From effective actions to the background geometry

A. Gorsky \textsuperscript{a,c}, V. Lysov \textsuperscript{a,b}

\textsuperscript{a} Institute of Theoretical and Experimental Physics, Moscow 117259, Russia

\textsuperscript{b} Institute of Physics and Technology, Moscow, Russia

\textsuperscript{c} W. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, USA

Abstract

We discuss how the background geometry can be traced from the one-loop effective actions in nonsupersymmetric theories in the external abelian fields. It is shown that upon the proper identification of the Schwinger parameter the Heisenberg-Euler abelian effective action involves the integration over the $AdS_3$, $S_3$ and $T^*S^3$ geometries, depending on the type of the external field. The interpretation of the effective action in the selfdual field in terms of the topological strings is found and the corresponding matrix model description is suggested. It is shown that the low energy abelian MHV one-loop amplitudes are expressed in terms of the type B topological string amplitudes in mirror to $T^*S^3$ manifold. We also make some comments on the relation between the imaginary part of the effective action and the branes in SU(2) as well as on the geometry of the contours relevant for the path integral.
1 Introduction

The duality between gauge theories and strings is presently under the attacks from the different directions. One of the most promising approaches involves the duality between N=4 SYM theory and IIB string theory in $AdS_5 \times S_5$ geometry with the additional flux of the four-form field [1]. The string tension is proportional to $\sqrt{\lambda}$ where $\lambda = g_{YM}^2 N$ is t'Hooft coupling hence the strong coupling limit in Yang-Mills theory can be treated almost classically on the stringy side. On the other hand the weak coupling limit on the gauge theory side in principle deserves the knowledge of the full quantum description of the string in this background - the problem which has not been solved yet.

If the correspondence is true it is desirable to derive the $AdS$ type geometries from the first principle starting from the perturbative Feynmann diagrams. The interesting step in this direction has been made in [2] where the first quantized picture was used to argue that the one-loop two and three point functions in the scalar theory can be naturally described in terms of the bulk-to-boundary propagators in $AdS_5$ integrated over position of the point in the bulk. It turned out that in the first quantized picture the Schwinger parameter serves as the radial coordinate in $AdS_5$ hence the integration over this variable corresponds to the integration over the interior of the fifth dimension. However starting from the four point function the situation appears to be more subtle and fully satisfactory picture is absent.

Therefore it is natural to clarify the situation using some different tractable object. The good candidate to work with is the effective action in the external field. It effectively involves the arbitrary order in the coupling constant and on the other hand the explicit answer can be derived. We shall consider such effective action from the background geometry perspective and shall argue that peculiar geometries emerge in a very natural way. In the particular case of the abelian theory with the constant electric and magnetic fields effective action will be formulated in terms of the $AdS_3$ and $S^3$ geometries where one of the coordinates shall be identified with the Schwinger parameter similar to discussion in [2]. Note that the first quantized picture we shall use below has been applied for the effective actions long time ago [3, 4].

More deep motivation for the choice of the effective action in the external field as the
convenient object to capture the geometry is as follows. The experience of the work with the supersymmetric theories suggests that a kind of the refined "holomorphic" object encodes the information about the external geometry. In the N=2 theories this role is played by the effective prepotential while in the N=1 theory the effective superpotential is the relevant object. In both cases the prepotential and superpotential can be read off from some Riemann surface embedded into the three dimensional noncompact Calabi-Yau manifold. The most efficient computational tool to get the explicit form of the effective actions in SUSY theories involves the topological strings (see [5, 6] for review). The open string topological A picture involves Chern-Simons type description while type B topological open string provides the corresponding matrix model. Both pictures get mapped into the closed string geometry upon the large N transition.

In this paper we shall consider nonsupersymmetric theories so at the first glance there is no such refined object at all and the situation looks hopeless. However it turns out that at least some part of the geometry can be read off from the effective actions in this case too. The proper geometric interpretation of the Schwinger parameter provides the important starting point. In the case of the selfdual background we will be able to go further and apply similar topological string ideas in the nonsupersymmetric case. The reason for such possibility can be naturally attributed to the residual supersymmetry known in the selfdual background. In fact we shall try to combine the geometrical interpretation of the Schwinger parameter with the picture familiar from the topological strings.

In the selfdual case we shall use the relation between the large N Chern-Simons theory and the Schwinger type calculations discussed in the context of the topological strings [7]. The gauge theory in the type A picture is realized on the worldvolume of D6 branes wrapped around $S^3$ considered as the Lagrangian submanifold in $T^*S^3$. Chern-Simons theory lives on $S^3$ and upon the large N transition it is dual to the theory on the resolved conifold with fluxes instead of branes [8]. The logarithm of the CS partition function at large N turns out to coincide with the scalar QED effective action in the selfdual field. This relation appears to be consistent with the identification of the Schwinger parameter in the wrapped picture. The similar representation will be found for the pure electric and magnetic backgrounds.

Surprisingly enough we can formulate the matrix model picture for the effective action in
the selfdual background at least at one loop. It is based on the matrix model representation of Chern-Simons theory on $S^3$ found in [10, 9] which corresponds to the geometry of the type B model identified with the mirror to $T^*S^3$. It turns out that the potential in the matrix model at least in one formulation involves the double trace terms. Let us emphasize that this matrix model is the nonsupersymmetric counterpart of the matrix model found by Dijkgraaf and Vafa [11] which describes the effective superpotential in N=1 SUSY YM theory. We shall make some comments concerning the proper interpretation of this matrix model in terms of the spectrum of the Dirac operator in the background fields.

In principle one could also consider a kind of the probe picture on the brane and use the representation of the effective action as the sum with the proper weights over the Wilson loops with the different boundary contours. These contours can be considered as the boundaries of the string worldsheets embedded into the ambient geometry. Roughly speaking in the probe picture the Schwinger parameter measures the lengths of the boundary contours and the effective action from the probe brane perspective can be considered as the back reaction of the string ending on it. To recognize the background geometry for the probe brane we first identify the integrands in the effective actions as the propagators of the particles with the different masses in $AdS_3$ and $S_3$ metrics. Therefore these geometries are expected to serve as the background for the probe D3 brane with the constant fields indeed. Such geometries are natural to emerge since, for instance, magnetic field amounts from the density of the D1 strings on the D3 worldvolume which yields the $AdS_3$ geometry around.

It is known from the textbooks that the one loop abelian effective action serves as the generating function for the low energy one-loop photon amplitudes. In particular the effective action in the selfdual background yields one loop MHV amplitudes with all ”+” or ”-” photons [12]. As a byproduct of our analysis we shall be able to develop an interesting type B topological string representation of such MHV amplitudes very much in the spirit of the recent discussion of nonabelian gluonic MHV amplitudes [13, 14]. In our case the mirror of $T^*S^3$ plays the role of the ”twistor” manifold. Moreover we shall derive the topological string interpretation for the generating function for MHV amplitudes with arbitrary number of photons involved. It turns out that such MHV amplitudes can be also derived from the perturbative expansions in
Chern-Simons theory on $S^3$ or in the corresponding matrix model.

The effective action in the electric field develops the imaginary part corresponding to the nonperturbative pair creation. We shall briefly discuss these issues from the point of view of our realization of the Schwinger parameter and make some relation with the quantization of the brane radii on SU(2) group manifoilds. We shall also present qualitative arguments implying that the curves with cusps are important in the path integral.

The paper is organized as follows. In Section 2 we shall briefly review Gopakumar’s arguments concerning the identification of the Schwinger parameters with the radial coordinate in $AdS_5$. In Section 3 we will show that the propagators in $AdS_5$ and $S^3$ type geometry are involved in the effective actions in the external abelian field. In Section 4 we shall find the precise background geometry for the selfdual external field and discuss the matrix models which substitute Dijkgraaf-Vafa model for nonsupersymmetric case at one loop level. In Section 5 we discuss the topological string representation for the abelian low energy one-loop MHV amplitudes. The relation between the spherical branes in $S^3$ and the pair production in the electric field is discussed in Section 6. Section 7 concerns the role of the contours with cusps in the path integral representation of the effective action. Some open questions and possible generalizations can be found in the last Section.

