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Height fluctuations for the stationary KPZ equation

P.L. Ferrari with A. Borodin, I. Corwin and B. Vető

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http://wt.iam.uni-bonn.de/~ferrari
Surface described by a height function $h(x, t)$, $x \in \mathbb{R}^d$ the space, $t \in \mathbb{R}$ the time

Models with local growth + smoothing mechanics

⇒ macroscopic growth velocity $v$ is a function of the slope only:

$$\frac{\partial h}{\partial t} = v(\nabla h)$$

Example: Isotropic growth

$$v(\nabla h) = v(0) \sqrt{1 + (\nabla h)^2}$$
Nematic liquid crystals: stable (black) vs metastable (gray) cluster

Takeuchi, Sano’10: PRL 104, 230601 (2010)
Nematic liquid crystals: stable (black) vs metastable (gray) cluster

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The Kardar-Parisi-Zhang (KPZ) equation is one of the models in the KPZ universality class, class of irreversible stochastic random growth models. 

The KPZ equation writes (by a choice of parameters) in one-dimension is

$$\partial_T h = \frac{1}{2} \partial_X^2 h + \frac{1}{2} (\partial_X h)^2 + \dot{W}$$

where $\dot{W}$ is the space-time white noise

Stationary initial conditions are any two-sided Brownian motion with drift fixed $b \in \mathbb{R}$. 

Kardar,Parisi,Zhang’86
The KPZ and SHE equations

- **KPZ equation**
  \[
  \partial_T h = \frac{1}{2} \partial_X^2 h + \frac{1}{2} (\partial_X h)^2 + \dot{W}
  \]

  ⇒ Problem in defining the object \((\partial_X h)^2\).
  For a way of doing it, see Hairer’s work [Hairer’11](#).

- Setting \(h = \ln Z\) (and ignoring the Itô-correction term) one gets the (well-defined) **Stochastic Heat Equation** (SHE):
  \[
  \partial_T Z = \frac{1}{2} \partial_T^2 Z + Z \dot{W}
  \]

- Given the solution of the SHE with initial condition \(Z(0, X) := e^{h(0, X)}\), one calls
  \[
  h(T, X) = \ln(Z(T, X))
  \]
  the **Cole-Hopf solution** of the KPZ equation.
The KPZ and SHE equations

- KPZ equation

\[ \partial_T h = \frac{1}{2} \partial_X^2 h + \frac{1}{2} \left[ (\partial_X h)^2 - \infty \right] + \dot{W} \]

⇒ Problem in defining the object \((\partial_X h)^2\).
   For a way of doing it, see Hairer’s work.

- Setting \(h = \ln Z\) (and ignoring the \(\text{Itô}-\text{correction term}\)) one gets the (well-defined) Stochastic Heat Equation (SHE):

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\[ h(T, X) = \ln (Z(T, X)) \]

the Cole-Hopf solution of the KPZ equation.
The Feynmann-Kac formula gives

\[ \mathcal{Z}(T, X) = \mathbb{E}_{T, X} \left( \mathcal{Z}_0(\pi(0)) : \exp \left\{ - \int_0^T ds \dot{W}(\pi(s), s) \right\} \right) \]

where the expectation is with respect Brownian paths, \( \pi \), backwards in time with \( \pi(T) = X \).

Interpretation: \( \mathcal{Z} \) is a partition function of the random directed polymer \( \pi \) with energy given by the white noise ”seen” by it. This is called Continuous Directed Random Polymer model (CDRP), the universal scaling limit of directed polymers.
Goal: obtain a reasonably explicit formula (solved problem) for

\[ \mathbb{P}(h(T, X) \leq s) \]

or the law of the process \( X \mapsto h(T, X) \) (open problem).

One possible approach: start with any directed polymer model which converges under an appropriate limit to the CDRP.
We consider now the following semi-discrete directed polymer model at positive temperature

- **Path measure** $P_0$: Continuous time one-sided simple random walk from $(0, 1)$ to $(t, N)$.

