence of eliminations); a MINLP (in place of an LP) is used to evaluate nodes in this search tree; and the descendents of a partial sequence $\pi'$ are all complete sequences that start with $\pi'$ (in place of all sequences than end in $\pi'$). Moreover, a variation of the winner elimination upper bound for IRV elections Cary. [2011] is used as an initial upper bound on the STV MOV (as described by Chilingirian et al. [2016]). Section describes the margin-stv algorithm in detail.

The winner elimination upper bound (on the IRV margin of victory) of Cary. [2011] finds the most efficient way to eliminate the apparent winner of an IRV election at each elimination round, returning the least-cost (involving the smallest number of vote changes) of these. Chilingirian et al. [2016] develop a version of this upper bound for use in STV elections. Figure outlines this STV variant of the winner elimination upper bound. Consider the example STV election of Table where $c_1$ was elected in the first round of counting,
\begin{verbatim}
1 \textit{weub} \leftarrow |B|
2 E \leftarrow \text{candidates (eventually) elected}
3 \textbf{for each} round of counting \textit{j} \textbf{do}
4 \quad \textbf{if} a candidate is eliminated in round \textit{j} \textbf{do}
5 \quad \quad c_j \leftarrow \text{candidate eliminated in round } j
6 \quad \quad v_j \leftarrow \text{number of votes in } c_j\text{’s tally in round } j
7 \quad \textbf{for each } w \in E \text{ that has not yet been elected by round } j \textbf{do}
8 \quad \quad w_j \leftarrow \text{number of votes in } w\text{’s tally in round } j
9 \quad \Delta \leftarrow \lceil w_j - v_j \rceil
10 \quad \textit{weub} \leftarrow \min(\Delta, \textit{weub})
11 \quad \textbf{if} \lceil w_j - \frac{1}{2} v_j \rceil \text{ is less than or equal to tallies of all candidates still standing (excluding } c_j) \textbf{do}
12 \quad \quad \textit{weub} \leftarrow \min(\lceil w_j - \frac{1}{2} v_j \rceil, \textit{weub})
13 \textbf{return} \textit{weub}
\end{verbatim}

Figure 2: The winner elimination upper bound of Cary [2011] applied to compute an upper bound (denoted \textit{weub}) on the MOV of an STV election (as used by Chilingirian et al. [2016]). The notation \textit{B} and \textit{E} denote the set of votes cast in the election, and the set of candidates elected to a seat, respectively.

c_2\text{ eliminated in the second, and } c_3\text{ elected in the third (at which point both available seats had been filled). Following the algorithm listed in Figure 2, the winner elimination upper bound (\textit{weub}) is initially set to the total number of votes cast in the election (Step 1), which is 60. The set of candidates that are (eventually) elected is } E = \{c_1, c_3\}. \text{ In the first round of counting, a candidate (} c_1) \text{ is elected. The algorithm moves on to the second round (Step 3), in which a candidate (} c_2) \text{ is eliminated with 10 votes (} v_2 = 10 \text{ in Step 6). In Step 7, we consider each candidate in } E \text{ that has not been elected by round 2—candidate } c_3—\text{ and determine how they could be eliminated in this round. Candidate } c_3 \text{ has 14 votes in round 2 (} w_2 = 14 \text{ in Step 8). We could certainly eliminate } c_3 \text{ by taking 4 votes from their tally (} \Delta = 4 \text{ in Step 9) and giving them to some other candidate (} c_2, \text{ for example). However, candidate } c_3 \text{ can still be eliminated in this round if we take 2 votes from their tally and give them to } c_2 \text{ (Steps 11 and 12), under the assumption that we can break the resulting tie between } c_3 \text{ and } c_2 \text{ in } c_2\text{’s favour. The winner elimination upper bound is set to } 2 \text{ in Step 12. The algorithm moves on to the last round of counting in which no candidate is eliminated. Steps 5 to 12 are skipped and a winner elimination upper bound of } 2 \text{ is returned. }

To the best of the authors knowledge, the work of Chilingirian et al. [2016] describes the only attempt to compute bounds on the MOV for STV elections (in its adaptation of the winner elimination upper bound of Cary [2011]). The algorithms we present in this paper are, to the best of our knowledge, the first attempts to compute exact margins in small STV elections and lower bounds on the MOV in larger instances. IRV elections, in contrast, have received more consideration. Blom et al. [2016] and Magrino et al. [2011] present algorithms for computing exact margins in IRV elections. A number of works have presented algorithms for computing lower and upper bounds on IRV margins (see Cary [2011] and Sarwate et al. [2013]).

The focus of this paper is the computation of the MOV for STV elections. This MOV is defined as the smallest number of vote manipulations (the replacing of the ranking of a vote with an alternate ranking) required to ensure that a different set of candidates is elected (i.e., at least one candidate in this set is replaced with one that was not originally elected to a seat). Similar questions have been considered for alternate voting rules. The complexity of manipulating an election with bribery is considered by Faliszewski et al. [2011], under a number of voting schemes: Condocert-based; approval voting; scoring rules; veto rules; and plurality. Their aim is to find a manipulation to achieve a desired election result, while minimising the cost of bribes given to voters for changing their vote. If the cost of bribing a voter to change their vote is 1, this least cost set of bribes is equivalent to the margin of victory, as we have defined it. Kaczmarczyk and Faliszewski [2016] consider a variant of the bribery problem (destructive shift bribery) in which voters can be bribed to denote

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the position of a candidate \( c \) in their ranking by \( p \) positions (i.e., to move a candidate down in their ranking by \( p \) positions) in a bid to ensure that \( c \) does not win the election. Kaczmarszczak and Faliszewski \citeyear{KaczmarszczakFaliszewski2016} analyse the complexity of the destructive shift bribery problem for a number of voting rules (\( k \)-Approval, Borda, Copeland and Maximin). Polynomial-time algorithms are presented for computing the smallest set of desired bribes in the case of the \( k \)-Approval, Borda, and Maximin rules, while the problem is shown to be NP-complete for the Copeland rule. Schürmann \citeyear{Schurmann2017} describe a model-checking-based approach for the computation of margins in elections using the D'Hondt method, applying their approach to the 2015 Danish national parliamentary elections. The STV elections we consider in this paper vary considerably from the computation of margins in elections using the D'Hondt method, applying their approach to the 2015 Danish national parliamentary elections. The STV elections we consider in this paper vary considerably from the

5 Simple Upper Bounds on the STV MOV

Figure \[2\] presents an algorithm for computing an upper bound (a winner elimination upper bound) on the STV MOV. In elections where all seats are filled by a candidate prior to any eliminations taking place (e.g., an election with 2 seats, 4 candidates, and outcome \([c_1, 1) (c_2, 1) (c_3, 0) (c_4, 0)]\)), this algorithm is not able to reduce the upper bound from its original value (the total number of votes cast in the election). In these instances, we introduce a simple bound on the STV MOV, computed as follows. Consider an election \( E = (C, B, Q, N, E) \). The STV counting algorithm of Figure \[1\] selects candidates to a seat once the number of votes in their tally reaches or exceeds \( Q \). To change the outcome of \( E \), with winning candidates \( E \subset C \), we must find a series of vote manipulations that elects a candidate \( c \in C \setminus E \) to a seat. It is clear that we can elect \( c \in C \setminus E \), with primary vote \( f_p(c) \) (the total number of votes in \( B \) in which \( c \) is ranked first), if we take \( Q - f_p(c) \) votes away from other candidates and give them to \( c \) (we replace the ranking of these votes by a ranking that preferences \( c \) first). We compute \( Q - f_p(c) \) for each \( c \in C \setminus E \) and take the smallest result as an upper bound on the STV MOV. We call this bound the Simple-STV upper bound.

