Collective oscillations of a quasi one dimensional Bose condensate under damping

Fatkhulla Kh Abdullaev 1, 2, Ravil M. Galimzyanov 2, * and Khayotullo Ismatullaev 2, 3

1 Dipartamento di Fisica "E.R. Caianiello", Università di Salerno, I-84081 Baronissi (SA), Italy
2 Physical-Technical Institute of the Academy of Sciences, G.Mavlyanov 2-b, Tashkent 700 084, Uzbekistan
3 Institute of Electronics of the Academy of Sciences, F.Khodjaev 33, Tashkent 700 125, Uzbekistan

E-mail: ravil@uzsci.net

Abstract. Influence of the damping on collective oscillations of a one-dimensional trapped Bose gas in the mean field regime has been studied. Using the phenomenological damping approach developed by L.P. Pitaevskii, modified variational equations for the parameters of the condensate wave function is derived. Analytical expressions for the condensate parameters in equilibrium state have been obtained. Bistability in nonlinear oscillations of the condensate under periodic variations of the trap potential is predicted. The predictions of the modified variational approach are confirmed by full numerical simulations of the 1D GP equation with the damping.

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* To whom correspondence should be addressed
1. Introduction

The dynamics of a one-dimensional trapped ultra-cold Bose gas has attracted considerable attention for last years \[1\]. Recently has been achieved regimes where 1D Bose gas with a zero-range two-body interaction in the Tonks -Girardeau (TG)regime \[2, 3, 4\] with fermionic behavior becomes visible. An investigation of the transition from the mean field regime to the TG regime was performed in \[5\]. Experimentally 1D regime has been realized in \[6\].

Measurements of the collective oscillations of such a system should give a lot of information about the BEC dynamics. In particular this is important in the analysis of the condensate dynamics in a magnetic waveguide, being a fundamental atom optical element \[7\]. Performed by this time theoretical descriptions have mainly dealt with conservative systems, e.g. see \[8\], where collective oscillations of a quasi 1D Bose-Einstein condensate(BEC) in the low and high density regimes were investigated. The damping of the radial BEC oscillations in a cylindric trap, connected with the parametric resonance and leading to the energy transfer from collective oscillations to longitudinal sound waves has been studied in \[9\]. The dissipation inheres in real systems. So it is of interest to investigate theoretically effect of damping on collective oscillations of a one-dimensional trapped repulsive Bose gas.

We consider here the problem using the phenomenological approach developed by L.P. Pitaevskii \[10\] and employed later in \[11\].

2. The model

The dynamics of a trapped one dimensional repulsive Bose gas with the damping is described in the framework of the 1D Gross-Pitaevskii equation \[10, 11\]

\[
i\hbar \phi_t = (1 + i\gamma)(-\frac{\hbar^2}{2m}\phi_{xx} + V(x, t)\phi + g_{1D}|\phi|^2\phi - \mu\phi),
\]

with the total number of atoms \( N = \int |\phi|^2 dx \). This equation is obtained in the case of a highly anisotropic external potential under the assumption that the transversal trapping potential is harmonic: \( V(y, z) = m\omega_\perp^2(y^2 + z^2)/2 \) and \( \omega_\perp \gg \omega_x \). Under such conditions we can consider the solution of 3D equation to have the form \( U(x, y, z; t) = R(y, z)\phi(x, t) \) where \( R_0^2 = m\omega_\perp \exp(-m\omega_\perp \rho^2/\hbar)/(\pi\hbar) \). Averaging in the radial direction (i.e. integrating over the transversal variables) we come to equation \(1\) describing the dynamics of the gas in longitudinal direction. The potential \( V(x, t) \) is assumed to be \( V(x, t) = m\omega_x^2x^2F(t) \), where \( F(t) \) describes the time dependence of the potential, which we consider here as \( F(t) = 1 + h\sin(wt) \). The effective one dimensional mean field nonlinearity coefficient \( g_{1D} = 2\hbar a_s\omega_\perp \), where \( a_s \) is the atomic scattering length. \( a_s > 0 \) corresponds to the Bose gas with a repulsive interaction between atoms and \( a_s < 0 \) to an attractive interaction.
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We have the following dimensionless form of the equation (1): 

