Accelerated Proximal Point Method and Forward Method for Monotone Inclusions

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Abstract
This paper proposes an accelerated proximal point method for maximally monotone operators. The proof is computer-assisted via the performance estimation problem approach. The proximal point method includes various well-known convex optimization methods, such as the proximal method of multipliers and the alternating direction method of multipliers, and thus the proposed acceleration has wide applications. Numerical experiments are presented to demonstrate the accelerating behaviors. In addition, this paper shows that the proposed acceleration applies to the forward method for cocoercive operators.

Keywords
Proximal point method · Forward method · Acceleration · Maximally monotone operators · Worst-case performance analysis

Mathematics Subject Classification (2010)
90C25 · 90C30 · 90C60 · 68Q25 · 49M25 · 90C22

1 Introduction
A fundamental tool for finding a root of a monotone operator is the proximal point method [41, 54]. The monotone operator theory is particularly of interest, since it is closely related to convex functions and convex minimization [5, 11, 55]. For example, the proximal point method is useful when solving ill-conditioned problems or dual problems. In particular, the augmented Lagrangian method (i.e., the method of multipliers) [32, 51] and the alternating direction method of multipliers (ADMM) [22, 23] are instances of the proximal point method applied to dual problems [20, 21, 53].

To improve the efficiency of the proximal point method, accelerating its worst-case rate has been of interest both in theory and in applications (see e.g., [1, 3, 4, 12, 25, 30, 38]). In specific, inspired by Nesterov’s fast gradient method [46, 47], Güler [30] accelerated the worst-case rate of the proximal point method for convex minimization with respect to the cost function. This yields the fast rate $O(1/i^2)$ where $i$ denotes the number of iterations, compared to the $O(1/i)$ rate of the proximal point method. However, this acceleration has not been theoretically generalized to the monotone inclusion problem, and only somewhat empirical accelerations, e.g., via the relaxation and the inertia (i.e., an implicit version of the heavy ball method [50], or equivalently, Nesterov’s and Güler’s accelerating technique [30, 46, 47]) in [1, 3, 4, 12, 25], have been studied. Therefore, this paper studies accelerating the worst-case rate of the proximal point method with respect to the fixed-point residual for maximally monotone operators. This provides the fast $O(1/i^2)$ rate, which improves upon the rate $O(1/i)$ of the proximal point method [7, 28]. The proof is computer-assisted via the performance estimation problem (PEP) approach [18] and its extensions [17, 19, 27, 28, 33, 34, 35, 36, 37, 56, 57, 58, 59].

Under the additional strong monotonicity condition, the proximal point method has a linear rate in terms of the fixed-point residual [54], while the proposed acceleration is not guaranteed to have such linear

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rate. Therefore, this paper further employs a restarting technique \( (e.g., [45, Section II.4][48, Section 5.1]) \) under the strong monotonicity condition, which can yield a linear rate faster than that of the proximal point method.

The proposed acceleration of the proximal point method has wide applications; it provides an acceleration to the proximal method of multipliers \([53]\), the Douglas-Rachford splitting method \([16,39]\), and ADMM \([22,23]\). The proposed result also applies to a preconditioned proximal point method such as the Chambolle-Pock method \([8,10,31]\) \( (i.e., \) a preconditioned ADMM\), yielding an accelerated Chambolle-Pock method.

The proximal point method is also known as the backward method, and for operators with the cocoercivity \( (a stronger condition than the monotonicity)\), the forward method is widely used, which includes the gradient method for convex minimization problem. As a by-product of this paper’s work on accelerating the proximal point method, this paper shows that the proposed acceleration also applies to the forward method for the cocoercive operators. Existing works on accelerating the forward method can be found, for example, in \([2,40]\).

Section 2 reviews maximally monotone operators, the proximal point method and its known accelerations. Section 3 studies the PEP with respect to the fixed-point residual for monotone inclusion problems. Section 4 proposes a new accelerated proximal point method using the PEP. Section 5 considers a restarting technique to yield a fast linear rate, under the additional strongly monotone assumption. Section 6 applies the proposed acceleration to well-known instances of the proximal point method, such as the proximal method of multipliers, the Douglas-Rachford splitting method, and ADMM. Section 6 also provides numerical experiments. Section 7 presents that the proposed approach also accelerates the forward method for cocoercive operators, and Section 8 concludes.

2 Problem and Method

2.1 Monotone Inclusion Problem

Let \( \mathcal{H} \) be a real Hilbert space equipped with inner product \( \langle \cdot , \cdot \rangle \), and associated norm \( || \cdot || \). A set-valued operator \( M : \mathcal{H} \to 2^\mathcal{H} \) is ** monotone if

\[
\langle Mx - My, x - y \rangle \geq 0 \quad \text{for all } x, y \in \mathcal{H}. \tag{1}
\]

A monotone operator is ** maximal if it is not strictly contained in any other monotone operator on \( \mathcal{H} \). Let \( \mathcal{M}(\mathcal{H}) \) be the class of maximally monotone operators on \( \mathcal{H} \). In addition, a set-valued operator \( M : \mathcal{H} \to 2^\mathcal{H} \) is ** \( \mu \)-strongly monotone for \( \mu \in \mathbb{R}_{++} \), if

\[
\langle Mx - My, x - y \rangle \geq \mu ||x - y||^2 \quad \text{for all } x, y \in \mathcal{H}. \tag{2}
\]

Let \( \mathcal{M}_\mu(\mathcal{H}) \) be the class of maximally and \( \mu \)-strongly monotone operators on \( \mathcal{H} \). Also, define \( \mathcal{B}(\mathcal{H}, \mathcal{G}) = \{A : \mathcal{H} \to \mathcal{G} \mid A \text{ is linear and bounded} \} \) for a real Hilbert space \( \mathcal{G} \) equipped with inner product \( \langle \cdot , \cdot \rangle \), and let \( L^* \in \mathcal{B}(\mathcal{G}, \mathcal{H}) \) be the adjoint of \( L \in \mathcal{B}(\mathcal{H}, \mathcal{G}) \) that satisfies \( \langle Lx, y \rangle = \langle x, L^*y \rangle \) for all \( x \in \mathcal{H} \) and \( y \in \mathcal{G} \).

This paper considers the monotone inclusion problem:

\[
\text{Find } x \in \mathcal{H} \quad \text{subject to } \quad 0 \in Mx, \tag{3}
\]

where \( M \in \mathcal{M}(\mathcal{H}) \) \( (\text{and } M \in \mathcal{M}_\mu(\mathcal{H})) \). This includes convex problems and convex-concave problems; a subdifferential \( \partial f \) of a closed proper convex function \( f : \mathcal{H} \to \mathbb{R} \cup \{\infty\} \) is maximally monotone \([43]\). Let \( \mathcal{F}(\mathcal{H}) \) be the class of closed proper convex functions on \( \mathcal{H} \).

We assume that the optimal set \( X_*(M) := \{ x \in \mathcal{H} : 0 \in Mx \} \) is nonempty. We also assume that the distance between an initial point \( x_0 \) and some optimal point \( x_* \in X_*(M) \) is bounded as

\[
||x_0 - x_*|| \leq R \quad \text{for a constant } R > 0. \tag{4}
\]
2.2 Proximal Point Method and Its Worst-Case Rates

Proximal point method was first introduced to convex optimization by Martinet [41], which is based on the proximal mapping by Moreau [44]. The method was later extended to monotone inclusion problem by Rockafellar [54]. The proximal point method for maximally monotone operators includes the augmented Lagrangian [32,51], the proximal method of multipliers [53], the Douglas-Rachford splitting method [16,39], and the alternating direction method of multipliers (ADMM) [22,23], so studying its worst-case convergence behavior and acceleration is important, which is of main interest in this paper.

The proximal mapping [44] (or the resolvent operator) of an operator \( M \) is defined as
\[
J_M := (I + M)^{-1},
\]
where \( I : H \to H \) is an identity operator, i.e., \( I(x) = x \) for all \( x \in H \). The resolvent operator \( J_M \) is single-valued and firmly nonexpansive for \( M \in M(H) \) [42]. The proximal point method [41,54] generates a sequence \( \{x_i\} \) by iteratively applying the resolvent operator with a positive real number \( \lambda \) as below.

**Proximal Point Method**

**Input:** \( M \in M(H), x_0 \in H, \lambda \in \mathbb{R}_{++} \).

**For** \( i = 0, 1, \ldots \)

\[
x_{i+1} = J_{\lambda M}(x_i).
\]

In [7, Proposition 8], the worst-case rate of the proximal point method with respect to the fixed-point residual
\[
||x - J_{\lambda M}(x)||^2
\]
was found to satisfy
\[
||x_i - x_{i-1}||^2 \leq \frac{R^2}{i}
\]
for \( i \geq 1 \). Very recently in [28], this was improved to
\[
||x_i - x_{i-1}||^2 \leq \left(1 - \frac{1}{i}\right)^{i-1} \frac{R^2}{i},
\]
which is exact when \( \dim H \geq 2 \); such exact worst-case case with \( \dim H = 2 \) given in [28] will be visited at the end of Section 4. The bound (8) is \((1 - \frac{1}{i})^{i-1}\)-times (and asymptotically e-times, where \( e \) is Euler’s number) lower than (7). When we additionally assume the \( \mu \)-strong monotonicity, the proximal point method has a linear rate [5, Example 2.31] [54]
\[
||x_i - x_{i-1}||^2 \leq \left(\frac{1}{1 + \lambda \mu}\right)^{2i} \lambda^2 ||Mx_0||^2
\]
for \( i \geq 1 \), which is exact considering the case \( x_0 = 1 \) and \( Mx = \mu x \) with \( \dim H = 1 \).

