Existence results for mean field equations
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EXISTENCE RESULTS FOR MEAN FIELD EQUATIONS

WEIYUE DING, JÜRGEN JOST, JIAYU LI AND GUOFANG WANG

Abstract. Let $\Omega$ be an annulus. We prove that the mean field equation

$$-\Delta \psi = \frac{e^{-\beta \psi}}{\int_\Omega e^{-\beta \psi}} \quad \text{in} \ \Omega$$

$$\psi = 0 \quad \text{on} \ \partial \Omega$$

admits a solution for $\beta \in (-16\pi, -8\pi)$. This is a supercritical case for the Moser-Trudinger inequality.

1. Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$. In this paper, we consider the following mean field equation

$$-\Delta \psi = \frac{e^{\beta \psi}}{\int_\Omega e^{\beta \psi}}, \quad \text{in} \ \Omega,$$

$$\psi = 0, \quad \text{on} \ \partial \Omega,$$

for $\beta \in (-\infty, +\infty)$. (1.1) is the Euler-Lagrange equation of the following functional

$$(1.2) \quad J_\beta(\psi) = \frac{1}{2} \int_\Omega |\nabla \psi|^2 + \frac{1}{\beta} \log \int_\Omega e^{\beta \psi}$$

in $H^{1,2}_0(\Omega)$. This variational problem arises from Onsager’s vortex model for turbulent Euler flows. In that interpretation, $\psi$ is the stream function in the infinite vortex limit, see [MP, p256ff]. The corresponding canonical Gibbs measure and partition function are finite precisely if $\beta > -8\pi$. In that situation, Caglioti et al. [CLMP1] and Kiessling [K] showed the existence of a minimizer of $J_\beta$. This is based on the Moser-Trudinger inequality

$$(1.3) \quad \frac{1}{2} \int_\Omega |\nabla \psi|^2 \geq \frac{1}{8\pi} \log \int_\Omega e^{8\pi \psi}, \quad \text{for any} \ \psi \in H^{1,2}_0(\Omega),$$

which implies the relevant compactness and coercivity condition for $J_\beta$ in case $\beta > -8\pi$. For $\beta \leq -8\pi$, the situation becomes different as described in [CLMP1]. On the unit disk, solutions blow up if one approaches $\beta = -8\pi$ - the critical case for (1.3)-(see also [CLMP2] and [Su]), and more generally, on starshaped domains, the Pohozaev identity yields a lower bound on the possible values of $\beta$ for which
solutions exist. On the other hand, for an annulus, [CLMP1] constructed radially symmetric solutions for any \( \beta \), and the construction of Bahri-Coron [BC] makes it plausible that solutions on domains with non-trivial topology exist below \(-8\pi\). Thus, for \( \beta \leq -8\pi \), \( J_\beta \) is no longer compact and coercive in general, and the existence of solution depends on the geometry of the domain.

In the present paper, we thus consider the supercritical case \( \beta < -8\pi \) on domains with non-trivial topology.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth, bounded domain whose complement contains a bounded region, e.g. \( \Omega \) an annulus. Then (1.1) has a solution for all \( \beta \in (-16\pi, -8\pi) \).

The solutions we find, however, are not minimizers of \( J_\beta \)-those do not exist in case \( \beta < 8\pi \), since \( J_\beta \) has no lower bound-but unstable critical points. Thus, these solutions might not be relevant to the turbulence problem that was at the basis of [CLMP1] and [K].

Certainly we can generalize Theorem 1.1 to the following equation

\[
-\Delta \psi = \frac{K e^{-\beta \psi}}{\int_\Omega K e^{-\beta \psi}}, \quad \text{in } \Omega,
\]

\[
\psi = 0, \quad \text{on } \partial \Omega,
\]

which was studied in [CLMP2]. Here \( K \) is a positive function on \( \Omega \).

