The simplest nearest-neighbor spin systems on regular graphs: Time dynamics of the mean coverage function

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Abstract
We establish a characterization of the class of the simplest nearest neighbor spin systems possessing the mean coverage function (mcf) that obeys a second order differential equation, and derive explicit expressions for the mcf's of the above models. Based on these expressions, the problem of ergodicity of the models is studied and bounds for their spectral gaps are obtained.

1 Introduction and Summary

It is commonly acknowledged that even in the case of a simple infinitesimal interaction mechanism, a description of the transient behavior of an interacting particle system (IPS) is an intractable mathematical problem in the theory of Markov processes. In view of this, there is a continuing interest, both in theory and applications, in seeking solvable (in some sense) models of IPS. One of the important functionals of IPS is undoubtedly, their mean coverage function. In the present paper we continue to study the behavior in time of the mean coverage function of a class of IPS called the simplest nearest neighbor spin systems (SNNSS) on s-regular graphs. Namely, developing the approach of Granovsky and Rozov [12], we establish a characterization of the class of SNNSS that posses a mean coverage function satisfying a second order differential equation. This is the main result of the present paper, stated in the Theorem, Section 3. The theorem asserts that the above class consists of the following four different modifications of the basic voter model: noisy voter model, noisy voter model with threshold = 2 (or 3) on 2- (resp.3) regular graphs, a special case of a general threshold = 2 model in one dimension and a degenerate model with threshold=s on s-regular graphs.

It should be noted that the first of these models is the unique SNNSS that has a mean coverage function satisfying a first order differential equation. This was proven in [12]. In Section 3 we derive explicit expressions for the mean coverage functions of the above four models, by solving the corresponding second order differential equations. The formulae obtained show that adding a constant noise to flip rates results in considerable change in transient behavior of the process. This matter is discussed in Section 5.

The next two sections are based on the aforementioned formulae for the mean coverage function. Section 4 is devoted to the mean density function. We prove here that, when started from the product Bernoulli measure, the mean density functions of the above processes do not depend on the size of the graph. This remarkable property is used for the study of ergodicity in the next section. Section 4 contains also a historical sketch of research related to the
subject.
It is clear that transient behavior of the mean coverage function, which is of interest in itself, also provides information on the long-time properties of the process considered. In view of this, the last Section 5 is devoted to ergodicity and bounding the spectral gap for the class of models defined in Theorem. We give here a positive answer on the open problem about ergodicity of threshold = 3 noisy voter model on 3-regular graphs, for some values of parameters. Based on the expressions derived in Section 3, we obtain the upper bounds for the spectral gap of the four SNNSS. These upper bounds are compared with the lower bounds given by the $\epsilon - M > 0$ condition (for references see [14], p.31).

Finally, note that in the course of the proof of the Theorem we derived identities that hold for a coverage of sites of a regular graph by 0’s and 1’s. These identities might be helpful in the study of other problems related to time dynamics of SNNSS.

Most of the notation and language of our paper have been adopted from the seminal monograph on IPS [14], by Liggett.

## 2 Background

We consider throughout the paper a SNNSS on a $s$-regular graph $G$ of finite size $N$, with the set of vertices(sites) $V = \{x\}$. Recall that a graph is called $s$-regular if each of its vertices has $s$ neighbors. By SNNSS we mean a time homogeneous Markov process $\varphi_t, \quad t \geq 0$ with state space $\mathcal{X}_N = \{0,1\}^V = \{\eta\}$ and the infinitesimal time dynamics given by (2.2) below. The elements $\eta = \{\eta(x), \ x \in V\}$ of $\mathcal{X}_N$ are called configurations. We will say that a site $x \in V$ is occupied(resp., empty) in the configuration $\eta \in \mathcal{X}_N$, if $\eta(x)$ is 1 (resp., 0). The SNNSS are featured by the property that the flip rate $c(x, \eta)$ of a spin at a site $x \in V$ in a configuration $\eta \in \mathcal{X}_N$ depends only on the number $k(x, \eta)$ of occupied neighbors of $x$ in the configuration $\eta$. Formally,

$$c(x, \eta) = \lambda_k(1 - \eta(x)) + \mu_k \eta(x), \quad k = k(x, \eta), \quad x \in V, \quad \eta \in \mathcal{X}_N,$$

where $\lambda_k, \ k = 0,1,\ldots,s$ (resp., $\mu_k, \ k = 0,1,\ldots,s$ ) are the rates of the infinitesimal transitions $0 \rightarrow 1$ (resp., $1 \rightarrow 0$) at a given site in a given configuration. Finally, denoting by $\eta_x$ the configuration obtained from $\eta$ by flipping the spin at the site $x$, the above assumptions conform to the following infinitesimal time dynamics of $\varphi_t, \quad t \geq 0$:

$$Pr(\varphi_{t+\Delta t} = \eta_x | \varphi_t = \eta) = c(x, \eta) \Delta t + o(\Delta t), \quad \Delta t \geq 0, \quad t \geq 0, \quad x \in V, \quad \eta \in \mathcal{X}_N,$$
where $\frac{\phi(\Delta t)}{\Delta t} \to 0$, as $\Delta t \to 0$.

So, to compare with a variety of the so called biased models, ( see e.g. Madras, Schinazi and Schonmann [16]) SNNSS is a spatially homogeneous process.

It is known that in the above setting the process $\phi_t, \ t \geq 0$ is fully defined by the $2s + 2$ parameters $\lambda_k \geq 0, \mu_k \geq 0, \ k = 0, \ldots, s$. Namely, the generator $\Omega$ of the process is given by

$$
\Omega f(\eta) = \sum_{x \in V} c(x, \eta)(f(\eta_x) - f(\eta)), \ f \in C(\mathcal{X}_N), \ \eta \in \mathcal{X}_N,
$$

where $C(\mathcal{X}_N)$ is the class of bounded functions $f : \mathcal{X}_N \to R$.

Denote $\phi_{i}^{(n)}(\eta), \ t \geq 0, \ \eta \in \mathcal{X}_N$ the SNNSS starting from a configuration $\eta$ and $M_{f}^{(n)}(t) = E f(\varphi_{i}^{(n)}), \ t \geq 0, \ f \in C(\mathcal{X}_N), \ \eta \in \mathcal{X}_N$. As in Granovsky and Rozov [12], our starting point will be the following assertion that is a straightforward consequence of the Hille- Yosida theorem.

**Proposition 1.** The function $M_{f}^{(n)}(t), \ t \geq 0$ satisfies, for all $\eta \in \mathcal{X}_N$, a linear differential equation of order $l, \ (l \geq 1)$

$$
\frac{d^l M_{f}^{(n)}(t)}{dt^l} = \sum_{i=0}^{l-1} A_i \frac{d^i M_{f}^{(n)}(t)}{dt^i} + B, \ t \geq 0, \ \eta \in \mathcal{X}_N
$$

with coefficients $A_i, \ i = 0, \ldots, l - 1$ and $B$ that do not depend on $\eta \in \mathcal{X}_N$ and $t \geq 0$, iff the generator $\Omega$ of the Markov process considered obeys the condition

$$
\Omega^l f = \sum_{i=0}^{l-1} A_i \Omega^i f + B, \ \eta \in \mathcal{X}_N,
$$

where $\Omega^i := \Omega(\Omega^{i-1}), \ i = 0, 1, \ldots$.

Our subsequent study of the characterization problem described in the previous section is based on the fact that (2.4) is equivalent to (2.5).

### 3 Main result

The coverage of the graph $G$ by a configuration $\eta \in \mathcal{X}_N$ is the function $|\eta| : \mathcal{X}_N \to R^+, \ |\eta| = \sum_{x \in V} \eta(x)$ and $M^{(n)}(t) := E|\varphi_{i}^{(n)}|, \ t \geq 0, \ \eta \in \mathcal{X}_N$ is called the mean coverage function of the process $\varphi_{i}^{(n)}, \ t \geq 0$. The function $M^{(n)}(t), t \geq 0$ is one of the most important functionals in applications. In Granovsky, Rolski, Woyczinski and Mann [1] and Belitsky, Granovsky [3] the function was studied in the context of adsorption - desorption process given by
It was observed there that the function $M^{(n)}(t), t \geq 0,$ has a saddle point, under certain conditions on parameters of the process.

