An open mapping theorem for nonlinear operator equations associated with elliptic complexes

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ABSTRACT
Let \( \{A^i, E^i\} \) be the elliptic complex on an \( n \)-dimensional smooth closed Riemannian manifold \( X \) with the first-order differential operators \( A^i \) and smooth vector bundles \( E^i \) over \( X \). We consider nonlinear operator equations, associated with the parabolic differential operators \( \partial_t + \Delta^i \), generated by the Laplacians \( \Delta^i \) of the complex \( \{A^i, E^i\} \), in special Bochner–Sobolev functional spaces. We prove that under reasonable assumptions regarding the nonlinear term the Fréchet derivative \( A_i' \) of the induced nonlinear mapping is continuously invertible and the map \( A_i \) is open and injective in chosen spaces.

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1. Introduction
Let \( X \) be a Riemannian \( n \)-dimensional smooth compact closed manifold and \( E^i \) be smooth vector bundles over \( X \). Denote by \( C^\infty_{E^i}(X) \) the space of all smooth sections of the bundle \( E^i \). Consider an elliptic complex of the first-order differential operators \( A^i \) on \( X \),

\[
0 \longrightarrow C^\infty_{E^0}(X) \overset{A^0}{\longrightarrow} C^\infty_{E^1}(X) \overset{A^1}{\longrightarrow} \cdots \overset{A^{N-1}}{\longrightarrow} C^\infty_{E^N}(X) \longrightarrow 0,
\]

where \( A^i \circ A^{i-1} \equiv 0 \). In this case, it is equivalent to say that the Laplacians \( \Delta^i = (A^i)^* A^i + A^{i-1}(A^{i-1})^* \), \( i = 0, 1, \ldots, N \), of the complex are the second-order strongly elliptic differential operators on \( X \) where operator \( (A^i)^* \) is formal adjoint to \( A^i \) [see (5) below]. For \( i < 0 \) and \( i \geq N \) we assume that \( A^i = 0 \).

Inspired by typical nonlinear problems of the Mathematical Physics, see, for instance [1,2], we consider a family of nonlinear parabolic equations, associated with the complex \( \{A^i, E^i\} \). With this purpose, we denote by \( M_{ij} \) two bilinear bi-differential operators of zero order (see [3] or [4]),

\[
M_{i,1}(\cdot, \cdot) : (E^{i+1}, E^i) \rightarrow E^i, \quad M_{i,2}(\cdot, \cdot) : (E^i, E^i) \rightarrow E^{i-1}.
\]

We set for a differentiable section \( v \) of the bundle \( E^i \)

\[
N^i(v) =: M_{i,1}(A^i v, v) + A^{i-1} M_{i,2}(v, v).
\]

Let now \( X_T \) be a cylinder, \( X_T = X \times [0, T] \), where the time \( T > 0 \) is finite. Then, for any fixed positive number \( \mu \) the operators \( \partial_t + \mu \Delta^i \) are parabolic on \( X \times (0, +\infty) \) (see [5]).

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Consider the following initial problem: given section \( f \) of the induced bundle \( E^i(t) \) (the variable \( t \) enters into this bundle as a parameter) and section \( u_0 \) of the bundle \( E^i \), find a section \( u \) of the induced bundle \( E^i(t) \) and a section \( p \) of the induced bundle \( E^{i-1}(t) \) such that

\[
\begin{align*}
\partial_t u + \mu \Delta^i u + N^i(u) + A^{i-1} p &= f & \text{in } X \times (0, T), \\
(A^{i-1})^* u &= 0 & \text{in } X \times [0, T], \\
(A^{i-2})^* p &= 0 & \text{in } X \times [0, T], \\
u(x, 0) &= u_0 & \text{in } X.
\end{align*}
\] (4)

Recently such a problem was considered in the weighted Hölder spaces over \( \mathbb{R}^n \times [0, T] \), where the weights provide prescribed asymptotic behaviour at the infinity with respect to the space variables (see [6]); it was proved that image of the operator \( A_i \), induced by (4), is open in these spaces.

If \( n = 2 \) or \( n = 3 \) and complex (1) is the de Rham complex, \( \{ A^i, E^i \} = \{ d^i, \Lambda^i \} \) with the exterior differentials acting on sections of the bundle of the exterior differential forms on \( X \), then for \( i = 1 \) and a suitable choice of the nonlinear term we may treat (4) as the initial problem for the well known Navier–Stokes equations for incompressible fluid over the manifold \( X \) (see, for instance, [7]). Note that if \( i = 1 \) then the equation with respect to \( p \) is actually missing because \( (A^{-1})^* = 0 \).

Of course, there are plenty of papers devoted to Navier–Stokes equations on Riemannian manifolds beginning from the pioneer work [8]. However, we will not concentrate our efforts on aspects of Hydrodynamics.

In contrast to [6] we consider this problem in special Sobolev–Bochner type spaces, cf. [9] for the de Rham complex on the torus \( \mathbb{T}^3 \) in the case where \( i = 1 \). Namely, using the standard tools, such as the interpolation Gagliardo–Nirenberg inequalities, Gronwall type lemmas and Faedo–Galerkin approximations, see, for instance, [1,2], we show that suitable linearizations of our problem have one and only one solution and nonlinear problem (4) has at most one solution in the constructed spaces. Applying the implicit function theorem for Banach spaces we prove that the image of \( A_i \) is open in selected spaces (thus, obtaining the so-called ‘open mapping theorem’). However, we do not discuss here a much more delicate question on the existence of solution to nonlinear problem (4) aiming at the maximal generality of the nonlinear term \( N^i(u) \) for the open mapping theorem to be true. It is worth to note that even for the existence of weak (distributional) solutions to (4) one has to impose rather restrictive but reasonable additional assumptions on the nonlinearity \( N^i \).

### 2. Functional spaces and basic inequalities

We need to introduce appropriate functional spaces. Namely, let \( dx \) be a volume form on \( X \) and a \( (\cdot, \cdot)_{x,i} \) denotes a Riemannian metric in the fibres of \( E_i \). As usual, we equip each bundle \( E^i \) with a smooth bundle homomorphism \( *_i : E^i \rightarrow E^{i*} \) defined by \( <*_i u, v >_{x,i} = (v, u)_{x,i} \) for all \( u, v \in E^i \). Then we can consider the space \( C^\infty_i(X) \) with the unitary structure

\[
(u, v)_i = \int_X (u, v)_{x,i} \, dx
\]

and the Lebesgue space \( L^2_i(X) \) with the norm \( \| u \|_i = \sqrt{\langle u, u \rangle_i} \). In this case the formal adjoint to \( A^i \) operator \( (A^i)^* : C^\infty_{E^{i+1}}(X) \rightarrow C^\infty_{E^{i}}(X) \) is defined in the following way for the sections \( u \in E^{i+1} \) and \( v \in E^i \):

\[
((A^i)^* u, v)_i := (u, A^i v)_{i+1}.
\] (5)

Let \( \{ U_i \}_{i=1}^N \) be a finite open cover of \( M \) by coordinate neighbourhoods over which \( E^i \) is trivial and \( \{ \psi_i \}_{i=1}^N \) a corresponding partition of unity,

\[
\psi_i \in C^\infty_i(X), \quad \psi_i(x) \geq 0,
\]
supp \psi_l \subset U_l, \quad \sum_{l=1}^{N} \psi_l(x) \equiv 1 \text{ on } X.

As usual, denote by \( W^s_{i,p}(X), s \in \mathbb{Z}_+, 1 \leq p \leq \infty \), the Sobolev space under the smooth vector bundle \( E^i \) (see, for instance, [10]). It is a Banach space of sections \( u \in L^p_i(X) \) with the norm

\[
\| u \|_{W^s_{i,p}(X)} = \left( \sum_{l=1}^{N} \sum_{|\alpha| \leq s} \| \partial^{\alpha} (\psi_l u) \|_{L^p(X)}^p \right)^{1/p}.
\]

In particular, it is a Hilbert space for \( p = 2 \), we denote it by \( H^s_i(X) \):

\[
\| u \|_{L^2_i(X)} = \left( \sum_{l=1}^{N} \sum_{|\alpha| \leq s} \| \partial^{\alpha} (\psi_l u) \|_{L^2(X)}^2 \right)^{1/2}.
\]

\[
\Delta^i : H^{s+2}_i(X) \to H^s_i(X)
\]

is Fredholm:

(1) the kernel of operator (7) equals to the finite-dimensional space \( \mathcal{H}^i \);
(2) given \( v \in H^s_i(X) \) there is a form \( u \in H^{s+2}_i(X) \) such that \( \Delta^i u = v \) if and only if \( (v, h)_i = 0 \) for all \( h \in \mathcal{H}^i \);
(3) there exists a pseudo-differential operator \( \varphi^i \) on \( X \) such that the operator

\[
\varphi^i : H^s_i(X) \to H^{s+2}_i(X),
\]

induced by \( \varphi^i \), is linear bounded and with the identity \( I \) we have

\[
\varphi^i \Delta^i = I - \Pi^i \text{ on } H^{s+2}_i(X), \quad \Delta^i \varphi^i = I - \Pi^i \text{ on } H^s_i(X)
\]

\[
\text{Proof: } \text{See, for instance, [4, Theorem 2.2.2].}
\]

For \( m \in \mathbb{N} \) we denote by \( \tilde{\nabla}^m_i \) an elliptic differential operator of order \( m \), connected with the complex (1),

\[
\tilde{\nabla}^m_i := \begin{cases} (\Delta^i)^{m/2}, & m \text{ is even}, \\ (A^i \oplus (A^{i-1})^\ast)(\Delta^i)^{(m-1)/2}, & m \text{ is odd}. \end{cases}
\]

It is easy to see that kernel of \( \tilde{\nabla}^m_i \) for all \( m \in \mathbb{N} \) equals to \( \mathcal{H}^i \). Then it follows from the ellipticity of (10), that there exist a parametrix \( \tilde{\varphi}^m_i \) such that

\[
I = \tilde{\nabla}^m_i \tilde{\varphi}^m_i + \Pi^i
\]
with identity operator $I$ (see, for instance, [11, Chapter 10]). Then for each $m \in \mathbb{N}$ and $1 \leq p \leq \infty$ we can consider the completion of the space $C_i^\infty(X)$ by the norm

$$
\|u\|_{\tilde{V}^m_{i,p}} = \left\{ \begin{array}{ll}
\left( \|\Delta^{m/2} u\|_{L^p_i(X)}^p + \|\Pi^i u\|_{L^p_i(X)}^p \right)^{1/p}, & m \text{ is even}, \\
\left( \|\Delta^{(m-1)/2} u\|_{L^p_{i+1}(X)}^p + \|\Pi^i u\|_{L^p_{i+1}(X)}^p \right)^{1/p} + \|\Delta^{(m-1)/2} u\|_{L^p_{i-1}(X)}^p + \|\Pi^i u\|_{L^p_{i-1}(X)}^p \right)^{1/p}, & m \text{ is odd}.
\end{array} \right.
$$

(11)

Also we may define in a standard way the linear elliptic pseudo-differential operator $\nabla^s_i = (\Delta^{s/2})$ of order $s \in \mathbb{R}_+$ on the section $E_i$, see, for instance, [12]. As above, there exist a parametrix $\varphi^i_m$ such that

$$
I = \nabla^m_i \varphi^i_m + \Pi^i.
$$

For $u \in C_i^\infty(X)$ we denote by $\|u\|_{V^s_{i,p}}$ the norm

$$
\|u\|_{V^s_{i,p}} = \left( \|\nabla^s_i u\|_{L^p_{i}(X)}^p + \|\Pi^i u\|_{L^p_{i}(X)}^p \right)^{1/p}.
$$

(12)

**Lemma 2.2:** For each $m \in \mathbb{Z}_+$, $s \in \mathbb{R}_+$ and $1 < p < \infty$ there are constants $\tilde{C}_1$, $\tilde{C}_2$, $C_1$ and $C_2$, such that

$$
\tilde{C}_1 \|u\|_{\tilde{V}^m_{i,p}} \leq \|u\|_{W^s_{i,p}(X)} \leq \tilde{C}_2 \|u\|_{\tilde{V}^m_{i,p}}
$$

(13)

$$
C_1 \|u\|_{V^s_{i,p}} \leq \|u\|_{W^s_{i,p}(X)} \leq C_2 \|u\|_{V^s_{i,p}}
$$

(14)

for all $u \in C_i^\infty(X)$.

