The maximal running time of hypergraph bootstrap percolation

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Abstract

We show that for every $r \geq 3$, the maximal running time of the $K_{r+1}$-bootstrap percolation in the complete $r$-uniform hypergraph on $n$ vertices $K_r^n$ is $\Theta(n^r)$. This answers a recent question of Noel and Ranganathan in the affirmative, and disproves a conjecture of theirs. Moreover, we show that the prefactor is of the form $r^{-r}e^{O(r)}$ as $r \to \infty$.

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1 Introduction

1.1 Background

Bootstrap percolation is a widely studied model for spreading infection or information on a graph. Given an initial set of infections, this process iteratively adds infections if a certain local pattern of infections is already present. While the original statistical physics perspective suggests selecting initial infections at random, interesting computational and extremal combinatorial questions arise when considering deterministic instances. The most classical extremal problems are determining the minimal number of infections necessary for infecting the entire graph and the maximal time until stationarity or complete infection.

These maximal times have been studied in several settings, including grids \cite{2} and hypercubes \cite{5,8}. Bootstrap percolation in which one infects the edges of a graph when they complete a copy of a fixed graph is referred to graph bootstrap percolation or weak saturation. In that setting, the maximal times were studied in \cite{1,3,6}. Finally, our setting of interest is the more general hypergraph bootstrap percolation, where maximal times were recently investigated by Noel and Ranganathan \cite{7}.

1.2 Model and result

For a positive integer $r \geq 2$ and an $r$-uniform hypergraph $H$ (or $r$-graph for short), we identify $H$ with its edge set and denote by $V(H)$ the set of vertices of $H$. In this paper, we pay particular attention to the complete $r$-graph on $n$ vertices denoted $K_r^n$, especially the $r$-graph $K_{r+1}^r$, which we denote $F_r$ for short.
For a positive integer \( r \geq 2 \) and an \( r \)-graph \( F \), the \( F \)-bootstrap process in \( K_n^r \) is a monotone cellular automaton described as follows. At the beginning, a subgraph \( G_0 \) of \( K_n^r \) is infected. At every step \( i \geq 1 \), we define

\[
G_i = G_{i-1} \cup \{ e \in K_n^r : \text{there is a copy } F' \subset K_n^r \text{ of } F \text{ such that } e = F' \setminus G_{i-1} \}.
\]  

The *running time* until stationarity (that is, the first step \( i \) at which \( G_i = G_{i+1} \)) of the \( F \)-bootstrap percolation with initially infected set \( F \) is denoted by \( T(F, G_0, n) \). In [7] it was proved that

\[
\max_{G_0 \subseteq K_n^r} T(K_m^r, G_0, n) = \Theta(n^r)
\]

for all fixed \( m \geq r + 2 \geq 5 \). In this paper, we are interested in the *maximum running time* of the \( F_r \)-bootstrap percolation \( T_r(n) \) defined as

\[
T_r(n) = \max_{G_0 \subseteq K_n^r} T(F_r, G_0, n).
\]

In this case, [7] only established that for all \( r \geq 3 \), \( T_r(n) \geq c_r n^{r-1} \) for some \( c_r > 0 \). Our work closes the gap between this lower bound and the trivial upper bound \( T_r(n) \leq \binom{n}{r} \). More precisely, we answer Question 5.2 from [7] in a strong sense, and disprove Conjecture 5.1 from the same paper suggesting that \( T_3(n) = O(n^2) \).

**Question 1** (Question 5.2 from [7]). Does there exist an integer \( r_0 \) such that, if \( r \geq r_0 \), then \( T_r(n) = \Theta(n^r) \)?

Our main result is the following lower bound.

**Theorem 2.** For every \( r \geq 3 \) and \( n \geq 2n^2 \), we have

\[
\frac{n^r}{2^{r+3}r^r} \leq T_r(n) \leq \frac{n^r e^r}{r^r}.
\]

Let us note that, while our result shows that the constant prefactor is \( r^{-r}e^{O(r)} \) as \( r \to \infty \), our construction has not been optimised for this purpose, but rather for the clarity of the exposition. We also observe that the condition \( n \geq 2n^2 \) is imposed mostly for technical reasons and is only used in [2].

