Comparing two approaches to Hawking radiation of Schwarzschild–de Sitter black holes

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Abstract
We study two different ways to analyze the Hawking evaporation of a Schwarzschild–de Sitter black hole. The first one uses the standard approach of surface gravity evaluated at the possible horizons. The second method derives its results via the generalized uncertainty principle (GUP) which offers yet a different method to look at the problem. In the case of a Schwarzschild black hole it is known that this method affirms the existence of a black hole remnant (minimal mass $M_{\text{min}}$) of the order of Planck mass $m_{\text{pl}}$ and a corresponding maximal temperature $T_{\text{max}}$ also of the order of $m_{\text{pl}}$. The standard $T(M)$ dispersion relation is, in the GUP formulation, deformed in the vicinity of Planck length $l_{\text{pl}}$ which is the smallest value the horizon can take. We generalize the uncertainty principle to Schwarzschild–de Sitter spacetime with the cosmological constant $\Lambda = \frac{1}{m_{\text{pl}}^2}$ and find a dual relation which, compared to $M_{\text{min}}$ and $T_{\text{max}}$, affirms the existence of a maximal mass $M_{\text{max}}$ of the order $(m_{\text{pl}}/m_{\Lambda}) m_{\text{pl}}$, minimum temperature $T_{\text{min}} \sim m_{\Lambda}$. As compared to the standard approach we find a deformed dispersion relation $T(M)$ close to $l_{\text{pl}}$ and in addition at the maximally possible horizon approximately at $r_{\Lambda} = 1/m_{\Lambda}$. $T(M)$ agrees with the standard results at $l_{\text{pl}} \ll r \ll r_{\Lambda}$ (or equivalently at $M_{\text{min}} \ll M \ll M_{\text{max}}$).

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1. Introduction
In recent years, the generalized uncertainty principle (GUP), the uncertainty relation which includes also gravity effects, has gained popularity [1–3]. Especially, in the context of black holes [4] and their evaporation [5–7] GUP has proved to be the harbinger of new, maybe partly also expected, effects in the context of quantum gravity. Compared to the standard Hawking radiation GUP deforms the standard $T(M)$ relation near the Planck length to the extent that
the Planck length becomes the smallest possible length scale in this context. One can interpret this result also in a different way: there exists a minimum mass (which is the black hole remnant) of the order of Planck mass which corresponds to a maximum temperature, also of the order of Planck mass [6, 8]. This fits neatly into the picture of dimensional analysis based on the Newtonian constant $G$. It is expected that all scales given by $G$, i.e. $l_{pl} = 1.61 \times 10^{-33}$ cm, $t_{pl} = 5.39 \times 10^{-44}$ s, $m_{pl} = 1.22 \times 10^{19}$ GeV and $\rho_{pl} = 5.16 \times 10^{93}$ g cm$^{-3}$ (density) are the extreme or limiting values which can be attained in a physical situation. It is also expected that at these scales, special effects of quantum gravity will show up.

Seen from a certain perspective, the early stage of quantum theory resembles the current state of art of what we call quantum gravity. With respect to the former, the important harbingers of the (those days, new) quantum theory were Planck’s black body radiation formula and the uncertainty relation $\Delta x \Delta p \geq \frac{1}{2}$ derived in the early days without the help of Schwarz inequality. Today quantum gravity seems to offer a very similar state of affairs which, of course, does not imply that there do not already exist aspirants for a complete quantum gravity theory. Hawking’s theory of black hole evaporation [5] is not only a quantum mechanical effect, but the radiation of black holes is also a perfect black body radiation. Second, the above-mentioned generalized uncertainty principle, which includes gravity effects, has been derived in different contexts: string theory and non-commutative quantum theory. Recently, a simpler derivation of this uncertainty relation has been found which agrees fully with the previous findings [1, 6].

Since GUP offers a robust tool to probe into quantum mechanics of black holes, it is interesting to raise the question, what will actually happen if another constant enters the Einstein’s equations. This is not a remote possibility as the recently discovered acceleration of the universe [9] cannot be explained without altering either the Einstein tensor $G_{\mu\nu}$ or the energy–momentum tensor for cosmology (including the equation of state). Opting for the first possibility, any new constant in $G_{\mu\nu}$ is independent of the situation to which we apply the Einstein’s equations and therefore a new constant of gravity. It is worth pointing out that the evidence for the need to change the standard gravity is growing. Observation of standard candles like type Ia supernova [10] and other key observations in relation with baryon acoustic oscillations [11], cosmic microwave background radiation [12], large scale structure [13] and weak lensing [14] led us to the conclusion that the expansion of the universe as compared to the standard Friedmann model is accelerated. All evidence is in agreement with a positive cosmological constant. Notably, the observations seem to favor the equation of state $p = -\rho$ which comes along with the gravity theory including the cosmological constant $\Lambda$.

If the positive cosmological constant $\Lambda$ explains the recently discovered accelerated stage of the universe, this constant is, beside the Newtonian constant $G = m_{pl}^{-2}$, the second constant of gravity. It is legitimate to put forth the question: how does the mass scale $m_\Lambda = \sqrt{\Lambda} \ll m_{pl}$ and length scale $r_\Lambda = 1/m_\Lambda \gg l_{pl}$ alter our expectations for quantum gravity. In the present paper we first find the Hawking approach to Schwarzschild–de Sitter black hole radiation. In the second step we elaborate on the evaporation of this black hole utilizing the generalized uncertainty principle (GUP) with $\Lambda$ and confirm the results by black body radiation. We compare the $T(M)$ dispersion relation which we derive from GUP including $\Lambda$ with the standard expression obtained from surface gravity calculated at an event horizon. They both agree for the intermediate mass range, i.e., masses much bigger than Planck mass, but much smaller than $M_{max} \sim (m_{pl}/m_\Lambda)m_{pl}$. This is what one would expect from GUP which now deforms the standard $T(M)$ relations at $l_{pl}$ (corresponding to the mass of black hole remnant $M_{min} \sim m_{pl}$) and at $r_\Lambda$ (corresponding to $M_{max}$). A careful analysis performed in this paper reveals the following picture: at masses close to Planck mass $T(M)$ follows the behavior found in [6] (here $\Lambda$ does not play any significant role), this is taken over by the standard
Hawking, i.e., $T(M) = m^2_{\text{pl}}/(8\pi M)$. As $M$ becomes bigger, the effects of $\Lambda$ become more important. They can still be described by the standard approach, i.e., calculating the surface gravity at a horizon where $\Lambda$ enters now explicitly the expression for $T(M)$. For even higher masses GUP modifies this standard picture to the extent that there exists a maximum mass of the order $M_{\text{max}}$ beyond which no positive definite solutions exist for $T$. This means that we have a minimum temperature $T_{\text{min}} = T(M_{\text{max}}) \sim m_{\Lambda}$. In short, GUP results into the existence of a maximum temperature corresponding to a minimum mass and a minimum temperature corresponding to a maximum mass. The latter results is due to $\Lambda$. We can replace the mass by length in which case we have a minimum (mentioned already above) and maximum length. The latter is $r_{\Lambda}$.

The outline of the paper is as follows. In section 2 we will determine the full and approximated expressions for the two horizons in the Schwarzschild–de Sitter case. Section 3 is devoted to the standard treatment of Hawking radiation of Schwarzschild–de Sitter black holes via the surface gravity calculated at the horizons. We will show that only the first horizon gives physically viable results as $T(M)$ calculated at the second horizon violates the condition $\partial T/\partial M < 0$. Section 4 contains the discussion of GUP applied to the black hole evaporation. We briefly touch the case $\Lambda = 0$ to show explicitly the major steps involved in the derivation. Then the generalization to $\Lambda$ will be transparent. We apply the uncertainty relation with $\Lambda$ to the black hole evaporation and find that, as far as the order of magnitude is concerned, for intermediate masses it agrees with results derived in section 2. We show the existence of $T_{\text{min}}$ and $M_{\text{max}}$. In section 5 we confirm the results obtained in section 4 by yet different methods. Section 6 discusses a different effect of $\Lambda$ in a temperature perceived at a distance. In section 7 we summarize our findings. The two appendices are included for the reader’s convenience and to facilitate the reading of the text.

