On the geometry of non-trivially embedded branes

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Abstract

We present a formal supersymmetric solution of type IIB supergravity generalizing previously known solutions corresponding to D3 branes to geometries without an orthogonal split between parallel and transverse directions. The metric is given implicitly as one with respect to which a certain connection is compatible. The case of the deformed conifold is discussed in detail.

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1. Introduction and motivation

The AdS/CFT correspondence in cases with lower supersymmetries involving the two- and three-dimensional AdS spaces is much less understood than the others [1]. In particular, the near-horizon geometry of black strings in simple five-dimensional supergravity (obtained by compactification of M-theory on a Calabi-Yau threefold $X$) is given by $AdS_2 \times S^2$. Here we meet a little puzzle discussed also in [2]. The spectrum of the supergravity theory is defined by $H^2(X, \mathbb{Z})$, and when considering a magnetic string solution, one naturally writes down a string coupled to $(h_{11}(X) - 1)$ vectors and a graviphoton. However as pointed out in [3], the theory on the string is governed by a much larger lattice defined by $H^2(P, \mathbb{Z})$, where $P$ is the four-cycle around which the M-theory fivebrane is wrapped. In particular, this accounts for a very large entropy. This information about the cycle is not reflected in the KK spectrum though. As seen in [4], the target space of the dual $(0, 4)$ conformal field theory is factorized. The coupling to supergravity is governed mostly by the so called universal sector, while the numerous modes making up the theory on the two-dimensional worldsheet are in the entropic sector. Apparently, the usual procedure of compactifying M-theory and then looking for the solutions misses the deformations of the cycle. In doing so we simply find a solution corresponding to the universal sector [1], and schematically we can write

$$[\text{compactification, solution}] = \text{entropic sector}.$$ 

Finding a complete solution with all the modes seems to be extremely hard. Instead, we try to give a description of the 10d geometry corresponding to the situation in [6]: a D-brane wrapping a non-trivial cycle in an internal Calabi-Yau space, shrinking to zero-size at certain points of the moduli space thus turning into a massless BPS black hole from the four dimensional non-compact space point of view [2].

D3 branes on conifold points have been discussed extensively in the AdS literature recently [7,8,9,10]. However only the situation where the branes are transverse to a CY manifold, and can be thought of as spacetime-filling, has been addressed there. We will be concerned with a rather different geometrical setup, the case of a D3 wrapping a 3-cycle in an internal CY threefold.

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1 The full entropy is recovered by loop corrections to supergravity [5].

2 Note that $N$ D3-branes do not form a bound state on the conifold. To ensure the reliability of supergravity we are tacitly assuming that $N$ is large.
In the next two sections, we will outline a procedure for finding formal supersymmetric solutions under some general assumptions about the metric in the absence of D3. Unfortunately, the metric of the solution is only given implicitly as one with respect to which a certain connection is compatible.

As a specific case, we will consider type IIB supergravity on a non-compact Calabi-Yau threefold which we will call the deformed conifold $C(\varepsilon)$. In doing this we are able to avoid the question of extra modes since in our limiting case the three-fold is approximated by $T^*S^3$ and $S^3$ is rigid. The deformed conifold, which will be described in the following in some detail, is topologically a 6d cone over an $S^2 \times S^3$ base, whose apex is replaced by an $S^3$.

Before turning to the concrete example of D3 on a shrinking $S^3$ cycle in section 5, we discuss the geometry of the singular and deformed conifolds of \cite{11}. Using the machinery of coset-space geometry \cite{12,13}, we are able to give an explicit form of the deformed conifold metric presented implicitly in \cite{11}. A possible generalization to M2 and M5 cases is discussed in section 6.

2. Branes on cycles

We present a method for constructing formal supersymmetric solutions of type IIB supergravity corresponding to D3 branes, which generalizes previously known solutions to geometries without an orthogonal split between directions parallel and transverse to the brane. In particular, we do not assume that the D3 is spacetime-filling but our method covers this situation as a special case.

In the presence of the D3 the 10d geometry will get deformed to account for the back reaction due to the brane. This back reaction is captured by the warping factor, a function of the coordinates transverse to the brane. There is considerable amount of literature on the subject of supergravity solutions corresponding to branes (see e.g. \cite{14,15}). These solutions assume that in the absence of D3 there is an orthogonal split of the 10d metric along parallel and transverse to the brane directions:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n$$  \hspace{1cm} (2.1)

where $\eta_{\mu\nu}$ is the (flat) metric on the cycle around which the D3 will wrap (which in that case is the whole 4d Minkowski spacetime) and $g_{mn}(y)$ is the transverse metric. In the presence of the D3, the geometry is modified

$$ds^2 = \Delta_{\perp}(y)\eta_{\mu\nu}dx^\mu dx^\nu + \Delta_{\parallel}(y)g_{mn}(y)dy^m dy^n$$  \hspace{1cm} (2.2)
where $\Delta_\perp(y)$, $\Delta_{\parallel}(y)$ are the warping factors (which turn out to be related) and are functions only of the coordinates transverse to the brane.

