Corrigendum: Characteristic functions under series and parallel connection of quantum graphs (2015 J. Math. Phys. A: Math. Theor. 48 365201)

V Pivovarchik
South Ukrainian National Pedagogical University, Staroportofrankovskaya str., 26, Odessa 65020, Ukraine
E-mail: v.pivovarchik@paco.net

We found a misprint in the article ‘Characteristic functions under series and parallel connection of quantum graphs, 2015 J. Math. Phys. A: Math. Theor. 48 365201’.

Equation (5.4) must look as follows:

$$
\Phi_{NN}(\lambda) = \Delta_1(\lambda)\Delta_2(\lambda)
\times \left( \begin{array}{c} \frac{\Phi_{DD}^1(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi_{DD}^2(\lambda)}{\Delta_2(\lambda)} \\ \frac{\Phi_{DD}^1(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi_{DD}^2(\lambda)}{\Delta_2(\lambda)} \end{array} \right) \left( \begin{array}{c} \frac{\Phi_{NN}^1(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi_{NN}^2(\lambda)}{\Delta_2(\lambda)} \\ \frac{\Phi_{NN}^1(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi_{NN}^2(\lambda)}{\Delta_2(\lambda)} \end{array} \right),
$$

$$
- \left( \begin{array}{c} \frac{\Phi_{DN}^1(\lambda)}{\Delta_1(\lambda)} - \frac{\Phi_{DN}^2(\lambda)}{\Delta_2(\lambda)} \\ \frac{\Phi_{DN}^1(\lambda)}{\Delta_1(\lambda)} - \frac{\Phi_{DN}^2(\lambda)}{\Delta_2(\lambda)} \end{array} \right) \left( \begin{array}{c} \frac{\Phi_{ND}^1(\lambda)}{\Delta_1(\lambda)} - \frac{\Phi_{ND}^2(\lambda)}{\Delta_2(\lambda)} \\ \frac{\Phi_{ND}^1(\lambda)}{\Delta_1(\lambda)} - \frac{\Phi_{ND}^2(\lambda)}{\Delta_2(\lambda)} \end{array} \right).
$$
Characteristic functions under series and parallel connection of quantum graphs

V Pivovarchik

South Ukrainian National Pedagogical University, Staroportofrankovskaya str., 26, Odessa 65020, Ukraine

E-mail: v.pivovarchik@paco.net

Received 1 July 2015
Accepted for publication 22 July 2015
Published 13 August 2015

Abstract
For a graph consisting of parallel connected subgraphs we express the characteristic function of the boundary value problem with generalized Neumann conditions at both joining points via characteristic functions of different boundary problems on the subgraphs.

Keywords: Dirichlet boundary condition, Neumann boundary condition, Kirchhoff’s condition, boundary value problem, spectral parameter

1. Introduction
We consider boundary value problems generated by the Sturm–Liouville equation on a connected compact metric graph with continuity and Kirchhoff’s conditions at interior vertices and Robin or Dirichlet conditions at pendant vertices (i.e. vertices of degree 1). The potentials in the Sturm–Liouville equations and constants in the conditions are assumed to be real, and such that the corresponding operator is self-adjoint.

To describe the spectrum of a boundary value problem on metric graphs it is possible to use characteristic functions or spectral determinants.

It was proved in [2] that the characteristic function $\Phi_N(\lambda)$ of the boundary value problem on a graph $G$ which consists of two subgraphs $G_1$ and $G_2$ with the generalized Neumann (i.e. continuity + Kirchhoff’s) boundary condition at the only cut-vertex (see [11], page 54 for a definition) $v$ of $G$ satisfies

$$\Phi_N(\lambda) = \Phi_N^1(\lambda)\Phi_N^2(\lambda) + \Phi_D^1(\lambda)\Phi_D^2(\lambda),$$

(1.1)

where $\Phi_N^j(\lambda)$ ($j = 1, 2$) are the characteristic functions of boundary value problems on subgraphs $G_1$ and $G_2$ with Neumann conditions at $v$, and $\Phi_D^j(\lambda)$ are the characteristic functions of boundary value problems on $G_1$ and $G_2$ with Dirichlet condition at $v$.

Formula (1.1) was proved in [2] (theorem 2.1) for trees but the proof is quite the same for any separable graph. Earlier this formula was obtained for the so-called spectral determinants.
In case of Neumann conditions at pendant vertices and under the condition of no loops and multiple edges the spectral determinant is nothing but our characteristic function. Also it should be mentioned that in [4] a definition of spectral determinant introduced in [5] was used which is not quite accurate. The solutions \( \psi_{\alpha,\beta} \) and \( \psi_{3\alpha} \) defined by (1)–(3) in [5] not always exist, e.g. in case of \( \gamma = \alpha = 0, \beta = \pi \) (in terms of [5]) these solutions are absent. Therefore we use the characteristic functions language in the present paper to be mathematically rigorous. Formula (1.1) for the graph \( P_2 \) (a path with two edges) has been used for solving the so-called inverse three spectra problem [6] and the Hochstadt–Lieberman problem [7] and for describing spectral problems generated by equations of Stieltjes strings on trees in [8–10].

A graph of connectivity 1 (see [1], page 75 or [11], page 72 for a definition), i.e. a separable graph can be considered as a series connection of its subgraphs and (1.1) can be used.

Of course, formula (1.1) is not valid if two subgraphs are parallel connected. In this case the graph consisting of the two subgraphs has connectivity at least 2. In the present paper we consider a parallel connection of subgraphs into a graph of connectivity 2 aiming to deduce an analogue of (1.1) for parallel connection of subgraphs. The main result is presented by theorem 5.2.

Since any graph can be constructed using series and parallel connections of edges, the results of this paper can be used in the synthesis of electrical circles.

