Higher algebraic $K$-theory of group actions with finite stabilizers

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Abstract

We prove a decomposition theorem for the equivariant $K$-theory of actions of affine group schemes $G$ of finite type over a field on regular separated noetherian algebraic spaces, under the hypothesis that the actions have finite geometric stabilizers and satisfy a rationality condition together with a technical condition which holds e.g. for $G$ abelian or smooth.

We reduce the problem to the case of a $GL_n$-action and finally to a split torus action.

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1 Introduction

The purpose of this paper is to prove a decomposition theorem for the equivariant $K$-theory of actions of affine group schemes of finite type over a field on a regular separated noetherian algebraic spaces. Let $X$ be a regular connected separated noetherian scheme with an ample line

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bundle, $K_0(X)$ its Grothendieck ring of vector bundles. Then the kernel of the rank morphism $K_0(X) \rightarrow \mathbb{Z}$ is nilpotent ([SGA6], VI, Théorème 6.9), so the ring $K_0(X)$ is indecomposable, and remains such after tensoring with any indecomposable $\mathbb{Z}$-algebra.

The situation is quite different when we consider the equivariant case. Let $G$ be an algebraic group acting on a noetherian separated regular scheme, or algebraic space, $X$ over a field $k$, and consider the Grothendieck ring $K_0(X, G)$ of $G$-equivariant perfect complexes. This is the same as the Grothendieck group of $G$-equivariant coherent sheaves on $X$, and coincides with the Grothendieck ring of $G$-equivariant vector bundles if all $G$-coherent sheaves are quotients of locally free coherent sheaves (which is the case, e.g. when $G$ is finite or smooth and $X$ is a scheme). Assume that the action of $G$ on $X$ is connected, that is, there are no nontrivial invariant open and closed subschemes of $X$. Still, $K_0(X, G)$ will usually decompose, after inverting some primes; for example, if $G$ is a finite group and $X = \text{Spec} \mathbb{C}$, then $K_0(X, G)$ is the ring of complex representations of $G$, which becomes a product of fields after tensoring with $\mathbb{Q}$.

In [Vi1] the second author analyzes the case that the action of $G$ on $X$ has finite reduced geometric stabilizers. Consider the ring of representations $R(G)$, and the kernel $\mathfrak{m}$ of the rank morphism $\text{rk} : K_0(X, G) \rightarrow \mathbb{Z}$. Then $K_0(X, G)$ is an $R(G)$-algebra; he shows that the localization morphism

$$K_0(X, G) \otimes \mathbb{Q} \rightarrow K_0(X, G)_\mathfrak{m}$$

is surjective, and that the kernel of the rank morphism $K_0(X, G)_\mathfrak{m} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ is nilpotent. Furthermore he conjectures that $K_0(X, G) \otimes \mathbb{Q}$ splits as a product of the localization $K_0(X, G)_\mathfrak{m}$ and some other ring, and formulates a conjecture about what the other factor ring should be when $G$ is abelian and the field is algebraically closed of characteristic 0. The proofs of the results in [Vi1] depend on an equivariant Riemann-Roch theorem, whose proof was never published by the author; however, all of the results have been proved and generalized in [EG].

The case that $G$ is a finite group is studied in [Vi2]. Assume that $k$ contains all $n$th roots of 1, where $n$ is the order of the group $G$. Then the author shows that, after inverting the order of $G$, the $K$-theory ring $K_* (X, G)$ of $G$-equivariant vector bundles on $X$ (which is assumed to be a scheme in that paper) is canonically the product of a finite number of rings, expressible in terms of ordinary $K$-theory of appropriate subschemes of fixed points of $X$. Here $K_* (X, G) = \bigoplus_i K_i (X, G)$ is the graded higher $K$-theory ring. The precise formula is as follows.

Let $\sigma$ be a cyclic subgroup of $G$ whose order is prime to the characteristic of $k$; then the subscheme $X^\sigma$ of fixed points of $X$ under the actions of $\sigma$ is also regular. The representation ring $R(\sigma)$ is isomorphic to the ring $\mathbb{Z}[t]/(t^n - 1)$, where $t$ is a generator of the group of characters $\text{hom}(\sigma, k^*)$. We call $R(\sigma)$ the quotient of the ring $R(\sigma)$ by the ideal generated by the element $\Phi_n(t)$, where $\Phi_n$ is the $n$th cyclotomic polynomial; this is independent of $t$. The ring $R(\sigma)$ is isomorphic to the ring of integers in the $n$th cyclotomic field. Call $N_G(\sigma)$ the normalizer of $\sigma$ in $G$; the group $N_G(\sigma)$ acts on the scheme $X^\sigma$, and, by conjugation, on the group $\sigma$. Consider the induced actions of $N_G(\sigma)$ on the $K$-theory ring $K_* (X^\sigma)$, and on the ring $R(\sigma)$.

Choose a set $\mathcal{C}(G)$ of representatives for the conjugacy classes of cyclic subgroups of $G$ whose order is prime to the characteristic of the field. The statement of the main result of [Vi2] is as follows.

**Theorem** There is a canonical ring isomorphism

$$K_* (X, G) \otimes \mathbb{Z} [1/|G|] \simeq \prod_{\sigma \in \mathcal{C}(G)} \left( K_* (X^\sigma) \otimes \tilde{R}(\sigma) \right)^{N_G(\sigma)} \otimes \mathbb{Z} [1/|G|].$$

In the present paper we generalize this decomposition to the case in which $G$ is an algebraic group scheme of finite type over a field $k$, acting on a noetherian regular separated algebraic space $X$ over $k$. Of course, we cannot expect a statement exactly like the one for finite groups,
expressing equivariant $K$-theory simply in terms of ordinary $K$-theory of the fixed point sets. For example, when $X$ is the Stiefel variety of $r$-frames in $n$-space, then the quotient of $X$ by the natural free action of $\text{GL}_r$ is the Grassmannian of $r$-planes in $n$-space, and $K_0(X, \text{GL}_r) = K_0(X/\text{GL}_r) = \mathbb{Z}$.

Let $X$ be a noetherian regular separated algebraic space of over $k$ with an action of an affine group scheme $G$ of finite type over $k$. We consider the Waldhausen $K$-theory group $K_\ast(X, G)$ of complexes of quasicoherent $G$-equivariant sheaves on $X$ with coherent bounded cohomology. This coincides on one hand with the Waldhausen $K$-theory group $K_\ast(X, G)$ of the subcategory of complexes of quasicoherent $G$-equivariant flat sheaves on $X$ with coherent bounded cohomology and hence has a natural ring structure given by the total tensor product and on the other with the Quillen group $K'_\ast(X, G)$ of coherent equivariant sheaves on $X$; furthermore, if every coherent equivariant sheaf on $X$ is the quotient of a locally free equivariant coherent sheaf, it also coincides with the Quillen group $K'^\ast_{naive}(X, G)$ of coherent locally free equivariant sheaves on $X$. These $K$-theories and their relationships are discussed in the Appendix.

Our result is as follows. First we have to see what plays the role of the cyclic subgroups of a finite group. This is easy; the group schemes whose rings of representations are of the form $k[t]/(t^n - 1)$ are not the cyclic groups, in general, but their Cartier duals, that is, the group schemes which are isomorphic to the group scheme $\mu_n$ of $n$th roots of 1 for some $n$. We call these group schemes “dual cyclic”. If $\sigma$ is a dual cyclic group, we can define $R\sigma$ as before. A dual cyclic subgroup $\sigma$ of $G$ is called essential if $X^\sigma \neq \emptyset$.

The correct substitute for the ordinary $K$-theory of the subspaces of invariants is the geometric equivariant $K$-theory $K_\ast(X, G)_{\text{geom}}$, which is defined as follows. Call $N$ the least common multiple of the orders of all the essential dual cyclic subgroups of $G$. Call $S_1$ be the multiplicative subset of the ring $R(G)$ consisting of elements whose virtual rank is a power of $N$; then $K_\ast(X, G)_{\text{geom}}$ is the localization $S_1^{-1}K_\ast(X, G)$. Notice that $K_1(X, G)_{\text{geom}} \otimes \mathbb{Q} = K_1(X, G)_\mathbb{Q}$, with the notation above. Moreover, if every coherent equivariant sheaf on $X$ is the quotient of a locally free equivariant coherent sheaf, by $[\mathbb{E} G]$, we have an isomorphism of rings

$$K_0(X, G)_{\text{geom}} \otimes \mathbb{Q} = A^*_G(X) \otimes \mathbb{Q}$$

where $A^*_G(X)$ denotes the direct sum of $G$-equivariant Chow groups of $X$.

We prove the following. Assume that the action of $G$ on $X$ is connected. Then the kernel of the rank morphism $K_0(X, G)_{\text{geom}} \to \mathbb{Z}[1/N]$ is nilpotent (see Corollary 5.2). This is remarkable; we have made what might look like a small step toward making the equivariant $K$-theory ring indecomposable, and immediately get an indecomposable ring. Indeed, $K_\ast(X, G)_{\text{geom}}$ “feels like” the $K$-theory ring of a scheme; we want to think of $K_\ast(X, G)_{\text{geom}}$ as what the $K$-theory of the quotient $X/G$ should be, if $X/G$ were smooth, after inverting $N$ (see Conjecture 5.8).

Furthermore, consider the centralizer $C_G(\sigma)$ and the normalizer $N_G(\sigma)$ of $\sigma$ inside $G$. The quotient $w_G(\sigma) = N_G(\sigma)/C_G(\sigma)$ is contained inside the group scheme of automorphisms of $\sigma$, which is a discrete group, so it is also a discrete group. It acts on $R(\sigma)$, by conjugation, and it also acts on the equivariant $K$-theory ring $K_\ast(X^\sigma, C_G(\sigma))$, and on the geometric equivariant $K$-theory ring $K_\ast(X^\sigma, C_G(\sigma))_{\text{geom}}$ (see Corollary 2.3).

We say that the action of $G$ on $X$ is sufficiently rational when the following two conditions are satisfied. Let $\overline{k}$ be the algebraic closure of $k$.

1. each essential dual cyclic subgroup $\sigma \subseteq G_{\overline{k}}$ is conjugate by an element of $G(\overline{k})$ to a dual cyclic subgroup of $G$;
2. if two essential dual cyclic subgroups of $G$ are conjugate by an element of $G(\overline{k})$, they are also conjugate by an element of $G(k)$.
Obviously every action over an algebraically closed field is sufficiently rational. Furthermore if $G$ is $\text{GL}_m$, $\text{SL}_m$, $\text{Sp}_m$ or a totally split torus, then any action of $G$ is sufficiently rational over an arbitrary base field (Proposition 2.3). If $G$ is a finite group, then the action is sufficiently rational when $k$ contains all $n$-th roots of 1, where $n$ is the least common multiple of the orders of the cyclic subgroups of $k$ of order prime to the characteristic, whose fixed point subscheme is nonempty. Denote by $\mathcal{C}(G)$ a set of representatives for essential dual cyclic subgroup schemes, under conjugation by elements of the group $G(k)$. Here is the statement of our result.

**Main Theorem.** Let $G$ be an affine group scheme of finite type over a field $k$, acting on a noetherian separated regular algebraic space $X$. Assume the following three conditions.

(a) The action has finite geometric stabilizers.

(b) The action is sufficiently rational.

(c) For any essential cyclic subgroup $\sigma$ of $G$, the quotient $G/C_G(\sigma)$ is smooth.

Then $\mathcal{C}(G)$ is finite, and there is a canonical isomorphism of $R(G)$-algebras

$$K_*(X,G) \otimes \mathbb{Z}[1/N] \cong \prod_{\sigma \in \mathcal{C}(G)} (K_*(X^\sigma, C_G(\sigma)))_{\text{geom}} \otimes \mathbb{R}(\sigma)^{w_G(\sigma)}.$$

Conditions (a) and (b) are clearly necessary for the theorem to hold. We are not sure about (c). It is rather mild, as it is satisfied, for example, when $G$ is smooth (this is automatically true in characteristic 0) or when $G$ is abelian. A weaker version of condition (c) is given in Subsection 5.2.

In case that $G$ is abelian over an algebraically closed field of characteristic 0, the Main Theorem implies Conjecture 3.6 in [Vi1]. When $G$ is a finite group, and the base field contains enough roots of 1, as in the statement of Theorem 1, then the conditions of the Main Theorem are satisfied; since the natural maps $K_*(X^\sigma, C_G(\sigma))_{\text{geom}} \to K_*(X^\sigma)_{C_G(\sigma)}$ become isomorphisms after inverting the order of $G$ (Proposition 5.7) the Main Theorem implies [Vi2] Theorem 1. However, the proof of the main theorem here is completely different from the proof of Theorem 1 in [Vi2].

As B. Toen pointed out to us, a weaker version of our main theorem (with $\mathbb{Q}$-coefficients and assuming $G$ smooth, acting with finite reduced stabilizers) follows from his Théorème 3.15 in [To1]; the étale techniques he uses in proving this result make it impossible to avoid tensoring with $\mathbb{Q}$. See also [To2].

Here is an outline of the paper.

First we define the homomorphism (Section 2.2). Next, in Section 3, we prove the result when $G$ is a totally split torus. Here the basic tool is the result of Thomason, which gives a generic description of the action of a torus on a noetherian separated algebraic space, and we prove the result by noetherian induction, using the localization sequence for the $K$-theory of equivariant coherent sheaves. Like in [Vi2], the difficulty here is that the homomorphism is defined via pullbacks, and thus it does not commute with the pushforwards intervening in the localization sequence. This is solved by producing a different isomorphism between the two groups in questions, using pushforwards instead of pullbacks, and then relating this to our map, via the self-intersection formula.

The next step is to prove the result in case $G = \text{GL}_n$; here the key point is a result of Merkurjev which links the equivariant $K$-theory of a scheme with a $\text{GL}_n$-action to the equivariant $K$-theory of the action of a maximal torus. This is carried out in Section 4.

Finally (Section 5) we reduce the general result to the case of $\text{GL}_n$, by considering an embedding $G \subseteq \text{GL}_n$, and the induced action of $\text{GL}_n$ on $Y = \text{GL}_n \times^G X$. It is at this point...
that condition (c) enters, allowing a clear description of \( Y^\sigma \) where \( \sigma \) is an essential dual cyclic subgroup of \( G \) (Proposition 5.6).

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2 General constructions

Notation. If \( S \) is a separated noetherian scheme, \( X \) is a noetherian separated \( S \)-algebraic space (which will be most of the time assumed to be regular) and \( G \) is a flat affine group scheme over \( S \) acting on \( X \), we denote by \( K_*(X,G) \) (respectively, \( K'_*(X,G) \)) the Waldhausen \( K \)-theory of the complicial biWaldhausen (\( \mathbb{B}^n \)-Tr) category \( \mathcal{W}_{1,X,G} \) of complexes of quasicoherent \( G \)-equivariant \( \mathcal{O}_X \)-Modules with bounded coherent cohomology (respectively, the Quillen \( K \)-theory of \( G \)-equivariant coherent \( \mathcal{O}_X \)-Modules).

As shown in the Appendix, if \( X \) is regular \( K_*^r(X,G) \) is isomorphic to \( K'_*(X,G) \) and has a canonical graded ring structure. When \( X \) is regular, the isomorphism \( K_*^r(X,G) \cong K'_*(X,G) \) will then allow us to switch between the two theories when needed.

2.1 Morphisms of actions and induced maps on \( K \)-theory

Let \( S \) be a scheme. By an action over \( S \) we mean a triple \((X,G,\rho)\) where \( X \) is an \( S \)-algebraic space, \( G \) is a group scheme over \( S \) and \( \rho : G \times_S X \to X \) is an action of \( G \) on \( X \) over \( S \). A morphism of actions \((f,\phi) : (X,G,\rho) \to (X',G',\rho')\) is a pair of \( S \)-morphisms \( f : X \to X' \) and \( \phi : G \to G' \), where \( \phi \) is a morphism of \( S \)-group schemes, such that the following diagram commutes

\[
\begin{array}{ccc}
G \times_S X & \xrightarrow{\rho} & X \\
\phi \times f \downarrow & & \downarrow f \\
G' \times_S X' & \xrightarrow{\rho'} & X'
\end{array}
\]

Equivalently, \( f \) is required to be \( G \)-equivariant with respect to the given \( G \)-action on \( X \) and the \( G \)-action on \( X' \) induced by composition with \( \phi \).

A morphism of actions \((f,\phi) : (X,G,\rho) \to (X',G',\rho')\) induces an exact functor \((f,\phi)^* : \mathcal{W}_{3,X',G'} \to \mathcal{W}_{3,X,G}\) where \( \mathcal{W}_{3,Y,H} \) denotes the complicial biWaldhausen category of complexes of \( H \)-equivariant flat quasi-coherent Modules with bounded coherent cohomology on the \( H \)-algebraic space \( Y \) (see Appendix). Let \((\mathcal{E}^*,\varepsilon^*)\) be an object of \( \mathcal{W}_{3,X',G'} \), i.e. \( \mathcal{E}^* \) is a complex of \( G' \)-equivariant flat quasi-coherent \( \mathcal{O}_{X'} \)-Modules with bounded coherent cohomology and for any \( i \)

\[
\varepsilon^i : \text{pr}_2^* \mathcal{E}^i \xrightarrow{\sim} \rho'^* \mathcal{E}^i
\]

is an isomorphism satisfying the usual cocycle condition. Here, \( \text{pr}_2 : G' \times_S X' \to X' \) denotes the obvious projection and similarly for \( \text{pr}_2 : G \times_S X \to X \). Since

\[
\rho^* f^* \mathcal{E}^* = (f \rho)^* \mathcal{E}^* = (\phi \times f)^* \rho'^* \mathcal{E}^*
\]

and

\[
\text{pr}_2^* f^* \mathcal{E}^* = (\phi \times f)^* \text{pr}_2^* \mathcal{E}^*
\]
we define \((f, \phi)^* (E^*, \varepsilon^*) \cong (f^* E^*, (\phi \times f)^* (\varepsilon^*)) \in W_{3, X, G}\) (the cocycle condition for each \((\phi \times f)^* (\varepsilon^*)\) following from the same condition for \(\varepsilon^*\)); \((f, \phi)^*\) is then defined on morphisms in the only natural way. \((f, \phi)^*\) is an exact functor and, if \(X\) and \(X'\) are regular so that the Waldhausen \(K\)-theory of \(W_{3, X, G}\) (respectively, of \(W_{3, X', G'}\)) coincides with \(K_* (X, G)\) (resp., \(K_* (X', G')\)) (see Appendix), it defines a ring morphism

\[
(f, \phi)^* : K_* (X', G') \longrightarrow K_* (X, G).
\]

A similar argument shows that if \(f\) is flat, \((f, \phi)\) induces a morphism

\[
(f, \phi)^* : K'_* (X', G') \longrightarrow K'_* (X, G).
\]

**Example 2.1** Let \(G\) and \(H\) be group schemes over \(S\) and \(X\) be an \(S\)-algebraic space. Suppose moreover:

1. \(G\) and \(H\) act on \(X\):

2. \(G\) acts on \(H\) by \(S\)-group schemes automorphisms (i.e., it is given a morphism \(G \rightarrow \text{Aut}_{\text{GrSch}/S} (H)\) of group functors over \(S\)));

3. The two preceding actions are compatible, i.e. for any \(S\)-scheme \(T\), any \(g \in G(T)\), \(h \in H(T)\) and \(x \in X(T)\) we have

\[
g \cdot (h \cdot x) = h^g \cdot (g \cdot x)
\]

where \((g, h) \mapsto h^g\) denotes the action of \(G(T)\) on \(H(T)\).

If \(g \in G(S)\) and \(g_T\) denotes its image via \(G(S) \rightarrow G(T)\), let us define a morphism of actions \((f_g, \phi_g) : (X, H) \rightarrow (X, H)\) as:

\[
f_g (T) : X (T) \longrightarrow X (T) : x \mapsto g_T \cdot x
\]

\[
\phi_g (T) : H (T) \longrightarrow H (T) : h \mapsto h^{g_T}
\]

This is an isomorphism of actions and induces an action of the group \(G(S)\) on \(K'_* (X, H)\) and on \(K_* (X, H)\). This applies, in particular, to the case where \(X\) is an algebraic space with a \(G\) action and \(G \supset H\), \(G\) acting on \(H\) by conjugation.

**2.2 The basic definitions and results**

Let \(G\) be a linear algebraic \(k\)-group scheme \(G\) acting with finite geometric stabilizers on a regular noetherian separated algebraic space \(X\) over \(k\).

We denote by \(R(G)\) the representation ring of \(G\).

A (Cartier) **dual cyclic subgroup** of \(G\) over \(k\) is a \(k\)-subgroup scheme \(\sigma \subseteq G\) such that there exists an \(n > 0\) and an isomorphism of \(k\)-groups \(\sigma \cong \mu_{n, k}\). If \(\sigma, \rho\) are dual cyclic subgroups of \(G\) and \(L\) is an extension of \(k\), we say that \(\sigma\) and \(\rho\) are conjugate over \(L\) if there exists \(g \in G(L)\) such that \(g \sigma(g)^{-1} = \rho(L)\) (where \(H(L) = H \times_{\text{Spec} k} \text{Spec} L\) for any \(k\)-group scheme \(H\) as \(L\)-subgroup schemes of \(G(L)\)).

