FRACTIONAL EQUATIONS
WITH INDEFINITE NONLINEARITIES

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Abstract. In this paper, we consider a fractional equation with indefinite nonlinearities
\((-\Delta)^{\alpha/2} u = a(x_1) f(u)\)
for 0 < \alpha < 2, where a and f are nondecreasing functions. We prove that there is no positive bounded solution. In particular, in the case \(a(x_1) = x_1\) and \(f(u) = u^p\), this remarkably improves the result in [15] by extending the range of \(\alpha\) from \([1, 2)\) to \((0, 2)\), due to the introduction of new ideas, which may be applied to solve many other similar problems.

1. Introduction. The fractional Laplacian in \(\mathbb{R}^n\) is a nonlocal pseudo-differential operator, assuming the form
\[
(-\Delta)^{\alpha/2} u(x) = C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy
\]
where \(\alpha\) is any real number between 0 and 2 and P.V. stands for the Cauchy principal value. In order the integral to make sense, we require \(u \in L^{\alpha} \cap C^{1,1}_{\text{loc}}\), where
\[
L_{\alpha} = \{ u \in L^{1}_{\text{loc}}(\mathbb{R}^n) | \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \}.
\]
We will assume that \(u\) satisfies this condition through out the paper.

2010 Mathematics Subject Classification. Primary: 35J60; Secondary: 35B06.
Key words and phrases. The fractional Laplacian, indefinite nonlinearities, method of moving planes, monotonicity, non-existence of positive solutions.
The first author is partially supported by the Simons Foundation Collaboration Grant for Mathematicians 245486.
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The third author is partially supported by NSF DMS 1500468.
The non-locality of the fractional Laplacian makes it difficult to investigate. To circumvent this difficulty, Caffarelli and Silvestre [4] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions. For a function \( u : \mathbb{R}^n \to \mathbb{R} \), consider its extension \( U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) that satisfies
\[
\begin{align*}
\text{div}(y^{1-\alpha} \nabla U) &= 0, \quad (x, y) \in \mathbb{R}^n \times [0, \infty), \\
U(x, 0) &= u(x).
\end{align*}
\]
Then it can be shown that
\[
(-\Delta)^{\alpha/2} u(x) = -C_{n,\alpha} \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}, \quad x \in \mathbb{R}^n.
\]
This extension method has been applied successfully to study equations involving the fractional Laplacian, and a series of fruitful results have been obtained (see [3] [15] and the references therein).

In [3], among many interesting results, when the authors considered the properties of the positive solutions for
\[
(-\Delta)^{\alpha/2} u(x) = u^p(x), \quad x \in \mathbb{R}^n,
\]
they first used the above extension method to reduce the nonlocal problem into a local one for \( U(x, y) \) in higher dimensional half space \( \mathbb{R}^n \times [0, \infty) \), then applied the method of moving planes to show the symmetry of \( U(x, y) \) in \( x \), and hence derived the non-existence in the subcritical case:

**Proposition 1.** (Brandle-Colorado-Pablo-Sanchez) Let \( 1 \leq \alpha < 2 \). Then the problem
\[
\begin{align*}
\text{div}(y^{1-\alpha} \nabla U) &= 0, \quad (x, y) \in \mathbb{R}^n \times [0, \infty), \\
- \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial U}{\partial y} &= U^p(x, 0), \quad x \in \mathbb{R}^n
\end{align*}
\]
has no positive bounded solution provided \( p < (n + \alpha)/(n - \alpha) \).

Then they took trace to obtain

**Corollary 1.** Assume that \( 1 \leq \alpha < 2 \) and \( 1 < p < \frac{n+\alpha}{n-\alpha} \). Then equation (2) possesses no bounded positive solution.

A similar extension method was adapted in [15] to obtain the nonexistence of positive solutions for an indefinite fractional problem:

**Proposition 2.** (Chen-Zhu) Let \( 1 \leq \alpha < 2 \) and \( 1 < p < \infty \). Then the equation
\[
(-\Delta)^{\alpha/2} u = x_1 u^p, \quad x \in \mathbb{R}^n
\]
possesses no positive bounded solution.

The common restriction \( \alpha \geq 1 \) is due to the approach that they need to carry the method of moving planes on the solutions \( U \) of the extended problem
\[
\text{div}(y^{1-\alpha} \nabla U) = 0, \quad (x, y) \in \mathbb{R}^n \times [0, \infty).
\]
Due to the presence of the factor \( y^{1-\alpha} \), they have to assume that \( \alpha \geq 1 \), and it seems that this condition cannot be weakened if one wants to carry out the method of moving planes on extended equation (5). However, this obstacle does not appear in equation (4), hence one may expect to be able to remove the condition if working on it directly.

