ISOMORPHIC AND ISOMETRIC STRUCTURE OF THE OPTIMAL DOMAINS FOR HARDY-TYPE OPERATORS

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Abstract. We investigate structure of the optimal domains for the Hardy-type operators including, for example, the classical Cesàro, Copson and Volterra operators as well as for some of their generalizations. We prove that, in some sense, the abstract Cesàro and Copson function spaces are closely related to the space $L^1$, namely, they contain “in the middle” a complemented copy of $L^1[0,1]$, asymptotically isometric copy of $\ell^1$ and also can be renormed to contain an isometric copy of $L^\infty[0,1]$. Moreover, the generalized Tandori function spaces are quite similar to $L^\infty$ because they contain an isometric copy of $\ell^\infty$ and can be renormed to contain an isometric copy of $L^\infty[0,1]$. Several applications to the metric fixed point theory will be given. Next, we prove that the Cesàro construction $X \mapsto CX$ does not commute with the truncation operation of the measure space support. We also study whether a given property transfers between a Banach function space $X$ and the space $TX$, where $T$ is the Cesàro or the Copson operator. In particular, we find a large class of properties which do not lift from $TX$ into $X$ and prove that the abstract Cesàro and Copson function spaces are never reflexive, are not isomorphic to a dual space and do not have the Radon–Nikodym property in general.

1. Introduction

In 1925 G. H. Hardy [35] proved the following inequality, which today is usually called the classical Hardy inequality

$$\int_0^\infty \left( \frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f(x)^p dx,$$

where $1 < p < \infty$ and $f$ is a nonnegative real-valued Lebesgue measurable function (see [53, Chapter 3] for more details). This inequality can be reformulated in the following way

the Hardy operator $f \mapsto \frac{1}{x} \int_0^x f(t) dt$ maps $L^p[0,\infty)$ continuously into itself.

Given an operator $T \in \mathcal{L}(Y,X)$, where $X$ and $Y$ are Banach function spaces, it is natural to ask whether there is a Banach function space, say $Z$, such that $T: Z \to X$ is also bounded and $Z$ is the largest, in the sense of inclusion, Banach function space with this property. This situation can be summarized by the following diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{T} & Z \\
\downarrow & & \downarrow \iota_T \\
X & \xleftarrow{t} & \end{array}
$$

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Under some technical assumptions \cite[p. 196]{21}, \( Z \) is the space of all measurable functions \( f \) such that \( T |f| \in X \), equipped with the norm \( \|f\|_Z = \|T|f|\|_X \). In other words, the space \( Z \) is the maximal or optimal domain for the operator \( T \) considered with values in the fixed space \( X \) and throughout this paper we adopt the convention to denote it by \( TX \). This point of view turned out to be helpful and fruitful in the study of such classes of operators like kernel operators (special cases of operators in this class are, for example, the Volterra, Cesàro, Copson, Poisson or Riemann–Liouville operator), differential operators, convolutions, Fourier transform and the Sobolev embedding (see \cite[and references given there]{65}).

The classical Cesàro and Copson function spaces appeared in a natural way as the optimal domains of the Hardy operator and its conjugate operator, respectively (see \cite[22, 25, 54] and [64]). For this reason and also to avoid the use of the term “Hardy space”, which is usually reserved for certain spaces of holomorphic functions on the unit disc (interestingly, introduced by F. Riesz in 1923 also to honour of Hardy), we will call mentioned operator the Cesàro operator reserved for certain spaces of holomorphic functions on the unit disc (interestingly, introduced by F. Riesz in 1923 also to honour of Hardy), we will call mentioned operator the Cesàro operator introduced by Sinnamon. Namely, for a symmetric space \( X \) on \( I = [0, \infty) \) such that the Cesàro operator \( C \) is bounded on \( X \) we can identify \( CX \) with \( X^\downarrow \) (see \cite[73]; see also \cite[Section 3]{33} for some additional remarks).

From an isomorphic point of view, the abstract Cesàro function spaces \( CX \) as well as the abstract Copson function spaces \( C^\delta X \) are a kind of a nontrivial mixture of the Banach function space \( X \) and \( L^1 \) in which the properties of both spaces manifest themselves. Following this idea, we will look for “the best possible” copies of the space \( \ell^1 \) and \( L^1[0, 1] \) in the Cesàro and Copson function spaces, and also of the space \( \ell^\infty \) in \( Ces_\infty := CL^\infty \). We apply our results, among others, to the fixed point theory which is a very wide branch of functional analysis and has been developed for several decades (see \cite[44]{34}). It has many applications, for example, in nonlinear analysis as well as in integral and differential equations. In particular, the question whether a Banach space \( X \) has or fails the (weak) fixed point property for nonexpansive mappings is a fundamental in this area.

In \cite[Theorem 1 and 2]{6} Astashkin–Maligranda proved that the Cesàro function spaces \( Ces_p := CL_p \) for \( 1 \leq p \leq \infty \) if \( I = [0, 1] \) and \( 1 < p < \infty \) if \( I = [0, \infty) \) fail to have the fixed point property for nonexpansive mappings. In contrast, it was proved by Cui–Hudzik \cite[18], Cui–Hudzik–Li \cite[19] and Cui–Meng–Pluciennik \cite[20] that their sequence counterparts, i.e., the Cesàro sequence spaces \( ces_p := CL_p \), have this property whenever \( 1 < p < \infty \). We will show that the abstract Cesàro and Copson function spaces on two separable measure spaces \( [0, 1] \) and \( [0, \infty) \) contain an order asymptotically isometric copy of \( \ell^1 \) (the notions of asymptotic isometries are intermediate between the isomorphic and isometric theory) and thus, by the Dowling–Lennard result, fail to have the fixed point property in general. In the case of Cesàro function spaces this result can be seen as a generalization of the Astashkin–Maligranda result from \cite[6]. In fact, the main idea to find an asymptotically isometric copy of \( \ell^1 \) (which, by the way, were introduced precisely to show that certain spaces fail to have the fixed point property) remains the same but the argument is much more sophisticated and works in full generality. An analogous result for the Copson function spaces is new even for \( X = L^p \). On the other hand, we also prove that nontrivial Tandori function spaces \( X \) contain an order isomorphically isometric copy of \( \ell^\infty \) and consequently even fail to have the weak fixed point property.

The second important problem we consider is the question whether “some” property can be transferred from a simpler structure to more complicated one and vice versa. This type of problems has been successfully considered for many constructions. For example, we can mention three of such questions: \( 1^0 \). \( (X, E) \mapsto E(X) \), where \( X \) is a Banach space, \( E \) is a Banach function space and \( E(X) \) is a Köthe–Bochner space, \( 2^0 \). \( (X, Y) \mapsto \mathcal{F}(X, Y) \), where \( X \) and \( Y \) are symmetric
spaces and \( \mathfrak{F} \) is an interpolation functor (see references in [56]), and \( 3^0 \). \( X \mapsto X^{(*)} \), where \( X \) is a Banach function space and \( X^{(*)} \) is the so-called symmetrization of \( X \) (see references in [19]).

We will also consider this problem but for the Cesàro and Copson construction \( X \mapsto TX \) presenting a large class of properties that never transfer from the space \( TX \) into \( X \).

Finally, we will examine the Cesàro construction \( X \mapsto CX \) itself. More precisely, we show that this construction does not commutate in general with the truncation operation \( X \mapsto X|_{[0,1]} \) highlighting in this way the difference between the Cesàro function spaces defined on \([0,1]\) and \([0,\infty)\). The question whether two operations commutate has been often investigated. For example, the symmetrization operation \( X \mapsto X^{(*)} \) commutates with the Calderón–Lozanovskiǐ construction \( \rho(X,Y) \) (in particular with the pointwise product \( X \circ Y \)) and with the pointwise multipliers \( M(X,Y) \) (in particular with the Köthe dual \( X' \)), see [19]. Furthermore, the Cesàro construction \( X \mapsto CX \) comutates with the interpolation functor \( \mathfrak{F} \) having the homogeneity property (see [56, Theorem 6]).

It is worth to mention that we are able to prove most of the results without the assumption that the Cesàro or Copson operator is bounded on \( X \), the assumption which is present in almost all results of this type.

The paper is organized as follows. After an introduction we collect some necessary definitions, basic facts and notations in Section 2. Here we also recall the duality theorem of Leśnik–Maligranda from [54], the Lindenstrauss–Tzafriri [57] and Boyd [16] results on interpolation because we will use them frequently.

In Section 3 we will provide some basic results regarding nontriviality of the abstract Copson function spaces (Lemma 3.1 and Corollary 3.2). We discuss also the difference between the condition \( TX \neq \{0\} \) and the fact that the operator \( T \) is bounded on \( X \), where \( T \) is the Cesàro or the Copson operator (Example 3.3).

Section 4 starts with two lemmas (Lemma 4.1 and Lemma 4.2) which will play a crucial role later on. In particular, they show that the nontrivial Cesàro and Copson function spaces contain “in the middle” a complemented copy of \( L^1[0,1] \). Next, we prove that a Banach space \( X \), which contains a complemented copy of a space \( Y \), can always be renormed to contain an isometric copy of the space \( Y \) (Theorem 4.4). As a corollary we obtain immediately that the Cesàro and Copson function spaces can be renormed to contain an isometric copy of \( L^1[0,1] \) and that the Tandori function spaces can be renormed to contain an isometric copy of \( L^\infty[0,1] \). Finally, in Theorem 4.5 and Theorem 4.6 we present the main result of this section, namely, that the Cesàro and Copson function spaces always contain an order asymptotically isometric copy of \( \ell^1 \).

Since the generalized Tandori function spaces \( \tilde{X} \) and the space \( Ces_{\infty} \) are never order continuous [54] Theorem 1 (e)] it follows that they contain an isomorphic copy of \( \ell^\infty \). Nevertheless, we prove that \( \tilde{X} \) and \( Ces_{\infty} \) even always contain an order isometric copy of \( \ell^\infty \) (Proposition 4.8 and Proposition 4.9).

Next, in Section 5, we try to compare the Cesàro function spaces defined on \([0,1]\) and on \([0,\infty)\) and we show that the Cesàro construction \( X \mapsto CX \), and the truncation operation of the measure space support \( X \mapsto X|_{[0,1]} \) does not commutate in general (Lemma 5.1). This fact explains, in a sense, quite surprising differences between some results obtained for the Cesàro function spaces on a finite and infinite interval (see also [2, 54] and [56]).

In Section 6, we analyze a problem of transfer of properties between \( X \) and \( TX \), where \( T = C \) or \( T = C^* \). We give an example of properties which lift from the Banach function space \( X \) to \( TX \) and vice versa (Corollary 6.1 and Lemma 6.3). Next, using the results of Bessaga–Pełczyński and Talagrand we obtain that the Cesàro and Copson function spaces are not isomorphic to a dual space and do not have the Radon–Nikodym property (Corollary 6.4). Moreover, we include an example of a certain class of Banach function spaces which contain “in the middle” an isomorphic copy of a Banach function space \( Y \), but the construction \( X \mapsto TX \), in a sense, forgets about this copy (Lemma 6.2). The presented comparison of this example with the result
from \([48]\) can be instructive. Furthermore, we give a large class of properties (including, for example, order continuity, \(p\)-concavity and the Dunford–Pettis property) which do not transfer from \(TX\) to \(X\) (Theorem 6.6).

The main result in Section 7 is Theorem 7.1, which states that the abstract Cesàro and Copson function spaces fail to have the fixed point property in general. We prove that, under additional assumptions, these spaces cannot even be renormed to have the fixed point property (Corollary 7.4). We also conclude that the generalized Tandori function spaces \(\tilde{X}\) and the space \(\text{Ces}_\infty\) fail to have the weak fixed point property (Proposition 7.5).

Section 8 presents a certain way to generalize the results from the previous sections. We show that the methods developed by us in Sections 4 and 7 also work for a wider class of operators, e.g., for the weighted Cesàro operator \(\mathcal{H}_w\) and its conjugate \(\mathcal{H}_w^*\) (Theorem 8.2 and Theorem 8.3). In particular, we prove that an abstract Volterra spaces \(VX\) fail to have the fixed point property as well (Corollary 8.4).

Finally, the Appendix is devoted to the analysis of a certain objects, specifically, two functions \(F_X\) and \(G_X\), that appeared in the proof of Theorems 4.3 and 4.6. We finish this section with a few examples (Example 9.3). In the first one we give some rather exotic examples of the function \(F_X\) and in the next we will justify that the order continuity of a symmetric space \(X\) is not crucial for the continuity of the function \(F_X\).

2. Notation and preliminaries

2.1. Banach function spaces and symmetric spaces. Denote by \(m\) the Lebesgue measure on \(I\), where \(I = [0, 1]\) or \(I = [0, \infty)\), and by \(L^0 = L^0(I)\) the set of all equivalence classes of real-valued Lebesgue measurable functions defined on \(I\). A Banach function space (or a Banach ideal space) \(X = (X, \|\cdot\|_X)\) on \(I\) is understood to be a Banach space \(X\) such that \(X\) is a linear subspace of \(L^0(I)\) satisfying the so-called ideal property, which means that if \(f, g \in L^0(I), |f(t)| \leq |g(t)|\) for almost all \(t \in I\) and \(g \in X\), then \(f \in X\) and \(\|f\|_X \leq \|g\|_X\). If it is not stated otherwise we assume that a Banach function space \(X\) contains a function \(f_0 \in X\) which is positive almost everywhere (in short, \(a.e.\)) on \(I\) (such a function is called the weak unit in \(X\)), which means that \(\text{supp}(X) = I\). Sometimes we will write \(X[0, 1]\) or \(X[0, \infty)\) to clearly indicate that a Banach function space \(X\) is defined on \(I = [0, 1]\) or on \(I = [0, \infty)\), respectively. We say that a Banach function space \(X\) is nontrivial if \(X \neq \{0\}\).

For two Banach function spaces \(X\) and \(Y\) on \(I\), the symbol \(X \hookrightarrow \neg \neg M Y\) denotes the fact that the inclusion \(X \subset Y\) is continuous with the norm not bigger than \(M\), i.e., there exists a constant \(M > 0\) (we will call it the embedding constant) such that \(\|f\|_Y \leq M \|f\|_X\) for all \(f \in X\). If the embedding \(X \hookrightarrow \neg \neg M Y\) holds with some (maybe unknown) constant \(M > 0\) we simply write \(X \hookrightarrow Y\) and \(\|f\|_Y \lesssim \|f\|_X\). Recall also that for two Banach function spaces \(X\) and \(Y\) the inclusion \(X \subset Y\) is always continuous. Moreover, \(X = Y\) (resp. \(X \equiv Y\)) means that the spaces \(X\) and \(Y\) have the same elements and their norms are equivalent (resp. equal). If the spaces \(X\) and \(Y\) are isomorphic (resp. are isometric under the isometry \(\lambda \cdot \text{id}\), where \(\lambda > 0\)), then we write \(X \simeq Y\) (resp. \(X \cong Y\)).

Let us remind that the Köthe dual space (or associated space) \(X' = X'(I)\) of a Banach function space \(X\) on \(I\) is defined as

\[
X' := \{f \in L^0(I): \|f\|_X' = \sup_{g \in X, \|g\|_X \leq 1} \int_I |f(x)g(x)| \, dx < \infty\}.
\]

The Köthe dual space is again a Banach function space. Moreover, \(X \uparrow X'' := (X')'\) and \(X = X''\) if and only if the norm in \(X\) has the Fatou property (in short \(X \in (FP)\)), i.e., if for
any sequence \((f_n) \subset X\) with \(0 < f_n \uparrow f\) almost everywhere on \(I\) such that \(\sup_{n \in \mathbb{N}} \|f_n\|_X < \infty\), we have \(f \in X\) and \(\|f_n\|_X \uparrow \|f\|_X\).

A function \(f \in X\), where \(X\) is a Banach function space on \(I\), is said to have an order continuous norm in \(X\) if for any decreasing sequence of sets \(A_n \subset I\) with empty intersection, we have \(\|f \chi_{A_n}\|_X \to 0\) as \(n \to \infty\) (see [13, Proposition 3.5, p. 15]). By \(X_a\) we denote the subspace of all functions with order continuous norm in \(X\). A Banach function space \(X\) on \(I\) is said to be order continuous (we write \(X \in (OC)\) for short) if every element of \(X\) has an order continuous norm, that is, if \(X_a = X\). The subspace \(X_a\) is always closed in \(X\) (cf. [13, Th. 3.8, p. 16]). If \(X\) is an order continuous Banach function space then \(X^* = X'\) (see [13, Theorem 4.1, p. 20]). Moreover, a Banach function space on \(I\) with the Fatou property is reflexive if and only if both \(X\) and \(X'\) are order continuous (cf. [13, Corollary 4.4, p. 23]).

