θ-dependence of QCD at Finite Isospin Density

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We probe the θ-dependence of QCD at finite isospin chemical potential μ_I using the effective chiral Lagrangian approach. The phase diagram in the θ, μ_I plane is constructed and described in detail in terms of chiral and pion condensates. The physics at θ ≈ π is investigated in both the normal and superfluid phases. Finally, the behaviour of the gluon condensate at finite μ_I is computed.

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I. INTRODUCTION

The θ-parameter of gauge theories has long attracted attention as it is a probe of the topological properties of the theory. In almost every context, from pure Yang-Mills theories to QCD the θ dependence of the theory is highly non-trivial and, frequently, non-analytic. In particular, in QCD with two flavours and equal non-zero quark masses it is believed that the so called Dashen’s phenomenon - a first order phase transition, characterized by spontaneous breaking of CP, occurs at θ = π.

In this paper, we investigate the influence of finite isospin chemical potential μ_I on the θ dependence of two flavor QCD. Besides pure academic interest, the main physical motivation for such a study is the attempt to understand the cosmological phase transition when θ, being non-zero and large at the very beginning of the phase transition, slowly relaxes to zero, as the axion resolution of the strong CP problem suggests. Of course, in real world, we are mostly interested in the effects of θ on matter at finite baryon, rather than isospin, density. Indeed, if isospin asymmetric matter presently exists in nature (say in neutron stars), it is accompanied by a large baryon density. However, analytical control over QCD is absent at moderate baryon density and appears only at asymptotically large baryon chemical potential, where one expects the color-superconducting state to be realized. Nevertheless, one may resort to QCD-like theories, such as N_c = 2 QCD at finite baryon density and N_c = 3 QCD at finite isospin density, where analytical control is present, to gain some insight into real dense QCD.

Due to the axial anomaly, the θ parameter of QCD is intimately tied to the quark mass matrix and may be incorporated into the effective chiral Lagrangian. There also exists a well-known procedure for including the effects of finite μ_I into the QCD chiral Lagrangian. We, thus, expect that we may adequately describe QCD at finite isospin density and θ ≠ 0 in the effective Lagrangian approach, as long as μ_I is much smaller than the mass of the lightest non-Goldstone boson (in QCD, the mass of the ρ meson, m_ρ).

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Using the above approach, we obtain a wide range of information about the phase diagram of two flavor QCD in the $\mu_I, \theta$ plane. We show that the transition to the superfluid, isospin breaking, phase occurs at $\mu_I$ equal to the $\theta$ dependent pion mass, $m_\pi(\theta)$. This implies that for fixed $\mu_I$ of order of the pion mass, the $\theta$ dependence of the theory becomes non-analytic. Two second order phase transitions, accompanied by a jump in the topological susceptibility, occur as $\theta$ relaxes from 2$\pi$ to 0.

We compute the $\theta$ dependence of chiral and pion condensates, as well as $\langle iG\tilde{G}\rangle$ and the topological susceptibility, in normal and superfluid phases. We find that the $\theta$ dependence in the superfluid phase near $\theta = \pi$ is much smoother than in the normal phase. In particular, for $m_u = m_d$, we show that the first order phase transition across $\theta = \pi$ present in the normal phase, disappears in the superfluid phase.

Finally, we discuss a few $\theta$ unrelated issues. Most importantly, we compute the $\theta$ dependence of the gluon condensate $\langle \frac{1}{32\pi^2}G^a_{\mu\nu}G^{a\mu\nu}\rangle$ on the isospin chemical potential in the superfluid phase. The gluon condensate decreases with density near the normal to superfluid phase transition, but, counter-intuitively, increases for $m_\pi \ll \mu_I \ll m_\rho$. We also evaluate a novel vacuum expectation value, which appears in the superfluid phase: $\langle i\bar{u}\gamma_0\gamma_5d\rangle$. This density, being nonzero even at $\theta = 0$, nonetheless has never been discussed in the literature previously. This density, itself, breaks the isospin symmetry, and so may be considered as an additional order parameter.

We note that the above agenda has also recently been implemented to study the properties of $N_c = N_f = 2$ QCD in the presence of non-zero $\theta$ at finite baryon and isospin density. Most of the results of the present study are in direct correspondence with the work\cite{13}. This is a consequence of the fact that the chiral Lagrangians describing $N_c = 3, N_f = 2$ QCD and the pion sector of the $N_c = 2, N_f = 2$ QCD are identical. Besides adapting the work\cite{13} to the $N_c = 3$ context, we presently discuss in some detail the theoretically interesting case of exactly degenerate quark masses, which was not analyzed in\cite{13}.

We hope that the results of this study would be of interest for lattice simulations. Indeed, the determinant of the Dirac operator is real and positive in QCD at non-zero isospin chemical potential and $\theta = 0$. The determinant remains real at $\mu_I \neq 0, \theta = \pi$. Thus, we hope that the $\mu_I$ dependence of the gluon condensate and the topological susceptibility at $\theta = 0$, can be explicitly checked on the lattice. This is a unique chance to study the gluon degrees of freedom and their dependence on light quark masses. The corresponding study might be important for the extrapolation procedure which has to be used in order to achieve the chiral limit. Moreover, we hope that the disappearance of Dashen’s phenomenon at $\theta = \pi$ in the superfluid phase can also be confirmed by lattice simulations.

