Epistemological obstacles in mathematical abstraction on abstract algebra

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Abstract. The general abstract algebra is a material that is still considered difficult by students. To know and understand the abstract algebra is needed mathematical abstraction. Mathematical abstraction ability is very prominent in abstract algebra lectures. In this research, the mathematical abstraction is seen based on RBC theory (Recognizing, Building-with, Constructing). This process of abstraction will be done in the process of epistemological obstacles. Epistemology obstacles are learning obstacles due to dissatisfaction, generalization processes are too broad, or languages apart. This research uses qualitative with case study approach. Participant in this study was a student who had studied abstract algebra. Additionally given in relation to abstract algebra, the student is interviewed in from the answer. The results of this study indicate had epistemology obstacles in every epistemic action in recognizing, building-with and construction. The learning obstacles that arises is the inappropriate use of the intuition and is too broad to generalize.

1. Introduction

The abstract algebra course is one of the most difficult subjects [1]. Group theory is a topic in abstract algebra that talks about a group [2]. Group theory is one of the materials that belong to the mathematical component in the formal aspect [3]. components of group theory are definitions, theorems, lemmas, and proofs. The group itself is a set with binary operations that must meet associative, have an identity, and have inverse [4,5]. Several studies on group theory reveal difficulties in learning group theory [6,8]. There are also several studies providing learning solutions such as with the method of mathematical card games [2], APOS [6,9,10], m-APOS [11,12], used of RBC theory [13,14].

In this study, the group theory to be seen is how students solve problems. The given problem is in the form of group homomorphism verification relating to the definition of homomorphism and group properties. With the giving of this problem is expected a process of thinking of student abstraction can be known on stage. In previous research, the misconception of some group concepts, such as group, is considered to be the same as a set, a subgroup is considered equal to a subset. The main cause is that students are still unfamiliar with the new concept that the group is a set that should stick with binary operations [15]. A new mathematical concept constructed from earlier mathematical concepts is a process of mathematical abstraction. Several studies have also found that the group theory course of thinking ability that stands out is the ability of abstraction [1,16].
Abstraction can be viewed as a natural process undertaken by the mind of a person who focuses on certain aspects and ignores certain things in decision making [17]. Piaget [18] introduces three kinds of abstractions: empirical abstractions, pseudo-empirical abstractions, and reflective abstractions. Empirical abstractions focus on objects and properties of objects such as color and weight, whereas false empirical abstractions focus more on actions on objects and action properties such as counting marbles, whereas in reflective abstractions focuses on mental actions on mental concepts such as commutative nature on the calculation of integers [17,19].

Abstraction has two meanings, first as a process of 'describing' a situation, and secondly a concept as a result of a process [17]. According to [20] abstraction occurs when from some object then in "disturb" characteristic or nature of the object that is considered not important, and finally only note or taken important properties shared. According to classical cognitive psychology [14], the main characteristic of abstraction is the sifting of the same or common nature of a set of real examples. In the classical approach, abstraction is considered the intrinsic nature of the new object. The ability of mathematical abstraction is the ability to reorganize previously crafted mathematical knowledge into a new mathematical structure. The term reorganization into a new structure implies that a mathematical relationship exists that includes (a) creating a new hypothesis (b) rediscovering a new mathematical generalization, proof or strategy to solve the problem. Such actions require high-level theoretical thinking. Nevertheless, empirical thinking can't be ignored. In Realistic Mathematics Education (RME) to build new mathematical knowledge based on previous mathematical knowledge is the same as vertical mathematical terminology. Vertical mathematization is an activity in which mathematical elements are collected, organized, organized, developed and so on into new elements, often in a more abstract or more formal form than the original.

The classical cognitive psychology approach is also performed by [21] with his idea of reflexive abstraction, which deals with the categorization of mental operations and abstraction of mental objects. The result of a reflexive abstraction is a scheme, building block of knowledge at each developmental level. Reflexive abstraction fuses the schema of the corresponding patterns of action. This process leads to a consistent and logical constructive theoretical model. Following Piaget, some mathematics educators offer a description of the mechanic's process in which students change their focus from concrete to abstract [22]. For most educators, abstractions proceed from a set of mathematical objects or processes and focus more on the special properties and relationships of objects, rather than to the object itself. The abstraction result consists of a class containing all objects that have special properties, which have a special relationship. The process of abstraction is a process of decontextualization because it does not pay attention to both the objects and some characteristics and relationships that are owned, even related to the realization or a particular representation. This process is linear, starting from objects to the class or structure, which will be called an object at a higher level. In the classical approach, the abstract is regarded as the intrinsic nature of the new object, nevertheless, this trait can't be obtained directly.

