EINSTEIN AND SCALAR FLAT RIEMANNIAN METRICS

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ABSTRACT. We find the critical point equation of the squared $L^2$-norm of the scalar curvature of constant volume Riemannian metrics, and show that its critical points are Einstein, or scalar flat Riemannian metrics.

1. Introduction

A Riemannian manifold $(M^n, g)$ is Einstein if it satisfies the tensorial relation

\[ r_g = \frac{s_g}{n} g, \]

where $r_g$ and $s_g$ are the Ricci and scalar curvature tensors, respectively. By using the trace of the differential Bianchi, this relation implies that $s_g$ must be constant when the dimension $n$ of $M$ is three, or larger. If $n = 2$, all metrics satisfy the tensorial relation above. In that case, the Einstein condition is strengthened to requiring that $s_g$ be constant also.

It is quite remarkable that there are closed manifolds $M$ that carry Einstein metrics with scalar curvatures of opposite signs [8]. In fact, the examples of such found by Catanese and LeBrun are products of Kähler surfaces, and the metrics on the surface factors are Kähler-Einstein relative to complex structures whose corresponding first Chern classes are negative, and positive, respectively.

The extremal metrics of Calabi [7] are the critical points of the squared $L^2$-norm of the scalar curvature functional among metrics that represent a given positive cohomology class. If the said class is a multiple of $c_1$ (which, therefore, must be a signed class), when $c_1 < 0$, or when $c_1 > 0$ and the manifold is a surface with reductive automorphism group, the extremal metric exists, and is given by an Einstein metric [4, 15], [13]. The Einstein metrics of [8] are found appealing to these results, once the complex structures with $c_1 < 0$ or $c_1 > 0$ in the underlying manifolds have been identified. (Naturally, these complex structures cannot be homotopically equivalent.) But more to the point of this note, this suggests that, in general, Einstein metrics might appear as critical points of the squared $L^2$-norm of the scalar curvature functional among Riemannian metrics of fixed volume.

It is surprising that the critical points of this latter functional have been described only in particular cases. In this note, we remedy that, and find that in addition to Einstein, the set of critical points includes the scalar flat metrics, Einstein or not, as is to be expected of absolute minimizers. The argument we provide for reaching this conclusion is a dimensionless reinterpretation of the critical point equation for the Calabi extremal metrics on Riemann surfaces.

If one has a path segment of almost Hermitian structures, if the metrics at both ends of the path exhibit differences in sign for some metric tensor, then somewhere

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in between some topological condition should arise as you cross the point where the sign changes. This simple minded thought was (and is) the main motivation for closing the gap in the description of the critical points of the aforementioned functional. In light of the results in [3] alluded to above, the Einstein equation is perhaps too strong of a metric tensor to keep in mind for this idea to work.

2. THE ENERGY $S(g)$ OF A RIEMANNIAN METRIC

Let $M$ be a closed manifold of dimension $n$. We let $\mathcal{M}$ be the set of all Riemannian metrics. This is an open subset of the space of symmetric 2-tensors $S^2(M)$ over $M$. The set of metrics of volume $v = 1$ will be denoted by $\mathcal{M}_v$.

The infinite dimensional manifold $\mathcal{M}_1$ is topologized using a Sobolev norm of sufficiently high order. At $g \in \mathcal{M}_1$, its tangent space $T_g \mathcal{M}_1$ consists of the space of symmetric 2-tensors whose trace is orthogonal to the constants. Indeed, if the metric $g$ is deformed infinitesimally in the direction of the symmetric two tensor $h$, then the volume form $d\mu_g$ varies according to the formula

$$\frac{d}{dt} d\mu_{g+th} \mid_{t=0} = \frac{1}{2} \text{tr} h \, d\mu_g,$$

and thus, $h$ is in the tangent space if, and only if, it satisfies the said condition.

We let $S^p(M)$ be the bundle of symmetric $p$-tensors on $M$. Covariant differentiation defines a map

$$\nabla^p : S^p(M) \to \Omega^1 M \otimes S^p(M),$$

which can be composed with the symmetrization operator to define

$$\delta^* : S^p(M) \to S^{p+1}(M).$$

The metric dual of $\delta^*_g$ defines the operator

$$\delta_g : S^{p+1}(M) \to S^p(M).$$

It plays a central role in the variational expression of the scalar curvature.

