Finite difference/spectral approximations for the two-dimensional time Caputo-Fabrizio fractional diffusion equation

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Abstract
The main contribution of this work is to construct and analyze stable and high order schemes to efficiently solve the two-dimensional time Caputo-Fabrizio fractional diffusion equation. Based on a third-order finite difference method in time and spectral methods in space, the proposed scheme is unconditionally stable and has the global truncation error $O(\tau^3 + N^{-m})$, where $\tau$, $N$ and $m$ are the time step size, polynomial degree and regularity in the space variable of the exact solution, respectively. It should be noted that the global truncation error $O(\tau^2 + N^{-m})$ is well established in [Li, Lv and Xu, Numer. Methods Partial Differ. Equ. (2019)]. Finally, some numerical experiments are carried out to verify the theoretical analysis. To the best of our knowledge, this is the first proof for the stability of the third-order scheme for the Caputo-Fabrizio fractional operator.

Keywords: Caputo-Fabrizio fractional operator; Spectral approximation; Stability and convergence

1. Introduction
In recent years, the Caputo-Fabrizio fractional (CF-fractional) operator [6] is attracting considerable attentions and has gained great popularity. Because of the non-singular kernel of the CF-fractional operator, it has widespread applications. For example, the non-Darcian flow and solute transport in porous media [24] and groundwater flow within an unconfined aquifer [11] are modelled by using the CF-fractional derivative in fluid dynamics. Modelling electro-magneto-hydrodynamic thermo-fluidic transport of biofluids with the Caputo-Fabrizio derivative are discussed in physics fields [1]. In control systems, the authors consider the wave movement on the surface of shallow water with the Caputo-Fabrizio derivative [2].

The various numerical methods for fractional differential equations with the CF-fractional derivative are proposed. For example, numerical approximation of the time CF-fractional derivative and application to groundwater model have been discussed in [3, 9]. The second-order algorithms for Fokker-Planck equation with the CF-fractional derivative is developed in [8] by finite difference method. A local discontinuous Galerkin method is studied for numerically solving the fractal mobile/immobile transport equation with the CF-fractional derivative [23]. In [17], a tau method is described for the CF-fractional differential equations where the shifted Legendre polynomials operational matrix of CF-fractional operator is obtained. Recently, a fully discrete scheme for fractional Cattaneo equation based on CF-fractional derivative is proposed in [14], where the convergence rate of the scheme is $O(\tau^2 + N^{-m})$. However, it seems that achieving a third-order accurate scheme for CF-fractional operator is not an easy task. Herein, we devote to providing effective and high accurate numerical schemes with $O(\tau^3 + N^{-m})$ for the following two-dimensional time CF-fractional diffusion equation [8], for $0 < \gamma < 1$

$$^C_0D^\gamma_t u(x, y, t) = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2}, \quad (x, y, t) \in \Omega \times (0, T), \quad (1.1)$$

with the initial condition $u(x, y, 0) = u_0(x, y)$ and the homogeneous Dirichlet boundary conditions in $\Omega = (-1, 1)^2$. Here the time CF-fractional derivative is defined by [6]

$$^C_0D^\gamma_t u(t) = \frac{1}{1-\gamma} \int_0^t u'(s)e^{-\sigma(t-s)}ds, \quad \sigma = \frac{\gamma}{1-\gamma}, \quad \gamma \in (0, 1). \quad (1.2)$$

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Remark 1.1. Here, we mainly focus on the model [1, 4], since the corresponding theory and numerical experiments can be extended to the two-dimensional space-time fractional diffusion equation [21].

\[ C_0^F D_t^\gamma u(x, y, t) = \frac{\partial\alpha u(x, y, t)}{\partial|x|} + \frac{\partial^\beta u(x, y, t)}{\partial|y|^\beta}, \quad x \in \Omega, \quad t \in (0, T), \]

where the Riesz fractional derivative with \( 1 < \alpha, \beta < 2 \) are given in [18].

The outline of this paper is organized as follows. In Section 2, we propose a third-order finite difference scheme for temporal discretization. The stability and convergence analysis for the one-dimensional time CF-fractional diffusion equation with the semi-discrete scheme are given in section 3. In section 4, we provide spectral methods for the space discretization and derive an optimal error estimation. In the next section, the convergence analysis for the two-dimensional time CF-fractional diffusion equation is discussed. Some numerical examples are given in section 6, which verify the theoretical analysis. The final section is the summary of the paper.

2. Time discretization

Nowadays, there are already two types of second-order discretization schemes for Caputo-Fabrizio fractional operator: the first type is given in [8] based on the Fourier transform method and fractional linear multistep method [7, 13]; and the second type is a L1 formula using linear interpolation approximation [14]. Based on the idea of [5, 12, 16], we employ the linear interpolation and quadratic interpolation approximation (L1–2 formula) to discrete the CF-fractional derivative, which derives a third-order discretization scheme.

Let the time step size \( t_n = n\tau, \ n = 0, 1, \ldots, M \) with \( \tau = \frac{T}{M} \). The linear interpolation approximation is applied at the first grid point for CF-fractional derivative

\[
C_0^F D_t^\gamma u(t_1) = \frac{1}{1 - \gamma} \int_0^{t_1} u'(s)e^{-\sigma(t_1-s)}ds = \frac{1}{1 - \gamma} \int_0^{t_1} H_1'(s)e^{-\sigma(t_1-s)}ds + r_1^\gamma, (2.1)
\]

where we use \( \int_0^{t_1} e^{-\sigma(t_1-s)}ds = \frac{1-e^{-\sigma t}}{\sigma} \) and linear interpolation is

\[
H_1(s) = u(t_0) + \frac{u(t_1) - u(t_0)}{\tau}(s - t_0),
\]

which interpolates the function \( u \) at the two points \( \{t_0, t_1\} \),

\[
H_1(t_0) = u(t_0), \quad H_1(t_1) = u(t_1),
\]

and \( r_1^\gamma \) is the first step truncation error.

We use the quadratic interpolation approximation for CF-fractional derivative with \( n \geq 2 \), i.e.,

\[
C_0^F D_t^\gamma u(t_n) = \frac{1}{1 - \gamma} \int_0^{t_n} u'(s)e^{-\sigma(t_n-s)}ds
\]

\[
= \frac{1}{1 - \gamma} \left( \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} u'(s)e^{-\sigma(t_n-s)}ds + \int_{t_{n-1}}^{t_n} u'(s)e^{-\sigma(t_n-s)}ds \right) (2.2)
\]

\[
= \frac{1}{1 - \gamma} \left( \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} H_1'(s)e^{-\sigma(t_n-s)}ds + \int_{t_{n-1}}^{t_n} H_1'(s)e^{-\sigma(t_n-s)}ds \right) + r_n^\gamma,
\]

it leads to

\[
C_0^F D_t^\gamma u(t_n) = \frac{1}{(1 - \gamma)\sigma^2 \tau^2} \left\{ \sum_{j=1}^{n-1} \left[ a_j u(t_{n-j-1}) + b_j u(t_{n-j}) + c_j u(t_{n-j+1}) \right] \\
+ \left( \sigma + 1 + \frac{\sigma + 1}{2} e^{-\sigma \tau} \right) u(t_{n-2}) + (2 - 2\sigma - 2e^{-\sigma \tau}) u(t_{n-1}) \\
+ \left( 3\sigma + 1 + \frac{\sigma - 1}{2} e^{-\sigma \tau} \right) u(t_{n-1}) \right\} + r_n^\gamma
\]
with the coefficients

\[ a_j + b_j + c_j = 0; \quad a_j = -\left(\frac{\sigma \tau}{2} + 1\right) e^{-\sigma \tau} + \left(\frac{3\sigma \tau}{2} + 1\right) e^{-\sigma(j+1)\tau}, \]
\[ b_j = 2e^{-\sigma \tau} - 2(\sigma \tau + 1) e^{-\sigma(j+1)\tau}; \quad c_j = \left(\frac{\sigma \tau}{2} - 1\right) e^{-\sigma \tau} + \left(\frac{\sigma \tau}{2} + 1\right) e^{-\sigma(j+1)\tau}. \]  

(2.3)

Here

\[ H_j(s) = u(t_j) - \frac{u(t_j) - u(t_j-1)}{\tau} (t_j - s) - \frac{u(t_j+1) - 2u(t_j) + u(t_j-1)}{2 \tau^2} (s - t_j), \]
\[ H_n(s) = u(t_n) - \frac{u(t_n) - u(t_n-1)}{\tau} (t_n - s) - \frac{u(t_n+1) - 2u(t_n) + u(t_n-2)}{2 \tau^2} (s - t_n), \]

which interpolate the function \( u \) at the three points \( \{t_j, t_j, t_{j+1}\}, 1 \leq j \leq n-1 \) and \( \{t_{n-2}, t_{n-1}, t_n\} \) respectively, i.e.,

\[ H_j(t_{j-1}) = u(t_{j-1}), \quad H_j(t_j) = u(t_j), \quad H_j(t_{j+1}) = u(t_{j+1}), \]
\[ H_n(t_{n-2}) = u(t_{n-2}), \quad H_n(t_{n-1}) = u(t_{n-1}), \quad H_n(t_n) = u(t_n), \]

and \( r_\tau^n \) is the truncation error of the approximation scheme.

