AN ISOMETRY THEOREM FOR GENERALIZED PERSISTENCE MODULES

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Abstract. In recent work, generalized persistence modules have proved useful in distinguishing noise from the legitimate topological features of a data set. Algebraically, generalized persistence modules can be viewed as representations for the poset algebra. The interplay between various metrics on persistence modules has been of wide interest, most notably, the isometry theorem of Bauer and Lesnick for (one-dimensional) persistence modules. The interleaving metric of Bubenik, de Silva and Scott endows the collection of representations of a poset with values in any category with the structure of a metric space. This metric makes sense for any poset, and has the advantage that post-composition by any functor is a contraction. In this paper, we prove an isometry theorem using this interleaving metric on a full subcategory of generalized persistence modules for a large class of posets.

1. Introduction

1.1. Persistent Homology. Informally, a generalized persistence module is a representation of a poset $P$ with values in a category $D$. That is, if $D$ is a category, a generalized persistence module $M$ with values in $D$ assigns an object $M(x)$ of $D$ for each $x \in P$, and a morphism $M(x \leq y)$ in $\text{Mor}_D(M(x), M(y))$ for each $x, y \in P$ with $x \leq y$ satisfying

$$M(x \leq z) = M(y \leq z) \circ M(x \leq y)$$

whenever $x, y, z \in P$ with $x \leq y \leq z$.

Perhaps surprisingly, the study of such objects is useful in topological data analysis. Persistent homology uses generalized persistence modules to attempt to discern the topological properties of a finite data set. We briefly summarize the algorithm applied to a point cloud of data in the persistent homology setting. This will lead to one-dimensional (generalized) persistence modules, where the poset $P = (0, \infty)$ or $\mathbb{R}$. This part of the discussion corresponds to $F$ and then $H \circ F$ in our schematic below. For a more extensive introduction, see [ZC05], [ELZ02] or [Oud15]. The typical workflow for persistent homology is as follows:

![Diagram](https://via.placeholder.com/150)

(In the above, $\text{Simp}$ denotes the category of abstract simplicial complexes) Suppose, for example, we wish to decide whether a data set $D \subseteq \mathbb{R}^2$ should be more correctly interpreted as an annulus or a disk. In order to decide between the two candidates, one calculates the homology of a filtration of simplicial complexes associated to the data set. This uses the Vietoris-Rips complex $(C_\epsilon)_{\epsilon > 0}$.

Specifically, for each $\epsilon > 0$, we let $C_\epsilon$ be the abstract simplicial complex whose $k$-simplices are determined by data points $x_1, x_2, \ldots, x_{k+1} \in D$ where $d(x_i, x_j) \leq \epsilon$ for all $1 \leq i, j \leq k + 1$. Clearly, for $\sigma \leq \tau$ in $(0, \infty)$, there is an inclusion of simplicial complexes $C_\sigma \hookrightarrow C_\tau$, thus we obtain a filtration of simplicial complexes indexed by $(0, \infty)$. Therefore, the assignment $F : \epsilon \rightarrow C_\epsilon$ is a
representation of the poset $(0, \infty)$ (in fact, a $P$-space) taking values in $Simp$. That is to say, $F$ is a generalized persistence module for $P = (0, \infty)$ and $D = Simp$. Since we wish to distinguish between an annulus and a disk, we apply the first homology functor $H_1(\cdot, K)$ to $F$ (where $K$ is some field), to obtain the representation of $P$ with values in $K$-mod, $\epsilon \to H_1(C_\epsilon, K)$.

Thus, the assignment $H_1(\cdot, K) \circ F$ given by $\epsilon \to H_1(C_\epsilon, K)$ is a one-dimensional persistence module. As $\epsilon$ increases generators for $H_1$ are born and die, as cycles appear and become boundaries. In persistent homology, one takes the viewpoint that true topological features of the data set can be distinguished from noise by looking for generators of homology which "persist" for a long period of time. Informally, one "keeps" an indecomposable summands of $H_1(\cdot, K) \circ F$ when it corresponds to a wide interval. Conversely, cycles which disappear quickly after their appearance (narrow ones) are interpreted as noise and disregarded.

This technique has been widely successful in topological data analysis (see, for example, [Car09], [CSEH07], [CdO12], [SG07], [CIdSZ08], [CCR13], [HNH+16], and [GPCI15]). Typically, the category of persistence modules with values in $K$-mod is given a metric-like structure. So-called soft stability theorems, which involve the continuity of the composition $H \circ F$ in the schematic, have been proven. Philosophically, these results have established the utility of this method from the perspective of data analysis (see, for example, [CSEH07]). Hard stability theorems, on the other hand, concern the continuity of $J$ in our schematic.

1.2. Algebraic Stability. One special type of hard stability theorem is an algebraic stability theorem. In such a theorem, one endows a collection of generalized persistence modules with two metric structures, and an automorphism $J$ is shown to be a contraction or an isometry. This situation can be fit into our previous workflow diagram by choosing the topological space of invariants to be the collection of generalized persistence modules itself, endowed with the alternate metric structure. Of particular interest is the case when $J$ is the identity function and the metrics are an interleaving metric and a bottleneck metric. Algebraic stability theorems of this type are common (see [Les11], [BL16], [BL13], and [CZ09]). While in the literature, the word "interleaving" is frequently used to describe slightly different metrics, we believe that the interleaving metric suggested by Bubenik, de Silva and Scott (see [BdS13]) has the advantage of being both most general, and categorical in nature. This interleaving metric makes sense on any poset $P$, and reduces to the interleaving metric of [BL13] when $P = (0, \infty)$. Alternatively, a bottleneck metric is nothing more than a way of extending a metric defined on a set $\Sigma$ to the collection of all $\mathbb{Z}_{\geq 0}$-valued functions with finite support on $\Sigma$. In this context, this is applied to the decomposition of a generalized persistence module into its indecomposable summands with their corresponding multiplicities.

1.3. Connections to Finite-dimensional Algebras. This paper concerns algebraic stability studied using techniques from the representation theory of algebras. Such representations appear because one-dimensional persistence modules arising from data always admit the structure of a representation of a finite totally ordered set. This fact comes from the simple observation that the one-dimensional persistence module given by $F : \epsilon \to C_\epsilon$ is necessarily a step function. More precisely, let

$$P_n = \{\epsilon_1 < \epsilon_2 < \ldots < \epsilon_n\} = \{\epsilon \in (0, \infty) : C_\epsilon \neq \lim_{\tau \to \epsilon^-} C_\tau\}.$$  

By definition, $F$ is constant on all intervals of the form $[\epsilon_i, \epsilon_{i+1})$. Thus, clearly, both $F$ and $H_1(\cdot, K) \circ F$ admit the structure of a generalized persistence module for $P = P_n$. When we restrict the structure of a one-dimensional persistence module to $P_n$, we say informally that we are discretizing. In this sense, generalized persistence modules for finite totally ordered sets are the discrete analogue of one-dimensional persistence modules. At this point the authors wish to point out two issues arising when one discretizes. First, a finite data set $D$ gives rise to not only a generalized persistence module, but also to its algebra. Thus, a priori two persistence
modules may not be able to be compared simply because they are not modules for the same algebra. Second, information about the width of the interval \([\epsilon_i, \epsilon_{i+1})\) in relevant to the analysis, but seems to be lost. Both of these issues are not addressed in this paper, though they are dealt with successfully in [MM17].

While one-dimensional persistence modules will always discretize to a generalized persistence module for a finite totally ordered set, representations of many other infinite families of finite posets also have a physical interpretation in the literature (see [BL16], [CZ09], [EH14]). For example, multi-dimensional persistence modules (see [CZ09]) will discretize in an analogous fashion to representations of a different family of finite posets. This is relevant because there is a categorical equivalence between the generalized persistence modules for a finite poset \(P\) with values in \(K\)-mod, and the module category of the finite-dimensional \(K\)-algebra \(A(P)\), the poset (or incidence) algebra of \(P\). The module theory (representation theory) of such algebras has been widely studied (see, for example [ACMT05], [Bac72], [Cib89], [Fei76], [Kle75], [BdlPS11], [Lou75], [Naz81], [Yuz81], [IK17], and many others). Thus, by passing to the jump discontinuities of a filtration of simplical complexes one may apply techniques from the representation theory of finite-dimensional algebras.

This perspective, however, suggests the need for caution. While it is well-known that the set of isomorphism classes of indecomposable modules for the algebra \(A(P_n)\) is finite, this situation is far from typical. In fact, for a generic finite poset \(P\), the representation theory of the algebra \(A(P)\) is undecidable in the sense of first order logic.

In the above, generic means for all but those on a known list. In particular, for all \(P\) not on the list, the algebra \(A(P)\) has infinitely many isomorphism classes of indecomposable modules. Indeed, this is the case for the algebras associated to many of the posets which arise when one discretizes in a situation pertinent to topological data analysis. This typically happens for multi-dimensional persistence modules (see [CZ09]), for example.

Because of this, studying arbitrary generalized persistence modules in complete generality is hopeless. Indeed, if a possibly infinite poset discretizes to a finite poset \(P\), and the module category for \(A(P)\) is undecidable, the same holds for generalized persistence modules for the original poset. Moreover, our intuition from persistent homology tells us that indecomposable modules should come with a notion of widths which can be measured, in order to decide whether they should be kept or interpreted as noise. In order to reconcile these two issues, we pass from the full category of all \(A(P)\)-modules, to a more manageable full subcategory where we can make sense of what it means for indecomposable modules to be ”wide.” This suggests the following template for a representation-theoretic algebraic stability theorem:

Let \(P\) be a finite poset of some prescribed type, and let \(K\) be a field. Choose a full subcategory \(\mathcal{C} \subseteq A(P)\)-mod, and let \(D\) and \(D_B\) be two metrics on \(\mathcal{C}\) where:

1. \(D\) is the interleaving distance of [BdS13] restricted to \(\mathcal{C}\), and
2. \(D_B\) is a bottleneck metric on \(\mathcal{C}\) which incorporates some algebraic information.

Prove that

\[
(\mathcal{C}, D) \xrightarrow{\text{Id}} (\mathcal{C}, D_B)
\]

is an isometry.

In addition, the class of posets covered should contain all the posets \(P_n, n \in \mathbb{N}\). In addition, the category \(\mathcal{C}\) should reduce to the full module category when \(P = P_n\). When this is the case, the theorem should be a discrete version of the classical isometry theorem [BL13]. If possible, elements of \(\mathcal{C}\) should have a nice physical description.
1.4. Main Results. Our algebraic stability theorem is stated below.

**Theorem 1.** Let $P$ be an $n$-Vee and let $C$ be the full subcategory of $A(P)$-modules consisting of direct sums of convex modules. Let $(a, b) \in \mathbb{N} \times \mathbb{N}$ be a weight and let $D$ denote interleaving distance (corresponding to the weight $(a, b)$) restricted to $C$.

Let $W(M) = \min\{\epsilon : \text{Hom}(M, M_{\Lambda}) = 0, \Lambda \in \mathcal{T}(P), h(\Gamma), h(\Lambda) \leq \epsilon\}$, and let $D_B$ be the bottleneck distance on $C$ corresponding to the interleaving distance and $W$. Then, the identity is an isometry from $(C, D) \xrightarrow{Id} (C, D_B)$.

Of course, much of the language in the Theorem has not yet been defined. The collection of $n$-Vees generalizes $\{P_m\}$ in the sense that a 1-Vee is exactly a finite totally ordered set. Such a theorem is very much in the flavor of classical algebraic stability theorems (see [BL16],[Les11],[BdS13]). It is common, for example, for the bottleneck metric restricted to indecomposables to be the interleaving metric. When $P$ is a 1-Vee and the choice of weight is $(1, 1)$, Theorem 1 is a discrete analogue of the standard isometry theorem of Bauer and Lesnick [BL13], though with a different notion of width, and with the interleaving metric of [BdS13]. Indeed, both Theorems 2 and 1 can be viewed as extensions of the discrete analogue to the classical isometry theorem [BL13].

In the statement Theorem 1, $W$ corresponds to our choice for the width function. We take our inspiration for $W$ from [BL16] and [Les11], but do not use the thickness of the support of a module $M$ in a direction. Instead, our width is defined in terms of algebraic conditions, although the two agree in the case of one-dimensional persistence modules. Our choice of the category $C$ is natural both from the perspective of persistent homology and from that of representation theory. Once some parameters are fixed, the collection of interleavings between two elements of $C$ has the structure of an affine variety (see Proposition 37 and Examples 11, 10, and 12). The interleaving distance between two generalized persistence modules is the smallest value of a parameter for which the corresponding variety of interleavings in non-empty (see Remark 4 and Example 12).

While certainly motivated by stability theorems in topological data analysis, the authors take the viewpoint that such a theorem need not make explicit reference to a data set. This paper will be organized as follows: in Section 3 we give a brief survey of the relevant background information, in Section 4 we define the class of posets in which we will work, and in Section 5 we investigate the action of the collection of translations on the set of homomorphisms between convex modules. Then, in Sections 6 we concentrate on 1-Vees, that is, totally ordered finite sets. In Section 6 in particular, we owe much to Bauer and Lesnick [BL13]. Then, in Section 7 we prove our main results. After the proof of the main results we include some examples.

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3. Preliminaries

3.1. Generalized Persistence Modules. Recall that if $P$ is a poset and $D$ is a category, a generalized persistence module $M$ with values in $D$ assigns an object $M(x)$ of $D$ for each $x \in P$, and a morphism $M(x \leq y)$ in $\text{Mor}_D(M(x), M(y))$ for each $x, y \in P$ with $x \leq y$ satisfying

$$M(x \leq z) = M(y \leq z) \circ M(x \leq y)$$

whenever $x, y, z \in P$ and $x \leq y \leq z$.

Let $D^P$ denote the collection of generalized persistence modules for $P$ with values in $D$. If $F, G \in D^P$, a morphism from $F$ to $G$ is a collection of morphisms $\{\phi(x)\}$, with $\phi(x) \in \text{Mor}_D(F(x), G(x))$ for all $x \in P$, such that for all $x \leq y$ we have a commutative diagram below for each $x \leq y$ in $P$.
With these morphisms, \( \mathcal{D}^P \) is a category. Equivalently, one could regard the poset \( P \) as a thin category. Then, a generalized persistence module will correspond to a covariant functor from \( P \) to \( \mathcal{D} \), and morphisms in \( \mathcal{D}^P \) will be natural transformations. When \( P \) is \((0, \infty) \) or \( \mathbb{R} \), we say that the elements of \( \mathcal{D}^P \) are one-dimensional persistence modules. In this paper, \( \mathcal{D} \) will always be \( \text{Simp} \) or \( K \)-mod.

### 3.2. Representation Theory of Algebras

In this subsection we give a brief summary of \( K \)-algebras and their representations (modules). For a more expansive introduction, see [ARS97], [Ben98a], [Ben98b]. Throughout, let \( K \) denote a field. If \( R \) is a \( K \)-algebra, by an \( R \)-module, we mean a finite-dimensional, unital, left \( R \)-module. The category \( R \text{-mod} \) consists of \( R \)-modules together with \( R \)-module homomorphisms.

Recall that an \( R \)-module \( M \) is indecomposable if it is not isomorphic to a direct sum of two of its proper submodules. The category \( R \text{-mod} \) is an abelian Krull-Schmidt category. That is, every module can be written as a direct sum of indecomposable modules in a unique way up to order and isomorphism. Moreover, the decomposition of modules is compatible with respect to homomorphisms in the following sense.

**Proposition 1.** Let \( R \) be a \( K \)-algebra, and let \( M, N \) be \( R \)-modules. Say, \( M \cong \bigoplus M_i \) and \( N \cong \bigoplus N_j \). Then, as vector spaces,

\[
\text{Hom}(M, N) \cong \bigoplus_{i,j} \text{Hom}(M_i, N_j).
\]

This says that any module homomorphism can be \( f : M \to N \) can be factored into a matrix of module homomorphisms \( f_{ij} : M_i \to N_j \).

### 3.2.1. Bound Quivers and their Representations

**Definition 2.** A quiver \( Q = (Q_0, Q_1, t, h) \) is an ordered tuple, where \( Q_0, Q_1 \) are disjoint sets, and \( t, h : Q_1 \to Q_0 \).

We call elements of \( Q_0 \) vertices, and elements of \( Q_1 \) arrows. The functions \( t \) and \( h \) denote the tail (start) and head (end) of the arrows. Thus, clearly \( Q \) is exactly a directed set. We will always suppose the sets \( Q_0, Q_1 \) are finite.

**Example 1.** Below are two quivers.
Quiver $A$ corresponds to $Q_0 = \{1, 2, 3, 4, 5\}, Q_1 = \{a, b, c, d\}$, for an appropriate choice of the functions $h, t$. Similarly, quiver $B$ corresponds to the sets $Q_0 = \{1, 2, 3, 4\}$, and $Q_1 = \{a, b, c, d, e, f, g\}$.

**Definition 3.** A path is a sequence of arrows $p = a_1 \ldots a_n$ where $t(a_i) = h(a_{i+1})$. The length of the path is the number of terms in the sequence $p$. In addition, at each vertex $i$ there is a "lazy" path $e_i$ of length 0 at the vertex $i$. We extend the functions $h, t$ to paths, by defining $t(p) = t(a_n)$ and $h(p) = h(a_1)$. In addition, $t(e_i) = h(e_i) = i$. An oriented cycle is a path $p$ of length greater than or equal to one with $t(p) = h(p)$.

Consider quiver $B$ in Example 1. Then $g$ and $abcd$ are oriented cycles, while $ggf$ is a path which is not an oriented cycle. Quiver $A$ has no oriented cycles.

**Definition 4.** A representation $V$ of a quiver $Q$ is a family $V = (\{V(i)\}_{i \in Q_0}, \{V(a)\}_{a \in Q_1})$, where $V(i)$ is a $K$-vector space for every $i \in Q_0$, and $V(a) : V(t(a)) \rightarrow V(h(a))$ is a $K$-linear map for every $a \in Q_1$.