The scalar curvature $s_g$ varies according to the expression

$$\frac{d}{dt} s_{g+th} \mid_{t=0} = \Delta_g(\text{tr} h) + \delta_g(\delta^*_g h) - (r_g, h)_g,$$

where $\Delta_g$ and $r_g$ are the Laplacian and Ricci tensor of $g$, respectively. The $g$-trace of any two tensor $h$ is just the pointwise inner-product $(g, h)_g$, so for instance, $s_g = (g, r_g)_g$.

The gradient of the Riemannian functional of Hilbert,

$$\mathcal{M} \xrightarrow{H} \mathbb{R},$$

$$g \mapsto \int_M s_g d\mu_g,$$

is well-known. By (2) and (3), we easily obtain that

$$\frac{d}{dt} H(g + th) \mid_{t=0} = \frac{d}{dt} \int s_{g+th} d\mu_{g+th} \mid_{t=0} = \int \left( \frac{s_g}{2} g - r_g, h \right)_g d\mu_g,$$

and if we restrict the domain of $H$ to $\mathcal{M}_v$, the ensuing Lagrange multiplier allows us to conclude that its critical points are the Einstein metrics, those that satisfy the tensorial equation (1) above. Notice that the fact that this holds when $n = 2$ reflects the fact that the functional (5) is constant in this case, $4\pi \chi(M)$, by the
Gauss-Bonnet theorem. The uniformization theorem produces a metric of constant scalar curvature in the conformal class, an Einstein representative of the class.

A natural Riemannian functional with more flexible critical points is given by

\[ M_1 \to \mathbb{R} \]
\[ g \mapsto S(g) = \int_M s_g^2 d\mu_g. \]

We think of this as the stored energy of \( M \) when being in the state defined by the metric \( g \), and ask for the states of \( M \) of minimum energy, which are found among the critical points of \( S \). Notice that by the extreme case of the Cauchy-Schwarz inequality, the critical states of the Hilbert functional \( H \) are subsumed into the critical points of \( S \). In spite of the ample amount of research involving the functional \( S \), its critical points has only been described in particular cases (see [6, page 133], [1, Proposition 1.1], [7, \( n=2 \]), or the more recent extension in [9]). We derive them in complete generality here.

By (2) and (4), we see that

\[ \frac{d}{dt} \int s_{g+th}^2 \big|_{t=0} = \int 2s_g(\Delta_g (\text{trace } h) + \delta_g (\delta_g h) - (r_g, h)_g + \frac{1}{4} s_g \text{trace } h) d\mu_g \]
\[ = \int \left( 2\Delta_g s_g + \frac{s_g^2}{2} \right) g + 2\nabla_g ds_g - 2s_gr_g, h \right)_g d\mu_g. \]

Lemma 1. A Riemannian metric \( g \in M_1 \) is a critical point of (6) if, and only if, we have that

\[ \nabla_g S = \left( 2\Delta_g s_g + \frac{s_g^2}{2} \right) g + 2\nabla_g ds_g - 2s_gr_g = \lambda_g g, \]

where

\[ \lambda_g = \frac{n - 4}{2n\mu_g(M)} \int s_g^2 d\mu_g = \frac{n - 4}{2n} \int s_g^2 d\mu_g. \]

Proof. By (7), we must have \( \nabla_g S \) must be orthogonal to all symmetric two-tensors of trace orthogonal to the constants, so parallel to \( g \). The value of the proportionality constant follows by computing the trace of the resulting tensorial identity \( \nabla_g S \), and integrating the resulting functions over \( M \) with respect to the measure defined by \( g \).

It is not immediate that the scalar curvature \( s_g \) of a metric satisfying the critical equation \( \nabla_g S \) must be a constant, but there are precedents when the dimension of \( M \) is either 2 or 4. In the latter case, \( s_g \) is proved to be constant [6] as an application of the maximum principle, and as such, the argument has a somewhat “local nature” flavor. (Besse’s main interest seems to have been on manifolds of dimension 4, perhaps the reason why his result was only derived in that case.) The argument used in dimension 2 has, by contrast, a “global” flavor built into it. We pause to analyze the details of these two results.