From (2.1) and (2.2), the finite difference operators for CF-fractional derivative can be defined by

\[ L_t^n u^1 = \frac{1}{(1 - \gamma)\sigma \tau} \left( u^1 - u^0 \right) (1 - e^{-\sigma \tau}) = \bar{\beta}_0 \tilde{\alpha}_0^{-1} (u^1 - u^0), \quad n = 1, \]

with \( \tilde{\alpha}_0 := (1 - \gamma)\sigma \tau, \quad \bar{\beta}_0 := 1 - e^{-\sigma \tau}; \)

and

\[ L_t^n u^n = \frac{1}{(1 - \gamma)\sigma^2 \tau^2} \left\{ \begin{aligned}
&\sum_{j=1}^{n-1} \left( a_j u^{n-j-1} + b_j u^{n-j} + c_j u^{n-j+1} \right) \\
&+ \left( \frac{\sigma \tau}{2} - 1 + \left( \frac{\sigma \tau}{2} + 1 \right) e^{-\sigma \tau} \right) u^{n-2} + \left( 2 - 2\sigma \tau - 2e^{-\sigma \tau} \right) u^{n-1} \\
&+ \left( \frac{3\sigma \tau}{2} - 1 + \left( 1 - \frac{\sigma \tau}{2} \right) e^{-\sigma \tau} \right) u^n \end{aligned} \} \quad \forall n \geq 2. \]

Let \( \alpha_0 := (1 - \gamma)\sigma^2 \tau^2, \quad \beta_0 := c_1 + \frac{3\sigma \tau}{2} - 1 + (1 - \frac{\sigma \tau}{2}) e^{-\sigma \tau} \), the above equation can be rewritten as

\[ L_t^n u^2 = \alpha_0^{-1} \left\{ (a_1 + \frac{\sigma \tau}{2} - 1 + \left( \frac{\sigma \tau}{2} + 1 \right) e^{-\sigma \tau} ) u^0 + (b_1 + 2 - 2\sigma \tau - 2e^{-\sigma \tau} ) u^1 + \beta_0 u^2 \right\}, \]

\[ L_t^n u^3 = \alpha_0^{-1} \left\{ a_2 u^0 + (a_1 + b_2 + \frac{\sigma \tau}{2} - 1 + \left( \frac{\sigma \tau}{2} + 1 \right) e^{-\sigma \tau} ) u^1 + (b_1 + c_2 + 2 - 2\sigma \tau - 2e^{-\sigma \tau} ) u^2 + \beta_0 u^3 \right\}, \]

and for all \( n \geq 4 \)

\[ L_t^n u^n = \alpha_0^{-1} \left\{ a_{n-1} u^0 + (b_{n-1} + a_{n-2}) u^1 + \sum_{j=3}^{n-2} (a_{j-1} + b_j + c_{j+1}) u^{n-j} \right. \]
\[ + \left. (a_1 + b_2 + c_3 + \frac{\sigma \tau}{2} - 1 + \left( \frac{\sigma \tau}{2} + 1 \right) e^{-\sigma \tau} ) u^{n-2} + (b_1 + c_2 + 2 - 2\sigma \tau - 2e^{-\sigma \tau} ) u^{n-1} + \beta_0 u^n \right\}. \]

According to the above equations, we have

\[ L_t^n u^1 = \bar{\beta}_0 \tilde{\alpha}_0^{-1} (u^1 - u^0), \quad n = 1, \]
\[ L_t^n u^n = \beta_0 \alpha_0^{-1} \left( u^n - \sum_{i=1}^{n} d_{n-i} u^{n-i} \right) \quad \forall n \geq 2. \]  

(2.4)

Here the coefficients are defined by

\[ d_0^2 = \left( -a_1 - \frac{\sigma \tau}{2} + 1 + \left( \frac{\sigma \tau}{2} + 1 \right) e^{-\sigma \tau} \right) \beta_0^{-1}, \quad d_1^2 = -(b_1 + 2 - 2\sigma \tau - 2e^{-\sigma \tau}) \beta_0^{-1}; \]
\[ d_0^3 = -a_2 \beta_0^{-1}, \quad d_1^3 = \left( -a_1 - b_2 - \frac{\sigma \tau}{2} + 1 - \left( \frac{\sigma \tau}{2} + 1 \right) e^{-\sigma \tau} \right) \beta_0^{-1}, \quad d_2^3 = -(b_1 + c_2 + 2 - 2\sigma \tau - 2e^{-\sigma \tau}) \beta_0^{-1}; \]
and for all $n \geq 4$
\[
d_0^n = -a_{n-1}^{-1} \beta_0^{-1}, \quad d_1^0 = (-a_{n-2} - b_{n-1}) \beta_0^{-1}, \quad d_{i-1}^n = (-a_{i-1} - b_i - c_{i+1}) \beta_0^{-1}, \quad i = 3, 4, \ldots, n - 2,
\]
\[
d_{n-2}^n = (-a_1 - b_2 - c_3 - \frac{\sigma \tau}{2} + 1 - \frac{\sigma \tau + 1}{2}) \beta_0^{-1}, \quad d_{n-1}^n = -(b_1 + c_2 + 2 - 2\sigma \tau - 2e^{-\sigma \tau}) \beta_0^{-1}.
\]

Lemma 2.1. For any $\gamma \in (0, 1)$, it holds
\[
|r_1^n| \leq \frac{1}{8(1 - \gamma)} \max_{t \in (0, T)} |\partial_t^2 u(t)| \gamma^2 \leq c_{u, \gamma} \gamma^2,
\]
\[
|r_1^n| \leq \frac{\sqrt{3}}{27(1 - \gamma)} \max_{t \in (0, T)} |\partial_t^3 u(t)| \gamma^3 \leq c_{u, \gamma} \gamma^3 \quad \forall n \geq 2,
\]

where $c_{u, \gamma}$ is a constant independent on $\tau$.

Proof. (i) From (2.1) and integration by parts, we have
\[
r_1^n = \frac{1}{1 - \gamma} \int_0^{t_1} (u'(s) - H_1'(s)) e^{-\sigma(t - s)} ds = \frac{-\sigma}{1 - \gamma} \int_0^{t_1} (u(s) - H_1(s)) e^{-\sigma(t - s)} ds.
\]
Using Taylor series expansion, there exists
\[
u(s) - H_1(s) = \frac{\partial_t^2 u (\zeta)}{2!} (s - t_0)(s - t_1), \quad t_0 \leq \zeta \leq t_1.
\]
Since $|(s - t_0)(s - t_1)|$ can be bounded by $\frac{1}{4} \gamma^2$ for all $s \in (t_0, t_1)$, we get
\[
|r_1^n| \leq \frac{\sigma}{2(1 - \gamma)} \max_{t \in (0, T)} |\partial_t^2 u(t)| \int_0^{t_1} |(s - t_0)(s - t_1)| e^{-\sigma(t - s)} ds
\]
\[
\leq \frac{\sigma}{8(1 - \gamma)} \max_{t \in (0, T)} |\partial_t^2 u(t)| \gamma^2 \left( \frac{1}{1 - \gamma} - \frac{e^{-\sigma t}}{\sigma} \right) \leq \frac{1}{8(1 - \gamma)} \max_{t \in (0, T)} |\partial_t^2 u(t)| \gamma^2.
\]

(ii) According to (2.2) and integration by parts, one has
\[
r_1^n = \frac{1}{1 - \gamma} \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (u'(s) - H_j'(s)) e^{-\sigma(t - s)} ds + \frac{1}{1 - \gamma} \int_{t_{n-1}}^{t_n} (u'(s) - H_n'(s)) e^{-\sigma(t - s)} ds
\]
\[
= \frac{-\sigma}{1 - \gamma} \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (u(s) - H_j(s)) e^{-\sigma(t - s)} ds + \frac{-\sigma}{1 - \gamma} \int_{t_{n-1}}^{t_n} (u(s) - H_n(s)) e^{-\sigma(t - s)} ds.
\]
Using Taylor series expansion, we derive
\[
u(s) - H_j(s) = \frac{\partial_t^3 u (\xi)}{3!} (s - t_{j-1})(s - t_j)(s - t_{j+1}), \quad t_{j-1} \leq \xi \leq t_{j+1},
\]
\[
u(s) - H_n(s) = \frac{\partial_t^3 u (\eta)}{3!} (s - t_{n-2})(s - t_{n-1})(s - t_n), \quad t_{n-2} \leq \eta \leq t_n.
\]
Hence
\[
|r_1^n| \leq \frac{\sigma}{6(1 - \gamma)} \max_{t \in (0, T)} |\partial_t^3 u(t)| \left( \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} |(s - t_{j-1})(t_j - s)(t_{j+1} - s)| e^{-\sigma(t - s)} ds \right)
\]
\[
+ \int_{t_{n-1}}^{t_n} |(s - t_{n-2})(t_{n-1} - s)(t_n - s)| e^{-\sigma(t - s)} ds
\]
\[
\leq \frac{\sqrt{3} \sigma}{27(1 - \gamma)} \max_{t \in (0, T)} |\partial_t^3 u(t)| \gamma^3 \int_0^T e^{-\sigma(t - s)} ds
\]
\[
= \frac{\sqrt{3} \sigma}{27(1 - \gamma)} \max_{t \in (0, T)} |\partial_t^3 u(t)| \gamma^3 \left( \frac{1}{1 - \gamma} - \frac{e^{-\sigma t}}{\sigma} \right) \leq \frac{\sqrt{3}}{27(1 - \gamma)} \max_{t \in (0, T)} |\partial_t^3 u(t)| \gamma^3,
\]

since $(s - t_{j-1})(t_j - s)(t_{j+1} - s)$ for $s \in (t_{j-1}, t_j)$ and $(s - t_{n-2})(t_{n-1} - s)(t_n - s)$ for $s \in (t_{n-1}, t_n)$ can be bounded by $\frac{2\sqrt{3}}{3} \gamma^3$. The proof is completed.
3. Stability and convergence of semidiscrete scheme for 1D

We consider the following one-dimensional time CF-fractional diffusion equation on a finite domain $\Omega = (-1, 1)$,

$$
\frac{C^\alpha F}{\partial \tau} D_t^\gamma u(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2}, \quad x \in \Omega, \quad t \in (0, T],
$$

(3.1)

subject to the following initial and boundary conditions:

- $u(x, 0) = u_0(x), \quad x \in \Omega$,
- $u(-1, t) = u(1, t) = 0, \quad 0 \leq t \leq T$.

From (2.1) and (2.2), (2.4), we have

$$
r^\gamma_i = \frac{C^\alpha F}{\partial \tau} D_t^\gamma u(t_n) - L_i^\gamma u(t_n) \forall n \geq 1.
$$

(3.2)

Then we can rewrite (3.1) as

$$
L_i^\gamma u(t_n) - \partial_x^2 u(t_n) = -r^\gamma_i \forall n \geq 1.
$$

(3.3)

For the simplification, we use $u^n$ to denote $u^n(x)$ and denote $u^n(x)$ as an approximation of $u(x, t_n)$. Then we have the following time discrete schemes of (3.1):

$$
L_i^\gamma u^n - \partial_x^2 u^n = 0 \forall n \geq 1,
$$

(3.4)

i.e.,

$$
u^1 - \tilde{\alpha}_0 \tilde{\beta}_0^{-1} \partial_x^2 v^1 = v^0, \quad n = 1;
$$

$$
u^n - \alpha_0 \beta_0^{-1} \partial_x^2 v^n = \sum_{i=1}^{n} d^n_{n-i} u^{n-i} \forall n \geq 2,
$$

(3.5)

where the coefficients $\tilde{\alpha}_0, \beta_0, \alpha_0, \beta_0, d^n_{n-i}$ are given in (2.3) and (2.4).