II. THE CHIRAL LAGRANGIAN

The low energy dynamics of $N_f = 2$ QCD are governed by the chiral Lagrangian for the pion field $U \in SU(2)$. A well known procedure exists to incorporate into this Lagrangian the effects of a finite $\theta$ parameter\cite{2,3}. A method for introducing a finite isospin chemical potential $\mu_I$ is also well-developed\cite{11,12}. To lowest order in quark mass and derivatives, the chiral Lagrangian reads,

$$\mathcal{L} = \frac{1}{4}f_\pi^2\text{Tr}(\nabla_\mu U \nabla_\mu U^\dagger) - \Sigma \text{ReTr}(MU)$$ (1)
where the flavor covariant derivatives are defined as,
\[ \nabla_0 U = \partial_0 U - \frac{1}{2} \mu_I [\tau^3, U], \quad \nabla_i U = \partial_i U \]
\[ \nabla_0 U^\dagger = \partial_0 U + \frac{1}{2} \mu_I [U^\dagger, \tau^3], \quad \nabla_i U^\dagger = \partial_i U^\dagger \]

We work in Euclidean space. Here the \( \theta \) parameter of QCD has been incorporated directly into the quark mass matrix,
\[ M = e^{-i\theta/N_f} \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \]

We keep \( m_u \neq m_d \) on purpose: as is known \( m_u = m_d \) is a very singular limit when one discusses \( \theta \) dependence, see below. The coefficient \( \Sigma \) is determined by the chiral condensate in the limit \( m \to 0^+, \theta = 0, \mu_I = 0 \),
\[ \Sigma = -\frac{\langle \bar{\psi} \psi \rangle_0}{2N_f} \]
as will be confirmed below. In our notations the chiral condensate includes the sum over all flavors, \( \langle \bar{\psi} \psi \rangle = \sum_f \langle \bar{\psi}_f \psi_f \rangle \).

Due to pseudo-reality of SU(\(N_f=2\)), one may, to this order in chiral perturbation theory, incorporate all effects of \( \theta, m_u, m_d \) into a common real quark mass via a redefinition,
\[ U = L \tilde{U} R^\dagger, \quad L = R^\dagger = e^{i\alpha \tau^3/2} \]
\[ \cos \alpha = \frac{(m_u + m_d) \cos(\theta/2)}{\sqrt{(m_u + m_d)^2 \cos^2(\theta/2) + (m_u - m_d)^2 \sin^2(\theta/2)}} \]
\[ \sin \alpha = \frac{(m_u - m_d) \sin(\theta/2)}{\sqrt{(m_u + m_d)^2 \cos^2(\theta/2) + (m_u - m_d)^2 \sin^2(\theta/2)}} \]

Our parameter \( \alpha \) is related to the commonly used Witten’s variables \( \phi_u, \phi_d [2] \), via,
\[ \phi_u = \theta/2 - \alpha, \quad \phi_d = \theta/2 + \alpha \]
\[ \phi_u + \phi_d = \theta, \quad m_u \sin \phi_u = m_d \sin \phi_d \]

After such a transformation, the Lagrangian \( \text{(11)} \) takes the form,
\[ \mathcal{L} = \frac{1}{4} f_\pi^2 \text{Tr}(\nabla_\mu \tilde{U} \nabla_\mu \tilde{U}^\dagger) - m(\theta) \Sigma \text{Re} \text{Tr}(\tilde{U}) \]

with,
\[ m(\theta) = \frac{1}{2} \left( (m_u + m_d)^2 \cos^2(\theta/2) + (m_u - m_d)^2 \sin^2(\theta/2) \right)^{1/2} \]

**III. PHASE DIAGRAM**

Our next step is to find the classical minimum of the effective Lagrangian \( \text{(10)} \) to determine the phase diagram. First let’s study the theory at zero chemical potential and fixed \( \theta \).
The classical minimum, is then given by, $\tilde{U} = 1$, and the lowest lying excitations correspond to a triplet of pions, with $\theta$ dependent mass,

$$m^2_\pi(\theta) = \frac{m(\theta)\langle \bar{\psi}\psi \rangle_0}{f^2_\pi} \quad (12)$$

The pion mass $m_\pi$ acquires a dependence on $\theta$ through the effective quark mass parameter $m(\theta)$. As we shall see, the whole phase diagram turns out to be determined by the parameter $m_\pi(\theta)$. We note that $m_\pi(\theta)$ reaches its maximum at $\theta = 0$ and minimum at $\theta = \pi$. Moreover, for $m_u = m_d$, $\theta = \pi$, $m_\pi$ vanishes to first order in $m_q$.

Now let’s turn on finite $\mu_I$. At fixed $\theta$, the phase diagram contains two phases: normal and superfluid. The transition from the normal phase to the superfluid phase occurs at the critical chemical potential $\mu_I = m_\pi(\theta)$. In the normal phase, $\langle \tilde{U} \rangle = 1$. In the superfluid phase, the $U(1)_I$ symmetry is spontaneously broken and,

$$\langle \tilde{U} \rangle = \lambda(\theta) + i\sqrt{1 - \lambda^2}(\tau^1 \cos \phi + \tau^2 \sin \phi) \quad (13)$$

where the variable $\phi$ labels the $U(1)_I$ degeneracy of the vacuum, and we have introduced the parameter $\lambda$ to describe both the normal and superfluid phase,

$$\lambda(\theta) = \begin{cases} 
1 & \text{normal phase} \\
\frac{m^2(\theta)}{\mu_I^2} & \text{superfluid phase}. 
\end{cases} \quad (14)$$

As expected, at $\theta = 0$ we reproduce the known results $[11, 12]$. At $\theta \neq 0$ the phase diagram looks the same as at $\theta = 0$, with the important replacement, $m^2_\pi \rightarrow m^2_\pi(\theta)$. This is a very natural conclusion. Indeed, at $\theta \neq 0$ pions still carry isospin number. Hence, their energy is lowered at finite isospin chemical potential. As soon as $\mu_I$ reaches the vacuum pion mass $m_\pi(\theta)$, Bose-condensation occurs leading to spontaneous breaking of $U(1)_I$ symmetry.

