Orientations without forbidden patterns on three vertices

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Abstract

Given a set $F$ of oriented graphs, a graph $G$ is a $Forb_e(F)$-graph if it admits an $F$-free orientation. Skrien showed that proper-circular arc graphs, nested interval graphs and comparability graphs, correspond to $Forb_e(F)$-graph classes for some set $F$ of orientations of $P_3$. Building on these results, we exhibit the list of all $Forb_e(F)$-graph classes when $F$ is a set of oriented graphs on three vertices. Structural characterizations for these classes are provided, except for the so-called perfectly-orientable graphs and the transitive-perfectly-orientable graphs, which remain as open problems.

1 Introduction

Given a set $F$ of oriented graphs, Skrien defined the class of $F$-graphs to be the class of graphs that admit an $F$-free orientation [16]. As stated in [9], we believe that this definition might be misleading in the sense that the class of $F$-graphs is negatively defined with respect to $F$. For this reason, we propose to invert this definition. Given a class of oriented graphs $O$, an $O$-graph is a graph that admits an orientation that belongs to $O$. In other words, the class of $O$-graphs is the family of underlying graphs of $O$. We denote this class of graphs by $U_O$. For instance, it is well-known that a graph $G$ is 2-edge-connected if and only if it admits a strongly connected orientation [5]. So, if $O$ is the class of strongly connected oriented graphs, then $U_O$ is the class of 2-edge-connected graphs.

Consider a pair of (oriented) graphs $G$ and $H$. If $G$ is homomorphic to $H$, we will write $G \rightarrow H$, and if $G$ is an induced (oriented) subgraph of $H$, we will write $G < H$. Given a

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set of oriented graphs, $F$, we denote by $Forb(F)$ the class of oriented graphs $G$ such that $H \not\rightarrow G$ for every $H \in F$. An embedding is a homomorphism $\varphi: G \rightarrow H$ such that $G$ is isomorphic to its image, $H[\varphi[V(G)]]$. So, $G$ embeds into $H$ if and only if $G < H$. We extend the previously introduced notation and denote by $Forb_e(F)$ the class of oriented graphs such that $H \not< G$ for every $H \in F$. In particular, the class of $Forb_e(F)$-graphs is the class of graphs that admit an $F$-free orientation. Notice that this class corresponds to the class of $F$-graphs in the sense of Skrien [16]. We will often write $Forb_e(H)$ instead of $Forb_e(\{H\})$.

Following Skrien’s notation, we will use $B_1$, $B_2$, and $B_3$ to denote the orientations of $P_3$, see Figure 1. Also in [16], Skrien proved structural characterizations of $Forb_e(F)$-graphs for every $F \subseteq \{B_1, B_2, B_3\}$, except for $\{B_1\}$ and $\{B_2\}$; notice that $Forb_e(B_1)$- and $Forb_e(B_2)$-graphs are actually the same class, known as perfectly-orientable graphs (p.o. graphs for short). We say that a graph is a transitive-perfectly-orientable graph if it admits a $\{B_1, \overrightarrow{C_3}\}$-free orientation, or equivalently, a $\{B_2, \overrightarrow{C_3}\}$-free orientation. In other words, a transitive-perfectly orientable graph is a graph that admits an orientation where the out-neighbourhood of every vertex is a tournament, and every tournament is transitively oriented. Clearly, this is a subclass of perfectly-orientable graphs.

![Figure 1: All possible orientations of non-empty graphs on three vertices.](image)

Studying the structure of $B_1$-free orientable graphs has caught the interest of several authors. In particular, Hartinger and Milanic, and the same authors with Brešar and Kos, have thoroughly studied this family in a series of papers [6, 11, 12]. They have nice results when the problem is restricted to certain families, e.g., they showed that a cograph is perfectly orientable if and only if it is $K_{2,3}$-free. Nonetheless, characterizing the class of perfectly orientable graphs through forbidden induced subgraphs remains an open problem in the
general case.

From the algorithmic point of view, Urrutia and Gavril found a polynomial time algorithm to recognize perfectly orientable graphs [19]. Furthermore, in [4], the authors show that for any subset $F$ of $\{B_1, B_2, B_3\}$, there is a polynomial time algorithm to determine if a graph admits an $F$-free orientation. They do so by reducing each of these problems to 2-SAT. Recall that in the classic article [1], 2-SAT is solved by proceeding on an auxiliary digraph constructed from the 2-SAT instance. By using these two techniques, we extend the aforementioned result from [4] to any subset of $\{B_1, B_2, B_3, T_3\}$, where $T_3$ is the transitive tournament of order 3. Instead of reducing our problem to 2-SAT, we give an explicit construction of an auxiliary digraph $D^+$. Then, we follow the same procedure used in [1] on $D^+$. Thus, we show a certifying polynomial time algorithm to determine if a graph belongs to $U_{\text{Forb}_e(F)}$, for any set $F \subseteq \{B_1, B_2, B_3, T_3\}$.

In addition to the algorithm mentioned above, in this paper we extend Skrien’s work by proposing characterizations of $\text{Forb}_c(F)$-graphs when $F$ is any set of oriented graphs on three vertices, except for $\{C_3, B_1\}$ and $\{B_1\}$, where $C_3$ denotes the directed 3-cycle. Probably the most interesting case is the family of $\text{Forb}_c(T_3)$-graphs, for which we provide a characterization in terms of forbidden homomorphic images of a family of graphs. The characterization of $U_{\text{Forb}_c(T_3)}$ results surprisingly natural, and the obstructions are obtained by “reverse-engineering” the construction of the constraint digraph $D^+$. These characterizations build up to our main result, which we now state.

**Theorem 1.** The following classes, and their intersections with complete multipartite graphs, are all infinite families of $\text{Forb}_c(F)$-graphs, where $F$ is a set of non-empty oriented graphs on three vertices (d.u.o. stands for “disjoint union of”).

1. Perfectly orientable graphs.
2. Comparability graphs.
3. Odd closed strip hom.-free graphs.
4. D.u.o. proper circular-arc graphs.
5. Trivially perfect graphs.
6. Transitive-perfectly orientable graphs.
7. D.u.o. unicyclic graphs.
8. D.u.o triangle-free unicyclic graphs.
9. 3-colourable comparability graphs.
10. Triangle-free graphs.
11. Clusters.
12. D.u.o. proper Helly circular-arc graphs.
13. D.u.o. triangle-free proper circular-arc graphs.
14. D.u.o. paths and cycles.
15. D.u.o. paths and cycles but no triangles.
16. D.u.o. triangles and stars.
17. Star forests.
18. Stars and empty graphs.
19. Matchings with isolated vertices.
20. Empty graphs and $K_2$.
21. Bipartite graphs.
22. Complete bipartite graphs.
The rest of the paper is organized as follows. In Section 2 we define the constraint digraph of a given graph $G$ and a set $F \subseteq \{B_1, B_2, B_3, T_3\}$. Then, we use this constraint digraph to present an algorithm to recognize $\text{Forb}_e(F)$-graphs, when $F \subseteq \{B_1, B_2, B_3, T_3\}$. In Section 3 we characterize $\text{Forb}_e(F)$-graphs for most of the cases not covered in [16]. Section 4 is devoted to characterize the class $\mathcal{U}_{\text{Forb}_e(T_3)}$, where we also use the construction of the constraint digraph of Section 2. Finally, in Section 5 we prove Theorem 1 and in Section 6 we describe the intersections of complete multipartite graphs with classes listed in Theorem 1. Conclusions and some open problems are presented in Section 7.

2 Constraint Digraph

We refer the reader to [3] and [5] for undefined basic terms. We denote the oriented graphs on three vertices as in Figure 1. Given a set $A$, we define $A \times 1 = A$ and $A \times 0 = \emptyset$. For a statement $P$, we denote by $1_{[P]}$ the truth value of $P$. In other words, $1_{[P]} = 1$ if $P$ is true, and $1_{[P]} = 0$ otherwise.

We say that any set $F \subseteq \{B_1, B_2, B_3, T_3\}$ is a simple set. For a graph $G$ and a simple set $F$. We construct the constraint digraph $D^+$ associated to $G$ and $F$ as follows. The vertex set, $V^+$, of $D^+$ is the set $\{(x, y) : xy \in E_G\}$; notice that for every edge $xy \in E_G$, both $(x, y)$ and $(y, x)$ belong to $V^+$. We define the following sets of arcs:

- $A_1 = \{((y, x), (z, y)) : xy \in E_G, yz \in E_G, zx \notin E_G\}$,
- $A_2 = \{((y, x), (y, z)) : xy \in E_G, yz \in E_G, zx \notin E_G\}$,
- $A_3 = \{((x, y), (z, y)) : xy \in E_G, yz \in E_G, zx \notin E_G\}$ \cup \{((y, x), (y, z)) : xy \in E_G, yz \in E_G, zx \notin E_G\}$, and
- $A_t = \{((x, y), (y, z)) : xy \in E_G, yz \in E_G, zx \in E_G\}$ \cup \{((x, y), (x, z)) : xy \in E_G, yz \in E_G, zx \in E_G\}$.

Finally, we define the arc set, $A^+$, of $D^+$ as

$$A^+ = (A_1 \times 1_{[B_1 \in F]}) \cup (A_2 \times 1_{[B_2 \in F]}) \cup (A_3 \times 1_{[B_3 \in F]}) \cup (A_t \times 1_{[T_3 \in F]}).$$

In this section we will use the constraint digraph for a recognition algorithm of certain families of $\text{Forb}_e(F)$-graphs. We will also use $D^+$ in Section 4 to find a structural characterization of $\mathcal{U}_{\text{Forb}_e(T_3)}$.

