Propagators of the Dirac fermions in de Sitter expanding universe

Ion I. Cotăescu

West University of Timișoara, V. Parvan Avenue 4 RO-300223 Timișoara, Romania

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Abstract

The propagators of the Dirac fermions are studied on both the configuration and momentum representations on the expanding portion of the (1+3)-dimensional de Sitter spacetime considering a fixed vacuum of Bunch-Davies type. In the configuration representation the method of Koksma and Prokopec [J. F. Koksma and T. Prokopec, Class. Quant. Grav. 26, 125003 (2009)] is applied recovering thus the form of the propagators in the massive case but obtaining a different result for the left-handed massless fermions. The principal new result reported here is the integral representation of the Feynman propagators of the massive and left-handed massless Dirac fields one needs for calculating the Feynman diagrams of our de Sitter QED in Coulomb gauge [I.I. Cotaescu and C. Crucean Phys. Rev. D. 87, 044016 (2013)] or of a larger QFT.

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*Electronic address: i.cotaescu@e-uvt.ro
I. INTRODUCTION

Important pieces of the quantum field theory on curved spacetimes are the two-point functions that can be calculated either as propagators by using mode expansions or by looking for new hypotheses complying with the general relativistic covariance as, for example, that of the maximal symmetry of the two-point functions on the hyperbolic spacetimes, i.e. de Sitter (dS) and anti-de Sitter ones [1].

The propagator of the Dirac fermions on the dS spacetime in configuration representation was derived first by Candelas and Reine which integrated the Green equation of this field [2]. The same propagator was calculated later as mode sum by Koskma and Prokopec in a more general context of spacetimes of arbitrary dimensions approaching to the dS one [3]. On the other hand, we developed the dS QED in Coulomb gauge [4] where we know the Dirac quantum modes in different bases [5–7] and we need to derive the Feynman propagators for calculating physical effects. Obviously, its expression in configuration representation is included in the general result of Ref. [3] but it must be extracted in the particular case of the (1 + 3)-dimensional dS spacetime and a fixed vacuum.

However, this is not enough for calculating Feynman diagrams for which we need a suitable integral representation of these propagators which should include the effects of the Heaviside functions resulted after computing the chronological products. Since we have not yet a such integral representation our principal objective here is to find it analyzing the structure of the fermion propagators investigated so far.

This is a good opportunity for reviewing the entire procedure of Ref. [3] of deriving the propagators of the Dirac field in configuration representation on the expanding portion of the dS spacetime. We assume that the covariant Dirac field is quantized canonically [8] and the vacuum is fixed being of the Bunch-Davies type [9,10] as in our dS QED [4]. Our goal is to present all the calculation details discussing the specific properties of the quantities under consideration as can be deduced from the general formulas of the theory of modified Bessel functions [11,12]. Thus we recover the results of Ref. [3] for the massive fermions in the particular case of the (1 + 3)-dimensional dS manifold but we obtain a different result for the left-handed massless fermions. Based on these investigations we propose a suitable integral representation of the Feynman propagators of the Dirac fermions we need for calculating Feynman diagrams in our dS QED [4] or in a larger QFT.
This paper is a piece of work having five sections. In the next one we review the fundamental solutions of the Dirac equation minimally coupled to the dS gravity in the massive and massless cases. We point out that these form complete systems of orthonormalized spinors allowing one to write down the mode expansion of the Dirac field. The third section is devoted to the anti-commutator matrix-functions and propagators defined as mode sums which can be calculated applying the method Ref. [3]. In the next one we discuss the form of the fermion propagators in the configuration representation in the massive and massless cases as resulted from the technical ingredients presented in the Appendices A and B. Here we observe that our propagator of the left-handed massless fermions is different from that of Ref. [3]. The last section is devoted to our principal new proposal namely, the integral representation of the Feynman propagators of the massive and massless left-handed Dirac fields that can be used in applications.

II. FUNDAMENTAL SPINOR SOLUTIONS

Let us first revisit some basics properties of the fundamental solutions of the Dirac equation minimally coupled to the gravity of the (1+3)-dimensional de Sitter expanding universe. In what follows we consider the normalized solutions of positive and negative frequencies of the spin basis [7] since those of the helicity basis [5] are not defined in rest frames.