A dual cyclic subgroup \(\sigma \subseteq G\) is said to be **essential** if \(X^\sigma \neq \emptyset\).

We say that the action of \(G\) on \(X\) is **sufficiently rational** if:

1. any two essential dual cyclic subgroups of \(G\) are conjugated over \(k\) iff they are conjugated over an algebraic closure \(\overline{k}\) of \(k\);
2. any essential dual cyclic subgroup \( \sigma \) of \( G(\overline{k}) \) is conjugated over \( \overline{k} \) to a dual cyclic subgroup of the form \( \sigma(\overline{\tau}) \) where \( \sigma \subseteq G \) is (essential) dual cyclic.

We denote by \( \mathcal{C}(G) \) a set of representatives for essential dual cyclic subgroups of \( G \) with respect to the relation of conjugacy over \( k \).

Remark 2.2 Note that if the action is sufficiently rational and \( \rho \), \( \sigma \) are essential dual cyclic subgroups of \( G \) which are conjugate over an algebraically closed extension \( \Omega \) of \( k \), then they are also conjugate over \( k \).

Proposition 2.3 Any action of \( \text{GL}_n \), \( \text{SL}_n \), \( \text{Sp}_{2n} \) or of a split torus is sufficiently rational.

Proof. If \( G \) is a split torus, Condition 1 is clear, because \( G \) is abelian, while it follows from the rigidity of diagonalizable groups that any subgroup scheme of \( G_\overline{k} \) is in fact defined over \( k \). Let \( \sigma \subseteq \text{GL}_m \) be a dual cyclic subgroup. Since \( \sigma \) is diagonalizable, we have an eigenspace decomposition

\[
V = k^m = \bigoplus_{\chi \in \widehat{\sigma}} V_\chi^\sigma
\]

such that the \( \chi \) with \( V_\chi \neq 0 \) generate \( \widehat{\sigma} \). Conversely, given a cyclic group \( C \) and a decomposition

\[
V = \bigoplus_{\chi \in \widehat{C}} V_\chi
\]

such that the \( \chi \) with \( V_\chi \neq 0 \) generate \( \widehat{C} \), there is a corresponding embedding of the Cartier dual \( \sigma \) of \( C \) into \( \text{GL}_n \) with \( V_\chi = V_\chi^\sigma \) for each \( \chi \in C = \widehat{\sigma} \). Now, if \( \sigma \subseteq \text{GL}_m(\overline{k}) \) is a dual cyclic subgroup defined over \( \overline{k} \), we can apply an element of \( \text{GL}_m(\overline{k}) \) to make the \( V_\chi \) defined over \( k \), and then \( g\sigma g^{-1} \) will be defined over \( k \). If \( \sigma \subseteq \text{GL}_m \) and \( \tau \subseteq \text{GL}_m \) are dual cyclic subgroups which are conjugate over \( \overline{k} \), pick an element of \( \text{GL}_m(\overline{k}) \) sending \( \sigma \) to \( \tau \). This induces an isomorphism \( \phi : \overline{\sigma} \simeq \overline{\tau} \), which by rigidity will be defined over \( k \). Then if \( \chi \) and \( \chi' \) are characters which correspond under the isomorphism of \( \widehat{\sigma} \) and \( \widehat{\tau} \) induced by \( \phi \), then the dimension of \( V_\chi^\sigma \) is equal to the dimension of \( V_{\chi'}^\tau \), so we can find an element \( g \) of \( \text{GL}_m \) which carries each \( V_\chi^\sigma \) onto the corresponding \( V_{\chi'}^\tau \); conjugation by this element carries \( \sigma \) onto \( \tau \). For \( \text{SL}_m \) the proof is very similar, if we remark that to give a dual cyclic subgroup \( \sigma \subseteq \text{SL}_m \subseteq \text{GL}_m \) correspond to giving a decomposition

\[
V = k^m = \bigoplus_{\chi \in \widehat{\sigma}} V_\chi^\sigma
\]

such that the \( \chi \) with \( V_\chi^\sigma \neq 0 \) generate \( \widehat{\sigma} \), with the condition \( \prod_{\chi \in \widehat{\sigma}} \chi^{\dim V_\chi^\sigma} = 1 \in \widehat{\sigma} \). For \( \text{Sp}_m \subseteq \text{GL}_{2m} \), a dual cyclic subgroup \( \sigma \subseteq \text{Sp}_m \) gives a decomposition

\[
V = k^{2m} = \bigoplus_{\chi \in \widehat{\sigma}} V_\chi^\sigma
\]

such that the \( \chi \) with \( V_\chi^\sigma \neq 0 \) generate \( \widehat{\sigma} \), with the condition that for \( v \in V_\chi^\sigma \) and \( v' \in V_{\chi'}^\sigma \) the symplectic product of \( v \) and \( v' \) is always 0, unless \( \chi \chi' = 1 \in \widehat{\sigma} \). Both conditions then follow rather easily from the fact that any two symplectic forms over a vector space are isomorphic.

Let \( N_{(G,X)} \) denote the least common multiple of the orders of essential dual cyclic subgroups of \( G \). Notice that \( N_{(G,X)} \) is finite: since the action has finite stabilizers, the group scheme of
stabilizers is quasifinite over $X$, therefore the orders of the stabilizers of the geometric points of $X$ are globally bounded.

We define $\Lambda_{(G,X)} = \mathbb{Z} \left[ \frac{1}{N_{(G,X)}} \right]$. If $H \subseteq G$ is finite, we also write $\Lambda_{H}$ for $\mathbb{Z} \left[ \frac{1}{|H|} \right]$. Note that, if $\sigma \subseteq G$ is dual cyclic, then $\Lambda_{\sigma} = \Lambda_{(\sigma, \text{Speck})}$ and if moreover $\sigma$ is essential $\Lambda_{\sigma} \subseteq \Lambda_{(G,X)}$.

If $H \subseteq G$ is a subgroup scheme and $A$ is a ring, we write $R(H)_A$ for $R(H) \otimes \mathbb{Z} A$. We denote by $rk_H : R(H) \rightarrow \mathbb{Z}$ and by $rk_{H,\Lambda_{(G,X)}} : R(H)_{\Lambda_{(G,X)}} \rightarrow \Lambda_{(G,X)}$ the rank ring homomorphisms.

We let $K'_* (X,G)_{\Lambda_{(G,X)}} = K'_*(X,G) \otimes \Lambda_{G,X}$ and $K_*(X,G)_{\Lambda_{(G,X)}} = K_*(X,G) \otimes \Lambda_{G,X}$ (for the notation, see the beginning of this Section). Recall that $K_*(X,G)_{\Lambda_{(G,X)}}$ is an $R(G)$-algebra via the pullback $R(G) \simeq K_0(\text{Spec}, G) \rightarrow K_0(X,G)$ and that $K_*(X,G) \simeq K'_*(X,G)$ since $X$ is regular (Appendix).

If $\sigma$ is a dual cyclic subgroup of $G$ of order $n$, the choice of a generator $t$ for the dual group $\hat{\sigma} \doteq Hom_{\text{GrSch}/k} (\sigma, G_{m,k})$ determines an isomorphism

$$R(\sigma) \simeq \frac{\mathbb{Z} [t]}{(tn - 1)}.$$ 

Let $p_{\sigma}$ be the canonical ring surjection

$$\mathbb{Z} [t] \left( \frac{tn - 1}{n} \right) \rightarrow \prod_{d|n} \mathbb{Z} [t] \left( \frac{t}{\Phi_d} \right)$$

and $\tilde{p}_{\sigma}$ the induced surjection

$$\mathbb{Z} [t] \left( \frac{tn - 1}{n} \right) \rightarrow \mathbb{Z} [t] \left( \frac{t}{\Phi_n} \right),$$

where $\Phi_d$ is the $d$-th cyclotomic polynomial. If $m_{\sigma}$ is the kernel of the composition

$$R(\sigma) \simeq \frac{\mathbb{Z} [t]}{(tn - 1)} \rightarrow \mathbb{Z} [t] \left( \frac{t}{\Phi_n} \right),$$

the quotient ring $R(\sigma)/m_{\sigma}$ does not depend on the choice of the generator $t$ for $\hat{\sigma}$.

**Notation.** We denote by $\tilde{R}(\sigma)$ the quotient $R(\sigma)/m_{\sigma}$.

We remark that if $\sigma$ is dual cyclic of order $n$ and $t$ is a generator of $\hat{\sigma}$, there are isomorphisms

$$R(\sigma)_{\Lambda_{\sigma}} \simeq \frac{\Lambda_{\sigma} [t]}{(tn - 1)} \simeq \prod_{d|n} \frac{\Lambda_{\sigma} [t]}{\Phi_d}$$

(1)

Let $\pi_{\sigma} : R(G)_{\Lambda_{(G,X)}} \rightarrow \tilde{R}(\sigma)_{\Lambda_{(G,X)}}$ be the canonical ring homomorphism. The $\sigma$-localization $K'_* (X,G)_{\sigma}$ of $K'_*(X,G)_{\Lambda_{(G,X)}}$ is the localization of the $R(G)_{\Lambda_{(G,X)}}$-module $K'_*(X,G)_{\Lambda_{(G,X)}}$ at the multiplicative subset $S_{\sigma} = \pi_{\sigma}^{-1}(1)$. The $\sigma$-localization $K_*(X,G)_{\sigma}$ is defined in the same way. If $H \subseteq G$ is a subgroup scheme, we also write $R(H)_{\sigma}$ for $S_{\sigma}^{-1} \left( R(H)_{\Lambda_{(G,X)}} \right)$.

If $\sigma$ is the trivial group, we denote by $K'_*(X,G)_{\text{geom}}$ the $\sigma$-localization of $K'_*(X,G)_{\Lambda_{(G,X)}}$ and call it the geometric part or geometric localization of $K'_*(X,G)_{\Lambda_{(G,X)}}$. Note that $\pi_1$ coincides with the rank morphism $rk_{G,\Lambda_{(G,X)}} : R(G)_{\Lambda_{(G,X)}} \rightarrow \Lambda_{(G,X)}$. Same definition for $K_*(X,G)_{\text{geom}}$.

Let $N_G(\sigma)$ (respectively, $C_G(\sigma) \subseteq N_G(\sigma)$) be the normalizer (resp., the centralizer) of $\sigma$ in $G$; since $\text{Aut}(\sigma)$ is a finite constant group scheme,

$$w_G(\sigma) = \frac{N_G(\sigma)}{C_G(\sigma)}$$

is also a constant group scheme over $k$ associated to a finite group $w_G(\sigma)$.

**Lemma 2.4** Let $H$ be a $k$-linear algebraic group, $\sigma \simeq \mu_{n,k}$ a normal subgroup and $Y$ an algebraic space with an action of $H/\sigma$. Then there is a canonical action of $w_H(\sigma)$ on $K'_*(Y, C_H(\sigma))$. 

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Proof. Let us first assume that the natural map
\[ H(k) \rightarrow w_H(\sigma) \]  
(2)
is surjective (which is true, for example, if \( k \) is algebraically closed). Since \( C_H(\sigma)(k) \) acts trivially on \( K'_*(Y, C_H(\sigma)) \) and, by Example 2.1, \( H(k) \) acts naturally on \( K'_*(Y, C_H(\sigma)) \), we may use (2) to define the desired action.

In general (2) is not surjective and we proceed as follows. Suppose we can find a closed immersion of \( k \)-linear algebraic groups \( H \hookrightarrow H' \) such that

(i) \( \sigma \) is normal in \( H' \);
(ii) \( H'/C_{H'}(\sigma) \simeq W_H(\sigma) \);
(iii) \( H'(k) \rightarrow w_H(\sigma) \) is surjective.

Consider the open and closed immersion \( Y \times C_H(\sigma) \hookrightarrow Y \times H \); this induces an open and closed immersion \( Y \times C_H(\sigma) C_{H'}(\sigma) \hookrightarrow Y \times C_H(\sigma) H' \) whose composition with the \( \acute{e} \)tale cover \( Y \times C_H(\sigma) H' \rightarrow Y \times H H' \) is easily checked (e.g. on geometric points) to be an isomorphism. Therefore

\[ K'_*(Y \times H H', C_{H'}(\sigma)) \simeq K'_*(Y \times C_H(\sigma) C_{H'}(\sigma), C_{H'}(\sigma)) \simeq K'_*(Y, C_H(\sigma)) \]

where the last isomorphism is given by Morita equivalence theorem ([8], Proposition 6.2). By (i) and (iii) we can apply the argument at the beginning of the proof and get an action of \( w_H(\sigma) \) on \( K'_*(Y \times H H', C_{H'}(\sigma)) \) and therefore on \( K'_*(Y, C_H(\sigma)) \) as desired. It is not difficult to check that this action does not depend on the chosen immersion \( H \hookrightarrow H' \).

Finally, let us prove that there exists a closed immersion \( H \hookrightarrow H' \) satisfying conditions (i)-(iii) above. First choose a closed immersion \( j : H \hookrightarrow GL_{n,k} \) for some \( n \). Clearly

\[ H/C_H(\sigma) \hookrightarrow GL_{n,k}/C_{GL_{n,k}}(\sigma) \]

and, embedding \( \sigma \) in a maximal torus of \( GL_{n,k} \), it is easy to check that

\[ GL_{n,k}(k) \rightarrow (GL_{n,k}/C_{GL_{n,k}}(\sigma))(k) \]
is surjective. Now define \( H' \) as the inverse image of \( H/C_H(\sigma) \) in the normalizer \( N_{GL_{n,k}}(\sigma) \).

Corollary 2.5 There is a canonical action of \( w_G(\sigma) \) on \( K'_*(X^\sigma, C_G(\sigma)) \) which induces an action on \( K'_*(X^\sigma, C_G(\sigma))_{geom} \).

Proof. Since \( C_G(\sigma) = C_{N_G(\sigma)}(\sigma) \), lemma 2.4 applied to \( Y = X^\sigma \) (respectively, \( Y = \text{Spec} k \)) and \( H = N_G(\sigma) \), yields an action of \( w_G(\sigma) \) on \( K'_*(X^\sigma, C_G(\sigma)) \) (respectively, on \( K_0(\text{Spec} k, C_G(\sigma)) = R(C_G(\sigma)) \)). The multiplicative system \( S_1 = \text{rk}^{-1}(1) \) is preserved by this action so that there is an induced action on the ring \( S_1^{-1}R(C_G(\sigma)) \). The pullback

\[ K_0(\text{Spec} k, C_G(\sigma)) \rightarrow K_0(X^\sigma, C_G(\sigma)) \]
is \( w_G(\sigma) \)-equivariant and then \( w_G(\sigma) \) acts on \( K'_*(X^\sigma, C_G(\sigma))_{geom} \).

Remark 2.6 If \( Y \) is regular, Lemma 2.4 gives also an action of \( w_H(\sigma) \) on \( K_*(Y, C_H(\sigma)) \), since \( K_*(Y, C_H(\sigma)) \simeq K'_*(Y, C_H(\sigma)) \) (Appendix). In particular, since by [8], Prop. 3.1, \( X^\sigma \) is regular, Corollary 2.3 still holds for \( K_*(X^\sigma, C_G(\sigma))_{geom} \).
Note also that the embedding of $k$-group schemes $W_G(\sigma) \hookrightarrow \text{Aut}_k(\sigma)$ induces, by Example 2.1, an action of $w_G(\sigma)$ on $K_0(\text{Spec}(k),\sigma) = R(\sigma)$.

The product in $\sigma$ induces a morphism of $k$-groups

$$\sigma \times C_G(\sigma) \to C_G(\sigma)$$

which in its turn induces a morphism

$$m_\sigma : K_* (X^\sigma, C_G(\sigma)) \to K_* (X^\sigma, \sigma \times C_G(\sigma)).$$

**Lemma 2.7** If $H \subseteq G$ is a subgroup scheme and $\sigma$ is contained in the center of $H$, there is a canonical ring isomorphism

$$K_* (X^\sigma, \sigma \times H) \cong K_* (X^\sigma, H) \otimes R(\sigma).$$

**Proof.** Since $\sigma$ acts trivially on $X^\sigma$, we have an equivalence (SGA3, I 4.7.3)

$$(\sigma \times H) - \text{Coh}_{X^\sigma} \cong \bigoplus_{\tilde{\sigma}} (H - \text{Coh}_{\tilde{X}^\sigma}) \quad (3)$$

(/where $\tilde{\sigma}$ is the character group of $\sigma$) which induces an isomorphism

$$K'_* (X^\sigma, \sigma \times H) \cong K'_* (X^\sigma, H) \otimes R(\sigma).$$

and we conclude since $K_* (Y, H) \cong K'_* (Y, H)$ and $K_* (X^\sigma, \sigma \times H) \cong K'_* (X^\sigma, \sigma \times H)$ (Appendix). [\hfill \Box]

For any essential dual cyclic subgroup $\sigma \subseteq G$, let $\Lambda = \Lambda_{(G,X)}$ and consider the composition

$$K_* (X,G)_\Lambda \to K_* (X, C_G(\sigma))_\Lambda \to K_* (X^\sigma, C_G(\sigma))_\Lambda \xrightarrow{m_\sigma} K_* (X^\sigma, C_G(\sigma))_\Lambda \otimes_{\Lambda} R(\sigma)_\Lambda \quad (4)$$

where the first map is induced by group restriction the last one is the geometric localization map tensored with the projection $R(\sigma)_\Lambda \to \bar{R}(\sigma)_\Lambda$ and we have used Lemma 2.7 with $H = C_G(\sigma)$; the second map is induced by restriction along $X^\sigma \hookrightarrow X$ which is a regular closed immersion (Th5, Prop. 3.1) therefore has finite Tor-dimension so that the pullback on $K$-groups is well defined (see the Appendix). It is not difficult to show that the image of (4) is actually contained in the invariant submodule

$$\left( K_* (X^\sigma, C_G(\sigma))_\text{geom} \otimes_{\Lambda} \bar{R}(\sigma)_\Lambda \right)^{w_G(\sigma)}$$

so that we get a map

$$\psi_{\sigma,X} : K_* (X,G)_\Lambda \to \left( K_* (X^\sigma, C_G(\sigma))_\text{geom} \otimes_{\Lambda} \bar{R}(\sigma)_\Lambda \right)^{w_G(\sigma)}.$$ 

Our basic map will be:

$$\Psi_{X,G} = \prod_{\sigma \in \mathcal{C}(G)} \psi_{\sigma,X} : K_* (X,G)_\Lambda \to \prod_{\sigma \in \mathcal{C}(G)} \left( K_* (X^\sigma, C_G(\sigma))_\text{geom} \otimes_{\Lambda} \bar{R}(\sigma)_\Lambda \right)^{w_G(\sigma)}. \quad (5)$$

Note that $\Psi_{X,G}$ is a morphism of $R(G)$-algebras as a composition of morphisms of $R(G)$-algebras.

The following technical lemma will be used in Propositions 3.5 and 4.6.
Lemma 2.8 Let $G$ be a linear algebraic $k$-group acting with finite stabilizers on a noetherian separated $k$-algebraic space $X$ and $\Lambda = \Lambda_{(G,X)}$. Let $H \subseteq G$ be a subgroup and $\sigma$ an essential dual cyclic subgroup contained in the center of $H$. Consider the composition

$$K'_\sigma(Y^\sigma, H)_\Lambda \longrightarrow K'_\sigma(Y^\sigma, H)_\Lambda \otimes_\Lambda R(\sigma)_\Lambda \longrightarrow K'_\sigma(Y^\sigma, H)_{\text{geom}} \otimes_\Lambda \tilde{R}(\sigma)_\Lambda$$

(6)

where the first morphism is induced by the product morphism $\sigma \times H \to H$ (recall Lemma 2.7) and the second is the tensor product of the geometric localization morphism with the projection $R(\sigma)_\Lambda \to \tilde{R}(\sigma)_\Lambda$. Then, if $\{G \}$ factors through $K'_\sigma(Y^\sigma, H)_\Lambda \to K'_\sigma(Y^\sigma, H)_\sigma$, yielding a morphism

$$\theta_{H,\sigma}: K'_\sigma(Y^\sigma, H)_\sigma \to K'_\sigma(Y^\sigma, H)_{\text{geom}} \otimes_\Lambda \tilde{R}(\sigma)_\Lambda.$$ (7)

Proof. Let $S_1$ (resp. $S_\sigma$) be the multiplicative subset in $R(H)_\Lambda$ consisting of elements going to 1 via the homomorphism $rk_{H,\Lambda}: R(H)_\Lambda \to \Lambda$ (resp., $R(H)_\Lambda \to \tilde{R}(\sigma)_\Lambda$). Observe that $K'_\sigma(X^\sigma, H)_\Lambda \otimes_\Lambda R(\sigma)_\Lambda$ (resp. $K'_\sigma(X^\sigma, H)_{\text{geom}} \otimes_\Lambda \tilde{R}(\sigma)_\Lambda$) is canonically an $R(H)_\Lambda \otimes R(\sigma)_\Lambda$-module (resp. an $S_1^{-1}R(H)_\Lambda \otimes \tilde{R}(\sigma)_\Lambda$-module) and therefore an $R(H)$ module via the coproduct ring morphism

$$\Delta_\sigma: R(H)_\Lambda \longrightarrow R(H)_\Lambda \otimes R(\sigma)_\Lambda$$

(resp. via the ring morphism

$$f_\sigma: R(H)_\Lambda \xrightarrow{\Delta_\sigma} R(H)_\Lambda \otimes R(\sigma)_\Lambda \longrightarrow S_1^{-1}R(H)_\Lambda \otimes \tilde{R}(\sigma)_\Lambda).$$

If we denote by $A'$ the $R(H)_\Lambda$-algebra $f_\sigma: R(H)_\Lambda \to S_1^{-1}R(H)_\Lambda \otimes \tilde{R}(\sigma)_\Lambda$, it is enough to show that the localization homomorphism

$$l'_\sigma: A' \longrightarrow S_1^{-1}(A')$$

is an isomorphism, because in this case the morphism $[\beta]$ will be induced by the $S_\sigma$-localization of $[\beta]$.

