THE PICARD GROUPS OF UNITAL INCLUSIONS OF UNITAL 
C*-ALGEBRAS INDUCED BY INVOLUTIVE EQUIVALENCE 
BIMODULES

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Abstract. Let $A$ be a unital $C^*$-algebra and $X$ an involutive $A-A$-equivalence bimodule. Let $A \subset C_X$ be the unital inclusion of unital $C^*$-algebras induced by $X$. We suppose that $A' \cap C_X = C_1$. We shall compute the Picard group of the unital inclusion $A \subset C_X$.

1. Introduction

Let $A$ be a unital $C^*$-algebra and $X$ an $A-A$-equivalence bimodule. Following [7], we say that $X$ is involutive if there exists a conjugate linear map $x \mapsto x^\#$ on $X$ such that

1. $(x^\#)^\# = x$, $x \in X$,
2. $(a \cdot x \cdot b)^\# = b^* \cdot x^\# \cdot a^*$, $x \in X$, $a, b \in A$,
3. $A \langle x, y^\# \rangle = \langle x^\#, y \rangle_A$, $x, y \in X$,

where $A(-, -)$ and $(-, -)_A$ are the left and the right $A$-valued inner products on $X$, respectively. We call the above conjugate linear map an involution on $X$. For each $A-A$-equivalence bimodule $X$, $\tilde{X}$ denotes its dual $A-A$-equivalence bimodule. For each $x \in X$, $\tilde{x}$ denotes the element in $\tilde{X}$ induced by $x$. For each involutive $A-A$-equivalence bimodule $X$, let $L_X$ be the linking $C^*$-algebra for $X$ defined in Brown, Green and Rieffel [1]. Following [7], we define the $C^*$-subalgebra $C_X$ of $L_X$ by

$$C_X = \{ \begin{bmatrix} a & x \\ x^\# & a \end{bmatrix} | a \in A, x \in X \}.$$ 

We regard $A$ as a $C^*$-subalgebra of $C_X$, that is, $A = \{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} | a \in A \}$.

In [5], we defined the Picard of unital inclusion of unital $C^*$-algebras $A \subset C$. We denote it by $\text{Pic}(A, C)$. In this paper, we shall compute $\text{Pic}(A, C_X)$ under the assumption that $A' \cap C_X = C_1$. Let us explain the strategy of computing $\text{Pic}(A, C_X)$. Let $f_A$ be the homomorphism of $\text{Pic}(A, C_X)$ to $\text{Pic}(A)$ defined in [3], where $\text{Pic}(A)$ is the Picard group of $A$. We compute $\text{Ker} f_A$ and $\text{Im} f_A$, the kernel of $f_A$ and the image of $f_A$, respectively and we construct a homomorphism $g_A$ of $\text{Im} f_A$ to $\text{Pic}(A, C_X)$ with $f_A \circ g_A = \text{id}_{\text{Pic}(A)}$. We can compute $\text{Pic}(A, C_X)$ in the above way.

2. Preliminaries

We recall the definition of the Picard group for a unital inclusion of unital $C^*$-algebras $A \subset C$. Let $Y$ be a $C-C$-equivalence bimodule and $X$ its closed subspace satisfying Conditions (1), (2) in [11 Definition 2.1]. Let $\text{Equi}(A, C)$ be the set of all such pairs $(X, Y)$ as above. We define an equivalence relation “$\sim$” as follows:

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For \((X, Y), (Z, W) \in \text{Equi}(A, C)\), \((X, Y) \sim (Z, W)\) in \text{Equi}(A, C) if and only if there is a \(C - C\)-equivalence bimodule isomorphism \(\Phi\) of \(Y\) onto \(W\) such that the restriction of \(\Phi\) to \(X\), \(\Phi|_X\), is an \(A - A\)-equivalence bimodule isomorphism \(X\) onto \(Z\). We denote by \([X, Y]\), the equivalence class of \((X, Y)\) in \text{Equi}(A, C). Let \(\text{Pic}(A, C) = \text{Equi}(A, C)/\sim\). We define the product in \(\text{Pic}(A, C)\) as follows: For \((X, Y), (Z, W) \in \text{Pic}(A, C)\)

\[ [X, Y][Z, W] = [X \otimes_A Z, Y \otimes_C W], \]

where the \(A - A\)-equivalence bimodule \(X \otimes_A Z\) is identified with the closed subspace “\(X \otimes_C Z\)” of \(Y \otimes_C W\) by \([5\text{ Lemma 3.1}]\) and “\(X \otimes_C Z\)” is defined by the closure of linear span of the set

\[ \{x \otimes z \in Y \otimes_C W \mid x \in X, z \in Z\} \]

by \([5]\) and easy computations, \(Y \otimes_C W\) and its closed subspace \(X \otimes_A Z\) satisfy Conditions (1), (2) in \([11\text{ Definition 2.1}]\) and \(\text{Pic}(A, C)\) is a group. We regard \((A, C)\) as an element in \(\text{Equi}(A, C)\) in the evident way. Then \([A, C]\) is unit element in \(\text{Equi}(A, C)\) in \(\text{Pic}(A, C)\). For any element \((X, Y) \in \text{Equi}(A, C)\), \((\tilde{X}, \tilde{Y}) \in \text{Equi}(A, C)\) and \([\tilde{X}, \tilde{Y}]\) is the inverse element of \([X, Y]\) in \(\text{Pic}(A, C)\). We call the group \(\text{Pic}(A, C)\) defined in the above, the Picard group of the unital inclusion of unital \(C^*\)-algebras \(A \subset C\).

Let \(f_A\) be the homomorphism of \(\text{Pic}(A, C)\) to \(\text{Pic}(A)\) defined by

\[ f_A([X, Y]) = [X] \]

for any \((X, Y) \in \text{Equi}(A, C)\).

3. Kernel

Let \(A\) be a unital \(C^*\)-algebra and \(X\) an involutive \(A - A\)-equivalence bimodule. Let \(A \subset C_X \) be the unital inclusion of unital \(C^*\)-algebras induced by \(X\) and we suppose that \(A' \cap C_X = C_1\). Let \(f_A\) be the homomorphism of \(\text{Pic}(A, C_X)\) to \(\text{Pic}(A)\) defined by

\[ f_A([M, N]) = [M] \]

for any \((M, N) \in \text{Equi}(A, C_X)\). In this section, we compute \(\text{Ker} f_A\). Let \((M, N) \in \text{Equi}(A, C_X)\). We suppose that \([M, N] \in \text{Ker} f_A\). Then \([M] = [A]\) in \(\text{Pic}(A)\) and by \([5\text{ Lemma 7.5}]\), there is a \(\beta \in \text{Aut}_0(A, C_X)\) such that

\[ [M, N] = [A, N_\beta] \]

in \(\text{Pic}(A, C_X)\) where \(\text{Aut}_0(A, C_X)\) is the group of all automorphisms \(\beta\) such that \(\beta(a) = a\) for any \(a \in A\) and \(N_\beta\) is the \(C_X - C_X\)-equivalence bimodule induced by \(\beta\) which is defined in \([5\text{ Section 2}]\). By the above discussions, we obtain the following lemma.

**Lemma 3.1.** With the above notation,

\[ \text{Ker} f_A = \{[A, N_\beta] \in \text{Pic}(A, C_X) \mid \beta \in \text{Aut}_0(A, C_X)\}. \]

Let \(\text{Aut}(A, C_X)\) be the group of all automorphisms \(\alpha\) of \(C_X\) such that the restriction of \(\alpha\) to \(A\), \(\alpha|_A\), is an automorphism of \(A\). Then \(\text{Aut}_0(A, C_X)\) is a normal subgroup of \(\text{Aut}(A, C_X)\). Let \(\pi\) be the homomorphism of \(\text{Aut}(A, C_X)\) to \(\text{Pic}(A, C_X)\) defined by

\[ \pi(\alpha) = [M_\alpha, N_\alpha] \]

for any \(\alpha \in \text{Aut}(A, C_X)\), where \((M_\alpha, N_\alpha)\) is the element in \(\text{Equi}(A, C_X)\) induced by \(\alpha \in \text{Aut}(A, C_X)\) (See \([5\text{ Section 3}]\)). By Lemma \([5.1]\) \(\pi(\text{Aut}_0(A, C_X)) = \text{Ker} f_A\) and \([5\text{ Lemma 3.4}]\),

\[ \text{Ker} f_A \cap \text{Aut}_0(A, C_X) = \text{Int}(A, C_X) \cap \text{Aut}_0(A, C_X), \]
where $\text{Int}(A, C_X)$ is the group of all $\text{Ad}(u)$ such that $u$ is a unitary element in $A$. Hence

$$\ker \pi \cap \text{Aut}_0(A, C_X) = \{\text{Ad}(u) \in \text{Aut}_0(A, C_X) \mid u \text{ is a unitary element in } A\} = \{\text{Ad}(u) \in \text{Aut}_0(A, C_X) \mid u \text{ is a unitary element in } A' \cap A\}.$$ 

Since $A' \cap C_X = C_1$, $A' \cap A = C_1$. Thus we can see that $\ker \pi \cap \text{Aut}_0(A, C_X) = \{1\}$. It follows that we can obtain that the following lemma.

**Lemma 3.2.** With the above notation, $\ker f_A \cong \text{Aut}_0(A, C_X)$.

Let $\text{Aut}^\theta_0(X)$ be the group of all involutive $A - A$-equivalence bimodule automorphisms of $X$. Let $E^A$ be the conditional expectation from $C_X$ onto $A$ defined by

$$E^A\left[\begin{array}{c} a \\ x^2 \\ a \end{array}\right] = \left[\begin{array}{c} a \\ 0 \\ a \end{array}\right]$$

for any $a \in A$, $x \in X$. Then $E^A$ is of Watatani index-finite type by [7, Lemma 3.4].

**Lemma 3.3.** With the above notation, $E^A = E^A \circ \beta$ for any $\beta \in \text{Aut}_0(A, C_X)$.

**Proof.** Let $\beta \in \text{Aut}_0(A, C_X)$. Then $E^A \circ \beta$ is also a conditional expectation from $C_X$ onto $A$. Since $A' \cap C_X = C_1$, by Watatani [17 Proposition 1.4.1], $E^A = E^A \circ \beta$. □

**Lemma 3.4.** With the above notation, for any $\beta \in \text{Auto}^\theta_0(A, C_X)$, there is the unique $\theta \in \text{Aut}^\theta_0(X)$ such that

$$\beta\left[\begin{array}{c} a \\ x^2 \\ a \end{array}\right] = \left[\begin{array}{c} a \\ \theta(x) \\ a \end{array}\right]$$

for any $a \in A$, $x \in X$.

**Proof.** For any $x \in X$, let

$$\beta\left[\begin{array}{c} 0 \\ x^2 \\ 0 \end{array}\right] = \left[\begin{array}{c} b \\ y^2 \\ b \end{array}\right],$$

where $b, y \in X$. Then by Lemma 3.3

$$\left[\begin{array}{c} b \\ 0 \\ b \end{array}\right] = (E^A \circ \beta)\left[\begin{array}{c} 0 \\ x^2 \\ 0 \end{array}\right] = E^A\left[\begin{array}{c} 0 \\ x^2 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right].$$

Hence $b = 0$. Thus

$$\beta\left[\begin{array}{c} 0 \\ x^2 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ y^2 \\ 0 \end{array}\right].$$