We denote by $M$ the de Sitter expanding universe of radius $\frac{1}{\omega}$ where the notation $\omega$ stands for its Hubble constant. We choose the moving chart $\{x\} = \{t, \vec{x}\}$ of the conformal time, $t \in (-\infty, 0]$, Cartesian coordinates and the line element

$$ds^2 = \frac{1}{(\omega t)^2} \left( dt^2 - d\vec{x} \cdot d\vec{x} \right),$$

which covers the expanding part of the de Sitter manifold. In addition, we use the non-holonomic frames defined by the tetrad fields which have only diagonal components,

$$e^0_0 = -\omega t, \quad e^i_j = -\delta^i_j \omega t, \quad \hat{e}^0_0 = -\frac{1}{\omega t}, \quad \hat{e}^i_j = -\delta^i_j \frac{1}{\omega t}. \quad (2)$$

In this tetrad-gauge, the massive Dirac field $\psi$ of mass $m$ and its Dirac adjoint $\bar{\psi} = \psi^+ \gamma^0$ satisfy the field equations $(D_x - m)\psi(x) = 0$ and, respectively, $\bar{\psi}(x)(\bar{D}_x - m) = 0$ given by the Dirac operator

$$D_x = -i\omega t \left( \gamma^0 \partial_t + \gamma^i \partial_i \right) + \frac{3i\omega}{2} \gamma^0, \quad (3)$$
and its adjoint
\[ \bar{D}_x = \left( \gamma^0 \frac{\partial}{\partial t} + \gamma^i \frac{\partial}{\partial x^i} \right) i \omega t - \frac{3i\omega}{2} \gamma^0, \]
whose derivatives act to the left.

The quantum Dirac field may be expanded in terms of fundamental spinors of positive and negative frequencies in different representations. Here we consider the momentum representation where the plane wave solutions \( U_{\vec{p},\sigma} \) and \( V_{\vec{p},\sigma} \) depend on the momentum \( \vec{p} \) and arbitrary polarization \( \sigma \). These spinors form an orthonormal basis satisfying the orthogonality relations
\[ \langle U_{\vec{p},\sigma}, U_{\vec{p}',\sigma'} \rangle = \langle V_{\vec{p},\sigma}, V_{\vec{p}',\sigma'} \rangle = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \]
(5)
\[ \langle U_{\vec{p},\sigma}, V_{\vec{p}',\sigma'} \rangle = \langle V_{\vec{p},\sigma}, U_{\vec{p}',\sigma'} \rangle = 0, \]
(6)
with respect to the relativistic scalar product [5]
\[ \langle \psi, \psi' \rangle = \int d^3x \sqrt{|g|} \bar{\psi}(x) \gamma^0 \psi(x) = \int d^3x (-\omega t)^{-3} \bar{\psi}(x) \gamma^0 \psi(x), \]
(7)
and the completeness condition [5]
\[ \int d^3p \sum_{\sigma} \left[ U_{\vec{p},\sigma}(x, \vec{x}) U_{\vec{p},\sigma}^+(x, \vec{x}') + V_{\vec{p},\sigma}(x, \vec{x}) V_{\vec{p},\sigma}^+(x, \vec{x}') \right] = (-\omega t)^3 \delta^3(\vec{x} - \vec{x}'), \]
(8)
In this representation the Dirac field may be expanded as
\[ \psi(t, \vec{x}) = \psi^{(+)}(t, \vec{x}) + \psi^{(-)}(t, \vec{x}) = \int d^3p \sum_{\sigma} \left[ U_{\vec{p},\sigma}(x) a(\vec{p}, \sigma) + V_{\vec{p},\sigma}(x) b^+(\vec{p}, \sigma) \right], \]
(9)
assuming that the particle \((a, a^\dagger)\) and antiparticle \((b, b^\dagger)\) operators satisfy the canonical commutation relations [5, 8],
\[ \{ a(\vec{p}, \sigma), a^+(\vec{p}', \sigma') \} = \{ b(\vec{p}, \sigma), b^+(\vec{p}', \sigma') \} = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}'). \]
(10)
Thus we obtain a good quantum theory where the one-particle operators conserved via Noether theorem become just the generators of the corresponding isometries [8].

The plane wave solutions can be derived as in Refs. [5, 7] by solving the Dirac equation in the standard representation of the Dirac matrices (with diagonal \( \gamma^0 \)). It is convenient to express here these solutions in terms of modified Bessel functions \( K_\nu \) instead of Hankel
functions as in Refs. [7,8]. Thus, by choosing new suitable phase factors we may write

\[
U_{\vec{p},\sigma}(t, \vec{x}) = \sqrt{\frac{p}{\pi \omega}} (\omega t)^2 \left( \begin{array}{c}
K_{\nu_-}(ipt) \xi_{\sigma} \\
K_{\nu_+}(ipt) \frac{\vec{p} \cdot \xi_{\sigma}}{p} \xi_{\sigma}
\end{array} \right) e^{i\vec{p} \cdot \vec{x}} \frac{(2\pi)^3}{2} \tag{11}
\]

\[
V_{\vec{p},\sigma}(t, \vec{x}) = -\sqrt{\frac{p}{\pi \omega}} (\omega t)^2 \left( \begin{array}{c}
K_{\nu_-}(-ipt) \frac{\vec{p} \cdot \eta_{\sigma}}{p} \eta_{\sigma} \\
K_{\nu_+}(-ipt) \eta_{\sigma}
\end{array} \right) e^{-i\vec{p} \cdot \vec{x}} \frac{(2\pi)^3}{2} , \tag{12}
\]

