CLASSIFICATION OF HIGHER MOBILITY LINKAGES

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Abstract. We provide a complete classification of paradoxical \( n \)-linkages, \( n \geq 6 \) whose mobility is \( n - 4 \) or higher containing \( R \), \( P \) or \( H \) joints. We also explicitly write down strong necessary conditions for \( nR \)-linkages of mobility \( n - 5 \).

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1. Introduction

Closed linkages are a crucial and basic object for the modern theory of kinematics, and in particular computational kinematics [GA90, CPY15, MS00, Sel05]. Algebraic and geometric methods are classically very often used for studying closed linkages, for instance, Sylvester, Kempe, Cayley and Chebyshev.

A closed loop linkage is a mechanical device consisting of rigid bodies coupled together by joints and forming a single loop. Each joint allows for a different type of motion, namely, revolute, translational along a fixed direction, helical, where a rotation angle and a translation length are dependent. To these we call \( R \), \( P \), \( H \), joints respectively.

The mobility (see definition 3.2 for a precise description) of a closed linkage is generically \( n - 6 \) by the Chebychev-Grübler-Kutzbach criterion [Kut29, Sel05], where \( n \) is the number of 1-degree-of-freedom (dof) joints in the closed loop. A closed linkage whose mobility is higher than expected is called a paradoxical linkage or an overconstrained linkage. Mobile linkages with 4 joints of type \( R \), \( P \) or \( H \) have been classified in [Del22]. In [ALS15], the authors settle the classification problem for 5-linkages with at least one helical joint, which complements the classification of mobile linkages with 5 joints of type \( R \), \( P \) or \( H \). Since the first mobile linkage with only revolute joints (6\( R \)-linkage) of Sarrus [Sar53], numerous paradoxical 6\( R \)-linkages were discovered (see reviews in [CY12, Die95, Bak02, Bak03, Li15]) but the classification of the paradoxical 6\( R \)-linkages is still open. Furthermore, the classification of paradoxical linkages with \( n \geq 6 \) joints of type \( R \), \( P \) or \( H \) is open too.

However, for linkages with more than six joints or mobility higher than two there has not been a systematic study yet. Our results are a major step further in the solution of this problem.

Key words and phrases. Linkages, Mobility, Bond Theory.
Theorem 1.1 (Main Theorem I, see 5.6, 5.11 and 5.14). A 6-linkage of mobility 2 is one of the following cases:

1. All joints are revolute with zero offsets and equal Bennett ratios;
2. There are exactly two P-joints and two pairs of neighboring parallel R or H-joints.
3. All axes of revolute or helical joints are parallel.

Theorem 1.2 (Main Theorem II, see 5.1, 5.15, 5.16 and 5.18). The mobility is at most $n - 3$.

1. An $n$-linkage, where $n > 6$, has mobility $n - 4$ if and only if all its revolute and helical axes are parallel.
2. An $n$-linkage, where $n > 5$, has mobility $n - 3$ only if all the axes of revolute or helical joints are parallel or concurrent.

When the mobility is $n - 5$ we have the following necessary conditions:

Theorem 1.3 (Main Theorem III, see 6.3). On any $nR$-linkage, where $n > 6$, of mobility $n - 5$, there are parallel neighboring axes or Bennett conditions.

The main tool we use is the freezing of a joint in a linkage with enough mobility. This means keeping one joint fixed at a generic position, enabling the use of induction based on linkages with a smaller number of joints and lower mobility. We also use a new formulation of the Bennett and concurrent conditions on revolute axes in terms of points in the absolute quadric cone, see lemma 3.8, as well as new considerations on bond theory for linkages with higher mobility, see section 4, and an extension of the $abc$–lemma to dual quaternions, see lemma 2.4.

### 2. Preliminaries

The real quaternions, $\mathbb{H}$, are the unique 4-dimensional associative division algebra over $\mathbb{R}$. An element in $\mathbb{H}$ can be uniquely written as $p = p_0 + p_1i + p_2j + p_3k$, where $p_i \in \mathbb{R}$ and $i^2 = j^2 = k^2 = ijk = -1$ are the usual Hamiltonian relations and its conjugate is $p^* := p_0 - p_1i - p_2j - p_3k$. The algebra of quaternions is normed and the norm is a multiplicative function given by

$$N : \mathbb{H} \rightarrow \mathbb{R}$$

$$p \mapsto pp^* = p_0^2 + p_1^2 + p_2^2 + p_3^2.$$

The quaternions are a good model for the 3D rotation group $SO(3)$, but it turns out that it is not enough to model the whole group of direct isometries of $\mathbb{R}^3$, $SE(3)$. For that we need dual quaternions. Let $\mathbb{DH}$ be the 8-dimensional associative algebra over $\mathbb{R}$ given by

$$\mathbb{DH} := \mathbb{H}[e]/(e^2).$$

That is, an element in $\mathbb{DH}$ is of the form $h = p + cd$ where $p$ and $d$ are quaternions and $c^2 = 0$. The quaternions $p$ and $d$ are called the primal and dual parts of $h$, respectively. The dual quaternions are normed and the norm is given by

$$N : \mathbb{DH} \rightarrow \mathbb{D}$$

$$p + cd \mapsto (p + cd)(p + cd)^* = pp^* + c(pd^* + dp^*)$$

$$= p_0^2 + p_1^2 + p_2^2 + p_3^2 + c(p_0d_0 + p_1d_1 + p_2d_2 + p_3d_3),$$

where $\mathbb{D} = \{a + be \mid a, b \in \mathbb{R} \text{ and } c^2 = 0\}$. The subset of unit dual quaternions, $\mathbb{DH}^u$, given by elements with norm 1, has a group structure and is diffeomorphic to the tangent bundle of the unit sphere $S^3$, $TS^3 \cong S^3 \times \mathbb{R}^3$.

The dual quaternions act on $\mathbb{R}^3$ and, in fact, restricting the action to the unit dual quaternions yields a surjection to $SE(3)$. Such a map is not injective, however, since any two unit dual quaternions differing by a sign map to the same isometry. Hence, it is natural to identify $h$ with $-h$ and, therefore, get a one-to-one correspondence

$$S^* := \{h \in \mathbb{F}^7 \mid N(h) \in \mathbb{R}^+\} \sim SE(3).$$
Classically, it is usual to write $S^*$ as $S \setminus E$ where $S = \{(p_0 \colon p_1 \colon p_2 \colon p_3 : d_0 : d_1 : d_2 : d_3) \in \mathbb{P}^7 | p_0 d_0 + p_1 d_1 + p_2 d_2 + p_3 d_3 = 0\}$ is the Study Quadric. Inside $S$ is the exceptional linear 3-space $E = \{(p_0 \colon p_1 \colon p_2 \colon p_3 : d_0 : d_1 : d_2 : d_3) \in \mathbb{P}^7 | p_0 = p_1 = p_2 = p_3 = 0\} = \{eh | h \in S\}.$

The subset $S \setminus E$ has a group structure inherited from the one in the unit dual quaternions. It is a classic result that $S \setminus E$ is isomorphic to $SE(3)$. See [HS10, Section 2.4].

In fact, the isomorphism can be given by $\psi : S \setminus E \rightarrow SE(3)$ such that $p + ed \mapsto (p + ed)(1 + ex)(p^* - ed^*)$.

Here, a point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is written in the form $1 + ex = (1, x_1i, x_2j, x_3k)$.

**Example 2.1.** The table 1 shows the correspondence between points in $S \setminus E$ and isometries of $\mathbb{R}^3$.

| Points in $S \setminus E$ | Isometries in $\mathbb{R}^3$ |
|---------------------------|-----------------------------|
| $(1 : 0 : 0 : 0 : t_1/2 : t_2/2 : t_3/3)$ | Translation by $(t_1, t_2, t_3)$ |
| $(t : u_1 : u_2 : u_3 : 0 : 0 : 0)$ where $t = \cot(\frac{\pi}{2})$ and $N(u) = 1$ | Rotation by $\alpha$ around the axis $u = (u_1, u_2, u_3)$. |

Very often we work in the complexification of the real algebra $\mathbb{D}H$. In fact, we can extend the dual quaternions to the algebra of complex dual quaternions, $\mathbb{D}H_{\mathbb{C}}$, defined as $\mathbb{D}H \otimes \mathbb{C}$. Inside $\mathbb{D}H_{\mathbb{C}}$ there is the subalgebra of complex quaternions, $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes \mathbb{C}$. The following lemma says that it has a linear structure.

**Lemma 2.2.** There is an algebra isomorphism, $\mathbb{H}_{\mathbb{C}} \cong M_2(\mathbb{C})$.

**Proof.** This is a special case of Wedderburn’s theorem, see [Wed08], and follows directly from the fact that $\mathbb{H}_{\mathbb{C}}$ is a simple algebra over $\mathbb{C}$. \hfill \Box

**Lemma 2.3** ($abc$–lemma). Suppose that $a$, $b$ and $c$ are norm zero complex quaternions. Then $abc = 0 \implies ab = 0$ or $bc = 0$.

