Threshold theorem in isolated quantum dynamics with local stochastic control errors

Manaka Okuyama¹, Kentaro Ohki², and Masayuki Ohzeki¹,³,⁴

¹Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan
²Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan
³Institute of Innovative Research, Tokyo Institute of Technology, Yokohama 226-8503, Japan and
⁴Sigma-i Co., Ltd., Tokyo 108-0075, Japan

(Dated: July 15, 2021)

Abstract

We investigate the effect of local stochastic control errors in the time-dependent Hamiltonian on isolated quantum dynamics. The control errors are formulated as time-dependent stochastic noise in the Schrödinger equation. For any local stochastic control errors, we establish a threshold theorem that provides a sufficient condition to obtain the target state, which should be determined in noiseless isolated quantum dynamics, as a relation between the number of measurements required and noise strength. The theorem guarantees that if the sum of the noise strengths is less than the inverse of computational time, the target state can be obtained through a constant-order number of measurements. If the opposite is true, the required number of measurements increases exponentially with computational time. Our threshold theorem can be applied to any isolated quantum dynamics such as quantum annealing, adiabatic quantum computation, the quantum approximate optimization algorithm, and the quantum circuit model.
I. INTRODUCTION

Recent advances in experimental techniques have enabled the experimental realization of quantum dynamics, and it has become increasingly important to understand quantum dynamics. In particular, recent efforts to realize quantum computation are progressing rapidly [1, 2]; hence, the precise control of quantum dynamics is required.

Although isolated quantum dynamics is ideally described by the Schrödinger equation, the influence of the external environment cannot be ignored in experimental systems. The external environment has two main effects on isolated quantum systems: the influence from a heat bath and control errors of the Hamiltonian. Here, we consider a case in which the influence of the heat bath can be eliminated. Therefore, the dynamics of the target quantum system can be realized if the Hamiltonian can be controlled in an ideal manner. However, it is difficult to control the Hamiltonian without errors in experimental systems. Then, a natural question arises: is isolated quantum dynamics robust to the effects of control errors? If the properties of isolated quantum dynamics dramatically change because of control errors, then it will be difficult to control the target quantum system in experimental systems, even if we can eliminate the effect of the heat bath. For example, quantum annealing [3–7] or adiabatic quantum computation [8–10] utilizes isolated quantum dynamics for computation, but there is no established theory of quantum error correction [11–15]. Additionally, while the quantum error correction theory is well established in the quantum circuit model [16–18], it is currently very difficult to implement it experimentally at a large scale. Therefore, it is vital to investigate the influence of control errors on isolated quantum dynamics from the perspective of quantum computation.

There are two main types of control errors that can occur in the time-dependent Hamiltonian. One is time-invariant noise [19–21], which acts as a bias. This type of errors modifies the Hamiltonian and will not be discussed in the present study. The other, which we will focus on, is stochastic control noise [22–24]. We formulate stochastic noise in unitary dynamics as time-varying stochastic noise. The time evolution of the system is described by the stochastic differential equation [25]. The present study examines whether it is possible to obtain the target state, which should be determined in noiseless time evolution, in the presence of stochastic control errors. For this purpose, we establish a threshold theorem that provides a sufficient condition for obtaining the target state in the Schrödinger equation with local stochastic control errors. Our threshold theorem clarifies that the number of measurements required to obtain the target state strongly depends on the computational
time and noise strength.

II. THRESHOLD THEOREM IN ISOLATED QUANTUM DYNAMICS WITH LOCAL STOCHASTIC CONTROL ERRORS

We consider the following isolated quantum dynamics:

\[ i \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle, \]

where \(0 \leq t \leq T\). By using the measurement basis \(|n\rangle\), we expand the final state \(|\psi(T)\rangle\) as

\[ |\psi(T)\rangle = \sum_n C_n |n\rangle. \]

We are interested in the \(m\)th eigenstate \(|m\rangle\) of the measurement basis, and its probability amplitude \(C_m\) at the final state \(|\psi(T)\rangle\) is given by

\[ C_m = 1 - \epsilon, \]

where \(0 < \epsilon < 1\) (without loss of generality, we have adjusted the global phase of \(|\psi(T)\rangle\) so that \(C_m\) is a positive real number). Then, if the number of measurements \(r\) satisfies

\[ r \gg \frac{1}{(1 - \epsilon)^2}, \]

we succeed in obtaining the target state \(|m\rangle\).

