Characterizing nonbilocal correlation: a geometric perspective

R. Muthuganesan\textsuperscript{1,2} \cdot S. Balakrishnan\textsuperscript{3} \cdot V. K. Chandrasekar\textsuperscript{1}

Received: 23 October 2021 / Accepted: 17 May 2022 / Published online: 14 June 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
Exploiting the notion of measurement-induced nonlocality (Luo and Fu in Phys Rev Lett 106:120401, 2011), we introduce a new measure to quantify the nonbilocal correlation. We establish a simple relation between the nonlocal and nonbilocal measures for the arbitrary pure input states. Considering the mixed states as inputs, we derive two upper bounds of affinity-based nonbilocal measure. Finally, we have studied the nonbilocality of a different combinations of input states.

Keywords Entanglement \cdot Quantum correlation \cdot Nonlocality \cdot Nonbilocality \cdot Projective measurements \cdot Nonlocality

1 Introduction

Nonlocality, the most fundamental and intriguing feature of a composite quantum system, is a direct consequence of the superposition principle, which creates a distinction between the behavior of the quantum and classical systems [1]. Nonlocality is referred as “spooky-action-at-a-distance” by Einstein [2] and Schrodinger [3]. Understanding this perplexing phenomenon in a simplest composite system, namely the bipartite
system, is a fundamental issue and of practical importance in developing quantum technologies. In the realm of Bell’s nonlocality [4], the presence of nonlocal character or entanglement is witnessed by the violation of Bell inequality. In the earliest quantum information theory era, it is believed that the entanglement is the complete manifestation of nonlocality of a quantum system.

Since the inception of Werner’s work [5] and quantum discord [6], it has been debated whether the entanglement can manifest the nonlocal aspects of a quantum system or not. Buscemi showed that entanglement can be the complete manifestation of nonlocality. In other words, “all entangled quantum states are nonlocal” [7]. On the other hand, Werner constructed a mixed state family, which admits the local hidden variable model even the state is entangled [5]. Further, it is shown that the presence of noise or mixedness is responsible for the destroying nonlocal correlation between local constituents of the composite system, and hence, some of the mixed entangled states behave locally [8]. Ollivier and Zurek introduced a measure called, quantum discord to quantify the quantum correlations beyond entanglement. It can capture the correlation between the marginal states which cannot be grasped by the entanglement [6]. Recently, a new variant of nonlocality, called measurement-induced nonlocality (MIN), is introduced [9]. MIN is based on the fact that local disturbance due to locally invariant von Neumann projective measurements on the marginal state can influence globally. MIN is quite different from the entanglement and violation of the Bell inequalities, more importantly it goes beyond entanglement.

In quantum entanglement swapping experiment, the independence between the multi-sources induces the nonlocal behavior of probability distributions and is called nonbilocal correlations [10, 11]. This kind of correlation is demonstrated and captured using nonlinear inequalities and one important class of these inequalities is the so-called binary-input-and-output bilocality inequality which is known as the bilocality inequality [10, 11]. In recent times, the considerable progress has been made in this context [12–21]. Gisin et al. have shown that pair of entangled states can violate the bilocality inequality, implying that tensorizing states may possess nonlocal correlation [22].

When two bipartite states with vanishing correlation are combined, it is always interesting to check that the tensorizing state possess any nonlocal advantage or not. It is shown recently that the combining two quantum systems exhibit better quantum advantages than the individual system. This is known as superactivation of nonlocality, symbolized as $0 + 0 > 0$ and it cannot occur in the classical world. The superactivation of nonlocality provides an answer for “can the state $\rho \otimes \rho$ be nonlocal if the $\rho$ is local.” Recently, Palazuelos explained the superactivation of quantum nonlocality in the sense of violating certain Bell inequalities with an entangled bound state [23]. The same study is carried out in the context of tensor networks [24, 25]. Further, the superactivation is also considered for arbitrary entangled states by allowing local preprocessing on the tensor product of different quantum states ($\rho \otimes \rho'$) [26] and symbolized as $1 + 0 > 1$. With our best knowledge, the quantification of this nonbilocal correlation is limited in the literature.

To quantify the nonbilocal correlation, we extend the notion of bipartite measurement-induced nonlocality (MIN) to a bilocal quantum system. This paper proposes a new version of the nonbilocal correlation measure. The relation between the nonlocal and
nonbilocal correlation measure is established, and it is shown that nonbilocality is always greater than the nonlocal correlation. Further, the upper bound of the bilocal correlation measure is obtained for the arbitrary mixed input states. To validate the properties of nonbilocal measure, we study the proposed quantity for a few examples.

This paper is organized as follows. In Sect. 2, we review the notion of measurement-induced nonlocality and provide the definition of affinity-based MIN. In Sect. 3, we introduce a nonbilocal measure based on the affinity induced metric and we derive the analytical formula of the measure when the input states are pure. Considering the input states are arbitrary mixed states, the upper bounds of bilocal correlation are presented in Sect. 4. In Sect. 5, the proposed measure studied for a few examples. Finally, in Sect. 6 we present the conclusion.