### 1.3 Outline of the proof

Theorem 2 is proved as follows. We start with a simple construction showing that \( T_3(2(4k - 3) + 1) \geq 8k^2 - 12k + 4 \) for any \( k \geq 1 \) (see Section 3), which reproves the corresponding result of [7]. The construction (see Fig. 2) consists of two vertex-disjoint paths, each containing \( 4k - 3 \) vertices, and an additional vertex \( v_1^3 \). The edges in the initial graph are: \( v_1^3 \) together with any 2-edge in some of the paths, \( v_1^3 \) together with the initial vertices of the two paths, and some 3-edges that do not contain \( v_1^3 \). This is done so that, firstly, only one edge is infected per time step, secondly, all edges which become infected contain \( v_1^3 \) and one vertex in each path, and moreover, the process is reversible in the sense that, if we replace the edge joining the beginnings of the two paths by the last edge infected by the process, the same edges become infected but in reverse order.

Based on this construction, we prove Theorem 2 for \( r = 3 \) by gluing \( 2k - 1 \) instances of it, as shown in Fig. 1 (see Proposition 4). Namely, using an auxiliary vertex \( v_2^3 \), we propagate the infection from the end of the process for the vertex \( v_1^3 \) to the end of the same process for a new vertex \( v_3^3 \) instead of \( v_1^3 \), which will then run the other way around. Repeating this procedure \( 2k - 2 \) times, we obtain a cubic running time. The key feature of this strategy is that no edges that do not contain \( v_1^3 \) are created in the first construction, so we do not alter the configuration induced by the first \( 2(4k - 3) \) vertices. Moreover, the same gluing technique allows us to extend the construction from \( r \) to \( r + 1 \) by adding a vertex to all edges, adding all edges that do not contain this vertex, and then gluing copies of this construction that differ only in this special vertex while keeping the remaining ones unchanged.
Let step of the first stage for the vertex \( v_3 \) and the first step of the first stage for the vertex \( v_3 \). On this intermediate step, only the edge \( v_{2k-1}^{1}v_{2k-1}^{2}v_3 \) is infected. Note that 3-edges are represented by smaller homothetic copies of the corresponding triangles.

2 Proof of Theorem 2

To start with, we fix a positive integer \( k = k(n) \) to be chosen appropriately later. Then, for all integers \( i \geq 1 \), we define the vertex sets

\[ A_i = \{v_i^1, v_i^2, \ldots, v_i^{k-3}\}, \quad A_i^* = \bigcup_{j=1}^{i} A_j. \]

In particular, the sets \((A_i)_{i \geq 1}\) are disjoint while the sets \((A_i^*)_{i \geq 1}\) form an increasing sequence with respect to inclusion. Next, we consider a particular case of \( F_r \)-bootstrap percolation where a sequence of edges can be infected both in the given and in a reverse order under a suitable initial condition.

Definition 3. Let \( r \geq 3 \) be an integer and \( H \) be an \( r \)-graph with vertex set \( V(H) \subset A^*_i \) and edge set \( E(H) \). Given a sequence \((e_i)_{i=0}^{T} \) of edges in \( E(H) \), we say that \( H \) is \((e_i)_{i=0}^{T}\)-sequential if the \( F_r \)-bootstrap percolation process in the complete \( r \)-graph on \( V(H) \) with initially infected \( r \)-graph:

(i) \( H \) runs for \( T \) steps, infecting only the \( r \)-edge \( e_i \) at step \( i \in [1,T] \);

(ii) \( H \setminus \{e_0\} \) is stationary (does not infect any \( r \)-edge);

(iii) \( (H_1 \cup \{e_T\}) \setminus \{e_0\} \) infects only the \( r \)-edge \( e_{T-i} \) at step \( i \in [1,T] \).

Definition 3 is central in the key construction presented in the next proposition.

Proposition 4. Fix integers \( r \geq 3 \) and \( k \geq 2 \). Let \( T_1 \geq 2 \) be an integer and \( H_1 \) be an \( r \)-graph with vertices \( A^*_{r-1} \cup \{v_1^r\} \). Suppose that \( H_1 \) is \((e_i)_{i=0}^{T_1}\)-sequential with \( v_1^r \in \bigcap_{i=0}^{T_1} e_i \). For all \( j \in [1,2k-1] \), let \( H_{2j-1} \) be a copy of \( H_1 \setminus \{e_0\} \) obtained by replacing \( v_1^r \) by \( v_{2j-1}^r \). Moreover, define \( e_i^{-} = e_i \setminus \{v_1^r\} \), and for all \( j \in [1,k-1] \), let

\[ H_{4j-2} = \{ e \subset \{v_{4j-3}^r, v_{4j-2}^r, v_{4j-1}^r\} \cup e_T^{-}; |e| = r, \{v_{4j-3}^r, v_{4j-1}^r\} \not\subset e, e_T^{-} \not\subset e \} \]

\[ H_{4j} = \{ e \subset \{v_{4j-1}^r, v_{4j}^r, v_{4j+1}^r\} \cup e_0^{-}; |e| = r, \{v_{4j-1}^r, v_{4j+1}^r\} \not\subset e, e_0^{-} \not\subset e \} \]
Applying this to $H_{i}$ the complete $H$ and proceed as follows.