Our main objective will be to describe the class of SNNSS satisfying (2.4) with $f = |\eta|$ and $l = 2.$ For $l = 1$ the problem was posed and solved in [12].

We introduce some more notation. Denote

$$g_i(\eta) := \Omega^i(|\eta|), \ i = 0, 1, \ldots, \eta \in \mathcal{X}_N$$

(3.6)

to obtain from (2.3)

$$g_1(\eta) = \sum_{x \in V} c(x, \eta)(1 - 2\eta(x)), \ \eta \in \mathcal{X}_N.$$  

(3.7)

In view of our objective, we will need to unlock the structure of $g_2.$

Let $D$ be a nonempty subset of $V.$ We will say that $y \in V$ is a neighbor of $D : y \sim D,$ if $y \notin D$ and $y$ is a neighbor of at least one site in $D,$ and we denote $\delta_1(D)$ the set of all neighbors of the subset $D.$ In particular, by $\delta_1(x)$ we denote the neighborhood of $x \in V.$ We also define $\delta_i(x) = \delta_1(\delta_{i-1}(x)), \ i = 1, 2, \ldots, \delta_0(x) = \{x\}, \ x \in V.$

Next, for any $x \in V$ define the difference operator $\Delta_x : C(\mathcal{X}_N) \rightarrow C(\mathcal{X}_N)$

$$\Delta_x f(\eta) = f(\eta_x) - f(\eta), \ f \in C(\mathcal{X}_N), \ x \in V$$  

(3.8)

and write $\Delta^{(2)}_{x,y} f = \Delta_x \Delta_y f, \ f \in C(\mathcal{X}_N), \ x, y \in V.$ Then, by our definition (3.6) and (2.3) we have

$$g_2(\eta) = \sum_{x \in V} c(x, \eta)\Delta_x g_1(\eta), \ \eta \in \mathcal{X}_N.$$  

(3.9)

Further, it follows from (3.7) and (2.4) that

$$\Delta_y g_1(\eta) = \sum_{x \in \delta_1(y)} (1 - 2\eta(x))\Delta_y c(x, \eta) -$$

$$\left( c(y, \eta_y) + c(y, \eta) \right)(1 - 2\eta(y)), \ \eta \in \mathcal{X}_N,$$  

(3.10)

for any $y \in V.$ Since $\Delta_{x,y} = \Delta_{y,x}, \ x, y \in V,$ (3.10) implies the important fact that

$$\Delta^{(2)}_{x,y} g_1(\eta) = 0, \ \eta \in \mathcal{X}_N,$$  

(3.11)

whenever $x \notin \delta_1(y) \cup \delta_2(y)$ and $x \neq y.$ This and (3.9) give
\[\Delta_y g_2(\eta) = \Delta_y \left[ c(y, \eta) \Delta_y g_1(\eta) \right] + \sum_{x \in \delta_1(y) \cup \delta_2(y)} \Delta_y \left[ c(x, \eta) \Delta_x g_1(\eta) \right], \quad y \in V, \ \eta \in \mathcal{X}_N. \quad (3.12)\]

For the proof of our main result, stated in the Theorem in the sequel, we need to impose the following two conditions on \(s\)-regular graphs \(G\) considered.

(i.) First, we assume that \(G\) is triangular free graph, which means that if \(x, y, z \in V : y, z \sim x\), then \(y, z\) are not neighbors. The second condition is a technical one.

(ii.) We assume the existence of a pair of vertices \(y, z \in V\) s.t. \(z \in \delta_3(y)\) and the two sets of vertices \(E_{1,2} := \delta_1(y) \cap \delta_2(z)\) and \(E_{2,1} := \delta_2(y) \cap \delta_1(z)\) are singletons.

Observe that the conditions (i) and (ii) are satisfied e.g., when \(G\) is an \(s\)-regular tree or \(G = \mathbb{Z}^d, \ d \geq 1\).

For the purpose of establishing our characterization result we employ a technique that is presented below. We start with the notations adopted from [3] and [12]. Denote \(n_k^{(i)} = n_k^{(i)}(\eta), \ k = 0, \ldots, s, \ i = 0, 1\) the number of occupied (i=1) (resp., empty (i=0)) sites having \(k\) occupied neighbors in a configuration \(\eta \in \mathcal{X}_N\), and let \(n_k^{(0)} = n_k^{(0)}(\eta), n_k^{(1)} = n_k^{(1)}(\eta), \ k = 0, \ldots, s\) denote the corresponding sets of vertices \(x \in V\) in a configuration \(\eta \in \mathcal{X}_N\). Finally, we denote \(V^{(i)} = V^{(i)}(\eta), i = 0, 1\) the set of all empty (resp. occupied) sites in \(\eta \in \mathcal{X}_N\).

Then \(g_1\) defined by (3.7) can be expressed as

\[g_1(\eta) = \sum_{k=0}^{s} (\lambda_k n_k^{(0)} - \mu_k n_k^{(1)}), \quad \eta \in \mathcal{X}_N. \quad (3.13)\]

The following identities that are valid for any \(s\)-regular graph will be crucial for our subsequent study:

\[P = P(\eta) := \sum_{x \in n_s^{(0)}} \Delta_x n_0^{(1)} = \sum_{x \in n_s^{(3)}} \Delta_x n_s^{(0)}, \quad \eta \in \mathcal{X}_N \quad (3.14)\]

\[Q_0 = Q_0(\eta) := \sum_{x \in V^{(0)}} \Delta_x n_s^{(0)} = -n_s^{(0)} + n_{s-1}^{(0)}, \quad \eta \in \mathcal{X}_N \quad (3.15)\]

\[Q_1 = Q_1(\eta) := \sum_{x \in V^{(1)}} \Delta_x n_s^{(0)} = -sn_s^{(0)} + n_{s}^{(1)}, \quad \eta \in \mathcal{X}_N \quad (3.16)\]

\[\sum_{x \in n_s^{(0)}} \Delta_x n_s^{(0)} = -n_s^{(0)}, \quad \sum_{x \in n_0^{(1)}} \Delta_x n_0^{(1)} = -n_0^{(1)}, \quad \eta \in \mathcal{X}_N. \quad (3.17)\]
The proof of the identities can be obtained after some thought from the preceding definitions. We also write

\[ R_i = R_i(\eta) := \sum_{x \in V^{(i)}} \Delta_x n_0^{(1)}, \quad i = 0, 1, \quad \eta \in \mathcal{X}_N. \]  

(3.18)

Let \( \bar{\eta} \) be the configuration obtained by flipping the spins at all sites \( x \in V \) in a configuration \( \eta \in \mathcal{X}_N \). Then we have \( n_k^{(i)}(\bar{\eta}) = n_{s-k}^{(1-i)}(\eta) \), \( k = 0, 1, \ldots, s \), \( i = 0, 1 \), \( \eta \in \mathcal{X}_N \) and, consequently,

\[ R_i(\eta) = Q_{1-i}(\bar{\eta}), \quad \eta \in \mathcal{X}_N, \quad i = 0, 1. \]  

(3.19)

Now we are in a position to state the following

**Lemma.** The identity (in \( \eta \))

\[ n_s^{(1)} + n_{s-1}^{(0)} - n_1^{(1)} - n_0^{(0)} = (n_s^{(0)} - n_0^{(1)})F_1 + |\eta|F_2 + F_3, \quad \eta \in \mathcal{X}_N, \]  

(3.20)

where \( F_i, \ i = 1, 2, 3 \) are coefficients that do not depend on \( \eta \), holds iff \( s = 2, 3 \). In both cases of \( s, F_2 = 2, F_3 = -N, \) while

\[ F_1 = \begin{cases} 
-3, & \text{if } s = 2 \\
-2, & \text{if } s = 3.
\end{cases} \]  

(3.21)

**Proof.** We put in (3.20) first \( \eta = \emptyset \) and then \( \eta = \bar{\emptyset} \) to find \( F_2 \) and \( F_3 \). Now consider the case \( s \geq 3 \). Due to the fact that the graph considered is triangular free, we have for any \( x \sim y, \ x, y \in V, \)

\[ n_s^{(1)}(\emptyset_{x,y}) = n_{s-1}^{(0)}(\emptyset_{x,y}) = n_1^{(1)}(\emptyset_{x,y}) = n_0^{(0)}(\emptyset_{x,y}) = 0, \ n_0^{(0)}(\emptyset_{x,y}) = N - 2s, \ n_1^{(1)}(\emptyset_{x,y}) = 2. \]  

(3.22)

Substituting this in (3.20), gives \( -2 - (N - 2s) = 4 - N \), which says that (3.20) does not hold for \( s > 3 \).