**Proof.** Let $X = \{x \in \mathbb{H}_{\mathbb{C}} | abx = 0\}$ and $Y = \{x \in \mathbb{H}_{\mathbb{C}} | bxx = 0\}$. Clearly, $Y \subseteq X$. Both $X$ and $Y$ are right ideals of $\mathbb{H}_{\mathbb{C}}$ and, therefore, linear subspaces of $M_2(\mathbb{C})$. Suppose that $ab \neq 0$. Then $X \neq M_2(\mathbb{C})$ and $X$ and $Y$ are 2–dimensional. Hence, $X = Y$. Since $c \in X$ by hypothesis, it follows that $c \in Y$. \hfill \Box

The $abc$–lemma is one of the main tools in our results concerning linkages of mobility 2 or higher. It was originally proven in [Li15] to help with classifying the bond diagrams for 6R-linkages with mobility 1 with at most four connections. We shall make extensive usage of this result. There is, in fact, a generalisation of the $abc$–lemma for complex dual quaternions that we will also use:

**Lemma 2.4** (Extended $abc$–lemma). Suppose that $a$, $b$ and $c$ are norm zero complex dual quaternions, none of which is a multiple of $e$. Then $abc = 0$ if and only if at least one of the following is true:

1. $ab = 0$;
2. $bc = 0$;
3. $ab$ and $bc$ are both multiples of $e$. 

TABLE 1. Correspondence between elements in $S \setminus E$ and $SE(3)$. 

| Points in $S \setminus E$ | Isometries in $\mathbb{R}^3$ |
|---------------------------|-----------------------------|
| $(1 : 0 : 0 : 0 : t_1/2 : t_2/2 : t_3/3)$ | Translation by $(t_1, t_2, t_3)$ |
| $(t : u_1 : u_2 : u_3 : 0 : 0 : 0)$ where $t = \cot(\frac{\pi}{2})$ and $N(u) = 1$ | Rotation by $\alpha$ around the axis $u = (u_1, u_2, u_3)$. |
Proof. It is clear that \( ab = 0 \) or \( bc = 0 \) implies \( abc = 0 \). Suppose that both \( ab \) and \( bc \) are multiples of \( \epsilon \).
Then, for every \( x \in \mathbb{DH}_C \), we have that \( abxbc = 0 \). We have thus to prove that there exists an element \( x \) for which \( bxb = b \). Any norm zero complex dual quaternion is of the form \( u(i - i)v \) (by [HLSS15, Lemma 1]) where \( u \) and \( v \) are invertible dual quaternions and the \( i \) is the complex imaginary unit. It follows that one only has to show that there is an element \( x \) such that \( (i - i)vux(i - i) = (i - i) \). We take \( x = \frac{1}{2} (uv)^* \).
To prove the converse we look at the left annihilator of \( bc \) in \( \mathbb{DH}_C \). If \( bc \) is not a multiple of \( \epsilon \) we have,
\[
a \in \text{Ann}_u(bc) = \mathbb{DH}_C \overline{bc} \subseteq \mathbb{DH}_C \overline{b},
\]
which proves our claim. \( \square \)

3. Linkages

Algebraically, we use the following definition of a linkage, which is taken from [ALS15]:

**Definition 3.1.** Let \( n \in \mathbb{N} \). An \( n \)-linkage is a finite sequence of joints \((j_1, \ldots, j_n)\) where each joint is represented by an element in \( S \setminus E \) in the following way:

**If \( j_k \) is an R-joint:** it is represented by the map \( m_k : \mathbb{P}^1 \to S \setminus E \) such that \( t_k \mapsto t_k - h_k \). The element \( h_k \) is a fixed unit dual quaternion such that \( h_k^2 = -1 \) and of the form \( h_k = (0 : p_1 : p_2 : p_3 : 0 : d_1 : d_2 : d_3) \). Hence, \( t_k - h_k = (t_k : p_1 : p_2 : p_3 : 0 : d_1 : d_2 : d_3) \). The parameter \( t_k \) corresponds to the angle of rotation 2arccot(\( t_k \)).

**If \( j_k \) is a P-joint:** it is represented by the map \( m_k : \mathbb{P}^1 \setminus \{0\} \to S \setminus E \) such that \( t_k \mapsto t_k - ep_k \). The element \( p_k \) is a fixed unit quaternion such that \( p_k^2 = -1 \) and is of the form \( p_k = (0 : p_1 : p_2 : p_3 : 0 : 0 : 0) \). Hence, \( t_k - ep_k = (t_k : 0 : 0 : 0 : p_1 : p_2 : p_3) \). The parameter \( t_k \) corresponds to the extent of translation.

**If \( j_k \) is a H-joint:** it is represented by the map \( m_k : \mathbb{R} \to S \setminus E \) where \( \alpha_k \mapsto (1 - \epsilon g_k \alpha_k p_k)(\cot(\epsilon g_k) - h_k) \). The number \( g_k \in \mathbb{R} \) is a fixed constant called the pitch and \( h_k = p_k + \epsilon d_k \) is such that \( h_k^2 = -1 \).

Moreover, the elements \( m_1, \ldots, m_n \) satisfy the loop equation,
\[
(1) \quad m_1 \cdots m_n \equiv 1.
\]

The set of all admissible positions of the linkage is called the configuration set of \( L \). This corresponds to the real solutions of the loop equation and is denoted by \( K_L \). Hence, for instance, a linkage consisting of 6 revolute joints has a configuration space in \( (\mathbb{P}^1)^6 \).

**Definition 3.2.** The Zariski closure of all the complex configurations of an \( n \)-linkage is an algebraic variety which is denoted by \( \overline{K_L} \) and its dimension is called the mobility.

Here by dimension we mean the complex dimension and we are more interested in the special complex solutions in \( K_L \) which will be introduced in the next section. We say that a linkage \( L \) is mobile if \( \dim \overline{K_L} > 0 \) and, in particular, we assume that no joint is frozen. Namely, we always consider the component of \( \overline{K_L} \) with highest dimension where no joint is frozen. Otherwise, we can treat the linkage as a reduced linkage with less number of joints.

**Remark 3.3.** Although we consider the complex dimension for the configuration space of \( L \), it is true that its real dimension is the same provided that there exists a real non-singular point in the configuration space.

**Denavit-Hartenberg Parameters and the Absolute Quadric Cone:**

We present geometric conditions of consecutive rotation joints. First, we start by introducing the set of Denavit-Hartenberg parameters of consecutive rotation joints. For indices \( i = 1, \ldots, n \), let \( l_i \) be the rotation axis of the \( i \)-th joint. The angle \( \phi_i \) is defined as the angle of the direction vectors of \( l_i \) and \( l_{i+1} \)
(with some choice of orientation). We also set \( c_i := \cos(\phi_i) \) and \( w_i = \cot(\frac{\phi_i}{2}) = \frac{\cos(\phi_i) + 1}{\sin(\phi_i)} \). The number \( d_i \) is defined as the orthogonal distance of the lines \( l_i \) and \( l_{i+1} \). Note that \( d_i \) may be negative; this depends on some choice of orientation of the common normal, which we denote by \( n_i \). We recall two definitions [HSS13a, Definition 3 and Definition 4] as:
Definition 3.4 ([HSS13a]). For a sequence $h_i, h_{i+1}, \ldots, h_j$ of consecutive joints, we define the coupling space $L_{i,i+1,\ldots,j}$ as the linear subspace of $\mathbb{R}^8$ generated by all products $h_k h_{k+1} \cdots h_{k+i}$, $1 \leq k_1 < \ldots < k_s \leq j$. (Here, we view dual quaternions as real vectors of dimension eight.) The empty product is included, its value is 1.

Definition 3.5 ([HSS13a]). The dimension of the coupling space $L_{i,i+1,\ldots,j}$ will be called the coupling dimension. We denote it by $l_{i,i+1,\ldots,j} = \dim L_{i,i+1,\ldots,j}$.

We also recall the [HSS13a, Theorem 1] as:

Theorem 3.6 ([HSS13a]). If $h_1, h_2, \ldots, h_n$ are rotation quaternions such that $h_i$ and $h_i+1$ are not compatible for $i = 1, \ldots, n-1$, the following statements hold true:

- All coupling dimensions $l_{1,\ldots,i}$ with $1 \leq i \leq n$ are even.
- The equation $L_{1,2} = 4$ always holds. Moreover, $L_{1,2} \subset S$ if and only if the axes of $h_1$ and $h_2$ are concurent.
- If $\dim L_{1,2,3} = 4$, then the axes of $h_1, h_2$ and $h_3$ are concurent.
- If $\dim L_{1,2,3} = 6$, then the axes of $h_1, h_2$ and $h_3$ satisfy the Bennett conditions: the normal feet of $h_1$ and $h_3$ on $h_2$ coincide and the normal distances $d_i$ and twisted angles $\alpha_i$ between consecutive axes are related by $d_1 \sin \alpha_2 = \pm d_2 \sin \alpha_1$.

Remark 3.7. Three axes being concurent can be thought of as a degenerate Bennett condition, where the normal distances are zero. Three axes being all parallel can be thought of as a degenerate Bennett condition, where the twist angles are zero.

We introduce a useful interpretation of the above results. Any plane in $\mathbb{R}^3$ is given by an equation $a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 = 0$ which is uniquely determined up to scalar multiplication. The coefficients of this equation are in one to one correspondence with points in $\mathbb{P}^3$, the dual projective space, except for the case when $a_0 = a_1 = a_2 = 0$ and $a_3 = 1$ which corresponds to the plane at infinity.

We define the dual absolute quadric cone $A$: $(x_0^2 + x_1^2 + x_2^2) = 0 \subset \mathbb{P}^3$. Let $e = (0 : 0 : 0 : 1)$. Then $e \in A$ is the only real point of $A$ as well as its only singular point and we have a bijection between oriented lines in $\mathbb{R}^3$ and points in $A \setminus e$.