We proceed to present the recognition algorithm. Given an input set $F \subseteq \{B_1, B_2, B_3, T_3\}$ and a graph $G$, this algorithm finds an $F$-free orientation of $G$, or outputs that it is not possible to find one. We begin by observing some properties of the constraint digraph $D^+$.
Proposition 2. Let $G$ be a graph and $F \subseteq \{B_1, B_2, B_3, T_3\}$. Then, in $D^+$, $(x, y) \rightarrow (z, w)$ if and only if $(w, z) \rightarrow (y, x)$.

Proof. Proving one implication is enough to prove the whole statement. Observe that $\{(x, y), (z, w)\} \in A^+$ if and only if $\{(x, y), (z, w)\} \in A_i$ for each $i \in \{1, 2, 3, 4\}$. We will prove the statement for the case when $\{(x, y), (z, w)\} \in A_1$, the other cases follow the same line of argumentation. If $\{(x, y), (z, w)\} \in A_1$ then $w = x$, $yx \in E_G$, $xz \in E_G$ and $zy \notin E_G$. Thus $zx \in E_G$, $xy \in E_G$ and $yz \notin E_G$, therefore $\{(x, z), (y, x)\} \in A_1$. Hence, $\{(w, z), (y, x)\} \in A_1$ if and only if $\{(x, y), (z, w)\} \in A_1$.

From here, the following two propositions are easy to obtain.

Proposition 3. Let $G$ be a graph and $F \subseteq \{B_1, B_2, B_3, T_3\}$. There is a directed path from $(x, y)$ to $(z, w)$ in $D^+$ if and only if there is a directed path from $(w, z)$ to $(y, x)$ in $D^+$.

Proof. Proceed by induction over the length of the directed path. Notice that Proposition 2 is the base case. Use again Proposition 2 in the inductive step.

Let $D$ be a digraph and let $\overrightarrow{D}$ be the digraph obtained from $D$ by reversing every arc. A digraph $D$ is skew-symmetric if it is isomorphic to $\overrightarrow{D}$.

Proposition 4. Let $G$ be a graph and $F \subseteq \{B_1, B_2, B_3, T_3\}$. The constraint digraph of $G$ and $F$ is skew-symmetric.

Proof. Let $D$ be a digraph. Let $D^+$ be the constraint digraph of $G$ and $F$. Consider the function $\varphi : V^+ \rightarrow V^+$ defined by $\varphi((x, y)) = (y, x)$. By Proposition 2 we conclude that $\varphi$ is a digraph isomorphism between $D^+$ and $\overrightarrow{D}$.

By the isomorphism shown in the previous proof, every strong component $S$ in $D^+$ has a dual component, $\overline{S}$ (which might be equal to $S$), induced by the vertices of the form $(y, x)$ where $(x, y) \in S$. By Proposition 3 a strong component $S_1$ reaches another one $S_2$, if and only if $\overline{S_2}$ reaches $\overline{S_1}$. A well-known algorithm of Tarjan [13] generates the strong components of a digraph in reverse topological order (i.e. if $S_1$ reaches $S_2$ then $S_2$ is generated before $S_1$).

Let us go back to the construction of the constraint digraph. Suppose that we want to find an $F$-free orientation of $G$. An arc $\{(x, y), (z, w)\}$ in $D^+$ tells us that, in order to achieve such an orientation, if we orient the edge $xy$ from $x$ to $y$, then we must orient the edge $zw$ from $z$ to $w$. Inductively, if there is a path from $(x, y)$ to $(z, w)$ and we orient the edge $xy$ from $x$ to $y$ then we must orient the edge $zw$ from $z$ to $w$. Thus, if $(x, y)$ and $(y, x)$ belong to the same strong component, $G$ does not admit an $F$-free orientation. In fact the reverse implication is also true. To see this, we consider the famous 2-SAT Algorithm due to Tarjan [11], and use it on the constraint digraph $D^+$ associated to a graph $G$ and a set $\{B_1, B_2, B_3, T_3\}$. This procedure is described in Algorithm 1.

Clearly, Algorithm 1 stops by determining that there exists a strong component $S$ such that $S = \overline{S}$ only if there is a vertex $(x, y) \in V^+$ in the same strong component as $(y, x)$. Otherwise a $\{\text{true}, \text{false}\}$-colouring of $D^+$ is obtained, which induces an $F$-free orientation of $G$. We prove this fact in the following proposition.
Algorithm 1: Recognition of $F$-free orientable graphs

**Input:** A graph $G$ and a set $F \subseteq \{B_1, B_2, B_3, T T_3\}$.

**Output:** A $\{\text{true}, \text{false}\}$-colouring of the vertices in $D^+$, or a strong component $S$ such that $S = \overline{S}$.

1. Construct the constraint digraph $D^+$ associated to $G$ and $F$;
2. Generate the strong components of $D^+$ in reverse topological order;
3. for each strong component $S$ of $D^+$ do
   4. if $S = \overline{S}$ then
      5. return $S$;
   else if vertices in $S$ are not marked then
      7. mark each vertex in $S$ true and each vertex in $\overline{S}$ false;
8. return the $\{\text{true}, \text{false}\}$-colouring of the vertices of $D^+$

Proposition 5. Let $G$ be a graph and $F \subseteq \{B_1, B_2, B_3, T T_3\}$. If Algorithm 1 outputs a $\{\text{true}, \text{false}\}$-colouring of the vertices in $D^+$, then vertices with colour true induce an $F$-free orientation of $G$.

**Proof.** Clearly, if $(x, y)$ is marked with true, then $(y, x)$ is marked with false. Also, every vertex receives one and only one truth colour. Hence the true-coloured vertices of $D^+$ induce an orientation of $G$; that is, if $(x, y)$ is marked true, then $xy$ is oriented as $(x, y)$. We now prove that this orientation is an $F$-free orientation of $G$. To do so, we must prove that for any two oriented edges $(x, y), (w, z) \in V^+$ that induce an oriented graph in $F$, then at least one is marked with false. By construction of $A^+$, it must happen that if $(x, y)$ and $(w, z)$ induce an oriented graph in $F$ then $(x, y) \rightarrow (z, w)$ and $(w, z) \rightarrow (y, x)$. Hence, we show that if $(x, y)$ is marked with true and $(x, y) \rightarrow (z, w)$, then $(z, w)$ is also marked with true. Since the algorithm marks all the vertices in the same strong component at once, it suffices to show that for any two strong components $S_1$ and $S_2$ of $D^+$, if $S_1$ is true-coloured and $S_1$ reaches $S_2$, then $S_2$ is also true-coloured. Suppose that $S_1$ is marked with true and it reaches $S_2$, but $S_2$ is false-coloured. Since $S_1$ reaches $S_2$, then $S_2 < S_1$, where $<$ is the reverse topological order of the strong components of $D^+$. Since $S_2$ is marked with false it means that $S_2$ was processed before $S_2$ (i.e. $S_2 < S_2$). Analogously, we see that $S_1 < S_1$. The transitivity of $<$ implies that $S_2 < S_1$. Since $S_1$ reaches $S_2$, by Proposition 3 $S_2$ reaches $S_1$, then $S_1 < S_2$. The previous inequalities yield the following chain, $S_1 < S_2 < S_2 < S_1$, from which we conclude that $S_1 = S_2$; equivalently $S_1 = S_2$. This contradicts that the algorithm does not assign two different truth values to the same component. Therefore if $S_1$ reaches $S_2$ and $S_1$ is marked with true, $S_2$ is marked with true as well.

These results build up to the following one.

Theorem 6. Let $G$ be a graph and $F \subseteq \{B_1, B_2, B_3, T T_3\}$. The following are equivalent:

- $G$ admits an $F$-free orientation.
• There are no vertices \((x, y), (y, x) \in V^+\) contained in the same strong connected component of \(D^+\).

• For any strong component \(S\), \(S \cap \overline{S} = \emptyset\) (i.e. \(S \neq \overline{S}\)).

**Proof.** The equivalence between the second and third item is trivial. On the paragraph preceding Algorithm 1 it was shown that the second statement implies the first one. The remaining implication is proved by Algorithm 1 and Proposition 5. \(\square\)

The order of \(D^+\) is \(2m\), where \(m\) is the number of edges of \(G\). Also note that \(d_{D^+}((x, y)) \leq d_G(x) + d_G(y)\) so, \(|A^+| \leq m\Delta(G) \leq mn\). Since the general step of Algorithm 1 runs in \(O(|V^+|)\) time and Tarjan’s Algorithm \([18]\) runs in \(O(|V^+| + |A^+|)\) time, our algorithm runs in \(O(mn)\) once \(D^+\) is constructed — to construct \(D^+\) we must process all sets of 3 vertices, which takes cubic time in \(|V|\). These arguments together with Theorem 6 show the following statement holds.

**Corollary 7.** Given an input graph \(G\) and an input set \(F \subseteq \{B_1, B_2, B_3, T_3\}\), it is in \(P\) to test if \(G\) admits an \(F\)-free orientation. In particular, Algorithm 1 decides if \(G\) admits an \(F\)-free orientation in cubic time.

