ON KENMOTSU MANIFOLDS WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The aim of the present paper is to study the properties of locally and globally $\phi$-concircularly symmetric Kenmotsu manifolds endowed with a semi-symmetric metric connection. First, we will prove that the locally $\phi$-symmetric and the globally $\phi$-concircularly symmetric Kenmotsu manifolds are equivalent. Next, we will study three dimensional locally $\phi$-symmetric, locally $\phi$-concircularly symmetric and locally $\phi$-concircularly recurrent Kenmotsu manifolds with respect to such connection and will obtain some geometrical results. In the end, we will construct a non-trivial example of Kenmotsu manifold admitting a semi-symmetric metric connection and validate our results.

Keywords: Kenmotsu manifolds, $\phi$-symmetric manifolds, $\eta$-parallel Ricci tensor, semi-symmetric metric connection, concircular curvature tensor.

1. Introduction

The product of an almost contact manifold $M$ and the real line $\mathbb{R}$ carries a natural almost complex structure. However, if one takes $M$ to be an almost contact metric manifold and suppose that the product metric $G$ on $M \times \mathbb{R}$ is Kähler, then the structure on $M$ is cosymplectic [19] and not Sasakian. On the other hand, Oubina [25] pointed that if the conformally related metric $e^{2t}G$, $t$ being the coordinates on $\mathbb{R}$ is Kähler, then $M$ is Sasakian and vice versa.

In [34], Tanno classified almost contact metric manifolds whose automorphism group possesses the maximum dimension. For such manifold $M$, the sectional curvature of the plane section containing $\xi$ is constant, say $c$. If $c > =, and < 0$, then $M$ is said be a homogeneous Sasakian manifold of constant sectional curvature, product of a line or a circle with Kähler manifold of constant holomorphic sectional curvature, and warped product space $\mathbb{R} \times_f C^n$, respectively. In 1972, Kenmotsu [23] characterized the geometrical properties of the manifold when $c < 0$, called Kenmotsu manifold. The geometrical properties of this manifold have been studied
by many geometers, for instance (see, [3], [7]-[11], [15], [16], [22], [26], [33], [36], [40], [41]).

In general, a geodesic circle (a curve whose first curvature is constant and second curvature is identically zero) does not transform into a geodesic circle by the conformal transformation

\[ \tilde{g}^{ij} = \psi^2 g^{ij} \]

of the fundamental tensor \( g^{ij} \). A transformation which preserves the geodesic circle was first introduced by Yano [37]. The conformal transformation (1.1) satisfying the partial differential equation

\[ \psi_{;ij} = \phi g_{ij} \]

change a geodesic circle into a geodesic circle. Such transformation is known as the concircular transformation and the geometry which leads with such transformation is known as the concircular geometry [37].

A (1, 3) type tensor \( C \) which remains invariant under the transformation (1.1), for an \( n \)-dimensional Riemannian manifold \( M \), given by

\[ C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)} [g(Y,Z)X - g(X,Z)Y] \]

for all vector fields \( X, Y \) and \( Z \) on \( M \) is known as a concircular curvature tensor [37], where \( R, r, \) and \( \nabla \) are the Riemannian curvature tensor, the scalar curvature, and the Levi-Civita connection, respectively. In view of (1.2), it is obvious that

\[ (\nabla_W C)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{dW}{n(n-1)} [g(Y,Z)X - g(X,Z)Y] \]

The importance of the concircular transformation and the concircular curvature tensor are well known in the differential geometry of \( F \)-structures such as complex, almost complex, Kähler, almost Kähler, contact and almost contact structures ([37], [6], [35]). In a recent paper, Ahsan and Siddiqui [1] have studied the application of concircular curvature in general relativity and cosmology.

Let \( (M,g) \) be a Riemannian manifold of dimension \( n \). A linear connection \( \tilde{\nabla} \) on \( (M,g) \), whose torsion tensor \( \tilde{T} \) of type (1, 2) is defined by

\[ \tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \]

For arbitrary vector fields \( X \) and \( Y \) on \( M \) are said to be torsion free or symmetric if \( \tilde{T} \) vanishes, otherwise it is non-symmetric. If the connection \( \nabla \) satisfies \( \nabla g = 0 \) on \( (M,g) \), then it is called metric connection, otherwise it is non-metric. In [17], Friedmann and Schouten introduced the notion of semi-symmetric linear connection on a differentiable manifold. Hayden [18] introduced the idea of semi-symmetric linear connection with non-zero torsion on a Riemannian manifold. The systematic study of the semi-symmetric metric connection on the Riemannian manifold was
introduced by Yano [38]. He proved that a Riemannian manifold endowed with a semi-symmetric metric connection has vanishing curvature tensor with respect to the semi-symmetric metric connection if and only if it is conformally flat. This result was generalized for vanishing Ricci tensor of the semi-symmetric metric connection by T. Imai ([20], [21]). Various geometrical and physical properties of this connection have been studied by many authors among whom are ([2]-[4], [12]-[14], [27]-[31], [39]). Motivated by the above studies, the authors will continue to study the properties of the Kenmotsu manifolds equipped with a semi-symmetric metric connection. The present paper is organized in the following manner:

After the introduction in Section 1, we will notify you on the basic results of the Kenmotsu manifolds and the semi-symmetric metric connection in Section 2 and Section 3, respectively. In section 4, we will start the study of globally $\phi$-conicircularly symmetric Kenmotsu manifold and prove that the manifold is $\eta$-Einstein as well as locally $\phi$-symmetric. The following sections deal with the study of locally $\phi$-symmetric, locally $\phi$-conicircularly symmetric, Ricci semisymmetric, $\eta$-parallel Ricci tensor and locally $\phi$-conicircularly recurrent Kenmotsu manifolds equipped with a semi-symmetric metric connection. In the last section, we will construct an example of three dimensional Kenmotsu manifold admitting a semi-symmetric metric connection to verify some results of our paper.

