Cyclic and Inductive Calculi are Equivalent

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Abstract—Brotherston and Simpson [citation] have formalized and investigated cyclic reasoning, reaching the important conclusion that it is at least as powerful as inductive reasoning (specifically, they showed that each inductive proof can be translated into a cyclic proof). We add to their investigation by proving the converse of this result, namely that each inductive proof can be translated into an inductive one. This, in effect, establishes the equivalence between first order cyclic and inductive calculi.

I. INTRODUCTION

Cyclic proofs are often a convenient replacement for inductive. They do not require the discovery of induction hypotheses that are usually stronger than the conjecture at hand, but rather proceed with simple case analysis steps and applications of simple laws, allowing “smaller” instances of the original conjecture to be used as lemmas. While cyclic proof methods still require a high level of ingenuity, they appear to lead to clearer proofs, and seem to lead to more successful automation when implemented in automated reasoning tools.

Under these circumstances, formal investigation of cyclic reasoning becomes very important. In [7], Brotherston and Simpson have provided a rigorous and sound treatment of first-order infinite descent reasoning, identifying cyclic proofs as a special case of practical importance. The infinite descent calculus in [7] is shown to be sound and complete; most importantly, cyclic proofs are shown to be a valid replacement for inductive proofs, via a translation scheme that converts every inductive proof into a cyclic one. While this would be sufficient to justify the use of cyclic reasoning in practice, it does not complete its formal investigation.

In the present paper, we prove Conjecture 7.7 of [7], which states that every cyclic proof can be converted into an inductive one. The proof is subtle and laborious, and while of possibly no immediate practical importance, it establishes the equivalence of inductive and cyclic reasoning, hopefully making a contribution towards a more wide-spread adoption of the latter in mechanical and automated reasoning tools.

Our paper is organized as follows. Section 2 discusses related work, giving a brief overview of the developments in [7]. Section 3 introduces the notations and terminology that will be used in the rest of the paper. Section 4 presents the proof that each cyclic proof can be converted into an inductive one. Section 5 concludes.

II. RELATED WORK

The present development builds heavily on Brotherston and Simpson’s work [4]–[7], which we describe it here in some detail. The work is structured in four main parts. First, a first order language is defined, in the spirit of [3], and then augmented with inductive definitions based on Martin-Löf’s “ordinary productions” [10]. A standard interpretation is defined for this language, where the inductive component is interpreted as the least fixed point of a monotone operator, cf. [11] Second, an inductive sequent calculus, named LKID, is defined. This calculus adopts the original rules of Gentzen’s LK calculus [15], and adapts its induction rules from [10]. The calculus is proved to be sound and complete with respect to Henkin semantics, and to enjoy a cut-elimination property, due to the particular formulation of the induction rules, which allow new formulas to be introduced in a proof as induction hypotheses. Due to this peculiarity, LKID no longer enjoys the subformula property that is instrumental in proving several significant properties of the LK calculus. Third, an infinite descent calculus, LKID∗ω, is introduced. This calculus is obtained by weakening the induction rules of LKID into case-split rules, which essentially no longer have a placeholder for an induction hypothesis. However, proofs are now allowed to be infinite, so long as every infinite branch in the proof tree exhibits a progress condition, namely that case-split rules of inductive predicates from the same set of mutually inductive predicates occur infinitely often along the path. LKID∗ω is proved sound and complete and, due to its potentially infinite proofs, stronger than LKID (but also interesting only from a theoretical perspective). Fourth, a finitary restriction is identified for LKID∗ω proofs, where proof trees are regular, and can thus be represented as graphs with possible cycles; each infinite path will yield a cycle in the graph representation, and the progress condition translates into every cycle going across at least one application of a case-split rule. The restricted calculus, of practical importance due to its finitary nature, is denoted as CLKID∗ω; its proofs are called cyclic proofs, and represent the focus of the current paper. The cyclic calculus is proved to be sound, and at least as powerful as the inductive calculus LKID; the proof is achieved via a translation scheme that converts every inductive proof into a cyclic one. Conjecture 7.7 in [7] hypothesizes that the converse is also true, that is, every CLKID∗ω proof can be converted into a LKID proof. Our paper presents a proof of this conjecture.

An approach similar to cyclic reasoning is presented for

1 A Henkin semantics is also defined at this point, and used to prove a completeness result for the infinite descent calculus; this, however, falls outside the scope of this paper.
the $\mu$-calculus in [12], [13]. A comprehensive overview of infinite descent, and approaches to using it in proof automation are given in [16], [17]. Cyclic reasoning-like methods have also been used in automated provers, especially for program verification, cf. [2], [9], [11], [14].

## III. Preliminaries

### A. First-order logic with inductive definitions

Following [7], we consider a fixed language $\Sigma$ with inductive predicate symbols $P_1, \ldots, P_n$. Terms of $\Sigma$ are defined, as usual; we write $t(x_1, \ldots, x_k)$ for a term whose variables are contained in $\{x_1, \ldots, x_k\}$. The formulas of the logic are the usual formulas of first-order logic.$^2$ We denote by $\pi$ the sequence of variables $x_1, \ldots, x_k$, and by $\tilde{t}(\pi)$ the sequence of terms $t_1(x_1, \ldots, x_k), \ldots, t_l(x_1, \ldots, x_k)$; the values $k$ and $l$ will be usually understood from the context.

**Definition 1 (Inductive definition set):** An inductive definition set $\Phi$ for $\Sigma$ is a finite set of productions of the form:

$$Q_1(\pi_1(\pi)) \ldots Q_k(\pi_k(\pi)) P_j(\tilde{t}_j(\pi)) \ldots P_m(\tilde{t}_m(\pi))$$

where $Q_1, \ldots, Q_k$ are ordinary predicate symbols, $j_1, \ldots, j_m, i \in \{1, \ldots, n\}$, $\pi$ is a set of distinguished variables, and the lengths of the sequences $\pi_1(\pi), \ldots, \pi_k(\pi), \tilde{t}_1(\pi), \ldots, \tilde{t}_m(\pi)$ match the arities of the predicate symbols.