2 Free fields and AdS

In this section we briefly review the arguments of Gopakumar [2] concerning the representation of the loop diagrams in the scalar theory in four dimensions in terms of the tree diagrams in $AdS_5$. He adopted the first quantized language for the calculation of two and three point functions and, for instance, the two point function can be presented as

$$\Gamma(k) = \int_0^\infty \frac{d\tau \tau^2}{\tau^{d/2+1}} \int d\alpha e^{-\tau\alpha(1-\alpha)k^2}$$

where the exponential factor comes from the worldline correlator of two vertex operators

$$< e^{i k x(\tau_1)} e^{-i k x(\tau_2)} = e^{-k^2 G(\tau_1, \tau_2)}$$

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and the worldline propagators reads as

$$G(\tau_1, \tau_2) = -\frac{\tau_{12}(\tau - \tau_{12})}{\tau}$$

(3)

The variable $\tau$ corresponds to the invariant length of the path $\tau = \int e(t)dt$ in the Polyakov like formulation of the particle action

$$S = \int_0^1 (e^{-\frac{1}{2}x^2} + m^2 e)dt$$

(4)

It is the particle counterpart of the stringy Liouville mode and therefore similar to the string case is a good candidate for the additional fifth dimension.

In the space representation the answer has the structure of the heat kernel

$$<x|e^{t\Delta}|y> = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(x-y)^2}{4t}}$$

(5)

and therefore using the relation between the heat kernel and the bulk-boundary propagator

$$K(t) = \int d\rho \rho^{d/2-3} e^{-\rho e^{\Delta/4}}$$

(6)

in can be brought modulo the overall factor to the form

$$\Gamma(x_1, x_2) = \int dz_0 z_0^{-(d+1)} K(x_1, z)K(x_2, z)$$

(7)

corresponding to the tree representation of the two-point function in $AdS_5$. Let us note that in this two-point case we can just consider the product of two free propagators from the very beginning

$$G(x_1 - x_2) = \int ds s^{-(d/2+1)} e^{-\frac{(x_1 - x_2)^2}{4s}}$$

(8)

and introduce the new parameter

$$s_{\text{tot}}^{-1} = s_1^{-1} + s_2^{-1}$$

(9)

which allow to treat the product in terms of $AdS_5$ geometry.

These simple arguments were shown to work for the three point functions but some modification for the higher point functions is needed. It is worth noting that the effective radius in $AdS_5$ geometry in this case is not related to the gauge coupling constant contrary to the standard gauge/string correspondence where $\sqrt{g^2 N_c} = \frac{R_{AdS}}{\alpha'}$. 

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3 Effective actions in the external fields and the propagators in the curved background

3.1 Propagators in $AdS_3$ and $S^3$

After the reviewing the free field case let us turn now to our main example - the theory in the external electromagnetic field. We shall try to recognize the background geometry in the loop calculations in the external field. Let us note that the external field to some extend provides the IR regularization of the theory. On the other hand as we shall see later it gives rise to the effective $AdS_3$ or $S^3$ type geometry with the radii related to the coupling constant. Moreover it shall be clear that these nontrivial geometries emerge after the summation over all orders in the external field and can not be seen at any fixed order.

Let us start with the effective action for the fermion with the mass $m$ arising from the determinant in the external field

$$S_{eff} = \log \text{det}(i\partial - eA - m) = \int \frac{dT}{T} \exp(-Tm) \int dt T r < x|\exp(-tD)|x >$$

The equivalent representation in the first quantized picture looks as follows

$$S_{eff} = \sum_{\text{paths} C} \exp(-mL(C)) \exp(-i\Phi(C)) < W(C) >$$

where

$$W(C) = \text{Tr} P \exp(i e \oint_C Adx)$$

and the factor $\exp(-i\Phi(C))$ is responsible for the spin impact on the path integral. For instance in two dimensions it just counts the number of selfintersections $\nu(C)$ of the contour $C$ with the weight $(-1)^{\nu(c)}$. In higher dimensions it can be described via a Wess-Zumino terms on $CP_n$ geometry, for instance $CP_1$ for the fermions in three dimensions [15] and $CP_3$ for the four dimensional fermions [16]. This form involves the summation over the Wilson loops and is appropriate for the derivation of the stringy picture. In particular, in the strong coupling limit each Wilson loop can be considered as the boundary of the open string worldvolume extended into the radial $AdS$ direction.
The general Euler-Heisenberg action for the fermions when both magnetic and electric fields are involved reads as (see, for instance, the recent review [17])

\[ L_{\text{eff}} = \int_0^\infty ds \frac{s^3}{3} e^{-sm^2} [es\text{ctg}(esa) \times es\text{bcth}(esb) - 1 - \frac{e^2s^2}{3}(b^2 - a^2)] \] (13)

where \(a, b\) are the standard invariants \(ab = E \cdot H, a^2 - b^2 = E^2 - H^2\). In what follows we shall omit the subtraction terms which can be restored in a trivial manner. The similar expressions are known for the particles of the different spins, for instance, the explicit calculation amounts to the following effective action for the spin S particle in the electric field

\[ L_{\text{eff,scal}} = \int_0^\infty ds \frac{s^3}{3} eEs \sin(eEs) \sin(2S + 1) eEs e^{-sm^2} \] (14)

It reduces for the fermion to

\[ L_{\text{eff,f}} = \int_0^\infty ds \frac{s^3}{3} seE\text{ctg}(eEs)e^{-sm^2} \] (15)

and for the scalar to

\[ L_{\text{eff,scal}} = \int_0^\infty ds \frac{s^3}{3} eEs \sin(eEs) e^{-sm^2} \] (16)

It is also convenient to present the expression for the effective action for spin 1/2 particle in the constant selfdual field G of electric type which we shall intensively use in what follows

\[ L_{\text{eff}} = \int_0^\infty ds \frac{s^3}{3} e^{sm^2} (esG\text{cth}(esG))^2 \] (17)

while for the scalar we have

\[ L_{\text{eff}} = \int_0^\infty ds \frac{s^3}{3} e^{sm^2} \left(\frac{esG}{\sin(esG)}\right)^2 \] (18)

Let us emphasize that there are two types of the selfdual background; one is similar to the electric case while the second to the magnetic one. There is the standard pair production mechanism in the electric version of the selfdual field.

Let us argue now that the effective action naturally involves the propagators in \(AdS_3\) and \(S_3\) geometry very much in a spirit of Gopakumar’s calculation. Consider the propagator of the massless particle on the SL(2,R) group manifold which coincides with \(AdS_3\). The propagator is defined as

\[ G(x, y) = -i \int d\tau < x | e^{-\tau \Delta_{\text{AdS}_3}} | y > \] (19)
where $\Delta_{AdS_3}$ in the Laplace-Beltrami operator on $SL(2,R)$. The propagator can be calculated using the expansion of the transition amplitude over the characters of $SL(2,R)$ unitary continuous representations. There are fundamental series of such representations with $j = -1/2 + i\nu/2$ and complementary one with $-1 < j < 0$. Summing over two series one arrives at the following expression

$$G(x, y) = \theta \coth \theta$$

(20)

were $\theta$ is defined via

$$\cosh \theta = \frac{1}{2} tr(g_x^{-1}g_y)$$

(21)

and the following parametrization of the group manifold is assumed

$$g_x = \frac{1}{z_0} \begin{pmatrix} z_1 - z_2 & 1 \\ z_1^2 - z_2^2 - z_0^2 & z_1 + z_2 \end{pmatrix}$$

(22)

In this parametrization we have the metric in the standard form

$$ds^2 = tr(g_x^{-1}dg_x)^2 = \frac{dz_0^2 - dz_1^2 + dz_2^2}{z_0^2}$$

(23)

Hence we immediately recognize the ingredient of the effective action for the fermion in the external magnetic field as the propagator of the massless mode in $AdS_3$ background. To compare this observation with variables in [2] we can identify

$$\cosh \theta = \frac{1}{2} tr(g_x^{-1}g_y) = \frac{z_0^2 + w_0^2 + |z - w|^2}{2z_0w_0} = \cosh \xi$$

(24)

For the massless scalar particle in $d = 2$ we obtain $\Delta = 0$ and its propagator on $AdS_3$

$$G_{scat}(z_0, w_0, z, w) = \left( \frac{\xi^2}{\xi^2 - 1} \right)^{\frac{1}{2}} = \coth \theta$$

(25)

The natural question is if the relation to $AdS_3$ we found is seen for the arbitrary spins as well. Let us consider the exact expression for the propagator on the AdS introducing the new variable $\mu^2 = 1 - m^2$

$$G(\theta) = \sum_{l=0}^{l=\infty} \left( \frac{1}{l + 1 + \mu} + \frac{1}{l + 1 - \mu} \right)(e^{(l+1)\theta} - e^{-(l+1)\theta})$$

(26)

Removing the divergency in this sum we obtain the following answer

$$G(\theta) = \frac{\theta \cosh(\mu \theta)}{\sinh \theta} + \sum_{l=1}^{\mu-1} \frac{\sinh(l\theta)}{(\mu - l) \sinh \theta}$$

(27)
The leading \( \theta \to \infty \) term in this expression is close to the leading term in the one-loop external field expression for the particle of spin \( S \) related to the mass as

\[
(2S)^2 = 1 - m^2 \tag{28}
\]

which means that the higher spins in the theory in some sense correspond to the tachyonic modes in the bulk.