2. Preliminaries

Let $M$ be an $n(=2m+1)$-dimensional connected almost contact metric manifold with an almost contact structure $(\phi, \xi, \eta, g)$, that is, $M$ admits a $(1, 1)$-type tensor field $\phi$, a $(1, 0)$-type vector field $\xi$, a 1-form $\eta$, and a compatible Riemannian metric $g$ satisfies

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X)$$

for all $X, Y \in T(M)$, where $T(M)$ denotes the tangent space of $M$ [5]. If an almost contact metric manifold $M$ satisfies

$$\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X$$

for all $X, Y \in T(M)$, then $M$ is called a Kenmotsu manifold [23]. From (2.1)-(2.3), it can be easily prove that

$$\nabla_X \xi = X - \eta(X)\xi$$

and

$$\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in T(M)$. Let $S$ denote the Ricci tensor of $M$. It is noticed that $M$ satisfies the following relations.

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$
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\[(2.7)\] \[R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi\]

and

\[(2.8)\] \[S(X, \xi) = -(n - 1)\eta(X)\]

for all \(X, Y \in T(M)\). The curvature tensor \(R\) in a 3-dimensional Kenmotsu manifold \(M\) assumes the form

\[R(X, Y)Z = \left(\frac{r + 4}{2}\right) [g(Y, Z)X - g(X, Z)Y] - \left(\frac{r + 6}{2}\right) [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi]\]

After contracting \(X\) it becomes

\[S(Y, Z) = \frac{1}{2} [(r + 2)g(Y, Z) - (r + 6)\eta(X)\eta(Y)]\]

for all \(X, Y \in T(M)\).

An \(n\)-dimensional Kenmotsu manifold \((M, g)\) is said to be an \(\eta\)-Einstein manifold if its non-vanishing Ricci-tensor \(S\) takes the form

\[(2.11)\] \[S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)\]

for all \(X, Y \in T(M)\), where \(a\) and \(b\) are smooth functions on \((M, g)\). If \(b = 0\) and \(a\) is constant, then \(\eta\)-Einstein manifold becomes Einstein manifold. Kenmotsu [23] proved that if \((M, g)\) is an \(n\)-dimensional \(\eta\)-Einstein manifold, then \(a + b = -(n - 1)\).

3. Semi-symmetric metric connection on Kenmotsu manifold

Let \(M\) be an \(n\)-dimensional Kenmotsu manifold endowed with a Riemannian metric \(g\). A linear connection \(\tilde{\nabla}\) on \((M, g)\) is said to be a semi-symmetric metric connection [38] if the torsion tensor \(\tilde{T}\) of the connection \(\tilde{\nabla}\) and the Riemannian metric \(g\) satisfies

\[(3.1)\] \[\tilde{T}(X, Y) = \eta(Y)X - \eta(X)Y\]

and

\[(3.2)\] \[\tilde{\nabla}g = 0\]

for all \(X, Y \in T(M)\). The Levi-Civita connection \(\nabla\) and the semi-symmetric metric connection \(\tilde{\nabla}\) on \((M, g)\) are connected by

\[(3.3)\] \[\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi\]

for all \(X, Y \in T(M)\) [38]. From (2.1), (2.2) and (3.3), it follows that

\[(3.4)\] \[(\tilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - \eta(X)\eta(Y) + g(X, Y).\]
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The curvature tensors $R$ and $\tilde{R}$ with respect to $\nabla$ and $\tilde{\nabla}$, respectively, are connected by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X + g(X,Z)AY - g(Y,Z)AX,$$

where $\alpha$ is a tensor field of type $(0,2)$ and $A$, a tensor field of type $(1,1)$, are related by

$$\alpha(Y,Z) = g(AY,Z) = (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}g(Y,Z),$$

for all $X, Y, Z \in T(M)$ [38]. From (2.1), (2.5), (3.5) and (3.6), it follows that

$$\tilde{R}(X,Y)Z = R(X,Y)Z - 3g(Y,Z)X + 3g(X,Z)Y + 2\eta(Y)\eta(Z)X - 2\eta(X)g(Y,Z)\xi - 2\eta(Y)g(X,Z)\xi.$$  

Contracting (3.7) along $X$, we get

$$\tilde{S}(Y,Z) = S(Y,Z) - (3n - 5)g(Y,Z) + 2(n - 2)\eta(Y)\eta(Z),$$

which becomes

$$\tilde{r} = r - n(3n - 7) - 4.$$ 

Here $\tilde{S}$ and $\tilde{r}$ denote the Ricci tensor and the scalar curvature with respect to the connection $\tilde{\nabla}$. Replacing $Z$ by $\xi$ in (3.8) and using (2.8), we have

$$\tilde{S}(Y,\xi) = -2(n - 1)g(Y,\xi).$$

Thus we can state:

**Proposition 3.1.** Let $M$ be an $n$-dimensional, $n \geq 3$, Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. Then $\xi$ is an eigen vector of $\tilde{S}$ corresponding to the eigenvalue $-2(n - 1)$.