2. Horizons of a de Sitter black hole

In the subsequent section we will derive the Hawking radiation Schwarzschild–de Sitter black hole via the surface gravity $\kappa$ taken at the horizon $r_c$, i.e., $\kappa(r_c)$. Therefore it makes sense to dwell a little bit on the two horizons existing in the Schwarzschild–de Sitter case (the elements of the Schwarzschild–de Sitter metric are given in appendix B). The starting point here is the horizon condition given by [15]

$$g^{rr}(r_c) = 0.$$ (1)

This condition in the Schwarzschild–de Sitter metric (see appendix B) is

$$1 - \frac{2r_s}{r_c} - \frac{1}{3} \frac{r_c^2}{r_A^2} = 0.$$ (2)

where

$$r_A = \frac{1}{m_A} \equiv \frac{1}{\sqrt{\Lambda}}, \quad r_s \equiv GM.$$ (3)

Equation (2) can be transformed into a third-order polynomial equation, namely

$$r_c^3 - (3r_A^2)r_c + 6r_A^2 = 0.$$ (4)

In appendix A we have sketched the solution of a third-order polynomial using an auxiliary angle $\phi$. In the case of equation (4) the relevant quantities $p, q, D$ and $R$ read

$$p = -3r_A^2, \quad q = 6r_A^2, \quad D = -r_A^2 + 9r_A^4 < 0, \quad R = r_A.$$ (5)
Hence, we can deduce that the polynomial under consideration corresponds to the case (i) described in appendix A. The auxiliary angle can be now defined as
\[
\cos \phi = \frac{6r_A r_s^3}{2 r_s^3} = 3 \left( \frac{r_s}{r_A} \right)
\]
and the solutions are parametrized with the help of trigonometric functions and their inverses in the following form:
\[
\begin{align*}
  r_1 &= -2r_A \cos \left( \frac{1}{3} \cos^{-1} \left( \frac{3r_s}{r_A} \right) \right) \\
  r_2 &= -2r_A \cos \left( \frac{1}{3} \left( \cos^{-1} \left( \frac{3r_s}{r_A} + 2\pi \right) \right) \right) \\
  r_3 &= -2r_A \cos \left( \frac{1}{3} \left( \cos^{-1} \left( \frac{3r_s}{r_A} + 4\pi \right) \right) \right).
\end{align*}
\]
Several conclusions, regarding these solution will be later of importance. We start to note that the solutions forbid any result for which \(3r_s r_A > 1\). As a consequence, the maximum value of \(r_s\) is given by \(\frac{r_s^{max}}{r_A} = 1\) or, equivalently by
\[
\frac{r_s^{max}}{r_A} = \frac{1}{3}.
\]
We then obtain a maximum mass, i.e. \(M \leq M_{max}\) which ensures the existence of horizons. With \(G = \frac{1}{m_{pl}^2}\) we can write it as
\[
M_{max} = \frac{1}{3} m_{pl}^2.
\]
The next issue of concern is the existence of a maximum horizon, i.e. the largest value \(r_3\) can assume while varying the mass \(M\). Calculating \(r_1(M_{max})\) gives
\[
\begin{align*}
  r_1(M_{max}) &= -2r_A, \\
  r_2(M_{max}) &= r_3(M_{max}) = r_c^{max} = r_A.
\end{align*}
\]
From the above we conclude that \(r_1\) is an unphysical (indeed, it is always negative), and henceforth we keep only \(r_2\) and \(r_3\) as the relevant physical horizons. Note that this is not the maximal horizon in the absolute sense as \(r_2\) can take values larger than \(r_c^{max}\). Indeed, we will see below that for \(M \ll M_{max}, r_2\) tends to \(\sqrt{3}r_A\). However, in the context of the generalized uncertainty principle, discussed in section 4, it will turn out that \(r_c^{max}/\sqrt{10}\) is the largest horizon for a Schwarzschild–de Sitter black hole with a well-defined temperature \(T\). Yet another way to confirm the above results is by first solving for the mass \(M\) from the horizon condition (2) which gives
\[
M(r_c) = \frac{r_c}{2} m_{pl}^2 - \frac{1}{6} \frac{r_c^3 m_{pl}^2}{r_A}
\]
and then looking for a local maximum according to \(\partial M/\partial r_c = 0\). Solving this equation for the variable \(r_c\) results in \(r_c^{max}\) as before. Replacing the previous result in (11), we also obtain as before the maximum mass defined in equation (9). Here we can see once again that \(M_{max}\) is associated with the maximal horizon \(r_A\).

Until now we have approximated the exact solutions (7) for the extreme values of the mass, i.e., when \(M\) approaches its maximum value \(M_{max}\). However, it will be equally important to approximate these solutions for intermediate values of the mass. We can first re-write our solution as
\[
r_2 \approx -2r_A \cos \left( \frac{5\pi}{6} - \frac{r_s}{r_A} - \frac{3}{2} \left( \frac{r_s}{r_A} \right)^3 \right)
\]
and we can see once again that \(M_{max}\) is associated with the maximal horizon \(r_A\).
which can be used to approximate it up to the first-order correction, i.e. \( r_2 \approx \sqrt{3} r_A - r_c \). Note that this horizon remains nonzero even if we put \( M \to 0 \) which makes it doubtful that it is a black hole horizon with a proper temperature. There is another peculiarity associated with this horizon as it decreases with increasing mass. It is therefore possible that \( r_2 \) and \( r_3 \) meet at a certain value of the mass. As mentioned before, they do that at \( M = M_{\text{max}} \). We can repeat a similar procedure for \( r_3 \) obtaining

\[
r_3 \approx 2r_s \left( 1 + \frac{4}{3} \frac{r_3^2}{r_A^2} \right)^{-1/2}.
\]

The correction term proportional to \( \frac{r_3^4}{r_A^4} \) is very small in almost the whole range of the masses, except in the case when the mass tends to the maximum value given in (9) (for which the approximated version above is not valid). It is often convenient to parametrize \( r_s = \omega r_A \), where \( \omega \) can take any value between 0 and \( \frac{1}{3} \) in agreement with (8). Then equation (13) reads

\[
r_3 \approx 2\omega r_A + \frac{8}{3} \omega^3 r_A.
\]

Even if \( \omega \to \omega_{\text{max}} = 1/3 \), the correction term is only of the order \( 10^{-1} \).

3. Classical Hawking radiation of a de-Sitter black hole

The idea of this section is the study of the temperature as a function of mass for a black hole in the Schwarzschild–de Sitter metric. We derive this relation by first calculating the surface gravity \( \kappa \) at the horizon \( r_c \) and relating it to the temperature \( T \) of the black hole by \( T = \frac{\kappa}{2\pi} \) (see, [15] and [16]). The word ‘classical’ refers here exactly to this procedure and we use it to distinguish it from the results obtained via the generalized uncertainty relation (GUP) in section 4. Following the arguments of [15] the surface gravity of a black hole is defined as \( \kappa = V a \) where all quantities are evaluated at the horizon \( r_c \). Here \( a \) is the invariant scalar acceleration and \( V \) is the redshift factor, which for static observers is equal to the proportionality factor between the time-like Killing vector and the 4-velocity \( K^\mu = V(x) U^\mu \). Above, \( K^\mu \) is the time-like Killing vector and \( U^\mu \) is the 4-velocity. For the Schwarzschild–de Sitter metric we obtain explicitly

\[
K^\mu = (1, 0, 0, 0), \quad U^\mu = \left( \left( \frac{2r_s}{r} - \frac{1}{3} \frac{r^2}{r_A^2} \right)^{-1/2}, 0, 0, 0 \right)
\]

and therefore the redshift factor is given by

\[
V = \sqrt{1 - \frac{2r_s}{r} - \frac{1}{3} \frac{r^2}{r_A^2}}.
\]

For the complete evaluation of the surface gravity we need the scalar 4-acceleration \( a \) which we have derived explicitly in appendix B. In the Schwarzschild–de Sitter metric we obtain

\[
a = \frac{r_s}{r} - \frac{1}{3} \frac{r^2}{r_A^2}.
\]

Hence the surface gravity takes the following simple expression:

\[
\kappa(r_c) = \left| \frac{r_s}{r_c} - \frac{1}{3} \frac{r_c^2}{r_A^2} \right|.
\]
We will comment about the absolute value in this expression in the following sub-section. Expression (18) can be further simplified such that $\kappa$ is a function of the horizons alone [17]. This can be achieved by replacing equation (11) into (18). The final formula is then

$$\kappa(r_c) = \frac{1}{2r_c^2} \left( r_c - \frac{1}{3} \frac{r_c^3}{r_A^2} \right) - \frac{1}{2r_c^2} \left( r_c - \frac{1}{2} \frac{r_c^3}{r_A^2} \right).$$

(19)

Viewing $\kappa$ as a function of the horizon or alternatively as a function of mass (see (18)) we note that $\kappa(r_{c_{\text{max}}}) = \kappa(M_{\text{max}}) = 0$ which obviously implies $T(M_{\text{max}}) = 0$ by the virtue of $T = \kappa/2\pi$. By the same identification between surface gravity and temperature the full $T-M$ relation can be spelled out as

$$T(M) = \frac{\kappa(r_3)}{2\pi} = \frac{1}{2\pi} \left( \cos \left( \frac{1}{2} \left( \cos^{-1} \left( \frac{3}{r_3^3} \right) + 4\pi \right) \right) \right) - \frac{1}{4r_A \cos \left( \frac{1}{2} \left( \cos^{-1} \left( \frac{3}{r_A^3} \right) + 4\pi \right) \right)}.$$

(20)

In the case of $r_c = r_3$ the absolute value, which appears in (19), is not necessary as $T(M_{\text{max}}) = 0$. We will discuss the case $\kappa(r_2)$ in a suitable approximation in the following sub-section. However, already here we note that this case is physically not without inconsistencies. As $M$ increases, the temperature will decrease in this case, but the horizon will become smaller and therefore also the entropy.