**Proof:** Follows immediately from the Gårding’s inequality, definitions of operators $\tilde{V}^m_{i,p}$, $\nabla^s_i$ and the fact that

$$
\|\nabla^s_i u\|_{L^p_{i}(X)} \leq c \|u\|_{W^s_{i,p}(X)}
$$

with some positive constant $c$, see, for instance, [12] or [13, Proposition 2.4].

Actually, it follows from (13) and (14) that we can equip the space $W^s_{i,p}(X)$ with the norm (11) or, for $1 < p < \infty$, (12).

In the sequel, we need the Gronwall’s Lemma (see [21]). Let us recall it.

**Lemma 2.3:** Let $A$, $B$ and $Y$ be real-valued functions defined on a segment $[a, b]$. Assume that $B$ and $Y$ are continuous and that the negative part of $A$ is integrable on $[a, b]$. If moreover $A$ is nondecreasing, $B$ is non-negative and $Y$ satisfies the integral inequality

$$
Y(t) \leq A(t) + \int_a^t B(s) Y(s) \, ds
$$

for all $t \in [a, b]$, then

$$
Y(t) \leq A(t) \exp \left( \int_a^t B(s) \, ds \right) \text{ for all } t \in [a, b].
$$
Let now $V_i^s = H_i^s(X) \cap S_{(A_i-1)}$ stand for the space of all the sections $u \in H_i^s(X)$ satisfying $(A_i^{-1})^s u = 0$ in the sense of the distributions in $X$. Denote by $L^2(I,H_i^s(X))$ the Bochner space of $L^2$-mappings

$$u(t) : I \to H_i^s(X),$$

where $I = [0, T]$, see, for instance, [1]. It is a Banach space with the norm

$$\|u\|_{L^2(I,H_i^s(X))}^2 = \int_0^T \|u\|_{H_i^s(X)}^2 \, dt.$$

We want to introduce the suitable Bochner–Sobolev type spaces, see, for instance, [9] for the de Rham complex in the degree $i = 1$ over the torus $\mathbb{T}^3$. As problem (4) is inspired by the models of Hydrodynamics, where $u, f, p$ represent velocity, outer force and pressure, respectively, for $s \in \mathbb{Z}_+$ denote by $B^{k,2s+2}_{i,vel}(X_T)$ the space of sections of the induced bundle $E^i(t)$ over $X_T$ such that

$$u \in C(I, V_i^{k+2s}) \cap L^2(I, V_i^{k+2s+1})$$

and

$$\nabla_i^m \partial_i^j u \in C(I, V_i^{k+2s-m-2j}) \cap L^2(I, V_i^{k+2s+1-m-2j})$$

for all $m + 2j \leq 2s$. It is a Banach space with the norm

$$\|u\|_{B^{k,2s+2}_{i,vel}(X_T)}^2 := \sum_{m+2j \leq 2s, 0 \leq i \leq k} \|\nabla_i^m \partial_i^j u\|_{C(I,L_i^2)}^2 + \|\nabla_i^{l+1} \nabla_i^m \partial_i^j u\|_{L^2(I,L_i^2)}^2.$$

Similarly, for $s, k \in \mathbb{Z}_+$, we define the space $B^{k,2s+2}_{i,for}(X_T)$ to consist of all sections

$$f \in C(I, H_i^{2s+k}(X)) \cap L^2(I, H_i^{2s+k+1}(X))$$

with the property that

$$\nabla_i^m \partial_i^j f \in C(I, H_i^{k+2s-m-2j}(X)) \cap L^2(I, H_i^{k+2s-m-2j+1}(X))$$

provided $m + 2j \leq 2s$.

If $f \in B^{k,2s+2}_{i,for}(X_T)$, then actually

$$\nabla_i^m \partial_i^j f \in C(I, H_i^{k+2s-m}(X)) \cap L^2(I, H_i^{k+1+2(s-j)-m}(X))$$

for all $m$ and $j$ satisfying $m + 2j \leq 2s$. We endow the spaces $B^{k,2s+2}_{i,for}(X_T)$ with the natural norms

$$\|f\|_{B^{k,2s+2}_{i,for}(X_T)}^2 := \sum_{m+2j \leq 2s, 0 \leq i \leq k} \|\nabla_i^m \partial_i^j f\|_{C(I,L_i^2)}^2 + \|\nabla_i^{l+1} \nabla_i^m \partial_i^j f\|_{L^2(I,L_i^2)}^2.$$

Finally, the spaces for the section $p$ are $B^{k+1,2s+2}_{i-1,pre}(X_T)$. By definition, they consist of all sections $p$ from the space $C(I, H_i^{2s+k+1}(X)) \cap L^2(I, H_i^{2s+k+2}(X))$ such that $A^{-1} p \in B^{k,2s+2}_{i,for}(X_T), (A^{-2})^s p = 0$ and

$$(p,h)_{L^2_{i-1}(X)} = 0$$

for all $h \in H_i^{1-1}$. We equip this space with the norm

$$\|p\|_{B^{k+1,2s+2}_{i-1,pre}(X_T)} = \|A^{-1} p\|_{B^{k,2s+2}_{i,for}(X_T)}.$$
Consider now bi-differential operator
\[ B_t(w, v) = M_{i,1}(A^i w, v) + M_{i,1}(A^i v, w) + A^{i-1}(M_{i,2}(w, v) + M_{i,2}(v, w)) \] (16)
such that
\[ |M_{i,1}(u, v)| \leq c_{i,1}|u| |v|, \quad |M_{i,2}(u, v)| \leq c_{i,2}|u| |v| \text{ on } X \] (17)
with some positive constants \( c_{i,j} \). In the sequel we will always assume that (17) holds.

**Theorem 2.4:** Suppose that \( s \in \mathbb{N}, k \in \mathbb{Z}_+, 2s + k > \frac{n}{2} - 1 \). Then the mappings
\[
\begin{align*}
\nabla_i^m : & \quad B_{i,\text{for}}^{k,2(s-1),s-1}(X_T) \to B_{i,\text{for}}^{k-m,2(s-1),s-1}(X_T), \quad m \leq k \\
\Delta^i : & \quad B_{i,\text{vel}}^{k,2s,s}(X_T) \to B_{i,\text{vel}}^{k,2s,s}(X_T), \\
\partial_i : & \quad B_{i,\text{vel}}^{k,2s,s}(X_T) \to B_{i,\text{vel}}^{k,2s,s}(X_T),
\end{align*}
\]
are continuous. Besides, if \( w, v \in B_{i,\text{vel}}^{k+2,2(s-1),s-1}(X_T) \) then the mappings
\[
\begin{align*}
B_i(w, \cdot) : & \quad B_{i,\text{vel}}^{k+2,2(s-1),s-1}(X_T) \to B_{i,\text{for}}^{k,2(s-1),s-1}(X_T), \\
B_i(w, \cdot) : & \quad B_{i,\text{vel}}^{k,2s,s}(X_T) \to B_{i,\text{for}}^{k,2(s-1),s-1}(X_T),
\end{align*}
\]
are continuous, too. In particular, for all \( w, v \in B_{i,\text{vel}}^{k+2,2(s-1),s-1}(X_T) \) there is positive constant \( c_{i,k} \) independent on \( v \) and \( w \), such that
\[
\|B_i(w, v)\|_{B_{i,\text{for}}^{k,2(s-1),s-1}(X_T)} \leq c_{i,k}\|w\|_{B_{i,\text{vel}}^{k+2,2(s-1),s-1}(X_T)}\|v\|_{B_{i,\text{vel}}^{k+2,2(s-1),s-1}(X_T)}. \] (19)

**Proof:** The proof is based on the following version of Gagliardo–Nirenberg inequality, see [14] or [15, Theorem 3.70] for the Riemannian manifolds.

**Lemma 2.5:** For all \( v \in L_i^{p_0}(X) \cap L_i^2(X) \) such that \( \nabla_i^j v \in L_i^{p_0}(X) \) and \( \nabla_i^{m_0} v \in L_i^{r_0}(X) \) we have
\[
\|\nabla_i^j v\|_{L_i^{p_0}(X)} \leq C \left( \left( \|\nabla_i^{m_0} v\|_{L_i^{p_0}(X)} + \|v\|_{L_i^2(X)} \right)^a \|v\|_{L_i^{r_0}(X)}^{1-a} + \|v\|_{L_i^2(X)} \right) \] (20)
with a positive constant \( C = C_{j_0, m_0}^{(n)}(p_0, r_0, q_0) \) independent on \( v \), where
\[
\frac{1}{p_0} = \frac{j_0}{n} + a \left( \frac{1}{r_0} - \frac{m_0}{n} \right) + \frac{(1-a)}{q_0} \text{ and } \frac{j_0}{m_0} \leq a \leq 1, \] (21)
with the following exceptional case: if \( 1 < r_0 < +\infty \) and \( m_0 - j_0 - n/r_0 \) is a non-negative integer then the inequality is valid only for \( \frac{j_0}{m_0} \leq a < 1 \).