4. **Globally $\phi$-concircularly symmetric Kenmotsu manifold with a semi-symmetric metric connection**

In this section, we will study the properties of the globally $\phi$-concircularly symmetric Kenmotsu manifold equipped with a semi-symmetric metric connection $\nabla$ and prove our result in the form of theorems.

**Definition 4.1.** A Kenmotsu manifold $M$ of dimension $n$ is said to be locally $\phi$-symmetric with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if the non-vanishing curvature tensor $\tilde{R}$ satisfies the relation

$$\phi^2(\tilde{\nabla}_W \tilde{R})(X,Y)Z = 0$$

for all vector fields $X$, $Y$, $Z$ and $W$ orthogonal to $\xi$. 
This notion was introduced by Takahashi [32] for Sasakian manifold.

**Definition 4.2.** An $n$-dimensional Kenmotsu manifold $M$ is said to be a globally $\phi$-concircularly symmetric manifold with respect to $\nabla$ if the non-zero concircular curvature tensor $C$ satisfies

(4.1) $\phi^2((\nabla_W C)(X,Y)Z) = 0$

for all vector fields $X, Y, Z, W \in T(M)$.

**Definition 4.3.** An $n$-dimensional Kenmotsu manifold $M$ equipped with the semi-symmetric metric connection $\tilde{\nabla}$ is said to be a globally $\phi$-concircularly symmetric Kenmotsu manifold with respect to $\tilde{\nabla}$ if the non-vanishing concircular curvature tensor $\tilde{C}$ with respect to $\tilde{\nabla}$ satisfies

(4.2) $\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = 0$

for arbitrary vector fields $X, Y, Z$ and $W$. Here $\tilde{C}$ is a concircular curvature tensor [37] with respect to $\tilde{\nabla}$ and is defined by

(4.3) $\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)} [g(Y,Z)X - g(X,Z)Y]$.

**Theorem 4.1.** An $n$-dimensional, $n \geq 3$, globally $\phi$-concircularly symmetric Kenmotsu manifold $M$ equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is an $\eta$-Einstein manifold.

**Proof.** We suppose that $M$ is a globally $\phi$-concircularly symmetric Kenmotsu manifold with respect to a semi-symmetric metric connection $\tilde{\nabla}$. Then we have

$\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = 0$.

In view of (2.1), the above equation becomes

$-(\tilde{\nabla}_W \tilde{C})(X,Y)Z + \eta((\tilde{\nabla}_W \tilde{C})(X,Y)Z)\xi = 0$.

Equation (1.3) along with above equation give

$-g((\tilde{\nabla}_W \tilde{R})(X,Y)Z, U) + \frac{d\tilde{r}(W)}{n(n-1)} [g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]$

$+ \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\eta(U) - \frac{d\tilde{r}(W)}{n(n-1)} [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(U) = 0$.

Replacing $X = U = e_i$, where $\{e_i, i = 1, 2, 3, ..., n\}$, be an orthonormal basis of the tangent space at each point of the manifold $M$ and then summing over $i, 1 \leq i \leq n$, we get

$-(\tilde{\nabla}_W \tilde{S})(Y,Z) + \frac{d\tilde{r}(W)}{n} g(Y,Z) + \eta((\tilde{\nabla}_W \tilde{R})(\xi,Y)Z)$

$- \frac{d\tilde{r}(W)}{n(n-1)} [g(Y,Z) - \eta(Y)\eta(Z)] = 0$. 
Putting $Z = \xi$ in the above equation and using (2.1), we get

\[(4.4) \quad -(\tilde{\nabla}_W \tilde{S})(Y, \xi) + \frac{d\tilde{r}(W)}{n} \eta(Y) + \eta((\tilde{\nabla}_W \tilde{R})(\xi, Y)\xi) = 0.\]

In view of (2.1), (2.2), (2.4), (2.6), (2.7), (3.3) and (3.7), we conclude that

\[\eta((\tilde{\nabla}_W \tilde{R})(\xi, Y)\xi) = 0\]

and hence the equation (4.4) becomes

\[(4.5) \quad (\tilde{\nabla}_W \tilde{S})(Y, \xi) = \frac{d\tilde{r}(W)}{n} \eta(Y).\]

Substituting $Y = \xi$ in (4.5) and using (2.1) and (2.8), we get $d\tilde{r}(W) = 0$. This implies that $\tilde{r}$ is a constant. So from (4.5), we obtain

\[(4.6) \quad (\tilde{\nabla}_W \tilde{S})(Y, \xi) = 0.\]

It is well known that

\[(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi).\]

In view of (2.1), (2.2), (2.4), (2.5), (2.8), (3.3), (3.4), (3.10) and (4.6), above equation takes the form

\[S(Y, W) = (n - 3)g(Y, W) - 2(n - 2)\eta(Y)\eta(W).\]

Hence the statement of the Theorem 4.1 is proved. \(\square\)

From the above equation, it is clear that $r = (n - 1)(n - 4)$. Hence the scalar curvature under consideration is constant. Thus we have

**Corollary 4.1.** An $n$-dimensional, $n > 3$, globally $\phi$-concircularly symmetric Kenmotsu manifold $M$ equipped with a semi-symmetric metric connection $\tilde{\nabla}$ possesses a constant scalar curvature.

