An explicit upper bound for the first $k$-Ramanujan prime

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Abstract

In this paper we establish an explicit upper bound for the first $k$-Ramanujan prime $R_1^{(k)}$ by using a recent result concerning the existence of prime numbers in small intervals.

1 Introduction

Let $k \in (1, \infty)$. The PNT implies that $\pi(x) - \pi(x/k) \to \infty$ as $x \to \infty$ and Shevelev [8] introduced the $n$th $k$-Ramanujan prime as follows.

Definition. Let $k > 1$ be real. For every $n \in \mathbb{N}$, let

$$R_n^{(k)} = \min\{m \in \mathbb{N} \mid \pi(x) - \pi(x/k) \geq n \text{ for every real } x \geq m\}.$$

It is easy to show that this number is prime and it is called the $n$th $k$-Ramanujan prime.

In this paper we give an explicit upper bound for the first $k$-Ramanujan prime $R_1^{(k)}$ for small $k$. In order to do this, we first give some known results on the existence of prime numbers in short intervals.

2 On the existence of prime numbers in short intervals

Bertrand’s postulate states that for every $n \in \mathbb{N}$ there is always a prime in the interval $(n, 2n]$. Now, we note some improvements of this result. In 2003, Ramaré & Saouter [5] showed that for every $x \geq 10726905041$ the interval $(x, x + x/28313999]$ always contains a prime number. This was improved by Dusart [3] in 2010 by showing that for every $x \geq 396738$ there is always a prime number $p$ with

$$x < p \leq x \left(1 + \frac{1}{25 \log^2 x}\right).$$

(1)

In 2014, Trudgian [9] proved that for every $x \geq 2898239$ there exists a prime number $p$ such that

$$x < p \leq x \left(1 + \frac{1}{111 \log^2 x}\right).$$

Recently, in [1] it is shown that the following result holds.
Proposition 2.1. For every $x \geq 58837$ there is a prime number $p$ such that

$$x < p \leq x \left(1 + \frac{1.188}{\log^3 x}\right).$$

3 On an upper bound for the first $k$-Ramanujan prime

Let $n \in \mathbb{N}, c > 0$ and $x_0 > 0$ so that for every $x \geq x_0$ there is a prime $p$ such that

$$x < p \leq x \left(1 + \frac{c}{\log^3 x}\right). \tag{2}$$

Then, we obtain the following result.

Proposition 3.1. Let $x \geq x_0$ and $k = 1 + c/\log^n x$. Then

$$R_1^{(k)} \leq kx.$$

Proof. Let $y \geq kx$. From (2) we obtain the existence of a prime $p$ in

$$\left(\frac{y}{k}, \frac{y}{k} \left(1 + \frac{c}{\log^n (y/k)}\right)\right).$$

Since $y/k \geq x$, we get

$$k \geq 1 + \frac{c}{\log^n (y/k)},$$

so that $p \in (y/k, y]$. \qed

Corollary 3.2. For every

$$k \in \left(1, 1 + \frac{c}{\log^n x_0}\right),$$

we have

$$R_1^{(k)} \leq k \cdot \exp \left(\sqrt[1/k-1]{c}\right).$$

Proof. Define $x \in \mathbb{R}$ so that

$$k = 1 + \frac{c}{\log^n x}.$$

Then $x \geq x_0$ and by using Proposition 3.1 we get

$$R_1^{(k)} = R_1^{(1+c/\log^n x)} \leq x \left(1 + \frac{c}{\log^3 x}\right) = k \cdot \exp \left(\sqrt[1/k-1]{c}\right).$$

This proves our corollary. \qed

4 A characterisation for $k$-Ramanujan primes

We obtain the following useful characterisation for the first $k$-Ramanujan prime.

Proposition 4.1. Let $N \in \mathbb{N}$. Then $p_N$ is the first $k$-Ramanujan prime iff the following two conditions are fulfilled:

(a) For every $n \geq N$, we have

$$\frac{p_{n+1}}{p_n} \leq k.$$

(b) We have

$$\frac{p_N}{p_{N-1}} > k.$$
Proof. Let $p_N = R_1^{(k)}$. To show (a), we assume that there is an integer $n \geq N$ so that $p_{n+1}/p_n > k$. Let $x = kp_n$. Then $p_n < x < p_{n+1}$, so that 

$$\pi(x) - \pi(x/k) = n - n = 0. \quad (3)$$

Since $x > p_N = R_1^{(k)}$, the equation (3) contradicts the definition of $R_1^{(k)}$. So, we proved (a). To show (b), we assume that $p_N/p_{N-1} \leq k$. Since $p_N = R_1^{(k)}$, there is a $x_0 \in [p_{N-1}, p_N)$ so that $\pi(x_0) - \pi(x_0/k) = 0$. Since we have $x_0/k < p_N/k \leq p_{N-1}$, we get 

$$0 = \pi(x_0) - \pi\left(\frac{x_0}{k}\right) > \pi(p_{N-1}) - \pi(p_{N-1}) = 0,$$

which gives a contradiction.