2 Measurement-induced nonlocality

To capture the bipartite quantum correlation beyond entanglement, Luo and Fu introduced a new measure of quantum correlation called measurement-induced nonlocality (MIN) using locally invariant projective measurement. It is originally defined as maximal square of Hilbert–Schmidt norm of difference of pre- and post-measurement states and is defined as [9]

\[ N(\rho) = \max_{\Pi^a} \|\rho - \Pi^a(\rho)\|^2, \tag{1} \]

where the maximization is taken over the von Neumann projective measurements on subsystem \( a \), \( \Pi^a(\rho) = \sum_k (\Pi^a_k \otimes 1^b)\rho(\Pi^a_k \otimes 1^b) \), and \( \Pi^a = \{\Pi^a_k\} = \{|k\rangle\langle k|\} \) being the projective measurements on the subsystem \( a \), which do not change the marginal state \( \rho^a \) locally, i.e., \( \Pi^a(\rho^a) = \rho^a \). Here, \( \|\mathcal{O}\| = \sqrt{\text{Tr}\mathcal{O}^\dagger\mathcal{O}} \) is the Hilbert–Schmidt norm of operator \( \mathcal{O} \). The dual of this quantity is geometric discord (GD) of the given state \( \rho \) and it is defined as [27, 28]

\[ D(\rho) = \min_{\Pi^a} \|\rho - \Pi^a(\rho)\|^2. \tag{2} \]

In general, if \( \rho^a \) is nondegenerate, then the optimization is not required and the above measures are equal. This is more general than the conventionally mentioned quantum nonlocality related to Bell’s version of nonlocality. Apart from the quantification of bipartite quantum correlation, this quantity provides a novel classification scheme for bipartite states, and is a useful resource quite different from entanglement. In particular, MIN is a more secured resource for cryptographic communication due to the invariance of marginal states. Due to its geometric nature, MIN is easy to compute compared with quantum discord [6] and realizable. Nevertheless, both the MIN and geometric discord are not useful quantifiers of quantum correlation due to the local ancilla problem, which is pointed out by Piani [29]. A natural way to circumvent this issue is to modify the definition of MIN using any contractive distance measure. One such distance measure between the states \( \rho \) and \( \sigma \) is defined as

\[ d_A(\rho, \sigma) = 1 - A(\rho, \sigma), \tag{3} \]
where $A(\rho, \sigma) = \text{Tr}\left(\sqrt{\rho} \sqrt{\sigma}\right)$ is the affinity between the states. Analogous to fidelity [30], the affinity is also a measure of closeness between the states $\rho$ and $\sigma$ [31, 32] and shares all the properties of fidelity [30]. Also, affinity is useful in the quantification of nonclassical correlations [33, 34] and quantum coherence [35]. It is worth mentioning that the affinity between the states is realizable using the quantum circuit. Due to its realization, affinity-based measures may have a good impact in the research of quantum information.

The affinity-based MIN is defined as [34]

$$N_{A}^{\text{MIN}}(\rho) = \max_{\Pi_{bc}} d_{A}(\rho_{ab} \otimes \rho_{cd}, \Pi_{bc}(\rho_{ab} \otimes \rho_{cd})) = 1 - \min_{\Pi_{bc}} \text{Tr}\left(\sqrt{\rho} \Pi_{bc}(\sqrt{\rho})\right).$$

Here also the optimization is taken over von Neumann projective measurements. It is worth reiterating that the $N_{A}(\rho)$ fixes the local ancilla problem using the multiplicative property of affinity. Hence, $N_{A}^{\text{MIN}}(\rho)$ is a faithful quantifier of quantum correlation or quantumness of the system. Further, we have shown that affinity-based MIN is closely related to local quantum uncertainty [36] and interferometric power of a quantum state [37].

### 3 Nonbilocality measure

In this section, we introduce the notion and measure of nonbilocality using affinity. We consider two input states $\rho^{ab}$ (shared between $a$ and $b$) and $\rho^{cd}$ (shared between $c$ and $d$) in the separable composite finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_{a} \otimes \mathcal{H}_{b} \otimes \mathcal{H}_{c} \otimes \mathcal{H}_{d}$. Then nonbilocal measure is defined as

$$N_{A}(\rho_{ab} \otimes \rho_{cd}) = \max_{\Pi_{bc}} d_{A}(\rho_{ab} \otimes \rho_{cd}, \Pi_{bc}(\rho_{ab} \otimes \rho_{cd})) = 1 - \min_{\Pi_{bc}} A(\rho_{ab} \otimes \rho_{cd}, \Pi_{bc}(\rho_{ab} \otimes \rho_{cd})).$$

where maximization/minimization is taken over the von Neumann projective measurement $\Pi_{bc} = \{\Pi_{bc}^{kl}\}$, which leaves $\rho^{bc} = \text{Tr}_{ad}(\rho_{ab} \otimes \rho_{cd})$ invariant locally, $d_{A}(\cdot, \cdot)$ is affinity-induced metric and the post-measurement state is $\Pi_{bc}(\sqrt{\rho_{ab} \otimes \rho_{cd}}) = \sum_{k,l}(1^{a} \otimes \Pi_{bc}^{kl} \otimes 1^{d})\sqrt{\rho_{ab} \otimes \rho_{cd}}(1^{a} \otimes \Pi_{bc}^{kl} \otimes 1^{d})$. Here, $\rho^{b} = \sum_{i} \lambda_{i} |i_{b}\rangle\langle i_{b}|$ and $\rho^{c} = \sum_{j} \lambda_{j} |j_{c}\rangle\langle j_{c}|$ are the marginal states of $\rho_{bc}$, if any one of the states is nondegenerate, then the measurement takes the form $\Pi_{bc} = \{\Pi_{bc}^{b} \otimes \Pi_{bc}^{c}\}$.

It is worth mentioning that above-defined quantity satisfies all the necessary axioms of a quantum correlation measure. Like MIN, the nonbilocare measure is also a useful resource for cryptographic communication. Next, we demonstrate some interesting properties of nonbilocality measure given by Eq. (5).

(i) $N_{A}(\rho_{ab} \otimes \rho_{cd})$ is nonnegative,

(ii) $N_{A}(\rho_{ab} \otimes \rho_{cd}) = 0$ for any product states $\rho_{ab} = \rho^{a} \otimes \rho^{b}$ and $\rho_{cd} = \rho^{c} \otimes \rho^{d}$.