When we focus on a convex minimization of \( f \in F(H) \), [58, Conjecture 4.2] conjectures that the proximal point method satisfies
\[
||x_i - x_{i-1}||^2 \leq \frac{R^2}{i^2}
\]
for \( i \geq 1 \), which is faster than (8) for maximally monotone operators. In addition, the \( O(1/i) \) worst-case rate of the proximal point method with respect to the cost function was studied in [29, Theorem 2.1], and this was improved by a constant 2 in [58, Theorem 4.1] for \( i \geq 1 \)
\[
f(x_i) - f(x_*) \leq \frac{R^2}{4\lambda i} \text{ for all } x_* \in X_*(\partial f).
\]
Remark 2.1 The results for the proximal point method can be applied to a preconditioned proximal point method. Let \( L \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \) be invertible such that \( L^{-1} = L^* \). Then, \( L^{-1} \circ M \circ L \) is maximally monotone for \( M \in \mathcal{M}(\mathcal{H}) \) \cite[Corollary 23.27]{Bauschke2011}, and the corresponding proximal point method is

\[
x_{i+1} = J_{\lambda L^{-1} \circ M \circ L}(x_i) = (I + \lambda L^{-1} \circ M \circ L)^{-1} x_i.
\]

Introducing \( \tilde{x}_i = Lx_i \) and \( P = LL^{-1} \) yields the following equivalent preconditioned proximal point method \cite{Lin2012}

\[
\tilde{x}_{i+1} = (P + \lambda M)^{-1} P \tilde{x}_i.
\]

So, for example, the inequality (7) provides the bound for the preconditioned proximal point method (13)

\[
||L^{-1}(\tilde{x}_i - \tilde{x}_{i-1})||^2 \leq \frac{R^2}{i}
\]

for \( i \geq 1 \), where \( ||L^{-1}(\tilde{x}_0 - \tilde{x}_*)|| \leq R \) and \( 0 \in ML^{-1}\tilde{x}_* \). This is particularly useful when considering the Chambolle-Pock method \cite{Chambolle2011, Pock2011} \cite[Lemma 2.2]{Nestergeneralized}, which is a preconditioned (or linearized) version of ADMM and thus is an instance of a preconditioned proximal point method.

### 2.3 Existing Accelerations for Proximal Point Method

This section reviews existing accelerations of proximal point method for convex minimization with respect to the cost function. To the best of our knowledge, there are no other type of proximal point methods that guarantee accelerated worst-case rates.

For convex minimization, Güler \cite{Guler1999} developed the following two accelerated versions, inspired by Nesterov’s fast gradient method \cite{Nesterov1983, Nesterov1983a}. The following is the first accelerated version of the proximal point method in \cite{Guler1999} which is an instance of FISTA \cite{Beck2009}; the original version in \cite{Guler1999} includes some variation.

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**Güler’s First Accelerated Proximal Point Method in \cite[Sec. 2]{Guler1999}**

Input: \( f \in \mathcal{F}(\mathcal{H}), \ x_0 = y_0 \in \mathcal{H}, \ \lambda \in \mathbb{R}_{++}, \ t_0 = 1 \).

For \( i = 0, 1, \ldots \)

\[
x_{i+1} = J_{\lambda \partial f}(y_i),
\]

\[
l_{i+1} = \frac{1 + \sqrt{1 + 4t_i^2}}{2},
\]

\[
y_{i+1} = x_{i+1} + \frac{t_i - 1}{t_{i+1}} (x_{i+1} - x_i).
\]

The sequence generated by the Güler’s first accelerated proximal point method satisfies \cite[Theorem 2.3]{Guler1999} \cite[Theorem 4.4]{Beck2009} for \( i \geq 1 \)

\[
f(x_i) - f(x_*) \leq \frac{R^2}{2\lambda t_i^2} \leq \frac{R^2}{\lambda(t+1)^2} \text{ for all } x_* \in X(\partial f).
\]

The following is another accelerated proximal point method by Güler \cite{Guler1999}, which the formulation is similar to those of the optimized gradient methods \cite{O茨lowski2018, Otslowski2018a, Otslowski2018b}.

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**Güler’s Second Accelerated Proximal Point Method in \cite[Appendix]{Guler1999}**

Input: \( f \in \mathcal{F}(\mathcal{H}), \ x_0 = y_0 \in \mathcal{H}, \ \lambda \in \mathbb{R}_{++}, \ t_0 = 1 \).

For \( i = 0, 1, \ldots \)

\[
x_{i+1} = J_{\lambda \partial f}(y_i),
\]

\[
l_{i+1} = \frac{1 + \sqrt{1 + 4t_i^2}}{2},
\]

\[
y_{i+1} = x_{i+1} + \frac{t_i - 1}{t_{i+1}} (x_{i+1} - x_i) + \frac{t_i}{t_{i+1}} (x_{i+1} - y_i).
\]

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4
The sequence generated by Güler’s second accelerated proximal point method satisfies [30, Theorem 6.1] for $i \geq 1$

\[
    f(x_i) - f(x^*) \leq \frac{R^2}{4M^2_{i-1}} \leq \frac{R^2}{\lambda(i+1)^2} \text{ for all } x^* \in X^*(\partial f),
\]

which is twice smaller than (15).

2.4 Main Contribution

To accelerate the worst-case rate of the proximal point method for maximally monotone operators, the relaxation and the inertia (i.e., an implicit version of the heavy ball method [50], or equivalently, Nesterov’s and Güler’s accelerating technique [30, 46, 47]) have been studied in [1, 3, 4, 12, 25], but none of them guarantee accelerated rates. Therefore, the main contribution of this paper is to develop a method that has a fast $O(1/i^2)$ rate with respect to the fixed-point residual, improving upon the $O(1/i)$ rate of the proximal point method in (7) and (8).

This paper considers the following general proximal point method with step coefficients \( \{h_{i+1,k+1}\}_{k=0}^i \) for reusing previous and current updates \( \{x_{i+1} - y_k\}_{k=0}^i \). This includes the proximal point method, the accelerated methods via the relaxation and the inertia [1, 3, 4, 12, 25], and the proposed accelerated method.

| General Proximal Point Method |
|--------------------------------|
| Input: \( M \in \mathcal{M}(\mathcal{H}), y_0 \in \mathcal{H}, \lambda \in \mathbb{R}_+ \). |
| For \( i = 0, 1, \ldots \) |
| \( x_{i+1} = J_\lambda M(y_i) \), |
| \( y_{i+1} = y_i + \sum_{k=0}^{i+1} h_{i+1,k+1}(x_{k+1} - y_k) \). |

This paper next uses the PEP approach to find the choice of \( \{h_{i+1,k+1}\}_{k=0}^i \) that guarantees the accelerated rate. While the formulation of the general proximal point method is inefficient in general, the proposed accelerated method with the specific choice of \( \{h_{i+1,k+1}\}_{k=0}^i \) found by PEP has an efficient equivalent form, similar to the other accelerated methods with the relaxation and/or the inertia [1, 3, 4, 12, 25].

In addition, this paper shows that the above approach, constructing the accelerated proximal point method, similarly applies to the forward method, yielding similar accelerated forward method for cocoercive operators.

3 Performance Estimation Problem for Maximally Monotone Operators

This section uses the performance estimation problem (PEP) approach [18, 58, 59] to analyze the general proximal point method for maximally monotone operators, in terms of the fixed-point residual (6). This was recently studied in [28] for the proximal point method, providing the exact rate (8); the same authors [27] also used the PEP to study the exact worst-case rate for the ergodic sequence of the (relaxed) proximal point method for the variational inequalities. Similarly, [58] used PEP to analyze the worst-case rate of the proximal point method for convex minimization in terms of the fixed-point residual and the cost function, yielding (10) and (11), respectively.

Building upon [18, 27, 28, 58, 59], the worst-case rate of the general proximal point method after \( N \) iterations for decreasing the fixed-point residual (6) under the initial distance condition (4) can be
This is an infinite-dimensional problem due to the constraint \( M \in \mathcal{M}(\mathcal{H}) \), which is impractical to solve. PEP in [18] further introduced a series of steps that reformulate such impractical problem into a tractable problem, which we apply to (17) step by step below.

