A geometric construction of the Riemann scalar curvature in Regge calculus

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Received 15 May 2008, in final form 18 July 2008
Published 16 September 2008
Online at stacks.iop.org/CQG/25/195017

Abstract
The Riemann scalar curvature plays a central role in Einstein’s geometric theory of gravity. We describe a new geometric construction of this scalar curvature invariant at an event (vertex) in a discrete spacetime geometry. This allows one to constructively measure the scalar curvature using only clocks and photons. Given recent interest in discrete pre-geometric models of quantum gravity, we believe it is ever so important to reconstruct the curvature scalar with respect to a finite number of communicating observers. This derivation makes use of a new fundamental lattice cell built from elements inherited from both the original simplicial (Delaunay) spacetime and its circumcentric dual (Voronoi) lattice. The orthogonality properties between these two lattices yield an expression for the vertex-based scalar curvature which is strikingly similar to the corresponding hinge-based expression in Regge calculus (deficit angle per unit Voronoi dual area). In particular, we show that the scalar curvature is simply a vertex-based weighted average of deficits per weighted average of dual areas.

PACS numbers: 04.60.Nc, 02.40.Ky

(Some figures in this article are in colour only in the electronic version)
Given this curvature invariant’s pivotal role in the theory of general relativity, we believe it is important to understand how to construct this geometric object locally at a chosen event in an arbitrary curved spacetime. Given recent interest in discrete pre-geometric models of quantum gravity, it is ever so important to reconstruct the curvature scalar with respect to a finite number of observers and photons [1, 2]. Even though we do have familiar discrete representations of each of the 20 components of the Riemann curvature tensor in terms of geodesic deviation or parallel transport around closed loops [3–5], and apart from the sterile act of simply taking the trace of the Riemann tensor, we are not aware of such a chronometric construction of the scalar curvature.

In this paper we provide such a discrete geometric description of this scalar curvature invariant utilizing the approach of Regge calculus [6–8], and the convergence-in-mean of Regge calculus was rigorously demonstrated [9]. In the spirit of quantum mechanics and recent approaches to quantum gravity, our construction uses only clocks and photons local to an event on an observer’s world line. Furthermore, this construction is based on a finite number of observers (clocks) exchanging a finite amount of information via photon ranging and yields the scalar curvature naturally expressed in terms of Voronoi and Delaunay lattices [10]. It has been shown that these lattices naturally arise in Regge calculus [11–21]. This construction further emphasizes the fundamental role that Voronoi and Delaunay lattices have in the discrete representations of spacetime which is perhaps not so surprising given its preponderant role in self-evolving and interacting structures in nature [10]. In this analysis we introduce a new hybrid (half Voronoi, half Delaunay) simplex which we argue is fundamental to Regge calculus [21] and perhaps fundamental to any discrete representation of classical and quantum gravity.

Consider the familiar simplicial representation of the geometry of spacetime common in Regge calculus [6, 7]. Here the spacetime is composed of a countable number simplicies. Each 4-simplex is endowed with a flat Minkowski spacetime interior. This lattice is a four-dimensional spacetime Delaunay lattice. By construction, the curvature in this lattice spacetime does not reside in its 4-simplicies, nor in its tetrahedra; however, the curvature is concentrated on each of its two-dimensional triangle hinges, $h$. Each of these hinges is the meeting place of three or more 4-simplicies. In the traditional description of Regge calculus, this hinge-based curvature is viewed as a conic singularity; however, it has been shown that the areas $h^*$ of the Voronoi lattice dual to the Delaunay simplicial lattice provides a natural area to distribute the curvature [21, 20]. The Voronoi lattice is constructed in the usual way by utilizing the circumcentric dual of the Delaunay lattice [10].

The key to our derivation of the Riemann scalar curvature is the identification $I_h \equiv I_v$ of the usual hinge-based expression, the Regge calculus version of the Hilbert action principle [6, 21] with its corresponding vertex-based expression. We begin with the Hilbert action in a $d$-dimensional continuum spacetime, which is expressible as an integral of the Riemann scalar curvature over the proper $d$-volume of the spacetime:

$$I = \frac{1}{16\pi} \int R \, dV_{\text{proper}}. \quad (1)$$

On our lattice spacetime, and following the standard techniques of Regge calculus, we can approximate this action as a sum over the triangular hinges $h$:

$$I \approx I_h = \frac{1}{16\pi} \sum_{\text{hinges}, h} R_h \Delta V_h, \quad (2)$$

where $R_h$ is the scalar curvature invariant associated with the hinge, and $\Delta V_h$ is the proper 4-volume in the lattice spacetime associated with the hinge $h$. Following earlier work by the
authors [21], this curvature will be defined explicitly below. Though, non-standard in Regge calculus, we may also express the action in terms of a sum over the vertices of the simplicial \(d\)-dimensional Delaunay lattice spacetime:

\[
I \approx \sum_{\text{vertices, } v} R_v \Delta V_v.
\]

