Necessary and sufficient conditions for the local creation of quantum discord

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Received 13 December 2012, in final form 17 February 2013
Published 28 March 2013
Online at stacks.iop.org/JPhysA/46/155301

Abstract
Quantum discord (QD) is one of the main quantum correlations in quantum information theory. In this paper, we show that a local channel cannot create QD for zero QD states of size $d \geq 3$ if and only if either it is a completely decohering channel or it is a nontrivial isotropic channel. For the qubit case this property is an additional characteristic of the completely decohering channel or the commutativity-preserving unital channel. In particular, the exact forms of the completely decohering channel and the commutativity-preserving unital qubit channel are proposed. Consequently, our results confirm and improve the conjecture proposed by Hu et al for the case of $d \geq 3$ and improve the result proposed by Streltsov et al for the qubit case.

PACS numbers: 03.65.Ud, 03.65.Db, 03.65.Yz
(Some figures may appear in colour only in the online journal)

1. Introduction

The characterization of quantum correlated composite quantum systems is an important topic in quantum information theory [1–6]. Different approaches for characterizing the quantum correlations were studied in the past two decades [2–23]. Recently, much interest has been devoted to the study of quantum correlations that may arise without entanglement, such as quantum discord (QD) [4], measurement-induced nonlocality (MIN) [6] and quantum deficit [24], etc. These quantum correlations can still be a resource for a number of quantum information applications [6, 7, 25–27]. So, this makes it important to better understand the dynamics of these quantum correlations under local noises (or operations).
Recently, the action of QD under the effect of a local channel has been discussed (see [28–33]). In the qubit case, Streltsov et al showed in [28] that a qubit channel that preserves commutativity is either unital, i.e. mapping maximal mixed state to maximal mixed state, or a completely decohering channel, i.e. nullifying QD in any state. In [29], for the $m \otimes 3$ system, it is showed that a channel $\Lambda$ acting on the second subsystem cannot create QD for zero QD states if and only if $\Lambda$ is either a completely decohering channel or an isotropic channel. And it is conjectured that this result is also valid for any $m \otimes n$ system with $n \geq 3$. In [29, 30], the authors proved that, for any $m \otimes n$ system with $mn < +\infty$, a channel $\Lambda$ (acting on the second subsystem) transforms the zero QD states to zero QD states if and only if $\Lambda$ preserves commutativity. The goal of this paper is to propose an explicit form of a commutativity-preserving channel, from which we will (i) give a positive answer to the above conjecture raised in [29], (ii) propose an exact form of the commutativity-preserving unital channel for the qubit system and (iii) present an explicit form of a ‘completely decohering channel’ for any system.

Besides fundamental interest, our results may result in useful applications. The concrete form of the commutativity-preserving unital qubit channel may lead to a number of experimental tasks based on QD since the qubit state is a direct resource in various quantum information processing tasks. From our results, we know exactly whether a local channel can create or nullify quantum correlation measured by QD. We will show that we have more choices of local channels for the qubit case in the issue of preserving zero QD in the state: except for the isotropic channels, there exist some non-isotropic channels which also cannot create QD for the qubit case, while only the isotropic channels cannot create QD for the higher dimensional case. In addition, for both qubit and the higher dimension cases, we provide several equivalent methods of determining whether a given local channel can create QD.

The paper is organized as follows. In section 2, we review the definitions of a quantum channel and QD, and fix some terminology. In section 3, we discuss the local channels that cannot create QD for zero QD states. This section is divided into two subsections. Subsection 3.1 deals with the qudit case with $d \geq 3$. The conjecture in [29] is confirmed and several new equivalent conditions for a local channel that cannot create QD for zero QD states are presented (theorem 1). Consequently, local channels that preserve zero QD states in both directions are characterized (proposition 1), which are precisely the character of the nontrivial isotropic channels. Subsection 3.2 is devoted to the qubit case. The exact form of the unital qubit commutativity-preserving channel is characterized (theorem 2), which improves the result in [28]. In addition, the qubit channels that preserve zero QD states in both directions are characterized (proposition 2). For clarity, the Bloch picture of these channels are proposed. Moreover, comparing theorem 1 with theorem 2, we find that the qubit case is different from the higher dimensional case: there exist many channels that cannot create QD while they are neither the completely decohering channels nor the isotropic channels. In section 4, we deal with the local channels that nullify QD in any state. It is shown that a local channel nullifies QD in any state if and only if it is a completely decohering channel (proposition 3) and the explicit form of the completely decohering channel is derived (proposition 4). A summary and some related questions are listed at the end.

2. Definitions and terminologies

Let $H$ be a complex Hilbert space describing a quantum system, $\dim H = n < +\infty$. Let $\mathcal{B}(H)$ be the space of all linear operators on $H$, and $S(H)$ the set consisting of all quantum states acting on $H$. Recall that a quantum channel (or channel, for short) is described by a
trace-preserving completely positive linear map \( \Lambda : \mathcal{B}(H) \to \mathcal{B}(H) \) that admits a form of Kraus operator representation, i.e.
\[
\Lambda(\cdot) = \sum_i X_i(\cdot)X_i^\dagger,
\]
where \( X_i \) are operators acting on \( H \) with \( \sum_i X_i^\dagger X_i = I \).

