Magnifying absolute instruments for optically homogeneous regions

Tomáš Tyc
Institute of Theoretical Physics and Astrophysics,
Masaryk University, Kotlářská 2, 61137 Brno, Czech Republic
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Most optical instruments, including a simple lens or sophisticated camera lenses, have various types of aberrations. There exist, however, optical instruments that are free of aberrations and provide sharp (stigmatic) images of all points in some 3D region of space within geometrical optics; such devices are called absolute instruments [1]. A prototype of an absolute instrument is Maxwell’s fish eye, a device designed in 1854 by J. C. Maxwell [2]. It uses positive refractive index and images stigmatically the whole space. Another type of absolute instrument is based on materials with a negative refractive index. It was proposed by J. Pendry in 2000 [3] and later realized experimentally [4]. Remarkably, both Maxwell’s fish eye and Pendry’s lens are not limited by diffraction and provide sub-wavelength resolution [3–7].

Among absolute instruments there is a class of a particular interest, namely devices whose object and image spaces are optically homogeneous regions, i.e., regions with a uniform refractive index. Until recently, the only known such devices were plane mirrors [1]. This has been changed by a recent excellent work of J. C. Miñano [8] who proposed several new absolute instruments imaging homogeneous regions and also showed that some well-known optical devices such as Eaton lens or Luneburg lens [9] are in fact absolute instruments as well. All of these devices have unit magnification, giving an image of the same size as the original object, and no magnifying absolute instrument for homogeneous regions has been known. We proposed a magnifying absolute instrument recently [10] based on a numerically found refractive index with certain special properties, but our colleague Klaus Bering has later shown analytically that such an index in fact does not exist [11].

Here we present several magnifying absolute instruments that provide stigmatic images of homogeneous regions of 3D space with an arbitrary magnification. They are all based on the same idea and provide either real or virtual images. This is the first proposal of a magnifying absolute instrument for homogeneous regions that employs isotropic materials with positive refractive index. We will explain our idea first on a particular example of a magnifying absolute instrument resembling Eaton lens [12], and then proceed to other devices.

Our device consists of two distinct regions (see Fig. 1). The first region (we will call it region I) is a sphere of unit radius, the second region (II) occupies the space between two hemispheres with the radii 1 and \( R > 1 \), respectively, lying in the half-space \( y > 0 \). Both regions are filled with a spherically symmetric refractive index that we will denote by \( n_I(r) \) and \( n_{II}(r) \), respectively. The indices are chosen such that \( n_I(1) = n_{II}(1) = R \) and \( n_{II}(R) = 1 \). The medium surrounding the lens is composed of two parts as well. In the region \( r \geq 1, y < 0 \) (region III) the refractive index is equal to \( R \) while in the region \( r \geq R, y > 0 \) (region IV) the refractive index is equal to unity. Thus the index of the lens matches that of the surrounding medium, apart from the annulus \( 1 < r < R, y = 0 \), and also the indexes at the border between regions I and II match each other.

The refractive index \( n_{II}(r) \) is chosen such that a light ray incident from region IV to region II is bent towards the center, eventually crossing the border between regions II and I. There is a large variety of refractive index profiles that achieve this, one option is to choose \( n_{II}(r) = [1 + c(r - 1)(R - r)]R/r \) with \( c > 0 \) sufficiently large, which we have also used in our ray tracing simulations with \( c = 1 \). We then design the refractive index \( n_I(r) \) in region I such that the light ray coming from region II is bent further and leaves region I for region III in exactly opposite direction than was the original direction of the ray in region IV, see Fig. 2. The performance of the device is thus similar to the performance of Eaton lens; we can therefore call it “magnifying Eaton lens”. As we shall see, the difference is that the impact parameter of the outgoing ray is \( R \) times smaller than the impact pa-
rameter of the incoming ray, and this fact is responsible for the lens magnification.