- **Random media**: $B_1, B_2, \ldots, B_N$ be independent standard Brownian motions. The energy is given by

$$-E(\pi) = B_1(t_1) + (B_2(t_2) - B_2(t_1)) + \ldots + (B_N(t) - B_N(t_{N-1}))$$

- **Boltzmann weight**: $P(\pi) = Z(t, N)^{-1} e^{-E(\pi)} P_0(\pi)$

$$Z(t, N) := \int_{0 < t_1 < t_2 < \ldots < t_{N-1} < t} e^{B_1(t_1) + (B_2(t_2) - B_2(t_1)) + \ldots + (B_N(t) - B_N(t_{N-1}))} \, dt_1 \ldots dt_{N-1}.$$
Recall the partition function

\[ Z(t, N) = \int_{0 < t_1 < t_2 < \ldots < t_{N-1} < t} e^{B_1(t_1) + (B_2(t_2) - B_2(t_1)) + \ldots + (B_N(t) - B_N(t_{N-1}))} dt_1 \ldots dt_{N-1}. \]

- **Law of large numbers**: for any \( \kappa > 0 \),

\[ f(\kappa) := \lim_{N \to \infty} \frac{1}{N} \ln Z(\kappa N, N) = \inf_{t > 0} (\kappa t - (\ln \Gamma)'(t)). \]

\[ 0'\text{Connell-Yor}'01;Moriarty,0'Connell'07 \]

- **Fluctuations**: in agreement with KPZ universality conjecture, for some known \( c(\kappa) > 0 \),

\[ \lim_{N \to \infty} \mathbb{P} \left( \frac{\ln Z(\kappa N, N) - Nf(\kappa)}{c(\kappa)N^{1/3}} \leq r \right) = F_{\text{GUE}}(r) \]

where \( F_{\text{GUE}} \) is the GUE Tracy-Widom distribution function

\[ \text{Borodin,Corwin,Ferrari}'12 \]
Recall that

\[ Z(t, N) := \int_{0 < t_1 < t_2 < \ldots < t_{N-1} < t} e^{B_1(t_1) + (B_2(t_2) - B_2(t_1)) + \ldots + (B_N(t) - B_N(t_{N-1}))} dt_1 \ldots dt_{N-1}. \]

The quantity \( u(t, N) := e^{-t} Z(t, N) \) satisfies

\[ \partial_t u(t, N) = (u(t, N - 1) - u(t, N)) + u(t, N) \dot{B}_N(t) \]

with initial condition \( u(0, N) = \delta_{1,N} \).

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  \]
  with initial condition \( u(0, N) = \delta_{1,N} \).
- Its continuous analogue is the CDRP, where \( P_0 \) is the law of a Brownian Bridge from \((0, 0)\) to \((T, X)\), and the random noise is white noise \( \dot{W} \). Its partition function \( Z(T, X) \) satisfy
  \[
  \partial_T Z = \frac{1}{2} \partial_X^2 Z + Z \dot{W}
  \]
  with initial conditions \( Z(0, X) = \delta_0(X) \).
Q: How to get a stationary situation?
A: Use Burke-type results

0’Connell, Yor’01; Seppäläinen, Valkó’10

(1) Replace $B_1(t)$ with $B_1(t) + at$

(2) Add boundary weights at $(−1, n)$ given by $\omega_{−1,n} \sim − \ln \Gamma(\alpha)$ for $n \geq 2$ and $\omega_{−1,1} = 1$.

⇒ This gives the partition function $Z(t, N)$

(3) Stationarity is recovered with $a = \alpha$
To recover the CDRP from the semi-discrete model

**Step 1:** We find an expression, with $\alpha > a$, for

$$\mathbb{E}(e^{-uZ(t,N)})$$

**Step 2:** Take the scaling

$$t = \sqrt{TN} + X, \quad a = \sqrt{N/T} + 1/2 + b, \quad \alpha = \sqrt{N/T} + 1/2 + \beta$$

and by Quastel, Remenik, Moreno-Flores

$$\frac{Z(\sqrt{TN} + X, N)}{C(N, X, T)} \Rightarrow Z_{b,\beta}(T, X)$$

with $C$ an explicit function, $Z_{b,\beta}(0, X) = \exp(B(X))$ with the Brownian motion $B$ having a drift $b$ on $\mathbb{R}_+$ and $\beta$ on $\mathbb{R}_-$. 

