TOTAL DOMINATOR COLORING NUMBER OF MIDDLE GRAPHS

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Abstract. A total dominator coloring of a graph $G$ is a proper coloring of $G$ in which each vertex of the graph is adjacent to every vertex of some color class. The total dominator chromatic number of a graph is the minimum number of color classes in a total dominator coloring. In this article, we study the total dominator coloring on middle graphs by giving several bounds for the case of general graphs and trees. Moreover, we calculate explicitly the total dominator chromatic number of the middle graph of several known families of graphs.

Keywords: Total dominator coloring, Total dominator chromatic number, Total domination number, Middle graph.

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1. Introduction

All graphs considered in this paper are non-empty, finite, undirected and simple. For standard graph theory terminology not given here we refer to [16]. For a simple graph $G = (V(G), E(G))$ we will denote the open neighbourhood and the closed neighbourhood of a vertex $v \in V(G)$ by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively, the minimum and maximum degree of $G$ by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively, and the induced subgraph by $S \subset V(G)$ by $G[S]$.

The notion of domination is well studied in graph theory and the literature on this subject has been surveyed in the two books [2, 4], see also [15] and [12]. A famous generalization of domination is the notion of total domination, see [6], [14] and [13]. A total dominating set $S$ of a graph $G$ is a subset of $V(G)$ such that for each vertex $v$ we have $N_G(v) \cap S \neq \emptyset$. The total domination number $\gamma_t(G)$ of $G$ is the minimum cardinality of a total dominating set of $G$.

Similarly to the notion of domination, also the notion of coloring of graphs has been intensively studied.
Definition 1.1. A proper coloring of a graph $G$ is a function from $V(G)$ to a set of colors such that any two adjacent vertices have a different color. The chromatic number $\chi(G)$ of $G$ is the minimum number of colors needed in a proper coloring of $G$.

In a proper coloring of a graph, a color class is a set consisting of all those vertices assigned the same color. If $f$ is a proper coloring of $G$ with the coloring classes $V_1, \ldots, V_\ell$ such that every vertex in $V_i$ has color $i$, we write simply $f = (V_1, \ldots, V_\ell)$.

Similarly to the notion of chromatic number, we can recall the notion of edge chromatic number.

Definition 1.2. The edge chromatic number, sometimes also called the chromatic index, of a graph $G$ is the smallest number of colors necessary to color each edge of $G$ such that no two edges incident on the same vertex have the same color. In other words, it is the number of distinct colors in a minimum edge coloring and it is denoted by $\chi'(G)$.

Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources and it has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization.

Motivated by the relation between coloring and domination, the notion of total dominator colorings was introduced in [7]. For more information see [5, 8–11].

Definition 1.3 ([7]). A Total Dominator Coloring, briefly TDC, of a graph $G$ with a positive minimum degree is a proper coloring of $G$ in which each vertex of $G$ is adjacent to every vertex of some color class. The total dominator chromatic number $\chi^t_d(G)$ of $G$ is the minimum number of color classes in a TDC of $G$.

Example 1.4. Consider $G = P_4$ with $V(G) = \{v_1, \ldots, v_4\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4\}$. If we consider $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4\}$, then $f = (V_1, V_2)$ is a proper coloring of $G$ but it is not a total dominator coloring, since $v_1$ is not adjacent to every vertex of some color class. On the other hand, if we consider $V_1 = \{v_1, v_4\}, V_2 = \{v_2\}$ and $V_3 = \{v_3\}$, then $f = (V_1, V_2, V_3)$ is a total dominator coloring of $G$.

Definition 1.5. Let $f = (V_1, \ldots, V_\ell)$ be a total dominator coloring of a graph $G$. If a vertex $v \in V(G)$ satisfies $V_i \subseteq N_G(v)$, then $v$ is called a common neighbour of $V_i$ or we say that $V_i$ totally dominates $v$. In this case, we write $v \succ_t V_i$, otherwise, we write $v \not\succ_t V_i$.

The set of all common neighbours of $V_i$ with respect to $f$ is called the common neighbourhood of $V_i$ in $G$ and denoted by $CN_G(V_i)$ or simply by $CN(V_i)$. A vertex $v$ is called a private neighbour of $V_i$ with respect to $f$ if $v \succ_t V_i$ and $v \not\succ_i V_j$ for all $j \neq i$. Moreover, for any total
dominator coloring $f = (V_1, \ldots, V_\ell)$ of a graph $G$, we have
\[
\bigcup_{i=1}^{\ell} CN(V_i) = V(G). \tag{1}
\]

In [3], the authors introduced the notion of the middle graph $M(G)$ of a graph $G$ as an intersection graph on $V(G)$.

**Definition 1.6.** Let $G = (V(G), E(G))$ be a simple graph. The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices $x, y$ in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds

1. $x, y$ are in $E(G)$ and $x, y$ are adjacent in $G$.
2. $x$ is in $V(G)$, $y$ is in $E(G)$, and $x, y$ are incident in $G$.

It is obvious that $M(G)$ contains the line graph $L(G)$ as induced subgraph, and that if $G$ is a graph of order $n = |V(G)|$ and size $m = |E(G)|$, then $M(G)$ is a graph of order $n + m$ and size $2m + |E(L(G))|$ which is obtained by subdividing each edge of $G$ exactly once and joining all the adjacent edges of $G$ in $M(G)$.

In order to avoid confusion throughout the paper, we fix a “standard” notation for the vertex set and the edge set of $M(G)$. Assume $V(G) = \{v_1, \ldots, v_n\}$, then we set $V(M(G)) = V(G) \cup M$, where $M = \{m_{ij} \mid v_i v_j \in E(G)\}$ and $E(M(G)) = \{v_i m_{ij}, v_j m_{ij} \mid v_i v_j \in E(G)\} \cup E(L(G))$.

The goal of this paper is to study the total dominator chromatic number of middle graphs. In Section 2, we describe some useful bounds involving total dominator chromatic number. In Section 3, we prove a series of lemmas that will play an important role in the rest of the paper. In Section 4, we calculate the total dominator chromatic number of the middle graph of several known families of graphs. In Section 5, we describe bounds for the total dominator coloring number of the middle graph of trees.

2. Useful bounds

We start this section by recalling some known bounds for total dominator coloring numbers.

**Theorem 2.1 ([7]).** For any connected graph $G$ of order $n$ with $\delta(G) \geq 1$,
\[
\max\{\chi(G), \gamma_t(G), 2\} \leq \chi_d^t(G) \leq n.
\]
Furthermore, $\chi_d^t(G) = 2$ if and only if $G$ is a complete bipartite graph, and $\chi_d^t(G) = n$ if and only if $G$ is a complete graph.