## 3 Graph properties and small forbidden orientations

In this section, we study families of \(Forb_e(F)\)-graphs when \(F\) consists of oriented graphs on three vertices. In \([16]\), Skrien studied the cases when \(F\) is a set of orientations of \(P_3\). For this reason, we study \(Forb_e(F)\)-graphs when either \(T_1 + T_2 \in F\) or \(F\) contains at least one orientation of \(C_3\). We begin by observing the following simple lemma.

**Lemma 8.** Let \(F\) be a set of oriented graphs and consider any graph \(H\). If \(F_H\) is the set of all orientations of \(H\), then the class of \(Forb_e(F \cup F_H)\)-graphs is the intersection of \(U_{Forb_e(F-F_H)}\) and \(H\)-free graphs.

**Proof.** If \(G\) is an \(H\)-free graph, and \(G'\) is an \((F - F_H)\)-free orientation of \(G\), then \(G'\) is an \((F \cup F_H)\)-free orientation of \(G\). On the other hand, if \(G \in U_{Forb_e(F \cup F_H)}\) and \(G'\) is an \((F \cup F_H)\)-free orientation of \(G\), then \(G'\) is \((F - F_H)\)-free and \(F_H\)-free. So, \(G \in U_{Forb_e(F - F_H)}\) and \(G\) is \(H\)-free. \(\square\)

In particular, since \(T_1 + T_2\) is the unique orientation of \(K_1 + K_2\), then the class of \(Forb_e(F \cup \{T_1 + T_2\})\)-graphs is the intersection of \(U_{Forb_e(F - \{T_1 + T_2\})}\) and complete multipartite graphs. Similarly, the class of \(Forb_e(F \cup \{3\overrightarrow{3}, T_3\})\)-graphs is the intersection of \(U_{Forb_e(F - \{3\overrightarrow{3}, T_3\})}\) and triangle-free graphs.

**Proposition 9.** Let \(F\) be a set of oriented graphs on 3 vertices. If \(F \subseteq \{B_1, B_2, B_3, T_3, T_1 + T_2\}\) or \(\{3\overrightarrow{3}, T_3\} \subseteq F\), then it is in \(P\) to test if an input graph admits an \(F\)-free orientation.

**Proof.** This statement follows from Corollary 7 and Lemma 8. \(\square\)
It is direct to verify that if the set of forbidden oriented graphs consists of connected graphs, then the associated hereditary property is closed under disjoint unions. Thus, it suffices to study connected graphs.

Forbidden oriented graphs | Graph family
-------------------------|------------------
$B_1, B_2, B_3$         | Complete graphs. |
$B_1, B_2$              | Proper circular-arc graphs. |
$B_1, B_3$              | Nested interval graphs. |
$B_2, B_3$              | Nested interval graphs. |
$B_1$                   | Open |
$B_2$                   | Open |
$B_3$                   | Comparability graphs. |

Table 1: On the left we see a set of forbidden oriented graphs, and on the right, the family it characterizes. This table is taken from [16].

Skrien’s results from [16] are included in Table 3. Recall that he found an alternative characterization for all sets containing orientations of $P_3$, except for perfectly orientable graphs. Bang-Jensen, Huang and Prisner also studied p.o. graphs, in particular, they proved the following result in [4].

**Proposition 10.** [4] Every graph with exactly one induced cycle of length greater than 3 is perfectly orientable.

This result can be equivalently restated as follows: Every triangle-free graph is perfectly orientable if it has only one induced cycle. With a simpler proof than the one found in [4], we prove the biconditional version of this result, which is a corollary to the following proposition.

**Proposition 11.** The following statements are equivalent for a connected graph $G$:

1. $G$ admits a $\{B_1, T_3\}$-free orientation,
2. $G$ admits an orientation such that $d^+(x) \leq 1$ for every vertex $x \in V_G$,
3. there is function $f : V_G \to V_G$ such that $E_G = \{xy : x \neq y, f(x) = y\}$,
4. $G$ is unicyclic,
5. $G$ has no more edges than vertices.

**Proof.** It is not hard to notice that the first two items are equivalent, and so are the second and third one. It is also straightforward to show that if $G$ has no more edges than vertices, then $G$ is unicyclic (recall that $G$ is connected), so 4 is an implication of 5. Now we prove that the second item implies the fifth one. Let $D_G$ be an orientation of $G$ such that $d^+(x) \leq 1$
for every vertex $x$ of $G$. Consider the function $i: A_{D_G} \to V_G$ where $i((x, y)) = x$. Since $d^+(x) \leq 1$, $i$ is an injective function. Thus $|E_G| = |A_{D_G}| \leq |V_G|$. To conclude the proof we show that if $G$ is unicyclic, it admits an $\{B_1, T_3\}$-free orientation. If $G$ is a tree, root $G$ in any vertex and orient the edges from descendent to ancestor. If $G$ is a cycle, orient $G$ in a cyclic way. In any other case, let $C$ by the only cycle in $G$. Orient $C$ in a cyclic way. Notice that $G/C$ is a tree. Root $G/C$ in the vertex corresponding to $C$. Orient the edges in $G/C$ from descendent to ancestor. We have oriented all edges in $G$ now, and it it not hard to notice that this orientation is $\{B_1, T_3\}$-free.

**Corollary 12.** A graph $G$ admits a $\{B_1, \overrightarrow{C_3}, T_3\}$-free orientation if and only if $G$ is unicyclic and triangle free.

**Proof.** Suppose $G$ admits a $\{B_1, \overrightarrow{C_3}, T_3\}$-free orientation. Clearly, $G$ is triangle-free and by Proposition 11, $G$ is also a unicyclic graph. On the other hand, consider a triangle-free unicyclic graph $G$. By Proposition 11, it admits a $\{B_1, T_3\}$-free orientation $D_G$. Since $G$ is triangle-free, $D_G$ is $\{B_1, \overrightarrow{C_3}, T_3\}$-free.

Another subclass of perfectly orientable graphs is the class of graphs that admit a $\{B_1, T_1 + T_2\}$-free orientation.

**Proposition 13.** The following statements are equivalent for a graph $G$:

1. $G$ admits a $\{B_1, T_1 + T_2\}$-free orientation.
2. $G$ admits a $\{B_2, T_1 + T_2\}$-free orientation.
3. $G$ is a $K_{2,3}$-free complete multipartite graph.

**Proof.** The equivalence between the first two items is obvious. By Lemma 8 a graph $G$ admits a $\{B_1, T_1 + T_2\}$-free orientation if and only if it is a perfectly-orientable complete multipartite graph. In [11], the authors showed that a cograph is perfectly orientable graph if and only if it is $K_{2,3}$-free. The claim now follows since complete multipartite graphs are cographs.

By further restricting the family described in Proposition 13, we immediately obtain the following simple corollary.

**Corollary 14.** The following statements are equivalent for a graph $G$:

1. $G$ admits a $\{B_1, T_1 + T_2, T_3, \overrightarrow{C_3}\}$-free orientation.
2. $G$ admits a $\{B_2, T_1 + T_2, T_3, \overrightarrow{C_3}\}$-free orientation.
3. $G$ is a $K_{2,3}$-free complete bipartite graph.
4. $G$ is a star or $G = C_4$. 

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When $F = \{B_3, T_3, C_3\}$, the class $\mathcal{U}_{\text{Forb}(F)}$ has a well-known characterization, which is a particular case of the Gallai-Hasse-Roy-Vitaver Theorem.

**Proposition 15.** A graph is bipartite if and only if it admits an $\{B_3, T_3, C_3\}$-free orientation.

In [16], Skrien shows that a graph is a proper circular arc graph if and only if it admits a $\{B_1, B_2\}$-free orientation. A proper circular-arc graph is a graph that admits an intersection model where no arc is contained in another. A family of sets $\mathcal{A}$ is said to have the Helly property, if for any subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that for any two sets $A, B \in \mathcal{B}$, $A \cap B \neq \emptyset$, then the intersection of all sets in $\mathcal{B}$ is non-empty. A (proper) Helly circular-arc graph is a graph that admits an intersection model that satisfies the Helly property (and no arc is contained in another). We extend Skrien’s result to proper Helly circular-arc graphs.

**Proposition 16.** A graph $G$ admits a $\{B_1, B_2, C_3\}$-free orientation if and only if $G$ is a proper Helly circular-arc graph.

**Proof.** Let $G$ be a graph that admits a $\{B_1, B_2, C_3\}$-free orientation. By line two of Table 3, we know that $G$ must be a proper circular-arc graph. Corollary 5 in [13], shows that a proper circular-arc graph is a proper Helly circular-arc graph if it contains neither the Hajós graph nor a 4-wheel as an induced subgraph. It is not hard to notice that neither of those graphs admit a $\{B_1, B_2, C_3\}$-free orientation. Thus, since $G$ is a proper circular-arc graph, $G$ must be a proper Helly circular-arc graph.

In [14], it is proved that a model of a proper circular-arc graph is the model of a proper Helly circular-arc graph if and only if no two nor three arcs cover its circle. Consider a proper Helly circular-arc graph $G$. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a model of $G$ where no three arcs cover the circle. Moreover, we can assume that no end points of the arcs in $\mathcal{A}$ coincide. Let us denote by $l_i$ the anti-clockwise end point of $A_i$, and by $r_i$ the clockwise end point. We denote by $D_G$ the following orientation of $G$. Consider an edge $A_iA_j \in E_G$. By moving in a clockwise motion around the circle, we see the endpoints of $A_i$ and $A_j$ form the sequence $[l_i, l_j, r_i, r_j]$ or $[l_j, l_i, r_j, r_i]$. We orient $A_iA_j$ form $A_i$ to $A_j$ when we see $[l_i, l_j, r_i, r_j]$, in the other case we orient it from $A_j$ to $A_i$. Bearing in mind that there are no three arcs that cover the circle, it is easy to see $D_G$ is $\{B_1, B_2, C_3\}$-free.

