A perspective on Black Hole Horizons from the Quantum Charged Particle

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Abstract. Black hole apparent horizons possess a natural notion of stability, whose spectral characterization can be related to the problem of the stationary quantum charged particle. Such mathematical relation leads to an “analyticity conjecture” on the dependence of the spectral properties on a complex “fine-structure-constant” parameter, that can reduce the study of the spectrum of the (non-selfadjoint) MOTS-stability operator to that of the (selfadjoint) Hamiltonian of the quantum charged particle. Moreover, this perspective might open an avenue to the spinorial treatment of apparent horizon (MOTS-)stability and to the introduction of semiclassical tools to explore some of the qualitative aspects of this black hole spectral problem.

1. The analogy: black hole apparent horizons and quantum charged particles

Given a black hole apparent horizon section $S$, the systematic study of the spectral problem of the so-called stability operator $L_S$ \cite{1} of marginally outer trapped surfaces (MOTS), namely

$$L_S \psi \equiv \left[ -\Delta + 2\Omega^a D_a - \left( |\Omega|^2 - D_a \Omega^a - \frac{1}{2} R_S + G_{ab} k^a \ell^b \right) \right] \psi = \lambda \psi,$$

has been proposed in \cite{2} as a methodology to explore aspects of the black hole horizon geometry, possibly leading to insights into the black hole stability/instability problem (paraphrased, following Kac’s spectral discussion \cite{3}, as “can one hear the stability of a black hole horizon?”).

The general resolution of the MOTS-spectral problem in (1) represents a challenging task. In this sense, any hint relating this problem to a better known and controlled problem is of clear relevance. This is precisely the context of Ref. [4], where a relation is presented between the MOTS-spectral problem and the study of the stationary states of a spin-0 quantum charged particle moving in the presence of electric and magnetic fields.

To make this statement more precise, we start by briefly describing the terms in the MOTS-stability operator $L_S$. Let us consider a $n$-dimensional spacetime with metric $g_{ab}$, associated Levi-Civita connection $\nabla_a$ and Einstein curvature tensor $G_{ab}$. Let $S$ be a codimension-2 spacelike closed (compact without boundary) surface with induced metric $q_{ab}$. $L_S$ contains both intrinsic and extrinsic geometry elements. The intrinsic geometry ones associated with $q_{ab}$ are $D_a$, $\Delta = D^a D_a$ and $R_S$, respectively the Levi-Civita connection, the scalar Laplacian operator and the Ricci scalar on $S$. Let us also introduce the volume form $\epsilon_{a_1...a_{n-2}}$ with associated measure $dS$. Regarding the extrinsic geometry, we introduce first two null vectors $\ell^a$ and $k^a$ spanning the normal bundle $T^\perp S$, normalized as $k^a \ell_a = -1$. Then the Hájíček form $\Omega_a = -k^b q_a^c \nabla_c \ell_b$.
encodes part of the extrinsic geometry of \( S \), in particular providing a connection in the normal bundle. Physically, \( \Omega_a \) represents an angular momentum density through the (Komar) expression \( J = \int_S \Omega_a \phi^a dS \), where \( \phi^a \) is an axial Killing vector on \( S \). The term \( G_{ab} k^a \ell^b \) is related to ambient spacetime dynamics through the Einstein equations \( G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \). Defining the (outgoing) expansion on \( S \) as \( \theta^{(f)} = q^{ab} \nabla_a \ell_b \), the MOTS condition on \( S \) is characterized as \( \theta^{(f)} = 0 \), whereas the MOTS-stability operator is defined [4] through the variation of this expansion \( L_S \psi \equiv \delta_{\ell k} \theta^{(f)} \), where \( \psi \) is a function on \( S \) characterizing the surface deformation along \( k^a \). Note that \( L_S \) is (formally) non-selfadjoint, namely due to the 2 \( \Omega^a D_a \psi \) term.