Here we will not assume that there is the “nice” split of the form (2.1). Instead we will take the 10d metric in the absence of D3 to be of the form

$$ds^2 = g_{\mu\nu}(x,\theta)dx^\mu dx^\nu + f^2(U)dU^2 + \sum_{a=1}^{5} (e(x,\theta,U)^{a}_{\mu}dx^\mu + e(x,\theta,U)^{a}_{i}d\theta^i)^2$$  \hspace{1cm} (2.3)

where $\{x^\mu; \mu = 0, \ldots 3\}$ are the coordinates parametrizing a non-trivially embedded cycle $\mathcal{C}$, $U$ is a “radial” coordinate in the transverse space and $\{\theta^i; i, \ldots 5\}$ are “angular” coordinates in the transverse space. The metric on $\mathcal{C}$ is $g_{\mu\nu}$ and it does not depend on $U$. Note that $e(x,\theta,U)^{a}_{\mu}$ encodes the deviations from orthogonally-split geometries. We will further assume that

$$e(x,\theta,U = 0)^{a}_{\mu} = e(x,\theta,U = 0)^{a}_{i} = 0,$$  \hspace{1cm} (2.4)

so that $\mathcal{C}$ is at $U = 0$. If we wish, we may consider $U$ as a collective label for a set of “radial ” coordinates $U_1, U_2, \ldots$ which enter metric (2.3) as $f_1^2(U_1)dU_1^2 + f_2^2(U_2)dU_2^2 + \ldots$

For simplicity we will drop the $\theta$ dependence of $g_{\mu\nu}$ in the following. All our arguments of section 3 go through for $\theta$-dependent $g_{\mu\nu}$ as well. Let us remark that if we keep the $\theta$ dependence, it appears that in the flat D3 limit we may recover warped $AdS_5 \times wS^5$ products. The possibility of such supergravity vacua has been known for some time [16,17], however these haven’t appeared as brane near-horizon limits so far. This discussion may provide a brane realization of such vacua.

The metric of (2.3) is of some generality. A trivial example would be the case where the 10d metric is that of a direct sum of 4d Minkowski plus a 6d cone, with $U$ being the distance (in the ten-dimensional sense) from the apex. Another example, which will be discussed in the following, is the geometry of the deformed conifold near the $S^3$ at the apex.

3. The solution

In this section we will discuss how does (2.3) change in the presence of a D3 along $\mathcal{C}$. As already emphasized, the solution will be given implicitly: the metric in the presence of D3 is the metric with respect to which the connection of equation (3.8) below, is metric compatible. This statement makes sense since for every torsion-free connection with $SO(N)$ holonomy (where $N$ is the dimension of the manifold) there is an essentially
unique metric with respect to which the connection is compatible. We can see this as follows: take the metric at a given point to be some constant symmetric $N \times N$ matrix. Parallel-transport it using the connection to define the metric at any other point. The absence of inconsistencies under parallel transport along closed loops is equivalent to the requirement of $SO(N)$ holonomy. The metric constructed in this way is defined up to rigid $GL(N, \mathbb{R})$ coordinate transformations.

We want to warn the reader that as we do not have the charge distribution explicitly, nothing excludes the possibility that this solution corresponds to a completely “smeared” D3. In that case it would be wrong to think of the D3 as wrapping $\mathcal{C}$. It is more correct to say that $\mathcal{C}$ will be identified with the horizon in the presence of the D3.

3.1. Conventions

- $\mu, \nu$ are curved indices for the directions along the D3.
- $m, n$ are curved indices for the coordinates transverse to the D3 including $U$.
- $i, j$ are curved indices for the coordinates transverse to the D3 excluding $U$.
- $M, N$ are ten-dimensional curved indices.
- $\alpha, \beta$ are flat indices for the directions along the D3.
- $a, b$ are flat indices corresponding to the directions transverse to the D3 excluding $U$.
- $\bullet$ is the flat index corresponding to $U$.
- $A, B$ are ten-dimensional flat indices.

3.2. The solution

It will be useful to give the coframe version of (2.3):

$$ds^2 = \sum_{\alpha=0}^{3} (e^\alpha)^2 + \sum_{a=1}^{5} (e^a)^2 + (e^\bullet)^2 \quad (3.1)$$

where

- $e^\alpha = e(x)^\alpha_\mu dx^\mu$; $e^\bullet = e(U)^\bullet_\mu dU = f(U)dU$
- $e^a = e(x, \theta, U)^a_\mu dx^\mu + e(x, \theta, U)^a_i d\theta^i \quad (3.2)$

The fields of IIB are the graviton $g_{MN}$, a complex scalar $\tau$ parametrizing an $SL(2, \mathbb{R})/U(1)$ coset space, a pair of two-forms $B^1_{MN}$ which form an $SL(2, \mathbb{R})$ doublet, a self-dual four-form $A^{(4)}$ with field strength $F^{(5)}$, and two complex-Weyl fermions: a gravitino $\psi_M$ and a dilatino $\lambda$.

