Skewness induced by gravity

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Abstract

Large-scale structures, observed today, are generally believed to have grown from random, small-amplitude inhomogeneities, present in the early Universe. We investigate how gravitational instability drives the distribution of these fluctuations away from the initial state, assumed to be Gaussian. Using second order perturbation theory, we calculate the skewness factor, $S_3 \equiv \langle \delta^3 \rangle / \langle \delta^2 \rangle^2$. Here the brackets, $\langle \ldots \rangle$, denote an ensemble average, and $\delta$ is the density contrast field, smoothed with a low pass spatial filter. We show that $S_3$ decreases with the slope of the fluctuation power spectrum; it depends only weakly on $\Omega$, the cosmological density parameter. We compare perturbative calculations with N-body experiments and find excellent agreement over a wide dynamic range. If galaxies trace the mass, measurements of $S_3$ can be used to distinguish models with Gaussian initial conditions from their non-Gaussian alternatives.

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1. Introduction

Long before first estimates of the skewness in counts of galaxies became available, Peebles (1980) showed how gravity can generate skewness in a random, initially Gaussian-distributed density field in an Einstein-de Sitter cosmological model. We have recently extended this calculation to $\Omega \neq 1$ (Bouchet et al. 1992, hereafter Paper I). In this Letter, we develop the formalism further in order to bridge the gap between theoretical concepts and observable quantities. We distinguish the mass density contrast field, $\delta \rho / \rho$, from the spatially smoothed field, $\delta$. While the former is not directly observable, the latter may be. Indeed, if the galaxies trace the mass, the moments of the frequency distribution in the counts of galaxies provide weighted averages of $\xi_m$ — the $m$-point density correlation functions$^*$,

\[
\langle \delta^m \rangle = \int \frac{dv_1 \ldots dv_m}{v^m} \xi_m(x_1, \ldots, x_m),
\]

rather than the proper moments, $\langle (\delta \rho / \rho)^m \rangle = \xi_m(0)$. Here $x_i$ are comoving spatial coordinates, $dv_i \equiv F(x_i) d^3 x_i$, while $F$ is a filter that determines the shape and volume, $v \equiv \int d^3 x F(x)$, of a resolution element. In order to take the smoothing process into account, we calculate moments of a weighted average of $\delta \rho(x) / \rho$,

\[
\delta(x) \equiv \int \frac{\delta \rho}{\rho}(x') F(x - x') d^3 x'.
\]

The filters considered here are spherically symmetric, sweep a unit volume and have a finite effective comoving half-width, $R$:

\[
\int F(x) d^3 x = 1, \quad \text{and} \quad \int F(x) x^2 d^3 x = R^2.
\]

We derive analytic expressions for the skewness of the smoothed field $\delta$ in models with various spectra of primeval fluctuations. We consider two kinds of filters, used by the observers and N-body simulators – the Gaussian and the “top hat”, or a sphere with a fixed comoving radius. We show that the resulting skewness is sensitive to the slope of the spectrum of primeval fluctuations, as well as to the properties of the filter, contrary to incorrect claims made in the literature recently. We then test the limits of our approach and compare perturbative results with N-body experiments. We end by listing some still unresolved problems and summarizing our results.

$^*$ We neglect the shot noise terms. All our calculations are made in the fluid limit.
2. Perturbation theory

Our calculations are based on the standard perturbative expansion for $\delta \rho / \rho$ in a Friedman universe, filled with non-relativistic pressureless fluid and zero cosmological constant (cf. §18 in Peebles 1980). We assume that to first order in perturbation theory, $\delta \rho / \rho$ is a random Gaussian field. To first order, all its statistical properties are therefore determined by the power spectrum, $P(k) \equiv \int \xi_2(x) \exp(ik \cdot x) \frac{d^3k}{(2\pi)^3}$. Perturbative calculations also assume that the amplitude of the density fluctuations, measured by their variance, is small: $\langle \delta^2 \rangle \ll 1$. The linear order term in the perturbative expansion for $\langle \delta^2 \rangle$ is

