Parametrizations of collinear and $k_T$-dependent parton densities in a proton

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Abstract

A new type of parametrization for parton distribution functions in a proton, based on their $Q^2$-evolution at large and small $x$ values, is constructed. In our analysis, the valence and nonsinglet parts obey the Gross-Llewellyn-Smith and Gottfried sum rules, respectively. For the singlet quark and gluon densities momentum conservation is taken into account. Then, using the Kimber-Martin-Ryskin prescription, we extend the consideration to Transverse Momentum Dependent (TMD, or unintegrated) gluon and quark distributions in a proton, which currently plays an important role in a number of phenomenological applications. The analytical expressions for the latter, valid for both low and large $x$, are derived for the first time.

Keywords: QCD evolution, parton density functions in a proton, Kimber-Martin-Ryskin approach
1 Introduction

The parton (quark and gluon) distribution functions (PDFs) in a proton are necessary part of any theoretical study performed within the Quantum Chromodynamics (QCD). They encode the information on the non-perturbative structure of a proton and directly related to the calculated cross sections (or other observables) via certain QCD factorization theorem. The QCD evolution leads to their essential dependence on the probing scale $Q^2$, which can be described by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [1]. Usually, the latter is solved numerically with leading order (LO), next-to-leading (NLO) or even next-to-next-to-leading (NNLO) accuracy, where number of corresponding phenomenological parameters of initial parton distributions are fitted at HERA, LHC and fixed target experiments at various $(x, Q^2)$ ranges. Large uncertainties for many processes at the LHC originate, of course, mainly from our restricted knowledge of the parton distributions (see, for example, [5] and references therein). Thus, studying the proton PDFs from both theoretical and experimental points of view is an important and urgent task.

In the present paper we continue with the idea [6] to present more information on the PDFs from the theoretical side. The approach consists of the two basic steps. First, we find asymptotics of solutions of the DGLAP equations for the parton densities at small and large values of the Bjorken variable $x$. Second, we combine these two solutions and then interpolate between them to obtain the analytical expressions for PDFs over the full range of $x$.

In a sense, this is not a very new idea. A similar approach has been proposed [7, 8] about of 50 years ago. However, in the present paper the parametrizations are constructed in a rather different way. In particular, following [6], they include important subasymptotic terms which are fixed exactly by the momentum conservation and also by the Gross-Llewellyn-Smith and Gottfried sum rules (see [9] and [10], respectively). Such calculations will be performed for the first time providing the community with new type of parametrization of gluon and quark densities in a proton valid at low and large $x$. Moreover, we extend our consideration [11, 12] and derive analytical expressions for Transverse Momentum Dependent (TMD) parton distributions using the Kimber-Martin-Ryskin (KMR) framework [13, 14]. These quantities are known to be a very suitable tool to investigate a less inclusive processes which proceed at high energies with large momentum transfer and/or containing multiple hard scales (see, for example, review [15] and references therein). Our main motivation is that up to now there is no analytical expressions for gluon and especially quark TMDs (both sea and valence) valid in a wide $x$ region.

The analysis of the present paper is limited to the LO in the perturbation theory, which is reasonable [16] for those processes at the LHC for which the NLO corrections are not known at present. Moreover, most of phenomenological applications involving TMDs are currently performed at the LO also (see, for example, [12, 17–20] and references therein). On the other hand, the consideration of PDFs at LO is the necessary first step in studying PDFs and TMDs at higher orders. These higher order corrections can be treated like those [8, 21, 22].

The outline of our paper is following. In Section 2 we describe our theoretical input. Sections 3 and 4 contain low $x$ and large $x$ PDF asymptotics. Parametrizations of parton densities, their properties and numerical results for PDFs are given in Section 5. Section 6

\footnote{Several recent PDF fits, as well as the references to the previous studies, can be found [2–4].}

\footnote{In our previous study [11, 12], only small $x$ limit have been considered and phenomenological model for large $x$ region have been applied.}
is devoted to TMD parton densities in the Kimber-Martin-Ryskin framework. Section 7 contains our conclusions. Most complicated calculations are presented in Appendices.

2 Theoretical input

In this section we briefly present the theoretical part of our analysis. The reader is referred to [23] for more details.

The deep-inelastic scattering (DIS) $l + N \rightarrow l' + X$, where $l$ and $N$ are the incoming lepton and nucleon, and $l'$ is the outgoing lepton, in one of the basic processes for studying of the nucleon structure. The DIS cross-section can be split to the lepton $L^\mu\nu$ and hadron $F^\mu\nu$ parts

$$d\sigma \sim L^\mu\nu F^\mu\nu.$$ (1)

The lepton part $L^\mu\nu$ is evaluated exactly, while the hadron one, $F^\mu\nu$, can be presented in the following form

$$F^\mu\nu = \left( -g^\mu\nu + \frac{q^\mu q^\nu}{q^2} \right) F_1(x, Q^2) + \left( p^\mu + \frac{(pq)}{q^2} q^\mu \right) \left( p^\nu + \frac{(pq)}{q^2} q^\nu \right) F_2(x, Q^2)$$

$$+ \ i\varepsilon_{\mu\nu\alpha\beta} p^\alpha x \frac{q^2}{q^2} F_3(x, Q^2) + \ldots ,$$ (2)

where the symbol $\ldots$ stands for those parts which depend on the nucleon spin. The functions $F_k(x, Q^2)$ with $k = 1, 2$ and 3 are the DIS structure functions (SFs) and $q$ and $p$ are the photon and parton momenta. Moreover, the two variables

$$Q^2 = -q^2 > 0 , \ x = \frac{Q^2}{2(pq)}$$ (3)

determine the basic properties of the DIS process. Here, $Q^2$ is the “mass” of the virtual photon and/or $Z/W$ boson, and the Bjorken variable $x$ ($0 < x < 1$) is the part of the hadron momentum carried by the scattering parton (quark or gluon).

2.1 Mellin transform

The Mellin transform diagonalizes the $Q^2$ evolution of the parton densities. In other words, the $Q^2$ evolution of the Mellin moment with certain value $n$ does not depend on the moment with another value $n'$.

The Mellin moments $M_k(n, Q^2)$ of the SF $F_k(x, Q^2)$

$$M_k(n, Q^2) = \int_0^1 dx \ x^{n-2} F_k(x, Q^2)$$ (4)

can be represented as the sum

$$M_k(n, Q^2) = \sum_{a=q,\bar{q},g} C^a_k(n, Q^2/\mu^2) A_a(n, \mu^2),$$ (5)

where $C^a_k(n, Q^2/\mu^2)$ are the coefficient functions and $A_a(n, \mu^2) = < N| O^a_{\mu_1, \ldots, \mu_n} | N >$ are the matrix elements of the Wilson operators $O^a_{\mu_1, \ldots, \mu_n}$, which in turn are process independent.
Phenomenologically, the matrix elements $A_a(n, \mu^2)$ are equal to the Mellin moments of the PDFs $f_a(x, \mu^2)$, where $f_a(x, \mu^2)$ are the distributions of quarks ($a = q_i$), antiquarks ($a = \bar{q}_i$) with $i = 1...6$ and gluons ($a = g$), i.e.

$$A_a(n, \mu^2) \equiv f_a(n, \mu^2) = \int_0^1 dx \, x^{-2} f_a(x, \mu^2).$$

(6)

The coefficient functions $C^a_k(n, Q^2/\mu^2)$ are represented by

$$C^a_k(n, Q^2/\mu^2) = \int_0^1 dx \, x^{-2} \tilde{C}^a_k(x, Q^2/\mu^2)$$

(7)

and responsible for the relationship between SFs and PDFs. Indeed, in the $x$-space the relation (5) is replaced by

$$F_k(x, Q^2) = \sum_{a=q,\bar{q},g} \tilde{C}^a_k(x, Q^2/\mu^2) \otimes f_a(x, \mu^2),$$

(8)

where $\otimes$ denotes the Mellin convolution

$$f_1(x) \otimes f_2(x) \equiv \int_x^1 \frac{dy}{y} f_1(y)f_2 \left(\frac{x}{y}\right).$$

(9)

Applying (5) and (8), one can fit the shapes of PDFs $f_a(x, \mu^2)$, which are process-independent and use them later for other processes. Note that the factorization scale $\mu^2$ is often taken as $\mu^2 = Q^2$. Here we will follow this choice.

2.2 Quark densities

The distributions of the $u$ and $d$ quarks contain the valence and the sea parts:

$$f_{q_1} \equiv f_u = f^V_u + f^S_u, \quad f_{q_2} \equiv f_d = f^V_d + f^S_d.$$  

(10)

The distributions of the other quark flavors and of all the antiquarks contain the sea parts only:

$$f_{q_j} = f^S_{q_j}, \quad (j = 3...6), \quad f_{\bar{q}_i} = f^S_{\bar{q}_i} \quad (i = 1...6).$$  

(11)

It is useful to define the combinations of quark densities of the valence part $f_V$, the sea one $f_S$ and the singlet one $f_{SI}$:

$$f_V = f^V_u + f^V_d, \quad f_S = \sum_{i=1}^{6} \left( f^S_{q_i} + f^S_{\bar{q}_i} \right), \quad f_{SI} = \sum_{i=1}^{6} (f_{q_i} + f_{\bar{q}_i}) = f_V + f_S.$$  

(12)

Because the PDFs, which contribute to the structure functions, are accompanied by some numerical factors, there are also nonsinglet parts

$$f_{\Delta_{ij}} = (f_{q_i} + f_{\bar{q}_i}) - (f_{q_j} + f_{\bar{q}_j}),$$

(13)

3All parton densities are multiplied by $x$, i.e. in the LO the structure functions are some combinations of the parton densities.

4Here we consider all quark flavors. Really, heavy quarks factorize out when $\sqrt{Q^2}$ becomes less than their masses, and we should exclude them from the $Q^2$-region.
which contain difference of densities of quarks and antiquarks with different values of charges.