2. Characteristic functions

Characteristic functions of boundary value problems on graphs are natural generalizations of characteristic functions of boundary value problems on an interval. For an interval we use the following four characteristic functions. If \( s(\lambda, x) \) is the solution of the Sturm–Liouville equation with a real valued potential \( q(x) \in W^2_2(0, I) \):

\[-y'' + q(x)y = \lambda^2 y, \quad x \in (0, I),\tag{2.1}\]

which satisfies the conditions \( s(\lambda, 0) = s'(\lambda, 0) = 0 \) and \( c(\lambda, x) \) is the solution which satisfies \( c(\lambda, 0) - 1 = c'(\lambda, 0) = 0 \), then we call \( \Phi_{DD}(\lambda) \overset{\text{def}}{=} s(\lambda, I) \) the Dirichlet–Dirichlet characteristic function because the set zeros of \( \Phi_{DD}(\lambda) \) coincides with the spectrum of the problem generated by (2.1) and the boundary conditions

\[y(0) = y(I) = 0.\]

We call \( \Phi_{DN}(\lambda) \overset{\text{def}}{=} s'(\lambda, I) \) the Dirichlet–Neumann characteristic function because the set of zeros of \( \Phi_{DN}(\lambda) \) coincides with the spectrum of the problem generated by (2.1) and the boundary conditions

\[y(0) = y'(I) = 0,\]

we call \( \Phi_{ND}(\lambda) \overset{\text{def}}{=} c(\lambda, I) \) the Neumann–Dirichlet characteristic function because the set of zeros of \( \Phi_{ND}(\lambda) \) coincides with the spectrum of the problem generated by (2.1) and the boundary conditions

\[y'(0) = y(I) = 0,\]

we call \( \Phi_{NN}(\lambda) \overset{\text{def}}{=} c'(\lambda, I) \) the Neumann–Neumann characteristic function because the set of zeros of \( \Phi_{NN}(\lambda) \) coincides with the spectrum of the problem generated by (2.1) and the boundary conditions

\[y'(0) = y'(I) = 0.\]
Now let us consider boundary value problems on a compact connected graph. Let $G$ be a metric graph with $g$ edges. We denote by $v_i$ the vertices of $G$, by $d(v_i)$ their degrees, by $e_i$ the edges of $G$ and by $l_i$ their lengths. An arbitrary vertex $v$ is chosen as the root. Since as it will be clear below our results do not depend on the orientation of the edges, we fix an arbitrary orientation but for the sake of convenience we assume that the root $v$ has only outgoing incident edges. Local coordinates for edges identify the edge $e_j$ with the interval $[0, l_j]$ so that the local coordinate increases in the direction of the edge. This means that the root $v$ has the local coordinate 0 on each incident edge. All the other interior vertices $v$ may have outgoing edges, with local coordinates 0, and incoming edges $e_j$ with local coordinates $l_j$. Functions $y_j$ on the edges are subject to the scalar Sturm–Liouville equations:

$$y_{j}'' + q_j(x_j)y_j = \lambda^2 y_j,$$

(2.2)

where $q_j$ is a real-valued function which belongs to $L_2[0, l_j]$. For an edge $e_j$ incident to a pendant vertex which is not the root we impose self-adjoint boundary conditions

$$y_{j}'(l_j) + \beta_j y_j(l_j) = 0,$$

or

$$y_{j}(l_j) + \beta_j y_j(l_j) = 0,$$

(2.4)

where $\beta_j \in \mathbb{R} \cup \{\infty\}$. The case $\beta_j = \infty$ corresponds to Dirichlet boundary condition $y_j(0) = 0$ or $y_j(l_j) = 0$.

For each interior vertex $v$ which is not the root with incoming edges $e_j$, where $j \in E^+(v)$, $E^+(v)$ is the set of indices of incoming into $v$ edges and outgoing edges $e_k$, where $k \in E^-(v)$, $E^-(v)$ is the set of indices of outgoing from $v$ edges, continuity conditions are

$$y_j(l_j) = y_k(0), \quad j \in E^+(v), \quad k \in E^-(v)$$

(2.5)

and Kirchhoff’s condition is

$$\sum_{k \in E^-(v)} y_k'(0) = \sum_{j \in E^+(v)} y_j'(l_j).$$

(2.6)

At the root $v$, we impose the continuity conditions

$$y_j(0) = y_k(0), \quad j \in E^-(v), \quad k \in E^+(v)$$

(2.7)

for all edges $e_j$ and $e_k$ outgoing from $v$ and Kirchhoff’s condition

$$\sum_{k \in E^-(v)} y_k'(0) = 0,$$

(2.8)

where the sum in the left-hand side is taken over all edges outgoing from $v$.

We will call the pairs of conditions (2.5), (2.6) and (2.7), (2.8) generalized Neumann conditions for an interior vertex. It is clear that being imposed at a pendant vertex these conditions are reduced to the usual Neumann condition (conditions (2.3) or (2.4) with $\beta_j = 0$).

Let us denote by $s_j(\lambda, x_j)$ the solution of the Sturm–Liouville equation (2.2) on an edge $e_j$ which satisfies the conditions $s_j(\lambda, 0) = s_j'(\lambda, 0) = 0$ and by $c_j(\lambda, x_j)$ the solution which satisfies the conditions $c_j(\lambda, 0) = 1 = c_j'(\lambda, 0) = 0$. Then the characteristic function, i.e. an entire function whose zeros coincide with the spectrum of the problem can be expressed via $s_j(\lambda, l_j)$, $s_j'(\lambda, l_j)$, $c_j(\lambda, l_j)$ and $c_j'(\lambda, l_j)$. To do it we introduce the following system of vector functions $\psi_j(\lambda, \vec{x}) = (0, 0, ..., c_j(\lambda, x_j), ..., 0)^T$ and
ψ_{j+n}(λ, \bar{x}) = \{0, 0, ..., \eta_j(λ, x_j), ..., 0\}^T for j = 1, 2, ..., g, where g is the number of edges in the graph, \bar{x} = \{x_1, x_2, ..., x_g\}^T. As in [3], we denote by L_j (j = 1, 2, ..., 2g) the linear functionals generated by (2.2)–(2.8). Then \Phi(λ) = (L_j(ψ_k(λ, \bar{I})))_{j,k}^{2g} where \bar{I} = \{t_1, t_2, ..., t_g\}^T is the characteristic matrix which represents the system of linear equations describing the continuity and Kirchhoff conditions for the interior vertices. Then we call

