On Cartesian product of matrices

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Abstract

Recently, Bapat and Kurata [Linear Algebra Appl., 562(2019), 135-153] defined the Cartesian product of two square matrices $A$ and $B$ as $A \otimes B = A \otimes J + J \otimes B$, where $J$ is the all one matrix of appropriate order and $\otimes$ is the Kronecker product. In this article, we find the expression for the trace of the Cartesian product of any finite number of square matrices in terms of traces of the individual matrices. Also, we establish some identities involving the Cartesian product of matrices. Finally, we apply the Cartesian product to study some graph-theoretic properties.

Keywords: Cartesian product, Kronecker product, Hadamard product, Trace.

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1 Introduction and terminology

By $M_{m,n}$, we denote the class of all matrices of size $m \times n$. Also, by $M_n$, we denote the class of all square matrices of order $n$. For $M \in M_n$, we write $m_{ij}$ or $M_{ij}$ to denote the $ij$–th element of $M$. By $J$ and $I$, we mean the matrix of all one’s and vector of all one’s, respectively of suitable order. Similarly 0 denotes the zero matrix or the vector. We will mention their order wherever its necessary. Throughout this article, we denote the sum of all entries of a matrix $A$ by $S_A$ and the sum of the entries of $i$–th row of $A$ by $A_i$. The
inertia of a square matrix $M$ with real eigenvalues is the triplet $(n_+(M), n_0(M), n_-(M))$, where $n_+(M)$ and $n_-(M)$ denote the number of positive and negative eigenvalues of $M$, respectively, and $n_0(M)$ is the algebraic multiplicity of 0 as an eigenvalue of $M$.

The Kronecker product of two matrices $A$ and $B$ of sizes $m \times n$ and $p \times q$, respectively, denoted by $A \otimes B$ is defined to be the $mp \times nq$ block matrix

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{pmatrix}. $$

The Hadamard product of two matrices $A$ and $B$ of the same size, denoted by $A \odot B$ is defined to be the entrywise product $A \odot B = [a_{i,j}b_{i,j}]$.

Bapat and Kurata [2] defined the Cartesian product of two square matrices $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$ as $A \otimes B = A \otimes \mathbb{I}_n + J_m \otimes B$. The authors proved the Cartesian product to be associative. We use $A^{[k]}$ to mean $\underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}}$.

If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $A \otimes B$ can be considered as a block matrix with $i, j$–th block $a_{ii} \mathbb{I}_n + B$, $i = 1, 2, \ldots, m$, in other words $A \otimes B$ is the matrix obtained from $A$ by replacing $a_{i,j}$ by $a_{ii} \mathbb{I}_n + B$. It can be observed that $a_{i,j} + b_{p,q}$ is the $p, q$–th entry of the $i, j$–th block of $A \otimes B$.

All graphs considered here are finite, undirected, connected and simple. The distance between two vertices $u, v \in V(G)$ is denoted by $d_{uv}$ and is defined as the length of a shortest path between $u$ and $v$ in $G$. The distance matrix of $G$ is denoted by $\mathcal{D}(G)$ and is defined by $\mathcal{D}(G) = (d_{uv})_{u,v \in V(G)}$. Since $\mathcal{D}(G)$ is a real symmetric matrix, all its eigenvalues are real. For a column vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, we have

$$x^T \mathcal{D}(G)x = \sum_{1 \leq i < j \leq n} d_{ij}x_ix_j. $$

The Wiener index $W(G)$ of a graph is the sum of the distances between all unordered pairs of vertices of $G$, in other words $W(G) = \frac{\sum_{i,j} \mathcal{D}(G)_{ij}}{2}$. The distance spectral radius $\rho^D(G)$ of $G$ is the largest eigenvalue of its distance matrix $\mathcal{D}(G)$. The transmission, denoted by $Tr(v)$ of a vertex $v$ is the sum of the distances from $v$ to all other vertices in $G$.

The Cartesian product $G_1 \square G_2$ of two graphs $G_1$ and $G_2$ is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$ and in which two vertices $(u, u')$ and $(v, v')$ are adjacent if and only if either $u = v$ and $u'$ is adjacent to $v'$ in $G_2$, or $u' = v'$ and $u$ is
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adjacent to $v$ in $G_1$. Let $Gu*Hv$ denote the graph obtained from two graphs $G$ and $H$
by identifying a vertex $u$ from $G$ with a vertex $v$ from $H$.

