Conformal SO(2,4) Transformations of the One-Cusp Wilson Loop Surface

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Abstract

By applying the conformal SO(2,4) transformations to the elementary one-cusp Wilson loop surface we construct various two-cusp and four-cusp Wilson loop surface configurations in $AdS_5$ and demonstrate that they solve the string equations of the Nambu-Goto string action. The conformal boosts of the basic four-cusp Wilson loop surface with a square-form projection generate various four-cusp Wilson loop surfaces with projections of the rescaled square, the rhombus and the trapezium, on which surfaces the classical Euclidean Nambu-Goto string actions in the IR dimensional regularization are evaluated.

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1 Introduction

The AdS/CFT correspondence [1] has more and more revealed the deep relations between the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and the string theory in $AdS_5 \times S^5$, where classical string solutions play an important role [2, 3, 4]. The energies of classical strings have been shown to match with the anomalous dimensions of the gauge invariant operators, while an open string ending on a curve at the boundary of $AdS_5$ has been analyzed to study the strong coupling behavior of the Wilson loop in the gauge theory [5, 6, 7].

Alday and Maldacena have used the AdS/CFT correspondence to compute the planar 4-gluon scattering amplitude at strong coupling in the $\mathcal{N} = 4$ SYM theory [8] and found agreement with the result of a conjectured form regarding the all-loop iterative structure and the IR divergence of the perturbative gluon amplitude [9]. The 4-gluon scattering amplitude has been evaluated as the string theory computation of the 4-cusp Wilson loop composed of 4 lightlike segments in the T-dual coordinates, where a certain open string solution in $AdS_5$ space is found to minimize the area of the string surface whose boundary conditions are determined by the massless gluon momenta, and a dimensional regularization is used to regularize the IR divergence. Before the IR regularization the worldsheet surface of this particular solution [8] is related by a certain conformal SO(2,4) transformation to the 1-cusp Wilson loop surface found in [10] (see also [11]).

The non-leading prefactor of the gluon amplitude has been studied [12] and the IR structure of $n$-gluon amplitudes has been fully extracted from a local consideration near each cusp [13], where the 1-cusp Wilson loop solution is constructed even in the presence of the IR regularization. By computing the 1-loop string correction to the 1-cusp Wilson loop solution, the 1-loop coefficient in the cusp anomaly function $f(\lambda)$ of the gauge coupling $\lambda$ has been derived as consistent with the energy of a closed string with large spin $S$ in $AdS_5$ [14]. Moreover, the 2-loop coefficient in $f(\lambda)$ has been presented [15] to agree with the results of [16, 17] for the strong coupling solution of the BES equation [18] in the gauge theory side. Based on the string sigma-model action a whole class of string solution for the 4-gluon amplitude has been constructed [19] under the constraint that the Lagrangian evaluated on the string solution takes a constant value. Applying the dressing method [20] used for the study of the giant magnons and their bound or scattering states [21, 22] to the elementary 1-cusp Wilson loop solution of [10], new classical solutions for Euclidean worldsheets in $AdS_5$ [23] have been constructed, where the surfaces end on complicated, timelike curves at the boundary of $AdS_5$. Several investigations associated with planar gluon amplitudes have been presented [24, 25, 26, 27, 28, 29, 30].

In ref. [8] the planar 4-gluon amplitude at strong coupling has been constructed by deriving the classical string sigma-model action evaluated on the 4-cusp Wilson loop surface whose edge traces out a rhombus on the projected two-dimensional plane at the boundary of $AdS_5$. Based on the Nambu-Goto string action we will apply various conformal SO(2,4) transformations to the elementary 1-cusp Wilson loop solution of [10]. We will show how the obtained string surface configurations satisfy the string equations of motion derived from the Nambu-Goto string action. We will observe that there appear various kinds of Wilson loop solutions which are separated into the 2-cusp Wilson loop solutions and the 4-cusp ones.
2 SO(2) × SO(4) transformations of the 1-cusp Wilson loop solution

We consider the 1-cusp Wilson loop solution of [10], where the open string world surface ends on two semi infinite lightlike lines and is given by

\[ r = \sqrt{2} \sqrt{y_0^2 - y_1^2} \]  

(1)

embedded in an \( AdS_3 \) subspace of \( AdS_5 \) with the metric written in the T-dual coordinates by [8]

\[ ds^2 = -\frac{dy_0^2 + dy_1^2 + dr^2}{r^2}. \]  

(2)

Here we take the static gauge where \((y_0, y_1)\) are regarded as worldsheet directions to write the Nambu-Goto action

\[ S = \frac{R^2}{2\pi} \int dy_0dy_1 \frac{1}{r^2} \sqrt{D}, \quad D = 1 + \left( \frac{\partial r}{\partial y_1} \right)^2 - \left( \frac{\partial r}{\partial y_0} \right)^2, \]  

(3)

from which the equation of motion for \( r \) is given by

\[ \frac{2\sqrt{D}}{r^3} = \partial_0 \left( \frac{\partial_0 r}{r^2 \sqrt{D}} \right) - \partial_1 \left( \frac{\partial_1 r}{r^2 \sqrt{D}} \right). \]  

(4)

The solution (1) is confirmed to satisfy the eq. (4) with \( \sqrt{D} = i \), which implies that the Lagrangian is purely imaginary when it is evaluated on the solution (1). Then the amplitude \( A \sim e^{iS} \) has an exponential suppression factor. The Poincare coordinates in \( AdS_5 \) with the boundary \( r = 0, \)

\[ ds^2 = -\frac{dy_0^2 + dy_1^2 + dy_2^2 + dy_3^2 + dr^2}{r^2} \]  

(5)

are related to the embedding coordinates \( Y_M (M = -1, 0, \cdots, 4) \) on which the conformal SO(2,4) transformation is acting linearly by the following relations

\[ Y^\mu = \frac{y^\mu}{r}, \quad \mu = 0, \cdots, 3, \]

\[ Y_{-1} + Y_4 = \frac{1}{r}, \quad Y_{-1} - Y_4 = \frac{r^2 + y_0 y^0}{r}, \]

\[ -Y_2^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1. \]  

(6)

The elementary 1-cusp solution (1) is expressed in terms of \( Y_M \) as

\[ Y_0^2 - Y_{-1}^2 = Y_1^2 - Y_4^2, \quad Y_2 = Y_3 = 0. \]  

(7)