Quantitatively, the $\theta$ dependence of the Goldstone mass $m_\pi(\theta)$ implies that the transition to superfluid phase is shifted to a smaller chemical potential $\mu_I$, compared to $\theta = 0$. In the limiting case, when $m_u = m_d$ and $\theta = \pi$, the transition occurs in the vicinity of $\mu = 0$ (see Section IV for a more precise discussion). For physical values, $m_d = 7\text{MeV}$, $m_u = 4\text{MeV}$, the transition at $\theta = \pi$ occurs at $\mu = \left(\frac{m_u - m_d}{m_u + m_d}\right)^{\frac{1}{2}} m_\pi(0) \sim 70\text{MeV}$.

We now wish to describe the phase diagram in terms of different condensates and densities. This can be achieved by the standard procedure of introducing sources into the chiral Lagrangian. We find that chiral condensates depend on $\mu_I$, $\theta$ in the following way,

$$\langle \bar{u}u \rangle = \frac{1}{2}\langle \bar{\psi}\psi \rangle_0 \lambda(\theta) \cos\left(\frac{\theta}{2} - \alpha\right), \quad \langle \bar{d}d \rangle = \frac{1}{2}\langle \bar{\psi}\psi \rangle_0 \lambda(\theta) \cos\left(\frac{\theta}{2} + \alpha\right) \quad (15)$$

$$i\langle \bar{u}\gamma_5 u \rangle = -\frac{1}{2}\langle \bar{\psi}\psi \rangle_0 \lambda(\theta) \sin\left(\frac{\theta}{2} - \alpha\right), \quad i\langle \bar{d}\gamma_5 d \rangle = -\frac{1}{2}\langle \bar{\psi}\psi \rangle_0 \lambda(\theta) \sin\left(\frac{\theta}{2} + \alpha\right)$$

while the pion condensate, which exists only in the superfluid phase and spontaneously breaks the $U(1)_I$ symmetry, takes the form,

$$i\langle \bar{u}\gamma_5 d \rangle = \frac{1}{2}\langle \bar{\psi}\psi \rangle_0 \sqrt{1 - \lambda^2(\theta)} \cos\left(\frac{\theta}{2}\right), \quad \langle \bar{u}d \rangle = \frac{1}{2}\langle \bar{\psi}\psi \rangle_0 \sqrt{1 - \lambda^2(\theta)} \sin\left(\frac{\theta}{2}\right)$$

Notice that once $\theta \neq 0$, the $P$ odd condensate $i\langle \bar{q}\gamma_5 q \rangle$ appears in addition to the usual $P$ even condensate $\langle \bar{q}q \rangle$. Similarly, in the superfluid phase at $\theta \neq 0$, the $P$ even condensate $\langle \bar{u}d \rangle$
exists alongside the ordinary $P$ odd pion condensate, $i\langle \bar{u}\gamma_5 d \rangle$. This is a direct consequence of explicit parity violation by the $\theta$ term.

We may also compute the following charge densities from our chiral Lagrangian,

$$ n_I = \frac{1}{2} \langle \bar{\psi}\gamma^{0}\tau^{3}\psi \rangle = f_{\pi}^{2}\mu_{I}(1 - \lambda(\theta)^{2})$$  \hspace{1cm} (16) $$ n^\pi_{A} = i\langle \bar{u}\gamma^{0}\gamma^{5}d \rangle = -f_{\pi}^{2}\mu_{I}\lambda(\theta)\sqrt{1 - \lambda^{2}(\theta)\cos(\alpha)}$$  \hspace{1cm} (17) $$ n^\pi_{A} = \langle \bar{u}\gamma^{0}d \rangle = -f_{\pi}^{2}\mu_{I}\lambda(\theta)\sqrt{1 - \lambda^{2}(\theta)\sin(\alpha)}$$  \hspace{1cm} (18)

All the charge densities vanish in the normal phase. In the superfluid phase, a non-zero isospin density appears, $n_I = \frac{1}{2} \langle \bar{\psi}\gamma^{0}\tau^{3}\psi \rangle$. This is precisely the density, which one expects to induce by applying an isospin chemical potential $\mu_I$. At $\theta = 0$ it coincides with the previous result $[13]$. In addition, we also obtain non-vanishing axial charge densities, $n^\pi_{A} = i\langle \bar{u}\gamma^{0}\gamma^{5}d \rangle$ and $n^\pi_{A} = \langle \bar{u}\gamma^{0}d \rangle$. Notice that $n^\pi_{A}$ does not vanish already at $\theta = 0$, nevertheless, it was never discussed previously in the literature. The quantity $n^\pi_{A}$ is the axial charge density, corresponding to off-diagonal generators of the $SU(2)_A$ group, which is both spontaneously and explicitly broken.

The density $n^\pi_{A}$ spontaneously breaks the $U(1)$ symmetry and, hence, may be considered as an order parameter alongside the pion condensate, $\langle \pi^{\prime} \rangle = i\langle \bar{u}\gamma_5 d \rangle$. Note that there was no explicit chemical potential conjugate to $n^\pi_{A}$ in the Lagrangian - once $U(1)_I$ is already spontaneously broken by $\langle \pi^\prime \rangle$, $n^\pi_{A}$ is induced automatically. The reader is referred to the paper $[13]$ on $N_c = N_f = 2$ QCD for a few arguments, which intuitively explain why in a system with nonvanishing $n_I$ and $\langle \pi^{\prime} \rangle$, the second order parameter $n^\pi_{A}$ automatically appears. The quantitative behaviour of these two order parameters is somewhat different. The pion condensate monotonically increases with $\mu_I$ after the normal to superfluid phase transition, and $\langle \pi^{\prime} \rangle \to -\frac{1}{2} \langle \bar{\psi}\psi \rangle_0$ for $\mu_I \gg m_\pi$. On the other hand, the new charge density $n^\pi_{A}$ first increases after the phase transition, reaches a peak at $\mu_I = 3^{1/4}m_\pi$, and then decreases to 0 for $\mu \gg m_\pi$. Of course, we always consider only $\mu_I \ll m_\rho$.