**Theorem 2.2 ([7]).** For any connected graph $G$ of order $n$ with $\delta(G) \geq 1$,
\[
\chi_d^t(G) \leq \gamma_t(G) + \min\{\chi(G[V(G) - S]) \mid S \text{ is a min-TDS of } G\},
\]
Moreover, \( \chi_t^d(G) \leq \gamma_t(G) + \chi(G) \).

In Theorem 2.2 we have to consider a minimum in order to get a better bound. In fact, if we consider \( G = P_6 \) with vertex set \( \{v_1, \ldots, v_6\} \), then \( S_1 = \{v_2, v_3, v_4, v_5\} \) and \( S_2 = \{v_1, v_2, v_5, v_6\} \) are two minimal TDS sets of \( P_6 \), with \( \chi(G[V(G) - S_1]) = 1 \) and \( \chi(G[V(G) - S_2]) = 2 \).

By Theorem 2.1, we have the following result.

**Theorem 2.3.** For any connected graph \( G \) of order \( n \geq 2 \) and size \( m \),
\[
\max \{\chi(M(G)), \gamma_t(M(G))\} \leq \chi_t^d(M(G)) \leq n + m - 1.
\]

From [7], it follows that for any graph \( G \), with connected components \( G_1, \ldots, G_w \), which has no isolated vertex, we have
\[
\max_{i=1}^w \{\chi_t^d(G_i)\} + 2w - 2 \leq \chi_t^d(G) \leq \sum_{i=1}^w \chi_t^d(G_i).
\]

Moreover, since \( M(G) = M(G_1) + \cdots + M(G_w) \), we obtain the following result.

**Theorem 2.4.** For any graph \( G \) with connected components \( G_1, \ldots, G_w \) which has no isolated vertex, we have
\[
\max_{i=1}^w \{\chi_t^d(M(G_i))\} + 2w - 2 \leq \chi_t^d(M(G)) \leq \sum_{i=1}^w \chi_t^d(M(G_i)).
\]

Therefore, it is sufficient to verify the total dominator chromatic number of connected graph. The next theorem describes bounds for the total domination number of the middle graph.

**Theorem 2.5 ( [13]).** Let \( G \) be a connected graph with \( n \geq 3 \) vertices. Then
\[
\left\lceil \frac{2n}{3} \right\rceil \leq \gamma_t(M(G)) \leq n - 1.
\]

As an immediate consequence of Theorems 2.3 and 2.5, we have the following result.

**Corollary 2.6.** Let \( G \) be a connected graph with \( n \geq 3 \) vertices. Then
\[
\chi_t^d(M(G)) \geq \left\lceil \frac{2n}{3} \right\rceil.
\]

**Problem 2.7.** Classify all connected graphs \( G \) such that \( \chi_t^d(M(G)) = \left\lceil \frac{2n}{3} \right\rceil \).

## 3. First general results

We start the section by recalling the notion of an independent set of vertices that is closely related to the notion of coloring.
**Definition 3.1.** An independent set is a set of vertices in a graph $G$ that no two of which are adjacent. A maximum independent set is an independent set of largest possible size for a given graph $G$. This size is called the independence number of $G$ and it is denoted by $\alpha(G)$.

**Lemma 3.2.** For any connected graph $G$ of order $n \geq 2$, we have

$$\alpha(M(G)) = n.$$  

*Proof.* Let $S$ be an independent set of $M(G)$. Since $V = \{v_1, \ldots, v_n\}$ is an independent set of $M(G)$, then $\alpha(M(G)) \geq n$. On the other hand, consider $M = \{m_{ij}|v_iv_j \in E(G)\}$ and $t = |M \cap S|$. Since if $m_{ij} \in S$, then $v_i, v_j \notin S$, we have $|V \cap S| \leq n - t$, and hence that $|S| = |V \cap S| + |M \cap S| \leq n - t + t = n$. As a consequence, $\alpha(M(G)) = n$. □

We can now state several lemmas that will play an important role in the rest of the paper. Assume that $G$ is a simple graph, $V(M(G)) = V \cup M$, where $V = V(G) = \{v_1, \ldots, v_n\}$, and $M = \{m_{ij}|v_iv_j \in E(G)\}$.

**Lemma 3.3.** Let $f = (V_1, \ldots, V_{\ell})$ be a TDC of $M(G)$. Then

1. For any vertex $v_i \in V$, if $v_i \succ_{i} V_k$ for some $1 \leq k \leq \ell$, then $|V_k| = 1$ and $V_k = \{m_{ij}\}$ for some $j$.
2. For any $m_{ij} \in M$, if $m_{ij} \succ_{i} V_k$ for some $1 \leq k \leq \ell$, then $1 \leq |V_k| \leq 2$.
3. For any $1 \leq k \leq \ell$, we have $|V_k| \leq \alpha(M(G)) = n$.

*Proof.* 1) If $v_i \succ_{i} V_k$, then $V_k \subseteq N_{M(G)}(v_i) \subseteq M$. This implies that if $|V_k| \geq 2$, then there exist $j \neq r$ such that $m_{ij}, m_{ir} \in V_k$, but this is impossible by the definition of TDC.

2) If $m_{ij} \succ_{i} V_k$, then $V_k \subseteq N_{M(G)}(m_{ij})$. This implies that if $|V_k| \geq 3$, then there exist $s \neq r$ such that $m_{is}, m_{ir} \in V_k$ or there exists $p \neq j$ such that $v_i, v_{ip} \in V_k$. However, both cases are impossible by the definition of TDC.

3) This is a consequence of the definition of independent set and Lemma 3.2. □

**Lemma 3.4.** Let $f = (V_1, \ldots, V_{\ell})$ be a TDC of $M(G)$. Then

1. if $|V_i| \geq 3$, then $|CN(V_i)| = 0$.
2. $\bigcup_{|V_i| \leq 2} CN(V_i) = V(M(G))$.

*Proof.* 1) It is a consequence of the fact that if $v \in V(M(G))$ and $v \succ_{i} V_k$, then $|V_k| \leq 2$, by Lemma 3.3.

2) It is a direct consequence of relation (1) and the previous part of the lemma. □

Given $f = (V_1, \ldots, V_{\ell})$ a TDC of $M(G)$, we will denote by $A_i = \{V_k \mid |V_k| = i\}$.
Lemma 3.5. Let \( f = (V_1, \ldots, V_\ell) \) be a TDC of \( M(G) \). Then

1. \(|A_1| \geq \left\lceil \frac{m}{2} \right\rceil \).
2. \(|A_1| + |A_2| \leq \ell \).

Proof. 1) Since \( f \) is a TDC, then for every \( i = 1, \ldots, n \), there exists \( 1 \leq k \leq \ell \) such that \( v_i \succ_t V_k \). By Lemma 3.3, \( V_k = \{m_{ij}\} \) for some \( j \). Now since \( v_i \not\succ_t \{m_{ij}\} \) for \( t \neq i, j \), then for every two vertices \( v_i, v_j \), there exists at least one \( k \) such that \( |V_k| = 1 \), and hence \( |A_1| \geq \left\lceil \frac{m}{2} \right\rceil \).