The article have been organized as follows. In Section 2, we discuss some existing
results involving Kronecker product of matrices and Cartesian product of graphs. In
Section 3, we find trace of various compositions of matrices involving Cartesian product.
Again in Section 4, we obtain some identities involving Cartesian product of matrices and
find some applications in graph theory.

2 Preliminaries

Kronecker product has been extensively studied in the literature. Some of the interesting
properties of the Kronecker product are given below.

Lemma 2.1. [5] If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $\text{tr}(A \otimes B) = \text{tr}(A) \times \text{tr}(B)$.

Lemma 2.2. [5] If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $(A \otimes B)^T = A^T \times B^T$.

Lemma 2.3. [5] If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $(A \otimes B)^* = A^* \times B^*$.

Lemma 2.4. [5] For $A \in \mathbb{M}_m$, $B \in \mathbb{M}_n$, and $a, b \in \mathbb{C}$, $aA \otimes bB = abA \otimes B$.

Lemma 2.5. [5] For matrices $A, B, C$ and $D$ of appropriate sizes

\[(A \otimes B)(C \otimes D) = (AC) \otimes (BD).\]

Lemma 2.6. [5] For any $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, there exist a permutation matrix $P$ such that

\[P^{-1}(A \otimes B)P = B \otimes A.\]

For more results on Kronecker product, we refer [3]. The Cartesian product of two
graphs have been studied by many researchers. Here we are interested in Cartesian
product of two matrices because for any two connected graphs $G_1$ and $G_2$, the distance
matrix of $G_1 \Box G_2$ equals to the Cartesian product of the distance matrices of $G_1$ and $G_2$,
i.e. $D(G_1 \Box G_2) = D(G_1) \otimes D(G_2)$. Zhang and Godsil [6] found the distance inertia of the
Cartesian product of two graphs.

Theorem 2.7. [6] If $G$ and $H$ are two connected graphs, where $V(G) = \{u_1, \ldots, u_m\}$
and $V(H) = \{v_1, \ldots, v_n\}$, then, the inertia of distance matrix of $G \Box H$ is $(n_+(Gu_m * 
Hu_n), (m-1)(n-1) + n_0(Gu_m * Hu_n), n_-(Gu_m * Hu_n))$. 


Corollary 2.8. [6] Let $T_1$ and $T_2$ be two trees on $m$ and $n$ vertices, respectively. Then the distance inertia of $T_1 \square T_2$ is $(1, (m-1)(n-1), m+n-2)$.

3 Trace of Cartesian product

Here we consider different compositions and products involving Cartesian product of matrices and evaluate their trace.

Lemma 3.1. If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $tr(A \otimes B) = n.\text{tr}(A) + m.\text{tr}(B)$.

Proof. We have
\[
\text{tr}(A \otimes B) = \text{tr}(A \otimes J_n + J_n \otimes B) = \text{tr}(A \otimes J_n) + \text{tr}(J_m \otimes B) = \text{tr}(A) \times \text{tr}(J_n) + \text{tr}(J_m) \times \text{tr}(B) \quad \text{[using Lemma 2.1]}
\]
\[
= n.\text{tr}(A) + m.\text{tr}(B).
\]

\[\blacksquare\]

Theorem 3.2. If $A_i \in \mathbb{M}_{n_i}$ and $k_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$, then
\[
tr(k_1 A_1 \otimes k_2 A_2 \otimes \cdots \otimes k_n A_n) = (\Pi_{i=1}^n n_i) \sum_{i=1}^n k_i \text{tr}(A_i) \frac{n_i}{n}.
\]

Proof. We prove the result by induction on $n$. For $n = 1$, there is nothing to prove. For $n = 2$, the result follows from Lemma 3.1. Suppose the result holds for $n = \ell \leq n - 1$. That is
\[
tr(k_1 A_1 \otimes k_2 A_2 \otimes \cdots \otimes k_\ell A_\ell) = (\Pi_{i=1}^\ell n_i) \sum_{i=1}^\ell k_i \text{tr}(A_i) \frac{n_i}{n_i}. \quad (3.1)
\]