Let $A$ denote the $R(H)_\Lambda$-algebra

$$\lambda_1 \otimes 1: R(H)_\Lambda \longrightarrow S_1^{-1}R(H)_\Lambda \otimes \tilde{R}(\sigma)_\Lambda$$

where $\lambda_1: R(H)_\Lambda \to S_1^{-1}R(H)_\Lambda$ denotes the localization homomorphism. It is a well known fact that there is an isomorphism of $R(H)_\Lambda$-algebras $\varphi: A' \to A$: this is exactly the dual assertion to “the action $H \times \sigma \to \sigma$ is isomorphic to the projection on the second factor $H \times \sigma \to \sigma$.” Therefore, we have a commutative diagram

$$
\begin{array}{ccc}
A' & \xrightarrow{\varphi} & A \\
\downarrow l'_\sigma & & \downarrow \psi \\
S_1^{-1}A' & \xrightarrow{S_1^{-1}\varphi} & S_1^{-1}A
\end{array}
$$

where $l_\sigma$ denotes the localization homomorphism and it is enough to prove that $l_\sigma$ is an isomorphism. To see this, note that the ring $\tilde{R}(\sigma)_\Lambda$ is a free $\Lambda$-module of finite rank (equal to $\phi(|\sigma|)$, $\phi$ being the Euler function) and there is a norm homomorphism

$$N: \tilde{R}(\sigma)_\Lambda \longrightarrow \Lambda$$

sending an element to the determinant of the $\Lambda$-endomorphism of $\tilde{R}(\sigma)_\Lambda$ induced by multiplication by this element; obviously, we have

$$N^{-1}(\Lambda^*) = \left(\tilde{R}(\sigma)_\Lambda\right)^*.$$
Analogously, there is a norm homomorphism
\[ N' : A' = S_1^{-1}R(H)_\Lambda \otimes \widetilde{R}(\sigma)_\Lambda \longrightarrow S_1^{-1}R(H)_\Lambda \]
and
\[ N^{-1}((S_1^{-1}R(H)_\Lambda)^*) = \left(S_1^{-1}R(H)_\Lambda \otimes \widetilde{R}(\sigma)_\Lambda\right)^*. \]

There is a commutative diagram
\[
\begin{array}{ccc}
S_1^{-1}R(H)_\Lambda \otimes \widetilde{R}(\sigma)_\Lambda & \xrightarrow{N'} & S_1^{-1}R(H)_\Lambda \\
\downarrow \text{rk}_{H,\Lambda} \otimes \text{id} & & \downarrow \text{rk}_{H,\Lambda} \\
\widetilde{R}(\sigma)_\Lambda & \xrightarrow{N} & \Lambda
\end{array}
\]
and, by definition of \( S_1 \), we get \( \text{rk}_{H,\Lambda}(\Lambda^*) = (S_1^{-1}R(H)_\Lambda)^* \). Therefore, by definition of \( S_\sigma \), \( S_\sigma/1 \) consist of units in \( A \) and we conclude.

The following Lemma, which is an easy consequence of a result of Merkurjev, will be the main tool in reducing the proof of the main theorem from \( G = \text{GL}_{n,k} \) to its maximal torus \( T \).

**Lemma 2.9** Let \( X \) be a noetherian separated algebraic space over \( k \) with an action of a split reductive group \( G \) over \( k \) such that \( \pi_1(G) \) ([Me], 1.1) is torsion free. Then, if \( T \) denotes a maximal torus in \( G \), the canonical morphism
\[ K'_*(X,G) \otimes_{R(G)} R(T) \longrightarrow K'_*(X,T) \]
is an isomorphism.

**Proof.** Let \( B \supseteq T \) be a Borel subgroup of \( G \). Since \( R(B) \simeq R(T) \) and \( K'_*(X,B) \simeq K'_*(X,T) \) ([Th4], proof of Th. 1.13, p. 594), by [Me], Prop. 4.1, the canonical ring morphism
\[ K'_*(X,G) \otimes_{R(G)} R(T) \longrightarrow K'_*(X,T) \]
is an isomorphism.

Since Merkurjev states his theorem for a scheme, we briefly indicate how it extends to a noetherian separated algebraic space \( X \) over \( k \). By [Th1] Lemma 4.3, there exists an open dense \( G \)-invariant separated subscheme \( U \subset X \). Since Merkurjev’s map commutes with localization, by the localization sequence and noetherian induction it is enough to know the result for \( U \). And this is given in [Me], Prop. 4.1. Note that by [Me], 1.22, \( R(T) \) is flat (actually free) over \( R(G) \) and therefore the localization sequence remain exact after tensoring with \( R(T) \).

The following is Lemma 3.2 in [Vi2]. It will be used frequently in the rest of the paper and it is stated here for the convenience of the reader:

**Lemma 2.10** Let \( W \) be a finite group acting on the left on a set \( A \) and let \( B \subseteq A \) be a set of representatives for the orbits. Assume that \( W \) acts on the left on a product of abelian groups of the type \( \prod_{\alpha \in A} M_\alpha \) in such a way that
\[ sM_\alpha = M_{s\alpha} \]
for any \( s \in W \).

For each \( \alpha \in B \), let us denote by \( W_\alpha \) the stabilizer of \( \alpha \) in \( W \). Then the canonical projection
\[ \prod_{\alpha \in A} M_\alpha \longrightarrow \prod_{\alpha \in B} M_\alpha \]
induces an isomorphism
\[ \left( \prod_{\alpha \in A} M_\alpha \right)^W \longrightarrow \prod_{\alpha \in B} (M_\alpha)^{W_\alpha}. \]
3 The main theorem: the split torus case

In this section $T$ will be a split torus over $k$.

**Proposition 3.1** Let $T' \subset T$ a closed subgroup scheme (diagonalizable, by [SGA3], IX 8.1), finite over $k$. Then the canonical morphism

$$\delta : R(T')_{\Lambda_{T'}} \rightarrow \prod_{\sigma \text{ dual cyclic } \sigma \subseteq T'} \widetilde{R}(\sigma)_{\Lambda_{T'}}$$

is a ring isomorphism.

**Proof.** Since both $R(T')_{\Lambda_{T'}}$ and $\prod \widetilde{R}(\sigma)_{\Lambda_{T'}}$ are free $\Lambda_{T'}$-modules of finite rank, it is enough to prove that for any nonzero prime $p \nmid |T'|$ the induced morphism of $F_p$-vector spaces

$$R(T')_{\Lambda_{T'}} \otimes_{\mathbb{Z}} F_p \rightarrow \prod_{\sigma \text{ dual cyclic } \sigma \subseteq T'} \widetilde{R}(\sigma)_{\Lambda_{T'}} \otimes_{\mathbb{Z}} F_p$$

is an isomorphism. Now, for any finite abelian group $A$ we have an equality $|A| = \sum_{A \twoheadrightarrow C} \varphi(|C|)$ where $\varphi$ denotes the Euler function, $|H|$ denotes the order of the group $H$ and the sum is extended to all cyclic quotients of $A$; applying this to the group of characters $\hat{T}'$ (so that the corresponding cyclic quotients $C$ are exactly the group of characters $\hat{\sigma}$ for $\sigma$ dual cyclic subgroups of $T'$) we see that the ranks of both sides in (8) coincide with $|T'|$ and it is then enough to prove that (8) is injective. Define a morphism

$$f : \prod_{\tau \text{ dual cyclic } \tau \subseteq T'} \widetilde{R}(\tau)_{\Lambda_{T'}} \rightarrow \prod_{\sigma \text{ dual cyclic } \sigma \subseteq T'} R(\sigma)_{\Lambda_{T'}}$$

of $R(T')_{\Lambda_{T'}}$-modules by requiring for any dual cyclic subgroup $\sigma \subseteq T'$, the commutativity of the following diagram

$$\begin{array}{ccc}
\prod_{\tau \text{ dual cyclic } \tau \subseteq T'} \widetilde{R}(\tau)_{\Lambda_{T'}} & \xrightarrow{f} & \prod_{\sigma \text{ dual cyclic } \sigma \subseteq T'} R(\sigma)_{\Lambda_{T'}} \\
\downarrow \quad \text{Pr}_\sigma & & \downarrow \quad \text{res}_\sigma \text{pr}_\sigma \\
\prod_{\sigma \subseteq \sigma} \widetilde{R}(\tau)_{\Lambda_{T'}} & \xleftarrow{\varphi} & R(\sigma)_{\Lambda_{T'}}
\end{array}$$

where $\text{Pr}_\sigma$ and $\text{pr}_\sigma$ are the obvious projections and $\varphi$ is the isomorphism

$$R(\sigma)_{\Lambda_{T'}} \xrightarrow{\text{res}_\sigma} \prod_{\tau \subseteq \sigma} R(\tau)_{\Lambda_{T'}} \xrightarrow{(\text{pr}_\tau)} \prod_{\tau \subseteq \sigma} \widetilde{R}(\tau)_{\Lambda_{T'}}$$

induced by (1). Obviously, $f \circ \delta$ coincides with the map

$$\prod_{\sigma \text{ dual cyclic } \sigma \subseteq T'} \text{res}_\sigma^{T'} : R(T')_{\Lambda_{T'}} \rightarrow \prod_{\sigma \text{ dual cyclic } \sigma \subseteq T'} R(\sigma)_{\Lambda_{T'}}$$

so we are reduced to proving that

$$R(T')_{\Lambda_{T'}} \otimes_{\mathbb{Z}} F_p \rightarrow \prod_{\sigma \text{ dual cyclic } \sigma \subseteq T'} R(\sigma)_{\Lambda_{T'}} \otimes_{\mathbb{Z}} F_p$$
is injective i.e. that if $A$ is a finite abelian group and $p \nmid |A|$ then
\[
\varphi : \mathbb{F}_p[A] \longrightarrow \prod_{C \in \{\text{cyclic quotients of } A\}} \mathbb{F}_p[C]
\] (9)
is injective. If $\hat{A} = Hom_{\text{AbGrps}}(A, \mathbb{C}^\ast)$ denotes the complex characters group of $A$, $R(\hat{A}) \cong \mathbb{Z}[A]$ and
\[
\varphi = \prod_{\hat{C}} \text{res}_{\hat{C}}^\hat{A} : R(\hat{A}) \longrightarrow \prod_{\hat{C} \in \{\text{cyclic subgroups of } \hat{A}\}} R(\hat{C})
\]
Since $p \nmid |A|$ it is enough to prove that if $\xi \in R(\hat{A}) \otimes \mathbb{Z}[1/|A|]$ has image via
\[
\text{res}_{\hat{C}}^\hat{A} \otimes \mathbb{Z}[1/|A|] : R(\hat{A}) \otimes \mathbb{Z}[1/|A|] \longrightarrow R(\hat{C}) \otimes \mathbb{Z}[1/|A|]
\]
contained in $p \left( R(\hat{A}) \otimes \mathbb{Z}[1/|A|] \right)$ for each cyclic $\hat{C} \subseteq \hat{A}$, then $\xi \in p \left( R(\hat{A}) \otimes \mathbb{Z}[1/|A|] \right)$.

By [Se] p. 73, there exists $(\theta'_{\hat{C}})_{\hat{C}} \in \prod_{\hat{C} \in \{\text{cyclic subgroups of } \hat{A}\}} R(\hat{C}) \otimes \mathbb{Z}[1/|A|]$ such that
\[
1 = \sum_{\hat{C}} \left( \text{ind}_{\hat{A}}^\hat{C} \otimes \mathbb{Z}[1/|A|] \right) (\theta'_{\hat{C}});
\]
therefore
\[
\xi = \sum_{\hat{C}} \xi \left( \text{ind}_{\hat{A}}^\hat{C} \otimes \mathbb{Z}[1/|A|] \right) (\theta'_{\hat{C}}) = \sum_{\hat{C}} \left( \text{ind}_{\hat{A}}^\hat{C} \otimes \mathbb{Z}[1/|A|] \right) \left( \theta'_{\hat{C}} \left( \text{res}_{\hat{C}}^\hat{A} \otimes \mathbb{Z}[1/|A|] \right) (\xi) \right)
\]
(by the projection formula) and we conclude.

**Remark 3.2** The proof of Prop. [Vi2] is similar to the proof of Prop. (1.5) of [Vi2] which is however incomplete; that is why we have decided to give all details here.

**Corollary 3.3** (i) If $\sigma \neq \sigma'$ are dual cyclic subgroups of $T$, we have $\overline{R}(\sigma)_{\sigma'} = 0$ and $\overline{R}(\sigma)_{\sigma} = \overline{R}(\sigma)$;

(ii) If $T' \subseteq T$ is a closed subgroup scheme, finite over $k$ and $\sigma$ is a dual cyclic subgroup of $T$, we have $R(\sigma')_{\sigma} = 0$ if $\sigma \not\subseteq T'$;

(iii) If $T' \subseteq T$ is a closed subgroup scheme, finite over $k$, the canonical morphism of $R(T)$-algebras
\[
R \left( T' \right)_{\Lambda_{T'}} \longrightarrow \prod_{\sigma \text{ dual cyclic } \sigma \subseteq T'} R \left( T' \right)_{\sigma}
\]
is an isomorphism.

**Proof.** (i) Suppose $\sigma \neq \sigma'$ and let $T' \subseteq T$ be the closed subgroup scheme of $T$ generated by $\sigma$ and $\sigma'$. The obvious morphism $\pi : R(T)_{\Lambda_{T'}} \rightarrow \overline{R}(\sigma)_{\Lambda_{T'}} \times \overline{R}(\sigma')_{\Lambda_{T'}}$ factors through $R(T'')_{\Lambda_{T'}} \rightarrow \overline{R}(\sigma)_{\Lambda_{T'}} \times \overline{R}(\sigma')_{\Lambda_{T'}}$, which is an epimorphism by Proposition [3.1]. If $\xi \in R(T)_{\Lambda_{T'}}$ with $\pi(\xi) = (0,1) \otimes 1$, we have
\[
\xi \in S_{\sigma'} \cap \ker(R(T)_{\Lambda_{T'}} \rightarrow \overline{R}(\sigma)_{\Lambda_{T'}}).
\]
Then $\overline{R}(\sigma)_{\sigma'} = 0$. The second assertion is obvious.

(ii) and (iii) follow immediately from (i) and Proposition [3.1]. ■
Now let $X$ be a regular noetherian separated $k$-algebraic space on which $T$ acts with finite stabilizers and let $\Lambda \doteq \Lambda(T,X)$. Obviously, $\mathcal{C}(T)$ is just the set of essential dual cyclic subgroups of $T$, since $T$ is abelian.

**Proposition 3.4** (i) If $j_\sigma : X^\sigma \hookrightarrow X$ denotes the inclusion, the pushforward $(j_\sigma)_*$ induces an isomorphism

$$K'_*(X^\sigma, T)_{\sigma} \rightarrow K'_*(X, T)_{\sigma}$$

(ii) The canonical localization morphism

$$K'_*(X, T)_\Lambda \rightarrow \prod_{\sigma \in \mathcal{C}(T)} K'_*(X, T)_{\sigma}$$

is an isomorphism and the product on the left is finite.

**Proof.** (i) The proof is the same as that of [Th5] Th. 2.1 where the use of [Th5] Cor. 1.3 has to be substituted by that of our Cor. 3.3 (ii) above since we use a localization different from Thomason’s.

(ii) By the generic slice theorem for torus actions ([Th1], Prop. 4.10), there exists a $T$-invariant nonempty open subspace $U \subset X$, a closed (necessarily diagonalizable) subgroup $T'$ of $T$ and a $T$-equivariant isomorphism

$$U \cong T/T' \times (U/T) \simeq \tilde{R} R(\sigma).$$

By noetherian induction and the localization sequence for $K'$-groups ([Th3], Theorem 2.7), the statement for $U$ implies that for $X$.

Again using noetherian induction, Thomason’s generic slice theorem for torus actions and (i), one similarly shows that the product $\prod_{\sigma \in \mathcal{C}(T)} K'_*(X, T)_{\sigma}$ is finite. ■

By Prop. 3.4, there is an induced isomorphism (of $\mathbb{R}(T)$-modules, not a ring isomorphism due to the composition with pushforwards)

$$\prod_{\sigma \in \mathcal{C}(T)} K'_*(X^\sigma, T)_{\sigma} \rightarrow K'_*(X, T)_\Lambda$$

As shown in Lemma 2.8, the product morphism $\sigma \times T \rightarrow T$ induces a morphism

$$\theta_{T, \sigma} : K'_*(X^\sigma, T)_{\sigma} \rightarrow K'_*(X^\sigma, T)_{\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda.$$

**Proposition 3.5** For any $\sigma \in \mathcal{C}(T)$, $\theta_{T, \sigma}$ is an isomorphism.

**Proof.** We will write $\theta_{X, \sigma}$ for $\theta_{T, \sigma}$ in order to emphasize the dependence of the map on the space. We proceed by noetherian induction on $X^\sigma$. Let $X' \subseteq X^\sigma$ be a $T$-invariant closed subspace and let us suppose that ([Th3]) is an isomorphism with $X$ replaced by any $T$-invariant proper closed subspace $Z$ of $X'$. By Thomason’s generic slice theorem for torus actions ([Th1], Prop. 4.10), there exists a $T$-invariant nonempty open subscheme $U \subset X'$, a (necessarily diagonalizable) subgroup $T'$ of $T$ and a $T$-equivariant isomorphism

$$U^\sigma \equiv U \cong T/T' \times (U/T) \simeq (U/T) \times T'.$$
Since $U$ is nonempty and $T$ acts on $X$ with finite stabilizers, $T'$ is finite over $k$ and, obviously, $\Lambda T' \subseteq \Lambda$. Let $Z^\sigma \equiv Z \setminus U$. Since

$$
\begin{align*}
\theta_{Z,\sigma} & \downarrow \\
K'_*(Z^\sigma, T)_\sigma & \rightarrow \ K'_*(X^\sigma, T)_\sigma \\
\theta_{V',\sigma} & \downarrow \\
K'_*(Z^\sigma, T)_{\sigma,\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda & \rightarrow \ K'_*(X^\sigma, T)_{\sigma,\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda \\
\theta_{U,\sigma} & \downarrow \\
K'_*(U^\sigma, T)_\sigma & \rightarrow \ K'_*(U^\sigma, T)_{\sigma,\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda
\end{align*}
$$

is commutative, by induction hypothesis and the five-lemma it will be enough to show that $\theta_{U,\sigma}$ is an isomorphism. By Morita equivalence theorem (Th3, Proposition 6.2) and Th1 Lemma 5.6, $K'_*(U, T) \simeq K'_*(U/T) \otimes_R T'$, so it is enough to prove that

$$
\theta_{\text{Spec}k,\sigma} : K'_0(\text{Spec}k, T')_\sigma = R(T')_\sigma \rightarrow K'_0(\text{Spec}k, T')_{\sigma,\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda = R(T')_{\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda
$$

is an isomorphism. But this follows immediately from Cor. 3.3 (i) and (iii).

Combining Proposition 3.3 with 10 we get an isomorphism

$$
\Phi_{X,T} : \prod_{\sigma \in \mathcal{C}(T)} K'_*(X^\sigma, T)_{\sigma,\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda \rightarrow K'_*(X, T)_\Lambda. \quad (12)
$$

The following lemma is a variant of Th5, Lemme 3.2, that already proves the statement below after tensoring with $\mathbb{Q}$.

**Lemma 3.6** Let $X$ be a noetherian regular separated algebraic space over $k$ on which a split $k$-torus acts with finite stabilizers and let $\sigma \in \mathcal{C}(T)$. Let $X^\sigma$ denote the regular $\sigma$-fixed subscheme, $j_\sigma : X^\sigma \hookrightarrow X$ the regular closed immersion (Th5, Proposition 3.1) and $N(j_\sigma)$ the corresponding locally free conormal sheaf. Then, for any $T$-invariant algebraic subspace $Y$ of $X^\sigma$, the cap-product

$$
\lambda_{-1}(N(j_\sigma)) \cap (-) : K'_*(Y, T)_\sigma \rightarrow K'_*(Y, T)_\sigma
$$

is an isomorphism.

