A sign-reversing involution for an extension of Torelli’s Pfaffian identity

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Abstract

We evaluate the hyperpfaffian of a skew-symmetric \(k\)-ary polynomial \(f\) of degree \(k/2 \cdot (n-1)\). The result is a product of the Vandermonde product and a certain expression involving the coefficients of the polynomial \(f\). The proof utilizes a sign reversing involution on a set of weighted, oriented partitions. When restricting to the classical case when \(k = 2\) and the polynomial is \((x_j - x_i)^{n-1}\), we obtain an identity due to Torelli.

1 Introduction

The Pfaffian of a skew-symmetric matrix is commonly defined as the square root of the determinant. Note that if the order of the matrix is odd, then the determinant vanishes and the Pfaffian is zero. Hence we assume that the order is even. Similar to the determinant (of any square matrix) being expressed as a sum over all perfect matchings of the complete bipartite graph, the Pfaffian has an explicit expression as a sum over all perfect matchings of the complete graph.

Barvinok \[1\] extended the notion of the Pfaffian to the hyperpfaffian. Instead of considering matchings of the complete graph, consider set partitions of the set \([n] = \{1, 2, \ldots, n\}\) into blocks of equal size \(k\). Let \(\Pi_{n,k}\) denote the set of such set partitions. Furthermore, let \(k\) be an even integer and \(n\) a multiple of \(k\). Let \(f\) be a \(k\)-ary skewsymmetric function defined on the set \([n]^k\). For a \(k\)-element subset \(B = \{b_1 < b_2 < \cdots < b_k\}\) of \([n]\) write \(f(B) = f(b_1, b_2, \ldots, b_k)\). Lastly, define the sign \((-1)^\tau\) of a partition \(\tau = \{B_1, B_2, \ldots, B_{n/k}\}\) in \(\Pi_{n,k}\) to be the sign of the permutation \(b_{1,1}, b_{1,2}, \ldots, b_{1,k}, b_{2,1}, b_{2,2}, \ldots, b_{2,k}, b_{3,1}, \ldots, b_{n/k,1}, \ldots, b_{n/k,k}\), where the \(i\)th block \(B_i\) is given by \(B_i = \{b_{i,1} < b_{i,2} < \cdots < b_{i,k}\}\). Then the hyperpfaffian is defined by the sum

\[
Pf(f) = \sum_{\tau} (-1)^\tau \cdot \prod_{i=1}^{n/k} f(B_i),
\]

where the sum is over all partitions \(\tau = \{B_1, B_2, \ldots, B_{n/k}\}\) in \(\Pi_{n,k}\); see \[1\] Section 3].

In the case when the function is a skew-symmetric polynomial \(f\) in \(k\) variables of degree \(k/2 \cdot (n-1)\), we evaluate the hyperpfaffian; see Theorem \[1\]. The result is a product of the Vandermonde product and an expression of the coefficients of the polynomial \(f\). We prove this using a sign reversing involution that cancels most of the terms, leaving only the terms corresponding to the Vandermonde determinant. The proof can be made completely combinatorial by combining the last step with Ira Gessel’s sign reversing involution in his proof of the Vandermonde identity \[3\]. In the classical Pfaffian case, that is, when \(k = 2\), our identity yields a nice expression, generalizing an identity due to Torelli \[5\].

In the last section we state some open questions about the hyperpfaffian, among them what identities does it satisfy.
2 The hyperpfaffian in connection with the exterior algebra

To give more motivation for the hyperpfaffian we introduce the exterior algebra. Recall that $f$ is a skew-symmetric function if for all permutations $\sigma$ in $S_k$ we have that

$$f(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(k)}) = (-1)^{\sigma} \cdot f(i_1, i_2, \ldots, i_k),$$

where $(-1)^{\sigma}$ denotes the sign of the permutation $\sigma$. Observe that if two of the entries $i_1, i_2, \ldots, i_k$ are equal, then the function value $f(i_1, i_2, \ldots, i_k)$ is equal to zero.

Let $\Lambda$ denote the exterior algebra in the variables $t_1, t_2, \ldots, t_n$. For $S = \{s_1 < s_2 < \cdots < s_m\}$ a subset of $[n]$, let $t_S$ denote the exterior product $t_S = t_{s_1} \wedge t_{s_2} \wedge \cdots \wedge t_{s_m}$. Observe that for two sets $S$ and $T$ that share at least one element, we have that $t_S \wedge t_T = 0$. Also note that if at least one of the two sets $S$ and $T$ have even cardinality, then the elements $t_S$ and $t_T$ commute, that is, $t_S \wedge t_T = t_T \wedge t_S$.