Now
\[
\text{tr}(k_1 A_1 \otimes k_2 A_2 \otimes \cdots \otimes k_\ell A_\ell \otimes k_{\ell+1} A_{\ell+1}) = k_{\ell+1} n_{\ell+1} tr(k_1 A_1 \otimes k_2 A_2 \otimes \cdots \otimes k_{\ell+1} A_{\ell+1}) + (\Pi_{i=1}^\ell k_i n_i) \text{tr}(A_{\ell+1}) \quad \text{[using Lemma 3.1]}
\]
\[
= k_{\ell+1} n_{\ell+1} (\Pi_{i=1}^\ell n_i) \sum_{i=1}^\ell k_i \text{tr}(A_i) \frac{n_i}{n_i} + (\Pi_{i=1}^\ell k_i n_i) \text{tr}(A_{\ell+1}) \quad \text{[using (3.1)]}
\]
\[
= (\Pi_{i=1}^{\ell+1} n_i) \sum_{i=1}^{\ell+1} k_i \text{tr}(A_i) \frac{n_i}{n_i}.
\]

Hence the result follows by induction. \[\blacksquare\]

As immediate corollary of the above theorem we get the following result.
Corollary 3.3. For $A \in \mathbb{M}_n$, $\text{tr}(A^k) = k \cdot n^{k-1} \text{tr}(A)$.

Proposition 3.4. If $A, B \in \mathbb{M}_n$, then

$$\text{tr}((A + B) \otimes (A - B)) = 2n \cdot \text{tr}(A).$$

Proof. From Lemma 3.1, we have

$$\text{tr}((A + B) \otimes (A - B)) = n \cdot \text{tr}(A - B) + n \cdot \text{tr}(A + B)$$

$$= n[\text{tr}(A) - \text{tr}(B) + \text{tr}(A) + \text{tr}(B)]$$

$$= 2n \cdot \text{tr}(A).$$

Proposition 3.5. For $A \in \mathbb{M}_m, b_i \in \mathbb{M}_n; i = 1, 2, \ldots, k$, then

$$\text{tr}(A \otimes (b_1 \otimes b_2 \otimes \cdots \otimes b_k)) = n^{k-1} \text{tr}(A) \sum_{i=1}^{k} \text{tr}(b_i).$$

Proof. We have

$$\text{tr}(A \otimes (b_1 \otimes b_2 \otimes \cdots \otimes b_k))$$

$$= \text{tr}(A) \cdot \text{tr}(b_1 \otimes b_2 \otimes \cdots \otimes b_k) \quad \text{[using Lemma 2.1]}$$

$$= \text{tr}(A) \cdot n^{k-1} \sum_{i=1}^{k} \text{tr}(b_i) \quad \text{[using Theorem 3.2]}$$

$$= n^{k-1} \text{tr}(A) \sum_{i=1}^{k} \text{tr}(b_i).$$

Theorem 3.6. If $A_i \in \mathbb{M}_{n_i}$ for $i = 1, 2, \ldots, t$, then

$$\text{tr}[(A_1 \otimes A_2 \otimes \cdots \otimes A_\ell) \otimes (A_{\ell+1} \otimes A_{\ell+2} \otimes \cdots \otimes A_m) \otimes \cdots \otimes (A_r \otimes A_{r+1} \cdots A_t)]$$

$$= \prod_{p=1}^{t} n_p \sum_{i=1}^{\ell} \frac{\text{tr}(A_i)}{n_i} \sum_{j=\ell+1}^{m} \frac{\text{tr}(A_j)}{n_j} \cdots \sum_{k=r}^{t} \frac{\text{tr}(A_k)}{n_k}.$$
\[ \left( \prod_{i=1}^{t} \sum_{i=1}^{\ell} \frac{\text{tr}(A_i)}{n_i} \right) \left( \prod_{j=\ell+1}^{m} \sum_{j=\ell+1}^{m} \frac{\text{tr}(A_j)}{n_j} \right) \cdots \left( \prod_{k=r}^{t} \sum_{k=r}^{t} \frac{\text{tr}(A_k)}{n_k} \right) \]  