With the same method, we may also handle the equation

\[
(1.4) \quad \Delta u - c + c K e^u = 0, \quad \text{for } 0 \leq c < \infty
\]

on a compact Riemann surface \( \Sigma \) of genus at least 1, where \( K \) is a positive function

(1.4) can also be considered as a mean field equation because it is the Euler-Lagrange equation of the functional

\[
(1.5) \quad J_\varepsilon(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 + c \int_\Sigma u - c \log \int_\Sigma K e^u.
\]

Because of the term \( c \int_\Sigma u \), \( J_\varepsilon \) remains invariant under adding a constant to \( u \), and therefore we may normalize \( u \) by the condition

\[
\int_\Sigma K e^u = 1
\]

which explains the absence of the factor \( (\int K e^u)^1 \) in (1.4). \( c < 8\pi \) again is a subcritical case that can easily be handled with the Moser-Trudinger inequality. The critical case \( c = 8\pi \) yields the so-called Kazdan-Warner equation [KW] and was treated in [DJLW] and [NT] by giving sufficient conditions for the existence of a minimizer of \( J_{8\pi} \). Here, we construct again saddle point type critical points to show
**Theorem 1.2.** Let $\Sigma$ be a compact Riemann surface of positive genus. Then (1.4) admits a non-minimal solution for $8\pi < c < 16\pi$.

Now we give a outline of the proof of the Theorems. First from the non-trivial topology of the domain, we can define a minimax value $\alpha_\beta$, which is bounded below by an improved Moser-Trudinger inequality, for $\beta \in (-16\pi, -8\pi)$. Using a trick introduced by Struwe in [St1] and [St2], for a certain dense subset $\Lambda \subset (-16\pi, -8\pi)$ we can overcome the lack of a coercivity condition and show that $\alpha_\beta$ is achieved by some $u_\beta$ for $\beta \in \Lambda$. Next, for any fixed $\tilde{\beta} \in (-16\pi, -8\pi)$, considering a sequence $\beta_k \in \Lambda$ tending to $\tilde{\beta}$, with the help of results in [BM] and [LS] we show that $u_{\beta_k}$ subconverges strongly to some $u_\beta$ which achieves $\alpha_\beta$.

After completing our paper, we were informed that Struwe and Tarantello [ST] obtained a non-constant solution of (1.4), when $\Sigma$ is a flat torus with fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, $K \equiv 1$ and $c \in (8\pi, 4\pi^2)$. In this case, it is easy to check that our solution obtained in Theorem 1.2 is non-constant.

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2. **Minimax values**

Let $\rho = -\beta$ and $u = -\beta \psi$. We rewrite (1.1) as

\begin{equation}
-\Delta u = \frac{\rho e^u}{\int_{\Omega} e^u}, \quad \text{in } \Omega,
\end{equation}

\begin{equation}
u = 0, \quad \text{on } \partial \Omega,
\end{equation}

and (1.2) as

\begin{equation}
J_\rho(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \rho \log \int_{\Omega} e^u
\end{equation}

for $u \in H^{1,2}_0(\Omega)$.

It is easy to see that $J_\rho$ has no lower bound for $\rho \in (8\pi, 16\pi)$. Hence, to get a solution of (1.1) for $\rho \in (4\pi, 16\pi)$, we have to use a minimax method. First, we define a center of mass of $u$ by

\begin{equation}
m_c(u) = \frac{\int_{\Omega} x e^u}{\int_{\Omega} e^u}.
\end{equation}

Let $B$ be the bounded component of $\mathbb{R}^2 \setminus \Omega$. For simplicity, we assume that $B$ is the unit disk centered at the origin. Then we define a family of functions

\begin{equation}
h : D \to H^{1/2}_0(\Omega)
\end{equation}

satisfying

\begin{equation}
\lim_{r \to 1} J_\rho(h(r, \theta)) \to -\infty
\end{equation}
and
\[ \lim_{r \to 1} m_c(h(r, \theta)) \text{ is a continuous curve enclosing } B. \]

Here \( D = \{(r, \theta) | 0 \leq r < 1, \theta \in [0, 2\pi]\} \) is the open unit disk. We denote the set of all such families by \( D_\rho \). It is easy to check that \( D_\rho \neq \emptyset \). Now we can define a minimax value
\[ \alpha_\rho := \inf_{h \in D_\rho} \sup_{u \in h(D)} J_\rho(u). \]

The following lemma will make crucial use of the non-trivial topology of \( \Omega \), more precisely of the fact that the complement of \( \Omega \) has a bounded component.