The following example introduces inductive predicates that shall be later used in our example proofs. In our examples, we prefer a functional notation for our predicates, that is, we write $(P t_1 \ldots t_k)$ instead of $P(t_1, \ldots, t_k)$.

**Example 1:** We define the predicates $\text{nat}$ and $t\text{plus}$, representing natural numbers, and tail-recursive addition.

| $\text{nat}$ | $\text{nat}\ x$ | $\text{nat}\ (S\ x)$ | $\text{tplus}\ x\ (S\ y)\ z$ | $\text{tplus}\ (S\ x)\ y\ z$ |
|-------------|-----------------|---------------------|-----------------------------|-----------------------------|

### B. A proof system for induction (LKID)

We write sequents of the form $\Gamma \vdash \Delta$, where $\Gamma$ and $\Delta$ are finite sets of formulas, and use the notation $\Gamma[\theta]$ to mean that the substitution $\theta$ of terms for free variables is applied to all formulas in $\Gamma$. For first order logic, we use the standard sequent calculus rules given in Figure 1. However, we do not add equality rules to our incarnation of this calculus, since an equality predicate, can be defined via inductive rules, if needed. We read the comma in sequents as set union, and thus the contraction and exchange rules are not necessary.

We augment the rules in Figure 1 with rules for introducing atomic formulas involving inductive predicates on the left and right of sequents.

First, for each production $\Phi_i, r \in \Phi$, each necessarily in the format of Definition [1] we include a corresponding sequent calculus right introduction rule for $P_i$:

$$\frac{\Gamma \vdash Q_1(\pi_1(\pi)), \ldots, \Gamma \vdash Q_k(\pi_k(\pi)), \Delta}{\Gamma \vdash P_j(\tilde{t}_j(\pi)), \ldots, \Gamma \vdash P_m(\tilde{t}_m(\pi)), \Delta}$$

$(P, R_r)$

Here $\pi$ is assumed to be a sequence of terms of the same length as the sequence $\pi$ of variables explicitly denoted in the production, and the occurrences of $\tilde{\pi}$ in the rule above represent the substitution $[\tilde{\pi}/\pi]$.

The left-introduction rules for inductively defined predicates represent in fact induction rules. The definition of these rules requires the notion of mutual dependency as defined in [10], and reproduced in the following definition:

**Definition 2 (Mutual dependency):** Let $R \subseteq \{P_1, \ldots, P_n\} \times \{P_1, \ldots, P_n\}$ be defined as the least relation satisfying the property that $(P_i, P_j) \in R$ whenever there exists a rule in $\Phi$ such that $P_i$ occurs in the rule’s conclusion, and $P_j$ occurs among its premises. Denote by $R^*$ the reflexive-transitive closure of $R$. Then, two predicates $P_i$ and $P_j$ are mutually dependent if both $(P_i, P_j) \in R^*$ and $(P_j, P_i) \in R^*$.

To obtain an instance of the left-introduction rule for any inductive predicate $P_j$, we first associate with every inductive predicate $P_i$ a tuple $\pi_i$ of size equal to the arity of $P_i$. Furthermore, we associate to every predicate $P_i$ that is mutually dependent with $P_j$ an induction hypothesis $F_i$ (which is, essentially, an arbitrary formula), possibly containing some of the induction variables $\pi_i$. Then, we define for each $i \in \{1, \ldots, n\}$:

$$G_i = \begin{cases} F_i & \text{if } P_i \text{ and } P_j \text{ are mutually dependent} \\ P_i(\pi_i) & \text{otherwise} \end{cases}$$

An instance of the induction rule for $P_j$ has the following schema:

$$\frac{\text{minor premises}}{\Gamma, P_j(\pi), \vdash \Delta}$$

$(P_{JN})$

where the premise $\Gamma, P_j(\pi), \vdash \Delta$ is called the major premise of the rule instance, and for each production $\Phi$, of the form (1), having in its conclusion a predicate $P_i$ that is mutually dependent with $P_j$, there is a corresponding minor premise:

$$\frac{\Gamma, Q_1(\pi_1(\pi)), \ldots, Q_k(\pi_k(\pi)), G_j, P_i(\tilde{t}_j(\pi)) \vdash F_j(\tilde{t}_j(\pi)/z_j)}{\Gamma, P_i(\tilde{t}_j(\pi)/z_j), \vdash F_j(\tilde{t}_j(\pi)/z_j)}$$

where $\tilde{\pi}$ is a vector of distinct fresh variables (i.e. not occurring in $\text{FV}(\Gamma \cup \Delta \cup \{P_j(\tilde{t}_j(\pi))\}_{1 \leq l \leq m})$) of the same length as the vector $\pi$ of variables explicitly identified in the production.

All the formulas of the calculus LKID, with the exception of the cut rule, have a principal formula, which is, by definition, the formula occurring in the lower sequent of the inference which is not in the cedents $\Gamma$ and $\Delta$. Every inference, except...
Formulas of a cut inference are called \( \Delta \) are called auxiliary formulas. Weakenings, has one or more auxiliary formulas, which are the formulas \( A \) and \( B \), occurring in the upper sequent(s) of the inference. The formulas which occur in the cedents \( \Gamma \) or \( \Delta \) are called side formulas of the inference. The two auxiliary formulas of a cut inference are called cut formulas.

**Example 2 (Induction rule for natural numbers):**

\[
\begin{align*}
\Gamma \vdash F(t) & \quad \Gamma, F(p) \vdash F(0) \\
\Gamma, F(t) & \vdash F(S(p)) \\
\Gamma, F(t) & \vdash F(t) \\
\Gamma, F(t) & \vdash F(t) \\
\Gamma, \text{nat } t & \vdash \Delta \quad \text{(natIND)}
\end{align*}
\]

where \( F(z) \) is the induction hypothesis, parameterized on an induction variable \( z \), and \( p \) is a fresh variable, i.e. \( p \notin FV(\Gamma \cup \Delta) \).