Now, let (a) and (b) be true. To show that $p_N = R_1^{(k)}$, we show first that $p_N \geq R_1^{(k)}$. Let $x \geq p_N$. We assume that $\pi(x) - \pi(x/k) = 0$. Then there exists an integer $n \geq N$ such that $p_n \leq x/k < x < p_{n+1}$. Hence, 

$$\frac{p_{n+1}}{p_n} > \frac{x}{x/k} = k,$$

which contradicts (1). Now, we prove that $p_N \leq R_1^{(k)}$. Let $x = kp_{N-1}$. Then, 

$$p_{N-1} < x < p_N. \quad (4)$$

Hence, we obtain 

$$\pi(x) - \pi\left(\frac{x}{x/k}\right) = N - 1 - \pi(p_{N-1}) = 0.$$

It follows that $R_1^{(k)} > x > p_{N-1}$. So $R_1^{(k)} \geq p_N$. \hfill \Box

5 Numerical results

In the following proposition we derive an explicit $p$ such that $R_1^{(k)} = p$ for the case $k = 1.0008968291$.

Proposition 5.1. We have 

$$R_1^{(1.0008968291)} = 58889 = p_{5950}.$$

Proof. Let $x_0 = 58837$, $c = 1.188$ and $n = 3$. Then 

$$1.0008968291 \leq 1 + \frac{1.188}{\log^2 58837}. \leq 58890$$

Using Proposition 2.1 and Corollary 3.2, we obtain that the inequality 

$$R_1^{(1.0008968291)} \leq 1.0008968291 \cdot \exp\left(\sqrt[3]{\frac{1.188}{0.0008968291}}\right) \leq 58890$$

holds. Since $R_1^{(1.0008968291)}$ is a prime number, we obtain 

$$R_1^{(1.0008968291)} \leq 58889.$$

On the other hand we have 

$$\pi(58888) - \pi\left(\frac{58888}{1.0008968291}\right) = 0,$$

hence $R_1^{(1.0008968291)} > 58888$. \hfill \Box

Remark. (a) If $k \geq 5/3$, then $R_1^{(k)} = 2$ (see [2] Prop. 2.5(ii))

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If $k \in [1.0008968291, 5/3)$, then, using Proposition 5.1 we obtain

$$m := \max\{n \geq 2 \mid p_n/p_{n-1} > k\} = \max\{n \in \{2, \ldots, 5950\} \mid p_n/p_{n-1} > k\}.$$  

By Proposition 5.1 it follows $R_1^{(k)} = p_m$.

By using Remark (b) and a computer, we obtain the following

**Corollary 5.2.** (a) If

$$k \in \left[1.0008968291, \frac{p_{5950}}{p_{5949}}\right),$$

then $R_1^{(k)} = 58889$.

(b) For every $1 \leq n \leq 44$ we define the numbers $a(n)$ by

| n  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| a(n) | 3  | 5  | 7  | 10 | 12 | 16 | 31 | 35 | 47 | 48 | 63 |
| p_{a(n)} | 5  | 11 | 17 | 29 | 37 | 53 | 127| 149| 211| 223| 307|

| n  | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| a(n) | 67 | 100| 218| 264| 298| 328| 368| 430| 463| 591| 651|
| p_{a(n)} | 331| 541| 1361|1693|1973|2203|2503|2999|3427|4861|

| n  | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| a(n) | 739 | 758| 782| 843| 891| 929| 1000|1184|1230|1316|1410|
| p_{a(n)} | 5623| 5779| 5981| 6521| 6947| 7283| 8501| 9587|10007|10831|11777|

| n  | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| a(n) | 1832| 2226| 3386| 3645| 3794| 3796| 4523| 4613| 4755| 5009| 5950|
| p_{a(n)} | 15727| 19661| 31469| 34123| 35671| 35729| 43391| 44351| 45943| 48731| 58889|

If $1 \leq n \leq 43$ and

$$k \in \left[\frac{p_{a(n)+1}}{p_{a(n)-1}}, \frac{p_{a(n)}}{p_{a(n)-1}}\right),$$

then $R_1^{(k)} = p_{a(n)}$.

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