In particular, a channel \( \Lambda \) acting on an \( n \)-dimensional quantum system is called an isotropic channel if it has the form
\[
\Lambda(\cdot) = t\Gamma(\cdot) + (1-t)\text{Tr}(\cdot)\frac{I}{n},
\]
where \( \Gamma \) is either a unitary operation or is unitarily equivalent to transpose (also see in [29]). Parameter \( t \) is chosen to make sure that \( \Lambda \) is a trace-preserving completely positive linear map. If \( t \) in equation (2) is nonzero, we call \( \Lambda \) a nontrivial isotropic channel. It is known by [29], \( -\frac{1}{n\cdot t} \leq t \leq \frac{1}{n\cdot t} \) when \( \Gamma \) is unitarily equivalent to transpose. If \( t = 0 \), \( \Lambda \) is the completely depolarizing channel, namely, \( \Lambda(S(H)) = \{ \frac{1}{n}I \} \). A channel \( \Lambda \) is called a completely decohering channel (or semi-classical channel) if \( (\Lambda(B))(H) \) is commutative. In general, the completely depolarizing channel is viewed as a special case of the completely decohering one.

QD, as a quantum correlation of a bipartite system, was initially introduced by Ollivier and Zurek [4] and by Henderson and Vedral [5]. We denote by \( A + B \) the bipartite system shared by Alice and Bob. Let \( H_A \) and \( H_B \) be the complex Hilbert spaces that describe the subsystem of Alice and Bob, respectively. Then \( H_A \otimes H_B \) corresponds to the composite system \( A + B \). Recall that, for a state \( \rho \in S(H_A \otimes H_B) \), the QD of \( \rho \) (up to part \( B \)) is defined by
\[
D_B(\rho) := \min_{I_B} \text{Tr}(I(\rho) - I(\rho|\Pi^B)),
\]
where the minimum is taken over all local von Neumann measurements \( \Pi^B \), \( I(\rho) := S(\rho_A) + S(\rho_B) - S(\rho) \) is interpreted as the quantum mutual information, \( S(\rho) := -\text{Tr}(\rho \log \rho) \) is the von Neumann entropy, \( I(\rho|\Pi^B) := S(\rho_A) - S(\rho|\Pi^B) \), \( S(\rho|\Pi^B) := \sum_k p_k S(\rho_k) \) and \( \rho_k = \frac{1}{p_k} (I_A \otimes \Pi^B_k) \rho (I_A \otimes \Pi^B_k) \) with \( p_k = \text{Tr}[(I_A \otimes \Pi^B_k) \rho (I_A \otimes \Pi^B_k)] \), \( k = 1, 2, \ldots, \dim H_B \).

Let \( \phi : \mathcal{B}(H) \to \mathcal{B}(H) \) be a map. Throughout this paper, we say that (i) \( \phi \) preserves normality if \( \phi \) maps normal operators to normal operators, namely, \( A \in \mathcal{B}(H) \) is normal implies that \( \phi(A) \) is normal (here, an operator \( A \in \mathcal{B}(H) \) is called a normal operator if \( AA^\dagger = A^\dagger A \)); (ii) \( \phi \) preserves normality in both directions if \( A \in \mathcal{B}(H) \) is normal if and only if \( \phi(A) \) is normal; (iii) \( \phi \) preserves commutativity (or \( \phi \) is a commutativity-preserving map) if \( [A, B] = AB - BA = 0 \) implies \( [\phi(A), \phi(B)] = 0 \) for any \( A, B \in \mathcal{B}(H) \); (iv) \( \phi \) preserves commutativity in both directions if \( [A, B] = 0 \Leftrightarrow [\phi(A), \phi(B)] = 0 \) for any \( A, B \in \mathcal{B}(H) \); (v) \( \phi \) preserves commutativity for Hermitian operators (resp. quantum states) if \( [A, B] = 0 \) implies \( [\phi(A), \phi(B)] = 0 \) for Hermitian operators (resp. quantum states) \( A \) and \( B \) in \( \mathcal{B}(H) \) (resp. \( S(H) \)); (vi) \( \phi \) preserves commutativity in both directions for Hermitian operators (resp. quantum states) if \( [A, B] = 0 \Leftrightarrow [\phi(A), \phi(B)] = 0 \) holds for Hermitian operators (resp. quantum states) \( A \) and \( B \) in \( \mathcal{B}(H) \) (resp. \( S(H) \)). We say that a local channel cannot create QD for zero QD states if \( D_B(\rho) = 0 \Rightarrow D_B((I_A \otimes \Lambda)\rho) = 0 \), where \( I_A \) denotes the identity map acting on part \( A \). For any \( A \in \mathcal{B}(H) \), \( A^\dagger \) denotes the transpose of \( A \) relative to an arbitrarily fixed basis.

3. Local channels that cannot create QD for zero QD states

This section is devoted to discussing the local channels that cannot create QD for zero QD states. We first consider the qudit case with \( d \geq 3 \) and then discuss the qubit case.
3.1. The qudit case (d ≥ 3)

The following is the main result of this subsection.

Theorem 1. Let \( H_A \) and \( H_B \) be complex Hilbert spaces with \( \dim H_A = m \geq 2 \) and \( \dim H_B = n \geq 3 \), and let \( \Lambda \) be a channel acting on the subsystem \( B \). Then the following statements are equivalent.

1. \( D_B(\rho) = 0 \Rightarrow D_B((I_A \otimes \Lambda)\rho) = 0 \).
2. \( \Lambda \) preserves commutativity for Hermitian operators.
3. Either (a) \( \Lambda \) is a completely decohering channel or (b) \( \Lambda \) is a nontrivial isotropic channel.

It is easily checked that the maps of the form (a) in item (3) are channels. The equivalence of (1) and (3) implies that a local channel cannot create QD for zero QD states if and only if it is a completely decohering channel, or it is an isotropic channel. Therefore, our theorem 1 particularly solves affirmatively the conjecture proposed in [29].