where \( p = |\vec{p}| \) and \( \nu_{\pm} = \frac{1}{2} \pm i \mu \), with \( \mu = \frac{m}{\omega} \). The the Pauli spinors \( \xi_{\sigma} \) and \( \eta_{\sigma} = i \sigma_2 (\xi_{\sigma})^* \) must be correctly normalized, \( \xi_{\sigma}^+ \xi_{\sigma'}^+ = \eta_{\sigma}^+ \eta_{\sigma'}^+ = \delta_{\sigma \sigma'} \), satisfying the completeness condition [13]

\[
\sum_{\sigma} \xi_{\sigma}^+ \xi_{\sigma} = \sum_{\sigma} \eta_{\sigma}^+ \eta_{\sigma} = 1_{2 \times 2} . \tag{13}
\]

In Ref. [5] we used the Pauli spinors of the helicity basis in which the direction of the spin projection is just that of the momentum \( \vec{p} \). However, we can project the spin on an arbitrary direction, independent on \( \vec{p} \), as in the case of the spin basis where the spin is projected on the third axis of the rest frame such that \( \xi_{\frac{1}{2}} = (1,0)^T \) and \( \xi_{-\frac{1}{2}} = (0,1)^T \) for particles and \( \eta_{\frac{1}{2}} = (0,-1)^T \) and \( \eta_{-\frac{1}{2}} = (1,0)^T \) for antiparticles [7].

The form of these spinors suggests us to introduce the auxiliary 4 \( \times \) 4 matrix-functions

\[
W_{\pm}(x) = \left( \begin{array}{cc}
K_{\nu_{\pm}}(ix) & 0 \\
0 & K_{\nu_{\pm}}(ix)
\end{array} \right), \quad \forall x \in \mathbb{R} , \tag{14}
\]

which have the obvious properties \( \tilde{W}_{\pm}(x) = W_{\pm}(x)^* = \gamma^5 W_{\mp}(-x) \gamma^5 = W_{\mp}(-x) \) and satisfy

\[
Tr [W_{\pm}(x)W_{\mp}(-x)] = \frac{1}{\pi x} , \tag{15}
\]

as it results from Eq. (A3). With their help we can write the fundamental spinors in a simpler form as

\[
U_{\vec{p},\sigma}(t, \vec{x}) = \sqrt{\frac{p}{\pi \omega}} (\omega t)^2 W_{-}(pt) \gamma(\vec{p}) u_{\sigma} \tag{16}
\]

\[
V_{\vec{p},\sigma}(t, \vec{x}) = \sqrt{\frac{p}{\pi \omega}} (\omega t)^2 W_{-}(-pt) \gamma(\vec{p}) v_{\sigma} , \tag{17}
\]

depending on the nilpotent matrix

\[
\gamma(\vec{p}) = \frac{\gamma^0 p - \vec{\gamma} \cdot \vec{p}}{p} , \tag{18}
\]
and the 4-dimensional spinors

\begin{align}
  u_\sigma &= \begin{pmatrix} \xi_\sigma \\ 0 \end{pmatrix}, \\
  v_\sigma &= \begin{pmatrix} 0 \\ \eta_\sigma \end{pmatrix}
\end{align}

(19)

that allow us to define the usual projector matrices

\begin{align}
  \pi_+ &= \sum_\sigma u_\sigma \bar{u}_\sigma = \frac{1 + \gamma^0}{2}, \\
  \pi_- &= \sum_\sigma v_\sigma \bar{v}_\sigma = \frac{1 - \gamma^0}{2},
\end{align}

(20)

that form a complete system since \( \pi_+ \pi_- = 0 \) and \( \pi_+ + \pi_- = 1 \). All these auxiliary quantities will help us to perform easily the further calculations either by using the form

\begin{align}
  W_\pm(x) &= \pi_+ K_{\nu\pm}(x) + \pi_- K_{\nu\mp}(x),
\end{align}

(21)

and simple rules as \( \gamma(\vec{p})^2 = 0 \), \( \gamma(\vec{p}) \gamma(-\vec{p}) = 2 \gamma(\vec{p}) \gamma^0 \), \( \gamma(\vec{p}) \pi_+ \gamma(\vec{p}) = \pm \gamma(\vec{p}) \), etc., or resorting to algebraic codes on computer.