6 Computing Minimal Manipulations: A MINLP

Given an IRV election and a sequence of candidate eliminations \( \pi' \), Magrino et al. \citeyear{Magrino+2011} present a linear program (LP) for computing the smallest number of vote manipulations required to ensure candidates are eliminated in the order specified in \( \pi' \). In this section, we present a mixed-integer non-linear program (MINLP) that, given an STV election \( E = (C, B, Q, N, E) \), and a candidate order \( \pi \) (a sequence of candidate eliminations and elections), computes the smallest number of vote manipulations (changes to votes in \( B \)) required to realise \( \pi \). We re-use some notation and constraints introduced in the LP of Magrino et al. \citeyear{Magrino+2011}. The votes in \( B \) form the original profile of the election. The MINLP below, denoted \textsc{DistanceToSTV}, introduces variables indicating which votes in \( B \) are to be changed, and what their ranking will be in a new or modified profile. The constraints of the following MINLP are designed to enforce a specific candidate order \( \pi \) by modifying the smallest number of votes in \( B \) (where required).

6.1 Notation

- \( s, S \) A signature \( s \in S \) is a partial or total ranking over the candidates in \( C \); \( S \) is the set of all possible partial or total rankings over \( C \) (including those that do not appear on a vote in \( B \)).
- \( N_s \) Number of votes with signature \( s \in S \) cast in election \( E \) (i.e., the number of votes in \( B \) whose ranking matches signature \( s \)).
Known upper bound on the number of vote manipulations required to realise $\pi$ (such as the winner elimination upper bound of Figure 2 or the Simple-STV upper bound of Section 5).

The set of rounds of counting in the election—in each round a candidate is either elected to a seat or eliminated.

Integer number of votes in $B$ that are modified so that their new signature is $s \in S$.

Integer number of votes whose signature in $B$ (the original profile) is $s \in S$, but are modified to something other than $s$ in the new profile.

Integer number of votes with signature $s \in S$ in the new election profile.

Transfer value of votes being redistributed from an elected candidate in round $j$.

Number of votes eligible for transfer, from the candidate elected to a seat in round $j$, to candidates that are still standing at the end of round $j$.

Binary variable with a value of 1 if the tally of candidate $i$ exceeds or equals the quota at the start of round $j$, and 0 otherwise.

Floating point number of votes in the tally of candidate $i$ at the start of round $j$.

Floating point number of votes with signature $s \in S$ in the tally of candidate $i$ at the start of round $j$.

Floating point number of votes with signature $s \in S$ transferred to candidate $i$ from the candidate elected or eliminated in round $j$.

The subset of vote signatures that could possibly be in the tally of candidate $i \in C$ at the start of round $j$ (this can be inferred on the basis of the order of candidates in $\pi$).

The candidate that is elected or eliminated in round $j$ (according to $\pi$).

The set of candidates that appear before candidate $i$ in the preference ordering of signature $s \in S$ that are still standing (have not been elected or eliminated) at the end of round $j$.

The set of candidates still standing at the start of round $j$.

The subset of candidates in $C$ that are elected to a seat according to $\pi$.

The subset of candidates in $C$ that are eliminated according to $\pi$.

6.2 Objective

The objective of $\text{DISTANCESTV}$ is to minimise the number of votes in $B$ whose signature is changed.

$$z = \min \sum_{s \in S} p_s$$

6.3 Constraints

Constraints (6) and (7) appear in the LP of Magrino et al. [2011]. Constraint (6) ensures that the total number of votes in the new election profile (after manipulation) is equal to the total number of votes in the original profile. Constraint (7) defines the number of votes with signature $s \in S$ in the new profile, $y_s$.

$$\sum_{s \in S} m_s = \sum_{s \in S} p_s$$

$$y_s = N_s + p_s - m_s \quad \forall s \in S$$

Constraint (5) ensures that candidates who have achieved a quota by round $j$, have also achieved a quota by round $j + 1$. Constraints (9) and (10) ensure that that number of votes a candidate $i$ has in their tally at the start of round $j$ is greater than or equal to $Q$ if $i$ has achieved a quota by round $j$ ($q_{ij}$ is 1) and less than $Q$ otherwise ($q_{ij}$ is 0, and $\epsilon \ll 1$).
\[ q_{i,j-1} \leq q_{ij} \quad \forall i \in \mathcal{C}, j \in \mathcal{R} \quad (8) \]
\[ v_{ij} \geq q_{ij} Q \quad \forall i \in \mathcal{C}, j \in \mathcal{R} \quad (9) \]
\[ v_{ij} \leq (1 - q_{ij}) (Q - \epsilon) + q_{ij} |\mathcal{B}| \quad \forall i \in \mathcal{C}, j \in \mathcal{R} \quad (10) \]

Constraints (11) and (12) define the number of votes of signature \( s \in \mathcal{S} \) in candidate \( i \)'s tally at the start of round \( j \). Where \( j > 1 \), this is equal to the number of votes of signature \( s \in \mathcal{S} \) in candidate \( i \)'s tally at the start of the previous round, plus the number of votes of signature \( s \in \mathcal{S} \) distributed to \( i \) from the candidate elected (or eliminated) in round \( j - 1 \). As all votes of a single signature \( s \) will reside in the tally of only one candidate at any point in time, at least one of \( y_{i,j-1,s} \) and \( d_{i,j-1,s} \) will be zero in each instance of Constraint (12). Although the order in which candidates are elected and eliminated is fixed (to that defined by \( \pi \)), the round in which a candidate is given, via distribution, votes of certain signatures can vary. Consider a candidate \( c \) whose tally has reached or exceeded the quota in round \( j \). Candidate \( c \) may have to wait several rounds to be elected to a seat, if one or more other candidates have also reached a quota by round \( j \), and have more votes in their tallies (they are elected before \( c \)). While candidate \( c \) is waiting to be elected, they are not given any additional votes during the distribution of the surpluses of elected candidates—these votes skip \( c \) and are given to the next eligible candidate in their ranking. As the candidate order \( \pi \) does not prescribe exactly when an elected candidate achieves a quota, we must support the possibility that votes of certain signatures \( s \in \mathcal{S} \) can be distributed to a candidate in one of a number of different rounds. Constraint (13) sums the total number of votes for candidate \( i \) at the start of round \( j \).

\[ y_{i,s} = y_s \quad \forall i \in \mathcal{C}, s \in \mathcal{E}_{ij} \quad (11) \]
\[ y_{i,s} = y_{i,j-1,s} + d_{i,j-1,s} \quad \forall i \in \mathcal{C}, j \in \mathcal{R}, j > 1, s \in \mathcal{E}_{ij} \quad (12) \]
\[ v_{ij} = \sum_{s \in \mathcal{E}_{ij}} y_{i,s} \quad \forall i \in \mathcal{C}, j \in \mathcal{R} \quad (13) \]

Constraint (14) ensures that the candidate elected to a seat in round \( j \) has more (or an equal number of) votes in their tally at the start of round \( j \) than all other candidates who are still standing (have not been eliminated or elected before round \( j \)). Conversely, a candidate who is eliminated in round \( j \) must have fewer (or an equal number of) votes in their tally at the start of round \( j \) than all other candidates who are still standing (Constraint (15)). Moreover, a candidate can only be eliminated in a round if no other remaining candidate has a quota’s worth of votes in their tally at the start of the round (Constraint (16)).

\[ v_{\pi[j],j} \geq v_{kj} \quad \forall j \in \mathcal{R}, \pi[j] \in \pi^+, k \in \mathcal{S}_j \setminus \{\pi[j]\} \quad (14) \]
\[ v_{\pi[j],j} \leq v_{kj} \quad \forall j \in \mathcal{R}, \pi[j] \in \pi^-, k \in \mathcal{S}_j \setminus \{\pi[j]\} \quad (15) \]
\[ q_{ij} = 0 \quad \forall j \in \mathcal{R}, \pi[j] \in \pi^-, i \in \mathcal{S}_j \quad (16) \]

