Improved Deterministic Algorithms for Non-monotone Submodular Maximization

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Abstract

Submodular maximization is one of the central topics in combinatorial optimization. It has found numerous applications in the real world. In the past decades, a series of algorithms have been proposed for this problem. However, most of the state-of-the-art algorithms are randomized. There remain non-negligible gaps with respect to approximation ratios between deterministic and randomized algorithms in submodular maximization.

In this paper, we propose deterministic algorithms with improved approximation ratios for non-monotone submodular maximization. Specifically, for the matroid constraint, we provide a deterministic $0.283 - o(1)$ approximation algorithm, while the previous best deterministic algorithm only achieves a $1/4$ approximation ratio. For the knapsack constraint, we provide a deterministic $1/4$ approximation algorithm, while the previous best deterministic algorithm only achieves a $1/6$ approximation ratio. For the linear packing constraints with large widths, we provide a deterministic $1/6 - \epsilon$ approximation algorithm. To the best of our knowledge, there is currently no deterministic approximation algorithm for the constraints.

1 Introduction

Submodular maximization refers to the problem of maximizing a submodular set function under some specific constraint. The submodularity of the objective functions captures the effect of diminishing returns in the economy and therefore the problem has found numerous applications in the real world, including influence maximization [19], sensor placement and feature selection [17, 18], information gathering [22] and machine learning [9, 20].

Due to its remarkable significance, submodular maximization has been studied over the past forty years. The difficulty of the problem is different depending on whether the objective function is monotone.

For the monotone case, it is well-known that the problem can not be approximated within a ratio better than $1 - 1/e$ even under the cardinality constraint [26, 11]. On the other hand, a natural greedy algorithm achieves a $1 - 1/e$ approximation ratio for the cardinality constraint [27] and $1/2$ ratio for the matroid constraint [13]. The greedy algorithm may have an arbitrarily bad approximation ratio for the knapsack constraint, but by combining the enumeration technique, it can be augmented to be an $1 - 1/e$ approximation algorithm [21, 29]. It has remained a longstanding

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open question whether the problem admits a $1 - 1/e$ approximation ratio for the matroid constraint. In 2008, Vondrák [30] answered this question affirmatively by proposing the famous continuous greedy algorithm. Later, the technique was further generalized and applied to other constraints such as the linear packing constraints [23] and general constraints with the down-closed property [12]. Unlike the greedy algorithm, the continuous greedy algorithm requires a sampling process and therefore is inherently a randomized algorithm.

For the non-monotone case, the problem can not be approximated within a ratio better than 0.491 under the cardinality constraint and 0.478 under the matroid constraint [14]. The best known algorithm for the aforementioned constraints achieves a 0.385 approximation ratio [6]. The algorithm is randomized since it applies the continuous greedy technique.

When randomness is not allowed, existing algorithms in the literature for the non-monotone case suffer from inferior approximation ratios. Currently, the best deterministic algorithm for the cardinality constraint achieves a $1/e$ approximation ratio [5]. The best one for the matroid constraint has an approximation ratio of 1/4 [16]. The best one for the knapsack constraint has an approximation ratio of 1/6 [15]. For linear packing constraints, there are currently no deterministic algorithms with constant ratios to the best of our knowledge.

The gap between deterministic and randomized algorithms inspires us to design better deterministic algorithms for the problem. This is of great interest for two reasons. From a theoretical viewpoint, it is an interesting and important question whether randomness is essentially necessary for submodular maximization. From a practical viewpoint, randomized algorithms only work well in the average case, while deterministic algorithms still work in the worst case. In this sense, deterministic algorithms are more robust and hence more suitable in some fields such as the security domain.

1.1 Our Contribution

In this paper, we provide several improved deterministic algorithms for non-monotone submodular maximization subject to different constraints.

- For the matroid constraint, we present a deterministic $0.283 - o(1)$ approximation algorithm with running time $O(n^2k^2)$.
- For the knapsack constraint, we present a deterministic $1/4$ approximation algorithm and a slightly faster deterministic $1/4 - \epsilon$ approximation algorithm. Our knapsack algorithm borrows the idea of the Simultaneous Greedy algorithm in [2] and the Twin Greedy algorithm in [16]. We also present a tight example showing that this approach can not reach an approximation ratio better than 1/4 even under a cardinality constraint. Hence, our analysis for the algorithm, as well as the analysis in [16], is tight.
- For the linear packing constraints, we present a deterministic $1/6 - \epsilon$ approximation algorithm when the width $W = \min \{b_i/A_{ij} \mid A_{ij} > 0\}$ satisfies $W = \Omega(\ln m/\epsilon^2)$. Our algorithm can be adapted to solve the instances with arbitrary widths when the number of constraints $m$ is constant.

We make a more detailed comparison between our and previous results in Table 1.

1.2 Related Work

To better illustrate the improvement of our results, this subsection provides a list of results in the literature on non-monotone submodular maximization under a matroid constraint, a knapsack constraint, and linear packing constraints.
| Constraint | Reference               | Ratio                  | Complexity                      | Type |
|-----------|-------------------------|------------------------|---------------------------------|------|
| Matroid   | Buchbinder et al. [8]   | $0.283 - o(1)$         | $\mathcal{O}(nk)$               | Rand |
| Matroid   | Buchbinder and Feldman  | 0.385                  | $\mathcal{O}(n^2 k^2)$          | Rand |
| Matroid   | Mirzasoleiman et al. [25]| $1/6 - \epsilon$      | $\mathcal{O}(nk + k/\epsilon)$ | Det |
| Matroid   | Lee et al. [24]         | $1/4 - \epsilon$       | $\mathcal{O}(nk)$               | Det |
| Matroid   | Han et al. [16]         | $1/4 - \epsilon$       | $\mathcal{O}(nk)$               | Det |
| Matroid   | Han et al. [16]         | $1/4 - \epsilon$       | $\mathcal{O}(nk)$               | Det |
| Matroid   | Theorem 1               | $0.283 - o(1)$         | $\mathcal{O}(nk)$               | Det |
| Knapsack  | Amanatidis et al. [1]   | 0.171                  | $\mathcal{O}(n \log n)$         | Rand |
| Knapsack  | Buchbinder and Feldman  | 0.385                  | $\mathcal{O}(n^5)$               | Rand |
| Knapsack  | Amanatidis et al. [2]   | 1/7                    | $\mathcal{O}(n^4)$               | Det |
| Knapsack  | Gupta et al. [15]       | 1/6                    | $\mathcal{O}(n^5)$               | Det |
| Knapsack  | Theorem 4               | 1/4                    | $\mathcal{O}(n^4)$               | Det |
| Knapsack  | Theorem 4               | $1/4 - \epsilon$       | $\mathcal{O}(nk)$               | Det |
| Packing   | Buchbinder and Feldman  | 0.385                  | $\mathcal{O}(n^5)$               | Rand |
| Packing   | Theorem 7               | $1/6 - \epsilon$       | $\mathcal{O}(n^5)$               | Det |

Table 1: Approximation algorithms for non-monotone submodular maximization under a matroid, a knapsack and linear packing constraints. The packing constraints have either a constant $m$ or a large width. “Complexity” refers to query complexity. “Rand” is short for “Randomized” and “Det” is short for “Deterministic”.

For the matroid constraint, the best randomized algorithm is based on the continuous greedy technique and achieves a 0.385 approximation ratio [6]. However, this algorithm suffers from high query complexity. There exist different randomized algorithms called Random Greedy that achieve $1/4$ and $0.283 - o(1)$ approximation ratios using $\mathcal{O}(nk)$ queries [8], where $n$ is the total number of the elements and $k$ is the rank of the matroid. When randomness is not allowed, Mirzasoleiman et al. [25] proposed a deterministic algorithm that achieves a $1/6 - \epsilon$ approximation ratio and uses $\mathcal{O}(nk + k/\epsilon)$. Lee et al. [24] proposed a deterministic $1/4 - \epsilon$ approximation algorithm via the local search technique. The algorithm uses $\mathcal{O}(n^4 \log n)$ queries. The best deterministic algorithm in the literature is called Twin Greedy [16], which achieves a $1/4$ approximation ratio using $\mathcal{O}(nk)$ queries and a $1/4 - \epsilon$ ratio using $\mathcal{O}(nk)$ queries.

For the knapsack constraint, the same randomized 0.385 approximation algorithm [6] also works by using a large number of queries. A different randomized algorithm [1] uses nearly linear queries at a cost of a 0.171 approximation ratio. The best deterministic algorithm for the problem achieves a $1/6$ approximation ratio and uses $\mathcal{O}(n^5)$ queries [15].

For the linear packing constraints, no constant approximation is possible when the number of constraints $m$ is an input. When $m$ is constant or the width of the constraints is large, constant approximation is possible [24, 10, 6]. Currently, the best known algorithm is again based on the continuous greedy algorithm and has an approximation ratio of 0.385 [6]. To the best of our knowledge, there is no deterministic algorithm for the problem in the literature.

### 1.3 Organization

In Section 2, we formally introduce the problem of non-monotone submodular maximization under a matroid constraint and a knapsack constraint. In Section 3, we propose a deterministic 0.283 – $o(1)$ approximation algorithm for the matroid constraint. In Section 4, we propose deterministic
algorithms for the knapsack constraint with different approximation ratios and query complexity. In Section 5, we propose a deterministic $1/6 - \epsilon$ approximation algorithm for the linear packing constraints when the width of the constraints is large. In Section 6, we conclude the paper and list some future directions.

## 2 Preliminaries

In this section, we state the problems studied in this paper.