As an example, we consider the electron-proton scattering, where the corresponding SF has the form

$$F_{ep}^2(x, Q^2) = 6\sum_{i=1}^6 e_i^2 \left( f_{q_i}(x, Q^2) + f_{\bar{q}_i}(x, Q^2) \right).$$

(14)

In the four-quark case (when $b$ and $t$ quarks are separated out), we will have

$$F_{ep}^2(x, Q^2) = \frac{5}{18} f_{SI}(x, Q^2) + \frac{1}{6} f_{\Delta}(x, Q^2),$$

(15)

where

$$f_{\Delta} = \sum_{q_i=u,c} \left( f_{q_i}(x, Q^2) + f_{\bar{q}_i}(x, Q^2) \right) - \sum_{q_i=d,s} \left( f_{q_i}(x, Q^2) + f_{\bar{q}_i}(x, Q^2) \right).$$

(16)

### 2.3 DGLAP equations

The PDFs obey the DGLAP equations [1]:

$$\frac{d}{d \ln Q^2} f_i(x, Q^2) = -\frac{1}{2} \sum_b \gamma_{NS}(x) \otimes f_i(x, Q^2), \quad i = NS, V,$$

$$\frac{d}{d \ln Q^2} f_a(x, Q^2) = -\frac{1}{2} \sum_b \gamma_{ab}(x) \otimes f_b(x, Q^2), \quad a, b = SI, g,$$

(17)

where $\gamma_i(x)$ and $\gamma_{ab}(x)$ are the so-called splitting functions. Anomalous dimensions (ADs) $\gamma_{ab}(n)$ of the twist-two Wilson operators $\mathcal{O}_{\mu_1,\ldots,\mu_n}^a$ in the brackets $b$ are the Mellin transforms of the corresponding splitting functions

$$\gamma_{ab}(n) = \int_0^1 dx \ x^{n-2} \gamma_{ab}(x), \quad f_a(n, \mu^2) = \int_0^1 dx \ x^{n-2} f_a(x, \mu^2).$$

(18)

At the LO of perturbation theory, ADs $\gamma_{ab}(n)$ have the following form [22]:

$$\gamma_{ab}(n) = a_s(Q^2) \gamma_{ab}^{(0)}(n), \quad \gamma_{NS}^{(0)}(n) = \gamma_{qq}^{(0)}(n), \quad a_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi} = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2_{LO})},$$

$$\gamma_{NS}^{(0)}(n) = 8C_F \left( S_1(n) - \frac{3}{4} - \frac{1}{2n(n+1)} \right),$$

$$\gamma_{qq}^{(0)}(n) = -4f \frac{n^2 + n + 2}{n(n+1)(n+2)}, \quad \gamma_{gg}^{(0)}(n) = -4C_F \frac{n^2 + n + 2}{n(n^2 - 1)},$$

$$\gamma_{gg}^{(0)}(n) = 8C_A \left( S_1(n) - \frac{11}{12} - \frac{1}{n(n-1)} - \frac{1}{(n+1)(n+2)} \right) + \frac{4f}{3},$$

(19)

where $C_A = N$, $C_F = (N^2 - 1)/(2N)$ for SU(N) group and $f$ is the number of active (massless) quarks and

$$S_1(n) = \sum_{m=1}^\infty \frac{1}{m} = \Psi(n+1) + \gamma_E,$$

(20)

with Euler $\Psi$-function and Euler constant $\gamma_E$. 
In the Mellin moment space, the DGLAP equation becomes to be the standard renormalization group equation. At LO we have
\begin{equation}
\frac{d}{d \ln Q^2} f_i(n, Q^2) = -\frac{a_s(Q^2)}{2} \gamma_{NS}^{(0)}(n) f_i(n, Q^2), \quad i = V, NS, \tag{21}
\end{equation}
\begin{equation}
\frac{d}{d \ln Q^2} f_a(n, Q^2) = -\frac{a_s(Q^2)}{2} \sum_{b=SI,g} \gamma_{ab}^{(0)}(n) f_b(n, Q^2), \quad a = SI, g. \tag{22}
\end{equation}

To solve (22), it is better to move to the ± components \cite{22,24}, that leads to the diagonal form:
\begin{equation}
\frac{d}{d \ln Q^2} f_{\pm}(n, Q^2) = -\frac{a_s(Q^2)}{2} \gamma_{\pm}^{(0)}(n) f_{\pm}(n, Q^2), \tag{23}
\end{equation}
where
\begin{equation}
\gamma_{\pm}^{(0)}(n) = \frac{1}{2} \left[ \gamma_{qq}^{(0)}(n) + \gamma_{gg}^{(0)}(n) \pm \sqrt{(\gamma_{qq}^{(0)}(n) - \gamma_{gg}^{(0)}(n))^2 + 4\gamma_{qg}^{(0)}(n)\gamma_{gq}^{(0)}(n)} \right]. \tag{24}
\end{equation}
The solutions of (21) and (23) have the following form:
\begin{equation}
f_a(n, \mu^2) = f_a(n, Q_0^2) e^{-d_a(n)s}, \quad a = V, NS, \pm, \tag{25}
\end{equation}
where \( Q_0^2 \) is some initial scale and
\begin{equation}
d_a(n) = \frac{\gamma_a^{(0)}(n)}{2\beta_0}, \quad s = \ln \frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}. \tag{26}
\end{equation}
The singlet quark and gluon densities can be expressed through their “±” components as
\begin{equation}
f_a(n, Q^2) = f_{a,+}(n, Q^2) + f_{a,-}(n, Q^2), \quad f_{a,\pm}(n, Q^2) = f_{a,\pm}(n, Q_0^2) e^{-d_{\pm}(n)s}, \tag{27}
\end{equation}
where
\begin{align}
f_{q,+}(n, Q_0^2) &= f_{q}(n, Q_0^2) - f_{q,-}(n, Q_0^2), \quad f_{q,-}(n, Q_0^2) = f_{q}(n, Q_0^2) \alpha_n + f_{q}(n, Q_0^2) \beta_n, \\
f_{g,+}(n, Q_0^2) &= f_{g}(n, Q_0^2) - f_{g,-}(n, Q_0^2), \quad f_{g,-}(n, Q_0^2) = f_{g}(n, Q_0^2) \alpha_n - f_{g}(n, Q_0^2) \varepsilon_n \tag{28}
\end{align}
and
\begin{align}
\alpha_n &= \frac{\gamma_{qq}^{(0)}(n) - \gamma_{gg}^{(0)}(n)}{\gamma_{-}^{(0)}(n) - \gamma_{+}^{(0)}(n)}, \quad \beta_n = \frac{\gamma_{qg}^{(0)}(n)}{\gamma_{-}^{(0)}(n) - \gamma_{+}^{(0)}(n)}, \quad \varepsilon_n = \frac{\gamma_{gq}^{(0)}(n)}{\gamma_{-}^{(0)}(n) - \gamma_{+}^{(0)}(n)}. \tag{29}
\end{align}

\subsection{2.4 Special cases}

Special case of parton evolution is the case \( n = 1 \) for the valence part, which corresponds to number \( N_V \) of structural quarks in considered hadron. For example, for proton \( N_V = 3 \). So, we have
\begin{equation}
\int_0^1 dx \frac{1}{x} f_V(x, \mu^2) = N_V, \tag{30}
\end{equation}
which is the so-called Gross-Llewellyn-Smith sum rule \cite{9}.

Indeed, for this case, \( \gamma_{NS}^{(0)}(n) = 0 \) and
\begin{equation}
f_a(n = 1, Q^2) = f_a(n = 1, Q_0^2), \quad a = V, NS. \tag{31}
\end{equation}

For the NS part, the corresponding sum rule, so-called Gottfried sum rule \cite{10}, is called as
\begin{equation}
\int_0^1 dx \frac{1}{x} f_{NS}(x, Q^2) = N_{NS}(Q^2) = 3I_G(Q^2), \tag{32}
\end{equation}
with \(^{25}\)

\[ I_G(Q_c^2) = 0.705 \pm 0.078, \quad Q_c^2 = 4 \text{ GeV}^2. \]  

(33)

We note that the result (32) is correct in the case of flavor-symmetric sea. Moreover, \(I_G(Q^2)\) has only very weak \(Q^2\)-dependence (see \(^{26}\)), which comes beyond LO from the so-called analytic continuation \(^{26,27}\) of the corresponding Wilson coefficients. So, the values of the Gottfried sum rule \(^{10}\) can be taken below as

\[ I_G(Q^2) \approx I_G(Q_c^2) = 0.705. \]  

(34)

For the sea quark and gluon densities the special case is the \(n = 2\), that corresponds to the conservation of total momentum carried by quarks and gluons, i.e.

\[ \int_0^1 dx \left( f_{SI}(x, Q^2) + f_g(x, Q^2) \right) = \int_0^1 dx \left( f_{SI}(x, Q_0^2) + f_g(x, Q_0^2) \right) = 1, \]  

(35)
i.e.

\[ f_{SI}(n = 2, Q^2) + f_g(n = 2, Q^2) = f_{SI}(n = 2, Q_0^2) + f_g(n = 2, Q_0^2) = 1. \]  

(36)

Consider the case \(n = 2\) more accurately. We have

\[ \gamma_{qq}^{(0)}(n = 2) = -\gamma_{gg}^{(0)}(n = 2) = \frac{16C_F}{3}, \quad \gamma_{gg}^{(0)}(n = 2) = -\gamma_{qq}^{(0)}(n = 2) = \frac{4f}{3}, \]  

(37)

and, thus,

\[ \gamma_{-}^{(0)}(n = 2) = 0, \quad \gamma_{+}^{(0)}(n = 2) = \frac{4}{3} \left( 4C_F + f \right), \quad \alpha_{n=2} = \beta_{n=2} = 1 - \varepsilon_{n=2} = \frac{f}{4C_F + f}. \]  

(38)

Using these values, we obtain

\[ f_{SI,-}(2, Q^2) = \frac{f}{4C_F + f} \left( f_{SI}(2, Q_0^2) + f_g(2, Q_0^2) \right) e^{-d_-(n=2)s} = \frac{f}{4C_F + f}, \]

\[ f_{g,-}(2, Q^2) = \frac{4C_F}{4C_F + f} \left( f_{SI}(2, Q_0^2) + f_g(2, Q_0^2) \right) e^{-d_-(n=2)s} = \frac{4C_F}{4C_F + f}, \]  

(39)

because \(f_{SI}(2, \mu_0^2) + f_g(2, \mu_0^2) = 1\) and \(d_-(n = 2) = \gamma_-(n = 2)/(2\beta_0) = 0\). Thus, the “−”-components are \(Q^2\)-independent. Moreover,

\[ f_{SI,-}(2, Q^2) + f_{g,-}(2, Q^2) = f_{SI}(2, Q_0^2) + f_g(2, Q_0^2) = 1, \]  

(40)
i.e. the sum of the “−” components of the singlet and gluon densities is responsible for the momentum conservation. For the “+” components we have

\[ f_{SI,+}(n, Q^2) = \frac{1}{4C_F + f} \left( 4C_F f_{SI}(2, Q_0^2) - f f_{g}(2, Q_0^2) \right) e^{-d_+(n=2)s}, \]

\[ f_{g,+}(n, Q^2) = \frac{1}{4C_F + f} \left( f f_{g}(2, Q_0^2) - 4C_F f_{SI}(2, Q_0^2) \right) e^{-d_+(n=2)s} \]  

(41)

and, thus,

\[ f_{SI,+}(2, Q^2) + f_{g,+}(2, Q^2) = 0, \]  

(42)
i.e. the sum of the “+” components of the singlet and gluon densities is exactly zero.