**Proof.** We proceed by noetherian induction on closed $T$-invariant subspaces $Y$ of $X^\sigma$. The statement is trivial for $Y = \emptyset$, so let us suppose $Y$ nonempty and

$$
\lambda_{-1}(N(j_\sigma)) \cap (-) : K'_*(Z, T)_\sigma \rightarrow K'_*(Z, T)_\sigma
$$

an isomorphism for any proper $T$-invariant closed subspace $Z$ of $Y$. By Thomason’s generic slice theorem for torus actions (Th1, Prop. 4.10), there exists a $T$-invariant nonempty open subscheme $U \subset Y$, a closed (necessarily diagonalizable) subgroup $T'$ of $T$ and a $T$-equivariant isomorphism

$$
U^\sigma \equiv U \simeq T'/T' \times (U/T).
$$

Since $U$ is nonempty and $T$ acts on $X$ with finite stabilizers, $T'$ is finite over $k$. Using the localization sequence and the five-lemma, we reduce ourselves to showing that

$$
\lambda_{-1}(N(j_\sigma)) \cap (-) : K'_*(U, T)_\sigma \rightarrow K'_*(U, T)_\sigma
$$

is an isomorphism. For this, it is enough to show that (the restriction of) $\lambda_{-1}(N(j_\sigma))$ is a unit in $K_0(U, T)_\sigma \simeq K_0(U/T)_\Lambda \otimes R(T')_\sigma$ (Th3, Proposition 6.2). Decomposing $N(j_\sigma)$ according to the characters of $T'$ we may write, shrinking $U$ if necessary,

$$
N(j_\sigma) = \bigoplus_{\rho \in T'} \mathcal{O}_{U/T}^\rho \otimes \mathcal{L}_\rho
$$

\[16\]
where $\mathcal{L}_\rho$ is the line bundle attached to the $T'$-character $\rho$ and $r_\rho \geq 0$ and therefore $\lambda_{-1}(\mathcal{N}(j_\sigma)) = \prod_{\rho \in \mathcal{T}'} (1 - \rho)^{r_\rho}$ in $K_0(U/T) \otimes R(T')$. The localization map $R(T')_\Lambda \to R(T')_\sigma \simeq \tilde{R}(\sigma)_\Lambda$ coincides with the composition

$$R(T')_\Lambda \xrightarrow{\pi_\sigma} R(\sigma)_\Lambda \xrightarrow{p_\sigma} \tilde{R}(\sigma)_\Lambda$$

of the restriction to $\sigma$ followed by the projection (Cor. 3.3) and then

$$\bigg(\text{id}_{K_0(U/T)_\Lambda} \otimes \pi_\sigma\bigg)(\mathcal{N}(j_\sigma)) = \bigoplus_{\chi \in \mathcal{I}(U/T)} \mathcal{O}^{\mathcal{T}_X}_U \otimes \mathcal{L}_\chi$$

in $K_0(U/T)_\Lambda \otimes R(\sigma)_\Lambda$, where the summand omits the trivial character since the decomposition of $\mathcal{N}(j_\sigma)$ according to the characters of $\sigma$ has vanishing fixed subsheaf $\mathcal{N}(j_\sigma)_0$ (see, e.g., [Th3], Prop. 3.1). Therefore

$$\lambda_{-1}\bigg(\bigg(\text{id}_{K_0(U/T)_\Lambda} \otimes \pi_\sigma\bigg)(\mathcal{N}(j_\sigma))\bigg) = \prod_{\chi \in \mathcal{I}(U/T)} (1 - \chi)^{\mathcal{T}_X}$$

and it is enough to show that the image of $1 - \chi$ in $\tilde{R}(\sigma)_\Lambda$ via $p_\sigma$ is a unit for any nontrivial character $\chi$ of $\sigma$. Now, the image of such a $\chi$ in

$$\tilde{R}(\sigma)_\Lambda \simeq \frac{\Lambda[t]}{(\Phi_{|\sigma|})}$$

($\Phi_{|\sigma|}$ being the $|\sigma|$-th cyclotomic polynomial) is of the form $[t^l]$ for some $1 \leq l < |\sigma|$, where $[-]$ denotes the class mod $\Phi_{|\sigma|}$; therefore the cokernel of the multiplication by $1 - [t^l]$ in $\Lambda[t]/(\Phi_{|\sigma|})$ is

$$\frac{\Lambda[t]}{(\Phi_{|\sigma|}, 1 - t^l)} = 0$$

since $\Phi_{|\sigma|}$ and $(1 - t^l)$ are relatively prime in $\Lambda[t]$, for $1 \leq l < |\sigma|$. Thus $1 - [t^l]$ is a unit in $\Lambda[t]/(\Phi_{|\sigma|})$ and we conclude. □

We are now able to prove our main theorem for $G = T$:

**Theorem 3.7** If $X$ is a regular noetherian separated $k$-algebraic space, then

$$\Psi_{X,T} : K_*(X, T)_\Lambda \rightarrow \bigotimes_{\sigma \in \mathcal{C}(T)} K_*(X^\sigma, T)_{\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda$$

is an isomorphism of $R(T)$-algebras.

**Proof.** Recall (Appendix) that $K_*(X, T) \simeq K'_*(X, T)$ and $K_*(X^\sigma, T) \simeq K'_*(X^\sigma, T)$, since both $X$ and $X^\sigma$ are regular ([Th5], Proposition 3.1). Since $\Phi_{X,T}$ is an isomorphism of $R(T)$-modules, it is enough to show that the composition $\Psi_{X,T} \circ \Phi_{X,T}$ is an isomorphism. A careful inspection of the definitions of $\Psi_{X,T}$ and $\Phi_{X,T}$, easily reduce the problem to proving that, for any $\sigma \in \mathcal{C}(T)$, the composition

$$K'_*(X^\sigma, T)_\sigma \xrightarrow{j_*} K'_*(X, T)_\sigma \xrightarrow{j_*} K'_*(X^\sigma, T)_\sigma$$

is an isomorphism, $j_* : X^\sigma \hookrightarrow X$ being the natural inclusion. Since $j_\sigma$ is regular, there is a self-intersection formula

$$j^* \circ j_{\sigma*} (-) = \lambda_{-1}(\mathcal{N}(j_\sigma)) \cap (-) \quad (13)$$

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$N(j_\sigma)$ being the conormal sheaf associated to $j_\sigma$ and we conclude by lemma [13]. To prove the self-intersection formula [13], we adapt [115], proof of Lemme 3.3. First of all, by Proposition [3.4] (i), $j_{\sigma*}$ is an isomorphism so it is enough to prove that $j_{\sigma*}j_{\sigma*}^*j_{\sigma*}(-) = j_{\sigma*}(N(j_\sigma))\cap(-))$. By projection formula (Appendix, Proposition 3.5), we have

$$j_{\sigma*}j_{\sigma*}^*j_{\sigma*}(-) = j_{\sigma*}j_{\sigma*}^*(1)\cap j_{\sigma*}(-) = j_{\sigma*}j_{\sigma*}^*(O_X)\cap j_{\sigma*}(-) = j_{\sigma*}(O_{X^\sigma})\cap j_{\sigma*}(-) = j_{\sigma*}(j_{\sigma*}(O_{X^\sigma})\cap(-))$$

Now, as explained in the Appendix, to compute $j_{\sigma*}(O_{X^\sigma})$ we choose a complex $F^*$ of flat quasi-coherent $G$-equivariant Modules on $X$ which is quasi-isomorphic to $O_{X^\sigma}$ and then

$$j_{\sigma*}(O_{X^\sigma}) = [j_{\sigma*}(F^*)] = [F^*\otimes O_{X^\sigma}] = \sum_i (-1)^i [H^i(F^*\otimes O_{X^\sigma})].$$

But $F^*$ is a flat resolution of $O_{X^\sigma}$, so $H^i(F^*\otimes O_{X^\sigma}) = \text{Tor}_i^O_{X^\sigma}(O_{X^\sigma},O_{X^\sigma}) \simeq \bigwedge^i N(j_\sigma)$, where the last isomorphism (SGA6, VII, 2.5) is natural hence $T$-equivariant. Therefore $j_{\sigma*}(O_{X^\sigma}) = \lambda_{-1}(N(j_\sigma))$ and we conclude.  

4 The main theorem: the case of $G = \text{GL}_{n,k}$

In this section we will use the result for $\Psi_{X,T}$ to deduce the same result for $\Psi_{X,\text{GL}_{n,k}}$.

**Theorem 4.1** Let $X$ be a noetherian regular separated algebraic space over a field $k$ on which $G = \text{GL}_{n,k}$ acts with finite stabilizers. Then the map defined in [3]

$$\Psi_{X,G}: K_*(X,G)_{\Lambda(G,X)} \rightarrow \prod_{\sigma \in \mathcal{C}(G)} \left(K_*\left(X^\sigma, C(\sigma)\right)_{\text{geom}} \otimes_{\Lambda(G,X)} \bar{R}(\sigma)_{\Lambda(G,X)}\right)^{w_{\mathcal{C}(\sigma)}(\sigma)}$$

is an isomorphism of $R(G)$-algebras and the product on the right is finite.

Throughout this section, entirely devoted to the proof of Theorem 4.1, we will simply write $G$ for $\text{GL}_{n,k}$, $\Lambda$ for $\Lambda(G,X)$ and $T$ for the maximal torus of diagonal matrices in $\text{GL}_{n,k}$.

First of all, let us observe that we can choose each $\sigma \in \mathcal{C}(G)$ contained in $T$. Moreover, $\Lambda(T,X) = \Lambda(G,X)$.

We need the following three preliminary lemmas [1.2, 1.3 and 4.4].

If $\sigma,\tau \subset T$ are dual cyclic subgroups, they are conjugate under the $G(k)$-action iff they are conjugated via an element in the Weyl group $S_n$. For any group scheme $H$ with a dual cyclic subgroup $\sigma \subset H$, we denote by $m_{H,\sigma}^T$ the kernel of $R(H)_\Lambda \rightarrow \bar{R}(\sigma)_\Lambda$ and by $R(H/\Lambda,\sigma)$ the completion of $R(H)_\Lambda$ with respect to the ideal $m_{H,\sigma}^T$.

The following Lemma is essentially a variant of Lemma 2.3 for $\sigma$-localizations.

**Lemma 4.2** Let $G = \text{GL}_{n,k}$, $T$ the maximal torus of $G$ consisting of diagonal matrices and $X$ an algebraic space on which $G$ acts with finite stabilizers.

(i) for any essential dual cyclic subgroup $\sigma \subset T$, the morphisms

$$\omega_{\sigma,\text{geom}}: K_*(X^\sigma, C_G(\sigma))_{\text{geom}} \otimes_{R(C_G(\sigma))_{\Lambda}} R(T)_\Lambda \rightarrow K_*(X^\sigma, T)_{\text{geom}}$$

$$\omega_{\sigma}: K_*(X^\sigma, C_G(\sigma))_{\Lambda} \otimes_{R(C_G(\sigma))_{\Lambda}} R(T)_\Lambda \rightarrow K_*(X^\sigma, T)_{\Lambda}$$
induced by $T \mapsto C_G(\sigma)$ are isomorphisms;

(ii) for any essential dual cyclic subgroup $\sigma \subseteq T$,

$$(m_{\sigma_C G(\sigma)})^N : K'_* (X^\sigma, C_G(\sigma))_\sigma = 0, \quad N \gg 0$$

and the morphism induced by $T \mapsto C_G(\sigma)$

$$\tilde{\omega}_\sigma : K'_* (X^\sigma, C_G(\sigma))_\sigma \otimes_{R(C_G(\sigma))} R(T)_\Lambda \to K'_* (X^\sigma, T)_\sigma$$

is an isomorphism.

**Proof.** (i) Since $C_G(\sigma)$ is isomorphic to a product of general linear groups over $k$ and $T$ is a maximal torus in $C_G(\sigma)$, by Lemma 2.9, the canonical ring morphism

$$K'_* (X, C_G(\sigma)) \otimes_{R(C_G(\sigma))} R(T) \to K'_* (X, T)$$

is an isomorphism. If $H \subseteq G$ is a subgroup scheme, we denote by $S^H_\tau$ the multiplicative subset of $R(H)_\Lambda$ consisting of the elements sent to 1 by the canonical ring homomorphism $R(H)_\Lambda \to \hat{R}(\sigma)_\Lambda$. By \([\text{15}]\), $\omega_\sigma$ coincides with the composition

$$K'_* (X^\sigma, C_G(\sigma))_\sigma \otimes_{R(C_G(\sigma))} R(T)_\Lambda \cong$$

$$K'_* (X^\sigma, T) \otimes_{R(C_G(\sigma))} \left( \left( S^G_C(\sigma) \right)^{-1} R(C_G(\sigma)) \right)^\Lambda R(T)_\Lambda \to$$

$$\to K'_* (X^\sigma, T) \otimes_{R(C_G(\sigma))} \left( S^T_C(\sigma) \right)^{-1} R(T)_\Lambda \cong K'_* (X^\sigma, T)_\sigma$$

where

$$\nu_\sigma : \left( S^G_C(\sigma) \right)^{-1} R(C_G(\sigma)) \otimes_{R(C_G(\sigma))} R(T)_\Lambda \to \left( S^T_C(\sigma) \right)^{-1} R(T)_\Lambda$$

is induced by $T \mapsto C_G(\sigma)$ and the last isomorphism follows from \([\text{15}]\); the same is true for $\omega_\sigma_{\text{geom}}$. Therefore it is enough to prove that $\nu_\sigma$ and

$$\nu_{\sigma, \text{geom}} : \left( S^G_C(\sigma) \right)^{-1} R(C_G(\sigma)) \otimes_{R(C_G(\sigma))} R(T)_\Lambda \to \left( S^T_C(\sigma) \right)^{-1} R(T)_\Lambda$$

are isomorphisms, i.e. if $S_\tau$ denotes the image of $S^G_C(\sigma)$ via the restriction map

$$R(C_G(\sigma)) \to R(T)_\Lambda$$

that $S^T_\tau/1$ consists of units in $(S_\tau)^{-1} R(T)_\Lambda$, for $\tau = 1$ and $\tau = \sigma$.

If $\Delta_\sigma$ denotes the Weyl group of $C_G(\sigma)$, which is a product of symmetric groups, we have

$$R(C_G(\sigma)) \cong R(T)^{\Delta_\sigma}$$

and therefore

$$\left( S^G_C(\sigma) \right)^{-1} R(C_G(\sigma)) \cong \left( (S_\tau)^{-1} R(T)_\Lambda \right)^{\Delta_\sigma},$$

since $R(T)$ is torsion free. Moreover, there is a commutative diagram

$$\begin{array}{ccc}
(S^G_C(\sigma))^{-1} R(C_G(\sigma))_{\Lambda} & \xleftarrow{\psi} & (S_\tau)^{-1} R(T)_{\Lambda} \\
\tilde{R}(\tau)_\Lambda \cong (S^T_\tau)^{-1} R(\tau)_{\Lambda} & \xleftarrow{\phi} & (S^T_\tau)^{-1} R(T)_{\Lambda}
\end{array}$$
where ψ is induced by \( \bar{\pi}_\sigma \) and the isomorphism \( \bar{R}(\tau)_\Lambda \simeq (S_\tau^T)^{-1}R(\tau)_\Lambda \) is obtained from Proposition 3.1 and Corollary 3.3. If we define the map

\[
M : (S_\tau)^{-1}R(T)_\Lambda \longrightarrow \left( (S_\tau)^{-1}R(T)_\Lambda \right)^{\Delta_\sigma}
\]

\[
\xi \mapsto \prod_{g \in \Delta_\sigma} g : \xi,
\]

it is easily checked that if for \( \xi \in (S_\tau)^{-1}R(T)_\Lambda \), \( \xi \) is a unit if \( M(\xi) \) is a unit and that \( \psi(M(\xi)) = 1 \) implies \( \xi \) is a unit in \( \left( (S_\tau)^{-1}R(T)_\Lambda \right)^{\Delta_\sigma} \). But \( \varphi \) is \( \Delta_\sigma \)-equivariant and therefore \( S_\tau^T/1 \) consists of units in \( (S_\tau)^{-1}R(T)_\Lambda \), for \( \tau = 1 \) or \( \sigma \).

(ii) Since \( R(C_G(\sigma)) \to R(T) \) is faithfully flat, by (i), it is enough to prove that

\[
(m_\sigma^T)^N K'_*(X^\sigma, T)_\sigma = 0 \quad \text{for} \quad N \gg 0.
\]

But (17) can be proved using the same technique as in the proof of, e.g., Proposition 3.5, i.e. noetherian induction together with Thomason’s generic slice theorem for torus actions.

The second part of (ii) follows, arguing as in (i), from the fact that (16) is an isomorphism since

\[
K'_*(X^\sigma, C_G(\sigma))_\sigma \otimes_{R(C_G(\sigma))_\Lambda} R(C_G(\sigma))_\Lambda,\sigma \simeq K'_*(X^\sigma, C_G(\sigma))_\sigma, \quad K'_*(X^\sigma, T)_\sigma \otimes_{R(T)_\Lambda} R(T)_\Lambda,\sigma \simeq K'_*(X^\sigma, T)_\sigma.
\]

If \( \sigma, \tau \subset T \) are dual cyclic subgroups conjugated under \( G(k) \), they are conjugate through an element of the Weyl group \( S_n \) and we write \( \tau \approx S_n \sigma \); moreover, we have \( m^G = m^T \) because conjugation by an element in \( S_n \) (actually, by any element in \( G(k) \)) induces the identity morphism on \( K \)-theory and in particular on the representation ring. Then there are canonical maps

\[
\overline{R(G)_{\Lambda,\sigma} \otimes_{R(G)_\Lambda} R(T)_\Lambda} \longrightarrow \prod_{\tau \approx S_n \sigma} \overline{R(T)_{\Lambda,\tau}} \quad (18)
\]

\[
\overline{R(C_G(\sigma))_{\Lambda,\sigma} \otimes_{R(C_G(\sigma))_\Lambda} R(T)_\Lambda} \longrightarrow \overline{R(T)_{\Lambda,\sigma}} \quad (19)
\]

**Lemma 4.3** The maps (13) and (14) are isomorphisms.

**Proof.** Since \( R(G) = R(T)^{S_n} \to R(T) \) is finite, the canonical map \( \overline{R(G)_{\Lambda,\sigma} \otimes_{R(G)_\Lambda} R(T)_\Lambda} \to \overline{R(T)^m G} \) (where \( \overline{R(T)^m G} \) denotes the \( m^G \)-adic completion of the \( R(G)_\Lambda \)-module \( R(T)_\Lambda \)) is an isomorphism. Moreover, \( R(G)_\Lambda = (R(T)_\Lambda)^{S_n} \) implies that

\[
\sqrt{m^G_\sigma R(T)_\Lambda} = \bigcap_{\tau \approx S_n \sigma} \sqrt{m^T_\tau} = \bigcap_{\tau \approx S_n \sigma} m^T_\tau,
\]

and by Corollary 3.3 (i), \( m^T_\tau + m^T_{\tau'} = R(T)_\Lambda \) if \( \tau \neq \tau' \). By the Chinese remainder lemma, we conclude that (18) is an isomorphism.

Arguing in the same way, we get that the canonical map

\[
\overline{R(C_G(\sigma))_{\Lambda,\sigma} \otimes_{R(C_G(\sigma))_\Lambda} R(T)_\Lambda} \longrightarrow \prod_{\tau \approx \Delta_\sigma \sigma} \overline{R(T)_{\Lambda,\tau}}
\]

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is an isomorphism, where $\Delta_{\sigma} = S_n \cap C_G(\sigma)$ is the Weyl group of $C_G(\sigma)$ with respect to $T$ and we write $\tau \approx_{\Delta_{\sigma}} \sigma$ to denote that $\tau$ and $\sigma$ are conjugate through an element of $\Delta_{\sigma}$. But $\Delta_{\sigma} \subset C_G(\sigma)$ so that $\tau \approx_{\Delta_{\sigma}} \sigma$ iff $\tau = \sigma$ and we conclude that (13) is an isomorphism. \[\square\]

**Lemma 4.4** For any essential dual cyclic subgroup $\sigma \subseteq G$, the canonical morphism

\[
R\left(\hat{G}\right)_{\Lambda,\sigma} \longrightarrow R\left(\hat{C_G(\sigma)}_{\Lambda,\sigma}\right)
\]

is a finite étale Galois cover (SGA1, Exp. V) with Galois group $w_G(\sigma)$.