Constraint (17) defines the value of variable \( d_{i,s} \) (the total number of votes of signature \( s \) distributed to candidate \( i \) in round \( j \) after candidate \( \pi[j] \) is elected or eliminated). Recall that the set \( \mathcal{E}_{ij} \) denotes the subset of signatures \( s \in \mathcal{S} \) that could possibly be in candidate \( i \)'s tally at the start of round \( j \). The set \( \mathcal{E}_{\pi[j],j} \cap \mathcal{E}_{i,j+1} \) contains only those signatures that could possibly be distributed from \( \pi[j] \) (the candidate elected or eliminated in round \( j \)) to candidate \( i \)—the subset of signatures that could have been in \( \pi[j] \)'s tally in round \( j \) and in the tally of candidate \( i \) at the start of round \( j + 1 \). If \( \pi[j] \) is eliminated (\( \pi[j] \in \pi^- \)), any votes distributed to another candidate are distributed at their current value (\( d_{i,s} = y_{\pi[j],s} \) for all \( s \in \mathcal{E}_{\pi[j],j} \cap \mathcal{E}_{i,j+1} \)). If \( \pi[j] \) is elected, candidate \( i \) may receive their votes of signature \( s \in \mathcal{E}_{\pi[j],j} \cap \mathcal{E}_{i,j+1} \) (at a reduced value) only if \( i \) does not already have a quota’s worth of votes and is either the next preferred candidate in \( s \), or all candidates that appear between \( \pi[j] \) and \( i \) in \( s \) already have a quota’s worth of votes.

\[ d_{i,s} = \begin{cases} y_{\pi[j],s} & \text{if } \pi[j] \in \pi^- \\ \tau_j y_{\pi[j],s} (1 - q_{ij}) \prod_{k \in \mathcal{B}_{i,j}} q_{kj} & \text{if } \pi[j] \in \pi^+ \end{cases} \quad \forall i \in \mathcal{C}, j \in \mathcal{R}, s \in \mathcal{E}_{\pi[j],j} \cap \mathcal{E}_{i,j+1} \quad (17) \]
Upon the election of a candidate $c$, a subset of the votes in their tally (those for which there is a next preferred candidate that is still standing—not yet eliminated or elected—and whose tally has not already reached or exceeded the quota) are distributed to one or more alternate candidates. These votes are called transferable votes. The remainder become exhausted (there is no such ‘next preferred’ candidate for these votes). The number of transferable votes in a candidate $c$’s tally, upon their election in round $j$, denoted $\rho_j$, is defined in Constraint (18). The transfer value assigned to these votes is dependent on both the size of $c$’s surplus (the value of votes in their tally minus the quota) and the total value of the transferable votes in $c$’s tally (the votes that have a valid next preferred candidate). This transfer value $\tau_j$, and its relationship to the quantity of transferable votes in an elected candidate’s tally $\rho_j$, is defined in Constraint (19). Both of these variables are relevant only in rounds in which a candidate is elected to a seat, and a surplus is distributed.

$$\rho_j = \sum_{i \in S_j} \sum_{s \in \mathcal{P}_{\pi[j],j} \cap \mathcal{P}_{i,j+1}} y_{\pi[j],j,s} (1 - q_{ij}) \prod_{k \in \mathcal{R}_{sij}} q_{kj} \quad \forall j \in R, \pi[j] \in \pi^+ \quad (18)$$

$$\tau_j \rho_j = v_{\pi[j],j} - Q \quad \forall j \in R, \pi[j] \in \pi^+ \quad (19)$$

Constraint (19) yields incorrect transfer values—that are greater than one—in the event that the total value of the transferable votes in an elected candidate $c$’s tally is less than their surplus. This can occur if a large portion of $c$’s votes become exhausted once they are elected. In this circumstance, the transfer value of $c$’s transferable votes is set to 1 (i.e., these votes are distributed at their current value). Constraint (19) is thus rewritten as shown in Constraint (20).

$$\tau_j = \begin{cases} 
1 & \text{if } \rho_j \leq v_{\pi[j],j} - Q \\
 v_{\pi[j],j} - Q & \text{otherwise} 
\end{cases} \quad \forall j \in R, \pi[j] \in \pi^+ \quad (20)$$

### 7 Computing Lower Bounds for Partial Candidate Orders

Magrino et al. [2011] present an LP for computing the smallest number of vote manipulations required to realise a sequence of candidate eliminations in a given IRV election. This LP, when applied to a partial elimination sequence $\pi'$ (under the assumption that all candidates not in $\pi'$ have already been eliminated and their votes distributed to candidates in $\pi'$), computes a lower bound on the manipulations required to realise an elimination sequence ending in $\pi'$. Similarly, we can apply $\text{DISTANCEToSTV}$ to a partial sequence of candidate elections and eliminations $\pi'$ (a partial candidate order) to compute a lower bound on the manipulations required to realise a complete order (including all candidates) that starts with $\pi'$.

Consider the STV election of Table 1 with candidates $c_1$, $c_2$, $c_3$, and $c_4$. Given the partial order $[(c_1,0), (c_3,1)]$, $\text{DISTANCEToSTV}$ computes the smallest number of vote manipulations required to ensure that:

- Candidate $c_1$ has the fewest votes in the 1st round of counting (fewer votes than $c_2$, $c_3$, and $c_4$);
- No candidate has a quota’s worth of votes in their tally in the 1st round;
- Candidate $c_3$ has a quota’s worth of votes in their tally in the 2nd round; and,
- Candidate $c_4$ has the most votes in their tally in the 2nd round (more votes than $c_2$ and $c_4$).

For any complete order $\pi$ that starts with $[(c_1,0), (c_3,1)]$, $\text{DISTANCEToSTV}$ will ensure that the above constraints hold in addition to constraints that enforce the remaining elections and eliminations in $\pi$. For any partial order $\pi'$, the set of constraints enforced by $\text{DISTANCEToSTV}$ is a subset of those enforced for any complete order starting with $\pi'$. Consequently, solving $\text{DISTANCEToSTV}$ for $\pi'$ yields a lower bound on the manipulations required to realise any complete order starting with $\pi'$. This property of the $\text{DISTANCEToSTV}$ MINLP allows us to form a branch-and-bound algorithm for computing the MOV in STV elections.
7.1 A Simple Lower Bounding Rule

Blom et al. [2016] present two methods for computing lower bounds on the degree of manipulation required to realise IRV elimination sequences that end in a partial order \( \pi' \), without requiring the solving of an LP. We adapt the logic underlying these rules to develop a lower bounding rule applicable to partial orders in STV elections. Given a partial order \( \pi' \) (a sequence of candidate eliminations and elections), this rule computes a lower bound on the number of vote manipulations required to realise any complete order starting with \( \pi' \).

Consider an STV election \( \mathcal{E} = (C, B, Q, N, E) \) and a partial order \( \pi' \). We can infer from \( \pi' \), the set of vote signatures \( s \in S \) that could potentially lie in the tally of each candidate \( c \in C \) in each round of election or elimination \( j \) in \( \pi' \) (this set is denoted \( \mathcal{P}_{c,j} \), as defined in Section 6.1). Consequently, we can infer the maximum possible number of votes that could lie in the tally of each candidate \( c \) in each round \( j \) (the total number of votes cast with a signature \( s \in \mathcal{P}_{c,j} \), \( V_{c,j}^{\text{max}} \), under the assumption that no manipulation has yet taken place (Equation 21). Recall that \( N_s \) denotes the number of votes in \( B \) that have been cast with signature \( s \in S \). Irrespective of when a candidate \( c \) is elected or eliminated, \( c \)'s tally will contain at least all cast votes in which they are the first preference (their primary vote has fewer votes than all candidates in \( S \) of round \( j \), (Equation 23). To ensure that \( c \) has a quota’s worth of votes at the start of round \( j \), we must modify at least \( l_q(c_j, \pi') \) votes (Equation 23).

\[
V_{c,j}^{\text{max}} = \sum_{s \in \mathcal{P}_{c,j}} N_s \quad \forall c \in C, j \in \{1, \ldots, |\pi'|\} \tag{21}
\]

\[
V_{c,j}^{\text{min}} = f_p(c) \quad \forall c \in C \tag{22}
\]

We consider each election and elimination in \( \pi' \). Let \( c_j \) denote the candidate elected or eliminated in round \( j \) of \( \pi' \). If \( c_j \) is elected, \( c_j \) must have a quota’s worth of votes in their tally (unless the number of seats left to be filled in round \( j \) equals the number of candidates still standing at the start of round \( j \)). To ensure that \( c_j \) has a quota’s worth of votes at the start of round \( j \), we must modify at least \( l_q(c_j, \pi') \) votes (Equation 23).

