Abstract

The main object of this article is to present an extension of the zero-inflated Poisson-Lindley distribution, called of zero-modified Poisson-Lindley. The additional parameter $\pi$ of the zero-modified Poisson-Lindley has a natural interpretation in terms of either zero-deflated/inflated proportion. Inference is dealt with by using the likelihood approach. In particular the maximum likelihood estimators of the distribution’s parameter are compared in small and large samples. We also consider an alternative bias-correction mechanism based on Efron’s bootstrap resampling. The model is applied to real data sets and found to perform better than other competing models.

Keywords: Poisson-Lindley distribution; Estimation; Bootstrap; Monte Carlo simulation; Bias correction.

1. Introduction

In many areas of statistical applications like insurance, medicine, ecology and biology, for example, we come across situations where the zeros show up in the count data with a greater or a lesser tendency. Correspondingly, one needs to adapt a count data model by inflating, deflating or truncating the probability associated with a zero count. In this sense many generalizations or modifications have been considered in the literature, see, for example, Zuur et al. (2009). Also, in Plackett (1953) the zero-truncated Poisson distribution is presented. A class of zero-modified Poisson models is discussed in Dietz & Bohning (2000). Already, in Borah & Deka Nath (2001) the Poisson-Lindley (PL) distribution has been further studied with only inflation of probability at zero.

The Poisson-Lindley (PL) distribution (Sankaran, 1970) is a generalized Poisson distribution (Lindley, 1958; Consul, 1989) with probability mass function (pmf) given by

$$\Pr(Y = k) = \frac{\theta^2 (k + \theta + 2)}{(\theta + 1)^{k+3}}, \quad \theta > 0, \quad k \in \mathbb{N}.$$
The mean and variance of the PL distribution are given, respectively, by
\[ E(Y) = \frac{\theta + 2}{\theta(\theta + 1)} \quad \text{and} \quad \text{Var}(Y) = \frac{\theta^3 + 4\theta^2 + 6\theta + 2}{\theta^2(\theta + 1)^2}. \]

Also is possible shown that \( \text{Var}(Y)/E(Y) > 1 \), i.e, the PL distribution is over-dispersed, for more details see, Ghitany & Al-Mutairi (2009).

Besides, some properties of this Inflated Poisson Lindley (IPL) distribution are discussed and Oluyede & Pararai (2007) examine and study relations in zero-adjusted models. Besides, relations for reliability measures in the adjusted and unadjusted models are established and appropriate comparisons including the relative error are presented. Conceicão et al. (2017) present a new family of distributions for count data, the so called zero-modified power series (ZMPS). Also, the Hurdle distribution version of the ZMPS distribution is presented.

In practice, count data are often overdispersed so that alternative distributions such as the Poisson-Lindley may be more appropriate than the Poisson distribution. However, if we are studying count data with excess zeros the PL distribution isn’t more adequate. This paper proposes an extension of the PL distribution called of zero-modified Poisson-Lindley (ZMPL). This extension takes into account the inclusion of an additional parameter \( \pi \) which has a natural interpretation in terms of either zero-inflated or zero-deflated proportion.

It is well-known that the MLEs are widely used to estimate the unknown parameters of the probability distributions due to their various desirable properties; for example, the MLEs are asymptotically unbiased, consistent, and asymptotically normal. However, many of these properties depend on an extremely large sample sizes. Those properties, such as unbiasedness, may not be valid for small or even moderate sample sizes, which are more practical in real data applications. Therefore, some bias-corrected techniques for the MLEs are desired in practice, especially when the sample size is small. In this paper, we consider an alternative bias-corrected maximum likelihood (ML) estimators based on Efron’s bootstrap resampling. Indeed, such a bias-correction technique has been applied successfully for parameter estimation in other distributions and models; see, for example, Efron & Tibshirani (1986).

The rest of this article is organized as follows. In Section 2, we present the ZMPL distribution and some of its properties. Section 3 we briefly discuss point and interval estimation by the ML method and of their corrected version for the ZMPL distribution. Numerical results from Monte Carlo simulation experiments are presented and discussed in Section 4. Section 5 two well-known data sets are considered for an empirical comparison. The paper ends with a discussion of extensions and other contexts with the ideas could be implemented in Section 6.

2. Zero-Modified Poisson-Lindley (ZMPL) distribution

A random variable \( X \) is said to have the ZMPL distribution if its probability mass function is given by
\[
\Pr(X = k) = \begin{cases} 
\pi + (1 - \pi) \frac{\theta^2 (\theta + 2)}{(\theta + 1)^2}, & k = 0, \\
(1 - \pi) \frac{\theta^2 (k + \theta + 2)}{(\theta + 1)^2}, & k \in \mathbb{N}^+.
\end{cases}
\]  

We will denote this distribution as \( \text{ZMPL}(\theta, \pi) \). For the parameter \( \pi \) it is presupposed that \( \frac{\theta^2 (\theta + 2)}{\theta^2 + \theta + 1} \leq \pi \leq 1 \). The parameter \( \pi \) is called zero-modification parameter and different values lead to different modifications of the ZMPL distribution:
If \( \pi = -\frac{\theta^2 + \theta}{\theta + 3 + \theta^2} \), then the distribution (1) becomes the zero-truncated Poisson-Lindley distribution Ghitany et al. (2008), where the parameter \( \pi \) cancels out and no longer appears as a model parameter, i.e., there is no chance at all of getting a zero observation into the sample;

For \( \pi \in \left( -\frac{\theta^2 + \theta}{\theta + 3 + \theta^2}, 0 \right) \), this yields a zero-deflated Poisson-Lindley distribution. That is, less zeros occur, than expected under the Poisson-Lindley distribution. Such models are denoted as zero-deflated Poisson-Lindley distribution. Zero-deflated rarely arise in practice;

If \( \pi = 0 \), than the corresponding ZMPL distribution is the usual Poisson-Lindley distribution (Sankaran, 1970);

For \( \pi \in (0, 1) \), this yields a zero-inflated Poisson-Lindley distribution, which is a Poisson-Lindley distribution with a proportion of additional zeros (Borah & Deka Nath, 2001);

If \( \pi = 1 \), than the corresponding zero-modified distribution is the degenerated at zero one.

