On quasi-separative semigroups

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ABSTRACT. We built some congruences on semigroups, from where a decomposition of quasi-separative semigroups was obtained.

1 Introduction

The research of separative semigroups was being begun from the famous paper of Hewitt and Zuckerman [3, 5], where, in particular, they proved that any commutative separative semigroup is isomorphic to a semilattice of cancellative semigroups. An generalization for the noncommutative case has been made by Burmistrovich [1] and independently by Petrich [4]. Drazin [2] introduced the term 'quasi-separativity' and studied connections between it and others semigroup properties (inversity, regularity etc).

We shall follow the terminology proposed by Drazin:

Definition 1 A semigroup \( S \) is called separative\(^1 \) if

\[
\begin{align*}
  x^2 &= xy \\
y^2 &= yx
\end{align*}
\implies x = y
\]

and

\[
\begin{align*}
  x^2 &= yx \\
y^2 &= xy
\end{align*}
\implies x = y
\]

for all \( x, y \in S \).

A semigroup \( S \) is called quasi-separative if

\[
x^2 = xy = yx = y^2 \implies x = y
\]

for all \( x, y \in S \).

Drazin also showed that in the definition of quasi-separativity we can replace \( [2] \) by the next condition

\[
x^2 = xy = y^2 \implies x = y.
\]

It often simplifies considerably proofs of assertions.

\(^1\)Burmistrovich [1] called it weakly cancellative
The main result of this paragraph is an extension of the Burmistrovich’s theorem (Theorem 3): any quasi-separative semigroup is decomposable into a semilattice of subsemigroups, which are called *quasi-cancellative* by us. With this aim we previously build certain congruences on arbitrary semigroup (Theorem 1); they give semilattice decompositions in the quasi-separative case. As a corrolary, in Sect. 4 we consider an intermediate class of semigroups (*weakly balanced semigroups*) between separative and quasi-separative ones and discuss the connections between them.

## 2 Relation Ω

Let $S$ be an arbitrary semigroup. By analogy with [2], we define two binary relations $E(a), F(a) \subset S \times S$ for every element $a \in S$:

$$E(a) = \{(x, y) \mid ax = ay\}, \quad F(a) = \{(x, y) \mid xa = ya\}.$$  

The next properties of these relations are obvious:

$$E(b) \subset E(ab) \quad (4)$$
$$F(a) \subset F(ab) \quad (5)$$
$$bE(ab) \subset E(a) \quad (6)$$
$$F(ab)a \subset F(b) \quad (7)$$

(here and below for a binary relation $R \subset S \times S$ and for an element $x \in S$ the relation $\{(x, y) \mid (a, b) \in R\}$ is denoted by $\text{xR}$; analogously, $Rx$).

In what follows, the main tools for our studying will be the relations $\Omega \subset S \times S$, which satisfy the next conditions:

$$\forall a \quad \Omega \cap E(a) = \Omega \cap F(a) \quad (8)$$
$$\forall a, b \quad b(\Omega \cap E(ab)) \subset \Omega \quad (9)$$
$$\forall a, b \quad (\Omega \cap F(ab))a \subset \Omega \quad (10)$$

and the equivalences $\sim_{\Omega}$ on $S$ corresponding to these relations:

$$a \sim_{\Omega} b \iff \Omega \cap E(a) = \Omega \cap E(b).$$

According to [8], such definition is equal to the following:

$$a \sim_{\Omega} b \iff \Omega \cap F(a) = \Omega \cap F(b).$$

**Lemma 1** For all elements $a, b \in S$ and a relation $\Omega$ which satisfies the conditions (8)-(10)

$$\Omega \cap E(a) \subset \Omega \cap E(ab) \cap E(ba).$$
Proof. By (8) we have:

$$\Omega \cap E(a) = (\Omega \cap E(a)) \cap (\Omega \cap F(a)).$$

From here, using (11) and (12), we get:

$$\Omega \cap E(a) \subset (\Omega \cap E(b)) \cap (\Omega \cap F(ab)) = \Omega \cap E(ba) \cap E(ab)$$

(the last equality follows from (8)). ■

Our first result is fulfilled for an arbitrary semigroup:

**Theorem 1** Let $\Omega \subset S \times S$ satisfies the conditions (8)-(10). Then the equivalence $\sim_\Omega$ is a congruence on $S$.