Further, the nonbilocal measure also vanishes for classical-quantum state $\rho_{ab} = \sum_{i} \rho^{i}_{a} \otimes p_{i} |i_{b}\rangle\langle i_{b}|$ and $\rho_{cd} = \sum_{j} q_{j} |j_{c}\rangle\langle j_{c}| \otimes \rho^{j}_{d}$.
(iii) $N_A(\rho_{ab} \otimes \rho_{cd})$ is locally unitary invariant in the sense that

$$N_A((U_{ab} \otimes U_{cd})\rho_{ab} \otimes \rho_{cd}(U_{ab} \otimes U_{cd})^\dagger) = N_A(\rho_{ab} \otimes \rho_{cd}),$$

where $U_{ab} = U_a \otimes U_b$ and $U_{cd} = U_c \otimes U_d$ are the local unitary operators.

(iv) $N_A(\rho_{ab} \otimes \rho_{cd})$ is positive, at least any one of the input states is entangled.

(v) Although $N_A(\rho_{ab}) = N_A(\rho_{cd}) = 0$, nevertheless $N_A(\rho_{ab} \otimes \rho_{cd}) > 0$.

Since the properties (i) - (v) are inherited directly from the properties of affinity-based $\text{MIN}$, we omit the proving process. Next, we establish a simple relation between the nonbilocal measure and $\text{MIN}$ (nonlocal).

**Theorem 1** The nonbilocal measure and affinity-based $\text{MIN}$ are connected as

$$N_A(\rho_{ba} \otimes \rho_{ab}) \geq N_{\text{MIN}}^A(\rho). \quad (7)$$

**Proof** To prove this, first we recall the definition of affinity-based nonbilocal measure,

$$N_A(\rho_{ba} \otimes \rho_{ab}) = \max_{\Pi^{aa}} d_A(\rho_{ba} \otimes \rho_{ab}, \Pi^{aa}(\rho_{ba} \otimes \rho_{ab})),$$

$$= 1 - \min_{\Pi^{aa}} A(\rho_{ba} \otimes \rho_{ab}, \Pi^{aa}(\rho_{ba} \otimes \rho_{ab})),$$

$$\geq \max_{\Pi^a} d_A(\rho_{ba} \otimes \rho_{ab}, (\Pi^a \otimes \Pi^a)(\rho_{ba} \otimes \rho_{ab})),$$

$$= 1 - \min_{\Pi^a} \text{Tr}(\sqrt{\rho_{ab}} \sqrt{\rho_{ab}}) (\Pi^a (\sqrt{\rho_{ab}}) \otimes \Pi^a (\sqrt{\rho_{ab}})),$$

$$= 1 - \min_{\Pi^a} \text{Tr}(\sqrt{\rho_{ab}} \Pi^a (\sqrt{\rho_{ab}}))^2,$$

$$\geq 1 - \min_{\Pi^a} \text{Tr}(\sqrt{\rho_{ab}} \Pi^a (\sqrt{\rho_{ab}})),$$

$$= N_{\text{MIN}}^A(\rho),$$

where the first inequality follows from the fact that $\Pi^a \otimes \Pi^a$ is not necessarily optimal and the second inequality is due to the square of the affinity which is always equal to or less than the affinity. Hence, the theorem is proved. The above theorem provides a closer connection between the nonbilocal and nonlocal measures, implying that the nonbilocal measure is always greater than $\text{MIN}$. Next, we compute the nonbilocal measure for pure input states.

**Theorem 2** Let $|\Psi_{ab}\rangle = \sum_i \sqrt{s_i} |i_a i_b\rangle$ and $|\Psi_{cd}\rangle = \sum_j \sqrt{r_j} |j_c j_d\rangle$ are the pure input states, then

$$N_A(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 1 - \sum_{i,j} s_i^4 r_j^4, \quad (8)$$

where $s_i$ and $r_j$ are Schmidt coefficients of $|\Psi_{ab}\rangle$ and $|\Psi_{cd}\rangle$, respectively.

The proof of the theorem is given in the appendix.
4 Nonbilocal correlation for mixed states

To compute nonbilocality of any arbitrary mixed input states, first, we recall some basic notation in the operator Hilbert space. Let $L(H_\alpha)$ be the Hilbert space of linear operators on $H_\alpha(\alpha = a, b, c, d)$ with the inner product $\langle X|Y \rangle = \text{Tr}X^\dagger Y$. An arbitrary $m \times n$ dimensional bipartite state can be written as

$$\sqrt{\rho_{ab}} = \sum_{i,j} \lambda_{ij}^{ab} X_i \otimes Y_j,$$

where $\{X_i : i = 0, 1, \cdots, m^2 - 1\}$ and $\{Y_j : j = 0, 1, \cdots, n^2 - 1\}$ are the orthonormal operator bases of the subsystem $a$ and $b$, respectively, and satisfies the relation $\text{Tr}X_k X_l = \delta_{kl}$, and $\lambda_{ij}^{ab} = \text{Tr}(\sqrt{\rho_{ab}} X_i \otimes Y_j)$ are real elements of matrix $\Lambda_{ab}$. Similarly, one can define the orthonormal operator bases as $\{P_k : k = 0, 1, \cdots, u^2 - 1\}$ and $\{Q_l : l = 0, 1, \cdots, v^2 - 1\}$ for another input state $\rho_{cd}$ with $u$ and $v$ are the dimensions of the marginal systems $c$ and $d$, respectively. Then, the state $\rho_{cd}$ is defined as

$$\sqrt{\rho_{cd}} = \sum_{k,l} \lambda_{kl}^{cd} P_k \otimes Q_l,$$

where $\lambda_{kl}^{cd} = \text{Tr}(\sqrt{\rho_{cd}} P_k \otimes Q_l)$ are the matrix elements of matrix $\Lambda_{cd}$. Then, the bilocal state is written as

$$\sqrt{\rho_{ab} \otimes \rho_{cd}} = \sum_{i,j} \sum_{k,l} \lambda_{ij}^{ab} \lambda_{kl}^{cd} X_i \otimes Y_j \otimes P_k \otimes Q_l.$$

**Theorem 3** For any arbitrary bilocal input states represented in Eq. (9), the upper bound of nonbilocal measure is

$$N_A(\rho_{ab} \otimes \rho_{cd}) \leq 1 - \sum_{s=1}^{nu} \mu_s,$$

where $\mu_s$ are the eigenvalues of the matrix $\Lambda_{ab,cd} \Lambda_{ab,cd}^t$ arranged in increasing order and $t$ denotes the transpose of a matrix.