### 3.1. First-order corrections of $\Lambda$ in Hawking radiation

In the approximate version $r_3 \approx 2r_2$ formula (19) simplifies considerably and gives us the first-order correction to the standard Hawking expression

$$\kappa(r_3) = \frac{m_A^2}{4M} - \frac{m_A^2}{m_{pl}^2} M$$

(21)

The $T-M$ relation reads

$$T(M) = \frac{m_{pl}^2}{8\pi M} - \frac{1}{2\pi} \frac{m_A^2}{m_{pl}^2} M$$

(22)

valid for every $M$, except as $M \to M_{\text{max}}$ given in (9). In fact, the case of the maximum value is included already in the equation $\kappa(r_{c_{\text{max}}}) = \kappa(M_{\text{max}}) = 0$. Note that in (22) we have not used the absolute value. The reason is that for every mass the temperature defined in this way (for the horizon $r_3$) is positive. This in turn is a consequence of the equivalence principle [16] because a local inertial observer in comparison with a static one perceives a positive scalar acceleration calculated from (17). The opposite happens for the horizon $r_2$. In that case, for every value taken by $r_2$, except that obtained in (10), the surface gravity given in (19) would be negative without taking the absolute value. The reason for this behavior is again due to the equivalence principle because in this case the local inertial observer is moving while $r$ increases. Therefore the static observer has a negative scalar acceleration. For example the maximum value of $r_2$ is $\sqrt{3}r_A$, in which case the result (12) replaced in (19), gives $\kappa = -\frac{1}{\sqrt{3}} m_A = -\frac{1}{\sqrt{3}} m_A$. On the other hand, for intermediate values of the mass, replacing $r_2 \approx \sqrt{3}r_A - r_s$ in (19) leads to

$$\kappa(r_2) \approx \left| -\frac{2r_A^2 + 2\sqrt{3}r_A r_s - r_s^2}{2(\sqrt{3}r_A - r_s)r_A^3} \right| \approx \frac{r_A - \sqrt{3}r_s}{r_A(\sqrt{3}r_A - r_s)}.$$

(23)

The temperatures associated with the above results is, respectively

$$T(M \ll M_{\text{max}}) \approx \frac{1}{2\pi \sqrt{3}} m_A$$

(24)
and
\[ T(M) \approx \frac{1}{2\pi} \frac{r_A - \sqrt{3}r_s}{r_A(\sqrt{3}r_A - r_s)}. \tag{25} \]

Apparently the behavior of the temperature function given in (25) is correct. In fact as \( M \) increases, the temperature decreases, i.e., \( \frac{dT}{dM} < 0 \) as it should be in view of the fact that the heat capacity of the black hole is negative. The same is accomplished by (22). Recall that we have insisted here on the absolute value because the acceleration of the static observer with respect to a local one is negative. Without the absolute value in expression (23) we would have negative temperatures and a positive slope \( \frac{dT}{dM} > 0 \) for the function \( T(M) \). However, a different inconsistency appears now in the entropy behavior. An increase of mass in \( r_2 \approx \sqrt{3}r_A - r_s \) implies a decrease of the horizon and as a consequence of that a decrease in the standard entropy value [15]. With these arguments in mind it is reasonable to discard the temperature function due to the horizon \( r_2 \).

### 3.2. Consequences

In the following section we will elaborate on the problem of Hawking radiation of a Schwarzschild–de Sitter black hole from the point of view of the generalized uncertainty principle (GUP). It therefore makes sense to collect here the important results we obtained the preceding sections. The results (20) and (22) associated with \( \kappa(r_3) \) represent the correct physical behavior. The heat capacity of the black hole is negative and the horizon and the entropy increases with the mass. The latter aspect is missing for \( \kappa(r_2) \). The Schwarzschild radius \( 2r_s \) has a maximum allowed value \( 2r_A/3 \) (see equation (8)) corresponding to a maximum mass given in (9) which gives a maximum allowed horizon \( r_c^{\text{max}} = r_A \) in equation (13). At the maximum mass (or horizon) the Hawking temperature becomes zero. This we can interpret as a minimum temperature in this case. We mention this explicitly since \( T_{\text{min}} \) will come out nonzero using the GUP approach below.

### 4. Hawking radiation via the generalized uncertainty principle (GUP)

In this section we consider the Hawking radiation via GUP developed for the case \( \Lambda = 0 \) in [1] and [6] (see also [18] and [19]). The main result is the deformation of the \( T-M \) dispersion relation close to the Planck length \( l_{\text{Pl}} \) which turns out to be now the minimum possible horizon. Another way of expressing this result is to say that the black hole mass \( M \) has a remnant of the order of Planck mass \( m_{\text{Pl}} \) (this defines also the minimum possible mass). With the inclusion of \( \Lambda \) we have seen in the preceding section that there exist a maximum horizon and maximum mass. The simple question which we pose here in connection with \( \Lambda \) is whether the \( T-M \) relation gets modified also close to \( r_c^{\text{max}} \) (or, which is equivalent, close to \( M_{\text{max}} \)). In this context it is worth noting that gravity with \( \Lambda \) displays often a duality. Where the Newtonian constant \( G \) sets a minimum (maximum) allowed value, the cosmological constant \( \Lambda \) restricts the range of a parameter by setting a maximum (minimum). An example is the range of validity of the Newtonian limit [20]. Here the distance \( r \) is limited by
\[ 2r_s \approx r_3 \ll r \ll r_2 \approx \sqrt{3}r_A \tag{26} \]

for the intermediate mass range \( M \). Another example of such duality is encountered in the motion of a test particle in the Schwarzschild–de Sitter metric [21]. The equation of motion can be brought into the form containing an effective potential \( U_{\text{eff}} \) which depends parametrically on \( r_s, r_A \) and the angular momentum per mass \( r_l \). The effective potential has generically three
local extrema: a maximum close to $2r_s$, a minimum in which the planets move and, due to $\Lambda$, a second maximum. To avoid that the first local maximum and the local minimum coincide to form a saddle point, one has to respect the inequality

$$r_i > r_i^{\text{min}} = 2\sqrt{3}r_s.$$  \hfill(27)

On the other hand, if we insist that the local minimum and maximum do not degenerate to a saddle point, we have to satisfy

$$r_i < r_i^{\text{max}} = \left(\frac{3}{4}\right)^{1/3} \left(r_s^2 r_A\right)^{1/3}.\hfill(28)$$

It is not unreasonable to expect that the Hawking radiation of Schwarzschild–de Sitter black hole displays similar duality features.

4.1. GUP with $\Lambda = 0$

It makes sense to have first a brief glimpse at the case $\Lambda = 0$. The generalized uncertainty principle (GUP) [1–3] and the discussion of Hawking radiation within its framework [6, 4, 7] have gained some popularity in the last few years. Therefore while discussing the case $\Lambda = 0$ we will only give the main steps which are of importance in generalizing it to $\Lambda \neq 0$.

The steps involved in deriving the uncertainty relation with gravity are [1, 6] (i) $\Delta x_{\text{grav}} \sim (|\vec{F}_{\text{grav}}|/m)L^2$, where $L$ is the typical length/time scale and here $|\vec{F}_{\text{grav}}/m| = r_i/r^2$, (ii) $E$ being the photon’s energy is the source of gravity felt by the probed particle; $E = p \sim \Delta p$ which is the uncertainty in momentum of the latter, (iii) $r \sim L$ taken together with the previous steps gives now $\Delta x_{\text{grav}} \sim G_N \Delta p$ which is to be added to the standard uncertainty relation resulting in

$$\Delta x \gtrsim \frac{1}{2\Delta p} + \frac{\Delta p}{2m_{\text{pl}}^2}.$$  \hfill(29)

It is evident that the method of obtaining (29) is heuristic (we follow here [1]). This is, however, not a drawback as the same result is obtained within string theory (see the papers by Veneziano in [2]) and non-commutative geometry (see the papers by Maggiore in [2]). This shows that the heuristic line of arguments is indeed valid and has the advantage of being also model independent.

