ON A HYPERBOLIC-PARABOLIC MIXED TYPE EQUATION

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Abstract. In this paper, the hyperbolic-parabolic mixed type equation
\[ \frac{\partial u}{\partial t} = \Delta A(u) + \text{div}(b(u)), \quad (x,t) \in \Omega \times (0,T), \]
with the homogeneous boundary condition is considered. We find that only a part of the boundary condition is able to ensure the posedness of the solutions. By introducing a new kind of entropy solution matching the part boundary condition in a special way, we obtain the existence of the solution by the BV estimate method, and establish the stability of the solutions by the Kruzkov bi-variables method.

1. Introduction. Consider the equation
\[ \frac{\partial u}{\partial t} = \Delta A(u) + \text{div}(b(u)), \quad (x,t) \in \Omega \times (0,T), \tag{1} \]
with the homogeneous boundary condition, where \( \Omega \subset \mathbb{R}^N \) is an open bounded domain with the appropriately smooth boundary \( \partial \Omega \). We assume that
\[ A(u) = \int_0^u a(s)ds, \quad a(s) \geq 0, a(0) = 0. \]
Equation (1) is of hyperbolic-parabolic mixed type. If \( A \equiv 0 \), it becomes the well-known conservation law equation, which belongs to the hyperbolic equations. The equation of the flow in a porous medium,
\[ \frac{\partial u}{\partial t} = \Delta u^m, \]
and the stationary boundary layer theory equation
\[ \frac{\partial w}{\partial x} + v_0(x) \frac{\partial w}{\partial y} = \nu \sqrt{w} \frac{\partial^2 w}{\partial y^2}, \tag{2} \]
are two special cases of equation (1), which belong to the degenerate parabolic equations. Here, in equation (2), \( v_0(x) \) is an initial velocity, and \( \nu \) is the viscous coefficient of the flow. One can refer to [22] for details. For the Cauchy problem of equation (1), there are considerable attention on its well-posedness, see [1, 2, 3, 4, 5, 6, 7, 17, 27, 28, 29, 33, 35, 37, 38] and the references therein.

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While considering the initial-boundary value problem of equation (1), usually we need the initial condition
\[ u(x, 0) = u_0(x), \; x \in \Omega. \] (3)

However, can we give the Dirichlet homogeneous boundary condition as follows?
\[ u(x, t) = 0, \; (x, t) \in \partial \Omega \times (0, T). \] (4)

When the equation is weakly degenerate, it is well known that we can give the Dirichlet homogeneous boundary condition (4) [33]. When the equation is strongly degenerate, there are two ways to deal with the corresponding problem. One is based on the $BV$ analysis technique. In general, instead of the whole boundary $\partial \Omega$, only a part of the boundary $\Sigma_p \subseteq \partial \Omega$ on which the trace of $u$ can be endowed in the traditional sense [31, 32, 34]:
\[ u(x, t) = 0, \; (x, t) \in \Sigma_p \times (0, T). \] (5)

The other way is that, the entropy solutions are defined only in $L^\infty$ space, the existence of the traditional trace (which was called the strong trace in [16]) on the boundary is not guaranteed, so the boundary condition is not directly shown in the traditional way as (4), it is implicitly contained in a family entropy inequalities [8, 13, 16, 18-21, 26].

The advantage of the first way lies in that, we can figure out the portion of the boundary on which we can impose the boundary value, whereas the rest portion of the boundary is free from the limitation of the boundary condition.

If the domain $\Omega = \mathbb{R}^N_+$ is the half space of $\mathbb{R}^N$, in our previous work [37], we studied the initial-boundary value problem of equation (1) in the half space
\[ \frac{\partial u}{\partial t} = \Delta A(u) + \text{div}(b(u)), \; (x, t) \in \mathbb{R}^N_+ \times (0, T). \]
We have proved that if $b'_N(0) < 0$, we can give the general Dirichlet boundary condition
\[ u(x, t) = 0, \; (x, t) \in \partial \mathbb{R}^N_+ \times (0, T), \]
which is satisfied in a particular weak sense. But if $b'_N(0) \geq 0$, then no boundary condition is necessary, the solution of the equation is free from any limitation of the boundary condition.

In this paper, we continue to investigate how to find a suitable homogeneous boundary condition as (5) by the first way. According to [37], if we regard equation (1) as a “linear” degenerate elliptic equation [10-11, 23-24]), then we know that $\Sigma_p$ defined in (5) should be as
\[ \Sigma_p = \{ x \in \partial \Omega : b'_N(0)n_i(x) < 0 \}. \] (6)
In this study, we will introduce a new kind of entropy solution to match the above homogeneous boundary condition (5). The aim of this paper is to get the well-posedness of the new kind of entropy solutions.

2. Main results. For small $\eta > 0$, let
\[ S_\eta(s) = \int_0^s h_\eta(\tau)d\tau, \; h_\eta(s) = \frac{2}{\eta} \left( 1 - \frac{|s|}{\eta} \right)_+. \]
Obviously $h_\eta(s) \in C(\mathbb{R})$, and
\[ h_\eta(s) \geq 0, \quad |sh_\eta(s)| \leq 1, \quad |S_\eta(s)| \leq 1; \lim_{\eta \to 0} S_\eta(s) = \text{sgns}, \lim_{\eta \to 0} sS_\eta(s) = 0. \]
Let
\[ \Sigma_{1\eta k} = \{ x \in \partial \Omega, \ S_\eta(k) b_i(0) - b_i(k) n_i(x) > 0 \}, \]
\[ \Sigma_{2\eta k} = \{ x \in \partial \Omega, \ S_\eta(k) b_i(0) - b_i(k) n_i(x) \leq 0 \}. \]
Here and in what follows, \( \{n_i\}_{i=1}^N \) is the inner normal vector of \( \Omega \). Clearly, \( \partial \Omega = \Sigma_{1\eta k} \cup \Sigma_{2\eta k} \).

Let
\[ \Sigma_1 = \bigcup_{\forall \eta \geq 0, \forall k \in \mathbb{R}} \Sigma_{1\eta k}, \quad \Sigma_2 = \partial \Omega \setminus \Sigma_1. \]

Now, we can consider two cases.

Case I. If \( \Sigma_1 \neq \emptyset \) is a real subset of \( \partial \Omega \), we will show that if equation (1) is degenerate on the boundary, instead of the usual Dirichlet homogeneous boundary condition (3), we only can give the part boundary condition as
\[ \gamma u \big|_{\Sigma_1 \times (0,T)} = 0, \]
to get the well-posedness of the initial-boundary value problem of equation (1).

In fact, by the definition of \( \Sigma_{1\eta k} \), we know that
\[ 0 < S_\eta(k) b_i(0) - b_i(k) n_i(x) = -k S_\eta(k) b'_i(\zeta) n_i(x), \]
where \( \zeta \in (k,0) \). If we let \( \eta \to 0 \). Then
\[ b'_i(\zeta) n_i(x) < 0. \]
Let \( k \to 0 \). We know that
\[ b'_i(0) n_i(x) < 0, \]
which is in accordance with (6).

**Definition 2.1.** If \( \Sigma_1 \neq \emptyset \) is a subset of \( \partial \Omega \), a function \( u \) is said to be the entropy solution of equation (1) with the initial condition (3) and the homogeneous boundary condition (7), provided the following conditions are true.