**Proof:** Indeed, under the hypothesis of Lemma, we have from (13) and (14)
\[
\|\nabla_i^j v\|_{L_i^{p_0}(X)} \leq c_1 \|v\|_{W_i^{j_0, p_0}(X)} \leq c_2 \|v\|_{\tilde{W}_i^{j_0, p_0}} \\
\leq \begin{cases} 
& c_3 \left( \|\Delta_i^{j_0/2} u\|_{L_i^{p_0}(X)} + \|\tilde{\Pi}_i^{j_0} u\|_{L_i^{p_0}(X)} \right), \quad \text{if } j_0 \text{ is even}, \\
& c_3 \left( \|\Delta_i^{j_0-(j_0-1)/2} u\|_{L_i^{p_0}(X)} + \|\tilde{\Pi}_i^{j_0} u\|_{L_i^{p_0}(X)} \right) + \|(\Delta_i^{j_0-1})^s \Delta_i^{(j_0-1)/2} u\|_{L_i^{p_0}(X)} + \|\tilde{\Pi}_i^{j_0} u\|_{L_i^{p_0}(X)} \right), \quad \text{if } j_0 \text{ is odd},
\end{cases}
\] (22)
with positive constants $c_1$, $c_2$ and $c_3$. In each local card $U_l$ we get
\[
\| (A^i)^{j_0/2} v \|_{L^p_0(X)} |U_l| \leq c_1 (\| M_{j_0,l} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)} + \| N_{j_0,l} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)})
\leq \tilde{c}_1 (\| \nabla^{j_0} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)} + \| v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)})
\]
\[
(23)
\]
if $j_0$ is even, and
\[
\left( \| A^i (A^j)^{(j_0-1)/2} v \|_{L^p_{i+1}(X)} + \| (A^{i-1})^a (A^j)^{(j_0-1)/2} v \|_{L^p_{i-1}(X)} \right) |U_l|
\leq c_2 (\| \tilde{M}_{j_0,l} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)} + \| \tilde{N}_{j_0,l} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)})
\]
\[
\leq c_2 (\| \nabla^{j_0} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)} + \| v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)})
\]
\[
(24)
\]
if $j_0$ is odd, with constants $c_1$, $\tilde{c}_1$, $c_2$, $\tilde{c}_2 > 0$, where $M_{j_0,l}, N_{j_0,l}, \tilde{M}_{j_0,l}, \tilde{N}_{j_0,l}$ and $\tilde{T}_{j_0,l}$ are some matrices with infinitely differentiable coefficients and $u_{j_0,l}$ is the representation of $v$ in $U_l$ (see, for instance, [4, Chapter 1]). Applying the Gagliardo–Nirenberg inequality (see [14] or [15]) we have
\[
\| \nabla^{j_0} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)} \leq c \left( \| \nabla^{m_0} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)} + \| u_{j_0,l} \|_{L^{1-a}} + \| \nabla^{j_0} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)} \right)
\]
\[
(25)
\]
with constant $c > 0$, where indexes are subordinate to (21). On the other hand, Gårding’s inequality implies
\[
\| \nabla^{m_0} v u_{j_0,l} \|_{L^p_0(\mathbb{R}^n)} \leq c \left( \| \nabla^{m_0} v \|_{L^p_0(\mathbb{R}^n)} + \| \Pi_{m_0} v \|_{L^p_0(\mathbb{R}^n)} \right)
\]
\[
(26)
\]
with positive constant $c$. It follows from the ellipticity of $\nabla^{m_0}_i$ that dimension of the space $H^{m_0}_i$ is finite, then $\| \Pi_{m_0} v \|_{L^p_0(\mathbb{R}^n)} \leq c \| \Pi_{m_0} v \|_{L^p_0(X)}$ with constant $c > 0$, since all tow norms are equivalent on finite dimension space, and, moreover, there exist a constant $\tilde{c} > 0$ such that
\[
\| \Pi_{m_0} v \|_{L^p_i(X)} \leq \tilde{c} \| v \|_{L^p_i(X)}.
\]
\[
(27)
\]
Summing up by $I$ the inequalities (23)–(26) and taking into account (27) we receive (20).

We need now Young’s inequality: given any $N = 1, 2, \ldots$, it follows that
\[
\sum_{j=1}^N a_j \leq \sum_{j=1}^N \frac{a_j^{p_j}}{p_j}
\]
\[
(28)
\]
for all positive numbers $a_j$ and all numbers $p_j \geq 1$ satisfying $\sum_{j=1}^N 1/p_j = 1$.

Next, the first three linear operators are continuous by the very definition of the function spaces.

Let us prove the (18). We start with $s = 1$ and argue by the induction. The space $B^{k+2,0,0}_{i,rel}(X_T)$ is continuously embedded into the spaces $C(I, V^{i}_{k+2})$ and $L^2(I, V^{i}_{k+2,1})$.

First, we note that, by the Sobolev embedding theorem for any $k, s \in \mathbb{Z}_+$ satisfying
\[
k - s > n/2,
\]
\[
(29)
\]
there exists a constant $c(k, s)$ depending on the parameters, such that
\[
\| u \|_{C^s_i(X)} \leq c(k, s) \| u \|_{H^i_k(X)}
\]
for all $u \in H^i_k(X)$. 

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Assume that \( n \geq 3 \) (for \( n = 2 \) proof is similar with simpler arguments). Then by (20)

\[
\|u\|_{L^{\frac{2n}{n+2}}_i(X)} \leq c \left( \|\nabla_i u\|_{L^2_i(X)} + \|u\|_{L^2_i(X)} \right)
\]

with the Gagliardo–Nirenberg constant \( c \), because

\[
p_0 = \frac{2n}{n-2} \text{ if } a = 1, \ j_0 = 0, \ m_0 = 1 \text{ and } r_0 = 2.
\]

If, in addition, \( m' \geq n/2 - 1 \) then by (20) and the hypothesis of the lemma,

\[
\|u\|_{L^2_n(X)} \leq c \left( \|\nabla^m_i u\|_{L^2_i(X)} \|\nabla u\|_{L^{\frac{2m'}{m'+1}}_i(X)} + \|u\|_{L^2_i(X)} \right)
\]

with the Gagliardo–Nirenberg constant \( c \), because

\[
\frac{1}{n} = \left( \frac{1}{2} - \frac{m'}{n} \right) a + \frac{1-a}{2}, \quad a = \frac{n-2}{2m'} \in (0, 1).
\]

Using by (17), Hölder’s inequality and Gårding’s inequality we have

\[
\|B_i(w, u)\|_{L^2_i(X)}^2 \leq \tilde{c} \left( \|w\|_{L^2_i(X)}^2 \left( \|\nabla_i u\|_{L^{\frac{2n}{n+2}}_i(X)} + \|u\|_{L^2_i(X)} \right) \right.
\]

\[
\left. + \left( \|\nabla_i w\|_{L^{\frac{2n}{n+2}}_i(X)} + \|\nabla_i w\|_{L^2_i(X)} \right) \|u\|_{L^2_i(X)}^2 \right)
\]

with a positive constant \( \tilde{c} \), independent of \( u \) and \( w \). Then, with \( m' = k + 2 > n/2 - 1 \) in (31),

\[
\|w\|_{L^2_{k+2}(X)}^2 \leq c_1 \left( \|\nabla^{k+2} w\|_{L^2_i(X)} \|w\|_{L^2_i(X)} + \|w\|_{L^2_i(X)} \right)^2 \leq c_2 \|w\|_{H^k_{k+2}(X)}^2,
\]

\[
\|\nabla_i u\|_{L^{\frac{2n}{n+2}}_i(X)}^2 \leq \tilde{c}_1 \left( \|\nabla_i u\|_{L^2_i(X)} + \|\nabla_i u\|_{L^2_i(X)} \right)^2 \leq \tilde{c}_2 \|u\|_{H^k_{k+2}(X)}^2,
\]

with positive constants \( c_1, \ c_2, \ \tilde{c}_1, \ \tilde{c}_2 \) independent of \( u \) and \( w \). Then we have

\[
\|B_i(w, u)\|_{L^2_i(X)}^2
\]

\[
\leq c_1 \left( \|w\|_{H^k_{k+2}(X)}^2 \left( \|u\|_{L^2_i(X)}^2 + \|u\|_{L^2_i(X)} \right) \right.
\]

\[
+ \left( \|w\|_{H^k_{k+2}(X)}^2 \left( \|w\|_{L^2_i(X)}^2 + \|w\|_{L^2_i(X)} \right) \right)
\]

\[
\leq \left( \|\nabla^{k+2} w\|_{L^2_i(X)}^2 + \|\nabla w\|_{L^2_i(X)}^2 \right) \|u\|_{H^k_{k+2}(X)}^2
\]

\[
\leq c_2 \|w\|_{H^k_{k+2}(X)}^2 \|u\|_{H^k_{k+2}(X)}^2,
\]

the positive constants \( c_1 \) and \( c_2 \) being independent of \( u \) and \( w \), and so

\[
\|B_i(w, u)\|_{C(I, L^2_i(X))} \leq c \|u\|_{C(I, H^k_{k+2}(X))} \|w\|_{C(I, H^k_{k+2}(X))}^2
\]

with constant \( c > 0 \).
For $n \geq 3$, by Gårding’s and Hölder’s inequalities we also have

$$
\| \nabla_i B_i(w, u) \|_{L_i^2(X)}^2 \leq \tilde{c} \left( \| w \|_{L_i^2(X)}^2 \| \nabla_i^2 u \|_{L_i^2(X)}^{2n} + \| \nabla_i w \|_{L_i^2(X)}^{2n} \| \nabla_i u \|_{L_i^2(X)}^{2n} + \| w \|_{L_i^2(X)}^{2n} \| u \|_{L_i^2(X)}^{2n} \right)
$$

with a constant $c$ independent of $u$ and $w$. On combining (33) and (34) we deduce that, for $n \geq 3$,

$$
\| B_i(w, u) \|_{L^2(I, H_i^l(X))}^2 \leq c \left( \| u \|_{H_i^{k+2}(X)}^2 \| w \|_{H_i^{k+2}(X)}^2 + \| u \|_{H_i^{k+3}(X)}^2 \| w \|_{H_i^{k+3}(X)}^2 \right).
$$

Inequalities (33) and (35) provide that the operator $B_i(w, \cdot)$ maps $B_i^{0,0,0}(X_T)$ continuously to $B_i^{0,0,0}(X_T)$ if $k > n/2 - 3$.

Next, for any $0 \leq k' \leq k_0, k_0 = k$ or $k_0 = k + 1$, similarly to (34), using the Hölder and Gårding’s inequality with a number $q = q(k', l) > 1$ we obtain

$$
\| \nabla_i^{k'} B_i(w, u) \|_{L_i^2(X)}^2 \leq \sum_{l=0}^{k'} \sum_{j=0}^{k'+1-l} \sum_{m=0}^{l} \left( C_{k', l}^{j, m} \| \nabla_i^m w \|_{L_i^{2n}(X)}^{2n} \| \nabla_i^j u \|_{L_i^{2n}(X)}^{2n} \right)
$$

with positive coefficients $C_{k', l}^{j, m}, \tilde{C}_{k', l}^{j, m}$.

If $0 \leq k' \leq k_0$, then we take $q = q(k', 0) = n \frac{n}{n-2}$ and use (28) and (31) with $m' = k + 2$, to obtain

$$
\| \nabla_i^{k+1} u \|_{L_i^{2n}(X)}^2 \| w \|_{L_i^{2n}(X)}^2 \leq c \left( \| \nabla_i^{k+1} u \|_{L_i^{2n}(X)}^2 + \| \nabla_i^{k+2} u \|_{L_i^{2n}(X)}^2 \right)^2 \left( \| \nabla_i^{k+2} w \|_{L_i^{2n}(X)}^2 \| w \|_{L_i^{2n}(X)}^2 + \| w \|_{L_i^{2n}(X)}^2 \right)^2
$$

for each $0 \leq j \leq k' + 1$, with positive constants $\tilde{c}, c$ independent on $u, w$. 