$$\sigma^2 = \int \frac{d^3k}{(2\pi)^3} P(k) W_k^2,$$  
(4)

where $W_k = \int F(x) \exp(ik \cdot x) d^3x$ is the smoothing window function. We consider three windows: $W_k \equiv 1$ (no filtering at all), $W_k = (3/kR) j_1(kR)$, a top hat, and $W_k = \exp(-k^2R^2/2)$, a Gaussian. Here and below $j_\ell$ denotes a spherical Bessel function of order $\ell$. Deviations from Gaussian behaviour, induced by gravity, appear at the second and higher orders and are fully determined by $P(k)$. We define the skewness factor as the ratio of skewness to variance squared, $S_3 = \frac{\langle \delta^3 \rangle}{\langle \delta^2 \rangle^{3/2}}$. To lowest non-vanishing order, $S_3$ is given by (Juszkiewicz & Bouchet 1991)

$$S_3 = \int \frac{d^3k d^3k'}{(2\pi)^6} P(k) P(k') W_k W_{k'} W_{|k-k'|} T(k,k') + O(\sigma^2),$$  
(5)

and the function $T(k,k')$ is given by

$$T(k,k') = 4 + 4\kappa(\Omega) - 6\mu(k/k') + \left[2 - 4\kappa(\Omega)\right] P_2(\mu),$$  
(6)

where $\mu = k \cdot k' / kk'$, and $P_2$ is a Legendre polynomial, while $\kappa$ is a slowly varying function of the current value of $\Omega$. For densities in the range $0.05 \leq \Omega \leq 3$,

$$\kappa(\Omega) \approx (3/14) \Omega^{-2/63},$$  
(7)

(Paper I). The expression for $\langle \delta^3 \rangle$ in $k$ space, with $\Omega = 1$, but without smoothing, was first derived by Fry (1984), in a truly seminal paper. Goroff et al. (1986), whose analysis was also restricted to the $\Omega = 1$ case, were first to include the filters $W_k$. In the absence of filtering ($W_k \equiv 1$), the dipole and quadrupole terms in equation (6) integrate to zero, and we obtain

$$S_3 = 4 + 4\kappa(\Omega) \approx 34/7 + \frac{34}{7}(\Omega^{-2/63} - 1),$$  
(8)

where the approximate form applies for $0.05 \leq \Omega \leq 3$. The first term on the right hand side, $34/7$, reproduces the the Peebles (1980) result for $\Omega = 1$. The second term, found by Bouchet et al. (1992) is the “curvature correction”, which arises when $\Omega \neq 1$. This term is always small, and $S_3$ is essentially insensitive to $\Omega$ (this was pointed out
independently by Martel & Freudling 1991). To second order, the growth rates of the variance and skewness are such that for any comoving smoothing scale, the ratio \( \langle \delta^3 \rangle / \langle \delta^2 \rangle^2 \) remains constant, if \( \Omega = 1 \). In an open universe, \( S_3 \) grows extremely slowly as \( \Omega \) decreases with time. In the \( \Omega \to 0 \) limit, \( \kappa \to 1/4 \) (Paper I), and \( S_3 \to 5 \), a tiny increase compared to \( 34/7 \approx 4.9 \).

Equation (8) can be regarded as an approximation, valid in the regime when the density gradients across the filtering scale \( R \) are small. Indeed, let us consider a density field with a large coherence length, \( R_c \equiv \sigma/\sigma_1 \), where

\[
\sigma_1^2 = \int k^2 P(k) W_k^2 d^3k / (2\pi)^3
\]

is the variance of the density gradient. In the limit \( R \ll R_c \), we can Taylor expand \( W_k W_{k'} W_{|k-k'|} \) about the origin, \( k = k' = 0 \), and then evaluate the integral (5) termwise. Using the normalization conditions (3), and integrating over \( \mu \), we obtain

\[
S_3 = 4 + 4\kappa(\Omega) - 2(R/R_c)^2 + O(R/R_c)^4 .
\]

(9)

The first two terms above describe the “local field” contribution, as in eq.(8). The third term is the tidal correction. Tidal effects, associated with the density gradients, tend to lower \( S_3 \). This decrease is stronger for fields with smaller coherence length. In the next section we will see that this effect is also present for pure power-law spectra: \( S_3 \) is anticorrelated with the relative amount of small-scale power.