Proof of theorem 1. (3)⇒(2) and (2)⇒(2)′′ are obvious. (2)′′⇒(2) holds since two Hermitian matrices \( A, B \) are commutative if and only if \( [A^+, B^+] = [A^+, B^-] = [B^+, A^-] = 0 \), where \( A^+ \) and \( A^- \) are the positive and the negative part of \( A \) respectively, \( B^+ \) and \( B^- \) are the positive and the negative part of \( B \) respectively (note that in such a decomposition, \( A^+ - 0 \), \( B^+ \)′′′ ≥ 0 and \( [A^+, A^-] = [B^+, B^-] = 0 \).

By [34, corollary 1], we know that if \( \phi : B(H_B) \rightarrow B(H_B) \) is a Hermitian-preserving linear map (namely, \( \phi(A^\dagger) = \phi(A)^\dagger \) for every \( A \)), then \( \phi \) preserves commutativity if and only if it preserves normality, and in turn, if and only if it preserves commutativity for Hermitian operators. Hence, (2)⇔(2)′⇔(2)′′ is immediate since the channels are Hermitian-preserving.

(2)⇒(3). Denote by \( \mathcal{H}_n \) the real linear space of all \( n \times n \) Hermitian complex matrices. Assume \( \Lambda \) as in equation (1). Then \( \Lambda \) is a commutativity-preserving map on \( \mathcal{H}_n \). Let \( \mathcal{M}_n \) be the algebra of all \( n \times n \) matrices. By [34, theorem 3], if \( \phi \) is a Hermitian-preserving (i.e. \( \phi(A^\dagger) = \phi(A)^\dagger \)) linear map on \( \mathcal{M}_n \) which also preserves commutativity for Hermitian matrices, then either \( \phi(\mathcal{M}_n) \) is commutative, or there exist a unitary matrix \( U \), a linear functional \( f \) on \( \mathcal{M}_n \) and a real number \( t \) such that \( \phi \) has one of the following forms: (i) \( \phi(A) = tUA^\dagger f(A)I \) for all \( A \) in \( \mathcal{M}_n \); (ii) \( \phi(A) = tUA^\dagger U^\dagger + f(A)I \) for all \( A \) in \( \mathcal{M}_n \). Note that \( \Lambda \) is a Hermitian-preserving linear map. Therefore, either \( \Lambda \) is commutative, or there exists a unitary operator \( U \in B(H_B) \) and a real number \( t \) such that \( \Lambda \) has one of the following forms: (i) \( \Lambda(A) = tUA^\dagger f(A)I \) for all \( A \) in \( B(H_B) \). (ii) \( \Lambda(A) = tUA^\dagger U^\dagger + f(A)I \) for all \( A \) in \( B(H_B) \). Note that \( B(H_B) \) can be regarded as a Hilbert space with the Hilbert–Schmidt inner product

\[
\langle X|Y \rangle := \text{Tr}(X^\dagger Y).
\]

It turns out that there exists an operator \( W \in B(H_B) \) such that \( f(A) = \text{Tr}(WA) \) holds for all \( A \in B(H_B) \). Considering the action of \( \Lambda \) on \( S(H_B) \) one obtains that \( t + nf(\rho) = 1 \) for all \( \rho \in S(H_B) \). It follows that \( \text{Tr}(W\rho) = \frac{1}{n}I_B \) for all \( \rho \) in \( S(H_B) \). Consequently, \( \text{Tr}(WA) = \frac{1-t}{n}I_B \) holds for all \( A \in B(H_B) \), which implies that

\[ W = \frac{1-t}{n}I_B. \]
From [29], $t$ satisfies $\frac{1}{n+1} \leq t \leq 1$ when $\Lambda(A) = iUAU^* + \frac{1-t}{n} \text{Tr}(A)B$ and $\frac{1}{n+1} \leq t \leq 1$ when $\Lambda(A) = tUA^TU^* + \frac{1-t}{n} \text{Tr}(A)B$, which guarantees that $\Lambda$ is completely positive. That is, $\Lambda$ is an isotropic channel. If $t \neq 0$, it is the nontrivial isotropic channel, i.e. the case of item (b); If $t = 0$, it is obvious that $\Lambda$ is a complete depolarizing channel, i.e. a special case of item (a).

If $\Lambda(\mathcal{B}(H_B))$ is commutative, then $\Lambda$ is a completely decohering channel. That is, (a) of item (3) holds.

(1)$\Rightarrow$(2). Let $\{|i\rangle\}$ be an orthonormal basis of $H_A$. Then any state $\rho$ acting on $H_A \otimes H_B$ can be represented by

$$\rho = \sum_{i,j} E_{ij} \otimes B_{ij},$$

where $E_{ij} = |i\rangle\langle j|$ and $B_{ij}$ are operators acting on $H_B$, and

$$(I_A \otimes \Lambda)\rho = \sum_{i,j} E_{ij} \otimes \Lambda(B_{ij}).$$

We proved in [35] that $D_B(\rho) = 0$ if and only if $B_{ij}$ are mutually commuting normal operators. It follows from $D_B(\rho) = 0 \Rightarrow D_B((I_A \otimes \Lambda)\rho) = 0$ that $\Lambda$ preserves normality, and thus, preserves commutativity for Hermitian operators according to [34, corollary 1]. In fact, for any normal operator $A \in \mathcal{B}(H_B)$, there exist positive operators $C, D \in \mathcal{B}(H_B)$ such that $A, C, D$ are mutually commuting and

$$\rho_0 = \frac{1}{\text{Tr}(C+D)}(E_{11} \otimes C + E_{12} \otimes A + E_{21} \otimes A^* + E_{22} \otimes D)$$

is a state. Moreover, by the result in [35] mentioned above, we have $D_B(\rho_0) = 0$. Thus $D_B((I_A \otimes \Lambda)\rho_0) = 0$, which implies that $\Lambda(A), \Lambda(C), \Lambda(D), \Lambda(A^*) = \Lambda(A)^*$ are mutually commuting normal operators. In particular, $\Lambda(A)$ is normal.