Applying (29) to black hole evaporation [6] consists essentially in identifying $\Delta x$ with the Schwarzschild radius $r_s$ [22] ($2r_s$ in our notation) as well as $\Delta p \sim p = E$ with the temperature up to a factor. This turns out to be the surface gravity $T_s = \kappa$ such that $T = T_s/2\pi$. The result is a quadratic equation in $T_s$

$$T_s^2 - 4MT_s + \frac{m_{\text{pl}}^2}{2} = 0.$$  \hfill(30)

from which it follows that:

$$T_s(M) = 2M\left(1 - \sqrt{1 - \frac{m_{\text{pl}}^2}{4M^2}}\right) \to \frac{m_{\text{pl}}^2}{4M}.$$  \hfill(31)

where we have chosen already a solution with the correct limit at large $M$ as indicated in the above equation. Two conclusions are in order. First, the temperature is well defined only if

$$M > M_{\text{min}} = M_{\text{remnant}} = \frac{m_{\text{pl}}}{2}.$$  \hfill(32)
which defines the minimum mass and the black hole remnant. Second, the existence of a minimum mass sets a scale for the maximally possible temperature $T_{\text{max}}$ via (31) and $T = T_*/2\pi$. It reads

$$T_{\text{max}} = \frac{M_{\text{min}}}{\pi} = \frac{m_{\text{pl}}}{2\pi}. \quad (33)$$

Yet another interpretation of the above results refers to the length scales involved. The existence of a black hole remnant is equivalent to say that the Schwarzschild horizon cannot be smaller than the Planck scale $l_{\text{pl}} = 1/m_{\text{pl}}$ [1, 6, 19, 23], i.e., $2r_{\text{min}} = l_{\text{pl}}$. This can be easily verified by re-writing (31) as

$$T_* = 2r_s \left( 1 - \sqrt{1 - \frac{1}{4} \left( \frac{l_{\text{pl}}}{r_s} \right)^2} \right) \quad (34)$$

which is well defined for $r_s > r_{\text{min}}$. Note that this minimum length scale is exactly what one would expect from quantum gravity. However, we should not forget that any estimation deduced from an uncertainty relation like (30) remains an order of magnitude estimate, having at the same time the advantage of being model independent. Choosing the right branch between the two solutions of the quadratic equation (30) has, as mentioned above, to do with the right limit for large masses which is known by the Hawking formula. However, even without knowing this limit explicitly, we could discriminate the physical solution from the non-physical one by using arguments based on the negative heat capacity of the Schwarzschild black hole, i.e., insisting on $\frac{\partial T}{\partial M} < 0$. The latter is a consequence of $\frac{\partial S}{\partial T} < 0$. This together with $\frac{\partial S}{\partial M} > 0$ allows us to conclude that $\frac{\partial T}{\partial M} < 0$ on very general grounds. This expectation is satisfied only if we choose the right physical solution of (30). Indeed, we then obtain

$$\frac{dT_*}{dM} = \frac{2}{\sqrt{1 - \frac{1}{4} \left( \frac{m_{\text{pl}}}{M} \right)^2}} \left( \sqrt{1 - \frac{1}{4} \left( \frac{m_{\text{pl}}}{M} \right)^2} - 1 \right) < 0 \quad (35)$$

since we have $\sqrt{1 - \frac{1}{4} \left( \frac{m_{\text{pl}}}{M} \right)^2} \leq 1$. As we have already seen in subsection 4.1 such general restriction are often not unimportant to exclude a possible solution.

### 4.2. GUP with $\Lambda \neq 0$

The generalized uncertainty principle with $\Lambda = 0$ bears interesting results in agreement with expectation from quantum gravity. The dispersion relation $T(M)$ gets modified near the Planck radius $2r_{\text{min}} = l_{\text{pl}}$ as compared to the standard Hawking result. We can paraphrase this also by stating that there exists a minimum mass $M_{\text{min}}$ which corresponds to a maximum temperature $T_{\text{max}}$. Motivated by the duality encountered in gravity theory with $\Lambda$ (see equations (27) and (28)), we can speculate that the generalized uncertainty principle with $\Lambda \neq 0$ will give us a dual relation where $T(M)$, as compared to (22), is modified close to $M \sim M_{\text{max}}$ (equation (9)) (or, which is equivalent, close to $r_{\text{max}}$ from equation (8)). This should give us a $T_{\text{min}} \neq 0$ given by the scales of $\Lambda$. Anticipating our results, we mention here that this is indeed the case and we obtain $T_{\text{min}} \sim m_{\Lambda}$.

We have seen that the generalized uncertainty principle can be obtained easily from the gravitational force. One can repeat this heuristic approach with GUP including $\Lambda \neq 0$. Since this uncertainty relation is new, we will check the results emerging from it by comparing it with (i) standard results for $T(M)$ and (ii) independent results in the context of black body radiation in section 5. Both these checks will show that the new GUP relation is consistent. To
repeat the steps leading to GUP from the previous section we need the gravitational potential \( \Phi \) for a spherically symmetric mass distribution with \( \Lambda \)

\[
\Phi = \frac{r_s}{r} - \frac{1}{6} \frac{r^3}{r_s^2}.
\]

(36)

Then following the arguments from the last sub-section the gravitational force per mass attributed to \( \Lambda \) is

\[
|\vec{F}_\Lambda| = \frac{1}{3} \frac{\Lambda L}{m} \text{, where } L \text{ is again a typical length scale in the problem under consideration.}
\]

The corresponding displacement is

\[
\Delta x_\Lambda \sim \frac{1}{3} \frac{\Lambda L^3}{m^2}.
\]

We use now the additional assumption \( L \sim \frac{1}{\Delta p} \) [24]. This assumption is equivalent to say that the precision of the momentum is inversely proportional to the typical length scale and can be found, e.g. in [25, 26] in connection with wave packets. It is analog to similar assumptions like \( \Delta t \sim E^{-1} \) in the context of estimating the pion mass in Yukawa’s theory [27] or \( \Delta x \sim p^{-1} \) in case we want to estimate the precision of the position [28]. Therefore we can write \( \Delta x_\Lambda \sim \frac{1}{3} \frac{m^2}{\Lambda \Delta p^3} \) such that the proposed relation for GUP with the inclusion of the cosmological constant is

\[
\Delta x \gtrsim \frac{1}{2\Delta p} + \Delta x_\Lambda, \quad \Delta x \gtrsim \frac{1}{2\Delta p} + \frac{\Delta p}{2m_{pl}} - \frac{\gamma m^2_\Lambda}{3 \Delta p^3},
\]

(37)

where we have taken into account the relative sign difference between the cosmological constant contribution and the standard Newtonian part [20]. We also include a factor \( \gamma \sim O(1) \) which accounts for the fact that we are dealing with orders of magnitude estimates. In comparing the results with (22) for masses smaller than \( M_{\text{max}} \), \( \gamma \) should come out of the order of 1. If this is not the case, something would be wrong with the uncertainty relation (37). As in the previous sub-section in the context of Hawking radiation the uncertainty in position is associated with the event horizon. In the case of Schwarzschild–de Sitter metric, we should, in principle, take the full expression (13). It will turn out, however, that it is sufficient to use the approximation \( 2r_s \). Then the generalized uncertainty applied to black hole evaporation gives an equation which generalizes (30)

\[
\frac{2M}{m_{pl}^2} \approx \frac{1}{2T} - \frac{T_s}{2m_{pl}^2} - \frac{\gamma m^2_\Lambda}{3 T^3}.
\]

(38)

It is worth noting that for high temperatures, the results of the previous sub-section for \( \Lambda = 0 \) are recovered from (38). Therefore, \( T_{\text{max}} \) in conjunction with \( M_{\text{min}} \) also follows from the above equation. For small temperatures (38) can be approximated to

\[
\frac{2M}{m_{pl}^2} \approx \frac{1}{2T} - \frac{\gamma m^2_\Lambda}{3 T^3},
\]

(39)

which amounts to solve a third-order polynomial of the form

\[
T^3 - \left( \frac{m_{pl}^2}{4M} \right) T^2 + \frac{\gamma m^2_\Lambda m_{pl}^2}{6 M} = 0.
\]

(40)

To solve this equation we refer to appendix A. In connection with (40) the following auxiliary constants are needed \( r = -\frac{m_{pl}^2}{4M}, s = 0, t = \frac{-\gamma m^2_\Lambda m_{pl}^2}{6 M} \) to obtain the reduced form of (40) which is reached by the shift

\[
y = T + \frac{r}{3} = T_s - \frac{m_{pl}^2}{12M},
\]

(41)

where \( y \) is the solution of the reduced third-order equation given in appendix A. The coefficients of the reduced equation can be calculated explicitly. They are \( p = -\frac{m_{pl}^2}{48M^2} \) and

\[
q = \frac{m^4}{M} \left( -\frac{1}{864 M^2} + \frac{\gamma m^2_\Lambda}{6 m_{pl}^2} \right),
\]

