Doubly-gauge-invariant formalism of brane-world cosmological perturbations

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Abstract
We review the doubly gauge invariant formalism of cosmological perturbations in the Randall-Sundrum brane world. This formalism leads to four independent equations describing the evolution of scalar perturbations. Three of these equations are differential equations written in terms of gauge invariant variables on the brane only, and the other is an integro-differential equation describing non-locality due to bulk gravitational waves. At low energy the evolution of the scalar-type cosmological perturbations in the brane-world cosmology differs from that in the standard cosmology only by non-local effects due to bulk gravitational waves.

1 Introduction
The idea that our four-dimensional world may be a timelike surface, or a world-volume of a 3-brane, in a higher dimensional spacetime has been attracting a great deal of physical interests. As shown by Randall and Sundrum [1], the 4-dimensional Newton’s law of gravity can be reproduced on a 4-dimensional timelike hypersurface with positive tension in a 5-dimensional AdS background despite the existence of the infinite fifth dimension.

Moreover, cosmological solutions in the Randall-Sundrum brane world scenario were found [2, 3, 4, 5]. In these solutions, the standard Friedmann equation is restored at low energy, if a parameter in the solutions is small enough. If the parameter is not small enough, it affects cosmological evolution of our universe as dark radiation [6]. Hence, the parameter should be very small in order that the brane-world scenario should be consistent with nucleosynthesis [7]. On the other hand, in ref. [8], it was shown that 5-dimensional geometry of all these cosmological solutions is the Schwarzschild-AdS (S-AdS) spacetime and that the parameter is equivalent to the mass of the black hole. Therefore, the 5-dimensional bulk geometry should be the S-AdS spacetime with a small mass, which is close to the pure AdS spacetime. Moreover, black holes with small mass will evaporate in a short time scale [9]. Thus, it seems a good approximation to consider the pure AdS spacetime as a 5-dimensional bulk geometry for the brane-world cosmology.

For the AdS bulk spacetime, the brane world scenario can reproduce the standard cosmology as evolution of a homogeneous isotropic universe at low energy. Hence, this scenario may be considered as a realistic cosmology and it seems effective to look for observable consequences of this scenario. For this purpose, cosmic microwave background (CMB) anisotropy is a powerful tool. Therefore, we would like to give theoretical predictions of the brane-world scenario on the CMB anisotropy. There are actually many papers on this subject [10, 11, 12, 13, 14, 15, 16, 17]. However, the calculation of the CMB spectrum is not an easy task. The main difficult problems are the following two: (i) how to give the initial condition; (ii) how to evolve perturbations. As for the first problem, there is essentially the same issue even in the standard cosmology. In this paper we shall concentrate on the problem (ii).

2 Doubly gauge-invariant formalism
In this section we review some important points of the gauge-invariant formalism of gravitational perturbations in the bulk and the doubly gauge-invariant formulation of the perturbed junction condition. For details, see refs. [10, 11].
2.1 Master equation in the bulk

Now let us consider gravitational perturbations in $D$-dimensional maximally-symmetric spacetimes since in the simple brane world scenario the background bulk geometry is known to be an AdS spacetime, one of three maximally symmetric spacetimes. Since a general motion of homogeneous, isotropic ($D-2$)-brane breaks the symmetry of the $D$-dimensional maximally symmetric spacetime to that of a ($D-2$)-dimensional constant-curvature space, we consider the following decomposition of the background spacetime.

$$g_{MN}^{(0)} = \gamma_{ab}dx^a dx^b + r^2 \Omega_{ij}dx^i dx^j,$$

where $\Omega_{ij}$ is a metric of a ($D-2$)-dimensional constant-curvature space, $\gamma_{ab}$ is a 2-dimensional metric depending only on the 2-dimensional coordinates $\{x^a\}$, and $r$ also depends only on $\{x^a\}$.

Now let us analyze metric perturbations around the maximally symmetric background. First, let us expand metric perturbations by harmonics on the $(D-2)$-dimensional constant-curvature space. Second, let us construct gauge-invariant variables from coefficients of the harmonics expansion. This procedure is done by analyzing the gauge transformation of the coefficients and taking gauge-invariant linear combinations of them. Thirdly, we can solve the constraint equations to obtain master variables from the gauge-invariant variables. The constraint equations are, of course, a part of $D$-dimensional perturbed Einstein equation. Finally, we can rewrite the remaining components of Einstein equation in terms of the master variables to obtain master equations.

