Variation of constants formulae for forward and backward stochastic Volterra integral equations

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Abstract

In this paper, we provide variation of constants formulae for linear (forward) stochastic Volterra integral equations (SVIEs, for short) and linear Type-II backward stochastic Volterra integral equations (BSVIEs, for short) in the usual Itô’s framework. For these purposes, we define suitable classes of stochastic Volterra kernels and introduce new notions of the products of adapted $L^2$-processes. Observing the algebraic properties of the products, we obtain the variation of constants formulae by means of the corresponding resolvent. Our framework includes SVIEs with singular kernels such as fractional stochastic differential equations. Also, our results can be applied to general classes of SVIEs and BSVIEs with infinitely many iterated stochastic integrals. The duality principle between generalized SVIEs and generalized BSVIEs is also proved.

Keywords: Stochastic Volterra integral equations; backward stochastic Volterra integral equations; variation of constants formulae.

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1 Introduction

In this paper, we provide variation of constants formulae for linear stochastic Volterra integral equations (SVIEs, for short) and linear backward stochastic Volterra integral equations (BSVIEs, for short). First of all, consider the following SVIE:

\[ X(t) = \varphi(t) + \int_0^t j(t,s)X(s) \, ds + \int_0^t k(t,s)X(s) \, dW(s), \quad t \in (0,T), \quad (1.1) \]

where \( W \) is a one-dimensional Brownian motion, \( j \) and \( k \) are given deterministic functions called the kernels, and \( \varphi \) is a given adapted process called the free term. If the kernels \( j(t,s) \) and \( k(t,s) \) do not depend on the first time-parameter \( t \), and if the free term is of the form \( \varphi(t) = x_0 + \int_0^t b(s) \, ds + \int_0^t \sigma(s) \, dW(s) \) for some constant \( x_0 \) and adapted processes \( b \) and \( \sigma \), then SVIE (1.1) is reduced to a linear stochastic differential equation (SDE, for short)

\[
\begin{cases}
  \text{d}X(t) = \{j(t)X(t) + b(t)\} \, dt + \{k(t)X(t) + \sigma(t)\} \, dW(t), \quad t \in (0,T), \\
  X(0) = x_0.
\end{cases}
\]

More importantly, SVIE (1.1) includes a class of fractional SDEs of the following form:

\[
\begin{cases}
  C^{D_0^+}_\alpha X(t) = j(t)X(t) + b(t) + \{k(t)X(t) + \sigma(t)\} \frac{dW(t)}{dt}, \quad t \in (0,T), \\
  X(0) = x_0,
\end{cases}
\]

where \( C^{D_0^+}_\alpha \) denotes the Caputo fractional differential operator of order \( \alpha \in (0,1] \). Indeed, by the definition, an adapted process \( X \) is called a solution to the above fractional SDE if it satisfies the integral equation

\[
X(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \{j(s)X(s) + b(s)\} \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \{k(s)X(s) + \sigma(s)\} \, dW(s), \quad t \in (0,T),
\]

where \( \Gamma(\alpha) \) is the Gamma function (see [36]). Thus, the above fractional SDE can be seen as SVIE (1.1) with the singular kernels

\[
j(t,s) = \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} j(s), \quad k(t,s) = \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} k(s),
\]

and the free term

\[
\varphi(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b(s) \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s) \, dW(s).
\]

Fractional differential systems are suitable tools to describe the dynamics of systems with memory effects and hereditary properties. There are many applications of fractional calculus in a variety of research fields including mathematical finance, physics, chemistry, biology, and other applied sciences. For detailed accounts of theory and applications of fractional calculus, see for example [8, 10, 24, 26] and the references cited therein.

It is well-known that a variation of constants formula for deterministic Volterra integral equations is an important tool in many qualitative theory such as the linear perturbation theory (see Chapter 11 of [11]), the invariant manifold theory (see [9]), and the linear-quadratic optimal control theory (see [15]). In the first half of this paper, we extend the variation of constants formula to the stochastic case (1.1).

A difficulty to solve SVIE (1.1) comes from the term \( \int_0^t k(t,s)X(s) \, dW(s) \). In order to treat the stochastic convolution term, Berger and Mizel [5] introduced an extended stochastic integral for anticipated integrands, and gave a variation of constants formula in terms of the anticipated stochastic integral. Also, Øksendal and Zhang [20, 21] studied SVIE (1.1), where the free term \( \varphi \) is not necessarily adapted, and the stochastic integral is taken in the Skorohod sense. They showed variation of constants formulae for (1.1) in extended frameworks of the functional process approach and the generalized white noise functional (Hida distribution) approach. Unfortunately, the frameworks of [5, 20, 21] cannot be applied to fractional SDEs due to some
where \( j \) parameters of (BSVIEs, for short), respectively. The difference between Type-I and Type-II BSVIEs is the role of the time-

Equations (1.3) and (1.4) are called linear Type-I and Type-II backw ard stochastic Volterra integral equations

adapted, fractional SDEs with delay and “noisy memory” (see Example 3.1). This class of SVIEs includes

integrals (see equation (3.1)), which is beyond the framework of \([5, 20, 21]\). This class of SVIEs includes

linear SVIEs with singular kernels such as fractional SDEs; in Example 3.25, we apply our general

above expressions, and all stochastic integrals are understood in the usual Itô’s sense. Second, our framework

main advantages of our frameworks are three fold. First, we do not need any anticipated calculus in the

\( Q \)-space of adapted

\( J \)- and the \( \ast \)-products, respectively (see Proposition 3.5 and Proposition 3.10). Then, we

show that the solution to the linear equation (1.2) is given by the variation of constants formula

\[ X = \varphi + J \ast X + K \ast X. \]  

(1.2)

Here, the \( \ast \)-product \( J \ast X \) with \( J = j \) corresponds to the convolution \( \int_0^t j(t,s)X(s)\,ds \) in terms of the Lebesgue integral, and the \( \ast \)-product \( K \ast X \) with \( K(t) = \int_0^t k(t,s)\,dW(s) \) corresponds to the stochastic convolution \( \int_0^t k(t,s)X(s)\,dW(s) \) in terms of the stochastic integral. The \( \ast \)-product for general adapted processes is defined by means of the Wiener–Itô chaos expansion (see Definition 3.4), and it has a representation by infinitely many iterated stochastic integrals in the classical Itô’s sense (see Proposition 3.6). We introduce suitable classes of stochastic Volterra kernels \( J \) and \( K \) and show that these spaces become Banach algebras with respect to the \( \ast \) and the \( \ast \)-products, respectively (see Proposition 3.3 and Proposition 3.10). Then, we show that the solution to the linear equation (1.2) is given by the variation of constants formula

\[ X = \varphi + Q \ast \varphi + R \ast \varphi, \]

where \( Q \) and \( R \) are resolvents of \( J \) and \( K \) in terms of the \( \ast \) and the \( \ast \)-products (see Theorem 3.22). The main advantages of our frameworks are three fold. First, we do not need any anticipated calculus in the above expressions, and all stochastic integrals are understood in the usual Itô’s sense. Second, our framework includes linear SVIEs with singular kernels such as fractional SDEs; in Example 3.25, we apply our general results to a time-fractional version of the Black–Scholes SDE and give an explicit expression of the solution. Third, we can apply our analysis to a more general class of SVIEs with (infinitely many) iterated stochastic integrals (see equation (3.1)), which is beyond the framework of \([5, 20, 21]\). This class of SVIEs includes fractional SDEs with delay and “noisy memory” (see Example 3.1).

Next, consider the following equations:

\[ Y(t) = \tilde{\psi}(t) + \int_t^T \{ j(s,t) \top Y(s) + k(s,t) \top Z(t,s) \} \,ds - \int_t^T Z(t,s) \,dW(s), \ t \in (0,T), \]

(1.3)

and

\[ Y(t) = \tilde{\psi}(t) + \int_t^T \{ j(s,t) \top Y(s) + k(s,t) \top Z(t,s) \} \,ds - \int_t^T Z(t,s) \,dW(s), \ t \in (0,T), \]

(1.4)

where \( j \) and \( k \) are given deterministic kernels, and \( \tilde{\psi} \) is a given (not necessarily adapted) stochastic process. Equations (1.3) and (1.4) are called linear Type-I and Type-II backward stochastic Volterra integral equations (BSVIEs, for short), respectively. The difference between Type-I and Type-II BSVIEs is the role of the time-parameters of \( Z(t,s) \) and \( Z(t,s) \) in the drivers. On the one hand, Type-I BSVIE (1.3) is an equation for a pair \((Y(\cdot), Z(\cdot, \cdot))\) with \( Z(t,s) \) defined on \( 0 < t < s < T \), and we call it an adapted solution to (1.3) if \( Y \) is adapted, \( Z(t,\cdot) = (Z(t,s))_{s \in (t,T)} \) is adapted for each \( t \in (0,T) \), and the equality (1.3) holds for a.e. \( t \in (0,T) \), a.s. Nonlinear Type-I BSVIEs were introduced by Lin [19] and Yong [37, 39]. Recently, many authors have applied Type-I BSVIEs to stochastic control and mathematical finance where the time-inconsistency is taken into account. For example, time-inconsistent dynamic risk measures and time-inconsistent recursive utilities can be modeled by the solutions to Type-I BSVIEs (see \([48, 18, 41, 30]\)). Time-inconsistent stochastic control problems related to Type-I BSVIEs were studied by Wang and Yong [31] and the author [12]. On the other hand, Type-II BSVIE (1.4) is an equation for \((Y(\cdot), Z(\cdot, \cdot))\) with \( Z(t,s) \) defined on \( (t,s) \in (0,T)^2 \), and we call it an adapted M-solution to (1.4) if \( Y \) is adapted, \( Z(t,\cdot) = (Z(t,s))_{s \in (0,T)} \) is adapted for each \( t \in (0,T) \), and the equality (1.4) together with the relation \( Y(t) = E[Y(t)] + \int_0^t Z(t,s) \,dW(s) \) hold for a.e. \( t \in (0,T) \), a.s. Type-II BSVIEs and the concept of adapted M-solutions were first introduced by Yong [39] and applied to the duality principle appearing in stochastic control problems for (forward) SVIEs. Since then, Type-II BSVIEs have been found important to study stochastic control problems of SVIEs (see \([6, 27, 28, 35, 33, 34, 49, 29, 18, 14]\).
Specifically, Chen and Yong [6] and Wang [33] showed that the optimal controls for linear-quadratic control problems of SVIEs are characterized by using the adapted M-solutions of linear Type-II BSVIEs. See also our previous papers [13, 14] for the counterparts in the infinite horizon setting.

For linear Type-I BSVIEs, Hu and Øksendal [16], Wang, Yong and Zhang [32], Ren, Coulibaly and Aman [25] and the author [13] obtained explicit formulae of the solutions in different settings. However, due to the difficulty of the appearance of the term $Z(s,t)$ in the driver, explicit solutions of linear Type-II BSVIEs have not been obtained in the literature, except for some concrete examples. In the latter half of this paper, we focus on linear Type-II BSVIEs. We provide a variation of constants formula for a general class of linear Type-II BSVIEs and give explicit expressions of the adapted M-solutions. For this purpose, we introduce new notions of the products which we call the backward $\ast$-product and the backward $\ast$-product for adapted processes (see Definition 4.3 and Definition 4.8). Then, regarding a generalized class of linear Type-II BSVIEs as a simple algebraic equation including (iterated) martingale representation operations, we obtain the variation of constants formula (see Theorem 4.12). Furthermore, we provide a duality principle between linear SVIEs and linear Type-II BSVIEs with infinitely many iterated stochastic integrals, which extends the result of Yong [39] to a more general framework (see Theorem 4.15).

The established variation of constants formulae and the duality principle in this paper are natural extensions of the counterparts in the infinite horizon setting.

We summarize the contributions of this paper.

- We introduce novel kinds of products for adapted $L^2$-processes and show their fundamental properties.
- We introduce and study the corresponding resolvents, which play crucial roles for solving SVIEs and BSVIEs.
- We prove variation of constants formulae for SVIEs and BSVIEs, which provide explicit expressions of the Wiener–Itô chaos expansions of the solutions.
- We provide useful sufficient conditions for the existence of the resolvent with respect to the $\ast$-product.

This paper is organized as follows. In Section 2 we recall the Wiener–Itô chaos expansion and introduce the notion of deterministic Volterra kernels, which play fundamental roles in our study. Section 3 is concerned with linear SVIEs. In Section 3.1 and Section 3.2 we introduce the notion of the $\ast$- and the $\ast$-products, respectively. Resolvents with respect to these products are studied in Section 3.3. Then, in Section 3.4 we show the variation of constants formula for (generalized) SVIEs, together with an application to the fractional Black–Scholes SDE. Section 4 is concerned with linear BSVIEs. Introducing the notions of the backward $\ast$- and the backward $\ast$-products in Section 4.1 and Section 4.2, respectively, we show in Section 4.3 the variation of constants formula for (generalized) BSVIEs. Also, we show the duality principle between SVIEs and BSVIEs in the general framework. Lastly, in Section 5 we study the existence of the resolvent with respect to the $\ast$-product.

**Notations**

$(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, and $W$ is a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. For each $s \geq 0$, $F^s = (\mathcal{F}_t^s)_{t \geq s}$ denotes the $\mathbb{P}$-augmentation of the filtration generated by $(W(t) - W(s))_{t \geq s}$. When $s = 0$, we denote $F^0 = (\mathcal{F}_t^0)_{t \geq 0}$ by $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$. $\mathbb{E}[\cdot]$ denotes the expectation. For each $t \geq 0$, $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$ is the conditional expectation given by $\mathcal{F}_t$.

For each $d_1, d_2 \in \mathbb{N}$, we denote the space of $(d_1 \times d_2)$-matrices by $\mathbb{R}^{d_1 \times d_2}$. We define $\mathbb{R}^{d_1} := \mathbb{R}^{d_1 \times 1}$, that is, each element of $\mathbb{R}^{d_1}$ is understood as a column vector. For each $A \in \mathbb{R}^{d_1 \times d_2}$, $A^\top$ denotes the transpose of $A$. $|A| := \sqrt{\text{tr}(AA^\top)}$ denotes the Frobenius norm, and $|A|_{\text{op}}$ denotes the operator norm of $A$ as an linear operator from $\mathbb{R}^{d_2}$ to $\mathbb{R}^{d_1}$ with respect to the Euclidean norms. For each $d \in \mathbb{N}$, $I_d \in \mathbb{R}^{d \times d}$ is the identity matrix. For each $n \in \mathbb{N}$ and $0 \leq S < T < \infty$, we denote by $\Delta_n(S,T)$ the set of $n$-tuples $(t_1, t_2, \ldots, t_n)$ such that
that \( T > t_1 > t_2 > \cdots > t_n > S \). For each \( t = (t_1, \ldots, t_n) \in \Delta_n(S, T) \), we sometimes use the notations \( \max t := t_1 \) and \( \min t := t_n \). We note that \( \Delta_1(S, T) = (S, T) \). Also, we define \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

For each \( 0 \leq S < T < \infty \), \( d_1, d_2 \in \mathbb{N} \) and \( n \in \mathbb{N} \), we define the following spaces:

- \( L^2(\Delta_n(S, T); \mathbb{R}^{d_1 \times d_2}) \): the set of \( \mathbb{R}^{d_1 \times d_2} \)-valued measurable (deterministic) functions \( \xi \) on \( \Delta_n(S, T) \) such that \( \|\xi\|_{L^2(\Delta_n(S, T))} < \infty \), where

\[
\|\xi\|_{L^2(\Delta_n(S, T))} := \left( \int_S^T \int_S^{t_1} \cdots \int_S^{t_{n-1}} |\xi(t_1, t_2, \ldots, t_n)|^2 \, dt_n \cdots dt_1 \right)^{1/2}.
\]

- \( L^2(\Delta_n(S, T); \mathbb{R}^{d_1 \times d_2}) \): the set of \( \mathbb{R}^{d_1 \times d_2} \)-valued \( \mathcal{F} \otimes \mathcal{B}(\Delta_n(S, T)) \)-measurable random fields \( \xi \) on \( \Omega \times \Delta_n(S, T) \) such that \( \|\xi\|_{L^2(\Delta_n(S, T))} < \infty \), where

\[
\|\xi\|_{L^2(\Delta_n(S, T))} := \mathbb{E} \left( \left( \int_S^T \int_S^{t_1} \cdots \int_S^{t_{n-1}} |\xi(t_1, t_2, \ldots, t_n)|^2 \, dt_n \cdots dt_1 \right)^{1/2} \right).
\]

- \( L^2(\Delta_n(S, T); \mathbb{R}^{d_1 \times d_2}) \): the set of \( \xi \in L^2(\Delta_n[S, T]; \mathbb{R}^{d_1 \times d_2}) \) such that \( \xi(t_1, t_2, \ldots, t_n) \) is \( \mathcal{F}_{t_1}^S \)-measurable for any \( (t_1, t_2, \ldots, t_n) \in \Delta_n(S, T) \). We denote \( \|\xi\|_{L^2(\Delta_n(S, T))} := \|\xi\|_{L^2(\Delta_n(S, T))} \).

- \( L^2(\Delta_n(S, T); \mathbb{R}^{d_1 \times d_2}) \): the set of \( \xi \in L^2(\Delta_n[S, T]; \mathbb{R}^{d_1 \times d_2}) \) such that \( \xi(t_1, t_2, \ldots, t_n) \) is \( \mathcal{F}_{t_1}^\infty \)-measurable for any \( (t_1, t_2, \ldots, t_n) \in \Delta_n(S, T) \). We denote \( \|\xi\|_{L^2(\Delta_n(S, T))} := \|\xi\|_{L^2(\Delta_n(S, T))} \).

When \( n = 1 \), we denote the spaces \( L^2(\Delta_1(S, T); \mathbb{R}^{d_1 \times d_2}) \), \( L^2(\Delta_1(S, T); \mathbb{R}^{d_1 \times d_2}) \) and \( L^2(\Delta_1(S, T); \mathbb{R}^{d_1 \times d_2}) \) by \( L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), \( L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) and \( L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), respectively.

Here we summarize the notations which we will introduce in the following sections. Let \( 0 \leq S < T < \infty \) and \( d_1, d_2, d_3 \in \mathbb{N} \) be fixed.

- For each \( n \in \mathbb{N}_0 \), \( \mathfrak{M}_n : L^2(\Delta_n(S, T); \mathbb{R}^{d_1 \times d_2}) \to L^2(\Delta_n(S, T); \mathbb{R}^{d_1 \times d_2}) \) and \( \mathfrak{M}_n : L^2(\Delta_n(S, T); \mathbb{R}^{d_1 \times d_2}) \to L^2(\Delta_n(S, T); \mathbb{R}^{d_1 \times d_2}) \) denote the \( n \)-th iterated stochastic integral operators. See Definition 2.4

- For each \( \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) and \( \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), \( \mathfrak{M}_n[\xi] \) and \( \mathfrak{M}_n[\xi] \) denote the integrands appearing in the \( n \)-th Wiener–Itô chaos expansions of \( \xi \) and \( \xi \), and \( \mathfrak{M}_n[\xi] := \mathfrak{M}_n[\mathfrak{M}_n[\xi]] \), \( \mathfrak{M}_n[\xi] := \mathfrak{M}_n[\mathfrak{M}_n[\xi]] \). See Proposition 2.2

- For each \( n \in \mathbb{N}_0 \), \( \mathfrak{V}_{n+1}(S, T; \mathbb{R}^{d_1 \times d_2}) \) denotes the space of \( (n + 1) \)-parameters deterministic Volterra kernels. See Definition 2.3

- For each \( f : \Delta_{m+1}(S, T) \to \mathbb{R}^{d_1 \times d_2} \) and \( g : \Delta_{n+1}(S, T) \to \mathbb{R}^{d_2 \times d_1} \) with \( m, n \in \mathbb{N}_0 \), \( f \circ g : \Delta_{m+n+1}(S, T) \to \mathbb{R}^{d_1 \times d_2} \) denotes the \( \circ \)-product. See Definition 2.6

- \( K_{\mathcal{F}}(S, T; \mathbb{R}^{d_1 \times d_2}) \) denotes the space of \( \ast \)-Volterra kernels. See Definition 3.2

- For each \( K \in K_{\mathcal{F}}(S, T; \mathbb{R}^{d_1 \times d_2}) \) and \( \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), \( K \ast \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) denotes the \( \ast \)-product. See Definition 3.3

- For each \( f \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) and \( f \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) with \( m, n \in \mathbb{N}_0 \), \( f \ast g : \Omega \times \Delta_{n+m+2}(S, T) \to \mathbb{R}^{d_1 \times d_2} \) denote the \( \ast \)-products. See Definition 3.7

- \( J_{\mathcal{F}}(S, T; \mathbb{R}^{d_1 \times d_2}) \) denotes the space of \( \ast \)-Volterra kernels. See Definition 4.3

- For each \( n \in \mathbb{N}_0 \), \( \mathcal{M}_n : L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \to L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) denotes the \( n \)-th martingale representation operator. See Definition 4.11

- For each \( K \in K_{\mathcal{F}}(S, T; \mathbb{R}^{d_1 \times d_2}) \) and \( \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), \( K \ast \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) denotes the backward \( \ast \)-product. See Definition 4.3

- For each \( J \in J_{\mathcal{F}}(S, T; \mathbb{R}^{d_1 \times d_2}) \) and \( \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), \( J \ast \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) denotes the backward \( \ast \)-product. See Definition 4.8

In the following sections, we fix \( 0 \leq S < T < \infty \) and \( d \in \mathbb{N} \).
2 Preliminaries

2.1 The Wiener–Itô chaos expansion

For each \( \xi \in L^2(\Delta_{n+1}(S,T); \mathbb{R}^{d_1 \times d_2}) \) with \( d_1, d_2 \in \mathbb{N} \) and \( n \in \mathbb{N} \), the iterated stochastic integral

\[
\int_S^t \int_S^{t_1} \cdots \int_S^{t_{n-1}} \xi(t, t_1, \ldots, t_n) \, dW(t_n) \cdots dW(t_1), \quad t \in (S, T),
\]

is well-defined as an element of \( L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \). Furthermore, the following isometry holds:

\[
\mathbb{E} \left[ \left| \int_S^T \int_S^{t_1} \cdots \int_S^{t_{n-1}} \xi(t, t_1, \ldots, t_n) \, dW(t_n) \cdots dW(t_1) \right|^2 \right]^{1/2} = \| \xi \|_{L^2(\Delta_{n+1}(S,T))}.
\]

From the Wiener–Itô chaos expansion in Itô [17], we see that each adapted process \( \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) has an orthogonal expansion in terms of iterated stochastic integrals with deterministic integrands. We introduce the following notation.

**Definition 2.1.** Let \( d_1, d_2 \in \mathbb{N} \).

(i) We define operators \( \mathfrak{M}_n : L^2(\Delta_{n+1}(S,T); \mathbb{R}^{d_1 \times d_2}) \to L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), \( n \in \mathbb{N}_0 \), by

\[
\mathfrak{M}_0[f_0](t) := f_0(t), \quad t \in (S, T),
\]

for each \( f_0 \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), and

\[
\mathfrak{M}_n[f_n](t) := \int_S^t \int_S^{t_1} \cdots \int_S^{t_{n-1}} f_n(t, t_1, \ldots, t_n) \, dW(t_n) \cdots dW(t_1), \quad t \in (S, T),
\]

for each \( f_n \in L^2(\Delta_{n+1}(S,T); \mathbb{R}^{d_1 \times d_2}) \) and \( n \in \mathbb{N} \).

