THE CONVOLUTION THEOREM OF HÁJEK AND LE CAM - REVISITED

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Abstract. The present paper establishes convolution theorems for regular estimators when the limit experiment is non-Gaussian or of infinite dimension with sparse parameter space. Applications are given for Gaussian shift experiments of infinite dimension, the Brownian motion signal plus noise model, Levy processes which are observed at discrete times and estimators of the endpoints of densities with jumps. The method of proof is also of interest for the classical convolution theorem of Hájek and Le Cam. As technical tool we present an elementary approach for the comparison of limit experiments on standard Borel spaces.

1. Introduction

The classical convolution theorem of Hájek-Le Cam and Inagaki, see Hájek(1970) and also Inagaki (1970), states that the asymptotic distribution \( Q \) of a sequence of properly normalized regular estimators is a convolution product \( Q = \nu * P \) of the limit distribution \( P \) of the asymptotically efficient estimators. The convolution theorem was first proved under finite dimensional local asymptotic normality (LAN). It is nowadays part of textbooks like Bickel et al (1993), Pfanzagl (1994) and Witting and Müller-Funk (1995), Le Cam and Yang (2000), for instance. The history was summarized by Le Cam, see Yang (1999). Le Cam pointed out that the case of infinite dimensions beyond the Gaussian case is still open in general, see also Le Cam (1994) and the recent discussion of Pöazelberger, Schachermayer and Strasser (2000). It is the aim of the present paper to reconsider the convolution theorem again under very general conditions which cover some results for infinite dimensions. Also we deal with the estimation of infinite dimensional parameters when the parameter space is not full. Based on Basu’s theorem (about ancillary statistics) we present another proof of the convolution which is also of interest for classical situations. We will make some comments about different kind of proofs of the convolution theorem. Bickel et al (1993) gave an analytic proof via characteristic functions, see also Droste and Wefelmeyer (1984). Another proof uses invariant means, see Pfanzagl (1994) or van der Vaart (1988, 1989). In a certain sense this proof is similar to Boll’s proof of the convolution theorem for limit experiments, see Boll (1955) or Strasser (1985), Sect. 38. Other proofs use the Markov Kakutani fix point theorem, see Heyer (1979), Le Cam (1994) or Pöazelberger, Schachermayer and Strasser (2000). A very elegant proof of the convolution theorem runs via limit
experiments, see Le Cam (1994), Theorem 1, Torgersen (1991), p. 401 ff and van der Vaart (1991). Under LAN the sequence of underlying experiments converges weakly to a limit shift experiment \( \{ P \ast \varepsilon_\vartheta : \vartheta \in \Theta \} \). The (normalized) sequence of regular estimators converges to \( Q \ast \varepsilon_\vartheta \) whenever \( \vartheta \) is the true local parameter. Results about the comparison of experiments show that \( \{ P \ast \varepsilon_\vartheta : \vartheta \in \Theta \} \) is more informative than \( \{ Q \ast \varepsilon_\vartheta : \vartheta \in \Theta \} \), see also Section 4. Then a Boll type convolution theorem yields the result \( Q = \nu \ast P \). Our approach is of this type but only uses mostly elementary arguments.

We will make some further comments about the literature. Early convolution theorems for infinite dimensional parameter spaces were considered by Moussatat (1976), Millar (1983, 1985), Strasser (1985) and Le Cam (1994) mostly for cylinder measures. Van der Vaart (1991) gave a proof for Gaussian shifts. Von den Heuvel and Klaassen (1999) and Sen (2000) presented the connection to the Bayesian framework. Schick and Susarla (1990) established an infinite dimensional convolution theorem with application to the Kaplan Meier estimator. Further convolution results about semiparametric estimation can be found in McNeney and Wellner (2000). The connection between risk inequalities, spread inequalities and the convolution theorem was discussed in Pfanzagl (2000). Beran (1997) studied the relationship between bootstrap convergence and the convolution theorem. On page 17 he mentioned that the convolution theorem works analogously to Basu’s theorem which is also central in our proof.

The paper is organized as follows. Section 1 introduces first the main ideas for LAN sequences of experiments and univariate parameters. This part is also of interest for teachers of statistics courses since only elementary tools are required. As mentioned about the proof is based on two steps. In a first step it is pointed out that the asymptotic distributions \( Q \ast \varepsilon_\vartheta \) of a regular sequence of estimators \( T_n : \Omega_n \to \mathbb{R} \) lead to an experiment \( \{ Q \ast \varepsilon_\vartheta : \vartheta \in \Theta \} \) which is a randomization of a limit shift experiment (and less informative). In the second step a comparison of \( \{ Q \ast \varepsilon_\vartheta : \vartheta \in \Theta \} \) with the limit experiment yields the convolution theorem. The main ingredients of the proof are Le Cam’s third Lemma and Basu’s theorem about ancillary statistics. As we will see in our sections 3 and 4 this methodology and the present method of proof also works for the asymptotically equivariant estimation of parameters of infinite dimension. In this case the sample space and the parameter space typically do not coincide and we have no full shift experiments. Thus we take care about the convolution theorem when only a sparse parameter space is available. Two applications are given for estimators of infinite dimensional parameters. Example 4 deals with the convolution theorem for the signal plus noise model with Brownian noise on \( C[0,1] \). Example 5 works for non Gaussian Levy processes.