If \( 1 \leq l \leq k' \leq k_0 \) then we may apply (20) to each factor in the typical summand
\[
\| \nabla^m_i w \|^2_{L^q_i(X)} \| \nabla^j_i u \|^2_{L^q_i(X)}
\]
with \( 0 \leq j \leq k' + 1 - l \), \( 0 \leq m \leq l \) and entries \( q = q(k', l), \alpha_j = \alpha_j^{(l)} \), satisfying
\[
\begin{align*}
\frac{j}{k_0 + 2} &\leq \alpha_1 = \frac{n}{k_0 + 2} \left( \frac{1}{2} - \frac{1}{2q} + \frac{j}{n} \right) < 1 \\
\frac{m}{k + 2} &\leq \alpha_2 = \frac{n}{k + 2} \left( \frac{1}{2q} + \frac{m}{n} \right) < 1.
\end{align*}
\]
Relations (38) are actually equivalent to the following:
\[
\begin{align*}
\frac{j}{k_0 + 2} &\leq \alpha_1 = \frac{n}{k_0 + 2} \left( \frac{1}{2} - \frac{1}{2q} + \frac{j}{n} \right) < 1 \\
\frac{m}{k + 2} &\leq \alpha_2 = \frac{n}{k + 2} \left( \frac{1}{2q} + \frac{m}{n} \right) < 1.
\end{align*}
\]
The lower bounds are always true if \( q > 1 \) and so, these inequalities are reduced to
\[
\frac{1}{2} + \frac{j - k_0 - 2}{n} < \frac{1}{2q} < \frac{k + 2 - m}{n}, \quad q > 1.
\]
The segment for \( \frac{1}{2q} \) is not empty because
\[
\frac{1}{2} + \frac{j - k_0 - 2}{n} < \frac{k + 2 - m}{n}
\]
provided by the assumptions \( k + 3 > n/2, 0 \leq k' \leq k_0, \) and \( j + m \leq k' + 1 \). Moreover, as
\[
\frac{1}{2} + \frac{j - k_0 - 2}{n} < \frac{1}{2}, \quad \frac{k + 2 - m}{n} > 0,
\]
we see that there is a proper \( q > 1 \) to achieve (39) and (40).

Then, similarly to (37),
\[
\begin{align*}
\| \nabla^m_i w \|^2_{L^q_i(X)} \| \nabla^j_i u \|^2_{L^q_i(X)} &\leq \tilde{c} \left( \| \nabla_i^{k_0+2} u \|_{L^q_i(X)}^{\frac{k_0+2}{n}} \| u \|_{L^q_i(X)}^{1-\alpha_1} + \| u \|_{L^q_i(X)} \right)^2 \\
&\times \left( \| \nabla_i^{k+2} w \|_{L^q_i(X)}^{\frac{k+2}{n}} \| w \|_{L^q_i(X)}^{1-\alpha_2} + \| w \|_{L^q_i(X)} \right)^2 \\
&\leq c \| u \|^2_{H^{k_0+2}_i(X)} \| w \|^2_{H^{k+2}_i(X)}
\end{align*}
\]
with positive constants \( \tilde{c}, c \) independent on \( u, w \).

Hence, (37) and (41) yield
\[
\| B_i(w, u) \|_{C(I, H^k_i(X))} \leq c \| u \|^2_{C(I, H^{k+2}_i(X))} \| w \|^2_{C(I, H^{k+2}_i(X))},
\]

(42)
\[ \|B_t(w, u)\|_{L^2(I, H^{k+1}_t(X))}^2 \leq \tilde{c} \left( \|u\|_{C(I, H^{k+2}_t(X))}^2 \|w\|_{L^2(I, H^{k+3}_t(X))}^2 + \|w\|_{C(I, H^{k+2}_t(X))}^2 \|u\|_{L^2(I, H^{k+3}_t(X))}^2 \right), \]  

(43) 

with positive constants \(c, \tilde{c}\) independent on \(u, w\).

Now (42) and (43) imply that the mapping \(B_t(w, \cdot)\) maps \(B^{k+2,0,0}_i(X_T)\) continuously to \(B^{k,0,0}_i(X_T)\) for any \(k > n/2 - 3\) if \(n \geq 3\) and bound (19) hold true for \(s = 1\).

Next, we argue by the induction. Assume that for some \(s' \geq 1\) the mapping \(B_t(w, \cdot)\) maps \(B^{k+2,2(s'-1),s'-1}_i(X_T)\) continuously to \(B^{k,2(s'-1),s'-1}_i(X_T)\) for any \(k > n/2 - 2s' - 1\) and bound (19) holds true for \(s = s'\). Then the space \(B^{k+2,2s',s'}_i(X_T)\) is continuously embedded to the space \(B^{k+4,2(s'-1),s'-1}_i(X_T)\) and, by the inductive assumption, \(B_t(w, \cdot)\) maps \(B^{k+4,2(s'-1),s'-1}_i(X_T)\) continuously to \(B^{k+2,2(s'-1),s'-1}_i(X_T)\) for any \((k+2) > n/2 - 2s' - 1\) or the same, \(k > n/2 - 2(s' + 1) - 1\). Moreover, bound (19) holds true for \(s = s'\) and with \(k + 2\) instead of \(k\).

It is left to check the behaviour of the partial derivatives \(\partial^j \nabla^k B_t(w, u)\) with \(k_0 \leq k + 1\). By the very definition of space \(B^{k+2,2s',s'}_i(X_T)\), the partial derivatives \(\partial^j_t u, \partial^j_t w\) belong to \(C(I, H^{k+2(\cdot - j)}_t(X))\) and \(L^2(I, H^{k+3+2(\cdot - j)}_t(X))\).

By the Leibniz rule,

\[ \partial_t B_t(w, u) = B_t(\partial_t w, u) + B_t(w, \partial_t u). \]

Then for any \(0 \leq k' \leq k_0, k_0 = k = k + 1, 0 \leq i' \leq s'\), similarly to (34)

\[ \|\nabla^k B_t(w, u)\|_{L^2_t(X)}^2 \leq \sum_{j=0}^{i'} \sum_{l=0}^{k'} \sum_{m=0}^{k'+1-l} \left( C^{j,m,j'}_{k',l',r} \|\partial_t^j \nabla^m w\|_{L^2_t(X)}^{2q} \|\partial_t^{i'-j} \nabla^l u\|_{L^2_t(X)}^{2q} \right) \]

(44)

with positive coefficients \(C^{j,m,j'}_{k',l',r}, C^{j,m,j'}_{k',l',r}\).

Similarly to (37), if \(0 \leq k' \leq k + 1\) then we take \(q = q(k', 0, s') = \frac{n}{n-2}\) and use (28) and (31) with \(m' = 2s' + k + 2\), to obtain

\[ \|\partial_t^{i'} \nabla^{k'+1} u\|_{L^2_t(X)}^{2q} \|w\|_{L^2_t(X)}^{2q} = \|\partial_t^{i'} \nabla^{k'+1} u\|_{L^2_t(X)}^{2q} \|w\|_{L^2_t(X)}^{2q} \leq \tilde{c} \left( \|\partial_t^{i'} \nabla^{k'+2} u\|_{L^2_t(X)}^{2q} \|w\|_{L^2_t(X)}^{2q} \right)^2 \]

\[ \leq \tilde{c} \left( \|\partial_t^{i'} u\|_{L^2_t(X)}^{2q} \|w\|_{L^2_t(X)}^{2q} \right)^2 \]

(45)

with positive constants \(\tilde{c}, c\) independent on \(u, w\).
Again, similarly to (41), if
\[
1 \leq l \leq k' \leq k_0, \quad 1 \leq j' \leq s',
\]
\[
0 \leq j \leq k' + 1 - l, \quad 0 \leq m \leq l,
\]
then we may apply (20) to each factor in the typical summand
\[
\| \partial_t^{s' - j'} \nabla_i^j u \|^2_{L_t^{2q} (X)} \| \partial_t^{j'} \nabla_i^m w \|^2_{L_t^{2q} (X)}
\]
with entries satisfying
\[
\begin{cases}
\frac{1}{2q} = \frac{j}{k_0 + 2 + 2j'} + \frac{1}{2} - \frac{k_0 + 2 + 2j'}{n} \alpha_1 + \frac{1}{2} - \frac{1}{2q}, \\
\frac{q-1}{2q} = \frac{m}{n} + \frac{1}{2} - \frac{k + 2 + 2(s' - j')}{n} \alpha_2 + \frac{1}{2} - \frac{1}{2q}, \\
\frac{1}{k_0 + 2 + 2j'} \leq \alpha_1 < 1, \quad \frac{m}{k + 2 + 2(s' - j')} \leq \alpha_2 < 1.
\end{cases}
\] (46)

Relations (46) are actually equivalent to the following:
\[
\frac{j}{k_0 + 2 + 2j'} \leq \alpha_1 = \frac{n}{k_0 + 2 + 2j'} \left( \frac{1}{2} - \frac{1}{2q} + \frac{j}{n} \right) < 1, \tag{47}
\]
\[
\frac{m}{k + 2 + 2(s' - j')} \leq \alpha_2 = \frac{n}{k + 2 + 2(s' - j')} \left( \frac{1}{2q} + \frac{m}{n} \right) < 1. \tag{48}
\]

The lower bounds are always true if \( q > 1 \) and so, these inequalities are reduced to
\[
\frac{1}{2} + \frac{j - k_0 - 2 - 2j'}{n} < \frac{1}{2q} < \frac{k + 2 + 2(s' - j') - m}{n}, \quad q > 1.
\]
The segment for \( \frac{1}{2q} \) is not empty because
\[
\frac{1}{2} + \frac{j - k_0 - 2 - 2j'}{n} \leq \frac{k + 2 + 2(s' - j') - m}{n},
\]
provided by the assumptions \( k > n/2 - 2(s' + 1) - 1 \), \( 0 \leq k' \leq k_0 \) and \( j + m \leq k' + 1 \). Moreover, as
\[
\frac{1}{2} + \frac{j - k_0 - 2 - 2j'}{n} < \frac{1}{2}, \quad \frac{k + 2 + 2(s' - j') - m}{n} > 0,
\]
we see that there is a proper \( q > 1 \) to achieve (47) and (48).