2) It is a direct consequence of \( |A_1| + |A_2| \leq \sum_{i \geq 1} |A_i| = \ell \). \( \square \)

Since the middle graph \( M(G) \) contains the line graph \( L(G) \), we can relate their total dominator chromatic numbers.

Theorem 3.6. Let \( G \) be connected graph of order \( n \geq 2 \), size \( m \geq 2 \) and \( \delta(G) \geq 1 \). Then

\[
\chi^d(M(G)) \geq \chi^d(L(G)).
\]

Proof. To fix the notation, assume that \( V(G) = V = \{v_1, \ldots, v_n\} \). Then \( V(M(G)) = V \cup M \), where \( M = V(L(G)) = \{m_{ij} = v_i v_j \in E(G)\} \). Let \( f = (V_1, \ldots, V_\ell) \) be a TDC of \( M(G) \). For all \( 1 \leq i \leq \ell \), consider \( W_i = V_i \setminus V \) and \( g = (W_1, \ldots, W_\ell) \). Then \( g \) define a coloring of the line graph \( L(G) \). If for every \( m_{ij} \in M \), there exists \( i \leq k \leq \ell \) such that \( m_{ij} \succ_t V_k \) and \( V_k \cap M \neq \emptyset \), then \( g \) is clearly a TDC of \( L(G) \). Assume now there exists \( m_{ij} \in M \) such that \( m_{ij} \succ_t V_{k_1}, \ldots, V_{k_r} \), but \( V_{k_1} \cup \cdots \cup V_{k_r} \subseteq V \). Since \( G \) is connected and \( m \geq 2 \), this implies \( N_{M(G)}(m_{ij}) \cap M = \{m_1, \ldots, m_p\} \neq \emptyset \). For every \( 1 \leq j \leq p \), assume that \( m_{ij} \in V_{t_j} \). If for some \( 1 \leq j \leq p \), we have \( V_{t_j} \cap M \subseteq N_{M(G)}(m_{ij}) \), then there is nothing to do (in this case \( m_{ij} \succ_t W_{t_j} \)). Assume now that for all \( 1 \leq j \leq p \), we have \( V_{t_j} \cap M \not\subseteq N_{M(G)}(m_{ij}) \). This implies that \( |V_{t_j} \cap M| \geq 2 \). In particular, \( V_{t_1} \cap M \not\subseteq N_{M(G)}(m_{ij}) \) and \( |V_{t_1} \cap M| \geq 2 \). Redefine \( W_{k_1} = \{m_1\} \) and \( W_{t_1} = V_{t_1} \setminus (V \cup \{m_1\}) \) (in this way \( m_{ij} \succ_t W_{t_1} \)). If we apply this procedure to all \( m_{ij} \in M \) such that \( m_{ij} \succ_t V_{k_1}, \ldots, V_{k_r} \), with \( V_{k_1} \cup \cdots \cup V_{k_r} \subseteq V \), we have that the obtained \( g \) is a TDC of \( L(G) \). This implies that \( \chi^d(M(G)) \geq \chi^d(L(G)) \). \( \square \)

Problem 3.7. Classify all connected graphs \( G \) such that \( \chi^d(M(G)) = \chi^d(L(G)) \).

4. Middle graph of known families

In this section, we calculate the total dominator chromatic number of the middle graph of several known families of graphs.

Theorem 4.1. For any star graph \( K_{1,n} \) on \( n+1 \) vertices, with \( n \geq 3 \),

\[
\chi^d(M(K_{1,n})) = n + 1.
\]

Proof. To fix the notation, assume \( V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\} \) and \( E(K_{1,n}) = \{v_0 v_1, \ldots, v_0 v_n\} \). Then \( V(M(K_{1,n})) = V(K_{1,n}) \cup M \) where
$M = \{m_i \mid 1 \leq i \leq n\}$. Let $f = (V_1, \ldots, V_\ell)$ be a minimal TDC of $M(K_{1,n})$. Since $K_{n+1} \cong M(K_{1,n})[v_0, m_1, \ldots, m_n] \subseteq M(K_{1,n})$, we have $\ell \geq n + 1$. Consider $V_i = \{m_i\}$ for $1 \leq i \leq n$, $V_{n+1} = V(K_{1,n})$ and $g = (V_1, \ldots, V_{n+1})$. By construction, $g$ is a TDC of $M(K_{1,n})$, and hence, $\chi^t_d(M(K_{1,n})) = n + 1$.

**Theorem 4.2.** For any double star graph $S_{1,n,n}$ on $2n + 1$ vertices, with $n \geq 1$,

$$\chi^t_d(M(S_{1,n,n})) = 2n + 1.$$  

**Proof.** To fix the notation, assume that $V(S_{1,n,n}) = \{v_0, v_1, \ldots, v_{2n}\}$ and $E(S_{1,n,n}) = \{v_0v_1, v_iv_{n+i} \mid 1 \leq i \leq n\}$. Then $V(M(S_{1,n,n})) = V(S_{1,n,n}) \cup M$, where $M = \{m_i, m_{i(n+i)} \mid 1 \leq i \leq n\}$. Let $f = (V_1, \ldots, V_\ell)$ be a TDC of $M(S_{1,n,n})$. Since the subgraph of $M(S_{1,n,n})$ induced by $\{m_i, v_{n+i} \mid 1 \leq i \leq n\}$ is isomorphic to a complete graph of order $n + 1$, then $\chi^t_d(M(S_{1,n,n})) \geq n + 1$. Without loss of generality, we can assume that $m_i \in V_i$, for $1 \leq i \leq n$, and $v_0 \in V_{n+1}$.

Now since $N_{M(S_{1,n,n})}(v_{n+i}) = \{m_{i(n+i)}\}$, then each $m_{i(n+i)}$ belong to a color class composed of only one element. This implies that we need at least another $n$ colors for a total dominator coloring of $S_{1,n,n}$. Hence $\chi^t_d(M(S_{1,n,n})) \geq 2n + 1$.