**Definition 1 (Submodular Function).** Given a finite ground set $N$ of $n$ elements, a set function $f : 2^N \rightarrow \mathbb{R}$ is submodular if for all $S, T \subseteq N$,

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

Equivalently, $f$ is submodular if for all $S \subseteq T \subseteq N$ and $u \in N \setminus T$,

$$f(S \cup \{u\}) - f(S) \geq f(T \cup \{u\}) - f(T).$$

For convenience, we use $f(S + u)$ to denote $f(S \cup \{u\})$, $f(u | T)$ to denote the marginal value $f(T + u) - f(T)$ of $u$ with respect to $T$, and $f(S | T)$ to denote the marginal value $f(S \cup T) - f(T)$. The function $f$ is non-negative if $f(S) \geq 0$ for all $S \subseteq N$. $f$ is monotone if $f(S) \leq f(T)$ for all $S \subseteq T \subseteq N$.

**Definition 2 (Matroid).** A matroid system $\mathcal{M} = (N, \mathcal{I})$ consists of a finite ground set $N$ and a collection $\mathcal{I} \subseteq 2^N$ of the subsets of $N$, which satisfies the following three properties.

- $\emptyset \in \mathcal{I}$.
- If $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.
- If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists an element $u \in B \setminus A$ such that $A \cup \{u\} \in \mathcal{I}$.

For a matroid system $\mathcal{M} = (N, \mathcal{I})$, each $A \in \mathcal{I}$ is called an independent set. If $A$ is additionally maximal inclusion-wise, it is called a base. All bases of a matroid have an equal size, which is known as the rank of the matroid. In this paper, we use $k$ to denote the rank of a matroid.

**Definition 3 (Knapsack).** Given a finite ground set $N$, assume there is a budget $B$, and each element $u \in N$ is associated with a cost $c(u) > 0$. For set $S \subseteq N$, its cost $c(S) = \sum_{u \in S} c(u)$. We say $S$ is feasible if $c(S) \leq B$. The knapsack can be written as $\mathcal{I} = \{S \mid c(S) \leq B\}$.

If $c(u) = 1$ for all $u \in N$ in the knapsack and let $B = k$, the knapsack reduces to $\mathcal{I} = \{S \mid |S| \leq k\}$. This is called the cardinality constraint or the uniform matroid, as it satisfies the definition of a matroid.

**Definition 4 (Linear Packing Constraints).** Given a finite ground set $N$, a matrix $A \in [0, 1]^{m \times n}$, and a vector $b \in [1, \infty)^m$. For set $S \subseteq N$, the linear packing constraints can be written as $\mathcal{I} = \{S \mid Ax_S \leq b\}$, where $x_S$ stands for the characteristic vector of the set $S$. Let $W = \min\{b_i / A_{ij} \mid A_{ij} > 0\}$. It is known as the width of the packing constraints.

When $m = 1$, the linear packing constraints reduce to the knapsack constraint.

**Definition 5 (Constrained Submodular Maximization).** The constrained submodular maximization problem has the form

$$\max\{f(S) \mid S \in \mathcal{I}\}.$$
In this paper, the constraint $\mathcal{I}$ is assumed to be a matroid constraint, a single knapsack constraint, or linear packing constraints respectively. The objective function $f$ is assumed to be non-negative, non-monotone, and submodular. Besides, $f$ is accessed by a value oracle that returns $f(S)$ when $S$ is queried. The efficiency of any algorithm for the problem is measured by the number of queries it uses.

3 Deterministic Approximation for Matroid Constraint

In this section, we present a deterministic $(0.283 - o(1))$-approximation algorithm for submodular maximization under a matroid constraint. Our algorithm is obtained by derandomizing the Random Greedy algorithm in [8], using the technique from [5].

To gain some intuition, we first describe the Random Greedy algorithm in [8]. For convenience, we add a set $D$ of $2k$ “dummy elements” to the original instance $(N, f, \mathcal{M})$ to obtain a new instance $(N', f', \mathcal{M}')$ in the following way.

- $N' = N \cup D$.
- $f'(S) = f(S \setminus D)$ for every set $S \subseteq N'$.
- $S \in \mathcal{I}'$ if and only if $S \setminus D \in \mathcal{I}$ and $|S| \leq k$.

Clearly, the new instance and the old one refer to the same problem. Thus, in the remaining part of this section, we assume that any instance always contains the dummy elements as defined above.

The existence of dummy elements allows us to assume that the optimal solution is a base of $\mathcal{M}$ by adding dummy elements to it if it is not initially. Another ingredient of the Random Greedy algorithm is the well-known exchange property of matroids, which is stated in Lemma 1.

**Lemma 1 ([28]).** If $A$ and $B$ are two bases of a matroid $\mathcal{M} = (N, \mathcal{I})$, then there exists a one-to-one function $g : A \rightarrow B$ such that

- $g(u) = u$ for every $u \in A \cap B$.
- For every $u \in A$, $B \cup \{u\} \setminus \{g(u)\} \in \mathcal{I}$.

The Random Greedy algorithm is depicted as Algorithm 1. It was shown in [8] to be a $(0.283 - o(1))$-approximation algorithm.

**Algorithm 1** Random Greedy for Matroid [8]

1: **Input** $N, f, \mathcal{M}$.
2: Initialize $S_0$ to be an arbitrary base containing only dummy elements in $D$.
3: for $i = 1$ to $k$ do
4:    Let $M_i \subseteq N \setminus S_{i-1}$ be a base of $\mathcal{M}$ containing only elements of $N \setminus S_{i-1}$ and maximizing $\sum_{u \in M_i} f(u | S_{i-1})$.
5:    Let $g_i$ be a function mapping each element of $M_i$ to an element of $S_{i-1}$ such that $S_{i-1} + u - g_i(u) \in \mathcal{I}$ for every $u \in M_i$.
6:    Let $u_i$ be a uniformly random element from $M_i$.
7:    $S_i \leftarrow S_{i-1} + u_i - g_i(u_i)$.
8: **return** $S_k$. 
We next explain how to use the technique from [5] to derandomize Algorithm 1. The resulting algorithm is presented as Algorithm 2. It mimics Algorithm 1 by explicitly maintaining a distribution $D_i$ in each iteration $i$, which is supported on the subsets that the algorithm will visit. The key point of Algorithm 2 is that it ensures that the size of the support of $D_i$ grows polynomially with $i$ instead of exponentially as in Algorithm 1.

Formally, we use a multiset of pairs $\{(p_j, S_j)\}$ to represent distribution $D_i$, where $S_j$ is a subset and $p_j$ is a probability. In this representation, we allow $p_j = p_{j'}$, $S_j = S_{j'}$ or both for $j \neq j'$. The sum of $p_j$ equals to 1. The support of $D_i$ consists of distinct $S_j$ and is denoted by $\text{supp}(D_i)$. The number of pairs in $D_i$ is denoted by $|D_i|$. Clearly, $|\text{supp}(D_i)| \leq |D_i|$.

Algorithm 2 updates the distribution and constructs $D_i$ from $D_{i-1}$ by finding an extreme-point solution of the linear program (P). The variables of (P) are $x(u, S)$ for $u \in M_i$ and $S \in \text{supp}(D_{i-1})$, which can be interpreted as the probability of adding $u$ into $S$. The constraints of (P) can be divided into six groups. The last two groups ensure that $x(u, S)$ is a legal probability vector for each $S \in \text{supp}(D_{i-1})$. The first group guarantees that the addition of those $u$ to $S$ with positive $x(u, S)$ attains an average marginal value of elements in $M_i$. The second constraint guarantees that the removal of $g_i, S(u)$ from $S$ causes a loss of at most an average marginal value. The third group of constraints says that any element outside $S$ will be added into $S$ with probability at most $1/k$ and the fourth group of constraints says that any element in $S$ will be kicked out with probability at least $1/k$. These constraints characterize the requirements for the elements to be selected in an iteration. It is easy to see that $x(u, S) = 1/k$ for every $S \in \text{supp}(D_{i-1})$ and $u \in M_i$ is a feasible solution of (P) and this is exactly what the Random Greedy algorithm does. However, this causes the size of the support of $D_i$ to grow exponentially with $i$. By choosing extreme-point solutions of (P), Algorithm 2 can ensure that it grows polynomially rather than exponentially.

There are two details about Algorithm 2 to which we need to pay attention. First, for $A = M_i$ and $B = S$, let $g_{i, S}$ be the mapping defined in Lemma 1. In line 5, Algorithm 2 needs to construct such mappings explicitly. By invoking a perfect matching search algorithm for bipartite matching, they can be found in polynomial time. Second, in line 7, if $u \in S$, $S + u - g_{i, S}(u)$ reduces to $S - g_{i, S}(u)$.

We now introduce several useful properties that Algorithm 2 can guarantee.

**Lemma 2** ([8]). Assume that $f$ is submodular. For any subset $T$ and a distribution $D$,

$$\mathbb{E}_{S \sim D}[f(T \cup S)] \geq f(T) \cdot \min_{u \in S} \frac{\Pr_{D}[u \notin S]}{\Pr_{D}[u \notin S]}. $$

**Lemma 3.** For every iteration $i = 1, 2, \ldots, k$ of Algorithm 2, the following properties hold:

1. The assignment $y(u, S) = 1/k$ for every $u \in M_i$ and $S \in \text{supp}(D_{i-1})$ is a feasible solution of (P).

2. The sum of the probabilities in $D_i$ equals to 1, and therefore $D_i$ is a valid distribution.

3. $|D_i| \leq n + 3k + 2 + |D_{i-1}|$. Thus, $|D_i| \leq ni + 3ki + 2i + 1 = O(ni)$.