(ii) We define operators \( \mathfrak{N}_n : L^2(\Delta_{n+2}(S,T); \mathbb{R}^{d_1 \times d_2}) \to L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), \( n \in \mathbb{N}_0 \), by

\[
\mathfrak{N}_0[g_0](t, s) := g_0(t, s), \quad (t, s) \in \Delta_2(S,T),
\]

for each \( g_0 \in L^2(\Delta_2(S,T); \mathbb{R}^{d_1 \times d_2}) \), and

\[
\mathfrak{N}_n[g_n](t, s) := \int_S^t \int_S^{t_1} \cdots \int_S^{t_{n-1}} g_n(t, t_1, \ldots, t_n, s) \, dW(t_n) \cdots dW(t_1), \quad (t, s) \in \Delta_2(S,T),
\]

for each \( g_n \in L^2(\Delta_{n+2}(S,T); \mathbb{R}^{d_1 \times d_2}) \) and \( n \in \mathbb{N} \).

In the present framework, the Wiener–Itô chaos expansion theorem in Itô [17] can be stated as follows.

**Proposition 2.2** (The Wiener–Itô chaos expansion for stochastic processes). Let \( d_1, d_2 \in \mathbb{N} \).

(i) The space \( L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \) is decomposed into the infinite orthogonal sum:

\[
L^2(S, T; \mathbb{R}^{d_1 \times d_2}) = \bigoplus_{n=0}^{\infty} \mathfrak{N}_n[L^2(\Delta_{n+1}(S,T); \mathbb{R}^{d_1 \times d_2})].
\]

More precisely, for any \( \xi \in L^2(S, T; \mathbb{R}^{d_1 \times d_2}) \), there exists a unique sequence \( \{ \mathfrak{F}_n[\xi] \}_{n \in \mathbb{N}_0} \) with \( \mathfrak{F}_n[\xi] \in L^2(\Delta_{n+1}(S,T); \mathbb{R}^{d_1 \times d_2}) \) for each \( n \in \mathbb{N}_0 \) such that

\[
\sum_{n=0}^{\infty} \| \mathfrak{F}_n[\xi] \|_{L^2(\Delta_{n+1}(S,T))}^2 < \infty.
\]
We introduce the spaces of deterministic Volterra kernels and investigate their fundamental properties.

2.2 Deterministic Volterra kernels

Definition 2.3. Define the spaces

\[ \mathcal{W}_n[\xi], \mathcal{W}_n[\Xi] \]

for \( n \geq 0 \) with

\[ \mathcal{W}_n[\xi], \mathcal{W}_n[\Xi] := \mathcal{M}_n[\xi], \mathcal{M}_n[\Xi], \]

which are the projections of \( \xi \in L^2_{\mathcal{F}}(S, T; \mathbb{R}^{d_1 \times d_2}) \) and \( \Xi \in L^2_{\mathcal{F},\mathcal{F}^*(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_2})} \) on the \( n \)-th chaos for each \( n \in \mathbb{N}_0 \), respectively.

2.2.1

The space \( L^2_{\mathcal{F},\mathcal{F}^*}(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_2}) \) is decomposed into the infinite orthogonal sum:

\[ L^2_{\mathcal{F},\mathcal{F}^*}(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_2}) = \sum_{n=0}^{\infty} \mathcal{W}_n[\xi] \]

where the infinite sum in the right-hand side converges in \( L^2_{\mathcal{F},\mathcal{F}^*}(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_2}) \). Furthermore, the following isometry holds:

\[ \|\xi\|_{L^2_{\mathcal{F},\mathcal{F}^*}(\Delta_2(S, T))}^2 = \sum_{n=0}^{\infty} \|\mathcal{W}_n[\xi]\|_{L^2_{\mathcal{F},\mathcal{F}^*}(\Delta_2(S, T))}^2. \]

We sometimes write

\[ \mathcal{W}_n[\xi] := \mathcal{M}_n[\xi], \mathcal{W}_n[\Xi] := \mathcal{M}_n[\Xi], \]

which are the projections of \( \xi \in L^2_{\mathcal{F}}(S, T; \mathbb{R}^{d_1 \times d_2}) \) and \( \Xi \in L^2_{\mathcal{F},\mathcal{F}^*}(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_2}) \) on the \( n \)-th chaos for each \( n \in \mathbb{N}_0 \), respectively.

2.2 Deterministic Volterra kernels

We introduce the spaces of deterministic Volterra kernels and investigate their fundamental properties.

Definition 2.3. Define the spaces \( \mathcal{V}_{n+1}(S, T; \mathbb{R}^{d \times d}), n \in \mathbb{N}_0 \), of \((n+1)\)-parameters deterministic Volterra kernels by

\[ \mathcal{V}_{n+1}(S, T; \mathbb{R}^{d \times d}) := \{ k : \Delta_{n+1}(S, T) \to \mathbb{R}^{d \times d} \mid k \text{ is measurable and } \|k\|_{\mathcal{V}_{n+1}(S, T)} < \infty \}, \]

where \( \|k\|_{\mathcal{V}_n(S, T)} := \text{ess sup}_{t \in (S, T)} |k(t)|_{\text{op}} \) for \( n = 0 \), and

\[ \|k\|_{\mathcal{V}_{n+1}(S, T)} := \text{ess sup}_{t \in (S, T)} \left( \int_{\Delta_n(t, T)} |k(s, t)|^2_{\text{op}} \, ds \right)^{1/2} \]

for \( n \in \mathbb{N} \). Here and elsewhere, for each matrix \( A \in \mathbb{R}^{d \times d} \), \( |A|_{\text{op}} \) denotes the operator norm of \( A \) as an linear operator on \( \mathbb{R}^d \). (Recall that \( |A| := \sqrt{\text{tr}(AA^T)} \) denotes the Frobenius norm of \( A \).)
Remark 2.5. From the above lemma, we see that kernels. For example, the fractional kernel

\[ \int_{V} f(t, s) g(t, s) \, ds = \int_{V} \frac{f(t, s)}{t-s} \, ds \]

for a.e. \((t_0, t_1, t_2, \ldots, t_n) \in \Delta_{n+1}(S, T)\). Then \(k\) is in \(V_{n+1}(S, T; \mathbb{R}^{d \times d})\), and it holds that \(\|k\|_{V_{n+1}(S, T)} \leq \|a\|_{L^2((0, T-S)^n; \mathbb{R})}\).

**Remark 2.5.** From the above lemma, we see that \(V_{n+1}(S, T; \mathbb{R}^{d \times d})\) includes \(((n + 1)\)-parameters) singular kernels. For example, the fractional kernel \(k(t, s) = \frac{1}{t-s} (t-s)^{\alpha-1} \) with \(\alpha \in (\frac{1}{2}, 1]\) is in \(V_{2}(S, T; \mathbb{R})\).

We use the following notation frequently in this paper.

**Definition 2.6.** For each \(f : \Delta_{m+1}(S, T) \rightarrow \mathbb{R}^{d_1 \times d_2}\) and \(g : \Delta_{n+1}(S, T) \rightarrow \mathbb{R}^{d_2 \times d_3}\) with \(m, n \in \mathbb{N}\) and \(d_1, d_2, d_3 \in \mathbb{N}\), we define the \(\triangleright\)-product \(f \triangleright g : \Delta_{m+n+1}(S, T) \rightarrow \mathbb{R}^{d_1 \times d_3}\) by

\[ (f \triangleright g)(t_0, t_1, \ldots, t_{m+n}) := f(t_0, t_1, \ldots, t_m) g(t_m, t_{m+1}, \ldots, t_{m+n}) \]

for \((t_0, t_1, \ldots, t_{m+n}) \in \Delta_{m+n+1}(S, T)\). When \(m = 0\), \(f \triangleright g : \Delta_{n+1}(S, T) \rightarrow \mathbb{R}^{d_1 \times d_3}\) is defined by

\[ (f \triangleright g)(t, t_1, \ldots, t_n) := f(t) g(t, t_1, \ldots, t_n) \]

for \((t, t_1, \ldots, t_n) \in \Delta_{n+1}(S, T)\). When \(n = 0\), \(f \triangleright g : \Delta_{m+1}(S, T) \rightarrow \mathbb{R}^{d_1 \times d_3}\) is defined by

\[ (f \triangleright g)(t_0, t_1, \ldots, t_m) := f(t_0, t_1, \ldots, t_m) g(t_m) \]

for \((t_0, t_1, \ldots, t_m) \in \Delta_{m+1}(S, T)\).

**Lemma 2.7.** (i) For each \(k_1 \in V_{m+1}(S, T; \mathbb{R}^{d \times d})\) and \(k_2 \in V_{n+1}(S, T; \mathbb{R}^{d \times d})\) with \(m, n \in \mathbb{N}_0\), the \(\triangleright\)-product \(k_1 \triangleright k_2\) is in \(V_{m+n+1}(S, T; \mathbb{R}^{d \times d})\) and satisfies

\[ \|k_1 \triangleright k_2\|_{V_{m+n+1}(S, T)} \leq \|k_1\|_{V_{m+1}(S, T)} \|k_2\|_{V_{n+1}(S, T)}. \]

(ii) For each \(k \in V_{n+1}(S, T; \mathbb{R}^{d \times d})\) and \(f \in L^2(\Delta_{n+1}(S, T); \mathbb{R}^{d \times d})\) with \(m, n \in \mathbb{N}_0\) and \(d_1 \in \mathbb{N}\), the \(\triangleright\)-product \(k \triangleright f\) is in \(L^2(\Delta_{m+n+1}(S, T); \mathbb{R}^{d \times d})\) and satisfies

\[ \|k \triangleright f\|_{L^2(\Delta_{m+n+1}(S, T))} \leq \|k\|_{V_{m+1}(S, T)} \|f\|_{L^2(\Delta_{n+1}(S, T))}. \]

**Proof.** (i) Noting the inequality \(|AB|_{\text{op}} \leq |A|_{\text{op}} |B|_{\text{op}}\) for \(A, B \in \mathbb{R}^{d \times d}\) and using the Fubini theorem, we see that, for a.e. \(t \in (S, T)\),

\[ \int_{\Delta_{m+n}(t, T)} [(k_1 \triangleright k_2)(s, t)]_{\text{op}}^2 \, ds = \int_{\Delta_{m}(t, T)} \int_{\Delta_{n}(t, T)} |k_1(s_1, \max s_2) k_2(s_2, t)|^2_{\text{op}} \, ds_1 \, ds_2 \]

\[ \leq \int_{\Delta_{n}(t, T)} \int_{\Delta_{m}(t, T)} |k_1(s_1, \max s_2)|^2_{\text{op}} |k_2(s_2, t)|^2_{\text{op}} \, ds_1 \, ds_2 \]

\[ = \int_{\Delta_{n}(t, T)} \left\{ \int_{\Delta_{m}(t, T)} |k_1(s_1, \max s_2)|^2_{\text{op}} \, ds_1 \right\} |k_2(s_2, t)|^2_{\text{op}} \, ds_2 \]

\[ \leq \|k_1\|_{V_{m+1}(S, T)}^2 \int_{\Delta_{n}(t, T)} |k_2(s_2, t)|^2_{\text{op}} \, ds_2 \]

\[ \leq \|k_1\|_{V_{m+1}(S, T)}^2 \int_{\Delta_{n}(t, T)} |k_2(s_2, t)|^2_{\text{op}} \, ds_2 \]

\[ \leq \|k_1\|_{V_{m+1}(S, T)}^2 \|k_2\|_{V_{n+1}(S, T)}^2 < \infty. \]

This implies that \(k_1 \triangleright k_2 \in V_{m+n+1}(S, T; \mathbb{R}^{d \times d})\) and \(\|k_1 \triangleright k_2\|_{V_{m+n+1}(S, T)} \leq \|k_1\|_{V_{m+1}(S, T)} \|k_2\|_{V_{n+1}(S, T)}\).
(ii) Noting the inequality $|AB| \leq |A|_{op}|B|$ for $A \in \mathbb{R}^{d \times d}$ and $B \in \mathbb{R}^{d \times d_1}$, we can prove the assertion (ii) by the same way as in (i).

**Lemma 2.8.** Let $k_n \in \mathcal{V}_{n+1}(S,T; \mathbb{R}^{d \times d})$ for each $n \in \mathbb{N}_0$. Suppose that $\sum_{n=0}^{\infty} ||k_n||_{\mathcal{V}_{n+1}(S,T)} < \infty$. Then $\sum_{n=0}^{\infty} ||k_n||^2_{L^2(\Delta_{n+1}(S,T))} < \infty$. Consequently, the infinite sum of the iterated stochastic integrals $\sum_{n=0}^{\infty} \mathbb{W}_n[k_n]$ converges in $L^2_\mathbb{P}(S,T; \mathbb{R}^{d \times d})$.

**Proof.** By Lemma 2.3, for each $n \in \mathbb{N}$, $k_n$ is in $L^2(\Delta_{n+1}(S,T); \mathbb{R}^{d \times d})$, and it holds that $||k_n||_{L^2(\Delta_{n+1}(S,T))} \leq \sqrt{d(T-S)} ||k_n||_{\mathcal{V}_{n+1}(S,T)}$. Furthermore, by the assumption, we have $\sup_{n \in \mathbb{N}_0} ||k_n||_{\mathcal{V}_{n+1}(S,T)} < \infty$. Therefore, we have

$$\sum_{n=0}^{\infty} ||k_n||^2_{L^2(\Delta_{n+1}(S,T))} \leq d(T-S) \sum_{n=0}^{\infty} ||k_n||^2_{\mathcal{V}_{n+1}(S,T)} \leq d(T-S) \sup_{n \in \mathbb{N}_0} ||k_n||_{\mathcal{V}_{n+1}(S,T)} \sum_{n=0}^{\infty} ||k_n||_{\mathcal{V}_{n+1}(S,T)} < \infty.$$  

This completes the proof.

3 **Stochastic Volterra integral equations**

In this section, we investigate linear SVIE (3.1). More generally, we consider SVIEs with infinitely many iterated stochastic integrals of the following form:

$$X(t) = \varphi(t) + \int_S^t J(t,s)X(s) \, ds + \sum_{n=1}^{\infty} \int_S^t \int_S^{t_1} \cdots \int_S^{t_{n-1}} k_n(t,t_1,\ldots,t_n)X(t_n) \, dW(t_n) \cdots dW(t_1), \quad t \in (S,T),$$

with suitable choices of kernels $J \in L^2(\Delta_2(S,T); \mathbb{R}^{d \times d})$ and $k_n \in \mathcal{V}_{n+1}(S,T; \mathbb{R}^{d \times d})$, $n \in \mathbb{N}$. In this paper, we call the above equation a *generalized SVIE*. This class of SVIEs includes fractional SDEs with “noisy memory”, as demonstrated in the following example.

**Example 3.1.** Consider the following fractional equation of order $\alpha \in \left(\frac{1}{2}, 1\right)$:

$$\left\{\begin{array}{ll}
\text{C}_\alpha D_\alpha X(t) = j(t)X(t) + X_1(t) + X_2(t) + b(t) + \{k(t)X(t) + X_1(t) + X_2(t) + \sigma(t)\} \, dW(t), & t \in (0,T), \\
X_1(t) = \int_0^t \ell_1(t,s)X(s) \, ds, & X_2(t) = \int_0^t \ell_2(t,s)X(s) \, dW(s), & t \in (0,T), \\
X(0) = x_0,
\end{array}\right.$$  

(3.2)

where $j(t), k(t), \ell_1(t,s), \ell_2(t,s)$ are $\mathbb{R}^{d \times d}$-valued, deterministic and bounded coefficients, $b, \sigma \in L^2_\mathbb{P}(0,T; \mathbb{R}^d)$, and $x_0 \in \mathbb{R}^d$. This is a fractional SDE which has a delay $X_1(t) = \int_0^t \ell_1(t,s)X(s) \, ds$ and a “noisy memory” $X_2(t) = \int_0^t \ell_2(t,s)X(s) \, dW(s)$. The term noisy memory was first introduced by Dahl et al. [7] in the classical SDEs framework. By the definition, an adapted process $X$ is said to be a solution to (3.2) if it satisfies the integral equation

$$X(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{j(s)X(s) + \int_0^s \ell_1(s,r)X(r) \, dr + \int_0^s \ell_2(s,r)X(r) \, dW(r) + b(s)\right\} \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{k(s)X(s) + \int_0^s \ell_1(s,r)X(r) \, dr + \int_0^s \ell_2(s,r)X(r) \, dW(r) + \sigma(s)\right\} \, dW(s)$$

for $t \in (0,T)$. Applying the stochastic Fubini theorem, we see that the above integral equation becomes

$$X(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b(s) \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s) \, dW(s)$$

9
\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t \left\{ (t-s)^{\alpha-1} j(s) + \int_s^t (t-r)^{\alpha-1} \ell_1(r, s) \, dr + \int_s^t (t-r)^{\alpha-1} \ell_1(r, s) \, dW(r) \right\} X(s) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} k(t_1) + \int_{t_1}^t (t-s)^{\alpha-1} \ell_2(s, t_1) \, ds \right\} X(t_1) \, dW(t_1)
+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^{t_1} (t-t_1)^{\alpha-1} \ell_2(t_1, t_2) X(t_2) \, dW(t_2) \, dW(t_1), \quad t \in (0, T).
\]

Thus, the fractional SDE with noisy memory (3.2) can be seen as a generalized SVIE (3.1) with the free term
\[
\varphi(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b(s) \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s) \, dW(s)
\]
and the kernels
\[
J(t, s) = \frac{1}{\Gamma(\alpha)} \left\{ (t-s)^{\alpha-1} j(s) + \int_s^t (t-r)^{\alpha-1} \ell_1(r, s) \, dr + \int_s^t (t-r)^{\alpha-1} \ell_1(r, s) \, dW(r) \right\},
\]
\[
k_1(t, t_1) = \frac{1}{\Gamma(\alpha)} \left\{ (t-t_1)^{\alpha-1} k(t_1) + \int_{t_1}^t (t-s)^{\alpha-1} \ell_2(s, t_1) \, ds \right\},
\]
\[
k_2(t, t_1, t_2) = \frac{1}{\Gamma(\alpha)} (t-t_1)^{\alpha-1} \ell_2(t_1, t_2), \quad k_n = 0, \quad n \geq 3.
\]

Note that since \(J(t, s)\) is \(\mathcal{F}_t^\ast\)-measurable, \(J\) is in \(L_{F, s}^2 \left( \Delta_2(0, T); \mathbb{R}^{d \times d} \right)\).

In order to investigate generalized SVIE (3.1), we write it as an algebraic equation on \(L_2^2(S, T; \mathbb{R}^d)\) as follows:
\[
X = \varphi + J \ast X + K \ast X, \tag{3.3}
\]
where \(J \ast X\) corresponds to the convolution in terms of the Lebesgue integral, and \(K \ast X\) with \(K = \sum_{n=1}^{\infty} \mathcal{M}_n [k_n]\) corresponds to the infinite sum of the stochastic convolutions with respect to the iterated stochastic integrals.

### 3.1 *-product

In this subsection, we introduce a suitable class of \(K\) and define the *-product \(K \ast X\) appearing in equation (3.3).

**Definition 3.2.** We define the space \(\mathcal{K}_F(S, T; \mathbb{R}^{d \times d})\) of *-Volterra kernels by
\[
\mathcal{K}_F(S, T; \mathbb{R}^{d \times d}) := \{ K \in L_2^2(S, T; \mathbb{R}^{d \times d}) | \mathcal{S}_n [K] \in \mathcal{V}_{n+1}(S, T; \mathbb{R}^{d \times d}), \quad \forall n \in \mathbb{N}_0, \quad \| K \|_{\mathcal{K}_F(S, T)} < \infty \},
\]
where
\[
\| K \|_{\mathcal{K}_F(S, T)} := \sum_{n=0}^{\infty} \| \mathcal{S}_n [K] \|_{\mathcal{V}_{n+1}(S, T)}.
\]

**Remark 3.3.** We emphasize that each *-Volterra kernel \(K \in \mathcal{K}_F(S, T; \mathbb{R}^{d \times d})\) is an \(\mathbb{R}^S\)-adapted and square-integrable stochastic process (with one time-parameter), see Lemma 2.8. Intuitively speaking, the Volterra structure corresponds to the adaptability of the process \(K\). Since each \((\mathcal{V}_{n+1}(S, T; \mathbb{R}^{d \times d}), \| \cdot \|_{\mathcal{V}_{n+1}(S, T)})\) is a Banach space, we see that \((\mathcal{K}_F(S, T; \mathbb{R}^{d \times d}), \| \cdot \|_{\mathcal{K}_F(S, T)})\) is a Banach space.

**Definition 3.4.** For each *-Volterra kernel \(K \in \mathcal{K}_F(S, T; \mathbb{R}^{d \times d})\) and \(\xi \in L_2^2(S, T; \mathbb{R}^{d \times d_1})\) with \(d_1 \in \mathbb{N}\), we define the *-product \(K \ast \xi \in L_2^2(S, T; \mathbb{R}^{d \times d_1})\) by the following Wiener–Itô chaos expansion:
\[
\mathcal{S}_n [K \ast \xi] := \sum_{k=0}^{n} \mathcal{S}_{n-k} [K] \circ \mathcal{S}_k [\xi], \quad n \in \mathbb{N}_0.
\]
The following lemma shows that the $\ast$-product is well-defined and makes the space $\mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})$ a Banach algebra.