We claim that $H_{i}$ be the complete $H$ and proceed as follows.

**Proposition 5.** Let $k \geq 2$. There exist $r$-edges $(e_{1})_{T_{i}=0}$ all containing $v_{1}^{i}$ and a $(e_{1})_{T_{i}=0}$-sequential $r$-graph $H_{1}$ with $V(H_{1}) = A_{n-1}^{*} \cup \{v_{1}^{i}\}$ and $T_{1} = (2k-1)r^{-3}(8k^{2} - 12k + 6) - 2$.

The base case $r = 3$ of Proposition 3 is left to Section 3.

**Proof of Proposition 5 assuming the case $r = 3$.** We argue by induction on $r$, the base being established in Section 3. Assume the induction hypothesis to be true for some $r \geq 3$. Then, Proposition 4 provides us with an $(e_{1})_{T_{i}=0}$-sequential $r$-graph denoted $H''$ with $V(H'') = A_{r}^{*}$ and $T = (2k-1)T_{1} + 4(k-1) = (2k-1)r^{-2}(8k^{2} - 12k + 6) - 2$.

Let $K$ be the complete $(r + 1)$-graph with $V(K) = A_{r}^{*}$. Define the $(r + 1)$-graph $H_{r+1}^{*} = \{e \cup \{v_{r+1}^{i}\} : e \in H'', V(K) \}$. We claim that $H_{r+1}^{*}$ is $(e_{1} \cup \{v_{r+1}^{i}\})_{T_{i}=0}$-sequential. To see this, observe that $H_{r+1}^{*} \setminus \{e_{0} \cup \{v_{r+1}^{i}\}\} \supset \{H_{r+1}^{*} \setminus \{e_{0} \cup \{v_{r+1}^{i}\}\}, \{H_{r+1}^{*} \cup \{v_{r+1}^{i}\}\} \setminus \{e_{0} \cup \{v_{r+1}^{i}\}\}\}$.

Applying this to $H' \in \{H_{r+1}^{*} \setminus \{e_{0} \cup \{v_{r+1}^{i}\}\}, \{H_{r+1}^{*} \cup \{v_{r+1}^{i}\}\} \setminus \{e_{0} \cup \{v_{r+1}^{i}\}\}\}$, it follows from Definition 3 that $H_{r+1}^{*}$ is $(e_{1} \cup \{v_{r+1}^{i}\})_{T_{i}=0}$-sequential, as desired.

We are now ready to conclude the proof of Theorem 2.

**Proof of Theorem 2.** Fix $r \geq 3$ and set $k = [(n/r + 3)/4] \geq n/(4r) - 1/4$. For the lower bound, we apply Proposition 5 and then Proposition 4 to obtain an $r$-graph $H'$ with $|A_{r}^{*}| = (r - 1)(4k - 3) + (4k - 3) \leq n$ vertices such that the $F_{r}$-bootstrap percolation process on the complete graph on $A_{r}^{*}$ with initially infected $r$-graph $H'$ runs for

$$T = (2k-1)r^{-2}(8k^{2} - 12k + 6) - 2 \geq (2k-1)r \geq \left(\frac{n}{2r} - \frac{3}{2}\right)^{r} \geq \left(\frac{n}{2r}\right)^{r}\left(1 - \frac{3}{2} \cdot \frac{3}{r}\right)^{3} = \frac{1}{8} \left(\frac{n}{2r}\right)^{r}. \quad (2)$$

steps, since $n \geq 2r^{2}$ and $r \geq 3$. Thus, the lower bound of Theorem 2 is witnessed by the $r$-graph obtained by adding $n - |A_{r}^{*}| \geq 0$ isolated vertices to $H'$. The upper bound simply states that the process cannot run more than $(n!)^{r} \leq n^{r}/r! \leq (ne/r)^{r}$ steps.

In the sequel, we call a vertex **odd** (resp. **even**) if its subscript is odd (resp. even).
Proof of Proposition 4. First of all, let us show that the process in $H \setminus \{e_0\}$ is stationary. Suppose for contradiction that this is not the case, and suppose that some $(r+1)$-tuple of vertices $f$ spans exactly $r$ edges. To begin with, $f$ cannot contain two non-consecutive vertices in $A_r$ since they are not contained in a common $r$-edge of $H$. Moreover, $f$ cannot be entirely contained in $A_r$ since otherwise $H_1 \setminus \{e_0\}$ would not be stationary. Two cases remain:

- Suppose that $|f \cap A_r| = 1$. Then the vertex in $f \cap A_r$ must be even (otherwise, we obtain a contradiction with the stationarity of $H_1 \setminus \{e_0\}$). However, even vertices in $A_r$ are not contained in edges with $r-1$ other vertices in $A_{r-1}$, so this case is impossible.