Interval graphs are particular instances of circular-arc graphs. In [16], Skrien showed that nested interval graphs correspond to $\text{Forb}_e(\{B_1, B_3\})$-graphs. A nested interval graph is a graph $G$ that admits an intersection model that consists of nested intervals of the real line. Equivalently, a graph $G$ is a nested interval graph if and only if it is $\{P_4, C_4\}$-free [8]. These graphs are also called trivially-perfect graphs.

**Proposition 17.** For a graph $G$ the following statements are equivalent:

1. $G$ admits a $\{B_1, B_3\}$-free orientation.
2. $G$ admits a $\{B_1, B_3, C_3\}$-free orientation.
3. $G$ is a nested interval graph.

4. $G$ is a trivially perfect graph.

Proof. The equivalent between the first and third statement is proved in [16]. The equivalence between the last two items is argued above this proposition. Clearly, the second statement is a particular case of the first one. To conclude the proof we show that the third statement implies the second one. This is immediate considering the orientation defined by the inclusion of the nested intervals.

Since every graph admits an acyclic orientation, every graph admits a $\overrightarrow{C_3}$-free orientation. On the contrary, not every graph admits a $T_3$-free orientation. Recall that a graph is locally bipartite if the open neighbourhood of every vertex induces a bipartite graph.

**Proposition 18.** For any graph $G$ the following statements hold:

- if $G$ is 3-colourable, then it admits a $T_3$-free orientation,
- if $G$ admits a $T_3$-free orientation, then it is $K_4$-free,
- if $G$ admits a $T_3$-free orientation, then it is locally bipartite.

Proof. Let $G$ be graph with a proper colouring $(V_0, V_1, V_2)$. By orienting the edges of $G$ from $V_i$ to $V_{i+1}$, with subindices taken modulo 3, we obtain a $T_3$-free orientation of $G$. In order to prove the second item, it suffices to notice that $K_4$ does not admit a $T_3$-free orientation. Let $D_G$ be a $T_3$-free orientation of a graph $G$. For any vertex $x \in V_G$, the sets $N^+_D_G(x)$ and $N^-_{D_G}(x)$ are a partition of $N_G(x)$. Since $D_G$ is $T_3$-free, $N^+_D_G(x)$ and $N^-_{D_G}(x)$ are independent sets.

As we will see later, the statements in the previous proposition are far from being necessary and sufficient conditions for a graph $G$ to admit a $T_3$-free orientation. For the moment, recall the well known result of Mycielski stating that the chromatic number on triangle-free graphs is unbounded [15]. Thus, there are graphs with arbitrary large chromatic number that admit a $T_3$-free orientation. Nonetheless, for perfect graph, the first condition of the previous proposition actually characterizes graphs admitting a $T_3$-free orientation.

**Proposition 19.** A perfect graph $G$ admits a $T_3$-free orientation if and only if it is 3-colourable.

Proof. Consider a perfect graph $G$. By Proposition [18] if $G$ is 3-colourable it admits a $T_3$-free orientation. On the other hand, suppose that $G$ admits a $T_3$-free orientation. By Proposition [18] $G$ is $K_4$-free. Since $G$ is perfect, $G$ is 3-colourable.

**Corollary 20.** A graph admits a $\{T_1 + T_2, T_3\}$-free orientation if and only if it is a complete 3-partite graph.
Proof. By part 1 of Lemma 8, the class $\mathcal{U}_{\text{Forb}(T_1+T_2,T_3)}$ is the intersection of $\mathcal{U}_{\text{Forb}(T_3)}$ and complete multipartite graphs. Since complete multipartite graphs are perfect graphs, the claim follows by Proposition 19.

Since comparability graphs are perfect graphs, the following propositions stem from Proposition 19.

**Proposition 21.** A graph admits a $\{B_3,T_3\}$-free orientation if and only if it is a 3-colourable comparability graph.

Proof. If a graph $G$ admits a $\{B_3,T_3\}$-free orientation, then it is a comparability graph. Thus, $G$ is a perfect graph that admits a $T_3$-free orientation. By Proposition 19, $G$ is a 3-colourable comparability graph. Now suppose that $G$ is a 3-colourable comparability graph. Since $G$ is perfect, it is $K_4$-free. Consider the partial order of the vertices, $<$, induced by the edges of $G$. Let $X_1 = \{x \in V_G: x \text{ is } <\text{-minimal}\}$, $X_3 = \{x \in V_G: x \text{ is } <\text{-maximal}\}$ and $X_2 = V_G - (X_1 \cup X_3)$. It follows from the construction of $X_i$, $i \in \{1,2,3\}$, and the fact that $G$ is $K_4$-free, that the sets $X_i$ is an independent set for $i \in \{1,2,3\}$. Orient the edges from $X_1$ to $X_2$, from $X_2$ to $X_3$ and from $X_3$ to $X_1$; name this orientation $D_G$. Clearly, $D_G$ is $T_3$-free. In order to show that $D_G$ is also $B_3$-free, consider three vertices $x, y, z \in V_G$, that induce a path on $G$. Since $\{x,y,z\}$ does not induce a triangle, it may not happen that $x < y < z$. Thus $x < y$ and $z < y$, or $y < x$ and $y < z$. Then $\{x,y,z\}$ induces either a $B_1$ or $B_2$ in $D_G$. Concluding that $D_G$ is a $\{B_3,T_3\}$-free orientation of $G$.

Before proceeding to study the non perfect graphs that admit a $T_3$-free orientation, allow us to study three very simple subclasses.

**Proposition 22.** A graph $G$ admits a $\{B_1,B_2,T_3\}$-free orientation if and only if $\Delta(G) \leq 2$. Equivalently, $G$ admits a $\{B_1,B_2,T_3\}$-free orientation if and only if $G$ is a disjoint union of paths and cycles.

Proof. Recall that $\Delta(G) \leq 2$ if and only if $G$ is a disjoint union of paths and cycles. Suppose that there is a vertex $x \in V_G$ with at least three distinct neighbours, $y, z, w$. Let $D_G$ be an orientation of $G$. Without loss of generality, $y$ and $z$ will be in-neighbours of $x$ in $D_G$. If $yz \in E_G$ then $\{x,y,z\}$ will induce a $T_3$ in $D_G$. On the other hand, if $yz \not\in E_G$, $\{x,y,z\}$ will induce a $B_1$ in $D_G$. Thus if $\Delta(G) \geq 3$, $G$ does not admit a $\{B_1,B_2,T_3\}$-free orientation. To conclude the proof, consider a disjoint union of paths and cycles $G$. By orienting every cycle and path of $G$ in a directed way, we obtain a $\{B_1,B_2,T_3\}$-free orientation of $G$.

**Proposition 23.** A connected graph $G$ admits a $\{B_1,B_3,T_3\}$-free orientation if and only if $G$ is a star or a triangle.

Proof. It is trivial to find a $\{B_1,B_3,T_3\}$-free orientation of a star or a triangle. Recall that a connected graph $G$ is a star if and only if $G$ is $\{P_1,C_4,C_3\}$-free. Notice that neither $P_1$ nor $C_4$ admit a $\{B_1,B_3\}$-free orientation. Thus if $G$ does not contain a triangle and admits a $\{B_1,B_3,T_3\}$-free orientation, $G$ is a star. On the contrary, if $G$ contains a triangle, observe that neither of the three connected supergraphs of $C_3$ on four vertices, admit a $\{B_1,B_3,T_3\}$-free orientation. Thus, if $G$ contains a triangle $C$, then $G = C$.
Corollary 24. A graph $G$ admits a $\{B_1, B_3, T_3, \overrightarrow{C}_3\}$-orientation if and only if it is a star forest. Equivalently, $G$ admits a $\{B_2, B_3, T_3, \overrightarrow{C}_3\}$-orientation if and only if it is a star forest.

4 $\text{Forb}_e(T_3)$-graphs

The following results build up to characterize the family of graphs that admit a $\{T_3\}$-free orientation.

Proposition 25. Consider a set of tournaments $F$ and a $\text{Forb}_e(F)$-graph $H$. If a graph $G$ admits a homomorphism $\phi: G \to H$, then $G$ admits an $F$-free orientation.

Proof. Consider an $F$-free orientation $D_H$ of $H$. We obtain an orientation $D_G$ of $G$ in the following way, there is an arc $(x, y)$ in $D_G$ if and only if $(\phi(x), \phi(y))$ is an arc in $D_H$. Since $\phi$ is a graph homomorphism, by the way we chose to orient the edges of $G$, $\phi$ induces a digraph homomorphism $\phi_D: D_G \to D_H$. Thus, every tournament $T$ in $D_G$, can be embedded in $D_H$. Since $F$ consists of tournaments and $D_H$ is an $F$-free orientation of $H$, $D_G$ is also an $F$-free orientation of $G$.

Recall that, if a graph $G$ admits a homomorphism to another graph $H$, we write $G \to H$, and $G \not\to H$ otherwise. If $\mathcal{F}$ is a set of graphs, we write $\mathcal{F} \not\to H$, if $G \not\to H$ for every graph $G \in \mathcal{F}$.