Further, if \( s = 3 \) and \( \eta = \emptyset_{x}, \ x \in V, \) then

\[ n_3^{(1)}(\emptyset_{x}) = n_2^{(0)}(\emptyset_{x}) = n_1^{(1)}(\emptyset_{x}) = n_0^{(0)}(\emptyset_{x}) = 0, \ n_0^{(0)}(\emptyset_{x}) = N - 4, \ n_1^{(1)}(\emptyset_{x}) = 1. \]  

(3.23)

The latter implies \( F_1 = -2 \).
In the case $s = 2$ we have $n_{s-1}^{(0)}(\emptyset_{x,y}) = 2$, and the same argument as before gives $F_1 = -3$. Finally, it is left to show that the identity (3.20) indeed holds for $s = 2,3$. We use the relationship

$$2|\eta| - N = |\eta| - (N - |\eta|) = \sum_{k=0}^{s} (n_{k}^{(1)} - n_{k}^{(0)})$$

to obtain for $s = 3$

$$n_{3}^{(1)} + n_{2}^{(0)} - n_{1}^{(1)} - n_{0}^{(0)} + 2(n_{3}^{(0)} - n_{1}^{(0)}) - 2|\eta| + N = \sum_{k=0}^{3} (kn_{k}^{(1)} - (s-k)n_{k}^{(1)}) = 0, \ \eta \in X_N, \ (3.24)$$

where the last equation follows from the identity $\sum_{k=1}^{s} k(n_{k}^{(0)} + n_{k}^{(1)}) = s|\eta|, \ \eta \in X_N$ that is valid for all $s$-regular graphs. The same argument proves the assertion for $s = 2$. ♣

Finally, we will distinguish the following modifications of the Basic Voter model:

**Noisy Voter Model.** $\lambda_k - \lambda_{k-1} = \mu_{k-1} - \mu_k = d, \ k = 1, \ldots, s$. The model was introduced in [12] and intensively studied in [11]. Here the noise is given by the two parameters $h_1 = \lambda_0, \ h_2 = \mu_0 - sd$ added to the basic voter model (see [14],[15]): $\lambda_k = kd, \ \mu_k = (s-k)d, \ k = 0,1,\ldots, s$

Note that in [14] Ex.2.5, p.136, it is considered a general (i.e. not necessarily the nearest neighbor) version of voter model with noise.

**Noisy Voter Model with Threshold** $= q \ (1 \leq q \leq s)$.

$$\lambda_k = \mu_{k+s-q+1} = h \geq 0, \ k = 0, \ldots, q - 1, \ \lambda_k = \mu_{k-q} = h + a \geq 0, \ k = q, \ldots, s. \ (3.25)$$

This is the simplest case of a nonlinear voter model. In the case $h = 0$ (the absence of noise), the model was suggested by Cox and Durrett in [7]. (For updated references see [15]). In [4] it was also considered the threshold voter model with noise added to the death rates only. If $q = s$, then by scaling all the rates by the factor $(2h + a)^{-1}$ the model becomes the nearest neighbor Majority Vote Process ([14], Ex.4.3(e),p.33 and Ex. 2.12,p. 140).

**Generalized Threshold Model with threshold** $= q \ (1 \leq q \leq s)$. The model is obtained from the previous one by adding a constant either to $s - q + 1$ birth rates $\lambda_k, \ k = q, \ldots, s$, or to $s - q + 1$ death rates $\mu_k, \ k = 0, \ldots, q - s$. Explicitly,

$$\lambda_k = \mu_{k+s-q+1} = h \geq 0, \ k = 0, \ldots, q - 1, \ \lambda_k = h + a \geq 0, \ k = q, \ldots, s,$$

$$\mu_k = h + b \geq 0, \ k = 0, \ldots, s - q. \ (3.26)$$
Theorem
The mean coverage function $M^{(n)}(t), \ t \geq 0$ of a SNNSS $\varphi_t, \ t \geq 0$ satisfies, for all $\eta \in X_N$, a second order linear differential equation

$$\frac{d^2 M^{(n)}(t)}{dt^2} = A_1 \frac{dM^{(n)}(t)}{dt} + A_0 M^{(n)}(t) + B, \ t \geq 0, \ \eta \in X_N \quad (3.27)$$

with coefficients $A_0, A_1, B$ that do not depend on $\eta \in X_N$ and $t \geq 0$, iff $\varphi_t, \ t \geq 0$ is one of the following four models $(C_1) - (C_4)$:

$(C_1)$ A noisy voter model.

$(C_2)$ A generalized threshold model with threshold $= s$ and $h = ab = 0$ or $h = 0, \ a = b$.

$(C_3)$ A threshold noisy voter model with threshold $= s$, when $s = 2, 3$

$(C_4)$ A generalized threshold model with threshold $s = 2$ and $h, a, b : h(a + b) = ab, \ h \geq 0, \ h + a \geq 0, \ h + b \geq 0$.

Proof. By virtue of Proposition 1, $(3.27)$ is equivalent to

$$g_2(\eta) = A_1 g_1(\eta) + A_0 |\eta| + B, \ \eta \in X_N, \quad (3.28)$$

The main difficulty is to prove that $(3.28)$ implies one of the four conditions $(C_1) - (C_4)$ on the rates of $\varphi_t, \ t \geq 0$

If $(3.28)$ holds, then, by $(3.11)$

$$\Delta^{(2)}_{y,z} g_2(\eta) = A_1 \Delta^{(2)}_{y,z} g_1(\eta) = 0, \ y \in V, \ z \in \delta_3(y), \ \eta \in X_N. \quad (3.29)$$

From the other hand, we get from $(3.12)$ and $(3.11)$

$$\Delta^{(2)}_{y,z} g_2(\eta) = \sum_{x \in \delta_1(y) \cup \delta_2(y)} \Delta^{(2)}_{y,z} c(x, \eta) \Delta_{x} g_1(\eta), \ y \in V, \ z \in \delta_3(y), \ \eta \in X_N. \quad (3.30)$$

In view of $(3.11)$ this gives

$$\Delta^{(2)}_{y,z} g_2(\eta) = \sum_{x \in E_{1,2}} \Delta_y c(x, \eta) \Delta^{(2)}_{x,z} g_1(\eta) + \sum_{x \in E_{2,1}} \Delta_z c(x, \eta) \Delta^{(2)}_{x,y} g_1(\eta), \ z \in \delta_3(y), \ \eta \in X_N, \quad (3.31)$$

where we denoted $E_{1,2} := \delta_1(y) \cap \delta_2(z)$ and $E_{2,1} := \delta_2(y) \cap \delta_1(z)$. We also derive from $(3.11)$

$$\Delta^{(2)}_{x,z} g_1(\eta) = \sum_{u \in \delta_1(x) \cap \delta_1(z)} (1 - 2\eta(u)) \Delta^{(2)}_{u,z} c(u, \eta), \ x \in \delta_2(z), \ \eta \in X_N \quad (3.32)$$

and
\[ \Delta^{(2)}_{x,y}g_1(\eta) = \sum_{u \in \delta_1(x) \cap \delta_1(y)} (1 - 2\eta(u))\Delta^{(2)}_{x,y}c(u, \eta), \quad x \in \delta_2(y), \quad \eta \in \mathcal{X}_N. \] (3.33)

We substitute now these expressions in (3.31) to obtain

\[ \Delta^{(2)}_{y,z}g_2(\eta) = \sum_{x \in E_{1,2}} \Delta_y[c(x, \eta)] \sum_{u \in \delta_1(x) \cap \delta_1(z)} (1 - 2\eta(u))\Delta^{(2)}_{x,y}c(u, \eta)] + \sum_{x \in E_{2,1}} \Delta_z[c(x, \eta)] \sum_{u \in \delta_1(x) \cap \delta_1(y)} (1 - 2\eta(u))\Delta^{(2)}_{x,y}c(u, \eta)], \quad y \in V, \quad z \in \delta_3(y), \quad \eta \in \mathcal{X}_N. \] (3.34)