### 3 Low \(x\) asymptotics

According to \(^{12}\), singlet quark density \(f_{SI}(x, Q^2)\) contains the valence part \(f_V(x, Q^2)\) and sea part \(f_S(x, Q^2)\).
3.1 Nonsinglet and valence parts

At small-$x$ values the NS and valence parts have the following asymptotics\(^5\): 

\[
 f_i(x) \to A_i(s) x^{\lambda_i}, \quad i = V, NS,
\]

where 

\[
 A_i(s) = A_i(0) e^{-d_i(1-\lambda_i)s}, \quad A_i(0) \equiv A_i, \quad d_i(n) = \frac{\gamma^0_{NS}(n)}{2\beta_0},
\]

\(\lambda_i\) and \(A_i(0)\) are free parameters and \(\Psi(n+1)\) is Euler \(\Psi\)-function. From the Regge calculus, the constant \(\lambda_i \approx 0.3 \div 0.5\). Moreover, the \(Q^2\) evolution of this parton density shows that \(\lambda_i\) should be \(Q^2\) independent\(^7\).

3.2 Singlet part

It was pointed out\(^2\) that the HERA small-$x$ data can be well interpreted in terms of the so-called doubled asymptotic scaling (DAS) phenomenon related to the asymptotic behavior of the DGLAP evolution discovered many years ago\(^3\). The study\(^2\) was extended\(^3\) to include the finite parts of anomalous dimensions of Wilson operators and Wilson coefficients\(^3\). This led to predictions\(^3\) of the small-$x$ asymptotic form of PDFs in the framework of the DGLAP dynamics, which were obtained starting at some \(Q_0^2\) with the flat function 

\[
f_a(x, Q_0^2) = A_a,
\]

where \(A_a\) are free parameters which have to be determined from the data. We refer to the approach of\(^3\) as generalized DAS approximation. In this approach the flat initial conditions\(^3\) determine the basic role of the AD singular parts as in the standard DAS case, whereas the contributions coming from AD finite parts and Wilson coefficients can be considered as corrections which are, however, important for achieving better agreement with experimental data.

Hereafter we consider for simplicity only the LO approximation\(^6\). The small-$x$ asymptotic expressions for sea quark and gluon densities \(f_a(x, \mu^2)\) can be written as follows:

\[
 f_a(x, Q^2) = f_a^+(x, Q^2) + f_a^-(x, Q^2),
\]

\[
 f_q^+(x, Q^2) = A_q^+ \tilde{I}_0(\sigma) e^{-2\sigma_x} + O(\rho), \quad A_q^+ = A_q + C A_q, \quad C = \frac{C_F}{C_A} = \frac{4}{9},
\]

\[
 f_q^-(x, Q^2) = A_q^- \tilde{I}_1(\sigma) e^{-2\sigma_x} + O(\rho), \quad A_q^- = \frac{2}{3} A_q^+, \quad \varphi = \frac{\rho}{C_A} = \frac{2}{3};
\]

\[
 f_g^-(x, Q^2) = A_g^+ e^{-d_x} + O(x), \quad A_g^- = -C A_g,
\]

\[
 f_g^-(x, Q^2) = A_g^- e^{-d_x} + O(x),
\]

where \(C_A = N_c, \quad C_F = (N_c^2 - 1)/(2N_c)\) for the color \(SU(N_c)\) group, \(\tilde{I}_\nu(\sigma)\) and \(\tilde{I}_\nu(\sigma)\) \((\nu = 0, 1)\) are the combinations of the modified Bessel functions (at \(s \geq 0\), i.e. \(\mu^2 \geq Q_0^2\)) and usual Bessel functions (at \(s < 0\), i.e. \(\mu^2 < Q_0^2\)):

\[
 \tilde{I}_\nu(\sigma) = \begin{cases} \rho^{\nu} I_\nu(\sigma), & \text{if } s \geq 0; \\ (-\rho)^{-\nu} J_\nu(\sigma), & \text{if } s < 0. \end{cases}, \quad \tilde{I}_\nu(\sigma) = \begin{cases} \rho^{-\nu} I_\nu(\sigma), & \text{if } s \geq 0; \\ \tilde{\rho}^{-\nu} J_\nu(\sigma), & \text{if } s < 0. \end{cases}
\]

\[
 I_\nu(\sigma) = \sum_{m=0}^{\infty} \frac{1}{k!(k+\nu)!} \sigma^{2k+\nu}, \quad J_\nu(\sigma) = \sum_{m=0}^{\infty} \frac{(-1)^k}{k!(k+\nu)!} \sigma^{2k+\nu},
\]

\(^5\)In the standard DAS approximation\(^3\) only the AD singular parts were used.

\(^6\)Both the LO and NLO results and their applications can be found in\(^3\) and\(^3\), respectively.
where $\tilde{T}_0(\sigma) = \tilde{I}_0(\sigma)$ and
\[
\sigma = 2\sqrt{\frac{\tilde{d}_+}{\ln \left(\frac{1}{x}\right)}}, \quad \rho = \frac{\sigma}{2\ln(1/x)}, \quad \tilde{\sigma} = 2\sqrt{-\tilde{d}_+ s \ln \left(\frac{1}{x}\right)}, \quad \tilde{\rho} = \frac{\tilde{\sigma}}{2\ln(1/x)}, \quad (48)
\]
and
\[
\hat{d}_+ = -\frac{4C_A}{\beta_0} = -\frac{12}{\beta_0}, \quad \tilde{d}_+ = 1 + \frac{4f(1 - C)}{3\beta_0} = 1 + \frac{20f}{27\beta_0}, \quad d_- = \frac{4Cf}{3\beta_0} = \frac{16f}{27\beta_0}, \quad (49)
\]
are the singular and regular parts of the anomalous dimensions and $\beta_0 = 11 - (2/3)f$ is the first coefficient of the QCD $\beta$-function in the $\overline{\text{MS}}$-scheme. The results for the parameters $A_a$ and $Q_a^2$ can be found in [34]; they were obtained for $\alpha_s(M_Z) = 0.1168$.

It is convenient to show the following expressions:
\[
\beta_0 \hat{d}_+ = -4C_A, \quad \beta_0 \tilde{d}_+ = \frac{C_A}{3} \left(11 + 2\varphi(1 - 2C)\right), \quad \beta_0 d_- = \frac{4Cf}{3} = \frac{4C_F\varphi}{3}. \quad (50)
\]

### 4 Large $x$ asymptotics

The large $x$ asymptotics of the valence, nonsinglet and sea quark densities have the following form (see [7,28] and Appendix A):
\[
\begin{align*}
&f_i(x, Q^2) \approx \frac{B_i(s)}{\Gamma(1 + \nu_i(s))} (1 - x)^{\nu_i(s)}, \quad i = NS, V, \\
&f_a(x, Q^2) = \sum_{\pm} f_{a,\pm}(x, Q^2), \quad a = q, g, \\
&f_{q,-}(x, Q^2) \approx \frac{B_-(s)}{\Gamma(1 + \nu_-(s))} (1 - x)^{\nu_-(s)}, \\
&f_{g,-}(x, Q^2) \approx \frac{B_-(s)}{\Gamma(2 + \nu_-(s))} \left[\ln(1/(1 - x)) + \hat{c} + \Psi(\nu_- + 2)\right] (1 - x)^{\nu_-(s)+1}, \\
&f_{g,+}(x, Q^2) \approx \frac{B_+(s)}{\Gamma(1 + \nu_+(s))} (1 - x)^{\nu_+(s)}, \\
&f_{q,+}(x, Q^2) \approx -\frac{K_+}{\Gamma(2 + \nu_+(s))} \left[\ln(1/(1 - x)) + \hat{c} + \Psi(\nu_+ + 2)\right] (1 - x)^{\nu_+(s)+1}, \quad (51)
\end{align*}
\]

where
\[
\nu_i(s) = \nu_i(0) + r_i s, \quad r_i = \frac{4C_i}{\beta_0}, \quad B_i(s) = B_i(0) e^{-p_is}, \quad p_i = r_i \left(\gamma_E + \hat{c}_i\right), \quad i = NS, V, \pm,
\]
\[
(52)
\]
with $B_i(0)$ and $\nu_i(0)$ being free parameters, $\gamma_E$ is the Euler constant and
\[
\begin{align*}
&\nu_j(0) \sim 3, \quad j = V, NS, -, \quad \nu_+(0) = \nu_j(0) + 1. \quad (54)
\end{align*}
\]
\[
\begin{align*}
&K_+ = \frac{f}{2(C_A - C_F)}, \quad K_- = \frac{C_F}{2(C_A - C_F)}, \quad \hat{c} = \gamma_E + \frac{C_A\hat{c}_+ - C_F\hat{c}_-}{C_A - C_F}, \quad (53)
\end{align*}
\]
The constant $\nu_i(0)$ can be estimated from the quark counting rules [35] as
\[
\nu_j(0) \sim 3, \quad j = V, NS, -, \quad \nu_+(0) = \nu_j(0) + 1. \quad (54)
\]
The relation \( \nu_+ (s) = \nu_- (s) + 1 \) leads to the smallness of the term \( f_{q,+} (x, Q^2) \). So, at large \( x \) we have
\[
f_{q,+} (x, Q^2) \approx 0 \quad \text{and, thus,} \quad f_q (x, Q^2) \approx f_{q,-} (x, Q^2).
\]
Moreover, the expressions (51) and (52) demonstrate the fall of the parton densities at large \( x \) when \( Q^2 \) increases.

\section{5 Parametrizations}

Here we present parametrizations of the nonsinglet and singlet quark and gluon densities constructed similar to ones obtained earlier \cite{6} in the valence case.

\subsection{5.1 Nonsinglet and valence parts}

The nonsinglet and valence quark part \( f_i (x, Q^2) \), where \( i = V, NS \), can be represented in the following form\footnote{Similar studies were carried out in \cite{36} (see also the review \cite{23}).}
\[
f_i (x, Q^2) = \left[ A_i (s) x^{\lambda_i} (1 - x) + \frac{B_i (s) x}{\Gamma (1 + \nu_i (s))} + D_i (s) x (1 - x) \right] (1 - x)^{\nu_i (s)},
\]
which is constructed as a combination of the small \( x \) and large \( x \) asymptotics and an additional term proportional to \( D_i (s) \), which is subasymptotics in both these regions. The \( Q^2 \)-dependence of the parameters in (56) is given by (51) and (44). The \( Q^2 \)-dependence of magnitude \( D_i (s) \) is determined by the corresponding sum rules (see below).