For a fixed quiver $Q$ and field $K$, the collection of all representations of $Q$ is a category with morphisms given below.

**Definition 5.** Let $Q$ be a quiver, and let $V, W$ be representations of $Q$. A morphism from $V$ to $W$, $\phi : V \rightarrow W$ is a collection of linear maps $\{\phi(i)\}_{i \in Q_0}$ with $\phi(i) : V(i) \rightarrow W(i)$ such that the diagram below commutes for all $a \in Q_1$

$$
\begin{array}{ccc}
V(t(a)) & \xrightarrow{V(a)} & V(h(a)) \\
\phi(t(a)) \downarrow & & \downarrow \phi(h(a)) \\
W(t(a)) & \xrightarrow{W(a)} & W(h(a))
\end{array}
$$

We denote by $Rep(Q)$, the category of $K$-representations of the quiver $Q$. When $\phi : V \rightarrow W$ is a morphism from $V$ to $W$ and $\phi(i)$ is invertible for all $i$, then we say $\phi$ is an isomorphism. If this is the case, we say that $V$ and $W$ are isomorphic.

**Definition 6.** If $V$ is a representation of a quiver $Q$ we say the support of $V$ is the set of all vertices $i \in Q_0$, such that $V(i)$ is not the zero vector space.

More generally, the *dimension vector of $V$* is the non-negative integer vector $(\dim_K(V(i)))$. Viewing the dimension vector of $V$ as a function from $Q_0$ to the non-negative integers, the support of $V$ is exactly the support of this function.

**Definition 7.** Let $Q$ be a quiver. The path algebra $KQ$ is the $K$-vector space with basis consisting of all paths (including those of length zero). We define multiplication in $KQ$ as the $K$-linear extension of concatenation of paths.

That is, if $p, q$ are paths, then $p \cdot q = pq$, if $pq$ is a path, and zero otherwise. If $t(p) = a, h(p) = b$, we define $p e_a = p = e_b p$. By extending $K$-linearly, we obtain a ring structure on $KQ$. It is easy to see that $KQ$ is finite-dimensional if and only if $Q$ has no oriented cycles.

The (two-sided) ideal $J$ in $KQ$ generated by the arrows is the radical of the ring $KQ$. For $n \in \mathbb{N}$, let $J^n$ denote the $n$th power of the radical $J$. When $Q$ has no oriented cycles, $J$ is a nilpotent ideal. We say an ideal $I$ is admissible if $J^n \subseteq I \subseteq J^2$, for some $n$. The elements of $I$ are called *relations*. If $Q$ is a quiver, and $I$ is an admissible ideal, we say $(Q, I)$ is a *bound quiver*.

**Definition 8.** Let $(Q, I)$ be a bound quiver. Then, $Rep(Q, I)$ denotes the collection of all representations $V$ in $Rep(Q)$ satisfying all the relations in $I$.

Then, $Rep(Q, I)$ with morphisms in $Rep(Q)$ forms a category.
Proposition 9. Let \((Q, I)\) be a bound quiver. Then, there exists a natural equivalence between \(\text{Rep}(Q)\) and \(KQ\)-mod, that restricts to \(\text{Rep}(Q, I)\) and \(KQ/I\).

Gabriel proved that when \(K\) is algebraically closed any finite-dimensional \(K\)-algebra is (Morita equivalent to) an algebra of the form \(KQ/I\). Thus, up to equivalence, the study of the module category of \(K\)-algebras (when \(K\) is algebraically closed) is the study of representations of bound quivers.

3.2.2. Poset Algebras. We will now define the algebra whose module theory is equivalent to generalized persistence modules with values in \(K\)-mod. This will be \(A(P)\), the poset algebra (or incidence algebra) of the poset \(P\).

Definition 10. Let \(P\) be a finite poset. Let \(Q_P\) be the quiver with \(Q_0 = P\). There is an \(a \in Q_1\) with \(t(a) = x, h(a) = y\) if,

(i.) \(x < y\), and
(ii.) there is no \(t \in P\), with \(x < t < y\).

The quiver \(Q_P\) is called the Hasse quiver of the poset \(P\). The Hasse quiver of \(P\) is exactly the lattice of the poset with arrows corresponding to minimal proper relations.

Example 2. The quivers below are the Hasse quivers for three finite posets.

Quiver A in Example 1 is also the Hasse quiver of a poset.

Note that if \(Q_P\) is the Hasse quiver for \(P\), and there is an arrow going from one vertex to another, it is necessarily unique. Because no ambiguity is possible, we may draw the Hasse quiver of a poset with arrows unlabeled. A finite quiver \(Q\) is the Hasse quiver of a poset if and only if has no oriented cycles, and for all arrows \(p, q\) in \(Q_P\) satisfying \(t(p) = t(q)\) and \(h(p) = h(q)\), then \(p = q\). Note that \(x \leq y\) in \(P\) if and only if there is a path \(q\) in the Hasse quiver with \(t(q) = x\) and \(h(q) = y\).

Definition 11. Let \(KQ_P\) denote the Hasse quiver of the poset \(P\). Then, the parallel ideal \(I_P\) is the two-sided ideal in \(KQ_P\) generated by all the relations equating any two paths \(p, q\) in \(Q_P\) satisfying \(t(q) = t(p)\) and \(h(q) = h(p)\).

For example, the poset in Example 2 E has parallel ideal generated by the element \(ca - db\). The Hasse quiver for C and D have trivial parallel ideals.
Definition 12. The poset algebra $A(P)$ is the bound quiver algebra

$$A(P) = KQ_P/I_P.$$  

By the equivalence in Proposition 9, we now see that the generalized persistence modules for a finite set $P$ with values in $K$-mod are the same as the modules for the poset algebra $A(P)$. This is because $Rep(Q_P, I_P)$ corresponds exactly to the definition of generalized persistence modules for $D = K$-mod, where the commutativity of the triangle below corresponds to the statement that $M$ satisfies all relations in $I_P$.

\[
\begin{array}{ccc}
M(x) & \xrightarrow{M(x \leq z)} & M(z) \\
\downarrow M(x \leq y) & & \downarrow M(y \leq z) \\
M(y) & & \\
\end{array}
\]

Thus, from this point forward, we pass freely between generalized persistence modules and modules for the corresponding poset algebra.

3.2.3. Representation Type. For this subsubsection only, let $K$ be algebraically closed. Informally, an arbitrary $K$-algebra $A$ is said to be of wild representation type if its module category contains a copy of the module category of all finite dimensional $K$-algebras. Rather surprisingly, this happens frequently.

Example 3. Poset $D$ in Example 2 is a poset whose algebra is of wild representation type.

When $A$ is of wild representation type, the classification of its modules up to isomorphism is hopeless. In contrast, the module category for $A$ may be of finite type, or of tame type. Finite representation type means that there are a finite number of isomorphism classes of indecomposable modules (like $A(P_n)$). Informally, if $A$ has tame representation type there are infinitely many isomorphism classes of indecomposable $A$-modules (though they are parametrized reasonably). It has been shown that every algebra is either finite, tame or wild. In particular, complete lists of poset algebras of finite representation type are known (see [Lou75], [DS]). The posets that arise when one discretizes generalized persistence modules for $P = \mathbb{R}^n$ are typically wild.

3.3. Interleaving Metrics on $P$ and $P^+$. We begin with the construction of the interleaving metric of Bubenik, de Silva and Scott (see [BdS13]).

Definition 13. Let $P$ be a finite poset and $\mathcal{T}(P^-)$ be the collection of endomorphisms of the poset $P$ with the additional property that $\Lambda p \geq p$ for all $p \in P$. We call the elements of $\mathcal{T}(P^-)$ translations.

Explicitly, a function $\Lambda : P \to P$ is an element of $\mathcal{T}(P^-)$ if and only if

$$x \leq y \implies \Lambda x \leq \Lambda y, \text{ and } \Lambda p \geq p, \text{ for all } p \in P.$$  

It is easy to see that the set $\mathcal{T}(P^-)$ is itself a poset under the relation $\Lambda \leq \Gamma$ if for all $p \in P$, $\Lambda p \leq \Gamma p$. Moreover $\mathcal{T}(P^-)$ is totally ordered if and only if $P$ is totally ordered, and $\mathcal{T}(P^-)$ is a monoid under functional composition. Let $d$ be any metric on a finite poset $P$, we define a height function $h = h(d)$ on $\mathcal{T}(P^-)$.

Definition 14. For $\Lambda \in \mathcal{T}(P^-)$ set $h(\Lambda) = \sup\{d(x, \Lambda x) : x \in P\}$

Of course, since $P$ is finite, we may replace supremum with maximum. Proceeding as in [BdS13], let $\mathcal{D}$ be any category. Then, $\mathcal{T}(P^-)$ acts on $\mathcal{D}^P$ on the right by the formulae

$$(F \cdot \Gamma)(p) = F(\Gamma p), \text{ and } (F \cdot \Gamma)(p \leq q) = F(\Gamma p \leq \Gamma q), \text{ for } \Gamma \in \mathcal{T}(P^-), F \in \mathcal{D}^P.$$  

Similarly, $\mathcal{T}(P^-)$ acts on morphisms in $\mathcal{D}^P$, by acting inside the argument.
Definition 15. Let $F, G \in \mathcal{D}^P$ and let $\Gamma, \Lambda$ be translations on $P$. A $(\Gamma, \Lambda)$-interleaving between $F$ and $G$ is a pair of morphisms in $\mathcal{D}^P$, $\phi : F \rightarrow G\Lambda$, $\psi : G \rightarrow F\Gamma$ such that the following diagrams commute:

\[
\begin{array}{ccc}
F & \xrightarrow{\phi} & F\Gamma \Lambda \\
\downarrow & & \downarrow \psi \\
G\Lambda & \xleftarrow{\phi^\Gamma} & G
\end{array}
\]

The two horizontal maps in the diagram above are given by the formulae:

for all $p \in P$, $F(p \leq \Gamma\Lambda p)$, and $G(p \leq \Lambda\Gamma p)$ respectively.

Note that two persistence modules are $(1, 1)$-interleaved, where $1$ is identity translation, if and only if they are isomorphic.

Definition 16 ([BdS13]). Given any metric $d$ on $P$, we define $D = D(d)$ by the formula;

\[
D(M, N) := \inf\{\varepsilon : \exists (\Gamma, \Lambda)$-interleaving with $\sup_{p \in P} d(p, \Gamma p), \sup_{p \in P} d(p, \Lambda p) \leq \varepsilon\} = \inf\{\varepsilon : \exists (\Gamma, \Lambda)$-interleaving with $h(\Lambda), h(\Gamma) \leq \varepsilon\}.
\]

From Bubenik, de Silva and Scott ([BdS13]), we know that $D$ is a Lawvere metric on $\mathcal{D}^P$, and for any category $F$, and functor $R : \mathcal{D} \rightarrow F$, post-composition by $R$ is a contraction from $\mathcal{D}^P$ to $\mathcal{F}^P$.

With hard stability theorems in mind, the fact that post-composition by any functor induces a contraction is particularly noteworthy. Still, independent of the choice of metric $d$ on $P$, without modification the resulting Lawvere metric $D = D(d)$ need not be a proper metric, simply because the collection of translations is not be rich enough to provide interleavings between arbitrary generalized persistence modules. This is unfortunate, since $\mathcal{T}(P^-)$ is defined naturally for any poset $P$. The failure comes from the fact that finite posets will always have fixed points.

Definition 17. We say that $p \in P$ is a fixed point of $P$, if $\Lambda p = p$ for all $\Lambda$ in $\mathcal{T}(P^-)$

Remark 1. Note that if $p_1, p_2, \ldots, p_n$ are maximal elements in $P$, then any maximal element in $\bigcap (-\infty, p_i]$ is necessarily a fixed point of $P$. This is relevant because one can easily show that if $M, N$ are two $A(P)$ modules, and $dim_K(M(p)) \neq dim_K(N(p))$ for some fixed point $p \in P$, then $D(M, N) = \infty$, where $D = D(d)$, and $d$ is any metric on $P$.

If $p$ is a fixed point of $P$ and $dim_K(M(p)) < dim_K(N(p))$. Then the diagram below does not commute for any morphisms $\phi, \psi$ and any translations $\Lambda, \Gamma$, since the composition cannot have full rank as required. Thus, $D(M, N) = \infty$.

\[
\begin{array}{ccc}
N(p) & \xrightarrow{\phi_{p}} & M(\Lambda p) = M(p) \\
\downarrow \phi_{p} & & \downarrow \psi_{\Lambda p} = \psi_{p} \\
N(p) & \xrightarrow{\psi} & N(\Gamma\Lambda p) = N(p)
\end{array}
\]

In particular, this says that if $p$ is a fixed point of $P$, with $p \in \text{Supp}(M), p \notin \text{Supp}(N)$, then $D(M, N) = \infty$. Because of this, there is no hope of realizing any honest metric as an interleaving metric on any finite poset. For example, for poset $C$ of Example 2, and for any choice of metric $d$, the resulting interleaving metric (on isomorphism classes of modules) is the infinite discrete Lawvere metric.

With this in mind, we make the following modification. We set $P^+ = P \cup \{\infty\}$ with added relations $p \leq \infty$, for all $p \in P$. We may now view $A(P)$-mod as the full subcategory of $A(P^+)$-modules where all objects are supported in $P$. Now there exist $(P^+)$ interleaveings between any
two \textit{A}(P)-modules. Note that the Hasse quiver for \(P^+\) is simply the Hasse quiver for \(P\) with added edges connecting maximal elements of \(P\) to \(\infty\). We now build the metric \(d\), attaching positive weights to each edge of the Hasse quiver of \(P^+\). Continuing with poset \(C\) from Example 2, we now have one of the below.

In the democratic case (on the right), the arrows in the Hasse quiver of \(P^+\) which were actually in Hasse quiver for \(P\) are labeled with one weight, while the "new" arrows are all labeled with a different value. Of particular interest is when \((a, b) \in \mathbb{N} \times \mathbb{N}\) (see Remark 2 below). We will confine our attention to this case in this paper (For an analysis of the non-democratic case, see [MM17]). When the Hasse quiver for \(P^+\) is as above, we will say that \((a, b)\) is a weight.

**Definition 18.** Now, we let \(d_{a,b}\) denote the weighted graph metric on the Hasse quiver of \(P^+\), and let \(D\) be a category. Then, \(D = D(d_{a,b})\) is the interleaving metric corresponding to the weight \((a, b)\) on \(D^P\).

With this modification, since any two generalized persistence modules can be interleaved, \(D\) defines the structure of a finite metric space on the isomorphism classes of elements of \(D^P\). We will now write \(T(P)\) for \(T((P^+)^-\)) and from this point forward, we suspend all posets at infinity.

**Remark 2.** Ultimately, we wish to consider a function defined on a full subcategory of isomorphism classes of \(A(P)\)-modules equipped with the interleaving distance \(D = D(d_{a,b})\). Of course, if the function \(J\) in our workflow diagram takes values in a metric space \(X\) with finite diameter, one can always choose \((a, b) = (\text{diam}(X), \text{diam}(X))\) to make the function a contraction. Thus, in future work, we endow the weightspace \(\mathbb{N} \times \mathbb{N}\) with the lexicographic ordering, and will consider minimal weights \((a, b)\) such that the function in question is a contraction for \(D = D(d_{a,b})\).

When the category \(D\) is \(K\)-modules, \(D = D(d_{a,b})\) will be our interleaving distance on the category \(C \subseteq D^P \cong A(P)\text{-mod}\). We now endow the set of isomorphism classes of \(A(P)\)-modules with the other metric structure.

### 3.4 Bottleneck Metrics

A bottleneck metric provides an alternate metric structure on the set of isomorphism classes of \(A(P)\)-modules, or indeed any subcategory \(C\) generated by a fixed collection of indecomposable modules. The construction begins with a metric \(d_2\) on a set \(\Sigma\), where \(\Sigma\) is a subset of isomorphism classes of indecomposable \(A(P)\)-modules. Additionally, we require a function \(W : \Sigma \to (0, \infty)\), compatible with \(d_2\) in the sense that for all \(\sigma_1, \sigma_2 \in \Sigma\),

\[|W(\sigma_1) - W(\sigma_2)| \leq d_2(\sigma_1, \sigma_2)\]

Following [BL13], [BL16], we define a matching between two multisets \(S, T\) of \(\Sigma\) to be a bijection \(f : S' \to T'\) between multisubsets \(S' \subseteq S\) and \(T' \subseteq T\). For \(\epsilon \in (0, \infty)\), we say a matching \(f\) is an \(\epsilon\)-matching if the following conditions hold:

\[|W(\sigma) - W(\sigma')| \leq \epsilon\]
Then the bottleneck distance between $W$ measures the size of an element of $\Sigma$, we call $W(\sigma)$ the width of $\sigma$. Thus, in an $\epsilon$-matching, elements of $S$ and $T$ which are actually identified are within $\epsilon$, while all those not identified have width at most $\epsilon$.

Given a $A(P)$-module $M$, the barcode of $M$, $B(M)$ is the multiset of the isomorphism classes of indecomposable summands of $M$ with their corresponding multiplicities. Thus, $B(M)$ is precisely a multiset of elements in $\Sigma$, when $\Sigma$ is the set of all isomorphism classes of indecomposable modules.

**Definition 19.** Let $S, T$ be two finite multisubsets of any set $\Sigma$. Suppose $d_2$ and $W$ are compatible. Then the bottleneck distance between $S$ and $T$ is defined by,

$$D_B(S, T) = \inf\{\epsilon \in \mathbb{R} : \text{there exists an } \epsilon\text{-matching between } S, T\}$$

Let $\Sigma$ be any fixed subset of isomorphism classes of indecomposable $A(P)$-modules. If $M, N$ are $A(P)$-modules with the property that every indecomposable summand of $M$ or $N$ is isomorphic to an element of $\Sigma$, then we may identify $M, N$ with their barcodes $B(M), B(N)$, two multisubsets of $\Sigma$. Then, set

$$D_B(M, N) := D_B(B(M), B(N)).$$

While there are many examples of bottleneck metrics in the literature, in this paper, we will choose $d_2$ to be the interleaving metric corresponding to the weight $(a, b)$ restricted to $\Sigma$, where $\Sigma$ is the set of convex modules. Our width will be an algebraic analogue of the width of the support of a one-dimensional persistence module. In the next subsection, we define our subcategory $C$.