**Proposition 3.5.** $(\mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}), ||\cdot||_{\mathcal{K}_\mathcal{F}(S, T)}, \ast)$ is a (real) unitary Banach algebra with the unit $I_d$, where $I_d \in \mathbb{R}^{d \times d}$ denotes the identity matrix which can be viewed as an element of $\mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})$. Furthermore, $(L_2^\ast(S, T; \mathbb{R}^d), \|\cdot\|_{L_2^\ast(S, T)})$ is a left Banach module over $\mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})$ with respect to the $\ast$-product. In other words, $(\mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}), ||\cdot||_{\mathcal{K}_\mathcal{F}(S, T)})$ is a Banach space, and for each $K, K_1, K_2, K_3 \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})$, $\xi, \xi_1, \xi_2 \in L_2^\ast(S, T; \mathbb{R}^d)$ and $\alpha \in \mathbb{R}$, the following hold:

$$K_1 \ast K_2 \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \text{ and } ||K_1 \ast K_2||_{\mathcal{K}_\mathcal{F}(S, T)} \leq ||K_1||_{\mathcal{K}_\mathcal{F}(S, T)} ||K_2||_{\mathcal{K}_\mathcal{F}(S, T)},$$

$$(K_1 \ast K_2) \ast K_3 = K_1 \ast (K_2 \ast K_3),$$

$$K_1 \ast (K_2 + K_3) = K_1 \ast K_2 + K_1 \ast K_3,$$

$$(K_1 + K_2) \ast K_3 = K_1 \ast K_3 + K_2 \ast K_3,$$

$$\alpha(K_1 \ast K_2) = (\alpha K_1) \ast K_2 = K_1 \ast (\alpha K_2),$$

$$K \ast \xi \in L_2^\ast(S, T; \mathbb{R}^d) \text{ and } ||K \ast \xi||_{L_2^\ast(S, T)} \leq ||K||_{\mathcal{K}_\mathcal{F}(S, T)} ||\xi||_{L_2^\ast(S, T)},$$

$$(K_1 \ast K_2) \ast \xi = K_1 \ast (K_2 \ast \xi),$$

$$K \ast (\xi_1 + \xi_2) = K \ast \xi_1 + K \ast \xi_2,$$

$$(K_1 + K_2) \ast \xi = K_1 \ast \xi + K_2 \ast \xi,$$

$$\alpha(K \ast \xi) = (\alpha K) \ast \xi = K \ast (\alpha \xi),$$

$$I_d \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}), ||I_d||_{\mathcal{K}_\mathcal{F}(S, T)} = 1, K \ast I_d = I_d \ast K = K, \text{ and } I_d \ast \xi = \xi.$$

**Proof.** Let $K, K_1, K_2, K_3 \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}), \xi, \xi_1, \xi_2 \in L_2^\ast(S, T; \mathbb{R}^d)$ and $\alpha \in \mathbb{R}$. By the triangle inequality, Lemma 2.7 and Young’s convolution inequality, we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} ||\hat{\mathcal{S}}_{n-k}[K_1] \ast \hat{\mathcal{S}}_k[K_2]||_{V_{n+1}(S, T)} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} ||\hat{\mathcal{S}}_{n-k}[K_1] \ast \hat{\mathcal{S}}_k[K_2]||_{V_{n+1}(S, T)} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} ||\mathcal{S}_{n-k}[K_1]||_{V_{n+1}(S, T)} ||\mathcal{S}_k[K_2]||_{V_{n+1}(S, T)} \leq \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} ||\mathcal{S}_n[K_1]||_{V_{n+1}(S, T)} ||\mathcal{S}_n[K_2]||_{V_{n+1}(S, T)} = ||K_1||_{\mathcal{K}_\mathcal{F}(S, T)} ||K_2||_{\mathcal{K}_\mathcal{F}(S, T)} < \infty.$$

Therefore, $K_1 \ast K_2 \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})$ is well-defined and satisfies

$$||K_1 \ast K_2||_{\mathcal{K}_\mathcal{F}(S, T)} \leq ||K_1||_{\mathcal{K}_\mathcal{F}(S, T)} ||K_2||_{\mathcal{K}_\mathcal{F}(S, T)}.$$ 

Similarly, we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left|\left|\hat{\mathcal{S}}_{n-k}[K] \ast \hat{\mathcal{S}}_k[\xi]\right|\right|^2_{L_2^\ast(\Delta_{n+1}(S, T))} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left|\left|\hat{\mathcal{S}}_{n-k}[K] \ast \hat{\mathcal{S}}_k[\xi]\right|\right|^2_{L_2^\ast(\Delta_{n+1}(S, T))} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left|\left|\hat{\mathcal{S}}_{n-k}[K] \ast \hat{\mathcal{S}}_k[\xi]\right|\right|^2_{L_2^\ast(\Delta_{n+1}(S, T))} \leq \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \left|\left|\mathcal{S}_n[K]\right|\right|^2_{L_2^\ast(\Delta_{n+1}(S, T))} ||\mathcal{S}_n[\xi]||_{L_2^\ast(\Delta_{n+1}(S, T))} \leq ||K||_{\mathcal{K}_\mathcal{F}(S, T)} ||\xi||_{L_2^\ast(S, T)}^2 \leq \infty.$$

Therefore, $K \ast \xi \in L_2^\ast(S, T; \mathbb{R}^d)$ is well-defined and satisfies

$$||K \ast \xi||_{L_2^\ast(S, T)} \leq ||K||_{\mathcal{K}_\mathcal{F}(S, T)} ||\xi||_{L_2^\ast(S, T)}.$$
Furthermore, for each \( n \in \mathbb{N}_0 \),
\[
\mathcal{F}_n[(K_1 * K_2) * K_3] = \sum_{k=0}^{n} \mathcal{F}_{n-k}[K_1 * K_2] \triangleright \mathcal{F}_k[K_3]
\]
\[
= \sum_{k=0}^{n} \left( \sum_{\ell=0}^{n-k} \mathcal{F}_{n-k-\ell}[K_1] \triangleright \mathcal{F}_\ell[K_2] \right) \triangleright \mathcal{F}_k[K_3]
\]
\[
= \sum_{k=0}^{n} \sum_{\ell=0}^{n-\ell} \mathcal{F}_{n-\ell}[K_1] \triangleright (\mathcal{F}_{\ell-k}[K_2] \triangleright \mathcal{F}_k[K_3])
\]
\[
= \sum_{\ell=0}^{n} \left( \sum_{k=0}^{n-\ell} \mathcal{F}_{n-\ell}[K_1] \triangleright \mathcal{F}_\ell[K_2] \right) \triangleright \mathcal{F}_k[K_3]
\]
\[
= \sum_{\ell=0}^{n} \mathcal{F}_{n-\ell}[K_1] \triangleright \mathcal{F}_\ell[K_2 * K_3]
\]
\[
= \mathcal{F}_n[K_1 * (K_2 * K_3)],
\]
and hence \((K_1 * K_2) * K_3 = K_1 * (K_2 * K_3)\). Similarly, we can show that \((K_1 * K_2) * \xi = K_1 * (K_2 * \xi)\). The remaining assertions are clear. 

Although the \(*\)-product is defined by means of the Wiener–Itô chaos expansion, it can be represented in terms of iterated stochastic integrals.

**Proposition 3.6.** For each \(*\)-Volterra kernel \( K \in K_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \) and \( \xi \in L^2_\mathcal{F}(S, T; \mathbb{R}^{d \times d_1}) \) with \( d_1 \in \mathbb{N} \), it holds that
\[
(K * \xi)(t) = \mathcal{F}_0[K](t)(\xi)(t) + \sum_{n=1}^{\infty} \int^t_S \int^{t_1}_S \cdots \int^{t_{n-1}}_S \mathcal{F}_n[K](t, t_1, \ldots, t_n) dW(t_n) \cdots dW(t_1), \quad t \in (S, T),
\]
where the infinite sum in the right-hand side converges in \( L^2_\mathcal{F}(S, T; \mathbb{R}^{d \times d_1}) \).

**Proof.** We define \( \Phi^K_n[\xi](t) := \mathcal{F}_0[K](t)\xi(t), \quad t \in (S, T), \)
\[
\Phi^K_n[\xi](t) := \int^t_S \int^{t_1}_S \cdots \int^{t_{n-1}}_S \mathcal{F}_n[K](t, t_1, \ldots, t_n) \xi(t_n) dW(t_n) \cdots dW(t_1), \quad t \in (S, T),
\]
for each \( n \in \mathbb{N} \). Note that the integrand in the above iterated stochastic integral is in \( L^2_\mathcal{F}(\Delta_{n+1}(S, T); \mathbb{R}^{d \times d_1}) \), and thus \( \Phi^K_n[\xi] \in L^2_\mathcal{F}(S, T; \mathbb{R}^{d \times d_1}) \) is well-defined. By using the isometry of the stochastic integral, similar calculations as in Lemma 2.2 show that \( \|\Phi^K_n[\xi]\|_{L^2_\mathcal{F}(S, T)} \leq \|\mathcal{F}_n[K]\|_{L^2_\mathcal{F}(\Delta(S, T))} \|\xi\|_{L^2_\mathcal{F}(S, T)} \) for each \( n \in \mathbb{N}_0 \). Thus, each \( \Phi^K_n \) is a bounded linear operator on \( L^2_\mathcal{F}(S, T; \mathbb{R}^{d \times d_1}) \) with the operator norm \( \|\Phi^K_n\|_{op} \leq \|\mathcal{F}_n[K]\|_{op} \|\xi\|_{L^2_\mathcal{F}(S, T)} \). Since \( \sum_{n=0}^{\infty} \|\mathcal{F}_n[K]\|_{L^2_\mathcal{F}(\Delta)} = \|K\|_{L^2_\mathcal{F}(S, T)} < \infty \), the infinite sum in the right-hand side of (3.4) converges in \( L^2_\mathcal{F}(S, T; \mathbb{R}^{d \times d_1}) \). Noting the continuity of the linear operator \( \Phi^K_n \) and considering the Wiener–Itô chaos expansion of \( \xi \in L^2_\mathcal{F}(S, T; \mathbb{R}^{d \times d_1}) \), we have
\[
\Phi^K_n[\xi] = \Phi^K_n \left( \sum_{m=0}^{\infty} \mathcal{W}_m[\xi] \right) = \sum_{m=0}^{\infty} \Phi^K_n[\mathcal{W}_m[\xi]], \quad n \in \mathbb{N}_0,
\]
where \( \mathcal{W}_m[\xi] := \mathcal{W}_m[\mathcal{F}_m[\xi]] \) for each \( m \in \mathbb{N}_0 \). Observe that, for each \( n, m \in \mathbb{N} \),
\[
\Phi^K_n[\mathcal{W}_m[\xi]](t) = \int^t_S \int^{t_1}_S \cdots \int^{t_{n-1}}_S \mathcal{F}_n[K](t, t_1, \ldots, t_n)
\]
\[
\times \left( \int^{t_n}_{S} \int^{t_{n+1}}_{S} \cdots \int^{t_{n+m-1}}_{S} \mathcal{W}_m[\xi](t_n, t_{n+1}, \ldots, t_{n+m}) dW(t_{n+m}) \cdots dW(t_{n+1}) \right) dW(t_n) \cdots dW(t_1)
\]
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This completes the proof.

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Phi_n^K [\mathcal{M}_m [\mathcal{I}]] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{M}_{n+m} [\mathcal{I}] \mathcal{I}
= \sum_{n=0}^{\infty} \mathcal{M}_n [\mathcal{I}] \mathcal{I}
= \sum_{n=0}^{\infty} \mathcal{M}_n [\mathcal{I}] \mathcal{I}
= \mathcal{N}_n [\mathcal{I}]. \]

This completes the proof.

\[ 3.2 \text{-product} \]

Next, we introduce a suitable class of \( J \) and define the \(*\)-product \( J \times \mathcal{I} \) appearing in equation \( 3.3 \).

**Definition 3.7.** For each \( j \in L_{\mathcal{F}}^2 (\Delta_{n+2}(S,T); \mathbb{R}^{d_1 \times d_2}) \), \( f \in L_{\mathcal{F}}^2 (S,T; \mathbb{R}^{d_2 \times d_3}) \) with \( n \in \mathbb{N}_0 \) and \( d_1, d_2, d_3 \in \mathbb{N}_0 \), we define the \(*\)-product \( j \times f : \Omega \times \Delta_{n+1}(S,T) \to \mathbb{R}^{d_1 \times d_3} \) by

\[ (j \times f)(t_0, t_1, \ldots, t_n) := \int_S^{t_n} j(t_0, t_1, \ldots, t_n, s) f(s) \, ds \]

for \( (t_0, t_1, \ldots, t_n) \in \Delta_{n+1}(S,T) \). Also, for each \( g \in L_{\mathcal{F}}^2 (\Delta_{m+2}(S,T); \mathbb{R}^{d_2 \times d_3}) \) with \( m \in \mathbb{N}_0 \), we define the \(*\)-product \( j \times g : \Omega \times \Delta_{m+1}(S,T) \to \mathbb{R}^{d_1 \times d_3} \) by

\[ (j \times g)(t_0, t_1, \ldots, t_{n+m+1}) := \int_{t_{n+1}}^{t_{n+m+1}} j(t_0, t_1, \ldots, t_n, s) g(s, t_{n+1}, \ldots, t_{n+m+1}) \, ds \]

for \( (t_0, t_1, \ldots, t_{n+m+1}) \in \Delta_{m+1}(S,T) \).

Recall that \( L_{\mathcal{F}}^2 (\Delta \Delta_2(S,T); \mathbb{R}^{d_1 \times d_3}) \) is the space of \( \Xi \in L_{\mathcal{F}}^2 (\Delta \Delta_2(S,T); \mathbb{R}^{d_1 \times d_3}) \) with \( \Xi(t,s) \) being \( \mathcal{F}_t \)-measurable for each \( (t,s) \in \Delta_2(S,T) \). Also, recall the definitions of \( \mathcal{M}_n \) and \( \mathcal{K}_n \) (see Definition \( 2.1 \)).

**Lemma 3.8.** For each \( \Xi \in L_{\mathcal{F}}^2 (\Delta \Delta_2(S,T); \mathbb{R}^{d_1 \times d_3}) \) and \( \xi \in L_{\mathcal{F}}^2 (S,T; \mathbb{R}^{d_2 \times d_3}) \) with \( d_1, d_2, d_3 \in \mathbb{N}_0 \), the \(*\)-product \( \Xi \times \xi \) is in \( L_{\mathcal{F}}^2 (S,T; \mathbb{R}^{d_1 \times d_3}) \), and the Wiener–Itô chaos expansion satisfies

\[ \mathcal{N}_n [\mathcal{I} \times \xi] = \sum_{k=0}^{n} \mathcal{N}_{n-k} [\mathcal{I}] \mathcal{I} \mathcal{I}, n \in \mathbb{N}_0. \]

Furthermore, for each \( \Xi_1 \in L_{\mathcal{F}}^2 (\Delta \Delta_2(S,T); \mathbb{R}^{d_1 \times d_3}) \) and \( \Xi_2 \in L_{\mathcal{F}}^2 (\Delta \Delta_2(S,T); \mathbb{R}^{d_2 \times d_3}) \), the \(*\)-product \( \Xi_1 \times \Xi_2 \) is in \( L_{\mathcal{F}}^2 (\Delta \Delta_2(S,T); \mathbb{R}^{d_1 \times d_3}) \), and the Wiener–Itô chaos expansion satisfies

\[ \mathcal{N}_n [\mathcal{I}_1 \times \mathcal{I}_2] = \sum_{k=0}^{n} \mathcal{N}_{n-k} [\mathcal{I}_1] \mathcal{I}_1 \mathcal{I}_2 \mathcal{I}_2, n \in \mathbb{N}_0. \]
Proof. We prove the first assertion. The second can be proved by the same way. Note that $\Xi(t, s)$ is $\mathcal{F}_t^s$-measurable, and hence it is independent of $\mathcal{F}_s$. By using Minkowski’s inequality and Hölder’s inequality, we have

$$
E \left[ \int_S \left( \int_t^T |\Xi(t, s)| |\xi(s)| \, ds \right)^2 \, dt \right]^{1/2} \leq \int_S E \left[ \int_t^T |\Xi(t, s)|^2 |\xi(s)|^2 \, ds \right]^{1/2} \, dt
$$

$$
= \int_S E \left[ \int_t^T |\Xi(t, s)|^2 \, dt \right]^{1/2} E \left[ |\xi(s)|^2 \right]^{1/2} \, ds
$$

$$
= \left( \int_S E \left[ \int_t^T |\Xi(t, s)|^2 \, dt \right] \, ds \right)^{1/2} \left( \int_S E \left[ |\xi(s)|^2 \right] \, ds \right)^{1/2}
$$

$$
< \infty.
$$

Thus, the $*$-product $\Xi * \xi \in \mathcal{L}_F^2(S, T; \mathbb{R}^{d_1 \times d_3})$ is well-defined, and the operations $\Xi \mapsto \Xi * \xi$ and $\xi \mapsto \Xi * \xi$ are continuous. Noting the adaptedness, we have $\Xi * \xi \in \mathcal{L}_F^2(S, T; \mathbb{R}^{d_1 \times d_3})$. Furthermore, by using the stochastic Fubini’s theorem, we have

$$
(\Xi * \xi)(t) = \left\{ \left( \sum_{n=0}^{\infty} \mathbb{W}_n[\Xi] \right) \ast \left( \sum_{m=0}^{\infty} \mathbb{W}_m[\xi] \right) \right\}(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \int_S \int_t^T \cdots \int_t^{t_{n-k-1}} \mathcal{F}_{n-k}[\Xi](t, t_1, \ldots, t_{n-k}, s) \, dW(t_{n-k}) \cdots dW(t_1)
$$

$$
\times \int_S \int_t^T \cdots \int_t^{t_{n-k+1}} \mathcal{F}_{k}[\xi](s, t_{n-k+1}, \ldots, t_n) \, dW(t_n) \cdots dW(t_{n-k+1}) \, ds
$$

$$
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \int_S \int_t^T \cdots \int_t^{t_{n-k-1}} \int_S \int_t^T \cdots \int_t^{t_{n-k+1}} \mathcal{F}_{n-k}[\Xi](t, t_1, \ldots, t_{n-k}, s) \mathcal{F}_{k}[\xi](s, t_{n-k+1}, \ldots, t_n) \, dW(t_n) \cdots dW(t_1)
$$

$$
\times \int_S \int_t^T \cdots \int_t^{t_{n-k-1}} \mathcal{F}_{n-k}[\Xi](t, t_1, \ldots, t_{n-k}, s) \, dW(t_{n-k}) \cdots dW(t_1)
$$

$$
= \sum_{n=0}^{\infty} \int_S \int_t^T \cdots \int_t^{t_{n-k-1}} \int_S \int_t^T \cdots \int_t^{t_{n-k+1}} \mathcal{F}_{n-k}[\Xi](t, t_1, \ldots, t_{n-k}, s) \mathcal{F}_{k}[\xi](s, t_{n-k+1}, \ldots, t_n) \, dW(t_n) \cdots dW(t_1)
$$

$$
\times \int_S \int_t^T \cdots \int_t^{t_{n-k-1}} \mathcal{F}_{n-k}[\Xi](t, t_1, \ldots, t_{n-k}, s) \, dW(t_{n-k}) \cdots dW(t_1),
$$

which implies that $\mathfrak{F}_n[\Xi * \xi] = \sum_{k=0}^{n} \mathfrak{F}_{n-k}[\Xi] \ast \mathfrak{F}_k[\xi]$ for any $n \in \mathbb{N}_0$. This completes the proof. 

Definition 3.9. We define the space $\mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})$ of $*$-Volterra kernels by

$$
\mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) := \{ J \in L^2_F, (\Delta_2(S, T); \mathbb{R}^{d \times d}) \mid \| J \|_{\mathcal{J}_\mathcal{F}(S, T)} < \infty \},
$$

where

$$
\| J \|_{\mathcal{J}_\mathcal{F}(S, T)} := \left( \int_{\Delta_{+2}(S, T)} \mathcal{F}_n[J](t) \, dt \right)^{1/2}.
$$

The following proposition shows algebraic properties of the space $\mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})$ and the $*$-product.

Proposition 3.10. $(\mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}), \| \cdot \|_{\mathcal{J}_\mathcal{F}(S, T)}, *)$ is a (real) Banach algebra without unit. Furthermore, $(L^2_F(S, T; \mathbb{R}^d), \| \cdot \|_{L^2_F(S, T)})$ and $(\mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}), \| \cdot \|_{\mathcal{K}_\mathcal{F}(S, T)})$ are left Banach modules over $\mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})$ with respect to the $*$-product. In other words, $(\mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}), \| \cdot \|_{\mathcal{J}_\mathcal{F}(S, T)}, *)$ is a Banach space, and for each $J, J_1, J_2, J_3 \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}), \xi, \xi_1, \xi_2 \in L^2_F(S, T; \mathbb{R}^d), K, K_1, K_2 \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})$ and $\alpha \in \mathbb{R}$, the following hold:

$$
J_1 \ast J_2 \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \text{ and } \| J_1 \ast J_2 \|_{\mathcal{J}_\mathcal{F}(S, T)} \leq \| J_1 \|_{\mathcal{J}_\mathcal{F}(S, T)} \| J_2 \|_{\mathcal{J}_\mathcal{F}(S, T)},
$$

(3.5)
Similarly, for each \( \xi \in L^2_{\delta}(S, T; \mathbb{R}^d) \) and \( \| J \ast \xi \|_{H^s(S, T)} \leq \| \| J \|_{H^s(S, T)} \| \xi \|_{L^2(S, T)}, \) \( \alpha(J \ast J) = (\alpha J) \ast (\alpha J), \)

\[
J \ast \xi \in L^2_{\delta}(S, T; \mathbb{R}^d) \quad \text{and} \quad \| J \ast \xi \|_{L^2(S, T)} \leq \| \| J \|_{H^s(S, T)} \| \xi \|_{L^2(S, T)},
\]

(3.6)

\[
J \ast J \ast \xi = J \ast \xi,
\]

\[
J \ast (\xi_1 \ast \xi_2) = J \ast \xi_1 \ast J \ast \xi_2,
\]

\[
\alpha(J \ast J) = (\alpha J) \ast (\alpha J),
\]

\[
J \ast K \in L^2_{\delta}(S, T; \mathbb{R}^{d \times d}) \quad \text{and} \quad \| J \ast K \|_{K_\alpha(S, T)} \leq \| \| J \|_{H^s(S, T)} \| \| K \|_{K_\alpha(S, T)},
\]

(3.7)

\[
J \ast (K_1 \ast K_2) = J \ast K_1 \ast J \ast K_2,
\]

\[
\alpha(J \ast K) = (\alpha J) \ast (\alpha K).
\]

Proof. We prove (3.6), (3.8) and (3.7). In this proof, we use the notation

\[
\| f \|_{L^2(\Delta_n(S, T))} := \left( \int_{\Delta_n(S, T)} |f(t)|^2_{\text{op}} \, dt \right)^{1/2}
\]

for each \( f \in L^2(\Delta_n(S, T); \mathbb{R}^{d \times d}) \). First, we observe that, for each \( j \in L^2(\Delta_{n+2}(S, T); \mathbb{R}^{d \times d}) \) and \( f \in L^2(\Delta_{m+2}(S, T); \mathbb{R}^d) \) with \( n, m \in \mathbb{N}_0 \),

\[
\| J \ast f \|^2_{L^2(\Delta_{n+m+2}(S, T))} \leq \int_{S} \int_{S} \int_{S} \cdots \int_{S} \int_{S} \left( \int_{T_{n+1}} \left( \int_{T_{n+1}} \cdots \int_{T_{n+m}} \left| j(t_0, t_1, \ldots, t_n, s) \right|_{\text{op}} f(s, t_{n+1}, \ldots, t_{n+m+1}) \right) ds \right)^2 dt_{n+m+1} \cdots dt_1 \, d t_0
\]

\[
= \int_{\Delta_{m+1}(S, T)} \int_{\Delta_{n+1}(\max r, T)} \left( \int_{\Delta_{n+1}(\max r, T)} \left| j(t, s) \right|_{\text{op}} f(s, r) \right)^2 ds \, dr
\]

\[
\leq \int_{\Delta_{m+1}(S, T)} \left( \int_{\max r} T_{n+1} \left( \int_{\Delta_{n+1}(s, T)} \left| j(t, s) \right|_{\text{op}} f(s, r) \right)^2 dt \right)^{1/2} ds \, dr
\]

\[
\leq \int_{\Delta_{m+1}(S, T)} \left( \int_{\max r} T_{n+1} \left( \int_{\Delta_{n+1}(s, T)} \left| j(t, s) \right|_{\text{op}} f(s, r) \right)^2 dt ds \right) \left( \int_{\max r} T_{n+1} \left( \int_{\Delta_{n+1}(s, T)} \left| f(s, r) \right|^2 \right) ds \right) \, dr
\]

\[
\leq \| J \ast f \|^2_{L^2(\Delta_{n+m+2}(S, T))} \| f \|^2_{L^2(\Delta_{n+m+2}(S, T))},
\]

where we used Minkowski’s inequality in the second inequality and Hölder’s inequality in the third inequality.

