Remarks on Leibniz algebras *

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Abstract

In this paper, first we construct a Lie 2-algebra associated to every Leibniz algebra via the skew-symmetrization. Furthermore, we introduce the notion of the naive representation for a Leibniz algebra in order to realize the abstract operations as a concrete linear operation. At last, we study some properties of naive cohomologies.

1 The skew-symmetrization of a Leibniz algebra

In this section, we construct a Lie 2-algebra from a Leibniz algebra via the skew-symmetrization. The notion of Leibniz algebras was introduced by Loday in [12], which is a vector space $g$, endowed with a linear map $[\cdot, \cdot]_g : g \otimes g \to g$ satisfying

$$[x, [y, z]_g]_g = [[x, y]_g, z]_g + [y, [x, z]_g]_g, \quad \forall \; x, y, z \in g.$$ (1)

The left center is given by

$$Z(g) = \{ x \in g \mid [x, y]_g = 0, \forall y \in g \}. \quad (2)$$

It is obvious that $Z(g)$ is an ideal and the quotient Leibniz algebra $g/Z(g)$ is actually a Lie algebra since $[x, x]_g \in Z(g)$, for all $x \in g$.

A Lie 2-algebra is a categorification of a Lie algebra, which is equivalent to a 2-term $L\infty$-algebra (see [1], [13] for more details).

Definition 1.1. A Lie 2-algebra is a graded vector space $g = g_1 \oplus g_0$, together with linear maps $\{l_k : \wedge^k g \to g, k = 1, 2, 3\}$ of degrees $\deg(l_k) = k - 2$ satisfying the following equalities:

(a) $l_1 l_2(x, a) = l_2(x, l_1(a))$,
(b) $l_2(l_1(a), b) = l_2(a, l_1(b))$,

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(c) $l_2(x, l_2(y, z)) + c.p. = l_1 l_3(x, y, z),$

(d) $l_2(x, l_2(y, a)) + l_2(y, l_2(a, x)) + l_2(a, l_2(x, y)) = l_3(x, y, l_1(a)),$

(e) $l_2(l_2(x, y), z, w) + c.p. = l_2(l_3(x, y, z), w) + c.p.,$

for all $x, y, z, w \in \mathfrak{g}_0,$ $a, b \in \mathfrak{g}_1,$ where $c.p.$ means cyclic permutations.

Given a Leibniz algebra $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$, introduce the following skew-symmetric bracket on $\mathfrak{g}$:

$$[x, y] = \frac{1}{2} ([x, y]_\mathfrak{g} - [y, x]_\mathfrak{g}), \quad \forall x, y \in \mathfrak{g},$$

and denote by $J_{x,y,z}$ the corresponding Jacobiator, i.e.

$$J_{x,y,z} = [x, [y, z]] + [y, [z, x]] + [z, [x, y]].$$

**Proposition 1.2.** Let $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$ be a Leibniz algebra.

(i) For all $x, y, z \in \mathfrak{g},$ we have

$$J_{x,y,z} = \frac{1}{4} ([z, [y, x]]_\mathfrak{g} + [x, [z, y]]_\mathfrak{g} + [y, [x, z]]_\mathfrak{g}).$$

(ii) $J_{x,y,z} \in Z(\mathfrak{g}),$ i.e. $J_{x,y,z}$ is in the left center of $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}).$

(iii) For all $x, y, z, w \in \mathfrak{g},$ we have

$$[x, [y, z, w]] - [y, [x, z, w]] + [z, [x, y, w]] - [w, [x, y, z]] - J_{[x,y,z],w} + J_{[z,x,w],y} - J_{[y,z,w],x} - J_{[y,w,x],z} - J_{[z,w,x],y} = 0.$$  