\phi_N(λ) := \det(Φ(λ))

the characteristic function of problem (2.2)–(2.8). We are interested also in the problem generated by the same equations and all boundary and matching conditions the same, but with the condition

y_j(0) = 0 \quad j \in E^-(v) (2.9)

instead of (2.8) at v. We denote the characteristic function of problem (2.2)–(2.7), (2.9), by \Phi_D(λ). We call (2.7), (2.9) the generalized Dirichlet condition. In case when v is a pendant vertex, conditions (2.7), (2.9) are reduced to the usual Dirichlet boundary condition. Let us assume that the graph G is separable, i.e. of connectivity 1 and let the root v be the second vertex incident with e_g (see figure 1). We divide our graph G into two subgraphs G_1 and G_2 having the only common vertex v. Denote by \Phi'_i(λ) (i = 1, 2) the characteristic function corresponding to subgraph G_i with Neumann condition at v and by \Phi''_i(λ) the characteristic function corresponding to subgraph G_i with Dirichlet condition at v.

Formula (1.1) describes series connection of subgraphs. In order to describe the characteristic functions for series and parallel connected subgraphs we need to generalize the notions of \Phi_{DD}, \Phi_{DN}, \Phi_{ND} and \Phi_{NN} introduced in the beginning of this section for boundary value problems on an interval.

Let the vertices v_in and v_out be pendant vertices and let the edge e_in be outgoing away from v_in and the edge e_out be incoming to v_out. Denote by \(w \neq v_{\text{out}}\) the second vertex incident with e_out. Let w have no incoming edges and denote the outgoing from w edges by e_{out} = \{e_2, ..., e_{out}\}.

Let us consider the Dirichlet–Dirichlet problem (2.2)–(2.7), (2.9) which in our terms is as follows:

\[ y_1(0) = 0 \quad (2.10) \]
at \( v_m \),
\[
y_j'(0) + \beta_j y_j(0) = 0, \tag{2.11}
\]
or
\[
y_j'(l_j) + \beta_j y_j(l_j) = 0, \tag{2.12}
\]
at each pendant vertex except of \( v_h \) and \( v_{\text{out}} \). For each interior vertex \( v \) being not \( w \) with incoming into edges \( e_j \) and outgoing \( v \) edges \( e_k \) the continuity conditions are
\[
y_j(l_j) = y_k(0), \quad j \in E^+(v), \quad k \in E^-(v)
\tag{2.13}
\]
and Kirchhoff’s condition is
\[
\sum_{k \in E^-(v)} y_k'(0) = \sum_{j \in E^+(v)} y_j'(l_j). \tag{2.14}
\]
At \( w \) we have
\[
y_{g-d(w)+1}'(0) = y_{g-d(w)+2}(0) = \ldots = y_{g-1}'(0) = y_g(0), \tag{2.15}
\]
and
\[
y_k'(0) = -\sum_{k=g-d(w)+1}^{g-1} y_k'(0) \tag{2.16}
\]
and
\[
y_k(l_k) = 0 \tag{2.17}
\]
at \( v_{\text{out}} \).
We substitute
\[
y_i = B_i c_i(\lambda, x_i) + A_i s_i(\lambda, x_i)
\]
into (2.10)–(2.17) and obtain
\[
B_1 = 0, \tag{2.18}
\]
\[
A_j + \beta_j B_j = 0, \tag{2.19}
\]
\[
B_j c_j(\lambda, l_j) + A_j s_j(\lambda, l_j) + \beta_j \left(B_j c_j(\lambda, l_j) + A_j s_j(\lambda, l_j)\right) = 0. \tag{2.20}
\]
We can also write the continuity and Kirchhoff’s condition at \( \tilde{v} \) as
\[
B_{g-d(w)+1} = B_{g-d(w)+2} = \ldots = B_{g-1}, \tag{2.21}
\]
\[
B_{g-1} = B_g, \tag{2.22}
\]
\[
A_g = -\sum_{k=g-d(w)+1}^{g-1} A_k, \tag{2.23}
\]
\[
B_g c_g(\lambda, l_g) + A_g s_g(\lambda, l_g) = 0. \tag{2.24}
\]
These equations can be written in a matrix form:

\[
\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
\end{pmatrix}
\begin{pmatrix}
B_1 \\
B_2 \\
\vdots \\
B_{g-1} \\
A_1 \\
A_2 \\
\vdots \\
A_{g-1}
\end{pmatrix}
= \begin{pmatrix}
0 \\
\ddots \\
\ddots \\
\ddots \\
\cdots \\
\cdots \\
\cdots \\
\end{pmatrix}.
\] (2.25)

Here \( s_j = s_j(\lambda, l_j), s'_j = s'_j(\lambda, l_j), c_j = c_j(\lambda, l_j), c'_j = c'_j(\lambda, l_j) \).

The determinant of the matrix in (2.24) is \( \Phi_{DP}(\lambda) \).

Let us delete equations (2.18), (2.22), (2.23) and (2.24) from (2.25) and set \( B_1 = B_g = A_g = 0 \) and \( A_1 = 1 \). Then we obtain a nonhomogeneous \((g-4) \times (g-4)\) system of linear algebraic equation with respect to unknowns \( B_2, B_3, \ldots, B_{g-1}, A_2, A_3, \ldots, A_{g-1} \):

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A_2 \\
\vdots \\
A_{g-1}
\end{pmatrix}
= \begin{pmatrix}
\times \\
\times \\
\times \\
\times \\
\times
\end{pmatrix}.
\] (2.26)

Definition 2.1. Let \( G \) be a connected graph with pendant vertices \( v_{in} \) and \( v_{out} \) and let \( L \) be the matrix of the Dirichlet–Dirichlet problem (2.10)–(2.17) on this graph (the matrix in (2.25)). We denote by \( \Delta(\lambda) \) the determinant of the principal submatrix of the matrix in (2.25) obtained by deleting the rows and columns number 1, \( g, g+1 \) and \( 2g \) (i.e. the matrix in (2.26)) and call it the reduced determinant of the graph \( G \).