Section 2 discusses variance inequalities for unbiased estimators which may lead to asymptotic convolution theorems whenever the asymptotic distributions are Gaussian. The core of the paper is the comparison of limit experiments in Section 3. Under mild regularity assumptions a convolution theorem is established for limit experiments. The proof of the convolution theorem is complete if we are now citing Le Cam’s theorem that limit experiments are more informative than the limit \( \{ Q \ast \varepsilon_\vartheta : \vartheta \in \Theta \} \) of the estimators. This part is the topic of Section 4. Here we give an elementary proof of Le Cam’s theorem for limit experiments on standard Borel spaces. Moreover, the kernels required for the comparison of experiments are constructed which transform one experiment in the other one. Notice that these
kernels are needed in Section 3. An interesting example is given when the endpoints of a distribution with jumps of the densities have to be estimated.

Let us start with our introductory example. Let \((P_t)\), be a family of distributions on \((\Omega, \mathcal{A})\) given by a real parameter \(t\). For an increasing sample size of \(n\) independent observations we like to estimate the parameter \(t\) of the model. Under regularity assumptions different estimators will be compared locally around a fixed point \(t = \vartheta_0\). Introduce by \(t = \vartheta_0 + \vartheta_\sqrt{n}\) a further local parameter \(\vartheta \in \mathbb{R}\) and consider the local model

\[
P_{n, \vartheta} := P^n_{\vartheta_0 + \vartheta_\sqrt{n}}
\]
on \((\Omega^n, \mathcal{A}^n)\) with independent replications.

**Regularity of the model.** Let \(\Theta \subset \mathbb{R}\) with \(0 \in \Theta\) be the local parameter space. Suppose that the model is local asymptotically normal (LAN), i.e. there exists some \(\sigma > 0\) and a central sequence of random variables \(X_n: \Omega^n \to \mathbb{R}\) with

\[
\log \frac{dP_{n, \vartheta}}{dP_{n, 0}} - [\vartheta X_n - \vartheta^2 \sigma^2 / 2] \to 0
\]
in \(P_{n, 0}\)-probability for \(\vartheta \in \Theta\), where

\[
\mathcal{L}(X_n|P_{n, 0}) \to N(0, \sigma^2)
\]
holds as \(n \to \infty\). Recall that LAN implies

\[
\mathcal{L}(X_n|P_{n, \vartheta}) \to N(\vartheta \sigma^2, \sigma^2)
\]
weakly under the parameter \(\vartheta\) and the experiments

\[
(\Omega^n, \mathcal{A}^n, \{P_{n, \vartheta}\}) \to (\mathbb{R}, \mathcal{B}, \{N(\vartheta \sigma^2, \sigma^2)\})
\]
are weakly convergent in the sense of Le Cam, see also Strasser (1985).

Consider now a sequence of estimators \(T_n: \Omega^n \to \mathbb{R}\) of the parameter \(t\). It is well known that the asymptotic efficiency of \(T_n\) typically implies that the linearization

\[
\sqrt{n}(T_n - \vartheta_0) - \frac{X_n}{\sigma^2} \to 0
\]
holds in \(P_{n, 0}\)-probability. Recall that local asymptotic minimax estimators and Fisher efficient estimators at \(\vartheta_0\) have this property. Recall that \(T_n\) is called Fisher efficient if \(n \text{Var}_{\vartheta_0}(T_n)\) reaches the asymptotic Cramer-Rao bound of the experiment \(\{P_{n, \vartheta}\}\).

**Definition 1.** A sequence of estimators \(T_n\) is called \(\Theta\)-regular if

\[
\mathcal{L}(\sqrt{n}(T_n - (\vartheta_0 + \vartheta / \sqrt{n}))|P_{n, \vartheta}) \to Q
\]
converges weakly as \(n \to \infty\) for all \(\vartheta \in \Theta\) where \(Q\) does not depend on \(\vartheta\).
The classical Hájek-Le Cam convolution theorem states that the asymptotic distribution $Q$ of a sequence of $\mathbb{R}$-regular estimators is more spread out than $N(0, \frac{1}{\sigma^2})$. Although various different proofs exist, also in textbooks, we will present a slightly different proof which indicates the crucial steps of more general convolution theorems with restricted parameter sets $\Theta$ of infinite dimensions and also for non-Gaussian limit experiments. Our method of proof will first be presented for the classical one-dimensional LAN case.

**Theorem 2.** Suppose that LAN holds for the parameter set $\Theta \subset \mathbb{R}$ where the closure of $\Theta$ has non empty interior, $\overline{\Theta} \neq \emptyset$. Let $T_n$ be a sequence of $\Theta$-regular estimators, i.e. $\mathcal{L}(\sqrt{n}(T_n - \vartheta_0)|P_{n,\vartheta}) \rightarrow Q * \varepsilon_\vartheta$ holds for each $\vartheta \in \Theta$. Then

(a) For each $\vartheta$ the vector of random variables

$$ (X_n, \sqrt{n}(T_n - \vartheta_0) - \frac{X_n}{\sigma^2}) $$

are convergent in distribution to a pair of independent random variables $(X, Z)$ on $\mathbb{R}^2$. Let $\nu$ denote the weak limit distribution of the second component $Z$ under $\vartheta = 0$, i.e.

$$ \mathcal{L}(\sqrt{n}(T_n - \vartheta_0) - \frac{X_n}{\sigma^2}|P_{n,0}) \rightarrow \nu. $$

Then (1.9) also holds under $P_{n,\vartheta}$ with the same limit $\nu$ independent of $\vartheta \in \Theta$.