In [10], when studying equation (2), the authors applied the method of moving planes directly to it without making an extension and thus obtain
Proposition 3. Assume that $0 < \alpha < 2$ and $u$ is a nonnegative solution of equation (2). Then

(i) In the critical case $p = \frac{n+\alpha}{n-\alpha}$, $u$ is radially symmetric and monotone decreasing about some point.

(ii) In the subcritical case $1 < p < \frac{n+\alpha}{n-\alpha}$, $u \equiv 0$.

This greatly improves the result in Corollary 1 by extending the range of $\alpha$ from $(1,2)$ to $(0,2)$.

Since its introduction, this direct method of moving planes has been applied to solve various problems involving fractional operators including fully nonlinear ones such as the fractional p-Laplacians (see [16] [9] [12] [6] [28] [23]). In this paper, we will modify the direct method of moving planes introduced in [10], so that it can be applied to equation (4) here without going through extension. There are several difficulties.

Usually, to carry on the method of moving planes, one needs to assume that the solution $u$ vanishes at $\infty$. For equation (2), without assuming $\lim_{|x| \to \infty} u(x) = 0$, in the critical and subcritical cases, one can exploit the Kelvin transform $v(x) = \frac{1}{|x|^{n-\alpha}} u\left(\frac{x}{|x|^2}\right)$ to derive

$$(-\Delta)^{\alpha/2} v(x) = \frac{1}{|x|^\gamma} v^p(x), \quad \text{with } \lim_{|x| \to \infty} v(x) = 0. \quad (6)$$

Here $\gamma \geq 0$ and the coefficient $\frac{1}{|x|^{\gamma}}$ possesses the needed monotonicity, so that one can carry on the method of moving planes on the transformed equation (6).

Now for equation (4), due to the presence of $x_1$, the coefficient of the transformed equation does not have the required monotonicity, and this renders the Kelvin transform useless.

To assume $\lim_{|x| \to \infty} u(x) = 0$ is impractical, because when in the process of applying this Liouville Theorem (nonexistence of solutions) in the blowing up and re-scaling arguments to establish a priori estimate, the solution of the limiting equation is known to be only bounded. Hence it is reasonable to assume that $u$ is bounded when we consider equation (4). Without the condition $\lim_{|x| \to \infty} u(x) = 0$, in order to use the method of moving planes, we introduce an auxiliary function. As we will explain in the next section, the situation in the fractional order equation is quite different and more difficult than the one in the integer order equation, and to overcome these difficulties, we introduce some new ideas when we move the planes along $x_1$ direction all the way up to $\infty$. Specifically, we consider a more general equation than (4):

$$(-\Delta)^{\alpha/2} u = a(x_1) f(u). \quad (7)$$

Using the direct method of moving planes, we obtain

Theorem 1. Let $0 < \alpha < 2$. Suppose $u$ is a positive bounded solution of equation (7). Assume that

$(H1)$: $a(t) \in C^s(\mathbb{R})$ for some $s \in (0,1)$ and $a(t)$ is nondecreasing in $\mathbb{R}$.

$(H2)$: $a(t) > 0$ somewhere for $t > 0$ and $\lim_{t \to -\infty} \sup a(t)|t|^\alpha \leq 0$.

$(H3)$: $f$ is locally Lipschitz and nondecreasing in $(0,\infty)$. Moreover, $f(0) = 0$ and $f > 0$ in $(0,\infty)$.

Then $u$ is monotone increasing in $x_1$ direction.
Remark 1. (i) Upon the completion of the work, we noticed that a similar theorem was obtained in [1]. However, our method is different than theirs and our conditions are weaker.

In order the moving of plane has a stating point, in [1], the authors simple assumed that
\[ a(t) \leq 0, \text{ for } t \leq 0 \]
under which they were able to apply a maximum principle to show that
\[ w_\lambda(x) \geq 0 \quad \forall x \in \Sigma_\lambda \]
with \( \lambda = 0 \). (Please see the definition of \( w_\lambda \) and \( \Sigma_\lambda \) in the next section.)

While in this paper, we remarkably weaken the condition on \( a(t) \). Instead of (8), we assume that
\[ \limsup_{t \to -\infty} |a(t)|^\alpha \leq 0. \]
If \( a(t) \) satisfies (8), then obviously it satisfies our condition. We also allow \( a(t) \) to be positive even for all \( t \leq 0 \) provided it decays fast enough, say faster than \( \frac{1}{|t|^\alpha} \) as \( t \to -\infty \). Under this weaker condition, a simple maximum principle does not apply. To start moving the plane, we estimate the decay rate of \( (-\Delta)^{\alpha/2} w_\lambda(x) \) as \( x_1 \to -\infty \), which involves more delicate calculations.