We define a map $\theta$ on $X$ by

$$\theta(x) = y,$$

where $y$ is the element in $X$ defined as above. Then clearly $\theta$ is linear and since

$$\beta\left[\begin{array}{c} 0 \\ x^2 \\ 0 \end{array}\right] = \beta\left[\begin{array}{c} 0 \\ x^2 \\ 0 \end{array}\right]^* = \left[\begin{array}{c} 0 \\ y^2 \\ 0 \end{array}\right],$$

we obtain that

$$\theta(x^2) = y^2 = \theta(x)^2.$$ 

Hence $\theta$ preserves the involution $\sharp$. Also, for any $a \in A$, $x \in X$,

$$\left[\begin{array}{c} 0 \\ \theta(a \cdot x) \\ \theta((a \cdot x)^2) \end{array}\right] = \beta\left[\begin{array}{c} 0 \\ a \cdot x \\ a \cdot x^2 \end{array}\right] = \beta\left[\begin{array}{c} 0 \\ a \\ a \cdot x \end{array}\right] = \left[\begin{array}{c} 0 \\ a \\ a \cdot \theta(x) \end{array}\right].$$

\[3\]
Hence \( \theta(a \cdot x) = a \cdot \theta(x) \) for any \( a \in A, x \in X \). Similarly \( \theta(x \cdot a) = \theta(x) \cdot a \) for any \( a \in A, x \in X \). Furthermore, for any \( x, y \in X \),
\[
\begin{bmatrix}
A(\theta(x), \theta(y)) & 0 \\
0 & A(\theta(x), \theta(y))
\end{bmatrix}
= \begin{bmatrix}
0 & \theta(x) \\
\frac{\theta(x)^y}{\theta(x)^y} & 0
\end{bmatrix}
\begin{bmatrix}
0 & \theta(y) \\
\frac{\theta(y)^y}{\theta(y)^y} & 0
\end{bmatrix}
= \beta\left( \begin{bmatrix}
\frac{\theta(x)^y}{\theta(x)^y} & 0 \\
0 & \frac{\theta(y)^y}{\theta(y)^y}
\end{bmatrix}\right)
= \beta\left( \begin{bmatrix}
A(x, y) & 0 \\
0 & A(x, y)
\end{bmatrix}\right)
= \left( \begin{bmatrix}
0 & y^y \\
0 & A(x, y)
\end{bmatrix}\right).
\]
Hence \( A(\theta(x), \theta(y)) = A(x, y) \) for any \( x, y \in X \). Similarly for any \( x, y \in X \),
\[
\begin{bmatrix}
\langle \theta(x), \theta(y) \rangle_A & 0 \\
0 & \langle \theta(x), \theta(y) \rangle_A
\end{bmatrix}
= \begin{bmatrix}
(x, y)_A & 0 \\
0 & (x, y)_A
\end{bmatrix},
\]
Hence \( \langle \theta(x), \theta(y) \rangle_A = (x, y)_A \) for any \( x, y \in X \). Thus \( \theta \in A \text{Aut}^2_A(X) \). Next, let \( \theta \in A \text{Aut}^3_A(X) \). Then let \( \beta \) be a map on \( C_X \) defined by
\[
\beta\left( \begin{bmatrix}
a & x \\
x^a & a
\end{bmatrix}\right) = \begin{bmatrix}
a & \theta(x) \\
\frac{x}{\theta(x)^x} & a
\end{bmatrix}
\]
for any \( a \in A, x \in X \). Then by easy computations, \( \beta \in \text{Auto}_0(A, C_X) \). Therefore, we obtain the conclusion.

Corollary 3.5. With the above notation, \( \text{Auto}_0(A, C_X) \cong A \text{Aut}^2_A(X) \).

Proof. This is immediate by Lemma 3.4.

Proposition 3.6. With the above notation, \( \text{Ker}_{FA} \cong T_1 \).

Proof. This is immediate by Lemma 3.2 Corollary 3.3 and the above discussions.

4. A RESULT ON STRONGLY MORITA EQUIVALENT UNITAL INCLUSIONS OF UNITAL C*-ALGEBRAS

In this section, we shall prove the following result: Let \( H \) be a finite dimensional C*-Hopf algebra and \( H^0 \) its dual C*-Hopf algebra. Let \((\rho, u)\) and \((\sigma, v)\) be twisted coactions of \( H^0 \) on unital C*-algebras \( A \) and \( B \), respectively. Let \( A \subset A \times_{\rho,u} H \) and \( B \subset B \times_{\sigma,v} H \) be unital inclusions of unital C*-algebras. We suppose that they are strongly Morita equivalent with respect to \( A \times_{\rho,u} H \) and \( B \times_{\sigma,v} H \)-equivalence bimodule \( Y \) and its closed subspace \( X \). And we suppose that \( A' \cap (A \times_{\rho,u} H) = C_1 \). Then there are a twisted coaction \((\gamma, w)\) of \( H^0 \) on \( B \) and a twisted coaction \( \lambda \) of \( H^0 \) on \( X \) satisfying the following:

1. \((\rho, u)\) and \((\gamma, w)\) are strongly Morita equivalent with respect to \( \lambda \),
2. \( B \times_{\sigma,v} H = B \times_{\gamma,w} H \),
3. \( Y \cong X \times_{\lambda} H \) as \( A \times_{\rho,u} H \) and \( B \times_{\sigma,v} H \)-equivalence bimodules.
In the next section, we shall use this result in the case of \( Z_2 \)-actions, where \( Z_2 = \mathbb{Z}/2\mathbb{Z} \). We shall use the results in [12] in order to prove the above result. First we recall [12].

Let \( H \) be a finite dimensional \( C^* \)-Hopf algebra. We denote its comultiplication, counit and antipode by \( \Delta, \epsilon \), and \( S \), respectively. We shall use Sweedler’s notation
\[
\Delta(h) = h(1) \otimes h(2)
\]
for any \( h \in H \) which suppresses a possible summation when we write comultiplications. We denote by \( N \) the dimension of \( H \). Let \( H^0 \) be the dual \( C^* \)-Hopf algebra of \( H \). We denote its comultiplication, counit and antipode by \( \Delta^0, \epsilon^0 \) and \( S^0 \), respectively. There is the distinguished projection \( e \in H \). We note that \( e \) is the Haar trace on \( H^0 \). Also, there is the distinguished projection \( \tau \) in \( H^0 \) which is the Haar trace on \( H \). Since \( H^0 \) is finite dimensional, \( H^0 \cong \otimes_{k=1}^K M_{d_k}(\mathbb{C}) \) as \( C^* \)-algebras, where \( M_n(\mathbb{C}) \) is the \( n \times n \) matrix algebra over \( \mathbb{C} \). Let
\[
\{ w_{ij}^k \mid k = 1, 2, \ldots, K, i, j = 1, 2, \ldots, d_k \}
\]
be a basis of \( H \) satisfying Szymański and Peligrad’s [10] Theorem 2.2,2], which is called a system of \textit{comatrix units} of \( H \), that is, the dual basis of a system of matrix units of \( H^0 \).

Let \( A \) be a unital \( C^* \)-algebra and \( (\rho, u) \) a twisted coaction of \( H^0 \) on \( A \), that is, \( \rho \) is a weak coaction of \( H^0 \) on \( A \) and \( u \) is a unitary element in \( A \otimes H^0 \otimes H^0 \) satisfying that
\begin{enumerate}
\item \( (\rho \otimes \text{id}) \circ \rho = \Delta \rho \circ (\text{id} \otimes \Delta^0) \circ \rho , \)
\item \( (u \otimes 1^0)(\text{id} \otimes \Delta^0 \otimes \text{id})(u) = (\rho \otimes \text{id} \otimes \text{id})(u)(\text{id} \otimes \text{id} \otimes \Delta^0)(u) , \)
\item \( (\text{id} \otimes h \otimes \epsilon^0)(u) = (h \otimes \epsilon^0 \otimes h)(u) = \epsilon^0(h)1 \) for any \( h \in H . \)
\end{enumerate}
For a twisted coaction \( (\rho, u) \) of \( H^0 \) on \( A \), we can consider the twisted action of \( H \) on \( A \) and its unitary element \( \tilde{\rho} \) defined by
\[
\tilde{\rho}(x, h, l) = (\text{id} \otimes h \otimes l)(u)
\]
for any \( x \in A, h, l \in H \). We call it the twisted action of \( H \) on \( A \) induced by \( (\rho, u) \). Let \( A \rtimes_{\rho, u} H \) be the twisted crossed product of \( A \) by the twisted action of \( H \) induced by \( (\rho, u) \). Let \( x \rtimes_{\rho, u} h \) be the element in \( A \rtimes_{\rho, u} H \) induced by \( x \in A \) and \( h \in H \). Let \( \tilde{\rho} \) be the dual coaction of \( H \) on \( A \rtimes_{\rho, u} H \) defined by
\[
\tilde{\rho}(x \rtimes_{\rho, u} h) = (x \rtimes_{\rho, u} h(1)) \otimes h(2)
\]
for any \( x \in A, h \in H \). Let \( E_{1}^{\rho, u} \) be the canonical conditional expectation from \( A \rtimes_{\rho, u} H \) onto \( A \) defined by
\[
E_{1}^{\rho, u}(x \rtimes_{\rho, u} h) = \tau(h)x
\]
for any \( x \in A, h \in H \). Let \( \Lambda \) be the set of all triplets \((i, j, k)\), where \( i, j = 1, 2, \ldots, d_k \) and \( k = 1, 2, \ldots, K \) with \( \sum_{k=1}^K d_k^2 = N \). Let \( W_{I}^{\rho, u} = \sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k \) for any \( I = (i, j, k) \in \Lambda \). By [9] Proposition 3.18, \( \{ (W_{I}^{\rho, u}, W_{I}^{\rho, u}) \}_{I \in \Lambda} \) is a quasi-basis for \( E_{1}^{\rho, u} \).

Let \( A \) and \( B \) be unital \( C^* \)-algebras and let \( (\rho, u) \) and \( (\sigma, v) \) be twisted coactions of \( H^0 \) on \( A \) and \( B \), respectively. Let \( A \rtimes_{\rho, u} H \) and \( B \rtimes_{\sigma, v} H \) be the twisted crossed products of \( A \) and \( B \) by \( (\rho, u) \) and \( (\sigma, v) \), respectively. We denote them by \( C \) and \( D \), respectively. Then we obtain unital inclusions of unital \( C^* \)-algebras, \( A \subset C \) and \( B \subset D \). We suppose that \( A \subset C \) and \( B \subset D \) are strongly Morita equivalent with respect to a \( C \sim D \)-equivalence bimodule \( Y \) and its closed subspace \( X \). We also suppose that \( A' \cap C = C_1 \). Then \( B' \cap D = C_1 \) by [11] Lemma 10.3. And by [11] Theorem 2.9, there are a conditional expectation \( F^B \) from \( D \) onto \( B \) and a conditional expectation \( E^X \) from \( Y \) onto \( X \) with respect to \( E_1^{\rho, u} \) and \( F^B \) satisfying Conditions (1)–(6) in [11] Definition 2.4. Since \( B' \cap D = C_1 \), by Watatani [12] Proposition 1.4.1], \( F^B = E_{1}^{\sigma, v} \), the canonical conditional expectation from \( D \) onto \( B \). Furthermore, by [11] Section 6], we can see that the unital inclusions
Lemma 4.1. $y \in C$ for any expectations from $C$ of unital $Y$ where we regard $D$ for any $y$.

Also, we define the action of $Y$ on $X$ with respect to the $C$-algebras, respectively. Let $Y = C \otimes_A X \otimes_B D$. Let $E^Y$ be the conditional expectation from $Y$ onto $Y$ with respect to $E^Y_{\rho,u}$ and $E^Y_{\sigma,v}$ defined by

$$E^Y(c \otimes x \otimes d) = \frac{1}{N} c \cdot x \cdot d$$

for any $c \in C$, $d \in D$, $x \in X$, where $E^Y_{\rho,u}$ and $E^Y_{\sigma,v}$ are the canonical conditional expectations from $C$ and $D$ onto $C$ and $D$, respectively.

We regard $Y$ as a closed subspace of $Y$ by the injective linear map $\phi$ from $Y$ into $Y$ defined by

$$\phi(y) = \sum_{i,j \in A} W^y_{ij} \otimes E^X(W^y_{ij} \cdot y \cdot W^y_{ij}) \otimes \tilde{W}^X_{ij}$$

for any $y \in Y$. By Remark 3.1, for any $\psi \in H^0$, $y \in Y$,

$$\psi \cdot \mu = \sum_{i \in A} [\psi \cdot \hat{\rho} W^\psi_{ii}] \cdot E^X(W^\psi_{ii} \cdot y).$$

Also, we define the action of $H^0$ on $D$ induced by $\beta$ as follows: For any $\psi \in H^0$, $y, z \in Y$,

$$\psi \cdot \beta (y, z) = (S^0(\psi_{ij}^{(1)}) \cdot \mu y, \psi_{ij}^{(2)} \cdot \mu z)_D,$$

where we regard $D$ as the linear span of the set $\{(y, z) \mid y, z \in Y\}$.

Lemma 4.1. For any $y \in Y$, $\tau \cdot \mu = E^X(y)$.