**Example 3 (Induction rule for tplus):**

\[
\begin{align*}
\Gamma, \text{nat } u & \vdash F(u,v,w) \\
\Gamma, F(t_1,t_2,t_3) & \vdash F(t_1,t_2,t_3) \\
\Gamma, F(t_1,t_2,t_3) & \vdash tplus t_1 t_2 t_3 \vdash \Delta \quad \text{(tplusIND)}
\end{align*}
\]

where \( F(z_1,z_2,z_3) \) is the induction hypothesis, parameterized on induction variables \( z_1, z_2, z_3 \), and \( u,v,w \) are fresh variables, i.e. \( u,v,w \notin FV(\Gamma \cup \Delta \cup F(z_1,z_2,z_3)) \).

A derivation tree is a tree of sequents in which each parent is obtained as the conclusion of an inference rule with its children as premises. A proof in LKID is a finite derivation tree all of whose branches end in an axiom (i.e. a proof rule without premises).

**Example 4:** Figure 3 depicts the proof tree for the sequent \( \text{nat } k \vdash \text{tplus } k \not 0 k \). The induction is nested. To obtain the proof, we first apply the (natIND) rule, with the induction hypothesis \( \text{tplus } z \not 0 z \). The first and third premise can then be discharged immediately, whereas the second premise requires further induction on the predicate \( \text{tplus} \). For the second round of induction, the induction hypothesis is \( \text{tplus } z_1 (S z_2) z_3 \).

**C. A cyclic proof system CLKID**

The proof rules of CLKID\( ^{\omega} \) are the rules of LKID, except that for each inductive predicate \( P_i \) of \( \Sigma \), the induction rule \( (P_i \text{IND}) \) is replaced by the case-split rule

\[
\Gamma, \exists \forall \Delta, \exists \forall \Delta \vdash \Delta \quad \text{(\exists \forall R)}
\]

where for each production of the form (1) having predicate \( P_i \) in its conclusion, there is a corresponding case distinction

\[
\Gamma[t/x], \Delta, \exists \forall \Delta \vdash \Delta \quad \text{(\exists \forall R)}
\]

where \( \overline{y} \) is a vector of distinct fresh variables of the same length as \( \overline{x} \).

**Example 5 (Case-split for nat and tplus):** The rules (natL) and (tplusL) represent the case-split rules for the predicates \( \text{nat} \) and \( \text{tplus} \).

\[
\Gamma[0/y] \vdash \Delta[0/y] \quad \Gamma[S p/y], \text{nat } p \vdash \Delta[S p/y] \quad \text{(natL)}
\]

\[
\Gamma[0/y], x/y_1, y_2/y_3 \vdash \Delta[0/y], x/y_1, y_2/y_3 \quad \text{tplus } x (S y) z \vdash \Delta[0/y], x/y_1, y_2/y_3 \quad \text{(tplusL)}
\]

The notion of proof for the CLKID\( ^{\omega} \) depends on a progress condition that needs to distinguish between multiple occurrences of the same inductive predicate in a sequent. To make that possible, we tag induction predicates by natural numbers; thus \( P^{\alpha} \) denotes the tagged version of inductive predicate \( P \), where \( \alpha \) is a natural number. A sequent where all inductive predicate occurrences are tagged is called a tagged sequent.
Definition 3 (Tagged proof tree): A tagged proof tree is a proof tree where each sequent is tagged according to the following rules:

- The endsequent is tagged such that each inductive predicate occurrence has a distinct tag.
- Inferences preserve the tags of predicates in side formulas.
- If an inductive predicate $P^\alpha$ is the principal formula in an inference, the predicates in the auxiliary formulas inherit the tag $\alpha$.
- In a cut inference, the cut formula may be a tagged predicate that appeared as a formula in the antecedent of an ancestor node.
- General formulas may appear in cut inferences only if their inductive predicates have fresh tags (i.e., not appearing at ancestor nodes).

Definition 4 (Proof in CLKID$^\omega$): A proof in CLKID$^\omega$ is a tagged derivation tree where each leaf is either (a) the tagged version of an axiom; or (b) it is cyclic, that is, it contains a sequent that is identical (tags included) to one of its ancestors called companion node, and with the added property that on the path from a companion to its corresponding cyclic leaf there is at least one application of a case-split rule.

Our notion of cyclic proof is slightly different, but equivalent to the notion of normalized cyclic proof in [7]. In the progress condition, the use of tags ensure that multiple occurrences of the same inductive predicates are distinguished, so that a cycle is obtained by repeated case analysis of inductive predicates from the same set of mutually inductive predicates. In [7], this is ensured by defining the notion of trace for infinite LKID$^\omega$ proofs. The progress condition can be relaxed in a variety of ways, leading to simpler proofs in practice; however, this is outside the scope of this work.

In what follows, we shall assume that proof trees are implicitly tagged, and we shall not mention the tags unless they are relevant to the context.

We denote proofs by $D = (S, R, P, C)$, where $S$ is the set of sequents appearing in $D$, $R$ is function mapping every sequent $\Gamma \vdash \Delta \in S$ to the inference rule that was applied to it in the proof, $P$ is a function mapping every sequent to its set of premises, and $R$ is a partial function mapping cyclic leaves into their companions.