This implies that

\[
\| J \ast f \|^2_{L^2(\Delta_{n+m+2}(S, T))} \leq \| J \ast f \|^2_{L^2(\Delta_{n+m+2}(S, T))} \| f \|^2_{L^2(\Delta_{n+m+2}(S, T))}.
\]

Similarly, for each \( f \in L^2(S, T; \mathbb{R}^d) \), it holds that

\[
\| J \ast f \|^2_{L^2(\Delta_{n+2}(S, T))} \leq \| J \ast f \|^2_{L^2(\Delta_{n+2}(S, T))} \| f \|^2_{L^2(\Delta_{n+2}(S, T))}.
\]

Also, for each \( j_1 \in L^2(\Delta_{n+2}(S, T); \mathbb{R}^{d \times d}) \) and \( j_2 \in L^2(\Delta_{m+2}(S, T); \mathbb{R}^{d \times d}) \) with \( n, m \in \mathbb{N}_0 \),

\[
\| j_1 \ast j_2 \|^2_{L^2(\Delta_{n+m+2}(S, T))} \leq \| j_1 \|^2_{L^2(\Delta_{n+2}(S, T))} \| j_2 \|^2_{L^2(\Delta_{m+2}(S, T))}.
\]
By the same way, we can show that, for each \( j \in L^2(\Delta_{n+2}(S,T); \mathbb{R}^{d \times d}) \) and \( k \in V_{m+1}(S,T; \mathbb{R}^{d \times d}) \) with \( n, m \in \mathbb{N}_0 \),

\[
\|j * k\|_{V_{n+m+1}(S,T)} \leq \|j\|_{L^2(\Delta_{n+2}(S,T)), \text{op}} \|k\|_{V_{n+1}(S,T)}.
\]

Fix \( J_1, J_2 \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \). Noting Lemma 3.8 the above observations and Young’s convolution inequality yield that

\[
\|J_1 * J_2\|_{\mathcal{J}_F(S,T)} = \sum_{n=0}^{\infty} \|\tilde{F}_n[J_1 * J_2]\|_{L^2(\Delta_{n+2}(S,T)), \text{op}}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \|\tilde{F}_{n-k}[J_1] * \tilde{F}_k[J_2]\|_{L^2(\Delta_{n+2}(S,T)), \text{op}}
\]
\[
\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \|\tilde{F}_{n-k}[J_1] * \tilde{F}_k[J_2]\|_{L^2(\Delta_{n+2}(S,T)), \text{op}}
\]
\[
\leq \sum_{n=0}^{\infty} \|\tilde{F}_n[J_1]\|_{L^2(\Delta_{n+2}(S,T)), \text{op}} \sum_{n=0}^{\infty} \|\tilde{F}_n[J_2]\|_{L^2(\Delta_{n+2}(S,T)), \text{op}}
\]
\[
= \|J_1\|_{\mathcal{J}_F(S,T)} \|J_2\|_{\mathcal{J}_F(S,T)}.
\]

Thus, \( 3.5 \) holds. For each \( J \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) and \( \xi \in L^2_F(S,T; \mathbb{R}^d) \), the isometry yields that

\[
\|J * \xi\|_{L^2_F(S,T)}^2 = \sum_{n=0}^{\infty} \|\tilde{F}_n[J * \xi]\|_{L^2(\Delta_{n+1}(S,T))}^2
\]
\[
= \sum_{n=0}^{\infty} \|\sum_{k=0}^{\infty} \tilde{F}_{n-k}[J] * \tilde{F}_k[\xi]\|_{L^2(\Delta_{n+1}(S,T))}^2
\]
\[
\leq \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \|\tilde{F}_{n-k}[J] * \tilde{F}_k[\xi]\|_{L^2(\Delta_{n+1}(S,T))} \right)^2
\]
\[
\leq \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \|\tilde{F}_{n-k}[J]\|_{L^2(\Delta_{n+k+2}(S,T)), \text{op}} \|\tilde{F}_k[\xi]\|_{L^2(\Delta_{n+k+1}(S,T))} \right)^2
\]
\[
\leq \left( \sum_{n=0}^{\infty} \|\tilde{F}_n[J]\|_{L^2(\Delta_{n+2}(S,T)), \text{op}} \|\tilde{F}_n[\xi]\|_{L^2(\Delta_{n+1}(S,T))} \right)^2
\]
\[
= \|J\|_{\mathcal{J}_F(S,T)} \|\xi\|_{L^2_F(S,T)}^2.
\]

which implies \( 3.6 \). Lastly, for each \( J \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) and \( K \in \mathcal{K}_F(S,T; \mathbb{R}^{d \times d}) \),

\[
\|J * K\|_{\mathcal{K}_F(S,T)} = \sum_{n=0}^{\infty} \|\tilde{F}_n[J * K]\|_{V_{n+1}(S,T)}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \|\tilde{F}_{n-k}[J] * \tilde{F}_k[K]\|_{V_{n+1}(S,T)} \right)
\]
\[
\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \|\tilde{F}_{n-k}[J] * \tilde{F}_k[K]\|_{V_{n+1}(S,T)}
\]
\[
\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \|\tilde{F}_{n-k}[J]\|_{L^2(\Delta_{n-k+2}(S,T)), \text{op}} \|\tilde{F}_k[K]\|_{V_{k+1}(S,T)}
\]
\[ \leq \sum_{n=0}^{\infty} ||\mathfrak{F}_n[J]||_{L^2(\Delta_{n+2}(S,T))} \|_\op \sum_{n=0}^{\infty} ||\mathfrak{F}_n[K]||_{V_{n+1}(S,T)} \]
\[ = ||J||_{\mathcal{J}(S,T)}||K||_{\mathcal{K}(S,T)}. \]

Thus, (3.7) holds.

For each \( J_1, J_2, J_3 \in \mathcal{J}(S,T; \mathbb{R}^{d \times d}) \), Fubini’s theorem yields that
\[(J_1 * J_2) * J_3(t,s) = \int_s^t (J_1 * J_2)(t,r)J_3(r,s) \, dr \]
\[= \int_s^t \int_r^t J_1(t,u)J_2(u,r) \, du J_3(r,s) \, dr \]
\[= \int_s^t J_1(t,u) \int_s^u J_2(u,r)J_3(r,s) \, dr \, du \]
\[= \int_s^t J_1(t,u)(J_2 * J_3)(u,s) \, du \]
\[= (J_1 * (J_2 * J_3))(t,s) \]
for \((t,s) \in \Delta_2(S,T)\). Thus, we have \((J_1 * J_2) * J_3 = J_1 * (J_2 * J_3)\). Similarly, we can show that \((J_1 * J_2) * \xi = J_1 * (J_2 * \xi)\) and \((J_1 * J_2) * K = J_1 * (J_2 * K)\) for \( \xi \in L_2^d(S,T; \mathbb{R}^d) \) and \( K \in \mathcal{K}(S,T; \mathbb{R}^{d \times d}) \). The other assertions are clear from the definition. We complete the proof.

Next, we investigate algebraic relationships between the \(*\)- and the \(*\)-products. As before, for each \(*\)-Volterra kernel \( K \in \mathcal{K}(S,T; \mathbb{R}^{d \times d}) \) and \( \Xi \in L_2^d(\Delta_2(S,T); \mathbb{R}^{d \times d_1}) \) with \( d_1 \in \mathbb{N} \), we define the \(*\)-product \( K * \Xi \in L_2^d(\Delta_2(S,T); \mathbb{R}^{d \times d_1}) \) by the Wiener–Itô chaos expansion
\[ \mathfrak{F}_n[K * \Xi] := \sum_{k=0}^{n} \mathfrak{F}_{n-k}[K] \triangleright \mathfrak{F}_k[\Xi], \quad n \in \mathbb{N}_0. \]

By Proposition 3.6 we see that
\[(K * \Xi)(t,s) = \mathfrak{F}_0[K](t)\Xi(t,s) + \sum_{n=1}^{\infty} \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} \mathfrak{F}_n[K](t,t_1,\ldots,t_n)\Xi(t_n,s) \, dW(t_n) \cdots dW(t_1) \]
for \((t,s) \in \Delta_2(S,T)\). Similar to Proposition 3.5 \((\mathcal{J}(S,T; \mathbb{R}^{d \times d}), || \cdot ||_{\mathcal{J}(S,T)}\) is a left Banach module over \((\mathcal{K}(S,T; \mathbb{R}^{d \times d}), || \cdot ||_{\mathcal{K}(S,T)}, \ast)\). Furthermore, the following proposition shows that the order of the \(*\)- and the \(*\)-products is exchangeable.

**Proposition 3.11.** For each \( K \in \mathcal{K}(S,T; \mathbb{R}^{d \times d}), J \in \mathcal{J}(S,T; \mathbb{R}^{d \times d}) \) and \( \xi \in L_2^d(S,T; \mathbb{R}^{d \times d_1}) \) with \( d_1 \in \mathbb{N} \), it holds that
\[ K * (J * \xi) = (K * J) * \xi \] (3.8)
and
\[ J * (K * \xi) = (J * K) * \xi. \] (3.9)

**Proof.** By the definition of the \(*\)-product and Lemma 3.8 we have, for each \( n \in \mathbb{N}_0 \),
\[ \mathfrak{F}_n[K * (J * \xi)] = \sum_{k=0}^{n} \mathfrak{F}_{n-k}[K] \triangleright \mathfrak{F}_k[J * \xi] \]
\[= \sum_{k=0}^{n} \mathfrak{F}_{n-k}[K] \triangleright \left( \sum_{\ell=0}^{k} \mathfrak{F}_{k-\ell}[J] * \mathfrak{F}_\ell[\xi] \right) \]
\[= \sum_{k=0}^{n} \sum_{\ell=0}^{k} \left( \mathfrak{F}_{n-k}[K] \triangleright \mathfrak{F}_{k-\ell}[J] \right) * \mathfrak{F}_\ell[\xi] \]
\[
\begin{align*}
&= \sum_{\ell=0}^{n} \left( \sum_{k=0}^{n} \mathfrak{F}_{n-k}[K] \triangleright \mathfrak{F}_{k-\ell}[J] \right) \ast \mathfrak{F}_{\ell}[\xi] \\
&= \sum_{\ell=0}^{n} \left( \sum_{k=0}^{n-\ell} \mathfrak{F}_{n-k}[K] \triangleright \mathfrak{F}_{k}[J] \right) \ast \mathfrak{F}_{\ell}[\xi] \\
&= \sum_{\ell=0}^{n} \mathfrak{F}_{n-\ell}[K \ast J] \ast \mathfrak{F}_{\ell}[\xi] \\
&= \mathfrak{F}_{n}[(K \ast J) \ast \xi].
\end{align*}
\]
Thus, (3.8) holds. Similarly,
\[
\begin{align*}
\mathfrak{F}_{n}[J \ast (K \ast \xi)] &= \sum_{k=0}^{n} \mathfrak{F}_{n-k}[J] \ast \mathfrak{F}_{k}[K \ast \xi] \\
&= \sum_{k=0}^{n} \mathfrak{F}_{n-k}[J] \ast \left( \sum_{\ell=0}^{k} \mathfrak{F}_{k-\ell}[K] \triangleright \mathfrak{F}_{\ell}[\xi] \right) \\
&= \sum_{k=0}^{n} \sum_{\ell=0}^{k} \mathfrak{F}_{n-k}[J] \ast \mathfrak{F}_{k-\ell}[K] \triangleright \mathfrak{F}_{\ell}[\xi] \\
&= \sum_{\ell=0}^{n} \left( \sum_{k=0}^{n-\ell} \mathfrak{F}_{n-k}[J] \ast \mathfrak{F}_{k}[K] \right) \triangleright \mathfrak{F}_{\ell}[\xi] \\
&= \sum_{\ell=0}^{n} \left( \sum_{k=0}^{n-\ell} \mathfrak{F}_{n-\ell-k}[J] \ast \mathfrak{F}_{k}[K] \right) \triangleright \mathfrak{F}_{\ell}[\xi] \\
&= \mathfrak{F}_{n}[(J \ast K) \ast \xi].
\end{align*}
\]
Thus, (3.9) holds.

**Remark 3.12.** By the associative properties of the \(\ast\) - and the \(\ast\)-products, we may use the notations \(K_{1} \ast K_{2} \ast K_{3}\), \(J_{1} \ast J_{2} \ast J_{3}\), \(J \ast K \ast \xi\), \(K \ast J \ast \xi\), and so on.

### 3.3 \(\ast\)-resolvent, \(\ast\)-resolvent and \((\ast, \ast)\)-resolvent

As in the classical theory on deterministic Volterra equations (see the textbook [11]), the resolvent with respect to the \(\ast\) - and the \(\ast\)-products play central roles in the study of linear SVIEs.

**Definition 3.13.** (i) For each \(\ast\)-Volterra kernel \(K \in \mathcal{K}_{\mathbb{F}}(S,T;\mathbb{R}^{d \times d})\), we say that a \(\ast\)-Volterra kernel \(R \in \mathcal{K}_{\mathbb{F}}(S,T;\mathbb{R}^{d \times d})\) is a \(\ast\)-resolvent of \(K\) if it satisfies the following resolvent equation:

\(R = K + K \ast R = K + R \ast K\).

(ii) For each \(\ast\)-Volterra kernel \(J \in \mathcal{J}_{\mathbb{F}}(S,T;\mathbb{R}^{d \times d})\), we say that a \(\ast\)-Volterra kernel \(Q \in \mathcal{J}_{\mathbb{F}}(S,T;\mathbb{R}^{d \times d})\) is a \(\ast\)-resolvent of \(J\) if it satisfies the following resolvent equation:

\(Q = J + J \ast Q = J + Q \ast J\).

First, we show that the \(\ast\) - and the \(\ast\)-resolvents are unique if they exist. More precisely, the following proposition holds.
Proposition 3.14. \( (i) \) Let \( * \)-Volterra kernels \( K, R_1, R_2 \in \mathcal{K}_F(S,T; \mathbb{R}^{d \times d}) \) be given. Suppose that \( R_1 = K + K * R_1 \) and \( R_2 = K + R_2 * K \). Then \( R_1 = R_2 \). In particular, each \( * \)-Volterra kernel has at most one \( * \)-resolvent.

(ii) Let \( * \)-Volterra kernels \( J, Q_1, Q_2 \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) be given. Suppose that \( Q_1 = J + J * Q_1 \) and \( Q_2 = J + Q_2 * J \). Then \( Q_1 = Q_2 \). In particular, each \( * \)-Volterra kernel has at most one \( * \)-resolvent.

The above proposition holds for any algebra (see Lemma 3.3 in [11]). Here, we give a proof for readers’ convenience.

Proof of Proposition 3.14. We prove (i). The assertion (ii) is proved by the same way. By the assumption, we have \( R_2 * (R_1 - K) = R_2 * K * R_1 = (R_2 - K) * R_1 \). Cancelling out \( R_2 * R_1 \), one gets \( R_2 * K = K * R_1 \). Thus, \( R_2 = K + R_2 * K = K + K * R_1 = R_1 \).

Remark 3.15. Let \( J \in L^2(\Delta_2(S,T); \mathbb{R}^{d \times d}) \) be a deterministic \( * \)-Volterra kernel, and suppose that \( Q \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) is a \( * \)-resolvent of \( J \). Then we see that \( \mathbb{E}[Q] \) is also a \( * \)-resolvent of \( J \). By the uniqueness of the \( * \)-resolvent, we have \( Q = \mathbb{E}[Q] \), and thus \( Q \) is deterministic.

Actually, the \( * \)-resolvent always exists.

Proposition 3.16. Every \( * \)-Volterra kernel \( J \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) has a unique \( * \)-resolvent \( Q \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \).

Proof. The uniqueness follows from Proposition 3.14. We show the existence. First, we assume that \( \|J\|_{\mathcal{J}_F(S,T)} < 1 \). By Proposition 3.10, we have \( \|J^n\|_{\mathcal{J}_F(S,T)} \leq \|J\|_{\mathcal{J}_F(S,T)}^n \) for any \( n \in \mathbb{N} \), where \( J^n \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) denotes the \((n-1)\)-fold \( * \)-product of \( J \) by itself. By the assumption, \( Q := \sum_{n=1}^{\infty} J^n \) converges in \( \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \). Observe that

\[
J + J * Q = J + J * \left( \sum_{n=1}^{\infty} J^n \right) = J + \sum_{n=1}^{\infty} J * J^n = \sum_{n=1}^{\infty} J^n = Q.
\]

Similarly, we have \( J + Q * J = Q \). Thus, \( Q \) is the \( * \)-resolvent of \( J \).

Now we consider the general case. For each \( \sigma > 0 \), define \( J_\sigma \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) by \( J_\sigma(t,s) := e^{-\sigma(t-s)} J(t,s) \) for \((t,s) \in \Delta_2(S,T)\). It is easy to see that

\[
\mathcal{F}_n[J_\sigma](t_0, t_1, \ldots, t_{n+1}) = e^{-\sigma(t_0-t_n)} \mathcal{F}_n[J](t_0, t_1, \ldots, t_{n+1})
\]

for \((t_0, t_1, \ldots, t_{n+1}) \in \Delta_{n+2}(S,T)\) and \( n \in \mathbb{N}_0 \). Noting the definition of \( \|J\|_{\mathcal{J}_F(S,T)} \) (see Definition 3.9), by the dominated convergence theorem, we see that \( \|J_\sigma\|_{\mathcal{J}_F(S,T)} \) tends to zero as \( \sigma \to \infty \). Let \( \sigma > 0 \) be such that \( \|J_\sigma\|_{\mathcal{J}_F(S,T)} < 1 \). Then there exists a unique \( * \)-resolvent \( Q_\sigma \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) of \( J_\sigma \). For \((t,s) \in \Delta_2(S,T)\), we have

\[
(J_\sigma * Q_\sigma)(t,s) = \int_s^t J_\sigma(t,r) Q_\sigma(r,s) \, dr = \int_s^t e^{-\sigma(t-r)} J(t,r) Q_\sigma(r,s) \, dr = e^{-\sigma(t-s)} \int_s^t J(t,r) e^{\sigma(r-s)} Q_\sigma(r,s) \, dr
\]

and

\[
(Q_\sigma * J_\sigma)(t,s) = \int_s^t Q_\sigma(t,r) J_\sigma(r,s) \, dr = \int_s^t Q_\sigma(t,r) e^{-\sigma(r-s)} J(r,s) \, dr = e^{-\sigma(t-s)} \int_s^t e^{\sigma(r-t)} Q_\sigma(t,r) J(r,s) \, dr.
\]

Therefore, the \( * \)-Volterra kernel \( Q \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) defined by \( Q(t,s) := e^{\sigma(t-s)} Q_\sigma(t,s) \) for \((t,s) \in \Delta_2(S,T)\) satisfies

\[
Q(t,s) = J(t,s) + \int_s^t J(t,r) Q(r,s) \, dr = J(t,s) + \int_s^t Q(t,r) J(r,s) \, dr
\]

for \((t,s) \in \Delta_2(S,T)\), and thus it is the \( * \)-resolvent of \( J \). This completes the proof. \( \Box \)
The case of the -resolvent is more difficult. We provide useful sufficient conditions for the existence of the -resolvent in Section [5].

The next lemma provides an explicit expression of the Wiener–Itô chaos expansion of the -resolvent.

**Lemma 3.17.** Suppose that a -Volterra kernel \( K \in \mathcal{K}_F(S, T; \mathbb{R}^{d \times d}) \) satisfies \( \mathcal{F}_0[K] = 0 \) (which is equivalent to \( \mathbb{E}[K] = 0 \)). If \( R \in \mathcal{C}_F(0, T; \mathbb{R}^{d \times d}) \) is a -resolvent of \( K \), then the Wiener–Itô chaos expansion is given by \( \mathcal{F}_0[R] = 0 \) and

\[
\mathcal{F}_n[R] = \sum_{i=1}^n \sum_{1 \leq m_1, m_2, \ldots, m_i \leq n} \mathcal{F}_{m_1}[K] \triangleright \mathcal{F}_{m_2}[K] \triangleright \ldots \triangleright \mathcal{F}_{m_i}[K], \quad n \in \mathbb{N}.
\]

**Proof.** By the definition, the Wiener–Itô chaos expansion of \( R \) satisfies the following equations:

\[
\mathcal{F}_n[R] = \mathcal{F}_n[K] + \sum_{k=0}^n \mathcal{F}_{n-k}[K] \triangleright \mathcal{F}_k[R], \quad n \in \mathbb{N}_0.
\]

Since \( \mathcal{F}_0[K] = 0 \), we have \( \mathcal{F}_1[R] = \mathcal{F}_1[K] \), and \( \mathcal{F}_n[R] = \mathcal{F}_n[K] + \sum_{k=1}^{n-1} \mathcal{F}_{n-k}[K] \triangleright \mathcal{F}_k[R] \) for \( n \geq 2 \). By the induction, we get the assertion. \( \square \)

Next, we introduce the notion of the \((*,*)\)-resolvent.

**Definition 3.18.** For each -Volterra kernel \( J \in \mathcal{J}_F(S, T; \mathbb{R}^{d \times d}) \) and -Volterra kernel \( K \in \mathcal{K}_F(S, T; \mathbb{R}^{d \times d}) \), we say that a pair \((Q, R) \in \mathcal{J}_F(S, T; \mathbb{R}^{d \times d}) \times \mathcal{K}_F(S, T; \mathbb{R}^{d \times d})\) is a \((*,*)\)-resolvent of \((J, K)\) if it satisfies the following resolvent equations:

\[
\begin{cases}
Q = J + J * Q + K * Q = J + Q * J + R * J, \\
R = K + J * R + K * R = K + Q * K + R * K.
\end{cases}
\]

The following is concerned with the uniqueness of the \((*,*)\)-resolvent.

**Lemma 3.19.** Let \((J, K) \in \mathcal{J}_F(S, T; \mathbb{R}^{d \times d}) \times \mathcal{K}_F(S, T; \mathbb{R}^{d \times d})\) be fixed. Suppose that \((Q_1, R_1), (Q_2, R_2) \in \mathcal{J}_F(S, T; \mathbb{R}^{d \times d}) \times \mathcal{K}_F(S, T; \mathbb{R}^{d \times d})\) satisfy

\[
\begin{cases}
Q_1 = J + J * Q_1 + K * Q_1, \\
R_1 = K + J * R_1 + K * R_1,
\end{cases}
\quad \text{and} \quad
\begin{cases}
Q_2 = J + Q_2 * J + R_2 * J, \\
R_2 = K + Q_2 * K + R_2 * K.
\end{cases}
\]

Then \( Q_1 = Q_2 \) and \( R_1 = R_2 \). In particular, the pair \((J, K)\) has at most one \((*,*)\)-resolvent.

**Proof.** By Proposition 3.10 and Proposition 3.11, we have

\[
Q_2 * (Q_1 - J) = Q_2 * (J * Q_1 + K * Q_1) = Q_2 * J * Q_1 + Q_2 * K * Q_1 = (Q_2 - J - R_2 * J) * Q_1 + Q_2 * K * Q_1.
\]

Cancelling out \( Q_2 * Q_1 \), we get

\[
Q_2 * J + Q_2 * K * Q_1 = J * Q_1 + R_2 * J * Q_1. \tag{3.10}
\]

Similarly,

\[
Q_2 * (R_1 - K) = Q_2 * (J * R_1 + K * R_1) = Q_2 * J * R_1 + Q_2 * K * R_1 = (Q_2 - J - R_2 * J) * R_1 + Q_2 * K * R_1.
\]

Cancelling out \( Q_2 * R_1 \), we get

\[
Q_2 * K + Q_2 * K * R_1 = J * R_1 + R_2 * J * R_1. \tag{3.11}
\]

Also,

\[
R_2 * (Q_1 - J) = R_2 * (J * Q_1 + K * Q_1) = R_2 * J * Q_1 + R_2 * K * Q_1 = R_2 * J * Q_1 + (R_2 - K - Q_2 * K) * Q_1.
\]
Proof. 