**Proof.** We prove the lemma by induction on $i$. Assume that it holds for every $1 \leq i' < i$. First, the correctness of the first property can be verified by directly plugging the assignment into (P). Next, notice that the sum of the probabilities in $D_i$ is

$$\sum_{S \in \text{supp}(D_{i-1})} \Pr_{D_{i-1}}[S] \sum_{u \in M_i} x(u, S) = \sum_{S \in \text{supp}(D_i)} \Pr_{D_{i-1}}[S] = 1.$$
Algorithm 2 Derandomization of Random Greedy for Matroid

1: **Input** \( N, f, \mathcal{M} \).
2: Initialize \( \mathcal{D}_0 = \{(1, S)\} \), where \( S \) is an arbitrary base containing only dummy elements in \( D \).
3: **for** \( i = 1 \) to \( k \) **do**
4: \( \text{Let } M_i \subseteq N \text{ be a base of } \mathcal{M} \text{ maximizing } \sum_{u \in M_i} \mathbb{E}_{S \sim \mathcal{D}_{i-1}}[f(u \mid S)]. \)
5: \( \text{Construct mapping } g_i,S \text{ for every } S \in \text{supp}(\mathcal{D}_{i-1}), \text{ where } g_i,S \text{ denotes the mapping defined in Lemma 1 by plugging } A = M_i \text{ and } B = S. \)
6: \( \text{Find an extreme point solution of the following linear formulation:} \)

\[
\begin{align*}
\sum_{u \in M_i} \mathbb{E}_{S \sim \mathcal{D}_{i-1}}[x(u, S) \cdot f(u \mid S)] & \geq \frac{1}{k} \sum_{u \in M_i} \mathbb{E}_{S \sim \mathcal{D}_{i-1}}[f(u \mid S)] \\
\sum_{u \in M_i} \mathbb{E}_{S \sim \mathcal{D}_{i-1}}[x(u, S) \cdot f(g_i,S(u) \mid S \setminus \{g_i,S(u)\})] & \leq \frac{1}{k} \sum_{u \in M_i} \mathbb{E}_{S \sim \mathcal{D}_{i-1}}[f(g_i,S(u) \mid S \setminus \{g_i,S(u)\})] \\
\mathbb{E}_{S \sim \mathcal{D}_{i-1}}[x(u, S) \cdot 1[u \notin S]] & \leq \frac{1}{k} \mathbb{Pr}_{S \sim \mathcal{D}_{i-1}}[u \notin S], \quad \forall u \in M_i \\
\mathbb{E}_{S \sim \mathcal{D}_{i-1}}[x(g_i^{-1}(u), S) \cdot 1[u \in S]] & \geq \frac{1}{k} \mathbb{Pr}_{S \sim \mathcal{D}_{i-1}}[u \in S], \quad \forall u \in N \\
\sum_{u \in M_i} x(u, S) & = 1, \quad \forall S \in \text{supp}(\mathcal{D}_{i-1}) \\
x(u, S) & \geq 0, \quad \forall u \in M_i, S \in \text{supp}(\mathcal{D}_{i-1})
\end{align*}
\]

7: \( \text{Construct a new distribution:} \)

\[
\mathcal{D}_i \leftarrow \{(x(u, S) \cdot \mathbb{Pr}_{\mathcal{D}_{i-1}}[S], S + u - g_i,S(u)) \mid u \in M_i, S \in \text{supp}(\mathcal{D}_{i-1}), x(u, S) > 0\}.
\]

8: **return** \( \arg \max_{S \in \text{supp}(\mathcal{D}_k)} f(S) \).

Thus, the second property holds. Finally, recall that for an extreme point solution \( x \), the number of variables \( x(u, S) \) that are strictly greater than zero is no more than the number of (tight) constraints. Besides, observe that the number of constraints in (P) at iteration \( i \) is at most \( n + 3k + 2 + |\text{supp}(\mathcal{D}_{i-1})| \leq n + 3k + 2 + |\mathcal{D}_{i-1}|. \) Since \( S + u - g_i,S(u) \) is added to \( \mathcal{D}_i \) only when \( x(u, S) > 0 \), the size of \( \mathcal{D}_i \) is also no more than \( n + 3k + 2 + |\mathcal{D}_{i-1}|. \)

**Lemma 4.** For every element \( u \in N \setminus D \) and \( 0 \leq i \leq k \),

\[
\mathbb{Pr}_{S \sim \mathcal{D}_i}[u \notin S] \geq \frac{1}{2} \left( 1 + \left( 1 - \frac{2}{k} \right)^i \right).
\]

**Proof.** For a fixed \( u \in N \setminus D \),

\[
\mathbb{Pr}_{S \sim \mathcal{D}_i}[u \in S] = \sum_{S \in \text{supp}(\mathcal{D}_{i-1}) : u \notin S} \mathbb{Pr}_{\mathcal{D}_{i-1}}[S] \cdot x(u, S) + \sum_{S \in \text{supp}(\mathcal{D}_{i-1}) : u \in S} \mathbb{Pr}_{\mathcal{D}_{i-1}}[S] \cdot \sum_{u' \in M_i - g_i^{-1}(u)} x(u', S).
\]

The first part equals to zero if \( u \notin M_i \), and if \( u \in M_i \) it holds that

\[
\sum_{S \in \text{supp}(\mathcal{D}_{i-1}) : u \notin S} \mathbb{Pr}_{\mathcal{D}_{i-1}}[S] \cdot x(u, S) = \sum_{S \in \text{supp}(\mathcal{D}_{i-1})} \mathbb{Pr}_{\mathcal{D}_{i-1}}[S] \cdot x(u, S) \cdot 1[u \notin S]
\]

7
The last inequality holds since $x$ is a feasible solution of (P)

The second part satisfies

\[
\sum_{S \in \text{supp}(D_{i-1})} \Pr_{D_{i-1}}[S] \cdot \sum_{u' \in M_i} x(u', S) \\
= \sum_{S \in \text{supp}(D_{i-1})} \Pr_{D_{i-1}}[S] \cdot \left(1 - x(g_{i,S}^{-1}(u), S)\right) \\
= \Pr_{D_{i-1}}[u \in S] - \sum_{S \in \text{supp}(D_{i-1})} \Pr_{D_{i-1}}[S] \cdot x(g_{i,S}^{-1}(u), S) \cdot 1[u \in S] \\
\leq (1 - 1/k) \Pr_{D_{i-1}}[u \in S].
\]

The last inequality holds since $x$ is a feasible solution of (P).

Let $p_{i,u} = \Pr_{S \sim D_i}[u \in S]$. By the above argument,

\[
p_{i,u} \leq (1 - p_{i-1,u})/k + p_{i-1,u}(1 - 1/k) = p_{i-1,u}(1 - 2/k) + 1/k.
\]

For $u \in N \setminus D$, $p_{0,u} = 0$. It is easy to show by induction that

\[
p_{i,u} \leq 0.5 \cdot (1 - (1 - 2/k)^i).
\]

The lemma follows immediately.

Let $O$ be the optimal solution. We have

**Lemma 5.** For every iteration $i = 1, 2, \ldots, k$ of Algorithm 2,

\[
E_{S \sim D_i} [f(S)] \geq \left(1 - \frac{2}{k}\right) \cdot E_{S \sim D_{i-1}} [f(S)] + \frac{(1 + (1 - 2/k)^{i-1})}{2k} \cdot f(O).
\]

**Proof.** On the one hand,

\[
\sum_{u \in M_i} E_{S \sim D_{i-1}} [x(u, S) \cdot f(u \mid S)] \geq \frac{1}{k} \sum_{u \in M_i} E_{S \sim D_{i-1}} [f(u \mid S)] \\
\geq \frac{1}{k} \sum_{u \in O} E_{S \sim D_{i-1}} [f(O \cup S) - f(S)] \\
\geq \frac{1}{k} \sum_{S \sim D_{i-1}} [0.5 \cdot (1 + (1 - 2/k)^{i-1}) \cdot f(O) - f(S)].
\]

The first inequality holds since $x$ is a feasible solution of (P). The second is due to the choice of $M_i$. The third is due to submodularity. The last follows from Lemmas 2 and 4.
On the other hand,
\[
\sum_{u \in M_i} E_{s \sim D_{i-1}} [x(u, S) \cdot f(g_i, s(u) \mid S \setminus \{g_i, s(u)\})]
\]
\[
\leq \frac{1}{k} \sum_{u \in M_i} E_{s \sim D_{i-1}} [f(g_i, s(u) \mid S \setminus \{g_i, s(u)\})]
\]
\[
= \frac{1}{k} E_{s \sim D_{i-1}} \left[ \sum_{u \in M_i} (f(S) - f(S \setminus \{g_i, s(u)\})) \right]
\]
\[
\leq \frac{1}{k} E_{s \sim D_{i-1}} [f(S)].
\]

The first inequality holds since \(x\) is a feasible solution of (P). The last inequality is due to submodularity.

Finally, by combining the above inequalities,
\[
E_{s \sim D_{i}} [f(S)]
\]
\[
= \sum_{u \in M_i} E_{s \sim D_{i-1}} [x(u, S) \cdot f(S + u - g_i, s(u))]
\]
\[
\geq \sum_{u \in M_i} E_{s \sim D_{i-1}} [x(u, S) \cdot (f(S + u) + f(S - g_i, s(u)) - f(S))]
\]
\[
= \sum_{u \in M_i} E_{s \sim D_{i-1}} [x(u, S) \cdot (f(S) + f(u \mid S) - f(g_i, s(u) \mid S \setminus \{g_i, s(u)\}))]
\]
\[
= E_{s \sim D_{i-1}} [f(S)] + \sum_{u \in M_i} E_{s \sim D_{i-1}} [x(u, S) \cdot f(u \mid S)] - \sum_{u \in M_i} E_{s \sim D_{i-1}} [x(u, S) \cdot f(g_i, s(u) \mid S \setminus \{g_i, s(u)\})]
\]
\[
\geq E_{s \sim D_{i-1}} [f(S)] + \frac{1}{k} E_{s \sim D_{i-1}} [0.5 \cdot (1 + (1 - 2/k)^{i-1}) \cdot f(O) - f(S)] - \frac{1}{k} E_{s \sim D_{i-1}} [f(S)]
\]
\[
= \left(1 - \frac{2}{k}\right) \cdot \frac{E_{s \sim D_{i-1}} [f(S)] + (1 + (1 - 2/k)^{i-1})}{2k} \cdot f(O).
\]

The following theorem provides a theoretical guarantee for Algorithm 2.

