Nonequilibrium Critical Behavior for Electron Tunneling through Quantum Dots in an Aharonov-Bohm Circuit

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Double quantum dots can provide an experimental realization of the 2 impurity Kondo model which exhibits a non-Fermi liquid quantum critical point (QCP) at a special value of its parameters. We generalize our recent study of double quantum dots in series to a parallel configuration with an Aharonov-Bohm flux. We present an exact universal result for the finite temperature and finite voltage conductance \( G[V,T] \) along the crossover from the QCP to the low energy Fermi liquid phase. Compared to the series configuration, here generically \( G[V,T] \neq G[-V,T] \), leading to current rectification.

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I. INTRODUCTION

It is now well established that quantum dots (QDs) behave as Kondo impurities at low temperatures.\(^{2,3}\) Whereas many theoretical tools are available to address linear transport, the nonequilibrium regime is far less studied, although it is typically addressed in experiment.\(^{4}\) A solution of nonequilibrium transport through a 1-channel Kondo impurity was achieved in Ref. [1]; however exact results were obtained only for a specific point in the parameters space (Toulouse limit). Another important development in this direction was the application of the Bethe-ansatz and finding of many body scattering states.\(^{5,6}\) Recently we found exact results for nonlinear transport close to a QCP in a double dot in series realizing the two impurity Kondo model (2IKM).\(^{7,8}\)

The 2IKM consists of two impurity spins coupled to two channels of conduction electrons and, at the same time, interacting with each other through an exchange interaction \( K \). Jones et al.\(^{9}\) observed that a QCP at \( K = K_c \) separates a local singlet from a Kondo-screened phase, where \( K_c \) is of the order of the Kondo temperature \( T_K \). The exact critical behavior was found using conformal field theory (CFT) and abelian bosonization methods. Implications of the 2IKM for transport through double QDs were studied in Refs. [12-15].

The presence of a sharp quantum phase transition in the 2IKM became questionable soon after its discovery; in the mean field study in Ref. [10] it was pointed out that the true QCP is restricted to the case of a special particle hole (P-H) symmetry assumed in Ref. [8]. This was confirmed by numerical renormalization group calculations.\(^{16}\) P-H symmetry breaking was later associated with two relevant potential scattering perturbations.\(^{16-17}\) Thus, in real systems the critical behavior for \( K = K_c \) can be observed only above a certain crossover energy scale, denoted here as \( T_{c,L}^{R} \). In order to obtain reliable predictions for QDs it is crucial to include the extra relevant perturbations associated with potential scattering in a real calculation. We achieved this task for a double QD using the method developed by Gan.\(^{18}\) The finding of exact crossover results including P-H symmetry breaking remains an open problem for the alternative proposed realization of the 2IKM by Zarán et al.\(^{17}\) Compared to their QD system involving at least three leads, our system has only two leads making the nonequilibrium behavior more tractable.

In this paper we generalize our previous results to a generic configuration ranging from series to parallel QD attached to two leads; see Fig. (1). In this generic configuration transport from left to right occurs via different interfering paths. A particular feature of our results distinguishes the generic case from the series case: in the generic case the finite voltage conductance \( G[V] \) has the property \( G[V] - G[-V] \neq 0 \), leading to current rectification, similar to a diode. This effect results from interactions and is absent in a noninteracting Landauer description.\(^{21}\) An additional aim of this paper is to provide important details on the calculation for the general series or parallel cases.

The outline of the paper is as follows. In Sec. [II] the double QD system is presented and mapped to the 2IKM. In Sec. [III] the conductance is calculated at the QCP using CFT methods, neglecting the effect of potentials scattering. In Sec. [IV] we consider deviations from the QCP due to variations of \( K \) from \( K_c \), and calculate the finite temperature crossover for the linear conductance using a mapping of the P-H symmetric 2IKM to the Ising model with a boundary magnetic field. We also apply this mapping for the QD system proposed by Zarán et al.\(^{17}\) as a realization of the 2IKM. In Sec. [V] potential scattering is incorporated in the Hamiltonian close to the QCP, and in the crossover formula for the linear conductance. In Sec. [VI] the full nonequilibrium problem at finite voltage and temperature in the vicinity of the critical point is addressed. Sec. [VII] contains conclusions. We relegate details on the calculation of the nonlinear conductance using Keldysh Green functions (GFs) to the appendix.

II. MODEL

The physical system under consideration is shown schematically in Fig. (1). It consists of left (L) and right
(R) leads tunnel coupled to two quantum dots 1, 2, with tunneling amplitudes $t_{iL/R}$, ($i = 1, 2$). We assume that both dots are in the Kondo regime, with gate voltages adjusted to give an odd number of electrons and the $t_{iL/R}$ are sufficiently weak compared to the charging energy, $U$, so that charge fluctuations can be ignored. We write the effective spin-1/2 moments as $S_i$ and $S_2$. We will be primarily interested in the case where $t_{1L} = t_{2R} = t_1$, $t_{2L}$ so that the left lead is primarily coupled to dot 1 and the right lead to dot 2 since only in this case will the QCP occur. Note that in the extreme case where $t_{1R} = t_{2L} = 0$, this reduces to the series configuration analyzed, for example, in Ref. [1]. The fluxes $\Phi_L$ and $\Phi_R$ are introduced in the triangular plaquettes as shown.

In the standard fashion, the conduction-electron channels that couple to the impurity are reduced to one-dimensional left moving Dirac fields $\psi_{i\alpha}(x)$, where $i = L, R$ and $\alpha = \uparrow, \downarrow$ are the lead and spin indices, respectively. We assume that a single mode in each lead couples to both impurities. Here we have linearized the conduction-electron dispersion around the Fermi level: $\epsilon_k = \hbar v_F k$, where $\epsilon_k$ and $k$ are measured relative to the Fermi level and Fermi wave number, respectively. $x$ is a fictitious position variable conjugate to $k$. We set $\hbar = v_F = 1$.

We discuss the different terms which will appear in the model Hamiltonian, Eq. (2.2). An exchange interaction

$$K_{12} \sim \frac{t_{12}^2}{U}$$

between the impurity spins is generated by the interimpurity tunneling $t_{12}$. The impurity spins are also Kondo-coupled to the conduction-electron spin density at the origin:

$$\vec{s}_j = \psi_{j\alpha}^\dagger \vec{S}_j \frac{\sigma^\beta}{2} \psi_{j\beta}, \quad i, j = L, R = 1, 2.$$  

(repeated spin indices summed)

In addition there are potential scattering (PS) terms $\propto \psi_{j\alpha}^\dagger \psi_{j\alpha}$ (repeated spin indices summed). The system is driven out of equilibrium by a source drain voltage $V$. Thus, the Hamiltonian $H$ is

$$H = H_0 + H_V + K_{12} \vec{S}_1 \cdot \vec{S}_2 + H_K + H_{PS} + H',$$

$$H_0 = \int_{-\infty}^{\infty} dx \psi_{j\alpha}^\dagger \epsilon^j \psi_{j\alpha},$$

$$H_V = \frac{eV}{2} \int_{-\infty}^{\infty} dx \psi_{j\alpha}^\dagger (\tau^z)^i \psi_{j\alpha},$$

$$H_K = \sum_{\ell=1,2} \left( J^{(\ell)} i \vec{s}_j \cdot \vec{S}_\ell \right),$$

$$H_{PS} = \psi_{j\alpha}^\dagger V^j \psi_{j\alpha},$$

with repeated lead and spin indices summed. To be complete one has to add the terms

$$H' = V'^j \psi_{j\alpha}^\dagger \cdot (\vec{S}_1 \times \vec{S}_2) + \psi_{j\alpha}^\dagger V'^j \psi_{j\alpha}(\vec{S}_1 \cdot \vec{S}_2).$$

However, close to the QCP the first (second) term of $H'$ has a similar effect as $H_K (H_{PS})$. Therefore, up to a correction to the actual coupling constants, energy scales, and to the critical value of different parameters at the QCP, all of which we are not able to determine exactly, it is legitimate to drop $H'$.

The Kondo interaction induces, via the Ruderman-Kittel-Kasuya-Yosida (RKKY) mechanism, an additional contribution to the inter-impurity exchange, $J = K_{12} + K_{RRKKY}$, where

$$K_{RRKKY} = 2\langle S_1 = \uparrow, S_2 = \uparrow | H_K \frac{1}{-H_0} H_K | S_1 = \uparrow, S_2 = \downarrow \rangle$$

$$= 4 \sum_{k_1 > 0, k_2 < 0} \frac{1}{-(\epsilon_{k_2} - \epsilon_{k_1})} \text{tr}\{J^{(1)} J^{(2)}\}.$$  

With the parametrization of $J^{(\ell)}$ given in Eq. (2.6), $\text{tr}\{J^{(1)} J^{(2)}\} = 4J^2 \sin^2(2\theta) \cos^2 \frac{\Phi}{2}$. Using $\sum_k = \nu \int dx$, where $\nu$ is the density of states in the leads, and restricting the band width to $|\epsilon_k| < U$, beyond which the effective spin description breaks down, one obtains a ferromagnetic contribution

$$K_{RRKKY} \sim -(\nu J)^2 U \sin^2(2\theta) \cos^2 \frac{\Phi}{2}.$$  

We estimate the potential scattering amplitudes by

$$V_L^i = \frac{t_{1L}^2 + t_{2L}^2}{U}, \quad (i = L, R)$$

$$V_R^i = \frac{t_{1L} t_{1R} e^{i\phi_L} + t_{2L} t_{2R} e^{-i\phi_R}}{U}$$

$$+ c' t_{1L} t_{12} t_{2L} e^{i(\phi_L - \phi_L)} + t_{2L} t_{12} t_{1R},$$  

(2.4)  

where $c'$ is a constant factor of order 1.

Until Sec. VLA we will assume the parity symmetry

$$S_1 \leftrightarrow S_2, \quad L \leftrightarrow R.$$  

(2.5)  