To prove the theorem, first, we compute the post-measured state. Here, the measurement operators are $\Pi^{bc} = \{1^a \otimes \Pi_h^{bc} \otimes 1^d\}$ and we have

$$\Pi^{bc}(\sqrt{\rho_{ab} \otimes \rho_{cd}}) = \sum_h \sum_{ijkl} \lambda_{ij}^{ab} \lambda_{kl}^{cd} X_i \otimes \Pi_h^{bc} (Y_j \otimes P_k) \Pi_h^{bc} \otimes Q_l$$

$$= \sum_h \sum_{ijkl} \lambda_{ij}^{ab} \lambda_{kl}^{cd} \gamma_{hjk} \gamma_{hj'k'} X_i \otimes Y_j' \otimes P_k' \otimes Q_l,$$
where $\gamma_{hjk} = \text{Tr}\Pi^h (Y_j \otimes P_k)$ are the elements of matrix $\Gamma$. After the straightforward calculation, the affinity between pre- and post-measurement states is

$$A\left(\sqrt{\rho_{ab} \otimes \rho_{cd}}, \Pi^{bc} (\sqrt{\rho_{ab} \otimes \rho_{cd}})\right) = \sum_h \sum_{ijj'kk'1} \gamma_{ij}^{ab} \gamma_{kj}^{cd} \gamma_{ij}^{ab} \gamma_{kj}^{cd} = \Gamma \Lambda_{ab,cd} \Lambda^t_{ab,cd} \Gamma^t,$$  \hspace{1cm} (12)

where $\Gamma$ is $nu \times n^2 u^2$ dimensional matrix. Then,

$$N_A(\rho_{ab} \otimes \rho_{cd}) = 1 - \min_{\Pi^{bc}} A(\rho_{ab}, \Pi^{bc}(\rho_{ab})) \leq 1 - \sum_{s=1}^{nk} \mu_s,$$  \hspace{1cm} (13)

where $\mu_s$ are the eigenvalues of the matrix $\Lambda_{ab,cd} \Lambda^t_{cd}$ listed in increasing order. Hence, the theorem is proved.

**Theorem 4** If the marginal state $\rho^b$ is nondegenerate, the nonbilocal measure $N_A(\rho_{ab} \otimes \rho_{cd})$ due to the measurement $\Pi^{bc}$ has the upper bound as

$$N_A(\rho_{ab} \otimes \rho_{cd}) \leq 1 - A(\rho_{ab}, \Pi^b(\rho_{ab})) \times \left(\sum_{\tau=1}^u \mu_{\tau}\right),$$  \hspace{1cm} (14)

where $\mu_{\tau}$ are the eigenvalues of matrix $\Lambda_{cd} \Lambda^t_{cd}$ arranged in an increasing order and $A(\rho_{ab}, \Pi^b(\rho_{ab}))$ is the affinity between the state $\rho_{ab}$ and post-measurement state $\Pi^b(\sqrt{\rho_{ab}})$.

To show this, we recall that if the marginal state is nondegenerate, then the optimization is not required. Here, the state $\rho^b$ is nondegenerate and the optimization is taken over $\Pi^c$ alone. The measurement operator is defined as $\Pi^b \otimes \Pi^c = \{\Pi_j^b \otimes \Pi_k^c\} = \{|jb\rangle \langle jb| \otimes \Pi_k^c\}$. The nonbilocality measure is

$$N_A(\rho_{ab} \otimes \rho_{cd}) = 1 - \min_{\Pi^{bc}} A(\rho_{ab} \otimes \rho_{cd}, \Pi^{bc}(\rho_{ab} \otimes \rho_{cd}))$$

$$= 1 - \min_{\Pi^{bc}} \text{Tr} \sqrt{\rho_{ab} \otimes \rho_{cd}} \cdot \Pi^{bc}(\sqrt{\rho_{ab} \otimes \rho_{cd}})$$

$$= 1 - \min_{\Pi^c} \text{Tr} \sqrt{\rho_{ab} \otimes \rho_{cd}} \cdot (\Pi^b \otimes \Pi^c)(\sqrt{\rho_{ab} \otimes \rho_{cd}})$$

$$= 1 - \text{Tr} \sqrt{\rho_{ab} \otimes \rho_{cd}} \cdot \Pi^b(\sqrt{\rho_{ab}}) \cdot \min_{\Pi^c} \text{Tr} \sqrt{\rho_{cd}} \Pi^c(\sqrt{\rho_{cd}}),$$  \hspace{1cm} (15)

where the quantity $\text{Tr} \sqrt{\rho_{ab} \otimes \rho_{cd}}$ is the affinity between the state $\rho_{ab}$ and post-measurement state $\Pi^b(\sqrt{\rho_{ab}})$. Following the optimization procedure given in [34], we write the second quantity as

$$\min_{\Pi^c} \text{Tr} \sqrt{\rho_{cd}} \Pi^c(\sqrt{\rho_{cd}}) = \min_{C} \text{Tr} C \Lambda_{cd} \Lambda^t_{cd} C^t.$$  \hspace{1cm} (16)
Then, we have

\[ N_A(\rho_{ab} \otimes \rho_{cd}) = 1 - A(\rho_{ab}, \Pi^b(\sqrt{\rho_{ab}})) \min_C \text{Tr} \Lambda_{cd} \Lambda_{cd}^t C_t \]

\[ \leq 1 - A(\rho_{ab}, \Pi^b(\rho_{ab})) \times \sum_{\tau=1}^{u} \mu_{\tau}, \quad (17) \]

where \( \mu_{\tau} \) are the eigenvalues of matrix \( \Lambda_{cd} \Lambda_{cd}^t \) arranged in an increasing order.