3. Power-law clustering models

We will now study power law spectra, \( P \propto k^n \). We set the smoothing length to unity and use dimensionless wavenumbers. Two kinds of filters are considered. We begin with the top hat. To separate the variables \( k \) and \( k' \) in the multiple integral (5), we use the addition theorem for Bessel functions *,

\[
\hat{j}_1(\omega) = \frac{\omega}{kk'} \sum_{\ell=1}^{\infty} (2\ell + 1) j_\ell(k) j_\ell(k') \frac{dP_\ell}{d\mu} ,
\]

(10)

where \( \omega = |k' - k| \), and \( P_\ell(\mu) \) are Legendre polynomials of order \( \ell \). After substituting the above expansion to equation (5) and integrating over \( \mu \), the skewness factor can be expressed as

\[
S_3 = 4 + 4\kappa(\Omega) + \sum_{\ell=1}^{\infty} (6 + 4\ell) \frac{\beta(n, \ell)}{B^2(n, 0)} ,
\]

(11)

* The formula for \( \hat{j}_1 \) can be derived by differentiating a similar expression for \( j_0 \), given in Abramowitz & Stegun (1964, eq. [10.1.45]); for the full derivation, see Watson (1944).
where $\beta(n, \ell) = B^2(n, \ell)$ if $\ell$ is even, and $\beta = -B(n+1, \ell) B(n-1, \ell)$ otherwise. The factors $B(n, \ell)$ are integrals of spherical Bessel functions,

$$B(-m, \ell) \equiv \int_0^\infty \frac{j_1(z) j_{\ell+1}(z)}{z^m} \, dz = \frac{\pi m! \left( \frac{\ell-m+1}{2} \right)!}{2^{m+1} \left( \frac{\ell+m}{2} \right)! \left( \frac{\ell+m+3}{2} \right)! \left( \frac{m-\ell}{2} \right)!}.$$  

(12)

For $-3 \leq n < 1$, the series converges. Summing it requires some knowledge of the properties of the gamma function and a high threshold against boredom. The reward is the simple final result,

$$S_3 = 4 + 4\kappa(\Omega) - (3 + n).$$  

(13)

This expression is extremely weakly sensitive to $\Omega$: the first two terms above are, in fact, identical to those on the right-hand side of equation (8). The skewness parameter is, however, strongly sensitive to the spectral slope, $n$.

### TABLE 1: Scale-free initial conditions

| $n$ | $F(x)^a$ | $S_3$ (perturbation theory) | $S_3$ (N-body) | $\sigma$ |
|-----|----------|----------------------------|----------------|---------|
| $-3$ | G        | $34/7$                      | $4.9$          | $-\quad-$|
| $-2$ | G        | $(4\pi + 9\sqrt{3})/7$    | $4.0$          | $3.8 \pm 0.2^e$ | $0.7$ |
| $-1$ | G        | $(20\pi - 12\sqrt{3})/7\sqrt{3}$ | $3.5$ | $3.6 \pm 0.2^e$ | $0.7$ |
| $0$  | G        | $8(4\pi\sqrt{3} - 17)/7\sqrt{3}$ | $3.1$ | $3.0 \pm 0.2^e$ | $0.7$ |
| $1$  | G        | $8(11\pi - 12\sqrt{3})/21\sqrt{3}$ | $3.0$ | $-\quad-$ | $-\quad-$ |
| $-1$ | T        | $(34/7) - 2$               | $2.9$          | $2.9 \pm 0.1^b$ | $0.6$ |
| $0$  | T        | $(34/7) - 3$               | $1.9$          | $1.8 \pm 0.3^c$ | $0.7$ |
| $1$  | T        | $1.9^d$                    | $1.9 \pm 0.1^b$ | $0.7$ |

$a$ G = Gaussian filter; T = top hat.

