Response to “Comment on ‘Primordial magnetic seed field amplification by gravitational waves’”

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Here we respond to the comment by Tsagas [1] on our paper [2]. We show that the results in that comment are flawed and cannot be used for drawing conclusions about the nature of magnetic field amplification by gravitational waves, and give further support that the results of [2] are correct.

PACS numbers: 98.80Cq

I. INTRODUCTION

In Ref. [2] it was shown, using proper second order covariant gauge-invariant perturbation theory, that the results in Ref. [1] concerning gravitational wave amplification of weak magnetic fields gave amplification rates incorrect by several orders of magnitude. The author of the comment [1] claims that our main result in Ref. [2] does not apply outside the Hubble radius, and that our numerical estimates and conclusions are therefore invalid as a result of neglecting oscillatory terms. Further claims that our approach is not truly gauge-invariant and also not mathematically complete as a result of neglecting the issue of constraints are also made. However, all these claims will be refuted in what follows. Moreover, the comment contains misleading statements (addressing some of our assumptions and approximations) which are in contrast to what was done (and indeed clearly stated) in our paper. In particular, we did not employ ideal magnetohydrodynamics as the comment states; it was only assumed that the electric field at first order is a consequence of requiring an homogeneous magnetic field interacting with gravity waves. Ideal magnetohydrodynamics was only used in Ref. [3].

II. THE ISSUE OF THE ELECTRIC CURL

In section II of the comment [1], the author argues that because of Eq. (1) it is not possible a priori to assume that $\langle \text{curl} E_a \rangle \perp = 0$ initially even though $\text{curl} E_a = 0$ initially. However, it follows from Maxwell’s equations that $\langle \text{curl} E_a \rangle \perp = -\Theta \text{curl} E_a + \text{curl} \text{curl} B_a$, and since initially curl $B_a$ vanishes (and therefore also curl curl $B_a$) because the first-order magnetic field $\dot{B}_a$ is homogeneous, one concludes that also $(\text{curl} E_a) \perp = 0$ when the interaction between gravitational waves and the first-order magnetic field is turned on. When these constraints are used in the governing equation for curl $E_a$ (see Eq. 2 in [1]), one indeed finds that in the spatially flat case the generated electric field stays curl-free if it was initially so.

The author of the comment came to the opposite conclusion by using the relation

$$\text{curl} \text{curl} B_a = -D^2 B_a + \mathcal{R}_{ab} B^b. \tag{1}$$

However, $B_a$ is not a gauge-invariant variable at second order, and thus cannot be used for the purpose of analyzing the appropriateness of the initial conditions. Let us make this important point very clear: the usual splitting of the magnetic field $B_a$ into a first-order homogeneous part and a second-order term, $B_a = \dot{B}_a + B_a^{(2)}$, as the author of the comment assumed, leads to critical inconsistencies when commutation relations are involved (see also Eqs. (15) and (16) in [2]). Taking the spacetime background to be spatially flat and inserting the said splitting in the right hand side of Eq. (1) yields

$$\text{curl} \text{curl} B_a = -D^2 B_a^{(2)} + \mathcal{R}^{(1)}_{ab} \dot{B}_b, \tag{2}$$

where $\mathcal{R}^{(1)}_{ab}$ denotes the first-order contribution to the 3-Ricci tensor. On the other hand, if one first employs the splitting for the term curl $B_a$, and then applies Eq. (1), one obtains

$$\text{curl} \text{curl} B_a = \text{curl} \text{curl} B_a^{(2)} = -D^2 B_a^{(2)} + \mathcal{R}^{(0)}_{ab} B_b^{(2)} = -D^2 B_a^{(2)}, \tag{3}$$

which differs from the result (2). This ambiguity, due to the gauge problem with the magnetic field splitting in relation to the use of the commutator relation, renders Eq. (1) in [1] meaningless and invalidates the conclusion drawn from it.

We stress that we neglected the electric current at all orders in our paper [2], which is potentially physically unsound. However, the inclusion of a first-order current re-
quires the inclusion of a first-order inhomogeneous magnetic field, which was beyond the scope of the paper \[2\]. Currents and velocity perturbations have been explicitly taken into account in our follow-up paper \[3\] by making use of ideal magnetohydrodynamics. In this case, we indeed have \((\operatorname{curl} E^\omega) \neq 0\), it was shown that this term only contributes at very small scales to the generated magnetic field. On large scales, the results for the generated magnetic field agree with the ones found previously (compare Eq. (50) of \[3\] with Eq. (49) of \[2\]).

