Gapped phases of quantum wires

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We investigate possible nontrivial phases of a two-subband quantum wire. It is found that inter- and intra-subband interactions may drive the electron system of the wire into a gapped state. If the nominal electron densities in the two subbands are sufficiently close to each other, then the leading instability is the inter-subband charge-density wave (CDW). For large density imbalance, the interaction in the inter-subband Cooper channel may lead to a superconducting instability. The total charge-density mode, responsible for the conductance of an ideal wire, always remains gapless, which enforces the two-terminal conductance to be at the universal value of $2e^2/h$ per occupied subband. On the contrary, the tunneling density of states (DOS) in the bulk of the wire acquires a hard gap, above which the DOS has a non-universal singularity. This singularity is weaker than the square-root divergency characteristic for non-interacting quasiparticles near a gap edge due to the “dressing” of massive modes by a gapless total charge density mode. The DOS for tunneling into the end of a wire in a CDW-gapped state preserves the power-law behavior due to the frustration the edge introduces into the CDW order. This work is related to the vast literature on coupled 1D systems, and most of all, on two-leg Hubbard ladders. Whenever possible, we give derivations of the important results by other authors, adopted for the context of our study.

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From a theorist’s point of view, electrons in quantum wires should provide a simplest realization of a Luttinger liquid. Indeed, as the motion is confined in the direction transverse to the axis of a wire, the system is effectively one-dimensional; the electron-electron interaction is strong enough (typically, of the order of the Fermi energy) for the interaction effects not to be washed by the temperature; and the state-of-the-art wires (at least the semiconductor version of them) are clean enough for disorder effects to be sufficiently weak. A crucial experimental test for the existence of the Luttinger-liquid state in quantum wires would be provided by tunneling into a wire, either in the middle or into the end. As is well-known, the tunneling density of states (DOS) of a Luttinger liquid reveals a pseudogap behavior, i.e., it is suppressed at energies close to the Fermi-energy, which should result in a power-law bias-dependence of the tunneling current, and in a power-law temperature dependence of the Ohmic conductance. Finite-bias and finite-temperature measurements have already been performed on a very special realization of quantum wires – carbon nanotubes, and observed non-linear current-voltage dependences were interpreted in terms of the Luttinger-liquid theory. Features of resonant tunneling, characteristic for a Luttinger liquid, have recently been observed on GaAs quantum wires prepared by cleaved edge overgrowth technique. Also, a Luttinger-liquid behavior has been reported in tunneling into InSb wires naturally grown in a porous material (asbestos). Tunneling pseudogap of a quantum wire has been described in terms of a Luttinger-liquid model both for a single- and multi-subband wire, the latter system exhibiting a smooth healing of the pseudogap as the number of the occupied channels increases. In anticipation of more and better controlled tunneling experiments on quantum wires, and also from a general point of view, we would like to ask if there are any processes which could open a true gap, rather than a pseudogap, in the electron spectrum of a wire, and if yes, what are the properties of the corresponding gapped phases.

To this end, we consider in this paper a two-subband quantum wire, having in mind semiconductor nanostructures studied recently in, e.g., Refs. To some approximation, this system is similar to other two important classes of 1D two-band systems studied extensively over the last few years, i.e., two-leg Hubbard ladders and (single-wall) carbon nanotubes (for an account of earlier results on coupled 1D systems, see, e.g., Refs.). Studies of Hubbard ladders identified inter-subband scattering processes capable of driving the system into a gapped state. Phase diagrams of a generic ladder, containing a multitude of gapped states, were constructed in Refs. A similarly formulated problem, with applications to a 1D system with electron and hole bands (valence-fluctuation problem), was investigated some time ago in Refs. The goal of the present paper is two-fold. First of all, we would like to understand which of the gapped phases, found in Hubbard ladders, have a chance to occur in quantum wires. The main difference between these two systems is that the Coulomb interaction in wires is (i) (supposed to be) purely repulsive; (ii) relatively long-ranged (even in the presence of a metallic gate); (iii) relatively well-known at distances larger than the lattice spacing (which is the range relevant for quantum wires); this imposes constraints on the choice of coupling constants for the Hubbard model. Also, because the electron wavelength in wires is larger than the lattice spacing, Umklapp scattering is unimportant. All these constraints reduce the variety of possible gapped states to (a) inter-subband charge-density wave (CDW), and (b) superconducting state. [We will come back to a more detailed description of these states shortly.] The main difference between carbon nanotubes and quantum wires is that the former, because of its special crystal structure, has two conducting subbands with commensurate Fermi-momenta. Although, as we will show, a quantum
wire with nominally different subband Fermi-momenta may be driven into the commensurate state by inter-subband backscattering, this process occurs in a competition with other processes and requires special analysis.

Having determined which gapped phase can in principle occur in a quantum wire, we focus on the calculation of measurable quantities in each of these phases, which is the second goal of the present paper. Our main emphasis is on the tunneling density of states, which turns out to exhibit an unusual threshold behavior due to coexistence of gapped and gapless modes and also be sensitive to the presence of open boundaries. In addition, we consider the two-terminal conductance both in the absence and in the presence of impurities.

Although this paper is not supposed to be a review, we present, when possible, derivations of important results by other authors, e.g., Refs. 26-28, adopted to the context of our study. Hopefully, this would help a reader, who is not an expert in the field, to understand connections between different approaches.

Having formulated the goals and scope of this paper, we now return to a generic two-subband quantum wire with incommensurate Fermi-momenta \( k_{1F} \) and \( k_{2F} \) in subbands 1 and 2, respectively. In the basis of occupied transverse states, this becomes the problem of two Luttinger liquids coupled by inter-subband interactions. To understand possible phases of such a system, one should consider all possible scattering processes involving electrons from different subbands, i.e., forward scattering, backscattering, and “Cooper scattering” (cf. Figures 1-3). Forward scattering simply renormalizes parameters of Luttinger liquids formed by electrons of each subbands but does not result in new phases, although it does change the conditions for occurrence of new phases. (To be more precise, forward scattering between Luttinger liquids in different subbands is responsible for the crossover into the Fermi-liquid state, but this crossover occurs smoothly as the number of channels increases).

The momentum transfer in a backscattering event involving electrons of different subbands is equal to \( k_{1F} \pm k_{2F} \). If \(|k_{1F} - k_{2F}| \gg T/\min\{v_{1F}, v_{2F}\}\), then there are no final states available for electrons involved in such a process (here \( v_{1F} \) and \( v_{2F} \) are the Fermi velocities in the two subbands). Thus, if the temperature \( T \) is low enough, interchannel backscattering is forbidden. However, it may become energetically favorable for a system to equalize the charge densities, and hence the Fermi momenta, of different subbands. This may occur if the Fermi-momenta difference is small enough and the amplitude of backscattering is large enough. After the densities are adjusted, backscattering becomes possible. As a result, inter-subband charge-density wave (CDW) phase may be formed, in which charge densities of the subbands form a staggered pattern, see Fig. 4. Similarly to classical charge-density waves, this phase is very sensitive to a random potential, resulting in pinning of the CDW and strong suppression of conductance with disorder.

The “Cooper” scattering event, on the other hand, always conserves momentum and energy. In this process, two electrons starting in, e.g., subband 1 with momenta \( k_{1F} \) and \( -k_{2F} \), scatter on each other and end up in the other subband, also with opposite momenta \( k_{2F} \) and \( -k_{1F} \). “Cooper scattering” can be considered as formation of a fluctuational Cooper pair in one of the subband followed by its tunneling into the other one. When kinetic energy gain due to such tunneling overcomes the Coulomb repulsion, the wire is in a Cooper (or superconducting) phase. This phase is characterized by locking of fluctuating charge currents in different subbands to each other as well as by spin gaps in each of the subbands. The Cooper phase is favored when Fermi-momenta imbalance is largest, i.e., when the second subband just starts to fill up. Disorder has a less pronounced effect on Cooper phase than on CDW one, similarly to what happens in higher dimensions.

It is important to emphasize here that inter-subband backscattering and Cooper scattering block only modes corresponding to relative charge- and spin-excitations, but leave the center-of-mass charge mode free. As a result, the conductance remains at the universal value of \( 2e^2/h \) per occupied subband in clean CDW and Cooper phases.

Despite being ideal conductors, both the CDW and Cooper phases are characterized by the truly gapped behavior of the tunneling density of states at energies below corresponding gaps. This is so because a 1D electron is a convolution of charge and spin collective excitations, and if some of these excitations acquire a gap, the entire electron acquires it as well.

Somewhat surprisingly, we find that tunneling into the end of the CDW wire is quite different. A tunnel barrier at the end of the wire distorts charge-density wave profile and creates a static semi-soliton. This allows tunneling into the end to occur even at energies below the bulk CDW gap (in the lowest order in the barrier’s transparency).

In Section VI, we consider, for illustrative purposes, the density of states of the “Mott phase”, which occurs in a single- or multi-subband wire subject to an external periodic potential. In the case of a wire, formed in a semiconductor heterostructure, this potential may be provided by an additional electrostatic gate of periodic shape. Varying the potential applied to this gate, one can tune electrons of the wire into the half-filling condition (one electron per unit cell of the periodic potential). Unlike the two strong-coupling phases mentioned above, the Mott phase, which is described by the half-filled Hubbard model, does not conduct current because its total charge fluctuations are gapped by the external potential.
II. HAMILTONIAN OF A TWO-SUBBAND QUANTUM WIRE

A. Classification of scattering processes

Electrons in a quantum wire are described by the following Hamiltonian

\[ H = \sum_s \int d^3r \Psi_s^\dagger(r) \left( -\frac{1}{2m} \nabla_r^2 - \mu + V_{\text{conf}}(r_\perp) \right) \Psi_s(r) \]

\[ + \frac{1}{2} \sum_{s,s'} \int d^3r d^3r' U(r - r') \Psi_s^\dagger(r) \Psi_{s'}(r') \Psi_s(r), \]

where \( s \) is the spin index, \( V_{\text{conf}}(r_\perp) \) is the confining potential in the transverse direction, and \( U(r) \) is the electron-electron interaction potential. The Fermi-wavelength of electrons is assumed to be much larger than the lattice spacing of the underlying crystal structure. Because of that, we do not consider umklapp processes, in which electron momentum is transferred to the lattice (except for in Ch. VIB). Hamiltonian (1) is Galilean-invariant, and hence our subsequent calculations have to preserve this invariance as well. We will return to this important point later on in our discussion.

If the chemical potential in the leads is such that only two lowest subbands of transverse quantization are occupied, the electron wavefunction is given by

\[ \Psi_s(r) = \sum_{n=1}^{2} \phi_n(r_\perp) \psi_{ns}(x), \]  

where \( \phi_n(r_\perp) \) are the orthogonal wavefunctions of transverse quantization, chosen to be real. In this basis, the kinetic part of Hamiltonian (1) becomes

\[ H_0 = \sum_{n,s} \int dx \psi_{ns}^\dagger(x) \left( -\frac{\partial_x^2}{2m} - \mu + \epsilon_n \right) \psi_n(x) \]

where \( \epsilon_n \) is the energy of the \( n \)-th transverse subband.

To describe low-energy excitations in the \( n \)-th subband, we expand the longitudinal part of the \( \Psi \)-operator, \( \psi_s(x) \), in terms of right- and left-moving excitations, residing around the Fermi-points of the \( n \)-th channel:

\[ \psi_{ns}(x) = R_{ns}(x)e^{i\kappa_{nF}x} + L_{ns}(x)e^{-i\kappa_{nF}x}. \]

In this representation, the interaction (four-fermion) part of Hamiltonian (1) reduces to a sum of two terms. The first one, \( U_{\text{intra}} \), describes the interaction of electrons within the same subband, and contains forward and backward scattering processes. The second one, \( U_{\text{inter}} \), describes the inter-subband interaction. It splits naturally into forward (\( U^F \)), backward (\( U^B \)), and Cooper (\( U^C \)) parts.

\[ U_{\text{inter}} = U^F + U^B + U^C. \]

The momentum transfer between subbands is limited by the width of the Fermi distribution function, and is therefore small at low temperatures. Forward scattering involves no momentum transfer between subbands (cf. Fig. II). This process is also an example of a direct process, in a sense that electrons stay in the same subband, as is evident from the explicit expression for \( U^F \)

\[ U^F = \frac{1}{2} \sum_{n\neq m} \int_{x,x'} M^{nm\dagger}(x - x') \sum_{s,s'} \left[ R_{ns}^\dagger(x) R_{ns}(x) + L_{ns}^\dagger(x) L_{ns}(x) \right] \times \left[ R_{ms'}^\dagger(x') R_{ms'}(x') + L_{ms'}^\dagger(x') L_{ms'}(x') \right], \]

where the direct matrix element is given by

\[ M^{nm\dagger}(x - x') = \int_{r_\perp r'_\perp} U(r - r') \phi_n^\dagger(r_\perp) \phi_m^\dagger(r'_\perp); \]

and \( \int_{z,z'} = \int dz \int dz' \).
By “backward scattering”, we understand processes with a non-zero momentum transfer \( \delta k = k_1F \pm k_2F \) between subbands. These processes can be divided further into direct and exchange parts

\[
U^B = U_d^B + U_x^B.
\]

In an exchange process [see Figs.2,3], electrons change subbands. Two parts of backscattering can be written as (for the sake of brevity, we omit here the \( x \rightarrow x' \)-dependence of the matrix elements):

\[
U_d^B = \frac{1}{2} \sum_{n \neq m} \sum_{s,s'} \int_{x,x'} M^{(nm)}_d \{ R^\dagger_{ns}(x) L_{ns}(x) L^\dagger_{ms'}(x') R_{ms'}(x') \} e^{2i(k_mF - k_nF)(x+x')}
\]

\[
+ (R \leftrightarrow L) e^{-2i(k_mF - k_nF)(x+x')} \]

and

\[
U_x^B = -\frac{1}{2} \sum_{n \neq m} \sum_{s,s'} \int_{x,x'} M^{(nm)}_x \{ \{ R^\dagger_{ns}(x) R_{ns'}(x') L^\dagger_{ms'}(x') R_{ms}(x) \} e^{i(k_mF - k_nF)(x-x')} \}
\]

\[
+ (R \leftrightarrow L) e^{-i(k_mF - k_nF)(x-x')} \}
\]

\[
+ \{ R^\dagger_{ns}(x) R_{ns'}(x') L^\dagger_{ms'}(x') L_{ms}(x) \} e^{i(k_mF + k_nF)(x-x')} + (R \leftrightarrow L) e^{-i(k_mF + k_nF)(x-x')} \}
\]

where the exchange matrix element is

\[
M^{(nm)}_x (x - x') = \int_{\mathbf{r}_\perp, \mathbf{r}'_\perp} U(|\mathbf{r} - \mathbf{r}'|) \phi_n(\mathbf{r}_\perp) \phi_m(\mathbf{r}'_\perp) \phi_n(\mathbf{r}_\perp') \phi_m(\mathbf{r}_\perp').
\]

FIG. 1. Example of inter-subband forward scattering. Filled (empty) circles denote initial (final) states of electrons. Dashed lines with arrows indicate “direction”of the scattering.

