CANONICAL VERTEX FORMALISM IN DT THEORY OF TORIC CALABI-YAU 4-FOLDS

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Abstract. Motivated by previous computations of Y. Cao, M. Kool and the author, we propose square roots and sign rules for the vertex and edge terms that compute Donaldson-Thomas invariants of a toric Calabi-Yau 4-fold, and prove that they are canonical, exploiting the combinatorics of plane and solid partitions.

1. Introduction

1.1. DT theory of Calabi-Yau 4-folds. Donaldson-Thomas invariants were firstly defined in Thomas’ thesis [31] as a way to count stable sheaves on projective Calabi-Yau 3-folds, and were recently extended to Calabi-Yau 4-folds in the seminal work of Cao-Leung [14].

Let $X$ be a quasi-projective variety, $\text{Hilb}^n(X, \beta)$ the Hilbert scheme of closed subschemes $Z \subset X$ with proper support in homology class $[Z] = \beta \in H_2(X, \mathbb{Z})$ and $\chi(O_Z) = n$ and denote by $\mathcal{I}$ the universal ideal sheaf on $X \times \text{Hilb}^n(X, \beta)$. By the work of Huybrechts-Thomas [19], the Atiyah class gives an obstruction theory on $\text{Hilb}^n(X, \beta)$

\[
\mathbb{R}\mathcal{H}om_{\pi}(\mathcal{I}, \mathcal{I})^\vee_0[-1] \to L_{\text{Hilb}^n(X, \beta)},
\]

where $(\cdot)_0$ denotes the trace-free part, $\mathbb{R}\mathcal{H}om_{\pi} = \mathbb{R}\pi_* \circ \mathbb{R}\mathcal{H}om_{\pi}$, $\pi : X \times \text{Hilb}^n(X, \beta) \to \text{Hilb}^n(X, \beta)$ and $L_{\text{Hilb}^n(X, \beta)}$ is the truncated cotangent complex.

If $X$ is a smooth projective 3-fold, the obstruction theory (1.1) is perfect, so $M$ carries a virtual fundamental class; Donaldson-Thomas invariants are defined by integrating insertions against it. Unfortunately, if $X$ is a smooth projective 4-fold, the obstruction theory fails to be perfect, so the machineries of Behrend-Fantechi [1] and Li-Tian [21] do not produce a virtual class. Nonetheless, if $X$ is a projective Calabi-Yau 4-fold, Borisov-Joyce [6] constructed a virtual fundamental class

\[
[H_{\text{Hilb}^n(X, \beta)}]^\text{vir} \in H_{2n}(\text{Hilb}^n(X, \beta), \mathbb{Z})
\]

which depends on a choice of orientation of $\text{Hilb}^n(X, \beta)$, i.e. a choice of square root of the isomorphism

\[
Q : \mathcal{L} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_{\text{Hilb}^n(X, \beta)},
\]

induced by Serre duality pairing, where $\mathcal{L} := \det(\mathbb{R}\mathcal{H}om_{\pi}(\mathcal{I}, \mathcal{I}))$ is the determinant line bundle. The existence of orientations was proved for arbitrary compact Calabi-Yau 4-folds by Cao-Gross-Joyce [8] and in the non-compact setting by Bojko [2]. Recently Oh-Thomas [27] proposed an alternative (algebraic) construction of the virtual cycle

\[
[H_{\text{Hilb}^n(X, \beta)}]^\text{vir} \in \mathbb{A}_n \left( \text{Hilb}^n(X, \beta), \mathbb{Z} \left[ \frac{1}{2} \right] \right),
\]

which coincides with Borisov-Joyce virtual cycle under the cycle map and proved a virtual localization formula. Morally, the virtual class is obtained (at least locally) as the zero locus of an isotropic section of an $SO(r, \mathbb{C})$-bundle over a smooth ambient space.

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1Here we denote by Calabi-Yau variety $X$ a smooth complex quasi-projective variety with $K_X \cong \mathcal{O}_X$ and $b_2(X) = 0$. 
1.2. Hilbert schemes of toric CY 4-folds. It is in general very difficult to compute Donaldson-Thomas type invariants of a Calabi-Yau 4-fold. Here we assume that $X$ is a toric Calabi-Yau 4-fold and denote by $T = \{ t_1t_2t_3t_4 = 1 \} \subset (\mathbb{C}^*)^4$ the subtorus preserving the Calabi-Yau form of $X$. The $T$-action naturally lifts to $\text{Hilb}^n(X, \beta)$, making the obstruction theory (1.1) $T$-equivariant by [30, Thm. 53]. Despite being almost never the Hilbert scheme $\text{Hilb}^n(X, \beta)$ proper, its fixed locus $\text{Hilb}^n(X, \beta)^T$ is reduced, 0-dimensional (Lemma 2.1) and the induced obstruction theory is trivial (Prop. 2.3), so we can define invariants $T$-equivariantly by means of Oh-Thomas virtual localization theorem (cf. [27, Thm. 7.1, Rem. 7.4]).

**Definition 1.1.** Let $\gamma \in H^*_T(\text{Hilb}^n(X, \beta))$. The $T$-equivariant Donaldson-Thomas invariants of $X$ are

\[
\text{DT}_n(X, \beta; \gamma) = \sum_{Z \in \text{Hilb}^n(X, \beta)^T} \sqrt{e^T(-T^\text{vir}_Z)} \cdot |Z| \in \frac{\mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)},
\]

where $T^\text{vir} \in K^0_{T}(\text{Hilb}^n(X, \beta))$ is the class of the dual of the obstruction theory (1.1) and $\lambda_i = e^T(t_i)$.

These invariants had been intensively studied (e.g. [11, 9, 15, 12]), where $\gamma$ is respectively a primary, descendant, tautological or $K$-theoretic insertion.

Here we denoted by $\sqrt{e^T(\cdot)}$ the ($T$-equivariant) square root Euler class defined by Edidin-Graham in [16] (cf. also [27, Sec. 7]). The definition of this class is far from being explicit since relies on the chosen orientation of $\text{Hilb}^n(X, \beta)$. To ease our life, we define the square root of a $T$-representation.

**Definition 1.2.** Let $V \in K^0_T(pt)$ be a virtual $T$-representation. We say that $T \in K^0_T(pt)$ is a square root of $V$ if

\[
V = T + \overline{T} \in K^0_T(pt),
\]

where $\overline{(\cdot)}$ denotes the dual $T$-representation.

Let $V$ admit a square root $T$ in $K^0_T(pt)$. Its square root Euler class satisfies

\[
\sqrt{e^T}(V) = \pm e^T(T),
\]

where the sign depends on the chosen orientation. Therefore, we can compute the invariants (1.2) by explicitly finding a square root of the virtual tangent bundle, at the price of introducing a (non-explicit!) sign at every $T$-fixed point of $\text{Hilb}^n(X, \beta)^T$.

Finding the correct signs given a square root of the virtual tangent space is one of the major difficulties of the theory, as the number of possible choices of signs rapidly increases with the number of fixed points, which makes the application of the virtual localization formula not feasible — we recall that the virtual localization formula is one of the rare tools at our disposal to perform computations in DT theory. The main goal of this paper is to propose various square roots of the virtual tangent bundle with sign rules, which would make the invariants (1.2) effectively computable.

As evidence to our proposals, we show that they

- are canonical (Theorem 2.8, 2.14, 2.17),
- play well with the dimensional reduction studied in [12, Sec. 2.1] (Remark 2.7, 2.13, 2.18),
- are consistent with previous computations [9, 10, 11, 12, 24, 26], both in the math and physics literature.

To each fixed point in $\text{Hilb}^n(X, \beta)^T$, we associate some combinatorial data using solid partitions, which can be displayed as arrangements of boxes in $\mathbb{Z}_{\geq 0}^4$ (cf. Section 2.1). We exploit the combinatorics of solid partitions to propose our sign rules and to prove they are canonical. It is important to notice that these signs carry independent interesting information on the combinatorics of solid partitions — a field where not much is currently known (see Section 1.4).