(2)$\Rightarrow$(1). Since $\Lambda$ is a Hermitian-preserving linear map, (1) implies that $\Lambda$ preserves normality and commutativity. Therefore, if $B_{ij}$ are mutually commuting normal operators, then $\Lambda(B_{ij})$ is also mutually commuting normal operators. Now, by [35], it is obvious that $D_B(\rho) = 0 \Rightarrow D_B((I_A \otimes \Lambda)\rho) = 0$.

Furthermore, we have

**Proposition 1.** Let $H_A$ and $H_B$ be complex Hilbert spaces with $\dim H_A = m \geq 2$ and $\dim H_B = n \geq 3$, and let $\Lambda$ be a channel acting on the subsystem $B$. Then the following statements are equivalent.

1. $D_B(\rho) = 0 \Leftrightarrow D_B((I_A \otimes \Lambda)\rho) = 0$.
2. $\Lambda$ preserves commutativity in both directions for Hermitian operators.
3. $\Lambda$ preserves commutativity in both directions.
4. $\Lambda$ preserves normality in both directions.
5. $\Lambda$ preserves commutativity in both direction for quantum states.
6. $\Lambda$ is a nontrivial isotropic channel.

**Proof.** (2)$\Leftrightarrow$(2'), (2)$\Leftrightarrow$(2''), (3)$\Rightarrow$(1), (3)$\Rightarrow$(2) and (3)$\Rightarrow$(2') are obvious.

(2)$\Leftrightarrow$(2'') is easily checked by the fact that $A$ is normal if and only if it can be written as $A = X + iY$ with $X$ and $Y$ Hermitian and $[X, Y] = 0$.

(1)$\Rightarrow$(3). According to (1)$\Leftrightarrow$(3) of theorem 1, $\Lambda$ admits the form of item (3) in theorem 1.

It is clear that the case (a) of item (3) cannot occur since the completely decohering channel nullifies QD in any state [30]. So, $\Lambda$ is a nontrivial isotropic channel.
According to the proof of (2)⇒(3) in theorem 1, we know that Λ admits the form as in item (3) of theorem 1. It is immediate that the case of 'Λ(ℬ(ℋ_B)) is commutative' cannot occur.

3.2. The qubit case

We now turn to the discussion of the qubit case, that is, \( \dim ℋ_B = 2 \). We will show that the form of commutativity-preserving unital channel for the qubit system is different from the higher dimensional case. Thus, the local channel of the qubit system that preserves zero QD states has different forms from those of higher dimensional systems.

Compared with theorem 1, the main result of this subsection is the following.

**Theorem 2.** Let \( ℋ_A \) and \( ℋ_B \) be complex Hilbert spaces with \( \dim ℋ_A = m \geq 2 \) and \( \dim ℋ_B = 2 \), and let \( Λ \) be a channel acting on the subsystem \( B \). Then the following statements are equivalent.

1. \( DB(ρ) = 0 \Rightarrow DB((IA ⊗ Λ)ρ) = 0 \).
2. \( Λ \) preserves commutativity for Hermitian operators.
   - \( (2') \) \( Λ \) preserves commutativity.
   - \( (2''') \) \( Λ \) preserves commutativity for quantum states.
3. Either (a) \( Λ \) is a completely decohering channel; or (b) for any orthonormal basis \( \{|e_1⟩, |e_2⟩\} \) of \( ℋ_B \), there exist a unitary operator \( U \in ℳ(ℋ_B) \), real numbers \( 0 \leq λ \leq 1 \) and complex numbers \( α, β, γ \) so that
   \[
   \left( \begin{array}{cc}
   αβ & γ \\
   γ^∗ & α^∗
   \end{array} \right)
   \] is contractive, such that, with respect to the space decomposition \( ℋ_B = ℂ|e_1⟩ ⊕ ℂ|e_2⟩ \),
   \[
   Λ\left( \begin{array}{cc}
   a_{11} & a_{12} \\
   a_{21} & a_{22}
   \end{array} \right)
   = U\left( \begin{array}{cc}
   λa_{11} + (1 − λ)a_{22} & 0 \\
   0 & (1 − λ)a_{11} + λa_{22}
   \end{array} \right) U^† + a_{12}X + a_{21}X^† U^†,
   \] for all \( A = \left( \begin{array}{cc}
   a_{11} & a_{12} \\
   a_{21} & a_{22}
   \end{array} \right) \), where
   \[
   X = \left( \begin{array}{cc}
   \sqrt{λ(1 − λ)}α & λβ \\
   (1 − λ)γ & −\sqrt{λ(1 − λ)}α
   \end{array} \right)
   \] with \( |β| + |γ| \neq 0 \), \( β \neq 0 \) when \( λ = 1 \) and \( γ \neq 0 \) when \( λ = 0 \).

Theorem 2 depicts the commutativity-preserving unital qubit channel in detail, which improves theorem 1 in [28] proposed by Streltsov et al.

**Proof of theorem 2.** We only need to check the implication (2)⇒(3). By a lemma (see the appendix), and noting that \( Λ \) is completely positive and trace-preserving, we can know that \( Λ \) is a completely decohering channel if the range of \( Λ \) is commutative.