[using Theorem 3.2]

\[ = \prod_{p=1}^{t} \sum_{i=1}^{\ell} \frac{\text{tr}(A_i)}{n_i} \sum_{j=\ell+1}^{m} \frac{\text{tr}(A_j)}{n_j} \cdots \sum_{k=r}^{t} \frac{\text{tr}(A_k)}{n_k} . \]

\[ \textbf{Theorem 3.7.} \text{ If } A_i \in \mathbb{M}_{n_i}; i = 1, 2, \ldots, t, \text{ then} \]

\[ \text{tr} \left[ (A_1 \otimes A_2 \otimes \cdots \otimes A_{\ell}) \otimes (A_{\ell+1} \otimes A_{\ell+2} \otimes \cdots \otimes A_m) \otimes \cdots \otimes (A_r \otimes A_{r+1} \otimes \cdots \otimes A_t) \right] \]

\[ = \prod_{p=1}^{t} \left[ \prod_{i=1}^{\ell} \frac{\text{tr}(A_i)}{n_i} + \prod_{j=\ell+1}^{m} \frac{\text{tr}(A_j)}{n_j} + \cdots + \prod_{k=r}^{t} \frac{\text{tr}(A_k)}{n_k} \right]. \]

\[ \text{Proof.} \text{ Using Theorem 3.2 and then Lemma 2.1, we get} \]

\[ \text{tr} \left[ (A_1 \otimes A_2 \otimes \cdots \otimes A_{\ell}) \otimes (A_{\ell+1} \otimes A_{\ell+2} \otimes \cdots \otimes A_m) \otimes \cdots \otimes (A_r \otimes A_{r+1} \otimes \cdots \otimes A_t) \right] \]

\[ = \prod_{p=1}^{t} \left[ \prod_{i=1}^{\ell} \frac{\text{tr}(A_i)}{n_i} + \prod_{j=\ell+1}^{m} \frac{\text{tr}(A_j)}{n_j} + \cdots + \prod_{k=r}^{t} \frac{\text{tr}(A_k)}{n_k} \right]. \]

\[ \textbf{4 Some identities and applications} \]

From the definition of Cartesian product of two matrices, we get following remarks.

\[ \textbf{Remark 4.1.} \text{ If } A \text{ and } B \text{ are square matrices and } k \in \mathbb{C}, \text{ then } kA \otimes kB = k(A \otimes B). \]

\[ \textbf{Remark 4.2.} \text{ For } A \in \mathbb{M}_{n} \text{ and any } k \in \mathbb{C}, \text{ } k \otimes A = A + kJ_n = A \otimes k. \]

For any square matrices A and B, from the definitions of Kronecker product and Cartesian product, it can be observed that if \( a_{i,j}b_{p,q} \) is an entry of \( A \otimes B \) then the corresponding entry of \( A \otimes B \) is \( a_{i,j} + b_{p,q} \). Thus from Lemma 2.6, we see that if \( P^{-1}(A \otimes B)P = B \otimes A \), then for the same \( P \), we get \( P^{-1}(A \otimes B)P = B \otimes A \). Thus we get the following result.

\[ \textbf{Remark 4.3.} \text{ If } A \text{ and } B \text{ are square matrices, then } A \otimes B \text{ is permutation similar to } B \otimes A. \]
Proposition 4.4. For $A \in \mathbb{M}_m, B \in \mathbb{M}_n$, $(A \odot B)^T = A^T \odot B^T$.

Proof. By definition we have

$$A \odot B = A \otimes J_n + J_m \otimes B$$

which implies

$$(A \odot B)^T = (A \otimes J_n + J_m \otimes B)^T$$

$$= (A \otimes J_n)^T + (J_m \otimes B)^T$$

$$= A^T \otimes J_n^T + J_m^T \otimes B^T$$

[using Lemma 2.2]

$$= A^T \otimes J_n + J_m \otimes B^T$$

$$= A^T \odot B^T.$$  

Hence the result. 

By repeated application of Proposition 4.4, we get the following result as a corollary.