For $X = \mathbb{C}^4$, our proposals generalize the ones already discussed by Nekrasov-Piazzalunga [26, Sec. 2.4] for $\beta = 0$ and by Cao-Kool and the author [12, Rem. 1.18] when the $T$-invariant subscheme...
$Z \subset X$ is supported in at most two $T$-invariant lines. Here we deal with the full theory, where $Z$ can be supported in four $T$-invariant lines.

1.3. Relation to String Theory. Donaldson-Thomas invariants of Calabi-Yau 4-folds — and their $K$-theoretic refinement — appear in String Theory as the result of supersymmetric localization of $U(1)$ super-Yang–Mills theory with matter on a Calabi–Yau 4-fold, studied by Nekrasov-Piazzalunga [24, 26] and by Bonelli-Fasola-Tanzini-Zenkevich [5] with an ADHM-type construction. In these cases, the authors study the quantum mechanics of a system of $D0$-$D8$ branes. Our results would allow to easily perform computations in the presence of points and curves, that is for a system of $D0$-$D2$-$D8$ branes.

1.4. Magnificent four. MacMahon [22] classically solved the combinatorial problem of enumerating partitions and plane partitions. Remarkably, his conjectural formula for solid partitions fails and no closed expression is expected to exist. From a modern geometric point of view, the enumeration of partitions and plane partitions appears as a suitable limit of $K$-theoretic invariants of Hilb$^n(\mathbb{C}^2)$ and Hilb$^n(\mathbb{C}^3)$, exploiting respectively the smooth structure and the symmetric perfect obstruction theory — cf. [28]. In the case of Hilb$^n(\mathbb{C}^4)$ the Borisov-Joyce/Oh-Thomas virtual structure crucially depends on a choice of orientation, which is responsible for the ambiguity of sign at each fixed point — a phenomenon that does not arise in lower dimension — and partially explains the failure of MacMahon’s conjecture, where the signs are not playing any rôle. For $d \geq 5$ there are no known virtual structures on Hilb$^n(\mathbb{C}^d)$ and no direct analogues of the combinatorial formulas in dimensions 2,3,4 seem to hold — see [10]. This may indicate that dimension 4 is special and we believe that our sign conjectures could play an important rôle in Combinatorics as well; quoting Nekrasov’s Magnificent four [24]

"The adjective ‘Magnificent’ reflects this author’s conviction that the dimension four is the maximal dimension where the natural albeit complex-valued probability distribution exists."

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Denote by $T = \{ t_1 t_2 t_3 t_4 = 1 \} \subset (\mathbb{C}^*)^4$ the subtorus preserving the Calabi-Yau form of $X$. The $(\mathbb{C}^*)^4$-action and the $T$-action naturally lift on $\text{Hilb}^n(X, \beta)$, whose fixed locus is reduced and 0-dimensional.

**Lemma 2.1** ([11, Lemma 2.1, 2.2]). We have an isomorphism of schemes

$$\text{Hilb}^n(X, \beta)^T = \text{Hilb}^n(X, \beta)^{(\mathbb{C}^*)^4},$$

which consists of finitely many reduced points.

We recap the description of the $(\mathbb{C}^*)^4$-fixed locus, which is completely analogous to [23, Sec. 4.2] in the setting of toric 3-folds. For an extensive treatment in the case of toric 4-folds, look at [11, Sec. 2.1].

Let $Z \in \text{Hilb}^n(X, \beta)^{(\mathbb{C}^*)^4}$ and $I$ be its ideal sheaf; $Z \subset X$ is preserved by the torus action, hence it must be supported on the $(\mathbb{C}^*)^4$-fixed points (corresponding to vertices $\alpha \in V(X)$) and $(\mathbb{C}^*)^4$-invariant lines of $X$ (corresponding to edges $\alpha \beta \in E(X)$). Since $I$ is $(\mathbb{C}^*)^4$-fixed on each open subset, $I$ must be defined on $U_\alpha$ by a monomial ideal

$$I_\alpha = I|_{U_\alpha} \subset \mathbb{C}[x_1, x_2, x_3, x_4],$$

and may also be viewed as a solid partition $\pi_\alpha$.

(2.1) $$\pi_\alpha = \left\{ (k_1, k_2, k_3, k_4), \prod_{i=1}^4 x_i^{k_i} \not\in I_\alpha \right\} \subset \mathbb{Z}_{\geq 0}^4.$$  

The associated subscheme of $I_\alpha$ is (at most) 1-dimensional. The corresponding partition $\pi_\alpha$ may be infinite in the direction of the coordinate axes. If the solid partition is viewed as a box diagram in $\mathbb{Z}^4$, the vertices in (2.1) are determined by the interior corners of the boxes, the corners closest to the origin.

The asymptotics of $\pi_\alpha$ in the coordinate directions are described by four ordinary finite-size plane partitions. In particular, in the direction of the $(\mathbb{C}^*)^4$-invariant curve $L_{\alpha \beta}$ given by $\{ x_2 = x_3 = x_4 = 0 \}$, we have the plane partition $\lambda_{\alpha \beta}$ with the following diagram

$$\lambda_{\alpha \beta} = \left\{ (k_2, k_3, k_4) : x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4} \not\in I_\alpha, \forall k_1 \geq 0 \right\}$$

$$= \left\{ (k_2, k_3, k_4) : x_2^{k_2} x_3^{k_3} x_4^{k_4} \not\in I_{\alpha \beta} \right\} \subset \mathbb{Z}_{\geq 0}^3,$$

where

$$I_{\alpha \beta} = I|_{U_\alpha \cap U_\beta} \subset \mathbb{C}[x_1^{\pm 1}, x_2, x_3, x_4].$$

The vertices of $\lambda_{\alpha \beta}$ defined above are the interior corners of the squares of the associated plane partition.

In summary, a $(\mathbb{C}^*)^4$-fixed ideal sheaf can be described in terms of the following data:

(i) a finite-size plane partition $\lambda_{\alpha \beta}$ assigned to each edge $\alpha \beta \in E(X)$;

(ii) a (possibly infinite) solid partition $\pi_\alpha$ assigned to each vertex $\alpha \in V(X)$, such that the asymptotics of $\pi_\alpha$ in the three coordinate directions is given by the plane partitions $\lambda_{\alpha \beta}$ assigned to the corresponding edges.

Let $Z \in \text{Hilb}^n(X, \beta)^T$ correspond to the partition data $\{ \pi_\alpha, \lambda_{\alpha \beta} \}_{\alpha, \beta}$. We see

$$\beta = \sum_{\alpha \beta \in E(X)} |\lambda_{\alpha \beta}| |I_{\alpha \beta}| \in H_2(X, \mathbb{Z}),$$

where $|\lambda_{\alpha \beta}|$ denotes the size of the plane partition $\lambda$, the number of the boxes in the diagram.
For a (possibly infinite) solid partition \( \pi \) such that its asymptotic plane partitions \( \lambda_i, i = 1, 2, 3, 4 \), are of finite size \( |\lambda_i| < \infty \), we define the renormalized volume \( |\pi| \) as follows. We set

\[
|\pi| := \# \{ \pi \cap [0, \ldots, N]^4 \} - (N + 1) \sum_{i=1}^4 |\lambda_i|, \quad N \gg 0.
\]

The renormalized volume is independent of the cut-off \( N \) as long as \( N \) is sufficiently large. We will say that a solid partition is point-like if all the asymptotics \( \lambda_i = 0, i = 1, \ldots, 4 \) and curve-like if at least one \( \lambda_i \neq 0 \). See [20] for a similar discussion of the renormalization of infinite plane partitions and its interpretation in terms of melting crystals.

To conclude, let \( \mathbf{m} = (m_2, m_3, m_4) \), \( \lambda \) a finite-size plane partition and set

\[
f_{\mathbf{m}}(\lambda) = \sum_{(i,j,k) \in \lambda} (1 - m_2 \cdot i - m_3 \cdot j - m_4 \cdot k).
\]

By [11, Lemma 2.4], if a \( T \)-invariant closed subscheme \( Z \subset X \) corresponds to a partition data \( \{ \pi_\alpha, \lambda_{\alpha\beta} \}_{\alpha, \beta} \), then

\[
\chi(\mathcal{O}_Z) = \sum_{\alpha \in V(X)} |\pi_\alpha| + \sum_{\alpha \beta \in E(X)} f_{\alpha\beta}(\lambda_{\alpha\beta}),
\]

where \( m_{\alpha\beta} \) is the multidegree of the normal bundle of the \( T \)-invariant line \( L_{\alpha\beta} \).