Example 6 (Cyclic proof): Figure 2 depicts a cyclic proof tree for the sequent $\forall k \cdot \text{nat} k \rightarrow \text{tplus} k 0 k$. In this tree, the leaf nodes where the notations $(\text{Cyclic}_1)$ and $(\text{Cyclic}_2)$ appear as rule names represent cyclic leaves, and the nodes marked with $\clubsuit 1$ and $\clubsuit 2$ represent the corresponding companion nodes. With this example, we want to explore how we can obtain an inductive proof from a cyclic one. Intuitively, as we start from the end sequent, we can mimic the application of rules in the cyclic proof to build the inductive proof, up to the point where we encounter a casesplit rule. The only choice there is to apply an induction rule. A possible candidate for the induction hypothesis is $\text{tplus} 0 0 p$. But that’s not good enough because we want to have a correspondence between the vertices in the inductive proof and the vertices in the cyclic proof, where for each vertex $v$ in the inductive proof that is not a translation artefact, there is a $v'$ in the cyclic proof such that if the sequent at $v$ is $\Gamma \vdash \Delta$, then the sequent at $v'$ is $\Gamma' \subseteq \Gamma, \vdash \Delta$. And clearly $\text{tplus} 0 0 p$ alone will not give us $\text{nat} p$ at $\bigtriangleup 2$. Therefore we set the induction hypothesis to $\text{tplus} 0 0 p \land \text{nat} p$. This generates a translation artefact, $\text{tplus} 0 0 p, \text{nat} p \vdash \text{nat} (S p)$, which can be easily discharged by a right introduction of nat followed by Axiom. We continue building the inductive proof by mimicking the cyclic one. For example, for the premise $\bigtriangleup 2$ in the inductive proof, we can get a derivation tree marked by the dotted polygon, that is isomorphic to the one marked by the dotted polygon in the cyclic proof. There are two leaf nodes in this dotted tree. One corresponds to the node where the cyclic rule is applied, which we can discharge by Axiom. The other corresponds to another casesplit in the cyclic proof. We apply induction in the same way and complete the proof.

IV. EQUIVALENCE OF LKID AND CLKID$^\omega$

In this section we present the main result of our paper, consisting of a translation scheme that converts cyclic proofs into inductive ones, and prove its correctness. We first introduce the necessary terminology.

Given a cyclic proof tree $D = (S, R, P, C)$, we say that a node $\Gamma \vdash \Delta$ is a case-split node, or a c-node if $R(\Gamma \vdash \Delta)$ is a case-split rule with the inductive predicate $P_i$ as a principal formula; we shall denote $P_i$ as $cs(\Gamma \vdash \Delta)$. We call the premises of a c-node descendants of case-split nodes, or d-nodes. By abuse of notation, we shall use $cs(\Gamma \vdash \Delta)$ to denote the principal formula of the inference that has $\Gamma \vdash \Delta$ as a premise when $\Gamma \vdash \Delta$ is a d-node. In our examples we shall mark c- and d-nodes by the symbols $\spadesuit 2$ and $\bigtriangleup 2$, respectively. The descendant-of-case-split-rooted-up-to-new-case-split tree, or dc-tree, rooted at a d-node $\Gamma_0 \vdash \Delta_0$ is the derivation tree obtained from the subtree of $D$ whose root is $\Gamma_0 \vdash \Delta_0$, by removing all subtrees rooted at all d-nodes distinct from $\Gamma_0 \vdash \Delta_0$. That is, no internal node of a dc-tree is a c-node, and its frontier contains only leaves of D and c-nodes. Given a dc-tree $T$, we say that $T$’s rules are applicable (or, for short, that $T$ is applicable) to a sequent $\Gamma \vdash \Delta$ if a correct derivation tree $T'$ can be formed with the endsequent $\Gamma \vdash \Delta$, such that there exists an isomorphism between $T$ and $T'$ that preserves the rules and their principal formulas across every pair of isomorphic elements.

If a node is both a d- and a c-node, then the dc-tree rooted at this node is empty. An empty dc-node is applicable to any sequent, leaving it unchanged.

Example 7 (dc-tree): In Figure 2 the dotted polygon encircles a dc-tree rooted at the sequent marked by $\bigtriangleup 2$. On its frontier, there is a c-node, marked with $\clubsuit 2$. In Figure 3 the dotted polygon represents the result of applying the dc-tree given in the previous figure to the sequent $\text{tplus} 0 0 p, \text{nat} p \vdash$
We denote by \( \mathcal{F}(P(T)) \) the set of predicates that are mutually inductive with \( P(T) \). If \( P(T) \) is tagged, the predicates in \( \mathcal{F}(P(T)) \) will have the same tags. The set of inductive predicates occurring in the antecedents of sequents of a proof tree may be partitioned into several disjoint families, identified tagged predicates. We shall denote the set of such families by \( \mathcal{F}(\mathcal{P}(\mathcal{D})) \), for a given cyclic proof \( \mathcal{D} \). Given a sequent \( \Theta \models \Gamma, P(T) \vdash \Delta \), we call \( \mathcal{E}(\Theta, P(T)) \) \( \models \Gamma \rightarrow \Delta \) the extract of \( \Theta \) w.r.t. \( P(T) \). The newly introduced conjunction, disjunction, and implication connectives are called distinguished connectives, and shall have special treatment in our translation scheme, as compared to the original connectives that appear in \( \Gamma \) and \( \Delta \).

A cyclic proof is in canonical form if all its cyclic leaves are c-nodes. A non-canonical proof can be easily converted by first unwinding the proof tree by pasting a copy of each companion-rooted subtree to the corresponding cyclic leaf. Doing this \textit{ad infinitum} will result in an infinite tree, representing in fact an LKID\( ^\infty \) proof tree, cf. [7]. On each infinite path, some c-node will occur infinitely often. Now, cut a finite “tip” of the infinite tree, so that its frontier cuts across the second occurrence of the infinitely-often occurring c-node on every infinite path. Then, the finite “tip” is a cyclic proof in canonical form.

Let \( \mathcal{D} = (S, R, P, C) \), and let \( P(T_1(\pi)) \) be a tagged inductive predicate such that various renamings \( P(T_1(\pi)) \), for some fresh set of variables \( \pi \), appear as principal formulas in c-nodes of \( S \). Let

\[
K_i = \left\{ \mathcal{E}(\Theta, P(T_1(\pi)))[\pi/\Theta] \mid \Theta \in S \text{ is a c-node with } cs(\Theta) = P(T_1(\pi)) \right\}
\]

and \( \mathcal{H}_i = P(T_1(\pi)) \cup K_i \), where \( \mathcal{H}_i \) is the induction hypothesis for
$P_\mathcal{T}(\mathcal{I}(\pi))$ extracted from $\mathcal{D}$. Here, the conjunctions appearing at the top level of $\mathcal{H}(\pi)$ are distinguished connectives, as are the connectives introduced by the extract notation.