Solving this system we find

\[
B_{g-1}^D = \frac{\Delta_{B_{g-1}}^D(\lambda)}{\Delta(\lambda)}, \quad A_{g-k}^D = \frac{\Delta_{A_{g-k}}^D(\lambda)}{\Delta(\lambda)} \quad k = 1, 2, \ldots, d(w) - 1,
\] (2.27)

where \( \Delta_{B_{g-1}}^D \) and \( \Delta_{A_{g-k}}^D \) are the corresponding cofactors:

\[
\Delta_{B_{g-1}}^D(\lambda) = \det \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}.
\]
\[ \Delta^D_{A_{g-1}}(\lambda) = \det \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \]

\[ \Delta^D_{A_{g-2}}(\lambda) = \det \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \]

and so on up to \( \Delta^D_{A_1}(\lambda) \).

We use upper index \( D \) to underline that the formulae correspond to the case of Dirichlet boundary condition at \( v_m \). Notice that \( \Delta \) does not depend on the condition at \( v_m \) while it is the determinant of the matrix obtained from the matrix of (2.27) by deleting among others the first column and the \((g+1)\) th column.

On the other hand, it is easy to notice that these cofactors appear in the expansion of the determinant

\[ \Phi_{DD}(\lambda) = c_g(\lambda, l_g) \Delta^D_{B_{g-1}}(\lambda) + s_g(\lambda, l_g) \sum_{k=g-d(v)+1}^{g-1} \Delta^D_{A_k}(\lambda) \]  \hspace{1em} (2.28)

because it is easy to see that \( \Delta^D_{B_{g-1}}(\lambda) = \Delta^D_{B_{g-1}}(\lambda) \) and \( \Delta^D_{A_k}(\lambda) = \Delta^D_{A_k}(\lambda) \).

In the same way we arrive at

\[ \Phi_{DN}(\lambda) = c'_g(\lambda, l_g) \Delta^D_{B_{g-1}}(\lambda) + s'_g(\lambda, l_g) \sum_{k=g-d(v)+1}^{g-1} \Delta^D_{A_k}(\lambda). \]  \hspace{1em} (2.29)

Now we consider the same problem but with Neumann condition \( y'_j(0) = 0 \) at \( v_m \). Then we obtain \( A_1 = 0 \) instead of (2.18) and

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
B_1 \\
B_2 \\
\vdots \\
B_{g-1} \\
B_g \\
A_1 \\
A_2 \\
\vdots \\
A_{g-1} \\
A_g
\end{pmatrix} = 0
\]
instead of (2.25). In the same way as (2.27) we obtain
\[
B_{k-1}^N = \frac{\Delta_{\mathcal{H}_{k-1}}^N(\lambda)}{\Delta(\lambda)}, \quad A_{k}^N = \frac{\Delta_{\mathcal{A}_{k-1}}^N(\lambda)}{\Delta(\lambda)}, \quad k = 1, 2, \ldots, d(w) - 1, \tag{2.30}
\]
where \(\Delta_{\mathcal{H}_{k-1}}^N\) and \(\Delta_{\mathcal{A}_{k-1}}^N\) are the corresponding cofactors. Upper index \(N\) we use to underline that the formulae correspond to the case of Neumann boundary condition at \(v\). Also we have
\[
\Phi_{\text{ND}}(\lambda) = c_{\text{ND}}(\lambda, l_g) \Delta_{\mathcal{H}_{k-1}}^N(\lambda) + s_{\text{ND}}(\lambda, l_g) \sum_{k=g-d(w)+1}^{g-1} \Delta_{\mathcal{A}_k}^N(\lambda),
\]
\[
\Phi_{\text{NN}}(\lambda) = c_{\text{NN}}(\lambda, l_g) \Delta_{\mathcal{H}_{k-1}}^N(\lambda) + s_{\text{NN}}(\lambda, l_g) \sum_{k=g-d(w)+1}^{g-1} \Delta_{\mathcal{A}_k}^N(\lambda).
\]

3. Series connection

Let us consider two connected graphs \(G_j (j = 1, 2)\). We choose two vertices \(v_{\text{in}}^j\) and \(v_{\text{out}}^j\) in each of them as the entrance and exit vertices. In this Section we investigate series connection of these graphs. We denote by \(\Phi_{\text{NN}}^j\) the characteristic function of the boundary value problem on the graph \(G_j\) with Neumann boundary conditions at \(v_{\text{in}}^j\) and \(v_{\text{out}}^j\). At all the other interior vertices generalized Neumann conditions (continuity and Kirchhoff conditions) are imposed while any self-adjoint conditions of the form (2.3) or (2.4) are imposed at pendant vertices. In the same way, we denote by \(\Phi_{\text{ND}}^j\) the characteristic function of the boundary value problem on the graph \(G_j\) with Neumann boundary condition at \(v_{\text{in}}^j\) and Dirichlet boundary condition at \(v_{\text{out}}^j\), by \(\Phi_{\text{DN}}^j\) the characteristic function of the boundary value problem on the graph \(G_j\) with Dirichlet boundary condition at \(v_{\text{in}}^j\) and Neumann boundary condition at \(v_{\text{out}}^j\), respectively.

If we connect \(v_{\text{out}}^1\) with \(v_{\text{in}}^2\), then we obtain a new graph \(G = G_1 \cup G_2\) with a cut-vertex \(v_{\text{in}}^1 = v_{\text{out}}^2\). Let us denote by \(\Phi_{\text{NN}}^1\), \(\Phi_{\text{ND}}^1\), \(\Phi_{\text{DN}}^1\) and \(\Phi_{\text{DD}}^1\) the characteristic functions of the problems on \(G\) with Neumann conditions at \(v_{\text{in}}^1\) and \(v_{\text{out}}^2\), with Neumann conditions at \(v_{\text{in}}^1\) and Dirichlet at \(v_{\text{out}}^2\), with Dirichlet condition at \(v_{\text{in}}^1\) and Neumann at \(v_{\text{out}}^2\) and with Dirichlet conditions at \(v_{\text{in}}^1\) and \(v_{\text{out}}^2\), respectively.