Let us make an SO(2,4) transformation defined by

\[
\begin{pmatrix}
Y_0 \\
Y_{-1}
\end{pmatrix} = P \begin{pmatrix}
Y'_0 \\
Y'_{-1}
\end{pmatrix}, \quad \begin{pmatrix}
Y_1 \\
Y_2 \\
Y_3 \\
Y_4
\end{pmatrix} = Q \begin{pmatrix}
Y'_1 \\
Y'_2 \\
Y'_3 \\
Y'_4
\end{pmatrix}
\]  

(8)
with
\[ P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \equiv P_1, \quad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \equiv Q_1. \] (9)

This SO(2) × SO(4) rotation of the elementary 1-cusp solution (7) generates a configuration
\[ Y_0'Y_{-1}' = Y_1'Y_4', \] (10)
which is equivalently expressed in terms of the Poincare coordinates as
\[ r = \sqrt{y_0^2 - y_1^2 - \frac{y_0 - y_1}{y_0 + y_1}} = \sqrt{A}, \] (11)
where the prime has been suppressed for convenience. When the string configuration (11) is
substituted into (4), \( \sqrt{D} \) is so compactly given by \( i/y_0 + y_1 \) that the right hand side (RHS) of (4) is separated into two parts for the region
\( y_0 + y_1 > 0 \)
\[ \frac{1}{iA^{3/2}} \left[ \left( y_1 + 2y_0 + \frac{y_1}{(y_0 + y_1)^2} \right) + \left( y_0 + 2y_1 + \frac{y_0}{(y_0 + y_1)^2} \right) \right] \]
\[ + \frac{3}{2iA^{5/2}} \left[ -y_0(y_0 + y_1) + \frac{y_1}{y_0 + y_1} \right] \frac{\partial A}{\partial y_0} + \left( -y_1(y_0 + y_1) + \frac{y_0}{y_0 + y_1} \right) \frac{\partial A}{\partial y_1}, \] (12)
whose second \( 1/A^{5/2} \) part becomes proportional to \( 1/A^{3/2} \) and then the equation of motion (4) is satisfied. For the region \( y_0 + y_1 < 0 \) the string equation is similarly satisfied. The solution (11) shows that the surface ends on the lines specified by \( y_0 = y_1, y_0 = -y_1 \pm 1 \) where two cusps are located at \( (y_0, y_1) = (\pm 1/2, \pm 1/2) \).

The SO(2,4) transformations given by \( P = 1_2, 2 \times 2 \) unit matrix, \( Q = Q_1 \) and \( P = -i\sigma_2 \) that interchanges \( Y_0 \) and \( Y_{-1} \), \( Q = Q_1 \) produce the following configurations
\[ Y_0'^2 - Y_{-1}'^2 = -2Y_1'Y_4', \quad Y_0'^2 - Y_{-1}'^2 = 2Y_1'Y_4' \] (13)
respectively, which turn out to be
\[ r^2 = y_0^2 - (y_1 + 1)^2 \pm 2\sqrt{y_0^2 + (y_1 + 1)^2 - 1}, \] (14)
\[ r^2 = y_0^2 - (y_1 - 1)^2 \pm 2\sqrt{y_0^2 + (y_1 - 1)^2 - 1}. \] (15)

In order to show that the latter surface equation obeys the string equation (4) we parametrize \( r \) as \( r = \sqrt{y_0^2 - (y_1 - 1)^2 + 2\sqrt{B}} \equiv \sqrt{A} \) for the plus sign, and \( B \equiv y_0^2 + (y_1 - 1)^2 - 1 \). In this case \( \sqrt{D} \) is again a pure imaginary \( \sqrt{D} = i/\sqrt{B} \). The RHS of (4) has also two parts
\[ \frac{1}{iA^{3/2}} \sqrt{B} \left[ 2B + (y_1 - 1)^2 + y_0^2 \right] + \frac{3}{iA^{5/2}} \sqrt{B} \left[ (y_1 - 1)^2(1 - \sqrt{B})^2 - y_0^2(1 + \sqrt{B})^2 \right], \] (16)
whose second part again becomes proportional to \( 1/A^{3/2} \) and combines with the first part to yield \( 2i/A^{3/2} \sqrt{B} \) which is just the left hand side (LHS) of (4).
For the plus sign of (15) at the boundary of $AdS_3$, $r = 0$, the surface ends on two lines $y_0 = -y_1 + \sqrt{2} + 1, y_0 = y_1 - (\sqrt{2} + 1)$ in $y_1 \geq 1 + 1/\sqrt{2}$ and two lines $y_0 = -y_1 - (\sqrt{2} - 1), y_0 = y_1 + \sqrt{2} - 1$ in $y_1 \leq 1 - 1/\sqrt{2}$. The region in the outside of the circle defined by $y_0^2 + (y_1 - 1)^2 = 1$ is allowed and there are two cusps located at $(y_0, y_1) = (0, \sqrt{2} + 1), (0, -\sqrt{2} + 1)$, where one semi infinite lightlike line and one finite lightlike line meet at each cusp for $y_0 \geq 0$ and the allowed region specified by $r^2 \geq 0$ is separated into $y_0 \geq 0$ part and $y_0 \leq 0$ part. For the minus sign of (15) the surface ends on two lines $y_0 = -y_1 + \sqrt{2} + 1, y_0 = y_1 + \sqrt{2} - 1$ in $y_0 \geq 1/\sqrt{2}$ and two lines $y_0 = -y_1 - (\sqrt{2} - 1), y_0 = y_1 - (\sqrt{2} + 1)$ in $y_0 \leq -1/\sqrt{2}$. There are two cusps located at $(y_0, y_1) = (\pm \sqrt{2}, 1)$. The string surface ends on the two semi infinite lightlike lines which emerge from the one cusp $(\sqrt{2}, 1)$ for the region $y_0 \geq \sqrt{2}$. The former surface (14) is similarly shown to be a two-cusp Wilson loop solution.

