Entropic Dynamics: The Schrödinger equation and its Bohmian limit

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Abstract
In the Entropic Dynamics (ED) derivation of the Schrödinger equation the physical input is introduced through constraints that are implemented using Lagrange multipliers. There is one constraint involving a “drift” potential that correlates the motions of different particles and is ultimately responsible for entanglement. The purpose of this work is to deepen our understanding of the corresponding multiplier $\alpha'$. Its main effect is to control the strength of the drift relative to the fluctuations. We show that ED exhibits a symmetry: models with different values of $\alpha'$ can lead to the same Schrödinger equation; different “microscopic” or sub-quantum models lead to the same “macroscopic” or quantum behavior. In the limit of large $\alpha'$ the drift prevails over the fluctuations and the particles tend to move along the smooth probability flow lines. Thus ED includes the causal or Bohmian form of quantum mechanics as a special limiting case.

1 Introduction
Entropic Dynamics (ED) is a framework that allows the formulation of dynamical theories as applications of entropic methods of inference [1]. In the application of ED to derive the Schrödinger equation for $N$ particles the physical input is introduced through constraints that are implemented using Lagrange multipliers [2]–[5]. There is one set of $N$ constraints, one for each particle, that control the quantum fluctuations. The central role played by the corresponding multipliers $\alpha_n$ ($n = 1 \ldots N$) is well understood: they serve to regulate the flow of time, and the differences among the $\alpha_n$ are associated to differences in the mass of the particles. There is another constraint involving a “drift” potential that correlates the motions of different particles. The drift potential contributes to the phase of the wave function; it is ultimately responsible for such quantum effects as interference and entanglement. The corresponding multiplier $\alpha'$ is not nearly as well understood and the purpose of this work is to fill this gap.

*Presented at MaxEnt 2015, the 35th International Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering (July 19–24, 2015, Potsdam NY, USA).
We begin with a brief overview of ED following the presentation found in [4]. Even at this stage it is clear that the role of \( \alpha' \) is to control the strength of the drift relative to the fluctuations. We show that ED exhibits a symmetry: models with different values of \( \alpha' \) can lead to the same Schrödinger equation or, to put it differently, different “microscopic” or sub-quantum models lead to the same “macroscopic” or quantum behavior. Then we argue that the single-valuedness of the quantum wave function restricts the values that \( \alpha' \) may take. We conclude by showing in the limit of large \( \alpha' \) the drift motion prevails over the fluctuations so that the particles tend to move along the smooth lines of probability flow. Thus ED includes the causal or Bohmian form of quantum mechanics as a special limiting case. Finally we show that ED allows the construction of a hybrid theory — a dynamics with quantum fluctuations but no quantum potential [12]. The Bohmian limit of this hybrid theory is fully equivalent to classical mechanics.

2 Entropic Dynamics

As discussed in [4] we consider the ED of \( N \) particles living in a flat Euclidean space \( \mathbf{X} \) with metric \( \delta_{ab} \). In ED particles have definite positions \( x^a_n \) and it is their unknown values that we wish to infer. (The index \( n = 1 \ldots N \) denotes the particle and \( a = 1, 2, 3 \) the spatial coordinate.) The position of the system in configuration space \( \mathbf{X}_N = \mathbf{X} \times \ldots \times \mathbf{X} \) is denoted \( x^A \) where \( A = (n, a) \).

The main assumption is that motion is continuous which means that it can be analyzed as a sequence of short steps. The method of maximum entropy is used to find the probability \( P(x'|x) \) that the system will take a short step from \( x^A \) to \( x'^A = x^A + \Delta x^A \).

The information about the motion is introduced through constraints. The fact that particles move by taking infinitesimally short steps from \( x^a_n \) to \( x'^a_n = x^a_n + \Delta x^a_n \) is imposed through \( N \) independent constraints,

\[
\langle \Delta x^a_n \Delta x^b_n \rangle \delta_{ab} = \kappa_n , \quad (n = 1 \ldots N) .
\]

where we shall eventually take the limit \( \kappa_n \rightarrow 0 \). Correlations among the particles are imposed through one additional constraint,

\[
\langle \Delta x^A \rangle \partial_A \phi = \sum_{n=1}^{N} \langle \Delta x^a_n \rangle \frac{\partial \phi}{\partial x^a_n} = \kappa' ,
\]

where \( \phi \) is the drift potential, and \( \partial_A = \partial/\partial x^A = \partial/\partial x^a_n \). \( \kappa' \) is another small but for now unspecified position-independent constant. Eq.\( \frac{2}{2} \) is a single constraint; it acts on the \( 3N \)-dimensional configuration space and is ultimately responsible for such quantum effects as interference and entanglement.

