MIXING TIME AND CUTOFF FOR ONE DIMENSIONAL PARTICLE SYSTEMS

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Abstract. We survey recent results concerning the total-variation mixing time of the simple exclusion process on the segment (symmetric and asymmetric) and a continuum analog, the simple random walk on the simplex with an emphasis on cutoff results. A Markov chain is said to exhibit cutoff if on a certain time scale, the distance to equilibrium drops abruptly from 1 to 0. We also review a couple of techniques used to obtain these results by exposing and commenting some elements of proof.

1. A short introduction to Markov chains

1.1. Definition of a Markov chain. A stochastic process \((X_t)_{t \geq 0}\) indexed by \(\mathbb{R}_+\) with value in a state-space \(\Omega\) is said to be a Markov process if at each time \(t \geq 0\), the distribution the future \((X_{t+u})_{u \geq 0}\), conditioned to that of the past \((X_s)_{s \in [0,t]}\) is only determined by its present state \(X_t\). This is equivalent to say that for every bounded measurable function \(F : \Omega^{\mathbb{R}_+} \to \mathbb{R}\) there exists \(G : \Omega \to \mathbb{R}\) such that

\[
\mathbb{E}\left[F((X_{t+s})_{s \geq 0}) \mid (X_u)_{u \in [0,t]}\right] = G(X_t). \tag{1.1}
\]

The assumption (1.1) can be interpreted as an absence of memory of the process and is called the Markov Property (we refer to [41, Chap. III] for an introduction to Markov processes). Markov chains are Markov processes which are right continuous for the discrete topology on \(\Omega\), meaning that \((X_t)\) always remains some time in its current state before jumping always from it

\[\forall t \geq 0, \quad \inf\{s, X_{t+s} \neq X_t\} > 0 .\]

Remark 1.1. The name Markov chains also (and perhaps more frequently) refers to discrete time Markov processes, that is processes indexed by \(\mathbb{Z}_+\) rather than \(\mathbb{R}_+\), see for instance [39]. Let us mention that the all the continuous time Markov chains mentioned in this paper are equivalent to discrete time Markov chains - in the sense that they can be obtained by composing a discrete time Markov chain with an homogeneous Poisson process on \(\mathbb{R}\) - even when the considered state-space is infinite. In particular they are càdlàg and do not display accumulation of jumps (a phenomenon called explosion see [45, Chapter 4]). We study these processes in continuous time rather than discrete time mostly for practical and aesthetic reasons, but the results remain valid for the discrete time version of the chains (and the adaptation of the proof from one setup to another is straightforward see for instance [23, Appendix B]). While some references we mention, such as [46], mention only the discrete time version of the chains, we always transpose the cited the cited results in the continuous time setup for a better presentation.
1.2. Markov semigroup, generator, invariant measures and reversibility.

The distribution of a Markov chain \((X_t)_{t \geq 0}\) is determined by two inputs:

(A) The distribution of its initial condition \(X_0\), which is a probability distribution on \(\Omega\), which we denote by \(\mu\).

(B) The rules of evolution of the future given the present, that is, the mapping \((\mathbb{R}^+ \to \mathbb{R}) \to (\Omega \to \mathbb{R})\), that associates \(G\) to \(F\) in Equation (1.1). It can be encoded in an operator acting on functions defined on \(\Omega\), the generator of the Markov chain.

Since, in the present paper, we are interested in statements which are valid for every initial distribution \(\mu\), when introducing examples of Markov chains, we are going to specify only their generator.

In finite state-space. Let us start by defining the generator of a Markov chain in the simpler case when the state-space is finite (the reader can find in [31, Chapter 20] a more substantial introduction and proofs of the results mentioned in this section). An important intermediate step is the definition of Markov semigroup \((P_t)_{t \geq 0}\) associated with the Markov chain. It is a sequence of \(\Omega \times \Omega\) matrices such that satisfy the semigroup property \(P_{s+t} = P_s P_t\) (where matrix multiplication is considered) and such that for every \(x, y \in \Omega\) and \(s, t \geq 0\), when \(\mathbb{P}[X_s = x] > 0\), then

\[
\mathbb{P}[X_{t+s} = y \mid X_s = x] = P_t(x, y).
\]  

(1.2)

Note that \((P_t)_{t \geq 0}\) jointly with the initial distribution fully determines the finite dimensional distributions of the process since the iteration of (1.2) yields

\[
\mathbb{P}\left[ X_0 = x_0, X_{t_1} = x_1, X_{t_1+t_2} = x_2, \ldots, X_{\sum_{i=1}^{k} t_i} = x_k \right] = \mathbb{P}[X_0 = x_0] P_{t_1}(x_0, x_1) P_{t_2}(x_1, x_2) \cdots P_{t_k}(t_{k-1}, t_k).
\]  

(1.3)

The semigroup property together with our assumption that \((X_t)\) is càdlàg imply that there exists an \(\Omega \times \Omega\) matrix \(\mathcal{L}\) - the generator of the Markov chain - which is such that for all \(t \geq 0\)

\[
\forall t > 0, \quad P_t = e^{\mathcal{L}t} := \sum_{k=1}^{\infty} \frac{t^k}{k!} \mathcal{L}^k.
\]

Note that when we have for \(x, y \in \Omega, x \neq y\)

\[
\begin{cases}
\mathcal{L}(x, y) = \lim_{t \to 0} \frac{1}{t} P_t(x, y), \\
-\mathcal{L}(x, x) = \lim_{t \to 0} \frac{1}{t} (1 - P_t(x, x)),
\end{cases}
\]

(1.4)

so that \(\mathcal{L}(x, y)\) represent the rate at which our Markov chains jumps from \(x\) to \(y\) while \(-\mathcal{L}(x, x)\) corresponds to the rate at which the chain jumps away from \(x\). In practice when introducing the generator of a Markov chain, we simply write its action (by left multiplication) on \(\mathbb{R}\) valued functions on \(\Omega\). That is

\[
\mathcal{L} f(x) := \sum_{y \in \Omega} \mathcal{L}(x, y) f(y) = \sum_{y \in \Omega \setminus \{x\}} \mathcal{L}(x, y) [f(y) - f(x)].
\]
We focus on the case of irreducible Markov chains, that is, we assume that every state of $\Omega$ can be reached from any other states with a finite number of jumps. Formally, for each $x, y$ there exists $k \geq 1$ and a sequence $x_0, x_1, \ldots, x_k$ with $x_0 = x$ and $x_k = y$ such that
\[
\forall i \in [1, k], \quad L(x_{i-1}, x_i) > 0.
\]
This condition immediately implies that $P_s(x, y) > 0$ for every $x, y \in \Omega$. If $L$ is irreducible, and $P_\mu$ denotes the law of the Markov chain with generator $L$ and initial distribution $\mu$, then there exists a unique probability $\pi$ on $\Omega$ such that $P_\pi(X_t = x) = \pi(x)$ for every $x$. Such a probability is called the invariant distribution of the Markov chain. Considering $\pi$ as a (line) vector on $\Omega$ this is equivalent to either of the two relations below
\[
\begin{cases}
\forall t > 0, \quad \pi P_t = \pi,
\pi L = 0.
\end{cases}
\tag{1.5}
\]
The convergence theorem for irreducible finite state-space Markov chains states (see for instance [31, Theorems 4.9 and 20.1]) that the invariant probability measures $\pi$ is also the limit distribution for $X_t$ when $t \to \infty$. More precisely for any probability $\mu$ on $\Omega$ we have
\[
\lim_{t \to \infty} P_\mu(X_t = x) = \pi(x).
\tag{1.6}
\]
We want to investigate quantitative aspect of this convergence. For the Markov chains in this paper, the stationary measure satisfy the so-called detailed balance condition
\[
\forall x, y \in \Omega, \quad \pi(x)L(x, y) = \pi(y)L(y, x),
\tag{1.7}
\]
where we use the notation
\[
[a, b] := \lfloor a, b \rfloor \cap \mathbb{Z}.
\tag{1.8}
\]
It can be easily checked that (1.7) implies (1.5), but there are irreducible Markov chains for which the stationary probability does not satisfy (1.7). Markov chains for which the stationary measure satisfies (1.7) are called reversible.