(ii): The local version of (7) when \( \alpha = 2 \) was considered in [2], [27], [17], to just mention a few. Note that our assumptions on \( a(t) \) are even more general than those in these literatures.

By comparing the solution \( u \) with the first eigenfunction at a unit ball far away from the origin, we derive a contradiction and hence prove

**Theorem 2.** *Besides the conditions in Theorem 1, further assume that\[ a(t) \to \infty \quad \text{as} \quad t \to \infty.\]Then equation (7) possesses no positive bounded solution.*

In Section 2, we used a *direct method of moving planes* to derive the monotonicity of solutions along \( x_1 \) direction and prove Theorem 1. In Section 3, we establish the nonexistence of positive solutions and obtain Theorem 2. The last section is the appendix which provides the proof for an elementary lemma.

We would like to mention that another effective method in dealing with equations involving fractional Laplacian is the *integral equation approach*. One first show that the fractional equation is equivalent to an integral equation, then use the *method of moving planes in integral forms* introduced in [13] to investigate qualitative properties of the solutions. We mention a few related work on classification of solutions [7] [17] [18] [32] [33], integral equations [14] [22], non-Lipchitz cases [21], over-determined problems [20] [29], asymptotic [24] [25], symmetry [30] [31] [32] [36] and references there in. We would also like to mention a recent fundamental result in [26] in which a Bocher type theorem and some maximum principles were established for fractional super harmonic functions.

2. **Monotonicity of solutions.** Consider
\[ (-\Delta)^{\alpha/2} u(x) = a(x_1) f(u(x)), \quad x \in \mathbb{R}^n. \]

We will use the *direct method of moving planes* to show that every positive solution must be strictly monotone increasing along \( x_1 \) direction and thus prove Theorem 1.
Let
\[ T_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda, \text{ for some } \lambda \in \mathbb{R} \} \]
be the moving planes,
\[ \Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 < \lambda \} \]
be the region to the left of the plane, and
\[ x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n) \]
be the reflection of \( x \) about the plane \( T_\lambda \).

Assume that \( u \) is a solution of pseudo differential equation (9). To compare the values of \( u(x) \) with \( u_\lambda(x) \equiv u(x^\lambda) \), we denote
\[ w_\lambda(x) = u_\lambda(x) - u(x). \]
From the assumption (H1), it follows that
\[ \left( -\triangle \right)^{\alpha/2} w_\lambda(x) = \left( a(x^\lambda) - a(x_1) \right) f(u_\lambda) + a(x_1) \left( f(u_\lambda) - f(u) \right) \geq a(x_1) M(\lambda, x) w_\lambda(x), \tag{10} \]
where
\[ M(\lambda, x) = \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)}. \]
Under our assumption that \( u \) is bounded and \( f(\cdot) \) is locally Lipschitz continuous and nondecreasing,
\[ M(\lambda, x) \text{ is bounded and nonnegative.} \tag{11} \]
We want to show that
\[ w_\lambda \geq 0 \quad \forall \ x \in \Sigma_\lambda \text{ and for all } \lambda \in (-\infty, \infty). \]
To this end, usually a contradiction argument is used. Suppose \( w_\lambda \) has a negative minimum in \( \Sigma_\lambda \), then one would derive a contradiction with inequality (10). However, here we only assume that \( u \) is bounded, which cannot prevent the minimum of \( w_\lambda(x) \) from leaking to \( \infty \). To overcome this difficulty, for integer order equations (see [27])
\[ -\triangle u = x_1 u^p(x), \quad x \in \mathbb{R}^n, \]
an auxiliary function was introduced:
\[ \bar{w}_\lambda(x) = \frac{w_\lambda(x)}{g(x)} \quad \text{with } g(x) \to \infty, \text{ as } |x| \to \infty. \]
Now
\[ \lim_{|x| \to \infty} \bar{w}_\lambda(x) = 0 \]
and hence \( \bar{w}_\lambda \) can attain its negative minimum in the interior of \( \Sigma_\lambda \). The corresponding left hand side of (10) becomes
\[ -\triangle w_\lambda = -\triangle \bar{w}_\lambda \cdot g - 2\nabla \bar{w}_\lambda \cdot \nabla g - \bar{w}_\lambda \cdot \triangle g. \tag{12} \]
At a minimum of \( \bar{w}_\lambda \), the middle term on the right hand side vanishes since \( \nabla \bar{w}_\lambda = 0 \). This makes the analysis easier. However, the fractional counter part of (12) is
\[ -2C \int_{\mathbb{R}^n} \frac{(\bar{w}_\lambda(x) - \bar{w}_\lambda(y))(g(x) - g(y))}{|x - y|^{n+\alpha}} dy + \bar{w}_\lambda \cdot (-\triangle)^{\alpha/2} g. \]
At a minimum of \( \bar{w}_\lambda \), the middle term on the right hand side (the integral) neither vanish nor has a definite sign. This is the main difficulty encountered by the fractional nonlocal operator, and to circumvent which, we introduce a different
auxiliary function and estimate \((-\triangle)^{\alpha/2}w_\lambda\) in an entirely new approach. We believe that this new idea may be applied to study other similar problems involving fractional operators.