Our immediate aim is to find conditions on the parameters of a SNNSS, imposed by the requirement

\[ \Delta^{(2)}_{y,z}g_2(\eta) = 0, \quad y \in V, \quad z \in \delta_3(y), \quad \eta \in \mathcal{X}_N. \] (3.35)

Let, in accordance with the assumption (ii) on \( G \), the vertices \( y, z \) in (3.34) be such that

\[ E_{1,2} = \{u_1\}, \quad E_{2,1} = \{u_2\}, \] (3.36)

where \( u_1, u_2 \in V \). Then, in view of the above definition of the vertices \( u_1, u_2 \), (3.34) becomes

\[ \Delta^{(2)}_{y,z}g_2(\eta) = (\Delta_y[c(u_1, \eta)])(1 - 2\eta(u_2))\Delta^{(2)}_{u_1,z}c(u_2, \eta) + (\Delta_z[c(u_2, \eta)])(1 - 2\eta(u_1))\Delta^{(2)}_{u_2,y}c(u_1, \eta), \quad \eta \in \mathcal{X}_N, \quad z \in \delta_3(y). \] (3.37)

The last expression will be our main tool in the subsequent study.

We see from (3.37) that

\[ \Delta^{(2)}_{y,z}g_2(\eta_y) = -\Delta^{(2)}_{y,z}g_2(\eta), \quad \Delta^{(2)}_{y,z}g_2(\eta_z) = -\Delta^{(2)}_{y,z}g_2(\eta), \quad z \in \delta_3(y), \quad \eta \in \mathcal{X}_N. \]

In view of this we set in (3.37), \( \eta(y) = \eta(z) = 0 \). We also agree to write \( \Delta(\bullet)_k = (\bullet)_{k+1} - (\bullet)_k \) and \( \Delta^{(2)}(\bullet)_k = (\bullet)_{k+2} - 2(\bullet)_{k+1} + (\bullet)_k \), where \( \bullet \) is either \( \lambda \) or \( \mu \).

Now we will be attempting to find the explicit form of (3.37) in the following three cases of \( \eta \in \mathcal{X}_N \) that exhaust all the possibilities. For brevity, we denote \( k_i = k(u_i, \eta), \quad \eta \in \mathcal{X}_N \). It is important to note that \( \delta_1(u_1) \cap \delta_1(u_2) \) is the empty set, since \( G \) is triangular free.

**Case 1.** \( \eta \in \mathcal{X}_N : \eta(u_1) = \eta(u_2) = 0. \)

\[ \Delta^{(2)}_{y,z}g_2(\eta) = (\Delta\lambda_{k_1})(\Delta^{(2)}\lambda_{k_2}) + (\Delta\lambda_{k_2})(\Delta^{(2)}\lambda_{k_1}), \quad z \in \delta_3(y), \quad \eta \in \mathcal{X}_N. \] (3.38)
where \(0 \leq k_1, k_2 \leq s - 2\).

**Case 2.** \(\eta \in \mathcal{X}_N: \eta(u_1) = \eta(u_2) = 1\).

\[
\Delta_{y,z}^{(2)} g_2(\eta) = \left(\Delta \mu_{k_1}\right) \left(\Delta^{(2)} \mu_{k_2-1}\right) + \left(\Delta \mu_{k_2}\right) \left(\Delta^{(2)} \mu_{k_1-1}\right), \quad z \in \delta_3(y), \quad \eta \in \mathcal{X}_N, \quad (3.39)
\]

where \(1 \leq k_1, k_2 \leq s - 1\).

**Case 3.** \(\eta \in \mathcal{X}_N: \eta(u_1) = 0, \quad \eta(u_2) = 1\).

\[
\Delta_{y,z}^{(2)} g_2(\eta) = \left(\Delta \lambda_{k_1}\right) \left(\Delta^{(2)} \mu_{k_2}\right) + \left(\Delta \mu_{k_2}\right) \left(\Delta^{(2)} \lambda_{k_1-1}\right), \quad (3.40)
\]

where \(1 \leq k_1 \leq s - 1, \quad 0 \leq k_2 \leq s - 2\).

We now know from the three cases considered, that the condition (3.33) implies

\[
\left(\Delta \lambda_{k_1}\right) \left(\Delta^{(2)} \lambda_{k_2}\right) + \left(\Delta \lambda_{k_2}\right) \left(\Delta^{(2)} \lambda_{k_1}\right) = 0, \quad 0 \leq k_1, k_2 \leq s - 2 \quad (3.41)
\]

\[
\left(\Delta \mu_{k_1}\right) \left(\Delta^{(2)} \mu_{k_2-1}\right) + \left(\Delta \mu_{k_2}\right) \left(\Delta^{(2)} \mu_{k_1-1}\right) = 0, \quad 1 \leq k_1, k_2 \leq s - 1 \quad (3.42)
\]

\[
\left(\Delta \lambda_{k_1}\right) \left(\Delta^{(2)} \mu_{k_2}\right) + \left(\Delta \mu_{k_2}\right) \left(\Delta^{(2)} \lambda_{k_1-1}\right) = 0, \quad 1 \leq k_1 \leq s - 1, \quad 0 \leq k_2 \leq s - 2. \quad (3.43)
\]

Setting in (3.41), (3.42) \(k_1 = k_2 = k\) gives

\[
\left(\Delta \lambda_{k}\right) \left(\Delta^{(2)} \lambda_{k}\right) = 0, \quad 0 \leq k \leq s - 2 \quad (3.44)
\]

\[
\left(\Delta \mu_{k}\right) \left(\Delta^{(2)} \mu_{k-1}\right) = 0, \quad 1 \leq k \leq s - 1. \quad (3.45)
\]

Using this fact, we multiply the equations (3.41), (3.42) by \(\Delta \lambda_{k_1}\) and by \(\Delta \mu_{k_1}\) correspondingly, to obtain

\[
\left(\Delta \lambda_{k_1}\right) \left(\Delta^{(2)} \lambda_{k_2}\right) = 0, \quad 0 \leq k_1, k_2 \leq s - 2 \quad (3.46)
\]

\[
\left(\Delta \mu_{k_1}\right) \left(\Delta^{(2)} \mu_{k_2-1}\right) = 0, \quad 1 \leq k_1, k_2 \leq s - 1. \quad (3.46)
\]

Since, by our definition, \(\Delta^{(2)}(\bullet)_{k_2} = \Delta(\bullet)_{k_2+1} - \Delta(\bullet)_{k_2}\), the equations (3.46) imply for \(s > 2\)

\[
\Delta^{(2)} \lambda_k = 0, \quad 0 \leq k \leq s - 3. \quad (3.47)
\]

\[
\Delta^{(2)} \mu_k = 0, \quad 1 \leq k \leq s - 2. \quad (3.47)
\]
Thus, we have for $s > 2$

\[
\Delta \lambda_k : = d_{\lambda}, \quad k = 0, \ldots, s - 2 \\
\Delta \mu_k : = d_{\mu}, \quad k = 1, \ldots, s - 1
\]  

(3.48)

Finally, in view of (3.47) and (3.48), we obtain from (3.41)-(3.43) for all $s \geq 2$

\[
d_{\lambda} \Delta^{(2)} \lambda_{s-2} = 0 \\
d_{\mu} \Delta^{(2)} \mu_0 = 0 \\
(\Delta \lambda_{s-1}) (\Delta^{(2)} \mu_0 + (\Delta \mu_0) (\Delta^{(2)} \lambda_{s-2}) = 0.
\]  

(3.49)

Summarizing the preceding argument we conclude that (3.48) together with (3.49) are necessary and sufficient for (3.35). Our next step will be devoted to show that the conditions (3.48), (3.49) on the parameters $\lambda_k, \mu_k, k = 0, \ldots, s$ imply one of the conditions $(C_1)-(C_4)$. Assume first that in (3.49) $d_{\lambda} \neq 0$. Then we should have $\Delta^{(2)} \lambda_{s-2} = \Delta \lambda_{s-1} - \Delta \lambda_{s-2} = 0$, and, consequently, in view of (3.48), $\Delta \lambda_{s-1} = d_{\lambda} \neq 0$. Hence, in view of the last equation in (3.49), we obtain

\[
\Delta^{(2)} \mu_0 = \Delta^{(2)} \lambda_{s-2} = 0.
\]  