**Lemma 2.1.** \( \alpha_\rho > -\infty \) for any \( \rho \in (8\pi, 16\pi) \).

**Remark.** It is an interesting question whether \( \alpha_{16\pi} = -\infty \).

To prove Lemma 2.1, we use the improved Moser-Trudinger inequality of [CL] (see also [A]). Here we have to modify a little bit.

**Lemma 2.2.** Let \( S_1 \) and \( S_2 \) be two subsets of \( \Omega \) satisfying \( \text{dist}(S_1, S_2) \geq \delta_0 > 0 \) and \( \gamma_0 \in (0, 1/2) \). For any \( \epsilon > 0 \), there exists a constant \( c = c(\epsilon, \delta_0, \gamma_0) > 0 \) such that
\[ \int_\Omega e^u \leq c \exp\left\{ \frac{1}{32\pi - \epsilon} \int_\Omega |\nabla u|^2 + c \right\} \]
holds for all \( u \in H^{1,2}_0(\Omega) \) satisfying
\[ \int_{S_1} e^u \geq \gamma_0 \quad \text{and} \quad \frac{\int_{S_1} e^u}{\int_{S_2} e^u} \geq \gamma_0. \]

**Proof.** The Lemma follows from the argument in [CL] and the following Moser-Trudinger inequality
\[ \frac{1}{2} \int_\Omega |\nabla u|^2 - 8\pi \log \int_\Omega e^u \geq c \]
for any \( u \in H^{1,2}_0(\Omega) \), where \( c \) is a constant independent of \( u \in H^{1,2}_0(\Omega) \). \( \square \)

We will discuss the inequality (\*) and its application in another paper.

**Proof of Lemma 2.1.** For fixed \( \rho \in (8\pi, 16\pi) \) we claim that there exists a constant \( c_\rho \) such that
\[ \sup_{u \in h(D)} J_\rho(u) \leq c_\rho, \quad \text{for any } h \in D_\rho. \]

Clearly (2.6) implies the Lemma. By the definition of \( h \), for any \( h \in D_\rho \), there exists \( u \in h(D) \) such that
\[ m_c(u) = 0. \]

We choose \( \epsilon > 0 \) so small that \( \rho < 16\pi - 2\epsilon \). Assume (2.6) does not hold. Then we have sequences \( \{h_i\} \subset D_\rho \) and \( \{u_i\} \subset H^{1,2}_0(\Omega) \) such that \( u_i \in h_i(D) \) and
\[ m_c(u_i) = 0 \]
\[ \lim_{i \to +\infty} J(u_i) = -\infty. \]

We have the following Lemma.
Lemma 2.3. There exists $x_0 \in \tilde{\Omega}$ such that

\[(2.9) \quad \lim_{i \to \infty} \frac{\int_{B_{1/2}(x_0) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}} \to 1.\]

Proof. Set

\[A(x) := \lim_{i \to \infty} \frac{\int_{B_{1/4}(x) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}}.\]

Assume that the Lemma were false, then there exists $x_0 \in \tilde{\Omega}$ such that

\[A(x_0) < 1 \quad \text{and} \quad A(x_0) \geq A(x) \quad \text{for any } x \in \Omega.\]

It is easy to check $A(x_0) > 0$, since $\Omega$ can be covered by finite many balls of radius $1/4$. Let $\gamma_0 = A(x_0)/2$. Recalling (2.8) and applying lemma 2.2, we obtain

\[(2.10) \quad \frac{\int_{\Omega \setminus B_{1/2}(x_0)} e^{u_i}}{\int_{\Omega} e^{u_i}} \to 0\]

as $i \to \infty$, which implies (2.9). □

Now we continue to prove Lemma 2.1. (2.9) implies

\[
\frac{\int_{\Omega} xe^{u_i}}{\int_{\Omega} e^{u_i}} - x_0 = \frac{\int_{\Omega} (x - x_0)e^{u_i}}{\int_{\Omega} e^{u_i}}
= \frac{\int_{B_{1/2}(x_0)} (x - x_0)e^{u_i}}{\int_{\Omega} e^{u_i}} + o(1)
\]

which, in turn, implies that $m_x(u_i) - x_0 < 2/3$. This contradicts (2.7). □

Lemma 2.4. $\alpha_\rho/\rho$ is non-increasing in $(8\pi, 16\pi)$.