However our results are not restricted to this case, as will be discussed in Sec. VLA. Parity implies $t_{1L} = t_{2R} \equiv$
Indeed, using Eq. (2.6) we obtain
\[
H \sim \left( \frac{t_1 e^{i\Phi/2}}{t_2} \right).
\]
This leads us to parameterize the hermitian exchange matrices by
\[
J^{(1)} = \tilde{J}, \quad J^{(2)} = \tau^x \tilde{J} \tau^x, \quad \tilde{J} = J(1 + \cos(2\theta)\tau^z + \sin(2\theta)(\cos(\Phi/2)\tau^z - \sin(\Phi/2)\tau^y)),
\]
where
\[
\theta = |\arctan(t_2/t_1)|, \quad J \sim \frac{t_1^2 + t_2^2}{U}. \quad (2.7)
\]
Parity symmetry for the PS amplitudes implies \(V_L^R = V_R^L, V_L^L = V_R^R\) (Im\(V_R^R = 0\)). We can estimate
\[
V_L^L \sim \frac{t_1^2 + t_2^2}{U}, \quad V_R^L \sim \frac{t_1 t_2}{U} \cos \frac{\Phi}{2} + e^{i(\frac{t_1^2 + t_2^2}{2} - \frac{t_1^2}{2} - \frac{t_2^2}{2})} t_{12}. \quad (2.8)
\]
It is convenient to define even and odd channels \(\psi_{e,o} = \psi_{Eo} / \sqrt{2}\), in terms of which the parity transformation reads
\[
\psi_e \rightarrow \psi_e, \quad \psi_o \rightarrow -\psi_o.
\]
The most general form of \(H_K + H_{PS}\) consistent with parity is
\[
H_K = J_e \psi_e^\dagger \sigma^3 \psi_{e,\beta} \cdot (\vec{S}_1 + \vec{S}_2) + J_0 \psi_{o,\alpha}^\dagger \sigma^3 \psi_{o,\beta} \cdot (\vec{S}_1 + \vec{S}_2) + [J_m \psi_e^\dagger \sigma^3 \psi_{o,\beta} + h.c.] \cdot (\vec{S}_1 - \vec{S}_2),
\]
\[
H_{PS} = V_e \psi_e^\dagger \psi_{e,\alpha} + V_o \psi_{o,\alpha}^\dagger \psi_{o,\alpha}. \quad (2.9)
\]
Indeed, using Eq. (2.9) we obtain \(H_K + H_{PS}\) in this form with
\[
J_{e,o} = \frac{\tilde{J}_L + \tilde{J}_R \mp (J_R + J_L)}{2} = J[1 \pm \sin(2\theta)\cos(\Phi/2)],
\]
\[
J_m = \frac{\tilde{J}_L - \tilde{J}_R + J_R - J_L}{2} = J[\cos(2\theta) - i\sin(2\theta)\sin(\Phi/2)] = |J_m|e^{i\phi_m},
\]
\[
V_{e,o} = \frac{V_L^L + V_R^R \pm (V_R^L + V_L^L)}{2}, \quad (2.10)
\]
where
\[
\phi_m = -\arctan(\tan(2\theta)\sin(\Phi/2)). \quad (2.11)
\]
For finite flux \(J_m\) has an imaginary part. To recover real coupling constants in \(H_K\) we remove this phase by a redefinition of the fields
\[
\psi_e \rightarrow \psi_e' = e^{-i\phi_m/2} \psi_e, \quad \psi_o \rightarrow \psi_o' = e^{i\phi_m/2} \psi_o. \quad (2.12)
\]
In the \(\psi_{e,o}'\) basis, \(H_K\) has real coupling constants, \(J_{e,o} J_o\) and \(|J_m|\), and it corresponds to the notation in Ref. [10].

It is convenient to define \(\psi_j' = \frac{\psi_{e} + \psi_{o}}{\sqrt{2}}\). Equivalently, the fields \(\psi_j' (j = 1, 2)\) are related to the L-R basis by the rotation \(\psi_i = (Me^{i\tau^x \phi_m/2}M)^j \psi_j'\), where \(M = \tau^x \tau^y\).

As will be discussed in Sec. (VI A), observability of the QCP in this system is restricted to the regime \(t_1 \gg t_2\), or equivalently small \(\theta\) (see Eq. (2.7)). In this limit the two impurity Kondo physics is especially transparent, since each QD is coupled essentially to one lead. \(K\) can be tuned by means of \(t_{12}\). For \(K > K_c\) the impurities are locked into a singlet, while for \(K = 0\) each impurity is Kondo-screened by the nearby lead. In the case of exact P-H symmetry, occurring for \(V_2 = 0\), those points in the \(K\)-parameters space are separated by a QCP at a critical value \(K = K_c \sim TK_\Delta\).

### III. Conductance at the Fixed Point

In Ref. [1] the conductance of a series double QD was calculated using the tunneling current operator. In this paper until Sec. (VI) we use the Kubo linear conductance formula written in terms of bulk current correlation function. The reason for taking this different approach here is that it relates the conductance to correlation functions in certain field theories, that can be addressed using boundary conformal field theory or integrability methods. This allow us to express the conductance of double QDs described by the 2IKM in terms of correlation functions in the boundary Ising field theory.

The linear conductance can be calculated from the Kubo formula
\[
G = \lim_{L \to \infty} \lim_{\omega \to 0} \frac{e^2}{h\omega(2L)^2} \int_{-L}^{L} dx \int_{-L}^{L} dy \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \times \langle J(r, \tau) J(r', 0) \rangle.
\]
Here \(r\) is the physical coordinate; see Fig. [2]. It should be distinguished from the fictitious coordinate \(x\) labeling the chiral fermions \(\psi_{io}(x)\). We define the chiral current densities in each lead \(j_L(x) = \psi^\dagger L_{io}(x) \psi_{Lio}(x)\), \(j_R(x) = \psi^\dagger R_{io}(x) \psi_{Rio}(x)\). The bulk current operator \(J(r)\) can be written as
\[
J(r) = -\left\{ \begin{array}{cc} j_R(r) - j_R(-r) & r > 0 \\ j_L(r) - j_L(-r) & r < 0 \end{array} \right. .
\]
It is useful to define the odd current \(j_o(x) = j_L(x) - j_R(x)\), since \(\int_{-L}^{L} dx J(r) = \int_{-L}^{L} dx j_o(x) \mathrm{sgn}(x)\). The conductance is given in terms of the odd current correlator,
\[
G = \lim_{L \to \infty} \lim_{\omega \to 0} \frac{e^2}{h\omega(2L)^2} \int_{-L}^{L} dx \int_{-L}^{L} dy \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \times \langle j_o(x, \tau) j_o(y, 0) \mathrm{sgn}(xy) \rangle. \quad (3.1)
\]

The odd current \(j_o(x) = \psi^\dagger R_{io}(x) \tau^x \psi_{Rio}(x)\) corresponds to the z-component of the flavor current of the fermions in
the L-R basis. We define the flavor current in terms of the fermions $\psi'_{j\alpha}$ after the rotation Eq. (2.12),

$$j^f = \psi'_{j\alpha} \frac{\partial^2}{2} \psi'_{j\alpha}. \quad \text{(repeated indices summed)}$$ (3.2)

The transformation Eq. (2.12) amounts to a rotation in the flavor sector,

$$j_o = 2[\cos \phi_m(j^f)^2 - \sin \phi_m(j^f)^2].$$ (3.3)

Consider the weak coupling limit $J \to 0$. It corresponds to a trivial boundary condition (BC) $\psi_L(x = 0^+) = \psi_R(x = 0^-)$, $\psi_R(x = 0^+) = \psi_R(x = 0^-)$, describing free fermions with full reflection at the boundary. Also this BC makes apparent the continuity of the chiral fields $\psi_L$, $\psi_R$ at $x = 0$. Accordingly, the odd current correlator is given by

$$\langle j_o(x, \tau) j_o(y, 0) \rangle_{J=0} = -\frac{1}{\pi^2} \frac{1}{(\tau + i(x-y))^2}.$$

To calculate the odd current correlator at the nontrivial fixed point, we apply CFT methods and the Bose-Ising representation used in Ref. [10]. In this representation the 4 fermions $\psi_{i\sigma}$ are represented using a coset construction in terms of three Wess-Zumino-Witten (WZW) nonlinear $\sigma$ models, $SU(2)_1^{\text{charge}i} \times SU(2)_2^{\text{charge}i} \times SU(2)_2^{\text{spin}}$, together with a $\mathbb{Z}_2$ Ising model. The currents of the two $SU(2)_1$ $\sigma$ models are associated with the charge of each species $\psi'_{1\sigma}$ and $\psi'_{2\sigma}$. The current of the $SU(2)_2$ model is associated with the total spin.

Following Ref. [10] one may write down representations of the various operators in the free fermion theory as product of charge (or isospin) bosons, the total spin boson, and the Ising field. The $k = 2$ WZW model has primary fields of spin $j = 0$ (identity operator 1), $j = 1/2$ (fundamental field $g_0$), and $j = 1$ (denoted $\vec{\phi}$). The $k = 1$ WZW model only has the identity operator and the $j = 1/2$ primary, $h_A$. Their scaling dimension is given by $\Delta = \frac{4(j+1)}{2k}$. The Ising model has three primary fields: the identity operator 1 ($\Delta = 0$), the Ising order parameter $\sigma$ ($\Delta = 1/16$), and the energy operator $\epsilon$ ($\Delta = 1/2$). For example the fermion field is written in this representation as

$$\psi'_{i\sigma} \propto (h_i)_1 g_0 \sigma.$$ (3.4)

Consider the OPE of the fundamental field $g \times g = 1 + \vec{\phi}$. The operator under consideration is a spin singlet, $\langle \vec{\chi}_j \rangle (x, \tau) \chi_j (y, 0) \rangle = \frac{1}{\tau + i(x-y)}$. First consider $(j^f)^z = \frac{1}{2} (\psi'^{1\alpha}_1 \psi'^{1\alpha}_a - \psi'^{1\alpha}_a \psi'^{1\alpha})$. This is just the charge difference between flavors, represented by $I_1^z - I_2^z$, where $I_i$ is the $SU(2)_1^{\text{charge}i}$ current, $(i = 1, 2)$. For the operator $(j^f)^z = \psi'^{1\alpha}_1 \psi'^{1\alpha}_a$, we use Eq. (3.2),

$$\psi'^{1\alpha}_1 (x) \psi'^{1\alpha}_a (x) \propto \lim_{x' \to x} g^{1\alpha} (x') g_0 (x) (h_1)_1^{1\alpha} (x') (h_2)_1 (x) \sigma (x') \sigma (x).$$

The three factors have dimensions which add correctly to $1/2$. The representation of other operators can be determined using the operator product expansion (OPE). For the Ising model the OPE gives

$$g \times g = 1 + \epsilon, \quad \sigma \times \epsilon \to \sigma, \quad \epsilon \times \epsilon \to 1.$$

This OPE is equivalent to that of the $k = 2$ WZW model with the identifications $\sigma \leftrightarrow g$ and $\epsilon \leftrightarrow \vec{\phi}$.

Using the OPE, symmetry considerations, and consistency of scaling dimensions, we shall determine the representation of the odd current $j_o$. The latter is related in Eq. (3.3) to the flavor current operators $(j^f)^z = (j^f)^y$. First consider $(j^f)^z = \frac{1}{2} (\psi'^{1\alpha}_1 \psi'^{1\alpha}_a - \psi'^{1\alpha}_a \psi'^{1\alpha})$. This is just the charge difference between flavors, represented by $I_1^z - I_2^z$, where $I_i$ is the $SU(2)_1^{\text{charge}i}$ current, $(i = 1, 2)$. For the operator $(j^f)^z = \psi'^{1\alpha}_1 \psi'^{1\alpha}_a$, we use Eq. (3.4),

$$\psi'^{1\alpha}_1 (x) \psi'^{1\alpha}_a (x) \propto \lim_{x' \to x} g^{1\alpha} (x') g_0 (x) (h_1)_1^{1\alpha} (x') (h_2)_1 (x) \sigma (x') \sigma (x).$$

Consider the OPE of the fundamental field $g \times g = 1 + \vec{\phi}$. The operator under consideration is a spin singlet, $\langle \vec{\chi}_j \rangle (x, \tau) \chi_j (y, 0) \rangle = \frac{1}{\tau + i(x-y)}$. First consider $(j^f)^z = \frac{1}{2} (\psi'^{1\alpha}_1 \psi'^{1\alpha}_a - \psi'^{1\alpha}_a \psi'^{1\alpha})$. This is just the charge difference between flavors, represented by $I_1^z - I_2^z$, where $I_i$ is the $SU(2)_1^{\text{charge}i}$ current, $(i = 1, 2)$. For the operator $(j^f)^z = \psi'^{1\alpha}_1 \psi'^{1\alpha}_a$, we use Eq. (3.2),

$$\psi'^{1\alpha}_1 (x) \psi'^{1\alpha}_a (x) \propto \lim_{x' \to x} g^{1\alpha} (x') g_0 (x) (h_1)_1^{1\alpha} (x') (h_2)_1 (x) \sigma (x') \sigma (x).$$