### III. SCALES AND AMPLIFICATION

When discussing super-horizon scales in section III of his comment, the author correctly points out that the solution Eq. (7) in Ref. \[2\] of the interaction term \(I(\tau)\) in the case of dust is valid as long as \(x = 2^\ell \tau^{1/3}/(a_0 H_0) \ll 1\) holds (similarly for the case of radiation). However, since the gravitational wave number is defined as \(\ell = 2 \pi a/\lambda_{GW}\) and the Hubble length \(\lambda_H = 1/H\), we also have \(x = 2 \pi (\lambda_H/\lambda_{GW})_0 \tau^{1/3}\), and therefore the condition \(x \ll 1\) will sooner or later break down as \(\tau\) grows, even though the ratio \((\lambda_H/\lambda_{GW})_0\) is small for a particular gravity wave mode with wavenumber \(\ell\). The solutions Eq. (4) and Eq. (5) in \[1\] therefore describe the evolution of an initially super-horizon magnetic mode [meaning \((\lambda_H/\lambda_{GW})_0 \ll 1\)] only correctly up to the time of horizon crossing (at best). Thereafter one has to rely on the solutions Eqs. (49) and (50) given in \[2\].

The fact that we did not display the solutions of the generated magnetic field for the dust and radiation case in full detail when dealing with the modes with non-zero wavenumber, \(\ell \neq 0\), seems to have led to some confusion. To arrive at the solutions Eqs. (49) and (50) in \[2\] we used generic initial conditions leading to complicated expressions, and they were thus not stated explicitly in \[2\]; however, all the relevant terms for the amplification have been given and the nature of the non-displayed expressions has been circumscribed in \[2\]. To improve on this, we give in the following the full solution for the total (first- plus second-order) magnetic field in the radiation era assuming initial conditions \(I_{(1)}(\tau = 1) = \sigma_0(\ell) B_0\) and \(I_{(2)}(\ell) = 0\) for the interaction variable. The latter condition simply means that \((\sigma(\ell)/H)_0 = 2 \sigma_0(\ell)\) has been chosen. With these initial conditions, Eq. (50) in \[2\] gives

\[
\frac{B_{\text{Rad}}^{(\ell)}}{B_0} = \left(\frac{a_0}{a}\right)^2 \left\{ 1 + \left(\frac{\sigma}{H}\right)_0 \frac{5}{L^2} \right. \\
+ \frac{a_0}{a} \left(\frac{\sigma}{H}\right)_0 \left[ \cos \left(\frac{L a}{a_0}\right) \frac{5 \sin L - L^2 \sin L - 5 L \cos L}{L^3} \right] \\
\left. + \sin \left(\frac{L a}{a_0}\right) \frac{L^2 \cos L - 5 L \sin L - 5 \cos L}{L^3} \right\},
\]

where \(L \equiv \ell/(a_0 H_0) = 2\pi (\lambda_H/\lambda_{GW})_0\). As expected, the generated magnetic field is proportional to the initial shear anisotropy \((\sigma/H)_0\); it starts to grow from a zero value until it saturates to a constant value (modulo adiabatic decay); this is also evident from FIG. 1. Note that the solution \(4\) is exact and valid for all finite wavelengths at all times. In the infinite-wavelength limit, \(\ell \to 0\), the solution for the magnetic field goes over into

\[
\frac{B_{\text{Rad}}^{(\ell)}}{B_0} = \left(\frac{a_0}{a}\right)^2 \left\{ 1 + \frac{2}{3} \left(\frac{\sigma}{H}\right)_0 \left[ \frac{a_0}{a} - 1 \right] \\
+ \frac{5}{6} \left(\frac{\sigma}{H}\right)_0 \left[ \left(\frac{a_0}{a}\right)^2 - 1 \right] \right\},
\]

which is of course the same as Eq. (47) in \[2\], where identical initial conditions had been used.

A numerical analysis of the gravito-magnetic interaction (cf. FIG. 1) demonstrates that the generated magnetic field starts from zero, undergoes super-adiabatic growth until it begins to oscillate around a value which depends on the initial conditions but is the same during both the radiation or dust/reheating eras. Whilst the super-adiabatic growth is approximately described by the infinite-wavelength solutions Eqs. (46) and (47) in \[2\], the saturation effect is only present in the finite-wavelength solutions Eqs. (49) and (50) in \[2\] [see also Eq. (4) above].

What then is the emerging magnetic field at the end of the day? Returning to Eq. (3) and dividing by the energy density of the background radiation, one obtains

![FIG. 1: The dimensionless induced magnetic fields plotted against dimensionless time \(\tau\) for dust and radiation, taking \(L = 0.01\). For given initial conditions the generated magnetic field saturates at a value described by our result (6). The super-adiabatic growth is approximately depicted by the infinite-wavelength solutions (46) and (47) in \[2\].](image.png)
where the wavenumber indices have been suppressed and the resonant condition \( \lambda_{GW} \sim \lambda_B \) — the gravity wave-length \( \lambda_{GW} \) matches the size \( \lambda_B \) of the magnetic region — has been used. This agrees with our result Eq. (51) given in [2]. As pointed out in our paper, the case of dust is very similar and leads to the same result \( \text{Eq. (6)} \) for the maximally resulting magnetic field.