Furthermore, let $f(S)$ denote the function value $f(s_1, s_2, \ldots, s_k)$.

Luque and Thibon expressed the hyperpfaffian in terms of the exterior algebra [1, Equation (79)]. We include a proof for completeness.

**Proposition 2.1.** The hyperpfaffian of the skew-symmetric function $f$ defined on the set $[n]^k$ is the unique scalar given by the equation

$$\left( \sum_S f(S) \cdot t_S \right)^{n/k} = (n/k)! \cdot \text{Pf}(f) \cdot t_{[n]},$$

where the sum is over all $k$-element subsets of the set $[n]$.

**Proof.** Begin by noting that the sign of a partition $\tau = \{B_1, B_2, \ldots, B_{n/k}\}$ is the unique scalar $(-1)^{\tau}$ such that $t_{B_1} \wedge t_{B_2} \wedge \cdots \wedge t_{B_{n/k}} = (-1)^{\tau} \cdot t_{[n]}$. Now expand the power in the proposition to obtain that

$$\left( \sum_S f(S) \cdot t_S \right)^{n/k} = \sum_{B_1} \cdots \sum_{B_{n/k}} f(B_1) \cdots f(B_{n/k}) \cdot t_{B_1} \cdots t_{B_{n/k}},$$

where each sum is over all $k$-element subsets of $[n]$. Observe that the product in the exterior algebra is zero if two of the sets have a common element. Hence the sum reduces to a sum over all ordered partitions of $[n]$. Ordered here refers to the set of blocks having a linear order. But given a partition in $\Pi_{n,k}$ there are $(n/k)!$ ways to obtain an ordered partition. Hence the sum reduces to $(n/k)! \cdot t_{[n]}$ times the right hand side of equation (114), proving the result. \qed

**Lemma 2.2.** Let $f$ be a skew-symmetric function on the set $[n]^k$ and let $\sigma$ be a permutation on the set $[n]$. Then the function $g(i_1, i_2, \ldots, i_k) = f(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k))$ is skew-symmetric and the two hyperpfaffians differ by the sign $(-1)^{\sigma}$, that is, $\text{Pf}(g) = (-1)^{\sigma} \cdot \text{Pf}(f)$.

**Proof.** It is straightforward to observe that $g$ is skew-symmetric. It is enough to prove the identity for the adjacent transposition $\sigma = (j, j + 1)$. Let $u_i = t_{\sigma(i)}$, that is, a reordering of the basis of the exterior algebra. We claim that $g(\sigma(S)) \cdot u_{\sigma(S)} = f(S) \cdot t_S$. If neither $j$ nor $j + 1$ belong to the set $S$, there is nothing to prove. If only one of them belongs to $S$, yet again, there is nothing to prove. Finally, if both $j$ and $j + 1$ belong to $S$, we have that $g(\sigma(S)) = -f(S)$, and $u_{\sigma(S)} = -t_S$, and the two signs cancel. Hence the two sums $\sum_S g(S) \cdot u_S$ and $\sum_S f(S) \cdot t_S$ are equal. Now the result follows from the definition of the hyperpfaffian and that $u_{[n]} = -t_{[n]}$. \qed

For more information regarding the hyperpfaffian and its applications, see Redelmeier [7].
3 Preliminaries

A weak composition \( \vec{r} \) of an integer \( m \) is a vector \((r_1, r_2, \ldots, r_k)\) whose entries are non-negative integers and their sum is \( m \). The entries are called parts. For a composition \( \vec{r} \) into \( k \) parts we let \( x^{\vec{r}} \) denote the monomial \( x_1^{r_1}x_2^{r_2}\cdots x_k^{r_k} \). Furthermore, let the symmetric group \( S_k \) act on compositions into \( k \) parts by reordering the parts.

Let \( f(x_1, x_2, \ldots, x_k) \) be a homogeneous polynomial of degree \( k/2 \cdot (n-1) \). The polynomial \( f \) can be expressed as

\[
f(x_1, x_2, \ldots, x_k) = \sum_{\vec{r}} a_{\vec{r}} \cdot x^{\vec{r}},
\]

where the sum is over all weak compositions \( \vec{r} \) of \( k/2 \cdot (n-1) \) into \( k \) parts. Furthermore, assume that the polynomial \( f \) is skew-symmetric. This implies that the coefficients satisfy that \( a_{\sigma \vec{r}} = (-1)^{\sigma} \cdot a_{\vec{r}} \).