The following lemma defines the simplest case of our translation scheme and establishes its correctness.

**Lemma 1:** Consider a canonical cyclic proof $\mathcal{D} = (S, R, P, C)$ with the following properties: (a) the endsequent is a $c$-node, and (b) $\text{Fam}(\mathcal{D})$ is a singleton. The following are true.

- Denote by $P_\mathcal{T}(\mathcal{I}_i)$ the tagged predicate representing the principal formula of $\mathcal{D}$'s endsequent. We apply the $(P, \text{IND})$ inference rule to the same endsequent, with $\mathcal{H}_{i,j}$ as an induction hypothesis for each predicate $P_\mathcal{T}(\mathcal{I}_{i,j}) \in \mathcal{F}(P_\mathcal{T}(\mathcal{I}_i))$. We then apply elimination rules for all the distinguished connectives in the consequents of all the resulting minor premises, we obtain an inductive derivation tree $\mathcal{D}'$ whose frontier has sequents of the form $\mathcal{H}_{\Theta} \vdash \Delta$, where $\Theta = \Pi \vdash \Delta$ is a $d$-node in $\mathcal{D}$, and $\mathcal{H}_{\Theta}$ is the induction hypothesis extracted from $\mathcal{D}$ for the tagged predicate that represents the case-analysis descendant of $c_0(\Theta)$ in $\Theta$.

- Each sequent $\mathcal{H}_{\Theta} \vdash \Delta$ on the frontier of $\mathcal{D}'$ will be fully
discharged by applying the corresponding dc-tree rooted at \( \Gamma \vdash \Delta \) in \( \mathcal{D} \), followed by elimination of remaining distinguished connectives.

**Proof:** Suppose the end sequent is \( \Gamma, P_1(\mathbf{u}) \vdash \Delta \). The major premise is \( \Gamma, \mathcal{H}_r(\mathbf{u}, \mathbf{z}_x \Gamma) \vdash \Delta \). Denote the root node of \( \mathcal{D} \) by \( \text{root} \). We have that \( \mathcal{E}(\text{root}, P_1(\mathbf{u})) \) is in \( \mathcal{H}_r \). We apply appropriately many \((\vee L), (\wedge L), (\wedge R), (\vee R)\) rules followed by \((W_k)\) to get \( \Gamma, (\bigvee \Delta) \vdash \Delta \), which can be trivially discharged.

For every production rule \( \Phi_{j,r} \), whose conclusion is a predicate what is mutually dependent with \( P_i \), we have a minor premise

\[
\Gamma, Q_1(\pi_1(\mathbf{y})), \ldots, Q_k(\pi_k(\mathbf{y})),
G_{j_1}[\mathbf{f}_1(\mathbf{y})/z_{j_1}], \ldots, G_{m}[\mathbf{f}_m(\mathbf{y})/z_{j_m}]
\vdash \mathcal{H}_r[\mathbf{f}_i(\mathbf{y})/z_i], \Delta
\]

where \( \mathbf{y} \) is a vector of distinct fresh variables.

let \( S = \{ k | 1 \leq k \leq m \land P_k, \mathbf{z}_x \in \mathcal{F}(P_i) \} \).

we apply \((W_k)\) and \((\bigvee L)\) appropriate times on those \( G_{j_x}, x \in S \) to remove the distinguished connectives and get:

\[
\Omega, Q_1(\pi_1(\mathbf{y})), \ldots, Q_k(\pi_k(\mathbf{y})),
P_{j_1}[\mathbf{f}_1(\mathbf{y})/z_{j_1}], \ldots, P_{j_m}[\mathbf{f}_m(\mathbf{y})/z_{j_m}]
\vdash \mathcal{H}_r[\mathbf{f}_i(\mathbf{y})/z_i]
\]

where \( \Omega = (\bigcup_{x \in S} \mathcal{K}_{j_x}[\mathbf{f}_x(\mathbf{y})/z_{j_x}]) \).

We apply \((\bigvee R)\) appropriately many times to break all the distinguished connectives and get branches of the form:

\[
\Omega, Q_1(\pi_1(\mathbf{y})), \ldots, Q_k(\pi_k(\mathbf{y})),
P_{j_1}[\mathbf{f}_1(\mathbf{y})/z_{j_1}], \ldots, P_{j_m}[\mathbf{f}_m(\mathbf{y})/z_{j_m}],
\vdash F[\mathbf{f}_i(\mathbf{y})/z_i]
\]

where \( F \in \mathcal{K}_r \cup P_2(\mathbf{z}_x) \).

Suppose \( F = P_2(\mathbf{z}_x) \), notice that all the premises for the right introduction rule corresponding to \( \Phi_{i,r} \) are in the context, so we can discharge it by applying the right introduction rule, and then the (Ax) rule for all the premises.

Otherwise, \( F \in \mathcal{K}_r \). Therefore we have a \( \Theta \) such that \( \mathcal{E}(\Theta, P_1(\mathbf{f}_1(\mathbf{y}))) \vdash \mathcal{E}(\Theta, \mathbf{y}) \), and \( S(\Theta) = \Gamma, P_1(\mathbf{f}_1(\mathbf{y})) \vdash \Delta \). So \( F[\mathbf{f}_i(\mathbf{y})/z_i] = \bigvee \Gamma \vdash \bigvee \Delta \).

We apply the \((\rightarrow R), (\bigvee L), (\bigvee R)\) appropriate times to remove the distinguished connectives and get the sequent:

\[
\Omega, \Gamma[\mathbf{f}_i(\mathbf{y})/\mathbf{y}], Q_1(\pi_1(\mathbf{y})), \ldots, Q_k(\pi_k(\mathbf{y})),
P_{j_1}[\mathbf{f}_1(\mathbf{y})/z_{j_1}], \ldots, P_{j_m}[\mathbf{f}_m(\mathbf{y})/z_{j_m}],
\vdash \Delta[\mathbf{f}_i(\mathbf{y})/\mathbf{y}]
\]

which is denoted by \( L \).