Corollary 4.5. For square matrices $A_i$ for $i = 1, 2, \ldots, n$,

$$(A_1 \odot A_2 \odot \cdots \odot A_n)^T = A_1^T \odot A_2^T \odot \cdots \odot A_n^T.$$  

Proceeding as in Proposition 4.4 and using Lemma 2.3, we get the following result.

Proposition 4.6. For $A \in \mathbb{M}_m, B \in \mathbb{M}_n$, $(A \odot B)^* = A^* \odot B^*$.

By repeated application of Proposition 4.6, we get the following result as a corollary.

Corollary 4.7. For square matrices $A_i$ for $i = 1, 2, \ldots, n$,

$$(A_1 \odot A_2 \odot \cdots \odot A_n)^* = A_1^* \odot A_2^* \odot \cdots \odot A_n^*.$$  

Theorem 4.8. If $A \in \mathbb{M}_m, B \in \mathbb{M}_n$, then $A \odot B$ is symmetric if and only if $A$ and $B$ are both symmetric.

Proof. If $A$ and $B$ are both symmetric, then $A^T = A$ and $B^T = B$. Now

$$(A \odot B)^T = A^T \odot B^T$$  \quad [by Proposition 4.4]

$$= A \odot B.$$  

Therefore $A \odot B$ is symmetric.
Conversely, suppose that \( A \otimes B \) is symmetric. Then 1,1 block of \( A \otimes B \) must be symmetric. But 1,1 block of \( A \otimes B \) is \( a_{1,1}J_n + B \) which is symmetric if and only if \( B \) is symmetric. Again since \( A \otimes B \) is symmetric, the 1,1 entry of any \( i,j \)-th block of \( A \otimes B \) must be same as 1,1 entry of \( j,i \)-th block of \( A \otimes B \). That is \( a_{i,j} + b_{1,1} = a_{j,i} + b_{1,1} \) for all \( i, j = 1, 2, \ldots, n \). Which implies that \( A \) is symmetric.

**Theorem 4.9.** If \( A \in \mathbb{M}_m, B \in \mathbb{M}_n \), then \( A \otimes B \) is skew-symmetric if and only if \( A \) and \( B \) are both skew-symmetric.

**Proof.** If \( A \) and \( B \) are both skew-symmetric, then \( A^T = -A \) and \( B^T = -B \). Now

\[
(A \otimes B)^T = A^T \otimes B^T \quad \text{[by Proposition 4.4]}
\]

\[
= (-A) \otimes (-B)
\]

\[
= (-A) \otimes J_n + J_m \otimes (-B)
\]

\[
= -A \otimes J_n - J_n \otimes B \quad \text{[by Lemma 2.4]}
\]

\[
= -A \otimes B.
\]

Therefore \( A \otimes B \) is skew-symmetric.

The other direction is similar to that of the proof of Theorem 4.8.

**Theorem 4.10.** If \( A \in \mathbb{M}_m \) and \( B \in \mathbb{M}_n \), the \( A \otimes B \) is a diagonal matrix if and only if

\( A = kJ_m \) and \( B = -kJ_n \) for some \( k \in \mathbb{C} \). Furthermore in that case \( A \otimes B = 0 \).

**Proof.** If \( A = kJ_m \) and \( B = -kJ_n \) for some \( k \in \mathbb{C} \), then

\[
A \otimes B = kJ_m \otimes J_n + J_m \otimes (-kJ_n)
\]

\[
= 0. \quad \text{[using Lemma 2.4]}
\]

Again if \( A \otimes B \) is a diagonal matrix, then we must have

\[
a_{i,i} + b_{p,q} = 0 \text{ for } i = 1, 2, \ldots, m \text{ and } p, q = 1, 2, \ldots, n; \ p \neq q,
\]

\[
a_{i,j} + b_{p,p} = 0 \text{ for } i, j = 1, 2, \ldots, m; \ i \neq j \text{ and } p, q = 1, 2, \ldots, n;
\]

\[
a_{i,j} + b_{p,q} = 0 \text{ for } i = 1, 2, \ldots, m; \ i \neq j \text{ and } p = 1, 2, \ldots, n; \ p \neq q.
\]

Solving all those equations we see that all entries of \( A \) are equal (say \( k \)) and all entries of \( B \) are also equal (\( -k \)). Thus we get our required result.