**Theorem 1.** Algorithm 2 achieves a \((1 + e^{-2})/4 - O(1/k^2)\) approximation ratio using \(O(n^2 k^2)\) queries.

**Proof.** We prove by induction that
\[
E[f(S_i)] \geq \frac{1}{4} \left[ 1 + \left( \frac{2(i + 1)}{k} - 1 \right) \left( 1 - \frac{2}{k} \right)^{i-1} \right] \cdot f(O).
\]

Clearly, the claim holds for \(i = 0\). Assume that the claim holds for every \(0 \leq i' < i\). By the above lemma,
\[
E_{s \sim D_{i}} [f(S)] \geq \left(1 - \frac{2}{k}\right) \cdot \frac{E_{s \sim D_{i-1}} [f(S)] + 0.5(1 + (1 - 2/k)^{i-1})}{k} \cdot f(O)
\]

\[
E_{s \sim D_{i}} [f(S)] \geq \left[ 1 + \left( \frac{2(i + 1)}{k} - 1 \right) \left( 1 - \frac{2}{k} \right)^{i-1} \right] \cdot f(O).
\]
\[
\geq \left(1 - \frac{2}{k}\right) \cdot \frac{1}{4} \left[1 + \left(\frac{2i}{k} - 1\right) \left(1 - \frac{2}{k}\right)^{i-2}\right] \cdot f(O) + \frac{1 + (1 - 2/k)^{i-1}}{2k} \cdot f(O)
\]
\[
= \frac{1}{4} \left[1 + \left(\frac{2(i+1)}{k} - 1\right) \left(1 - \frac{2}{k}\right)^{i-1}\right] \cdot f(O).
\]

Therefore,
\[
E_{S \sim \mathcal{D}_k}[f(S)] \geq \frac{1}{4} \left[1 + \left(\frac{2(k+1)}{k} - 1\right) \left(1 - \frac{2}{k}\right)^{k-1}\right] \cdot f(O)
\]
\[
\geq 1 + \frac{e^{-2} (1 + 2/k) (1 - 4/k)}{4} \cdot f(O)
\]
\[
= \left(1 + \frac{e^{-2}}{4} - O(1/k^2)\right) \cdot f(O).
\]

Finally, since $|\mathcal{D}_i| = O(in)$, Algorithm 2 makes $O(n^2i)$ queries at iteration $i$. Therefore, the total number of queries made during all the iterations is $O(n^2k^2)$.

\[\square\]

4 Deterministic Approximation for Knapsack Constraint

In this section, we present deterministic approximation algorithms for submodular maximization under a knapsack constraint. In Section 4.1, we present the Twin Greedy algorithm, which is originally designed for the matroid constraint in [16] and for a mechanism design version of submodular maximization in [2]. It returns a set with a $1/4$ approximation ratio and uses $O(n^2)$ queries. In Section 4.2, we combine the threshold technique with the Twin Greedy algorithm and obtain the so-called Threshold Twin Greedy algorithm, which returns a set with a $1/4 - \epsilon$ approximation ratio and uses $\tilde{O}(n/\epsilon)$ queries. However, the sets returned by these two algorithms may be infeasible. Therefore, in Section 4.3, we introduce the enumeration technique to turn them into feasible solutions with the same approximation ratio. In Section 4.4, we present a tight example that shows that the Twin Greedy algorithm can not achieve an approximation ratio better than $1/4$ even under the cardinality constraint. Hence, our analysis for the algorithm, as well as the analysis in [16], is tight.

4.1 The Twin Greedy Algorithm

In this section, we present a deterministic $1/4$ approximation algorithm for the knapsack constraint. The formal procedure is presented as Algorithm 3. It maintains two disjoint candidate solutions $S_1$ and $S_2$ throughout its execution. At each round, when there remain unpacked elements and at least one feasible candidate solution, it determines a pair $(k, u)$ such that $u$ has the largest density with respect to $S_k$ ($k \in \{1, 2\}$). Then, $u$ is added into $S_k$ when its marginal value to $S_k$ is positive, otherwise, the algorithm will terminate immediately. Note that the algorithm may return an infeasible set, since each candidate may pack one more element that violates the knapsack constraint. We will handle this issue in Section 4.3 by a standard enumeration technique.

We begin to analyze Algorithm 3 by first defining some notations. Assume that Algorithm 3 ran for $t$ rounds in total. In each round, an element was selected. Thus, there are $t$ elements selected. Some of them were added to $S_1$, and the others were added to $S_2$. For $i = 1, 2, \ldots, t$, let $u_i$ be the element selected by Algorithm 3 at round $i$. For $k = 1, 2$ and $i = 1, 2, \ldots, t$, let $S_k^i$ be the $k$-th
The last two inequalities are due to submodularity. Next, summing over all $u_j$ we have

\[
\sum_{j: u_j \in S_k \setminus O} f(u_j \mid S_k^{j-1}) \geq c(S_k \setminus O) \cdot f(O \setminus (S_1^m \cup S_2^m) \mid S_k).
\]

The last two inequalities are due to submodularity. Next, summing over all $u_j \in S_k \setminus O$, we have

\[
\sum_{j: u_j \in S_k \setminus O} f(u_j \mid S_k^{j-1}) \geq \frac{c(S_k \setminus O)}{c(O \setminus (S_1^m \cup S_2^m))} \cdot f(O \setminus (S_1^m \cup S_2^m) \mid S_k).
\]
Since \( c(S_k) \geq B \geq c(O) \), we have \( c(S_k \setminus O) \geq c(O \setminus S_k) = c(O \setminus S_k^m) \geq c(O \setminus (S_1^m \cup S_2^m)) \). Thus,
\[
\sum_{j: u_j \in S_k \setminus O} f(u_j | S_k^j) \geq f(O \setminus (S_1^m \cup S_2^m) | S_k).
\]

\[\square\]

**Lemma 7.** For any \( k \in \{1, 2\} \), let \( \ell = 3 - k \) be the index of the other candidate solution. For fixed \( i \in \{1, 2, \ldots, t\} \), if \( c(S_i^{k-1}) < B \), then for any subset \( T \subseteq S_i^j \), \( f(T | S_i^k) \leq \sum_{j: u_j \in T} f(u_j | S_i^{j-1}) \).

**Proof.** For any \( u_j \in T \subseteq S_i^j \), since \( u_j \) was added into \( S_\ell \) instead of \( S_k \) and \( c(S_i^{j-1}) \leq c(S_i^{k-1}) < B \), we have
\[
\frac{f(u_j | S_i^{j-1})}{c(u_j)} \geq \frac{f(u_j | S_i^{j-1})}{c(u_j)}.
\]
Thus, \( f(u_j | S_i^{j-1}) \geq f(u_j | S_i^{j-1}) \). Besides, \( S_i^{j-1} \subseteq S_i^k \) since round \( j - 1 \) is before round \( i \). Then,
\[
\sum_{j: u_j \in T} f(u_j | S_i^j) \geq \sum_{j: u_j \in T} f(u_j | S_i^j) \geq \sum_{j: u_j \in T} f(u_j | S_i^j) \geq f(T | S_i^k).
\]

The last two inequalities are due to submodularity. \[\square\]

The following theorem provides a theoretical guarantee for Algorithm 3.

**Theorem 2.** Algorithm 3 achieves a \( 1/4 \) approximation ratio (though the output may be infeasible) and uses \( O(n^2) \) queries.

**Proof.** By our notations, we need to show that \( f(S^*) \geq \frac{1}{4} f(O) \). Since \( f(S^*) = \max\{f(S_1), f(S_2)\} \geq (f(S_1) + f(S_2))/2 \), it suffices to show that \( f(S_1) + f(S_2) \geq \frac{1}{2} f(O) \). We prove this by case analysis, according to whether the candidate solutions are feasible.

**Case 1:** Both candidate solutions are feasible at the end of Algorithm 3, i.e., \( c(S_1) < B \) and \( c(S_2) < B \).

In this case, by Lemma 6, we have
\[
\begin{align*}
    f(O \setminus S_2 | S_1) &= f(O \setminus (S_1 \cup S_2) | S_1) \leq 0, \\
    f(O \setminus S_1 | S_2) &= f(O \setminus (S_1 \cup S_2) | S_2) \leq 0.
\end{align*}
\]

Besides, by plugging \( T = O \cap S_\ell \subseteq S_\ell \) into Lemma 7, we have
\[
\begin{align*}
    f(O \cap S_2 | S_1) &\leq \sum_{j: u_j \in O \cap S_2} f(u_j | S_i^{j-1}), \\
    f(O \cap S_1 | S_2) &\leq \sum_{j: u_j \in O \cap S_1} f(u_j | S_i^{j-1}).
\end{align*}
\]

Therefore, by combining the above inequalities,
\[
\begin{align*}
    f(S_1) + f(S_2) &\geq \sum_{j: u_j \in O \cap S_1} f(u_j | S_i^{j-1}) + \sum_{j: u_j \in O \cap S_2} f(u_j | S_i^{j-1}) \\
    &\geq f(O \setminus S_1 | S_2) + f(O \setminus S_2 | S_1) + f(O \cap S_2 | S_1) + f(O \cap S_1 | S_2) \\
    &\geq f(O | S_1) + f(O | S_2)
\end{align*}
\]

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Therefore, by combining the above inequalities, 
\[ f(S_1) + f(S_2) = f(O \cup S_1) - f(S_1) + f(O \cup S_2) - f(S_2) \geq f(O) - (f(S_1) + f(S_2)). \]

The last two inequalities hold due to submodularity and the fact that \( S_1 \cap S_2 = \emptyset \). Thus, \( f(S_1) + f(S_2) \geq \frac{1}{2}f(O) \).

**Case 2:** At least one candidate solution is not feasible at the end of Algorithm 3, i.e., \( c(S_1) \geq B \) or \( c(S_2) \geq B \).