**Theorem 4.2.** Let $M$ be an $n$-dimensional, $n \geq 3$, Kenmotsu manifold admits a semi-symmetric metric connection $\tilde{\nabla}$. Then the globally $\phi$-concircularly symmetric manifold and the locally $\phi$-symmetric manifold with respect to $\tilde{\nabla}$ coincide.

**Proof.** We suppose that the manifold $M$ is globally $\phi$-concircularly symmetric with respect to a semi-symmetric metric connection $\tilde{\nabla}$. Since $r$ is constant on $M$ and therefore $\tilde{r}$ is also constant. The covariant derivative of (4.3) gives

\[(4.7) \quad (\tilde{\nabla}_W \tilde{C})(X, Y)Z = (\tilde{\nabla}_W \tilde{R})(X, Y)Z.\]
In view of (3.3), (3.4) and (3.7), we get

\[
(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\tilde{\nabla}_W R)(X,Y)Z + 4\{\nabla W \eta(Y) - \eta(Y)\eta(W) + g(Y,W)\} \eta(Z)X
\]
\[+ 2\{\nabla W \eta(X) - \eta(X)\eta(W) + g(X,W)\} \eta(Y)Z \]
\[+ 2\{\nabla W \eta(Z) - \eta(Z)\eta(W) + g(Z,W)\} \eta(X)Y\]
\[+ 2\{\nabla W \eta(X) - \eta(X)\eta(W) + g(X,W)\} \eta(Y)X \]
\[+ 2\{\nabla W \eta(Z) - \eta(Z)\eta(W) + g(Z,W)\} \eta(X)Y\]
\[+ 2\{\nabla W \eta(Z) - \eta(Z)\eta(W) + g(Z,W)\} \eta(Y)X\]
\[+ 2\{\nabla W \eta(X) - \eta(X)\eta(W) + g(X,W)\} \eta(Y)Z\]
\[+ 2\{\nabla W \eta(W) - \eta(W)\eta(X) + \eta(Y)g(X,Z)\}\]
\[+ 4\{\nabla W \eta(Y) + \eta(Y)\eta(W) - g(Y,W)\}\xi.
\]

(4.8)

Using (2.4) and (2.5) in (4.8), we obtain

\[
(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\tilde{\nabla}_W R)(X,Y)Z + 4\{\nabla W \eta(Y) - \eta(Y)\eta(W) + g(Y,W)\} \eta(Z)X
\]
\[+ 4\{\nabla W \eta(X) - \eta(X)\eta(W) + g(X,W)\} \eta(Y)Z \]
\[+ 4\{\nabla W \eta(Z) - \eta(Z)\eta(W) + g(Z,W)\} \eta(X)Y\]
\[+ 4g(Y,Z)\{\nabla W \eta(W) - \eta(W)\eta(X) + \eta(Y)g(X,Z)\}\]
\[+ 4\{\nabla W \eta(Y) + \eta(Y)\eta(W) - g(Y,W)\}\xi.
\]

(4.9)

If X, Y, Z and W are orthogonal to ξ then from above equation, we get

\[
(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\tilde{\nabla}_W R)(X,Y)Z + 4g(Y,Z)g(X,W)\xi.
\]

In view of (4.7) and (4.10), we have

\[
(\tilde{\nabla}_W \tilde{C})(X,Y)Z = (\tilde{\nabla}_W R)(X,Y)Z + 4g(Y,Z)g(X,W)\xi.
\]

Operating $\phi^2$ on either sides of the above equation and then using (2.1) we get

\[
\phi^2(\tilde{\nabla}_W \tilde{C})(X,Y)Z = \phi^2(\tilde{\nabla}_W R)(X,Y)Z
\]

for all vector fields X, Y, Z and W orthogonal to ξ. From the equations (4.7) and (4.9), it is clear that the equation (4.11) satisfies for all vector fields X, Y, Z and W on $\mathcal{M}$. Hence the statement of the Theorem 4.2 is proved.

Remark 4.1. The last equation shows that a locally $\phi$-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is always globally $\phi$-concurrently symmetric manifold. Thus we conclude that on a Kenmotsu manifold locally $\phi$-symmetric and globally $\phi$-symmetric manifolds are equivalent corresponding to the connection $\tilde{\nabla}$.

5. Three dimensional locally $\phi$-symmetric Kenmotsu manifolds with respect to the semi-symmetric metric connection

This section deals with the study of the locally $\phi$-symmetric Kenmotsu manifold $\mathcal{M}$ with respect to a semi-symmetric metric connection $\tilde{\nabla}$. Now, we will consider a 3-dimensional locally $\phi$-symmetric Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ and prove the following:
Theorem 5.1. A 3-dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is locally $\partial$-symmetric with respect to the connection $\tilde{\nabla}$ if and only if $dr(W) = 0$, $W$ is an orthonormal vector field to $\xi$.