(3)

It is the Riemann scalar curvature \((R_v)\) at the vertex \(v\) that appears in this expression that we seek in this paper, and it is the equivalence between (2) and (3) that will yield it. But first we must use the orthogonality inherent between the Voronoi and Delaunay lattices to determine the relevant 4-volumes \((\Delta V_v, \Delta V_h)\).\(^1\)

Consider a vertex \(v\) in the Delaunay lattice, and consider a triangle hinge \(h\) having vertex \(v\) as one of its three corners. We define \(A_{hv}\) to be the fraction of the area of hinge \(h\) closest to vertex \(v\) than to its other two vertices (figure 1). Dual to each triangle hinge, and in particular to triangle \(h\), is a unique co-dimension 2 area, \(A^*_h\), in the Voronoi lattice. This area necessarily lies in a \((d-2)\)-dimensional hyperplane orthogonal to the two-dimensional plane defined by the triangle \(h\). The number of vertices of the dual \((d-2)\)-polygon, \(h^*\), is equal to the number of \(d\)-dimensional simplicies hinging on triangle \(h\), and is always greater than or equal to three. If we join each of three vertices of hinge \(h\) with all of vertices of \(h^*\) with new edges, then we naturally form a \(d\)-dimensional proper volume associated with a vertex \(v\) and hinge \(h\). This \(d\)-dimensional polytope is a hybridization of the Voronoi and Delaunay lattices, they completely tile the lattice spacetime without gaps or overlaps, and they inherit their rigidity from the underlying simplicial lattice:

\[
\Delta V_{hv} = \frac{2}{d(d-1)} A_{hv} A^*_h.
\]

(4)

The simplicity of this expression (the factorization of the simplicial spacetime and its dual) is a direct consequence of the inherent orthogonality between the Voronoi and Delaunay lattices, and its impact on this calculation, and in Regge calculus as a whole, cannot be overstated. These \(d\)-cells are the Regge-calculus hybrid versions of the reduced Brillouin cells commonly found in solid state physics, though they are hybrid because they are coupled to their dual structures in the underlying atomic lattice. We view these as the fundamental building blocks of lattice gravity and at the Planck scale perhaps the Regge calculus version of Leibniz’s Monads—\textit{Vinculum Substantiale}. The scalar factor in this expression, which depends on the dimension of the lattice, was derived in the appendix of an earlier paper [21]. Furthermore we obtain the complete proper \(d\)-volume, \(\Delta V_v\), by linearly summing (4) over each of the triangles \(h\) in the Delaunay lattice sharing vertex \(v\):

\[
\Delta V_v = \sum_{h_{hv}} \Delta V_{hv} = \frac{2}{d(d-1)} \sum_{h_{hv}} A_{hv} A^*_h.
\]

(5)

We now can re-express the Regge–Hilbert action at a vertex in terms of these hybrid blocks:

\[
I_v = \frac{1}{16\pi} \sum_v R_v \sum_{h_{hv}} \frac{2}{d(d-1)} A_{hv} A^*_h.
\]

(6)

\(^1\) Although the primary concern of the authors is to apply these results to the four-dimensional pseudo-Riemannian geometry of spacetime, our equations are valid for any Riemann geometry of dimension \(d\). Therefore, in the text and equations to follow we will explicitly use the symbol \(d\) to represent the dimensionality of the geometry, the reader interested in general relativity can simply set \(d = 4\).
The triangle hinge $h$ to the left is partitioned into three areas. The shaded region ($A_{hv}$) represents the portion of the triangle that is closer to the lower vertex than its other two vertices. The darkened and pronounced vertex appearing in each of the three line drawings of this figure is the circumcenter of the hinge, $h$. Each hinge has its corresponding two-dimensional dual Voronoi area ($A^*_{h}$) shown in the upper right part of the figure as a pentagonal shaped polygon, and illustrate this dual area as ‘encircling’ the $d$-dimensional ’kite’ hinge. In bottom right portion of the figure, the ‘kite’ hinge is connected to its dual Voronoi polygon by ($4 \times 6 = 24$) new lattice edges—thus forming the Voronoi–Delaunay hybrid $d$-cell which is fundamental to our derivation, the derivation of the Hilbert action in Regge calculus, and, we believe, fundamental to any discrete representation of gravitation. Each of these edges, as well as the edges of the Voronoi area are algebraic functions of the original Regge simplicial (Delaunay) lattice spacetime, and accordingly we have not added or subtracted any degrees of freedom. These new hybrid $d$-cells provide a new, and proper tiling of the lattice spacetime.