We next analyze the properties of the above coefficients.

**Lemma 3.1.** Let $\gamma \in (0, 1), n \geq 2$. Then the coefficients in the scheme (3.4) satisfy

1. $\beta_0 = \frac{3\sigma \tau}{2} - 1 + (1 + \frac{\sigma \tau}{2}) e^{-2\sigma \tau} > 0, \quad \alpha_0 > 0, \quad \tilde{\alpha}_0 > 0, \quad \tilde{\beta}_0 > 0,

2. $\sum_{i=1}^{n} d^n_{n-i} = 1$,

3. $a_j < a_{j+1}, \quad 1 \leq j \leq n - 1$,

4. $d^n_{n-i} > 0, \quad 1 \leq i \leq n$.

**Proof.**

(1) It is simple to show that by Taylor series expansion

$$
\beta_0 = \frac{3\sigma \tau}{2} - 1 + (1 + \frac{\sigma \tau}{2})(1 - 2\sigma \tau + 2\sigma^2 \tau^2 + \frac{4}{3} \sigma^3 \tau^3 + \cdots) = \sigma^2 \tau^2 - \frac{1}{3} \sigma^3 \tau^3 - \frac{2}{3} \sigma^4 \tau^4 + \cdots > 0.
$$

(2) The property can be proved by the mathematical induction. For $n = 2$, we have

$$
d_0^2 + d_1^2 = \left(-a_1 - \frac{\sigma \tau}{2} + \left(\frac{\sigma \tau}{2} + 1\right) e^{-\sigma \tau} - b_1 - 2 + 2\sigma \tau + 2e^{-\sigma \tau}\right) \beta_0^{-1}
$$

$$
= \left((\frac{\sigma \tau}{2} + 1) e^{-\sigma \tau} - \frac{3\sigma \tau}{2} + \frac{\sigma \tau}{2} + 1 - \left(\frac{\sigma \tau}{2} + 1\right) e^{-\sigma \tau} - 2e^{-\sigma \tau}
\right) \beta_0^{-1}
$$

$$
+ (2\sigma \tau + 2)e^{-2\sigma \tau} - 2 + 2\sigma \tau + 2e^{-\sigma \tau}\right) \beta_0^{-1}
$$

$$
= \left((\frac{\sigma \tau}{2} + 1) e^{-2\sigma \tau} + \frac{3\sigma \tau}{2} - 1\right) \left(\frac{\sigma \tau}{2} + 1\right) e^{-2\sigma \tau} + \frac{3\sigma \tau}{2} - 1\right)^{-1} = 1.
$$

Assuming that

$$
\sum_{i=1}^{j} d_{j-i}^i = 1, \quad j = 3, \ldots, n - 1,
$$

and using $a_{n-1} + b_{n-1} + c_{n-1} = 0$ in (2.3). Then we get

$$
\sum_{i=1}^{n} d_{n-i}^n = \sum_{i=1}^{n-1} d_{n-1-i}^i + (-a_{n-1} - b_{n-1} - c_{n-1}) \beta_0^{-1} = 1.
$$
(3) From (2.3), it is easy to get
\[ a_j - a_{j+1} = -\frac{\sigma \tau}{2} + 1) e^{-\sigma j \tau} + (2\sigma \tau + 2)e^{-\sigma (j+1) \tau} - \left( \frac{3\sigma \tau}{2} + 1 \right) e^{-\sigma (j+2) \tau} \]
\[ = e^{-\sigma (j+2) \tau} \left( -\frac{\sigma \tau}{2} + 1 \right) e^{2\sigma \tau} + (2\sigma \tau + 2)e^{\sigma \tau} - \left( \frac{3\sigma \tau}{2} + 1 \right) \]
\[ = e^{-\sigma (j+2) \tau} - \sigma^3 \tau^3 - \frac{11}{12} \sigma^4 \tau^4 \ldots < 0. \]

(4) By using Taylor series expansion, the desired results can be obtained, see Lemma 7.1 of the Appendix. The proof is completed.

Lemma 3.2. Let
\[ S_i = \sum_{j=1}^{i-1} d_j, \quad \forall i \geq 2 \quad \text{and} \quad S_i = 0, \quad i \leq 1. \]
Then
\[ 0 \leq S_i < S_{i+1} < 1 - e^{-\sigma T} \quad \forall i \geq 1. \]

Proof. It is obviously to get 0 \leq S_1 < S_2. We next prove S_2 < S_3. From the coefficients of (3.5), we have
\[ S_2 = d_2 = d_2^3 + c_2 \beta_0^{-1}, \quad S_3 = d_3^1 + d_3^2. \]
Using \( a_j + b_j + c_j = 0 \) in (2.3) and Taylor series expansion, it yields
\[ S_2 - S_3 = c_2 \beta_0^{-1} - d_3^3 = (a_1 - a_2 + \frac{\sigma \tau}{2} - 1) \left( \frac{\sigma \tau}{2} + 1 \right) e^{-\sigma \tau} < \frac{11}{12} \sigma^3 \tau^3 e^{-3\sigma \tau} < 0. \]

According to (2), (3) of Lemma 3.1 and the coefficients of (3.5), one has
\[ S_i = 1 - d_0^i \quad \text{and} \quad S_i - S_{i+1} = d_{i+1}^i - d_0^i = (a_{i-1} - a_i) \beta_0^{-1} < 0. \]
On the other hand, taking \( n > i \), we have \( S_i < S_n = 1 - d_0^0 < 1 - e^{-\sigma T} \), since
\[ d_0^n = -a_{n-1} \beta_0^{-1} = e^{-n \sigma \tau} \left( \frac{3^2 \tau^2}{12} + \frac{3^4 \tau^4}{8} + \cdots \right) \beta_0^{-1} \]
\[ < e^{-n \sigma \tau} \left( \frac{3^2 \tau^2}{12} + \frac{3^4 \tau^4}{8} + \cdots \right) \left( \frac{3^2 \tau^2 - \frac{1}{3} \sigma^3 \tau^3 - \frac{2}{3} \sigma^4 \tau^4 \cdots} \right)^{-1} \]
\[ > e^{-n \sigma \tau} \geq e^{-\sigma \tau}. \]
The proof is completed.

3.1. Stability analysis of the semidiscrete scheme for 1D

Let \((\cdot, \cdot)\) denote \((\cdot, \cdot)_{L^2(\Omega)}, \|\cdot\|\) denote \(\|\cdot\|_{L^2(\Omega)}\) and \(\|\cdot\|_k\) denote \(\|\cdot\|_{H^k(\Omega)}\). The variational formulation of (3.5) with the homogeneous boundary conditions reads: find \(u^n \in H^1_0(\Omega)\), such that
\[ (u^1, v) + \tilde{a}_0 \beta_0^{-1}(\partial_x u^1, \partial_x v) = (u^0, v) \quad \forall v \in H^1_0(\Omega), \quad n = 1; \]
\[ (u^n, v) + \alpha_0 \beta_0^{-1}(\partial_x u^n, \partial_x v) = \sum_{i=1}^{n} d_{n-1}^i(u^{n-i}, v) \quad \forall v \in H^1_0(\Omega) \quad \forall n \geq 2. \]

Theorem 3.1. The semi-discrete problem (3.6) is unconditionally stable for all \(\tau > 0\) and the following estimate holds:
\[ \|u^n\| \leq \|u^0\| \quad \forall n \geq 1. \]

Proof. We will prove the result by mathematical induction. For \(n = 1\), there exists
\[ (u^1, v) + \tilde{a}_0 \beta_0^{-1}(\partial_x u^1, \partial_x v) = (u^0, v) \quad \forall v \in H^1_0(\Omega). \]
Taking \(v = u^1\) and using the Schwarz inequality, we obtain
\[ \|u^1\| \leq \|u^0\|. \]

For \(n = 2\), it yields
\[ (u^2, v) + \alpha_0 \beta_0^{-1}(\partial_x u^2, \partial_x v) = \sum_{i=1}^{2} d_{2-i}^i(u^{2-i}, v) \quad \forall v \in H^1_0(\Omega). \]
Taking \( v = u^2 \) and using the Schwarz inequality, Lemma 3.1 one has
\[
\|u^2\| \leq \sum_{i=1}^{2} d_{2-i}^2 \|u^2-i\| = d_1^2 \|u^1\| + d_0^2 \|u^0\| \leq d_1^2 \|u^0\| + d_0^2 \|u^0\| = \|u^0\|.
\]
Supposing
\[
\|u^j\| \leq \|u^0\|, \quad j = 3, \ldots, n - 1,
\]
and taking \( v = u^n \) in (3.6), we have
\[
(u^n, u^n) + a_0 \beta_0^{-1} (\partial_x u^n, \partial_x u^n) = \sum_{i=1}^{n} d_{n-i}^n (u^{n-i}, u^n),
\]
From Lemma 3.1 it leads to
\[
\|u^n\| \leq \sum_{i=1}^{n} d_{n-i}^n \|u^{n-i}\| \leq \sum_{i=1}^{n} d_{n-i}^n \|u^0\| = \|u^0\|.
\]
The proof is completed.