**Corollary 4.11.** There exist no square matrices \( A, B \) such that \( A \otimes B = I \).
Theorem 4.12. If \( A, C \in \mathbb{M}_m \), \( B, D \in \mathbb{M}_n \), then \( A \otimes B = C \otimes D \) if and only if \( C = A - kI_m \) and \( D = B + kI_n \) for some \( k \in \mathbb{C} \).

**Proof.** If \( C = A - kI_m \) and \( D = B + kI_n \) for some \( k \in \mathbb{C} \), then

\[
C \otimes D = (A - kI_m) \otimes (B + kI_n) = (A - kI_m) \otimes I_n + I_m \otimes (B + kI_n) = A \otimes I_m - kI_m \otimes I_n + I_m \otimes B + I_m \otimes kI_n = A \otimes B.
\]

Conversely, suppose that \( A \otimes B = C \otimes D \). Then every block of \( A \otimes B \) equals to the corresponding block of \( C \otimes D \),

i.e. \( a_{i,j}I_n + B = c_{i,j}I_n + D \) for \( i, j = 1, 2, \ldots, m \).

Which implies that \( a_{i,j} + b_{p,q} = c_{i,j} + d_{p,q} \) for any \( i, j, p, q = 1, 2, \ldots, m \) and \( p, q = 1, 2, \ldots, n \). That is \( a_{i,j} - c_{i,j} = d_{p,q} - b_{p,q} \) for any \( i, j, p, q = 1, 2, \ldots, m \) and \( p, q = 1, 2, \ldots, n \). Therefore we must have \( A - C = \lambda I_m \) and \( D - B = \lambda I_n \) for some \( \lambda \in \mathbb{C} \). Hence the theorem follows.

Theorem 4.13. If \( A, B \in \mathbb{M}_n \), then \( A \otimes B = B \otimes A \) if and only if \( B = A + kI_n \) for some \( k \in \mathbb{C} \).

**Proof.** If \( B = A + kI_n \), then by direct calculation we have

\[
A \otimes B = B \otimes A = A \otimes A + kI_n^2.
\]

Now suppose \( A \otimes B = B \otimes A \). Then \( a_{i,j} + b_{p,q} = c_{i,j} + a_{p,q} \) for all \( i, j, p, q = 1, 2, \ldots, n \). Therefore

\[
\sum_{p,q=1}^{n} (a_{i,j} + b_{p,q}) = \sum_{p,q=1}^{n} (b_{i,j} + a_{p,q})
\]

which gives \( n^2a_{i,j} + S_B = n^2b_{i,j} + S_A \)

i.e. \( b_{i,j} = a_{i,j} + \frac{S_B - S_A}{n^2} \) for all \( i, j = 1, 2, \ldots, n \).

Thus \( B = A + kI_n \) for \( k = \frac{S_B - S_A}{n^2} \).

Theorem 4.14. If \( A, B, C, D \in \mathbb{M}_n \), then

(i) \( (A \otimes B)(C \otimes D) = AC \otimes BD + A \mathbb{I}_n \otimes \mathbb{J}_n B + \mathbb{J}_n C \otimes D \mathbb{I}_n \).
(ii) \((A \otimes B) \circ (C \otimes D) = (A \circ C) \otimes (B \circ D) + A \otimes D + C \otimes B\).

**Proof.** (i) We have

\[
(A \otimes B)(C \otimes D) = (A \otimes J_n + J_n \otimes B)(C \otimes J_n + J_n \otimes D)
\]
\[
= (A \otimes J_n)(C \otimes J_n) + (A \otimes J_n)(J_n \otimes B) + (J_n \otimes B)(C \otimes J_n)
\]
\[
+ (J_n \otimes B)(J_n \otimes D)
\]
\[
= AC \otimes J_n^2 + A J_n \otimes J_n B + J_n C \otimes B J_n + J_n^2 \otimes BD
\]
\[
= AC \otimes BD + A J_n \otimes J_n B + J_n C \otimes D J_n.
\]