**Step 1.** As usual, in the first step of the method of moving planes, we show that, for \(\lambda\) close to negative infinity, we have

\[
w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda.
\]

To this end, one usually uses a contradiction argument. Suppose there is a negative minimum \(x^o\) of \(w_\lambda\), then one would try to show that

\[
(-\triangle)^{\alpha/2}w_\lambda(x^o) < a(x^o)M(\lambda, x^o)w_\lambda(x^o)
\]

which is a contradiction to inequality (10). In order \(w_\lambda\) to possess such a negative minimum, one obvious condition to impose on it is \(\lim_{|x| \to \infty} w_\lambda(x) = 0\), which is almost equivalent to \(\lim_{|x| \to \infty} u(x) = 0\). However, this condition is too strong in practice. The non-existence of solutions of (9) is used as an important ingredient in obtaining a priori estimate on the solutions by applying a blowing-up and re-scaling argument, and the solutions of the limiting equations are bounded, but may not goes to zero at infinity. Hence it is more reasonable to assume that the solutions are bounded, hence \(w_\lambda\) is bounded.

Different from the logarithmic auxiliary function chosen in [27] and [15], we choose the auxiliary function as

\[
g(x) = |x - Re_1|^\sigma, \quad \bar{w}_\lambda(x) = \frac{w_\lambda(x)}{g(x)},
\]

where

\[
R = \lambda + 1, \quad e_1 = (1, 0, \cdot, 0),
\]

and \(\sigma\) is a small positive number to be chosen later.

Obviously, \(\bar{w}_\lambda\) and \(w_\lambda\) have the same sign and

\[
\lim_{|x| \to \infty} \bar{w}_\lambda(x) = 0.
\]

Now suppose (13) is violated, then there exists a negative minimum \(x^o\) of \(\bar{w}_\lambda\), at which we compute:

\[
(-\triangle)^{\alpha/2}w_\lambda(x^o) = CPV \int_{\Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+\alpha}} dy + C \int_{\mathbb{R}^n \setminus \Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+\alpha}} dy
\]

\[
= CPV \int_{\Sigma_\lambda} \left\{ \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+\alpha}} + \frac{w_\lambda(x^o) + w_\lambda(y)}{|x^o - y^\lambda|^{n+\alpha}} \right\} dy
\]

\[
= CPV \int_{\Sigma_\lambda} [\bar{w}_\lambda(x^o) - \bar{w}_\lambda(y)]g(y) \left( \frac{1}{|x^o - y|^{n+\alpha}} - \frac{1}{|x^o - y^\lambda|^{n+\alpha}} \right) dy
\]

\[
+ C\bar{w}_\lambda(x^o) \left\{ \int_{\Sigma_\lambda} \frac{2g(x^o)}{|x^o - y^\lambda|^{n+\alpha}} dy
\]

\[
+ PV \int_{\Sigma_\lambda} [g(x^o) - g(y)] \left( \frac{1}{|x^o - y|^{n+\alpha}} - \frac{1}{|x^o - y^\lambda|^{n+\alpha}} \right) dy \right\}
\]
Lemma 2.1. Assume that 
where 
To calculate \( g(x^o) \) with a positive constant \( C \)
where 
Here we have used the anti-symmetry property 

In 

We split 

Evaluate the integral in 

By elementary calculus,

From this Lemma, we have

The proof is elementary, and will be given in the Appendix.

By elementary calculus,

with a positive constant \( c_1 \).