It is plain to see that in the light of the above (being in unison with comments already made in our paper [2]), the criticism brought forward by the author of the comment (see Eqs. (8) and (9) therein) is not correct. Also, Eq. (9) of [1] is algebraically wrong and cannot be used to draw conclusions about the validity of the results in [2].

IV. GAUGE-INVARIANCE AND LINEARITY

The author of the comment claims that obtaining the second-order generated magnetic field via integration of the gauge-invariant variable \( \beta_a = \dot{B}_{<a>} + 2\Theta B_a/3 \) is in itself not a gauge-invariant procedure. However, if we expand the magnetic field \( B_a = B_a^{(1)} + B_a^{(2)} + \ldots \) into first, second and higher order parts, then we obtain \( \beta_a = \dot{B}_{<a>}^{(2)} + 2\Theta B_a^{(2)}/3 \), and \( \beta_a \) thus evidently only describes the second-order part (higher order terms are neglected as usual). To get \( B_a^{(2)} \) from an integration is harmless since, as we pointed out in section II above and also in section IIC of our paper [2], the gauge issue arises if commutator relations have to be calculated, which is not the case here.

The author of the comment further writes in the second paragraph of section IV [1] that there was no new information in our second-order approach in comparison to his own first-order approach to the problem at hand [4]. This makes little sense, since the two methods are fundamentally different in nature, and also since the governing equations for the generated magnetic field differ in both cases. Moreover, when solving numerically the relevant weak-field equations [see Eqs. (61) and (62) in [2]] for the magnetic field, taking identical initial conditions as discussed above, one obtains different results (cf. FIG. 2). In contrast to the magnetic field obtained in our second-order approach (see FIG. 1), the weak-field magnetic field shows large oscillations but no saturation.

V. SUMMARY

In this response to the criticism brought forward by the comment [1] of Tsagas, we pointed out the following:

- The author’s argumentation regarding the curl of the electric field is untenable since it is based on a relation (Eq. (1) in [1]) which is not valid at second order;
- The setting we employed was not that of ideal magnetohydrodynamics;
- Further strengthening our main results in [2] with numerical experiments demonstrates that there was no improper assessment of scale in [2].
- Equation (9) in [1] is algebraically wrong, and can
Fig. 2: The generated magnetic field in the weak-field approximation in dimensionless units. The induced magnetic fields are plotted against the dimensionless time variable $\tau$ for the dust and radiation cases taking $L \equiv \ell/(a_0 H_0) = 2\pi(\lambda_H/\lambda_{GW})_0 = 0.01$.

Thus not be used for drawing conclusions about the validity of the expressions in [2].

- Although our formalism explicitly makes use of second order perturbation theory, the author of the comment claims that our numerical results are comprised because we neglected a third-order term [see (10) in [1]];

- The generated magnetic field obtained by means of the weak-field approximation (see [1]) differs significantly from the one obtained with our formalism, the former shown to give rise to unreasonable amplification rates;

- The second-order variable $\beta_a = \dot{B}_{<a>} + 2\Theta B_a/3$ contains only the second-order magnetic field $B_a^{(2)}$, and that an integration of $\beta_a$ with respect to time yields the evolution of $B_a^{(2)}$.

In conclusion, the criticism contained in the comment has thus been shown to be unfounded. Rather, it should be pointed out that, if treated using the proper gauge-invariant covariant second order perturbation theory, the interaction between magnetic fields and gravitational waves can give rise to interesting effects and be a possible source of a boost of weak seed fields as energy is transferred between the different degrees of freedom.

[1] C.G. Tsagas, [gr-qc/0503042](http://arxiv.org/abs/gr-qc/0503042), to appear in Phys. Rev. D (2007).

[2] G. Betschart, C. Zunckel, P. K. S. Dunsby and M. Marklund, Phys. Rev. D 72, 123514 (2005).

[3] C. Zunckel, G. Betschart, P. K. S. Dunsby and M. Marklund, Phys. Rev. D 73, 103509 (2006).

[4] C. G. Tsagas, P. K. S. Dunsby and M. Marklund, Phys. Lett. B 561, 17 (2003).

[5] Observe that in section V. of [2] the suffix 0 denoted the end of reheating, whereas here it has the suffix $RH$. 