3.2. Error estimates of the semidiscrete scheme for 1D

Now we carry out an error analysis for the the semi-discrete problem (3.6). The first step solution \( u^1 \) has second order accuracy. In fact, from (3.3) and (3.4), we obtain
\[
\frac{1 - e^{-\sigma \tau}}{(1 - \gamma) \sigma \tau} (u(1) - u^1) - \partial_x^2 (u(1) - u^1) = -r^1.
\]
Taking the inner product of both sides with \( u(1) - u^1 \) and using \( 0 < \frac{1 - e^{-\sigma \tau}}{(1 - \gamma) \sigma \tau} < \frac{1}{\gamma} \) and Lemma 2.1 with \( |r^1| \leq c \tau^2 \), it yields
\[
\|u(1) - u^1\| \leq c(1 - \gamma) \tau^2.
\]
In order to establish a global third-order accurate scheme, we design the sub-stepping scheme within the interval \((0, t_1)\) as follows: let \( k \) be the smallest integer such that \( k \geq \frac{1}{\sqrt{\gamma}} \), and set the sub-step size
\[
\tau_1 = \frac{\tau}{k}.
\]
Now we apply the first step scheme with the sub-time step size \( \tau_1 \)
\[
\frac{1}{(1 - \gamma) \sigma \tau_1} \sum_{j=0}^{m-1} b_j (u^{1,(m-j)} - u^{1,(m-j-1)}) - \partial_x^2 u^{1,(m)} = 0, \quad m = 1, 2, \ldots, k
\]
with \( u^{1,(0)} = u_0 \), where \( b_j = e^{-\sigma j \tau_1} - e^{-\sigma (j+1) \tau_1} \). It is derived that \( u^1 = u^{1,(k)} \) is an approximation to \( u(t_1) \) with accuracy
\[
\|u(t_1) - u^1\| \leq c \tau_1^2 \leq c \tau^3.
\]

**Theorem 3.2.** Let \( u \) be the exact solution of (3.1), \( \{u^n\}_{n=2}^M \) be the semi-discrete solution of (3.6) with the initial \( u^0 = u(0) \), and \( u^1 = u^{1,(k)} \) be given in (3.7). Suppose \( \partial^3_x u \in L^\infty((0,T]; L^2(\Omega)) \), then the following error estimate holds:
\[
\|u(t_n) - u^n\| \leq c e^{\sigma T} \|\partial^3_x u\|_{L^\infty(L^2(\Omega))} \tau^3, \quad \forall n \geq 2.
\]
where \( \|u\|_{L^\infty(L^2(\Omega))} := \sup_{t \in [0,T]} \|u(t)\|_{L^2(\Omega)} \).

**Proof.** Let \( e^n = u(t_n) - u^n \) with \( e^0 = 0 \). Combining (3.3) and (3.4), we derive
\[
(e^n, v) + a_0 \beta_0^{-1} (\partial_x e^n, \partial_x v) = \sum_{i=1}^{n-1} d_{n-i}^n (e^{n-i}, v) - a_0 \beta_0^{-1} (r^n, v) \quad \forall v \in H^1_0(\Omega).
\]
Taking \( v = e^n \) and using the Schwarz inequality, we obtain
\[
\|e^n\| \leq \sum_{i=1}^{n-1} d_{n-i}^n \|e^{n-i}\| + a_0 \beta_0^{-1} \|r^n\| \leq \sum_{i=1}^{n-1} d_{n-i}^n \|e^i\| + r_{\text{max}}.
\]
According to Lemma 2.1 and 3.8, there exists
\[ r_{\text{max}} = \alpha_0 \beta_0^{-1} \max_{2 \leq n \leq M} \| r^n \| \leq c_{\gamma} \| \partial_t^3 u \|_{L^\infty(\Omega)^3} \] and \( \| e^1 \| \leq r_{\text{max}}. \)

From (3.9) and Lemma 3.2 it leads to
\[
\| e^n \| \leq \sum_{i=1}^{n-1} d_i \| e^i \| + r_{\text{max}} \leq \left( 1 + \sum_{i=1}^{n-1} d_i \right) \| e^1 \| + \left( 1 + \sum_{i=1}^{n-1} d_i \right) r_{\text{max}} \leq (1 + S_n + S_n^2 + \cdots + S_n^{n-1}) r_{\text{max}} < \frac{1}{1 - S_n} r_{\text{max}} < e^{\sigma T}, r_{\text{max}} \leq c_{\gamma} \sigma^3 \| \partial_t^3 u \|_{L^\infty(\Omega)^3} \tau^3.
\]

The proof is completed.

4. Space discretization and error estimates for 1D

4.1. A Galerkin spectral method in space for 1D

Let \( P_N(\Omega) \) be the space of all polynomials of degree less than or equal to \( N \) with respect to \( x \), and \( P^0_N(\Omega) = H^1_0(\Omega) \cap P_N(\Omega) \). Let \( \pi_N \) be the \( H^1_0 \)-orthogonal projection operator from \( H^1_0(\Omega) \) into \( P^0_N(\Omega) \), which is defined for \( \psi \in H^1_0(\Omega) \) such that
\[
(\partial_x \pi_N \psi, \partial_x v_N) = (\partial_x \psi, \partial_x v_N) \quad \forall v_N \in P^0_N(\Omega). \tag{4.1}
\]

It is known that the following estimate holds \[4, 20\]:
\[
\| \psi - \pi_N \psi \|_l \leq c N^{l-m} \| \psi \|_m \quad \forall \psi \in H^m(\Omega) \cap H^1_0(\Omega), \ m \geq 1, \ l = 0, 1. \tag{4.2}
\]

First, we consider the first step full discrete solution, which is obtained by the spectral approximation to the sub-time stepping problems (3.7), find \( u^{N(1)}_{\text{m}}(t) \in P^0_N(\Omega) \) such that for all \( v_N \in P^0_N(\Omega), \)
\[
\frac{1}{1 - \gamma \sigma^3} \sum_{j=0}^{m-1} b_j \left( u^{N(m-j)}_N - u^{N(m-j-1)}_N, v_N \right) + (\partial_x u^{N(m)}_N, \partial_x v_N) = 0, \ m = 1, 2, \ldots, k \tag{4.3}
\]

with \( u^{N(0)}_N = \pi_N u^0 \). Then set \( u^{N(k)}_N = u^{N(k)}_N \). The following error estimate can be derived for this first step solution:
\[
\| u(t_1) - u^{N(k)}_N \| \leq c \left( N^{-m} \| C \|_0 \tau^3 \| \partial_t^3 u \|_{L^\infty(\Omega)^3} + N^{-m} \tau^3 \| \partial_t^3 u \|_{L^\infty(\Omega)^3} \right), \tag{4.4}
\]

where \( \| u \|_{L^\infty(H^{m}(\Omega))} := \sup_{t \in [0,T]} \| u(\cdot, t) \|_{H^{m}(\Omega)} \).

The proof of the above estimate \[4.4\] follows in a similar manner of Theorem 4.1, we omit it here and only derive the error estimate of the spectral method for the time steps \( n \geq 2. \)

The spectral discretization of the weak problem (3.6) reads: find \( u_N \in P^0_N(\Omega) \) such that
\[
(u_N, v_N) + \alpha_0 \beta_0^{-1} (\partial_x u_N, \partial_x v_N) = \sum_{i=1}^{n} d_{n-i} \left( u_{n-i}^N, v_N \right) \quad \forall v_N \in P^0_N(\Omega). \tag{4.5}
\]

**Theorem 4.1.** Let \( u \) be the exact solution of (3.7), \( \{ u_N \}_{n=1}^{M} \) be the full discrete solution of (4.5) with the first step solution \( u_N \) given by (4.3) and the initial condition \( u^0 = \pi_N u^0 \). Suppose \( \partial_t^3 u \in L^\infty((0,T]; H^{m}(\Omega)) \), \( m \geq 1. \) Then the following error estimate holds:
\[
\| u(t_n) - u_N \| \leq c_{\gamma} \sigma^3 \left( N^{-m} \| C \|_0 \tau^3 \| \partial_t^3 u \|_{L^\infty(\Omega)^3} + N^{-m} \| u \|_{L^\infty(\Omega)^3} \right), \tag{4.6}
\]

where \( c_{\gamma} \) is a constant independent on \( \tau. \)
Proof. From (3.3), (2.4) and (4.1), we have
\[
(u(t_n), v_N) + \alpha_0 \delta_0^{-1}(\partial_x u(t_n), \partial_x v_N) = \sum_{i=1}^{n} d_{n-i}^n(u(t_{n-i}), v_N) - \alpha_0 \delta_0^{-1}(r_n^N, v_N) \quad \forall v_N \in P_N^0(\Omega),
\]
which is equal to
\[
(\pi_N^1 u(t_n), v_N) + \alpha_0 \delta_0^{-1}(\pi_N^1 \partial_x u(t_n), \partial_x v_N) = (\pi_N^1 u(t_n) - u(t_n), v_N) + \sum_{i=1}^{n} d_{n-i}^n(\pi_N^1 u(t_{n-i}), v_N)
\]
\[
\quad + \sum_{i=1}^{n} d_{n-i}^n(u(t_{n-i}) - \pi_N^1 u(t_{n-i}), v_N) - \alpha_0 \delta_0^{-1}(r_n^N, v_N).
\]
Subtracting the above identity from (4.5) with \(e_N^n := u_N^n - \pi_N^1 u(t_n)\), we get
\[
(e_N^n, v_N) + \alpha_0 \delta_0^{-1}(\partial_x e_N^n, \partial_x v_N) = \sum_{i=1}^{n} d_{n-i}^n(e_N^{n-i}, v_N) + \alpha_0 \delta_0^{-1}(\delta_0^n, v_N), \quad n \geq 2
\]
with
\[
\delta_0^n = (I_d - \pi_N^1) L_i^1 u(t_n) + r_n^1 = (I_d - \pi_N^1) (C_0^1 D_i^1 u(t_n) - r_n^1) + r_n^1,
\]
where \(I_d\) is the identity operator, \(L_i^1 u(t_n)\) is given in (2.4) and the last identity holds by (3.2).

According to the triangular inequality and (4.2), Lemma 2.1, we obtain
\[
\|\delta_0^n\| \leq \|(I_d - \pi_N^1) C_0^1 D_i^1 u(t_n)\| + \|(I_d - \pi_N^1) r_n^1\| + \|r_n^1\|
\]
\[
\leq c_\gamma \left( N^{-m} \|C_0^1 D_i^1 u\|_{L^\infty(H^m(\Omega))} + N^{-m} \tau^3 \|D_i^1 u\|_{L^\infty(H^m(\Omega))} + \tau^3 \|D_i^1 u\|_{L^\infty(L^2(\Omega))} \right), \quad n \geq 2.
\]
Taking \(v_N = e_N^n\) in (4.6), and using the Schwarz inequality, it yields
\[
\|e_N^n\| \leq \sum_{i=1}^{n} d_{n-i}^n \|e_N^{n-i}\| + \alpha_0 \delta_0^{-1}\|\delta_0^n\| \leq \sum_{i=1}^{n-1} d_{n-i}^n \|e_N^n\| + 2(1 - \gamma)\|\delta_0^n\|,
\]
where \(1 - \gamma < \alpha_0 \delta_0^{-1} < 2(1 - \gamma)\) with \(\sigma > 1\).