If the other conformal transformations are performed by $P = P_1, Q = 1_4, 4 \times 4$ unit matrix and $P = P_1$,

$$Q = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \equiv Q_2,$$

(17)

that interchanges $Y_1$ and $Y_4$, we have two curves

$$-2Y_0'Y_1' = Y_1'^2 - Y_2'^2, \quad 2Y_0'Y_1' = Y_1'^2 - Y_4'^2,$$

(18)

which are expressed in terms of the Poincare coordinates as

$$r^2 = (y_0 + 1)^2 - y_1^2 \pm 2\sqrt{(y_0 + 1)^2 + y_1^2 - 1}, \quad (19)$$

$$r^2 = (y_0 - 1)^2 - y_1^2 \pm 2\sqrt{(y_0 - 1)^2 + y_1^2 - 1}, \quad (20)$$

respectively. These expressions are compared with (14) and (15) under the exchange of $y_0$ and $y_1$. The two curves (19) and (20) also obey the string eq. (4) in the same way as (15).

For the plus sign of (19) the surface ends on two lines $y_0 = -y_1 + \sqrt{2} - 1, y_0 = y_1 - (\sqrt{2} + 1)$ in $y_1 \geq 1/\sqrt{2}$ and two lines $y_0 = -y_1 - (\sqrt{2} + 1), y_0 = y_1 + \sqrt{2} - 1$ in $y_1 \leq -1/\sqrt{2}$ which meet at two cusps $(-1, \pm \sqrt{2})$ respectively. For the minus sign of (19) the surface ends on two lines $y_0 = -y_1 + \sqrt{2} - 1, y_0 = y_1 + \sqrt{2} - 1$ in $y_0 \geq -1 + 1/\sqrt{2}$ and two lines $y_0 = -y_1 - (\sqrt{2} + 1), y_0 = y_1 - (\sqrt{2} + 1)$ in $y_0 \leq -1 - 1/\sqrt{2}$ which meet at two cusps $(\sqrt{2} - 1, 0)$ and $(-\sqrt{2} - 1, 0)$ respectively. Similarly for the plus sign of (20) two cusps are located at $(1, \pm \sqrt{2})$ and for the minus sign there are two cusps $(\pm \sqrt{2} + 1, 0)$.

The remaining SO(2,4) isometry generated by $P = 1_2$ and $Q = Q_2$ yields a configuration $Y_0'^2 - Y_1'^2 = Y_4'^2 - Y_2'^2$ with a slight sign difference from the starting solution (7). In the Poincare coordinates it is given by

$$r^2 = y_0^2 - y_1^2 \pm \sqrt{2(y_0^2 + y_1^2) - 1}, \quad (21)$$

whose surface ends on four lines $y_0 = y_1 \pm 1, y_0 = -y_1 \pm 1$ which meet at two cusps $(0, \pm 1)$ for the plus sign and at two cusps $(\pm 1, 0)$ for the minus sign. The string surface (21) can be also confirmed to obey the string equation (4) in a similar way to the solution (15).
Now we consider the conformal SO(2,4) transformations that generate a non-zero value of \( y_2 \). First we set \( P = P_1 \) and

\[
Q = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0
\end{pmatrix} \equiv Q_3(\alpha)
\]  

(22)

to have

\[
Y_0'Y_1' = -\frac{\cos 2\alpha}{2}(Y_1^2 - Y_2^2) + \sin 2\alpha Y_1'Y_2', \quad Y_4' = 0,
\]

(23)

which give a surface

\[
\begin{align*}
r &= \sqrt{1 + y_0^2 - y_1^2 - y_2^2} \equiv \sqrt{A}, \\
y_0 &= -\frac{\cos 2\alpha}{2}(y_1^2 - y_2^2) + \sin 2\alpha y_1y_2.
\end{align*}
\]

(24)

At \( \alpha = \pi/4 \) this surface reduces to

\[
r = \sqrt{(1 - y_1^2)(1 - y_2^2)}, \quad y_0 = y_1y_2,
\]

(25)

which show that the Wilson loop at the boundary is composed with four cusps and four lightlike lines and takes a square form for its projection on the \((y_1, y_2)\) plane [8].

We choose a static gauge that \((y_1, y_2)\) are the worldsheet coordinates for the Euclidean open string surface to express the Nambu-Goto action as

\[
S = \frac{R^2}{2\pi} \int dy_1dy_2 \sqrt{\frac{\partial_i \partial_i y_0}{r^2}},
\]

\[
D = 1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_i r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2.
\]

(26)

The equation of motion for \( y_0 \) is given by

\[
\partial_i \left( \frac{\partial_i y_0}{\sqrt{D}} \right) = \partial_1 \left( \frac{\partial_1 r C}{r^2 \sqrt{D}} \right) - \partial_2 \left( \frac{\partial_2 r C}{r^2 \sqrt{D}} \right), \quad C \equiv \partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0
\]

(27)

and the equation of motion for \( r \) takes a form

\[
\frac{2}{r^3} \sqrt{D} = -\partial_i \left( \frac{\partial_i r}{r^2 \sqrt{D}} \right) + \partial_1 \left( \frac{\partial_2 y_0 C}{r^2 \sqrt{D}} \right) - \partial_2 \left( \frac{\partial_1 y_0 C}{r^2 \sqrt{D}} \right).
\]

(28)

The insertion of the expression (24) into \( C \) in (27) and \( D \) in (26) yields \( C = -2 \cos 2\alpha y_1y_2 + \sin 2\alpha (y_1^2 - y_2^2)/\sqrt{A} \) and \( D = 1 \). When the surface (24) is substituted into the string equation (27) its RHS can be so rewritten by \((\partial_2 r/r)\partial_1(C/r\sqrt{D}) - (\partial_1 r/r)\partial_2(C/r\sqrt{D})\) that it is evaluated as \(-2y_0(y_1^2 + y_2^2 - 2)/A^2\) which equals to the LHS of (27). The RHS of (28) is expressed as sum of two parts

\[
\frac{1}{A^{3/2}}[2 - 3(y_1^2 + y_2^2)] + \frac{3}{A^{5/2}}[(y_0(-\cos 2\alpha y_1 + \sin 2\alpha y_2) - y_1)^2 + (y_0(\cos 2\alpha y_2 + \sin 2\alpha y_1) - y_2)^2 - (2 \cos 2\alpha y_1y_2 + \sin 2\alpha (y_1^2 - y_2^2))^2],
\]

(29)
whose second part becomes proportional to $1/A^{3/2}$ and is summed up with the first part into $2/A^{3/2}$, which is the LHS of \((28)\). For $\alpha = \pi/2$ or ($\alpha = 0$) the string surface \((24)\) is given by

\[
y_0 = \frac{y_1^2 - y_2^2}{2}, \quad \left( y_0 = -\frac{y_1^2 - y_2^2}{2} \right),
\]

\[
r = \sqrt{\left( 1 + \frac{y_1 + y_2}{\sqrt{2}} \right) \left( 1 - \frac{y_1 + y_2}{\sqrt{2}} \right) \left( 1 + \frac{y_1 - y_2}{\sqrt{2}} \right) \left( 1 - \frac{y_1 - y_2}{\sqrt{2}} \right)},
\]

from which we see that the Wilson loop at the boundary $r = 0$ has a square-form projection on the $(y_1, y_2)$ plane characterized by the four lines, $y_2 = \pm y_1 + \sqrt{2}, y_2 = \pm y_1 - \sqrt{2}$ and contains four cusps located at $(y_0, y_1, y_2) = (1, \pm \sqrt{2}, 0), (1, 0, \pm \sqrt{2})$. This square in the $(y_1, y_2)$ plane is produced by making a $\pi/4$-rotation of the square of the 4-cusp solution \((25)\).