Assume that for some \( m \in \{1, 2, \ldots, \ell\} \), \( S_1 \) first became infeasible at the end of round \( m \), that is, \( c(S_1^{m-1}) < B \) but \( c(S_1^m) \geq B \), and \( c(S_2^m) < B \). By Lemma 6,

\[ f(O \setminus S_1 | S_2) = f(O \setminus (S_1 \cup S_2) | S_2) \leq \max \left\{ 0, \sum_{j:u_j \in S_2 \setminus O} f(u_j | S_2^{j-1}) \right\} \leq \sum_{j:u_j \in S_2 \setminus O} f(u_j | S_2^{j-1}). \]

Again by Lemma 6 and the fact that \( S_1 = S_1^m \),

\[ f(O \setminus S_2^m | S_1) = f(O \setminus (S_1^m \cup S_2^m) | S_1) \leq \sum_{j:u_j \in S_1 \setminus O} f(u_j | S_1^{j-1}). \]

Next, by submodularity and plugging \( k = 2, \ell = 1, T = O \cap S_1 \subseteq S_1^m \) into Lemma 7,

\[ f(O \cap S_1 | S_2) \leq f(O \cap S_1 | S_2^m) \leq \sum_{j:u_j \in O \cap S_1} f(u_j | S_2^{j-1}). \]

By plugging \( k = 1, \ell = 2, T = O \cap S_2^m \subseteq S_2^m \) into Lemma 7,

\[ f(O \cap S_2^m | S_1) = f(O \cap S_2^m | S_1^m) \leq \sum_{j:u_j \in O \cap S_2^m} f(u_j | S_2^{j-1}) \leq \sum_{j:u_j \in O \cap S_2} f(u_j | S_2^{j-1}). \]

Therefore, by combining the above inequalities,

\[ f(S_1) + f(S_2) = \sum_{j:u_j \in S_1 \setminus O} f(u_j | S_1^{j-1}) + \sum_{j:u_j \in O \cap S_1} f(u_j | S_1^{j-1}) + \sum_{j:u_j \in S_2 \setminus O} f(u_j | S_2^{j-1}) + \sum_{j:u_j \in O \cap S_2} f(u_j | S_2^{j-1}) \geq f(O \setminus S_2^m | S_1) + f(O \cap S_1 | S_2) + f(O \setminus S_1 | S_2) + f(O \cap S_2^m | S_1) \geq f(O | S_1) + f(O | S_2) = f(O \cup S_1) - f(S_1) + f(O \cup S_2) - f(S_2) \geq f(O) - (f(S_1) + f(S_2)). \]

The last two inequalities hold due to submodularity and the fact that \( S_1 \cap S_2 = \emptyset \). Thus, \( f(S_1) + f(S_2) \geq \frac{1}{2}f(O) \).

Finally, Algorithm 3 runs at most \( n \) rounds and makes \( O(n) \) queries at each round. Thus, it makes \( O(n^2) \) queries in total.

**4.2 The Threshold Twin Greedy Algorithm**

In this section, we accelerate the Twin Greedy algorithm by applying the threshold technique [4]. The modified algorithm is called the Threshold Twin Greedy algorithm and is formulated as Algorithm 4.
Algorithm 4 maintains a set of thresholds for each element and stores them decreasingly in a priority queue. At each round, Algorithm 4 only considers the first element in the queue. It first removes the element from the queue and then compares its marginal density with its threshold. If the marginal density is at least \((1 - \epsilon)\) of the threshold, the element will be added to the current solution. Otherwise, the threshold will be updated to the marginal density and the element will be reinserted into the queue as long as it has not been reinserted into the queue for \(\Omega(\log n)\) times.

| Algorithm 4 Threshold Twin Greedy for Knapsack |
|------------------------------------------------|
| 1: Input \(N, f, c, B, \epsilon.\)             |
| 2: \(S_1 \leftarrow \emptyset, S_2 \leftarrow \emptyset, J \leftarrow \{1, 2\}.\) |
| 3: \(\Delta(u) \leftarrow f(u)\) for \(u \in N.\) |
| 4: Maintain a priority queue \(Q\) where elements in \(N\) are sorted in decreasing order by their keys, and the key of \(u \in N\) is initialized to \(\Delta(u)/c(u)\). |
| 5: \(q(u) \leftarrow 0\) for \(u \in N.\) \(\triangleright q(u)\) records the number of times \(u\) is reinserted into \(Q\) |
| 6: \(D \leftarrow \emptyset.\) \(\triangleright D\) records the elements that has been reinserted into \(Q\) \(\frac{2\ln(n/\epsilon)}{\epsilon}\) times |
| 7: while \(N \neq \emptyset, J \neq \emptyset\) and \(Q \neq \emptyset\) do |
| 8: Remove the element \(u\) from \(Q\) with the maximum key. |
| 9: \(k \leftarrow \arg\max_{k' \in J} f(u \mid S_{k'}).\) |
| 10: if \(f(u \mid S_{k}) \leq 0\) then \(\triangleright k \in J\) iff \(c(S_{k}) < B\) |
| 11: break |
| 12: if \(f(u \mid S_{k}) \geq (1 - \epsilon) \cdot \Delta(u)\) then |
| 13: \(S_{k} \leftarrow S_{k} \cup \{u\}.\) |
| 14: \(N \leftarrow N \setminus \{u\}.\) |
| 15: if \(c(S_{k}) \geq B\) then |
| 16: \(J \leftarrow J \setminus \{k\}.\) |
| 17: else |
| 18: if \(q(u) \leq \frac{2\ln(n/\epsilon)}{\epsilon}\) then |
| 19: \(\Delta(u) \leftarrow f(u \mid S_{k}).\) |
| 20: Reinsert \(u\) into \(Q\) with key \(\Delta(u)/c(u)\). |
| 21: \(q(u) \leftarrow q(u) + 1.\) |
| 22: else |
| 23: \(D \leftarrow D \cup \{u\}.\) |
| 24: return \(\arg\max\{f(S_1), f(S_2)\}\). |

In this way, if the element was added into the solution, it means that Algorithm 4 selects an element with the largest density at this round, up to a \((1 - \epsilon)\) factor. Otherwise, the threshold of this element will decrease by a factor of at least \(1 - \epsilon\). In other words, the threshold decreases exponentially and it takes \(O(\log n)\) queries to determine whether Algorithm 4 should select an element.

We begin to analyze Algorithm 4 by first defining some notations. Assume that Algorithm 4 added \(t\) elements into \(S_1\) and \(S_2\) in total, denoted by \(\{u_1, u_2, \ldots, u_t\}\). Some of them were added to \(S_1\), and the others were added to \(S_2\). For \(i = 1, 2, \ldots, t - 1\), \(u_i\) was added right before \(u_{i+1}\). But \(u_i\) and \(u_{i+1}\) are not necessarily added in two successive rounds. For \(i = 1, 2, \ldots, t\), let \(\Delta_i(u)\) be the value of \(\Delta(u)\) at the beginning of the round where \(u_i\) is added. For \(k = 1, 2\) and \(i = 1, 2, \ldots, t\), let \(S^i_k\) be the \(k\)-th candidate solution after \(u_i\) is added and \(S_k\) denote the \(k\)-th candidate solution after \(u_t\) is added. Thus, \(S^i_k = S_k \cap \{u_1, u_2, \ldots, u_i\}\). Let \(S^*\) be the set returned by Algorithm 3, \(O\) be the optimal solution, and \(O' = O \setminus D\), where \(D\) is defined in Algorithm 4.
We first show that the marginal value of $O \setminus O'$ with respect to $S_k$ is small.

**Lemma 8.** For any $k \in \{1, 2\}$, we have $f(O \setminus O' \mid S_k) \leq \epsilon f(O)$.

**Proof.** Since $O \setminus O' \subseteq D$, this means that each $u \in O \setminus O'$ has been reinserted into $Q$ for $\frac{2 \ln(n/\epsilon)}{\epsilon}$ times. Since $\Delta(u)$ decreases by a factor of at least $1 - \epsilon$ each time $u$ is reinserted into $Q$, we have $f(u \mid S_k) \leq (1 - \epsilon) \frac{\ln(n/\epsilon)}{\epsilon} f(u) \leq \frac{2}{n} f(u) \leq \frac{\epsilon}{5} f(O)$. By submodularity, $f(O \setminus O' \mid S_k) \leq \sum_{u \in O \setminus O'} f(u \mid S_k) \leq f(O)$. \qed

Next, we introduce two useful properties (Lemmas 9 and 10) that Algorithm 4 can guarantee. These two lemmas are parallel to Lemmas 6 and 7, by replacing $O$ with $O' = O \setminus D$.

**Lemma 9.** For any $k \in \{1, 2\}$, we have

- If $c(S_k) < B$, then $f(O' \setminus (S_1 \cup S_2) \mid S_k) \leq 0$.
- If $c(S_k) \geq B$, let $m$ be the smallest index such that $S_k^m = S_k$, then we have $(1 - \epsilon) \cdot f(O' \setminus (S_k^m \cup S_2^m) \mid S_k) \leq \sum_{j: u_j \in S_k \setminus S_k^m} f(u_j \mid S_k^{j-1})$.

**Proof.**

**Case** $c(S_k) < B$. If $O' \setminus (S_1 \cup S_2) = \emptyset$, then the claim holds trivially. If $O' \setminus (S_1 \cup S_2) \neq \emptyset$, since $O' \cap D = \emptyset$, it means that each $u \in O' \setminus (S_1 \cup S_2)$ is still in the queue $Q$. Since $c(S_k) < B$, by line 10 of Algorithm 4, this means that $f(u \mid S_k) \leq 0$ for each $u \in O \setminus (S_1 \cup S_2)$. By submodularity, $f(O' \setminus (S_1 \cup S_2) \mid S_k) \leq 0$.

**Case** $c(S_k) \geq B$. For any $u_j \in S_k \setminus O'$ and $u \in O' \setminus (S_k^m \cup S_2^m)$, since $u_j$ is the $j$-th element added into $S_k$ and $u$ was in the queue $Q$ but not selected at that time, we have

$$\frac{f(u_j \mid S_k^{j-1})}{c(u_j)} \geq \frac{(1 - \epsilon) \cdot \Delta_j(u_j)}{c(u_j)} \geq \frac{(1 - \epsilon) \cdot \Delta_j(u)}{c(u)} \geq \frac{(1 - \epsilon) \cdot f(u \mid S_k^{j-1})}{c(u)}.$$

The second inequality holds since $u_j$ has the maximum key in $Q$. The third inequality holds by submodularity and the observation that there must exist a $k' < k$ such that $\Delta_j(u) = f(u \mid S_k^{j-1})$.