When the state-space is a continuum. When our state-space is a continuum the above description of the generator as a matrix cannot be used. In that case the semi-group associated to the Markov chain $(P_t)_{t \geq 0}$ is a sequence of probability kernels such that for every bounded measurable function $f$ on $\Omega$ every $s$ and $t$ we have
\[
\mathbb{E}[f(X_{t+s}) \mid X_s] = P_t f(X_s) \quad \text{with} \quad P_t f(x) := \int_\Omega f(y) P_t(x, dy).
\tag{1.9}
\]
Informally $P_t(x, A)$ is the probability that $X_{s+t} \in A$ knowing that $X_s = x$. In analogy with (1.3), the semi-group $(P_t)_{t \geq 0}$ jointly with the initial distribution determines fully the finite dimensional distributions of $(X_t)_{t \geq 0}$. The generator of the Markov chain $L$ can be defined in analogy with (1.4) by
\[
Lf := \lim_{t \to 0} \frac{P_t f - f}{t}.
\tag{1.10}
\]
\[\text{Strictly speaking, the relation (1.9) does not uniquely defines } (P_t)_{t \geq 0}, \text{ since one can modify } P_t(x, \cdot) \text{ for a set of } x \text{ which is visited with probability zero but this is not a relevant issue for our discussion.}\]
For a general Markov processes, the limit on the r.h.s. in (1.10) may not exist for every bounded measurable $f$, the set of functions for which the limit (1.10) does exists is called the domain of the generator. In this paper however, we are going to consider only Markov chains with uniformly bounded jump rates so we won’t have to worry about this. The conditions for existence and uniqueness of a stationary probability distribution and for a convergence such as (1.6) in continuous state-space are very far from being as nice as in the finite case (see for instance [41, Chapter 3]). In this survey, we consider only chains for which the stationary measure exists and is unique. They also satisfy the continuum counterpart of (1.7), that is to say that the operator $\mathcal{L}$ is self-adjoint in $L_2(\pi)$.

1.3. Total variation distance and mixing time. In order to quantify the convergence to equilibrium (1.6), we need a notion of distance on the set $M_1(\Omega)$ of probability measure on $\Omega$, equipped with a $\sigma$-algebra (which is simply the power set $P(\Omega)$ when $\Omega$ is finite). We consider the total-variation distance, which quantifies how well two variables with different distributions can be coupled. Given $\alpha, \beta \in M_1(\Omega)$, the total-variation distance between $\alpha$ and $\beta$ is defined by

$$\|\alpha - \beta\|_{TV} := \sup_{A \subset \Omega} |\alpha(A) - \beta(A)|,$$

where the supremum is taken over measurable sets. The following equivalent characterizations of the total-variation distance helps to have a better grasp of the notion. It is a sort of $L_1$ distance which measures how well two random variables can be coupled.

**Proposition 1.2.** If $\Omega$ is finite or countable we have

$$\|\alpha - \beta\|_{TV} := \frac{1}{2} \sum_{x \in \Omega} |\alpha(x) - \beta(y)|$$

If $\nu$ is a measure on $\Omega$ such that both $\alpha$ and $\beta$ are absolutely continuous w.r.t. $\nu$ then

$$\|\alpha - \beta\|_{TV} := \frac{1}{2} \int_\Omega \left| \frac{d\alpha}{d\nu} - \frac{d\beta}{d\nu} \right| \nu(dx).$$

We have

$$\|\alpha - \beta\|_{TV} := \min_{X_1 \sim \alpha, X_2 \sim \beta} \mathbb{P}[X_1 = X_2]$$

where the minimum is taken over the set of all probability distribution $\mathbb{P}$ on $\Omega \times \Omega$ which have marginal laws $\alpha$ and $\beta$.

The total-variation distance to equilibrium of the Markov chain with generator $\mathcal{L}$ and stationary measure $\pi$ at time $t$ is given by

$$d(t) := \sup_{\mu \in M_1(\Omega)} \|\mathbb{P}_\mu(X_t \in \cdot) - \pi\|_{TV}.$$

where $\mathbb{P}_\mu$ the law of the Markov chain with generator $\mathcal{L}$ and initial measure $\mu$. A standard coupling argument is sufficient to show that $d(t)$ is non-decreasing as a
function of $t$. Given $\varepsilon \in (0,1)$, the mixing time associated to the threshold $\varepsilon$ or $\varepsilon$-mixing time of the Markov chain $X_t$ is given by

$$T_{\text{mix}}(\varepsilon) := \inf\{t > 0 : d(t) \leq \varepsilon\} = \sup\{t > 0 : d(t) > \varepsilon\}.$$ 

It indicates how long it takes, for a Markov chain starting from an arbitrary initial condition, to get close to its equilibrium measure. Note that when $\Omega$ is finite and the chain is irreducible, (1.6) guarantees that $\lim_{t \to \infty} d(t) = 0$ so that $T_{\text{mix}}(\varepsilon) < \infty$ for all $\varepsilon$. For chains with continuum state, it is relevant to study the mixing time in the form defined above only if there is a unique stationary probability measure $\lim_{t \to \infty} d(t) = 0$.

**Remark 1.3.** In the case when $d(t) \not\to 0$ some relevant variant of the mixing time can be defined by considering a restriction on the initial condition, for instance by restricting $x$ to a compact subset of $\Omega$, see e.g. [5, 13].

1.4. **Organization of the paper.** The main object of this paper is to survey some results and methods concerning the mixing time of some Markovian one-dimensional particle systems (with discrete and continuum state-space). In Section 2 we introduce these processes. In Section 3 we expose some results obtained with co-authors in the past decade, and propose a short survey of related research. In Section 4 we review a couple of pivotal ideas, which first appeared in [46] (in a slightly different form) and show how they can be combined to obtain (non-optimal) upper bound on the mixing time. In Section 5 we discuss the technical refinements that are required to improve these bounds into optimal results.

**Remark 1.4.** In both Section 4 and Section 5, we have made the choice to focus exclusively on upper-bound estimates for the mixing time. For the theorems presented in this survey - and in most instances of mixing time problems - this is generally thought to be the hardest part of the result.

**Some comments on notation.** In the remainder of the paper, we always use the letter $\pi$ (with superscripts and subscripts to underline the dependence in parameters) to denote the equilibrium measure of each of the considered Markov chains, so that the meaning of, say, $\pi_N$ or $\pi_{N,k}$ will depend on the context. When several Markov chains with different initial distribution are considered, we may use a superscript to underline the initial distribution (for instance $(X^x_t)$ denotes a Markov chain starting from the stationary distribution). If the initial distribution is a Dirac mass $\delta_x$ with $x \in \Omega$, we write $X^x_t$ rather than $X^\delta_x_t$.

2. **One dimensional particle systems and interface models**

The Markov chains introduced in this section model the motion of particles in a one dimensional space. In each instance, we do not introduce a single chain but rather a sequence of chains, which are indexed by one or two parameters, which correspond to the size of the system and/or the number of particles. We want to understand the evolution of the mixing time when these parameters diverge to infinity.
2.1. The interchange process on the segment. The symmetric interchange process on the segment. For \( N \geq 2 \) we let \( S_N \) denote the symmetric group, that is, the set of permutations on \( N \) elements. For \( i \neq j \), we let \( \tau_{i,j} \) denote the transposition which exchanges the position of \( i \) and \( j \). We define the (symmetric) interchange process on the segment \([1, N]\) (recall (1.8)) as the Markov chain on \( S_N \) with generator

\[
\mathcal{L}^{(N)} f(\sigma) := \frac{1}{2} \sum_{i=1}^{N-1} [f(\sigma \circ \tau_{i,i+1}) - f(\sigma)].
\]

It takes little efforts to check that the Markov chain described above is irreducible, and that the uniform probability on \( S_N \) satisfies the detailed balanced condition (1.7). A more intuitive description of the process, which we denote by \((\sigma_t)\) can be obtained using Equation (1.4): it jumps away from its current state with rate \((N-1)/2\) (that is to say, the time between consecutive jumps are IID exponential variables of mean \(2/(N-1)\)) and when it jumps, it chooses uniformly among the permutation obtained by composing on the right with a transposition of the form \(\tau_{i,i+1}\) for \(i \in [1, N]\), or in other words, it interchanges the value of two randomly chosen consecutive coordinates.

An alternative description is to say that \(\sigma_t\) is updated with a rate \(N-1\) (which is twice the previous rate). At an update time \(t\), one coordinate \(i \in [1, N-1]\) is chosen uniformly at random, and \(\sigma_t\) is re-sampled by choosing uniformly at random in the set \(\Theta(i, \sigma_{t-})\), where \(\sigma_{t-}\) is used to denote the left limit at \(t\) and

\[
\Theta(i, \sigma) := \{\sigma' \in S_N : \forall j \in [1, N] \setminus\{i, i+1\}, \sigma'(j) = \sigma(j)\} = \{\sigma, \sigma \circ \tau_{i,i+1}\}.
\]

Note that with this description, at each update, the value of \(\sigma_t\) remains unchanged with probability \(1/2\). This second description might seem at first sight less natural than the first one, but turns out to be more convenient to construct monotone couplings, see Section 4.1.

