Extended BRS Symmetry and Gauge Independence in On-Shell Renormalization Schemes

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Abstract

Extended BRS symmetry is used to prove gauge independence of the fermion renormalization constant $Z_2$ in on-shell QED renormalization schemes. A necessary condition for gauge independence of $Z_2$ in on-shell QCD renormalization schemes is formulated. Satisfying this necessary condition appears to be problematic at the three-loop level in QCD.

In on-shell schemes, the fermion mass renormalization $Z_m$ and wave function renormalization $Z_2$ have been observed to be gauge parameter independent in explicit two-loop QED and QCD calculations [1]. Gauge parameter independence of $Z_2$ is phenomenologically significant because it implies that the difference between the (fermion) anomalous dimension of heavy quark effective theories [2] and QCD is gauge independent.

An extension of BRS symmetry, which allows variations of the gauge parameter to be included as part of the symmetry transformations [3], will be applied to the gauge parameter dependence of $Z_2$. This approach results in an extension of Slavnov-Taylor identities, allowing gauge dependence to be formulated algebraically. Previous application of these techniques resulted in a proof of the gauge independence of the mass renormalization $Z_m$ to all orders in on-shell QED and QCD renormalization schemes [4]. We will prove the gauge parameter independence of $Z_2$ in on-shell schemes for QED and formulate a necessary condition for gauge independence in QCD which appears problematic beyond the two-loop level. This complements earlier work on gauge independence of $Z_2$ in QED resulting in the (dimensionally-regularized) relation [5]

$$\frac{\partial Z_2}{\partial \xi} \sim \int d^D k \frac{1}{k^4} = 0 \quad (1)$$

where $\xi$ is the gauge parameter and the massless tadpole is zero in dimensional regularization. Since the QED result (1) cannot be extended to QCD, our extended BRS symmetry proof for QED provides a new approach to formulating questions of gauge independence of $Z_2$ in QCD.

The QED Lagrangian in the auxiliary field formalism [6] for covariant gauges is

$$\mathcal{L} = -\frac{1}{4} F^2 + \bar{\psi} (i \gamma^\mu - m) \psi + \frac{\xi}{2} B^2 + B \partial \cdot A - c \partial^2 c \quad (2)$$

where $F$ is the field strength and $B$ is the auxiliary gauge field. This Lagrangian is invariant under the BRS symmetry

$$\delta A_\mu = c \partial_\mu c \quad , \quad \delta \bar{\psi} = ic gc \bar{\psi} \quad , \quad \delta c = 0$$

$$\delta B = 0 \quad , \quad \delta \psi = -ic gc \psi \quad , \quad \delta \bar{c} = 0 \quad (3)$$

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where \( \epsilon \) is a global grassmann quantity. The auxiliary field formalism guarantees nilpotence of the BRS transformations without invoking equations of motion.

An extension of BRS symmetry that includes gauge parameter variations introduces a new term in the Lagrangian

\[ \mathcal{L} \rightarrow \mathcal{L} + \frac{\chi}{2} \bar{c} B \]  

where \( \chi \) is a global grassmann variable. Although \( \chi \) will be set to zero after functional differentiation, it is still important to recognize that since \( \chi \) is a global Grassmann quantity, it does not change the dynamics of any process with zero ghost number. The modified Lagrangian (4) is invariant under the following extended BRS symmetry [3]

\[
\begin{align*}
\delta^+ A_\mu &= \epsilon \partial_\mu c, \quad \delta^+ \bar{\psi} = i \epsilon g c \bar{\psi}, \quad \delta^+ c = 0 \\
\delta^+ B &= 0, \quad \delta^+ \psi = -i \epsilon g c \psi, \quad \delta^+ \bar{c} = B \\
\delta^+ \xi &= \epsilon \chi, \quad \delta^+ \chi = 0
\end{align*}
\]  

As for BRS symmetry, the extended BRS symmetry (5) implies the following relation for the effective action \( \Gamma \).