The corresponding cumulative distribution function (c.d.f.) is given by

\[
\Pr(X \leq k) = \begin{cases} 
0, & \text{if } k < 0; \\
\pi + (1 - \pi) \left\{ 1 - \frac{[k + 1]!}{\theta^{[k + 1]}} \right\}, & \text{if } k \geq 0,
\end{cases}
\]

where \([k]\) is the integer part of \( k \). The survival function of \( X \), when \( \frac{1}{\theta + 1} < 1 \), is

\[
\Pr(X \geq k) = \sum_{m=\lfloor k \rfloor}^{\infty} \Pr(m; \theta, \pi) = \begin{cases} 
1, & \text{if } k \leq 0; \\
(1 - \pi) \left\{ \frac{(\theta + 1)^{[k + 1]}}{(\theta + 1)^{[k + 1]}} \right\}, & \text{if } k > 0.
\end{cases}
\]

Let \( X \) denotes a random variable with probability mass function given in (1). The quantile function, say \( Q_X(p) \), defined by \( F_X^{-1}(p) \) is given by

\[
Q_X(p; \theta, \pi) = \begin{cases} 
0, & \text{if } p_\pi < 0, \\
Q_Y(p_\pi; \theta), & \text{if } p_\pi \geq 0,
\end{cases}
\]

where \( p_\pi = (p - \pi)/(1 - \pi) \) and \( Q_Y(p_\pi; \theta) \) is the quantile function of a Poisson-Lindley distribution. Borah & Deka Nath (2001) present some properties of the ZMPL distribution.

Fisher index of dispersion (Johnson et al., 2005, p. 163) of the random variable \( X \), defined by \( \text{FI}(X) = \text{Var}(X)/\text{E}(X) \) is given by

\[
\text{FI}(X) = \left( \frac{\text{E}(X^2)}{\text{E}(X)} - \text{E}(X) \right) = \pi \mu_{PL} + \frac{(\theta^3 + 4 \theta^2 + 6 \theta + 2)}{\theta(\theta + 1)(\theta + 2)} = \pi \mu_{PL} + \text{FI}(Y),
\]

where \( \mu_{PL} \) and \( \text{FI}(Y) \) are, respectively, the mean and the Fisher index of dispersion of a Poisson-Lindley distribution. Thus, the ZMPL distribution presents underdispersion when \( \theta > \sqrt{2} \) and \( \pi \in \left[ -\frac{\theta^2 + 2}{\theta + 3 + \theta^2}, 0 \right) \); and overdispersion when \( \pi \in [0, 1) \).
2.1. R package

The theoretical results has been implemented into a piece of statistical software: the zmpl package for R (R Core Team, 2017). To install this package, the R code below must be used.

```r
devtools::install_github("zmpldistribution/zmpl")
```

This package contain a collection of utilities for analyzing data from ZMPL distributions. Some of the functions are: dzmpl(), pzmpl(), qzmpl(), rzmpl(), mle(), fi.zmpl() and grad.test().

3. Inference

This section is concerned with the estimation of the two parameters of interest. We consider two estimation methods, namely, moments and maximum likelihood.

3.1. Method of moments estimator

Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) from the ZMPL distribution with probability mass as given in (1). The sample arithmetic and quadratic means are defined by

\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{and} \quad s^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2,
\]

respectively.

We have that the first two moments of \( X \) are given by \( E(X) = (1 - \pi)(\theta + 2)/[\theta(\theta + 1)] \) and \( E(X^2) = (1 - \pi)[(\theta + 2)^2 + 2]/[\theta^2(\theta + 1)] \). We can use the arithmetic and quadratic means such that

\[
E(X) = \bar{x} \quad \text{and} \quad E(X^2) = s^2,
\]

and then solve them for \( \theta \) and \( \pi \), the moment estimators of \( \theta \) and \( \pi \) are defined as

\[
\tilde{\theta} = \frac{(2\bar{x} - s^2) + \sqrt{(s^2 - 2\bar{x} + 2\bar{x}s^2)}}{(s^2 - \bar{x})} \quad \text{and} \quad \tilde{\pi} = 1 - \frac{\tilde{\theta}(\tilde{\theta} + 1)\bar{x}}{\tilde{\theta} + 2},
\]

where \( \bar{x} = s^2 \), if and only if \( x_i = 0 \) or \( 1 \) for all \( i = 1, 2, \ldots n \). A data set where all observations are zeros and ones is not worth analyzing for this distribution. This situation, of course, will not lead to any estimate of \( \theta \). However, such situation may arise in a simulation experiment when \( n \) and/or \( \theta \) are very small or \( \pi \) is great. For this reason, we will assume throughout this paper that \( \bar{x} \neq s^2 \).