- Suppose that $f \cap A_r = \{v_{i,r}, v_{i+1,r}\}$ for some $i \in [1, 4k - 4]$. We further assume that $i = 4j$ is divisible by 4, the other cases being treated identically. Since the only edges containing $v_{i,j}$ are those in $H_{4j}$, necessarily $f = \{v_{i,j}, v_{i,j+1}\} \cup e_0 \overline{e}_0$. However, in this case the two edges $\{v_{i,j}\} \cup e_0 \overline{e}_0$ and $\{v_{i,j+1}\} \cup e_0 \overline{e}_0$ are both missing, so we obtain the desired contradiction.

Now, for every $j \in [1, 2k - 1]$ and every $i \in [0, T_1]$, denote $e_{i,j} = e_i^- \cup \{v_{2j-1}\}$. In particular, $e_{i,1} = e_i$. Further set

$$H_\infty = H \cup \{e_{i,j} : i \in [0, T_1], j \in [1, 2k - 1]\} \cup \{\{v_{i,j-2}\} \cup e_{T_1,j}^- : j \in [1, k - 1]\} \cup \{\{v_{i,j}\} \cup e_0^- : j \in [0, k - 1]\}.$$  

We claim that $H_\infty$ is also stationary. As above, the $r+1$ tuple $f$ spanning exactly $r$ edges of $H_\infty$ falls in one of the following two cases:

- Suppose that $|f \cap A_r| = 1$. Then, the vertex in $f \cap A_r$ must be even as otherwise we obtain a contradiction with the fact that $H_1 \cup \{e_{i,1} : i \in [0, T_1]\}$ is stationary (which is true by the definition of $T_1$). However, even vertices in $A_r$ are contained in only one edge with $r-1$ other vertices in $A_{r-1}$, so this case is impossible.

- Suppose that $f \cap A_r = \{v_{i,r}, v_{i+1,r}\}$ for some $i \in [1, 4k - 4]$. Again, we assume that $i = 4j$. Since the only edges containing $v_{i,j}$ are those in $H_{4j} \cup \{v_{i,j}\} \cup e_0$, necessarily $f = \{v_{i,j}, v_{i,j+1}\} \cup e_0 \overline{e}_0$. However, $f$ already spans a copy of $F_r$ in $H_\infty$, which leads to a contradiction.

Moreover, since $H \subseteq H_\infty$, all edges infected by $H$ are contained in $H_\infty$.

From Definition 3 one may conclude that for every $i \in [0, T_1]$, the only copies of $F_r$ in $H_1 \cup \{e_{i,j}\}_{j=1}^{T_1}$ that $e_i$ participates in are spanned by $e_{i-1} \cup e_i$ (if $i > 0$) and by $e_i \cup e_{i+1}$ (if $i < T_1$). Moreover, note that:

- for every $j \in [1, 2k - 1]$, $v_{i,2j-1}$ is contained in a common edge of $H_\infty$ with $v_{i,2j-2}$ (if $j \geq 2$) and with $v_{i,2j}$ (if $j \leq 2k - 2$), but with no other vertex in $A_r$,

- no edge among $(e_{i,j})_{i=1}^{T_1-1}$ participates in a copy of $F_r$ in $H_\infty$ with any of $v_{i,2j-2}$ and $v_{i,2j}$,

- for every $j \geq 2$, $e_{0,j}$ spans a copy of $F_r$ in $H_\infty$ together with $v_{i,2j-2}$ and $e_{1,j}$ (if $j$ is even) or $e_{T_1-1,j}$ (if $j$ is odd) only; for every $j \leq 2k - 2$, $e_{0,i,j}$ spans a copy of $F_r$ in $H_\infty$ together with $v_{i,2j}$ and $e_{T_1-1,j}$ (if $j$ is odd) or $e_{1,j}$ (if $j$ is even) only.