### 3.5. The Category Generated by Convex Modules

Since a finite poset $P$ may have the property that $A(P)$-mod is of wild representation type, a characterization of all of the isomorphism classes of its indecomposable modules might not be possible. Moreover, an indecomposable module is not determined by its support (see Definition 6). Let $\Omega$ denote the set of isomorphism classes of indecomposable $A(P)$-modules. Clearly, the function

$$\Omega \xrightarrow{\text{Supp}} \mathcal{P}(P),$$

which sends $M \xrightarrow{\text{Supp}} \text{Supp}(M)$, its support may have infinite (and unknowable) domain, but always has finite range. Motivated by one-dimensional persistent homology, we normalize taking the perspective that the width of an indecomposable should be determined only by its support. We, therefore restrict out attention to the category $C$ generated by an appropriate set $\Sigma$ of indecomposable thin modules. A module is thin if its dimension vector consists of only zeros and ones.

**Definition 20.** An indecomposable module $M$ is convex, if it thin, and if it is isomorphic to a module $M'$ where $M'$ satisfies

for all $x, y \in \text{Supp}(M')$, with $x \leq y$, the linear map $M'(x \leq y)$ is given by $\text{Id}_K$.

Let $C$ be the full subcategory of $A(P)$-modules which are direct sums only of convex modules. This is the full subcategory of $A(P)$-modules that we will focus on. We note that in the literature, convex modules are sometimes called interval modules (see [BL16]). We use convex instead to avoid confusion with either subsets of the poset $P$, or elements of its poset algebra $A(P)$. In particular, some convex modules are supported in an honest interval in the poset, while others are not.

Clearly, when restricted to the set of isomorphism classes of convex modules, the function $M \rightarrow \text{Supp}(M)$ is one-to-one. Of course, the function is not onto, as not every subset of $P$ is the support of a convex module. One easily checks that if $S \subseteq P$, then there exists a convex module $M$ (unique up to isomorphism) with $\text{Supp}(M) = S$ if and only if
(i) For all \( s_1, s_2 \in S \) there exists an unoriented path in the Hasse quiver of \( P \) that connects \( s_1 \) and \( s_2 \) staying entirely within \( S \), and

(ii) For all \( s_1, s_2 \in S \) the set \( \{ p \in P : s_1 \leq p \leq s_2 \} = [s_1, s_2] \subseteq S. \)

In the above, an unoriented path is a product of paths and their formal inverses. If \( S \) satisfies (i), we say \( S \) is connected, and if \( S \) satisfies (ii), we say \( S \) is interval convex. Regardless of the representation type of the poset \( P \), \( \Sigma = \{ [\sigma] : \sigma \text{ is convex} \} \) is finite.

It is well known that if \( P \) has no crowns (a subposet of a certain form), then every indecomposable thin \( A(P) \)-module is a convex module [ACMT05]. On the other hand, when \( P \) has non-trivial cohomology, many indecomposable thin modules will not be convex (see Example 4 below from [Fei76]). While the class of posets we will restrict to in the next section contain many posets of wild representation type, they all have the property that every indecomposable thin is convex.

**Example 4.** Consider the posets given below.

The convex modules for the algebra with poset \( F \) have supports given by the following subsets;

\[
\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 6\}, \{5, 6\}, \{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 4\}, \\
\{2, 4, 6\}, \{3, 4, 5\}, \{4, 5, 6\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{2, 3, 4, 5\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}, \{1, 2, 3, 4, 5\}, \\
\{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}.
\]

Poset \( G \) has the property that when \( K \) is infinite, there are infinitely many non-isomorphic indecomposable thin modules modules with full support. Of course, there is exactly one convex module with full support.

Convex modules are of interest in representation theory. For example, in a large class of algebras, each simple modules can be associated to a collection (in fact, a poset) of convex modules in such a way that the representation type of the algebra can be determined in an effective fashion [DN92], [DN99], [Rin91]. In persistent homology similar classes of generalized persistence modules have frequently been used (see [CO16], [BL16]). From the perspective of representation theory, it is easy to see that all of the indecomposable projectives and injectives, and all simples modules are convex. Moreover, since convex modules are uniquely determined by their support, we agree with the sentiment in [BL16], that \( C \) is the correct categorical framework for generalized persistence modules for \( P \), when \( P \) has arbitrary representation type.

4. A Particular Class of Posets

In this section we confine our discussion to a certain class of finite posets. Though easy to describe, most such posets are of wild representation type (see the discussion in Subsection 3.2). We will restrict to \( C \), the full subcategory of \( A(P) \)-modules which are isomorphic to a direct sum of convex modules.

Let \( P \) be a finite poset such that:

1. \( P \) has a unique minimal element \( m \),
2. for every maximal element \( M_i \in P \), the interval \([m, M_i]\) is totally ordered, and
(3) \([m, M_i] \cap [m, M_j] = \{m\}\) for all \(i \neq j\).

As a technical convenience, we sometimes also assume

(4) their exists an \(i_0\) with \([m, M_{i_0}]\) \(> [m, M_i]\), for all \(i \neq i_0\).

That is, \(P\) is a tree which branches only at the its unique minimal element and has one totally ordered segment longer than the others.

**Definition 21.** If \(P\) satisfies conditions (1), (2), (3), we say \(P\) is an \(n\)-Vee, where \(n\) denotes the number of maximal elements in \(P\). If, in addition, \(P\) satisfies (4) we say that \(P\) is an asymmetric \(n\)-Vee.

Clearly, a \(1\)-Vee is exactly a finite totally ordered set. It is easy to see that every \(1\)-Vee is an asymmetric. We will prove our isometry theorem for \(n\)-Vees.

**Example 5.** Poset \(D\) in Example 2 is an asymmetric \(3\)-Vee (with wild representation type).

**Remark 3.** The convex modules for \(n\)-Vees have some nice properties. Note that if \(P\) is any finite poset, then, the following two statements are equivalent:

(i) \(P\) has a unique minimal element \(m\) and every maximal interval in \(P\), \([m, M_i]\) is totally ordered.

(ii) the support of every convex module has a unique minimal element.

That is to say, finite posets satisfying only properties (1) and (2) in the definition for \(n\)-Vees are precisely those posets for which the support of a convex module always has a unique minimal element. The proof is easy, but we include it.

**Proof.** First, if \(P\) is as above, from the characterization of convex modules in Subsection 3.5 it is clear that the support of each convex module has a unique minimal element. On the other hand, for a contradiction suppose \(P\) satisfies (ii), but not (i). Let \(S \subseteq P\) denote the support of a potential convex module. If \(P\) has at least two minimals, then set \(S = P\). Thus it must be the case that \(P\) has a unique minimal \(m\). If there is a maximal interval \([m, M_j]\) contained in \(P\) with \([m, M_i]\) not totally ordered. Then, there exist \(x, y \in [m, M_j]\) with \(x, y\) not comparable. But then \(S = [x, M_j] \cup [y, M_j]\) is the support of a convex module contradicting (ii).

We will now establish some properties of the collection of translations of an asymmetric \(n\)-Vee. Much (but not all) carries over to (general) \(n\)-Vees (see the end of the proof of Theorem 1).

**Lemma 22.** Let \(P\) be an asymmetric \(n\)-Vee, and let \((a, b)\) be any weights. Let \(d = d_{a,b}\) denote the weighted graph metric on the Hasse quiver of \(P^+\) corresponding to \((a, b)\). Then,

(i) For each \(\epsilon \in \{h(\Lambda) : \Lambda \in \mathcal{T}(P)\}\), the set \(\{\Gamma \in \mathcal{T}(P) : h(\Gamma) = \epsilon\}\) has a unique maximal element \(\Lambda_\epsilon\).

(ii) The set \(\{\Lambda_\epsilon\}\) is totally ordered, and \(\Lambda_\epsilon \leq \Lambda_\delta\) if and only if \(\epsilon \leq \delta\).

(iii) If \(\Lambda, \Gamma \in \mathcal{T}(P)\) with \(h(\Lambda), h(\Gamma) \leq \epsilon\) then there exists a \(\Lambda_\delta\) with \(\Lambda, \Gamma \leq \Lambda_\delta\), and \(h(\Lambda_\delta) = \delta = \max\{h(\Lambda), h(\Gamma)\}\).

**Proof.** Let \(P\) be as above. First, say \(n > 1\), then \(P = \bigcup [m, M_i]\), with \([m, M_{i_0}]\) of maximal cardinality. Let \(T_i = [m, M_i] - 1\), so by hypothesis, \(T_{i_0} > T_i\) for all \(i \neq i_0\). Let \(T = \max\{T_i : i \neq i_0\}\) (note that if \(P\) was not asymmetric \(T = T_{i_0}\)). Let \(\epsilon \in \{h(\Lambda) : \Lambda \in \mathcal{T}(P)\}\) and suppose \(h(\Lambda) = \epsilon\). If \(\Lambda m > m\), then \(\epsilon \geq aT + b\), since;

- if \(\Lambda m = \infty\), then \(h(\Lambda) = aT_{i_0} + b\),
- if \(\Lambda m \in (m, M_{i_0}]\), then \(h(\Lambda) \geq aT + b\), and
- if \(\Lambda m \in (m, M_i], i \neq i_0\), then \(h(\Lambda) = aT_{i_0} + b\).
Therefore, if $\epsilon < aT + b$, $\Lambda m = m$. Then, $\Lambda \leq \Lambda_\epsilon$, where

$$\Lambda_\epsilon(x) = \begin{cases} m, & \text{if } x = m \\ \max\{y \in (m, M_i] \cup \{\infty\} : d(x, y) \leq \epsilon\}, & \text{if } x \in (m, M_i] \end{cases}$$

On the other hand, if $aT_0 + b > \epsilon > aT + b$, then $\Lambda m \in [m, M_{i_0}]$, and $\Lambda((m, M_{i_i})) = \infty$ for $i \neq i_0$. In this case, $\Lambda \leq \Lambda_\epsilon$, where

$$\Lambda_\epsilon(x) = \begin{cases} \infty, & x \in (m, M_{i_0}], i \neq i_0 \\ \max\{y \in (m, M_{i_0}] \cup \{\infty\} : d(x, y) \leq \epsilon\}, & x \in [m, M_{i_0}] \end{cases}$$

Lastly, if $\epsilon = aT_0 + b$, then $\Lambda \leq \Lambda_\epsilon$, where $\Lambda_\epsilon(x) = \infty$, for all $x$.

Note that the formulae above are well defined, since $[m, M_i] \cap [m, M_j] = \{m\}$ for all $i \neq j$. Now, suppose that $n = 1$. Then $\Lambda \leq \Lambda_n$, where $\Lambda_n(x) = \max\{y \geq x : d(x, y) \leq \epsilon\}$ for any $\epsilon$. This proves (i). The expressions for $\Lambda_n$ show that (ii) holds. Now let $\Lambda, \Gamma \in \mathcal{T}(P)$ with $h(\Lambda), h(\Gamma) \leq \epsilon$, and suppose $\max\{h(\Lambda), h(\Gamma)\} = \delta$. Without loss of generality, say $h(\Lambda) = \delta, h(\Gamma) \leq \delta$. Then, $\Lambda \leq \Lambda_\delta$ and $\Gamma \leq \Lambda_{h(\Gamma)} \leq \Lambda_\delta$, by (i), (ii) as required.

The important observation is that although $\mathcal{T}(P)$ is not totally ordered, (for $n > 1$) it is directed in such a way that one may pass to a larger translation without increasing the height. In contrast, for an arbitrary finite poset $P$, $\mathcal{T}(P)$ will still be a directed set (because we suspended at infinity). It may be the case, however, that for all $\Lambda_\delta$, $\Gamma \leq \Lambda_{\delta 0}, h(\Lambda_{\delta 0}) > \kappa > \max\{h(\Lambda), h(\Gamma)\}$. That is to say, one may have to pay a price when passing to any larger common translation. Lemma 22 shows that this does not happen for asymmetric $n$-Vees. We are now ready to define the width of a convex module.

**Lemma 23.** Let $P$ be an asymmetric $n$-Vee, and let $(a, b)$ be a weight. Then for all $I$ convex, the following are equal;

(i) $W(I) = W_1(I) = \min\{\epsilon : \exists \Lambda, \Gamma \in \mathcal{T}(P), h(\Lambda), h(\Gamma) \leq \epsilon, \text{ and } \text{Hom}(I, I\Lambda \Gamma) = 0\}$

(ii) $W_2(I) = \min\{\epsilon : \exists \Lambda \in \mathcal{T}(P), h(\Lambda) \leq \epsilon, \text{ and } \text{Hom}(I, I\Lambda^2) = 0\}$.

(iii) $W_3(I) = \min\{\epsilon : \exists \Lambda_\epsilon \in \mathcal{T}(P) \text{ with } \text{Hom}(I, I\Lambda_\epsilon^2) = 0\}$.

Before proving Lemma 23, we note that for any $I$ convex and for any $\theta \in \mathcal{T}(P)$,

$$\text{Hom}(I, I\theta) \neq 0 \iff \exists x \in \text{Supp}(I), \theta x \in \text{Supp}(I) \iff \theta x' \in \text{Supp}(I), \text{ for } x' \text{ minimal in } \text{Supp}(I).$$

This follows from general properties of module homomorphisms, and the observation in Remark 3 that convex modules for $n$-Vees have unique minimal elements. (See the Section 5 for a detailed analysis of homomorphisms and translations)

Using this fact, we see that if $\Lambda \leq \Gamma$ and $\text{Hom}(I, I\Lambda) = 0$, then $\text{Hom}(I, I\Gamma) = 0$. Thus this condition defining $W$ produces an interval in $\{h(\Lambda) : \Lambda \in \mathcal{T}(P)\}$. We will now prove Lemma 23.

**Proof.** Let $\Lambda, \Gamma \in \mathcal{T}(P)$ with $h(\Lambda), h(\Gamma) \leq \epsilon$, and suppose $\text{Hom}(I, I\Lambda \Gamma) = 0$, and $\delta = \max\{h(\Lambda), h(\Gamma)\}$. Then, by Lemma 22, there exists $\Lambda_\delta$, with $h(\Lambda_\delta) = \delta$ and $\Lambda \leq \Lambda_\delta$. Then $\Lambda \Gamma \leq \Lambda_\delta^2$ so $\text{Hom}(I, I\Lambda_\delta^2) = 0$, so $W_3(I) \leq W_2(I) \leq W(I)$. But $\mathcal{T} \implies \inf(S) \geq \inf(T)$, thus $W_3(I) \geq W_2(I) \geq W(I)$, so all are equal. With this equivalence established, we define the width of a convex module.

**Definition 24.** Let $P$ be an asymmetric $n$-Vee and let $(a, b)$ be a weight. Let $I$ be convex. Then,

$$W(I) = W_1(I) = \min\{\epsilon : \exists \Lambda, \Gamma \in \mathcal{T}(P), h(\Lambda), h(\Gamma) \leq \epsilon, \text{ and } \text{Hom}(I, I\Lambda \Gamma) = 0\}.$$

While this definition of the width of a module is formulated algebraically, and is natural considering the structure of $\mathcal{T}(P)$, it is not without complication. Intuitively $\frac{1}{2}|\text{Supp}(I)|$ (or perhaps $\frac{1}{2}|\text{Supp}(I)|$) is a first approximation of $W(I)$. Indeed, this is the discrete analogue of the width
used in the classical isometry theorem [BL13], as their work corresponds to translations that are exactly constant shifts. This discrete analogue of this is the choice of weights \((a, b) = (1, 1)\) on a 1-Vee. For an \(n\)-Vee, however, modules with smaller support may happen to have large widths or the opposite. For example, if \(P\) is a 2-Vee and \(I\) is the simple convex module supported at \(m\), then \(W(I) = aT + b\). In contrast, if \(x \in (m, M_i)\) and \(J\) is the convex module supported at \(x\), \(W(J) = a\). Moreover, any convex module supported at \(M_i\) for some \(i\) necessarily has width greater or equal to \(b\). This is relevant, as no relation between \(a\) and \(b\) is specified.

The following Proposition will prove useful in Section 6 when we produce an explicit matching for 1-Vees. This result is an analogue of the corresponding statement in [BL13].

**Proposition 25.** Let \(P\) be an asymmetric \(n\)-Vee, let \(A = \bigoplus_j A_i, C = \bigoplus_j C_j\) be in \(C\). For any module \(M\), let \(B(M)\) denote the barcode of \(M\) viewed as a multiset, and let \(\Lambda \in T(P)\). Then,

1. If \(A \xrightarrow{f} C\) is an injection, then for all \(d \in P\), the set
   \[
   \{| i : [-, d] \text{ is a maximal totally ordered subset of Supp}(A_i) | \} \leq \{| j : [-, d] \text{ is a maximal totally ordered subset of Supp}(C_j) | \},
   \]
2. If \(A \xrightarrow{g} C\) is a surjection, then for all \(b \in P\),
   \[
   \{| j : [b, -] \text{ is a maximal totally ordered subset of Supp}(C_j) | \} \leq \{| i : [b, -] \text{ is a maximal totally ordered subset of Supp}(A_i) | \}.
   \]
3. If \(A\) and \(C\) are \((\Lambda, \Lambda)\)-interleaved, and \(A \xrightarrow{\phi} CA\) is one of the homomorphisms, then for all \(I\) in \(B(\ker(\phi))\), \(W(I) \leq h(\Lambda)\).
4. If \(A\) and \(C\) are \((\Lambda, \Lambda)\)-interleaved, and \(A \xrightarrow{\phi} CA\) is one of the homomorphisms, then for all \(J\) in \(B(\cok(\phi))\), \(W(J) \leq h(\Lambda)\).

Before proving the Proposition 25, we state a Lemma.