Corollary 26. For every set of tournaments $F$, there is a set of graphs $\mathcal{F}$ such that for any graph $G$, $G$ admits a $T_3$-free orientation if and only if $\mathcal{F} \not\to G$.

Proof. By Proposition 25 an example of such a set, is the set of graphs that do not admit an $F$-free orientation.

This corollary motivates the characterization we propose of $\text{Forb}_e(T_3)$-graphs: We describe a set of graphs $\mathcal{F}$ such that a graph $G$ admits a $T_3$-free orientation if and only if $\mathcal{F} \not\to G$.

We begin by introducing some definitions. Consider two parallel paths on the plane $P$ and $Q$. A strip is a graph $G$ obtained from $P$ and $Q$ as follows. First, add one edge joining the initial vertices of $P$ and $Q$, and one joining the end vertices of $P$ and $Q$. Then, triangulate the region between $P$ and $Q$ in such a way that every new edge is incident with one vertex in $P$ and one in $Q$. We call $P$ and $Q$ the bounding paths of $G$. In the top of Figure 4, we illustrate an example of a strip.

A closed strip is obtained from a strip $G$ with bounding paths $P$ and $Q$ by identifying the first and final vertices of $P$, and the first and final vertices of $Q$. Similarly, a Möbius strip is obtained by identify the first vertex of $P$ with the final vertex of $Q$, and the first vertex of $Q$ with the final vertex of $P$. We will abuse nomenclature and call $P$ and $Q$ the bounding paths of the closed (resp. Möbius) strip — notice that the quotients of $P$ and $Q$ are cycles in the corresponding closed strip. In Figure 4, we depict an example of a closed strip and of a Möbius strip.
Figure 2: On the top, an example of a strip $S$ with bounding paths $P$ and $Q$. On the bottom, an example of a closed strip (left) and an example of a Möbius strip (right) obtained from $S$.

Allow us to discuss some particular cases of strips. Suppose that one of the bounding paths of a strip $S$ is trivial, i.e., it is a path on one vertex. In this case, the closed strip obtained from $S$ is a wheel. An even strip is a strip with an even number of triangles; otherwise we say it is an odd strip. Similarly, an even closed (resp. Möbius) strip is a closed (resp. Möbius) strip obtained from an even strip; otherwise we say it is an odd closed (resp. Möbius) strip. It is not hard to notice that the number of triangles in a strip $S$ equals the number of $PQ$-edges minus 1 (where $P$ and $Q$ are the bounding paths of $S$). Thus, a closed (resp. Möbius) strip with bounding paths $P$ and $Q$ is even if and only if there an even number of $PQ$-edges.

Lemma 27. Let $G$ be a graph. If there is a homomorphism $S \to G$ where $S$ is an odd closed strip or an even Möbius strip, then $G$ does not admit a $T_3$-free orientation.

Proof. By Proposition 25 it suffices to show that neither odd closed strips nor even Möbius strips admit a $T_3$-free orientation. Consider a closed strip $S$ with bounding paths $P$ and $Q$, and let $e_0, \ldots, e_{n-1}, e_0$ be the $PQ$-edges indexed according to a clockwise ordering. Suppose that $S$ admits a $T_3$-free orientation $S'$. Since all triangles in $S$ must be oriented cyclically in $S'$, if $e_i$ is oriented from $P$ to $Q$, then $e_{i+1}$ must be oriented from $Q$ to $P$ (indices taken modulo $n$). Inductively, $e_0$ forces an orientation of $e_{n-1}$, which by the previous argument, must be opposite to the orientation of $e_0$. These restrictions are compatible if and only if $S$ is an even closed strip. Thus, odd closed strips do not admit a $T_3$-free orientation. With similar arguments we see that a Möbius strip is $T_3$-free orientable if and only if it is an odd Möbius strip. The claim follows.

Our characterization of $T_3$-free orientable graphs asserts that the converse implication of
Lemma 27 holds. To do so, it will be convenient to describe homomorphisms from closed strips to a graph $G$ by means of certain sequences of edges in $G$.

Notice that a strip $S$ with bounding paths $P$ and $Q$, can be described by its sequence of $PQ$-edges (ordered from left to right), and by indicating for each of these edges which end vertex belongs to $P$ and which to $Q$. To be precise, we represent a strip $S$ as a sequence of edges $p_1q_1, \ldots, p_nq_n$ with the following properties: The sets $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_n\}$ induce two disjoint paths; the intersection of $\{p_i, q_i\}$ and $\{p_{i+1}, q_{i+1}\}$ is $\{p_i\}$ or $\{q_i\}$; and either $p_i = p_{i+1}$ (so $p_i$ is considered to be a pivot) or $p_ip_{i+1}$ is an edge. For instance, the sequence $ax, ay, by, cy, cz$ represents the following strip.

This representation of strips by means of edge sequences, provides a simple way of describing homomorphisms from strips to graphs. Consider a graph $G$ and let $p_1q_1, \ldots, p_nq_n$ be a sequence of edges of $G$. This sequence defines a homomorphism from a strip $S$ to $G$ if for every $i \in \{1, \ldots, n-1\}$ the following statements hold:

1. Either $p_i = p_{i+1}$ and $q_iq_{i+1} \in E(G)$, or $q_i = q_{i+1}$ and $p_ip_{i+1} \in E(G)$.

2. If $p_n = p_1$ and $q_n = q_1$, then the sequence defines a homomorphism from a closed strip to $G$.

3. If $p_n = q_1$ and $q_n = p_1$, then the sequence defines a homomorphism from a Möbius strip to $G$.

Moreover, the parity of the corresponding strip is the same as the parity of $n-1$ (the length of the sequence).

Recall that $D^+ = (V^+, A^+)$ denotes the constraint digraph defined in Section 2. By definition of $A^+$, it follows that if $F = \{T_3\}$, then every arc in $A^+$ is symmetric. Thus, we may think of $D^+$ as a graph, and so, for any graph $G$ we denote by $G^+$ the constraint graph of $G$ and $\{T_3\}$. Now, we observe that paths in $G^+$ translate to homomorphisms of strips to $G$. To do so, we will use the previous description of homomorphism from strips.

**Lemma 28.** Consider a graph $G$ and a pair $xy$ and $zw$ of edges in $G$. If there is an $(x,y)(z,w)$-path of even (resp. odd) length in $G^+$, then there is a sequence $p_1q_1, \ldots, p_nq_n$ of edges of $G$ that defines a homomorphism from some odd (resp. even) strip to $G$. Furthermore, we can choose $p_1 = x$, $q_1 = y$, and $p_n = z$ and $q_n = w$ (resp. $p_n = w$ and $q_n = z$).

**Proof.** We proceed by induction over the length of the $(x,y)(z,w)$-path. Regarding the furthermore statement, in this paragraph we show that $\{p_1, q_1\} = \{x, y\}$ and $\{z, w\} = \{p_n, q_n\}$ — we take care of the vertex equalities in the second paragraph. The base case is
when the \((x, y)(z, w)\)-path is an edge. Thus, by definition of \(E^+\), the set vertices \(\{x, y, z, w\}\) induces a triangle and the claim is obvious. Consider now an \((x, y)(z, w)\)-path \(W\) of length at least two, and let \((a, b)\) be the vertex before \((z, w)\) in \(W\). So, there is an odd (resp. even) sequence \(S = p_1q_1, \ldots, p_{n-1}q_{n-1}\) of edges of \(G\), such that \(\{p_1, q_1\} = \{x, y\}\) and \(\{p_{n-1}, q_{n-1}\} = \{a, b\}\). Since there is an edge \((a, b)(z, w)\) in \(G^+\), then \(\{a, b, z, w\}\) induces a triangle in \(G\), so \(\{a, b\} \cap \{z, w\} = \{v\}\). We extend the previous sequence \(S\) to an even (resp. odd) sequence \(S' = p_1q_1, \ldots, p_{n-1}q_{n-1}, p_nq_n\) where \(p_n\) and \(q_n\) are defined depending on the value of \(v\). Let \(u\) be the unique vertex in \(\{z, w\} \setminus \{v\}\). If \(v = p_{n-1}\) then \(p_n = v\) and \(q_n = u\); otherwise \(p_n = u\) and \(q_n = v\). The fact that \(S'\) defines a homomorphism from an even (resp. odd) strip follows from the choice of \(p_n\) and \(q_n\), and from the fact that \(\{a, b, z, w\}\) induces a triangle (and from the induction hypothesis).

To prove the furthermore statement, first notice that we can choose, without loss of generality, \(p_1 = x\) and \(q_1 = y\). Given this choice, we follow a similar inductive argument as above to verify that \(p_n = z\) and \(q_n = w\) (resp. \(p_n = w\) and \(q_n = z\)). Let \(W\) be the \((x, y)(z, w)\)-path in \(G^+\), and \(S\) the constructed sequence of edges of \(G\). By construction of \(S\), for each edge \(p_iq_i\) in \(S\), there is a vertex \(v_i\) in \(W\) representing one orientation of \(p_iq_i\), i.e., \(v_i\) equals \((p_i, q_i)\) or equals \((q_i, p_i)\). With an inductive argument we can notice that the orientation of \(p_iq_i\) represented by \(v_i\) depends on the parity of \(i\), i.e., the first edge is represented by the orientation from \(p_1\) to \(q_1\) (by assumption), the second one from \(q_2\) to \(p_2\) (by definition of \(E^+\)), and inductively \(v_i = (p_i, q_i)\) when \(i\) is odd, and \(v_i = (q_i, p_i)\) when \(i\) is even (also by definition of \(E^+\)). Finally, since \(W\) is an \((x, y)(z, w)\)-path and its length \(\ell(W)\) is \(n - 1\), then \((z, w) = (p_n, q_n)\) if \(\ell(W)\) is odd, and \((z, w) = (p_n, q_n)\) if \(\ell(W)\) is even.