FIG. 2. Example of inter-subband backscattering: (a) direct, (b) exchange. Notations as in Fig.1.
Momentum conservation requires that the energy of at least one of the states, involved into direct backscattering, should be far away from the Fermi energy, which forbids this process at not too high temperatures \( (T \ll \left| k_{mF} - k_{nF}\right|, \text{min} \\{v_{Rm}, v_{Rn}\}) \). This is reflected in the presence of the exponential factors in front of the fermion operators in Eq. (12). The restriction can be lifted though, if the system prefers to gain energy from backscattering by equalizing the subband densities, so that \(k_{mF} = k_{nF}\).

Finally, we call “Cooper scattering” (Fig. 3) a process in which two electrons with zero total momentum (a fluctuational Cooper pair) hop from, e.g., channel \( m \) to channel \( n \), so that the total momentum \( Q = -k_m + k_m = 0 \) and \(-k_n + k_n = 0 \) is conserved. This process is also referred to as “Josephson coupling” or as “\( g_{00\pi} \) process” (see Fig. 3). The Hamiltonian of Cooper scattering is given by

\[
U^C = \frac{1}{2} \sum_{n \neq m} \sum_{s,s'} \int_{x,x'} M_x^{nm} \left[ R_{ns}^\dagger(x) L_{ns'}^\dagger(x') e^{i k_{nF}(x'-x)} + L_{ns}^\dagger(x) R_{ns'}^\dagger(x') e^{-i k_{nF}(x'-x)} \right]
\times \left[ R_{ms'}(x') L_{ms}(x) e^{i k_{mF}(x'-x)} + L_{ms'}(x') R_{ms}(x) e^{-i k_{mF}(x'-x)} \right].
\]

By construction, Cooper scattering is of the exchange type. In what follows, we use the following abbreviations: forward scattering \( \equiv FS \), direct backward scattering \( \equiv dBS \), exchange backward scattering \( \equiv xBS \), Cooper scattering \( \equiv CS \).

For a generic situation of \( k_n \neq k_m \), the only momentum-conserving inter-subband scattering processes are FS, xBS, and CS. The amplitudes of these processes depend on the ratio \( a/d \), where \( a \) is a typical transverse size of the wire (which determines the spatial extent of \( \phi_n(x,R) \)) and \( d \) is the interaction range. In the limit of \( a/d \to 0 \), the interaction potential can be taken out of integrals (13) and (14), upon which \( M_d \) remains finite, whereas \( M_x \) vanishes. It can be readily shown that for finite but small ratio \( a/d \), the exchange matrix element is small: \( M_x \sim (a/d)^2 M_d \). The long-range interaction thus discriminates against exchange processes. If (as it is most often the case) a wire is formed by means of a gate deposited over the 2D heterostructure, \( d \) is given by the distance to the gate, which screens the Coulomb interaction in the wire (see Fig. 4). Typically, \( a/d = 0.1 \sim 1 \).

![FIG. 4. Schematic view of a gated wire. The wire of typical transverse size \( a \) is separated by distance \( d \) from the metallic gate. Distance \( d \) determines the range of interaction among electrons inside the wire.](image)

**B. Bosonized form of the Hamiltonian**

We use the conventional bosonization procedure, in which

\[
R_{ns}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{i \sqrt{2\pi} (\varphi_{ns} - \theta_{ns})},
\]

\[
L_{ns}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{-i \sqrt{2\pi} (\varphi_{ns} + \theta_{ns})},
\]

and short-range cut-off \( \alpha \sim k_F^{-1} \). Boson fields \( \varphi_{ns} \) and \( \theta_{ns} \) with \( n = 1,2; \ s = \pm 1 \), are decomposed into charge-(\( \rho \)) and spin-(\( \sigma \)) collective modes

\[
\varphi_{ns} = \frac{1}{\sqrt{2}} (\varphi_{n\rho} + s\varphi_{n\sigma}),
\]

\[
\theta_{ns} = \frac{1}{\sqrt{2}} (\theta_{n\rho} + s\theta_{n\sigma}).
\]

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Parts of the Hamiltonian, corresponding to free motion and intra-subband interactions \((H_0 + U_{\text{intra}})\), take the well-known Luttinger-liquid form:

\[
H_{np} = \frac{1}{2} \int dx \left( v_{np} K_{np} (\partial_x \theta_{np})^2 + \frac{v_{np}}{K_{np}} (\partial_x \varphi_{np})^2 \right),
\]

\[
H_{n\sigma} = \frac{1}{2} \int dx \left( v_{n\sigma} K_{n\sigma} (\partial_x \theta_{n\sigma})^2 + \frac{v_{n\sigma}}{K_{n\sigma}} (\partial_x \varphi_{n\sigma})^2 \right)
+ \frac{2U(2k_F)}{(2\pi\alpha)^2} \int dx \cos[\sqrt{8\pi} \varphi_{n\sigma}],
\]

which describes independent charge- and spin-density excitations \((H_{np} \text{ and } H_{n\sigma}, \text{ respectively})\). The cosine term in \(\text{Eq.}(17)\) is due to backscattering within a single subband. Explicit expressions for the Luttinger-liquid parameters will be discussed later.

Upon bosonization, the three types of intersubband interactions take the following form:

\[
U_0 = \frac{2f_0}{\pi} \int dx \partial_x \varphi_1 \partial_x \varphi_2;
\]

\[
U_B^d = \frac{4f_{bs}}{\pi^2 \alpha^2} \int dx \cos \left[ \sqrt{2\pi} (\varphi_1 - \varphi_2) + 2(k_1F - k_2F)x \right]
\times \cos[\sqrt{2\pi} \varphi_{1\sigma}] \cos[\sqrt{2\pi} \varphi_{2\sigma}];
\]

\[
U_B^x = -\frac{1}{2} \int dx \left( \frac{b_1 + b_2}{\pi} (\partial_x \varphi_1 \partial_x \varphi_2 + \partial_x \varphi_1 \partial_x \varphi_2) + \frac{b_1 - b_2}{\pi} (\partial_x \theta_1 \partial_x \theta_2 + \partial_x \theta_1 \partial_x \theta_2) \right)
+ \frac{2}{\pi^2 \alpha^2} \cos[\sqrt{2\pi} (\theta_1 - \theta_2)] \cos[\sqrt{2\pi} \varphi_{1\sigma}] \cos[\sqrt{2\pi} \varphi_{2\sigma}]
- (b_1 - b_2) \sin[\sqrt{2\pi} \varphi_{1\sigma}] \sin[\sqrt{2\pi} \varphi_{2\sigma}];
\]

\[
U_C^d = \frac{4}{2\pi^2 \alpha^2} \int dx \left[ t_{sp} \cos[\sqrt{2\pi} (\theta_1 - \theta_2)] \cos[\sqrt{2\pi} \varphi_{1\sigma}] \cos[\sqrt{2\pi} \varphi_{2\sigma}]
+ t_{sp} \cos[\sqrt{2\pi} (\theta_1 - \theta_2)] \cos[\sqrt{2\pi} \varphi_{1\sigma}] \sin[\sqrt{2\pi} \varphi_{2\sigma}] \right].
\]

The corresponding amplitudes are given by

\[
f_0 = \int dx M^{(12)}_d(x),
\]

\[
f_{bs} = \int dx M^{(12)}_d(x) \cos[(k_1F + k_2F)x],
\]

\[
b_{1,2} = \int dx M^{(12)}_d(x) \cos[(k_1F \mp k_2F)x],
\]

\[
t_{sp} = \int dx M^{(12)}_d(x) \cos(k_1F x) \cos(k_2F x),
\]

\[
t_{sp} = \int dx M^{(12)}_d(x) \sin(k_1F x) \sin(k_2F x).
\]

In the last two lines, \(t_{sp}(t_{sp})\) are the amplitudes of singlet (triplet) Cooper processes.

The highly non-linear (cosine) terms in Eqs.\((13,14,21)\) signal potential instabilities of the ground state due to inter-subband interactions. For the dBS process \(\text{Eq.}\((13)\)\), this instability is of the charge-density-wave (CDW) type, quantity \(\varphi_1 - \varphi_2\) being the phase of the CDW (particle-hole) condensate. If subbands are equivalent \((k_1F = k_2F)\), the energy is minimized by adjusting the CDW-condensate phase is such a way that the cosine takes its minimum value \((-1, \text{ for repulsive interactions})\). For non-equivalent subbands \((k_1F \neq k_2F)\), the global minimization of the energy is impossible due to the position-dependent phase shift, and thus the CDW instability is suppressed. Nevertheless, if the energy gain due to opening of the CDW gap is large enough, the system may choose to adjust the subband densities,
where \( \rho \) various parts of the Hamiltonian reduce to 17

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\( \theta \) higher dimensions. Quantity accompanied by opening of spin gaps in each of the subbands, in analogy with a superconducting transition in amplitudes may be arbitrary, and the OAF instability is possible, at least may result from direct electron-electron interaction and some from exchange of virtual phonons. Hence, the ratios of ladder, amplitudes of various scattering processes may be determined by entirely different physics, e.g., some of them result from direct electron-electron interaction and some from exchange of virtual phonons. Hence, the ratios of amplitudes may be arbitrary, and the OAF instability is possible, at least \textit{a priori}.

Finally, the CS process [Eq. (21)] may lead to a superconducting instability (of both singlet and triplet types), accompanied by opening of spin gaps in each of the subbands, in a analogy with a superconducting transition in higher dimensions. Quantity \( \theta_{1\rho} - \theta_{2\rho} \) plays the role of the superconducting condensate phase. Inter-subband forward scattering [Eq. (15)] has an important role in developing a superconducting instability—it reduces electron repulsion in the relative charge-density fluctuation channel, making it possible for Cooper scattering to become relevant. The superconducting phase is also known as a “C1S0-phase” (meaning: one gapless charge mode and no gapless spin modes) or a “d-wave superconductor.” Indicating that the order parameter is odd upon interchanging the electrons, forming a Cooper pair, between subband. This particular instability received much attention recently as one of the models of HTC superconductivity. We also note in passing that the idea of superconductivity in a two-band system has a long history, starting from the 1968 paper by Fröhlich [33].

The idea, employed in earlier work, is that if the masses of electrons in two subbands are significantly different, there always–even in 3D–exists a gapless plasmon excitation (a direct analog of Langmuir-Tonks ion sound waves in plasma), which serves as a mediator of effective attraction. The superconducting phase in a 1D two-band system is already composed of gapless excitations and is not limited by the condition of different masses (although, as we will see shortly, there is no lack of other constraints).

Note also that \( U^B_a \) [19] and \( U^C \) [21] mix charge and spin modes, and thus spoil the spin-charge separation present in the Hamiltonian of a single subband.

Processes [13 - 21] have been written down in the literature in many different ways, so it is worth to make a connection to previous work here. Identification of our notations with the “g-ological” ones (used by Schulz in his two important papers [22]) is as follows: \( t_{1\rho} = g_{12} + g_{23}, t_{2\rho} = g_{23} - g_{12}, b_1 = g_{13}, \) and his \( g_{11} \)-process (backscattering with opposite spins) should be equated with our \( U(2k_F) > 2U(0) \) (for identical subbands), which is never the case for any physical \( U(\rho) \). In what follows, we will not therefore consider the OAF phase. (Note that for a Hubbard ladder, amplitudes of various scattering processes may be determined by entirely different physics, e.g., some of them result from direct electron-electron interaction and some from exchange of virtual phonons. Hence, the ratios of amplitudes may be arbitrary, and the OAF instability is possible, at least \textit{a priori}.)

One more connection is made by noting that (somewhat lengthy) Eq. (20) can be represented compactly as

\[
U^B_x = - \int dx \{(b_1 + b_2)(\rho_1 \rho_2 + S_1 S_2) + (b_1 - b_2)(j_1 \cdot j_2 + j_s \cdot j_s)\},
\]

where \( \rho_n \) (\( S_n \)) is charge (spin) density, and \( j_{cn} \) (\( j_{sn} \)) is charge (spin) current in the \( n \)-th subband, using notations of Emery, Kivelson, and Zachar [14, 15].

III. SPINLESS ELECTRONS

1. Model

In this section we consider a “toy” model of a two-subband system of spinless electrons, which contains all interesting effects we want to discuss and, at the same time, allows for a rather complete analytic treatment. In this model, various parts of the Hamiltonian reduce to.
\[
H_0 \to \tilde{H}_0 = \int dx \sum_n \left[ \frac{v_n}{2K_n} (\partial_x \varphi_n)^2 + \frac{v_nK_n}{2} (\partial_x \theta_n)^2 \right]; \quad (24)
\]

\[
U^F \to \tilde{U}^F = \frac{f_0}{\pi} \int dx \partial_x \varphi_1 \partial_x \varphi_2; \quad (25)
\]

\[
U^B_d \to \tilde{U}^B_d = \frac{f_{b_d}}{\pi^2 \alpha^2} \int dx \cos[\sqrt{4\pi}(\varphi_1 - \varphi_2) + 2(k_{1_F} - k_{2_F})x]; \quad (26)
\]

\[
U^B_x \to \tilde{U}^B_x = -\frac{1}{2\pi} \int dx \left[ (b_1 + b_2) \partial_x \varphi_1 \partial_x \varphi_2 + (b_1 - b_2) \partial_x \theta_1 \partial_x \theta_2 \right]; \quad (27)
\]

\[
U^C \to \tilde{U}^C = \frac{f_C}{2\pi^2 \alpha^2} \int dx \cos \sqrt{4\pi}(\theta_1 - \theta_2); \quad (28)
\]

so that

\[
\tilde{H} = \tilde{H}_0 + \tilde{U}^F + \tilde{U}^B_d + \tilde{U}^B_x + \tilde{U}^C. \quad (29)
\]

Amplitude \(f_C\) plays now the role of \(t_{sp}\) for spinless electrons. Analysis of potentially “dangerous” (in a sense of inducing instabilities) intersubband processes reduces to estimating the scaling dimensions of corresponding cosine operators in terms of the parameters of the harmonic part. In their turn, these parameters are related to the Fourier components of the electron-electron interaction potential. As it turns out, the latter relation is not that straightforward, and we will clarify this point in the next Section.