1Elsewhere, in the context of particles with spin, we will see that the potential \( \phi(x) \) can be given a natural geometric interpretation as an angular variable. Its integral over any closed loop is \( \oint d\phi = 2\pi n \) where \( n \) is an integer.
Already at this early stage we see that ED exhibits an epistemic symmetry that at first sight seems trivial: two rational agents in different epistemic states can be led to exactly the same inference. Indeed, an agent who imposes \(\phi, \kappa\)' with the pair \((\phi, \kappa') = (C\phi, C\kappa')\) where \(C\) is some arbitrary constant.

The result of maximizing entropy leads to

\[
P(x'|x) = \frac{1}{\zeta} \exp[-\sum_n \left(\frac{1}{2} \alpha_n \Delta x_n^a \Delta x_n^b \delta_{ab} - \alpha' \Delta x_n^a \frac{\partial \phi}{\partial x_n^a}\right)] ,
\]

where \(\zeta\) is a normalization constant and \(\alpha_n\) and \(\alpha'\) are Lagrange multipliers. In previous work we took advantage of the symmetry above and rescaled \(\alpha' \phi \rightarrow \phi\) which amounts to choosing \(C = 1/\alpha'\). Here we will keep \(\alpha'\) explicit.

The successive iteration of many infinitesimal steps to produce a finite change requires the introduction of time. As discussed in [2]-[5] entropic time is measured by the fluctuations themselves which leads to

\[
\alpha_n = \frac{m_n}{\eta \Delta t} ,
\]

where the particle-specific constants \(m_n\) will be called “masses” and \(\eta\) is a constant that fixes the units of time relative to those of length and mass. With this choice of \(\alpha_n\) a generic displacement can be expressed as an expected drift plus a fluctuation,

\[
\Delta x^A = b^A \Delta t + \Delta w^A .
\]

\(b^A(x)\) is the drift velocity,

\[
\langle \Delta x^A \rangle = b^A \Delta t \quad \text{with} \quad b^A = \frac{\eta \alpha'}{m_n} \delta^{AB} \partial_B \phi = \eta \alpha' m^{AB} \partial_B \phi ,
\]

where \(m^{AB}\) is the inverse of the “mass” tensor, \(m_{AB} = m_n \delta_{AB}\), and the fluctuations \(\Delta w^A\) satisfy,

\[
\langle \Delta w^A \rangle = 0 \quad \text{and} \quad \langle \Delta w^A \Delta w^B \rangle = \frac{\eta}{m_n} \delta^{AB} \Delta t = \eta m^{AB} \Delta t .
\]

These equations show that for very short steps, as \(\Delta t \rightarrow 0\), the fluctuations are much larger than the drift \(\langle \Delta w^A \rangle \sim \Delta t^{1/2}\) while \(\langle \Delta x^A \rangle \sim \Delta t\) and we have a Brownian motion. They also show that for fixed \(\phi\) the effect of the multiplier \(\alpha'\) is to enhance or suppress the drift \(b^A \Delta t\) relative to the fluctuations \(\Delta w^A\).

Having introduced a convenient notion of time through (4), the result of accumulating many changes is that the probability distribution \(\rho(x,t)\) in configuration space obeys a Fokker-Planck equation, (See e.g., [1]),

\[
\partial_t \rho = -\partial_A \left( \rho v^A \right) ,
\]

where \(v^A\) is the velocity of the probability flow in configuration space or current velocity,

\[
v^A = b^A + u^A \quad \text{and} \quad u^A = -\eta m^{AB} \partial_B \log \rho^{1/2}
\]
is the osmotic velocity. Since both $b^A$ and $u^A$ are gradients the current velocity is a gradient too,

$$v^A = m^{AB} \partial_B \Phi \quad \text{where} \quad \Phi = \eta \rho' - \eta \log \rho^{1/2}.$$  \hfill (10)