$b$ White (1992).

$c$ Bouchet & Hernquist (1992).

$d$ Corrected for finite grid effects.

$e$ Weinberg & Cole (1992).

We now consider the Gaussian filter, $W_k = \exp(-k^2/2)$. The variance and skewness are finite for $n > -3$. To evaluate the skewness integral (5) in this case, we used a coordinate transformation which reduces $(k - k')^2$ to a sum of squares. Similar multiple integrals are considered by Rice (1954). Our results for $\Omega = 1$ and several integer values of the power index $n$ are listed in Table 1. In Figure 1 we plot $S_3(n)$ for $\Omega = 1$ and the entire range $-3 \leq n \leq 1$, obtained by computing the integral (5) numerically. The case $n = 1$ was considered earlier by Grinstein & Wise (1984). Again, when $\Omega \neq 1$, there is a small additional “curvature term” in the expression for $S_3$. We write this additional term as

$$\Delta S_3(n, \Omega) \equiv \frac{6}{7} (\Omega^{-2/63} - 1) p_n,$$  

(14)
so that the Gaussian-smoothed skewness factor is \( S_3(n, \Omega) = S_3(n, 1) + \Delta S_3 \). For \( n = -3, -2, -1, \) and 0 we obtained \( p_n = 1, 1.04, 1.13, \) and 1.29, respectively. For \( n = -3, \) both \( \langle \delta^3 \rangle \) and \( \langle \delta^2 \rangle^2 \) diverge. However, their ratio is finite and equal to \( 4 + 4\kappa \).

Note that in all cases, discussed in this section, the dependence of \( S_3 \) on the relative amount of small-scale power is qualitatively similar to that in equation (9): \( S_3 \) is a decreasing function of \( n \).

**Figure 1.** The skewness factor for power law spectra and Gaussian smoothing. Rigorous perturbation theory agrees well with the N-body results, while the Zel’dovich approximation systematically underestimates \( S_3 \).

### 4. Comparison with N-body experiments

In Table 1 we summarize the results of several N-body experiments, used to study the evolution of skewness in the weakly nonlinear regime for scale-invariant initial conditions. The first column gives the power index, \( n \). The second column specifies the spatial filter used in the simulation. The third column contains our predictions for \( S_3 \), based on the perturbation theory. The fourth column gives \( \sigma \), measured at the same time and on the same smoothing scale in the simulation, as \( S_3 \). The examples \( n = +1 \) and \( -1 \) at the bottom of Table 1 were provided to us by Simon White, who calculated \( S_3 \), using the output from N-body simulations of Efstathiou *et al.* (1988). Similar results for \( S_3 \) were recently obtained by Bouchet & Hernquist (1992) and Lahav *et al.* (1993) in their simulations of later stages of clustering (\( \sigma \approx 1 \)). For a Gaussian filter, we used
the recent numerical results of Weinberg & Cole (1992; we quote sampling errors as estimated by David Weinberg, private communication).

We have also considered the cold dark matter model (regarded here as no more than an example of a spectrum with a scale-dependent slope). The skewness factor was calculated by numerical integration of eq. (5) and compared to an N-body simulation. For the simulation, we used the particle-in-cell code, described in Moutarde et al. (1991). This simulation involves \(64^3\) particles on a \(128^3\) grid. We assume the ‘standard’ spectrum with a Hubble constant of 50 km s\(^{-1}\) Mpc\(^{-1}\), \(\Omega = 1\), and no biasing. In Table 2 we list results for several smoothing widths \(R\), and for two kinds of filters. All measurements were made at the time, when the r.m.s. fluctuation in the density field, smoothed with an \(R = 16\) Mpc top hat, reached 0.92.