Then, similarly to (37),
\[
\| \partial_t^{s' - j'} \nabla_i^j u \|^2_{L_t^{2q} (X)} \| \partial_t^{j'} \nabla_i^m w \|^2_{L_t^{2q} (X)}
\leq \bar{c} \left( \| \partial_t^{s' - j'} \nabla_i^{k_0+2+2j'} u \|_{L_t^{2q} (X)}^{\alpha_1} \| \partial_t^{j'} u \|_{L_t^{2q} (X)}^{1 - \alpha_1} + \| \partial_t^{s' - j'} u \|_{L_t^{2q} (X)}^{2} \right)^2
\cdot \left( \| \partial_t^{j'} \nabla_i^{k + 2 + 2(s' - j')} w \|_{L_t^{2q} (X)}^{\alpha_2} \| \partial_t^{j'} w \|_{L_t^{2q} (X)}^{1 - \alpha_2} + \| \partial_t^{j'} w \|_{L_t^{2q} (X)}^{2} \right)^2
\leq c \| \partial_t^{s' - j'} u \|^2_{H_t^{k_0+2+2j'} (X)} \| \partial_t^{j'} w \|^2_{H_t^{k+2+2(s' - j')} (X)}
\] (49)
with positive constants \( \bar{c}, c \) independent on \( u, w \).
Lemma 2.6: Let $s, k \in \mathbb{Z}_+$ for all $u$, we have

$$\| \partial_t^{s'} B_i(w, u) \|_{C(I, H^i_1(X))}^2 \leq \sum_{j=0}^{s'} \left( c_{j} \| \partial_t^{s'-j} u \|_{C(I, H^{k+2+2j'}_1(X))}^2 \| \partial_t^j w \|_{C(I, H^{k+2+2(j'-j)}_1(X))}^2 + \tilde{c}_j \| \partial_t^{s'-j} w \|_{L^2(I, H^{k+2+2j'}_1(X))}^2 \| \partial_t^j u \|_{L^2(I, H^{k+2+2(j'-j)}_1(X))}^2 \right),$$

with positive constants $c_j$ and $\tilde{c}_j$, $0 \leq j' \leq s'$, independent on $u$, $v$.

Now (50) and (51) imply that the mapping $B_i(w, \cdot)$ maps $B_{i,for}^{k+2+2s',s'}(X_T)$ continuously to $B_{i,vel}^{k,2(s-1),s-1}(X_T)$ if $n \geq 3$. Moreover, by (28), bound (19) holds true for $s = s' + 1$. This finishes the proof of inequality (19) and the continuity of operator $B_i(w, \cdot) : B_{i,vel}^{k+2+2(s-1),s-1}(X_T) \to B_{i,for}^{k,2(s-1),s-1}(X_T)$, for $n \geq 3$ and for all $k \in \mathbb{Z}_+$ and $s \in \mathbb{N}$ satisfying $2s + k > n/2 - 1$.

The boundedness of the operator $B_i(w, \cdot) : B_{i,vel}^{k,2(s-1),s-1}(X_T) \to B_{i,for}^{k,2(s-1),s-1}(X_T)$ now follows from the definition of the spaces. \[ \square \]

Let us introduce now the Helmholtz type projection $P^i$ from $B_{i,for}^{k,2(s-1),s-1}(X_T)$ to the kernel of operator $(A^{i-1})^*$. 

**Lemma 2.6:** Let $s, k \in \mathbb{Z}_+$. For each $i$ the pseudo-differential operator $P^i = (A^i)^*A^i\phi^i + \Pi^i$ on $X$ induce continuous map

$$P^i : B_{i,for}^{k,2(s-1),s-1}(X_T) \to B_{i,vel}^{k,2(s-1),s-1}(X_T),$$

such that

$$P^i \circ P^i u = P^i u, \quad (P^i u, v)_{L^2_1(X)} = (u, P^i v)_{L^2_1(X)}, \quad (P^i u, (I - P^i)u)_{L^2_1(X)} = 0$$

for all $u, v \in B_{i,for}^{k,2(s-1),s-1}$.

**Proof:** Indeed, as $A^{i+1} \circ A^i = 0$,

$$P^i \circ P^i u = ((A^i)^*A^i\phi^i + \Pi^i) \circ ((A^i)^*A^i\phi^i + \Pi^i) u = ((A^i)^*A^i\phi^i(A^i)^*A^i\phi^i + \Pi^i) u = ((A^i)^*A^i\phi^i + \Pi^i) u.$$

It follows from Theorem 2.1 that

$$P^i = I - A^{i-1}(A^{i-1})^*\phi^i,$$

and then

$$(P^i u, v)_{L^2_1(X)} = (P^i u, P^i v + A^{i-1}(A^{i-1})^*\phi^i v)_{L^2_1(X)} = (P^i u, P^i v)_{L^2_1(X)} = (u, P^i v)_{L^2_1(X)}.$$
because \((A^{i-1})^*P^i = 0\). On the other hand,
\[
(P^i u, (I - P^i)u)_{L^2_i(X)} = (P^i u, u)_{L^2_i(X)} - (P^i u, P^i u)_{L^2_i(X)} = 0.
\]

Finally, the continuity of \(P^i : B_{t,\text{for}}^{k,2(s-1),s-1} (X_T) \to E_{t,\text{vel}}^{k,2(s-1),s-1} (X_T)\) follows from Theorem 2.4 and the commutative relations \(P^i \partial_j^i = \partial_j^i P^i\) with \(j \leq s - 1\).

3. An open mapping theorem

Consider now the linearisation of problem (4): given sufficiently regular section \(f, w\) of the induced bundle \(E_i(t)\) and section \(u_0\) of the bundle \(E_i\), find sufficiently regular sections \(u\) and \(p\) of the induced bundles \(E_i(t)\) and \(E_{i-1}(t)\) which satisfy
\[
\begin{align*}
\partial_t v + \mu \Delta^i v + B_i(w, v) + A^{i-1} p &= f \quad \text{in } X_T, \\
(A^{i-1})^* v &= 0 \quad \text{in } X_T, \\
(A^{i-2})^* p &= 0 \quad \text{in } X_T, \\
v(x, 0) &= v_0 \quad \text{in } X. 
\end{align*}
\]

(53)

Now we want to show that (53) has one and only one solution in the spaces, introduced in the first paragraph. We start with the following simple corollary of the Hodge Theorem 2.1.

**Corollary 3.1:** Let \(F \in B_{t,\text{for}}^{k,2(s-1),s-1} (X_T)\) satisfy \(P^i F = 0\) in \(X_T\). Then there is a unique section \(p \in B_{i-1,\text{pre}}^{k+1,2(s-1),s-1} (X_T)\) such that (15) holds and
\[
A^{i-1} p = F \text{ in } X \times [0, T].
\]

**Proof:** Under the hypotheses of Theorem, the section
\[
p = (A^{i-1})^* \phi^i F
\]
is a solution of (54). Indeed,
\[
A^{i-1} p = A^{i-1} (A^{i-1})^* \phi^i F,
\]
but it follows from Theorem 2.1 and Lemma 2.6 that
\[
F = A^{i-1} (A^{i-1})^* \phi^i F + P^i F = A^{i-1} (A^{i-1})^* \phi^i F,
\]
because \(P^i F = 0\), then \(A^{i-1} p = F\) and \((A^{i-2})^* p = 0\) by the construction of the solution. Moreover
\[
(p, h)_{L^2_{t-1}(X)} = ((A^{i-1})^* \phi^i F, h)_{L^2_{t-1}(X)} = (\phi^i F, A^{i-1} h)_{L^2_t(X)} = 0
\]
for all \(h \in H^{i-1}\), hence \(p \in B_{i-1,\text{pre}}^{k+1,2(s-1),s-1} (X_T)\).

Let now \(p_1, p_2 \in B_{i-1,\text{pre}}^{k+1,2(s-1),s-1} (X_T)\) are two solutions of (54). Then \(p = p_1 - p_2\) is solution too and \(A^{i-1} p = 0\). Since \(p \in B_{i-1,\text{pre}}^{k+1,2(s-1),s-1} (X_T)\), then \((A^{i-2})^* p = 0\) and \(p\) actually belong to the harmonic space \(H^{i-1}\). However, as
\[
(p, h)_{L^2_{t-1}(X)} = 0
\]
for all \(h \in H^{i-1}\), then \(p \equiv 0\).

Let us recall the following useful lemma by J.-L. Lions.
Lemma 3.2: Let $V$, $H$ and $V'$ be Hilbert spaces such that $V'$ is the dual to $V$ and the embeddings $\mathcal{V} \subset H \subset V'$ are continuous and everywhere dense. If $u \in L^2(I, V)$ and $\partial_t u \in L^2(I, V')$ then

$$\frac{d}{dt} \|u(\cdot, t)\|_H^2 = 2 \langle \partial_t u, u \rangle$$

(55)

and $u$ is equal almost everywhere to a continuous mapping from $[0, T]$ to $H$.

Proof: See [2, Ch. III, §1, Lemma 1.2].

Theorem 3.3: Let $n \geq 2$ and suppose that $w \in L^2(I, V_i^1) \cap L^2(I, L_i^\infty(X)) \cap L^\infty(I, L_i^n(X))$. Given any pair $(f, u_0) \in L^2(I, (V_i^1)' \times V_i^0)$, there is a unique section $u \in C(I, V_i^0) \cap L^2(I, V_i^1)$ with $\partial_t u \in L^2(I, (V_i^1)'),$ satisfying

$$\begin{cases}
\frac{d}{dt} (u, v)_{L_i^2(X)} + \mu (A^1 u, A^1 v)_{L_i^2(X)} = \langle f - B_i(w, u), v \rangle, \\
u(\cdot, 0) = u_0
\end{cases}$$

(56)

for all $v \in V_i^K$ with $k \geq n/2$.

Proof: The proof is based on the standard method by Faedo–Galerkin, energy estimates and Gagliardo–Nirenberg inequality, see, for instance, [2,16] or [17].

Theorem 3.4: Let $n \geq 2$, $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $2s + k > n/2$, and $w \in B_i^{k+2s,s}_{i,vel}(X_T)$. Then (53) induces a bijective continuous linear mapping

$$A_{w,j} : B_i^{k,2s,s}_{i,vel}(X_T) \times B_i^{k+1,2(s-1),s-1}_{i-1,pre}(X_T) \rightarrow B_i^{k,2(s-1),s-1}_{i,for}(X_T) \times V_i^{2s+k},$$

(57)

which admits a continuous inverse $A_{w,j}^{-1}$.

Note, if $2s + k > n/2 - 1$, then the space $B_i^{k,2s,s}_{i,vel}(X_T)$ is continuously embedded into the space $L^2(I, L_i^\infty(X)) \cap L^\infty(I, L_i^n(X))$. Indeed, it follows from (29) that $L^2(I, H_i^{k+2s+1}(X)) \hookrightarrow L^2(I, L_i^\infty(X))$. On the other hand, (31) gives $C(I, H_i^{k+2s}(X)) \hookrightarrow L^\infty(I, L_i^n(X))$, then

$$B_i^{k,2s,s}_{i,vel}(X_T) \hookrightarrow L^2(I, L_i^\infty(X)) \cap L^\infty(I, L_i^n(X))$$

(58)

with $2s + k > n/2 - 1$.

Proof: Again, we may follow the standard scheme of solving nonlinear parabolic equations, see, for instance, [1,2,16–18].

We begin with a simple lemma.