Given these elements, and as shown in [4], the MOTS-stability operator is related to the Hamiltonian of the quantum charged particle through the identifications

\[
\Omega_a \leftrightarrow \frac{i e}{\hbar c} A_a \quad , \quad R_S \leftrightarrow \frac{4mc}{\hbar^2} \phi \quad , \quad G_{ab} k^a \ell^b \leftrightarrow - \frac{2m}{\hbar^2} V ,
\]

where \( A_a \) and \( \phi \) correspond to the vector and scalar potentials of given magnetic and electric fields on \( S \), whereas \( V \) represents an external mechanical potential. Indeed, \( L_S \) becomes

\[
\frac{\hbar^2}{2m} L_S \leftrightarrow \hat{H} = - \frac{\hbar^2}{2m} \Delta + \frac{i e}{mc} A^a D_a + \frac{i e}{2mc} D_a A^a + \frac{e^2}{2mc^2} A_a A^a + e\phi + V = \frac{1}{2m} \left( -i h D - \frac{e}{c} \right)^2 + e\phi + V ,
\]

where \( \hat{H} \) is the Hamiltonian of the spin-0 quantum charged particle if the formal parameter \( h \) is interpreted as the Planck constant (over \( 2\pi \)) and \( m \) and \( e \) as the mass and charge of the particle. The main points in [4] are, first, to bring attention to the fact that the derivative and Hájiček terms in [4] can be combined as a perfect square into a single connection term, as follows

\[
L_S \psi = \left( - (D - \Omega)^2 + \frac{1}{2} R_S - G_{ab} k^a \ell^b \right) \psi .
\]

This underlines the role of \( (D_a - \Omega_a) \) as the relevant connection in the problem (actually a connection in the normal bundle \( T^+ S \)). And secondly, pushing forward this simple remark, Ref. [4] establishes a structural connection between the MOTS-stability and quantum charged particle problems beyond the purely formal analogy embodied in the correspondences [2]. Specifically, both problems possess and Abelian gauge symmetry defined by the transformations

\[
\text{Quantum Charged Particle:} \quad A_a \rightarrow A_a - D_a \sigma \quad , \quad \psi \rightarrow e^{i e \sigma / (c h)} \psi \quad , \quad \text{MOTS:} \quad \Omega_a \rightarrow \Omega_a - D_a \sigma \quad , \quad \psi \rightarrow e^{-\sigma} \psi .
\]

The first one defines the compact \( U(1) \)-gauge symmetry of electromagnetism, whereas the second one defines a non-compact \( \mathbb{R}^+ \)-gauge symmetry corresponding to the invariance of the MOTS-geometry under a rescaling of the null vectors (namely a boost-transformation in the normal direction): \( \ell^a = e^{-\sigma} f^a, k^a = e^\sigma k^a \). In particular, this identifies the Hájiček form as the gauge potential of this \( \mathbb{R}^+ \)-abelian symmetry. Moreover, as apparent in [4] this gauge potential \( \Omega_a \) (and therefore the black hole angular momentum) is introduced into the MOTS-problem via a minimal coupling mechanism, i.e. by substitution \( D_a \rightarrow (D_a - \Omega_a) \) in the non-rotating problem. This is exactly the mechanism for switching on magnetic (and electric) fields in the quantum particle. We can summarize these gauge symmetries considerations in the following lemma [4]:

**Lemma 1 (MOTS-gauge transformations).**

Under the null normal rescaling \( \ell^a = f \ell^a, k^a = f^{-1} k^a \), with \( f > 0 \):

1. The expansion and Hájiček form transform as: \( \theta^{(f)} = f \theta^{(f)} \) and \( \Omega_a = \Omega_a + D_a (\ln f) \).
ii) The MOTS-stability operator transforms covariantly: $(L_S)'\psi = f L_S(f^{-1}\psi)$, where $(L_S)'\psi = \delta\psi(-\nu)^{\theta(\ell)}$.

iii) The MOTS-eigenvalue problem is invariant under the additional eigenfunction transformation, $\psi' = f \psi$. That is, $L_S\psi = \lambda \psi$ goes to $(L_S)'\psi' = \lambda \psi'$.

It is interesting to note that the gauge-connection $D^\Omega_a = D_a - \Omega_a$ is intimately related to the “covariant derivative” introduced in the GHP formalism \[5\] to define a notion of “Fermi-Walker” transport of a vector along the surface $S$. In particular, a vector $\gamma k^a$ is “Fermi-transported” along $S$ if $(D_a - \Omega_a)\gamma = 0$ (such transport is in general non-integrable on $S$ and therefore dependent on the path). More generally the GHP formalism, based on a choice of two null directions at each spacetime point and naturally adapted to the study of codimension-2 surfaces, could offer insights into the MOTS-stability problem through its concepts of spin- and boost-weighted quantities (the latter directly connected with the notion of MOTS-gauge symmetry).