Let us perform the SO(2,4) transformations specified by $P = 1_2, Q = Q_3(\pi/4)$ and $P = -i\sigma_2, Q = Q_3(\pi/4)$ to derive two string configurations

\[
Y_0^{r_2} - Y_{-1}^{r_2} = -2Y_1^{r_1}Y_2^{r_1}, \quad Y_0^{r_2} - Y_{-1}^{r_2} = 2Y_1^{r_1}Y_2^{r_1}
\]

with $Y_1^r = 0$, which are further represented by

\[
y_0 = \sqrt{1 - 2y_1y_2}, \quad r = \sqrt{2 - (y_1 + y_2)^2}, \quad \left( y_0 = \sqrt{1 + 2y_1y_2} \equiv \sqrt{B}, \quad r = \sqrt{2 - (y_1 - y_2)^2} \equiv \sqrt{A}. \right)
\]

In order to see how the configuration \((33)\), for instance, satisfies the string equations we calculate $C$ and $\sqrt{D}$ to be compact expressions $C = -(y_1^2 - y_2^2)/\sqrt{AB}, \sqrt{D} = 1/\sqrt{B}$. The RHS of \((27)\) is estimated as

\[
- \partial_1 \left( \frac{(y_1 - y_2)(y_1^2 - y_2^2)}{A^2} \right) - \partial_2 \left( \frac{(y_1 - y_2)(y_1^2 - y_2^2)}{A^2} \right) = -\frac{2(y_1 - y_2)^2}{A^2},
\]

which agrees with the LHS of \((27)\). The RHS of \((28)\) can be separated into a $1/A^{3/2}$ part and a $1/A^{5/2}$ part as

\[
\frac{1}{A^{3/2}\sqrt{B}}\left[ 2B - (y_1 - y_2)^2 - 2(y_1^2 + y_2^2) \right] + \frac{1}{A^{5/2}\sqrt{B}}\left[ 6(y_1 - y_2)^2B - 3(y_1^2 - y_2^2)^2 \right],
\]

whose second part becomes $3(y_1 - y_2)^2/3A^{3/2}B$ which makes \((35)\) equal to $2/A^{3/2}\sqrt{B}$, the LHS of \((28)\).

The projection of the surface \((33)\) at the boundary of $AdS_5$ on the $(y_1, y_2)$ plane is composed of two separated parallel lines, $y_2 = y_1 + \sqrt{2}$ on which $y_0 = |\sqrt{2}y_1 + 1|$ and $y_2 = y_1 - \sqrt{2}$ on which $y_0 = |\sqrt{2}y_1 - 1|$. In the region defined by $A > 0$, that is, the region between the two parallel lines, $B$ also takes a positive value. From one cusp located at $(y_0, y_1, y_2) = (0, -1/\sqrt{2}, 1/\sqrt{2})$ two semi infinite lightlike lines emerge on a plane specified by $y_2 = y_1 + \sqrt{2}$, while on a plane $y_2 = y_1 - \sqrt{2}$ two semi infinite lightlike lines intersect transversally at the other cusp located at $(0, 1/\sqrt{2}, -1/\sqrt{2})$. Thus the Wilson loop is composed of two separated parts each of which is represented by two semi infinite lightlike lines with
a cusp. Therefore the solution (33) as well as (32) is regarded as a two-cusp Wilson loop solution.

There remain two conformal transformations defined by

\[ P = \frac{1}{2}, \quad Q = Q_3(\pi/2) \text{ and } P = 1, \quad Q = Q_3(0), \]

which produce the following configurations

\[ Y_0'^2 - Y_{-1}'^2 = Y_1'^2 - Y_2'^2, \]
\[ Y_0'^2 - Y_{-1}'^2 = Y_1'^2 - Y_2'^2 \]

with \( Y'_4 = 0 \), which are respectively written by

\begin{align*}
Y_0 &= \sqrt{1 - y_1^2 + y_2^2}, \\
Y_0 &= \sqrt{1 + y_1^2 - y_2^2} \equiv \sqrt{B}, \\
r &= \sqrt{2 - 2y_1^2} \equiv \sqrt{A}.
\end{align*}

For the surface (39) \( C \) and \( \sqrt{D} \) are evaluated as

\[ C = 2y_1y_2/\sqrt{AB}, \quad \sqrt{D} = 1/\sqrt{B}. \]

In this case the demonstration of (27) is simpler than the above cases due to \( \partial_1 r = 0 \). The eq. (28) is also confirmed to hold in a way similar to (35).

In ref. [14] the solution (37) has been analyzed in the string sigma-model action in the conformal gauge and the leading 1-loop correction to it has been computed together with the 1-loop correction to the starting 1-cusp solution (1), and further the 2-loop correction to the latter solution has been studied [15]. In ref. [14] the solution of (37) was presented by

\begin{align*}
Y_0 &= \frac{1}{\sqrt{2}} \cosh(\alpha \sigma + \beta \tau), \\
Y_{-1} &= \frac{1}{\sqrt{2}} \cosh(\alpha \tau - \beta \sigma), \\
Y_1 &= \frac{1}{\sqrt{2}} \sinh(\alpha \sigma + \beta \tau), \\
Y_2 &= \frac{1}{\sqrt{2}} \sinh(\alpha \tau - \beta \sigma), \\
Y_3 &= Y_4 = 0,
\end{align*}

where \( \alpha^2 + \beta^2 = 2 \), the Euclidean worldsheet coordinates \( (\tau, \sigma) \) take values in the infinite interval. The parametrization (40) is equivalently transformed in terms of the Poincare coordinates into

\begin{align*}
r &= \frac{\sqrt{2}}{\cosh(\alpha \tau - \beta \sigma)}, \\
y_0 &= \frac{\cosh(\alpha \sigma + \beta \tau)}{\cosh(\alpha \tau - \beta \sigma)}, \\
y_1 &= \frac{\sinh(\alpha \sigma + \beta \tau)}{\cosh(\alpha \tau - \beta \sigma)}, \\
y_2 &= \tanh(\alpha \tau - \beta \sigma),
\end{align*}

which indeed satisfies the eq. (39).