Let \( \mathbf{vp} \) be the premise of \( \Theta \) in \( \mathcal{D} \) that corresponds to the production rule \( \Phi_{i,r} \), then \( \mathbf{vp} \) is a d-node, and according to
the casesplit rule schema, \( \mathcal{S}(vp) \) is
\[
\Gamma \vdash \Delta = \mathcal{S}(vp),
\]
Let \( \Gamma \vdash \Delta = \mathcal{S}(vp) \), then \( L[\pi/\gamma] = \Delta, \Gamma \vdash \Delta \), exactly the shape we want.

In the process of applying the corresponding dc-tree to the frontier, there can be 3 cases when we reach the leaf of the dc-tree. If we hit a leaf, it’s discharged. If we reach a casesplit, it’s necessarily on some case-analysis descendant of \( cs(\Theta) \), say \( P_\varphi \), \( \mathcal{H} \) must contain an extracted induction hypothesis for \( P_\varphi \). Therefore we can trivially discharge it. If we reach a cyclic leaf, since the cyclic proof is canonical, it must be identical to some \( c \)-node. By the same argument for the casesplit case, we can discharge it.

---

**Example 8:** Consider the following inductive definitions:

\[
\begin{array}{cccccc}
\alpha & 0 \quad & a \quad & x & \gamma & y \\
\beta \quad & a \quad & y & \quad & \quad & \quad \\
\gamma & 0 \quad & b \quad & x \quad & \quad & \quad \\
\delta \quad & b \quad & \quad & \quad & \quad & \quad \\
\end{array}
\]

and the sequent \( a \cdot x \vdash eqo \cdot x, eqo \cdot y \). Figure 4 presents a cyclic proof for this sequent, whereas Figures 5 and 6 present its inductive proof. The inductive proof is obtained by simply mechanically applying the translation scheme described in Lemma 1 and is intentionally complicated so as to showcase the scheme in its generality. First, we note in Figure 4 that the
c- and d-nodes are marked with ♣ symbol and ◊ symbols. Dc-trees are easy to identify, as most of them have a single branch. For instance, the node marked with ◊15 is the root of the d-tree that applies the rules (eqoR2) and (Subst). The cyclic proof contains several shortcuts. Repeating, non-cyclic nodes are marked with the \( \heartsuit \) symbol, in order to save space. They simply indicate that the corresponding proof subtree can be pasted at the corresponding (Idem) node to obtain a full blown canonical proof.

To create an inductive proof, we need to create induction hypotheses first. For instance, the induction hypothesis for \( a \times y \) would be created from all the c-nodes that have a variant of \( a \times y \) in the antecedent. Thus, \( \mathcal{H}_a \) is

\[
\begin{align*}
&\left((\text{eqe } x (S y) \lor \text{eqo } x (S y))\right)\land \\
&\left((\text{eqo } x (S y) \lor \text{eqo } (S x) (S^2 y))\right)\land \\
&\left((\text{eqo } x (S x) (S^2 y) \lor \text{eqo } (S x) (S^3 y))\right)\land \\
&\left((\text{eqo } (S^2 x) (S^3 y) \lor \text{eqe } x (S y))\right)
\end{align*}
\land a \times y
\]

Induction hypotheses for \( b \times y \) and \( c \times y \) can be obtained in a similar manner, and they appear in the consequents of the minor premises in the inductive proof. In the general case, the predicate itself should be part of its own induction hypothesis. However, this is not always necessary, as the predicate instance does not always play a role in the proof. Since it is the case with the current proof, to simplify the proof, we have omitted the predicates \( a \times y, b \times y \) and \( c \times y \) from their inductive hypotheses.

Once the induction hypotheses have been created, the inductive principle can be applied, creating 7 minor premises. Each of the premises is shown separately. To each premise we can apply rules that eliminate distinguished connectives that appear in the consequent of sequents. After this step, all resulting inductive sequents will have a correspondent in the cyclic proof. The correspondence is indicated by using the same markings. For instance, node \( \heartsuit14 \) in the cyclic proof corresponds to node \( \diamondsuit14 \) of the inductive proofs. The inductive sequent \( \diamondsuit14 \) looks very similar to the corresponding cyclic one, but has an extra induction hypothesis, the one for \( c \times y \), in its antecedent. Obviously, the sequence of rules applied to the cyclic \( \diamondsuit14 \) can also be applied to the inductive \( \heartsuit14 \), resulting in a node that corresponds to \( \clubsuit2 \). This sequent can be discharged by further elimination of distinguished connectives.

It is clear why this discharge is possible. The consequent of \( \clubsuit2 \) was used in creating an induction hypothesis for \( b \times y \), and is present in all antecedents of the subtree rooted at \( \diamondsuit14 \). This type of arrangement is used consistently, and will lead to discharging all minor premises using the dc trees contained in the cyclic tree.

Given a cyclic proof \( D \), and a family of tagged mutually inductive predicates \( \Psi \), consider the result of removing from \( D \) all the dc-trees whose root \( \Theta \) has the property that \( cs(\Theta) \in \Psi \). This set is in general a set of cyclic derivation trees. Consider a tree \( T \) in this set, and let us examine its frontier. There are two types of reasons why \( T \) is no longer a valid cyclic proof:

1. the companion of a cyclic leaf now appears in a different tree of the set (i.e. the cyclic leaf is broken); or
2. a node that used to be the root of a sub-proof has now become a leaf – we shall call such nodes open. Now, we can fix the broken cyclic leaves by pasting at that node a copy of the subtree rooted at the old companion. Doing this repeatedly will eliminate the broken cyclic leaves, while possibly increasing the number of open nodes. We denote by \( \text{Erase}(D, \Psi) \) the set that contains all the derivation trees obtained by removing \( D \) all the dc-trees whose root \( \Theta \) has the property that \( cs(\Theta) \in \Psi \), and then repairing the broken cyclic nodes of the resulting subtree.