On the other hand, since $g = (V_1, \ldots, V_{2n+1})$, where $V_i = \{m_i\}$, $V_{n+i} = \{m_{i(n+i)}\}$ for $1 \leq i \leq n$ and $V_{2n+1} = V(S_{1,n,n})$, is a TDC of $M(S_{1,n,n})$, then $\chi^t_d(M(S_{1,n,n})) \leq 2n + 1$ and hence $\chi^t_d(M(S_{1,n,n})) = 2n + 1$. □

**Theorem 4.3.** For any path $P_n$ of order $n \geq 3$,

$$\chi^t_d(M(P_n)) = \begin{cases} n & \text{if } 3 \leq n \leq 7 \\ n - 1 & \text{if } n = 8 \\ \left\lceil \frac{2n}{3} \right\rceil + 2 & \text{otherwise} \end{cases}$$

**Proof.** Assume $V = V(P_n) = \{v_1, \ldots, v_n\}$. Then $V(M(P_n)) = V \cup M$, where $M = \{m_i(n+i) \mid 1 \leq i \leq n-1\}$. Notice that if $f = (V_1, \ldots, V_\ell)$ is a TDC of $M(P_n)$, then we can always assume that $V_1 = \{m_1\}$ and $V_2 = \{m_{(n-1)}\}$. Moreover, since defining $V_i = \{m_{(i-1)}\}$, for $i = 3, \ldots, n - 1$, and $V_n = V$ always gives a TDC of $M(P_n)$, then $\chi^t_d(M(P_n)) \leq n$.

Assume first that $n = 3$. Since the induced subgraph $M(P_3)[m_{12}, v_2, m_{23}]$ is isomorphic to $K_3$, then $\chi^t_d(M(P_3)) \geq 3$, and hence $\chi^t_d(M(P_3)) = 3$.

Assume that $n = 4$. By Corollary 2.6, $\chi^t_d(M(P_4)) \geq 3$. If $\chi^t_d(M(P_4)) = 3$, this would force $V_3 = V \cup \{m_{23}\}$, but this is impossible. This implies that $\chi^t_d(M(P_4)) = 4$.

Fix $n = 5$. By Corollary 2.6, $\chi^t_d(M(P_5)) \geq 4$. If $\chi^t_d(M(P_5)) = 4$, then we can assume $m_{23} \in V_3$ and $m_{34} \in V_4$, but then we could not color $v_3$. This implies that $\chi^t_d(M(P_5)) = 5$.

When $n = 6$, by Corollary 2.6, $\chi^t_d(M(P_6)) \geq 4$. If $\chi^t_d(M(P_6)) = 4$, then we can assume $m_{23} \in V_3$ and $m_{34} \in V_4$, but then we could not
color \(v_3\). This implies that \(\chi_d'(M(P_6)) \geq 5\). If \(\chi_d'(M(P_6)) = 5\), then we can assume \(m_{23} \in V_3, m_{34} \in V_4\) and \(v_3 \in V_5\). However, this implies that \(m_{56} \not\in V_i\) for all \(i = 1, 3, 4, 5\), but this is impossible, and hence, \(\chi_d'(M(P_6)) = 6\).

Consider \(n = 7\). By Corollary 2.6, \(\chi_d'(M(P_7)) \geq 5\). If \(\chi_d'(M(P_7)) = 5\), then we can assume \(m_{23} \in V_3, m_{34} \in V_4\) and \(v_3 \in V_5\). This implies that \(m_{67} \not\in V_i\) for all \(i = 1, 3, 4, 5\), but this is impossible, and hence, \(\chi_d'(M(P_7)) \geq 6\). If \(\chi_d'(M(P_7)) = 6\), then we can assume \(m_{23} \in V_3, m_{34} \in V_4\) and \(v_3 \in V_5\), and hence that \(m_{67} \not\in V_6\). This forces \(m_{45} \in V_3\). If \(m_{45} \in V_3\), then \(m_{12} \not\in V_i\) for all \(i = 2, \ldots, 6\), and hence \(m_{45} \in V_5\). As a consequence, \(v_4 \in V_3\), but this implies \(m_{12} \not\in V_i\) for all \(i = 2, \ldots, 6\). This shows that \(\chi_d'(M(P_7)) = 7\).

Assume that \(n = 8\). By Corollary 2.6, \(\chi_d'(M(P_8)) \geq 6\). If \(\chi_d'(M(P_8)) = 6\), then we can assume \(m_{23} \in V_3, m_{34} \in V_4\) and \(v_3 \in V_5\), and hence that \(m_{78} \not\in V_6\). This forces \(m_{45} \in V_3\). If \(m_{45} \in V_3\), then \(m_{12} \not\in V_i\) for all \(i = 2, \ldots, 6\), and hence \(m_{45} \in V_5\). As a consequence, \(v_4 \in V_3\), but this implies \(m_{12} \not\in V_i\) for all \(i = 2, \ldots, 6\). This shows that \(\chi_d'(M(P_8)) \geq 7\). On the other hand, if we consider \(V_1 = \{m_{12}\}, V_2 = \{m_{78}\}, V_3 = \{m_{23}\}, V_4 = \{m_{34}, m_{56}\}, V_5 = \{m_{45}\}, V_6 = \{m_{67}\}\) and \(V_7 = V\), then \(f = (V_1, \ldots, V_7)\) is a TDC of \(M(P_8)\), and hence, \(\chi_d'(M(P_8)) = 7\).

Finally, consider the case \(n \geq 9\). By the description of total dominating sets from [13], we have that \(\gamma_t(M(P_n)) = \left\lceil \frac{2n}{3} \right\rceil\) and that if \(S\) is a total dominating set of \(M(P_n)\), then \(M(P_n)[V \setminus S]\) is the disjoint union of graphs isomorphic to \(K_1\) and \(P_3\). This fact together with Theorem 2.2 and Corollary 2.6 implies that

\[
\left\lceil \frac{2n}{3} \right\rceil + 2 \geq \chi_d'(M(P_n)) \geq \left\lceil \frac{2n}{3} \right\rceil.
\]

Consider \(f = (V_1, \ldots, V_7)\) be a minimal TDC of \(M(P_n)\), and \(S\) a minimal total dominating set of \(M(P_n)\). Notice that \(|S| = \left\lceil \frac{2n}{3} \right\rceil\). By [13, Lemma 2.1], we can assume that \(S = \{m_{i_1(i_1+1)}, \ldots, m_{i_{\left\lceil \frac{2n}{3} \right\rceil}(\left\lceil \frac{2n}{3} \right\rceil+1)\} \subseteq M\), and that each element of \(S\) belong to a different color class. Without loss of generalities, assume that \(m_{i_j(i_j+1)} \in V_j\).