**Lemma 26.** Let \(P\) be an asymmetric \(n\)-Vee, say \(P = \bigcup [m, M_j]\), with \([m, M_j]\) totally ordered. Let \(m_i = \min(m, M_i)\), and let \(I_j\) be the left ideal in \(A(P)\) generated by \(\{m_i : i \neq j\}\). Then,

1. For any \(M\) convex,
   \[
   M/I_jM = \begin{cases} 
   0, & \text{if Supp}(M) \cap [m, M_j] = \phi \\
   \text{the convex module with support given by Supp}(M) \cap [m, M_j] \text{ otherwise.} 
   \end{cases}
   \]
2. For \(A, B \in C\), if \(f\) is a homomorphism \(A \xrightarrow{f} B/I_jB\), then \(f\) factors through \(A/I_jA\).

**Proof.** (i) obvious. Statement (ii) is clear, since for \(f : A \to B/I_jB, w \in I_j, f(w \cdot a) = w \cdot f(a) = 0\). \(\square\)

Note that if \(n = 1\), the left ideal \(I_i\) is identically zero, but the above is still true. We now prove Proposition 25.

**Proof.** Let \(A, C\) be as above. For all \(i\), let \(A_i = A(P)x_i, x_i \in \text{Supp}(A_i)\), and let \([x_i, X_i]\) be a maximal connected totally ordered subset of \(\text{Supp}(A_i)\) (We do not suppose \(x_i = m\)). Similarly, let \(y_j\) be such that \(C_j = A(P)y_j\). For \(i, j\) let \(f_j^i : A_i \to C_j\). Now suppose \(A \xrightarrow{f} C\) is an injection. Fix \(i_0\) and \([x_{i_0}, X_{i_0}]\) be maximal contained in \(\text{Supp}(A_{i_0})\). Since \(f_0 = (f_0^i)_i : A_{i_0} \to \bigoplus_j C_j\) is an inclusion, for any \(t \in [x_{i_0}, X_{i_0}]\), there exists \(j(t)\) with \(f_0^i(x_{i_0}) \neq 0\). Since \(f_0^j(x_{i_0})\) is a homomorphism, \(f_0^j(x_{i_0}) \neq 0 \implies f_0^j(x_{i_0}) \neq 0\), and it is that case that there exists \(\ell > X_{i_0}\), with \(\ell \in \text{Supp}(C_{j_0})\). Set \(j_0 = j(X_{i_0})\). Therefore,

\[
\{ j : [x_{i_0}, X_{i_0}] \subseteq \text{Supp}(C_j) \text{ and for all } \ell, \ell > X_{i_0} \implies \ell \notin \text{Supp}(C_j) \} \neq \phi.
\]
Now, for \( d \in P \), let

\[
\begin{align*}
  j(d) &= \{ j : [-, d] \subseteq \text{Supp}(C_j), \ell \notin \text{Supp}(C_j) \text{ for } \ell > d \}, \text{ and } \\
  i(d) &= \{ i : [-, d] \subseteq \text{Supp}(A_i), \ell \notin \text{Supp}(A_i) \text{ for } \ell > d \}.
\end{align*}
\]

Clearly, \( i(d) \neq \phi \implies j(d) \neq \phi \). Now, let \( d \in P, d \neq m \) with \( i(d) \neq \phi \). Say \( d \in (m, M_k] \). Then,

\[
\bigoplus_{i \in i(d)} A_i / I_k A_i \hookrightarrow \bigoplus_{j \in j(d')} C_j / I_k C_j \hookrightarrow C / I_k C \implies \\
\bigoplus_{i \in i(d)} (A_i / I_k A_i) \hookrightarrow \bigoplus_{j \in j(d')} (C_j / I_k C_j) \hookrightarrow C / I_k C
\]

where the above inclusions are induced from \( f \) and the inclusion of a submodule into a larger module respectively. Thus, \( |i(d)| \leq |j(d)| \). If \( d = m \), then,

\[
\bigoplus_{i \in i(m)} A_i \hookrightarrow \bigoplus_{j \in j(m)} C_j \hookrightarrow C \implies \bigoplus_{i \in i(m)} A_i(m) \hookrightarrow \bigoplus_{j \in j(m)} C_j(m) \hookrightarrow C(m),
\]

so \( |i(m)| \leq |j(m)| \). This proves (i). The proof of (ii) is similar, though one inducts on the the cardinality of \( S = \{ b : [b, \infty) \} \) is a maximal totally ordered subset of \( A_i \) for some \( A_i \).

Now we prove (iii). For a contradiction, suppose there exists an \( I \in B(\ker(\phi)) \) with \( \text{Hom}(I, I\Lambda^2) \neq 0 \). But then the diagram below commutes.

\[
\begin{array}{ccc}
I & \xleftarrow{\phi|_I} & I\Lambda^2 \\
\downarrow{\phi|_I} & & \downarrow{\psi} \\
C_j & & \\
\end{array}
\]

Thus, \( \psi\Lambda\phi(I) \neq 0 \), a contradiction. This proves (iii).

Now let \( J \in B(\text{cok}(\phi)) \). For a contradiction, suppose \( W(J) > h(\Lambda) \). But then there exists \([x, X]\) a maximal subinterval in \( \text{Supp}(J) \) with \( \Lambda^2 x \leq X \). Let \( \{ b_x + \text{im}(\phi), \ldots b_X + \text{im}(\phi) \} \) be the corresponding basis elements for \( J \). But then there exists \( j \) such that

1. \([x, X] \subseteq \text{Supp}(C_j\Lambda) \), and
2. \( C_j\Lambda(y) \notin \text{im}(\phi) \), for \( x \leq y \leq X \).

Then, \( \Lambda x, \Lambda X \in \text{Supp}(C_j) \Rightarrow \Lambda^2 \Lambda x = \Lambda \Lambda^2 x \leq \Lambda X \) which is in the support of \( C_j \). Therefore \( W(C_j) \geq h(\Lambda) \). But then, the following diagram commutes.

\[
\begin{array}{ccc}
C_j(\Lambda x) & \xrightarrow{(C_j\Lambda^2)(\Lambda x)} & (C_j\Lambda)(\Lambda^2 x) \\
\downarrow{J\Lambda(\Lambda x)} & & \downarrow{(C_j\Lambda)(\Lambda^2 x)} \\
& & J\Lambda(\Lambda x)
\end{array}
\]

But then \( (C_j\Lambda^2)(\Lambda x) \in \text{im}(\phi\Lambda)(\Lambda x) \), a contradiction. This proves (iv) and finishes the proof. \( \square \)

The Example below shows that (i),(ii) in the Proposition 25 cannot be extended from maximal totally ordered intervals to convex subsets.
Example 6. Consider the 2-Vee \([m, M_1] \cup [m, M_2]\), where \(m < x < M_1\) and \(m < y < z < M_2\). Let \(C_1\) be the convex module supported on \([m, x, M_1]\), and \(C_2\) be the convex module supported on \([m, y, z, M_2]\). Say \(C_1\) has basis \(\{e_m, e_x, e_{M_1}\}\) and \(C_2\) has basis \(\{f_m, f_y, f_z, f_{M_2}\}\). Then the submodule of \(C_1 \oplus C_2\) with basis \(\{e_m + f_m, e_x, e_{M_1}, f_y, f_z, f_{M_2}\}\) is isomorphic to the convex module with full support. Thus, let \(A_1\) be the convex module with full support. Then, \(A_1 \hookrightarrow C_1 \oplus C_2\), and while one can make the claim in the Proposition for each maximal totally ordered subset of the support of \(A_1\) separately, one cannot do so simultaneously.

In the next section we study homomorphisms and translations and their properties in \(C\).

5. HOMOMORPHISMS AND TRANSLATIONS

In this section we investigate the relationship between homomorphisms and translations in the category \(C\). In the interest of generality, we will relax our hypotheses on the poset \(P\). In this section, unless otherwise specified, \(P\) is any finite poset. The functions defined in Definitions 27, 28 are analogues of functions used by Bauer and Lesnick [BL13]. In this context, however, they fail to preserve \(W\), and may annihilate a convex module.

Note that if \(S \subseteq P\) is non-empty and interval convex, then it canonically determines the isomorphism class of an element of \(C\) under the identification;

\[
S \rightarrow \bigoplus M_i, \text{ where } \text{Supp}(M_i) \text{ is the } i\text{th connected component of } S.
\]

We use this in the definition below.

Definition 27. Let \(P\) be any finite poset and \(M\) be convex. Say \(\text{Supp}(M) = \bigcup_i [a_i, b_i]\), where \([a_i, b_i]\) are maximal intervals in \(\text{Supp}(M)\), and let \(\Gamma \in \mathcal{T}(P)\). Then, \(M^{+\Gamma}\) is the element of \(C\) given by \(\text{Supp}(M^{+\Gamma}) = S = \bigcup_i [\Gamma a_i, b_i]\).

That is, \(M^{+\Gamma}\) is the direct sum determined by \(S = \bigcup_i [\Gamma a_i, b_i]\). (Note that if \(\Gamma a_i \not\subseteq b_i\), then \([\Gamma a_i, b_i]\) is empty.) One easily checks that,

(i) If \(P\) is an \(n\)-Vee, \(M^{+\Gamma}\) is convex, or 0, and

(ii) For a general poset \(P\), \(M^{+\Gamma}\) is a submodule of \(M\).

Moreover, for \(i \in P\),

\[
M^{+\Gamma}(i) = \sum_x \text{im}(M(x \leq \Gamma x \leq i)) = \text{im}(M(x_0 \leq \Gamma x_0 \leq i)) \text{ for any } x_0 \leq \Gamma x_0 \leq i.
\]

That is, \(\theta = \theta_i \in M^{+\Gamma}(i) \implies \theta \in \text{im}(M(x_0 \leq \Gamma x_0 \leq i)) \text{ for any } x_0 \leq \Gamma x_0 \leq i.\) Now, for \(M \in C\) arbitrary, set

\[
M^{+\Gamma} = \bigoplus_i M_i^{+\Gamma}, \text{ where } M = \bigoplus_i M_i.
\]

If \(P\) has the property that for all \(i \in P\), \((-\infty, i]\) is totally ordered, one can still find \(x = x(i)\) such that \(M^{+\Gamma}(i) = \text{im}(M(x \leq \Gamma x \leq i))\) is still valid. Thus, in particular, the result holds for as \(n\)-Vee. Note that if \((-\infty, i]\) is not totally ordered, then \(M^{+\Gamma}(i) = \sum_x \text{im}(M(x \leq \Gamma x \leq i))\). When \(P\) is an \(n\)-Vee, we now make a dual definition.

Definition 28. Let \(P\) be an \(n\)-Vee and let \(M\) be a convex module. Say \(\text{Supp}(M) = \bigcup_i [x_i, X_i]\) where each \(x \leq X_i \leq M_i\). (Recall that since \(P\) is an \(n\)-Vee, the support of each convex module has a
minimal element. So $X_i \neq x$ for more than one $i$ implies $x = m.)$ Let $\Gamma \in \mathcal{T}(P)$. Then, $M^{-\Gamma}$ is the convex modules with

$$\text{Supp}(M^{-\Gamma}) = \bigcup_i [x, X_i^{N(i)}], \text{ where } X_i^{N(i)} = \max \{y : \Gamma y \leq X_i, y \geq x\}.$$ 

Note that if no such $X_i^{N(i)}$ exists, $M^{-\Gamma} = 0$

One easily checks that,
(i) If $M$ is convex, $M^{-\Gamma}$ is either identically zero or convex, and
(ii) $M^{-\Gamma}$ is a quotient of $M$.

Because of (i) we may extend our definition from $\Sigma$ to $C$. For $M \in C$, set

$$M^{-\Gamma} = \bigoplus_t M_t^{-\Gamma}, \text{ where } M = \bigoplus_t M_t.$$ 

Notice that the assignment $M \rightarrow M^+\Gamma$ moves the left endpoints of the support of a module to the right, while $M \rightarrow M^{-\Gamma}$ moves the right endpoints of the support to the left. A physical characterization of $M^{-\Gamma}$ is possible, but will prove unnecessary for our purposes.

We now prove a useful proposition.

**Proposition 29.** Let $P$ be an $n$-Vee, and let $I, M \in C$. Let $(\phi, \psi)$ be a $(\Lambda, \Lambda)$-interleaving between $I$ and $M$. Say $\phi : I \rightarrow M\Lambda$. Then,

(i) $I^{-\Lambda^2}$ is a quotient of both $I$ and $\text{im}(\phi)$, and
(ii) $M^{+\Lambda^2}\Lambda$ is a submodule of both $M\Lambda$ and $\text{im}(\phi)$.

**Proof.** First, by the comments above, $I^{-\Lambda^2}$ is a quotient of $I$. Now, since $I$ and $M$ are $(\Lambda, \Lambda)$-interleaved, $\psi\Lambda \circ \phi = (I \rightarrow I\Lambda^2)$. Therefore,

$$(\psi\Lambda)(\text{im}(\phi)) = I^{-\Lambda^2},$$

and hence $I^{-\Lambda^2}$ is a homomorphic image, and hence a quotient of $\text{im}(\phi)$. This proves (i).

We now prove (ii). First, already $M^{+\Gamma}$ is a submodule of $M$. Moreover, $C \leq D$ implies $\tau$, $C\tau \leq D\tau$, for any $\tau \in \mathcal{T}(P)$. Hence, $M^{+\Lambda^2}\Lambda$ is a submodule of $M\Lambda$. It remains to show that $M^{+\Lambda^2}\Lambda$ is a submodule of $\text{im}(\phi)$. Let $i \in P, i \in \text{Supp}((M^{+\Lambda^2}\Lambda)).$ Then, as a vector spaces,

$$((M^{+\Lambda^2}\Lambda)(i)) \subset (\text{im}(\phi))(i) \oplus_K (\text{coker}(\phi))(i).$$

Let $\theta_i \in ((M^{+\Lambda^2}\Lambda)(i))$. Note that $\theta_i = \theta'_{\Lambda i}$. At the $i$ level, $\theta_i = a_i + b_i$ with $a_i \in (\text{im}(\phi))(i)$ and $b_i \in (\text{coker}(\phi))(i)$. Then,

$$\theta_i \in \text{im}((M(x \leq \Lambda^2 x \leq \Lambda i))) \implies \theta'_{\Lambda i} = M(x \leq \Lambda^2 x \leq \Lambda i)(a_x + b_x), a_x \in \text{im}(\phi), b_x \in \text{coker}(\phi) \implies$$

$$M(x \leq \Lambda^2 x \leq \Lambda i)(a_x) = a_i + \alpha_i, \text{ with } \alpha_i \in \text{im}(\phi), \text{ and}$$

$$M(x \leq \Lambda^2 x \leq \Lambda i)(b_x) = -\alpha_i + b_i.$$ 

But, by Proposition 25, $W(\text{coker}(\phi)) < h(\Lambda)$, thus $-\alpha_i + b_i = 0$, and therefore $\alpha_i = b_i = 0$. Hence, $\theta_i$ was fully contained in $(\text{im}(\phi))(i)$.

Thus,

$$\text{im}((M(x \leq \Lambda^2 x \leq \Lambda i))) \subset (\text{im}(\phi))(i) \text{ for all } i, \text{ hence}$$

$$((M^{+\Lambda^2}\Lambda)(i)) \subset (\text{im}(\phi))(i) \text{ for all } i.$$
Therefore,
\[(M^+\Lambda^2)\Lambda \subseteq \text{im}(\phi).\]

This proves (ii). \(\square\)

We will now consider the action of \(\mathcal{T}(P)\) on \(\mathcal{C} \cup \{0\}\). We first point out that, in general, the monoid \(\mathcal{T}(P)\) need not act on \(\Sigma \cup \{0\}\).

**Example 7.** Let \(P\) be the poset \(E\) in Example 2, and let \(\Lambda\) be the translation \(1 \rightarrow 1, 2 \rightarrow 4, 3 \rightarrow 4, 4 \rightarrow \infty, \infty \rightarrow \infty\). Let \(J\) be the convex module with support equal to \(\{2, 3, 4\}\). Then, \(J\Lambda \cong S \oplus T\), where \(S\) is the simple supported on \(\{2\}\), and \(T\) is the simple supported on \(\{3\}\). Alternatively, let \(P\) be the poset \(1, 2 \leq 3\), with \(1, 2\) not comparable. Let \(J\) be the convex module with full support, and \(\Lambda\) be given by \(1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow \infty, \infty \rightarrow \infty\). Then, \(J\Lambda\) is again a direct sum of two convex modules.

Example 7 shows that the action of \(\mathcal{T}(P)\) on \(\mathcal{C} \cup \{0\}\) need not restrict to \(\Sigma \cup \{0\}\). In Lemma 31 we will see that when \(P\) is an \(n\)-Vee, however, the action does restrict. First, a quick observation.

**Lemma 30.** Let \(P\) be any poset with a unique minimal element \(m\), and suppose \(\Lambda \in \mathcal{T}(P)\) with \(\Lambda m = m\). Then for all convex \(J\) with \(m \in \text{Supp}(J)\), \(J\Lambda\) is convex.

*Proof.* Let \(P\) be as above, \(M\) be convex, with \(m \in \text{Supp}(M)\). Let \(\Lambda\) be a translation with \(\Lambda m = m\). Clearly \(M\Lambda\) is thin. Let \(t_1, t_2\) be in the support of \(M\Lambda\), and suppose \(t_1 \leq t \leq t_2\). Then, \(m t_1, m t_2 \in \text{Supp}(M)\) \(\implies [m t_1, m t_2] \subseteq \text{Supp}(M)\), since \(M\) is convex. Since \(\Lambda t \in [m t_1, m t_2]\), \(t\) is in the support of \(M\), so \([m, M]\) is convex. Now, since \(\Lambda m = m\), \(m \leq x\) for all \(x\), \(\text{Supp}(M\Lambda)\) is connected. \(\square\)

**Lemma 31.** Let \(P\) be an \(n\)-Vee, \(I\) a convex module and \(\Lambda \in \mathcal{T}(P)\). Then, \(I\Lambda\) is either the zero module or convex.