Let \( \Gamma_{n,k} \) denote the set of all increasing weak compositions of \( k/2 \cdot (n-1) \) into \( k \) distinct parts. That is, the set \( \Gamma_{n,k} \) is given by

\[
\Gamma_{n,k} = \left\{ (r_1, r_2, \ldots, r_k) \in \mathbb{N}^k : 0 \leq r_1 < r_2 < \cdots < r_k, \sum_{i=1}^{k} r_i = k/2 \cdot (n-1) \right\}.
\]

Hence we can write the skew-symmetric polynomial \( f \) in the form

\[
f(x_1, x_2, \ldots, x_k) = \sum_{\vec{r} \in \Gamma_{n,k}} \sum_{\sigma \in S_k} (-1)^{\sigma} \cdot a_{\vec{r}} \cdot x^{\sigma \vec{r}}. \tag{3.1}
\]

We define an oriented partition to be a partition where each block is endowed with a linear order. Let \( T_{n,k} \) denote the set of all oriented partitions \( \rho \) of the set \([n]\) where each block has cardinality \( k \). That is, for an oriented partition \( \rho = \{C_1, C_2, \ldots, C_{n/k}\} \), each block \( C_i \) is a list \( C_i = (c_{i,1}, c_{i,2}, \ldots, c_{i,k}) \).

Observe that the number of oriented partitions is given by \( |T_{n,k}| = (k!)^{n/k} \cdot \Pi_{n,k} = n!/(n/k)! \).

This can be directly observed by taking a permutation on \( n \) elements and dividing into \( n/k \) blocks of size \( k \). Permuting the \( n/k \) blocks, yields the same oriented partition. Also observe that since \( k \) is even, all the \((n/k)!\) permutations yielding the same oriented partition have the same sign. We define this sign to be the sign of the oriented partition, denoted \((-1)^{\rho}\). More explicitly, the sign of \( \rho \) is given by the sign of the permutation

\[
\pi(\rho) = (c_{1,1}, \ldots, c_{1,k}, c_{2,1}, \ldots, c_{2,k}, c_{3,1}, \ldots, c_{n/k,k}) \tag{3.2}
\]

By removing the linear order on each block from the oriented partition \( \rho \), we obtain a partition \( \tau \). We note that the sign of the oriented partition \( \rho \) and the sign of the partition \( \tau \) are related by

\[
(-1)^{\rho} = (-1)^{\tau} \cdot (-1)^{\sigma_1} \cdot (-1)^{\sigma_2} \cdots (-1)^{\sigma_{n/k}}, \tag{3.3}
\]

where \( \sigma_i \) is the permutation on the set \( \{c_{i,1}, c_{i,2}, \ldots, c_{i,k}\} \) that orders the \( i \)th block, that is, \( \sigma_i(c_{i,1}) < \sigma_i(c_{i,2}) < \cdots < \sigma_i(c_{i,k}) \).

Let \( R_{n,k} \) denote the collection of sets of size \( n/k \) of compositions in \( \Gamma_{n,k} \), where all the parts of the compositions are distinct. Let \( \beta = \{\vec{r}_1, \ldots, \vec{r}_{n/k}\} \) denote such a set in \( R_{n,k} \). Observe that the sum of all the entries of the compositions is given by \( n/k \cdot k/2 \cdot (n-1) \) which is the sum \( 0 + 1 + \cdots + (n-1) \).

Hence we conclude that the underlying parts of the compositions of \( \beta \) are the integers \( 0 \) through \( n-1 \). Thus we view \( \beta \) as an oriented set partition of the elements \( \{0, \ldots, n-1\} \) into \( n/k \) blocks of
size \( k \) in which each block is a composition in \( \Gamma_{n,k} \). Define the sign of \( \beta = \{ \bar{r}_1, \ldots, \bar{r}_{n/k} \} \in R_{n,k} \) with \( \bar{r}_i = (r_{i,1}, \ldots, r_{i,k}) \), denoted by \((-1)^\beta\), to be the sign of the permutation

\[
\pi(\beta) = r_{1,1}, \ldots, r_{1,k}, \ r_{2,1}, \ldots, r_{2,k}, \ r_{3,1}, \ldots, r_{n/k,k},
\]

where \( \pi(\beta) \) is a permutation of the elements \( \{0, 1, \ldots, n-1\} \).