The presented analogy between the MOTS-stability problem and the quantum charged particle offers the possibility of transferring tools and concepts from one problem to the other. Here we will comment: on the possibility of addressing the MOTS-spectral problem by importing the expertise in the stationary spectrum of the quantum charged particle through an analytic continuation procedure, on the possibility of introducing semi-classical tools into the study of black horizon geometry and, finally, on an approach to a spinorial formulation of MOTS-stability by mimicking Pauli’s and Dirac’s steps in the introduction of spinors in the quantum charged particle. We will conclude with a brainstorming list of possible directions of further study.

2. MOTS-spectrum analyticity in the fine structure constant $\alpha$

Let us introduce the “fine structure constant” $\alpha \equiv \frac{e^2}{\hbar c}$ and set $\hbar = m = c = 1$. We define the operator family $L[\sqrt{\alpha}]$ on the square root of this “fine-structure-constant” complex parameter $\alpha$, as

\[
L[\sqrt{\alpha}] = -\frac{1}{2}(D - i\sqrt{\alpha}\Omega)^2 - \frac{\alpha}{4}R_S - \frac{1}{2}G_{ab}k^a\theta^b = -\frac{1}{2}\Delta + i\sqrt{\alpha}(\Omega \cdot D + \frac{1}{2}D \cdot \Omega) + \frac{\alpha}{2}\Omega^2 - \frac{\alpha}{4}R_S - \frac{1}{2}G(k, \ell),
\]

so that the quantum charged particle corresponds to a real positive $\alpha = 1$ (normalized as $e^2 = 1$), whereas (half) the MOTS-stability operator corresponds to a real negative fine structure constant $\alpha = -1$. More precisely, we make branch choices $H = L[\sqrt{\alpha} = 1]$ and $L_S[2 = L[\sqrt{\alpha} = -1]]$.

The operator family $L[\sqrt{\alpha}]$ provides an “analytical continuation” in $\sqrt{\alpha}$ from the self-adjoint quantum charged particle Hamiltonian $\hat{H}$ to the non-selfadjoint MOTS-stability operator $L_S$. This naturally raises the following question: can we recover the MOTS-spectrum ($\alpha = -1$) as an analytic extension of the quantum charged particle spectrum ($\alpha = 1$) self-adjoint problem? If this is indeed possible, we can perform the following “self-adjoint trick” to reduce the resolution of the non-selfadjoint spectral problem associated with $L_S$ to a self-adjoint one: first multiply formally $\Omega_a$ by $\sqrt{\alpha}$, then rotate $\sqrt{\alpha}$ in the complex plane to $i\sqrt{\alpha}$ to produce a self-adjoint operator whose spectral problem can be explicitly solved using standard techniques, and finally perform a “back rotation” $\sqrt{\alpha} \rightarrow \frac{1}{i}\sqrt{\alpha}$ in the explicitly obtained eigenvalues.

Answering the question above in its full generality sets a difficult problem in perturbation theory of linear operators \[4\]. We formulate the following conjecture \[4\] as an open problem:

**Analyticity Conjecture.** Given an orientable closed surface $S$ and the one-parameter family of operators $L[\sqrt{\alpha}]$ defined in \[4\], (in the complex $\sqrt{\alpha}$), the MOTS-spectrum ($\sqrt{\alpha} = 1$) can be recovered as an “analytic continuation” of the quantum charged particle spectrum ($\sqrt{\alpha} = 1$).

In order to provide some support on the validity of this conjecture, we consider an explicit example that we can fully solve and that presents the essential qualitative features to be expected
In the generic case. This simple example provides an analogue in our context to the Landau levels for the quantum charged particle in a constant magnetic field in \( \mathbb{R}^3 \). We take as \( S \) a topological sphere \( S^2 \) endowed with the round metric \( g_{ab} = r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \). On \( S^2 \), the Hodge decomposition of the Hájíček form is \( \Omega_a = e^a b D_b \omega + D_a \zeta \). We make the simplest non-trivial choice \( \omega = \sqrt{\alpha} \cos \theta, \zeta = 0 \) (with \( \sqrt{\alpha} \in \mathbb{R} \)) that leads to \( \Omega = \sqrt{\alpha} \sin^2 \theta d\phi \). Finally, we assume vacuum \( (G_{ab} + \Lambda g_{ab} = 8\pi T^a_{ab} = 0) \). With these choices, the relevant terms in (11) are

\[
R_S = \frac{2}{r^2}, \quad 2\Omega^a D_a \psi = \frac{2\sqrt{\alpha}}{r^2} \partial_r \psi, \quad |\Omega|^2 = \frac{\alpha}{r^2} \sin^2 \theta, \quad D^a \Omega_a = 0, \quad G_{ab}k^a \ell^b = \Lambda. \tag{7}
\]