An argument similar to the one used in Theorem 3.2 shows that
\[
\|e_N^n\| \leq e^{\sigma T} \cdot 2(1 - \gamma)\|\delta_0^n\|,
\]
i.e.,
\[
\|e_N^n\| \leq 2(1 - \gamma)c_\gamma e^{\sigma T} \left( N^{-m} \|C_0^1 D_i^1 u\|_{L^\infty(H^m(\Omega))} + N^{-m} \tau^3 \|D_i^1 u\|_{L^\infty(H^m(\Omega))} + \tau^3 \|D_i^1 u\|_{L^\infty(L^2(\Omega))} \right).
\]
Finally, we use the triangle inequality and (4.2) to conclude
\[
\|u(t_n) - u_N^n\| \leq \|e_N^n\| + \|u(t_n) - \pi_N^1 u(t_n)\|
\]
\[
\leq c_\gamma e^{\sigma T} \left( N^{-m} \|C_0^1 D_i^1 u\|_{L^\infty(H^m(\Omega))} + N^{-m} \|u\|_{L^\infty(H^m(\Omega))}
\quad + N^{-m} \tau^3 \|D_i^1 u\|_{L^\infty(L^2(\Omega))} + \tau^3 \|D_i^1 u\|_{L^\infty(L^2(\Omega))} \right).
\]
The proof is completed. 

4.2. A Legendre collocation method in space for 1D

The Gauss-Lobatto numerical quadrature is used to evaluate the integrals in space. Let \(L_N(x)\) denote the Legendre polynomial of degree \(N\), \(\{x_i : i = 0, ..., N\}\) are the Legendre-Gauss-Lobatto points, which are the roots of the polynomials \((1 - x^2)L_N^1(x)\); \(\{\omega_i : i = 0, ..., N\}\) are the weights such that the following quadrature holds,
\[
\int_{-1}^{1} \varphi(x)dx = \sum_{i=0}^{N} \varphi(x_i) \omega_i \quad \forall \varphi \in P_{2N-1}(\Omega).
\]
(4.7)

The discrete inner product is defined by
\[
(\phi, \psi)_N := \sum_{i=0}^{N} \phi(x_i) \psi(x_i) \omega_i,
\]
and let \( \| \phi \| := (\phi, \phi)^{1/2} \). It is well known that the following estimates hold \([19]\): \[
\| \varphi \| \leq \| \varphi \| \leq \sqrt{3} \| \varphi \| \quad \forall \varphi \in \mathbb{P}_N(\Omega),
\]
\[
| (\varphi, v_N) - (\varphi, v_N) | \leq c N^{-m} \| \varphi \| \| v_N \| \quad \forall \varphi \in H^m(\Omega), \; \forall v_N \in \mathbb{P}_N(\Omega), \; m \geq 1.
\]
We first consider the first step full discrete scheme of \([3.7]\): find \( u_N^{1(m)} \in \mathbb{P}_N(\Omega) \) such that \[
\tilde{a}_N(u_N^{1(m)}, v_N) = \tilde{F}_N(v_N) \quad \forall v_N \in \mathbb{P}_N(\Omega), \quad m = 1, 2, \ldots, k
\]
with \( u_N^1 = u_N^{1(1)} \). Here the bilinear form \( \tilde{a}_N(\cdot, \cdot) \) and functional \( \tilde{F}_N(\cdot) \), respectively, are defined by \[
\tilde{a}_N(u_N^{1(m)}, v_N) := b_0(u_N^{1(m)}, v_N)_N + (1 - \gamma) \sigma \tau_1 (\partial_x u_N^{1(m)}, \partial_x v_N)_N,
\]
\[
\tilde{F}_N(v_N) := \sum_{i=0}^{m-2} (b_i - b_{i+1})(u_N^{1(m-i-1)}, v_N)_N + b_{m-1}(u_N^{1(0)}, v_N)_N.
\]
The spectral collocation approximation of \([3.6]\): find \( u_N^p \in \mathbb{P}_N(\Omega) \) with \( n \geq 2 \) such that \[
a_N(u_N^p, v_N) = \mathcal{F}_N(v_N) \quad \forall v_N \in \mathbb{P}_N(\Omega),
\]
where the bilinear form \( a_N(\cdot, \cdot) \) and functional \( \mathcal{F}_N(\cdot) \), respectively, are defined by \[
a_N(u_N^p, v_N) := (u_N^p, v_N)_N + \alpha_0 \beta_0^{-1}(\partial_x u_N^p, \partial_x v_N)_N,
\]
\[
\mathcal{F}_N(v_N) := \sum_{i=1}^{n} d_{n-i}^p(u_N^{n-i}, v_N)_N.
\]
The proof of the estimate \([4.10]\) follows in a similar manner of Theorem 4.2, we omit it here and only give the error estimate for the problem \([4.11]\).

**Theorem 4.2.** Let \( u \) be the exact solution of \([3.1]\), \( \{ u_N^M \}_{n=2} \) be the solution of the problem \([4.11]\) with the first step solution \( u_N^1 \) given by \([4.10]\). Suppose \( \partial_t^1 u \in L^\infty((0, T]; H^m(\Omega)), m \geq 1 \). For \( \forall n \geq 2 \), the following error estimate holds:

\[
\| u(t_n) - u_N^n \| \leq c_2 \left( f^3 \| \partial_t^1 u \|_{L^\infty(\mathbb{T}^2)} + N^{-m} \| u \|_{L^\infty(H^m(\Omega))} + N^{-m} \| C_F D^1_t u \|_{L^\infty(H^m(\Omega))} + N^{-m} \| \partial_t^1 u \|_{L^\infty(H^m(\Omega))} \right).
\]

**Proof.** Let \( e_N^n = u_N^n - \pi_N^1 u(t_n) \). From \([4.11]\), we have \[
a_N(e_N^n, v_N) = (e_N^n, v_N)_N + \alpha_0 \beta_0^{-1}(\partial_x e_N^n, \partial_x v_N)_N
\]
\[
= (u_N^n, v_N)_N + \alpha_0 \beta_0^{-1}(\partial_x u_N^n, \partial_x v_N)_N - (\pi_1^N u(t_n), v_N)_N - \alpha_0 \beta_0^{-1}(\partial_x \pi_1^N u(t_n), \partial_x v_N)_N
\]
\[
= \sum_{i=1}^{n} d_{n-i}^p(u_N^{n-i}, v_N)_N - (\pi_1^N u(t_n), v_N)_N - \alpha_0 \beta_0^{-1}(\partial_x \pi_1^N u(t_n), \partial_x v_N)_N,
\]
which is equal to \[
(e_N^n, v_N)_N + \alpha_0 \beta_0^{-1}(\partial_x e_N^n, \partial_x v_N)_N = \sum_{i=1}^{n} d_{n-i}^p(e_N^{n-i}, v_N)_N + (\varepsilon_1^n, v_N)_N + (\varepsilon_2^n, v_N)_N,
\]
where \[
(\varepsilon_1^n, v_N)_N = (u(t_n) - \pi_1^N u(t_n), v_N)_N - \sum_{i=1}^{n} d_{n-i}^p(u(t_{n-i}) - \pi_1^N u(t_{n-i}), v_N)_N,
\]
\[
(\varepsilon_2^n, v_N)_N = -(u(t_n), v_N)_N + \sum_{i=1}^{n} d_{n-i}^p(u(t_{n-i}), v_N)_N - \alpha_0 \beta_0^{-1}(\partial_x \pi_1^N u(t_n), \partial_x v_N)_N.
\]

Next we estimate \( (\varepsilon_1^n, v_N)_N \) and \( (\varepsilon_2^n, v_N)_N \). According to \([2.4]\) and \([3.2]\), it yields \[
(\varepsilon_1^n, v_N)_N = \left( (I_d - \pi_1^N) u(t_n) - \sum_{i=1}^{n} d_{n-i}^p(u(t_{n-i}) \right) v_N)_N
\]
\[
= (I_d - \pi_1^N) (C_F D^1_t u(t_n) - r_n^p), v_N)_N.
\]
According to (4.9), triangle inequality and Schwarz inequality, we obtain
\[
| (\varepsilon_1^n, v_N)_N | \leq | (\varepsilon_1^n, v_N) | + | (\varepsilon_1^n, v_N) - (\varepsilon_1^n, v_N) |
\]
\[
\leq \alpha_0 \beta_0^{-1} | (I_N - \pi_N)(C^F D^*_u(t_n) - r^n), v_N) |
\]
\[
+ \alpha_0 \beta_0^{-1} N^{-1} \| (I_N - \pi_N)(C^F D^*_u(t_n) - r^n) \|_1 \| v_N \|.
\]
Using (4.2), triangle inequality and Lemma 2.1, it leads to
\[
| (\varepsilon_1^n, v_N)_N | \leq c_\gamma \alpha_0 \beta_0^{-1} \left( N^{-m} \| C^F D^*_u \|_{L^\infty(\Omega)} + N^{-m} \| \partial_3^3 \|_{L^\infty(\Omega)} \right) \| v_N \|.
\] (4.13)