(b) For fixed $\vartheta$ the limit distribution of (1.8) is given by

$$ N(\vartheta\sigma^2, \sigma^2) \otimes \nu $$

and the convolution theorem

$$ Q * \varepsilon_\vartheta = \nu * N(\vartheta, \frac{1}{\sigma^2}) $$

holds for all $\vartheta \in \Theta$.

(c) The sequence $T_n$ is asymptotically efficient at $\vartheta_0$, i.e. (1.10) holds in $P_{n,0}$ probability, iff the weak limit $Q * \varepsilon_\vartheta$ of $\sqrt{n}(T_n - \vartheta_0)$ w.r.t. $P_{n,\vartheta}$ is $N(\vartheta\sigma^2, \sigma^2)$ at least for one $\vartheta \in \Theta$.

An elegant proof of the convolution theorem is based on the third Lemma of Le Cam which is summarized for LAN experiments.

**Lemma 3.** Suppose that $S_n : \Omega^n \rightarrow \mathbb{R}$ is a further sequence of statistics with weak limit law

$$ \mathcal{L}((X_n, S_n)|P_{n,0}) \rightarrow \mu_0 $$

on $\mathbb{R}^2$. Then (1.12) is weakly convergent also under each sequence $P_{n,\vartheta}$ to the distribution $\mu_\vartheta$ on $\mathbb{R}^2$ with $\mu_\vartheta \ll \mu_0$ and

$$ \frac{d\mu_\vartheta}{d\mu_0}(x,y) = \exp(\vartheta x - \frac{1}{2} \vartheta^2\sigma^2). $$
Observe that formula (1.13) is a direct consequence of Le Cam’s third lemma of the form given by Hájek, Šidák and Sen (1999), 7.1.4 and the references given in the proof of Lemma 23 below. After these preparations we will indicate the crucial steps of the proof of the convolution theorem.

Proof. (of Theorem 2)

Step 1. According to (1.3) and (1.7) the pair of random variables (1.9) is tight on $\mathbb{R}^2$ for $\vartheta = 0$. Then we find a subsequence $\{m\} \subset \mathbb{N}$ such that

$$ (1.14) \quad (X_m, \sqrt{m} (T_m - \vartheta_0)) \to (X, T) $$

is distributional convergent to some $(X, T)$ under $P_{n,0}$ along our subsequence. If $\mu_0$ denotes the distribution of $(X, T)$ under $\vartheta = 0$ then Le Cam’s third Lemma implies that the distributional convergence along $\{m\}$ of (1.14) also holds under $P_{n,\vartheta}$ with limit law $\mu_\vartheta$ given by (1.13) for each $\vartheta \in \Theta$. Let $\Pi_i : \mathbb{R}^2 \to \mathbb{R}$ denote the projections on the coordinates for $i = 1, 2$. By (1.3) and (1.7) we have

$$ (1.15) \quad L(\Pi_1 | \mu_\vartheta) = N(\vartheta, \frac{1}{\sigma^2}) \quad \text{and} \quad L(\Pi_2 | \mu_\vartheta) = Q \ast \varepsilon_\vartheta $$

for all $\vartheta \in \Theta$.

Step 2. By Neyman’s criterion about sufficiency the projection $\Pi_1$ is $\{\mu_\vartheta : \vartheta \in \Theta\}$ sufficient, confer (1.13). Within the language of comparison experiments this means that

$$ \{N(0, \frac{1}{\sigma^2}) \ast \varepsilon_\vartheta : \vartheta \in \Theta\} $$

is more informative as $\{Q \ast \varepsilon_\vartheta : \vartheta \in \Theta\}$. For full parameter spaces $\Theta = \mathbb{R}$ the convolution theorem (1.11) follows from Boll’s convolution theorem of shift experiments. Boll’s theorem requires an analytic proof which sometimes uses fixed point methods. We will substitute this part by more elementary arguments which give us the full result also when $\Theta$ is not full. For the details we refer to the proof of Theorem 12.

The proof runs as follows. It is easy to see that

(i) $\Pi_1$ is sufficient and boundedly complete for $\{\mu_\vartheta : \vartheta \in \Theta\}$. Observe that the densities of $\mu_\vartheta$ can be extended to the parameter space $\Theta$ and that the bounded completeness has to be checked for $\Theta$ only by continuity arguments, see Remark 15. Without restrictions we may assume that $0 \in \Theta$ holds. Otherwise the family (1.13) can be shifted due to the translation invariance of the Gaussian shift family. Also we can find an open set $\Theta_1 \subset \Theta$ with $\Theta_1 + \Theta_1 \subset \Theta$ where $\{\mu_\vartheta : \vartheta \in \Theta_1\}$ is boundedly complete as required in the proof of Theorem 12.

(ii) It is not hard to check that $\Pi_2 - \frac{\Pi_1}{\sigma^2}$ is ancillary w.r.t. $\Theta_1$. Details can be found in (3.10)-(3.12) of the proof of Theorem 12 below. Consequently, Basu’s theorem, see Pfanzagl (1994), implies that

$$ (1.16) \quad \Pi_1 \text{ and } \Pi_2 - \frac{\Pi_1}{\sigma^2} $$
are independent under $\mu_0$ for each $\vartheta \in \Theta_1$, i.e. (1.10) and the convolution theorem (1.11) hold.