However, it is difficult to completely control the time-dependent Hamiltonian \(\hat{H}(t)\) without encountering control errors in experimental systems. We incorporate the control errors of \(\hat{H}(t)\) into the Schrödinger equation as noise that occurs stochastically at each moment. Because we consider isolated quantum dynamics, control errors should also be described as a unitary time evolution. It is well known that norm-preserving stochastic noise can be described by the Stratonovich process [22, 23]. Thus, we express the Schrödinger equation with stochastic noise as follows:

\[ id|\phi(t)\rangle = \left( \hat{H}(t)dt + \sum_k \hat{H}_{\text{error},k}(t) \circ dW_k(t) \right) |\phi(t)\rangle, \]

where \(\hat{H}_{\text{error},k}(t)\) describes a stochastic control error, \(W_k(t)\) describes standard Brownian motion, and the symbol “ \(\circ\)” denotes the Stratonovich interpretation. The equivalent Ito process is given by

\[ id|\phi(t)\rangle = \left( \hat{H}(t)dt + \sum_k \hat{H}_{\text{error},k}(t) \cdot dW_k(t) \right) |\phi(t)\rangle \]

\[ -\frac{i}{2} \sum_k \hat{H}_{\text{error},k}^2(t) |\phi(t)\rangle dt, \]

where \(\hat{H}_{\text{error},k}^2(t)\) denotes the square of the stochastic control error at each moment.

3
where the symbol “•” denotes the Ito interpretation. We note that the norm of $|\phi(t)\rangle$ is always preserved in time evolution.

Furthermore, we assume that $\hat{H}_{\text{error},k}(t)$ satisfies the local error condition

$$\hat{H}_{\text{error},k}^2(t) = g_k^2(t)\hat{I},$$

(7)

where $\hat{I}$ is the identity operator and $g_k(t)$ is an arbitrary time-dependent function. This condition assumes that stochastic control errors occur only locally and not globally. For example, when $\hat{H}_{\text{error},k}(t)$ is constructed from the product of a Pauli operator, the local error condition is satisfied.

Under these settings, we examine whether it is possible to obtain the target state. If local stochastic control errors have a devastating effect on isolated quantum dynamics, the target state cannot be obtained in experimental systems. Additionally, if the number of measurements required to obtain the target state depends on the problem size $N$, local stochastic control errors have a serious influence on the difficulty of the problem. Our threshold theorem, which clearly addresses these issues, is described as follows.

**Theorem 1.** We are interested in the $m$th eigenstate $|m\rangle$ of the measurement basis and its probability amplitude $C_m$ at the final state $|\psi(T)\rangle$ in the noiseless Schrödinger equation (1) is given by

$$C_m = 1 - \epsilon,$$

(8)

where $0 < \epsilon < 1$. If $\epsilon$ and the number of instances $r$ satisfy

$$r \gg \frac{1}{\delta^2},$$

(9)

$$r \gg \frac{1}{\alpha^2},$$

(10)

$$\epsilon + \delta + \alpha e^{\frac{1}{4} \int_T^T \sum_k g_k^2(t)dt} < 1,$$

(11)

for $\delta > 0$ and $\alpha > 0$, then the target state $|m\rangle$ is always included in the results of $r$ measurements in the Schrödinger equation with stochastic local control errors $\hat{H}_{\text{error},k}(t)$.

Our threshold theorem states that the number of measurements required depends strongly on the strength of the noise. For simplicity, we consider a case in which the strength of the noise is time-independent: $g_k(t) = g_k$. Then, from Eq. (11), the following condition must be satisfied for the number of measurements to be bounded by a constant order:

$$\frac{T}{2} \sum_k g_k^2 = O(1).$$

(12)
Thus, for any local stochastic control errors, if the sum of the noise strengths is less than the inverse of the computational time, the target state can be obtained through a constant-order number of measurements. Conversely, if the sum of the noise strengths is greater than the inverse of the computational time, the number of measurements required increases exponentially with respect to the computational time. In conclusion, local stochastic control errors can have a serious impact on the difficulty of the problem, depending on the noise strength.

For example, we apply our threshold theorem to quantum annealing. We consider a case in which the computational time $T$ is given by a polynomial of the problem size $N$:

$$T = O(N^a),$$

which is efficiently solved by quantum annealing. Then, we must suppress the noise as the problem size increases:

$$\frac{1}{2} \sum_k g_k^2 = O(N^{-a}).$$

Otherwise, from Eq. (11), the number of measurements required increases exponentially with respect to the problem size:

$$r \gg e^{N^a \sum g_k^2}.$$

Therefore, when noise suppression fails, stochastic control errors change an efficient quantum-annealing-based solution to the problem into an inefficient solution.