(42)
The parametric solution depends on the sign of \( q \), which depends only on the variable \( M \). We denote the branch point by \( M_{q=0} \), which can be found by setting \( q = 0 \). We find

\[
M_{q=0} = \frac{1}{12\sqrt{\gamma} m_A} m^2_{pl} \tag{43}
\]

such that \( M < M_{q=0} \) for \( q < 0 \) and \( M > M_{q=0} \) for \( q > 0 \). The existence of real solution, i.e. \( T_{+}(M) \) or \( y \) depends crucially on \( D \) in appendix A. In the case under consideration it reads

\[
D \equiv \frac{1}{4} m^6_{pl} m^2_{A} \left( \frac{\gamma^2}{36} \left( \frac{m^2_{A}}{m^2_{pl}} \right) - \frac{\gamma}{3(864)} \left( \frac{m^2_{pl}}{M^2} \right) \right). \tag{44}
\]

It can be demonstrated that for \( D > 0 \) there are no physical solutions of the associated third-order equation and only \( D < 0 \) is of interest for us. A limit on the value of \( M \) is set by putting \( D = 0 \). We find from \( D = 0 \), \( M_{\text{max}} \),

\[
M_{\text{max}} = \frac{1}{6\sqrt{2\gamma} m_{A}} m^2_{pl} \tag{45}
\]

such that \( M < M_{\text{max}} \) if \( D < 0 \). Later in the text we will find \( \gamma = 5/9 \) by comparing the GUP solution \( T_{+}(M) \) to that found in section 3. In other words, we have also \( D(M > M_{\text{max}}) \) \( > 0 \). The real solution in the case \( p < 0 \) and \( D > 0 \) (see case (ii) in appendix A) is \( y_1 = -2R \cosh \phi/3 \) which is positive definite if \( R < 0 \). The latter implies \( q < 0 \) and from this we conclude that \( M < M_{q=0} \). However, as we will show below, we have \( M_{\text{max}} > M_{q=0} \) which is in contradiction to \( D(M > M_{\text{max}}) \) \( > 0 \). Opting for \( q > 0 \) (i.e. \( R > 0 \)) the solution is \( T_{1+} = y_1 + R(1 - 2 \cosh \phi/3) \) which is always negative since the smallest value of \( \cosh x \) is 1. A remark about the three different mass scales is in order. We have

\[
M_{\text{max}} > M_{\text{max}} > M_{q=0}, \tag{46}
\]

where \( M_{\text{max}} \) is the value found in (9) in connection with \( r_{s}^{\text{max}} \). Nevertheless, all these values are of the same order of magnitude as

\[
M_{q=0} \approx \frac{M_{\text{max}}}{2\sqrt{2}}. \tag{47}
\]

The correction to \( 2r_{s} \) at \( M = M_{\text{max}} \) given in (13) as \( 4r_{s}^3/3r_{A}^4 \) is suppressed by one order of magnitude as compared to 1. This justifies the use of \( 2r_{s} \) as an approximation in equation (39).

It will be convenient from now on to parametrize the mass \( M \) by a parameter \( \zeta \) defined by

\[
M = \frac{M_{\text{max}}}{\zeta}, \tag{48}
\]

where \( \zeta = 1 \) corresponds to \( M_{\text{max}} \). The branch point corresponding to \( q = 0 \) can be now characterized by \( \zeta > \sqrt{2} \) for \( q < 0 \) and \( 1 < \zeta < \sqrt{2} \) for \( q > 0 \).

### 4.2.1. The branch \( q > 0 \)

The parameter \( R \) given in appendix A, which depends on the sign of \( q \) is simply

\[
R = \frac{1}{12} \frac{m^3_{pl}}{M}. \tag{49}
\]

Obviously, case (i) from appendix A applies in this case. Hence the auxiliary angle, as \( D < 0 \) and \( p < 0 \), can be calculated as

\[
\cos \phi = -1 + 144\gamma \frac{M^2 m^2_{A}}{m^4_{pl}}. \tag{50}
\]
The zeros of the reduced third-order equation in terms of parameter $\zeta$ in (48) can be easily found to be

$$y_1 = -\sqrt{2\gamma m_A \zeta} \cos \left( \frac{1}{3} \cos^{-1} \left( -1 + \frac{2}{\zeta^2} \right) - \frac{1}{2} \right)$$  \hspace{1cm} (51)

$$y_2 = -\sqrt{2\gamma m_A \zeta} \cos \left( \frac{1}{3} \cos^{-1} \left( -1 + \frac{2}{\zeta^2} \right) + 2\pi \right)$$  \hspace{1cm} (52)

$$y_3 = -\sqrt{2\gamma m_A \zeta} \cos \left( \frac{1}{3} \cos^{-1} \left( -1 + \frac{2}{\zeta^2} \right) + 4\pi \right).$$  \hspace{1cm} (53)

From equations (41) and (48) it is possible to find the explicit solutions for the surface gravity $T$.

$$T_{1\ast}(\zeta) = -\sqrt{2\gamma m_A \zeta} \left( \cos \left( \frac{1}{3} \cos^{-1} \left( -1 + \frac{2}{\zeta^2} \right) - \frac{1}{2} \right) \right)$$  \hspace{1cm} (54)

$$T_{2\ast}(\zeta) = -\sqrt{2\gamma m_A \zeta} \left( \cos \left( \frac{1}{3} \cos^{-1} \left( -1 + \frac{2}{\zeta^2} \right) + 2\pi \right) - \frac{1}{2} \right)$$  \hspace{1cm} (55)

$$T_{3\ast}(\zeta) = -\sqrt{2\gamma m_A \zeta} \left( \cos \left( \frac{1}{3} \cos^{-1} \left( -1 + \frac{2}{\zeta^2} \right) + 4\pi \right) - \frac{1}{2} \right).$$  \hspace{1cm} (56)

It remains to discuss which of the above solutions is physical (bearing in mind that the real temperature $T$ is $T = T_{\ast}/2\pi$). It is easy to show that $T_1 = T_{1\ast}/2\pi$ is negative. The right choice between the solutions can be done by the requirement that the deformed $T(M)$ relation must smoothly match the classical result for moderate masses, i.e. equation (22) (which in turn for even smaller masses goes over to the standard Hawking formula). This is impossible if the deformed solution increases with mass as (22) has negative heat capacity. One can show that $\partial T_3/\partial M$ is always positive and therefore can be discarded as a physical solution. To see that, it suffices to calculate $T_2$ as well as $T_3$ at two different points. We start with $\zeta = 1$ ($M = M_\text{max}$) where we consider the surface gravity as a function of $\zeta$. We get

$$T_{2\ast}(1) = T_{3\ast}(1) = T_{\text{min\ast}} = \sqrt{2\gamma m_A}.$$  \hspace{1cm} (57)

If we can establish that the physical solution is $T_2$, this would imply the existence of a minimum temperature due to $\Lambda$ in conjunction with $M_\text{max}$ at a horizon approximately $r_\Lambda/3$, namely

$$T_{\text{min}} = \frac{T_{2\ast}(1)}{2\pi} = \frac{T_{\text{min\ast}}}{2\pi} = \frac{\sqrt{2\gamma}}{2\pi} m_A \approx 0.225\sqrt{2\gamma} m_A.$$  \hspace{1cm} (58)

At $\zeta \to \sqrt{2}$, where the sub-index ‘−’ implies the limit taken from the left, we have $T_{2\ast}(\sqrt{2}) = 2.73\sqrt{\gamma} m_A$ and $T_{3\ast} = \sqrt{\gamma} m_A$. These results are equivalent to $T_2(\sqrt{2}) = 0.4348\sqrt{\gamma} m_A > T_{\text{min}}$ and $T_3(\sqrt{2}) = 0.159\sqrt{\gamma} m_A < T_{\text{min}}$. Hence $T_3$ a monotonically increasing function with $M$ and therefore $T_2$ is the physical solution. This establishes $T_{\text{min}}$ in (58) as a genuine minimal value of temperature.