The projection of the surface (39) at the boundary of \( AdS_5 \), \( r = 0 \) on the \((y_1, y_2)\) plane is composed of two separated parallel lines, \( y_2 = 1 \) and \( y_2 = -1 \) on which \( y_0 \) is specified by the same equation \( y_0 = |y_1| \). From a different viewpoint the projection of the surface (39) on the \((y_0, y_1)\) plane at a fixed value of \( y_2 \) in \(-1 < y_2 < 1\) is described by a hyperbolic curve \( y_0 = \sqrt{y_1^2 + (1 - y_2^2)} \), while that projection at the boundary value \( y_2 = 1 \) or \( y_2 = -1 \) becomes two semi infinite lightlike lines intersecting transversely at a cusp located at \((y_0, y_1, y_2) = (0, 0, 1)\) or \((0, 0, -1)\). Thus the solution (39) as well as (38) has two cusps in the same way as (33) and (32).
3 Conformal boosts of the 4-cusp Wilson loop solution

We turn to the conformal boost transformations of the 4-cusp solution \( Y_0 Y_{-1} = Y_1 Y_2, Y_3 = Y_4 = 0 \) and see whether the transformed configurations are solutions of the string equations for the Nambu-Goto action. The boost in the (-1,4) plane given by

\[
Y_4 = \gamma (Y_4' - v Y_{-1}'), \quad Y_{-1} = \gamma (Y_{-1}' - v Y_4')
\]

with \( \gamma = 1/\sqrt{1-v^2} \) is performed to yield \( Y_4' - v Y_{-1}' = 0 \) and \( \gamma Y_0'(Y_{-1}' - v Y_4') = Y_4'Y_{-1}' \), which are represented in terms of the Poincare coordinates as

\[
y_0' = \gamma (1+v) y_1' y_2', \quad r' = \sqrt{\frac{1-v}{1+v} + y_0'^2 - y_1'^2 - y_2'^2} \equiv \sqrt{A}.
\]

Alternatively the boost (42) can be described by

\[
r' = \sqrt{\frac{1-v}{1+v}} r, \quad y_1' = \sqrt{\frac{1-v}{1+v}} y_1, \quad y_2' = \sqrt{\frac{1-v}{1+v}} y_2
\]

owing to the relation \( 1/r' = Y_{-1}' + Y_4' = \gamma (1+v)/r \). Below the prime will be omitted. For the eq. (27) \( C \) is given by \( C = -\gamma (1+v)(y_1^2 - y_2^2)/\sqrt{A} \), while \( D \) takes a simple value \( D = 1 \). The RHS of (27) is computed by

\[
-\frac{2\gamma (1+v)}{A^3} \left[ A \left( \gamma^2(1+v)^2y_1y_2(y_1^2 + y_2^2) - 2y_1y_2 \right) \\
+ (y_1^2 - y_2^2) \left( -\gamma^2(1+v)^2y_1^2 - 1 \right) y_2 \partial_1 A + (\gamma^2(1+v)^2y_2^2 - 1) y_1 \partial_2 A \right],
\]

whose \( 1/A^3 \) part vanishes owing to the symmetric form of \( A \) for the exchange of \( y_1 \) and \( y_2 \). The \( 1/A^2 \) part becomes equal to the LHS of (27). The RHS of the other string equation (28) is calculated by

\[
\frac{1}{A^{3/2}} \left[ 2 - 3\gamma^2(1+v)^2(y_1^2 + y_2^2) + \frac{3(y_1^2 + y_2^2)}{A^{5/2}} \gamma^4(1+v)^4y_1^2y_2^2 + 1 - \gamma^2(1+v)^2(y_1^2 + y_2^2) \right],
\]

whose second \( 1/A^{5/2} \) part is so simplified into \( 3(y_1^2 + y_2^2)/(1+v)/(A^{3/2}(1-v)) \) that (46) becomes \( 2/A^{3/2} \) which is just the LHS of (28).

Since \( r' \) of (43) is represented with the prime by

\[
r' = \sqrt{\frac{1+v}{1-v}} \sqrt{\left( \frac{1-v}{1+v} - y_1^2 \right) \left( \frac{1-v}{1+v} - y_2^2 \right)},
\]

the projection of the string surface on the \((y_1', y_2')\) plane at the \( AdS_5 \) boundary is the square with side length \( 2\sqrt{(1-v)/(1+v)} \equiv 2a \), which is compared to the square with side length 2 for the starting basic solution (25). Following the IR dimensional regularization of ref. [8] we define the following regularized Nambu-Goto action

\[
S = \frac{\sqrt{AdS}}{2\pi} \int dy_1dy_2 \sqrt{D} r^{2+\epsilon}.
\]
with $d = 4 - 2\epsilon, c_d = 2^{4\epsilon} \pi^{3\epsilon} \Gamma(2 + \epsilon), \lambda_d = \lambda \mu^{2\epsilon}/(4\pi e^{-\gamma})\epsilon, \gamma = -\Gamma'(1)$, where $\lambda_d$ is described by the IR cutoff scale $\mu$ and the dimensionless four dimensional coupling $\lambda$ to match the field theory side. Substituting the solution (47) into the action (48) and making a variable transformation (44) to carry out an integral over the inside of the square we have

$$
-iS = B_\epsilon \int_{-1}^{1} dy_1 dy_2 \frac{1}{[(1 - y_1^2)(1 - y_2^2)]^{1+\epsilon}} = B_\epsilon \frac{\pi \Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2} - \epsilon\right)^2}
$$

with $B_\epsilon = \sqrt{\lambda_d c_d}/2\pi \epsilon$, where $i$ is due to the Euclidean worldsheet and a double pole appears when $\epsilon < 0 \to 0$. From the structure of the action (26) if $r$ and $y_\mu$ are solutions of the string equations, then the rescaling configuration specified by $ar$ and $ay_\mu$ with an arbitrary constant $a$ also satisfies the string equations. The solution (44) exhibits a rescaling solution.