The first step is to reformulate the problem (17) into a finite-dimensional problem. [56, Fact 1] implies that one can replace \( M \in \mathcal{M}(\mathcal{H}) \) in (17) by a set of inequality constraints (1) for \( M \in \mathcal{M}(\mathcal{H}) \) on the finite number of pairs of points \( \{x_1, \ldots, x_N, x_\ast\} \) without strictly relaxing the problem (17). In specific, such constraints are

\[
\langle q_i - q_j, x_i - x_j \rangle \geq 0,
\]

for all \( i, j \in \{1, \ldots, N, \ast\} \), with additional variables \( q_i \in Mx_i \) for \( i = 1, \ldots, N \) and \( q_\ast = 0 \in Mx_\ast \). Then the resulting equivalent problem of (17) is

\[
\max_{x_1, \ldots, x_N, x_\ast \in \mathcal{H}, q_1, \ldots, q_N \in \mathcal{H}} \frac{1}{R^2} \|x_N - y_{N-1}\|^2
\]

subject to

\[
\langle q_i - q_j, x_i - x_j \rangle \geq 0, \quad i < j = 1, \ldots, N, \\
\langle q_i, x_i - x_\ast \rangle \geq 0, \quad i = 1, \ldots, N, \\
x_{i+1} = y_i - \lambda q_{i+1}, \quad i = 0, \ldots, N - 1, \\
y_{i+1} = y_i - \lambda \sum_{k=0}^1 h_{i+1,k+1} q_{k+1}, \quad i = 0, \ldots, N - 2, \\
\|y_0 - x_\ast\|^2 \leq R^2.
\]

Further removing \( x_i \) and using the change of variables

\[
g_i := \frac{\lambda}{R} q_i, \quad i = 1, \ldots, N,
\]

simplify the problem (19) as

\[
\max_{y_0, \ldots, y_{N-1}, x_\ast \in \mathcal{H}, g_1, \ldots, g_N \in \mathcal{H}} \|g_N\|^2
\]

subject to

\[
\frac{1}{R} \langle g_i - g_j, y_{i-1} - Rg_i - y_{j-1} + Rg_j \rangle \geq 0, \quad i < j = 1, \ldots, N, \\
\frac{1}{R} \langle g_i, y_{i-1} - Rg_i - x_\ast \rangle \geq 0, \quad i = 1, \ldots, N, \\
y_{i+1} = y_i - R \sum_{k=0}^1 h_{i+1,k+1} g_{k+1}, \quad i = 0, \ldots, N - 2, \\
\|y_0 - x_\ast\|^2 \leq R^2.
\]
As in [18, 28, 58, 59], we next introduce the Gram matrix

\[
Z = \begin{bmatrix}
\|g_1\|^2 & \langle g_1, g_2 \rangle & \cdots & \langle g_1, g_N \rangle & \frac{1}{\mathcal{H}} \langle g_1, y_0 - x_+ \rangle \\
\langle g_1, g_2 \rangle & \|g_2\|^2 & \cdots & \langle g_2, g_N \rangle & \frac{1}{\mathcal{H}} \langle g_2, y_0 - x_+ \rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\langle g_1, g_N \rangle & \cdots & \|g_N\|^2 & \frac{1}{\mathcal{H}} \langle g_N, y_0 - x_+ \rangle & \frac{1}{\mathcal{H}} \|y_0 - x_+\|^2 \\
\frac{1}{\mathcal{H}} \langle g_1, y_0 - x_+ \rangle & \cdots & \frac{1}{\mathcal{H}} \langle g_N, y_0 - x_+ \rangle & \frac{1}{\mathcal{H}} \|y_0 - x_+\|^2 & \frac{1}{\mathcal{H}} \|y_0 - x_+\|^2
\end{bmatrix} \tag{22}
\]

to relax the problem as

\[
\max_{Z \in S^{N+1}_+} \text{tr}\{ u_N u_N^T Z \} \tag{23}
\]
subject to \(\text{tr}\{ A_{i,j}(h) Z \} \leq 0, \quad i < j = 1, \ldots, N,\)
\(\text{tr}\{ B_i(h) Z \} \leq 0, \quad i = 1, \ldots, N,\)
\(\text{tr}\{ C Z \} \leq 1,\)

where \(u_i\) is an \(i\)th standard basis in \(\mathbb{R}^{N+1}\) and

\[
\begin{align*}
A_{i,j}(h) &:= (u_i - u_j) \odot (u_i - u_j) - (u_i - u_j) \odot \sum_{l=1}^{j-1} \sum_{k=0}^{l} h_{l+1,k+1} u_{l+1} u_{k+1}, \\
B_i(h) &:= u_i u_i^T - u_i \odot u_{N+1} + u_i \odot \sum_{l=0}^{i-1} \sum_{k=0}^{l} h_{l+1,k+1} u_{l+1} u_{k+1}, \\
C &:= u_{N+1} u_{N+1}^T
\end{align*}
\]

with the outer product operator \(u \odot v := \frac{1}{\mathcal{H}}\langle u v^T + v u^T \rangle\). If \(\dim \mathcal{H} \geq N + 1\), the problems (17) and (23) are equivalent, based on the following lemma similar to [56, Lemma 1].

**Lemma 3.1** If \(\dim \mathcal{H} \geq N + 1\), then

\[Z \in S^{N+1}_+ \iff \exists \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_N, \frac{1}{\mathcal{H}}(y_0 - x_+) \in \mathcal{H} \text{ such that } Z = \text{expression of (22)}.\]

For simplicity in later analysis, we discard some constraints as

\[
\max_{Z \in S^{N+1}_+} \text{tr}\{ u_N u_N^T Z \} \tag{24}
\]
subject to \(\text{tr}\{ A_{i-1,i}(h) Z \} \leq 0, \quad i = 2, \ldots, N,\)
\(\text{tr}\{ B_N(h) Z \} \leq 0,\)
\(\text{tr}\{ C Z \} \leq 1,\)

which does not affect the result in the paper, i.e., the optimal values of (23) and (24) are found to be numerically equivalent for the method proposed in this paper. Finally, we construct the associated Lagrangian dual of (24)

\[
\mathcal{B}_D(h) := \min_{a_2, \ldots, a_N, b_N, c \in \mathbb{R}} c \tag{D}
\]
subject to \(\sum_{i=2}^{N} a_i A_{i-1,i}(h) + b_N B_N(h) + c C - u_N u_N^T \geq 0,\)
\(a_2, \ldots, a_N, b_N, c \geq 0,\)

where \(a_2, \ldots, a_N, b_N, c\) are dual variables associated with the constraints of (24), respectively. Then, for any given \(h\) for the general proximal point method, one can compute its (upper bound of) worst-case fixed-point residual by numerically solving (D) using any SDP solver. For some choices of \(h\) as for the proximal point method in [28], it might be possible to analytically solve (D), i.e., [28] analytically solved (D) for the proximal point method yielding the rate (8). This paper provides another choice of \(h\) that provides an analytical solution to (D) with an accelerated rate.
4 Accelerating the Proximal Point Method for Maximally Monotone Operators

Using the dual problem (D), this section develops an accelerated version of the proximal point method via PEP:

\[ \min_h \mathcal{B}_D(h), \]  

(HD)

which is studied in [17, 18, 19, 33, 34, 35, 36] for certain classes of problems and methods. The problem is non-convex but convex for the variables \((a_2, \ldots, a_N, b_N, c)\) given \(h\) and for the variables \((c, h)\) given \((a_2, \ldots, a_N, b_N)\), so we used a variant of alternating minimization that alternatively optimizes over \((a_2, \ldots, a_N, b_N, c)\) given \(h\) and over \((c, h)\) given \((a_2, \ldots, a_N, b_N)\) to find a minimizer using a SDP solver [13, 26]. Inspired by numerical results, the following lemma specifies a feasible point of (HD) analytically, which we conjecture to be a minimizer of (HD).

**Lemma 4.1** The following

\[
    h_{i,k} = \begin{cases} 
        \frac{2k}{N+1}, & i = 1, \ldots, N-1, \quad k = 1, \ldots, i-1, \\
        \frac{2}{i+1}, & i = 1, \ldots, N-1, \quad k = i, 
    \end{cases} 
\]

\[
    a_i = \frac{2(i-1)i}{N^2}, \quad i = 2, \ldots, N, \quad b_N = \frac{2}{N}, \quad c = \frac{1}{N^2} \tag{25}
\]

is a feasible point of (D) and (HD).

**Proof** It is obvious that \(a_2, \ldots, a_N, b_N, c\) are nonnegative, so we are only left to show the positive semidefinite condition in (D). Since

\[
\begin{align*}
    \sum_{i=2}^{N} a_i A_{i-1,i}(h) + b_N B_N(h) + cC - u_N^T u_N \\
    &= \sum_{i=2}^{N} \frac{2(i-1)i}{N^2} \left[ (u_{i-1} - u_i) \odot (u_{i-1} - u_i) - (u_{i-1} - u_i) \odot \left( \frac{2(i-1)}{i} u_{i-1} - \sum_{k=0}^{i-3} \frac{2(k+1)}{(i-1)k} u_{k+1} \right) \right] \\
    &+ \frac{2}{N} \left[ u_N^T u_N - u_N \odot u_{N+1}^T + u_N \odot \sum_{l=0}^{N-2} \left( \frac{2(l+1)}{l+2} u_{l+1} - \sum_{k=0}^{l-1} \frac{2(k+1)}{(l+1)(l+2)} u_{k+1} \right) \right] \\
    &+ \frac{1}{N^2} u_{N+1}^T u_{N+1} - u_N^T u_N \\
    &= \sum_{i=2}^{N-1} \left[ \frac{2(i-1)i}{N^2} + \frac{2i(i+1)}{N^2} \left( 1 - \frac{2i}{i+1} \right) \right] u_i u_i^T + \frac{2(N_1-1)N}{N^2} + \frac{2}{N-1} u_N^T u_N + \frac{1}{N^2} u_{N+1}^T u_{N+1} \\
    &+ \sum_{i=2}^{N-1} \left[ \frac{2(i-1)i}{N^2} \left( -2 + \frac{2(i-1)}{i} \right) + \frac{2i(i+1)}{N^2} \frac{2(i-1)}{i(i+1)} \right] u_i u_i^T \\
    &+ \frac{2(N-1)}{N} \left[ -2 + \frac{2(N-1)}{N} \right] u_{N-1} \odot u_N - \frac{2}{N} u_N^T u_{N+1} \\
    &+ \sum_{k=0}^{N-3} \left[ -\frac{2(i-1)i}{N^2} \frac{2(k+1)}{(i-1)k} + \frac{2i(i+1)}{N^2} \frac{2(k+1)}{i(i+1)} \right] u_{k+1} \odot u_i \\
    &+ \sum_{k=0}^{N-3} \left[ \frac{2(N-1)}{N^2} \frac{2(k+1)}{(N-1)N} + \frac{2}{N} \frac{2(k+1)}{k+2} - \sum_{l=k+1}^{N-2} \frac{2(k+1)}{(l+1)(l+2)} \right] u_{k+1} \odot u_N \\
    &= u_N^T u_N + \frac{1}{N^2} u_{N+1}^T u_{N+1} - \frac{2}{N} u_N^T u_{N+1} \\
    &= \left( u_N - \frac{1}{N} u_{N+1} \right) \left( u_N - \frac{1}{N} u_{N+1} \right)^T \geq 0,
\end{align*}
\]