2. Galilean invariance and Pauli principle: single-subband Luttinger liquid

To begin with, we consider the simplest case when there is no intersubband interaction and the Hamiltonian is given by the sum of two single-subband Hamiltonians \(H^0\). (As our discussion is referred now to a single subband, we suppress temporarily the subband index.) For a given effective 1D interaction potential \(U(x)\), the Luttinger-liquid parameters (\(K\) and \(v\)) depend on the \(q = 0\) and \(q = 2k_F\) Fourier components of \(U\), as well on bare Fermi velocity \(v_F\):

\[
K = K[U(0)/v_F, U(2k_F)/v_F], \quad (30)
\]

\[
v = v_FV[U(0)/v_F, U(2k_F)/v_F], \quad (31)
\]

where \(K(x, y)\) and \(V(x, y)\) are some dimensionless functions of their arguments. Relations \(30, 31\) have to satisfy (i) the Pauli principle and (ii) Galilean invariance. The Pauli principle for spinless fermions means that for the case of contact interaction, i.e., when \(U(0) = U(2k_F)\), the system should behave as if there is no interaction at all. Accordingly, \(K = 1\) and \(v = v_F\) for this case, or

\[
K(x, x) = V(x, x) = 1. \quad (32)
\]

Galilean invariance stipulates that \(Kv = v_F\), or

\[
K(x, y)V(x, y) = 1, \forall x, y. \quad (33)
\]

Physically, condition \(33\) comes about either by requiring that the shift of the ground state energy due to the motion of a system as a whole does not depend on the interaction \(34\) or by requesting that the dc conductivity of a uniform system \(35\) does not depend on interactions (Peierls theorem) \(36\). (Also, one can use the interaction-invariance of the persistent current in a ring threaded by the Aharonov-Bohm flux).

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\(^{0}\) Here we consider a uniform Luttinger liquid. The role of reservoirs, to which the wire is attached to, will be discussed in Sec.V.
Conventional bosonization of the \textit{g}-ology Hamiltonian (see, e.g., review\cite{1}) leads to

\begin{align*}
K &= \sqrt{\frac{2\pi v_F + g_4 - g_2}{2\pi v_F + g_4 + g_2}}, \\
v &= v_F \sqrt{(1 + \frac{g_4 - g_2}{2\pi v_F})(1 + \frac{g_4 + g_2}{2\pi v_F})}. \tag{34}
\end{align*}

In terms of the Fourier components of the interaction potential, the \textit{g}-parameters are expressed as \(g_4 = U(0)\) (right-right and left-left amplitude) and \(g_2 = U(0) - U(2k_F)\) (right-left amplitude), and Eq.\,(34) gives

\begin{align*}
K &= \left[ 1 + \frac{U(2k_F)}{2v_F} \right]^{-1/2}, \\
v &= v_F \left[ 1 + \frac{U(2k_F)}{2\pi v_F} \right]^{1/2} \cdot \left[ 1 + \frac{2U(0) - U(2k_F)}{2\pi v_F} \right]^{1/2}. \tag{wrong}
\end{align*}

One can see that expressions above do not satisfy conditions (32), (33). Indeed, it follows from Eq.\,(wrong) that \(v \neq v_F\) for \(U(0) = U(2k_F)\) and that \(Kv \neq v_F\) as long as \(U(2k_F) \neq 0\). Usually, spinless Luttinger liquid model does not include backscattering explicitly. The rationale for such a simplification is that for spinless particles in 1D this process is indistinguishable from forward scattering, see, e.g., Ref.\cite{14}. We do not find this approach satisfactory, as it is clear that the behavior of the system should be determined both by forward and backward amplitudes. Also, correct expressions for \(K\) and \(v\) should include both \(U(0)\) and \(U(2k_F)\), otherwise the Pauli principle cannot be satisfied. This argument can also be re-phrased in terms of direct and exchange contributions to the self-energy\cite{1}.

What did we do wrong to arrive at the Luttinger-liquid model which does not satisfy two basic physical principles? As one can show by using the Ward identities (conservation laws) for the system of interacting electrons with \textit{linear} spectrum\cite{2}, the problem occurs already at the level of fermions and is thus not inflicted by some subtleties of bosonization. Rather, it is a manifestation of an \textit{anomaly}, i.e., a violation of the conservation law caused by regularization, which one is forced to use in a model with linear and unbound spectrum\cite{2}.

One way to deal with this problem is to replace Eqs.\,(wrong) by expressions which do not follow directly from the original fermion Hamiltonian with linear spectrum, but do satisfy all necessary criteria. This is an accord with the point of view\cite{2} that one should consider \(K\) and \(v\) as phenomenological parameters, which are renormalized from their bare values by irrelevant or marginal operators neglected in the course of linearization. It turns out that one can find \textit{exact} expressions for \(K\) and \(v\), satisfying a minimal set of requirements.

First, we notice that the Pauli principle requires \(K(x,y)\) to be a function of either \(x - y\) or \(x/y\). The latter choice contradicts to the requirement that \(K\) must have Taylor expansions both around \(x = 0\) and \(y = 0\). Therefore,

\begin{align*}
K(x,y) &= \kappa(x - y), \tag{35} \\
\kappa(0) &= 1. \tag{36}
\end{align*}

Then we notice that the model with forward scattering only, i.e., the original Luttinger model\cite{2}, does respect Galilean invariance. Therefore one can take the Luttinger model expression for \(K\) as the correct one, which means that

\begin{equation}
K(x,0) = 1/\sqrt{1+x}. \tag{37}
\end{equation}

Combining Eq.\,(36) with Eq.\,(37), we see that

\begin{align*}
K(x,y) &= 1/\sqrt{1+x-y}, \tag{38} \\
\mathcal{V}(x,y) &= \sqrt{1+x-y}, \tag{39}
\end{align*}

or

\begin{align*}
K &= \left[ 1 + \frac{U(0) - U(2k_F)}{2\pi v_F} \right]^{-1/2}, \tag{40} \\
v &= v_F \left[ 1 + \frac{U(0) - U(2k_F)}{2\pi v_F} \right]^{1/2}. \tag{41}
\end{align*}

The physical meaning of Eqs.\,(40),(41) is obvious: the effective interaction is equal to backscattering minus forward scattering. One can check that a Luttinger-liquid model with parameters given by Eqs.(40),(41) reproduces correctly
results for a 1D electron system, obtained without linearization but in the limit of weak interactions. For instance, the (inverse) compressibility of a Luttinger liquid, parametrized by $K$ and $v$ from (42), is given by

$$\frac{1}{\chi} = \frac{\pi v}{K} = \pi v_F + U(0) - U(2k_F).$$

(42)

As one can check, Eq. (42) coincides with the inverse compressibility of electrons with a quadratic spectrum obtained in the Hartree-Fock approximation. A perturbative (linear in $U$) form of Eqs. (40, 41) has recently been derived in 43. It can also be read off from the tunneling exponent of a 1D system with a quadratic spectrum.

3. Galilean invariance and Pauli principle: coupled subbands

Now we allow for harmonic coupling between subbands, i.e., take into account intersubband forward [Eq.(25)] and exchange backscattering [Eq.(27)]. For the contact interaction, the amplitudes of these two processes coincide: $v_1 = v_2 = f_0$ and, as a result, inter-subband interaction drops out. The Pauli principle is thus satisfied. Intersubband exchange backscattering does violate Galilean invariance, and the correction procedure, similar to that for a single subband, is necessary. We will not do it here however, because for a long-range interaction ($a/d \ll 1$), the violation is “weak”: the deviation from the Galilean-invariant result is proportional to the exchange amplitudes, which are small compared to the direct ones.

A. Nearly equivalent subbands

First, we discuss the CDW-instability, which may occur if the density equilibration between subbands is energetically favorable. To simplify the discussion, we consider the case of a long range interaction ($a/d \ll 1$), when amplitudes of exchange processes are small. In the leading order, $f_C = b_{1,2} = 0$ and the only “dangerous” process to be considered is direct intersubband backscattering. Furthermore, we assume that subbands are nearly equivalent and put $v_1 = v_2 = v_{2F}$ and $K_1 = K_2$ but keep $\delta k_F = k_{1F} - k_{2F}$ in Eq. (20) finite. It is convenient to introduce symmetric and antisymmetric combinations of boson fields

$$\varphi_{\pm} = \frac{\varphi_1 + \varphi_2}{\sqrt{2}}; \quad \theta_{\pm} = \frac{\theta_1 + \theta_2}{\sqrt{2}},$$

(43)

which correspond to fluctuations of total (+) and relative (-) subband charge and current. In terms of these fields,

$$H = H_+ + H_-,$n$$

$$H_+ = \frac{1}{2} \int dx \left\{ \frac{v_+}{K_+} (\partial_x \varphi_+)^2 + v_+ K_+ (\partial_x \theta_+)^2 \right\},$$

(44)

$$H_- = \frac{1}{2} \int dx \left\{ \frac{v_-}{K_-} (\partial_x \varphi_-)^2 + v_- K_- (\partial_x \theta_-)^2 + \frac{f_{bs}}{\pi^2 a^2} \cos[\sqrt{8} \pi \varphi_- + 2 \delta k_F x] \right\},$$

(45)

where

$$K_+ = \left[ 1 + \frac{2U(0) - U(2k_F)}{\pi v_F} \right]^{-1/2}; \quad K_- = \left[ 1 - \frac{U(2k_F)}{\pi v_F} \right]^{-1/2},$$

(46)

and $v_{\pm} = v_F/K_{\pm}$. Note that $K_- > 1$ for $U(2k_F) > 0$, which signals effective attraction in the (-) channel.

1. Collective adjustment of densities as a commensurate-incommensurate transition

To understand how the CDW-instability works, we consider first a model situation of $K_- < 1$, so that operator $\cos[\sqrt{8} \pi \varphi_-]$ is relevant in the RG sense. Finite $\delta k_F$ stops the RG-flow at scale $\ell \sim 1/\ln|\delta k_F|\alpha$, thus precluding the system from reaching its strong-coupling limit. However, this consideration does not take into account the possibility of a collective density readjustment between subbands. Such a readjustment may occur, if the kinetic energy loss $\Omega = v_F \delta k_F$ is compensated by the gain in the potential energy due to opening of the gap in the (-) channel. In other
words, when the difference in electron densities is sufficiently small, the total energy is minimized by equating the densities and opening the charge gap. Obviously, such a process cannot be considered at the level of single-particle description of transverse quantization. Instead, one should now treat eigenstates and eigenenergies of the wire as being determined by a self-consistent procedure, involving both single-particle and many-body effects.

The mechanism described above can be considered as a commensurate-incommensurate transition. The incommensurability, defined as \( I = L^{-1} \left( \int dx \partial_x \phi_- \right) \), where \( L \) is the length of the wire, is known to have a threshold behavior[44]. \( I \sim \sqrt{\Omega^2 - \Omega_c^2} \Theta(\Omega - \Omega_c) \), where \( \Omega_c = \sqrt{2\pi K_{-CDW}} \) and the expression for the gap follows from mapping on the exactly solvable model of a Heisenberg spin chain[45]. \( \Delta_{CDW} \sim (f_{bs})^{1/2(1-K_-)} \). As follows from the definition of the incommensurability, \( I = 0 \) implies \( \delta k_F = 0 \), i.e., equal subband densities. Therefore, the re-adjustment takes place if \( \Omega < \Omega_c \). Backscattering is then enabled and relevant (for \( K_- < 1 \)), even for a non-zero initial value of \( \delta k_F \).

What is the physical meaning of this instability? A simple picture can be obtained in the limit of strong (both inter- and intra-subband) interactions, when the potential energy dominates over quantum fluctuations. In this case, electrons of each of the subbands form a regular lattice (Wigner crystal). Boson fields \( \phi_n \) also have periodic structures with period equal to \( \sqrt{\pi} \) (recall that a shift of \( \sqrt{\pi} \) corresponds to adding one electron to the system). For \( f_{bs} > 0 \), the energy of intrasubband repulsion

\[
  U_{bs} \cos[\sqrt{\pi} \phi_-] = -U_{bs} \cos[\sqrt{\pi} (\phi_1 - \phi_2 + \sqrt{\pi}/2)]
\]

is minimized by a relative phase shift of \( \sqrt{\pi}/2 \) between the subbands, which corresponds to a shift of electron lattices by half-a-period. This is an inter-subband charge-density wave (CDW).

![FIG. 5. An illustration of the charge density wave in two coupled subbands. A staggered configuration lowers the energy due to short-range repulsion, if the densities are commensurate.](image)

2. Competition between CDW and Cooper channels

Let us now suppose that the density re-adjustment did occur, i.e., \( \delta k_F = 0 \), but Cooper scattering is also present, so that the Hamiltonian of the (-)-channel is

\[
  H_- = \frac{1}{2} \int dx \left\{ \frac{v_-}{K_-} (\partial_x \phi_-)^2 + v_- K_- (\partial_x \theta_-)^2 + \frac{U_{bs}}{\pi^2 q^2} \cos \sqrt{8\pi \phi_-} + \frac{f_C}{2\pi^2 q^2} \cos \sqrt{8\pi \theta_-} \right\}.
\]

(48)

Which of the two instabilities–CDW or superconductivity–wins? The situation of this type, when cosines of both mutually conjugated fields (\( \phi_- \) and \( \theta_- \)) are present, was analyzed by Schulz and Giamarchi[46]. They found that the result is very sensitive not only to the value of \( K_- \), which determines the scaling dimensions of the fields, but also to the ratio of amplitudes, \( f_C/f_{bs} \). As \( K_- > 1 \) for repulsive \( U(x) \), it may seem that superconductivity is favored over CDW. The situation is not that straightforward, however. For example, consider the situation of weak and long-range interactions, i.e., assume that \( U(0), U(2k_F) \ll v_F \) and \( a \ll d \). Because the interaction is weak, both processes are almost marginal, CDW being on the slightly irrelevant and superconductivity on the slightly relevant side. For long-range interactions, \( f_{bs} \sim U(2k_F) \) and \( f_C \sim (a/d)^2 U(0) \). Modeling \( U(x) \) by

\[
  U(x) = \left\{ \begin{array}{ll}
    e^2 / \epsilon x, & \text{for } x < d; \\
    0, & \text{for } x > d,
  \end{array} \right.
\]

we get \( U(2k_F)/U(0) \sim \ln(k_Fa)/\ln(d/a) \). Thus

\[
  \frac{f_C}{f_{bs}} \sim \left( \frac{a}{d} \right)^2 \ln d/a \ln k_F a \ll 1.
\]

(49)

The RG-equation for \( K_- \)[46]

\[
  \frac{d}{dt} K_- = f_C^2 - f_{bs}^2
\]

(50)
shows that $K_-$ decreases, if $|f_C| < |f_{sn}|$. Even if initially $K_-(0) > 1$, the situation with $K_-(l) < 1$, when CDW is relevant, will be reached in the process of renormalization. For weak and long-ranged interactions, CDW thus wins over superconductivity.

If interactions are not sufficiently weak and/or long-ranged, only a full RG solution can determine the leading instability. We will not analyze the general case here.