\[
l_q(c_j, \pi') = \max(0, Q - V_{c,j}^{\text{max}}) \tag{23}
\]

If candidate \( c_j \) is eliminated, \( c_j \) must have fewer votes than all other candidates still standing at the start of round \( j \), denoted \( S_j \), and no remaining candidate can have a quota’s worth of votes. To ensure that \( c_j \) has fewer votes than all candidates in \( S_j \), we must modify at least \( l_{l1}(c_j, \pi') \) votes (Equation 24). To ensure that no remaining candidate has a quota’s worth of votes at the start of round \( j \), we must modify at least \( l_{l2}(c_j, \pi') \) votes (Equation 25). To ensure that \( c_j \) is eliminated in round \( j \), we must modify at least \( l_e = \max(l_{l1}, l_{l2}) \) votes.

\[
l_{l1}(c_j, \pi') = \arg \max_{c \in S_j \setminus \{c_j\}} \max(0, V_{c,j}^{\text{min}} - V_{c,j}^{\text{max}}) \tag{24}
\]

\[
l_{l2}(c_j, \pi') = \arg \max_{c \in S_j} \max(0, V_{c,j}^{\text{min}} - Q) \tag{25}
\]

Equation 25 will ensure that no candidate has more than a quota’s worth of votes in round \( j \). To ensure that a candidate has less than a quota, their tally value must be less than or equal to \( Q - \epsilon \), where \( \epsilon \ll 1 \). As we are computing a lower bound on required vote manipulation, we ignore the \( \epsilon \) term for simplicity.

A lower bound on the degree of manipulation required to realise a complete order starting with \( \pi' \) is computed by taking the maximum of \( l_q \) and \( l_e \) for each candidate elected, and eliminated, in \( \pi' \).

\[
\text{lb}(\pi') = \arg \max_j \left\{ \begin{array}{ll}
  l_q(c_j, \pi') & \text{if } c_j \text{ is elected} \\
  l_e(c_j, \pi') & \text{if } c_j \text{ is eliminated}
\end{array} \right. \tag{26}
\]

**Example 1.** Consider the STV election of Table 1 and the candidate order \( \pi = [(c_3, 0), (c_1, 1), (c_2, 1), (c_4, 0)] \), where candidate \( c_3 \) is eliminated in Round 1, \( c_1 \) is elected in Round 2, and \( c_2 \) is elected Round 3. We now use our lower bounding rule to compute a lower bound on the number of vote manipulations required to realise this order of elections and eliminations. The original winners in this election are candidates \( c_1 \) and \( c_3 \) (\( E = \{c_1, c_3\} \)). Given the cast votes listed in Table 1a, we determine where these vote signatures would lie (i.e., in which candidates tally), in each round of counting (until all seats have been filled), assuming \( \pi \).
is realised (Table 4. Prior to any manipulation of the cast votes, the maximum value of each candidates tally, in each round, is listed in Table 7. The minimum value of each candidates tally, in each round, is equal to their primary vote. Here, $V_{c_1}^{\text{min}} = 26$, $V_{c_2}^{\text{min}} = 10$, $V_{c_3}^{\text{min}} = 9$, and $V_{c_4}^{\text{min}} = 15$. We consider each election and elimination in $\pi$. To eliminate $c_3$ in Round 1, $c_4$ must have fewer votes than all other candidates. Here, $l_{q}(c_3, \pi) = 0$ (no manipulation is required to ensure $c_3$ has the fewest votes). No candidate can have a quota’s worth of votes in Round 1. In this example, the quota is 21 votes, $l_{q}(c_3, \pi) = 5$, and $l_{q}(c_4, \pi) = 5$. To elect $c_1$ in Round 2, we must ensure that $c_1$ has a quota in Round 2. Here, $l_{q}(c_1, \pi) = \max(0, Q - V_{c_1}^{\text{max}}) = \max(0, 21 - 26) = 0$. To elect $c_2$ in Round 3, we must ensure that $c_2$ has a quota in Round 3. Here, $l_{q}(c_2, \pi) = \max(0, Q - V_{c_2}^{\text{max}}) = \max(0, 21 - 10) = 11$. Consequently, a lower bound on the manipulation required to realise $\pi$ is $\ell_b(\pi) = \max\{l_{e}(c_3, \pi), l_{q}(c_1, \pi), l_{q}(c_2, \pi)\} = \max\{5, 0, 11\} = 11$.

We describe, in Section 8, how this lower bounding rule can be used to avoid solving the $\text{DISTANCE}_{\text{STV}}$ MINLP for some partial orders explored by the $\text{margin-stv}$ algorithm.

### 8 Computing Margins in STV Elections ($\text{margin-stv}$)

Figure 8 defines our $\text{margin-stv}$ algorithm for computing the MOV of an STV election. The $\text{margin-stv}$ algorithm maintains an initially empty frontier of partial candidate orders $F$ (Step 1). An upper bound $UB$ on the STV MOV is computed in Step 2. In our implementation of $\text{margin-stv}$ we use the minimum of the winner elimination upper bound of Figure 2 and the Simple-STV upper bound of Section 4. The frontier $F$ is populated with partial candidate orders of size one (the election or elimination of a single candidate)—each of which is assigned a lower bound on required manipulations with $\text{DISTANCE}_{\text{STV}}$—in Steps 4 – 9. Partial orders with a lower bound less than the current upper bound $UB$ are added to $F$ in Step 9.

Example 2. Consider the STV election of Table 7. The winner elimination upper bound is 2 votes and the Simple-STV bound is 6 votes. In Step 2, the current upper bound $UB$ is initialised to 2. The partial candidate orders $\{[c_1, 0]\}, \{[c_1, 1]\}, \{[c_2, 0]\}, \{[c_2, 1]\}, \{[c_3, 0]\}, \{[c_3, 1]\}, \{[c_4, 0]\}, \text{ and } \{[c_4, 1]\}$ are created, and assigned lower bounds by $\text{DISTANCE}_{\text{STV}}$ of 11, 0, 6, 11, 6, 12, 8, and 6, respectively. Only $\{[c_1, 1]\}$ is added to $F$ in Step 9. Figure 4 visualises the search tree explored by $\text{margin-stv}$ in this example.

Once the frontier $F$ is populated with partial orders of size one, Steps 10 to 14 expand these orders in the search for an alternate outcome for the election that requires the smallest degree of manipulation to realise. Partial orders with the smallest assigned lower bound are expanded in turn. A partial order $\pi'$ is expanded by first removing it from the frontier (Step 12). For each candidate $c$ not mentioned in $\pi'$, two new partial
Figure 3: The margin-stv algorithm for computing the MOV of an STV election $E = (C, B, Q, N, E)$ with candidates $C$, votes $B$, quota $Q$, seats $N$, and winners $E \subset C$.

orders are formed in which $c$ is elected ($\pi' + [(c, 1)]$) and eliminated ($\pi' + [(c, 0)]$) in successive iterations of the loop in Steps 18 to 26. If the new partial order $\pi$ is valid (i.e., it does not elect the same candidates elected in the original winners set $E$), it is evaluated with $\text{DistanceToSTV}$ (Step 21). If $\pi$ contains all candidates (it is a complete order), it represents an alternate election outcome requiring $l$ vote manipulations to realise ($l$ is computed by $\text{DistanceToSTV}$ in Step 21). If $l$ is less than the current upper bound, $UB$, $UB$ is replaced with $l$ (Step 23). If $\pi$ is not a complete order, it is added to $F$ if $l$ is less than $UB$ (Step 26).

We repeatedly expand the partial order with the smallest assigned lower bound, updating the frontier $F$, until $F$ is empty (there are no remaining partial orders that can be expanded). At this point, the algorithm terminates and returns the current upper bound $UB$ as the election MOV in Step 15.