Furthermore, for every odd (resp. even) $j \in [1, 2k - 2]$, $v_{i,2j} \cup e_{T_1}^-$ (resp. $v_{i,2j} \cup e_0^-$) spans a copy of $F_r$ in $H_\infty$ together with $v_{i,2j-1}$ and $v_{i,2j+1}$ only. We conclude that in the sequence of edges

$$\{(e_{i,1})_{i=0}^{T_1}, v_{i,2} \cup e_{T_1}^-, (e_{T_1-1,i,2})_{i=0}^{T_1}, v_{i,2} \cup e_0^-, (e_{i,3})_{i=0}^{T_1}, \ldots, (e_{T_1-1,i,2k-2})_{i=0}^{T_1}, v_{i,2k-4} \cup e_0^-, (e_{i,2k-1})_{i=0}^{T_1}\},$$

(3) every edge except the first one and the last one participate in exactly two copies of $F_r$ in $H_\infty$, one spanning together with the edge preceding it and another one with the edge succeeding it. Moreover, the first and the last edges participate in exactly one copy of $F_r$ in $H_\infty$, spanning together with $e_{1,1}$ and with $e_{T_1-1,2k-1}$, respectively. Recalling Definition 3 this completes the proof of Proposition 4 since (3) is an enumeration of all $T + 1$ edges in $H_\infty \setminus (H \setminus \{e_0\})$. □
In order to prove Proposition 5 for $r = 3$, our goal is to construct a sequential 3-graph $H_1$ with vertices $A_2^* \cup \{v_1^3\}$ and running time $8k^2 - 12k + 4$. For all $i \in [1, k - 1]$, define the disjoint edge sets

$$
\begin{align*}
\mathcal{E}_{1,i} &= \{v_{2i-1}^1v_{2i}^2v_{2i+1}^2 : j \in [2i - 1, 4k - 2 - 2i]\}, \\
\mathcal{E}_{2,i} &= \{v_{1}^1v_{2i+1}^3v_{4k-2i}^2 : j \in [2i - 1, 4k - 2 - 2i]\}, \\
\mathcal{E}_{3,i} &= \{v_{4k-1-2i}^1v_{2j}^2v_{j+1}^2 : j \in [2i + 1, 4k - 2 - 2i]\}, \\
\mathcal{E}_{4,i} &= \{v_{1}^1v_{j+1}^3v_{2i+1}^2 : j \in [2i + 1, 4k - 2 - 2i]\}, \\
\mathcal{E}_i &= \mathcal{E}_{1,i} \cup \mathcal{E}_{2,i} \cup \mathcal{E}_{3,i} \cup \mathcal{E}_{4,i}
\end{align*}
$$

(see Fig. 2 for an illustration of $\mathcal{E}_1$). Also, denote $\mathcal{E} = \bigcup_{i=1}^{k-1} \mathcal{E}_i$. This is the set of edges with vertices in $A_2^*$ that we infect at the beginning.

In addition to $\mathcal{E}$, some edges containing $v_1^3$ are also infected initially (see Fig. 2). Denote

$$
\mathcal{F} = \bigcup_{j=1}^{4k-4} \bigcup_{l=1}^{2} \{v_j^1v_{j+1}^3v_{1}^3\}. 
$$

(5)

In other words, the set $\mathcal{F}$ consists of all triplets containing the vertex $v_1^3$ and a consecutive pair of vertices in one of the paths $v_1^3v_2^1\ldots v_{4k-3}^1$ and $v_1^3v_2^1\ldots v_{4k-3}^2$. Finally, we need the *ignition edge* $e_0 = v_1^1v_2^1v_1^3$ to start the growth process.

**Lemma 6.** The set $\mathcal{E} \cup \mathcal{F}$ contains no 3 edges with a total of 4 vertices.

**Proof.** Let us first consider 4-tuples of vertices in $A_2^*$. Note that all edges in $\mathcal{E}$ contain two odd vertices and one even vertex. Hence, a set of 4 odd vertices in $A_2^*$ spans no edges, and a set of 4 vertices in $A_2^*$ with at most 2 of them odd spans at most 2 edges. Finally, a set of 4 vertices in $A_2^*$ with 3 of them odd cannot span 3 edges since at least two of the odd vertices are either both in $A_1$, or both in $A_2$, and hence cannot participate in the same edge as they are not consecutive.

Now, consider 4-tuples containing the vertex $v_1^3$. On the one hand, $A_1$ and $A_2$ are independent sets in the 3-graph $\mathcal{E}$, and moreover the graph with edges $\{e \setminus v_1^3 : v_1^3 \in e \in \mathcal{F}\}$ contains no triangles. Thus,
every 4-tuple containing \( v_1^3 \) and three vertices in \( A_1 \) (respectively in \( A_2 \)) spans no edges of \( \mathcal{E} \) and at most 2 edges of \( \mathcal{F} \). It remains to observe that every 4-tuple containing \( v_1^3 \) and at least one vertex from both \( A_1 \) and \( A_2 \) cannot span more than 1 edge of \( \mathcal{F} \) (so 2 edges in total) since \( \mathcal{F} \) contains no edges intersecting both \( A_1 \) and \( A_2 \). □

We are now ready to conclude the proof of Proposition 5 for \( r = 3 \) by showing the following statement, illustrated in Fig. 2 for \( s = 1 \) and in Fig. 1 for \( s = k - 1 \). The proof is a straightforward but tedious verification.