B. Non-equivalent subbands: Renormalization Group

Now we consider a generic situation of non-equivalent subbands, when $\delta k_F$ is not small enough for the density re-adjustment to occur. We find that a strong imbalance between Fermi-velocities of occupied subbands actually helps superconducting instability to develop (see case B below), despite the fact that a naive scaling dimension estimate does not show this. This effect follows from the next-to-leading order perturbative RG calculations, which we present here.

Because $\delta k_F \neq 0$, we neglect the dBS process [Eq. (26)] from the outset but keep the Cooper one [Eq. (28)]. For long-range interaction, one can also neglect the xBS process, Eq.(27), whose amplitude is small for this case: $b_1 \propto (a/d)^2$. Its inclusion is straightforward ($U^B_x$ is quadratic) but does not lead to any qualitatively new results, while complicating the analysis significantly. The Hamiltonian then reads

$$\tilde{H} = \tilde{H}_0 + \tilde{U}^F + \tilde{U}^C.$$  

Because $\tilde{U}^C$ contains $\theta$-fields, it is convenient to switch from the Hamiltonian to the Lagrangian approach and to integrate out the $\varphi$-fields. The quadratic part of the resulting action is diagonalized by the following transformation

$$\left( \begin{array}{c} \bar{\theta}_1 \\ \bar{\theta}_2 \end{array} \right) = \left( \begin{array}{cc} \mu_1 & 0 \\ 0 & \mu_2 \end{array} \right) \left( \begin{array}{cc} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{array} \right) \left( \begin{array}{c} \sqrt{v_1 K_1} \bar{\theta}_1 \\ \sqrt{v_2 K_2} \bar{\theta}_2 \end{array} \right),$$  

where

$$\tan 2\beta = \frac{u_0^2}{v_1^2 - v_2^2}, \quad u_0 = \left[ 2 f_0 \sqrt{v_1 K_1 v_2 K_2 / \pi} \right]^{1/2},$$  

and

$$\mu_1 = \frac{\cos \beta}{\sqrt{v_1 K_1}} - \frac{\sin \beta}{\sqrt{v_2 K_2}},$$  
$$\mu_2 = \frac{\sin \beta}{\sqrt{v_1 K_1}} + \frac{\cos \beta}{\sqrt{v_2 K_2}}.$$  

In terms of new fields, the action is given by

$$S = \frac{1}{2} \int dx d\tau \left[ \sum_n R_n \left\{ \frac{1}{u_n^2} (\partial_\tau \bar{\theta}_n)^2 + u_n (\partial_x \bar{\theta}_n)^2 \right\} + \frac{f_C}{\pi^2 \alpha^2} \cos \sqrt{4\pi}(\bar{\theta}_1 - \bar{\theta}_2) \right],$$  

where

$$u_n^2 = \frac{1}{2} \left( v_1^2 + v_2^2 \pm \sqrt{(v_1^2 - v_2^2)^2 + u_0^4} \right)$$  

are the velocities of new collective modes and $R_n = 1/(u_n \mu_n^2)$ are the new stiffness coefficients.

We are now ready to perform the momentum-shell RG, i.e., to expand perturbatively in coupling constant $f_C$ and integrate out high-energy modes with 2-momentum $k$ within a thin strip $\Lambda - d\Lambda \leq k \leq \Lambda$ ($d\Lambda / \Lambda \ll 1$). The first-order contributions renormalize $f_C$, whereas the second-order ones renormalize stiffnesses $R_n$. The main difference between our case and the conventional RG-treatment of the sine-Gordon action (see, e.g., Ref.[3]) is that the $f_C^2$-contribution produces (among others) mixed gradient terms of the type $\partial_\tau \bar{\theta}_1 \partial_\tau \bar{\theta}_2$ ($\nu = \tau, x$), which are absent in bare action (53). To eliminate these terms, we transform fields one more time:

$$\left( \begin{array}{c} \bar{\theta}_1 \\ \bar{\theta}_2 \end{array} \right) = \left( 1 + \frac{\beta'}{t'} \right)^{1/2} \left( \begin{array}{cc} \cosh \beta' & t' \sinh \beta' \\ -\frac{\beta'}{t' \sinh \beta'} & \cosh \beta' \end{array} \right) \left( \begin{array}{c} \bar{\theta}_1 \\ \bar{\theta}_2 \end{array} \right),$$  

(57)
are chosen in such a way that the coefficients in front of mixed gradient terms vanish. When written in terms of \( \theta_n \), the action is brought into its original form but with renormalized parameters. Resulting RG equations read

\[
\frac{d}{dl} R_1 = -\bar{f}^2 \frac{1}{R_1} \left( \frac{4\gamma^2}{R_1^2(1+\gamma^2)} + \frac{2}{R_2^2(1+\gamma^2)} + \frac{\gamma}{R_1 R_2} \right),
\]

\[
\frac{d}{dl} R_2 = -\bar{f}^2 \frac{1}{R_2} \left( \frac{4}{R_1^2(1+\gamma^2)} + \frac{2\gamma^2}{R_2^2(1+\gamma^2)} + \frac{1}{R_1 R_2} \right),
\]

\[
\frac{d}{dl} \gamma = \bar{f} \frac{1-\gamma^2}{R_1 R_2},
\]

\[
\frac{d}{dl} \bar{f} = \left( 2 - \frac{1}{R_1} - \frac{1}{R_2} \right) \bar{f},
\]

where \( \bar{f} = (1/\pi) f_C \sqrt{u_1^{-2} + u_2^{-2}} \) is the dimensionless coupling constant and \( \gamma = u_1/u_2 \). To the \( \bar{f}^2 \)-accuracy, all terms multiplying \( \bar{f}^2 \) on the right-hand-side of the first three equations above have to be treated as constants determined by the initial conditions. The system of RG-equations has an obvious integral of motion

\[
C = \frac{x^2}{c_1} + \frac{y^2}{c_2} - \bar{f}^2,
\]

where

\[
x \equiv 1 - 1/R_1, \quad y \equiv 1 - 1/R_2,
\]

and \( c_1, c_2 \) are the coefficients in front of \( \bar{f}^2 \) in Eqs. (52-60), respectively. Note also that \( x = (c_1/c_2) y + p \) (we denote \( x = x(l), \ y = y(l) \), whereas initial values are denoted by sub-index 0, i.e., \( x(0) = x_0, \) etc.). Constants of motion \( C \) and \( p \) are determined by initial conditions.

The flow described by (58-64) is quite similar to that of a canonical Kosterlitz-Thouless system: \( x \) and \( y \) increase with \( \bar{f} \) regardless of its sign. If \( x_0, \ y_0 > 0 \), \( \bar{f} \) grows unrestrictedly, flowing into the strong-coupling regime with a gap in the \( \theta_1 - \theta_2 \) channel. Such initial conditions correspond to \( R_{1,2} > 1 \), i.e., to the attractive interaction in the \( \theta_n \)-channels. It is worth emphasizing here that due to the presence of inter-subband forward scattering, such effective attraction may arise in a purely repulsive system, as we shall demonstrate shortly (case \( A \) below). Another relevant limit is represented by the “repulsive” case (case \( B \)), where initially \( x_0 < 0, \ y_0 < 0 \). For a strong repulsion (\( R_{1,2} \ll 1 \), \( \bar{f} \) quickly renormalizes to zero and the resulting phase is a two-subband Luttinger liquid. However, there is a region of anomalously small \( x_0, y_0 \sim f_0 \) (which requires strong inter-subband scattering), where Cooper scattering may still be important. One finds that if

\[
(c_1 + c_2) \bar{f}_0^2 > (x_0 + y_0)^2
\]

The Cooper process wins over repulsion and initially negative variables \( x, y \) change sign during renormalization. In this case \( f_c \) decreases with \( l \), initially but then passes through a minimum and flows finally into strong-coupling regime \( \bar{f} \geq 1 \). Equation (65) is a condition for the development of superconducting fluctuations in the system with purely repulsive interactions.

Now we again apply our analysis to a wire with weak and long-range interactions. Two limits are possible.

**Case A:** \( \Delta_{CDW}/k_F \ll v_{1F} - v_{2F} \ll U_0 \).

The first inequality allows one to neglect direct backscattering, which leads to inter-subband CDW, whereas the second one allows to consider subbands as “nearly equivalent”. Denoting \( v_\ell = v_{1\ell} - v_{2\ell} \), and the \( 2k_F \)-component of the interaction potential in the \( n \)-th subband by \( U_n^{(2)} \), one finds

\[
\frac{1}{R_1} + \frac{1}{R_2} = 2 - \frac{U_1^{(2)} + U_2^{(2)}}{2\pi v_F} - \frac{\delta v_F U_2^{(1)}}{2\pi v_F} \leq 2,
\]
which corresponds to effective attraction. Thus, thanks to inter-subband forward scattering, the Cooper process is relevant in the system with purely repulsive interaction. Note that a small velocity imbalance \( \delta v_F > 0 \) enhances the relevance of Cooper scattering. (As our second subband is chosen to have a higher energy, \( \delta v_F \) is always positive.)

**Case B: \( v_{1F} - v_{2F} \gg U(0) \). Van Hove singularity.**

In this limit an analytic solution is also possible. Generally, one finds that \( 1/R_1 + 1/R_2 > 2 \), which corresponds to effective repulsion. Neither CDW nor superconducting instability can develop, and the resulting phase is a two-subband Luttinger liquid. This is not true, however, in the limit of a strong velocity imbalance, when \( v_{2F}/v_{1F} \ll 1 \), i.e., when the second subband just opens for conduction. Then \( u_n \approx v_n \) and \( \gamma = u_1/u_2 \sim v_{1F}/v_{2F} \gg 1 \). Hence \( c_1 \sim \gamma \) and it follows from (21) that Cooper process wins over repulsion, if \( f_C > U(0)/\sqrt{\gamma} \). For long-range interactions, the last inequality reduces to

\[
\frac{v_{2F}}{v_{1F}} = \left( \frac{a}{d} \right)^4.
\]

The physics of this scenario is well-known - interactions are enhanced due to the large value of the density of states \((\propto 1/v_{2F})\) in the upper subband (Van Hove singularity). We should warn here that our calculations do not describe the very onset of conduction in the upper subband, because its proper description requires accounting for the nonlinearity of the electron spectrum, which is beyond our bosonization analysis. However, such a calculation was performed by Balents and Fisher, who analyzed the case of a contact interaction. They found that superconducting fluctuations are indeed enhanced in this limit.

We thus see that superconducting fluctuations do have a good chance to overcome the electron-electron repulsion and drive the system into a strong-coupling phase with the gap in the spectrum of relative current fluctuations, \( \theta_1 - \theta_2 \).

There is another important feature of the RG-flow described by Eqs. (11)-(12): the interaction tries to equilibrate densities in the subbands. This is seen from equation (13): \( dv/dl \) is proportional to \( 1 - \gamma^2 \), which makes \( \gamma = 1 \) a stable fixed point. \( \gamma \) tends to increase, if initially \( \gamma_0 < 1 \), and it tends to decrease, if \( \gamma_0 > 1 \).

**IV. ELECTRONS WITH SPINS**

Guided by the results of the previous Section, we now comment briefly on what happens if spin is included. As should be clear from the complexity of Eqs. (11)-(21), this question has no simple answer. For a quantum wire with \( 0 < U(2k_F) < U(0) \), possible phases are again (i) Luttinger liquid, (ii) inter-subband CDW, and (iii) Cooper phase (superconductor). As with spinless electrons, subbands must be nearly equivalent in order for the CDW phase to occur, whereas the Cooper phase needs effective attraction in the relative charge-density excitation channel. When neither of these conditions is met, a two-subband Luttinger liquid is realized. At different degrees of generality, renormalization group analysis of the model defined by Eqs. (11)-(21) has been performed in the past and we refer to papers [4]-[17] for a detailed description.

1. **Long-range interactions**

If the interaction is long-range and weak, a considerable simplification occurs. In this case, amplitudes of forward intra- and inter-subband processes are the same [see discussion after Eq. (22)] and a simple perturbative estimate of scaling dimension \( \delta_C \) of the Cooper process (21) is possible.

For weak interactions, \( K_{n\sigma} = 1 - (2U(0) - U_{2k_F}^{(n)})/2\pi v_nF \), and \( f_0 = U(0) \). The \( SU(2) \)-invariance requires that \( K_{n\sigma} = 1 \). One finds

\[
\delta_C = 2 - \frac{U_{2k_F}^{(2)} + U_{2k_F}^{(2)}}{4\pi v_F} - \frac{\delta v_F U_{2k_F}^{(1)}}{v_F 4\pi v_F} \leq 2.
\]

Observe that this result coincides with Eq. (66) upon replacing \( U(0) \to 2U(0) \). Thus, Cooper scattering is relevant for repulsive long-range interactions (and assuming also that \( 0 < \delta v_F < U_0 \)).

However, if \( K_{n\sigma} \) is sufficiently close to its non-interacting value, i.e., to unity, backscattering is strong \( (f_{bs} \gg t_{sp}, t_{tp}) \), and \( \delta k_F \) is small, the CDW-channel can take over the Cooper one, similarly to the scenario described in Sec. III A 3.
2. Electron ladder

There is an important question where an RG consideration is very helpful. Suppose that condition (68) is satisfied and thus tunneling of fluctuational Cooper pairs is relevant. What happens to spin excitations? To answer this question, we relax the $SU(2)$-invariance condition and perform the RG calculation for two nearly equivalent subbands so that $v_{n\nu} = v_{n\sigma}, K_{n\nu} = K_{n\sigma}$, where $\nu = \rho, \sigma$. Nevertheless, we assume that $\delta k_F$ is still finite and neglect direct backscattering [19], similarly to Subsection III A. The problem then becomes identical to that of two coupled equivalent chains coupled by the interaction ("electron ladder"). Also, for the sake of simplicity, we consider only the singlet backscattering (19), similarly to Subsection III A. The problem the n becomes identical to that of two coupled equivalent chains coupled by the interaction ("electron ladder").