\[
\begin{align*}
\partial_{\xi} \frac{\delta^2 \Gamma}{\delta \bar{\psi}(y) \delta \psi(x)} &= + \frac{\delta^3 \Gamma}{\delta \psi(y) \delta K \delta \bar{\psi}(x)} \frac{\delta^2 \Gamma}{\delta \bar{\psi}(y) \delta \psi(x) \delta \bar{\psi}(x)} + \frac{\delta^2 \Gamma}{\delta \psi(y) \delta \psi(x) \delta \bar{\psi}(x) \delta \bar{\psi}(x)} \\
&+ \frac{\delta^3 \Gamma}{\delta \psi(y) \delta \psi(x) \delta \bar{\psi}(x) \delta \bar{\psi}(x)} + \frac{\delta^3 \Gamma}{\delta \psi(y) \delta \psi(x) \delta \bar{\psi}(x) \delta \bar{\psi}(x)} \\
&+ \frac{\delta^3 \Gamma}{\delta \psi(y) \delta \psi(x) \delta \bar{\psi}(x) \delta \bar{\psi}(x)} \delta K \delta \chi
\end{align*}
\]  

Transforming to momentum space and defining

\[
\begin{align*}
\delta^2 \Gamma &\equiv \int \frac{d^4q}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} e^{-iq(y-z)-i\ell(w-z)} F(q, \ell, -q - \ell) \\
\delta^2 \Gamma &\equiv \int \frac{d^4q}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} e^{-iq(x-z)-i\ell(w-z)} \bar{F}(q, \ell, -q - \ell)
\end{align*}
\]

results in the final form needed for studying the gauge dependence of the fermion propagator \( S_F \) in QED [4].

\[
\frac{\partial}{\partial \xi} S_F^{-1}(p) = S_F^{-1}(p) \left[ F(p, -p, 0) + \bar{F}(-p, p, 0) \right]
\]

Note that the Green functions \( F(p, -p, 0) \) and \( \bar{F}(p, -p, 0) \) cannot have single particle poles.

In on-shell renormalization schemes the bare mass \( m_0 \) and the renormalized mass \( M \) are related through the condition

\[
S_F^{-1}(p) \bigg|_{\hat{p} = M} = 0
\]

This results in the definition of the mass renormalization constant.

\[
\frac{m_0}{M} = Z_m
\]

The wave function renormalization constant \( Z_2 \) is the residue of \( S_F \) at the \( \hat{p} = M \) pole.

\[
Z_2 = \lim_{\hat{p} = M} (\hat{p} - M) S_F(p)
\]

\(^{1}\) An implicit coordinate integration is associated with the \( \chi \) derivative.
Perturbative expansions of $Z_m$ and $Z_2$ have been calculated to two-loop order in a scheme which dimensionally
regulates both the infrared and ultraviolet divergences, resulting in explicitly gauge independent expressions for
QED and QCD \[1\].

The mass renormalization $Z_m$, and hence $M$, has been proven to be gauge independent to all orders of pertur-
bation theory \[1, 4\]. Thus when both sides of (14) are divided by $p / M$ along with the gauge independence of
QED and QCD \[1\].

Perturbative expansions of $Z$ formalism. Since the
vertex function $\Gamma_{\mu}$ is defined by

$$\int d^4x e^{ip\cdot x} \langle 0 | T(B(x)B(0)) | O \rangle = 0 \tag{18}$$

$$\int d^4x e^{ip\cdot x} \langle 0 | T(B(x)A_{\mu}(0)) | O \rangle = \frac{p_{\mu}}{p^2} \equiv G_{\mu}(p) \tag{19}$$

$$\int d^4x e^{ip\cdot x} \langle 0 | T(A_{\mu}(x)A_{\nu}(0)) | O \rangle = i \left[ -\frac{g^{\mu\nu}}{p^2} + (1 - \xi) \frac{p^{\mu}p^{\nu}}{p^4} \right] \equiv D^{\mu\nu}(p) \tag{20}$$

BRS symmetry implies that (18) and (19) are valid to all orders in pertur-
bation theory \[4, 7\]. Thus when both sides of (11) are divided by
$p / M$ along with the gauge independence of
QED and QCD \[1\].