**Proof.** Let $a, b, c \in S$ and $a \sim_\Omega b$. Obviously, for the proving the right compatibility of $\sim_\Omega$ it is enough to verify the inclusion

$$\Omega \cap E(ac) \subset \Omega \cap E(bc).$$

Let $(x, y) \in \Omega \cap E(ac)$. Owing to $\Omega$ $(cx, cy) \in \Omega$. On the other hand, (9) implies the inclusion $(cx, cy) \in cE(ac) \subset E(a)$. Therefore,

$$(cx, cy) \in \Omega \cap E(a) = \Omega \cap E(b).$$

From here it follows that $(x, y) \in \Omega \cap E(bc)$.

Similarly, by (7) and (10) the left compatibility can be proved. ■

**Example.** Let $S$ be a commutative semigroup, $\Omega = S \times S$. Then the conditions of Theorem 1 are true and the equivalence

$$a \sim b \iff E(a) = E(b)$$

is a congruence relation.

## 3 A decomposition of quasi-separative semigroups

In this section we apply the preceding theorem to quasi-separative semigroups.

Note that the definition of quasi-separativity in the form (3) may be formulated in terms of the relations $E(a)$ and $F(a)$:

$$(a, b) \in E(a) \cap F(b) \implies a = b$$

(11)

for all $a, b \in S$. 3
Theorem 2 Let $\Omega$ be a relation on quasi-separative semigroup $S$ which satisfies the conditions (8)-(10). Then $S/\sim_\Omega$ is a semilattice.

Proof. First, show that $S/\sim_\Omega$ is a band. In order to verify this statement it is sufficient to justify that the equality $\Omega \cap E(a) = \Omega \cap E(a^2)$ is right for any $a \in S$.

An inclusion

$$\Omega \cap E(a) \subset \Omega \cap E(a^2)$$

at once follows from Lemma 1. Conversely, if $(x, y) \in \Omega \cap E(a^2)$, then $(ax, ay) \in E(a)$. Moreover, owing to (9)

$$(ax, ay) \in a(\Omega \cap E(a^2)) \subset \Omega,$$

hence, $(ax, ay) \in \Omega \cap E(a)$. By Lemma 1

$$(ax, ay) \in \Omega \cap E(ax) \cap E(xa) \cap E(ya) \cap E(ay),$$

whence, in particular,

$$(ax, ay) \in \Omega \cap E(ax) \cap E(ay) = \Omega \cap E(ax) \cap F(ay)$$

by the condition (8). From (11) we obtain $ax = ay$, that is $(x, y) \in E(a)$. Therefore, $\Omega \cap E(a^2) \subset \Omega \cap E(a)$ and the first part of Theorem is proved.

Now we shall prove that $S/\sim_\Omega$ is commutative, viz. that $\Omega \cap E(ab) = \Omega \cap E(ba)$. The successive using the properties (4), (8) and (5) gives us:

$$\Omega \cap E(ab) \subset \Omega \cap E(bab) = \Omega \cap F(bab) \subset \Omega \cap F((ba)^2) = \Omega \cap E((ba)^2).$$

Since, as proved above, $S/\sim$ is a band, then

$$\Omega \cap E(ab) \subset \Omega \cap E(ba).$$

Analogously,

$$\Omega \cap E(ba) \subset \Omega \cap E(ab),$$

what completes the proof of Theorem. ■

Next assertion gives us a preliminary information about $\sim_\Omega$-classes. Denote by $\Delta_T$ the diagonal of Cartesian square $T \times T$.