(i) Note that
\[ R_2 \ast J + R_2 \ast J \ast Q_1 = K \ast Q_1 + Q_2 \ast K \ast Q_1. \] 
(3.12)

Lastly,
\[ R_2 \ast (R_1 - K) = R_2 \ast (J \ast R_1 + K \ast R_1) = R_2 \ast J \ast R_1 + R_2 \ast K \ast R_1 = R_2 \ast J \ast R_1 + (R_2 - K - Q_2 \ast K) \ast R_1. \]

Cancelling out \( R_2 \ast R_1 \), we get
\[ R_2 \ast K + R_2 \ast J \ast R_1 = K \ast R_1 + Q_2 \ast K \ast R_1. \] 
(3.13)

By (3.10) and (3.12), we see that

\[ Q_2 \ast J + R_2 \ast J = J \ast Q_1 + K \ast Q_1. \]

Thus, we obtain
\[ Q_2 = J + Q_2 \ast J + R_2 \ast J = J + J \ast Q_1 + K \ast Q_1 = Q_1. \]

Furthermore, by (3.11) and (3.13), we see that
\[ Q_2 \ast K + R_2 \ast K = J \ast R_1 + K \ast R_1. \]

Thus, we obtain
\[ R_2 = K + Q_2 \ast K + R_2 \ast K = K + J \ast R_1 + K \ast R_1 = R_1. \]

This completes the proof. \( \square \)

We note that the \((\ast, \ast)\)-resolvent is defined as a pair of kernels. The next proposition shows that the \((\ast, \ast)\)-resolvent is constructed by the \ast-resolvent and the \ast-resolvent.

Proposition 3.20. Let a \ast-Volterra kernel \( J \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \) and a \ast\ast-Volterra kernel \( K \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \) be given.

(i) Let \( Q_1 \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \) be the \ast-resolvent of \( J \), and suppose that \( K + Q_1 \ast K \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \) has a \ast\ast-resolvent \( R_1 \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \). Then \((Q, R) \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \times \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})\) defined by \( Q := Q_1 + R_1 \ast Q_1 \) and \( R := R_1 \) is the \((\ast, \ast)\)-resolvent of \((J, K)\).

(ii) Suppose that \( K \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \) has a \ast\ast-resolvent \( R_2 \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \), and let \( Q_2 \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \) be the \ast-resolvent of \( J + R_2 \ast J \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \). Then \((Q, R) \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \times \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})\) defined by \( Q := Q_2 \) and \( R := R_2 + Q_2 \ast R_2 \) is the \((\ast, \ast)\)-resolvent of \((J, K)\).

Proof. (i) Note that \( Q_1 = J + J \ast Q_1 \) and \( R_1 = K + Q_1 \ast K + (K + Q_1 \ast K) \ast R_1 \). By Proposition 3.5, Proposition 3.10, and Proposition 3.11 we have
\[
J \ast R = J \ast R_1 = J \ast K + J \ast Q_1 \ast K + J \ast K \ast R_1 + J \ast Q_1 \ast K \ast R_1 = (J \ast Q_1) \ast K + (J \ast Q_1) \ast (K \ast R_1) = Q_1 \ast K + Q_1 \ast K \ast R_1,
\]
and hence
\[
K \ast J \ast R + K \ast R = K \ast Q_1 \ast K + Q_1 \ast K \ast R_1 + K \ast R_1 = K \ast Q_1 \ast K \ast R_1 = R_1 = R.
\]

Consequently, we get
\[ R = K \ast J \ast R + K \ast R. \] 
(3.14)
Furthermore, we have
\[ J + J \ast Q + K \ast Q = J + J \ast (Q_1 + R_1 \ast Q_1) + K \ast (Q_1 + R_1 \ast Q_1) \]
\[ = J + J \ast Q_1 + (K + J \ast R_1 + K \ast R_1) \ast Q_1 \]
\[ = Q_1 + R_1 \ast Q_1 = Q, \]
and thus
\[ Q = J + J \ast Q + K \ast Q. \] (3.15)
Similarly, from the equalities \( Q_1 = J + Q_1 \ast J \) and \( R_1 = K + Q_1 \ast K + R_1 \ast (K + Q_1 \ast K) \), we have
\[ K + Q \ast K + R \ast K = K + (Q_1 + R_1 \ast Q_1) \ast K + R_1 \ast K \]
\[ = K + Q_1 \ast K + R_1 \ast (K + Q_1 \ast K) \]
\[ = R_1 = R, \]
and thus
\[ R = K + Q \ast K + R \ast K. \] (3.16)
Furthermore, we have
\[ J + Q \ast J + R \ast J = J + (Q_1 + R_1 \ast Q_1) \ast J + R_1 \ast J \]
\[ = J + Q_1 \ast J + R_1 \ast (J + Q_1 \ast J) \]
\[ = Q_1 + R_1 \ast Q_1 = Q, \]
and thus
\[ Q = J + Q \ast J + R \ast J. \] (3.17)
By (3.14), (3.15), (3.16) and (3.17), we see that \((Q, R)\) is the \((\ast, \ast)\)-resolvent of \((J, K)\).

(ii) The claim (ii) can be proved by the same way as (i) by inverting the roles of \(J\) and \(K\), \(Q\) and \(R\), and the \(\ast\)-product and the \(\ast\)-product.

\[ \square \]

**Remark 3.21.** From Proposition 3.20 (ii) and Proposition 3.16, if \(K \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})\) has an \(\ast\)-resolvent, then \((J, K)\) has a \((\ast, \ast)\)-resolvent for any \(J \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})\). The existence of the \(\ast\)-resolvent will be discussed in Section 5.

### 3.4 A variation of constants formula for generalized SVIEs

Now we are ready to show the variation of constants formula for generalized SVIEs.

**Theorem 3.22.** Let a \(\ast\)-Volterra kernel \(J \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})\) and a \(\ast\)-Volterra kernel \(K \in \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})\) be fixed, and suppose that \((J, K)\) has a \((\ast, \ast)\)-resolvent \((Q, R) \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \times \mathcal{K}_\mathcal{F}(S, T; \mathbb{R}^{d \times d})\). Then for any free term \(\varphi \in L^2_S(S, T; \mathbb{R}^d)\), the generalized SVIE
\[ X = \varphi + J \ast X + K \ast X \]
has a unique solution \(X \in L^2_S(S, T; \mathbb{R}^d)\). This solution is given by the variation of constants formula:
\[ X = \varphi + Q \ast \varphi + R \ast \varphi. \]

**Proof.** Define \(X \in L^2_S(S, T; \mathbb{R}^d)\) by \(X = \varphi + Q \ast \varphi + R \ast \varphi\). By Proposition 3.3, Proposition 3.10 and Proposition 3.11, we have
\[ J \ast X = J \ast (\varphi + Q \ast \varphi + R \ast \varphi) = (J + J \ast Q) \ast \varphi + (J \ast R) \ast \varphi \]
and
\[ K \ast X = K \ast (\varphi + Q \ast \varphi + R \ast \varphi) = (K \ast Q) \ast \varphi + (K + K \ast R) \ast \varphi. \]
Combining the above two equalities and using the resolvent equations, we get
\[ J \ast X + K \ast X = (J + J \ast Q + K \ast Q) \ast \varphi + (K + R + K \ast R) \ast \varphi = Q \ast \varphi + R \ast \varphi. \]
This implies that \( X = \varphi + J \ast X + K \ast X \).

Conversely, if \( X \in L^2_\mathbb{F}(S,T;\mathbb{R}^d) \) satisfies \( X = \varphi + J \ast X + K \ast X \), then we have
\[ Q \ast \varphi = Q \ast (X - J \ast X - K \ast X) = (Q - Q \ast J) \ast X - (Q \ast K) \ast X \]
and
\[ R \ast \varphi = R \ast (X - J \ast X - K \ast X) = -(R \ast J) \ast X + (R - R \ast K) \ast X. \]
Combining the above two equalities and using the resolvent equations, we get
\[ Q \ast \varphi + R \ast \varphi = (Q - Q \ast J - R \ast J) \ast X + (R - Q \ast K - R \ast K) \ast X = J \ast X + K \ast X. \]
This implies that \( X = \varphi + Q \ast \varphi + R \ast \varphi \). This completes the proof.

Noting the stochastic integral representation of the \( \ast \)-product (see Proposition 3.40), we immediately get the following corollary.

**Corollary 3.23.** Let \( J \in \mathcal{F}(S,T;\mathbb{R}^{d \times d}) \) and \( K = \sum_{n=1}^{\infty} \mathbb{M}_n[k_n] \in \mathcal{K}_\mathbb{F}(S,T;\mathbb{R}^{d \times d}) \) with \( k_n \in \mathcal{V}_{n+1}(S,T;\mathbb{R}^{d \times d}) \), \( n \in \mathbb{N} \), be fixed, and suppose that \((J, K)\) has a \((\ast, \ast)\)-resolvent \((Q, R) \in \mathcal{F}(S,T;\mathbb{R}^{d \times d}) \times \mathcal{K}_\mathbb{F}(S,T;\mathbb{R}^{d \times d}) \). Then for any free term \( \varphi \in L^2_\mathbb{F}(S,T;\mathbb{R}^d) \), the generalized SVIE
\[
X(t) = \varphi(t) + \int_S^t J(t,s)X(s) \, ds + \sum_{n=1}^{\infty} \int_S^t \int_S^{t_1} \cdots \int_S^{t_{n-1}} k_n(t,t_1,\ldots,t_n)X(t_n) \, dW(t_n) \cdots dW(t_1), \quad t \in (S,T),
\]
has a unique solution \( X \in L^2_\mathbb{F}(S,T;\mathbb{R}^d) \). This solution is given by the variation of constants formula:
\[
X(t) = \varphi(t) + \int_S^t Q(t,s)\varphi(s) \, ds + \sum_{n=1}^{\infty} \int_S^t \int_S^{t_1} \cdots \int_S^{t_{n-1}} r_n(t,t_1,\ldots,t_n)\varphi(t_n) \, dW(t_n) \cdots dW(t_1), \quad t \in (S,T),
\]
where \( r_n := \mathbb{F}_n[R] \in \mathcal{V}_{n+1}(S,T;\mathbb{R}^{d \times d}) \) for \( n \in \mathbb{N} \). Here, the infinite sum in the right-hand side converges in \( L^2_\mathbb{F}(S,T;\mathbb{R}^d) \).

**Corollary 3.24.** Let deterministic kernels \( j \in L^2(\Delta_2(S,T);\mathbb{R}^{d \times d}) \) and \( k \in \mathcal{V}_2(S,T;\mathbb{R}^{d \times d}) \) be fixed. Let \( q \in L^2(\Delta_2(S,T);\mathbb{R}^{d \times d}) \) be the \( \ast \)-resolvent of \( j \), and assume that \( \mathbb{M}_1[k + q \ast k] \) has a \( \ast \)-resolvent. Then for any free term \( \varphi \in L^2_\mathbb{F}(S,T;\mathbb{R}^d) \), the SVIE
\[
X(t) = \varphi(t) + \int_S^t j(t,s)X(s) \, ds + \int_S^t k(t,s)X(s) \, dW(s), \quad t \in (S,T),
\]
has a unique solution \( X \in L^2_\mathbb{F}(S,T;\mathbb{R}^d) \). This solution is given by the variation of constants formula:
\[
X(t) = \varphi(t) + \int_S^t q(t,s)\varphi(s) \, ds
+ \sum_{n=1}^{\infty} \int_S^t \int_S^{t_1} \cdots \int_S^{t_{n-1}} (k + q \ast k)^{\ast n}(t,t_1,\ldots,t_n)\left\{ \varphi(t_n) + \int_S^{t_n} q(t_n,s)\varphi(s) \, ds \right\} dW(t_n) \cdots dW(t_1), \quad t \in (S,T).
\]
Here, \((k + q \ast k)^{\ast n} \in \mathcal{V}_{n+1}(S,T;\mathbb{R}^{d \times d})\) denotes the \((n - 1)\)-fold \( \vartriangleright \)-product of \( k + q \ast k \in \mathcal{V}_2(S,T;\mathbb{R}^{d \times d}) \) by itself, and the infinite sum in the right-hand side converges in \( L^2_\mathbb{F}(S,T;\mathbb{R}^d) \).
Proof. Note that \( \mathfrak{M}_1[k] + q \ast \mathfrak{M}_1[k] = \mathfrak{M}_1[k + q \ast k] \). Thus, by Lemma 3.17 the \(*\)-resolvent of \( \mathfrak{M}_1[k] + q \ast \mathfrak{M}_1[k] \) is given by \( R = \sum_{n=1}^{\infty} \mathfrak{M}_n[(k + q \ast k)_{[n]}] \in K_F(S, T; \mathbb{R}^d \times d) \). Defining \( Q \in J_{F}(S, T; \mathbb{R}^d \times d) \) by \( Q := q + R \ast q \), by Proposition 3.20 we see that \((Q, R)\) is the \((\ast, \ast)\)-resolvent of \((j, \mathfrak{M}_1[k])\). Thus, by Theorem 3.22 the SVIE \( X = \varphi + j \ast X + \mathfrak{M}_1[k] \ast X \) has a unique solution \( X \in L^2_F(S, T; \mathbb{R}^d) \), and the solution is given by

\[
X = \varphi + Q \ast \varphi + R \ast \varphi
= \varphi + (q + R \ast q) \ast \varphi + R \ast \varphi
= \varphi + q \ast \varphi + R \ast (\varphi + q \ast \varphi).
\]

Noting Proposition 3.16 we get the assertion. \(\)

Example 3.25. Consider the following one-dimensional linear fractional SDE of order \( \alpha \in (\frac{1}{2}, 1] \):

\[
\begin{aligned}
\left\{ \begin{array}{l}
\mathcal{D}^\alpha_0 X(t) = \mu X(t) + \sigma X(t) \frac{dW(t)}{dt}, \quad t \in (0, T), \\
X(0) = x_0,
\end{array} \right.
\end{aligned}
\]

where \( \mu, \sigma, x_0 \in \mathbb{R} \) are constants. When \( \alpha = 1 \), the above equation becomes the Black–Scholes SDE:

\[
\begin{aligned}
\left\{ \begin{array}{l}
dX(t) = \mu X(t) dt + \sigma X(t) dW(t), \quad t \in (0, T), \\
X(0) = x_0,
\end{array} \right.
\end{aligned}
\]

and the solution is the geometric Brownian motion \( X(t) = x_0 \exp((\mu - \sigma^2/2)t + \sigma W(t)) \). Thus, (3.18) can be seen as a time-fractional Black–Scholes SDE. Note that (3.18) is equivalent to the following SVIE:

\[
X(t) = x_0 + \int_0^t j(t, s)X(s) ds + \int_0^t k(t, s)X(s) dW(s), \quad t \in (0, T),
\]

where \( j(t, s) = \frac{\sigma}{1(\alpha)}(t-s)^{\alpha-1} \) and \( k(t, s) = \frac{\sigma^2}{1(\alpha)}(t-s)^{\alpha-1} \). Here we use some fundamental calculus for the fractional kernel and the Mittag–Leffler function (see Section 1.2 in the textbook [23]). The \(*\)-resolvent \( q \in L^2(\Delta_2(0, T); \mathbb{R}) \) of \( j \) is given by \( q(t, s) = \mu f(t-s) \), where \( f(t) = t^{\alpha-1}E_{\alpha,\alpha}(\mu t^\alpha) \), and \( E_{\beta,\gamma}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\beta k + \gamma)} \) is the Mittag–Leffler function. Furthermore, it holds that \( (k + q \ast k)(t, s) = \sigma f(t-s) \) for \( (t, s) \in \Delta_2(0, T) \), and thus the \(*\)-resolvent \( R \in K_F(0, T; \mathbb{R}) \) of \( \mathfrak{M}_1[k + q \ast k] \) is given by

\[
R(t) = \sum_{n=1}^{\infty} \sigma^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t-t_1)f(t_1-t_2)\cdots f(t_{n-1}-t_n) dW(t_n)\cdots dW(t_1), \quad t \in (0, T).
\]

Here, the existence of the \(*\)-resolvent \( R \) follows from the results in Section 5. Observe that \( 1 + \int_0^t q(t, s) ds = 1 + \mu \int_0^t f(t-s) ds = E_{\alpha,1}(\mu t^\alpha) =: E_{\alpha}(\mu t^\alpha) \). Consequently, by Corollary 3.24 the solution \( X \in L^2_F(S, T; \mathbb{R}) \) to the fractional Black–Scholes SDE (3.18) is given by

\[
X(t) = x_0 \left\{ E_{\alpha}(\mu t^\alpha) + \sum_{n=1}^{\infty} \sigma^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t-t_1)f(t_1-t_2)\cdots f(t_{n-1}-t_n) dW(t_n)\cdots dW(t_1) \right\}
\]

for \( t \in (0, T) \). In particular, if \( \mu = 0 \), we have

\[
X(t) = x_0 \left\{ 1 + \sum_{n=1}^{\infty} \left( \frac{\sigma}{\Gamma(\alpha)} \right)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} (t-t_1)^{\alpha-1}(t_1-t_2)^{\alpha-1}\cdots(t_{n-1}-t_n)^{\alpha-1} dW(t_n)\cdots dW(t_1) \right\}
\]

for \( t \in (0, T) \).
4 Backward stochastic Volterra integral equations

In this section, we investigate linear Type-II BSVIEs. First of all, consider the following Type-II BSVIE:

\[ Y(t) = \tilde{\psi}(t) + \int_t^T \{ j(s,t)^\top Y(s) + k(s,t)^\top Z(s,t) \} \, ds - \int_t^T Z(t,s) \, dW(s), \quad t \in (S,T), \]  

where \( \tilde{\psi}(\cdot) \in L_2^2(S,T;\mathbb{R}^d) \) is a given free term such that \( \tilde{\psi}(t) \) is \( \mathcal{F}_t^S \)-measurable for each \( t \in (S,T) \), and \( j \in L_2^2(\Delta_2(S,T);\mathbb{R}^{d \times d}) \) and \( k \in V_2(\mathcal{S},\mathbb{R}^{d \times d}) \) are given deterministic kernels. We say that a pair \( (Y(\cdot),Z(\cdot,\cdot)) \) is an adapted \( M \)-solution of BSVIE \( \text{(11)} \) if \( Y \in L_2^2(S,T;\mathbb{R}^d) \), \( Z : \Omega \times (S,T)^2 \to \mathbb{R}^d \) are measurable, \( Z(t,\cdot) \) is \( \mathbb{F}^S \)-adapted for a.e. \( t \in (S,T) \), \( \mathbb{E}[\int_S^T \int_S^T |Z(t,s)|^2 \, dt \, ds] < \infty \), and the equation, together with the relation \( Y(t) = \mathbb{E}_S[Y(t)] + \int_S^t Z(t,s) \, dW(s) \) hold for a.e. \( t \in (S,T) \), a.s. (see [39]). Note that the term \( Z(t,s) \) for \( (t,s) \in \Delta_2(S,T) \) is determined by \( Y \) via the martingale representation theorem. Thus, we can write \( Z = \mathcal{M}_1[Y] \) in \( \Delta_2(S,T) \) by means of the martingale representation operator \( \mathcal{M}_1 : L_2^2(S,T;\mathbb{R}^d) \to L_2^2(\Delta_2(S,T);\mathbb{R}^d) \), which will be defined below. Therefore, taking the conditional expectation \( \mathbb{E}_t \) in \text{(1.1)} \), we see that the above equation is equivalent to the following equations:

\[
\begin{aligned}
Y(t) &= \mathbb{E}_t[\tilde{\psi}(t)] + \mathbb{E}_t \left[ \int_t^T j(s,t)^\top Y(s) \, ds \right] + \mathbb{E}_t \left[ \int_t^T k(s,t)^\top \mathcal{M}_1[Y](s,t) \, ds \right], \\
\int_t^T Z(t,s) \, dW(s) &= \Theta^Y(t) - \mathbb{E}_t[\Theta^Y(t)], \\
\end{aligned}
\]

where the process \( \Theta^Y(t) := \tilde{\psi}(t) + \int_t^T j(s,t)^\top Y(s) \, ds, \quad t \in (S,T), \) is in \( L_2^2(S,T;\mathbb{R}^d) \) and depends on \( Y \). Note that the above two equations are decoupled in the sense that if \( Y \in L_2^2(S,T;\mathbb{R}^d) \) solves the first equation, then the term \( Z(t,s) \) for \( (t,s) \in (S,T)^2 \setminus \Delta_2(S,T) \) is determined by \( Y \) as the integrand of the martingale representation theorem for \( \Theta^Y(t) \). Consequently, Type-II BSVIE \( \text{(11)} \) can be seen as an integral equation for \( Y \in L_2^2(S,T;\mathbb{R}^d) \) (not for the pair \( (Y(\cdot),Z(\cdot,\cdot)) \)) including the martingale representation operator:

\[ Y(t) = \psi(t) + \int_t^T j(s,t)^\top Y(s) \, ds + \int_t^T k(s,t)^\top \mathcal{M}_1[Y](s,t) \, ds, \quad t \in (S,T), \]

where \( \psi \in L_2^2(S,T;\mathbb{R}^d) \) is defined by \( \psi(t) := \mathbb{E}_t[\tilde{\psi}(t)] \) for \( t \in (S,T) \).

**Definition 4.1.** Let \( d_1, d_2 \in \mathbb{N} \). For each \( n \in \mathbb{N}_0 \), we define the martingale representation operator \( \mathcal{M}_n : L_2^2(S,T;\mathbb{R}^{d_1 \times d_2}) \to L_2^2(\Delta_{n+1}(S,T);\mathbb{R}^{d_1 \times d_2}) \) via the martingale representation theorem inductively as follows:

for each \( \xi \in L_2^2(S,T;\mathbb{R}^{d_1 \times d_2}) \),

\[ \mathcal{M}_0[\xi](t_0) := \xi(t_0) \]

for \( t_0 \in (S,T) \), and

\[ \mathcal{M}_n[\xi](t_0,t_1,\ldots,t_n) = \mathbb{E}_S[\mathcal{M}_n[\xi](t_0,t_1,\ldots,t_n)] + \int_S^{t_n} \mathcal{M}_{n+1}[\xi](t_0,t_1,\ldots,t_n,t_{n+1}) \, dW(t_{n+1}) \]

for \( (t_0,t_1,\ldots,t_n) \in \Delta_{n+1}(S,T) \) and \( n \in \mathbb{N} \).

**Remark 4.2.**

(i) For each \( \xi \in L_2^2(S,T;\mathbb{R}^{d_1 \times d_2}) \) and \( n \in \mathbb{N}_0 \), it holds that

\[ \mathbb{E}_S[\mathcal{M}_n[\xi](t_0,t_1,\ldots,t_n)] = \mathcal{F}_n[\xi](t_0,t_1,\ldots,t_n), \quad (t_0,t_1,\ldots,t_n) \in \Delta_n(S,T). \]

See [43] below.