1. \( u \) satisfies
\[ u \in BV(Q_T) \cap L^\infty(Q_T), \quad \frac{\partial}{\partial x_i} \int_0^u \sqrt{a(s)} ds \in L^2(Q_T). \]

2. For any \( \varphi_1, \varphi_2 \in C^2(\overline{Q}_T), \varphi_1 \geq 0, \nabla \varphi_1 \big|_{\partial \Omega \times (0,T)} = 0, \varphi_1 \big|_{\partial \Omega \times [0,T]} = \varphi_2 \big|_{\partial \Omega \times [0,T]} \), and \( \text{supp} \varphi_2 \subset \overline{\Omega} \times (0,T) \), with \( k \in \mathbb{R} \) and small \( \eta > 0 \), \( u \) satisfies
\[ \iint_{Q_T} \left[ I_\eta(u-k) \varphi_{1t} - B'_\eta(u,k) \varphi_{1x_1} + A_\eta(u,k) \Delta \varphi_1 \right] dxdt \]
\[ - \iiint_{Q_T} S'_\eta(u-k) \nabla \int_0^u \sqrt{a(s)} ds \big| \varphi_1 dxdt \]
\[ + S_\eta(k) \int_{Q_T} [u \varphi_{2t} - (b_i(u) - b_i(0)) \varphi_{2x_i} + A(u) \Delta \varphi_2] dxdt \]
\[ + S_\eta(k) \int_0^T \int_{\Sigma_{1\eta k}} ((b_i(0) - b_i(k)) n_i \varphi_1) dt ds \geq 0. \]

3. The homogeneous boundary value is true in the sense of trace,
\[ \gamma u \big|_{\Sigma_1 \times (0,T)} = 0. \]

4. The initial value is true in the sense of
\[ \lim_{t \to 0} \int_\Omega | u(x,t) - u_0(x) | dx = 0. \]
Here the pairs of equal indices imply a summation from 1 up to \( N \), and
\[
B_{\eta}(u, k) = \int_0^u b_1(s) s_\eta(s - k) ds, \quad A_{\eta}(u, k) = \int_0^u a(s) s_\eta(s - k) ds,
\]
\[
I_{\eta}(u - k) = \int_0^{u - k} s_\eta(s) ds.
\]

Case II. If \( \Sigma_1 = \emptyset \), no any other boundary value condition is necessary. In other words, the solution of equation (1) is completely controlled by the initial condition. Then Definition 2.1 can be re-stated as

**Definition 2.2.** A function \( u \) is said to be the entropy solution of equation (1), with the initial value (3), if the following three conditions are true.

1. \( u \) satisfies
   \[
   u \in BV(Q_T) \cap L^\infty(Q_T), \quad \frac{\partial}{\partial x_i} \int_0^u \sqrt{a(s)} ds \in L^2(Q_T).
   \]

2. For any \( \varphi \in C^2_0(Q_T) \), \( \varphi \geq 0 \), \( k \in R \), and small \( \eta > 0 \), \( u \) satisfies
   \[
   \int_{Q_T} \left[ I_{\eta}(u - k) \varphi_t - B_{\eta}(u, k) \varphi_x + A_{\eta}(u, k) \Delta \varphi \right] dx dt
   - \int_{Q_T} s_{\eta}'(u - k) | \nabla \int_0^u \sqrt{a(s)} ds |^2 \varphi dx dt \geq 0.
   \] (11)

3. The initial value is true in the sense of
   \[
   \lim_{t \to 0} \int_{\Omega} u(x, t) - u_0(x) | dx = 0.
   \]

On the one hand, if equation (1) has a classical solution \( u \), multiplying (1) by \( \varphi_1 s_{\eta}(u - k) \) and integrating over \( Q_T \), we are able to show that \( u \) satisfies Definition 2.1. Multiplying (1) by \( \varphi s_{\eta}(u - k) \) and integrating over \( Q_T \), we are able to show that \( u \) satisfies Definition 2.2.

On the other hand, in the first case of that \( \Sigma_1 \neq \emptyset \) is a real subset of \( \partial \Omega \), let \( \eta \to 0 \) in (8). One has
\[
\int_{Q_T} |u - k| \varphi_1 t - \text{sgn}(u - k)(b_i(u) - b_i(k)) \varphi_1 x_i + \text{sgn}(u - k)(A(u) - A(k)) \Delta \varphi_1 | dx dt
+ \text{sgn}(k) \int_{Q_T} (u \varphi_2 t - (b_i(u) - b_i(0)) \varphi_2 x_i + A(u) \Delta \varphi_2) dx dt
- s_\eta(k) \int_0^T \int_{\Sigma_{1k}} [(b_i(0) - b_i(k))] n_i \varphi_1 dt d\sigma \geq 0.
\]

Let \( \varphi_2 = 0 \) and so \( \varphi_1 \big|_{\partial \Omega \times (0, T)} = 0 \). Then
\[
\int_{Q_T} |u - k| \varphi_1 t - \text{sgn}(u - k)(b_i(u) - b_i(k)) \varphi_1 x_i | dx dt
+ \int_{Q_T} \text{sgn}(u - k)(A(u) - A(k)) \Delta \varphi_1 dx dt \geq 0.
\]

In the second case of that \( \Sigma_1 = \emptyset \), let \( \eta \to 0 \) in (11). We have
\[
\int_{Q_T} |u - k| \varphi t - \text{sgn}(u - k)(b_i(u) - b_i(k)) \varphi x_i | dx dt
\]
Thus if 

Thus if \( u \) is the entropy solution in Definition 2.1, then \( u \) is an entropy solution defined in \[3, 27, 37\] et al.

**Theorem 2.3.** Suppose that \( A(s) = C^3 \), \( b_i(s) = C^2 \), and \( u_0(x) \in L^\infty(\Omega) \). Suppose that

\[
A'(0) = a(0) = 0.
\]

If \( \Sigma_1 \neq \emptyset \) is a subset of \( \partial \Omega \), then equation (1) with the initial-boundary value conditions (3) and (7) has an entropy solution in the sense of Definition 2.1. If \( \Sigma_1 = \emptyset \), then equation (1) with the initial condition (3) has an entropy solution in the sense of Definition 2.2.

We start the proof by considering the following regularized problem

\[
\frac{\partial u}{\partial t} = \Delta A(u) + \varepsilon \Delta u + \sum_{i=1}^{N} \partial b_i(u) \frac{\partial}{\partial x_i}, \quad \text{in } Q_T = \Omega \times (0, T),
\]

with

\[
u(x, 0) = u_0(x), \quad x \in \Omega,
\]

\[
u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T).
\]

We denote the solution of problem (12)-14 as \( u_\varepsilon \). Let \( u \) be the solution of the problem (1)-3-7. If it is the limit of the approximate solution \( u_\varepsilon \), we say \( u \) is a viscous solution. Since the solution \( u \) is in the BV sense, the trace of \( u \) on the boundary \( \partial \Omega \) can always be defined in the traditional way. Now, if \( u \) is a viscous solution, we know that on the whole boundary \( \partial \Omega \), it has

\[
u = \lim_{\varepsilon \to 0} u_\varepsilon = 0, \quad x \in \partial \Omega.
\]

However, for a general entropy solution \( u \) in the sense of Definition 2.1, it satisfies

\[
\gamma u \mid_{\Sigma_1 \times (0, T)} = 0.
\]

Whereas on the other part \( \Sigma_2 \times (0, T) \), it is unclear if \( u \) is equivalent to 0 or not. Based on this observation, we will prove the following stabilities.