Remark 2.1. The process described above is one of the many examples of random walks on \(S_N\). This family of process has attracted attention, since the origin of the study of mixing times, due to the connection it has with the problem of card shuffling (see [31, Chapter 8] and references therein). The symmetric interchange process, which we have considered here on the segment can be generalized: the study of the mixing properties for the interchange process on an arbitrary graphs has been an active field of research, see for instance [7, 35, 18] and references therein.

The biased interchange process. We consider a variant of the process which induce a bias towards more “ordered” permutations, that is, favors moves which drive the chain “closer” to the identity permutation. The set \(\Theta(i, \sigma)\) is composed of two elements. We let \(\sigma^{(i,+)}\) be the element of \(\Theta(i, \sigma)\) such that \(\sigma^{(i,+)}(i) < \sigma^{(i,+)}(i+1)\) and \(\sigma^{(i,-)}\) denote the element of \(\Theta(i, \sigma)\) such that \(\sigma^{(i,-)}(i) > \sigma^{(i,-)}(i+1)\) (intuitively \(\sigma^{(i,+)}\) is the permutation which is more ordered). Letting \(p \in (1/2, 1)\) \((p = 1/2\) corresponds to the symmetric case considered above, the case \(p \in (0, 1/2)\) is equivalent to \(p \in (1/2, 1)\) after reverting the order of the coordinates) and setting \(q := 1-p\), we define
the generator of the biased interchange process of the segment

\[ \mathcal{L}_N^{(p)} f(\sigma) := \sum_{i=1}^{N-1} \left[ p \left( f(\sigma(i,+)) - f(\sigma) \right) + q \left( f(\sigma(i,-)) - f(\sigma) \right) \right] \]

The introduction of a bias drastically modifies the stationary distribution. We let

\[ D(\sigma) = \sum_{1 \leq i < j \leq N} 1_{\{\sigma(i) > \sigma(j)\}} \]

Setting \( \lambda := p/q (\lambda > 1) \), the probability measure \( \pi_N^{(p)} \) defined by

\[ \pi_N^{(p)}(\sigma) = \frac{\lambda^{-D(\sigma)}}{\sum_{\sigma' \in \mathcal{S}_N} \lambda^{-D(\sigma')}} \]

satisfies the detailed balance condition for \( \mathcal{L}_N^{(p)} \). As \( \lambda > 1 \) the measure \( \pi_N^{(p)} \) concentrates most of its mass in a small neighborhood of the identity (more precisely \( D(\sigma) \) is typically of order \( N \) under \( \pi_N^{(p)} \) while it is of order \( N^2 \) under the uniform measure).

2.2. The exclusion process on the segment. This Markov chain models the evolution of particles diffusing on a segment and subject to exclusion: each site can host at most one particle. Letting \( N \) denote the length of the segment. A particle configuration is encoded by a sequence of 0 and 1 on the segments, ones and zeros indicating respectively the presence/absence of particle on a site. The space of configurations with a fixed number of particle \( k \) is defined by

\[ \Omega_{N,k} := \left\{ \xi, [1, N] \rightarrow \{0, 1\} : \sum_{i=1}^{N} \xi(i) = k \right\} \]

Given \( \xi \in \Omega_{N,k} \) and distinct \( i, j \in [1, N] \), we set \( \xi^{(i,j)} = \xi \circ \tau_{i,j} \) and define the generator of the Symmetric Simple Exclusion Process (or SSEP) to be

\[ \mathcal{L}_{N,k} f(\xi) := \frac{1}{2} \sum_{i=1}^{N-1} \left[ f(\xi \circ \tau_{i,i+1}) - f(\xi) \right] \]

An intuitive way to describe the above Markov chain is to say that each particle particle performs an independent, continuous time nearest neighbor random walk with jump rate \( 1/2 \) to the left and to the right, but that any jump which would result in either, a particle moving out of the segment (that is a jump to the site 0 or \( N + 1 \)) or two particles occupying the same site (that is a jump of a particle to an already occupied site) are canceled (see Figure 1). The uniform probability on \( \Omega_{N,k} \) satisfies the detailed balance condition (1.7).
Given \( p \in (1/2, 1) \) we can also define the \textit{Asymmetric Simple Exclusion Process} (or ASEP) which is a similar process on \( \Omega_{N,k} \), but where the particles perform a random walk with respective jump rates \( p \) and \( q \) to the right and to the left. The corresponding generator is

\[
\mathcal{L}_{N,k}^{(p)} f(\xi) := \sum_{i=1}^{N-1} p 1_{\{ \xi(i) > \xi(i+1) \}} [f(\xi \circ \tau_{i,i+1}) - f(\xi)] + \sum_{i=1}^{N-1} q 1_{\{ \xi(i) < \xi(i+1) \}} [f(\xi \circ \tau_{i,i+1}) - f(\xi)].
\]

Here also the introduction of the bias yields a modification of the stationary probability. The probability which satisfy the detailed balance condition is given by (recall \( \lambda = p/q \))

\[
\pi^{(p)}_{N,k}(\xi) := \frac{\lambda^{-A(\xi)}}{\sum_{\xi' \in \Omega_{N,k}} \lambda^{-A(\xi')}}
\]

where \( A(\xi) := \sum_{i=1}^{N} (N - i)\xi(i) - \frac{k(k-1)}{2} \) denotes the (minimal) number of particle moves that separates \( \xi \) from the configuration \( 1_{[N-k+1,N]} \) with all particles packed to the right of the segment. Note that \( A(\xi) \) is typically of order 1 under \( \pi^{(p)}_{N,k} \) whereas it is of order \( N^2 \) under the uniform measure.

2.3. \textbf{The corner-flip dynamics.} We consider a Markov chains that models the motion of interface which are subject only to local moves. The one-dimensional interface is the graph of a one dimensional nearest neighbor path which belongs to the state space

\[
\Xi_{N,k} := \{ \zeta, [0, N] \to \mathbb{Z} : \zeta(0) = 0, \zeta(N) = N - 2k, \forall i \in [0, N - 1], |\zeta(i+1) - \zeta(i)| = 1 \}. \tag{2.2}
\]

We introduce a Markov chain on \( \Xi_{N,k} \), that only change the coordinates of \( \zeta \) one at a time. Given \( \zeta \in \Xi_{N} \) and \( i \in [1, N - 1] \) we introduce \( \zeta^{(i)} \) to be the element of \( \Omega_{N} \) for which only the coordinate at \( i \) has been changed (see Figure \ref{fig:corner-flip}).

\[
\begin{cases}
\zeta^{(i)}(j) := \zeta(j) & \text{if } j \neq i, \\
\zeta^{(i)}(i) := \zeta(i) - 2 & \text{if } \zeta(i+1) = \zeta(i-1) := \zeta(i) - 1, \\
\zeta^{(i)}(i) := \zeta(i) + 2 & \text{if } \zeta(i+1) = \zeta(i-1) := \zeta(i) + 1, \\
\zeta^{(i)}(i) := \zeta(i) & \text{if } |\zeta(i+1) - \zeta(i-1)| = 2.
\end{cases}
\]

The generator of the symmetric corner flip dynamic is given by

\[
\mathcal{G}_{N,k} := \frac{1}{2} \sum_{i=1}^{N-1} [f(\zeta^{(i)}) - f(\zeta)].
\]

A way to visualized this dynamics is to say that each “corner” displayed by the the graph of the \( \zeta \) is flipped with rate 1/2. The uniform measure on \( \Xi_{N,k} \) satisfies the detailed balance condition \( \text{(1.7)} \).
Figure 1. Graphical representation of the exclusion process and of the corner-flip dynamics with $k = 6$ and $N = 14$. Particles represented by circle, jump to the right with rate $p$ and to the left with rate $q = 1 - p$ ($p = 1/2$ in the symmetric case), jumps are canceled if a particle tries to jump on an already occupied site. After applying the transformation $h$ given in (2.5), we obtain the corner flip dynamics, where each downward pointing corner on our interface is flipped up with rate $p$ and each upward pointing corner is flipped down with rate $q$. The quantity $A(\zeta)$ which is the number of up-flips that needs to be performed to reach the maximal configuration $\wedge$ (represented as a thin solid line of the figure) is equal to 22.