The very similar argumentation for the electric field involves the propagator of the corresponding modes in \( S^3 \) geometry coinciding with the SU(2) group manifold. For instance, the propagator of the massless mode in \( S^3 \) can be derived from the summation over the characters of the unitary irreducible SU(2) representations

\[
G(\theta) = \sum_j \frac{2j + 1}{j(j + 1)} \chi_j(\theta) \tag{29}
\]

amounting to

\[
G(\theta) = \theta \cot \theta \tag{30}
\]

It is involved into the effective action for the fermion in the external electric field where the following parametrization is implied

\[
cos \theta = \frac{1}{2} tr g_x^{-1} g_y \tag{31}
\]

The inspection of the generic effective action in (E,H) fields shows that the product of \( AdS_3 \) and \( S^3 \) propagators is involved while for the external selfdual fields the product of two \( AdS_3 \) or \( S^3 \) propagators is relevant. Hence generically the constant abelian field background feels the \( AdS_3 \) or \( S^3 \) type gravitational background around. Similar to Gopakumar’s calculation the effective actions involve the integration over the boundary conditions for propagators in the background geometry.

Let us briefly discuss the two-point function in the external field and consider the simplest case of the massless scalar theory in the selfdual external field. This choice is motivated by the very simple form of the scalar particle propagator in the external field \( F_{\mu\nu}F_{\nu\rho} = -f^2 \delta_{\mu\rho} \) in the gauge \( xA = 0 \)

\[
G(x,f) = \left( \frac{ef}{4\pi} \right)^2 \int_0^\infty \frac{d\tau}{\sinh^2(ef\tau)} e^{-ef/4x^2\coth(ef\tau)} \tag{32}
\]
Hence the two-point function in $x$ representation is just
\[ \Pi(x) = \frac{e^{-ef/2x^2}}{4\pi^2x^4} \] (33)

In the free case the bulk-boundary propagators in $AdS_5$ are involved into the two-point functions while in the case of the external selfdual field the bulk-bulk propagators in $AdS_3$ are relevant hence it is desirable to explain how $AdS_5$ geometry is restored if the external field is switched off. To this aim let us look at the propagator in the external field which after the change of variables $\xi = ef \cdot \coth(ef\tau)$ can be brought to the following form
\[ G(x, f) = \int_{ef}^\infty d\xi e^{-x^2\xi} \] (34)
which exactly coincides with the free particle case modulo the restriction in the integration region. In the external field the proper time variable $\tau$ corresponding to the radial coordinate in $AdS$ varies in the limit $\infty > \tau > ef$ instead of $0 < \tau < \infty$ in the free case. Hence the external field plays the role of the natural regularization with respect to the radial coordinate in $AdS$ and it is clear that the full $AdS_5$ can not emerge in this case.

The explicit expression for the two point function for the scalar field in the external selfdual field in the first quantized picture
\[ \Pi_2(p) = \int \frac{dT}{T} \int D\alpha \exp\left\{ \frac{\sinh(efT\alpha)}{2ef} - \frac{2\sinh^2(\alpha T/2)}{(ef)^2T}p^2 \right\} \] (35)
can be obtained after the simple calculation
\[ \Pi_2(p) = \int_0^\infty \frac{(ef)^2dT}{\sinh(efT)} \int d\alpha \exp\left\{ \frac{\sinh(efT\alpha)}{2ef} - \frac{2\sinh^2(\alpha T/2)}{(ef)^2T}p^2 \right\} \] (36)

### 3.2 Relation to the 2D Yang-Mills theory

We have identified the ingredients of the effective action as the propagators in $AdS_3$ and $S_3$. Let us represent the massless propagators in one more way through the two dimensional Yang-Mills theory on the disc. Let us consider now the effective action of the fermion in the pure electric field. To derive the 2D interpretation of the effective action let us start with the representation of the $S^3$ propagator discussed in [18] in terms of the two-dimensional Yang-Mills theory with SU(2) gauge group on the disc.
To make the correspondence exact, one introduces the amplitude of the two-dimensional Yang-Mills theory on a disk with radial coordinate $x^0$, $0 \leq x^0 \leq T$, and angular $x^1$, $0 \leq x^1 \leq L$, of area $\mathcal{A} = LT/2$, and a holonomy at its boundary $C = \partial \Sigma$,

$$U = P \exp i \int_C A.$$  

The partition function on the disk is [19]

$$\mathcal{Z}[U; g^2 \mathcal{A}] = \int D\mathcal{A}_\mu \delta \left( P e^{i \int_C dx A(x)} , U \right) e^{-\frac{1}{g^2} \int_\Sigma d^2x \sqrt{\det g_{\mu\nu}} Tr F^2} = \sum_j (2j + 1) \chi_j[U] e^{-g^2 A j(j+1)/2},$$

where and $\chi_j[U]$ are characters for the spin-$j$ representation of the gauge group. Thus,

$$\mathcal{Z}[U; g^2 \mathcal{A} = 2\tau] = < -v' | e^{-\tau H} | v >,$$

where $U = g^{-1}_v g_v$ and $H$ is SU(2) Casimir operator. The propagator of the massless particle on the SU(2) group manifold is given by the integral of the wave functional in two-dimensional Yang-Mills theory on the disc with respect to its area

$$G_{S_3}(\theta) = \int_0^\infty d\tau \mathcal{Z}[U; 2\tau].$$

Therefore we can represent the effective action of the fermion in the electric field as the partition function of the SU(2) YM theory on the disc integrated over the area of the disc and boundary holonomy

$$S_{\text{eff}}(E) = \int ds dA s^{-3} e^{-sm^2} Z_{2dYM}(arccos(es E), A)$$

One could have in mind also the instantonic realization of the same partition function of the 2d YM partition function related to its saturation by a sum over the classical saddle points in the path integral [20]. The instantons under consideration are solutions to the Yang-Mills equations of motion on the two-dimensional disc with the boundary conditions set by the holonomy $tr U[A(x^0 = T, x^1)] = 2 \cos \tilde{\theta}$, where $\tilde{\theta} = \pi - \theta$. The classical configurations in the $A^0(x^0, x^1) = 0$ gauge correspond to the straight paths connecting the initial and final points and read in the topological charge-$\ell$ sector

$$A^1_\ell(x^0, x^1) = x^0 (\sigma_3/2)(\tilde{\theta} + 2\pi \ell) / \mathcal{A}.$$  

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The action evaluated on these instanton solutions reads

$$S[A_\ell] = 2(\hat{\theta} + 2\pi \ell)^2/(g^2 A),$$

and summation over $l$ amounts to the equivalent representation for the transition element in YM theory

$$<0|e^{i\tau L^2}|v> = \frac{(-i\pi \tau)^{-3/2}}{\pi \sin \theta} \frac{\partial}{\partial \hat{\theta}} \sum l e^{-i(\hat{\theta} + 2\pi \ell)\tau/4}$$

(42)

The representations via characters and via instantons are related to each other by the Poisson resummation formula and amount to the same expression for the partition function.

The picture emerging for the pure magnetic background is very similar to the electric case with SL(2, R) two dimensional Yang-Mills theory instead of SU(2) one. The effective action reads now as

$$S_{\text{eff}}(H) = \int ds dA s^{-3} e^{-sm^2} Z_{\text{YM,SL}}(2, R)(A, \arccosh(esE))$$

(43)

in terms of partition function of Yang-Mills theory. We shall discuss the emergence and role of the AdS$_2$ geometry in the magnetic case later.

In the selfdual electric background case we have the following representation for the fermion effective action

$$S_{\text{eff}}(E) = \int ds dA s^{-3} e^{-sm^2} Z_{\text{YM,SU}}^2(2, SU(2))(A, \arccos(esE))$$

(44)

In the next section we shall demonstrate that the background geometry can be derived for the selfdual case very explicitly.