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3 We would like to thank G. Moore for explaining this to us.
The supersymmetry transformations are parametrized by a complex-Weyl spinor $\epsilon$ and, in a background with all fields set to zero except for the graviton and the four-form, only the gravitino transformation is not identically zero

$$\delta \epsilon \psi_A = D(\Omega)_A \epsilon + \frac{i}{4 \times 5!} \Gamma^{A_1 \ldots A_5} F_{A_1 \ldots A_5} \Gamma_A \epsilon$$

(3.3)

where

$$D(\Omega)_A = \partial_A + \frac{1}{4} \Omega_A^{BC} \Gamma_{BC}; \quad \partial_A = e_A^M \partial_M$$

(3.4)

and $\Omega^A_B$ is the connection one-form corresponding to the 10d metric in the presence of the D3.

Our ansatz for the five-form is given in terms of one function of $U$, $C(U)$

$$F_{a_0 a_1 a_2 a_3} = \varepsilon_{a_0 a_1 a_2 a_3} C(U); \quad F_{a_1 a_2 a_3 a_4 a_5} = \varepsilon_{a_1 a_2 a_3 a_4 a_5} \cdot C(U)$$

(3.5)

with all other components equal to zero. We then find

$$\Gamma^{A_1 \ldots A_5} F_{A_1 \ldots A_5} \Gamma_a = -i 5! C(U) \Gamma_a (\rho^{(6)} + \rho^{(4)})$$

(3.6)

$$\Gamma^{A_1 \ldots A_5} F_{A_1 \ldots A_5} \Gamma_a = i 5! C(U) \Gamma_a (\rho^{(6)} + \rho^{(4)})$$

where

$$\rho^{(4)} := i \Gamma^{\alpha=0 \ldots \Gamma^{\alpha=3}}; \quad \rho^{(6)} := -i \Gamma^{a=1 \ldots \Gamma^{a=5}}$$

(3.7)

are the “parallel” and “transverse” chirality operators. Let $\omega^A_B$ be the connection one-form associated to the metric (3.1). Our ansatz for the 10d metric in the presence of the D3 will be given implicitly by requiring that it be associated to the connection $\Omega^A_B$ given by

$$\Omega^A_B = \omega^A_B + e^A \partial_B \Delta(U)_{(A)} - e_B \partial^A \Delta(U)_{(B)}$$

(3.8)

where

$$\Delta(U)_{(A)} = \begin{cases} \Delta(U)_{||}, & A = \alpha \\ \Delta(U)_{\perp}, & A = a \end{cases}$$

(3.9)

are two warping factors.

In [7] a relation identical to (3.8) holds, arising from a rescaling of the vielbeins $e^A \rightarrow \Delta_{(A)} e^A$. Our case however is very different as it does not imply that the associated metrics are related by such a rescaling. The reason is that, as follows from (3.1), (3.2), the connection $\omega^A_B$ generally has non zero components of the form $\omega^a_{\alpha a}$ mixing transverse with parallel directions.
We will now assume that
\[ \rho^{(4)} \epsilon = \rho^{(6)} \epsilon = \epsilon \]  
(3.10)
the susy transformations then read:
\[
0 = [D(\omega) + \Gamma_{\alpha\bullet}(\partial_U \ln \Delta_{||} + \frac{1}{2} C)] \epsilon \\
0 = [D(\omega)_{a} - \frac{1}{2} C\delta_{a\bullet} + \Gamma_{a\bullet}(\partial_U \ln \Delta_{\perp} - \frac{1}{2} C)] \epsilon
\]
(3.11)
Setting
\[
C(U) = 2\partial_U \ln \Delta(U)_{\perp} \\
\Delta(U)_{\perp} = \frac{1}{\Delta(U)_{||}} \\
\epsilon = \Delta(U)_{||} \hat{\epsilon}
\]
(3.12)
equation (3.11) reduces to
\[
D(\omega)_{A} \hat{\epsilon} = 0,
\]
(3.13)
i.e. the solution preserves some supersymmetry provided the geometry (3.1) in the absence of D3 admits a covariantly constant spinor. The integrability of (3.13) is equivalent to the requirement of Ricci-flatness for the geometry in the absence of D3.
\[
Ric(\omega)_{AB} = 0
\]
(3.14)
To check the consistency of our ansatz with Einstein equations
\[
Ric(\Omega)_{AB} = \frac{1}{4 \times 4!} F_{A1...A4} F_{B_{1}...A_{4}}
\]
(3.15)
we simply substitute (3.8) into (3.15) taking (3.14) into account. The result is [18, 7]:
\[
D(\omega)_{U} D(\omega)^{U} \Delta_{\perp} (U)^{2} = 0
\]
(3.16)
This is the condition that \( \Delta_{\perp} (U)^{2} \) is harmonic. In proving the above we used the fact that
\[
\omega^{\bullet}_{\alpha} = 0
\]
(3.17)
We can see this as follows. From (3.1) we get
\[
0 = d(e(U)^{U} \partial U) = de^\bullet = -\omega^{\bullet}_{\alpha} e_{\alpha} - \omega_{a} e^{a}
\]
(3.18)
therefore the only possibly nonzero components of \( \omega^{\bullet}_{\alpha} \) are of the form \( \omega^{\bullet}_{\alpha} \). On the other hand
\[
de^{\alpha} = -\omega^{\alpha}_{\bullet} e^\bullet + \ldots = -\omega_{\alpha}^{\bullet} e^{\alpha} \wedge e^\bullet + \ldots
\]
(3.19)
But since \( \partial_U e^{\alpha} = 0 \), \( de^{\alpha} \) cannot have a piece proportional to \( dU \) and we conclude that \( \omega^{\bullet}_{\alpha} = 0 \).
4. Geometry of conifolds