Theorem 2.1 of [2] mentioned in Introduction applied in our case gives us

Theorem 3.1.
\[
\Phi_{\text{NN}}(\lambda) = \Phi_{\text{NN}}^1(\lambda)\Phi_{\text{DD}}^2(\lambda) + \Phi_{\text{ND}}^1(\lambda)\Phi_{\text{SN}}^2(\lambda), \tag{3.1}
\]
\[
\Phi_{\text{ND}}(\lambda) = \Phi_{\text{NN}}^1(\lambda)\Phi_{\text{DD}}^2(\lambda) + \Phi_{\text{ND}}^1(\lambda)\Phi_{\text{SN}}^2(\lambda), \tag{3.2}
\]
\[
\Phi_{\text{DN}}(\lambda) = \Phi_{\text{DN}}^1(\lambda)\Phi_{\text{DD}}^2(\lambda) + \Phi_{\text{DD}}^1(\lambda)\Phi_{\text{SN}}^2(\lambda), \tag{3.3}
\]
\[
\Phi_{\text{DD}}(\lambda) = \Phi_{\text{DN}}^1(\lambda)\Phi_{\text{DD}}^2(\lambda) + \Phi_{\text{DD}}^1(\lambda)\Phi_{\text{SN}}^2(\lambda). \tag{3.4}
\]
Using these identities we immediately obtain an analogue of the Lagrange identity:

Corollary 3.1.
\[
\Phi_{\text{ND}}(\lambda)\Phi_{\text{DN}}(\lambda) - \Phi_{\text{NN}}(\lambda)\Phi_{\text{DD}}(\lambda) \tag{3.5}
\]
\begin{equation}
= \left( \Phi_{ND}(\lambda)\Phi_{DN}(\lambda) - \Phi_{NN}(\lambda)\Phi_{DD}(\lambda) \right) \left( \Phi_{SN}(\lambda)\Phi_{SN}(\lambda) - \Phi_{SN}(\lambda)\Phi_{SN}(\lambda) \right).
\end{equation}

4. Auxiliary results

Denote by \( s_{ij}(\lambda, x_{ij}) \) the solution of the equation
\begin{equation}
-\gamma_{ij}'' + g_{ij}(x_{ij})y_{ij} = \lambda^2 y_{ij}, \quad j = 1, 2
\end{equation}
on the edge of \( G_j \) incident with \( v^j_{in} \) which satisfies \( s_{ij}(\lambda, 0) = s_{ij}(\lambda, 0) - 1 = 0 \) and by \( c_{ij}(\lambda, x_{ij}) \) the solution which satisfies \( c_{ij}(\lambda, 0) - 1 = c'_{ij}(\lambda, 0) = 0 \).

Let us consider a boundary value problem on \( G_j \) which consists of equations
\begin{equation}
-\gamma_{ij}'' + g_{ij}(x_{ij})y_{ij} = \lambda^2 y_{ij}, \quad i = 1, 2, \ldots, g_i,
\end{equation}
continuity and Kirchhoff conditions at all interior vertices, conditions \( y_{ij}(0) = y'_{ij}(0) - 1 = 0 \) at \( v^j_{in} \) and no condition at \( v^j_{out} \). There exists a solution of this problem, maybe not unique, of the form \( Y_j(\lambda, \vec{x}_i) = (y_{ij,1}(\lambda, x_{ij,1}), y_{ij,2}(\lambda, x_{ij,2}), y_{ij,3}(\lambda, x_{ij,3}), \ldots, y_{ij, g_i}(\lambda, x_{ij, g_i}))^T \), where \( \vec{x}_i = [x_{ij,1}, x_{ij,2}, \ldots, x_{ij, g_i}]^T \) is the coordinate vector corresponding to \( G_j \). In the same way we define the solution of the problem generated by equations (4.1), conditions \( y_{ij}(0) - 1 = y'_{ij}(0) = 0 \) at \( v^j_{in} \) and no condition at \( v^j_{out} \) : \( U_j(\lambda, \vec{x}_i) = (u_{ij,1}(\lambda, x_{ij,1}), u_{ij,2}(\lambda, x_{ij,2}), u_{ij,3}(\lambda, x_{ij,3}), \ldots, u_{ij, g_i}(\lambda, x_{ij, g_i}))^T \).

If we substitute \( Y_j(\vec{x}_i) \) into equations (2.10)–(2.16) written for \( G_j \), we obtain
\begin{equation}
y_{g_i,1}(\lambda, x_{g_i,1}) = \frac{\Delta_{R_{g_i,1}}^N(\lambda)}{\Delta(\lambda)} - c_{g_i,1}(\lambda, x_{g_i,1}) + \sum_{k=g-d(u)+1}^{d-1} \frac{\Delta_{R_{g_i,1}}^N(\lambda)}{\Delta(\lambda)} s_{g_i,1}(\lambda, x_{g_i,1}),
\end{equation}
where \( \Delta(\lambda) \) is the reduced determinant of \( G_i \).

In the same way
\begin{equation}
u_{g_i,1}(\lambda, x_{g_i,1}) = \frac{\Delta_{N_{g_i,1}}^N(\lambda)}{\Delta(\lambda)} - c_{g_i,1}(\lambda, x_{g_i,1}) + \sum_{k=g-d(u)+1}^{d-1} \frac{\Delta_{N_{g_i,1}}^N(\lambda)}{\Delta(\lambda)} s_{g_i,1}(\lambda, x_{g_i,1}).
\end{equation}