We note that the new order parameter $n^\pi_{A}$ vanishes, in the limit $m_q \to 0$. We expect that in the regime of asymptotically large $\mu_I$, where analytical control is present, and both $n_I$ and $\langle \pi^{\prime} \rangle$ are believed to be non-vanishing, one can explicitly show that $n^\pi_{A}$ will also appear once $m_q \neq 0$ is considered.

### IV. THETA DEPENDENCE

So far we have mostly focused on the $\mu_I$ dependence at fixed $\theta$. In this section we would like to focus more on the $\theta$ dependence, drawing the phase diagram in the $(\theta, \mu_I)$ plane. We will also pay particularly careful attention to the physics near $\theta = \pi$.

We begin by briefly reviewing the well-known $\theta$ dependence at $\mu = 0$. The grand canonical potential $\Omega(\theta)$ is,

$$ \Omega(\theta, \mu = 0) = -f_{\pi}^{2}m_\pi^{2}(\theta)$$  \hspace{1cm} (19)

By differentiating $\Omega(\theta)$ we can compute correlation functions of $G\bar{G}$,

$$ \frac{\partial \Omega}{\partial \theta} = \langle \frac{g^{2}G\bar{G}}{32\pi^{2}} \rangle$$  \hspace{1cm} (20) $$ -\frac{\partial^{2} \Omega}{\partial \theta^{2}} = \chi = -\int d^{4}x \langle T\frac{g^{2}G\bar{G}}{32\pi^{2}}(x)\frac{g^{2}G\bar{G}}{32\pi^{2}}(0) \rangle_{conn}$$  \hspace{1cm} (21)
At $\mu = 0$ we find,
\[
\frac{\langle i g^2 G\tilde{G} \rangle}{32\pi^2}_{\mu = 0} = -\frac{1}{4} \frac{m_u m_d}{m(\theta)} \sin(\theta) \langle \tilde{\psi}\psi \rangle_0
\]
\[
\chi(\mu = 0) = \frac{1}{4} \frac{m_u m_d}{m(\theta)} \left( \cos(\theta) + \frac{m_u m_d}{4m(\theta)^2} \sin^2(\theta) \right) \langle \tilde{\psi}\psi \rangle_0
\]
(22)

Expressions (22) reflect the well-known strong $\theta$ dependence in the region $m_u \approx m_d = m_q$, $\theta \approx \pi$. Let’s introduce the asymmetry parameter, $\epsilon = \frac{|m_u - m_d|}{m_u + m_d}$ and assume $\epsilon \ll 1$. The $CP$ odd order parameter $\langle iG\tilde{G} \rangle$, though apparently smooth for $\epsilon \neq 0$, experiences a steep crossover in the region $|\theta - \pi| \sim \epsilon$. Correspondingly, the topological susceptibility $\chi$ has a sharp peak around $\theta = \pi$ of width $\Delta \theta \sim \epsilon$ and height $\chi(\pi)/|\chi(0)| = 1/\epsilon$.

Such behaviour of the $CP$ odd order parameter $\langle iG\tilde{G} \rangle$ strongly suggests that for $m_u = m_d$, spontaneous breaking of $CP$ symmetry occurs at $\theta = \pi$. This situation, known as Dashen’s phenomenon, has been extensively studied in QCD with $N_f = 3$ and $N_f = 2$ [1, 2, 3, 4, 5]. For $N_f = 3$ with $m_s \gg m_u, m_d$ it is believed that spontaneous $CP$ breaking occurs at $\theta = \pi$ for $|m_u - m_d| m_s < m_u m_d$.

For $N_f = 2$, the key observation [4, 5] is that Dashen’s phenomenon is not under complete theoretical control in the effective Lagrangian (11). Indeed, for a moment, we fix $m_u = m_d$. Then, for general $\theta$, the mass term explicitly breaks the symmetry of the effective Lagrangian (11) from $SU(2)_L \times SU(2)_R$ to $SU(2)_V$. However, for $\theta = \pi$, the mass term in the effective Lagrangian vanishes, restoring the symmetry to $SU(2)_L \times SU(2)_R$ and giving rise to apparently massless goldstones: $m_\pi^2(\theta = \pi) = 0$. Yet, no such symmetry restoration occurs in the fundamental microscopic QCD Lagrangian at $\theta = \pi$. This contradiction is resolved by including higher order (quadratic) mass terms in the effective Lagrangian, which would explicitly break $SU(2)_A$ even at $\theta = \pi$ [5]. It is precisely these terms, which control the physics of Dashen’s phenomenon.

In this paper we would like to consider two different regimes. In the first regime, one may neglect the higher order mass terms by considering fixed $\frac{|m_u - m_d|}{m_u + m_d} \neq 0$ and sufficiently small $m_q$. Of course, in such a regime one automatically excludes the regions of parameter space where Dashen’s transition is realized, and may discuss only the quantitatively steep crossover in the normal phase. The second regime that we discuss is obtained by considering the exactly degenerate case $m_u = m_d$. We show that this second regime exhibits the Dashen’s transition in the normal phase, which disappears in the superfluid phase.

### A. Crossover Regime

In this section we discuss the regime in which the leading order chiral Lagrangian (11) accurately describes the physics for all $\theta$. Here we give only a brief summary of the results concerning this regime, for further discussion see [13].

If the leading order (12) pion mass at $\theta = \pi$, $m_\pi^2(\theta = \pi) \propto |m_u - m_d|$, is sufficiently large one may neglect the higher order mass terms in the effective chiral Lagrangian. For any fixed $\frac{|m_u - m_d|}{m_u + m_d} \neq 0$ this is achieved by considering sufficiently small $m_q$. If the higher order mass terms are largely saturated by a third quark of mass $m_{u,d} \ll m_s \ll \Lambda_{QCD}$, one requires,
\[
\frac{|m_u - m_d|}{m_u + m_d} \gg \frac{m_{u,d}}{m_s} \sim \frac{m_\pi^2(\theta = 0)}{M_\eta^2}
\]
(23)
FIG. 1: Phase diagram of $N_f = 2$ QCD as a function of $\mu_I$ and $\theta$. Here, $\epsilon = \frac{m_u - m_d}{m_u + m_d} = 0.01$. A rapid crossover occurs in the normal phase at $\theta = \pi$, which becomes a first order phase transition, when $m_u = m_d$.