To design the refractive index $n_1(r)$, we employ the standard method for solving the inversion problem [9, 13]. We will characterize rays in the lens by the quantity $L = nr \sin \alpha$ analogous to mechanical angular momentum, where $\alpha$ is the angle between the radius vector and the ray [14]. Angular momentum is conserved and motion of a particle is planar in central potentials, and the same holds for light ray in a spherically symmetric refractive index. Consider a ray propagating in the plane $xy$ horizontally (i.e., in the direction of $x$-axis) in region IV and entering region II of the lens at point A. Since $n = 1$ in region III, the angular momentum $L$ is equal to the impact parameter of the incoming ray. The polar angle $h$ swept by the ray in region II before entering region I (say, at point B) can be calculated by the expression [15]:

$$h(L) = L \int_1^R \frac{dr}{r \sqrt{|r n_1(r)|^2 - L^2}}$$

(1)

$h(L)$ is at the same time the change of the ray direction during propagation in region II from point A to B. This is because the product $nr$ is the same at both points A and B, so is $L = nr \sin \alpha$, and therefore the angle $\alpha$ between the ray and radius vector is the same at B as is in A.

The scattering angle $\chi$ (change of ray direction) corresponding to motion in region I must therefore be

$$\chi(L) = \pi - h(L),$$

(2)

which ensures that the total change of ray direction during motion in regions II and I is $\pi$. Solving the inversion problem, we then arrive at the following implicit equation for the refractive index $n_1(r)$ [9, 13]:

$$n_1(r) = R \exp \left[ \frac{1}{\pi} \int_{r n_1(r)}^R \frac{\chi(L) dL}{\sqrt{L^2 - |r n_1(r)|^2}} \right].$$

(3)

This way the refractive index is expressed analytically, although not explicitly. The refractive index in regions I and II is shown in Fig. 4.

FIG. 2: Magnifying Eaton lens with $R = 2$. Light rays incident on it from region IV are changed into rays in region III propagating in the opposite direction, with impact parameters reduced by the factor of $R$.

Now we have to show that the device we have just described is indeed an absolute magnifying instrument, i.e., it provides stigmatic image of some 3D region of space. First we note that from conservation of $L$ it follows that the impact parameter of the outgoing ray in region II is $R$-times smaller than the impact parameter of the incoming ray in region IV. Second we note that although we considered a horizontal ray in the $xy$-plane in our construction, the lens will have a similar effect on most other rays too, i.e., rays incident on it from region IV will be changed to rays moving in the opposite direction in region III and having $R$-times smaller impact parameters. This is caused by the spherical symmetry of the refractive index in regions I and II. There will also be rays for which this does not happen, namely the ones that at some point cross the interface between regions II and III, but still for an infinite number of rays the lens does the job it is designed for.

Now consider a collection of rays emerging from some point A at radius vector $\vec{r}_A$ in region III and incident on region I of the lens. As we have seen, these rays will be transformed by the lens into rays propagating in region IV, each parallel to the original ray in region III. Therefore the lines on which these outgoing rays lie intersect at the point $A'$ with radius vector $\vec{r}_{A'} = -R\vec{r}_A$, which way becomes the virtual image of the point A (see Fig. 3), and the magnification of the device is clearly equal to $R$.

As can be shown, the refractive index in region I diverges for $r \to 0$. To avoid this singularity, we can modify the lens by utilizing the fact that Luneburg lens [9] equipped with a spherical mirror on its surface has the same effect on the incoming rays [8] as Eaton lens. Imagine we place a mirror on the part of the interface between regions I and II, allowing the rays in region I to be reflected before re-entering region II. We again require that the outgoing rays move in the opposite direction with respect to the incoming rays, but now rays with small impact parameters do not have to make a rapid turn near the origin at it was the case with magnifying Eaton lens.
FIG. 4: Refractive index in regions I and II of magnifying Eaton lens (thick red), Luneburg lens (thin black) and magnifying invisible sphere (dashed blue) with $R = 2$.

FIG. 5: Magnifying Luneburg lens with $R = 2$. A spherical mirror covers part of the interface between regions I and II. The effect on rays coming from region IV is the same as that of magnifying Luneburg lens and therefore it also provides a magnified virtual image.

lens, and therefore the index will not need a singularity there. It can be shown by simple geometrical considerations that in this case the required scattering angle in region I corresponding to the ray segment between point of entrance to region I and the point of incidence on the mirror is

$$\chi(L) = \arcsin \frac{L}{R} - \frac{h(L)}{2},$$

which, after substitution to Eq. (3) leads to the refractive index shown in Fig. 4 for $R = 2$. Ray tracing in the magnifying Luneburg lens is shown in Fig. 5.