**Table 2: \(S_3\) for a variable slope spectrum (CDM)**

| \(R^a\) | \(F(x)^b\) | \(S_3\) (perturbative) | \(S_3\) (N-body) | \(\sigma(R)\) |
|---|---|---|---|---|
| 5.76 | G | 3.70 ± 0.07\(^c\) | 3.95 ± 0.15\(^d\) | 1.30 |
| 8.12 | G | 3.60 ± 0.07 | 3.70 ± 0.20 | 0.92 |
| 11.52 | G | 3.50 ± 0.07 | 3.50 ± 0.15 | 0.64 |
| 16.28 | G | 3.40 ± 0.07 | 3.30 ± 0.10 | 0.44 |
| 23.00 | G | 3.30 ± 0.07 | 3.20 ± 0.10 | 0.29 |
| 32.60 | G | 3.25 ± 0.10 | 3.05 ± 0.10 | 0.18 |
| 11.52 | T | 3.15 ± 0.10 | 3.80 ± 0.38 | 1.30 |
| 23.00 | T | 2.75 ± 0.10 | 3.05 ± 0.30 | 0.64 |
| 46.00 | T | 2.25 ± 0.25 | 2.45 ± 0.25 | 0.29 |
| 65.20 | T | 2.00 ± 0.50 | 2.20 ± 0.20 | 0.18 |

\(^a\) Smoothing scale in Mpc.
\(^b\) G = Gaussian filter; T = top hat.
\(^c\) Monte-Carlo integration errors.
\(^d\) Sampling errors.

We were able to find only two cases of seemingly serious disagreement between perturbative predictions and numerical experiments.

The first case is the \(n = 1\) and a top hat filter, when the perturbative series (13) diverges: \(S_3 = \infty\). Meanwhile, N-body experiments (Lahav et al. 1993, White 1992) give \(S_3 \approx 2\). This ‘discrepancy’ is easy to understand. The divergence is caused by the \(P(k) \propto k\) behaviour at large wavenumbers. Such a spectrum is not reproduced in simulations for wavenumbers larger than the Nyquist frequency, \(k_N\), defined by the particle grid. To model this effect, we replaced the scale-free \(n = 1\) spectrum, with

\[
P(k) = \begin{cases} 
A k, & \text{if } k \leq k_N; \\
A k_N, & \text{otherwise},
\end{cases}
\]

\begin{equation}
(15)
\end{equation}
where $A$ is a constant, and $k_N$ matches the grid used by Efstathiou et al. (1988). After this modification, the skewness integral (5) converged to $S_3 = 1.9$, in perfect agreement with the N-body results (Table 1).

The second case of seeming discrepancy is the $S_3$ obtained by Park (1991) for $n = -1$ and a Gaussian filter. The source of problem in this case is the Zel’dovich (1970) approximation (hereafter ZA), used by Park instead of an N-body code to calculate particle trajectories. ZA does not conserve momentum at second order in perturbation theory. This leads to an incorrect form for $T(k,k')$, with $\kappa$ in equation (6) set to zero (Grinstein & Wise 1986; Paper I). Using this incorrect expression, we were able to reproduce Park’s result (Figure 1). Clearly, ZA systematically underestimates $S_3$ and disagrees rather badly with the rigorous perturbation theory and with the N-body results.

5. Discussion

Our results, summarized in equation (14) and Table 1, show that the skewness parameter is a decreasing function of the slope of the power spectrum; it is also sensitive to the shape of the window function used.

Gravitationally induced skewness was also investigated by Coles & Frenk (1991, hereafter CF), who claim that $S_3 \approx 3$ is a universal constant. The perturbative as well as N-body results, discussed here do not support this claim. The cause of our differences with CF is their failure to recognize that the perturbative results they use are not universal but instead are spectrum- and filter-specific. For example, their equation (11), quoted from Grinstein & Wise (1987, hereafter GW), is valid only for $n = 1$ and a Gaussian filter. More importantly, the GW formula is inapplicable to the N-body simulations, conducted by CF, because the spectra and filters assumed do not match each other; for the same reason the GW formula cannot be meaningfully compared with the $S_3$, estimated from QDOT counts, considered by CF. Despite our quantitative differences, we do agree with CF on the qualitative level, that a scale-independent $S_3$ can be a signature of Gaussian initial conditions and a simple power-law spectrum (see the last paragraph in this section).