Before providing the proof, we emphasize that our threshold theorem is only a sufficient condition for any isolated quantum dynamics. Incorporating the structure of the problem might improve our result.

### III. PROOF OF THRESHOLD THEOREM

From Eq. (6), the time evolution of the expectation of the state is described as

$$\frac{d}{dt} \mathbb{E}[|\psi(t)\rangle] = \left(\hat{H}(t) - \frac{i}{2} \sum_k g_k^2(t)\right) \mathbb{E}[|\psi(t)\rangle].$$

Then, the following relation holds:

$$e^{\frac{i}{\hbar} \int_0^T \sum_k g_k^2(t) dt} \mathbb{E}[|\psi(T)\rangle] = |\psi(T)\rangle.$$
We describe the state in one instance of Eq. (6) as $|\phi_i(t)\rangle$ and expand it at the final time $T$ as

$$|\phi_i(T)\rangle = \sum_n C_{i,n} |n\rangle.$$  

(18)

From Eq. (17), we immediately find

$$e^{\frac{1}{2} \int_0^T \sum_k g_k^2(t) dt} E[C_{i,m}] = C_m = 1 - \epsilon.$$  

(19)

Then, using the Chernoff–Hoeffding inequality [26, 27], we have

$$\Pr \left[ \left| \frac{r}{r} \sum_{i=1}^r \Re C_{i,m} - (1 - \epsilon) \right| > \delta_1 \right] \leq 2 \exp \left( -\frac{r \delta_1^2}{2} \right),$$  

(20)

$$\Pr \left[ \left| \frac{r}{r} \sum_{i=1}^r \Im C_{i,m} \right| > \delta_2 \right] \leq 2 \exp \left( -\frac{r \delta_2^2}{2} \right).$$  

(21)

where $C_{i,m} = \Re C_{i,m} + i\Im C_{i,m}$ and $\delta_1, \delta_2 > 0$. In the following, we set $\delta_1 = \delta_2 = \delta$ and always consider the case in which

$$r \gg \frac{1}{\delta^2}.$$  

(22)

Then, from Eqs. (20) and (21), the following relations hold:

$$\left| \frac{r}{r} \sum_{i=1}^r \Re C_{i,m} - (1 - \epsilon) \right| \leq \delta,$$  

(23)

$$\left| \frac{r}{r} \sum_{i=1}^r \Im C_{i,m} \right| \leq \delta.$$  

(24)

These inequalities play an important role as constraints.

Next, we consider the case in which the probability amplitude of the target state in the $r$ instances of Eq. (6) satisfies

$$\frac{1}{r} \sum_{i=1}^r |C_{i,m}|^2 > \alpha^2.$$  

(25)

Then, if the number of instances $r$ satisfies

$$r \gg \frac{1}{\alpha^2},$$  

(26)

we can obtain the correct result because the target state is always included in the measurement results.
In the following, we prove that for \( r \gg 1/\delta^2 \) and \( \epsilon + \delta + \alpha \exp(\int_0^T \sum_k g_k^2(t)dt/2) < 1 \), Eqs. (23), (24), and (25) are always compatible. In order to accomplish this, we consider

\[
\frac{1}{r} \sum_{i=1}^{r} |C_{i,m}|^2 \leq \alpha^2, \tag{27}
\]

and investigate a necessary condition for Eqs. (23), (24), and (27) to be compatible. Under Eq. (27), the absolute value of \( \sum_{i=1}^{r} \Re C_{i,0} \) takes the maximum value \( ra \) when \( \Re C_{i,0} = \alpha \) and \( \Im C_{i,0} = 0 \). Thus, for Eqs. (23), (24), and (27) to hold simultaneously, the following inequality must be satisfied:

\[
\alpha e^{\frac{1}{2} \int_0^T \sum_k g_k^2(t)dt} \geq 1 - \epsilon - \delta. \tag{28}
\]

We stress that this inequality is a necessary condition for Eqs. (23), (24), and (27) to be compatible. In other words, for \( \epsilon + \delta + \alpha \exp(\int_0^T \sum_k g_k^2(t)dt/2) < 1 \) and \( r \gg 1/\delta^2 \), Eqs. (23), (24), and (27) do not hold simultaneously, while Eqs. (23) and (24) always hold. This implies that Eqs. (24), (25), and (27) are always compatible for \( \epsilon + \delta + \alpha \exp(\int_0^T \sum_k g_k^2(t)dt/2) < 1 \) and \( r \gg 1/\delta^2 \).

In summary, for \( \epsilon + \delta + \alpha \exp(\int_0^T \sum_k g_k^2(t)dt/2) < 1 \), \( r \gg 1/\delta^2 \), and \( r \gg 1/\alpha^2 \), the target state \( |m\rangle \) can be obtained from \( r \) measurements, which is proof of the threshold theorem.