(ii) Here

\[
(A \otimes B) \circ (C \otimes D) = (A \otimes J_n + J_n \otimes B) \circ (C \otimes J_n + J_n \otimes D)
\]
\[
= (A \otimes J_n) \circ (C \otimes J_n) + (A \otimes J_n) \circ (J_n \otimes D) + (J_n \otimes B) \circ (C \otimes J_n)
\]
\[
+ (J_n \otimes B) \circ (J_n \otimes D)
\]
\[
= (A \circ C) \otimes J_n + A \otimes D + C \otimes B + J_n \otimes (B \circ D)
\]
\[
= (A \circ C) \otimes (B \circ D) + A \otimes D + C \otimes B.
\]

**Proposition 4.15.** For matrices \(A, B, C\) of suitable orders,

\[
(A + B) \otimes C = \frac{1}{2}[A \otimes C + B \otimes C + (A + B) \otimes J_n]
\]

and

\[
A \otimes (B + C) = \frac{1}{2}[A \otimes B + A \otimes C + J \otimes (B + C)].
\]

**Proof.** We prove only the first result as the second one can be proved similarly. If \(A\) and \(B\) are matrices of same order (say \(m\)) and matrix \(C\) is of order \(n\), then

\[
(A + B) \otimes C = (A + B) \otimes J_n + J_m \otimes C
\]
\[
= A \otimes J_n + B \otimes J_n + J_m \otimes C. \quad \text{[since } \otimes \text{ is distributive]} \quad (4.2)
\]

From (4.2), we have

\[
(A + B) \otimes C = A \otimes J_n + B \otimes C \quad (4.3)
\]
and

\[(A + B) \odot C = A \odot C + B \odot \mathbb{J}_n. \quad (4.4)\]

Now adding (4.3) and (4.4), we get

\[2((A + B) \odot C) = A \odot C + B \odot C + (A + B) \odot \mathbb{J}_n.\]

Hence the result follows.

**Theorem 4.16.** If \( A_i \in \mathbb{M}_m, \ B_i \in \mathbb{M}_n \) for \( i = 1, 2, \ldots, k \), then

\[
\left( \sum_{i=1}^{k} A_i \right) \odot \left( \sum_{i=1}^{k} B_i \right) = \sum_{i=1}^{k} (A_i \odot B_i).
\]

**Proof.** We prove the result by induction on \( k \). For \( k = 1 \), the result is trivial. For \( k = 2 \), we have

\[
(A_1 + A_2) \odot (B_1 + B_2) = (A_1 + A_2) \odot \mathbb{J}_n + \mathbb{J}_m \odot (B_1 + B_2)
\]

\[
= A_1 \odot \mathbb{J}_n + A_2 \odot \mathbb{J}_n + \mathbb{J}_m \odot B_1 + \mathbb{J}_m \odot B_2 \quad \text{[since \( \odot \) is distributive]}
\]

\[
= A_1 \odot B_1 + A_2 \odot B_2. \quad (4.5)
\]

Thus the result holds for \( k = 2 \). Suppose the identity holds for \( k = 1, 2, \ldots, \ell \ < k \), then

\[
\left( \sum_{i=1}^{\ell+1} A_i \right) \odot \left( \sum_{i=1}^{\ell+1} B_i \right) = \left( \sum_{i=1}^{\ell} A_i \right) \odot \left( \sum_{i=1}^{\ell} B_i \right) + A_{\ell+1} \odot B_{\ell+1} \quad \text{[by (4.5)]}
\]

\[
= \sum_{i=1}^{\ell} (A_i \odot B_i) + A_{\ell+1} \odot B_{\ell+1} \quad \text{[by induction hypothesis]}
\]

\[
= \sum_{i=1}^{\ell+1} (A_i \odot B_i).
\]

Hence the result follows.

Using Theorem 4.16 repeatedly, we get the following general result.