This condition is, indeed, realized in the true physical world. If, on the other hand, the higher order terms are controlled by a light $\eta'$ (as motivated by $N_c \to \infty$), one needs to consider,

$$\frac{|m_u - m_d|}{m_u + m_d} \gg \frac{m(\bar{\psi}\psi)_0}{f_{\pi}^2 M_{\eta'}^2} \sim \frac{m_\pi^2(\theta = 0)}{M_{\eta'}^2}$$

(24)

Let us now turn on finite $\mu_I$. Once conditions (23), (24) are met, all the results of previous sections hold for any $\theta$. In particular, the transition to the superfluid phase occurs at $\mu = m_\pi(\theta)$ (see Fig. 1).

Thus, for $\mu_I < m_\pi(\theta = \pi)$ the normal phase is realized for all $\theta$, while for $\mu_I > m_\pi(\theta = 0)$ we are entirely in the superfluid phase. Finally, if we fix $\mu_I$ with $m_\pi(\theta = \pi) < \mu_I < m_\pi(\theta = 0)$ and vary $\theta$ from 0 to $2\pi$ we encounter two phase transitions: from normal to superfluid phase and then back to normal. Thus, the $\theta$ dependence becomes non-analytic in this region! Since the normal to superfluid phase transition is second order, we expect the topological susceptibility, $\chi$ to be discontinuous across the phase boundary. The transitions between normal and superfluid phases occur at $\theta = \theta_c$ and $\theta = 2\pi - \theta_c$, with the critical $\theta_c$ given by $m_\pi(\theta_c) = \mu_I$.

In the superfluid phase, the free energy density reads,

$$\Omega(\theta) = -\frac{1}{2} f_{\pi}^2 \mu_I^2 \left( 1 + \frac{m_\pi^4(\theta)}{\mu_I^4} \right).$$

(25)

Clearly, the $\theta$ dependence in the superfluid phase is different from that in the normal phase (19). This is most clearly seen by computing topological density and topological suscepti-
bility in the superfluid phase,

\[ \langle i g^2 \tilde{G} \tilde{G} \rangle = \frac{m_u m_d}{4 f_\pi^2 \mu_I^2} \langle \bar{\psi} \psi \rangle_0^2 \sin(\theta), \]
\[ \chi = -\frac{m_u m_d}{4 f_\pi^2 \mu_I^2} \langle \bar{\psi} \psi \rangle_0^2 \cos(\theta). \]  

(26)

The corresponding expressions should be compared with (22) describing the normal phase. Focusing for a moment on \( \mu_I > m_\pi(\theta = 0) \), we see that the \( \theta \) dependence is very smooth: there is no sign of rapid crossover in \( \langle i G \tilde{G} \rangle \) near \( \theta = \pi \) and the large peak in the susceptibility \( \chi \) disappears. Moreover, as \( \mu_I \) increases, the \( \theta \) dependence is suppressed, as expected. This smooth \( \theta \) dependence at \( \theta \sim \pi \) in the superfluid phase should be contrasted with sharp behavior in the normal phase discussed above, see eq. (22). As has been explained in detail in the parallel study on \( N_c = 2 \) QCD [13], the disappearance of the “Dashen’s crossover” as \( \mu_I \) increases is accomplished in the following way. First, when \( m_\pi(\theta = \pi) < \mu_I < m_\pi(\theta = 0) \), the crossover splits into two second order normal to superfluid phase transitions. These phase transitions replace the peak in the topological susceptibility \( \chi \) by finite jumps in \( \chi \) at the transition points:

\[ \frac{\chi(\theta_e^+) - \chi(\theta_e^-)}{|\chi(0)|} = \frac{m_u m_d m_\pi^2(0) \langle \bar{\psi} \psi \rangle_0^2}{4 f_\pi^2 \mu_I^2} \sin^2(\theta_c) \]  

(27)

Finally, once \( \mu_I > m_\pi(\theta = 0) \) no phase transitions can be triggered by varying \( \theta \), and the “Dashen’s crossover” becomes entirely washed out.

We conclude this section by noting that we can use our results for the topological susceptibility \( \chi \) and the chiral condensate \( \langle \bar{\psi} \psi \rangle \) to study how the Ward Identities get saturated in different phases with arbitrary \( \theta \). [3, 14, 15, 16],

\[ \chi = -\int d^4 x (T g^2 \tilde{G} \tilde{G} (x) g^2 \tilde{G} \tilde{G}(0))_{\text{conn}} = \frac{1}{N_f^2} \langle \bar{\psi} M \psi \rangle + O(M^2) \]  

(28)

\[ O(M^2) = -\frac{1}{N_f^2} \int d^4 x \langle T \bar{\psi} \gamma_5 M \psi(x) \bar{\psi} \gamma_5 M \psi(0) \rangle_{\text{conn}}. \]

This Ward Identity is related to the axial anomaly and, thus, should not be affected by infra-red effects, such as finite chemical potential. One can explicitly check that our results imply that at \( \theta = 0 \), the identity (28) is, indeed, straightforwardly satisfied both in the normal and superfluid phases. However, at \( \theta \neq 0 \), in the superfluid phase, one must include the \( O(M^2) \) term in (28) on the same footing as the \( O(M) \) term for the Ward Identity to be satisfied. The reader is referred to the work [13] where \( N_c = 2 \) case was discussed in detail. In the present case with \( N_c = 3 \) the saturation of the Ward Identities goes precisely in the same way as in [13], and therefore, we do not need to repeat it here.