2.2. The vertex formalism. Denote by \( \mathcal{I} \) the universal ideal sheaf of the universal subscheme \( \mathcal{Z} \subset X \times \text{Hilb}^n(X, \beta) \). The obstruction theory \( \mathbf{R} \text{Hom}_\mathcal{I}(\mathcal{I}, \mathcal{I})[1] \to \text{Hilb}^n(X, \beta) \) is naturally \( T \)-equivariant by [30, Cor. 4.4] and endowed by the \( T \)-equivariant Serre quadratic pairing. Over a point \( Z \in \text{Hilb}^n(X, \beta) \) the fiber of the virtual tangent space is

\[
T^\text{vir}_Z = \mathbf{R} \text{Hom}_\mathcal{I}(\mathcal{I}, \mathcal{I})[1]|_Z = \mathbf{R} \text{Hom}(\mathcal{I}_Z, \mathcal{I}_Z)[1],
\]

where \( \mathcal{I}_Z \) is the ideal sheaf of \( Z \). To a \( T \)-fixed point \( Z \) corresponds a partition data \( \{ \pi_\alpha, \lambda_{\alpha\beta} \} \) with \( \alpha \in V(X), \alpha \beta \in E(X) \); denote by

\[
Z_\alpha = H^0(Z|_{U_\alpha}, \mathcal{O}_Z|_{U_\alpha}) = \sum_{(i,j,k) \in \pi} t_1^i t_2^j t_3^k t_4^l,
\]

\[
Z_{\alpha\beta} = \sum_{(a,b,c) \in \lambda} t_1^a t_2^b t_3^c t_4^d,
\]

where the line \( L_{\alpha\beta} \) is defined by \( \{ x_j = x_k = x_l = 0 \} \). Recall the vertex formalism\(^2\) developed in [11, Sec. 2.4]

\[
V_\alpha = Z_\alpha + Z_{\alpha\beta} - \frac{Z_{\alpha\beta}}{t_2 t_3 t_4} + \sum_{i=1}^4 \frac{F_{\alpha\beta}(t_i, t_i', t_i''')}{1 - t_i},
\]

\[
F_{\alpha\beta} = -Z_{\alpha\beta} + \frac{Z_{\alpha\beta}}{t_2 t_3 t_4} + \sum_{i=1}^4 \frac{F_{\alpha\beta}(t_i, t_i', t_i''')}{1 - t_i},
\]

\[
E_{\alpha\beta} = \frac{t_1^{-1} F_{\alpha\beta}(t_2 t_3 t_4)}{1 - t_1^{-1}} - \frac{F_{\alpha\beta}(t_2 t_3 t_4)}{1 - t_1^{-1}},
\]

where \( \{ t_1, t_1', t_1'', t_1''' \} = \{ 1, 2, 3, 4 \} \) and for a set of indices \( I \), we set \( P_I = \prod_{\alpha \in I} (1 - t_\alpha) \). This vertex formalism allows us to compute the virtual tangent space at a \( T \)-fixed point.

**Proposition 2.2** ([11, Prop. 2.11]). Let \( X \) be a toric Calabi-Yau 4-fold, \( \beta \in H_2(X, \mathbb{Z}) \) and \( Z \in \text{Hilb}^n(X, \beta)^T \). Then

\[
T^\text{vir}_Z = \sum_{\alpha \in V(X)} V_\alpha + \sum_{\alpha \beta \in E(X)} E_{\alpha\beta}.
\]

\(^2\)We only write down \( F_{\alpha\beta} \) and \( E_{\alpha\beta} \) when \( L_{\alpha\beta} \cong \mathbb{P}^1 \) is given by \( \{ x_2 = x_3 = x_4 = 0 \} \), i.e. the leg along the \( x_1 \)-axis. The other cases follow by symmetry.
Anticipating some results from Section 2.4, 2.5 we prove that the obstruction theory induced on
the $T$-fixed locus is trivial.

**Proposition 2.3.** Let $X$ be a toric Calabi-Yau 4-fold and $\beta \in H_2(X, \mathbb{Z})$. Then the induced
obstruction theory on $\text{Hilb}^n(X, \beta)^T$ is trivial. In particular, for $Z \in \text{Hilb}^n(X, \beta)^T$, the virtual
tangent space $T_Z^{vir}$ is $T$-movable.

**Proof.** By Lemma 2.10, 2.15 there exist $T$-movable square roots $v_\alpha, e_\alpha \beta$ for any $\alpha \in V(X), \alpha \beta \in E(X)$, which implies that also $V_\alpha, E_\alpha \beta$ and $T_Z^{vir}$ are $T$-movable by Proposition 2.2. We have an identity in $T$-equivariant $K$-theory

$$T_Z^{vir} = \text{Ext}^1(I_Z, I_Z) - \text{Ext}^2(I_Z, I_Z) + \text{Ext}^3(I_Z, I_Z) \in K_T^0(\text{pt}).$$

It follows by Lemma 2.1 that $\text{Ext}^1(I_Z, I_Z)^T = \text{Ext}^3(I_Z, I_Z)^T = 0$, by which we conclude that $\text{Ext}^2(I_Z, I_Z)^T = 0$ as well. \hfill $\square$

In the remainder of the paper, we will exhibit (several) explicit square roots $v_\alpha, e_\alpha \beta$ of $V_\alpha, E_\alpha \beta$
reducing the DT invariants in (1.2) to

$$\text{DT}_n(X, \beta; \gamma) = \sum_{Z \in \text{Hilb}^n(X, \beta)^T} \prod_{\alpha \in V(X)} (-1)^{\sigma(Z, v_\alpha)} e_\alpha \cdot \prod_{\alpha \beta \in E(X)} (-1)^{\sigma(Z, e_\alpha \beta)} e_\alpha \beta \cdot \gamma |_Z,$$

and propose explicit canonical signs $(-1)^{\sigma(Z, v_\alpha)}, (-1)^{\sigma(Z, e_\alpha \beta)}$.

### 2.3. The vertex term: points.

To each fixed point $Z \in \text{Hilb}^n(C^4)^T$ corresponds a solid partition $\pi$ of size $n$. Denote by $Z_\pi, V_\pi$ the vertex terms $Z_\alpha, V_\alpha$ in (2.3), (2.5). By $T$-equivariant Serre duality, we know that $V_\pi$ admits a square root. We set

$$v_\pi^i = Z_\pi - \overline{P_{ijkl} Z_\pi Z_{\pi}},$$

which enjoys

$$V_\pi = v_\pi^i + \overline{v_\pi},$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. For $i = 4$, it recovers the square root already found in [26, 12]. The next lemma was already proven by Nekrasov-Piazzalunga [26], whose proof we sketch for completeness and to introduce useful notation.

**Lemma 2.4** ([26, Sec. 2.4.1]). Let $\pi$ be a point-like solid partition and $i = 1, \ldots, 4$. Then $v_\pi^i$ is $T$-movable.