**Theorem 5** If the marginal states \( \rho^b \) and \( \rho^c \) are nondegenerate and the dimension of the \( \rho^c \) is \( u = 2 \), then the closed formula of nonbilocal measure \( N_A(\rho_{ab} \otimes \rho_{cd}) \) is expressed as

\[ N_A(\rho_{ab} \otimes \rho_{cd}) = 1 - A(\rho_{ab}, \Pi^b(\rho_{ab})) \times \|\lambda_{cd}\| + \lambda_{\min}, \quad (18) \]

where \( \lambda_{cd} = (\lambda_{cd}^{00}, \lambda_{cd}^{01}, \ldots, \lambda_{cd}^{0(v^2-1)}) \) is a \( v^2 \) dimensional row vector, and \( \Lambda = ((\lambda_{cd}^{kl})_{k=1,2,3; \; l=0,1,\ldots,v^2-1}) \) is a \( 3 \times v^2 \) dimensional matrix and \( \lambda_{\min} \) is the least eigenvalues of \( \Lambda^t \Lambda \).

Using the completeness relation \( \sum_k \Pi_k^c = 1^c \), we show that \( c_{0k} = -c_{1k} (k = 1, 2, 3) \), then \( \sum_k c_{0k}^2 = 1 \). Then, the vector \( c = \sqrt{2}(c_{0k} \ c_{0k} \ c_{0k}) \) with \( |c| = 1 \). Now,

\[ C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}, \quad (19) \]

and

\[ \Lambda_{cd} = \begin{pmatrix} \lambda_{cd} \\ \Lambda \end{pmatrix}, \quad (20) \]

where \( \lambda_{cd} = (\lambda_{cd}^{00}, \lambda_{cd}^{01}, \ldots, \lambda_{cd}^{0(v^2-1)}) \) is a \( v^2 \) dimensional row vector, and \( \Lambda = ((\lambda_{cd}^{kl})_{k=1,2,3; \; l=0,1,\ldots,v^2-1}) \) is a \( 3 \times v^2 \) dimensional matrix. We have

\[ \min_C \text{Tr} \Lambda_{cd} \Lambda_{cd}^t C_t = \|\lambda_{cd}\| + \lambda_{\min}, \quad (21) \]

where \( \lambda_{\min} \) is the least eigenvalue of matrix \( cRR^t c^t \). Then, we have computed the closed formula of nonbilocal measure

\[ N_A(\rho_{ab} \otimes \rho_{cd}) = 1 - A(\rho_{ab}, \Pi^b(\sqrt{\rho_{ab}})) \times (\|\lambda_{cd}\| + \lambda_{\min}) \quad (22) \]

to complete the proof.
5 Illustrations

In this section, we compute the affinity-based measurement-induced nonbilocality for some input states and compare with the Hellinger distance $N_{He}(\cdot)$ [38] and Hilbert–Schmidt norm $N_{HS}(\cdot)$ [39] nonbilocal measure.

**Example 1** Let $|\Psi_{ab}\rangle = |00\rangle$ and $|\Psi_{cd}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ are the two input states. According to Theorem 2, the nonbilocal measure is

$$N_A(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 1 - \sum_{i, j} s_i^4 r_j^4. \quad (23)$$

The Schmidt coefficients for $|\Psi_{ab}\rangle$ are 0 and 1. Similarly, $|\Psi_{cd}\rangle$ has the Schmidt coefficients $1/\sqrt{2}$ and $1/\sqrt{2}$. Then, $N_A(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 0.5$. The above example validates the property (iv) of the $N_A(\rho_{ab} \otimes \rho_{cd})$.

**Example 2** The input state is $|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \otimes (|00\rangle + |11\rangle)/\sqrt{2}$. Then,

$$N_A(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 1 - 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{4}. \quad (24)$$

As a consequence of Theorem 2, the affinity-based nonbilocal measure is equal to Hellinger distance $N_{He}(\cdot)$ and Hilbert–Schmidt norm $N_{HS}(\cdot)$-based nonbilocal measures are equal, i.e., $N_A(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = N_{He}(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = N_{HS}(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle)$.

**Example 3** Next, we consider the combination of maximally entangled state as

$$\rho_{ab} = \frac{1}{3}(|\Psi^+\rangle \langle \Psi^+| + |\Psi^-\rangle \langle \Psi^-| + |\Phi^+\rangle \langle \Phi^+|),$$

where $|\Psi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Phi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. In straightforward, the square root of $\rho_{ab}$ is

$$\sqrt{\rho_{ab}} = \frac{1}{\sqrt{3}}(|\Psi^+\rangle \langle \Psi^+| + |\Psi^-\rangle \langle \Psi^-| + |\Phi^+\rangle \langle \Phi^+|).$$