Assume that the range of \( Λ \) is not commutative. Take an orthonormal basis \( \{|e_1⟩, |e_2⟩\} \) of \( ℋ_B \) and denote \( E_{ij} = |e_i⟩⟨e_j| \), \( i, j = 1, 2 \). Then the lemma in the appendix ensures that there exist a unitary operator \( U \) on \( ℋ_B \) and nonnegative real numbers \( λ_1, λ_2, μ_1, μ_2 \) with \( λ_1 + μ_1 = λ_2 + μ_2 \) such that, with respect to the space decomposition \( ℋ_B = ℂ|e_1⟩ ⊕ ℂ|e_2⟩ \),

\[
Λ(E_{11}) = U\left( \begin{array}{cc}
λ_1 & 0 \\
0 & λ_2
\end{array} \right) U^†, \quad Λ(E_{22}) = U\left( \begin{array}{cc}
μ_1 & 0 \\
0 & μ_2
\end{array} \right) U^†.
\]
As $\text{Tr}(\rho(E_{11})) = \text{Tr}(\rho(E_{22})) = 1$, we must have $\lambda_1 = \mu_2 = 1 - \lambda_2 = 1 - \mu_1 \geq 0$ and $\lambda_1 + \mu_1 = 1$. Let $\lambda_1 = \lambda$. Then $0 \leq \lambda \leq 1$ and

$$\Lambda(E_{11}) = U \begin{pmatrix} \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix} U^\dagger,$$

$$\Lambda(E_{22}) = U \begin{pmatrix} 1 - \lambda & 0 \\ 0 & \lambda \end{pmatrix} U^\dagger.$$ (6)

Note that $\text{Tr}(\Lambda(E_{11})) = 0$. So, there are complex numbers $x, y, z$ such that $\Lambda(E_{12}) = U \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} U^\dagger$ and $\Lambda(E_{21}) = \Lambda(E_{12})^\dagger$.

It is well known by a theorem of Choi [36] that the map $\Lambda$ is completely positive if and only if the block matrix $[\Lambda(E_{ij})]$ is positive. It follows from equations (6) and (7) that

$$\begin{pmatrix} \Lambda(E_{11}) & \Lambda(E_{12}) \\ \Lambda(E_{21}) & \Lambda(E_{22}) \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 - \lambda \\ x & y \\ z & 0 \end{pmatrix} \begin{pmatrix} 1 - \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \begin{pmatrix} 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & U^\dagger \end{pmatrix}.$$

Let

$$J = \begin{pmatrix} \lambda & 0 \\ 0 & 1 - \lambda \\ x & y \\ z & -x \end{pmatrix}.$$

Then $\Lambda$ is completely positive if and only if $J \succeq 0$, and in turn, from theorem 1.1 in [37], if and only if there exists a contractive matrix $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix}$ such that

$$\begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{1 - \lambda} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \begin{pmatrix} \sqrt{1 - \lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda(1 - \lambda)} \alpha + \lambda \beta & \lambda \beta \\ (1 - \lambda) \gamma - \sqrt{\lambda(1 - \lambda)} \eta \end{pmatrix}.$$

It follows that $\eta = -\alpha$ and

$$X = U^\dagger \Lambda(E_{12}) U = \begin{pmatrix} \sqrt{\lambda(1 - \lambda)} \alpha + \lambda \beta \\ (1 - \lambda) \gamma - \sqrt{\lambda(1 - \lambda)} \eta \end{pmatrix}.$$

If $|\beta| + |\gamma| = 0$, or $\beta = 0$ when $\lambda = 1$, or $\gamma = 0$ when $\lambda = 0$, then the channel reduces to a completely decohering one. Now it is clear that $\Lambda$ has the form of equation (5). This completes the proof. $\square$

It is worth highlighting that (i) a channel with the form as in equation (5) does not preserve commutativity in both directions necessarily, which is different from the higher dimensional case; (ii) the isotropic qubit channel is only a special case of (b) in item (3) of theorem 2. The map $\Lambda$ has the form $\Lambda(A) = tU AU^\dagger + \frac{1}{4} \text{Tr}(A) I$ for all $A$ if and only if $t = 2\lambda - 1$, $\beta = \frac{\alpha + \gamma}{2}$ and $\alpha = \gamma = 0$; $\Lambda$ has the form $\Lambda(A) = t U AU^\dagger + \frac{1}{4} \text{Tr}(A) I$ for all $A$ if and only if $t = 2\lambda - 1$, $\gamma = \frac{\alpha + \gamma}{2}$ and $\alpha = \beta = 0$. So there are many commutativity-preserving unital channels for the qubit system that are neither isotropic nor completely decohering. This is quite different from the case of $n \geq 3$ as stated in theorem 1.

Going further, we have

**Proposition 2.** Let $H_A$ and $H_B$ be complex Hilbert spaces with $\dim H_A = m$ and $\dim H_B = 2$, and let $\Lambda$ be a channel acting on the subsystem $B$. Then the following statements are equivalent.
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(1) $D_B(\rho) = 0 \iff D_B((I_2 \otimes A)\rho) = 0$.
(2) $\Lambda$ preserves commutativity in both directions for Hermitian operators.

(2') $\Lambda$ preserves commutativity in both directions.
(2'') $\Lambda$ preserves normality in both directions.
(2'''') $\Lambda$ preserves commutativity in both directions for quantum states.

(3) $\Lambda$ has the form as equation (5) with $X$ satisfying $\lambda \neq \frac{1}{2}, |\beta| + |\gamma| \neq 0; \gamma \neq 0$ when $\lambda = 0; \beta \neq 0$ when $\lambda = 1; (\lambda|\beta| \neq (1 - \lambda)|\gamma|$ when $\alpha = 0, \beta \gamma \neq 0$ and $\lambda(1 - \lambda) \neq 0; \text{and}\lambda|\beta| \neq (1 - \lambda)|\gamma|$ or $\lambda \beta \alpha \neq (1 - \lambda)\gamma \bar{\alpha}$ when $\alpha \beta \gamma \neq 0$ and $\lambda(1 - \lambda) \neq 0$.

Proof. We only need to check that (2)$\iff$(3).

(2)$\implies$(3). As $\Lambda$ preserves commutativity in both directions, $\Lambda$ has the form as equation (5) and is injective. The injectivity reveals that $X$ satisfies the conditions as in term (3) above. That is, (3) holds.