We can also define an asymmetric version of the dynamics which favors flipping the corners in one direction. Given $\zeta \in \Xi_N$ and $i \in [i, i+1]$ we define $\zeta^{(i, \pm)}$ to be respectively the “highest” and “lowest” path in the set $\{\zeta^{(i)}, \zeta\}$ (the set is possibly a singleton, so that we may have $\zeta^{(i, +)} = \zeta^{(i, -)}$)

$$
\begin{cases}
\zeta^{(i, \pm)}(j) := \zeta(j) & \text{if } j \neq i, \\
\zeta^{(i, +)}(i) := \max(\zeta(i), \zeta^{(i)}(i)), \\
\zeta^{(i, -)}(i) := \min(\zeta(i), \zeta^{(i)}(i))
\end{cases}
$$

and define the generator of the asymmetric corner-flip dynamics as

$$
\mathcal{L}^{(p)}_{N,k} := \sum_{i=1}^{N-1} p \left[ f(\zeta^{(i,+)}) - f(\zeta) \right] + q \left[ f(\zeta^{(i,-)}) - f(\zeta) \right].
$$

For the asymmetric corner flip, the reversible measure $\pi^{(p)}_{N,k}$ is defined by

$$
\pi^{(p)}_{N,k}(\zeta) := \frac{\lambda^{-A(\zeta)}}{\sum_{\zeta' \in \Xi_N} \lambda^{-A(\zeta')}}.
$$
where $A(\zeta)$ denote the halved geometric area lying between $\zeta$ and the highest path in $\Omega_{N,k}$ defined by

$$\wedge(i) = \min(i, 2(N - k) - i)$$

that is

$$A(\zeta) := \frac{1}{2} \sum_{i=1}^{N-1} (\wedge(i) - \zeta(i)).$$

2.4. Random walk on the simplex. Let us finally consider a Markov chain for which the state-space is a continuum. We let $X_N$ denote the $N-1$-dimensional simplex, defined by

$$X_N := \{ (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} : 0 \leq x_1 \leq \cdots \leq x_{N-1} \leq N \}.$$

We introduce a dynamic which is a continuum analog of the symmetric exclusion, the coordinates $x_1, \ldots, x_{N-1}$ can be thought as the positions of $N-1$ particles on the segment $[0, N]$. The generator of the dynamics, is given by

$$L_N f(x) = \sum_{i=1}^{N-1} \int_0^1 [f(x^{(u,i)}) - f(x)] du$$

for $f : X_N \to \mathbb{R}$ bounded and measurable, where $x^{(u,i)} \in X_N$ is defined by

$$\begin{cases} x_j^{(u,i)} := x_j & \text{for } j \neq i, \\ x_i^{(u,i)} := ux_{i+1} + (1-u)x_{i-1}, \end{cases}$$

with the convention that $x_0 = 0$ and $x_N = N$. In words, at rate one, each coordinate is re-sampled uniformly in its possible range of value, which is the segment $[x_{i-1}, x_{i+1}]$ (see Figure 2).

**Figure 2.** Graphical representation of the random walk on the simplex ($N = 7$). When the position of a particle is updated ($x_5$ on the picture), it is re-sampled uniformly in the interval delimited by the neighboring particles (that is $[x_4, x_6]$), with the convention that $x_0 = 0$ and $x_N = N$.

The uniform probability on $X_N$, $\pi_N$ defined by

$$\pi_N(dx) := \frac{(N-1)!}{N^{N-1}} 1_{x \in X_N} dx_1 \ldots dx_{N-1}$$

is stationary for $L_N$, and that the generator is self adjoint in $L^2(\pi_N)$.

Let us present an explicit construction of the Markov chain $(X_t^x)_{t \geq 0}$ with initial distribution $\delta_x$ using auxiliary random variables. Such a construction is referred to as a graphical construction and turns out to be very convenient to work with (see for instance Section 4.1 below). It follows the following steps:
(i) To each coordinate $i \in [1, N-1]$, we associate an independent rate-1 Poisson clock process $(T_n(i))_{n \geq 1}$ (the increments of $T(i)$ are IID exponential variables of mean 1) and a sequence of uniform random variables on $[0, 1]$, $(U_n(i))_{n \geq 1}$.

(ii) We set $X^x(0) = x$. The process is càdlàg and $(X^x(t))_{t \geq 0}$ remains constant on the each of the open intervals of the set $(0, \infty) \setminus \{T_n(i), n \geq 1, i \in [1, N-1]\}$.

(iii) At time $t = T_n(i)$, we determine $X^x(t)$ from $X^x(t_-)$ (the left limit at $t$) by setting

$$X_i(t) := U_n(i)X_{i+1}(t_-) + (1 - U_n(i))X_{i-1}(t).$$

The other coordinates are unchanged: $X_j(t) = X_j(t_-)$ for $j \neq i$.

The reader can check that, for any bounded measurable function $f$ and every $x \in \mathbb{X}_N$

$$\lim_{t \to 0} \frac{\mathbb{E}[f(X^x(t))] - f(x)}{t} = L_Nf(x).$$

(2.4)

2.5. Correspondences. The Markov chains presented in Section 2.1, 2.2 and 2.3 are very much related to one another. Let us first describe the correspondence between particle system and discrete interfaces. Let us consider $\zeta : \Omega_{N,k} \to \Xi_{N,k}$ defined by

$$h(\zeta)(x) := \sum_{y=1}^{x} (1 - 2\zeta(x)).$$

(2.5)

It is immediate to check that $h(\zeta) \in \Xi_{N,k}$ for every $\zeta \in \Omega_{N,k}$ and that $h$ is a bijection (we have $h^{-1}(\zeta)(x) = \frac{1+((x-1)-\zeta(x))}{2}$). Furthermore we have $\mathcal{L}_{N,k} \circ h = h \circ \mathcal{L}_{N,k}$ and as a consequence, if $(\eta_t)_{t \geq 0}$ is a Markov chain with generator $\mathcal{L}_{N,k}^{(p)}$, then its image $h(\eta_t)_{t \geq 0}$ is a Markov chain on $\Xi_{N,k}$ with generator $\mathcal{L}_{N,k}^{(p)}$ (this is of course also true in the symmetric case, when $p = 1/2$).

The corner-flip representation of the exclusion process, can be convenient for reasoning since it allows for better visual representation of an order relation which is conserved by the dynamics (see Section 1.1).

Another useful - although not bijective - correspondence is the one between the interchange process and the exclusion process. Given $k \in [1, N-1]$ we define $\zeta^{(k)} : \mathcal{S}_N \to \Omega_{N,k}$ as

$$\zeta^{(k)}(\sigma) = 1_{[N-k+1,N]} \circ \sigma.$$

Since $\mathcal{L}_{N,k}^{(p)} \circ \zeta^{(k)} = \zeta^{(k)} \circ \mathcal{L}_{N,k}^{(p)}$, if $(\sigma_t)_{t \geq 0}$ is a Markov chain on $\mathcal{S}_N$ with generator $\mathcal{L}_{N,k}^{(p)}$, then $(\zeta^{(k)}(\sigma_t))_{t \geq 0}$ is a Markov chain on $\Omega_{N,k}$ with generator $\mathcal{L}_{N,k}^{(p)}$. The whole sequence of projections $(\zeta^{(k)}(\sigma))_{k=1}^{N-1}$ allows to recover $\sigma$ since we have

$$\sigma(i) = N - k + 1 \iff \zeta^{(k)}(i) - \zeta^{(k-1)}(i) = 1.$$ (2.6)

3. Review of mixing time results for one dimensional particle systems

3.1. The cutoff phenomenon. Let us now survey a few results concerning the mixing time of the Markov chains introduced in the previous section. For all of
these processes, an asymptotic equivalent to the mixing time $T^{(N)}_{\text{mix}}(\varepsilon)$ (or $T^{(N,K)}_{\text{mix}}(\varepsilon)$ we have several parameters) is obtained in the limit when the parameter $N$ (or $N$ and $k$) tends to infinity. A striking common feature of all these results is that in the asymptotic equivalent of $T^{(N)}_{\text{mix}}(\varepsilon)$, there is no dependence in $\varepsilon$ ($T^{(N)}_{\text{mix}}(\varepsilon)$ depends on $\varepsilon$ but this dependence only appear in higher order terms). In particular we have for any $\varepsilon > 0$

$$\lim_{N \to \infty} \frac{T^{(N)}_{\text{mix}}(\varepsilon)}{T^{(N)}_{\text{mix}}(1 - \varepsilon)} = 1.$$  

This mean that on a certain time scale, the distance to equilibrum drops abruptly from 1 to 0. This phenomenon is known as cutoff and is believed to hold for a wide class of Markov chain (we refer to [12] and [31, Chapter 18] and references therein for an historical introduction to cutoff). Cutoff is most of the time delicate to prove. For many Markov chains, while a short argument allow to identify the mixing time up to a constant multiplicative factor (cf. Section 4), much more effort is usually needed to obtain asymptotically matching upper and lower bounds.