*Proof.* First, from the proof of Lemma 30, if non-zero \(I\Lambda\) is in \(C\). We now proceed in cases. First, suppose \(m \in \text{Supp}(I)\). If \(\Lambda m = m\), then \(m\) is in the support of \(I\Lambda\), so \(I\Lambda\) is convex. On the other hand, if \(\Lambda m \in (m, M_j]\), then for all \(j \neq i, (m, M_j] \cap \text{Supp}(I\Lambda) = \phi\). But then \(\text{Supp}(I\Lambda) \subseteq [m, M_i]\), hence it is convex or zero, since it is interval convex. If \(m \notin \text{Supp}(I)\), then \(I\) is supported in \((m, M_j]\) for some \(j\) and the result follows. \(\square\)

We will now work towards the characterization of homomorphism between convex modules when \(P\) is an \(n\)-Vee. In the interest of generality, we begin with an arbitrary finite poset \(P\).

**Definition 32.** Let \(I, M\) be convex. Let \(\{e_x : x \in \text{Supp}(I)\}, \{f_x : x \in \text{Supp}(M)\}\) be \(K\)-bases for \(I, M\) respectively. Consider the linear function \(\Phi_{I,M}\), defined by

\[
\Phi_{I,M}(e_y) = \begin{cases} f_y, & \text{if } y \in \text{Supp}(I) \cap \text{Supp}(M) \\ 0 & \text{otherwise} \end{cases}
\]

By inspection, \(\Phi_{I,M}\) is a non-zero module homomorphism if and only if \(\text{Supp}(I) \cap \text{Supp}(M)\) satisfies,

(i) \(\text{Supp}(I) \cap \text{Supp}(M) \neq \phi\)
(ii) \(x \in \text{Supp}(I) \cap \text{Supp}(M), y \geq x, y \in \text{Supp}(M) \implies y \in \text{Supp}(I), \) and
(iii) \(x \in \text{Supp}(I) \cap \text{Supp}(M), y \leq x, y \in \text{Supp}(I) \implies y \in \text{Supp}(M).\)

Note that even when it is not a module homomorphism, \(\Phi_{I,M}\) can be viewed as the linear extension of \(\chi(\text{Supp}(I) \cap \text{Supp}(M))\), the characteristic function on the intersection of the supports of \(I\) and \(M\).

The following two lemmas will allow us to conclude that when \(P\) is an \(n\)-Vee, up to a \(K\)-scalar, this is the only possible module homomorphisms from \(I\) to \(M\).
Lemma 33. Let $P$ be any finite poset, and let $I, M$ be convex. Let $S \subseteq \text{Supp}(I) \cap \text{Supp}(M)$, with $S$ nonempty. Suppose that there exists an $N \in C$ with $\text{Supp}(N) = S$. Then, $N$ is isomorphic to the image of a non-zero module homomorphism from $I$ to $M$ if and only if

(a) for all $x \in S$, if $y \in \text{Supp}(I)$ with $y \leq x$, then $y \in S$, and
(b) for all $x \in S$, if $y \in \text{Supp}(M)$ with $y \geq x$, then $y \in S$.

Proof. $S$ corresponds to the support of a non-zero quotient module of $I$ if and only if $S$ satisfies (a). Similarly, $S$ corresponds to a non-zero submodule of $M$ if and only if $S$ satisfies (b). Since any homomorphism can be factored into an injection after a surjection, the result follows.

Lemma 34. Let $P$ be an $n$-Vee. Let $I, M$ be convex modules. Then, $\text{Hom}(I, M) \cong K$ or $0$ (as a vector space).

Proof. First, let $P$ be any finite poset, and $I, M$ be convex. Suppose that $g$ is any non-zero homomorphism from $I$ to $M$. Then, by Lemma 33, $\text{im}(g) = I/\ker(g)$ has support equal to $S \subseteq \text{Supp}(I) \cap \text{Supp}(M)$ satisfying (a), (b) from Lemma 33. We will show that any such $S$ is a union of connected components of $\text{Supp}(I) \cap \text{Supp}(M)$. Since $g$ is non-zero, $S$ is non-empty. Now, let $s \in S$ and suppose that $y \in \text{Supp}(I) \cap \text{Supp}(M)$ with $y \geq s$. Then, by (b), $y \in S$. Similarly, if $y \in \text{Supp}(I) \cap \text{Supp}(M)$ with $y \leq s$, then by (a), $y \in S$. Therefore $S$ contains the connected component of $s$ in $\text{Supp}(I) \cap \text{Supp}(M)$. The result follows.

Now, if $P$ is an $n$-Vee, $\text{Supp}(I) \cap \text{Supp}(M)$ is connected, so $S$ must be the full intersection. As above, let $\{e_x : x \in \text{Supp}(I)\}, \{f_x : x \in \text{Supp}(M)\}$ be $K$ bases for $I, M$ respectively.

Then,

$$g(e_z) = \begin{cases} 
  c_z f_z, & \text{if } z \in \text{Supp}(I) \cap \text{Supp}(M), c_z \in K \\
  0 & \text{otherwise,}
\end{cases}$$

where $\text{Supp}(I), \text{Supp}(M)$ satisfy conditions (i), (ii) and (iii) from below Definition 32. Clearly, every non-zero quotient of $I$ must have support containing the minimal element $t$ of $\text{Supp}(I)$. Then, since $I, M$ are convex,

$$g(I(t \leq y))e_t = ge_y = M(t \leq y)ge_t = M(t \leq y)c_t f_t = c_t f_y = c_y f_y.$$ 

Therefore, $g = c_t \Phi_{I,M}$. Of course, if $g$ is identically zero, $g$ is still in the span of $\Phi_{I,M}$.

We now investigate the action of $T(P)$ on $\text{Hom}(I, M)$, when $I, M$ are convex. From the observation in the proof of Lemma 34, we see that for $P$ arbitrary, $\text{Hom}(I, M)$ will have dimension equal to the number of connected components of $\text{Supp}(I) \cap \text{Supp}(M)$. Still, for a fixed translation $\Lambda \in T(P), \text{Hom}(I\Lambda, M\Lambda)$ may be trivial (even when $I\Lambda, M\Lambda$ are non-zero). We now state a condition which ensures that $\text{Hom}(I, M) \cdot \Lambda \neq 0$. This can be done more generally, but we state the result only for $P$ an $n$-Vee.

Lemma 35. Let $P$ be an $n$-Vee, and let $I, M$ be convex. Let $\Lambda \in T(P)$. Say $\text{Hom}(I, M) \neq 0$ and there exists $t$ with $\Lambda t \in \text{Supp}(I) \cap \text{Supp}(M)$. Then $\text{Hom}(I\Lambda, M\Lambda) \neq 0$.

Proof. Since $\Lambda t \in \text{Supp}(I) \cap \text{Supp}(M)$, $I\Lambda$ and $M\Lambda$ are not zero. Also, by Lemma 34, $\text{Supp}(I)$ and $\text{Supp}(M)$ satisfy the conditions (i), (ii) and (iii) from below Definition 32. Since $I\Lambda, M\Lambda$ are convex, it is enough to show that the above still holds for $\text{Supp}(I\Lambda)$ and $\text{Supp}(M\Lambda)$. Again, $t \in \text{Supp}(I\Lambda) \cap \text{Supp}(M\Lambda)$, hence the intersection is nonempty. Now let $z \in \text{Supp}(I\Lambda) \cap \text{Supp}(M\Lambda)$, with $w \in \text{Supp}(M\Lambda), w \geq z$. Then, $\Lambda z \in \text{Supp}(I) \cap \text{Supp}(M), \Lambda w \in \text{Supp}(M)$, and $\Lambda w \geq \Lambda z$. Therefore, $\Lambda z \in \text{Supp}(I)$, so $z \in \text{Supp}(I\Lambda)$. The last requirement is proved similarly.

Note that conditions (ii) and (iii) are clearly inherited from $I, M$. The authors point out that the hypothesis above that the intersection of supports coincides with the image of the translation is required even on a totally ordered set (see Example 8 below).
Example 8. Let $P$ be the totally ordered set $\{1, 2, 3, 4, 5, 6\}$ with its standard ordering, and let $\Lambda$ send $1 \to 2, 2 \to 3, 3 \to 4, 4 \to 5, 5 \to 6, 6 \to 6$. Let $I$ and $M$ be the convex modules supported on $\{4, 5, 6\}$ and $\{3, 4\}$ respectively. Note that $\text{Hom}(I, M) \neq 0$, $I\Lambda$ and $M\Lambda$ are supported on $\{4, 5\}$ and $\{2, 3\}$ respectively. Clearly, the supports of $I\Lambda$ and $M\Lambda$ are disjoint, so $\text{Hom}(I\Lambda, M\Lambda) = 0$.

Notation 36. Let $P$ be an $n$-Vee, $\Lambda \in \mathcal{T}(P)$ and say $I$ is convex. We write $\Phi^I_\Lambda$ for $\Phi_{I, I\Lambda}$, as $I\Lambda$ is either zero or convex. For $I \in C$, we write $\Phi^I_M$ for the canonical homomorphism as well, since it is necessarily diagonal. (Of course, as mentioned above, even if $I\Lambda$ is not trivial, it may be the case that $\Phi^I_\Lambda$ is identically zero.)

We can now show that when $P$ is an $n$-Vee the collection of interleavings between two elements of $C$ will have the structure of an affine variety (not necessarily irreducible). Though the result still holds for more general posets, our proof is an application of the results of this section. Some examples are provided in Section 8

Proposition 37. Let $P$ be an $n$-Vee and let $I = \bigoplus I_s$, $M = \bigoplus M_t$ be two elements of $C$. Let $\Lambda, \Gamma \in \mathcal{T}(P)$. Then the collection of $(\Lambda, \Gamma)$-interleavings between $I$ and $M$ has the structure of an affine variety.

Indeed, as stated above the result holds for any finite poset, though when $P$ is an $n$-Vee the variety has a simpler description. We sketch the proof. Let $P, I, M, \Lambda$ be as above, and let $\phi, \psi$ be any interleaving between $I$ and $M$. Thus, we obtain the commutative triangles below.

\[
\begin{array}{ccc}
I & \xrightarrow{\phi^I_\Lambda} & \Gamma \Lambda \\
\phi \downarrow & & \downarrow \psi \\
M \Lambda & \xrightarrow{\psi \Lambda} & M
\end{array}
\]

Therefore, as matrices of module homomorphisms;

\[
[\psi^s_\Lambda] \cdot [\phi^s_t] = [\Phi^s_{I_s, \Lambda}], \text{ and } [\phi^t_s \Gamma] \cdot [\psi^s_t] = [\Phi^t_{M_t, \Gamma}]
\]

where $\phi, \psi$ decompose into their component homomorphisms $\phi^s_t : I_s \to M_t \Lambda$ and $\psi^s_t : M_t \to I_s \Gamma$ respectively. By Lemma 34, $\phi^s_t, \psi^s_t$ are in the span of $\Phi_{I_s, M_t \Lambda}$ and $\Phi_{M_t, I_s \Gamma}$ respectively. Hence, if $\text{Hom}(I_s, M_t \Lambda)$ is not identically zero, $\phi^s_t = \lambda^s_t \Phi_{I_s, M_t \Lambda}$, where $\lambda^s_t \in K$, with a similar result holding for $\psi^s_t$. In addition, $(\lambda \Phi_{I_s, B})A_0 = \lambda (\Phi_{A_0, B}A_0)$ for all scalars $\lambda$, translations $A_0$, and all $A, B$ convex.

Therefore, the interleavings between $I$ and $M$ correspond to the algebraic set given by values of $\lambda^s_t, \mu^s_t$ satisfying all quadratic relations obtained by evaluating the matrix equations above at all elements of $P$.

More precisely, first suppose $\text{Hom}(I_s, M_t \Lambda), \text{Hom}(M_t, I_s \Lambda) = 0$ for all $s, t$. In this case, the variety of interleavings $V^{\Lambda, \Gamma}(I, M)$ is given by

\[
V^{\Lambda, \Gamma}(I, M) = \begin{cases}
\text{the zero variety, if } W(I_s), W(M_t) \leq \max \{h(\Lambda), h(\Gamma)\} \text{ for all } s, t \\
\text{the empty variety, otherwise.}
\end{cases}
\]

The above cases correspond to whether or not setting all morphisms identically equal to zero corresponds to an admissible interleaving between $I$ and $M$. On the other hand, suppose some of the relevant spaces of homomorphisms above are non-zero. Then, let $r^s_t, q^s_t$ be given by

\[
r^s_t = \lambda^s_t \cdot \text{dim}_K(\text{Hom}(I_s, M_t \Lambda)), \text{ and } q^s_t = \mu^s_t \cdot \text{dim}_K(\text{Hom}(M_t, I_s \Lambda)).
\]

Also, let

\[
r^s_t = r^s_t \cdot \text{dim}_K(\text{Hom}(I_s \Lambda, M_t \Lambda^2)), \text{ and } q^s_t = q^s_t \cdot \text{dim}_K(\text{Hom}(M_t \Lambda, I_s \Lambda^2)).
\]
Let $R$ denote the $|T| \times |S|$ matrix $R = [r_{ij}^s \Phi_{I_i,M_j \Lambda}]$. Similarly, let $Q$ denote the $|S| \times |T|$ matrix $Q = [q_{ij}^s \Phi_{M_j,I_i \Lambda}]$. Also, set $\bar{R} = [\bar{r}_{ij}^s \Phi_{I_i,M_j \Lambda}], \bar{Q} = [\bar{q}_{ij}^s \Phi_{M_j,I_i \Lambda}]$.

Then, since $\phi_{ij}^s = \lambda_{ij}^s \Phi_{I_i,M_j \Lambda}$ and $\psi_{ij}^s = \mu_{ij}^s \Phi_{M_j,I_i \Lambda}$, the homomorphisms $\phi_i$ and $\psi_i$ correspond to an interleaving if and only if the equations below are satisfied, when evaluated at all elements of the poset $P$.

$$ (1) \quad \bar{Q} \cdot R = [\bar{q}_{ij}^s \Phi_{M_j,I_i \Lambda \Lambda^2}] \cdot [\bar{r}_{ij}^s \Phi_{I_i,M_j \Lambda^2}] = [\Phi_{I_i,M_j \Gamma}]$$

Therefore, in this situation $V^\Lambda,\Gamma(I,M)$ is the affine algebraic set with coordinate ring given by $K[\{\lambda_{ij}^s : \text{Hom}(I_i, M_j \Lambda) \neq 0\}, \{\mu_{ij}^s : \text{Hom}(M_j, I_i \Gamma) \neq 0\}]$ modulo the ideal given by all identities from (1). For some computations, see Examples 10, 11.

When $P$ is not an $n$-Vee (or at least a tree branching only at a unique minimal element), the collection of interleavings still admits the structure of a variety, though the description is more cumbersome.

**Remark 4.** Using Proposition 37, we may visualize the interleaving distance between two elements of $C$ as follows. Let $(a, b)$ be any weight, and let $I, M \in C$. For each $\epsilon \in \{h(\Lambda)\}$, let $V_\epsilon(I, M)$ denote the variety of $(\Lambda, \Lambda_\epsilon)$-interleavings between $I$ and $M$. Then,

$$ D(I, M) = \min\{\epsilon : \text{the variety } V^{\Lambda,\Lambda_\epsilon}(I, M) \text{ is non-empty}\}. $$

For some computations see Example 12.

We now observe that our width gives rise to a bottleneck metric when $P$ is an $n$-Vee. For this $W$ must be compatible with the interleaving distance in the sense of Subsection 3.4.

**Proposition 38.** Let $P$ be an $n$-Vee, and let $(a, b)$ be weights. Let $D = D_{a,b}$ be the interleaving distance, and $W$ be the width function. Then, for $I, M$ convex,

$$ |W(I) - W(J)| \leq D(I, J). $$

The proof, which proceeds in cases, is omitted. Since $W$ and $D$ are compatible on $\Sigma$, we obtain a bottleneck metric on the category $C$ (see Subsection 3.4). Let $D_B$ denote this bottleneck metric. In the next section we will prove an isometry theorem for 1-Vees.

### 6. ISOMETRY THEOREM FOR FINITE TOTALLY ORDERED SETS

We now prove the isometry theorem for finite totally ordered sets. We will fix notation in this section for our poset. Let $P = \{m < m_1 < m_2 < \cdots < n = M_1\} = [m,n]$ be totally ordered (a 1-Vee), and fix any weight $(a, b)$. Note that, in this section only, $n$ does not correspond to the number of maximal elements in $P$. We begin with some preliminary observations.

**Lemma 39.** Let $P = \{m < m_1 < m_2 < \cdots < n = M_1\} = [m,n]$, and suppose $\Lambda$ be a power of a maximal translation with given height. Then,

1. $\text{im}(\Lambda) \cap P = [\Lambda(m), n]$.
2. If $i \in [\Lambda(m), n)$, then $\Lambda^{-1}(i)$ is a singleton.
3. $\Lambda i = \Lambda j \in P \implies i = j$ or $\Lambda i = \Lambda j = n$.

The result follows from the form of the maximal translation $\Lambda$ (see the proof of Lemma 22). Note that the power of a maximal translation need not be maximal. Moreover, $h(\Lambda^2)$ need not be $2h(\Lambda)$. The following Lemma follows from our characterization of the homomorphisms between convex modules in the last section (see Lemma 34).
\textbf{Lemma 40.} If $I,J$ are convex modules for $P = \{m < m_1 < m_2 < \ldots < n = M_1\} = [m,n]$, then \Hom(I,J) \neq 0 if and only if the endpoints of \Supp(I) = [x,X] and \Supp(J) = [y,Y] satisfy
\[ y \leq x \leq Y \leq X. \]

As previously mentioned, any homomorphism is a scalar in $K$ times $\Phi_{I,J}$ (see Definition 32).

\textbf{Lemma 41.} Let $P$ be as above, and suppose $\Lambda = \Lambda_e$ is a maximal translation. Let $A$ and $B$ be convex, and suppose $AA, BA \neq 0$ and \Hom(A,B) \neq 0. Then \Hom(A\Lambda, B\Lambda) \neq 0.