In addition, we have to keep track of the spin-density sector, which contain the cosine term corresponding to intra-subband spin backscattering [17]. For convenience, we denote the amplitude of this term by $g_\sigma$, its initial value being $g_\sigma(0) = U(2k_F)$. After tedious but straightforward calculations we arrive at the following system of RG equations:

$$K_{{\rho}_+} = \left[ K_{\rho}^{-2} + 2f_0/\pi v_F \right]^{-1/2}. \quad (69)$$

Relative current fluctuations $\theta_{{\rho}_-} = (\theta_{1\rho} - \theta_{2\rho})/\sqrt{2}$ are described by the sine-Gordon theory [$t_{sp}$ term in (21)] with

$$K_{{\rho}_-}^{-2} = \left[ K_{\rho}^{-2} - 2f_0/(\pi v_F) \right]^{-1/2} \quad \text{and} \quad v_{{\rho}_-} = v_\rho[1 - 2f_0K_\rho/\pi v_F]^{1/2}. \quad (70)$$

In addition, we have to keep track of the spin-density sector, which contain the cosine term corresponding to intra-subband backscattering (17). For convenience, we denote the amplitude of this term by $g_\sigma$, its initial value being $g_\sigma(0) = U(2k_F)$. After tedious but straightforward calculations we arrive at the following system of RG equations:

\[
\begin{align*}
\frac{d}{dt} \tilde{g} &= 2(1 - K_\sigma)\tilde{g} - \tilde{t}^2, \\
\frac{d}{dt} \tilde{t} &= (2 - K_\sigma - \frac{1}{K_{\rho}} - \tilde{g})\tilde{t}, \\
\frac{d}{dt} (1 - K_\sigma) &= \frac{1}{2} (\tilde{g}^2 + \tilde{t}^2), \\
\frac{d}{dt} (1 - \frac{1}{K_{\rho}}) &= \tilde{t}^2,
\end{align*}
\]

where $\tilde{g} = g_\sigma/(\pi v_F)$ and $\tilde{t} = t_{sp}/(\pi v_F)$. Let us recall what happens in the absence of Cooper tunneling first and set $t_{sp} = 0$ everywhere in this system. In the weak-coupling limit, $K_\sigma = (1 - \tilde{g})^{-1/2} \approx 1 + \tilde{g}/2$ and (71) becomes $d\tilde{g}/dt = -\tilde{g}^2$, which gives $\tilde{g}_t = \tilde{g}_0/(1 + \tilde{g}_0\ell)$. For repulsive interactions ($\tilde{g}_0 > 0$), $\tilde{g} \propto \ell^{-1} \to 0$ as $\ell \to \infty$: intra-subband backscattering is marginally irrelevant. Observe now that when the Cooper process is present and relevant, i.e., when $\tilde{t}$ increases, the flow of $\tilde{g}$ is modified: the $\tilde{t}^2$-term on the right-hand-side of Eq. (71) changes the sign of $d\tilde{g}/dt$. Hence $\tilde{g}_t$ is bound to become negative in the process of renormalization and grows unrestrictedly in its absolute value. Intra-subband spin backscattering is thus driven relevant by singlet-pair tunneling, which results in pinning of $\varphi_\sigma$ in Eq. (17) and opening of the spin gap. Thus, similar to the true superconducting state in higher dimensions, the Cooper phase is characterized by gaps in both charge- and spin-channels. The only massless excitations are those of the total charge channel. This phenomenon is not restricted to the degenerate electron ladder but rather is a generic feature of the system of coupled subbands and/or chains, see, e.g., Refs. [13, 17].

V. CONDUCTANCE

Having realized the importance of inter-subband interactions, we now proceed with the analysis of its effect on observable properties of quantum wires. The first property we consider is conductance $G$.

A. No disorder

Our results for the conductance of a clean wire can be understood from the following simple considerations. The dc conductance of a single-subband wire is equal to $2e^2/h$ regardless of the interactions in the wire [14, 24]. Consider now a wire with several subbands occupied. Those interband interactions, which do not open gaps, lead only to a renormalization of Luttinger-liquid parameters. As these parameters do not enter the final result for $G$, the conductance remains quantized in units of $2e^2/h$ per occupied subband. Other processes, such as direct backscattering and
Cooper scattering, may open gaps in channels of relative charge fluctuations as well as in spin channels. Neither of these gaps, however, affects the center-of-mass of the electron fluid through the wire, which continues to move along the wire unrestrictedly. Therefore, $G$ remains unrenormalized by this type of interactions as well.

Now we demonstrate the proof of the statements made above. Consider the case of a superconducting instability, when the cosine term of Cooper scattering in Eq. (33) is relevant and the $\theta_{\rho_\perp}$-field is thus gapped. Gaussian fluctuations of the gapped field can be described by expanding the relevant cosine term around its minimum value:

$$\left(4f_C / \pi^2 \alpha^2\right) \cos[\sqrt{2\pi}(\theta_{1\rho} - \theta_{2\rho})] \approx \text{const} + m^2(\theta_{1\rho} - \theta_{2\rho})^2,$$

where $m$ is the mass of the field. The strong-coupling (superconducting) phase corresponds to $m \neq 0$, whereas in the Luttinger-liquid phase $m = 0$. Expanding $\cos[\sqrt{2\pi} \varphi_{m,n}]$ around their minima as well, we find that at the Gaussian level charge and spin modes decouple again. Note though that now these are massive modes.

As spin excitations do not affect charge transport, we concentrate on the charge sector of the theory, whose Hamiltonian is given by the sum of Eqs. (14) and \((72)\). To simplify notations, we suppress index $\rho$ in this section, so that $\theta_{\rho} \rightarrow \theta_-$, etc. Using $v_n K_n = v_n F$, we write the charge Hamiltonian as

$$H_\rho = \frac{1}{2} \int dx \left\{ \frac{v_1 F + v_2 F}{2} \left[ (\partial_x \varphi_+)^2 + (\partial_x \varphi_-)^2 \right] + \frac{1}{2} \left( \frac{v_1}{K_1} + \frac{v_2}{K_2} \right) \left[ (\partial_x \varphi_+)^2 + (\partial_x \varphi_-)^2 \right] \right.$$

$$+ \frac{2f_0}{\pi} \left[ (\partial_x \varphi_+)^2 - (\partial_x \varphi_-)^2 \right] + (v_1 F - v_2 F) \partial_x \varphi_+ \partial_x \varphi_+ + \left( \frac{v_1}{K_1} - \frac{v_2}{K_2} \right) \partial_x \varphi_+ \partial_x \varphi_- + m^2 \theta_-^2 \right\}. \tag{76}$$

The total charge current is given by $j = e \sqrt{2}/\pi \sum_n \partial_t \varphi_n = e(2/\sqrt{\pi}) \partial_t \varphi_+$. The conductivity is related to the retarded Green’s function $G_{++}(x,t) = -i\Theta(t)[\langle \varphi_+(x,t), \varphi_+(0,0) \rangle]$, which is coupled to another Green’s function $G_{--}(x,t) = -i\Theta(t)[\langle \varphi_-(x,t), \varphi_+(0,0) \rangle]$ by the following equations of motion

$$(-i\partial_t)^2 G_{++} = \frac{v_1 + v_2}{2} \delta(x) \delta(t) - \frac{1}{2} \partial_x \left\{ \frac{v_1^2}{K_1} + \frac{v_2^2}{K_2} + \frac{2f_0}{\pi} (v_1 + v_2) \right\} \partial_x G_{++}$$

$$- \frac{1}{2} \partial_x \left\{ \frac{v_1^2}{K_1} - \frac{v_2^2}{K_2} - \frac{2f_0}{\pi} (v_1 - v_2) \right\} \partial_x G_{--};$$

$$(-i\partial_t)^2 G_{--} = \frac{v_1 - v_2}{2} \delta(x) \delta(t) - \frac{1}{2} \partial_x \left\{ \frac{v_1^2}{K_1} + \frac{v_2^2}{K_2} - \frac{2f_0}{\pi} (v_1 + v_2) \right\} \partial_x G_{--}$$

$$- \frac{1}{2} \partial_x \left\{ \frac{v_1^2}{K_1} - \frac{v_2^2}{K_2} + \frac{2f_0}{\pi} (v_1 - v_2) \right\} \partial_x G_{++}$$

$$- m^2 \left\{ \frac{1}{2} \left( \frac{v_1}{K_1} + \frac{v_2}{K_2} \right) - \frac{2f_0}{\pi} \right\} G_{--} - m^2 \frac{1}{2} \left( \frac{v_1}{K_1} - \frac{v_2}{K_2} \right) G_{++}. \tag{78}$$

In the massless limit $(m = 0)$, this system of equations is solved readily. In order to model the effect of non-interacting electron reservoirs, which the wire is attached to, we assume that $K_n, v_n$ vary with $x$ adiabatically and approach their non-interacting values of $K_n = 1, v_n = v_F$ for $x \rightarrow \pm \infty$. In the zero-frequency limit, the solution is particularly simple: $G_{++}(x, \omega \rightarrow 0) = 1/2i\omega$, $G_{--}(x, \omega = 0) = 0$. As a result, conductivity is $x$-independent, and the conductance is simply $G = 2 \times 2e^2/h$.

In order to see the effect of the gap, we consider first the case of equivalent subbands (“electron ladder”), introduced in Sec. 4.2. One observes immediately that conditions $v_1 = v_2$, $K_1 = K_2$ lead to complete decoupling of equations for $G_{++}$ and $G_{--}$. As a result, the total charge mode $\varphi_+$ is not affected by the gap. Taking the boundary condition for $K$ and $v$ into account gives again the universal result $G = 4e^2/h$. The result for the "electron ladder" thus gives us a hint that $G$ remains at its universal value despite the presence of the gap in the relative charge channel. In order to prove this statement in the general case, we neglect for the moment the boundary conditions for $K_n$ and $v_n$, i.e., consider a uniform wire with two coupled subbands. System (78) is then solved by Fourier transformation. The key feature of the result for $G_{++}(q, \omega)$ is that it still has a pole corresponding to a massless mode $\omega \propto q$, despite the presence of the massive term. The conductivity becomes

$$\sigma(q, \omega) \sim \frac{\omega F(\omega, q)}{(\omega^2 - q^2)(\omega^2 + \omega_m^2 - q^2)^2}, \tag{79}$$
where \( \bar{v}, \bar{u} \) are some renormalized velocities, \( \omega_m \) is some energy proportional to \( m^2 \), and \( F(\omega, q) \) is a smooth function of its arguments. As a result, \( \sigma(q, \omega) = \bar{G} \delta(q) \) in the limit \( \omega \to 0 \), where \( \bar{G} \) has a meaning of the conductance. Because we neglected the boundary conditions corresponding to presence of non-interacting leads in this calculation, \( \bar{G} \) depends on all interaction parameters – \( v_n, K_n \), and \( m^2 \). and is of course different from \( 4e^2/h \). It is clear though that once the boundary conditions are restored, this non-universal value is replaced by the universal factor of \( 4e^2/h \). The only other possibility is \( G = 0 \), which, however, is ruled out by the fact that \( G_{++}(q, \omega) \) has a massless pole.

We thus conclude that the conductance of a clean wire remains at the universal quantized value irrespective of whether the relative charge mode is gapped or not. The case of a CDW instability can be treated in a similar manner.

### B. Disordered wire

A disordered two-band system in the presence of interaction-induced instabilities was considered by Orignac and Giamarchi\(^{22}\) and by Egger and Gogolin\(^{23}\). Our discussion of a disordered two-subband wire follows largely these two papers.

Results of the subsequent analysis can be summarized as follows. If Cooper scattering opens a gap, the system does not become a real superconductor: a single weak impurity splits eventually the wire into two disconnected halves at low enough energies, and even a weak random potential leads to localization of electrons, similar to the case of a gapless Luttinger liquid. Nevertheless, effects of disorder are less pronounced than for a gapless Luttinger liquid. On the contrary, the CDW-state is more sensitive to disorder than a gapless Luttinger liquid.

#### 1. Spinless electrons

We begin by considering a single impurity described as a potential perturbation \( w(x, r_\perp) \). The impurity causes backscattering of electrons within the occupied subbands, as well as inter-subband backscattering. The amplitudes of corresponding processes are given by

\[
W_n(2k_nF) = \int dx dr_\perp w(x, r_\perp) \phi_n^2(r_\perp) \cos(2k_nF x), \quad n = 1, 2; 
\]

\[
W_{\text{inter}} = \int dx dr_\perp w(x, r_\perp) \phi_1(r_\perp) \phi_2(r_\perp) \cos \left[ (k_{1F} + k_{2F}) x \right].
\]

If \( w \) varies slowly across the wire, then \( W_{\text{inter}} \ll W_n \) due to the orthogonality of transverse wavefunctions, and we consider intra-subband backscattering first. The bosonized form of intra-subband backscattering is

\[
W_{\text{intra}}^n = \frac{W_n(2k_nF)}{\pi \alpha} \cos[\sqrt{4\pi} \varphi_\alpha] = \frac{W_n(2k_nF)}{\pi \alpha} \cos[\sqrt{2\pi} (\varphi_\alpha \pm \varphi_-)], \quad n = 1, 2;
\]

so the total backscattering operator is given by

\[
W_{\text{intra}} = \sum_{n=1,2} W_{\text{intra}}^n = \frac{W_2(2k_2F) - W_1(2k_1F)}{\pi \alpha} \sin[\sqrt{2\pi} \varphi_+] \sin[\sqrt{2\pi} \varphi_-]
+ \frac{W_2(2k_2F) + W_1(2k_1F)}{\pi \alpha} \cos[\sqrt{2\pi} \varphi_+] \cos[\sqrt{2\pi} \varphi_-].
\]

Note that \( W_{\text{intra}} \) is local in space and thus cannot change the RG-flows of bulk parameters of the wire. Depending on these bulk parameters, however, \( W_{\text{intra}} \) will either grow, splitting eventually the wire into two disconnected pieces, or decay, in which case the impurity effectively disappears.

(i) Cooper phase.

In the Cooper phase, \( \theta_- \) is gapped, hence \( \varphi_- \) fluctuates strongly. [This follows from the fact that \( \theta_- \) and \( \varphi_- \) are canonically conjugated fields, see Sec. IV.] On the first sight, it may seem that these strong fluctuations render \( W_{\text{intra}} \) to zero. To see that it is not so, consider a second-order impurity contribution, e.g.,

\[
\left( \frac{W_1(2k_1F)}{\pi \alpha} \right)^2 \int d\tau \int d\tau' \langle e^{i\sqrt{2\pi} \varphi_- (\tau)} e^{-i\sqrt{2\pi} \varphi_- (\tau')} \rangle \cos[\sqrt{2\pi} \varphi_+(\tau)] \cos[\sqrt{2\pi} \varphi_+(\tau')]
\]

(84)
As it will be explained in more details in Sec. [IV], the correlator of $\varphi_-$-fields in the Cooper phase decays exponentially, i.e., $\langle e^{i\sqrt{2}\pi \varphi_-(\tau)} e^{-i\sqrt{2}\pi \varphi_-(\tau')} \rangle \sim e^{-\Delta_{SC}|\tau - \tau'|}$, where $\Delta_{SC}$ is the Cooper gap in the $\theta_-$-channel. As a result, the double integration over $\tau, \tau'$ in Eq. (84) is effectively contracted into a single one, the result of integration being

$$\Delta_{SC}^{-1} \langle W(2k_F)/\pi \alpha \rangle^2 \int d\tau \cos(\sqrt{8\pi \varphi_+}(\tau)).$$

(85)

The mechanism of generating higher order impurity backscattering was discovered in [8]. $(\sqrt{\delta_\tau}$ under the cosine indicates that this is a two-particle backscattering process.) Following the RG-calculations of Kane and Fisher [4], one finds that impurity backscattering, generated in this way, becomes relevant for $K_+ < 1/2$. Note that this requires rather strong electron repulsion. For weaker repulsion, backscattering amplitude scales to zero and the wire retains a universal conductance of $2e^2/h$.