(3.50)

By the same argument, (3.50) should also hold under the assumption $d_{\mu} \neq 0$. (3.50) together with (3.48) is equivalent to saying that the flip rates $\lambda_k, \quad k = 0, \ldots, s$ and $\mu_k, \quad k = 0, \ldots, s$ form arithmetical progressions:

\[
\Delta \lambda_k = d_{\lambda}, \quad k = 0, \ldots, s - 1 \\
\Delta \mu_k = d_{\mu}, \quad k = 0, \ldots, s - 1.
\]  

(3.51)

So, assuming $d_{\lambda} \neq 0$, one has

\[
\Delta_y c(x, \eta) = (1 - 2 \eta(y))(d_{\lambda}(1 - \eta(x)) + d_{\mu} \eta(x)), \quad x \in \delta_1(y), \quad x, y \in V, \quad \eta \in \mathcal{X}_N.
\]  

(3.52)

Now (3.52) and (3.10) imply

\[
\Delta_y g_1(\eta) = (1 - 2 \eta(y))(sd_{\lambda} - 2k(y, \eta)(d_{\mu} + d_{\lambda}) - \lambda_0 - \mu_0), \quad y \in V, \quad \eta \in \mathcal{X}_N,
\]  

(3.53)

\[
\Delta^{(2)} y_2 c(x, \eta) = 0, \quad y_2 \in \delta_2(y), \quad x \in V, \quad \eta \in \mathcal{X}_N,
\]  

(3.54)
and
\[ \Delta_{y_2,y}^{(2)} g_1(\eta) = 0, \quad y_2 \in \delta_2(y), \quad \eta \in \mathcal{X}. \]  \hfill (3.55)

Hence, by virtue of (3.28) it follows from (3.55) that
\[ \Delta_{y_2,y}^{(2)} g_2(\eta) = 0, \quad y_2 \in \delta_2(y), \quad \eta \in \mathcal{X}. \]  \hfill (3.56)

By (3.51) the latter is equivalent to
\[ \Delta_{y_2,y}^{(2)} \left( c(y_1, \eta) \Delta_{y_1} g_1(\eta) \right) = -4(d_\lambda + d_\mu)(1 - 2\eta(y_1))(1 - 2\eta(y_2)) \left( (1 - \eta(y_1))d_\lambda - \eta(y_1)d_\mu \right) = 0, \]
\[ y_2 \in \delta_2(y), \quad y_1 \in \delta_1(y), \quad \eta \in \mathcal{X}. \]  \hfill (3.57)

This implies \( d_\lambda + d_\mu = 0 \), which by (3.51), corresponds to the noisy voter model (C_1). Since the same conclusion is valid under the assumption \( d_\mu \neq 0 \), it is left to assume that \( d_\lambda = d_\mu = 0 \).

In this case it follows from (3.48) that the parameters of the SNNSS are of the form
\[ \lambda_k = \lambda, \quad k = 0, \ldots, s - 1, \quad \mu_k = \mu, \quad k = 1, \ldots, s, \quad \lambda_s = \lambda + a, \quad \mu_0 = \mu + b, \]  \hfill (3.58)

where \( a, b \in \mathbb{R} \) are such that \( \lambda + a \geq 0, \quad \mu + b \geq 0 \).

Note that in the case considered all three conditions (3.49) are satisfied, because (3.58) implies \( \Delta_{y_2,y}^{(2)} \lambda_{s-2} = a, \quad \Delta_{y_2,y}^{(2)} \mu_0 = -b \). In view of the relationships \( \sum_{k=0}^{s} n_k^{(0)} = N - |\eta| \) and \( \sum_{k=0}^{s} n_k^{(1)} = |\eta| \), (3.13) yields for the model (3.58)
\[ g_1(\eta) = \lambda N - (\lambda + \mu)|\eta| + an_s^{(0)} - bn_0^{(1)}, \quad \eta \in \mathcal{X}. \]  \hfill (3.59)

By (2.3) we also have
\[ \Omega(n_k^{(i)}) = \lambda \sum_{x \in V^{(0)} \setminus n_k^{(0)}} \Delta_x n_k^{(i)} + \lambda_s \sum_{x \in n_k^{(0)}} \Delta_x n_k^{(i)} + \mu \sum_{x \in V^{(1)} \setminus n_k^{(1)}} \Delta_x n_k^{(i)} + \]
\[ \mu_0 \sum_{x \in n_0^{(1)}} \Delta_x n_k^{(i)}, \quad \eta \in \mathcal{X}, \quad k = 0, \ldots, s, \quad i = 0, 1. \]  \hfill (3.60)

Next, we apply (3.60) for \( n_s^{(0)} \) and \( n_0^{(1)} \) to obtain, with the help of the identities (3.14) - (3.17),
\[ \Omega(n_s^{(0)}) = \lambda Q_0 + \mu Q_1 - an_s^{(0)} + bP, \quad \eta \in \mathcal{X}, \]  \hfill (3.61)

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and
\[
\Omega(n_0^{(1)}) = \lambda R_0 + \mu R_1 - bn_0^{(1)} + aP, \quad \eta \in \mathcal{X}_N. \tag{3.62}
\]

Now we derive from (3.59) the expression for \( g_2 \) that we will be working with:
\[
g_2(\eta) = - (\lambda + \mu) g_1(\eta) + \lambda a Q_0 - \mu b R_1 + \mu a Q_1 - \lambda b R_0 - a^2 n_s^{(0)} + b^2 n_0^{(1)}, \quad \eta \in \mathcal{X}_N. \tag{3.63}
\]
(3.63) and (3.59) show that for the model (3.58) the relationship (3.28) holds iff
\[
T(\eta) := \lambda a Q_0 - \mu b R_1 + \mu a Q_1 - \lambda b R_0 - a^2 n_s^{(0)} + b^2 n_0^{(1)} - (an_s^{(0)} - bn_0^{(1)}) A_2 - B_2 |\eta| - C_2 = 0, \quad \eta \in \mathcal{X}_N, \tag{3.64}
\]
where \( A_2, B_2, C_2 \) are coefficients that do not depend on \( \eta \in \mathcal{X}_N \). We put in (3.64) first \( \eta = \emptyset \) and then \( \eta = \emptyset \) to obtain \( C_2 = -\lambda b N \) and \( B_2 = \lambda b + \mu a \).

We will treat separately the case \( s \geq 3 \) and the case \( s = 2 \). Since \( a = b = 0 \) leads to a particular case of noisy voter model, we suppose in the sequel that \( a^2 + b^2 \neq 0 \). Let some fixed \( y, z \in V \) obey the condition (ii), and \( u_1, u_2 \in V \) are defined as in (3.66).

The case \( s \geq 3 \). Consider the following two configurations: \( \eta_1 \), defined by \( \eta_1(y) = \eta_1(u_1) = 0, \eta_1(v) = 1 \), for all \( v \neq u_1, y \), and \( \eta_2 = (\eta_1)_{u_2} \). It is easy to figure out the following relationships
\[
Q_0(\eta_i) = 2, \quad i = 1, 2, \quad Q_1(\eta_1) = N - 2s, \quad Q_1(\eta_2) = N + 1 - 3s, \quad R_0(\eta_i) = R_1(\eta_i) = 0, \quad n_s^{(0)}(\eta_i) = n_0^{(1)}(\eta_i) = 0, \quad i = 1, 2. \tag{3.65}
\]
Substituting (3.65) in (3.64) gives
\[
2\lambda a + \mu a (N - 2s) - (N - 2)(\lambda b + \mu a) + \lambda b N = 0 \tag{3.66}
\]
and
\[
2\lambda a + \mu a (N + 1 - 3s) - (N - 3)(\lambda b + \mu a) + \lambda b N = 0, \tag{3.67}
\]
which implies
\[
\mu a(s - 1) = (a + b)\lambda, \quad a\mu(3s - 4) = \lambda(3b + 2a). \tag{3.68}
\]
By the same argument, applied to the configurations \( \bar{\eta}_1, \bar{\eta}_2 \) we also get
\( \lambda b(s - 1) = (a + b)\mu, \quad \lambda b(3s - 4) = \mu(3a + 2b). \)  

We will find all solutions of (3.68) and (3.69). First we see that \( \lambda \mu \neq 0 \) implies \( a = b \) and consequently, \( \lambda = \mu > 0, \ s = 3 \). This gives the threshold voter model (C3). If \( \lambda \mu = 0 \), then we should have \( \lambda a = \lambda b = \mu a = \mu b = 0 \). By (3.64) this implies \( a = b \) or \( ab = 0 \). In the first case, we have \( \lambda = \mu = 0 \), which is again (C3), while in the second case, \( \lambda = \mu = ab = 0 \), which is (C2).