Two separate ideas about context include the context of mathematical objects and the set of external factors. Students who do abstraction little by little ignore the context of various mathematical objects. However, a set of external factors can influence this process of abstraction. In the cognitive approach, the context that can affect the process of abstraction is considered as a set of external factors. Characteristics of abstraction as a process that occurs in a complex context that requires tasks, tools, and other artifacts; personal history of participants; and physical and social settings.

In this study, it will be seen that abstraction in context takes precedence over aspects of the process rather than results based on [14] will be seen 3 actions epistemic: Recognizing, Building-with, Constructing (RBC). Epistemic actions are often specified in suitable settings. Therefore, settings with rich social interactions are good frameworks for observing epistemic actions. Recognizing takes place when the learner recognizes that a specific previous knowledge construct is relevant to the problem he or she is dealing with. Building-with action comprising the combination of constructs in order to achieve a localized goal, such as the realization of a strategy or a justification or a solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction.
Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct [23].

In the learning process of students in gaining knowledge, an individual often experiences obstacles or learning obstacles. Learning obstacles consists of three kinds, namely: Ontogenic obstacles, didactical obstacles, and epistemological obstacles [24]. Learning obstacles ontogenic are psychological, instrumental, and conceptual. Ontogenic obstacles psychological is unpreparedness of children in learning due to psychological aspects such as motivation and low interest to the material being studied. Ontogenic obstacles instrumental is a technical difficulty that causes the child can't fully follow the situation that occurs in learning as a result of not understanding the technical things are key from a learning process. While conceptual ontogenic obstacles are a type of difficulty with regard to the conceptual levels contained in the designs less compatible with the state of the child seen from previous learning experiences. Concerned or conceptual demands can cause children to lose their learning orientation so that they are prestigious. Conversely, the conceptual challenge is too low to be the cause of the underachiever in learning. The structural sequence of material (representing the interconnectedness of concepts) and functional sequence (representing the continuity of thought processes) have an impact on one's learning process. Similarly, the present stage of the material (less detailed or too detailed) can have a meaningful impact on the learning process. Learning obstacles that result from sequence and presentation factors are called didactical obstacles. The learning obstacles of epistemological are more due to the limited context used for the first time a concept is learned. epistemological obstacles consist of a tendency to rely on intuitive experience tricks, a tendency to generalize and to be caused by the use of natural language. Epistemological obstacles can lead to stagnation of scientific knowledge, and even a decline in one's knowledge.

Nevertheless basically in the formation of a knowledge occurs very complex, through the system of interaction. One such subsystem consists of teachers, students, and knowledge systems [25]. When a learner finds obstacles in his or her learning experience, it may be possible that the interaction system, the learning process that occurs, the nature of the teacher's teaching, the nature of the subject matter, the genetic factors and personal development. This shows that there is an overlap between the various constraints, due to the complex nature of the construction of such knowledge.

Based on the above explanation, the problems that arise in abstract algebra learning especially group theory is in shaping new knowledge of concepts within group theory. The meaning of the process of forming the new concept is the process of abstraction. The process of abstraction seen in this research using RBC theory (Recognizing, Building-with, Constructing). In the process of abstraction will form new mathematical objects, in the process, the goal is not always achieved because of learning obstacles. Therefore, to ascertain whether there is learning obstacles or will not be seen in this study. Learning obstacles seen is learning obstacle epistemology.

2. Methodology
This research method using qualitative research with case study approach. Participant in this study was someone who a 3rd-grade student and had attended abstract algebra lectures on group theory. Student are given questions about group theory based on indicators of abstraction ability. After obtained the answer is analyzed to see abstraction ability. Student who experience epistemological obstacles are interviewed to find out what obstacles are going on. In this study, the learning obstacles that appears only a student.