Proof. We first observe that if $J(u) \leq 0$, then $\log \int_{\Omega} e^u > 0$ which implies that

\[J_\rho(u) \geq J_{\rho'}(u) \quad \text{for } \rho' \geq \rho.
\]

Hence $D_{\rho} \subset D_{\rho'}$ for any $16\pi > \rho' \geq \rho > 8\pi$. On the other hand, it is clear that

\[
\frac{J_{\rho}}{\rho} - \frac{J_{\rho'}}{\rho'} = \frac{1}{2} \left( \frac{1}{\rho} - \frac{1}{\rho'} \right) \int_{\Omega} |\nabla u|^2 \geq 0.
\]

if $\rho' \geq \rho$. Hence we have

\[
\frac{\alpha_\rho}{\rho} \geq \frac{\alpha_{\rho'}}{\rho'}
\]

for $16\pi > \rho' \geq \rho > 8\pi$. □
3. Existence for a dense set

In this section we show that $\alpha_{\rho}$ is achieved if $\rho$ belongs to a certain dense subset of $(8\pi, 16\pi)$ defined below.

The crucial problem for our functional is the lack of a coercivity condition, i.e., for a Palais-Smale sequence $u_i$ for $J_\rho$, we do not know whether $\int_{\Omega} |\nabla u_i|^2$ is bounded.

We first have the following lemma.

**Lemma 3.1.** Let $u_i$ be a Palais-Smale sequence for $J_\rho$, i.e., $u_i$ satisfies
\begin{equation}
|J_\rho(u_i)| \leq c < \infty
\tag{3.1}
\end{equation}
and
\begin{equation}
dJ_\rho(u_i) \to 0 \text{ strongly in } H^{-1,2}(\Omega)
\tag{3.2}
\end{equation}
If, in addition, we have
\begin{equation}
\int_{\Omega} |\nabla u_i|^2 \leq c_0, \quad \text{for } i = 1, 2, \ldots
\tag{3.3}
\end{equation}
for a constant $c_0$ independent of $i$, then $u_i$ subconverges to a critical point $u_0$ for $J_\rho$ strongly in $H^{-1,2}_0(\Omega)$.

**Proof.** The proof is standard, but we provide it here for convenience of the reader.

Since $\int_{\Omega} |\nabla u_i|^2$ is bounded, there exists $u_0 \in H^{-1,2}_0(\Omega)$ such that
(i) $u_i$ converges to $u_0$ weakly in $H^{-1,2}_0(\Omega)$,
(ii) $u_i$ converges to $u_0$ strongly in $L^p(\Omega)$ for any $p > 1$ and almost everywhere,
(iii) $e^{u_i}$ converges to $e^{u_0}$ strongly in $L^p(\Omega)$ for any $p \geq 1$.

From (i)-(iii), we can show that $dJ(u_0) = 0$, i.e., $u_0$ satisfies
\[-\Delta u_0 = \rho e^{u_0} \text{ in } \Omega.
\]
Testing $dJ_\rho$ with $u_i - u_0$, we obtain
\[o(1) = (dJ_\rho(u_i) - dJ_\rho(u), u_i - u_0) = \int_{\Omega} |\nabla(u_i - u_0)|^2 - \rho \int_{\Omega} \frac{e^{u_i}}{\int_{\Omega} e^{u_i}} - \frac{e^{u_0}}{\int_{\Omega} e^{u_0}}(u_i - u_0) = \int_{\Omega} |\nabla(u_i - u_0)|^2 + o(1),
\]
by (i)-(iii). Hence $u_i$ converges to $u_0$ strongly in $H^{-1,2}_0(\Omega)$.