This is our central result for QED: the gauge dependence of the wave function renormalization constant is related
to the on-shell properties of the Green function $F(p, -p, 0) + \tilde{F}(p, -p, 0)$. In particular, if this Green function is
zero on-shell, then $Z_2$ is gauge independent.

Before studying the on-shell behaviour of $F(p, -p, 0) + \tilde{F}(p, -p, 0)$ we review some aspects of the auxiliary field
formalism. Since the $B$ field and $\partial \cdot A$ are mixed in the Lagrangian (2) the quadratic part of the Lagrangian must
be diagonalized, leading to the free field propagators

$$\int d^4x e^{ip\cdot x} \langle 0 | T(B(x)B(0)) | O \rangle = 0 \tag{18}$$

$$\int d^4x e^{ip\cdot x} \langle 0 | T(B(x)A_{\mu}(0)) | O \rangle = \frac{p_{\mu}}{p^2} \equiv G_{\mu}(p) \tag{19}$$

$$\int d^4x e^{ip\cdot x} \langle 0 | T(A_{\mu}(x)A_{\nu}(0)) | O \rangle = i \left[ -\frac{g^{\mu\nu}}{p^2} + (1 - \xi) \frac{p^{\mu}p^{\nu}}{p^4} \right] \equiv D^{\mu\nu}(p) \tag{20}$$

As illustrated in Figure 1, the (QED) Green function $F(p, -p, 0)$ is easily written in terms of one-particle
irreducible Green functions

$$F(p, -p, 0) = \int d^4Dk \Gamma_{\mu}(k, p)G_{\mu}(k)S_{F}(p + k)\tilde{D}(k^2) \tag{21}$$

where $\tilde{D}(k^2)$ is the ghost propagator (which for QED corresponds to the free field result) and the fermion-photon
vertex function $\Gamma_{\mu}$ is defined by

$$S_{F}(p)\Gamma_{\mu}(p, k)S_{F}(p + k)D^{\mu\nu}(k) = \int d^4x \int d^4y e^{ik\cdot x + ip\cdot y} \langle 0 | T[\psi(x)A_{\mu}(x)\bar{\psi}(y)] | O \rangle \tag{22}$$

Substituting (13) and the (free-field) ghost propagator into (21) and using the Ward identity for the vertex function

$$k^\mu \Gamma_{\mu}(p, k) = S_{F}^{-1}(p + k) - S_{F}^{-1}(p) \tag{23}$$

simplifies the expression for $F(p, -p, 0)$.

$$F(p, -p, 0) = iS_{F}^{-1}(p) \int d^4Dk \frac{1}{k^4}S_{F}(p + k) - i \int d^4Dk \frac{1}{k^4} \tag{24}$$
Figure 1: Feynman diagram expressing $F(p, -p, 0)$ in terms of one-particle irreducible functions represented by the solid circles. Dashed lines represent the ghost field, and the dotted line represents the auxiliary field $B$. Composite operators coupled to the currents are represented by the partially-filled circles.

The second term in the above equation is a massless tadpole which is zero in dimensional regularization, leading to the final expression for $F(p, -p, 0)$ in QED.

$$F(p, -p, 0) = iS_F^{-1}(p) \int d^Dk \frac{1}{k^4} S_F(p + k)$$

(25)

In the on-shell scheme infrared and ultraviolet divergences are dimensionally regulated, so the integral in (25) is finite on-shell. Thus the $S_F^{-1}(p)$ prefactor in (25) implies that $F(p, -p, 0)$ is zero at the $p / = M$ mass-shell. This argument can be trivially extended to $\bar{F}(p, -p, 0)$, and we conclude that to all orders in QED

$$F(p, -p, 0) + \bar{F}(p, -p, 0) \bigg|_{p = M} = 0$$

(26)

and hence from the result (17) we have proven the gauge independence of the QED renormalization constant $Z_2$ in mass-shell schemes.

