SELFCONSISTENT DESCRIPTION OF A THERMAL PION GAS

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Abstract

We examine a hot pion gas by including medium modifications of the two-body scattering amplitude as well as mean-field effects selfconsistently. In contrast to earlier calculations, the in-medium T-matrix is rather close to the free one while the mean-field potential agrees well with lowest-order estimates. We also discuss the validity of the quasiparticle approximation. It is found that it is reliable for temperatures up to $\sim 150$ MeV. Above this temperature off-shell effects in the pion selfenergy become important, especially if the pions are strongly out of chemical equilibrium.

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1 Introduction

In ultrarelativistic heavy ion collisions several hundred particles are produced in the final state. In the midrapidity region at CERN energies of 200 GeV/A most of them are pions (with a pion-to-baryon ratio of $\sim 6 : 1$). The lifetime of the pionic fireball is a few fm/c and the freeze-out volume is typically $10^3 \text{ fm}^3$. The theoretical understanding of such a thermal 'pion gas' is currently of great interest in connection with possible signals from the quark-gluon plasma (QGP) phase transition [1]. On the other hand a hot and dense Bose gas is also interesting from a many-body point of view since one may expect interesting correlations associated with the statistics.

Several aspects of the thermal pion gas have been discussed previously, including mean-field effects [4] and medium modifications of the $\pi\pi$ cross section [4]. It is the purpose of the present paper to give a unified description of these processes by using a reliable model for the vacuum $\pi\pi$ interaction and requiring selfconsistency at the two-body level. This leads to a Brueckner scheme, familiar from the microscopic many-body theory of nuclear matter. Throughout the discussion we shall assume that the gas is in thermal but not necessarily in chemical equilibrium ($\mu_\pi \neq 0$). The latter seems to be required from fits to pion $p_T$-spectra, which yield $\mu_\pi \sim m_\pi$ [4]. The assumption of thermal equilibrium, on the other hand, seems reasonable from the following simple estimate: at 200 GeV/A the freeze-out density $n_\pi$ of pions produced at midrapidity is $\sim 0.3 \text{ fm}^{-3}$. With $N_\pi \approx 400$ this yields a freeze-out radius $R_f \sim 7 \text{ fm}$. Taking an average $\pi\pi$ cross section $\langle \sigma \rangle = 15 \text{ mb}$ gives a mean free path $\lambda = 1/n_\pi \langle \sigma \rangle \sim 2.2 \text{ fm}$ and hence the mean number of collisions is $\sim 3$. Thermalization is further corroborated in a scenario where initially a quark-gluon plasma (QGP) is formed. Here QCD string breaking models as well as parton cascade simulations also yield thermal equilibration which should survive during the hadronization.
2 The Vacuum $\pi\pi$ Interaction

The starting point for our description of the hot pion gas is the vacuum $\pi\pi$ interaction model of ref. [5] which is based on meson exchange. Here the basic meson-meson interaction is constructed from an effective meson Lagrangian with phenomenological form factors at the vertices. From these vertices two-body pseudopotentials are constructed including the most important $s$- and $t$-channel meson exchanges. Among these the $t$-channel $\rho$ exchange between $\pi\pi$ states and the $s$-channel $\rho$ pole term will be most important for our discussion. We employ the Blankenbecler-Sugar (BbS) reduction [6] of the 4-dimensional Bethe-Salpeter equation maintaining covariance [7]. The $\pi\pi T$-matrix for given angular momentum $J$ and isospin $I$ is obtained as

$$T_{\pi\pi}^{JI}(Z, q_1, q_2) = V_{\pi\pi}^{JI}(Z, q_1, q_2) + \int_{0}^{\infty} dk \, k^2 \frac{4\omega_k^2}{\omega_k^2 Z^2 - 4\omega_k^2 + i\eta} \, V_{\pi\pi}^{JI}(Z, q_1, k) \, G_{\pi\pi}^0(Z, k) \, T_{\pi\pi}^{JI}(Z, k, q_2),$$