For every line $l$ in $\mathbb{R}^3$, there is a corresponding dual line $l^*$ in $\mathbb{P}^3$ which intersects $A$ in two non-real points. These two points correspond to the two possible orientations of the line $l$.

A joint is the same as an oriented line in $\mathbb{R}^3$. To a joint $h_i$ and its half turn $-h_i$, we associate the points $p_i$ and $p_i^*$ in $A \setminus e$, respectively.

Lemma 3.8. (1) The joints axes of $h_1$ and $h_2$ are concurent if and only if the four points $p_1, p_1^*, p_2,$ and $p_2^*$ are on a conic in $A \setminus e$.

(2) The joints $h_1, h_2$ and $h_3$ satisfy a Bennett condition if and only if $p_1, p_2, p_2^*, p_3$ are on a conic in $A \setminus e$.

(3) If the joints $h_1, h_2, h_3$ satisfy a Bennett condition and the joints $h_1, h_2, h_4$ satisfy a Bennett condition, then joints $h_1, h_2, h_3$ satisfy a Bennett condition.

Proof. (1) The axes of $h_1$ and $h_2$ span a plane in $\mathbb{P}^3$. This plane is a point in the dual space contained in the dual lines $l_1^*$ and $l_2^*$. Hence, $l_1^*$ and $l_2^*$ also span a plane. The intersection of this plane with $A \setminus e$ is a conic that contains in particular $p_1, p_1^*, p_2,$ and $p_2^*$. When the two axes are parallel the conic is degenerate.

(2) We translate the lemma in coordinates. The statements remains unchanged if we apply an Euclidean isometry an all 3 lines $h_1$, $h_2$ and $h_3$ simultaneously. Therefore, we may assume w.l.o.g. that $h_2$ is the first axis and that the common normal between $h_1$ and $h_2$ is the third axis. The point $p_2$ corresponding to $h_2$ is $(0 : 1 : i : 0)$, and the point $p_2^*$ corresponding to $h_2$ with reversed orientation is $(0 : 1 : -i : 0)$. Assume $h_1$ corresponds to the point $p_1 = (x_1 : y_1 : z_1 : w_1)$, and that $h_3$ corresponds to the point $p_3 = (x_3 : y_3 : z_3 : w_3)$. Note that $x_1^2 + y_1^2 + z_1^2 = x_3^2 + y_3^2 + z_3^2 = 0$. Because the common normal between $h_1$ and $h_2$ is the third axis, we can choose projective coordinates such that $x_1, y_1$ are purely imaginary (i.e. real multiples of $i$) and $z_1$ is real. The
orbit of rotation of \( h_3 \) around \( h_2 \) is contained in the plane spanned by \( p_2, p_2^- \) and \( p_3 \). Hence the first statement in the lemma is not changed if we apply a rotations around \( h_2 \) to \( h_3 \). Therefore, we may assume w.l.o.g. that the common normal between \( h_2 \) and \( h_3 \) is parallel to the third axis and that \( x_3, y_3 \) are purely imaginary and \( z_3 \) is real.

We now compute the angles between \( l_2, l_1 \) and \( l_2, l_3 \). W.l.o.g., we may assume \( x_1^2 + y_1^2 = 1 \), \( z_1 = i \). If \( \alpha_1 \) is the angle between \( l_2 \) and \( l_1 \), then we have \( x_1 = \sin(\alpha_1), y_1 = \cos(\alpha_1) \). Similarly, if \( \alpha_2 \) is the angle between \( l_2 \) and \( l_3 \), then \( x_2 = \sin(\alpha_2), y_3 = \cos(\alpha_2) \).

Computation of distances and the offset: the normal distance \( d_1 \) between \( h_2 \) and \( h_1 \) is the imaginary part of \( w_1 \). Hence we have \( w_1 = d_1 i \) and \( w_3 = d_2 i + a \). Collecting everything about \( p_1, p_2 \) and \( p_3 \), we get:

\[
\begin{align*}
p_1 &= (\sin(\alpha_1) : \cos(\alpha_1) : 1 : d_1 i) \\
p_2 &= (0 : 1 : i : 0) \\
p_3 &= (\sin(\alpha_2) : \cos(\alpha_2) : 1 : d_2 i + a).
\end{align*}
\]

Now it is clear that the first statement of the lemma is equivalent to

\[
x_1 w_3 - x_3 w_1 = \sin(\alpha_1) o + (\sin(\alpha_1) d_2 - \sin(\alpha_2) d_1) i = 0
\]

and this is equivalent to the second statement.

(3) We can get it using the second statement.

\[ \square \]

**Lemma 3.9.** Let \( L \) be an \( nR \)-linkage with mobility \( m \geq 2 \) and joints \( \{h_1, \ldots, h_n\} \). Suppose that freezing joint \( h_n \) in a general position gives a linkage \( L' \) such that \( \{h_{n-1}, h_1, h_2\} \) satisfy a Bennett condition. Then,

\[ \{h_{n-1}, h_n, h_1, h_2\} \]

are concurrent.

**Proof.** By lemma 3.8, the associated points \( p_{n-1}, p_1, p_1^-, p_2 \) are contained on a conic, \( X \). We now rotate \( h_{n-1} \) around the fixed \( h_n \). From this rotation we get a conic \( Y \) containing the points \( p_n, p_{n-1} \). Since \( Y \subseteq X \) and these are irreducible and equi-dimensional, \( X = Y \). Hence, \( p_1, p_{n-1} \) and \( p_1^-, p_n \) are two sets of coplanar points. This is equivalent to say that the axes of \( h_1, h_n \) and \( h_{n-1} \). Finally, \( h_2 \) is concurrent with these since it satisfies a Bennett condition with \( h_1, h_{n-1} \).

\[ \square \]

### 4. Bond Theory

In this section, we define bonds associated to a linkage. We use it only for the results in section 6.

**Definition 4.1.** Let \( L \) be a linkage and \( J \) the set of joints of \( L \). Let \( S \subseteq J \) be a non-empty set of joints of \( L \). We say that \( S \) forms a **chain** if it consists of a sequence of consecutive joints. Notice that \( S \) is a chain if an only if \( J \setminus S \) is a chain.

Let \( L \) be a linkage with \( n \) joints. To an ordered chain \( S = \{j_r, \ldots, j_s\} \), where indices are taken mod \( n \), we associate the map \( \Phi_S: K_L \to \mathbb{D} \mathbb{H} \) given by \( m_r(t_r) \cdots m_s(t_s) \). We define \( \Phi_{J \setminus S}: K_L \to \mathbb{D} \mathbb{H} \) to be the map satisfying the loop equation eq. (1), that is,

\[ \Phi_S(t) \Phi_{J \setminus S}(t) \in \mathbb{C}^* \quad \forall t \in K_L. \]

More explicitly,

\[ \Phi_{J \setminus S}(t) = m^{*}_{r-1}(t) \cdots m^{*}_{r}(t) m^{*}_{n}(t) \cdots m^{*}_{s+1}(t) = (m_{s+1}(t) \cdots m_{r-1}(t))^*. \]

That is,

\[ \Phi_{J \setminus S}(t) = m_{s+1}(t) \cdots m_{r-1}(t). \]
The images of the maps $\Phi_S$ and $\Phi_{\beta_0}$ correspond to the relative motion between the link $r$ and link $s + 1$. We want to point out that it is also very interesting to look at what is the motion behavior at the compactification of $SE(3)$, see [HSS13a], i.e., the complex solutions of $\overline{K_L}$: We extend $\Phi_S$ to $\overline{K_L}$ and define the set of bonds of $L, B$, as

$$B = \{ t \in \overline{K_L} \mid \Phi_j(t) = 0 \}.$$  

The following lemma is adapted from [HSS13a] which is a consequence of the Affine Dimension Theorem [Har77, Proposition I.7.1] and is the basic consequence of higher mobility from the perspective of bonds:

**Lemma 4.2.** The set $B$ is an algebraic hypersurface of $\overline{K_L}$. Moreover, its complex dimension is one less than the mobility of the linkage.

Each bond component of an $m$-dimensional irreducible component of $\overline{K_L}$ has dimension $m - 1$. In this paper, we always consider such irreducible components where no joint is frozen. There might be lower dimensional components in $\overline{K_L}$.

**Lemma 4.3.** Suppose $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve given by

$$(i - h_s)(t_1 - h_{s+1})(t_2 - h_{s+2})(i - h_{s+3}) = 0$$

where $h_i$ are non-parallel R-joints. Then, there is a Bennett condition between $(h_s, h_{s+1}, h_{s+2})$ or $(h_{s+1}, h_{s+2}, h_{s+3})$.

**Proof.** Since $C$ is a curve, there is a surjective map $C \to \mathbb{P}^1$ to one of the components, say, to the first one. In particular, the pre-image of $i \in \mathbb{P}^1$ is $(i, t)$ for some fixed $t \in \mathbb{P}^1$. That is, we have, over $i$,

$$(i - h_s)(t_1 - h_{s+1})(t_2 - h_{s+2})(i - h_{s+3}) = 0.$$ 

By lemma 2.4, it follows that $(i - h_{s+1})(t' - h_{s+2})(i - h_{s+3}) = 0$, since $h_s$ and $h_{s+1}$ are not parallel. By theorem 3.6, this is equivalent to a Bennett condition between $(h_{s+1}, h_{s+2}, h_{s+3})$. $\square$

Define the subset of bonds $\tilde{B}$ to be the ones of the form $(\ast, i, \beta_k, i, \ast)$ where $N(m_k(\beta_k)) \neq 0$.