\subsection{5.2 Sea and gluon parts}

The sea and gluon parts can be represented as combinations of the \( \pm \) terms:
\[
f_j (x, Q^2) = \sum_{\pm} f_{j, \pm} (x, Q^2), \quad j = q, g,
\]
\[
f_{q,-} (x, Q^2) = \left[ A_q e^{-d_s} (1 - x)^{m_{q,-}} + \frac{B_- (s) x}{\Gamma (1 + \nu_-(s))} + D_- (s) x (1 - x) \right] (1 - x)\nu_-(s),
\]
\[
f_{g,-} (x, Q^2) = \left[ A_g e^{-d_s} (1 - x)^{m_{g,-}} + \frac{B_- (s) x}{\Gamma (2 + \nu_-(s)) \left[ \ln (1/(1 - x)) + \hat{c} + \Psi (\nu_- + 2) \right]} + D_- (s) x (1 - x) \right] (1 - x)\nu_-(s) + 1,
\]
\[
f_{g,+} (x, Q^2) = \left[ A_g \tilde{I}_0 (\sigma) e^{-d_s} (1 - x)^{m_{g,+}} + \frac{B_+ (s) x}{\Gamma (1 + \nu_+(s))} + D_+ (s) x (1 - x) \right] (1 - x)\nu_+(s),
\]
\[
f_{q,+} (x, Q^2) = A_q \tilde{I}_1 (\sigma) e^{-d_s} (1 - x)^{m_{q,+} + 1},
\]
where \( K_- \) and \( \hat{c} \) are shown in (53). We note that one can set \( m_{q,-} = m_{g,+} = 2 \) and \( m_{q,+} = m_{g,-} = 1 \). In this case, small-\( x \) asymptotics is suppressed for large \( x \) for comparison with the subasymptotic behavior of \( \sim D_\pm (x) \). Moreover, the small-\( x \) asymptotics will contain the same powers of the factor \( (1 - x) \) for quarks and gluons.
We would like to note that the valence quarks contribute to the “−”-component but not to “+”-one. So,
\[ f_{SI}(x, Q^2) = f_{SI−}(x, Q^2) + f_{SI+}(x, Q^2), \]
\[ f_{SI−}(x, Q^2) = f_q−(x, Q^2) + f_V(x, Q^2), \quad f_{SI+}(x, Q^2) = f_q+(x, Q^2). \] (58)
The parameters involved in (56) — (58) can be fitted, for example, from the comparison with known parametrizations of NNPDF group [37] and/or taking into account the sum rules shown in the next section.

5.3 Properties of parameterizations

The obtained above parametrizations of parton distributions in a proton should obey sum rules given by (30) and (32).

5.3.1 Gross-Llewellyn-Smith and Gottfried sum rules

The additional relations between the parameters in (56) stems from the LO Gross-Llewellyn-Smith sum rule [9] and Gottfried sum rule [10]:
\[ \int_0^1 \frac{dx}{x} f_i(x, Q^2) = N_i, \quad i = V, NS, \quad N_V = 3, \quad N_{NS} = 3 I_G, \] (59)
where the value of $I_G$ can be found in (34).

So, we have the following relations:
\[ N_i = A_i(s) \frac{\Gamma(\lambda_i) \Gamma(2 + \nu_i(s))}{\Gamma(\lambda_i + 2 + \nu_i(s))} + \frac{B_i(s)}{\Gamma(2 + \nu_i(s))} + \frac{D_i(s)}{2 + \nu_i(s)}, \] (60)
i.e.
\[ D_i(s) = (2 + \nu_i(s)) \left[ N_i - A_i(s) \frac{\Gamma(\lambda_i) \Gamma(2 + \nu_i(s))}{\Gamma(\lambda_i + 2 + \nu_i(s))} - \frac{B_i(s)}{\Gamma(2 + \nu_i(s))} \right]. \] (61)
The valence and NS densities at low and large $x$ asymptotics are proportional each other. So, we can apply the following notations:
\[ \nu_V(s) \approx \nu_{NS}(s), \quad \lambda_V \approx \lambda_{NS}. \] (62)

5.3.2 Momentum conservation

The momentum conservation (35) leads to the following relations:
\[ 1 = G_V(s) + G_q−(s) + G_g−(s) \quad 0 = G_q+(s) + G_g+(s), \] (63)
where
\[ \int_0^1 dx f_i(x, Q^2) = G_i(s), \quad (i = V, NS), \quad \int_0^1 dx f_{a±}(x, Q^2) = G_{a±}^+(s), \quad (a = q, g). \] (64)
So, we have

\[ G_i(s) = A_i(s) \frac{\Gamma(\lambda_i + 1)\Gamma(2 + \nu_i(s))}{\Gamma(\lambda_i + 3 + \nu_i(s))} + \frac{B_i(s)}{\Gamma(3 + \nu_i(s))} + \frac{D_i(s)}{(2 + \nu_i(s))(3 + \nu_i(s))}, \]

\[ G^+_q(s) = A^+_q \Phi_0(m_{q,+} + \nu_+(s)) e^{-\hat{\alpha}_q s} + \frac{B_+(s)}{\Gamma(3 + \nu_+(s))} + D_+(s) \frac{\Gamma(3/2)\Gamma(2 + \nu_+(s))}{\Gamma(7/2 + \nu_+(s))}, \]

\[ G^-_q(s) = \frac{A_q}{1 + \nu_-(s) + m_{q,-}} e^{-d_- s} + \frac{B_-(s)}{\Gamma(3 + \nu_-(s))} + D_-(s) \frac{\Gamma(3/2)\Gamma(2 + \nu_-(s))}{\Gamma(7/2 + \nu_-(s))}, \]

\[ G^+_q(s) = A^+_q \Phi_1(1 + m_{q,+} + \nu_+(s)) e^{-\hat{\alpha}_q s}. \] (65)

where

\[ \Phi_0(\nu(s)) = \int_0^1 dx I_0(\sigma)(1 - x)^{\nu(s)} = \sum_{l=0}^\infty C^\nu_l (-1)^l \frac{e^{(ds)/(l+1)}}, \]

\[ \Phi_1(\nu(s)) = \int_0^1 dx \rho I_1(\sigma)(1 - x)^{\nu(s)} = \sum_{l=0}^\infty C^\nu_l (-1)^l \frac{e^{(ds)/(l+1)}}, \] (66)

with

\[ C^\nu_l = \frac{\Gamma(\nu+1)}{\nu!} \Gamma(\nu+1-l), \quad d = |\hat{d}|, \] (67)

and \( \hat{d} < 0 \) is defined above in (50). For \( \nu = 1, 2, \) and \( 3 \), we have:

\[ \Phi_j(1) = e^{ds} - \frac{1}{2 - j} e^{ds/2}, \quad j = 0, 1, \]

\[ \Phi_j(2) = e^{ds} - (1 + j) e^{ds/2} + \frac{1}{3 - 2j} e^{ds/3}, \]

\[ \Phi_j(3) = e^{ds} - \frac{3}{2 - j} e^{ds/2} + (1 + 2j) e^{ds/3} - \frac{1}{4 - 3j} e^{ds/4}. \] (68)

We would like to note that comparing (61) and (65) at \( i = V, NS \) we have

\[ G_i(s) = \frac{1}{3 + \nu_i(s)} \left[ N_V - (1 - \lambda_i) A_i(s) \Gamma(\lambda_i) \Gamma(3 + \nu_i(s)) \frac{B_i(s)}{\Gamma(3 + \nu_i(s))} + B_i(s) \right]. \] (69)

Moreover,

\[ D_-(s) = \frac{\Gamma(7/2 + \nu_-(s))}{\Gamma(3/2)\Gamma(2 + \nu_-(s))} \left[ 1 - G_V(s) - G^-_q(s) - \overline{G}^-_q(s) \right], \]

\[ D_+(s) = -\frac{\Gamma(7/2 + \nu_+(s))}{\Gamma(3/2)\Gamma(2 + \nu_+(s))} \left[ G^+_q(s) + \overline{G}^+_q(s) \right], \] (70)

where

\[ \overline{G}^-_q(s) = \frac{A_q}{1 + \nu_-(s) + m_{q,-}} e^{-d_- s} + \frac{B_-(s)}{\Gamma(3 + \nu_-(s))}, \]

\[ \overline{G}^+_q(s) = A^+_q \Phi_0(m_{q,+} + \nu_+(s)) e^{-\hat{\alpha}_q s} + \frac{B_+(s)}{\Gamma(3 + \nu_+(s))}. \] (71)
If the argument $\nu$ of $\Phi_k(\nu)$ is large, which is the case (see Section 5.4. below), then we have the approximation

$$
\Phi_0(\nu) \approx \frac{1}{1 + \nu} I_0(\sigma_\nu), \quad \sigma_\nu = \sigma \text{ with } \ln(1/x) \to \Psi(2 + \nu) + \gamma_E \approx \ln(1 + \nu) + \gamma_E,
$$

$$
\Phi_1(\nu) \approx \frac{\rho_\nu}{1 + \nu} I_1(\sigma_\nu), \quad \rho_\nu = \rho \text{ with } \ln(1/x) \to \Psi(2 + \nu) + \gamma_E \approx \ln(1 + \nu) + \gamma_E, \quad (72)
$$

where $\sigma$ and $\rho$ are given in section 3.2 and $\gamma_E$ is the Euler constant. The evaluating the results (72) can be found in Appendix B. Then, at $s = 0$ we will have relations between $A_\pm^a$, $B_\pm(0)$ and $D_\pm(0)$:

$$
G_i(s = 0) = \frac{1}{3 + \nu(0)} \left[ N_i - (1 - \lambda_V) A_i(0) \frac{\Gamma(\lambda_V)\Gamma(3 + \nu(0))}{\Gamma(\lambda_V + 3 + \nu(0))} + \frac{B_i(0)}{\Gamma(3 + \nu(0))} \right],
$$

$$
G_+^g(s = 0) = \frac{A_+^g}{1 + \nu_+(0) + m_{g,+}} + \frac{B_+(0)}{\Gamma(3 + \nu_+(0))} + D_+(0) \frac{\Gamma(3/2)\Gamma(2 + \nu_+(0))}{\Gamma(7/2 + \nu_+(0))},
$$

$$
G_-^g(s = 0) = \frac{A_+^g}{1 + \nu_+(0) + m_{g,-}} + \frac{B_-(0)}{\Gamma(3 + \nu_-(0))} + D_-(0) \frac{\Gamma(3/2)\Gamma(2 + \nu_-(0))}{\Gamma(7/2 + \nu_-(0))},
$$

$$
G_-^q(s = 0) = \frac{A_+^q}{2 + \nu_-(0) + m_{q,-}} + \frac{B_-(0)}{\Gamma(4 + \nu_-(0))} + D_-(0) \frac{\Gamma(3/2)\Gamma(2 + \nu_-(0))}{\Gamma(7/2 + \nu_-(0))},
$$