Similarly we define the corresponding solutions for \( G_2 : \ Y_2(\lambda, \vec{x}_2) = (y_{2,1}(\lambda, x_{2,1}), y_{2,2}(\lambda, x_{2,2}), y_{2,3}(\lambda, x_{2,3}), \ldots, y_{2, g_2}(\lambda, x_{2, g_2}))^T \) which satisfies equations
\begin{equation}
-\gamma_{2,ij}'' + g_{2,ij}(x_{2,ij})y_{2,ij} = \lambda^2 y_{2,ij}, \quad i = 1, 2, \ldots, g_2,
\end{equation}
and conditions \( y_{2,ij}(0) = y'_{2,ij}(0) - 1 = 0 \) at \( v^2_{in} \) and no condition at \( v^2_{out} \) and \( U_2(\lambda, \vec{x}_2) = (u_{2,1}(\lambda, x_{2,1}), u_{2,2}(\lambda, x_{2,2}), u_{2,3}(\lambda, x_{2,3}), \ldots, u_{2, g_2}(\lambda, x_{2, g_2}))^T \) which satisfies \( u_{2,ij}(0) - 1 = u'_{2,ij}(0) = 0 \) at \( v^2_{in} \) and no condition at \( v^2_{out} \). Then
\begin{equation}
y_{g_2,1}(\lambda, x_{g_2,1}) = \frac{\Delta_{R_{g_2,1}}^D(\lambda)}{\Delta(\lambda)} - c_{g_2,1}(\lambda, x_{g_2,1}) + \sum_{k=g-d(u)+1}^{d-1} \frac{\Delta_{R_{g_2,1}}^D(\lambda)}{\Delta(\lambda)} s_{g_2,1}(\lambda, x_{g_2,1}),
\end{equation}
\begin{equation}
u_{g_2,1}(\lambda, x_{g_2,1}) = \frac{\Delta_{N_{g_2,1}}^D(\lambda)}{\Delta(\lambda)} - c_{g_2,1}(\lambda, x_{g_2,1}) + \sum_{k=g-d(u)+1}^{d-1} \frac{\Delta_{N_{g_2,1}}^D(\lambda)}{\Delta(\lambda)} s_{g_2,1}(\lambda, x_{g_2,1}),
\end{equation}
where \( \Delta(\lambda) \) is the reduced determinant of \( G_2 \).
5. Parallel connection

Now we consider parallel connection of $G_1$ with $G_2$, i.e. a graph $G$ obtained by connection $v_{in}^1$ with $v_{out}^1$ and $v_{in}^2$ with $v_{out}^2$. To simplify situation let us assume that $v_{in}^j$ and $v_{out}^j$ are pendant vertices of $G_j$ for $j = 1, 2$, respectively, (see figure 2).

Let us denote by $\Phi_{NN}(\lambda)$, $\Phi_{ND}(\lambda)$, $\Phi_{DN}(\lambda)$ and $\Phi_{DD}(\lambda)$ the characteristic functions of the problems on $G$ with Neumann conditions at $v_{in}^1 = v_{in}^2$ and $v_{out}^1 = v_{out}^2$, with Neumann conditions at $v_{in}^1 = v_{in}^2$, and Dirichlet at $v_{out}^1 = v_{out}^2$, with Dirichlet condition at $v_{in}^1 = v_{in}^2$ and Neumann at $v_{out}^1 = v_{out}^2$ and with Dirichlet conditions at $v_{in}^1 = v_{in}^2$ and at $v_{out}^1 = v_{out}^2$, respectively.

Again theorem 2.1 of [2] mentioned in introduction used in our terms gives us

\[
\Phi_{DD}(\lambda) = \Phi_{DD}^1(\lambda)\Phi_{DD}^2(\lambda),
\]

\[
\Phi_{DN}(\lambda) = \Phi_{DN}^1(\lambda)\Phi_{DD}^2(\lambda) + \Phi_{DD}^1(\lambda)\Phi_{DN}^2(\lambda),
\]

\[
\Phi_{ND}(\lambda) = \Phi_{ND}^1(\lambda)\Phi_{DD}^2(\lambda) + \Phi_{DD}^1(\lambda)\Phi_{ND}^2(\lambda).
\]

Our aim is to express $\Phi_{NN}(\lambda)$ via $\Phi_{NN}^1(\lambda)$, $\Phi_{ND}^i(\lambda)$, $\Phi_{DN}^i(\lambda)$ and $\Phi_{DD}^i(\lambda)$ ($i = 1, 2$).

**Theorem 5.2.** Let a connected graph $G$ consist of two parallel connected subgraphs $G_1$ and $G_2$ and let $G_1$ and $G_2$ have only two common vertices which are pendant if we consider $G_1$ and $G_2$ separately. Then the characteristic function of Neumann–Neumann problem on $G$

\[
\Phi_{NN}(\lambda) = \Delta_1(\lambda)\Delta_2(\lambda) \left( \frac{\Phi_{DN}^1(\lambda)}{\Delta_1(\lambda)} - \frac{\Phi_{DN}^2(\lambda)}{\Delta_2(\lambda)} \right) \left( \frac{\Phi_{ND}^1(\lambda)}{\Delta_1(\lambda)} - \frac{\Phi_{ND}^2(\lambda)}{\Delta_2(\lambda)} \right),
\]

(5.4)
where $\Delta_l(\lambda)$ are the reduced determinants of the subgraphs $G_j$ $(j=1,2)$, $\Phi^i_{\text{ND}}$ and $\Phi^i_{\text{NN}}$ $(j=1,2)$ are the characteristic functions of the Neumann–Dirichlet and Dirichlet–Neumann problems on $G_j$, respectively.

**Proof.** Let us look for solution of Neumann–Neumann problem on $G$ in the form \( R_1 Y_1 + R_3 U_1, R_2 Y_2 + R_3 U_2 \), where $R_j$ are constants. Then the continuity condition at $v^1_{\text{in}} = v^2_{\text{in}}$ is

\[
R_1 = R_2.
\]

The Kirchhoff condition at $v^1_{\text{in}} = v^2_{\text{in}}$ is

\[
R_1 = -R_2.
\]

Continuity condition at $v^1_{\text{out}} = v^2_{\text{out}}$ is

\[
R_1 y_{v_{1,1}}(\lambda, l_{v_{1,1}}) + R_3 u_{v_{1,1}}(\lambda, l_{v_{1,1}}) = R_2 y_{v_{2,2}}(\lambda, l_{v_{2,2}}) + R_2 u_{v_{2,2}}(\lambda, l_{v_{2,2}}).
\]

The Kirchhoff condition at $v^1_{\text{out}} = v^2_{\text{out}}$ is

\[
R_1 y_{v_{1,1}}(\lambda, l_{v_{1,1}}) + R_3 u_{v_{1,1}}(\lambda, l_{v_{1,1}}) + R_2 y_{v_{2,2}}(\lambda, l_{v_{2,2}}) + R_2 u_{v_{2,2}}(\lambda, l_{v_{2,2}}) = 0.
\]

It is clear that if $y_{v_{1,1}}(\lambda, l_{v_{1,1}}) = 0$, then $Y_1$ is an eigenvector of the problem on $G_1$ with Dirichlet conditions at $v^1_{\text{in}}$ and at $v^1_{\text{out}}$. It means that the set of zeros of $y_{v_{1,1}}(\lambda, l_{v_{1,1}})$ coincides with the set of zeros of $\Phi^i_{\text{DD}}(\lambda)$.