In the case \( m = 0 \) (when \( \mu = 0 \)) it is convenient to consider the chiral representation of the Dirac matrices (with diagonal \( \gamma^5 \)) and the chart \( \{ t, \vec{x} \} \). Then the fundamental solutions can be written in the helicity basis of the left-handed massless Dirac field as,

\begin{align}
  U_{0\vec{p},\lambda}(t, \vec{x}) &= \lim_{\mu \to 0} \frac{1 - \gamma^5}{2} U_{\vec{p},\lambda}(t, \vec{x}) \\
  &= \left( \frac{-\omega t}{2\pi} \right)^{3/2} \begin{pmatrix} (1/2 - \lambda) \tilde{\xi}_\lambda(\vec{p}) \\ 0 \end{pmatrix} e^{-ipt + i\vec{p} \cdot \vec{x}}
\end{align}

(22)

\begin{align}
  V_{0\vec{p},\lambda}(t, \vec{x}) &= \lim_{\mu \to 0} \frac{1 - \gamma^5}{2} V_{\vec{p},\lambda}(t, \vec{x}) \\
  &= \left( \frac{-\omega t}{2\pi} \right)^{3/2} \begin{pmatrix} (1/2 + \lambda) \tilde{\eta}_\lambda(\vec{p}) \\ 0 \end{pmatrix} e^{ipt - i\vec{p} \cdot \vec{x}}
\end{align}

(23)

where \( U_{\vec{p},\lambda} \) and \( V_{\vec{p},\lambda} \) are the fundamental solutions of the helicity basis of the massive Dirac field as given in Ref. [5]. We observe that the only non-vanishing components are either of positive frequency and \( \lambda = -1/2 \) or of negative frequency and \( \lambda = 1/2 \), as in Minkowski spacetime. This is because the massless Dirac equation is conformal covariant such that the dS spinors are just the Minkowski ones multiplied with the conformal factor \((-\omega t)^{3/2}\).
III. ANTI-COMMUTATOR AND GREEN MATRIX-FUNCTIONS

Let us consider the partial anti-commutators matrix-functions of positive and negative frequencies [5],

\[ S^{(\pm)}(t, t', \vec{x} - \vec{x}') = -i\{\psi^{(\pm)}(t, \vec{x}), \bar{\psi}^{(\pm)}(t', \vec{x}')\} , \]  

which satisfy the Dirac equation in both sets of variables,

\[ (D_x - m)S^{(\pm)}(t, t', \vec{x} - \vec{x}') = S^{(\pm)}(t, t', \vec{x} - \vec{x}')(\bar{D}_{x'} - m) = 0 . \]  

The total anti-commutator matrix-function [5]

\[ S(t, t', \vec{x} - \vec{x}') = -i\{\psi(t, \vec{x}), \bar{\psi}(t', \vec{x}')\} = S^{(+)}(t, t', \vec{x} - \vec{x}') + S^{(-)}(t, t', \vec{x} - \vec{x}') \]  

has similar properties and, in addition, satisfy the equal-time condition

\[ S(t, t', \vec{x} - \vec{x}') = -i\gamma^0(-\omega t)^3\delta^4(\vec{x} - \vec{x}') \]  

resulted from Eq. (8).

In the quantum theory of fields it is important to study the Green functions related to the partial or total anti-commutator matrix-functions. We introduce the retarded (R) and advanced (A) Green functions [13],

\[ S_R(t, t', \vec{x} - \vec{x}') = \theta(t - t')S(t, t', \vec{x} - \vec{x}') \]  
\[ S_A(t, t', \vec{x} - \vec{x}') = -\theta(t' - t)S(t, t', \vec{x} - \vec{x}') \]  

while the Feynman propagator is defined in usual manner as [13],

\[ S_F(t, t', \vec{x} - \vec{x}') = -i\langle 0|T[\psi(x)\bar{\psi}(x')]|0 \rangle \]  
\[ = -\theta(t - t')S^{(+)}(t, t', \vec{x} + \vec{x}') + \theta(t' - t)S^{(-)}(t, t', \vec{x} - \vec{x}') . \]  

From the above definitions we find that these Green functions satisfy the Green equation that in the conformal chart has the form [5],

\[ (D_x - m)S_{F/R/A}(t, t', \vec{x} - \vec{x}') = S_{F/R/A}(t, t', \vec{x} - \vec{x}')(\bar{D}_{x'} - m) = (\omega t)^3\delta^4(x - x'). \]  

However, the Green equation has an infinite set of solutions corresponding to various initial conditions. Here we are interested to study only the Green functions \(S_R, S_A\) and \(S_F\) which will be called propagators in what follows.
The anti-commutator matrix-functions can be calculated with the help of the fundamental spinors (5) and (6) obtaining similar expressions,

\[ iS^{(+)}(t, t', \bar{x} - \bar{x}') = \sum_\sigma \int d^3p \, U_{\bar{\bar{\sigma}}, \sigma}(t, \bar{x}) \bar{U}_{\bar{\bar{\sigma}}, \sigma}(t', \bar{x}') \]

\[ = \frac{\omega^3}{8\pi^4} (tt')^2 \int d^3p \, pe^{i\vec{p} \cdot (\bar{x} - \bar{x}')} W_-(pt) \gamma(\bar{p}) W_+(\bar{p}t'), \quad (32) \]