**Lemma 4.4.** Let $\beta, \beta' \in \tilde{B}$ be bonds such that $\beta_k \neq \beta'_k$. Let $S \subseteq J$ be the chain of R-joints $\{ j_{k-1}, j_k, j_{k+1} \}$. If $e \Phi_S(\beta) = e \Phi_S(\beta') = 0$, then $S$ is a set of parallel joints.

**Proof.** The equation $(i - p_{k-1})(t - p_k)(i - p_{k+1}) = 0$ is linear in $t \in \mathbb{P}^1$. Hence, if the coefficients are not all identically zero, it has at most one solution. Since $\beta \neq \beta'$ are two solutions, it follows that the coefficients of the equation are identically zero, that is,

$$\Phi_S(\beta) = \Phi_S(\beta') = 0 \quad \iff \quad \begin{cases} 
(i - p_{k-1})(i - p_{k+1}) = 0 \\
(i - p_{k-1})p_k(i - p_{k+1}) = 0 
\end{cases}$$

The first equation implies $h_{k-1} \parallel h_{k+1}$. From the second equation we then get $h_k \parallel h_{k+1}$. $\square$

The following definition is adapted from [HSS13a, LSS18]:

**Definition 4.5.** Let $\beta$ be a bond of $L$. Recall that $\beta$ is attached to a joint $j_k$ if $N(m_k(\beta)) = 0$, where $m_k = t_k - c_{p_k}$ if $j_k$ is a $P$-joint and $m_k = t_k - h_k$ if $j_k$ is an $R$-joint. We call $A_\beta \subseteq J$ the subset of joints to which $\beta$ is attached. Hence, $\beta \in B$ if and only if there is a $k$ such that $j_k \in A_\beta$.

It was proven in [LSS18] that the mobility of a specific joint in a linkage is equivalent to the existence of a bond attached to that joint. Hence, for a mobile linkage, there is always some $\beta$ for which $A_\beta$ is non-empty. In fact, it is known that $|A_\beta| \geq 2$ in general. See [HSS13a]. In addition, in [HSS13a, Proposition 1.1], it is also proved that the dimension of the set of all bonds is $m - 1$ if the linkage has mobility $m$.

We can see these properties from the following example which is a concrete known mobile $P4R$-linkage taken from [MB97, MR95].
Example 4.6. This is an example of a $P4R$-linkage with two pairs of parallel neighbor axes. It is constructed using the geometry constraints from [MB97, MR95]. Set 
\[ h_0 = ej + ek, \quad h_1 = k + ej, \quad h_2 = k + ei, \quad h_3 = j + ei + 2ek, \quad h_4 = j + ek. \]
Both sides of 
\[ (t_0 - h_0)(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) \equiv 1 \]
are written as 8-dimensional vectors, which yields a system of 7 equations. The 5th coordinate is a redundant condition because it is already satisfied when the other six equations are fulfilled due to the Study condition [HS10]. In order to exclude “unwanted” solutions, that is, those such that 
\[ t_0(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1) = 0 \]
in general we add an extra equation 
\[ t_0(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)u - 1 = 0, \]
Using Gröbner basis on Maple, an elimination ideal with respect to $u$ (the Rabinowitsch trick, [Rab29]) is 
\[ I = \langle t_1 - t_3, t_1 + t_2, t_1 + t_4, t_0t_1 + t_1^2 + t_0 + 1 \rangle \]
which defines the closure of the configuration set $\overline{K_L}$. The set of bonds are then defined by adding the equation 
\[ t_0(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1) = 0. \]
They are 
\[ \{ t_0 = 0, t_1 = i, t_2 = -i, t_3 = i, t_4 = -i \} \]
and its conjugate, where $i$ is the complex imaginary unit.

Figure 1. Five configurations of a $P4R$-linkage.

5. $n$-linkages with mobility at least $n - 4$

In this section we explore the geometric constraints that arise from requiring higher mobility in a linkage, i.e., mobility at least $n - 4$ for an $n$-linkage.

**Proposition 5.1.** The maximum possible mobility of an $n$-linkage where $n \geq 4$ is $n - 3$.

**Proof.** By the classification of 4-linkages, see [Del22], this is well known. We proceed by strong induction on $n$. Let $L$ be a linkage with mobility $m$ and $n$ joints, with $m \geq n - 2$. If we freeze one joint, then we get a linkage $L'$ of mobility $m - 1$ and $n - l$ active joints, where $l \geq 1$. This is a contradiction by the induction hypothesis. \hfill $\Box$

**Proposition 5.2.** A closed $n$-linkage with $n \geq 5$ and $P$ or $R$ joints has mobility $n - 3$ if and only if the motions of all joints are in one of the two three dimensional groups, $SE(2)$ and $SO(3)$.

**Proof.** We prove the result by induction on $n$. If $n = 5$, the result is well known, see [ALS15]. We proceed by strong induction on $n$. Let $L$ be a linkage with mobility $m$ and $n$ joints, with $m = n - 3$. If we freeze one joint, then we get a linkage $L'$ of mobility $m - 1$ and $n - l$ active joints, where $l \geq 1$. By the induction hypothesis and proposition 5.1, it follows that $l = 1$. Moreover, all the motions of joints of $L'$ are in $SE(2)$ and $SO(3)$. The motion of the joint we froze must all be in the same group due to the closure condition. \hfill $\Box$
Remark 5.3. For $n = 4$, the statement is not true by several examples: The Bennett 4R-linkage is one of them. Also, we cannot include linkages with helical joints since there are many counter-examples. See Example 5.4.

Example 5.4. Consider an $nR$-linkage, $L$, where all revolute axes are parallel. All the joint motions are in $SE(2)$, therefore the mobility of $L$ is $n - 3$. Let $L'$ be a new linkage formed by replacing the $R$-joints by $H$-joints with the same pitch. We claim that there is an isomorphism of configurations of $L$ and $L'$, therefore the mobility is the same. If we have a configuration of $L'$, then we can ignore all the translations perpendicular to the plane where all the motions are taking place and this gives a configuration of $L$. In the other direction, we add translations perpendicular to this plane, proportional to the rotation angles. A priori, it is not clear that these translations add up to zero. But they do, since the sum of all rotation angles in $L$ is constant.

A closed linkage has mobility $n - 4$ if the motions of all joints are in the Schoenflies subgroup of $SE(3)$ which has dimension 4. However, the converse is not true. For instance, Goldberg 5R and 6R with mobility 2. We completely classify higher mobility $n$-linkages with mobility $n - 4$. The 5-linkages with mobility 1 is classified in [ALS15]. We give the classification for 6R-linkages with mobility 2 in section 5.1. We give the classification for 6-linkages with mobility 2, $P$-joints and $H$-joints in section 5.2 containing families that are not in the Schoenflies subgroup. We give the classification for $nR$-linkages ($n > 6$) with mobility $n - 4$ or higher in section 5.3 where there is no an $nR$-linkage with mobility $n - 4$. We give the classification for $n$-linkages ($n > 6$) with mobility $n - 4$ or higher in section 5.4 where the $n$-linkages has mobility $n - 4$ iff their joints are in the Schoenflies subgroup.

Recall that if $L$ has two neighbouring $R$-joints with equal axes or two neighbouring $P$-joints with equal directions, then we say that $L$ is degenerate. We also assume that $L$ is not degenerate, and that no joint parameters are constant during motion (the highest dimensional components) of the linkage (otherwise one could easily make $n$ smaller).

Throughout this section, we use the following fact when we freeze one joint of $n$-linkage with mobility $n - 4$

Lemma 5.5. Let $L$ be an $n$-linkage, where $n > 5$, of mobility $n - 5$, such that all joints actually move. If we freeze one joint of $L$ at a general position, then there is at most one more joint being frozen simultaneously in the resulting linkage $L'$.

Proof. If there are another two or more joints being frozen simultaneously, then the resulting linkage $L'$ has $m \leq n - 3$ joints with mobility $n - 5$ which is impossible. \hfill $\Box$

5.1. 6R-linkages. In this section, we classify paradoxical 6R-linkage with mobility at least 2. It is known that all axes of the 6R-linkage with mobility at least 3 are concurrent (spherical linkage) or are parallel (planar linkage). If a 5R-linkage with mobility 2, then, by [HSS13a], it is either spherical or planar. If a 4R-linkage with mobility 1, then, by [Del22], it can be spherical, planar or Bennett. For a Bennett 4R-linkage, its Denavit-Hartenberg parameters fulfill:

$$b_1 = b_2 = b_3 = b_4,$$

$$c_1 = c_3, \quad c_2 = c_4,$$

$$o_1 = o_2 = o_3 = o_4 = 0.$$  \hfill (2)

If a 5R-linkage with mobility 1, then, by [Kar98, HSS13a], either all axes are concurrent or it is a Goldberg 5R-linkage. For a Goldberg 5R-linkage, its Denavit-Hartenberg parameters fulfill:

$$b_1 = b_2 = b_3 = b_4,$$

$$c_1 = c_4,$$

$$o_2 = o_3 = o_4 = 0,$$  \hfill (3)

and some further complicated equational conditions which are not used in this paper, see [Die95]. We give the classification paradoxical 6R-linkage of mobility 2 in the following theorem.
Theorem 5.6. Let $L = [h_1, h_2, h_3, h_4, h_5, h_6]$ be a 6R-linkage with mobility 2. Then its Denavit-Hartenberg parameters are: all Bennett ratios are same, i.e., $b_s = b_{s+1}$, all offsets are zeros, i.e., $a_s = 0$, and the cosines ($c_s$) of twist angles ($\alpha_s$) fulfill one of two following cases (by a cyclic shift of indices):

1. $c_1 = c_4$, $c_2 = c_6$, $c_3 = c_5$. This is a composition of two Bennett Linkages and one common link and two joints.
2. $c_1 = c_4$, $c_2 = c_5$, $c_3 = c_6$.