The master equation in general $D$-dimensions was first obtained in ref. [8] and confirmed in ref. [9]. The master equations for generic values of the $(D-2)$-dimensional momentum $k$ (eg. $k \neq 0$ for the $K=0$ case) are of the following form.

$$r^{\alpha+\beta} \nabla^\alpha[r^{-\alpha}\nabla_\alpha(r^{-\beta}\Phi)] - (k^2 + \gamma K)r^{-2}\Phi = 0,$$

where $\Phi$ represents one of master variables $\Phi_{(S)}$, $\Phi_{(V)}$ or $F_{(T)}$, $\nabla_\alpha$ is the 2-dimensional covariant derivative compatible with the metric $\gamma_{ab}$, and $K$ is the curvature constant of the $(D-2)$-dimensional constant-curvature space. Two equivalent sets of constants ($\alpha$, $\beta$, $\gamma$) are listed in Table 1. Master equations for some exceptional values of $k$ can be found in ref. [8].

| $\Phi$ | $\alpha$ | $\beta$ | $\gamma$ | $\alpha$ | $\beta$ | $\gamma$ |
|--------|----------|---------|---------|----------|---------|---------|
| $\Phi_{(S)}$ | $D-4$ | 1 | 0 | $-(D-6)$ | $D-4$ | $2(D-5)$ |
| $\Phi_{(V)}$ | $D-2$ | 0 | $-(D-3)$ | $-(D-4)$ | $D-3$ | $D-3$ |
| $F_{(T)}$ | $D$ | $-(D-3)$ | $-2(D-2)$ | $-(D-2)$ | 2 | 2 |

Table 1: Two sets of values of ($\alpha$, $\beta$, $\gamma$)

In the next sections we consider the $K=0$ case only. In this case the exceptional value of $k$ is $k=0$. For $k=0$, the corresponding perturbations have the plane symmetry and, thus, the generalized Birkoff’s theorem guarantees that the perturbed bulk geometry is a S-AdS spacetime. Hence, the perturbation with $k=0$ is actually perturbation of the mass parameter of the S-AdS spacetime around the pure AdS and can be understood as dark radiation on the brane [8]. Hence, we shall concentrate on perturbations with non-zero $k$. This treatment is, of course, justified by the fact that perturbations with different $k$ are decoupled from each other at the linearized level.

As an example, let us consider the case with $D=5$ and $K=0$. This example is relevant for the 5-dimensional brane world with spatially flat background brane.

In this case we decompose perturbations by harmonics on a 3-dimensional flat space. As shown in Table 2, we have, for example, a gauge-invariant variable which transforms as a 2-dimensional scalar and a 3-dimensional scalar. (In Table 2 $Y$ is a scalar harmonics, $V_{(T)i}$ is a transverse vector harmonics and $T_{(T)ij}$ is a transverse traceless tensor harmonics.) We also have a variable which transforms as a 2-dimensional symmetric tensor and a 3-dimensional scalar. Hence, we have $(1+3) \times 1$ gauge-invariant degrees of freedom.

\[\text{In the latter paper, they extended the master equation of vector and tensor perturbations to more general background without maximal symmetry.}\]
for perturbations which transform as 3-dimensional scalars. Similarly, we have $2 \times (3 - 1)$ and $1 \times (6 - 3 - 1)$ gauge-invariant degrees of freedom for perturbations which transform as 3-dimensional transverse vectors and transverse traceless tensors, respectively. Therefore, the total number of gauge-invariant degrees of freedom is 10. However, the number of degrees of freedom of gravitons in 5-dimensions is 5. So, we have too much gauge-invariant variables compared to the number of gravitons.

On the other hand, after solving constraint equations, all we have are master variables which transforms as 2-dimensional scalars. Hence, as shown in Table 2, the total number of reduced degrees of freedom is 5. Therefore, the master variables concisely describes gravitons in 5-dimensions.

Table 2: Number of degrees of freedom

|                | 3-D scalar | 3-D T vector | 3-D TT tensor | # of variables |
|----------------|------------|--------------|---------------|----------------|
| 2-D scalar     | $F Y \delta_{ij}$ | $F_a V_{(T)i}$ | $F_{(T)j} T_{(T)ij}$ | 10             |
| 2-D vector     | $F_{ab} Y$ | $(1 + 3) \times 1$ | $2 \times (3 - 1)$ | 1              |
| 2-D tensor     | (symmetric) | $(6 - 3 - 1)$ | $(6 - 3 - 1)$ | 5              |
| Master variables | $\Phi_{(S)}$ | $\Phi_{(V)}$ | $F_{(T)}$ | 5              |

2.2 Perturbed junction condition

Having the description of the bulk gravitational waves, what we have to do is to investigate Israel junction condition [18].

First, let us represent the world volume of a $(D - 2)$-brane in a $D$-dimensional spacetime by the parametric equations

$x^M = Z^M(y)$,  \hspace{1cm} (3)

where $x^M$ ($M = 0, \cdots, D - 1$) and $y^\mu$ ($\mu = 0, \cdots, D - 2$) are $D$-dimensional coordinates and $D - 1$ parameters, which play a role of $(D - 1)$-dimensional coordinates on the brane world-volume. Next, let us consider perturbations of the functions $Z^M$ and the $D$-dimensional metric $g_{MN}$. Then, we can calculate perturbations of the induced metric and the extrinsic curvature of the hypersurface as functions of $y^\mu$. Next, we can express the perturbed junction condition in terms of these perturbations and matter perturbations on the brane. Finally, by applying the perturbed junction condition to the homogeneous isotropic background motion of the $(D - 2)$-brane and performing the harmonic expansion as in the previous subsections, we can obtain junction conditions for gauge-invariant variables and master variables.