Throughout the paper, we will accept the convention that whenever we take a subset \(A \subset I\), we mean that \(A\) is a Lebesgue measurable set. For a function \(f \in L^0(I)\) we define the support of \(f\) as

\[
\text{supp}(f) := \{x \in I : f(x) \neq 0\}.
\]

For a measurable function \(w : I \to (0, \infty)\) (the weight on \(I\)) and for a Banach function space \(X\) on \(I\), the weighted Banach function space \(X(w) = X(w)(I)\) is defined as

\[
X(w) := \{f \in L^0(I) : fw \in X\},
\]

with the norm \(\|f\|_{X(w)} = \|fw\|_X\). It is clear that \(X(w)\) is a Banach function space on \(I\) and \(X(w)' = X'(1/w)\).

For a function \(f \in L^0(I)\) we define the distribution function \(d_f(\lambda) := m(\{t \in I : |f(t)| > \lambda\})\) for \(\lambda > 0\). We say that two functions \(f, g \in L^0(I)\) are equimeasurable when they have the same distribution functions, i.e., \(d_f \equiv d_g\). By a symmetric space (symmetric Banach function space or rearrangement invariant Banach function space) on \(I\) we mean a Banach function space \(E = (E, \|\cdot\|_E)\) on \(I\) with the additional property that for any two equimeasurable functions \(f, g \in L^0(I)\) if \(f \in E\) then \(g \in E\) and \(\|f\|_E = \|g\|_E\). In particular, \(\|f\|_E = \|f^*\|_E\), where \(f^*(t) := \inf\{\lambda > 0 : d_f(\lambda) \leq t\}\) for \(t \geq 0\).

For general properties of Banach lattices, Banach function spaces and symmetric spaces we refer to the books by Bennett–Sharpley [13], Kantorovich–Akilov [43], Krein–Petunin–Semenov [51], Lindenstrauss–Tzafriri [57], Maligranda [61], Meyer-Nieberg [63], and Wnuk [75].

### 2.2. Cesàro, Copson and Tandori function spaces

For a Banach function space \(X\) on \(I\) the abstract Cesàro function space \(CX = CX(I)\) is defined as

\[
CX := \{f \in L^0(I) : C|f| \in X\}
\]

with the norm \(\|f\|_{CX} = \|C|f|\|_X\), where \(C\) denotes the Cesàro operator (sometimes also called the Hardy operator)

\[
C : f \mapsto Cf(x) := \frac{1}{x} \int_0^x f(t)dt \quad \text{for} \quad 0 < x \in I.
\]

The Copson and Tandori spaces are directly related to the Cesàro spaces. For a Banach ideal space \(X\) on \(I\) we define the abstract Copson function space \(C^*X = C^*X(I)\) as

\[
C^*X := \{f \in L^0(I) : C^*|f| \in X\}
\]

with the norm \(\|f\|_{C^*X} = \|C^*|f|\|_X\), where \(C^*\) denotes the conjugate operator (in the sense of Köthe) to the Cesàro operator \(C\), which will be called the Copson operator, that is

\[
C^* : f \mapsto C^*f(x) := \int_{I \cap [x, \infty)} \frac{f(t)}{t} dt \quad \text{for} \quad x \in I,
\]

\[
C^*(f) := \int_{I \cap [x, \infty)} \frac{f(t)}{t} dt.
\]
and the abstract Tandori function space \( \tilde{X} = X(I) \) as

\[
\tilde{X} := \{ f \in L^0(I) : \tilde{f} \in X \} \quad \text{with the norm} \quad \| f \|_{\tilde{X}} := \| \tilde{f} \|_X ,
\]

where by the nonincreasing majorant \( \tilde{f} \) of a given function \( f \) we understand by

\[
\tilde{f}(x) := \operatorname{esssup}_{t \in I \cap x} |f(t)| \quad \text{for} \quad x \in I.
\]

The abstract Cesàro function spaces are simply a generalization of the well-known classical Cesàro spaces \( \text{Ces}_p[0,1] \) and \( \text{Ces}_p[0,\infty) \). Indeed, if we take \( X = L^p \), where \( 1 \leq p \leq \infty \), then \( \text{Ces}_p = C/L^p \) (note, that in the case when \( p = 1 \) we have \( \text{Ces}_1[0,1] = L^1(\ln(1/t)) \) and \( \text{Ces}_1[0,\infty) = \{ 0 \} \)). The space \( \text{Ces}_{\infty}[0,1] \) appeared already in 1948 and it is known as the Korenblyum–Krein–Levin space \( K \) (see [50, 75] p. 26 and p. 61 and [77] pp. 469–471).

Various properties of these spaces have been studied by Astashkin in [4], Astashkin–Maligranda in [6, 12], Hassard–Hussein in [36], Kamińska–Kubiak in [41], Kubiak in [52], Shiue in [68] and Sy–Zhang–Lee in [76]. Taking \( X = L^\Phi \), \( X = \Lambda_\varphi \) or \( X = M_\varphi \) we obtain the Cesàro–Orlicz, Cesàro–Lorentz and Cesàro–Marcinkiewicz spaces, respectively, which have been studied intensively by Astashkin–Leśnink–Maligranda in [5], Kiwerski–Kolwicz in [45–47], and Kiwerski–Tomaszewski in [48]. General consideration of this construction when \( X \) is a Banach function space or sometimes a symmetric space were initiated in [54] and [55]. More recently, the structure of these spaces, especially in their general form, is quite popular to study among various researchers such as Astashkin–Leśnink–Maligranda [5], Curbera–Ricker [22], Delgado–Soria [25], and Kiwerski–Tomaszewski [48].

Note that Cesàro function spaces \( CX \) are never symmetric nor reflexive. Nevertheless, at least when \( X \) is a symmetric space, there are some connections and similarities with the classical theory of normed ideal spaces and symmetric spaces. For example, it has been shown in [48, Theorem 3] that order continuity property “transfers” quite well between \( X \) and \( CX \). Moreover, \( \text{Ces}_{\infty} \) and \( \text{ces}_{\infty} \) are isomorphic, see [5, Theorem 3.1] (this is analogous to the well-known Pelczyński result [66], which states that spaces \( L^\infty \) and \( \ell^\infty \) are isomorphic). Furthermore, \( \ell^1 \) has the Schur property but is not isomorphic to \( \ell^1 \) [3, Theorem 3.1]. Of course, there are also big differences if we compare the results obtained in the cases of a finite and infinite interval. For example, this differences can be seen in results on the Köthe duality for abstract Cesàro function spaces in [54] Theorems 3, 4 and 5] (cf. also Theorem A below) or in the interpolation results proved in [56].

It is worth mentioning here that the study of the classical Cesàro sequence spaces \( \text{ces}_p = C^p \) for \( 1 < p \leq \infty \) began much earlier and many results have been obtained, see [7] and [12] and the references therein.

Copson function spaces \( \text{Cop}_p = C^*L^p \) and Copson sequence spaces \( \text{cop}_p = C^*\ell^p \) have appeared already in Bennett’s memoir [14], pp. 25–28 and p. 123]. Furthermore, Astashkin–Maligranda used Copson function spaces \( \text{Cop}_p \) to describe their interpolation results, see [9, Section 2]. The abstract Copson spaces have been studied by Leśnink–Maligranda in [56]. For some connections between the Cesàro and Copson function spaces and even their iterations \( CCX \) and \( C^*C^*X \) we refer to [56] Theorem 1 (a) and (b)].

Leśnink and Maligranda suggested in [54] to call \( \tilde{X} \) the generalized Tandori spaces since Tandori [74] proved in 1954 that \( (\text{Ces}_{\infty}[0,1])' = \tilde{L}^1[0,1] \). Moreover, these spaces appeared earlier but without such name, e.g., in [7] and [55]. The Tandori spaces are related to the Köthe duality of Cesàro spaces. Many special cases of this general construction have been studied by Alexiewicz [2], Astashkin–Maligranda [7], Bennett [14], Jagers [39], Kamińska–Kubiak [41] and Luxemburg–Zaanen [58]. The general Tandori spaces \( \tilde{X} \) have been studied by Leśnink–Maligranda in [54–56] and the following Köthe duality result was proved in [54] Theorems 3, 5 and 6].
Theorem A. If $X$ is a Banach function space on $I = [0, \infty)$ such that the Cesàro operator $C$ and the dilation operator $\sigma_\tau$ (for some $0 < \tau < 1$) are bounded on $X$, then
\begin{equation}
(CX)' = \widetilde{X}'.
\end{equation}
Furthermore, if $X$ is a symmetric space on $I = [0, 1]$ with the Fatou property such that both operators $C$ and $C^*$ are bounded on $X$, then
\begin{equation}
(CX)' = \widetilde{X}(w) \quad \text{where} \quad w: [0, 1) \ni x \mapsto \frac{1}{1-x}.
\end{equation}

The dilation operator $\sigma_\tau$ for $\tau > 0$ is defined by $\sigma_\tau f(x) := f(x/\tau)$ for $0 < x < \infty$ and
\begin{align*}
\sigma_\tau f(x) := \begin{cases} 
 f(x/\tau), & \text{if } x < \min\{1, \tau\} \\
 0, & \text{if } \tau \leq x < 1
\end{cases}
\end{align*}
for $0 < x \leq 1$. This operator is bounded in any symmetric space $X$ on $I$ and $\|\sigma_\tau\|_{X \to X} \leq \max\{1, \tau\}$ (see [13], p. 148, and [51], pp. 96–98). The Boyd indices of a symmetric space $X$ are defined by
\begin{align*}
p(X) := \lim_{\tau \to \infty} \frac{\ln \tau}{\ln \|\sigma_\tau\|_{X \to X}} \quad \text{and} \quad q(X) := \lim_{\tau \to 0^+} \frac{\ln \tau}{\ln \|\sigma_\tau\|_{X \to X}}.
\end{align*}
Let us mention that these numbers can be different for $X$ on $I = [0, 1]$ and for $X$ on $I = [0, \infty)$, but always we have estimates $1 \leq p(X) \leq q(X) \leq \infty$ (see [51], [57] and [60]).

We will use the following result from the Lindenstrauss–Tzafriri book [57], Proposition 2.b.3, p. 132.

Theorem B. If $X$ is a symmetric space on $I$, then there are constants $A, B > 0$ such that
\begin{equation}
L^p \cap L^q \xrightarrow{A} X \xrightarrow{B} L^p + L^q,
\end{equation}
for every $p, q > 0$ satisfying $1 \leq p < p(X)$ and $q(X) < q \leq \infty$, where $p(X)$ and $q(X)$ are the Boyd indices of the space $X$,
\begin{equation}
L^p \cap L^q := \{ f \in L^0(I) : \|f\|_{L^p \cap L^q} = \max\{\|f\|_{L^p}, \|f\|_{L^q}\} < \infty\},
\end{equation}
and
\begin{equation}
L^p + L^q := \{ f \in L^0(I) : \|f\|_{L^p + L^q} = \inf_{g \in L^p, h \in L^q} \{\|g\|_L + \|h\|_{L^q}\} < \infty\}.
\end{equation}
Moreover, if $p(X) = 1$ (resp. $q(X) = \infty$) then we can take $p = 1$ (resp. $q = \infty$) in (2.3).

Let us recall the important result about boundedness of the Cesàro operator (cf. [53], Theorem 17, p. 130).

Theorem C. Let $X$ be a symmetric space on $I$. Then
\begin{enumerate}
\item the Cesàro operator $C$ is bounded on $X$ if and only if $p(X) > 1$,
\item the Copson operator $C^*$ is bounded on $X$ if and only if $q(X) < \infty$.
\end{enumerate}

Throughout the article we will use the following notation: the norm of the function $f_\lambda : I \ni x \mapsto \frac{1}{x} \chi_{[\lambda, m(I)]}(x)$, where $0 < \lambda \in I$, in a Banach function space $X$ on $I$, will be denoted by $\| \frac{1}{x} \chi_{[\lambda, m(I)]}(x) \|_{X(I)}$, i.e.,
\begin{align*}
\| f_\lambda \|_{X(I)} := \left\| \frac{1}{x} \chi_{[\lambda, m(I)]}(x) \right\|_{X(I)},
\end{align*}
and the norm of the function \((C \vert f \vert)\chi_A: I \ni x \mapsto \frac{1}{x} \int_0^x |f(t)|\, dt \chi_A(x)\), where \(A \subset I\), will be denoted by

\[
\left\| \frac{1}{x} \int_0^x |f(t)|\, dt \chi_A(x) \right\|_{X(I)} := \|(C \vert f \vert)\chi_A\|_{X(I)}.
\]

Recall that if \(X\) is a Banach function space on \(I\) and \(X \in (FP)\), then the Cesàro operator \(C\) is bounded on \(X\) if and only if the Copson operator \(C^*\) is bounded on \(X'\) and \(\|C\|_{X \rightarrow X} = \|C^*\|_{X' \rightarrow X'}\) (see [59, Remark 1 (iv)]). Note also that if \(X\) is a Banach function space on \(I\), then the assumption \(C: X \rightarrow X\) is in fact equivalent to the statement that the Cesàro operator \(C\) is bounded on \(X\) (see [15]). Clearly, if the operator \(C\) is bounded on \(X\), then \(X \hookrightarrow CX\). Therefore, the space \(CX\) is nontrivial with \(\text{supp}(CX) = \text{supp}(X) = I\). We will now collect some other useful facts about an abstract Cesàro function space \(CX\), which are proved in [5, the proof of Proposition 2.2], [54, Theorem 1 (a) and (b)] and [48, Lemma 2].

**Theorem D.** Let \(X\) be a Banach function space on \(I\). Then

(i) \(CX[0, 1]\) is nontrivial if and only if \(\chi_{[\lambda, 1]} \in X\) for some \(0 < \lambda < 1\).

(ii) \(CX[0, \infty)\) is nontrivial if and only if \(\frac{1}{x} \chi_{[\lambda, \infty)}(x) \in X\) for some \(\lambda > 0\).

In particular, \(\lambda, m(I) \subset \text{supp}(CX)\) for some \(0 < \lambda < m(I)\).

If \(X\) is a Banach function space on \(I\) such that the Cesàro operator \(C\) is bounded on \(X\) or \(X\) is a symmetric space on \([0, 1]\) or \(X\) is a symmetric space on \([0, \infty)\) with \(CX[0, \infty) \neq \{0\}\), then

(iii) \(\chi_{[\lambda, 1]} \in X\) for all \(0 < \lambda < 1\) if \(I = [0, 1]\),

(iv) \(\frac{1}{x} \chi_{[\lambda, \infty)}(x) \in X\) for all \(\lambda > 0\) if \(I = [0, \infty)\).

In particular, \(\text{supp}(CX) = \text{supp}(X) = I\). Let us emphasize also that if \(X\) is a symmetric space on \([0, 1]\), then \(CX\) is always nontrivial.

### 3. Some auxiliary results

We will give below a few simple but useful facts about the Copson spaces.

**Lemma 3.1.** Let \(X\) be a Banach function space on \(I\). Then the Copson space \(C^*X\) is nontrivial if and only if \(\chi_{[0, \lambda]} \in X\) for some \(0 < \lambda < m(I)\).

**Proof.** Assume that \(C^*X \neq \{0\}\). Then there exists \(f \in C^*X\) with \(|f(x)| > 0\) for \(x \in A \subset I\) and \(m(A) > 0\). Of course, we can also find \(\lambda > 0\) such that

\[
\int_{\lambda}^{m(I)} \frac{|f(t)|}{t} \, dt := \eta > 0.
\]

Therefore, we obtain that

\[
\eta \chi_{[0, \lambda]}(x) = \int_{\lambda}^{m(I)} \frac{|f(t)|}{t} \, dt \chi_{[0, \lambda]}(x)
\]

\[
\leq \int_{x}^{m(I)} \frac{|f(t)|}{t} \, dt \chi_{[0, \lambda]}(x) \leq C^* |f| (x) \in X,
\]

so \(\chi_{[0, \lambda]} \in X\).

If \(\chi_{[0, \lambda]} \in X\) for some \(0 < \lambda < m(I)\), then for \(0 < a < \lambda\) we have

\[
\left\| \chi_{[a, \lambda]} \right\|_{C^*X} \leq \left\| \left( \int_a^\lambda \frac{dt}{t} \right) \chi_{[a, \lambda]} \right\|_{X} = \ln(\frac{\lambda}{a}) \left\| \chi_{[0, \lambda]} \right\|_{X} < \infty,
\]

which means that \(C^*X \neq \{0\}\).

\(\square\)
Corollary 3.2. (1) The Copson space $C^*X$ is always nontrivial whenever $X$ is a symmetric space.

(2) If $X$ is a Banach function space on $I$ such that the operator $C^*$ is bounded on $X$, then \( \text{supp}(C^*X) = I \). In particular, the Copson space $C^*X$ is nontrivial. Moreover,

(i) \( L^\infty[0,1]_{[0,\lambda]} \hookrightarrow X[0,1] \) for all \( 0 < \lambda < 1 \) and, in addition, \( L^\infty[0,1] \hookrightarrow X[0,1] \) if $X$ has the Fatou property,

(ii) \( L^\infty_{\text{fin}}[0,\infty) \subset X[0,\infty) \) and, in addition, \( L^\infty_b[0,\infty) \hookrightarrow X[0,\infty) \) if $X$ has the Fatou property,

where \( L^\infty_{\text{fin}}(I) := \{ f \in L^\infty(I) : m(\text{supp}(f)) < \infty \} \) and \( (L^\infty(I))_b = L^\infty(I)_b \) is the closure of \( L^\infty_{\text{fin}}(I) \) in $L^\infty(I)$.