**Theorem 2.4.** Suppose that \( A(s) = C^2 \) and \( b_i(s) = C^1 \). If \( \Sigma_1 \neq \emptyset \) is a subset of \( \partial \Omega \), let \( u \) and \( v \) be solutions of equation (7) with the different initial values \( u_0(x), v_0(x) \in L^\infty(\Omega) \) respectively. Suppose that

\[
\gamma u(x, t) = f(x, t), \quad \gamma v = g(x, t), \quad (x, t) \in \partial \Omega \times (0, T),
\]

and in particular,

\[
\gamma u = \gamma v = 0, \quad x \in \Sigma_1.
\]

Suppose that the distance function \( d(x) = \text{dist}(x, \partial \Omega) \) satisfies that

\[
|\Delta d| \leq c, \quad \frac{1}{\lambda} \int_{\Omega_\lambda} dxdt \leq c,
\]

where \( \lambda \) is a small constant, and \( \Omega_\lambda = \{ x \in \Omega, d(x, \partial \Omega) < \lambda \} \). Then there holds

\[
\int_{\Omega} |u(x, t) - v(x, t)| \ dx \leq \int_{\Omega} |u_0 - v_0| \ dx + c \cdot \sup_{\Sigma_2 \times (0, T)} |f(x, t) - g(x, t)|,
\]

where \( (x, t) \in R^{N+1} \), \( c \sup_{\Sigma_2 \times (0, T)} |f(x, t) - g(x, t)| \) is in the sense of \( N \)-dimensional Hausdorff measure, and \( \sup \) is the usual essential supremum.
Theorem 2.5. Suppose that $A(s)$ is $C^2$ and $b_i(s)$ is $C^1$. If $\Sigma_1 = \emptyset$, let $u$ and $v$ be solutions of equation (41) with the different initial values $u_0(x)$, $v_0(x) \in L^\infty(\Omega)$ respectively. Suppose that the distance function $d(x)$ satisfies (17), and 

$$\gamma u(x,t) = f(x,t), \ \gamma v = g(x,t), \ (x,t) \in \partial \Omega \times (0,T).$$

Then it has

$$\int_\Omega |u(x,t) - v(x,t)| \ dx \leq \int_\Omega |u_0 - v_0| \ dx + c \sup_{\partial \Omega \times (0,T)} |f(x,t) - g(x,t)|.$$

Clearly, if $u, v$ appearing in Theorem 2.4-Theorem 2.5 are the two viscous solutions, then we have

$$\int_\Omega |u(x,t) - v(x,t)| \ dx \leq \int_\Omega |u_0 - v_0| \ dx.$$

3. Existence of the solution.

Lemma 3.1. \cite{9} Assume that $\Omega \subset R^N$ is an open bounded set and let $f_k, f \in L^q(\Omega)$, as $k \to \infty$, $f_k \to f$ weakly in $L^q(\Omega)$ for $1 \leq q < \infty$. Then there holds

$$\lim_{k \to \infty} \inf \| f_k \|_{L^q(\Omega)} \geq \| f \|_{L^q(\Omega)}.$$

We now consider the regularized problem (12)-(14). There is a classical solution $u_\varepsilon \in C^2(Q_T) \cap C^3(Q_T)$ of this problem provided that $A$ and $b_i$ satisfy the assumptions in Theorem 2.3, one can refer to the eighth chapter of \cite{12} for details.

Firstly, since $u_0(x) \in L^\infty(\Omega)$ is smooth, by the maximum principle, we have

$$|u_\varepsilon| \leq \| u_0 \|_{L^\infty} \leq M.$$

Secondly, let us make the $BV$ estimates of $u_\varepsilon$. To the end, we begin with the local coordinates of the boundary $\partial \Omega$.

Let $\delta_0 > 0$ be small enough such that

$$E^{\delta_0} = \{ x \in \Omega : \text{dist}(x, \Sigma) \leq \delta_0 \} \subset \bigcup_{\tau=1}^n V_\tau,$$

where $V_\tau$ is a region, on which one can introduce local coordinates

$$y_k = F_k^\tau(x)(k = 1, 2, \cdots, N), y_N |_{\partial \Omega} = 0,$$

with $F^\tau_k$ appropriately smooth and $F^\tau_N = F^N_\tau$, such that the $y_N$-axes coincides with the inner normal vector.

Lemma 3.2. \cite{11} Let $u_\varepsilon$ be the solution of the approximate problem (12)-(14). If the assumptions of Theorem 2.3 are true, then we have

$$\varepsilon \int_{\partial \Omega} \frac{\partial u_\varepsilon}{\partial n} \ d\sigma \leq c_1 + c_2 \left( |\nabla u_\varepsilon|_{L^1(\Omega)} + |\frac{\partial u_\varepsilon}{\partial t}|_{L^1(\Omega)} \right),$$

with constants $c_i$ $(i = 1, 2)$ are independent of $\varepsilon$.

Theorem 3.3. Let $u_\varepsilon$ be the solution of the approximate problem (13)-(14). If the assumptions of Theorem 2.3 are true, then it has

$$|\text{grad} u_\varepsilon|_{L^1(\Omega)} \leq c,$$

where $c$ is independent of $\varepsilon$, and

$$|\text{grad} u_\varepsilon|^2 = \sum_{i=1}^N \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^2 + \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2.$$
Proof. Differentiate \((12)\) with respect to \(x_s, s = 1, 2, \cdots, N, N + 1, x_{N+1} = t,\) and sum up for \(s\) after multiplying the resulting relation by \(\frac{S_n(\|\text{gradu}\|)}{|\text{gradu}|}.\) In what follows, we simply denote \(u_s\) by \(u,\) and \(\partial\Omega\) by \(\Sigma.\) Integrating it over \(\Omega\) yields
\[
\int_{\Omega} \frac{\partial u_{x_s}}{\partial t} u_{x_s} \frac{S_n(\|\text{gradu}\|)}{|\text{gradu}|} \, dx = \int_{\Omega} \frac{\partial}{\partial t} \int_0^{\|\text{gradu}\|} S_n(\tau) d\tau \, dx
= \frac{d}{dt} \int_{\Omega} I_n(\|\text{gradu}\|) \, dx.
\]
(19)

As usual, pairs of the indices of \(s\) imply a summation from 1 to \(N + 1,\) pairs of the indices of \(i, j\) imply a summation from 1 to \(N,\) and \(\{n_i\}_{i=1}^N\) is the inner normal vector of \(\Omega.\)

\[
\int_{\Omega} \Delta(a(u)u_{x_s}) u_{x_s} \frac{S_n(\|\text{gradu}\|)}{|\text{gradu}|} \, dx = \int_{\Omega} \frac{\partial}{\partial x_i} [a'(u)u_{x_s} u_{x_s} + a(u)u_{x_s,x_s}] \frac{S_n(\|\text{gradu}\|)}{|\text{gradu}|} \, dx
= \int_{\Omega} \frac{\partial}{\partial x_i} (a'(u)u_{x_s} u_{x_s}) \frac{S_n(\|\text{gradu}\|)}{|\text{gradu}|} \, dx
+ \int_{\Omega} a'(u)u_{x_s} \frac{\partial}{\partial x_i} I_n(\|\text{gradu}\|) \, dx
= \int_{\Omega} \frac{\partial}{\partial x_i} (a'(u)u_{x_s}) \|\text{gradu}\| S_n(\|\text{gradu}\|) \, dx
- \int_{\Sigma} a'(u)u_{x_s} n_i I_n(\|\text{gradu}\|) \, d\sigma
- \int_{\Sigma} I_n(\|\text{gradu}\|) \frac{\partial}{\partial x_i} (a'(u)u_{x_s}) \, d\sigma
- \int_{\Sigma} a'(u)u_{x_s} n_i I_n(\|\text{gradu}\|) \, d\sigma,
\]

\[
\int_{\Omega} \frac{\partial}{\partial x_i} (a(u)u_{x_s,s}) u_{x_s} \frac{S_n(\|\text{gradu}\|)}{|\text{gradu}|} \, dx = \int_{\Omega} \frac{\partial}{\partial x_i} (a(u)u_{x_s,s}) \frac{\partial}{\partial \xi_s} I_n(\|\text{gradu}\|) \, dx
= - \int_{\Sigma} a(u)u_{x_s,s} n_i \frac{\partial}{\partial \xi_s} I_n(\|\text{gradu}\|) \, d\sigma
- \int_{\Omega} a(u) \frac{\partial^2 I_n(\|\text{gradu}\|)}{\partial \xi_s \partial \xi_p} u_{x_s,s} u_{x_p} \, dx.
\]
where \(\xi_s = u_{x_s}.\) So we have
\[
\varepsilon \int_{\Omega} \Delta u_{x_s} u_{x_s} \frac{S_n(\|\text{gradu}\|)}{|\text{gradu}|} \, dx
= - \varepsilon \int_{\Sigma} \frac{\partial I_n(\|\text{gradu}\|)}{\partial x_i} n_i d\sigma
- \varepsilon \int_{\Omega} \frac{\partial^2 I_n(\|\text{gradu}\|)}{\partial \xi_s \partial \xi_p} u_{x_s,s} u_{x_p} \, dx.
\]
\[ \int_\Omega \nabla (\tilde{b}(u) u_x) u_x \frac{S_n(|\text{grad} u|)}{|\text{grad} u|} \, dx \]