Lemma 3.5: If $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $2s + k > n/2 - 1$ then (57) is an injective continuous linear mapping.
\textbf{Proof:} Indeed, the continuity of $A_{n,t}$ follows from Theorem 2.4. Let
\begin{equation}
(u, p) \in B_{l, \text{vel}}^{k,2s} (X_T) \times B_{l, \text{vel}}^{k+1,2(s-1),s-1} (X_T),
\end{equation}
\begin{equation}
A_{n,t}(u, p) = (f, u_0) \in B_{l, \text{for}}^{k,2(s-1),s-1} (X_T) \times V_i^{2s+k}.
\end{equation}
As $(A^{i-1})^* u = 0$ in $X_T$ we have
\begin{equation}
(A^{i-1} u, u)_{H^2_i(X)} = \|A^i u\|_{H^2_{i+1}(X)}^2 \quad (59)
\end{equation}
and
\begin{equation}
(A^{i-1} p, u)_{H^2_i(X)} = (p, (A^{i-1})^* u)_{H^2_{i-1}(X)} = 0. \quad (60)
\end{equation}
As $2s + k + 1 > n/2$, by the Sobolev embedding theorems, see (29), the space $B_{l, \text{vel}}^{k,2s} (X_T)$ is continuously embedded into $L^2(I, L_{\infty}^n (X))$. Then formulas (55), (59) and (60) readily imply that $u$ is a weak solution to (53) granted by Theorem 3.3, i.e. (56) is fulfilled.

Let us continue with the proof of the surjectivity.

\textbf{Lemma 3.6:} Let $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $2s + k > n/2$ and $w \in B_{l, \text{vel}}^{k,2s} (X_T)$. Then for each $(f, u_0) \in B_{l, \text{for}}^{k,2(s-1),s-1} (X_T) \times V_i^{2s+k}$ there is a solution $u \in B_{l, \text{vel}}^{k,2s} (X_T)$ to (56).

\textbf{Proof:} Let $(f, u_0)$ be an arbitrary data in $B_{l, \text{for}}^{k,2(s-1),s-1} (X_T) \times V_i^{2s+k}$ and let $\{u_m\}$ be the sequence of the corresponding Faedo–Galerkin approximations, namely,
\begin{equation}
u_m = \sum_{j=1}^{M} g_j^{(m)} (t) b_j (x),
\end{equation}
where the system $\{b_j\}_{j \in \mathbb{N}}$ is a $L^2_i (X)$-orthogonal basis in $V_i^0$ and the functions $g_j^{(m)}$ satisfy the following relations
\begin{equation}
(\partial_t u_m, b_j)_{L^2_i(X)} + \mu (A^i u_m, A^i b_j)_{L^2_i(X)} + (B_i (w, u_m), b_j)_{L^2_i(X)} = (f, b_j), \quad (61)
\end{equation}
\begin{equation}
u_m (x, 0) = u_{0,m} (x)
\end{equation}
for all $0 \leq j \leq m$ with the initial datum $u_{0,m}$ from the linear span $\mathcal{L} ([b_j]_{j=1}^{m})$ such that the sequence $\{u_{0,m}\}$ converges to $u_0$ in $H_0$. For instance, as $\{u_{0,m}\}$ we may take the orthogonal projection onto the linear span $\mathcal{L} ([b_j]_{j=1}^{m})$.

The scalar functions $F_j(t) = (f(\cdot, t), b_j)$ belong to $C^{s-1} [0, T] \cap H^s [0, T]$, and the components
\begin{equation}
C_{k,j}^{(m)} (t) = \mu (A^i b_k, A^i b_j)_{L^2_i(X)} + (B_i (w(\cdot, t), b_k), b_j)_{L^2_i(X)}
\end{equation}
\begin{equation}
= \mu (A^i b_k, A^i b_j)_{L^2_i(X)} + (M_{1,1} (A^i w, b_k), b_j)_{L^2_i(X)} + (M_{1,1} (A^i b_k, w), b_j)_{L^2_i(X)} \quad (62)
\end{equation}
don’t belong to $C^s [0, T] \cap H^{s+1} [0, T]$. Since $w \in B_{l, \text{vel}}^{k,2s} (X_T)$, formula (62) means that the entries of the matrix $\exp \int_0^f C_{k,j}^{(m)} (\tau) \, d\tau$ belong actually to $C^{s+1} [0, T] \cap H^{s+2} [0, T]$ and then the components of the vector $g^{(m)}$ belong to $C^s [0, T] \cap H^{s+1} [0, T]$.

Again we assume that $n \geq 3$ (for $n = 2$ arguments are similar). Let $s = 1$ and $k \in \mathbb{Z}_+$ satisfying $k > n/2 - 2$. If we multiply the equation corresponding to index $j$ in (61) by $\frac{d g_j^{(m)}}{d \tau}$ then, after the summation with respect to $j$, we obtain for all $\tau \in [0, T]$:
\begin{equation}
\|\partial_t u_m\|^2_{L^2_i(X)} + \frac{\mu \, d}{d \tau} \|A^i u_m\|^2_{L^2_i(X)}
\end{equation}
\((f, \partial_{\tau} u_m)_{L^2_t(X)} - (M_{i,1}(A^i u_m, w, \partial_{\tau} u_m)_{L^2_t(X)} - (M_{i,1}(A^i w, u_m, \partial_{\tau} u_m)_{L^2_t(X)} \cdot (63)\)

It follows from (17) and from the Hölder inequality with \(q_1 = \infty, q_2 = 2, q_3 = 2\) and \(p_1 = \frac{2n}{n-2}, p_2 = n, p_3 = 2\) that

\[
\left| (M_{i,1}(A^i u_m, w, \partial_{\tau} u_m)_{L^2_t(X)} + (M_{i,1}(A^i w, u_m, \partial_{\tau} u_m)_{L^2_t(X)} \right| \\
\leq c \left( \|A^i u_m\|^2_{L^2_t(X)} \|w\|^2_{L^2_t(X)} + \|u_m\|^2_{L^2_t(X)} \|A^i w\|^2_{L^2_t(X)} \|\partial_{\tau} u_m\|^2_{L^2_t(X)} \right) \\
\leq c_1 \left( \|A^i u_m\|^2_{L^2_t(X)} \|w\|^2_{L^2_t(X)} + \|u_m\|^2_{L^2_t(X)} \|A^i w\|^2_{L^2_t(X)} \|\partial_{\tau} u_m\|^2_{L^2_t(X)} \right) + \frac{1}{2} \|\partial_{\tau} u_m\|^2_{L^2_t(X)} \quad (64)\]

with positive constants \(c\) and \(c_1\). The last estimation is due to the Young’s inequality (28). By the inequality (31) with \(m' = k + 1\) and Lemma 2.2 there are constants \(c, c_1, c_2 > 0\) such that

\[
\|A^i w\|^2_{L^2_t(X)} \leq c_2 \left( \|\nabla^i w\|^2_{L^2_t(X)} + \|w\|^2_{L^2_t(X)} \right) \leq c_2 \left( \|\nabla^{k+1} w\|^2_{L^2_t(X)} + \|w\|^2_{L^2_t(X)} \right) \leq c \|w\|^2_{H^k(X)}. \quad (65)\]

On the other hand (30) gives

\[
\|u_m\|^2_{L^2_t(X)} \leq c \left( \|\nabla u_m\|^2_{L^2_t(X)} + \|u_m\|^2_{L^2_t(X)} \right)^2 \leq c_1 \left( \|A^i u_m\|^2_{L^2_t(X)} + \|u_m\|^2_{L^2_t(X)} \right), \quad (66)\]

with positive constants \(c, c_1\). The last estimate is the consequence of Gårding’s inequality

\[
\|u_m\|^2_{H^k_t(X)} \leq c \left( \|A^i u_m\|^2_{L^2_t(X)} + \|(A^{i-1}u)m\|^2_{L^2_t(X)} + \|u_m\|^2_{L^2_t(X)} \right) \]

with a constant \(c > 0\), however \((A^{i-1}u)m = 0\).

After the integration of (63) with respect to \(\tau \in I_t = [0, t]\) we arrive at the following:

\[
2\|\partial_{\tau} u_m\|^2_{L^2_t(I_t, L^2_t(X))} + \mu \|A^i u_m\|^2_{L^2_t(I_t, L^2_t(X))} \leq \mu \|A^i u_0 m\|^2_{L^2_t(X)} + 2\|f\|^2_{L^2_t(I_t, L^2_t(X))} \|\partial_{\tau} u_m\|^2_{L^2_t(I_t, L^2_t(X))} + \frac{1}{2} \|\partial_{\tau} u_m\|^2_{L^2_t(I_t, L^2_t(X))} \]

\[
+ c \int_0^t \left( \|w\|^2_{L^2_t(X)} \|A^i u_m\|^2_{L^2_t(X)} + \|u_m\|^2_{L^2_t(X)} \|A^i w\|^2_{L^2_t(X)} \right) \, d\tau \\
\leq \mu \|A^i u_0 m\|^2_{L^2_t(I_t, L^2_t(X))} + 4\|f\|^2_{L^2_t(I_t, L^2_t(X))} + \|\partial_{\tau} u_m\|^2_{L^2_t(I_t, L^2_t(X))} \]

\[
+ c_1 \int_0^t \|w\|^2_{H^{k+2} t(X)} \left( \|A^i u_m\|^2_{L^2_t(X)} + \|u_m\|^2_{L^2_t(X)} \right) \, d\tau, \quad (67)\]

with positive constants \(c, c_1\) independent on \(w\) and \(m\), the last bound being a consequence of the Sobolev Embedding Theorems and inequalities (64)–(66).
Lemma 3.7: Suppose \( k \in \mathbb{Z}_+ \) satisfying \( k > n/2 - 2 \). If \((f, u_0) \in B^{k,0,0}_{L^2_{\text{vol}}}(X_T) \times V^k_i \) then

\[
\| \nabla^k_i u_m \|^2_{C(L^2_{\text{vol}})} + \mu \| \nabla^{k+1}_i u_m \|^2_{L^2_{\text{vol}}(L^2_i)} \leq c_k(\mu, w, f, u_0) \tag{68}
\]

for any \( 0 \leq k' \leq k + 2 \), the constants \( c_k(\mu, w, f, u_0) > 0 \) depending on \( k' \) and \( \mu \) and the norms \( \| w \|_{B^{k,0,0}_{L^2_{\text{vol}}}(X_T)} \), \( \| f \|_{B^{k,0,0}_{L^2_{\text{vol}}}(X_T)} \), \( \| u_0 \|_{V^{k+2}} \) but not on \( m \).

**Proof:** We argue by induction. Let \( k = 0 \) and \( k' = 0 \), substituting \( u_m \) into (56) instead of \( \nu \) and \( u \) we have

\[
\frac{d}{dt} \| u_m \|^2_{L^2_i(X)} + 2\mu \| \nabla_i u_m \|^2_{L^2_i(X)} = 2 \left( (B_i(w, u_m) - f, u_m)_{L^2_i(X)} \right)
\]

for all \( t \in [0, T] \). Using Hölder inequality, we get

\[
2 \left| (f, u_m)_{L^2_i(X)} \right| \leq \frac{2}{\mu} \| f \|^2_{L^2_i(X)} + \frac{\mu}{2} \| u_m \|^2_{L^2_i(X)} \tag{70}
\]

and similarly

\[
2 \left| (B_i(w, u_m), u_m)_{L^2_i(X)} \right| \leq \frac{c}{\mu} \| B_i(w, u_m) \|^2_{L^2_i(X)} + \frac{\mu}{c} \| u_m \|^2_{L^2_i(X)} \tag{71}
\]

with an arbitrary positive constant \( c \).