Since by Lemma 2.4 $\rho \to \alpha_{\rho}/\rho$ is non-increasing in $(8\pi, 16\pi)$, $\rho \to \alpha_{\rho}/\rho$ is a.e. differentiable. Set
\begin{equation}
\Lambda := \{\rho \in (8\pi, 16\pi) | \alpha_{\rho}/\rho \text{ is differentiable at } \rho \}
\tag{3.4}
\end{equation}
$\tilde{\Lambda} = [8\pi, 16\pi]$, see [St1]. Let $\rho \in \Lambda$ and choose $\rho_k \nearrow \rho$ such that
\begin{equation}
0 \leq \lim_{k \to \infty} \frac{1}{(\rho - \rho_k)}(\frac{\alpha_{\rho}}{\rho} - \frac{\alpha_{\rho_k}}{\rho_k}) \leq c_1
\tag{3.5}
\end{equation}
for some constant $c_1$ independent of $k$. 

Lemma 3.2. \( \alpha_\rho \) is achieved by a critical point \( u_\rho \) for \( J_\rho \) provided that \( \rho \in \Lambda \).

Proof. Assume, by contradiction, that the Lemma were false. From Lemma 3.1, there exists \( \delta > 0 \) such that

\[
\|dJ_\rho(u)\|_{H^{-1,2}(\Omega)} \geq 2\delta
\]

in

\[
N_\delta := \{ u \in H^{1,2}_0(\Omega) \mid \int_\Omega |\nabla u|^2 \leq c_2, |J_\rho(u) - \alpha_\rho| < \delta \}.
\]

Here, \( c_2 \) is any fixed constant such that \( N_\delta \neq \emptyset \). Let \( X_\rho : N_\delta \to H^{1,2}_0(\Omega) \) be a pseudo-gradient vector field for \( J_\rho \) in \( N_\delta \), i.e. a locally Lipschitz vector field of norm \( \|X_\rho\|_{H^{1,2}_0} \leq 1 \) with

\[
\langle dJ_\rho(u), X_\rho(u) \rangle < -\delta.
\]

See [P] for the construction of \( X_\rho \).

Since

\[
\|dJ_\rho(u) - dJ_{\rho_k}(u)\| = \|dJ_{\rho_k} - \frac{\rho_k}{\rho} dJ_{\rho_k}(u)\| + \|(1 - \frac{\rho_k}{\rho})dJ_{\rho_k}(u)\|
\]

\[
\leq \frac{1}{2}(1 - \frac{\rho_k}{\rho}) \int |\nabla u|^2 + c(1 - \frac{\rho_k}{\rho}) \int |\nabla u|^2 \to 0
\]

uniformly in \( \{ u \mid \int_\Omega |\nabla u|^2 \leq c_2 \} \), \( X_\rho \) is also a pseudo-gradient vector field for \( J_{\rho_k} \) in \( N_\delta \) with

\[
\langle dJ_{\rho_k}(u), X_\rho(u) \rangle < -\delta / 2,
\]

for \( u \in N_\delta \), provided that \( k \) is sufficiently large.

For any sequence \( \{h_k\} \), \( h_k \in D_{\rho_k} \subset D_\rho \) such that

\[
\sup_{u \in h_k(D)} J_{\rho_k}(u) \leq \alpha_{\rho_k} + \rho - \rho_k
\]

and all \( u \in h_k(D) \) such that

\[
J_\rho(u) \geq \alpha_\rho - (\rho - \rho_k)
\]

we have the following estimate

\[
\frac{1}{2} \int_\Omega |\nabla u|^2 = \rho \cdot \rho_k \frac{J_{\rho_k}(u) - J_\rho(u)}{\rho_k - \rho} \leq \rho \cdot \rho_k \frac{2(\rho - \rho_k)}{\rho - \rho_k} + (\rho + \rho_k)
\]

\[
\leq C
\]

by (3.5), (3.9) and (3.10), where \( C = (16\pi)^2c_1 + 32\pi \).
Now we consider in $N_\delta$ the following pseudo-gradient flow for $J_\rho$. First choose a Lipschitz continuous cut-off function $\eta$ such that $0 \leq \eta \leq 1$, $\eta = 0$ outside $N_\delta$, $\eta = 1$ in $N_{\delta/2}$. Then consider the following flow in $H^{1,2}_c(\Omega)$ generated by $\eta X_\rho$

$$\frac{\partial \phi}{\partial t}(u,t) = \eta(\phi(u,t))X_\rho(\phi(u,t))$$
$$\phi(u,0) = u.$$

By (3.7) and (3.8), for $u \in N_{\delta/2}$, we have

$$(3.12) \quad \frac{d}{dt} J_\rho(\phi(u,t))|_{t=0} \leq -\delta$$

and

$$(3.13) \quad \frac{d}{dt} J_{\rho_k}(\phi(u,t))|_{t=0} \leq -\delta/2$$

for large $k$.