**Proof.** Without loss of generality, suppose that $i = 4$. We prove the statement by induction on the size of $\pi$. If $|\pi| = 1$, then $v_\pi^4$ has no constant terms. Suppose now that the claim holds for all solid partitions $\pi$ of size $|\pi| \leq n$. Consider a solid partition $\tilde{\pi}$ of size $|\tilde{\pi}| = n + 1$; this can be seen as a solid partition $\pi$ of size $n$ with an extra box over it, corresponding to a $Z^4$-lattice point $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$. We have

$$Z_{\tilde{\pi}} = Z_\pi + t^\mu$$

and

$$v_\pi^4 = v_\pi^4 + t^\mu - \overline{P_{ijkl} t^{\mu} Z_\pi Z_\pi Z_{\pi} Z_{\pi} + 1} = v_\pi^4 + t^\mu - \overline{P_{ijkl} (t^{\mu} Z_\pi Z_{\pi} + t^{-\mu} Z_\pi + 1)}.$$

By induction, we know that $v_\pi^4$ is $T$-movable. Consider the subdivision $\tilde{\pi} = \pi' \sqcup \pi''$ of the boxes of
$\tilde{\pi}$, where $\pi'$ corresponds to the lattice points $\nu$ such that $\nu \leq \mu$ and $\pi''$ corresponds to the lattice
points \( \nu \) such that \( \nu \not\leq \mu \). Here, given \( \nu = (\nu_1, \nu_2, \nu_3, \nu_4) \), we say that \( \nu \leq \mu \) if \( \nu_i \leq \mu_i \) for all \( i = 1, \ldots, 4 \). Denote by

\[
Z_{x'} := \sum_{\nu \in \pi'} t^\nu = \sum_{i \leq \mu_1, j \leq \mu_2, k \leq \mu_3, l \leq \mu_4} t_i^1 t_j^2 t_k^3 t_l^4,
\]

\[
Z_{x''} := \sum_{\nu \in \pi''} t^\nu.
\]

By construction, \( Z_{\bar{x}} = Z_{x'} + Z_{x''} \). We want to prove that

\[
(t^\nu + \frac{1}{T_{123}} - \frac{1}{T_{123}} t^\mu Z_{x'} - \frac{1}{T_{123}} t^\mu Z_{x''} - \frac{1}{T_{123}} t^{-\mu} Z_{x'} - \frac{1}{T_{123}} t^{-\mu} Z_{x''})^\text{fix} = 0,
\]

where by \((\cdot)^\text{fix}\) we denote the invariant factors under the \( T \)-action\(^3\). For a set of indices \( I \), denote by \( \delta_I \) the function which is 1 if only if all the indices are equal. The contribution of each summand is

\[
(t^\nu + \frac{1}{T_{123}})^\text{fix} = 1 + \delta_{\mu_1, \mu_2, \mu_3, \mu_4},
\]

\[
(\frac{1}{T_{123}} t^\mu Z_{x'})^\text{fix} = 0,
\]

\[
(\frac{1}{T_{123}} t^{-\mu} Z_{x'})^\text{fix} = 0,
\]

\[
(\frac{1}{T_{123}} t^\mu Z_{x''})^\text{fix} = \sum_{i=0}^{\mu_4} \delta_{\mu_1, \mu_2, \mu_3, i},
\]

\[
(\frac{1}{T_{123}} t^{-\mu} Z_{x''})^\text{fix} = 1 - \sum_{i=0}^{\mu_4 - 1} \delta_{\mu_1, \mu_2, \mu_3, i},
\]

by which we conclude the induction step. \( \square \)

We propose a sign rule\(^4\) for the sign in (2.8), relative to the square root \( v_4^i \).

**Conjecture 2.5.** Let \( \pi \) be a solid partition corresponding to a \( T \)-fixed point in \( \text{Hilb}^n(\mathbb{C}^4) \). Then the sign relative to the square root \( v_4^i \) is \((-1)^{\sigma_i(\pi)}\), where

(2.9) \[
\sigma_i(\pi) = |\pi| + \# \{ (a_1, a_2, a_3, a_4) \in \pi : a_j = a_k = a_l < a_i \}
\]

and \( \{ i, j, k, l \} = \{ 1, 2, 3, 4 \} \).

**Remark 2.6.** For \( i = 4 \), the sign rule (2.9) was proposed by Nekrasov-Piazalunga in the physics literature in [26], as the result of supersymmetric localization in string theory\(^5\). It was also proposed in [12], based on explicit low order computations. This sign rule is consistent with the previous computations of [9, 10, 11, 12, 24, 26].

**Remark 2.7.** Let \( \pi \) corresponds to a \( T \)-invariant closed subscheme \( Z \subset \mathbb{C}^4 \) supported in the hyperplane \( \{ x_i = 0 \} \subset \mathbb{C}^4 \), for \( i = 1, \ldots, 4 \). Then \( \sigma_i(\pi) = |\pi| \), which is consistent with the dimensional reduction studied in [12, Sec. 2.1].

We prove now that the sign rule (2.9) is canonical, meaning that it does not really depend on choosing a preferred \( x_i \)-axis, as one should expect from the correct sign rule.

**Theorem 2.8.** Let \( \pi \) a point-like solid partition. For every \( i, j = 1, \ldots, 4 \) we have

\[
(-1)^{\sigma_i(\pi)} e^T(-v_4^j) = (-1)^{\sigma_j(\pi)} e^T(-v_4^i).
\]

\(^3\)In other words, for a virtual \( T \)-representation \( F \in K^0_T(\text{pt}) \), \((F)^\text{fix}\) is its constant term when viewing \( F \) as a Laurent polynomial in the torus coordinates.

\(^4\)After a first draft of the paper was written, Kool-Rennemo [20] announced a proof for this sign rule.

\(^5\)In [26] the sign rule is slightly different due to a slightly different choice of square root.
Proof. \(v^i_\pi - v^j_\pi\) are both square roots of \(\sqrt{\pi}\) and are \(T\)-movable by Lemma 2.4. To prove the claim, we need to write
\[
v^i_\pi - v^j_\pi = U^\pi - \overline{U^\pi},
\]
for a \(U_\pi \in K^{\text{pt}}(\text{pt})\) and compute the parity of \(\text{rk} U^\text{mov}_\pi\); in fact
\[
\frac{e^T(v^i_\pi)}{e^T(v^j_\pi)} = \frac{e^T(U^\text{mov}_\pi)}{e^T(U^\text{mov}_\pi)} = (-1)^{\text{rk} U^\text{mov}_\pi}.
\]
Without loss of generality, suppose \(i = 4\) and \(j = 3\); we prove the claim by induction on the size of \(\pi\). If \(|\pi| = 1\), we have \(Z_\pi = 1\) and
\[
v^4_\pi - v^3_\pi = \overline{P}_{12}(t_3^{-1} - t_4^{-1}) = \overline{P}_{12}t_3^{-1} - \overline{P}_{12}t_3^{-1},
\]
where used that \(t_1t_2t_3t_4 = 1\); clearly \(\text{rk}(\overline{P}_{12}t_3^{-1}) = 0\). Suppose now the claim holds for any partition of size \(|\pi| \leq n\) and consider a solid partition \(\tilde{\pi}\), obtained by a solid partition \(\pi\) of size \(n\) by adding a box whose lattice coordinates are \(\mu = (\mu_1, \mu_2, \mu_3, \mu_4)\). We have
\[
v^4_\pi - v^3_\pi = v^4_\pi - v^4_\tilde{\pi} + \overline{P}_{12}(t_3^{-1} - t_4^{-1}) + \overline{P}_{12}(t_3^{-1} - t_4^{-1})(\overline{Z^\mu_\pi} + Z^\mu_\tilde{\pi}) = v^4_\pi - v^3_\tilde{\pi} + \overline{P}_{12}(t_3^{-1} - t_4^{-1}) + \overline{P}_{12}(t_3^{-1} - t_4^{-1})(\overline{Z^\mu_\pi} + Z^\mu_\tilde{\pi}),
\]
and by the induction step
\[
e^T(v^4_\pi - v^3_\pi) = (-1)^{\sigma_4(\pi) - \sigma_3(\pi)},
\]
e^T(\overline{P}_{12}(t_3^{-1} - t_4^{-1})) = 1.
\]
The final piece to compute is of the form
\[
\overline{P}_{12}(t_3^{-1} - t_4^{-1})(\overline{Z^\mu_\pi} + Z^\mu_\tilde{\pi}) = (W_1 + \overline{W_1})(\overline{W_2} - \overline{W_2}) = (W_1W_2 - \overline{W_1W_2}) + (\overline{W_1W_2} - W_1\overline{W_2}),
\]
with
\[
W_1 = t^{-\mu}Z_\pi,
\]
\[
W_2 = t_3^{-1}\overline{P}_{12}.
\]
Since \(\text{rk}W_1W_2 = \text{rk}\overline{W_1W_2} = 0\), we just need to compute the parity of \(\text{rk}(W_1W_2)^{\text{fix}} + \text{rk}(\overline{W_1W_2})^{\text{fix}}\).