For the Cooper-phase case, $\cos(\sqrt{2\pi \delta \varphi_-}) \approx 1$, we see that the first term in (86) gives the strongest contribution to backscattering, which is relevant already for $K_+ < 2$. The second term requires more work. To the second order in the amplitude of this term, an expression similar to (84) is generated, but now it involves the following average

$$\langle \sin(\sqrt{2\pi \delta \varphi_-}(\tau)) \sin(\sqrt{2\pi \delta \varphi_-}(\tau')) \rangle \sim \frac{\Delta_{CDW}}{\Delta_0} \sinh(\tau - \tau'),$$

(87)

where $\Delta_{CDW}$ is the CDW gap, $\Delta_0$ is the ultraviolet energy cutoff, and $K_0(x)$ is the modified Bessel function, $[K_0(x) \sim x^{-1/2}]$ for $x \gg 1$. This result is due to the fact that in the massive phase

$$\langle e^{i\sqrt{2\pi \delta \varphi_-}(\tau)} e^{i\sqrt{2\pi \delta \varphi_-}(\tau')} \rangle \sim \exp[-\Delta_{CDW}/\Delta_0 \pm K_0(\Delta_{CDW}(\tau - \tau')).$$

(88)

As a result, correlation functions of sines and cosines of massive fields are different: the first ones decay exponentially with distance, whereas the second ones reach constant values. Thus the double integral over $\tau, \tau'$ in (84) can be reduced to the single one again, and, similarly to the Cooper-phase case, two-particle impurity backscattering, relevant for $K_+ < 1/2$, is generated.

It is also straightforward to analyze the effect of inter-subband impurity scattering

$$W_{\text{inter}} = \frac{2W_{\text{inter}}(k_F + k_F)}{\pi \alpha} \cos(\sqrt{2\pi \varphi_+}) \cos(\sqrt{2\pi \varphi_-}).$$

(89)

Similarly to Eq. (84), we have to average over the strongly fluctuating $\theta_-$-field, which generates again the two-particle backscattering term $\sim \cos(\sqrt{8\pi \varphi_+})$, relevant for $K_+ < 1/2$. Hence, the perturbative correction to the conductance of a CDW wire behaves as

$$-\delta G_{CDW} \propto w^2 \epsilon K_+ - 2 + (w^2/\Delta_{CDW})^2 \epsilon^4 K_+ - 2,$$

(90)

where $\epsilon = \max\{T, \text{bias}\}$ and where we have also indicated the order of the impurity potential. Please note that the exponent of the weak-link counterpart of (90), derived in Eq. (140) of Section [VII], is not related to the leading exponent $K_+ - 2$ in Eq. (90) by the conventional duality relation [4]. We conjecture that this violation of duality signals phase transition separating regimes of weak and strong tunneling.

For the Cooper-phase case, $\cos(\sqrt{2\pi \theta_-})$ is replaced by $\sin(\sqrt{2\pi \theta_-})$. Hence, $W_{\text{inter}}$ also generates the effective two-particle term in the second order of perturbation theory (cf. the analysis of the second term in (86)). The correction to the conductance is given by

$$-\delta G_{SC} \propto (w^2/\Delta_{SC})^2 \epsilon^4 K_+ - 2.$$

(91)
To summarize, the Cooper phase is insensitive to a single impurity as long as $K_+ > 1/2$, whereas the CDW one is stable only for $K_+ > 2$.

We now turn to the case of a weak random potential produced by many impurities. To establish the boundary between delocalized and localized regimes for the case of weak disorder, it suffices to replace $\epsilon \rightarrow 1/L$ in Eqs. \cite{90,91} and to multiply $\delta G$ by the total number of impurities, proportional to $L$. Depending on whether $\delta G$ increases or decreases with $L$, the wire is in the localized or delocalized phase. By doing so, one concludes that a wire is localized for $K_+ < 3$, if it is in the CDW phase, and for $K_+ < 3/4$, if it is in the Cooper phase. For comparison, a (spinless) single-subband Luttinger liquid is localized for $K < 3/2$.

2. Electrons with spins

The bosonized form of impurity backscattering is given by

$$W_{\text{intra}}^{1(2)} = \frac{4W_{1(2)}(2K_{F1,2})}{2\pi \alpha} \cos[\sqrt{\pi}(\varphi_{\rho+} \pm \varphi_{\rho-})]$$

$$\times \cos[\sqrt{\pi}(\varphi_{\sigma+} \pm \varphi_{\sigma-})];$$

$$W_{\text{inter}} = \frac{4W_{\text{inter}}(k_{F1} + k_{F2})}{2\pi \alpha} \cos[\sqrt{\pi}(\varphi_{\rho+} + \theta_{\rho-})]$$

$$\times \cos[\sqrt{\pi}(\theta_{\sigma-} + \varphi_{\sigma+})],$$

where $+(-)$ in the argument of cosines refers to the 1st (2nd) subband and all operators are evaluated at the position of the impurity.

In the Cooper phase, the $\theta_{\sigma-}$ and $\varphi_{\sigma \pm}$-modes are gapped, whereas the conjugated modes, i.e., $\varphi_{\rho-}$ and $\theta_{\sigma \pm}$, exhibit strong fluctuations. Integrating out $\varphi_{\rho-}$ and $\varphi_{\sigma \pm}$, we find

$$W_{\text{intra}} \sim (\Delta_{SC}^{-1}[W(2k_{1,2} F)])^2 \cos[\sqrt{4\pi} \varphi_{\rho+}],$$

which is relevant for $K_{\rho+} < 1$. The same is true for $W_{\text{inter}}$, where strong fluctuations of $\theta_{\sigma-}$ produce a similar operator. The correction to the conductance behaves as

$$- \delta G_{SC} \propto (w/\Delta_{SC})^2 e^{2K_{\rho+}^{-2}},$$

i.e., as if we were dealing with a single-channel Luttinger liquid, characterized by parameter $K_{\rho+}$, subject to an effectively reduced impurity potential. Weak random potential leads to localization for $K_{\rho+} < 3/2$.

In the CDW-state, the situation is different. In this case, the $\varphi_{\rho-}$ and $\varphi_{\sigma \pm}$-modes are gapped, whereas the $\theta_{\rho-}$ and $\theta_{\sigma \pm}$-modes fluctuate strongly. As a result, intersubband backscattering is renormalized into $\cos[\sqrt{4\pi} \varphi_{\rho+}]$, as in the Cooper phase, but intra-subband one remains unchanged and is determined by the dynamics of the only gapless $\varphi_{\rho+}$ mode:

$$W_{\text{intra}} \propto \cos[\sqrt{4\pi} \varphi_{\rho+}] .$$

Pinning of $\varphi_{\rho+}$ at the impurity site leads to the suppression of the conductance. $W_{\text{intra}}$ is relevant, i.e., an impurity eventually splits the wire into two disconnected halves for $K_{\rho+} < 4$. The correction to the conductance behaves as

$$- \delta G_{CDW} \propto w^2 \epsilon^{K_{\rho+}^{-2}} .$$

This is to be contrasted with the case of a gapless Luttinger liquid, when the impurity is relevant only for repulsive interactions ($K < 1$). This reflects the fact that a real (gapped) charge-density-wave is pinned stronger than the fluctuating one (Luttinger liquid). Finally, weak random potential localizes the CDW-wire for $K_{\rho+} < 6$. It is worth pointing out here that such large values of critical $K_{\rho+}$ separating localized and extended regimes, imply strong effective attraction between charge fluctuations in the $\rho+ \text{ channel}$. It might well be that a charge segregation, instead of the CDW instability, will take place for such a strong attraction.

Thus, for electrons with spin, the Cooper phase is more stable to impurities than the CDW one, similar to a spinless case, but both are unstable in the physically relevant region of $K_{\rho+} < 1$.

\[20\]
VI. SINGLE-PARTICLE DENSITY OF STATES

We now turn to the discussion of tunneling into a quantum wire. To the leading order in barrier transparency $\mathcal{T}$, the differential tunneling conductance is

$$G(V) = \frac{dI}{dV} = |\mathcal{T}|^2 \rho_T \rho(eV),$$  

(98)

where $\rho_T$ is the density of states (DOS) in the contact (which we assume to be energy-independent), $\rho(e)$ is the DOS of the wire, and $V$ is the applied voltage. When the wire is in the gapless Luttinger-liquid phase, $\rho(e) \propto |e|^\beta$. This behavior has recently been observed in tunneling into carbon nanotubes.\[^{26}\] Tunneling into the edge of a fractional quantum Hall system also exhibits a power-law current-voltage dependence,\[^{27}\] which might be an indication of a chiral Luttinger-liquid state at the edge. Here, however, the situation is not that straightforward, and other explanations, different from a chiral Luttinger liquid, have also been suggested.\[^{28}\]

Suppose now that a two-subband quantum wire is in one of the possible gapped phases, i.e., CDW or Cooper phase. The goal of this Section is to analyze what would a tunneling experiment show in this case. The answer turns out to depend crucially on the geometry of the experiment. If the tunneling contact probes the interior of the wire, the gapped behavior is predicted: $G(V) = 0$ for $eV < \Delta$, where $\Delta$ is the appropriate energy gap. For $eV > \Delta$ the behavior is non-universal: the threshold behavior of $\rho$ is determined by gapless charge and spin modes. More surprisingly, tunneling into the end of the CDW-wire exhibit a gapless behavior, similar to the Luttinger-liquid case. The tunneling exponent though is different from that for the gapless phase.

A. Tunneling preliminaries

The local single-particle (or tunneling) density of states is given by

$$\rho(\omega, x) = -\frac{1}{\pi} \text{Im} \{G_{rel}(\omega, x)\},$$  

(99)

where $G_{rel}(\omega, x)$ is a Fourier transform of retarded Green’s function $G_{rel}(t, x) = -i \Theta(t) \Sigma_s \langle \{ \Psi_s(t, x), \Psi_s^\dagger(0, x) \} \}$. Representing the electron of the $n$-th subband as a sum of right- and left-movers and accounting for the orthogonality of transverse wavefunctions, the Green’s function becomes $G_{rel}(t, x) = \sum_n \left[ G_{rel}^R(t, x) + G_{rel}^L(t, x) \right]$, where the summation is over all occupied subbands and $G_{rel}^N(t, x) = -i \Theta(t) \Sigma_s \langle \{ N_{ns}(t, x), N_{ns}^\dagger(0, x) \} \}$. The contribution of the off-diagonal terms $\sim \langle R_n L_m^\dagger \rangle$ is less singular and is thus neglected. $\rho(\omega, x)$ is a sum of contributions from right and left movers of all occupied subbands. To find $\rho(\omega, x)$, it is convenient to calculate first the Matsubara Green’s function

$$G^R(\tau, x) = -\langle T_\tau R_s(\tau, x) R_s^\dagger(0, 0) \rangle,$$  

(100)

and then to make the analytic continuation to real frequencies. Left- and right-moving fermions give identical contributions to $\rho$, thus the result obtained from (100) is simply multiplied by a factor of two at the end.

The key feature of gapped phases in multisubband 1D systems is the co-existence of gapped and gapless modes, which also makes the calculations to be slightly less trivial. The single-particle Green’s function under these circumstances has recently been considered in Ref.\[^{29-31}\], and our analysis follows largely these two papers.

B. Warm-up: DOS of a half-filled Hubbard chain

To warm up, we consider the simplest system in which gapped and gapless modes co-exist – a single-band Hubbard chain at half-filling. In the context of nanostructure physics, such a system is produced by imposing an artificial periodic potential of period $a_0$ over a quantum wire.\[^{32}\] At half-filling, the Fermi-momentum $k_F = \pi/2a_0$ is commensurate with the reciprocal lattice spacing, which gives rise to Umklapp scattering. An Umklapp process occurs as simultaneous backscattering of two right- or left-moving electrons, the total momentum transferred to the lattice being $\pm 4 \times \pi/2a_0 = \pm 2\pi/a_0$. This process is responsible for opening of the (Mott-Hubbard) gap in the charge-excitation spectrum. On the other hand, spin excitations remain gapless, and are described by the $SU(2)$-invariant Luttinger-liquid Hamiltonian ($K_S = 1$).

The corresponding Hamiltonian of the charge sector is
\[ H = \frac{1}{2} \int dx \left\{ v_{\rho} K_{\rho} (\partial_x \theta_\rho)^2 + \frac{v_{\rho}}{K_{\rho}} (\partial_x \varphi_\rho)^2 + g \cos[\sqrt{8\pi} \varphi_\rho] \right\}, \quad (101) \]

where the last (cosine) term represents Umklapp scattering. Substituting Eq. (13) into Eq. (101), one finds that \( G^R \) factorizes into a product of spin and charge parts

\[
G^R(\tau, x) = -\frac{s_n(\tau)}{2\pi} F_\sigma(\tau, x) F_\rho(\tau, x); \\
F_\nu = \langle \exp[i \sqrt{\frac{\nu}{2}} (\varphi_\nu(1) - \varphi_\nu(0))] \rangle \exp[-i \sqrt{\frac{\nu}{2}} (\theta_\nu(1) - \theta_\nu(0))]; \nu = \rho, \sigma, \quad (102)
\]

where shorthand notations \( 1 \equiv (\tau, x) \), \( 0 \equiv (0, 0) \) have been used. \( F_\sigma \) is gapless, whereas \( F_\rho \) contains massive fields.