Proof. Fix a 2-dimensional irreducible component of $K_L$. Suppose we freeze one joint. Since $L$ has mobility 2, we still have a mobile linkage $L'$. The number of joints in $L'$ is either 4 or 5.

Case I: Suppose there is one joint such that when we freeze it at a generic position, another joint gets frozen as well. We can assume that this is joint 1. As the resulting linkage $L'$ is a 4R-linkage, this $L'$ must be a Bennett linkage or a linkage with four concurrent axes. It is not possible that another non-neighbor joint gets frozen, because, by lemma 3.9, all six axes must be concurrent which contradicts the mobility 2. Therefore, it is only possible that one of the neighbors of joint 1 is frozen and we can assume it is joint 2. We claim that the relative positions of axes 3 and 6 does not change when we move $L$. To prove this claim, we move $L$ and freeze $h_1$ in another position. Again, $h_2$ gets frozen and we call the resulting 4R-linkage $L''$. Assume that $L'$ is a Bennett Linkage. Then, $L'$ and $L''$ have the same Denavit-Hartenberg parameters since they are both mobile 4R-linkages with partially coinciding parameters. If $L'$ is not a Bennett Linkage, then $h_3, h_4, h_5, h_6$ are concurrent and we can replace it by a spherical joint (S) and this is a 2RS-linkage with mobility at least 1 since, otherwise, axes 1 and 2 would not move. Call the center of the spherical joint $o$. We can compute the orbit of $o$ in the frame of the link with axes $h_1$ and $h_2$. If $o$ is not on the joint $h_1$, then the orbit is a circle around $h_1$ and similarly for $h_2$. These two circles must coincide, otherwise the 2RS-linkage would not be mobile and we conclude that $L$ is a degenerate linkage. On the other hand, if $o$ is on $h_1$, then all axes are concurrent or one joint does not move.

Finally, since the relative positions of joints 3 and 6 does not change, we can introduce an extra link connecting them and the linkage $L$ is the composition of two Bennett Linkages stacked on top of each other, having one link and two joints in common. See fig. 2 for an example.

Case II: Suppose no other joint gets frozen when we freeze any joint at a generic position. The linkage $L'$ becomes a Goldberg 5R-linkage. The equational conditions of case II can be obtained by cyclically freezing joint by joint using the Denavit-Hartenberg parameter constraints in eq. (3).

We show the following numeric examples of 6R-linkage of mobility 2.

Example 5.7. Here is an example of a 6R-linkage without any parallel neighbor axes with mobility 2. It is constructed using factorization of motion polynomials as in [HSS13b]. In fact, it is combined with two Bennett 4R-linkages which have two consecutive rotational joints in common. Set

$$
\begin{align*}
  h_1 &= i, \\
  h_2 &= \left( \frac{1}{3} + \frac{4}{9} \epsilon \right) i + \left( \frac{2}{3} + \frac{2}{9} \epsilon \right) j - \left( \frac{2}{3} - \frac{4}{9} \epsilon \right) k, \\
  h_3 &= \left( \frac{41}{105} + \frac{4288}{11025} \epsilon \right) i + \left( \frac{88}{105} - \frac{16}{11025} \epsilon \right) j - \left( \frac{8}{21} - \frac{872}{2205} \epsilon \right) k, \\
  h_4 &= \left( \frac{33}{35} + \frac{68}{1225} \epsilon \right) i - \left( \frac{6}{35} - \frac{274}{1225} \epsilon \right) j - \left( \frac{2}{7} - \frac{12}{245} \epsilon \right) k, \\
  h_5 &= \left( \frac{1093}{1365} + \frac{313072}{1863225} \epsilon \right) i - \left( \frac{52}{105} + \frac{16}{11025} \epsilon \right) j + \left( \frac{92}{273} - \frac{149572}{372645} \epsilon \right) k, \\
  h_6 &= \left( \frac{29}{39} + \frac{340}{1521} \epsilon \right) i - \left( \frac{2}{3} - \frac{2}{9} \epsilon \right) j + \left( \frac{2}{3} - \frac{536}{1521} \epsilon \right) k.
\end{align*}
$$
We have that the four joints defined by \( h_1, h_2, h_3, h_4 \) are a Bennett 4R-linkage. At the same configuration the \( h_5, h_6, h_4 \) are another Bennett 4R-linkage [HSS13b]. Using Gröbner basis from Maple, an elimination ideal for the Zariski closure \( \overline{K_L} \) (See [LS15]) is

\[
I = (2t_2 + 2t_3 + 1, t_6 + t_5 + 1, t_1 t_3 + t_1 t_5 + t_3 t_5 - 1, \\
2t_1 t_3 + 2t_1 t_5 + 2t_3 t_4 + 2t_4 t_5 - 2t_3 + 3t_4 - t_5 - 1, \\
2t_1 t_3^2 + 2t_1^2 t_4 - t_1 t_3 + 3t_1 t_4 - 2t_3^2 + 3t_3 t_4 + t_1 - t_3 + 2t_4 - 1).
\]

By adding the equation \((t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)(t_5^2 + 1) = 0\) and using primary decomposition, the 1-dimensional bonds are defined by the ideals:

\[
B_1 = (t_1 t_3 - t_1 t_6 - t_1 - t_3 t_6 - t_3 - 1, t_2 + t_3 + 1/2, t_4 + t_5 + t_6 + 1, t_5^2 + 1), \\
B_2 = (t_1 t_6^2 + 3t_1 t_6 + 13/4t_1 + t_4 t_6^2 + 2t_4 t_6 + 2t_5 - 1/2t_6^2 - 5/4t_6 - 1/4, t_2 + t_4, \\
t_1 t_4 - t_1 t_6 - 3/2t_1 - t_4 t_6 - t_4 + 1/2t_6 - 1/2, t_3 - t_4 + 1/2, t_3^2 + 1, t_5 + t_6 + 1), \\
B_3 = (t_1 + 1/3t_6 - 1/3, t_2 + t_6, t_3 - t_6 + 1/2, t_5 + t_6 + 1, t_5^2 + 1), \\
B_4 = (t_1 + t_3 + t_2 + t_3 + 1/2, t_3^2 + 1, t_3 t_4 + t_3 t_6 - t_4 t_6 + 1/2t_4 + 1/2t_6 + 1, \\
t_3 t_6^2 + t_3 - t_4 t_6 + 4t_4 + 1/2t_6^2 - 1/4t_6 - 1/2, \\
t_1^2 t_6 - t_1^2 t_6 + 5/4t_4 - t_1 t_6^2 + 1/2t_4 t_6 + t_4 + 5/4t_6 + t_6 + 1, t_5 + t_6 + 1), \\
B_5 = (t_1 - t_6 - 1, t_2 + t_3 + 1/2, t_3 t_4 + t_3 t_6 - t_4 t_6 + 1/2t_4 + 1/2t_6 + 1, \\
t_5 + t_6 + 1, t_3^2 + 2t_6 + 2), \\
B_6 = (t_2 + t_6 + 3/2, t_3 - t_6 - 1, t_4 - 1/3t_6 - 2/3, t_5 + t_6 + 1, t_6^2 + 2t_6 + 2).
\]

Five configurations of the mobility 2 linkages are shown in fig. 2.

![Figure 2. Five configurations of the 6R-linkage in example 5.7.](image)

**Example 5.8.** There is a cheap way to obtain another linkage when the original linkage has a Bennett condition. This is call the isomeric mechanism by [Woh91]. Namely, one can replace a joint axis (which is in the middle of the Bennett condition) by another new axis where the new axis and the original three axis make a Bennett loop. There are six Bennett conditions in example 5.7. Using the isomeric trick for some Bennett conditions, we will obtain 6R-linkages with frozen joints which are not mainly interesting here. One can obtain other three isomeric mechanisms by replacing one (or two) of the joints \( h_1 \) and \( h_4 \) by new joint axes \( h_1' \) and \( h_4' \)

\[
h_1' = \left( \frac{101}{117}, \frac{112}{1521}, \epsilon \right) i + \frac{4}{9} \left( \frac{28}{117} - \frac{404}{1521} \right) k, \\
h_4' = \left( \frac{3301}{4095}, \frac{240628}{1863225}, \epsilon \right) i + \left( \frac{86}{315} + \frac{274}{1225}, \epsilon \right) j - \left( \frac{430}{819} - \frac{117232}{372645} \right) k.
\]

With this replacement, we have no Bennett 4R-linkage for any consecutive four joints. Using Gröbner basis from Maple, we can have same arguments as example 5.7.
5.2. 6-linkages. In this section we classify 6-linkages of mobility 2 or higher containing revolute joints and at least a prismatic or helical joint. Mobile linkages with 4 joints of type $R$, $P$ or $H$ have been classified in [Dei22]. Mobile linkages with 5 joints of type $R$, $P$ or $H$ have been classified in [ALS15].

The following result settles the case for $P4R$-linkages:

**Theorem 5.9** ([ALS15, Theorem 6]). Let $L$ be a 5-linkage with at least one $P$-joint and all other joints of type $R$. Then one of two following cases occur:

1. All axes of $R$-joints are parallel.
2. Suppose $j_0$ is the unique $P$-joint of $L$. Then, $h_1 \parallel h_2$ and $h_3 \parallel h_4$. $L$ has mobility 1, and $t_1 = \pm t_2$ and $t_3 = \pm t_4$ is fulfilled on the configuration curve.