We split \( I_2 \) into three parts:

To calculate \((-\Delta)^{\alpha/2}g(x^o)\), we simply employ the following

**Lemma 2.1.** Assume that \( \gamma < \alpha \).

\[ (-\Delta)^{\alpha/2}(|x-a|^\gamma) = C_\gamma |x-a|^{\gamma-\alpha}, \]  

where \( C_\gamma \) is a constant continuously depending on \( \gamma \),

\[ C_\gamma : \begin{cases} 
> 0, & \text{if } \alpha-n<\gamma<0; \\
= 0, & \text{if } \gamma=0 \text{ or } \alpha-n; \\
< 0, & \text{if } 0<\gamma<\alpha. 
\end{cases} \]

The proof is elementary, and will be given in the Appendix. From this Lemma, we have

\[ (-\Delta)^{\alpha/2}g(x^o) = \frac{C_\sigma}{|x^o-\text{Re}_1|^{n-\sigma}} = \frac{C_\sigma g(x^o)}{|x^o-\text{Re}_1|^\sigma}, \]

where \( C_\sigma \) can be made as small as we wish for sufficiently small \( \sigma \). Evaluate the integral in \( I_{21} \) in two regions

\[ D_1 = \Sigma_\lambda \cap (|y| \leq K|x^o|) \quad \text{and} \quad D_2 = \Sigma_\lambda \cap (|y| > K|x^o|). \]

In \( D_1 \), due to our choice of \( R = \lambda + 1 \),
\[ |g(x^o) - g(y)| \leq \frac{|\nabla g(\xi)||x^o-y|}{|\xi-\text{Re}_1|^{1-\sigma}} \leq C_2 |\nabla g(x^o)||x^o-y| \leq C_2 |g(x^o)|. \]
we derive a constant depending only on \( x \). Consequently,

\[
\left| \int_{D_1} \frac{g(x^o) - g(y)}{|x^o - y|^\alpha} dy \right| \\
\leq C_2 \sigma g(x^o) \left( \int_{D_1} \frac{1}{|x^o - y|^\alpha} dy \right) \\
\leq \frac{C_2 \sigma g(x^o)}{|x^o_1 - \lambda|^\alpha}. \tag{19}
\]

On \( D_2 \), notice that \( |g(x^o) - g(y)| \leq |g(x^o)| + |g(y)| \leq C_3 |y|^\sigma \) and \( |x^o - y|^\alpha \geq |x^o - y| \sim |y| \), we derive

\[
\left| \int_{D_2} \frac{g(x^o) - g(y)}{|x^o - y|^\alpha} dy \right| \leq C_3 \left( \int_{D_2} \frac{|y|^\sigma}{|y|^\alpha} dy \right) \leq \frac{C_3}{(K|x^o|)^{\alpha - \sigma}}. \tag{20}
\]

Combining (19) and (20), we arrive at

\[
|I_{21}| \leq (C_2 \sigma + \frac{C_3}{K^{\alpha - \sigma}}) \frac{g(x^o)}{|x^o_1 - \lambda|^\alpha}. \tag{21}
\]

Taking into account of (14), (15), (16), and (18), we obtain

\[
(-\Delta)^{\alpha/2} w_\lambda(x^o) \leq C \bar{w}_\lambda(x^o) (I_1 + I_2) \\
\leq C \bar{w}_\lambda(x^o) \left(C_1 + C_\sigma - C_2 \sigma - \frac{C_3}{K^{\alpha - \sigma}} \right) \frac{g(x^o)}{|x^o_1 - \lambda|^\alpha} \\
\leq C w_\lambda(x^o) \frac{1}{|x^o_1 - \lambda|^\alpha}. \tag{22}
\]

To derive the last inequality above, we choose \( K \) large, then let \( \sigma \) be sufficiently small (hence \( C_\sigma \) becomes sufficiently small), which implies \( I_1 + I_2 > 0 \) in the mean time.

Combining (22) with (10), we arrive at, for \( \lambda \leq 0 \),

\[
C < a(x^o_1)|x^o_1 - \lambda|^\alpha M(\lambda, x^o) \leq a(x^o_1)|x^o_1|^\alpha M(\lambda, x^o) \tag{23}
\]

with a positive constant \( C \). This is impossible under our condition (H2) taking into account that \( M(\lambda, x) \) is bounded and nonnegative.

Now we have completed Step 1. That is, we have shown that for all \( \lambda \) close to negative infinity, it holds

\[
w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda.
\]

**Step 2.** The above inequality provides a starting point to move the plane. Now we move plane \( T_\lambda \) towards the right as long as the inequality holds. We will show that \( T_\lambda \) can be moved all the way to infinity. More precisely, let

\[
\lambda_0 = \sup \{ \lambda \mid w_\lambda(x) \geq 0, \ x \in \Sigma_\mu, \ \mu \leq \lambda \},
\]

and we will show that \( \lambda_0 = \infty \).