Next, using the Sobolev embedding theorem we have in each local card \( U_l \)

\[
\| w_{U_l} \|_{L^\infty(\mathbb{R}^n)} \leq c \| w_{U_l} \|_{H^{k+2}(\mathbb{R}^n)},
\]

where \( w_{U_l} \) is the representation of \( w \) in \( U_l \) and \( c \) is a positive constant. Then, under Lemma 2.2 we get

\[
\int_0^t \| B_i(w, u_m) \|^2_{L^2_i(X)} ds \leq c \left( \int_0^t \| w \|^2_{H^k(X)} \| \nabla_i u_m \|^2_{L^2_i(X)} ds + \| w \|^2_{L^2_i(H^k(X))} \right) \tag{73}
\]

From Theorem 3.3 and the Sobolev embedding theorem, it follows that

\[
\| w \|^2_{L^2_i(H^k(X))} \| u_m \|^2_{C(L^2_{\text{vol}})} \leq c(w) \left( \| u_0 \|^2_{L^2_i(X)} + \| f \|^2_{L^2_i(V^k_i)} \right)
\]

where \( c(w) \) is a positive constant depending on \( \| w \|^2_{L^2_i(H^k(X))} \).

Next, let \( u_{0,m} \) is an orthogonal projection on the linear span \( L(\{b_j \}) \in V^k_i \), then

\[
\lim_{m \to +\infty} \| u_0 - u_{0,m} \|_{H^k_i(X)} = 0, \quad \| u_{0,m} \|_{H^k_i(X)} \leq \| u_0 \|_{H^k_i(X)}
\]

for each \( k' \in \mathbb{Z}_+ \).

Then, after integration of (69) over the interval \([0, t]\) and taking into account (70), (71) and (73) we have

\[
\| u_m \|^2_{L^2_i(X)} + \mu \int_0^t \| \nabla_i u_m \|^2_{L^2_i(X)} ds \leq c \left( \| u_0 \|^2_{L^2_i(X)} + \frac{2}{\mu} \| f \|^2_{L^2_i(V^k_i)} + \frac{4}{\mu} \int_0^t \| w \|^2_{H^k_i(X)} \| \nabla_i u_m \|^2_{L^2_i(X)} ds \right) \tag{74}
\]
for all \( t \in [0, T] \). Here the constant \( c_{0,0} \) depends on \( \|w\|_{B^{0,2,1}_{\text{vel}}(X_T)}, \|u_0\|_{C(I, L^2_2(X))}^2 \) and \( \|f\|_{L^2(I, L^2_2(Y))}^2 \) only. It follows from the Gronwall’s Lemma 2.3 that

\[
\|u_m\|_{C(I, L^2_2(X))}^2 + \mu \|\nabla u_m\|_{L^2(I, L^2_2(X))}^2 \leq c_1(\mu, w, f, u_0)
\]

(75)

with a constant \( c_1(\mu, w, f, u_0) \) depending on \( \mu \) and \( \|w\|_{B^{0,2,1}_{\text{vel}}(X_T)}, \|f\|_{B^{0,2,1}_{\text{vel}}(X_T)} \) and \( \|u_0\|_{V^1_2} \) only.

Now, substituting \( \nabla_i u_m \) into (56) instead of \( v \) with \( r \in \mathbb{Z}_+ \) and \( u_m \) instead of \( u \) we have

\[
\frac{d}{dt} \|\nabla_i^{r-1} u_m\|_{L^2_2(X)}^2 + 2\mu \|\nabla_i^{r+1} u_m\|_{L^2_2(X)}^2 = 2 \left( \nabla_i^{r-1}(B_i(w, u_m) - f), \nabla_i^{r+1} u_m \right)_{L^2_2(X)}
\]

(76)

for all \( t \in [0, T] \). Furthermore, using H"older inequality, we get

\[
2 \left( \nabla_i^{r-1} f, \nabla_i^{r+1} u_m \right)_{L^2_2(X)} \leq \frac{2}{\mu} \|\nabla_i^{r-1} f\|_{L^2_2(X)}^2 + \frac{\mu}{2} \|\nabla_i^{r+1} u_m\|_{L^2_2(X)}^2,
\]

(77)

and similarly

\[
2 \left( \nabla_i^{r-1} B_i(w, u_m), \nabla_i^{r+1} u_m \right)_{L^2_2(X)} \leq \frac{c}{\mu} \|\nabla_i^{r-1} B_i(w, u_m)\|_{L^2_2(X)}^2 + \frac{\mu}{c} \|\nabla_i^{r+1} u_m\|_{L^2_2(X)}^2
\]

(78)

with an arbitrary positive constant \( c \).

Now, combining (76) for \( r = 1 \) and (77), (78), (73) with integration over the interval \([0, t]\), we arrive at the estimate

\[
\|\nabla u_m(\cdot, t)\|_{L^2_2(X)}^2 + \mu \int_0^t \|\nabla_i^2 u_m(\cdot, s)\|_{L^2_2(X)}^2 \, ds \leq c \left( \|\nabla_i u_0, 0\|_{L^2_2(X)}^2 \right)
\]

\[
+ \frac{2}{\mu} \|f\|_{L^2(I, L^2_2(Y))}^2 + c_{0,0} + \frac{4}{\mu} \int_0^t \|w\|_{H^2_2(X)}^2 \|\nabla_i u_m\|_{L^2_2(X)}^2 \, ds
\]

(79)

for all \( t \in [0, T] \). Here the constant \( c_{0,0} \) depends on \( \|w\|_{B^{0,2,1}_{\text{vel}}(X_T)}, \|u_0\|_{L^2_2(X)}^2 \) and \( \|f\|_{L^2(I, L^2_2(Y))}^2 \) only.

At this point inequality (79) and Gronwall’s Lemma 2.3 yield

\[
\|\nabla u_m\|_{C(I, L^2_2(X))}^2 + \mu \|\nabla_i^2 u_m\|_{L^2(I, L^2_2(X))}^2 \leq c_1(\mu, w, f, u_0)
\]

(80)

with a constant \( c_1(\mu, w, f, u_0) \) depending on \( \mu \) and \( \|w\|_{B^{0,2,1}_{\text{vel}}(X_T)}, \|f\|_{B^{0,2,1}_{\text{vel}}(X_T)} \) and \( \|u_0\|_{V^1_2} \) only.

Now, the Sobolev embedding theorem and H"older inequality yield

\[
\int_0^t \|\nabla_i B_i(w, u_m)\|_{L^2_2(X)}^2 \, ds \leq c \left( \int_0^t \|w\|_{H^2_2(X)}^2 \|\nabla_i^2 u_m\|_{L^2_2(X)}^2 \, ds \right)
\]

\[
\quad + \|w\|_{L^2(I, H^2_2(X))}^2 \|u_m\|_{C(I, H^2_2(X))}^2 + \|\nabla_i^2 w\|_{C(I, L^2_2(X))}^2 \|u_m\|_{L^2(I, L^2_2(X))}^2
\]

(81)
for some positive constant \(c\). On combining (76) for \(r = 2\) and (81), (77), (78) with integration over the interval \([0, t]\) we obtain

\[
\| \nabla_i^2 u_m(\cdot, t) \|_{L_i^2(X)}^2 + \mu \int_0^t \| \nabla_i^2 u_m(\cdot, s) \|_{L_i^2(X)}^2 \, ds \leq \| \nabla_i^2 u_{0,m} \|_{L_i^2(X)}^2
\]

\[
+ \frac{2}{\mu} \| \nabla f \|_{L^2(I, L_i^2(X))}^2 + c_{0,0} + c_{1,0} \frac{2}{\mu} \int_0^t \| w \|_{H_i^2(X)}^2 \| \nabla_i^2 u_m \|_{L_i^2(X)}^2 \, ds
\]

\[
+ c_{1,0} \frac{2}{\mu} \left( \| w \|_{L^2(I, H_i^2(X))}^2 \| u_m \|_{C(I, H_i^2(X))}^2 + \| \nabla_i^2 w \|_{L^2(I, H_i^2(X))}^2 \| u_m \|_{L^2(I, H_i^2(X))}^2 \right).
\]

(82)

From inequalities (80), (82) and Gronwall’s Lemma 2.3 it follows readily that

\[
\| \nabla_i^2 u_m \|_{C(I, L_i^2(X))}^2 + \mu \| \nabla_i^2 u_m \|_{L^2(I, L_i^2(X))}^2 \leq c_2(\mu, w, f, u_0),
\]

where \(c_2(\mu, w, f, u_0)\) is a constant depending on \(\mu\) and \(\| w \|_{B_{\text{vel}}^{k,1}(X_T)}, \| f \|_{B_{\text{vel}}^{0,0}(X_T)}\) and \(\| u_0 \|_{V_i^2}\), only.

Assume that the sequence \(\{u_m\}\) is bounded in the space \(B^{k,0,0}_{\text{vel}}(X_T) \times V_i^{k+2}\), with \(k = k' + 2\), i.e.

\[
\| \nabla_i^k u_m \|_{C(I, L_i^2(X))}^2 + \mu \| \nabla_i^{k+1} u_m \|_{L^2(I, L_i^2(X))}^2 \leq c_{k'}(\mu, w, f, u_0),
\]

(83)

if \(0 \leq k'' \leq k' + 2\), where the constants \(c_{k'}(\mu, w, f, u_0)\) depend on \(\mu\) and the norms \(\| w \|_{B_{\text{vel}}^{k,1}(X_T)}, \| f \|_{B_{\text{vel}}^{0,0}(X_T)}, \| u_0 \|_{V_i^{k+2}}\) but not on \(m\). Then, combining (77), (78) with integration over the time interval \([0, t]\), we get

\[
\| \nabla_i^{k'+3} u_m(\cdot, t) \|_{L_i^2(X)}^2 + \mu \int_0^t \| \nabla_i^{k'+4} u_m(\cdot, s) \|_{L_i^2(X)}^2 \, ds
\]

\[
\leq \| \nabla_i^{k'+3} u_{0,m} \|_{L_i^2(X)}^2 + \| \nabla_i^{k'+2} f \|_{L^2(I, L_i^2(X))}^2 + \frac{2}{\mu} \| \nabla_i^{k'+2} B_i(w, u_m) \|_{L^2(I, L_i^2(X))}^2.
\]

(84)