**Lemma 7.** The 3-graph \( H_1 = \mathcal{E} \cup \mathcal{F} \cup \{e_0\} \) is \((e_i)_{i=0}^{T_1}\)-sequential for

\[
\sum_{j=1}^{s-1} (16(k - j) - 4) < i \leq \sum_{j=1}^{s} (16(k - j) - 4)
\]

(which clearly exists since \( T_1 = \sum_{j=1}^{k-1} (16(k - j) - 4) \)). Moreover, define

\[
\alpha = \alpha(i) = i - \sum_{j=1}^{s-1} (16(k - j) - 4) \in [1, 16(k - s) - 4].
\]

Then,

- if \( \alpha \in [1, 4(k - s)] \), then \( e_i = v_{2s-1}^1v_{2s-1+\alpha}^2v_1^3 \);
- if \( \alpha \in [4(k - s) + 1, 8(k - s)] \), then \( e_i = v_{6s-4k-1+\alpha}^1v_{4k-2s-1}^2v_1^3 \);
- if \( \alpha \in [8(k - s) + 1, 12(k - s) - 2] \), then \( e_i = v_{10s-4k-2s-1}^1v_{4k-10s-1-\alpha}^2v_1^3 \);
- if \( \alpha \in [12(k - s) - 1, 16(k - s) - 4] \), then \( e_i = v_{16k-14s-3-\alpha}^1v_{2s+1}^2v_1^3 \).

Note that the sequence of 3-edges indicated in Lemma 7 may be alternatively described in the following more intuitive way (see Fig. 2). At the beginning, the process first infects the edges \( v_1^1v_j^2v_1^3 \) for \( j \in [2, 4k - 3] \), then \( v_1^1v_{4k-3}^2v_1^3 \) for \( j \in [2, 4k - 3] \), then \( v_1^1v_{4k-3}^2v_1^3 \) for \( j \in [3, 4k - 4] \) in reverse order, then \( v_j^1v_1^2v_1^3 \) for \( j \in [3, 4k - 4] \) in reverse order. At this point, \( s \) increases from 1 to 2. The process continues to propagate in a similar way except that, every time \( s \) increases, the “ends” shift two vertices towards the middle (so \( v_1^3 \) plays the role of \( v_1^1 \) when \( s \) is incremented for the first time).

**Proof.** To begin with, the \( F_3 \)-bootstrap percolation process on the complete 3-graph with vertices \( A_2^3 \cup \{v_1^3\} \) and initially infected 3-graph \( H_1 \setminus \{e_0\} = \mathcal{E} \cup \mathcal{F} \) is stationary by Lemma 6. Now, set \( G_{-1} = \mathcal{E} \cup \mathcal{F}, \ G_0 = H_1 \) and let the sequence \((G_i)_{i \geq 1}\) be defined as in (1). We will show by induction on \( i \geq 0 \) that \( G_i \setminus G_{i-1} = \{e_i\} \).

For \( i = 0 \) the statement is clear by construction. Suppose that for some \( i \geq 0 \), the induction hypothesis is satisfied for \( i \). Then, all edges in \( G_{i+1} \setminus G_i \) must span 4 vertices with the edge \( e_i \), and in particular must intersect \( e_i \) in 2 vertices. We consider 4 cases according to the interval containing \( \alpha = \alpha(i) \). For convenience, we extend the notation by setting \( s(0) = 1 \) and \( \alpha(0) = 0 \) (so that \( \alpha(i) = 0 \) only when \( s(i) = 1 \), which is consistent with \( e_0 = v_{2s-1}^1v_{2s-1+\alpha}^2v_1^3 \)).