By the consideration of Fourier transform we see that the factor $\nu$ of (1.11) is unique. If we now turn to (1.10), we see that the limit distributions (1.14) must be the same $\mu_0$ for all subsequences $\{m\}$ when (1.14) holds. By tightness this fact proves the convergence of (1.14) along $n \in \mathbb{N}$ first for $\vartheta = 0$ and by Lemma 3 for all $\vartheta$. These arguments finish the proof of part (a) and (b). Part (c) is now trivial since the efficiency of $T_n$ corresponds to $\nu = \varepsilon_0$. \hfill \Box

Along these lines of the proof more general convolution theorems will be established.

- The sample space and the parameter space may be of infinite dimension.
- The parameter set $\Theta$ may be a restricted set (not a full vector space).
- If we turn to limit experiments other distributions than Gaussian distributions are allowed.

The extended convolution theorems of Section 3 have various application. As application we will consider two examples with parameter and sample spaces of infinite dimension which can be treated by the new convolution theorem.

**Example 4.** [Signal plus noise model with fixed finite sample size] Let $B_1, \ldots, B_n$ denote $n$ independent standard Brownian motions with values in $C[0, 1]$. Our observations are the processes

\[(1.17) \quad X_i(t) = B_i(t) + \int_0^t \vartheta(u)du, \quad 0 \leq t \leq 1, \quad 1 \leq i \leq n,\]

where the parameter $\vartheta$ belongs to a class square integrable functions $\Theta \subset L^2[0, 1]$ with respect to the uniform distribution on $[0, 1]$. We like to estimate the signal $h(\vartheta)$,

\[(1.18) \quad h : \Theta \rightarrow C[0, 1], \quad h(\vartheta)(t) = \int_0^t \vartheta(u)du\]

by estimators $T(X_1, \ldots, X_n)$,

\[(1.19) \quad T : C[0, 1]^n \rightarrow C[0, 1].\]

The natural estimator is $S$,

\[(1.20) \quad S(X_1, \ldots, X_n) = \frac{1}{n} \sum_{i=1}^n X_i.\]

At a fixed point $\vartheta_0 \in \Theta$ this estimator can be compared with competing estimators $T$. For this purpose introduce in addition to the global parameter $\vartheta_0$ another local parameter $\eta \in L^2[0, 1]$ by

\[(1.21) \quad \vartheta = \vartheta_0 + \frac{\eta}{\sqrt{n}}.\]

Two different classes of estimator will be studied.

(a) (Equivariant estimation). Suppose that the estimator fulfills
(1.22) \[ \mathcal{L} \left( \sqrt{n}(T - h(\vartheta_0)) \bigg| \vartheta_0 + \frac{\eta}{\sqrt{n}} \right) = \mathcal{L} \left( \sqrt{n}(T - h(\vartheta_0)) \bigg| \vartheta_0 \right) * \varepsilon_{\eta} \]

for \( \eta \) with \( \vartheta_0 + \frac{\eta}{\sqrt{n}} \in \Theta \). Under mild assumptions about the size of \( \Theta \) the convolution theorem

(1.23) \[ \mathcal{L} \left( \sqrt{n}(T - h(\vartheta_0)) \bigg| \vartheta_0 \right) = \nu * \mathcal{L} \left( \sqrt{n}(S - h(\vartheta_0)) \bigg| \vartheta_0 \right) \]

holds on \((C[0,1])\), see Example 17 below for details.

(b) (Unbiased estimation). Under various assumption it can be shown that \( S \) is the best unbiased estimator which is Fisher efficient in the sense that \( S \) attains the nonparametric Cramer-Rao bound, see Janssen (2003) for related results.

Recall that a real Levy process \((Z_t)_{t \geq 0}\) is a stochastically continuous process with independent stationary increments.

**Example 5.** Let \((Z_t)_{t \geq 0}\) be a Levy process with absolutely continuous distributions \( \mathcal{L}(Z_t) \ll \lambda \) for all \( t > 0 \). At a given sequence of discrete times \( 0 < t_1 < t_2 < \ldots \) the Levy process serves as error distribution of our observations

(1.24) \[ X = (Z_{t_i} + \vartheta_i)_{i \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \]

with unknown parameters \( \vartheta = (\vartheta_i)_{i \in \mathbb{N}} \in \Theta \subset \mathbb{R}^\mathbb{N} \). We are going to estimate the parameter \( \vartheta \) by equivariant estimators \( T(X), T : \mathbb{R}^\mathbb{N} \rightarrow \mathbb{R}^\mathbb{N} \), with

(1.25) \[ \mathcal{L}(T(X)|\vartheta) = \mathcal{L}(T(X)|0) * \varepsilon_{\vartheta} \]

for \( \vartheta \in \Theta \). Under the assumptions the identity \( S = \text{id} \) is the best equivariant estimator in the sense that the convolution theorem

(1.26) \[ \mathcal{L}(T(X)|\vartheta) = \nu * \mathcal{L}(S(X)|\vartheta) \]

holds for all \( \vartheta \in \Theta \). The details are presented in Example 18.

**Definition 6.** Suppose that there exist some \( \vartheta^0 = (\vartheta^0_i)_{i \in \mathbb{N}} \in \mathbb{Q}^\mathbb{N} \), such that

(1.27) \[ \{(\vartheta_i)_{i \in \mathbb{N}} \in \mathbb{Q}^\mathbb{N} : \vartheta_i = \vartheta^0_i \text{ finally}\} \subset \Theta \]

holds.