**Proof.** Since $R\left(T\right)$ is flat over $R\left(G\right) = R\left(T\right)^{S_n}$, we have

\[
R\left(\hat{G}\right)_{\Lambda,\sigma} \simeq R\left(\hat{G}\right)_{\Lambda,\sigma} \otimes_{R(G)} R\left(T\right)_{\Lambda,\sigma} \simeq \left( R\left(\hat{G}\right)_{\Lambda,\sigma} \otimes_{R(G)} R\left(T\right)_{\Lambda}\right)^{S_n} \simeq \left( \prod_{\tau \text{ dual cyclic}} R\left(\hat{T}\right)_{\Lambda,\tau}\right)^{S_n},
\]

the last isomorphism being given in Lemma 4.3. By Lemma 2.10, we get

\[
R\left(\hat{G}\right)_{\sigma} \simeq \left( R\left(\hat{T}\right)_{\sigma}\right)^{S_n,\sigma}
\]

where $S_n$ acts on the set of dual cyclic subgroups of $T$ which are $S_n$-conjugated to $\sigma$ and $S_n,\sigma$ denotes the stabilizer of $\sigma$. Analogously, denoting by $\Delta_{\sigma}$ the Weyl group of $C_G(\sigma)$, by Lemma 4.2 (ii) we have

\[
R\left(\hat{C_G(\sigma)}_{\Lambda,\sigma}\right) \simeq R\left(\hat{C_G(\sigma)}_{\Lambda,\sigma} \otimes_{R(C_G(\sigma))_{\Lambda}} R\left(T\right)_{\Lambda}\right)^{\Delta_{\sigma}} \simeq \left( R\left(\hat{C_G(\sigma)}_{\Lambda,\sigma} \otimes_{R(C_G(\sigma))_{\Lambda}} R\left(T\right)_{\Lambda}\right)\right)^{\Delta_{\sigma}}
\]

where the last isomorphism is given by Lemma 4.3. From the exact sequence

\[
1 \rightarrow \Delta_{\sigma} \rightarrow S_{n,\sigma} \rightarrow w_G(\sigma) \rightarrow 1
\]

we conclude that

\[
R\left(\hat{G}\right)_{\Lambda,\sigma} \simeq \left( R\left(\hat{C_G(\sigma)}_{\Lambda,\sigma}\right)\right)^{w_G(\sigma)}.
\]

By SGA1 Prop. 2.6, Exp. V, it is now enough to prove that the stabilizers of geometric points (i.e. the inertia groups of points) of $Spec\left(R\left(\hat{C_G(\sigma)}_{\Lambda,\sigma}\right)\right)$ under the $w_G(\sigma)$-action are trivial.

First of all, let us observe that $Spec\left(\hat{R}(\sigma)_{\Lambda}\right)$ is a closed subscheme of $Spec\left(R\left(\hat{C_G(\sigma)}_{\Lambda,\sigma}\right)\right)$. This can be seen as follows. It is obviously enough to show that if $s$ denotes the order of $\sigma$, the map

\[
\pi_{\sigma} : R\left(C_G(\sigma)\right)_{\Lambda} \longrightarrow R\left(\sigma\right)_{\Lambda} = \frac{\Lambda\left[t\right]}{\left(t^s - 1\right)}
\]

is surjective. First consider the case where $\sigma$ is contained in the center of $G$. Since $R\left(\sigma\right)_{\Lambda}$ is of finite type over $\Lambda$, we may as well show that for any prime \footnote{Recall that $\sigma$ is essential hence $s$ is invertible in $\Lambda$.} $p | s$ the induced map

\[
\pi_{\sigma,p} : R\left(C_G(\sigma)\right)_{\Lambda} \otimes \mathbb{F}_p \longrightarrow R\left(\sigma\right)_{\Lambda} \otimes \mathbb{F}_p
\]

is surjective. Note that if $E$ denote the standard $n$-dimensional representation of $G$, $\pi_{\sigma}$ sends $\bigwedge^t E$ to $\binom{n}{t} t^s$. If $p | n$, $\pi_{\sigma,p}$ is surjective (in fact $\pi_{\sigma}(E) = nt$ and $n$ is invertible in $\mathbb{F}_p$). If $p | n$, let us write $n = qm$, with $q = p^i$ and $p \nmid m$. Since $(s, q) = 1$, $t^q$ is a ring generator of $R\left(\sigma\right)_{\Lambda}$ and
to prove $\pi_{\sigma,p}$ is injective it is enough to show that $p \nmid \binom{n}{q}$. This is elementary since the binomial expansion of 
\[(1 + X)^n = (1 + X^q)^m\]
in $\mathbb{F}_p [X]$, yields $\binom{n}{q} = m$ in $\mathbb{F}_p$. For a general $\sigma \subseteq T$, let $C_G (\sigma) = \prod_{i=1}^{l} \text{GL}_{d_i,k}$, where $\sum d_i = n$ and let $\sigma_i$ denote the image of $\sigma$ in $\text{GL}_{d_i,k}$, $i = 1, \ldots, l$. Since $\sigma \subseteq \prod_{i=1}^{l} \sigma_i$ is an inclusion of diagonalizable groups, the induced map
\[R \left( \prod_{i=1}^{l} \sigma_i \right) = \bigotimes_{i=1}^{l} R(\sigma_i) \longrightarrow R(\sigma)\]
is surjective (e.g. [SGA3], tome II). But $R (C_G (\sigma))_{\Lambda} \rightarrow R(\sigma)_{\Lambda}$ factors as
\[R (C_G (\sigma))_{\Lambda} \rightarrow \bigotimes_{i=1}^{l} R(\sigma_i)_{\Lambda} \longrightarrow R(\sigma)_{\Lambda}\]
and also the first map is surjective (by the previous case, since $\sigma_i$ is contained in the center of $\text{GL}_{d_i,k}$ and $|\sigma_i|$ divides $|\sigma|$). This proves that Spec $\left( \tilde{R}(\sigma)_{\Lambda} \right)$ is a closed subscheme of Spec $\left( R(C_G (\sigma))_{\Lambda,\sigma} \right)$.

Since $R (C_G (\sigma))_{\Lambda,\sigma}$ is the completion of $R (C_G (\sigma))_{\Lambda}$ along the ideal
\[\ker \left( R(C_G (\sigma))_{\Lambda} \rightarrow \tilde{R}(\sigma)_{\Lambda} \right),\]
any nonempty closed subscheme of Spec $\left( R(C_G (\sigma))_{\Lambda,\sigma} \right)$ meets the closed subscheme Spec $\left( \tilde{R}(\sigma)_{\Lambda} \right)$.

To prove that $w_G (\sigma)$ acts freely on the geometric points of Spec $\left( R(C_G (\sigma))_{\Lambda,\sigma} \right)$ it is then enough to show that it acts freely on the geometric points of Spec $\left( \tilde{R}(\sigma)_{\Lambda} \right)$.

Actually, more is true: the map $q : \text{Spec} \left( \tilde{R}(\sigma)_{\Lambda} \right) \rightarrow \text{Spec} (\Lambda)$ is a $(\mathbb{Z}/s\mathbb{Z})^*$-torsor. In fact, if $\text{Spec} (\Omega) \rightarrow \text{Spec} (\Lambda)$ is a geometric point, the corresponding geometric fiber of $q$ is isomorphic to the spectrum of
\[\prod_{\alpha_i \in \tilde{\mu}_s(\Omega)} \frac{\Omega[t]}{(t - \alpha_i)} \simeq \prod_{\alpha_i \in \tilde{\mu}_s(\Omega)} \Omega\]
and $(\mathbb{Z}/s\mathbb{Z})^*$ acts by permutation on the primitive roots $\tilde{\mu}_s(\Omega)$, by $\alpha \mapsto \alpha^k$, $(k, s) = 1$. In particular, the action of the subgroup $w_G (\sigma) \subset (\mathbb{Z}/s\mathbb{Z})^*$ on Spec $\left( \tilde{R}(\sigma)_{\Lambda} \right)$ is free.  

**Proposition 4.5** The canonical morphism
\[K'_* (X, G)_{\Lambda} \longrightarrow \prod_{\sigma \in \mathcal{C}(G)} \left( K'_* (X^\sigma, C_G (\sigma))_{\sigma} \right)^{w_G(\sigma)}\]
is an isomorphism.

\[2\text{Recall that the constant group scheme associated to } (\mathbb{Z}/s\mathbb{Z})^* \text{ is isomorphic to } \text{Aut}_k (\sigma).\]
Proof. By Lemma 4.3, the canonical ring morphism
\[ K'_*(X, G) \otimes_{R(G)} R(T) \rightarrow K'_*(X, T) \]
is an isomorphism. Since \( R(G) \rightarrow R(T) \) is faithfully flat, it is enough to show that
\[ K'_*(X, T)_\Lambda \simeq K'_*(X, G)_\Lambda \otimes_{R(G)_\Lambda} R(T)_\Lambda \rightarrow \prod_{\sigma \in \mathcal{C}(G)} (K'_*(X^\sigma, C_G(\sigma))_\sigma)^{w_G(\sigma)} \otimes_{R(G)_\Lambda} R(T)_\Lambda \]
is an isomorphism. By Proposition 3.4 (ii), we are left to prove that
\[ \text{is an isomorphism. By Proposition 3.4 (ii), we are left to prove that} \]
is an isomorphism. By Proposition 3.4 (ii), we are left to prove that
\[ \prod_{\sigma \in \mathcal{C}(G)} (K'_*(X^\sigma, C_G(\sigma))_\sigma)^{w_G(\sigma)} \otimes_{R(G)_\Lambda} R(T)_\Lambda \simeq \prod_{\sigma \in \text{dual cyclic}} K'_*(X, T)_\sigma \quad (21) \]
For any \( \tau \in \mathcal{C}(G) \) (\( \tau \subseteq T \), as usual), we have
\[ K'_*(X^\tau, C_G(\tau))_\tau \otimes_{R(G)_\Lambda, \tau} R(T)_\Lambda, \tau \simeq \left( K'_*(X^\tau, C_G(\tau))_\tau \otimes_{R(C_G(\tau))_\Lambda, \tau} R(C_G(\tau))_\Lambda, \tau \right) \otimes_{R(G)_\Lambda, \tau} R(T)_\Lambda, \tau \]
\[ \simeq \left( K'_*(X^\tau, C_G(\tau))_\tau \otimes_{R(G)_\Lambda, \tau} R(C_G(\tau))_\Lambda, \tau \right) \otimes_{R(G)_\Lambda, \tau} R(T)_\Lambda, \tau \]
By Lemma 4.4, for any \( R(G)_\Lambda, \tau \)-module \( M \), we have
\[ M \otimes_{R(G)_\Lambda, \tau} R(C_G(\tau))_\Lambda, \tau \simeq w_G(\tau) \times M \]
since a torsor is trivial when base changed along itself. Therefore
\[ K'_*(X^\tau, C_G(\tau))_\tau \otimes_{R(G)_\Lambda, \tau} R(T)_\Lambda, \tau \simeq w_G(\tau) \times \left( K'_*(X^\tau, C_G(\tau))_\tau \otimes_{R(C_G(\tau))_\Lambda, \tau} R(T)_\Lambda, \tau \right) \quad (22) \]
with \( w_G(\tau) \) acting on l.h.s. by left multiplication on \( w_G(\tau) \). Applying Lemma 4.2 (ii), to the l.h.s., we get
\[ K'_*(X^\tau, C_G(\tau))_\tau \otimes_{R(G)_\Lambda, \tau} R(T)_\Lambda, \tau \simeq w_G(\tau) \times K'_*(X^\tau, T)_\tau \]
and taking invariants with respect to \( w_G(\tau) \),
\[ \left( K'_*(X^\tau, C_G(\tau))_\tau \otimes_{R(G)_\Lambda, \tau} R(T)_\Lambda, \tau \right)^{w_G(\tau)} \simeq K'_*(X^\tau, T)_\tau \quad (23) \]
Comparing (21) to (23), we are reduced to proving that for any \( \sigma \in \mathcal{C}(G) \) there is an isomorphism
\[ (K'_*(X^\sigma, C_G(\sigma))_\sigma)^{w_G(\sigma)} \otimes_{R(G)_\Lambda} R(T)_\Lambda \simeq \prod_{\tau \in \text{dual cyclic}} \left( K'_*(X^\tau, C_G(\tau))_\tau \otimes_{R(G)_\Lambda, \tau} R(T)_\Lambda, \tau \right)^{w_G(\tau)} \]
Since \( R(T)_\Lambda, \tau \) is flat over \( R(G)_\Lambda, \tau \) and \( w_G(\tau) \) acts trivially on it, we have (SGA1)
\[ \left( K'_*(X^\tau, C_G(\tau))_\tau \otimes_{R(G)_\Lambda, \tau} R(T)_\Lambda, \tau \right)^{w_G(\tau)} \simeq \left( K'_*(X^\tau, C_G(\tau))_\Lambda, \tau \right)^{w_G(\tau)} \otimes_{R(G)_\Lambda, \tau} R(T)_\Lambda, \tau, \]
By Lemma 4.3, we have isomorphisms
\[ (K'_*(X^\sigma, C_G(\sigma))_\sigma)^{w_G(\sigma)} \otimes_{R(G)_\Lambda} R(T)_\Lambda \]
Lemma 4.2; \( C \triangleq \) where:

\[
\text{T is an isomorphism. Moreover, } R(G)K(\text{recall that } g \text{ is independent on the choice of product is an isomorphism.}) \approx \\text{K}_G(X^\tau, C_G(\tau))_\tau \text{ whose restriction to invariants }
\]

\[
\left( \text{K}_G(X^\tau, C_G(\sigma))_\sigma \right)^{\omega_G(\sigma)} \approx \left( \text{K}_G(X^\tau, C_G(\tau))_\tau \right)^{\omega_G(\tau)}
\]

is independent on the choice of \( g \). Therefore we have a canonical isomorphism

\[
\left( \text{K}_G(X^\sigma, C_G(\sigma))_\sigma \right)^{\omega_G(\sigma)} \otimes_{R(G)} R(T)_\Lambda \approx \prod_{\tau \text{ dual cyclic } \scriptstyle \tau \approx S_n^\ast} \left( \text{K}_G(X^\tau, C_G(\tau))_\tau \right)^{\omega_G(\tau)} \otimes_{R(G)} R(T)_\Lambda
\]

as desired. \( \blacksquare \)

Since \( K_*(X,G) \approx K'_*(X,G) \) and \( K_*(X^\sigma, C_G(\sigma)) \approx K'_*(X^\sigma, C_G(\sigma)) \), comparing Proposition 4.5 with (4.4), we see that the proof of Theorem 4.3 can be completed by the following

**Proposition 4.6** For any \( \sigma \in \mathcal{C}(G) \), the morphism given by Lemma 2.8 and induced by the product \( C_G(\sigma) \times \sigma \rightarrow C_G(\sigma) \)

\[
\theta_{C_G(\sigma),\sigma} : \text{K}_G(X^\sigma, C_G(\sigma))_\sigma \rightarrow \text{K}_G(X^\sigma, C_G(\sigma))_{\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda
\]

is an isomorphism.

**Proof.** To simplify the notation we write \( \theta_\sigma \) for \( \theta_{C_G(\sigma),\sigma} \). As usual, we may suppose \( \sigma \) contained in \( T \). Since \( C_G(\sigma) \) is isomorphic to a product of general linear groups over \( k \), we can take \( T \) as its maximal torus and by Lemma 2.9, the canonical ring morphism

\[
\text{K}_G(X,C_G(\sigma))_\sigma \otimes_{R(G)} R(T)_\Lambda \rightarrow \text{K}_G(X,T)_\sigma
\]

is an isomorphism. Moreover, \( R(C_G(\sigma)) \rightarrow R(T) \) being faithfully flat, it is enough to prove that \( \theta_\sigma \otimes \text{id}_{R(T)} \) is an isomorphism. To prove this, let us consider the commutative diagram

\[
\begin{array}{ccc}
\text{K}_G(X^\sigma, C_G(\sigma))_\sigma \otimes_{R(G)} R(T)_\Lambda & \xrightarrow{\theta_\sigma \otimes \text{id}} & \left( \text{K} \approx \text{K}_G(X^\sigma, C_G(\sigma))_{\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda \right) \otimes_{R(G)} R(T)_\Lambda \\
\omega_\sigma & \downarrow & \gamma_\sigma \\
\text{K}_G(X^\sigma, T)_\sigma & \xrightarrow{\theta_{T,\sigma}} & \text{K}_G(X^\sigma, T)_{\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda
\end{array}
\]

where:

- \( \Delta_{C_G(\sigma)} : R(C_G(\sigma))_\Lambda \rightarrow R(C_G(\sigma))_\Lambda \otimes \tilde{R}(\sigma)_\Lambda \) (induced by the product \( C_G(\sigma) \times \sigma \rightarrow C_G(\sigma) \));

- \( \omega_\sigma \) is the canonical map induced by the inclusion \( T \hookrightarrow C_G(\sigma) \) and is an isomorphism by Lemma 4.3;

- \( \theta_{T,\sigma} \) is an isomorphism as shown in the proof of Theorem 3.7.
Applying this to $M \beta_2$ second factor $pr_2$, where we have denoted by $f : H$ using Lemma 4.2, we get a canonical isomorphism to the general fact that “an action $H \times Y \to Y$ is isomorphic over $X$ to the projection on the second factor $pr_2 : H \times Y \to Y$”, for any group scheme $H$ and any algebraic space $Y$. From (25) we get an isomorphism

\[
\alpha : \left( R(C_G(\sigma)_\Lambda \otimes \widetilde{R}(\sigma)_\Lambda \right) \otimes_{R(C_G(\sigma))_\Lambda \otimes \widetilde{R}(\sigma)_\Lambda} R(T)_\Lambda \longrightarrow R(T)_\Lambda \otimes \widetilde{R}(\sigma)_\Lambda
\]
where \( \left( R(C_G(\sigma))_\Lambda \otimes \tilde{R}(\sigma)_\Lambda \right)' \) denotes \( R(C_G(\sigma))_\Lambda \)-algebra

\[
\Delta_{C_G(\sigma)} : R(C_G(\sigma))_\Lambda \to R(C_G(\sigma))_\Lambda \otimes \tilde{R}(\sigma)_\Lambda.
\]

Therefore, if we denote by \( \left( R(T)_\Lambda \otimes \tilde{R}(\sigma)_\Lambda \right)'' \) the \( R(C_G(\sigma))_\Lambda \otimes \tilde{R}(\sigma)_\Lambda \)-algebra

\[
(res \otimes \text{id}) \circ \alpha_T : R(C_G(\sigma))_\Lambda \otimes \tilde{R}(\sigma)_\Lambda \to R(T)_\Lambda \otimes \tilde{R}(\sigma)_\Lambda,
\]

the composition

\[
\left( K'_*(X^\sigma, C_G(\sigma))_{\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda \right)' \otimes_{R(C_G(\sigma))_\Lambda} R(T)_\Lambda
\]

\[
\cong K'_*(X^\sigma, T)_{\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda
\]

is an isomorphism and it can be easily checked to coincide with \( \tilde{\gamma}_\sigma \).

\section{The main theorem: the general case}

In this Section, we use Theorem \ref{thm:main} to deduce the same result for the action of a linear algebraic \( k \)-group \( G \), having finite stabilizers, on a regular separated noetherian \( k \)-algebraic space \( X \). We will write \( \Lambda \) for \( \Lambda_{(G,X)} \).