(1)

where $k = |\vec{k}|$ etc.; $Z$ is the CMS energy and $G_{\pi\pi}^0(Z, k)$ the vacuum two-pion propagator in the CMS frame with pions of momenta $\vec{k}$ and $-\vec{k}$ (the Lohse et al. model also contains coupling to the $K\bar{K}$ channel which has been omitted for brevity in eq. (1); but which is included in the calculation). In the BbS form the two-pion propagator is given by

$$G_{\pi\pi}^0(Z, k) = \frac{1}{\omega_k} \frac{1}{Z^2 - 4\omega_k^2 + i\eta},$$

(2)

with $\omega_k^2 = k^2 + m_{\pi}^2$. This model gives a good description of the phase shifts and inelasticities up to $\sim 1.5$ GeV which is more than sufficient for our purposes.

3 Selfconsistency

The most obvious medium modification of the $\pi\pi$ scattering in the gas surrounding is a change in momentum weight of the intermediate two-pion propagator, first studied in ref. [3]. Identifying the CMS frame with the thermal reference frame one has

$$G_{\pi\pi}(Z, k; \mu_{\pi}, T) = \frac{1}{\omega_k} \frac{1 + 2f_k(\mu_{\pi}, T)}{Z^2 - 4\omega_k^2},$$

(3)

with $f_k$$\mu_{\pi}, T) = \frac{1 + 2f_k(\mu_{\pi}, T)}{Z^2 - 4\omega_k^2}$. This model gives a good description of the phase shifts and inelasticities up to $\sim 1.5$ GeV which is more than sufficient for our purposes.
where $f_k = (\exp[(\omega_k - \mu_\pi)/T] - 1)^{-1}$ is the thermal Bose factor and $\mu_\pi$ the chemical potential. The identification of the CMS frame with the thermal frame simplifies our calculations considerably (allowing for a relative velocity between the two frames we find that it effectively acts like a change in $\mu_\pi$ which is a parameter for us anyway). At fixed $T$ the pion chemical potential fixes the density via

$$n_\pi(\mu_\pi, T) = g_\pi \int \frac{d^3q}{(2\pi)^3} f_\pi(\mu_\pi, T),$$

(4)

where $g_\pi = 3$ is the isospin degeneracy factor. We use a temperature range of 100-200 MeV. A temperature of 100 MeV is roughly the lowest temperature from thermal fits to $p_T$-spectra at the AGS. On the other hand 200 MeV should be an upper limit for purely hadronic models since one expects the phase transition into a QGP around this value. While thermal equilibrium seems a reasonable assumption it is not clear whether chemical equilibration is reached during the evolution of the pion gas. Indeed, fits of the CERN $p_T$-spectra could be improved with $\mu_\pi \sim 125$ MeV [4]. Hence we discuss both cases $\mu_\pi = 0$ and $\mu_\pi = 125$ MeV. At the same temperature the latter gives a higher pion density reaching a maximal density $n_\pi \sim 0.7$ fm$^{-3}$ at 200 MeV.

There is a second effect which needs to be considered [2]. The $\pi\pi$ interaction introduces a pion selfenergy $\Sigma_\pi$ (‘mean field’) which changes the single-pion dispersion relation

$$\omega_k^2 = m_\pi^2 + k^2 + \Sigma_\pi(\omega_k, k; \mu_\pi, T)$$

(5)

as a function of density and temperature. In terms of the forward-scattering amplitude $M_{\pi\pi}$, $\Sigma_\pi$ is expressed as [2, 3]:

$$\Sigma_\pi(\omega, k; \mu_\pi, T) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} f_p(\mu_\pi, T) M_{\pi\pi}(k^{(4)}, p^{(4)}).$$

(6)

Relating the forward-scattering amplitude to the T-Matrix as

$$M_{\pi\pi}(E_{\text{cms}}) = (2\pi)^3 E_{\text{cms}}^2 T_{\pi\pi}(E_{\text{cms}}),$$