the given point is a feasible point of (HD). \(\square\)
Before providing the worst-case rate of the general proximal point method with \( h \) in (25), using Lemma 4.1, we develop its efficient formulation below that has a low computational cost per iteration, comparable to that of the proximal point method.

### Proposed Accelerated Proximal Point Method for Maximally Monotone Operators

**Input:** \( M \in \mathcal{M}(\mathcal{H}), x_0 = y_0 = y_{-1} \in \mathcal{H}, \lambda \in \mathbb{R}_{++} \).

For \( i = 0, 1, \ldots \)

\[
x_{i+1} = J_M(y_i), \quad y_{i+1} = x_{i+1} + \frac{i}{i+2}(x_{i+1} - x_i) - \frac{i}{i+2}(x_i - y_{i-1}).
\]

### Proposition 4.1

The sequences \( \{x_i\} \) and \( \{y_i\} \) generated by the general proximal point method with step coefficients \( \{h_{i,k}\} \) in (25) are identical to the corresponding sequence generated by the proposed accelerated proximal point method.

**Proof** We use induction, and for clarity we use the notation \( x'_0, x'_1, \ldots \) and \( y'_0, y'_1, \ldots \) for the general proximal point method with (25). It is obvious that \( x'_0 = x_0 = y'_0 = y_0, x'_1 = x_1 = y_1 \), and we have

\[
y'_1 = y'_0 + h_{1,1}(x'_1 - y'_0) = x'_1 = y_1.
\]

Similarly, it is obvious that \( x'_2 = x_2 \), and we have

\[
y'_2 = y'_1 + \sum_{k=0}^{1} h_{2,k+1}(x'_{k+1} - y'_k) = y_1 + \frac{4}{3}(x_2 - y_1) - \frac{3}{3}(x_1 - y_0)
\]

\[
= x_2 + \frac{1}{3}(x_2 - x_1) - \frac{1}{3}(x_1 - y_0) = y_2.
\]

Assuming \( x'_l = x_l \) for \( l = 0, \ldots, i \) and \( y'_l = y_l \) for \( l = 0, \ldots, i \), we have

\[
y'_{i+1} = y'_i + \sum_{k=0}^{i} h_{i+1,k+1}(x'_{k+1} - y'_k)
\]

\[
= y_i + \frac{2(i+1)}{i+2}(x_{i+1} - y_i) + \sum_{k=0}^{i-1} \left( -\frac{2(k+1)}{(i+1)(i+2)} \right) (x_{k+1} - y_k)
\]

\[
= y_i + \left( 1 + \frac{i}{i+2} \right) (x_{i+1} - y_i) + \frac{i}{i+2} \sum_{k=0}^{i-1} \left( -\frac{2(k+1)}{i(i+1)} \right) (x_{k+1} - y_k)
\]

\[
= x_{i+1} + \frac{i}{i+2} (x_{i+1} - y_i) + \frac{i}{i+2} (y_{i} + y_{i-1} - 2 x_{i})
\]

\[
= x_{i+1} + \frac{i}{i+2} (x_{i+1} - x_i) - \frac{i}{i+2} (x_i - y_{i-1}) = y_{i+1},
\]

where the fourth equality uses

\[
y_i = y_{i-1} + \frac{2i}{i+1}(x_i - y_{i-1}) + \sum_{k=0}^{i-2} \left( -\frac{2(k+1)}{i(i+1)} \right) (x_{k+1} - y_k)
\]

\[
= y_{i-1} + \frac{2i}{i+1} (x_i - y_{i-1}) + \sum_{k=0}^{i-1} \left( -\frac{2(k+1)}{i(i+1)} \right) (x_{k+1} - y_k).
\]

\( \square \)

The proposed accelerated method has the inertia term \( \frac{2i}{i+1}(x_{i+1} - x_i) \), similar to Nesterov’s acceleration [46,47] and Guler’s methods [30]. However, the proposed method also has a correction term \( -\frac{i}{i+2}(x_i - y_{i-1}) \), which is essential to guarantee an accelerated rate below. Note that without such correction term, the accelerated method can diverge, which we provide an example at the end of this section.
Theorem 4.1 Let $M \in \mathcal{M}(\mathcal{H})$ and let $x_0, y_0, x_1, y_1, \ldots \in \mathcal{H}$ be generated by the proposed accelerated proximal point method. Assume that $\|x_0 - x^*\| \leq R$ for a constant $R > 0$ and for some $x^* \in X_*(M)$. Then for any $i \geq 1$,

$$
\|x_i - y_{i-1}\|^2 \leq \frac{R^2}{i^2}.
$$

(27)

Proof Using Lemma 4.1, the general proximal point method with $h$ (25) satisfies

$$
\frac{1}{R^2\nu} \|x_N - y_{N-1}\|^2 \leq B_D(h) = \frac{1}{N^2}.
$$

(28)

Since the the iterates of the method are recursive and do not depend on a given $N$, the bound (28) generalizes to the intermediate iterates of the method. By Proposition 4.1, the proposed accelerated proximal point method also satisfies the bound (28), which concludes the proof.

The bound (8) of the proximal point method was found to be exact in [28] by specifying a certain operator $M$ achieving the bound (8) exactly; that is, for given $N$, the proximal point method exactly achieves the bound (8) for the operator

$$
M \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\lambda \sqrt{N-1}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},
$$

(29)

with an initial point $x_0 = [1 0]^T$. Such exact analysis is important since it reveals the worst-case behavior of the iterates of the method. However, we were not able to show that the bound (27) of the proposed method is exact, which we leave as future work. Instead, we compared the behavior of the iterates of the proximal point method and its accelerated variants on the operator $M$ in (29). Figure 1 compares the proximal point method, G"uler's first accelerated method with $M$ instead of $\partial f$ (i.e., an instance of the inertia method) and the proposed accelerated method, with an initial point $x_0 = [1 0]^T$ and the optimal point $x^* = 0$. Note that the G"uler's first method is almost equivalent to the proposed accelerated method without the correction term $-\frac{1}{i+2}(x_i - y_{i-1})$, and this exhibits diverging behavior in Figure 1. Figure 1 illustrates that the correction term greatly helps the iterates to rapidly converge by reducing the radius of the orbit of the iterates, compared to other methods.

Fig. 1 Solving a worst-case monotone inclusion problem of the proximal point method with $M$ (29) with $N = 100$; (left) the fixed-point residual vs. iteration, (right) the trajectory of the iterates $x_0 = [x_{0,1}, x_{0,2}]^T$ (markers are displayed every 5th iterations).
5 Restarting the Accelerated Proximal Point Method for Strongly Monotone Operators

For strongly monotone operators, the proximal point method has a linear rate (9), whereas the proposed accelerated method is not guaranteed to have such fast rate. Technically, one should be able to find an accelerated method for strong monotone operators via PEP, as we did for the monotone operators in the previous section. However, the resulting PEP problem, a reminiscent of (HD), is much more difficult to solve, and we leave it as future work. Instead, we consider a fixed restarting technique in [45, Section 11.4][48, Section 5.1] that restarts an accelerated method with a sublinear rate every certain number of iterations to yield a fast linear rate, particularly for $M \in M_{\mu}(H)$ in this section.