Suppose the contrary, \( \lambda_0 < \infty \). Then by the definition of \( \lambda_0 \), there exists a sequence of numbers \( \{ \lambda_k \} \), with \( \lambda_k \downarrow \lambda_0 \), and \( x^k \in \Sigma_{\lambda_k} \), such that

\[
w_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} w_{\lambda_k} < 0. \tag{24}
\]
By (10) and (22), there exists a constant $c_0 > 0$, such that
\[
\frac{c_0}{|x_1^k - \lambda_k|^\alpha} \leq a(x_1^k)M(\lambda, x_1^k).
\] (25)

Denote
\[
x = (x_1, y), \quad x^k = (x_1^k, y^k) \quad \text{and} \quad u_k(x) = u(x_1, y - y^k).
\]
Then $u_k(x)$ satisfies the same equation as $u(x)$ does. Notice that both $u_k$ and $(-\Delta)^{\alpha/2}u_k$ are bounded, by Sobolev embeddings and regularity arguments, for some $\epsilon_1 > 0$, one can derive a uniform $C^{\alpha + \epsilon_1}$ estimate on $\{u_k\}$ (for example, see [11]), and hence concludes that there is a nonnegative ($\not\equiv 0$) function $u_0$, such that
\[
u_k(x) \to u_0(x) \quad \text{and} \quad (-\Delta)^{\alpha/2}u_0 = a(x_1)f(u_0), \quad x \in \mathbb{R}^n.
\]

Let
\[
w_{k, \lambda}(x) = u_k(x^k) - u_k(x) = u_{k, \lambda}(x) - u_k(x)
\]
and
\[
w_{0, \lambda}(x) = u_0(x^k) - u_0(x) = u_{0, \lambda}(x) - u_0(x).
\]
Then obviously
\[
w_{k, \lambda}(x) \to w_{0, \lambda}(x), \quad \text{as} \quad k \to \infty,
\]
and due to the boundedness of $\{x_1^k\}$, there exists a subsequence (still denoted by $\{x_1^k\}$) which converges to $x_1^0$. Hence
\[
w_{0, \lambda_0}(x_1^0, 0) = \lim_{k \to \infty} w_{\lambda_k}(x^k) \leq 0.
\]

On the other hand, for each $x \in \Sigma_{\lambda_0}$, we have
\[
0 \leq w_{k, \lambda_0}(x) \to w_{0, \lambda_0}(x).
\]
Furthermore, from the equation
\[
(-\Delta)^{\alpha/2}w_{0, \lambda_0}(x) = a(x_1^0)f(u_{0, \lambda_0}) - a(x_1)f(u_0), \quad x \in \Sigma_{\lambda_0}, \quad (26)
\]
By the strong maximum principle, we must have
\[
w_{0, \lambda_0}(x) > 0 \quad \text{for each} \quad x \in \Sigma_{\lambda_0}, \quad (27)
\]
or
\[
w_{0, \lambda_0}(x) \equiv 0 \quad \text{for each} \quad x \in \Sigma_{\lambda_0}.
\] (28)

If (28) holds, then $a(x_1^0) = a(x_1)$ in $\Sigma_{\lambda_0}$, which contradicts the assumption (H1) and (H2). Therefore, (27) must be true, which implies that
\[
x_1^0 = \lambda_0.
\]
It follows that
\[
|x_1^k - \lambda_k| \to 0, \quad \text{as} \quad k \to \infty.
\]
This contradicts (25).

Hence, we must have
\[
\lambda_0 = \infty.
\] (29)
It follows that
\[
w_{\lambda}(x) \geq 0, \quad \forall \ x \in \Sigma_{\lambda}, \quad \text{for all} \ \lambda \in (-\infty, \infty).
\]
Or equivalently, $u$ is monotone increasing in $x_1$ direction.
This completes the proof of Theorem 1.
3. Non-existence of solutions. In the previous section, we have shown that positive solutions of the equation

\[ (-\Delta)^{\alpha/2}u = a(x_1)f(u), \quad x \in \mathbb{R}^n \]  

(30)

are monotone increasing along \( x_1 \) direction. Base on this, we will derive a contradiction, and hence prove the non-existence.

