The q-deformed quark model: quantum isospin and q-hadrons

M. Klein-Kreisler and M. Ruiz-Altaba†

Instituto de Física
Universidad Nacional Autónoma de México
A.P. 20-364, 01000 México D.F.

We consider the possibility that the $SU(2)$ isospin symmetry, exact in strong interactions but only approximate in nature, is in fact a quantum group. Using a doublet of $q$-quarks, we build the wavefunctions of $\pi$-mesons, nucleons and $\Delta$ baryons. We redo the usual quark model computation for the magnetic moments and mass relations, and everything fits with experimental data. We find, nevertheless, that it is impossible to parametrize successfully the large mass difference between the charged and neutral pions with the single $q$ of the quantum group $U_q(sl(2))$. 

† (martin@teorical.ifisicacu.unam.mx, marti@teorical0.ifisicacu.unam.mx)
1. Introduction

One of the most beautiful and old ideas in particle physics is the isospin invariance of strong interactions, discovered by Heisenberg. It constitutes one of the keystones of the quark model, which later became current algebra and eventually 't Hooft’s standard model of strong interactions, namely quantum chromodynamics. Its success in a variety of predictions makes it a fundamental conceptual tool for the understanding of many low-energy phenomena and properties of hadrons. The quantitative predictions, however, are corrected by the electromagnetic interactions and, of course, the full QCD. It is amusing and instructive to consider whether a slight deformation of $SU(2)$ can yield better fits with experiment. Of course, any such deformation will introduce one (or more) additional parameters and thus predictivity, along with physical understanding, will be lost or trivialized unless enough care is exercised. A beautiful opportunity for deforming the $SU(2)$ of isospin is provided by the quantum group of Leningrad, Drinfeld and Jimbo, which is the technical context we shall work in \[.\] The basic elements of the theory of quantum groups we need are briefly reviewed in section 2, along with some general comments on the possibility of applying quantum groups to four-dimensional physics. This seems, in principle, non-sense, but, as we shall argue, we can bypass all objections we could think of in the present context, with some minor caveats. Section 3 is devoted to the construction of the hadron states using the quark $q$-isospin doublets. Section 4 presents our computations for the baryons, which go through very smoothly, whereas section 5 is devoted to the pions, where difficulties arise. It is interesting that, contrary to naive expectations, the $q$-deformed $SU(2)$ of isospin cannot accommodate the large mass difference of pions without invoking explicit electromagnetic corrections: it is not true that the extra parameter $q$ allows one to fit anything. Throughout this letter, we restrict ourselves to a world with only one family (i.e. doublet) of quarks, so we make believe that the strange and even heavier quarks do not exist or, more soundly, are essentially completely decoupled from the static properties of hadrons we investigate. The final section 6 sums up our conclusions and perspectives.

2. The quantum isospin group and its possible relevance to physics

Given the $SU(2)$ algebra, it is possible to $q$-deform it to obtain the quantum group $U_q(sl(2))$, with generators $I_+, I_-$ and $I_z$, subject to the commutation relations

$$[I_3, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = \frac{q^{I_3} - q^{-I_3}}{q - q^{-1}}$$  \hspace{1cm} (2.1)
In the limit \( q \to 1 \), one recovers the usual \( SU(2) \) with the help of L’Hôpital’s rule. Let us point out the well-known but crucial fact that the mere commutation relations \((2.1)\) constitute only, per se, a curious and rather ugly redefinition of the generators of \( SU(2) \). No new physics may arise from just \( q \)-deforming (that is, \( q \)-redefining) the algebra. The meat of the matter lies in the asymmetric co-products

\[
\Delta (I_{\pm}) = I_{\pm} \otimes q^{I_3} + q^{-I_3} \otimes I_{\pm}, \quad \Delta (I_3) = I_3 \otimes 1 + 1 \otimes I_3 \quad (2.2)
\]

The choice \((2.1)\) and \((2.2)\) is the canonical one for the quantum deformation of \( SU(2) \), and its salient feature is the asymmetry in the coproducts for \( I_{\pm} \), which disappears in the classical \( q \to 1 \) limit. It is possible to redefine the generators such that the algebra \((2.1)\) is the classical one, but then the coproducts \((2.2)\) become complicated and remain asymmetric. Inversely, the usual (classical) \( SU(2) \) can be defined by \((2.1)\), but then the coproduct is complicated and symmetric.