4 Effective actions in the selfdual field and the topological strings

4.1 Scalar case

In this section we shall find the relation between the effective actions in the selfdual background and the topological strings very much in the spirit of the previous discussions in the SUSY case.
We shall exploit the relation between the large \( N \) CS theory on \( S^3 \) which represents the topological open string A model and topological string models related to it by the large \( N \) transition or mirror symmetry. Upon the large \( N \) transition the topological A model corresponding to A branes wrapped around \( S^3 \) Lagrangian submanifold in \( T^*S^3 \) gets mapped into the topological closed string in the resolved conifold with fluxes in \( P^1 \)'s instead of branes [8]. In the mirror dual topological B model topological branes are wrapped around 2-cycles in the mirror Calabi-Yau manifold and upon the large \( N \) transition are replaced by the blown up 3-cycles on the modified Calabi-Yau manifold. The latter has been used in [11] to derive the effective superpotential in N=1 SYM theory.

Let us recall the relevant facts concerning CS theory on \( S^3 \). The partition function of CS theory on \( S^3 \) has been calculated long time ago [21] and reads as

\[
Z_{CS}(N, k, S^3) = e^{i\pi N(N-1)/8} \frac{1}{(N+k)^{N/2}} \sqrt{\frac{N+k}{N}} \prod_{j=1}^{N-1} (2\sin \frac{j\pi}{N+k})^{N-j}
\]

The theory can be treated as the open topological type A string field theory [22] on \( T^*S^3 \) where the coupling constant is identified with

\[
g_s = \frac{i}{k+N}.
\]

In the brane setup it can be considered as the worldvolume theory on the \( N \) topological A branes wrapped around the Lagrangian submanifold in \( T^*S^3 \). The theory implies the topological expansion

\[
Z_{CS}(N, k, S^3) = \sum_{g, h} C_{g, h, N^2-2g} \lambda^{2g-2+h}
\]

over the worldvolumes with \( h \) holes and \( g \) handles, where \( \lambda = g_s N \) is the standard t’Hooft coupling. In what follows we shall be mainly interested in the planar limit

\[
F_0(\lambda) = \sum_h F_{0, h} \lambda^h
\]

The relation between the CS partition function and the topological string amplitudes has been found in [8] where it was shown that amplitudes coincide with the proper expansion terms of \( \log Z_{CS}(N, k, S^3) \). On the other hand such terms can be found via a kind of the Schwinger like loop calculations [7, 8]. From the dual gravity perspective such terms counts
the contribution to the $W^g R^2$ terms in the N=2 effective action where $W$ corresponds to the graviphoton background.

We shall use the observation [7] that the effective action of the scalar in the abelian selfdual field of electric type coincides with the $\log Z_{CS}(N, k, S^3)$ at $N \to \infty$

$$\log Z_{CS}(N = \infty, k, S^3) = \int_0^\infty dse^{-sk(s/2 \sin s/2)^2}$$

(49)

The identification of the parameters looks as follows

$$k_{CS} = m^2/2e_f$$

(50)

hence the large level $k$ corresponds to the weak field limit. The limit $N \to \infty$ can be thought of as arising from the infinite number of topological branes wrapped around $S^3$. These correspond to the strings inside D3 brane in the probe picture which provide the constant selfdual background field. The level $k$ expansion of the free energy of CS theory in this limit also coincides with the genus expansion of the free energy of c=1 noncritical bosonic string at the selfdual radius [25]. In the framework of type A model open topological string on $T^* S^3$ can undergo the topological large N transition to the resolved conifold without branes. The Kahler class of the corresponding $S^2$ is fixed by the t’Hooft coupling.

It is most convenient for our purposes to use the mirror type B topological model which is related to CS theory as follows. First, let us recall the construction of the manifold mirror to $T^* S^3$ which can be presented as

$$xu + yv = \mu$$

(51)

and can be considered as $T^2 \times \mathbb{R}$ fibration over $\mathbb{R}^3$. The mirror transforms $T^* S^3$ to the blowup of

$$xy = (e^u - 1)(e^v - 1)$$

(52)

along the locus $x = y = (e^u - 1) = (e^v - 1) = 0$ by inserting a $P^1$. The imaginary parts of $u$ and $v$ in the mirror manifold are T dual to the one-cycles on the torus fiber in A model. After the mirror transform $N$ type B branes get wrapped around $P^1$.

It was observed in [10, 9] that the large N CS theory on $S^3$ can be described by the matrix model defined on the type B branes wrapped around $P^1$. The theory on their worldvolumes
reads as
\[ S = \int_{P^1} Trv\bar{Du} \] (53)
where \( u \) and \( v \) are normal coordinates to the brane worldvolumes. The theory on \( P^1 \) should be considered as emerged after the gluing of two halves of the sphere with nontrivial map between boundaries. The gluing operator \( U \) fixes the superpotential of the corresponding matrix model and in the case under consideration it reads as
\[ U = exp\left(\frac{1}{g_s} \int_{P^1} \omega Tr u^2\right) \] (54)
where \( \omega \) is a (1,1) form. The measure in the corresponding matrix model turns out to be unitary
\[ d_H u = \prod_i du_i \left(\prod_{i \leq j} 2\sin\left(\frac{u_i - u_j}{2}\right)^2\right) \] (55)

The matrix model can be solved in the usual way introducing the density of eigenvalues \( \rho(e^u) \) and the corresponding resolvent \( v(e^u) \). In the single cut solution resolvent fixes the Riemann surface where \( u \) and \( v \) are defined [9]
\[ (e^v - 1)(e^{v+u} - 1) + e^t - 1 = 0 \] (56)
where t’Hooft parameter is
\[ t = \frac{1}{2\pi i} \int_A vdu \] (57)
This Riemann surface corresponds to the nontrivial part of the manifold obtained from B model upon the large N transition.

What the coincidence of the effective action in the selfdual background and \( \log Z_{CS}(N, k, S^3) \) could teach us? The first lesson is that now we can precisely determine the background geometry from the one-loop effective action in nonsupersymmetric theory that was one of our purposes. The case of the scalar effective action is the most transparent one however later we shall consider the similar picture for the spinor particle. The Kahler moduli of \( P^1 \) in Calabi-Yau can be identified with the dimensionless combination of the particle mass and the external field hence the topological amplitudes in type B theory correspond to the expansion of the effective action in terms of the external field. The role of the Schwinger parameter is quite clear - it
defines the radii of $S^2$ spheres inside $S^3$ in the type A geometry while in the Type B model it corresponds to the eigenvalues of the holonomies.

Moreover now we can develop the matrix model representation for the effective action of the scalar in the selfdual field in nonsupersymmetric theory which reads as

$$Z_{MM}(g_s) = \frac{1}{VolU(N)} \int d_H M \exp \left( \frac{1}{2g_s} Tr M^2 \right)$$

(58)

where the integral is over the Hermitian matrixes with the Haar measure. This matrix integral can be traded for the matrix integral with the standard Hermitian measure but with the additional potential term corresponding to the double trace operators [9]

$$Z_{MM}(g_s) = \frac{1}{VolU(N)} \int d_{Herm} M \exp \left( \frac{1}{2g_s} Tr M^2 + V(M) \right)$$

(59)

where

$$2V(M) = \sum_{k=1}^{\infty} a_k \sum_{s=0}^{2k} (-1)^s C^2_s Tr M^s Tr M^{2k-s}$$

(60)

where $a_k$ can be expressed in terms of Bernoulli numbers $a_k = \frac{B_{2k}}{k(2k)!}$. The perturbation theory in terms of the vacuum expectation values of the double trace operators calculated with the Gaussian measure can be developed similar to [9].

Let us emphasize that the very precise background geometry in the selfdual case can be naturally attributed to the residual supersymmetry of the quantum fluctuations in the selfdual case. It is natural to expect that the remarkable relations between two loop and one loop effective actions found in [44] can be related to the topological string interpretation found in our paper. This issue is under investigation now.

### 4.2 On the matrix model picture

Since we relate one loop effective action to the matrix integral the standard questions enherited from the matrix model technique can be reformulated in our case. It is instructive to compare the effective action in the selfdual background with the matrix model description of $N=1$ theory [11] and chiral effective action in QCD. We could expect two different matrix realizations; one is the large $N$ matrix model while the second is Kontsevich type model with finite size matrixes.
We shall indeed see two possible pictures in what follows and discuss some key ingredients like resolvent and loop equation on the matrix model side from the gauge theory perspective postponing more detailed analysis for the future work.