Proof. If $X$ is a symmetric space on $I$, then \( \chi_{[0,\lambda]} \in X \) for all \( 0 < \lambda < m(I) \), so \( C^*X \neq \{0\} \), see Lemma 3.1.

It is also clear that if the operator $C^*$ is bounded on $X$ then $X \hookrightarrow C^*X$ and consequently $\text{supp}(C^*X) = I$.

(i) Of course, the condition $L^\infty[0,1]_{[0,\lambda]} \hookrightarrow X$ is equivalent to \( \chi_{[0,\lambda]} \in X \). Take \( 0 < \lambda < 1 \) and let \( f_0 \) be a weak unit in $X$. Then \( \int_{[\lambda]} \frac{|f_0(t)|}{tdt} \leq \delta > 0 \) and proceeding as in the first part of the proof of Lemma 3.1 we get $\chi_{[0,\lambda]} \in X$. If, additionally, $X \in (FP)$, then we conclude that $\chi_{[0,1]} \in X$, i.e., \( L^\infty[0,1] \hookrightarrow X \).

(ii) Similarly, as in the case (i), above we obtain that \( L^\infty_{\text{fin}}[0,\infty) \subset X[0,\infty) \) (note only that \( L^\infty_{\text{fin}} \) is not complete whence the inclusion \( L^\infty_{\text{fin}} \subset L^\infty_b \) is not continuous). Suppose now that $X \in (FP)$ and take \( f \in L^\infty_b[0,\infty) \). To show that \( f \in X \) it is enough to take a sequence \( (f_n) \subset L^\infty_{\text{fin}}[0,\infty) \) with \( 0 \leq f_n \uparrow f \).

Many results, although we can probably say that almost all, in the theory of Cesàro and Copson function spaces are proved under the assumption that at least one of the operators $C$ or $C^*$ is bounded on $X$. As mentioned in the introduction, we are able to prove our results under the essentially weaker assumption (actually the weakest possible one) that the Cesàro or Copson function space is nontrivial. In this context, it seems reasonable to give several examples discussing the difference between these two assumptions, because many naturally appearing spaces have the property that the operator $T$, where $T = C$ or $T = C^*$, is not bounded on $X$ but $TX \neq \{0\}$ or even $\text{supp}(TX) = I$.

Example 3.3. (a) Consider, as in [54] Example 2], the space $L^p(w_1)$ on $[0,\infty)$, where \( 1 < p < \infty \) and

\[
 w_1(x) = \frac{1}{1 - x} \chi_{[0,1]}(x) + \chi_{[1,\infty)}(x).
\]

Then $\text{supp}(X) = [0,\infty)$, $\text{supp}(CX) = [1,\infty)$ and $\text{supp}(C^*X) = [0,1]$. Put $X = L^1(w_2)$, where

\[
 w_2 : I \ni x \mapsto \frac{1}{x}.
\]

Then \( \chi_{[0,\lambda]} \notin X \) for every \( 0 < \lambda < m(I) \), so $C^*X = \{0\}$. Finally, take $X = L^\infty(w_3)$, where

\[
 w_3 = \text{id}_I : I \ni x \mapsto x.
\]

Then $CX \equiv L^1$. On the other hand, if $C : X \to X$ is bounded, then $X \hookrightarrow CX$ but $L^\infty(w_3) \not\hookrightarrow L^1$ (just take $f(x) = 1/x$), so $C$ is not bounded on $L^\infty(w_3)$ and $\text{supp}CX = I$.

(b) It is easy to see that the space $\text{Ces}l_{1}[0,1]$ is just a weighted $L^1(w)[0,1]$ space, where $w(t) = \ln(1/t)$ for \( 0 < t \leq 1 \). Indeed, we have

\[
 \int_0^1 \left( \frac{1}{x} \int_0^x |f(t)| \, dt \right) dx = \int_0^1 \left( \int_t^1 \frac{dx}{x} \right) |f(t)| \, dt = \int_0^1 |f(t)| \ln\left( \frac{1}{t} \right) \, dt,
\]
see [7, Theorem 1 (a)]. Therefore, despite the fact that the Cesàro operator $C$ is not bounded on $L^1[0,1]$ (cf. Theorem C), we see again that $\text{supp}(\text{Ces}_1[0,1]) = [0,1]$. Thus, if $f \in \text{Ces}_1[0,1]$ and $\text{supp}(f) \subseteq [a,b]$, where $0 < a < b < 1$, then

$$\ln(\frac{1}{b})\|f\|_{L^1[0,1]} \leq \|f\|_{\text{Ces}_1[0,1]} \leq \ln(\frac{1}{a})\|f\|_{L^1[0,1]},$$

see also [6, Lemma 1, inequality (4)]. Equality (3.1) shows by the way that $\text{Ces}_1[0,\infty) = \{0\}$, cf. [7, Theorem 1 (a)].

(c) Clearly, $\text{Cop}_1 \equiv L^1$ and $\text{Cop}_\infty \equiv L^1(1/t)$ because

$$\|f\|_{\text{Cop}_1} = \int_{I} (\int_{x}^{m(I)} \frac{|f(t)|}{t} dt) dx = \int_{I} (\int_{0}^{t} \frac{|f(t)|}{t} dt) dt = \|f\|_{L^1},$$

and

$$\|f\|_{\text{Cop}_\infty} = \sup_{x \in I} \int_{x}^{m(I)} \frac{|f(t)|}{t} dt = \int_{0}^{m(I)} \frac{|f(t)|}{t} dt = \|f\|_{L^1(1/t)}.$$

Again, $\text{supp}(\text{Cop}_\infty) = I$ but the Copson operator $C^*$ is not bounded on $L^\infty$, see Theorem C.

(d) Let $X$ be a symmetric space on $[0,1]$ with $p(X) = 1$. Then the Cesàro operator $C$ is not bounded on $X$ but $CX \neq \{0\}$, cf. Theorem C and D. For example, if $X$ is the Zygmund space $L \log L[0,1]$ (see [13, Definition 6.1, p. 243]), then $p(X) = 1$ (see [13, Theorem 6.5, p. 247]).

(e) Consider a symmetric space $X$ on $I$ such that the Copson operator $C^*$ is not bounded on $X$. Then, by Lemma 3.1 $C^*X \neq \{0\}$, because $\chi_{[0,\lambda]} \in X$ for each $0 < \lambda < m(I)$. In particular, we can take $X = L^\Phi$, where $L^\Phi$ is the Orlicz space generated by the Orlicz function $\Phi$ which does not satisfy the suitable $\Delta_2$-condition. Since $\Phi \notin \Delta_2$, so $q(L^\Phi) = \infty$ (see [57, Proposition 2.5, p. 139] and [60, Theorem 3.2, p. 22]) and consequently $C^*$ is not bounded on $X$, cf. Theorem C.

(f) Suppose that $X$ is the Orlicz space $L^\Phi$ generated by the Orlicz function

$$\Phi(x) = x \log(1 + x).$$

First, note that $p(L^\Phi) = \alpha_{\Phi}$, where $\alpha_{\Phi}$ is the lower Orlicz-Matuszewska index of $\Phi$ (see [57, Proposition 2.5, p. 139 and Remark 2, p. 140]). Moreover, it is not difficult to calculate that $\alpha_{\Phi} = 1$ (cf. [60, pp. 7–21]). Consequently, the operator $C$ is not bounded on $X$, cf. Theorem C. We claim that $CX \neq \{0\}$. It is clear when $I = [0,1]$. In fact, $L^\Phi$ is a symmetric space, so $\chi_{[0,\lambda]} \in X$ for each $0 < \lambda < 1$ and we can apply Theorem D. If $I = [0,\infty)$, according to Theorem D, we need to show that $(f_\lambda : x \mapsto \frac{1}{\lambda} \chi_{[\lambda,\infty)}(x)) \in L^\Phi$ for some $\lambda > 0$. Recall that $f \in L^\Phi$, whenever $\int_{0}^{\infty} \Phi(\gamma \|f(x)\|) dx < \infty$ for some $\gamma > 0$ (see [61]). We have

$$\int_{0}^{\infty} \Phi(\|f_\lambda(x)\|) dx = \int_{\lambda}^{\infty} \frac{1}{x} \log \left(1 + \frac{1}{x}\right) dx \leq \int_{\lambda}^{\infty} \frac{1}{x^2} dx = \frac{1}{\lambda} < \infty,$$

and the claim follows.

4. COPIES OF $\ell^1$, $\ell^\infty$, $L^1[0,1]$ AND $L^\infty[0,1]$ IN CESÀRO, COPSON AND TANDORI FUNCTION SPACES

The norms of the Cesàro and Copson function spaces are generated by a positive sublinear operator $T$, where $T$ stands for the Cesàro or the Copson operator, and by the norm of a Banach function space $X$. Thus, in a sense, the space $TX$ is a nontrivial mix of the space $L^1$ and $X$ and some similarities to both these spaces can be found in $TX$. We will make these statements more precise showing first that the Cesàro and Copson function spaces contain “good” copies of $L^1[0,1]$ and $\ell^1$. 

10
Lemma 4.1. Let $X$ be a Banach function space on $I$ such that $CX \neq \{0\}$. Then there are numbers $0 < a < b < m(I)$ such that

$$
(4.1) \quad \left\| \frac{1}{x} \chi_{[b,m(I)]}(x) \right\|_X \|f\|_{L^1(I)} \leq \|f\|_{CX} \leq \left\| \frac{1}{x} \chi_{[a,m(I)]}(x) \right\|_X \|f\|_{L^1(I)},
$$

for all $f \in CX$ with $\text{supp}(f) \subset [a, b]$. In particular, the space $CX$ contains a complemented copy of $L^1[0,1]$.

**Proof.** We will give only a sketch of the proof because, in fact, this lemma is just a reformulation of Proposition 2.2 in [5] (cf. also [12] Theorem 5.1 (b)).

Let $I = [0,1]$. First of all, $\chi_{[\lambda,1]} \in X$ for some $0 < \lambda < 1$ due to nontriviality of the space $CX$, see Theorem D. Take $a = \lambda$ and choose a number $b \in (a,1)$. Then $\frac{1}{x} \chi_{[a,1]}(x) \in X$ and, from the ideal property, also $\frac{1}{x} \chi_{[b,1]}(x) \in X$. Now, for such numbers $0 < a < b < 1$ and $f \in CX$ with $\text{supp}(f) \subset [a, b]$ it is obvious that

$$
\frac{1}{x} \left\| f \right\|_{L^1[0,1]} \chi_{[b,1]}(x) \leq \frac{1}{x} \int_0^x |f(t)| \, dt \leq \frac{1}{x} \left\| f \right\|_{L^1[0,1]} \chi_{[a,1]}(x),
$$

for any $0 < x \in I$. Thus,

$$
(4.2) \quad \left\| \frac{1}{x} \chi_{[b,1]}(x) \right\|_X \left\| f \right\|_{L^1[0,1]} \leq \left\| f \right\|_{CX} = \left\| \frac{1}{x} \int_0^x |f(t)| \, dt \right\|_X \leq \left\| \frac{1}{x} \chi_{[a,1]}(x) \right\|_X \left\| f \right\|_{L^1[0,1]},
$$

and (i) follows.

At this point, it is clear that $\{f \in L^1[0,1]: \text{supp}(f) \subset [a, b]\} \simeq L^1[0,1]$ and this copy of $L^1[0,1]$ is in fact complemented because the projection $P: f \mapsto f \chi_{[a,b]}$ is bounded.

In the case when $I = [0, \infty)$ the proof is completely analogous. $\square$

**Lemma 4.2.** Let $X$ be a Banach function space on $I$ such that $C^*X \neq \{0\}$. Then there are numbers $0 < a < b < m(I)$ such that

$$
(4.3) \quad \left\| \chi_{[0,a]} \right\|_X \left\| f \right\|_{L^1(1/t)(I)} \leq \left\| f \right\|_{C^*X} \leq \left\| \chi_{[0,b]} \right\|_X \left\| f \right\|_{L^1(1/t)(I)},
$$

for $f \in C^*X$ with $\text{supp}(f) \subset [a, b]$. In particular, the space $C^*X$ contains a complemented copy of $L^1[0,1]$.

**Proof.** We will give the proof only if $I = [0,1]$. The remaining case is analogous.

Suppose $I = [0,1]$. Thanks to the assumption that the Copson function space $C^*X$ is nontrivial we get $\chi_{[\lambda,1]} \in X$ for some $0 < \lambda < 1$, see Lemma 5.1. Take $b = \lambda$ and choose a number $a \in (0, b)$. If $f \in C^*X$ and $\text{supp}(f) \subset [a, b]$, then we have

$$
C^* |f| (x) = \int_x^1 \frac{|f(t)|}{t} \, dt \geq \int_a^b \frac{|f(t)|}{t} \, dt \chi_{[0,a]}(x) = \left\| f \right\|_{L^1(1/t)(0,1]} \chi_{[0,a]}(x).
$$

Moreover,

$$
C^* |f| (x) = \int_x^1 \frac{|f(t)|}{t} \, dt \leq \int_a^b \frac{|f(t)|}{t} \, dt \chi_{[0,b]}(x) = \left\| f \right\|_{L^1(1/t)(0,1]} \chi_{[0,b]}(x).
$$

Putting together the above inequalities we obtain (4.3).

The last part of this lemma is clear since

$$
\{f \in L^1(1/t)[0,1]: \text{supp}(f) \subset [a, b]\} = \{f \in L^1[0,1]: \text{supp}(f) \subset [a, b]\} \simeq L^1[0,1],
$$

whenever $0 < a < b < 1$ (because $\frac{1}{t} \left\| f \chi_{[a,b]} \right\|_{L^1[0,1]} \leq \left\| f \chi_{[a,b]} \right\|_{L^1(1/t)[0,1]} \leq \frac{1}{a} \left\| f \chi_{[a,b]} \right\|_{L^1[0,1]}$ for $f \in L^1(1/t)[0,1]$) and it is enough to take the projection $P: f \mapsto f \chi_{[a,b]}$. $\square$
If additionally the Cesàro or the Copson operator is bounded on the Banach function space $X$, then we can deduce a little stronger versions of Lemma 4.1 and Lemma 4.2 respectively. More precisely, if the Cesàro operator $C$ is bounded on $X$, then it follows from Theorem D that $\text{supp}(CX) = I$ and consequently, Lemma 4.1 holds true for all $0 < a < b < m(I)$. Of course, due to Corollary 3.2, analogous remark holds true also for every nontrivial Copson function space $C^*X$.

It is clear, that every nontrivial Cesàro and Copson function space contains also a complemented copy of $\ell^1$ (simply because the space $L^1[0,1]$ contains such a copy). Moreover, James’s distortion theorem for $\ell^1$ states that a Banach space $X$ contains an isomorphic copy of $\ell^1$ if and only if it contains an almost isometric copy of $\ell^1$, that is, for every $0 < \varepsilon < 1$, there exists a sequence $(x_n) \subset X$ such that $(1 - \varepsilon) \sum_{n=1}^{\infty} |\alpha_n| \leq \|\sum_{n=1}^{\infty} \alpha_n x_n\|_X \leq \sum_{n=1}^{\infty} |\alpha_n|$, for all $\alpha = (\alpha_n) \in \ell^1$. Therefore, as an immediate conclusion from the complemented version of James’s distortion theorem [32, Theorem 2] we obtain the following result.

**Corollary 4.3.** Let $T = C$ or $T = C^*$. If $X$ is a Banach function space on $I$ such that $TX \neq \{0\}$, then the space $TX$ contains a complemented almost isometric copy of $\ell^1$. In particular, the space $TX$ is not reflexive.

It turns out that we can prove even a stronger versions of our Lemmas 4.1 and 4.2 and Proposition 2.2 from [5].

**Theorem 4.4.** Let $X$ be a Banach space and assume that $X$ contains a complemented copy of a Banach space $Z$. Then there exists an equivalent norm on $X$ such that $X$ contains an isomorphic copy of $Z$. In particular, if $X$ is a Banach function space and $T = C$ or $T = C^*$, then:

(i) the space $TX$ can be renormed to contains an isometric copy of $L^1[0,1]$, whenever $TX \neq \{0\}$.

(ii) every nontrivial Tandori function space $\tilde{X}$ can be renormed to contain an isometric copy of $L^\infty[0,1]$.