If \( \alpha \in [0, 4(k - s)] \), then \( e_i = v_{2s-1}^1v_{2s-1+\alpha}^2v_1^3 \). One may easily check that the vertices in \( A_2^* \) that complete the pair \( v_{2s-1}^1v_{2s-1+\alpha}^2 \) to an edge in \( G_{i-1} \) are

\[
\begin{cases}
 v_{2s+\alpha}^2 & \text{if } \alpha = 0, \\
 v_{2s-2+\alpha}^2, v_{2s+\alpha}^2 & \text{if } \alpha \in [1, 4(k - s) - 1], \\
 v_{2s-2+\alpha}^1, v_{2s}^1 & \text{if } \alpha = 4(k - s). 
\end{cases}
\]
The vertices that complete the pair $v_{2s-1}^3$ to an edge in $G_{i-1}$ are
\[
\begin{cases}
(v_{2s-1+j}^2)_{j\in[0,\alpha-1]}, v_{1s}^3 & s = 1, \\
(v_{2s-1+j}^2)_{j\in[0,\alpha-1]}, v_{1s}^3, (v_{2s-2}^3)_{j\in[2s-1]∪[2k-s+1,2k-1]} & s ≥ 2.
\end{cases}
\]

Finally, the vertices that complete the pair $v_{2s-1+\alpha}^3$ to an edge in $G_{i-1}$ are
\[
\begin{cases}
v_{2s+\alpha}^3 & \alpha = 0, \\
v_{2s+\alpha}^3, v_{2s+\alpha}^3 & \alpha ∈ [1, 4(k-s) - 1], \\
v_{2s+\alpha}^3, v_{1s}^3 & \alpha = 4(k-s).
\end{cases}
\]

However, for $\alpha ≠ 0$, $\{v_{2s+\alpha}^3\} \cup e_i$ already spans a copy of $F_3$ in $G_i$, so the only edge that is generated from $G_i$ is $e_{i+1}$.

If $\alpha ∈ \{4(k-s) + 1, 8(k-s)\}$, then $e_i = v_{6s-4k-1+\alpha}^3 v_{4k-2s-1}^3$. One may easily check that the vertices in $A_2^\alpha$ that complete the pair $v_{6s-4k-1+\alpha}^3 v_{4k-2s-1}^3$ to an edge in $G_{i-1}$ are
\[
\begin{cases}
v_{6s-4k-2+\alpha}^3, v_{6s-4k+\alpha}^3 & \alpha ≠ 8(k-s), \\
v_{4k-2s-2}^3, v_{4k-2s-2}^3 & \alpha = 8(k-s).
\end{cases}
\]

The vertices that complete the pair $v_{6s-4k-1+\alpha}^3 v_{1s}^3$ to an edge in $G_{i-1}$ are
\[
\begin{cases}
v_{4k-4}^3 & \alpha = 8(k-s), \\
v_{6s-4k-2+\alpha}^3, v_{6s-4k+\alpha}^3 & \text{otherwise}.
\end{cases}
\]

Finally, the vertices that complete the pair $v_{4k-2s-1}^3$ to an edge in $G_{i-1}$ are
\[
\begin{cases}
(v_{5-4k+j}^1)_{j\in[4(k-1),\alpha-1]}, v_{4k-4}^3 & s = 1, \\
(v_{6s-4k-1+j}^1)_{j\in[4(k-s),\alpha-1]}, (v_{2j-1}^3)_{j\in[1,s-1]∪[2k-s+1,2k-1]}, v_{4k-2s-2}^3, v_{4k-2s}^3 & s ≠ 1.
\end{cases}
\]

Then, the only vertices that complete at least two pairs are
\[
\begin{cases}
v_{4k-2s-2}^3, v_{4k-2s-2}^3 & \alpha = 8(k-s), \\
v_{6s-4k+\alpha}^3, v_{6s-4k-2+\alpha}^3 & \alpha ≠ 8(k-s).
\end{cases}
\]

However, $\{v_{6s-4k-2+\alpha}^3\} \cup e_i$ already spans a copy of $F_3$ in $G_i$, so the only edge that is generated from $G_i$ is $e_{i+1}$.

If $\alpha ∈ \{8(k-s) + 1, 12(k-s) - 2\}$, then $e_i = v_{4k-2s-1}^3 v_{12k-10s-1-\alpha}^3$. One may easily check that the vertices in $A_2^\alpha$ that complete the pair $v_{4k-2s-1}^3 v_{12k-10s-1-\alpha}^3$ to an edge in $G_{i-1}$ are
\[
\begin{cases}
v_{4k-2s-2}^3, v_{2s+2}^3 & \alpha = 12(k-s) - 2, \\
v_{12k-10s-2-\alpha}^3, v_{12k-10s-\alpha}^3 & \alpha ≠ 12(k-s) - 2.
\end{cases}
\]

The vertices that complete the pair $v_{4k-2s-1}^3 v_{1s}^3$ to an edge in $G_{i-1}$ are
\[
\begin{cases}
(v_{2k-11}^2)_{j\in[8s-8,\alpha-1]}, v_{4k-3}^3, v_{4k-4}^3 & s = 1, \\
(v_{2j-1}^3)_{j\in[2s,s-1], v_{2k-10s-1}^3} & s ≠ 1.
\end{cases}
\]
The vertices that complete the pair \(v_{12k-10s-\alpha}^2 v_1^3\) to an edge in \(G_{i-1}\) are

\[
(v_{2j-1})_{j \in [1,s] \cup [2k-s+1,2k-1]}, v_{12k-10s-2-\alpha}^2, v_{12k-10s-\alpha}^2.
\]