(7)

where $E_{\text{cms}} = \sqrt{s} = [(p + k)^\mu (p + k)_\mu]^{1/2}$ is the CMS energy of the two colliding pions,
one can transform the selfenergy expression into

\[
\Sigma_{\pi}(\omega, k; \mu_{\pi}, T) = \frac{\pi}{k} \int_{0}^{\infty} dp \frac{p}{\omega_p} f_p(\mu_{\pi}, T) \int_{E_{\text{min}}}^{E_{\text{max}}} dE_{\text{cms}} E_{\text{cms}}^3 T_{\pi\pi}(E_{\text{cms}}) .
\]  (8)

Here we have restricted ourselves to the on-shell T-Matrix neglecting the dependence on the total momentum \( \vec{P} = \vec{k} + \vec{p} \) of the pair. Thus the energy integration bounds are given as

\[
E_{\text{max/min}} = (\omega^2 + \omega_p^2 + 2\omega \omega_p - k^2 - p^2 \pm 2kp)^{1/2} .
\]  (9)

For the forward-scattering T-Matrix we take the spin-isospin weighted sum including partial waves up to \( J = 2 \):

\[
T_{\pi\pi}(E_{\text{cms}}) = \frac{1}{4\pi} \sum_{I,J=0}^{2} \frac{(2I + 1)}{3} (2J + 1) T_{JI\pi\pi}(E_{\text{cms}}) ,
\]  (10)

which saturates the cross section in the relevant energy range.

It should be noted that the rate \( \Gamma_k = -2Im\Sigma_{\pi}/(2(k^2 + m_{\pi}^2)^{1/2}) \) deduced from eq. (8) is not entirely consistent with that obtained from the collision term of the bosonic Boltzmann equation. A correct account of the Bose correlations of the two interacting pions will modify the occupancy factor in (8) [9]. It turns out, however, that this is a small effect [10].

It is now evident that the pion selfenergy and the T-Matrix should be combined in a selfconsistent Brueckner scheme as indicated in Fig. 1. This implies that the selfenergy (8) is to be calculated from the in-medium T-Matrix which, on the other hand, should be obtained from the in-medium 2\( \pi \) propagator including the pion selfenergy:

\[
G_{\pi\pi}(Z, k; \mu_{\pi}, T) = (1 + 2f_k(\mu_{\pi}, T)) \int \frac{id\omega}{2\pi} D_{\pi}(\omega, k) D_{\pi}(Z - \omega, k) ,
\]  (11)

where

\[
D_{\pi}(\omega, k) = \left[ \omega^2 - m_{\pi}^2 - k^2 - \Sigma_{\pi}(\omega, k; \mu_{\pi}, T) \right]^{-1}.
\]  (12)

Together with the in-medium scattering equation,

\[
T_{\pi\pi}(\mu_{\pi}, T) = V_{\pi\pi} + V_{\pi\pi} G_{\pi\pi}(\mu_{\pi}, T) T_{\pi\pi}(\mu_{\pi}, T) ,
\]  (13)
equations (8), (11) and (12) define the selfconsistency problem (see also Fig. 1). It should be noticed that the pseudopotential $V_{\pi\pi}$ remains unchanged.

In the following we will discuss two different methods of calculating $G_{\pi\pi}$.

4 Quasiparticle Approximation

The quasiparticle approximation (QPA) is valid if the lifetime is long or more precisely if the quasiparticle energy

$$e_k \equiv (m_\pi^2 + k^2 + Re \Sigma_\pi(e_k, k))^{1/2}$$  (14)

is much larger than its decay width $\Gamma_k$. In this case the energy-dependence of $Re \Sigma_\pi$ is expanded to first order around the 'quasiparticle pole' $e_k$ as

$$Re \Sigma_\pi(\omega, k) \approx Re \Sigma_\pi(e_k, k) + \frac{\partial Re \Sigma_\pi(\omega, k)}{\partial \omega^2}|_{e_k} (\omega^2 - e_k^2) .$$  (15)