**Proof.** First, let $P: X \to X$ be a projection onto $Y_1 \subset X$ and $T$ be an isomorphism from $Z$ onto $Y_1$. Consequently, we have the following diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{T^{-1}} & Z \\
\downarrow & & \\
Y_1 & \xrightarrow{P} & X
\end{array}
$$

and

$$
\begin{array}{ccc}
X & \xrightarrow{T^{-1}} & Z \\
\downarrow & & \\
Y_2 & \xrightarrow{id_X-P} & X
\end{array}
$$

that is, $X \simeq Y_1 \oplus Y_2 \simeq Z \oplus Y_2$. We will introduce a new norm $\|\cdot\|_X$ on the space $X$ which is defined as

$$
\|x\|_X := \|T^{-1}Px\|_Z + \|\text{id}_X - P\|_X.
$$

(4.4)

It turns out that this norm is equivalent to the original one. In fact,

$$
\|x\|_X \leq \|T^{-1}\|_{Y \to Z} \|Px\|_Y + (1 + \|P\|) \|x\|_X
\leq \|T^{-1}\|_{Y \to Z} \|P\| \|x\|_X + (1 + \|P\|) \|x\|_X
\leq (1 + \|P\| + \|T^{-1}\|_{Y \to Z} \|P\|) \|x\|_X.
$$

On the other hand,

$$
\|x\|_X = \|Px + (\text{id}_X - P)x\|_X
\leq \|Px\|_Y + \|\text{id}_X - P\|_X
\lesssim \|T^{-1}Px\|_Z + \|\text{id}_X - P\|_X \|x\|_X = \|x\|_X.
$$
Combining the above inequalities, we see that the norm $\|\cdot\|_X$ is equivalent to the norm $\|\cdot\|_X$. Moreover, if $y \in Y$ then

\[(4.5)\quad \|y\|_X = \|T^{-1}Py\|_Z + \|(\id_X - P)y\|_X = \|T^{-1}Py\|_Z = \|T^{-1}y\|_Z,\]

which means that $T : Z \to (Y, \|\cdot\|_X)$ is an isometry.

(i) This is clear due to Lemmas 4.1 and 4.2 and the first part of the proof.

(ii) Let’s start with a simple observation. If $X$ is a Banach function spaces on $I$, then nontriviality of the space $\tilde{X}$ is equivalent to the statement that $\chi_{[0,\lambda]} \in X$ for some $0 < \lambda < m(I)$. In fact, if $\tilde{X} \neq \{0\}$, so we can find an element $f \in \tilde{X}$ such that $|f(x)| > 0$ for $x \in A \subset I$ and $m(A) > 0$. Setting $B_n := \{x \in A : |f(x)| > 1/n\}$, where $n \in \mathbb{N}$, we see that there exists $n_0 \in \mathbb{N}$ with $m(B_{n_0}) > 0$. Therefore, we have

\[
\frac{1}{n_0} \chi_{[0,m(B_{n_0})]} \leq \tilde{f} \in X,
\]

that is, $\chi_{[0,\lambda]} \in X$ for $\lambda = m(B_{n_0})$. The second implication is clear. Now, since $\tilde{X} \neq \{0\}$, it follows that $(0, \lambda) \subset \text{supp}(\tilde{X})$ for some $0 < \lambda < m(I)$. Using the same argument as in [5, Proposition 2.2] but for $0 < a < b < \lambda$ we can prove that the Tandori space $\tilde{X}$ contains a complemented copy of $L^\infty[0,1]$. To finish the proof it is enough to apply once again the first part.

Let us note that if $X$ is a Banach function space on $I$ with $CX \neq \{0\}$, then exactly as we just did in Theorem 4.3, we get the following diagram

\[(4.6)\quad CX \xrightarrow{P} Y_1 \xrightarrow{\id} L_1[0,1] \leftarrow Q L_1[a,b]\]

where $P : f \mapsto f|_{[a,b]}$ for $0 < a < b < m(I)$ is a bounded projection, $Y_1 = L^1[a,b] := \{f \in L^1(I) : \text{supp}(f) \subset [a,b]\}$, the mapping $Q$ is a linear isometry between $L^1[a,b]$ and $L^1[0,1]$ and

\[
\|f\|_{CX} = \|f\chi_{[a,b]}\|_{L^1} + \|f\chi_{(0,a)\cup(b,m(I))}\|_{CX},
\]

for $f \in CX$. Now, if we take $X = L^p$ for $1 \leq p < \infty$ if $I = [0,1]$ and $1 < p < \infty$ if $I = [0,\infty)$, then we obtain the Astashkin–Maligranda result from [10, Lemma 4] concerning analogous renormings of classical Cesàro function spaces $Ces_p$.

Recall that a Banach function space $X$ contains an order asymptotically isometric copy of $\ell^1$, whenever there is a sequence $(f_n) \subset X$ with pairwise disjoint supports and a sequence $(\varepsilon_n) \subset (0,1)$ such that $\varepsilon_n \to 0$ and

\[(4.7)\quad \sum_{n=1}^{\infty} (1 - \varepsilon_n) |\alpha_n| \leq \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_{X} \leq \sum_{n=1}^{\infty} |\alpha_n|,
\]

for each $\alpha = (\alpha_n) \in \ell^1$. This notion was introduced by Dowling–Lennard in [29, Definition 1.1] and used to show that every nonreflexive subspace of $L^1[0,1]$ fails the fixed point property.

The notion of an asymptotically isometric copy of $\ell^1$ is closely related to an almost isometric copy of $\ell^1$ and consequently to James’s distortion theorem (see [28, Question] and [44, p. 270]). However, Dowling–Johnson–Lennard–Turett [28] gave an example of a renorming of the space $\ell^1$ which contains no asymptotically isometric copy of $\ell^1$. We will further extend the class of spaces which contains an asymptotically isometric copy of $\ell^1$ showing that nontrivial Cesàro and Copson function spaces always contain such a copy. Note that for $Ces_p$-spaces such a claim has been proved in [6, Theorems 1 and 2]. It would seem naturally to look for a generalization of
this result for symmetric spaces first (or even first for Orlicz spaces). Amazingly, it turns out that symmetry of the space $X$ is not important in our proof.

Before giving the proof, for $X$ being a Banach function space on $I$, let us define a function $F_X$ as follows

\[(4.8) \quad F_X := F_X[0,1]: I \ni \lambda \to \left\| \frac{1}{x} \chi(\lambda,1)(x) \right\|_X \in [0, \infty],\]

if $I = [0,1]$, and

\[(4.9) \quad F_X := F_X[0,\infty]: I \ni \lambda \to \left\| \frac{1}{x} \chi(\lambda,\infty)(x) \right\|_X \in [0, \infty],\]

if $I = [0, \infty)$.

**Theorem 4.5.** Let $X$ be a Banach function space on $I$ such that the Cesàro function space $CX$ is nontrivial. Then the space $CX$ contains an order asymptotically isometric copy of $\ell^1$.

**Proof.** Suppose $I = [0,1]$. Since $CX \neq \{0\}$, so $\chi_{[\lambda_0,1]} \in X$ for some $0 < \lambda_0 < 1$, see Theorem D. For each $\lambda_0 < a < 1$ set

$$\Omega_a := \{\lambda \in (\lambda_0,1): F_X(\lambda) = F_X(a)\}.$$ 

Of course, card $(\Omega_a) \geq 1$. Let us now consider the following two cases.

(a) Assume that card $(\Omega_a) = 1$ for every $a \in (\lambda_0,1)$. Obviously, the function $F_X$ is nonincreasing in the interval $[\lambda_0,1]$, whence it contains at most countably many points of discontinuity. Let $\lambda_0 < a_0 < 1$ be a point of continuity of the function $F_X$. Take a sequence $(a_n) \subset (\lambda_0, a_0)$ such that $a_n \uparrow a_0$ as $n \to \infty$ and put

$$g_n := \frac{\chi(a_n,a_{n+1})}{\left\| \chi(a_n,a_{n+1}) \right\|_{CX}}.$$ 

From the definition $\text{supp}(g_n) = (a_n, a_{n+1}) \subset (a_n, a_0)$ and $\text{supp}(g_n) \cap \text{supp}(g_m) = \emptyset$ if $n \neq m$, $m,n \in \mathbb{N}$. Using the right-hand side of the estimate (4.1) we have

\[(4.10) \quad \left\| \chi(a_n,a_{n+1}) \right\|_{CX} \leq \left\| \frac{1}{x} \chi(a_n,1)(x) \right\|_X \left\| \chi(a_n,a_{n+1}) \right\|_{L^1[0,1]} \]

\[(4.11) \quad = \left\| \frac{1}{x} \chi(a_n,1)(x) \right\|_X (a_{n+1} - a_n) = F_X(a_n)(a_{n+1} - a_n). \]

Furthermore, using left-hand side of (4.11) and the above estimate, since elements $g_n$ are mutually disjoint, we obtain

$$\left\| \sum_{n=1}^{\infty} \alpha_n g_n \right\|_{CX} \geq \left\| \frac{1}{x} \chi_{[a_0,1]}(x) \right\|_X \sum_{n=1}^{\infty} |\alpha_n g_n|_{L^1[0,1]} \left\| \chi(a_n,a_{n+1}) \right\|_{L^1[0,1]} \left\| \chi(a_n,a_{n+1}) \right\|_{CX}$$

$$= F_X(a_0) \sum_{n=1}^{\infty} \alpha_n g_n \left\| \chi(a_n,a_{n+1}) \right\|_{CX} = F_X(a_0) \sum_{n=1}^{\infty} \alpha_n \left\| \chi(a_n,a_{n+1}) \right\|_{CX}$$

$$\geq F_X(a_0) \sum_{n=1}^{\infty} \frac{|\alpha_n| (a_{n+1} - a_n)}{F_X(a_n)} (a_{n+1} - a_n) = \sum_{n=1}^{\infty} \frac{F_X(a_0)}{F_X(a_n)} |\alpha_n|,$$

for each $\alpha = (\alpha_n) \in \ell^1$. Denote

$$\theta_n := \frac{F_X(a_0)}{F_X(a_n)}.$$
Since card \((\Omega_0) = 1\), it follows that
\[
F_X(a_0) = \left\| \frac{1}{x} \chi_{(a_0,1]}(x) \right\|_X > \left\| \frac{1}{x} \chi_{(a_0,1]}(x) \right\|_X = F_X(a_0).
\]
Consequently, \((\theta_n) \subset (0,1)\) and, thanks to continuity of the function \(F_X\) at the point \(a_0\), we have that \(\theta_n \to 1\) as \(n \to \infty\). Finally, put
\[
\varepsilon_n := 1 - \theta_n.
\]
Then \((\varepsilon_n) \subset (0,1), \varepsilon_n \to 0\) as \(n \to \infty\) and
\[
\text{Let us now consider the following two cases.}
\]
\begin{align*}
\text{(a) Assume that there is } a \in (\lambda_0,1) \text{ with card } (\Omega_a) > 1. \text{ Therefore, there are numbers } a_1, a_2 \in (\lambda_0,1) \text{ such that } a_1 \neq a_2, \text{say } a_1 < a_2, \text{and } F_X(a_1) = F_X(a_2). \text{ Thus, for each number } a_3 \text{ with } a_1 < a_3 < a_2, \text{by the monotonity of the norm, we have}
\end{align*}
\[
F_X(a_1) \geq F_X(a_3) \geq F_X(a_2),
\]
which means the function \(F_X\) is constant on the interval \((a_1, a_2)\), i.e. \((a_1, a_2) \subset \Omega_a\). Following the same way as in case (a) we get easily that the space \(CX\) contains even an order isometric copy of \(\ell^1\).

\text{20}. The proof when \(I = [0, \infty)\) is the same as in the previous case. The only difference, of course, lies in the consideration of the function
\[
F_X : [0, \infty) \ni \lambda \mapsto \left\| \frac{1}{x} \chi_{(\lambda,\infty]}(x) \right\|_X \in (0, \infty].
\]

It is clear that due to the similarities occurring in Lemma 4.1 and Lemma 4.2, a result analogous to Theorem 4.5 will be rather expected also in the case of the Copson function spaces. It will be convenient to start with the following natural modification of the previously introduced function \(F_X\), namely,
\[
G_X := G_X(I) : I \ni \lambda \mapsto \left\| \chi_{[0,\lambda]} \right\|_X \in (0, \infty],
\]
where \(X\) is a Banach function space on \(I\).

**Theorem 4.6.** Let \(X\) be a Banach function space on \(I\) such that the Copson function space \(C^*X\) is nontrivial. Then the space \(C^*X\) contains an order asymptotically isometric copy of \(\ell^1\).

**Proof.** With minor changes the proof is similar as that of Theorem 4.5. Note, however, that the structure of the proof itself seems to be dual to the previous one. Details are provided for the convenience of the reader.

\text{10}. Suppose \(I = [0,1]\). Since \(C^*X \neq \{0\}\), there is \(0 < \lambda_0 < 1\) with \(\chi_{[0,\lambda_0]} \in X\), see Lemma 3.1. For each \(b \in (0, \lambda_0)\) set
\[
\Omega_b := \{ \lambda \in (0, \lambda_0) : G_X(\lambda) = G_X(b) \}.
\]
Of course, card \((\Omega_b) \geq 1\). Let us now consider the following two cases.

\text{(a) Assume that card } (\Omega_b) = 1 \text{ for every } 0 < b < 1. \text{ Obviously, the function } G_X \text{ is nondecreasing on the interval } [0, \lambda_0], \text{whence it contains at most countably many points of discontinuity.
Let $b_0 \in (0, \lambda_0)$ be a point of continuity of the function $G_X$. Take a sequence $(b_n) \subset (0, \lambda_0)$ such that $b_n \downarrow b_0$ as $n \to \infty$ and put

$$h_n := \frac{X(b_{n+1}, b_n)}{\|X(b_{n+1}, b_n)\|_{C^*X}}.$$ 

From the definition $\text{supp}(h_n) = (b_{n+1}, b_n) \subset (b_0, b_n)$ and $\text{supp}(h_n) \cap \text{supp}(h_m) = \emptyset$ if $n \neq m$, $m, n \in \mathbb{N}$. Using the right-hand side of the estimate (4.3) we have

$$\|X(b_{n+1}, b_n)\|_{C^*X} \leq \|X[0, b_n]\|_X \|X(b_{n+1}, b_n)\|_{L^1(1/t)[0, 1]} = G_X(b_n) \|X(b_{n+1}, b_n)\|_{L^1(1/t)[0, 1]}.$$ 

Furthermore, using left-hand side of (4.3) and the above estimate, since elements $h_n$ are mutually disjoint, we obtain

$$\sum_{n=1}^{\infty} \alpha_n h_n \geq \|X[0, b_0]\|_X \sum_{n=1}^{\infty} \alpha_n h_n \|X(b_{n+1}, b_n)\|_{L^1(1/t)[0, 1]} = G_X(b_0) \sum_{n=1}^{\infty} |\alpha_n| \|X(b_{n+1}, b_n)\|_{L^1(1/t)[0, 1]} = G_X(b_0) \sum_{n=1}^{\infty} |\alpha_n|,$$

for each $\alpha = (\alpha_n) \in \ell^1$. Denote

$$\theta_n := \frac{G_X(b_0)}{G_X(b_n)}.$$ 

Since $\text{card} (\Omega_{b_0}) = 1$, it follows that

$$G_X(b_n) = \|X[0, b_n]\|_X > \|X[0, b_0]\|_X = G_X(b_0).$$

Consequently, $(\theta_n) \subset (0, 1)$ and, thanks to continuity of the function $G_X$ at the point $b_0$, we have that $\theta_n \to 1$ as $n \to \infty$. Finally, put

$$\varepsilon_n := 1 - \theta_n.$$ 

Then $(\varepsilon_n) \subset (0, 1)$, $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\sum_{n=1}^{\infty} \alpha_n h_n \geq \sum_{n=1}^{\infty} (1 - \varepsilon_n) |\alpha_n|.$$ 

The second of the estimates we need is trivial. Note that $\|h_n\|_{C^*X} = 1$, so

$$\sum_{n=1}^{\infty} |\alpha_n| \|h_n\|_{C^*X} = \sum_{n=1}^{\infty} |\alpha_n|.$$ 

Thus, combining the inequalities (4.17) and (4.18), we finish the proof in that case.

(b) Assume that there is $b \in (0, \lambda_0)$ with $\text{card} (\Omega_b) > 1$. Therefore, there are numbers $b_1, b_2 \in (0, 1)$ such that $b_1 \neq b_2$, say $b_1 < b_2$, and $G_X(b_1) = G_X(b_2)$. Thus, for each number $b_3$ with $b_1 < b_3 < b_2$, by the monotonicity of the norm, we have

$$G_X(b_1) \leq G_X(b_3) \leq G_X(b_2),$$

which means the function $G_X$ is constant on the interval $(b_1, b_2)$, i.e., $(b_1, b_2) \subset \Omega_b$. Following the same way as in case (a) we get easily that the space $C^*X$ contains even an order isometric copy of $\ell^1$.

2°. If $I = [0, \infty)$, we follow the same way as in 1°. \hfill $\square$
Proposition 4.8. In the context of Theorem 4.4 and Theorem 4.5 a natural question arises: maybe the Cesàro function space $CX, \|\cdot\|_{CX}$ always contains an isometric copy of $\ell^1$ or $L^1[0,1]$? In general, the answer is no. Indeed, if the Banach function space $X$ is rotund, then the space $CX$ is also rotund, see [16 Lemma 2]. Therefore, for example, if we take $X = L^p$ for $1 < p < \infty$, then $\text{Ces}_p$ is rotund. Consequently, the space $\text{Ces}_p$ cannot contain an isometric copy of neither $\ell^1$ nor $L^1[0,1]$, because they are not rotund. However, let us remind that the space $(CX, \|\cdot\|_{CX})$ can always be renormed to contain an isometric copy of $L^1[0,1]$, cf. Theorem 4.4 (i).