As in the proof of Lemma 2.4, consider the subdivision \(\tilde{\pi} = \pi \cup \pi''\), with \(Z_\pi = Z_\pi' + Z_\pi''\), where
\[
Z_\pi' = \sum_{i \leq \mu_1, j \leq \mu_2, k \leq \mu_3, l \leq \mu_4} t_1^i t_2^j t_3^k t_4^l,
\]
\[
Z_\pi'' = \sum_{\nu \in \pi''} t^\nu.
\]
We have
\[
(W_1W_2)^{\text{fix}} = (t_3^{-1}\overline{P}_{12}t^{-\mu}Z_\pi' + t_3^{-1}\overline{P}_{12}t^{-\mu}Z_\pi'')^{\text{fix}}.
\]
As in the proof Lemma 2.4 we compute
\[
(t_3^{-1}\overline{P}_{12}t^{-\mu}Z_\pi'')^{\text{fix}} = 0,
\]
\[
(t_3^{-1}\overline{P}_{12}t^{-\mu}Z_\pi')^{\text{fix}} = \sum_{k=0}^{\mu_3} \sum_{l=0}^{\mu_4} \delta_{\mu_1, \mu_2, k, l}.
\]
Notice now that
\[
\overline{W_2} = t_3^{-1}\overline{P}_{12},
\]
thus, by symmetry, we have
\[
(W_1W_2)^{\text{fix}} = \sum_{k=0}^{\mu_3} \sum_{l=0}^{\mu_4} \delta_{\mu_1, \mu_2, k, l}.
\]
We compute the parity
\[
\text{rk}(W_1W_2)^{\text{fix}} + \text{rk}(W_1^*W_2)^{\text{fix}} = \sum_{k=0}^{\mu_3} \sum_{l=0}^{\mu_4-1} \delta_{\mu_1,\mu_2,k,l} + \sum_{k=0}^{\mu_4-1} \sum_{l=0}^{\mu_3} \delta_{\mu_1,\mu_2,k,l}
\]
which gives
\[
\sum_{l=0}^{\mu_4-1} \delta_{\mu_1,\mu_2,\mu_3,l} + \sum_{k=0}^{\mu_3-1} \delta_{\mu_1,\mu_2,k,\mu_4} \mod 2.
\]
Notice that
\[
\sigma_4(\pi) = \sum_{i=0}^{\mu_4-1} \delta_{\mu_1,\mu_2,\mu_3,i} + \sum_{i=0}^{\mu_3-1} \delta_{\mu_1,\mu_2,k,\mu_4},
\]
which concludes the proof.

2.4. The vertex term: curves. Set \( A = (A_1, A_2, A_3, A_4) \), where \( A_i \) are finite plane partitions. Denote by \( P_A \) the collection of (possibly infinite) solid partitions, whose asymptotic profile is given by \( A \). To any solid partition \( \pi \in P_A \) corresponds a \( T \)-invariant closed subscheme \( Z \subset \mathbb{C}^4 \); denote by \( Z_\pi, Z_{A_i}, V_\alpha \) in (2.3), (2.4), (2.5). \( T \)-equivariant Serre duality implies that \( V_\pi \) admits a square root; set
\[
V_\pi^e = Z_\pi - P_{jkl}Z_\pi Z_{\pi E} + \sum_{j \neq i, j=1}^{4} f_{ij} \frac{1}{1-t_j} + \left( -Z_{A_i} + P_{jkl}(Z_\pi Z_{A_j} - Z_\pi Z_{A_l}) + \frac{P_{jkl}}{1-t_j} Z_{A_j} Z_{A_l} \right),
\]
where \( \{ i, j, k, l \} = \{ 1, 2, 3, 4 \} \). Setting \( i = 4 \), we recover the explicit square root choice studied in [12]. We redistribute now the vertex terms in a different way. Write, for \( N \gg 0 \)
\[
Z_\pi = Z_{\pi_{\text{nor}}} + \sum_{i=1}^{4} Z_{A_i} \frac{t_i}{1-t_i} t_i^{N},
\]
\[
Z_{A_i} = Z_{A_i} \sum_{n=0}^{N} t_i^n,
\]
\[
\frac{Z_{A_i}}{1-t_i} = Z_{A_i} + \frac{Z_{A_i}}{1-t_i} t_i^{N+1}.
\]
Here the (point-like) solid partition \( \pi_{\text{nor}} \) is the cut-off for \( N \gg 0 \) of the (possibly curve-like) solid partition \( \pi \). If \( \pi \) is a point-like solid partition, then simply \( \pi_{\text{nor}} = \pi \), while \( \pi \) is simply the cut-off for \( N \gg 0 \) of the curve-like solid partition corresponding to the infinite leg along the \( x_i \)-axis containing \( \pi \). Using the above expressions, we can express the vertex terms as
\[
V_\pi^e = V_\pi^e_{\text{nor}} - \sum_{a=1}^{4} V_{\pi_a}^a + A_\pi^e + B_\pi^e - C_\pi^e + D_\pi^e,
\]
where
\[
A_\pi^e = -P_{jkl} \sum_{a \neq b} \frac{Z_{A_a}}{1-t_a} t_a^{-1} t_b^{-1} t_j t_k \frac{Z_{A_b}}{1-t_b} (t_a t_b^{N+1})^{N+1} - P_{1234} \sum_{a \neq i} \frac{Z_{A_a}}{1-t_a} t_a^{-1} (t_{a_i} t_i^{N+1} - t_i^{N+1} t_{a_i}^{-1}),
\]
\[
B_\pi^e = -P_{jkl} \sum_{a \neq i} \left( \frac{Z_{A_a}}{1-t_a} t_{a_i} t_i^{N+1} (Z_{\pi_{\text{nor}}} - Z_{\pi_a}) + \frac{Z_{A_a}}{1-t_a} t_i^{-(N+1)} (Z_{\pi_{\text{nor}}} - Z_{\pi_a}) \right)
\]
\[
- P_{1234} \frac{Z_{A_a}}{1-t_a} t_i^{-(N+1)} (Z_{\pi_{\text{nor}}} - Z_{\pi_a}),
\]
\[ C_i^\prime = T_{123} \left( Z_\pi (Z_{\rho^{\text{nor}}} - Z_{\tau}) + \sum_{a \neq i} \frac{Z_{\lambda_a}}{1 - t_a} \rho^{(N+1) Z_{\pi_a}} \right), \]

where \{ i, j, k, l \} = \{ 1, 2, 3, 4 \}. Motivated by the above expression, we define a new square root of \( V_\pi \)

\[ \sqrt{V_\pi} = \sqrt{V_\pi} \cdot C_z^\prime + \overline{C_z^\prime}. \]

**Remark 2.9.** It is an equivalent problem to find the sign rule for \( \sqrt{V_\pi} \) or for \( V_\pi \). In fact, such two sign rules will differ just by \((-1)^{rk(C_z^\prime)^{\text{nor}}\rho} \), which is completely determined by the solid partition \( \pi \). Moreover, if the \( T \)-invariant closed subscheme \( Z \subset \mathbb{C}^4 \) corresponding to a solid partition \( \pi \) is not supported in the \( T \)-invariant line \( \{ x_j = x_k = x_l = 0 \} \), we have that \( \sqrt{V_\pi} = V_\pi \).

**Lemma 2.10.** Let \( \pi \) be a curve-like solid partition and \( i = 1, \ldots, 4 \). Then \( \sqrt{V_\pi} \), \( \sqrt{V_\pi} \)' are \( T \)-movable.

**Proof.** By Lemma 2.4 \( \sqrt{V_{\text{prod}}} \), \( \sqrt{V_{\text{nor}}} \) are \( T \)-movable, for \( a = 1, \ldots, 4 \). For \( N \gg 0 \), we clearly have \((A_\pi^a)^{\text{fix}} = (B_\pi^a)^{\text{fix}} = 0\).

We propose a sign rule for the sign in \((2.8)\), relative to the square root \( \sqrt{V_\pi} \).