\[ = \sum_{i=1}^N \int_\Omega \frac{\partial}{\partial x_i} (b'_i(u)) |\text{grad} u| S_n(|\text{grad} u|) \, dx \]

\[ + \sum_{i=1}^N \int_\Omega b'_i(u) \frac{\partial I_n(|\text{grad} u|)}{\partial x_i} \, dx \]

\[ = \sum_{i=1}^N \int_\Omega \frac{\partial}{\partial x_i} (b'_i(u)) [(|\text{grad} u| S_n(|\text{grad} u|) - I_n(|\text{grad} u|)) \, dx \]

\[ - \int_\Sigma b'_i(u) I_n(|\text{grad} u|) n_i \, d\sigma. \]

From (19) to (20), by the assumption \(a(0) = 0\), we have

\[ \frac{d}{dt} \int_\Omega I_n(|\text{grad} u|) \, dx = \int_\Omega \frac{\partial}{\partial x_i} (a'(u) u_{x_i}) [(|\text{grad} u| S_n(|\text{grad} u|) - I_n(|\text{grad} u|)) \, dx \]

\[ - \int_\Omega a(u) \frac{\partial^2 I_n(|\text{grad} u|)}{\partial x_i \partial x_j} u_{x_i,x_j} u_{x_j,x_i} \, dx - \varepsilon \int_\Omega \frac{\partial^2 I_n(|\text{grad} u|)}{\partial x_i \partial x_j} u_{x_i,x_j} u_{x_j,x_i} \, dx \]

\[ + \sum_{i=1}^N \int_\Omega \frac{\partial}{\partial x_i} (b'_i(u)) [|\text{grad} u| S_n(|\text{grad} u|) - I_n(|\text{grad} u|)) \, dx \]

\[ - \left[ \int_\Sigma a'(u) u_{x_i} n_i I_n(|\text{grad} u|) \, d\sigma + \int_\Sigma b'_i(u) I_n(|\text{grad} u|) n_i \, d\sigma \right] + \varepsilon \int_\Sigma \frac{\partial I_n(|\text{grad} u|)}{\partial x_i} n_i \, d\sigma. \] (21)

By observing that on \(\Sigma\), there holds

\[ -b'_i(u) \frac{\partial u}{\partial n} n_i = \varepsilon \Delta u + \Delta A(u), \quad u = 0, \]

then the surface integrals in (21) can be rewritten as

\[ S = - \left[ \int_\Sigma b'_i(u) I_n(|\text{grad} u|) n_i \, d\sigma + \varepsilon \int_\Sigma \frac{\partial I_n(|\text{grad} u|)}{\partial x_i} n_i \, d\sigma \right] + \int_\Sigma a'(u) u_{x_i} n_i I_n(|\text{grad} u|) \, d\sigma \]

\[ = -\varepsilon \int_\Sigma \left[ \frac{\partial I_n(|\text{grad} u|)}{\partial x_i} n_i - \Delta u \frac{I_n(|\text{grad} u|)}{\partial n} \right] \, d\sigma \]

\[ + \int_\Sigma a(u) \left[ \frac{\partial I_n(|\text{grad} u|)}{\partial x_i} n_i - \Delta u \frac{I_n(|\text{grad} u|)}{\partial n} \right] \, d\sigma \]

\[ = -\varepsilon \int_\Sigma \left[ \frac{\partial I_n(|\text{grad} u|)}{\partial x_i} n_i - \Delta u \frac{I_n(|\text{grad} u|)}{\partial n} \right] \, d\sigma. \]

Note that

\[ u_{x_{N+1}} \mid_\Sigma = u_1 \mid_\Sigma = 0, \]
Proof of Theorem 2.3.
We now prove that
\[ \lim_{\eta \to 0} S = -\varepsilon \int_{\Sigma} \text{sgn}(\frac{\partial u}{\partial \nu})(u_{x,j}n_jn_i - \Delta u) \, d\sigma. \]
We use the local coordinates on \( V_\tau, \tau = 1, 2, \cdots, n \):
\[ y_k = F^k(x), k = 1, 2, \cdots, N, \quad y_m|_\Sigma = 0. \]
By a direct computation [31] on \( \Sigma \cap V_\tau \), we have
\[ u_{x,x_j} n_j n_i = \sum_{k=1}^{N} u_{y_k} F^N x_i F^k x_j + \sum_{k=1}^{N-1} u_{y_k} F^N x_i F^k x_j + u_{y_m} F^m x_i x_j, \]
where \( F^k = F^k \). By the fact of that the inner normal vector is
\[ \bar{n} = - \left( \frac{\partial F^N}{\partial x_1}, \cdots, \frac{\partial F^N}{\partial x_N} \right) = -\text{grad}F^N, \]
and
\[ u_{x,x_j} n_j n_i - \Delta u = u_{y_m} \left( \frac{F^m x_i x_j F^N x_j - F^m x_i x_j}{|\text{grad}F^N|^2} \right). \]
By Lemma 3.2, we see that \( \lim_{\eta \to 0} S \) can be estimated by \( |\text{grad}u|_{L^1(\Omega)} \).
Thus, letting \( \eta \to 0 \), and noticing that
\[ \lim_{\eta \to 0} ||\text{grad}u||_{S_{\eta} ||\text{grad}u|| - I_{\eta} ||\text{grad}u||} = 0, \]
we have
\[ \frac{d}{dt} \int_{\Omega} |\text{grad}u| \, dx \leq c_1 + c_2 \int_{\Omega} |\text{grad}u| \, dx. \]
By Gronwall’s inequality, we get
\[ \int_{\Omega} |\text{grad}u| \, dx \leq c. \tag{22} \]
By (12) and (22), it is easy to see that
\[ \int_{Q_T} (a(u_\varepsilon) + \varepsilon) |\nabla u_\varepsilon|^2 \, dx \, dt \leq c. \tag{23} \]
Thus, there exists a subsequence \( \{u_{\varepsilon_n}\} \) of \( u_\varepsilon \) and a function \( u \in BV(Q_T) \cap L^\infty(Q_T) \) such that \( u_{\varepsilon_n} \to u \) a.e. on \( Q_T \).

Proof of Theorem 2.3. We now prove that \( u \) is a generalized solution of the problem [1-3]-[7]. For any \( \varphi(x,t) \in C^1_0(Q_T) \), it has
\[ \iint_{Q_T} \left[ \frac{\partial}{\partial x_i} \int_0^u \sqrt{a(s)} \, ds \right] \varphi(x,t) \, dx \, dt \]
\[ = - \iint_{Q_T} \int_0^{u_*} \sqrt{a(s)} \, ds \left[ \varphi(x,t) \right] \, dx \, dt. \]
The above equality is also true for any \( \varphi(x,t) \in L^2(Q_T) \). By Hölder’s inequality, from [23] we have
\[ \frac{\partial}{\partial x_i} \int_0^u \sqrt{a(s)} \, ds \to \frac{\partial}{\partial x_i} \int_0^u \sqrt{a(s)} \, ds \text{ weakly in } L^2(Q_T), \quad i = 1, 2, \cdots, N. \]
This implies that
\[ \frac{\partial}{\partial x_i} \int_0^u \sqrt{a(s)} ds \in L^2(Q_T), \quad i = 1, 2, \ldots, N. \]

Thus \( u \) satisfies (1) in Definition 2.1.