Given a Riemannian metric \( g \), we denote by \( \pi_g \) the \( L^2 \)-projection operator of \( L^2 \)-functions onto the constants. If \( g \in M_1 \) is a critical point of (6), its scalar curvature \( s_g \) must satisfy the equation

\[ (2n - 2)\Delta_g s_g + \frac{n - 4}{2} (s_g^2 - \pi_g(s_g^2)) = 0. \]
When \( n = 4 \), a solution \( g \) to (10) must be such that \( s_g \) is a harmonic, and therefore, constant by the maximum principle.

In general, a solution of (10) is the constant \( s_g = \pi_g(s_g) = \pm \sqrt{\pi_g(s_g^2)} \) if, and only if, \( s_g^2 - \pi_g(s_g^2) \) has a zero of order three. For (10) implies that this zero of order three must be a zero of infinite order, and the vanishing of this function would then be a consequence of Aronszajn’s unique continuation theorem for solutions to elliptic equations of order two \([3]\). Or said differently, if there exists a point \( p \) in \( M \) where the function \( u_g = s_g - \pi_g(s_g) \) vanishes to order three, then \( u_g \) vanishes to infinite order at the said point, and therefore, it must be the zero function. Indeed, in terms of the function \( u_g \), (10) is given by the equivalent expression

\[
(2n - 2)\Delta_g u_g + \frac{n - 4}{2}(u_g^2 + 2u_g\pi_g(s_g) + (\pi_g(s_g))^2 - \pi_g(s_g^2)) = 0,
\]

which then implies that the constants \( \pi_g(s_g^2) \) and \( (\pi_g(s_g))^2 \) would coincide, and by iterated differentiation followed by evaluation at \( p \), the point \( p \) would be a zero of infinite order for \( u \), and so \( u_g \equiv 0 \). Notice that if we knew that \( \pi_g(s_g^2) \) and \( (\pi_g(s_g))^2 \) coincide, by a mere integration of the equation above, we would conclude that \( \|u_g\|_{L^2} = 0 \), and so \( u_g \equiv 0 \).

But if \( s_g \) is a solution to (10), the existence of a point where \( s_g^2 - \pi_g(s_g^2) \) has a zero of order three is hard to prove on its own, if at all possible. It is the case that the function \( s_g \) is constant, but the proof of this fact requires information contained in equation (8) itself, which is somewhat lost in passing to its trace equation (10).

This subtle point is illustrated well in the known proof that \( s_g \) is constant when \( n = 2 \), an argument that, ironically, is more elaborate than the \( n = 4 \) case above.

Indeed, let \( M \) be a differentiable surface. By passing to a double covering if necessary, let us assume that \( M \) is oriented. Then \( M \) can be provided with a compatible complex structure \( J \), which is defined by taking an orthonormal \( g \)-frame \( \{e_1, e_2\} \) and declaring that \( Je_1 := e_2 \). This makes of \( M \) a complex manifold of dimension 1, and the metric \( g \) is Kähler with Kähler form \( \omega(\cdot, \cdot) = g(J\cdot, \cdot) \). We have the identity \(-2i\partial \bar{\partial} f = (\Delta f)\omega \) for any function \( f \).

For dimensional reasons, we then have that \( \rho_g = \frac{n}{2}\omega_g \), where \( \rho_g \) is the Ricci form of \( g \). By (10), \( 2\Delta s_g = s_g^2 - \pi_g s_g^2 \), and so \( \Delta^2 s_g = s_g\Delta s_g - (\nabla s_g, \nabla s_g) \). Therefore

\[
\Delta^2 s_g + 4(\rho_g, i\partial \bar{\partial} s_g) = -((\nabla s_g, \nabla s_g)),
\]

an equation that we rewrite as

\[
4((\partial^g)^\#)(\partial^g)^\# s_g = \Delta^2 s_g + 4(\rho, i\partial \bar{\partial} s_g) + (\nabla s_g, \nabla s_g) = 0,
\]