An explicit illustration of the on-shell behaviour of $F(p, -p, 0) + \bar{F}(p, -p, 0)$ in the regularization scheme [1] to one-loop order requires evaluation of the diagram in Figure 2. In terms of the integrals (with the convention $D = 4 + 2\epsilon$)

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{[(k - p)^2 - m_0^2]^{\alpha \beta} k^2} = I[\alpha, \beta]$$

(27)

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu}{[(k - p)^2 - m_0^2]^{\alpha \beta} k^2} = p^\mu J[\alpha, \beta]$$

(28)

we find the one-loop expression for $F(p, -p, 0) + \bar{F}(p, -p, 0)$.

$$F(p, -p, 0) + \bar{F}(p, -p, 0) = 2ig^2 \left[ m_0 (p^2 - m_0) J(1, 2) + (p^2 + m_0^2) J(1, 2) - I(1, 1) \right]$$

(29)

and hence the on-shell behavior of $F + \bar{F}$ to one-loop order is given by

$$\lim_{p = M} \left[ F(p, -p, 0) + \bar{F}(p, -p, 0) \right] = 2ig^2 \lim_{p = M = m_0} \left[ 2m_0^2 J(1, 2) - I(1, 1) \right]$$

(30)

The desired on-shell values for the integrals in (30) can be reduced to evaluation of a single class of scalar integrals.

$$\Lambda[\alpha, \beta] = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[k^2 + 2p \cdot k]^{\alpha \beta} k^2}$$

(31)
Figure 2: Feynman diagram for one-loop contributions to $F(p, -p, 0)$. Dashed lines represent the ghost field, and the dotted line represents the auxiliary field $B$. Composite operators coupled to the currents are represented by the partially-filled circles.

a particular example being a relation between $J(\alpha, \beta)$ and $\Lambda(\alpha, \beta)$

$$\lim_{p=m_0} J(\alpha, \beta) = \frac{1}{2m_0^2} [\Lambda(\alpha, \beta - 1) - \Lambda(\alpha - 1, \beta)]$$

(32)

The integration by parts technique [8] for these on-shell integrals leads to recursion relations among the $\Lambda(\alpha, \beta)$. The identities

$$0 = \int d^D k \frac{\partial}{\partial k^\mu} \left( \frac{k^\mu}{[k^2 + 2p \cdot k]^\alpha} k^{2\beta} \right)$$

(33)

$$0 = \int d^D k \frac{\partial}{\partial k^\mu} \left( \frac{k^\mu}{[k^2 + 2p \cdot k]^\alpha} k^{2\beta} \right)$$

(34)

lead to the recursion relations

$$0 = -\beta \Lambda(\alpha - 1, \beta + 1) + (\beta - \alpha) \Lambda(\alpha, \beta) - 2\alpha m_0^2 \Lambda(\alpha + 1, \beta) + \alpha \Lambda(\alpha + 1, \beta - 1)$$

(35)

$$0 = (D - 2\beta - \alpha) \Lambda(\alpha, \beta) - \alpha \Lambda(\alpha + 1, \beta - 1)$$

(36)

The recursion relation (36) can also be obtained from dimensional analysis. These recursion relations allow the on-shell behaviour of the one-loop integrals, after setting mass tadpoles to zero, to be reduced to the fundamental dimensional regularization result

$$\Lambda(\alpha, 0) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 - m_0^2]^\alpha} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(-\epsilon) 2^{\alpha - 2 - \epsilon} m_0^{2\epsilon} m_0^{-\alpha} \Gamma(\alpha - 2 - \epsilon)}{\Gamma(\alpha)}$$

(37)

Using the above techniques it is simple to find the on-shell integrals required in (30).