The remaining sectors of the theory other than the Ising model remain unaffected. Correlation functions of factors belonging to sectors other than the Ising model have the form dictated by conformal invariance, $(O_\Delta (x, \tau) O_\Delta (y, 0)) = \frac{1}{(\tau + i(x-y))^2}$, where $\Delta$ is the scaling dimension of $O$. This form remains valid both at the trivial and nontrivial fixed points. On the other hand
correlation functions of fields from the Ising sector do depend on BC. There is a general formula for correlation function of primary operators for a BC obtained by fusion with a primary \( a_{\Delta} ^{24.25} \)

\[
\langle \mathcal{O}_\Delta (x, \tau) \mathcal{O}_\Delta (y, 0) \rangle = \frac{1}{(\tau + i(x - y))^{2\Delta}} \times \left\{ \begin{array}{ll}
\frac{s_{\Delta}^2}{s_{\Delta}^2 + s_{\Delta}^2} & xy > 0 \\
\frac{s_{\Delta}^2}{s_{\Delta}^2 + s_{\Delta}^2} & xy < 0
\end{array} \right.
\]

(3.5)

Here \( S_j^\pm \) are elements of the modular \(-\)matrix. For the Ising model this is given by

\[
S = \begin{pmatrix}
1/2 & 1/2 & 1/\sqrt{2} \\
1/2 & 1/2 & -1/\sqrt{2} \\
1/\sqrt{2} & -1/\sqrt{2} & 0
\end{pmatrix}
\]

where the first, second and third rows and columns are labeled by the fields with scaling dimension 0, 1/2, 1/16, respectively. The change in BC in the IfKM from trivial to nontrivial fixed points corresponds to fusion with the spin operator in the Ising sector. Setting \( \Delta = 1/2, a = 1/16 \) we have \( s_{\Delta}^2 = -1 \). Hence Eq. (3.5) gives

\[
\langle \mathcal{O}_\Delta (x, \tau) \mathcal{O}_\Delta (y, 0) \rangle_{\text{free}} = \langle \mathcal{O}_\Delta (x, \tau) \mathcal{O}_\Delta (y, 0) \rangle_{\text{fixed}} \cdot \text{sgn}(xy). \quad (3.6)
\]

where up to a normalization factor \( \langle \mathcal{O}_\Delta (x, \tau) \mathcal{O}_\Delta (y, 0) \rangle_{\text{fixed}} \propto \frac{1}{\tau + i(x - y)} \). We may interpret this as a phase shift of \( \pi/2 \) that the energy operator \( \epsilon(x) \) undergoes at \( x = 0 \). We proceed to evaluate the odd current correlation function. Since the crossed terms \( \langle (j^f)^\mp (j^f)^\pm \rangle \) vanish we obtain

\[
\langle j_o(x, \tau) j_o(y, 0) \rangle_{\text{free}} = \langle j_o(x, \tau) j_o(y, 0) \rangle_{\text{fixed}} \propto \left\{ \begin{array}{ll}
1 & xy > 0 \\
\cos(2\phi_m) & xy < 0
\end{array} \right.
\]

(3.7)

One can use the Kubo formula Eq. (3.1) to calculate the conductance. However a calculation is unnecessary: Curiously, one obtains exactly the same result for the conductance. However a calculation is unnecessary:

\[
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\]

**A. \( T = 0 \) phase diagram**

Having found the conductance at the QCP at \( K = K_c \), we shall consider the surrounding FL fixed points and draw a phase diagram. Here and until Sec. (VI) we consider the P-H symmetric model. In this model charge transfer between the leads leading to finite current is allowed by the exchange interaction in \( H_K \). The main role of \( \theta = \arctan(t_2/t_1) \) and flux \( \Phi \) is to modify the crossover scales \( T_K \) and \( K_c \). We plot in Fig. (3) the phase diagram at fixed \( \theta \) as function of \( K \) and flux. The NFL state occurs along the curve \( K = K_c[\Phi] \), where \( K_c \sim T_K[J_o, J_o, J_m] \) and \( J_o, J_o, J_m \) depend on flux through Eqs. (2.10). This curve is characterized by a finite conductance \( G = G_0 \) at \( \Phi \neq 0, 2\pi \). It separates the \( K > K_c \) local singlet phase from the \( K < K_c \) Kondo-screened phase.

The conductance vanishes in both FL phases. At \( K > K_c \) the system remains in its weak coupling limit, corresponding to weakly transmitting tunnel junctions. At \( K < K_c \) a Kondo-screened phase is developed and the two channels \( \psi_i \) and \( \psi_o \) participate in the screening of the combined spin–1 impurity. In the effective FL description both the even and odd channels acquire a phase shift of \( \delta_o = -\delta_o = \pi/2 \). The conductance vanishes as a result of destructively interference between even and odd channels: an incoming electron from the left lead \( \psi L = \psi L^{\text{in}} + \psi L^{\text{in}} \) scatters into the outgoing state \( \psi R^{\text{out}} = e^{2i\delta_o} \psi R^{\text{out}} \) in the left lead, corresponding to full reflection. (The situation is reversed if a \( \pi/2 \) phase shift occurs only in one channel). We point out that when we include P-H symmetry breaking the conductance is finite in the FL phases.

In the Hamiltonian Eq. (2.9) the condition \( J_m \neq 0 \) is required to mix the impurity singlet and triplet subspaces of the Hilbert space. At \( J_m = 0 \) the transition at \( K = K_c \) corresponds to a level crossing between those subspaces. We point out that this special situation occurs in our system for the symmetric point \( t_1 = t_2, (\theta = \pi/4), \) and at zero flux. In this case, when \( K < K_c \) the conductance is \( G = G_0 \), since the odd channel is decoupled, \( \delta_o = 0 \), and as a result of Kondo effect in the even channel \( \delta_e = \pi/2 \).

**IV. UNIVERSAL CROSSOVER AS FUNCTION OF INTER-IMPURITY INTERACTION \( K \)**

In the previous section we calculated the conductance at the critical value of the inter-impurity exchange interaction \( K = K_c \), and assuming P-H symmetry. In this situation the system flows from weak coupling (\( J = 0 \)) to aNFL fixed point, corresponding to free BC in the Ising sector. At finite \( \{K - K_c\} \) the system flows to another fixed points as illustrated in Fig. (4). Depending on the sign of \( K - K_c \), those two states correspond to fixing the boundary spin in a semi-infinite Ising chain to point up or down. Note that whereas both the attrac-
and contribute to the ground state degeneracy only at
by the impurity spin states. The latter are decoupled
tive and

restricted by an energy scale

operators emerging from other sectors of the theory.

C. Energy correlator at finite boundary field

In a bulk CFT a typical local operator is a product of left and right moving factors \( \phi(x) = \phi_L(x)\phi_R(x) \) where we suppress the time variable. The Ising model has three primary bulk operators denoted \( \mathcal{O}_\Delta \), \( (\Delta = 1/2, 1/16, 0) \). In the presence of a boundary at \( x = 0 \) one can formulate the theory in terms of left moving fields only, \( \phi(x) = \phi_L(x)\phi_R(-x), \ x > 0 \). For example, \( \mathcal{O}_{1/2}(x) = \epsilon(x)\epsilon(-x) \). In particular, at \( \tau = 0, \ y = -x \) the correlator of the left moving Ising fields \( \epsilon \) at any \( h \) is related to the one-point function of the bulk energy operator of

\[ H_{\text{sing}} = \frac{1}{2} \int_{-\infty}^{\infty} dx \chi(x)i\partial_x \chi(x) + H_B, \quad H_B = h\sigma_B. \]  
At \( h = 0 \) this model corresponds to free BC, expressed by the continuity of the chiral Majorana fermion field \( \chi(x) \) at \( x = 0 \). \( h \) is an external magnetic field acting on the boundary spin \( \sigma_B \) only. Clearly \( h = \pm \infty \) implies fixed BC. The boundary spin can be written as

\[ \sigma_B = i\chi(x = 0)\sigma. \]  
Here \( \sigma \) is an additional Majorana fermionic boundary degree of freedom which anticommutes with \( \chi \) and satisfies \( \sigma^2 = 1/2 \).

The bulk energy operator of the Ising model corresponds to a mass term \( m \chi \bar{\chi} \), which is a product of a left- and a right-moving Majorana fields. Therefore the left moving factor of the energy operator, which is the field we refer to as the energy operator, is just the free Majorana fermion \( \epsilon(x) \sim \chi(x) \) with dimension \( \Delta = 1/2 \). Note that \( \chi \) was introduced most naturally within free BC, while \( \epsilon \) was introduced to represent free fermions at the fixed BC fixed point. Indeed, for free BC of the Ising model \( \chi(x) \) is continuous and \( \epsilon(x) \) undergoes a \( \pi/2 \) phase shift at \( x = 0 \) [see Eq. (3.3)]. Hence,

\[ \epsilon(x) = \text{sgn}(x)\chi(x). \]  

A. Boundary Ising model

It is well known that the scaling limit of the two dimensional classical Ising model at its bulk critical point is described by a free massless Majorana field theory. Here we consider the two dimensional model with a boundary, which is equivalent to the quantum semi-infinite chain. After unfolding the model in the standard fashion\(^{22}\) we obtain a left moving Majorana fermion on the infinite line,

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the boundary Ising model, \( \left\langle \epsilon(x,0)\epsilon(-x) \right\rangle_h = \langle \mathcal{O}_{1/2}(x) \rangle_h \). The one-point function of the bulk energy operator was calculated using the integrability of the boundary Ising model with the result \( \langle \mathcal{O}_{1/2}(x) \rangle_h = \int_{-\infty}^{\infty} du \frac{e^{2iux}}{2\pi} \frac{\hbar^2/2-u}{1+e^{\beta u}} \). (4.5)

Here \( \beta = T^{-1} \) is the inverse temperature. More generally consider the correlation function \( C_h(x,y,\tau) = \left\langle \epsilon(x,\tau)\epsilon(y,0) \right\rangle_h \). Consider a perturbative calculation of \( C_h(x,y,\tau) \) in \( H_B \). It can be shown \( 26,27,28 \) that (i) the correction vanishes for \( xy > 0 \), (ii) for \( xy < 0 \) the correction is a function of \( z = \tau + i(x - y) \). This implies that we can analytically continue the one-point function to find \( C_h(x,y,\tau) \),

\[
C_h(x,y,\tau) = \langle \mathcal{O}_{1/2}(x) \rangle_h |_{x \rightarrow -iz}, \quad x > 0, y < 0. \quad (4.6)
\]

For \( x < 0, y > 0 \) one can use \( C_h(x,y,\tau) = -C_h(y,x,-\tau) \), where the \(-\) sign arises from the fermionic nature of \( \epsilon \).