Proof. By routine computations, we obtain the lemma. Indeed, by Remark 3.1, for any $y \in Y$,

$$\tau \cdot \mu = \sum_{i \in A} [\tau \cdot \hat{\rho} W^\tau_{ii}] \cdot E^X(W^\tau_{ii} \cdot y)$$

$$= \sum_{i,j,k} [\tau \cdot \hat{\rho} (\sqrt{d_k} \cdot \rho \cdot u w_{ij}^k)^*] \cdot E^X(\sqrt{d_k} \cdot \rho \cdot u w_{ij}^k \cdot y)$$

$$= \sum_{i,j,l,j_2,j_3 \in A} [\tau \cdot \hat{\rho} (\hat{u}(S(w_{j_1,j_2}^k), w_{ij_1}^k)^* \cdot \rho \cdot u w_{j_{23}}^k \cdot \sqrt{d_k} \cdot \rho \cdot u w_{j_{23}}^k)]$$

$$\cdot E^X(\sqrt{d_k} \cdot \rho \cdot u w_{ij}^k \cdot y)$$

$$= \sum_{i,j,l,j_2,j_3 \in A} \sqrt{d_k} [\tau \cdot \hat{\rho} (\hat{u}(S(w_{j_1,j_2}^k), w_{ij_1}^k)^* \cdot \rho \cdot u w_{j_{23}}^k)] \cdot E^X(\sqrt{d_k} \cdot \rho \cdot u w_{ij}^k \cdot y)$$

$$= \sum_{i,j,l,j_2,j_3 \in A} \sqrt{d_k} [\hat{u}(S(w_{j_1,j_2}^k), w_{ij_1}^k)^* \cdot \rho \cdot u w_{j_{23}}^k \cdot \tau(w_{j_{23}}^k)]$$

$$\cdot E^X(\sqrt{d_k} \cdot \rho \cdot u w_{ij}^k \cdot y)$$

$$= \sum_{i,j,l,j_2,j_3 \in A} \sqrt{d_k} [\hat{u}(S(w_{j_1,j_2}^k), w_{ij_1}^k)^* \cdot \rho \cdot u \tau(w_{j_{23}}^k)] \cdot E^X(\sqrt{d_k} \cdot \rho \cdot u w_{ij}^k \cdot y).$$
Since \( \tau \circ S^0 = \tau, \tau(w_{ijj}^k) = \tau(S(w_{ijj}^k)) = (\tau \circ S^0)(w_{ijj}^k) = \tau(w_{ijj}^k). \) Hence

\[
\tau \cdot y = \sum_{i,j_1,j_2,k} \sqrt{d_k} (\tilde{u}(S(w_{ijj}^k), w_{ij_1}^k, w_{ij_2}^k) \cdot E^X((\sqrt{d_k} \times_{\rho,u} w_{ij_1}^k \tau(w_{ijj}^k)) \cdot y)) = \sum_{i,j_1,j_2,k} \sqrt{d_k} (\tilde{u}(S(w_{ijj}^k), w_{ij_1}^k, w_{ij_2}^k) \cdot E^X((\sqrt{d_k} \times_{\rho,u} \tau(w_{ijj}^k)) \cdot y)) = \sum_{i,j_1,j_2,k} d_k (\tilde{u}(S(w_{ijj}^k), w_{ij_1}^k, w_{ij_2}^k) * \times_{\rho,u} 1) \cdot E^X(y).
\]

Since \( \tau = \tau^*, \tau(w_{ijj}^k) = \tau^*(w_{ijj}^k) = \tau(S(w_{ijj}^k)) = \tau(w_{ijj}^k). \) Hence

\[
\tau \cdot y = \sum_{i,j_1,j_2,k} d_k (\tilde{u}(S(w_{ijj}^k), w_{ij_1}^k, w_{ij_2}^k) \cdot \tau(w_{ijj}^k)) * \times_{\rho,u} 1) \cdot E^X(y) = \sum_{i,j_1,j_2,k} d_k (\tilde{u}(S(\tau(w_{ijj}^k))), \times_{\rho,u} 1) \cdot E^X(y) = \sum_{i,j_1,j_2,k} d_k (\tilde{u}(S(\tau(w_{ijj}^k))) \cdot E^X(y).
\]

Since \( \epsilon \circ S = \epsilon, \epsilon(S(\tau(w_{ijj}^k))) = \tau(w_{ijj}^k)1. \) Hence

\[
\tau \cdot y = \sum_{j_1,j_2,k} d_k (\tilde{u}(\tau(w_{ijj}^k))) E^X(y) = \sum_{j_1,j_2,k} d_k (\tilde{u}(w_{ijj}^k)1) E^X(y) = N\tau(\epsilon) E^X(y)(y) = E^X(y)
\]
since \( \epsilon = \epsilon(\sum_{j,k} d_k w_{ijj}^k) \). Therefore, we obtain the conclusion. \( \square \)

We recall that the unital inclusions of unital \( C^* \)-algebras, \( C \subset C_1 \) and \( D \subset D_1 \) are strongly Morita equivalent with respect to \( Y_1 \) and its closed subspace \( Y \). Also, \( C \subset C_1 \) and \( D \subset D \times_\beta H^0 \) are strongly Morita equivalent with respect to the \( C_1 - D \times_\beta H^0 \)-equivalence bimodule \( Y \times_\mu H^0 \) and its closed subspace \( Y_1 \), where \( Y \times_\mu H^0 \) is the crossed product of \( Y \) by the coaction \( \mu \) and it is a \( C_1 - D \times_\beta H^0 \)-equivalence bimodule (See [13]). Hence the unital inclusions \( D \subset D_1 \) and \( D \subset D \times_\beta H^0 \) are strongly Morita equivalent with respect to the \( D - D \times_\beta H^0 \)-equivalence bimodule \( \tilde{Y}_1 \otimes C_1(Y \times_\mu H^0) \) and its closed subspace \( \tilde{Y} \otimes C_1(Y \times_\mu H^0) \). Then since \( \tilde{Y} \otimes C_1 \) is isomorphic to \( D \) as \( D - D \times_\beta H^0 \)-equivalence bimodule, we can see that there is an isomorphism \( \Psi \) of \( D_1 \) onto \( D \times_\beta H^0 \) which is defined as follows: Since \( Y \) is a \( C_1 - D \)-equivalence bimodule, there are elements \( y_1, \ldots, y_n \in Y \) such that \( \sum_{i=1}^n \langle y_i, y_i \rangle_D = 1 \). Let \( \Psi \) be the map from \( D_1 \) to \( D \times_\beta H^0 \) defined by

\[
\Psi(d) = \sum_{i,j} \langle d \cdot \tilde{y}_i \otimes y_i, \tilde{y}_j \otimes y_j \rangle_{D \times_\beta H^0}
\]

for any \( d \in D_1 \). By [13] Section 5], \( \Psi \) is an isomorphism of \( D_1 \) onto \( D \times_\beta H^0 \) satisfying that \( \Psi(d) = d \) for any \( d \in D \) and that \( E^0_1 \circ \Psi = E^0_2 \), where \( E^0_1 \) is a canonical conditional expectation from \( D \times_\beta H^0 \) onto \( D \) and \( E^0_2 \) is the canonical conditional expectation from \( D_1 \) onto \( D \).

**Remark 4.2.** \( \tilde{Y} \) is a closed subspace of \( \tilde{Y}_1 \) by the inclusion \( \tilde{\phi} \) defined by

\[
\tilde{\phi}(\tilde{y}) = \tilde{\phi}(y)
\]

for any \( y \in Y \). Also, \( Y \) is a closed subspace \( Y \times_\mu H^0 \) by the inclusion defined by

\[
Y \longrightarrow Y \times_\mu H^0 : y \mapsto y \times_\mu 1^0.
\]
Lemma 4.3. With the above notation, \( \Psi(e_B) = \Psi(1 \times_\beta \tau) = 1 \times_\beta \tau \).

Proof. The lemma can be proved by routine computations. Indeed, we note that \( \tilde{Y} \) is regarded as a closed subspace of \( \tilde{Y}_1 \) by the inclusion \( \bar{\phi} \) and \( Y \) is regarded as a closed subspace \( Y \times_\beta 1^0 \) of \( Y \times_\beta H^0 \). Then

\[
\Psi(e_B) = \Psi(1 \times_\beta \tau) = \sum_{i,j} \langle (1 \times_\beta \tau) \cdot (\tilde{y}_i \otimes y_i), \tilde{y}_j \otimes y_j \rangle_{D \times_\beta H^0} \]

\[
= \sum_{i,j} \langle [y_i \cdot (1 \times_\beta \tau)] \otimes y_i, \tilde{y}_j \otimes y_j \rangle_{D \times_\beta H^0} \]

\[
= \sum_{i,j} \langle y_i, [y_i \cdot (1 \times_\beta \tau)] \rangle_{C \times_\beta H^0}. \]

We note that \( \tilde{y}_i, \tilde{y}_j \in \tilde{Y} \subset \tilde{Y}_1 \) and that \( y_i, y_j \in Y = Y \times_\mu 1^0 \subset Y \times_\mu H^0 \). Hence

\[
\Psi(e_B) = \sum_{i,j} \langle y_i \times_\mu 1^0, c_i \langle \phi(y_i) \cdot (1 \times_\beta \tau), \phi(y_j) \rangle \cdot y_j \rangle_{D \times_\beta H^0}. \]

Furthermore, let \( \{(u_k, u^*_l)\} \) and \( \{(v_l, v^*_l)\} \) be quasi-bases for \( E_1^{\sigma,u} \) and \( E_1^{\sigma,v} \), respectively. Then

\[
\phi(y_i) \cdot (1 \times_\beta \tau) = \phi(y_i) : e_B = \sum_{k,l} u_k \otimes E_X(u^*_k \cdot y_i \cdot v_l) \otimes \tilde{u}_l \cdot e_B \]

\[
= \sum_{k,l} u_k \otimes E_X(u^*_k \cdot y_i \cdot v_l) \otimes \tilde{E}_B(v_l) \]

\[
= \sum_{k,l} u_k \otimes E_X(u^*_k \cdot y_i \cdot v_l E_B(v^*_l)) \otimes \tilde{1}_D \]

\[
= \sum_{k,l} u_k \otimes E_X(u^*_k \cdot y_i) \otimes \tilde{1}_D. \]

Hence since \( 1 \times_\beta \tau \) is a projection in \( D_1 \),

\[
c_i \langle y_i \cdot (1 \times_\beta \tau), y_j \rangle = c_i \langle y_i \cdot (1 \times_\beta \tau), y_j \cdot (1 \times_\beta \tau) \rangle \]

\[
= \sum_{k,l} c_i \langle u_k \otimes E_X(u^*_k \cdot y_i) \otimes \tilde{1}_D, u_l \otimes E_X(u^*_l \cdot y_j) \otimes \tilde{1}_D \rangle \]

\[
= \sum_{k,l} c_i \langle u_k \cdot A(E_X(u^*_k \cdot y_i) \otimes \tilde{1}_D, E_X(u^*_l \cdot y_j) \otimes \tilde{1}_D), u_l \rangle \]

\[
= \sum_{k,l} c_i \langle u_k \cdot A(E_X(u^*_k \cdot y_i), B(\tilde{1}_D, \tilde{1}_D), E_X(u^*_l \cdot y_j)), u_l \rangle \]

\[
= \sum_{k,l} c_i \langle u_k E^A(c \langle u^*_k \cdot y_i, E_X(u^*_l \cdot y_j) \rangle), u_l \rangle \]

\[
= \sum_{k,l} c_i \langle u_k E^A(u^*_k c \langle y_i, E_X(u^*_l \cdot y_j) \rangle), u_l \rangle \]

\[
= \sum_{l} c \langle y_i, E_X(u^*_l \cdot y_j) \rangle c_A u^*_l. \]
Thus
\[
\Psi(e_B) = \sum_{i,j,l} \langle y_i \times_\mu 1^0, c(y_i, E^X(u^*_1 \cdot y_j)) e_A u^*_1 \times_\mu 1^0 \rangle_{D \times_\beta H^0}
\]
\[
= \sum_{i,j,l} \langle c(E^X(u^*_1 \cdot y_j)) \cdot (y_i \times_\mu 1^0), e_A u^*_1 \times_\mu 1^0 \rangle_{D \times_\beta H^0}
\]
\[
= \sum_{i,j,l} \langle E^X(u^*_1 \cdot y_j) \cdot (y_i, y_j), e_A u^*_1 \cdot y_j \rangle_{D \times_\beta H^0}
\]
\[
= \sum_{i,j} \langle u_i e_A \cdot E^X(u^*_1 \cdot y_j), y_j \rangle_{D \times_\beta H^0}.
\]

Since we identify \( e_A \) with \( 1 \times_\beta \tau \), we obtain that
\[
u_i e_A \cdot E^X(u^*_1 \cdot y_j) = (u_i \times_\beta 1^0)(1 \times_\beta \tau) \cdot (E^X(u^*_1 \cdot y_j) \times_\mu 1^0)
\]
\[
= (u_i \times_\beta \tau) \cdot (E^X(u^*_1 \cdot y_j) \times_\mu 1^0).
\]

By Lemma 4.4, \( E^X(u^*_1 \cdot y_j) = \tau' \times_\mu (u^*_1 \cdot y_j) \), where \( \tau' = \tau \). Thus
\[
u_i e_A \cdot E^X(u^*_1 \cdot y_j) = (u_i \times_\beta \tau) \cdot \tau' \times_\mu (u^*_1 \cdot y_j) \times_\mu 1^0
\]
\[
= u_i [\tau] \cdot \tau' \times_\mu (u^*_1 \cdot y_j) \times_\mu 1^0
\]
\[
= u_i [\tau'] \times_\mu (u^*_1 \cdot y_j) \times_\mu \tau
\]
\[
= u_i E^X(u^*_1 \cdot y_j) \times_\mu \tau.
\]

It follows by [11] Lemma 5.4] that
\[
\Psi(e_B) = \sum_{j,l} \langle u_i E^X(u^*_1 \cdot y_j) \times_\mu \tau, y_j \rangle_{D \times_\beta H^0} = \sum_j \langle y_j \times_\mu \tau, y_j \times_\mu 1^0 \rangle_{D \times_\beta H^0}
\]
\[
= \sum_j \tau \gamma(y_j, y_j) = \sum_j \tau \gamma(y_j, y_j) = \tau \gamma(y_j, y_j).
\]

Therefore we obtain the conclusion. \( \square \)

Let \( (Y \times_\mu H^0)_\Psi \) be the \( C_1 - D_1 \)-equivalence bimodule induced by the \( C_1 - D \times_\beta H^0 \)-equivalence bimodule \( Y \times_\mu H^0 \) and the isomorphism \( \Psi \) of \( D_1 \) onto \( D \times_\beta H^0 \). Let \( E^1 \) be the linear map from \( Y \times_\mu H^0 \) onto \( Y \) defined by
\[
E^1(y, \times_\mu \psi) = \psi(e) \cdot y
\]
for any \( y \in Y, \psi \in H^0 \), where \( y \times_\mu \psi \) is the element in \( Y \times_\mu H^0 \) induced by \( y \in Y, \psi \in H^0 \). Then \( E^1 \) is a conditional expectation from \( Y \times_\mu H^0 \) onto \( Y \) with respect to \( E^0 \) and \( E_1 \), the canonical conditional expectation from \( D_1 \times_\beta H^0 \) onto \( D_1 \) by [11] Proposition 4.1. Let \( E^1 \cdot \Psi \) be the linear map from \( (Y \times_\mu H^0)_\Psi \) onto \( Y \) induced by \( E^0 \) and \( \Psi \).