1. Bosonic calculation

To calculate \( F_\rho \) in the bosonic language, we adopt the semiclassical approximation, i.e., expand the cosine in \( (101) \) around its minimum to the second order in fluctuations. This is equivalent to replacing \( g \cos[\sqrt{8\pi} \varphi_\rho] \rightarrow m^2 \varphi_\rho^2 \), which defines the mass \( m^2 \equiv 4\pi g \). Now the averaging is straightforward:

\[
F_\rho(\tau, x) = \exp \left\{ -\frac{\pi}{2} \int \frac{d^2 k}{(2\pi)^2} (1 - \cos[\vec{k} \cdot \vec{\varphi}]) \left( G_\varphi(k) + G_\varphi^0(k) + 2i G_\varphi(k) \frac{k_i k_j}{vK k_i^2 + m^2} \right) \right\}, \quad (103)
\]

where \( \vec{k} \equiv (k_0, k_1) = (\omega_n, q) \), \( \vec{\varphi} \equiv (\tau, x) \) and

\[
G_\varphi(\omega_n, q) \equiv \langle T_\tau \varphi_\rho(\vec{\varphi}) \rangle_{\omega_n, q} = \frac{vK}{v^2 q^2 + \omega_n^2 + m^2 vK}. \quad (104)
\]

The Green’s function of \( \theta_\rho \)-fields can be written as

\[
G_\theta(\omega_n, q) \equiv \langle T_\tau \theta_\rho(\vec{\theta}) \rangle_{\omega_n, q} = \frac{v}{K} \left( \frac{1}{v^2 q^2 + \omega_n^2} + \frac{\omega_n^2}{v^2 q^2 + \omega_n^2} \right) = G_{\theta 1}^{(1)}(\omega_n, q) + G_{\theta 2}^{(2)}(\omega_n, q), \quad (105)
\]

where \( \tilde{m}^2 \equiv m^2 vK \). The first term in Eq. (105) is just a free Green’s function, whereas the second one is present only in the strong-coupling phase and contains an infrared divergence at \( q \rightarrow 0 \). This divergence is often explained by the “uncertainty principle”: in a gapped phase, a position-like field \( (\varphi_\rho) \) acquires an average value, hence its canonical conjugate, momentum-like field \( (\theta_\rho) \) fluctuates strongly, hence the average \( \langle \theta_\rho \theta_\rho \rangle \) diverges. Let us analyze the Fourier transform of \( G_\theta \) in more details, and define

\[
I(x, \tau) = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} (1 - \cos[\vec{k} \cdot \vec{\varphi}]) G_{\theta 2}^{(2)}(\omega, q). \quad (106)
\]

Changing to polar coordinates \( \omega_n = k \cos \phi, q = k \sin \phi, \tau = z \cos \theta, x = z \sin \theta \), we get

\[
I(z, \theta) = \frac{\tilde{m}^2 z^2}{(2\pi)^2} 2 \int_0^{2\pi} d\phi \frac{\cos^2 \phi}{\sin^2 \phi} \cos^2(\phi - \theta) \int_0^\infty dy \frac{1 - \cos y}{y^2 + \tilde{m}^2 z^2 \cos^2(\phi - \theta)}. \quad (107)
\]

The integral over \( \phi \) diverges at \( \phi = 0 \), which, if taken literally, means that \( I = \infty \) and thus \( F_\rho(\tau, x = 0) = 0 \) for any finite \( \tau \). However, this divergence is absent at \( \theta = \pi/2 \), which corresponds to \( \tau = 0, z = x \). Let us therefore continue the calculation at this special point. Despite the cancellation of the infrared divergence, the integral is still controlled by the region of small \( \phi \): \( \phi \sim 1/(\tilde{m} x) \ll 1 \). Expanding \( \sin \phi \sim \phi \) and extending the limits of angular integration to \( \pm \infty \), we find that \( I(z, \theta = \pi/2) = \tilde{m} x/4 + O(1/\tilde{m} x) \) for \( \tilde{m} x \gg 1 \). Collecting regular contributions from \( G_\varphi \) and \( G_{\theta 1}^{(1)} \), we find that the equal-time exponential correlator of \( \theta_\rho \)-fields is given by

\[
\langle e^{i \alpha \theta_\rho(x, 0)} e^{-i \alpha \theta_\rho(0, 0)} \rangle = \exp \left[ -\frac{\alpha^2}{4\tilde{m}^2} \ln(x^2/\alpha^2) - \frac{\alpha^2}{4\tilde{m}^2} \right]. \quad (108)
\]

An important feature here is that the expected exponential decay of this correlator is modified by the power-law prefactor, given by the usual Luttinger-liquid correlator. Symbolically, \( \langle e^{i \alpha \theta_\rho(x, 0)} e^{-i \alpha \theta_\rho(0, 0)} \rangle_{\tilde{m} \neq 0} = \)
\[ \langle e^{i \theta(x,0)} e^{-i \theta(0,0)} \rangle_{m=0} = e^{-\bar{m}x}. \] One should be careful in using Eq. (108): Luttinger-liquid parameter \( K \), which appears here, should in fact be understood as the strong-coupling fixed-point value, \( K^* \), which is often unknown.

Fortunately, the fixed-point value of \( K \) is known for a half-filled Hubbard chain: \( K^* = 1/2 \). For \( \bar{m}x \gg 1 \) we then obtain

\[
\langle e^{i \sqrt{8 \pi} \varphi(x)} e^{-i \sqrt{8 \pi} \varphi(0)} \rangle \rightarrow \text{const},
\]

\[
\langle e^{i \sqrt{8 \pi} \theta(x)} e^{-i \sqrt{8 \pi} \theta(0)} \rangle \sim \exp[-\pi \bar{m}x/4]/\sqrt{x},
\]

and the full Green’s function behaves as

\[
G^R(0, x) \propto \exp[-\pi \bar{m}x/4] \times 1/\sqrt{x}.
\]

in agreement with Ref. 2. The second \( x^{-1/2} \)-factor in Eq. (110) is due to gapless spin excitations.

So far, all calculations have been straightforward. Now we would like to argue that the infrared divergence of \( I(z, \theta \neq 0) \) is an artifact of the semiclassical approximation, which ignores degeneracy of \( \cos[\sqrt{8 \pi} \varphi] \) with respect to a uniform shift \( \varphi \rightarrow \varphi + \sqrt{2}N \) with integer \( N \). The proper theory of both massive and massless phases should be Lorentz-invariant. We thus propose that the correct result, valid for any \( x = \sqrt{x^2 + v^2 T^2} \), is given by Eq. (108) where \( x \) is replaced by Euclidian distance \( z: x \rightarrow z \). Therefore,

\[
\langle e^{i \theta(x,\tau)} e^{-i \theta(0,0)} \rangle = \exp \left[ -\frac{a^2}{4 \pi K} \ln \left( \frac{x^2 + v^2 T^2}{\sigma^2} \right) - \frac{a^2}{4K} \bar{m} \sqrt{x^2 + v^2 T^2} \right].
\]

Similar arguments in favor of such replacement were given by Voit 26.

We now use Eq. (111) to evaluate Eq. (103), and find [compare with (109)]

\[
F_\sigma(\tau, 0) \sim \sqrt{\frac{\alpha}{v |\tau|}} \exp[-\pi \bar{m}v|\tau|/4].
\]

The spin sector average is non-zero and universal (thanks to \( K_\sigma = 1 \)), therefore \( F_\sigma(\tau, 0) \sim \sqrt{\alpha/v |\tau|} \). The corresponding DOS will be calculated later, see Eq. (123). The correctness of the procedure described above is verified in the next Section.

2. Re-fermionization.

To check that our proposition makes sense, we now switch gears and derive Eq. (112) in a completely different way. To this end, we use the Luther-Emery refermionization procedure 24, which works for \( K = 1/2 \), i.e., at the fixed point of a half-filled Hubbard chain. This procedure begins with an innocuous looking transformation \( \varphi_\rho = \varphi/\sqrt{2} \); \( \theta_\rho = \sqrt{2} \theta \), which changes Umklapp scattering in Eq. (109) into backscattering of some auxiliary particles (solitons): \( \cos(\sqrt{8 \pi} \varphi_\rho) = \cos(\sqrt{4 \pi} \varphi) \). Right- and left-going solitons are defined by

\[
\psi_\pm = \frac{1}{\sqrt{2 \pi \alpha}} \exp \{ \pm i \sqrt{\pi}(\varphi \mp \theta) \}.
\]

In terms of new bosons, the original fermion operator (13) becomes

\[
R_s = \frac{e^{-i \pi/8}}{\sqrt{2 \pi \alpha}} \exp[i \pi \sqrt{\pi/2} (\varphi - \theta)] \exp(i \sqrt{\pi/2} \varphi) \exp(-i \sqrt{\pi} \theta).
\]

It can also be written in terms of solitons

\[
R_s = e^{i \pi/8} \exp[i \pi \sqrt{\pi/2} (\varphi - \theta)] \exp(-i \sqrt{\pi} \varphi) \psi_+,
\]

where \( \varphi \) is expressed in terms of soliton density fluctuations as \( (1/\sqrt{\pi}) \theta_\rho \varphi =: \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \). The usefulness of these formal manipulations is based on the fact that Hamiltonian (104) is quadratic in massless solitons \( \psi_\pm \), and the mass (or the gap \( \Delta \)) in their spectrum is determined by coupling constant \( g; \Delta = g/(2 \pi \alpha) \). Due to the presence of
Hence, the DOS of massive fermions is given by the long-time asymptotics of Bessel functions to evaluate the DOS at $t_0$ from Eq. (117), it can be shown that
\[ F(0, \omega) = -\frac{i\omega_n + vq}{\omega_n^2 + \Delta^2 + v^2 q^2}. \] (116)

At $x = 0$, \[ F(0, 0) = -\frac{1}{v} \int d\omega_n \frac{\omega_n e^{i\omega_n \tau}}{\sqrt{\omega_n^2 + \Delta^2}} = -\frac{1}{v} \partial_\tau \int d\omega_n \frac{\cos(\omega_n \tau)}{\sqrt{\omega_n^2 + \Delta^2}} = \frac{\Delta}{v} \text{sgn}(\tau) K_1(\Delta |\tau|). \] (117)

Asymptotically, \[ F(0, \omega) \sim (\Delta/v \sqrt{\Delta^2/\tau}) e^{-\Delta |\tau|/v}, \] in agreement with our earlier proposition (112).

It follows from Eqs. (116)-(117) that upon continuing to real frequencies it follows from Eqs. (116)-(117) that upon continuing to real frequencies
\[ \rho_{\text{ret}}(\omega) = -\frac{i}{2\pi v} \Theta(\omega - \Delta) \frac{\omega}{\sqrt{\omega^2 - \Delta^2}}. \] (118)

Up to a factor of $1/2$, which is due to the fact that this is the contribution of right-movers only, the obtained result is just the DOS of free massive particles with dispersion $\epsilon(q) = \sqrt{v^2 q^2 + \Delta^2}$. Since there are no particles above energy $\Delta$ at zero temperature, there are no interaction corrections to the density of states as well. Another way of deriving this result consists in using the Ising-model representation of $\varphi$, $\theta$ fields (see Ref. 60 for details). In this representation, \[ \exp(-i\sqrt{\pi} \varphi) = \sigma_1 \mu_2 - i\mu_1 \sigma_2, \]
\[ \exp(i\sqrt{\pi} \varphi) = \mu_1 \mu_2 + i\sigma_1 \sigma_2, \] (119)
where $\mu_i$ ($\sigma_i$) are order (disorder) fields of the $d = 2$ Ising model, whose correlators are known. At long times, i.e., when $\Delta \tau \gg 1$,
\[ \langle \mu_i(\tau) \mu_j(0) \rangle \sim \delta_{ij} (\mu)^2, \]
\[ \langle \sigma_i(\tau) \sigma_j(0) \rangle \sim \delta_{ij} K_0(\Delta |\tau|). \] (120)

As a result, \[ F(\omega) = sgn(\tau) K_0(\Delta |\tau|). \] Because of the condition $\Delta \tau \gg 1$, there is no discrepancy between Eqs. (120) and (117), since the leading asymptotic term of $K_\nu(x)$ is $\nu$-independent. Hence one again finds a square-root singularity in $\rho_{\text{ret}}(\omega)$ for $\omega - \Delta \ll \Delta$. The correspondence with the Ising model allows one to estimate neglected terms as $e^{-3\Delta \tau}$.

A more illuminating way to understand the square-root singularity is provided by the real-time calculation. Starting from Eq. (117), it can be shown that
\[ \text{Im}[F_{\text{ret}}(\omega)] = \frac{\Delta}{4\pi v} \int_{-\infty}^{\infty} dt e^{i\omega t} \left( K_1(-i\Delta t) - K_1(i\Delta t) \right) \]
\[ = \frac{\Delta}{2\pi v} \int_0^\infty dt \sin(\omega t) Y_1(\Delta t) = \Theta(\omega - \Delta) \frac{\omega}{2v \sqrt{\omega^2 - \Delta^2}}, \] (121)
where $Y_1(x)$ is the Bessel function of the second kind. Since asymptotically $Y_1(x) \sim \sin(x)/\sqrt{x}$, the origin of the singularity at $\omega = \Delta$ can be easily understood. For $\omega - \Delta \ll \Delta$, the integrand of (121) oscillates very slowly, with period $t_0 = 2\pi/(\omega - \Delta)$. The integral is thus determined by long times, $t \approx t_0$, and can be estimated as $\int_0^{t_0} 1/\sqrt{t} \sim \sqrt{t_0}$. We see that the threshold behavior of the DOS is determined by times much longer than $1/\Delta$, which justifies our use of the long-time asymptotics of Bessel functions to evaluate the DOS at $\omega \approx \Delta$.

3. DOS of a physical electron

To find the density of states of a physical electron, we have to convolute Eq. (118) with the contribution of the gapless spin mode:
\[ \rho(\omega, x = 0) = \int \frac{d\epsilon}{2\pi} F_\epsilon(\epsilon, x = 0) F_\sigma(\omega_n - \epsilon, x = 0)|_{\omega_n = -i\omega} \]
\[ = \frac{2}{\pi} \int_0^\infty \text{dIm}[F_{\text{ret}}(\epsilon)] \text{Im}[F_{\text{ret}}(\omega - \epsilon)]. \] (122)
Since $F_n(\epsilon_n) \sim \sqrt{\alpha/iv\epsilon_n}$, we find
\[\rho(\omega, x = 0) = \frac{2}{\pi v} \int \frac{\alpha}{v} \Theta(\omega - \Delta) \int_{\Delta}^{\omega} \frac{d\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} \frac{1}{\sqrt{\omega - \epsilon}}.\] (123)

For $\omega - \Delta \ll \Delta$, where our derivation is valid,
\[\rho(\omega, x = 0) = \frac{\pi}{v} \sqrt{\frac{q}{v}} \Theta(\omega - \Delta).\] (124)

Instead of a square-root singularity of Eq. (118), the DOS of a physical electron exhibits a regular behavior approaching a finite value at the threshold. This modification is due to dressing of the gapped char-ge mode by gapless spin excitation-ns. At energies much above the gap DOS increases, $\rho(\omega, x = 0) \sim \sqrt{\omega}$, which means that the spectral weight is shifted to higher energies. The energy-independent electron DOS near the threshold was obtained in (124).