Proof. From (2.9) and (3.7), we get

$$R(X, Y)Z = \left(\frac{r-2}{2}\right)\{g(Y, Z)X - g(X, Z)Y\}$$

$$+ \left(\frac{r+2}{2}\right)\left[\eta(Y)g(X, Z)\xi + \eta(X)\eta(Z)Y\right] - \eta(X)g(Y, Z)\xi - \eta(Y)\eta(Z)X.$$ (5.1)

Taking covariant differentiation of (5.1) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along $W$, we have

$$(\tilde{\nabla}_W R)(X, Y)Z = \frac{dr(W)}{2}\left\{g(Y, Z)X - g(X, Z)Y - \eta(X)\eta(Z)Y\right\}$$

$$+ \left\{-g(X, Z)\eta(Y) + g(Y, Z)\eta(X)\right\}\xi + \left\{\eta(Y)\eta(Z)\right\}X$$

$$+ \left(\frac{r+2}{2}\right)\left\{g(X, Z)(\tilde{\nabla}_W \eta)(Y)\xi + g(X, Z)\eta(Y)(\tilde{\nabla}_W \eta)Y\right\} - g(Y, Z)(\tilde{\nabla}_W \eta)(X)\xi - g(Y, Z)\eta(X)\tilde{\nabla}_W \eta\xi$$

$$+ \eta(Z)(\tilde{\nabla}_W \eta)(X)Y + \eta(X)(\tilde{\nabla}_W \eta)(Z)Y$$

$$- \eta(Z)(\tilde{\nabla}_W \eta)(Y)X - \eta(Y)(\tilde{\nabla}_W \eta)(Z)X].$$ (5.2)

In consequence of (3.3) and (3.4), (5.2) becomes

$$(\tilde{\nabla}_W R)(X, Y)Z = \frac{dr(W)}{2}\left\{g(Y, Z)X - g(X, Z)Y - g(X, Z)\eta(Y)\xi\right\} - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi + \eta(Y)\eta(Z)X$$

$$+ \left(\frac{r+2}{2}\right)\left\{-\eta(X)g(Y, Z)\{\nabla_W \xi + W - \eta(W)\xi\} - g(Y, Z)\{(\nabla_W \eta)(X) - \eta(X)\eta(W) + g(X, W)\}\xi$$

$$+ \eta(Z)\{(\nabla_W \eta)(X) - \eta(X)\eta(W) + g(X, W)\}Y + \eta(X)\{(\nabla_W \eta)(Z) - \eta(W)\eta(Z) + g(Z, W)\}Y$$

$$- \eta(Z)\{(\nabla_W \eta)(Y) - \eta(Y)\eta(W) + g(Y, W)\}X - \eta(Y)\{(\nabla_W \eta)(Z) - \eta(Z)\eta(W) + g(Z, W)\}X$$

$$+ g(X, Z)\{(\nabla_W \eta)(Y) - \eta(Y)\eta(W) + g(Y, W)\}\xi$$

$$+ g(X, Z)\eta(Y)\{(\nabla_W \xi + W - \eta(W)\xi\}].$$ (5.3)

Let us suppose that the vector fields $X, Y, Z$ and $W$ are orthogonal to $\xi$, therefore
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(5.3) becomes
\[
(\nabla_W \tilde{R})(X,Y)Z = \frac{dr(W)}{2} \{g(Y,Z)X - g(X,Z)Y\} \\
+ \left( \frac{r + 2}{2} \right) [g(X,Z) \{(\nabla_W \eta)(Y) + g(Y,W)\} \\
- g(Y,Z) \{(\nabla_W \eta)(X) + g(X,W)\}] \xi.
\]

Operating $\phi^2$ on both sides of (5.4) and then using (2.1) and (2.2), we obtain
\[
(5.5) \quad \phi^2((\nabla_W \tilde{R})(X,Y)Z) = -\frac{dr(W)}{2} \{g(Y,Z)X - g(X,Z)Y\}.
\]

From the equation (5.5), it is obvious that the manifold $M$ is locally $\phi$-symmetric Kenmotsu manifold with respect to $\nabla$ if and only if $dr(W) = 0$. Hence the statement of the Theorem 5.1 is proved.

6. Three dimensional Locally $\phi$-concircularly symmetric Kenmotsu manifold with a semi-symmetric metric connection

Definition 6.1. A Kenmotsu manifold $M$ is said to be locally $\phi$-concircularly symmetric with respect to the semi-symmetric metric connection $\nabla$ if its concircular curvature tensor $\tilde{C}$ satisfies
\[
\phi^2((\nabla_W \tilde{C})(X,Y)Z) = 0
\]
for all vector fields $W$, $X$, $Y$ and $Z$ orthogonal to $\xi$.

Theorem 6.1. A 3-dimensional Kenmotsu manifold $M$ with respect to the semi-symmetric metric connection $\nabla$ is locally $\phi$-concircularly symmetric manifold with respect to the connection $\nabla$ if and only if the scalar curvature $r$ is constant.

Proof. From (2.9) and (3.7), it follows that
\[
\tilde{R}(X,Y)Z = \left( \frac{r - 2}{2} \right) \{g(Y,Z)X - g(X,Z)Y\} \\
+ \left( \frac{r + 2}{2} \right) [\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
+ \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\} \xi].
\]