**Proof.** The first conclusion is obtained by straightforward computations. For any $w \in \mathfrak{g},$ by (4) and the fact that for all $x \in \mathfrak{g}, [x, x]_\mathfrak{g} \in Z(\mathfrak{g}),$ we have

$$[J_{x,y,z}, w]_\mathfrak{g} = \frac{1}{4} ([z, [y, x]]_\mathfrak{g} + [x, [z, y]]_\mathfrak{g} + [y, [x, z]]_\mathfrak{g} + [w, [x, y, z]]_\mathfrak{g})$$

$$= \frac{1}{4} ([z, [y, x]]_\mathfrak{g} - [y, [z, x]]_\mathfrak{g} - [z, [x, y]]_\mathfrak{g} + [w, [x, y, z]]_\mathfrak{g})$$

$$= 0,$$

which implies that $J_{x,y,z} \in Z(\mathfrak{g}).$ At last, since the bracket $[\cdot, \cdot]$ given by (3) is skew-symmetric, we have

$$[x, [y, z, w]] - [y, [x, z, w]] + [z, [x, y, w]] - [w, [x, y, z]] - J_{[x,y,z],w} + J_{[z,x,w],y} - J_{[y,z,w],x} - J_{[y,w,x],z} - J_{[z,w,x],y}$$

$$= [x, [y, z, w]] - [y, [x, z, w]] + [z, [x, y, w]] - [w, [x, y, z]].$$

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The proof is finished. ■

Next, for a Leibniz algebra \((\mathfrak{g}, [\cdot, \cdot])\), we consider the graded vector space \(G = Z(\mathfrak{g}) \oplus \mathfrak{g}\), where \(Z(\mathfrak{g})\) is of degree 1, \(\mathfrak{g}\) is of degree 0. Define a degree \(-1\) differential \(l_3 = i : Z(\mathfrak{g}) \to \mathfrak{g}\), the inclusion. Define a degree 0 skew-symmetric bilinear map \(l_2\) and a degree 1 totally skew-symmetric trilinear map \(l_3\) on \(G\) by

\[
\begin{align*}
    l_2(x, y) &= \llbracket x, y \rrbracket = \frac{1}{2}(\llbracket x, y \rrbracket - [y, x]_{\mathfrak{g}}) \quad \forall x, y \in \mathfrak{g}, \\
    l_2(x, c) &= -l_2(c, x) = \llbracket x, c \rrbracket = \frac{1}{2}(\llbracket x, c \rrbracket - [c, x]_{\mathfrak{g}}) \quad \forall x \in \mathfrak{g}, \ c \in Z(\mathfrak{g}), \\
    l_2(c_1, c_2) &= 0 \quad \forall c_1, c_2 \in Z(\mathfrak{g}), \\
    l_3(x, y, z) &= J_{x,y,z}
\end{align*}
\]  

(7)

The following theorem is our main result in this section, which says that one can obtain a Lie 2-algebra via the skew-symmetrization of a Leibniz algebra.

**Theorem 1.3.** With the above notations, \((G, l_1, l_2, l_3)\) is a Lie 2-algebra.

**Proof.** By the definition of \(l_1\), \(l_2\) and \(l_3\), it is obvious that Conditions (a)-(d) in Definition 1.1 hold. By (iii) in Proposition 1.2, Condition (e) also holds. Thus, \((G, l_1, l_2, l_3)\) is a Lie 2-algebra. ■

## 2 Representations of Leibniz algebras

The theory of representations and cohomologies of Leibniz algebras was introduced and studied in [13]. Especially, faithful representations and conformal representations of Leibniz algebras were studied in [3] and [9] respectively. See [4] [7] for more applications of cohomologies of Leibniz algebras.

**Definition 2.1.** A representation of the Leibniz algebra \((\mathfrak{g}, [\cdot, \cdot])\) is a triple \((V, l, r)\), where \(V\) is a vector space equipped with two linear maps \(l : \mathfrak{g} \to \mathfrak{gl}(V)\) and \(r : \mathfrak{g} \to \mathfrak{gl}(V)\) such that the following equalities hold:

\[
\begin{align*}
    l_{[x, y]_{\mathfrak{g}}} &= [l_x, l_y], \quad r_{[x, y]_{\mathfrak{g}}} = [r_x, r_y], \quad r_y \circ l_x = -r_y \circ r_x, \quad \forall x, y \in \mathfrak{g}.
\end{align*}
\]  

(8)
The resulting cohomology is denoted by 

The resulting cohomology is denoted by $H^\bullet(\mathfrak{g}; l, r).