(ii) Noting the isometry, we have \( \| \mathcal{M}_{n+1}[\xi] \|_{L_2^2(\Delta_{n+1}(S,T))} \leq \| \mathcal{M}_n[\xi] \|_{L_2^2(\Delta_{n+1}(S,T))} \) for any \( n \in \mathbb{N}_0 \) and \( \xi \in L_2^2(S,T;\mathbb{R}^{d_1 \times d_2}) \). By the induction, we see that \( \mathcal{M}_n : L_2^2(S,T;\mathbb{R}^{d_1 \times d_2}) \to L_2^2(\Delta_{n+1}(S,T);\mathbb{R}^{d_1 \times d_2}) \) is a bounded linear operator with the operator norm \( \| \mathcal{M}_n \|_{\text{op}} \leq 1 \).
We will consider the following generalized equation including (infinitely many) martingale representation operators:

\[
Y(t) = \psi(t) + \mathbb{E}_t \left[ \int_t^T J(s, t)^T Y(s) \, ds \right] + \sum_{n=1}^{\infty} \int_{\Delta_n(t, T)} k_n(s, t)^T M_n[Y](s, t) \, ds, \quad t \in (S, T),
\]

where \( \psi \in L^2_\mathbb{F}(S, T; \mathbb{R}^d) \), \( J \in \mathcal{J}_\mathbb{F}(S, T; \mathbb{R}^{d \times d}) \) and \( K := \sum_{n=1}^{\infty} \mathbb{M}_n[k_n] \in \mathcal{K}_\mathbb{F}(S, T; \mathbb{R}^{d \times d}) \). We call this equation a generalized BSVIE. The solution we are looking for is the adapted process \( Y \in L^2_\mathbb{F}(S, T; \mathbb{R}^d) \) satisfying (4.2). In order to investigate generalized BSVIE (4.2), we write it as an algebraic equation on \( L^2_\mathbb{F}(S, T; \mathbb{R}^d) \) as follows:

\[
Y = \psi + J^\top * Y + K^\top * Y,
\]

where \( J^\top * Y \) corresponds to the term \( \mathbb{E}_t[\int_t^T J(s, t)^T Y(s) \, ds] \), and \( K^\top * Y \) corresponds to the infinite sum of the integrals including martingale representation operators. Then, we provide a variation of constants formula for (4.3). Furthermore, we show the duality principle between a generalized SVIE and a generalized BSVIE.

### 4.1 Backward ∗-product

In this subsection, we investigate the term \( K^\top * Y \) appearing in equation (4.3).

**Definition 4.3.** For each ∗-Volterra kernel \( K \in \mathcal{K}_\mathbb{F}(S, T; \mathbb{R}^{d \times d}) \) and \( \xi \in L^2_\mathbb{F}(S, T; \mathbb{R}^{d \times d_1}) \) with \( d_1 \in \mathbb{N} \), we define the backward ∗-product \( K \leftarrow \xi \in L^2_\mathbb{F}(S, T; \mathbb{R}^{d \times d_1}) \) by the following Wiener–Itô chaos expansion:

\[
\mathfrak{F}_n[K \leftarrow \xi](t_0, t) := \sum_{k=n}^{\infty} \int_{\Delta_k(n, t_0, T)} \mathfrak{F}_k[K](s, t_0) \mathfrak{F}_k[\xi](s, t) \, ds
\]

for \((t_0, t) = (t_0, t_1, \ldots, t_n) \in \Delta_{n+1}(S, T)\) and \( n \in \mathbb{N}_0 \), where \( \int_{\Delta_0(t_0, T)} \alpha \, ds := \alpha \) for each vector \( \alpha \).

The following lemma ensures the well-definedness of the backward ∗-product.

**Lemma 4.4.** For each ∗-Volterra kernel \( K \in \mathcal{K}_\mathbb{F}(S, T; \mathbb{R}^{d \times d}) \) and \( \xi \in L^2_\mathbb{F}(S, T; \mathbb{R}^{d \times d_1}) \) with \( d_1 \in \mathbb{N} \), the backward ∗-product \( K \leftarrow \xi \in L^2_\mathbb{F}(S, T; \mathbb{R}^{d \times d_1}) \) is well-defined and satisfies

\[
\|K \leftarrow \xi\|_{L^2_\mathbb{F}(S, T)} \leq \|K\|_{\mathcal{K}_\mathbb{F}(S, T)} \|\xi\|_{L^2_\mathbb{F}(S, T)}.
\]

Furthermore, if \( K_1, K_2 \in \mathcal{K}_\mathbb{F}(S, T; \mathbb{R}^{d \times d}) \), then \( K_1 \leftarrow K_2 \) is in \( \mathcal{K}_\mathbb{F}(S, T; \mathbb{R}^{d \times d}) \) and satisfies

\[
\|K_1 \leftarrow K_2\|_{\mathcal{K}_\mathbb{F}(S, T)} \leq \|K_1\|_{\mathcal{K}_\mathbb{F}(S, T)} \|K_2\|_{\mathcal{K}_\mathbb{F}(S, T)}.
\]

**Proof.** By using Minkowski’s inequality and Hölder’s inequality, we have, for each \( n \in \mathbb{N}_0 \),

\[
\left\{ \int_{\Delta_{n+1}(S, T)} \left( \sum_{k=n}^{\infty} \int_{\Delta_{k-n}(\max t, T)} |\mathfrak{F}_k \leftarrow \mathfrak{F}_n[K](s, \max t) \mathfrak{F}_k[\xi](s, t)| \, ds \right)^2 \, dt \right\}^{1/2}
\]

\[
\leq \sum_{k=n}^{\infty} \left\{ \int_{\Delta_{n+1}(S, T)} \left( \int_{\Delta_{k-n}(\max t, T)} |\mathfrak{F}_k \leftarrow \mathfrak{F}_n[K](s, \max t) \mathfrak{F}_k[\xi](s, t)| \, ds \right)^2 \, dt \right\}^{1/2}
\]

\[
\leq \sum_{k=n}^{\infty} \left\{ \int_{\Delta_{n+1}(S, T)} \int_{\Delta_{k-n}(\max t, T)} |\mathfrak{F}_k \leftarrow \mathfrak{F}_n[K](s, \max t)|_{\text{op}} \, ds \int_{\Delta_{k-n}(\max t, T)} |\mathfrak{F}_k[\xi](s, t)| \, ds \, dt \right\}^{1/2}
\]

\[
\leq \sum_{k=n}^{\infty} \left\{ \int_{\Delta_{n+1}(S, T)} \int_{\Delta_{k-n}(\max t, T)} |\mathfrak{F}_k \leftarrow \mathfrak{F}_n[K](s, \max t)|_{\text{op}}^2 \, ds \int_{\Delta_{k-n}(\max t, T)} |\mathfrak{F}_k[\xi](s, t)|^2 \, ds \, dt \right\}^{1/2}
\]

\[
\leq \sum_{k=n}^{\infty} \|\mathfrak{F}_k \leftarrow \mathfrak{F}_n[K]\|_{L^2(\Delta_{k-n}(S, T))} \|\mathfrak{F}_k[\xi]\|_{L^2(\Delta_{k-n}(S, T))}.
\]
This implies that
\[ \| \tilde{\mathcal{F}}_n[K \ast \xi] \|_{L^2(\Delta_{n+1}(S,T))} \leq \sum_{k=n}^{\infty} \| \tilde{\mathcal{F}}_{k-n}[K] \|_{\mathcal{V}_{k-n+1}(S,T)} \| \tilde{\mathcal{F}}_k[\xi] \|_{L^2(\Delta_{k+1}(S,T))}. \]

Therefore, by Young’s convolution inequality and the isometry,
\[ \sum_{n=0}^{\infty} \| \tilde{\mathcal{F}}_n[K \ast \xi] \|_{L^2(\Delta_{n+1}(S,T))}^2 \leq \left( \sum_{n=0}^{\infty} \| \tilde{\mathcal{F}}_{k-n}[K] \|_{\mathcal{V}_{n+1}(S,T)} \| \tilde{\mathcal{F}}_k[\xi] \|_{L^2(\Delta_{k+1}(S,T))} \right)^2 \]
\[ \leq \left( \sum_{n=0}^{\infty} \| \tilde{\mathcal{F}}_n[K] \|_{\mathcal{V}_{n+1}(S,T)} \right)^2 \sum_{n=0}^{\infty} \| \tilde{\mathcal{F}}_n[\xi] \|_{L^2(\Delta_{n+1}(S,T))}^2 \]
\[ = \| K \|_{\mathcal{V}_0(S,T)} \| \xi \|_{L^2(S,T)}^2 < \infty. \]

This implies the first assertion. Similarly, we can show that
\[ \| \tilde{\mathcal{F}}_n[K_1 \ast K_2] \|_{\mathcal{V}_{n+1}(S,T)} \leq \sum_{k=n}^{\infty} \| \tilde{\mathcal{F}}_{k-n+1}[K_1] \|_{\mathcal{V}_{k-n+1}(S,T)} \| \tilde{\mathcal{F}}_k[K_2] \|_{\mathcal{V}_{k+1}(S,T)} \]
for each \( n \in \mathbb{N}_0 \), and thus
\[ \| K_1 \ast K_2 \|_{\mathcal{K}_0(S,T)} = \sum_{n=0}^{\infty} \| \tilde{\mathcal{F}}_n[K_1 \ast K_2] \|_{\mathcal{V}_{n+1}(S,T)} \]
\[ \leq \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \| \tilde{\mathcal{F}}_{k-n}[K_1] \|_{\mathcal{V}_{k-n+1}(S,T)} \| \tilde{\mathcal{F}}_k[K_2] \|_{\mathcal{V}_{k+1}(S,T)} \sum_{n=0}^{\infty} \| \tilde{\mathcal{F}}_n[\xi] \|_{\mathcal{V}_{n+1}(S,T)} \]
\[ = \| K_1 \|_{\mathcal{K}_0(S,T)} \| K_2 \|_{\mathcal{K}_0(S,T)}. \]

Hence, the last assertion holds.

The following proposition shows fundamental algebraic properties of the backward \( * \)-product.

**Proposition 4.5.** For each \( K, K_1, K_2 \in \mathcal{K}_\mathcal{F}(S,T; \mathbb{R}^{d \times d}) \), \( \xi, \xi_1, \xi_2 \in L^2_d(S,T; \mathbb{R}^{d \times d_1}) \) with \( d_1 \in \mathbb{N} \), and \( \alpha \in \mathbb{R} \), the following hold:

\[ I_d \ast \xi = \xi, \]
\[ (K_1 + K_2) \ast \xi = K_1 \ast \xi + K_2 \ast \xi, \]
\[ K \ast (\xi_1 + \xi_2) = K \ast \xi_1 + K \ast \xi_2, \]
\[ \alpha(K \ast \xi) = (\alpha K) \ast \xi = K \ast (\alpha \xi), \]
\[ K_1 \ast (K_2 \ast \xi) = (K_2 \ast K_1^\top) \ast \xi. \]

**Proof.** We prove \( K_1 \ast (K_2 \ast \xi) = (K_2 \ast K_1^\top) \ast \xi \). The others are trivial from the definition. Note that \( \tilde{\mathcal{F}}_n[K] = \tilde{\mathcal{F}}_n[K^\top] \) for any \( n \in \mathbb{N}_0 \) and \( K \in \mathcal{K}_\mathcal{F}(S,T; \mathbb{R}^{d \times d}) \). For each \( n \in \mathbb{N}_0 \) and \( t \in \Delta_{n+1}(S,T) \), we have
\[ \tilde{\mathcal{F}}_n[K \ast (K_2 \ast \xi)](t) \]
\[ = \sum_{k=n}^{\infty} \int_{\Delta_{k-n}(\max t, T)} \tilde{\mathcal{F}}_{k-n}[K_1](s, \max t) \tilde{\mathcal{F}}_k[K_2 \ast \xi](s, t) \, ds \]
\[ = \sum_{k=n}^{\infty} \int_{\Delta_{k-n}(\max t, T)} \tilde{\mathcal{F}}_{k-n}[K_1](s, \max t) \left\{ \sum_{\ell=k}^{\infty} \int_{\Delta_{\ell-k}(\max s, T)} \tilde{\mathcal{F}}_{\ell-k}[K_2](r, \max s) \tilde{\mathcal{F}}_{\ell}[\xi](r, s, t) \, dr \right\} \, ds \]
Hence, we have

\[ (K_1 \ast (K_2 \ast \xi)) = \mathcal{F}_0[K](t) \xi(t) + \sum_{n=1}^{\infty} \int_{\Delta_n(t,T)} \mathfrak{F}_n[K](s,t) \mathcal{M}_n[\xi](s,t) \, ds, \quad t \in (S,T), \]

(4.4)

where the infinite sum in the right-hand side converges in \( L^2_\mathcal{F}(S,T; \mathbb{R}^{d \times d_1}) \).

Proof. First, we show that, for any \( k, n \in \mathbb{N}_0 \), and \( (s_1, \ldots, s_k, t_0, t) = (s_1, \ldots, s_k, t_0, t_1, \ldots, t_n) \in \Delta_{n+k+1}(S,T) \),

\[ \mathfrak{F}_n[M_k[\xi](s_1, \ldots, s_k, \cdot)](t_0, t) = \mathfrak{F}_{n+k}[\xi](s_1, \ldots, s_k, t_0, t). \]

(4.5)

Clearly, for \( k = 0 \), (4.5) holds for any \( n \in \mathbb{N}_0 \). Assume that (4.5) holds for any \( n \in \mathbb{N}_0 \) for some \( k \in \mathbb{N}_0 \).

By the definition of the martingale representation operator \( \mathcal{M}_{k+1} \), we have, for \( (s, s_k) = (s_0, \ldots, s_k-1, s_k) \in \Delta_{k+1}(S,T) \),

\[ \mathcal{M}_k[\xi](s, s_k) = \mathbb{E}_S[M_k[\xi](s, s_k)] + \int_S \mathcal{M}_{k+1}[\xi](s, s_k, t) \, dW(t) \]

\[ = \mathbb{E}_S[M_k[\xi](s, s_k)] + \int_S \sum_{n=0}^{s_k} \mathbb{M}_n[M_{k+1}[\xi](s, s_k, \cdot)](t) \, dW(t) \]

\[ = \mathbb{E}_S[M_k[\xi](s, s_k)] + \sum_{n=0}^{s_k} \int_S \mathbb{M}_n[M_{k+1}[\xi](s, s_k, \cdot)](t) \, dW(t) \]

\[ = \mathbb{E}_S[M_k[\xi](s, s_k)] + \int_S \mathfrak{F}_0[M_{k+1}[\xi](s, s_k, \cdot)](t) \, dW(t) \]

\[ + \sum_{n=1}^{s_k} \int_S \int_S \ldots \int_S \mathfrak{F}_n[M_{k+1}[\xi](s, s_k, \cdot)](t_1, \ldots, t_n) \, dW(t_n) \cdots dW(t_1) \, dW(t). \]

From this, together with the assumption of the induction, we have, for any \( n \in \mathbb{N}_0 \) and \( (s, s_k, t_0, t) = (s_0, \ldots, s_k-1, s_k, t_0, t_1, \ldots, t_n) \in \Delta_{n+k+2}(S,T) \),

\[ \mathfrak{F}_n[M_{k+1}[\xi](s, s_k, \cdot)](t_0, t) = \mathfrak{F}_{n+1}[M_k[\xi](s, \cdot)](s_k, t_0, t) = \mathfrak{F}_{n+k+1}[\xi](s, s_k, t_0, t). \]

By the induction, we see that (4.5) holds for any \( k, n \in \mathbb{N}_0 \).

We write the right-hand side of (4.4) by \( \sum_{k=0}^{\infty} \Psi^K_k[\xi](t) \), where \( \Psi^K_k[\xi](t) := \mathfrak{F}_0[K](t) \xi(t), t \in (S,T), \) and

\[ \Psi^K_k[\xi](t) := \int_{\Delta_k(t,T)} \mathfrak{F}_k[K](s,t) \mathcal{M}_k[\xi](s,t) \, ds, \quad t \in (S,T), \]

(4.3)

Next, we prove a representation of the backward \(*\)-product in terms of martingale representation operators.

**Proposition 4.6.** For each \(*\)-Volterra kernel \( K \in \mathcal{K}_H(S,T; \mathbb{R}^{d \times d}) \) and \( \xi \in L^2_\mathcal{F}(S,T; \mathbb{R}^{d \times d_1}) \) with \( d_1 \in \mathbb{N} \), it holds that

\[ (K \ast \xi)(t) = \mathfrak{F}_0[K](t)\xi(t) + \sum_{n=1}^{\infty} \int_{\Delta_n(t,T)} \mathfrak{F}_n[K](s,t) \mathcal{M}_n[\xi](s,t) \, ds, \quad t \in (S,T), \]
for $k \in \mathbb{N}$. Noting that $\|M_k[\xi]\|_{L^2(\Delta_{k+1}(S,T))} \leq \|\xi\|_{L^2(S,T)}$, we see that

$$
\|\Psi^K_k[\xi]\|_{L^2(S,T)}^2 = \mathbb{E}\left[\int_S^T \left| \mathcal{F}_k[K](s,t)M_k[\xi](s,t) \right|^2 dt \right] \\
\leq \mathbb{E}\left[\int_S^T \left( \int_{\Delta_{k}(t,T)} |\mathcal{F}_k[K](s,t)|^2_{op} ds \right) \left( \int_{\Delta_{k}(t,T)} |M_k[\xi](s,t)|^2 ds \right) dt \right] \\
\leq \|\mathcal{F}_k[K]\|_{V_{k+1}(S,T)}^2 \|M_k[\xi]\|_{L^2(\Delta_{k+1}(S,T))}^2 \\
\leq \|\mathcal{F}_k[K]\|_{V_{k+1}(S,T)}^2 \|\xi\|_{L^2(S,T)}^2.
$$

This implies that, for each $k \in \mathbb{N}_0$, $\Psi^K_k$ is a bounded linear operator on $L^2(S,T;\mathbb{R}^{d \times d})$ with the operator norm $\|\Psi^K_k\|_{op} \leq \|\mathcal{F}_k[K]\|_{V_{k+1}(S,T)}$. Noting that $\sum_{k=0}^{\infty} \|\mathcal{F}_k[K]\|_{V_{k+1}(S,T)} = \|K\|_{K'}(S,T) < \infty$, the infinite sum $\sum_{k=0}^{\infty} \Psi^K_k[\xi]$ converges in $L^2(S,T;\mathbb{R}^{d \times d})$. Furthermore, by the stochastic Fubini theorem and (4.5), we have

$$
\mathcal{F}_n[\Psi^K_k[\xi]](t_0, t) = \int_{\Delta_k(t_0,T)} \mathcal{F}_k[K](s,t_0)\mathcal{F}_n[M_k[\xi](s,\cdot)](t_0, t) ds \\
= \int_{\Delta_k(t_0,T)} \mathcal{F}_k[K](s,t_0)\mathcal{F}_{n+k}[\xi](s, t_0, t) ds
$$

for $(t_0, t) = (t_0, t_1, \ldots, t_n) \in \Delta_{n+1}(S,T)$ and $k, n \in \mathbb{N}_0$. Thus, we get

$$
\mathcal{F}_n\left[\sum_{k=0}^{\infty} \Psi^K_k[\xi]\right](t_0, t) = \sum_{k=0}^{\infty} \mathcal{F}_n[\Psi^K_k[\xi]](t_0, t) \\
= \sum_{k=0}^{\infty} \int_{\Delta_k(t_0,T)} \mathcal{F}_k[K](s,t_0)\mathcal{F}_{n+k}[\xi](s, t_0, t) ds \\
= \sum_{k=n}^{\infty} \int_{\Delta_{k-n}(t_0,T)} \mathcal{F}_{k-n}[K](s,t_0)\mathcal{F}_k[\xi](s, t_0, t) ds \\
= \mathcal{F}_n[K \rightharpoonup \xi](t_0, t)
$$

for $(t_0, t) = (t_0, t_1, \ldots, t_n) \in \Delta_{n+1}(S,T)$ and $n \in \mathbb{N}_0$. This implies that the equality $K \rightharpoonup \xi = \sum_{k=0}^{\infty} \Psi^K_k[\xi]$ holds in $L^2(S,T;\mathbb{R}^{d \times d})$, and we complete the proof. □

The following is a duality principle with respect to the $*$-product and the backward $*$-product.

**Proposition 4.7.** For each $*$-Volterra kernel $K \in K_\mathcal{F}(S,T;\mathbb{R}^{d \times d})$ and $\varphi, \psi \in L^2(S,T;\mathbb{R}^{d})$, it holds that

$$
\langle K \star \varphi, \psi \rangle_{L^2(S,T)} = \langle \varphi, K^T \rightharpoonup \psi \rangle_{L^2(S,T)}.
$$

Here, for each $\xi_1, \xi_2 \in L^2(S,T;\mathbb{R}^{d})$, $\langle \xi_1, \xi_2 \rangle_{L^2(S,T)} := \mathbb{E}\left[\int_S^T \langle \xi_1(t), \xi_2(t) \rangle dt\right]$ denotes the inner product in the Hilbert space $L^2(S,T;\mathbb{R}^{d})$.

**Proof.** By the isometry, together with the Fubini theorem, we have

$$
\langle K \star \varphi, \psi \rangle_{L^2(S,T)} = \sum_{n=0}^{\infty} \langle \mathcal{F}_n[K \star \varphi], \mathcal{F}_n[\psi] \rangle_{L^2(\Delta_{n+1}(S,T))} \\
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \langle \mathcal{F}_n-k[K], \mathcal{F}_k[\varphi], \mathcal{F}_n[\psi] \rangle_{L^2(\Delta_{n+1}(S,T))} \\
= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \langle \mathcal{F}_n-k[K], \mathcal{F}_k[\varphi], \mathcal{F}_n[\psi] \rangle_{L^2(\Delta_{n+1}(S,T))}
$$

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Thus, we get the assertion. \(\square\)

### 4.2 Backward *-product

Next, we investigate the term \(J^r Y\) appearing in equation \(4.3\).