**Lemma 4.4.** With the above notation, \( E^1 \cdot \Psi \) is a conditional expectation from \( (Y \times_\mu H^0)_\Psi \) onto \( Y \) with respect to \( E^0 \) and \( E^1 \).

**Proof.** We shall show that Conditions (1)-(6) in [11] Definition 2.4] hold. Let \( y, z \in Y, c \in C, d \in D \) and \( \psi \in H^0 \).
\begin{align*}
(1) & \quad E^1((c \times_\beta \psi) \cdot y) = E^1(c \times_\beta \psi) \cdot (y \times_\mu 1^0) = E^\mu E^1(c \cdot [\psi(1)_1, y] \times_\mu \psi(2)) \\
& = c \cdot [\psi(1)_1, y] \psi(2) \cdot e = c \cdot \psi(e) \cdot y = \psi(e) \cdot y.
\end{align*}
On the other hand,
\[ E_2^{\mu,\nu}(c \times_\beta \psi) \cdot y = \psi(e) c \cdot y. \]
Hence Condition (1) holds.

(2)
\[ E_1^{\mu,\psi}(c \cdot (y \times_\mu \psi)) = E_1^{\mu,\psi}((c \cdot y) \times_\mu \psi) = c \cdot y \psi(e) = \psi(e) c \cdot y. \]
On the other hand,
\[ c \cdot E_1^{\mu,\psi}(y \times_\mu \psi) = c \cdot \psi(e) y = \psi(e) c \cdot y. \]
Hence Condition (2) holds.

(3)
\[ E_2^{\mu,\nu}(c \times_\beta H^0(y \times_\mu \psi, z \times_\mu 1^0)) = E_2^{\mu,\nu}(c \times_\beta H^0(y \times_\mu \psi, z \times_\mu 1^0)) = E_2^{\mu,\nu}(c(y, [S^0(\psi_1^* \psi) \times_\beta \psi_2^]) = c(y, [S^0(\psi_1^* \psi) \times_\beta \psi_2^]) \psi_2(e) = c(y, [S^0(\psi_1^* \psi) \times_\beta \psi_2^]) \psi_2(e) = c(y, [e(S^0(\psi_1^* \psi) \times_\beta \psi_2^]) = c(y, \psi(e) y, z) = \psi(e) c(y, z). \]
Hence Condition (3) holds.

(4)
\[ E_1^{\mu,\psi}(y \cdot (d \times_\beta \psi)) = E_1^{\mu,\psi}(y \cdot \Psi(d \times_\beta \psi)) = y \cdot E_1^{\mu,\psi}(\Psi(d \times_\beta \psi)) = y \cdot E_2^{\mu,\nu}(d \times_\beta \psi). \]
Hence Condition (4) holds.

(5)
\[ E_1^{\mu,\psi}((y \times_\mu \psi) \cdot d) = E_1^{\mu,\psi}((y \times_\mu \psi) \cdot \Psi(d)) = E_1^{\mu,\psi}(y \times_\mu \psi) \cdot \Psi(d) = E_1^{\mu,\psi}(y \times_\mu \psi) \cdot d. \]
Hence Condition (5) holds.

(6)
\[ E_2^{\mu,\nu}(y \times_\mu \psi, z)_{D \times_\beta H^0} = E_2^{\mu,\nu}((y \times_\mu \psi, z \times_\mu 1^0)_{D \times_\beta H^0}) = E_2^{\mu,\nu}((y \times_\mu \psi, z \times_\mu 1^0)_{D \times_\beta H^0}) = E_2^{\mu,\nu}((\psi_1^* \cdot \beta \langle y, z \rangle_D) \times_\beta \psi_2^) = \psi(e) \langle y, z \rangle_D. \]
On the other hand,
\[ (E_1^{\mu,\psi}(y \times_\mu \psi, z \times_\mu 1^0)_{D \times_\beta H^0} = \Psi^{-1}(\langle \psi(e) y, z \rangle_{D \times_\beta H^0}) = \psi(e) \langle y, z \rangle_D. \]
Hence Condition (6) holds. Therefore, we obtain the conclusion. \( \square \)

**Lemma 4.5.** With the above notation, for any \( y \in Y \),
\[ E_1^{\mu,\psi}(e_A \cdot y \cdot e_B) = \frac{1}{N} E(X(y)). \]
Proof. By the definition of $E^{\mu,\Psi}_1$ and Lemma 4.6, 

$$E^{\mu,\Psi}_1(e_A \cdot y \cdot e_B) = E^\mu_1((1 \rtimes_{\beta} \tau) \cdot y \cdot (1 \rtimes_{\beta} \tau)).$$

Also, 

$$(1 \rtimes_{\beta} \tau) \cdot y \cdot (1 \rtimes_{\beta} \tau) = (1 \rtimes_{\beta} \tau) \cdot (y \rtimes_{\mu} 1^0) \cdot (1 \rtimes_{\beta} \tau) = (1 \rtimes_{\beta} \tau) \cdot (y \rtimes_{\mu} \tau') = [\tau(y) \rtimes_{\mu} \tau'] = \tau(y) \rtimes_{\mu} \tau,$$

where $\tau' = \tau$. Hence 

$$E^{\mu,\Psi}_1(e_A \cdot y \cdot e_B) = E^\mu_1([\tau(y) \rtimes_{\mu} \tau]) = [\tau(y)\tau(e)] = \frac{1}{N} E^X(y)$$

by Lemma 4.7. \hfill $\Box$

Proposition 4.6. With the above notation, there is a $C_1-D_1$-equivalence bimodule isomorphism $\theta$ of $Y_1$ onto $(Y \rtimes_{\mu} H^0)\Psi$ such that $E^{\mu,\Psi}_1 = E^\Psi \circ \theta$.

Proof. This is immediate by Lemma 4.5 and [11, Theorem 6.13]. \hfill $\Box$

Next, modifying the discussions of [12, Section 5], we shall show that there is a $C^*$-Hopf algebra automorphism $f^0$ of $H^0$ such that

$$\widehat{\beta} \circ \Psi = (\Psi \circ f^0) \circ \widehat{\sigma},$$

where $\widehat{\beta}$ is the dual coaction of $\beta$ and $\widehat{\sigma}$ is the second dual coaction of $(\sigma, v)$.

Lemma 4.7. With the above notation, $\Psi |_{B' \cap D_1}$, the restriction of $\Psi$ to $B' \cap D_1$ is an isomorphism of $B' \cap D_1$ onto $B' \cap (D \rtimes_{\beta} H^0)$.

Proof. It suffices to show that $\Psi(d) \in B' \cap (D \rtimes_{\beta} H^0)$ for any $d \in B' \cap D_1$. For any $d \in B' \cap D_1$, $b \in B$,

$$\Psi(d)b = \Psi(d)(b) = \Psi(b(d))b = \Psi(b)\Psi(d) = b\Psi(d).$$

Hence $\Psi(d) \in B' \cap (D \rtimes_{\beta} H^0)$ for any $d \in B' \cap D_1$. \hfill $\Box$

By Lemma 4.8. $B' \cap D_1 = 1 \rtimes_{\sigma,v} 1 \rtimes_{\beta} H^0$. Also, we have the next lemma.

Lemma 4.8. With the above notation, $B' \cap (D \rtimes_{\beta} H^0) = 1 \rtimes_{\sigma,v} 1 \rtimes_{\beta} H^0$.

Proof. We note that $\psi \cdot_{\mu} x = e^0(\psi)x$ for any $\psi \in H^0$, $x \in X$ by [12, Lemma 3.2]. Thus by the definition of $\beta$, $\psi \cdot (b \rtimes_{\sigma,v} 1) = e^0(\psi)(b \rtimes_{\sigma,v} 1)$ for any $\psi \in H^0$, $b \in B$ (See [12, Section 4]). Hence in the same way as in the proof of [12, Lemma 5.8], we obtain the conclusion. \hfill $\Box$

Since $\Psi(1 \rtimes_{\beta} \tau) = 1 \rtimes_{\beta} \tau$ by Lemma 4.8 and $\Psi(d) = d$ for any $d \in D$, in the same way as in [12, Lemma 5.6], we can see that there is an isomorphism $\widehat{\Psi}$ of $D_2$ onto $D \rtimes_{\beta} H^0 \rtimes_{\beta} H$ satisfying that

$$\widehat{\Psi}|_{D_1} = \Psi,$$

$$E^\beta_{e_2} \circ \widehat{\Psi} = \Psi \circ E^\sigma_{e_2},$$

$$\widehat{\Psi}(1 \rtimes_{\beta} 1 \rtimes_{\beta} e) = 1 \rtimes_{\beta} 1 \rtimes_{\beta} e,$$

where $\widehat{\beta}$ is the second dual coaction of $(\sigma, v)$, $\widehat{\beta}$ is the dual coaction of $\beta$, $D_2 = D_1 \rtimes_{\beta} H$ and $E^\sigma_{e_2}$ and $E^\beta_{e_2}$ are the canonical conditional expectations from $D_2$ and $D \rtimes_{\beta} H^0 \rtimes_{\beta} H$ onto $D_1$ and $D \rtimes_{\beta} H^0$, respectively. Furthermore, in the same way as in the above or [12, Section 5], $\widehat{\Psi}|_{B' \cap D_2}$ is an isomorphism of $B' \cap D_2$ onto $D' \cap (D \rtimes_{\beta} H^0 \rtimes_{\beta} H)$. Since

$$B' \cap D_1 = B' \cap (D \rtimes_{\beta} H^0) = 1 \rtimes_{\sigma,v} 1 \rtimes_{\beta} H^0$$
by Lemma 4.8 we identify \( B' \cap D_1 \) and \( B' \cap (D \times_\beta H^0) \) with \( H^0 \). Let \( f^0 = \Psi|_{B' \cap D_1} \) and we regard \( f^0 \) as a \( C^* \)-algebra automorphism of \( H^0 \). By the proof of [12, Lemma 5.9], we can see that

\[
N^2(E_1^{\sigma,v} \circ E_1^{|\tau|})(1 \times_\beta \psi \times_\beta 1)(1 \times_\beta 1_0 \times_\beta e)(1 \times_\beta \tau \times_\beta 1)(1 \times_\beta 1_0 \times_\beta h) = \psi(e),
\]

\[
N^2(E_1^{|\tau|} \circ E_2^{|\beta|})(1 \times_\beta \psi \times_\beta 1)(1 \times_\beta 1_0 \times_\beta e)(1 \times_\beta \beta \times_\beta 1)(1 \times_\beta 1_0 \times_\beta h) = \psi(e),
\]

for any \( h \in H, \psi \in H^0 \). Hence in the same way as in the proof of [12, Lemma 5.9], we can see that \( f^0 \) is a \( C^* \)-Hopf algebra automorphism of \( H^0 \).

**Lemma 4.9.** With the above notation, \( \widehat{\beta} \circ \Psi = (\Psi \otimes f^0) \circ \widehat{\sigma} \).

**Proof.** This can be proved in the same way as in the proof of [12, Lemma 5.10]. \( \square \)

**Lemma 4.10.** With the above notation, \( \widehat{\beta}(1_D \times_\beta \tau) \) is Murray-von Neumann equivalent to \( (1_D \times_\beta \tau) \otimes 1^0 \) in \( (D \times_\beta H^0) \otimes H^0 \).