In view of (1.2) and (6.1), we get
\[
\tilde{C}(X,Y)Z = \left( \frac{r - 2}{2} \right) \{g(Y,Z)X - g(X,Z)Y\} \\
+ \left( \frac{r + 2}{2} \right) [\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
+ \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\} \xi] \\
+ \frac{r}{6} \{g(Y,Z)X - g(X,Z)Y\}.
\]
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Taking covariant derivative of (6.2) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along $W$, we have

\[
(\tilde{\nabla}_W \tilde{C})(X,Y)Z = \frac{dr(W)}{2} [g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y \\
+ \{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\} \xi - \eta(Y)\eta(Z)X] \\
+ \left(\frac{r+2}{2}\right) [g(X,Z)(\tilde{\nabla}_W \eta)(Y)\xi + g(X,Z)\eta(Y)\tilde{\nabla}_W \xi \\
- g(Y,Z)(\tilde{\nabla}_W \eta)(X)\xi - g(Y,Z)\eta(X)\tilde{\nabla}_W \xi + \eta(X)(\tilde{\nabla}_W \eta)(Z)Y \\
- \eta(Z)(\tilde{\nabla}_W \eta)(Y)X - \eta(Y)(\tilde{\nabla}_W \eta)(Z)X + \eta(Z)(\tilde{\nabla}_W \eta)(Y)Y] \\
+ \frac{dr(W)}{6} \{g(Y,Z)X - g(X,Z)Y\}. \tag{6.3}
\]

Let us consider that the vector fields $X$, $Y$ and $Z$ are orthonormal to $\xi$ and therefore (6.3) converts into the form

\[
(\tilde{\nabla}_W \tilde{C})(X,Y)Z = \frac{dr(W)}{2} \{g(Y,Z)X - g(X,Z)Y] \\
+ \left(\frac{r+2}{2}\right) \{g(X,Z)(\tilde{\nabla}_W \eta)(Y) - g(Y,Z)(\tilde{\nabla}_W \eta)(X)\} \xi \\
+ \frac{dr(W)}{6} \{g(Y,Z)X - g(X,Z)Y\}. \tag{6.4}
\]

Using (3.4) in (6.4), we obtain

\[
(\tilde{\nabla}_W \tilde{C})(X,Y)Z = \frac{2dr(W)}{3} \{g(Y,Z)X - g(X,Z)Y\} + \left(\frac{r+2}{2}\right) [g(X,Z)(\tilde{\nabla}_W \eta)(Y) \\
- g(X,Z)\eta(Y)\eta(W) - g(Y,Z)(\tilde{\nabla}_W \eta)(X) + g(Y,W)g(X,Z) \\
- g(Y,Z)g(X,W) + g(Y,Z)\eta(X)\eta(W)] \xi. \tag{6.5}
\]

Applying $\phi^2$ on both sides of (6.5) and using (2.1), we get

\[
\phi^2 \left(\tilde{\nabla}_W \tilde{C})(X,Y)Z\right) = \frac{2dr(W)}{3} \{g(Y,Z)X - g(X,Z)Y\}. \tag{6.6}
\]

This proved the statement of the Theorem 6.1. \qed

From the Theorem 5.1 and the Theorem 6.1, we can state the following:

**Corollary 6.1.** A 3-dimensional Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is locally $\phi$-concircularly symmetric with respect to the connection $\tilde{\nabla}$ if and only if it is locally $\phi$-symmetric with respect to $\tilde{\nabla}$.

[7]
Theorem 7.1. A 3-dimensional Ricci semisymmetric Kenmotsu manifold with respect to a semi-symmetric metric connection $\tilde{\nabla}$ possesses a constant scalar curvature.

Proof. Let us consider a 3-dimensional Kenmotsu manifold $M$ equipped with a semi-symmetric metric connection $\tilde{\nabla}$ which satisfies $\tilde{R}(X,Y) \cdot \tilde{S} = 0$, that is, $M$ is Ricci semisymmetric with respect to $\tilde{\nabla}$ and then we have

$$\tilde{S}(\tilde{R}(X,Y)Z,W) + \tilde{S}(Z,\tilde{R}(X,Y)W) = 0.$$  \hfill (7.1)

Replacing $X$ by $\xi$ in (7.1), we get

$$\tilde{S}(\tilde{R}(\xi,Y)Z,W) + \tilde{S}(Z,\tilde{R}(\xi,Y)W) = 0.$$  \hfill (7.2)

From (5.1), it is obvious that

$$\tilde{R}(\xi,Y)Z = -2\{g(Y,Z)\xi - \eta(Z)Y\}.$$  \hfill (7.3)

By virtue of (3.10), (7.2) and (7.3), we obtain

$$\eta(Z)\tilde{S}(Y,W) + 4\eta(W)g(Y,Z) + \eta(W)\tilde{S}(Z,Y) + 4\eta(Z)g(Y,W) = 0.$$  \hfill (7.4)

Let $\{e_i\}, i = 1, 2, 3$, is an orthonormal basis of the tangent space at each point of the manifold $M$. Putting $Y = Z = e_i$ in (7.4) and taking summation over $i$, $1 \leq i \leq 3$, we get

$$\tilde{r} + 12\eta(W) = 0.$$  \hfill (\tilde{r} + 12)\eta(W) = 0.

Since $\eta(W) \neq 0$, in general, therefore $\tilde{r} = -12$ (constant). This proved the statement of the Theorem 7.1. \hfill $\Box$

In consequence of the Theorem 6.1 and Theorem 7.1, we state:

Corollary 7.1. If a 3-dimensional Kenmotsu manifold $M$ with respect to a semi-symmetric metric connection $\tilde{\nabla}$ satisfies the condition $\tilde{R}(X,Y) \cdot \tilde{S} = 0$, then $M$ is locally $\phi$-symmetric as well as locally $\phi$-concircularly symmetric with respect to $\tilde{\nabla}$, respectively.