Consider now a proof tree \( T \in \text{Erase}(D, \Psi) \), for some cyclic proof \( D \), and some family of mutually inductive predicates \( \Psi \). Denote by \( O \) the set of open leaves in \( T \), and let \( H \) be the formula

\[
\bigwedge_{r, A \in O} \left( \bigwedge_{\Gamma \rightarrow \Delta} \bigvee \Delta \right),
\]

where the connectives introduced in \( H \) are distinguished.

Then, the tree \( T' \) obtained from \( T \) by adding \( H \) to the antecedent of every sequent can be easily converted into a cyclic proof by elimination rules for the distinguished connectives at the formerly open nodes. We shall denote the cyclic proof thus obtained by \( \text{Fix}(T) \).

**Lemma 2:** Consider a canonical cyclic proof \( D = (S, R, P, C) \) whose endsequent is a c-node. Denote by \( P_i(T_i) \) the tagged predicate representing the principal formula of \( D \)'s endsequent. We apply the \((P, \text{IND})\) inference rule to the same endsequent, with \( \mathcal{H}_i \) as an induction hypothesis for each predicate \( P_j(T_j) \in F(P_i(T_i)) \). We then apply elimination rules for all the distinguished connectives in the consequents of all the resulting minor premises, we obtain an inductive derivation tree \( D' \). There are two types of nodes on the frontier of \( D \):

- Sequents of the form \( \mathcal{H}, \Gamma \vdash \Delta \), where \( \Theta \equiv \Gamma \vdash \Delta \) is a d-node in \( D \), and \( \mathcal{H} \) is the induction hypothesis extracted from \( D \) for the tagged predicate that represents the case-analysis descendant of \( \text{cs}(\Theta) \) in \( \Theta \). Each such sequent will be fully discharged by applying the corresponding \( dc \)-tree rooted at \( \Gamma \vdash \Delta \) in \( D \), followed by elimination of remaining distinguished connectives.
- Sequents of the form \( \mathcal{H}, \Gamma \vdash \Delta \), where \( \Theta \equiv \Gamma \vdash \Delta \) is a c-node in \( D \), such that \( \text{cs}(\Theta) \notin F(P_i) \); however, \( \Theta \) is on the frontier of a \( dc \)-tree rooted at some sequent \( \Theta' \), with \( P_j(T_j) = \text{cs}(\Theta') \in F(P_i(T_i)) \) and \( \mathcal{H} \) being the induction hypothesis used in the application of \((P, \text{IND})\) for the case-analysis descendant of \( P_j(T_j) \) residing in \( \Theta' \). Moreover \( \mathcal{H}, \Gamma \vdash \Delta \) is the endsequent of the cyclic proof \( \text{Fix}(T) \) where \( T \) is the derivation tree rooted at \( \Theta \) in \( \text{Erase}(D, F(P_i(T_i))) \).

**Proof:** For d-node frontiers, discharge them by methods described in Lemma 1. For c-node frontier case. We must have collected \( \Delta[z_i/u] \) at some c-node \( \Theta' \) in \( D \), where \( \text{cs}(\Theta') \in F(P_i(T_i)) \), and \( \Theta' \) has a premise \( vp \), where \( S(vp) = \Gamma \vdash \Delta \). Now \( \Theta \) is on the frontier of a dc-tree rooted at \( \Theta' \), and \( \mathcal{H} \) is the induction hypothesis used in the application of \((P, IND)\) for the case-analysis descendant of \( P_j(T_j) \) residing in \( \Theta' \). In
fact, $\Theta$ is a premise of $\Theta'$. In $\text{Erase}(D, \mathcal{F}(P_i(t_i)))$, $\Theta$ is not a descendant of any other node. Therefore it is the endsequent of the cyclic proof $\text{Fix}(T)$, where $T$ is the derivation tree rooted at $\Theta$.

It is worthy of noting at this point that the presence of the predicate $P_i(t_i)$ in its own inductive hypothesis is indeed necessary, so as to ensure that for each $d$-node, its corresponding inductive node is a strict weakening. This furthermore ensures that any sequence of rules applicable to the $d$ node is also applicable to its corresponding inductive node. For instance, let us examine closely the cyclic proof in Figure 2 and its inductive translation given in Figure 3. In this proof, the inductive hypothesis for the predicate $\text{nat}$ is $\text{nat} \ k \land t\text{plus} \ 0 \ k$. Had we not added $\text{nat} \ k$ to the inductive hypothesis, the weakening just above the right premise of the cut rule would not have been possible. That is to say that the $d$-tree rooted at the $\text{nat} \ 2$ node in the cyclic proof would not have been applicable to the corresponding node in the inductive proof, invalidating both Lemmas [1] and [2].

**Theorem 1 (Former Conjecture 7.7):** Let $D$ be a canonical CLKID$\omega$ proof with endsequent $\Gamma \vdash \Delta$. Then, there exists an LKID proof of $\Gamma \vdash \Delta$.