On the other hand, from (4.12), (2.4) and (4.7), we have
\[
(\varepsilon_2^n, v_N)_N = -\alpha_0 \beta_0^{-1} (L^*_u(t_n), v_N)_N - \alpha_0 \beta_0^{-1} (\partial_3 \pi_N u(t_n), \partial_3 v_N)_N
\]
\[
= -\alpha_0 \beta_0^{-1} (L^*_u(t_n), v_N)_N - \alpha_0 \beta_0^{-1} (\partial_3 \pi_N u(t_n), \partial_3 v_N)_N \quad \forall v_N \in \mathbb{P}_H(\Omega).
\]
Furthermore, we get from (3.1)
\[
(\partial_3 u(t_n), \partial_3 v_N)_N = - (C^F D^*_u(t_n), v_N)_N.
\]
Using the above equation and (4.11), it implies that
\[
(\varepsilon_2^n, v_N)_N = \alpha_0 \beta_0^{-1} (L^*_u(t_n), v_N)_N - \alpha_0 \beta_0^{-1} (L^*_u(t_n), v_N)_N + \alpha_0 \beta_0^{-1} (C^F D^*_u(t_n) - L^*_u(t_n), v_N)_N.
\]
According to (4.9), (3.2), and Schwarz inequality, we obtain
\[
| (\varepsilon_2^n, v_N)_N | \leq \alpha_0 \beta_0^{-1} \left( N^{-m} \| C^F D^*_u \|_{L^\infty(\Omega)} + N^{-m} \| \partial_3^3 \|_{L^\infty(\Omega)} \right) \| v_N \|
\]
\[
\leq \alpha_0 \beta_0^{-1} \left( N^{-m} \| C^F D^*_u \|_{L^\infty(\Omega)} + N^{-m} \| \partial_3^3 \|_{L^\infty(\Omega)} \right) \| v_N \|
\]
\[
\leq c_\gamma \alpha_0 \beta_0^{-1} \left( N^{-m} \| C^F D^*_u \|_{L^\infty(\Omega)} + N^{-m} \| \partial_3^3 \|_{L^\infty(\Omega)} \right) \| v_N \|.
\]
Combining the above equation and (4.13), it yields
\[
| (\varepsilon_1^n, v_N)_N | + | (\varepsilon_2^n, v_N)_N | \leq \varepsilon_{\max} \| v_N \|
\]
with
\[
\varepsilon_{\max} = 2 c_\gamma (1 - \gamma) \left( N^{-m} \| C^F D^*_u \|_{L^\infty(\Omega)} + N^{-m} \| \partial_3^3 \|_{L^\infty(\Omega)} \right) \| v_N \| + c_\gamma \alpha_0 \beta_0^{-1} \left( N^{-m} \| C^F D^*_u \|_{L^\infty(\Omega)} + N^{-m} \| \partial_3^3 \|_{L^\infty(\Omega)} \right) \| v_N \|.
\]
where $1 - \gamma < \alpha_0 \beta_0^{-1} < 2(1 - \gamma)$ with $\sigma \tau \leq 1$. Taking $v_N = e_N^n$ in (4.12), we have
\[
\| e_N^n \| \leq \sum_{i=1}^{n} d_n \| e_N^{n-i} \| \| e_N \| + \varepsilon_{\max} = \sum_{i=1}^{n-1} d_n \| e_N^{n-i} \| \| e_N \| + \varepsilon_{\max}.
\]
Using (4.8) and an argument similar to the one used in Theorem 3.2, it shows that
\[
\| e_N^n \| \leq \| e_N^n \| \leq e^{\sigma T} \cdot \varepsilon_{\max}.
\]
From the above equation and (4.2), we have
\[
\| u(t_n) - u_N \| \leq c_\gamma (\tau^3 \| \partial_3^3 \|_{L^\infty(\Omega)} + N^{-m} \| \partial_3 \|_{L^\infty(\Omega)} + N^{-m} \| C^F D^*_u \|_{L^\infty(\Omega)} + N^{-m} \| \partial_3^3 \|_{L^\infty(\Omega)} \right) \| v_N \|.
\]
The proof is completed. □

5. Space discretization and error estimates for 2D

Let the domain $\Omega = (-1, 1)^2$. From (3.2), we can rewrite (1.1) as the following form
\[
L^*_u(t_n) - \partial_3^2 u(t_n) - \partial_3^3 u(t_n) = -r^n.
\]
For the simplification, we use $u^n$ to denote $u^n(x, y)$ and denote $u^n(x, y)$ as an approximation of $u(x, y, t_n)$. An argument similar in (3.5), we have the following time discrete schemes of (1.1):
\[
L^*_u u^n - \partial_3^2 u^n - \partial_3^3 u^n = 0 \quad \forall n \geq 1,
\]
i.e.,

\[ u^1 - \bar{\alpha}_0 \bar{\beta}_0^{-1} \partial_x^2 u^1 - \bar{\alpha}_0 \bar{\beta}_0^{-1} \partial_y^2 u^1 = u^0, \quad n = 1; \]

\[ u^n - \alpha_0 \beta_0^{-1} \partial_x^2 u^n - \alpha_0 \beta_0^{-1} \partial_y^2 u^n = \sum_{i=1}^{n} d_{n-i} u^{n-i} \quad \forall n \geq 2. \]

(5.1)

where the coefficients \( \bar{\alpha}_0, \beta_0, \alpha_0, \beta_0, d_{n-i} \) are given in (2.3) and (2.4). Here we apply the first step scheme with the sub-time step size \( \tau_1 \).

\[
\frac{1}{(1-\gamma)\sigma \tau_1} \sum_{j=0}^{m-1} b_j (u^{1,(m-j)} - u^{1,(m-j-1)}) - \partial_x^2 u^{1,(m)} - \partial_y^2 u^{1,(m)} = 0, \quad m = 1, 2, \ldots, k.
\]

(5.2)

The variational formulation of (5.1) with the homogeneous boundary conditions reads: find \( u^n \in H^1_0(\Omega) \), such that

\[
(u^1, v) + \bar{\alpha}_0 \bar{\beta}_0^{-1} (\partial_x u^1, \partial_x v) + \bar{\alpha}_0 \bar{\beta}_0^{-1} (\partial_y u^1, \partial_y v) = (u^0, v) \quad \forall v \in H^1_0(\Omega), \quad n = 1;
\]

\[
(u^n, v) + \alpha_0 \beta_0^{-1} (\partial_x u^n, \partial_x v) + \alpha_0 \beta_0^{-1} (\partial_y u^n, \partial_y v) = \sum_{i=1}^{n} d_{n-i} (u^{n-i}, v) \quad \forall v \in H^1_0(\Omega) \quad \forall n \geq 2.
\]

(5.3)

5.1. A Galerkin spectral method in space for 2D

We denote by \( P_N \) the space of algebraic polynomials of degree less than or equal to \( N \) with respect to \( x \) and \( y \), and by \( P_N^0 \) the subspace of those polynomials that vanish on the boundary of \( \Omega \). The \( H^1_0 \)-orthogonal projection operator \( \pi_N \) is from \( H^1_0(\Omega) \) upon \( P_N^0(\Omega) \) such that

\[
(\nabla \pi_N \psi, \nabla v_N) = (\nabla \psi, \nabla v_N) \quad \forall v_N \in P_N^0(\Omega).
\]

We consider the first step full discrete solution, which is obtained by the spectral approximation to the sub-time stepping problems (5.2), find \( u^{1,(m)}_N \in P_N^0(\Omega) \) such that for all \( v_N \in P_N^0(\Omega) \),

\[
\frac{1}{(1-\gamma)\sigma \tau_1} \sum_{j=0}^{m-1} b_j \left( u^{1,(m-j)}_N - u^{1,(m-j-1)}_N, v_N \right) + (\partial_x u^{1,(m)}_N, \partial_x v_N) + (\partial_y u^{1,(m)}_N, \partial_y v_N) = 0.
\]

(5.4)

The spectral discretization of the weak problem (5.3) reads: find \( u^N_N \in P_N^0(\Omega) \) such that

\[
(u^N_N, v_N) + \alpha_0 \beta_0^{-1} (\partial_x u^N_N, \partial_x v_N) + \alpha_0 \beta_0^{-1} (\partial_y u^N_N, \partial_y v_N) = \sum_{i=1}^{n} d_{n-i} (u^{n-i}_N, v_N) \quad \forall v_N \in P_N^0(\Omega).
\]

(5.5)

**Theorem 5.1.** Let \( u \) be the exact solution of (1.1), \( \{u^N_N\}_{n=0}^{\infty} \) be the full discrete solution of (5.5) with the first step solution \( u^N_N \) given by (5.4) and the initial condition \( u^0 = \pi_N u^0 \). Suppose \( \partial_x^2 u \in L^\infty((0,T);H^m(\Omega)), m \geq 1 \). Then the following error estimate holds:

\[
\|u(t_n) - u^N_N\| \leq c_\gamma e^{\sigma T} \left( N^{-m}\|D^m u\|_{L^\infty(\Omega)} + N^{-m}\|u\|_{L^\infty(\Omega)} \right)
\]

\[
+ N^{-m} \tau^3 \|\partial_x^3 u\|_{L^\infty(\Omega)} + \tau^3 \|\partial_x^2 u\|_{L^2(\Omega)}
\]

where \( c_\gamma \) is a constant independent on \( \tau \).

**Proof.** Using the ideas of the proof of Theorem 4.1, we can similarly prove this theorem; the details are omitted here. \( \square \)

5.2. A Legendre collocation method in space for 2D

Let \( \{x_i : i = 0, \ldots, N\}, \{y_j : j = 0, \ldots, N\} \) be the Legendre-Gauss-Lobatto points with respect to the direction \( x \) and \( y \), which are the zeros of the polynomials \( (1-x^2)L_N^0(x), (1-y^2)L_N^0(y) \), respectively, \( \{\omega_i : i = 0, \ldots, N\}, \{\omega_j : j = 0, \ldots, N\} \) are the corresponding weights such that the following quadrature holds

\[
\int_{-1}^{1} \int_{-1}^{1} \varphi(x,y) dx dy = \sum_{i=0}^{N} \sum_{j=0}^{N} \varphi(x_i, y_j) \omega_i \omega_j \quad \forall \varphi \in P_{2N-1}(\Omega),
\]
and the discrete inner product
\[(\phi, \psi)_N := \sum_{i=0}^N \sum_{j=0}^N \phi(x_i, y_j) \psi(x_i, y_j) \omega_i \omega_j.\]

Then the following estimates are well known \[19\]
\[\|\varphi\| \leq \|\varphi\|_N \leq 3\|\varphi\| \quad \forall \varphi \in P_N(\Omega),\]
\[|(\varphi, v_N) - (\varphi, v_N)_N| \leq cN^{-m}\|\varphi\|_m\|v_N\| \quad \forall \varphi \in H^m(\Omega) \quad \forall v_N \in P_N(\Omega), \quad m \geq 1.\]

We first consider the first step full discrete scheme of (5.2): find \(u_{N,1}^{1(m)} \in P_{N-1}^0(\Omega)\) such that
\[\overline{a}_N(u_{N,1}^{1(m)}, v_N) = \overline{F}_N(v_N) \quad \forall v_N \in P_{N-1}^0(\Omega), \quad m = 1, 2, \ldots, k\] (5.6)
with \(u_{N,1}^{1} = u_{N,1}^{1(k)}\). Here the bilinear form \(\overline{a}_N(\cdot, \cdot)\) and functional \(\overline{F}_N(\cdot)\), respectively, are defined by
\[\overline{a}_N(u_{N,1}^{1(m)}, v_N) := b_0(u_{N,1}^{1(m)}, v_N)_N + (1 - \gamma)\sigma \tau_1(\partial_x u_{N,1}^{1(m)}, \partial_x v_N)_N + (1 - \gamma)\sigma \tau_1(\partial_y u_{N,1}^{1(m)}, \partial_y v_N)_N,\]
\[\overline{F}_N(v_N) := \sum_{i=0}^{m-2} (b_i - b_{i+1})(u_{N,1}^{1(m-i-1)}, v_N)_N + b_{m-1}(u_{N,1}^{1(0)}, v_N)_N.\]