Suppose that \(\ell = \left\lceil \frac{2n}{3} \right\rceil\). Since \(n \geq 9\), then \(n - 1 > \left\lceil \frac{2n}{3} \right\rceil\), and the induced subgraph \(M(P_n)[V \setminus S]\) has a subgraph \(G\) isomorphic to \(P_3\) of the form \(M(P_n)[v_r, m_{r(r+1)}, v_{r+1}]\), for some \(r\). Since \(G\) needs at least two colors, this implies that \(|\{V_i \mid |V_i| \geq 2\}| \geq 2\). Assume that \(V_p\) and \(V_s\) have cardinality bigger than 2 for some \(1 \leq p < s \leq \ell\). This implies that \(m_{(i_p-1)i_p}, m_{(i_p+1)(i_p+2)}, m_{(i_s-1)i_s}\) and \(m_{(i_s+1)(i_s+2)}\) all belong to color classes made of only one element, otherwise one between \(v_{i_p}, v_{i_p+1}, v_{i_s}\) and \(v_{i_s+1}\) is not a common neighbour of any \(V_i\). This implies that \(m_{(i_p-1)i_p}, m_{(i_p+1)(i_p+2)}, m_{(i_s-1)i_s}, m_{(i_s+1)(i_s+2)} \in S\), and hence that \(|S| > \left\lceil \frac{2n}{3} \right\rceil\). This implies that \(\ell \geq \left\lceil \frac{2n}{3} \right\rceil + 1\).
Assume that \( \ell = \left\lceil \frac{2n}{3} \right\rceil + 1 \). Let \( S' = S \cup \{w\} \), where \( w \in V_t \). Since \( n \geq 9 \), then \( n-1 > \left\lceil \frac{2n}{3} \right\rceil + 1 \), and the induced subgraph \( M(P_n)[V \setminus S'] \) has a subgraph \( G \) isomorphic to \( P_3 \) of the form \( M(P_n)[v_r, m_{r(i+1)}, v_{r+1}] \), for some \( r \). Using the same argument as the case \( \ell = \left\lceil \frac{2n}{3} \right\rceil \), we have that one between \( m_{r(i+1)} \) and \( v_r \) belongs to \( V_t \) and the other to a \( V_p \) for some \( i \leq p \leq \ell - 1 \). In addition, \( |V_j| = 1 \) for all \( j \neq \ell, p \), and \( m_{(r-1)i_p} \) and \( m_{(r+1)(i_p+2)} \) belong to color classes made of only one element. This implies that \( v_{i_p}, v_{i_p+1} \in V_t \), and hence that one between \( m_{(r-1)i_p} \) and \( m_{(r+1)(i_p+2)} \) is not a common neighbour of any \( V_i \). This implies that \( \ell \geq \left\lceil \frac{2n}{3} \right\rceil + 2 \), and hence that \( \chi_d(M(P_n)) = \left\lceil \frac{2n}{3} \right\rceil + 2 \). \( \square \)

**Lemma 4.4.** For any \( n \geq 5 \), we have

\[
\chi_d(M(P_n)) \leq \chi_d(M(C_n)) \leq n.
\]

**Proof.** To fix the notation, assume \( V(P_n) = V(C_n) = V = \{v_1, \ldots, v_n\} \), \( E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \) and \( E(C_n) = E(P_n) \cup \{v_1v_n\} \). Then \( V(M(P_n)) = V \cup M \), where \( M = \{m_{(i+1)} \mid 1 \leq i \leq n-1\} \), and \( V(M(C_n)) = V \cup M \cup \{m_{1n}\} \).

If we define \( V_1 = \{m_{12}\}, V_2 = \{m_{23}, m_{1n}\}, V_i = \{m_{i(i+1)}\} \) for all \( i = 3, \ldots, n-1 \), and \( V_n = V \), then \( (V_1, V_2, \ldots, V_5) \) be a minimal TDC of \( M(C_n) \). This shows that \( \chi_d(M(C_n)) \leq n \). Let \( f = (V_1, V_2, \ldots, V_s) \) be a minimal TDC of \( M(C_n) \). Since \( s \leq n \), without loss of generality, we can assume that \( m_{1n} \in V_1 \) and \( |V_1| \geq 2 \). Define \( g = (V_1', V_2, \ldots, V_s) \), where \( V_1' = V_1 \setminus \{m_{1n}\} \). By construction \( g \) is a coloring of \( M(P_n) \). This implies that \( v_1, v_n \not\in V_1 \). Moreover, if \( m_{12} \supseteq V_1, \) then \( m_{12} \supseteq V_1' \), and similarly for \( m_{n(n-1)} \). This implies that \( g \) is a TDC of \( M(P_n) \), and hence \( \chi_d(M(C_n)) \geq \chi_d(M(P_n)) \). \( \square \)

**Theorem 4.5.** For any cycle \( C_n \) of order \( n \geq 3 \),

\[
\chi_d(M(C_n)) = \begin{cases} 4 & \text{if } n = 3 \\ n & \text{if } n = 4, 5 \\ \left\lceil \frac{2n}{3} \right\rceil + 2 & \text{otherwise} \end{cases}
\]

**Proof.** To fix the notation, assume \( V(C_n) = V = \{v_1, \ldots, v_n\} \) and \( E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_1v_n\} \). Then \( V(M(C_n)) = V \cup M \), where \( M = \{m_{i(i+1)} \mid 1 \leq i \leq n-1\} \) and \( \{m_{1n}\} \).

Assume that \( n = 3, 4 \). Then \( M(C_n) \supseteq K_3 \), so \( \chi_d(M(C_n)) \geq 3 \). If \( \chi_d(M(C_n)) = 3 \), then \( |A_1| = 0 \) which is a contradiction by Lemma 3.5. So \( \chi_d(M(C_n)) \geq 4 \). Now since \( (\{m_{12}\}, \{m_{23}\}, \{m_{1n}\}, V) \) and \( (\{m_{12}\}, \{m_{23}, m_{1n}\}, \{m_{34}\}, V) \) are TDC of \( M(C_3) \) and \( M(C_4) \) respectively, we have \( \chi_d(M(C_n)) = 4 \).

By Lemma 4.4, if \( n = 5 \) then \( \chi_d(M(C_5)) = 5 \).

Assume now that \( n \geq 6 \) and \( n \neq 8 \). By the description of total dominating sets from [13], we have that \( \gamma_t(M(C_n)) = \left\lceil \frac{2n}{3} \right\rceil \) and that we have that if \( S \) is a total dominating set of \( M(C_n) \), then \( M(C_n)[V \setminus S] \)
is the disjoint union of graphs isomorphic to $K_1$ and $P_3$. This fact together with Theorem 2.2 and Corollary 2.6 implies that
\[
\left\lceil \frac{2n}{3} \right\rceil + 2 \geq \chi_d^f(M(C_n)) \geq \left\lfloor \frac{2n}{3} \right\rfloor.
\]
On the other hand, by Theorem 4.3 and Lemma 4.4
\[
\left\lceil \frac{2n}{3} \right\rceil + 2 = \chi_d^f(M(P_n)) \leq \chi_d^f(M(C_n))
\]
and hence, $\chi_d^f(M(C_n)) = \left\lceil \frac{2n}{3} \right\rceil + 2$, for all $n \geq 6$ except for $n = 8$.