To prove our main result, recall that \(G\) admits a \(T_3\)-free orientation if and only if for each edge \(xy \in E(G)\) the vertices \((x, y), (y, x) \in V^+\) are in different connected components of \(G^+\) (Theorem 6).

**Theorem 29.** Let \(\mathcal{F}\) be the set of all odd closed strips and even Möbius strips. For a graph \(G\), the following statements are equivalent:

1. \(G\) admits a \(T_3\)-free orientation.
2. There is no homomorphism \(F \rightarrow G\) whenever \(F \in \mathcal{F}\).
3. There is no homomorphism from an odd closed strip to \(G\).
4. There is no homomorphism from an even Möbius strip to \(G\).

**Proof.** Clearly, the second statement is equivalent to the conjunction of the third and fourth statements. So, by showing that the third and fourth statements are equivalent, we will prove that the second, third and fourth statements are equivalent. Also, by Lemma 27, the first statement implies the remaining statements. To conclude the proof we will show that the second statement implies the first one.

To prove the equivalence between the last two statements, it suffices to show that for any odd closed (resp. even Möbius) strip \(S\), there is an even Möbius (resp. odd Möbius) strip \(S'\)
such that $S' \to S$. Suppose that $S$ is an odd closed strip, and let $S_0$ be an odd strip such that $S$ is a quotient of $S_0$. Let $P = p_1 \ldots p_n$ and $Q = q_1 \ldots q_m$ be the bounding paths of $S_0$. Recall that the region between $P$ and $Q$ is triangulated in $S_0$. This implies that either $p_{n-1}q_m \in E(S_0)$ or $q_{m-1}p_n \in E(S_0)$; without loss of generality we assume that $p_{n-1}q_m \in E(S_0)$. Consider the even strip $S_1$ with bounding paths $R = r_1 \ldots r_n$ and $T = t_1 \ldots t_{m+1}$ with the following adjacencies. For each $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, m-1\}$ there is an edge $r_ir_j$ if and only if there is an edge $p_ip_j$. The remaining $RT$-edges are $r_nr_{m+1}$, $r_mr_{m-1}$ and $r_tm_{m+1}$. Clearly, $S_1$ has exactly one more triangle than $S_0$, so the Möbius strip $S'$ defined by $S_1$ is an odd Möbius strip. To see that $S' \to S$, first consider the mapping $\varphi: V(S_1) \to V(S_0)$ defined by $r_i \mapsto p_i$ and $t_j \mapsto q_j$ for every $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, m-1\}$, and $r_n \mapsto q_m$, $t_m \mapsto p_{n-1}$ and $t_{m+1} \mapsto p_n$. We illustrate this mapping and construction as follows.

![Diagram](image)

By definition of the edge set of $S_1$, the mapping $\varphi$ is a homomorphism from $S_1$ to $S_0$. Furthermore, notice that $S'$ is defined from $S_1$ by identifying $r_n$ and $t_1$, and identifying $t_{m+1}$ and $r_1$. Also, $S$ is defined from $S_0$ by identifying $p_n$ and $p_1$, and $q_m$ and $q_1$. It is not hard to notice that $\varphi$ commutes with these vertex identifications. Thus, the mapping $\varphi$ factors to a homomorphism $S \to S'$. This shows that for every odd closed strip $S$, there is an even Möbius strip $S'$ such that $S' \to S$. With a similar construction one can show that for every even Möbius strip $S'$, there is an odd closed strip $S$ such that $S \to S'$. Therefore, the third and fourth statements are equivalent.

To conclude the proof we show that the second statement implies the first one. To do so, we prove the contrapositive statement. Suppose that $G$ does not admit a $T_3$-free orientation. So, by Lemma 28 there is an $(x, y)(y, x)$-path in $G^+$ for some edge $xy$ of $G$. Thus, applying Lemma 28 to $(x, y)$ and $(y, x)$ yields an edge sequence $p_1q_1, \ldots, p_nq_n$ of $G$ such that $p_1 = x$ and $q_1 = y$, and either $p_n = y$ and $q_n = x$ if $n$ is even, or $p_n = x$ and $q_n = y$. So, if the length of the edge sequence $S$ is even, then $S$ defines a homomorphism from an even Möbius strip to $G$, and if the length is odd, then $S$ defines a homomorphism from an odd closed strip to $G$. Therefore, $G$ does not admits a $T_3$-free orientation then $F \to G$ for some $F \in \mathcal{F}$. This concludes the proof.

## 5 Proof of Theorem 1

We first prove that for each class $\mathcal{C}$ listed in Theorem 1 there is a set of oriented graphs on three vertices $F$, such that $\mathcal{C}$ is the class of $\text{Forb}_e(F)$-graphs. Then, we show that for each set $F$ of oriented graphs on three vertices, there is a class $\mathcal{C}$ listed in Theorem 1 such that $\mathcal{U}_{\text{Forb}_e(F)}$ equals $\mathcal{C}$, equals the intersection of $\mathcal{C}$ and triangle-free graphs, or equals the intersection of $\mathcal{C}$ and complete-multipartite graphs.
5.1 Part 1

We list the graphs in the same order as in Theorem 1. In each case we provide the set of forbidden oriented graphs, together with the corresponding reference. As mentioned before, we have no characterization for two of these classes.

1. **Perfectly orientable graphs.** By definition [16], these graphs are $F orb_e(B_1)$-graphs, and equivalently $F orb_e(B_2)$-graphs.

2. **Comparability graphs.** These graphs are $F orb_e(B_3)$-graphs [16]. This class also corresponds to $F orb_e(B_3, \overrightarrow{C}_3)$ (definition).

3. **Odd closed strip hom.-free graphs.** These graphs are $F orb_e(T_3)$-graphs (Theorem 29).

4. **Disjount union of proper circular-arc graphs.** These graphs are $F orb_e(B_1, B_2)$-graphs [16] — see Table 3.

5. **Trivially perfect graphs.** These graphs are $F orb_e(B_1, B_2)$-graphs [16] — see Table 3.

6. **Transitive-perfectly orientable graphs** By definition, these graphs are $F orb_e(B_1, \overrightarrow{C}_3)$-graphs, and equivalently $F orb_e(B_2, \overrightarrow{C}_3)$-graphs.

7. **Disjoint union of unicyclic graphs.** These graphs are $F orb_e(B_1, T_3)$-graphs, and equivalently $F orb_e(B_2, T_3)$-graphs (Proposition I).

8. **Disjount union of triangle-free unicyclic graphs.** These graphs are $F orb_e(B_2, \overrightarrow{C}_3)$-graphs, and equivalently $F orb_e(B_2, T_3, \overrightarrow{C}_3)$-graphs (Direct implication of Proposition I).

9. **3-colourable comparability graphs.** These graphs are $F orb_e(B_3, T_3)$-graphs (Proposition 21).

10. **Triangle-free graphs.** These graphs are $F orb_e(\overrightarrow{C}_3, T_3)$-graphs (trivial).

11. **Clusters.** These graphs are $F orb_e(B_1, B_2, B_3)$-graphs (trivial).

12. **Disjoint union of proper Helly circular-arc graphs.** These graphs are $F orb_e(B_1, B_2, \overrightarrow{C}_3)$-graphs (Proposition 16).

13. **Disjoint union of triangle-free proper circular-arc graphs.** These graphs are $F orb_e(B_1, B_2, T_3, \overrightarrow{C}_3)$-graphs (Implication of Lemma 8 and [16] — see Table 3).

14. **Disjoint unions of paths and cycles.** These graphs are $F orb_e(B_1, B_2, T_3)$-graphs (Proposition 22).
15. **Disjoint unions of paths and cycles but no triangles.** These graphs are $Forb_e(B_1, B_2, T_3, \overrightarrow{C}_3)$-graphs (Direct implication of Proposition [22]).

16. **Disjoint union of stars and triangles.** These graphs are $Forb_e(B_1, B_3, T_3)$-graphs (Proposition [23]).

17. **Star Forests.** These graphs are $Forb_e(B_1, B_3, T_3, \overrightarrow{C}_3)$-graphs (Corollary [24]).

18. **Stars and empty graphs.** These graphs are $Forb_e(B_1, B_2, B_3, T_3, \overrightarrow{C}_3)$-graphs (Lemma [8] and Corollary [24]).

19. **Matchings with isolated vertices.** These graphs are $Forb_e(B_1, B_2, B_3, T_3, \overrightarrow{C}_3)$-graphs (trivial).

20. **Empty graphs and $K_2$.** These graphs are $Forb_e(B_1, B_2, B_3, T_3, \overrightarrow{C}_3)$-graphs (trivial).

21. **Bipartite graphs.** These graphs are $Forb_e(B_3, T_3, \overrightarrow{C}_3)$-graphs (Proposition [15]).