We start with a general fact

\begin{proposition}
Let \( X \) be a regular noetherian separated \( k \)-algebraic space on which a linear algebraic \( k \)-group \( G \) acts with finite stabilizers. Then there exists an integer \( N > 0 \) such that if \( a_1, \ldots, a_N \in K_0(X,G)_{\text{geom}} \) have rank zero on each connected component of \( X \), then the multiplication by \( \prod_{i=1}^N a_i \) on \( K'_*(X,G)_{\text{geom}} \) is zero.
\end{proposition}

In particular

\begin{corollary}
Let \( X \) be a regular noetherian separated \( k \)-algebraic space with a connected action of a linear algebraic \( k \)-group \( G \) having finite stabilizers. Then the geometric localization

\[
\text{rk}_{0,\text{geom}} : K_0(X,G)_{\text{geom}} \to \Lambda
\]

of the rank morphism has a nilpotent kernel.
\end{corollary}

\textbf{Proof of Prop. 5.3.} Let us choose a closed immersion \( G \hookrightarrow \text{GL}_{n,k} \) (for some \( n > 0 \)). By Morita equivalence,

\[
K'_*(X,G) \simeq K'_*(X \times^G \text{GL}_{n,k}, \text{GL}_{n,k})
\]

and

\[
K_0(X,G) \simeq K_0(X \times^G \text{GL}_{n,k}, \text{GL}_{n,k}).
\]
Moreover, $\Lambda_{(X \times^{G} \text{GL}_{n,k}, \text{GL}_{n,k})} = \Lambda$. Let $\xi = x/s \in K^*_s (X,G)_{\text{geom}}$, with $x \in K^*_s (X,G)_\Lambda$ and $s \in \text{rk}^{-1}(1)$ where $\text{rk} : R(G) \to \Lambda$ is the rank morphism, $a_i = \alpha_i/s_i$, with $\alpha_i \in K_0(X,G)_\Lambda$ and $s_i \in \text{rk}^{-1}(1)$ for $i = 1, \ldots, N$. Let us consider the elements $x/1$ in $K^*_s (X \times^{G} \text{GL}_{n,k}, \text{GL}_{n,k})_{\text{geom}}$, $\alpha_i/1$ in $K_0(X \times^{G} \text{GL}_{n,k}, \text{GL}_{n,k})_{\text{geom}}$ for $i = 1, \ldots, N$. Since the canonical homomorphism

$$K^*_s (X \times^{G} \text{GL}_{n,k}, \text{GL}_{n,k})_{\text{geom}} \to K^*_s (X,G)_{\text{geom}}$$

is a morphism of modules over the ring morphism

$$K_0(X \times^{G} \text{GL}_{n,k}, \text{GL}_{n,k})_{\text{geom}} \to K_0(X,G)_{\text{geom}},$$

if the theorem holds for $G = \text{GL}_{n,k}$ and $N$ is the corresponding integer, the product $\prod_i \alpha_i/1$ in $K_0(X,G)_{\text{geom}}$ annihilates $x/1 \in K^*_s (X,G)_{\text{geom}}$ and a fortiori $\prod_i a_i$ annihilates $\xi$ in $K^*_s (X,G)_{\text{geom}}$. So, we may assume $G = \text{GL}_{n,k}$. Let $T$ be the maximal torus of diagonal matrices in $G$. By Lemma 4.2 (i) with $\sigma = 1$, there are isomorphisms

$$\omega_{1,\text{geom}} : K_0(X,\text{GL}_{n,k})_{\text{geom}} \otimes_{R(\text{GL}_{n,k})_{\Lambda}} R(T)_{\Lambda} \simeq K_0(X,T)_{\text{geom}},$$

$$\omega_{2,\text{geom}}: K_0(X,\text{GL}_{n,k})_{\text{geom}} \otimes_{R(\text{GL}_{n,k})_{\Lambda}} R(T)_{\Lambda} \simeq K^*_s (X,T)_{\text{geom}}.$$

Since $R(\text{GL}_{n,k}) \to R(T)$ is faithfully flat and the diagram

$$\begin{array}{ccc}
K_0(X,\text{GL}_{n,k})_{\text{geom}} \otimes_{R(\text{GL}_{n,k})_{\Lambda}} R(T)_{\Lambda} & \to & \Lambda \\
\downarrow_{\omega_{2,\text{geom}}} & & \downarrow_{\omega_{2,\text{geom}}} \\
K_0(X,T)_{\text{geom}} & \to & \Lambda
\end{array}$$

commutes, we reduce ourselves to proving the proposition for $G = T$, a split torus.

To handle this case, we proceed by noetherian induction on $X$. By [Th1], Prop. 4.10, there exists a $T$-invariant nonempty open subscheme $j : U \hookrightarrow X$, a closed diagonalizable subgroup $T'$ of $T$ and a $T$-equivariant isomorphism

$$U \simeq T/T' \times (U/T).$$

Since $U$ is nonempty and $T$ acts on $X$ with finite stabilizers, $T'$ is finite over $k$ and $K^*_s (U,T) \simeq K^*_s (U/T) \otimes_{R(T')} R(T')$, by Morita equivalence theorem ([Th3], Proposition 6.2). Let $i : Z \hookrightarrow X$ be the closed complement of $U$ in $X$ and $N'$ an integer satisfying the proposition for both $Z$ and $U$. Consider the geometric localization sequence

$$K^*_s (Z,T)_{\text{geom}} \xrightarrow{i^*} K^*_s (X,T)_{\text{geom}} \xrightarrow{j^*} K^*_s (U,T)_{\text{geom}}$$

and let $\xi \in K^*_s (X,T)_{\text{geom}}$. Let $a_1, \ldots, a_{2N'} \in K_0(X,T)_{\text{geom}}$. By our choice of $N'$,

$$j^*(a_{N'+1} \cdots a_{2N'} \cup \xi) = 0,$$

thus $a_{N'+1} \cdots a_{2N'} \cap \xi = i_*(\eta)$ for some $\eta$ in $K^*_s (Z,T)_{\text{geom}}$. By projection formula we get

$$a_1 \cdots a_{2N'} \cup \xi = i_* (i^* (a_1) \cdots i^* (a_{N'}) \cup \eta)$$

which is zero by our choice of $N'$ and by the fact that rank morphisms commutes with pullbacks. Thus, $N \doteq 2N'$ satisfies our proposition. ■
Remark 5.3 By Corollary 5.2, \( K_*(X,G)_{\text{geom}} \) is isomorphic to the localization of \( K_*(X,G)_\Lambda \) at the multiplicative subset \( (\text{rk}_0)^{-1}(1) \), where \( \text{rk}_0 : K_0(X,G)_\Lambda \to \Lambda \) is the rank morphism. Therefore, if \( X \) is regular, \( K_*(X,G)_{\text{geom}} \) depends only on the quotient stack \( [X/G] \) ([L-MB]) and not on its presentation as a quotient.

The main theorem of this paper is:

**Theorem 5.4** Let \( X \) be a noetherian regular separated algebraic space over a field \( k \) and \( G \) a linear algebraic \( k \)-group with a sufficiently rational action on \( X \) having finite stabilizers. Suppose moreover that for any essential dual cyclic \( k \)-subgroup scheme \( \sigma \subseteq G \), the quotient algebraic space \( G/C_G(\sigma) \) is smooth over \( k \) (which is the case if, e.g., \( G \) is smooth or abelian). Then \( C(G) \) is finite and the map defined in (5)

\[
\Psi_{X,G} : K_*(X,G)_\Lambda \longrightarrow \prod_{\sigma \in C(G)} \left( K_*(X^\sigma,C(\sigma))_{\text{geom}} \otimes \tilde{R}(\sigma)_\Lambda \right)^{wG(\sigma)}
\]

is an isomorphism of \( R(G) \)-algebras.

Remark 5.5 In the next subsection we will also give less restrictive hypotheses on \( G \) under which Theorem 5.4 still holds.

Note also that if \( X \) has the "G-equivariant resolution property" (i.e. any \( G \)-equivariant coherent sheaf is the \( G \)-equivariant epimorphic image of a \( G \)-equivariant locally free coherent sheaf) then in Theorem 5.4 one can replace our \( K_* \) with Quillen \( K \)-theory of \( G \)-equivariant locally free coherent sheaves. This happens, for example, if \( X \) is a scheme and \( G \) is smooth or finite ([[324]]).
Note that \( \Lambda_{(Y,\GL_{n,k})} = \Lambda \).
Consider the morphism defined in (3)
\[
\Psi_{X,G} : K_*(X,G)_\Lambda \rightarrow \prod_{\sigma \in C(G)} \left( K_*(X^\sigma, C_G(\sigma))_{\text{geom}} \otimes \widetilde{R}(\sigma)_\Lambda \right)^{w_G(\sigma)} \; ;
\]
By Theorem 4.1, the map
\[
\Psi_{Y,\GL_{n,k}} : K_*(Y,\GL_{n,k})_\Lambda \rightarrow \prod_{\rho \in C(\GL_{n,k})} \left( K_*(Y^\rho, C_{\GL_{n,k}}(\rho))_{\text{geom}} \otimes \widetilde{R}(\rho)_\Lambda \right)^{w_{\GL_{n,k}}(\rho)}
\]
is an isomorphism and by Morita equivalence theorem (\[\text{Th3}\], Proposition 6.2) \( K_*(Y,\GL_{n,k})_\Lambda \simeq K_*(X,G)_\Lambda \). We will prove the theorem by constructing an isomorphism
\[
\prod_{\rho \in C(\GL_{n,k})} \left( K_*(Y^\rho, C(\rho))_{\text{geom}} \otimes \widetilde{R}(\rho)_\Lambda \right)^{w_{\GL_{n,k}}(\rho)} \rightarrow \prod_{\sigma \in C(G)} \left( K_*(X^\sigma, C_G(\sigma))_{\text{geom}} \otimes \widetilde{R}(\sigma)_\Lambda \right)^{w_G(\sigma)}
\]
commuting with the \( \Psi \)'s and Morita isomorphisms.

Let \( \alpha : C(G) \rightarrow C(\GL_{n,k}) \) be the natural map. If \( Y^\rho \neq \emptyset \), there exists a dual cyclic subgroup \( \sigma \subseteq G \), \( \GL_{n,k} \)-conjugate to \( \rho \) (and \( X^\sigma \neq \emptyset \)); therefore \( Y^\rho = \emptyset \) unless \( \rho \in \text{im}(\alpha) \) and we may restrict the first product in (28) to those \( \rho \) in the image of \( \alpha \) and suppose \( \text{im}(\alpha) \subseteq C(G) \) as well. The following proposition describes the \( Y^\rho \)'s which appear:

**Proposition 5.6** Let \( X \) be a noetherian regular separated algebraic space over a field \( k \) and \( G \) a linear algebraic \( k \)-group with a sufficiently rational action on \( X \) having finite stabilizers. Suppose moreover that for any essential dual cyclic \( k \)-subgroup scheme \( \sigma \subseteq G \), the quotient algebraic space \( G/C_G(\sigma) \) is smooth over \( k \). Let \( G \hookrightarrow \GL_{n,k} \) a closed embedding and \( \rho \in \text{im}(\alpha) \) an essential dual cyclic subgroup and \( Y = \GL_{n,k}^G X \) the algebraic space quotient for the left diagonal action of \( G \). If \( C_{\GL_{n,k},G}(\rho) \subseteq C(G) \) denotes the fiber \( \alpha^{-1}(\rho) \), then:

(i) choosing for each \( \sigma \in C_{\GL_{n,k},G}(\rho) \) an element \( u_{\rho,\sigma} \in \GL_{n,k}(k) \) such that \( u_{\rho,\sigma} \sigma u_{\rho,\sigma}^{-1} = \rho \) (in the obvious functor-theoretic sense), determines a unique isomorphism of algebraic spaces over \( k \)

\[
j_\rho : \prod_{\sigma \in C_{\GL_{n,k},G}(\rho)} N_{\GL_{n,k}}(\sigma) \times_{N_G(\sigma)} X^\sigma \rightarrow Y^\rho;
\]

(ii) \( C_{\GL_{n,k},G}(\rho) \) is finite.

**Proof.** (ii) follows from (i) since \( Y^\rho \) is quasi-compact. The proof of (i) requires several steps.

(a) **Definition of \( j_\rho \).** If \( \sigma \in C_{\GL_{n,k},G}(\rho) \), let \( N_\sigma \) be the presheaf on the category \( \text{Sch}_k \) of \( k \)-schemes which associates to \( T \rightarrow \text{Spec}k \) the set

\[
N_\sigma(T) = \frac{N_{\GL_{n,k}}(\sigma)(T) \times X^\sigma(T)}{N_G(\sigma)(T)};
\]

since \( N_G(\sigma) \) acts freely on \( N_{\GL_{n,k}}(\sigma) \times X^\sigma \) (on the left), the flat sheaf associated to \( N_\sigma \) is \( N_{\GL_{n,k}}(\sigma) \times_{N_G(\sigma)} X^\sigma \). Let \( \tilde{Y}^\rho \) be the presheaf on \( \text{Sch}_k \) which associates to \( T \rightarrow \text{Spec}k \) the set

\[
\tilde{Y}^\rho(T) = \left\{ [A,x] \in \frac{\GL_{n,k}(T) \times X(T)}{G(T)} \mid \forall T' \rightarrow T, \forall r \in \rho(T'), [r A_T', x_{T'}] = [A_{T'}, x_{T'}] \right\} ;
\]
the flat sheaf associated to \( \hat{Y}^\rho \) is \( Y^\rho \) (e.g. [DG], II, §1, n. 3). If \( u_{\rho,\sigma} \in GL_{n,k}(k) \) is such that 
\[ u_{\rho,\sigma}\sigma u_{\rho,\sigma}^{-1} = \rho \]
(in the obvious functor-theoretic sense), the presheaf map

\[ \hat{j}_{\rho,\sigma} : N_{\sigma} \to \hat{Y}^\rho \]
\[ \hat{j}_{\rho,\sigma}(T) : N_{\sigma}(T) \ni [B,x] \mapsto [u_{\rho,\sigma}B,x] \in \hat{Y}^\rho (T) \]
is easily checked to be well-defined. Let \( j_{\rho,\sigma} : N_{GL_{n,k}(\sigma)} \times^{N_G(\sigma)} X^\sigma \to Y^\rho \) denote the associated sheaf map and define \( r_{A,T} g^{-1} = A_T \)
\[ gx_T = x_T. \]

Then, \( A^{-1} \rho A \) defines (functorially over \( T_0 \)) a dual cyclic subgroup scheme \( \sigma'_0 \) of \( G(T_0) \) over \( T_0 \).
Since \( \sigma'_0 \) is isomorphic to some \( \mu_n.T_0 \), it descends to a dual cyclic subgroup \( \sigma' \) of \( G \) over \( k \) which is \( GL_{n,k} \)-conjugate to \( \rho \) since \( T_0 \to \Spec\Omega \) has a section and \( GL_{n,k} \) satisfies our rationality condition (RC) (see Remark 2.2 (i)). By definition of \( C_{GL_{n,k},G}(\rho) \) there exists a unique \( \sigma \in C_{GL_{n,k},G}(\rho) \) which is \( G \)-conjugated to \( \sigma' \) over \( k \) i.e.
\[ g\sigma'g^{-1} = \sigma \]
(functorially) for some \( g \in G(k) \). Since \( \sigma \in C_{GL_{n,k},G}(\rho) \), there is an element \( u \in GL_{n,k}(k) \) such that \( u\sigma u^{-1} = \rho \). Therefore \( u^{-1}Ag^{-1} \) restricted to \( T_0 \) is in \( N_{GL_{n,k}(\sigma)}(T_0) \), \( gx \in X^\sigma(T_0) \) and if 
\[ [u^{-1}Ag^{-1}, gx] \sim \]
denotes the element in \( (N_{GL_{n,k}(\sigma)} \times^{N_G(\sigma)} X^\sigma)(\Omega) \) represented by the element \( [u^{-1}Ag^{-1}, gx] \) in \( N_{\sigma}(T_0) \), we have \( j_{\rho,\sigma}(\Omega) ([u^{-1}Ag^{-1}, gx]) = \xi \) by definition of \( j_{\rho,\sigma} \). Thus \( j_{\rho}(\Omega) \) is surjective.

Now, let \( \eta \in (N_{GL_{n,k}(\sigma)} \times^{N_G(\sigma)} X^\sigma)(\Omega) \) (respectively, \( \eta' \in (N_{GL_{n,k}(\sigma')} \times^{N_G(\sigma')} X^{\sigma'})(\Omega) \)) for \( \sigma \) and \( \sigma' \) in \( C_{GL_{n,k},G}(\rho) \). Choosing a common refinement, we can assume there exists a \( fppf \) cover \( T_0 \to \Spec\Omega \) such that \( \eta \) (resp. \( \eta' \)) is represented by an element \( [B,y] \in N_{\sigma}(T_0) \) (resp. \( [B',y'] \in N_{\sigma'}(T_0) \)). If \( j_{\rho}(\Omega)(\eta) = j_{\rho}(\Omega)(\eta') \), there exists a \( fppf \) cover \( T_1 \to T_0 \) such that 
\[ [u_{\rho,\sigma}B,y] = [u_{\rho,\sigma}B',y'] \]
in \( GL_{n,k}(T_1) \times X(T_1)/G(T_1) \) i.e. there is an element \( g \in G(T_1) \) such that
\[ u_{\rho,\sigma}Bg^{-1} = u_{\rho,\sigma}B' \]
in \( GL_{n,k}(T_1) \)
\[ gy = y' \]
in \( X(T_1) \).

Then it is easy to check that \( \sigma = g^{-1}\sigma'g \) over \( T_1 \) and, as in the proof of surjectivity of \( j_{\rho}(\Omega) \), since \( T_1 \to \Spec\Omega \) has a section and \( G \) satisfies our rationality condition (RC), \( \sigma \) and \( \sigma' \) are \( G \)-conjugated over \( k \) as well and therefore \( \sigma = \sigma' \) as elements in \( C_{GL_{n,k},G}(\rho) \). In particular, 
\( g \in N_G(\sigma)(T_1) \) and \( [B,y] = [B',y'] \in N_{\sigma}(T_1) \). Since \( T_1 \to \Spec\Omega \) is still a \( fppf \) cover, we have \( \eta = \eta' \) and \( j_{\rho}(\Omega) \) is injective.

(c) Each \( j_{\rho,\sigma} \) is a closed and open immersion. It is enough to show that each \( j_{\rho,\sigma} \) is an open immersion because in this case it is also a closed immersion, \( Y^\rho \) being quasi-compact. Since \( N_{GL_{n,k}(\rho)} \) acts on both \( \prod_{\sigma \in C_{GL_{n,k},G}(\rho)} N_{GL_{n,k}(\sigma)} \times^{N_G(\sigma)} X^\sigma \) and \( Y^\rho \) and \( j_{\rho,\rho} \) is equivariant, it will be enough to prove that \( j_{\rho,\rho} \) is an open immersion. We will prove first that \( j_{\rho,\rho} \) is injective and unramified and then conclude the proof by showing that it is also flat (in fact, an étale injective map is an open immersion).
(c₁) \( j_{ρ,ρ} \) is injective and unramified. It is enough to show that the inverse image under \( j_{ρ,ρ} \) of a geometric point is a (geometric) point. Consider the commutative diagram

\[
\begin{array}{ccc}
N_{GL_{n,k}} (ρ) \times X^ρ & \xrightarrow{l} & GL_{n,k} \times X \\
p \downarrow & & \downarrow π \\
N_{GL_{n,k}} (ρ) \times N_G(ρ) X^ρ & \xrightarrow{i_ρ \circ j_{ρ,ρ}} & Y
\end{array}
\]

where \( l \) and \( i_ρ : Y^ρ \hookrightarrow Y \) are the natural inclusions and \( p, π \) the natural projections. Let \( y_0 \) be a geometric point of \( Y \) in the image of \( i_ρ \circ j_{ρ,ρ} \); using the action of \( N_{GL_{n,k}} (ρ) \) on \( N_{GL_{n,k}} (ρ) \times N_G(ρ) X^ρ \) and \( Y^ρ \), we may suppose that \( y_0 \) is of the form \([1, x_0] ∈ Y^ρ(Ω), \) with \( Ω \) an algebraically closed field and \( x_0 ∈ X^ρ(Ω) \). Obviously, \((1, x_0) ∈ N_{GL_{n,k}} (ρ) \times N_G(ρ) X^ρ(Ω)\) is contained in \( j_{ρ,ρ}^{-1}(y_0) \) and, by faithful flatness of \( p \), \( j_{ρ,ρ}^{-1}(y_0) = (1, x_0) \) if

\[
p^{-1} \left( (1, x_0) \right) = \pi^{-1} (y_0) \cap \left( N_{GL_{n,k}} (ρ) \times N_G(ρ) X^ρ \right).
\]

But \( G(Ω) \simeq π^{-1} (y_0) \) via \( g ↦ g^{-1} (g, x_0) \) and \( N_G(ρ)(Ω) \simeq p^{-1} \left( (1, x_0) \right) \) via \( h ↦ (h^{-1}, hx_0) \), therefore (29) follows from \( N_G(ρ) = N_{GL_{n,k}} (ρ) \cap G \).

(c₂) \( j_{ρ,ρ} \) is flat. This fact is proved in the next subsection where we also single out a more general technical hypothesis for the action of \( G \) on \( X \) under which Proposition 5.6 still holds.