**Remark 5.10.** Let $L$ be a 5-linkage with at least one $P$-joint and all other joints of type $R$. If there is just one $P$-joint in $L$, then the mobility of $L$ is 2. If there are two $P$-joints or more in $L$, then the mobility of $L$ is either 2 where the directions of all $P$-joints are parallel to the plane (say $M$) which is perpendicular to the axes $R$-joints or 1 where there are at least two $P$-joints such that their directions are not parallel to the plane $M$.

Besides theorem 5.6, we solve the classification of linkages with 6 joints of mobility 2 with joints type $R$ and $P$:

**Theorem 5.11.** Let $L$ be a 6-linkage of mobility 2 with at least one $P$-joint and all other joints of type $R$. Then one of two following cases occur:

1. All axes of $R$-joints are parallel and $L$ has at least two $P$-joints.
2. Suppose $j_0$ and $j_3$ are the only two $P$-joints of $L$. Then, $h_1 \parallel h_2$ and $h_5 \parallel h_4$. $L$ has mobility 2, $p_0 \parallel p_3$, and $t_1 = \pm t_2$ and $t_3 = \pm t_4$ is fulfilled on the configuration curve.

**Proof.** Case I: Suppose $L$ has one $P$-joint. Since $L$ has mobility 2, we can find an $R$-joint such that the $P$-joint is still active when we freeze this $R$-joint. If we freeze such an $R$-joint, we still have a mobile linkage $L'$. If no other joint is frozen, then we have two pairs of parallel joints by theorem 5.9 and remark 5.10. Assume that $L$ has two pairs of parallel joints, we can freeze the $P$-joint of $L$ which results in a $5R$-linkage with two pairs of parallel joints with different directions. By the classification of $5R$-linkages, [Kar98, HSS13a], this is impossible. On the other hand, if another $R$-joint is frozen, then the resulting mobile 4-linkage $L'$ will become a 4-linkage with at least one $P$-joint. Then all axes of $R$-joints are parallel in $L'$ by [Dei22]. Therefore, if we freeze the $P$-joint, all axes of $R$-joints are parallel in $L$. In this case, the direction of the $P$-joint must be perpendicular to the direction of the axes of $R$-joints, and the mobility of $L$ is 3 which contradicts our assumption.

Case II: Suppose $L$ has two $P$-joints. We freeze one $P$-joint (say joint $j_2$) and call the resulting linkage $L'$. If the other $P$-joint is frozen too, then the resulting 4$R$-linkage is mobile only when all axes are parallel. Assume that the other $P$-joint must be active. If there is another $R$-joint being frozen simultaneously, by lemma 5.5, then the axes of $R$-joints of the $L'$ must be parallel because it is 4-linkage with a $P$-joint and the direction of the $P$-joint is parallel to the plane which is perpendicular to the axes of $R$-joints in $L'$. And the axes of $R$-joints of the $L$ must be parallel by spherical projection. Then we have contradiction that either the mobility of $L$ is 3 when joint $j_n$ is active or joint $j_n$ is not an active joint in $L$. Assume that no other joint is frozen, then, by theorem 5.9 and remark 5.10, we have two pairs of parallel joints in $L'$. We can freeze the other $P$-joint in $L$ to get another $P4R$-linkage, $L''$. We claim the two $P$-joints have the same direction. Otherwise, they would span a plane and if we freeze an $R$-joint in $L$, then a neighboring $R$-joint needs to be frozen as well. And, the two remaining $R$-joints must generate a circular translation which is not containing in the plane spanned by the motion of the $P$-joints which is impossible. Therefore, in this case, we have two possibilities, either all axes of $R$-joints are parallel where the translation parameters of the two $P$-joints are dependent or there are two pairs of parallel $R$-joints as in theorem 5.9 where the two translations have the same direction.

Case III: Third, if $L$ has $r = 3$ or more $P$-joints, then all rotation axes are parallel. Because a spherical linkage of $L$ with $6 - r \leq 3$ revolute joints is necessarily degenerate: if all three (or fewer) joints are actually moving, then all axes are coinciding. Therefore, in this case, the mobility of $L$ is 2
when there are at least two \( P \)-joints such that their directions are not parallel to the plane which is perpendicular to the axes \( R \)-joints.

**Remark 5.12.** Let \( L \) be a 6-linkage of mobility 2 or 3 with at least one \( P \)-joint and all other joints of type \( R \). If there is just one \( P \)-joint in \( L \), then the mobility of \( L \) is 3 when the direction of the \( P \)-joint is parallel to the plane (say \( M \)) which is perpendicular to the axes \( R \)-joints (all are parallel). If there are two \( P \)-joints or more in \( L \) and the axes of \( R \)-joints in \( L \) are parallel, then the mobility of \( L \) is either 3 where the directions of all \( P \)-joints are parallel to the plane (say \( M \)) which is perpendicular to the axes \( R \)-joints or 2 where there are at least two \( P \)-joints such that their directions are not parallel to the plane \( M \).

**Figure 3.** Five configurations of a \( PRRPRR \)-linkage.

Mobile linkages with 5 joints of type \( R \), \( P \) or \( H \) have been classified in [ALS15].

**Theorem 5.13 ([ALS15, Theorem 10]).** Let \( L \) be a non-degenerate mobile 5-linkage with \( R \)-, \( P \)-, and \( H \)-joints, with at least one \( H \)-joint, such that all joints actually move. Up to cyclic permutation, the following cases are possible.

1. All axes of \( R \)- and \( H \)-joints are parallel.
2. There is one \( P \)-joint \( j_0 \), all other joints are of type \( H \) or \( R \), \( h_1 || h_2 \) and \( h_3 || h_4 \).

Recall from [ALS15], for a \( n \)-linkage \( L \) with at least one helical joint, we say \( L_r \) that every \( H \)-joint is replaced by an \( R \)-joint with same axis, and in family \( L_p \) that every \( H \)-joint is replaced by a \( P \)-joint with the direction parallel to the axis of the \( H \)-joint. We solve the case for linkages with 6 joints of mobility 2 with joints type \( R \), \( P \) or \( H \):  

**Theorem 5.14.** Let \( L \) be a 6-linkage of mobility 2 or higher with at least one \( H \)-joint, such that all joints actually move. Up to cyclic permutation, the following cases are possible.

1. All axes of \( R \)- and \( H \)-joints are parallel.
2. There are two \( P \)-joints \( j_0 \) and \( j_3 \) and \( p_0 || p_3 \), all other joints are of type \( H \) or \( R \), \( h_1 || h_2 \) and \( h_4 || h_5 \).

**Proof.** Let \( r \) be the number of neighbouring blocks of equal axes of the spherical projection \( L_s \). The proof proceeds by case distinction on \( r \).

- **Case** \( r = 1 \): Then all axes of \( L_s \) are equal, hence all axes of \( H \)- and \( R \)-joints of \( L \) are parallel; this is possibility (1) of the theorem, and the motion of each joint belongs to a same Schoenflies motion subgroup of \( SE(3) \).
- **Case** \( r = 2 \): Then each of the two blocks of \( R \)-joints in \( L_s \) has at least two joints, because a single joint could not move. The linkage \( L_r \) is movable and therefore has at least four joints that actually move. In particular, it cannot happen that all axes of \( L_r \) that actually move are parallel. After removing the joints that remain fixed, \( L_r \) still has two blocks of parallel axes. By comparing with theorem 5.11, it follows that \( L_r \) is a \( PRRPRR \) linkage; if, say, \( j_0 \) and \( j_3 \) are prismatic joints, then \( h_1 || h_2 \) and \( h_4 || h_5 \). This is possibility (2) of the theorem.
- **Case** \( r \geq 3 \): If there is at least one group of joints of the spherical projection \( L_s \) with only one \( R \)-joint, then this joint cannot move. Hence the corresponding \( H \)- or \( R \)-joint of \( L \) does not move either, contradicting our assumption. Assume that there is no group of joints of the spherical projection \( L_s \) with only one \( R \)-joint. By freezing one joint of \( L \), at most two joints are frozen
simultaneously, say $L_f$ with a helical joint is the resulting linkage. If $L_f$ is a 5-linkage, then $L_f$ has three groups of joints of the spherical projection, which is impossible. If $L_f$ is a 5-linkage, then $L_f$ has three groups of joints of the spherical projection, which is impossible. If $L_f$ is a 4-linkage with a helical joint, then $L_f$ has at least two groups of joints of the spherical projection, which is impossible too.

5.3. $nR$-linkages. In this section, we classify paradoxical $nR$-linkage with mobility of $n - 4$, where $n > 6$.

**Theorem 5.15.** Let $L$ be an $nR$-linkage, where $n > 6$, of mobility $n - 4$ or higher, then all axes are concurrent. Moreover, $L$ has mobility $n - 3$.

**Proof.** We prove this result by induction on the number of active joints.

For $n = 7$, we assume that $L$ is a 7R-linkage with mobility 3. Then, freezing the joint $j_n$ of $L$, we get a linkage $L'$ with mobility 2. If no other joint is frozen, by theorem 5.6, the $(h_2, h_1, h_{n-1})$ and $(h_1, h_{n-1}, h_{n-2})$ satisfy Bennett conditions and, in fact, all the Bennett ratios are equal. As we freeze the joint $j_n$ at an arbitrary position, we get that $h_2, h_1, h_n, h_{n-1}$ are concurrent by lemma 3.9. Therefore, all Bennett ratios are equal to zero and $L'$ has mobility 3. Then $L$ has mobility 4, which is a contradiction. If another joint is frozen, $L'$ becomes a 5R-linkage with mobility at least 2. Then all axes of $L'$ are concurrent. We can freeze another joint of $L$ different from $j_n$ and, in a similar way, we conclude that all axes of the resulting linkage are concurrent. Therefore, all axes are concurrent and the mobility of $L$ is 4.