Proposition 1 If $S$ is a quasi-separative semigroup, then each $\sim_\Omega$-class $T$ satisfies the next condition for all $a \in T$:

$$\Omega \cap E(a) \cap (T \times T) \subset \Delta_T.$$
Proof. Indeed, let \((x, y) \in \Omega \cap E(a) \cap (T \times T)\). Since \(x \sim_\Omega y \sim_\Omega a\), then
\[
(x, y) \in \Omega \cap E(a) = \Omega \cap E(x) \cap F(y),
\]
whence, by (11), we have \(x = y\). ■

Definition 2 We call a semigroup \(S\) quasi-cancellative if the condition
\[
\begin{align*}
\forall x, y \in S^1 & \quad xby = xcy \iff yxb = yxc \iff byx = cyx \\
ab & = ac.
\end{align*}
\]
implies \(b = c\).

Obviously, every right- or left-cancellative semigroup is quasi-cancellative.

Our main result on structure of quasi-separative semigroups is the next

Theorem 3 A semigroup is quasi-separative if and only if it is a semilattice of quasi-separative quasi-cancellative semigroups.

Proof. Necessity. Denote a binary relation \(\Omega_S\) on \(S\):
\[
\Omega_S = \{(x, y) \mid \forall a, b \in S^1 \quad axb = aby \iff xba = yba \iff bax = bay\} \quad (12)
\]
and verify the conditions \([9]-[10]\) for it. Since for \(a = 1\) we have:
\[
xb = yb \iff bx = by,
\]
for any pair \((x, y) \in \Omega_S\), then obviously, \([9]\) holds. Now we prove that \(\Omega_S\) is left compatibility, from where \([5]\) will follow.

Let \((x, y) \in \Omega_S, b \in S\). To prove that \((bx, by) \in \Omega_S\) one needs to check the fulfilment of the implications:
\[
\forall c, d \in S^1 \quad cbxd = cbyd \iff dcxb = dcby \iff bxdc = bydc.
\]

The implication \(cbxd = cbyd \iff dcxb = dcby\) immediately follows from the definition of \(\Omega_S\). Let \(dcxb = dcby\). From \([12]\) we obtain:
\[
\begin{align*}
dcxb = dcby & \implies xdcb = ydcb.
\end{align*}
\]
Therefore
\[
(bxdc)^2 = bx(dcbx)dc = b(xdcby)dc = (bydc)^2
\]
and quasi-separativity implies \(bxdc = bydc\).

Similarly, if \(bxdc = bydc\), then
\[
bxdc = bydc \implies dcbx = dcby \implies xdcb = ydcb.
\]

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Hence 
\[(cbxd)^2 = cbx(dcbx)d = c(bxdc)byd = (cbyd)^2\]
and \(cbxd = cbyd\).

In the same way right compatibility is checked, and so the condition \(\text{(11)}\) is fulfilled.

Thus, \(S/\sim_{\Omega_S}\) is a commutative band by Theorem \(2\). It remains to show that its components are quasi-cancellative.

Let suppose that the conditions of Definition \(2\) hold for some elements \(a, b, c, d\) from the \(\sim_{\Omega_S}\)-class \(T \subset S\). It means that 
\[(c, d) \in \Omega_T \cap E(a) \subset \Omega_S \cap E(a) = \Omega_S \cap E(b).\]

Hence \(bc = bd\). Moreover, \(\text{(8)}\) implies \(cb = db\). In particular, replacing \(b\) in the obtained equations by \(c\) and \(d\), we get:
\[c^2 = cd = dc = d^2,\]
whence \(c = d\).

**Sufficiency.** It is easy to see that any semilattice of quasi-separative semigroups is also quasi-separative. ■

### 4 Corollaries and Examples

In this section we show that Theorem \(3\) implies the theorem of Burmistrovich on the separative semigroups and obtain an assertion about certain intermediate class of semigroups.

**Proposition 2** Every separative quasi-cancellative semigroup is cancellative.

**Proof.** Let \(S\) be separative and quasi-cancellative, \(a, b, c \in S, ab = ac\). By Lemma 1 from \(\text{[1]}\) for all \(x, y \in S\)

\[xby = xcy \implies byx = cyx \implies yxb = yxc \implies xby = xcy.\]

So, by quasi-separativity \(b = c\).