We compared our calculations with results of N-body experiments to see if the perturbative series, truncated at second order, can lead to sensible results over a broad enough dynamic range. To our own surprise, the agreement between the two sets of results remained excellent even when the “small parameter” in the expansion, $\sigma$, was close to unity. Apart from incorrectly conducted or misinterpreted simulations, all discrepancies appear to be within the sampling errors of numerical experiments. We also note that the agreement with N-body results was systematically better for the Gaussian filter than for the top hat, most likely because of high frequency sidelobes, which made the top hat-smoothed $S_3$ sensitive to strongly non-linear fluctuations at small scales.
Remarkably, the qualitative properties of the weakly nonlinear clustering, described above, appear to hold in the nonlinear regime as well. When $\sigma \gg 1$, at least for $\Omega = 1$ and scale-free initial conditions, the relation between $\langle \delta^3 \rangle$ and $\langle \delta^2 \rangle$ is well described by the semi-empirical formula

$$\xi_3(x_1,x_2,x_3) = Q [\xi_2(x_1,x_2) \xi_2(x_3,x_2) + \text{sym.}], \tag{16}$$

where $Q = Q(n)$ is a parameter, dependent on the initial spectral index only (Peebles 1980; Efstathiou et al. 1988). Adopting the Ansatz (16) is equivalent to setting $T(k,k') = 3Q$ in the skewness integral (5). For a top hat filter, methods described in §3 yield

$$S_3 \approx 3Q + \frac{\nu^2 (3 + \nu)^2 Q}{(5 - \nu)^2 (2 - \nu)^4 (3 - \nu)^4}, \tag{17}$$

where $\nu$ is the spectral index in the strongly-nonlinear regime, related to the initial slope by the so-called scaling solution, $\nu = -6/(n + 5)$ (Peebles 1980; if $n \geq -3$, then $0 \geq \nu \geq -3$). Substituting $Q(n)$, measured in N-body experiments (Efstathiou et al. 1988), we get $S_3 = 4.5, 2.9, and 2.3$ for $n = -1, 0, and +1$, respectively. Similar results were recently obtained in N-body experiments of Weinberg & Cole (1992), who directly measured $S_3(n)$ in the nonlinear regime, and by Fry et al. (1993), who found that $Q(n)$ decreases with $n$. To summarise: the scaling $\langle \delta^3 \rangle \propto \sigma^4$ seems to hold equally well both for $\sigma \ll 1$ and $\sigma \gg 1$; moreover, in both regimes $S_3$ is a decreasing function of $n$.

Preliminary results from observations appear to be consistent with a scale independent $S_3$ (Saunders et al. 1991; Coles & Frenk 1991; Park 1991; Bouchet et al. 1991, 1993; Gaztaña 1992; Lahav et al. 1993) This is exactly what is expected in the standard gravitational instability picture with Gaussian initial conditions and a simple power-law spectrum. It is even possible that strongly non-Gaussian models, which give $S_3$, diverging like $\sigma^{-1}$ instead of being constant, can already be excluded (Silk & Juszkiewicz 1991). However, before reaching dramatic conclusions, more work is needed. For example, the absence of rich galaxy clusters in the IRAS survey is likely to cause a systematic underestimate of $S_3$. We need to understand how $S_3$ in the matter distribution relates to the skewness in galaxy counts, as matter and galaxies may be distributed differently. Finally, it is necessary to account for the effect of redshift space distortion on $S_3$. Of all of the unresolved problems listed above, the latter admits the most straightforward solution, and we plan to report on this in near future.

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