In this way, we need to evaluate the last summand on the right-hand side of (84). For all \(0 \leq k' \leq k\), similarly to (36), we have

\[
\| \nabla_i^{k'+2} B_i(w, u_m) \|_{L^2(I, L_i^2(X))}^2
\]

\[
\leq c_0 \frac{2}{\mu} \int_0^t \| \nabla_i^{k'+3} u_m(\cdot, s) \|_{L_i^2(X)}^2 \| w(\cdot, s) \|_{H_i^2(X)}^2 \, ds
\]

\[
+ \frac{k'+1}{\mu} \sum_{j=0}^{k'-1} c_j \| \nabla_i^{k'+j} u_m \|_{L^2(I, L_i^2(X))}^2 \| w \|_{L^2(I, H_i^{k'+j+1}(X))}^2
\]

\[
+ \frac{k'+1}{\mu} \sum_{j=1}^{k'-1} c_j \| \nabla_i^{k'+j} w \|_{L^2(I, L_i^2(X))}^2 \| u_m \|_{L^2(I, H_i^{k'+j+1}(X))}^2
\]

\[
+ c_0 \frac{2}{\mu} \| w \|_{L^2(I, H_i^{k'+3}(X))}^2 \| u_m \|_{L^2(I, L_i^2(X))}^2.
\]

(85)
with positive constants \(c_j\). All terms on the right-hand side of this inequality can be estimated due to the inductive assumption of (83) and the Sobolev embedding theorem. From (83)–(85), it follows that

\[
\| \nabla_i^{k+3} u_m(\cdot, t) \|^2_{L_i^2(X)} + \mu \int_0^t \| \nabla_i^{k+4} u_m(\cdot, s) \|^2_{L_i^2(X)} \, ds \leq \| \nabla_x^{k+3} u_0 \|^2_{L_i^2(X)}
+ \| \nabla_i^{k+2} f \|^2_{L^2(I, L_i^2(X))} + c_{k+2} \frac{2}{\mu} \int_0^t \| \nabla_i^{k+3} u_m(\cdot, s) \|^2_{L_i^2(X)} \| w(\cdot, s) \|^2_{H_i^2(X)} \, ds
+ R_{k+3}(\mu, w, f, u_0),
\]

for all \( t \in [0, T] \), the remainder \( R_{k+3}(w, f, u_0) \) depends on \( \mu \) and \( \| w \|_{p_{vel}^{k+1,2}(X_T)} \), \( \| f \|_{p_{for}^{k+1,0}(X_T)} \) and \( \| u_0 \|_{V^{k+3}} \) but not on the index \( m \). When combined with the induction hypothesis of (83), the latter estimate implies that the assertion of the lemma is true for all \( k \in \mathbb{Z}_+ \).

It follows from Lemma (3.7) that the sequence \( \{u_m\} \) is bounded in the space \( C(I, H_i^{k+3}(x)) \cap L^2(I, H_i^{k+3}(x)) \) if the data \((f, u_0)\) belong to \( p_{for}^{k,0}(X_T) \times V_{i,m}^{k+2} \). Hence it follows that we may extract a subsequence \( \{u_{m'}\} \), such that

1. for any \( j \) satisfying \( j \leq k + 3 \), the sequence \( \{\nabla_j u_{m'}\} \) converges weakly in \( L^2(I, L_i^2(x)) \).
2. the sequence \( \{u_{m'}\} \) converges *-weakly in \( L^\infty(I, H_i^{k+2}(x)) \cap L^2(I, H_i^{k+3}(x)) \) to an element \( u \).

On the other hand, applying Lemma 2.3 to the inequality (67) we see that the sequence \( \{\partial_t u_m\} \) is bounded in the space \( L^2(I, L_i^2(X)) \). In particular, we may extract a subsequence \( \{\partial_t u_{m'}\} \) such that \( \{\partial_t u_{m'}\} \) converges weakly in \( L^2(I, V_i^0) \) to an element \( \tilde{u} \in L^2(I, V_i^0) \).

By the very construction and Theorem 3.3, the section \( u \) is the unique solution to (59) from the space \( L^\infty(I, V_i^{k+2}) \cap L^2(I, V_i^{k+3}) \cap C(I, V_i^0) \). Hence, Lemma 3.2 yields \( \nabla_j u \in C^2(I, L_i^2(X)) \) if \( j \leq k + 2 \). Moreover, by (42), the section \( B_i(w, u) \) belongs to \( C(I, H_i^k(X)) \cap L^2(I, H_i^{k+1}(X)) \).

Actually, (56) imply that

\[
\partial_t u = -\mu \Delta_i^1 u + P^i(B_i(w, u) - f) \text{ in } X_T
\]

in the sense of distributions. According to Lemma 3.1, the projection \( P^i \) maps \( C(I, H_i^k(X)) \cap L^2(I, H_i^{k+1}(X)) \) continuously into \( C(I, V_i^k) \cap L^2(I, V_i^{k+1}) \). Then the section \( \partial_t u \) belongs to \( C(I, V_i^k) \cap L^2(I, V_i^{k+1}) \).

We have thus proved that (56) admits a unique solution \( u \in B_{vel}^{k,2,1}(X_T) \) for any data \((f, u_0)\) in \( B_{for}^{k,0,0}(X_T) \times V_i^{k+2} \).

Now, it follows from Lemma 2.6 that

\[
(I - P^i)(f - B_i(w, u)) = A^{i-1}(A^{i-1})^* (f - B_i(w, u))
\]

and then the Corollary 3.1 implies that there is a unique function \( p \in B_{i-1}^{k+1,0,0}(X_T) \) such that

\[
A^{i-1} p = (I - P^i)(f - B_i(w, u)) \text{ in } X_T.
\]
Adding (87) and (88) we conclude that the pair
\[(u, p) \in B_{i,vel}^{k,2s+1}(X_T) \times B_{i,for}^{k+1,0,0}(X_T)\]
is the unique solution to (53) related to the datum \((f, u_0) \in B_{i,for}^{k,0,0}(X_T) \times V_i^{2+k}\). This implies the surjectivity of the mapping \(A_{w,i}\) for \(s = 1\) and for any \(k \in \mathbb{Z}_+\).

We finish the proof of the theorem with induction in \(s \in \mathbb{N}\). More precisely, assume that the assertion of the theorem concerning the surjectivity of the mapping \(A_{w,i}\) holds for some \(s = s' \in \mathbb{N}\) and any \(k \in \mathbb{Z}_+.\) Let \((f, u_0) \in B_{i,vel}^{k,2s',s'}(X_T) \times V_i^{2(s'+1)+k}\). It is clear that
\[B_{i,for}^{k,2s',s'}(I) \times V_i^{2(s'+1)+k} \hookrightarrow B_{i,for}^{k+2,2(s'-1),s'-1}(X_T) \times V_i^{2s'+k+2},\]
and we see that according to the induction assumption there is a unique solution \((u, p)\) to (53) which belongs to \(B_{i,vel}^{k+2,2s',s'}(X_T) \times B_{i,for}^{k+3,2(s'-1),s'-1}(X_T)\).

By Theorem 2.4 and Lemma 2.6, the sections \(\Delta^i u, B_i(w, u)\) and \(P^i(f - B_i(w, u))\) belong to \(B_{i,for}^{k,2s',s'}(X_T)\), and so the derivative \(\partial_t u\) is in this space, too, because of (87). As a consequence, (88) implies \(A^{i-1}p \in B_{i,for}^{k,2s',s'}(X_T)\), and so \(p \in B_{i-1,pre}^{k+1,2s',s'}(X_T)\).

Thus, the pair \((u, p)\) actually belongs to \(B_{i,vel}^{k,2(s' + 1),s' + 1}(X_T) \times B_{i,for}^{k+2,2s',s'}(X_T)\), i.e. the mapping \(A_{w,i}\) of (57) is surjective for all \(k \in \mathbb{Z}_+\) and \(s \in \mathbb{N}\).

Finally, as the mapping \(A_{w,i}\) is bijective and continuous, the continuity of the inverse \(A_{w,i}^{-1}\) follows from the inverse mapping theorem for Banach spaces.

Now we may formulate the main results of this paper.

**Theorem 3.8:** Let \(n \geq 2\), \(s \in \mathbb{N}\) and \(k \in \mathbb{Z}_+, 2s + k > n/2\). Then (4) induces an injective continuous nonlinear mapping
\[A_i : B_{i,vel}^{k,2s}(X_T) \times B_{i,for}^{k+1,2(s-1),s-1}(X_T) \to B_{i,for}^{k,2(s-1),s-1}(X_T) \times H_i^{2s+k}\]
which is moreover open.

**Proof:** Indeed, the continuity of the mapping \(A_i\) is clear from Theorem 2.4.

Moreover, suppose that
\[(u, p) \in B_{i,vel}^{k,2s}(X_T) \times B_{i,for}^{k+1,2(s-1),s-1}(X_T),\]
\[A_i(u, p) = (f, u_0) \in B_{i,for}^{k,2(s-1),s-1}(X_T) \times H_i^{2s+k}.\]

Let us show that Problem (4) has at most one solution \((u, p)\) in the space \(B_{i,vel}^{k,2s}(X_T) \times B_{i,for}^{k+1,2(s-1),s-1}(X_T)\). Indeed, let \((u', p')\) and \((u'', p'')\) be any two solutions to (4) from the declared function space, i.e. \(A_i(u', p') = A_i(u'', p'')\) and sections \(u = u' - u''\) and \(p = p' - p''\) satisfies (4) with zero data \((f, u_0) = (0, 0)\). Moreover, as the left side of (4) is integrable with square we have
\[\frac{d}{dt}\|u\|^2_{L^2_t(X)} + 2\mu\|A^i u\|^2_{L^2_t(X)} = \left((B_i(u'', u'') - B_i(u', u'))(u), u\right)_{L^2_t(X)},\]
with the condition \(u(\cdot, 0) = 0\). Hence it follows from (17), Lemma 2.3 and Hölder’s inequality that \(u \equiv 0\). On the other hand, Corollary 3.1 implies \(p' = p''\). So, the operator \(A_i\) of (89) is injective.

Finally, it is easy to see that the Frechet derivative \(A_i'(w, p_0)\) of the nonlinear mapping \(A\) at an arbitrary point
\[(w, p_0) \in B_{i,vel}^{k,2s}(X_T) \times B_{i,for}^{k+1,2(s-1),s-1}(X_T)\]
coinsides with the continuous linear mapping \(A_{w,i}\) of (57). By (58) and Theorem 3.4, \(A_{w,i}\) is an invertible continuous linear mapping from \(B_{i,vel}^{k,2s}(X_T) \times B_{i,for}^{k+1,2(s-1),s-1}(X_T)\) to \(B_{i,for}^{k,2(s-1),s-1}(X_T) \times H_i^{2s+k}.\)
Both the openness of the mapping $A_i$ and the continuity of its local inverse mapping now follows from the implicit function theorem for Banach spaces, see for instance [19, Theorem 5.2.3, p. 101].

For the de Rham complex over the torus $T^3$ at the degree $i = 1$ Theorem 3.8 was proved in [9]; actually this situation corresponds to the Navier–Stokes equations for incompressible fluid in the periodic setting, see [20].

It is worth to note that for an Existence Theorem related to even weak (distributional) solutions to (4) one should necessarily assume that the bilinear forms $M_{i,1}$ have additional properties. For example, in the above case for the de Rham complex this is the vanishing property of the so-called trilinear form, see [1,2,17]. This means that the open mapping theorem is only a first step toward an Existence Theorem for regular solutions to (4).

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