Theorem 4.5 gives also some information about the generalized Tandori function spaces $\tilde{X}$. In short, they are quite similar to $L^\infty$.

Corollary 4.7. Let $X$ be a Banach function space on $I = [0, \infty)$ such that the space $X'$ is order continuous and $X$ has the Fatou property (which is true, for example, if $X$ is a reflexive space). Assume also that the Copson operator $C^*: X \to X$ is bounded and the dilation operator $\sigma\tau: X \to X$ is bounded for some $\tau > 1$. Then the Tandori function space $\tilde{X}$ contains an isomorphic copy of $L^1[0,1]$ and $C[0,1]^\ast$.

Proof. Because $X' \in (OC)$ then $C (X') \in (OC)$. Note that, since $X \in (FP)$, $C^*: X \to X$ if and only if $C: X' \to X'$ and $\sigma\tau: X \to X$ if and only if $\sigma\tau\sigma\tau: X' \to X'$, see for example [49 Remark 1]. Consequently, applying Theorem A (2.1) for the space $X'$, we get

$$(4.19) \quad (C (X'))' = (C(X'))' = \tilde{X}'' = \tilde{X},$$

with equivalent norms. The space $C(X')$ contains an order asymptotically isometric copy of $\ell^1$ via Theorem 4.5. By the Dilworth–Girardi–Hagler result [26 Theorem 2], the dual space $(C (X'))'$ contains an isometric copy of $L^1[0,1]$ and an isometric copy of $C[0,1]^\ast$. Thus, by equality (4.19), the Tandori space $\tilde{X}$ contains an isomorphic copy of $L^1[0,1]$ and $C[0,1]^\ast$. $\square$

The similarity between the Tandori function spaces $\tilde{X}$ and $L^\infty$ becomes even clearer in the context of the following result.

Proposition 4.8. Let $X$ be a Banach function space on $I$ such that the Tandori function space $\tilde{X}$ is nontrivial. Then the Tandori function space $\tilde{X}$ contains an order isomorphically isometric copy of $\ell^\infty$.

Proof. Since $\tilde{X} \neq \{0\}$, it follows that there exists $0 < a \in I$ such that $\chi[0,a) \in X$ (see [54 Theorem 1 (c)]). Put

$$f_0 = \frac{\chi[0,a)}{\|\chi[0,a)\|_X}.$$  

Let $a_n = (1 - \frac{1}{2^n+1})a$ and $A_n = (a_n - \delta_n, a_n + \delta_n)$, where $\delta_n = (a_{n+1} - a_n)/2$ and $n \in \mathbb{N}$. Denote

$$B_1 = [0, a/2) \cup \bigcup_{n=1}^\infty A_n.$$  

Then the set $[0, a) \setminus B_1$ consists of infinitely many pairwise disjoint intervals, say, $[0, a) \setminus B_1 = \bigcup_{n=1}^\infty C_n^{(1)}$. Let

$$B_2 = C_1^{(1)} \cup C_3^{(1)} \cup C_5^{(1)} \cup \ldots = \bigcup_{n=1}^\infty C_{2n-1}^{(1)}.$$  

Again, the set $[0, a) \setminus (B_1 \cup B_2)$ consists of infinitely many pairwise disjoint intervals, say, $[0, a) \setminus (B_1 \cup B_2) = \bigcup_{n=1}^\infty C_n^{(2)}$. Next, let

$$B_3 = C_1^{(2)} \cup C_3^{(2)} \cup C_5^{(2)} \cup \ldots = \bigcup_{n=1}^\infty C_{2n-1}^{(2)}.$$  

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We proceed analogously, defining the sequence of sets \((B_n)_{n=1}^{\infty}\). Put \(f_n := f_0 \chi_{B_n}\). Note that 
\[ 0 \leq f_n \leq f_0 \text{ and } \text{supp}(f_n) \cap \text{supp}(f_m) = \emptyset \text{ for each } n \neq m. \]

Moreover,
\[ \tilde{f}_n = \tilde{f}_0 = f_0, \]
whence
\[ \|f_n\|_X = \|\tilde{f}_n\|_X = \|\tilde{f}_0\|_X = \|f_0\|_X = 1 \text{ and } \|f_0\|_X = \|\tilde{f}_0\|_X = \|f_0\|_X = 1. \]

Applying Theorem 1 from [38] we conclude that the space \(\tilde{X}\) contains an order isomorphically isometric copy of \(\ell^\infty\). \(\square\)

The problem of describing the Cesàro–Orlicz function spaces containing an order isomorphically isometric copy of \(\ell^\infty\) has been considered in [47]. Although formally the case of the space \(\text{Ces}_\infty\) (which is not isomorphic to \(\ell^\infty\) – see [7, Theorem 7]) has been excluded there, it follows that the argument used in [47, Theorems 3 and 4 case (B2)] can be applied to get the below result. We will give, however, the direct and simple proof without referring to the structure of the Orlicz spaces.

**Proposition 4.9.** The space \(\text{Ces}_\infty\) contains an order isomorphically isometric copy of \(\ell^\infty\).

**Proof.** For a start, let us recall that
\[
(C_{\text{Ces}_\infty})_a = \{ f \in \text{Ces}_\infty : \lim_{x \to 0^+} \frac{1}{x} \int_0^x |f(t)| \, dt = 0 \},
\]
see [38, Remark 19]. Put \(f_0 := \chi_{[0,1]}\). Then \(\|f_0\|_{C_{\text{Ces}_\infty}} = 1\) and \(\text{dist}(f_0, (C_{\text{Ces}_\infty})_a) = 1\). In fact, it follows from (4.20) that
\[
\text{dist}(f_0, (C_{\text{Ces}_\infty})_a) := \inf_{h \in (C_{\text{Ces}_\infty})_a} \|f_0 - h\|_{C_{\text{Ces}_\infty}} = \inf_{h \in (C_{\text{Ces}_\infty})_a} \sup_{0 \leq t \in I} C \, |f_0 - h|(t) \\
\geq \inf_{h \in (C_{\text{Ces}_\infty})_a} \sup_{0 \leq t \in I} (C \, |f_0|(t) - C \, |h|(t)) \\
\geq \inf_{h \in (C_{\text{Ces}_\infty})_a} \lim_{t \to 0^+} (C \, |f_0|(t) - C \, |h|(t)) = \lim_{t \to 0^+} C \, |f_0|(t) = 1.
\]

Now, we can once again use Theorem 2 from [38] and finish the proof. \(\square\)

5. **ON DIFFERENCES IN THE CESÁRO CONSTRUCTION ON \([0,1]\) AND \([0,\infty)\)**

We begin with a short discussion. Recall that following the standard definitions we should define a truncation of the Banach function space \(X\) on \(I\) to a set \(A \subset I\) as \(X|_A = \{ f \in X : \text{supp}(f) \subset A \}\), where \(\text{supp}(f) := \{ x \in I : f(x) \neq 0 \}\). However, when applying the Cesàro construction to the truncated space the situation is more delicate. If we would like to follow the above definition, then we should write that \(C(X[0,\infty]|_{[0,1]})\) is the space of all functions \(f \in L^0[0,\infty)\) (because the space \(X[0,\infty]|_{[0,1]}\) contains functions from \(L^0[0,\infty)\) such that \(C \, |f| \in X[0,\infty]|_{[0,1]}\), i.e., \(f \in X[0,\infty)\)) and \(\text{supp}(C \, |f|) \subset [0,1]\). But the last condition is never satisfied, whenever \(0 \neq f \in L^0[0,\infty)\), so this is meaningless. That is why we will proceed in the following way. For a Banach function space \(X\) on \([0,\infty)\) and a subset \(A \subset [0,\infty)\) we define the **truncation of the space \(X[0,\infty)\) to a set \(A\)** as
\[
X[0,\infty]|_A := \{ f \in L^0[0,1] : f \chi_A \in X[0,\infty) \},
\]
with the norm \( \|f\|_{X(0,\infty), A} = \|f|_A\|_{X(0,\infty)} \) (that is, we look at an element \( f \) as a function defined on \([0, 1]\), but compute its norm as if it was still defined on \([0, \infty)\) and equal zero on \([0, \infty) \setminus A\), if \( A \subset [0, 1] \) and
\[
X[0, \infty)|_A := \{ f \in X[0, \infty) : \text{supp}(f) \subset A \},
\]
with the norm \( \|f\|_{X[0,\infty), A} = \|f|_A\|_{X(0,\infty)} \), if \( A \not\subset [0, 1] \). Moreover, if \( X \) is a Banach function space on \([0, 1]\) and \( A \subset [0, 1] \), then we can use the definition given at the beginning, that is, the truncation of the space \( X[0, 1] \) to the set \( A \) as
\[
X[0, 1]|_A := \{ f \in X[0, 1] : \text{supp}(f) \subset A \},
\]
with the norm \( \|f\|_{X[0,1), A} = \|f|_A\|_{X[0,1]} \). Note that if we use the following notation
\[
\mathcal{X}[0,1] := X[0,\infty)|_{[0,1]} = \{ f \in L^0[0,1] : f|_{[0,1]} \in X[0,\infty) \},
\]
with the norm \( \|f\|_{\mathcal{X}[0,1]} = \|f|_{X[0,\infty), [0,1]} \), then the functor \( X \mapsto \mathcal{X} \) is a way to associate to the space \( X \) defined on \( I = [0, \infty) \) its “natural” counterpart defined on \( I = [0, 1] \). Observe also that if \( A \subset [0, 1] \), then the space \( X[0,\infty)|_A \) is actually isometrically isomorphic to the space \( Y := \{ f \in X[0,\infty) : \text{supp}(f) \subset A \} \) with the norm \( \|f\|_Y = \|f\|_{X[0,\infty)} \) via the mapping \( J : f \mapsto f|_A \). Moreover, it is clear, that \( X[0,\infty)|_{[0,1]} \) is a symmetric space, whenever \( X[0,\infty) \) is a symmetric space.

Clearly, \( L^p[0,\infty)|_{[0,1]} = L^p[0,1] \). In contrast, for the Cesàro function spaces the situation is quite different. In fact, if \( f \in Ces_p[0,\infty) \) and \( \text{supp}(f) \subset [0, 1] \), then
\[
\|f\|_{Ces_p[0,\infty)} = \left\| f \right\|_{Ces_p[0,1]} + \frac{1}{p-1} \|f\|_{L^1[0,1]} ,
\]
i.e., \( Ces_p[0,\infty)|_{[0,1]} = Ces_p[0,1] \cap L^1[0,1] \) for \( 1 < p < \infty \), see \[7\] Remark 5]. In the next lemma we will show an analogue of equality \( (5.3) \) for abstract Cesàro function spaces \( CX \). However, let us notice now that according to definitions \( (5.1) \) and \( (5.2) \), we have
\[
CX[0,\infty)|_{[0,1]} = \{ f \in L^0[0,1] : C(|f|_{X[0,1]}) \in X[0,\infty) \},
\]
with the norm \( \|f\|_{CX[0,\infty), [0,1]} = \|C(|f|_{X[0,1]})\|_{X[0,\infty)} \), and
\[
C(X[0,\infty)|_{[0,1]}) := C(X[0,1]) = \{ f \in L^0[0,1] : (C|f|)_{X[0,1]} \in X[0,\infty) \},
\]
with the norm \( \|f\|_{C\langle CX[0,\infty), [0,1]\rangle} = \|C(|f|_{X[0,1]})\|_{X[0,\infty)} \) (cf. with the definition of the Cesàro operator \( C \) and the abstract Cesàro space \( CX \)). Roughly speaking, if \( X \) is a Banach function space on \([0, \infty)\), then we always have two ways (in general, nonequivalent) to obtain the Cesàro function space on \([0, 1]\) – first applying the functor \( X \mapsto \mathcal{X} \) and then the Cesàro construction or vice versa. All this also means that the Cesàro construction \( X \mapsto CX \) is significantly different for \( I = [0, 1] \) and \( I = [0, \infty) \).

The equality \( (5.3) \) from Lemma \( 5.1 \) below is an abstract version of equality \( (5.5) \) from \[7\] and means that the functor \( X \mapsto \mathcal{X} \) does not commutate in general with the Cesàro construction \( X \mapsto CX \) (that is, \( \mathcal{X}X[0,1] = CX[0,1] \cap L^1[0,1] \)). This result explains also, in some sense, a rather surprising difference in the description of the Köthe duality of Cesàro function spaces \( CX \) on \( I = [0, 1] \) and on \( I = [0, \infty) \), cf. \[53\] Theorems 3, 5 and 6] or Theorem A.

**Lemma 5.1.** Let \( X \) be a Banach function space on \( I \) such that either the Cesàro operator \( C \) is bounded on \( X \) or \( X \) is a symmetric space on \([0, 1]\) or \( X \) is a symmetric space on \([0, \infty) \) with \( CX[0,\infty) \neq \{0\} \). Then the following embedding
\[
CX(I)|_{[0,\lambda]} \hookrightarrow L^1(I)|_{[0,\lambda]}
\]
holds for $0 < \lambda < m(I)$, but in general not for $\lambda = 1$ if $I = [0, 1]$, and not for $\lambda = \infty$ if $I = [0, \infty)$. Moreover, the following equalities hold

(5.9) \[ CX[0, \infty)|(0,1) = C(X[0, \infty)|(0,1)) \cap L^1[0, 1], \]

and

(5.10) \[ CX[0, \infty)|(0,\lambda) = (C(X[0, \infty)|(0,1)))|(0,\lambda) \quad \text{for} \quad 0 < \lambda < 1. \]

Finally, if $X$ is a symmetric space on $[0, \infty)$ with $q(X) < \infty$, then $C(X[0, \infty)|(0,1)) \neq CX[0, \infty)|(0,1)$ and the space $C(X[0, \infty)|(0,1))$ is never a subspace of $CX[0, \infty)|(0,1)$.

Proof of the embedding (5.8). Take $f \in CX(I)$ with supp$(f) \subset [0, \lambda]$, where $0 < \lambda < m(I)$. In this case we must ensure that $(f_\lambda: I \ni x \mapsto \frac{1}{x}X(\lambda,m(I))(x)) \in X$. But this immediately follows from Theorem D and our assumptions. Keeping this in mind we have the following inequalities

\[
\|f\|_{CX(I)} := \left\| \frac{1}{x} \int_0^x |f(t)| \, dt \right\|_{X(I)} = \left\| \frac{1}{x} \int_0^x |f(t)| \, dt X(0,\lambda)(x) + \frac{1}{x} \int_0^\lambda |f(t)| \, dt X(\lambda,\lambda)(x) \right\|_{X(I)} \\
\geq \left\| \frac{1}{x} \int_0^\lambda |f(t)| \, dt X(\lambda,\lambda)(x) \right\|_{X(I)} = \left\| \frac{1}{x} X(\lambda,\lambda)(x) \right\|_{X(I)} \|f\|_{L^1[I](0,\lambda]}. 
\]

Therefore, $CX(I)|(0,\lambda) \rightarrow L^1[I](0,\lambda]$. Counterexamples for embeddings $CX \rightarrow L^1$ when either $\lambda = 1$ or $\lambda = \infty$ can be found in [7, Theorem 1 (d)]. Note, however, that for example $Ces_{\infty}[0, 1] \rightarrow L^1[0, 1]$, see [7] Theorem 1 (d).

Proof of the equality (5.9). Take $f \in C(X[0, \infty)|(0,1)) \cap L^1[0, 1]$. To prove this equality we will need to know that $(f_{\lambda=\lambda}: [0, \infty) \ni x \mapsto \frac{1}{x}X(1,\infty)(x)) \in X[0, \infty)$. Again, our assumptions together with Theorem D ensure that this is indeed the case. Since

\[
\|f\|_{C(X[0, \infty)|(0,1))} \leq \left\| \frac{1}{x} \int_0^x |f(t)| \, dt X(0,\lambda)(x) \right\|_{X(0,\infty)} + \left\| \frac{1}{x} \int_0^1 |f(t)| \, dt X(1,\lambda)(x) \right\|_{X(0,\infty)} \\
= \left\| f \right\|_{C(X[0, \infty)|(0,1))} + \left\| \frac{1}{x} X(1,\lambda)(x) \right\|_{X(0,\infty)} \|f\|_{L^1[0, 1]} 
\]

it follows that

$C(X[0, \infty)|(0,1)) \cap L^1[0, 1] \rightarrow CX[0, \infty)|(0,1]$.

Next, we will show the reverse embedding. Take $f \in CX[0, \infty)$ with supp$(f) \subset [0, 1]$. Again, as above, we have $\frac{1}{x}X(1,\lambda)(x) \in X$, whence

\[
\|f\|_{C(X[0, \infty))} \geq \max \left\{ \left\| (C[f]) X(0,\lambda) \right\|_{X(0,\infty)} , \left\| \frac{1}{x} \int_0^1 |f(t)| \, dt X(1,\lambda)(x) \right\|_{X(0,\infty)} \right\} \\
= \max \left\{ \left\| f \right\|_{C(X[0, \infty)|(0,1))} , \left\| \frac{1}{x} X(1,\lambda)(x) \right\|_{X(0,\infty)} \|f\|_{L^1[0, 1]} \right\}. 
\]

In consequence,

$CX[0, \infty)|(0,1) \rightarrow C(X[0, \infty)|(0,1)) \cap L^1[0, 1]$, which proves the equality (5.9).