The Bloch form of $\sqrt{\rho_{ab}}$ can be written as

$$\sqrt{\rho_{ab}} = \frac{1}{4} \left[ \sqrt{3} (I \otimes I) + \frac{1}{\sqrt{3}} (\sigma_x \otimes \sigma_x) + \frac{1}{\sqrt{3}} (\sigma_y \otimes \sigma_y) + \frac{1}{\sqrt{3}} (\sigma_z \otimes \sigma_z) \right],$$

where $\sigma_i$ are Pauli’s spin matrices. Here, the eigenprojective measurements are $\Pi^{bc} = \{|\Psi^+\rangle \langle \Psi^+|, |\Psi^-\rangle \langle \Psi^-|, |\Phi^+\rangle \langle \Phi^+|, |\Phi^-\rangle \langle \Phi^-|\}$. Then, the nonbilocal measure is computed as

$$N_A(\rho_{ba} \otimes \rho_{ab}) = 1 - \min_{\Gamma} \Gamma \Lambda_{ab,cd} A_{ab,cd}^\Gamma \Gamma^\Gamma \geq 1 - \frac{7}{12} = \frac{5}{12}. \quad (25)$$
The affinity-based MIN is \(N_A^{\text{MIN}}(\rho) = 1/6 < N(\rho_{ba} \otimes \rho_{ab})\), resulting the consequence of Theorem 1. Here, we observe that \(N_A(\rho_{ba} \otimes \rho_{ab}) = N_{He}(\rho_{ba} \otimes \rho_{ab}) > N_{HS}(\rho_{ba} \otimes \rho_{ab})\).

**Example 4** In this case, the input states are

\[
\rho_{ab} = \rho_{cd} = \frac{1}{2} |0\rangle\langle 0 | \otimes |0\rangle\langle 0 | + \frac{1}{2} |1\rangle\langle 1 | \otimes |1\rangle\langle 1 |
\]  

(26)

shared between \(a\) and \(b\). The MIN of the above state is zero. We obtain

\[
A_{ab} = A_{cd} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]  

(27)

Here, we choose the optimal von Neumann measurements as \(\{\Pi^{bc}\} = \{H^{\otimes 2} |ij\rangle\langle ij| H^{\otimes 2}\}\) with \(i, j = 0, 1\) and \(H\) is

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]  

(28)

the popular single-qubit Hadamard gate. After the straightforward calculation, we have

\[
N_A(\rho_{ba} \otimes \rho_{ab}) = 1 - \min_{\Gamma} \Gamma A_{ab,cd} A_{ab,cd}^t \Gamma^t = \frac{3}{4}.
\]  

(29)

Here, we notice that \(N_A(\rho_{ba} \otimes \rho_{ab}) = N_{He}(\rho_{ba} \otimes \rho_{ab}) = 3/4\) and \(N_{HS}(\rho_{ba} \otimes \rho_{ab}) = 3/16\).

**6 Conclusion**

Nonlocality is often related to the entanglement or violation of Bell’s inequality. In this article, we have employed the nonlocality in different notions, namely measurement-induced nonlocality (MIN). Extending the definition of affinity-based MIN, we have introduced a new variant of quantifier to quantify the nonlocal correlation contained in tensorizing a local state with itself called nonbilocal correlation and also demonstrated the computational properties of the proposed measure. A closer connection between the affinity-based measurement-induced nonlocality and measurement-induced non-bilocality is also derived. An analytical formula of nonbilocal measure is derived when the input states are pure. Moreover, two upper bounds of nonbilocal measure are also obtained for the mixed input states. For illustration, we have studied the nonbilocality for different input states.
Like MIN, the nonbilocal measure may also useful resource for remote state preparation, quantum dense coding and cryptographic communication, and hence, the proposed nonbilocal measure provides more insight into quantum information theory.

Acknowledgements This work was financially supported by the Council of Scientific and Industrial Research (CSIR), Government of India, under Grant No. 03(1444)/18/EMR-II.

Data availability Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Appendix

Let $|\Psi_{ab}\rangle = \sum_i s_i |i_a i_b\rangle$ and $|\Psi_{cd}\rangle = \sum_j r_j |j_c j_d\rangle$ are the two pure input states with $s_i$ and $r_j$ are the respective Schmidt coefficients of input states.

Noting that

$$\rho_{ab} \otimes \rho_{cd} = |\Psi_{ab}\rangle \langle \Psi_{ab}| \otimes |\Psi_{cd}\rangle \langle \Psi_{cd}|$$

We get

$$= \sum_{i i' j j'} s_i s'_i r_j r'_j |i'_a\rangle \otimes |i'_b\rangle \otimes |j'_c\rangle \otimes |j'_d\rangle.$$

One can compute the marginal state

$$\rho_{bc} = \text{Tr}_{ad}(|\Psi_{ab}\rangle \langle \Psi_{ab}| \otimes |\Psi_{cd}\rangle \langle \Psi_{cd}|) = \sum_{ij} s_i^2 r_j^2 |i_b j_c\rangle \langle i_b j_c|.$$

For pure state $\sqrt{\rho} = \rho$. From the above equation, the post-measurement state $\Pi^b c(\sqrt{\rho_{ab}} \otimes \rho_{cd})$ can be rewritten as

$$\Pi^b c(\sqrt{\rho_{ab}} \otimes \rho_{cd}) = \Pi^b c(\rho_{ab} \otimes \rho_{cd})$$

$$= \sum_{hk} (1^a \otimes \Pi^b_{hk} \otimes 1^d) \left( \sum_{ii' jj'} s_i s'_i r_j r'_j |i'_a\rangle \otimes |i'_b\rangle \otimes |j'_c\rangle \otimes |j'_d\rangle \right)$$

$$= \sum_{hk} (1^a \otimes \Pi^b_{hk} \otimes 1^d) \left( \sum_{ii' jj'} s_i s'_i r_j r'_j |i'_a\rangle \otimes |i'_b\rangle \otimes \Pi^b_{hk} |j'_c\rangle |j'_d\rangle \otimes |j'_d\rangle \right)$$

$$= \sum_{hk} \sum_{ii' jj'} s_i s'_i r_j r'_j |i'_a\rangle \otimes U |h_b k_c\rangle \langle h_b k_c| U^\dagger |i'_b j'_c\rangle \langle i'_b j'_c| U |h_b k_c\rangle \langle h_b k_c| U^\dagger \otimes |j'_d\rangle \langle j'_d|.$$