2. **Unbiased estimation, variance inequalities and preliminary versions of the convolution theorem**

In this section we will start with finite sample results for locally unbiased estimation of vector valued statistical functionals. It is shown that variance inequalities within convex classes of estimators are linked to preliminary versions of the convolution theorem. Moreover, it is shown that the existence of a sequence of locally minimum variance estimators already imply a convolution theorem whenever the underlying estimators are jointly asymptotically normal.

Let \( g : \Theta \rightarrow W \) denote a statistical functional with values in a real vector space \( W \). Suppose that \( I \subset \{ f : W \rightarrow \mathbb{R} \} \) is a set of linear functions and let \( \mathcal{B} := \sigma(f : f \in I) \) denote the \( \sigma \)-field generated by \( I \) on \( W \). Let \( E = (\Omega, \mathcal{A}, \{ P_{\vartheta} : \vartheta \in \Theta \}) \)
Suppose that (b) holds. Then for each estimator \( S \) for all \( x \) as linear function. Suppose that \( \Omega = V \).

Example 8. The following classes \( \mathcal{K} \) of estimators have the extended convexity property:

(a) The estimators \( T \) where \( f(T) \) is unbiased for our functional \( g \) and all \( \vartheta \in \Theta \).

(b) Let \( \Theta \) be a subset of a linear space \( V \) such that \( g \) has an extension \( g : V \to W \) as linear function. Suppose that \( \Omega = V \) and let \( T \) be the strictly equivariant estimators (for \( g \)), i.e.

\[
T(x + \vartheta) = T(x) + g(\vartheta)
\]

for all \( x \in V \) and all \( \vartheta \in \Theta \).
Lemma [7] is a preliminary version of a convolution theorem which is expressed by the variance decomposition. The convolution theorem

\[(2.2) \quad \mathcal{L}(T|\vartheta_0) = \mathcal{L}(S|\vartheta_0) * \mathcal{L}(T-S|\vartheta_0)\]

holds whenever \((f(S))_f\) and \((f(T-S))_f\) are independent under \(\vartheta_0\) for each \(f \in I\). This will not be true in general but if the vector \((f(S), f(T-S))\) is jointly Gaussian then the components are independent since they are uncorrelated. If \(I\) is a linear space of functions the convolution theorem for the \(f\)-marginals then implies \((2.2)\). This simple observation leads to an asymptotic convolution theorem of asymptotically normal locally unbiased estimators.

Consider a sequence of experiments

\[(2.3) \quad E_n = (\Omega_n, \mathcal{A}_n, \{P_{n,\vartheta} : \vartheta \in \Theta\})\]

and a sequence \(g_n : \Theta \to W\) of statistical functionals. For each \(n\) let \(K_n\) be a class of unbiased estimators of \(g_n\) at \(\vartheta_0\) so that assumption (A) holds. Suppose as in Lemma [7] (a) that \(S_n\) is a minimum variance estimator at \(\vartheta_0\) in \(K_n\) for each \(n\).

**Theorem 9.** Suppose that \(I\) is a linear subspace of functions on \(W\). Let \(T_n \in K_n\) be a competing sequence of estimators such that the joint distribution

\[(2.4) \quad a_n(f(S_n) - E_{\vartheta_0}(f(S_n)), f(T_n) - E_{\vartheta_0}(f(T_n))) \to (f(S), f(T))\]

is weakly convergent under \(\vartheta_0\) to a centered Gaussian random variable \((f(S), f(T))\) for all \(f \in I\), where \(a_n > 0\) denotes a normalizing sequence. Assume that \(S\) and \(T\) are \(W\)-valued random variables. Let in addition

\[(2.5) \quad \text{Var}_{\vartheta_0}(a_n f(S_n)) \to \text{Var}(f(S)) \text{ and } \text{Var}_{\vartheta_0}(a_n f(T_n)) \to \text{Var}(f(T))\]

hold for all \(f \in I\) as \(n \to \infty\). Then the convolution theorem in the sense of equation (2.2) holds for the asymptotic distributions on \(W\).

**Proof.** In a first step the convolution theorem

\[(2.6) \quad \mathcal{L}(f(T)|\vartheta_0) = \mathcal{L}(f(S)|\vartheta_0) * \mathcal{L}(f(T-S)|\vartheta_0)\]

is proved for all univariate marginals given by \(f \in I\). For this purpose it is enough to show that

\[(2.7) \quad \text{Cov}(f(S), f(T-S)) = 0\]

since \((f(S), f(T)-f(S))\) is Gaussian. On a new probability space we may find random variables \(X_n, X, Y_n\) and \(Y\) with distributions

\[\mathcal{L}(X_n, Y_n) = \mathcal{L}(a_n(f(T_n) - E(f(T_n))), a_n(f(S_n) - E(f(S_n))))\]

and

\[\mathcal{L}(X, Y) = \mathcal{L}(f(T), f(S))\]

with \((X_n, Y_n) \to (X, Y)\) almost surely, see Dudley (1989), p.325. Our assumptions together with (2.5) imply by Vitali’s theorem the \(L_2\)-convergence of \(X_n \to X\) and \(Y_n \to Y\). Thus

\[E((X_n - Y_n)Y_n) \to E((X - Y)Y) = \text{Cov}(f(S), f(T-S))\]

holds. Since \(f(S_n)\) is a minimum variance estimator the covariance principle of Lemma [7] implies

\[E((X_n - Y_n)Y_n) = 0\]

for each \(n\) and (2.7) holds. Since \(I\) is a linear space the Cramer Wold device then implies via (2.6) the convolution equation (2.2). □
Remark 10. Under the assumptions of Theorem 9 the univariate marginals \( f(S) \) and \( f(T - S) \) are independent for all \( f \in I \).