We insert this in Eq. (1), separate variables as \( \Psi = S(\theta)e^{im\phi} \) and write \( \lambda = \lambda_R + i\lambda_I \). The imaginary part of the equation produces \( \lambda_I = 2\sqrt{\alpha}m/r^2 \), whereas the real part \( (x = \cos \theta) \) is

\[
\frac{d}{dx} \left((1 - x^2) \frac{d}{dx}\right) S + \left(((\lambda_R + \Lambda)r^2 + \alpha - 1) - \alpha x^2 - \frac{m^2}{1 - x^2}\right) S = 0. \tag{8}
\]

Solutions to this equation are given [7] by the (prolate) spheroidal harmonics \( S_{lm}(\sqrt{\alpha}, \cos \theta) \), with values \( \lambda_{lm}(\sqrt{\alpha}) = (\lambda_R + \Lambda)r^2 + \alpha - 1 \). The solutions of the MOTS-problem are therefore

\[
\lambda = \frac{\lambda_{lm}(\sqrt{\alpha}) + 1 - \alpha}{r^2} - \Lambda + i\frac{2\sqrt{\alpha}m}{r^2}, \quad \psi_{lm}(\theta, \phi) = S_{lm}(\sqrt{\alpha}, \cos \theta)e^{im\phi}. \tag{9}
\]

To study the corresponding quantum charged particle problem, we make \( \sqrt{\alpha} \rightarrow i\sqrt{\alpha} \) in \( \Omega_a \). Denoting the new eigenvalues as \( \tilde{\lambda} \) and the eigenfunctions by \( \tilde{\psi} \) and repeating the steps above, we first obtain \( \tilde{\lambda}_1 = 0 \) (the operator is now self-adjoint), whereas the equation on \( x \) is now

\[
\frac{d}{dx} \left((1 - x^2) \frac{d}{dx}\right) S + \left(((\tilde{\lambda}_R + \Lambda)r^2 + 2\sqrt{\alpha}m - \alpha - 1) + \alpha x^2 - \frac{m^2}{1 - x^2}\right) S = 0. \tag{10}
\]

This is now the equation for the (oblate) spheroidal harmonics [7] with solutions \( \tilde{S}_{lm}(i\sqrt{\alpha}, \cos \theta) = S_{lm}(i\sqrt{\alpha}, \cos \theta) \) and where the corresponding \( \tilde{\lambda}_{lm}(i\sqrt{\alpha}) = (\tilde{\lambda}_R + \Lambda)r^2 + 2\sqrt{\alpha}m - \alpha - 1 \) satisfy \( \tilde{\lambda}_{lm}(i\sqrt{\alpha}) = \lambda_{lm}(i\sqrt{\alpha}) \) (note that \( \lambda_{lm}(i\sqrt{\alpha}) \) are real numbers [7]). The solution is now

\[
\tilde{\lambda} = \frac{\lambda_{lm}(i\sqrt{\alpha}) + 1 + \alpha - 2\sqrt{\alpha}m}{r^2} - \Lambda, \quad \tilde{\psi}_{lm}(\theta, \phi) = S_{lm}(i\sqrt{\alpha}, \cos \theta)e^{im\phi}. \tag{11}
\]

This is in agreement with the “analyticity conjecture” formulated above, since both eigenvalues and eigenfunctions in the solution (9) to the non-selfadjoint problem are indeed recovered from the solutions of the self-adjoint case (11), when applying back the shift \( \sqrt{\alpha} \rightarrow \frac{1}{i}\sqrt{\alpha} \).

3. Semi-classical tools in the MOTS-spectral problem. Towards an action principle for MOTS-stability

The possibility of reducing the spectral problem of \( L_S \) to that of a selfadjoint operator, admitting in addition the interpretation of a quantum Hamiltonian, opens the possibility of considering a particular semi-classical approach to the study of \( L_S \). Inverting the quantization rule \( p_i \rightarrow -iD_i \) (with \( \hbar = 1 \)), the Hamiltonian \( \tilde{H}(\sqrt{\alpha}) \equiv L[\sqrt{\alpha}] \) leads to the classical Hamiltonian function

\[
H_c[\sqrt{\alpha}](x, p) = (p - \sqrt{\alpha}\Omega)^2 + \frac{1}{2} R_S - G(k, \ell). \tag{12}
\]

Then, we can use \( H_c[\sqrt{\alpha}](x, p) \) as the starting point to obtain approximate solutions to the spectral problem of \( \tilde{H}(\sqrt{\alpha}) \) by employing semi-classical tools such as the WKB techniques [8].
Under the assumption of the validity of the “analyticity conjecture”, the relevant expressions approximating the MOTS-problem would be then obtained by evaluating \( \sqrt{\alpha} \rightarrow -i \).