Note that, \( s^2 = \bar{x}^2 - \bar{x}^2 \) and thus we can write the moment estimator of \( \theta \) as follows

\[
\bar{\theta} = \frac{[2\bar{x} - (s^2 - \bar{x}^2)] + \sqrt{(s^2 - \bar{x}^2)(1 + 2\bar{x} - 2\bar{x})}}{(s^2 - \bar{x}^2 - \bar{x})},
\]

where \( s^2 = (1/n) \sum_{i=1}^{n} (x_i - \bar{x})^2 \). An iterative method for finding the maximum likelihood (ML) estimators has to be employed. To start the iterative procedure, we could use the initial value obtained by the method of moments.
3.2. Maximum likelihood estimators

Let $x_1, x_2, \ldots, x_n$ be a random sample from ZMPL distribution. Then, the corresponding likelihood function for $A = (\theta, \pi)^T$ is given by

$$L(A; x) = \prod_{i=1}^{n} \left( \frac{1 - (1 - \pi)^{\theta}(\theta + 2)}{\theta(\theta + 1)^3} \right)^{I_0(x_i)} \left( \frac{1 - \pi}{\theta + 1} \right)^{1-I_0(x_i)} \left( \frac{\theta^2(x_i + \theta + 2)}{(\theta + 1)^{x_i+3}} \right)^{1-I_0(x_i)}$$

$$= \left\{ \frac{\pi + (1 - \pi)^{\theta}(\theta + 2)}{\theta(\theta + 1)^3} \right\}^{\sum_{i=1}^{n} I_0(x_i)} \prod_{i=1}^{n} \left( \frac{1 - \pi}{\theta + 1} \right)^{1-I_0(x_i)} \left( \frac{\theta^2(x_i + \theta + 2)}{(\theta + 1)^{x_i+3}} \right)^{1-I_0(x_i)}$$

$$= \left\{ \frac{\pi + (1 - \pi)^{\theta}(\theta + 2)}{\theta(\theta + 1)^3} \right\}^{n_0} \left\{ \frac{(1 - \pi)^{\theta^2}}{(\theta + 1)^3} \right\}^{n - n_0} \prod_{i=1}^{n} \left( \frac{(x_i + \theta + 2)}{(\theta + 1)^{x_i+3}} \right)^{1-I_0(x_i)}$$

where $n_0 = \sum_{i=1}^{n} I_0(x_i)$ is the number of zeros in sample and $I_0(\cdot)$ is once again the indicator function of the set A. Hence, the respective log-likelihood obtained from (2) can be expressed as

$$\ell(A; x) = n_0 \log \left( \frac{\pi + (1 - \pi)^{\theta}(\theta + 2)}{\theta(\theta + 1)^3} \right) + (n - n_0) \log \left( \frac{(1 - \pi)^{\theta^2}}{(\theta + 1)^3} \right) + \sum_{i=1}^{n} [1 - I_0(x_i)] \log(x_i + \theta + 2) - \log(\theta + 1) \sum_{i=1}^{n} [1 - I_0(x_i)] x_i.$$ 

(3)

We obtain the score by taking derivatives of the corresponding log-likelihood function with respect to the unknown parameters as

$$\ell'_\pi = \frac{n_0 \left( 1 - \frac{(\theta + 2)^2}{(\theta + 1)^3} \right)}{\pi + (1 - \pi)^{\theta(\theta + 2)}} - \frac{n - n_0}{(1 - \pi)} \right),$$

$$\ell'_\theta = \frac{n_0 (1 - \pi) \left( \frac{(\theta + 2)^2}{(\theta + 1)^3} \right)}{\pi + (1 - \pi)^{\theta(\theta + 2)}} - \frac{(n - n_0) (\theta - 2)}{\theta(\theta + 1)}$$

$$+ \sum_{i=1}^{n} \frac{[1 - I_0(x_i)]}{(x_i + \theta + 2)} - \frac{1}{(\theta + 1)} \sum_{i=1}^{n} [1 - I_0(x_i)] x_i.$$ 

(4)

From (4), the maximum likelihood of $\pi$ is $\hat{\pi} = \left[ 1 - \left( 1 - \frac{n_0}{n} \right) \frac{(\theta + 1)^2}{(\theta + 1)^3} \right]$. Because the solution of the equation $\ell'_\theta = 0$ obtained in (3) doesn’t have a closed-form, we maximize the log-likelihood function given in (3) on $\theta$ by using a non-linear optimization algorithm for determining the maximum likelihood estimates of $\theta$.

Now, we obtain the respective expected Fisher information matrix by taking derivatives of the elements of the score vector given in (4) with respect to the unknown parameters as

$$i(A) = \begin{bmatrix} i_{\theta\theta} & i_{\theta\pi} \\ i_{\pi\theta} & i_{\pi\pi} \end{bmatrix}.$$ 

(5)
with

\[
i_{\pi\pi} = n \left\{ \frac{1 - \theta^2/(\theta+2)^2}{\pi + (1-\theta^2/(\theta+2)^2)^{1/2}} \right\}^2 + \left\{ 1 - \left( \frac{1 - \theta^2/(\theta+2)^2}{\pi + (1-\theta^2/(\theta+2)^2)^{1/2}} \right) \right\}^2 \left( 1 - \pi \right)^2 \right\} ,
\]

\[
i_{i\pi} = i_{\pi\theta} = n \left\{ \frac{3\theta^2 - 4\theta}{(\theta + 1)^2} - \frac{3\theta^2}{(\theta + 1)^4} \right\}^2 + n \left\{ 1 - \left( \frac{1 - \theta^2/(\theta+2)^2}{\pi + (1-\theta^2/(\theta+2)^2)^{1/2}} \right) \right\} \left( 1 - \pi \right)^2 \right\} ,
\]

\[
i_{\theta\theta} = \frac{n(1-\pi)^2}{\pi + (1-\theta^2/(\theta+2)^2)^{1/2}} \left\{ \frac{3\theta^3 - 7\theta^2 + 4\theta + 2}{\theta^2(\theta+1)^2} \right\}^2 - n(1-\pi) \left\{ \frac{\theta^2}{\theta + 1} \Phi \left( \frac{1}{\theta + 1}; 1; \theta \right) \right\} ,
\]

where \( \Phi \left( \frac{1}{\theta + 1}; 1; \theta \right) \) is the Lerch zeta-function. An integral representation is given by \( \Phi \left( \frac{1}{\theta + 1}; 1; \theta \right) = \int_0^1 \frac{\theta \ln(1-\alpha)}{\theta + 1-\alpha} \mathrm{d}u \).