Example 11. Let \( P \) be a set of probability measures on a measurable space \((\Omega, \mathcal{A})\) and let \( X \) be a \( W \)-valued random variable. Suppose that \( g : P \to W \) is the mean functional in the sense that \( f(g(P)) = E_P(f(X)) \) holds for all \( f \in I \) and every \( P \in P \). We like to estimate \( g_n(P^n) := g(P) \) for \( P_n := \{P^n : P \in P\} \). The model is given by independent copies \( X_1, X_2, \ldots \) of \( X \) with unknown law \( \mathcal{L}(X_1, \ldots, X_n) = P^n \) for each \( n \). Consider the class \( \mathcal{K}_n \) of \( W \)-valued unbiased estimators \( T_n = T_n(X_1, \ldots, X_n) \) with \( E_P(f(T_n)) = f(g(P)) \) for all \( f \in I \) and all \( P \in P \). Let \( P_0 \in P \) be fixed. Let \( I \) always denote a linear space of functions.

(a) Let \( P \) be a convex set such that the marginal distributions \( \{\mathcal{L}(f|P) : P \in P\} \) are complete for each \( f \in I \). Then the \( W \)-valued mean \( S_n = \frac{1}{n} \sum_{i=1}^n X_i \) is a minimum variance estimator in \( \mathcal{K}_n \) at \( P_0 \) in the sense of Lemma 7(a). Recall that the order statistics of \( f(X_1), \ldots, f(X_n) \) are sufficient and complete, see Pfanzagl (1994), Sect. 1.5. Thus the theorem of Lehmann and Scheffe can be applied.

(b) Let \( T_n \) be a competing sequence of estimators in \( \mathcal{K}_n \) which admit a linearization at \( P_0 \) with

\[
(2.8) \quad n \text{Var}_{P_0} \left( f \left( T_n - \sum_{i=1}^n a_{ni} X_i \right) \right) \to 0
\]
as \( n \to \infty \) for all \( f \). Assume that \( a_{ni} \) are reals with

\[
(2.9) \quad n \sum_{i=1}^n a_{ni}^2 \to \kappa > 0 \quad \text{and} \quad n^\frac{1}{2} \max_{1 \leq i \leq n} |a_{ni}| \to 0 \quad \text{as} \quad n \to \infty.
\]

Then together with the Cramer-Wold device the central limit theorem of Lindeberg and Feller implies that

\[
(2.10) \quad (\sqrt{n} (f(S_n) - g(P_0)), \sqrt{n} (f(T_n) - g(P_0))) \to \left( Y_f^{(S)}, Y_f^{(T)} \right)
\]
is weakly convergent under \( P_0^n \) to a centered Gaussian random variable for all \( f \in I \). On the space \((\mathbb{R}^I, \mathcal{B}^I)\) the convolution theorem

\[
(2.11) \quad \mathcal{L} \left( \left( Y_f^{(T)} \right)_{f \in I} \right) = \mathcal{L} \left( \left( Y_f^{(S)} \right)_{f \in I} \right) \ast \mathcal{L} \left( \left( Y_f^{(T)} - Y_f^{(S)} \right)_{f \in I} \right)
\]
holds. If in addition \( Y_f^{(S)} = f(S), Y_f^{(T)} = f(T) \) arise from \( W \)-valued random variables \( S \) and \( T \) then the convolution theorem holds on \( W \).

(c) Let \( P \) denote all continuous distributions on the unit interval \([0,1]\). We estimate the distribution function \( g(P) = F \) of \( P \) in the space of cad lag functions \( W = D[0,1] \) on \([0,1]\). The set \( I \) is the linear space generated by the projections \( h \mapsto h(t) \) for \( t \in [0,1] \). The minimum variance estimator is the empirical distribution function \( S_n = F_n \). The Gaussian limit process \( S \) on \( W \) is the transformed Brownian bridge \( t \mapsto B_0(F(t)) \), where \( B_0(\cdot) \) denotes a standard Brownian bridge. It can be shown that under our regularity assumptions the limit process of \( t \mapsto \sqrt{n} (T_n(t) - F(t)) \) can be realized by some process \((T(t))_t\) in \( D[0,1] \) and under the conditions of part (b) the convolution theorem holds for \( T \) and \( S \) on \( D[0,1] \).
3. THE CONVOLUTION THEOREM FOR LIMIT EXPERIMENTS

In general there will not be finite sample optimal estimators (or they turn out to be unknown) which may serve as benchmark for the underlying sequence \( T_n \). At this stage an asymptotic solution is presented within the limit experiment where a convolution theorem can be presented under fairly general condition.

In a first step linear functionals are estimated where the parameter space \( \Theta \subset V_0 \) is part of a vector space \( V_0 \). Below let \( V, W \) always denote measurable vector spaces with \( \sigma \)-fields \( \mathcal{B}(V) \) and \( \mathcal{B}(W) \) where \( (W, \mathcal{B}(W)) \) is a standard Borel space (i.e. \( W \) is a measurable set of a polish space and \( \mathcal{B}(W) \) is the Borel \( \sigma \)-field).