By a similar argument to Lemma 6, we have

$$\sum_{j: u_j \in S_k \setminus O'} f(u_j \mid S_k^j) \geq \frac{c(S_k \setminus O')}{c(O' \setminus (S_k^m \cup S_2^m))} \cdot (1 - \epsilon) \cdot f(O' \setminus (S_k^m \cup S_2^m) \mid S_k).$$

Since $c(S_k) \geq B \geq c(O')$, we have $c(S_k \setminus O') \geq c(O' \setminus S_k) = c(O' \setminus S_k^m) = c(O' \setminus (S_k^m \cup S_2^m))$. Thus,

$$\sum_{j: u_j \in S_k \setminus O'} f(u_j \mid S_k^{j-1}) \geq (1 - \epsilon) \cdot f(O' \setminus (S_k^m \cup S_2^m) \mid S_k).$$ \qed

**Lemma 10.** For any $k \in \{1, 2\}$, let $\ell = 3 - k$ be the index of the other candidate solution. For fixed $i \in \{1, 2, \ldots, t\}$, if $c(S_k^{i-1}) < B$, then for any subset $T \subseteq S_k^i$, $f(T \mid S_k^i) \leq \sum_{j: u_j \in T} f(u_j \mid S_k^{j-1})$.

**Proof.** This is a restatement of Lemma 7. \qed

The following theorem provides a theoretical guarantee for Algorithm 4.

**Theorem 3.** Algorithm 4 achieves a $1/4 - \epsilon$ approximation ratio (though the output may be infeasible) and uses $O((n/\epsilon) \log(n/\epsilon))$ queries.
Proof. We need to show that \( f(S^*) \geq (\frac{1}{2} - \epsilon) f(O) \). Since \( f(S^*) = \max\{f(S_1), f(S_2)\} \geq (f(S_1) + f(S_2))/2 \), it suffices to show that \( f(S_1) + f(S_2) \geq (\frac{1}{2} - 2\epsilon) f(O) \). We prove this by case analysis, according to whether the candidate solutions are feasible.

**Case 1:** Both candidate solutions are feasible at the end of Algorithm 3, i.e., \( c(S_1) < B \) and \( c(S_2) < B \).

In this case, by Lemma 9, we have
\[
\begin{align*}
  f(O' \setminus S_2 | S_1) &= f(O' \setminus (S_1 \cup S_2) | S_1) \leq 0, \\
  f(O' \setminus S_1 | S_2) &= f(O' \setminus (S_1 \cup S_2) | S_2) \leq 0.
\end{align*}
\]

Besides, by plugging \( T = O' \cap S_\ell \subseteq S_\ell \) into Lemma 10, we have
\[
\begin{align*}
  f(O' \cap S_2 | S_1) &\leq \sum_{j: u_j \in O' \cap S_2} f(u_j | S_2^{i-1}), \\
  f(O' \cap S_1 | S_2) &\leq \sum_{j: u_j \in O' \cap S_1} f(u_j | S_1^{i-1}).
\end{align*}
\]

Therefore, by combining the above inequalities,
\[
\begin{align*}
  f(S_1) + f(S_2) &\geq \sum_{j: u_j \in O' \cap S_1} f(u_j | S_1^{i-1}) + \sum_{j: u_j \in O' \cap S_2} f(u_j | S_2^{i-1}) \\
  &\geq f(O' \setminus S_1 | S_2) + f(O' \cap S_2 | S_1) + f(O' \setminus S_2 | S_1) + f(O' \cap S_1 | S_2) \\
  &\geq f(O' | S_1) + f(O' | S_2).
\end{align*}
\]

By Lemma 8,
\[
\begin{align*}
  f(S_1) + f(S_2) + 2\epsilon \cdot f(O) &\geq f(O' | S_1) + f(O' | S_2) + f(O \setminus O' | S_1) + f(O \setminus O' | S_2) \\
  &\geq f(O | S_1) + f(O | S_2) \\
  &= f(O \cup S_1) - f(S_1) + f(O \cup S_2) - f(S_2) \\
  &\geq f(O) - (f(S_1) + f(S_2)).
\end{align*}
\]

The last two inequalities hold due to submodularity and the fact that \( S_1 \cap S_2 = \emptyset \). By rearranging the inequality, \( f(S_1) + f(S_2) \geq (\frac{1}{2} - \epsilon) f(O) \).

**Case 2:** At least one candidate solution is not feasible at the end of Algorithm 3, i.e., \( c(S_1) \geq B \) or \( c(S_2) \geq B \).

Assume that for some \( m \in \{1, 2, \ldots, t\} \), \( S_1 \) first became infeasible when \( u_m \) is added, that is, \( c(S_1^{m-1}) < B \) but \( c(S_1^m) \geq B \), and \( c(S_2^m) < B \). By Lemma 9,
\[
\begin{align*}
  f(O' \setminus S_1 | S_2) &= f(O' \setminus (S_1 \cup S_2) | S_2) \\
  &\leq \max \left\{ f(O \setminus S_2 | S_1) \right\} \leq \sum_{u_j \in S_2 \setminus O'} f(u_j | S_2^{i-1}).
\end{align*}
\]

Again by Lemma 9 and the fact that \( S_1 = S_1^m \),
\[
(1 - \epsilon) \cdot f(O' \setminus S_2^m | S_1) = (1 - \epsilon) \cdot f(O' \setminus (S_1^m \cup S_2^m) | S_1) \leq \sum_{u_j \in S_1 \setminus O'} f(u_j | S_1^{i-1}).
\]
Next, by submodularity and plugging $k = 1, \ell = 2, T = O' \cap S_1 \subseteq S_1^m$ into Lemma 10,

$$f(O' \cap S_1 | S_2) \leq f(O' \cap S_1 | S_2^m) \leq \sum_{j: u_j \in O' \cap S_1} f(u_j | S_1^{j-1}).$$

By plugging $k = 1, \ell = 2, T = O' \cap S_2^m \subseteq S_2^m$ into Lemma 10,

$$f(O' \cap S_2^m | S_1) = f(O' \cap S_2^m | S_1^m) \leq \sum_{j: u_j \in O' \cap S_2^m} f(u_j | S_2^{j-1}) \leq \sum_{j: u_j \in O' \cap S_2} f(u_j | S_2^{j-1}).$$

Therefore, by combining the above inequalities,

$$f(S_1) + f(S_2) = \sum_{j: u_j \in S_1 \setminus O'} f(u_j | S_1^{j-1}) + \sum_{j: u_j \in O' \cap S_1} f(u_j | S_1^{j-1}) + \sum_{j: u_j \in S_2 \setminus O'} f(u_j | S_2^{j-1}) + \sum_{j: u_j \in O' \cap S_2} f(u_j | S_2^{j-1})$$

$$\geq (1 - \epsilon) \cdot f(O' \cap S_2^m | S_1) + f(O' \cap S_1 | S_2) + (O' \setminus S_1 | S_2) + f(O' \setminus S_2^m | S_1)$$

$$\geq f(O' | S_1) + f(O' | S_2) - \epsilon \cdot f(O' \cap S_2^m | S_1)$$

$$\geq f(O' | S_1) + f(O' | S_2) - \epsilon \cdot \sum_{j: u_j \in O' \cap S_2} f(u_j | S_2^{j-1})$$

$$\geq f(O' | S_1) + f(O' | S_2) - \epsilon \cdot f(S_2).$$

Next, by Lemma 8,

$$(1 + \epsilon)(f(S_1) + f(S_2)) + 2\epsilon \cdot f(O) \geq f(O' | S_1) + f(O' | S_2) + f(O \setminus O' | S_1) + f(O \setminus O' | S_2)$$

$$\geq f(O | S_1) + f(O | S_2)$$

$$= f(O \cup S_1) - f(S_1) + f(O \cup S_2) - f(S_2)$$

$$\geq f(O) - (f(S_1) + f(S_2)).$$

The last two inequalities hold due to submodularity and the fact that $S_1 \cap S_2 = \emptyset$. By rearranging the inequality, $f(S_1) + f(S_2) \geq \frac{1 - 2\epsilon}{1 + \epsilon} f(O)$.

Finally, since each element will be reinserted to the queue $O(\log(n/\epsilon)/\epsilon)$ times and there are $n$ elements, the total number of queries made by Algorithm 4 is $O((n/\epsilon) \log(n/\epsilon))$. 

### 4.3 Obtaining Approximation-Preserving Feasible Solutions

This section is dedicated to turning the sets returned by Algorithm 3 and Algorithm 4 into feasible solutions with the same approximation ratios. The formal procedure is presented as Algorithm 5. To get some intuitions, recall that the solutions of Algorithm 3 and Algorithm 4 become infeasible only when the last added element violates the knapsack constraint. Thus, by throwing away the last element, the solutions become feasible. However, this may cause a huge loss in the approximation ratio if the element has a significant value. By enumerating all feasible solutions of size at most two, we can find elements of large values. As a result, the remaining elements, including the last one added by Algorithm 3 or Algorithm 4, will not be too large. Thus, we can safely throw them away. The overall procedure incurs an additional $n^2$ factor in the query complexity due to the enumeration step.

Formally, we show the following theorem.
Algorithm 5 Enumeration (Threshold) Twin Greedy for Knapsack

1: **Input** \( N, f, c, B \).
2: for all \( E \subseteq N \) with \( |E| \leq 2 \) and \( c(E) \leq B \) do
3: Let \( D = \{u \in N \setminus E \mid f(u \mid E) > \frac{1}{2}f(E)\} \).
4: \( G_E = (\text{Threshold-})\text{Twin-Greedy}(N \setminus (E \cup D), f(\cdot \mid E), c(\cdot), B - c(E)) \).
5: Let \( R_E = G_E \) if \( c(G_E) \leq B - c(E) \) and otherwise \( R_E = G_E \setminus \{u_E\} \), where \( u_E \) is the last element added into \( G_E \).
6: Let \( S_E = E \cup R_E \).
7: return \( \arg\max_E \{f(S_E)\} \).