The main purpose of this section is to provide a specific example of the geometrical setup under which our solution of the previous section is valid. Indeed we show (see subsec. 4.5) that the near-horizon limit of the deformed conifold is a particular case to which our method applies.

None of this section is new. The results are in principle contained in previous works [11,12,13]. However we want to draw the attention of the reader to two points. Equation (4.15) below, contains a term (the one proportional to $B$) which is usually omitted from discussions in the literature related to $T^{1,1}$ spaces. However this term appears naturally in the metric of the deformed conifold, explicitly presented in equation (4.35). In section 5 of [11] the metric is given implicitly in terms of 7 variables (one radius and 6 Euler angles) in a form which makes it difficult to distinguish 6 independent ones. For these reasons we think the discussion of this section is useful.

4.1. $T^{p,q}$ spaces

A cone $C_{d+1}$ in $d + 1$ spacetime dimensions over a d-dimensional base $X_d$ is given by

$$ds^2 = d\rho^2 + \rho^2 g_{ij} dx^i dx^j$$  \hspace{1cm} (4.1)

The cone $C_{d+1}$ has the property that the vector $\frac{\partial}{\partial \rho}$ is conformally killing. The metric $g_{ij}$; $i,j = 1 \ldots d$ determines the geometry of the base. $C_{d+1}$ is Ricci-flat iff $X_d$ is Einstein with cosmological constant $(d - 1)$ and is irregular at $\rho = 0$ unless $X_d = S^d$.

Our situation corresponds to $d=5$ and we will consider the base to be a $T^{1,1}$ space. A $T^{1,1}$ space is a particular example of $T^{p,q}$ spaces [19]. These can be thought of as $U(1)$ fibrations over $S^2 \times S^2$. Let $0 \leq \phi_i \leq 2\pi , 0 \leq \theta_i \leq \pi , i = 1, 2$ parametrize the two $S^2$ and let $0 \leq \psi_i \leq 4\pi$ be the coordinate on the $U(1)$ fiber. The line element of $T^{p,q}$ is then given by

$$ds^2 = \lambda_1 (d\psi + p\cos \theta_1 d\phi_1 + q\cos \theta_2 d\phi_2)^2$$
$$+ \lambda_2 (\sin^2 \theta_1 d\phi_1^2 + d\theta_1^2) + \lambda_3 (\sin^2 \theta_2 d\phi_2^2 + d\theta_2^2)$$  \hspace{1cm} (4.2)

where the first term on the rhs is the vertical displacement along the fibre and the other two terms are the line elements on the $S^2$’s. By “forgetting” one of the $S^2$’s the $T^{1,1}$ space can be thought of as an $S^3$ fibration over $S^2$ -the base being the $S^2$ we “forget” and
the fibre being a $U(1)$ fibration over the other $S^2$. This fibration is actually trivial and therefore $T^{1,1}$ is topologically $S^2 \times S^3$.

If the following algebraic conditions are met

$$
\Lambda \lambda_1 = \frac{p^2}{2} \left( \frac{\lambda_1}{\lambda_2} \right)^2 + \frac{q^2}{2} \left( \frac{\lambda_1}{\lambda_3} \right)^2
$$

$$
= \left( \frac{\lambda_1}{\lambda_2} \right)^2 - \frac{p^2}{2} \left( \frac{\lambda_1}{\lambda_2} \right)^2
$$

$$
= \left( \frac{\lambda_1}{\lambda_3} \right)^2 - \frac{q^2}{2} \left( \frac{\lambda_1}{\lambda_3} \right)^2
$$

(4.3)

equation (1.2) describes an Einstein manifold of cosmological constant $\Lambda$.

In the case $p = q = 1$ and $\Lambda = 4$ the cone over $T^{1,1}$ is Ricci-flat and (4.3) implies

$$
\lambda_1 = \frac{1}{9}; \quad \lambda_2 = \lambda_3 = \frac{1}{6}
$$

(4.4)

The spaces $T^{p,q}$ also have a coset description as $SU(2) \times SU(2)/U(1)$, which is another way to see the $S^2 \times S^3$ topology. It will pay off to make a digression on the geometry of coset spaces which will eventually help us give a useful description of the deformed conifold. For a more comprehensive account one should consult the literature [12,13].

4.2. The geometry of coset spaces.

In this section we use techniques of coset spaces to give a generalization of the metric (1.2). This generalized metric for $T^{1,1}$ will appear naturally in the following when we discuss the deformed conifold.