3.2. Mixing time results. The SSEP and the Interchange Process. We group the results concerning the exclusion process and the interchange process as their proofs share a lot of common ideas. We start with the symmetric case. The lower bounds in the result below have been proved by Wilson in [46] while the upper bounds have been obtained by the author in [25].

**Theorem 3.1.** For any sequence $k_N$ such that $\lim \min(k_N, N - k_N) = \infty$ the mixing time of symmetric exclusion process on $[1, N]$ with $k_N$ particle satisfies, for any $\varepsilon \in (0, 1)$

$$\lim_{N \to \infty} \frac{T^{\text{SSEP},(N,k_N)}_{\text{mix}}(\varepsilon)}{N^2 \log \left[\min(k_N, N - k_N)\right]} = \frac{1}{\pi^2}$$  

(3.1)

For the interchange process on on $[1, N]$ we have for any $\varepsilon \in (0, 1)$

$$\lim_{N \to \infty} \frac{T^{\text{IP},(N)}_{\text{mix}}(\varepsilon)}{N^2 \log N} = \frac{1}{\pi^2}.$$

In view of the correspondence discussed in Section 2.5, the first part of the result, that is (3.1), is also valid for the corner-flip dynamics introduced in Section 2.3.

The random walk on the simplex. The process is in a sense very similar to the the simple exclusion process with a positive density of particle. However the methods developed in [25, 46] - and more generally, many of the techniques concerning upper bounds for mixing time - rely on the fact that the state-space is discrete. The following, proved in [4], is one of the few cutoff results that have been proved for a Markov chain evolving in a continuum (see [19] for another example).

**Theorem 3.2.** For the random walk on the simplex $X_N$ we have for any $\varepsilon \in (0, 1)$

$$\lim_{N \to \infty} \frac{T^{\text{RW},(N)}_{\text{mix}}(\varepsilon)}{N^2 \log N} = \frac{1}{\pi^2}.$$
Upper and lower bounds of the right order - that is $N^2 \log N$ - but without the right constant factor has been proved prior to the above theorem in [38] (see also [37] for similar results in a periodic setting).

The ASEP and the Biased Interchange Process. The introduction of a bias has the effect of making the system mix considerably faster: a time of order $N$ is required instead of $N^{2+o(1)}$ in the symmetric case (this was proved in was proved in [2]). In [22] jointly with C. Labbé, we were able to identify the sharp asymptotic of the mixing time, proving cutoff both for the ASEP and for the biased interchange process.

Theorem 3.3. For any $p \in (1/2, 1]$ and any sequence $(k_N)$ such that

$$\forall N \geq 2, k_N \in [1, N-1] \quad \text{and} \quad \lim_{N \to \infty} \frac{k_N}{N} = \alpha \in [0, 1],$$

the mixing time of asymmetric exclusion process with $k_N$ particle satisfies, for every $\varepsilon \in (0, 1)$

$$\lim_{N \to \infty} \frac{T_{\text{ASEP},(p,N,k_N)}^{\text{mix}}(\varepsilon)}{N} = \frac{(\sqrt{\alpha} + \sqrt{1-\alpha})^2}{2p-1}.$$

For the biased interchange process on segment we have for every $\varepsilon \in (0, 1)$

$$\lim_{N \to \infty} \frac{T_{\text{BIP},(p,N)}^{\text{mix}}(\varepsilon)}{N} = \frac{2}{2p-1}.$$  

Note that the expression for the mixing time in the above result diverges when $p$ tends to $1/2$. In [30, 23] the crossover regime between the symmetric and asymmetric case is investigated. The right order of magnitude for the mixing time is established [30] while [23] proves cutoff results.

3.3. Review of related works. Cutoff window and cut of profile. The results above concern the first order asymptotics of the mixing time. However, one can aim for results with a finer precision. For instance one can try to estimate the order of magnitude of $T_{\text{mix}}^{(N)}(\varepsilon) - T_{\text{mix}}^{(N)}(1-\varepsilon)$ (say for a fixed $\varepsilon \in (0, 1/2)$, this could theoretically depend on the value of $\varepsilon$, but in practice it does not for most chains), a quantity called the width of the cutoff window. One can further refine the picture and look for the limit of the distance to equilibrium $d^{(N)}(t)$ after recentering the picture on $t = T_{\text{mix}}^{(N)}(1/2)$ and rescaling time by the cutoff window width. This is called the cutoff profile, and is the finest degree of description of convergence to equilibrium. For the SSEP on the circle - which is the closest cousin for the exclusion SSEP on the segment - the cutoff window - of order $N^2$ - and profile have been identified in [24, 26]. In the asymmetric case, the cutoff window - of order $N^{1/3}$ - and profile have been identified in [3] (in the case where the density of particle is positive).

The exclusion process with open boundary condition. We have considered above dynamics where the number of particle is conserved. It is possible to consider the case of open boundaries, where particle can enter and exit the segment on the left and on the right. In that case, the equilibrium and dynamical behavior of the system depends a lot on the value chosen for the exit and entrance rate of the particle at the
left and right boundary. Mixing time results for the exclusion of the segment with a variety of boundary condition are proved in [15], where several open questions and conjectures are also displayed. One of these conjecture is solved in [13], where it shown that in the maximal current phase, for the totally asymmetric exclusion process (TASEP) the mixing time in that case is of order $N^{3/2}$. A similar result is predicted to hold for the asymmetric exclusion process on the circle, and the corresponding lower bound on the mixing time can be deduced from the results in [1].

The exclusion process in a random environment. Another variant of the process has been considered where the bias that each particles feels depends on the site on which it lies. That is to say $p$ varies with $i$. In [42, 29] the case of IID random biases has been considered. This is a multi-particle version of the the classical Random Walk in a Random Environment (RWRE) (see e.g. [44, 20] for seminal references and [14] for a study of mixing time of RWRE). The works [29, 42] show that the presence of inhomogeneities in the environment can slow down the convergence to equilibrium.

The exclusion process in higher dimension. The symmetric exclusion process on a higher dimensional rectangle or torus has also been investigated. Proving result beyond dimension one turns out to be more difficult since monotonicity (in the sense of Section 4.1), which is a tool of crucial importance, cannot be used. It has been shown in [34] that the exclusion process in that case continues with a mixing time of order $N^2 \log k$ (see also [36, 48] for earlier functional inequalities which implies that the mixing time is of order $N^2 \log N$ when there is a density of particles).

More general interfaces The mixing of one dimensional interfaces have been studied very much beyond the case of the corner-flip dynamics. In [10, 6, 8, 27, 28, 47], the case of interfaces interacting with a substrate has been considered. The references [11, 8, 16, 9] investigate the mixing time of higher dimensional interfaces. In [3], interfaces with real valued height functions are considered beyond the case of the random walk on the simplex. Let us finally mention [13] which proves cutoff for Gaussian interfaces (the lattice free field) in arbitrary dimension.