**Conjecture 2.11.** Let \( \pi \) be a curve-like solid partition. Then the sign relative to the square root \( \sqrt{V_\pi} \) is \((-1)^{\sigma_i(\pi)}\), where

\[ \sigma_i(\pi) = |\pi| + \# \{ (a_1, a_2, a_3, a_4) \in \pi: a_j = a_k = a_l < a_i \} - \sum_{\text{leg}} \# \{ (a_1, a_2, a_3, a_4) \in \text{leg}: a_j = a_k = a_l < a_i \} \]

and \( \{ i, j, k, l \} = \{ 1, 2, 3, 4 \} \), where leg denote the curve-like solid partitions obtained by translating the plane partitions \( \lambda \) along the \( x_i \)-axis.

**Remark 2.12.** For \( i = 4 \) the sign rule \((2.11)\) was proposed in [12] for at most two non-empty legs, based on explicit low order computations. This sign rule is consistent with the computations of [11, 12].

**Remark 2.13.** Let \( \pi \) corresponds to a \( T \)-invariant closed subscheme \( Z \subset \mathbb{C}^4 \) supported in the hyperplane \( \{ x_i = 0 \} \subset \mathbb{C}^4 \), for \( i = 1, \ldots, 4 \). Then \( \sigma_i(\pi) = |\pi| \) which is consistent with the dimensional reduction studied in [12, Sec. 2.1].

We prove now that the sign rule \((2.11)\) is canonical, meaning that it does not really depend on choosing a preferred \( x_i \)-axis.

**Theorem 2.14.** Let \( \pi \) a curve-like solid partition. For every \( i, j = 1, \ldots, 4 \) we have

\[ (-1)^{\sigma_i(\pi)} \rho^T (\sqrt{V_\pi}) = (-1)^{\sigma_i(\pi)} \rho^T (\sqrt{V_\pi}). \]

**Proof.** \( \sqrt{V_\pi} \) and \( \sqrt{V_\pi} \) are both square roots of \( V_\pi \) and are \( T \)-movable by Lemma 2.10. The difference of the two sections is

\[ \sqrt{V_\pi} = \sqrt{V_\pi} + \frac{4}{A_\pi^i - A_\pi^i + B_\pi^i - B_\pi^i} \]

By Theorem 2.8, we know that

\[ e^T (\sqrt{V_\pi} - \sqrt{V_\pi}) = (-1)^{\sigma_i(\pi) - \sigma_j(\pi)}, \]

which satisfies

\[ \sigma_i(\pi) - \sigma_j(\pi) = \sigma_i(\pi) - \sigma_j(\pi) - \sum_{a=1}^{4} (\sigma_i(\pi_a) - \sigma_j(\pi_a)). \]
To conclude, with a simple computation it is possible to show that
\[
A^j_n - A^j_2 = U_{A,i,j,\pi} - U_{A,i,j,\pi},
\]
\[
B^j_n - B^j_2 = U_{B,i,j,\pi} - U_{B,i,j,\pi},
\]
where, for \( N \gg 0 \), we have \( \text{rk}(U_{A,i,j,\pi})^{\text{mov}} = \text{rk}(U_{A,i,j,\pi})^{\text{fix}} = 0 \) and \( \text{rk}(U_{B,i,j,\pi})^{\text{mov}} = \text{rk}(U_{B,i,j,\pi})^{\text{fix}} = 0 \).

2.5. The edge term. In this section we study square roots and propose a sign rule for the edge term 2.7. For simplicity, let’s assume that the edge \( \alpha \beta \in E(X) \) corresponds to the \( \mathbb{P}^1 \) given, in the local coordinates of \( U_a \), by \( \{ x_2 = x_3 = x_4 = 0 \} \), with normal bundle
\[
N_{\mathbb{P}^1/X} \cong \mathcal{O}(m_2) \oplus \mathcal{O}(m_3) \oplus \mathcal{O}(m_4),
\]
satisfying \( m_2 + m_3 + m_4 = -2 \); we set \( m = (m_2, m_3, m_4) \). We also fix a finite-size plane partition \( \lambda \), corresponding to the profile of the non-reduced \( T \)-fixed leg. The discussion for the legs along the other directions will be completely analogous. Denote here by \( Z_\lambda, E_\lambda \) the edge terms \( Z_{\alpha \beta}, E_{\alpha \beta} \) in (2.4), (2.7), where
\[
Z_\lambda = \sum_{(j,k,l) \in \lambda} t^j_2 t^k_3 t^l_4.
\]
Denote by \( (\cdot) : K^0_T(\mathbb{P}^1) \to K^0_T(\mathbb{P}^1) \) the map sending
\[
V \mapsto V(t_1^{-1}, t_2 t_1^{-m_2}, t_3 t_1^{-m_3}, t_4 t_1^{-m_4}).
\]
The edge term admits a square root; set
\[
e^j_\lambda = t_1^{-1} \frac{f^j_\lambda}{1-t_1^{-1}} = \frac{\tilde{f}^j_\lambda}{1-t_1^{-1}},
\]
\[
f^j_\lambda = -Z_\lambda + \widetilde{P}_{kl} Z_\lambda Z_\lambda,
\]
which enjoys
\[
E_\lambda = e^j_\lambda + \overline{e^j_\lambda},
\]
where \( \{ j,k,l \} = \{ 2,3,4 \} \). Setting \( j = 4 \), we recover the explicit square root choice studied in [12].

Lemma 2.15. Let \( \lambda \) be a finite plane partition and \( j = 2,3,4 \). Then \( e^j_\lambda \) is \( T \)-movable.

Proof. Without loss of generality, suppose that \( j = 4 \). Write \( f^4_\lambda = \sum \nu t^\nu \), where the sum is over \( \nu = (\nu_2, \nu_3, \nu_4) \) and we set \( t^\nu = t^\nu_2 t^\nu_3 t^\nu_4 \). We have
\[
(2.12) \quad \frac{1}{1-t_1^{-1}}(t_1^{-1} t^\nu - t^\nu_2 t_3^{-m_2}) = \begin{cases} -t^\nu \sum_{i=0}^{\text{mv}} t_i^i & \text{mv} \leq 0, \\ 0 & \text{mv} = 1, \\ t^\nu_2^{-1} \sum_{i=0}^{\text{mv}-2} t_i^i & \text{mv} \geq 2 \end{cases}
\]
where \( \text{mv} \) denotes the standard scalar product in \( \mathbb{Z}^3 \). Therefore the contribution to the \( T \)-fixed part of each \( t^\nu_2 \) is
\[
(2.13) \quad \text{rk} \left( \frac{1}{1-t_1^{-1}}(t_1^{-1} t^\nu - t^\nu_2 t_3^{-m_2}) \right)^{\text{fix}} = \begin{cases} -\sum_{i=0}^{\text{mv}} \delta_{i,2,2,2,2,2} & \text{mv} \leq 0, \\ 0 & \text{mv} = 1, \\ \sum_{i=1}^{\text{mv}} \delta_{i,2,2,2,2,2} & \text{mv} \geq 2 \end{cases}
\]
\[
(2.14) \quad \begin{cases} -1 & \nu_2 = \nu_3 = \nu_4 \geq 0, \\ 0 & \nu_2 = \nu_3 = \nu_4 \leq -1, \\ 1 & \text{else.} \end{cases}
\]
Denote by $W_l$ the sub-representation of $f^4_\lambda$ corresponding to the irreducible $T$-representation $(t_2 t_3 t_4)^l$, for $l \in \mathbb{Z}$. Equation (2.13) translates into

$$\text{rk} \sum_\nu \left( \frac{1}{1 - t_1^{-1} \nu^* - t_1^{-m_\nu^*}} \right) = \sum_{l \geq 0} \left( \text{rk} W_{r-l-1} - \text{rk} W_l \right).$$

Notice that $f^4_\lambda - \overline{W}(t_2 t_3 t_4)^{-1}$ is the 3-fold vertex of [23, Eqn. (12)] in the variables $t_2, t_3, t_4$, which is $T_0$-movable for $T_0 = \{ t_2 t_3 t_4 = 1 \} \subset (\mathbb{C}^*)^3$ (cf. [23, pag. 1279]). This implies that for any $l \in \mathbb{Z}$

$$\text{rk} W_l = \text{rk} W_{r-l-1},$$

by which we conclude the proof. \hfill \Box

We propose a sign rule for the sign in (2.8), relative to the square root $e^4_\lambda$.