In contrast to the finite dimensional case the parameter space and the sample spaces do not coincide, see Example 4 where \( V_0 = L^2[0, 1] \) is a Hilbert space and \( V = C[0, 1] \) is the path space of Brownian motion. However, \( V_0 \) can be embedded in \( V \). Assume that there exists a linear injective function

\[
h : V_0 \longrightarrow V
\]

and let \( f : V \rightarrow W \) be a linear measurable function. Then we are going to estimate the functional \( g \)

\[
g : \Theta \longrightarrow W , \quad g := f \circ h|_{\Theta}
\]

Under regularity assumptions the limit experiment of \( E_n, \) see (2.3), has the form

\[
E = (V, \mathcal{B}(V), \{P * \varepsilon_{h(\vartheta)} : \vartheta \in \Theta\})
\]

for some distribution \( P \) on \((V, \mathcal{B}(V))\), see Section 4 for the notion of weak convergence of experiments. After an appropriate normalization the limit distributions of estimators \( T_n : V \rightarrow W \) under \( P_{n, \vartheta} \) often have the form \( Q * \varepsilon_{g(\vartheta)} \) where \( Q \) is some distribution \( Q \) on \((W, \mathcal{B}(W))\). This leads to the experiment

\[
F = (W, \mathcal{B}(W), \{Q * \varepsilon_{g(\vartheta)} : \vartheta \in \Theta\}) .
\]

Under mild regularity conditions a convolution theorem holds as we will see below. Typically \( F \geq F \) is more informative in the sense of Le Cam. This means that \( F \) is a randomization of \( E \). Under additional assumptions this can be expressed via kernels. Suppose now that there exists a kernel \( K \)

\[
K : V \times \mathcal{B}(W) \longrightarrow [0, 1] , \quad (x, B) \longmapsto K(x, B)
\]

with

\[
Q * \varepsilon_{g(\vartheta)}(\cdot) = K \left( P * \varepsilon_{h(\vartheta)} \right)(\cdot) := \int K(x, \cdot) P * \varepsilon_{h(\vartheta)}(dx)
\]

for all \( \vartheta \in \Theta \). The construction of kernels is done in Section 4 for standard Borel spaces.

Let \( \Pi_1 : V \times W \rightarrow V \) and \( \Pi_2 : V \times W \rightarrow W \) denote the projections. Then we find a distribution \( \mu_{\vartheta} \) on \((V \times W, \mathcal{B}(V) \otimes \mathcal{B}(W))\) with

\[
\mathcal{L}(\Pi_1|_{\mu_{\vartheta}}) = P * \varepsilon_{h(\vartheta)} , \quad \mathcal{L}(\Pi_2|_{\mu_{\vartheta}}) = Q * \varepsilon_{g(\vartheta)}
\]

and conditional law \( \mathcal{L}(\Pi_2|\Pi_1 = x) = K(x, \cdot) \). For \( A \in \mathcal{B}(V) \) and \( B \in \mathcal{B}(W) \) we may define

\[
\mu_{\vartheta}(A \times B) = K \times (P * \varepsilon_{h(\vartheta)})(A \times B) := \int_A K(x, B) P * \varepsilon_{h(\vartheta)}(dx).
\]
Theorem 12. Let $\Theta_1 \subset \Theta$ be a subset with $\Theta_1 + \Theta_1 \subset \Theta$ such that \{ $P \cdot \varepsilon_h(\vartheta) : \vartheta \in \Theta_1$ \} is boundedly complete. Then the following assertions hold.

(a) The distribution $\nu := \mathcal{L}(\Pi_2 - f(\Pi_1) | \mu_\vartheta)$ does not depend on the $\vartheta \in \Theta_1$, i.e. $\Pi_2 - f(\Pi_1)$ is an ancillary statistic w.r.t. $\Theta_1$. Moreover, $\Pi_1$ and $\Pi_2 - f(\Pi_1)$ are $\mu_\vartheta$-independent for all $\vartheta \in \Theta_1$.

(b) For each $\vartheta \in \Theta$ we have

$$Q \ast \varepsilon_{g(\vartheta)} = \nu \ast \mathcal{L}(f | P) \ast \varepsilon_{g(\vartheta)}.$$

Remark 13. Under the present assumptions the distribution of the randomized estimator $x \mapsto K(x, \cdot)$ of the functional $g$ of (3.2) is more spread out than the distribution of the point estimator $x \mapsto f(x)$. The kernel $K$ of (3.5) may be chosen to be the convolution kernel $K(x, \cdot) = \nu \ast \varepsilon_{f(x)}$ and it is unique $P \ast \varepsilon_{h(\vartheta)}$ a.e. for all $\vartheta \in \Theta_1$.

Proof. Without restrictions we may assume that $V_0 = V$ and $g = f$ hold with the identity $h$ on $V_0$. Choose $\vartheta, \tau \in \Theta_1$. In a first step we will prove that

$$K(x, \cdot) = K(x + \vartheta, \cdot) \ast \varepsilon_{-f(\vartheta)}, \quad P \ast \varepsilon_\tau \text{ a.e.}$$

holds. To see this we consider $B \in \mathcal{B}(W)$. By (3.6) we obtain

$$Q \ast \varepsilon_{f(\tau)}(B) = Q \ast \varepsilon_{f(\tau + \vartheta)}(B + f(\vartheta)) = \int K(x, B + f(\vartheta)) P \ast \varepsilon_{\tau + \vartheta} (dx) = \int K(x + \vartheta, B + f(\vartheta)) P \ast \varepsilon_\tau (dx).$$

On the other hand $Q \ast \varepsilon_{f(\tau)}(B) = \int K(x, B) P \ast \varepsilon_\tau (dx)$ holds for all $\tau \in \Theta_1$. Bounded completeness now implies equality in (3.10) for the restriction on countable families of $\tau$’s. Since $\mathcal{B}(W)$ is countably generated the result holds for the whole $\sigma$-algebra.