### 3.3. Bias-corrected estimators

A methodology for bias-correcting estimators is by bootstrap resampling (Efron, 1979). Let be a random sample \( x = (x_1, x_2, \ldots, x_n)^T \) from the random variable \( X \) with common distribution \( \mathcal{D} \). Define \( \theta = g(\mathcal{D}) \) a function of \( \mathcal{D} \) known as parameter and consider \( \tilde{\theta} = s(x) \) an estimator of \( \theta \). We consider that \( \mathcal{D} \) belongs to a parametric family which is known and has finite dimension, \( \mathcal{D}_\nu \). Using a consistent estimator for \( \nu(\mathcal{D}_\nu) \) we can obtain a parametric estimate for \( \mathcal{D} \). Therefore, we can write the bias of the estimator \( \tilde{\theta} = s(x) \) as \( B_{\mathcal{D}_\nu}(\tilde{\theta}, \theta) = E_{\mathcal{D}_\nu}(s(x)) - g(\mathcal{D}) \). Then the bootstrap bias estimate can be expressed as

\[
B_{\mathcal{D}_\nu}(\tilde{\theta}, \theta) = E_{\mathcal{D}_\nu}(s(x)) - g(\mathcal{D}).
\]

Finally, the bootstrap bias estimate, calculated from the \( B \) replicates of \( \tilde{\theta} \), is \( B_{\mathcal{D}_\nu}(\tilde{\theta}, \theta) = \tilde{\theta}^* - s(x) \). Then a second order bias-corrected estimator is given by

\[
\hat{\theta}_{bc} = s(x) - B_{\mathcal{D}_\nu}(\tilde{\theta}, \theta) = 2\tilde{\theta} - \tilde{\theta}^*.
\]

where \( \tilde{\theta}^* = \frac{1}{B} \sum_{b=1}^B \tilde{\theta}_b^* \) is a approximate of approximate the expected value \( E_{\mathcal{D}_\nu}(s(x)) \).

### 3.4. Confidence Interval (CI)

#### 3.4.1. Asymptotic Confidence Interval (aCI)

Under some regularity conditions, \( \hat{\lambda} \) is a consistent estimator of \( \lambda \) and it has a distribution that is asymptotically normal. The, \( \sqrt{n}(\hat{\lambda} - \lambda) \to \mathcal{N}_2(0, j(\lambda)^{-1}) \), as \( n \to \infty \), where \( j(\lambda) = \lim_{n \to \infty} \frac{1}{n} \hat{i}(\lambda) \), with \( i(\lambda) \) being the expected Fisher information matrix in (5) and \( \to \) denotes convergence in distribution to. Note that \( i(\lambda)^{-1} \) is a consistent estimator of the asymptotic variance-covariance matrix of \( \lambda \). In practice, one may approximate the expected Fisher information matrix by its observed version, whereas the elements of the diagonal of the inverse of this matrix can be used to approximate the corresponding standard errors (see Efron & Hinkley, 1978, for details about the use of observed versus expected Fisher information matrices).

As a result above, the asymptotic 100(1 - \( \alpha \))% confidence intervals (aCI) for \( \theta \) and \( \pi \) are given by, respectively...
where \( \hat{\theta} \) and \( \hat{\pi} \) are the estimated asymptotic standard error (se) of the maximum likelihood estimator of \( \theta \) and \( \pi \), respectively, and \( z_{\alpha/2} \) is the \( (\alpha/2) \)th quantile of the standard normal distribution.

### 3.4.2. Percentile Confidence Intervals (pCI)

With the empirical distribution \( \hat{D} \) of \( \hat{\theta} \) obtained by bootstrap, one can construct percentile confidence intervals (pCI), with approximate coverage \( 1 - \alpha \), \( 0 < \alpha < 1/2 \), by computing, by computing the percentiles \( \alpha/2 \) and \( 1 - \alpha/2 \) of \( \hat{D} \). The pCI is given by

\[
[\hat{\theta} - z_{\alpha/2} \text{se}(\hat{\theta}); \hat{\theta} + z_{\alpha/2} \text{se}(\hat{\theta})].
\]

After arranging in increasing order the \( B \) bootstrap replicates of \( \hat{\theta} \), \( \hat{\theta}_b^* \), we calculate the lower and upper limits of the percentiles interval as the integer parts of \( B \cdot (\alpha/2) \) and \( B \cdot (1 - \alpha/2) \), respectively.

### 3.5. Hypothesis test

Consider the hypotheses \( H_0 : \pi = 0 \) and \( H_1 : \pi \neq 0 \). The interest lies in testing the null hypothesis (PL distribution) against the alternative hypothesis (ZMPL distribution).

#### 3.5.1. Gradient test

Let \( x = (x_1, x_2, \ldots, x_n)^\top \) be a random sample of size \( n \) from the ZMPL distribution, each \( x_i, i = 1, 2, \ldots, n \), having p.m.f. (1). The gradient statistic, \( S_g \), is given by

\[
S_g = n \hat{\pi}^2 (\hat{\theta}^2 + 3\hat{\theta} + 1),
\]

where \( \hat{\lambda} \) is unrestricted maximum likelihood estimator of \( \lambda \). Asymptotically, \( S_g \) has a central chi-square distribution with one degree of freedom under \( H_0 \).

### 4. Simulation

In this section, we conduct a study based on Monte Carlo (MC) simulations to assess the performance of the ML estimators and of their corrected version. The simulations were conducted using R language (R Core Team, 2017). The ML estimators of the parameters \( \theta \) and \( \pi \) were obtained by maximizing the log-likelihood function using the BFGS method by package maxLik (Henningsen & Toomet, 2011). For each MC replication and for each ML estimate of the parameters of the model, we obtained interval estimates of the asymptotic type, of the percentile type (bootstrap). All intervals were obtained by estimating the two limits independently. The scenario of this simulation study considers 5 000 MC replications and 1 000 bootstrap replications in each case, sample sizes \( n = 35, 60, 90 \) and 120. The values of \( \theta \) and \( \pi \) were fixed at \( \theta \in \{1.5, 2.0\} \) and \( \pi \in \{-0.10, 0.00, 0.10\} \).