**Step 3:** Take the $\beta \to b$ limit through analytic continuation
Theorem (For simplicity, case of drift $b = 0$, position $X = 0$.)

Let $h(T, X)$ be the stationary solution to the KPZ equation and let $K_0$ denote the modified Bessel function. Then, for $T > 0$, $\sigma = (2/T)^{1/3}$ and $S \in \mathbb{C}$ with positive real part,

$$
\mathbb{E} \left[ 2\sigma K_0 \left( 2 \sqrt{S \exp \left\{ \frac{T}{24} + h(T, 0) \right\}} \right) \right] = f(S, \sigma),
$$

where the function $f$ is explicit.
Define on $\mathbb{R}_+$ the function

$$Q(x) = \frac{-1}{2\pi i} \int_{-\frac{1}{4\sigma} + i\mathbb{R}} \, dw \, \frac{\sigma \pi S^{-\sigma w}}{\sin(\pi \sigma w)} e^{-w^3/3 + wx} \frac{\Gamma(\sigma w)}{\Gamma(-\sigma w)},$$

and the kernel

$$\bar{K}(x, y) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{4\sigma} + i\mathbb{R}} \, dw \int_{\frac{1}{4\sigma} + i\mathbb{R}} \, dz \, \frac{\sigma \pi S^{\sigma(z-w)}}{\sin(\sigma \pi (z-w))} \frac{e^{z^3/3 - zy}}{e^{w^3/3 - wx}} \frac{\Gamma(-\sigma z)}{\Gamma(\sigma z)} \frac{\Gamma(\sigma w)}{\Gamma(-\sigma w)}.$$

Let $\gamma_E = 0.577\ldots$ be the Euler constant, define

$$f(S, \sigma) = - \det(\mathbb{I} - \bar{K}) \left[ \sigma(2\gamma_E + \ln S) \right]
+ \left\langle (\mathbb{I} - \bar{K})^{-1}(\bar{K}1 + Q), 1 \right\rangle
+ \left\langle (\mathbb{I} - \bar{K})^{-1}(1 + Q), Q \right\rangle.$$ 

where the determinants and scalar products are all meant in $L^2(\mathbb{R}_+)$. 
Corollary

For any $r \in \mathbb{R}$, we have

\[
\mathbb{P} \left( h(T, 0) \leq -\frac{T}{24} + r \left( \frac{T}{2} \right)^{1/3} \right) = \frac{1}{\sigma^2} \frac{1}{2\pi i} \int_{-\delta+i\mathbb{R}} \frac{d\xi}{\Gamma(-\xi)\Gamma(-\xi+1)} \int_{\mathbb{R}} dx \, e^{x\xi/\sigma} f \left( e^{-\frac{x+r}{\sigma}}, \sigma \right)
\]

for any $\delta > 0$ and where $\sigma = \left( \frac{2}{T} \right)^{1/3}$.

There is another representation obtained in Sasamoto, Imamura ’12. It is obtained by (non-rigorous) replica approach, but equality after the replica step of the computation has been verified.
Corollary (For simplicity, here just $b = 0$ and $X = 0$)

For any $r \in \mathbb{R}$,

$$
\lim_{T \to \infty} \mathbb{P} \left( h(T, 0) \leq -\frac{T}{24} + r(T/2)^{1/3} \right) = F_0(r),
$$

where $F_0$ is the Baik-Rains distribution given by

$$
F_0(r) = \frac{\partial}{\partial r} \left( g(r) F_{\text{GUE}}(r) \right),
$$

with $F_{\text{GUE}}$ is the GUE Tracy-Widom distribution and $g(r)$ is an explicitly known function.

Results for one-point distribution in other KPZ models

Baik, Rains’00; Sasamoto, Imamura’04; Prähofer, Spohn’04; Ferrari, Spohn’05

Results for multi-point distributions

Baik, Ferrari, Péché’10; Ferrari, Spohn, Weiss’15
To get the large time limit we do not employ the inversion formula.