### C. Direct calculation of the energy correlator

For the present problem the desired correlator can be computed directly as will be done in this subsection. We turn to a calculation of the Majorana Green function (GF) \( G(\tau, x, y) = -\left\langle \chi(x, \tau)\chi(y, 0) \right\rangle \) at finite \( h \) and temperature \( T = \beta^{-1} \). From Eq. (4.3), the energy correlator is

\[
\left\langle \epsilon(x,\tau)\epsilon(y,0) \right\rangle_h = -G(\tau, x, y)\text{sgn}(xy). \quad (4.7)
\]

For \( h = 0 \), \( G(\tau, x, y) \) is a free fermion GF,

\[
G^{(0)}(\tau, x, y) = \frac{1}{2\pi} \frac{-\pi/\beta}{\sin(\pi(\tau + i(x-y)))} = \frac{1}{\beta} \sum_{n} e^{-i\omega_n \tau} \chi^{(0)}(i\omega_n, x, y) = \frac{i}{\beta} \sum_{n} e^{-i\omega_n (\tau + i(x-y))} \times \theta(-\omega_n)\theta(x-y) - \theta(\omega_n)\theta(y-x)],
\]

where \( \omega_n = \frac{\pi}{\beta}(1 + 2n) \). Since the interaction in Eq. (4.3) is quadratic in fermion fields, we may sum up the perturbation series in the boundary magnetic field exactly, giving

\[
G(i\omega_n, x, y) = G^{(0)}(i\omega_n, x, y) + \hbar^2 G^{(0)}(i\omega_n, x, 0)G_0(i\omega_n)G^{(0)}(i\omega_n, 0, y). \quad (4.8)
\]

Here \( G_0(\omega_n) = -\int_0^\beta d\tau e^{i\omega_n \tau} (a(\tau)a) \) is the \( a \) propagator. When \( a \) is decoupled, its propagator is given by \( g^{(0)}(\omega_n) = (i\omega_n)^{-1} \). Eq. (4.8) becomes exact when \( G_0(\omega_n) \) is calculated to infinite order in \( h \). This is accomplished by the self energy \( \Sigma_0(\omega_n) = \hbar^2 G^{(0)}(i\omega_n, 0, 0) = -i\hbar^2\text{sgn}(\omega_n)/2 \). Thus

\[
G_0(i\omega_n) = (i\omega_n + i\hbar^2\text{sgn}(\omega_n))/2 \quad (4.9)
\]

Plugging this result in Eq. (4.3) yields the result

\[
G(i\omega_n, x, y) = G^{(0)}(i\omega_n, x, y) + ie^{\omega_n(x-y)} \times \sum_{s=\pm 1} \theta(s\omega_n)\theta(sy)\theta(-sx) \frac{\hbar^2}{\omega_n + h^2\text{sgn}(\omega_n)/2}. \quad (4.10)
\]

When \( xy > 0 \) there is no dependence on \( h \). To compare Eq. (4.10) in the nontrivial region \( xy < 0 \) with the result obtained by analytic continuation of the one point function of the energy operator, Eq. (4.5), we write the Fourier transform of Eq. (4.10) into

\[
G(\tau, x, y) = \int_{-\infty}^{\infty} du \frac{e^{i\beta(x-y)}}{2\pi} \frac{i\hbar^2\text{sgn}(x-y)/2 - u}{1 + e^{\beta u}} \frac{i\hbar^2\text{sgn}(x-y)/2 + u}. \quad (4.11)
\]

valid for \( x \cdot y < 0 \). One arrives at the same result using Eqs. (4.5), (4.6), and (4.7). In this notation the integration variable \( u \) is related to the momentum of the particles used in the form factors method.

### D. Finite temperature conductance

At finite temperature the conductance is obtained by analytic continuation

\[
G = \lim_{L \rightarrow \infty} \lim_{\omega \rightarrow 0} \frac{ie^2}{\hbar\omega(2L)^2} \int_{-L}^{L} dx \int_{-L}^{L} dy \text{sgn}(xy) \times \int_{-\beta/2}^{\beta/2} d\tau e^{-i\omega_n \tau} \left( j_o(x, \tau)j_o(y, 0) \right)_{i\omega_n \rightarrow -i\omega_n + i\nu_n}. \quad (4.11)
\]

where \( \nu_n = \frac{2\pi n}{\beta} \). For \( h = 0 \) the finite temperature odd current correlator, \( \langle j_o(x, \tau)j_o(y, 0) \rangle_{\text{free}} \) is given by Eq. (4.3) where \( \langle j_o(x, \tau)j_o(y, 0) \rangle_{j_o} = \frac{1}{\beta^2 \sin^2(\pi(x-y))} \). At finite \( h \) we use Eq. (4.3) and

\[
\langle (j^1)^2(x, \tau)(j^1)^2(y) \rangle = 0, \quad (4.11)
\]

Using the Bose Ising representation of the flavor currents, given in Table I and Eq. (4.7), we obtain

\[
\langle j_o(x, \tau)j_o(y, 0) \rangle_h = -4\cos^2 \phi_m (G^{(0)}(\tau, x, y))^2 + 4\sin^2 \phi_m (j^1(x, \tau)^2(y) + (j^1)^2(x, \tau)^2(y)).
\]

Compared to free BC, the odd current correlator obtains an additional term

\[
\langle j_o(x, \tau)j_o(y, 0) \rangle_h = \langle j_o(x, \tau)j_o(y, 0) \rangle_{\text{free}} - 2\sin^2 \phi_m \frac{G^{(0)}(\tau, x, y) - G^{(0)}(\tau, x, y)}{\beta \sin(\pi(x-y))}. \quad (4.12)
\]

The first term contributes \( 2\sin^2 \phi_m \) to the conductance. The second term is nonvanishing only for \( xy < 0 \),
as can be seen from Eq. (11). Note that $G(t, x, y) - G(0)(t, x, y) \to_{t, x, y} \to_{t, x, y} -2G(0)(t, x, y)\delta(-xy)$, hence

$$\langle j_0(x, \tau)j_0(y, 0) \rangle_{\omega = \infty} = \langle j_0(x, \tau)j_0(y, 0) \rangle_{\text{fixed}},$$

as expected. At finite $T$ and $h$ the contribution of the second term to the conductance is given by $2ReG_1$ where

$$G_1 = \lim_{\omega \to 0} \lim_{L \to \infty} \frac{-ie^2}{h\omega(2L)^2} \int_0^L dx \int_0^L dy \times \langle j_0(x)j_0(y) \rangle_{\omega = \omega + i0^+},$$

where $\langle j_0(x)j_0(y) \rangle_{\omega = \omega + i0^+} = \langle j_0(x)j_0(y) \rangle_h - \langle j_0(x)j_0(y) \rangle_{\text{free}}$. This correlator can be expressed as a Matsubara sum

$$\langle j_0(x < 0)j_0(y > 0) \rangle_{\nu_n}^{(1)} = \frac{4\sin^2\phi_m}{\pi^2} \int_{-\beta/2}^{\beta/2} d\tau e^{i\nu_n \tau} \times \sum_{m, l} \theta(\omega_m)\theta(\omega_l) e^{-i(\omega_m + \omega_l)(\tau + i(y-x))} \frac{ih^2}{\omega_l + i\hbar^2/2} \frac{\sin^2\phi_m(2\pi)^2}{\beta \nu_n(y-x)^2} \sum_{l=0}^{n-1} \frac{\beta \hbar^2}{2\pi(1+2l) + \beta \hbar^2} \sum_{l=0}^{n-1} \frac{4\pi}{2\pi(1+2l) + \beta \hbar^2} = \psi(z) = d\log \Gamma(z)/dz,$$

Performing the analytic continuation $i\nu_n \to \omega + i0^+$, sending $\omega \to 0$, and performing the spatial integrations we obtain

$$G/G_0 = 1 - F[T/T^*], \quad F[t] = \frac{1}{4\pi t} Re \psi_1 \left( \frac{1}{2} + \frac{1}{4\pi t} \right), \quad \psi_1(z) = d^2 \log \Gamma(z)/dz^2,$$

where $\psi_1(z) = d^2 \log \Gamma(z)/dz^2$ is the trigamma function and $G_0$ and $T^*$ are given in Eqs. (3.9) and (4.10). The scaling function $F[t]$ has the properties $F[0] = 1$ and $F[\infty] = 0$. A signature of a NFL is the existence of relevant operators in the Hamiltonian with scaling dimension $\Delta < 1$. The QD setup discussed here allows to observe that the inter-impurity interaction is such a relevant perturbation with $\Delta = 1/2$. According to Eq. (4.13) the crossover from $G \sim G_0$ to insulating FL state as function of $K - K_c$ occurs at a value of $|K - K_c|$ which scales with temperature as $T^{1/2}$.

E. Conductance in the model of Zaránd et. al. 17

We pause here to comment on an application of the Ising model with boundary magnetic field for a different double QD model proposed by Zaránd et. al. 17 as a realization of the 2IKM. We will show that in this system the full crossover of the conductance as function of $K$ in the P-H symmetric point can be expressed in terms of the one point function of the spin operator of the boundary Ising model.

Consider a modified QD system with an additional lead B coupled only to $S_2$ as in Fig. (5). Transport takes place between the left and right leads, where lead B acts as a screening channel for $S_2$. The analysis of Sec. (III) goes through, and the conductance is given by Eq. (3.1), where the odd current is still written as

$$j_0(x) = \psi_0(z)^x \psi_{1, j^x} + \psi_{1, j^x} \psi_0(z)^x, \quad j = 1, 2 = L, R.$$ The next step in Sec. (III) was to rewrite $j_0(x)$ in the basis $\psi'$ which is natural in the representation of the 2IKM Hamiltonian. For the present system this basis is

$$\psi_1' = \frac{t_{1L} \psi_L + t_{1R} \psi_R}{\sqrt{t_{1L}^2 + t_{1R}^2}}, \quad \psi_2' = \psi_B.$$ A third fermion $\psi_3' = \frac{-t_{1L} \psi_L + t_{1R} \psi_R}{\sqrt{t_{1L}^2 + t_{1R}^2}}$ is decoupled from the impurities. We specialize to $t_{1L} = t_{1R}$. In this basis the correlator $\langle j_0(x, \tau)j_0(y, 0) \rangle$ occurring in the Kubo formula factorizes into the product of GFs for $\psi_1'$ and $\psi_2'$, where the latter is a free fermion GF. For the GF of $\psi_1'$ we use the bosonization formula Eq. (3.9), where the only factor which is sensitive to the critical point is the Ising spin operator, leading to

$$\langle j_0(x, \tau)j_0(y, 0) \rangle = \frac{1}{\pi^2} \langle \sigma(x, \tau)\sigma(y, 0) \rangle,$$ For finite $h$ and $T$ Eqs. (4.11) and (4.14) express the conductance in terms of the 2-point function for the chiral spin operator at finite magnetic field $h$. Following the analysis leading to Eq. (4.10), the 2-point function for the chiral spin operator is related to the one point function of the bulk spin operator by analytic continuation,

$$\langle \sigma(x, \tau)\sigma(y, 0) \rangle = \langle \sigma_1(x)\sigma_0(y) \rangle_h, \quad x > 0, y < 0.$$ The calculation of the one point function of the Ising spin at finite magnetic field $h$ was addressed using integrability and the form factors method. Different than the case of the energy operator, a closed expression for $\langle \sigma_1(x)\sigma_0(y) \rangle_h$ is not available. In the limiting cases $h = 0$
and \( h = \pm \infty \) CFT methods can be used. For BC obtained by fusion with operator \( a \), Eq. (3.35) gives \( \Delta = 1/16 \)

\[
\langle \sigma(x, \tau) \sigma(y, 0) \rangle = \begin{cases} 
1/(t + u(x - y)/x^2) & \text{if } x > 0 \\
1/(t + u(x - y)/x^2) & \text{if } x < 0 
\end{cases}
\]  
(4.15)

where \( S(a) = S^{1/16} / S^{1/16} \). It is easy to calculate the conductance with Eq. (4.15), with the result \( G = \frac{2}{\pi} (1 - S(a)) \). At weak coupling \( J = 0 (a = 0) \) we have \( S(0) = 1, G = 0 \). This is also the result for the BC obtained by starting at the QCP and setting \( k = 0, G = \epsilon^2/h \). In the Kondo screened phase \( (a = 1/2) \) we have \( S(1/2) = -1, G = \epsilon^2/h \). We leave for a future work to apply Eq. (4.15) in order to interpolate between those values of \( G \) at finite temperature and \( h \propto (K - K_c) \). The additional difficulty for this system arises due to the presence of the \( \sigma \) GF rather than the \( \epsilon \) GF.