If \( \Sigma_1 \neq \emptyset \) is a subset of \( \Sigma \), let \( \varphi_1 \in C^2(Q_T) \), \( \varphi_1 \geq 0 \), and \( \text{supp} \varphi_1 \subset \Omega \times (0, T) \), \( \nabla \varphi_1 |_{\Sigma \times (0, T)} = 0 \). Multiplying (12) by \( \varphi_1 S_\eta(u_\varepsilon - k) \), and integrating it over \( Q_T \), we obtain
\[ \int_{Q_T} \frac{\partial u_\varepsilon}{\partial t} \varphi_1 S_\eta(u_\varepsilon - k) dx dt = \int_{Q_T} \Delta A(u_\varepsilon) \varphi_1 S_\eta(u_\varepsilon - k) dx dt + \varepsilon \int_{Q_T} \Delta u_\varepsilon \varphi_1 S_\eta(u_\varepsilon - k) dx dt + \sum_{i=1}^N \int_{Q_T} \frac{\partial b_i(u_\varepsilon)}{\partial x_i} \varphi_1 S_\eta(u_\varepsilon - k) dx dt. \]

Let us calculate every term in (24) by integration by parts as follows
\[ \int_{Q_T} \frac{\partial u_\varepsilon}{\partial t} \varphi_1 S_\eta(u_\varepsilon - k) dx dt = -\int_{Q_T} I_\eta(u_\varepsilon - k) \varphi_1 dx dt, \]
\[ \varepsilon \int_{Q_T} \Delta u_\varepsilon \varphi_1 S_\eta(u_\varepsilon - k) dx dt = -\varepsilon \int_0^T \sum_\Sigma \nabla u_\varepsilon \cdot \vec{n} \varphi_1 S_\eta(u_\varepsilon - k) dtd\sigma \]
\[ -\varepsilon \int_{Q_T} \nabla u_\varepsilon (S_\eta(u_\varepsilon - k) \nabla \varphi_1 + \varphi_1 S_\eta(u_\varepsilon - k) \nabla u_\varepsilon) dx dt \]
\[ = \varepsilon S_\eta(k) \int_0^T \sum_\Sigma \nabla u_\varepsilon \cdot \vec{n} \varphi_1 dtd\sigma - \varepsilon \int_{Q_T} \nabla u_\varepsilon S_\eta(u_\varepsilon - k) \nabla \varphi_1 dx dt \]
\[ -\varepsilon \int_{Q_T} \nabla u_\varepsilon |^2 S_\eta'(u_\varepsilon - k) \varphi_1 dx dt, \]
\[ \int_{Q_T} \Delta A(u_\varepsilon) \varphi_1 S_\eta(u_\varepsilon - k) dx dt = S_\eta(k) \int_0^T \sum_\Sigma \nabla A(u_\varepsilon) \cdot \vec{n} \varphi_1 dtd\sigma \]
\[ -\int_{Q_T} \nabla A(u_\varepsilon)(S_\eta(u_\varepsilon - k) \nabla \varphi_1 + \varphi_1 S_\eta(u_\varepsilon - k) \nabla u_\varepsilon) dx dt \]
\[ = S_\eta(k) \int_0^T \sum_\Sigma \nabla A(u_\varepsilon) \cdot \vec{n} \varphi_1 dtd\sigma - \int_{Q_T} \nabla A(u_\varepsilon) S_\eta(u_\varepsilon - k) \nabla \varphi_1 dx dt \]
\[ -\int_{Q_T} a(u_\varepsilon) |\nabla u_\varepsilon|^2 S_\eta'(u_\varepsilon - k) \varphi_1 dx dt. \]

Since \( \nabla \varphi_1 |_{\Sigma \times (0, T)} = 0 \), we deduce that
\[ \int_{Q_T} \frac{\partial b_i(u_\varepsilon)}{\partial x_i} \varphi_1 S_\eta(u_\varepsilon - k) dx dt \]
\[ = -\int_0^T \sum_\Sigma [b_i(u_\varepsilon) - b(k)] n_i \varphi_1 S_\eta(u_\varepsilon - k) dtd\sigma \]
\[ -\int_{Q_T} [b_i(u_\varepsilon) - b(k)] \frac{\partial \varphi_1}{\partial x_i} S_\eta(u_\varepsilon - k) + \varphi_1 S_\eta(u_\varepsilon - k) \frac{\partial u_\varepsilon}{\partial x_i} dx dt \]
\[ = S_\eta(k) \int_0^T \sum_\Sigma \varphi_1 [b_i(0) - b(k)] n_i d\sigma dt - \int_{Q_T} B_\eta^i(u_\varepsilon, k) \varphi_1 x_i dx dt. \]
From [24]-25, we have
\[ \begin{align*}
& \int_{Q_T} I_\eta(u_x - k) \varphi_{11} dx dt + \int_{Q_T} A_\eta(u_x, k) \Delta \varphi_1 dx dt \\
& - \int_{Q_T} B_\eta(u_x, k) \varphi_{11} dx dt - \int_{Q_T} a(u_x) \left| \nabla u_x \right|^2 S'_\eta(u_x - k) \varphi_1 dx dt \\
& - \varepsilon \int_{Q_T} \nabla u_x \cdot \nabla \varphi_1 S_\eta(u_x - k) dx dt \leq \int_{Q_T} \left| \nabla u_x \right|^2 S'_\eta(u_x - k) \varphi_1 dx dt \\
& + \varepsilon S_\eta(k) \int_0^T \int_{\Sigma_{1n_k}} (b_i(0) - b_i(k)) n_i \varphi_1 dtd\sigma + S_\eta(k) \int_0^T \int_{\Sigma_{2n_k}} \nabla \varphi_1 dtd\sigma \\
& + S_\eta(k) \int_0^T \int_{\Sigma_{3n_k}} (b_i(0) - b_i(k)) n_i \varphi_1 dtd\sigma = 0. \tag{26}
\end{align*} \]

Taking \( \varphi_2 \in C^2(\bar{Q}_T) \), \( \varphi_1 |_{\partial \Omega \times [0,T]} = \varphi_2 |_{\partial \Omega \times [0,T]} \), and supp\( \varphi_2 \subset \bar{\Omega} \times (0,T) \), we have
\[ \begin{align*}
S_\eta(k) \int_0^T \int_{\Sigma_{1n_k}} \nabla A(u_x) \cdot \nabla \varphi_1 dtd\sigma + \varepsilon S_\eta(k) \int_0^T \int_{\Sigma_{2n_k}} \nabla \varphi_1 dtd\sigma \\
= S_\eta(k) [\varepsilon \int_{Q_T} \int_{\Sigma_{1n_k}} \frac{\partial u_x}{\partial x_i} \frac{\partial \varphi_2}{\partial x_i} dx dt + \int_{Q_T} \nabla A(u_x) \cdot \nabla \varphi_2 dx dt \\
- \int_{Q_T} (b_i(u_x) - b_i(0)) \frac{\partial \varphi_2}{\partial x_i} dx dt + \int_{Q_T} u_x \frac{\partial \varphi_2}{\partial t} dx dt], \tag{27}
\end{align*} \]

\[ \int_{Q_T} \nabla A(u_x) \cdot \nabla \varphi_2 dx dt = - \int_0^T \int_{\Omega} A(0) \frac{\partial \varphi_2}{\partial x_i} n_i dtd\sigma - \int_{Q_T} A(u_x) \Delta \varphi_2 dx dt \\
= - \int_{Q_T} A(u_x) \Delta \varphi_2 dx dt, \tag{28} \]