Consider a Lie group $G$ and a subgroup $H \subset G$ generated by $\{ H_i; i = 1 \ldots dimH \}$ such that $G$ is generated by $\{ H_i, E_a; a = 1 \ldots dimG - dimH \}$ and

$$
[H_i, H_j] = c_{ij}^k H_k;
$$

$$
[H_i, E_a] = c_{ia}^b E_b;
$$

$$
[E_a, E_b] = c_{ab}^d E_d + c_{ab}^i H_i
$$

(4.5)

We call left cosets the elements of the form $g.H$, $g \in G$. To parametrize the coset we choose a particular group element in each coset which we call the coset representative $L(\phi^a)$.

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4 There are various way to see this. See for example [8].
Here \( \{\phi^\alpha, \alpha = 1 \ldots \dim G - \dim H\} \) are coordinates on \( G/H \). The element \( L^{-1}\partial_\alpha L \), where \( \partial_\alpha := \partial/\partial\phi^\alpha \), is in the Lie algebra of \( G \). The expansion
\[
L^{-1}\partial_\alpha L = e^a_\alpha E_a + \omega^i_\alpha H_i
\] (4.6)
defines the vielbein \( e^a_\alpha(\phi) \) and the \( H \)-connections \( \omega^i_\alpha(\phi) \). Coset manifolds \( G/H \) have at least an isometry \( G' \times N(H)/H \), where the latter factor is the normalizer (the largest subgroup of \( G \) of which \( H \) is a normal subgroup) and \( G = G' \times (U(1) \text{ factors common to } G \text{ and to } N(H)/H) \).

A metric on \( G/H \) preserving the isometries is given by
\[
g_{\alpha\beta} = h_{ab} e^a_\alpha e^b_\beta
\] (4.7)
where \( h_{ab} \) is an \( H \)-invariant tensor
\[
c_{ia}c_{ub} + c_{ib}c_{ac} = 0
\] (4.8)
As an application, we can reproduce the metric (4.2) for \( p = q = 1 \) as follows. Parametrizing using Euler angles the group element of \( SU(2) \times SU(2) \) can be written as
\[
e^{i\sigma_3\phi_1/2} e^{i\sigma_2\theta_1/2} e^{i\sigma'_3\phi_2/2} e^{i\sigma'_2\theta_2/2} e^{i(\sigma_3 + \sigma'_3)(\psi_1 + \psi_2)/4} e^{i(\sigma_3 - \sigma'_3)(\psi_1 - \psi_2)/4}
\] (4.9)
where \( \{\sigma_i\} \) and \( \{\sigma'_i\} \) obey the algebra of the Pauli matrices with \( [\sigma, \sigma'] = 0 \). We take the coset representative to be
\[
L = e^{i\sigma_3\phi_1/2} e^{i\sigma_2\theta_1/2} e^{i\sigma'_3\phi_2/2} e^{i\sigma'_2\theta_2/2} e^{i(\sigma_3 + \sigma'_3)\psi/2}
\] (4.10)
Let
\[
E_{1,2} = i/2 \sigma_{1,2}; \quad E_{3,4} = i/2 \sigma'_{1,2};
E_5 = i/4(\sigma_3 + \sigma'_3); \quad H = i/4(\sigma_3 - \sigma'_3)
\] (4.11)
From (4.10) we get
\[
e^1 = -\sin\theta_1 d\phi_1
\]
\[e^2 = d\theta_1
\]
\[e^3 = \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2
\]
\[e^4 = \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2
\]
\[e^5 = d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2
\] (4.12)
Since the coset representation \( c_{ia}^b \) of \( H \) (cf. (4.5)) is block diagonal
\[
c_{ia}^b = \frac{i}{2} \begin{pmatrix} -\sigma_2 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]  
the most general \( H \)-invariant tensor \( h_{ab} \) is of the form
\[
h_{ab} = \begin{pmatrix} A\mathbb{I}_2 & B\mathbb{I}_2 & 0 \\ B\mathbb{I}_2 & C\mathbb{I}_2 & 0 \\ 0 & 0 & D \end{pmatrix}
\]  
Substituting to (4.7) we get the \( T^1 \), \( T^2 \) metric
\[
ds^2 = D(e^5)^2 + A((e^1)^2 + (e^2)^2) + C((e^3)^2 + (e^4)^2) + 2B(e^1e^3 + e^2e^4)
\]
\[
= D(d\psi + \cos\theta_1d\phi_1 + \cos\theta_2d\phi_2)^2 + A(\sin^2\theta_1d\phi_1^2 + d\theta_1^2) + C(\sin^2\theta_2d\phi_2^2 + d\theta_2^2)
\]
\[
+ 2B[\cos\psi(d\theta_1d\theta_2 - d\phi_1d\phi_2\sin\theta_1\sin\theta_2) + \sin\psi(\sin\theta_1d\phi_1d\theta_2 + \sin\theta_2d\phi_2d\phi_1)]
\]  
Note that for \( B \neq 0 \) the metric above cannot be Einstein. Indeed one finds for the \( a = 2, b = 4 \) component of the Ricci tensor \( R_{ab} \)
\[
R_{24} = \frac{-B^2(\cos\theta_1 - \cos\theta_2)\csc\theta_1\csc\theta_2}{8(A - B)^{3/2}(A + B)^{3/2}}
\]  
For \( B = 0 \) the metric reduces to the standard metric (4.2) with \( p = q = 1 \). For simplicity we will consider the case \( A = C \). It is useful to make a redefinition
\[
e^a \rightarrow g^a := P^a_{\ b}e^b; \ P := \begin{pmatrix} \frac{1}{\sqrt{2}}\mathbb{I}_2 & -\frac{1}{\sqrt{2}}\mathbb{I}_2 & 0 \\ \frac{1}{\sqrt{2}}\mathbb{I}_2 & \frac{1}{\sqrt{2}}\mathbb{I}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]  
In this basis (4.13) becomes diagonal
\[
ds^2 = D(g^5)^2 + (A + B)[(g^3)^2 + (g^4)^2] + (A - B)[(g^1)^2 + (g^2)^2]
\]  
We remark that the first two terms describe a squashed \( S^3 \). Indeed if we define
\[
T := L_1\sigma_1L_2^\dagger\sigma_1
\]  
we find
\[
d\Omega_3^2 := \frac{1}{2}Tr(dT^\daggerdT) = \frac{1}{2}(g^5)^2 + (g^3)^2 + (g^4)^2
\]  
Since \( T \) is an \( SU(2) \) matrix, the metric above is the standard round sphere metric. Comparing with (4.18) we conclude that the first two terms are the line element of a (squashed) \( S^3 \). Moreover the last two terms
\[
ds_2^2 := (g^1)^2 + (g^2)^2
\]  
should describe a surface which is topologically an \( S^2 \) fibered over the squashed \( S^3 \).
4.3. The Ricci-flat cone over $T^{1,1}$ as a Kähler manifold.