**Definition 4.8.** For each \(\Xi \in L_{F,c}^2(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_2})\) and \(\xi \in L_{\mathcal{F}}^2(S, T; \mathbb{R}^{d_2 \times d_3})\) with \(d_1, d_2, d_3 \in \mathbb{N}\), we define the backward *-product \(\Xi \tilde{\ast} \xi\) by

\[
(\Xi \tilde{\ast} \xi)(t) := \mathbb{E}_t \left[ \int_t^T \Xi(s, t)\xi(s) \, ds \right], \quad t \in (S, T).
\]

Also, for each \(\Xi_1 \in L_{F,c}^2(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_2})\) and \(\Xi_2 \in L_{F,c}^2(\Delta_2(S, T); \mathbb{R}^{d_2 \times d_3})\), we define

\[
(\Xi_1 \tilde{\ast} \Xi_2)(t, s) := \mathbb{E}_t \left[ \int_t^T \Xi_1(r, t)\Xi_2(r, s) \, dr \right], \quad (t, s) \in \Delta_2(S, T).
\]

**Lemma 4.9.** For each \(\Xi \in L_{F,c}^2(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_2})\) and \(\xi \in L_{\mathcal{F}}^2(S, T; \mathbb{R}^{d_2 \times d_3})\) with \(d_1, d_2, d_3 \in \mathbb{N}\), the backward *-product \(\Xi \tilde{\ast} \xi\) is in \(L_{\mathcal{F}}^2(S, T; \mathbb{R}^{d_1 \times d_3})\), and the Wiener–Itô chaos expansion satisfies

\[
\mathfrak{S}_n[\Xi \tilde{\ast} \xi](t_0, t) = \sum_{k=0}^{n} \int_{t_0}^{t} \mathfrak{S}_{k-n}[\Xi](s, t_0)\mathfrak{S}_k[\xi](s, t) \, ds,
\]

for each \(n \in \mathbb{N}_0\) and \((t_0, t) = (t_0, t_1, \ldots, t_n) \in \Delta_{n+1}(S, T)\). Furthermore, for each \(\Xi_1 \in L_{F,c}^2(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_2})\) and \(\Xi_2 \in L_{F,c}^2(\Delta_2(S, T); \mathbb{R}^{d_2 \times d_3})\), the backward *-product \(\Xi_1 \tilde{\ast} \Xi_2\) is in \(L_{F,c}^2(\Delta_2(S, T); \mathbb{R}^{d_1 \times d_3})\), and the Wiener–Itô chaos expansion satisfies

\[
\mathfrak{S}_n[\Xi_1 \tilde{\ast} \Xi_2](t_0, t) = \sum_{k=0}^{n} \int_{t_0}^{t} \mathfrak{S}_{k-n}[\Xi_1](s, t_0)\mathfrak{S}_k[\Xi_2](s, t) \, ds,
\]

for each \(n \in \mathbb{N}_0\) and \((t_0, t) = (t_0, t_1, \ldots, t_n, t_{n+1}) \in \Delta_{n+2}(S, T)\).

**Proof.** We prove the first assertion. The second can be proved similarly. By using the conditional Hölder’s inequality and noting that \(\Xi(s, t)\) is independent of \(\mathcal{F}_t\), we have

\[
\mathbb{E} \left[ \int_S^T \mathbb{E}_t \left[ \int_t^T \Xi(s, t)\xi(s) \, ds \right] \, dt \right]^2 \leq \mathbb{E} \left[ \int_S^T \mathbb{E}_t \left[ \int_t^T |\Xi(s, t)||\xi(s)| \, ds \right] \, dt \right]^2 \leq \mathbb{E} \left[ \int_S^T \mathbb{E}_t \left[ \int_t^T \Xi(s, t)^2 \, ds \right] \mathbb{E}_t \left[ \int_t^T |\xi(s)|^2 \, ds \right] \, dt \right].
\]
This implies that $\Xi^{\ast}_{\omega} \xi \in L^2_{\mathbb{F}}(S,T;\mathbb{R}^{d_1 \times d_3})$, and that the operations $\Xi \mapsto \Xi^{\ast}_{\omega} \xi$ and $\xi \mapsto \Xi^{\ast}_{\omega} \xi$ are continuous. Observe that

$$\Xi^{\ast}_{\omega} \xi = \left( \sum_{m=0}^{\infty} \mathcal{M}_m[\Xi] \right)^{\ast} \left( \sum_{n=0}^{\infty} \mathcal{M}_n[\xi] \right) = \sum_{m,n=0}^{\infty} \mathcal{M}_m[\Xi]^{\ast} \mathcal{M}_n[\xi].$$

For each $m, n \in \mathbb{N}_0$ and $t \in (S, T)$, we have

$$\mathcal{M}_m[\Xi]^{\ast} \mathcal{M}_n[\xi](t) = \mathbb{E}_t \left[ \int_t^T \mathcal{M}_m[\Xi](s, t) \mathcal{M}_n[\xi](s) \, ds \right]$$

$$= \int_t^T \mathbb{E}_t \left[ \left( \int_t^s \mathcal{M}_m[\Xi](s, t_1, \ldots, t_m, t) \, dW(t_1) \right) \cdots \left( \int_t^s \mathcal{M}_n[\xi](s, t_1, \ldots, t_n) \, dW(t_n) \right) \right] \, ds.$$

If $m > n$, we get

$$\mathcal{M}_m[\Xi]^{\ast} \mathcal{M}_n[\xi](t) = 0.$$

On the other hand, if $m \leq n$, we have

$$\mathcal{M}_m[\Xi]^{\ast} \mathcal{M}_n[\xi](t)$$

$$= \int_t^T \int_t^s \mathcal{M}_m[\Xi](s, t_1, \ldots, t_m, t) \cdots \int_t^s \mathcal{M}_n[\xi](s, t_1, \ldots, t_n) \, dW(t_n) \cdots dW(t_{n+1}) \, dt_{n+1} \cdots dt_1 \, ds$$

$$= \int_t^T \int_t^s \mathcal{M}_m[\Xi](s, t_1, \ldots, t_m, t) \cdots \int_t^s \mathcal{M}_n[\xi](s, t_1, \ldots, t_n) \, dW(t_n) \cdots dW(t_{n+1}) \, dt_{n+1} \cdots dt_1 \, ds$$

$$= \int_t^T \int_t^s \mathcal{M}_m[\Xi](s, t_1, \ldots, t_m, t) \cdots \int_t^s \mathcal{M}_n[\xi](s, t_1, \ldots, t_n) \, dW(t_n) \cdots dW(t_{n+1}) \, dt_{n+1} \cdots dt_1 \, ds$$

$$= \int_t^T \int_t^s \cdots \int_t^s \Delta_{m+1}(t,T) \mathcal{M}_m[\Xi](s, t) \mathcal{M}_n[\xi](s, t_1, \ldots, t_{n-m}) \, dW(t_{n-m}) \cdots dW(t_1),$$

where we used the stochastic Fubini theorem in the last equality. Consequently, we obtain

$$(\Xi^{\ast}_{\omega}\xi)(t)$$

$$= \sum_{m \leq n} \int_t^T \int_t^s \cdots \int_t^s \mathcal{M}_m[\Xi](s, t) \mathcal{M}_n[\xi](s, t_1, \ldots, t_{n-m}) \, dW(t_{n-m}) \cdots dW(t_1)$$

$$= \sum_{n=0}^{\infty} \int_t^T \cdots \int_t^T \left\{ \sum_{k=n}^{\infty} \mathcal{M}_k[\Xi](s, t) \mathcal{M}_n[\xi](s, t_1, \ldots, t_{n}) \, ds \right\} \, dW(t_n) \cdots dW(t_1).$$
This implies that
\[
\mathfrak{F}_n[\Xi \Delta_n \xi](t_0, t_1, \ldots, t_n) = \sum_{k=n}^{\infty} \int_{\Delta_{k-n+1}(t_0, t)} \mathfrak{F}_{k-n}[(s, t_0) \mathfrak{F}_k[\xi](s, t_1, \ldots, t_n)] ds
\]
for each \( n \in \mathbb{N}_0 \) and \((t_0, t_1, \ldots, t_n) \in \Delta_{n+1}(S, T)\). This completes the proof.

**Proposition 4.10.** For each \( J, J_1, J_2, J_3 \in \mathcal{F}_p(S, T; \mathbb{R}^{d \times d}) \), \( \xi, \xi_1, \xi_2 \in L^2_p(S, T; \mathbb{R}^{d \times d_1}) \) with \( d_1 \in \mathbb{N} \), \( K \in \mathcal{K}_p(S, T; \mathbb{R}^{d \times d}) \) and \( \alpha \in \mathbb{R} \), the following hold:

\[
\begin{align*}
J_1 \ast J_2 \ast J_3 &= J_1 \ast (J_2 \ast J_3) = J_1 \ast J_2 \ast J_3, \\
(J_1 + J_2) \ast J_3 &= J_1 \ast J_3 + J_2 \ast J_3, \quad \alpha(J_1 \ast J_2) = (\alpha J_1) \ast J_2 = J_1 \ast (\alpha J_2),
\end{align*}
\]

\[
\begin{align*}
J \ast (\xi_1 + \xi_2) &= J \ast \xi_1 + J \ast \xi_2, \\
J \ast (J_1 + J_2) \ast \xi &= J \ast J_1 \ast \xi + J \ast J_2 \ast \xi, \quad \alpha(J \ast \xi) = (\alpha J) \ast \xi = J \ast (\alpha \xi),
\end{align*}
\]

\[
\begin{align*}
J_1 \ast (J_2 \ast J_3) &= (J_1 \ast J_2) \ast J_3, \\
J_1 \ast (J_2 \ast \xi) &= (J_1 \ast J_2) \ast \xi, \\
J \ast (K \ast \xi) &= (K \ast J) \ast \xi, \\
K \ast (J \ast \xi) &= (K \ast J) \ast \xi.
\end{align*}
\]

**Proof.** Noting Lemma 4.9, the assertions (4.6), (4.7) and (4.8) can be proved by the same way as in Lemma 4.4 and thus we omit the proofs. We prove (4.10), (4.11) and (4.12). The equality (4.9) can be proved by the same way as (4.10), and the other assertions are clear from the definition.

First, we prove (4.10). Noting the measurability of \( J_1 \) and using Fubini’s theorem, we see that, for each \( t \in (S, T) \),

\[
(J_1 \ast (J_2 \ast \xi))(t) = \mathbb{E}_t \left[ \int_t^T J_1(s, t) (J_2 \ast \xi)(s) ds \right]
\]
\[
= \mathbb{E}_t \left[ \int_t^T \int_s^T J_1(s, t) J_2(r, s) \xi(r) dr ds \right]
\]
\[
= \mathbb{E}_t \left[ \int_t^T J_1(s, t) \int_s^T J_2(r, s) \xi(r) dr ds \right]
\]
\[
= \mathbb{E}_t \left[ \int_t^T \left( \int_s^T J_2(r, s) J_1(s, t) \xi(r) dr \right) dr \right]
\]
\[
= \mathbb{E}_t \left[ \int_t^T (J_2 \ast J_1)(r, t) \xi(r) dr \right]
\]
\[
= ((J_2 \ast J_1) \ast \xi)(t).
\]

Thus, (4.10) holds.
Next, we prove (4.11). By Lemma [4.3] for each \( n \in \mathbb{N}_0 \) and \((t_0, t) = (t_0, t_1, \ldots, t_n) \in \Delta_{n+1}(S, T)\),

\[
\mathfrak{F}_n[J \mapsto (K \mapsto \xi)](t_0, t)
\]

\[
= \sum_{k=n}^{\infty} \int_{\Delta_{k-n}(t_0, T)} \sum_{\ell=k}^{\infty} \mathfrak{F}_{k-\ell} \mathfrak{F}_k[J](r, s) dJ(r, s, t, s, t) ds
\]

\[
= \sum_{k=n}^{\infty} \int_{\Delta_{k-n}(t_0, T)} \sum_{\ell=k}^{\infty} \mathfrak{F}_{k-\ell} \mathfrak{F}_k[J](r, s) dJ(r, s, t, s, t) ds
\]

\[
= \sum_\ell \int_{\Delta_{t-n}(t_0, T)} \mathfrak{F}_\ell \left( \sum_{k=n}^{\infty} \mathfrak{F}_{k-\ell} \mathfrak{F}_k[J](r, s) dJ(r, s, t, s, t) ds \right) dr
\]

Thus, (4.11) holds.

Lastly, we prove (4.12). By Lemma [4.3] and Lemma [3.8] for each \( n \in \mathbb{N}_0 \) and \( t \in \Delta_{n+1}(S, T)\),

\[
\mathfrak{F}_n[K \mapsto (J \mapsto \xi)](t)
\]

\[
= \sum_{k=n}^{\infty} \int_{\Delta_{k-n}(t_0, T)} \sum_{\ell=k}^{\infty} \mathfrak{F}_{k-\ell} \mathfrak{F}_k[K](r, s) dK(r, s, t, s, t) ds
\]

\[
= \sum_\ell \int_{\Delta_{t-n}(t_0, T)} \mathfrak{F}_\ell \left( \sum_{k=n}^{\infty} \mathfrak{F}_{k-\ell} \mathfrak{F}_k[K](r, s) dK(r, s, t, s, t) ds \right) dr
\]

Thus, (4.12) holds, and we finish the proof.

The following is a duality principle with respect to the \(*\)-product and the backward \(*\)-product.

**Proposition 4.11.** For each \(*\)-Volterra kernel \( J \in \mathcal{J}_\mathcal{F}(S, T; \mathbb{R}^{d \times d}) \) and \( \varphi, \psi \in L_2^\mathcal{F}(S, T; \mathbb{R}^d) \), it holds that

\[
\langle J \ast \varphi, \psi \rangle_{L_2^\mathcal{F}(S, T)} = \langle \varphi, J^\ast \ast \psi \rangle_{L_2^\mathcal{F}(S, T)}.
\]

**Proof.** By using Fubini’s theorem, we have

\[
\langle J \ast \varphi, \psi \rangle_{L_2^\mathcal{F}(S, T)} = \mathbb{E} \left[ \int_T \langle (J \ast \varphi)(t), \psi(t) \rangle dt \right]
\]

\[33\]
Thus, we get the assertion.

4.3 A variation of constants formula for generalized BSVIEs

Now we show the variation of constants formula for generalized BSVIEs.

**Theorem 4.12.** Let a \(*\)-Volterra kernel \( J \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \) and a \(*\)-Volterra kernel \( K \in \mathcal{K}_F(S,T; \mathbb{R}^{d \times d}) \) be fixed, and suppose that \((J, K)\) has a \((*, *)\)-resolvent \((Q, R) \in \mathcal{J}_F(S,T; \mathbb{R}^{d \times d}) \times \mathcal{K}_F(S,T; \mathbb{R}^{d \times d})\). Then for any free term \( \psi \in L^2_F(S,T; \mathbb{R}^d) \), the generalized BSVIE

\[
Y = \psi + J^{\uparrow \downarrow} Y + K^{\uparrow \downarrow} Y
\]

has a unique solution \( Y \in L^2_F(S,T; \mathbb{R}^d) \). This solution is given by the variation of constants formul a:

\[
Y = \psi + Q^{\uparrow \downarrow} \psi + R^{\uparrow \downarrow} \psi.
\]

**Proof.** Define \( Y \in L^2_F(S,T; \mathbb{R}^d) \) by \( Y = \psi + Q^{\uparrow \downarrow} \psi + R^{\uparrow \downarrow} \psi \). By Proposition 4.5, Proposition 4.10 and the resolvent equations, we have

\[
J^{\uparrow \downarrow} Y = J^{\uparrow \downarrow} (\psi + Q^{\uparrow \downarrow} \psi + R^{\uparrow \downarrow} \psi)
= J^{\uparrow \downarrow} \psi + (Q \ast J)^{\uparrow \downarrow} \psi + (R \ast J)^{\uparrow \downarrow} \psi
= (J + Q \ast J + R \ast J)^{\uparrow \downarrow} \psi
= Q^{\uparrow \downarrow} \psi
\]

and

\[
K^{\uparrow \downarrow} Y = K^{\uparrow \downarrow} (\psi + Q^{\uparrow \downarrow} \psi + R^{\uparrow \downarrow} \psi)
= K^{\uparrow \downarrow} \psi + (Q \ast K)^{\uparrow \downarrow} \psi + (R \ast K)^{\uparrow \downarrow} \psi
= (K + Q \ast K + R \ast K)^{\uparrow \downarrow} \psi
= R^{\uparrow \downarrow} \psi.
\]

Thus, we get \( Y = \psi + J^{\uparrow \downarrow} Y + K^{\uparrow \downarrow} Y \).

Conversely, if \( Y \in L^2_F(S,T; \mathbb{R}^d) \) satisfies \( Y = \psi + J^{\uparrow \downarrow} Y + K^{\uparrow \downarrow} Y \), then we have

\[
Q^{\uparrow \downarrow} \psi = Q^{\uparrow \downarrow} (Y - J^{\uparrow \downarrow} Y - K^{\uparrow \downarrow} Y)
= Q^{\uparrow \downarrow} Y - (J \ast Q)^{\uparrow \downarrow} Y - (K \ast Q)^{\uparrow \downarrow} Y
= (Q - J \ast Q - K \ast Q)^{\uparrow \downarrow} Y
= J^{\uparrow \downarrow} Y
\]
and

\[ R^\top \star \psi = R^\top \star (Y - J^\top \star Y - K^\top \star Y) \]
\[ = R^\top \star Y - (J \ast R)^\top \star Y - (K \ast R)^\top \star Y \]
\[ = (R - J \ast R - K \ast R)^\top \star Y \]
\[ = K^\top \star Y. \]

Thus, we get \( Y = \psi + Q^\top \star \psi + R^\top \star \psi \). This completes the proof. \( \square \)

From the above result, together with the stochastic integral representation of the backward \( \ast \)-product (see Proposition 4.10), we immediately get the following corollary.

**Corollary 4.13.** Let \( J \in \mathcal{J}_\Phi(S,T;\mathbb{R}^{d \times d}) \) and \( K = \sum_{n=1}^{\infty} \mathcal{M}_n[k_n] \in \mathcal{K}_\Phi(S,T;\mathbb{R}^{d \times d}) \) with \( k_n \in \mathcal{V}_{n+1}(S,T;\mathbb{R}^{d \times d}) \), \( n \in \mathbb{N} \), be fixed, and suppose that \((J,K)\) has a \((\ast,\ast)\)-resolvent \((Q,R) \in \mathcal{J}_\Phi(S,T;\mathbb{R}^{d \times d}) \times \mathcal{K}_\Phi(S,T;\mathbb{R}^{d \times d})\). Then for any free term \( \psi \in L_2^J(S,T;\mathbb{R}^d) \), the generalized BSVIE

\[ Y(t) = \psi(t) + \mathbb{E}_t \left[ \int_t^T J(s,t)^\top Y(s) \, ds \right] + \sum_{n=1}^{\infty} \int_{\Delta_{n}(t,T)} k_n(s,t)^\top \mathcal{M}_n[Y](s,t) \, ds, \ t \in (S,T), \]

has a unique solution \( Y \in L_2^J(S,T;\mathbb{R}^d) \). This solution is given by the variation of constants formula:

\[ Y(t) = \psi(t) + \mathbb{E}_t \left[ \int_t^T Q(s,t)^\top \psi(s) \, ds \right] + \sum_{n=1}^{\infty} \int_{\Delta_{n}(t,T)} r_n(s,t)^\top \mathcal{M}_n[\psi](s,t) \, ds, \ t \in (S,T), \]

where \( r_n := \mathfrak{F}_n[R] \in \mathcal{V}_{n+1}(S,T;\mathbb{R}^{d \times d}) \) for \( n \in \mathbb{N} \). Here, the infinite sum in the right-hand side converges in \( L_2^J(S,T;\mathbb{R}^d) \).

**Corollary 4.14.** Let deterministic kernels \( j \in L^2(\Delta_2(S,T);\mathbb{R}^{d \times d}) \) and \( k \in \mathcal{V}_2(S,T;\mathbb{R}^{d \times d}) \) be fixed. Let \( q \in L^2(\Delta_2(S,T);\mathbb{R}^{d \times d}) \) be the \( \ast \)-resolvent of \( j \), and assume that \( \mathcal{M}_1[k + q \ast k] \) has a \( \ast \)-resolvent. Then for any free term \( \psi \in L_2^J(S,T;\mathbb{R}^d) \), the Type-II BSVIE

\[ Y(t) = \psi(t) + \mathbb{E}_t \left[ \int_t^T j(s,t)^\top Y(s) \, ds \right] + \int_t^T k(s,t)^\top \mathcal{M}_1[Y](s,t) \, ds, \ t \in (S,T), \]

has a unique solution \( Y \in L_2^J(S,T;\mathbb{R}^d) \). This solution is given by the variation of constants formula:

\[ Y(t) = \psi(t) + \sum_{n=1}^{\infty} \int_{\Delta_{n}(t,T)} (k + q \ast k)^\top \mathcal{M}_n[\psi](r,t) \, dr \]
\[ + \int_t^T q(s,t)^\top \mathbb{E}_t \left[ \psi(s) + \sum_{n=1}^{\infty} \int_{\Delta_{n}(s,T)} (k + q \ast k)^\top \mathcal{M}_n[\psi](r,s) \, dr \right] \, ds, \ t \in (S,T). \]

**Proof.** By the arguments in the proof of Corollary 3.24, we see that \((Q,R)\) defined by \( R = \sum_{n=1}^{\infty} \mathcal{M}_n[(k + q \ast k)^\top] \in \mathcal{K}_\Phi(S,T;\mathbb{R}^{d \times d}) \) and \( Q = q + R \ast q \in \mathcal{J}_\Phi(S,T;\mathbb{R}^{d \times d}) \) is the \((\ast,\ast)\)-resolvent of \((j,\mathcal{M}_1[k])\). Thus, by Theorem 4.12, the BSVIE \( Y = \psi + J^\top \star \psi + 2\mathbb{M}_1[k]^\top \star Y \) has a unique solution \( Y \in L_2^J(S,T;\mathbb{R}^d) \), and the solution is given by

\[ Y = \psi + Q^\top \star \psi + R^\top \star \psi \]
\[ = \psi + (q + R \ast q)^\top \star \psi + R^\top \star \psi \]
\[ = \psi + q^\top \star \psi + q^\top \star (R^\top \star \psi) + R^\top \star \psi \]
\[ = \psi + R^\top \star \psi + q^\top \star (\psi + R^\top \star \psi). \]

Here, we used Proposition 4.10. Noting Proposition 4.16, we get the assertion. \( \square \)
By the variation of constants formulae, together with the duality principles for the $\ast$- and the $\ast$-products (see Proposition 3.17 and Proposition 4.11), we get the following duality principle between generalized SVIEs and generalized BSVIEs.

**Theorem 4.15.** Let a $\ast$-Volterra kernel $J \in \mathcal{F}(S,T;\mathbb{R}^{d\times d})$ and a $\ast$-Volterra kernel $K \in \mathcal{K}_F(S,T;\mathbb{R}^{d\times d})$ be fixed, and suppose that $(J,K)$ has a $(\ast,\ast)$-resolvent $(Q,R) \in \mathcal{F}(S,T;\mathbb{R}^{d\times d}) \times \mathcal{K}_F(S,T;\mathbb{R}^{d\times d})$. For each $\varphi,\psi \in L_2^F(S,T;\mathbb{R}^d)$, let $X,Y \in L_2^F(S,T;\mathbb{R}^d)$ be the solutions of the generalized SVIE

$$X = \varphi + J \ast X + K \ast X$$

and the generalized BSVIE

$$Y = \psi + J^\top \ast Y + K^\top \ast Y,$$

respectively. Then it holds that

$$\langle X,\psi \rangle_{L_2^F(S,T)} = \langle \varphi,Y \rangle_{L_2^F(S,T)}.$$

**Proof.** By Theorem 3.22 and Theorem 4.12 $X$ and $Y$ are given by $X = \varphi + Q \ast \varphi + R \ast \varphi$ and $Y = \psi + Q^\top \ast \psi + R^\top \ast \psi$, respectively. Therefore, by Proposition 4.7 and Proposition 4.11, we get

$$\langle X,\psi \rangle_{L_2^F(S,T)} = \langle \varphi,Q \ast \varphi + R \ast \varphi,\psi \rangle_{L_2^F(S,T)} = \langle \varphi,\psi,Q^\top \ast \psi + R^\top \ast \psi \rangle_{L_2^F(S,T)} = \langle \varphi,Y \rangle_{L_2^F(S,T)}.$$

Thus, the assertion holds.