Proof of the equality (5.10). Of course, if $X$ and $Y$ are Banach function spaces on $I$, then

(5.11) \[ (X \cap Y)|_A \equiv X|_A \cap Y|_A \quad \text{for every} \quad A \subset I, \]
and because \( \|f\|_{(X|A)} = \|f\|_{(X|B)} = \|f\|_{(X|A \cap B)} = \|f\|_{(X|A \cap B)} \), so

\[
(X|A)_B = X|A \cap B \quad \text{for every } A, B \subseteq I.
\]

Combining the above equalities with the embedding (5.5) we get

\[
CX[0, \infty)|_{[0,\lambda]} \equiv (CX[0, \infty)|_{[0,1]}),
= (C(X[0, \infty)|_{[0,1]} \cap L^1[0, 1])|_{[0,\lambda]},
= (C(X[0, \infty)|_{[0,1]}))|_{[0,\lambda]}.
\]

This gives the equality (5.10).

Finally, suppose that \( q(X) < \infty \). Then as in [7, Theorem 1 (d)] we can show that the function

\[
f(x) = \frac{1}{1-x} \quad \text{for } 0 \leq x < 1,
\]

belongs to the space \( C(X[0, \infty)|_{[0,1]} \). In fact, by (2.3) Theorem B, we have \( L^1 \cap L^q[0, \infty) \overset{\text{A}}{\hookrightarrow} X[0, \infty) \) for \( q(X) < q < \infty \). Therefore, using equation (5.11) we have

\[
L^q[0, 1] = L^1 \cap L^q[0, 1] = (L^1 \cap L^q[0, \infty))|_{[0,1]} \overset{\text{A}}{\hookrightarrow} X[0, \infty)|_{[0,1]}.
\]

Moreover, \( \int_0^1 (\frac{1}{x} \ln(\frac{1}{1-x}))^q dx < \infty \) (cf. [7, Theorem 1 (d), p. 334]), whence

\[
\|f\|_{C(X[0,\infty)|_{[0,1]}}^q = \|C|f||_{X[0,\infty)|_{[0,1]}}^q \lesssim \|C|f||_{L^q[0,1]}^q
= \int_0^1 \left( \frac{1}{x} \int_0^x \frac{dt}{1-t} \right)^q dx = \int_0^1 \left( \frac{1}{x} \ln(\frac{1}{1-x}) \right)^q dx < \infty.
\]

Of course, \( f \notin L^1[0, 1] \) and this ends the proof of this lemma. \( \square \)

Let us also note that if \( X = L^\infty[0, \infty) \), then \( q(X) = \infty \) and \( L^\infty[0, \infty)|_{[0,1]} \equiv L^\infty[0, 1] \). Therefore, \( C(X[0, \infty)|_{[0,1]} \equiv C \infty[0, 1] \) and, in view of equality (5.9) and embedding \( C \infty[0, 1] \hookrightarrow L^1[0, 1] \) (see [7, Theorem 1 (d)]), we have

\[
CX[0, \infty)|_{[0,1]} = C \infty[0, \infty)|_{[0,1]} = C \infty[0, 1] \cap L^1[0, 1] = C \infty[0, 1].
\]

This means that it can happen that \( C(X[0, \infty)|_{[0,1]} = CX[0, \infty)|_{[0,1]} \) and the assumption about the Boyd index in the last part of the above lemma cannot be omitted.

6. ON A TRANSFER OF PROPERTIES BETWEEN \( X \) AND \( TX \)

Inclusions and equalities between Cesàro spaces \( CX \) and Copson spaces \( C^*X \) are collected, for example, in [56, Theorem 1] (see also [7] and [14]). Recall that if \( X \) is a Banach function space on \([0, \infty)\) such that both operators \( C \) and \( C^* \) are bounded on \( X \), then

\[
C^*X \overset{A}{\hookrightarrow} CX \overset{B}{\hookrightarrow} C^*X,
\]

where \( A = \|C\|_{X \rightarrow X} \) and \( B = \|C^*\|_{X^* \rightarrow X} \), that is, \( CX = C^*X \), see [56, Theorem 1 (iii)]. However, if \( I = [0, 1] \), then the situation is a bit more complicated. More precisely, if \( X \) is a Banach function space on \([0, 1] \) such that the Cesàro and Copson operators are bounded on \( X \) and \( L^\infty[0, 1] \hookrightarrow X \hookrightarrow L^1[0, 1] \) (for example, if \( X \) is a symmetric space), then

\[
CX[0,1] \cap L^1[0,1] = C^*X[0,1],
\]

see [56, Theorem 1 (vi) and (vii)]. Therefore, at least when we consider Banach function spaces on \([0, \infty) \) such that both operators \( C \) and \( C^* \) are bounded on \( X \), all results regarding the isomorphic structure of the Cesàro function spaces “transfer” almost trivially to the case of the Copson function spaces and vice versa. However, it may not be the case if \( I = [0, 1] \). After all, we will prove the following
Corollary 6.1. Let $X$ be a symmetric space such that both operators $C$ and $C^*$ are bounded on $X$. Then the Copson space $C^*X$ is order continuous if and only if $X$ is order continuous.

Proof. First of all, $X$ is order continuous if and only if $CX \subseteq [0,1] \cap L^1[0,1] \subseteq (OC)$. Take $f_0 \in L^1[0,1]$ such that $f_0 \notin X_0$ but $f_0 \in L^1[0,1]$. Without loss of generality, we can also assume that $f_0 = f_0^*$, see [17] Lemma 2.6. From the boundedness of the Cesàro operator $C$ it follows that $C(f_0) \in X$. Moreover, the element $f_0$ is a nonincreasing function and $X_a$ is an order ideal of $X$ (see [13] Theorem 3.8, p. 116)). Therefore, $f_0 \notin C(X_a) = (CX)_{a+b}$, see [48] Theorem 16. In summary, $f_0 \in (CX \setminus (CX)_{a+b}) \subseteq [0,1]$. But, in view of the equality (6.2), this means that $f_0 \in C^*X \setminus (C^*X)_{a+b}$, i.e., $C^*X \notin (OC)$. □

It may happen that applying the construction $TX$, where $T = C$ or $T = C^*$, we lose some information about the original space $X$. We will give rather general example of this kind.

The idea behind the next lemma is simple. Every Cesàro and Copson function space contain “in the middle” an isomorphic copy of $L^1[0,1]$ (cf. Lemma 6.1 and Lemma 6.2). Therefore, up to equivalence of norms, we can change the space $X$ “in the middle” (cf. the equality (6.3)) and still get the Cesàro or Copson function space equal to the original one (cf. the equality (6.2)).

Lemma 6.2. Let $T = C$ or $T = C^*$. Define the Banach function space $Z = Z[0,1]$ as

$$Z[0,1] := X_{[a,b]} \oplus Y_{[a,b]} \oplus X \quad \text{for} \quad 0 < a < b < 1,$$

where $X$ and $Y$ are Banach function spaces on $[0,1]$ such that $L^\infty[0,1] \hookrightarrow Y \hookrightarrow X$. Then

$$TZ[0,1] = TX[0,1],$$

Proof of the embeddings $CZ \hookrightarrow CX$ and $C^*X \hookrightarrow C^*Z$. In the proof of this part we need only the assumption that $Y \hookrightarrow X$. Indeed, then

$$Z[0,1] = X_{[a,b]} \oplus Y_{[a,b]} \oplus X_{[a,b]} \hookrightarrow X_{[a,b]} \oplus X_{[a,b]} \oplus X_{[a,b]} = X[0,1].$$

Note that just by the definition if $E$ and $F$ are Banach function spaces on $I$ and $E \hookrightarrow F$, then $TE \hookrightarrow TF$. Consequently, $TZ \hookrightarrow TX$.

Proof of the embedding $CX \hookrightarrow CZ$. First, observe that if $W$ is a Banach function space on $[0,1]$ such that supp($CW) = [0,1]$, then

$$CW[0,1]_{[a,b]} = L^1[0,1]_{[a,b]} = L^1[a,b] \quad \text{for} \quad 0 < a < b < 1,$$

due to Lemma 4.1. Take $f \in CX$ and denote $f_1 = f \chi_{[0,a]}, f_2 = f \chi_{[a,b]}$ and $f_3 = f \chi_{[b,1]}$. We need to show that $f_1, f_2, f_3 \in CZ$. Note that $(C(f_1)) \chi_{[a,b]} = (C(f)) \chi_{[a,b]} \in X$. Moreover, $(C(f_1)) \chi_{[a,b]} \in L^\infty[0,1]$, so $(C(f_1)) \chi_{[a,b]} \in Y$, because $L^\infty[0,1] \hookrightarrow Y$. Next, observe that $(C(f_1)) \chi_{[b,1]} \leq (C(f)) \chi_{[b,1]} \in X$. Thus $C(f_1) \in Z$. Since $L^\infty[0,1] \hookrightarrow Y \hookrightarrow X$, so $L^\infty[0,1] \hookrightarrow Z$ and supp$(CX) = supp(CZ) = [0,1]$. In consequence, by equality (6.5), we have $f_2 \in CX_{[a,b]} \subseteq L^1[0,1]_{[a,b]} = CZ_{[a,b]}$. Finally, $C(f_3) = C(f) \chi_{[b,1]} \leq C(f) \chi_{[b,1]} \in X_{[b,1]} = Z_{[b,1]}$. Consequently, $f \in CZ$, but this means that $CX \hookrightarrow CZ$.

Proof of the embedding $C^*X \hookrightarrow C^*Z$. Let $f \in C^*X$ and set $f_1 = f \chi_{[0,a]}, f_2 = f \chi_{[a,b]}$ and $f_3 = f \chi_{[b,1]}$. Since $C^*(f_1) \leq C^*(f) \in X$, it follows that $(C^*(f_1)) \chi_{[0,a]} \in Z$. Moreover, $(C^*(f_1)) \chi_{[a,b]} \equiv 0$, whence $f_1 \in CZ$. Just as above we conclude that supp$(C^*X) = supp(C^*Z) = [0,1]$. Thus,

$$f_2 \in C^*X_{[a,b]} = L^1[0,1]_{[a,b]} = C^*Z_{[a,b]}.$$ Moreover, $C^*(f_3) \leq C^*(f) \in X$, so
\((C^* | f_3 | \chi_{[0,a]} \in Z \) and \((C^* | f_3 | \chi_{[b,1]} \in Z \). Finally, note that \(C^* | f | \in X \) and the function \(C^* | f | \) is nonincreasing, thus \(C^* | f_3 | \chi_{[a,b]} (x) = \int_0^1 \frac{|f(t)|}{t} dt = C^* | f | (b) < \infty \) for each \(x \in [a,b] \). Therefore, we obtain that \((C^* | f_3 | \chi_{[a,b]} \in L^\infty [0,1] [a,b] \to Y [a,b] \) and this ends the proof. \(\square\)

The above lemma can be viewed as an abstract version of Example 1 from [56]. In particular, if \(X \) is a symmetric space on \([0, 1] \), then \(L^\infty [0, 1] \to X \to L^1 [0, 1] \) and

\[ T(L^1 [0, a] \oplus X |_{[a, b]} \oplus L^1 [b, 1]) \simeq L^1 [0, 1], \]

but

\[ T(X |_{[0, a]} \oplus L^\infty [a, b] \oplus X |_{[b, 1]}) = TX [0, 1]. \]

In connection with the above lemma, the following simple observation is worth noting.

**Lemma 6.3.** Let \(T = C \) or \(T = C^* \). Assume that \(X \) and \(Y \) are symmetric spaces on \(I \) such that

1. the Cesàro operator \(C \) is bounded on \(X \) and \(Y \), respectively, if \(T = C \);
2. both operators \(C \) and \(C^* \) are bounded on \(X \) and \(Y \), respectively, if \(T = C^* \).

Then \(X = Y \) if and only if \(TX = TY \).

**Proof.** If \(X = Y \), then \(X \hookrightarrow Y \) and \(Y \hookrightarrow X \), so \(TX \hookrightarrow TY \) and \(TY \hookrightarrow TX \). Thus, \(TX = TY \).

Suppose that \(T = C \). If \(CX \hookrightarrow CY \), then thanks to boundedness of the Cesàro operator and the symmetry of \(X \), we have

\[ \|f\|_Y = \|f^*\|_Y \leq \|C(f^*)\|_Y \lesssim \|C(f^*)\|_X \leq \|C\|_{X \to X} \|f^*\|_X \lesssim \|f\|_X, \]

that is, \(X \hookrightarrow Y \). Changing the roles of \(X \) and \(Y \) we can show the reverse embedding \(Y \hookrightarrow X \).

Let \(T = C^* \) and assume that \(C^* X \hookrightarrow C^* Y \). If \(I = [0, \infty) \), then \(C^* X = CX \) and there is nothing to prove, see the equality (6.1). On the other hand, if \(I = [0, 1] \), then \(C^* X = CX \cap L^1 \), see the equality (6.2). Since and \(X \hookrightarrow C^* X \), it follows that

\[ X \hookrightarrow C^* X \hookrightarrow C^* Y = CY \cap L^1 \hookrightarrow CY. \]

Moreover, \(\|f\|_Y = \|f^*\|_Y \leq \|C(f^*)\|_Y \) and consequently

\[ \|f\|_Y \leq \|C(f^*)\|_Y = \|f^*\|_Y \lesssim \|f^*\|_X = \|f\|_X, \]

that is, \(X \hookrightarrow Y \). \(\square\)

If we try to reformulate Lemma 6.3 using isomorphism instead of the “equalities”, then this result is not longer true. For example, if \(X = L^\infty \) and \(Y = L^\infty (t) \), then \(CY \equiv L^1 \) and \(X \not\simeq Y \) but of course \(L^1 \) is not isomorphic to \(Ces\infty \) (because \(L^1 \) is separable and \(Ces\infty \) is not). On the other hand, if \(X = L^1 [0, 1] \) and \(Y = L^\infty (t) [0, 1] \), then \(CX \simeq L^1 [0, 1] \), \(CY \equiv L^1 [0, 1] \) and \(CX \not\simeq CY \) but \(X \) is not isomorphic to \(Y \).

Generally, some isomorphic as well as isometric properties inherit well from \(X \) to the Cesàro space \(CX \) (for example, order continuity [56], Fatou property [54, Theorem 1 (d)] and rotundity [46]). However, there are properties, like reflexivity, which Cesàro function spaces never have (cf. Corollary 6.3). In other words, certain properties never transfer from \(X \) to \(CX \). Below we present next two properties of this kind.

**Corollary 6.4.** Let \(T = C \) or \(T = C^* \). Suppose that \(X \) is an order continuous Banach function space on \(I \) with \(TX \not\equiv \{0\} \). Then

1. \(TX \) is not isomorphic to a dual space.
2. \(TX \) does not have the Radon–Nikodym property.
Proof. Part (i) of the proof is the same as in [10] Theorem 3. We will give the details for a sake of completeness.

(i) We argue by contradiction. Suppose that $TX$ is isomorphic to a dual space, i.e. there exist a Banach function space $Y$ with $(TX, \|\cdot\|_{TX}) \simeq Y^*$. By Theorem [4.4] we can find an equivalent norm, say $\|\cdot\|'$, on the space $TX$ such that $(TX, \|\cdot\|')$ contains a closed subspace isometric to $L^1[0,1]$. Of course, $(TX, \|\cdot\|') \simeq Y^*$. It follows from the definition that if $X \in (OC)$ then $TX \in (OC)$, cf. also [56] Lemma 1 (a). Thus, our assumptions show that $(TX, \|\cdot\|') \in (OC)$. Now, we can apply the well known fact that a Banach function space is separable if and only if it is order continuous and the measure $\mu$ is separable [13 Theorem 5.5] to conclude that $(TX, \|\cdot\|')$ is also separable. Applying the Bessaga–Pelczyński result, see [15], it follows that $(TX, \|\cdot\|')$ has the Krein–Milman property. Therefore, every closed bounded set in $(TX, \|\cdot\|')$ is a closed convex hull of its extreme points. On the other hand, the closed unit ball in $L^1[0,1]$ has no extreme points. This contradiction ends the proof.

(ii) This case follows from the Talagrand theorem, see [63, Corollary 5.4.21], which states that a separable Banach lattice is isomorphic to a dual Banach lattice if and only if it has the Radon–Nikodym property.

The above Corollary has been proved for $Ces_p$-spaces in [10] Theorem 3 and in [11] Corollaries 5.1 and 5.5 (using duality arguments). Moreover, part (i) of the above result for the Cesàro function spaces $CX$, where $X$ is an order continuous symmetric space such that the Cesàro operator is bounded on $X$, is included in [5] Proposition 5.3. Interestingly, it may happen that the Cesàro function space $CX$ for a nonseparable space $X$ is isomorphic to a dual space. It was proved in [5], the equality (2.9) and Theorem 5.1] that

$$(\ell^1)^* = (\ell^1)' = ces_{\infty} \simeq Ces_{\infty},$$

which means that $Ces_{\infty}$ is isomorphic to a dual space.