\textbf{Proof.} Let $s \in \Supp(A) \cap \Supp(B)$. If $s \in \text{im}(\Lambda)$ we are done by Lemma 35. Otherwise, $[x,Y] \cap \text{im}(\Lambda)$ is empty, where $\Supp(A) = [x,X]$, $\Supp(B) = [y,Y]$ with $y \leq x \leq Y \leq X$ as in Lemma 40. But then, by Lemma 39, $[y,Y]$ is disjoint from the image of $\Lambda$, therefore, $B\Lambda = 0$, a contradiction. \hfill \square

Lastly, the following is an easy consequences of the results of the previous section (see Definitions 27, 28).

\textbf{Lemma 42.} Let $P$ be totally ordered. Then the following are equivalent:

(i) $\Hom(J, JA^2) \neq 0$
(ii) there is an $x \in \Supp(J)$ with $A^2x \in \Supp(J)$
(iii) $J+A^2 \neq 0$
(iv) $(J+A^2)\Lambda \neq 0$
(v) $J-A^2 \neq 0$.

We are now ready to prove that every interleaving induces a matching of barcodes when $P$ is a 1-Vee. This is very much an algebraic reformulation of the results of Bauer and Lesnick in [BL13] applied to our framework. We will make use of their canonical matchings of barcodes induced by injective or surjective module homomorphisms.

\textbf{Definition 43.} (see Section 4 in [BL13]) Let $P$ be totally ordered, and let $I = \bigoplus_{s \in S} I_s, M = \bigoplus_{t \in T} M_t$ be in $C$. Let $f$ be a module homomorphism from $I \xrightarrow{f} M$. Then,

(i) if $f$ is surjective, let $\Theta(f)$ from $B(M)$ to $B(I)$ be the canonical matching of barcodes.
(ii) if $f$ is injective, let $\Theta(f)$ from $B(I)$ to $B(M)$ be the canonical matching of barcodes.

Recall from [BL13], $\Theta$ is categorical on injections or surjections. That is, $f = g \circ h$, $f, g, h$ surjections implies $\Theta(f) = \Theta(h) \circ \Theta(g)$. And dually, $f = g \circ h$, $f, g, h$ injections implies $\Theta(f) = \Theta(g) \circ \Theta(h)$.

The authors wish to emphasize that the above statements holds for any permissible enumeration on each barcode. That is to say, for each module $M$, all isomorphic elements of the barcode $B(M)$ may be enumerated arbitrarily. This enumeration is then fixed. In one instance, it will be convenient (though not necessary) to choose explicitly an enumeration for a particular barcode.

We now establish some additional properties of convex modules for 1-Vees.

\textbf{Lemma 44.} Let $P = [m,n]$ be a 1-Vee, $\Lambda$ a maximal translation on $P$. Let $\Sigma$ be the set of isomorphism classes of convex modules. Let $F,G$ be the functions
\[ F,G : \Sigma \rightarrow \Sigma \cup \{0\} \]
where $F(\sigma) = \sigma^{-A^2}$ and $G(\sigma) = \sigma^{+A^2}$.

Let $\Sigma_0 = \{\sigma : W(\sigma) > h(\Lambda)\}$, and $\Sigma$ be $\Sigma_0 \cap \{\sigma \in \Sigma : \Supp(\sigma) = [x,X], \text{ with } A^2x = n\}$.

(i) $F(\Sigma_0) \subset \Sigma$, and $F$ is one-to-one on $\Sigma_0$.
(ii) $G(\Sigma_0) \subset \Sigma$, and $G$ is one-to-one on $\Sigma_0 - \Sigma$. Also, $G(\Sigma) = \{\sigma_n\Lambda\}$, where $\sigma_n$ is the convex module with support $[n]$.\hfill 23
Proof. We will show that if \( \sigma_1, \sigma_2 \in \Sigma_0 \) with \( F(\sigma_1) \cong F(\sigma_2) \), then \( \sigma_1 \cong \sigma_2 \). Since convex modules are characterized by their supports, say \( F(\sigma_1), F(\sigma_2) \) have shared support \([x, X']\). Then \( X' \) is maximal such that \( \Lambda^2 X' \leq X_1 \) and also such that \( \Lambda^2 X' \leq X_2 \) where \( \sigma_1, \sigma_2 \) have support given by \([x, X_1]\), and \([x, X_2]\) respectively. But then by Lemma 39, \( X_1 = X_2 \). Then \( \sigma_1 \cong \sigma_2 \). This proves (i).

For (ii), we’ll prove the contrapositive. Suppose \( \sigma_1, \sigma_2 \in \Sigma_0 - \bar{\Sigma} \) have supports given by \([x_1, X_1]\), \([x_2, X_2]\) respectively. Suppose \( \Lambda^2 x_1 < \Lambda^2 x_2 \leq n \), then by Lemma 39, \( \Lambda x_1 < \Lambda x_2 \). Then, again by Lemma 39, \( G(\sigma_1) = [\Lambda x_1, \cdot], G(\sigma_2) = [\Lambda x_2, \cdot] \), which are distinct. On the other hand, if \( x_1 = x_2, X_1 < X_2 \leq n \), then \( \sigma_1^\Lambda, \sigma_2^\Lambda \) have supports given by \( [\Lambda^2 x_1, X_1], [\Lambda^2 x_1, X_2] \) respectively. But then, since only \( X_2 \) is possibly equal to \( n \), the right endpoint of the support of \( G(\sigma_2) \) is strictly larger than the right endpoint of the support of \( G(\sigma_1) \).

Clearly, if \( \sigma \in \bar{\Sigma} \), the support of \( \sigma \) is \([x, n]\), with \( \Lambda^2 x = n \). Then, by inspection, \( G(\sigma) = \sigma_n \Lambda \). Moreover, it is clear from the proof that \( G^{-1}(\sigma_n \Lambda) \subseteq \bar{\Sigma} \).

\( \square \)

**Proposition 45.** Let \( P \) be totally ordered and let \( C \) be the full subcategory of \( A(P) \)-modules consisting of direct sums of convex modules. Let \((a, b) \in \mathbb{N} \times \mathbb{N}\) be a weight and let \( D \) denote interleaving distance (corresponding to the weight \((a, b)\)) restricted to \( C \).

Let \( W(M) = \min \{ \epsilon : \text{Hom}(M, M \Gamma \Lambda) = 0, \Gamma, \Lambda \in \mathcal{T}(P), h(\Gamma), h(\Lambda) \leq \epsilon \} \), and let \( D_B \) be the bottleneck distance on \( C \) corresponding to the interleaving distance and \( W \). Then, the identity is an isometry from

\[
(C, D) \xrightarrow{\text{Id}} (C, D_B).
\]

This corresponds to the case that \( P \) is a 1-Vee in Theorem 1. The result follows from Theorem 2. We will proceed in the same fashion as [BL13]. Before continuing, we point out that Theorem 2 (and later Theorem 1) do not say that every interleaving is diagonal (see Examples 10, 11). Instead, they simply constrain the isomorphism classes of modules which admit an interleaving.

**Theorem 2.** Let \( P \) be totally ordered \((P \text{ is a 1-Vee})\) and let \( I = \bigoplus_{s \in S} I_s, M = \bigoplus_{t \in T} M_t \) be in \( C \). Let \( \Lambda = \Lambda_\epsilon \in \mathcal{T}(P) \) be maximal with \( h(\Lambda) = \epsilon \). Suppose there exists a \((\Lambda, \Lambda)\)-interleaving between \( I \) and \( M \). Then there exists a \(h(\Lambda)\) matching from \( B(I) \) to \( B(M) \).

The proof of the Theorem will consist of three parts.

1. If \( W(I_s) > h(\Lambda) \), then \( I_s \) is matched.
2. If \( W(M_t) > h(\Lambda) \), then \( M_t \) is matched.
3. If \( I_s \) and \( M_t \) are matched (independent of \( W \)), then there is a \((\Lambda, \Lambda)\)-interleaving between \( I_s \) and \( M_t \).

Our matching is a slight modification of the matching in [BL13]. It is given by the following composition (see Definition 43)

\[
B(I) \xrightarrow{\Theta(\rho)^{-1}} B(im(\phi)) \xrightarrow{\Theta(\iota)} B(M \Lambda) \xrightarrow{B(M, MA)} B(M),
\]

where \( \iota \) is the inclusion from \( im(\phi) \) into \( M \Lambda \), \( \rho \) is the surjection from \( I \) to \( im(\phi) \), and \( B(M, MA) \) is the natural inclusion of barcodes induced from \( M \Lambda \) to \( M \) given by \( B(M, MA)(M_t \Lambda) = M_t \).

Note that in [BL13], it was only necessary to take the matching as far as \( B(M, MA) \). There, that was justified since the assignment which we call \( B(M, MA) \) was a bijection between barcodes which preserved \( W \). In the present context neither of these properties hold. Specifically, \( |B(M, MA)| \) may be strictly smaller than \( |B(M)| \). Moreover, either of \( W(M_t), W(M_t \Lambda) \) may be strictly larger than the other. The detailed schematic below displays all relevant convex modules. This will be useful in the proof.
We now prove Theorem 2.

**Proof.** First, say $I_s \in B(I)$ with $W(I_s) > h(\Lambda)$. Then, by Lemma 42, $I_s^{-\Lambda^2} \neq 0$. Additionally, by Proposition 29, and since induced matchings are categorical for surjections, we obtain the commutative triangle of barcodes below.

$$
\begin{array}{ccc}
I & \xrightarrow{\text{im}\phi} & B(I) \\
I^{-\Lambda^2} & \xrightarrow{\text{im}\phi} & B(I^{-\Lambda^2}) \\
\end{array}
$$

But the induced matching $B(I^{-\Lambda^2}) \to B(I)$ sends $I_s^{-\Lambda^2} \xrightarrow{\Theta} I_s$ up to isomorphism, therefore $I_s$ is matched with an element of $B(\text{im}(\phi))$. That is, $I_s \in \text{im}(\Theta(\rho))$. But then, since $\Theta(\iota)$ and $B(\Phi^\Lambda_M)$ are injections of barcodes, $I_s$ is matched with some $M_t \in B(M)$. This establishes (1).

Next, suppose $M_t \in B(M)$ with $W(M_t) > h(\Lambda)$. Then, by Lemma 42, $M_t^{+\Lambda^2} \Lambda \neq 0$. Moreover, by Proposition 29, and since induced matchings are categorical for injections, we obtain a commutative diagram of barcodes for any choice of admissible enumeration. It is convenient to specify a particular enumeration for $B(M^{+\Lambda^2} \Lambda)$. This is done as follows;

- For $\sigma \in B(M^{+\Lambda^2} \Lambda)$, $\sigma \nleq \sigma_n \Lambda$ (see Lemma 44), there is no restriction on the enumeration restricted to $\{\sigma\}$.
- For $\sigma \in B(M^{+\Lambda^2} \Lambda)$, $\sigma \cong \sigma_n \Lambda$, enumerate $\{\sigma\}$, by $\sigma_1 = G(\tau_1) \leq G(\tau_2) = \sigma_2$ if and only if $\tau_1 \leq \tau_2$.

With this choice of enumeration, we obtain the commutative diagram below.

$$
\begin{array}{ccc}
im(\phi) & \xrightarrow{M \Lambda} & B(im(\phi)) \\
M^{+\Lambda^2} \Lambda & \xrightarrow{B(M \Lambda)} & B(M) \\
\end{array}
$$

Since $B(\rho)$ is an injection of barcodes, this proves (2).
We now prove (3). First, note that if $I_s$ and $M_t$ are matched with $h(I_s), h(M_t) \leq \epsilon$, then setting $\phi, \psi$ both equal to zero, we obtain a $(\Lambda, \Lambda)$-interleaving between $I_s$ and $M_t$. Therefore, let
\[
S' = \{ s : h(I_s) > \epsilon \}, \text{ and } T' = \{ t : h(M_t) > \epsilon \}.
\]

We will write $I_s \mathrel{\updownarrow} M_t$ when $I_s$ and $M_t$ are matched. It remains to show that if $I_s \mathrel{\updownarrow} M_t$, then there is a $(\Lambda, \Lambda)$-interleaving between $I_s$ and $M_t$ when,

(a) $s \in S', t \notin T'$,
(b) $s \notin S', t \in T'$, or
(c) $s \in S', t \in T'$

Note that because of the asymmetry associated with the matching, the cases (b.) and (c.) are not identical. Let the supports of $I_s, I_s, M_t, M_t, \Lambda$ be given by $[w, W], [x, X], [y, Y]$, and $[z, Z]$ respectively. When $s \in S'$, let $X^0$ be maximal such that $\Lambda^2 X^0 \leq X$. That is, $I_s^\Lambda \Lambda^2$ has support given by $[x, X^0)$. Similarly, when $t \in T'$, let $y^0 = \Lambda z$, so then $M_t^\Lambda \Lambda^2 \Lambda$ has support given by $[y^0, Y]$. Note that if $z \notin im(\Lambda)$, we have that $\Lambda z < \Lambda^2 y$ and $y = m$.

Proceeding as in [BL13], by Proposition 25, if $I_s \mathrel{\updownarrow} M_t$, then we have the relations
\[
y \leq x \leq Y \leq X.
\]

Hence, there is a non-zero homomorphism from $I_s \to M_t \Lambda$. Therefore, set $\Phi_{I_s, M_t} = \chi([x, Y]) = \phi'. This will be one of our interleaving morphisms. We next define our second interleaving morphism. We must show that if one of (a), (b), or (c) is satisfied, we have the relations,
\[
w \leq z \leq W \leq Z.
\]

By inspection, it suffices to show the following statements:

(i) If $t \in T'$, then $w \leq z$.
(ii) If $s \in S'$, then $z \leq W$ and $W \leq Z$.
(iii) If $s \in S'$ and $t \notin T'$, then $w \leq z$.
(iv) If $s \notin S'$ and $t \in T'$, then $z \leq W$ and $W \leq Z$.

We now prove (i) through (iv). First, if $t \in T'$, then $\Lambda z = y_0$. Also, $w$ is minimal such that $\Lambda w \geq x$. As $x \leq y_0, x \leq \Lambda z$, and so $w \leq z$ by minimality. This proves (i).

Next, say $s \in S'$. Then, $z \leq \Lambda y$ by definition. Also, since $x \leq y$ we have that $\Lambda y \leq \Lambda x$. As $W$ is maximal such that $\Lambda W \leq X$, and $s \in S'$, we have $\Lambda^2 x \leq X$. Therefore, $\Lambda x \leq W$. Since $\Lambda(\Lambda x) \leq X, \Lambda x \leq W$. Therefore $z \leq \Lambda y \leq \Lambda x \leq W$ as required. Continuing, since $s \in S', X_0$ is maximal such that $\Lambda^2 X_0 = X$. By the maximality of $W, \Lambda X_0 = W$. But then we have $W = \Lambda X_0 \leq \Lambda Y \leq Z$, since $im(\phi)$ includes into $I^\Lambda \Lambda^2$. This proves (ii).

Now, suppose $s \in S', t \notin T'$. If $x \geq \Lambda m$, then $\Lambda w = x$, and so $\Lambda^3 w = \Lambda^2 x \leq X$, since $s \in S'$. But then, $W \geq \Lambda^2 w$. Hence, since $t \notin T'$, we have $\Lambda^2 w \leq W \leq Y < \Lambda^2 z$. The result follows from monoticity. On the other hand, if $x < \Lambda m$, then $w = m$, so $w \leq z$. Thus we have shown (iii).

Lastly, say $s \notin S', t \in T'$. We must establish $z \leq W$ and $W \leq Z$. First, since $t \in T'$, we have that $\Lambda z = y_0 \leq Y \leq Z$. Since $W$ is maximal with $\Lambda W \leq X$, it follows that $z \leq W$. Next, note that if $t \in T'$, then $\Lambda W = X$, since $\Lambda W \neq X \implies X \notin im(\Lambda)$. Then $\Lambda^2 z = \Lambda(\Lambda z) \leq Z$, so $y \geq \Lambda z$. Therefore $X$ is in $im(\Lambda)$ so it must be the case that $\Lambda W = X$. But then since $s \notin S', t \in T'$, we have $\Lambda W = X < \Lambda^2 x \leq \Lambda^2 y_0 = \Lambda^3 z \leq \Lambda Z$. Therefore, $\Lambda W \neq n$, so by monoticity $W < Z$ as required. This proves (iv).

Thus, we have shown that if $s \in S'$ or $t \in T'$,
\[
w \leq z \leq W \leq Z.
\]

Therefore, set $\Phi_{M_t, I_s} = \chi([z, W]) = \psi'. This will be our second interleaving morphism. It now remains only to show that
\[
\psi' \Lambda \circ \phi' = \Phi_{I_s}^{\Lambda^2}, \text{ and } \phi' \Lambda \circ \psi' = \Phi_{M_t}^{\Lambda^2}.
\]
Thus, we have;
\[
\phi' \Lambda = \chi([x, Y])\Lambda = \chi([w, Y^*]), \quad \text{and} \quad \psi' \Lambda = \chi([z, W])\Lambda = \chi([y, W^*]),
\]
where \(Y^*\) is maximal such that \(\Lambda Y^* \leq Y\), and \(W^*\) is the maximal with \(\Lambda W^* \leq W\).

We now proceed to establish the required commutativity conditions. First, say \(s \in S'\). We will show that \(\psi' \Lambda \circ \phi' = \Phi_{I_s, I_s, A^2} = \chi([x, W^*])\). Note that, by definition, \(\psi' \Lambda \circ \phi'\) is a composition of module homomorphisms, and, hence, a module homomorphism. Therefore, by Lemma 34, we need only show that the linear map \(\chi([x, W^*])\) is non-zero at any vertex. To do this, we will establish that \(x \leq W^*\) (that is, \(\chi([x, W^*])\) is non-zero at \(x\)). But, \(s \in S' \implies \Lambda^2 x \leq X\). As \(W^*\) is maximal with \(\Lambda^2 W^* = X\), the inequality follows. Now say \(s \notin S'\). Then, \(\psi' \Lambda \circ \phi' = 0\) as required.