where \( \partial^g f \) is defined by the identity \( g(\partial^g f, \cdot) = \partial f \). Thus, the vector field \( \partial^g s_g \) is holomorphic. If \( M \) is either a hyperbolic or a parabolic Riemann surface, \( \partial^g s_g = 0 \), and \( s_g \) is constant. In the elliptic case, \( \partial^g s_g = 0 \) also but for a different and less elementary reason: The Kazdan-Warner invariant \([10]\) vanishes, and this measures the obstruction of \( g \) to being conformally equivalent to the standard metric. It follows that \( g \) itself is the standard metric, and \( s_g \) is constant. Thus, for any closed surface \( M \), a critical point of (10) must be a metric of constant scalar curvature (as proved in \([7]\) in the oriented case).

The point of the 2-dimensional argument above is that (10) leads to (11) by using the dimensional identity \( \rho_g = \frac{n}{2}\omega_g \), and (11) is the critical point equation (8) written as a symmetric nonlinear operator in \( g \) acting on \( s_g \). However, it is not
the case that all solutions of (10) (as an equation in \( s_g \)) are constant. Indeed, the ordinary differential equation
\[
2\ddot{u} = c - u^2, \quad c \text{ constant}
\]
has plenty of periodic solutions, and, for instance, on flat tori of all dimensions, the partial differential equation
\[
2\Delta_g u - (u^2 - c) = 0
\]
has many nonconstant solutions in addition to \( u = \sqrt{c} \). On the surface \( M \), the critical point of (6) singles out a metric whose scalar curvature is the constant solution to this equation, out of all the metrics of fixed volume in the conformal class of the starting one. This is the uniformization theorem for the conformal class that the initial metric defines. Once we pass to a double cover, dimensional reasons make the Hessian \( \nabla_g \) of a function \( f \), \( s_g \) in particular, \( J \)-invariant.

Our proof that the critical points of (6) have constant scalar curvature is a dimensionless reinterpretation of the argument above for \( n = 2 \). We use the differential Bianchi identity in the role that \( r_g = (s_g/2)g \) plays then.

**Lemma 2.** The scalar curvature \( s_g \) of a critical metric \( g \) of the functional (6) is constant.

**Proof.** We consider the variational expression (6) for tensors of the form \( h_\varphi = \nabla_g \varphi \in T_g M_1 \), \( \varphi \) a function on \( M \). Let \( g_t = g_{t,\varphi} \) be a family of metrics such that \( g_0 = g \) and \( \dot{g}_0 = h_\varphi \). By Bochner formula, we have that
\[
\delta(h_\varphi) = -\Delta_g d\varphi + r_g^{ij} \nabla_j \varphi,
\]
where \( \Delta_g \) is the Hodge Laplacian on forms, and since
\[
\delta g r_g = -\frac{1}{2} ds_g,
\]
we obtain the identity
\[
-\delta \delta(h_\varphi) = -\delta \nabla_g^*(h_\varphi) = \Delta_g^2 \varphi + r_g \cdot \nabla \nabla \varphi + \frac{1}{2} (\nabla_g s_g, \nabla g \varphi).
\]
Proceeding as in (7), by a simple dualization we obtain that
\[
\frac{d}{dt} \int s_{g+th}^2 \big|_{t=0} = \int 2 s_g (\Delta_g(\text{trace } h) + \delta g (\delta g h) - (r_g, h) g + \frac{1}{4} s_g \text{trace } h) d\mu_g
\]
\[
= -4 \int s_g \left( \Delta_g^2 \varphi + r_g \cdot \nabla \nabla \varphi + \frac{1}{2} (\nabla_g s_g, \nabla g \varphi) \right) d\mu_g
\]
\[
= -4 \int s_g (\nabla_g d)^* (\nabla_g d) \varphi d\mu_g,
\]
where the adjoint is taken with respect to the metric \( g \). If \( g \) is a critical metric of (6), this expression must vanish. If we take \( \varphi = s_g \), we conclude that the squared \( L^2 \)-norm of the \( \nabla_g ds_g \) vanishes, so \( \nabla_g ds_g = 0 \), and \( s_g \) is harmonic, and therefore, constant. \( \square \)

In the argument above, the variation of the metric \( g \) in the direction of a Hessian is analogous to the conformal deformation of a Kähler metric on a real oriented surface. In both cases, the deformed metrics differ from the original one by the action of a one parameter group of diffeomorphism, and in the latter case, the Kähler forms of all the resulting metrics represent a fixed Kähler class relative to the complex structure that the conformal metric defines. In higher dimensions
though, even if \( M \) is assumed to be complex of Kähler type, the resulting variations do not have to be Kähler as the Hessian does not need to be invariant with respect to the complex structure.