Before we come to the description of the deformed conifold let us summarize some of the results of [11]. In particular we will see how the Ricci-flat cone over $T^{1,1}$ can be thought of as a singular, non compact Calabi-Yau threefold.

Consider the cone in $\mathbb{C}^4$ given by

$$\sum_{A=1}^{4} (w^A)^2 = 0 \quad (4.22)$$

Let us define a radial coordinate $\rho$ by

$$\rho^2 = trWW^\dagger \quad (4.23)$$

where $W := \frac{1}{\sqrt{2}} (w^i \sigma_i + iw^4 \mathbb{I}_2)$. The base of the cone is described by

$$detW = 0; \quad \rho^2 = constant \quad (4.24)$$

A Kähler metric deriving from an $SU(2) \times SU(2)$-invariant Kähler potential $K(\rho^2)$ reads

$$ds_C^2 = |tr W^\dagger dW|^2 K(\rho^2)'' + tr (dW^\dagger dW) K(\rho^2)' \quad (4.25)$$

Ricci-flatness determines the Kähler potential to be proportional to $\rho^{4/3}$.

Let us parametrize (4.24) as

$$W = \rho Z; \quad Z = L_1.Z^{(0)}.L_2^\dagger \quad (4.26)$$

where $L_i, i = 1, 2$ are $SU(2)$ matrices. In terms of Euler angles

$$L_j = \begin{pmatrix} \cos \theta_j e^{i(\psi_j + \phi_j)/2} & -\sin \theta_j e^{-i(\psi_j - \phi_j)/2} \\ \sin \theta_j e^{i(\psi_j - \phi_j)/2} & \cos \theta_j e^{-i(\psi_j + \phi_j)/2} \end{pmatrix} \quad (4.27)$$

and

$$Z^{(0)} = \frac{1}{2} (\sigma_1 + i\sigma_2) \quad (4.28)$$

Substituting to (4.25) (after a redefinition of the radial coordinate and by setting $\psi := \psi_1 + \psi_2$) we find

$$ds_C^2 = (d\rho)^2 + ds_X^2 \quad (4.29)$$

with $ds_X^2$ the $T^{1,1}$ metric previously given in (4.2), for the unique choice of constants $\lambda_1 = 1/9, \lambda_2 = \lambda_3 = 1/6$ which give a Ricci-flat metric on the cone. The fact that the base is $T^{1,1}$ can also be seen directly from (4.20) by noting that there is a transitive $SU(2) \times SU(2)$ action with a $U(1)$ stabilizer embedded symmetrically in the two $SU(2)$ factors

$$L_1 \rightarrow L_1 U; \quad L_2 \rightarrow L_2 U^\dagger; \quad U := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in U(1) \quad (4.30)$$
4.4. The deformed conifold: smooth noncompact $CY_3$

In this section we will describe the deformation of the conifold. The apex is replaced by an $S^3$. Insisting on Ricci-flatness we get a smooth noncompact $CY_3$. The $\rho = constant \neq \varepsilon$ surfaces are still $T^{1,1}$ spaces whose geometry is described by the generalized metric (4.13). We also examine the geometry near the apex.

