REAL SPECTRAL TRIPLES ON 3-DIMENSIONAL NONCOMMUTATIVE LENS SPACES.

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Abstract. We study almost real spectral triples on quantum lens spaces, as orbit spaces of free actions of cyclic groups on the spectral geometry on the quantum group $SU_q(2)$. These spectral triples are given by weakening some of the conditions of a real spectral triple. We classify the irreducible almost real spectral triples on quantum lens spaces and we study unitary equivalences of such quantum lens spaces. We show that if $r$ is coprime to $p$, the $C^*$-algebras corresponding to the quantum lens spaces $L_q(p, r)$ and $L_q(p, 1)$ are isomorphic. Also, we show that all such quantum lens spaces are principal $U(1)$-fibrations over quantum teardrops.

Lens spaces, orbit spaces of free actions of cyclic groups on odd-dimensional spheres, were first introduced in 1884 by Walther Dyck \[14\]. Lens spaces are interesting because they are some of the simplest manifolds exhibiting the difference between homotopy type and homeomorphism type. In this article we study almost real spectral triples on quantum lens spaces, as orbit spaces of free actions of cyclic groups on the spectral geometry on the quantum group $SU_q(2)$, as constructed in \[10\]. These spectral triples are given by weakening some of the conditions of a real spectral triple, just like in \[10\].
We classify the irreducible spectral geometries on quantum lens spaces and we study unitary equivalences of such quantum lens spaces.

We show that if \( r \) is coprime to \( p \), the \( C^* \)-algebras corresponding to the quantum lens spaces \( L_q(p,r) \) and \( L_q(p,1) \) are isomorphic. In the commutative (non-quantum) case, the lens spaces \( L(p,r) \) and \( L(p,r') \) are homeomorphic if and only if \( r \cdot r' \equiv \pm 1 \mod p \) or \( r \pm r' \equiv 0 \mod p \). Also, we show that all such quantum lens spaces are principal \( U(1) \)-fiber bundles over quantum teardrops, generalizing a result from [4]. We discuss the implications of this for the construction of spectral triples on the quantum teardrop.

1. The equivariant spectral triple on \( SU_q(2) \)

We recall the construction of the equivariant real spectral triple on \( SU_q(2) \) from [10]. This is not a real spectral triple in the sense of [7], since the opposite algebra only commutes with the algebra up to compact operators. This was done in order to cope with certain “no go-theorems”, which showed that it was impossible for a \( su_q(2) \)-equivariant spectral to satisfy all conditions of a real spectral triple [10, Remark 6.6].

Let \( q \) denote a real number, \( 0 < q < 1 \). Let \( \mathcal{A}(SU_q(2)) \) be the \( * \)-algebra generated by the two elements \( a, b \), satisfying the following relations:

\[
\begin{align*}
(1a) & \quad ba = qab \\
(1b) & \quad b^*a = qab^* \\
(1c) & \quad bb^* = b^*b \\
(1d) & \quad a^*a + q^2b^*b = 1 \\
(1e) & \quad aa^* + bb^* = 1.
\end{align*}
\]

From these relations it follows that \( a^*b = qba^* \), \( a^*b^* = qb^*a^* \) and \( [a, a^*] = (q^2 - 1)bb^* \). If \( q = 1 \), we recover the generators of \( SU(2) \) as a commutative space.

The Hopf \( * \)-algebra \( U_q(\mathfrak{su}(2)) \), for which we require the spectral triple to be equivariant, is generated over \( \mathbb{C} \) by elements \( e, f, k, k^{-1} \), satisfying:

\[
\begin{align*}
(2) & \quad kk^{-1} = 1, \quad ek = qke, \quad kf = qfk, \quad k^2 - k^{-2} = (q - q^{-1})(fe - ef),
\end{align*}
\]

with the coproduct given by

\[
\begin{align*}
(3) & \quad \Delta k = k \otimes k, \quad \Delta e = e \otimes k + k^{-1} \otimes e, \quad \Delta f = f \otimes k + k^{-1} \otimes f,
\end{align*}
\]

and antipode given by

\[
\begin{align*}
(4) & \quad Sk = k^{-1}, \quad Sf = -qf, \quad Se = -q^{-1}e.
\end{align*}
\]

This algebra has irreducible finite dimensional representations \( \sigma_l \) on the \( 2l+1 \) dimensional vector space \( V_l \), labeled by half-integers \( l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \) [20]. Let \( |l,m> \), (where \( m = -l, -l+1, \ldots l \)) be a basis for the irreducible \( U_q(\mathfrak{su}(2)) \)-module \( V_l \). Define \([n] = [n]_q = q^n - q^{-n} \). The representation \( \sigma_l \) on \( V_l \) is given
by
\[
\sigma_l(k) | l, m \rangle = q^m | l, m \rangle,
\]
(5)
\[
\sigma_l(f) | l, m \rangle = \sqrt{|l - m|} q^{l + m + 1} | l, m + 1 \rangle,
\]
\[
\sigma_l(e) | l, m \rangle = \sqrt{|l - m|} q^{l + m} | l, m - 1 \rangle.
\]
We can define two commuting actions of $\mathfrak{su}(2)$ on the algebra $A(SU_q(2))$, as follows. One is the usual left action $\triangleright$, defined by
\[
k \triangleright a = q^{\frac{k}{2}} a, \quad k \triangleright a^* = q^{-\frac{k}{2}} a^*, \quad k \triangleright b = q^{\frac{k}{2}} b, \quad k \triangleright b^* = q^{-\frac{k}{2}} b^*,
\]
(6)
\[
f \triangleright a = 0, \quad f \triangleright a^* = - q b^*, \quad f \triangleright b = a, \quad f \triangleright b^* = 0,
\]
\[
e \triangleright a = b, \quad e \triangleright a^* = 0, \quad e \triangleright b = 0, \quad e \triangleright b^* = - q^{-1} a^*.
\]
The other action, commuting with the previous one, is given by applying the order 2 automorphism $\theta$ of $\mathfrak{su}(2)$, which maps $e$ to $-f$ and maps $k$ to $k^{-1}$, and the inverse of the antipode to the usual right action. As can be checked, this gives a left representation $\cdot$, commuting with (6):
\[
k \cdot a = q^{\frac{k}{2}} a, \quad k \cdot a^* = q^{-\frac{k}{2}} a^*, \quad k \cdot b = q^{\frac{k}{2}} b, \quad k \cdot b^* = q^{-\frac{k}{2}} b^*,
\]
(7)
\[
f \cdot a = 0, \quad f \cdot a^* = q b^*, \quad f \cdot b = 0, \quad f \cdot b^* = - a,
\]
\[
e \cdot a = - b^*, \quad e \cdot a^* = 0, \quad e \cdot b = q^{-1} a^*, \quad e \cdot b^* = 0.
\]
There is a vector space basis $e_{klm}$ of $A(SU_q(2))$, given by monomials of the form
\[
e_{klm} := \begin{cases} a^k b^l b'^m & k \in \mathbb{Z}, k \geq 0, l, m \in \mathbb{N} \\ b^l b'^m a^* - k & k \in \mathbb{Z}, k < 0, l, m \in \mathbb{N} \end{cases}
\]
(8)

There is a so-called Haar state, which is the unique [34, Theorem 4.2] bi-invariant linear functional $\psi$ on the $C^*$-completion of $A(SU_q(2))$, given by [34, Appendix A1]:
\[
\psi(e_{klm}) = \begin{cases} 0 & \text{when } k > 0 \text{ or } l \neq m \\ \frac{1 - q^2}{1 - q^{m+2}} & \text{when } k = 0 \text{ and } l = m 
\end{cases}
\]

Just as in the commutative ($q \to 1$) case, the GNS-representation space $V = L^2(SU_q(2), \psi)$ with respect to the Haar state $\psi$ has a Peter-Weyl decomposition
\[
V = \bigoplus_{2l=0}^{\infty} V_l \otimes V_l,
\]
by a suitable analogue of the Peter-Weyl theorem for compact quantum groups [36, Section 6]. We abbreviate a vector $|lm\rangle \otimes |ln\rangle \in V_l \otimes V_l$ by $|lmmn\rangle$.

On this space we have two commuting left $\mathfrak{su}(2)$-actions, $\lambda, \rho$, given on the subspace $V_l \otimes V_l$ by
\[
\lambda := \text{id} \otimes \sigma_l, \quad \rho := \sigma_l \otimes \text{id}.
\]
(9)
These two commuting actions are an extension to $q \neq 1$ of the classical case where we can identify

$$SU(2) = S^3 = \text{Spin}(4)/\text{Spin}(3) = (SU(2) \times SU(2))/SU(2),$$

with the action of $\text{Spin}(4)$ on $SU(2)$ given by $(g,h)x = gxh^{-1}$.