\[ i\psi_t + \frac{1}{2}\psi_{xx} - \frac{x^2}{2}F(t)\psi - g|\psi|^2\psi + \mu\psi = \]

\[ = i\gamma\left(-\frac{1}{2}\psi_{xx} + \frac{x^2}{2}F(t)\psi + g|\psi|^2\psi - \mu\psi\right) = R(\psi, \psi^*), \quad (2) \]

by setting: 

\[ t = \omega_x t, \quad l = \sqrt{\hbar/(m\omega_x)}, \quad x = x/l, \quad \psi = \sqrt{\frac{a_s}{\omega_\perp/\omega_x}} \phi, \]

with \( g = \pm 1 \) for the repulsive and attractive two-body interactions respectively.

3. Variational analysis

To describe collective oscillations of a Bose gas under damping we employ the variational approach. For the wavefunction \( \psi(x, t) \) we use the gaussian trial function

\[ \psi(x, t) = A(t)\exp(-\frac{x^2}{2a^2(t)}) - \frac{ib(t)x^2}{2} - i\varphi(t), \quad (3) \]

where \( A, a, b \) and \( \varphi \) are the amplitude, width, chirp and linear phase, respectively.

Equation (2) can be obtained from the variational equations

\[ \frac{\partial L}{\partial \psi^*} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \psi_x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \psi_t} + \frac{\delta L}{\delta \psi^*} = 0, \quad (4) \]

where \( L \) is the Lagrangian density, \( L \equiv L(x, t) \), of a conservative system, given by

\[ L = i\left(\psi_t\psi^* - \psi^*_t\psi\right) - \frac{1}{2}|\psi_x|^2 - \left(\frac{x^2}{2}F(t) - \mu\right)|\psi|^2 - \frac{g}{2}|\psi|^4 \quad (5) \]

and \( L_R \) is defined as \( \partial L_R/\partial \psi^* = -R(\psi, \psi^*) \), where \( R(\psi, \psi^*) \) is the right side of equation (2).

Inserting trial function (3) into equation (5) and averaging it as

\[ \bar{L} = \int L(x, t)dx \quad (6) \]

we obtain the averaged Lagrangian of the conservative system in terms of the trial function parameters:

\[ \frac{\bar{L}}{\sqrt{\pi}} = \frac{A^2a^3b_t}{4} + A^2a\varphi_t - \frac{A^2}{4a} - \frac{A^2a^3b^2}{4} - \]

\[ -\frac{FA^2a^3}{4} - \frac{gA^4a}{2\sqrt{2}} + \mu A^2a. \quad (7) \]

Using equation (2) and its conjugate, we obtain a system of equations for the variational parameters \( \eta_i \) [13, 14]:

\[ \frac{\partial \bar{L}}{\partial \eta_i} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \eta_{it}} = \int dx\left(R\frac{\partial \psi^*}{\partial \eta_i} + R^*\frac{\partial \psi}{\partial \eta_i}\right). \quad (8) \]
Inserting (3) and (7) into equation (8) we derive the following system of ordinary differential equations (ODE):

\[
\frac{d(A^2a)}{dt} = \gamma A^2 + \frac{\gamma A^2a^3b^2}{2} + \frac{\gamma F A^2a^3}{2} + \sqrt{2}gA^4a - 2\gamma \mu A^2a,
\]

\[
\frac{d(A^2a^3)}{dt} = -2A^2a^3b - \frac{\gamma A^2a}{2} + 3\gamma A^2a^5b^2 + \frac{\sqrt{2}gA^4a}{2} - 2\gamma \mu A^2a^3,
\]

\[
\frac{db}{dt} = \frac{2\gamma b}{a^2} - \frac{1}{a^4} + b^2 + F - \frac{gA^2}{\sqrt{2}a^2},
\]

\[
\frac{d\varphi}{dt} = -\frac{\gamma b}{2} + \frac{1}{2a^2} + \frac{5gA^2}{4\sqrt{2}} - \mu.
\]

This system of equations can also be obtained from the modified form of the conservation law for the number of atoms and by using the moments method, as shown in Appendix. Let us rewrite the system using the notations \(x = a^2\), \(y = A^2\):

\[
x_t = -2xb - \gamma + \gamma x^2b^2 + \gamma Fx^2 - \frac{\gamma gyx}{\sqrt{2}},
\]

\[
y_t = yb + \frac{\gamma y}{x} + \frac{5\gamma gy^2}{2\sqrt{2}} - 2\gamma \mu y,
\]

\[
b_t = \frac{2\gamma b}{x} - \frac{1}{x^2} + b^2 + F - \frac{gy}{\sqrt{2}x}.
\]

This ODE system is the main result of this section.