22. **Complete bipartite graphs.** These graphs are $Forb_e(T_1+T_2, T_3, \overrightarrow{C}_3)$-graphs (trivial).

23. **Complete 3-partite graphs.** These graphs are $Forb_e(T_1+T_2, T_3)$-graphs (Corollary [20]).

24. **$K_{2,3}$-free complete multipartite graphs.** These graphs are $Forb_e(B_1, T_1+T_2)$-graphs, or equivalently $Forb_e(B_2, T_1+T_2)$-graphs (Proposition [13]).

25. **Complete multipartite graphs.** These graphs are $Forb_e(T_1+T_2)$-graphs (Lemma [8]).

26. **All graphs.** These graphs are $Forb_e(T_3)$-graphs (every graph admits an acyclic orientation).

**Intersection with complete multipartite graphs.** So far, we have shown that each class $\mathcal{C}$ listed in Theorem [1] is a class of $Forb_e(F)$-graphs for some finite set $F$ of non-empty oriented graphs on three vertices. The fact that the intersection of $\mathcal{C}$ and complete multipartite graphs is a class of $Forb_e(F)$-graphs (for some finite set $F$ of non-empty oriented graphs on three vertices) follows from Lemma [8].

### 5.2 Part 2

We present this part of the proof as a series of tables. The leftmost column of each table lists graphs in the forbidden set $F$; the mid-column contains the name of the class $U_{Forb_e(F)}$ and
the corresponding list number of Theorem 1—an asterisks means that the class is finite; and the last column contains the corresponding reference. We begin with those sets that contain exactly one graph.

| Oriented graphs in $F$ | Forb$_e(F)$-graphs                  | Reference     |
|------------------------|------------------------------------|---------------|
| $B_1$                  | 1. Perfectly orientable graphs     | Definition    |
| $B_2$                  | 1. Perfectly orientable graphs     | Definition    |
| $B_3$                  | 2. Comparability graphs            | Skrien        |
| $C_3$                  | 21. All graphs                     | Trivial       |
| $T_3$                  | 3. Odd closed strip hom.free graphs| Theorem       |
| $T_1 + T_2$            | 19. Complete multipartite graphs   | Trivial       |

Table 2: Sets containing one oriented graphs on three vertices.

From now on, we only consider sets that do not contain $T_1 + T_2$. We will treat these cases separately.

| Oriented graphs in $F$ | Forb$_e(F)$-graphs                  | Reference     |
|------------------------|------------------------------------|---------------|
| $B_1, B_2$             | 4. Proper circular-arc graphs      | Skrien        |
| $B_1, B_3$             | 5. Trivially perfect graphs         | Skrien        |
| $B_1, C_3$             | 6. Transitive-perfectly orientable graphs | Definition |
| $B_1, T_3$             | 7. Disjoint union of unicyclic graphs | Proposition |
| $B_2, B_3$             | 5. Trivially perfect graphs         | Skrien        |
| $B_2, C_3$             | 6. Transitive-perfectly orientable graphs | Definition |
| $B_2, T_3$             | 7. Disjoint union of unicyclic graphs | Proposition |
| $B_3, C_3$             | 2. Comparability graphs            | Definition    |
| $B_3, T_3$             | 9. 3-colourable comparability graphs | Proposition |
| $C_3, T_3$             | 10. Triangle-free graphs            | Trivial       |

Table 3: Sets containing two oriented graphs on three vertices, but not $T_1 + T_2$.

The tables displayed in this section show that if $(T_1 + T_2) \not\subseteq F$ or $|F| = 1$, then the class of Forb$_e(F)$-graphs is either finite or listed in Theorem 1. By Lemma 8, if $(T_1 + T_2) \in F$, then the class $\mathcal{U}_{Forb_e(F)}$ is the intersection of $\mathcal{U}_{Forb_e(F-(T_1+T_2))}$ and complete multipartite graphs. Thus, the claim of Theorem 1 holds.
Oriented graphs in $F$ | $Forb_e(F)$-graphs | Reference
---|---|---
$B_1, B_2, B_3$ | 11. Clusters. | Trivial.
$B_1, B_2, C_3$ | 12. Proper Helly circular-arc graphs. | Proposition 16
$B_1, B_2, T_3$ | 13. Disjoint union of paths and cycles. | Proposition 22
$B_1, B_3, C_3$ | 5. Trivially perfect graphs. | Proposition 17
$B_1, B_3, T_3$ | 14. Disjoint union of triangles and stars. | Proposition 23
$B_1, C_3, T_3$ | 8. Disjoint union of triangle-free unicyclic graphs. | Corollary 12
$B_2, B_3, C_3$ | 5. Trivially perfect graphs. | Proposition 17
$B_2, B_3, T_3$ | 14. Disjoint union of triangles and stars. | Proposition 23
$B_2, C_3, T_3$ | 8. Disjoint union of triangle-free unicyclic graphs. | Corollary 12
$B_3, C_3, T_3$ | 16. Bipartite graphs. | Proposition 15

Table 4: Sets containing three oriented graphs on three vertices, but not $T_1 + T_2$. 

| Oriented graphs in $F$ | $Forb_e(F)$-graphs | Reference |
---|---|---
$B_1, B_2, B_3, C_3$ | 11. Clusters | Trivial |
$B_1, B_2, B_3, T_3, *K_3, K_2$ and $K_1$ | 11. D.u.o. triangle-free proper circular-arc graphs | Lemma 8 + Table 3
$B_1, B_3, C_3, T_3$ | Star forest | Corollary 24
$B_2, B_3, C_3, T_3$ | Star forest | Corollary 24
$B_1, B_2, B_3, C_3, T_3$ | Matchings with isolated vertices | Lemma 8

Table 5: Sets containing four or five oriented graphs on three vertices, but not $T_1 + T_2$ nor both orientations of the triangle.

### 6 Complete multipartite graphs

For the sake of completeness, we comment on the intersection of classes listed in Theorem 1 and complete multipartite graphs.

**Proposition 30.** For a complete multipartite graph, the following statements are equivalent:

1. $G$ admits a $\{B_1, \overrightarrow{C_3}, T_1 + T_2\}$-free orientation.
2. $G$ admits a $\{B_2, \overrightarrow{C_3}, T_1 + T_2\}$-free orientation.
3. $G$ is a transitive-perfectly orientable graph.
4. $G$ is a complete graph, a complete graph minus two non-incident edges, or a complete split graph.
5. $G$ is a $\{K_{2,3}, K_{2,2,2}\}$-free complete multipartite graph.

Proof. The equivalence between first three items follow from Proposition 16 and from Lemma 8. The last two statements are evidently equivalent. It is straightforward to find a $\{B_1, \overrightarrow{C_3}, T_1 + T_2\}$-free orientation of a graph described in item 4, so the fourth statement implies the first three. On the other hand, it is not hard to notice that neither $K_{2,3}$ nor $K_{2,2,2}$ admit a $\{B_1, \overrightarrow{C_3}, T_1 + T_2\}$-free orientation, so the first statement implies the last two. This concludes the proof. 

Proposition 31. For a complete multipartite graph, the following statements are equivalent:

1. $G$ admits a $\{B_1, B_2, \overrightarrow{C_3}, T_1 + T_2\}$-free orientation.

2. $G$ is a proper Helly circular-arc graph.

3. $G$ is an empty graph, a complete graph, a complete graph minus an edge, or $C_4$.

4. $G$ is a $\{K_{1,3}, K_{1,2,2}\}$-free complete multipartite graph.

Proof. The equivalence between the first two items follows from Proposition 16 and Lemma 8. The last two statements are evidently equivalent. It is immediate to find a $\{B_1, B_2, \overrightarrow{C_3}, T_1 + T_2\}$-free orientation of an empty graph, a complete graph, a complete graph minus an edge or a complete graph minus two non-incident edges. On the other hand, it is not hard to see that neither $K_{1,3}$ nor $K_{1,2,2}$ admit such an orientation. So, the equivalence between the four statements holds. 

Proposition 32. For a complete multipartite graph, the following statements are equivalent:

1. $G$ is a proper-circular arc graph.

2. $G$ admits a $\{B_1, B_2, T_1 + T_2\}$-free orientation.

3. $G$ is an empty graph, a complete graph, a complete graph minus an edge, or a complete graph minus two non-incident edges.

4. $G$ is a $\{K_{1,3}, K_{2,2,2}\}$-free complete multipartite graph.

Proof. The equivalence between the first two items follows from Table 3 and Lemma 8. The last two statements are evidently equivalent. On the one hand, it is immediate to find a $\{B_1, B_2, T_1 + T_2\}$-free orientation of an empty graph, a complete graph, a complete graph minus an edge or a complete graph minus two non-incident edges. On the other one, it is not hard to see that neither $K_{1,3}$ nor $K_{2,2,2}$ admit such an orientation. So, the equivalence between the four statements holds. 

Proposition 33. For a graph $G$, the following statements are equivalent:

1. $G$ is a trivially perfect complete multipartite graph.
2. $G$ admits a $\{B_1, B_3, T_1 + T_2\}$-free orientation.

3. $G$ admits a $\{B_1, B_3, \overrightarrow{C}_3, T_1 + T_2\}$-free orientation.

4. $G$ is a $C_4$-free complete multipartite graph.

5. $G$ is an empty graph, a complete graph or a complete graph minus an edge.