**Theorem 3.** The Euler-Lagrange equation for the functional (6) is given by the tensorial equation

\[
2s_g \left( \frac{s_g}{n} g - r_g \right) = 0. 
\]

On a connected closed manifold \( M \), a metric \( g \) is a critical if, and only if, it is either Einstein, or scalar flat, or both.

**Proof.** By Lemma 2, the scalar curvature \( s_g \) is constant. By this, (8) simplifies to the tensorial equation (12).

Let \( g \) be a critical metric. If \( n = 2 \), then \( r_g = \frac{s_g}{2} g \) and so Einstein. If \( n > 2 \), if \( s_g \) is nonzero at some point in \( M \), \( s_g \) must be a nonzero constant, and \( g \) must be Einstein, in a neighborhood of the said point of \( M \). By Aronszajn’s unique continuation theorem [3], (10) implies that \( s_g \) is a nonzero constant globally, and \( g \) is Einstein. Finally, Ricci or scalar flat metrics satisfy (12). \( \square \)

3. **Some remarks on the critical values of \( S \)**

In dimension two, the set of critical values of (6) can be described completely, a direct consequence of the uniformization theorem.

**Theorem 4.** For any closed surface \( M \), the functional (6) has exactly one critical value given by \( 16\pi^2 \chi(M)^2 \), \( \chi(M) \) the Euler characteristic of \( M \). For a given conformal class of metrics on \( M \), this critical value is realized by an Einstein metric \( g \) in the class of scalar curvature \( s_g = 4\pi\chi(M) \), and this metric in the class is unique up to isometries.

This result will have no strong counterpart in higher dimension, as the Catanese-LeBrun example [8] alluded to before shows. Though the entire picture remains unclear, even the nongeneric positive case is substantially more complicated.

**Lemma 5.** If \( M \) carries critical metrics of (6) of nonnegative scalar curvature, then the set of scalar curvatures of all such is bounded above, and if \( n \geq 3 \), it could contain infinitely many elements. If \( M \) carries a metric \( g \) of nontrivial nonnegative scalar curvature, and \( n \geq 3 \), then 0 must be a critical value of (6).

**Proof.** If \( r_g = \frac{s_g}{n} g \geq 0 \), then by Bishop comparison theorem we have that

\[
v_g(M) \leq \omega_n (\text{diam}(M, g))^n,
\]

where \( \omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \), \( \Gamma \) the Gamma function. If \( s_g > 0 \), by Myers’ theorem, we have that

\[
\text{diam}(M, g) \leq \frac{\pi}{\sqrt{s_g n(n-1)}},
\]

and so \( s_g \) must be bounded above since \( v_g(M) = 1 \).

On \( M = \mathbb{S}^2 \times \mathbb{S}^3 \), Wang and Ziller [14] construct a countably infinite set of volume one Einstein metrics \( g_n \) whose scalar curvatures \( s_{g_n} \) are positive, and such that \( s_{g_n} \searrow 0 \). In the limit, the metrics collapse.
The latter assertion is well known [6, Theorem 4.32 (ii)]. We sketch a proof here for completeness. The conformal Laplacian
\[ L_g = 4 \frac{n-1}{n-2} \Delta_g + s_g \]
is a strictly positive self-adjoint operator, and therefore, the Yamabe conformal invariant \( Y(M, [g]) \) is strictly positive. On the other hand, \( M \) carries a metric \( \tilde{g} \) with negative scalar curvature (see, for instance, [5] and [4]), and so \( Y(M, [\tilde{g}]) < 0 \).