Let $\sigma = (2/T)^{1/3}$ and the rescaled height function $\tilde{h} = \sigma (h(T, 0) + T/24)$. Let $S = e^{-r/\sigma}$. Then

$$
\mathbb{E} \left( 2\sigma K_0 \left( e^{(\tilde{h} - r)/(2\sigma)} \right) \right) = \int_{\mathbb{R}} dx \mathbb{P}(\tilde{h} \leq x) e^{(x-r)/(2\sigma)} K_1 \left( e^{(x-r)/(2\sigma)} \right)
$$

$$
\approx \int_{-\infty}^{r} \mathbb{P}(\tilde{h} \leq x) \to F_{\text{GUE}}(r) g(r).
$$
For $s \in \mathbb{R}$, define

$$\mathcal{R} = s + \int_{s}^{\infty} dx \int_{0}^{\infty} dy \text{Ai}(x + y),$$

$$\Psi(y) = 1 - \int_{0}^{\infty} dx \text{Ai}(x + y),$$

$$\Phi(x) = \int_{0}^{\infty} d\lambda \int_{s}^{\infty} dy \text{Ai}(x + \lambda)\text{Ai}(y + \lambda) - \int_{0}^{\infty} dy \text{Ai}(y + x).$$

Let $P_{s}(x) = 1 \{x > s\}$ and the Airy kernel

$$K_{\text{Ai}}(x, y) = \int_{0}^{\infty} d\lambda \text{Ai}(x + \lambda)\text{Ai}(y + \lambda).$$

Define the function

$$g(s) = \mathcal{R} - \left\langle (1 - P_{s}K_{\text{Ai}}P_{s})^{-1}P_{s}\Phi, P_{s}\Psi \right\rangle.$$
• How to get the main result: consider the semidirected polymer model.

**Step 1:** Start with $\alpha > a$ and add an extra (independent) weight $\omega(-1, 1) \sim - \ln \Gamma(\alpha - a)$. Thus

$$\tilde{Z}(t, N) \equiv Z(t, N)e^{\omega(-1,1)}$$

In this setting we get first a formula of the form (see later)

$$\mathbb{E} \left[ e^{-u\tilde{Z}(t,N)} \right] = \det(\mathbb{1} + K_u)$$
Step 2: An elementary explicit computation (recall \( \tilde{Z}(t, N) \equiv Z(t, N)e^{\omega(-1,1)} \)) gives then

Corollary

For \( \alpha > a \),

\[
\mathbb{E} \left[ 2 \left( u Z(t, N) \right)^{\frac{\alpha-a}{2}} K_{-\left(\alpha-a\right)} \left( 2 \sqrt{u Z(t, N)} \right) \right] = \Gamma(\alpha - a) \mathbb{E} \left[ e^{-u \tilde{Z}(t, N)} \right],
\]

where \( K_\nu \) is the modified Bessel function of order \( \nu \).
Step 3: In

\[
\mathbb{E} \left[ 2\left( u Z(t, N) \right)^{\frac{\alpha-a}{2}} K_{-(\alpha-a)} \left( 2\sqrt{u Z(t, N)} \right) \right] = \Gamma(\alpha - a) \mathbb{E} \left[ e^{-u \tilde{Z}(t, N)} \right],
\]

taking \( N \to \infty \) under the scaling

\[ t = \sqrt{TN} + X, \quad a = \sqrt{N/T} + 1/2 + b, \quad \alpha = \sqrt{N/T} + 1/2 + \beta \]

leads to ...
Theorem

Let us denote by $Z_{b,\beta}(T,0)$ the solution to the SHE/KPZ equation with initial data $Z_0(X) = \exp(B(X))$, where $B(X)$ is a two-sided Brownian motion with drift $\beta$ to the left of 0 and drift $b$ to the right of 0, with $\beta > b$. Then, for $S > 0$,