We recall the definition of a $(\lambda,\rho)$-equivariant representation [10, Definition 3.2]:

**Definition.** Let $\lambda$ and $\rho$ be two commuting representations of a Hopf-algebra $H$ on a vector space $V$. A representation $\Pi$ of a $*$-algebra $A$ on $V$ is $(\lambda,\rho)$-equivariant if the following compatibility relations hold:

$$
\begin{align*}
(\lambda(h)\Pi(a))v \otimes w &= \pi(h^{(1)} \cdot a) \lambda(h^{(2)})v, \\
(\rho(h)\Pi(a))v \otimes w &= \pi(h^{(1)} \triangleright a) \rho(h^{(2)})v,
\end{align*}
$$

for all $h \in H$, $a \in A$ and $v \in V$.

In our case, the two commuting actions $\lambda$ and $\rho$ on the Hilbert space are given by (9).

In the case of $SU_q(2)$, we need to consider two copies [10] of the Hilbert space:

$$H = V \otimes \mathbb{C}^2 \simeq V \otimes V_{\frac{1}{2}} \simeq V_{\frac{1}{2}} \oplus \bigoplus_{2j=1}^{\infty} (V_{j+\frac{1}{2}} \otimes V_j) \oplus (V_{j-\frac{1}{2}} \otimes V_j)$$

For convenience, we rename the spaces on the right hand side as

$$H = H_0^\uparrow \oplus \bigoplus_{2j \geq 1} H_j^\uparrow \oplus H_j^\downarrow,$$

with $H_j^\uparrow \simeq (V_{j+\frac{1}{2}} \otimes V_j)$ and $H_j^\downarrow \simeq (V_{j-\frac{1}{2}} \otimes V_j)$.

The definition of the $(\lambda,\rho)$ action on $V$ in (9) can be amplified to $H$ in several ways. Following [10], we define:

**Definition.** The representations $\lambda$ and $\rho$ of $U_q(\text{su}(2))$ on $V$ are amplified to $\lambda'$ and $\rho'$ on $H$, as

$$
\lambda'(h) := \left(\lambda \otimes \sigma_{\frac{1}{2}}\right)(\Delta h) = \lambda(h^{(1)}) \otimes \sigma_{\frac{1}{2}}(h^{(2)}), \\
\rho'(h) := \rho(h) \otimes \text{id}.
$$

It is easy to check that $\lambda'$ and $\rho'$ commute.

We now define an explicit basis of $H$ which is suited to $(\lambda',\rho')$-equivariance, i.e. basis vectors are eigenvectors for $\lambda'(k)$ and $\rho'(k)$ and $\lambda'(f)$, $\rho'(f)$, $\lambda'(e)$ and $\rho'(e)$ are ladder operators. We use the shorthand $k^\pm = k \pm \frac{1}{2}$. Set

$$C_{j\mu} := q^{-\frac{(j+\mu)/2}{2j} \left[ j - \mu \frac{1}{2} \right]} \text{ and } S_{j\mu} := q^{\frac{(j-\mu)/2}{2j} \left[ j + \mu \frac{1}{2} \right]}.$$
Using the decomposition of $\mathcal{H}$ of (11), we define for $j = l + \frac{1}{2}$, $\mu = m - \frac{1}{2}$, $\mu = -j, \ldots, j$ and $n = -j + \frac{1}{2}, \ldots, j - \frac{1}{2}$:

$$|j\mu n \downarrow\rangle := C_{j\mu} |j^- \mu^+ n\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle + S_{j\mu} |j^- \mu^- n\rangle \otimes |\frac{1}{2}, +\frac{1}{2}\rangle,$$

with $m^\pm = m \pm \frac{1}{2}$, and $|j^- \mu^+ n\rangle \in V_{j^-}$. For $j = l - \frac{1}{2}$, $\mu$ as before and $n = -j - \frac{1}{2}, \ldots, j + \frac{1}{2}$:

$$|j\mu n \uparrow\rangle := -S_{j\mu} |j^+ \mu^+ n\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle + C_{j\mu} |j^+ \mu^- n\rangle \otimes |\frac{1}{2}, +\frac{1}{2}\rangle.$$

The action of the elements of $U_q(\mathfrak{su}(2))$ can now be calculated, and it is easily shown that the $|j\mu n \uparrow\rangle$ and $|j\mu n \downarrow\rangle$ are joint eigenvectors for $\lambda'(k)$ and $\rho'(k)$ with eigenvalues $q^\mu$ for $\lambda'$ and $q^n$ for $\rho'$.

Summarizing, we have the following definition for the Hilbert space $\mathcal{H}$:

**Definition.** Let $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and let $\mu = -j, -j + 1, \ldots, j$ and $n = -j - \frac{1}{2}, -j + \frac{1}{2}, \ldots, j + \frac{1}{2}$. The Hilbert space $\mathcal{H}$ of the real spectral triple $(A(SU_q(2)), \mathcal{H}, D, J)$ is spanned by basis vectors

$$|j\mu n \rangle := \begin{pmatrix} |j\mu n \uparrow\rangle \\ |j\mu n \downarrow\rangle \end{pmatrix}$$

with the convention that the $|j\mu n \downarrow\rangle$ component is zero when $n = \pm (j + \frac{1}{2})$ or $j = 0$.

An equivariant representation of the algebra $A(SU_q(2))$ on this Hilbert is given by:

**Proposition 1.1 (10).** The following representation $\Pi$ of $A(SU_q(2))$ on the Hilbert space $\mathcal{H}$ with orthonormal basis $|j\mu n \rangle$, is equivariant with respect to the $(\lambda', \rho')$-action of $U_q(\mathfrak{su}(2))$.

(18a) $\Pi (a) |j, \mu, n \rangle = \alpha_{j\mu n}^+ |j^+ \mu^+ n^+\rangle + \alpha_{j\mu n}^- |j^- \mu^+ n^+\rangle$

(18b) $\Pi (b) |j, \mu, n \rangle = \beta_{j\mu n}^+ |j^+ \mu^+ n^-\rangle + \beta_{j\mu n}^- |j^- \mu^+ n^-\rangle$

(18c) $\Pi (a^*) |j, \mu, n \rangle = \tilde{\alpha}_{j\mu n}^+ |j^+ \mu^+ n^-\rangle + \tilde{\alpha}_{j\mu n}^- |j^- \mu^+ n^-\rangle$

(18d) $\Pi (b^*) |j, \mu, n \rangle = \tilde{\beta}_{j\mu n}^+ |j^+ \mu^- n^+\rangle + \tilde{\beta}_{j\mu n}^- |j^- \mu^- n^+\rangle$
where $\alpha_{j\mu n}^\pm$ and $\beta_{j\mu n}^\pm$ are, up to phase factors depending only on $j$, the following triangular $2 \times 2$ matrices:

\begin{align}
(19a) & \quad \alpha_{j\mu n}^+ = q^{(\mu+n-\frac{1}{2})/2[j + \mu + 1]} \begin{pmatrix} q^{-j - \frac{1}{2}} \frac{[j+n+\frac{1}{2}]}{[2j+2]} & 0 \\ \frac{1}{2} \frac{[j-n+\frac{1}{2}]}{[2j+2]} & q^{-j} \frac{[j+n+\frac{1}{2}]}{[2j+1]} \end{pmatrix} \\
(19b) & \quad \alpha_{j\mu n}^- = q^{(\mu+n-\frac{1}{2})/2[j - \mu]} \begin{pmatrix} q^{j+1} \frac{[j-n+\frac{1}{2}]}{[2j+1]} & -q^{j} \frac{[j+n+\frac{1}{2}]}{[2j+1]} \\ 0 & q^{j+\frac{1}{2}} \frac{[j-n+\frac{1}{2}]}{[2j+2]} \end{pmatrix} \\
(19c) & \quad \beta_{j\mu n}^+ = q^{(\mu+n-\frac{1}{2})/2[j + \mu + 1]} \begin{pmatrix} [j-n+\frac{1}{2}] & 0 \\ -q^{-j-1} \frac{[j+n+\frac{1}{2}]}{[2j+1]} & q^{-j} \frac{[j-n+\frac{1}{2}]}{[2j+1]} \end{pmatrix} \\
(19d) & \quad \beta_{j\mu n}^- = q^{(\mu+n-\frac{1}{2})/2[j - \mu]} \begin{pmatrix} -q^{j+\frac{1}{2}} \frac{[j+n+\frac{1}{2}]}{[2j+1]} & -q^{j} \frac{[j-n+\frac{1}{2}]}{[2j+1]} \\ 0 & [j+n+\frac{1}{2}] \end{pmatrix}
\end{align}

and the remaining matrices are the hermitian conjugates

\[ \tilde{\alpha}_{j\mu n}^\pm = (\alpha_{j\mu n}^\mp)^\dagger, \quad \tilde{\beta}_{j\mu n}^\pm = (\beta_{j\mu n}^\mp)^\dagger. \]