Obviously, $(\mathbb{R}, 0, 0)$ is a representation of $\mathfrak{g}$, which is called the trivial representation. Denote the resulting cohomology by $H^\bullet(\mathfrak{g})$. Another important representation is the adjoint representation $(\mathfrak{g}, \text{ad}_L, \text{ad}_R)$, where $\text{ad}_L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and $\text{ad}_R : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ are defined as follows:

The resulting cohomology is denoted by $H^\bullet(\mathfrak{g}; \text{ad}_L, \text{ad}_R)$.

The graded vector space $\bigoplus_k C_k(\mathfrak{g}, \mathfrak{g})$ equipped with the graded bracket

is a graded Lie algebra, where $\alpha \circ \beta \in C^{p+q+1}(\mathfrak{g}, \mathfrak{g})$ is defined by

See [2, 3] for more details. In particular, for $\alpha \in C^2(\mathfrak{g}, \mathfrak{g})$, we have

Thus, $\alpha$ defines a Leibniz algebra structure if and only if $[\alpha, \alpha] = 0$.

For a representation $(V, l, r)$ of $\mathfrak{g}$, it is obvious that $(V, l, 0)$ is a representation of $\mathfrak{g}$ too. Thus, we have two semidirect product Leibniz algebras $\mathfrak{g} \ltimes_{(l, r)} V$ and $\mathfrak{g} \ltimes_{(l, 0)} V$ with the brackets $[\cdot, \cdot]_{(l, r)}$ and $[\cdot, \cdot]_{(l, 0)}$ respectively:

The right action $r$ induces a linear map $\tau : (\mathfrak{g} \oplus V) \to (\mathfrak{g} \oplus V)$ as follows:

The adjoint actions maps $\text{ad}_L$ and $\text{ad}_R$ on the Leibniz algebra $\mathfrak{g} \ltimes_{(l, 0)} V$ are given by

$$\text{ad}_L(x + u)(y + v) = [x, y]_\mathfrak{g} + l_xv,$$

$$\text{ad}_R(x + u)(y + v) = [y, x]_\mathfrak{g} + l_yu.$$
Proposition 2.3. With the above notations, $\mathbf{\tau}$ satisfies the following Maurer-Cartan equation on the Leibniz algebra $g \ltimes (l,0) V$:
\[
\partial \mathbf{\tau} - \frac{1}{2}[\mathbf{\tau}, \mathbf{\tau}] = 0,
\]
where $\partial$ is the coboundary operator for the adjoint representation of $g \ltimes (l,0) V$. Consequently, the Leibniz algebra $g \ltimes (l,0) V$ is a deformation of the Leibniz algebra $g \ltimes (l,0) V$ via the Maurer-Cartan element $\mathbf{\tau}$.

Proof. By direct computation, we have
\[
\partial \mathbf{\tau}(x + u, y + v, z + w) = \text{ad}_L(x + u)\mathbf{\tau}(y + v, z + w) - \text{ad}_L(y + v)\mathbf{\tau}(x + u, z + w) \nonumber \\
- \text{ad}_R(z + w)\mathbf{\tau}(x + u, y + v) - \mathbf{\tau}(x + u, [y + v, z + w]_{(l,0)}) \nonumber \\
+ \mathbf{\tau}(x + u, [y + v, z + w]_{(l,0)}) \nonumber \\
= l_x r_z v - l_y r_z u - r_z l_x v + r_{[y,z]_g} u - r_{[x,z]_g} v.
\]

On the other hand, we have
\[
[[\mathbf{\tau}, \mathbf{\tau}](x + u, y + v, z + w) = 2(\mathbf{\tau}(\mathbf{\tau}(x + u, y + v), z + w) - \mathbf{\tau}(x + u, \mathbf{\tau}(y + v, z + w)) \nonumber \\
+ \mathbf{\tau}(y + v, \mathbf{\tau}(x + u, z + w))) \nonumber \\
= 2 r_z r_y u.
\]

Thus, we have
\[
\left(\partial \mathbf{\tau} - \frac{1}{2}[\mathbf{\tau}, \mathbf{\tau}]\right)(x + u, y + v, z + w) = l_x r_z v - l_y r_z u - r_z l_x v + r_{[y,z]_g} u - r_{[x,z]_g} v - r_z r_y u \nonumber \\
= l_x r_z v - l_y r_z u - r_z l_x v + r_{[y,z]_g} u - r_{[x,z]_g} v + r_z l_y u \nonumber \\
= 0.
\]