$$
\mathbb{E} \left[ 2 \left( Se^{\frac{T}{24}} Z_{b,\beta}(T,0) \right)^{\frac{\beta-b}{2}} K_{-(\beta-b)} \left( 2\sqrt{Se^{\frac{T}{24}}} Z_{b,\beta}(T,0) \right) \right] = \Gamma(\beta-b) \det(1 - K_{b,\beta}) L^2(\mathbb{R}_+)$$

where $K_\nu(z)$ is the modified Bessel function of order $\nu$ and

$$
K_{b,\beta}(x,y) = \frac{1}{(2\pi i)^2} \int_{C_w} dw \int_{C_z} dz \frac{e^{z^3/3 - zy} \Gamma(\beta - \sigma z) \Gamma(\sigma w - b)}{\sin(\sigma \pi (z - w)) \Gamma(\sigma z - b) \Gamma(\beta - \sigma w)}
$$

where $\sigma = (2/T)^{1/3}$.
Step 4: Recover the stationary initial condition by taking the $\beta \downarrow b$ limit:

- r.h.s.: analytic continuation (to be singled out: a factor $1/ (\beta - b)$ from the Fredholm determinant)
- l.h.s.: analytic continuation and a-priori bound on the left-tail of $\ln Z_{b,\beta}$

Corwin, Hammond’13
Q: How to get the starting formula, namely

$$\mathbb{E} \left[ e^{-u\tilde{Z}(t,N)} \right] = \det(1 + Ku)$$

for the semi-directed polymer?
The configurations are elements on

Let \( q \in (0, 1) \) be fixed. Particle \( \lambda_k^{(m)} \) jumps to the right with rate

\[
\text{rate} (\lambda_k^{(m)}) = \frac{(1-q\lambda_{k-1}^{(m-1)} - \lambda_k^{(m)-1})(1-q\lambda_k^{(m)} - \lambda_{k+1}^{(m)})}{(1-q\lambda_k^{(m)} - \lambda_{k+1}^{(m)-1})}
\]
Set $q = e^{-\varepsilon}$ and look at time $t = \tau/\varepsilon^2$.

As $\varepsilon \to 0$,

In particular, $T_1^N = \ln Z(\tau, N)$ in distribution.

Borodin, Corwin’11
Goal: get a generating function for the \( q \)-Whittaker with specialization \( \rho(\alpha, 0, \gamma) \) with \( q \in (0, 1) \).

**Step 1:** Start with specialization \( \rho(0, \beta, 0) \) with \( \beta \) having a finite number of non-zero entries

\[
\mathbb{E} \left( \frac{1}{(\zeta q^{-\lambda_1^N}; q)_\infty} \right) = \mathbb{E} \left( \sum_{k \geq 0} \frac{\zeta^k q^{-k \lambda_1^N}}{(q; q)_k} \right) = \sum_{k \geq 0} \frac{\zeta^k \mathbb{E}(q^{-k \lambda_1^N})}{(q; q)_k} = \det(1 + K_\zeta)
\]

Remark: For the \( \rho(\alpha, 0, \gamma) \) case, our model, \( \mathbb{E}(q^{-k \lambda_1^N}) = \infty \) for \( k \geq k_0(q) \).

The key step which is non-rigorous in the replica-type approach is the exchange of \( \mathbb{E} \) and \( \sum_{k \geq 0} \).
Step 2: See that for general specializations, both lhs/rhs can be expanded in formal power series

$$\det(1 + K_\zeta) = \sum_{\lambda} R_\lambda p_\lambda(\rho(\alpha, \beta, \gamma)),$$

and by the full power of Macdonald processes

Borodin, Corwin ’11

$$\mathbb{E}\left(\frac{1}{(\zeta e^{-\lambda_1^N}; q)_\infty}\right) = \sum_{\lambda} L_\lambda p_\lambda(\rho(\alpha, \beta, \gamma))$$

with $R_\lambda$ and $L_\lambda$ independent of the $\rho(\alpha, \beta, \gamma)$.

Step 3: By Step 1, we have $R_\lambda = L_\lambda$. Using this and Step 2 for $\rho(\alpha, 0, \gamma)$ one gets the result.