Example 3 (Example 2 cont). The frontier, $F$, now contains one partial order $\pi' = [(c_1, 1)]$. The following partial orders are formed and evaluated with $\text{DistanceToSTV}$ in successive executions of Step 19 and 21:

- $[(c_1, 1), (c_2, 0)]$ Lower bound of 0
- $[(c_1, 1), (c_2, 1)]$ Lower bound of 12
- $[(c_1, 1), (c_3, 0)]$ Lower bound of 2
Figure 4: Complete search tree explored in the application of \texttt{margin-stv} to the STV election of Table 1 among candidates \(c_1, c_2, c_3,\) and \(c_4.\) The initial upper bound on the MOV in this example is 2 votes.

[\((c_1, 1), (c_3, 1)\)] \hspace{1cm} \text{Invalid as the original winners } c_1 \text{ and } c_3 \text{ are both elected}

[\((c_1, 1), (c_4, 0)\)] \hspace{1cm} \text{Lower bound of 3}

[\((c_1, 1), (c_4, 1)\)] \hspace{1cm} \text{Lower bound of 7}

Only \[\{(c_1, 1), (c_2, 0)\}\] is added to \(F\) in Step 26. At this point, \(F = \{\{(c_1, 1), (c_2, 0)\}\}, \) and \[\{(c_1, 1), (c_2, 0)\}\] is the next order expanded in Step 13. The following complete orders are formed in successive executions of Step 19 (in our implementation of \texttt{margin-stv}, we complete all partial orders formed in Step 19 with their 'obvious' ending if possible—i.e., if there is only one candidate remaining). All orders have a lower bound greater than the current upper bound of 2, and are pruned from consideration.

[\[(c_1, 1), (c_2, 0), (c_3, 0), (c_4, 1)\]\] \hspace{1cm} \text{\texttt{DISTANCESTV} evaluation of 3}

[\[(c_1, 1), (c_2, 0), (c_3, 1), (c_4, 0)\]\] \hspace{1cm} \text{Invalid as original winners } c_1 \text{ and } c_3 \text{ are elected}

[\[(c_1, 1), (c_2, 0), (c_4, 0), (c_3, 1)\]\] \hspace{1cm} \text{Invalid as original winners } c_1 \text{ and } c_3 \text{ are elected}

[\[(c_1, 1), (c_2, 0), (c_4, 1), (c_3, 0)\]\] \hspace{1cm} \text{\texttt{DISTANCESTV} evaluation of 6}

After pruning all partial orders from \(F,\) the frontier is empty and the algorithm returns the upper bound of 2 as the election MOV in Step 15. In this example, \texttt{margin-stv} has shown that there is no better manipulation (requiring fewer vote changes) than that inferred by the winner elimination upper bound.

In Steps 7 and 21, where \texttt{DISTANCESTV} is evaluated for a partial (or complete) order \(\pi,\) we can use the lower bounding rule of Section 7.1 to compute an initial lower bound for \(\pi, l'.\) If \(l' \geq UB,\) the current upper bound, we need not solve \texttt{DISTANCESTV} as \(\pi\) can immediately be pruned from consideration. If \(l' < UB,\) we can solve \texttt{DISTANCESTV} to get a second lower bound for \(\pi, l.\) We can then assign to \(\pi\) the maximum of the two lower bounds \(l'\) and \(l.\) In Section 9, we examine the relative performance of \texttt{margin-stv}, in terms of the number of \texttt{DISTANCESTV} MINLPs solved and the runtime of the algorithm, in the setting where this extra lower bounding computation is performed, and when it is omitted.
8.1 Implementation Details

We use SCIP [Achterberg, 2009] to solve each MINLP formed by margin-stv. In practice, we have found that some instances of DISTANCE TO STV, even for small election instances, can be difficult for SCIP to solve in reasonable time. Consequently, we terminate MINLP solves if the time since a last improving solution has been found reaches a pre-specified time limit. If a partial order \( \pi' \) is being evaluated, and the MINLP is terminated before an optimal solution is found, the best objective value (a lower bound on the optimal objective) is assigned to \( \pi' \) as its lower bound. If a complete order \( \pi \) is being evaluated, and the MINLP is terminated before finding an optimal solution, \( \pi \) is inserted into the frontier \( F \) with a lower bound equal to the best objective value of the MINLP, and the current upper bound is not revised (as would normally occur after evaluating a complete order). The result is that margin-stv may terminate with a frontier that contains a number of complete orders (that cannot be further expanded) with a smallest lower bound \( L \), and a current upper bound \( UB \) that is greater than \( L \) (i.e., the algorithm returns with a lower and upper bound on the MOV, but not an exact value). This occurs for a number of elections in our test set in Section 9.

It is clear from the evaluation of margin-stv, in Section 9, on small STV elections (with candidate numbers ranging from 4 to 13) that it will not scale to more realistic elections with dozens of candidates. To improve its scalability, we vary the algorithm so that a MIP relaxation of the DISTANCE TO STV is constructed to evaluate each partial and complete order formed in Steps 6 and 19 of Figure 3. We consider two types of relaxation. The first replaces each bilinear term present in the MINLP (these terms appear in Constraints (17) and (20)) with McCormick [1976] inequalities. Each bilinear term \( xy \), where \( x \) and \( y \) are continuous variables with domains \([x^L, x^U]\) and \([y^L, y^U]\), is replaced with variable \( z = xy \). Variable \( z \) is defined by Equations (27) to (30). The optimal solution of the relaxed DISTANCE TO STV is a lower bound on that of the MINLP. The MOV computed by margin-stv, in this context, is a lower bound on the true MOV.

\[
\begin{align*}
z &\geq y^L x + x^L y - x^L y^L \\
&\geq y^U x + x^U y - x^U y^U \\
z &\leq y^U x + x^L y - x^L y^U \\
z &\leq y^L x + x^U y - x^U y^L
\end{align*}
\]

Our second relaxation replaces each bilinear term present in the MINLP (these terms appear in Constraints (17) and (20)) with a piecewise linear relaxation (specifically, a relaxation proposed by Gounaris et al. [2009]). The result is again a MIP relaxation of the DISTANCE TO STV whose optimal solution is a lower bound on that of the MINLP. Given a product of two continuous variables \( x \) and \( y \) (a bilinear term \( xy \)), the piecewise linear relaxation we apply is defined as follows (replicated from the work of Gounaris et al. [2009]), where the variable \( z = xy \) replaces the bilinear term where it appears in the original MINLP. We divide the domain of \( x \) into \( K \) ranges \( \{[x_0, x_1], \ldots, [x_{k-1}, x_k]\} \) and introduce a binary variable \( \lambda_k \) to indicate whether \( x \) has a value in range \( [x_{k-1}, x_k] \). Constraint (31) ensures that only one \( \lambda_k \) can take on a value of 1. The domain of variable \( y \) is \([y^L, y^U]\). The variable \( \delta y_k \) defines the “deviation of variable \( y \) from its lower bound \( y^L \), if \( x \in [x_{k-1}, x_k] \)” and is set to 0 if the value of \( x \) lies outside of this range [Gounaris et al., 2009].

\[
\begin{align*}
\sum_{k=1}^{K} \lambda_k &= 1 \\
\sum_{k=1}^{K} x_{k-1} \lambda_k &\leq x \leq \sum_{k=1}^{K} x_k \lambda_k \\
y &= y^L + \sum_{k=1}^{K} \delta y_k \\
0 &\leq \delta y_k \leq (y^U - y^L) \lambda_k \\
\forall k
\end{align*}
\]
\[
\begin{align*}
z & \geq y^U x + \sum_{k=1}^{K} x_k \delta y_k - (y^U - y^L) \sum_{k=1}^{K} x_k \lambda_k \\
x & \leq y^U x + \sum_{k=1}^{K} x_{k-1} \delta y_k - (y^U - y^L) \sum_{k=1}^{K} x_{k-1} \lambda_k \\
z & \leq y^L x + \sum_{k=1}^{K} x_k \delta y_k \\
z & \geq y^L x + \sum_{k=1}^{K} x_k \delta y_k
\end{align*}
\]

In the DistanceToSTV MINLP, variables \( \tau_j \) and \( y_{\pi[j],j,s} \) (Constraint (17)), and \( \tau_j \) and \( \rho_j \) (Constraint (20)) participate in bilinear terms. We treat the transfer value of votes distributed from the candidate elected in round \( j \), \( \tau_j \), as \( x \) in the piecewise linear relaxation scheme presented above, with variables \( y_{\pi[j],j,s} \) (the total value of votes of signature \( s \) in the tally of candidate \( \pi[j] \) at the start of round \( j \)) and \( \rho_j \) (the number of transferable votes in the tally of the candidate elected in round \( j \)) as \( y \). In Section 4, we discretise the domain of \( \tau_j \) into varying numbers of segments (by varying \( K \)), and compare the resulting lower bounds (on margins) found by margin-stv to the true MOV found when no such relaxation is used.