**Proof.** By Lemmas 4.3, 4.9,

\[
\widehat{\beta}(1_D \times_\beta \tau) = \widehat{\beta}(\Psi(1_D \times_\beta \tau)) = (\Psi \otimes f^0)(\widehat{\sigma}(1_D \times_\beta \tau)).
\]

By [9, Proposition 3.19], \( \widehat{\sigma}(1_D \times_\beta \tau) \) is Murray-von Neumann equivalent to \( (1 \times_\beta \tau) \otimes 1^0 \) in \( D_1 \otimes H^0 \). Hence we obtain the conclusion by Lemma 4.3. \( \square \)

**Lemma 4.11.** With the above notation, \( \beta \) is saturated, that is, the action of \( H \) on \( D \) induced by \( \beta \) is saturated in the sense of Szymański and Peligrad [16].

**Proof.** By the definition of \( \widehat{\sigma} \),

\[
D_1(1_D \times_\beta \tau)D_1 = D_1.
\]

Since \( \Psi \) is an isomorphism of \( D_1 \) onto \( D \times_\beta H^0 \),

\[
(1_D \times_\beta H^0)(1_D \times_\beta \tau)(D \times_\beta H^0) = \Psi(D_1(1_D \times_\beta \tau)D_1) = \Psi(D_1) = D \times_\beta H^0
\]

by Lemma 4.1. Hence \( \beta \) is saturated. \( \square \)

Since \( \beta \) is saturated by Lemma 4.11, there is the conditional expectation \( E^{D_\beta} \) from \( D \) onto \( D_\beta \) defined by

\[
E^{D_\beta}(d) = \tau \circ d
\]

for any \( d \in D \) (See [16, Proposition 2.12]), where \( D_\beta \) is the fixed-point \( C^* \)-subalgebra of \( D \) for \( \beta \). Also, since \( \widehat{\beta}(1_D \times_\beta \tau) \) is Murray-von Neumann equivalent to \( (1 \times_\beta \tau) \otimes 1^0 \) in \( (D \times_\beta H^0) \otimes H^0 \) by Lemma 4.10, there is a twisted coaction \( (\gamma, w) \) of \( H^0 \) on \( D_\beta \) and an isomorphism \( \pi_D \) of \( D_\beta \) satisfying

\[
E_1^{\sigma,w} = E^{D_\beta} \circ \pi_D, \quad \psi \circ \pi_D(d) = \pi_D(\psi \circ d)
\]

for any \( d \in D, \psi \in H^0 \) by [9, Proposition 6.1, 6.4 and Theorem 6.4]. We identify \( D_\beta \times_{\gamma,w} H \) and \( E_1^{\sigma,w} \) with \( D \) and \( E^{D_\beta} \) by the above isomorphism \( \pi_D \), respectively. We show that \( B = D_\beta \). By the definition of \( \beta, B \subset D_\beta \). Let \( F \) be the conditional expectation of \( D_\beta \) onto \( B \) defined by \( F = E_1^{\sigma,w}|_{D_\beta} \), the restriction of \( E_1^{\sigma,w} \) to \( D_\beta \). Since \( E_1^{\sigma,w} \) is of Watatani index-finite type, there is a quasi-basis \( \{ (d_i, d_i^* ) \}_{i=1}^n \) for \( E_1^{\sigma,w} \). Then \( F \circ E_1^{\sigma,w} \) is also a conditional expectation from \( D_1 \) onto \( B \). Since \( B' \cap D_1 = C_1 \), by [17, Proposition 1.4.1],

\[
E_1^{\sigma,w} = F \circ E_1^{\sigma,w}.
\]

**Lemma 4.12.** With above notation, \( F \) is of Watatani index-finite type and its Watatani index, \( \text{Ind}_W(F) \in C_1 \).
Proof. We claim that \( \{(E_1^{\gamma,w}(d_i), E_1^{\gamma,w}(d'_i))\}_{i=1}^n \) is a quasi-basis for \( F \). Indeed, for any \( d \in D^3 \),
\[
\sum_{i=1}^n E_1^{\gamma,w}(d_i)F(E_1^{\gamma,w}(d'_i))d = \sum_{i=1}^n E_1^{\gamma,w}(d_i)(F \circ E_1^{\gamma,w})(d'_i E_1^{\gamma,w}(d)) \\
= \sum_{i=1}^n E_1^{\gamma,w}(d_i)(F \circ E_1^{\gamma,w})(d'_i E_1^{\gamma,w}(d)) \\
= \sum_{i=1}^n E_1^{\gamma,w}(d_i)E_1^{\sigma,v}(d'_i E_1^{\gamma,w}(d)) \\
= \sum_{i=1}^n E_1^{\gamma,w}(d_i)E_1^{\sigma,v}(d'_i E_1^{\gamma,w}(d)) \\
= E_1^{\gamma,w}(E_1^{\gamma,w}(d)) = d
\]
since \( E_1^{\sigma,v} = F \circ E_1^{\gamma,w} \) and \( E_1^{\gamma,w}(d) = d \) for any \( d \in D^3 \). Hence \( F \) is of Watatani index-finite type. Also, \( \text{Ind}_{W}(F) \in (D^3)' \cap D^3 \subset B' \cap D = C1 \) by [17] Proposition 1.2.8. □

Lemma 4.13. With the above notation, \( B = D^3 \).

Proof. It suffices to show that \( \text{Ind}_W(F) = 1 \). By [17] Proposition 1.7.1,
\[
\text{Ind}_W(E_1^{\sigma,v}) = \text{Ind}_W(F)\text{Ind}_W(E_1^{\gamma,w}).
\]
By [17] Proposition 3.18 \( \text{Ind}_W(E_1^{\sigma,v}) = \text{Ind}_W(E_1^{\gamma,w}) = N \). Hence \( \text{Ind}_W(F) = 1 \).
Therefore, we obtain the conclusion by [17]. □

Let \( Y^\mu = \{ y \in Y \mid \mu(y) = y \otimes 1^0 \} \). By [4] Theorem 4.9, there are a twisted coaction \( \lambda \) of \( H \) on \( Y^\mu \) and a Hilbert \( A \rtimes_{(\rho,u)} H - B \rtimes_{(\gamma,w)} H \)-bimodule isomorphism \( \pi_Y \) of \( Y^\mu \) onto \( H \) such that
\[
\psi \cdot_\mu \pi_Y(x \rtimes_\lambda h) = \pi_Y(\psi \cdot_\lambda (x \rtimes_\lambda h))
\]
for any \( x \in Y^\mu \), \( h \in H \), \( \psi \in H^0 \). Furthermore, by [4] Lemma 3.10, \( Y^\mu \) is an \( A - B \)-equivalence bimodule and hence \( \pi_Y \) is an \( A \rtimes_{(\rho,u)} H - B \rtimes_{(\gamma,w)} H \)-equivalence bimodule isomorphism. We identify \( Y \) with \( Y^\mu \rtimes_\lambda H \) by the isomorphism \( \pi_Y \).
Thus the twisted coactions \( (\rho,u) \) and \( (\gamma,w) \) are strongly Morita equivalent with respect to the twisted coaction \( \lambda \) of \( H \) on the \( A - B \)-equivalence bimodule \( Y^\mu \). We show that \( Y^\mu = X \).

Lemma 4.14. With the above notation, \( Y^\mu = X \).

Proof. By [12] Lemma 3.2, \( X \subset Y^\mu \). Also, for any \( y \in Y^\mu \), \( \tau \cdot_\mu y = \epsilon(\tau)y = y \).
On the other hand, by Lemma [11] \( \tau \cdot_\mu y = E^X(y) \). Hence \( y = E^X(y) \in X \). Thus we obtain that \( Y^\mu \subset X \). □

By the above discussions, we obtain the following theorem:

Theorem 4.15. Let \( H \) be a finite dimensional C^*-Hopf algebra and \( H^0 \) its dual C^*-Hopf algebra. Let \( (\rho,u) \) and \( (\sigma,v) \) be twisted coactions of \( H^0 \) on unital C^*-algebras \( A \) and \( B \), respectively. Let \( A \subset A \rtimes_{(\rho,u)} H \) and \( B \subset B \rtimes_{(\sigma,v)} H \) be unital inclusions of unital C^*-algebras. We suppose that they are strongly Morita equivalent with respect to \( A \rtimes_{(\rho,u)} H - B \rtimes_{(\sigma,v)} H \)-equivalence bimodule \( Y \) and its closed subspace \( X \). And we suppose that \( A' \cap (A \rtimes_{(\rho,u)} H) = C1 \). Then there are a twisted coaction \( (\gamma,w) \) of \( H^0 \) on \( B \) and a twisted coaction \( \lambda \) of \( H^0 \) on \( X \) satisfying the following:
(1) \( (\rho,u) \) and \( (\gamma,w) \) are strongly Morita equivalent with respect to \( \lambda \),
(2) \( B \rtimes_{(\sigma,v)} H = B \rtimes_{(\gamma,w)} H \),
(3) \( Y \cong X \rtimes_\lambda H \) as \( A \rtimes_{(\rho,u)} H - B \rtimes_{(\sigma,v)} H \)-equivalence bimodules.
5. Image

Let $A$ be a unital $C^*$-algebra and $X$ an involutive $A - A$-equivalence bimodule. Let $A \subset C_X$ be the unital inclusion of unital $C^*$-algebras induced by $X$. We suppose that $A' \cap C_X = C_1$. Let $f_A$ be the homomorphism of $\text{Pic}(A,C_X)$ onto $\text{Pic}(A)$ defined in Preliminaries, that is,

$$f_A([M,N]) = [M]$$

for any $(M,N) \in \text{Equi}(A,C_X)$. In this section, we shall compute $\text{Im} f_A$, the image of $f_A$.

Let $E_A$ be the conditional expectation from $C_X$ onto $A$ defined in Section 3 and let $\epsilon_A$ be the Jones projection for $E_A$. Since $E_A$ is of Watatani index-finite type by [7, Lemma 3.4], there is the $C^*$-basic construction of the inclusion $A \subset C_X$ for $E_A$, which is the linking $C^*$-algebra $L_X$ for $X$, that is,

$$L_X = \{ \begin{pmatrix} a & x \\ y & b \end{pmatrix} | a, b \in A, \ x, y \in X \}. \tag{5.1}$$

By [7] Lemma 2.6, we can see that there is the action $\alpha^X$ of $\mathbb{Z}_2$, the group of order two, on $C_X$ defined by

$$\alpha^X \left( \begin{pmatrix} a & x \\ x^2 & a \end{pmatrix} \right) = \begin{pmatrix} a & -x \\ -x^2 & a \end{pmatrix}$$

for any $\begin{pmatrix} a & x \\ x^2 & a \end{pmatrix} \in C_X$ and that $L_X \cong C_X \rtimes_{\alpha^X} \mathbb{Z}_2$ as $C^*$-algebras. We note that we regard an action $\beta$ of $\mathbb{Z}_2$ on a unital $C^*$-algebra $B$ as the automorphism $\beta$ of $B$ with $\beta^2 = \text{id}$ on $B$. We identify $L_X$ with $C_X \rtimes_{\alpha^X} \mathbb{Z}_2$. Let $M$ be an $A - A$-equivalence bimodule satisfying that

$$\tilde{M} \otimes_A X \otimes_A M \cong X$$

as involutive $A - A$-equivalence bimodules. Then by the proof of [6, Lemma 5.11], we can see that there is an element $(M,C_M) \in \text{Equi}(A,C_X)$, where $C_M$ is a $C_X - C_X$-equivalence bimodule induced by $M$, which is defined in [6] Section 5.]. Next, we show that

$$\tilde{M} \otimes_A X \otimes_A M \cong X$$

as involutive $A - A$-equivalence bimodules for any $(M,N) \in \text{Equi}(A,C_X)$. Let $(M,N)$ be any element in $\text{Equi}(A,C_X)$. Since $A' \cap C_X = C_1$, by [5] Lemma 4.1 there is the unique conditional expectation $E_M$ from $N$ onto $M$ with respect to $E_A$ and $E_A$. Let $N_1$ be the upward basic construction of $N$ for $E_M$ (See [11] Definition 6.5). Then by [11] Corollary 6.3, the unital inclusion $C_X \subset L_X$ is strongly Morita equivalent to itself with respect to $N_1$ and its closed subspace $N$. Hence by Theorem 4.15, there are an action $\gamma$ of $\mathbb{Z}_2$ on $C_X$ and an action $\lambda$ of $\mathbb{Z}_2$ on $N$ satisfying the following:

1. The actions $\alpha^X$ and $\gamma$ of $\mathbb{Z}_2$ on $C_X$ are strongly Morita equivalent with respect to the action $\lambda$ of $\mathbb{Z}_2$ on $N$,
2. $L_X = C_X \rtimes_{\alpha^X} \mathbb{Z}_2 = C_X \rtimes_{\gamma} \mathbb{Z}_2$,
3. $N_1 \cong N \rtimes_{\lambda} \mathbb{Z}_2$ as $L_X - L_X$-equivalence bimodules.