4.2.2. The branch $q < 0$. In this case the auxiliary angle is

$$\cos \phi = 1 - 144\gamma \frac{M^2 m_A^2}{m_{\text{pl}}^4}.$$  \hspace{1cm} (59)

Following the same procedure as above we arrive at:
\begin{align}
T_1' (\zeta) &= \sqrt{2} \gamma m_A \zeta \left( \cos \left( \frac{1}{3} \cos^{-1} \left( 1 - \frac{2}{\zeta} \right) \right) + \frac{1}{2} \right) \label{eq:60} \\
T_2' (\zeta) &= \sqrt{2} \gamma m_A \zeta \left( \cos \left( \frac{1}{3} \cos^{-1} \left( 1 - \frac{2}{\zeta} \right) + 2 \pi \right) + \frac{1}{2} \right) \label{eq:61} \\
T_3' (\zeta) &= \sqrt{2} \gamma m_A \zeta \left( \cos \left( \frac{1}{3} \cos^{-1} \left( 1 - \frac{2}{\zeta} \right) + 4 \pi \right) + \frac{1}{2} \right). \label{eq:62}
\end{align}

To check the continuity at the branch point it is necessary to evaluate these equations at \( \zeta \to \sqrt{2}^+ \), where the sub-index ‘+’ denotes the limit approached from the right. We obtain the following equalities
\begin{align}
T_2^* (\zeta \to \sqrt{2}^-) &= T_1' (\zeta \to \sqrt{2}^+) \quad \text{and} \\
T_3^* (\zeta \to \sqrt{2}^-) &= T_3' (\zeta \to \sqrt{2}^+). \label{eq:63}
\end{align}

which means that on this branch \( T_1' \) is the physical solution. After some expansion the \( T-M \) relation in the vicinity of \( \zeta = \sqrt{2} \) is approximately given by
\begin{equation}
T( M) \approx T_1' \approx \frac{1}{8} \pi m_{pl}^2 M - \frac{10}{9 \pi} \left( \frac{m_A}{m_{pl}} \right)^2 M. \label{eq:64}
\end{equation}

The fact that \( T_1' \) is the right physical choice can also be established by expanding the above results for \( \zeta \gg 1 \). From \( \ref{eq:60}, \ref{eq:61} \) and \( \ref{eq:62} \) it follows \( T_1' (\zeta \gg 1) \approx \frac{\sqrt{2} \pi}{m_A \zeta} \) and \( T_2 = T_3 \approx 0 \) confirming again that \( \ref{eq:60} \) is the right physical choice on this branch since replacing the definition of \( \zeta \) therein and using \( T = T_\star / 2 \pi \) we obtain the standard Hawking result, i.e.,
\begin{equation}
T( M) \approx T_1 = \frac{1}{8 \pi} m_{pl}^2 M. \label{eq:65}
\end{equation}

Note that our starting point has been the uncertainty relation \( \ref{eq:38} \) which we have already approximated in \( \ref{eq:39} \) for small temperature. Although, it is easy to see that the full uncertainty relation \( \ref{eq:38} \) approximated for large temperatures leads to the same results as discussed in sub-section 4.1 (this refers especially to the existence of \( T_{\max} \) and \( M_{\min} \)), the Hawking result \( \ref{eq:64} \) is the extreme low mass expansion which follows from the approximation \( \ref{eq:39} \).

For the intermediate mass range we would expect that we recover the functional form of equation \( \ref{eq:22} \). This will allow us to fix, in principle, the parameter \( \gamma \) and to make a consistency check of the uncertainty relation \( \ref{eq:37} \). Recall that arguments in connection with uncertainty relations involve orders of magnitude estimates. Therefore, if we recover form uncertainty relation the functional from of \( \ref{eq:22} \) such that in comparison with \( \ref{eq:40} \) we get \( \gamma \sim \mathcal{O}(1) \), then the uncertainty relation \( \ref{eq:37} \) is certainly consistent. We elaborate on these issues in more detail below.

### 4.2.3. The matching condition.

We start with
\begin{equation}
T_\star (M) = \frac{1}{12} m_{pl}^2 M - \frac{1}{6} m_{pl}^2 \cos \left( \frac{1}{3} \cos^{-1} \left( 1 - 2 \left( \frac{M}{M_{\max}^*} \right)^2 \right) \right) \label{eq:66}
\end{equation}

obtained from \( T_1' \). For \( M \ll M_{\max}^* \), we can expand \( \ref{eq:65} \) in powers of \( M / M_{\max}^* \) and divide \( T_\star \) by \( 2 \pi \) to arrive at
\begin{equation}
T_\star (M) \approx \frac{1}{8 \pi} m_{pl}^2 M - \frac{9}{10 \pi} \gamma \left( \frac{m_A}{m_{pl}} \right)^2 M. \label{eq:67}
\end{equation}

We see that the functional form is identical to \( \ref{eq:22} \). Demanding that in this mass region the two results be equal allows us to fix \( \gamma = 5/9 \). Indeed, we see that \( \gamma \) is of order 1 as it should.
be if the uncertainty relation with $\Lambda$ is physically relevant. Fixing the parameter $\gamma$ permits us to write
\[ T_{\text{min}} = 1.05 m_A \approx m_A, \quad M_{\text{max}}^* = \frac{1}{2\sqrt{10}} m_{\Lambda}. \]

Recall that we started probing into the Hawking radiation via the generalized uncertainty relation to see if by the inclusion of $\Lambda$ we get a relation dual to $T_{\text{max}}$ and $M_{\text{min}}$ which is a result of GUP with $\Lambda = 0$ (and also with $\Lambda \neq 0$ for large temperatures). Equation (67) is indeed such a dual result due to $\Lambda$.

4.2.4. GUP with $\Lambda \neq 0$ in the extreme value of the horizon. In equation (38) we used for the horizon the value $2r_s$ with the justification that $r_s(M_{\text{max}}^*) \approx 2r_s$ with correction being roughly $10^{-1}$. Nevertheless, this does not exclude the possibility that there exists a solution $T(M)$ if we start in equation (37) with a horizon bigger than $2r_s(M_{\text{max}}^*)$. For the sake of completeness, we probe into this matter by parametrizing the horizon as $\beta r_{\Lambda}$ where we are interested in the parameter $\beta$ around the value 1. The relevant equation, corresponding to (40), is now
\[ \beta r_{\Lambda} \approx \frac{1}{2 T_*} - \frac{5}{27} T_*^3. \]

The third-order equation takes the form
\[ T^3_* - \left( \frac{m_{\Lambda}}{2\beta} \right) T_*^2 + \frac{5}{27\beta} m_{\Lambda}^3 = 0 \]
with $r = -\frac{m_{\Lambda}}{2\beta}$ and $t = \frac{5}{27\beta}$ being the coefficients of the reduced third-order polynomial. According to appendix A we have
\[ y = T_* - \frac{m_{\Lambda}}{6\beta}, \quad p = -\frac{m_{\Lambda}^2}{12\beta^2}, \quad q = \frac{(-1 + 20\beta^2)m_{\Lambda}^3}{108\beta^3} \approx \frac{5}{27} m_{\Lambda}^3. \]

Using $p$ and $q$ we can evaluate $D$ which gives
\[ D = -\frac{1}{46656} \left( \frac{m_{\Lambda}}{\beta} \right)^6 + \frac{25}{2916} \frac{m_{\Lambda}^6}{\beta^2} > 0 \]
which is positive, at least for $1 < \beta < \sqrt{3}$ ($\beta = 1$ corresponds to the maximum value of the horizon $r_s$ whereas $\beta = \sqrt{3}$ to the maximum value of $r_s$). Thus we have now $D > 0$ and $p < 0$ which is case (ii) in appendix A. The parameter $R$ and the auxiliary angle come out to be $R = \frac{m_{\Lambda}}{6\beta}$ and $\cosh \phi = 20\beta^2$ which for $\beta = 1$ is $\phi = 3.68$. The real solution is
\[ T_* = R(1 - 2 \cosh(\phi/3)) \]
which is never positive. Indeed, the first step to get a real positive solution is to return to case (i) in appendix A which requires $D < 0$. Hence, putting $D(\beta) = 0$ gives $\beta = 1/\sqrt{10}$ in agreement with $2r_s(M_{\text{max}}^*) = r_{\Lambda}/\sqrt{10}$. In the context of GUP we therefore have a minimum and maximum horizon defined by
\[ r_{\text{min}} = l_{\text{pl}}, \quad r_{\text{max}} = 2r_s(M_{\text{max}}^*) = \frac{1}{\sqrt{10}} r_{\Lambda}. \]
5. An independent source of information on $T_{\text{min}}$ and $T_{\text{max}}$

In 1966 Sakharov found a maximum temperature of black body radiation to be of the order of Planck mass [29]

$$T_{\text{Sakharov}} \approx m_{\text{pl}}.$$  (73)

He based his results on very general arguments. This result is confirmed in equation (33) which is of the same order of magnitude as (73). Sakharov’s result bears a certain importance. Combined with Hawking’s formula for black hole evaporation $T = 1/(8\pi G N M)$, it implies independently of GUP the existence of a black hole remnant of the order of Planck mass. Indeed, the value of the maximal temperature is $\sim 10^{32}$ K and has only a physical relevance in black hole evaporation. We can show yet a third way, to establish this important result. This method is then also suitable to include $\Lambda$. The $-g_{00}$ component of the metric should be positive definite (see, chapter 84 in [30] for a general discussion). We can regard also the mass $M$ entering the Schwarzschild metric as energy which, in turn, can be replaced by the energy density $\rho$, i.e. $0 < -g_{00} = 1 - \frac{2GM}{R} = 1 - (8\pi/3)\rho R^2$. Hence $\rho < \frac{1}{8\pi} m_{\text{pl}}^2$. Using the Stefan–Boltzmann law $\rho = \sigma T^4$ gives [31] $T^4 < \frac{3}{8\pi} m_{\text{pl}}$. Finally, to get rid of the radius $R$ we employ the quantum mechanical result for black body radiation, $R > 1/T$ [31–33].