Finally for $n = 8$, by the previous argument, $8 \geq \chi_d^f(M(C_8)) \geq 7$. Let $f = (V_1, \ldots, V_\ell)$ be a minimal TDC of $M(C_n)$. If $\ell = 7$, then there exists $1 \leq i \leq 7$ such that $|V_i \cap M| \geq 2$. Without loss of generalities, assume $i = 1$ and $m_{12} \in V_1$. Since $f$ is a TDC, then we can assume $V_2 = \{m_{18}\}$ and $V_3 = \{m_{23}\}$. To finish the proof it is enough to discuss the case when $m_{67} \in V_1$ and when $m_{78} \in V_1$. Suppose that $m_{67} \in V_1$. This implies that we can assume $V_4 = \{m_{78}\}$ and $V_5 = \{m_{56}\}$. As a consequence $V_6 = \{m_{34}\}$ or $V_6 = \{m_{45}\}$. If $V_6 = \{m_{34}\}$, then $V_7 = \{m_{45}\}$ or $V_7 = \{v_5\}$. However in both cases, we do not have enough color classes to color all remaining vertices. If $V_6 = \{m_{45}\}$, then $V_7 = \{m_{34}\}$ or $V_7 = \{v_3\}$. However in both cases, we do not have enough color classes to color all remaining vertices. Suppose now that $m_{78} \in V_1$. This implies that we can assume $V_4 = \{m_{67}\}$. As a consequence $V_5 = \{v_2\}$ or $V_5 = \{v_3\}$ or $V_5 = \{m_{34}\}$, and similarly, $V_6 = \{m_{45}\}$ or $V_6 = \{m_{56}\}$. All these cases imply that $v_1, v_5 \in V_7$ and that $|V_1|, |V_7| \geq 3$. However, this implies that $m_{18}$ is not a common neighbour of any $V_i$. This shows that $\chi_d^f(M(C_8)) = 8$ \qed

**Remark 4.6.** The two inequalities of Lemma 4.4 are both strict, by Theorem 4.5.

**Theorem 4.7.** For any wheel $W_n$ of order $n \geq 4$,
\[
\chi_d^f(M(W_n)) = \begin{cases} 
5 & \text{if } n = 4 \\
n + 2 & \text{if } n \geq 5
\end{cases}
\]

**Proof.** To fix the notation, assume $V(W_n) = V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0v_1, v_0v_2, \ldots, v_0v_{n-1}\} \cup \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_1\}$. Then we have $V(M(W_n)) = V(W_n) \cup M$, where $M = \{m_i \mid 1 \leq i \leq n - 1\} \cup \{m_{i(i+1)} \mid 1 \leq i \leq n - 2\} \cup \{m_{i(n-1)}\}$.

Assume $n = 4$. Consider $V_1 = V$, $V_2 = \{m_1, m_{23}\}$, $V_3 = \{m_2\}$, $V_4 = \{m_3, m_{12}\}$ and $V_5 = \{m_{13}\}$. By construction, $f = (V_1, \ldots, V_5)$ is a TDC of $M(W_n)$, and hence $\chi_d^f(M(W_4)) \leq 5$. On the other hand, let $f = (V_1, \ldots, V_\ell)$ be a minimal TDC of $M(W_4)$. Since $K_4 \cong M(W_3)[v_0, m_1, \ldots, m_3] \subseteq M(W_4)$, then $\ell \geq 4$. If $\ell = 4$, this implies that, up to reordering the color classes, $V_1 = V$, $V_2 = \{m_1, m_{23}\}$, $V_3 = \{m_2, m_{13}\}$ and $V_4 = \{m_3, m_{12}\}$, but in this way $v_0 \not\sim V_i$ for
This implies that $\chi_d(M(W_4)) = 5$.

Assume that $n \geq 5$ is even. Consider $V_1 = V$, $V_2 = \{m_1, m_{23}, m_{45}, \ldots, m_{n(2)(n-1)}\}$, $V_3 = \{m_{34}, m_{56}, \ldots, m_{(n-1)n}\}$, $V_4 = \{m_{12}\}$ and, for any $i = 5, \ldots, n + 2$, $V_i = \{m_{i-3}\}$. By construction, $f = (V_1, \ldots, V_{n+2})$ is a TDC of $M(W_n)$, and hence $\chi_d(M(W_n)) \leq n + 2$. On the other hand, let $f = (V_1, \ldots, V_l)$ be a minimal TDC of $M(W_n)$. Since $K_n \cong M(W_n)[v_0, m_1, \ldots, m_{n-1}] \subseteq M(W_n)$, then $\ell \geq n$ and $v_0, m_1, \ldots, m_{n-1}$ all belong to different color classes. If each $m_i$ belongs to a color class of cardinality 1, then $\ell \geq n + 3$, contradicting the first part of the proof for this case. This implies that there exists at least one $m_i$ belonging to a color class with at least 2 elements. If there is only one such class, then we can assume, without loss of generality, that $m_1 \in V_1$ and $|V_1| \geq 2$. Since $f$ is a TDC, then at least one between $m_{12}$ and $m_{1(n-1)}$ has to belong to a color class of cardinality 1. Moreover, $m_{12}, m_{1(n-1)} \notin V_1$ and hence $\ell \geq n + 2$, proving our thesis. Assume that there exist $1 \leq i \leq n - 1$ such that $m_i, m_j$ belong to a color class with at least 2 elements. If $j - i = 1$ or $j - i = n - 2$, then we can assume, without loss of generality, that $i = 1, j = 2$, $m_1 \in V_1$ with $|V_1| \geq 2$ and $m_2 \in V_2$ with $|V_2| \geq 2$. This implies that two vertices between $v_1, v_2, m_{1(n-1)}, m_{12} \text{ and } m_{23}$ have to belong to a color class of cardinality 1 and hence $\ell \geq n + 2$, proving our thesis. If $2 \geq j - i \geq n - 3$, then at least one between $m_{i(i-1)}$ and $m_{i(i+1)}$ has to belong to a color class of cardinality 1, and similarly at least one between $m_{(j-1)j}$ and $m_{j(j+1)}$ has to belong to a color class of cardinality 1. This implies that $\ell \geq n + 2$, proving our thesis. This shows that $\chi_d(M(W_n)) = n + 2$.