4. Numerical simulations

We have carried out a series of time dependent simulations of the system within the variational approach using equation (11) and also by performing exact numerical calculations using equation (2). In our numerical calculations we discretize the problem in a standard way, with the time step \(dt\), and spatial step \(dx\), so \(\psi^k_j\) approximates \(\psi(jdx, kdt)\). More specifically we approximate the governing equation (2) with the following semi-implicit Crank-Nickolson scheme using split-step method [12]

\[
\frac{i(\psi^{k+1}_j - \psi^{k}_j)}{dt} + \frac{i(\psi^{k+\frac{1}{2}}_j - \psi^{k-\frac{1}{2}}_j)}{dt} + \frac{i(\psi^{k+\frac{1}{2}}_j - \psi^{k}_j)}{dt} = \frac{1}{2}H_1(\psi^{k+1}_j + \psi^{k+\frac{1}{2}}_j) + \frac{1}{2}H_2(\psi^{k+\frac{3}{2}}_j + \psi^{k+\frac{1}{2}}_j) + \frac{1}{2}H_3(\psi^{k+\frac{1}{2}}_j + \psi^{k}_j),
\]

where \(H_1 = 0.5(1 + i\gamma)(x^2F(t)/2 + g|\psi(x, t)|^2 - \mu)\), \(H_2 = -(1 + i\gamma)\partial^2 / \partial x^2\), \(H_3 = H_1\) and \(\psi^{k+\frac{3}{2}}, \psi^{k+\frac{1}{2}}\) are defined so that

\[
\frac{i(\psi^{k+\frac{1}{2}}_j - \psi^{k}_j)}{dt} = \frac{1}{2}H_1(\psi^{k+\frac{1}{2}}_j + \psi^{k}_j),
\]
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\[
\frac{i(\psi_j^{k+\frac{2}{3}} - \psi_j^{k+\frac{4}{3}})}{dt} = \frac{1}{2} H_2(\psi_j^{k+\frac{2}{3}} + \psi_j^{k+\frac{4}{3}}), \tag{13}
\]

\[
\frac{i(\psi_j^{k+1} - \psi_j^{k+\frac{2}{3}})}{dt} = \frac{1}{2} H_3(\psi_j^{k+1} + \psi_j^{k+\frac{2}{3}}). \tag{14}
\]

Solving (12) and (13) first at time \( t_k = kdt \) we produce intermediate solutions \( \psi_j^{k+\frac{2}{3}} \) and \( \psi_j^{k+\frac{4}{3}} \). The final solution for one time step \( dt \) is obtained from (14).

The results of numerical simulations of both PDE and ODE models are presented below.

Figure 1 shows time evolution of the width and the norm of the condensate for the case of \( F(t) = 1 \), i.e. when the external potential has no time perturbations. The figure is presented to compare the character of the width oscillations and the behavior of the norm depending on the values of the chemical potential \( \mu \) and the dissipative constant \( \gamma \). As seen, the frequency of the oscillations does not depend on the dissipative constant at all and depends weakly on the chemical potential, e.g. when \( \mu = 2, w_0 = 1.808 \) while when \( \mu = 3, w_0 = 1.77 \). For greater values of \( \gamma \), the oscillations damp faster. We see that the ODE leads to an equilibrium state the norm of which is 3-4 percent less than that of the PDE results.

From figure 1 it can be seen that damping process of the condensate eventually leads to the equilibrium state. This equilibrium state can also be obtained analytically by solving equation (10). Taking into consideration that in an equilibrium state \( x_t = 0, y_t = 0 \) and \( b = 0 \) the following expressions can be obtained:

\[
a^2 = \frac{4\mu + \sqrt{16\mu^2 + 60}}{10}, \]

\[
A^2 = \frac{4\sqrt{2}}{5} \mu - \frac{2\sqrt{2}}{5a^2}. \tag{15}
\]

Putting the value of chemical potential e.g. for \( \mu = 2 \) we find \( a = 1.383, A = 1.403 \) and \( N = A^2a\sqrt{\pi} = 4.825 \) which are confirmed by the PDE results.