In their algorithm for computing margins in IRV elections, [Magrino et al. 2011] recognise that their DistanceTo LP, defined to compute the smallest number of vote changes required to realise a particular (partial or complete) elimination order, need only consider a subset of possible vote signatures (in place of all possible partial and total orderings over the set of candidates). Given an elimination order \( \pi \), specifying which candidates are eliminated in each counting round, [Magrino et al. 2011] note that classes of ballot signatures, denoted equivalence classes, will behave in the same way (be transferred between candidates in the same way) as counting progresses. Consequently, each DistanceTo LP need only contain variables defining the number of votes of each equivalence class present in the modified election profile. The definition of equivalence classes presented by [Magrino et al. 2011] can be applied in the context of both IRV and STV elections, given an elimination sequence (for an IRV election) and a sequence of elections and eliminations (for an STV election). In our implementation of margin-stv, we define the DistanceToSTV MINLP (and its relaxations) over signature equivalence classes (i.e., the set \( S \) contains the signatures of all equivalence classes, in place of all partial and total orders over the set of candidates \( C \)).

9 Evaluation

We first evaluate margin-stv on a set of small STV elections. To do so, we take 16 IRV elections conducted in the US between 2007 and 2014, and re-imagine them as STV elections with 2 available seats. The number of candidates in these elections range between 4 and 13. All experiments in this paper have been conducted on a machine with an Intel Xeon ES-2440 2.40GHz 6 core processor, and 64GB of RAM.

Table 4 reports the results of margin-stv – without and with the use of the lower bounding rule of Section 4.1 – on 16 small IRV (re-imagined as STV) elections. We record: the number of candidates \(|C|\) in the election; the best computed upper bound on the MOV (the minimum of the winner elimination and Simple-STV upper bound); the MOV returned by margin-stv (or bounds on the MOV); the number of DistanceToSTV MINLPs solved by the algorithm; and the time taken by margin-stv to compute the MOV (in seconds). SCIP [Achterberg, 2009] is used to solve all DistanceToSTV MINLPs. We terminate each run of margin-stv after 12 hours, and infer a lower and upper bound on the MOV from the state of the frontier. Our algorithm did not terminate for three instances (13, 14, and 16) within 12 hours. In 11/16 instances, our algorithm could not produce an exact MOV, but a lower and upper bound on the MOV. In 7/11 instances, the lower and upper bound differed by only 1 vote (i.e., margin-stv was able to produce very tight bounds on the MOV). Table 4 shows that our lower bounding rule can, in some instances, reduce the runtime of the algorithm. For instances 1, 2, 10, and 12, runtimes are reduced by 107, 761, 106, and 1,635
| # | |C| |Best UB| |mov-stv| |MOV| |MINLPs| |Time| |mov-stv| |MOV| |MINLPs| |Time|
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|1 |4 |38,199 |30,654 – 37,908 |54 |2,528 |30,654 – 37,908 |35 |2,421 |
|2 |4 |5,683 |2,725 – 2,726 |17 |900 |2,725 – 2,726 |7 |139 |
|3 |4 |6,530 |6,359 – 6,360 |25 |464 |6,359 – 6,360 |19 |649 |
|4 |4 |2,527 |63 |18 |171 |63 |14 |169 |
|5 |4 |653 |59 |17 |20 |59 |11 |19 |
|6 |4 |763 |375 |23 |708 |375 |13 |1,086 |
|7 |5 |7,500 |7,499 – 7,500 |48 |213 |7,499 – 7,500 |30 |211 |
|8 |5 |1,079 |628 – 629 |45 |2,972 |628 – 629 |30 |2,972 |
|9 |5 |1,425 |659 – 660 |38 |2,331 |659 – 660 |26 |2,318 |
|10 |5 |422 |164 – 165 |42 |565 |164 – 165 |22 |459 |
|11 |7 |1,719 |441 |58 |1 |441 |39 |1 |
|12 |7 |2,289 |1,326 – 1,327 |2,247 |13,693 |1,326 – 1,327 |1,006 |12,058 |
|13 |8 |2,857 |1,216 – 1,576 |— |∞ |1,309 – 1,576 |— |∞ |
|14 |11 |1,178 |309 – 1,178 |— |∞ |351 – 1,178 |— |∞ |
|15 |11 |63 |4 |244 |107 |4 |190 |106 |
|16 |13 |2,188 |823 – 2,188 |— |∞ |0 – 2,188 |— |∞ |

Table 4: Application of margin-stv to 16 US IRV elections re-imagined as STV elections with 2 seats, reporting: the number of candidates (|C|); best upper bound on the MOV; computed MOV (or bounds on the MOV); number of MINLPs solved; and algorithm runtime (in seconds) (∞ represents a time out of 12 hours) without and with the use of the lower bounding rule of Sec. 7.1. Best results for bounds are bolded.

Our lower bounding rule can increase the runtime of the algorithm—for instances 3 and 6, runtimes are increased by 185 and 378 seconds. Our lower bounding rule consistently reduces the number of MINLPs solved, but if these MINLPs are quickly deemed infeasible by SCIP [Achterberg, 2009], the additional time spent computing lower bounds for each partial order visited becomes an overhead.

Table 5 reports the results of the relaxed margin-stv algorithm (with bilinear terms replaced by McCormick [1976] inequalities) in each STV instance of Table 4 without and with the lower bounding rule of Section 7.1. CPLEX 12.5 is used to solve all DistanceToSTV MIPs formed by our margin-stv relaxations. We report the lower bound on the MOV found by margin-stv alongside the exact MOV (or bounds on the exact MOV) reported in Table 4. The lower bounds found by margin-stv (with McCormick [1976] inequalities) are often significantly lower than the exact MOV. Our lower bounding rule is beneficial in this setting— it often discovers tighter lower bounds on partial and complete orders than the MIP relaxation of DistanceToSTV (in 7/16 instances, applying the lower bounding rule results in a tighter lower bound being found by relaxed margin-stv), and reduces runtimes significantly for larger instances.

Relaxed margin-stv (with McCormick [1976] inequalities and our lower bounding rule) can compute a lower bound on the margin in each of our 16 STV instances within 12 hours. However, in 5/16 instances a trivial lower bound of 0 is found. We consider whether solving a piecewise linear relaxation of each DistanceToSTV MINLP, as described in Section 8.1, results in margin-stv finding better lower bounds. Table 5 reports the results of piecewise-relaxed margin-stv (with the use of our lower bounding rule) given varying K (with higher values of K forming a more accurate relaxation of each DistanceToSTV MINLP). In general, the resulting MOV lower bounds increase as K increases. The counter-example is instance 16, for which margin-stv computes lower bounds of 760, 95, 17, and 0 for K = 5, 10, 15, and 20. As K increases, the MIP relaxations solved throughout the algorithm become more complex and time consuming to solve. Consequently, margin-stv makes less progress through the space of partial orders in instance 16 within the 12 hour time limit. In the K = 5 setting, margin-stv finds reasonable lower bounds in most instances. The exact MOV is found in 2 instances (11 and 15). In instance 13, we find a margin that lies within the bounds found on the MOV by the exact algorithm. Across instances 1 to 3, 6, 8 to 10, 12, and 16, the MOV
Table 5: Application of relaxed margin-stv (bilinear terms replaced with McCormick [1976] inequalities, without and with the lower bounding rule of Sec. 7.1) to the STV elections of Table 4, reporting: the number of candidates ($|C|$); the best known bounds on MOV from Table 4; lower bound on the MOV computed by margin-stv; number of MIPs solved; and algorithm runtime (in seconds). Best lower bounds are bolded.