By \( \nabla \varphi_1 |_{\Sigma \times (0,T)} = 0 \) and \( a(0) = 0 \), from (26)-(28) we have
\[ \begin{align*}
& \int_{Q_T} I_\eta(u_x - k) \varphi_{11} dx dt + \int_{Q_T} A_\eta(u_x, k) \Delta \varphi_1 dx dt \\
& - \varepsilon \int_{Q_T} \nabla u_x \cdot \nabla \varphi_1 S_\eta(u_x - k) dx dt \leq \int_{Q_T} a(u_x) \left| \nabla u_x \right|^2 S'_\eta(u_x - k) \varphi_1 dx dt \\
& + \varepsilon S_\eta(k) \int_0^T \int_{\Sigma_{1n_k}} (b_i(0) - b_i(k)) n_i \varphi_1 dtd\sigma + S_\eta(k) \int_0^T \int_{\Sigma_{2n_k}} \nabla \varphi_1 dtd\sigma \\
& + S_\eta(k) \int_0^T \int_{\Sigma_{3n_k}} (b_i(0) - b_i(k)) n_i \varphi_1 dtd\sigma \geq 0. \tag{29}
\end{align*} \]

By Lemma 3.1, we obtain
\[ \liminf_{\varepsilon \to 0} \int_{Q_T} S'_\eta(u_x - k) a(u_x) \frac{\partial u_x}{\partial x_i} \frac{\partial u_x}{\partial x_i} \varphi_1 dx dt. \]
Proof of Theorem 2.4.

Let \( \varepsilon \to 0 \) in (29). By (30), we get (8), while (9) is naturally concealed in the limiting process.

The proof of the initial condition (10) is similar to that as in [27, 35], so we omit the details.

If \( \Sigma_1 = \emptyset \), let \( \varphi \in C_0^2(Q_T) \), \( \varphi \ge 0 \), and \( \{ n_i \} \) be the inner normal vector of \( \Omega \).

Multiplying both sides of (12) by \( \varphi S_\eta(u_\varepsilon - k) \), and integrating it over \( Q_T \), by a similar calculation as in the case of \( \Sigma \neq \emptyset \), we derive that

\[
\int_{Q_T} I_\eta(u_\varepsilon - k) \varphi_1 dxdt + \int_{Q_T} A_\eta(u_\varepsilon, k) \Delta \varphi dxdt + \int_{Q_T} B_\eta'(u_\varepsilon, k) \varphi_\varepsilon dxdt \\
- \varepsilon \int_{Q_T} \nabla u_\varepsilon \cdot \nabla \varphi S_\eta(u_\varepsilon - k) dxdt - \varepsilon \int_{Q_T} |\nabla u_\varepsilon|^2 S_\eta'(u_\varepsilon - k) \varphi dxdt \\
- \int_{Q_T} a(u_\varepsilon) |\nabla u_\varepsilon|^2 S_\eta'(u_\varepsilon - k) \varphi dxdt = 0. \tag{31}
\]

By Lemma 3.1, it gives

\[
\liminf_{\varepsilon \to 0} \int_{Q_T} S_\eta'(u_\varepsilon - k) a(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_i} \varphi dxdt \\
\ge \int_{Q_T} S_\eta'(u - k) |\nabla \sqrt{a(s)}|^2 \varphi dxdt. \tag{32}
\]

Let \( \varepsilon \to 0 \) in (31). By (32), we arrive at (11).

Also, the proof of (10) is similar to that in [27, 35], we omit the details. \( \square \)

4. Proof of Theorem 2.4. Let \( \Gamma_u \) be the set of all jump points of \( u \in BV(Q_T), v \) be the normal of \( \Gamma_u \) at \( X = (x, t) \), \( u^+(X) \) and \( u^-(X) \) be the approximate limits of \( u \) at \( X \in \Gamma_u \) with respect to \( (v, Y - X) > 0 \) and \( (v, Y - X) < 0 \) respectively. For the continuous function \( p(u, x, t) \) and \( u \in BV(Q_T) \), we define

\[
\widehat{p}(u, x, t) = \int_0^1 p(\tau u^+ + (1 - \tau) u^-, x, t) d\tau,
\]

which is called the composite mean value of \( p \). For a given \( t \), we denote \( \Gamma_u^t, H^t, (v_1^t, \ldots, v_N^t) \) and \( u^+_t \) as all jump points of \( u(\cdot, t) \), Hausdorff measure of \( \Gamma_u^t \), the unit normal vector of \( \Gamma_u^t \), and the asymptotic limit of \( u(\cdot, t) \) respectively. Moreover, if \( f(s) \in C^1(R), u \in BV(Q_T) \), then \( f(u) \in BV(Q_T) \) and

\[
\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u) \frac{\partial u}{\partial x_i}, \quad i = 1, 2, \ldots, N, N + 1,
\]

where \( x_{N+1} = t \).

Lemma 4.1. Let \( u \) be a solution of equation (4). Then in the sense of Hausdorff measure \( H_N(\Gamma_u) \), we have

\[
a(s) = 0, \quad s \in I(u^+(x, t), u^-(x, t)) \quad a.e. \quad \text{on} \quad \Gamma_u,
\]

where \( I(\alpha, \beta) \) denotes the closed interval with endpoints \( \alpha \) and \( \beta \).
This lemma can be proved in a similar way as described [35], we omit the details.

The proof of Theorem 2.4. Let \( u, v \) be two entropy solutions of equation (1) with initial values

\[
 u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x).
\]

Suppose that \( u(x, t) = v(x, t) = 0, (x, t) \in \Sigma_1 \times (0, T) \).

By Definition 2.1, for any \( \varphi_1, \varphi_2 \in C^2(\Omega_T), \varphi_1 \geq 0, \varphi_1 |_{a_0 \times [0, T]} \neq \varphi_2 |_{a_0 \times [0, T]}, \)

supp\( \varphi_2 \), supp\( \varphi_1 \subset \Omega \times (0, T), \eta > 0, k, l \in \mathbb{R}, \) we have

\[
\int_{Q_T} \left[ I_{\eta}(u-k)\varphi_{1\tau} - B_{\eta}^i(u, k)\varphi_{1x_i} + A_{\eta}(u, k)\Delta \varphi_1 \right] \, dx \, dt
\]

\[
- \int_{Q_T} \left[ S_{\eta}'(u-k) \left| \nabla \int_0^t \sqrt{a(s)} \, ds \right|^2 \varphi_1 \right] \, dx \, dt
\]

\[
- S_{\eta}(k) [b_i(0) - b_i(k)] \int_0^T \int_{\Sigma_{1k}} \varphi_1 n_i \, dt \, d\sigma
\]

\[
+ S_{\eta}(k) \int_{Q_T} [u\varphi_{2\tau} - (b_i(u) - b_i(0))\varphi_{2x_i} + A(u)\Delta \varphi_2] \, dx \, dt \geq 0,
\]

\[
\int_{Q_T} \left[ I_{\eta}(v-l)\varphi_{1\tau} - B_{\eta}^i(v, l)\varphi_{1y_i} + A_{\eta}(v, l)\Delta \varphi_1 \right] \, dy \, d\tau
\]

\[
- \int_{Q_T} \left[ S_{\eta}'(v-l) \left| \nabla \int_0^t \sqrt{a(s)} \, ds \right|^2 \varphi_1 \right] \, dy \, d\tau
\]

\[
- S_{\eta}(l) [(b_i(0) - b_i(l)] \int_0^T \int_{\Sigma_{1l}} \varphi_1 n_i \, dt \, d\sigma
\]

\[
+ S_{\eta}(l) \int_{Q_T} [v\varphi_{2\tau} - (b_i(v) - b_i(0))\varphi_{2y_i} + A(v)\Delta \varphi_2] \, dy \, d\tau \geq 0,
\]