## 4 Main Theorem

Using the skew-symmetric polynomial given in equation \((3.1)\), we have the following identity.

**Theorem 4.1.** The hyperpfaffian \( \text{Pf}(f(x_S)) \) of order \( n \) is the product of the Vandermonde product with a signed sum of products of coefficients \( a_\pi \):

\[
\text{Pf}(f(x_S))_{S \in \binom{[n]}{k}} = \left( \sum_\beta (-1)^\beta \prod_{i=1}^{n/k} a_{\bar{r}_i} \right) \cdot \prod_{1 \leq i < j \leq n} (x_j - x_i),
\]

where the sum ranges over all partitions \( \beta \) in \( R_{n,k} \).

**Example 4.2.** When \( n = 12 \) and \( k = 4 \) there are there are 32 oriented partitions in \( R_{12,4} \). The coefficient in Theorem 4.1 is in this case given by

\[
a_{0,1,10,11}a_{2,3,8,9}a_{4,5,6,7} + a_{0,1,10,11}a_{2,4,7,9}a_{3,5,6,8} + a_{0,1,10,11}a_{2,5,6,9}a_{3,4,7,8} + a_{0,1,10,11}a_{2,5,7,8}a_{3,4,6,9} + a_{0,2,9,11}a_{1,4,7,10}a_{3,5,6,8} + a_{0,2,9,11}a_{1,4,7,10}a_{3,4,7,8} - a_{0,2,9,11}a_{1,4,7,10}a_{3,4,5,10} + a_{0,3,8,11}a_{1,2,9,10}a_{4,5,6,7} + a_{0,3,8,11}a_{1,2,9,10}a_{4,5,6,9} + a_{0,3,8,11}a_{1,2,9,10}a_{4,5,7,9} + a_{0,3,8,11}a_{1,2,9,10}a_{4,5,7,8} + a_{0,3,8,11}a_{1,2,9,10}a_{4,5,6,10} + a_{0,3,8,11}a_{1,2,9,10}a_{4,5,6,8} \]

Let \( W_{n,k} \) be the set of oriented partitions \( \rho = \{C_1, C_2, \ldots, C_{n/k}\} \) on the set \([n]\) with a composition \( \bar{w}_i = (w(c_{i,1}), \ldots, w(c_{i,k})) \in \Gamma_{n,k} \) assigned to each block \( C_i = (c_{i,1}, \ldots, c_{i,k}) \). We define the following notions for such a weighted oriented partition \( \rho \). Let \((-1)^\rho\) be the sign of \( \rho \), defined as in the previous section by the sign of the permutation \( \pi(\rho) \) from equation \((5.2)\). Let the coefficient \( c(\rho) \) denote the product \( \prod_{i=1}^{n/k} a_{\bar{w}_i} \), determined by the weight vectors of \( \rho \). Lastly, let \( w(\rho) \) denote the monomial \( \prod_{i=1}^{n/k} x_{C_i}^{w(c_{i,j})} \) where \( x_{C_i} = \prod_{j=1}^{k} x_{c_{i,j}} \).

**Lemma 4.3.** The following expansion holds for the hyperpfaffian:

\[
\text{Pf}(f(x_S))_{S \in \binom{[n]}{k}} = \sum_{\rho \in W_{n,k}} (-1)^\rho \cdot c(\rho) \cdot w(\rho).
\]

**Proof.** Expanding equation \((4.1)\) and applying equation \((3.1)\), we have

\[
\text{Pf}(f(x_S))_{S \in \binom{[n]}{k}} = \sum_{\tau \in \Pi_{n,k}} (-1)^\tau \cdot \prod_{i=1}^{n/k} \left( \sum_{\bar{r} \in \Gamma_{n,k}} \sum_{\sigma \in \Xi_k} (-1)^\sigma \cdot a_{\bar{r}} \cdot x^{\sigma_{\bar{r}}_{\bar{r}}} \right).
\]
Using the distributive law, expand the above product. We obtain an oriented, weighted partition $\rho$ for each term by orienting the elements in each block $B_i \in \tau$ by increasing size of the exponents of their associated variables. The composition $\bar{\tau}$ corresponds to the choice of weight vector for each block, and the permutation $\sigma$ will undo the orientation of the block to properly assign the weights as exponents. Multiplying the sign of $\sigma$ for each block with the sign of $\tau$ gives the sign of $\rho$ as described in equation (3.3) because for block $B_i$, $\sigma = \sigma_i^{-1}$.