V. UNIVERSAL CROSSOVER AT FINITE POTENTIAL SCATTERING

Until here we assumed P-H symmetry and emphasized that the crossover is in the universality class of the boundary Ising model. Now we shall consider the more general situation with potential scattering (PS). We will see that the Ising and charge \( SU(2)_1 \) sectors of the theory are coupled. However this coupling can be written in a simple quadratic form in the Majorana \( SO(8) \) representation that will be introduced below.

It is convenient to write \( H_{PS} \), defined in Eq. (2.2), in the \( \psi' \) basis [defined in Eq. (2.12)],

\[
H_{PS} = \frac{V_L}{2} + \frac{V_R}{2} (\psi'^r 1 \psi'^r 1 + \psi'^r 2 \psi'^r 2) + \text{Re}V_R \psi'^r 1 \psi'^r 2 + \psi'^r 2 \psi'^r 1 + V_A (\psi'^r 1 \psi'^r 1 - \psi'^r 2 \psi'^r 2) + V_B (\psi'^r 1 \psi'^r 2 - \psi'^r 2 \psi'^r 1),
\]  
(5.1)

where

\[
V_A = \frac{V_L - V_R}{2} \cos \phi_m - \text{Im} V_L \sin \phi_m,
\]

\[
V_B = \frac{V_L - V_R}{2} \sin \phi_m + \text{Im} V_L \cos \phi_m.
\]  
(5.2)

In the parity symmetric case \( V_A = V_B = 0 \).

At the QCP the PS terms describing charge transfer between channels \( \psi'^r 1 \) and \( \psi'^r 2 \) generate relevant perturbations. To see this consider their Bose-Ising representation (using Table II)

\[
\psi'^r 1 \psi'^r 2 + h.c. \sim (h_1)^r \tau^z (h_2) \epsilon, \\
\text{i} \psi'^r 1 \psi'^r 2 + h.c. \sim (h_1)^r (h_2) \epsilon.
\]  
(5.3)

At the non-trivial fixed point the energy operator \( \epsilon \) “disappears” by double fusion; hence one obtains two relevant boundary operators \((h_1)^r \tau^z (h_2) \) and \((h_1)^r(h_2) \), with dimension \( \Delta = 1/2 \). In the parity symmetric case only the first operator is allowed. These relevant operators have the dimension of a free fermion. Following Gan, a fermion representation emerges naturally in the \( SO(8) \) representation that we shall introduce in the next subsection. In order for these relevant operators to have bosonic statistics, in the \( SO(8) \) representation indeed they are written as a product of a bulk fermion with a local fermion with dimension \( \Delta = 0 \), which can be associated with a leftover impurity degree of freedom.

On the other hand the intra-channel PS terms lead to marginal operators at the QCP,

\[
\psi'^r 1 \psi'^r 2 \sim I_1 + I_2.
\]  
(5.4)

A. Fixed point Hamiltonian in \( SO(8) \) representation

Following Ref. [29] we bosonize the original theory and introduce four left moving bosonic fields: \( \psi'^r 1 \sim 1 \psi'^r 2 \sim \frac{1}{2 \pi} \partial_x \phi_{ja} \). In terms of the bosons we can write the fermions as \( \psi'^r 1 \sim F_{ja} e^{-i \phi_{ja}} \). The Klein factors \( F_{ja} \) take care of our sign convention required for products of exponentials of bosonic fields. They satisfy

\[
[F_{\mu}, N_{\nu}] = \delta_{\mu \nu} F_{\mu}, \quad \{F_{\mu}, F_{\nu}^r\} = 2 \delta_{\mu \nu}, \quad (F_{\mu} F_{\mu}^r = F_{\mu}^r F_{\mu} = 1), \quad \{F_{\mu}, F_{\nu}\} = 0,
\]  
(5.5)

and \( [F_{\mu}, \phi_{\nu}] = 0 \), where \( \mu, \nu = \{i, \alpha \} \) and \( N_{\mu} \) is the fermion number of species \( \mu \).

Subsequently 4 linear bosonic combinations are defined, corresponding to charge, spin, flavor, and difference of spin between the flavors,

\[
\phi_c = \frac{1}{2} \sum_{ja} \phi_{ja}, \quad \phi_s = \frac{1}{2} \sum_{ja} (\tau^z)^{ja} \phi_{ja}, \quad \phi_f = \frac{1}{2} \sum_{ja} (\tau^z)^{ja} \phi_{ja}, \quad \phi_X = \frac{1}{2} \sum_{ja} \tau^z \phi_{ja}.
\]

Since the exponents of these new bosons have dimension 1/2, we define new fermions \( \psi_A \sim F_A e^{-i \phi_A} \), \( A = c, s, f, X \). The new Klein factors satisfy Eq. (5.5) with \( \mu, \nu = c, s, f, X \). To fix a convention we define

\[
F_X^r F_{1}^r = F_{1}^r F_X^r, \quad F_X^r F_{2}^r = F_{2}^r F_{2}^r, \quad F_X^r F_{2}^r = F_{1}^r F_{2}^r.
\]

The free part of the Hamiltonian can be written equivalently in bosonic or fermionic form,

\[
H_0 = \sum_A \int dx (\partial_x \phi_A)^2 = \sum_A \int dx \psi'^r_A (i \partial_x) \psi_A.
\]

Taking the real and imaginary parts of those fermions we obtain 8 Majorana fermions

\[
\chi_A = \frac{\psi'^r_A + \psi_A}{\sqrt{2}}, \quad \chi_A = \frac{\psi'^r_A - \psi_A}{\sqrt{2} i}.
\]
One can establish a connection between the description of the 2IKM in terms of $SU(2)_{1}^{\text{charge 1}} \times SU(2)_{1}^{\text{charge 2}} \times SU(2)_{2}^{\text{spin}} \times Z_2$ with 8 Majorana fermions. The two $SU(2)_{1}$ groups can be represented in terms of two bosons $\phi_1^1, \phi_2^0$. The $SU(2)_{2}^{\text{spin}}$ current $\tilde{j}^s = \frac{1}{2} \psi_i^{\dagger} \tau^z_\beta \psi_{i\beta}$ has the representation $(j^s)^{x} = \psi_{x}^{\dagger} \psi_{x}$, $(j^s)^{+} = \sqrt{2} \chi^X \psi_{x}^{\dagger}$. Of particular interest for the present work, the flavor current Eq. (5.2) has the representation (see Table I)

\[(j_f)^{x} = \psi_{f}^{\dagger} \psi_{f}, \quad (j_f)^{+} = -\sqrt{2} i \psi_{f} \chi^X. \tag{5.6}\]

The Ising fermion $\chi$ can be identified with $\chi^X$. In fact the nontrivial BC involves only one out of the 8 Majorana fermions, reading $\chi^X(0^{-}) = -\chi^X(0^+)$. For a description of the physics relative to the nontrivial fixed point it is convenient to work with the continuous Ising fermion field

\[\chi(x) = \text{sgn}(x)\chi^X(x) = \epsilon(x)\text{sgn}(x). \tag{5.7}\]

Using Eq. (5.3), in the P-H symmetric case the relevant operator can be written as

\[\sigma_B = i\chi(x = 0)a = i(\text{sgn}(x)\chi^X(x))_{x=0} \cdot a. \tag{5.8}\]

Now consider the non P-H symmetric case. From the SO(8) representation of the flavor current Eq. (5.5), the two PS terms in Eq. (5.3), $(j^x)^{x,u}$, are written in the trivial fixed point as $i\chi^X \chi_{1,2}^X$. CFT methods tell us that the operators at the QCP are obtained from the operators at the trivial fixed point by double fusion with the spin operator of the Ising model. Having identified $\chi^X$ with the Ising fermion, double fusion gives $\chi^X \rightarrow 1 + \chi^X$. To obtain the correct bosonic statistics we argue that this fusion rule should be modified to

\[\chi^X \rightarrow a + \chi^X, \]

where $a$ is the local fermion appearing in Eq. (5.8). Hence the relevant PS operators at the QCP are

\[(h_1)^{\dagger} \tau^z (h_2) \sim i\chi^X_1 a, \tag{5.9}\]

\[(h_1)^{\dagger} (h_2) \sim i\chi^X_2 a. \]

Thus, $a$ couples the Ising sector with the charge sectors. The main argument in favor of this form is obtained by considering the self-correlation function of the relevant operators, e.g.,

\[\langle (h_1^\dagger h_2)(\tau)(h_1^\dagger h_2) \rangle \sim 2^{0}(\tau, 0, 0) \mathcal{G}_a(\tau), \]

at the P-H symmetric point. Fourier transforming Eq. (5.4) for $\mathcal{G}_a(\omega_n)$ we can deduce the behavior of $\mathcal{G}_a(\tau)$: in the limit $\tau \ll \hbar^2(\tau \gg \hbar^2)$, the correlator $\mathcal{G}_a(\tau)$ goes like $\tau^0(\tau^{-1})$. This implies that in these two limits the correlator $\langle (h_1^\dagger h_2)(\tau)(h_1^\dagger h_2) \rangle$ goes like $\tau^{-1}(\tau^{-2})$, respectively, as expected from an operator with scaling dimension $\Delta = \frac{3}{2}(1)$. This scaling behavior is obtained relying on the fact that $a$ contains the information about the crossover. It explains why $a$, and not some other decoupled local operator, should be coupled to $\chi^X$ in Eq. (5.1). On the contrary, the presence of an additional decoupled local operator at the QCP is ruled out as inconsistent with the ground state degeneracy. Away from the P-H symmetric point, the local operator $a$ becomes also sensitive to the deviation from the QCP due to potential scattering and $\mathcal{G}_a$ is modified relative to Eq. (4.9).