Parenthetically, functional form (124) remains valid when the spin channel is gapped as well. In this case the density of states is also gapless. In this case the density

\[F_\sigma(\omega - \Delta) \sim 2\pi \alpha \frac{1}{\sqrt{\omega}}\] (125)

Now we apply Eq. (108) to the correlator $F(\tau) = -\langle T R_{1,s}(\tau, 0) R_{1,s}(0, 0) \rangle$. In the Cooper phase, $\theta_{\rho-}$ and $\varphi_{\sigma \pm}$ are gapped, hence their conjugates are exponentially suppressed. As a consequence, e.g.,
\[(e^{i\sqrt{\pi}(\varphi_{\sigma-}(\tau) - \theta_{\rho-}(\tau))/2} e^{-i\sqrt{\pi}(\varphi_{\sigma-}(0) - \theta_{\rho-}(0))/2}) = C_\sigma \left(\frac{\alpha}{v_{\sigma} |\tau|} \right)^{1/8} \exp \left[ -\frac{\pi m_{\sigma} v_{\sigma} |\tau|}{16} \right].\] (126)

On the other hand, $\varphi_{\rho+}$ and $\theta_{\rho+}$ remain critical. As a result,
\[F(\tau) \propto \text{sgn}(\tau) |\tau|^{-\kappa} \exp \left[ -\frac{\pi}{16} (2m_{\sigma} v_{\sigma} + m_{\rho-} v_{\rho-}) |\tau| \right], \quad \kappa = 1 - \frac{1}{8}(3 + K_{\rho+} + 1/K_{\rho+}).\] (127)

Hence the DOS behaves as
\[\rho(\omega) \propto \frac{\Theta(\omega - \Delta_{SC})}{(\omega - \Delta_{SC})^\gamma}, \quad \gamma = 1 - \kappa = 1 - \frac{1}{8}(5 - K_{\rho+} - 1/K_{\rho+}),\] (128)

where $\Delta_{SC} = (\pi/16)(2m_{\sigma} v_{\sigma} + m_{\rho-} v_{\rho-})$. Note that $\gamma \leq 3/8 < 1/2$ for $K_{\rho+} \leq 1$.

Exactly at half-filling, when $\rho_+ \$ mode is also gapped due to Umklapp scattering and there are no more gapless modes, our procedure gives $F(\tau) \sim e^{-\Delta/\sqrt{|\tau|}}$, in agreement with recent exact result. The corresponding DOS is that of a free massive particle, $\rho_{Eff}(\omega) \sim \Theta(\omega - \Delta)/\sqrt{\omega - \Delta}.

Comparison of (128) with (124) shows that softening of the square-root singularity is pronounced weaker for a ladder than for a Hubbard chain, because a ladder has three gapped and only one gapless mode. As repulsion in the $\rho_+$ channel becomes stronger, i.e., as $K_{\rho+}$ decreases, the singularity becomes weaker and disappears at $K_{\rho+} = (5 - \sqrt{21})/2 \approx 0.2$. For even smaller $K_{\rho+}$, we have $\rho(\omega = \Delta_{SC}) = 0$. This behavior though is not very realistic as it requires very strong repulsion. For weak repulsion, i.e., when $K_{\rho+} \approx 1$,
\[\rho(\omega) \propto \Theta(\omega - \Delta_{SC})/|\omega - \Delta_{SC}|^{3/8},\] (129)

and the threshold singularity is still present albeit softened compared to the free-massive-particle case. We note that $\rho(\omega)$ (128) is similar to the DOS of high unoccupied subbands of a quantum wire, considered recently by Balents. \[\text{[25]}\]
One would expect that the long-range order of the Cooper phase affects tunneling. Indeed, one finds that pair correlations are determined by the Luttinger parameter of the total charge fluctuations only,

\[ \langle R_{1,s}(\tau)L_{1,-s}(\tau) L_{1,-s}(0)R_{1,s}(0) \rangle \sim \langle e^{-i\sqrt{\pi}\theta_{\nu+}(\tau)}e^{i\sqrt{\pi}\theta_{\nu+}(0)} \rangle \sim \tau^{-1/(2K_{\nu+})}, \]

whereas all other two-particle combinations decay exponentially. Thus, although the single-particle density of states is strictly zero, the two-particle one is not. In principle, this effect can be checked experimentally by tunneling into a two-channel wire from the superconducting tip—one should observe then a nonzero tunneling current of Cooper pairs. Its magnitude, however, will be much smaller than the current in a system of a normal tip and gapless wire, because the probability of two-particle (Cooper pair) tunneling \(|T|^4\) is exponentially smaller than that of single-particle tunneling \(|T|^2\).

**VII. TUNNELING INTO THE END OF A GAPPED WIRE**

Tunneling into the end of a Luttinger liquid is different from tunneling into the bulk. The reason for this difference is the open boundary condition \(\psi = 0\) for the electron wavefunction. For boson modes describing charge and spin displacements, this condition means pinning at the boundary. The difference between the edge and bulk tunneling was considered first theoretically by Kane and Fisher, and has recently been observed in experiments on tunneling into carbon nanotubes. A rigorous treatment of a Luttinger liquid with open boundary conditions, which involves re-formulation of the bosonization procedure, can be found in Refs. 14 and 16.

Suppose now that a two-subband wire is driven into a CDW state by direct backscattering processes accompanied by density adjustment, as described in Sec. III A 1. The relative mode of charge excitations is described by Hamiltonian (43), in which we put \(\delta K_F = 0\). From the equivalence of Eqs. (101) and (13) (with \(\delta K_F = 0\)), we expect the fixed-point value of \(K\) in the CDW-phase to be the same as for a half-filled Hubbard chain, i.e., \(K^* = 1/2\). The total charge mode \(\varphi_+\) remains gapless and plays the same role as the spin mode of the Hubbard chain [see Eqs. (124) and (22)] — it softens the threshold singularity of the DOS. For \(\omega < \Delta_{CDW}\), the density of states is equal to zero. Thus, if the tunneling contact probes the interior of the wire, a gapped behavior is observed.

However, the DOS at the end of a wire exhibits gapless behavior, i.e., \(\rho_{end} \propto |\omega|^\alpha\), as we will demonstrate in the rest of this section. Consequently, \(I(V) \propto |V|^{\alpha+1}\) for tunneling from a macroscopic (Fermi-liquid contact) into the end of the wire, and \(I(V) \propto |V|^{2\alpha+1}\) for tunneling through a barrier located somewhere within the wire.

This very different behavior of the DOS in the bulk and at the end of the wire can be understood physically for tunneling through a barrier located within the wire (cf. Fig. 4). Without the barrier, the CDW is free to slide and the conductance is the same as in the absence of any interactions. Squeezing one electron into the middle of the wire leads to creation of a soliton-like compression in one of the two modes, and to accompanying it “stretch” in the other mode, which requires an energy of the order of the charge gap \(\Delta_{CDW}\). However, such an excitation needs not be created when the barrier distorts the uniform profile of the CDW. Indeed, the boundary condition imposed by the barrier pins the mode \(\varphi_-\) at \(x = 0\) to a value which is different from the one in the bulk, \(\varphi_-(x \rightarrow \infty) = \sqrt{\pi/8}\) (the latter follows from the minimization of the CDW energy, as illustrated in Fig. 3). Therefore, the regular order of CDW is already frustrated near the barrier: there is a built-in compression in one of the modes, and depression in the other. The electron that tunnels through the barrier arrives into the “stretched” mode. Upon the proper shift of both modes, the system arrives into a state with the same energy but with a switched “polarity” of frustration. This consideration is true if the barrier is strong enough to destroy the CDW order in its vicinity, i.e., the barrier height is larger than \(\Delta_{CDW}\).

\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

1 \hspace{1cm} 2

**FIG. 6.** Illustration of the CDW in the vicinity of a barrier: while the bulk value for \(\varphi_-\) is \(\sqrt{\pi/8}\), the barrier forces \(\varphi_-\) to vanish at \(x = 0\).

We now give a derivation for the current through the barrier. To model the boundary conditions corresponding to a wire cut into two semi-infinite pieces, we choose the potential barrier in the form \(w(x, r) = \pi a W \delta(x)\), so that \(W_{\text{in}} = 1, W_{\text{out}} = 0\) [cf. Eqs. (80) and (81)]. To find the current in, e.g., the 1st subband, through the barrier at \(x = 0\)

\[
I \propto \lim_{t \rightarrow \infty} \langle \partial_t \varphi_1(x = 0, t) \rangle/e,
\]

(131)
we need to calculate the rate at which \( \varphi_1(x = 0, t) \) increases in the stationary limit in the presence of a potential drop \( -eV \int dx \left( \partial_x \varphi_+ / \sqrt{\pi} \right) \text{sgn}(x) \), proportional to the applied voltage \( V \) (due to their equivalence, it does not matter which of the two modes we use for measuring the current). Eq. (131) reduces the transport problem to that of the dynamics of a quantum particle \( q(t) \equiv \varphi_1(x = 0, t) \) subject to “damping” by all of the remaining bulk degrees of freedom, including those of the second mode. Therefore, we can employ methods of dissipative quantum mechanics to solve this problem.

The effective action of the boundary mode \( S_{eff} \) is obtained by tracing over these remaining degrees of freedom

\[
e^{-S_{eff}[q]} = \int \mathcal{D}[\varphi_1] \mathcal{D}[\varphi_2] \delta[q(\tau) - \varphi_1(x = 0, \tau)] \delta[\varphi_2(x = 0, \tau)] e^{-S},
\]

(132)

where

\[
S = \int dx \int d\tau \left[ \frac{1}{2v_F^2} (\partial_x \varphi_+)^2 + \frac{2}{2K} (\partial_x \varphi_+)^2 \right.
\]

\[
+ \frac{1}{2v_F^2} (\partial_x \varphi_-)^2 \left. + \frac{2}{2K} (\partial_x \varphi_-)^2 + \frac{4f_{bs}}{\\alpha^2} q^2 \right],
\]

(133)

(134)

where \( \pm \) combinations are defined in (43). Performing the integration, we get

\[
S_{eff}[q] = \frac{1}{2} \int d\tau \int d\tau' q(\tau) K(\tau - \tau') q(\tau'),
\]

(135)

which accounts for the influence of the bulk modes exactly. In (132) we assume a large barrier so that incoherent single electron passages through subband 1 determine the current, while subband 2 is fixed by the barrier, cf. (132). Simultaneous contributions from subband 2 represent coherence effects, and are of higher order in the barrier transmission coefficient. The dynamics of \( q \) is governed by

\[
S_0 = \int d\tau \left( W \cos 2\sqrt{\pi} \eta - \frac{eV}{\sqrt{2\pi}} \eta \right) + S_{eff},
\]

(136)

which includes the applied voltage. Eq. (136) describes a damped quantum particle on a tilted washboard potential. Tunneling between adjacent minima of the potential corresponds to single electron transfers.

For large \( W \), Eq. (136) maps onto a tight binding model with nearest neighbor hopping \( T \), whose value can in principle be deduced from \( W \), assuming a \( \delta \)-barrier : it is renormalized by damping compared to the bare value\( K \). In this analogy, (131) corresponds to the particle’s mobility, studied in Ref. In leading order \( \sim |T|^2 \), the result is

\[
I(V) = e|T|^2 \int_0^\infty dt \sin(eVt) \text{Im} e^{-w(t)},
\]

(137)

where

\[
w(t) = \int_0^\infty \frac{J(\omega)}{\omega^2} (1 - e^{-i\omega t}).
\]

(138)

The spectral function \( J(\omega) \) is related to the Fourier transform \( \mathcal{K}(\omega) \) of the kernel appearing in (137),

\[
J(\omega) = -\lim_{\eta \to 0} \text{Im} \mathcal{K}(-i\omega + \eta) = \frac{\omega}{K_+} + \frac{\sqrt{\omega^2 - \omega_0^2}}{K_-} \Theta(\omega - \omega_0),
\]

(139)

with \( \omega_0^2 = 8f_{bs}v_F/\\alpha^2 \). This yields

\[
I(V) \sim V^{1/K_+ - 1},
\]

(140)

which gives \( \alpha_d = 1/2K_+ - 1 \) at voltages \( V < \omega_0/e \). Above the gap, i.e, for \( V \gg \omega_0/e \), gapless behavior of a two-subband Luttinger liquid is restored \( I(V \gg \omega_0/e) \sim V^{1/K_+ + 1/K_- - 1} \).

The gapless behavior of fermion’s DOS at the end of the wire can be interpreted in terms of a midgap state in the \((-\) channel located near the boundary. This is a bound state with zero energy formed in the potential well created by the static distortion of the \( \varphi_- \) field subject to an open boundary condition \( \varphi_-(x = 0) = 0 \) (see Appendix
D of the paper by Fabrizio and Gogolin\cite{Fabrizio}). Qualitatively, this state is similar to the boundary state in a doped two-leg spin ladder, which represents free $S = 1/2$ spin induced by a charged impurity, see, e.g.,\cite{Fabrizio}. As a result, local density of states of the (−) channel at the end of the wire takes the form $\rho_-(\epsilon) \sim \lambda \delta(\epsilon) + \rho^{\text{reg}}$, where $\rho^{\text{reg}}$ stands for contribution of massive modes with energies above the CDW gap. On the other hand, the (+) mode remains gapless, and its end-chain DOS is given by $\rho_+(x = 0, \omega) \sim \omega^{1/2} K_+^{-1}$,\cite{Fabrizio, Fabrizio}. The factor of $1/2$ in the exponent is due to “factorization” of the electron operator into (±) modes. Another consequence of such factorization is that the DOS of a physical electron is a convolution of the DOS of the (±) channels [cf. Eq. (122)]

$$\rho_{\text{phys}}(x = 0, \omega) \sim \int_0^\omega d\epsilon \rho_+(x = 0, \epsilon) \rho_-(x = 0, \omega - \epsilon).$$

Therefore,

$$\rho_{\text{phys}}(x = 0, \omega) \sim \rho_+(x = 0, \omega) = \omega^{1/2} K_+^{-1},$$

which implies $\alpha_d = 1/(2K_+) - 1$, in agreement with the result of the explicit calculation of $I(V)$ presented above, Eq. (140).

\section*{VIII. EXPERIMENTAL CONSEQUENCES AND CONCLUSIONS}

Apart from the rather well-known Luttinger-liquid phase, a two-subband quantum wire may also exhibit either a CDW or superconducting (Cooper) phase. In both of these phases, certain modes of inter-subband charge- and spin-excitations are gapped, whereas the center-of-mass charge mode remains gapless. As a result, the conductance remains at the universal value of $2e^2/h$ per occupied subband, irrespective of whether the wire is in a gapless or gapped phase. However, the single-particle density of states in the middle of the wire has a hard gap. Above the gap, the DOS exhibits a non-universal threshold behavior $\rho(\omega) \sim \Theta(\omega - \Delta)(\omega - \Delta)^{-b}$, where $b \leq 1/2$. Softening of the threshold singularity is due to “dressing” of gapped modes by the remaining gapless one.

We find that the DOS for tunneling into the end of a wire in the CDW phase remains gapless, with the exponent determined by the center-of-mass mode only. This effect is due to frustration introduced into the CDW order by an open boundary (strong barrier).

Where should one look for such exotic phases of a quantum wire? We believe that quantum wires\cite{Fabrizio} prepared by cleaved edge overgrowth technique may be well-suited for observing the CDW phase. Indeed, the cross-section of such a wire is close to a square, which implies that the lowest states of transverse quantization should be close in energy. Hence, one-dimensional subbands can have close Fermi-momenta, which is a necessary condition for the formation of the CDW state. The Cooper phase, on the other hand, requires effective attraction in the relative charge channel, and has the best chance to occur when the second (upper) subband just opens for conduction. i.e., is near the Van Hove singularity.

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