Let us perform the following boost in the (0,4) plane for the 4-cusp solution (25)

$$
Y_4 = \gamma(Y'_4 - v Y'_0), \quad Y_0 = \gamma(Y'_0 - v Y'_4).
$$

We obtain a string surface described by

$$
r' = \sqrt{1 - \frac{2b}{\gamma} y_0 + y_0^2 - y_1^2 - y_2^2} \equiv \sqrt{A}, \quad b = \gamma v
$$

and $v y_0^2 - y_0 + \gamma y'_1 y'_2 = 0$ for which we choose

$$
y_0 = \frac{\gamma}{2b}(1 - \sqrt{1 - 4by'_1 y'_2}) \equiv \frac{\gamma}{2b}(1 - \sqrt{B}),
$$

which reduces to the starting solution (25) in the limit $v \to 0$. We consider the $b < 1$ case. Below the prime will be suppressed. For the eq. (27) $C$ is evaluated as $C = -\gamma(y_1^2 - y_2^2)/\sqrt{AB}$ from the involved expressions (51) and (52), while $D$ in (26) has many complicated terms but can be cast into a simple form $\sqrt{D} = 1/\sqrt{B}$ through $\gamma^2 = 1 + b^2$. The substitution of these expressions into the RHS of (27) leads to

$$
\frac{1}{\sqrt{A}} \left[ 2b \left(1 - \frac{\gamma^2}{2b^2}\right) \frac{y_0^2 + y_2^2}{\sqrt{B}} + 4y_1 y_2 + \frac{\gamma^2}{b}(y_1^2 + y_2^2) \right]
+ \frac{2\gamma}{\sqrt{A}}(y_1 - y_2) \left[ ((1 - \frac{\gamma^2}{2b^2}) \frac{b y_1}{\sqrt{B}} + y_1 + \frac{\gamma^2}{2b} y_2) \partial_2 A - \left(1 - \frac{\gamma^2}{2b^2}\right) \frac{b y_1}{\sqrt{B}} + y_2 + \frac{\gamma^2}{2b} y_1 \right] \partial_1 A
$$

whose second $1/A^3$ part vanishes owing to the symmetric expression of $A$ for the exchange of $y_1$ and $y_2$ and the remaining first part becomes coincident with the LHS of (27). In order to prove the eq. (28) we devote ourselves to the $1/A^{5/2}$ part of its RHS

$$
\frac{3}{A^{5/2} \sqrt{B}} \left[ \left(1 - \frac{\gamma^2}{2b^2}\right)^2 b^2 (y_1^2 + y_2^2) + 2b \sqrt{B} \left(1 - \frac{\gamma^2}{2b^2}\right) \left(2y_1 y_2 + \frac{\gamma^2}{2b}(y_1^2 + y_2^2)\right) \right]
+(1 - 4b y_1 y_2) \left(\frac{2\gamma^2}{b} y_1 y_2 + \left(1 + \frac{\gamma^4}{4b^2}\right) (y_1^2 + y_2^2) - \gamma^2 (y_1^2 - y_2^2)^2\right]
$$

which turns out to be $6b(2y_1 y_2 + \gamma^2 (y_1^2 + y_2^2)/2b)/A^{3/2} \sqrt{B}$, which further cancels against the remaining $1/A^{3/2}$ part to leave $2/A^{3/2} \sqrt{B}$, that is, the LHS of (28).
The insertion of (52) into (51) with \( r' = 0 \) leads to the projection of the Wilson loop on the \((y'_1, y'_2)\) plane, which is expressed as

\[
(y'_{1}^{2} + y'_{2}^{2}) - \frac{\gamma^{2}}{b^{2}}(y'_{1}^{2} + y'_{2}^{2})(1 - 2by'_{1}y'_{2}) + \frac{\gamma^{4}}{b^{4}}y'_{1}y'_{2}^{2} - \frac{4}{b}y'_{1}y'_{2} + \frac{1}{b^{2}} = 0. \tag{55}
\]

This equation can be factorized into

\[
\left[ \left( y'_{2} + \frac{y'_{1}}{b} \right)^{2} - \frac{1}{b^{2}} \right] [(y'_{2} + by')^{2} - 1] = 0, \tag{56}
\]

which gives four lines \( y'_{2} + y'_{1}/b = \pm 1/b, y'_{2} + y'_{1}/b = \pm 1 \) that form a rhombus on the \((y'_1, y'_2)\) plane.

The boost (50) gives \( 1/r' = Y'_{-1} + Y' = Y_{-1} + \gamma(Y_{4} + vY_{4}) \) and \( y'_{0}/r' = \gamma(Y_{0} + vY_{4}) \), which become \( 1/r' = (1 + by_{0})/r \) and \( y'_{0} = \gamma y_{0}r'/r \) through the starting solution (25). Thus the boost is represented in terms of the Poincare coordinates as

\[
y'_{0} = \frac{\gamma y_{1}y_{2}}{1 + by_{1}y_{2}}, \quad y'_{1} = \frac{y_{1}}{1 + by_{1}y_{2}}, \quad y'_{2} = \frac{y_{2}}{1 + by_{1}y_{2}}. \tag{57}
\]

The eq. (52) is changed into the first equation in (57) when the second and third equations in (57) are substituted into (52), while the eq. (51) becomes

\[
r' = \frac{\sqrt{(1 - y'_{1}^{2})(1 - y'_{2}^{2})}}{1 + by_{1}y_{2}}, \tag{58}
\]

which vanishes at \( y_{1} = \pm 1 \) and \( y_{2} = \pm 1 \). The four cusps of the Wilson loop are determined from (57) with \( y_{1} = \pm 1, y_{2} = \pm 1 \) to be located in the coordinates \((y'_{0}, y'_{1}, y'_{2})\) as

\[
\left( \frac{-\sqrt{1+b^{2}}}{1-b'}, \frac{1}{1-b'} \frac{1}{1-b} \right), \quad \left( \frac{\sqrt{1+b^{2}}}{1+b'}, \frac{1}{1+b'} \frac{1}{1+b} \right),
\]

\[
\left( \frac{\sqrt{1+b^{2}}}{1+b}, \frac{1}{1+b'} \frac{1}{1+b} \right), \quad \left( \frac{-\sqrt{1+b^{2}}}{1-b}, \frac{1}{1-b'} \frac{1}{1-b} \right). \tag{59}
\]