The maximal temperature obtained this way, namely $T < T_{\text{max}} = \sqrt{\frac{45}{8\pi} m_{\text{pl}}}$ is of the same order of magnitude as $T_{\text{max}}$ in equation (33). Repeating the same steps $\Lambda \neq 0$, i.e. for the Schwarzschild–de Sitter metric we can write

$$0 < \rho < \frac{3}{3} \frac{m_{\text{pl}}^2}{8\pi R^2} = \frac{1}{3} \frac{m_{\text{pl}}^2}{8\pi R^2} < m_{\text{pl}}^2 T^2 = \frac{1}{3} \frac{3}{8\pi} m_{\text{pl}}^2 m_{A}^2,$$

where we used again $R > 1/T$. One of these inequality, $\rho < \mathcal{F}$, gives us back $T_{\text{max}}$ with small correction due to $\Lambda$. The other one, $0 < \mathcal{F}$, can be translated into $T_{\text{min}}$ such that in the end we get $\frac{1}{2\rho} m_{A} = T_{\text{min}} < T < T_{\text{max}} \sim m_{\text{pl}}$ confirming the existence of a minimal and maximal temperature in a different way.

6. Gravitational redshift for the temperature with the presence of $\Lambda$

In this section we study yet another consequence of $\Lambda$ in the measurement of temperature at a distance $r^*_1$. This effect manifests itself in the gravitational redshift of thermal and electromagnetic radiation. To compare the two cases, we work first on the electromagnetic part. We follow here in parts [15]. In equation (16) we obtained the redshift factor $V = \sqrt{-K^\sigma K^\sigma}$ in the Schwarzschild–de Sitter metric which we can use in the standard relation of redshift for electromagnetic wavelength [15] for static observers

$$\lambda_2 = \frac{V_2}{V_1} \lambda_1 = Z_{\text{grav}} \lambda_1,$$  (74)

where a photon with wavelength $\lambda_1$ has been emitted at a distance $r^*_1$ and detected as $\lambda_2$ at $r^*_2$. It can be demonstrated that at the Killing horizons [15] the redshift factor is zero, i.e., we have $V(r \approx 2r_s) = V(r \approx \sqrt{3}r_A) = 0$ which is also evident from the explicit form

$$\lambda_2 = Z_{\text{grav}}(r^*_2, \lambda_1) = \left( \frac{1 - \frac{2r_s}{r_1}}{1 - \frac{2r_s}{r_1} - \frac{1}{3} \frac{r^*_2}{r_1}} \right) \lambda_1.$$  (75)

For a fixed mass satisfying the condition $r_s \ll r_A$, a photon emitted by a static observer 1 will be observed by static observer 2 at a distance $r \approx \sqrt{3}r_A$ with a wavelength $\lambda_2$. 


given by

$$\lambda_2 \approx \frac{V_2(r \approx \sqrt{3r_A})}{V_1} \lambda_1 \approx 0.$$  (76)

If this is true, there must be some distance $r_0$ after which the wavelength $\lambda_2$ begins to decrease in contrast to what happens in the case $\Lambda = 0$ where $Z_\text{grav} \to 1/V_1$ as $r_2^\ast$ becomes large. Suppose a photon is emitted at $r_1^\ast$ and detected as $\lambda_2$ at $r_2^\ast \to r_0$. If the photon is emitted at the same distance, but detected as $\lambda_2'$ at $r_2^\ast > r_0$, then $\lambda_2'$ is blueshifted as compared to $\lambda_2$. The distance $r_0$ can be found if we consider the function [20]

$$e^{\nu(r)} = 1 - \frac{2r_s}{r} - \frac{1}{3} \frac{r^2}{r_A^2}.$$  (77)

It has a local maximum at $r_0 = (3r_sr_A^2)^{1/3}$ where $r_0$ coincides with the distance after which a test body has no bound orbits in the Schwarzschild–de Sitter metric [21] (see, also equation (28)). We can summarize the results by saying that $\lambda_2$ increases as $r$ increases up to the value $r_0$ whereas $\lambda_2$ decreases as $r$ increases starting from $r_0$. The maximal redshift experienced by electromagnetic waves is

$$Z_\text{grav}^{\text{max}} = \frac{\sqrt{1 - 2\left(\frac{r_s}{r_A}\right)^{2/3}}}{V_1}.$$  (78)

after which $\lambda_2$ becomes smaller. Worth noting is the fact that $r_0$ is of astrophysical order of magnitude as it is a combination of a large and a small distance [21].

A similar procedure can be repeated for thermal radiation [15] starting with

$$T_2 = \frac{V_1}{V_2} T_1 = \frac{V_1}{V_2} \frac{a_1}{2\pi},$$  (79)

where $a_1$ is the invariant acceleration [15]. The observed temperature at a given distance $r_2^\ast$ is given by

$$T(r_2^\ast) = \lim_{r_1^\ast \to 2r} \frac{V_1 a_1}{2\pi V_2} = \frac{1}{V_2 2\pi} \frac{\kappa}{\kappa/2\pi} = \frac{\kappa/2\pi}{\sqrt{1 - \frac{2r_s}{r_2^\ast} - \frac{1}{3} \frac{r^2}{r_A^2}}}.$$  (80)

As $r_2^\ast \to (3r_sr_A^2)^{1/3}$, we obtain approximately

$$T(r_2^\ast \to r_0) \approx \frac{\kappa/2\pi}{\sqrt{1 - 2\left(\frac{r_s}{r_A}\right)^{2/3}}}$$  (81)

which corresponds to a minimal temperature at a finite distance. After $r_0$ the temperature increases with $T(r_2^\ast) \to \infty$ as $r_2^\ast \to \sqrt{3r_A}$ which shows the difference with the case of the photon’s wavelength. As before we summarize this result by saying that $T_2$ decreases as $r$ increases up to the value $r_0$ while $T_2$ increases starting from $r_0$.

### 7. Conclusions

As compared to the standard Hawking radiation formula $T(M) = m_{pl}^2/(8\pi M)$, the results from the generalized uncertainty principle [1, 6], with $\Lambda = 0$, give a slightly different picture which agrees, as far as the existence of some minimal/maximal physical quantities (mass, length, etc) is concerned, with expectations from quantum gravity:

1. There exists a black hole remnant with a mass $M_{\text{min}} \sim m_{pl}$ corresponding to a maximal temperature $T_{\text{max}} \sim m_{pl}$.
(2) There exists a minimum length (minimum horizon) \( r_{\text{sc}}^{\text{min}} \sim l_{\text{pl}} \).

(3) For large masses as compared to \( M_{\text{min}} \), \( T(M) \) goes over to the standard Hawking formula. For masses close to \( M_{\text{min}} \) (equivalently, for the horizon close to \( r_{\text{sc}}^{\text{min}} \)), \( T(M) \) gets deformed.

With the inclusion of \( \Lambda \), Einstein’s gravity becomes a two-scale theory which in our universe has a hierarchical structure: \( m_{\text{pl}} \gg m_{\Lambda}, r_{\Lambda} \gg r_{\text{sc}} \gg l_{\text{pl}} \), etc. It is known that \( \Lambda \) has a dual effect in the sense that if a quantity is restricted by some maximal (minimal) value connected to the Newtonian constant \( G \), \( \Lambda \) has the opposite effect, i.e., it introduces a minimal (maximal) restriction. Two such examples have been explicitly given in equations (27) and (28) which also demonstrate the fact that \( \Lambda \) has local effects [34]. It is then not unreasonable to ask if a generalized uncertainty principle with \( \Lambda \) displays dual effects to the points 1–3 above. To this end, we formulated an uncertainty relation with \( \Lambda \) along the same lines of arguments used in the standard GUP case. We applied it to Hawking radiation and found that (in doing so we emphasized certain aspects and neglected other [35] which would not change our results):

(4) The results 1–3 from above hold.

(5) There exists a maximum mass due to \( \Lambda \) whose value is \( M_{\text{max}} \sim (m_{\text{pl}}/m_{\Lambda})m_{\Lambda} \) corresponding a minimum temperature \( T_{\text{min}} \sim m_{\Lambda} \). We obtain the same result in section five by looking into black body radiation. This again confirms that the new GUP relation is consistent.

(6) There exists a maximum length (at least in black hole radiation context) \( r_{\text{sc}}^{\text{max}} \sim r_{\Lambda}/3 = 1/3\sqrt{\Lambda}. \) Beyond this value the GUP equation as applied to black hole evaporation does not have any solution.