So, the final results for $A_q$, $A_g$ and $B_\pm(0)$ can be obtained from experimental data for sea quark and gluon densities at $Q^2 = Q_0^2$ (i.e. for $s = 0$):

$$
f_j(x, Q_0^2) = \sum \pm f_{j,\pm}(x, Q_0^2), \quad j = q, g,
$$

$$
f_{g,+}(x, Q_0^2) = \left[ A_+^g(1 - x)^{m_{q,+}} + \frac{B_+(0) x}{\Gamma(1 + \nu_+(0))} + \frac{B_+(0) x}{\Gamma(1 + \nu_+(0))} \right] (1 - x)^{\nu_+(0)},
$$

$$
f_{q,-}(x, Q_0^2) = \left[ A_q^q(1 - x)^{m_{q,-}} + \frac{B_-(0) x}{\Gamma(1 + \nu_-(0))} + \frac{B_-(0) x}{\Gamma(1 + \nu_-(0))} \right] (1 - x)^{\nu_-(0)},
$$

$$
f_{g,-}(x, Q_0^2) = \left[ A_+^g(1 - x)^{m_{q,-}} + \frac{B_-(0) x}{\Gamma(2 + \nu_-(0))} + \frac{B_-(0) x}{\Gamma(2 + \nu_-(0))} \right] (1 - x)^{\nu_-(0) + 1}, \quad f_{q,+}(x, Q_0^2) = 0,
$$

with $\nu_+(0) = \nu_-(0) + 1$ and $\nu_-(0) \sim 3$.

### 5.4 Results for parton densities

From numerical analysis we have $B_-(s) = 0$, i.e. the large $x$ behavior is defined by valence quarks. Then the results for $f_{a,-}(x, Q^2)$ with $a = q$ or $g$ in (57) are strongly simplified:

$$
f_{q,-}(x, Q^2) = \left[ A_q e^{-d_-(s)}(1 - x)^{m_{q,-}} + D_-(s)x(1 - x) \right] (1 - x)^{\nu_-(s)},
$$

$$
f_{g,-}(x, Q^2) = \left[ A_g e^{-d_-(s)}(1 - x)^{\nu_-(s) + m_{q,-} + 1}. \right.
$$

Similar simplification has the place also for $f_{a,-}(x, Q_0^2)$ in (74). To have it, we should put $s = 0$ in the results (75).
Moreover, we have simplifications also for $G_{a}^{-}(s)$ with $a = q$ or $g$ in (65), for $G_{q}^{-}(s)$ in (71) and for $G_{a}^{-}(s = 0)$ in (73). Indeed, we should replace $G_{a}^{-}(s)$ in (65) by

$$G_{q}^{-}(s) = \frac{A_{q}}{1 + \nu_{-}(s) + m_{q,-}} e^{-d_{-} s} + D_{-}(s) \frac{\Gamma(3/2)\Gamma(2 + \nu_{-}(s))}{\Gamma(7/2 + \nu_{-}(s))},$$

and $G_{a}^{-}(s = 0)$ ($a = q, g$) in (73) by the results (76) with $s = 0$. The results $G_{q}^{-}(s)$ in (71) should be replaced by

$$G_{q}^{-}(s) = \frac{A_{q}}{2 + \nu_{-}(s) + m_{g,-}} e^{-d_{-} s}.$$

and $G_{a}^{-}(s = 0)$ ($a = q, g$) in (73) by the results (76) with $s = 0$. The results $G_{q}^{-}(s)$ in (71) should be replaced by

$$G_{q}^{-}(s) = \frac{A_{q}}{1 + \nu_{-}(s) + m_{g,-}} e^{-d_{-} s}.$$

The values of all parameters involved into derived expressions can be determined from the comparison with the known parametrizations of numerical solutions of DGLAP equations and/or taking into account the sum rules. In our analysis, we employ the latest parametrizations proposed by the NNPDF Collaboration, namely, NNPDF4.0 set [37]. Results of our fit are collected in Table 1. Additionally, in Fig. 1 we show the comparison between our PDFs (labeled as AKL) and corresponding results obtained by the MMHT’2014 [38] and NNPDF groups. We find a good agreement between our analytical derivation and relevant numerical analyses.

### 6 TMD parton densities in a proton

Now we turn to the derivation of analytical expressions for the TMD gluon and quark density functions in a proton. Our consideration is based on the KMR procedure [13], which is a formalism to construct the TMDs from the conventional PDFs. The key assumption is that the transverse momentum dependence of the parton distributions enters only at the last evolution step, so that conventional PDFs can be used up to this step. There are known differential and integral formulations of KMR approach in the literature (see [39] for more information and discussion). Below we derive expressions for the TMDs using both these schemes.

| $Q_0$, GeV | $A_V(0)$ | $A_q$ | $A_g$ | $B_V(0)$ | $B_-(0)$ | $B_+(0)$ |
|------------|----------|-------|-------|----------|----------|----------|
| **AKL**    | $\sqrt{0.43}$ | 3.0   | 0.95  | 0.77     | 100.0    | 0.0      | $13 \cdot 10^6$ |
| $m_{q,-}$  | $m_{q,+}$ | $m_{g,-}$ | $m_{g,+}$ | $\nu_V(0)$ | $\nu_-(0)$ | $\nu_+(0)$ |
| **AKL**    | 2.0      | 1.0   | 1.0   | 2.0      | 4.0      | 7.2      | 8.2      |

Table 1: The fitted values of various parameters involved in our analytical expressions for PDFs in a proton.
Figure 1: The gluon, valence, singlet and sea quarks densities in a proton calculated as a function of the longitudinal momentum fraction $x$ at hard scale $Q^2 = 4 \text{ GeV}^2$. The purple curve corresponds to the results obtained with AKL, the green and blue curves with NNPDF4.0 (LO) and MMHT'2014 (LO) parton density functions, respectively.
6.1 Differential formulation

In the differential formulation of KMR procedure, we have the TMD parton densities $f_a^{(d)}(x, k^2, Q^2)$, where $a = V, q$ or $g$ as

$$f_a^{(d)}(x, k^2, Q^2) = \frac{\partial}{\partial \ln k^2} \left[ T_a(Q^2, k^2) \hat{D}_a(x, k^2) \right],$$

where

$$f_a(x, k^2) = x \hat{D}_a(x, k^2).$$

Since

$$f_a(x, k^2) = \sum_\pm f_{a,\pm}(x, k^2), \quad a = q, g,$$

we see that the $f_a^{(d)}(x, k^2, Q^2)$ have similar form

$$f_a^{(d)}(x, k^2, Q^2) = \sum_\pm f_{a,\pm}^{(d)}(x, k^2, Q^2).$$

Using the expressions (56) — (57) for PDFs $f_a(x, k^2)$, we obtain for the TMDs. The complete results are shown in Appendix C. Here we present only the final results:

$$f_i^{(d)}(x, k^2, Q^2) = \beta_0 a_s(k^2) T_i(Q^2, k^2) \times \left\{ \left[ d_q R_q(\Delta) - r_V \ln \left( \frac{1}{1 - x} \right) \right] f_i(x, k^2) \right. \right.$$  

$$\left. - \left[ d_V(1 - \lambda_V) A_V e^{-d_V(1 - \lambda) s_2}(1 - x) \right. \right.$$  

$$\left. + \left[ \nu_V + r_V \Psi(1 + \nu_V(s_2)) \right] \frac{B_i(s_2) x}{\Gamma(1 + \nu_V(s_2))} (1 - x)^{\nu_V(s_2)} \right\}, \quad (i = V, NS),$$

$$f_{g,+}^{(d)}(x, k^2, Q^2) = \beta_0 a_s(k^2) T_g(Q^2, k^2) \times \left\{ \left[ d_g R_g(\Delta) - r_+ \ln \left( \frac{1}{1 - x} \right) \right] f_{g,+}(x, k^2) \right. \right.$$  

$$\left. - \left[ \hat{d} \right. \left. + T(\sigma_2) + \hat{d} \right. \left. + T(\sigma_2) \right] A_+ e^{-\hat{d} + s_2(1 - x)^{m_+}} \right.$$  

$$\left. + \left[ \nu_+ + r_+ \Psi(1 + \nu_+(s_2)) \right] \frac{B_+(s_2) x}{\Gamma(1 + \nu_+(s_2))} (1 - x)^{\nu_+(s_2)} \right\},$$

$$f_{g,+}^{(d)}(x, k^2, Q^2) = \beta_0 a_s(k^2) T_g(Q^2, k^2) \times \left\{ \left[ d_g R_g(\Delta) - r_+ \ln \left( \frac{1}{1 - x} \right) \right] f_{g,+}(x, k^2) \right. \right.$$  

$$\left. - \left[ \hat{d} \right. \left. + T(\sigma_2) + \hat{d} \right. \left. + T(\sigma_2) \right] A_+ e^{-\hat{d} + s_2(1 - x)^{m_+}} \right\},$$

$$f_{q,+}^{(d)}(x, k^2, Q^2) = \beta_0 a_s(k^2) T_q(Q^2, k^2) \times \left\{ \left[ d_q R_q(\Delta) - r_+ \ln \left( \frac{1}{1 - x} \right) \right] f_{q,+}(x, k^2) \right. \right.$$  

$$\left. - d_+ A_+ e^{-d_+ s_2(1 - x)^{\nu_+(s_2)+1+m_+}} \right\},$$

$$f_{q,-}^{(d)}(x, k^2, Q^2) = \beta_0 a_s(k^2) T_q(Q^2, k^2) \times \left\{ \left[ d_q R_q(\Delta) - r_- \ln \left( \frac{1}{1 - x} \right) \right] f_{q,-}(x, k^2) \right. \right.$$  

$$\left. - d_- A_- e^{-d_- s_2(1 - x)^{\nu_-(s_2)+m_-}} \right\},$$

$$f_{g,-}^{(d)}(x, k^2, Q^2) = \beta_0 a_s(k^2) T_g(Q^2, k^2) \times \left\{ \left[ d_g R_g(\Delta) - r_- \ln \left( \frac{1}{1 - x} \right) \right] f_{g,-}(x, k^2) \right. \right.$$  

$$\left. - d_- A_- e^{-d_- s_2(1 - x)^{\nu_-(s_2)+m_-+1}} \right\}, \quad (82)$$

16
where we neglected the derivations of $D_a(s)$ ($a = V, \pm$). Indeed, the magnitudes $D_a(s)$ of intermediate terms have slow $s$ dependence and their derivations can be neglected.