In figure 2 we can observe an interesting behavior of the norm. If an external perturbation is applied to a trapped BEC which is already in the equilibrium state then the norm of the condensate starts decreasing. In the figure the frequency of the periodical trap perturbation is taken to be equal to the eigenfrequency of the system determined from the figure 1 and the amplitude of time perturbations is taken as \( h = 0.06 \). We see that for smaller values of the dissipative constant \( \gamma \) the norm goes farther from the equilibrium state. That is, the dissipative constant is not simply the quantity which is responsible for diminishing the norm, but it is the constant keeping a condensate in the equilibrium state.

The width dynamics under main resonance with the initial wave packet taken in the equilibrium state is depicted in figure 3. As shown, in contrast to the norm the width oscillates near the previous point. As seen from the figure the amplitude of the width oscillations becomes stable by the time \( t = 140 \).
Performing ODE and PDE calculations for the frequencies which lie around the eigenfrequency of the BEC and measuring the amplitudes of the above-said stable oscillations we have plotted the values of the oscillation amplitude as a function of the frequency of the periodical trap perturbations with \( h = 0.03 \) and \( h = 0.06 \) in the cases \( \gamma = 0.01 \) and \( \gamma = 0.005 \) in figure [4].

We see that e.g. when \( h = 0.06 \) and \( \gamma = 0.005 \) the highest amplitude of oscillations is driven in the trap perturbation with the frequency \( w = 1.89 \) which is more than the eigenfrequency \( w_0 = 1.806 \). Bistability appears with the smaller values of \( \gamma \) in the vicinity of this critical frequency. For \( w = 1.88 \) we observe large oscillations with \( a_{osc} = 1.6 \), while at \( w = 1.91 \) we observe much smaller width oscillations with \( a_{osc} = 0.6 \). It can be seen that with the growth of trap perturbations the value of the critical frequency becomes greater.

Let us estimate parameters for the experiment. The magnetic trap can be taken with parameters \( \omega_\perp = 2\pi \times 40 \) Hz, \( \omega_x = 2\pi \) Hz, and the number of atoms of \(^{85}\)Rb \( N = 0.23 \times 10^4 \). For the external field \( B = 161.57 \), \( a_s = 0.3 \)nm (repulsive gas). Then, the eigenfrequency of the harmonically trapped BEC \( 2\pi \nu_0 = 1.808 \times \omega_x = 11.36 \) Hz. By applying the external perturbation \( F(t) = 1 + 0.06 \sin(2\pi \nu t) \) with \( 2\pi \nu = 11.36 \) Hz to the trapped BEC one will observe decreasing of the number of atoms by 20, 13 and 7 percent for \( \gamma = 0.1, 0.2 \) and 0.3 respectively. For the external perturbation with \( 2\pi \nu = 11.8 \) Hz, large oscillations will be observed with \( a_{osc} = 1.6l = 1.6\sqrt{\hbar/(m\omega_x)} = 17.4 \) \( \mu \)m, while at \( 2\pi \nu = 12 \) Hz one will observe much smaller oscillations with \( a_{osc} = 0.6l = 6.5 \) \( \mu \)m.

5. Conclusions

In this paper we study the collective oscillations of a quasi one dimensional Bose gas in the presence of dissipative effects. The modified Gross-Pitaevskii equation in the framework of the phenomenological approach \([10, 11]\) has been used. To describe evolution of oscillations we employ the modified variational approach taking into account the dissipation. We confirm results for the oscillations damping obtained from the system of equations for the wave function parameters, by the moments method and direct numerical simulations of the full GP equation.

The calculations show that damping oscillations of a BEC which eventually comes to an equilibrium state occur with the eigenfrequency which does not depend on the value of the dissipative constant \( \gamma \) and only depends on the chemical potential \( \mu \).

The expressions for the width and the norm of a condensate in an equilibrium state have been derived analytically.

We study the main resonance in condensate oscillations. For a BEC in an equilibrium state it is shown that making the external potential oscillate leads to decreasing of the condensate norm and the norm begins oscillating around the point which is less than the previous stationary one in the equilibrium state, whereas the condensate width oscillates near the previous point.