Fix $n \geq 8$. Suppose that all axes of any $(n - k)R$-linkage with $k \geq 1$ whose mobility is at least $n - 5$ are concurrent. Let $L$ be an $nR$-linkage with mobility at least $n - 4$. Then, freezing the joint $j_n$ of $L$, we get a $(n - k)R$-linkage $L'$ with mobility at least $n - 5$. By strong induction hypothesis, $L'$ has mobility $n - 4$ and all axes of $L'$ (and hence of $L$) are concurrent.

5.4. $n$-linkages. In this section, we classify paradoxical $n$-linkage with mobility $n - 4$, where $n > 6$. If the mobility of an $n$-linkage with at least one $P$-joint and other $R$-joints is $n - 3$, then all the revolute joints are parallel.

**Theorem 5.16.** Let $L$ be an $n$-linkage, where $n > 6$, of mobility $n - 4$ with at least one $P$-joint and other $R$-joints, such that all joints actually move, then all axes of $R$-joints are parallel and the number of $P$-joints is at least 2.

**Proof.** We prove this result by induction on the number of active joints. For $n = 7$, we have following cases:

**Case I:** Suppose $L$ has one $P$-joint. Since $L$ has mobility 3, we can find an $R$-joint such that the $P$-joint is still active when we freeze this $R$-joint. If we freeze such an $R$-joint, we still have a mobile linkage $L'$ with mobility 2. If no other joint is frozen, then we have contradiction by theorem 5.11. On the other hand, by lemma 5.5, if another $R$-joint is frozen, then the resulting mobile 5-linkage $L'$ has mobility 2 with one $P$-joint. Then all axes of $R$-joints are parallel in $L'$. The two frozen $R$-joints are either parallel to the axes of $R$-joints, i.e., all axes of $R$-joints in $L$ are parallel, or parallel to each other and being neighboring in $L$. Assuming that we have two groups of parallel joints in $L$. If we freeze the $P$-joint of $L$, the resulting linkage with at most six revolute joints has mobility 2 and two groups of parallel joints. This is impossible by the classification of 5R-linkages and 6R-linkages with mobility 2 theorem 5.6. Therefore, in this case, the direction of the $P$-joint must be perpendicular to the direction of the axes of $R$-joints, and the mobility of $L$ is 4 which contradicts our assumption.

**Case II:** Suppose $L$ has two $P$-joints. If we freeze a $P$-joint, we still have a mobile linkage $L'$ with mobility 2. If the other $P$-joint is frozen, then the resulting $L'$ is a 5R-linkage of mobility 2. Then all axes of $R$-joints must be parallel because the $P$-joint would change the intersecting point if all axes of $R$-joints were concurrent. On the other hand, if the other $P$-joint is not frozen, then the resulting linkage must be a 5-linkage by theorem 5.11 and all axes of $R$-joints are parallel in $L'$. The frozen $R$-joint must
be parallel to the axes of \( R \)-joints in \( L' \), i.e., all axes of \( R \)-joints in \( L \) are parallel and the direction of one \( P \)-joint is parallel to the plane that is perpendicular to the axes of \( R \)-joints which contradicts that the mobility of \( L \) is 3. Therefore, in this case, the directions of the two \( P \)-joints must be not parallel to the plane which is perpendicular to the axes of \( R \)-joints.

**Case III:** Suppose \( L \) has three \( P \)-joints. If we freeze a \( P \)-joint (which is named \( j_{m} \)), we still have a mobile linkage \( L' \) with mobility 2. The other two \( P \)-joints cannot be frozen simultaneously. Otherwise, the resulting \( L' \) is a 4\( R \)-linkage of mobility 2 which is impossible. On one hand, if another \( P \)-joint is frozen, then the resulting mobile 5-linkage \( L' \) has mobility 2 with one \( P \)-joint. Then all axes of \( R \)-joints are parallel in \( L' \). Then all axes of \( R \)-joints must be parallel in \( L \). On the other hand, if the two \( P \)-joints are both active, then either all axes of \( R \)-joints are parallel in \( L' \) or we have two pairs of parallel joints and two \( P \)-joints with same directions by theorem 5.11. Assuming we have only one group of parallel joints, then all axes of \( R \)-joints must be parallel in \( L \). Assume that we have two pairs of parallel joints as theorem 5.11, we can freeze another \( P \)-joint of \( L \) different from \( j_{m} \) and, in a similar way, we conclude that the directions of all three \( P \)-joints are same. Then \( L \) must be degenerated which is a contradiction. Therefore, in this case, all axes of \( R \)-joints are parallel in \( L \) and there are at least two \( P \)-joints such that their directions must be not parallel to the plane which is perpendicular to the axes of \( R \)-joints.

**Case IV:** If \( L \) has \( r = 4 \) or more \( P \)-joints, then all rotation axes are parallel. Because a spherical linkage of \( L \) with \( 6 - r \leq 3 \) revolute joints is necessarily degenerate: if all three (or fewer) joints are actually moving, then all axes are coinciding. Therefore, in this case, all axes of \( R \)-joints are parallel in \( L \) and there are at least two \( P \)-joints such that their directions must be not parallel to the plane which is perpendicular to the axes of \( R \)-joints.

Fix \( n \geq 8 \). Suppose that all axes of any \((n-\ell)\)-linkage of mobility \( n - 5 \) with at least one \( P \)-joint are parallel, where \( 2 \geq k \geq 1 \) by lemma 5.5, the number of \( P \)-joints is at least 2 when \( k = 1 \), the directions of \( P \)-joints are perpendicular to the direction of the axes of \( R \)-joints. Let \( L \) be an \( n \)-linkage of mobility \( n - 4 \) with at least one \( P \)-joint. Then, freezing an \( R \)-joint (say \( j_{n} \)) of \( L \) such that one \( P \)-joint is always active, we get a \((n-\ell)\)-linkage \( L' \) with mobility at least \( n - 5 \), where \( 2 \geq \ell \geq 1 \). Assuming \( \ell = 1 \), by strong induction hypothesis, all axes of \( L' \) (hence of \( L \)) are parallel and the number of \( P \)-joints is at least 2 in \( L' \) (hence in \( L \)). Assuming \( l = 1 \), by strong induction hypothesis, all axes of \( R \)-joints of \( L' \) (hence of \( L \)) are parallel and the number of \( P \)-joints is at least 2 in \( L' \) (hence in \( L \)). Assuming \( l = 2 \) where another joint (say joint \( j_{m} \)) is frozen, by strong induction hypothesis, all axes of \( R \)-joints of \( L' \) are parallel and the directions of \( P \)-joints are perpendicular to the direction of the axes of \( R \)-joints. If the joint \( j_{m} \) is a \( P \)-joint, then the axis of the frozen \( R \)-joint \( j_{n} \) must be parallel to all axes of \( R \)-joints of \( L' \). Hence, all axes of \( R \)-joints of \( L \) are parallel. As the directions of \( P \)-joints are perpendicular to the direction of the axes of \( R \)-joints in \( L' \) (hence of \( R \)-joints in \( L \)), the direction of the \( P \)-joint \( j_{m} \) must be perpendicular to the direction of the axes of \( R \)-joints in \( L \). Then the mobility of \( L \) is \( n - 3 \) which is a contradiction. If the joint \( j_{m} \) is an \( R \)-joint, then the axes of the frozen \( R \)-joints \( j_{m}, j_{n} \) must be parallel. We can freeze a \( P \)-joint of \( L \) such that \( j_{m} \) and \( j_{n} \) are not frozen simultaneously, and the resulting linkage we call \( L'' \). If \( L'' \) has \( P \)-joint, in a similar way, we conclude that all axes of \( R \)-joints in \( L'' \) are parallel. Then all axes of \( R \)-joints in \( L \) are parallel. Then the mobility of \( L \) is \( n - 3 \) which is a contradiction. If \( L'' \) has no \( P \)-joint, by theorem 5.15, we conclude that all axes of \( R \)-joints in \( L'' \) are parallel. Then all axes of \( R \)-joints in \( L \) are parallel. Then the mobility of \( L \) is \( n - 3 \) which is a contradiction.

\[ \square \]

**Theorem 5.17.** Let \( L \) be an 7-linkage of mobility 3 with at least one \( H \)-joint, such that all joints actually move, then all axes of \( R \)- and \( H \)-joints are parallel.

**Proof.** Let \( r \) be the number of neighbouring blocks of equal axes of the spherical projection \( L_{4} \). The proof proceeds by case distinction on \( r \).

**Case** \( r = 1 \): Then all axes of \( L_{4} \) are equal, hence all axes of \( H \)- and \( R \)-joints of \( L \) are parallel; this is possibility, and the motion of each joint belongs to a same Schoenflies motion subgroup of \( SE(3) \).