Assume $n$ is odd. Consider $V_1 = V$, $V_2 = \{m_{12}, m_{34}, \ldots, m_{(n-2)(n-1)}\}$, $V_3 = \{m_{23}, m_{45}, \ldots, m_{(n-1)n}\}$ and, for any $i = 4, \ldots, n + 2$, $V_i = \{m_{i-3}\}$. By construction, $f = (V_1, \ldots, V_{n+2})$ is a TDC of $M(W_n)$, and hence $\chi_d(M(W_n)) \leq n + 2$. On the other hand, let $f = (V_1, \ldots, V_l)$ be a minimal TDC of $M(W_n)$. Since $K_n \cong M(W_n)[v_0, m_1, \ldots, m_{n-1}] \subseteq M(W_n)$, then $\ell \geq n$ and $v_0, m_1, \ldots, m_{n-1}$ all belong to different color classes. If each $m_i$ belongs to a color class of cardinality 1, then $\ell \geq n + 2$, proving our thesis. Assume that there exists at least one $m_i$ belonging to a color class of cardinality bigger than 2. If there is only one such class, then we can assume, without loss of generality, that $m_1 \in V_1$ and $|V_1| \geq 2$. Since $f$ is a TDC, then at least one between $m_{12}$ and $m_{1(n-1)}$ has to belong to a color class of cardinality 1. Moreover, $m_{12}, m_{1(n-1)} \notin V_1$ and hence $\ell \geq n + 2$, proving our thesis. Assume that there exist $1 \leq i \leq n - 1$ such that $m_i, m_j$ belong to a color class of cardinality bigger than 2. If $j - i = 1$ or $j - i = n - 2$, then we can assume, without loss of generality, that $i = 1, j = 2$, $m_1 \in V_1$ with $|V_1| \geq 2$ and $m_2 \in V_2$ with $|V_2| \geq 2$. This implies that two vertices between $v_1, v_2, m_{1(n-1)}, m_{12} \text{ and } m_{23}$ have to belong to a color class of cardinality 1 and hence $\ell \geq n + 2$, proving our thesis.
then at least one between \( m_{(i-1)j} \) and \( m_{(i+1)j} \) has to belong to a color class of cardinality 1, and similarly at least one between \( m_{(j-1)j} \) and \( m_{(j+1)j} \) has to belong to a color class of cardinality 1. This implies that \( \ell \geq n+2 \), proving our thesis. This shows that \( \chi_d(M(W_n)) = n+2 \). □

**Theorem 4.8** ([1]). For any complete graph \( K_n \) of order at least 2,

\[
\chi'(K_n) = \begin{cases} 
  n - 1 & \text{if } n \text{ is even} \\
  n & \text{if } n \text{ is odd}.
\end{cases}
\]

**Theorem 4.9.** For any complete graph \( K_n \) on \( n \) vertices, with \( n \geq 2 \),

\[
\chi(M(K_n)) = n.
\]

**Proof.** To fix the notation, assume \( V(K_n) = V = \{v_1, \ldots, v_n\} \) and \( E(K_n) = \{v_iv_j \mid 1 \leq i < j \leq n\} \). Then we have \( V(M(K_n)) = V(K_n) \cup M \), where \( M = \{m_{ij} \mid 1 \leq i < j \leq n\} \).

Assume \( n \) is even. By Theorem 4.8, we can consider \( f_1 = (V_1, \ldots, V_{n-1}) \) a proper coloring of \( L(K_n) \). Since to color the induced subgraph \( M(K_n)[v_1, m_{12}, \ldots, m_{1n}] \subseteq M(K_n) \) we need \( n \) colors, then \( \chi(M(K_n)) \geq n \). On the other hand, \( g_1 = (V_1, \ldots, V_{n-1}, V) \) is a proper coloring of \( M(K_n) \), and hence \( \chi(M(K_n)) = n \).

Assume \( n \) is odd. By Theorem 4.8, we can consider \( f_2 = (V_1, \ldots, V_n) \) a proper coloring of \( L(K_n) \). Since to color the induced subgraph \( M(K_n)[v_1, m_{12}, \ldots, m_{1n}] \subseteq M(K_n) \) we need \( n \) colors, then \( \chi(M(K_n)) \geq n \). On the other hand, for each \( 1 \leq i \leq n \), \( d_{K_n}(v_i) = n-1 \). This implies that for each \( 1 \leq i \leq n \) there exists \( 1 \leq j \leq n \) such that \( m_{ik} \notin V_j \) for all \( 1 \leq k \leq n \). Let \( W_j = V_j \cup \{v_i \mid m_{ik} \notin V_j \text{ for all } 1 \leq k \leq n\} \). By construction \( g_2 = (W_1, \ldots, W_n) \) is a proper coloring of \( M(K_n) \), and hence \( \chi(M(K_n)) = n \). □

**Theorem 4.10.** For any complete graph \( K_n \) of order \( n \geq 2 \),

\[
n + 1 \leq \chi_d^i(M(K_n)) \leq n + \left\lfloor \frac{2n}{3} \right\rfloor - 1.
\]

Moreover, the bounds are tight.

**Proof.** To fix the notation, assume \( V(K_n) = V = \{v_1, \ldots, v_n\} \) and \( E(K_n) = \{v_iv_j \mid 1 \leq i < j \leq n\} \). Then \( V(M(K_n)) = V \cup M \) where \( M = \{m_{ij} \mid 1 \leq i < j \leq n\} \). By [13], we know that \( \gamma_i(M(K_n)) = \left\lceil \frac{2n}{3} \right\rceil \) and also that the sets

\[
S_0 = \{m_{(3i+1)(3j+2)}, m_{(3i+2)(3j+3)} \mid 0 \leq i \leq \left\lfloor n/3 \right\rfloor - 1\} & \text{ if } n \equiv 0 \pmod{3}, \\
S_1 = S_0 \cup \{m_{(n-1)n}\} & \text{ if } n \equiv 1 \pmod{3}, \\
S_2 = S_0 \cup \{m_{(n-2)(n-1)}, m_{(n-1)n}\} & \text{ if } n \equiv 2 \pmod{3},
\]

are minimal TDSs of \( M(K_n) \). Since to color the induced subgraph \( (M(K_n) - S_i)[v_1, m_{13}, \ldots, m_{1n}] \subseteq M(K_n) - S_i \) for each \( 0 \leq i \leq 2 \), we need \( n - 1 \) colors, then \( \chi(M(K_n) - S_i) \geq n - 1 \). We know that for every graph \( G \), \( \chi(G) \leq \Delta(G) + 1 \). Now since \( \Delta(M(K_n) - S_i) = n - 2 \),
we have \( \chi(M(K_n) - S_i) \leq n - 1 \), and hence \( \chi(M(K_n) - S_i) = n - 1 \).

By Theorem 2.2 we have \( \chi'_d(M(K_n)) \leq n + \left[ \frac{2n}{3} \right] - 1 \).

We claim that \( n + 1 \leq \chi'_d(M(K_n)) \). By absurd, assume \( \chi'_d(M(K_n)) = n = \chi(M(K_n)) \) and let \( f = (V_1, \ldots, V_n) \) be a minimal TDC of \( M(K_n) \).

By Theorems 4.8 and 4.9, this implies that \( |V_i| \geq 2 \) for each \( 1 \leq i \leq n \), which is a contradiction by Lemma 3.5. As a consequence, \( n + 1 \leq \chi'_d(M(K_n)) \). By Theorem 4.5, the bounds are tight when \( n = 3 \).