8. $\eta$-parallel Ricci tensor with respect to the semi-symmetric metric connection
A Ricci tensor $\tilde{S}$ of a Kenmotsu manifold $M$ equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is called $\eta$-parallel with respect to $\tilde{\nabla}$ if it is non-zero and satisfies

$$(\tilde{\nabla}_X \tilde{S})(\phi Y, \phi Z) = 0$$

for all vector fields $X$, $Y$ and $Z$ on $M$.

The notion of $\eta$-parallel Ricci tensor on a Sasakian manifold was introduced by M. Kon [24]. Since then, many authors studied the geometrical and physical properties of this tensor.

**Theorem 8.1.** If a 3-dimensional Kenmotsu manifold $M$ with respect to the semi-symmetric metric connection $\tilde{\nabla}$ possesses an $\eta$-parallel Ricci tensor, then the scalar curvature of $M$ is constant.

**Proof.** In view of (2.2), (2.9) and (3.8), we have

$$(\tilde{\nabla}_W \tilde{S})(\phi X, \phi Y) = \left(\tilde{\nabla}_W \eta\right)(X, \phi Y) - \tilde{\nabla}_W(\tilde{\nabla}_Y \eta)(X) - \tilde{\nabla}_Y(\tilde{\nabla}_W \eta)(X).$$

In view of (2.1), (2.3), (2.5), (3.3), (3.4), (8.1) and (8.3), it can be easily found that

$$d\tilde{\nabla}(W) \{g(X, Y) - \eta(X)\eta(Y)\} = 0,$$

which gives

$$d\tilde{\nabla}(W) = 0 \iff \tilde{\nabla} is constant.$$

Hence the statement of the Theorem 8.1 is proved.

In the light of the Theorem 6.1 and Theorem 8.1, we state the following corollary.

**Corollary 8.1.** If a 3-dimensional Kenmotsu manifold $M$ equipped with a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta$-parallel Ricci tensor, then the manifold is locally $\phi$-symmetric as well as locally $\phi$-concircularly symmetric with respect to $\tilde{\nabla}$, respectively.
9. Three dimensional locally $\phi$-concircularly recurrent Kenmotsu manifold with respect to the semi-symmetric metric connection

**Definition 9.1.** A Kenmotsu manifold $M$ equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is said to be $\phi$-concircularly recurrent with respect to $\tilde{\nabla}$ if there exists a non-zero 1-form $A$ on $M$ such that

$$(9.1) \quad \phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = A(W)\tilde{C}(X,Y)Z$$

for arbitrary vector fields $X$, $Y$, $Z$ and $W$, where $\tilde{C}$ is the concircular curvature tensor with respect to the semi-symmetric metric connection $\tilde{\nabla}$. If the 1-form $A$ vanishes identically on $M$, then the manifold $M$ with $\tilde{\nabla}$ is reduced to a locally $\phi$-concircularly symmetric manifold with respect to $\tilde{\nabla}$.

**Theorem 9.1.** If a 3-dimensional locally $\phi$-concircularly recurrent Kenmotsu manifold admits a semi-symmetric metric connection $\tilde{\nabla}$, then the curvature tensor with respect to $\tilde{\nabla}$ assumes the form (9.7).

**Proof.** From (3.9) and (5.5), we have

$$(9.2) \quad \phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = -\frac{d\tilde{r}(W)}{2} \{g(Y,Z)X - g(X,Z)Y\}.$$ 

On the other hand, from (1.3), it is seen that (for $n=3$)

$$(9.3) \quad (\tilde{\nabla}_W \tilde{C})(X,Y)Z = (\tilde{\nabla}_W \tilde{R})(X,Y)Z - \frac{d\tilde{r}(W)}{6} \{g(Y,Z)X - g(X,Z)Y\}.$$ 

Applying $\phi^2$ on both sides of (9.3), we get

$$(9.4) \quad \phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = \phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) - \frac{d\tilde{r}(W)}{6} \{g(Y,Z)\phi^2X - g(X,Z)\phi^2Y\}.$$ 

In consequence of (2.1), (9.1) and (9.2), it is obvious that

$$(9.5) \quad A(W)\tilde{C}(X,Y)Z = -\frac{d\tilde{r}(W)}{3} \{g(Y,Z)X - g(X,Z)Y\} - \frac{d\tilde{r}(W)}{6} \{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \xi.$$ 

Replacing $W$ with $\xi$ in (9.5), we get

$$(9.6) \quad \tilde{C}(X,Y)Z = -\frac{d\tilde{r}(\xi)}{3A(\xi)} \{g(Y,Z)X - g(X,Z)Y\} - \frac{d\tilde{r}(\xi)}{6A(\xi)} \{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \xi,$$

provided $A(\xi) \neq 0$. In view of (1.2) and (9.6), we have

$$(9.7) \quad \hat{\tilde{R}}(X,Y)Z = a \{g(Y,Z)X - g(X,Z)Y\} - b \{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \xi,$$

where $a = \left\{ \frac{\tilde{r}}{6} - \frac{d\tilde{r}(\xi)}{3A(\xi)} \right\}$, $b = \frac{d\tilde{r}(\xi)}{6A(\xi)}$ and $A$ is a non-zero 1-form. □
10. Example of a Kenmotsu manifold admitting a semi-symmetric metric connection

In this section, we will construct a non-trivial example of a Kenmotsu manifold admitting the semi-symmetric metric connection and after that we will validate our results.