Especially, if \( \varphi_1 \in C^2_0(Q_T), \varphi_2 \equiv 0 \), we have

\[
\int_{Q_T} \left[ I_{\eta}(u-k)\varphi_{1\tau} - B_{\eta}^i(u, k)\varphi_{1x_i} + A_{\eta}(u, k)\Delta \varphi_1 \right] \, dx \, dt
\]

\[
- \int_{Q_T} \left[ S_{\eta}'(u-k) \left| \nabla \int_0^t \sqrt{a(s)} \, ds \right|^2 \varphi_1 \right] \, dx \, dt \geq 0,
\]

\[
\int_{Q_T} \left[ I_{\eta}(v-l)\varphi_{1\tau} - B_{\eta}^i(v, l)\varphi_{1y_i} + A_{\eta}(v, l)\Delta \varphi_1 \right] \, dy \, d\tau
\]

\[
- \int_{Q_T} \left[ S_{\eta}'(v-l) \left| \nabla \int_0^t \sqrt{a(s)} \, ds \right|^2 \varphi_1 \right] \, dy \, d\tau \geq 0.
\]
We choose \( k = v(y, \tau), \ l = u(x, t), \ \varphi_1 = \psi(x, t, y, \tau) \) in (33) and (34), integrate it over \( Q_T \). It gives
\[
\iint_{Q_T} \iint_{Q_T} [I_\eta(u - v)(\psi_1 + \psi_\tau) - (B^i_\eta(u, v)\psi_x + B^i_\eta(v, u)\psi_y)] \, dx \, dt \, dy \, d\tau \\
+ \iint_{Q_T} \iint_{Q_T} [A_\eta(u, v)\Delta_x \psi + A_\eta(v, u)\Delta_y \psi] \, dx \, dt \, dy \, d\tau \\
- S'_\eta(u - v) \left( |\nabla_x \int_0^u \sqrt{a(s)}ds|^2 + |\nabla_y \int_0^v \sqrt{a(s)}ds|^2 \right) \psi \, dx \, dt \, dy \, d\tau. \quad (35)
\]
Here \( \Delta_x \) is the usual Laplacian operator corresponding to the variable \( x \), and \( \nabla_x \) is the gradient operator corresponding to the variable \( x \).

Clearly, it has
\[
\frac{\partial j_h}{\partial t} + \frac{\partial j_h}{\partial \tau} = 0, \quad \frac{\partial j_h}{\partial x_i} + \frac{\partial j_h}{\partial y_i} = 0, \quad i = 1, \ldots, N; \\
\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \tau} = \frac{\partial \phi}{\partial x_i} j_h, \quad \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = \frac{\partial \phi}{\partial x_i} j_h.
\]
Note that
\[
\lim_{\eta \to 0} B^i_\eta(u, v) = \lim_{\eta \to 0} B^i_\eta(v, u) = \text{sgn}(u - v)(b_1(u) - b_1(v)),
\]
as \( \eta \to 0 \), and
\[
\iint_{Q_T} \iint_{Q_T} [B^i_\eta(u, v)\psi_x + B^i_\eta(v, u)\psi_y] \, dx \, dt \, dy \, d\tau \\
\to \iint_{Q_T} \iint_{Q_T} \text{sgn}(u - v)[b_1(u) - b_1(v)]\phi_x \, j_h \, dx \, dt \, dy \, d\tau,
\]
as \( h \to 0 \). So we get
\[
\iint_{Q_T} \iint_{Q_T} \text{sgn}(u - v)[b_1(u) - b_1(v)]\phi_x \, j_h \, dx \, dt \, dy \, d\tau \\
\to \iint_{Q_T} \text{sgn}(u - v)[b_1(u) - b_1(v)]\phi_x \, dx \, dt.
\]
For the third and the fourth term in (35), one can see that
\[
\iint_{Q_T} \iint_{Q_T} [A_\eta(u, v)\Delta_x \psi + A_\eta(v, u)\Delta_y \psi] \, dx \, dt \, dy \, d\tau = \iint_{Q_T} \iint_{Q_T} \{A_\eta(u, v)(\Delta_x \phi_j \hbar + 2\phi_x \phi_{j, x}, + \phi\Delta_x \phi_j \hbar) + A_\eta(v, u)\phi\Delta_y \phi_j \hbar \} \, dx \, dt \, dy \, d\tau \\
= \iint_{Q_T} \iint_{Q_T} \{A_\eta(u, v)\Delta_x \phi_j \hbar + A_\eta(u, v)\phi_x \phi_{j, x} + A_\eta(v, u)\phi_x \phi_{j, y} \} \, dx \, dt \, dy \, d\tau \\
- \iint_{Q_T} \iint_{Q_T} \{a(u)\hat{S}_\eta(u - v) \frac{\partial u}{\partial x_i} - \int_u^v a(s)\hat{S}_\eta(s - v) \, ds \frac{\partial u}{\partial x_i} \phi_{j, x} \} \, dx \, dt \, dy \, d\tau,
\]
where
\[
a(u)\hat{S}_\eta(u - v) = \int_0^1 a(su^+ + (1 - s)u^-)S_\eta(su^+ + (1 - s)u^- - v) \, ds,
\]
By virtue of Lemma 4.1, we derive that
\[\int_u^v a(s)S\eta/(s-v)ds = \int_0^1 \int_{u+(1-s)v}^v a(\sigma)S\eta/(\sigma-su^+ - (1-s)v^-)d\sigma ds.\]

Note that
\[\int Q_T \int_{Q_T} S\eta'(u-v) \left( |\nabla_x I(x) \int_0^u \sqrt{a(s)} ds \right|^2 + |\nabla_y I(x) \int_0^u \sqrt{a(s)} ds \right|^2 \psi dx dy.
\]

By virtue of Lemma 4.1, we derive that
\[\int Q_T \int_{Q_T} S\eta'(u-v) \left( |\nabla_x I(x) \int_0^u \sqrt{a(s)} ds \right|^2 + |\nabla_y I(x) \int_0^u \sqrt{a(s)} ds \right|^2 \psi dx dy.
\]
We know that as $\eta \to 0$, from (35)-(36), and letting $\eta \to 0$, we have
\[ \lim_{\eta \to 0} A_\eta(u,v) = \lim_{\eta \to 0} A_\eta(v,u) = \text{sgn}(u-v)[A(u)-A(v)], \]
where
\[ \phi = \phi_x j_{hx} + \phi_y j_{hy}, \]
we have
\[ \lim_{\eta \to 0} [A_\eta(u,v)\phi_x, j_{hx} + A_\eta(u,v)\phi_y, j_{hy}] = 0. \quad (36) \]

From (35)-(36), and letting $\eta \to 0, h \to 0$ in (35), we get
\[
\begin{align*}
&-\int_{Q_T} \int_{Q_T} \int_0^1 \int_{su^+(1-s)u^-} [\sqrt{a(\sigma)} - \sqrt{a(su^+ + (1-s)u^-)}] \\
&\quad \times S_{\eta}(\sigma - su^+ - (1-s)u^-) d\sigma ds \frac{\partial u}{\partial x_i} j_h x_i \phi dx dy d\tau \to 0,
\end{align*}
\]

as $\eta \to 0$.