Let us first remind some points concerning the matrix model description of $N=1$ theory [11]. The corresponding Hermitian matrix model involves the tree superpotential

$$Z = \int dM \exp \frac{1}{g_s} Tr W_{\text{tree}}(M)$$

(61)

that is the potential of the matrix model coincides with the tree superpotential and the matrix $M$ is the image of the adjoint chiral field [11]. The effective superpotential can be expressed in terms of the partition function of this matrix model. The standard matrix model resolvent has the field theory counterpart

$$R(z) = Tr W^2 \frac{1}{z - \Phi}$$

(62)

where $\Phi$ is the adjoint chiral superfield. The Virasoro constraints get mapped into the Konishi anomalies and their generalizations [23, 24] on the gauge theory side moreover the loop equation has been mapped into the chiral ring relations in $N=1$ SYM theory with the adjoint matter [24]. The loop equation collecting all Virasoro constraints into a single equation yields the background geometry for the type B topological closed string.

In our nonsupersymmetric case we have no tree superpotential as well as the adjoint field at all so at the first sight it is unclear what substitutes the resolvent and anomaly relations in this case properly. We suggest that the counterpart of the matrix $M$ is just the Dirac operator $D(A)$ in the fermion case and $D^2(A)$ gets mapped into the $M^2$ in the scalar case. Hence the proper resolvent on the gauge theory side is

$$\frac{\delta S_{\text{eff}}}{\delta m}|_{m=z} = G_F(z) = Tr \frac{1}{D(A) - z}$$

(63)

for fermion and

$$G_S(z) = Tr \frac{1}{D^2 - z^2}$$

(64)

for scalars and trace is taken over the corresponding Hilbert space. To get the proper normalization on the matrix model side we could use the Casher-Banks relation for the chiral
condensate in the selfdual field

\[ \langle \overline{\Psi} \Psi \rangle \propto \rho(0) \propto \frac{f^2}{m} \]  

and take into account that

\[ G_F(z + i\epsilon) - G_F(z - i\epsilon) = \pi \rho(z) \]  

Hence it is natural to multiply the matrix model resolvent by the factor \( g_s^2 \).

Let us turn to the second ingredient of the matrix model namely the loop equations which have the chiral ring relations as a gauge theory counterpart in N=1 case. The loop equations can be derived from the change of variables in the matrix integral

\[ \delta M \propto f(M) \]  

and it is convenient to take \( f(M) = (z - M)^{-1} \). On the gauge theory side the corresponding variations involve the variation of the eigenvalues of the Dirac operator induced by the variation of the external field. However more convenient interpretation follows from the variation of the variable dual to the eigenvalue of the Dirac operator which is just the Schwinger parameter. The change of variables in the matrix integral can be related to the change of variables \( s \rightarrow g(s) \) in the path integral. Therefore the Ward identities in the matrix model can be translated in the dual representation to the invariance of the integral over the Schwinger parameter under the change of variable \( s \rightarrow g(s) \). Remind that our starting point was the identification of the Schwinger parameter as the coordinate in the background geometry. Therefore the Virasoro constraints can be rephrased as the invariance under the diffeomorphisms of the coordinate \( s \).

There are also another two matrix model realizations of effective actions involving the large N unitary matrices. The first one looks as \[ Z_{CS} = \int dU e^{1/g_s Tr(\log U)^2} \]  

and just this form makes the hidden quantum group structure in CS theory quite manifest. The second model is useful for the "crystallic" picture for Calabi- Yau and reads as \[ Z_{CS} = \int dU det \theta_{00}(U, q) \]  

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where
\[
\theta_{00}(e^{ix}, q) = \sum_m q^{m^2/2}e^{imx}
\] (70)

All matrix model realizations of the CS theory amount to the same answer and can be equally used in the selfdual case.

It was discussed recently that one could expect two different matrix realizations of the topological strings [29, 30, 31]. One model can be attributed to the theory on the large number of branes wrapped around the compact surface while the second Kontsevich type model involves the finite number of the noncompact branes which deform the complex geometry. Hence we could look for the second Kontsevich realization of the effective action in the selfdual field. To this aim it is convenient to use the relation to the c=1 model at the selfdual radius mentioned above. The corresponding Kontsevich models for this theory has been found in [33, 32] and reads as
\[
Z_{kon}(\mu, N) = \int dM e^{i\mu Tr M - (i\mu + N)tr log M}
\] (71)

The loop equation in this Kontsevich model or the corresponding chiral ring relation provides the conifold geometry where the type A model is defined on. Note that there is the natural deformation of the c=1 partition function by the operators corresponding to the tachyon modes. The emerging partition function with ”times” corresponding to the tachyonic modes included can be identified with the tau function of the Toda hierarchy. It would be interesting to consider the corresponding deformations of the effective action in the selfdual background. We plan to discuss this point elsewhere.

Let us emphasize that the matrix models we have discussed for the selfdual background have the same origin as ones suggested for the description of the Dirac operator spectrum in QCD (see [28], for a review). Once again there are two different matrix model realizations. The QCD is essentially nonabelian theory and the emergence of the large N matrix model can be considered as a kind of averaging over the instanton moduli space. This large N matrix model usually involves the Gaussian measure. However there is the second Kontsevich type representation in terms of unitary \(SU(N_f)\) matrixes representing the Goldstone pion modes
\[
Z_{QCD} = \int dU exp(\Sigma Tr(MU) + \nu log det U)
\] (72)
in the sector where \( \nu \) is the topological charge and \( \Sigma \) is the chiral condensate. The matrix resolvent has been identified with the resolvent of the Dirac operator in this case too. It is known that the resolvent obeys the algebraic equation defining the Riemann surface [28]. It has the following structure

\[
G^3(z) - 2zG(z) + G(z)(z^2 - \frac{1}{\Sigma^2}) - \frac{z}{\Sigma^2} = 0
\]  

(73)

and provides the geometry of the internal space for the low energy QCD similar to the discussion above.

### 4.3 Spin 1/2 case

Let us discuss the spinor case and start with mentioning the modifications which can be expected from the very beginning. First, note that the loop representation for the effective action implies that the spin factor corresponding to the WZW action in \( CP^3 \) manifold has to be taken into account. From the probe brane perspective the fundamental matter gets represented by D7 branes whose positions fix the masses of the particles and corresponding degrees of freedom amounted from the strings connecting D3 and D7 branes.

The modifications of the type A and type B pictures by the additional brane come as follows. In type A picture the matter representing brane yields the Wilson loop in Chern-Simons theory on \( S^3 \). Such Wilson loops have been discussed in [34] where the explicit answer for the corresponding partition function has been derived. On the B model side we have to insert the additional operators to the matrix model realization. Recall the expression for the effective action

\[
L_{eff} = \int_0^\infty \frac{ds}{s^3} e^{-sm^2}(esGc\text{th}(esG))^2
\]

(74)

and let us interpret it in the proper way. First remind that the factor \((esGc\text{th}(esG))^2\) is nothing but the product of two massless propagators in \( S^3 \). Just these propagators yield the Wilson loop in CS theory in type A picture.

In type B picture we deform the matrix model by the additional operators and it is quite
convenient to represent the effective action in the following way
\[ S_{\text{eff}}(m) = \langle \Psi_{2dYM}(s)|U(s,m)|\Psi_{2dYM}(s) \rangle \tag{75} \]

where we have exploited the relation between the particle propagators and the wave functionals in 2D YM theory on the disc. In this case the Schwinger parameter plays the role of the boundary holonomy which is the argument of the wave functional. Hence we arrive at the following picture; there are two discs with the single insertions on each and SU(2) connection on each which are connected by the cylinder. Each disc yields the corresponding wave function and the integration over the Schwinger parameter corresponds to the matrix element. Note that for the general spin J the integrand has the additional spin dependent terms
\[ (TrJ^2V(s))^2 = (\chi_J(s))^2 = \left( \frac{\sin(2J+1)s}{\sin s} \right)^2 \tag{76} \]

which fixes the structure of the inserted operator.

The natural question concerns the modification of the matrix model description in the spinor case. Hopefully in the one-loop case the answer is very simple - there is no need in the modification at all. The point is that the selfdual case enjoys the residual supersymmetry and the one loop effective actions are related as follows
\[ S_{1\text{-loop,spin}} = -2S_{1\text{-loop,scal}} + \left( \frac{ef}{8\pi} \right)^2 \log(m^2/\mu^2) \tag{77} \]

The second term in the r.h.s. amounts from the fermionic zero modes whose density in the self-dual fields is proportional to \((ef)^2\). It is responsible for the chiral condensate in the background field \([35]\)
\[ m < \bar{\Psi}\Psi > = \left( \frac{ef}{8\pi^2} \right)^2 \tag{78} \]

hence apart from zero modes the same matrix model can be used for the spinor case.