**Definition 4.11.** The friendship graph \( F_n \) of order \( 2n + 1 \) is obtained by joining \( n \) copies of the cycle graph \( C_3 \) with a common vertex.

**Theorem 4.12.** Let \( F_n \) be the friendship graph with \( n \geq 2 \). Then

\[
\chi'_d(M(F_n)) = 2n + 2.
\]

**Proof.** To fix the notation, assume \( V(F_n) = \{v_0, v_1, \ldots, v_{2n}\} \) and \( E(F_n) = \{v_0v_1, v_0v_2, \ldots, v_{2n-1}v_{2n}\} \). Then \( V(M(F_n)) = V(F_n) \cup M \), where \( M = \{m_i \mid 1 \leq i \leq 2n\} \cup \{m_{i(i+1)} \mid 1 \leq i \leq 2n-1 \text{ and } i \text{ is odd}\} \).

Let \( f = (V_1, \ldots, V_\ell) \) be a minimal TDC of \( M(F_n) \). Since the induced subgraph \( M(F_n)[v_0, m_1, \ldots, m_{2n}] \) is isomorphic to \( K_{2n+1} \), we have \( \ell \geq 2n + 1 \). Without loss of generality, we can assume that \( m_i \in V_i \) for \( 1 \leq i \leq 2n \) and \( v_0 \in V_{2n+1} \). This implies that \( v_i \succ_t \{m_i\} \) for \( 1 \leq i \leq 2n \) and \( \{m_i\} \) is an unique color class. Now since at least two color classes are needed to color the vertices \( v_i \) and \( m_{i(i+1)} \) for some \( i \), we have \( \ell > 2n + 1 \), which is a contradiction. So \( \ell \geq 2n + 2 \). On the other hand, if we consider \( V_i = \{m_i\} \) for \( 1 \leq i \leq 2n \), \( V_{2n+1} = \{v_0, v_1, \ldots, v_{2n}\} \), and \( V_{2n+2} = \{m_{i(i+1)} \mid 1 \leq i \leq 2n-1 \text{ and } i \text{ is odd}\} \), we have that \( (V_1, \ldots, V_{2n+2}) \) is a TDC of \( M(F_n) \), and hence \( \chi'_d(M(F_n)) = 2n + 2 \).

5. MIDDLE GRAPH OF TREES

In this section, we will describe bounds for the total dominator coloring number of the middle graph of trees.

**Theorem 5.1.** Let \( T \) be a tree of order \( n \geq 2 \). Then

\[
\chi'_d(M(T)) \leq n.
\]

**Proof.** Assume \( V(T) = \{v_1, \ldots, v_n\} \). Then \( V(M(T)) = V(T) \cup M \) where \( M = \{m_{v_j} \mid v_iv_j \in E(T)\} \). We give \( |M| = n - 1 \) colors to each element of \( M \) and a different color to each element in \( V(T) \). This coloring is a TDC of \( M(T) \) with \( n \) color classes. Hence \( \chi'_d(M(T)) \leq n \).

If we consider \( T \) a tree and we denote the set of leaves of \( T \) by \( \text{leaf}(T) = \{v \in V(T) \mid d_T(v) = 1\} \), then we have the following result.

**Theorem 5.2.** Let \( T \) be a tree with \( n \geq 3 \) vertices. Then

\[
\chi'_d(M(T)) \geq |\text{leaf}(T)| + 1.
\]
Proof. To fix the notation, assume leaf($T$) = \{v_1, \ldots, v_k\}, for some $k \leq n$ and $f = (V_1, \ldots, V_l)$ be a minimal TDC of $M(T)$. Since for each $i = 1, \ldots, k$, $N_M(T)(v_i) = \{m_{ij}\}$ for some $j$, we have that $\{m_{ij}\}$ is an unique color class so that $v_i \triangleright_1 \{m_{ij}\}$. This implies that $\ell \geq k$. On the other hand, since at least one color class is needed to color the vertices in $V(M(T)) = \{N_M(T)(v_i) | 1 \leq i \leq k\}$, we have $\chi^d(M(T)) \geq k + 1 = |\text{leaf}(T)| + 1$.

Remark 5.3. Notice that the inequality of Theorem 5.2 is sharp by Theorem 4.1.

Theorem 5.4. For any non-empty tree $T$ of order $n \geq 2$ with diam($T$) $\leq 3$,

$$\chi^d(M(T)) = n.$$

Proof. Assume $V = V(T) = \{v_1, \ldots, v_n\}$. Then $V(M(T)) = V \cup M$ where $M = \{m_{ij} | v_i v_j \in E(T)\}$.

If diam($T$) $= 1$, then $T \cong K_2$ and $M(T) \cong P_3$, and so $\chi^d(M(T)) = 2$. If diam($T$) $= 2$, then $n \geq 3$ and $T \cong K_{1,n-1}$ and so $\chi^d(M(T)) = n$, by Theorem 4.1.

If diam($T$) $= 3$, then $T$ is a tree which is obtained by joining central vertex $v$ of a tree $K_{1,p}$ and the central vertex $w$ of a tree $K_{1,q}$ where $p + q = n - 2$. Let leaf($T$) = \{v_i | 1 \leq i \leq n - 2\} be the set of leaves of $T$. Obviously $V(T) = \text{leaf}(T) \cup \{v, w\}$ and $|\text{leaf}(T)| = n - 2$. Define $v_{n-1} = v$ and $v_n = w$. Let $f = (V_1, \ldots, V_l)$ be a TDC of $M(T)$. Since, for each $1 \leq i \leq n - 2$, if $v_i \triangleright_1 V_{k_i}$ for some $1 \leq k_i \leq \ell$, then $|V_{k_i}| = 1$ and $V_{k_i}$ is an unique color class for each $1 \leq i \leq n - 2$, then $\chi^d(M(T)) \geq n - 2$. Since the induced subgraph of $M(T)$ by $\{m_{(n-1)n}, v_n\}$ is a complete graph of order 2, we conclude that we need another 2 colors, so $\chi^d(M(T)) \geq n$. On the other hand, since $g = (V_1, \ldots, V_n)$, where $V_i = \{m_{ij}\}$ for $1 \leq i \leq n - 2$, $1 \leq j \leq n - 1$, $V_{n-1} = \{m_{(n-1)n}\}$ and $V_n = V$, is a TDC of $M(T)$, we have $\chi^d(M(T)) \leq n$, and hence $\chi^d(M(T)) = n$.

Remark 5.5. Notice that the inequality of Theorem 5.1 is sharp by Theorem 5.4.

Remark 5.6. In general, the converse implication of Theorem 5.4 is not true. To see this it is enough to look at Theorem 4.2. In fact if $T = S_{1,n,n}$ with $n \geq 2$, then $\chi^d(M(T)) = |V(T)|$, even if diam($T$) $= 4$.

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