The final expression can be found in refs. [10]. (See also ref. [8].) Here, I would only like to stress one important aspect of the perturbed junction condition.

First, since we are supposed to be living on the brane, physics in our world must not be affected by the following $D$-dimensional gauge transformation in the higher dimensional bulk.

$x^M \rightarrow x^M + \xi^M(x)$.  \hspace{1cm} (4)

On the other hand, all observable quantities in our world must be invariant under the following $(D - 1)$-dimensional gauge transformation on the brane.

$y^\mu \rightarrow y^\mu + \zeta^\mu(y)$.  \hspace{1cm} (5)

What is important here is that these two kinds of gauge transformations are independent. One might expect that the $(D - 1)$-gauge transformation would be a part of the $D$-gauge transformation. In fact, as explicitly shown in ref. [11], it is not. Therefore, all physical quantities in our world on the brane must be invariant under these two independent gauge transformations. In particular, the junction condition must, and actually can, be written in terms of doubly-gauge-invariant variables only.
3 Integro-differential equation

We already have all basic equations. Namely, we have master equations in the bulk and doubly-gauge-invariant junction condition. The next task we have to do would be to simplify the basic equations to extract physics from them. In this paper we consider scalar perturbations for $D = 5$, $K = 0$.

Before showing the result, let us think about what would be expected. See figure 1. First, let us consider perturbations localized on the brane. In other words, matter perturbations. By definition, they propagate on the brane. However, at the same time, they can generate gravitational waves. Gravitational waves can, of course, propagate in the bulk spacetime, and may collide with the brane at a spacetime point different from the spacetime point at which the gravitational waves were produced ($t_1 \neq t_2$, $x_1 \neq x_2$). When they collide with the brane, they should alter evolution of perturbations localized on the brane. Hence, evolution of perturbations localized on the brane should be non-local. It must, and actually can, be described by some integro-differential equations.

Figure 1: Non-locality due to bulk gravitational waves

For scalar perturbations, the result of the simplification is three differential equations and an integro-differential equation on the brane [17]. Two of the three differential equations can be understood as the perturbed conservation equation, which is a general consequence of the junction condition. These two are, of course, exactly the same as the corresponding equations in the standard cosmology. The other of the three differential equations differs from the corresponding equation in the standard cosmology only by terms of order $O(\rho/\lambda, p/\lambda)$, where $\rho$ and $p$ are energy density and pressure of the background matter on the brane and $\lambda$ is the tension of the brane. Therefore, that equation reduces to the equation in the standard cosmology at low energy ($|\rho/\lambda| \ll 1, |p/\lambda| \ll 1$).

The last of the four equations is an integro-differential equation of the form

$$\tilde{R}(t) + \int dt' K(t, t') S(t') = 0,$$

(6)

if we assume that there is no gravitational waves coming from outside of the Poincare patch of the bulk AdS [17] where $\tilde{R}(t)$ is a linear combination of gauge-invariant metric and matter perturbations and their time-derivatives, $S(t)$ is a linear combination of gauge-invariant matter perturbations, and the kernel $K(t, t')$ is constructed from the retarded Green function of the master equation for bulk gravitational waves. Here, we mention that all physical quantities were constructed from coefficients of the 3-dimensional harmonic expansion and that the kernel $K(t, t')$ depends on $k^2$ quite non-trivially since the master equation depends on $k^2$, where $k$ is the 3-dimensional momentum vector. Thus, this equation indeed describes the non-locality due to bulk gravitational waves: the matter perturbation $S(t')$ at the time $t'$ generates gravitational waves and the gravitational waves, in turn, affects the evolution of the perturbation $\tilde{R}(t)$ on the brane at the different time $t$. The corresponding equation in the standard cosmology is, of course, local and differs from $\tilde{R} = 0$ only by a term of order $O(\rho/\lambda, p/\lambda)$.

Therefore, if we consider a low energy regime, where $|\rho/\lambda| \ll 1$ and $|p/\lambda| \ll 1$, then the four equations for scalar perturbations are almost the same as the corresponding equations in the standard cosmology. The only difference is the non-local term due to bulk gravitational waves. Since a large amount of bulk

\footnote{For the modification by gravitational waves coming from outside of the Poincare patch, or initial gravitational waves, see ref. [17].}
gravitational waves can be produced at an early stage of the brane universe and propagate in the bulk, there may be a possibility that the non-local effect is significant even at a low-energy stage. Further investigation is needed.

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