### 6.2 Integral formulation

Following investigations done in \([11]\), we can obtain the following results ($j = V, \ NS$):

\[
\begin{align*}
    f_j(x, k^2, Q^2) &= \beta_0 a_s(k^2) T_q(Q^2, k^2) \times \left\{ d_q R_q(\Delta) - r_v \ln \left( \frac{1}{1-x} \right) \right\} f_j(x, k^2) - \tilde{R}_j(x, k^2), \\
    f_a(x, k^2, Q^2) &= f_a^{(i)}(x, k^2, Q^2) + f_a^{(i)}(x, k^2, Q^2), \\
    f_a^{(i)}(x, k^2, Q^2) &= \beta_0 a_s(k^2) T_a(Q^2, k^2) \times \left\{ d_a R_a(\Delta) - r_- \ln \left( \frac{1}{1-x} \right) \right\} f_a^{(i)}(x, k^2) - \tilde{R}_a^{(i)}(x, k^2), \\
    f_a^{(i)}(x, k^2, Q^2) &= \beta_0 a_s(k^2) T_a(Q^2, k^2) \times \left\{ d_a R_a(\Delta) - r_+ \ln \left( \frac{1}{1-x} \right) \right\} \tilde{J}_a^{(i)}(x, k^2) - \tilde{R}_a^{(i)}(x, k^2) \tag{83}
\end{align*}
\]

where

\[
\begin{align*}
    \tilde{R}_j(x, k^2) &= \left[ d_V(1 - \lambda_V) A_j e^{-d_V(1-\lambda)V_x} \right] \\
    &+ \left[ p_V + r_V \Psi(1 + \nu_V(s_2)) \right] \frac{B_j(s_2) x}{\Gamma(1 + \nu_V(s_2))} (1 - x)^{\nu_V(s_2)+1}, \\
    \tilde{R}_q^{(i)}(x, k^2) &= \left( \tilde{d}_+ T_0(\hat{\sigma}_2) + \tilde{d}_+ T_1(\hat{\sigma}_2) \right) A_q^{(i)} e^{-\hat{d}_+ s_2 (1 - x)^{\nu_+(s_2)+m_q,+}}, \\
    \tilde{R}_q^{(i)}(x, k^2) &= \left( \tilde{d}_+ T_1(\hat{\sigma}_2) + \tilde{d}_+ T_0(\hat{\sigma}_2) \right) A_g^{(i)} e^{-\hat{d}_+ s_2 (1 - x)^{m_q,+}} \\
    &+ \left[ p_+ + r_+ \Psi(1 + \nu_V(s_2)) \right] \frac{B_g(s_2) x}{\Gamma(1 + \nu_V(s_2))} (1 - x)^{\nu_+(s_2)}, \\
    \tilde{R}_q^{(i)}(x, k^2) &= d_- A_q e^{-d_- s_2 (1 - x)^{\nu_-(s_2)+m_q,-}}, \\
    \tilde{R}_q^{(i)}(x, k^2) &= d_- A_g e^{-d_- s_2 (1 - x)^{\nu_-(s_2)+m_q,-}} \tag{84}
\end{align*}
\]

with

\[
\frac{x}{x_0} = x, \quad x_0 = 1 - \Delta, \quad \hat{\sigma}_2 = \sigma_2(x \rightarrow \bar{x}), \quad \hat{\rho}_2 = \rho_2(x \rightarrow \bar{x}) \tag{85}
\]

and

\[
\tilde{J}_a^{(i)}(x, k^2) = f_a^{(i)}(x, k^2) \text{ with } \sigma_2 \rightarrow \hat{\sigma}_2, \quad \rho_2 \rightarrow \hat{\rho}_2 \tag{86}
\]

As in \([82]\), we neglected contributions coming from the magnitudes $D_a(s)$, where $a = V, \pm$.

### 6.3 Sudakov form factors $T_a(Q^2, k^2)$

For the Sudakov form factors, we have \([12]\):

\[
T_a(Q^2, k^2) = \exp \left[ -d_a R_a(\Delta) s_1 \right] \tag{87}
\]
where

\[ s_1 = \ln \left( \frac{a_s(k^2)}{a_s(Q^2)} \right), \quad d_a = \frac{4C_a}{\beta_0}, \quad C_q = C_F, \quad C_g = C_A, \quad \beta_0 = \frac{C_A}{3} \left( 11 - 2\varphi \right), \]

\[ R_q(\Delta) = \ln \left( \frac{1}{\Delta} \right) - \frac{3x_0^2}{4} = \ln \left( \frac{1}{\Delta} \right) - \frac{3}{4}(1 - \Delta)^2, \]

\[ R_g(\Delta) = \ln \left( \frac{1}{\Delta} \right) - \left( 1 - \frac{\varphi}{4} \right)x_0^2 + \frac{1 - \varphi}{12}x_0^3(4 - 3x_0) = \]

\[ = \ln \left( \frac{1}{\Delta} \right) - \left( 1 - \frac{\varphi}{4} \right)(1 - \Delta)^2 + \frac{1 - \varphi}{12}(1 - \Delta)^3(1 + 3\Delta). \]  

(88)

6.4 Cut-off parameter \( \Delta \)

For the phenomenological applications, the cut-off parameter \( \Delta \) usually has one of two basic forms:

\[ \Delta_1 = \frac{k}{Q}, \quad \Delta_2 = \frac{k}{k + Q}, \]  

(89)

that reflects the two cases: \( \Delta_1 \) is in the strong ordering, \( \Delta_2 \) is in the angular ordering (see [39]). In all above cases, except the results for \( T_a(Q^2, k^2) \), we can simply replace the parameter \( \Delta \) by \( \Delta_1 \) and/or \( \Delta_2 \). For the Sudakov form factors, we note that the parameters \( \Delta_i \) (with \( i = 1, 2 \)) contribute to the integrand and, thus, their momentum dependence changes the results in (87). Perform the correct evaluation (see [12]) we have

\[ T^{(i)}_a(Q^2, k^2) = \exp \left[ -4C_a \int_{k^2}^{Q^2} \frac{dp^2}{p^2} a_s(p^2) R_a(\Delta_i) \right]. \]

(90)

The analytic evaluation of \( T^{(i)}_a(Q^2, k^2) \) is a very cumbersome procedure, which will be accomplished in the future. With the purpose of simplifying our analysis, below we use the numerical results for \( T^{(i)}_a(Q^2, k^2) \).

6.5 Results for TMD parton densities

Our results for the TMD quark and gluon densities in a proton, obtained in both differential and integral formulation of the KMR procedure, are shown in Fig. 2. Note that cut-off parameter \( \Delta \) is taken in the form corresponding to the angular ordering condition. We find that the difference between these two scenarios is originated at large parton transverse momenta only, whereas at low \( k_T \) they are coincide to each other. In addition, we plot here for comparison the results derived in other approaches. So, we show the TMD gluon distribution obtained from the numerical solution [40] of the Catani-Ciafaloni-Fiorani-Marchesini (CCFM) evolution equation [41], namely, JH’2013 set 2. The CCFM equation is smoothly interpolates between the small-\( x \) BFKL gluon dynamics and large-\( x \) DGLAP one, and JH’2013 set 2 gluon is often used in a different phenomenological applications (see, for example, [17–20]). Also we plot here the results for the TMD PDFs obtained within the KMR approach where the NNPDF4.0 set has been used as an input. One can see that all these TMDs have a different shape in \( k_T \) and a different overall normalization. The step behavior at low \( k_T \sim 1 \text{ GeV} \) is related to the special normalization condition usually applied in the KMR scheme (see [13] for more information). Studying the phenomenological consequences of observed differences is an important and interesting task, but, however, it is out of our present consideration.
Figure 2: The TMD gluon, valence, singlet and sea quarks densities in a proton calculated as a function of the parton transverse momentum $k_T$ at longitudinal momentum fraction $x = 0.001$ and hard scale $Q^2 = 10 \text{ GeV}^2$. The purple and green curves correspond to the results obtained with AKL sets, which were obtained from differential and integral formulation of KMR approach, respectively. The blue curve corresponds to the results obtained with TMD gluon distribution JH’2013 set 2 and with TMD quark distributions calculated numerically in the traditional KMR scenario, where the conventional parton densities from standard NNPDF (LO) set are used as an input.
7 Conclusion

In the paper we proposed an analytical expressions for the proton PDFs based on their exact asymptotics at small and large $x$ values. The derived parameterizations contain sub-asymptotic terms, which are fixed by momentum conservation and, in the nonsinglet and valence parts, by the Gross-Llewellyn-Smith and Gottfried sum rules (see [9] and [10], respectively. The rest of parameters is fixed by a comparison with recent numerical solution of the DGLAP equation done by the NNPDF group [37] and presented in Table 1. Then, our consideration has been extended to the parton densities, dependent on the transverse momentum (TMDs). These quantities are often used in the number of phenomenological applications and widely discussed in the literature at present. We employ the popular Kimber-Martin-Ryskin formalism [13, 14] and derive the TMD quark and gluon distributions in a proton within the both differential and integral formulation of the KMR scheme. In the calculations we considered the different treatments of kinematical constraint, reflecting the angular and strong ordering conditions. An analytical expressions for the PDFs and TMDs (in particular, for the quark TMDs), valid at both low and large $x$, were obtained for the first time. As a next step, we plan to use the present PDF sets to study PDF modifications in nuclei and, of course, to extend the present investigations beyond the LO. In these investigations we will follow [42,43] and [44], respectively.