One can then perform the folding integral (11) analytically which gives

$$G_{\pi\pi}(Z, k; \mu_\pi, T) = \frac{1}{\bar{\omega}_k} \frac{z_k^2 (1 + 2 f_k(\mu_\pi, T))}{Z^2 - 4\bar{\omega}_k^2}$$  (16)

with

$$z_k \equiv (1 - \frac{\partial Re \Sigma_\pi(\omega, k)}{\partial \omega^2}|_{e_k})^{-1} \text{ the polestrength} ,$$

$$\bar{\omega}_k^2 \equiv e_k^2 + i z_k Im \Sigma_\pi(e_k, k) \text{ quasipion dispersion relation} .$$  (17)

Together with this approximation eqs. (8),(11) and (13) are solved by iteration starting from the free pion dispersion relation and the vacuum T-Matrix. We keep the pion density $n_\pi$ fixed during the iteration by readjusting $\mu_\pi$ in each step (the final $\mu_\pi$ differs from the starting value $\mu_\pi(0)$ by a small amount). The selfconsistent results are shown in Fig. 2. Already for chemical equilibrium ($\mu_\pi(0) = 0$) we find a considerable reduction of the peak values in $Im T_{JI}$ as compared to the vacuum case in both s- and p-wave. This is in agreement with the lowest-order results from refs. [3, 11]. The reduction is mainly caused
by the Bose factors \((1 + 2f)\) leading to a stronger weighting of the lower pion energies.

For the same reason the near threshold region shows an enhancement over the vacuum T-Matrix especially in the s-wave. We define the pion 'optical potential' as

\[
V_\pi(k) \equiv \frac{\Sigma_\pi(e_k, k)}{2(k^2 + m_\pi^2)^{1/2}},
\]

which is shown in the lower part of Fig. 2. In agreement with refs. \[2, 8\] we find attraction in \(ReV_\pi\) for low momenta. The selfconsistent potential, however, is significantly weaker than in lowest order. The maximum in \(ImV_\pi\) is due to formation of the \(\rho\)-resonance, whereas the non-zero values at \(k = 0\) arise from s-wave interaction with thermally moving pions.

5 Off-shell Integration of \(G_{\pi\pi}\)

To check the reliability of the QPA we perform the same calculations as described in the previous section, but the quasiparticle two-pion propagator is now replaced by a numerical integration of (11) accounting for the full off-shell properties of \(\Sigma_\pi(\omega, k)\). Using the symmetry relation

\[
\Sigma_\pi(-\omega, k) = \Sigma_\pi(\omega, k),
\]

the in-medium propagator (11) can be written as

\[
G_{\pi\pi}(Z, k; \mu_\pi, T) = (1 + 2f_k(\mu_\pi, T)) \int \frac{id\omega}{Z/2} D_\pi(\omega, k) D_\pi(|Z - \omega|, k).
\]

Fig. 3 shows the selfconsistent results for a chemically equilibrated pion gas \((\mu_\pi^{(0)} = 0)\). At \(T = 125\) MeV the results coincide with the QPA within a few percent. For \(T > 150\) MeV the deviations become larger: the T-Matrices in s- and p-wave are now enhanced compared to the vacuum curve for most of the energy range (e.g. the peak value of the \(\rho\)-resonance increases by \(\approx 30\%\) at \(T = 200\) MeV). This clearly must be an off-shell effect. From the lower part of Fig. 3 one can conclude that a first-order expansion of \(Re\Sigma_\pi\) around \(e_k\) does not describe the energy dependence correctly; in addition \(\Gamma_k/e_k \sim 0.6\) around
\( k = 300 \text{MeV}/c \), \textit{i.e.} the quasiparticle lifetime becomes very short and renders the QPA invalid. The potentials \( V_{\pi}(k) \) are now very close to the lowest-order results. As compared to the QPA a considerable amount of attraction is restored in the low-momentum region of \( \text{Re}V_{\pi}(k) \). We also investigate a scenario with finite chemical potential \( \mu_{\pi}^{(0)} = 125 \text{MeV} \) as suggested by thermal fits to the SPS \( p_T \)-spectra \cite{4}. The s-wave \( \pi\pi \) interaction now shows a strong accumulation of strength near threshold (Fig. 4) which might lead to quasi-bound \( \pi^+\pi^- \) pairs as suggested in ref. \cite{11}. The latter possibility deserves further study.