Since
\[ \lim_{\eta \to 0} A_\eta(u,v) = \lim_{\eta \to 0} A_\eta(v,u) = \text{sgn}(u-v)[A(u)-A(v)], \]
we have
\[ \lim_{\eta \to 0} [A_\eta(u,v)\phi_x, j_{hx} + A_\eta(u,v)\phi_y, j_{hy}] = 0. \quad (36) \]

Let $\delta_\varepsilon$ be the mollifier, and
\[ \omega_\lambda \varepsilon = \omega_\lambda \ast \delta_\varepsilon(d), \]
where $\omega_\lambda \varepsilon(x) \in C^2_0(\Omega)$ is defined as follows: for any given small enough $0 < \lambda$, $0 \leq \omega_\lambda \leq 1$, $\omega|\partial \Omega = 0$ and
\[ \omega_\lambda (x) = 1, if \ d(x) = \text{dist}(x, \partial \Omega) \geq \lambda. \]

When $0 \leq d(x) \leq \lambda$, it has
\[ \omega_\lambda (d(x)) = 1 - \frac{(d(x)-\lambda)^2}{\lambda^2}. \]

Choose a suitable test function $\phi$ as (37) by
\[ \phi(x,t) = \omega_\lambda \varepsilon(x) \eta(t), \]
where $\eta(t) \in C_0^\infty(0,T)$. Then we have
\[
\begin{align*}
\omega_\lambda' \varepsilon(d) &= \int_{\{|s|<\varepsilon\} \cap \{0<d-s<\lambda\}} \omega_\lambda'(d-s) \delta_\varepsilon(s) ds \\
&= -\int_{\{|s|<\varepsilon\} \cap \{0<d-s<\lambda\}} \frac{2(d-s-\lambda)}{\lambda^2} \delta_\varepsilon(s) ds, \\
|\omega_\lambda' \varepsilon(d)| &\leq \frac{c}{\lambda}, \\
\omega_\lambda'' \varepsilon(d) &= \omega_\lambda'' \varepsilon(d) \ast \delta_\varepsilon(d) = -\frac{2}{\lambda^2} \int_{\{|s|<\varepsilon\} \cap \{0<d-s<\lambda\}} \delta_\varepsilon(s) ds.
\end{align*}
\]

We know that
\[
\begin{align*}
\Delta \phi &= \eta(t) \Delta (\omega_\lambda \varepsilon(d(x))) \\
&= \eta(t) \nabla (\omega_\lambda \varepsilon(d)) \nabla d \\
&= \eta(t)[\omega_\lambda'' \varepsilon(d)|\nabla d|^2 + \omega_\lambda' \varepsilon(d) \Delta d] \\
&= \eta(t)[-\frac{2}{\lambda^2} \int_{\{|s|<\varepsilon\} \cap \{0<d-s<\lambda\}} \delta_\varepsilon(s) ds + \omega_\lambda' \varepsilon(d) \Delta d].
\end{align*}
\]
By using the conditions \[17\] and using the fact of that \(|\nabla d| = 1\), from \[37\] we have
\[
\iint_{Q_T} |u(x, t) - v(x, t)| \phi_t dx dt + c \int_0^T \int_{\Omega_\lambda} \eta(t) |\omega_{A_{\Omega_\lambda}}(d)| \cdot u - v \cdot dx dt \geq 0, \tag{38}
\]
where \(\Omega_\lambda = \{x \in \Omega : d(x) < \lambda\}\). According to the definition of the trace of BV functions \[7\], by \[15\] and \[16\], when \(x \in \Sigma_0, \gamma v = \gamma v\), we let \(\lambda \to 0\) in \[38\] and have
\[
\lim_{\lambda \to 0} \int_0^T \int_{\Omega_\lambda} \eta(t) |\omega_{A_{\Omega_\lambda}}(d)| \cdot u - v \cdot dx dt \leq c \lim_{\lambda \to 0} \int_0^T \eta(t) \frac{1}{\lambda} \int_{\Omega_\lambda} |u - v| \cdot dx dt.
\]

Let \(0 < s < \tau < T\), and
\[
\eta(t) = \int_{s-t}^{\tau-t} \alpha_{\varepsilon}(\sigma) d\sigma, \quad \varepsilon < \min\{\tau, T - s\},
\]
where \(\alpha_{\varepsilon}(t)\) is the kernel of the mollifier with \(\alpha_{\varepsilon}(t) = 0\) for \(t \notin (-\varepsilon, \varepsilon)\). Then there is
\[
c \cdot \sup_{\Sigma_0 \times (0,T)} |f(x, t) - g(x, t)| + \int_{Q_T} |u(x, t) - v(x, t)| \cdot \eta'_t dx dt \geq 0.
\]
Let \(\varepsilon \to 0\). Then there holds
\[
|u(x, \tau) - v(x, \tau)|_{L^1(\Omega)} \leq |u(x, s) - v(x, s)|_{L^1(\Omega)} + c \cdot \sup_{\Sigma_0 \times (0,T)} |f(x, t) - g(x, t)|.
\]
Consequently, the desired result follows by letting \(s \to 0\). \(\square\)

5. **Proof of Theorem 2.5.**

*Proof.* If \(\Sigma_1 = \emptyset\), let \(u\) and \(v\) be two entropy solutions of equation \[1\] in the sense of Definition 2.2. Suppose that the initial values are
\[
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x),
\]
and the boundary values are the same as \[18\].

By Definition 2.2, for any \(\varphi \in C_0^2(Q_T), \varphi \geq 0\) and \(\eta > 0, k, l \in R\), we have
\[
\iint_{Q_T} \left[ I_\eta(u - k) \varphi_t - B^t_\eta(u, k) \varphi_{x_1} + A_\eta(u, k) \Delta \varphi \right] dx dt
- \iint_{Q_T} S^t_\eta(u - k) \cdot |\nabla \int_0^u \sqrt{a(s)} ds| \cdot |\varphi| dx dt \geq 0, \tag{39}\]
and
\[
\iint_{Q_T} \left[ I_\eta(v - l) \varphi_t - B^t_\eta(v, l) \varphi_{x_1} + A_\eta(v, l) \Delta \varphi \right] dy d\tau
- \iint_{Q_T} S^t_\eta(v - l) \cdot |\nabla \int_0^v \sqrt{a(s)} ds| \cdot |\varphi| dy d\tau \geq 0. \tag{40}\]
If we choose \( k = v(y, \tau), \ l = u(x, t), \ \varphi = \psi(x, t, y, \tau) \) in (39) and (40), integrate it over \( QT \), we find
\[
\int\int_{QT} \left[ u(x,t) - v(x,t) \right] \phi_t - \text{sgn}(u - v)(b_i(u) - b_i(v))\phi_{x_i} \, dxdt
\]
\[
+ \int\int_{QT} |A(u) - A(v)|\Delta \phi \, dxdt \geq 0.
\]
Similar as the proof of Theorem 2.4, we have
\[
\int\int_{QT} |u(x,t) - v(x,t)|\phi_t \, dxdt + c\int_{0}^{T} \int_{\Omega \setminus \Omega_{\lambda}} \eta(t)|\omega_{\lambda}^\prime(d)| \, |u - v| \, dxdt \geq 0.
\]
Let \( \lambda \to 0 \). We then have
\[
c \cdot \sup_{\Sigma \times (0,T)} |f(x,t) - g(x,t)| + \int_{0}^{T} \int_{\Omega \setminus \Omega_{\lambda}} \eta(t)|\omega_{\lambda}^\prime(d)| \, |u - v| \, dxdt \geq 0.
\]
Let \( 0 < s \leq \tau < T \), and
\[
\eta(t) = \int_{s-t}^{\tau-t} \alpha_\varepsilon(\sigma) \, d\sigma, \quad \varepsilon < \min\{\tau, T-s\}.
\]
It follows that
\[
c \cdot \sup_{\Sigma \times (0,T)} |f(x,t) - g(x,t)| + \int_{0}^{T} \left[ \alpha_\varepsilon(t-s) - \alpha_\varepsilon(t-\tau) \right] |u - v|_{L^1(\Omega)} \, dt \geq 0,
\]
Let \( \varepsilon \to 0 \). Then it gives
\[
|u(x,\tau) - v(x,\tau)|_{L^1(\Omega)} \leq |u(x,s) - v(x,s)|_{L^1(\Omega)} + c \cdot \sup_{\Sigma \times (0,T)} |f(x,t) - g(x,t)|.
\]
Consequently, the desired result follows by letting \( s \to 0 \). \( \square \)

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