4. Presentation of a few technical tools used to prove these results

We review of few key ingredients used in the proof of the results presented in the previous section. More precisely, to illustrate these techniques, we present a proof of non-optimal results concerning the mixing time of the simple exclusion process on the segment (symmetric and asymmetric), or rather, to its corner-flip representation. Although the presentation slightly differs, the argument found below is in spirit very similar to the one found in [46, Section 3]. The reasoning can be applied without much change to the interchange process (see Remark 4.7) but for clarity and conciseness we limit the exposition of details to the case of the exclusion process. We discuss in Section 5 how additional ideas are needed to improve on this non-optimal result.
4.1. **Order preservation.** Let \( \leq \) be a partial order relation on \( \Omega \). Given \( \alpha, \beta \in M_1(\Omega) \) we say \( \alpha \) is stochastically dominated by \( \beta \) (for the order \( \leq \)) and write \( \alpha \preceq \beta \) if one can construct - on the same probability space - a pair of \( \Omega \)-valued variables \( Z_\alpha \) and \( Z_\beta \) with respective distributions \( \alpha \) and \( \beta \) such that such that we have \( Z_\alpha \leq Z_\beta \) with probability one. The Markov chain with generator \( \mathcal{L} \) is said to be **order preserving** or **attractive** if its semigroup preserves stochastic ordering, that is to say that for any \( t > 0 \)

\[
\alpha \preceq \beta \Rightarrow \alpha P_t \preceq \beta P_t.
\]

An equivalent way of saying this is to say that the dynamic is order preserving if if for any \( x, y \in \Omega \) such that \( x \preceq y \), one can couple two Markov chains \((X^x_t)_{t \geq 0}\) and \((X^y_t)_{t \geq 0}\) with respective initial condition \( x \) and \( y \) in such a way that

\[
\forall t \geq 0, \quad X^x_t \leq X^y_t.
\]

**Order preservation for the corner-flip dynamics.**

We define \( \leq \) on \( \Xi_{N,k} \) to simply be the coordinate-wise order, that is

\[
\zeta \leq \zeta' \iff \forall i \in [1, 2N-1], \quad \zeta(i) \leq \zeta'(i).
\]

To show that the corner-flip dynamic on \( \Xi_{N,k} \) is order-preserving, we use a construction which is similar to that presented above Equation (2.4), using clock processes \((\mathcal{T}_n^{(i)})_{i \in [1, N-1], n \geq 0}\) (independent Poisson processes with mean one inter-arrival law) and accessory variables \((U_n^{(i)})_{i \in [1, N-1], n \geq 0}\) which are IDD uniform variable on the interval \([0, 1]\). The clock processes \((\mathcal{T}^{(i)})_{n \geq 0}\) determines when the update of the coordinate \( i \) is performed, and the variables \(U_n^{(i)}\) are used to determine whether the corner should be flipped up or down. Given \( \zeta \in \Xi_{N,k} \) we construct \((h_t^\zeta)\) as the unique cad-lag process which satisfy:

1. \( h_0^\zeta = \zeta \),
2. \((h_t^\zeta)_{t \geq 0}\) remains constant on intervals of \( \mathbb{R}^+ \setminus (\mathcal{T}_n^{(i)})_{i \in [1, N-1], n \geq 0} \).
3. If \( t = \mathcal{T}_n^{(i)} \) and \( h_{t-}^\zeta = \xi \) then
   - (A) Then if \( U_n^{(i)} \geq [1-p, 1) \) set \( h_t = \xi^{(i,+)} \)
   - (B) Then if \( U_n^{(i)} \geq [0, 1-p) \) set \( h_t = \xi^{(i,-)} \).

Since the set \( \{\mathcal{T}_n^{(i)}\}_{i \in [1, N-1], n \geq 0} \) display no accumulation point, \((h_t^\zeta)\) can be constructed by performing the updates sequentially. We can use this construction (using the same \( \mathcal{T} \) and \( U \)) to obtain a collection of processes \((h_t^\zeta)\), \( \zeta \in \Xi_{N,k} \) constructed on the same probability which is such that

\[
\zeta \leq \zeta' \implies \forall t \geq 0, \quad h_t^\zeta \leq h_t^{\zeta'}.
\]

The validity of (4.2) follows from the fact that each update is order preserving, which holds true because for any fixed \( i \) the applications \( \zeta \mapsto \zeta^{(i,\pm)} \) are order preserving.

A coupling such as the one presented above, where chains starting from all initial condition are constructed on a common probability space, is called a **grand coupling**. This type of construction using auxiliary variable is called the **graphical construction** and is quite common for interacting particle systems or spin systems (see for instance [32, Chap III.6]).
Remark 4.1. For the interchange process, we can use the order which corresponds to (4.1) after applying the correspondences of Section 2.5 and a similar construction allows to obtain a monotone grand coupling. The analog construction also provide a monotone grand coupling for the random walk on the simplex.

4.2. **Connection with the discrete heat equation.** Let us expose first how the evolution of the mean of simple observables - the height function in the symmetric case, the exponential of the height in the asymmetric case - can be described by a simple system of linear equation.

**In the symmetric case.** Given $\zeta \in \Xi_{N,k}$ we define $u^{\zeta}(t, \cdot)$, to be the recentered mean height of the interface at time $t$ for the corner-flip dynamic with initial condition $\zeta$:

$$u^{\zeta}(t, i) := \mathbb{E}\left[h^{\zeta}_t(i)\right]$$

(4.3)

For a real valued function $f$ defined on $[1, N-1]$, we define $\Delta_D f$ ($\Delta_D$ being the discrete Laplace operator with Dirichlet boundary condition) by

$$\Delta_D f(i) := f(i+1) + f(i-1) - 2f(i) \quad \text{for } i \in [1, N-1].$$

(4.4)

with the convention that $f(0) = 0$ and $f(N) = N-2k$. The function $u^{\zeta}$ is the unique solution of the following system of differential equations, that can be considered as a partial differential equation where the space variable is discrete ($\Delta_D$ acts on the second variable)

$$\partial_t u(t, i) = \frac{1}{2} \Delta_D u(t, i), \quad \forall i \in [1, N-1].$$

(4.5)

Setting $U^{(i)}(\zeta) := \zeta(i)$, Equation (4.5) is deduced from the identity (that can be checked from the definition of the generator)

$$\mathcal{L}_{N,k} U^{(i)}(\zeta, i) = \frac{1}{2} \Delta_D \zeta(i).$$

More precisely (4.5) is obtained by combining (1.10), the Markov property, the identity , and the fact that, $\Delta_D$ being an affine transformation, it commutes with the expectation, as follows

$$\partial_t u(t, i) = \mathbb{E}\left[\mathcal{L}_{N,k} U^{(i)}(h^{\zeta}_t)\right] = \mathbb{E}\left[\Delta_D h^{\zeta}_t(i)\right] = \Delta_D \left(\mathbb{E}\left[h^{\zeta}_t\right]\right)(i) = \Delta_D u(t, i).$$

(4.6)

The fact that $u^{\zeta}$ does not have zero boundary condition is not a problem since in computations, we consider the difference $u^{\zeta} - \bar{u}^{\zeta}$ which displays zero boundary condition. The Dirichlet Laplacian with zero boundary condition $\Delta_D^{(0)}$ is a linear operator that can easily be diagonalized. The family $(\sin(j))_{j=1}^{N-1}$ defined by $\sin(j)(i) := \sin\left(\frac{ij\pi}{N}\right)$ forms a base of eigenvectors of $\Delta_D^{(0)}$ in $\mathbb{R}^N$ and we have

$$\Delta_D^{(0)} \sin(j) = -2\gamma_N^{(j)} \sin(j) \quad \text{where} \quad \gamma_N^{(j)} = 1 - \cos\left(\frac{ij\pi}{N}\right).$$

(4.7)

Using Parceval’s inequality we obtain the following contractive estimates which we use to bound the mixing time.
Lemma 4.2. If \( u : [0, \infty) \times [1, N - 1] \) satisfies \( \partial_t u = \Delta_D^{(0)} u \), then we have for any \( t \geq 0 \)
\[
\sum_{i=1}^{N-1} u(t, i)^2 \leq e^{-2\gamma_i(N) t} \sum_{i=1}^{N-1} u(0, i)^2.
\]

In the asymmetric case. When \( p \neq 1/2 \), the quantity \( \mathcal{L}_{N,k}^{(p)} U^{(j)}(\zeta) \) cannot be expressed as a linear combination of \( U^{(j)}(\zeta), j \in [1, N] \) so that there is no way to recover a linear system analogous to (4.5) for the averaged heights.

However we can obtain something similar for the evolution of an averaged quantity related to the heights. The key idea which can be traced by to [17] (where it is used to derive hydrodynamic limits) is to apply the so-called discrete Cole-Hopf transform. We consider exponentials of heights rather than heights themselves. Recalling that \( \lambda = p/q \), we define
\[
V(\zeta, i) := \lambda \hat{\zeta}(i) \quad \text{and} \quad v^\zeta(t, i) := \mathbb{E}^{(p)} \left[ V(h^\zeta(i)) \right].
\]

Setting \( \varrho := (\sqrt{p} - \sqrt{q})^2 \), it can be checked from the definition of the generator that for every \( \zeta \) and \( i \in [1, N - 1] \) we have
\[
\mathcal{L}_{N,k}^{(p)} V(\zeta, i) = \sqrt{pq} \Delta_D V(\zeta, i) - \varrho V(\zeta, i) \tag{4.8}
\]
where this time \( \Delta_D \) denotes the Dirichlet Laplacian defined as in (4.4) but with boundary condition \( f(0) = 1 \) and \( f(N) = \lambda \hat{\zeta}^{-k} \) (we refer to [22, Section 3.3] for details on the computation leading to (4.8)). In (4.8) note that \( \mathcal{L}_{N,k}^{(p)} \) acts on the first coordinate while \( \Delta_D \) acts on the second one. As in (4.6), we obtain from (4.8) that \( v^\zeta \) satisfies
\[
\partial_t v(t, i) := (\sqrt{pq} \Delta_D - \varrho) v(t, i), \quad \forall i \in [1, N - 1], \tag{4.9}
\]
Again, the non-zero boundary condition for \( \Delta_D \) here is of no importance since in practice we are going to consider the difference \( v^\zeta - v^{\zeta'} \). As in the symmetric case the diagonalization of the operator with 0 boundary condition \( \Delta_D^{(0)} \) yields the following estimate.