\[ iS^{(-)}(t, t', \bar{x} - \bar{x}') = \sum_\sigma \int d^3p \, V_{\bar{\bar{\sigma}}, \sigma}(t, \bar{x}) \bar{V}_{\bar{\bar{\sigma}}, \sigma}(t', \bar{x}') \]

\[ = \frac{\omega^3}{8\pi^4} (tt')^2 \int d^3p \, pe^{i\vec{p} \cdot (\bar{x} - \bar{x}')} W_-(pt) \gamma(-\bar{p}) W_+(pt'), \quad (33) \]

after changing \( \vec{p} \to -\vec{p} \) in the last integral. Furthermore, we proceed as in Ref. [3] exploiting the Eqs. (A2) which allow one to write

\[ S^{(\pm)}(t, t', \bar{x} - \bar{x}') = \frac{1}{(-\omega t')} (D_x + m) \Sigma^{(\pm)}(t, t', \bar{x} - \bar{x}'), \quad (34) \]

where the new simpler matrix-functions

\[ i\Sigma^{(+)}(t, t', \bar{x} - \bar{x}') = \frac{\omega^3}{8\pi^4} (tt')^2 \int d^3p \, pe^{i\vec{p} \cdot (\bar{x} - \bar{x}')} W_+(pt) W_+(\bar{p}t'), \quad (35) \]

\[ i\Sigma^{(-)}(t, t', \bar{x} - \bar{x}') = -\frac{\omega^3}{8\pi^4} (tt')^2 \int d^3p \, pe^{i\vec{p} \cdot (\bar{x} - \bar{x}')} W_-(pt) W_+(pt'), \quad (36) \]

have the remarkable property

\[ \Sigma^{(-)}(t, t', \bar{x} - \bar{x}') = -\Sigma^{(+)}(t', t, \bar{x} - \bar{x}'). \quad (37) \]

Note that a similar representation can be written as

\[ S^{(\pm)}(t, t', \bar{x} - \bar{x}') = \bar{\Sigma}^{(\pm)}(t, t', \bar{x} - \bar{x}') (\bar{D}_x + m) \frac{1}{-\omega t'}, \quad (38) \]

by using the adjoint operator (4) and the new matrix-functions

\[ \bar{\Sigma}^{(\pm)}(t, t', \bar{x} - \bar{x}') = \gamma^5 \Sigma^{(\pm)}(t, t', \bar{x} - \bar{x}') \gamma^5. \quad (39) \]

IV. PROPAGATORS IN CONFIGURATION REPRESENTATION

The principal advantage of introducing the matrix-functions \( \Sigma^{(\pm)} \) is the opportunity of finding the expressions of the propagators in the configuration representation since their
integrals can be solved by using Eqs. (A5) \[3\]. Indeed, the integrals of Eq. (35) which are of the form
\[
I_{\pm}(t, t', \vec{x}) = \int d^3p e^{i\vec{p}\cdot\vec{x}} K_{\nu_{\pm}}(ipt) K_{\nu_{\pm}}(-ipt')
\]
(40)
can be calculated in spherical coordinates in momentum space with the third axis along $\vec{x}$. Solving first the angular integrals we remain with the radial one
\[
I_{\pm}(t, t', \vec{x}) = \frac{4\pi}{|\vec{x}|} \int_0^\infty dp K_{\nu_{\pm}}(ipt) K_{\nu_{\pm}}(-ipt') \sin p|\vec{x}|
\]
(41)
which has the form (A5). Unfortunately, the condition (A7) is not fulfilled since in this case we have $a = it$ and $b = -it'$ such that $\Re(a + b) = 0$. Therefore we must redefine these integrals substituting $t \to t - i\epsilon$ with a small $\epsilon > 0$. The new integrals
\[
I_{\pm}^\epsilon(t, t', \vec{x}) = \frac{4\pi}{|\vec{x}|} \int_0^\infty dp K_{\nu_{\pm}}(\epsilon p + ipt) K_{\nu_{\pm}}(-ipt') \sin p|\vec{x}|
\]
(42)
are now convergent and can be solved according to Eq. (A5). Thus we find the definitive form of the integrals of Eq. (35) that read
\[
I_{\pm}^\epsilon(t, t', \vec{x} - \vec{x}') = \frac{\pi^2}{2(tt')^{\frac{3}{2}}} \Gamma\left(\frac{3}{2} + \nu_{\pm}\right) \Gamma\left(\frac{3}{2} - \nu_{\pm}\right) F\left(\frac{3}{2} - \nu_{\pm}; \frac{3}{2} + \nu_{\pm}; 2; 1 + \chi_{\epsilon}\right)
\]
(43)
where
\[
\chi_{\epsilon} = \frac{(t - t' - i\epsilon)^2 - (\vec{x} - \vec{x}')^2}{4tt'},
\]
(44)
is related to the geodesic distance between the points $(t, \vec{x})$ and $(t', \vec{x}')$ \[10\]. Thus the structure of the matrix-functions $\Sigma^{(\pm)}$ and implicitly $S^{(\pm)}$ is completely determined. For example the matrix-function $\Sigma^{(+)}_{\epsilon}$ can be written in a closed form,
\[
\Sigma^{(+)}_{\epsilon}(t, t', \vec{x} - \vec{x}) = \frac{\omega^3}{16\pi^2 \sqrt{tt'}} \left[ \pi_{+} \Gamma\left(\frac{3}{2} + \nu_{+}\right) \Gamma\left(\frac{3}{2} - \nu_{+}\right) F\left(\frac{3}{2} - \nu_{+}; \frac{3}{2} + \nu_{+}; 2; 1 + \chi_{\epsilon}\right)
\right.
\]
\[
+ \pi_{-} \Gamma\left(\frac{3}{2} + \nu_{-}\right) \Gamma\left(\frac{3}{2} - \nu_{-}\right) F\left(\frac{3}{2} - \nu_{-}; \frac{3}{2} + \nu_{-}; 2; 1 + \chi_{\epsilon}\right]
\]
(45)
recovering thus the result of Ref. \[3\] for $D = 4$ and $a = (-\omega t)^{-1}$. Moreover, the matrix-function $\Sigma^{(-)}_{\epsilon}$ can be derived from Eq. (37) as
\[
\Sigma^{(-)}_{\epsilon}(t, t', \vec{x} - \vec{x}') = -\Sigma^{(+)}_{\epsilon}(t', t, \vec{x} - \vec{x}') = -\Sigma^{(+)}_{\epsilon}(t, t', \vec{x} - \vec{x}'),
\]
(46)
since the expression (45) is symmetric in $t$ and $t'$ except $\chi_{\epsilon}$ for which the change $t \leftrightarrow t'$ reduces to $\epsilon \to -\epsilon$. Finally, the matrix-functions $S^{(\pm)}$ have to be calculated according to Eqs. (34).
The massless propagators can be derived directly as the limits