We identify $N_1$ with $N \rtimes_{\lambda} \mathbb{Z}_2$. Let $\tilde{\alpha}^X$ be the dual action of $\alpha^X$, which is an action of $\mathbb{Z}_2$ on $L_X$. We regard $\tilde{\alpha}^X$ as an automorphism of $L_X$ with $(\tilde{\alpha}^X)^2 = \text{id}$ on $L_X$,
which is defined by

\[\tilde{\alpha}^X\left(\begin{bmatrix} a & x \\ x^* & a \end{bmatrix}\right) = \begin{bmatrix} a & x \\ x^* & a \end{bmatrix}\text{ for any }\begin{bmatrix} a & x \\ x^* & a \end{bmatrix} \in C_X,\]

\[\tilde{\alpha}^X\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} .\]

Let \((L_X)_{\tilde{\alpha}^X}\) be the involutive \(L_X - L_X\)-equivalence bimodule induced by \(\tilde{\alpha}^X\), that is, \((L_X)_{\tilde{\alpha}^X} = L_X\) as vector spaces over \(C\) and the left \(L_X\)-action and the left \(L_X\)-valued inner product on \((L_X)_{\tilde{\alpha}^X}\) are defined in the usual way. The right \(L_X\)-action and the right \(L_X\)-valued inner product on \((L_X)_{\tilde{\alpha}^X}\) are defined as follows: For any \(a \in L_X, x, y \in (L_X)_{\tilde{\alpha}^X}\),

\[x \cdot a = x\tilde{\alpha}^X(a), \quad \langle x, y \rangle_{L_X} = \tilde{\alpha}^X(x^* y) .\]

Furthermore, we define the involution \(\hat{\gamma}\) as follows: For any \(x \in (L_X)_{\tilde{\alpha}^X}\),

\[x^\flat = \tilde{\alpha}^X(x)^* .\]

Then by easy computations \((L_X)_{\tilde{\alpha}^X}\) is an involutive \(L_X - L_X\)-equivalence bimodule.

Let \(\hat{\lambda}\) be the dual action of \(\lambda\), which is an action of \(Z_2\) by linear automorphisms of \(N_1 = N \rtimes_{\lambda} Z_2\) such that

\[\tilde{\alpha}^X((L_X(m, n)) = L_X((\hat{\lambda}(m), \hat{\lambda}(n)), \]

\[\tilde{\gamma}((\langle m, n \rangle_{L_X}) = (\hat{\lambda}(m), \hat{\lambda}(n))_{L_X}\]

for any \(m, n \in N_1\), where we regard the action \(\hat{\lambda}\) as a linear automorphism of \(N_1\) with \(\hat{\lambda}^2 = \text{id}\) on \(N_1\). We note that

\[\hat{\lambda}(x \cdot m) = \tilde{\alpha}^X(x) \cdot \hat{\lambda}(m), \quad \hat{\lambda}(m \cdot x) = \hat{\lambda}(m) \cdot \tilde{\gamma}(x)\]

for any \(m \in N_1, x \in L_X\). Since \(L_X = C_X \rtimes_{\alpha} Z_2 = C_X \rtimes_{\gamma} Z_2\) and \(N_1 = N \rtimes_{\lambda} Z_2\), in the same way as after the proof of Lemma 13 and in the proof of Lemma 19 or by the discussions of [12, Section 5], there is an automorphism \(\kappa\) of \(L_X\) satisfying the following:

\[\tilde{\gamma} \circ \kappa = \kappa \circ \tilde{\alpha}^X, \quad \kappa|_{C_X} = \text{id}_{C_X}\]

Then \(\kappa|_{A' \cap L_X}\) is an automorphism of \(A' \cap L_X\). And by [8], [9], \(A' \cap L_X \cong C^2\). Since \(e_A \in A' \cap L_X, \kappa(e_A) = e_A\) or \(1 - e_A\). If \(\kappa(e_A) = e_A\), \(\kappa = \text{id}_{L_X}\) since \(\kappa|_{C_X} = \text{id}_{C_X}\).

Hence \(\tilde{\gamma} = \tilde{\alpha}^X\). If \(\kappa(e_A) = 1 - e_A\), \(\kappa = \tilde{\alpha}^X\) since \(\kappa = \tilde{\alpha}^X\) = \text{id} on \(C_X\). Hence \(\tilde{\gamma} \circ \tilde{\alpha}^X = \tilde{\alpha}^X \circ \tilde{\alpha}^X = \text{id}_{L_X}\). Thus \(\tilde{\gamma} = (\tilde{\alpha}^X)^{-1} = \tilde{\alpha}^X\). Then, we obtain the following:

**Lemma 5.1.** With the above notation,

\[\bar{N}_1 \otimes_{L_X} (L_X)_{\tilde{\alpha}^X} \otimes_{L_X} N_1 \cong (L_X)_{\tilde{\alpha}^X}\]

as \(L_X - L_X\)-equivalence bimodules.

**Proof.** We note that \(N_1 = N \rtimes_{\lambda} Z_2\). Let \(\pi\) be the linear map from \(\bar{N}_1 \otimes_{L_X} (L_X)_{\tilde{\alpha}^X} \otimes_{L_X} N_1\) to \((L_X)_{\tilde{\alpha}^X}\) defined by

\[\pi(\bar{m} \otimes x \otimes n) = \langle x^* \cdot m, \hat{\lambda}(n) \rangle_{L_X}\]

for any \(m, n \in N_1, x \in (L_X)_{\tilde{\alpha}^X}\), where we regard \(\langle x^* \cdot m, \hat{\lambda}(n) \rangle_{L_X}\) as an element in \((L_X)_{\tilde{\alpha}^X}\). We show that \(\pi\) is an involutive \(L_X - L_X\)-equivalence bimodule isomorphism of \(\bar{N}_1 \otimes_{L_X} (L_X)_{\tilde{\alpha}^X} \otimes_{L_X} N_1\) onto \((L_X)_{\tilde{\alpha}^X}\). By routine computations, we can see that \(\pi\) is well-defined. Since \(L_X \cdot N_1 = N_1\) by Brown, Mingo and Shen
Proposition 1.7] and \((L_X)_{\otimes X}\) is full with respect to the right \(L_X\)-valued inner product, \(\pi\) is surjective. For any \(m, n, m_1, n_1 \in N_1, x, x_1 \in L_X\),

\[
\langle \pi(m \otimes x \otimes n), \pi(m_1 \otimes x_1 \otimes n_1) \rangle_{L_X} = \langle \langle x^* \cdot m, \hat{\lambda}(n) \rangle_{L_X}, \langle x_1^* \cdot m_1, \hat{\lambda}(n_1) \rangle_{L_X} \rangle_{L_X}
\]

as involutive \((L_X)^{\Delta X} \otimes_{L_X} L_X\)-valued inner products. Thus we can obtain that \(\pi\) preserves involutions \(\hat{\lambda}\). Therefore, we obtain the conclusion. 

We regard \(e_A L_X\) as an \(A - L_X\)-equivalence bimodule in the usual way, where we identify \(e_A L_X e_A\) with \(A\). Also, we regard \(L_X e_A\) as an \(L_X - A\)-equivalence bimodule in the usual way. We note that \(L_X e_A \cong \tilde{e}_A L_X\) as \(L_X - A\)-equivalence bimodules by the map \(xe_A \in L_X e_A \mapsto e_A x^* \in \tilde{e}_A L_X\). In the same way as in \[7\] Section 3], we regard \(e_A L_X(1 - e_A)\) as an involutive \(A - A\)-equivalence bimodule.

**Lemma 5.2.** With the above notation,

\[
e_A L_X \otimes_{L_X} (L_X)_{\otimes X} \otimes_{L_X} L_X e_A \cong e_A L_X(1 - e_A) \cong X
\]
as involutive \(A - A\)-equivalence bimodules.

**Proof.** By \[7\] Theorem 3.11], we can see that \(e_A L_X(1 - e_A) \cong X\) as involutive \(A - A\)-equivalence bimodules. Let \(\pi\) be the linear map from \(e_A L_X \otimes_{L_X} (L_X)_{\otimes X} \otimes_{L_X} L_X e_A\) to \(e_A L_X(1 - e_A)\) defined by

\[
\pi(e_A x \otimes y \otimes z e_A) = e_A x y \tilde{\alpha}^X(z e_A) = e_A x y \hat{\lambda}(z)(1 - e_A)
\]
for any \( x, y, z \in L_X \). We note that \( \hat{\alpha}^X(e_A) = 1 - e_A \) by [7] Remark 2.7. Clearly \( \pi \) is surjective. For any \( x, y, z \in L_X \),

\[
A(\pi(e_A x \otimes y \otimes z e_A), \pi(e_A x_1 \otimes y_1 \otimes z_1 e_A)) = A(e_A xy \hat{\alpha}^X(z)(1 - e_A), e_A x_1 y_1 \hat{\alpha}^X(z_1)(1 - e_A)) = e_A xy \hat{\alpha}^X(z)(1 - e_A) \hat{\alpha}^X(z_1^* y_1^* x_1^* e_A).
\]

On the other hand,

\[
A(e_A x \otimes y \otimes z e_A, e_A x_1 \otimes y_1 \otimes z_1 e_A) = A(e_A x \cdot L_X (y \otimes z e_A, y_1 \otimes z_1 e_A), e_A x_1) = [e_A x \cdot L_X (y \otimes z e_A, y_1 \otimes z_1 e_A)] x_1^* e_A = e_A x \cdot L_X (y \otimes z A z_1^*, y_1) x_1^* e_A = e_A x \cdot L_X (y \hat{\alpha}^X(z A z_1^*), y_1) x_1^* e_A = e_A xy \hat{\alpha}^X(z A z_1^*) y_1^* x_1^* e_A = e_A xy \hat{\alpha}^X(z)(1 - e_A) \hat{\alpha}^X(z_1^*) y_1^* x_1^* e_A.
\]

Hence \( \pi \) preserves the left \( A \)-valued inner products. Similarly, we can see that \( \pi \) preserves the right \( A \)-valued inner products. Thus we can obtain that \( \pi \) is an \( A - A \) equivalence bimodule isomorphism by the remark after [3] Definition 1.1.18.

Furthermore,

\[
\pi((e_A x \otimes y \otimes z e_A)^2) = (e_A xy \hat{\alpha}^X(z)(1 - e_A))^2 = \hat{\alpha}^X(1 - e_A) z^* \hat{\alpha}^X(y^* x^*) (1 - e_A) = e_A z^* \hat{\alpha}^X(y^* x^*)(1 - e_A).
\]

On the other hand,

\[
\pi((e_A x \otimes y \otimes z e_A)^2) = \pi(e_A z^* \hat{\alpha}^X(y^*) \otimes x^* e_A) = e_A z^* \hat{\alpha}^X(y^* x^*)(1 - e_A).
\]

Hence \( \pi \) preserves the involutions \( \hat{\cdot} \). Therefore, we obtain the conclusion. \( \square \)

**Lemma 5.3.** With the above notation, \( e_A L_X \otimes_{L_X} C_X \cong A \) as \( A - A \) equivalence bimodules, where \( C_X \) is regarded as an \( L_X - A \) equivalence bimodule in the usual way and \( A \) is regarded as the trivial \( A - A \) equivalence bimodule.

**Proof.** Let \( \pi \) be the linear map from \( e_A L_X \otimes_{L_X} C_X \) to \( A \) defined by

\[
\pi(e_A e_A e_A b \otimes c) = e_A e_A e_A b \cdot c = E^A(a) e_A b \cdot c = E^A(a) E^A(b c)
\]

for any \( a, b, c \in C_X \). Clearly \( \pi \) is surjective. For any \( a, b, c, a_1, b_1, c_1 \in C_X \),

\[
A(\pi(e_A e_A e_A b \otimes c), \pi(e_A e_A e_A b_1 \otimes c_1)) = A(E^A(a) E^A(b), E^A(a_1) E^A(b_1 c_1)) = E^A(a) E^A(b) E^A(c_1 b_1^*) E^A(a_1^*).
\]

On the other hand,

\[
A(e_A e_A e_A b \otimes c, e_A e_A e_A b_1 \otimes c_1) = A(e_A e_A e_A b \cdot L_X (c, c_1), e_A a_1 e_A b_1) = A(e_A e_A e_A b \cdot c e_A c_1, e_A a_1 e_A b_1) = A(e_A E^A(a) E^A(b) c_1^* E^A(a_1) b_1) = E^A(a) E^A(b) E^A(c_1 b_1^*) E^A(a_1^*) e_A.
\]
Since we identify \( A \) with \( Ae_A \) by the map \( a \in A \mapsto ae_A \in Ae_A \), \( \pi \) preserves the left \( A \)-valued inner products. Similarly, we can see that \( \pi \) preserves the right \( A \)-valued inner products. Thus by the remark after [3, Definition 1.1.18], we obtain the conclusion. \( \Box \)

**Proposition 5.4.** For any \((M, N) \in \text{Equi}(A, C_X)\),
\[
X \cong \tilde{M} \otimes_A X \otimes_A M
\]
as involutive \( A - A \)-equivalence bimodules.