The Dirac operator of the spectral triple constructed in [10] Section 5 is given by:

\begin{equation}
D \begin{pmatrix} |j \mu n \uparrow\rangle \\ |j' \mu' n' \downarrow\rangle \end{pmatrix} = \begin{pmatrix} (2j + \frac{3}{2}) |j \mu n \uparrow\rangle \\ -(2j' + \frac{3}{2}) |j' \mu' n' \downarrow\rangle \end{pmatrix},
\end{equation}

The reality operator, constructed in [11] Section 6 is given by:

\begin{equation}
J \begin{pmatrix} |j \mu n \uparrow\rangle \\ |j' \mu' n' \downarrow\rangle \end{pmatrix} = \begin{pmatrix} i^{2(j+\mu+n)} |j, -\mu, -n \uparrow\rangle \\ i^{2(j'-\mu'-n')} |j', -\mu', -n' \downarrow\rangle \end{pmatrix}.
\end{equation}

Together, the algebra $A(SU_q(2))$, with representation as given in Proposition 11.1, Dirac operator $D$ given by (20), and reality operator $J$ given by (21), satisfy most conditions as stated in [7], with some slight modifications. These modifications are that $[\Pi(x), J\Pi(y^*)J^\dagger]$ and $[[D', \Pi(x)], J\Pi(y^*)J^\dagger]$ are not exactly 0, but lie in the two-sided ideal $K_q$ in $B(H)$ generated by the compact, positive, trace class operators

\begin{equation}
L_q : L_q |j, \mu, n\rangle = q^j |j, \mu, n\rangle.
\end{equation}

This means that the first order and reality conditions should be modified appropriately.

Also, it is currently unknown whether the Hochschild cycle condition and the Poincaré duality condition are satisfied. For a discussion of these conditions for $SU_q(2)$, see Remark 3.2.
2. Topological quantum lens spaces

Commutative lens spaces are defined as the quotient of an odd-dimensional sphere by a free action of a finite cyclic group. We will only consider 3-dimensional lens spaces here. Let $p, r_1, r_2$ be integers such that $\gcd(r_1, p) = \gcd(r_2, p) = 1$.

We view $S^3$ as the points $(z_1, z_2) \in \mathbb{C}^2$ such that $|z_1|^2 + |z_2|^2 = 1$. We can define an action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ on $S^3$ as

$$(z_1, z_2) \mapsto (e^{\frac{2\pi i r_1}{p}} z_1, e^{\frac{2\pi i r_2}{p}} z_2).$$

The lens space $L(p; r_1, r_2)$ is then defined as the quotient of $S^3$ by this action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$.

For 3-dimensional lens spaces we see that $L(p; r_1, r_2) = L(p; 1, r_1^{-1} r_2)$ by multiplying each component by $e^{\frac{2\pi i}{r_1} r_1^{-1}}$ (where $r_1^{-1}$ is meant mod $p$). We will thus write $L(p, r)$ for the lens space $L(p; r_1, r_2)$, with $r = r_1^{-1} r_2$.

Since $S^3 \simeq SU(2)$, it is natural to define quantum lens spaces as quotients of $SU_q(2)$ by free actions of finite cyclic groups (the analogy is $z_1 \leftrightarrow a$, and $z_2 \leftrightarrow b$). We want to consider quotients of the algebra $A(SU_q(2))$ under actions of the group $\mathbb{Z}/p\mathbb{Z}$. In order to do this, we need to embed $\mathbb{Z}/p\mathbb{Z}$ into $\text{Aut}(SU_q(2))$. Since there exists only one non-trivial outer automorphism of $A(SU_q(2))$ [13 Proposition 3.1], namely the map of order two induced by $a \mapsto a, b \mapsto b^*$, we can fit the automorphisms of $A(SU_q(2))$ into the following exact sequence:

$$0 \to \text{Inn}(A(SU_q(2))) \to \text{Aut}(A(SU_q(2))) \to \mathbb{Z}/2\mathbb{Z} \times U(1) \to 0,$$

with $U(1)$ acting by multiplying the $a$ and $b$ generators by complex numbers of modulus 1. To maximize the analogy with the commutative case, we consider actions of cyclic groups $\mathbb{Z}/p\mathbb{Z}$ embedded in $U(1)$ as outer automorphisms. Let $g$ denote a generator of $\mathbb{Z}/p\mathbb{Z}$, and let $r$ be an integer relatively prime to $p$. Let $\epsilon(r) = (r + 1) \mod 2$. We have:

**Proposition 2.1.** The following action of $\mathbb{Z}/p\mathbb{Z}$ on the algebra $A(SU_q(2))$:

$$v(g) \triangleright a = e^{\frac{2\pi i}{p} r a} a$$

$$v(g) \triangleright b = e^{\frac{2\pi i}{p} r b} b.$$

generates a unitary representation $v$ of $\mathbb{Z}/p\mathbb{Z}$ on $\mathcal{H}$ given by:

$$v(g) |j, \mu, n\rangle = e^{\frac{2\pi i}{p} ((1+r)\mu+(1-r)n+\frac{1}{2}\epsilon(r))} |j, \mu, n\rangle.$$

The presence of $\epsilon(r)$ guarantees that the expression in brackets is an integer.

**Proof.** Since $\mathbb{Z}/p\mathbb{Z}$ is a group, we demand that the action of $A$ on $\mathcal{H}$ is equivariant with respect to the group viewed as a Hopf algebra, with coproduct $\Delta(g) = g \otimes g$. From this it follows that $v(g) |j\mu n\rangle = c_{j\mu n} |j\mu n\rangle$, for some $c_{j\mu n} \in \mathbb{C}$, since otherwise this would not be compatible with the
equivariance condition $v(g)\Pi(a)v = \Pi(g(1)a)v(g(1))v$, where $g(1)a$ is $e^{2\pi i a}$, by Proposition 1.1. We can then calculate:

$$v(g)\Pi(a)|j\mu n\rangle = \Pi(e^{2\pi i p a})v(g)|j\mu n\rangle.$$ 

From (18) we see that the action of $a$ and $a^*$ on $\mathcal{H}$ leaves the difference $\mu - n$ constant, and $b$ leaves the sum $\mu + n$ constant.

We get the recurrence relations $v(g)|j\pm\mu+n+\rangle = e^{2\pi i p}v(g)|j\mu n\rangle$ and $v(g)|j\pm\mu+n-\rangle = e^{2\pi i p r}v(g)|j\mu n\rangle$. Solving these recurrence relations, we see that $v(g)|j\mu n\rangle = e^{2\pi i p ((\mu+n)+r(\mu-n))} + c$, where $c$ is any constant. In order to have $g^p = 1$, we see that $c = \epsilon(r)$ plus an additional integer, which can be set to zero.

**Proposition 2.2.** The above defined action on the $SU_q(2)$ extends to its $C^*$-algebra and is free in the sense of Ellwood [15].

Proof. First of all, we can easily translate the action of $\mathbb{Z}/p\mathbb{Z}$ to the right coaction of $C(\mathbb{Z}/p\mathbb{Z})$ on the $SU_q(2)$ algebra:

$$\Delta_R x = \sum_{g \in \mathbb{Z}/p\mathbb{Z}} (g \triangleright x) \otimes \delta_g.$$ 

Recall that that the freeness of a coaction of a Hopf algebra $H$ on a $C^*$-algebra $A$ means (for a right coaction) that the spans of $(A \otimes \text{id})\Delta_R(A)$ and $\Delta_R(A)(A \otimes \text{id})$ are dense in $A \otimes H$ for a minimal tensor product.

For the action, which defines lens spaces the freeness is easy to verify. Consider the identity:

$$(a a^* + b b^*)^r = 1,$$

which could be rewritten, using the commutation relations as:

$$a^r (a^*)^r + b P(a, a^*, b, b^*) = 1,$$

where $P$ is some polynomial in the generators. Therefore,

$$\Delta_R(a^r) ((a^*)^r \otimes 1) + \Delta_R(b) (P(a, a^*, b, b^*) \otimes 1) = \sum_{k=0}^{p-1} e^{2\pi i \frac{k}{r}} \otimes \delta_k = 1 \otimes f,$$

where $f$ is the function on $\mathbb{Z}/p\mathbb{Z}$:

$$f(k) = e^{2\pi i \frac{k}{r}}.$$

As for $r > 0$ and $p$ relatively prime the function $f$ generates the algebra $C(\mathbb{Z}/p\mathbb{Z})$ this finishes the proof.  