(3)$\implies$(2). By theorem 2, (3) implies that $\Lambda$ preserves commutativity. Moreover, the conditions ensure that $\Lambda$ is injective. For any $A \in \mathcal{B}(H_B)$, denote by $\{A\}'$ the commutant of $A$, that is, $\{A\}' = \{B \in \mathcal{B}(H_B) : AB = BA\}$. Then $\text{dim}\{\Lambda(A)\}' = \text{dim}\{A\}'$. This entails that $\text{dim}\{\Lambda(A), \Lambda(B)\} = 0 \implies [A, B] = 0$. So, $\Lambda$ preserves commutativity in both directions. \qed

Propositions 1 and 2 imply that a local channel neither creates nor vanishes the quantum correlation measured by QD if and only if it preserves commutativity in both directions, and in turn, if and only if it is one-to-one and it outputs commutative states whenever the input states are commutative.

At the end of this section, we give a geometric picture of commutative qubit states and commutativity-preserving qubit channels. It is well known that the Bloch ball, whose boundary is the Bloch sphere, corresponds to the space of all two-level density matrices. The surface represents all pure states while the interior of the Bloch sphere, the open Bloch ball, represents mixed states. In particular, the center of the sphere corresponds to the maximally mixed state.

Indeed an arbitrary density matrix can be parameterized as

$$\rho = \begin{pmatrix} \frac{1}{2} + z & x - iy \\ x + iy & \frac{1}{2} - z \end{pmatrix}$$

with $x^2 + y^2 + z^2 \leq \frac{1}{4}$. It is customary to regard this an expansion in terms of the Pauli matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, so that

$$\rho = \frac{1}{2} I + \vec{r} \cdot \vec{\sigma}.$$ (8)

The vector $\vec{r}$ is known as the Bloch vector.

Let

$$\rho_k = \begin{pmatrix} \frac{1}{2} + z_k & x_k - iy_k \\ x_k + iy_k & \frac{1}{2} - z_k \end{pmatrix}, \quad k = 1, 2.$$  

It is straightforward that $[\rho_1, \rho_2] = 0$ if and only if $x_1z_2 = x_2z_1, y_1z_2 = y_2z_1$ and $x_1y_2 = x_2y_1$. Equivalently, $[\rho_1, \rho_2] = 0$ if and only if $\vec{r}_1 = i\vec{r}_2$ for some real number $i$, where $\vec{r}_k$ denotes the Bloch vector of $\rho_k$, $k = 1, 2$.

**Observation 1.** Two qubit quantum states are commutative if and only if two Block vectors representing quantum states are collinear in the Bloch ball (see figures 1 and 2).

**Observation 2.** A qubit channel preserves commutativity in both directions if and only if it is a one-to-one transformation and it maps the line that is collinear with the center of the ball to the line that is collinear with the center of the ball (see figure 3). A completely decohering qubit channel maps the Bloch ball to a line that is collinear with the center of the ball (see figures 4 and 5).
Let $\rho$ be a qubit state with $\vec{r}_\rho = (x, y, z)$. Then a unitary evolution of $\rho$, i.e. $U \rho U^\dagger$ for some unitary matrix, corresponds to a rotation of the Bloch vector $\vec{r}_\rho$ around the center of the ball.

Write $\rho' = U \rho U^\dagger$ with $\vec{r}_\rho' = (\tilde{x}, \tilde{y}, \tilde{z})$, then $\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = x^2 + y^2 + z^2$. Conversely, if $\rho_1$ and $\rho_2$ are two qubit states with $\vec{r}_{\rho_1} = (x_1, y_1, z_1)$, $\vec{r}_{\rho_2} = (x_2, y_2, z_2)$ and $x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2$, then $\rho_1 = U \rho_2 U^\dagger$ for some unitary matrix $U$. For simplicity, for a given state $\rho$, we denote by $\Pi_U \vec{r}_\rho$ the rotation of the Bloch vector $\vec{r}_\rho$, which corresponds to the unitary evolution of the state $\rho$, $U \rho U^\dagger$. 

Figure 1. $\rho_1$ and $\rho_2$ are commutative.

Figure 2. $\rho_1$ and $\rho_2$ are not commutative.

Figure 3. $\Lambda_1$ and $\Lambda_2$ are commutativity-preserving unital channels, they transform commutative states to commutative ones.

Figure 4. $\Lambda_3$ and $\Lambda_4$ are completely decohering channels; the red lines are the image of all qubit states under these channels.
Observation 3. Let $\Lambda$ be a qubit channel as in equation (5), and let $\alpha = a + ib$, $\beta = c + id$ and $\gamma = e + if$, where $a$, $b$, $c$, $d$, $e$ and $f$ are real numbers, $i$ is the imaginary unit. Then

$$\tilde{r}_p = \Pi_U(x', y', z')$$

for all $\tilde{r}_p = (x, y, z)$, where $\sigma = \Lambda(\rho), x' = \lambda(xc - yd) + (1 - \lambda)(xe - yf), y' = \lambda(ey + dx) - (1 - \lambda)(cy + fx)$ and $z' = (2\lambda - 1)z + 2\sqrt{\lambda(1 - \lambda)}(xa - yb)$.

4. Local channels that nullify QD

In the following let us turn to the question of when a local channel nullifies QD in any state.