**Proof:** First, we isolate and remove from $D$ the subtrees rooted at the shallowest c-nodes (the ones that do not have other c-node ancestors). The result of the removal is a derivation tree that is valid in LKID. If we convert the cyclic c-node rooted proof trees that were removed into inductive proof trees, and then paste them back, we obtain a valid inductive proof for the original sequent. Thus, we can focus only on converting c-node rooted proof trees into inductive ones. Following Lemma [2], we apply the $(P_i, \text{IND})$ induction rule to the endsequent $\Theta \equiv \Gamma \vdash \Delta$, where $P_i = \text{cs}(\Theta)$ in $D$. After elimination of distinguished connectives, we obtain a valid inductive derivation tree $D'$. On the frontier of $D'$ we have sequents of the form $\mathcal{H}, \Gamma' \vdash \Delta'$ which can be in either of the following two situations. The first situation is that $\Gamma' \vdash \Delta'$ is the root of a $d$-tree $T$ in $D$, in which $\mathcal{T}$ is applicable to $\mathcal{H}, \Gamma' \vdash \Delta'$, and leads sequents that can be trivially discharged by application of elimination rules for distinguished connectives. Thus, part of the outstanding sequents after the application of the inductive principle will be discharged. The second situation is that $\Gamma' \vdash \Delta'$ is a c-node in $D$. In this case, $\mathcal{H}, \Gamma' \vdash \Delta'$ cannot be immediately discharged. However, each of the sequents that cannot be immediately discharged is in the set of endsequents of proofs in the set $\{\text{Fix}(T) | T \in \text{Erase}(D, P_i(t_i))\}$. Now, each of these proofs is strictly smaller than $D$, and can be recursively converted into an inductive proof by Lemma [2]. Each of the inductive proofs can be pasted at the right place in the current inductive derivation tree, to produce a valid inductive proof tree for the original endsequent, which proves the theorem. Most notably, our (meta)proof is in fact a cyclic proof!

## V. Further Work

In [7], the discussion before Conjecture 7.7 states that cut is probably not eliminable from CLKID$\omega$. This statement also appears as Conjecture 5.2.4 in [5]. We believe that this is true, since the sequent $\text{nat} \ k \vdash t\text{plus} \ 0 \ k$, which has a non-cut-free proof, as shown in one of our examples, also has a unique and infinite derivation in a cut-free CLKID. Both uniqueness and non-finiteness are easy to establish, and show that, since there exists a sequent with no cut-free proof, but provable in the presence of cut, the cut inference rule is not admissible in CLKID$\omega$.

Moreover, we believe that it is possible to define a notion of anchored cut similar to Gentzen’s (cf. [8]). Indeed, the only type of cuts that really are needed are the ones where the cut formulas are inductive predicates that have previously been subjected to case-analysis rules; all other cuts can be easily eliminated.

Thus, by defining free cuts as non-anchored cuts, a free-cut elimination result can be obtained, leading to a natural analogue of the subformula property. It would then naturally follow that, due to the translation scheme presented in this work, the induction calculus LKID would enjoy a form of subformula property too. This is contrary to popular belief (cf. [10]), but not necessarily surprising at this point, since it has become clear that the induction hypotheses, previously believed to be completely arbitrary, are made up exclusively from sequents appearing in the cyclic proof, and can, in the presence of a free-cut elimination result, be obtained by recombining parts of the endsequent.

## References

1. Peter Aczel. *Handbook of Mathematical Logic*, pages 739–782. North-Holland, 1977.

2. G. Barthe and S. Stratulat. Validation of the javacard platform with implicit induction techniques. In R. Nieuwenhuis, editor, *Rewriting Techniques and Applications*, 14th International Conference, RTA-03, LNCS 2706, pages 337–351, Valencia, Spain, June 9–11, 2003. Springer.

3. Jon Barwise. *An Introduction to First-Order Logic*, pages 5–46. Studies in Logic and the Foundations of Mathematics. North-Holland, 1977.

4. James Brotherston. Cyclic proofs for first-order logic with inductive definitions. In *Automated Reasoning with Analytic Tableaux and Related Methods: Proceedings of TABLEAUX*, volume 3702 of Lecture Notes in Artificial Intelligence, pages 78–92. Springer, 2005.

5. James Brotherston. *Sequent Calculus Proof Systems for Inductive Definitions*. PhD thesis, University of Edinburgh, 2006.

6. James Brotherston and Alex Simpson. Sequent calculi for induction and infinite descent. In *LICS*, pages 51–62, 2007.

7. James Brotherston and Alex Simpson. Sequent calculi for induction and infinite descent. *Journal of Logic and Computation*, 2010.

8. Samuel R. Buss. *An Introduction to Proof Theory*, pages 3–78. Studies in Logic and the Foundations of Mathematics. North-Holland, 1998.

9. Joxan Jaffar, Andrew E. Santos, and Razvan Voicu. A coinduction rule for entailment of recursively defined properties. In Peter J. Stuckey, editor, *CP*, volume 5202 of Lecture Notes in Computer Science, pages 493–508. Springer, 2008.

10. Per Martin-Löf. Hauptspitz for the intuitionistic theory of iterated inductive definitions. In J.E. Fenstad, editor, *Proceedings of the Second Scandinavian Logic Symposium*, pages 179–216. North-Holland, 1971.

11. Huu Hai Nguyen, Cristina David, Shengchao Qin, and Weingan Chin. Automated verification of shape and size properties via separation logic. In *In VMCAI*. Springer, 2007.

12. C. Sprenger and M. Dam. A note on global induction mechanisms in a $\mu$-calculus with explicit approximations. *Theoretical Informatics and Applications*, 37:365–399, 2003.
[13] C. Sprenger and M. Dam. On the structure of inductive reasoning: circular and tree-shaped proofs in the \( \mu \)-calculus. In *Proceedings of Foundations of Software Science and Computation Structures (FOS-SACS'03)*, volume 2620 of *Lecture Notes in Computer Science*, pages 425–440. Springer, 2003.

[14] Sorin Stratulat. Automatic 'descente infinie' induction reasoning. In Bernhard Beckert, editor, *TABLEAUX*, volume 3702 of *Lecture Notes in Computer Science*, pages 262–276. Springer, 2005.

[15] M.E. Szabo, editor. *The Collected Papers of Gerhard Gentzen*. North-Holland, 1969.

[16] Claus-Peter Wirth. History and future of implicit and inductionless induction: Beware the old jade and the zombie! In Dieter Hutter and Werner Stephan, editors, *Mechanizing Mathematical Reasoning*, volume 2605 of *Lecture Notes in Computer Science*, pages 192–203. Springer, 2005.

[17] Claus-Peter Wirth. A self-contained and easily accessible discussion of the method of descente infinie and fermat's only explicitly known proof by descente infinie. *CoRR*, abs/0902.3623, 2009. informal publication.