The spectral collocation approximation of (5.3): find \(u_n^0 \in P_N^0(\Omega)\) with \(n \geq 2\) such that
\[a_N(u_n^0, v_N) = F_N(v_N) \quad \forall v_N \in P_{N-1}^0(\Omega),\] (5.7)
where the bilinear form \(a_N(\cdot, \cdot)\) and functional \(F_N(\cdot)\), respectively, are defined by
\[a_N(u_n^0, v_N) := (u_n^0, v_N)_N + \alpha_0 \beta_0^{-1}(\partial_x u_n^0, \partial_x v_N)_N + \alpha_0 \beta_0^{-1}(\partial_y u_n^0, \partial_y v_N)_N,\]
\[F_N(v_N) := \sum_{i=1}^{n} d_{n-i}(u_{n-i}^0, v_N)_N.\]

**Theorem 5.2.** Let \(u\) be the exact solution of (1.1), \(\{u_N^n\}_{n=2}^M\) be the solution of the problem (5.7) with the first step solution \(u_{N,1}^{1}\) given by (5.6). Suppose \(\partial_t^2 u \in L^\infty((0, T]; H^m(\Omega)), m \geq 1.\) For \(\forall n \geq 2,\) the following error estimate holds:
\[\|u(t_n) - u_n^0\| \leq c_N \left( \gamma^3 \|\partial_t^3 u\|_{L^\infty(L^2(\Omega))} + N^{-m}\|u\|_{L^\infty(H^m(\Omega))} + N^{-m}\|\overline{CF}_D^u\|_{L^\infty(H^m(\Omega))} + N^{-m}\|\partial_t^3 u\|_{L^\infty(H^m(\Omega))} \right).\]

Proof. The proof of this result is quite similar to that given in Theorem 4.2 and so is omitted. \(\square\)

**Remark 5.1.** As pointed out in [10, 13], the regularity estimate is \(\|u\|_\mu \leq C\|f\|, \quad 1 \leq \mu \leq 1.5\) with \(\mu = \alpha\) or \(\mu = \beta\). Then the corresponding theory can be extended to solve two-dimensional time-space fractional diffusion equation [21]:
\[\frac{\partial}{\partial t}^\alpha D_t^\alpha u(x, y, t) = \frac{\partial^\alpha u(x, y, t)}{\partial|x|^\alpha} + \frac{\partial^\beta u(x, y, t)}{\partial|y|^\beta} + f(x, y, t), \quad x \in \Omega, \quad t \in (0, T],\] (5.8)
where \(1 < \alpha, \beta < 2\). The numerical experiments are carried out in Example 6.3, where the spectral collocation approximation of the space fractional derivative can be seen in (4.4) of [22].

### 6. Numerical experiments

We numerically verify the above theoretical results including convergent orders by the \(l_\infty\) norm and the discrete \(L^2\) norm. Without loss of generality, we add a force term \(f(x, t)\) and \(f(x, y, t)\) on the right hand side of (3.1) and (1.1), respectively. We express the function \(u_N^n\) in terms of the Lagrangian interpolations at Legendre-Gauss-Lobatto or Chebyshev-Gauss-Lobatto points
\[u_N^n(x) = \sum_{i=0}^N u_i^n l_i(x), \quad u_N^n(x, y) = \sum_{i=0}^N \sum_{j=0}^N u_{i,j}^n l_i(x)l_j(y), \quad l_i(x) = \prod_{j=0, j \neq i}^N \frac{x - x_j}{x_i - x_j},\]
where \(u_i^n := u_N^n(x_i), \quad u_{i,j}^n := u_N^n(x_i, y_j)\) at the mesh points \(\{x_i, y_i : i = 0, \ldots, N\}\).
6.1. Numerical results for 1D

Example 6.1. Consider the one-dimensional case of (1.1) or (3.1) on a finite domain \(-1 < x < 1, \ 0 < t < 1\) with the initial condition \(u(x,0) = \sin(2\pi x)\) and the homogeneous boundary conditions \(u(1,t) = u(-1,t) = 0\). The forcing function is
\[
f(x,t) = \frac{1}{1-\gamma} \sin(2\pi x) \left( \frac{3t^2}{\sigma} - \frac{6t}{\sigma^2} + \frac{6}{\sigma^3} (1 - e^{-\sigma t}) \right) + 4\pi^2(t^3 + 1) \sin(2\pi x).
\]
Then (3.1) has the exact solution \(u(x,t) = (t^3 + 1) \sin(2\pi x)\).

Table 1: The \(l_\infty\) norm and discrete \(L^2\)-norm for (4.11) approximated on Chebyshev-Gauss-Lobatto points with \(N = 20\).

| \(\tau\)   | \(\gamma = 0.3\) Rate | \(\gamma = 0.5\) Rate | \(\gamma = 0.7\) Rate |
|-----------|------------------------|------------------------|------------------------|
| 1/20      | 3.2362e-07             | 7.8914e-07             | 1.6837e-06             |
| 1/40      | 3.0934e-08             | 1.0757e-07             | 2.8750e-07             |
| 1/80      | 5.4436e-09             | 1.3848e-08             | 2.9575e-08             |
| 1/160     | 5.4548e-10             | 1.7269e-09             | 3.0034e-09             |

| \(\tau\)   | \(\gamma = 0.3\) Rate | \(\gamma = 0.5\) Rate | \(\gamma = 0.7\) Rate |
|-----------|------------------------|------------------------|------------------------|
| 1/20      | 3.8640e-07             | 9.4198e-07             | 2.0096e-06             |
| 1/40      | 5.1603e-08             | 1.2855e-07             | 2.8734e-07             |
| 1/80      | 6.6170e-09             | 1.6700e-08             | 2.9444e-08             |
| 1/160     | 6.1517e-10             | 2.0389e-09             | 2.9677e-09             |

Fig 1: The \(l_\infty\) norm and discrete \(L^2\)-norm to Example 6.1 approximated on Legendre-Gauss-Lobatto and Chebyshev-Gauss-Lobatto points with \(\tau = 1/800\) (LP: Legendre points, CP: Chebyshev points).

Table 1 shows that the \(l_\infty\) norm and the discrete \(L^2\)-norm at time \(T = 1\) and the numerical results confirm that the scheme (4.11) has the third-order global truncation error in time direction. Fig 1 indicates that the errors show an exponential decay since the errors are linear versus the polynomial degrees.

6.2. Numerical results for 2D

Example 6.2. Consider the two-dimensional problem (1.1) on a finite domain \(-1 < x, y < 1, \ 0 < t < 1\) with the initial condition \(u(x,y,0) = \sin(2\pi x) \sin(2\pi y)\) and the homogeneous boundary conditions. The forcing function is
\[
f(x,y,t) = \frac{1}{1-\gamma} \sin(2\pi x) \sin(2\pi y) \left( \frac{3t^2}{\sigma} - \frac{6t}{\sigma^2} + \frac{6}{\sigma^3} (1 - e^{-\sigma t}) \right) + 8\pi^2(t^3 + 1) \sin(2\pi x) \sin(2\pi y).
\]
Then (1.1) has the exact solution \(u(x,y,t) = (t^3 + 1) \sin(2\pi x) \sin(2\pi y)\).
Table 2: The $l_\infty$ norm and discrete $L^2$-norm for (5.7) approximated on Chebyshev-Gauss-Lobatto points with $N = 21$.

| $\tau$ | $\gamma = 0.3$ | Rate | $\gamma = 0.5$ | Rate | $\gamma = 0.7$ | Rate |
|--------|----------------|------|----------------|------|----------------|------|
| 1/20   | 1.6851e-07     |      | 4.1109e-07     |      | 8.7473e-07     |      |
| 1/40   | 2.2527e-08     | 2.9031| 5.6102e-08     | 2.8733| 1.2696e-07     | 2.7842|
| 1/80   | 2.9136e-09     | 2.9508| 7.3132e-09     | 2.9395| 1.7023e-08     | 2.8990|
| 1/160  | 2.9540e-10     | 3.3021| 9.1948e-10     | 2.9916| 2.2001e-09     | 2.9518|

Table 2 shows that the $l_\infty$ norm and the discrete $L^2$-norm at time $T = 1$ and the numerical results confirm that the scheme (5.7) has the third-order global truncation error in time direction. Fig 2 indicates that the errors show an exponential decay since the errors are linear versus the polynomial degrees.

6.3. Numerical results for 2D: Time-space fractional diffusion equation (5.8)

**Example 6.3.** Consider the two-dimensional time-space fractional diffusion equation (5.8) on a finite domain $-1 < x, y < 1$, $0 < t < 1$ with the initial condition $u(x, y, 0) = (\cos(\pi x) + 1)(\cos(\pi y) + 1)$ and the homogeneous boundary conditions. Then (5.8) has the exact solution

$$u(x, y, t) = (t^3 + 1)(\cos(\pi x) + 1)(\cos(\pi y) + 1).$$

Table 3 shows that the $l_\infty$ norm and the discrete $L^2$-norm at time $T = 1$ and the numerical results confirm that the scheme (5.8) has the third-order global truncation error in time direction. Fig 3 indicates that the errors show an exponential decay since the errors are linear versus the polynomial degrees.