Let $\Lambda$ be a channel acting on the subsystem $B$. It is obvious that if $\Lambda(B(H_B))$ is commutative, then $D_B((I_A \otimes \Lambda)\rho) = 0$ for any state $\rho \in S(H_A \otimes H_B)$, namely, $\Lambda$ nullifies QD in any state. Conversely, if $\Lambda$ nullifies QD in any state, one can check that $\Lambda(B(H_B))$ is commutative. In fact, writing $\rho = \sum_{i,j} E_{ij} \otimes B_{ij}$ as in equation (3), $D_B((I_A \otimes \Lambda)\rho) = 0$ yields that $\Lambda(B_{ij})$s are mutually commuting normal operators. For any $A \in B(H_B)$, there exist positive operators $P_1, P_2, P_3, P_4$ such that $A = P_1 - P_2 + i(P_3 - P_4)$. Then for any $B \in B(H_B)$, there exist positive numbers $a_i, i = 1, 2, 3, 4$, such that

$$\rho_0 = \frac{1}{\text{Tr}(a_1P_1 + a_2P_2)}(E_{11} \otimes a_1P_1 + E_{12} \otimes B + E_{21} \otimes B^\dagger + E_{22} \otimes a_2P_2)$$

and

$$\sigma_0 = \frac{1}{\text{Tr}(a_3P_3 + a_4P_4)}(E_{11} \otimes a_3P_3 + E_{12} \otimes B + E_{21} \otimes B^\dagger + E_{22} \otimes a_4P_4)$$

are states in $S(H_A \otimes H_B)$. It follows from $D_B((I_A \otimes \Lambda)\rho_0) = 0$ and $D_B((I_A \otimes \Lambda)\sigma_0) = 0$ that $\Lambda(A)$ and $\Lambda(B)$ are commuting normal operators, which implies that $\Lambda(B(H_B))$ is commutative. Thus, the following result is true.

Proposition 3. Let $H_A$ and $H_B$ be complex Hilbert spaces with $\dim H_A = m \geq 2$ and $\dim H_B = n \geq 2$, and let $\Lambda$ be a channel acting on the subsystem $B$. Then $D_B((I_A \otimes \Lambda)\rho) = 0$ for any state $\rho \in S(H_A \otimes H_B)$ if and only if $\Lambda$ is a completely decohering channel.

Finally, let us discuss the form of a completely decohering channel.

Proposition 4. Let $\Lambda$ be a channel acting on the system associated with an $n$-dimensional complex Hilbert space $H$, $n \geq 2$. Then $\Lambda$ is a completely decohering channel if and only if there exist an $n$-outcome POVM for the system $B$, i.e. $[W_i]_{i=1}^n \subseteq B(H)$ with $\sum W_i = I, W_i \geq 0, i = 1, 2, \ldots, n$, and an orthonormal basis, $\{|e_i\rangle\}$, of $H$, such that

$$\Lambda(A) = \sum_{i} \text{Tr}(W_iA)\langle e_i|\langle e_i|$$

for all $A$. (9)
Proof. The ‘if’ part is clear. We show the ‘only if’ part. By definition, $\Lambda$ is completely decohering implies that $\Lambda(B(H))$ is commutative. Then $\Lambda(A)$ is normal for any $A \in B(H)$ since $[\Lambda(A), \Lambda(A')]=[\Lambda(A), \Lambda(A)'] = 0$. Hence all elements in $\Lambda(B(H))$ are normal and mutually commutative. It follows that there exist positive linear functionals $f_i$ of $B(H)$, $i=1, 2, \ldots, n$ and an orthonormal basis $\{|e_i\rangle\}$ of $H$ such that

$$\Lambda(A) = \sum_i f_i(A) |e_i\rangle \langle e_i|.$$ 

Therefore, there exist positive operators $W_i \in B(H)$, $i=1, 2, \ldots, n$, so that $f_i(A) = \text{Tr}(W_i A)$ holds for any $A \in B(H)$. Since $\Lambda$ is a trace-preserving map, we have $\text{Tr}(A) = \sum_i \text{Tr}(W_i A) = \text{Tr}(\sum_i W_i) A$ holds for any $A \in B(H)$, which leads to $\sum_i W_i = I_B$. That is $\Lambda$ has the form as desired. $\square$

5. Conclusions

By the feature of a commutativity-preserving linear map, we obtained a clear picture of local channels that cannot create quantum discord (QD). We obtained an exact form of a local channel that cannot create QD for zero QD states. Consequently, the conjecture in [29] was confirmed and theorem 1 in [28] was improved. We also found that, remarkably, the qubit case is quite different from the higher dimensional case since there exist qubit local channels that are not isotropic channels while they preserve zero QD states in both directions as well. In addition, the geometric picture of the commutative qubit states in the Bloch ball was depicted.

We hope that our results will be useful in realizing quantum communication and quantum computation experimentally.

Our results lead to interesting questions for further study: what is the form of local channel $\Lambda_{a(b)} : B(H_{a(B)}) \to B(H_{a(B)})$ if $D_{A_{a(b)}}(\rho) = D_{A_{a(b)}}((\Lambda_a \otimes \Lambda_b)\rho)$ for all $\rho$? We conjecture that $\Lambda_{a(b)}$ is the unitary operation in such a case; however, the proof may be difficult since the calculation of QD is a hard work in general. Moreover, what is the form of a non-local channel $\Lambda : B(H_A \otimes H_B) \to B(H_A \otimes H_B)$ if it satisfies one of the following conditions: (1) $D_{A_{a(b)}}(\rho) = 0 \Rightarrow D_{A_{a(b)}}(\Lambda(\rho)) = 0$ (or $D_{A_{a(b)}}(\rho) = 0 \Leftrightarrow D_{A_{a(b)}}(\Lambda(\rho)) = 0$); (2) $D_{A_{a(b)}}(\rho) = D_{A_{a(b)}}(\Lambda(\rho))$?

Acknowledgments

This work is partially supported by the China Postdoctoral Science Foundation funded project (2012M520603), the Natural Science Foundation of China (11171249, 11101250) and the Research start-up fund for Doctors of Shanxi Datong University (2011-B-01). The authors thank Karol Zyczkowski and Yiu-Tung Poon for valuable discussions.