The dynamics described by the FP equation (8) is a standard diffusion. To describe a “mechanics” we require that the diffusion be “non-dissipative”. This is achieved by an appropriate readjustment or updating of the constraint (2) after each step $\Delta t$. The net effect is that the drift potential $\phi$, or equivalently $\Phi$, is promoted to a fully dynamical degree of freedom. The diffusion is said to be “non-dissipative” when the actual updating is implemented by imposing that a certain functional $\tilde{H} [\rho, \Phi]$ be conserved; in order to offset the entropic change $\rho \rightarrow \rho + \delta \rho$, one requires a change $\Phi \rightarrow \Phi + \delta \Phi$ such that

$$\tilde{H} [\rho + \delta \rho, \Phi + \delta \Phi] = \tilde{H} [\rho, \Phi].$$  \hfill (11)

As shown in [4] the requirement that $\tilde{H}$ be conserved for arbitrary choices of $\rho$ and $\Phi$ implies that the coupled evolution of $\rho$ and $\Phi$ is given by a conjugate pair of Hamilton’s equations,

$$\partial_t \rho = \frac{\delta \tilde{H}}{\delta \Phi} \quad \text{and} \quad \partial_t \Phi = - \frac{\delta \tilde{H}}{\delta \rho}.$$  \hfill (12)

When the “ensemble” Hamiltonian $\tilde{H}$ is chosen so that the first equation reproduces the FP equation (8), the second becomes a Hamilton-Jacobi equation. Arguments from information geometry [4] can be invoked to further specify the form of the functional $\tilde{H} [\rho, \Phi]$. They suggest that the natural choice of $\tilde{H}$ is

$$\tilde{H} [\rho, \Phi] = \int dx \left[ \frac{1}{2} \rho m^{AB} \partial_A \Phi \partial_B \Phi + \rho V + \xi m^{AB} \frac{1}{\rho} \partial_A \rho \partial_B \rho \right].$$  \hfill (13)

The first term in the integrand is the “kinetic” term that reproduces (8). The second term is the simplest possible non-trivial interaction, an energy term that is linear in $\rho$ and introduces the standard potential $V(x)$. The parameter $\xi = \hbar^2/8$ turns out to be crucial: it controls the relative contributions of the two potentials, it defines the value of what we call Planck’s constant $\hbar$, and it sets the scale that separates quantum from classical regimes.

To conclude this brief review of ED we combine $\rho$ and $\Phi$ into a single complex function,

$$\Psi = \rho^{1/2} \exp(i\Phi/\hbar).$$  \hfill (14)

The pair of Hamilton’s equations (12) can then be written as a single complex linear equation,

$$i \hbar \partial_t \Psi = -\frac{\hbar^2}{2} m^{AB} \partial_A \partial_B \Psi + V \Psi,$$  \hfill (15)

which we recognize as the Schrödinger equation.
3 Quantized circulation

An important question is whether the Fokker-Planck and Hamilton-Jacobi equations, eqs. (12), are fully equivalent to the Schrödinger equation. This point was first raised by Wallstrom [6] as an objection to Nelson’s stochastic mechanics [7] and concerns the single- or multi-valuedness of phases and wave functions. Wallstrom’s objection was that stochastic mechanics leads to phases $\Phi$ and wave functions $\Psi$ that are either both multi-valued or both single-valued. Both alternatives are unsatisfactory: quantum mechanics forbids multi-valued wave functions, while single-valued phases can exclude physically relevant states (e.g., states with non-zero angular momentum).