Here, the von Neumann projective measurement is expressed as

$$\Pi^b c = \{ \Pi^b_{hk} \equiv U |h_b k_c\rangle \langle h_b k_c| U^\dagger \}$$
Consequently,

\[
\sqrt{\rho_{ab} \otimes \rho_{cd} \Pi^{bc}} \left( \sqrt{\rho_{ab} \otimes \rho_{cd}} \right)
\]

\[
= \left( \sum_{ii'jj'} s_i s_{i'} r_j r_{j'} |i_a\rangle \otimes |i_b\rangle |j_c\rangle \otimes |j_d\rangle |i'_d\rangle \right) \left( \sum_{hk} \sum_{uu'vv'} s_u s_{u'} r_v r_{v'} \langle h_b k_c | U^\dagger | u_b v_c \rangle \langle u'_b v'_c | U | h_b k_c \rangle \langle h_b f_c | U^\dagger \otimes | v_d \rangle \langle v'_d | \right)
\]

\[
= \sum_{ii'jj'} \sum_{hk} \sum_{uu'vv'} s_i s_{i'} r_j r_{j'} s_u s_{u'} r_v r_{v'} \langle h_b k_c | U^\dagger | u_b v_c \rangle \langle u'_b v'_c | U | h_b k_f \rangle \langle i_a | \langle i'_a | \otimes \langle i_b j_c | \langle i'_b j'_c | U | h_b k_c \rangle \langle h_b k_c | U^\dagger \otimes | j_d | \langle j'_d | v_d \rangle \langle v'_d | \right).
\]

Then, the affinity between the pre- and post-measurement state is computed as

\[
A(\rho_{ab} \otimes \rho_{cd}, \Pi^{bc})(\sqrt{\rho_{ab} \otimes \rho_{cd}}) = \text{Tr} \sqrt{\rho_{ab} \otimes \rho_{cd} \Pi^{bc}} \left( \sqrt{\rho_{ab} \otimes \rho_{cd}} \right)
\]

\[
= \sum_{iujhk} s_i^2 s_u^2 r_j^2 r_h^2 \langle h_b k_c | U^\dagger | i_b j_c \rangle \langle i_b j_c | U | h_b k_c \rangle \langle u_b v_c | U | h_b k_c \rangle \langle h_b k_c | U^\dagger \otimes | v_d \rangle \langle v_d | \right)
\]

\[
= \sum_{hk} \left( \langle h_b k_c | U^\dagger \rho_{bc} U | h_b k_c \rangle \right)^2 .
\]

Then, the affinity between the pre- and post-measurement state is computed as

\[
A(\rho_{ab} \otimes \rho_{cd}, \Pi^{bc})(\rho_{ab} \otimes \rho_{cd}) = \text{Tr} \sqrt{\rho_{ab} \otimes \rho_{cd} \Pi^{bc}} \left( \sqrt{\rho_{ab} \otimes \rho_{cd}} \right)
\]

\[
= \sum_{iujhk} s_i^2 s_u^2 r_j^2 r_h^2 \langle h_b k_c | U^\dagger | i_b j_c \rangle \langle i_b j_c | U | h_b k_c \rangle \langle u_b v_c | U | h_b k_c \rangle \langle h_b k_c | U^\dagger \otimes | v_d \rangle \langle v_d | \right)
\]

\[
= \sum_{hk} \left( \langle h_b k_c | U^\dagger \rho_{bc} U | h_b k_c \rangle \right)^2 .
\]

The nonbilocal measure for pure state is

\[
N_A (|\Psi_{ab} \rangle \otimes |\Psi_{cd} \rangle) = \max_{\Pi^{bc}} D_A \left( \sqrt{\rho_{ab} \otimes \rho_{cd}}, \Pi^{bc} \left( \sqrt{\rho_{ab} \otimes \rho_{cd}} \right) \right)
\]

\[
= 1 - \min_{\Pi^{bc}} A(\rho_{ab} \otimes \rho_{cd}, \Pi^{bc}(\rho_{ab} \otimes \rho_{cd}))
\]

\[
= 1 - \min_{\Pi^{bc}} \sum_{hk} \left( \langle h_b k_c | U^\dagger \rho_{bc} U | h_b k_c \rangle \right)^2 ,
\]

where the optimization is over all von Neumann measurements given in Eq. (32), leaving the marginal state \( \rho^{bc} \) invariant. That is,

\[
\rho^{bc} = \sum_{hk} \langle h_b k_c | U^\dagger \rho_{bc} U | h_b k_c \rangle U | h_b k_c \rangle \langle h_b k_c | U^\dagger
\]

is a spectral decomposition of \( \rho^{bc} \) since \( \{ U | h_b k_c \} \) is an orthonormal base. Comparing the above equation with Eq. (31), we obtained

\[
N_A (|\Psi_{ab} \rangle \otimes |\Psi_{cd} \rangle) = 1 - \sum_{i,j} s_i^4 r_j^4 ,
\]

(33)
Hence, the theorem is proved. It is worth mentioning that the affinity-based nonbilocal measure for pure state is equal to the Hellinger distance and Hilbert–Schmidt norm-based nonbilocal measures [38, 39].