**Case** \( r \geq 2 \): Then each of the two blocks of \( R \)-joints in \( L_{4} \) has at least two joints, because a single joint could not move. The linkage \( L_{r} \) is movable and therefore has at least four joints that
actually move. After removing the joints that remain fixed, \(L_r\) still has two blocks of parallel axes. By comparing with theorem 5.16 and theorem 5.15, this is impossible when the mobility of \(L_r\) is at least 3.

\[\square\]

**Theorem 5.18.** Let \(L\) be an \(n\)-linkage, where \(n \geq 7\) of mobility \(n - 4\) with at least one \(H\)-joint, such that all joints actually move, then all axes of \(R\)- and \(H\)-joints are parallel.

**Proof.** We prove this result by induction on the number of active joints.

For \(n = 7\), it is true by theorem 5.17.

Fix \(n \geq 8\). Suppose that all axes of \(R\)- and \(H\)-joints are parallel for any \((n - k)\)-linkage with at least one \(H\)-joint, and \(1 \leq k \leq 2\), and the mobility is \(n - 5\). Let \(L\) be an \(n\)-linkage of mobility \(n - 4\) with at least one \(H\)-joint. Then, freezing an \(R\)-joint (or \(H\)-joint if there is no \(R\)-joint) \(j_s\) of \(L\) such that a helical joint is not frozen simultaneously. We get a \((n - k)\)-linkage \(L'\) with mobility \(n - 5\). By strong induction hypothesis, all axes of \(R\)- and \(H\)-joints of \(L'\) are parallel. Assume that \(k = 1\). The axis of the frozen joint \(j_s\) must be parallel to other axes by spherical projection. Hence, all axes of \(R\)- and \(H\)-joints are parallel in \(L\). Assume that \(k = 2\). There is another joint being frozen say \(j_r\). If the joint \(j_r\) is a \(P\)-joint, then we conclude that the axis of \(j_r\) must be parallel to the axes of \(R\)- and \(H\)-joints in \(L'\). Then all axes of \(R\)- and \(H\)-joints are parallel. If the joint \(j_r\) is not a \(P\)-joint, then the two axes of \(j_s\) and \(j_r\) must be parallel to each other. Assume that there are two groups of parallel axes. By comparing with theorem 5.16 and theorem 5.15, this is impossible when the mobility of \(L_r\) is at least \(n - 4\). Therefore, all axes of \(R\)- and \(H\)-joints are parallel in \(L\).

\[\square\]

6. \(nR\)-Linkages with Mobility \(n - 5\)

In this section we present necessary conditions that the geometry of a paradoxical \(nR\)-linkage with mobility \(n - 5\) must satisfy, where \(n > 6\).

**Lemma 6.1.** Let \(L\) be a \(7R\)-linkage of mobility 2 with no concurrent axes or Bennett conditions. Let \(\beta\) be a bond. Then, up to symmetry, we have \(A_\beta = \{j_1, j_4\}\).

**Proof.** We exclude all the possibilities where \(|A_\beta| \geq 3\) up to symmetry.

1. \(|A_\beta| \geq 6\): This is clearly not possible since in this case \(A_\beta\) is a chain.

2. \(|A_\beta| = 5\): There are two cases up to symmetry: \(A_\beta = \{j_1, j_2, j_3, j_4, j_6\}\) and \(A_\beta = \{j_1, j_2, j_3, j_5, j_6\}\). Suppose that \(A_\beta = \{j_1, j_2, j_3, j_4, j_6\}\). If there are parallel axes we are done. Otherwise, by lemma 2.4, it follows that \((i - h_4)(t_5 - h_5)(i - h_6) = 0\) and we have a Bennett condition between \((h_4, h_5, h_6)\). The argument is the same when \(A_\beta = \{j_1, j_2, j_3, j_5, j_6\}\) and we omit it.

3. \(|A_\beta| = 4\): There are three cases up to symmetry: \(A_\beta = \{j_1, j_2, j_3, j_5\}\), \(A_\beta = \{j_7, j_1, j_3, j_3\}\) and \(A_\beta = \{j_7, j_1, j_3, j_5\}\). In the first two cases, lemma 2.4 implies that there is a Bennett condition in the linkage. Suppose \(A_\beta = \{j_7, j_1, j_3, j_5\}\). If \(h_1\) and \(h_7\) are not parallel, then

\[(i - h_1)(t_2 - h_2)(i - h_3)(t_4 - h_4)(i - h_5) = 0\]

Since \(L\) has mobility 2, this defines a curve \(C \subset \mathbb{P}^1_{x_i} \times \mathbb{P}^1_{x_i}\). By lemma 4.3, we have a Bennett condition between \((h_1, h_2, h_3)\) or \((h_3, h_4, h_5)\)

4. If \(|A_\beta| = 3\), then no two such elements are consecutive: Suppose \(A_\beta = \{j_1, j_2, j_5\}\). Then, there are two far connections since \(h_1\) and \(h_2\) are not parallel. These are given by

\[C_1: (i - h_2)(t_3 - h_3)(t_4 - h_4)(i - h_5) = 0\]

\[C_2: (i + h_1)(t_7 + h_7)(t_6 + h_6)(i + h_5) = 0\]

At least one of \(C_1\) is a curve since \(L\) has mobility 2. Then there is a Bennett condition by lemma 4.3.

Assume \(A_\beta = \{j_1, j_3, j_5\}\): By lemma 2.4, the only possibility is that \((i - h_1)(t_2 - h_2)(i - h_3) \in E\). If that was the case then, \(i - h_5 \in E\) which is impossible.
(5) \(|A_\beta| = 2\): The previous arguments apply to exclude bonds such that \(A_\beta = \{j_1, j_2\}\). If \(A_\beta = \{j_1, j_3\}\) then there would be a Bennett condition between \((h_1, h_2, h_3)\).

\[\square\]

**Theorem 6.2.** Let \(L\) be a mobile 7R-linkage with mobility at most 2 and with no concurrent axes or Bennett conditions. Then, \(L\) has mobility 1.

**Proof.** By lemma 6.1, joints \(j_1\) and \(j_2\) are not simultaneously in \(A_\beta\). Hence, the projection

\[\pi: \mathbb{K}_L \to \mathbb{P}^1 \times \mathbb{P}^1\]

\[(t_1, \ldots, t_7) \mapsto (t_1, t_2)\]

is not surjective and the image is contained in a curve. It follows that \(t_1\) and \(t_2\) are algebraically dependent. In the same way \(t_2\) and \(t_3\) are algebraically dependent as well and so on and, therefore, all \(t_i\) are so. It follows that the mobility of \(L\) is 1.

\[\square\]

We now explain how to generalise this result to any number of joints.

**Theorem 6.3.** Let \(L\) be an nR-linkage with \(n \geq 7\) and mobility at most \(n - 5\) and no concurrent axes or Bennett conditions. Then \(L\) has mobility at most \(n - 6\).

**Proof.** We prove this result by induction on the number of active joints where the base case is theorem 6.2. Fix \(n \geq 8\). Suppose that only one joint gets frozen. Then we have an \((n - 1)R\)-linkage whose mobility is at least \(n - 6\) has concurrent axes or satisfy Bennett conditions. Let \(L\) be an \(nR\)-linkage with mobility at least \(n - 5\) without any concurrent axes. Then, freezing the joint \(j_n\) of \(L\), we get a linkage \(\tilde{L}\) with mobility at least \(n - 6\). By the induction hypothesis, \((h_2, h_1, h_{n-1})\) satisfy a Bennett condition. Notice that we can assume that these are the joints satisfying a Bennett condition since otherwise it would be trivial to prove that \(L\) satisfies a Bennett condition as well. The result now follows from lemma 3.9.

Suppose that two joints get frozen. If \(n = 8\), then we have a 6R with mobility 2. By theorem 5.6, we have Bennett conditions between all three consecutive joints. Hence, we have at least one Bennett condition in \(L\). If \(n \geq 8\), we get an \((n - 2)R\)-linkage of mobility \(n - 5\) which is impossible by theorem 5.15.

Suppose that \(s \geq 3\) joints get frozen. We have an \((n - s)R\)-linkage \(L'\), of mobility \(n - 6\). By theorem 5.15, \(s = 3\) and all the axes of \(L'\) are concurrent. The conclusion is clear for \(n \geq 10\). Otherwise, lemma 3.9 implies the result.

\[\square\]

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**References**

[ALS15] H. Ahmadinezhad, Z. Li, and J. Schicho, An algebraic study of linkages with helical joints.

[Bak02] J Eddie Baker, Displacement–closure equations of the unspecialised double-hooke’s-joint linkage, Mechanism and Machine Theory 37 (2002), no. 10, 1127–1144.

[Bak03] ———, Overconstrained six-bars with parallel adjacent joint-axes, Mechanism and Machine Theory 38 (2003), no. 2, 103–117.

[CPY15] Yan Chen, Rui Peng, and Zhong You, Origami of thick panels, Science 349 (2015), no. 6246, 396–400.

[CY12] Yan Chen and Zhong You, Spatial Overconstrained Linkages - The Lost Jade, Explorations in the History of Machines and Mechanisms (Teun Koetsier and Marco Ceccarelli, eds.), History of Mechanism and Machine Science, vol. 15, Springer Netherlands, 2012, pp. 535–550 (English).

[Del22] E. Delassus, The closed and deformable linkage chains with four bars, Bull. Sci. Math. 46 (1922), 283–304.

[Die95] Peter Dietmaier, Einfach übergeschlossene Mechanismen mit Drehgelenken, Habilitation thesis, Graz University of Technology, 1995.

[GAA90] C. Gosselin and J. Angeles, Singularity analysis of closed-loop kinematic chains, IEEE Transactions on Robotics and Automation 6 (1990), no. 3, 281–290.