IV. CONCLUSIONS

We have established a threshold theorem that provides a sufficient condition for obtaining the target state in isolated quantum dynamics with any local stochastic control errors. Our threshold theorem guarantees that if the sum of the noise strengths is less than the inverse of the computational time, the target state can be obtained through a constant-order number of measurements. However, if the sum of the noise strengths is larger than the inverse of the computational time, the number of measurements required increases exponentially with respect to the computational time.

Furthermore, we imposed the local error condition (7) on stochastic control errors. If this condition is broken, the simple relation (17) does not hold. Then, we cannot guarantee that the target state can be obtained by increasing the number of measurements. In other words, non-local stochastic control errors have a serious influence on isolated quantum dynamics.

In the present study, we considered only time-varying noise as a control error. However, time-invariant noise can also be considered as a control error \([19,21]\), and our threshold theorem cannot be applied to such noise. Time-invariant noise modifies the Hamiltonian. In order to obtain the
target state in experimental systems, such noise must be reduced to the limit to establish an error
correction theory for such noise, or a counterpart of our threshold theorem must be derived for
such noise.

Finally, it is worth mentioning that we have considered the worst-case scenario, and model-
dependent properties may reduce the number of measurements required. For example, in adiabatic
quantum computation, adiabatic time evolution suppresses the diabatic transition from the ground
state to other excited states. In such cases, the effect of stochastic control errors may also be
reduced.

The present work was financially supported by JSPS KAKENHI Grant Nos. 18H03303,
19H01095, 19K23418, 20H02168 and 21K13848 and by JST-CREST (No. JPMJCR1402) from
the Japan Science and Technology Agency.

[1] M. W. Johnson et al., Nature (London) 473, 194 (2011).
[2] F. Arute et al., Nature (London) 574, 505 (2019).
[3] T. Kadowaki and H. Nishimori, Phys. Rev. E 58, 5355 (1998).
[4] P. Ray, B. K. Chakrabarti, and A. Chakrabarti, Phys. Rev. B 39, 11828 (1989).
[5] J. Brooke, D. Bitko, T. F. Rosenbaum, and G. Aeppli, Science 284, 779 (1999).
[6] J. Brooke, T. F. Rosenbaum, and G. Aeppli, Nature (London) 413, 610 (2001).
[7] G. E. Santoro, R. Martoňák, E. Tosatti, and R. Car, Science 295, 2427 (2002).
[8] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, arXiv: quant-ph/0001106.
[9] D. Aharonov, W. van Dam, J. Kempe, Z. Landau, S. Lloyd, and O. Regev, SIAM J. Comput. 37, 166
   (2007).
[10] A. Mizel, D. A. Lidar, and M. Mitchell, Phys. Rev. Lett. 99, 070502 (2007).
[11] S. P. Jordan, E. Farhi, and P. W. Shor, Phys. Rev. A 74, 052322 (2006).
[12] D. A. Lidar. Phys. Rev. Lett. 100, 160506 (2008).
[13] G. Quiroz and D. A. Lidar, Phys. Rev. A 86, 042333 (2012).
[14] K. L. Pudenz, T. Albash, and D. A. Lidar, Nat. Commun. 5, 3243 (2014).
[15] D. Venturelli, S. Mandrá, S. Knysb, B. O’Gorman, R. Biswas, and V. Smelyanskiy, Phys. Rev. X 5,
   031040 (2015).
[16] P. W. Shor, Phys. Rev. A 52, R2493 (1995).
[17] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[18] J. Preskill, Quantum Inf. Comput. **13**, 181 (2013).

[19] J. Roland and N. J. Cerf, Phys. Rev. A **71**, 032330 (2005).

[20] S. Mandrà, G. G. Guerreschi, and A. Aspuru-Guzik, Phys. Rev. A **92**, 062320 (2015).

[21] S. Muthukrishnan, T. Albash, and D. A. Lidar, Phys. Rev. A **99**, 032324 (2019).

[22] E. Wong and M. Zakai, Ann. Math. Stat. **36**, 1560 (1965).

[23] C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer-Verlag, Berlin, 1999).

[24] A. Dutta, A. Rahmani, and A. del Campo, Phys. Rev. Lett. **117**, 080402 (2016).

[25] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, Berlin, 1983).

[26] H. Chernoff, Ann. Math. Stat. **23**, 493 (1952).

[27] W. Hoeffding, J. Am. Stat. Assoc. **58**, 13 (1963).