Suppose one restarts the proposed method every $k$ (inner) iterations by initializing the $(j+1)$th outer iteration $x_{j+1,0} = y_{j+1,0} = y_{j+1,-1}$ by $x_{j,k}$, where $x_{j,1}$ and $y_{j,1}$ denote iterates at the $j$th outer iteration and $t$th inner iteration for $j = 0, 1, \ldots$ and $t = -1, 0, 1, \ldots, k$. Using the rate (27) (with $R = ||x_{j,0} - x_*||$) and the strong monotonicity condition (2), we have

$$||x_{j,k} - y_{j,k-1}||^2 \leq \frac{||x_{j,0} - x_*||^2}{k^2} \leq \frac{1}{\mu k^2}||Mx_{j,0}||^2$$

(30)

for $j = 0, 1, \ldots$. Since $\frac{1}{k}(x_{j-1,k} - y_{j-1,k-1}) \in Mx_{j,0}$, we have a linear rate

$$||x_{j,k} - y_{j,k-1}||^2 \leq \frac{1}{\lambda^2 \mu k^2}||x_{j-1,k} - y_{j-1,k-1}||^2.$$  

(31)

For a given $N = jk$ total number of steps, the optimal choice of the restarting interval is

$$k_{opt} = \frac{e}{\lambda \sqrt{\mu}},$$

(32)

where $e$ is Euler’s number. The corresponding linear rate is $O((e^{\sqrt{\mu}/e} - 2)^N)$, whereas the proximal point method has the rate $O((1 + \lambda \mu)^{-2N})$ in (9). For a given $\mu \in \mathbb{R}_{++}$, restarting with a certain $\lambda \in \mathbb{R}_{++}$ satisfying $e^{\sqrt{\mu}/e} > 1 + \lambda \mu$ is guaranteed to have a rate faster than that of the proximal point method, under the strong monotonicity condition.

For numerical verification, we consider a toy problem that is a combination of the worst-case problems for $M(H)$ and $M_{\mu}(H)$:

$$M \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda \sqrt{N} - 1} & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

(33)

where we choose $N = 100$, $\lambda = 1$ and $\mu = 0.02$, which satisfies $e^{\sqrt{0.02}/e} \approx 1.05 > 1 + 0.02$. The optimal restarting interval is $k_{opt} \approx 19$, and we run 200 iterations in the experiment. Figure 2 compares the proximal point method, its accelerated variants, and the proposed accelerated method with restarting, with an initial point $x_0 = [1 0]^T$ and the optimal point $x_* = 0$. Figure 2 presents that the proximal point method has a linear rate that is faster than the proposed method (with a sublinear rate), while the restarting greatly accelerates the proposed method with a fast linear rate. In addition, examples in the next section present that the restarting can be useful even without strong monotonicity (but possibly with local strong monotonicity).

6 Applications of the Accelerated Proximal Point Method

As mentioned earlier, the proximal point method for maximally monotone operators include various well-known convex optimization methods. These include the augmented Lagrangian (i.e., the method of multipliers), the proximal method of multipliers, and ADMM. The augmented Lagrangian method is equivalent to the proximal point method directly solving the dual convex minimization problem [53], so Güler’s methods [30] already provide acceleration, whereas other instances of the proximal point method have no known accelerations yet. Thus, this section introduces accelerations to well-known instances of the proximal point method, which were not possible previously to the best of our knowledge (under this paper’s setting).
0 50 100 150 200

Fig. 2 Solving a strongly monotone inclusion problem with \( M \) (33); (left) the fixed-point residual vs. iteration, (right) the trajectory of the iterates \( x_i = [x_{i,1}, x_{i,2}]^T \) (markers are displayed every 5th iterations).

6.1 Accelerating the Proximal Point Method for Convex-Concave Saddle-Point Problem

Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be real Hilbert spaces equipped with inner product \((\cdot, \cdot)\). This section considers a convex-concave saddle-point problem

\[
\min_{u \in \mathcal{H}_1} \max_{v \in \mathcal{H}_2} \phi(u, v),
\]

where \( \phi(\cdot, v) \in \mathcal{F}(\mathcal{H}_1) \) and \( -\phi(u, \cdot) \in \mathcal{F}(\mathcal{H}_2) \). The saddle subdifferential operator of \( \phi \),

\[
\begin{bmatrix}
\partial_u \phi(u, v) \\
\partial_v (-\phi(u, v))
\end{bmatrix},
\]

is maximally monotone \([52]\), so \([54]\) applied the proximal point method on such operator to solve the convex-concave saddle-point problem. This section further applies the proposed acceleration to such proximal point method as below.

Accelerated Proximal Point Method for Convex-Concave Saddle-Point Problem

Input: \( \phi(\cdot, v) \in \mathcal{F}(\mathcal{H}_1), -\phi(u, \cdot) \in \mathcal{F}(\mathcal{H}_2), \hat{u}_0 \in \mathcal{H}_1, \hat{v}_0 \in \mathcal{H}_2, \)
\( x_0 = y_0 = y_{-1} = (\hat{u}_0, \hat{v}_0), \lambda \in \mathbb{R}^+ \).

For \( i = 0, 1, \ldots \)
\[
x_{i+1} = (u_{i+1}, v_{i+1}) = \arg \min_{u \in \mathcal{H}_1, v \in \mathcal{H}_2} \left\{ \phi(u, v) + \frac{1}{2\lambda}||u - \hat{u}_i||^2 - \frac{1}{2\lambda}||v - \hat{v}_i||^2 \right\},
\]
\[
y_{i+1} = (\hat{u}_{i+1}, \hat{v}_{i+1}) = x_{i+1} + \frac{1}{i+2} (x_{i+1} - x_i) - \frac{i}{i+2} (x_i - y_{i-1}).
\]

One primary use of this accelerated methods is the following convex-concave Lagrangian problem

\[
\min_{u \in \mathcal{H}_1} \max_{v \in \mathcal{H}_2} \{ L(u, v) := f(u) + \langle v, A u - b \rangle \},
\]

associated with the linearly constrained problem

\[
\min_{u \in \mathcal{H}_1} f(u)
\]
subject to \( A u = b \),

where \( A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) and \( b \in \mathcal{H}_2 \). The resulting method is called the proximal method of multipliers in \([53]\), and applying the proposed acceleration to this method leads to below.
Accelerated Proximal Method of Multipliers

```
Accelerated Proximal Method of Multipliers
Input: φ(·, v) ∈ F(ℋ1), −φ(·, v) ∈ F(ℋ2), u0 ∈ ℋ1, v0 ∈ ℋ2,
\quad x_0 = y_0 = y_{-1} = (u_0, v_0), \lambda \in \mathbb{R}_{++}.
For i = 0, 1, . . .
\quad u_{i+1} = \arg \min_{u \in \mathcal{H}_1} \left\{ L(u, \hat{v}_i) + \frac{\lambda}{2}||Au - b||^2 + \frac{1}{2\lambda}||u - \bar{u}_i||^2 \right\},
\quad x_{i+1} = (u_{i+1}, \hat{v}_i + \lambda(Au_{i+1} - b)),
\quad y_{i+1} = (u_{i+1}, \hat{v}_{i+1}) = x_{i+1} + \frac{i}{i+2}(x_{i+1} - x_i) - \frac{i}{i+2}(x_i - y_{i-1}).
```

Note that this method without the acceleration and the term \(\frac{1}{2\lambda}||u - \bar{u}_i||^2\) reduces to the augmented Lagrangian method. This method has an advantage over the augmented Lagrangian method and its accelerated variants; the primal iterate \(u_{i+1}\) is uniquely defined with a better conditioning.

**Example 6.1** We apply the accelerated proximal method of multipliers to a basis pursuit problem

\[
\min_{u \in \mathbb{R}^{d_1}} \ ||u||_1 \quad \text{subject to} \quad Au = b,
\]

where \(A \in \mathbb{R}^{d_2 \times d_1}\) and \(b \in \mathbb{R}^{d_2}\). In the experiment, we choose \(d_1 = 100, \ d_2 = 20\), and randomly generated \(A\). A true sparse \(u_{\text{true}}\) is randomly generated followed by a thresholding to sparsify nonzero elements, and \(b\) is then given by \(Au_{\text{true}}\). We run 100 iterations of the proximal method of multipliers and its variants with \(\lambda = 0.01\) and initial \(x_0 = 0\). Since the \(u_{i+1}\)-update does not have a closed form, we used a sufficient number of iterations to solve the \(u_{i+1}\)-update using the strongly convex version of FISTA [6] in [9, Theorem 4.10].

Figure 3 compares the proximal method of multipliers and its accelerated variants. Similar to Figure 1, Güler’s first accelerated version diverges, while the proposed method has accelerating behavior, compared to the non-accelerated version. The proposed method exhibits an oscillation in Figure 3 (and a subtle oscillation in Figure 1), which might be due to high momentum, owing from the acceleration, discussed in [49]. So in Figure 3 we heuristically restarted the method every 30 iterations to avoid such oscillation and accelerate, as suggested in [49]. Developing an approach to appropriately choosing a restarting interval or adaptively restarting the method as in [49] for such problem are left as future work.\(^1\)

6.2 Accelerating the Douglas-Rachford Splitting Method

This section considers a monotone inclusion problem in a form

\[
\text{Find}_{x \in \mathcal{H}} \quad \text{subject to} \quad 0 \in (M_1 + M_2)x
\]

for \(M_1, M_2 \in \mathcal{M}(\mathcal{H})\), where \(J_{\rho M_1}\) and \(J_{\rho M_2}\) are more efficient than \(J_{\rho(M_1 + M_2)}\) for a positive real number \(\rho\). For such problem, the Douglas-Rachford splitting method [16, 39] that iteratively applies the operator

\[
G_{\rho, M_1, M_2} := J_{\rho M_1} \circ (2J_{\rho M_2} - I) + (I - J_{\rho M_2})
\]

has been found to be effective in many applications including ADMM, which we discuss in the next section.