**Theorem 4.17.** For \( A_i \in \mathbb{M}_m, B_i \in \mathbb{M}_n, \ldots, C_i \in \mathbb{M}_\ell, \) for \( i = 1, 2, \ldots, k \), then

\[
\left( \sum_{i=1}^{k} A_i \right) \odot \left( \sum_{i=1}^{k} B_i \right) \odot \cdots \odot \left( \sum_{i=1}^{k} C_i \right) = \sum_{i=1}^{k} (A_i \odot B_i \odot \cdots \odot C_i).
\]

**Lemma 4.18.** If \( A \) and \( B \) are any square matrices, then

\[S_{A \odot B} = S_A S_B.\]
Proof. If \( A \in \mathbb{M}_m \) and \( B \in \mathbb{M}_n \), then the \( i,j \)-th block of \( A \otimes B \) is \( a_{i,j}B \) and \( S_{a_{i,j}B} = a_{i,j}S_B \). Therefore we get
\[
S_{A \otimes B} = S_B \sum_{i,j=1}^{m} a_{i,j} = S_B S_A.
\]
Hence the result follows. \( \square \)

**Theorem 4.19.** If \( A \in \mathbb{M}_m \) and \( B \in \mathbb{M}_n \), then
\[
S_{A \otimes B} = n^2S_A + m^2S_B.
\]

**Proof.** We have
\[
S_{A \otimes B} = S_{A \otimes I_n + I_m \otimes B} = S_{A \otimes I_n} + S_{I_m \otimes B} = S_A \times n^2 + m^2 \times S_B. \quad \text{[using Lemma 4.18].}
\]
Hence the theorem holds. \( \square \)

As a corollary of Theorem 4.19, we get the expression for the Wiener index of Cartesian product of two connected graphs.

**Corollary 4.20.** If \( G_1 \) and \( G_2 \) are two connected graphs of order \( m \) and \( n \) respectively, then
\[
W(G_1 \square G_2) = n^2W(G_1) + m^2W(G_2).
\]

As an application of above corollary we get the following result.

**Corollary 4.21.** If \( H \) is any fixed connected graph and \( G_1, G_2 \) are connected graphs of same order with \( W(G_1) \geq W(G_2) \), then
\[
W(H \square G_1) \geq W(H \square G_2),
\]
with equality if and only if \( W(G_1) = W(G_2) \).

**Theorem 4.22.** If \( A \in \mathbb{M}_m \) and \( B \in \mathbb{M}_n \), then \( A \circ B \) has constant row sum if and only if \( A \) and \( B \) both have constant row sums.

**Proof.** Let us consider any arbitrary row of \( A \circ B \). If the first entry of that row is \( a_{i,1} + b_{j,i} \), then the row sum of that row of \( A \circ B \) equals to
\[
(na_{i,1} + B_j) + (na_{i,2} + B_j) + \cdots + (na_{i,m} + B_j) = nA_j + mB_j, \quad (4.6)
\]
Now if $A$ and $B$ have constant row sums, then $A_i = \frac{S_A}{m}$ and $B_i = \frac{S_B}{n}$. Therefore, by (4.6), $A \otimes B$ has constant row sum equal to $\frac{n}{m}S_A + \frac{m}{n}S_B$.

Again if $A \otimes B$ has constant row sum (say $k$), then from (4.6) we get

$$nA_i + mB_j = k \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$ 

Keeping $i$ fixed, we see that $B_j$ is constant for $j = 1, 2, \ldots, n$. Similarly, keeping $j$ fixed we get $A_i$ is constant for $i = 1, 2, \ldots, m$. Hence, the theorem holds.

The following result is a reformulation of Theorem 4.22. Therefore, the proof is omitted.

**Theorem 4.23.** If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $\mathbb{I}_{mn}$ is an eigenvector of $A \otimes B$ if and only if $\mathbb{I}_m$ and $\mathbb{I}_n$ are eigenvectors of $A$ and $B$ respectively.

As an application of Theorem 4.22, we get the following result as a corollary.

**Corollary 4.24.** The Cartesian product $G_1 \square G_2$ of two connected graphs $G_1$ and $G_2$ is transmission regular if and only if $G_1$ and $G_2$ are both transmission regular.

From the proof of Theorem 4.22, we get a lower bound for the distance spectral radius of the Cartesian product of two connected graphs.

**Corollary 4.25.** If $G_1$ and $G_2$ are two connected graphs of order $m$ and $n$ respectively, then

$$\rho^D(G_1 \square G_2) \geq \frac{n}{m}W(G_1) + \frac{m}{n}W(G_2),$$

with equality if and only if $G_1$ and $G_2$ are both transmission regular.

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