**B. Phase Transition Regime**

In the present section, we would like to consider the degenerate case \( m_u = m_d = m \), which has not been discussed in the companion paper [13]. This regime is believed to support a first order phase transition across \( \theta = \pi \) at zero chemical potential [5, 6]. The discussion of the crossover regime in section IVA is highly suggestive of the fact that this phase transition
disappears in the superfluid phase. We shall now explicitly demonstrate this claim. We note that the point \( m_u = m_d \), \( \theta = \pi \) might be of importance for lattice fermions\(^4\, 17\), as it is equivalent to a theory where one quark mass is negative and \( \theta \) parameter is not explicitly present.

As already mentioned, in QCD with \( N_f = 2 \), one needs to include second order mass terms in the effective chiral Lagrangian in order to accurately describe physics near \( m_u = m_d \), \( \theta = \pi \). As argued in \([5]\), the dominant second order mass term is,

\[
V_2(U) = -l_7 \frac{\Sigma^2}{f_\pi^4} (\text{Im} \text{Tr}(MU))^2
\]  

(29)
as it contains \( \theta \) dependence different from the leading mass term. Including this term in our chiral Lagrangian, we obtain,

\[
\mathcal{L} = \frac{1}{4} f_\pi^2 \text{Tr}(\nabla_\mu U \nabla_\mu U^\dagger) - \Sigma \text{Re} \text{Tr}(MU) - l_7 \frac{\Sigma^2}{f_\pi^4} (\text{Im} \text{Tr}(MU))^2
\]  

(30)

Let us review the \( \theta \) dependence contained in (30) at \( \mu_I = 0 \). As is known, the physics at \( \mu_I = 0 \) in the neighborhood of \( \theta = \pi \), crucially depends on the sign of \( l_7 \). A number of arguments\([5]\) suggest that \( l_7 \) is positive, in particular, in the large \( N_c \) limit, \( l_7 \sim \frac{f_\pi^2}{2M_\pi' N_c} \). We shall assume \( l_7 > 0 \) for the rest of this work. In this case, the static classical minimum of (30) is given by,

\[
U = 1 \quad \text{for} \quad 0 \leq \theta < \pi \quad \text{and} \quad U = -1 \quad \text{for} \quad 0 < \theta \leq 2\pi.
\]  

At \( \theta = \pi \), the classical minimum is degenerate: \( U = \pm 1 \), signalling spontaneous breaking of the \( P,CP \) symmetries.

Computing the value of the \( CP \) order parameter, \( \langle \bar{G} \tilde{G} \rangle \), near \( \theta = \pi \),

\[
\langle i \frac{g^2 G \tilde{G}}{32 \pi^2} \rangle_{\theta = \pm \pi} = \pm \frac{m_\pi}{2} |\langle \bar{\psi} \psi \rangle_0| \tag{31}
\]

This is exactly the result one would derive by naively setting \( m_u = m_d \) in eq. (22). Thus, once \( m_u = m_d \), the rapid crossover discussed in the previous section becomes a phase transition.

The \( \theta \) dependent mass of the three degenerate goldstones becomes,

\[
m_\pi^2(\theta) = \frac{m_\pi^2 |\langle \bar{\psi} \psi \rangle_0|}{f_\pi^2} \cos(\theta/2) + \frac{2l_7 m_\pi^2 (\bar{\psi} \psi)_0}{f_\pi^4} \sin^2(\theta/2) \tag{32}
\]

We see that the goldstones pick up a small, but non-vanishing, mass at \( \theta = \pi \), due to the \( V_2 \) term in the chiral Lagrangian.

Turning on a finite chemical potential, we see that for \( |\mu_I| < m_\pi(\theta) \), we are in the normal phase, while for \( |\mu_I| > m_\pi(\theta) \), we are in the superfluid phase (see Fig. 2). The static minimum \( U \) of the Lagrangian (30) in both phases is again given by expression (13), except that now,

\[
\lambda(\theta) = \begin{cases} 
\text{sgn} \left( \cos(\theta/2) \right) & \text{normal phase} \\
\frac{m_\pi^2(0) \cos(\theta/2)}{\mu_I^2 - m_\pi^2(\pi) \sin^2(\theta/2)} & \text{superfluid phase}
\end{cases} \tag{33}
\]

The normal phase at finite \( \mu_I \) again has the same physical properties as at \( \mu_I = 0 \). In particular, a first order phase transition across \( \theta = \pi \) persists for \( |\mu_I| < m_\pi(\theta = \pi) \).
However, once $|\mu_I| > m_\pi(\theta = \pi)$, the Dashen’s transition splits into two second order normal to superfluid phase transitions. The $\theta$ dependence in the superfluid phase is very smooth. In particular, $P$ parity is not spontaneously broken at $\theta = \pi$: one may check,

$$\langle i \frac{g^2 G \tilde{G}}{32\pi^2} \rangle_{\theta = \pi} = 0$$  \hspace{1cm} (34)$$

It is amusing to note, that in the superfluid phase, spontaneous breaking of parity is shifted from $\theta = \pi$ to $\theta = 0$.

We observe that in the region $m_\pi(\theta = \pi) < \mu_I < m_\pi(\theta = 0)$, the $\theta$ dependence is essentially the same as in a theory with $l_7 < 0$ and $\mu_I = 0$. Indeed, in such a theory, one would have instead of a first order phase transition at $\theta = \pi$, two second order phase transitions just before and after $\theta = \pi$, accompanied by spontaneous breaking of the $SU(2)_V$ symmetry. This is similar to the picture that we obtain for $l_7 > 0$ and $m_\pi(\pi) < \mu_I < m_\pi(0)$, except that only the $U(1)_I$ subgroup of $SU(2)_V$ is broken spontaneously (the other generators of $SU(2)_V$ are explicitly broken by finite $\mu_I$).