A relevant question is: can we define an action from which the MOTS-stability problem can be derived variationally? The action for a (complex) scalar field coupled to an electromagnetic field provides the answer in the self-adjoint case, but its straightforward application to \( L_S \) fails. We have not been able to find a real action from which \( L_S \) emerges variationally. On the other hand, considering a complex scalar \( \psi \) on \( S \) we can introduce the following complex action

\[
S = \int_S dS \left( q^{ab}(D_a + \Omega_a)\psi^* (D_b - \Omega_b)\psi + \left( \frac{1}{2} R_S - G_{ab} k^a k^b \right) \psi^* \psi - F_F \psi^* - F_{\psi^*} \psi \right). \tag{13}
\]

Variating independently \( \psi^* \) and \( \psi \), the corresponding Euler-Lagrange equations are

\[
L_S \psi = \left( -(D - \Omega)^2 + \frac{1}{2} R_S - G_{ab} k^a k^b \right) \psi = F_F \psi,
\]

\[
(L_S)^\dagger \psi^* = \left( -(D + \Omega)^2 + \frac{1}{2} R_S - G_{ab} k^a k^b \right) \psi^* = F_{\psi^*}. \tag{14}
\]

Choosing \( F_F = \lambda \psi \), with \( \lambda \) a parameter to be determined, we recover the eigenvalue problem \([I]\). One recovers not only the elliptic problem for \( L_F \), related to the ingoing variation of the outgoing expansion \( L_S \psi = \delta_{-} \psi \theta^{(i)} \), but also the problem \((L_S)^\dagger \psi^* \) related to the outgoing variation of the ingoing expansion: \((L_S)^\dagger \psi^* = \delta_{+} \psi \theta^{(k)} - \kappa (\psi^* \ell) \theta^{(k)}\). This suggests that the relevant object in this context is actually a two-component vector related to second variations of the element of area. “Diagonal” second variations would correspond to Raychaudhuri equations, whereas the “non-diagonal” ones are expressed in terms of the MOTS-stability operator and its adjoint.

4. Spinors and MOTS-stability

Having a spinor (first-order) characterization of MOTS-stability would present various potential interests: i) inner boundary conditions for elliptic problems with a horizon (e.g. in Witten’s proof of mass positivity, approaches to Penrose-like inequalities...), ii) reduction of the spectral problem to that of a first-order operator and, more generally, iii) natural setting for studying MOTS-stability when studying spinorial fields propagating on a black hole spacetime background.

The analogy of the MOTS-stability operator with the Hamiltonian of the quantum charged particle provides a natural way for introducing spinors by mimicking Pauli’s and Dirac’s approaches to particles’ spin. Let us first introduce some notation. Considering a tetrad \( e^a_i \), such that \( g_{ab} e^a_i e^b_j = \eta_{ij} \), assuming a spin-structure we introduce gamma-matrices \( \gamma^a = e^a_i \gamma^i \) acting on spinors \( \Psi \) and satisfying the Clifford algebra \( \{ \gamma^a, \gamma^b \} = 2g^{ab} \mathbf{1} \). In the Riemannian case of the MOTS \( S \), we consider \( \{ \gamma^a, \gamma^b \} = 2q^{ab} \mathbf{1} \). The derivative connection on spinors is

\[
D_\alpha \Psi = \left( D_a + \frac{1}{8} \omega^a_{\beta \gamma} [\gamma_\beta, \gamma_\gamma] \right) \Psi, \quad \text{where} \quad \omega^a_{\beta \gamma} \text{ is the standard spin-connection associated with } e^a_i.
\]