Consider deforming (4.22) to

$$detW = -\varepsilon^2/2$$

(4.31)

We can again define a radial coordinate as in (4.23) but now $\rho$ is bounded below by $\varepsilon$. We can parametrize the $\rho = constant$ surfaces by

$$W_{\varepsilon} = \rho Z_{\varepsilon}; \quad Z_{\varepsilon} = L_1Z_{\varepsilon}^{(0)}.L_2^\dagger$$

(4.32)

where $L_i$’s are as before and

$$Z_{\varepsilon}^{(0)} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}; \quad a = \frac{1}{2} \left( \sqrt{1 + \frac{\varepsilon^2}{\rho^2}} + \sqrt{1 - \frac{\varepsilon^2}{\rho^2}} \right); \quad b = \frac{\varepsilon^2}{2\rho^2}a^{-1}$$

(4.33)

For $\rho \neq \varepsilon$ there is again a transitive $SU(2) \times SU(2)$ action with a $U(1)$ stabilizer. For $\rho = \varepsilon$ however, the stabilizer is enhanced to the whole of $SU(2)$

$$L_1 \rightarrow L_1U; \quad L_2 \rightarrow L_2\sigma_1U\sigma_1,$$

(4.34)

where $U \in SU(2)$. The surfaces $\rho = constant$ are again $T^{1,1}$ spaces except for the surface $\rho = \varepsilon$ which is an $S^3$.

The metric is obtained by substituting in (4.25) $W$ given by (4.32). The result is

$$ds^2 = [(\rho^2\gamma' - \gamma)(1 - \frac{\varepsilon^4}{\rho^4}) + \gamma] \left( (d\rho)^2 \frac{1}{\rho^2(1 - \frac{\varepsilon^4}{\rho^4})} + \frac{1}{4}(d\psi + \cos\theta_1d\phi_1 + \cos\theta_2d\phi_2)^2 \right)$$

$$+ \frac{\gamma}{4}(\sin^2\theta_1d\phi_1^2 + d\theta_1^2 + \sin^2\theta_2d\phi_2^2 + d\theta_2^2)$$

$$+ \gamma\frac{\varepsilon^2}{2\rho^2}[\cos\psi(d\theta_1d\theta_2 - d\phi_1d\phi_2\sin\theta_1\sin\theta_2) + \sin\psi(\sin\theta_1d\phi_1d\theta_2 + \sin\theta_2d\phi_2d\theta_1)]$$

(4.35)

\footnote{In the original version of this paper this equation had an error. We would like to thank I. Klebanov and M. Strassler for pointing it out and also for pointing out the errors it propagated to the next sections.}
where $\gamma := \rho^2 K'(\rho^2)$ and $\gamma' := \gamma'(\rho^2)$. The metric on the $\rho = \text{constant}$ slice is of the generalized form (4.13).

Requiring Ricci-flatness and correct asymptotic behaviour of the metric (4.35) leads to the following differential equation

$$\rho^2(\rho^4 - \epsilon^4)(\gamma^3)' + 3\epsilon^4\gamma^3 - 2\rho^8 = 0 \quad (4.36)$$

The general solution reads

$$\gamma = (c + \frac{\epsilon^4}{2}(sinh2\tau - 2\tau))^{1/3}(tanh\tau)^{-1} \quad (4.37)$$

where $c$ is a constant. In the limit $\rho/\epsilon \to \infty$ the constant $c$ can be dropped and the solution asymptotes its “cone” value. For $c \neq 0$, $\gamma$ diverges at $\rho = \epsilon$. From now on we set $c$ to zero.

4.5. The near-horizon limit $\rho \to \epsilon$

Let us examine in more detail the limit $\rho \to \epsilon$. We make a change of variables

$$\delta = \rho - \epsilon \quad (4.38)$$

so that

$$0 \leq \delta < \infty \quad (4.39)$$

We find that the metric (4.35) takes the form:

$$ds^2 \sim R_\epsilon^2[(dv)^2 + d\Omega_3^2 + \frac{v^2}{2}ds_2^2] \quad (4.40)$$

where

$$v := \sqrt{\frac{2\delta}{\epsilon}} \to 0 \quad (4.41)$$

and

$$R_\epsilon := \frac{1}{\sqrt{2}}\left(\frac{2\epsilon^4}{3}\right)^{1/6} \quad (4.42)$$

is the radius of the $S^3$ on the apex, $d\Omega_3^2$ is the round sphere element defined in (4.20) and $ds_2^2$ was defined in (4.21) and describes a fibre of $S^2$ topology.
5. D3 on $S^3$

We will now consider the near-horizon limit in the ten-dimensional sense $r \sim 0; \; \rho \sim \varepsilon$. The ten-dimensional metric in the absence of D3 is of the form (2.3), and our solution-generating method applies, only near the $S^3$ at the apex of the deformed conifold. We therefore want to emphasize that we only have a metric (implicitly) for the geometry near the horizon. Presumably there is a complete solution, corresponding to a D3 whose near-horizon limit coincides with the one given here but we were not able to obtain it. Of course the remark at the beginning of section 3 applies here as well: as we do not have the charge distribution explicitly, nothing excludes the possibility that this solution corresponds to a completely “smeared” D3.