Let $W_{n,k}^r$ denote the subset of $W_{n,k}$ with repeated weights, and let $W_{n,k}^d$ denote the complement, that is, partitions with distinct weights. We now create a sign-reversing involution $\phi$ for the set $W_{n,k}^r$ to narrow our focus to only partitions with distinct weights. Given a partition $\rho$ in $W_{n,k}^r$, let $(i, j)$ be the lexicographically smallest pair of elements in $[n]$ in which $w(i) = w(j)$. Define $\phi(\rho)$ by swapping $i$ and $j$, while leaving the weight vector for each block and the orientation unchanged.

**Lemma 4.4.** The function $\phi$ is a sign-reversing involution on the set $W_{n,k}^r$ which does not change the coefficient nor the monomial. That is, for an oriented partition $\rho$ we have that $\phi^2(\rho) = \rho$, $c(\phi(\rho)) = c(\rho)$, $w(\phi(\rho)) = w(\rho)$, but $(-1)^{\phi(\rho)} = -(-1)^\rho$.

**Proof.** By definition, it follows that $\phi$ is an involution, and that it leaves the coefficient and the monomial unchanged. To see that $\phi$ is sign-reversing, consider the consequences of swapping $i$ and $j$ within the permutation $\pi(\rho)$. We get that $\pi(\phi(\rho)) = (i, j) \circ \pi(\rho)$, hence the transposition changes the sign of the corresponding permutation as $\phi$ is applied. Thus, $(-1)^{\phi(\rho)} = -(-1)^\rho$.

Observe that the weighted oriented partition in Figure 1 has non-distinct powers of the pair of variables $x_1$ and $x_{12}$; and the pair $x_3$ and $x_9$, and that the pair $(1, 12)$ is lexicographic least. Hence this weighted oriented partition cancels with the oriented partition $\{(9, 12, 2, 4), (5, 3, 8, 10), (11, 1, 7, 6)\}$.

We now concentrate on weighted oriented partitions where the weights are distinct, that is, the set $W_{n,k}^d$. Note that this implies that the weights are 0 through $n-1$ and allows us to narrow our focus to weight vectors that make up an oriented partition in $R_{n,k}$.

For a weighted oriented partition $\rho$ in $W_{n,k}^d$ let $\sigma$ be the unique permutation such that $w(\rho) = x_1^{\sigma_1 - 1} x_2^{\sigma_2 - 1} \cdots x_n^{\sigma_n - 1}$. Furthermore, let $\beta$ be the set of weight vectors assigned to the blocks of $\rho$, that is, $\beta$ lies in $R_{n,k}$. Observe that this describes a bijection between $W_{n,k}^d$ and the Cartesian product of the symmetric group $\mathfrak{S}_n$ and the weight vectors $R_{n,k}$.
Lemma 4.5. The sign of a weighted oriented partition $\rho$ in $W^d_{n,k}$ factors as $(-1)^\rho = (-1)^\beta \cdot (-1)^\sigma$.

Proof. Define the permutation $\pi(\beta)'$ on $[n]$ such that $\pi(\beta)' = \pi(\beta)_i + 1$. Since $(-1)^{\pi(\beta)'} = (-1)^{\pi(\beta)}$, it is enough to observe that the permutation $\pi(\beta)'$ factors as $\sigma \circ \pi(\rho)$. □

Proof of Theorem 4.7. By combining Lemmas 4.3 through 4.5 we have that

$$\text{Pf}(f(x_S))_{S \in \binom{[n]}{k}} = \sum_{\rho \in W^d_{n,k}} (-1)^\rho \cdot c(\rho) \cdot w(\rho) = \left( \sum_{\beta} (-1)^\beta \cdot \prod_{i=1}^{n/k} a_{\vec{r}_i} \right) \cdot \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \cdot x_1^{\sigma_1 - 1} \cdots x_n^{\sigma_n - 1},$$

where the last sum is the Vandermonde determinant, which is equal to the Vandermonde product. □

An algebraic proof of Theorem 4.1 is as follows. Note that when setting two of the variables $x_i$ and $x_j$ equal, the hyperpfaffian vanishes by Lemma 2.2. Hence as a polynomial in $x_1$ through $x_n$, the Vandermonde product divides the hyperpfaffian. However, the two sides have same degree $n/k \cdot k/2 \cdot (n-1) = \binom{n}{2}$, and hence are equal up to a constant. By considering the coefficient of the term $x_2 \cdots x_{n-1}^n x_{n-1}^{-1}$, we obtain the constant $\sum_{\beta} (-1)^\beta \cdot \prod_{i=1}^{n/k} a_{\vec{r}_i}$.