The evaluation of point estimation was conducted based on the following measures for each sample size: the mean of estimates (omitted), bias, variance (omitted) and mean squared error. In what concerns interval estimation, we display the means of the empirical coverage probabilities, obtained from the relative frequencies of which the
true parameter value belongs to the intervals. The observed frequencies at which the lower limit of the interval was larger (smaller) than the true parameter values are also indicated.

Table 1 shows the results of numerical evaluation of point estimators of the parameters of ZMPL distribution. Note that the usual ML estimators of the parameters \( \theta \) and \( \pi \) are considerably more biased than their corrected versions via bootstrap. For \( n = 60 \), \( \pi = -0.10 \) and \( \theta = 1.5 \), we noted bias for \( \hat{\lambda} \) and \( \hat{\lambda}_{bc} \) equal to \((0.090, -0.042)\) and \((0.005, 0.018)\), respectively. That is, the uncorrected estimators \( \hat{\theta} \) and \( \hat{\pi} \) are, respectively, about 18 and 2 times more biased than the proposed corrected estimators. Moreover, by the asymptotic properties of the ML estimators, the bias of all the estimators decrease as the sample size increases. When \( \theta \) increases the estimates of the bias and mean squared error also increases for both estimators. Additionally, the bias of the uncorrected estimators \( \hat{\pi} \) is negative while that the bias of the corrected estimator \( \hat{\pi}_{bc} \) is positive. About the mean squared error, we verify that it decreases as the sample size increases in all estimators, which is numerical indicative of the consistency of the estimators.

Table 2 presents the results of numerical evaluation of interval estimators of the parameters of ZMPL distribution (for brevity, we only present results for \( \theta = 1.5 \)). For the parameter \( \theta \), nominal coverages of 0.95 and 0.99, generally, the interval aCI had the best empirical coverages. Now for the parameter \( \pi \), nominal coverages of 0.90 and 0.99, generally, the interval pCI had the best empirical coverages.

5. Applications

In this section we apply the proposed methodology to two real demand data sets. Here, we perform an exploratory data analysis and, based on it, we show the good fitting of the ZMPL distribution to the analyzed data. As the samples are large, we estimate the unknown parameters of the fitted models by the ML method (as discussed in Section 3).

5.1. Example 1: Inflation of zeros

In this section we have tried to fit Poisson distribution, Poisson-Lindley distribution and Zero-Modified Poisson-Lindley distribution to a biological data using maximum likelihood estimates. Here we use a dataset related to mammalian cytogenetic dosimetry lesions in rabbit lymphoblast induced by streptonigrin (NSC-45383), exposure \(-60 \mu g/Kg\). For more details about the data see Shanker & Fesshaye (2015). Table 3 presents some descriptive measures to the data set studied in this application. The skewness here is 2.42. This value implies that the distribution of the data is positively skewed. For the kurtosis, we have 10.70 implying that the distribution of the data is leptokurtic. The dispersion index shows that the data must be modeled by a overdispersed model (FI = 1.56). The proportion of zeros in the data set is 69%. Then, we have evidence that there is inflation of zeros. Thus, the use of the ZMPL distribution for fitting this data set appears justified.

| Measures | Minimum | Maximum | Mean | Variance | Skewness | Kurtosis | FI |
|----------|---------|---------|------|----------|----------|----------|----|
| Values   | 0.00    | 6.00    | 0.47 | 0.74     | 2.42     | 10.70    | 1.56 |
Table 4 provides the estimates of the model parameters and chi-squared test. The gradient statistic for testing \( H_0 : \pi = 0 \) is \( S_g = 114.49 \) and \( p \)-value < 0.001, i.e, the parameter \( \pi \) is statistically different from zero.

Table 4: Distribution of mammalian cytogenetic dosimetry lesions in rabbit lymphoblast induced by streptonigrin (NSC-45383), Exposure - 60 \( \mu g/Kg \) with expected frequency obtained by fitting Poisson, Poisson-Lindley and Zero-modified Poisson-Lindley distributions.

| No of mammalian cytogenetic dosimetry lesions | Observed frequency | Expected frequency |
|-----------------------------------------------|--------------------|--------------------|
| Poisson                                      | Zero-modified      | Poisson-Lindley    | Zero-modified      |
|                                              |                    |                    |                    |
| 0                                            | 413                | 374.0              | 413.0              | 413.0              |
| 1                                            | 124                | 177.4              | 116.0              | 133.6              | 123.4              |
| 2                                            | 42                 | 42.1               | 52.1               | 42.6               | 42.9               |
| 3                                            | 15                 | 6.6                | 15.6               | 13.3               | 14.5               |
| 4                                            | 5                  | 0.8                | 3.5                | 4.1                | 4.8                |
| 5                                            | 0                  | 0.1                | 0.6                | 1.2                | 1.6                |
| 6                                            | 2                  | 0.0                | 0.2                | 0.5                | 0.5                |
| Total                                        | 601                | 601                | 601                | 601                |
| Estimate of parameter                        | \( \hat{\theta} = 0.47421 \) | \( \hat{\theta} = 0.8989 \) | \( \hat{\theta} = 2.6854 \) | \( \hat{\theta} = 2.4098 \) |
| Confidence Interval                          | aCI(\( \theta \), 0.95) = (0.4192; 0.5293) | aCI(\( \theta \), 0.95) = (0.7304; 1.0675) | aCI(\( \theta \), 0.95) = (2.3619; 3.0088) | aCI(\( \theta \), 0.95) = (1.8904; 2.9290) |
| \( \chi^2 \)                                | 726.2816           | 42.19              | 9.6880             | 5.9064             |
| d.f                                          | 6                  | 6                  | 6                  | 6                  |
| p-value                                      | < 0.0001           | < 0.0001           | 0.1384             | 0.4338             |

The standardized differences (SD) are calculated of the following form

\[
\Delta_{sd} = \frac{\delta_{sd}}{\max\{\delta_{1sd}, \ldots, \delta_{msd}\}}, \quad i = 1, \ldots, m; s \in S,
\]

where \( \delta_{sd} = (\text{observed}_{sd} - \text{expected}_{sd}) \), \( \delta_{1sd} = (\delta_{11sd}, \ldots, \delta_{1msd}) \), \( m \) is the number of studied models, \( S \) is the support (common) of the distributions and \( |A| \) is the cardinality of a set \( A \). The SD plots are formed by standardized versus common distributions of distributions. We do the this when we are interested in to know what model that best fits the data.