Observe that if we replace $r$ by $r-p$ the action on the generators does not change. If we take $p$ even then $r$ is necessarily odd, and $\epsilon(r)$ is 0. If $r$ is
even, then \( p \) is necessarily odd, and \( r - p \) is odd, and on the Hilbert space the actions determined by \((p, r)\) and \((p, r - p)\) are equivalent since:

\[
e^{\frac{2\pi i}{r}((1+(r-p))\mu+(1-(r-p))n)} = \left( e^{\frac{2\pi i}{p}(\mu+n)} e^{-\frac{\pi i \epsilon(r)}{p}} \right) e^{\frac{2\pi i}{p}((1+r)\mu+(1-r)n+\frac{1}{2}\epsilon(r))}
= \left( -e^{-\frac{\pi i \epsilon(r)}{p}} \right) \rho(g),
\]

where we have used that \( n - \mu \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z} \). Since

\[
(26)
\left( -e^{-\frac{\pi i \epsilon(r)}{p}} \right)^p = 1,
\]

for odd \( p \), the actions are equivalent. For this reason, we can always take \( r \) odd, and \( \epsilon(r) = 0 \). Let \( K = 0, 1, 2, \ldots, p - 1 \). We define \( \mathcal{H}_K \) as the eigensubspace of \( \mathcal{H} \) for the action of \( g \) with eigenvalue \( e^{-\frac{\pi i K}{p}} \). Further, let \( L_q(p, r) \) be the subalgebra of \( \mathcal{A}(SU_q(2)) \) which is invariant under the action of \( \mathbb{Z}/p\mathbb{Z} \).

**Proposition 2.3.** For each \( K = 0, 1, 2, \ldots, p - 1 \) and each \( x \in L_q(p, r) \), \( \Pi(x)\mathcal{H}_K \subset \mathcal{H}_K \).

**Proof.** From (8) we know that \( \mathcal{A}(SU_q(2)) \) is generated as a vector space by the elements \( a^k b^{*m} \) and \( b^{*m} a^n \) for \( k, l, m, n \in \mathbb{N} \). Since \( p \) and \( r \) are taken to be relatively prime, we see from (24a) and (24b) that \( L(p, r) \) is generated by elements \( a^k b^{*m} \) with \( k + r(l - m) \equiv 0 \mod p \) and \( b^{*m} a^n \) with \( r(l - m) - n \equiv 0 \mod p \). From (18), we can calculate that the action of \( a^k b^{*m} \) on \( \mu \) and \( n \) is given by

\[
\mu \mapsto \mu + \frac{1}{2}k + \frac{1}{2}l - \frac{1}{2}m, \quad n \mapsto n + \frac{1}{2}k - \frac{1}{2}l + \frac{1}{2}m.
\]

Then if \( v(g)\langle j, \mu, n \rangle = e^{\frac{2\pi i K}{p}}\langle j, \mu, n \rangle \), we have \((1+r)\mu+(1-r)n \equiv K \mod p\) and it then follows that

\[
(1+r)(\mu + \frac{1}{2}k + \frac{1}{2}l - \frac{1}{2}m) + (1-r)(n + \frac{1}{2}k - \frac{1}{2}l + \frac{1}{2}m) = (1+r)\mu + (1-r)n + k + r(l - m) \equiv K \mod p.
\]

\( \square \)

**Proposition 2.4.** The equivariant real structure \( J \), as given in (21) satisfies:

\[
(27)
J\mathcal{H}_K = \mathcal{H}_{K'},
\]

where \( K + K' \equiv 0 \mod p \).

**Proof.** We have \( J\langle j, \mu n \rangle = c_{j\mu n} \langle j - \mu - n \rangle \) with \( c_{j\mu n} \) a complex number, and from

\[
(1+r)\mu + (1-r)n = K \mod p,
\]

it follows that

\[
(1+r) \cdot (-\mu) + (1-r) \cdot (-n) = -K \mod p.
\]

\( \square \)
3. Geometry of quantum lens spaces

We now turn to the construction of the almost real spectral triple of the quantum lens space. As stated at the end of Section 1, we modify some conditions of a real spectral triple, exactly as in [10], i.e. the real structure and the first order condition only hold up to the compact operators of positive trace class defined in (22).

Furthermore, it is unknown if the Hochschild cycle condition and the Poincaré duality are satisfied for $SU_q(2)$. This means that also for $L_{q}(p,r)$ we do not know if they are satisfied.

We call a structure, satisfying all conditions of [7], with the modification of the first order condition, and the removal of the Hochschild cycle condition and Poincaré duality an almost real spectral triple.

**Proposition 3.1.** Let $L_{q}(p,r)$, $q \in (0,1)$, be the quantum lens space as defined above. Then for any $K = 0, 1, \ldots p-1$, the Hilbert space $H_K \oplus H_{K'}$, where $K + K' \equiv 0 \mod p$, the reality structure $J$ and the Dirac operator $D$ taken as the restrictions of $J$ and $D$ from the $A(SU_q(2))$ spectral geometry constitute a spectral geometry over the quantum lens space $L_{q}(p,r)$.

**Proof.** Almost all the usual conditions for a real spectral triple are easily seen to carry over from the $SU_q(2)$ case. For completeness, we list them here, and give short arguments why they carry over.

- The compact resolvent and dimension growth follow from the fact that for $j$ big enough, there always exist $\mu$ and $n$ such that there are vectors $\langle j, \mu, n \rangle \in H_K$, and thus also in $H_{K'}$, hence the dimension growth is satisfied. The compact resolvent condition also follows from this, and the fact that the dimension of the kernel of $D$ still is finite dimensional.

- The commutation relations between $J$ and $D$ and the sign of $J^2$ are trivially the same as for $SU_q(2)$.

- The algebra satisfies the regularity condition: $[D,a]$ is a bounded operator on $H$ for all $a \in A$ and both $a$ and $[D,a]$ belong to the domain of smoothness $\bigcap_{k=1}^{\infty} \text{Dom}(\delta^k)$ of the derivation $\delta$, with $\delta(T) = [D,T]$. This condition is satisfied since $L_{q}(p,r)$ is a subalgebra of the algebra $A(SU_q(2))$ which satisfies this condition by [33] Proposition 2.1.

- The opposite algebra and first order condition are weakened as described in [10]: the opposite algebra condition $[a, Jb^*J^{-1}] = 0$ and the first order condition $[[D,a], Jb^*J^{-1}] = 0$ should hold up to compact operators in $\mathcal{K}_q$ as defined in (22). We should check whether this condition still holds for the lens space. First, observe that the $L_{q}$ operators are unchanged, since they only see the $j$ component, which does not play a role in the $v(g)$ action. In fact, due to [10] Proposition 7.2 for the generators $a, a^*, b, b^*$, the commutation relations are given by $[\pi(x), \pi^\alpha(y)] = L_q^2 A$, with $A$ a bounded operator depending on $x$ and $y$. Since for any $a, b$ in a ring $R$, we have
that if \([a,b] \in \mathcal{I}\), with \(\mathcal{I}\) an ideal of \(R\), we have \([a^m,b^n] \in \mathcal{I}\) by repeatedly applying \([a,b] \in \mathcal{I}\), the generators of \(L_q(p,r)\) as described in Proposition 2.3 also satisfy this condition.

\[\square\]

**Remark 3.2.** The Hochschild cycle condition, the analogue of having a nowhere vanishing volume form, is problematic in the same way as it is for \(SU_q(2)\) alone. Certainly, as shown in [27, Proposition 1.2], for \(SU_q(2)\) there are no nontrivial Hochschild cycles \(n\)-cycles for \(n > 1\). However, the original Hochschild cycle condition does not require the cycle to be nontrivial. It remains yet to be verified whether a (possibly modified) version of the Hochschild cycle condition holds for \(SU_q(2)\) and the quantum lens spaces. The “dimension drop” phenomenon alone can be addressed by using twisted Hochschild homology as in [19, Theorem 1.2], where an analogue of the volume form, \(dA\) was constructed in twisted Hochschild homology for \(SU_q(2)\). However, the relevant setup for the that would be this of modular Fredholm modules and modular spectral triples [30,31].

### 3.0.1. Irreducibility

Though any of the above listed spectral geometries for the quantum lens spaces is admissible from the point of view of the non-commutative axiomatic approach, not all correspond to spin structures on commutative lens spaces. We demand that our spectral triple be irreducible as the analogue of a connected manifold in commutative geometry. If we use [17, Definition 11.2] or [25, Definition 2.1], all real spectral triples above are irreducible, since the \(J\) operator interchanges the \(\mathcal{H}_K\) and \(\mathcal{H}_{K'}\) spaces. If we use the definition of [8, Remark 6 on p.163], only irreducibility with respect to the algebra action and Dirac operator are demanded. None of the above real spectral triples are then irreducible, however the cases where \(K = K' = 0\) and \(K = K' = p/2\) if \(p\) is even can be made irreducible by dropping one of the two copies of the Hilbert space, and setting the real spectral triple to be \((L_q(p,r), \mathcal{H}_K, D, J)\). Since \(J\mathcal{H}_K \subset \mathcal{H}_K\) in this case, this is a well-defined spectral triple, and irreducible.