Appendix. Commutativity-preserving linear maps on $\mathcal{M}_2$

In order to prove theorem 2, the lemma below is necessary.

**Lemma.** Let $\phi : \mathcal{M}_2 \to \mathcal{M}_2$ be a Hermitian-preserving linear map. Then $\phi$ preserves commutativity if and only if (a) either there exist two Hermitian matrices $W_1$ and $W_2$, and an orthonormal basis $\{|e_1\rangle, |e_2\rangle\}$ of $\mathbb{C}^2$, such that

$$\phi(A) = \text{Tr}(W_1 A) |e_1\rangle \langle e_1| + \text{Tr}(W_2 A) |e_2\rangle \langle e_2|.$$
or (b) there exist a unitary matrix $U$, and real numbers $\lambda_i, \mu_i$ with $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$ such that

$$
\phi(E_{11}) = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^\dagger, \tag{A.1}
$$

$$
\phi(E_{22}) = U \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} U^\dagger, \tag{A.2}
$$

where $E_{ij}$ denotes the $2 \times 2$ matrix with $(i, j)$-entry $1$ and others $0$.

**Proof.** Suppose that $\phi(I) \neq 0$. Or else, replace $\phi$ by $\psi$ with $\psi(A) = \phi(A) + \text{Tr}(A)I$.

There are two different cases: (1) $\phi(I)$ is not a scalar matrix and (2) $\phi(I) = \lambda I$ for some $\lambda \neq 0$.

If $\phi(I)$ is not a scalar matrix, there are two different subcases.

**Case 1:** rank $\phi(I) = 2$. In this case, since $[\phi(X), \phi(I)] = 0$ holds for all $X \in \mathcal{M}_2$, there exist two Hermitian matrices $W_1$ and $W_2$, and an orthonormal basis, $\{|e_1\}, \{|e_2\}\}$ of $\mathbb{C}^2$, such that

$$
\phi(A) = \text{Tr}(W_1A)|e_1\rangle|e_1\rangle + \text{Tr}(W_2A)|e_2\rangle|e_2\rangle. \tag{A.3}
$$

Consequently, $\phi(M_2)$ is commutative.

**Case 2:** rank $\phi(I) = 1$. We may assume that $\phi(I) = \gamma |\xi\rangle\langle\xi|$ for some unit vector $|\xi\rangle$ and real number $\gamma \neq 0$. Let $|\xi\rangle$ be a unit vector that is orthogonal to $|\xi\rangle$. Then

$$
\phi(M_2) \subseteq \{x|\xi\rangle\langle\xi| + y|\xi\rangle\langle\xi| : x, y \in \mathbb{C}\}.
$$

Since $\phi$ maps a Hermitian matrix into a Hermitian one and any matrix is a linear combination of Hermitian matrices, we can thus conclude that $\phi$ still has the form as in equation (A.3).

Conversely, if $\phi$ has the form as in equation (A.3), then it is clear that $\phi$ preserves commutativity.

We now suppose that $\phi(I) = \lambda I$ for some $\lambda \neq 0$. Since $[E_{11}, E_{22}] = 0$, we have $[\phi(E_{11}), \phi(E_{22})] = 0$. Thus there exists a $2 \times 2$ unitary matrix $U$ such that equations (A.1) and (A.2) hold. Moreover, $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$.

On the other hand, assume that $\phi$ satisfies equations (A.1) and (A.2) with $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$. Take real numbers $x, z, \alpha, \gamma$ and complex numbers $y, \beta$ so that $\phi(E_{12} + E_{21}) = U\left(\begin{smallmatrix} 0 & \gamma \\ \gamma^* & 0 \end{smallmatrix}\right)U^\dagger$ and $\phi(iE_{12} - iE_{21}) = U\left(\begin{smallmatrix} \alpha & \beta \\ \beta^* & -\alpha \end{smallmatrix}\right)U^\dagger$. Then

$$
\phi\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = U\left[\begin{pmatrix} \lambda_1 a_{11} + \mu_1 a_{22} & 0 \\ 0 & \lambda_2 a_{11} + \mu_2 a_{22} \end{pmatrix} + a_{12}X + a_{21}X^\dagger\right]U^\dagger
$$

holds for any matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where $X = U^\dagger \phi(E_{12})U = \frac{i}{2}\left(\begin{smallmatrix} \alpha & \beta \\ -\beta^* & -\alpha \end{smallmatrix}\right)$. In particular,

$$
\phi\left(\begin{pmatrix} a & u + iv \\ u - iv & c \end{pmatrix}\right) = U\left(\begin{pmatrix} a\lambda_1 + c\mu_1 + ux + va & uy + v\beta \\ uy + v\beta & a\lambda_2 + c\mu_2 + uz + vy \end{pmatrix}\right)U^\dagger
$$

for any Hermitian matrix $\begin{pmatrix} a & u + iv \\ u - iv & c \end{pmatrix}$. Note that two Hermitian matrices

$$
A = \begin{pmatrix} a & u + iv \\ u - iv & c \end{pmatrix}, B = \begin{pmatrix} d & u + iv \\ u - iv & f \end{pmatrix},
$$

are commuting if and only if $(a-c)v_2 = u_1(d-f), (a-c)v_2 = v_1(d-f)$ and $u_1v_2 = u_2v_1$.

Then, by the above fact and noting that $\lambda_1 - \lambda_2 = \mu_2 - \mu_1$, one can check that, for any Hermitian matrices $A, B, [A, B] = 0 \Rightarrow [\phi(A), \phi(B)] = 0$. So $\phi$ preserves commutativity for Hermitian matrices. Then by [34, corollary 1], we know that $\phi$ preserves commutativity. \qed
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