**Conjecture 2.16.** Let $\lambda$ be a finite plane partition. Then the sign relative to the square root $e^4_\lambda$ is $(-1)^{\sigma_i(\lambda)}$, where

$$\sigma_i(\lambda) = f_{\lambda T}(\lambda + |\lambda|m_i + \# \{ (a_2, a_3, a_4) \in \lambda: a_2 = a_k < a_i \},$$

and $\{ i, j, k \} = \{ 2, 3, 4 \}$.

We prove now that the sign rule (2.15) is canonical, meaning that it does not really depend on choosing a preferred $x_i$-axis.

**Theorem 2.17.** Let $\lambda$ be a finite plane partition. For every $i, j = 2, 3, 4$ we have

$$(-1)^{\sigma_i(\lambda)} e^{T}(e^4_\lambda) = (-1)^{\sigma_i(\lambda)} e^{T}(-e^4_\lambda).$$

**Proof.** Without loss of generality assume $i = 4, j = 3$. Say, for $k \in \mathbb{Z},$

$$A(k) = \begin{cases} -\sum_{i=0}^{k-1} t_1^i & k \leq 0, \\ 0 & k = 1, \\ t_1^{-1} \sum_{i=0}^{k-2} t_1^{-i} & k \geq 2 \end{cases}$$

and, for a $T$-representation $V$,

$$B(V) = \sum_{\nu \in V} t_\nu A(m_\nu) \in K_0^T(\text{pt}),$$

where the sum is over the weight spaces of $V$. We extend the definition of $B(V)$ by linearity to $K_0^T(\text{pt})$. By (2.12), we have

$$e^4_\lambda - e^3_\lambda = B(f^4_\lambda - f^3_\lambda),$$

Notice the decomposition

$$f^4_\lambda - f^3_\lambda = W_\lambda + \overline{W}(t_2 t_3 t_4)^{-1},$$

$$W_\lambda = Z_\lambda \overline{Z}_\lambda (t_4^{-1} - t_3^{-1}).$$

Then

$$e^4_\lambda - e^3_\lambda = B(W_\lambda) + B(\overline{W}(t_2 t_3 t_4)^{-1})$$

$$= B(W_\lambda) - B(W_\lambda),$$

by which we conclude that

$$e^{T}(e^4_\lambda - e^3_\lambda) = (-1)^{\text{rk} B(W_\lambda)^{\text{mov}}}. $$

We compute the parity of $\text{rk} B(W_\lambda)^{\text{mov}}$ by induction on the size of $\lambda$. If $|\lambda| = 1$, we clearly have

$$\text{rk} B(W_\lambda)^{\text{mov}} = m_4 + m_3 \mod 2.$$
Suppose now that the claim holds for all plane partition of size $|\lambda| \leq n$ and consider a plane partition $\tilde{\lambda}$ of size $|\tilde{\lambda}| = n + 1$; this can be seen as a plane partition $\lambda$ of size $n$ with an extra box over it, corresponding to a $\mathbb{Z}^3$-lattice point $\mu = (\mu_2, \mu_3, \mu_4)$. We have

$$B(W_\lambda) = B(W_\tilde{\lambda}) + B(Y_3) - B(Y_3) + B(t_4^{-1} - t_3^{-1}),$$

$$Y_i = t_i^{-1}(Z_{\lambda^{-\mu}} + \overline{Z_{\lambda^\mu}}) \quad i = 3, 4,$$

and by the inductive step

$$\text{rk } B(W_\lambda)^{\text{mov}} = \sigma_4(\lambda) - \sigma_3(\lambda) \mod 2,$$

$$\text{rk } B(t_4^{-1} - t_3^{-1})^{\text{mov}} = m_4 - m_3 \mod 2.$$

Clearly, $\text{rk } B(Y_4)^{\text{mov}} = \text{rk } B(Y_4)^{\text{fix}} \mod 2$. In fact,

$$\text{rk } B(Y_4) = \sum_{\nu \in \mathbb{Z}^3} (m(\mu - \nu + (0, 0, -1)) + m(\nu - \mu + (0, 0, -1)))$$

$$= -2m_4|\lambda|.$$

A simple analysis of $B(Y_4)^{\text{fix}}$ as in (2.13) yields

$$\text{rk } B(Y_4)^{\text{fix}} = \# \{ \nu \in \lambda: \mu_2 - \nu_2 = \mu_4 - \nu_3 = \mu_4 - \nu_4 + 1 \}$$

$$- \# \{ \nu \in \lambda: \mu_2 - \nu_2 = \mu_3 - \nu_3 = \mu_4 - \nu_4 - 1 \},$$

where $\nu = (\nu_2, \nu_3, \nu_4)$; in particular, it has to satisfy $\nu \leq \mu$. Therefore we can write it as

$$\text{rk } B(Y_4)^{\text{fix}} = \sum_{i=0}^{\mu_2} \sum_{j=0}^{\mu_3} \sum_{k=0}^{\mu_4} \left( \sum_{i=0}^{\mu_2} \sum_{j=0}^{\mu_3} \sum_{k=0}^{\mu_4} \delta_{\mu_2 - i, \mu_3 - j, \mu_4 - k + 1} - \delta_{\mu_2 - i, \mu_3 - j, \mu_4 - k - 1} \right)$$

$$= \sum_{i=0}^{\mu_2} \sum_{j=0}^{\mu_3} \sum_{k=0}^{\mu_4} \left( \sum_{i=0}^{\mu_2} \sum_{j=0}^{\mu_3} \sum_{k=0}^{\mu_4} \delta_{\mu_2 - i, \mu_3 - j, \mu_4 - k} - \sum_{k=1}^{\mu_4 + 1} \delta_{\mu_2 - i, \mu_3 - j, \mu_4 - k} \right)$$

By symmetry we may compute the difference

$$\text{rk } (B(Y_4) - B(Y_3))^{\text{fix}} = \sum_{i=0}^{\mu_2} \left( \sum_{j=0}^{\mu_3} \sum_{k=0}^{\mu_4 - 1} \delta_{\mu_2 - i, \mu_3 - j, \mu_4 - k} - \sum_{j=0}^{\mu_3} \sum_{k=0}^{\mu_4 - 1} \delta_{\mu_2 - i, \mu_3 - j, \mu_4 - k} \right)$$

$$- \sum_{k=0}^{\mu_4 - 1} \left( \sum_{j=0}^{\mu_3} \sum_{k=1}^{\mu_4 + 1} \delta_{\mu_2 - i, \mu_3 - j, \mu_4 - k} - \sum_{j=0}^{\mu_3} \sum_{k=1}^{\mu_4 + 1} \delta_{\mu_2 - i, \mu_3 - j, \mu_4 - k} \right)$$

Further analyzing which of these sums actually contribute to the rank, we finally get that

$$\text{rk } (B(Y_4) - B(Y_3))^{\text{fix}} = \begin{cases} 1 & \mu_2 = \mu_3 < \mu_4, \\ 1 & \mu_2 = \mu_4 < \mu_3, \\ 0 & \text{else.} \end{cases} \mod 2$$

Therefore we conclude that

$$\text{rk } B(W_\lambda)^{\text{mov}} = \sigma_4(\tilde{\lambda}) - \sigma_3(\tilde{\lambda}) \mod 2,$$

which finishes the inductive step. \(\square\)