| # | $|C|$ | Exact MOV Bounds | margin-stv MOV (LB) | MIPs solved | Time (s) | margin-stv with LB of Sec 7.1 MOV (LB) | MIPs solved | Time (s) |
|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 30,654 – 37,908 | 17,662 | 28 | 0.8 | 23,079 | 11 | 0.2 |
| 2 | 4 | 2,725 – 2,726 | 399 | 15 | 0.2 | 789 | 5 | 0.1 |
| 3 | 4 | 6,359 – 6,360 | 3245 | 28 | 0.4 | 6356 | 12 | 0.1 |
| 4 | 4 | 63 | 0 | 12 | 0.1 | 0 | 8 | 0.1 |
| 5 | 4 | 59 | 0 | 13 | 0.1 | 0 | 7 | 0.1 |
| 6 | 4 | 375 | 43 | 19 | 0.1 | 46 | 10 | 0.1 |
| 7 | 5 | 7,499 – 7,500 | 0 | 23 | 0.3 | 0 | 13 | 0.2 |
| 8 | 5 | 628 – 629 | 218 | 33 | 0.3 | 339 | 20 | 0.3 |
| 9 | 5 | 659 – 660 | 64 | 26 | 0.3 | 64 | 17 | 0.3 |
| 10 | 5 | 164 – 165 | 34 | 25 | 0.2 | 91 | 19 | 0.4 |
| 11 | 7 | 441 | 441 | 57 | 0.6 | 441 | 38 | 0.6 |
| 12 | 7 | 1,326 – 1,327 | 413 | 48 | 0.9 | 859 | 166 | 4 |
| 13 | 8 | 1,309 – 1,576 | 289 | 215 | 8 | 289 | 117 | 4 |
| 14 | 11 | 351 – 1,178 | 0 | 723 | 217 | 0 | 347 | 36 |
| 15 | 11 | 4 | 4 | 243 | 8 | 4 | 189 | 11 |
| 16 | 13 | 823 – 2,188 | 0 | — | ∞ | 0 | 21,520 | 23,243 |

lower bounds discovered are, on average, 14.4% below the exact MOV (or lower bound on the MOV found by the exact algorithm). In instances 4 and 5, piecewise-relaxed margin-stv finds trivial lower bounds of 0, however the exact MOV in these instances is small (63 and 59 votes, respectively).

9.1 Real-World STV Elections

In the preceding section we have evaluated margin-stv on a series of IRV elections (re-imagined as STV elections with 2 seats) held in the US between 2007 and 2014. The number of candidates in these elections range from 4 to 13. STV elections often involve a large number of candidates. In the 2012 election of senators to the ACT senate in Australia, 3 senators were elected in each of 3 districts. An STV election was held for each district, involving 20, 26, and 28 candidates. In the 2014 Victorian senate election in Australia, 5 senators were elected in each of 5 districts. The number of candidates in these elections ranged from 37 to 52. The use of margin-stv for elections of this size, without relaxation, is not feasible (due to the complexity of the DistanceToSTV MINLPs that must be solved). We demonstrate the value of our margin-stv algorithm for computing lower bounds on the margin in a number of real-world STV elections.

In addition to the use of MINLP relaxations McCormick [1976] inequalities and piecewise-linear relaxations), we parallelise the exploration of the frontier of partial candidate orders. In place of selecting one partial candidate order to expand (at Step 13 of Figure 3), we select the first $N_F$ (where $N_F$ is a parameter) orders in the frontier $F$. Each of these orders is expanded in parallel, and the frontier updated accordingly – the children of all expanded orders with DistanceToSTV evaluations that are less than the current upper bound are inserted into the frontier. This upper bound is revised, and orders pruned from the frontier, when a complete order is found with a DistanceToSTV evaluation that is lower than the current upper bound.

The number of variables and constraints in the DistanceToSTV models solved by margin-stv is determined by the number of candidates in the orders being evaluated. In most STV elections, the number of available seats is much less than the number of candidates. Consequently, these orders will contain lengthy sequences of candidate eliminations. To limit the memory and solve time requirements of the Dis-
Table 6: Application of piecewise-relaxed margin-stv (with the lower bounding rule of Sec. 7.1) to the STV elections of Table 4, reporting: the best known bounds MOV from Table 4; lower bound on the MOV computed by piecewise-relaxed margin-stv; number of MIPs solved; and algorithm runtime (in seconds).

| # | Exact MOV Bounds | $K = 5$ | $K = 10$ |
|---|------------------|---------|----------|
|   |                  | MOV     | MIPs     | Time (s) | MOV     | MIPs     | Time (s) |
|   |                  | LB solved |         |         | LB solved |         |         |
| 1 | 30,654 – 37,908  | 28,038  | 12       | 1        | 29,020  | 12       | 1        |
| 2 | 2,725 – 2,726    | 2,512   | 6        | 0.5      | 6,356   | 17       | 2        |
| 3 | 6,359 – 6,360    | 0       | 8        | 0.1      | 0       | 8        | 0.1      |
| 4 | 63               | 0       | 7        | 0.4      | 10      | 9        | 1        |
| 5 | 59               | 337     | 10       | 0.2      | 609     | 19       | 3        |
| 6 | 375              | 558     | 19       | 0.9      | 609     | 19       | 3        |
| 7 | 7,499 – 7,500    | 5,168   | 14       | 0.9      | 6,347   | 28       | 4        |
| 8 | 628 – 629        | 531     | 24       | 2        | 581     | 25       | 7        |
| 9 | 659 – 660        | 558     | 19       | 0.9      | 609     | 19       | 3        |
| 10| 164 – 165        | 133     | 19       | 4        | 141     | 19       | 3        |
| 11| 441              | 441     | 38       | 0.7      | 441     | 38       | 0.8      |
| 12| 1,326 – 1,327    | 1,049   | 486      | 37       | 1,211   | 698      | 83       |
| 13| 1,309 – 1,576    | 1,321   | 538      | 65       | 1,431   | 636      | 243      |
| 14| 351 – 1,178      | 53      | 84       | 22       | 602     | 3,008    | 2,900    |
| 15| 4                | 4       | 189      | 22       | 4       | 189      | 27       |
| 16| 823 – 2,188      | 760     | 7,025    | 35,583   | 95      | —        | ∞        |

| # | Exact MOV Bounds | $K = 15$ | $K = 20$ |
|---|------------------|----------|----------|
|   |                  | MOV      | MIPs     | Time (s) | MOV      | MIPs     | Time (s) |
|   |                  | LB solved |         |         | LB solved |         |         |
| 1 | 20,654 – 37,908  | 29,549   | 12       | 3        | 29,968   | 12       | 3        |
| 2 | 2,725 – 2,726    | 2,582    | 6        | 2        | 2,609    | 6        | 1        |
| 3 | 6,359 – 6,360    | 6,356    | 17       | 2        | 6,356    | 17       | 1        |
| 4 | 63               | 0        | 11       | 0.4      | 0        | 11       | 1        |
| 5 | 59               | 28       | 9        | 1        | 35       | 9        | 1        |
| 6 | 375              | 351      | 10       | 1        | 358      | 10       | 2        |
| 7 | 7,499 – 7,500    | 6,721    | 29       | 3        | 6,923    | 29       | 4        |
| 8 | 628 – 629        | 598      | 25       | 10       | 606      | 25       | 12       |
| 9 | 659 – 660        | 623      | 19       | 5        | 638      | 20       | 4        |
| 10| 164 – 165        | 147      | 19       | 6        | 155      | 19       | 6        |
| 11| 441              | 441      | 38       | 0.6      | 441      | 38       | 1        |
| 12| 1,326 – 1,327    | 1,221    | 700      | 105      | 1,257    | 758      | 185      |
| 13| 1,309 – 1,576    | 1,468    | 747      | 452      | 1,486    | 754      | 638      |
| 14| 351 – 1,178      | 785      | 10,511   | 15,732   | 878      | 13,786   | 28,909   |
| 15| 4                | 4        | 189      | 33       | 4        | 189      | 39       |
| 16| 823 – 2,188      | 17       | —        | ∞        | 0        | —        | ∞        |

TANCETO_{STV} models formed by margin-stv, we group these sequences of eliminated candidates together, ignoring constraints that enforce their relative elimination order (i.e., these candidates are effectively eliminated in a single round of counting). This allows us to take a 15 candidate order, for example, and reduce its complexity to that of a 5 candidate order (with 5 rounds of candidate election and elimination). Grouping eliminated candidates together not only reduces the number of constraints required, but also significantly reduces the number of vote equivalence classes that need to be considered, in each DISTANCETo_{STV} model. This grouping together of eliminated candidates adds an additional level of relaxation to the algorithm.