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A PDF asymptotics at large $x$ values

Using large $n$ expansion for $S_1(n)$ as

$$S_1(n) = \ln n + \gamma_E + O(n^{-1}),$$

of some auxiliary anomalous dimension $d_c = c_1 S_1(n) + c_2$, we see for the general renorm-group exponent

$$e^{-d_c s} = e^{-(c_1 (\ln n + \gamma_E) - c_2) s} + O(n^{-1}) = n^{-c_1 s} e^{-\tau_2 s} + O(n^{-1}), \quad \tau_2 = c_2 + c_1 \gamma_E. \quad (A2)$$

Let some PDF $f_c(x, Q^2)$ has the following $Q^2$-dependence

$$f_c(x, Q^2) = f_c(x, Q_0^2) e^{-d_c s} = f_c(x, Q_0^2) n^{-c_1 s} e^{-\tau_2 s}. \quad (A3)$$

It is convenient to represent our basic variable $s$ shown in Eq. (26) as

$$s = \bar{s}(Q^2) - \bar{s}(Q_0^2) \equiv \bar{s} - \bar{s}_0, \quad (A4)$$

when

$$\bar{s}(Q^2) = \ln[\ln(Q^2/\Lambda_{1,0}^2)]. \quad (A5)$$

When we can see that the result (A3) for PDF $f_c(x, Q^2)$ can be rewritten as

$$f_c(x, Q^2) = B_c n^{-\nu_1 - c_1 \bar{s}} e^{\nu_2 - \tau_2 \bar{s}}, \quad (A6)$$
with some free parameters $B_c$, $\nu_1$ and $\nu_2$. For the initial condition $f_c(x,Q_0^2)$, we have the same result but with the replace $s \to s_0$.

Now we consider the Mellin moment
\[ M_c(n) = \int_0^1 dx x^{n-1} (1-x)^{\nu_c} = \frac{\Gamma(n)\Gamma(\nu_c + 1)}{\Gamma(n + \nu_c + 1)} \] (A7)

and try to represent it in a form, similar to (A3) and (A13).

It is convenient to use Sterling formula
\[ \ln \Gamma(z) = z \ln z - z + \frac{1}{2} \ln \frac{2\pi}{z} + B_2 \frac{z}{2} + O(z^{-2}) \] (A8)

at large $z$ values, where $B_2$ is Bernoulli number.

So, for large $z$ and fixed $\delta$ values, we have after little algebra
\[ \ln \frac{\Gamma(z+\delta)}{\Gamma(z)} = \delta \ln z + \frac{\delta(\delta-1)}{2z} + O(z^{-2}) \] (A9)

and, thus,
\[ \frac{\Gamma(z)}{\Gamma(z+\delta)} = z^{-\delta} e^{\delta(1-\delta)/(2z)} + O(z^{-2}) = z^{-\delta} \left( 1 + \frac{\delta(1-\delta)}{2z} \right) + O(z^{-2}) \] (A10)

So, the Mellin moment $M_c(n)$ can be represented as
\[ M_c(n) = n^{-(\nu_c+1)} \left( 1 - \frac{\nu_c(1+\nu_c)}{2n} \right) + O(n^{-2}) \] (A11)

For PDF we have $((\alpha+1) \to \nu_1 + c_1 \bar{s})$
\[ f_c(x,Q^2) = B_c \frac{(1-x)^{\nu_c + c_1 \bar{s}}}{\Gamma(\nu_c + 1 + c_1 \bar{s})} e^{\nu_2 - \nu_2 s} = f_c(x,Q_0^2) \frac{\Gamma(\nu_c + 1 + c_1 \bar{s}_0)}{\Gamma(\nu_c + 1 + c_1 \bar{s})} e^{-\nu_2 s}, \] (A12)

where $\nu_c = \nu_1 - 1$. The constant $\nu_2$ may be neglected.

So, finally we have
\[ f_c(x,Q^2) = B_c \frac{(1-x)^{\nu_c(s)}}{\Gamma(\nu_c(s) + 1)} e^{-\nu_2 s}, \quad \nu_c(s) = \nu_c(0) + \bar{c}_2 s, \] (A13)

where
\[ \nu_c(0) = \nu_c + c_1 \bar{s}_0, \quad B_c = f_c(x,Q^2) \frac{\Gamma(\nu_c(0) + 1)}{(1-x)^{\nu_c(0)}} e^{c_2 \bar{s}_0} \] (A14)

A.1 $O(n^0)$ accuracy

At $O(n^0)$ accuracy,
\[ \gamma_{qg} = \gamma_{gq} = O(n^{-1}) \] (A15)

and, thus, the $Q^2$-evolutions of the singlet quark and gluon densities are not related each other.

The corresponding anomalous dimesions have the following form
\[ \gamma_a = 8C_F \left( \ln n + \gamma_E - \frac{3}{4} \right) + O(n^{-1}), \quad (a = NS, V, qg), \quad \gamma_a = 8C_A \left( \ln n + \gamma_E \right) - 2\beta_0 + O(n^{-1}) \] (A16)
and parton densities

\[ f_q(x, Q^2) = f_-(x, Q^2) + O(n^{-1}), \quad f_g(x, Q^2) = f_+(x, Q^2) + O(n^{-1}), \quad (A17) \]

have the large \( x \) asymptotics

\[ f_b(x, Q^2) = B_b(s) \left( \frac{1 - x}{\Gamma(v_b(s) + 1)} \right), \quad B_b(s) = B_b(0)e^{-p_b s}, \quad \nu_b(s) = \nu_b(0) + r_b s, \quad (b = NS, V, \pm) \quad (A18) \]

where where \( r_b \) and \( p_b \) are given in Eqs. (52).

### A.2 \( O(n^{-1}) \) accuracy

At \( O(n^{-1}) \) accuracy,

\[ \gamma_{qg} = -\frac{4f}{4n} + O(n^{-2}), \quad \gamma_{qq} = -\frac{4C_F}{4n} + O(n^{-2}), \]
\[ \gamma_{qg} = 8C_F \left( \ln n + \hat{c}_- + \frac{1}{2n} \right) + O(n^{-2}), \quad \gamma_{gg} = 8C_A \left( \ln n + \hat{c}_+ + \frac{1}{2n} \right) + O(n^{-2}) \quad (A19) \]

where \( \hat{c}_\pm \) are given in Eq. (53).

Using these results, we have

\[ \gamma_+ = \gamma_{gg} + O(n^{-2}), \quad \gamma_- = \gamma_{qq} + O(n^{-2}), \quad (A20) \]

i.e. the evolution of the “\( \pm \)”-components are same as in the previous subsection.

The corresponding projectors have the following from:

\[ \alpha = 1 + O(n^{-2}), \quad \beta = \frac{K_-}{n} \frac{1}{\ln n + \hat{c}}, \quad \hat{c} = \frac{K_+}{n} \frac{1}{\ln n + \hat{c}}, \quad (A21) \]

where \( K_\pm \) and \( \hat{c} \) are given in Eq. (53).

Thus, now the contributions of initial quarks (gluons) to the final quarks (gluons) during \( Q^2 \) evolution are same as in the previous subsection (see Eq. (A18)) but there are additional contributions: initial quarks to gluons and vice versa. The last contributions have unusual form \( \sim \frac{1}{n \ln n} \) at the large \( n \) values.

So, for the \( Q^2 \)-evolutions of the singlet quark and gluon densities we have

\[ f_a(n, Q^2) = \hat{f}_a^+(n, Q_0^2) e^{-d_+ s} + \hat{f}_a^-(n, Q_0^2) e^{-d_- s}, \quad (A22) \]

where

\[ \hat{f}_q^+(n, Q_0^2) = -\beta f_g(n, Q_0^2), \quad \hat{f}_q^-(n, Q_0^2) = f_q(n, Q_0^2) + \beta f_g^+(n, Q_0^2), \]
\[ \hat{f}_q^-(n, Q_0^2) = \epsilon f_q(n, Q_0^2), \quad \hat{f}_q^+(n, Q_0^2) = f_q(n, Q_0^2) - \epsilon f_g^+(n, Q_0^2), \quad (A23) \]

Thus, the NS and valence parton densities have the large \( x \) asymptotics, shown in Eq. (51) of main text. For the singlet quark and gluon densities the situation is different. Their large \( x \) asymptotics contain standard contributions: the “\( + \)”-component for gluon density and the “\( - \)”-component for sea quark density (see Eq. (A18), for example). But there are also the additional parts, \( \sim K_\pm \) in Eq. (51), coming from the contributions \( \sim \beta \) and \( \sim \epsilon \) in (A23). We will sketch a way to calculate the additional parts \( \overline{f}_\pm (x, Q^2) \) in Appendix B.
B Results at large $\nu$ values

To obtain the results $\overline{f}_\pm(x, Q^2)$ it is convenient to calculate the inverse Mellin transform of auxiliary function

$$f_A(n) = \frac{1}{n^{\nu+2}(\ln n + a)}. \quad (B1)$$

To do it, we expand the denominator of $f_A(n)$ as

$$f_A(n) = \frac{1}{n^{\nu+2}(\ln n + a)} = \sum_{m=0}^{\infty} \frac{(-1)^m \ln^m n}{n^{\nu+2}a^{m+1}} = \sum_{m=0}^{\infty} \frac{1}{a^{m+1}} \left( \frac{d}{d\nu} \right)^m \frac{1}{n^{\nu+2}}. \quad (B2)$$

Since

$$\int_0^1 dx x^{n-2} (1-x)^{\nu+1} \approx \frac{\Gamma(\nu + 2)}{(n-1)^{\nu+1}} \approx \frac{\Gamma(\nu + 2)}{n^{\nu+1}}, \quad (B3)$$

then the function $f_A(x)$, which is the inverse Mellin transform of $f_A(n)$,

$$f_A(n) = \int_0^1 dx x^{n-2} f_A(x), \quad (B4)$$

has the form

$$f_A(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \int_0^1 dx x^\mu f_A(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \sum_{k=0}^m C_k^m \left( \frac{d}{d\nu} \right)^{m-l} (1-x)^{\nu+1} \left( \frac{d}{d\nu} \right)^l \frac{1}{\Gamma(\nu + 2)}. \quad (B5)$$

In the r.h.s. we will have the powers of Polygamma functions

$$\Psi(\nu + 2) = \frac{d}{d\nu} \ln \Gamma(\nu + 2), \quad \Psi^{(m+1)}(\nu + 2) = \frac{d}{d\nu} \Psi^{(m)}(\nu + 2), \quad (m \geq 0). \quad (B6)$$

At large $\nu$-values, $\Psi(\nu + 2) \sim \ln(\nu + 2)$ and $\Psi^{(m)}(\nu + 2) \sim 1/(\nu + 2)^m, \quad (m \geq 1)$. So, we can neglect contributions from $\Psi^{(m)}(\nu + 2), \quad (m \geq 1)$ and obtain

$$\left( \frac{d}{d\nu} \right)^l \frac{1}{\Gamma(\nu + 2)} \approx \frac{(-1)^l \Psi^l(\nu + 2)}{\Gamma(\nu + 2)}. \quad (B7)$$

So, we have

$$f_A(x) \approx \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \left( \ln \frac{1}{1-x} + \Psi(\nu + 2) \right)^m \frac{(1-x)^{\nu+1}}{\Gamma(\nu + 2)} = \frac{1}{\ln \frac{1}{1-x} + a + \Psi(\nu + 2)} \frac{(1-x)^{\nu+1}}{\Gamma(\nu + 2)}. \quad (B8)$$