Finally it is worth questioning about the possible worldsheet description of the spinor effective action. To this aim it is necessary to identify the operator which has to be inserted on the topological string worldsheet. It turns out that it can be recognized in type A picture using the example treated in \([36]\). It was shown that the relevant deformation amounts to the multiplication of all string amplitudes by the factors
\[ \exp(-c \int dx \sum_{kl} \Lambda_k(x,\bar{x})\Lambda_l(x,\bar{x}) \frac{\partial}{\partial \theta_k} \frac{\partial}{\partial \theta_l}) \tag{79} \]
where $\theta$ is the angular variable along the isometry in $S^3$.

Consider two boundaries of the worldsheet localized in $S^3$ at branes at points $z_i = \gamma_i, \bar{\gamma}_i, \phi_i$ where the following string frame metric is implied here

$$ds^2 = Q_5(d\phi^2 + e^{2\phi}d\gamma)$$(80)

The quasiclassical expression for the deforming operator reads as [37]

$$\Lambda_i = -\frac{(\bar{\gamma}_i - \bar{x})e^{2\phi_i}}{1 + |\gamma_i - x|^2 e^{2\phi_i}}$$ (81)

If we assume that $\gamma_i = 0$ then the emerging integral over the target

$$\int d^2x \frac{(\bar{x})e^{2\phi_i}}{1 + |x|^2 e^{2\phi_1}} - \frac{(\bar{x})e^{2\phi_2}}{1 + |x|^2 e^{2\phi_2}}$$

amounts precisely to the factor $(-1 + (\phi_1 - \phi_2)\cot\gamma(\phi_1 - \phi_2))$ involved in the expression for the effective action in the electric field.

Let us comment on the meaning of operator $\Lambda$. It is chosen in such way that $\partial_2\Lambda$ is a primary operator of the worldsheet conformal algebra with dimension (0,1) as well as a primary of space-time conformal algebra with dimension (1,0). The insertion of the current $J(x)$ into the CFT correlator is equivalent to an insertion of the vertex operator

$$K(x) \propto \int dzk(z)\partial_2\Lambda(z, x)$$ (83)

in the string worldsheet, where $k(z)$ is the worldsheet current [37]. The worldsheet Lagrangian corresponding to this CFT marginal deformation is nonlocal

$$\delta S \propto \int dx \int dz_1 \int dz_2 k(z_1)k(z_2)\partial_{z_1}\Lambda(z_1, x)\partial_{z_2}\Lambda(z_2, x)$$ (84)

Let us emphasize that the worldsheet theory above involves the nonlocal operators corresponding to the double trace deformations so we meet here the touching surfaces once again.

### 4.4 Magnetic selfdual field

In the magnetic selfdual background the $S^3$ geometry is substituted by the $AdS_3$ and the propagators of the modes in $AdS_3$ are involved into the effective action in the type A picture.
As we have shown before the propagators in $AdS_3$ are related to the integrated partition function of 2D SL(2,R) Yang-Mills theory. Let us explain now that this relation provides a proper type B picture.

To this aim let us remind that the topological SL(2,R) Yang-Mills theory corresponds to the $AdS_2$ gravity and argue that $AdS_2$ submanifold in $AdS_3$ is naturally involved into the problem via the relation of 2D SL(2,R) YM theory with perturbed topological gravity in two dimensions. It is known [38] that at $g^2 = 0$ SL(2,R) YM theory is equivalent to the topological Jackiw-Teitelboim gravity with the action

$$S = \int d^2x \sqrt{det g_{\mu\nu}} \left( R(g) - \Lambda \right) \eta , \quad (85)$$

where $\eta$ is the dilaton field and $\Lambda$ is the cosmological constant. Solutions to the classical equations of motion give rise to the $AdS_2$ gravity coupled to the dilaton.

The gravity degrees of freedom are zweibein $e^a(a = 0, 1)$ and the spin connection $\omega$ which can be combined into the gauge field

$$A = E^a P_a + \omega L \quad (86)$$

where $P_a$ and $L$ are the generators of the translations and Lorentz transformations respectively. Due to the nonvanishing cosmological constant the Poincare algebra is deformed to

$$[\Lambda, P_a] = \epsilon^a_b P_b \quad [P_a, P_b] = \epsilon_{ab} \Lambda L \quad (87)$$

where $\Lambda$ is the cosmological constant. This space can be identified with the SL(2,R) algebra. The gauge curvature reads as

$$F = d\omega + \Lambda e \wedge e \quad (88)$$

and the condition of vanishing curvature is nothing but the equation of motion in the Jackiw-Teitelboim model

$$R = 2\Lambda \quad (89)$$

where $R$ is the Ricci curvature scalar. Hence the topological SL(2,R) 2D Yang-Mills theory

$$S_{top} = \int Tr \phi F \quad (90)$$
describes the dilaton gravity in two dimensions. The action enjoys the evident gauge invariance which is equivalent to the general coordinate invariance in two dimensions.

The theory has no Hamiltonian and the Gauss law constraint

$$\partial \phi + [A_1, \phi] = 0$$

(91)

generating the gauge transformations has to be imposed as equation on the physical gauge invariant states in the Hilbert space. It is clear that any functional of Wilson loop observable obeys the quantum constraint which plays the role of the Wheeler-De-Witt equation

$$[\partial_x \frac{\delta}{\delta A^i_1(x)} + \epsilon_{ij} A^j_1(x) \frac{\delta}{\delta A^i_1(x)}] \Psi(A) = 0$$

(92)

The physical SL(2,R) Yang-Mills theory whose partition function is involved into the effective action in the magnetic field corresponds to the insertion of the operator

$$\delta L = Tr \phi^2$$

(93)
in the action of worldsheet theory. The origin for this term to appear has been explained in the similar context in [40]. Integration over the area amounts to the peculiar wave functional in 2D gravity depending on the boundary holonomy.

To match with the effective action picture we have to identify the boundary holonomy in terms of the boundary paths. The comparison with the propagator immediately gives

$$tr g = \cosh(\epsilon H s)$$

(94)
hence we have to check if this relation is consistent with the worldvolume interpretation. Let us recall that the holonomy of the SL(2,R) connection on a disc yields the length of the boundary via the formula [39]

$$tr_{1/2} P exp \int_C A = 2 \cosh \frac{l(C)}{2}$$

(95)

Therefore we have to check if the boundary length is proportional to $\epsilon Es$ indeed. Qualitatively the proportionality to the Schwinger parameter is correct since it defines the length of the boundary trajectory.
That is we have arrived at the proper gluing picture for magnetic selfdual case as well. Indeed we have qualitatively represented the effective action of the spinor particle as gluing two $AdS_2$ manifolds with fixed boundary length perturbed by $Tr\phi^2$ operator each

$$S_{\text{eff,magn}}(m) = \langle \Psi_{SL(2,R)}(s)|U(s,m)|\Psi_{SL(2,R)}(s) \rangle$$ (96)

Note that since $AdS_2$ has two boundaries there are some concerns on this point which have to be clarified.

## 5 One-loop low energy MHV amplitudes

In this Section as a byproduct of the topological string picture for the effective action in the selfdual field found in this paper we shall develop the interesting interpretation of one-loop low energy maximal helicity violating (MHV) photon amplitudes. The idea to exploit the effective actions to derive MHV amplitudes at one and two loops was used in [12, 44]. The key point is to consider the limit when all photon momenta are small compared to the mass of the particle in the loop. In this limit the effective action serves as the generating function for the amplitude with the arbitrary number of the external photon legs. To derive the amplitude from the effective action one introduces the momenta and polarization for each external leg

$$F_{\mu\nu}^i = k_{i\mu}^i \epsilon_{i\nu}^i - k_{i\nu}^i \epsilon_{i\mu}^i$$ (97)

It is convenient to define

$$F_t = \sum_{i=1}^{N} F_i$$ (98)

and expand the one loop effective action in powers of $F_t$. To get the amplitude with $N$ external photons the effective action has to be expanded to the $N$-th power in the external field $F_t$ and only terms which involve each $F_i$ linearly are kept. To obtain the MHV amplitudes when polarizations of all photons are ”+” or ”-” it is necessary to consider the effective action in the selfdual background since selfdual fields have fixed chirality [41]. The expansion of the effective action amounts to the following answer for the $N$ photon low energy amplitudes at one loop in QED

$$\Gamma^1(k, \epsilon) = -\frac{2(2e)^N}{(4\pi)^2 m^{2N-4} c^1(N/2)\chi N}$$ (99)
where
\begin{equation}
c^1(n) = -\frac{B_{2n}}{2n(2n-2)}
\end{equation}

$B_n$ are Bernoulli numbers and $\chi_N$ is kinematical factor which can be presented in the spinor helicity notations
\begin{equation}
\chi_N = \frac{(N/2)!}{2^{N/2}} ([1^2][3^2][5^2] \ldots [(N-1)^2][N^2] + \text{perm})
\end{equation}

Similar expression can be found for scalar electrodynamics and the two-loop generalization has been derived [44].