The remaining part of this subsection will be devoted to conclude the proof of Theorem 5.4 using Proposition 5.6 (ii) allows one to define a canonical isomorphism

\[
\prod_{ρ ∈ C(GL_{n,k})} \left( K_*(Y^ρ, C_{GL_{n,k}} (ρ))_{geom} \otimes \tilde{R}(ρ)_Λ \right)^{wGL_{n,k}(ρ)} \simeq \\
\prod_{σ ∈ C(G)} \left( K_*(N_{GL_{n,k}} (σ) \times N_G(σ) X^σ, C_{GL_{n,k}} (σ))_{geom} \otimes \tilde{R}(σ)_Λ \right)^{wGL_{n,k}(σ)}
\]

next we show, using Lemma 2.10, that each factor in the r.h.s. is isomorphic to

\[
(K_*(C_{GL_{n,k}} (σ) \times C_G(σ) X^σ, C_{GL_{n,k}} (σ))_{geom} \otimes \tilde{R}(σ)_Λ)^{wG(σ)}.
\]

The conclusion (i.e. the isomorphism (28)) is then accomplished by establishing, for any regular noetherian separated algebraic space \( Z \) on which \( G \) acts with finite stabilizers, a "geometric" Morita equivalence

\[
K_*(GL_{n,k} \times^G Z, GL_{n,k})_{geom} \simeq K_*(Z, G)_{geom}.
\]

First of all, note that the choice of a family \( \{ u_{ρ,σ} | σ ∈ C_{GL_{n,k},G}(ρ) \} \) of elements \( u_{ρ,σ} ∈ GL_{n,k}(k) \) such that \( u_{ρ,σ} u_{ρ,σ}^{-1} = ρ \), which uniquely defines \( j_ρ \) in Proposition 5.6, also determines a unique family of isomorphisms

\[
\{ \text{int}(u_{ρ,σ}) : C_{GL_{n,k}} (ρ) → C_{GL_{n,k}} (σ) | σ ∈ C_{GL_{n,k},G}(ρ) \}
\]

(where \( \text{int}(u_{ρ,σ}) \) denotes conjugation by \( u_{ρ,σ} \)) and this family gives us an action of \( C_{GL_{n,k}} (ρ) \) on

\[
\prod_{σ ∈ C_{GL_{n,k},G}(ρ)} N_{GL_{n,k}} (σ) \times N_G(σ) X^σ
\]
(since $N_{GL_{n,k}}(\sigma)$, and then $C_{GL_{n,k}}(\sigma)$, acts naturally on $N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma$ by left multiplication on $N_{GL_{n,k}}(\sigma)$). With this action, $j_\rho$ becomes a $C_{GL_{n,k}}(\rho)$-equivariant isomorphism and since $\text{int}(u_{\rho,\sigma})$ induces an isomorphism $R(C_{GL_{n,k}}(\rho)) \simeq R(C_{GL_{n,k}}(\sigma))$ commuting with rank morphisms, $j_\sigma$ induces an isomorphism

$$K_*(Y^\rho, C_{GL_{n,k}}(\rho))_{\text{geom}} \otimes \overline{R}(\rho)_\Lambda \simeq \prod_{\sigma \in C_{GL_{n,k},G}(\rho)} K_*(N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma, C_{GL_{n,k}}(\sigma))_{\text{geom}} \otimes \overline{R}(\sigma)_\Lambda$$

which, by definition of the action of $N_{GL_{n,k}}(\rho)$ on each $N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma$, induces an isomorphism

$$\left( K_*(Y^\rho, C_{GL_{n,k}}(\rho))_{\text{geom}} \otimes \overline{R}(\rho)_\Lambda \right)^w_{GL_{n,k}(\rho)} \simeq \prod_{\sigma \in C_{GL_{n,k},G}(\rho)} \left( K_*(N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma, C_{GL_{n,k}}(\sigma))_{\text{geom}} \otimes \overline{R}(\sigma)_\Lambda \right)^w_{GL_{n,k}(\sigma)}$$

(30)

Now, if $j'_\rho$ is induced, as in Proposition 5.6, by another choice of a family $\{v_{\rho,\sigma} | \sigma \in C_{GL_{n,k},G}(\rho)\}$ of elements $v_{\rho,\sigma} \in GL_{n,k}(k)$ such that $v_{\rho,\sigma} v_{\rho,\sigma}^{-1} = \rho$, $v_{\rho,\sigma}^{-1} u_{\rho,\sigma} \in N_{GL_{n,k}}(\sigma)(k)$ and there is a commutative diagram

$$\begin{array}{ccc}
N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma & \xrightarrow{(v_{\rho,\sigma}^{-1} u_{\rho,\sigma})} & N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma \\
{\rho} \downarrow j_\rho & & \downarrow j'_\rho \\
Y^\rho & & Y^\rho
\end{array}$$

Therefore, the isomorphism (30) on the invariants is actually independent on the choice of the family $\{u_{\rho,\sigma} | \sigma \in C_{GL_{n,k},G}(\rho)\}$. Since $C_{GL_{n,k},G}(\rho) = \alpha^{-1}(\rho)$ and, as already observed, $Y^\rho = \emptyset$ unless $\rho \in \text{im}(\alpha)$, this gives us a canonical isomorphism

$$\prod_{\rho \in C(G)_{GL_{n,k}}} \left( K_*(Y^\rho, C_{GL_{n,k}}(\rho))_{\text{geom}} \otimes \overline{R}(\rho)_\Lambda \right)^w_{GL_{n,k}(\rho)} \simeq \prod_{\sigma \in C(G)} \left( K_*(N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma, C_{GL_{n,k}}(\sigma))_{\text{geom}} \otimes \overline{R}(\sigma)_\Lambda \right)^w_{GL_{n,k}(\sigma)}$$

Now, let us fix $\sigma \in C(G)$ and let us choose a set $A \subset N_{GL_{n,k}}(\sigma)(k)$ such that the classes in $w_{GL_{n,k}}(\sigma)$ of the elements in $A$ constitute a set of representatives for the $w_G(\sigma)$-orbits in $w_{GL_{n,k}}(\sigma)$; $A$ is a finite set. Since

$$C_{GL_{n,k}}(\sigma) \times^{C_G(\sigma)} X^\sigma \hookrightarrow N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma$$

is an open and closed immersion, the morphism

$$\prod_{A} C_{GL_{n,k}}(\sigma) \times^{C_G(\sigma)} X^\sigma \longrightarrow N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma$$

$$[C, x]_{A_i \in A} \longmapsto [A_i \cdot C, x]$$

(in the obvious functor-theoretic sense), which is easily checked to induce an isomorphism on geometric points, is an isomorphism. Therefore, there is an induced isomorphism

$$\prod_{A} C_{GL_{n,k}}(\sigma) \times^{C_G(\sigma)} X^\sigma, C_{GL_{n,k}}(\sigma)\}_{\text{geom}} \otimes \overline{R}(\sigma)_\Lambda$$

$$\simeq K_*(N_{GL_{n,k}}(\sigma) \times^{N_G(\sigma)} X^\sigma, C_{GL_{n,k}}(\sigma))_{\text{geom}} \otimes \overline{R}(\sigma)_\Lambda$$

32
Since \( w_{\text{GL}_{n,k}} (\sigma) \) acts transitively on \( \mathcal{A} \) with stabilizer \( w_{G} (\sigma) \), by Lemma 2.10, we get a canonical isomorphism:

\[
\left( K_* \left( N_{\text{GL}_{n,k}} (\sigma) \times N_G (\sigma) X^\sigma, C_{\text{GL}_{n,k}} (\sigma) \right)_{\text{geom}} \otimes \tilde{R} (\sigma)_\Lambda \right)^{w_{\text{GL}_{n,k}} (\sigma)} \\
\cong \left( K_* \left( C_{\text{GL}_{n,k}} (\sigma) \times C_G (\sigma) X^\sigma, C_{\text{GL}_{n,k}} (\sigma) \right)_{\text{geom}} \otimes \tilde{R} (\sigma)_\Lambda \right)^{w_{G} (\sigma)}.
\]

Since by Morita equivalence (\text{Th3}, Proposition 6.2),

\[
K_* \left( C_{\text{GL}_{n,k}} (\sigma) \times C_G (\sigma) X^\sigma, C_{\text{GL}_{n,k}} (\sigma) \right) \cong K_* \left( X^\sigma, C_{G} (\sigma) \right),
\]

(31)

to conclude the proof of Theorem 5.4, we need only to show that the natural morphism

\[
K_* \left( C_{\text{GL}_{n,k}} (\sigma) \times C_G (\sigma) X^\sigma, C_{\text{GL}_{n,k}} (\sigma) \right)_{\text{geom}} \cong K_* \left( X^\sigma, C_{G} (\sigma) \right)_{\text{geom}}
\]

(32)

induced by (31) is still an isomorphism. Since the diagram

\[
K_* \left( \text{GL}_{n,k} \times C_G (\sigma) X^\sigma, \text{GL}_{n,k} \right)_{\text{geom}} \xrightarrow{\alpha} K_* \left( C_{\text{GL}_{n,k}} (\sigma) \times C_G (\sigma) X^\sigma, C_{\text{GL}_{n,k}} (\sigma) \right)_{\text{geom}} \\
\xrightarrow{\gamma} \text{Morita equivalence}
\]

is commutative and, by Morita equivalence

\[
K_* \left( C_{\text{GL}_{n,k}} (\sigma) \times C_G (\sigma) X^\sigma, C_{\text{GL}_{n,k}} (\sigma) \right) \cong K_* \left( \text{GL}_{n,k} \times C_{\text{GL}_{n,k}} (\sigma) \times C_G (\sigma) X^\sigma, \text{GL}_{n,k} \right) \\
\cong K_* \left( \text{GL}_{n,k} \times C_{\text{GL}_{n,k}} (\sigma) X^\sigma, \text{GL}_{n,k} \right),
\]

to show \( \beta \) is an isomorphism it is enough to prove that for any regular separated algebraic space \( Z \) on which \( G \) acts with finite stabilizers, Morita equivalence induces an isomorphism

\[
K_* \left( \text{GL}_{n,k} \times Z, \text{GL}_{n,k} \right)_{\text{geom}} \cong K_* \left( Z, G \right)_{\text{geom}},
\]

(33)

since in this case both \( \alpha \) and \( \gamma \) are isomorphisms.

Let \( \pi : R (\text{GL}_{n,k}) \to R (G) \) is the restriction morphism, \( \rho : R (G) \to K_0 (Z, G) \) the pullback along \( Z \to \text{Spec} k \), \( \text{rk}' : R (\text{GL}_{n,k}) \to \Lambda \) and \( \text{rk} : R (G) \to \Lambda \) the rank morphisms, \( S' \doteq (\text{rk}')^{-1} (1), S \doteq (\text{rk})^{-1} (1) \) and \( T \doteq \pi (S') \subseteq S \); the following diagram commutes

\[
\begin{array}{ccc}
T^{-1} K_0 (Z, G)_\Lambda & \xrightarrow{\text{rk}_0 T} & \Lambda \\
\downarrow & \nearrow \text{rk}_{\text{geom}} \\
K_0 (Z, G)_{\text{geom}}
\end{array}
\]

where \( \text{rk}_{\text{geom}} \) and \( \text{rk}_0 T \) denote the localizations of the rank morphism \( \text{rk}_0 : K_0 (Z, G)_\Lambda \to \Lambda \). By Morita equivalence the natural map (which commutes with the induced rank morphisms)

\[
K_0 \left( \text{GL}_{n,k} \times^G Z, \text{GL}_{n,k} \right)_{\text{geom}} \to T^{-1} K_0 (Z, G)_\Lambda
\]

is an isomorphism and then, by Proposition 5.1, \( \ker (\text{rk}_{0, T} : T^{-1} K_0 (Z, G)_\Lambda \to \Lambda) \) is nilpotent. Now, if \( s \in S \), \( \text{rk}_{0, T} (\rho (s) / 1) = \text{rk} (s) = 1 \) and therefore

\[
T^{-1} K_0 (Z, G)_\Lambda \to K_0 (Z, G)_{\text{geom}}
\]
and
\[ K_0(\GL_{n,k} \times^G \Z, \GL_{n,k})_{\text{geom}} \to K_0(\Z, \GL_{n,k})_{\text{geom}} \]
are both isomorphisms. Since \( K_*(\GL_{n,k} \times^G \Z, \GL_{n,k})_\Lambda \) is naturally a \( K_0(\GL_{n,k} \times^G \Z, \GL_{n,k})_\Lambda \)-module and an \( R(\GL_{n,k})_\Lambda \)-module via the pullback ring morphism \( \rho' : R(\GL_{n,k})_\Lambda \to K_0(\GL_{n,k} \times^G \Z, \GL_{n,k})_\Lambda \), we have:
\[
K'_*(\GL_{n,k} \times^G \Z, \GL_{n,k})_{\text{geom}} \\
\cong K'_*(\GL_{n,k} \times^G \Z, \GL_{n,k})_\Lambda \otimes K_0(\GL_{n,k} \times^G \Z, \GL_{n,k})_\Lambda \\
K_0(\GL_{n,k} \times^G \Z, \GL_{n,k})_{\text{geom}} \\
\cong K'_*(\Z, \GL_{n,k})_\Lambda \otimes K_0(\GL_{n,k} \times^G \Z, \GL_{n,k})_\Lambda \\
K_0(\GL_{n,k} \times^G \Z, \GL_{n,k})_{\text{geom}} \\
\cong K'_*(\Z, \GL_{n,k})_\Lambda \otimes K_0(\GL_{n,k} \times^G \Z, \GL_{n,k})_\Lambda \\
K_0(\Z, \GL_{n,k})_{\text{geom}}
\]
which proves (32) and conclude the proof of theorem 5.4.

5.2 Hypotheses on \( G \)

In this subsection we conclude the proof of Proposition 5.6, showing that (this is part (c) of the proof)
\[
j_{\rho, \rho} : N_{\GL_{n,k}}(\rho) \times_{N_G(\rho)} X^\rho \to Y^\rho
\]
is flat. This is the only step in the proof of Proposition 5.6 where we make use of the hypothesis that the quotient algebraic space \( G/C_G(\rho) \) is smooth over \( k \). Actually, our proof will work under the following weaker hypothesis. Let \( S \) denote the spectrum of the dual numbers over \( k \)
\[ S = \text{Spec}(k[\varepsilon]) \]
and for any \( k \)-group scheme \( H \), we let \( \Pi^1(S, H) \) denote the \( k \)-vector space of isomorphism classes of pairs \((P \to S, y)\) where \( P \to S \) is an \( H \)-torsor and \( y \) is a \( k \)-rational point on the closed fiber of \( P \). Then Proposition 5.6, and hence Theorem 5.4, still holds with hypothesis (S) "for any essential dual cyclic subgroupscheme \( \sigma \subseteq G \), the quotient \( G/C_G(\sigma) \) is smooth" over \( k \)" replaced by the following:
(S') "for any essential dual cyclic \( k \)-subgroup scheme \( \sigma \subseteq G \), we have
\[
\dim \Pi^1(S, C_G(\sigma)) = \dim(\Pi^1(S, G))^\sigma
\]

First we will prove that \( j_{\rho, \rho} \) is flat assuming (S'). Since \( p : N_{\GL_{n,k}}(\rho) \times X^\rho \to N_{\GL_{n,k}}(\rho) \times_{N_G(\rho)} X^\rho \) is faithfully flat, it is enough to prove that \( j_{\rho, \rho} = j_{\rho, \rho} \circ p \) is flat. Let \( \pi : \GL_{n,k} \times X \to Y \) be the projection and
\[
f : \GL_{n,k} \times X \times G \to \GL_{n,k} \times X
\]
\[
(A, x, g) \mapsto (Ag^{-1}, gx).
\]
Consider the following cartesian squares
\[
\begin{array}{ccc}
U & \xrightarrow{u} & \pi^{-1}(Y^\rho) \hookrightarrow \GL_{n,k} \times X \\
\downarrow & & \downarrow \pi \\
N_{\GL_{n,k}}(\rho) \times X^\rho & \xrightarrow{\iota} & Y^\rho \hookrightarrow Y
\end{array}
\]
Since \( \pi \) is faithfully flat, it is enough to prove that \( u_\rho \) is flat. But the squares

\[
\begin{array}{ccc}
U & \leftrightarrow & \text{GL}_{n,k} \times X \times G \\
\downarrow & & \downarrow f \\
N_{\text{GL}_{n,k}}(\rho) \times X^\rho & \leftrightarrow & \text{GL}_{n,k} \times X \\
\text{pr}_{12} \downarrow & & \downarrow \pi \\
& & \text{pr}_1 \downarrow \\
\end{array}
\]

are cartesian and (in the obvious functor-theoretic sense)

\[
U = \{(A, x, g) \in \text{GL}_{n,k} \times X \times G \mid A^{-1} \rho A = \rho, \ x \in X^\rho \} \simeq N_{\text{GL}_{n,k}}(\rho) \times X^\rho \times G.
\]

Moreover, if \( P \equiv \{A \in \text{GL}_{n,k} \mid A^{-1} \rho A \subseteq G\} \), the map

\[
\pi^{-1}(Y^\rho) = \{(A, x) \in N_{\text{GL}_{n,k}}(\rho) \times X \mid A^{-1} \rho A \subseteq G, \ x \in X A^{-1} \rho A\} \longrightarrow P \times X^\rho
\]

\[
(A, x) \longmapsto (A, Ax)
\]

is an isomorphism. Therefore, we are reduced to prove that the map

\[
v_\rho : N_{\text{GL}_{n,k}}(\rho) \times X^\rho \times G \longrightarrow P \times X^\rho
\]

\[
(A, x, g) \longmapsto (Ag^{-1}, Ax)
\]

is flat. But, since the diagram

\[
\begin{array}{ccc}
N_{\text{GL}_{n,k}}(\rho) \times X^\rho \times G & \xrightarrow{v_\rho} & P \times X^\rho \\
\text{pr}_{13} \downarrow & & \downarrow \text{pr}_1 \\
N_{\text{GL}_{n,k}}(\rho) \times G & \xrightarrow{\Theta_\rho} & P
\end{array}
\]

where \( \Theta_\rho(A, g) = (Ag^{-1}) \), is easily checked to be cartesian, it is enough to show that \( f_\rho \) is flat.

To do this, let us observe that \( \rho \) acts by conjugation on \( \text{GL}_{n,k}/G \) (quotient by the \( G \)-action on \( \text{GL}_{n,k} \) by right multiplication) and we have a cartesian diagram

\[
\begin{array}{ccc}
P & \xleftarrow{\tau} & \text{GL}_{n,k} \\
\downarrow & & \downarrow \\
(\text{GL}_{n,k}/G)^\rho & \xleftarrow{\chi_\rho} & (\text{GL}_{n,k}/G)^\rho
\end{array}
\]

then \( \tau \) is a \( G \)-torsor and \( \Theta_\rho \) is \( G \)-equivariant. Thus, the commutative diagram in which the vertical arrows are \( G \)-torsors

\[
\begin{array}{ccc}
N_{\text{GL}_{n,k}}(\rho) \times G & \xrightarrow{\Theta_\rho} & P \\
\text{pr}_1 \downarrow & & \downarrow \tau \\
N_{\text{GL}_{n,k}}(\rho) & \xrightarrow{\chi_\rho} & (\text{GL}_{n,k}/G)^\rho
\end{array}
\]

where \( \chi_\rho(A) = [A] \in \text{GL}_{n,k}/G \), is cartesian and we may reduce ourselves to prove that \( \chi_\rho \) is flat. Now observe that \( N_{\text{GL}_{n,k}}(\rho) \) acts on the left of both \( N_{\text{GL}_{n,k}}(\rho) \) and \( (\text{GL}_{n,k}/G)^\rho \) in such a way that \( \chi_\rho \) is \( N_{\text{GL}_{n,k}}(\rho) \)-equivariant. Therefore it is enough to prove that \( \chi_\rho \) is flat when restricted to the connected component of the identity in \( N_{\text{GL}_{n,k}}(\rho) \) i.e. that the map

\[
\chi'_\rho : C_{\text{GL}_{n,k}}(\rho) \longrightarrow (\text{GL}_{n,k}/G)^\rho
\]

is flat. Now, \( C_{\text{GL}_{n,k}}(\rho) = (\text{GL}_{n,k})^\rho \), where \( \rho \) acts by conjugation, and both \( (\text{GL}_{n,k})^\rho \) and \( (\text{GL}_{n,k}/G)^\rho \) are smooth by [TH5] Prop. 3.1 (since \( \text{GL}_{n,k} \) and \( \text{GL}_{n,k}/G \) are smooth) and each fiber of \( \chi'_\rho \) has dimension equal to \( \dim (C_{G}(\rho)) \) because \( \chi'_\rho \) is \( C_{\text{GL}_{n,k}}(\rho) \)-equivariant for the
natural actions and all the fibers are obtained from \((\chi'_\rho)^{-1}([1]) = C_G(\rho)\) by the \(C_{GL_{n,k}}(\rho)\)-action. Therefore, \(\chi'_\rho\) is flat if
\[
\dim(C_{GL_{n,k}}(\rho)) = \dim(C_G(\rho)) + \dim((GL_{n,k}/G)^\rho).
\] (36)

Note that, in any case,
\[
\dim(C_{GL_{n,k}}(\rho)) \leq \dim(C_G(\rho)) + \dim((GL_{n,k}/G)^\rho).
\] (37)

Since \(GL_{n,k}\) is smooth, \(\dim((GL_{n,k}/G)^\rho) = \dim_k(T_1(GL_{n,k}/G)^\rho)\), where \(T_1\) denotes the tangent space at the class of \(1 \in GL_{n,k}\). Moreover, since \(\text{H}^1(S,GL_{n,k}) = 0\), there is an exact sequence of \(k\)-vector spaces
\[
0 \rightarrow \text{Lie}(G) \rightarrow \text{Lie}(GL_{n,k}) \rightarrow T_1(GL_{n,k}/G) \rightarrow \text{H}^1(S,G) \rightarrow 0
\] (38)

which, \(\rho\) being linearly reductive over \(k\), yields an exact sequence of \(\rho\)-invariants
\[
0 \rightarrow \text{Lie}(G)^\rho \rightarrow \text{Lie}(GL_{n,k})^\rho \rightarrow T_1(GL_{n,k}/G)^\rho \rightarrow \text{H}^1(S,G)^\rho \rightarrow 0
\] (39)

But \(GL_{n,k}\) is smooth, so
\[
\dim_k(\text{Lie}(GL_{n,k})^\rho) = \dim(GL_{n,k})^\rho = \dim(C_{GL_{n,k}}(\rho))
\]
and, since \(\text{Lie}(C_G(\rho)) = \text{Lie}(G) \cap \text{Lie}(C_{GL_{n,k}}(\rho))\), we get
\[
\dim_k(\text{Lie}(G)^\rho) = \dim_k(\text{Lie}(C_G(\rho))).\]

By (38), we get
\[
\dim_k(\text{H}^1(S,G)^\rho) = \dim_k(T_1(GL_{n,k}/G)^\rho) - \dim(C_{GL_{n,k}}(\rho)) + \dim_k(\text{Lie}(C_G(\rho)) = \\
\dim((GL_{n,k}/G)^\rho) - \dim(C_{GL_{n,k}}(\rho)) + \dim(C_G(\rho)) + \dim_k(\text{Lie}(C_G(\rho)) - \dim(C_G(\rho));
\] (40)

hence (38) is satisfied if
\[
\dim_k(\text{H}^1(S,G)^\rho) = \dim_k(\text{Lie}(C_G(\rho)) - \dim(C_G(\rho)).
\] (41)

But
\[
\dim_k(\text{H}^1(S,C_G(\rho))) = \dim_k(\text{Lie}(C_G(\rho)) - \dim(C_G(\rho)),
\]

by the exact sequence (analogous to (38) with \(G\) replaced by \(C_G(\rho)\))
\[
0 \rightarrow \text{Lie}(C_G(\rho)) \rightarrow \text{Lie}(GL_{n,k}) \rightarrow T_1(GL_{n,k}/C_G(\rho)) \rightarrow \text{H}^1(S,C_G(\rho)) \rightarrow 0
\]

hence (38) holds by hypothesis \((S')\).