To obtain a contribution $\sim K_-$ in Eqs. (65) and (73), it is convenient to calculate the following auxiliary integral

$$I(\mu) = \int_0^1 dx x^\mu f_A(x) = \int_0^1 dx \frac{x^\mu}{\ln \frac{1}{1-x} + a + \Psi(\nu + 2)} \frac{(1-x)^{\nu+1}}{\Gamma(\nu + 2)}. \quad (B9)$$

Expanding the denominator as in Eq. (B2), we have

$$I(\mu) = \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \int_0^1 dx x^\mu \left( \ln \frac{1}{1-x} + \Psi(\nu + 2) \right)^m \frac{(1-x)^{\nu+1}}{\Gamma(\nu + 2)}. \quad (B10)$$
As it was shown above in Eqs. (B5) and (B8) the integral in the r.h.s. has the following form

$$\int_0^1 dx x^\mu \left( \ln \frac{1}{1-x} + \Psi(\nu + 2) \right)^m \frac{(1-x)^{\nu+1}}{\Gamma(\nu+2)} \approx (-1)^m \frac{d}{d\nu} \int_0^1 dx x^\mu (1-x)^{\nu+1} \Gamma(\nu+2)$$

$$= (-1)^m \left( \frac{d}{d\nu} \right)^m \frac{\Gamma(\mu+1) \Gamma(\mu+\nu+3)}{\Gamma(\mu+\nu+3)} \approx \Psi^m(\mu + \nu + 3) \frac{\Gamma(\mu+1) \Gamma(\mu+\nu+3)}{\Gamma(\mu+\nu+3)}.$$  \hspace{1cm} (B11)

So, for the auxiliary integral $I(\mu)$ we have

$$I(\mu) \approx \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \Psi^m(\mu + \nu + 3) \frac{\Gamma(\mu+1) \Gamma(\mu+\nu+3)}{\Gamma(\mu+\nu+3)} = \frac{1}{a + \Psi(\mu + \nu + 3)} \frac{\Gamma(\mu+1) \Gamma(\mu+\nu+3)}{\Gamma(\mu+\nu+3)} \hspace{1cm} (B12)$$

Thus, the results $\sim K^-$ for $g^-_b(s)$ in Eq. (65) and for $g^-_g(s = 0)$ in Eq. (73), can be obtained from (B12) for $\mu = 1$.

To obtain the results (72) it is convenient to consider the following auxiliary integral

$$\Phi_j(\mu, \nu) = \int_0^1 dx x^\mu \rho^j I_j(\sigma) (1-x)^\nu.$$  \hspace{1cm} (B13)

Expanding Bessel function, we have

$$\Phi_j(\mu, \nu) = \int_0^1 dx x^\mu \sum_{k=0}^{\infty} \frac{(ds)^{k+j}}{k!(k+j)!} \left( \ln \frac{1}{x} \right)^k (1-x)^\nu.$$  \hspace{1cm} (B14)

As it was above in Eq. (B11), the integral in the r.h.s. can be rewritten as

$$\int_0^1 dx x^\mu \left( \ln \frac{1}{x} \right)^k (1-x)^\nu = (-1)^k \left( \frac{d}{d\mu} \right)^k \int_0^1 dx x^\mu (1-x)^\nu = (-1)^k \left( \frac{d}{d\mu} \right)^k \frac{\Gamma(\mu+1) \Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}.$$  \hspace{1cm} (B15)

Taking approximations (B6), we have

$$\int_0^1 dx x^\mu \left( \ln \frac{1}{x} \right)^k (1-x)^\nu = \left( \Psi(\mu + \nu + 2) - \Psi(\mu + 1) \right) \frac{\Gamma(\mu+1) \Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}.$$  \hspace{1cm} (B16)

and, thus, the integral $\Phi_j(\mu, \nu)$ is equal to

$$\Phi_j(\mu, \nu) = \rho^j I_j(\sigma) \frac{\Gamma(\mu+1) \Gamma(\nu+1)}{\Gamma(\mu+\nu+2)},$$  \hspace{1cm} (B17)

where

$$\sigma = 2\sqrt{ds(\Psi(\mu + \nu + 2) - \Psi(\mu + 1))}, \quad \rho = \frac{\sigma}{2(\Psi(\mu + \nu + 2) - \Psi(\mu + 1))}.$$  \hspace{1cm} (B18)

The results (72) for $\Phi_j(\nu)$ ($j = 0, 1$) can be obtained from Eq. (B17) at $\mu = 0$.

C Differential formulation of KMR approach

Now we can find the results for TMD parton densities without derivatives. Derivation of $T_a(Q^2, k^2)$ is as follows

$$\frac{\partial T_a(Q^2, k^2)}{\partial \ln k^2} = d_a \beta_0 a(k^2) R_a(\Delta) T_a(Q^2, k^2).$$  \hspace{1cm} (C1)
and derivations of conventional PDFs are as follows

\[
\frac{\partial f_{V}(x, k^2)}{\partial \ln k^2} = \beta_0 a_s(k^2) \left\{ r_V \ln(1 - x) f_V(x, k^2) - \left[ d_V (1 - \lambda_V) e^{-d_V(1-\lambda)s_2} (1 - x) + \left[ p_V + r_V \Psi(1 + \nu_V(s_2)) \right] \frac{B_V(s_2) x}{\Gamma(1 + \nu_V(s_2))} + D_V^* (s_2) x (1 - x)^{\nu_V(s_2)} \right] \right\},
\]

\[
\frac{\partial f_{g,+}(x, k^2)}{\partial \ln k^2} = \beta_0 a_s(k^2) \left\{ r_+ \ln(1 - x) f_{g,+}(x, k^2) - \left[ \hat{d}_+ T_1(\sigma_2) + \tilde{d}_+ I_0(\sigma_2) \right] A_+^+ e^{-\tilde{d}_+ s_2} \times (1 - x)^{m_{g,+}} + \left[ p_+ + r_+ \Psi(1 + \nu_+(s_2)) \right] \frac{B_+(s_2) x}{\Gamma(1 + \nu_+(s_2))} + D^*_g (s_2) x (1 - x)^{\nu_+(s_2)} \right\},
\]

\[
\frac{\partial f_{q,+}(x, k^2)}{\partial \ln k^2} = \beta_0 a_s(k^2) \left\{ r_+ \ln(1 - x) f_{q,+}(x, k^2) - \left[ \hat{d}_+ T_0(\sigma_2) + \tilde{d}_+ I_1(\sigma_2) \right] A_+^+ e^{-\tilde{d}_+ s_2} (1 - x)^{\nu_+(s_2)+1+m_{q,+}} \right\},
\]

\[
\frac{\partial f_{q,-}(x, k^2)}{\partial \ln k^2} = \beta_0 a_s(k^2) \left\{ r_- \ln(1 - x) f_{q,-}(x, k^2) - \left[ d_- A_q e^{d_- s_2} (1 - x)^{m_{q,-}} + D_q^* (s_2) x (1 - x)^{\nu_-(s_2)} \right] \right\},
\]

\[
\frac{\partial f_{g,-}(x, k^2)}{\partial \ln k^2} = \beta_0 a_s(k^2) \left\{ r_- \ln(1 - x) f_{g,-}(x, k^2) - d_- A_g e^{d_- s_2} (1 - x)^{\nu_-(s_2)+m_{q,-}+1} \right\}, (C2)
\]

where

\[
D^*_n(s) = \frac{d}{ds} D_n(s), \quad s_2 = \ln \left( \frac{a_s(Q_0^2)}{a_s(k^2)} \right), \quad \sigma_2 = \sigma(s \to s_2), \quad \rho_2 = \rho(s \to s_2). \quad (C3)
\]
So, the results for the TMD parton densities read the form (81) with

\[
\begin{aligned}
f^{(d)}_V(x, k^2, Q^2) &= \beta_0 a_s(k^2) T_q(Q^2, k^2) \times \left\{ d_q R_q(\Delta) - r_V \ln \left( \frac{1}{1 - x} \right) \right\} f_V(x, k^2) \\
&- \left[ d_V(1 - \lambda_V)^+ A_V e^{-d_V(1-\lambda)s_2}(1 - x) \\
&\quad + [p_V + r_V \Psi(1 + \nu_V(s_2))] \frac{B_V(s_2) x}{\Gamma(1 + \nu_V(s_2))} + D^*_v(s_2) x(1 - x) \right] (1 - x)^{\nu_V(s_2)} \right\}, \\
\end{aligned}
\]

\[
\begin{aligned}
f^{(d)}_{q,+}(x, k^2, Q^2) &= \beta_0 a_s(k^2) T_q(Q^2, k^2) \times \left\{ d_q R_q(\Delta) - r_+ \ln \left( \frac{1}{1 - x} \right) \right\} f_{q,+}(x, k^2) \\
&- \left[ d_+ T_1(\sigma_2) + d_+ T_0(\sigma_2) \right] A^+_q e^{-d_+ s_2}(1 - x)^{\nu_+(s_2)} \\
&\quad + [p_+ + r_+ \Psi(1 + \nu_+(s_2))] \frac{B_+(s_2) x}{\Gamma(1 + \nu_+(s_2))} + D^*_g(s_2) x(1 - x) \right] (1 - x)^{\nu_+(s_2)} \right\}, \\
\end{aligned}
\]

\[
\begin{aligned}
f^{(d)}_{q,-}(x, k^2, Q^2) &= \beta_0 a_s(k^2) T_q(Q^2, k^2) \times \left\{ d_q R_q(\Delta) - r_- \ln \left( \frac{1}{1 - x} \right) \right\} f_{q,-}(x, k^2) \\
&- \left[ d_- A_q e^{-d_- s_2}(1 - x)^{m_q-} + D^*_q(s_2) x(1 - x) \right] (1 - x)^{\nu_-(s_2)} \right\}, \\
\end{aligned}
\]

\[
\begin{aligned}
f^{(d)}_{g,-}(x, k^2, Q^2) &= \beta_0 a_s(k^2) T_g(Q^2, k^2) \times \left\{ d_g R_g(\Delta) - r_- \ln \left( \frac{1}{1 - x} \right) \right\} f_{g,-}(x, k^2) \\
&- d_- A_g e^{-d_- s_2}(1 - x)^{\nu_-(s_2) + m_g-} + D^*_g(s_2) x(1 - x) \right] (1 - x)^{\nu_-(s_2)} \right\}, \\
\end{aligned}
\]

\[(C4)\]

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