Proof of Theorem 2. Since \( a(x_1) \) is positive somewhere and nondecreasing, we may assume that \( a(x_1) \) is positive for \( (R - 2, \infty) \) for some large \( R \). Let \( \mathbb{B}_1(Re_1) \) be the unit ball centered at \( (R, 0, \cdots, 0) \). Let \( \phi \) be the first eigenfunction associated with \( (-\Delta)^{\alpha/2} \) in \( \mathbb{B}_1(Re_1) \):

\[
\begin{cases}
(-\Delta)^{\alpha/2}\phi(x) = \lambda_1 \phi(x) & x \in \mathbb{B}_1(Re_1), \\
\phi(x) = 0 & x \in \mathbb{B}_1^C(Re_1).
\end{cases}
\]

Let \( \xi_0 = \min_{\mathbb{B}_1(0)} u \). Since \( u \) is positive, then \( \xi_0 > 0 \) and \( m_0 = \frac{f(\xi_0)}{\sup_{\mathbb{B}_1} u} > 0 \). since \( u \) is monotone increasing in \( x_1 \) direction, it follows that

\[ (-\Delta)^{\alpha/2}u(x) \geq a(R - 1)m_0u \quad \text{in} \quad \mathbb{B}_1(Re_1) \]

It is assumed that \( a(x_1) \to \infty \) as \( x_1 \to \infty \), if one chooses \( R > 0 \) sufficiently large, we have

\[ (-\Delta)^{\alpha/2}u(x) \geq \lambda_1 u(x), \quad \forall x \in \mathbb{B}_1(Re_1). \]  

(31)

Let

\[ m = \max_{\mathbb{B}_1(Re_1)} \frac{\phi}{u} \quad \text{and} \quad v(x) = mu(x). \]

Then obviously,

\[
\begin{cases}
(-\Delta)^{\alpha/2}(v(x) - \phi(x)) \geq 0 & \forall x \in \mathbb{B}_1(Re_1) \\
v(x) - \phi(x) > 0 & \forall x \in \mathbb{B}_1^C(Re_1).
\end{cases}
\]

By the strong maximum principle, we must have

\[ v(x) > \phi(x), \quad \forall x \in \mathbb{B}_1(Re_1). \]

This contradicts the definition of \( v \), because at a maximum point \( x^o \), we have

\[ v(x^o) = \frac{\phi(x^o)}{u(x^o)}u(x^o) = \phi(x^o). \]

Therefore, equation (30) does not possess any positive solution, and hence we complete the proof of Theorem 2. \( \square \)

Appendix. Here we prove Lemma 2.1. Without loss of generality, we may choose \( a = 0 \).

In the definition

\[ (-\Delta)^{\alpha/2}(|x|^\gamma) = C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{|x|^\gamma - |y|^\gamma}{|x - y|^{n+\alpha}} dy, \]

let \( y = |x|z \), then it becomes

\[ |x|^{\gamma - \alpha} C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{1 - |z|^\gamma}{|x|^{n+\alpha}} dz = |x|^{\gamma - \alpha} C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{1 - |e - z|^{n+\alpha}}{|e - z|^{n+\alpha}} dz = |x|^{\gamma - \alpha} C_\gamma. \]

Here \( e \) is any unit vector in \( \mathbb{R}^n \).
To see the sign of \( C_\gamma \), we split the integral into two parts, then make change of variable \( z = \frac{y}{|y|^2} \) in the second part to arrive at

\[
C_\gamma = C_{n,\alpha} \text{P.V.} \left\{ \int_{B_1(0)} \frac{1 - |z|^\gamma}{|e - z|^{n+\alpha}} dz + \int_{B_1(0)} \frac{1 - |y|^{-\gamma}}{|e - y|^{n+\alpha}|y|^{n-\alpha}} dy \right\}
\]

\[
= C_{n,\alpha} \text{P.V.} \int_{B_1(0)} \frac{(1 - |z|^\gamma)(1 - |z|^{n-\gamma})}{|e - z|^{n+\alpha}} dz.
\]

Now the conclusion of Lemma 2.1 follows immediately.

REFERENCES

[1] B. Barrios, L. Del Pezzo, J. García-Melián and A. Quaas, A Liouville theorem for indefinite fractional diffusion equations and its application to existence of solutions, Disc. Cont. Dyn. Syst., 37 (2017), 5731–5746.

[2] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, Supperlinear indefinite elliptic problems and nonlinear Liouville theorems, Topol. Methods Nonl. Anal., 4 (1994), 59–78.

[3] C. Brandle, E. Colorado, A. de Pablo and U. Sanchez, A concave-convex elliptic problem involving the fractional Laplacian, Proc Royal Soc. of Edinburgh, 143 (2013), 39–71.

[4] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Comm. PDE., 32 (2007), 1245–1260.

[5] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure. Appl. Math, 62 (2009), 597–638.

[6] M. Cai and L. Ma, Moving planes for nonlinear fractional Laplacian equation with negative powers, Disc. Cont. Dyn. Sys. - Series A, 38 (2018), 4603–4615.

[7] W. Chen, Y. Fang and R. Yang, Liouville theorems involving the fractional Laplacian on a half space, Advances in Math., 274 (2015), 167–198.