Turn now to the topological string interpretation of such MHV amplitudes. To this aim let us use the relation to large $N$ CS theory discussed above and expand $\log Z_{CS}$ in inverse powers of level $k$. The $N$-th term $\mathcal{F}_N$ in the expansion corresponds to the $N$-th term in the expansion of the effective action in the selfdual background that is it corresponds to the amplitude with $N$ external legs. On the other hand CS theory can be mapped into the topological string amplitudes and the $l$-th term in the expansion of the CS partition function corresponds to the genus zero topological string amplitude with $l$ holes corresponding to the insertion of the vertex operators.

The situation has many common features with the recent approach to the calculation of the MHV amplitudes in N=4 SYM theory [13]. In nonabelian case the all ”+” MHV one-loop massless MHV amplitudes were calculated both in YM theory [42] and massless QED [43]. These amplitudes are related to the amplitudes of topological type B strings which are localized on the holomorphic genus zero curves in twistor target space [14]. The Minkowski coordinate parametrizes the moduli of the curve in the twistor space. Moreover as amplitude is localized on the complex line in the twistor space it corresponds to the point-like vertex in the Minkowski space.

In our abelian example the answer can be reformulated in terms of the topological type B string amplitudes in Calabi-Yau manifold once again. In our case the mirror of $T^*S^3$ plays the role of the ”twistor” manifold and the mass parametrizes the geometry. Hence we have here some analogue of the ”twistor” manifold for nonsupersymmetric case involving the mass parameter. In this low-energy limit we have the transparent localization of the amplitudes to
the point in the Minkowski space. Indeed the effective action in the selfdual field serves as the
generating function for all "+" low-energy MHV amplitudes. On the other hand in the low
energy limit of the large mass to get the amplitudes we just expand the effective action in terms
of the set of local operators multiplied by $m^{-2N}$. In the first quantized formulation it can be
thought of as the expansion of the Wilson loops in terms of the local operators. Therefore the
low-energy amplitudes are presumably localized on the complex lines in the "twistor" manifold.

The advantage of the massive case here is that the topological string picture can be ex-
tended to the higher loops immediately. We believe that the abelian low energy one-loop MHV
amplitudes could serve as a good toy model for the topological string interpretation of the
nonabelian MHV amplitudes in nonsupersymmetric case. Note also that we have type A pic-
ture for the amplitude as well described by large N Chern-Simons theory or closed topological
theory upon the large N geometrical transition. In particular the mass parameter effectively
measures the Kahler class of $P^1$ in the resolved conifold. Since we have developed the matrix
model description for type B topological strings corresponding to the effective actions it can
be used for the calculation of the one-loop MHV diagrams. The k-th terms of the expansion of
the matrix model partition function correspond to the one-loop abelian MHV amplitudes with
k external legs.

6 Imaginary part

In this Section we shall briefly comment on the relation of the $S^3$ geometry with the Schwinger
pair production. In the electric field the effective action develops the imaginary part which is
responsible for the probability of the pair creation. The probability looks as [45]

$$w = (eE)^2 \frac{2s + 1}{8\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{(2s+1)(n+1)}}{n^2} \exp \left( -\frac{n\pi M^2}{eE} \right)$$

(102)

where $s = 0$ or $1/2$ is the particle spin, $e$ – charge and $E$ – constant electric field. The
 corresponding stringy generalization has been found in [48] and the relevant brane geometry in
[47].
Let us recall the geometrical meaning of the Schwinger parameter in the calculation of the imaginary part and discuss the leading exponential factors first. The imaginary part in the path integral is saturated by the classical trajectories in the Euclidean space. Since upon the Wick rotation the electric field gets transformed into the magnetic one the trajectories are circles. To get the radii of the circles one could first integrate over the proper time in the path integral which amounts to the Schwinger parameter \( s = L/m \), where \( L \) is the length of the contour. The radii can be obtained by the minimization of the effective action for the negative radial mode (see, for instance, [46])

\[
S_{\text{eff}} = mL - eEA
\]

where \( L \) and \( A \) are the perimeter and the area of the closed particle Euclidean trajectory. Then the radius can be found and the probability up to the exponential accuracy reads as

\[
w \sim \exp(-S_{\text{eff}}^\text{min})
\]

Thus the Schwinger parameter takes the quantized values at the leading approximation.

Let us examine now the probability of this process from the \( S^3 \) geometry viewpoint. It is clear that the imaginary part in the effective action amounts from poles in the Schwinger parameter integration located at

\[
eEs_k = 2\pi k
\]

Remind that we have related the Schwinger parameter with the group coordinates via the relation \( trg = cosesH \). Remarkably enough these poles correspond precisely to the possible quantized positions of \( S_2 \) branes in \( S_3 \) [49]. These branes are stable due to the fluxes on their worldvolumes. Hence we could suggest that these \( SU(2) \) branes are involved into the tunneling process. The possible mechanism mentioned in [49] deals with the decay of D0 branes located at the group unit to the extended brane located at \( s_k \).

After the Euclidean rotation to \( AdS_3 \) the tunneling process can be embedded into the string picture as follows. The Euclidean circle above is just one of the boundaries of the cylinder extended along the radial \( AdS_3 \) coordinate while the length of the cylinder is proportional to the particle mass hence the area of the cylinder is just the first term in the effective action [47]. On the other hand the same geometry can be considered as the propagation of the rigid closed
string in the different channel. Then the Euclidean particle trajectory is just the fixed time closed string. Therefore in this channel the poles in the $AdS_3$ propagator can be identified with the closed string states.

It could be questioned how the arguments above can be extended to the strong coupling region and match the calculation of the vacuum expectation values of the Wilson loops in the strong coupling limit [50]. In that case the relevant surface has the disc topology with Wilson loop boundary. To get transition from the cylinder to disc topology let us remind the interpretation of the critical electric field found in [47]. Namely it was shown that when the finite string tension is taken into account the minimal surface deviates from the cylinder form and at the critical field

$$eE = \frac{1}{\alpha'}$$

(106)

the surface shrinks at one point transforming into two disconnected surfaces. Hence in the closed string channel the strong coupling calculation of the Wilson loops corresponds to the overcritical electric field.

7 On the geometry of paths

Since the effective actions can be represented via the sum over the closed paths weighted with the proper factors it is natural to ask which paths really dominate the sum. Here we would like to present a couple of arguments which imply that the paths with the cusps are relevant. The first argument involves the observation made above that the double-trace deformations of the worldsheet action turns out to be important. Moreover the matrix model describing the effective action in the selfdual case involves the double trace terms in the potential as well. It is wellknown that the double-trace deformations correspond to the touching of the string worldsheets.

The second argument involves the restrictions on the invariant lengths of the particle trajectories. The simplest example which one could have in mind involves the temperature. Namely, consider the classical particle trajectories in the Euclidean space-time in the electric field. If there is no temperature then the classical trajectories are just the circles. When the tempera-
ture is switched on the periodicity condition in the Euclidean time should be imposed. Hence there are two different situations with the radius of the circle corresponding to the classical motion is larger or smaller than $T/2$. In the former case we have specific situation when the circle becomes two arcs instead and the cusps emerge immediately.

In our case there is no temperature and therefore periodicity in the Euclidean time direction. However we have to integrate over the invariant lengths of trajectories and if the total length of the trajectory is fixed by the value of the Schwinger parameter then it imposes the restriction on the contour. Hence we are tempting to speculate that in the Euclidean geometry we have two arcs once again and the angle at the cusp is fixed by the length on the trajectory.