Finally, we would like to comment regarding the triple point $\theta = \pi, |\mu_I| = m_\pi(\pi)$ that appears in the phase diagram of Fig. 2. This triple point, the set of classical minima of the Lagrangian presents a sphere $S^2$. Such degeneracy is definitely accidental, and we expect that it will be lifted by higher order terms in the chiral Lagrangian (most likely $O(m^3), O(m^2 \mu)$ terms). These higher order terms also have the potential to change the phase diagram in the immediate vicinity of the triple point. However, we believe that once we are outside the window,

$$|\theta - \pi| < \frac{m}{\Lambda_{QCD}}, \hspace{1cm} |\frac{\mu_I}{m_\pi(\pi)} - 1| < \frac{m}{\Lambda_{QCD}}$$ \hspace{1cm} (35)$$

FIG. 2: Phase diagram of $N_f = 2$ QCD for $m_u = m_d$. Solid line indicates a first order phase transition, while dashed lines indicate second order phase transitions. The region near the triple point is subject to further investigation.
all the results described above are valid. Most importantly, Dashen’s phenomenon is present in the normal phase and disappears in the superfluid phase. Whether this disappearance occurs precisely at the triple point (which would be the most simple scenario) or through a more complicated series of phase transitions closely surrounding the triple point is still an open question. In order to answer this question one should classify all the terms in the effective chiral Lagrangian similar to the classic construction [18]. However, near $\theta = \pi$, the dimensional counting rule in such a Lagrangian should be based on the relation $m_{\pi}^2 \sim m_q^2$ which is the basis for the classification scheme presented in [18].$^1$ We did not attempt to analyze the corresponding problem of classification of higher order terms in the effective chiral Lagrangian with $\mu_I \neq 0, \theta \neq 0$ in the present study. As we already stated, outside the region $[\delta]$ the higher order corrections can not change our results.

To conclude the section: we have analyzed the effects of finite $\mu_I$ on the rapid crossover, which in the absence of chemical potential occurs at $\theta = \pi$ for $m_u \neq m_d$ and becomes a phase transition once $m_u = m_d$. In both cases, the crossover (first order phase transition) is replaced by two second order phase transitions as $\mu_I$ increases. We note that if $N_f > 2$ for the Dashen’s phenomenon to happen one does not require precise equality of the light quarks, $m_u = m_d$, rather it is sufficient if the quark masses are close enough $[2, 3]$. Our remark here is as follows: we believe that in the case $N_f > 2$ the pattern of the replacement of the first order phase transition by two second order phase transitions with increasing $\mu_I$ remains the same as described in the present section.

Our last remark: Dashen’s phenomenon as well as the $\theta$ dependence has been studied recently in [19] in a very different approach. The corresponding study had concentrated on the weak coupling regime when Euclidean space time volume $L$ is small in comparison with the Goldstone mass, $L \ll m_{\pi}^{-1}$. We emphasize that the results presented here are valid in the opposite regime $L \gg m_{\pi}^{-1}$ which corresponds to the physically relevant case.

V. GLUON CONDENSATE

Having determined the $\theta$ and $\mu_I$ dependence of different condensates and densities containing the quark degrees of freedom, one can wonder if similar results can be derived for the gluon condensate $\langle G_{\mu\nu}^2 \rangle$, which describes the gluon degrees of freedom. As is known, the gluon condensate represents the vacuum energy of the ground state in the limit $m_q = 0, \mu = 0$ and plays a crucial role in such models as the MIT Bag model, where a phenomenological “bag constant” $B$ describes the non-perturbative vacuum energy of the system. The question we would ideally want to answer: how will the gluon condensate $\langle G_{\mu\nu}^2 \rangle$ (bag constant $B$) depend on $\mu, \theta$ if the system is placed into dense matter? This question is relevant for a number of different studies such as the equation of state in the interior of neutron stars, see e.g. [20], or stability of dense strangelets [21]. Of course, it is difficult to answer this question in QCD at finite baryon density, however, the answer can be easily obtained in QCD with $\mu_I \ll m_\rho$, which is the subject of the present work.

$^1$ This phenomenon, when “naively” higher order corrections in $m_q$ start to play a crucial role has been previously observed in eq. [28] when Ward identities have been analyzed.

$^2$ It is actually possible that for $N_f = 2$ the phase transition also occurs already for a very small, but non-zero $m_u - m_d$.
We work in Minkowski space in this section. We start from the equation for the conformal anomaly,

$$\Theta_\mu^\nu = -\frac{bg^2}{32\pi^2} G^{a\mu}_{\nu\rho} G^{a\rho\nu} + \bar{\psi} M \psi$$

(36)

where we have taken the standard 1 loop expression for the $\beta$ function and $b = \frac{11}{3} N_c - \frac{2}{3} N_f = \frac{29}{3}$, for $N_c = 3$, $N_f = 2$. As usual, a perturbative constant is subtracted in expression (36).

Now, we can use the effective Lagrangian (1) to calculate the change in the trace of the anomaly,\( \langle G^{2}_{\mu\nu} \rangle \) due to a finite isospin chemical potential. The energy density $\epsilon$ and pressure $p$ are obtained from the grand canonical potential $\Omega$,

$$\epsilon = \Omega + \mu_I n_I, \quad p = -\Omega$$

(37)

Therefore, the conformal anomaly implies,

$$\langle \frac{bg^2}{32\pi^2} G^{a\mu}_{\nu\rho} G^{a\rho\nu} \rangle_{\mu,m,\theta} - \langle \frac{bg^2}{32\pi^2} G^{a\mu}_{\nu\rho} G^{a\rho\nu} \rangle_0 = -4 \left( \langle \pi, m, \theta \rangle - \langle \pi \rangle \right) - \mu_I n_I (\mu, m, \theta) + \langle \bar{\psi} M \psi \rangle_{\mu,m,\theta}$$