The question when a given property “transfers” also in the opposite direction, i.e. from $CX$ to $X$, imposes itself. However, as the next theorem will show, in the class of Banach function spaces the answer is basically always negative. Before we formulate this result we need the following definition.

Let $X$ be a Banach function space with the property $P$ (in short, $X \in (P)$). We will say that the property $P$ is good for the Cesàro construction if $P$ is invariant under equivalent renormings (that is, if $(X, \|\cdot\|) \in (P)$ and $\|\cdot\|'$ is an equivalent norm on $X$, then also $(X, \|\cdot\|') \in (P)$) and we can find two nontrivial symmetric spaces $X$ and $Y$ on $[0,1]$, which satisfy the following conditions:

- **(G1)** $X \in (P)$ but $Y \notin (P)$,
- **(G2)** $X \cap Y \notin (P)$,
- **(G3)** $X|A \in (P)$ for every $\emptyset \neq A \subset [0,1]$,
- **(G4)** $CX \in (P)$.

Similarly, we will say that the property $P$ is good for the Copson construction replacing the condition $(G4)$ with “$C^*X \in (P)$” in the above definition. Moreover, the property $P$ is good if is good for the Cesàro and Copson construction. The definitions given do not look particulary restrictive, however, we will give some examples of such properties.

**Example 6.5.** (a) Let us start with the fact that order continuity is good property. Take $P = OC$. First, note that if $X \in (OC)$, then $X|A \in (OC)$ for every $\emptyset \neq A \subset [0,1]$. Moreover, if $X \in (OC)$, then $TX \in (OC)$, see [56] Lemma 1 (a) and Corollary 6.1. It remains to find two nontrivial symmetric spaces on $[0,1]$ with properties (G1) and (G2). For example, let $X = L^1[0,1]$ and $Y$ be any symmetric space on $[0,1]$ with $Y \neq Y_0$ or let $Y = L^\infty[0,1]$ and $X$ be any order continuous symmetric space on $[0,1]$. It is clear that in both cases $X \cap Y \notin (OC)$.
(b) We will show that the Dunford–Pettis property (for short, DPP) is good property (see [11] p. 115 for the definition and [27] for related results). Kamińska–Mastyło proved in [42] that there are exactly two nonisomorphic symmetric spaces on $[0,1]$ with the Dunford–Pettis property, namely $L^1[0,1]$ and $L^\infty[0,1]$. Therefore, if $X = L^1[0,1]$ and $Y$ is a reflexive symmetric space on $[0,1]$, then $X \in (DPP)$, $Y \notin (DPP)$, $X \cap Y = Y \notin (DPP)$ and $CL^1[0,1] = Ces_1[0,1] \simeq L^1[0,1] \in (DPP)$. Moreover, $C^*L^1[0,1] = Cop_1[0,1] \simeq L^1[0,1] \in (DPP)$. Finally, a complemented subspaces of spaces with the Dunford–Pettis property also have it, so the condition (G3) is satisfied in an obvious way.

(c) Let $p \geq 1$ and suppose that $X$ is a Banach function space on $I$ which is $p$-concave with constant $L \geq 1$ (see [57] pp. 45–46 for the definition) and such that the Cesàro operator $C$ is bounded on $X$. First, we will show that then also the space $CX$ is $p$-concave with constant $L$. Recall, that the space $L^1(I)[0,x]$ for $0 < x \in I$ is 1-convex with constant 1 and $p$-concave with constant 1, that is,

\[
\left( \sum_{k=1}^{n} \left\| f_k \right\|_{L^1(I)[0,x]}^p \right)^{1/p} = \left( \sum_{k=1}^{n} \left( \int_0^x |f_k(t)| \, dt \right)^p \right)^{1/p} 
\]

\[
\leq \int_0^x \left( \sum_{k=1}^{n} |f_k(t)|^p \right)^{1/p} \, dt = \left\| \sum_{k=1}^{n} |f_k|^p \right\|_{L^1(I)[0,x]}
\]

for every $0 < x \in I$, see [57] Proposition 1.d.5, [62] Theorem 4.3 and the second part of the proof in [12] Theorem 4]. Therefore, we see immediately that

\[
\left( \sum_{k=1}^{n} (C |f_k|^p)^p \right)^{1/p} = \left( \sum_{k=1}^{n} \left( \frac{1}{x} \int_0^x |f_k(t)| \, dt \right)^p \right)^{1/p} 
\]

\[
\leq \frac{1}{x} \int_0^x \left( \sum_{k=1}^{n} |f_k(t)|^p \right)^{1/p} \, dt = C \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p}.
\]

Using the above inequality and $p$-concavity of the space $X$, we have

\[
\left( \sum_{k=1}^{n} \left\| f_k \right\|_{CX}^p \right)^{1/p} = \left( \sum_{k=1}^{n} \left\| C |f_k| \right\|_{X}^p \right)^{1/p} 
\]

\[
\leq L \left\| \sum_{k=1}^{n} (C |f_k|^p)^p \right\|_{X} 
\]

\[
\leq L \left\| C \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_{X} = L \left\| \sum_{k=1}^{n} |f_k|^p \right\|_{CX}^{1/p} 
\]

for all $f_1, f_2, ..., f_n \in CX$. Consequently, also the space $CX$ is $p$-concave with the same constant as for $X$. Taking, for example, $X = L^p[0,1]$ for $1 < p < \infty$ and $Y = L^\infty[0,1]$ we conclude that $p$-concavity is good property for the Cesàro construction.

To show that $p$-concavity is a good property also for the Copson construction we will prove first that if $X$ and $Y$ are $p$-concave Banach function spaces on $I$, then the space $X \cap Y$ is $p$-concave as well. Let $Z = X \cap Y$ and $\|f\|_Z = \max\{\|f\|_X, \|f\|_Y\}$. Take $f_1, f_2, ..., f_n \in Z$ and denote by $L_X, L_Y > 0$ the constants of $p$-concavity of $X$ and $Y$, respectively (cf. [57]). We have

\[
\left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_Z \geq \left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_X \geq \frac{1}{L_X} \left\| \sum_{k=1}^{n} \left\| f_k \right\|^p \right\|_{X}^{1/p}
\]

\[
\left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_Y \geq \left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_Y \geq \frac{1}{L_Y} \left\| \sum_{k=1}^{n} \left\| f_k \right\|^p \right\|_{Y}^{1/p}
\]

\[
\left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_Z \geq \left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_X \geq \frac{1}{L_X L_Y} \left\| \sum_{k=1}^{n} \left\| f_k \right\|^p \right\|_{X}^{1/p}
\]

\[
\left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_Y \geq \left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_Y \geq \frac{1}{L_X L_Y} \left\| \sum_{k=1}^{n} \left\| f_k \right\|^p \right\|_{Y}^{1/p}
\]
(6.6) \[ \left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_Z \geq \left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_Y \geq \frac{1}{L} \left( \sum_{k=1}^{n} \|f_k\|_Y^p \right)^{1/p}. \]

Therefore, setting \( L = \max \{L_X, L_Y\} \), we have

\[ \left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_Z \geq \frac{1}{2L} \left( \sum_{k=1}^{n} (\|f_k\|_X + \|f\|_Y)^p \right)^{1/p} \geq \frac{1}{2L} \left( \sum_{k=1}^{n} \max \{\|f\|_X, \|f\|_Y\}^p \right)^{1/p} = \frac{1}{2L} \left( \sum_{k=1}^{n} \|f_k\|_Z^p \right)^{1/p}. \]

But this means that the space \( Z \) is actually \( p \)-concave and the claim follows.

Now, it is clear that if \( X \) is \( p \)-concave symmetric space such that both operators \( C \) and \( C^* \) are bounded on \( X \), then the Copson space \( C^*X \) is also \( p \)-concave. Indeed, suppose that \( I = [0, 1] \), because if \( I = [0, \infty) \) there is nothing to prove, cf. the equality (6.4). Since \( L^1[0, 1] \) is \( p \)-concave with constant 1, it follows from the first part, that the space \( CX[0, 1] \) is also \( p \)-concave. Therefore, the space \( CX[0, 1] \cap L^1[0, 1] \) is \( p \)-concave as well. But in view of the equality (6.2) this means that also the Copson space \( C^*X[0, 1] \) is \( p \)-concave.

(d) Let \( \Upsilon(X) := \{p > 1: X \text{ contain an isomorphic copy of } \ell^p \} \). Then

\[ \Upsilon(Ces_p[0, 1]) = \Upsilon(L^p[0, 1]) \cup \Upsilon(L^1[0, 1]) \quad \text{for} \quad 1 \leq p < \infty, \]

see [7, Theorem 10] and [12, Theorem 5.5 and Fig. 1–2]. Now, if we set \( p > 1 \) and define a property \( P \) to means that the space \( X \) contains an isomorphic copy of \( \ell^p \), then \( P \) is also good property for the Cesàro construction.

**Theorem 6.6.** Let \( T = C \) or \( T = C^* \). Suppose that \( X \) is a Banach function space on \( I \) such that \( TX \) is nontrivial and let \( P \) be a good property. The implication: if \( TX \in (P) \) then \( X \in (P) \), does not hold in general.

**Proof.** We consider only the case when \( P \) is a good property for the Cesàro construction, because the proof in the second case is the same.

Since \( P \) is a good property, it follows that there are nontrivial symmetric spaces \( X \) and \( Y \) on \([0, 1]\) such that \( X \in (P) \) and \( Y \notin (P) \) (by property (G1)). Note that we only have two possibilities either \( Y \hookrightarrow X \) or \( X \not

\[ Y \not

\]

\[ X \not

\]

where the first inclusion follows from [13, Theorem 6.6, p. 77] (clearly, \( X \cap Y \) is a symmetric space). Consequently, \( X \cap Y \neq \{0\} \) and \( X \cap Y \notin (P) \) by property (G2). Moreover, \( X \cap Y \hookrightarrow X \)
and so the assumptions of Lemma 6.2 are satisfied also in this case for the spaces \( X \cap Y \) and \( X \). Now we can continue like in the previous case but instead of the space \( Y \) we take \( X \cap Y \). □

It seems interesting that when we restrict the class of spaces under consideration to class of symmetric spaces, then it may happen that the above theorem is not true. For example, it was proved by Kiwerski–Tomaszewski [43, Theorem 3] that a space \( X \) is order continuous if and only if \( CX \) is also order continuous, whenever \( X \) is a symmetric space such that the Cesàro operator is bounded on \( X \) (see also Corollary 6.1). Moreover, if \( I = [0, \infty) \) and we allow the situation that \( CX = \{0\} \), then it is easy to see that \( C(L^1 \cap L^\infty) = \{0\} \in (OC) \) but \( L^1 \cap L^\infty \notin (OC) \).

The results in the next section show that also fixed point properties do not transfer from \( X \) into the Cesàro or Copson function spaces.

7. Applications to the metric fixed point theory

A Banach space \( X = (X, \|\cdot\|_X) \) has the fixed point property (\( X \in (FPP) \) for short) if every nonexpansive mapping \( T : K \to K \), that is, the mapping satisfying
\[
\|T(x) - T(y)\| \leq \|x - y\| \quad \text{for all} \quad x, y \in K,
\]
on every nonempty, closed, bounded and convex subset \( K \) of \( X \), has a fixed point, i.e., there exist a point \( x_0 \in K \) such that \( T(x_0) = x_0 \). If the same holds for every nonempty, weakly compact and convex subset \( K \) of \( X \), we say that this space has the weak fixed point property (we write \( X \in (wFPP) \)). Of course, if the space \( X \) has the fixed point property, then \( X \) has the weak fixed point property and both properties are equivalent in the class of reflexive spaces. The spaces \( c_0, \ell^1, L^1[0, 1], L^\infty[0, 1], L^{p,1}[0, \infty) \) and \( C[0, 1] \) fail the fixed point property and the spaces \( \ell^\infty, c_0(\Gamma) \) and \( \ell^1(\Gamma) \), for \( \Gamma \) uncountable, cannot be even renormed to have the fixed point property, see Theorem 2, Corollary 3 and remark after Proposition 7 in [31]. However, \( c_0 \) and \( \ell^1 \) have the weak fixed point property but \( L^1[0, 1] \notin (wFPP) \), as it was proved by Alspach [3].

We are now ready to prove the main result of this section.

**Theorem 7.1.** Let \( T = C \) or \( T = C^* \). If \( X \) is a Banach function space on \( I \) such that \( TX \neq \{0\} \), then the space \( TX \) fails to have the fixed point property. Moreover,

(i) the space \( (TX)^* \) cannot be renormed to have the fixed point property,

(ii) the space \( (TX)^* \) fails to have the weak fixed point property.

**Proof.** The first claim that \( CX \notin (FPP) \) follows immediately from Theorem 4.5 and the Dowling–Lennard result from [29], which states that a Banach space which contains an asymptotically isometric copy of \( \ell^1 \) fails to have the fixed point property (see also [30, Theorem 2.3 and Corollary 2.11]). For the Copson space \( C^*X \) we apply Theorem 4.6 respectively.

(i) It is known that if a Banach space \( X \) contains complemented copy of \( \ell^1 \), then \( X^* \) cannot be renormed to have the fixed point property (see [31, Corollary 4]). Thus we should apply only Corollary 4.3.

(ii) Recall the Dilworth–Girardi–Hagler result [26, Theorem 2] which states that a Banach space \( X \) contains an asymptotically isometric copy of \( \ell^1 \) if and only if the dual space \( X^* \) contains an isometric copy of \( L^1[0, 1] \). Combining Theorem 4.5 with the Dilworth–Girardi–Hagler result we obtain that the space \( (CX)^* \) contains an isometric copy of \( L^1[0, 1] \). In view of Alspach result from [3] this means that \( (CX)^* \notin (wFPP) \). Again, in the case of the Copson space \( C^*X \) we simply use Theorem 4.6. □

By the Alspach result [3] and our Theorem 4.4 we easily obtain the following corollary.

**Corollary 7.2.** Let \( T = C \) or \( T = C^* \). Assume that \( X \) is a Banach function space on \( I \) with \( TX \neq \{0\} \). Then there is an equivalent norm on the space \( TX \) for which \( TX \) fails the weak fixed point property.
In the next remark we collect some known results concerning the (weak) fixed point property and copies of $\ell^\infty$.

**Remark 7.3.** Let $X$ be a Banach space.

(i) If $X$ contains an isomorphic copy of $\ell^\infty$, then $X$ cannot be renormed to have the fixed point property.

(ii) If $X$ contains an isometric copy of $\ell^\infty$, then $X$ fails to have the weak fixed point property.

**Proof.** (i) It follows from Pelczyński result [67] that a separable Banach space $X$ contains an isomorphic copy of $\ell^1$ if and only if $X^*$ contains an isomorphic copy of $\ell^1(\Gamma)$ for some uncountable set $\Gamma$. In particular, $(\ell^1)^* = \ell^\infty$ and consequently $\ell^\infty$ contains an isomorphic copy of $\ell^1(\Gamma)$. Moreover, by Dowling–Lennard–Turett result [31, Theorem 1] any renorming of the space $\ell^1(\Gamma)$ contains an asymptotically isometric copy of $\ell^1$. But a Banach space which contains an asymptotically isometric copy of $\ell^1$ fails the fixed point property [30]. Therefore, the space $X$ fails the fixed point property as well.

(ii) A classical result is that $\ell^\infty$ is the universal space for all separable Banach spaces, i.e., every separable Banach space $X$ can be isometrically embedded into $\ell^\infty$ (just take a dense subset $\{x_n : n \in \mathbb{N}\} \subset S(X)$ with $x_n x_n = 1$ for all $n \in \mathbb{N}$, where $\{x_n : n \in \mathbb{N}\} \subset S(X^*)$, and put $T : X \ni x \mapsto (x_n x)_{n=1}^\infty \in \ell^\infty$). Therefore, in particular, $\ell^\infty$ contains an isometric copy of $L^1[0,1]$. Again, by the Alspach result [3], $X \notin (wFPP)$.

Lozanovskii proved in [59] that a Banach function space $X$ is order continuous if and only if it contains no isomorphic copy of $\ell^\infty$. Moreover, if $X$ is a symmetric space and $C$ is bounded on $X$, then the space $CX$ is order continuous if and only if $X$ is also (see [48]). Consequently, by the above Remark 7.3 and Corollary 6.1 we have the following

**Corollary 7.4.** Let $T = C$ or $T = C^*$. If $X$ is a symmetric space on $I$ such that $C$ and $C^*$ are bounded on $X$ and $X$ is not order continuous, then $TX$ cannot be renormed to have the fixed point property.

**Proposition 7.5.** The spaces $Ces_1[0,1]$, $Ces_\infty$, $Cop_1$, $Cop_\infty$ and nontrivial Tandori function spaces $\tilde{X}$ fail to have the weak fixed point property.