Finally, observe that when the polynomial $f$ is replaced with a polynomial of degree less than $k/2 \cdot (n-1)$ the hyperpfaffian will be zero. This can be seen in two ways. Either, the only polynomial of degree less than $\binom{n}{2}$ which is divisible by the Vandermonde product is the zero polynomial. Alternatively, the sign reversing involution has no fixed points; that is, it cancels all the terms.

5 Application to the classical Pfaffian

Let us now focus on the $k = 2$ case. In this case, the oriented partitions devolve into directed matchings, and the compositions in $\Gamma_{n,2}$ have two parts with the sum $n-1$, hence they have the form $(i, n-1-i)$ from $i = 0, 1, \ldots, n/2 - 1$. This leads the skew polynomial $f$ to have the following form, in which we abbreviate the coefficients $a_{i,n-1-i}$ as simply $a_i$,

$$f(x, y) = \sum_{i=0}^{n-1} a_i \cdot x^i y^{n-1-i},$$

where $a_{n-1-i} = -a_i$. Since the only oriented partition in $R_{n,2}$ is $\{(0, n-1), (1, n-2), \ldots, (n/2 - 1, n/2)\}$, which has the sign $(-1)^{\binom{n}{2}} = 1$, Theorem 4.1 reduces to the following corollary.

Corollary 5.1. The Pfaffian $\text{Pf}(f(x_i, x_j))$ of order $n$ is the product of the first $n/2$ of the coefficients $a_i$ times the Vandermonde product:

$$\text{Pf}(f(x_i, x_j))_{1 \leq i < j \leq n} = \prod_{i=0}^{n/2 - 1} a_i \cdot \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

As a corollary we have the following identity due to Torelli [S], see also [H Equation (4.6)].

Corollary 5.2 (Torelli). When the skew-symmetric polynomial is the function $f(x, y) = (y - x)^{n-1}$, the Pfaffian is given by

$$\text{Pf}(f(x_i, x_j))_{1 \leq i < j \leq n} = (-1)^{\binom{n}{2}} \cdot \prod_{i=0}^{n/2 - 1} \binom{n-1}{i} \cdot \prod_{1 \leq i < j \leq n} (x_j - x_i).$$
It is enough to observe that \( a_i = (-1)^i \cdot \binom{n-1}{i} \).

6 Concluding remarks

Benjamin and Dresden [2] gave a different combinatorial proof of the Vandermonde identity than Gessel [3]. They gave a combinatorial interpretation of both sides and then used a sign reversing involution on the opposite side than Gessel. Is it possible to prove Corollary 5.1 or more generally, Theorem 4.1 by a similar technique?

Which other identities does the hyperpfaffian satisfy? See Knuth [4] and Tanner [5] for the expansion for products of two overlapping Pfaffians, and for applications of this identity. Can any of these results be generalized for hyperpfaffians? One such example is the following identity for compositions of the hyperpfaffians, proved by Luque and Thibon [6].

**Theorem 6.1.** Let \( k, n \) and \( p \) be three even positive integers such that \( n \) is a multiple of \( k \) and \( p \) is a multiple of \( n \). Let \( f \) be a skew-symmetric \( k \)-ary function on the set \([p]\). Define an \( n \)-ary function \( g \) by the hyperpfaffian of order \( n \), that is,

\[
g(i_1, \ldots, i_n) = \text{Pf}(f)(i_1, \ldots, i_n).
\]

Then the hyperpfaffian of order \( p \) of the function \( g \) is given by a constant times the hyperpfaffian of \( f \) of order \( p \), that is,

\[
\text{Pf}(g) = \frac{1}{(p/n)!} \cdot \binom{p/k}{n/k, \ldots, n/k} \cdot \text{Pf}(f),
\]

where there are \( p/n \) instances of \( n/k \) in the multinomial coefficient.

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