Using (2.29), (2.30) we obtain from (4.2), (4.3), (4.5) and (4.6) that

\[
y_{v_{1,1}}(\lambda, l_{v_{1,1}}) = \frac{\Phi^i_{\text{DD}}(\lambda)}{\Delta_1(\lambda)} u_{v_{1,1}}(\lambda, l_{v_{1,1}}) = \frac{\Phi^i_{\text{ND}}(\lambda)}{\Delta_1(\lambda)} y_{v_{2,2}}(\lambda, l_{v_{2,2}}) = \frac{\Phi^i_{\text{NN}}(\lambda)}{\Delta_1(\lambda)} u_{v_{2,2}}(\lambda, l_{v_{2,2}}) = \frac{\Phi^i_{\text{NN}}(\lambda)}{\Delta_2(\lambda)} y_{v_{1,1}}(\lambda, l_{v_{1,1}}) = \frac{\Phi^i_{\text{DD}}(\lambda)}{\Delta_2(\lambda)} u_{v_{1,1}}(\lambda, l_{v_{1,1}}) = \frac{\Phi^i_{\text{NN}}(\lambda)}{\Delta_2(\lambda)} y_{v_{2,2}}(\lambda, l_{v_{2,2}}) = \frac{\Phi^i_{\text{NN}}(\lambda)}{\Delta_2(\lambda)} u_{v_{2,2}}(\lambda, l_{v_{2,2}}).
\]

The determinant of system (5.5)–(5.8) is

\[
D(\lambda) \overset{\text{def}}{=} \text{det} \begin{bmatrix}
0 & 0 & 1 & -1 \\
1 & 1 & 0 & 0 \\
\Phi^i_{\text{DD}}(\lambda)/\Delta_1(\lambda) & \Phi^i_{\text{DD}}(\lambda)/\Delta_2(\lambda) & \Phi^i_{\text{ND}}(\lambda)/\Delta_1(\lambda) & \Phi^i_{\text{ND}}(\lambda)/\Delta_2(\lambda) \\
\Phi^i_{\text{NN}}(\lambda)/\Delta_1(\lambda) & \Phi^i_{\text{NN}}(\lambda)/\Delta_2(\lambda) & \Phi^i_{\text{NN}}(\lambda)/\Delta_1(\lambda) & \Phi^i_{\text{NN}}(\lambda)/\Delta_2(\lambda)
\end{bmatrix}
\]

\[
= \left( \frac{\Phi^i_{\text{DD}}(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi^i_{\text{DD}}(\lambda)}{\Delta_2(\lambda)} \right) \left( \frac{\Phi^i_{\text{ND}}(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi^i_{\text{NN}}(\lambda)}{\Delta_2(\lambda)} \right)
\]

\[
-(\frac{\Phi^i_{\text{NN}}(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi^i_{\text{NN}}(\lambda)}{\Delta_2(\lambda)}) \left( \frac{\Phi^i_{\text{DD}}(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi^i_{\text{DD}}(\lambda)}{\Delta_2(\lambda)} \right).
\]

It is clear that to obtain the characteristic function $\Phi_{\text{NN}}(\lambda)$ the determinant (5.6) must be multiplied by $(\Delta_1(\lambda)\Delta_2(\lambda))^p$ (with some $p \geq 1$) to be an entire function. Let us show that this $p$ must be 1.

To prove it we notice that $\Phi^i_{\text{DD}}(\lambda)$ is an entire function of exponential type $L^i = \sum_{k=1}^N l_k^i$, where $l_k^i$ are the lengths of the edges of the subgraph $G_j$ and $g_j$ is their
number. The function \( \Delta_j(\lambda) \) is an entire function of exponential type \( \sum_{k=2}^{6} t_k^{(j)} = L(t) - L^{(j)} - L_{x_j}^{(j)}. \)

The graph \( G_j \) consists of two edges \( e_1 \) and \( e_{x_j} \) and the subgraph \( G_0 \) (see figure 2) which are series connected. Then for \( e_1^{(j)} \) and \( e_{x_j}^{(j)} \) we have
\[
c_1j\left( \lambda, l_1^{(j)} \right) s_{1j}(\lambda, l_1^{(j)}) = 1, \tag{5.10}
\]
\[
c_{x_j}j\left( \lambda, l_{x_j}^{(j)} \right) s_{x_jj}(\lambda, l_{x_j}^{(j)}) = 1. \tag{5.11}
\]

Applying the analogue of Lagrange identity for series connected subgraphs (3.5) and using (5.7) and (5.8) we obtain
\[
\Phi_{ND}^{(j)}(\lambda) \Phi_{DN}^{(j)}(\lambda) - \Phi_{NN}^{(j)}(\lambda) \Phi_{DD}^{(j)}(\lambda)
\]
\[
= \Phi_{ND}^{(0)}(\lambda) \Phi_{DN}^{(0)}(\lambda) - \Phi_{NN}^{(0)}(\lambda) \Phi_{DD}^{(0)}(\lambda),
\]
where quantities with the zero upper index correspond to \( G_0 \). Thus \( \Phi_{ND}^{(j)}(\lambda) \Phi_{DN}^{(j)}(\lambda) - \Phi_{NN}^{(j)}(\lambda) \Phi_{DD}^{(j)}(\lambda) \) is an entire function of exponential type \( L(t) - L^{(j)} - L_{x_j}^{(j)}. \)