The SD-plot of the fitted ZMPL, PL, ZMP and Poisson models are shown in Figure 1. Based on the values of the Table 4, we conclude that the ZMPL distribution provides a better fit than the Poisson, ZMP and PL distributions.
5.2. Example 2: Deflation of zeros

This dataset gives the number of outbreaks of strikes in the UK coal mining industry in successive four-week periods, in the years 1948-1959 (Ridout & Besbeas, 2004). These data are only modestly underdispersed, with FI of 0.75 (see, Table 5). The proportion of zeros in the data set is 29%. Then, we have evidence that there is deflation of zeros. Thus, the use of the ZMPL distribution for fitting this data set appears justified.

Table 5: Descriptive Measures

| Measures | Minimum | Maximum | Mean  | Variance | Skewness | Kurtosis | FI |
|----------|---------|---------|-------|----------|----------|----------|----|
| Values   | 0.00    | 4.00    | 0.99  | 0.74     | 0.80     | 3.5      | 0.75|

Table 6 provides the estimates of the model parameters and chi-squared test. The gradient statistic for testing \( H_0 : \pi = 0 \) is \( S_g = 5275.1 \) and \( p \)-value < 0.001, i.e, the parameter \( \pi \) is statistically different from zero.

Figure 2 presents the SD-plot for the fitted models. Based on the values of the Table 6 and Figure 2, we observe that the ZMP and ZMPL models are competitive.
Table 6: Fitted frequencies, estimate of parameters and chi-square statistics from fitted four distributions to the strike outbreak data of Ridout & Besbeas (2004).

| No of outbreaks | Observed frequency | Poisson | Zero-modified Poisson | Poisson-Lindley | Zero-modified Poisson-Lindley |
|-----------------|--------------------|---------|-----------------------|-----------------|-------------------------------|
| 0               | 46                 | 57.76   | 46.00                 | 75.24           | 46.00                         |
| 1               | 76                 | 57.39   | 74.69                 | 40.55           | 77.79                         |
| 2               | 24                 | 28.51   | 27.27                 | 20.73           | 22.95                         |
| 3               | 9                  | 9.44    | 6.64                  | 10.23           | 6.63                          |
| ≥4              | 1                  | 2.35    | 1.21                  | 4.93            | 1.89                          |
| Total           | 156                | 156     | 156                   | 156             | 156                           |

Estimate of \( \hat{\theta} = 0.9936 \), \( \hat{\theta} = 0.7301 \), \( \hat{\theta} = 1.4010 \), \( \hat{\theta} = 2.9579 \).

Confidence Interval:
- \( \hat{\theta} = 0.9936 \): \( \text{aCI}(0.95) = (0.8372; 1.5800) \), \( \text{aCI}(0.95) = (0.5271; 0.9331) \)
- \( \hat{\theta} = 0.7301 \): \( \text{aCI}(0.95) = (-0.6526; -0.0691) \), \( \text{aCI}(0.95) = (1.1478; 1.6542) \)
- \( \hat{\theta} = 1.4010 \): \( \text{aCI}(0.95) = (1.0436; 3.8721) \), \( \text{aCI}(0.95) = (-1.9923; -0.7028) \)

| \( \chi^2 \) | d.f | p-value |
|--------------|-----|---------|
| 9.8986       | 4   | < 0.0001 |
| 1.2816       | 4   | 0.8628  |
| 44.748       | 4   | 0.04217 |
| 1.3404       | 4   | 0.8545  |

Figure 2: SD-plot for Ridout & Besbeas (2004) data.

6. Concluding remarks

The paper has introduced the zero-modified Poisson-Lindley distribution. The paper has derived maximum likelihood estimators and of their corrected version. Two applications of the ZMPL distribution illustrated the usefulness of the model. A main theme of the paper has been the comparison of the efficiency of the ML and of their corrected version. It is shown that the biases of the bias-corrected estimators have good finite-sample behavior, outperforming the ML estimator. Two different strategies for interval estimation were considered and numerically evaluated. Thus, zero-modified distributions are good candidates when excess of zeros appears. Zero-modified distributions should be the solution of this problem to avoid any over or under estimation of the measure of interest. In conclusion, it is believed that the ZMPL distribution and subsequent regression model may offer a very useful tool for analyzing data characterized with a large or small amount of zeros.
References

Borah, M., & Deka Nath, A. 2001. A study on the Inflated Poisson Lindley distribution. *Jour. Ind. Soc. Ag. Statistics*, 54(3), 317–323.

Conceição, K.S., Louzada, M.G., Andrade, M.G., & Helou, E.S. 2017. Zero-modified power series distribution and its Hurdle distribution version. *Journal of Statistical Computation and Simulation*, DOI: 10.1080/00949655.2017.1289529.

Consul, P.C. 1989. *Generalized Poisson Distribution Properties and Applications*. New York and Basel: Marcel Dekker, Inc.

Dietz, E., & Bohning, D. 2000. On estimation of the Poisson parameter in zero-modified Poisson models. *Computational Statistics & Data Analysis*, 34, 441–459.