In [21, Theorem 4], the Douglas-Rachford operator (40) was found to be a resolvent \(J_{M_{\rho, M_1, M_2}}\) of a maximally monotone operator

\[
M_{\rho, M_1, M_2} := G_{\rho, M_1, M_2}^{-1} - I.
\]

\(^1\) We found that adaptively restarting the method when the fixed-point residual increases seems to be a good option in practice.
In other words, the Douglas-Rachford splitting method is an instance of the proximal point method (with \( \lambda = 1 \)) as

\[
\nu_{i+1} = J_{M_\rho, M_1, M_2}(\nu_i) = G_{\rho, M_1, M_2}(\nu_i)
\]

for \( i = 0, 1, \ldots \). Therefore, we can apply the proposed acceleration to the Douglas-Rachford splitting method as below.

**Accelerated Douglas-Rachford Splitting Method**

**Input:** \( M_1, M_2 \in M(H) \), \( \eta_0 = \eta = \eta_{-1} \in H \), \( \rho \in \mathbb{R}_{++} \).

For \( i = 0, 1, \ldots \)

\[
\begin{align*}
\nu_{i+1} &= G_{\rho, M_1, M_2}(\eta_i) \\
\eta_{i+1} &= \nu_{i+1} + \frac{i}{i+2}(\nu_{i+1} - \nu_i) - \frac{i}{i+2}(\nu_i - \eta_i) - 1
\end{align*}
\]

Using (8) and (27), we have the following worst-case rates for the Douglas-Rachford splitting method and its accelerated variant. Finding exact bounds for the Douglas-Rachford splitting method and its variant is left as future work; [56] used PEP to analyze the exact worst-case rate of Douglas-Rachford splitting method under some additional conditions.

**Corollary 6.1** Assume that \( ||\nu_0 - \nu_*|| \leq R \) for some \( \nu_* \in X_*(M_\rho, M_1, M_2) \). The Douglas-Rachford splitting method satisfies

\[
||\nu_i - \nu_{i-1}||^2 \leq \left(1 - \frac{1}{i}\right)^{i-1}\frac{R^2}{i},
\]

and the proposed accelerated Douglas-Rachford splitting method satisfies

\[
||\nu_i - \eta_{i-1}||^2 \leq \frac{R^2}{i^2}.
\]

[20, 21] illustrated that ADMM is equivalent to the Douglas-Rachford splitting method on the dual problem, so we naturally develop an accelerated ADMM in the next section and provide numerical experiment of the accelerated ADMM and thus the accelerated Douglas-Rachford splitting method.
6.3 Accelerating the Alternating Direction Method of Multipliers (ADMM)

Let \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{G} \) be real Hilbert spaces equipped with inner product \( \langle \cdot, \cdot \rangle \). This section considers a linearly constrained convex problem

\[
\min_{x \in \mathcal{H}_1, z \in \mathcal{H}_2} f(x) + g(z) \tag{45}
\]

subject to \( Ax + Bz = c \),

where \( f \in \mathcal{F}(\mathcal{H}_1), g \in \mathcal{F}(\mathcal{H}_2), A \in \mathcal{B}(\mathcal{H}_1, \mathcal{G}), B \in \mathcal{B}(\mathcal{H}_2, \mathcal{G}) \) and \( c \in \mathcal{G} \). Its dual problem is

\[
\max_{\nu \in \mathcal{G}} \{ -f^*(-A^*\nu) - g^*(-B^*\nu) + \langle c, \nu \rangle \}, \tag{46}
\]

where \( f^*(y) := \sup_{x \in \mathcal{H}_1} \{ \langle y, x \rangle - f(x) \} \) and \( g^*(y) := \sup_{z \in \mathcal{H}_2} \{ \langle y, z \rangle - g(z) \} \) are the conjugate functions of \( f \) and \( g \), respectively. The dual problem (46) is equivalent to the following monotone inclusion problem

\[
\text{Find} \nu \in \mathcal{G} \quad \text{subject to} \quad 0 \in -A\partial f^*(-A^*\nu) - B\partial g^*(-B^*\nu) - c. \tag{47}
\]

We next use the connection between ADMM for solving (45) and the Douglas-Rachford splitting method for solving (47) in [14, Proposition 9] [55] to develop an accelerated ADMM, using the accelerated Douglas-Rachford splitting method in the previous section.

Denoting

\[
M_1 := -A\partial f^*(-A^*\cdot) - c \quad \text{and} \quad M_2 := -B\partial g^*(-B^*) \tag{48}
\]

converts the problem (47) into a form of the monotone inclusion problem (39). Then we use the following equivalent form of the accelerated Douglas-Rachford splitting method to solve (39) with (48):

\[
\begin{align*}
\zeta_{i+1} &= J_{\rho M_2}(\xi_i) \tag{49} \\
\xi_{i+1} &= J_{\rho M_1}(2\zeta_{i+1} - \xi_i) \\
\nu_{i+1} &= \eta_i + (\xi_{i+1} - \zeta_i) \\
\eta_{i+1} &= \nu_{i+1} + \frac{i}{i+2} (\nu_{i+1} - \nu_i) - \frac{i}{i+2} (\nu_i - \eta_{i-1})
\end{align*}
\]

for \( i = 0, 1, \ldots \). Replacing the resolvent operators of \( M_1 \) and \( M_2 \) in (48) by minimization steps yields

\[
\begin{align*}
\tilde{z}_{i+1} &= \arg\min_{z \in \mathcal{H}_2} \left\{ g(z) + \langle \eta_i, Bz \rangle + \frac{\rho}{2}\|Bz\|^2 \right\} \tag{50} \\
\zeta_{i+1} &= \eta_i + \rho B\tilde{z}_{i+1} \\
\tilde{x}_{i+1} &= \arg\min_{x \in \mathcal{H}_1} \left\{ f(x) + \langle \eta_i + 2\rho B\tilde{z}_{i+1}, Ax - c \rangle + \frac{\rho}{2}\|Ax - c\|^2 \right\} \\
\xi_{i+1} &= \eta_i + \rho (A\tilde{x}_{i+1} - c) + 2\rho B\tilde{z}_{i+1} \\
\nu_{i+1} &= \eta_i + \rho (A\tilde{x}_{i+1} + B\tilde{z}_{i+1} - c) \\
\eta_{i+1} &= \nu_{i+1} + \frac{i}{i+2} (\nu_{i+1} - \nu_i) - \frac{i}{i+2} (\nu_i - \eta_{i-1}).
\end{align*}
\]

By discarding \( \zeta_i \) and \( \xi_i \), and defining

\[
\begin{align*}
\tilde{\nu}_i := \nu_i - \rho (A\tilde{x}_i - c) \quad \text{and} \quad \tilde{\eta}_i := \eta_i - \rho (A\tilde{x}_i - c), \tag{51}
\end{align*}
\]

independently of the values of \( \zeta_i, \xi_i \), we have the following ADMM algorithm.

By discarding \( \zeta_i \) and \( \xi_i \), and defining

\[
\begin{align*}
\tilde{\nu}_i := \nu_i - \rho (A\tilde{x}_i - c) \quad \text{and} \quad \tilde{\eta}_i := \eta_i - \rho (A\tilde{x}_i - c), \tag{51}
\end{align*}
\]

independently of the values of \( \zeta_i, \xi_i \), we have the following ADMM algorithm.
for $i = 0, 1, \ldots$, we have

$$z_{i+1} = \arg \min_{z \in \mathcal{H}_1} \left\{ g(z) + \langle \hat{\eta}_i + \rho (A\hat{x}_i - c), Bz \rangle + \frac{\rho}{2} \|Bz\|^2 \right\} \quad (52)$$

$$= \arg \min_{z \in \mathcal{H}_2} \left\{ g(z) + \langle \hat{\eta}_i, A\tilde{x}_i + Bz - c \rangle + \frac{\rho}{2} \|A\tilde{x}_i + Bz - c\|^2 \right\}$$

$$\tilde{x}_{i+1} = \arg \min_{x \in \mathcal{H}_1} \left\{ f(x) + \langle \hat{\nu}_{i+1} + \rho Bz_{i+1}, Ax - c \rangle + \frac{\rho}{2} \|Ax + Bz_{i+1} - c\|^2 \right\}$$

$$= \arg \min_{x \in \mathcal{H}_1} \left\{ f(x) + \langle \hat{\nu}_{i+1} + \rho Bz_{i+1}, Ax + Bz_{i+1} - c \rangle + \frac{\rho}{2} \|Ax + Bz_{i+1} - c\|^2 \right\}$$

$$\hat{\nu}_{i+1} = \hat{\eta}_i + \rho (A\tilde{x}_i + Bz_{i+1} - c)$$

$$\eta_{i+1} = \frac{i}{i+2} (\hat{\nu}_{i+1} - \nu_i + \rho A(\tilde{x}_{i+1} - \tilde{x}_i)) - \frac{i}{i+2} (\hat{\nu}_i - \nu_{i-1} + \rho A(\tilde{x}_i - \tilde{x}_{i-1})).$$