**Remark 2.18.** Let $X = K_Y$ be the canonical bundle of a smooth projective toric 3-fold $Y$ and consider $\text{Hilb}^n(X, \beta)$, where $\beta \in H_2(X, \mathbb{Z})$ is a class pulled-back from $Y$. Consider a $\mathbb{T}$-fixed point $Z \in \text{Hilb}^n(X, \beta)^{\mathbb{T}}$ (corresponding to a partition data $\{ \pi_\alpha, \lambda_\alpha \}_{\alpha, \beta}$) scheme-theoretically supported on the zero section of $X \to Y$. Locally on the toric charts, label the fiber direction by $x_4$ and denote by $m_{\alpha, \beta}^n$ the degree of the normal bundle of $L_{\alpha, \beta}$ in the $x_4$-direction. Consider the square root of $T^{\mathbb{T}}_Z$ given by

$$v_Z = \sum_{\alpha \in V(X)} v_\alpha^4 + \sum_{\alpha, \beta \in E(X)} \phi_{\alpha, \beta}^4.$$
By Remark 2.12, the sign rules proposed for vertex and edge terms imply that the correct sign would be
\[
(-1)^{\sigma(Z,x_2)} = \prod_{\alpha \in V(X)} (-1)^{\sigma_i(\pi_\alpha)} \cdot \prod_{\alpha \beta \in E(X)} (-1)^{\sigma_i(\lambda_{\alpha \beta})}
= \prod_{\alpha \in V(X)} (-1)^{|\pi_\alpha|} \cdot \prod_{\alpha \beta \in E(X)} (-1)^{|\lambda_{\alpha \beta}| m_{\alpha \beta}' + f_{m_{\alpha \beta}}(\lambda_{\alpha \beta})}
= (-1)^{n+\epsilon_1(Y) \beta},
\]

where the last equality follows from (2.2) and
\[
\sum_{\alpha \beta \in E(X)} |\lambda_{\alpha \beta}| m_{\alpha \beta}' = - \sum_{\alpha \beta \in E(X)} |\lambda_{\alpha \beta}| \deg N_{Y/X}|_{L_{\alpha \beta}}
= - \sum_{\alpha \beta \in E(X)} |\lambda_{\alpha \beta}| \deg K_Y|_{L_{\alpha \beta}}
= \sum_{\alpha \beta \in E(X)} |\lambda_{\alpha \beta}| c_1(T_Y) \cdot [L_{\alpha \beta}]
= c_1(Y) \cdot \beta.
\]
The same sign was proposed in a similar setting for stable pair invariants [13, Prop. 4.2, Rmk. A.2], where such local geometries are studied, motivated by a choice of preferred orientation as in [7].

3. Refinements

3.1. \textit{K-theoretic invariants.} Let \( X \) be a smooth Calabi-Yau 4-fold and \( M \) a moduli space of compactly supported sheaves on \( X \). Oh-Thomas defined [27, Def. 5.9] a \textit{(twisted) virtual structure sheaf}
\[
\mathcal{O}_M^{\text{vir}} \in K_0(M),
\]
which depends on a chosen orientation and whose properties mimic the virtual structure sheaf with the Nekrasov-Okounkov twist in the classical 3-fold theory [25]. While in the 3-fold theory the twist is introduced to make DT invariants more symmetric, here it is actually necessary to be defined. We define the \( K \)-theoretic version of the invariants (1.2) by means of Oh-Thomas \( K \)-theoretic virtual localization theorem (cf. [27, Thm. 7.3]); such invariants are studied in [24, 26, 12, 4, 3, 5].

**Definition 3.1.** Let \( V \in K_0^T(\text{Hilb}^n(X, \beta)) \). The \( K \)-theoretic \( T \)-equivariant Donaldson-Thomas invariants of \( X \) are
\[
\text{DT}_n^K (X, \beta; V) = \sum_{Z \in \text{Hilb}^n(X, \beta)^T} \sqrt{\epsilon^T} (-T_Z^{\text{vir}}) \cdot V\big|_Z \in \frac{Q(t_1, t_2, t_3, t_4)}{(t_1 t_2 t_3 t_4 - 1)}.
\]
Here \( \epsilon^T \) is the \((T\text{-equivariant}) K\text{-theoretic Euler class}, which is defined as follows. Let \( X \) be a scheme and \( V \) a \( T \)-equivariant locally free sheaf on \( X \). We define
\[
\epsilon^T(V) := A^*V^\vee = \sum_{i \geq 0} (-1)^i A^iV^\vee \in K_0^T(X),
\]
and extend it by linearity to any class \( V \in K_0^T(X) \). Finally, \( \sqrt{\epsilon^T(\cdot)} \) is the \((T\text{-equivariant}) K\text{-theoretic square-root Euler class}; the complete description and construction of this class is in [27, Sec. 5.1]. Let \( V \) be a \( T \)-representation with a square root \( T \) in \( K_0^T(\text{pt}) \). Its \( K \)-theoretic square root Euler class satisfies
\[
\sqrt{\epsilon^T}(V) = \pm \epsilon^T(T) \otimes (\det T)^{\frac{1}{2}} \in K_0^T \left( \text{pt}, \mathbb{Z} \left[ \frac{1}{2} \right] \right).
\]
For an irreducible $T$-representation\(^6\) $t^\mu$, define

$$[t^\mu] = t^\tau - t^{-\tau} \in K_T^0(pt)$$

and extend it by linearity to any $V \in K_T^0(pt)$. It is shown in [18, Sec. 6.1] that

$$\epsilon_T(V) \otimes (\det V)^{\frac{1}{2}} = [V],$$

for any virtual $T$-representation $V \in K_T^0(pt)$. Therefore, given square roots $\nu_\alpha, e_{\alpha\beta}$ of $\nu_\alpha, E_{\alpha\beta}$, we have

$$\text{DT}_n^K(X, \beta; V) = \sum_{Z \in \text{Hilb}^n(X, \beta)} \prod_{\alpha \in V(X)} (-1)^{\sigma(Z, \nu_\alpha)} [-\nu_\alpha] \prod_{\alpha \beta \in E(X)} (-1)^{\sigma(Z, e_{\alpha\beta})} [-e_{\alpha\beta}] \cdot V|_Z.$$  

The operator $[,]$ satisfies $[t^{-\mu}] = -[t^\mu]$, the same multiplicative property of $e_T^\cdot()$; therefore, the results of Theorem 2.8, 2.14 and 2.17 hold the same replacing $e_T^\cdot()$ by $[,]$.

3.2. Elliptic invariants. An elliptic refinement of DT invariants was proposed in [18, Sec. 8.2] and studied in [4]. Set

$$\theta(p\; y) = -ip^{1/8}(y^{1/2} - y^{-1/2}) \prod_{n=1}^{\infty} (1 - p^n)(1 - yp^n)(1 - y^{-1}p^n),$$

$$\eta(p) = p^{\frac{\tau}{2}} \prod_{n \geq 1} (1 - p^n).$$

Set $p = e^{2\pi i \tau}$, with $\tau \in \mathbb{H} = \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \}$. Denoting $\theta(\tau|z) := \theta(e^{2\pi i \tau}; e^{2\pi i z})$, $\theta$ enjoys the modular behaviour

$$\theta(\tau|z + a + b\tau) = (-1)^{a+b} e^{-2\pi ibz} e^{-\pi i b^2 \tau} \theta(\tau|z), \quad a, b \in \mathbb{Z}.$$  

See [18, Sec. 8.1] and [17, Sec. 6] for related discussion on the modularity of these functions. For an irreducible $T$-representation $t^\mu$, define

$$\theta[t^\mu] = (i \cdot \eta(p))^{-1} \theta(p; t^\mu) \in K_T^0(pt)[p^{[p^{\frac{1}{2}}]}]$$

and extend it by linearity to any $V \in K_T^0(pt)$. Let $V \in K_T^0(\text{Hilb}^n(X, \beta))$. The elliptic $T$-equivariant Donaldson-Thomas invariants of $X$ are

$$\text{DT}_{n}^{\text{ell}}(X, \beta; V) = \sum_{Z \in \text{Hilb}^n(X, \beta)} \prod_{\alpha \in V(X)} (-1)^{\sigma(Z, \nu_\alpha)} \theta[-\nu_\alpha] \prod_{\alpha \beta \in E(X)} (-1)^{\sigma(Z, e_{\alpha\beta})} \theta[-e_{\alpha\beta}] \cdot V|_Z.$$  

Again, as $\theta[t^{-\mu}] = -\theta[t^\mu]$, the results of Theorem 2.8, 2.14 and 2.17 hold the same replacing $e_T^\cdot()$ by $\theta[.]$.

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\(^6\)To be precise, we should replace the torus $T$ with its double cover where the character $t^\tau$ is well-defined (cf. [25, Sec. 7.1.2]).
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