Efron, B. 1979. *Bootstrap Methods: Another Look at the Jackknife*. Ann. Statist., 1–26.

Efron, B., & Hinkley, D.V. 1978. Assessing the Accuracy of the Maximum Likelihood Estimator: Observed Versus Expected Fisher Information. *Biometrika*, 65(3), 457–482.

Efron, B., & Tibshirani, R. 1986. Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy. *Statistical science*, 54–75.

Ghitany, M. E., Al-Mutairi, D. K., & Nadarajah, S. 2008. Zero-truncated Poisson-Lindley distribution and its application. *Mathematics and Computers in Simulation*, 79, 279–287.

Ghitany, M.E., & Al-Mutairi, D. K. 2009. Estimation Methods for the discrete Poisson-Lindley distribution. *Journal of Statistical Computation and Simulation*, 1–9.

Henningsen, Arne, & Toomet, Ott. 2011. *maxLik*: A package for maximum likelihood estimation in R. *Computational Statistics*, 26(3), 443–458.

Johnson, N.L., Kemp, A.W., & Kotz, S. 2005. *Univariate Discrete Distribution*. New York: Wiley.

Lindley, D.V. 1958. Fiducial distributions and Baye's theorem. *J. Roy. Statist. Soc. B*, 20, 102–107.

Oluyede, B.O., & Pararai, M. 2007. A note on relations for reliability measures in zero-adjusted models. *Applied Mathematical Sciences*, 1(56), 2789–2798.

Plackett, R.L. 1953. The Truncated Poisson distribution. *Biometrics*, 9(4), 485–488.

R Core Team. 2017. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Ridout, M.S., & Besbeas, P. 2004. An empirical model for underdispersed count data. *Statistical Modelling*, 4(1), 77–89.

Sankaran, M. 1970. The discrete Poisson-Lindley distribution. *Biometrics*, 26, 145–149.

Shanker, R., & Fesshaye, H. 2015. On Poisson-Lindley Distribution and Its Applications to Biological Sciences. *Biometrics & Biostatistics International Journal*, 2, 1–5.

Zuur, A.F., Ieno, E.N., Walker, N.J., Saveliev, A.A., & Smith, G.M. 2009. *Mixed Effects Models and Extensions in Ecology with R*. Statistics for Biology and Health. Springer New York. Chap. Zero-Truncated and Zero-Inflated Models for Count Data, pages 261–293.
Table 1: Empirical bias and mean squared errors (in parentheses).

| $n$  | $\pi$ | $\hat{\theta}$ | $\hat{\theta}_{bc}$ | $\hat{\pi}$ | $\hat{\pi}_{bc}$ | $\hat{\theta}$ | $\hat{\theta}_{bc}$ | $\hat{\pi}$ | $\hat{\pi}_{bc}$ |
|------|-------|-----------------|----------------------|--------------|------------------|----------------|----------------------|--------------|------------------|
|     |       |                 |                      |              |                  |                |                      |              |                  |
| -0.10 | 0.163 | -0.029          | -0.080               | 0.075        | 0.322            | -0.103         | -0.139               | 0.073        |                  |
|      | (0.426) | (0.216) | (0.119) | (0.113) | (1.388) | (1.008) | (0.292) | (0.205) |                  |
| 35   | 0.00   | 0.198           | -0.024               | -0.082       | 0.076            | 0.367          | -0.045               | -0.146       | 0.058            |
|      | (0.487) | (0.219) | (0.114) | (0.097) | (1.527) | (1.228) | (0.275) | (0.199) |                  |
| 0.10 | 0.230 | -0.013          | -0.092               | 0.072        | 0.371            | -0.001         | -0.131               | 0.056        |                  |
|      | (0.572) | (0.271) | (0.121) | (0.104) | (1.511) | (1.244) | (0.236) | (0.155) |                  |
|      |       |                 |                      |              |                  |                |                      |              |                  |
| -0.10 | 0.090 | 0.005           | -0.042               | 0.018        | 0.193            | -0.042         | -0.080               | 0.027        |                  |
|      | (0.162) | (0.100) | (0.052) | (0.045) | (0.690) | (0.553) | (0.144) | (0.120) |                  |
| 60   | 0.00   | 0.096           | -0.002               | -0.043       | 0.023            | 0.212          | -0.042               | -0.085       | 0.022            |
|      | (0.179) | (0.107) | (0.050) | (0.046) | (0.617) | (0.385) | (0.107) | (0.070) |                  |
| 0.10 | 0.116 | 0.001           | -0.046               | 0.028        | 0.217            | -0.037         | -0.069               | 0.032        |                  |
|      | (0.228) | (0.134) | (0.051) | (0.047) | (0.717) | (0.541) | (0.099) | (0.071) |                  |
|      |       |                 |                      |              |                  |                |                      |              |                  |
| -0.10 | 0.056 | 0.010           | -0.029               | 0.003        | 0.120            | -0.018         | -0.048               | 0.017        |                  |
|      | (0.091) | (0.066) | (0.031) | (0.028) | (0.289) | (0.201) | (0.064) | (0.051) |                  |
| 90   | 0.00   | 0.062           | 0.009                | -0.028       | 0.010            | 0.127          | -0.025               | -0.050       | 0.015            |
|      | (0.105) | (0.076) | (0.031) | (0.029) | (0.317) | (0.218) | (0.059) | (0.046) |                  |
| 0.10 | 0.077 | 0.014           | -0.032               | 0.011        | 0.164            | -0.005         | -0.057               | 0.009        |                  |
|      | (0.126) | (0.088) | (0.029) | (0.027) | (0.428) | (0.311) | (0.061) | (0.046) |                  |
|      |       |                 |                      |              |                  |                |                      |              |                  |
| -0.10 | 0.045 | 0.018           | -0.023               | 0.000        | 0.090            | -0.004         | -0.039               | 0.006        |                  |
|      | (0.066) | (0.050) | (0.023) | (0.022) | (0.195) | (0.152) | (0.044) | (0.037) |                  |
| 120  | 0.00   | 0.048           | 0.015                | -0.024       | 0.003            | 0.111          | 0.004                | -0.044       | 0.002            |
|      | (0.074) | (0.056) | (0.023) | (0.022) | (0.240) | (0.179) | (0.043) | (0.036) |                  |
| 0.10 | 0.057 | 0.019           | -0.021               | 0.013        | 0.106            | -0.013         | -0.038               | 0.008        |                  |
|      | (0.087) | (0.064) | (0.022) | (0.023) | (0.250) | (0.187) | (0.037) | (0.031) |                  |
Table 2: Empirical coverage probability and empirical tails coverage probability (Left ; Right) of the confidence intervals for \( \theta = 1.5 \).