The requirement that the wave function $\Psi$ be single-valued amounts to imposing a quantized circulation condition,

$$\oint \Gamma \, d\ell \, A \partial_A \frac{\Phi}{\hbar} = 2\pi n,$$

(16)

where $\Gamma$ is any closed loop in configuration space and $n$ is an integer. On the other hand, we had earlier briefly mentioned that the drift potential $\phi$ is to be interpreted as an angle in which case, the integral over the closed loop $\Gamma$ gives

$$\oint \Gamma \, d\ell \, A \partial_A \phi = 2\pi n',$$

(17)

where $n'$ is an integer associated to the drift potential $\phi$. We do not discuss this issue in any detail except to note that this is true when particle spin is incorporated into the theory. Indeed, as shown by Takabayasi, a similar result holds for the hydrodynamical formulation of spinning particles [8]. But, if eq. (17) is true, then we can use eq. (10) to integrate the phase $d\Phi/\hbar$ over a closed path. Since $\rho$ is single-valued,

$$\oint \Gamma \, d\ell \, A \partial_A \log \rho = 0,$$

(18)

and we obtain

$$\oint \Gamma \, d\ell \, A \partial_A \frac{\Phi}{\hbar} \, dA = \frac{\eta \alpha'}{\hbar} \oint \Gamma \, d\ell \, A \partial_A \phi = 2\pi n' \frac{\eta \alpha'}{\hbar}.$$

(19)

Comparing (16) and (19) we conclude that

$$n' \frac{\eta \alpha'}{\hbar} = n$$

(20)

is an integer. Since this must simultaneously hold for all loops $\Gamma$ including loops with arbitrary values of $n'$ we conclude that

$$\eta \alpha' = N \hbar$$

(21)

where $N$ is an integer. This is precisely the quantization condition that leads to full equivalence between ED and the Schrödinger equation because it guarantees that wave functions will remain single-valued even for multi-valued phases.
4 The effect of $\alpha'$

The dynamics described by (12) and (13), or by the Schrödinger equation (15) is clearly independent of $\alpha'$ and therefore we have a symmetry. As we see in eqs.(6) and (7) different choices of $\alpha'$ lead to different Brownian motions at the sub-quantum or “microscopic” level. However, they all lead to the same evolution of $\rho$ and $\Phi$ and the same dynamics — the same Schrödinger equation — at the quantum or “macroscopic” level.

The Bohmian limit We can directly study the sub-quantum effect of $\alpha'$ in eqs.(6) and (7). It is, however, more instructive to rescale $\eta$ and write $\eta = \tilde{\eta}/\alpha'$. Under such rescaling the $\alpha'$ dependence has migrated from the drift to the fluctuations,

$$\langle \Delta x^A \rangle = \tilde{\eta} m^{AB} \partial_B \phi \Delta t \quad \text{and} \quad \langle \Delta w^A \Delta w^B \rangle = \frac{\tilde{\eta}}{\alpha'} m^{AB} \Delta t .$$ (22)

Increasing $\alpha'$ at fixed $\tilde{\eta}$ has the effect of suppressing the fluctuations while leaving the drift unaffected. In the limit $\alpha' \to \infty$ we expect the fluctuations to be negligible; the particles will follow smooth trajectories that do not resemble a Brownian motion at all.

From eq.(10) we have

$$\Phi = \tilde{\eta} \phi - \frac{\tilde{\eta}}{\alpha'} \log \rho^{1/2} ,$$ (23)

so that for large $\alpha'$

$$\Phi \to \tilde{\eta} \phi \quad \text{and} \quad v^A \to b^A .$$ (24)

Therefore, for $\alpha' \to \infty$ the current and the drift velocities coincide. Particles follow smooth trajectories that coincide with the lines of probability flow. This is exactly the kind of motion postulated by Bohmian mechanics [9]-[11].

We can therefore claim that, at least formally, entropic dynamics includes Bohmian mechanics as a special limiting case. However, there are important differences between ED and Bohmian mechanics that need to be emphasized. First, it is worth pointing out that the limit $\alpha' \to \infty$ is a tricky one because it is meant to be taken only after we take the limit $\Delta t \to 0$. This is what allows us to write differential equations such as (8). Thus, no matter how large the (fixed) value of $\alpha'$, entropic dynamics remains “entropic”. Even for large $\alpha'$ the dynamics is still driven by fluctuations and at sufficiently microscopic scales the expected motion is Brownian.