The case \( s = 2 \). Taking \( \eta_1 \) as above, gives

\[
Q_0(\eta_1) = 2, \quad Q_1(\eta_1) = N - 4, \quad R_1(\eta_1) = 2, \quad R_0(\eta_1) = n_2^{(0)}(\eta_1) = n_0^{(1)}(\eta_1) = 0. 
\]

Consequently, (3.64) implies \( (a + b)(\lambda - \mu) = 0 \). If \( a + b = 0 \), then (3.64) becomes

\[
\lambda a(Q_0 + R_0) + \mu a(Q_1 + R_1) - a^2(n_2^{(0)} - n_0^{(1)}) - \\
(n_2^{(0)} + n_0^{(1)})A_2a - a(\mu - \lambda)|\eta| - \lambda aN = 0, \quad \eta \in X_N. 
\]

Since in the case \( s = 2 \)

\[
Q_0 + R_0 = -2n_2^{(0)} - 2n_0^{(1)} + N - |\eta|, \quad \eta \in X_N 
\]

and

\[
Q_1 + R_1 = -2n_2^{(0)} - 2n_0^{(1)} + |\eta|, \quad \eta \in X_N, 
\]

we see that (3.71) implies \( a = 0 \), and, consequently, \( b = 0 \).

Let now \( \lambda = \mu \). Then, we employ (3.18), (3.15) and (3.16) to rewrite (3.64) as

\[
T(\eta) = \lambda \left[ aL(\eta) - bL(\bar{\eta}) \right] - a^2n_2^{(0)} + b^2n_0^{(1)} - \\
(an_2^{(0)} - bn_0^{(1)})\bar{A}_2 = 0, \quad \eta \in X_N, 
\]

where we denoted \( L(\eta) = n_1^{(0)} + n_2^{(1)} - |\eta| \) and \( \bar{A}_2 = A_2 + 3\lambda \). So,

\[
T(\eta) + T(\bar{\eta}) = (a - b)\left[ \lambda(L(\eta) + L(\bar{\eta})) - (a + b)(n_2^{(0)} + n_0^{(1)}) - \\
(n_2^{(0)} + n_0^{(1)})\bar{A}_2 \right] = 0, \quad \eta \in X_N. 
\]
First observe that if \( a = b \) and \( \lambda = \mu \) then we have the model \((C_3)\). Next, substituting in (3.75)

\[
L(\eta) + L(\bar{\eta}) = n_1^{(0)} + n_2^{(1)} + n_1^{(1)} + n_0^{(0)} - N = -n_2^{(0)} - n_0^{(1)},
\]

we have

\[
(n_2^{(0)} + n_0^{(1)})(\bar{A}_2 + a + b + \lambda) = 0, \quad \eta \in \mathcal{X}_N. \tag{3.76}
\]

Consequently, \( \bar{A}_2 = -a - b - \lambda \), which in view of (3.74) yields

\[
\lambda a(n_1^{(0)} + n_2^{(1)} + n_0^{(0)} - |\eta|) - \lambda b(n_1^{(1)} + n_0^{(0)} + n_0^{(1)} - N + |\eta|) + ab(n_2^{(0)} - n_0^{(1)}) = 0, \quad \eta \in \mathcal{X}_N. \tag{3.77}
\]

A specific feature of the case \( s = 2 \) is that the following identity holds

\[
2(n_2^{(0)} - n_0^{(1)}) = n_1^{(1)} - n_1^{(0)}, \quad \eta \in \mathcal{X}_N. \tag{3.78}
\]

So, we obtain from (3.77)

\[
(n_2^{(0)} - n_0^{(1)})(-\lambda a - \lambda b + ab) = 0, \quad \eta \in \mathcal{X}_N, \tag{3.79}
\]

which gives the model \((C_4)\).

This completes the proof of the necessity of the conditions \((C_1) - (C_4)\).

The proof that each of the conditions \((C_1) - (C_4)\) is sufficient for (3.27) is now simple. In the case \((C_1)\) it was shown in [12] that

\[
g_1(\eta) = \lambda_0 N - (\lambda_0 + \mu_s)|\eta|, \quad \eta \in \mathcal{X}_N, \tag{3.80}
\]

which implies \( g_2(\eta) = -(\lambda_0 + \mu_s)g_1(\eta), \eta \in \mathcal{X}_N \).

In the case \((C_3)\) we have \( s = 2, 3 \), \( \lambda_k = \mu_{k+1} = h \), \( k = 0, \ldots, s - 1 \), \( \lambda_s = \mu_0 := h + a \), where \( a \in R : h + a \geq 0 \). So, (3.63) becomes

\[
g_2(\eta) = -2hg_1(\eta) + ah(Q_0 - R_1 + Q_1 - R_0) - a^2(n_s^{(0)} - n_0^{(1)}), \quad \eta \in \mathcal{X}_N. \tag{3.81}
\]

By the Lemma, (3.13), (3.14) and (3.18) we further obtain for \( s = 2, 3 \)

\[
g_2(\eta) = -2hg_1(\eta) + ha((F_1 - s - 1)(n_s^{(0)} - n_0^{(1)}) + 2|\eta| - N) - a^2(n_s^{(0)} - n_0^{(1)}), \quad \eta \in \mathcal{X}_N. \tag{3.82}
\]
where $F_1$ is given by (3.21). Hence, in view of (3.59), we have in the both cases of $s$

$$g_2(\eta) = -(a + 8h)g_1 - 6h^2(2|\eta| - N), \quad \eta \in \mathcal{X}_N.$$  

(3.83)

Let now (C4) hold. With the help of (3.78) it is easy to verify that (3.74) indeed holds with

$$\tilde{A}_2 = -a - b - h.$$  

Finally, in the case of the model (C2) we have either $g_2(\eta) = -bg_1(\eta), \quad \eta \in \mathcal{X}_N$ or $g_2(\eta) = -ag_1(\eta), \quad \eta \in \mathcal{X}_N$. ♣

**Corollary 1.** The mean coverage functions $M^{(\eta)}(t), \quad \eta \in \mathcal{X}_N, \quad t \geq 0$ of the models (C$i$), $i = 1, 2, 3, 4$ are given by the expressions (D$i$), $i = 1, 2, 3, 4$ correspondingly:

(D1):

$$M^{(\eta)}(t) = (|\eta| - \frac{\lambda_0 N}{\lambda_0 + \mu_s}) \exp \left(- (\lambda_0 + \mu_s)t\right) + \frac{\lambda_0 N}{\lambda_0 + \mu_s}, \quad t \geq 0, \quad \eta \in \mathcal{X}_N$$  

(3.84)

(D2):

$$M^{(\eta)}(t) = -n_s^{(0)}(\eta) \exp(-at) + |\eta| + n_s^{(0)}(\eta), \quad t \geq 0, \quad \eta \in \mathcal{X}_N$$  

(3.85)

or

$$M^{(\eta)}(t) = n_0^{(1)}(\eta) \exp(-bt) + |\eta| - n_0^{(1)}(\eta), \quad t \geq 0, \quad \eta \in \mathcal{X}_N.$$  

(3.86)

(D3):

$$M^{(\eta)}(t) = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t} + \frac{N}{2}, \quad t \geq 0, \quad \eta \in \mathcal{X}_N,$$  

(3.87)

where

$$\alpha_i = \frac{8h + a - (-1)^i \sqrt{(8h + a)^2 - 48h^2}}{2}, \quad i = 1, 2.$$  