The point we wish to emphasize is that we want to use the full algebraic structure of the quantum group (along with its other unmentioned features), not only the awkward \((2.1)\). By this we mean that we really want to imagine for a moment that the action of, say, the isospin lowering generator \( I_- \) on a two-particle state distinguishes between the first and the second particles: indeed, since its action on the state is precisely \( \Delta (I_-) \), and \( \Delta \) is asymmetric, we must have a way of ordering the two particles so that the results we get are physically meaningful. In general, this seems impossible if the particles are asymptotic states. Even for them, the consistent use of a clever Drinfel’d twist allows the preservation of the CPT theorem with only Fermi or Bose statistics surviving \[3\] but the problem of interpretation remains. For our application of the quantum group to strong interactions, we shall limit ourselves, in this letter, to single particle properties \[4\] such as the mass and the magnetic moment. We do not solve the problem but bypass it.

Still, the same problem of ordering crops up again when we start talking about the quarks inside the hadron. We will really think that in a \( \pi^+ \) it makes sense to speak of the \( u \) quark as being before or after the \( \bar{d} \) quark. The two alternatives are not completely unrelated: the passage from one to the other is just a similarity transformation, essentially a messy braiding with the famous \( R \)-matrix. Since quarks are confined, we feel justified in applying to them funny prescriptions. This practice mimics closely the symmetrizations one does in the standard quark model to get the relevant wavefunctions for hadrons. Of course, the kets will be related to the conjugates of the bras with the extra braiding factors.
of the $R$-matrix, but after the party is over we still end up with expressions for the hadrons in terms of the quark states and the deformation parameter $q$. Whether $u$ or $\bar{d}$ comes first amounts to exchanging the coproduct $\Delta$ with its transpose $\Delta'$ (one is the conjugate to the other through the $R$-matrix) or, equivalently, to interchanging $q$ with $q^{-1}$. In any given analysis, we stick to a choice of conventions and keep in mind that the same computation by someone else with the same result might need the redefinition $q \leftrightarrow q^{-1}$. To illustrate the point with a touch of chutzpah, we shall use $\Delta'$ for mesons and $\Delta$ for baryons below.

The solution to the ordering problem for quarks inside a hadron is physically simple: we go to the infinite-momentum frame and work there. This agrees, of course, with the usual quark model framework [5]. What is new, is that we add an ordering to the quarks in this infinitely boosted frame. This trick has been used in field theory to derive an integrable two-dimensional (XXZ) model from QCD [6].

3. $q$-hadrons from $q$-quarks

The doublet of quarks $\left(\begin{array}{c}|u> \\ |d>\end{array}\right)$ sits in a fundamental irrep of $U_q(sl(2))$, so that $I_-|u> = |d>$ and so on. The doublet of antiquarks is $\left(\begin{array}{c}|\bar{d}> \\ -|\bar{u}>\end{array}\right)$ where the $-$ is conventional and allows one to read off the $G$-parity of pion states effortlessly (nothing of what we shall do depends on this sign). Taking the tensor product of these two doublets yields a $q$-isovector and a $q$-isosinglet. The highest weight in the $I^3 = 1$ $q$-triplet, is $|\pi^+ > = |u> \otimes |\bar{d}> = |ud>$. We choose the ordering convention quark-antiquark. Also, we choose $\Delta'$ as coproduct, so that the $I^3 = 0$ partner of $|\pi^+ >$ is proportional to

$$\Delta' (S_-) |u> \otimes |\bar{d}> = -q^{1/2} |u> \otimes |\bar{u}> + q^{-1/2} |d> \otimes |\bar{d}>$$

(3.1)

The normalization of this state requires the use of a conjugation, which reverses the order of the states in the tensor product, and thus involves the $R$-matrix. To avoid pathologies, the value of $q$ (not as a “symbol”) must be either real or a pure phase. The result is, predictably,

$$|\pi^0 > = \frac{1}{\sqrt{q + q^{-1}}} \left(-q^{1/2} |u\bar{u}> + q^{-1/2} |d\bar{d}>\right)$$

(3.2)

And, of course, $|\pi^- >= |d\bar{u}>$.  

3
The $\Delta$ spin-3/2 resonances (with coproduct $\Delta$ as advertised) are similarly obtained by acting on $|\Delta^{++}\rangle = |u uu\rangle$ with $(1 \otimes \Delta)\Delta(I_-) = (\Delta \otimes 1)\Delta(I_-)\rangle$. We find

$$
|\Delta^+\rangle = \frac{1}{\sqrt{q^2 + 1 + q^{-2}}}(q|duu\rangle + |udu\rangle + q^{-1}|uud\rangle)
$$

$$
|\Delta^0\rangle = \frac{1}{\sqrt{q^2 + 1 + q^{-2}}}(q^{-1}|udd\rangle + |dud\rangle + q^{-1}|ddu\rangle)
$$

(3.3)

and $|\Delta_-\rangle = |ddd\rangle$.