7. Conclusions

There are already some theoretical convergence results for time Caputo-Fabrizio fractional operator. We notice that the results are mainly first-order or second-order schemes. In this work, we first obtain the third-order discretization schemes by the linear interpolation and quadratic interpolation approximation, which is so called $L_1^2$ formula. The designed scheme is based on finite difference method and spectral method, which is unconditionally stable and has a truncation error $O(\tau^3 + N^{-m})$. The presented method is also extended to solve the two-dimensional time-space CF-fractional diffusion equation. Numerical experiments indicate that a third order temporal convergence and exponential convergence in space if the exact solution is smooth enough, which are in good agreement with the theoretical analysis.
Table 3: The $l_{\infty}$ norm and discrete $L^2$-norm for 5.8 approximated on Legendre-Gauss-Lobatto points for $\alpha = \beta = 1.5$ with $N = 12$.

| $\tau$ | $\gamma = 0.3$ | Rate | $\gamma = 0.5$ | Rate | $\gamma = 0.7$ | Rate |
|--------|----------------|------|----------------|------|----------------|------|
| 1/10   | 6.652e-05     |      | 1.5079e-04     |      | 2.7243e-04     |      |
| 1/20   | 9.650e-06     | 2.7852| 2.3188e-05     | 2.7011| 5.0166e-05     | 2.4411|
| 1/40   | 1.2588e-06    | 2.9385| 3.1506e-06     | 2.8797| 7.3545e-06     | 2.7700|
| 1/80   | 1.5534e-07    | 3.0185| 3.7753e-07     | 3.0610| 9.5823e-07     | 2.9402|

| $\tau$ | $\gamma = 0.3$ | Rate | $\gamma = 0.5$ | Rate | $\gamma = 0.7$ | Rate |
|--------|----------------|------|----------------|------|----------------|------|
| 1/10   | 6.0693e-05    |      | 1.3713e-04     |      | 2.4731e-04     |      |
| 1/20   | 8.8298e-06    | 2.7811| 2.1139e-05     | 2.6976| 4.5738e-05     | 2.4349|
| 1/40   | 1.1730e-06    | 2.9122| 2.8948e-06     | 2.8684| 6.7360e-06     | 2.7634|
| 1/80   | 1.4281e-07    | 3.0380| 3.6859e-07     | 2.9734| 8.9923e-07     | 2.9051|

Fig 3: The $l_{\infty}$ norm and discrete $L^2$ norm to Example 6.3 approximated on Legendre-Gauss-Lobatto and Chebyshev-Gauss-Lobatto points for $\gamma = 0.5$ with $\tau = 1/500$ (LP: Legendre points, CP: Chebyshev points).

Appendix

Lemma 7.1. Let $\gamma \in (0, 1)$, $n \geq 2$. Then the coefficients in the scheme 3.5 satisfy $d_{n-1}^i > 0$, $1 \leq i \leq n$.

Proof. For $n = 2, 3$,

\[
d_1^2 = -(b_1 + 2 - 2\sigma \tau - 2e^{-\sigma \tau}) \beta_0^{-1} = (-2e^{-\sigma \tau} + 2(\sigma \tau + 1)e^{-2\sigma \tau} - 2 + 2\sigma \tau + 2e^{-\sigma \tau}) \beta_0^{-1}
\]

\[
= e^{-2\sigma \tau}(2\sigma \tau + 2 + (2\sigma \tau - 2)e^{2\sigma \tau}) \beta_0^{-1} = e^{-2\sigma \tau}\left(\frac{4}{3}\sigma^3 + \frac{4}{3}\sigma^4 + \cdots \right) \beta_0^{-1} > 0,
\]

\[
d_0^2 = \left(-a_1 - \frac{\sigma^2}{2} + 1 - \left(\frac{\sigma^2}{2} + 1\right)e^{-\sigma \tau}\right) \beta_0^{-1}
\]

\[
= \left(\frac{\sigma^2}{2} + 1\right)e^{-\sigma \tau} - \left(\frac{\sigma^2}{2} + 1\right)e^{2\sigma \tau} - \frac{\sigma^2}{2} + 1 - \left(\frac{\sigma^2}{2} + 1\right)e^{-\sigma \tau}\right) \beta_0^{-1}
\]

\[
= e^{-2\sigma \tau}\left(-\frac{\sigma^2}{2} + 1 + \left(\frac{\sigma^2}{2} + 1\right)e^{2\sigma \tau}\right) \beta_0^{-1} = e^{-2\sigma \tau}\left(\sigma^2 + \frac{\sigma^3}{3} + \cdots \right) \beta_0^{-1} > 0,
\]

\[
d_2^2 = -(b_1 + b_2 + 2 - 2\sigma \tau - 2e^{-\sigma \tau}) \beta_0^{-1}
\]

\[
= \left(-2e^{-\sigma \tau} + (2\sigma \tau + 2)e^{-2\sigma \tau} - \left(\frac{\sigma^2}{2} + 1\right)e^{-2\sigma \tau} - \left(\frac{\sigma^2}{2} + 1\right)e^{-3\sigma \tau} - 2 + 2\sigma \tau + 2e^{-\sigma \tau}\right) \beta_0^{-1}
\]

\[
= e^{-3\sigma \tau}\left(\frac{3\sigma^2}{2} + 3\sigma^3 + \frac{3\sigma^4}{8} + \cdots \right) \beta_0^{-1} > 0,
\]
\[ d_1^3 = (-a_1 - b_2 - \frac{\sigma\tau}{2} + 1 - \left(\frac{\sigma\tau}{2} + 1\right) e^{-\sigma\tau}) \beta_0^{-1} \]
\[ = \left(\frac{\sigma\tau}{2} + 1\right) e^{-\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 1\right) e^{-2\sigma\tau} - 2e^{-2\sigma\tau} + (2\sigma\tau + 2) e^{-3\sigma\tau} - \frac{\sigma\tau}{2} + 1 - \left(\frac{\sigma\tau}{2} + 1\right) e^{-\sigma\tau}\right) \beta_0^{-1} \]
\[ = e^{-3\sigma\tau} \left(\frac{3\sigma\tau}{2} - 3\right) e^{\sigma\tau} + (2\sigma\tau + 2) + \left(1 - \frac{\sigma\tau}{2}\right) e^{3\sigma\tau}\right) \beta_0^{-1} \]
\[ = e^{-3\sigma\tau} \left(\sigma^3 + \frac{3\sigma^4}{4} + \cdots\right) \beta_0^{-1} > 0, \]
\[ d_0^3 = -a_2\beta_0^{-1} = \left(\frac{\sigma\tau}{2} + 1\right) e^{-2\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 1\right) e^{-3\sigma\tau}\right) \beta_0^{-1} \]
\[ = e^{-3\sigma\tau} \left(\frac{\sigma\tau}{2} + 1\right) e^{\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 1\right) \beta_0^{-1} = e^{-3\sigma\tau} \left(\sigma^2 + \frac{5\sigma^4}{12} + \cdots\right) \beta_0^{-1} > 0. \]

For \( n \geq 4, \)
\[ d_{n-1} = - (b_1 + c_2 + 2 - 2\sigma\tau - 2e^{-\sigma\tau}) \beta_0^{-1} = \left(\frac{3\sigma\tau}{2} + 1\right) e^{-2\sigma\tau} - \left(\frac{\sigma\tau}{2} + 1\right) e^{-3\sigma\tau} - 2 + 2\sigma\tau\right) \beta_0^{-1} \]
\[ = e^{-3\sigma\tau} \left(\frac{3\sigma\tau}{2} + 3\right) e^{\sigma\tau} - \left(\frac{\sigma\tau}{2} + 1\right) + (2\sigma\tau - 2) e^{3\sigma\tau}\right) \beta_0^{-1} \]
\[ = e^{-3\sigma\tau} \left(\frac{5\sigma^3}{4} \sigma^4 + \frac{21\sigma^4}{8} + \frac{23\sigma^5}{8}\right) \beta_0^{-1} > 0, \]
\[ d_{n-2} = (-a_1 - b_2 - c_3 - \frac{\sigma\tau}{2} + 1 - \left(\frac{\sigma\tau}{2} + 1\right) e^{-\sigma\tau}) \beta_0^{-1} \]
\[ = \left(1 - \frac{\sigma\tau}{2} - \left(\frac{3\sigma\tau}{2} + 3\right) e^{-2\sigma\tau} + \left(\frac{3\sigma\tau}{2} + 3\right) e^{-3\sigma\tau} - \left(\frac{\sigma\tau}{2} + 1\right) e^{-4\sigma\tau}\right) \beta_0^{-1} \]
\[ = e^{-4\sigma\tau} \left(1 - \frac{\sigma\tau}{2} e^{4\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 3\right) e^{2\sigma\tau} + \left(\frac{3\sigma\tau}{2} + 3\right) e^{3\sigma\tau} - \left(\frac{\sigma\tau}{2} + 1\right) \beta_0^{-1}\right) \]
\[ = e^{-4\sigma\tau} \left(\frac{11\sigma^3}{4} + \frac{41\sigma^4}{24} + \frac{119\sigma^5}{8}\right) \beta_0^{-1} > 0, \]
\[ d_{n-1} = (-a_{n-1} - b_{n-1} - c_{n+1}) \beta_0^{-1} \]
\[ = \left(\frac{\sigma\tau}{2} + 1\right) e^{-(i-1)\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 3\right) e^{-(i+1)\sigma\tau} + \left(\frac{3\sigma\tau}{2} + 3\right) e^{-(i+2)\sigma\tau}\right) \beta_0^{-1} \]
\[ = e^{-(i+2)\sigma\tau} \left(\frac{3\sigma\tau}{2} + 3\right) e^{\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 3\right) e^{2\sigma\tau} + \left(\frac{3\sigma\tau}{2} + 3\right) e^{3\sigma\tau} - \left(\frac{\sigma\tau}{2} + 1\right) \beta_0^{-1}\right) \]
\[ = e^{-(i+2)\sigma\tau} \left(a^{i+3} + 2a^i + 2a^5 + \cdots\right) \beta_0^{-1} > 0, \quad i = 3, 4, \ldots , n - 2, \]
\[ d_1 = (-a_{n-2} - b_{n-1}) \beta_0^{-1} = \left(\frac{\sigma\tau}{2} + 1\right) e^{-(n-2)\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 3\right) e^{-(n-1)\sigma\tau} + (2\sigma\tau + 2) e^{-n\sigma\tau}\right) \beta_0^{-1} \]
\[ = e^{-n\sigma\tau} \left(\frac{3\sigma\tau}{2} + 1\right) e^{\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 3\right) e^{2\sigma\tau} + (2\sigma\tau + 2) \beta_0^{-1} \]
\[ = e^{-n\sigma\tau} \left(\frac{13\sigma^3}{12} + \frac{23\sigma^4}{24} + \frac{41\sigma^5}{8}\right) \beta_0^{-1} > 0, \]
\[ d_0 = -a_{n-1} \beta_0^{-1} = \left(\frac{\sigma\tau}{2} + 1\right) e^{-(n-1)\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 1\right) e^{-n\sigma\tau}\right) \beta_0^{-1} \]
\[ = e^{-n\sigma\tau} \left(\frac{\sigma\tau}{2} + 1\right) e^{\sigma\tau} - \left(\frac{3\sigma\tau}{2} + 1\right) \beta_0^{-1} = e^{-n\sigma\tau} \left(\sigma^2 + \frac{5\sigma^3}{12} + \frac{\sigma^4}{8} + \cdots\right) \beta_0^{-1} > 0. \]

The proof is completed.

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