For even \(p\) we obtain two possible spin structures, for odd \(p\) just one, just as in the commutative case [16].

**Theorem 3.3.** The quantum lens space \(L_q(p,r)\) admits one irreducible almost real spectral triple if \(p\) is odd, and two if \(p\) is even. The spectral geometries are given in the two cases by

- \(p \text{ mod } 2 = 1\): \((L_q(p,r), \mathcal{H}_0, D, J)\).
- \(p = 2P\): \((L_q(p,r), \mathcal{H}_0, D, J)\) and \((L_q(p,r), \mathcal{H}_P, D, J)\).

with \(D\) the Dirac operator as described in [20] and \(J\) the operator given in [21].
The spectrum of the Dirac operator. Let us recall that the spectrum of the Dirac operator over $\mathcal{A}(SU_q(2))$ (with appropriate normalization) is given by:

$$D|j,\mu,n,\uparrow\rangle = \left(2j + \frac{3}{2}\right)|j,\mu,n,\uparrow\rangle$$

with multiplicity: $(2j + 1)(2j + 2)$

$$D|j,\mu,n,\downarrow\rangle = -\left(2j + \frac{1}{2}\right)|j,\mu,n,\downarrow\rangle$$

with multiplicity: $(2j + 1)2j$

Note that the spectrum is symmetric. Since in the construction of the spectral geometries on quantum lens spaces we keep the Dirac operator of $SU_q(2)$ (and only restrict the Hilbert space), the spectrum remains unchanged, only the multiplicities differ. We shall reduce the problem of computing these multiplicities to a number theoretic problem of solving congruence relations.

**Proposition 3.4.** The eigenvalues of $D$ belonging to an irreducible almost real spectral triple as in Theorem 3.3 with Hilbert space $\mathcal{H}_K$ ($K = 0$ or $K = \frac{1}{2}p$) are $2j + \frac{3}{2}$ and $-2j - \frac{1}{2}$, with respective multiplicity $N^+(j)$ and $N^-(j)$, (either of them could be 0, which means that the value is not present in the spectrum) where $N^\pm(j)$ denotes the number of solutions to the equation:

$$(28) \quad (1+r)\mu + (1-r)n \equiv K \mod p$$

with $-j \leq \mu \leq j$ and $-(j \pm \frac{1}{2}) \leq n \leq j \pm \frac{1}{2}$.

The exact calculation of the number of eigenvalues is a tedious task. The solution depends heavily on the properties of $(1+r)$ and $(1-r)$, in particular on the greatest common divisor of $1 \pm r$ and $p$. Although in each case the explicit solutions for $\mu$ and $n$ can be easily found, calculating the number of solutions for a given $j$ is rather difficult for an abstract choice of $r$ and $p$.

For illustration, we show here some pictures of what the spectra for small $p$ looks like. We represent the basis vectors $|j\mu n \uparrow\rangle, |j\mu n \downarrow\rangle$ of the Hilbert space $\mathcal{H}$ of the spectral triple on $SU_q(2)$ as defined in Proposition 1.1 as a lattice, with $\mu$ on the horizontal axis and $n$ on the vertical axis. We project the $j$ coordinate away, since it does not play a role for determining whether a vector lies in $\mathcal{H}_K$, only in confining the possible $\mu$ and $n$. We illustrate this by drawing lines through the allowed values (in the form of rectangles around the origin) for $j = 3$ and in the $\uparrow$ part of $\mathcal{H}$. We draw a circle $\bigcirc$ through the origin $\mu = 0, n = 0$.

The stars ($\bullet$) and diamonds ($\blacklozenge$) represent basis vectors of the $\mathcal{H}_K$ Hilbert space of the lens space $(L_q(p,r), \mathcal{H}_K, D, J)$. The circles ($\circ$) represent basis vectors of the Hilbert space of $SU_q(2)$ which are not part of the lens space. The stars are the allowed values for integer valued $j$, the diamonds are the allowed values for half-integer valued $j$. 
Example: the $L(p, p-1)$ lens space. This corresponds to the choice $r = p - 1$ or equivalently $r = -1$ (then $\epsilon(r) = 0$).

**Proposition 3.5.** The spectrum of the Dirac operator on $L(p, p - 1)$ is:
we consider the standard spin structure below. The multiplicity of the eigenvalue of

\[ \lambda_D = \begin{cases} 
2kp + 2l + \frac{1}{2}, & \text{with multiplicity: } 2(2k + 1)(kp + l) \\
(2k + 1)p + 2l + \frac{1}{2}, & \text{with multiplicity: } (2k + 2)((2k + 1)p + 2l) \\
-2kp - 2l - \frac{3}{2}, & \text{with multiplicity: } 2(2k + 1)(kp + l + 1) \\
-(2k + 1)p - 2l + \frac{1}{2}, & \text{with multiplicity: } (2k + 2)((2k + 1)p + 2l + 2)
\end{cases} 
\]

Note that the \( \lambda_D = \frac{1}{2} \) eigenvalue \((k = l = 0)\) has multiplicity 0.

\[ \lambda_D = \begin{cases} 
2kp + 2l + \frac{1}{2}, & \text{with multiplicity: } 2(2k + 1)(kp + l) \\
-2kp - 2l - \frac{3}{2}, & \text{with multiplicity: } 2(2k + 1)(kp + l + 1)
\end{cases} 
\]

Again, the \( \lambda_D = \frac{1}{2} \) eigenvalue has multiplicity 0 \((k = l = 0)\).

\[ \lambda_D = \begin{cases} 
(2k + 1)p + 2l + \frac{1}{2}, & \text{with multiplicity: } (2k + 2)((2k + 1)p + 2l) \\
-(2k + 1)p - 2l + \frac{1}{2}, & \text{with multiplicity: } (2k + 2)((2k + 1)p + 2l + 2)
\end{cases} 
\]

Remark 3.6. The result is the same as the result for the commutative case as described in [1, Theorem 5], when we set \( T = 1 \).

Proof. Since \( r = -1 \), \((1 + r)\mu = 0\) and so we only need to consider \( n \) in the analysis below. The multiplicity of the \( \mu \) factor for a given \( j \) is \( 2j + 1 \). When we consider the standard spin structure \( K = 0 \), we see that \( 2n \equiv 0 \mod p \).

Consider first the case \( p \) odd. \( 2n \equiv 0 \mod p \) means that either \( n \) is integer valued, and \( n = kp \) for some \( k \in \mathbb{Z} \), or \( n \) is half integer valued and \( n = \frac{1}{2}p + kp \) for some \( k \in \mathbb{Z} \). Since the spectrum is symmetric, we only need to consider the \( \uparrow \) part.

In case \( n \) is integer valued, \( j \) must be half-integer valued. When \( j < p - 1 \), there is only one possibility for \( n \), namely 0. When \( p - 1 < j < 2p - 1 \), we have 3 possibilities for \( n \): \(-p, 0, p\). In general, when \( kp - 1 < j < (k + 1)p - 1 \), \( k \) an integer > 0, \( j \) half-integer valued, we have \( 2k + 1 \) possibilities for \( n \). If \( n \) is half-integer valued, \( j \) must be integer-valued. The first value for \( j \) such that \( n = \frac{1}{2}p + kp \), \( k \in \mathbb{Z} \), is possible, is \( j = \frac{1}{2}p - \frac{1}{2} \). There are two possibilities for \( n \): \( \frac{1}{2}p \) and \(-\frac{1}{2}p \). We get 2 new possibilities of \( n \) if we increase \( j \) by \( p \).

Since the eigenvalue of \( D \) on a vector \( |j\mu \uparrow \rangle \) is \( 2j + \frac{3}{2} \), and \( j = \frac{1}{2}, \frac{3}{2}, \ldots \), or \( j = \frac{1}{2}p - \frac{1}{2}, \frac{1}{2}p + \frac{1}{2}, \ldots \), we get the asked spectrum for \( p \equiv 1 \mod 2 \), where
the first formula corresponds to \( j \) half-integer valued, and the second to \( j \) integer valued.

For \( p \) even and \( K = 0 \), we see that \( n \) half-integer valued is not a possibility, and the analysis for the \( n \) integer-valued case is the same as for \( p \) odd.