Lemma 4.3. If \( v \) satisfies \( \partial_t v = \sqrt{pq} \Delta_D^{(0)} v - \varrho v \) then we have
\[
\sum_{i=1}^{N-1} v(t, i)^2 \leq e^{-2(\gamma_i(N) + \varrho) t} \sum_{i=1}^{N-1} v(0, i)^2.
\]

4.3. Using the heat equations to obtain bounds on the mixing time. Let \( (h_t^{(1)}) \) and \( (h_t^{(2)}) \) be two ordered corner flip dynamics, that is, such that \( h_t^{(1)} \leq h_t^{(2)} \) for all \( t \). Using only Lemmas 4.2, 4.3 and order preservation we can control the coupling time of \( (h_t^{(1)}) \) and \( (h_t^{(2)}) \) defined by
\[
\tau := \inf \{ t > 0 : h_t^{(1)} = h_t^{(2)} \}. \tag{4.10}
\]
Proposition 4.4. If \( (h_1^{(1)}) \) and \( (h_1^{(2)}) \) be two ordered symmetric corner flip dynamics then for any \( t > 0 \) we have

\[
P[\tau > t] \leq k(N - 1)e^{-\gamma_1^{(N)} t}.
\]

If \( (h_1^{(1)}) \) and \( (h_1^{(2)}) \) be two ordered asymmetric corner flip dynamics with parameter \( p \) we have

\[
P[\tau > t] \leq k(N - 1)\lambda^{N/2 - 1} e^{-\rho t}.
\]

From these coupling estimates we can derive upper estimates on the mixing time

Corollary 4.5. We have

\[
T_{\text{ssep},(N,k_N)} \leq \frac{1}{\gamma_1^{(N)}} \log \left( \frac{2k(N - 1)}{\varepsilon} \right),
\]

\[
T_{\text{asep},(p_N,k_N)} \leq \frac{1}{\rho} \left( \frac{N}{2} - 1 \right) \log \lambda + \log \left( \frac{2k(N - 1)}{\varepsilon} \right)
\]

Remark 4.6. Replacing \( \gamma_1^{(N)} \) by an asymptotic equivalent \( \frac{\pi^2}{2N^2} \) we find that the upper bound on the SSEP mixing time is \( \frac{2N^2}{\pi^2} (\log N + \log k) (1 + o(1)) \) which is in the best case, a factor 4 always from the estimate given in Theorem 3.1. For the ASEP, our upper bound is asymptotically equivalent to \( \frac{\log \lambda}{2\rho} N \). Since we have for every \( p \in (1/2, 1) \)

\[
\frac{\log \lambda}{2\rho} > \frac{2}{2p - 1} = \max_{\alpha \in [0,1]} \frac{(\sqrt{\alpha} + \sqrt{1 - \alpha})^2}{2p - 1},
\]

in this case again the estimate is not sharp. The reason why the bounds in Corollary 4.5 are not sharp further discussed in Section 5.

Proof of Corollary 4.5. Using the correspondence of Section 2.5 we can reason with the corner flip dynamics since it has the same mixing time. In order to prove an upper bound on the mixing time, one must bound from above the distance between \( \pi \) and the stationary measure \( \pi \) for an arbitrary \( \zeta \in \Xi_{N,k} \). In order to transform this into a coupling problem, note that \( \pi = \mathbb{P}[h_t^\zeta \in \cdot] \) where, with some abuse of notation, we let \( h_t^\pi \) denote a Markov chain with initial condition \( \pi \).

Let us consider now three different dynamics, \( h_t^\zeta, h_t^\land \) and \( h_t^\rceil \), with respective initial condition \( \zeta, \land \) (defined in Section 3.1) and stationary. They are constructed on the same probability space and coupled in such a way that for all \( t \geq 0 \) (Section 4.1 gives such a coupling)

\[
h_t^\rceil \leq h_t^\land \quad \text{and} \quad h_t^\zeta \leq h_t^\rceil.
\]

Using (1.2) stationarity and union bound, we have

\[
\|\mathbb{P}[h_t^\zeta \in \cdot] - \pi\|_{TV} \leq \mathbb{P}[h_t^\zeta \neq h_t^\rceil] \leq \mathbb{P}[h_t^\zeta \neq h_t^\land] + \mathbb{P}[h_t^\rceil \neq h_t^\rceil] = \mathbb{P}[\tau_1 > t] + \mathbb{P}[\tau_2 > t],
\]

where we have set

\[
\tau_1 := \inf\{t : h_t^\zeta \neq h_t^\land\} \quad \text{and} \quad \tau_2 := \inf\{t : h_t^\rceil \neq h_t^\rceil\}.
\]
The tail distribution of \( \tau_1 \) and \( \tau_2 \) can be estimated using Proposition 4.4 and we obtain (let us now for the first time highlight the difference in \( p \))

\[
\begin{aligned}
\|P[h_t^e \in \cdot] - \pi\|_{TV} &\leq 2(N-1)k\epsilon^{-1(N_t)\lambda} \quad \text{in the symmetric case,} \\
\|P[h_t^e \in \cdot] - \pi\|_{TV} &\leq 2(N-1)k\lambda^{N/2-1}e^{-\rho t} \quad \text{in the asymmetric case.}
\end{aligned}
\]  

(4.12)

The reader can then check that the value of \( t \) which makes the r.h.s. in (4.12) equal to \( \epsilon \) is the claimed upper bound on the mixing time. \( \square \)

**Proof of Proposition 4.4** Let us start with the symmetric case. We set

\[
h_{t}^{(1,2)}(i) := h_{t}^{(2)}(i) - h_{t}^{(1)}(i) \quad \text{and} \quad u^{(1,2)}(t, i) := \mathbb{E} \left[ (h_{t}^{(2)} - h_{t}^{(1)})(i) \right].
\]

Since \( h_{t}^{(1)} \leq h_{t}^{(2)} \), we have \( h_{t}^{(1,2)}(i) \geq 0 \) for all \( i \), and if \( h_{t}^{(1)} \neq h_{t}^{(2)} \) the inequality must be strict for at least one value of \( i \). Since the minimal discrepancy between two values of \( \zeta(i) \) is 2, this implies that

\[
P[\tau > t] = P \left[ h_{t}^{(1)} \neq h_{t}^{(2)} \right] = P \left[ \sum_{i=1}^{N-1} (h_{t}^{(1,2)}(i) \geq 2 \right] \leq \frac{1}{2} \sum_{i=1}^{N-1} u^{(1,2)}(t, i). \quad (4.13)
\]

Combining Cauchy-Schwarz with Lemma Lemma 4.2, from (4.5) we know that \( u^{(1,2)} \) satisfies the assumption - we have

\[
\sum_{i=1}^{N-1} u^{(1,2)}(t, i) \leq \left( (N-1) \sum_{i=1}^{N-1} u^{(1,2)}(t, i)^2 \right)^{1/2} \leq e^{-\gamma N t} \left( (N-1) \sum_{i=1}^{N-1} u^{(1,2)}(0, i)^2 \right)^{1/2}
\]

and we can conclude using the fact \( u^{(1,2)}(0, i) \leq 2k \) since \( 2k \) is a bound for the maximal height difference between two elements in \( \Xi_{N, k} \).