We define a ten-dimensional radial coordinate $U$ and an angle $\theta$ by

$$
\begin{align*}
    r &= U \cos \theta; \quad \nu = \frac{U}{R_\varepsilon} \sin \theta
\end{align*}
$$

where $\nu$ was defined in (4.41). Taking (4.40) into account, we see that the metric is of the form (3.1) as advertised

$$
\begin{align*}
    ds^2 &= -dt^2 + R_\varepsilon^2 d\Omega_3^2 \\
        &\quad + dU^2 + U^2 \left( d\theta^2 + \cos^2 \theta d\Omega_2^2 + \frac{1}{2} \sin^2 \theta ds_2^2 \right)
\end{align*}
$$

where the first line contains the directions parallel to the D3 and the second line contains the transverse geometry. As already remarked below (4.21), $ds_2^2$ is the line element of an $S^2$ fibred over an $S^3$. The vielbein $\tilde{e}^A$ of the 10d metric in the presence of D3 has to be compatible with the connection given in (3.8) as already explained. In particular if we define

$$
    h^A := \tilde{e}^A - e^A
$$

we have

$$
D(\omega)h^A + e^A \wedge d\ln \Delta(A) + e^A \wedge h^B \partial_B \ln \Delta(A) + e_B \wedge h^B\partial^A \ln \Delta(B) = 0
$$

The geometry in the presence of D3 is given by equations (5.4), (3.16), but we will not be more explicit here. However the case of the shrinking cycle limit $\varepsilon \to 0 \ (R_\varepsilon \to 0)$ of the near-horizon geometry and the case of the flat D3 limit can be analyzed explicitly.
5.1. The $\varepsilon \to 0$ limit of the near-horizon geometry

As we see from (4.40), in the $\nu \to 0; \varepsilon \to 0 \ (R_\varepsilon \to 0)$ limit the near-horizon geometry becomes effectively four-dimensional and is split orthogonally between the directions parallel to the brane (the time direction) and the transverse directions. Equation (3.16) reduces to the harmonic condition in three spatial dimensions and we recover $AdS_2 \times S^2$ as usual.

5.2. The flat D3 limit.

We can examine the limit where the D3 is seen as flat, i.e. for $R_\varepsilon \to \infty$ with $U$ fixed. All curvatures vanish in this limit. Moreover, for $\delta \to \infty$ we are moving away from the brane and we expect to recover ten dimensional Minkowski space as a solution. Indeed in this case we easily see that $\Delta \to \text{constant}$.

6. M-branes, $M^{pq}$ spaces and T-duality

We conclude with a brief discussion on higher-dimensional conifolds. Indeed one can also consider conifolds with seven-dimensional bases in $M$-theory - as has been done extensively in the eighties in compactifications of the eleven-dimensional supergravity [20] but also very recently. We shall concentrate only on the example that is directly related to our previous discussion, namely $M^{10}$ (and its “T-dual” $M^{01}$). Just as $T^{p,q}$ these are constructed as $U(1)$ bundles [21,22]:

$$M^{pq} = M^{pq0} = \frac{S^5 \times S^3}{U(1)}.$$  (6.1)

Since odd spheres can be thought of as $U(1)$ bundles over projective spaces, the factoring leads to identification of the two fibers in (6.1) and as a result $M^{pq}$ can be thought as a $U(1)$ bundle over $\mathbb{CP}^2 \times S^2$ with the topology depending on the ratio of $p$ and $q$ (homotopically all these spaces are the same). While in general the isometry group is $SU(3) \times SU(2) \times U(1)$, we are interested in cases with larger symmetry - $SO(6) \times SO(3)$ for $M^{10} = S^5 \times S^2$ and $SU(3) \times SU(2) \times SU(2)$, for $M^{01} = \mathbb{CP}^2 \times S^3$.

A cone over $M^{10}$, $C(M^{10})$, can be almost everywhere described by a quadric in $\mathbb{CP}^6$ with two real planes intersecting it. Trying to resolve the singularity, we end up replacing the apex of the cone either by $S^2$ factor of by $S^5$. Differently from the previous case where only the deformation of $C(T^{1,1})$ (finite size $S^3$) was of interest for us here we get a “duality”
between the factors - when putting $M$-theory on the cone, we can get a three-dimensional back hole either by wrapping $M2$ on $S^2$, or by the dual procedure of wrapping $M5$ on $S^5$.

Similarly, a two-dimensional black hole can be constructed by wrapping D3 on the shrinking $S^3$ in $C(M^{01})$. A circle compactification of the previous case accompanied with $T$-duality should relate this to the $M2$ and $M5$ discussed above. Indeed, as known [23], $T$-duality untwists $U(1)$ bundles interchanging the bases of the two cones. All these cases have no supersymmetry preserved since both $M^{10}$ and $M^{01}$ admit no Killing spinors.

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