Putting together Eqs. (5.8) and (5.9), the correction to the fixed point Hamiltonian in SO(8) representation is

\[\delta H = i \left( \lambda_1 \chi^X_{2}(x) \text{sgn}(x) + \lambda_2 \chi^X_{1}(x) + \lambda_3 \chi^X_{2}(x) \right) a \bigg|_{x=0}, \tag{5.10}\]

with

\[\lambda_1 = c_1 \frac{K - K_e}{\sqrt{T_K}}, \quad \lambda_2, \lambda_3 = c_2 \sqrt{T_K} \nu(\text{Re}V_{R}^{e}, V_{B}), \tag{5.11}\]

where $V_{R}$ and $V_{B}$ are given in Eqs. (5.8) and (5.9), and $c_1$ and $c_2$ are constants of $O(1)$. This estimate of $\lambda_2, \lambda_3$ will be justified below; as we shall see, based on the dimension $\Delta = 1/2$ of the three relevant operators in Eq. (5.10) we obtain the crossover energy scale

\[T^* = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \equiv \lambda^2. \tag{5.12}\]

To estimate $\lambda_2$ and $\lambda_3$ we consider the renormalization group flow of the inter-channel potential scattering operators $\psi^{\dagger \alpha} \psi^{\dagger \alpha} \pm h.c.$). In the presence of these operators the flow to the QCP stops at energy scale $T^{*}_{LR}$. To estimate $T^{*}_{LR}$ we consider the renormalization of these operators in the perturbative regime at energy scales $D \gg T_K$ and then in the nonperturbative regime at energy scales $D \ll T_K$, respectively. (A related calculation for the 2 channel Kondo model appears in [31]). We assume that $K = K_e$. At the initial scale $D_0 \gg T_K$ the dimensionless bare value of these PS operators are $\Delta_0 = \nu(\text{Re}V_{R}^{e})$ and $\Delta_0' = \nu(\text{Re}V_{B})$; see Eq. (5.1). We assume $\Delta_0, \Delta_0' \ll 1$. Since in the weak coupling regime potential scattering does not renormalize, we have

\[\Delta(T_K) \sim \Delta_0, \quad \Delta^*(T_K) \sim \Delta_0'. \]

These can be viewed as the initial values of the coupling constants of the relevant perturbations $(h_1)^{\dagger} \tau^z (h_2)$ and $(h_1)^{\dagger} (h_2)$, respectively. Since these operators have dimension $1/2$, the dependence of their coupling constants on $D \ll T_K$ is described by

\[\frac{\Delta(D)}{\Delta(T_K)} \sim \frac{\Delta^*(D)}{\Delta^*(T_K)} \sim \left( \frac{T_K}{D} \right)^{1/2}. \]

The condition max{$\Delta(T^{*}_{LR}), \Delta^*(T^{*}_{PS})$} \sim 1 gives the estimate

\[T^{*}_{LR} \sim \max(T_K \Delta_0^2, T_K (\Delta_0')^2). \tag{5.13}\]
A more precise estimate would take into account higher order terms in the \( \beta \)-function for \( \Delta, \Delta' \). However, we expect that this would only change our estimate of \( T_{LR}^* \) by logarithmic factors.

Identifying \( T_{LR}^* \) with \( \lambda_2^2 + \lambda_3^2 \) in Eq. (5.12) gives the estimate for \( \lambda_2 \) and \( \lambda_3 \) given in Eq. (5.11). Under the condition \( \Delta_0, \Delta'_0 \ll 1 \) one has a wide energy range \( T_{LR}^* \ll D \ll T_K \) for the observation of the QCP. This can occur in a certain parameters regime, as we discuss in Sec. (VI A).

We point out that our estimate for the energy scale \( T_{LR}^* \), Eq. (5.13), which agrees with (17), is inconsistent with that of Sakai and Shimizu, who studied the 2IKM with finite transfer matrix between the impurities using numerical renormalization group.20 This discrepancy requires further investigation.

B. Linear conductance with potential scattering

We generalize the linear conductance calculation of Sec. (IV) for finite potential scattering. Using Eqs. (3.3) and (5.5) the odd current operator is

\[
j_o = 2i \chi_2 \left( \cos \phi_m \chi_1^f + \sin \phi_m \chi_2^X \text{sgn}(x) \right).
\]  

(5.14)

The operator \( a \) is now coupled to three free Majorana fields, and its GF, Eq. (4.10) generalizes to \( G_a(i\omega_n) = (i\omega_n + \lambda^2 \text{sgn}(\omega_n))/2 \), where \( \lambda^2 \) is defined in Eq. (5.12). Similarly,

\[
-\langle \chi_i(x) \chi_j(y) \rangle_{i\omega_n} = G^{(0)}(i\omega_n, x, y) \delta_{ij} + h_i h_j \delta G(i\omega_n, x, y),
\]

where

\[
\delta G(i\omega_n, x, y) = G^{(0)}(i\omega_n, x, 0) G_a(i\omega_n) G^{(0)}(i\omega_n, 0, y).
\]

Generalizing Eq. (4.12) we obtain the odd current correlator

\[
\langle j_o(x, \tau) j_o(y, 0) \rangle_{\lambda_1, \lambda_2, \lambda_3} = \langle j_o(x, \tau) j_o(y, 0) \rangle_{\text{free}} + (\sin^2 \phi_m (\lambda_1^2 + \lambda_2^2) - \cos^2 \phi_m (\lambda_2^2 + \lambda_3^2)) \times 4 G^{(0)}(i\omega_n, x, y) \delta G(i\omega_n, x, y).
\]

As a result the conductance has the scaling form

\[
G/G_0 = 1 - F[T/T^*] \sin^2 \phi_m (\lambda_1^2 + \lambda_2^2) - \cos^2 \phi_m (\lambda_2^2 + \lambda_3^2) \frac{\lambda_2^2}{\lambda^2 \sin^2 \phi_m}.
\]

(5.15)

We see that the conductance at the free fixed point (\( \lambda = 0 \)) is still given by \( G_0 = 2 \frac{e^2}{h} \sin^2 \phi_m \). At \( \lambda \to \infty \) the Fermi liquid conductance is

\[
G_{FL} = \frac{2 e^2}{h} (\lambda_3^2 + \cos^2 \phi_m \lambda_3^2) / \lambda^2.
\]

(5.16)

We may rewrite Eq. (5.15) as

\[
\frac{G - G_{FL}}{G_0 - G_{FL}} = 1 - F[T/T^*].
\]

C. Gan’s theory and its relation to boundary Ising model

Gan presented a solution of the 2IKM, constructing an effective Hamiltonian for a finite region in the phase diagram around the critical point by controlled projection.19 The effective Hamiltonian is solved exactly not only at the critical point but also for the surrounding Fermi-liquid phase. Excellent agreement was found with numerical renormalization group and CFT, in spite of the fact that the theory of Gan is not spin-SU(2) invariant. We shall substantiate the relation of Gan’s theory to the CFT by showing explicitly that the operators at the critical point have the same form for both theories.

In the next section we will use this approach to calculate the nonlinear conductance.

Gan theory uses the SO(8) representation, and the two impurity spins turn into a local fermion \( d \), where \( \{d, d^\dagger\} = 1 \). Defining two Majorana fermions \( a = d - d^\dagger/\sqrt{2} \) and \( b = \frac{d + d^\dagger}{\sqrt{2}} \), Gan’s Hamiltonian in the P-H symmetric case involves only the spin-flavor (\( X \)) sector, and can be written as \( H_G = H_G^{(0)} + \delta H_G \) where

\[
H_G^{(0)} = \frac{1}{2} \int dx \chi_2^X \partial_x \chi_2^X,
\]

\[
\delta H_G = 2i \sqrt{T_K} \chi_2^X (0) b - i(K - K_c) ab.
\]

(5.17)

We shall show that for energy scales \( \ll T_K \) this coincides with the Ising model Eq. (17). To see this suppose \( K = K_c \) and consider a mode expansion

\[
\chi_2^X(x) = \sum_k (\phi_k(x) \psi_k + h.c.), \quad b = \sum_k (u_k \psi_k + h.c.),
\]

where \( \{\psi_k, \psi_k^\dagger\} = \delta(k - k'), \{\psi_k, \psi_k\psi_k\} = 0 \), and where initially we choose \( \Lambda \gg T_K \) as an ultraviolet cutoff. In the basis of \( \psi_k \) Gan’s Hamiltonian is equal to \( H = \sum_k \epsilon_k \psi_k^\dagger \psi_k \). One can obtain a Schrödinger’s equation for the wave functions \( \varphi_k(x) \) and \( u_k \) by equating the expansions of \( [H_G, \chi_2^X(x)] = [H, \chi_2^X(x)] \) and \( [H, b] = [H, b] \).

One obtains

\[
2i \sqrt{T_K} \delta(x) u_k + i \partial_x \varphi_k(x) = \epsilon_k \varphi_k(x),
\]

\[
-2i \sqrt{T_K} \varphi_k(0) = \epsilon_k u_k.
\]

The solution is \( \varphi_k(x) \propto e^{i k x} [\theta(x) \varphi_k^{(+)}(x) + \theta(-x) \varphi_k^{(-)}(x)] \), \( \varphi_k(0) = \frac{1}{2} (\varphi_k^{(+)}(0) + \varphi_k^{(-)}(0)) \), \( u_k = \frac{1}{2 \epsilon_k} \sqrt{T_K} \varphi_k(0), \)

\( \varphi_k^{(+)}(0) / \varphi_k^{(-)}(0) = e^{-2i \delta}, \tan \delta = \frac{T_K}{\epsilon_k}, \epsilon_k = -k \) (note that we work with left movers). While at \( T_K = 0 \) we have the BC \( \chi_2^X(0^+) = \chi_2^X(0^-) \), we see from the wave function that the effect of the first boundary term in \( H_G \) is to modify this BC to \( \chi_2^X(0^+) = -\chi_2^X(0^-) \) for energies \( \ll T_K \). The key observation is that the following operator identity holds if one restricts the mode expansion of its LHS and RHS to energies below a cutoff \( \Lambda \ll T_K \),

\[
b = \frac{1}{\sqrt{T_K}} \chi_1(0),
\]

(5.18)
where $\chi_1(x) = \chi^X(x) \text{sgn}(x)$. Physically this means that at energy scales below $T_K$ the local operator $b$ is absorbed into the field $\chi^X$ and changes its BC. Using the operator identity Eq. (5.18), we see that the term $\propto K - K_c$ in $\delta H_G$ is equivalent to the boundary operator in the Ising model, Eq. (5.8).

This establishes the connection between Gan’s theory and the boundary Ising model arising from the CFT solution, showing that Gan’s anisotropic theory describes correctly also the vicinity of the isotropic fixed point.

VI. CROSSOVER AT FINITE BIAS

Gan’s formulation of the QCP in the SO(8) Majorana representation provides a direct way to calculate the nonlinear conductance at finite source drain voltage along the crossover from the NFL fixed point to the surrounding FL fixed points, including the P-H symmetry breaking. Relegating the details of the calculation based on the Keldysh technique to the appendix, our result is

$$G = G_0 + G_S F\left[\frac{T}{T^*}, \frac{eV}{T^*}\right] + G_A F'\left[\frac{T}{T^*}, \frac{eV}{T^*}\right],$$

$$F[t, v] = \frac{1}{4\pi} \text{Re} \psi_1 \left(\frac{1}{2} + \frac{1}{4\pi t} + \frac{i v}{2\pi t}\right),$$

$$F'[t, v] = \frac{1}{4\pi} \text{Im} \psi_1 \left(\frac{1}{2} + \frac{1}{4\pi t} + \frac{i v}{2\pi t}\right),$$

$$\frac{G_S}{2\pi \hbar} = -\lambda_2^2 \sin^2 \phi_m + \lambda_3^2 \cos^2 \phi_m + \lambda_2^2 (1 - 2 \sin^2 \phi_m),$$

$$\frac{G_A}{2\pi \hbar} = \sin(2\phi_m) \lambda_3 \sin \phi_m + \lambda_1 \cos \phi_m. \quad (6.1)$$

Here $T^*, G_0, \phi_m$ and $\theta$ are given in Eqs. (6.12), (6.9), (6.11) and (6.7); $\psi_1(z)$ is defined below Eq. (6.13). This result is valid for $eV, T, T^* \ll T_K$. When $T^* \gg T, eV$ the system is in the FL state and the nonlinear conductance coincides with the linear conductance, Eq. (6.10), $G_{FL} = G_0 + G_S$.

The scaling functions $F[t, v]$ and $F'[t, v]$ are symmetric and asymmetric in $v$, respectively; see Figs. (6) and (7). The asymmetric component is a new feature in the parallel QD, as compared to series QD where $\phi_m = 0$. Having $G[V] \neq G[-V]$ is a signature of interactions, since the Landauer noninteracting formula leads to $G[V] = G[-V]$. This leads to a universal rectification effect. This rectification effect is odd under parity, $\lambda_3 \rightarrow -\lambda_3$. Note however that it does not have a well defined transformation property with respect to $\Phi \rightarrow -\Phi$.