In the p-wave the changes of the \( \rho \)-resonance are appreciable: at highest temperatures the width decreases considerably which might be detectable via dilepton pairs coming from the midrapidity region.

6 Summary

Based on a selfconsistent Brueckner theory we have presented a numerical analysis of a hot, interacting pion gas in thermal equilibrium, which may be realized in future experiments at RHIC or LHC (at \( \sqrt{s} \geq 100 \text{GeV}/A \)). The selfconsistent Brueckner scheme accounts for statistical (Bose factors) as well as dynamical (selfenergy) modifications of the pion propagation in the gas. Using a realistic model for the vacuum \( \pi\pi \) interaction we have solved the non-linear problem by iteration. We have compared results from the QPA to those taking full account of the off-shell behavior in the pion selfenergy. It is found that the QPA breaks down for temperatures \( T \geq 150 \text{MeV} \) and finite chemical potentials, \textit{i.e.} high pion densities. In the full calculation we find a slight \textit{increase} of the in-medium T-matrix – in contrast to refs. \cite{3,11}, where only statistical modifications have been taken into account. The main cause of this enhancement is attributed to strong off-shell effects in the pion selfenergy. The single-particle potentials show considerable attraction for low pion momenta, although quantitatively somewhat less than our lowest-order results. As was shown recently by Koch and Bertsch \cite{12}, such an attraction is not able to explain the low-\( p_T \) enhancement in the pion spectra of current experiments. However, as suggested
by numerical simulations of the bosonic transport equation \[13\], an increase of the in-medium T-Matrix, as found in our calculations, is likely to ensure thermalization of the pionic fireball and hence to produce an excess of low-\(p_T\) pions through the \((1 + f)\) factors in the collision integral. This issue and the impact of finite baryon density in the pion gas will be addressed to a future publication.

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FIGURE CAPTIONS

**Fig. 1** Set of selfconsistent equations for an interacting pion gas:
upper part: Dyson equation for in-medium pion propagation
lower part: in-medium $\pi\pi$ T-Matrix equation.

**Fig. 2** Selfconsistent pion gas at $\mu_\pi^{(0)} = 0$ in QPA:
upper part: imaginary part of the $\pi\pi$ T-Matrix in the $JI=00$- and $JI=11$-channels for several temperatures (long-dashed lines: $T = 125$ MeV, short-dashed lines: $T = 150$ MeV, dotted lines: $T = 200$ MeV)
lower part: corresponding pion potentials (short-dashed lines: $T = 150$ MeV, dotted lines: $T = 200$ MeV; the dashed-dotted and dashed-double-dotted lines show the lowest-order results calculated with the vacuum T-Matrix for $T = 150$ MeV and $T = 200$ MeV, respectively).

**Fig. 3** Selfconsistent pion gas at $\mu_\pi^{(0)} = 0$ with full off-shell integration of $G_{\pi\pi}(eq. (20))$:
upper part: imaginary part of the $\pi\pi$ T-Matrix (see Fig. 2)
lower part: real part of the pion potentials (see Fig. 2) and real part of the off-shell pion selfenergy at $T = 200$ MeV.

**Fig. 4** Imaginary part of the selfconsistent $\pi\pi$ T-Matrix with full off-shell integration of $G_{\pi\pi}$ for $\mu_\pi^{(0)} = 125$ MeV (long-dashed lines: $T = 100$ MeV, short-dashed lines: $T = 150$ MeV, dotted lines: $T = 175$ MeV, full lines: free space).