**Proof.** Since $Ces_1[0,1] \equiv L^1(\log(1/|t|))[0,1]$ (see Example 3.3(b)), $L^1(\log(1/|t|))[0,1]$ is isometric to $L^1[0,1]$ and $L^1[0,1] \notin (wFPP)$ by the Alspach result [3], so $Ces_1[0,1] \notin (wFPP)$.

Arguing in the same way, $Cop_1 \equiv L^1$ and $Cop_\infty \equiv L^1(1/|t|)$ (see Example 3.3(c)) also fail the weak fixed point property.

If $\tilde{X} \neq \{0\}$, then the claim follows from Proposition 4.8 and Remark 7.3. For the space $Ces_\infty$, we apply Proposition 4.9 and Remark 7.3 respectively. □

8. Generalizations and applications of results

Until now, most of the results we have obtained for the Cesàro and Copson function spaces has been proven in the class of Banach function spaces and under the assumption about the nontriviality. It turns out that using previously developed methods we can transfer (without much effort) the most important results from Sections 4 and 7 to even more general optimal domains. We start with some definitions.

Denote by $\mathcal{H}_w$ the **weighted Cesàro operator** which is defined as

$$\mathcal{H}_w : f \mapsto \mathcal{H}_w f(x) := w(x) \int_0^x f(t) \, dt \quad \text{for} \quad t \in I,$$

where $w$ is a positive weight on $I$. For a Banach function spaces $X$ on $I$ by the **weighted Cesàro function space** $C_w X(I) = C_w X$ space we mean

$$C_w X := \{ f \in L^0 : \mathcal{H}_w |f| \in X \} \quad \text{with the norm} \quad \|f\|_{C_w X} = \|\mathcal{H}_w |f|\|_X.$$
These spaces for $X = L^p$, where $1 \leq p < \infty$, were studied by Kamińska-Kubiak [41] and by Kubiak in [52]. Observe, that the study of the spaces $C_{p,w} := \mathcal{H}_w L^p$ is more or less equivalent to study of the spaces $C L^p(w)$, that is, the Cesàro operator on the weighted $L^p$-spaces. Of course, if we take $w(x) = 1/x$ then $C_w X \equiv CX$. Moreover, if $w \equiv 1$ then $\mathcal{H}_w = V$, where $V$ denote the Volterra operator

$$V: f \mapsto Vf(x) := \int_0^x f(t)\,dt \quad \text{for} \quad t \in I.$$ 

Easy computations involving the Fubini’s theorem shows that a conjugate operator $\mathcal{H}_w^*$ to the weighted Cesàro operator $\mathcal{H}_w$ is given by the formula

$$\mathcal{H}_w^*: f \mapsto \mathcal{H}_w^* f(x) := \int_{I \cap [x,\infty)} w(t)f(t)\,dt \quad \text{for} \quad t \in I.$$ 

The space $C_w^* X(I) = C_w^* X$ associated with this operator can be called the weighted Copson function space. Again, if $w(t) = 1/t$ then $C_w^* X \equiv C^* X$ and if $w(t) \equiv 1$ then $C_w^* X \equiv V^* X$.

Note that

$$\|f\|_{C_w^* X} = \|w(x)\int_0^x |f(t)|\,dt\|_X = \|C |f|\|_{X(w)} = \|f\|_{C_Y^*},$$ 

that is, $C_w X \equiv CY$, where $Y = X(v)$ and $v(x) := xw(x)$. That is why it is easy to transfer claims about the Cesàro spaces $CX$ for $X$ being a Banach function space (rather not symmetric) to the spaces $C_w X$. Moreover, $C_w X \equiv V(\mathcal{X}(w))$.

It is easy to see that the space $C_w X$ is nontrivial if and only if $w(x)\chi_{[\lambda_0,m(I)]}(x) \in X$ for some $0 < \lambda_0 < m(I)$, cf. (8.1) and [54] Theorem 1 (a) and (b)]. Furthermore, if $C_w X$ is nontrivial then $\chi_{[0,\lambda_0]} \in X$ for some $0 < \lambda_0 < m(I)$, cf. Lemma. Keeping in mind this observation and following the proofs of Lemma 4.1 and Lemma 4.2 we can show that

**Lemma 8.1.** Let $X$ be a Banach function space on $I$.

(i) Assume that $C_w X(I) \neq \{0\}$. Then there exist $0 < \lambda_0 \in I$ with

$$\|w(x)\chi_{[b,m(I)]}(x)\|_{X(I)} \|f\|_{L^1[a,b]} \leq \|f\|_{C_w X(I)} \leq \|w(x)\chi_{[a,m(I)]}(x)\|_{X(I)} \|f\|_{L^1[a,b]},$$

for all $f \in C_w X(I)$ such that supp$(f) \subset [a, b]$, where $0 < \lambda_0 < a < b < m(I)$.

(ii) If $C_w^* X(I) \neq \{0\}$, then we can find $0 < \eta_0 \in I$ with

$$\|\chi_{[0,a]}\|_{X(I)} \|f\|_{L^1[w](a,b]} \leq \|f\|_{C_w^* X(I)} \leq \|\chi_{[0,b]}\|_{X(I)} \|f\|_{L^1[w](a,b]},$$

for all $f \in C_w^* X(I)$ such that supp$(f) \subset [a, b]$, where $0 < a < b < \eta_0 \leq m(I)$.

The next theorem, in the case of the weighted Cesàro function space $C_w X$, easily follows from the identification $C_w X \equiv CY$, where $Y = X(v)$ and $v(x) = xw(x)$ (cf. (8.1)), and Theorem 4.5. On the other hand, if $T = \mathcal{H}_w^*$, it is sufficient to use the same argument as in the proof of Theorem 4.6 and Lemma 8.1 (ii) instead of Lemma 4.2 (actually, the proof will be almost identical, because we can use the same function $G_X$). Summarizing the above discussion, we can obtain

**Theorem 8.2.** Let $X$ be a Banach function space on $I$ and $T = \mathcal{H}_w$ or $T = \mathcal{H}_w^*$. Then the nontrivial space $TX$ contains an order asymptotically isometric copy of $\ell^1$.

A similar result for the space $C_{p,w} := C_w L^p$, where $1 \leq p < \infty$, was obtained by Kubiak [52, Theorem 5.1].

Now, a direct consequence of Theorem 8.2 and the Dowling–Lenard–Turett result [30] (cf. proof of Theorem 7.1) is the following

**Theorem 8.3.** Let $X$ be a Banach function space on $I$ and $T = \mathcal{H}_w$ or $T = \mathcal{H}_w^*$. Then the space $TX$ fails the fixed point property whenever it is nontrivial.
As a direct application of the above considerations we can formulate the following

**Corollary 8.4.** The nontrivial Volterra space \( V^X \) contains an asymptotically isometric copy of \( \ell^1 \) and, consequently, fails the fixed point property.

Finally, let us mention also that \( \text{Vol}_1 := VL^1 \equiv L^1(1 - t) \) and \( \text{Vol}_\infty := VL^\infty \equiv L^1 \) (cf. Example 3.3 (b) and (c)) and consequently

**Theorem 8.5.** The spaces \( \text{Vol}_1 \) and \( \text{Vol}_\infty \) fail the weak fixed point property.

9. **Appendix**

In the proof of Theorem 4.5 it was necessary for the function \( F_X \) to be continuous in at least one point. It turned out that it is always continuous in uncountably many points. However, the question whether this function is actually continuous on the whole domain may be of independent interest.

**Lemma 9.1.** Let \( X \) be a Banach function space on \( I \) such that the operator \( C \) is bounded on \( X \). Assume that one of the following holds true

(i) the space \( X \) is order continuous,

(ii) the space \( X \) is symmetric and \( X \not
\overset{\rightarrow}{\leftarrow} \mathbb{L}^\infty \).

Then the function \( F_X \) is finitely valued and continuous for all \( 0 < x \in I \).

**Proof.** Actually, the proof of Lemma 4.1 shows that the function \( F_X \) is finitely valued. Moreover, \( \text{supp}(CX) = I \) because the Cesàro operator is bounded on \( X \), see Theorem D. It remains to prove that \( F_X \) is also continuous. We will consider two situations.

Assume that \( I = [0, 1] \). Let us fix \( 0 < \lambda_0 < 1 \) and take a sequence \( (\lambda_n)_{n=1}^\infty \subset [0, 1] \) such that \( \lambda_n \to \lambda_0 \). We will show that

\[
F_X(\lambda_n) \to F_X(\lambda_0),
\]

that is, the function \( F_X \) is continuous on \( (0, 1) \). Note first that there exist \( 0 < \varepsilon < \min\{\lambda_0, 1 - \lambda_0\} \) and \( N \in \mathbb{N} \) such that

\[
0 \leq |f_{\lambda_n}(x) - f_{\lambda_0}(x)| = \frac{1}{x} \chi(\min\{\lambda_0, \lambda_n\}, \max\{\lambda_0, \lambda_n\})(x) \leq \max_{n \geq N}\left\{ \frac{1}{\lambda_0}, \frac{1}{\lambda_n} \right\} \chi(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)(x),
\]

for \( n \geq N \) and \( 0 < x \leq 1 \). Put

\[
h_n := \frac{1}{x} \chi(\min\{\lambda_0, \lambda_n\}, \max\{\lambda_0, \lambda_n\}) \quad \text{and} \quad H := \max_{n \geq N}\left\{ \frac{1}{\lambda_0}, \frac{1}{\lambda_n} \right\} \chi(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon).
\]

Then \( h_n \to 0 \) almost everywhere on \([0, 1]\) as \( n \to \infty \), and it follows from (9.2) that \( 0 \leq h_n \leq H \). We claim that \( H \in X_\alpha \). In fact, if \( X \) is a symmetric space and \( X \not
\overset{\rightarrow}{\leftarrow} \mathbb{L}^\infty \), then \( X_\alpha = X_b \), where \( X_b \) is the closure in \( X \) of the set of bounded functions supported in sets of finite measure (see, for example, Theorem B in [38]). It is clear that \( H \) is such a function and so \( H \in X_\alpha \). However, if \( X \) is an order continuous Banach function space then the situation is a little bit different. Due to boundedness of the Cesàro operator and Theorem D, we can see that \( \chi_{[\lambda, 1]} \in X \) for all \( 0 < \lambda < 1 \). Therefore, \( H \in X = X_\alpha \) and the claim follows. Just from the definition of order continuity and (9.2) we obtain that

\[
0 \leq |F_X(\lambda_n) - F_X(\lambda_0)| = \left\| \frac{1}{x} \chi(\lambda_n, 1)(x) \right\|_X - \left\| \frac{1}{x} \chi(\lambda_0, 1)(x) \right\|_X \leq \left\| \frac{1}{x} \chi(\lambda_n, 1)(x) - \frac{1}{x} \chi(\lambda_0, 1)(x) \right\|_X = \left\| \left( \frac{1}{x} \chi(\min\{\lambda_0, \lambda_n\}, \max\{\lambda_0, \lambda_n\}) \right)(x) \right\|_X = \|h_n\|_X \to 0 \quad \text{as} \quad n \to \infty.
\]
This proves (9.1). In the missing case, when $\lambda_0 = 1$, the argument is essentially the same, so we will omit it.

Now suppose that $I = [0, \infty)$. Note only that in this case $\frac{1}{x} \chi_{(\lambda, \infty)}(x) \in X$ for each $0 < \lambda \in I$ and $\chi_{[a,b]} \in X$ for each $0 < a < b < \infty$, see Theorem D. Thus we can proceed as in the previous case. \hfill \square

It is not surprising that we can prove analogous lemma also for the function $G_X$.

**Lemma 9.2.** Let $X$ be a Banach function space on $I$ such that the Copson operator is bounded on $X$. Assume that one of the following holds true

(i) the space $X$ is order continuous,
(ii) the space $X$ is symmetric and $X \not\hookrightarrow L^\infty$.

Then the function $G_X$ is finitely valued and continuous for all $x \in I$.

**Proof.** We proceed as in the proof of Lemma 9.1 and use Corollary 3.2 instead of the proof of Lemma 4.2. \hfill \square

If we replace the assumption that the operator $T$, where $T = C$ or $T = C^*$, is bounded on $X$ by the assumption that $TX \neq \{0\}$, then using the same arguments as before we obtain that the function $F_X$ (resp. $G_X$) is finitely valued and continuous for all $x \in \text{int}(\text{supp}(TX))$.

The above lemma does not exclude the possibility that the function $F_X$ is continuous on $0 < x \in I$ but the space $X$ has trivial order continuous part. In fact, it is rather common for spaces with $X_0$ being trivial to own this property. We will now give some examples illustrating the discussion about the continuity of the function $F_X$.

**Example 9.3.** (a) Let $X$ be a Banach function space on $I$ and $w_0, w_1 : I \to (0, \infty)$ be two weights that differ only on the set of measure zero, i.e., $m(\{x \in I : w_0(x) \neq w_1(x)\}) = 0$. Then, of course, $X(w_0) \equiv X(w_1)$. In particular, if $X$ satisfies the assumptions of Lemma 9.1 and $w_1 = \mathcal{D}$, where

$$\mathcal{D} : I \ni x \mapsto \mathcal{D}(x) := \chi_{I/\mathbb{Q}}(x),$$

is a Dirichlet function, then $X \equiv X(\mathcal{D})$ and $\mathcal{D}$ is nowhere continuous function on $I$ but $F_{X(\mathcal{D})}$ is a continuous function for all $0 < x \in I$.

(b) Put

$$w_2 : [0, 1] \ni x \mapsto w_2(x) := 2\chi_\mathcal{C}(x) + \chi_{[0,1]}(x),$$

where $\mathcal{C}$ is the Smith–Volterra–Cantor set (or the fat Cantor set). The set of discontinuities of $w_2$ is the set $\mathcal{C}$, so it is uncountable and of positive measure. However, the set of discontinuities of the function $F_X$, where $Y := L^\infty(w_2)[0,1]$, is at most countable.

(c) Let $(q_n) \subset \mathbb{Q} \cap [0,1]$ be a sequence of rational numbers and put $Z := L^\infty(w_3)[0,1]$, where

$$w_3 : [0, 1] \ni x \mapsto w_3(x) := \sum_{\substack{q_n \in \mathbb{Q} \cap [0,1] \atop q_n < x}} 2^{-n}.$$

Then the function $F_Z$ is discontinuous at every rational number from the interval $[0,1]$ and continuous elsewhere.

(d) Let $X$ be a symmetric space such that $X \cong L^\infty$, i.e., the spaces $X$ and $L^\infty$ have the same elements and $\|f\|_X = A \|f\|_{L^\infty}$ for $f \in X$ and some constant $A > 0$. Then, with the same notation as in the proof of Lemma 9.1 we have

$$|F_X(\lambda_n) - F_X(\lambda_0)| = \left\| \frac{1}{x} \chi_{(\lambda_n, A)}(x) \right\|_X - \left\| \frac{1}{x} \chi_{(\lambda_0, A)}(x) \right\|_X = A \left| \frac{1}{\lambda_n} - \frac{1}{\lambda_0} \right| \to 0,$$

as $n \to \infty$. But this means that $F_X$ is continuous for $0 < x \in I$. Is also worth noting that if $X$ is a symmetric space on $[0,1]$ then the condition $X_a = \{0\}$ is equivalent to $X = L^\infty[0,1]$, see [48].
Theorem B], however, in the class of Orlicz spaces the condition \((L^\Phi) = \{0\}\) (that is, the Orlicz function \(\Phi\) takes also infinite values) is equivalent to \(L^\Phi \cong L^\infty[0,1]\). Moreover, for symmetric spaces on \([0,\infty)\) the aforementioned condition \(X^\Phi = \{0\}\) is equivalent to \(X \hookrightarrow L^\infty[0,\infty)\), see also [14, Theorem B].

(e) Let \(Y\) be a Banach function space on \(I\) such that \(Y \cong X \cap L^\infty\), where

\[
X \cap L^\infty := \{ f \in L^0 : \|f\|_Y := \max\{\|f\|_X, \|f\|_{L^\infty}\} < \infty \},
\]

and \(X\) is an order continuous Banach function space on \(I\). Of course, \(Y^\Phi = \{0\}\) but we can prove that the function \(F_Y\) is continuous for \(0 < x \in I\) (we will give the sketch of the proof only for \(I = [0,1]\) because the remaining case is the same). Indeed, we have

\[
|F_Y(x) - F_Y(0)| = \left\| \frac{1}{x} \chi_{(\lambda_n,1]}(x) \right\|_Y - \left\| \frac{1}{x} \chi_{(\lambda_0,1]}(x) \right\|_Y
= B \left\| \frac{1}{x} \chi_{(\lambda_n,1]}(x) \right\|_{X \cap L^\infty} - \left\| \frac{1}{x} \chi_{(\lambda_0,1]}(x) \right\|_{X \cap L^\infty},
\]

for some constant \(B > 0\). Example 9.3 (d) above and Lemma 9.1 implies that \(F_L\) and \(F_X\) are continuous functions for all \(0 < x \in I\). Therefore, \(F_Y = B \max\{F_X, F_{L^\infty}\}\) is also continuous for \(0 < x \in I\) as a maximum of two continuous functions and the claim follows.

The same examples can be considered also in the context of the function \(G_X\).

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