(3.88)

and

$$C_1 = \frac{\alpha_2(|\eta| - \frac{1}{2} N) + g_1(\eta)}{\alpha_2 - \alpha_1}, \quad C_2 = |\eta| - \frac{N}{2} - C_1, \quad \eta \in \mathcal{X}_N.$$  

(3.89)

(D4):

$$M^{(\eta)}(t) = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t} + N \frac{4h + a}{8h + a + b}, \quad t \geq 0, \quad \eta \in \mathcal{X}_N,$$  

(3.90)

where

$$\alpha_i = \frac{6h + a + b - (-1)^i \sqrt{4h^2 + (a + b)^2 + 8h(a + b)}}{2}, \quad i = 1, 2, \quad ab = h(a + b)$$  

(3.91)

and

$$C_1 = \frac{\alpha_2(|\eta| - \frac{N}{8h + a + b}) + g_1(\eta)}{\alpha_2 - \alpha_1}, \quad C_2 = |\eta| - N \frac{4h + a}{8h + a + b} - C_1, \quad \eta \in \mathcal{X}_N.$$  

(3.92)
Proof.
The assertions follow from the relationships \((E_i), i = 1, 2, 3, 4\) below that hold for the models \((C_i), i = 1, 2, 3, 4\) correspondingly.

\((E_1):\)
\[g_1(\eta) = \lambda_0 N - (\lambda_0 + \mu_s)\vert \eta \vert, \quad \eta \in \mathcal{X}_N\]  
(3.93)

\((E_2):\)
\[g_2(\eta) = -ag_1(\eta), \quad a \geq 0, \quad \eta \in \mathcal{X}_N\]  
(3.94)

or
\[g_2(\eta) = -bg_1(\eta), \quad b \geq 0, \quad \eta \in \mathcal{X}_N\]  
(3.95)

\((E_3):\)
\[g_2(\eta) = -(a + 8h)g_1 - 6h^2(2\vert \eta \vert - N), \quad \eta \in \mathcal{X}_N\]  
(3.96)

\((E_4):\)
\[g_2(\eta) = -(6h + a + b)g_1(\eta) - h(8h + a + b)\vert \eta \vert + hN(4h + a), \quad \eta \in \mathcal{X}_N.\]  
(3.97)

Namely, the expressions \((D_1)-(D_4)\) are obtained by solving the second order differential equations corresponding to \((E_1)-(E_4),\) under the initial conditions
\[M^{(\nu)}(0) = \vert \eta \vert, \quad \frac{dM^{(\nu)}(0)}{dt} = g_1(\eta), \quad \eta \in \mathcal{X}_N.\]  
(3.98)

4 The mean density function

Let \(\nu\) be a probability measure on the state space \(\mathcal{X}_N.\) Denote by \(\varphi^{(\nu)}_t, \quad t \geq 0\) the SNNSS starting from \(\nu\) (this means that the distribution of \(\varphi_0^{(\nu)}\) is \(\nu\)), and denote by
\[M^{(\nu)}(t) = E_{\nu}M^{(\nu)}(t), \quad t \geq 0\] the corresponding mean coverage function. The function \(w^{(\nu)}_N(t) = N^{-1}M^{(\nu)}(t), \quad t \geq 0\) is called the mean density coverage function corresponding to the initial distribution \(\nu.\)

Historical remark. The mean density function was studied in a number of papers. In addition to the previously mentioned literature that is immediately related to the context of the present paper, we outline now some adjacent topics of research. Special attention was devoted to the contact process. Gray [13] investigated the behavior of the population profile function \(p_t(x) := P(\varphi_t^{(\nu)}(x)) = 1, \quad x \in V, \quad t \geq 0,\) when \(G = Z\) and \(\eta\) is the empty
configuration flipped at the vertex \( x = 0 \). Belitsky [1] treated a special case of the previously mentioned adsorption-desorption process, when \( \mu_0 > 0, \mu_k = 0, k = 1,2 \) and \( G = Z \). It was proven in [1], that the function \( w_N^{(\nu_0)}(t), t \geq 0 \), where \( \nu_0 \) is the measure concentrated on the empty configuration, possesses a saddle point. This extends the result of [1]. Continuing the discussion in [1], Belitsky [1] relates the above phenomenon to the violation of the classical Langmuir law, known in physical chemistry. Note, that from [1], as well as [2], one can see how complicated is the structure of the iterations \( g_i, i \geq 1 \) of the generator of the process considered. This explains the difficulties in the study of the transient behavior of functionals of contact process even in the case \( G = Z \). The problem becomes much simpler in the framework of the mean-field theory, that corresponds to the case when \( G \) is a complete graph. In this case, a SNNSS conforms to the birth-death process (see Granovsky and Zeifman [10]). The limiting behavior, of the density process \( N^{-1}|\varphi_t| \), as \( N \to \infty \), when \( \varphi_t, t \geq 0 \) is the basic contact process, was extensively studied in the literature. For the most recent review of the topic see Durrett [8].

In conclusion, we mention two papers devoted to voter models. Cox [6] derived the limit of the density process for the basic voter model on the torus in \( Z^d \), under an appropriate time scaling. Mountford [17] considered a class of one-dimensional multitype IPS, that are featured by the following property of its generator \( \Omega : \)

\[
\sup_{n,\eta} |\Omega f_n(\varphi_t)| \leq \text{const},
\]

(4.99)

where \( f_n(\eta) = \sum_{x \in Z : |x| \leq n} \eta(x) \). He proved that under condition (4.99) the coverage process \( f_n(\eta_t), t \geq 0 \) is a martingale plus a term that is negligible as \( n \to \infty \). In this sense these models can be viewed as a generalization of the basic voter model. It should be noted that the condition (4.99) fails for all SNNSS (C1)-(C4), except only the case of the basic voter model. An important particular case of \( \nu \) is the product Bernoulli measure \( \nu_p, 0 \leq p \leq 1 \), defined by

\[
\nu_p(\eta) = \prod_{x \in V} p^{\eta(x)}(1-p)^{1-\eta(x)} = p^{||\eta||}(1-p)^{N-||\eta||}, \quad \eta \in \mathcal{X}_N.
\]

(4.100)

For a given \( 0 \leq p \leq 1 \), the measure \( \nu_p \) corresponds to the initial distribution on \( \mathcal{X}_N \), such that all spins are i.i.d. Bernoulli random variables. In view of this, \( M^{(\nu_p)}(t) = NE^{\varphi_t^{(\nu_p)}}(x), \forall x \in V, \eta \in \mathcal{X}_N \), and consequently, \( w_N^{(\nu_p)}(t) = E^{\varphi_t^{(\nu_p)}}(x), \forall x \in V, \eta \in \mathcal{X}_N \). Hence, the mean density \( w_N^{(\nu_p)}(t), t \geq 0 \) defines the marginal distribution of the process \( \varphi_t^{(\nu_p)} \) at any site \( x \in V \) at time \( t \geq 0 \). It turns out that the densities \( w_N^{(\nu_p)}(t), t \geq 0 \) corresponding to SNNSS’s (C1) – (C4), have the following remarkable property.
**Proposition 2.** The mean density functions \( w_N^{(\nu p)}(t), t \geq 0, 0 \leq p \leq 1 \) of models \((C_1) - (C_4)\) do not depend on \( N \).

**Proof.**

It follows from (3.28) that

\[
g_i(\eta) = A_1 g_{i-1}(\eta) + A_0 g_{i-2}(\eta), \quad i = 3, \ldots, \eta \in \mathcal{X}_N, \tag{4.101}
\]

where, in view of \((E_1) - (E_4)\), the coefficients \( A_1, A_0 \) do not depend neither on \( \eta \in \mathcal{X}_N \) nor \( N \).

We deduce from (3.7) that

\[
E(\nu p) g_1(\eta) = N \left[ E(\nu p) \left[ c(x, \eta)(1 - 2\eta(x)) \right] \right], \quad \forall x \in V. \tag{4.102}
\]

It is clear that the expected value in the RHS of (4.102) does not depend on \( N \), for any SNNSS.