The nucleon states require some attention. In the decomposition $1/2 \otimes 1/2 \otimes 1/2 = 1/2 \oplus 1/2 \oplus 3/2$ we wish to identify the nucleon doublet as a state symmetric in the two quarks with spin $\uparrow$. Although we have not written out the spin part of the wave-function, for the mesons it was something like $\uparrow \downarrow \uparrow$ (along with the appropriate symmetrization) and for the $\Delta$'s it is $\uparrow \uparrow \uparrow$. Now, if we look at the $\uparrow \uparrow \downarrow$ piece of the nucleon wave-function, we must require that the flavor counterpart be $q$-symmetric in the first two quarks (which are symmetric in the spin wavefunction). Using the fact that $\Delta(I_-)|uu\rangle = q^{-1/2}|ud\rangle + q^{1/2}|du\rangle$, we require the nucleon wave-functions to be

$$
|p\rangle \sim q^{-1/2}|udu\rangle + q^{1/2}|duu\rangle + P|uud\rangle
$$

$$
|n\rangle \sim q^{-1/2}|udd\rangle + q^{1/2}|dud\rangle + N|ddu\rangle
$$

(3.4)

with $P$ and $N$ to be determined from the doublet condition. After conjugating with the $R$-matrix and normalizing, the result is again simply

$$
|p\rangle = \frac{1}{\sqrt{1 + q^{-2} + (1 + q^2)^2}}[q^{-1}|udu\rangle + |duu\rangle - (1 + q^2)|uud\rangle]
$$

$$
|n\rangle = \frac{1}{\sqrt{1 + q^2 + (1 + q^{-2})^2}}[q|dud\rangle + |udd\rangle - (1 + q^{-2})|ddu\rangle]
$$

(3.5)

To end this section, note that the above wavefunctions are valid for $q$ real. Similar expressions hold for $q$ a pure phase (just the normalizations change slightly). In what follows, we shall always quote the results for $q$ real only.

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1 Do not confuse the baryons $\Delta^{++}$, $\Delta^+$, etc. with the coproducts $\Delta$ and $\Delta' = P \circ \Delta'$. 

4. $q$-baryons

We are now equipped to compute masses and magnetic moments. The mass operator $M$ and the charge operator $Q$ are postulated to have trivial coproducts (like $I_3$), and to be diagonal in the constituent quarks. Of course, the valence quark masses we use are different in each isospin multiplet, so for instance the value of $m_u$ from nucleons need not be the same as that from $\Delta$’s. We note

\[ m_{\Delta^{++}} = \langle \Delta^{++} | M \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes M | \Delta^{++} \rangle \tag{4.1} \]

and similarly for the other states, and find that the $q$ dependence drops out:

\[ m_{\Delta^{++}} = 3m_u, \quad m_{\Delta^+} = 2m_u + m_d, \quad m_{\Delta^0} = m_u + 2m_d, \quad m_{\Delta^-} = 3m_d \tag{4.2} \]

so that the prediction from $q$-isospin is the same as the usual one in the traditional quark model, namely

\[ m_{\Delta^{++}} - m_{\Delta^+} = m_{\Delta^+} - m_{\Delta^0} = m_{\Delta^0} - m_{\Delta^-} = m_u - m_d \equiv -\delta \tag{4.3} \]

Although the actual values of the parameters $m_u$ and $m_d$ are of no interest, their difference $\delta = m_d - m_u$ should be more or less glue-free and may be identified with the mass difference obtained from nucleons (again the $q$’s disappear):

\[ m_p = 2m_u + m_d, \quad m_n = m_u + 2m_d \tag{4.4} \]

Thus,

\[ \delta = m_n - m_p = 1.29\text{MeV} \tag{4.5} \]

which agrees with the spread in masses of the $\Delta$-resonances\[7\], as it should since the quark model works.

To compute the nucleon magnetic moments, we look at the piece of the wavefunction with spins aligned according to $\uparrow\uparrow\downarrow$, so that

\[ \mu_p = \langle p | \mu \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \mu | p \rangle \tag{4.6} \]

and similarly for $\mu_n$. We find the following gruesome expressions

\[ \mu_p = \frac{1}{1 + q^2 + (1 + q^2)^2} \left\{ 2(1 + q^2)^2 \mu_u - \left[ (1 + q^2)^2 - (1 + q^{-2}) \right] \mu_d \right\} \tag{4.7} \]

\[ \mu_n = \frac{1}{1 + q^2 + (1 + q^2)^2} \left\{ 2(1 + q^{-2})^2 \mu_d - \left[ (1 + q^{-2})^2 - (1 + q^2) \right] \mu_u \right\} \]

5
Therefore, using now the point-particle expression for the magnetic moment of a quark, \( \mu = Q/m \), we find the interesting

\[
\frac{\mu_n}{\mu_p} = -2 \frac{(m_u + m_d)(1 + q^{-2} - q^2) + 2q^2 m_u}{(m_u + 4m_d)(1 - q^{-2} + q^2) + 4q^{-2} m_d}
\]

Experimentally, \( \frac{\mu_n}{\mu_p} = -0.684979 \) and the quark model prediction is \(-2/3\). Under the (wrong) hypothesis that \( m_u = m_d \), the \( q \)-quark model fits the datum with \( q = 0.992 \). As emphasized repeatedly, this value of \( q \) is physically equivalent to the one obtained from \( \Delta' \), namely \( q = 1.008 \). Using the mass difference \( \delta \) noted above (4.5), we get instead \( q = 0.991 \).