Taking into account that \( \Delta_0(-\lambda) = \Delta_0(\lambda) \) and \( \Delta_0(\lambda) = \Delta(\lambda) \) we conclude that
\[
\frac{\Phi_{ND}^{(j)}(\lambda) \Phi_{DN}^{(j)}(\lambda) - \Phi_{NN}^{(j)}(\lambda) \Phi_{DD}^{(j)}(\lambda)}{\Delta_0(\lambda)} \rightarrow C, \quad \text{Im} \lambda \to \infty, \tag{5.12}
\]
where \( C \) is a real constant. Therefore, using (5.6) we obtain
\[
\Delta_1(\lambda) \Delta_2(\lambda) D(\lambda)
\]
\[
= \frac{\Phi_{ND}^{1}(\lambda) \Phi_{DN}^{1}(\lambda) - \Phi_{NN}^{1}(\lambda) \Phi_{DD}^{1}(\lambda)}{\Delta_1(\lambda)} \Delta_2(\lambda)
\]
\[
+ \Phi_{NN}^{1}(\lambda) \Phi_{DD}^{1}(\lambda) + \Phi_{DD}^{1}(\lambda) \Phi_{NN}^{1}(\lambda) + \Phi_{NN}^{2}(\lambda) \Phi_{DD}^{1}(\lambda) + \Phi_{DD}^{2}(\lambda) \Phi_{NN}^{1}(\lambda)
\]
\[
+ \Phi_{NN}^{2}(\lambda) \Phi_{DD}^{2}(\lambda) - \Phi_{NN}^{2}(\lambda) \Phi_{DD}^{2}(\lambda) \Delta_1(\lambda).
\]

Due to (5.12) and (5.13) we conclude that \( \Delta_1(\lambda) \Delta_2(\lambda) D(\lambda) \) is a meromorphic function with the set of zeros which coincides with the set of zeros of the entire function \( \Phi_{NN}(\lambda) \), i.e. the characteristic function of the Neumann–Neumann problem on the whole graph. To prove this we notice that
\[
\left| \Phi_{NN}(\lambda) \right| \left| \frac{\Delta_1(\lambda) \Delta_2(\lambda) D(\lambda)}{\Delta_1(\lambda) \Delta_2(\lambda)} \right| \left| e^{Im \lambda L(1 + o(1))} \right|
\]
where \( L = L^{(1)} + L^{(2)} = \sum_{k=1}^{p_1} n_k^{(1)} + \sum_{k=1}^{p_2} n_k^{(2)}, \) \( p_j \) is the number of vertices in \( G_j \) and the number of vertices in \( G \) is \( p_1 + p_2 - 2. \) Due to (5.12) and (5.14) we obtain
\[
\left| \Delta_1(\lambda) \Delta_2(\lambda) D(\lambda) \right| \left| e^{Im \lambda L(1 + o(1))} \right|
\]
The theorem is proved.
For the case of \( m \) parallel connected subgraphs we obtain

\[
\Phi_{NN} = \begin{pmatrix}
0 & 0 & \ldots & \ldots & 0 & 1 & -1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 & 1 & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & 0 & 1 & 0 & \ldots & 0 & -1 \\
1 & 1 & \ldots & \ldots & 1 & 0 & 0 & \ldots & \ldots & 0 \\
-\Phi_{DD} - \Phi_{DD}^2 & 0 & \ldots & 0 & \Phi_{ND} & \Phi_{ND}^2 & 0 & \ldots & 0 & \Phi_{NN} \\
-\Phi_{DD} & 0 & \ldots & 0 & \Phi_{ND} & \Phi_{ND}^2 & 0 & \ldots & 0 & \Phi_{NN} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_{DN} & 0 & \ldots & \ldots & \Phi_{DN}^m & \Phi_{DN}^2 & \Phi_{NN} & \ldots & \ldots & \Phi_{NN}^m \\
\Phi_{DN} & 0 & \ldots & \ldots & \Phi_{DN}^m & \Phi_{DN}^2 & \Phi_{NN} & \ldots & \ldots & \Phi_{NN}^m \\
\end{pmatrix}
\]

\[\prod_{j=1}^m \Delta_j.\]

Formulae (3.1)–(3.5) and (5.1)–(5.3), (5.14) remain true for problems generated by finite dimensional analogue of Sturm–Liouville equation, so-called Stieltjes string equation.

References

[1] Narasingh D 1974 *Graph Theory with Application to Engineering and Computer Science* ed A K Ghosh (New Delhi: Prentice-Hall)
[2] Law C-K and Pivovarchik V 2009 Characteristic functions of quantum graphs *J. Phys. A: Math. Theor.* 42 035302
[3] Pokornyj Y V, Penkin O M, Pryadiev I, Borovskikh A V, Lazarev K P and Shabrov S A 2004 *Differential Equations on Geometric Graphs* (Moscow: Fizmatgiz) in Russian
[4] Texier C 2008 On the spectrum of the Laplace operator of metric graphs attached at a vertex. Spectral determinant approach *J. Phys. A: Math. Theor.* 41 085207
[5] Desbois J 2000 Spectral determinant of Schrödinger operators on graphs *J. Phys. A: Math. Gen.* 33 L63
[6] Pivovarchik V N 1999 An inverse Sturm–Liouville problem by three spectra *Integral Eq. Operator Theory* 34 234–43
[7] Martinyuk O and Pivovarchik V 2010 On the Hochstadt–Liberman theorem *Inverse Problems* 26 035011
[8] Law C-K, Pivovarchik V and Wang W C 2013 A polynomial identity and its application to inverse spectral problems in Stieltjes strings *Oper. Matrices* 7 603–16
[9] Pivovarchik V, Rozhenko N and Tretter C 2013 Dirichlet–Neumann inverse spectral problem for a star graph of Stieltjes strings *Linear Algebra. Appl.* 439 2263–92
[10] Pivovarchik V and Tretter C Location and multiplicities of eigenvalues for a star graph of Stieltjes strings *J. Difference Eqs. Appl.* 21 383–402
[11] Tutte W T 2001 *Graph Theory* (Cambridge: Cambridge University Press)