Then, replacing $\tilde{x}_i$ by $x_{i+1}$ and reordering steps appropriately yield the following accelerated version of ADMM, which reduces to the standard ADMM when we let $\hat{\eta}_i = \hat{\nu}_i$ for $i = 0, 1, \ldots$

| Accelerated Alternating Direction Method of Multipliers |
|-----------------------------------------------------|
| Input: $f \in \mathcal{F}(\mathcal{H}_1)$, $g \in \mathcal{F}(\mathcal{H}_2)$, $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{G})$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{G})$, $x_0 \in \mathcal{H}_1$, $z_0 \in \mathcal{H}_2$, $\nu_0 \in \mathcal{G}$, $\rho \in \mathbb{R}^+$. For $i = 0, 1, \ldots$ |
| $x_{i+1} = \arg \min_{x \in \mathcal{H}_1} \left\{ f(x) + \langle \hat{\nu}_i, Ax + Bz_i - c \rangle + \frac{\rho}{2} \|Ax + Bz_i - c\|^2 \right\}$ |
| $\hat{\eta}_i = \left\{ \begin{array}{ll}
\hat{\nu}_i & i = 0, 1, \\
\hat{\nu}_i + \frac{i}{i+1} (\hat{\nu}_i - \hat{\nu}_{i-1} + \rho A(x_{i+1} - x_i)) - \frac{i}{i+1} (\hat{\nu}_{i-1} - \hat{\nu}_{i-2} + \rho A(x_i - x_{i-1})), & i = 2, 3, \ldots \\
\end{array} \right.$ |
| $z_{i+1} = \arg \min_{z \in \mathcal{H}_2} \left\{ g(z) + \langle \hat{\eta}_i, Ax_{i+1} + Bz - c \rangle + \frac{\rho}{2} \|Ax_{i+1} + Bz - c\|^2 \right\}$ |
| $\hat{\nu}_{i+1} = \hat{\eta}_i + \rho (A\tilde{x}_{i+1} + Bz_{i+1} - c)$ |

Since

$$\nu_i = \eta_{i-1} = \hat{\nu}_i - \hat{\eta}_{i-1} - \rho (Ax_{i+1} - Ax_i) = \rho (Ax_{i+1} + Bz_i - c),$$

we have the following worst-case rates with respect to the infeasibility for ADMM and its accelerated version, using (8) and (27). Remark 2.1 implies that the following results also apply to the Chambolle-Pock method [8,10,31] that is a preconditioned ADMM.

**Corollary 6.2** Assume that $\|\nu_0 + \rho A(x_0 - c) - \nu_0\| \leq R$ for some $\nu_0 \in X_*(M_{\rho,-A\partial f^*(-A^*)^{-1}c,-B\partial g(-B^*)})$ Alternating direction method of multipliers satisfies

$$\|Ax_{i+1} + Bz_i - c\|^2 \leq \left( 1 - \frac{1}{i} \right)^{i-1} \frac{R^2}{\rho^2 i^2},$$

and the proposed accelerated alternating direction method of multipliers satisfies

$$\|Ax_{i+1} + Bz_i - c\|^2 \leq \frac{R^2}{\rho^2 i^2}.$$  

The bound (54) is $(1 - \frac{1}{i})^{i-1}$-times (and $i$-times asymptotically) smaller than the known rate for ADMM in [14, Theorem 15], which originated from the bound (7). Finding exact bounds for the ADMM and its proposed variant is yet left as future work.

**Remark 6.1** Many existing rates for (preconditioned) ADMM consider the **ergodic** sequences $\{\tilde{x}_i\}$ and $\{z_i\}$, where $\tilde{x}_i := \frac{1}{i} \sum_{r=1}^i x_r$ and $z_i := \frac{1}{i} \sum_{r=1}^i z_r$ (see e.g., [8,10,14,15]). In particular, in [14, Theorem 15], ADMM is found to satisfy

$$\|Ax_{i+1} + Bz_i - c\|^2 \leq \frac{16R^2}{\rho^2 i^2}.$$
which is faster than the rate of the nonergodic sequence \(\{x_i, z_i\}\) of ADMM in (54) and is comparable to the rate of the proposed accelerated ADMM in (55). However, some numerical experiments in [10] illustrate that the performance of the nonergodic sequence can be faster than that of the ergodic sequence. We leave further understanding of the rates of the ergodic and nonergodic sequences of (preconditioned) ADMM and their relationship as future work.

Remark 6.2 [8,10,24] proposed accelerated variants of (preconditioned) ADMM under some additional conditions, while the proposed method does not require such conditions.

Example 6.2 We apply the accelerated ADMM to the problem

\[
\min_{x \in \mathbb{R}^{d_1}, z \in \mathbb{R}^{d_2}} \frac{1}{2} ||Hx - b||^2 + \gamma ||z||_1
\]

subject to \(Dx - z = 0\),

with a positive real number \(\gamma\), associated with the total-variation-regularized least-squares problem

\[
\min_{x \in \mathbb{R}^{d_1}} \frac{1}{2} ||Hx - b||^2 + \gamma ||Dx||_1,
\]

where \(H \in \mathbb{R}^{p \times d_1}, b \in \mathbb{R}^p\), and a matrix \(D \in \mathbb{R}^{d_2 \times d_1}\) is given as

\[
D = \begin{bmatrix}
1 & -1 & 0 & 0 & \cdots & 0 \\
0 & 1 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 0 & 1 & -1 & 0 \\
0 & \cdots & \cdots & 0 & 1 & -1 \\
\end{bmatrix}
\]

By letting \(f(x) = \frac{1}{2} ||Hx - b||^2\), \(g(z) = \gamma ||z||_1\), \(A = D\), \(B = -I\) and \(c = 0\), we have the following accelerated ADMM method:

\[
x_{i+1} = \arg \min_{x \in \mathbb{R}^{d_1}} \left\{ \frac{1}{2} ||Hx - b||^2 + (\hat{\nu}_i, Dx - z_i) + \frac{\rho}{2} ||Dx - z||^2 \right\}
\]

\[
= (H^\top H + \rho D^\top D)^{-1}(D^\top (\rho z_i - \hat{\nu}_i) + H^\top b)
\]

\[
\hat{\eta}_i = \begin{cases}
\hat{\nu}_i, & i = 0, 1, \\
\nu_i + \frac{1}{\tau_1}(\hat{\nu}_i - \nu_{i-1} + \rho D(x_{i+1} - x_i)) - \frac{1}{\tau_2}(\hat{\eta}_{i-1} - \hat{\eta}_{i-2} + \rho D(x_i - x_{i-1})), & i = 2, 3, \ldots
\end{cases}
\]

\[
z_{i+1} = \arg \min_{z \in \mathbb{R}^{d_2}} \left\{ \gamma ||z||_1 + (\hat{\eta}_i, Dx_{i+1} - z) + \frac{\rho}{2} ||Dx_{i+1} - z||^2 \right\} = S_\gamma \left(Dx_{i+1} + \frac{1}{\rho} \hat{\eta}_i\right)
\]

\[
\hat{\nu}_{i+1} = \hat{\eta}_i + \rho (Dx_{i+1} - z_{i+1}),
\]

where the soft-thresholding operator is defined as \(S_\gamma(z) := \max\{|z| - \tau, 0\} \odot \text{sign}(z)\) with the element-wise absolute value, maximum and multiplication operators, \(|\cdot|\), \(\max\{\cdot, \cdot\}\) and \(\odot\), respectively.

In the experiment, we choose \(d_1 = 100, d_2 = 99, p = 5\), and a true vector \(x_{\text{true}}\) is constructed such that a vector \(Dx_{\text{true}}\) has few nonzero elements. A matrix \(H\) is randomly generated and a noisy vector \(b\) is generated by adding randomly generated (noise) vector to \(Hx_{\text{true}}\). We choose the parameters \(\gamma = 3\) and \(\rho = 0.05\) in the experiment.

Figure 4 illustrates the fixed-point residual of ADMM and its accelerated variants. Interestingly, ADMM has a rate comparable to the \(O(1/i^2)\) rate of the proposed method. This does not contradict with the theory, and we leave further investigating the worst-case rate of ADMM under the Lipschitz continuity condition of \(\nabla f\); similar analysis but under different conditions can be found in [14,15]. Noticing the oscillating behavior of the proposed ADMM in Figure 4, we heuristically restarted the proposed method every 20 iterations, yielding a linear rate, without a strong monotonicity condition.\(^2\)

While [8] found restarting (reinitializing) a relaxed (preconditioned) ADMM not useful, our experiment suggests that restarting could be effective in some practical cases, as also illustrated in [24] for a different accelerated ADMM.

\(^2\) Since \(\nabla f\) is Lipschitz continuous, the operator \(M_1 = -D_0D^\top\) in (48) for the problem (56) is strongly monotone, but this is insufficient to guarantee a strong monotonicity of \(M_\rho, M_1, M_2\) (41) for the problem (56).
Fig. 4 Solving a total-variation-regularized least-squares problem (57); the fixed-point residual vs. iteration

7 Accelerated Forward Method for Cocoercive Operators

Inspired by the acceleration of the proximal point method (also known as the backward method), this section applies the proposed acceleration to the forward method for cocoercive operators. A single-valued operator $M : \mathcal{H} \to \mathcal{H}$ is $\beta$-cocoercive for $\beta \in \mathbb{R}^+_0$ if

$$\langle Mx - My, x - y \rangle \geq \beta \|Mx - My\|^2 \quad \text{for all } x, y \in \mathcal{H}. \quad (60)$$

Let $C_\beta(\mathcal{H})$ be the class of $\beta$-cocoercive operators on $\mathcal{H}$.