---

### 5 Existence of the $\ast$-resolvent

As we have seen in the above sections, the $(\ast,\ast)$-resolvent plays crucial roles for solving SVIEs and BSVIEs. Recall that every $\ast$-Volterra kernel has a unique $\ast$-resolvent (see Proposition 3.10). Also, the $(\ast,\ast)$-resolvent can be constructed by the $\ast$-resolvent and the $\ast$-resolvent (see Proposition 3.20). In this section, we focus on the existence of the $\ast$-resolvent. We borrow some ideas from Section 9.3 of the textbook [11], where the general theory on deterministic Volterra kernels is discussed.

First, we show the following lemma.

**Lemma 5.1.** Every $\ast$-Volterra kernel $K \in \mathcal{K}_F(S,T;\mathbb{R}^{d\times d})$ with $\|K\|_{\mathcal{K}_F(S,T)} < 1$ has a $\ast$-resolvent $R \in \mathcal{K}_F(S,T;\mathbb{R}^{d\times d})$. Furthermore, it holds that $R = \sum_{n=1}^{\infty} K^{\ast n}$, where $K^{\ast n}$ denotes the $(n-1)$-fold $\ast$-product of $K$ by itself, and the infinite sum converges in $\mathcal{K}_F(S,T;\mathbb{R}^{d\times d})$.

**Proof.** By Proposition 3.5, we have $\|K^{\ast n}\|_{\mathcal{K}_F(S,T)} \leq \|K\|_{\mathcal{K}_F(S,T)}$ for any $n \in \mathbb{N}$. Thus, by the assumption, the infinite sum $R := \sum_{n=1}^{\infty} K^{\ast n}$ converges in $\mathcal{K}_F(S,T;\mathbb{R}^{d\times d})$. Observe that

$$K + K \ast R = K + K \ast \left( \sum_{n=1}^{\infty} K^{\ast n} \right) = K + \sum_{n=1}^{\infty} K \ast K^{\ast n} = \sum_{n=1}^{\infty} K^{\ast n} = R.$$

Similarly, we have $K + R \ast K = R$. Thus, $R$ is the $\ast$-resolvent of $K$.

The above lemma is not so useful by its own right for the existence of the $\ast$-resolvent on the whole interval $(S,T)$ with arbitrary length. In order to ensure the existence of the $\ast$-resolvent in more general settings, we introduce the notion of the restriction and the concatenation of $\ast$-Volterra kernels with respect to subintervals of $(S,T)$.

**Definition 5.2.** For each $\ast$-Volterra kernel $K \in \mathcal{K}_F(S,T;\mathbb{R}^{d\times d})$ and a subinterval $(U,V)$ of $(S,T)$, we define the restricted $\ast$-Volterra kernel $K_{(U,V)} \in \mathcal{K}_F(U,V;\mathbb{R}^{d\times d})$ of $K$ on $(U,V)$ by the Wiener–Itô chaos expansion

$$\mathfrak{S}_n[K_{(U,V)}] := \mathfrak{S}_n[K]_{\Delta_{n+1}(U,V)}, \quad n \in \mathbb{N}_0,$$

where, for each function $f : \Delta_{n+1}(S,T) \to \mathbb{R}^{d\times d}$ with $n \in \mathbb{N}_0$, the function $f|_{\Delta_{n+1}(U,V)} : \Delta_{n+1}(U,V) \to \mathbb{R}^{d\times d}$ is the restriction of $f$ on $\Delta_{n+1}(U,V)$. 

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Remark 5.3.  
(i) Note that it does not hold \( K_{(U,V)}(t) = K(t) \) for \( t \in (U,V) \) in general.

(ii) Let \( 0 \leq S \leq U < V \leq T < \infty \). Suppose that a \( \ast \)-Volterra kernel \( K \in K_{\mathbb{F}}(S,T;\mathbb{R}^{d \times d}) \) has a \( \ast \)-resolvent \( R \in K_{\mathbb{F}}(S,T;\mathbb{R}^{d \times d}) \) on \((S,T)\). By the definition, it is easy to see that the restricted \( \ast \)-Volterra kernel \( K_{(U,V)} \in K_{\mathbb{F}}(U,V;\mathbb{R}^{d \times d}) \) has a \( \ast \)-resolvent on \((U,V)\), which is given by the restricted \( \ast \)-Volterra kernel \( R_{(U,V)} \in K_{\mathbb{F}}(U,V;\mathbb{R}^{d \times d}) \) of \( R \).

We prove the opposite direction of (ii) in the above remark. In other words, we provide a (non-trivial) concatenation procedure for the \( \ast \)-resolvent. The following is the main result of this section.

Theorem 5.4. Let \( K \in K_{\mathbb{F}}(S,T;\mathbb{R}^{d \times d}) \) be a \( \ast \)-Volterra kernel on \((S,T)\) with \( 0 \leq S < U < T < \infty \). Suppose that there exists \( \{U_k\}_{k=0}^m \subset \mathbb{R} \) with \( m \in \mathbb{N} \) such that \( S = U_0 < U_1 < \cdots < U_m = T \) and that, for every \( k \in \{0,1,\ldots,m-1\} \), the restricted \( \ast \)-Volterra kernel \( K_{(U_k,U_{k+1})} \in K_{\mathbb{F}}(U_k,U_{k+1};\mathbb{R}^{d \times d}) \) has a \( \ast \)-resolvent on \((U_k,U_{k+1})\). Then \( K \) has a \( \ast \)-resolvent on \((S,T)\).

Proof. It suffices to show the theorem when \( m = 2 \). Let \( S < U < T \) be fixed, and assume that \( K_{(S,U)} \in K_{\mathbb{F}}(S,U;\mathbb{R}^{d \times d}) \) (resp. \( K_{(U,T)} \in K_{\mathbb{F}}(U,T;\mathbb{R}^{d \times d}) \)) has a \( \ast \)-resolvent \( R_{(S,U)} \in K_{\mathbb{F}}(S,U;\mathbb{R}^{d \times d}) \) on \((S,U)\) (resp. \( R_{(U,T)} \in K_{\mathbb{F}}(U,T;\mathbb{R}^{d \times d}) \) on \((U,T)\)). Define \( \ast \)-Volterra kernels \( K_i, R_i \in K_{\mathbb{F}}(S,T;\mathbb{R}^{d \times d}) \), \( i = 1,2 \), on the whole interval \((S,T)\) by the Wiener–Itô chaos expansions

\[
\tilde{\mathfrak{f}}_n[K_1] := \tilde{\mathfrak{f}}_n[K(S,U)] \text{ on } \Delta_{n+1}(S,U), \quad \tilde{\mathfrak{f}}_n[K_1] := 0 \text{ on } \Delta_{n+1}(S,U) \setminus \Delta_{n+1}(S,U), \quad n \in \mathbb{N}_0, \\
\tilde{\mathfrak{f}}_n[K_2] := \tilde{\mathfrak{f}}_n[R(S,U)] \text{ on } \Delta_{n+1}(U,T), \quad \tilde{\mathfrak{f}}_n[K_2] := 0 \text{ on } \Delta_{n+1}(U,T) \setminus \Delta_{n+1}(U,T), \quad n \in \mathbb{N}_0, \\
\tilde{\mathfrak{f}}_n[R_1] := \tilde{\mathfrak{f}}_n[R(U,T)] \text{ on } \Delta_{n+1}(S,U), \quad \tilde{\mathfrak{f}}_n[R_1] := 0 \text{ on } \Delta_{n+1}(S,U) \setminus \Delta_{n+1}(S,U), \quad n \in \mathbb{N}_0, \\
\tilde{\mathfrak{f}}_n[R_2] := \tilde{\mathfrak{f}}_n[R(U,T)] \text{ on } \Delta_{n+1}(U,T), \quad \tilde{\mathfrak{f}}_n[R_2] := 0 \text{ on } \Delta_{n+1}(U,T) \setminus \Delta_{n+1}(U,T), \quad n \in \mathbb{N}_0.
\]

Define \( R \in K_{\mathbb{F}}(S,T;\mathbb{R}^{d \times d}) \) by

\[
R := K + K \ast R_1 + R_2 \ast K + R_2 \ast K \ast R_1.
\]

We will show that \( R \) is the \( \ast \)-resolvent of \( K \) on \((S,T)\).

First, we show that \( R_i \) is the \( \ast \)-resolvent of \( K_i \) on \((S,T)\) for \( i = 1,2 \). Observe that, on \( \Delta_{n+1}(S,T) \) with \( n \in \mathbb{N}_0 \),

\[
\tilde{\mathfrak{f}}_n[K_1 \ast R_1] = \sum_{k=0}^{n} \tilde{\mathfrak{f}}_{n-k}[K_1] \ast \tilde{\mathfrak{f}}_k[R_1] = \left( \sum_{k=0}^{n} \tilde{\mathfrak{f}}_{n-k}[K(S,U)] \ast \tilde{\mathfrak{f}}_k[R(U,T)] \right) \mathbb{1}_{\Delta_{n+1}(S,U)} \\
= \tilde{\mathfrak{f}}_n[K(S,U) \ast R(U,T)] \mathbb{1}_{\Delta_{n+1}(S,U)} = \tilde{\mathfrak{f}}_n[R(U,T) - K(S,U)] \mathbb{1}_{\Delta_{n+1}(S,U)} \\
= \tilde{\mathfrak{f}}_n[R_1 - K_1],
\]

and hence \( R_1 = K_1 + K_1 \ast R_1 \) on \((S,T)\). Similarly, we can show that \( R_1 = K_1 + R_1 \ast K_1 \) and \( R_2 = K_2 + K_2 \ast R_2 = K_2 + R_2 \ast K_2 \) on \((S,T)\). Thus, \( R_i \) is the \( \ast \)-resolvent of \( K_i \) on \((S,T)\) for \( i = 1,2 \).

Second, we show the following claim:

\[
R = K + K \ast R_1 + K_2 \ast R = K + R_2 \ast K + R \ast K_1.
\]  \( \text{(5.1)} \)

Since \( R_2 \) is the \( \ast \)-resolvent of \( K_2 \), we have

\[
K_2 \ast R = K_2 \ast K + K_2 \ast K_1 R_1 + K_2 \ast R_2 \ast K_2 + K_2 \ast R_2 \ast K \ast R_1 \\
= R_2 \ast K + R_2 \ast K \ast R_1.
\]

By inserting this formula into the definition of \( R \), we get the first equality in \( (5.1) \). On the other hand, since \( R_1 \) is the \( \ast \)-resolvent of \( K_1 \), we have

\[
R \ast K_1 = K \ast K_1 + K \ast R_1 \ast K_1 + R_1 \ast K_1 + R \ast K \ast R_1 \ast K_1 \\
= R_1 \ast K_1.
\]
By inserting this formula into the definition of \( R \), we get the second equality in (5.1).

Third, let us show the following claim:

\[
\delta_n[K] = \delta_n[K_1] \text{ on } \Delta_{n+1}(S,U) \text{ and } \delta_n[K] = \delta_n[K_2] \text{ on } \Delta_{n+1}(U,T),
\]

\[
\delta_n[R] = \delta_n[R_1] \text{ on } \Delta_{n+1}(S,U) \text{ and } \delta_n[R] = \delta_n[R_2] \text{ on } \Delta_{n+1}(U,T),
\]

(5.2)

for each \( n \in \mathbb{N}_0 \). The equalities for \( K \) are trivial from the definition. We show the equalities for \( R \). Let \( n \in \mathbb{N}_0 \). Observe that, on \( \Delta_{n+1}(S,U) \),

\[
\delta_n[K] = \delta_n[K_1],
\]

\[
\delta_n[K \ast R_1] = \sum_{k=0}^{n} \delta_{n-k}[K] \ast \delta_k[R_1] = \sum_{k=0}^{n} \delta_{n-k}[K_1] \ast \delta_k[R_1] = \delta_n[K_1 \ast R_1],
\]

\[
\delta_n[R_2 \ast K] = \sum_{k=0}^{n} \delta_{n-k}[R_2] \ast \delta_k[K] = 0,
\]

\[
\delta_n[R_2 \ast K \ast R_1] = \sum_{k=0}^{n} \delta_{n-k}[R_2] \ast \delta_k[K \ast R_1] = 0,
\]

and hence

\[
\delta_n[R] = \delta_n[K_1] + \delta_n[K_1 \ast R_1] = \delta_n[K_1 + K_1 \ast R_1] = \delta_n[R_1].
\]

On the other hand, on \( \Delta_{n+1}(U,T) \),

\[
\delta_n[K] = \delta_n[K_2],
\]

\[
\delta_n[K \ast R_1] = \sum_{k=0}^{n} \delta_{n-k}[K] \ast \delta_k[R_1] = 0,
\]

\[
\delta_n[R_2 \ast K] = \sum_{k=0}^{n} \delta_{n-k}[R_2] \ast \delta_k[K] = \sum_{k=0}^{n} \delta_{n-k}[R_2] \ast \delta_k[K_2] = \delta_n[R_2 \ast K_2],
\]

\[
\delta_n[R_2 \ast K \ast R_1] = \sum_{k=0}^{n} \delta_{n-k}[R_2] \ast \delta_k[R_1] = 0,
\]

and hence

\[
\delta_n[R] = \delta_n[K_2] + \delta_n[R_2 \ast K_2] = \delta_n[K_2 + R_2 \ast K_2] = \delta_n[R_2].
\]

Thus, the claim holds.

Lastly, we show the following claim:

\[
K \ast R_1 + K_2 \ast R = K \ast R \text{ and } R_2 \ast K + R \ast K_1 = R \ast K.
\]

(5.3)

By using (5.2), for each \( n \in \mathbb{N}_0 \), we have

\[
\delta_n[K \ast R_1] = \sum_{k=0}^{n} \delta_{n-k}[K] \ast \delta_k[R_1] = \sum_{k=0}^{n} \delta_{n-k}[K] \ast (\delta_k[R_1] \mathbb{1}_{\Delta_{k+1}(S,U)})
\]

\[
= \sum_{k=0}^{n} \delta_{n-k}[K] \ast (\delta_k[R] \mathbb{1}_{\Delta_{k+1}(S,U)})
\]

and, for a.e. on \( \Delta_{n+1}(S,T) \),

\[
\delta_n[K_2 \ast R] = \sum_{k=0}^{n} \delta_{n-k}[K_2] \ast \delta_k[R] = \sum_{k=0}^{n} (\delta_{n-k}[K_2] \mathbb{1}_{\Delta_{n-k+1}(U,T)}) \ast \delta_k[R]
\]

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Thus, we get
\[
\tilde{\mathfrak{f}}_n[K* R_1 + K_2* R] = \sum_{k=0}^{n} (\tilde{\mathfrak{f}}_n - k[K] \triangleright \mathfrak{f}_k[R])
\]
for each \(n \in \mathbb{N}_0\). Therefore, the first equality in (5.3) holds. Similarly, for each \(n \in \mathbb{N}_0\), we have
\[
\tilde{\mathfrak{f}}_n[R_2* K] = \sum_{k=0}^{n} (\tilde{\mathfrak{f}}_n - k[R_2] \triangleright \mathfrak{f}_k[K]) = \sum_{k=0}^{n} (\tilde{\mathfrak{f}}_n - k[R] \triangleright \mathfrak{f}_k[K])
\]
and, for a.e. on \(\Delta_{n+1}(S, T)\),
\[
\tilde{\mathfrak{f}}_n[R* K_1] = \sum_{k=0}^{n} (\tilde{\mathfrak{f}}_n - k[R] \triangleright \mathfrak{f}_k[K_1]) = \sum_{k=0}^{n} (\tilde{\mathfrak{f}}_n - k[R] \triangleright (\mathfrak{f}_k[K_1] \mathbb{1}_{\Delta_{k+1}(S, U)}))
\]
\[
= \sum_{k=0}^{n} (\tilde{\mathfrak{f}}_n - k[R] \triangleright \mathfrak{f}_k[K])
\]
\[
= \sum_{k=0}^{n} (\tilde{\mathfrak{f}}_n - k[R] \mathbb{1}_{\Delta_{n-k+1}(S, U)} \mathbb{1}_{\Delta_{n-k+1}(S, T)}) \triangleright \mathfrak{f}_k[K].
\]
Thus, we get
\[
\tilde{\mathfrak{f}}_n[R_2* K + R* K_1] = \sum_{k=0}^{n} (\tilde{\mathfrak{f}}_n - k[R] \triangleright \mathfrak{f}_k[K]) = \tilde{\mathfrak{f}}_n[R* K] \text{ a.e. on } \Delta_{n+1}(S, T)
\]
for each \(n \in \mathbb{N}_0\). Therefore, the second equality in (5.3) holds.

By Lemma 5.1 and Theorem 5.4, we get the following important corollaries.

**Corollary 5.5.** Let \(K \in K_{\mathbb{R}}(S, T; \mathbb{R}^{d \times d})\) be a \(*\)-Volterra kernel. Suppose that there exists \(\{U_k\}_{k=0}^{m} \subseteq \mathbb{R}\) such that \(S = U_0 < U_1 < \cdots < U_m = T\) and that, for every \(k \in \{0, 1, \ldots, m-1\}\), the restricted \(*\)-Volterra kernel \(K_{(U_k, U_{k+1})} \in K_{\mathbb{R}}(U_k, U_{k+1}; \mathbb{R}^{d \times d})\) satisfies 
\[|||K_{(U_k, U_{k+1})}|||_{K_{\mathbb{R}}(U_k, U_{k+1})} < 1.\]
Then \(K\) has a \(*\)-resolvent \(R \in K_{\mathbb{R}}(S, T; \mathbb{R}^{d \times d})\) on \((S, T)\).

**Corollary 5.6.** Let \(K \in K_{\mathbb{R}}(S, T; \mathbb{R}^{d \times d})\) be a \(*\)-Volterra kernel. Assume that
\[
\text{ess sup}_{t \in (S, T)} |\tilde{\mathfrak{f}}_0[K](t)|_{\text{op}} < 1
\]
and that there exists a nonnegative function \(a \in L^2(0, T - S; \mathbb{R})\) satisfying
\[
|\tilde{\mathfrak{f}}_n[K](t_0, t_1, \ldots, t_n)|_{\text{op}} \leq \prod_{i=1}^{n} a(t_i - t_{i-1}), \quad (t_0, t_1, \ldots, t_n) \in \Delta_{n+1}(S, T),
\]
for any \(n \in \mathbb{N}\). Then \(K\) has a \(*\)-resolvent \(R \in K_{\mathbb{R}}(S, T; \mathbb{R}^{d \times d})\) on \((S, T)\).
Proof. We denote $\delta := \text{ess sup}\{\|\mathfrak{S}_0[K](t)\|\}_{op} < 1$. Let $\varepsilon \in (0, 1)$ be a number such that $\delta + \frac{\varepsilon}{1 - \varepsilon} < 1$. Then there exists $\{U_k\}_{k=0}^m$ with $m \in \mathbb{N}$ such that $S = U_0 < U_1 < \cdots < U_m = T$ and that $\|a\|_{L^2(0,U_{k+1}-U_k)} \leq \varepsilon$ for every $k \in \{0,1,\ldots,m-1\}$. Fix an arbitrary $k \in \{0,1,\ldots,m-1\}$. Note that
\[
\|\mathfrak{S}_0[K(U_k,U_{k+1})]\|_{V_1(U_k,U_{k+1})} = \text{ess sup}_{t \in (U_k,U_{k+1})} |\mathfrak{S}_0[K](t)|_{op} \leq \delta.
\] Furthermore, noting Lemma 2.3, we see that
\[
\|\mathfrak{S}_n[K(U_k,U_{k+1})]\|_{V_{n+1}(U_k,U_{k+1})} \leq \|a\|_{L^2(0,U_{k+1}-U_k)} \leq \varepsilon^n
\] for any $n \in \mathbb{N}$. Thus, we have
\[
\|K(U_k,U_{k+1})\|_{K_p(U_k,U_{k+1})} = \sum_{n=0}^{\infty} \|\mathfrak{S}_n[K(U_k,U_{k+1})]\|_{V_{n+1}(U_k,U_{k+1})} \leq \delta + \frac{\varepsilon}{1 - \varepsilon} < 1.
\] By Corollary 5.5 we get the assertion. \(\square\)

Corollary 5.7. Let $K \in K_p(S,T;\mathbb{R}^{d \times d})$ be a $*$-Volterra kernel. Assume that
\[
\text{ess sup}_{t \in (S,T)} |\mathfrak{S}_0[K](t)|_{op} < 1
\] and that
\[
\sup_{n \in \mathbb{N}} \left\{ \left( n! \right)^{-\frac{n-2}{2p}} \text{ess sup}_{t \in (S,T)} \left( \int_{\Delta_n(t,T)} |\mathfrak{S}_n[K](s,t)|_{op}^p \, ds \right)^{1/p} \right\}^{1/n} < \infty
\] for some $p > 2$. Then $K$ has a $*$-resolvent $R \in K_p(S,T;\mathbb{R}^{d \times d})$ on $(S,T)$.

Proof. We denote $\delta := \text{ess sup}\{\|\mathfrak{S}_0[K](t)\|\}_{op} < 1$ and
\[
C_p := \sup_{n \in \mathbb{N}} \left\{ \left( n! \right)^{-\frac{n-2}{2p}} \text{ess sup}_{t \in (S,T)} \left( \int_{\Delta_n(t,T)} |\mathfrak{S}_n[K](s,t)|_{op}^p \, ds \right)^{1/p} \right\}^{1/n} < \infty.
\] Let $\varepsilon > 0$ be a number such that
\[
C_p \varepsilon^{\left(\frac{p-2}{2p}\right)} < 1 \quad \text{and} \quad \delta + C_p \varepsilon^{\left(\frac{p-2}{2p}\right)} < 1.
\] Take $\{U_k\}_{k=0}^m$ with $m \in \mathbb{N}$ such that $S = U_0 < U_1 < \cdots < U_m = T$ and that $U_{k+1} - U_k < \varepsilon$ for every $k \in \{0,\ldots,m-1\}$. Let $k \in \{0,\ldots,m-1\}$ be fixed. The restricted $*$-Volterra kernel $K(U_k,U_{k+1}) \in K_p(U_k,U_{k+1};\mathbb{R}^{d \times d})$ satisfies
\[
\|\mathfrak{S}_0[K(U_k,U_{k+1})]\|_{V_1(U_k,U_{k+1})} = \text{ess sup}_{t \in (U_k,U_{k+1})} |\mathfrak{S}_0[K](t)|_{op} \leq \delta
\] and, by Hölder’s inequality,
\[
\|\mathfrak{S}_n[K(U_k,U_{k+1})]\|_{V_{n+1}(U_k,U_{k+1})} \leq \left( \frac{(U_{k+1} - U_k)^n}{n!} \right)^{\frac{(p-2)}{2p}} \text{ess sup}_{t \in (S,T)} \left( \int_{\Delta_n(t,T)} |\mathfrak{S}_n[K](s,t)|_{op}^p \, ds \right)^{1/p} \leq (C_p \varepsilon^{\left(\frac{p-2}{2p}\right)})^n.
\] for each $n \in \mathbb{N}$. Thus, we have
\[
\|K(U_k,U_{k+1})\|_{K_p(U_k,U_{k+1})} = \sum_{n=0}^{\infty} \|\mathfrak{S}_n[K(U_k,U_{k+1})]\|_{V_{n+1}(U_k,U_{k+1})} \leq \delta + C_p \varepsilon^{\left(\frac{p-2}{2p}\right)} < 1.
\] By Corollary 5.5 we get the assertion. \(\square\)
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