(38)

Here, the subscript 0 on an expectation value means that it is evaluated at $\mu = m = 0, \theta = 0$. The good news is that we have already calculated all quantities on the right-hand side of eq. (38) - see expressions (15), (19), (25). Thus, in the normal phase we obtain,

$$\langle \frac{bg^2}{32\pi^2} G^{a\mu}_{\nu\rho} G^{a\rho\nu} \rangle_{\mu,m,\theta} - \langle \frac{bg^2}{32\pi^2} G^{a\mu}_{\nu\rho} G^{a\rho\nu} \rangle_0 = -3 m(\theta) \langle \bar{\psi} \psi \rangle_0$$

(39)

When $\theta = 0$, (39) reduces to the standard result [16], which was derived in a different manner. As expected, $\langle G^{2}_{\mu\nu} \rangle$ does not depend on $\mu$ in the normal phase. The superfluid phase is more exciting,

$$\langle \frac{bg^2}{32\pi^2} G^{a\mu}_{\nu\rho} G^{a\rho\nu} \rangle_{\mu,m,\theta} - \langle \frac{bg^2}{32\pi^2} G^{a\mu}_{\nu\rho} G^{a\rho\nu} \rangle_0 = f_\pi^2 \mu_I^2 \left( 1 + \frac{2 m_\pi^2(\theta)}{\mu_I^2} \right).$$

(40)

It is instructive to represent the same formula in a somewhat different way,

$$\langle \frac{bg^2}{32\pi^2} G^{a\mu}_{\nu\rho} G^{a\rho\nu} \rangle_{\mu,m,\theta} - \langle \frac{bg^2}{32\pi^2} G^{a\mu}_{\nu\rho} G^{a\rho\nu} \rangle_{\mu=0,m,\theta} = f_\pi^2 (\mu_I^2 - m_\pi^2(\theta)) \left( 1 - \frac{2 m_\pi^2(\theta)}{\mu_I^2} \right),$$

(41)

which makes contact with the fact that in the normal phase, when $\mu_I \leq m_\pi(\theta)$, the gluon condensate does not vary with $\mu_I$. However, for $\mu_I \geq m_\pi(\theta)$, the dependence of the gluon condensate $\langle G^{2}_{\mu\nu} \rangle$ on $\mu_I$ in the superfluid phase becomes rather interesting. The condensate decreases with $\mu_I$ for $m_\pi < \mu_I < 2^{1/4} m_\pi$ and increases afterwards. The qualitative difference in the behaviour of the gluon condensate for $\mu_I \approx m_\pi$ and for $m_\pi \ll \mu_I \ll m_\rho$ can be explained as follows. Right after the normal to superfluid phase transition occurs, the isospin density $n_I$ is small and our system can be understood as a weakly interacting gas of pions. The pressure of such a gas is negligible compared to the energy density, which comes mostly from pion rest mass. Thus, $\langle \Theta_\mu^\nu \rangle$ increases with $n_I$ and, according to the anomaly equation (36), $\langle G^{2}_{\mu\nu} \rangle$ decreases. A similar decrease in $\langle G^{2}_{\mu\nu} \rangle$ with baryon density is expected to occur in “dilute” nuclear matter (see [22] and review [23]). On the other hand, for $\mu_I > m_\pi$, energy density is approximately equal to pressure, and both are mostly due to self-interactions of the pion condensate. Luckily, the effective chiral Lagrangian (1) gives us control over these self-interactions as long as $\mu_I \ll m_\rho$. Such control is largely
absent in corresponding calculations of $\langle G_{\mu\nu}^2 \rangle$ in nuclear matter. As $\Delta \epsilon \sim \Delta p$, the trace $\langle \Theta_{\mu}^\mu \rangle$ decreases and the gluon condensate increases with isospin density. Such behaviour of $\langle G_{\mu\nu}^2 \rangle$ is quite unusual, as finite quark chemical potentials, on general grounds, are expected to suppress the gluons.

VI. CONCLUSION

The main purpose of this work was to investigate the phase diagram of $N_f = 2$ QCD at finite $\theta$ parameter and isospin density. We have found that the $\theta$ dependence becomes non-analytic: for fixed $\mu_I$ of order of the pion mass, two phase transitions of the second order occur as $\theta$ varies from $2\pi$ to 0. We have also demonstrated the conjecture originally presented in [13]: in the limit of degenerate quark masses, spontaneous $P$ breaking occurs in the normal phase, but is absent in the superfluid phase. For $m_u = m_d$, a first order transition across $\theta = \pi$ is present in the normal phase, but disappears in the superfluid phase by splitting into two second order normal to superfluid transitions. The precise details of the neighborhood of the triple point where such splitting takes place remain to be determined.

There are a few more interesting observations which deserve to be mentioned here:
a) Knowledge of $\theta$ dependence of different condensates allows one to calculate the topological susceptibility and other interesting correlation functions as a function of $\mu$. Corresponding Ward Identities at nonzero $\mu_I$ are satisfied in a quite nontrivial way, and can be tested on the lattice.
b) Physics of gluon degrees of freedom and $\mu_I$ dependence of the gluon condensate can also be tested on the lattice. The behavior of the gluon condensate as a function of $\mu_I$ is very nontrivial, as has been explained in the text. Nevertheless, our prediction is robust in a sense that it is based exclusively on the chiral dynamics and no additional assumptions have been made to derive the corresponding expression. Our formulae might be useful for the lattice simulations when one tries to extrapolate the results to the chiral limit at nonzero $\mu_I$.

Finally, we should emphasize that all results presented above are valid only for very small chemical potentials $\mu_I \ll \Lambda_{QCD}$ when the chiral effective theory is justified. For larger chemical potentials we expect a transition to a deconfined phase at $\mu_I \simeq 5\Lambda_{QCD}$ [24].

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