We complete the proof of Proposition 5.6 showing that \((S)\) implies \((S')\). Since \(C_G(\rho) \subseteq G\), we have a natural map
\[
\epsilon : \text{H}^1(S,C_G(\rho)) \rightarrow \text{H}^1(S,G)^\rho
\]
and by (40) and (37), we get
\[
\dim_k(\text{H}^1(S,G)^\rho) \geq \dim_k(\text{H}^1(S,C_G(\rho))).
\] (42)

Now, if \((S)\) holds, i.e. if \(G/C_G(\rho)\) is smooth, and \([P \rightarrow S, y]\) is a class in \(\text{H}^1(S,G)^\rho\), \(P/C_G(\rho) \rightarrow S\) is smooth and \(y\) induces a point in the closed fiber of \(P/C_G(\rho) \rightarrow S\); we may reduce the structure group to \(C_G(\rho)\), thus showing that \(\epsilon\) is surjective. By (42) we conclude that \(\epsilon\) is an isomorphism and this implies \((S')\).
5.3 Final remarks

**Proposition 5.7** Let $X$ be a noetherian regular separated algebraic space over $k$ and $G$ a finite group acting on $X$. There is a canonical isomorphism of $R(G)$-algebras

$$K_*(X,G)_{\text{geom}} \otimes \mathbb{Z}[1/|G|] \simeq K_*(X)^G \otimes \mathbb{Z}[1/|G|].$$

**Proof.** Since $\ker(rk : K_0(X) \to H^0(X, \mathbb{Z}[1/|G|]))$ is nilpotent by Corollary 5.2, the canonical homomorphism

$$\pi^* : K_*(X,G) \to K_*(X)^G$$

induces a ring homomorphism (still denoted by $\pi^*$):

$$\pi^* : K_*(X,G)_{\text{geom}} \otimes \mathbb{Z}[1/|G|] \to K_*(X)^G \otimes \mathbb{Z}[1/|G|].$$

Moreover, the functor

$$\pi_* : \mathcal{F} \mapsto \bigoplus_{g \in G} g^* \mathcal{F}$$

defined on coherent $\mathcal{O}_X$-modules, induces a homomorphism

$$\pi_* : K_*'(X)^G \otimes \mathbb{Z}[1/|G|] \to K_*'(X,G)_{\text{geom}} \otimes \mathbb{Z}[1/|G|]$$

and (recalling that $K_*(X,G) \simeq K_*'(X,G)$) we obviously get:

$$\pi^* \pi_*(\mathcal{F}) = |G| \cdot \mathcal{F}.$$

On the other hand, we have:

$$\pi_* \pi^*(\mathcal{F}) \simeq \mathcal{F} \otimes \pi_* \mathcal{O}_X.$$

But $rk(\pi_* \mathcal{O}_X) = |G|$ and therefore $\pi_* \pi^*$ is an isomorphism too, because of Corollary 5.2. As a corollary of this result and of Theorem 5.4, we recover Theorem 1 of [Vi2] which was proved there in a completely different way.

We conclude the paper with a conjecture expressing the fact that $K_*(X,G)_{\text{geom}}$ should be the $K$-theory of the quotient $X/G$, if $X/G$ is regular, after inverting the orders of all the essential dual cyclic subgroups of $G$:

**Conjecture 5.8** Let $X$ be a noetherian regular separated algebraic space over a field $k$ and $G$ a linear algebraic $k$-group acting on $X$ with finite stabilizers in such a way that the quotient $X/G$ exists as a regular algebraic space. Let $N$ denote the least common multiple of the orders of all the essential dual cyclic subgroups of $G$ and $\Lambda = \mathbb{Z}[1/N]$. If $p : X \to X/G$ is the quotient map, the composition

$$K_*(X/G)_\Lambda \overset{p^*}{\to} K_*(X,G)_\Lambda \to K_*(X,G)_{\text{geom}}$$

is an isomorphism.

**Remark 5.9** Bertrand Toen pointed out to us that if $X/G$ is smooth it follows from the results of [EG] that the composition

$$K_0(X/G) \otimes \mathbb{Q} \to K_0(X,G) \otimes \mathbb{Q} \to K_0(X,G)_{\text{geom}} \otimes \mathbb{Q}$$

is an isomorphism.
6 Appendix: Higher equivariant $K$-theory of noetherian regular separated algebraic spaces

In this Appendix we describe the $K$-theories we use in the paper and their relationships. We essentially follow the example of [Th-Tr], Section 3. Of [Th-Tr] we also adopt the language.

Let us remark that it is strongly probable that there exist equivariant versions of most of the results in [Th-Tr], Section 3. In particular, there should exist a higher $K$-theory of $G$-equivariant cohomologically bounded pseudocoherent complexes on $Z$ (respectively, of $G$-equivariant perfect complexes on $Z$) for any quasi-compact algebraic space $Z$ having most of the alternative models described in [Th-Tr], 3.5-3.12. The arguments below can also be considered as a first step toward an extension of [Th-Tr], 3.11-3.12 to the equivariant case on algebraic spaces. However, to keep the paper to a reasonable size, we have decided to give only the results we need and moreover we have made almost no attempt to optimize the hypotheses.

We would also like to mention the paper [J] (in particular Section 1) in which, among many other results, the general techniques of [Th-Tr] are used as guidelines for the $K$-theory of arbitrary Artin stacks.

We work in a slightly more general situation than required in the rest of the paper. Let $S$ be a separated noetherian scheme, $G$ be a group scheme affine over $S$ which is finitely presented, separated and flat over $S$. We denote by $G$-AlgSp$_{reg}$ the category of regular noetherian algebraic spaces separated over $S$ with an action of $G$ over $S$ and equivariant maps.

**Definition 6.1** If $X \in G$-AlgSp$_{reg}$, we denote by $K_\ast(X,G)$ (respectively, $K'_\ast(X,G)$, resp., $K_{naive}\ast(X,G)$) the Waldhausen $K$-theory of the complicial biWaldhausen category $W_{1,X}$ of complexes of quasi-coherent $G$-equivariant $O_X$-Modules with bounded coherent cohomology (respectively, the Quillen $K$-theory of the abelian category of $G$-equivariant coherent $O_X$-Modules, resp., the Quillen $K$-theory of the exact category of $G$-equivariant locally free coherent $O_X$-Modules).

**Proposition 6.2** Let $Z \to S$ be a morphism of noetherian algebraic spaces such that the diagonal $Z \to Z \times_S Z$ is affine, $H \to S$ an affine group space acting on $Z$. Let $\mathcal{F}$ be an equivariant quasicoherent sheaf on $Z$ of finite flat dimension; then there exists a flat equivariant quasicoherent sheaf $\mathcal{F}'$ on $Z$ together with a surjective $H$-equivariant homomorphism $\mathcal{F}' \to \mathcal{F}$.

In particular, if $Z$ is regular this holds for all equivariant quasicoherent sheaves $\mathcal{F}$ on $Z$.

The hypotheses of the previous Proposition insure that the usual morphism $Z \times_S H \to Z \times_S Z$ is affine. In fact, the projection $Z \times_S H \to Z$ is obviously affine, the projection $Z \times_S Z \to Z$ has affine diagonal, so this follows from the elementary fact that if $Z \to U \to V$ are morphisms of algebraic spaces, $Z \to V$ is affine and $U \to V$ has affine diagonal, then $Z \to U$ is affine. Consider the quotient stack $[Z/H]$ ([L-MB]); the argument above implies that the diagonal $Z \to Z \times_S Z$ is affine. Since an $H$-equivariant quasicoherent $O_Z$-Module is the same as a quasicoherent Module over $Z$, now Proposition 6.2 follows from the following more general result:

**Proposition 6.3** Let $S$ be a noetherian algebraic space, $\mathcal{X}$ a noetherian algebraic stack over $S$ with affine diagonal. Let $\mathcal{F}$ be a quasicoherent sheaf of finite flat dimension on $\mathcal{X}$; then there exists a flat quasicoherent sheaf $\mathcal{F}'$ on $\mathcal{X}$ together with a surjective homomorphism $\mathcal{F}' \to \mathcal{F}$.
Proof. Take an affine scheme $U$ with a flat morphism $f : U \to X$; then $f$ is affine, and in particular the pushforward $f_*$ on quasicoherent sheaves is exact. Consider a quasicoherent sheaf $\mathcal{F}$ on $X$ of finite flat dimension, with the adjunction map $\mathcal{F} \to f_* f^* \mathcal{F}$. This map is injective; call $Q$ its cokernel. Clearly the flat dimension of $f_* f^* \mathcal{F}$ is the same as the flat dimension of $\mathcal{F}$; we claim that the flat dimension of $Q$ is at most equal to the flat dimension of $\mathcal{F}$. Now, if there were a section $\mathcal{X} \to U$ of $f$, then the sequence

$$0 \to \mathcal{F} \to f_* f^* \mathcal{F} \to Q \to 0$$

would split, and this would be clear. However, to compute the flat dimension of $Q$ we can pull back to any flat surjective map to $\mathcal{X}$; in particular after pulling back to $U$ we see that $f$ acquires a section, and the statement is checked. Now $U$ is an affine scheme, so we can take a flat quasicoherent sheaf $\mathcal{P}$ on $U$ with a surjective map $u : \mathcal{P} \to f^* \mathcal{F}$. Call $\mathcal{F}'$ the kernel of the composition $f_* \mathcal{P} \to f_* f^* \mathcal{F} \to Q$; then $\mathcal{F}'$ surjects onto $\mathcal{F}$, and fits into an exact sequence $0 \to \mathcal{F}' \to f_* \mathcal{P} \to Q \to 0$. But $f_* \mathcal{P}$ is flat over $\mathcal{X}$, so the flat dimension of $\mathcal{F}'$ is less than the flat dimension of $Q$, unless $Q$ is flat. But since the flat dimension of $Q$ is at most equal to the flat dimension of $\mathcal{F}$, we see that the flat dimension of $\mathcal{F}'$ is less than the flat dimension of $\mathcal{F}$, unless $\mathcal{F}$ is flat. The proof is completed with a straightforward induction on the flat dimension of $\mathcal{F}$. ■

Theorem 6.4 Let $X$ be an object in $G$-$AlgSp_{reg}$. The obvious inclusions of the following complicial biWaldhausen categories induce homotopy equivalences on the Waldhausen $K$-theory spectra $K^{(i)}(X) \cong K(W_{i,X})$, $i = 1, 2, 3$. In particular, the corresponding Waldhausen $K$-theories $K_{s}^{(i)}(X,G)$ coincide.

(i) $W_{1,X} = (\text{complexes of quasi-coherent } G\text{-equivariant } \mathcal{O}_{X}\text{-Modules with bounded coherent cohomology})$;

(ii) $W_{2,X} = (\text{bounded complexes in } G\text{-Coh}_{X})$;

(iii) $W_{3,X} = (\text{complexes of flat quasi-coherent G-equivariant } \mathcal{O}_{X}\text{-Modules with bounded coherent cohomology})$

Moreover the Waldhausen $K$-theory of any of the categories above coincide with Quillen $K$-theory $K_{*}^{(i)}(X,G)$ of $G$-equivariant coherent $\mathcal{O}_{X}$-Modules.

Proof. By [Th2], 1.13, the inclusion of $W_{2,X}$ in $W_{1,X}$ induces an equivalence of $K$-theory spectra. Proposition 3.2 together with [Th-Tr], Lemma 1.9.5 (applied to $\mathcal{D} = (\text{flat } G\text{-equivariant } \mathcal{O}_{X}\text{-Modules})$ and $A = (G\text{-equivariant } \mathcal{O}_{X}\text{-Modules})$), implies that for any object $E^{*}$ in $W_{1,X}$ there exists an object $F^{*}$ in $W_{3,X}$ and a quasi-isomorphism $F^{*} \to E^{*}$. Therefore, by [Th-Tr], 1.9.7 and 1.9.8, the inclusion of $W_{3,X}$ in $W_{1,X}$ induces an equivalence of $K$-theory spectra.

The last statement of the Theorem follows immediately from [Th2], 1.13, p. 518. ■

Since any complex in $W_{3,X}$ is degreewise flat and $X$ is regular (hence boudedness of cohomology is preserved under tensor product$^3$), the tensor product of complexes makes the Waldhausen $K$-theory spectrum of $W_{3,X}$ into a functor $K^{(3)}$ from $G$-$AlgSp_{reg}$ to ring spectra, with product

$$K^{(3)} \wedge K^{(3)} \to K^{(3)},$$

$^3$In fact this is a non equivariant statement and a local property in the flat topology, so it reduces to the same statement for regular affine schemes which is elementary (see also [SGA6]).
exactly as described in [Th-Tr] 3.15. In particular, by Theorem 3.4, $K_*$ is a functor from $G$-$\text{AlgSp}_{\text{reg}}$ to graded rings. In the same way, tensor product with complexes in $W_{3,X}$ gives a pairing

$$K^{(3)} \wedge K^{(1)} \longrightarrow K^{(1)}$$

between the corresponding functors from $G$-$\text{AlgSp}_{\text{reg}}$ to spectra so that $K^{(1)}_*(X,G)$ becomes a module over the ring $K^{(3)}_*(X,G)$ functorially in $(X,G) \in G$-$\text{AlgSp}_{\text{reg}}$. We denote the corresponding cap-product by

$$\cap : K^{(3)}_*(X,G) \otimes K^{(1)}_*(X,G) \longrightarrow K^{(1)}_*(X,G)$$

which becomes the ring product in $K_*(X,G)$ with the identifications allowed by Theorem 3.4. Note that there is an obvious ring morphism $\eta : K^\text{naive}_*(X,G) \to K^{(3)}_*(X,G)$ and if $\cap^\text{naive} : K^\text{naive}_*(X,G) \otimes K^\text{naive}_*(X,G) \longrightarrow K^\text{naive}_*(X,G)$ denotes the usual “naive” cap-product on Quillen $K$-theories, there is a commutative diagram

$$
\begin{array}{ccc}
K^\text{naive}_*(X,G) \otimes K^\text{naive}_*(X,G) & \longrightarrow & K^\text{naive}_*(X,G) \\
\eta \otimes u & \downarrow & \downarrow u \\
K^{(3)}_*(X,G) \otimes K^{(1)}_*(X,G) & \longrightarrow & K^{(1)}_*(X,G)
\end{array}
$$

where $u$ is the isomorphism of Theorem 3.4. Because of that we will simply write $\cap$ for both the naive and non-naive cap-products.

Note that, as shown in [Th2] 1.13, p.519, $K'_*(-,G)$ (and therefore $K_*(X,G)$ under our hypotheses) is a covariant functor for proper maps in $G$-$\text{AlgSp}_{\text{reg}}$; on the other hand, since any map in $G$-$\text{AlgSp}_{\text{reg}}$ has finite Tor-dimension, $K_*(-,-)$ is a contravariant functor from $G$-$\text{AlgSp}_{\text{reg}}$ to (graded) rings. In fact, if $f : X \longrightarrow Y$ is a morphism in $G$-$\text{AlgSp}_{\text{reg}}$ the same argument in [Th-Tr], 3.14.1 shows that there is an induced pullback exact functor $f^* : W_{3,Y} \longrightarrow W_{3,X}$ and then we use theorem 3.2 to identify $K^{(3)}_*(-,-)$ with $K_*(-,-)$.

**Proposition 6.5** (Projection formula) Let $j : Z \longrightarrow X$ be a closed immersion in $G$-$\text{AlgSp}_{\text{reg}}$. Then, if $\alpha$ is in $K_*(X,G)$ and $\beta$ in $K'_*(Z,G)$, we have

$$j_*(j^*(\alpha) \cap \beta) = \alpha \cap j_*(\beta)$$

in $K'_*(X,G)$.

**Proof.** Since $j$ is affine, $j_*$ is exact on quasi-coherent Modules and therefore induces an exact functor of complicial biWaldhausen categories $j_* : W_{1,Z} \longrightarrow W_{1,X}$ (the condition of bounded coherent cohomology being preserved by regularity of $Z$ and $X$). Therefore, the maps

$$(\alpha, \beta) \longmapsto j_*(j^*(\alpha) \cap \beta),$$

$$(\alpha, \beta) \longmapsto \alpha \cap j_*(\beta)$$

from $K_*(X,G) \times K'_*(Z,G)$ to $K'_*(X,G) \simeq K_*(X,G)$ are induced by the exact functors $W_{3,X} \times W_{1,Z} \longrightarrow W_{1,X}$

$$(F^*, E^*) \longmapsto j_*(j^*(F^*) \otimes E^*),$$

$$(F^*, E^*) \longmapsto F^* \otimes j_*(E^*).$$

But for any equivariant quasi-coherent sheaf $\mathcal{F}$ on $X$ and $\mathcal{G}$ on $Z$, there is a natural (hence equivariant) isomorphism

$$j_*(j^*\mathcal{F} \otimes \mathcal{G}) \simeq \mathcal{F} \otimes j_*\mathcal{G}$$

which, again by naturality, induces an isomorphism between the two functors in (3); therefore we conclude by [Th-Tr], 1.5.4. ■
Remark 6.6 Since we only need the projection formula for (regular) closed immersion in this paper we have decided to state the result only in this case. However, since by [Th2] 1.13 p. 519, \( K_*(X,G) \) coincides also with Waldhausen \( K \)-theory of the category \( \mathcal{W}_{4,X} \) of complexes of \( G \)-equivariant quasi-coherent injective Modules on \( X \) with bounded coherent cohomology, and therefore, by Theorem 6.4, it also coincides with Waldhausen \( K \)-theory of the category \( \mathcal{W}_{5,X} \) complexes of \( G \)-equivariant quasi-coherent flasque Modules on \( X \) with bounded coherent cohomology, for any proper map \( f : X \to Y \) in \( G \)-AlgSp\(^{\text{reg}} \), we have an exact functor \( f_* : \mathcal{W}_{5,X} \to \mathcal{W}_{5,Y} \) which therefore gives a "model" for the push-forward \( f_* : K_*(X,G) \to K_*(Y,G) \) (cf. [Th-Tr], 3.16). Now, the proof of [Th-Tr] 3.17 should also give a proof of Proposition 6.5 with \( j \) replaced by any proper map in \( G \)-AlgSp\(^{\text{reg}} \), because it only uses [Th-Tr] 2.5.5, which obviously holds for \( X \) and \( Y \) noetherian algebraic spaces, and [SGA4] XVII, 4.2, which should give a canonical \( G \)-equivariant Godement flasque resolution of any complex of \( G \)-equivariant Modules on any algebraic space in \( G \)-AlgSp\(^{\text{reg}} \), since it is developed in a general topos.

It is very probable that theorem 6.4 and therefore the functoriality with respect to morphisms of finite Tor-dimension still hold without the regularity assumption on the algebraic spaces. On the other hand, it should also be true that with \( G \) and \( X \) as above (therefore \( X \) regular), the Waldhausen \( K \)-theory of the category of \( G \)-equivariant perfect complexes on \( X \) coincides with \( K'_*((X,G)) \). This last statement should follow, (with a bit of work to identify \( K'_*((X,G)) \) with the Waldhausen \( K \)-theory of \( G \)-equivariant pseudocoherent complexes with bounded cohomology on \( X \)) from [J], 1.6.2.

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