For the $\beta$-cocoercive operator, the following forward method (that iteratively applies the forward operator $I - \beta M$) is guaranteed to converge weakly to a solution [5, Theorem 26.14]; this section presents the exact sublinear rate for the forward method, and its acceleration.

**Forward Method**

Input: $M \in C_\beta(\mathcal{H}), \ y_0 \in \mathcal{H}$.

For $i = 0, 1, \ldots$

$$y_{i+1} = (I - \beta M)y_i.$$

Similar to (17), the worst-case rate of the general forward method (implicitly defined below similar to the general proximal point method) after $N$ iterations for decreasing the fixed-point residual (6) under the initial distance condition (4) can be computed by

$$\max_{M \in C_\beta(\mathcal{H})} \max_{x_1, \ldots, x_N \in \mathcal{H}, \ y_0, \ldots, y_{N-1} \in \mathcal{H}, \ x_i \in X_i(M)} \frac{1}{R^2} \|x_N - y_{N-1}\|^2 \quad (61)$$

subject to $x_{i+1} = (I - \beta M)y_i, \ i = 0, \ldots, N - 1,$

$$y_{i+1} = y_i + \sum_{k=0}^{i} h_{i+1, k+1}(x_{k+1} - y_k), \ i = 0, \ldots, N - 2,$$

$$\|y_0 - x_0\|^2 \leq R^2.$$

Similar to (19), we reformulate the problem (61) into a finite-dimensional problem. [56, Proposition 2] implies that the constraint $M \in C_\beta(\mathcal{H})$ can be replaced by a set of finite number of inequality constraints (60) for $M \in C_\beta(\mathcal{H}),$ which is

$$\langle \tilde{q}_i - \tilde{q}_j, y_i - y_j \rangle \geq \beta \|\tilde{q}_i - \tilde{q}_j\|^2$$
for all $i, j \in \{0, \ldots, N-1, \ast\}$, with additional variables $\tilde{q}_i \in M y_i$ for $i = 0, \ldots, N-1$ and $\tilde{q}_\ast = 0 \in M x_\ast$. Then, the corresponding equivalent problem of (61) is

$$\max_{x_1, \ldots, x_N, x_N \in \mathcal{H}, y_0, \ldots, y_{N-1} \in \mathcal{H}} \frac{1}{R^2}||x_N - y_{N-1}||^2$$

subject to

$$\langle \tilde{q}_i, y_i - y_j \rangle \geq \beta ||\tilde{q}_i||^2, \quad i < j = 0, \ldots, N-1,$$

$$\langle \tilde{q}_i, y_i - y_\ast \rangle \geq \beta ||\tilde{q}_i||^2, \quad i = 0, \ldots, N-1,$$

$$x_{i+1} = y_i - \beta \tilde{q}_i, \quad i = 0, \ldots, N-1,$$

$$y_{i+1} = y_i - \beta \sum_{k=0}^{i} h_{i+1,k+1} q_k, \quad i = 0, \ldots, N-2,$$

$$||y_0 - x_\ast||^2 \leq R^2.$$  

Interestingly, replacing $\beta$ and $\tilde{q}_i$ for $i = 0, \ldots, N-1$ by $\lambda$ and $q_{i+1}$ for $i = 0, \ldots, N-1$ respectively, and then using $x_{i+1} = y_i - \lambda q_{i+1}$ makes the problem (61) equivalent to (17). Therefore, the results on the proximal point method and its accelerated variant directly apply to the forward method and its accelerated variant as below.

**Theorem 7.1** Let $M \in \mathcal{C}_\beta(\mathcal{H})$ and let $y_0, y_1, \ldots \in \mathcal{H}$ be generated by the forward method. Assume that $||y_0 - x_\ast|| \leq R$ for a constant $R > 0$ and for some $x_\ast \in X_\ast(M)$. Then for any $i \geq 1$,

$$||M y_{i-1}||^2 \leq \left(1 - \frac{1}{i}\right)^{i-1} \frac{R^2}{\beta^2}.$$  

Starting from an initial point $y_0 = [1 \ 0]^\top$, the inequality (63) for $||M y_{N-1}||^2$ is exactly achieved by the operator

$$M \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\beta N} \begin{bmatrix} 1 & \sqrt{N-1} \\ -\sqrt{N-1} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

which is the Yosida approximation of the operator (29) of index $\beta = \lambda$.

**Remark 7.1** The Yosida approximation of $M$ of index $\lambda$ is

$$M_\lambda := \frac{1}{\lambda} (I - J_{\lambda M}).$$

This operator is $\lambda$-cocoercive and provides a forward operator interpretation of the resolvent operator as

$$J_{\lambda M} = (I + \lambda M)^{-1} = I - \lambda M_\lambda,$$

which is thus closely related to the equivalence between (17) and (61).

The following is the proposed accelerated forward method, which has the fast $O(1/i^2)$ rate.

**Proposed Accelerated Forward Method for Maximally Monotone Operators**

Input: $M \in \mathcal{C}_\beta(\mathcal{H})$, $x_0 = y_0 = y_{-1} \in \mathcal{H}$.

For $i = 0, 1, \ldots$

$$x_{i+1} = (I - \beta M) y_i,$$

$$y_{i+1} = x_{i+1} + \frac{i}{i + 2} (x_{i+1} - x_i) - \frac{i}{i + 2} (x_i - y_{i-1}).$$

**Theorem 7.2** Let $M \in \mathcal{C}_\beta(\mathcal{H})$ and let $x_0, y_0, x_1, y_1, \ldots \in \mathcal{H}$ be generated by the proposed accelerated forward method. Assume that $||x_0 - x_\ast|| \leq R$ for a constant $R > 0$ and for some $x_\ast \in X_\ast(M)$. Then for any $i \geq 1$,

$$||M y_{i-1}||^2 \leq \frac{R^2}{\beta^2 i^2}.$$  

19
Similar to Section 5 under additional strongly monotonicity, i.e., \( M \in \mathcal{M}_\mu(\mathcal{H}) \cap \mathcal{C}_\beta(\mathcal{H}) \) where \( \mu \leq \frac{1}{\gamma} \), the accelerated forward method with restarting can yield fast linear rate. Suppose one restarts the proposed accelerated forward method every \( k \) (inner) iterations by initializing the \((j+1)\)th outer iteration \( x_{j+1,0} = y_{j+1,0} = y_{j+1,-1} \) by \( y_{j,l} \), where \( x_{j,l} \) and \( y_{j,l} \) denote iterates at the \( j \)th outer iteration and \( l \)th inner iteration for \( j = 0, 1, \ldots \) and \( l = -1, 0, 1, \ldots, k \). Using the rate (67) (with \( R = ||x_{j,0} - x_*|| \)) and the strong monotonicity condition (2), we have a linear rate

\[
||M y_{j,k}||^2 \leq \frac{||x_{j,0} - x_*||^2}{\beta^2(k+1)^2} \leq \frac{1}{\mu \beta^2(k+1)^2} ||Mx_{j,0}||^2
\]

for \( j = 0, 1, \ldots \). For a given \( N = jk \) total number of steps, the (approximate) optimal choice of the restarting interval is

\[
k_{opt}' \approx \frac{e}{\beta \sqrt{\mu}},
\]

by using the upper bound \( \frac{1}{\mu \beta^2} ||Mx_{j,0}||^2 \) of (68) for simplicity. The corresponding linear rate is \( O((e^\beta \sqrt{\mu}/e)^{-2N}) \).

For numerical verification, Figure 5 considers two toy problems \( M \) (64) for \( N = 100 \) with 100 iterations, and

\[
M \begin{bmatrix} u \\ v \end{bmatrix} = \begin{pmatrix} \frac{1}{\beta N} \left( \begin{array}{c} 1 \\ \sqrt{N-1} \end{array} \right) \\ \begin{array}{c} -\sqrt{N-1} \\ 1 \end{array} \end{pmatrix} + \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]

for \( N = 100, \beta = 1 \) and \( \mu = 0.02 \). The optimal restarting interval is \( k_{opt}' \approx 19 \), and we run 200 iteration in the second experiment with (70). Figure 5 compares the forward method, Nesterov’s fast gradient method [46, 47] (with \( M \) instead of \( \nabla f \) for \( f \in \mathcal{F}(\mathcal{H}) \) with Lipschitz continuous gradient), and the proposed accelerated forward method (with and without restarting). As expected, the results in Figure 5 are similar to those of Figures 1 and 2.

![Fig. 5 Solving a worst-case monotone inclusion problem; the fixed-point residual vs. iteration with operators (left) \( M \) in (64) and (right) \( M \) in (70).](image)

8 Conclusion

This paper developed an accelerated proximal point method (i.e., an accelerated backward method) for maximally monotone operators, and an accelerated forward method for cocoercive operators, with respect to the fixed-point residual, using the computer-assisted performance estimation problem approach. Restarting technique was further employed to accelerate the proximal point method and the forward
method under the strong monotonicity condition. The proposed acceleration was applied to various instances of the proximal point method such as the proximal method of multipliers, the Douglas-Rachford splitting method, and the alternating direction method of multipliers, yielding accelerations both theoretically and practically.

We leave developing accelerations for more general or more specific classes of problems or methods as future work, possibly via the performance estimation problem approach; a comprehensive understanding of accelerations for the alternating direction method of multipliers with respect to various performance measures under various conditions are yet remain open.

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