[8] W. Chen and C. Li, Methods on Nonlinear Elliptic Equations, AIMS book series, vol. 4, 2010.

[9] W. Chen and C. Li, Maximum principle for the fractional p-Laplacian and symmetry of solutions, Adv. Math., 335 (2018), 735–758.

[10] W. Chen, C. Li and Y. Li, A direct method of moving planes for fractional Laplacian, Advances in Math., 308 (2017), 404–437.

[11] W. Chen, C. Li and Y. Li, A direct blow-up and rescaling argument on nonlocal elliptic equations, International J. Math., 27 (2016), 1650064, 20 pp.

[12] W. Chen, C. Li and G. Li, Maximum principles for a fully nonlinear fractional order equation and symmetry of solutions, Cal. Var. & PDEs, 56 (2017), Art. 29, 18 pp.

[13] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math., 59 (2006), 330–343.

[14] W. Chen, C. Li and B. Ou, Qualitative properties of solutions for an integral equation, Disc. Cont. Dyn. Sys., 12 (2005), 347–354.

[15] W. Chen and J. Zhu, Indefinite fractional elliptic problem and Liouville theorems, J. Diff. Equa., 260 (2016), 4758–4785.

[16] T. Cheng, Monotonicity and symmetry of solutions to fractional Laplacian equation, Disc. Cont. Dyn. Sys. - Series A, 37 (2017), 3587–3599.

[17] J. Dou and Y. Li, Classification of extremal functions to logarithmic Hardy-Littlewood-Sobolev inequality on the upper half space, Disc. Cont. Dyn. Sys. - Series A, 38 (2018), 3939–3953.

[18] Y. Fang and W. Chen, A Liouville type theorem for poly-harmonic Dirichlet problem in a half space, Advances in Math., 229 (2012), 2835–2867.

[19] B. Gidas, W. Ni and L. Nirenberg, Symmetry and the related properties via the maximum principle, Comm. Math. Phys., 68 (1979), 209–243.

[20] X. Han, G. Lu and J. Zhu, Characterization of balls in terms of Bessel-potential integral equation, J. Diff. Equa., 252 (2012), 1589–1602.

[21] F. Hang, On the integral systems related to Hardy-Littlewood-Sobolev inequality, Math. Res. Lett., 14 (2007), 373–383.

[22] F. Hang, X. Wang and X. Yan, An integral equation in conformal geometry, Ann. H. Poincare Nonl. Anal., 26 (2009), 1–21.
[23] S. Jarohs and T. Weth, Symmetry via antisymmetric maximum principles in nonlocal problems of variable order, *Annali di Mat. Pura Appl.*, 195 (2016), 273–291.

[24] Y. Lei, Asymptotic properties of positive solutions of the Hardy Sobolev type equations, *J. Diff. Equa.*, 254 (2013), 1774–1799.

[25] Y. Lei, C. Li and C. Ma, Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy Littlewood Sobolev system of integral equations, *Cal. Var. & PDEs*, 45 (2012), 43–61.

[26] C. Li, Z. Wu and H. Xu, *Maximum Principles and Bocher Type Theorems*, Proceedings of the National Academy of Sciences, June 20, 2018.

[27] C. S. Lin, On Liouville theorem and apriori estimates for the scalar curvature equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 27 (1998), 107–130.

[28] B. Liu, Direct method of moving planes for logarithmic Laplacian system in bounded domains, *Disc. Cont. Dyn. Sys. - Series A*, 38 (2018), 5339–5349.

[29] G. Lu and J. Zhu, An overdetermined problem in Riesz-potential and fractional Laplacian, *Nonlinear Analysis*, 75 (2012), 3036–3048.

[30] G. Lu and J. Zhu, The axial symmetry and regularity of solutions to an integral equation in a half space, *Pacific J. Math.*, 253 (2011), 455–473.

[31] G. Lu and J. Zhu, Symmetry and regularity of extremals of an integral equation related to the Hardy-Sobolev inequality, *Cal. Var. & PDEs*, 42 (2011), 563–577.

[32] L. Ma and D. Chen, A Liouville type theorem for an integral system, *Comm. Pure Appl. Anal.*, 5 (2006), 855–859.

[33] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ratton. Mech. Anal.*, 195 (2010), 455–467.

[34] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary, *J. de Math. Pures et Appl.*, 101 (2014), 275–302.

[35] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.*, 60 (2007), 67–112.

[36] R. Zhuo, W. Chen, X. Cui and Z. Yuan, Symmetry and non-existence of solutions for a nonlinear system involving the fractional Laplacian, *Disc. Cont. Dyn. Sys.*, 36 (2016), 1125–1141.

Received for publication January 2018.

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