The proof is finished. $\blacksquare$
Proof. Write $A = \xi \otimes u$, then we have

\[
(l^* \otimes 1 + 1 \otimes l)_x A(v) = (l^* \otimes 1 + 1 \otimes l)_x (\xi \otimes u + \xi \otimes l_x u)(v) = \langle l_x^* \xi, v \rangle u + \langle \xi, v \rangle l_x u = -\langle \xi, l_x v \rangle u + \langle \xi, v \rangle l_x u = [l_x, A](v).
\]

Therefore, we have

\[
\partial r(x, y) = (l^* \otimes 1 + 1 \otimes l)_x r(y) - r([x, y]_g) = [l_x, r_y] - r_{[x, y]_g} = 0,
\]

which implies that $r$ is a 1-cocycle. □

3 Naive representations of Leibniz algebras

It is known that the aim of a representation is to realize an abstract algebraic structure as a class of linear transformations on a vector space. Such as a Lie algebra representation is a homomorphism from $g$ to the general linear Lie algebra $gl(V)$. Unfortunately, the representation of a Leibniz algebra discussed above does not realize the abstract operation as a concrete linear operation. Therefore, it is reasonable for us to provide an alternative definition for the representation of Leibniz algebras. It is lucky that there is a god-given Leibniz algebra worked as a “general linear algebra” defined as follows:

Given a vector space $V$, then $(V, l = \text{id}, r = 0)$ is a natural representation of $gl(V)$, which is viewed as a Leibniz algebra. The corresponding semidirect product Leibniz algebra structure on $gl(V) \oplus V$ is given by

\[
\{A + u, B + v\} = [A, B] + Av, \quad \forall A, B \in gl(V), \quad u, v \in V.
\]

This Leibniz algebra is called omni-Lie algebra and denoted by $gl(V)$. The notion of an omni-Lie algebra was introduced by Weinstein in [15] as the linearization of a Courant algebroid. The notion of a Courant algebroid was introduced in [11], which has been widely applied in many fields both for mathematics and physics (see [6] for more details). Its Leibniz algebra structure also played an important role when studying the integrability of Courant brackets [5].

Notice that the skew-symmetric bracket,

\[
[[A + u, B + v]] = [A, B] + \frac{1}{2}(Av - Bu), \tag{13}
\]

which is obtained via the skew-symmetrization of $\{\cdot, \cdot\}$, is used in his original definition. As a special case in Theorem 1.3, $(gl(V) \oplus V, [[\cdot, \cdot]])$ is a Lie 2-algebra. Even though an omni-Lie algebra is not a Lie algebra, all Lie algebra structures on $V$ can be characterized by the Dirac structures in $\mathfrak{o}(V)$. In fact, the next proposition will show that every Leibniz algebra structure on $V$ can be realized as a Leibniz subalgebra of $\mathfrak{o}(V)$. For any $\varphi : V \rightarrow gl(V)$, consider its graph

\[
G_\varphi = \{\varphi(u) + u \in gl(V) \oplus V | \forall u \in V\}.
\]

Proposition 3.1. With the above notations, $G_\varphi$ is a Leibniz subalgebra of $\mathfrak{o}(V)$ if and only if

\[
[[\varphi(u), \varphi(v)]] = \varphi(\varphi(u)v), \quad \forall u, v \in V. \tag{14}
\]

Furthermore, under this condition, $(V, [[\cdot, \cdot]]_\varphi)$ is a Leibniz algebra, where the linear map $[\cdot, \cdot]_\varphi : V \otimes V \rightarrow V$ is given by

\[
[u, v]_\varphi = \varphi(u)v, \quad \forall u, v \in V. \tag{15}
\]
Proof. Since $\mathfrak{so}(V)$ is a Leibniz algebra, we only need to show that $G_{\varphi}$ is closed if and only if $[14]$ holds. The conclusion follows from

$$\{\varphi(u) + u, \varphi(v) + v\} = [\varphi(u), \varphi(v)] + \varphi(u)v.$$

The other conclusion is straightforward. The proof is finished. ■

Recall that a representation of a Lie algebra $g$ on a vector space $V$ is a Lie algebra homomorphism from $g$ to the Lie algebra $gl(V)$, which realizes an abstract Lie algebra as a subalgebra of a concrete Lie algebra. Similarly, for a Leibniz algebra, we suggest the following definition:

**Definition 3.2.** A naive representation of a Leibniz algebra $g$ on a vector space $V$ is a Leibniz algebra homomorphism $\rho : g \rightarrow \mathfrak{ol}(V)$.

According to the two components of $gl(V) \oplus V$, every linear map $\rho : g \rightarrow \mathfrak{ol}(V)$ can be split into two linear maps: $\phi : g \rightarrow gl(V)$ and $\theta : g \rightarrow V$. Then, we have

**Proposition 3.3.** A linear map $\rho = \phi + \theta : g \rightarrow gl(V) \oplus V$ is a naive representation of $g$ if and only if $(V, \phi, 0)$ is a representation of $g$ and $\theta : g \rightarrow V$ is a 1-cocycle for the representation $(V, \phi, 0)$.

**Proof.** On one hand, we have

$$\rho([x, y]_g) = \phi([x, y]_g) + \theta([x, y]_g).$$

On the other hand, we have

$$\{\rho(x), \rho(y)\} = \{\phi(x) + \theta(x), \phi(y) + \theta(y)\} = [\phi(x), \phi(y)] + \phi(x)\theta(y).$$

Thus, $\rho$ is a homomorphism if and only if

$$\phi([x, y]_g) = [\phi(x), \phi(y)], \quad (16)$$
$$\theta([x, y]_g) = \phi(x)\theta(y). \quad (17)$$

By definition, Equalities $[16]$ and $[17]$ are equivalent to that $(V, \phi, 0)$ is a representation and $\theta : g \rightarrow V$ is a 1-cocycle respectively. ■

**Theorem 3.4.** Let $(V, l, r)$ be a representation of the Leibniz algebra $g$. Then

$$\rho = (l^* \otimes 1 + 1 \otimes l) + r : g \rightarrow \mathfrak{ol}(V^* \otimes V)$$

is a naive representation of $g$ on $V^* \otimes V$.

**Proof.** By Proposition 2.4, $r : g \rightarrow gl(V)$ is a 1-cocycle on $g$ with the coefficients in the representation $(V^* \otimes V, l^* \otimes 1 + 1 \otimes l, 0)$. By Proposition 3.3, $\rho = (l^* \otimes 1 + 1 \otimes l) + r$ is a homomorphism from $g$ to $\mathfrak{ol}(V^* \otimes V)$. ■

A trivial naive representation $\rho_T$ of $g$ on $\mathbb{R}$ is defined to be a homomorphism

$$\rho_T = \phi + \theta : g \rightarrow \mathfrak{gl}(\mathbb{R}) \oplus \mathbb{R}$$

such that $\phi = 0$. By Proposition 3.3, we have

**Proposition 3.5.** Trivial naive representations of a Leibniz algebra are in one-to-one correspondence to $\xi \in g^*$ such that $\xi|_{[g, g]} = 0$.

The adjoint naive representation $\text{ad}$ is defined to be the homomorphism

$$\text{ad} = \text{ad}_L + \text{id} : g \rightarrow \mathfrak{gl}(g) \oplus g.$$
4 Naive cohomology of Leibniz algebras

Let \( \rho : g \rightarrow \mathfrak{o}(V) \) be a naive representation of the Leibniz algebra \( g \). It is obvious that \( \text{im}(\rho) \subset \mathfrak{o}(V) \) is a Leibniz subalgebra so that one can define a set of \( k \)-cochains by

\[
C^k(g; \rho) = \{ f : \otimes^k g \rightarrow \text{im}(\rho) \}
\]

and an operator \( \delta : C^k(g; \rho) \rightarrow C^{k+1}(g; \rho) \) by

\[
\delta c^k(x_1, \ldots, x_{k+1}) = \sum_{i=1}^k (-1)^{i+1} \{ \rho(x_i), c^k(x_1, \ldots, \hat{x}_i, \ldots, x_{k+1}) \} \\
+ (-1)^{k+1} \{ c^k(x_1, \ldots, x_k), \rho(x_{k+1}) \} \\
+ \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, [x_i, x_j] g, x_{j+1}, \ldots, x_{k+1}).
\]

Lemma 4.1. With the above notations, we have \( \delta^2 = 0 \).