$$J(1, 2) = \frac{i}{(4\pi)^{D/2} m_0^{2\epsilon}} \frac{\Gamma(-\epsilon) 2m_0^2(D - 3)}{m_0^2(D - 3)}$$

(38)

$$I(1, 1) = \frac{i}{(4\pi)^{D/2} m_0^{2\epsilon}} \frac{\Gamma(-\epsilon)}{(D - 3)}$$

(39)

and hence in the on-shell regularization scheme [1], the Green function $F + \bar{F}$ is zero on-shell to one-loop order, providing a specific example of our general result.
The gauge dependence of $Z_2$ in QCD can be formulated in a similar fashion. Analogous to (3) the Lagrangian for QCD becomes

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi} (i D - m) \psi + \frac{\xi}{2} B^2 + B \partial \cdot A - \bar{c} \partial^\mu D_\mu c + \frac{\chi}{2} \bar{c} B$$

(40)

which is invariant under an extended BRS symmetry

$$\delta^+ A_\mu = \epsilon D_\mu c , \quad \delta^+ \bar{\psi} = i \epsilon g c \bar{\psi} , \quad \delta^+ c = -\frac{1}{2} \epsilon g [c, \bar{c}]$$

$$\delta^+ B = 0 , \quad \delta^+ \bar{\psi} = -i \epsilon g c \bar{\psi} , \quad \delta^+ \bar{c} = B$$

(41)

$$\delta^+ \xi = \epsilon \chi , \quad \delta^+ \chi = 0$$

(42)

The extended BRS symmetry (41) implies the following identity for the effective action nearly identical in form to the QED identity (7)

$$0 = \frac{\delta \Gamma}{\delta K_\mu} \frac{\delta \Gamma}{\delta A_\mu} + \frac{\delta \Gamma}{\delta K} \frac{\delta \Gamma}{\delta \bar{\psi}} + \frac{\delta \Gamma}{\delta K} \frac{\delta \Gamma}{\delta \psi} + B \frac{\delta \Gamma}{\delta c} + \frac{\delta \Gamma}{\delta K} \frac{\delta \Gamma}{\delta c} + \frac{\partial \Gamma}{\partial \xi} \frac{\delta \Gamma}{\delta c}$$

(43)

where $K_\mu$ and $K_c$ are currents coupled to composite operators respectively coupled to the extended BRS variations of $A_\mu$ and $c$. Following the procedure used to develop (8) leads to a QCD expression in a similar form.

$$\frac{\partial}{\partial \xi} \frac{\delta^2 \Gamma}{\delta \bar{\psi}(y) \delta \psi(x)} = + \frac{\delta \Gamma}{\delta \psi(y) \delta K \delta \chi \delta \psi(x) \delta c} + \frac{\delta \Gamma}{\delta \psi(y) \delta \psi(x) \delta K \delta \chi}$$

(44)

After transforming to momentum space we find a result identical in form to (11).

$$\frac{\partial}{\partial \xi} S_F^{-1}(p) = S_F^{-1}(p) [F(p, -p, 0) + \tilde{F}(-p, p, 0)]$$

(45)

As in the QED case, we see that the necessary condition for gauge independence of $Z_2$ in QCD is for the Green function $F + \tilde{F}$ to be zero on shell. The distinction between QED and QCD occurs in the interactions, particularly the ghost-gluon interaction, which will contribute to $F(p, -p, 0)$. This is particularly evident at three loop level where diagrams (such as those in Figure 3) occur that cannot be related to the fundamental two- or three-point Green functions. Thus at three-loop level there is no simple extension of the result (21) from QED to QCD, and hence gauge independence of $Z_2$ in on-shell schemes seems problematic at the three-loop level and beyond in QCD.

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Figure 3: A three-loop QCD diagram contributing to \( F(p, -p, 0) \) which cannot be reduced to the the form \([21]\) composed of fundamental one-particle irreducible Green functions.

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