To prove the right cancellativity we ought to apply the Lemma 1 \(\text{[1]}\) to the equality \(ba = ca\) and to refer to the previous argumentation. ■

**Corollary 1** (Burmistrovich’s Theorem \(\text{[1]}\)) A semigroup is separative if and only if it is isomorphic to a semilattice of cancellative semigroups. ■
**Definition 3** A semigroup \( S \) is called weakly cancellative if for every \( a, b, x, y \in S \)

\[
\begin{align*}
ax &= ay \\
xb &= yb
\end{align*}
\]

implies \( x = y \).

We call a semigroup \( S \) weakly balanced, if (13) implies

\[
\begin{align*}
xa &= ya \\
xb &= yb
\end{align*}
\]

Obviously, every weakly cancellative semigroup is quasi-separative; but in general this is not hold in the weakly balanced case (for example, all commutative semigroups are weakly balanced). On the other hand, by above-mentioned Lemma 1 all separative semigroups are weakly balanced, so two next facts give a partial extension of Burmistrovich’s theorem to the more wide class of semigroups.

**Proposition 3** If \( S \) is a quasi-cancellative weakly balanced semigroup, then \( S \) is also weakly cancellative.

**Proof.** Let \( S \) be quasi-cancellative and weakly balanced, \( a, b, x, y \in S \) and 

\[
ax = ay, \quad xb = yb.
\]

If \( uxv = uyv \) for some elements \( u, v \in S^1 \), then by the weakly balancity from this last equality and from \( axv = ayv \) we obtain \( xuv = yuv \). Similarly, implications

\[
xvu = yvu \implies vux = vuy \implies uvx = uyv.
\]

can be obtained. Now \( x = y \) because of quasi-cancellativity. ■

**Corollary 2** Every quasi-separative weakly balanced semigroup is isomorphic to a semilattice of weakly cancellative semigroups. ■

We don’t know whether the converse to the Corollary is true. One can affirm only that a semilattice of weakly cancellative semigroups (which, evidently, is quasi-separative) satisfies the next condition:

\[
\begin{align*}
a^2x &= a^2y \\
xa^2 &= ya^2 \implies \begin{cases} 
ax &= ay \\
xa &= ya
\end{cases}
\]
\]

Really, it follows out of \( a^2x = a^2y \) that \( ax, ay, xa, ya \) contain in the same component of the semilattice. From the antecedent of (14) we have:

\[
\begin{align*}
(xa)(ax) &= (xa)(ay) \\
(ax)(a^2x) &= (ay)(a^2x)
\end{align*}
\]
Now weakly cancellativity implies \( ax = ay \) and, similarly, \( xa = ya \).

In conclusion we discuss the connections between considered classes of semigroups. They may be presented by a diagram:

\[
\begin{array}{c}
\text{Separativity} \Rightarrow \text{Quasi-separativity} \\
\quad \uparrow \\
\text{Weakly balancity} \Rightarrow \text{Quasi-separativity} \\
\quad \uparrow \\
\text{Cancellativity} \Rightarrow \text{Weak cancellativity} \Rightarrow \text{Quasi-separativity} \\
\quad \uparrow \\
\end{array}
\]

Now we shall show that all implications in this picture are strict.

Obviously, any commutative quasi-cancellative semigroup is cancellative. Hence not every separative semigroup is quasi-cancellative. From here it follows that all the vertical implications are strict.

Every completely simple semigroup is weakly cancellative, but not separative (if it is not a group). Hence the left horizontal implications are strict.

Bicyclic semigroup \( B = \langle a, b \mid ba = 1 \rangle \) is quasi-separative. Since \( B \) is simple, it cannot be decomposed into a nontrivial semilattice of its sub-semigroups. By Theorem 3 it is quasi-cancellative. On the other hand, the equalities

\[ b^2 \cdot 1 = b^2 \cdot ab, \quad 1 \cdot a = ab \cdot a \]

imply that \( B \) is not weakly balanced. From here it follows that the right horizontal implications are strict.
References

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