To check the symmetry properties of our results we considered the two impurity Anderson model for our model [Fig. (4)] to first order in the (intra-dot) interaction $U$. While at $U = 0$ we have $G[V] = G[-V]$, which follows from Landauer formula, to first order in $U$ we get a finite $G[V] - G[-V]$. This asymmetric behavior of the conductance follows from an asymmetric dependence of the occupation of the dots on voltage. This simple limit gives the same symmetry properties of $G[V] - G[-V]$ compared to the QCP, namely the rectification effect is odd under a parity, and does not have a well defined symmetry property with respect to $\Phi \rightarrow -\Phi$.

At energy scales comparable to $T_K$ the conductance has additional voltage and temperature dependence due to irrelevant operators at the QCP. The leading irrelevant operator is $H_{ir} = T_K^{-1/2} i \partial \chi(x) a |_{x=0}$, with dimension $\Delta = 3/2$. In the proposed realization of the 2IKM it leads to the conductance correction $\delta G \propto \sqrt{T_K}$, characteristic of a NFL fixed point. However in the present system the irrelevant operator gives a nonzero correction only to fourth order, leading to $\delta G \propto \left(\frac{T}{T_K}\right)^2$, as we outline below. The a-GF has an additional self energy $\Sigma^R = -i\omega^2/T_K$, and

$$G^R_a(\omega) = \frac{1}{\omega + i T^*/2} - \frac{1}{\omega + i (T^*/2 + \omega^2/T_K)}. \quad (6.12)$$

This has poles at $\omega = -i T_K\frac{1}{2}(1 \pm \sqrt{1 - \frac{4T^*}{T_K}})$. For $T^* \ll T_K$ we have

$$G^R_a(\omega) \approx \frac{1}{\omega + i T^*/2} - \frac{1}{\omega + i T_K}. \quad (6.13)$$

Qualitatively, the irrelevant correction at finite $T_K$ has the same form of the fixed point conductance with $T^*/2 \rightarrow T_K$. Indeed at energy scales smaller than $T^*$, the
latter has quadratic dependence on $T/T^*$ and $eV/T^*$, It should be pointed out that to fourth order in $H_{ex}$ it is no longer consistent to disregard more irrelevant operators of dimension $\Delta = 2$. However their inclusion leads only to the modification of the effective Kondo temperature in the corrections $(T/T_K)^2$ and $(eV/T_K)^2$.

In Figs. (8) and (9) the conductance is plotted in the parity-symmetric case at $\lambda_3 = 0$ and zero temperature as function of source drain voltage, for different ratios $\lambda_2/\lambda_1$. The generic behavior of $G[V]$ consists of a wide peak of width $T_K$ and height $G_0$, with a superimposed narrow structure (peak or dip) of width $T^*$, with height $G_S$ (relative to the background $G_0$). Note that $G_S$ is positive (negative) for $\lambda^2 \sin^2 \phi_m < (>) \lambda^2 \cos^2 \phi_m + \lambda^2 (1 - 2 \sin^2 \phi_m)$, leading to a narrow peak (dip). When $\lambda_3 = 0$ and $\lambda_1 \tan \phi_m \ll \lambda_2$, Eq. (6.1) predicts a peak amplitude close to the unitary limit $2e^2/h$. For this case, we mention that when $T^*_{LR}$ and $|K - K_1| \gtrsim T_K$, our results do not apply, and we expect a splitting of this peak as a function of $V$. We obtain this behavior on a qualitative level by going back to the high energy $E \sim T_K$ description with Eq. (5.17).

FIG. 8: Nonlinear conductance at $\lambda_3 = 0$ for $\lambda_2/\lambda_1 = 0, 1/4, 1/2$. The line shape consists of a narrow peak or dip structure of width $T^*$, superimposed on top of a wide peak of width $T_K$.

FIG. 9: Nonlinear conductance at $\lambda_3 = 0$ for $\lambda_2/\lambda_1 = 0, 1, 5$, reaching the unitary limit when the relevant perturbation is dominated by potential scattering, namely $\lambda_2 \gg \lambda_1$.

In Figs. (10) and (11) we plot the conductance under the same conditions except $\lambda_3 = 1/\sqrt{10}$ and $\lambda_3 = 1/\sqrt{2}$, respectively, showing asymmetric behavior. When $G_A > 0$ ($G_A < 0$) [defined in Eq. (6.1)], the slope of the conductance at $V = 0$, $\frac{d G}{d V}|_{V=0}$, is negative (positive). The sign of $G_A$ is changed under a parity transformation ($\lambda_3 \rightarrow -\lambda_3$), but it also depends on the sign of the combination $(\sin(2\phi_m)/(\lambda_2 \sin \phi_m + \lambda_1 \cos \phi_m))$.

FIG. 10: Nonlinear conductance at finite $\lambda_3 = 1/\sqrt{10}$ for $\lambda_2/\lambda_1 = 0, 1/4, 1/2$, showing asymmetric features.

FIG. 11: Nonlinear conductance at finite $\lambda_3 = 1/\sqrt{2}$ for $\lambda_2/\lambda_1 = 0, 1, 5$.

A. Observability

In this subsection we discuss the realizability of the critical point in real experiment. Dealing with a repulsive critical point, the first condition we are concerned with is the smallness of the relevant perturbations, $T^* \ll T_K$. Secondly, we shall list some marginal corrections.

In order to tune $K = K_r > 0$, it is needed to reduce the ferromagnetic contribution $K_{KKY}$ [Eq. (2.3)] compared to $K_{12}$ [Eq. (2.1)], either (i) by setting $\Phi = \pi$, which is sufficient in the ideal situation where the device is perfectly parity symmetric, or (ii) in the more generic and realistic case, where parity symmetry is only approximate, by creating large asymmetry

$$\frac{t_2}{t_1} \ll \frac{t_{12}}{U(\nu J)} \sim \sqrt{\frac{t_K}{U}} \frac{1}{\nu J}$$

(6.2)
The limit $t_2 = 0$ corresponds to the the series QD. In either case, using Eq. (2.11), the condition $K = K_e \sim T_K$ is achieved by tuning the inter-dot coupling $t_{12} \sim \sqrt{U T_K}$.

At $K = K_e$ ($\lambda_1 = 0$) Eqs. (5.12) and (5.11) give the crossover scale
\[ T_{LR}^* |K - K_e| = T_K \left( (\text{Re} V_L^R)^2 + (\nu V_B)^2 \right), \]
where $V_B$ and $V_L^R$ are given in Eqs. (5.2) and (2.8).

In the parity symmetric case (i) we have $V_B = 0$, and $V_L^R$ is real and dominated by its second term in Eq. (2.8) because $\Phi = \pi$, leading to
\[ T_{LR}^*/T_K \sim |\nu V_L^R|^2 \sim (\nu J)^2 \frac{T_K}{U} \ll 1, \]
as required for the validity of the critical theory.

In the more realistic case (ii), on top of Eq. (6.2) we bound $t_2/t_1$ from below,
\[ \sqrt{\frac{T_K}{U}} \sim t_{12}/U \lesssim t_2/t_1 \ll \sqrt{\frac{T_K}{U}}, \]
such that $V_L^R$ is dominated by the first term in Eq. (2.8).

In addition we demand approximate parity symmetry,
\[ \frac{|t_{1L} - t_{2L}|}{t_1} \ll \sqrt{\frac{T_K}{U}} (t_1/t_2), \quad \sin(\Phi - \Phi_R) \ll 1, \]
such that $|V_B| \ll \text{Re} V_L^R$. Here $t_1 = \frac{t_1 - i t_2}{\sqrt{2}}$ and $t_2 = \frac{-i t_1 + t_2}{\sqrt{2}}$. It leads to $T_{LR}^*/T_K \sim |\nu V_L^R|^2 \sim (\nu J)^2 \left( \frac{T_K}{U} \right) \ll 1$, as required for the validity of the critical theory.

Next we estimate the marginal corrections. Spin $SU(2)$ symmetry is broken by the Zeeman energy $E_Z = \gamma B \left( S_1^z + S_2^z \right)$. This leads to a marginal operator $\hat{O}$ which reads in the Bose-Ising representation $\hat{O} \hat{\epsilon}$. In GaAs QDs, the Zeeman energy is reduced due to a small g-factor: for the experimental conditions in Ref. 2, $K_0$ corresponds to a magnetic field of few tesla, or equivalently to $\sim 10^3$ flux quanta in a area of $\mu m^2$; for a magnetic field corresponding to $\Phi = \pi$ we have $\langle E_Z \rangle \sim 10^{-3} T_K$, leading to small marginal correction to the conductance.

Other marginal operators allowed at the QCP are the inter- and intra-channel PS, Eqs. (5.3) and (5.4). Those are expected to introduce small corrections of $O\left(\nu J^2\right)$ to the conductance. Part of those operators break parity symmetry. So, the parity symmetry Eq. (2.5) is not required to hold exactly. Indeed the QCP has been observed numerically for a broken parity Hamiltonian in Ref. [17].

VII. CONCLUSIONS

We studied double quantum dots in the vicinity of the quantum critical point of the 2-impurity Kondo model. In the P-H symmetric model we used a mapping to the boundary Ising model with finite boundary magnetic field, to calculate the finite temperature crossover of the conductance from the QCP to the stable fixed points. This method generalizes the CFT approach, which addresses only the vicinity of the fixed points. We used this method to relate the conductance of the proposed system of Zaránd et. al.[14] to the one-point function of the magnetization operator in the boundary Ising model which can be calculated numerically.

Using the method developed by Gan, we solved the general and experimentally relevant case with potential scattering, and found the nonlinear conductance at finite temperature along the multidimensional crossover from QCP to surrounding FL states. Compared to the series double QD, we found that in the general configuration the universal scaling function contains both symmetric and asymmetric terms in the source drain voltage, leading to a current rectification.

VIII. ACKNOWLEDGMENTS

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APPENDIX A: CALCULATION OF THE NONLINEAR CONDUCTANCE USING KELDYSH GREEN FUNCTION TECHNIQUE

We briefly recall basic concepts of the nonequilibrium formulation. Then the problem at hand will be addressed, and the calculation of the nonlinear conductance will be outlined.

One usually assumes that the system is in equilibrium at some initial time, taken here to be $t = -\infty$. A perturbation $H_1$ is turned on adiabatically in time, $H = H_0 + e^{iH_1} t$ to drive the system out of equilibrium. The expectation value of an operator such as the current $I$ is given by its trace in the Heisenberg picture at $t = 0$ weighted by the initial distribution function,
\[ \langle I \rangle = \text{Tr}\{e^{-\beta H_0} u_0^{t} (0, -\infty) \hat{I} u(t, 0, -\infty)\}, \]
where $u(t_0, t) = T \exp(-i \int_{t_0}^{t} dt' H(t'))$ and $T$ is the time ordering operator. In order to employ Wick’s theorem, one transforms to the interaction picture, $\langle I \rangle = \text{Tr}\{e^{-\beta H_0} u_1^{0} (0, -\infty) \hat{I} u_t (0, -\infty)\}$, where $u_t(t_0, t) = T \exp(-i \int_{t_0}^{t} dt' H_1(t'))$, and $\mathcal{O}_I(t) = e^{iH_0 t} \mathcal{O} e^{-iH_0 t}$.