Then, the only vertices that complete at least two pairs are

\[
\begin{cases}
  v_{4k-2s-2}^1, v_{2s+2}^2, & \alpha = 12(k-s) - 2 \\
  v_{12k-10s-2-\alpha}^1, v_{12k-10s-\alpha}^2, & \alpha \neq 12(k-s) - 2.
\end{cases}
\]

However, \(\{v_{12k-10s-\alpha}^2\} \cup e_i\) already spans a copy of \(F_3\) in \(G_i\), so the only edge that is generated from \(G_i\) is \(e_{i+1}\).

If \(\alpha \in [12(k-s) - 1, 16(k-s) - 4]\), then \(e_i = v_{16k-14s-3-\alpha}^1 v_{2s+1}^2 v_1^3\). One may easily check that the vertices in \(A^*_2\) that complete the pair \(v_{16k-14s-3-\alpha}^1 v_{2s+1}^2\) to an edge in \(G_{i-1}\) are

\[
\begin{cases}
  v_{2k}^1, & \alpha = 16(k-s) - 4, s = k - 1; \\
  v_{2s+2}^1, v_{2s+2}^2, & \alpha = 16(k-s) - 4, s \neq k - 1; \\
  v_{16k-14s-4-\alpha}^1, v_{16k-14s-2-\alpha}^2, & \alpha \neq 16(k-s) - 4.
\end{cases}
\]

The vertices that complete the pair \(v_{16k-14s-3-\alpha}^1 v_1^3\) to an edge in \(G_{i-1}\) are

\[
(v_{2j-1})_{j \in [2,s] \cup [2k-s,2k-1]}, v_{16k-14s-4-\alpha}^1, v_{16k-14s-2-\alpha}^2.
\]

Finally, the vertices that complete the pair \(v_{2s+1}^2 v_1^3\) to an edge in \(G_{i-1}\) are

\[
(v_{16k-14s-3-j}^1)_{j \in [12(k-s)-2,\alpha-1]}, (v_{2j-1}^1)_{j \in [1,s] \cup [2k-s+1,2k-1]}, v_{2s}^2, v_{2s+2}^2.
\]

Then, the only vertices that complete at least two pairs are

\[
\begin{cases}
  v_{2k}^1, & \alpha = 16(k-s) - 4, s = k - 1; \\
  v_{2s+2}^1, v_{2s+2}^2, & \alpha = 16(k-s) - 4, s \neq k - 1; \\
  v_{16k-14s-4-\alpha}^1, v_{16k-14s-2-\alpha}^2, & \alpha \neq 16(k-s) - 4.
\end{cases}
\]

However, \(\{v_{16k-14s-2-\alpha}^2\} \cup e_i\) already spans a copy of \(F_3\) in \(G_i\), so the only edge that is generated from \(G_i\) is \(e_{i+1}\), and the process stops when it infects \(e_{T_1}\).

It remains to show that the \(F_3\)-bootstrap percolation process on the complete 3-graph with vertices \(A^*_2 \cup \{v_3^1\}\) and initially infected 3-graph \(\mathcal{E} \cup \mathcal{F} \cup \{e_{T_1}\}\) infects the edges \(\{e_i\}_{i=0}^{T_1-1}\) in reverse order. Notice that in the process started from \(G_0\), every copy of \(F_3\) that becomes completely infected contains two edges in \(\mathcal{E} \cup \mathcal{F}\) and the edges \(e_i\) and \(e_{i+1}\) for some \(i \geq 0\). Therefore, starting from \(\mathcal{E} \cup \mathcal{F} \cup \{e_{T_1}\}\), the edges that do become infected are exactly \(\{e_i\}_{i=0}^{T_1-1}\). Moreover, all copies of \(F_3\) in \(G_{T_1}\) are spanned by some of the vertex sets \(\{e_{i-1} \cup e_i\}_{i=1}^{T_1}\), which implies that the bootstrap percolation with initial condition \(\mathcal{E} \cup \mathcal{F} \cup \{e_{T_1}\}\) infects the edges \(\{e_i\}_{i=0}^{T_1-1}\) in the reverse order.

\[\square\]

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**Note.** A few days after the current article appeared on arxiv, an independent proof of a version of Theorem 2 was found by Espunya Díaz, Janzer, Kronenberg and Lada 4.
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