$$S^{(\pm)}_{0\epsilon}(t, t', \vec{x} - \vec{x}') = \lim_{\mu \to 0} \frac{1 - \gamma^5}{2} S^{(\pm)}(t, t', \vec{x} - \vec{x}') \left( \frac{1 + \gamma^5}{2} \right).$$

(47)

According to Eq. (34) this can be put in the form

$$S^{(\pm)}_{0\epsilon}(t, t', \vec{x} - \vec{x}') = \frac{1 - \gamma^5}{2} \left[ \frac{1}{(-\omega t)} D_x \Sigma^{(\pm)}_{0\epsilon}(t, t', \vec{x} - \vec{x}') \right],$$

(48)

where

$$\Sigma^{(\pm)}_{0\epsilon}(t, t', \vec{x} - \vec{x}') = \lim_{\mu \to 0} \Sigma^{(\pm)}(t, t', \vec{x} - \vec{x}') = \pm i \frac{\omega^3}{16\pi^2} \frac{\sqrt{tt'}}{\chi_{\pm\epsilon}},$$

(since \(F(1, 2; 2; x) = (1 - x)^{-1}\) and \(\Gamma(1) = \Gamma(2) = 1\)). Thus we arrive at the simple final result

$$S^{(\pm)}_{0\epsilon}(t, t', \vec{x} - \vec{x}') = \frac{1 - \gamma^5}{2} \left[ \pm i \frac{\omega^3}{16\pi^2} \frac{1}{(-\omega t)} D_x \frac{\sqrt{tt'}}{\chi_{\pm\epsilon}} \right],$$

(49)

which is different from Eq. (25) of Ref. [3]. An interesting exercise is to recover this formula starting directly with the spinors of the helicity basis (22) and (23). A hint is given in the Appendix B.

In this section we succeeded to recover the results of Ref. [3] in the massive case for \(D = 4\) and \(a = (-\omega t)^{-1}\) but obtaining a different result for the left-handed massless fermions. However, in both these cases the propagators defined by Eqs. (28), (29) and (30) calculated in momentum representation depend explicitly by the Heaviside functions \(\theta\) such that they cannot be used for calculating transitions amplitudes in a perturbation theory as long as all the time integrals must be performed in the chronological order [13]. In the Minkowski spacetime this problem is completely solved in momentum representation where the Fourier transform of the Feynman propagator includes the effects of the chronological product according to the well-know method of contour integrals [13]. Let us see what happens in the dS case.