Proof. For all \( x \in g \) and \( u \in \text{im}(\rho) \), define

\[
l_x(u) = \{ \rho(x), u \}, \quad r_x(u) = \{ u, \rho(x) \}.
\]

By the fact that \( \mathfrak{o}(V) \) is a Leibniz algebra, we can deduce that \( \text{im}(\rho); l, r \) is a representation of \( g \) on \( \text{im}(\rho) \) in the sense of Definition 2.1. \( \delta \) is just the usual coboundary operator for this representation so that \( \delta^2 = 0 \).

Thus, we have a well-defined cochain complex \( (C^\bullet(g; \rho); \delta) \). The resulting cohomology is called the naive cohomology and denoted by \( H^\bullet_{\text{naive}}(g; \rho) \). In particular, \( H^\bullet_{\text{naive}}(g) \) and \( H^\bullet_{\text{naive}}(g; \mathfrak{o}) \) denote the naive cohomologies corresponding to the trivial naive representation and adjoint naive representation of \( g \) respectively.

Theorem 4.2. With the above notations, we have \( H^\bullet_{\text{naive}}(g) = H^\bullet(g) \).

Proof. If \( [g, g]_g = g \), there is only one trivial naive representation \( \rho = 0 \) by Proposition 3.5. In this case, all the cochains are also 0. Thus, \( H^\bullet_{\text{naive}}(g) = 0 \). On the other hand, under the condition \( [g, g]_g = g \), it is straightforward to deduce that for any \( \xi \in C^k(g) \), \( \delta \xi = 0 \) if and only if \( \xi = 0 \). Thus, \( H^\bullet(g) = 0 \).

If \( [g, g]_g \neq g \), any \( 0 \neq \xi \in g^* \) such that \( \xi|_{[g, g]_g} = 0 \) gives rise to a trivial naive representation \( \rho_T \). Furthermore, we have \( \text{im}(\rho_T) = \mathbb{R} \) and \( C^k(g) = \wedge^k \mathbb{R}^* \). Thus, the sets of cochains are the same associated to two kinds of representations. Since \( V \) is an abelian subalgebra in \( \mathfrak{o}(V) \), for any \( \xi \in C^k(g) \), we have

\[
\delta \xi(x_1, \ldots, x_{k+1}) = \sum_{i=1}^k (-1)^{i+1} \{ \rho_T(x_i), c^k(x_1, \ldots, \hat{x}_i, \ldots, x_{k+1}) \} \\
+ (-1)^{k+1} \{ c^k(x_1, \ldots, x_k), \rho_T(x_{k+1}) \} \\
+ \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, [x_i, x_j] g, x_{j+1}, \ldots, x_{k+1}) \\
= \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, [x_i, x_j] g, x_{j+1}, \ldots, x_{k+1}) \\
= \partial \xi(x_1, \ldots, x_{k+1}).
\]
Thus, we have $H^\bullet_{\text{naive}}(g) = H^\bullet(g)$.

For the adjoint naive representation $\mathfrak{ad} = \text{ad}_L + \text{id} : g \to \mathfrak{gl}(g) \oplus g$, any $k$-cochain $f$ is uniquely determined by a linear map $\tilde{f} : \otimes^k g \to g$ such that

$$f = (\text{ad}_L \circ \tilde{f}) : \otimes^k g \to \text{im}(\mathfrak{ad}).$$

**Theorem 4.3.** With the above notations, we have $H^\bullet_{\text{naive}}(g; \mathfrak{ad}) = H^\bullet(g; \text{ad}_L, \text{ad}_R)$.