For the asymmetric case we apply the reasoning to the exponential of the heights

\[
W(\zeta) := \sum_{i=1}^{N-1} V(\zeta, i) \quad \text{and} \quad W_{t}^{(1,2)} := W(h_{t}^{(2)}) - W(h_{t}^{(1)}) \quad (4.14)
\]

Note that since \( \zeta(i) \geq -k \) for all \( \zeta \) and \( i \), the minimal positive value of \( W_{t}^{(1,2)} \) is given by

\[
\delta_{\min} := \min_{\zeta' \geq \zeta, \zeta' \neq \zeta} W(\zeta') - W(\zeta) = (\lambda - 1)\lambda^{-k/2}.
\]

Repeating the reasoning in (4.13) in the asymmetric case, we obtain that

\[
P[\tau > t] \leq \mathbb{E} \left[ W_{t}^{(1,2)} \right] \delta_{\min}
\]

Now from (4.9), \( v^{(1,2)}(t, i) := \mathbb{E} \left[ V(h_{t}^{(2)}, i) - V(h_{t}^{(1)}, i) \right] \) satisfies the assumption Lemma 4.3 We obtain, using Cauchy-Schwarz inequality, that

\[
\mathbb{E} \left[ W_{t}^{(1,2)} \right] \leq \sum_{i=1}^{N-1} v^{(1,2)}(t, i) \leq e^{-\rho t} \left( (N-1) \sum_{i=1}^{N-1} v^{(1,2)}(0, i)^2 \right)^{1/2}.
\]
Now considering that the maximal possible height-difference is $2k$ and that the maximal possible value of $\zeta(i)$ is always smaller than $N - k$ we have for every $i \in \llbracket 1, N - 1 \rrbracket$

$$v^{(1,2)}(0, i) \leq \max_{\zeta' \in \Xi_{N,k}} \lambda^\zeta'(i)(1 - \lambda^{-k}) \leq k(\lambda - 1)\lambda^{N-2k-1}.$$  

Setting $\delta_{\text{max}} := (\lambda - 1)\lambda^{N-2k-1}$, we obtain that $\sum_{i=1}^{N-1} v^{(1,2)}(0, i)^2 \leq \delta_{\text{max}}^2 (N - 1)k^2$ so that

$$\mathbb{P}[\tau > t] \leq \frac{\delta_{\text{max}}^2}{\delta_{\text{min}}^2} (N - 1)k e^{-qt}$$

which is the desired result. \hfill \square

**Remark 4.7.** Note that the argument exposed in the section can also be used without changes for the interchange process. Indeed the correspondences exposed in Section 2.5 allows to associate, to the dynamic $\sigma_t$, $N - 1$ corner-flip dynamics ($h^{(k)}_t$), $k = 1, \ldots N$, defined by

$$h^{(k)}_t = h \circ \xi^{(k)} \circ \sigma_t$$

where the transformations $h$ and $\xi^{(k)}$ are those of Section 2.5. The observation (2.6) guarantees that two dynamics $\sigma^{(1)}_t$ and $\sigma^{(2)}_t$ are coupled when all the corresponding corner-flip dynamics are coupled, so that the analog of Proposition 4.4 is valid for the interchange process on the segment, with the factor $k(N - 1)$ replaced by $(N - 1)^3$. The reader can refer to [46, Section 3] and [22, Section 3.4] for more details in the symmetric and asymmetric cases respectively.

5. **Shortcomings and possible improvements of the reasoning above**

5.1. **For symmetric dynamics.** As mentioned in Remark 4.6, the upper-bound on the SSEP mixing time is suboptimal, off by a factor 4 in the case when $k$ and $N - k$ are of order $N$. There are two separate reasons for which the method does not yield an optimal result, each being accountable for a multiplicative factor 2. To illustrate this, let us mention [46, Section 8], where it is proved that for the monotone coupling inherited from the graphical construction (described in Section 4.1) the coupling time $\tau_1$ in (4.11) is of order $\frac{2}{\pi^2} N^2 \log k$. This results shows that not only the method above is off by a factor two to estimate the coupling time, but also, when comparing it to Theorem 3.1 that this coupling time itself does allow for a sharp estimate on the mixing time. This means that in order to improve the bound on the mixing time, we have to design a monotone coupling that makes the value of the coupling time $\tau$ as small as possible.

This becomes particularly obvious when the Random Walk on the simplex is considered (recall Section 2.4). If one considers the monotone grand coupling based on the graphical construction presented in Section 2.4 then trajectories starting with different initial conditions never coalesce ($\tau = \infty$ almost surely). Hence for this model, there can be no equivalent of Proposition 4.4: Any non trivial estimate on $\tau$ must rely on specific features of the coupling beyond monotonicity.
In \cite{4, 5, 23, 25, 24, 26}, refinements have been performed in order to obtain optimal estimates on the mixing time. This first one is the introduction of a coupling that is aimed at minimizing the coalescence time. The basic idea for the discrete model is to make the corner-flips performed by \( h_t^{(1)} \) and \( h_t^{(2)} \) less synchronized while preserving monotonicity so that the quantity

\[
A(t) := \sum_{i=1}^{N-1} (h_t^{(2)} - h_t^{(1)}),
\]

which is an integer-valued supermartingale, hits zero faster. Roughly speaking, this is achieved by having, at any given time, independent corner flips for coordinates at which \( h_t^{(2)}(x) > h_t^{(1)}(x) \), and synchronized corner flips for coordinates at which \( h_t^{(2)}(x) = h_t^{(1)}(x) \) (the couplings used in the continuous setup in \cite{4, 5} are based on an analogous intuition). The second key improvement is to use diffusion estimates in order to estimate the time when \( A(t) \) hits 0, instead of relying on Markov’s inequality. For the corner-flip dynamics, \( A(t) \) is a time changed random walk on \( \mathbb{Z}_+ \), and the hitting time of 0 can be precisely estimated if one has some control over its jump rate (see \cite{25, 24, 26}). This idea was considerably improved in \cite{4, 5, 23} where we need to estimate the hitting time of zero of a supermartingale which is not integer valued. The improvements comes from reasoning in terms of martingale brackets instead of jump rate.

5.2. For asymmetric dynamics. Remark 4.6 also underlines that the result of the previous section is also suboptimal in the asymmetric case. The reason for this is that the quantity \( W_t^{(2,1)} \) considered in (4.14) is typically much smaller than its average (by a factor which is exponential in \( N \)). Since this quantity has very wild fluctuation, it is not possible to apply to it the same technique as in the symmetric case. The proof of Theorem 3.2 presented in \cite{22} relies on two key ingredients:

(A) Hydrodynamic limits.
(B) The control of particle speed when the density is vanishing.

Hydrodynamic limits are an extensively studied topic for particle systems (see \cite{21}). The hydrodynamic limit of a system is the limit obtained for the evolution of the particle density after rescaling time and space. It usually takes the form of the solution to partial differential equation. In the case of the asymmetric exclusion process, is has been established (see \cite{40} where the result is proved in a much broader context) that the hydrodynamic limit - after rescaling time and space by \( N \) - is the solution of the equation

\[
\partial_t \rho = (2p - 1) \partial_x \left[ \rho(1 - \rho) \right].
\]

More precisely, for the exclusion on the segment, we have to consider some specific notion solution and boundary conditions (see \cite{22}, Section 5 for details). In this context, given any initial condition \( \rho_0 \), which satisfies

\[
\forall x \in [0, 1], \ 0 \leq \rho_0(x) \leq 1 \quad \text{and} \quad \int_{[0,1]} \rho(x) = \alpha
\]
(5.1) has a unique solution which stabilizes to the fixed point $1_{[1-\alpha,1]}$ after a time $\left(\frac{\sqrt{\alpha}+\sqrt{1-\alpha}}{2p-1}\right)^2$, indicating that at time $\left(\frac{\sqrt{\alpha}+\sqrt{1-\alpha}}{2p-1}\right)^2N$ the system is macroscopically at equilibrium.

What remains to check afterwards is that around that time the system is also at equilibrium in the total variation sense, which is a priori a much finer statement. The important point is to verify that the position of the leftmost particle and rightmost empty site match the indication given by the macroscopic profile (that is, are both $(1-\alpha)N+o(N)$), and this is where the point (B) comes into play (we refer to [22, Section 6] for more details).

Once we have proved that both the density of particle and the position of leftmost particle/rightmost empty site have reached there equilibrium, we still have not proved that the system is at equilibrium. However this information implies that with the notation of Section 4.3 when $t = t_{\alpha,N} := \left(\frac{\sqrt{\alpha}+\sqrt{1-\alpha}}{2p-1}\right)^2$ we have $W_{t}^{(1,2)} = \exp(o(N))\delta_{\min}$. Hence we can use, as a third step of our reasoning, the contraction estimate of Lemma 4.3 to show that coupling must occurs shortly after time $t_{\alpha,N}$.

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