When \( p = 2P \) is even and \( K = P \), we see that if \( P \) is even, then \( n \) is only integer-valued, and if \( P \) is odd, \( n \) is only half-integer valued. If \( P \) is even, we then have \( j \) is half-integer valued, and the first solution for \( n \) arises when \( j = P/2 - \frac{1}{2} \). We get two solutions for \( n \), \( P/2 \) and \( -P/2 \), in this case, and two new solutions for \( n \) when we increase \( j \) by \( P \). When \( P \) is odd, we need \( n \) to be half-integer valued, with the first two possibilities arising when \( j = P/2 - \frac{1}{2} \), and again, two new possibilities for \( n \) when we increase \( j \) by \( P \), hence the spectrum given above.

\[ \square \]

There is a second case which we can easily derive from the calculations above. If we look at equation (28), we see that in addition to the \( r = -1 \) case, the \( r = +1 \) case should easily be calculable. Classically, the \( L(p, p-1) \) and \( L(p, 1) \) lens spaces are isometric, the orientation is just reversed, so one expects the spin structures to be the same. We will show that this is the case for the quantum lens spaces in the next section.

4. Unitary equivalences

It is known [29] that two commutative lens spaces \( L(p, r) \) and \( L(p', r') \) are homeomorphic if and only if \( p = p' \) and \( r' r \equiv \pm 1 \mod p \), or \( r' \pm r \equiv 0 \mod p \).

It is also known that two 3-dimensional lens spaces are homeomorphic if and only if they are (Laplace) isospectral, see [24].

That these concepts are related for lens spaces can be intuitively understood by looking at diagrams as in Section 3.0.1. There we see that the diagram of \( L(p, r) \) is the same as the diagram of \( L(p, r') \), \( r' = \pm r \pm 1 \), up to rotation, mirroring and interchanging the stars and diamonds. For example, \( 3 \equiv -(5)^{-1} \mod 7 \), and we see in Figures 5 and 6 that they are the same if we rotate Figure 5 one quarter clockwise and flip the stars and diamonds. The \( L(5, 1) \) and \( L(5, -3) \) lens space diagrams of Figures 3 and 4 are very different however, and not related by mirroring and rotations by quarter-turns.

In [24], using results on these type of lattices from [37], it is then shown that these type of lattice isomorphism induce an isomorphism of the algebra of smooth functions.

In the noncommutative case when \( q \neq 1 \), the lens spaces \( L_q(p, r) \) and \( L_q(p, r') \) are shown to be unitary equivalent when \( r = \pm r' \), if we take care of the subtleties of equivariant representations.

**Theorem 4.1.** When \( q \in (0, 1) \), the quantum lens spaces given by \( (L_q(p, r), \mathcal{H}_K, D, J) \) and \( (L_q(p, r'), \mathcal{H}_K, D, J) \) are unitary equivalent if \( r' \equiv -r \mod p \). The unitary equivalence is implemented by the order order-two automorphism \( \sigma(a) = a, \sigma(b) = -b^* \) of \( \mathcal{A}(SU_q(2)) \) and the action \( U \) on the Hilbert
space given by $U|j\mu n\uparrow\rangle = -c_{j\mu n}|j^+n\mu \downarrow\rangle$ and $U|j\mu n\downarrow\rangle = c_{j\mu n}|j^-n\mu \uparrow\rangle$, with again $c_{j\mu n}$ a complex number of norm 1.

Proof. To show that this map is a unitary equivalence, we first study the equivalence of Hilbert spaces. If $v \in \mathcal{H}_K$ with $K = 0$ or $K = p/2$, we have $(1 + r)\mu + (1 - r)n \equiv 0 \mod p$ or $p/2$ respectively. If we take $r' = -r$, we see that $(1 + r')\mu + (1 - r')n = (1 - r)\mu + (1 + r)n$ and we see that if we interchange $\mu$ and $n$, $v \in \mathcal{H}_K$ is mapped to a vector $v \in \mathcal{H}'_K$ in the $K = 0$ or $K = p/2$ subspace of the $v$ action for $r' = -r$.

To define a compatible action of the algebra on $\mathcal{H}'_K$, we see from Proposition 1.1 that we need to interchange $b$ and $-b^*$. Now to show that this indeed gives a unitary equivalence on the algebra, we need to show that the $U^{-1}(\Pi'(-b))U = \Pi(b^*)$, and $U^{-1}(\Pi'(a))U = \Pi(a)$. However, since $U$ interchanges the $\uparrow$ and $\downarrow$ part, $\Pi'$ is not the representation as given in Proposition 1.1 because then $U^{-1}\Pi'(b)U$ is not $(\rho', \lambda')$-equivariant as described in Section 1. We need to demand that $\Pi'$ is equivariant with respect to a different $\mathfrak{su}(2)$-action $(\rho'', \lambda'')$, such that $U^{-1}\rho''(g)U = \rho'(g)$ and $U^{-1}\lambda''(g)U = \lambda'(g)$. \qed

Of course, from [23] it is implicit that as graph $C^*$-algebras $L_q(p, r)$ and $L_q(p, r')$ for $r$ and $r'$ coprime to $p$ are isomorphic. However, it is unclear whether this descends to an isomorphism on the level of smooth algebras, or whether this leads to a unitary equivalence.

Remark 4.2. Observe that even though the algebras are isomorphic to each other it is not obvious that the spectral triples are unitarily equivalent. This is because the construction of spectral triples over lens spaces is based on the restriction of an equivariant spectral triple over the full $\mathcal{A}(SU_q(2))$ algebra. As it is generally not true that a restriction of given spectral triple to two subalgebras results in unitary equivalent spectral triples\footnote{A trivial example is that of a torus with a nontrivial spin structure - its restriction to two different subalgebras of functions over a circle gives two spectral triples which correspond to two distinct spin structures over the circle.}. Only the above construction of unitary equivalence demonstrates that we indeed have at most 2 (and not more) real spectral triples over the quantum lens spaces.

It should be noted that using similar methods as in [23] one can show that any spectral triple over quantum lens spaces is a restriction of a spectral triple over $\mathcal{A}(SU_q(2))$ algebra. This does not guarantee, however, that the resulting spectral triple lifted to $\mathcal{A}(SU_q(2))$ is equivariant.

Remark 4.3. For the other type of equivalences in the commutative case, i.e. the $r \to r^{-1}$ case, we do not know if they give rise to unitary equivalences in the $q \neq 1$ case. For isomorphisms of $L_q(p, r)$ coming from the automorphisms of $\mathcal{A}(SU_q(2))$, as classified by [23], we can show that they do not give rise to isomorphisms between $L_q(p, r)$ and $L_q(p, r^{-1})$. Take for example an element of the form $a^*b^j b^m \in L_q(p, r)$, with $r(l - m) - 1 \equiv 0 \mod p$, i.e. $l - m \equiv r^{-1} \mod p$. We have $r^{-1}(l - m) \equiv (r^{-1})^2 \neq 1 \mod p$ if $r^{-1} \neq r$.\footnote{A trivial example is that of a torus with a nontrivial spin structure - its restriction to two different subalgebras of functions over a circle gives two spectral triples which correspond to two distinct spin structures over the circle.}
This means that if \( r^{-1} \neq r \), the identity automorphism is not a map from \( L_q(p, r) \) to \( L_q(p, r^{-1}) \). Also, we have \( r^{-1}(m - l) \equiv r^{-1} \cdot (-r^{-1}) \not\equiv 1 \mod p \) if \( r^{-1} \not\equiv -r \), hence the automorphism \( a \rightarrow a, b \rightarrow b^* \) is not a map from \( L_q(p, r) \) to \( L_q(p, r^{-1}) \). Hence the automorphisms of \( \mathcal{A}(SU_q(2)) \) do not give homomorphisms from \( L_q(p, r) \) to \( L_q(p, r^{-1}) \) if \( r^{-1} \not\equiv \pm r \). The same argument holds for \( L_q(p, r) \) and \( L_q(p, -r^{-1}) \).

5. Quantum Teardrops and Principal Fiber Bundles

In [4], it was shown that certain quantum lens spaces, namely the lens spaces \( L_q(p; 1, p) \) could be viewed as being principal \( U(1) \)-comodule algebras over the quantum teardrops \( WP_q(1, p) \). This was actually done on the level of the coordinate algebras. The \( L_q(p; 1, p) \) lens spaces do not fit into the framework described above, as we only study lens spaces were the coaction of the finite group action is free, which is not the case for lens spaces of the form \( L_q(p, r) \) with \( p, r \) not coprime, as can be deduced from the proof of Proposition 2.2.