Example 10.1. Let
\[ M = \{(x, y, z) \in \mathbb{R}^3 : x, y, z(\neq 0) \in \mathbb{R}\}, \]
be a three dimensional Riemannian manifold, where \((x, y, z)\) denotes the standard coordinates of a point in \(\mathbb{R}^3\). Let us suppose that
\[ e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z} \]
be a set of linearly independent vector fields at each point of the manifold \(M\) and therefore it forms a basis for the tangent space \(T(M)\). We also define the Riemannian metric \(g\) of the manifold by
\[ g(e_i, e_j) = \delta_{ij}, \quad \text{where} \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \]

Let us consider a 1-form \(\eta\) defined by
\[ \eta(Z) = g(Z, e_3) \]
for any \(Z \in T(M)\) and a tensor field \(\phi\) of type \((1,1)\) defined by
\[ \phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0. \]

By the linearity properties of \(\phi\) and \(g\), we can easily verify the following relations
\[ \phi^2 X = -X + \eta(X)e_3, \quad \eta(e_3) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \]
for arbitrary vector fields \(X, Y \in T(M)\). This shows that for \(\xi = e_3\), the structure \((\phi, \xi, \eta, g)\) defines an almost contact metric structure on \(M\).

If \(\nabla\) represents the Levi-Civita connection with respect to the Riemannian metric \(g\), then with the help of above relations, we can easily calculate the non-vanishing components of Lie bracket as:
\[ [e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2. \]

We recall the Koszul’s formula
\[ 2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(X, Z) - Z g(X, Y) - g([X, Y], Z) \]
for all vector fields \(X, Y, Z \in T(M)\). It is obvious from Koszul’s formula that
\[ \nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \]

From the above calculations, we can observe that \(\nabla_X \xi = X - \eta(X)\xi\) for \(\xi = e_3\). Thus the manifold \((M, g)\) is a Kenmotsu manifold of dimension 3 and the structure \((\phi, \eta, \xi, g)\) denotes the Kenmotsu structure on the manifold \(M\) [16].
In consequence of (3.3) and the above results, we can find that
\[
\nabla_{e_1} e_1 = -2e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 2e_1,
\]
\[
\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = -2e_3, \quad \nabla_{e_3} e_2 = 2e_2,
\]
\[
\nabla_{e_1} e_3 = 0, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_3} e_3 = 0
\]
and also the components of torsion tensor \(\hat{T}\) are
\[
\hat{T}(e_i, e_i) = \nabla_{e_i} e_i - \nabla_{e_i} e_i - [e_i, e_i] = 0, \quad \text{for } i = 1, 2, 3
\]
\[
\hat{T}(e_1, e_2) = 0, \quad \hat{T}(e_1, e_3) = e_1, \quad \hat{T}(e_2, e_3) = e_2.
\]
This shows that \(\hat{T} \neq 0\) and, therefore, by the equation (3.1), we can say that the linear connection defined in (3.3) is a semi-symmetric connection on \((M, g)\). By straightforward calculation, we can also find
\[
(\nabla_{e_2} g)(e_2, e_3) = 0, \quad (\nabla_{e_2} g)(e_3, e_1) = 0, \quad (\nabla_{e_2} g)(e_1, e_2) = 0
\]
and other components by symmetric properties. This demonstrates that the equation (3.2) is satisfied and hence the linear connection defined by (3.3) is a semi-symmetric metric connection on \(M\). Thus, we can say that the manifold \((M, g)\) is a 3-dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection defined by (3.3).

With the help of the above discussions, we can calculate the curvature and Ricci tensors of \(M\) with respect to the semi-symmetric metric connection \(\nabla\) as
\[
\widehat{R}(e_1, e_2)e_3 = 0, \quad \widehat{R}(e_1, e_3)e_3 = -2e_1, \quad \widehat{R}(e_3, e_2)e_3 = -2e_3,
\]
\[
\widehat{R}(e_3, e_1)e_1 = -2e_3, \quad \widehat{R}(e_2, e_1)e_1 = -4e_2, \quad \widehat{R}(e_2, e_3)e_3 = -2e_2,
\]
\[
\widehat{R}(e_1, e_2)e_2 = 0, \quad \widehat{S}(e_1, e_1) = -6, \quad \widehat{S}(e_2, e_2) = -2, \quad \widehat{S}(e_3, e_3) = -4
\]
and other components can be calculated by skew-symmetric properties. We can easily observe that the equation (3.10) is verified.

Next, we have to prove that the manifold \((M, g)\) is a Ricci semisymmetric with respect to the connection \(\nabla\), i.e., \(\widehat{R} \cdot \widehat{S} = 0\). For instance,
\[
(\widehat{R}(e_2, e_1) \cdot \widehat{S})(e_1, e_1) = 0, \quad (\widehat{R}(e_3, e_2) \cdot \widehat{S})(e_1, e_1) = 0, \quad (\widehat{R}(e_3, e_3) \cdot \widehat{S})(e_1, e_1) = 0
\]
In a similar way, we can verify other components. Also, we can prove that \(\tilde{r} = -12\) (constant) and hence the Theorem 7.1 is verified. Moreover, it can be easily seen that the Theorem 5.1, Theorem 6.1 and the Theorem 8.1 have been verified.

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