V. FEYNMAN PROPAGATORS

In dS spacetimes we also have a momentum representation but we do not know how to exploit it since in this geometry the propagators are functions of two time variables, \(t - t'\) and \(tt'\), instead of the unique variable \(t - t'\) of the Minkowski case. This situations generates new difficulties since apart a Fourier transform in \(t - t' \in \mathbb{R}\) a suplementary Mellin transform
for the new variable $tt' \in \mathbb{R}^+$ might be used. Obviously, this is not an effective manner for solving the problem of the Feynman propagator.

Under such circumstances, we come back to the well-known method of contour integrals proposing the following suitable integral representation of the Feynman propagator,

$$S_F(t,t',\vec{x}-\vec{x}') = \frac{\omega^3}{\pi^2} (tt')^2 \int d^3p \frac{e^{i\vec{p} \cdot (\vec{x}-\vec{x}')}}{(2\pi)^3} \int_{-\infty}^{\infty} ds |s| W_-(st) \frac{\gamma^0 s - \vec{\gamma} \cdot \vec{p}}{s^2 - p^2 + i\epsilon} W_+(-st').$$ \hspace{1cm} (50)

Here we introduced a supplemental integral in the complex $s$-plane which includes the effect of the Heaviside functions in a similar manner as in the flat case.

In order to verify that Eq. (50) is a correct representations of the Feynman propagator we observe that for large values of $s$ the we may approximate

$$|s| W_-(st)(\gamma^0 s - \vec{\gamma} \cdot \vec{p}) W_+(-st') \sim (\gamma^0 s - \vec{\gamma} \cdot \vec{p}) \frac{\pi}{2\sqrt{tt'}} e^{-is(t-t')}. \hspace{1cm} (51)$$

This asymptotic expression tends to 0 either for $t > t'$ and $\Im s \to -\infty$ or when $t < t'$ and $\Im s \to \infty$. Therefore, the integration over the real axis is equivalent either with that on the
contour $C_-$ for $t > t'$ or with the integral on the contour $C_+$ for $t' > t$. Applying the Cauchy theorem we obtain that the integral on the contour $C_-$ gives the term $-\theta(t - t')S^{(+)}$ while the integration on the contour $C_+$ yields the second term, $\theta(t' - t)S^{(-)}$. Thus we proved that the integral representation (50) gives the correct Feynman propagator (30), including the effect of the Heaviside functions.

The Feynman propagator of the left-handed massless fermions can be calculated as

$$S_0(t', \bar{x} - \bar{x}') = \lim_{\mu \to 0} \frac{1 - \gamma^5}{2} S_F(t, t', \bar{x} - \bar{x}') \left( 1 + \frac{\gamma^5}{2} \right),$$  \hspace{1cm} (52)$$

taking into account that for $\mu = 0$ we may use the particular functions (A4) arriving thus at the final result

$$S_0(t, t', \bar{x} - \bar{x}') = \frac{\omega^3}{(2\pi)^4} (tt')^\frac{3}{2} \int d^3p \int_{-\infty}^{\infty} ds \frac{1 - \gamma^5}{2} \frac{\gamma^0 s - \vec{\gamma} \cdot \vec{p}}{s^2 - p^2 + i\epsilon} e^{i\vec{p} \cdot (\bar{x} - \bar{x}) - is(t - t')},$$  \hspace{1cm} (53)$$

which is similar with that of the flat case since for $\omega \to 0$ we have $\omega t \to -1$.

The integral representations of the Feynman propagators we propose here are suitable for calculating Feynman diagrams where the integration over the supplemental variables $s$ will appear in each internal fermionic line. On the other hand, we know that the contributions of the electromagnetic field is similar as in the Minkowski case since the Maxwell equations are conformal invariant [4]. Thus after solving the space integrals generating Dirac $\delta$ functions and integrating over momenta we remain in each diagram with a time integral for each vertex and an integral for each internal line. Solving all these integrals we shall obtain the desired amplitudes in momentum representation.

**Appendix A: Modified Bessel functions and some integrals**

According to the general properties of the modified Bessel functions, $K_{\nu}(z) = K_{-\nu}(z)$ [12], we deduce that those used here, $K_{\nu_{\pm}}(z)$, with $\nu_{\pm} = \frac{1}{2} \pm i\mu$ are related among themselves through

$$[K_{\nu_{\pm}}(z)]^* = K_{\nu_{\mp}}(z^*), \hspace{1cm} \forall z \in \mathbb{C},$$  \hspace{1cm} (A1)$$

satisfy the equations

$$\left( \frac{d}{dz} + \frac{\nu_{\pm}}{z} \right) K_{\nu_{\pm}}(z) = -K_{\nu_{\mp}}(z),$$  \hspace{1cm} (A2)$$

and the identities

$$K_{\nu_{\pm}}(z)K_{\nu_{\pm}}(-z) + K_{\nu_{\pm}}(-z)K_{\nu_{\pm}}(z) = \frac{i}{\pi z},$$  \hspace{1cm} (A3)$$