It is clear that for any $h \in D_{\rho_k} h(r,\theta) \not\in N_\delta$ for $r$ close to 1. Hence $\phi(h,t) \in D_{\rho_k}$ for any $t > 0$. In particular, $\phi(\cdot,t)$ preserves the class of $h_k \in D_{\rho_k}$ with condition (3.9). On the other hand, for any $h \in D_\rho$ by definition

$$\sup_{u \in h(D)} J_\rho(u) \geq \alpha_\rho.$$ 

Hence for any $h_k \in D_{\rho_k}$ with condition (3.9), $\sup_{u \in h(D),t} J_\rho(u)$ is achieved in $N_{\delta/2}$, provided that $k$ is large enough. Consequently, by (3.12), we have

$$\frac{d}{dt} \sup\{J_\rho(u) | u \in \phi(h(D),t)\} \leq -\delta$$

for all $t \geq 0$, which is a contradiction. □

4. Proof of Theorem 1.1

From section 3, we know that for any $\bar{\rho} \in (8\pi,16\pi)$ there exists a sequence $\rho_k \not\to \bar{\rho}$ such that $\alpha_{\rho_k}$ is achieved by $u_k$. Consequently $u_k$ satisfies

$$(4.1) \quad -\Delta u_k = \rho_k \frac{e^{u_k}}{\int_{\Omega} e^{u_k}}, \quad \text{in } \Omega,$$
$$u_k = 0, \quad \text{on } \partial \Omega.$$ 

From Lemma 2.4, we have

$$(4.2) \quad J_\rho(u_k) = \alpha_{\rho_k} \text{ is bounded},$$

for some constant $\alpha_0 > 0$ which is independent of $k$. Let $v_k = u_k - \log \int_{\Omega} e^{u_k}$. Then $v_k$ satisfies

$$(4.3) \quad -\Delta v_k = \rho_k e^{v_k}$$

with

$$(4.4) \quad \int_{\Omega} e^{v_k} = 1.$$ 

By results of Brezis-Merle [BM] and Li-Shafir [LS] we have
**Lemma 4.1.** ([BM], [LS]) There exists a subsequence (also denoted by $v_k$) satisfying one of the following alternatives:

(i) \( \{v_k\} \) is bounded in \( L^\infty_{loc}(\Omega) \);
(ii) \( v_k \to -\infty \) uniformly on any compact subset of \( \Omega \);
(iii) there exists a finite blow-up set \( \Sigma = \{a_1, \cdots, a_m\} \subset \Omega \) such that, for any \( 1 \leq i \leq m \), there exists \( \{x_k\} \subset \Omega \), \( x_k \to a_i \), \( u_k(x_k) \to \infty \), and \( v_k(x) \to -\infty \) uniformly on any compact subset of \( \Omega \setminus \Sigma \). Moreover,

\[
\rho_k \int_{\Omega} e^{v_k} \to \sum_{i=1}^{m} 8\pi n_i
\]

where \( n_i \) is positive integer.

For our special functions \( v_k \), we can improve Lemma 4.1 as follows

**Lemma 4.2.** There exists a subsequence (also denoted by \( v_k \)) satisfying one of the following alternatives:

(i) \( \{v_k\} \) is bounded in \( L^\infty_{loc}(\Omega) \);
(ii) \( v_k \to -\infty \) uniformly on \( \Omega \);
(iii) there exists a finite blow-up set \( \Sigma = \{a_1, \cdots, a_m\} \subset \Omega \) such that, for any \( 1 \leq i \leq m \), there exists \( \{x_k\} \subset \Omega \), \( x_k \to a_i \), \( u_k(x_k) \to \infty \), and \( v_k(x) \to -\infty \) uniformly on any compact subset of \( \Omega \setminus \Sigma \). Moreover, (4.5) holds.