We now show the commutativity of the other triangle. First, suppose that \(t \in T'\). As above, we will show that \(\phi' \Lambda \circ \psi' = \Phi_{M_t, M_t, A^2} = \chi([y, Y^*])\). Again, we need only demonstrate that \(z \leq Y^*\). But \(t \in T' \implies \Lambda^2 z \leq Z\). Since \(Y^*\) is maximal with \(\Lambda^2 Y^* = Z\), the result follows. Again, if \(t \notin T'\), the result is trivial.

Therefore, if \(I_s \downarrow M_t\), then there is a \((\Lambda, \Lambda)\)-interleaving between \(I_s\) and \(M_t\) as required. This proves (3) and finishes the proof of the theorem. \(\square\)

In the next section we will use Theorem 2 to prove our main result.

7. PROOF OF MAIN RESULTS

Before proving the main results, we establish some useful facts. This first result will allow us to make a "half matching."

**Lemma 46.** Let \(S, T\) be sets with \(S\) finite, let \(x : S \to \mathcal{P}(T)\) be a function such that for all \(\phi \neq S_0 \subseteq S\),
\[
\left| \bigcup_{s \in S_0} x(s) \right| \geq |S_0|.
\]

Then, there exists a function \(F : S \to T\) such that \(F\) is an injection, and for all \(s, F(s) \in x(s)\).

**Proof.** We prove the result by induction on \(|S|\). If \(|S| = 1\), the result is trivial. Now say \(|S| > 1\) and the result holds for all sets with smaller cardinality. First, suppose there exists a non-empty subset \(S_0 \subseteq S\) such that
\[
\left| \bigcup_{s \in S_0} x(s) \right| = |S_0|.
\]

Let \(S_0\) be a minimal non-empty subset of \(S\) where equality holds. We will show that we can define an injection \(f\) from \(S_0\) to \(T\) with \(f(s) \in x(s)\). Pick \(s_0 \in S_0\), \(t_0 \in x(s_0)\) and set \(f(s_0) = t_0\). If \(S_0 = \{s_0\}\) we are done, so assume \(S_0 \neq \{s_0\}\). Then, let \(\overline{x} : S_0 - \{s_0\} \to \mathcal{P}(T)\), be defined by \(\overline{x}(s) = x(s) - \{t_0\}\). Now let \(S'\) be a non-empty subset of \(S_0 - \{s_0\}\). Then,
\[
\left| \bigcup_{s \in S'} \overline{x}(s) \right| = \left| \bigcup_{s \in S'} x(s) - \{t_0\} \right| = \left| \bigcup_{s \in S'} x(s) \right| - \{t_0\} \geq |S'| + 1 - 1 = |S'|
\]
by the minimality of \(S_0\). Thus, by induction, there exists a one-to-one function \(f : S_0 - \{s_0\} \to T\) such that \(f(s) \in \overline{x}(s)\). Clearly, \(f\) can be extended to an injection on all of \(S_0\). If \(S_0 = S\), set \(f = F\) and we are done. Otherwise, define
\[
\overline{x} : S - S_0 \to \mathcal{P}(T)\] be defined by \(\overline{x}(s) = x(s) - \{f(\sigma) : \sigma \in S_0\}\).

Now, let \(\overline{s}_1, \overline{s}_2, ..., \overline{s}_k \in S - S_0\). Clearly, for all \(i, x(\overline{s}_i) = \overline{x}(\overline{s}_i) \cup T_i\) for some set \(T_i \subseteq \{f(\sigma) : \sigma \in S_0\}\). Note that
\[
\left| \bigcup_{i \leq k} \overline{x}(\overline{s}_i) \right| < k \implies \left| \bigcup_{s \in S_0} x(s) \cup x(\overline{s}_1) \cup x(\overline{s}_2) \cup ... \cup x(\overline{s}_k) \right| < |S_0| + k
\]
a contradiction. Thus, by induction, there is an injection $\bar{f} : S - S_0 \rightarrow T$ with $\bar{f}(s) \in \bar{x}(s)$. By construction $F = f \cup \bar{f}$ is the desired function from all of $S$ to $T$.

On the other hand, if $S$ has the property that for all $S_0 \subseteq S$, $S_0 \neq \phi$,

$$\left| \bigcup_{s \in S_0} x(s) \right| > |S_0|,$$

pick $s_1 \in S, t_1 \in x(s_1)$ and set $f(s_1) = t_1$. Again, let

$$\bar{x} : S - \{s_1\} \rightarrow \mathcal{P}(T)$$

be defined by $\bar{x}(s) = x(s) - \{t_1\}$.

Then, for $S_0 \subseteq S - \{s_1\}$,

$$\left| \bigcup_{s \in S_0} \bar{x}(s) \right| = \left| \bigcup_{s \in S_0} x(s) - \{t_1\} \right| \geq |S_0| + 1 - 1 = |S_0|.$$  

Since $|S - \{s_1\}| < |S|$, the result holds by induction. \qed

**Example 9.** Let $S = \{1, 2, 3, 4, 5\}, T = \{a, b, c, d, e\}$ the function $x$ given by

$$1 \rightarrow \{a, b, d\}, 2 \rightarrow \{b, c, e\}, 3 \rightarrow \{a, c, d\}, 4 \rightarrow \{d\}, 5 \rightarrow \{e\}.$$  

A matching is constructed by setting $f(4) = d$, and $f(5) = e$. Then, one can choose any bijection from $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$. We glue $f$ and $\bar{f}$ to obtain an injection $F$ from $S$ to $T$.

Next we make a simple observation about interleavings.

**Lemma 47.** Let $P$ be any poset, $\Lambda, \Gamma \in T(P)$. Let $A, B, C, D$ be any $A(P)$-modules with $\phi, \psi$ a $(\Lambda, \Gamma)$-interleaving between $A \oplus B$ and $C \oplus D$. Then, if $\text{Hom}(A, DA) = 0 = \text{Hom}(C, BT)$, then $A, C$ are $(\Lambda, \Gamma)$-interleaved and $B, D$ are $(\Lambda, \Gamma)$-interleaved.

**Proof.** Note that we do not assume any modules are in the category $C$. For brevity, let $f_A, f_B$ denote the canonical homomorphism from $A \rightarrow A\Gamma$ and $B \rightarrow B\Gamma$ respectively. Similarly, let $g_C, g_D$ denote $C \rightarrow C\Lambda\Gamma$ and $D \rightarrow D\Lambda\Gamma$ respectively. By decomposing $\phi, \psi$ into their component homomorphisms, we have;

$$\begin{bmatrix} f_A & 0 \\ 0 & f_B \end{bmatrix} = \begin{bmatrix} \psi_A^C & \psi_A^D \\ \psi_B^A & \psi_B^D \end{bmatrix} \begin{bmatrix} \phi_A^C & \phi_A^D \\ \phi_B^C & \phi_B^D \end{bmatrix} = \begin{bmatrix} \psi_A^C \phi_A^C & \psi_A^C \phi_A^D + \psi_A^D \phi_B^D \\ \psi_B^A \phi_B^C & \psi_B^A \phi_B^D \end{bmatrix},$$

$$\begin{bmatrix} g_C & 0 \\ 0 & g_D \end{bmatrix} = \begin{bmatrix} \phi_C^A \Gamma & \phi_C^D \Gamma \\ \phi_D^A \Gamma & \phi_D^D \Gamma \end{bmatrix} \begin{bmatrix} \psi_C^A & \psi_C^D \\ \psi_D^A & \psi_D^D \end{bmatrix} = \begin{bmatrix} \phi_C^A \Gamma \psi_C^A & \phi_C^A \Gamma \psi_C^D + \phi_D^D \Gamma \psi_D^B \\ \phi_D^A \Gamma \psi_D^A & \phi_D^A \Gamma \psi_D^D \end{bmatrix}.$$  

Thus, by inspection, if we set $\phi_B^C, \psi_A^D = 0$, the required condition will still be satisfied. \qed

We point out that this does not say that the interleaving was initially diagonal (see Example 11).

**Corollary 48.** Let $P$ be a finite poset with a unique minimal element $m$. Let $X, Y \in C$, and $\Lambda$ be a translation. Suppose $X = \bigoplus_x X_s$, and $Y = \bigoplus_y Y_t$ are $(\Lambda, \Lambda)$-interleaved, and $\Lambda m = m$. Let $S_m = \{s \in S : X_s(m) \neq 0\}, T_m = \{t \in T : Y_t(m) \neq 0\}$. Then,

$$\bigoplus_{s \in S_m} X_s, \bigoplus_{t \in T_m} Y_t \text{ are } (\Lambda, \Lambda)-\text{interleaved, and } \bigoplus_{s \notin S_m} X_s, \bigoplus_{t \notin T_m} Y_t \text{ are } (\Lambda, \Lambda)-\text{interleaved.}$$  

**Proof.** This follows easily from Lemma 47. \qed

**Proposition 49.** Let $P$ be an $n$-Vee. Let $I = \bigoplus_{s \in S} I_s, M = \bigoplus_{t \in T} M_t$ be in $C$. Suppose for all $s, t, I_s$ and $M_t$ are supported at $m$. Let $\Lambda \in T(P)$ with $\Lambda m = m$. Suppose there exists a $(\Lambda, \Lambda)$-interleaving between $I$ and $M$. Then there exists a $h(\Lambda)$ matching (in the sense of Theorem 2) from $B(I)$ to $B(M)$. 

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Proof. First, we show that $|B(I)| = |B(M)|$. Since $m$ is fixed by $\Lambda$, the commutativity of the diagram below shows that $|B(I)| = \text{rank}(f) \leq \dim(M(m)) = |B(M)|$.

$$(\bigoplus_{s \in S} I_s)(m) \xrightarrow{\phi} (\bigoplus_{t \in T} M_t \Lambda)(m) = (\bigoplus_{t \in T} M_t)(m)$$

Thus, by symmetry $|B(I)| = |B(M)|$. Now, let $s \in S$. Since $I_s \to I_s \Lambda^2$ is nonzero, its image is in the image of $\psi \Lambda \phi_I$. Thus in particular, there exists a $t \in T$ with $\psi \Lambda_\phi^t \neq 0$. That is, $\text{Hom}(M \Lambda, I_s \Lambda^2) \circ \text{Hom}(I_s, M \Lambda) \neq 0$. But by Lemma 35 then $\text{Hom}(I_s \Lambda, M \Lambda^2)$ is also not equal to zero. So, $\text{Hom}(M_t, I_s \Lambda^2), \text{Hom}(I_s \Lambda, M_t \Lambda^2)$ are both nonzero, hence their composition is nonzero since it is defined at $m$. But then, up to a scalar, it is the composition $M_t \to M_t \Lambda^2$, since $\text{Hom}(M_t, M_t \Lambda^2) = K$ by Lemma 34. Thus, there is a $(\Lambda, \Lambda)$-interleaving between $I_s$ and $M_t$. We have shown that whenever $\psi \Lambda_\phi^t \neq 0$, there is a $(\Lambda, \Lambda)$-interleaving between $I_s$ and $M_t$.

Now for $s \in S$, let $x(s) = \{t \in T : \psi \Lambda_\phi^t \neq 0\}$. Let $S_0 \subseteq S$. Then, the diagram below commutes

$$(\bigoplus_{s \in S_0} I_s) \xrightarrow{\psi} (\bigoplus_{t \in x(s)} M_t)$$

Hence, by evaluation at $m$, $|S_0| = \text{rank}\{(\bigoplus_{s \in S_0} I_s)(m) \to (\bigoplus_{s \in S_0} I_s \Lambda^2)(m)\} \leq \bigcup_{s \in S_0} x(s)$. Then, by Lemma 46, there is a injection $f$ from $S$ to $T$ with $f(s) \in x(s)$ for all $s$. The result follows since $|S| = |T|$ and $t \in x(s)$ implies there is a $(\Lambda, \Lambda)$-interleaving between $I_s$ and $M_t$. \hfill \Box

We are now ready to prove our main results.

Proof of Theorem 1 First, let $P$ be an asymmetric $n$-Vee, $P = \bigcup\{m, M_i\}$ with $\|m, M_{i0}\| > \|m, M_i\|$ for $i \neq i_0$, and fix the weight $(a, b)$. We will prove that any $(\Lambda_1, \Gamma_1)$-interleaving between $I, M \in C$ produces an $\epsilon$-matching for $\epsilon = \max \{ h(\Lambda_1), h(\Gamma_1) \}$. Once this is established, $D_B \leq D$. For the other inequality, note that an $\epsilon$-matching yields (after inserting appropriate zero homomorphisms) a diagonal interleaving, thus $D \leq D_B$, and hence equality.

Let $I = \bigoplus_{s \in S} I_s, M = \bigoplus_{t \in T} M_t$ be in $C$. If $V$ is a partition of $P$, for $v \in V$, let

$$S_v = \{s \in S : \text{the minimal element of Supp}(I_s) is in } v\}$$

Similarly, define $T_v$. Now, suppose there is a $(\Lambda_1, \Gamma_1)$-interleaving between $I$ and $M$. Then, by Lemma 22, there exists $\Lambda = \Lambda_1, \Gamma_1 \leq \Lambda$, maximal, where $\epsilon = \max \{ h(\Lambda_1), h(\Gamma_1) \}$. By [BdS13] since $\Lambda_1, \Gamma_1 \leq \Lambda$, there exists a $(\Lambda, \Lambda)$-interleaving between $I$ and $M$.

First, if $\epsilon < aT + b$, then $\Lambda m = m$. In this case, consider the partition $V$ of $P$ given by

$$V = \{(m, M_i)\} \cup \{m\}, \text{ and set } S_m = S_{\{m\}}, S_i = S_{\{m, M_i\}}.$$

Similarly, set $T_m = T_{\{m\}}$ and $T_i = T_{\{m, M_i\}}$. Since $\Lambda m = m$, for all $M$ convex,

- $\text{Supp}(M) \subseteq (m, M_i), \text{MA} \neq 0 \implies \text{Supp MA} \subseteq (m, M_i)$, and
- $m \in \text{Supp}(M) \implies m \in \text{Supp}(MA)$.  

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Therefore, if \( s \in S_m, t \in T_i \), then \( \text{Hom}(I_s, M_t \Lambda) = 0 \). Similarly, if \( t \in T_m, s \in S_i \), then \( \text{Hom}(M_t, I_s \Lambda) = 0 \). Then, by Lemma 47, we may diagonalize, obtaining \((\Lambda, \Lambda)\)-interleavings between
\[
\bigoplus_{s \in S_m} I_s \quad \text{and} \quad \bigoplus_{t \in T_m} M_t,
\]
and also between \( \bigoplus_{s \notin S_m} I_s \) and \( \bigoplus_{t \notin T_m} M_t \).

We now diagonalize further. Again, since \( \Lambda m = m \), for each \( i \neq j, s \in S_i, t \in T_j \) we obtain interleavings between \( \text{Hom}(I_s, M_t \Lambda) = 0 \) as well as the symmetric condition. Therefore, by applying Lemma 47 repeatedly, we obtain interleavings between
\[
\bigoplus_{s \in S_v} I_s \quad \text{and} \quad \bigoplus_{t \in T_v} M_t \quad \text{for all} \quad v \in \mathcal{V}.
\]

Hence, by Proposition 49, we get a matching between the elements of the barcodes supported at \( m \). Also, for each \( i \), \( \Lambda_{[m,M_i]} \) is a maximal translation on a totally oriented set. Therefore, for each \( i \) we acquire a matching between those elements of the barcode in \( S_i \) and \( T_i \) by Theorem 2. Thus, an \( \epsilon \)-matching is produced piecewise.

Now, suppose \( \epsilon = h(\Lambda) \geq aT + b \). Then, for all convex modules \( M, M \Lambda \) is identically 0, or is a convex module supported in \([m,M_{i_0}]\). Then, \( M \Lambda = M \Lambda / \mathcal{I}_{i_0} M \Lambda \), thus any homomorphism from \( N \to M \Lambda \) factors through \( N / \mathcal{I}_{i_0} N \).

Consider the partition
\[
\mathcal{V} = \{[m,M_{i_0}]\} \cup \{(m,M_i) : i \neq i_0\}, \quad \text{and set} \quad S_i = S_{(m,M_i)}, S_m = S_{[m,M_{i_0}]}.
\]

Similarly, define \( T_m, T_i \). Then, for \( s \in S_m, t \in T_i \), \( \text{Hom}(I_s, M_t \Lambda) = 0 \), since \( M_t \Lambda = 0 \). Since the symmetric condition holds as well, again by Lemma 47 we obtain an interleaving between
\[
\bigoplus_{s \in S_m} I_s \quad \text{and} \quad \bigoplus_{t \in T_m} M_t, \quad \text{and between} \quad \bigoplus_{s \notin S_m} I_s \quad \text{and} \quad \bigoplus_{t \notin T_m} M_t.
\]

Since the latter interleaving corresponds to convex modules \( N \) with \( W(N) \leq \epsilon \), it suffices to match only convex modules with indices in \( S_m \) and \( T_m \).

However, the morphisms
\[
\bigoplus_{s \in S_m} I_s \xrightarrow{\phi} \bigoplus_{t \in T_m} M_t \Lambda, \quad \text{and} \quad \bigoplus_{t \in T_m} M_t \xrightarrow{\psi} \bigoplus_{s \in S_m} I_s \Lambda
\]
factor through \( \bigoplus_{s \in S_m} (I_s / \mathcal{I}_{i_0} I_s) \) and \( \bigoplus_{t \notin T_m} (M_t / \mathcal{I}_{i_0} M_t) \) respectively. Thus since \( \Lambda_{[m,M_{i_0}]} \) is maximal, again the result follows from Theorem 2.