| \( \theta \) | \( \pi \) |
|---|---|
| \( n \) | aCI(\(0.90\)) pCI(\(0.90\)) aCI(\(0.95\)) pCI(\(0.95\)) aCI(\(0.99\)) pCI(\(0.99\)) aCI(\(0.90\)) pCI(\(0.90\)) aCI(\(0.95\)) pCI(\(0.95\)) aCI(\(0.99\)) pCI(\(0.99\)) |
| 35 0.00 | 0.929 | 0.898 | 0.932 | 0.983 | 0.976 | 0.919 | 0.910 | 0.955 | 0.976 | 0.956 | 0.982 | 0.988 |
| (0.000;0.071) | (0.012;0.016) | (0.000;0.045) | (0.064;0.004) | (0.000;0.017) | (0.024;0.000) | (0.072;0.010) | (0.007;0.082) | (0.044;0.002) | (0.000;0.044) | (0.018;0.000) | (0.000;0.012) |
| 0.10 0.929 | 0.868 | 0.953 | 0.933 | 0.982 | 0.974 | 0.923 | 0.910 | 0.955 | 0.976 | 0.954 | 0.985 | 0.987 |
| (0.000;0.071) | (0.101;0.013) | (0.000;0.047) | (0.064;0.004) | (0.000;0.018) | (0.026;0.000) | (0.067;0.010) | (0.009;0.081) | (0.043;0.002) | (0.001;0.045) | (0.015;0.000) | (0.000;0.013) |

| 35 0.00 | 0.924 | 0.892 | 0.957 | 0.942 | 0.987 | 0.983 | 0.919 | 0.910 | 0.960 | 0.959 | 0.988 | 0.991 |
| (0.01;0.006) | (0.084;0.024) | (0.000;0.043) | (0.051;0.016) | (0.000;0.013) | (0.017;0.000) | (0.061;0.019) | (0.023;0.067) | (0.037;0.003) | (0.005;0.036) | (0.012;0.000) | (0.000;0.009) |
| 0.10 0.922 | 0.896 | 0.954 | 0.932 | 0.984 | 0.982 | 0.912 | 0.906 | 0.955 | 0.960 | 0.983 | 0.989 |
| (0.005;0.073) | (0.081;0.023) | (0.000;0.047) | (0.051;0.016) | (0.000;0.016) | (0.018;0.000) | (0.069;0.019) | (0.022;0.072) | (0.041;0.004) | (0.002;0.038) | (0.016;0.000) | (0.000;0.011) |

| 90 0.00 | 0.913 | 0.893 | 0.958 | 0.943 | 0.986 | 0.982 | 0.908 | 0.898 | 0.957 | 0.955 | 0.988 | 0.991 |
| (0.020;0.067) | (0.077;0.030) | (0.002;0.040) | (0.046;0.011) | (0.000;0.014) | (0.017;0.001) | (0.064;0.028) | (0.029;0.073) | (0.036;0.007) | (0.007;0.038) | (0.012;0.000) | (0.000;0.009) |
| 0.10 0.916 | 0.896 | 0.957 | 0.939 | 0.987 | 0.983 | 0.915 | 0.910 | 0.959 | 0.961 | 0.987 | 0.990 |
| (0.018;0.066) | (0.078;0.026) | (0.003;0.040) | (0.050;0.011) | (0.000;0.013) | (0.016;0.001) | (0.058;0.027) | (0.021;0.068) | (0.034;0.008) | (0.002;0.036) | (0.012;0.000) | (0.000;0.010) |

| 0.10 0.908 | 0.895 | 0.953 | 0.943 | 0.987 | 0.986 | 0.910 | 0.903 | 0.953 | 0.958 | 0.989 | 0.990 |
| (0.027;0.064) | (0.073;0.031) | (0.007;0.040) | (0.043;0.014) | (0.000;0.013) | (0.013;0.002) | (0.057;0.033) | (0.031;0.066) | (0.036;0.012) | (0.007;0.035) | (0.011;0.000) | (0.000;0.010) |

| 120 0.00 | 0.900 | 0.893 | 0.954 | 0.944 | 0.987 | 0.986 | 0.903 | 0.913 | 0.952 | 0.962 | 0.989 | 0.991 |
| (0.024;0.067) | (0.075;0.032) | (0.007;0.039) | (0.042;0.014) | (0.000;0.013) | (0.013;0.001) | (0.063;0.034) | (0.020;0.067) | (0.036;0.012) | (0.002;0.037) | (0.010;0.001) | (0.000;0.009) |
| 0.10 0.907 | 0.891 | 0.954 | 0.943 | 0.986 | 0.984 | 0.918 | 0.948 | 0.962 | 0.984 | 0.990 | 0.990 |
| (0.023;0.070) | (0.076;0.032) | (0.005;0.041) | (0.044;0.013) | (0.000;0.014) | (0.014;0.002) | (0.063;0.032) | (0.018;0.064) | (0.041;0.011) | (0.003;0.036) | (0.016;0.001) | (0.000;0.010) |