**Theorem 4.** Algorithm 5 achieves a \( 1/4 \) approximation ratio and uses \( O(n^4) \) queries if Twin Greedy is invoked. It achieves a \( 1/4 - \epsilon \) approximation ratio and uses \( O((n^3/\epsilon) \log(n/\epsilon)) \) queries if Threshold Twin Greedy is invoked.

**Proof.** We first note that \( c(S_E) \leq B \) for every \( S_E \), since by the definition of \( R_E \), we have \( c(R_E) \leq B - c(E) \).

Let \( O \) be the optimal solution. Assume that \( |O| > 2 \), since otherwise \( O \) is a candidate of \( E \) and will be found immediately. Next, order elements in \( O \) in a greedy manner such that \( o_1 = \arg\max_{o \in O} f(o) \), \( o_2 = \arg\max_{o \in O \setminus \{o_1\}} f(o \mid o_1) \), etc. Then, \( \{o_1, o_2\} \) is a candidate of \( E \) and will be visited during the for loop. In the following, we consider the round where \( E = \{o_1, o_2\} \) and show that the corresponding \( S_E \) achieves the desired ratio. This already suffices since the algorithm returns the maximum \( S_E \).

We claim that \( f(o \mid E) \leq f(E)/2 \) for any \( o \in O \setminus E \). This follows from \( f(o \mid E) \leq f(o \mid o_1) \leq f(o_2 \mid o_1) \), \( f(o \mid E) \leq f(o) \leq f(o_1) \) and \( f(o_1) + f(o_2 \mid o_1) = f(E) \). The claim implies that \( D \cap (O \setminus E) = \emptyset \) or equivalently \( O \setminus E \subseteq N \setminus (E \cup D) \). Besides, \( c(O \setminus E) \leq B - c(E) \). Thus, when the Twin Greedy algorithm is invoked, \( f(G_E \mid E) \geq f(O \setminus E \mid E)/4 \).

By the definition of \( R_E \),

\[
f(R_E \mid E) \geq f(G_E \mid E) - f(u_E \mid E)
\geq \frac{1}{4}f(O \setminus E \mid E) - \frac{1}{2}f(E)
= \frac{1}{4}f(O) - \frac{3}{4}f(E).
\]

The first inequality is due to the submodularity of \( f(\cdot \mid E) \). The second inequality holds since \( u_E \in N \setminus (E \cup D) \) and hence \( f(u_E \mid E) \leq f(E)/2 \). The equality holds since \( E \subseteq O \). Finally,

\[
f(S_E) = f(E) + f(R_E \mid E) \geq \frac{1}{4}f(O) + \frac{1}{4}f(E) \geq \frac{1}{4}f(O).
\]

When the Threshold Twin Greedy algorithm is invoked, the theorem follows from the same argument, which we omit for simplicity. \(\square\)

**4.4 A Tight Example for Twin Greedy**

In this section, we present a tight example in Theorem 5, showing that the Twin Greedy algorithm can not reach an approximation ratio better than \( 1/4 \) even under the cardinality constraint. Note that for the cardinality constraint, Twin Greedy always outputs a feasible solution. Thus, the enumeration technique is unnecessary in this case.
Theorem 5. There is an instance of non-monotone submodular maximization under a cardinality constraint such that if we run the Twin Greedy algorithm on this instance, the returned solution has a value of at most $1/4 + o(1)$ of the optimum.

Proof. Given a finite ground set $N$ of $n$ elements, arbitrarily choose two different elements $u_1, u_2 \in N$. For any $S \subseteq N$, let $T = S \setminus \{u_1, u_2\}$. Define a set function $f : 2^N \to \mathbb{R}_+$ as follows.

$$f(S) = \begin{cases} 0, & u_1, u_2 \in S \\ |T|, & u_1, u_2 \notin S \\ 1 + \epsilon + \frac{1}{2}|T|, & \text{otherwise} \end{cases}$$

It is easy to verify that $f$ is non-negative, non-monotone, and submodular.

Assume that the constraint parameter $k = |N| = n$ is an even number. Clearly, the set $N \setminus \{u_1, u_2\}$ is an optimal solution with value $f(N \setminus \{u_1, u_2\}) = k - 2$. On the other hand, the Twin Greedy algorithm will add $u_1$ into $S_1$ and $u_2$ into $S_2$ in the first two rounds. After that, the algorithm may reach such a state that half of the remaining elements $N \setminus \{u_1, u_2\}$ are added into $S_1$ and the other half is added into $S_2$. In this case, $f(S_1) = f(S_2) = 1 + \epsilon + \frac{1}{2}(k - 1) = \frac{k}{4} + \frac{1}{2} + \epsilon$. Hence, the returned solution has a value of at most $1/4 + o(1)$ of the optimum.

Finally, the instance can be easily generalized to arbitrary $k \leq n$ by adding “dummy” elements.

\[ \square \]

5 Deterministic Approximation for Linear Packing Constraints

In this section, we present a deterministic algorithm for submodular maximization under linear packing constraints with a large width. Our algorithm is obtained by combining the multiplicative-updates algorithm [3] for the monotone case and the technique from [15] for dealing with the lack of monotonicity. The multiplicative-updates algorithm is presented in Section 5.1. The overall algorithm is presented in Section 5.2.

5.1 The Multiplicative Updates Algorithm

The multiplicative updates algorithm is depicted as Algorithm 6. It takes a parameter $\lambda$ as input, which is set to $\lambda = e^{\epsilon W}$ in the analysis. It maintains a weight $w_i$ for the $i$-th constraint, which is updated in a multiplicative way. Intuitively, it can be regarded as running a greedy algorithm over a “virtual” knapsack constraint, where element $j$ has a dynamic cost $\sum_{i=1}^{m} A_{ij} w_i$ and the knapsack has a dynamic budget $\sum_{i=1}^{m} b_i w_i$.

Algorithm 6 Multiplicative Updates for Linear Packing Constraints

\begin{verbatim}
1: Input $N, f, A, b, \lambda$
2: $S \leftarrow \emptyset$
3: for $i = 1$ to $m$ do $w_i = 1/b_i$.
4: while $\sum_{i=1}^{m} b_i w_i \leq \lambda$ and $N \setminus S \neq \emptyset$ do
5:     $j \leftarrow \text{arg max}_{j \in N \setminus S} \frac{f(j|S)}{\sum_{i=1}^{m} A_{ij} w_i}$.
6:     if $f(j | S) \leq 0$ then break
7:     $S \leftarrow S \cup \{j\}$.
8:     for $i = 1$ to $m$ do $w_i = w_i \lambda^{A_{ij}/b_i}$.
9: if $Ax_S \leq b$ then return $S$.
10: else return $S \setminus \{j\}$, where $j$ is the last element added into $S$.
\end{verbatim}
Theorem 6 provides a theoretical guarantee for the set $S$ returned by Algorithm 6.

**Theorem 6.** For any fixed $\epsilon > 0$, assume that $W \geq \max\{\ln m / e^2, 1 / \epsilon\}$ and set $\lambda = e^{\epsilon W}$. Then, the set $S$ returned by Algorithm 6 is feasible and satisfies $f(S) \geq \frac{1}{2}(1 - 3\epsilon) \cdot f(S \cup C)$ for any set $C$ satisfying $Ax_C \leq b$. Algorithm 6 uses $O(n^2)$ queries.

**Proof.** We begin the proof by defining some notations. Assume that the algorithm added $t + 1$ elements in total during the `while` loop. For each $r = 1, 2, \ldots, t + 1$, let $S_r$ be the current solution immediately after the $r$-th iteration and $j_r$ be the element selected in the $r$-th iteration. For each $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, t + 1$, let $w_{ir}$ be the value of $w_i$ immediately after the $r$-th iteration and $\beta_r = \sum_{i=1}^{m} b_i w_{ir}$. Let $S$ be the set returned by Algorithm 6.

We first consider the feasibility of $S$. If $Ax_{S_{t+1}} \leq b$, then $S = S_{t+1}$ is certainly feasible. If $Ax_{S_{t+1}} > b$, then $S = S_t$. We will show that $S_t$ is feasible. Let $t'$ be the first iteration such that $S_{t'}$ is infeasible. Namely, $\sum_{j \in S_{t'}} A_{ij} > b_i$ for some $i$. Then, by the value of $w_{it'}$,

$$b_i w_{it'} = b_i w_0 \prod_{j \in S_{t'}} \lambda^{A_{ij}/b_i} = \lambda^{\sum_{j \in S_{t'}} A_{ij}/b_i} > \lambda.$$ 

This implies $\beta_t > \lambda$ and the `while` loop will terminate after this round. It implies that $t' = t + 1$ and therefore $S_t$ is feasible.

Next, we show that $f(S) \geq \frac{1}{2}(1 - \epsilon) \cdot f(S \cup C)$. First consider the case where the `while` loop terminates with $\sum_{i=1}^{m} b_i w_{ir(t+1)} \leq \lambda$. By the above argument, $\sum_{i=1}^{m} b_i w_{ir(t+1)} \leq \lambda$ implies that $S_{t+1}$ is feasible. Thus, $S = S_{t+1}$. If the `while` loop does not break in line 6, then $S = N$ and the theorem holds. If the `while` loop breaks in line 6, by submodularity, $f(C \mid S) \leq \sum_{w \in C \setminus S} f(w \mid S) \leq 0$. Then, $f(S) \geq f(S \cup C)$ and the theorem holds.