(7) For intermediate masses the \( T(M) \) dispersion relation derived via GUP goes over to the standard relation (22) derived via the surface gravity. To put it in different words, the fact that \( \gamma = 5/9 \) comes out of order of unity tells us that the GUP relation with \( \Lambda \) is correct. For even smaller masses this goes over to the Hawking formula \( T \propto 1/M \) which in turn gets replaced by the deformed relation for masses close to \( m_{\text{pl}} \) (see point 3 above). For masses close to \( M_{\text{max}} \) (equivalently, for the horizon close \( r_{\text{sc}}^{\text{max}} \)) \( T(M) \) also gets modified as compared to (22).

Some of the above results find an independent confirmation. Notable is, first of all, the paper by Sakharov [29] who in 1966 derived the maximal temperature to be of the order of Planck mass. His line of arguments are different from GUP (indeed, in 1966 Hawking radiation has not been discovered yet). In section 5 we also showed that the maximal and minimal temperature can be confirmed from the Schwarzschild–de Sitter metric by using the Stefan–Boltzmann law and a quantum mechanical restriction on \( R \), i.e., \( R > 1/T \).

It is clear that in 1966 a temperature of the order of Planck mass which in units of Kelvin is \( 10^{32} \) K was theoretically inaccessible in the sense that no available theory produced such a temperature. Evidently, extreme situations are asked for here. It is only with the advent of Hawking radiation that \( T_{\text{max}} \) makes phenomenologically sense. Quite similarly \( T_{\text{min}} \sim 10^{-29}K \) (we used the fact that today the preferable value of \( \Lambda \) is given by \( \rho_{\text{vac}} = \Lambda/(8\pi G) \approx 0.7\rho_{\text{crit}} \) requires equally an extreme situation and is a temperature which appears only in connection with black hole evaporation. Sakharov derived his \( T_{\text{max}} \) as a maximum temperature of black body radiation (we can apply it to black holes since the spectrum of the latter is that of black body radiation). It appears that \( T_{\text{min}} \) might enjoy also a broader interpretation as the minimal temperature which can be reached in nature, at least in principle.

If we look back, the speculations that the Planck length is the smallest length in nature were based on purely dimensional analysis. GUP confirms this expectation when applied to Hawking radiation. On the other hand GUP predicts also a maximal length of the order \( r_{\Lambda} \),
again the context of black holes. Quantum gravity effects become not only important at \( l_{\text{ql}} \), but evidently also at \( r_\Lambda \). Such a result could also have been guessed (as opposed to explicitly demonstrated as e.g. in GUP) on the basis of scale analysis and therefore we might speculate that the maximal length, as its minimal counterpart, has a broader meaning as a maximally possible length in nature. If so, there should be interesting consequences for cosmology in our universe which is right now dominated by \( \Lambda \). This is to say, \( r_\Lambda = \frac{1}{\sqrt{\Lambda}} = \frac{1}{\sqrt{3}} \left( \frac{\rho_{\text{crit}}}{\rho_{\text{vac}}} \right)^{-1/2} H_0^{-1} \), which means that the Hubble radius is almost \( r_\Lambda \) at the present epoch.

The example with the Hubble radius is also interesting from the perspective of cosmological coincidences. Not only the Hubble radius is dominated by \( \Lambda \), but it is also worth mentioning that the maximal mass (see equation (47)), which we found, is also close to the mass of the universe. Such coincidences might be of interest in the framework of different theories [36, 37].

Finally, it makes sense to compare our findings with similar results obtained elsewhere.

Appendix A. General solution of a third-order polynomial

In the present paper we have been using many times the parametric solution of a third-order polynomial. For the reader’s convenience and to set up general definition, we outline below the three different cases of the zeros of the third-order polynomial [42]. The standard equation of a third-order polynomial is

\[ x^3 + rx^2 + sx + t = 0. \]  

(A.1)

The reduced form of the third-order equation (A.1), requires the change of variable

\[ y \equiv x + \frac{r}{3} \]  

(A.2)

such that the reduced form is given by

\[ y^3 + py + q = 0. \]  

(A.3)

The corresponding coefficients read

\[ p = s - \frac{r^2}{3}, \quad q = \frac{2}{27} r^3 - \frac{rs}{3} + t. \]

It is necessary to establish some classification criteria for the solutions of the reduced third-order equation. These criteria are based on a parameter \( D \) defined by

\[ D \equiv \left( \frac{p}{3} \right)^3 + \left( \frac{q}{2} \right)^2. \]
A second important parameter $R$ entering the parametric solutions is

$$ R \equiv \text{sign}(q) \sqrt{|p|^3} $$

The solutions are parametrized by an auxiliary angle $\phi$ whose exact definition depends on the signs of $p$ and $D$. We distinguish three cases as follows:

Case (i) $p < 0$, $D \leq 0$

In this case, the auxiliary angle is defined as

$$ \cos \phi = \frac{q}{2R^3} $$

with the corresponding solutions all real and given by

$$ y_1 = -2R \cos \frac{\phi}{3}, \quad y_2 = -2R \cos \left( \frac{\phi}{3} + \frac{2\pi}{3} \right), \quad y_3 = -2R \cos \left( \frac{\phi}{3} + \frac{4\pi}{3} \right). $$

Case (ii) $p < 0$, $D > 0$

In this case, the auxiliary angle is

$$ \cosh \phi = \frac{q}{2R^3} $$

and the corresponding solutions are

$$ y_1 = -2R \cosh \frac{\phi}{3}, \quad y_2 = R \cosh \frac{\phi}{3} + i\sqrt{3}R \sinh \frac{\phi}{3}, $$

$$ y_3 = y_2^* = R \cosh \frac{\phi}{3} - i\sqrt{3}R \sinh \frac{\phi}{3}. $$

Case (iii) $p > 0$, $D > 0$

In this section, we define the auxiliary angle to be

$$ \sinh \phi = \frac{q}{2R^3} $$

with the explicit solutions

$$ y_1 = -2R \sinh \frac{\phi}{3}, \quad y_2 = R \sinh \frac{\phi}{3} + i\sqrt{3}R \cosh \frac{\phi}{3}, $$

$$ y_3 = y_2^* = R \sinh \frac{\phi}{3} - i\sqrt{3}R \cosh \frac{\phi}{3}. $$

Appendix B. Invariant scalar 4-acceleration for static observers

In agreement with [16], a typical static observer will have a 4-velocity given by (we use here natural units)

$$ u^\alpha = \frac{dx^\alpha}{d\tau} = (u^0, 0, 0, 0), \quad u^0 = \frac{dt}{d\tau} = (-g_{00})^{1/2}. $$

The observer’s proper 4-acceleration components will be

$$ a^\alpha = \frac{du^\alpha}{d\tau} = u^\alpha u_\nu^\alpha. $$
Explicitly, this equation reads \( a^\alpha = (u^\alpha, \nu + \Gamma^\alpha_{\sigma \nu} u^\sigma) u^\nu \) or \( a^\alpha = (u^\alpha_0 + \Gamma^\alpha_{00} u^0) u^0 \) for a local static observer with \( u^i = 0 \). In local static coordinates the condition \( u^0_0 = 0 \) is satisfied. Thus we obtain the simple expression \( a^\alpha = \Gamma^\alpha_{00} (\!\!- g^{00})^{-1} \) where we used the fact that \( u^0_0 u^0 = (\!\!- g^{00})^{-1} \).

For the explicit calculation of \( a^\alpha \), it is necessary to evaluate the Christoffel connection and use the fact that the metric is static, i.e., \( g_{\mu \nu, 0} = 0 \). Then we obtain
\[
a^\alpha = \frac{1}{2} g^{\alpha \mu} g_{00, \mu} g_{00}^{-1}.
\]

In the Schwarzschild–de Sitter metric all non-radial components vanish \( a^\theta = a^\phi = 0 \) and the only surviving component is [16]
\[
a^r = \frac{1}{2} g^{rr} g_{00, r} g_{00}^{-1}.
\]

The Schwarzschild–de Sitter metric is given by
\[
g_{rr} = 1 - \frac{2r_s}{r} - \frac{r^2}{3r_A^2}, \quad g^{00} = -g_{rr} = g_{00}^{-1}.
\]

Replacing (B.2) in (B.1) we arrive at
\[
a^r = \frac{1}{2} g_{00, r} = \frac{r_s}{r^2} - \frac{1}{3} \frac{r}{r_A^2}.
\]

The invariant acceleration can be calculated to be
\[
a = \sqrt{g_{\mu \nu} a^\mu a^\nu} = \frac{1}{2} \sqrt{g_{rr}(g_{00} g_{rr})^{-1}} \frac{dg_{00}}{dr},
\]
where we made use of \( g_{00} g_{rr} = -1 \).

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