We again see that the projection of the Wilson loop on the \((y'_1, y'_2)\) plane is a rhombus. Alternatively the cusp positions (59) are determined from the intersections of the four lines defined by (56). The four lightlike segments between the adjacent cusps characterize the four massless gluon momenta so that the parameter \( b \) is related with the ratio of the Mandelstam variables \( s, t \) as \( s/t = (1 + b^{2})/(1 - b^{2}) \). The classical Nambu-Goto action (48) evaluated on this 4-cusp Wilson loop solution is represented through (57) and (58) as

\[
-iS = \frac{\sqrt{\lambda_{d}c_{d}}}{2\pi} \int_{-1}^{1} dy_{1}dy_{2} \frac{1 - by_{1}y_{2}}{(1 + by_{1}y_{2})^3} \frac{1}{r^{2+\epsilon} \sqrt{B(y'_0)}} \tag{60}
\]

\[
= \frac{\sqrt{\lambda_{d}c_{d}}}{2\pi} \int_{-1}^{1} dy_{1}dy_{2} \frac{(1 + by_{1}y_{2})^\epsilon}{[(1 - y_{1}^2)(1 - y_{2}^2)]^{1+\frac{\epsilon}{2}}.}
\]
By expanding the integrand as a power series on \( b \) we evaluate the integral over \( y_1 \) and \( y_2 \) as

\[
\frac{\sqrt{\lambda_d c_d}}{2\pi} \pi \Gamma \left( -\frac{\epsilon}{2} \right)^2 F \left( \frac{1}{2}, \frac{1}{2}, 1 - \epsilon; b^2 \right).
\]

(61)

Here in view of (51) and (52) we present a string configuration using an arbitrary constant \( a \)

\[
r' = \sqrt{a^2 - 2ab y_0 + y_0^2 - y_1^2 - y_2^2} \equiv \sqrt{A}, \quad b = \gamma v,
\]

(62)

\[
y_0' = \frac{\gamma}{2b} (a - \sqrt{a^2 - 4by_1 y_2}) \equiv \frac{\gamma}{2b} (a - \sqrt{B}),
\]

(63)

whose \( v \to 0 \) limit reduces to (53) with \( a \) replaced by \( \sqrt{(1 - v)/(1 + v)} \). This string surface is confirmed to satisfy the string equations with \( C = -\gamma(y_1^2 - y_2^2)/\sqrt{AB} \) and \( \sqrt{D} = a/\sqrt{B} \). This solution is just a rescaling solution for (57) and (58) as expressed by

\[
y_0' = \frac{a y_1 y_2}{1 + by_1 y_2}, \quad y_1' = \frac{ay_1}{1 + by_1 y_2}, \quad y_2' = \frac{ay_2}{1 + by_1 y_2},
\]

(64)

\[
r' = \frac{a \sqrt{(1 - y_1^2)(1 - y_2^2)}}{1 + by_1 y_2}.
\]

(65)

The insertion of (64) into (62) leads to (63) and the eq. (63) becomes the first equation in (64) when the second and third equations are substituted. The classical action of this rescaling solution is similarly evaluated as

\[
-iS = \frac{\sqrt{\lambda_d c_d}}{2\pi} \int_{-1}^{1} dy_1 dy_2 \frac{a^2 (1 - by_1 y_2)}{(1 + by_1 y_2)^3} r^{2+\epsilon} \sqrt{B(y_1^2)}.
\]

(66)

which gives (61) with the \( \sqrt{\lambda_d c_d}/2\pi \) factor replaced by \( B_\epsilon = \sqrt{\lambda_d c_d}/2\pi a^\epsilon \), whose \( \epsilon \) expansion leads to the exponential form of the planar 4-gluon scattering amplitude at strong coupling of [8]. In ref. [8] for the 4-cusp Wilson loop solution the string sigma-model action was used, while based on the Nambu-Goto action we have demonstrated that the 4-cusp Wilson loop configuration indeed solves the string equation and evaluated the classical action on this 4-cusp solution.

There is another boost in the (-1,1) plane specified by

\[
Y_1 = \gamma(Y_1' - v Y_{1'}), \quad Y_{-1} = \gamma(Y_{-1}' - v Y_{-1}'),
\]

(67)

which produces a string surface

\[
y_0' = \frac{y_2'(y_1' - v)}{1 - vy_1'} \equiv \frac{y_2'(y_1' - v)}{B}, \quad r' = \sqrt{1 + y_0'^2 - y_1'^2 - y_2'^2} \equiv \sqrt{A}.
\]

(68)

This configuration is not symmetric under the interchange of \( y_1' \) and \( y_2' \). Although \( C \) is obtained by an involved form \( C = [-y_1(y_1 - v)/B + (1 - v^2)y_2^2/B^2]/\sqrt{A} \) here with unprimed
expressions, $\sqrt{D}$ is calculated to be a simple form $\sqrt{D} = \sqrt{1-\nu^2}/B$. The string equation (27) is proved as follows. The RHS of (27) is evaluated as

$$\frac{\nu}{A^2} \left[ \sqrt{1-\nu^2} \left( 2y_1 x - (1-y_1^2)\partial_1 x \right) - \frac{2B^2}{\sqrt{1-\nu^2}} \left( \frac{(1-v^2)(y_1-v)}{B^3} x + \left( \frac{(1-v^2)y_2^2(y_1-v)}{B^3} - y_1 \right) \frac{1-v^2}{B^3} \right) \right] + \frac{2\nu}{A^2} \left[ \sqrt{1-\nu^2} y_2 (1-y_1^2)\partial_1 A + \frac{B^2}{\sqrt{1-\nu^2}} \left( \frac{(1-v^2)y_2^2(y_1-v)}{B^4} - y_1 \right) \partial_2 A \right],$$

(69)

where $X = -y_1(y_1-v)/B^2 + (1-v^2)y_2^2/B^3$ and the $1/A^3$ part vanishes through $A = (1-y_1^2)(1-y_1^2(1-v^2)/B^2)$ in spite of the asymmetric expression. The remaining $1/A^2$ part agrees with the LHS of (27). For the string equation (28), the $1/A^{5/2}$ terms of the RHS are gathered together into a sum of terms with odd powers of $1/B$

$$\frac{3\sqrt{1-v^2}(1-y_1^2)}{A^{5/2}B} \left[ y_1^2 + \frac{(1-v^2)y_2^2(1-y_1^2)}{B^2} - \frac{(1-v^2)y_2^4}{B^4} \right],$$

(70)

which turns out to be a $1/A^{3/2}$ term as $3\sqrt{1-v^2}(y_1^2 + (1-v^2)y_2^2/B^2)/BA^{3/2}$. It combines with the remaining $1/A^{3/2}$ terms to be equated with $2\sqrt{1-v^2}/BA^{3/2}$, that is, the LHS of (28), where the $1/B^3$ terms are canceled out and the $1/B^2$ terms can be changed into a $1/B$ term.