**Proof.** By Lemmas 5.2, 5.1,
\[
X \cong e_A L_X \otimes_{L_X} (L_X)^* \otimes_{L_X} L_X e_A
\]
\[
\cong e_A L_X \otimes_{L_X} \overline{N}_1 \otimes_{L_X} (L_X)^* \otimes_{L_X} N_1 \otimes_{L_X} L_X e_A
\]
\[
\cong e_A L_X \otimes_{L_X} \overline{N}_1 \otimes_{L_X} L_X e_A \otimes_A X \otimes_A e_A L_X \otimes_{L_X} N_1 \otimes_{L_X} L_X e_A.
\]
as involutive \( A - A \)-equivalence bimodules. Since \( N_1 = C_X \otimes_A M \otimes_A \overline{C_X} \),
\[
e_A L_X \otimes_{L_X} N_1 \otimes_{L_X} L_X e_A = e_A L_X \otimes_{L_X} C_X \otimes_A M \otimes_A \overline{C_X} \otimes_{L_X} L_X e_A,
\]
where \( C_X \) is regarded as an \( L_X - A \)-equivalence bimodule. Hence by Lemma 5.3,
\[
e_A L_X \otimes_{L_X} N_1 \otimes_{L_X} L_X e_A \cong e_A L_X \otimes_{L_X} C_X \otimes_A M \otimes_A e_A L_X \otimes_{L_X} C_X
\]
\[
\cong A \otimes_A M \otimes_A A \cong M
\]
as \( A - A \)-equivalence bimodules. Therefore,
\[
X \cong [e_A L_X \otimes_{L_X} N_1 \otimes_{L_X} L_X e_A] \otimes_A X \otimes_A [e_A L_X \otimes_{L_X} N_1 \otimes_{L_X} L_X e_A]
\]
\[
\cong \tilde{M} \otimes_A X \otimes_A M
\]
as involutive \( A - A \)-equivalence bimodules. \( \Box \)

**Theorem 5.5.** Let \( A \) be a unital \( C^* \)-algebra and \( X \) an involutive \( A - A \)-equivalence bimodule. Let \( A \subset C_X \) be the unital inclusion of unital \( C^* \)-algebras induced by \( X \). We suppose that \( A' \cap C_X = C_1 \). Let \( f_A \) be the homomorphism of \( \text{Pic}(A, C_X) \) to \( \text{Pic}(A) \) defined by
\[
f_A([M, N]) = [M]
\]
for any \((M, N) \in \text{Equi}(A, C_X)\). Then the image of \( f_A \) is:
\[
\text{Im} f_A = \{ [M] \in \text{Pic}(A) \mid M \text{ is an } A - A \text{-equivalence bimodule with } X \cong \tilde{M} \otimes_A X \otimes_A M \text{ as involutive } A - A \text{-equivalence bimodules} \}.
\]

**Proof.** This is immediate by Proposition 5.4 and the proof of [6, Lemma 5.11]. \( \Box \)

6. A HOMOMORPHISM

In this section, we shall construct a homomorphism \( g \) of \( \text{Im} f_A \) to \( \text{Pic}(A, C_X) \) with \( f_A \circ g = \text{id} \) on \( \text{Im} f_A \). Let \( M \) be an \( A - A \)-equivalence bimodule with \( X \cong \tilde{M} \otimes_A X \otimes_A M \) as involutive \( A - A \)-equivalence bimodules. Let \( \Phi_M \) be an involutive \( A - A \)-equivalence bimodule isomorphism of \( \tilde{M} \otimes_A X \otimes_A M \) onto \( X \) and let \( \tilde{\Phi}_M \) be the involutive \( A - A \)-equivalence bimodule isomorphism of \( \tilde{M} \otimes_A \tilde{X} \otimes_A M \) onto \( \tilde{X} \) induced by \( \Phi_M \) (See [3, Section 5]). Let \( \Psi_M \) and \( \tilde{\Psi}_M \) be the \( A - A \)-equivalence bimodule isomorphism of \( X \otimes_A M \) onto \( M \otimes_A X \) and the \( A - A \)-equivalence bimodule
isomorphism of $\breve{X} \otimes_A M$ onto $M \otimes_A \breve{X}$ indexed by $\Phi_M$ and $\breve{\Phi}_M$, which are defined in [6 Section 5], respectively. Let $C_M$ be the linear span of the set

$$C_M^X = \{ \begin{bmatrix} m_1 \otimes x & m_2 \otimes x \end{bmatrix} \mid m_1, m_2 \in M, x \in X \}. $$

Also, let $C_M^X$ be the linear span of the set

$$X \otimes C_M = \{ \begin{bmatrix} m_1 \otimes x & m_2 \otimes x \end{bmatrix} \mid m_1, m_2 \in M, x \in X \}. $$

As mentioned in [6 Section 5], we identify $C_M$ with $C_M^X$ by $\Psi_M$ and $\breve{\Psi}_M$. In the same way as in [6 Section 5], we define the left $C_X$-action and the right $C_X$-action on $C_M$ as follows:

$$\begin{bmatrix} a & x \\ \overline{x} & a \end{bmatrix} \cdot \begin{bmatrix} m_1 & m_2 & y \\ m_2 \otimes \overline{y} & m_1 \end{bmatrix} = \begin{bmatrix} a \otimes m_1 + x \otimes m_2 \otimes \overline{y} & a \otimes m_2 \otimes y + x \otimes m_1 \\ \overline{x} \otimes m_1 + a \otimes m_2 \otimes \overline{y} & \overline{x} \otimes m_2 \otimes y + a \otimes m_1 \end{bmatrix},$$

$$\begin{bmatrix} m_1 & m_2 & y \\ m_2 \otimes \overline{y} & m_1 \end{bmatrix} \cdot \begin{bmatrix} a & x \\ \overline{x} & a \end{bmatrix} = \begin{bmatrix} m_1 \otimes a + m_2 \otimes y \otimes \overline{x} & m_1 \otimes x + m_2 \otimes y \otimes a \\ m_2 \otimes \overline{y} \otimes a + m_1 \otimes \overline{x} & m_2 \otimes \overline{y} \otimes x + m_1 \otimes a \end{bmatrix},$$

for any $a \in A$, $m_1, m_2 \in M$, $x, y \in X$. But we identify $A \otimes_A M$, $M \otimes_A A$ and $X \otimes_A \breve{X}$, $\breve{X} \otimes_A X$ with $M$ and $A$ by the isomorphisms defined by

$$a \otimes m \in A \otimes_A M \mapsto a \cdot m \in M,$$

$$m \otimes a \in M \otimes_A A \mapsto m \cdot a \in M,$$

$$x \otimes \overline{y} \in X \otimes_A \breve{X} \mapsto (x, y) \in A,$$

$$\overline{x} \otimes y \in \breve{X} \otimes_A X \mapsto (x, y)_A \in A,$$

respectively and we identify $X \otimes_A M$ and $\breve{X} \otimes_A M$ with $M \otimes_A X$ and $M \otimes_A \breve{X}$ by $\Psi_M$ and $\breve{\Psi}_M$, respectively. By the above identifications, the right hand-sides of the above equations are in $C_M$. Before we define a left $C_X$-valued inner product and a right $C_X$-valued inner product on $C_M$, we define a conjugate linear map on $C_M$,

$$\begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes \overline{x} & m_1 \end{bmatrix} \in C_m \mapsto \begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes \overline{x} & m_1 \end{bmatrix}^\dagger \in C_m$$

by

$$\begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes \overline{x} & m_1 \end{bmatrix} \mapsto \begin{bmatrix} \overline{m}_1 & x \otimes \overline{m}_2 \\ \overline{x} \otimes m_2 & m_1 \end{bmatrix}.$$
for any \(m_1, m_2 \in M, x \in X\). We define the left \(C_X\)-valued inner product and the right \(C_X\)-valued inner product as follows:

\[
C_X \left\{ \begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes x & m_1 \end{bmatrix}, \begin{bmatrix} n_1 & n_2 \otimes y \\ n_2 \otimes y & n_1 \end{bmatrix} \right\} = \begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes x & m_1 \end{bmatrix} \begin{bmatrix} n_1 & n_2 \otimes y \\ n_2 \otimes y & n_1 \end{bmatrix} = \begin{bmatrix} m_1 \otimes \tilde{n}_1 + m_2 \otimes x \otimes \tilde{y} \otimes \tilde{n}_2 + m_1 \otimes y \otimes \tilde{n}_2 + m_2 \otimes x \otimes \tilde{n}_1, m_1 \otimes y \otimes \tilde{n}_2 + m_2 \otimes x \otimes \tilde{n}_1 \\ m_2 \otimes \tilde{n}_1 + m_1 \otimes y \otimes \tilde{n}_2 + m_2 \otimes x \otimes \tilde{n}_1, m_2 \otimes \tilde{n}_1 + m_1 \otimes y \otimes \tilde{n}_2 + m_2 \otimes x \otimes \tilde{n}_1 \end{bmatrix}.
\]



for any \(m_1, m_2, n_1, n_2 \in M, x, y \in X\), where we regard the tensor product as a product on \(C_M\) in the formal manner. We denote it by \(\cdot\). Also, we identify \(A \otimes_A M, M \otimes_A A \) and \(X \otimes_A \tilde{X}, \tilde{X} \otimes_A X\) with \(M\) and \(A\) by the same isomorphisms as above and we identify \(X \otimes_A M\) and \(\tilde{X} \otimes_A M\) with \(M \otimes_A X\) and \(M \otimes_A \tilde{X}\) by \(\Psi_M\) and \(\Psi_{\tilde{M}}\). By the above identifications, we can define the left \(C_X\)-valued and the right \(C_X\)-valued inner products. In the same way as above, we can define the left \(C_X\)-action and the right \(C_X\)-valued action on \(C_M\) and the left \(C_X\)-valued inner product and the right \(C_X\)-valued inner product on \(C_M\). Since we identify \(C_M\) with \(C_M^1, C_M^2, \ldots\) by \(\Psi_M\) and \(\Psi_{\tilde{M}}\), we can see that \(C_M\) and \(C_M^i\) are \(C_X\)-\(C_X\)-equivalence bimodules by \(6\) Lemma 5.10 and that each of them agrees with the other by routine computations (See \(6\) Section 5). We identify \(C_M\) with \(C_M^1\) as \(C_X\)-\(C_X\)-equivalence bimodules by the isomorphisms \(\Psi_M\) and \(\Psi_{\tilde{M}}\) and we denote them by the same symbol \(C_M\). Furthermore, by \(6\) Lemma 5.11, \((M, C_M) \in \text{Equi}(A, C_C)\).

Let \(\Phi_M\) be another involutive \(A\)-\(A\)-equivalence bimodule isomorphism of \(\tilde{M} \otimes_A X \otimes_A M\) onto \(X\) and let \(\Phi_{M'}\) be the involutive \(A\)-\(A\)-equivalence bimodule isomorphism of \(\tilde{M} \otimes_A \tilde{X} \otimes_A M\) onto \(\tilde{X}\) induced by \(\Phi_M\). Let \(\Psi_M\) be the \(A\)-\(A\)-equivalence bimodule isomorphism of \(X \otimes_A M\) onto \(M \otimes_A X\) induced by \(\Phi_M'\) and let \(\Psi_M'\) be the \(A\)-\(A\)-equivalence bimodule isomorphism of \(\tilde{X} \otimes_A M\) onto \(M \otimes_A \tilde{X}\) induced by \(\Phi_{M'}\). Then we can identify \(C_M\) with \(C_M^1, C_M^2, \ldots\) by the isomorphisms \(\Psi_M'\) and \(\Psi_{\tilde{M}}\). Hence we can obtain an element in \(\text{Equi}(M, C_X)\) by the above identification. We denote the element by \((M, C_M')\).