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that guarantees the correct orthonormalization properties of the fundamental spinors. For
\( \mu = 0 \) we have to use the simple functions \[11\]
\[
K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}. \quad (A4)
\]
The integrals of the form \[12\] may be solved according to the following formula \[12\]
\[
\int_{0}^{\infty} dx \, x K_\nu(ax) K_\nu(bx) \sin cx = \frac{c}{8(ab)^{\frac{3}{2}}} \Gamma \left( \frac{3}{2} + \nu \right) \Gamma \left( \frac{3}{2} - \nu \right) F \left( \frac{3}{2} - \nu, \frac{3}{2} + \nu; 2; \frac{1-u}{2} \right) \quad (A5)
\]
where
\[
u = \frac{a^2 + b^2 + c^2}{2ab}. \quad (A6)
\]
Notice that these integrals are convergent for
\[
\Re(a + b) > 0 \quad (A7)
\]
if \( c \) is a real number.

Another useful integral we use here is
\[
\int_{0}^{\infty} dx \, e^{-ax} \sin bx = \frac{b}{a^2 + b^2}, \quad \Re(a) > 0. \quad (A8)
\]

**Appendix B: Projectors of helicity basis**

The propagators can be calculated in any spin basis since the result is independent on
the concrete spinors we use as long as the system of these spinors is complete satisfying Eq.
\[13\]. However, in the case of massless fermions the helcities are restricted so that the spinors
\[22\] and \[23\] are non-vanishing only for \( \lambda = -\frac{1}{2} \) and, respectively, \( \lambda = \frac{1}{2} \). In other words,
the system of spinors is incomplete such that we must use the suitable projectors when we
calculate the matrix-functions \[24\].

The normalized particle-type spinors of the helicity basis which satisfy \((\vec{\sigma} \cdot \vec{p}) \xi_\lambda(\vec{p}) = 2\lambda p \xi_\lambda(\vec{p})\) have the form
\[
\xi_{\frac{1}{2}}(\vec{p}) = \sqrt{\frac{p + p^3}{2p}} \left( \begin{array}{c} 1 \\ \frac{1}{p + p^3} \end{array} \right), \quad \xi_{-\frac{1}{2}}(\vec{p}) = \sqrt{\frac{p + p^3}{2p}} \left( \begin{array}{c} -p^3 + p^2 \\ p + p^3 \end{array} \right), \quad (B1)
\]
while the antiparticle spinors are defined usually as \(\eta_\lambda(\vec{p}) = i\sigma_2 \xi_\lambda(\vec{p})^*\) \[13\]. With their help
one may construct the projectors
\[
P_\lambda = \xi_\lambda(\vec{p}) \xi_\lambda(\vec{p})^+ = \eta_\lambda(\vec{p}) \eta_\lambda(\vec{p})^+ = \frac{1}{2} + \lambda \frac{\vec{\sigma} \cdot \vec{p}}{p}. \quad (B2)
\]
Now we have the ingredients we need for deriving the matrix-functions (24) of the massless Dirac field starting with the fundamental spinors (22) and (23). After a few manipulation we find

$$S_0^{(\pm)}(t,t',\vec{x}-\vec{x}') = i \left( \frac{\omega}{2\pi} \right)^3 (tt')^{\frac{3}{2}} \int d^3p \begin{pmatrix} 0 & P_\mp \frac{1}{2} \\ 0 & 0 \end{pmatrix} e^{\pm i\vec{p} \cdot (\vec{x}-\vec{x}') \mp ip(t-t')} ,$$  

(B3)

bearing in mind that here we work with the chiral representation of the Dirac matrices (with diagonal $\gamma^5$). Therefore, we may write

$$S_0^{(\pm)}(t,t',\vec{x}-\vec{x}') = \frac{1 - \gamma^5}{2} \left[ \pm i \left( \frac{\omega}{2\pi} \right)^3 \frac{1}{(-\omega t)} D_x(tt')^{\frac{3}{2}} I_0^{0}(t-t',\vec{x}-\vec{x}') \right]$$  

(B4)

where the integrals of the form

$$I_0^{0}(t,\vec{x}) = \int \frac{d^3p}{2p} e^{\pm i\vec{p} \cdot \vec{x} \mp ip t} = \frac{2\pi}{|\vec{x}|} \int_0^\infty dp \, e^{\pm ip t} \sin p|\vec{x}|$$  

(B5)

must be replaced by the convergent ones

$$I_0^{0}(t,\vec{x}) \rightarrow I_0^{0*}(t,\vec{x}) = \frac{2\pi}{|\vec{x}|} \int_0^\infty dp \, e^{\mp ip (t \mp i\epsilon)} \sin p|\vec{x}| = -\frac{2\pi}{(t \mp i\epsilon)^2 - |\vec{x}|^2} ,$$  

(B6)

which can be calculated according to Eq. (A8) leading to the previous result (49).

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