A straightforward approach to recast MOTS-stability in terms of a first-order condition, would be to take the “square-root” of \( L_S \) in the same spirit in which the square root of the Klein-Gordon equation of a field of mass \( m \), namely \((i\hbar)^2 \square + m^2 c^2) \Psi = 0 \), is the Dirac equation \((i\hbar \gamma^i D_i + mc) \Psi = 0 \). However, this does not work for \( L_S \), due to the non-constancy of the corresponding “mass terms”. As an alternative, we consider the second-order Pauli equation starting from the following key remark: the Laplacian (in Euclidean \( \mathbb{R}^3 \) and acting on spinors) can be written in two ways as follows (\( \sigma^i \) are Pauli matrices, the gamma matrices in this setting)

\[
\Delta = D^i D_i = (\sigma^i D_i)^2, \quad \{ \sigma^i, \sigma^j \} = 2\delta^{ij} \mathbf{1} \text{ (Clifford relations)} \tag{15}.
\]

The crucial point is that introducing the magnetic vector potential through minimal coupling actually depends on the starting version we choose for the Laplacian. In particular, the spin-magnetic field coupling term \( \frac{be}{2mc} \sigma^i B_i \), with gioromagnetic factor 2, is recovered when performing
\[ \Delta = (\sigma^i D_i)^2 \rightarrow (\sigma^i(D_i - ie A_i))^2. \]

In the curved MOTS case, Eq. \((15)\) is generalized through (an adaptation of) the Lichnerowicz-Weitzenböck formula (we denote \( F_{ab}^\Omega = D_a \Omega_b - D_b \Omega_a \))

\[
(i \gamma^a(D_a - \Omega_a))^2 = -(D_a - \Omega_a)^2 + \frac{1}{4} R_S + \frac{1}{4} [\gamma^a, \gamma^b] F_{ab}^\Omega. \tag{16}
\]

The MOTS-stability operator can then be written as

\[
L_S = (i \gamma^a(D_a - \Omega_a))^2 + \frac{1}{4} R_S - \frac{1}{4} [\gamma^a, \gamma^b] F_{ab}^\Omega - G_{ab} k^a \ell^b. \tag{17}
\]

From the Pauli equation perspective, the \( F_{ab}^\Omega \) term modifies the “spin-magnetic field” coupling term correcting the “giromagnetic factor” and then indicating that, from this point of view, the MOTS is a composite object. More importantly, although this is still a second-order operator, it indicates the relevance of the first-order operator defined from the codimension-2 Sen connection, \( i \gamma^a(D_a - \Omega_a) \), and its related spectral problem. Finally, Pauli’s second-order equation can be obtained from the Dirac first-order one by taking the (non-relativistic) limit \( c \to 0 \). This remark indicates a formal path to define a first-order spinorial operator from which the MOTS-stability operator \( L_S \) can be recovered through a limit procedure. This will be presented elsewhere \([9]\).

5. Perspectives

The analogy between stable MOTS and quantum particles (stable MOTS behave as charged particles with “negative fine-structure constant”) provides the seed for a research program relying on the transfer of knowledge between black hole and quantum charged particle physics.

We list some possible directions of research: i) self-adjoint “shortcut” to the spectral MOTS-problem through the analyticity conjecture; ii) applications to the MOTS-stability of Kerr; iii) MOTS-spectrum statistics and applications of a “L\(\Sigma\)-spectral zeta function \( \zeta_{L\Sigma}(s) \)” constructed by analytic continuation from the \( \zeta_{L\Sigma}(s) \) of the selfadjoint \( L[\sqrt{\gamma}] \); iv) semiclassical (and dynamical-systems) tools in the study of the MOTS-stability operator, in particular to construct approximate explicit expressions for MOTS-stability in Kerr, as well as generic “high-eigenvalue” asymptotics; v) spinor reformulation of MOTS-stability and applications to inner boundary conditions in elliptic problems relevant for the “mass problem” in General Relativity (positivity, Penrose-like inequalities, quasi-local gravitational mass); vi) variational derivation of MOTS-stability from an action functional and relation to Ginzburg-Landau theory on closed Riemannian manifolds; vii) higher-dimensional black hole horizons, with richer topologies and field content through the Hodge-decomposition \( \Omega = da + \delta \beta + \gamma \) (with \( \gamma \) harmonic). Finally, we mention other complementary issues: gauge-invariant expression of the charged particle ground state from Donsker-Varadhan theory \([4]\), possibility of a MOTS “Aharonov-Bohm-like effect”, signature of quasi-normal modes/superradiance, “second-quantization”-approach (motivated by a “many-particle” treatment of the Schrödinger equation), study of effective MOTS-deformation “degrees of freedom” and possible statistical-mechanics/thermodynamical properties of MOTS.

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