In this section, we show that the situation for the coordinate algebras \( \mathcal{O}(L_q(p, r)) \), with \( p, r \) coprime, resembles that of the commutative \( q = 1 \) case: the algebras \( \mathcal{O}(L_q(1, p)) \) are principal \( U(1) \)-comodule algebras over the quantum 2-sphere \( S_q^2 \simeq WP_q(1, 1) \), while for \( r \neq 1, \mathcal{O}(L_q(p, r)) \) does not admit such a fibration over the natural candidate base space, a quantum teardrop as defined in [4].

On the other hand, the situation on the \( C^* \)-algebra level is quite different. Due to the isomorphism \( C(SU_q(2)) \simeq C(SU_q(2)) \) for \( 0 \leq q < 1 \), at the level of \( C^* \)-algebras, as found by Woronowicz [35, Theorem A2.2], we can show that the algebra \( C(L_q(p, r)) \) is a \( U(1) \)-principal comodule algebra over the quantum teardrop \( WP_q(1, r) \).

The teardrop orbifold of Thurston which we denote by \( WP(r_1, r_2) \), can be defined as the quotient of \( S^3 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \) by the following (twisted) action of \( S^1 := \{t \in \mathbb{C} : |t|^2 = 1\} \):

\[
t \cdot (z_1, z_2) = (t^{r_1} z_1, t^{r_2} z_2).
\]

Of course, the teardrop \( WP(n, n) \) for any \( n > 0 \) is just the sphere \( S^2 \), but the the quotients where \( r_1 \neq r_2 \) are not manifolds anymore, but orbifolds (in fact, what is usually called a “bad” orbifold, meaning that it there doesn’t exist a finite covering by a simply connected manifold).

The quantum teardrop can be defined similarly, as the subalgebra of \( SU_q(2) \) invariant under a suitable action of \( U(1) \):

\[
t \cdot (\alpha, \beta) = (t^{r_1} \alpha, t^{r_2} \beta),
\]

with \( \alpha \) and \( \beta \) the generators of the \( SU_q(2) \) \( C^* \)-algebra. The invariant subalgebra is of course the algebra generated as a vector space by basis vectors of the form \( \alpha^k \beta^l \beta^m \), such that \( r_1 k + r_2 (l - m) = 0 \). In [4, Theorem 2.1], it was shown that this algebra is generated by elements \( a \) and \( b \), which satisfy
the relations:
\[ a = a^* \]
\[ bb^* = q^{2r_1 r_2} a^{r_1} \prod_{m=0}^{r_2-1} (1 - q^{2m} a) \]
\[ ab = q^{-2r_2} ba \]
\[ b^* b = a^{r_1} \prod_{m=1}^{r_2} (1 - q^{-2m} a). \]

It is not hard to see that \( a = \beta \beta^* \) and \( b = \alpha^{r_2} (\beta^*)^{r_1} \) satisfy these relations, and that these elements generate the invariant subalgebra. In [4] it was also shown that on the \( C^* \)-algebra level, the quantum teardrops \( \mathbb{WP}_q(1, r) \) and \( \mathbb{WP}_q(r_1, r) \) are isomorphic.

In order to describe our results on fiber bundles, we will switch to coactions for this section. A continuous coaction \( \rho \) for a Hopf algebra \( H \) acting on a \( C^* \)-algebra \( A \) is a map \( \rho : A \to A \otimes H \) that has the following properties:

- \( \rho \) is injective
- \( \rho \) is a comodule structure: \((1 \otimes \Delta) \circ \rho = (\rho \otimes 1) \circ \rho\), where \( \Delta \) is the coproduct of \( H \).
- Podleś condition: \( \rho(A)(1 \otimes H) = A \otimes H \).

This coaction can be used to define principal \( H \)-comodule algebras, which can be seen as a generalization of the concept of a principal fiber bundle to noncommutative geometry [20],[5],[32].

Let \( A \) be a \( C^* \)-algebra, with a coaction \( \rho : A \to A \otimes H \) by a Hopf algebra \( H \). Denote by \( B \) the coinvariant part of \( A \), i.e. the part where \( \rho(h)a = a \otimes 1 \). This is a quantum principal fibration, or principal \( H \)-comodule algebra if:

- The canonical map \( \text{can} : A \otimes_B A \to A \otimes H : a \otimes a' \mapsto a\rho(a') \) is a bijection.
- The map \( B \otimes A \to A : b \otimes a \mapsto ba \) splits as a left \( B \)-module and and a right \( H \)-comodule map. This is also called equivariant projectivity.

Because of the results of [9] and [4], a right \( H \)-comodule algebra \( A \) is principal if and only if there exists a strong connection, i.e. there exists a map \( \omega : H \to A \otimes H \) such that:

\[(29a) \] \( \omega(1) = 1 \otimes 1 \)
\[(29b) \] \( \mu \circ \omega = \eta \circ \epsilon \)
\[(29c) \] \( (\omega \otimes \text{id}) \circ \Delta = (\text{id} \otimes \rho) \circ \omega \)
\[(29d) \] \( (S \otimes \omega) \circ \Delta = (\sigma \otimes \text{id}) \circ (\rho \otimes \text{id}) \circ \omega, \)

where \( S, \eta, \epsilon, \Delta \) are the antipode, unit, counit and comultiplication of the Hopf algebra \( H \), and \( \sigma : A \otimes H \to H \otimes A \) is the flip.

A strong connection in the case where \( H = U(1) \) is particularly nice to work with, because of the following lemma:

**Lemma 5.1.** An algebra \( A \), with continuous coaction \( \rho : A \to A \otimes U(1) \), has a strong connection if and only if there exists elements \( \sum_i a_i \otimes b_i \), and
∑_i b'_i ⊗ a'_i such that ∑ a_i b_i = ∑ b'_i a'_i = 1, for a_i and a'_i of degree −1 and b_i and b'_i of degree +1 for the coaction.

The proof of this lemma is a slight generalization of a construction done in the proof of [4, Theorem 3.3]. One could also say that and algebra \( A \) is a principal \( U(1) \)-fiber bundle over \( A_0 \) if and only if it is strongly \( \mathbb{Z} \)-graded, i.e. there is a \( \mathbb{Z} \)-grading \( A = \oplus_{k \in \mathbb{Z}} A_k \) such that \( A_k A_l = A_{k+l} \). This immediately follows from the lemma above.

**Proof.** We construct a strong connection by induction. Define \( \omega \) by:

\[
\begin{align*}
\omega(1) &= 1 \otimes 1 \\
\omega(u^n) &= \sum_i a_i \omega(u^{n-1}) b_i \\
\omega(u^{-n}) &= \sum_i b'_i \omega(u^{-n+1}) a'_i,
\end{align*}
\]

for each \( n \geq 1 \). Condition (29a) is immediate. For \( n = 1 \), we see that because \( \sum a_i b_i = \sum b'_i a'_i = 1 = \eta(\epsilon(u)) \), condition (29b) is satisfied. Conditions (29c) and (29d) are also obvious by the definition.

Now suppose \( \omega(u^{n-1}) \) satisfies conditions (29b)–(29d). Then

\[
\mu(\omega(u^n)) = \mu(\sum_i a_i \omega(u^{n-1}) b_i) = 1,
\]

because \( \mu(\omega(u^{n-1})) = 1 \). We also see that

\[
(id \otimes \rho)(\omega(u^n)) = (id \otimes \rho)\left(\sum_i a_i \omega(u^{n-1}) b_i\right) = \sum_i a_i \omega(u^{n-1}) b_i \otimes u^n = \omega(u^n) \otimes u^n.
\]

The same argument also works for condition (29c), and the \( u^{-n} \) case.

If the conditions of the lemma are not satisfied, it cannot be a fibration, because the map can be either missing \( 1 \otimes u \) or \( 1 \otimes u^{-1} \) from its image. □

**Theorem 5.2.** The coordinate algebra \( \mathcal{O}(L_q(p,r)) \), with \( p,r \) coprime, is a quantum principal \( U(1) \)-fibration over the quantum teardrop \( \mathcal{O}(\mathbb{W}P_q(1,r)) \) when \( 0 < q \leq 1, \) if and only if \( r = 1 \).