**Lemma 6.1.** With the above notation, \([M, C_M] = [M, C_M']\) in \(\text{Pic}(A, C_X)\).

**Proof.** We can construct a \(C_X\)-\(C_X\)-equivalence bimodule isomorphism using the \(A\)-\(A\)-equivalence isomorphisms \(\Psi_M, \Psi_{\tilde{M}}, \Psi_M', \Psi_{\tilde{M}}\). Hence \(C_M\) and \(C_M'\) are isomorphic as \(C_X\)-\(C_X\)-equivalence bimodules by the \(C_X\)-\(C_X\)-equivalence bimodule isomorphism, which leaves the diagonal elements in \(C_M\) and \(C_M'\) invariant. Thus \([M, C_M] = [M, C_M']\) in \(\text{Pic}(A, C_X)\). \(\square\)

Let \(M_1\) be another \(A\)-\(A\)-equivalence bimodule with \(\tilde{M}_1 \otimes_A X \otimes_A M_1 \cong X\) as involutive \(A\)-\(A\)-equivalence bimodules. Let \([M_1, C_{M_1}]\) be the element in \(\text{Pic}(A, C_X)\) induced by \(M_1\) in the above.

**Lemma 6.2.** With the above notation, we suppose that \(M\) and \(M_1\) are isomorphic as \(A\)-\(A\)-equivalence bimodules. Then \([M, C_M] = [M_1, C_{M_1}]\) in \(\text{Pic}(A, C_X)\).
Proof. Since $M \cong M_1$ as $A - A$-equivalence bimodules, there is an $A - A$-equivalence bimodule isomorphism $\pi$ of $M_1$ onto $M$. Let $\Phi_M$ be an involutive $A - A$-equivalence bimodules isomorphism of $\tilde{M} \otimes_A X \otimes_A M$ onto $X$. Then $\Phi_M \circ (\tilde{\pi} \otimes \text{id}_X \otimes \pi)$ is an involutive $A - A$-equivalence bimodule isomorphism of $\tilde{M}_1 \otimes_A X \otimes_A M_1$ onto $X$, where $\tilde{\pi}$ is the $A - A$-equivalence bimodule isomorphism of $\tilde{M}_1$ onto $M$ defined by

$$\tilde{\pi}(\tilde{m}) = \pi(m)$$

for any $m \in M$. Let $[M, C_M]$ and $[M_1, C_{M_1}]$ be the element in $\text{Pic}(A, C_X)$ induced by $\Phi_M$ and $\Phi_M \circ (\tilde{\pi} \otimes \text{id}_X \otimes \pi)$. Let $(M, C_M)$ be the element in $\text{Equi}(A, C_X)$ obtained by using the isomorphism $\Phi_M$ and let $(M_1, C_{M_1})$ be the element in $\text{Equi}(A, C_X)$ obtained by using the isomorphism $\Phi_M \circ (\tilde{\pi} \otimes \text{id} \otimes \pi)$. Then by the definitions of $(M, C_M)$, $(M_1, C_{M_1})$ and Lemma 6.1, we obtain that $[M, C_M] = [M_1, C_{M_1}]$ in $\text{Pic}(A, C_X)$.

Let $g$ be the map from $\text{Im} f_A$ to $\text{Pic}(A, C_X)$ defined by

$$g([M]) = [M, C_M]$$

for any $[M] \in \text{Im} f_A$. By Lemmas 6.1 and 6.2, $g$ is well-defined.

Let $M$ and $K$ be $A - A$-equivalence bimodules with $\tilde{M} \otimes_A X \otimes_A M \cong X$ and $\tilde{K} \otimes_A X \otimes_A K \cong X$ as $A - A$-equivalence bimodules, respectively. Let $(M, C_M)$ and $(K, C_K)$ be the elements in $\text{Equi}(A, C_X)$ induced by $M$ and $K$, respectively. Also, let $(M \otimes_A K, C_{M \otimes_A K})$ be the element in $\text{Equi}(A, C_X)$ induced by $M \otimes A K$.

Lemma 6.3. With the above notation, $C_M \otimes_{C_X} C_K \cong C_{M \otimes_A K}$ as $C_X - C_X$-equivalence bimodules.

Proof. Let $\pi$ be the linear map from $C_M \otimes_{C_X} C_K$ onto $C_{M \otimes_A K}$ defined by

$$\pi\left(\begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes x \otimes 2 & m_1 \end{bmatrix} \otimes \begin{bmatrix} k_1 & y \otimes k_2 \\ \tilde{y} \otimes k_2 & k_1 \end{bmatrix}\right) = \begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes x \otimes 2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} k_1 & y \otimes k_2 \\ \tilde{y} \otimes k_2 & k_1 \end{bmatrix}$$

for any $\left[\begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes x \otimes 2 & m_1 \end{bmatrix}\right] \in C_M^X$ and $\left[\begin{bmatrix} k_1 & y \otimes k_2 \\ \tilde{y} \otimes k_2 & k_1 \end{bmatrix}\right] \in X C_K$, where we regard the tensor product as a product on $C_M$ in the formal manner. But we identify $A \otimes_A K$ and $X \otimes_A \tilde{X}$ with $K$ and $A$ by the isomorphisms defined by

$$a \otimes k \in A \otimes_A K \mapsto a \cdot k \in K,$$

$$x \otimes \tilde{y} \in X \otimes_A \tilde{X} \mapsto A(x, y) \in A,$$

$$\tilde{x} \otimes y \in \tilde{X} \otimes_A X \mapsto (x, y)_X \in A,$$

respectively. Furthermore, we identify $X \otimes_A K$ and $X \otimes_A \tilde{K}$ with $K \otimes_A X$ and $K \otimes_A \tilde{X}$ as $A - A$-equivalence bimodules by $\Psi_K$ and $\Psi_{\tilde{K}}$, which are defined as above, respectively. Thus,

$$\pi\left(\begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes x \otimes 2 & m_1 \end{bmatrix} \otimes \begin{bmatrix} k_1 & y \otimes k_2 \\ \tilde{y} \otimes k_2 & k_1 \end{bmatrix}\right) = \begin{bmatrix} m_1 \otimes k_1 + m_2 \otimes A(x, y) \cdot k_2 \\ m_2 \otimes A(x, y) \cdot k_1 \end{bmatrix} \cdot \begin{bmatrix} k_1 & y \otimes k_2 \\ \tilde{y} \otimes k_2 & k_1 \end{bmatrix}$$

for any $m_1 \otimes k_1 + m_2 \otimes A(x, y) \cdot k_2$ and $m_2 \otimes A(x, y) \cdot k_1$. Thus, we have $C_M \otimes_{C_X} C_K \cong C_{M \otimes_A K}$ as $C_X - C_X$-equivalence bimodules.
Then by routine computations,
\[ A(x, y^2) = (x^2, y)_A, \]
\[ \Psi_K(\tilde{x}^2 \otimes k_1) = \sum_{i=1}^{n} u_i \otimes \Phi_K(\tilde{u}_i \otimes x \otimes k_1)^2, \]
\[ \Psi_K(x \otimes k_1) = \sum_{i=1}^{n} u_i \otimes \Phi_K(\tilde{u}_i \otimes x \otimes k_1), \]
\[ \tilde{\Psi}_K(\tilde{y}^2 \otimes k_2) = \sum_{i=1}^{n} u_i \otimes \Phi_K(\tilde{u}_i \otimes y \otimes k_2)^3, \]
\[ \Psi_K(y \otimes k_2) = \sum_{i=1}^{n} u_i \otimes \Phi_K(\tilde{u}_i \otimes y \otimes k_2), \]
where \( \{u_i\}_{i=1}^{n} \) is a finite subset of \( K \) with \( \sum_{i=1}^{n} A(u_i, u_i) = 1 \) and \( \Phi_K \) and \( \tilde{\Psi}_K \) are as defined in the above. Hence \( \pi \) is a linear map from \( C_M \otimes_{C_X} C_K \) to \( C_M \otimes_{A K} \).

Next, we show that \( \pi \) is surjective. We take elements
\[ \begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes \tilde{x}^2 & m_1 \end{bmatrix} \in C_M, \quad \begin{bmatrix} k_1 & 0 \\ 0 & k_1 \end{bmatrix} \in \tilde{x} C_K. \]
Then
\[ \pi\left( \begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes \tilde{x}^2 & m_1 \end{bmatrix} \otimes \begin{bmatrix} k_1 & 0 \\ 0 & k_1 \end{bmatrix} \right) = \begin{bmatrix} m_1 \otimes k_1 & m_2 \otimes \Psi_K(x \otimes k_1) \\ m_2 \otimes \tilde{\Psi}_K(\tilde{x}^2 \otimes k_1) & m_1 \otimes k_1 \end{bmatrix}. \]
We also take elements
\[ \begin{bmatrix} 0 & m_2 \otimes y \\ m_2 \otimes \tilde{y}^2 & 0 \end{bmatrix} \in C_M, \quad \begin{bmatrix} k_2 & 0 \\ 0 & k_2 \end{bmatrix} \in \tilde{y} C_K. \]
Then
\[ \pi\left( \begin{bmatrix} 0 & m_2 \otimes y \\ m_2 \otimes \tilde{y}^2 & 0 \end{bmatrix} \otimes \begin{bmatrix} k_2 & 0 \\ 0 & k_2 \end{bmatrix} \right) = \begin{bmatrix} 0 & m_2 \otimes \Psi_K(y \otimes k_2) \\ m_2 \otimes \tilde{\Psi}_K(\tilde{y}^2 \otimes k_2) & 0 \end{bmatrix}. \]
Thus
\[ \pi\left( \begin{bmatrix} m_1 & m_2 \otimes x \\ m_2 \otimes \tilde{x}^2 & m_1 \end{bmatrix} \otimes \begin{bmatrix} k_1 & 0 \\ 0 & k_1 \end{bmatrix} + \begin{bmatrix} 0 & m_2 \otimes y \\ m_2 \otimes \tilde{y}^2 & 0 \end{bmatrix} \otimes \begin{bmatrix} k_2 & 0 \\ 0 & k_2 \end{bmatrix} \right) = \begin{bmatrix} m_1 \otimes k_1 & m_2 \otimes \Psi_K(x \otimes k_1 + y \otimes k_2) \\ m_2 \otimes \tilde{\Psi}_K(\tilde{x}^2 \otimes k_1 + \tilde{y}^2 \otimes k_2) & m_1 \otimes k_1 \end{bmatrix}. \]

Since \( \Psi_K \) and \( \tilde{\Psi}_K \) are isomorphisms of \( X \otimes_A K \) and \( \tilde{X} \otimes_A K \) onto \( K \otimes_A X \) and \( K \otimes_A \tilde{X} \), respectively, we can see that \( \pi \) is surjective. Furthermore, by the definitions of \( \pi \) and the left and the right \( A \)-valued inner products on \( C_M, C_K \) and \( C_M \otimes_{A K} \), we can easily see that \( \pi \) preserves the left and the right \( A \)-valued inner products. Indeed, let \( M, M_1 \subseteq C_M \) and \( K, K_1 \subseteq C_K \). Then
\[ c^X_M(\pi(M \otimes K), \pi(M_1 \otimes K_1)) = c^X_M( (M \cdot K), (M_1 \cdot K_1) ) = M \cdot K \cdot (M_1 \cdot K_1) = M \cdot K \cdot \tilde{K}_1 \cdot M_1. \]
Also,
\[ c_X(M \otimes K, M_1 \otimes K_1) = c_X(M \cdot c_X(K, K_1), M_1) = c_X(M \cdot K \cdot K_1, M_1) = M \cdot K \cdot K_1 \cdot M_1. \]
Hence \( \pi \) preserves the left \( C_X \)-valued inner products. Similarly, we can see that \( \pi \) preserves the right \( C_X \)-valued inner products. Therefore, \( \pi \) is a \( C_X - C_X \)-equivariance bimodule isomorphism of \( C_M \otimes C_X C_K \) onto \( C_{M \otimes A K} \) by the remark after [3, Definition 1.1.18].

Proposition 6.4. With the above notation, \( g \) is a homomorphism of \( \text{Im} f_A \) to \( \text{Pic}(A, C_X) \) with \( f_A \circ g = \text{id} \) on \( \text{Im} f_A \).

Proof. This is immediate by Lemma [3,6] and the definition of \( g \).

We give the main result of this paper.

Theorem 6.5. Let \( A \) be a unital \( C^\ast \)-algebra and \( X \) an involutive \( A - A \)-equivalence bimodule. Let \( A \subset C_X \) be the unital inclusion of unital \( C^\ast \)-algebras induced by \( X \). We suppose that \( \mathcal{N} \cap C_X = \mathcal{C} \). Let \( f_A \) be the homomorphism of \( \text{Pic}(A, C_X) \) to \( \text{Pic}(A) \) defined by \( f_A([M, N]) = [M] \) for any \( (M, N) \in \text{Equ}(A, C_X) \). Then \( \text{Pic}(A, C_X) \) is isomorphic to a semi-direct product group of \( T \) by the group
\[ \{ [M] \in \text{Pic}(A) \mid M \text{ is an } A - A \text{-equivalence bimodule with } X \cong \tilde{M} \otimes_A X \otimes_A M \text{ as involutive } A - A \text{-equivalence bimodules} \}. \]

Proof. This is immediate by Proposition [3,6], Theorem [5,5] and Proposition [6,4].

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