Proof. The equivalence between the first three items follows from Lemma 8 and Proposition 17. The equivalence between the fourth and fifth statements is immediate. Finally, recall that trivially perfect graphs are $\{C_4, P_4\}$-free graphs [8]. Thus, since every complete multipartite graph is $P_4$-free, we conclude that the first and fourth statements are equivalent, which concludes the proof.

Theorem 34. The following classes are all infinite families of $\text{Forb}_e(F)$-graphs, where $F$ is a set of non-empty oriented graphs on three vertices and $\text{Forb}_e(F)$ is a subclass of complete multipartite graphs.

1. Complete multipartite graphs.

2. Complete 3-partite graphs.

3. Complete bipartite graphs.

4. $K_{2,3}$-free complete multipartite graphs.

5. $\{K_{2,3}, K_{2,2,2}\}$-free complete multipartite graphs.

6. $\{K_{2,3}, K_{1,2,2}\}$-free complete multipartite graphs.

7. $C_4$-free complete multipartite graphs.

8. Complete graphs and empty graphs.

9. Empty graphs, stars, $C_3$ and $C_4$.

10. Empty graphs, stars and $C_4$.

11. Empty graphs, stars and $C_3$.

12. Empty graphs and stars.

13. Empty graphs and finitely many graphs.

Proof. We show that the intersection of complete multipartite graphs and each class is the list of Theorem 1 is listed above. We proceed according to the listing order in Theorem 1.

(Thm 1.1) The intersection of perfectly orientable graphs and complete multipartite graphs are $K_{2,3}$-free complete multipartite graphs (Proposition 13).

(Thm 1.2) The intersection of comparability graphs and complete multipartite graphs equals the class of complete multipartite graphs (every complete multipartite graph is a comparability graph).

(Thm 1.3) The intersection of odd closed strip hom.-free graphs and complete multipartite graphs equal the class of complete 3-partite graphs (Corollary 20).

(Thm 1.4) The intersection of d.u.o. proper circular-arc graphs and complete multipartite graphs equal the class of $\{K_{1,3}, K_{2,2,2}\}$-free complete multipartite graphs (Proposition 32).

(Thm 1.5) The intersection of trivially perfect graphs and complete multipartite graphs are either complete graphs or complete graphs minus an edge (Proposition 33).
The intersection of transitive-perfectly orientable graphs and complete multipartite graphs equal the class of \(\{K_{2,3}, K_{2,2,2}\}\)-free complete multipartite graphs (Proposition 30).

The intersection of d.u.o. of unicyclic graphs and complete multipartite graphs are either empty graphs, stars, \(C_3\) or \(C_4\) (immediate).

The intersection of d.u.o. of triangle-free unicyclic graphs and complete multipartite graphs are either empty graphs, stars or \(C_4\) (immediate).

The intersection of 3-colourable comparability graphs and complete multipartite graphs are complete 3-partite graphs (every complete multipartite graph is a comparability graph).

The intersection of triangle-free graphs and complete multipartite graphs are complete bipartite graphs (immediate).

The intersection of clusters and complete multipartite graphs are complete graphs and empty graphs (immediate).

The intersection of d.u.o. proper Helly circular-arc graphs and complete multipartite graphs are \(\{K_{1,3}, K_{1,2,2}\}\)-free complete multipartite graphs (Proposition 31).

The intersection of d.u.o. triangle-free proper circular-arc graphs and complete multipartite graphs is the class empty graphs and some finite set (immediate from Proposition 32).

The intersection of d.u.o. paths and cycles or d.u.o. paths and cycles but no triangles, with complete multipartite graphs are either empty graphs and some finite set of graphs (immediate).

The intersection of d.u.o. triangles and stars and complete multipartite graphs is the class of stars and \(C_3\). (trivial).

The intersections of star forests and of stars and empty graphs with complete multipartite graphs is the class of stars and empty graphs (trivial).

The intersection of matchings and isolated vertices with complete multipartite are either empty graphs or \(K_2\). (trivial).

The intersections of either empty graphs and \(K_2\), bipartite graphs, complete bipartite graphs, complete 3-partite graphs, \(K_{2,3}\)-free complete multipartite graph, complete multipartite graphs or of all graphs, with complete multipartite graphs can be trivially described (and are listed above).

7 Conclusions

Algorithm is a certifying one, i.e., given a graph \(G\), it outputs an \(F\)-free orientation of \(G\) if it has one, or it finds and obstruction to being a \(\text{Forb}_e(F)\)-graph, but these obstructions live in the constraint digraph \(D^+\), not in \(G\). The proofs of Lemma 28 and Theorem 29 yield a polynomial time extension of this algorithm (in the case when \(F = \{T_3\}\)) that outputs an obstruction that now lives in \(G\); namely it outputs a forbidden homomorphic pre-image \(W\) and a homomorphism \(\varphi: W \rightarrow G\). Various of the reductions to 2-SAT are examples of certifying algorithms that exhibit an obstruction that does not belong to the graph \(G\). A
technique similar to the reverse engineering in the proof of Lemma 28 could work to find obstructions in \( G \) for other cases.

We listed all families of \( \text{Forb}_e(F) \)-graphs where \( F \) consists of non-empty oriented graphs on three vertices. Finding nice characterizations of perfectly orientable graphs and transitive-perfectly orientable graphs remain as open problems.

**Problem 35.** Characterize transitive-perfectly orientable graphs.

We briefly observe the following structural property of transitive-perfectly orientable graphs.

**Proposition 36.** Every transitive-perfectly orientable graph \( G \) admits a partition into two induced chordal graphs.

**Proof.** Let \( G' \) be a \( \{B_1, \overrightarrow{C}_3\} \)-free orientation of a graph \( G \). In [2], the authors show that any \( \{B_1, \overrightarrow{C}_3\} \)-free oriented graph has dichromatic number at most 2 (this result is also a consequence of a stronger statement found in [17]). Let \( U \) and \( V \) be the two colour classes in such a colouring of \( G' \). Since \( G' \) is \( B_1 \)-free, and \( G'[U] \) and \( G'[V] \) have no directed cycles, then the underlying induced subgraphs, \( G[U] \) and \( G[V] \), are chordal graphs. The claim follows.

Theorem 1 together with Proposition 9 show that all classes of \( \text{Forb}_e(F) \)-graphs can be recognized in polynomial time — except for transitive-perfectly orientable graphs, whose recognition complexity remains an open problem.

**Theorem 37.** Let \( F \) be a set of oriented graphs on three vertices. If \( F \neq \{B_1, \overrightarrow{C}_3\} \) and \( F \neq \{B_2, \overrightarrow{C}_3\} \), then it is in \( P \) to recognize \( \text{Forb}_e(F) \)-graphs.

Clearly, a graph \( G \) admits a \( \{B_1, \overrightarrow{C}_3\} \)-free orientation if and only if it admits a \( \{B_2, \overrightarrow{C}_3\} \)-free orientation.

**Problem 38.** Determine the complexity of deciding if an input graph \( G \) admits a \( \{B_1, \overrightarrow{C}_3\} \)-free orientation. Equivalently, determine the complexity of recognizing transitive-perfectly orientable graphs.

As a final conclusion, let us to see how Skrien’s work [16] and this work relate to characterizations through forbidden ordered patterns. An ordered pattern is a graph, \( G \), together with a linear ordering of its vertices. Similar to the procedure followed in [16] and in this work, one can fix a finite set of ordered patterns, \( P \), and characterize those graphs that admit a \( P \)-free ordering. For instance, if \( P \) is the singleton \( \{(1 \leq 2 \leq 3), (12, 13)\} \), then a graph \( G \) admits a \( P \)-free ordering if and only if \( G \) is chordal. Our work of Section 3 is similar (but not as thorough and complete) to [7], where Feuilloley and Habib characterize all families of graphs defined by admitting a \( P \)-free ordering for any set of oriented graphs on three vertices, \( P \). On the other hand, the algorithm exhibited in Section 2 was motivated by [10], where Hell, Mohar and Rafiey propose a master algorithm that determines if an input graph, \( G \), admits a \( P \)-free ordering, for any fixed set of ordered patterns on three vertices.

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Given a set $F$ of orientations of $P_3$, Skrien studied classes of graphs that admit an $F$-free acyclic orientation \[16\]. So, a possible problem one may think of, is to extend Skrien’s work as we did in this manuscript. Turns out that this has been indirectly solved in \[7\] and \[10\]. Consider an acyclic oriented graph $H'$ with underlying graph $H$. Denote by $P_{H'}$ the set of all ordered patterns, $(H, \leq)$, such that for any edge $xy \in E(H)$ it holds that $(x, y) \in A(H')$ if and only if $x \leq y$. Given a set of acyclic oriented graphs, $F$, we denote by $P_F$ the union of all sets, $P_{H'}$, where $H' \in F$. It is not hard to observe that the following observation holds.

Observation 39. Let $F$ be a set of acyclic oriented graphs, and let $P_F$ be the set of ordered patterns defined above. A graph $G$ admits an acyclic $F$-free orientation if and only if it admits a $P_F$-free ordering.

In light of this observation, if $F$ is a set of acyclic oriented graphs on three vertices, then the class of graphs that admit an $F$-free acyclic orientation is characterized in \[7\]. Moreover, due to the algorithm of Hell et al. \[10\], the following statement follows.

Proposition 40. Let $F$ be any set of oriented graphs on three vertices. Recognizing if an input graph admits an $F$-free acyclic orientation can be done in polynomial time.

Proof. It follows directly from Observation 39 and Corollary 1 in \[10\].

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