We have also shown that in resonances the bistability appears with smaller values
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of the dissipative constant in the vicinity of the critical frequency which is above the eigenfrequency of the BEC.

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Appendix

Differentiating the norm of the condensate by time we can obtain the first equation of the system (10) as

$$\frac{dN}{dt} = \frac{d}{dt} \int |\psi|^2 dx = \int (\psi_\ast x^2 + \psi^\ast \psi_t) dx.$$  (A.1)

Inserting \(\psi_t\) from equation (2) into (A.1) we derive the modified form of the conservation law for the norm of the condensate [29].

$$\frac{dN}{dt} = \gamma \int |\psi_x|^2 dx + \gamma F \int x^2 |\psi|^2 dx +$$

$$+ 2\gamma g \int |\psi|^4 dx - 2\gamma \mu \int |\psi|^2 dx.$$  (A.2)

Substituting the Gaussian trial function (3) into this equation we obtain equation (10b).

Equations (10b) and (10c) can be obtained by the moments method. We obtain equation (10b) by calculating

$$\frac{d <x^2>}{dt} = \int (\psi_\ast x^2 \psi + \psi^\ast x^2 \psi_t) dx.$$  (A.3)

The substitution of the \(\psi_t\) from equation (2) leads to the following equation:

$$\frac{d <x^2>}{dt} = i \int x(\psi_\ast x^2 \psi - \psi^\ast x^2 \psi_t) dx + \gamma \int x^2 |\psi_x|^2 dx -$$

$$- \gamma \int |\psi|^2 dx + \gamma F \int x^4 |\psi|^2 dx +$$

$$+ 2\gamma g \int x^2 |\psi|^4 dx - 2\gamma \mu \int x^2 |\psi|^2 dx.$$  (A.4)

Substituting trial function (3) we get equation (10b). Equation (10c) is derived by calculating

$$\frac{d <p^2>}{dt} = - \frac{d}{dt} \int \psi^\ast \psi_{xx} dx =$$

$$= - \int (\psi_\ast \psi_{xx} + \psi^\ast \psi_{txx}) dx.$$  (A.5)

Integrating by parts and substituting \(\psi_t\) from equation (2) we obtain

$$\frac{d}{dt} \int |\psi_x|^2 dx = iF \int x(\psi_\ast x^2 \psi - \psi^\ast x^2 \psi_t) dx +$$

$$+ ig \int (\psi_\ast x^2 \psi - \psi^\ast x^2 \psi_t) dx +$$
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\[ + \gamma \int |\psi_{xx}|^2 dx - \gamma F \int |\psi|^2 dx + \]
\[ + \gamma F \int x^2 |\psi_x|^2 dx + 4 \gamma g \int |\psi|^2 |\psi_x|^2 dx + \]
\[ + \gamma g \int (\psi^* \psi_x^2 + \psi^2 \psi_x^2) dx - 2 \gamma \mu \int |\psi_x|^2 dx. \]  \hspace{1cm} (A.6)

Inserting the Gaussian ansatz into this equation and using results (10a) and (10b) we obtain equation (10c).

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Figure captions

**Figure 1.** Dynamics of the norm (a) and the width (b) of the repulsive BEC in a harmonic trap without perturbation. The upper and lower diagrams are plotted for the cases $\mu = 3$ and $\mu = 2$ respectively. Solid and dotted lines show PDE and ODE results.

**Figure 2.** The behavior of the norm of the trapped BEC when the trap starts oscillating with the amplitude $h = 0.06$. Before the trap oscillations BEC was in the equilibrium state. We compare the PDE and ODE system simulations. The solid lines stand for full numerical simulations of the PDE, while the dotted lines represent the ODE results.

**Figure 3.** BEC width versus time $t$ in the main resonance in the case $h = 0.06$, $\mu = 2$, $\gamma = 0.01$. Solid and dotted lines show the PDE and ODE results respectively.

**Figure 4.** Oscillation amplitude as predicted by the PDE (scatter) and ODE (line) models. Here $\gamma = 0.005$ (solid lines and squares) and $\gamma = 0.01$ (dotted lines and circles). The upper two lines are for the case $h = 0.06$, the lower two are for the case $h = 0.03$. The chemical potential is equal to 2.
amplitude of width oscillations, \( a \) vs frequency, \( w \)