The last argument concerns the two loop calculation of the effective action which generically provides the information about one loop anomalous dimensions of operators. It is convenient to compare the structure of the two-loop answer with the cusp anomalous dimensions representing the renormalization of the Wilson loop with cusps. It was calculated long time ago [51] and reads as

$$\Gamma_{\text{cusp}}(\alpha, \theta) = \alpha sN(\theta \coth \theta - 1) + O(\alpha^2)$$

It corresponds to the renormalization of the Wilson loop with cusp angle $\theta$ and serves as the generating function for the anomalous dimensions of the operators with the large Lorentzian spin.

Let us look more carefully at the two loop answer and focus for a moment on the theory with $N=2$ SUSY. Effective action in the SUSY case has the following structure in two loops [52]

$$S_{\text{eff,2loop}} = g^4 \int ds f(s)(es \Psi \coth(es \Psi))$$

where $f(s)$ is the some polynomial and $\Psi$ is $N=2$ superfield. One immediately recognizes the cusp anomalous dimension which implies that the renormalization of the Wilson loop with the cusp indeed is relevant. On the other hand the two-loop effective action in the selfdual background can be expressed through one-loop answer [44] providing one more argument along this line.

We have argued above that it is natural to conjecture that the contours with cusps are
relevant for the representation of the effective action in the first quantized picture. On the other hand we have found some counterpart of the cusps in the dual stringy picture via the double trace operators. Of course the arguments above are quite tentative hence the additional clarification of this issue is needed.

8 Discussion

In this paper we have analyzed to what extend the one-loop effective actions in the nonsupersymmetric theory feel the higher dimensional background geometry. This is among the first steps which would provide the stringy picture for the perturbative regime in the nonsupersymmetric gauge theory. It turns out that the effective actions amount to the relatively precise background geometry which depends on the choice of the abelian background field. Purely electric field feels the SU(2) part of background, purely magnetic case involves the $AdS_3$ geometry while the selfdual background fixes the full Calabi-Yau threefold geometry. The key point of such identification is the interpretation of the Schwinger parameter emerging in the loop calculation as the coordinate in the background geometry. Then the integrands in the loop integrals are nothing but the propagators of the different massive or massless modes in this geometry.

The most transparent picture emerges for the selfdual background when the full three dimensional complex manifold has been determined. It turns out that the link to the large N CS theory fixes it to be $T^*S^3$ or its mirror. The effective action in this case can be related to the topological string amplitude in type A or B models. Moreover it turns out to be possible to develop the matrix model description of the selfdual case which is some sence can be considered as the counterpart of Dijkgraaf-Vafa picture for the nonsupersymmetric case. The important point is that at least one matrix model involves a set of double trace operators which differ the model from the N=1 SUSY case.

As a byproduct we have found the interesting interpretation of the abelian MHV loop amplitudes in terms of the topological type B strings. The target space for the topological strings has been defined and the amplitudes can be calculated or from the perturbative expansion in
CS theory on the A model side either in the matrix model on the B model side. However in spite of the evident similarity with the twistor type picture for the nonabelian massless case the additional work is needed to clarify the possible interpretation of the mirror to $T^*S^3$ as a kind of the twistor manifold. Note that the phenomena of the generation of the local vertexes for the MHV amplitudes is transparent in the low energy limit.

Our analysis strongly suggests that the closed loops with self intersections which are holographically dual to the multitrace operators in the bulk are important in the path integral. Moreover the geometry is encoded in the loop equations for the Wilson loops and Ward identities for the integral over Schwinger parameter. We also discuss some features of the nonperturbative pair production in terms of background geometry. Let us emphasize that the nontrivial background geometry discovered behind the one-loop effective actions can not be globally seen at any finite order in the external field. However it is important that the gauge coupling can be made arbitrary small hence the emergence of the nontrivial background is a weak coupling phenomena.

In our abelian nonsupersymmetric case the external field plays the role similar to the composite field $S = TrW^2$ in N=1 theory. For the nonabelian nonsupersymmetric case the role of S is played by the composite colorless field $\sigma = TrF^2$ so the derivation of the Veneziano-Yankielowicz effective Lagrangian [53] via matrix model representation of the resolved conifold can be paralleled with the effective potential in nonsupersymmetric case found in [54]. Note also that $\beta$ function of the theory can be derived from the effective action in the strong field limit hence the difference between the signs in the abelian and nonabelian cases acquires the geometrical interpretation. It would be interesting to combine our geometric interpretation of one-loop answer with the explicit instanton counting in N=2 case [55].

It is clear that there are a lot of questions to be answered within our approach. Some of them have been already mentioned in the body of the paper. The most immediate one concerns the nonabelian generalization of the geometrical picture discussed in this work. We shall discuss the associated one-loop nonabelian phenomena including Nilesen-Olesen instability and low-energy effective Lagrangian in a forthcoming publication. We shall also discuss the relation of the higher loop corrections to the effective actions to the topological string interpretation of the
low energy abelian MHV amplitudes at higher loops.

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9 Appendix A

We shall argue now that the massless propagator in the $\text{AdS}_3$ can be formulated in the second quantized 2D worldsheet picture as two point function (see the discussion in [56]). In this approach instead of the inserting of the double trace operator on the worldsheet we choose the Rindler vacuum state. To this aim consider the two dimensional worldsheet field theory with the equation of motion

$$ (\partial_t^2 - \partial_x^2)\phi + m^2 \phi = 0 \quad (109) $$

whose solution has the following mode expansion

$$ \phi(x, t) = \int \frac{d\beta}{2\pi} (a^*(\beta)e^{-im(x\sinh\beta - t\cosh\beta)} + a(\beta)e^{im(x\sinh\beta - t\cosh\beta)}) \quad (110) $$

It is convenient to introduce Rindler coordinates

$$ x = r \cosh \theta, \quad t = r \sinh \theta $$

$$ -\infty < \theta < +\infty, \quad 0 < r < +\infty \quad (111) $$

in the space-time region $x > |t| > 0$. Let us perform the following Laplace transform with respect to the radial coordinate

$$ \lambda_\theta(\alpha) = \int dr e^{imr \sinh \alpha} (-\frac{1}{r} \partial_r + im \cosh \alpha) \phi(r, \theta) \quad (112) $$

Then the commutation relation for the Laplace transformed field reads as

$$ [\lambda(\alpha_1), \lambda(\alpha_2)] = i\tanh(\alpha_1 - \alpha_2)/2 \quad (113) $$
and the Hilbert space is spanned by vectors $a(\beta_n)\ldots a(\beta_1)|vac>$ where the vacuum state is defined as

$$a(\beta)|vac> = 0 \quad <vac|a^+(\beta) = 0$$ (114)

One can introduce the two point function

$$F(\alpha_1 - \alpha_2) = <vac|\lambda(\alpha_1)\lambda(\alpha_2)|vac>$$ (115)

and it appears that the explicit calculation amounts to the following answer [57]

$$F(\alpha - i\pi) = -\frac{1}{\pi}\alpha/2\coth(\alpha/2) + \text{singular terms}$$ (116)

The singular terms cancel in the difference $F(\alpha - i\pi) - F(0)$ which coincides with the propagator of the massless mode in $AdS_3$.

10 Appendix B

Let us consider the Lagrangian of the scalar field in the constant magnetic field

$$L = - (\partial_\mu + ieA_\mu)\phi^*(\partial_\mu - ieA_\mu)\phi$$ (117)

and take $A_1 = A_2 = A_4 = 0$, $A_2 = Hx_1$. Then the eigenvalue equation becomes

$$-E^2\phi_E = (\Delta - e^2H^2x_1^2 - 2ieHx_1^2\frac{\partial}{\partial x_2})\phi_E$$ (118)

yielding the solution to the eigenvalue problem

$$E^2 = 2eH(n + \frac{1}{2} + k_3^2)$$ (119)

The same calculation can be done for arbitrary spin and the answer will be the following

$$E^2 = 2eH(n + \frac{1}{2} + S_3)$$ (120)

hence the sum of all zero-point energies becomes

$$F = c\int dk_3 \sum_{n,S_3} \sqrt{2eH(n + \frac{1}{2} + S_3) + k_3^2}$$ (121)
In general this sum is divergent, however performing the $\epsilon$ regularization we obtain

$$F = \mu^{-2\epsilon}(\text{const}) \int_0^\infty d\tau \tau^{-2-\epsilon} \sum_{n,S_3} e^{-i\tau 2\epsilon H(n+1/2+S_3)}$$

amounting after the rotation of the contour to

$$F = \mu^{-2\epsilon}(\text{const}) \int_0^\infty ds \frac{\sinh(2S+1) eHs}{s^{2+\epsilon} \sinh^2(eHs)}$$

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