**Proof.** From Lemma 4.1, we only have to consider one more case in which blow-up points are in the boundary of \( \Omega \). There are two possibilities: One is bubbling too fast such that after rescaling we obtain a solution of \( \Delta u = e^u \) in a half plane; Another is bubbling slow such that after rescaling we obtain a solution of \( \Delta u = e^u \) in \( \mathbb{R}^2 \). One can exclude the first case. In the second case, one can follow the idea in [LS] to show that (4.5) holds. See also [L]. \( \square \)

**Proof of Theorem 1.1.** (4.4), (4.5) and \( \tilde{\rho} \in (8\pi, 16\pi) \) imply that cases (ii) and (iii) in Lemma 4.2 does not occur. Consequently \( \{v_k\} \) is bounded in \( L^\infty_{loc}(\Omega) \). Now we can again apply Lemma 2.2 as follows.

Let \( S_1 \) and \( S_2 \) be two disjoint compact subdomains of \( \Omega \). Since \( \{v_k\} \) is bounded in \( L^\infty_{loc}(\Omega) \), we have

\[
\frac{\int_{S_i} e^{v_k}}{\int_{\Omega} e^{v_k}} = \int_{S_i} e^{v_k} \geq c_0, \quad i = 1, 2
\]

for a constant \( c_0 = c_0(S_1, S_2, \Omega) > 0 \) independent of \( k \). Choosing \( \epsilon \) such that \( 16\pi - \tilde{\rho} > 2\epsilon \) and applying Lemma 2.2, with the help of (4.2), we obtain

\[
c \geq J_{\rho_k}(u_k) = \frac{1}{2} \int_{\Omega} |\nabla u_k|^2 - \rho_k \log \int_{\Omega} e^{u_k}
\geq \frac{1}{2} (1 - \tilde{\rho}_{16\pi - \epsilon/2}) \int_{\Omega} |\nabla u_k|^2
\geq \frac{1}{2} (1 - \tilde{\rho}_{16\pi - \epsilon/2}) \int_{\Omega} |\nabla u_k|^2
\]
which implies that $\int_{\Omega} |\nabla u_k|^2$ is bounded. Now by the same argument in the proof of Lemma 3.1, $u_k$ subconverges to $u_\beta$ strongly in $H^{1,2}_0(\Omega)$ and $u_\beta$ is a critical point of $J_\beta$. Clearly, $u_\beta$ achieves $\alpha_\beta$. This finishes the proof of Theorem 1.1.

\[Q.E.D.\]

Proof of Theorem 1.2. Since the proof is very similar to one presented above, we only give a sketch of the proof of Theorem 1.2. Let $\Sigma$ be a Riemann surface of positive genus. We embed $X : \Sigma \to \mathbb{R}^N$ for some $N \geq 3$ and define the center of mass for a function $u \in H^{1,2}(\Sigma)$ by

$$m_c(u) = \frac{\int_{\Sigma} X e^{u}}{\int_{\Sigma} e^{u}}.$$

Since $\Sigma$ is of positive genus, we can choose a Jordan curve $\Gamma^1$ on $\Sigma$ and a closed curve $\Gamma^2$ in $\mathbb{R}^N \setminus \Sigma$ such that $\Gamma^1$ links $\Gamma^2$. We know that $\inf_{u \in L^{1,2}(\Sigma)} J_c(u)$ is finite if and only if $c \in [0, 8\pi]$ (see [DJLW]). Now define a family of functions $h : D \to H^{1,2}(\Sigma)$ (as in section 2) satisfying

$$\lim_{r \to 1} J_{p}(h(r, \theta)) \to -\infty$$

and

$$\lim_{r \to 1} m_c(h(r, \theta))$$

as a map from $S^1 \to \Gamma^1$ is of degree 1.

Let $D_c$ denote the set of all such families. It is also easy to check that $D_c \neq \emptyset$. Set

$$\alpha_c := \inf_{h \in D_c} \sup_{u \in h(D)} J_c(u).$$

We first have

$$\alpha_c > -\infty,$$

using the fact that $\Gamma^1$ links $\Gamma^2$ and Lemma 2.2. Then by the same method as presented above, we can prove that $\alpha_c$ is achieved by some $u_c \in H^{1,2}(\Sigma)$, which is a solution of (1.4), for $c \in (8\pi, 16\pi)$. \[Q.E.D.\]

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