**Proof.** For basis vector \( \alpha^k \beta^l (\beta^*)^m \in L_q(p,r) \), we have \( k + r(l - m) = np \) for some \( n \in \mathbb{Z} \). Define the coaction \( \rho \) of \( U(1) \) to be \( \rho(\alpha^k \beta^l (\beta^*)^m) = \alpha^k \beta^l (\beta^*)^m \otimes u^n \), with \( n \) the number above. Clearly \( \mathbb{W}P_q(1,r) \) is the coinvariant subalgebra under this coaction.
Starting with the identities $(\alpha^*\alpha + q^2\beta^*\beta)^p = 1$, and $(\alpha\alpha^* + \beta^*\beta)^p = 1$, we see that by working out the parenthesis, we get the identities

\[(30a)\quad \alpha^p\alpha^p + \sum_{p_1=1}^p c_{p_1}(\alpha^*)^{p-p_1}\alpha^{p-p_1}(\beta^*)^{p_1}\beta^{p_1} = 1.\]

\[(30b)\quad \alpha^p\alpha^p + \sum_{p_1=1}^p c'_{p_1}\alpha^{p-p_1}(\alpha^*)^{p-p_1}(\beta^*)^{p_1}\beta^{p_1} = 1.\]

where the $c_{p_1}, c'_{p_1} \in \mathbb{R}$ are calculated by working out the identity, and depend on $q$. We can then define a strong connection by setting $\omega(1) = 1 \otimes 1$ and defining it inductively as:

\[\omega(u^n) = (\alpha^*)^p\omega(u^{n-1})\alpha^p + \sum_{p_1=1}^p c_{p_1}q^{p_1(p-p_1)}(\alpha^*)^{p-p_1}(\beta^*)^{p_1}\omega(u^{n-1})\alpha^{p-p_1}\beta^{p_1}.\]

If $r = 1$, we have $\rho(\alpha^{p-p_1}\beta^{p_1}) = \alpha^{p-p_1}\beta^{p_1} \otimes u$, so this clearly satisfies the conditions for $\sum_i a_i \otimes b_i$ stated above, and so defines a strong connection by Lemma 5.1.

For $r \neq 1$, but coprime to $p$, and $q \neq 0$, it is impossible to define a strong connection this way, since the $(\beta^*)^p$ part of the \[\text{(30a)}\] can never be split into two parts $a_1a_2$ such that $\rho(a_2) = a_2 \otimes u$. In fact, we can show that the canonical map \textbf{can} : $A \otimes_B A \to A \otimes H$, used in the definition of a quantum principal fibration, cannot be surjective if we restrict ourselves to coordinate algebra $\mathcal{O}(L_q(p, r))$, with $r \neq 1$ and $p, r$ coprime. To show this, we adapt the proof of [1 Theorem 3.2] to the case of lens spaces.

A basis for $\mathcal{O}(L_q(p, r)) \otimes \mathcal{O}(L_q(p, r))$ is given by

\[\alpha^k\beta^l\beta^m \otimes \alpha^{k'}\beta^{l'}\beta^{m'},\]

where $k, k' \in \mathbb{Z}$ and $l, l', m, m' \in \mathbb{N}$, with the usual convention that for $k < 0$: $\alpha^k := \alpha^{-k}$, and $k + r(l - m) \equiv 0 \mod p$, and $k' + r(l' - m') \equiv 0 \mod p$, or equivalently, $k' + r(l' - m') = np$, for some $n \in \mathbb{Z}$ The image of \textbf{can} of these basis element is then of the form

\[\alpha^k\alpha^{k'}\beta^{l+l'}\beta^{m+m'} \otimes u^n,\]

hence every element in the image of \textbf{can} must be a linear combination of these. We proceed to show that $1 \otimes u$ cannot lie in this image. An easy calculation shows that in order to have 1 in the image, we must have $k = -k'$, otherwise we always end up with some factor of $\alpha$ or $\alpha^*$ in the image. Similarly, we can assume that $l+l' = m+m'$. We also need $k' + r(l' - m') = p$. Because $r$ is coprime to $p$, we see that $k' \neq 0$. However, if $q \neq 0$, every finite sum adding up to 1 is always going to contain a term $k = k' = 0$, because of the decomposition $\alpha\alpha^* + \beta\beta^* = 1$ (or alternatively $\alpha^*\alpha + q^2\beta^*\beta = 1$). Hence $1 \otimes u$ does not lie in the image of \textbf{can}, and so \textbf{can} is not bijective, and $\mathcal{O}(L_q(p, r))$ is not a quantum principal fibration over $\mathcal{O}(\mathbb{W}_p^q(1, r))$. \hfill \square

If we look closely at the $q = 0$ case, we can see however:
Theorem 5.3. If \( q = 0 \), the algebra \( O(L_q(p,r)) \) is a quantum principal fibration over \( O(\mathbb{P}_q(1,r)) \).

Proof. We check the conditions of Lemma 5.1. Recall that the of \( L_q(p,r) \) over \( \mathbb{P}_q(1,r) \) for a basis element of the form \( \alpha^k \beta^l(\beta^*)^m \) is given by \( k + r(l - m) = np \), where \( n \in \mathbb{Z} \) is the grading. For the positively graded part, we have the obvious \( (\alpha^*)^p \alpha^p = 1 \).

The negatively graded part is more involved, but it can be shown that

\[
\sum_{p_1=0}^{p} a\alpha^{p-p_1} \beta^{p_1}(\alpha^*)(r-1)p_1 \alpha^{p_1}(\beta^*)^{p_1}(\alpha^*)^{p_{1}} = 1,
\]

which gives a correctly graded decomposition, with \( a_i = \alpha^{p-p_1} \beta^{p_1}(\alpha^*)(r-1)p_1 \), since \( (p - p_1) + rp_1 - (r - 1)p_1 = p \).

To show that the sum equals 1, first observe that \( \alpha^* \alpha = 1 \), hence we can reduce the sum to

\[
\sum_{p_1=0}^{p} \alpha^{p-p_1} \beta^{p_1}(\beta^*)^{p_1}(\alpha^*)^{p_{1}}.
\]

Then, observing that \( (\beta^*)^2 = \beta \), this can be further reduced to

\[
\alpha^p(\alpha^*)^p + \sum_{p_1=1}^{p} \alpha^{p-p_1} \beta^p(\alpha^*)^{p_1} + \beta^*.
\]

We can collapse it so something even simpler still, by observing that for all \( k > 0 \), we have \( \alpha^k(\alpha^*)^k = \alpha^{k-1}(1 - \beta^*)(\alpha^*)^{k-1} \). Hence

\[
\alpha^p(\alpha^*)^p + \sum_{p_1=1}^{p} \alpha^{p-p_1} \beta^p(\alpha^*)^{p_1} = \alpha \alpha^* = 1,
\]

and the sum reduces to \( \alpha \alpha^* + \beta \alpha = 1 \).

Thus we have proven that the conditions of Lemma 5.1 are satisfied, hence \( O(L_q(p,r)) \) is a quantum principal fibration over \( O(\mathbb{P}_q(1,r)) \).

Of course, because in the continuous case \( C(SU_q(2)) \simeq C(SU_0(2)) \) for \( 0 \leq q < 1 \), by [35] Theorem A2.2, this leads to the corollary:

Corollary 5.4. The quantum lens space \( C(L_q(p,r)) \) is quantum principal fiber bundle over quantum teardrop \( C(\mathbb{P}_q(1,l)) \).

We do not know what happens in the smooth \( q \neq 0 \) case.

Remark 5.5. The coaction for which the quantum teardrop is the coinvariant part can easily be translated into an action of a Hopf algebra, with which we can construct an equivariant real spectral triple on the quantum teardrop \( \mathbb{P}_q(1,r) \), coming from the real spectral triple on the quantum lens space \( L_q(p,r) \). This allows us to calculate for example the Dirac spectrum of the quantum teardrops, and compare them to the spectrum of the quantum sphere \( S^2_q \). Indeed, a similar approach has been implemented in [21], where an example of an odd spectral triple over quantum weighted projective...
spaces was constructed. However, when \( r \) is even, such a construction does not lead to an irreducible real spectral triple as discussed in Section 3.0.1.

Intuitively, the quantum teardrop \( WP_q(1,r) \) can be understood to be the limit \( p \to \infty \) of the quantum lens space \( L_q(p,r) \). Of course, creating a restriction of the spectral triple to the invariant subalgebra does not demonstrate that it is a noncommutative manifold. The classical teardrops are not manifolds and therefore the Dirac operator and the spectral triple for them make sense only when considered over the covering space. It remains open whether the orbifold-type singularities disappear when one considers \( q \)-deformed objects as some studies suggest (see [3, 4]).

However, when it comes to spectral triples we cannot, in contrast to [12, 13], claim that the spectral triple over the quantum lens space is projectable. This is not surprising, as no such projectability exists even for the \( SU_q(2) \) spectral triple considered as a noncommutative \( U(1) \) principal bundle over the standard Podleś sphere. The quantum teardrop can be obtained in a similar way as a subalgebra of the \( SU_q(2) \) algebra and moreover, we know that \( L_q(1,r) \) is a quantum principal fiber over this teardrop, which also means that the coaction is free. For this reason we should expect many similarities between these cases and therefore we believe that restriction of the Dirac operator to some invariant subspace of the Hilbert space, while being at the same time a valid proposal for a spectral triple, does not demonstrate that the noncommutative case is different from the classical one. For this reason we postpone the analysis of the case to future work.

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