We consider, in this section, the performance of margin-stv on two sets of STV elections (the 2012 and 2008 ACT senate elections). In each experiment reported in this section, the margin-stv algorithm has been afforded a time limit of 3 days. If this time limit is reached, and margin-stv has not converged, we report the best lower bounds on the MOV computed by the algorithm within that time period.
Ginninderra, 28 candidates, 3 seats, MOV upper bound of 19 votes

| K | N_F | MOV LB | MIPs solved | Time (s) |
|---|-----|--------|-------------|----------|
| 1 | 1   | 0      | 462         | 472      |
| 1 | 10  | 0      | 684         | 485      |
| 1 | 20  | 0      | 684         | 483      |
| 5 | 1   | 19     | 2,838       | 3,050    |
| 5 | 10  | 19     | 685         | 481      |
| 5 | 20  | 19     | 685         | 482      |

Brindabella, 20 candidates, 3 seats, MOV upper bound of 729 votes

| K | N_F | MOV LB | MIPs solved | Time (s) |
|---|-----|--------|-------------|----------|
| 1 | 1   | 0      | 2,503       | 1,638    |
| 1 | 10  | 0      | 4,143       | 345      |
| 1 | 20  | 0      | 6,177       | 371      |
| 5 | 1   | 0      | 588         | 396      |
| 5 | 10  | 0      | 3,179       | 298      |
| 5 | 20  | 0      | 6,137       | 412      |
| 10 | 1  | 194    | 28,704      | 26,534   |
| 10 | 10 | 194    | 18,146      | 2,639    |
| 10 | 20 | 194    | 12,092      | 1,226    |
| 20 | 1  | 219    | 39,217      | 79,466   |
| 20 | 10 | 220    | 24,670      | 10,170   |
| 20 | 20 | 219    | 15,146      | 3,670    |
| 40 | 1  | 294    | 133,058     | ∞        |
| 40 | 10 | 412    | 347,961     | ∞        |
| 40 | 20 | 297    | 82,529      | ∞        |

Table 7: Lower bounds on the MOV of the 2012 ACT senate elections, found by margin-stv with varying degrees of piecewise-relaxation (K). The first N_F = 1, 10, and 20, partial orders on the frontier are expanded in parallel. The number of MIPs solved by, and the runtime of margin-stv is recorded alongside the degree of piecewise-relaxation K (K = 1 reduces to the use of McCormick [1976] inequalities).

9.1.1 2012 ACT Senate

We first consider the 2012 election of senators to the ACT senate in Australia. In the Brindabella, Ginninderra, and Molonglo districts, 3 senators were elected from a set of 20, 28, and 26 candidates, respectively. The winner elimination upper bound procedure of Figure 2 computes an upper bound on the margin in each of these elections of 729, 19, and 250 votes. Note that the method of distributing surplus votes in the counting of STV elections varies across jurisdictions. We have modelled our algorithm on what is (arguably) the most straightforward variant – the Inclusive Gregorian method. We simulate the counting of each election considered in this section according to the algorithm of Figure 1, and compute margins (and bounds on the margins) for these elections under the assumption that this counting method is applied.

Tables 7 and 8 record the lower bounds on the MOV found by margin-stv in the 2012 Brindabella, Ginninderra, and Molonglo districts elections, alongside the number of MIPs solved, and time required, by the algorithm. Recall that each MIP formed by margin-stv is a piecewise-linear relaxation of the MINLP of Section 7. The parameter K controls the degree of relaxation applied, with K = 1 equivalent to the use of McCormick [1976] inequalities, and larger values of K producing tighter relaxations (and MIPs of greater complexity). Instances in which margin-stv did not converge (terminate) after a 3 day period were aborted, and the best found lower bound on the MOV (together with the number of MIPs solved by margin-stv up to the point at which it was aborted) is recorded. Tables 7 and 8 examine the performance of margin-stv given varying degrees of parallelisation of the algorithm (N_F). It is clear that if the MIPs formed by the algorithm
are not over-relaxed (resulting in trivial lower bounds on the MOV of 0 being computed), parallelisation of the algorithm leads to significant improvements in runtime.

We have computed lower and upper bounds on the MOV for the 2012 Brindabella, Ginninderra, and Molonglo district senate elections, of 412–729 votes, 19–19 votes, and 178–250 votes, respectively. For the Ginninderra district election, we have computed the exact MOV of 19 votes.

9.1.2 2008 ACT Senate

We next consider the 2008 election of senators to the ACT senate in Australia. In the Brindabella, Ginninderra, and Molonglo districts, 3 senators were elected from a set of 19, 27, and 40 candidates, respectively. The winner elimination upper bound procedure of Figure 2 computes an upper bound on the margin in each of these elections of 655, 1,344, and 4,227 votes. Table 9 records the lower bounds on the MOV found by margin-stv for these district elections, alongside the number of MIPs solved, time required by the algorithm, and degree of relaxation (K) applied. We have computed lower and upper bounds on the MOV for the 2008 Brindabella, Ginninderra, and Molonglo district senate elections, of 591–655 votes, 228–1,344 votes, and 25–4,227 votes, respectively. It is clear that margin-stv has difficulty in computing lower bounds on the MOV of STV elections with a very large number of candidates (greater than 30). The Molonglo district election of 2008 involves 40 candidates, with margin-stv unable to converge after 3 days of computation, producing what is likely to be a very weak MOV lower bound of 25 votes. Our algorithm has more success on the Brindabella election, determining that the MOV lies within a narrow range of votes (591 to 655).
Table 9: Lower bounds on the MOV of the 2008 ACT senate elections, found by margin-stv with varying degrees of piecewise-relaxation (K). The first $N_F = 20$ partial orders on the frontier are expanded in parallel. The number of MIPs solved by, and the runtime of, margin-stv is recorded alongside the degree of piecewise-relaxation $K$ ($K = 1$ reduces to the use of McCormick [1976] inequalities).

| K | MOV LB | MIPs solved | Time (s) |
|---|--------|-------------|----------|
| 1 | 0      | 12,308      | 723      |
| 5 | 119    | 111,188     | 47,815   |
| 10| 228    | 3,134,488   | $\infty$|

Ginninderra, 27 candidates, 3 seats, MOV upper bound of 1,344 votes

Brindabella, 19 candidates, 3 seats, MOV upper bound of 655 votes

Molonglo, 40 candidates, 3 seats, MOV upper bound of 4,227 votes

10 Concluding Remarks

In this paper we develop the first ever algorithm, margin-stv, for computing exact margins of victory in STV elections, assuming the use of the Inclusive Gregory method of surplus distribution. The algorithm is based on solving a mixed integer nonlinear formulation for the problem of computing a minimal manipulation to achieve a desired election order. The algorithm is able to compute exact margins of victory for small elections, e.g. less than 12 candidates and 2 seats. For larger real elections margin-stv is able to compute reasonable lower bounds within a few hours. The algorithm struggles on elections with large numbers of candidates, essentially since the search spaces grows as the factorial of the number of candidates, but provides an important first step in tackling this important and challenging problem.

As future work we plan to adapt the margin-stv algorithm to answer specific questions such as the influence of instances of multiple-voting on election outcomes, and the impact of losing votes, similar to our prior work Blom et al., 2016 on Instant Runoff Voting (IRV) elections.

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