Another interesting magnifying device can be derived by our method from the invisible sphere described in [13]. Here we require again that the rays leaving the lens propagate parallel to their original direction, but this time go forwards instead of backwards. The scattering angle in this case is $\chi(L) = 2\pi - h(L)$. If the lens should work well for rays with the direction close to $x$-axis, the border between regions III and IV now has to be in the $yz$ plane and similarly region II now lies at $x > 0$ instead of $y > 0$.

The virtual image of a point A is now formed at point $A'$ with $\vec{r}_A' = R\vec{r}_A$. Ray tracing in this lens is shown in Fig. 6, the refractive index is in Fig. 4.

The last device we will discuss is a magnifying absolute instrument that provides real images. Now we have to arrange regions I – IV in a somewhat different fashion, see Fig. 7 region I is given by the condition $r > R$, so it is the whole 3D space with the exception of the sphere of radius $R$. Region II occupies the space between two hemispheres with the radii 1 and $R > 1$, respectively, lying in the half-space $x > 0$. Region III is the unit hemisphere at $x > 0$ and region IV is a hemisphere of radius $R$ at $x < 0$. Refractive indices in regions II, III and IV are as before. Light rays now enter the lens (region II of it) from region III, i.e., from the inside. After having been bent in region II, they propagate in region I and we require that when they enter region IV, they are parallel to their original direction in region III, see Fig. 7.

To find the refractive index $n_I(r)$ that achieves this, we have to solve an “outer” inversion problem instead of the usual “inner” problem. This can be done by employing inversion in the sphere of radius $R$ which transforms the outer problem to the inner one. A careful analysis of the scattering angle $\chi'$ in the transformed problem reveals that $\chi'(L) = 4\arcsin(L/R) - h(L)$, which then gives the transformed refractive index $n'_I(r')$ as a function of the transformed radius $r' = R^2/r$ with the help of formula (3), but with the “$R$” omitted in front of the exponential since we require $n'_I(R) = 1$. The index $n_I(r)$ can then be calculated as

$$n_I(r) = \frac{R^2}{r^2} n'_I(R^2/r),$$

which follows from the equality of optical paths in the original and transformed region I.

To see that this device indeed creates a real magnified image, consider rays emerging from some point A at radius vector $\vec{r}_A$ in region III, see Fig. 8. The rays get to region IV assuming their original direction and head towards the point $A'$ at $\vec{r}_A' = R\vec{r}_A$. This point lies outside of region IV and therefore the image is virtual. However,
FIG. 7: Inside-out magnifying instrument with the regions marked. The rays enter the lens from region III inside the lens (the object space) and after making loops in region I, which now extends to infinity, enter region IV (image space) in their original direction.

FIG. 8: Image formation on the instrument from Fig. 7. Rays originating at point A in region III and converging in region IV to a point outside this region are reflected by the mirror to the real image A” of A.

we can take advantage of the fact that the rays are converging and place a mirror at the flat interface of region IV, see Fig. 8. This way the virtual image it turned into a real image at point A”, which is a mirror image of A’ in the plane x = 0. Making the mirror double-sided, also rays emerging from the point A to the left will contribute to forming the image.

In conclusion, we have proposed several absolute optical instruments that create magnified stigmatic images of homogeneous 3D region. They are all designed by the same general idea. Especially appealing is the lens giving a real magnified image and the magnifying Luneburg lens with its moderate refractive index range; for R not too large, it should be possible to realize the latter for near infrared or even visible light using e.g. graded index structures in silicon [16] or diamond. Further research will reveal whether some of these devices, e.g. the lens giving the real image, could provide sub-wavelength resolution similarly to Maxwell’s fish eye [5–7]. Magnifying absolute instruments could find their applications in various fields, for example in photolithography, but more importantly, our research has shown that such devices exist at all, something that was not clear until this date [1, 8].

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