Second, and perhaps even more important, there is a major philosophical difference: Bohmian mechanics attempts to provide an actual description of reality, a description of the ontology of the universe as it “really” is and as it “really” happens. In the Bohmian view the universe consists of real particles that have definite positions and their trajectories are guided by something real, the wave function $\Psi$ [9].
In contrast, ED is a purely epistemic theory. It does not attempt to describe the world. Its pragmatic goal is less ambitious and also more realistic: to make the best possible predictions on the basis of very incomplete information. In ED the particles also have definite positions and its formalism includes a function $\Phi$ that behaves as a wave. But $\Phi$ is a tool for reasoning; it is not meant to represent anything real. There is no implication that the particles move the way they do because they are guided by a pilot wave or because they are being pushed around by some stochastic force. In fact ED is silent on the issue of what causative power is responsible for the peculiar motion of the particles. What the probability $\rho$ and the phase $\Phi$ are designed to accomplish is to guide our inferences. They guide our expectations of where to find the particles but they do not exert any causal influence on the particles themselves.

**A hybrid theory and the classical limit** Equation (13) includes a parameter $\xi$ that regulates the strength of the quantum potential. Any non-zero value $\xi > 0$ yields a fully quantum mechanics, albeit with differing values of $\hbar$. The value $\xi = 0$ leads, however, to a qualitatively different theory. One might suspect that $\xi = 0$ gives classical mechanics but this is not so. According to equations (12) and (13) for $\xi = 0$ the probability $\rho$ follows the gradient of $\Phi$,

$$\frac{\partial \rho}{\partial t} = \delta \tilde{H} \delta \Phi = - \partial_A \left( \rho v^A \right) \quad \text{with} \quad v^A = m^{AB} \partial_B \Phi,$$

and $\Phi$ evolves according to the classical Hamilton-Jacobi equation,

$$\frac{\partial \Phi}{\partial t} = - \frac{\delta \tilde{H}}{\delta \rho} = - \frac{1}{2} m^{AB} \partial_A \Phi \partial_B \Phi - V.$$

Therefore the probability $\rho$ flows along the classical path. However, there is no implication that the particles themselves follow the classical paths. Indeed, at any instant of time the particles undergo the same fluctuations, eq.(22), that we would expect for any non-zero value of $\xi$.

The $\xi = 0$ model resembles classical mechanics in some respects and quantum mechanics in others; it is a hybrid theory. Just as in quantum mechanics the particles follow Brownian paths and the dynamics is a non-dissipative diffusion; they even satisfy an uncertainty principle [12]. On the other hand, just as in classical mechanics, the probability flows according to paths described by the classical Hamilton-Jacobi equation. One can even combine $\rho$ and $\Phi$ into a single complex function, $\Psi = \rho^{1/2} \exp(i\Phi/\hbar)$, and write the coupled evolution of $\rho$ and $\Phi$ in terms of a single complex equation that resembles a Schrödinger equation,

$$i\hbar \partial_t \Psi_k = - \frac{\hbar^2}{2} m^{AB} \partial_A \partial_B \Psi_k + V \Psi_k + \frac{\hbar^2}{2} m^{AB} \partial_A \partial_B |\Psi_k| \Psi_k.$$

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2 By “classical” mechanics we mean a Newtonian deterministic mechanics. The $\xi = 0$ theory could be called a “classical indeterministic mechanics” but this is not useful as it would broaden the meaning of the term ‘classical’ to cover any theory that is ‘not-quantum’. 
But this equation is not linear which means that a central feature of quantum behavior, the superposition principle, has been lost.

For the hybrid theory too we can take the Bohmian limit $\alpha' \to \infty$. Increasing $\alpha'$ at fixed $\tilde{\eta}$ has the same effect of suppressing the fluctuations so the particles follow smooth trajectories that coincide with the lines of probability flow. The one difference is that for $\xi = 0$ the lines of probability flow are determined by the classical Hamilton-Jacobi equation \[\text{(23)},\] and therefore the particles follow classical trajectories. We conclude that the Bohmian limit of the hybrid theory is classical mechanics. In other words, classical mechanics is related to the hybrid theory in exactly the same way as Bohmian mechanics is related to entropic dynamics.

Acknowledgments  We would like to thank M. Abedi, C. Cafaro, N. Caticha, S. DiFranzo, A. Giffin, S. Ipek, D.T. Johnson, K. Knuth, S. Nawaz, M. Reginatto, C. Rodriguez, and K. Vanslette, for many discussions on entropy, inference and quantum mechanics.

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