If \( P \) is an \( n \)-Vee but is not asymmetric, then we may not use Lemmas 22, and 23 explicitly. It is still the case, however, that for \( \epsilon \in \{h(\Gamma) : \Gamma \in \mathcal{T}(P)\} \), with \( \epsilon < aT + b \), the set \( \{\Lambda : h(\Lambda) = \epsilon\} \) has a unique maximal element. Moreover, the set \( \{\Lambda_{\epsilon} : \epsilon < aT + b\} \) is still totally ordered. Thus, if \( I, M \) are \((\Lambda, \Gamma)\)-interleaved with \( \max\{h(\Lambda), h(\Gamma)\} < aT + b \) the proof above still goes through. On the other hand, when \( P \) is not asymmetric, for all convex modules \( \sigma \), \( W(\sigma) \leq aT + b = aT_{i_0} + b \). Therefore, though there is not a unique translation with height corresponding to this value, interleavings of this height always produce empty matchings.

8. Examples

We conclude with some examples. First, in Example 10, we decompose an interleaving as in the proof of Theorem 1. We also compute the varieties (see Proposition 37) corresponding to two interleavings. Along the way, we construct some non-diagonal interleavings. In this section, if \( M \) is convex with support given by \( S \), we write \( M \sim S \).
Example 10. Let $P$ be the 2-Vee, $P = [m, x_3] \cup [m, y_6]$ and let $(a, b)$ be a weight. Let $\Lambda = \Lambda_a$. Consider the following convex modules.

Let $I = A \oplus B \oplus X$ and $M = C \oplus D \oplus Y \oplus Z$. We will decompose an arbitrary $(\Lambda, \Lambda)$-interleaving between $I$ and $M$ as in the proof of Theorem 1. Then, we will calculate the varieties (see Remark 37) corresponding to the “factored” interleavings the decomposition produces on the appropriate partition of the barcodes $B(I)$ and $B(M)$.

First, let $\phi', \psi'$ be any $(\Lambda, \Lambda)$-interleaving between $I$ and $M$. By Lemma 47, since $\Lambda m = m$, there exist

- a $(\Lambda, \Lambda)$-interleaving between $A \oplus B$ and $C \oplus D$, and
- a $(\Lambda, \Lambda)$-interleaving between $X$ and $Y \oplus Z$.

We treat these separately, referring to each in turn as $\phi, \psi$. First, we factor each $\phi, \psi$ into their corresponding summands, adopting the previous notation. For example $\phi_X^Y : X \to Y \Lambda$. Since $I, M \in \mathcal{C}$, we know that

$$\phi_X^Y = \lambda \Phi_{X,Y} \Lambda, \text{ and } \phi^X_Y = \lambda \Phi_{X \Lambda, Y \Lambda^2}.$$  

Of course, all other similar identities hold as well. We first concentrate on the modules supported in $(m, M)$. Thus, we have the diagrams below.

Therefore, we have the matrices of module homorphisms

$$\phi = \begin{bmatrix} \phi_X^Y = \alpha \Phi_{X,Y} \Lambda \\ \phi_Z^X = \beta \Phi_{X,Z} \Lambda \end{bmatrix} \text{ and } \psi = \begin{bmatrix} \psi_X^Y = \lambda \Phi_{Y,X \Lambda} \\ \psi_X^Z = \mu \Phi_{Z,X \Lambda} \end{bmatrix}.$$  

Since $\phi, \psi$ is a $(\Lambda_a, \Lambda_a)$-interleaving between $X$ and $Y \oplus Z$ equations (2) and (3) below must hold. First,

$$\Phi_X^{\Lambda^2} = \begin{bmatrix} \Phi_X^{\Lambda^2} \\ \Phi_Z^{\Lambda^2} \end{bmatrix} = \begin{bmatrix} (\psi_X^Y \Lambda) \phi_X^Y + (\psi_X^Z \Lambda) \phi_Z^X \\ (\lambda \alpha + \beta \mu) \Phi_X^{\Lambda^2} \end{bmatrix}.$$  

And then,

$$\Phi_{Y \oplus Z}^{\Lambda^2} = \begin{bmatrix} \Phi_Y^{\Lambda^2} \\ \Phi_Z^{\Lambda^2} \end{bmatrix} = \begin{bmatrix} \alpha \lambda \Phi_{Y,Y \Lambda^2} + \alpha \mu \Phi_{Z,Y \Lambda^2} \\ \beta \lambda \Phi_{Y,Z \Lambda^2} + \beta \mu \Phi_{Z,Z \Lambda^2} \end{bmatrix}. $$
Note that the above equations define the variety $V^{\Lambda,\Lambda}(X, Y \oplus Z)$, because in the notation of Proposition 37, no variables are deleted between $R$ and $\bar{R}$, and $Q$ and $\bar{Q}$ respectively. For computational purposes, it is convenient to arrange the relevant convex modules side by side:

\[
\begin{array}{ccc}
 y_5 & y_4 & y_3 \\
 X & Y & Z \\
 y_2 & y_1 \\
 X\Lambda & Y\Lambda & Z\Lambda \\
 y_2 & y_1 \\
 X\Lambda^2 & Y\Lambda^2 & Z\Lambda^2 \\
 y_5 & y_4 & y_3 \\

\end{array}
\]

Notice that evaluating (2) at all elements of $P$ we have

\[
\left( \Phi^A_{X^2} \right)(y_3) = [1] \quad \text{and} \quad \left( \Phi^A_{X} \right)(i) = [0] \quad \text{for any } i \neq y_3.
\]

Similarly, evaluating (3) we have

\[
\left( \Phi^A_{Y \oplus Z} \right)(y_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \left( \Phi^A_{Y \oplus Z} \right)(i) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } i \neq y_3.
\]

Therefore, we get the following system of equations:

\[
\lambda\alpha + \mu\beta = 1, \quad \alpha\lambda = 1, \quad \beta\lambda = 0.
\]

Thus, $V^{\Lambda,\Lambda}(X, Y \oplus Z)$ is the affine variety with coordinate ring $K[\lambda, \mu, \alpha, \beta]$ modulo the ideal $\langle \lambda\alpha + \mu\beta - 1, \alpha\lambda - 1, \beta\lambda \rangle$ (see Proposition 37).

This corresponds to one choice of parameter in $K^{*}$ and one in $K$.

Next consider the modules supported at $m$. We now compute the variety $V^{\Lambda,\Lambda}(A \oplus B, C \oplus D)$. Again, we must have the commutative triangles below.

\[
\begin{array}{ccc}
 A \oplus B & \xrightarrow{\Phi^A_{A \oplus B}} & (A \oplus B)\Lambda^2 \\
 \phi & \downarrow & \psi_A \\
 (C \oplus D)\Lambda & \xrightarrow{\phi} & (A \oplus B)\Lambda
\end{array}
\quad \quad \quad
\begin{array}{ccc}
 C \oplus D & \xrightarrow{\Phi^A_{C \oplus D}} & (C \oplus D)\Lambda^2 \\
 \psi & \downarrow & \phi_A \\
 (C \oplus D)\Lambda & \xrightarrow{\psi} & (A \oplus B)\Lambda
\end{array}
\]

Decomposing $\phi, \psi$ we have,

\[
\phi = \begin{bmatrix} \phi^A_A &= e\Phi_{A,CA} \\ \phi^B_A &= f\Phi_{B,CA} \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \psi^C_A &= i\Phi_{C,AA} \\ \psi^D_A &= j\Phi_{D,AA} \end{bmatrix}.
\]

Since $\phi, \psi$ is a $(\Lambda, \Lambda)$-interleaving between $A \oplus B$ and $C \oplus D$ (and since no variables are eliminated from $Q$ to $\bar{Q}$) we have,

\[
\Phi^A_{A \oplus B} = \begin{bmatrix} \Phi^A_A &= 0 \\ \Phi^B_A &= \Phi^B_{B,AA} \end{bmatrix} = \begin{bmatrix} (ei + gj)\Phi_{A,AA} \\ (ek + gl)\Phi_{A,BA} \end{bmatrix}
\]

Evaluating everything at $m$, we obtain

\[
\begin{bmatrix} ei + gj \\ ek + gl \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Since evaluation at any other element of the poset makes all equations trivial, this identity (along with the redundant one obtained from $\Phi^A_{C \oplus D}$) is the only necessary condition. Therefore, the space of $(\Lambda, \Lambda)$-interleavings between $A \oplus B$ and $C \oplus D$, $V^{\Lambda,\Lambda}(A \oplus B, C \oplus D)$ is $\text{GL}_2(K)$, the variety of invertible $2 \times 2$ matrices. In particular, many interleavings between $A \oplus B$ and $C \oplus D$ are as far from diagonal as possible.
By choosing a point in each variety separately, we obtain an interleaving between $I$ and $M$. Note that although we produced many interleavings, we did not classify the $(\Lambda, \Lambda)$-interleavings between $I$ and $M$. This is clear, as we passed from our original $\phi', \psi'$ to a pair of separate interleavings on a partition of each barcode. In the next example we compute the full variety of interleavings between two elements of $C$.

**Example 11.** Let $P$ be again be the 2-Vee, $P = [\lambda, x_3] \cup [\mu, y_5]$. Let $(a, b)$ be a weight, and set $\Lambda = \Lambda_{2a}$.

Let $I = A \oplus B$ and $M = C \oplus D$, and let $\Lambda = \Lambda_{2a}$. We will calculate $V_{\Lambda, \Lambda}(I, M)$, the variety corresponding to all $(\Lambda, \Lambda)$-interleavings between $I$ and $M$ (see Proposition 37). First, consider an arbitrary such interleaving, $\phi, \psi$. Of course, as always this yields the standard commutative triangles. Since $\lambda m = \mu$, there exist no non-zero morphisms from $A$ into any module not containing the minimal $m$. Moreover, $m \in \text{Supp}(\sigma \Lambda)$ if and only if $m \in \text{Supp}(\sigma)$. Hence, using our standard notation, it must be the case that $\phi_A$ and $\psi_C$ are identically zeros. Then, as matrices of module homomorphisms, we have

$$\phi = \begin{bmatrix} \phi^A_C & \phi^B_C \\ 0 & \phi^B_D \end{bmatrix} = \begin{bmatrix} \lambda \Phi^A_{C\Lambda} & \rho \Phi^B_{C\Lambda} \\ 0 & \mu \Phi^B_{D\Lambda} \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \psi^C_A & \psi^D_A \\ 0 & \psi^D_B \end{bmatrix} = \begin{bmatrix} \alpha \Phi^C_{A\Lambda} & \gamma \Phi^D_{A\Lambda} \\ 0 & \beta \Phi^D_{B\Lambda} \end{bmatrix}$$

Since $\phi$ and $\psi$ constitute a $(\Lambda, \Lambda)$-interleaving, we obtain equations (4) and (5) below.

\[
\Phi_I^A = \Phi_{AB}^A = \begin{bmatrix} \Phi_A^2 & 0 \\ 0 & \Phi_B^2 \end{bmatrix} = \begin{bmatrix} \alpha \lambda \Phi_{A, AA^2} & (\alpha \rho + \gamma \mu) \Phi_{B, AA^2} \\ 0 & \beta \mu \Phi_{B, BA^2} \end{bmatrix}
\]

\[
\Phi_M^A = \Phi_{CD}^A = \begin{bmatrix} \Phi_C^2 & 0 \\ 0 & \Phi_D^2 \end{bmatrix} = \begin{bmatrix} \lambda \alpha \Phi_{C, CA^2} & (\lambda \gamma + \rho \beta) \Phi_{D, DA^2} \\ 0 & \mu \beta \Phi_{D, DA^2} \end{bmatrix}
\]

Evaluating (4), we see that

\[
(\Phi_I^A)(m) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (\Phi_I^A)(y_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad (\Phi_I^A)(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for} \quad x \neq m, y_1
\]

From the evaluation at $y_1$, we get the restrictions:

$$\alpha \lambda = 1, \beta \mu = 1, \alpha \rho + \gamma \mu = 0,$$

the last of which only appears because $\Phi_{B, AA^2} \neq 0$.

When evaluating at $m$, we obtain the redundant constraint $\alpha \lambda = 1$. Since every homomorphism in (4) has support contained in $[\lambda, y_1]$, so there are no further restrictions from (4).

By inspection, evaluating (5) obtains no new conditions, since

$$\alpha \rho + \gamma \mu = 0 \iff \rho = -\lambda \gamma \mu \iff \lambda \gamma + \rho \beta = 0.$$

Therefore, $V_{\Lambda, \Lambda}(I, M)$ is the affine variety with coordinate ring $K[\lambda, \mu, \rho, \alpha, \beta, \gamma]$ modulo the ideal $(\lambda \alpha - 1, \mu \beta - 1, \alpha \rho + \gamma \mu)$.
Thus, the interleavings are parametrized by two elements of $K^*$ and one element of $K$.

We point out that in both Examples 10 and 11 there were additional degrees of freedom for interleavings not seen when one passes to a matching. In Example 12 we realize the interleaving distance as the minimum height of a translation with non-empty variety.

**Example 12.** Let $P$ be the 1-Vee $P = [m, x_3]$, and let $(a, b)$ be a weight, with $a < b$. Consider the convex modules

\[
A \sim [m, y_2] \quad \text{ and } \quad B \sim [m, y_1] \quad \text{ and } \quad [m]
\]

We will calculate the variety $V^\Lambda,\Lambda(I, M)$ for various choices of translations. We will use these calculations to point out the interleaving distance as in Remark 4. Whatever translations we consider, we always set

\[
\phi = \left[ \phi_B^A = \alpha \Phi_{A,B} \right] \quad \text{and} \quad \psi = \left[ \psi_A^B = \beta \Phi_{B,A} \right]
\]

First, let $\Lambda = \Lambda_0$, the identity translation. We can see that there is no $(\Lambda_0, \Lambda_0)$-interleaving between $A$ and $B$, since $\Phi_{A}^{\Lambda_0} \neq 0$ but $\text{Hom}(B, A\Lambda) = 0$. The variety of $(\Lambda_0, \Lambda_0)$-interleavings therefore, must be the empty variety. We recover this via the equation

\[
[\Phi_{A}^{\Lambda_0}] = \bar{Q} \cdot R = [0] \cdot [\alpha \Phi_{A,B}]
\]

(using the notation of Proposition 37) which has no solution, say when evaluated at $m$. Thus, for $\Lambda = \Gamma = \Lambda_0$, the variety $V^{\Lambda,\Gamma}(I, M)$ is empty, as required.

Now, say $\Lambda = \Lambda_a$. Since $a < b$, $\Lambda_a$ is the next largest maximal translations. Then, $A\Lambda \cong B$ and $B\Lambda \cong C$. Moreover, $\text{Hom}(A, B\Lambda)$, $\text{Hom}(B, A\Lambda) \cong K$. The space of interleavings are defined by the equations

\[
[\Phi_{A}^{\Lambda_a}] = \bar{Q} \cdot R = [\alpha \beta (\Phi_{B\Lambda,AA^2} \cdot \Phi_{A,B\Lambda})], \quad [\Phi_{B}^{\Lambda_a}] = \bar{R} \cdot Q = [0 \cdot \beta (\Phi_{A\Lambda,BA^2} \cdot \Phi_{B,AA})]
\]

Note that the variable $\alpha$ is absent from $\bar{R}$, since $\text{Hom}(A\Lambda, B\Lambda^2) = 0$. The first equation yields $\alpha \beta = 1$ by evaluating at $m$, and the second equation is consistent and trivially satisfied since $\Phi_{B}^{\Lambda_a} = 0$.

Therefore, the space of $(\Lambda_a, \Lambda_a)$-interleavings corresponds to the affine variety with coordinate ring $K[\alpha, \beta]$ modulo the ideal $\langle \alpha \beta - 1 \rangle$. This corresponds to a choice of one parameter in $K^*$. Also, since $\epsilon = a$ corresponds to the first non-zero variety, we can see that $D(A, B) = a$.

Now suppose that $\Lambda = \Gamma = \Lambda_{2a}$. Then, $A\Lambda \cong C$ and $B\Lambda \cong 0$. Also, $\text{Hom}(A, B\Lambda) = 0$ and $\text{Hom}(B, A\Lambda) \cong K$.

The space of interleavings are defined by the equations

\[
[\Phi_{A}^{\Lambda_{2a}}] = Q \cdot R = [0 \cdot 0 (\Phi_{B\Lambda,AA^2} \cdot \Phi_{A,B\Lambda})], \quad [\Phi_{B}^{\Lambda_{2a}}] = \bar{R} \cdot Q = [0 \cdot \beta (\Phi_{A\Lambda,BA^2} \cdot \Phi_{B,AA})].
\]
Since $\Phi^A_\Lambda, \Phi^A_\Lambda$ are both identically zero, any value of $\beta$ satisfies the above equations. Therefore, in this case $V^{\Lambda, \Gamma}(A, B)$ corresponds to the affine variety with coordinate ring $K[\beta]$. Of course, this corresponds to a choice of one parameter in $K$.

On the other hand, when $\Lambda = \Gamma = \Lambda_l$ for $l \geq 3a$, $A\Lambda = B\Lambda = 0$. In this case, the only interleaving between $A$ and $B$ is $\phi = \psi = 0$. That is to say, for such translations, $V^{\Lambda, \Gamma}(A, B)$ is the 0 variety.

Putting it all together, we obtain the curve below with values in the category of affine varieties. Only the jump discontinuities are labeled. In a slight abuse of notation, we write the coordinate rings instead of their corresponding varieties.

\[ K[\alpha, \beta]/\langle \alpha \beta - 1 \rangle \quad K[\beta] \quad 0 \]

\[ \begin{array}{c}
\emptyset \quad K^+ \quad K \quad 0 \quad \cdots \\
\epsilon \in [0, a) \quad \epsilon \in [a, 2a) \quad \epsilon \in [2a, 3a) \quad \epsilon \in [3a, \infty) \\
D(A, B)
\end{array} \]

**Remark 5.** In [MM17] we investigate the non-democratic choice of weights on a finite totally ordered set. Here we will show that from the perspective of topological data analysis, both potential problems associated with discretizing one-dimensional persistence modules can be overcome. Moreover, we recover the (classical) interleaving distance as a limit of discrete distances. In future work, we will study the geometric formulation of interleaving distances (see Remark 4 and Example 12).

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