We now return to the more familiar hinge-based Regge–Hilbert action ($I_h$) of (2). The proper 4-volume associated with hinge $h$ has been shown to be factorable in terms of the area of the triangle hinge and its corresponding dual Voronoi area [21]:

$$\Delta V_h = \frac{2}{d(d - 1)} A_h A^*_h. \quad (7)$$

Following the procedure discussed above, we can express the area of $h$ a sum of its circumcentrically-partitioned pieces (figure 1):

$$A_h = \sum_{v|h} A_{hv}. \quad (8)$$

Therefore the action per hinge (2) can be expressed as the following double summation:

$$I_h = \frac{1}{16\pi} \sum_h \sum_{v|h} R_h \left( \frac{2}{d(d - 1)} A_{hv} A^*_h \right). \quad (9)$$

A key step in this derivation is the ability to switch the order of summation, and fortunately action is unchanged if we reverse this order:

$$I_h = \frac{1}{16\pi} \sum_v \sum_{h|v} R_v \left( \frac{2}{d(d - 1)} A_{hv} A^*_h \right). \quad (10)$$

The vertex-based action of (6) must be equal to this hinge-based action of (10). We immediately obtain the desired expression for the Riemann scalar curvature at a vertex:

$$R_v = \frac{\sum_{h|v} R_h A^*_h A_{hv}}{\sum_{h|v} A^*_h A_{hv}} = \frac{\sum_{h|v} R_h A^*_h A_{hv}}{\sum_{h|v} A^*_h A_{hv}} / \frac{\sum_{h|v} A_{hv}}{\sum_{h|v} A_{hv}}. \quad (11)$$

Here we have divided the numerator and denominator by the same quantity leaving it unchanged. Both the numerator and denominator are in the form of a weighted average.
over the ‘Brillion kites’ \((A_{hv})\) at vertex \(v\). We define, in a natural way, the ‘kite weighted average’ at vertex \(v\) of any hinge-based quantity \(Q_h\) as follows:

\[
\langle Q \rangle_v \equiv \frac{\sum_{h_v} Q_h A_{hv}}{\sum_{h_v} A_{hv}}.
\] (12)

Given this definition, the scalar curvature invariant at vertex \(v\) can be expressed as a ‘kite-weighted average’ of the integrated hinge-based scalar curvature of Regge calculus:

\[
R_v = \frac{\langle R_h A_h^* \rangle_v}{\langle A_h^* \rangle_v},
\] (13)

where it was shown in [21] that the Riemann scalar curvature at the hinge \(h\) is expressible as the hinge’s curvature deficit \((\epsilon_h)\) per unit Voronoi area \((A_h^*)\) dual to \(h\):

\[
R_h = \frac{1}{d(d-1)} \frac{\epsilon_h}{A_h^*}.
\] (14)

Therefore the expression for the vertex-based scalar curvature invariant derived here is strikingly similar to the usual Regge calculus expression for the hinge-based scalar curvature invariant (14). The only difference is that the numerator and denominator of (14) is replaced by their kite-weighted averages:

\[
R_v = \frac{1}{d(d-1)} \frac{\langle \epsilon_h \rangle_v}{\langle A_h^* \rangle_v}.
\] (15)

In a four-dimensional spacetime the minimum number of events needed to measure the scalar curvature at a vertex \((v)\) is six. This occurs when the four-dimensional Voronoi cell dual to \(v\) is itself a 4-simplex. This corresponds to the minimum allowable number of simplicies in a Regge calculus spacetime lattice that can meet at vertex \(v\) and is consistent with earlier results on the minimum number of test particles needed to measure the 20 components of the Riemann tensor [3, 5]. Such a minimal 6-point scalar curvature detector can be constructed solely from null (laser) and timelike (clock) edges—the tools available to a spacetime surveyor [22] (figure 2). Such chronometric constructions, we believe will be useful in their applications to discrete models of quantum gravity.
Acknowledgments

We thank Renate Loll and Seth Lloyd for stimulating discussions which provided us the motivation to continue this research, and we are especially grateful to John A Wheeler for providing the inspiration and initial guidance into this field of spacetime geodesy. We would like to thank Florida Atlantic University’s Office of Research and the Charles E Schmidt College of Science for the partial support of this research.

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