The string solution (68) can be expressed as

$$r' = \frac{1}{1-vy_1^2} \sqrt{(1-y_1^2)^2 - (1-v^2)y_2^2},$$

(71)

so that the projection of the Wilson loop on the $(y_1', y_2')$ plane is a trapezium formed by the four lines $y_1' = \pm 1$ and $y_2' = \pm (vy_1' - 1)/\sqrt{1-v^2}$. In $(y_0, y_1', y_2')$ the four cusps are located at

$$\left( -\sqrt{\frac{1+v}{1-v}}, -1, \frac{1+v}{1-v} \right), \quad \left( \sqrt{\frac{1-v}{1+v}}, 1, \frac{1-v}{1+v} \right),$$

$$\left( \frac{1+v}{1-v}, -1, -\frac{1+v}{1-v} \right), \quad \left( -\frac{1-v}{1+v}, 1, -\frac{1-v}{1+v} \right),$$

(72)

which imply that the Wilson loop consists of the four lightlike segments. The boost (67) is alternatively expressed as

$$y_0' = \frac{y_1y_2}{\gamma(1+vy_1)}, \quad y_1' = \frac{v+y_1}{1+vy_1}, \quad y_2' = \frac{y_2}{\gamma(1+vy_1)},$$

(73)

which lead to (72) again and are substituted into the second eq. of (68) to be

$$r' = \frac{\sqrt{(1-y_1^2)(1-y_2^2)}}{\gamma(1+vy_1)}.$$

(74)

Then the action (48) can be evaluated on this classical solution as

$$-iS = \frac{\sqrt{\lambda_d c_d}}{2\pi} \int_{-1}^{1} dy_1 dy_2 \frac{1-v^2}{\gamma(1+vy_1)^3(1-v^2)\nu^{2+\epsilon}} B(y_1') \left( \sqrt{1+b^2 + by_1} \right)^\epsilon \frac{1}{\left[ (1-y_1^2)(1-y_2^2) \right]^{1+\epsilon}},$$

(75)
The integral is calculated by expanding the integrand as a power series on $b$ as

$$
\sqrt{\pi} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( -\frac{\epsilon}{2} \right)}{\Gamma \left( \frac{-\epsilon}{2} \right)^2} (1 + b^2)^{\epsilon/2} \sum_{n=0}^{\infty} \left( \frac{b^2}{1+b^2} \right)^n \frac{\Gamma \left( n - \frac{\epsilon}{2} \right)}{n!}.
$$

(76)

whose summation factor is evaluated as $\Gamma(-\epsilon/2)(1 - b^2/(1 + b^2))^{\epsilon/2}$ through $(1 - z)^\alpha = F(-\alpha, \beta, \beta, z)$ so that we obtain $\pi \Gamma(-\epsilon/2)^2 / \Gamma((1 - \epsilon)/2)^2$. This result is compared with (49) with $a = 1$ or (61) with $b = 0$, the Mandelstam variables $s = t$, which is consistent with the observation that the locations (72) of the four cusps indicate $s = t$.

Here we discuss the remaining boost in the $(0,1)$ plane defined by

$$
Y_1 = \gamma(Y_1' - vY_0'), \quad Y_0 = \gamma(Y_0' - vY_1'),
$$

(77)

which yields a string configuration

$$
y_0' = \frac{y_1'(y_2' + v)}{1 + vy_2'}, \quad r' = \sqrt{1 + y_0'^2 - y_1'^2 - y_2'^2}.
$$

(78)

For (78) the projection of the Wilson loop on the $(y_1', y_2')$ plane shows a trapezium in the same way as that for (68). Although this configuration is transformed into (68) under the interchanges $y_1' \leftrightarrow y_2'$ and $v \leftrightarrow -v$, we can show that it satisfies the string equations in the same way as the solution (68).

4 Conclusions

Starting with the elementary 1-cusp Wilson loop solution of [10], we have constructed various kinds of string configurations by performing the conformal SO(2,4) transformations in the embedding coordinates of $AdS_5$ and then rewriting the results back in the Poincare coordinates. Analyzing the obtained string surfaces to see where they end on at the $AdS_5$ boundary, we have read off the shapes of the Wilson loops. In order to see whether the conformal transformed string configurations are extrema of the worldsheet area we have demonstrated that they indeed solve the involved string equations of motion for the Nambu-Goto string action. In these demonstrations the string Lagrangian $\sqrt{D/r^2}$ does not take a constant value in our worldsheet coordinates but it is important that $\sqrt{D}$ takes a simple manageable expression.

We have made two types of SO(2) $\times$ SO(4) transformations that are characterized by two kinds of SO(4) rotations such that one does not change $Y_2$ and the other interchanges $Y_2$ and $Y_4$. We have observed that the former type of transformations generate a variety of 2-cusp Wilson loop solutions, while the latter type of them produce not only the 4-cusp Wilson loop solutions but also the 2-cusp Wilson loop solutions. The projection of the latter 2-cusp Wilson loop surfaces on the $(y_1, y_2)$ plane is two separated parallel lines, which is compared with the square-form projection of the 4-cusp Wilson loop surface. Applying the boost SO(2,4) transformations to the basic 4-cusp solution with the square-form projection we have constructed three kinds of 4-cusp solutions whose projections are the rescaled square, the rhombus and the trapezium.
By combining the conformal boost in the (0,4) plane and the rescaling we have derived a 4-cusp Wilson loop surface whose projection is a rescaled rhombus. Based on the Nambu-Goto action in the IR dimensional regularization we have obtained the classical Euclidean action evaluated on this 4-cusp solution and reproduced the same exponential expression of the 4-gluon amplitude as derived in [8] from the string sigma-model action.

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