5. \( q \)-pions

In contradistinction with the baryon case, the \( q \)'s in the meson wave-functions may show up in their mass formulae, but only if \( q \) is a phase. For real \( q \), from the wavefunction (3.2), we find

\[
m_{\pi^\pm} = m_{\pi^0} = m_u + m_d
\]

Meaning that in the \( q \)-deformed quark model, the neutral and charged pions remain degenerate. But for \( q \) a pure phase, we find

\[
m_{\pi^\pm} = m_u + m_d , \quad m_{\pi^0} = \frac{2}{q + q^{-1}} \left( q^{-1} |d \bar{d} \rangle + +q |u \bar{u} \rangle \right)
\]

where we have used, as advertised, the coproduct \( \Delta' \). Note that the prefactor is the inverse of \( \cos \alpha \), if \( q = e^{i \alpha} \), so it tends to make the neutral pion heavier than the charged ones. Also, the mass turns out to be complex unless \( m_u = m_d \), in which case all the \( q \)-dependence cancels out and we are left with the degenerate case again.

To solve this conundrum, one must include electromagnetic corrections. Very simply, we take the additional contribution to the mass formulae to be given by the electrostatic potential between the two quarks in the pion, as if they were hanging out at some distance \( R \) from each other. It turns out that

\[
\delta E m_{\pi^\pm} = 2E_\pm , \quad \delta E m_{\pi^0} = -\frac{4q + q^{-1}}{q + q^{-1}} E_0
\]

where \( E_x = \frac{4\pi \alpha}{9} \frac{1}{R_x} \) and we would expect \( R_x^{-1} \sim m_{\pi^x} \). Taking the experimental value of \( \Delta m_{\pi} = m_{\pi^\pm} - m_{\pi^0} = 4.59 \) MeV, takes us then to the real value \( q = 0.324 \).

The reader ought to be relieved that this numerology is over. The clear conclusion is that the low mass of the neutral pion cannot be naturally accommodated in the simple \( q \)-deformed isospin symmetry. Trying to explain why (or rather, parametrize how) the \( \pi^0 \) is lighter than the \( \pi^\pm \) was the original motivation for this work, and it seems to be impossible: the value of \( q \) derived from the nucleons’ magnetic moments does not account for it.
6. Conclusions and outlook

As noted in section 2 above, we have restricted ourselves to static predictions of the $(q$-deformed) quark model. They are satisfactory for the nucleons but not so for the pions. We can fit the parameter $q$ from the magnetic moments of the nucleons, and recover their masses in the usual way. It may be that the $\pi^0$ is not amenable to such a simple picture, exceptionally rather than generically. After all, it is the lightest hadron, an almost massless goldstone boson, with the extremely long lifetime involving the chiral anomaly, and so on. Let us emphasize that $q$ disappears from the mass formulae and can be brought in only through the very *ad hoc* electrostatic corrections, which work more or less in the usual quark model anyway. The extension to dynamics of $q$-isospin involves, as a first step, how to include the parameter $q$ in the amplitudes for $\pi - N$ scattering: there $q$ would just work as an additional parameter to better the fits between the quark model and data, but the conceptual difficulty of $q$-deforming hadronic interactions seems rather formidable (even in the light-cone).

It would be very nice if quantum groups provided an explanation for the celebrated $I = 1/2$ rule, but we have no inkling on how this may come about (back-of-the-envelope estimates with $q$ a root of unity do not work). The traditional wordplay “spin from isospin” is also food for thought, or sleep in the present context. We believe, however, that the test of $q$-isospin will come from its extension to a $q$-deformed flavor $SU(3)$, where the large mass difference between the strange quark and the light $q$-isospin doublet allows for much cleaner computations ($q$ should be rather different from one) and the pion mass difference is of no great import. At the very least, one should be able to come up with a $q$-deformed Gell-Mann–Okubo mass formula.

We believe that, despite the relative meagerness of our quantitative results, it is worthwhile to explore where, in four dimensions, quantum symmetries may show up. The more hidden the symmetry, the better, and thus quarks seem obvious candidates, for these explorations.

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