Now we only need to consider the case where the `while` loop terminates with $\sum_{i=1}^{m} b_i w_{ir(t+1)} > \lambda$. In this case, the `while` loop does not break in line 6, and $S_t$ is returned. We will show that $S_t$ satisfies the desired property. We proceed by bounding the value of $\beta_t$. For the lower bound, by the definition of $W$ and $\lambda = e^{\epsilon W}$, we have

$$\beta_t e^\epsilon = \sum_{i=1}^{m} b_i w_{it} \cdot (e^{\epsilon W})^{1/W} \geq \sum_{i=1}^{m} b_i w_{ir(t+1)} > e^{\epsilon W}.$$ 

Then, $\beta_t > e^{\epsilon(W-1)}$. For the upper bound, notice that for any $r = 1, 2, \ldots, t$,

$$\beta_r = \sum_{i=1}^{m} b_i w_{ir} = \sum_{i=1}^{m} b_i w_{i(r-1)} \cdot (e^{\epsilon W})^{A_{ijr}/b_i} \\ \leq \sum_{i=1}^{m} b_i w_{i(r-1)} \cdot \left(1 + \frac{\epsilon W A_{ijr}}{b_i} + \left(\frac{\epsilon W A_{ijr}}{b_i}\right)^2\right) \\ \leq \sum_{i=1}^{m} b_i w_{i(r-1)} + (\epsilon W + \epsilon^2 W) \sum_{i=1}^{m} A_{ijr} w_{i(r-1)} \\ = \beta_{r-1} + (\epsilon W + \epsilon^2 W) \sum_{i=1}^{m} A_{ijr} w_{i(r-1)}.$$ 

The first inequality holds since $e^x \leq 1 + x + x^2$ for any $x \in [0, 1]$. The second inequality holds since $WA_{ijr}/b_i \leq 1$ by definition and hence $(\epsilon W A_{ijr}/b_i)^2 \leq \epsilon^2 W A_{ijr}/b_i$. We continue with bounding the
value of $\sum_{i=1}^{m} A_{ij} \cdot w_{i(r-1)}$. By the choice of $j_r$ and submodularity, for any $j \in C \setminus S_{r-1}$,

$$
\frac{f(j_r \mid S_{r-1})}{\sum_{i=1}^{m} A_{ij} \cdot w_{i(r-1)}} \geq \frac{f(j \mid S_{r-1})}{\sum_{i=1}^{m} A_{ij} \cdot w_{i(r-1)}} \geq \frac{f(j \mid S_{t})}{\sum_{i=1}^{m} A_{ij} \cdot w_{i(r-1)}}.
$$

Since $AX_C \leq b$, it follows that

$$
\frac{f(j_r \mid S_{r-1})}{\sum_{i=1}^{m} A_{ij} \cdot w_{i(r-1)}} \geq \frac{\sum_{j \in C \setminus S_{r-1}} f(j \mid S_{t})}{\sum_{i=1}^{m} A_{ij} \cdot w_{i(r-1)}} \geq \frac{f(C \mid S_{t})}{\sum_{i=1}^{m} b_i \cdot w_{i(r-1)}} = \frac{f(C \mid S_{t})}{\beta_{r-1}}.
$$

Therefore,

$$
\sum_{i=1}^{m} A_{ij} \cdot w_{i(r-1)} \leq \frac{f(j_r \mid S_{r-1})}{f(C \mid S_{t})} \cdot \beta_{r-1}.
$$

Plugging it back into the recurrence of $\beta_r$, we obtain that

$$
\beta_r \leq \beta_{r-1} \cdot \left(1 + \frac{(\epsilon W + e^2 W) f(j_r \mid S_{r-1})}{f(C \mid S_{t})}\right) \leq \beta_{r-1} \cdot \exp \left(\frac{(\epsilon W + e^2 W) f(j_r \mid S_{r-1})}{f(C \mid S_{t})}\right).
$$

The last inequality is due to $1 + x \leq e^x$. By expanding the above recurrence and the fact that $\beta_0 = m \leq e^{\epsilon W}$, we have

$$
\beta_t \leq \beta_0 \prod_{r=1}^{t} \exp \left(\frac{(\epsilon W + e^2 W) f(j_r \mid S_{r-1})}{f(C \mid S_{t})}\right) \\
\leq \exp \left(e^2 W + (\epsilon W + e^2 W) \sum_{r=1}^{t} \frac{f(j_r \mid S_{r-1})}{f(C \mid S_{t})}\right) \\
\leq \exp \left(e^2 W + (\epsilon W + e^2 W) \frac{f(S_{t})}{f(C \mid S_{t})}\right).
$$

Combining with $\beta_t \geq e^{(W-1)}$, we obtain that

$$
\frac{\epsilon (W-1) - e^2 W}{\epsilon W + e^2 W} \leq \frac{f(S_{t})}{f(C \mid S_{t})}.
$$

Note that $(\epsilon (W-1) - e^2 W)/(\epsilon W + e^2 W) \geq (1 - 2\epsilon)/(1 + \epsilon) \geq 1 - 3\epsilon$. Thus, we get $f(S_{t}) \geq (1 - 3\epsilon) \cdot f(C \mid S_{t}) = (1 - 3\epsilon) \cdot f(S_{t} \cup C) = f(S_{t})$ and hence $f(S_{t}) \geq \frac{1}{2} (1 - 3\epsilon) \cdot f(S_{t} \cup C)$.

Finally, Algorithm 6 runs at most $n$ rounds and makes $O(n)$ queries at each round. Thus, it makes $O(n^2)$ queries in total.

\[\square\]

### 5.2 The Main Algorithm for Linear Packing Constraints

The multiplicative updates algorithm itself can not produce a solution with a constant approximation ratio, due to the lack of monotonicity. To overcome this difficulty, we apply the technique from [15]. The resulting algorithm is depicted as Algorithm 7. It first invokes the multiplicative updates algorithm to obtain a solution $S_1$. Then it runs the multiplicative updates algorithm again over the remaining elements $N \setminus S_1$ to obtain another solution $S_2$. These two solutions still can not guarantee any constant approximation ratios. To remedy this, the algorithm produces the third solution $S'_1$ by solving the unconstrained submodular maximization problem over $S_1$. This problem can be solved by an optimal deterministic 1/2-approximation algorithm [5] using $O(n^2)$ queries.
Our algorithm finally returns the maximum solution among $S_1$, $S_2$, and $S'_1$. We will show that this achieves a $1/6 - \epsilon$ approximation ratio.

\begin{algorithm}
1: \textbf{Input} $N, f, A, b, \epsilon$.
2: $S_1 \leftarrow$ Multiplicative-Updates($N, f, A, b, e^{W/3}$).
3: $S_2 \leftarrow$ Multiplicative-Updates($N \setminus S_1, f, A, b, e^{W/3}$).
4: $S'_1 \leftarrow$ Unconstrained-Submodular-Maximization($S_1, f$).
5: \textbf{return} $\max\{S_1, S_2, S'_1\}$.
\end{algorithm}

**Theorem 7.** For any fixed $\epsilon > 0$, assume that $W \geq \max\{9 \ln m/\epsilon^2, 3/\epsilon\}$. Algorithm 7 achieves a $1/6 - \epsilon$ approximation ratio and uses $O(n^2)$ queries.

**Proof.** Let $O \in \arg \max\{f(S) : Ax_S \leq b\}$. By Theorem 6, we have

\[ f(S_1) \geq \frac{1}{2}(1-\epsilon) \cdot f(S_1 \cup O) \quad \text{and} \quad f(S_2) \geq \frac{1}{2}(1-\epsilon) \cdot f(S_2 \cup (O \setminus S_1)). \]

If $f(S_1 \cap O) \geq \delta \cdot f(O)$, then $f(S'_1) \geq \frac{1}{2} \cdot f(S_1 \cap O) \geq \frac{\delta}{2} \cdot f(O)$. If $f(S_1 \cap O) \leq \delta \cdot f(O)$, then

\[ f(S_1) \geq \frac{1}{2} \cdot f(S_1 \cup O) \geq \frac{1-\epsilon}{2} \cdot (f(S_1 \cup O) + f(S_1 \cap O) - \delta \cdot f(O)). \]

We thus have that

\[ \max\{f(S_1), f(S_2)\} \geq \frac{1}{2} (f(S_1) + f(S_2)) \]
\[ \geq \frac{1-\epsilon}{4} \cdot (f(S_1 \cup O) + f(S_1 \cap O) + f(S_2 \cup (O \setminus S_1)) - \delta \cdot f(O)) \]
\[ \geq \frac{1-\epsilon}{4} \cdot (f(S_1 \cup S_2 \cup O) + f(O \setminus S_1) + f(S_1 \cap O) - \delta \cdot f(O)) \]
\[ \geq \frac{1-\epsilon}{4} \cdot (f(S_1 \cup S_2 \cup O) + f(O) - \delta \cdot f(O)) \]
\[ \geq \frac{(1-\epsilon)(1-\delta)}{4} \cdot f(O). \]

The third and fourth inequalities hold due to submodularity. The last inequality holds due to non-negativity. Therefore, the approximation ratio of the returned set is at least $\max\{\delta/2, (1-\epsilon)(1-\delta)/4\}$. Let $\delta = (1-\epsilon)/(3-\epsilon)$. We get that the approximation ratio is at least $(1-\epsilon)/6$.

Finally, since both the algorithm from [5] and Algorithm 6 make $O(n^2)$ queries, Algorithm 3 also makes $O(n^2)$ queries in total.

6 Conclusion and Future Work

In this paper, we propose deterministic algorithms with improved approximation ratios for non-monotone submodular maximization under a matroid constraint, a single knapsack constraint, and linear packing constraints, respectively. We also show that the analysis of our knapsack algorithms is tight.

A central open question in this field is whether deterministic algorithms can achieve the same approximation ratio as randomized algorithms. When the objective function is non-monotone, though our algorithms improve the best known deterministic algorithms, their approximation ratios
are still worse than the best randomized algorithms. When the function is monotone, the state-of-the-art deterministic algorithm for the matroid constraint achieves a 0.5008 approximation ratio [7], which is also smaller than the optimal $1 - 1/e$ ratio. It is very interesting to fill these gaps.

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