3. Result and Discussion
Here are the results of the answers and the results of interviews from student who experience epistemology obstacles. Problem given in the form of essays on group theory that contain indicators of abstraction ability. Here's a question that becomes an epistemological obstacle:

1. Let \( \emptyset : G \rightarrow H \) homomorphism function, prove if G komutatif group then \( \emptyset(G) \) komutatif group
Table 1. Interview.

| No. Interview | Interview | Answers Respondent |
|---------------|-----------|--------------------|
| **I.1**       | Try to explain the answer about the no.1! | Here, I will be proved if G group is commutative then ∅(G) commutative group |
| **I.2**       | What is homomorphism? | The function of homomorphism is a surjective and injective function, taken any \( y_1, y_2 \) elements in ∅(G), then there are \( x_1, x_2 \) in G. So \( y_1 = ∅(x_1) \) and \( y_2 = ∅(x_2) \) |
| **I.3**       | where got the equation \( ∅(x_1), ∅(x_2) = ∅(x_1 \cdot x_2) \)? | \( ∅(x_1) \cdot ∅(x_2) = ∅(x_1 \cdot x_2) \) from epimorphism |
| **I.4**       | Why \( ∅(x_1 \cdot x_2) = ∅(x_2 \cdot x_1) \)? | Because G is commutative so \( ∅(x_1 \cdot x_2) = ∅(x_2 \cdot x_1) \) |
| **I.5**       | Why \( ∅(x_2) \cdot ∅(x_1) = y_2 \cdot y_1 \) | because the function is injective \( ∅(x_2), ∅(x_1) = y_2 \cdot y_1 \) |

Based on the above answer, the student replied incorrectly. But the process he answered there are interesting steps to be studied more deeply. To deepen the answer was conducted an in-depth interview. Here is the result of analysis with RBC abstraction point of view from how to answer the problem and the reason for every step in the answer. At the answer step 1 there is a process of taking a member of the function result, which is known to be a group showing the lack of recognizing what is already known. The recognizing process in abstraction abilities indicates a lack of linkage between what is known and what will be shown. Epistemology obstacles shows the existence of obstacles in connecting the known with the taken as a step jump. This happens because the student is wrong to take the intuition. The built-with process in abstraction abilities in step 1 occurs weakness, where she building-with from a various knowledge that is not in accordance with the context of the problem.
Because epistemologically there is an obstacle that causes the lack of early knowledge in stepping. The process of constructing this step is yet to be seen. In step 2 to step 4 she shows the definition of homomorphism that is more than it should be, this shows the process of recognizing in abstraction abilities about the function of homomorphism less so profound, which appears even another requirement that is not a major requirement in homomorphism function. This suggests epistemological obstacles also, where knowledge of homomorphism is less powerful, the main thing about the definition of homomorphism does not arise. In step 3 and 4, it also shows the reinforcement of the term its surjective function not on the definition of homomorphism, but on the definition of isomorphism. This is reinforced by an interview that shows an understanding of her homomorphism, following his interview:

Researcher (R) : try to explain why take the function result $\emptyset(G)$ it?
Participant (P) : will be proved if G is a commutative group then $\emptyset(G)$ is also commutative, here function $\emptyset$ already known homomorphism function whose condition is surjective and injective.

From the results of these interviews on the construction process in the mathematical abstraction of homomorphism become truncated, there is missing. Judging from the epistemological obstacle of knowledge about homomorphism has occurred where there is a generalization process of the concept of isomorphism is too broad. As a result, there arises an unimportant concept of excess, which results in the intuitiveness of the step being incorrect.

In step 5 raises a clear word whose purpose is the same as the same term as it is unable to show the process. In step 6 and 7 new look recognition process in mathematical abstraction. The emergence of this 7th step shows an understanding of the function of homomorphism. The process of recognizing, building-with, and constructing this step appears simultaneously, although it can’t be seen in sequence. This stringing process occurs when generating the idea of function g being operated. The process of constructing also arises when elements in function g can be expressed in another form, which corresponds to the definition of homomorphism. Epistemologically this step does not appear significant, but the 5th step answer using the word clearly indicates an inaccurate generalization and intuitive process.

In step 7 and 8 is the main process of answering the question. In the step comes together with the process of recognizing, building-with, and constructing. The recognition process arises when giving commutative information in addition to step 8. The assembling and construct process occurs when the position changes of the elements within the function.

4. Conclusion and Implications
Based on the results and the above discussion can be concluded that abstraction ability can occur epistemology obstacles both in the phase of recognizing, building-with, and constructing. Epistemological obstacles may result from false intuition in determining prior knowledge. to answer the question, or to do a less precise generalization.

Implications that need to be done in the learning process is the need to provide different contexts in abstract algebra courses such as concrete and abstract concrete combination of formal and non formal.

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