References

1. Nielsen, M., Chuang, I.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2010)
2. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47, 777 (1935)
3. Schrödinger, E.: Discussion of probability relations between separated systems. Proc. Camb. Philos. Soc. 31, 555 (1935)
4. Bell, J.S.: On the Einstein Podolsky Rosen paradox. Physics 1, 195 (1964)
5. Werner, R.F.: Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model. Phys. Rev. A 40, 4277 (1989)
6. Ollivier, H., Zurek, W.H.: Quantum discord: a measure of the quantumness of correlations. Phys. Rev. Lett. 88, 017901 (2001)
7. Buscemi, F.: All entangled quantum states are nonlocal. Phys. Rev. Lett. 108, 200401 (2012)
8. Almeida, M.L., Pironio, S., Barrett, J., Toth, G., Acin, A.: Noise robustness of the nonlocality of entangled quantum states. Phys. Rev. Lett. 99, 040403 (2007)
9. Luo, S., Fu, S.: Measurement-induced nonlocality. Phys. Rev. Lett. 106, 120401 (2011)
10. Branciard, C., Gisin, N., Pironio, S.: Characterizing the nonlocal correlations of particles that never interacted. Phys. Rev. Lett. 104, 170401 (2010)
11. Branciard, C., Rosset, D., Gisin, N., Pironio, S.: Bilocal versus nonbilocal correlations in entanglement-swapping experiments. Phys. Rev. A 85, 032119 (2012)
12. Fritz, T.: Beyond Bell’s theorem: correlation scenarios. New J. Phys. 14, 103001 (2012)
13. Fritz, T.: Beyond Bell’s theorem II: scenarios with arbitrary causal structure. Commun. Math. Phys. 341, 391–434 (2016)
14. Wood, C.J., Spekkens, R.W.: The lesson of causal discovery algorithms for quantum correlations: causal explanations of Bell-inequality violations require fine-tuning. New J. Phys. 17, 033002 (2015)
15. Henson, J., Lal, R., Pusey, M.F.: Theory-independent limits on correlations from generalized Bayesian networks. New J. Phys. 16, 113043 (2014)
16. Chaves, R., Brask, J.B., Brunner, N.: Device-independent tests of entropy. Phys. Rev. Lett. 115, 110501 (2015)
17. Tavakoli, A., Skrzypczyk, P., Cavalcanti, D., Acín, A.: Nonlocal correlations in the star-network configuration. Phys. Rev. A 90, 062109 (2014)
18. Tavakoli, A.: Quantum correlations in connected multipartite Bell experiments. J. Phys. A Math. Theor. 49, 145304 (2016)
19. Tavakoli, A.: Bell-type inequalities for arbitrary noncyclic networks. Phys. Rev. A 93, 030101 (2016)
20. Chaves, R.: Polynomial Bell inequalities. Phys. Rev. Lett. 116, 010402 (2016)
21. Rosset, D., Branciard, C., Barnea, T.J., Putz, G., Brunner, N., Gisin, N.: Nonlinear Bell inequalities tailored for quantum networks. Phys. Rev. Lett. 116, 010403 (2016)
22. Gisin, N., Mei, Q.X., Tavakoli, A., Renou, M.O., Brunner, N.: All entangled pure quantum states violate the bilocality inequality. Phys. Rev. A 96, 020304 (2017)
23. Palazuelos, C.: Superactivation of quantum nonlocality. Phys. Rev. Lett. 109, 190401 (2012)
24. Cavalcanti, D., Almeida, M.L., Scarani, V., Acin, A.: Quantum networks reveal quantum nonlocality. Nat. Commun. 2, 184 (2011)
25. Cavalcanti, D., Rabelo, R., Scarani, V.: Nonlocality tests enhanced by a third observer. Phys. Rev. Lett. 108, 040402 (2012)
26. Masanes, L., Liang, Y.-C., Doherty, A.C.: All bipartite entangled states display some hidden nonlocality. Phys. Rev. Lett. 100, 090403 (2008)
27. Dakic, B., Vedral, V., Brukner, C.: Necessary and sufficient condition for nonzero quantum discord. Phys. Rev. Lett. 105, 190502 (2010)
28. Luo, S., Fu, S.: Geometric measure of quantum discord. Phys. Rev. A 82, 034302 (2010)
29. Piani, M.: Problem with geometric discord. Phys. Rev. A 86, 034101 (2012)
30. Jozsa, R.: Fidelity for mixed quantum states. J. Mod. Opt. 41, 2315 (1994)
31. Luo, S., Zhang, Q.: Informational distance on quantum-state space. Phys. Rev. A 69, 032106 (2004)
32. Bhattacharyya, A.: On a measure of divergence between two statistical populations defined by their probability distributions. Bull. Calcutta Math. Soc. 35, 99 (1943)
33. Muthuganesan, R., Chandrasekar, V.K.: Characterizing nonclassical correlation using affinity. Quantum Inf. Process. 18, 223 (2019)
34. Muthuganesan, R., Chandrasekar, V.K.: Measurement-induced nonlocality based on affinity. Commun. Theor. Phys. 72, 075103 (2020)
35. Muthuganesan, R., Chandrasekar, V.K., Sankaranarayann, R.: Quantum coherence measure based on affinity. Phys. Lett. A 394, 127205 (2021)
36. Girolami, D., Tufarelli, T., Adesso, G.: Characterizing nonclassical correlations via local quantum uncertainty. Phys. Rev. Lett. 110, 240402 (2013)
37. Girolami, D., Souza, A.M., Giovannetti, V., Tufarelli, T., Filgueiras, J.G.: Quantum discord determines the interferometric power of quantum states. Phys. Rev. Lett. 112, 210401 (2014)
38. Zhang, Y., He, K.: Quantifying measurement-induced nonbilocal correlation. Quantum Inf. Process. 20, 248 (2021)
39. Zhang, Y., He, Y., He, K.: Generalization of measurement-induced nonlocality in the bilocal scenario. Int. J Theor. Phys. 60, 2178 (2021)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.