Next, in (3.28) the coefficient \( B = NB_0 \), where, by \((E_1) - (E_4)\), the factor \( B_0 \) does not depend on \( N \). So, (3.28) implies \( E(\nu p) g_2(\eta) = Nq_2 \), where the factor \( q_2 \) does not depend on \( N \). This together with (4.101) and (4.102) gives \( E(\nu p) g_i(\eta) = Nq_i, \ i = 3, \ldots, \) where again the factors \( q_i, i = 3, \ldots, \) do not depend on \( N \). Finally, to complete the proof, we use the Hille-Yosida series expansion of the function \( M(\nu p)(t), t \geq 0 \).

**Remark 1.** It can be shown that even linear SNNSS given by \( \lambda_k = \lambda_0 + d_\lambda k, \mu_k = \mu_0 + d_\mu k, \) \( k = 0, 1, \ldots, s \) with \( d_\lambda \neq d_\mu \) have not the property stated in Proposition 2.

In view of Proposition 2, we write \( w^{(\nu p)}(t) := w_N^{(\nu p)}(t), t \geq 0, 0 \leq p \leq 1, \) \( N = s + 1, \ldots, \) for the models \((C_1) - (C_4)\). The explicit expressions for the functions \( w^{(\nu p)}(t), t \geq 0 \) is easy to obtain from \((E_1) - (E_4)\). Now the Trotter - Kurtz approximation theorem ([14]) and \((E_1) - (E_4)\) give immediately the following

**Corollary 2.** For models \((C_1) - (C_4)\) on a finite or infinite \( s \)-regular graph,

\[
Pr(\varphi_t^{(\nu p)}(x) = 1) = w^{(\nu p)}(t), \quad t \geq 0, \quad x \in V, \quad 0 \leq p \leq 1. \tag{4.103}
\]

In particular, in the presence of noise, the processes \((C_1), (C_3), (C_4)\) have ergodic marginals, in the sense that for each of these processes

\[
\lim_{t \to \infty} w^{(\nu p)}(t) \tag{4.104}
\]

exists and does not depend on \( 0 \leq p \leq 1 \).

The expressions \((D_1), (D_3)\) and \((D_4)\) give correspondingly the following values for the limit in (4.104):
Remark 2. Formulae (D₁)-(D₄) show the complicated influence of a constant additive noise on a transient behaviour of the process. In particular, note that under the absence of noise \( h = 0 \) we have in (3.88), (3.91) \( \alpha_2 = 0 \). It is also appropriate to mention that the processes (C₁)-(C₄) are either attractive or anti-attractive. The latter means that in the definition of attractiveness (14, p.132) the direction of inequalities for the flip rates is reversed.

5 Ergodicity and Spectral gap

**Ergodicity.** We will address the question of ergodicity of the processes (C₁)-(C₄) on a finite or infinite \( s \)-regular graph. The process (C₂) is, obviously, not ergodic. The processes (C₁), (C₃) with \( a \geq 0 \) and (C₄) with \( a, b \geq 0 \) are attractive. The key property of such processes is that ergodicity of their marginals implies the ergodicity of the process. A beautiful argument leading to this assertion is explained in [14](see Corollary 2.8, p. 75, and Corollary 2.4, p.136).

So, by Corollary 2, we get

**Corollary 3.** The following three processes are ergodic: (C₁) with \( h_1 + h_2 = \lambda_0 + \mu_s > 0 \), (C₃) with \( a \geq 0, h > 0 \) and (C₄) with \( a, b \geq 0, h > 0 \).

**Remark 3** i. The ergodicity of the first among the three models in Corollary 3 was proven in [12]. The ergodicity of the third one as well as the second one in the case \( s = 2 \), follows from the fact that these are attractive spin systems in one dimension with translation invariant and positive flip rates (see [14], Theorem 3.14, p.152). To the best of our knowledge, the established ergodicity of the second model in the case \( s = 3 \) answers an open question. We will explain below that the \( \epsilon - M > 0 \) condition in the case considered gives ergodicity for \( a < \frac{2}{3} \) only.

ii. It is interesting to observe that, by Corollary 2, the models considered have ergodic marginals also in the case when they are not attractive and even not ergodic. For example, this is true for (C₃), when \( h + a = 0, s = 2 \), in which case the process on \( Z_1 \) is not ergodic, having two different absorbing states \( \eta_i : \eta_i(x) = 0.5(1 + (-1)^{i+|x|}), i = 1, 2, x \in Z_1 \).

**Spectral Gap.** Recall (see for references [4], [5]) that the spectral gap \( \alpha > 0 \) of an exponentially ergodic Feller-Markov process \( \varphi_t, \quad t \geq 0 \) with an invariant measure \( \nu \) on state space \( \mathcal{X} \) is defined by

\[
\alpha = \sup_{\eta \in \mathcal{X}} \left\{ \beta > 0 : \sup_{\eta \in \mathcal{X}} |E_f(\varphi_{t}^{(\eta)}) - \int f d\nu| \leq Af \exp(-\beta t), \quad f \in C(\mathcal{X}), \quad t \geq 0 \right\},
\]

(5.106)
where $A_f \geq 0$ does not depend on $t \geq 0$.

It is plain that formulae (D1), (D3) and (D4) for the mean coverage function provide an estimate from above for spectral gaps of the corresponding processes. Namely, if $\tilde{\alpha}$ is the rate of exponential convergence to the equilibrium of the function $M^{(\eta)}(t), t \geq 0 = w^{(\nu_p)}(t), t \geq 0$, then \( \alpha \leq \tilde{\alpha} \). From the other hand, the celebrated $\epsilon - M > 0$ condition of ergodicity ([14], p.31) provides the lower bound of the spectral gap of any ergodic IPS on a finite or infinite graph: \( \alpha \geq \epsilon - M > 0 \). In the case of a SNNSS on a $s$-regular graph the quantities $\epsilon$ and $M$ are given by

$$
\epsilon = \min_{0 \leq k \leq s} \{\lambda_k + \mu_k\} \quad M = s \max_{0 \leq k \leq s} \{|\lambda_k - \lambda_{k-1}|, |\mu_k - \mu_{k-1}|\}.
$$

(5.107)

Below are the values of $\epsilon - M$ and $\tilde{\alpha}$ for the three ergodic models considered.

(C1):

$$
\epsilon - M = \lambda_0 + \mu_s = \tilde{\alpha}
$$

(5.108)

So,

$$
\alpha = \lambda_0 + \mu_s,
$$

(5.109)

if $\lambda_0 + \mu_s > 0$. This fact was observed in [12]. Moreover, it was proven in [10] that if $G$ is a complete graph, then the noisy voter model is the only one SNNSS with the property (5.109).

(C3):

$$
\tilde{\alpha} = \alpha_1,
$$

(5.110)

where $\alpha_1$ is given by (3.88), while

$$
\epsilon - M = \begin{cases} 
2h - sa, & \text{if } a > 0 \\
2h + (s + 1)a, & \text{if } a \leq 0,
\end{cases}
$$

(5.111)

where $s = 2, 3$. This says that in the case $a > 0$ the $\epsilon - M > 0$ condition is applicable for $s = 2$, if $h > a$, and for $s = 3$, if $h > 1.5a$. It is easy to verify that $\alpha_1 = \epsilon - M > 0$ iff $a = 0$ which is the trivial case of the noisy voter model.

(C4):

$$
\tilde{\alpha} = \alpha_1,
$$

(5.112)

where $\alpha_1$ is given by (3.91), while

$$
\epsilon - M = 2h + \min\{0, a, b\} - 2\max\{|a|, |b|\},
$$

(5.113)

where $h = \frac{ab}{a+b} > 0$.  

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Here again one can see that $\alpha_1 = \epsilon - M > 0$ is impossible unless $a = b = 0$.

The preceding discussion leads us to the following

**Conjecture.** Noisy Voter model is the only one NNSS for which $\alpha = \epsilon - M > 0$.

**Concluding remark.** It is demonstrated by our Theorem, that passing from the first order differential equation to the one of the second order does not enrich much the class of solvable (in the sense of the mean coverage function) SNNSS. A natural question arising in this connection is the characterization of SNNSS in the case of higher order differential equations. One can expect that the progress in this direction will lead to the discovery of wider classes of solvable SNNSS. The solution to this problem requires the analysis of the structure of generators $g_i$, as defined in (3.6), of higher orders. At the moment the problem looks intractable.

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