**Proof.** Since any $k$-cochain $f : \otimes^k g \to \text{im}(\mathfrak{ad})$ is uniquely determined by a linear map $\tilde{f} : \otimes^k g \to g$ via (19). Thus, there is a one-to-one correspondence between the sets of cochains associated to the two kinds representations via $f \leadsto \tilde{f}$. Furthermore, we have

$$\delta f(x_1, \ldots, x_{k+1})$$

$$= \sum_{i=1}^{k} (-1)^{i+1} \{ \mathfrak{ad}(x_i), f(x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1}) \}$$

$$+ (-1)^{k+1} \{ f(x_1, \ldots, x_k), \mathfrak{ad}(x_{k+1}) \}$$

$$+ \sum_{1 \leq i < j \leq k+1} (-1)^i f(x_1, \ldots, \widehat{x_i}, \ldots, x_{j-1}, [x_i, x_j]_g, x_{j+1}, \ldots, x_{k+1})$$

$$= \sum_{i=1}^{k} (-1)^{i+1} \left( [\text{ad}_L(x_i), \text{ad}_L f(x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1})] + \text{ad}_L(x_i) f(x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1}) \right)$$

$$+ (-1)^{k+1} \left( \text{ad}_L f(x_1, \ldots, x_k) + f(x_1, \ldots, x_k), \text{ad}_L(x_{k+1}) + x_{k+1} \right)$$

$$+ \sum_{1 \leq i < j \leq k+1} (-1)^i \left( \text{ad}_L f(x_1, \ldots, \widehat{x_i}, \ldots, x_{j-1}, [x_i, x_j]_g, x_{j+1}, \ldots, x_{k+1}) \right)$$

$$= \sum_{i=1}^{k} (-1)^{i+1} \left( \text{ad}_L [x_i, f(x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1})]_g + [x_i, f(x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1})]_g \right)$$

$$+ (-1)^{k+1} \left( \text{ad}_L [f(x_1, \ldots, x_k), x_{k+1}]_g + [f(x_1, \ldots, x_k), x_{k+1}]_g \right)$$

$$+ \sum_{1 \leq i < j \leq k+1} (-1)^i \left( \text{ad}_L f(x_1, \ldots, \widehat{x_i}, \ldots, x_{j-1}, [x_i, x_j]_g, x_{j+1}, \ldots, x_{k+1}) \right)$$

$$= \text{ad}_L \partial f(x_1, \ldots, x_{k+1}) + \partial f(x_1, \ldots, x_{k+1}).$$
Thus, we have $\delta f = 0$ if and only if $\partial f = 0$. Similarly, we can prove that $f$ is exact if and only if $f$ is exact. Thus, the corresponding cohomologies are isomorphic.

At last, we consider a naive representation $\rho$ such that the image of $\rho$ is contained in the graph $\mathcal{G}_\varphi$ for some linear map $\varphi : V \rightarrow \mathfrak{gl}(V)$ satisfying Eq. (14). In this case, $\rho$ is of the form $\rho = \varphi \circ \theta + \theta$, where $\theta : \mathfrak{g} \rightarrow V$ is a linear map. A $k$-cochain $f : \otimes^k \mathfrak{g} \rightarrow \text{im}(\rho)$ is of the form $f = \varphi \circ f + f$, where $f : \otimes^k \mathfrak{g} \rightarrow V$ is a linear map.

Define left and right actions in the sense of Definition 2.1 by

$$
\begin{align*}
  l_x u &= \text{pr}\{\rho(x), \varphi(u) + u\} = \varphi(\theta(x))u; \\
  r_x u &= \text{pr}\{\varphi(u) + u, \rho(x)\} = \varphi(u)\theta(x),
\end{align*}
$$

where $\text{pr}$ is the projection from $\mathfrak{gl}(V) \oplus V$ to $V$. Similar to Theorem 4.3, it is easy to prove that

**Theorem 4.4.** Let $\rho$ be a naive representation such that the image of $\rho$ is contained in the graph $\mathcal{G}_\varphi$ for some linear map $\varphi : V \rightarrow \mathfrak{gl}(V)$ satisfying Eq. (14). Then we have

$$
H^\bullet_{\text{naive}}(\mathfrak{g}; \rho) = H^\bullet(\mathfrak{g}; l, r),
$$

where $l$ and $r$ are given by (20) and (21) respectively.

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