By continuity of the Yamabe invariant, along the segment \((1-t)g + t\tilde{g}\), there exists a metric whose Yamabe invariant is zero. Its conformal class contains a metric of zero scalar curvature. This metric is a critical point of \( \text{(6)} \) of critical value 0. □

We close the note by discussing a family of Hermitian deformations of an Einstein metric on a manifold that exhibits different signs for its scalar and \( J \)-scalar curvatures, a bit intended to justify the comment we made at the end of §1.

Example 6. The Calabi-Eckmann manifold \( S^{2n+1} \times S^{2n+1} \) is the total space of a flat torus bundle over \( \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \), and thus, carries an integrable almost complex structure \( J \) compatible with the standard product metric \( g_n \). Suppose that this metric is dilated in the vertical directions by a factor \( \varepsilon^2 \). Then, the scalar and \( J \)-scalar curvature of the dilated metric \( g_n^\varepsilon \) are given by
\[
\begin{align*}
s_{g_n^\varepsilon} &= 4n(2n+1) + 4n(1 - \varepsilon^2), \\
s_{Jg_n^\varepsilon} &= 4n + 4n(2n+1)(1 - \varepsilon^2),
\end{align*}
\]
respectively. Notice that \( g_n = g_n^1 \).

It follows that as \( \varepsilon \searrow 0 \), the Einstein metric \( g_n \) is deformed and collapses in the Gromov-Hausdorff sense to the product of the standard Fubini-Study metrics on \( \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \). On the other hand, if we blow up the torus fiber letting \( \varepsilon \nearrow \infty \), the Einstein metric \( g_n \) transitions smoothly from metrics such that \( s_{g_n^\varepsilon} > 0 \) and \( s_{Jg_n^\varepsilon} > 0 \) to ones where both of these scalar tensors have negative values, the inequality \( s_{g_n^\varepsilon} > s_{g_n^\varepsilon} \) holding always.

References

[1] M. Anderson, Extrema of curvature functionals on the space of metrics on 3-manifolds, Calc. Var. and P.D.E. 5 (1997), pp. 199-269.
[2] T. Aubin, Equations du Type Monge-Ampère sur les Variétés Kählériennes Compactes, C. R. Acad. Sci. Paris 283A (1976), pp. 119-121.
[3] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations of second order, J. Math. Pures Appl., 36 (1957), pp. 235-249.
[4] T. Aubin, Métriques riemanniennes et courbure, J. Diff. Geom. 11 (1976), pp. 573-598.
[5] A. Avez, Valeur moyennée du scalaire de courbure sur une variété compacte, Applications relativistes, C.R. Acad. Sci. Paris, 256 (1063), pp. 5271-5273.
[6] A. L. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete; 3 Folge, Band 10, Springer-Verlag, 1987.
[7] E. Calabi, Extremal Kähler metrics, in Seminar on Differential Geometry (S. T. Yau Ed.), Annals of Mathematics Studies, Princeton University Press, 1982, pp. 259–290.
[8] F. Catanese & C. LeBrun, On the scalar curvature of Einstein manifolds. Math. Res. Lett. 4 (1997), pp. 843854.
[9] G. Catino, Critical metrics on the \( L^2 \)-norm of the scalar curvature, Proc. Amer. Math. Soc. 142 (2014), pp. 39813986.
[10] J.L. Kazdan & F.W. Warner, Curvature functions for compact two manifolds, Ann. of Math., 99 (1974), pp. 14-47.
[11] C.LeBrun & S.R. Simanca, On the Kähler Classes of Extremal Metrics, Geometry and Global Analysis (Sendai, Japan 1993), First Math. Soc. Japan Intern. Res. Inst. Eds. Kotake, Nishikawa & Schoen.

[12] S.B. Myers, Riemannian manifolds with positive mean curvature, Duke Math. J. 8 (1941), pp. 401-404.

[13] G. Tian, On Calabi’s Conjecture for Complex Surfaces with Positive First Chern Class, Inv. Math., 101 (1990), pp. 101-172.

[14] M. Wang & W. Ziller, Einstein Metrics on Principal Torus Bundles, J. Differential Geometry, 31 (1990), pp. 215-248.

[15] S.T. Yau, On the Curvature of Compact Hermitian Manifolds, Inv. Math. 25 (1974), pp. 213-239.

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