Following Keldysh, for a perturbative expansion of this quantity it is convenient to introduce 4 types of GFs $G_{\alpha\beta}^{\pm}(t, t')$:
\[ G^{11}(1, 1') = -i \langle \hat{T} \chi(1) \chi(1') \rangle, \]
\[ G^{12}(1, 1') = G^{21}(1, 1') = i \langle \chi(1') \chi(1) \rangle, \]
\[ G^{22}(1, 1') = G^{11}(1, 1') = -i \langle \hat{T} \chi(1) \chi(1') \rangle. \]
Here \( \hat{T} \) is the anti-time ordering operator. It is convenient to consider an alternative set of GFs, by defining the Keldysh GF matrix \( \mathcal{G} = \begin{pmatrix} G^R & G^< \\ 0 & G^A \end{pmatrix} \), where

\[
G^{R,A}(1,1') = \mp i\theta [\pm (t_1 - t_1')] \{(\chi(1), \chi(1'))_+ \}.
\]

Given a self energy, \( \Sigma = \begin{pmatrix} \Sigma^R & \Sigma^< \\ 0 & \Sigma^A \end{pmatrix} \), the Keldysh GF matrix has the expansion

\[
\mathcal{G}(\omega) = \mathcal{G}^{(0)}(\omega) + \mathcal{G}^{(0)}(\omega)\Sigma(\omega)\mathcal{G}^{(0)}(\omega) + ... ,
\]

where matrix multiplication in Keldysh space is understood and \( A(\omega) = \int dt e^{i\omega t} A(t) \). This leads to the Dyson equation for the advanced/retarded components of \( \mathcal{G} \).

\[
G^R(\omega) = G^{R(0)}(\omega) + G^{R(0)}(\omega)\Sigma^R(\omega)G^R(\omega),
\]

and to the Keldysh equation

\[
G^< = G^R \Sigma^< G^A + (1 + G^R \Sigma^R)G^{<(0)}(1 + \Sigma^A G^A).
\]

We now apply this scheme to our problem with

\[
H_0 = \sum_{j=1}^{2} \sum_{A=\epsilon,s,f,X} \frac{1}{2} \int_{-\infty}^{\infty} dx \chi_j^A \not{\partial_x} \chi_j^A + eV \sum_{\alpha} (N_{L\alpha} - N_{R\alpha}),
\]

\[
H_1 = \delta H_{G\alpha} + i\lambda_2 \chi_1^\alpha, a + i\lambda_3 \chi_2^\alpha, a,
\]

where \( N_{L\alpha} = \int dx \psi_{\alpha}^\dagger \psi_{\alpha} \), \( i = L, R \). Here \( H_0 \) is the \( J = 0 \) fixed point Hamiltonian, including the source drain voltage \( V \), and \( \delta H_{G\alpha} \) is given in Eq. (B17). It is more convenient to use \( \delta H_{G\alpha} \), which includes the local \( b \) operator, rather than the first term in \( \delta H \), \( i\lambda_1 \chi_2^X(x) \text{sgn}(x/a) \). Both formulations should give the same result for energy scales \( \ll T_K \), as we showed generally in Sec. V.C. At \( t = -\infty \) the system consists of two decoupled leads with equilibrium with different potentials. It is convenient to make a change of basis, in which the operator \( Y = \frac{1}{2} \sum_{\alpha} (N_{L\alpha} - N_{R\alpha}) = \frac{1}{2} \int_{-\infty}^{\infty} dx j_0(x) \) is diagonal. Using Eq. (A13) for \( j_0 \), we see that

\[
Y = \int_{-\infty}^{\infty} dx \alpha_+^\dagger = \alpha \int_{-\infty}^{\infty} dx \alpha_-^\dagger \alpha_+,
\]

where we defined new fermions \( \alpha \) and \( \beta \): \( \alpha = \frac{\alpha_+ - \alpha_-}{\sqrt{2}}, \beta = \frac{\beta_+ - \beta_-}{\sqrt{2}}, \) in terms of the 4 Majorana fermions \( \alpha_\pm \) and \( \beta_\pm \) given by

\[
\alpha_+ = \chi_2^f, \quad \beta_+ = \chi_2^X,
\]

\[
\alpha_- = -(\cos \phi_m \chi_1^f + \sin \phi_m \chi_2^X),
\]

\[
\beta_- = (\sin \phi_m \chi_1^f - \cos \phi_m \chi_2^X).
\]

We see that the voltage raises the chemical potential of the \( \alpha \) fermions by \( eV \), whereas the chemical potential for the \( \beta \) fermions remains zero. The system is at equilibrium at \( t = -\infty \) since in this case the \( \alpha \) and \( \beta \) fermion numbers are conserved. At \( t > -\infty \), \( H_1 \) leads to the current operator \( \hat{I} = i[Y, H_1] \) which drives the system out of equilibrium and is given by

\[
\hat{I} = -2i\sqrt{T_K} \sin \phi_m \alpha_+ (0)b - i\lambda_2 \cos \phi_m \alpha_+ (0)a - i\lambda_3 \alpha_- (0)a.
\]

We shall express the expectation value of the current by the Green functions \( \mathcal{G}_{\mu}(t) \) where the indices refer to the fermion local operators \( \nu = (a, b) = (1, 2) \) and \( \mu = (\alpha_+, \alpha_-, \beta_+, \beta_-) = (1, 2, 3, 4) \). Using Eq. (A3) the current expectation value reads

\[
\langle I(t = 0) \rangle = -2\sqrt{T_K} \sin \phi_m G^{<(0)}_{\alpha_+}(t = 0) - \lambda_2 \cos \phi_m G^{<(0)}_{\alpha_+}(t = 0) - \lambda_3 G^{<(0)}_{\alpha_-}(t = 0).
\]

We construct the exact GFs [appearing in Eq. (A5)] from the free GFs calculated at \( t = -\infty \) (\( H_1 = 0 \)): for \( \mu, \mu' = (\alpha_+, \alpha_-, \beta_+, \beta_-) = (1, 2, 3, 4) \), one finds

\[
(G^{R,A})_{\mu \mu'} = \pm \frac{i}{2} \delta_{\mu \mu'},
\]

\[
(G^{<})_{\alpha_+ \alpha_+} = i\overline{f}(\omega), (G^{<})_{\beta_+ \beta_+} = i\overline{f}(\omega), (G^{<})_{\alpha_- \alpha_+} = \pm \frac{1}{2} f(\omega - eV) - f(\omega + eV).
\]

Here \( f(x) = (1 + e^{x/T})^{-1}, \overline{f}(x) = \frac{1}{2} [f(x + eV) + f(x - eV)] \). Note that the voltage couples the two Majorana fermions \( \alpha_+ \) and \( \alpha_- \), and here we assumed a bandwidth \( \gg \omega, V, T \). The free GF for the local Majorana fermions \( \nu = (a, b) = (1, 2) \) is \( (G^{R})^{(0)}_{\nu \nu'} = \delta_{\nu \nu'}(\omega + i\delta)^{-1} \), where \( \delta \) is a positive infinitesimal. We write \( H_1 \) in a convenient form

\[
H_1 = -i\sqrt{T_K} \chi_1^f \chi_2^b \int_{-\infty}^{\infty} dx j_0(x) \sum_{\mu=1}^{2} \sum_{\nu=1}^{4} \{ \alpha_+, \alpha_-, \beta_+, \beta_- \} \Lambda_{\mu \nu}(a, b) \nu',
\]

where

\[
\Lambda_{11, 12, 13, 14} = (-\lambda_3, \lambda_2 \cos \phi_m, 0, -\lambda_2 \sin \phi_m), \Lambda_{12, 12, 13, 12} = 2\sqrt{T_K}(0, \sin \phi_m, 0, \cos \phi_m).
\]

We obtain the full GF \( G^{\mu \nu'}_{\alpha_+ \beta_-} \) for \( \nu, \nu' = a, b \) as follows. First suppose \( K = K_c \) (\( \lambda_1 = 0 \)); we denote the different GFs and self energies in this case as \( G, \Sigma \), respectively. At \( K = K_c \) the self energy matrix is

\[
\Sigma_{\alpha \mu \nu'} = -\Lambda_{\mu \nu} G^{(0)}_{\alpha \alpha}, \Sigma_{\beta \mu \nu'} = \Lambda_{\mu \nu} G^{(0)}_{\beta \beta}, \quad \text{(repeated indices summed)}.
\]

Eqs. (A6) and (A7) give \( \Sigma_{ab}^{R, R} = -\frac{i}{2}(\lambda_2^2 + \lambda_3^2), \Sigma_{ab}^{R, R} = -2iT_K, \Sigma_{ab}^{R, R} = \Sigma_{ba}^{R, R} = 0 \). Eq. (A12) gives \( G^{R}_{ba} = (\omega + i\lambda_2^2 + \lambda_3^2)^{-1}, G^{R}_{bb} = (\omega + 2iT_K)^{-1}, G^{R}_{ab} = G^{R}_{ba} = 0 \). For
energies $\ll T_K$ we can approximate $\tilde{G}_{ab}^{R} = (2i T_K)^{-1}$. For the lesser GF Eq. (A3) gives

$$G^< = G^R \Sigma^< G^A, \quad (A8)$$

where matrix equation and multiplication in $ab$ space is understood.

For $K \neq K_c$ the full matrix GF $G_{\nu\nu'}$ can be calculated from the series Eq. (A1) where $G^{(0)} \rightarrow G$ and

$$\Sigma^R = \Sigma^A = \sqrt{T_K} \lambda_1 \tau^y,$$

$\Sigma^< = 0$, where $\tau^y$ acts in $ab$ space. Eq. (A2) gives

$$G_{ab}^{R}(\omega) = (\omega + i\lambda^2/2)^{-1},$$
$$G_{bb}^{R}(\omega) = (2i T_K)^{-1} - (\lambda^2/4 T_K)(\omega + i\lambda^2/2)^{-1},$$
$$G_{ab(ab)}^{R}(\omega) = \mp(\lambda/2\sqrt{T_K})(\omega + i\lambda^2/2)^{-1}.$$

For $G^<$, since $\Sigma^< = 0$ we are left with the second term of Eq. (A3), which simplifies to [using Eq. (A8) and (A2)]

$$G^< = G^R \Sigma^< G^A,$$

where matrix equation and multiplication in $ab$ space is understood.

The GFs appearing in the current Eq. (A5) satisfy the Dyson equation

$$\mathcal{G}_{\mu}(t = 0) = \int \frac{d\omega}{2\pi} \sum_{\nu',\nu''} \mathcal{G}_{\nu'\nu''}(\omega) i \Lambda_{\mu\nu'} \mathcal{G}^{(0)}_{\nu''\mu}(\omega). \quad (A9)$$

To evaluate $\mathcal{G}_{\mu}^<$ we use the identity ($A B)^< = A^< B^A + A^R B^<$. We encounter two types of integrals for the current:

$$I'[V, T, \lambda^2] = i \int d\omega \frac{f(\omega - eV) - f(\omega + eV)}{\omega + i\lambda^2/2}$$
$$= \text{Im} \left( \frac{1}{2} + \frac{\lambda^2/2 + ieV}{2\pi T} \right),$$
$$I''[V, T, \lambda^2] = \int d\omega \frac{f(\omega) - \tilde{f}(\omega)}{\omega + i\lambda^2/2}$$
$$= \psi \left( \frac{1}{2} + \frac{\lambda^2/2 + ieV}{2\pi T} \right) - \text{Re} \psi \left( \frac{1}{2} + \frac{\lambda^2/2 + ieV}{2\pi T} \right).$$

Note that $I'[V] = -I'[-V]$, and $I''[V] = I''[-V]$. From these results one can readily obtain the result for the nonlinear conductance, Eq. (6.1).