Fix now some $\tau \in \Theta_1$. Then it is easy to see that the projection $\Pi_1$ is sufficient and boundedly complete w.r.t. to the model \{ $\mu_{\vartheta + \tau} : \vartheta \in \Theta_1$ \} of (3.7) on $V \times W$.

Next we will show that $\Pi_2 - f(\Pi_1)$ is an ancillary statistic w.r.t. $\Theta_1$. Consider $A \in \mathcal{B}(V), B \in \mathcal{B}(W)$ and an arbitrary $\vartheta \in \Theta_1$. Taking (3.10) into account we have

$$\mu_\tau \left( \{ \Pi_1 + \vartheta \in A, \Pi_2 + f(\vartheta) \in B \} \right) = \int 1_A(x + \vartheta) K(x, B - f(\vartheta)) P \ast \varepsilon_\tau (dx) = \int 1_A(x + \vartheta) K(x + \vartheta, B) P \ast \varepsilon_\tau (dx) = \mu_{\tau + \vartheta} \left( \{ \Pi_1 \in A, \Pi_2 \in B \} \right).$$

If we now write $\Pi_2 - f(\Pi_1) = (\Pi_2 + f(\vartheta)) - f(\Pi_1 + \vartheta)$ we see that

$$\mathcal{L}(\Pi_2 - f(\Pi_1) | \mu_{\tau + \vartheta}) = \mathcal{L}(\Pi_2 - f(\Pi_1) | \mu_\tau)$$

does not depend on $\vartheta \in \Theta_1$. Basu’s theorem, see Pfanzagl (1994), p.45, then implies the independence of the sufficient and boundedly complete statistic $\Pi_1$ and the ancillary statistic $\Pi_2 - f(\Pi_1)$.

Assertion (b) is then obvious since (3.7) and $\Pi_2 = (\Pi_2 - f(\Pi_1)) + f(\Pi_1)$ holds. Observe that (3.9) holds for all $\vartheta \in \Theta$ when we have equality for at least one $\vartheta$. □
For the rest of this section we like to study applications of Theorem 12. First we give sufficient conditions for bounded completeness of experiments on sample spaces with infinite dimension.

**Lemma 14.** Let $\mathcal{A}_n \subset \mathcal{A}$ denote an increasing sequence of $\sigma$-fields with $\mathcal{A}_0 := \sigma(\mathcal{A}_n : n \in \mathbb{N})$. Assume that there exists an increasing sequence of $\Theta_n \subset \Theta$ of parameter sets such that the $\sigma$-fields $\mathcal{A}_n$ are sufficient and boundedly complete w.r.t. $(\Omega, \mathcal{A}, \{P_\theta : \theta \in \Theta_n\})$ for each $n$. If $\{P_\theta : \theta \in \Theta\} \ll \{P_\theta : \theta \in \bigcup_{n=1}^\infty \Theta_n\}$ is dominated then $\mathcal{A}_0$ is boundedly complete w.r.t. $\{P_\theta : \theta \in \Theta\}$.

**Proof.** It is sufficient to prove boundedly completeness for $\Theta_0 = \bigcup_{n=1}^\infty \Theta_n$. Consider a bounded $\mathcal{A}_0$-measurable function $f : \Omega \rightarrow \mathbb{R}$ with $E_{\theta}(f) = 0$ for all $\theta \in \Theta_0$. By the assumption of sufficiency there exists for each $n \in \mathbb{N}$ a version of the conditional expectation

$$f_n := E_{\theta}(f|\mathcal{A}_n) = E_{P_\theta}(f|\mathcal{A}_n) \ P_\theta \ a.e.$$

which is independent of $\theta \in \Theta_n$. Thus $\int f_n dP_\theta = 0$ holds for all $\theta \in \Theta_n$. We conclude $f_n = 0$ $P_\theta$ a.e. for all $\theta \in \Theta_n$, since the $\sigma$-field $\mathcal{A}_n$ is boundedly complete w.r.t. $\{P_\theta : \theta \in \Theta_n\}$. For fixed $\theta_0 \in \Theta_m$ then $f_n$ vanishes $P_{\theta_0}$ a.e. for all $n \geq m$. On the other hand the martingale convergence theorem implies $f_n \rightarrow f$ $P_{\theta_0}$ a.e. and $f$ vanishes $P_{\theta_0}$ a.e.

**Remark 15.** (a) Let $P \ll \lambda^d$ be absolutely continuous on $\mathbb{R}^d$ without zeros $\hat{P} \neq 0$ of the Fourier transform. If the parameter space $\Theta \subset \mathbb{R}^d$ is dense in $\mathbb{R}^d$ then the shift family $\{P * \varepsilon_\theta : \theta \in \Theta\}$ is boundedly complete. Recall from Hewitt and Ross (1963), Sect. 19 and 20, that $\theta \mapsto dP_{*\varepsilon_\theta}/d\lambda^d$ is continuous in $L_1(\lambda^d)$. Since $\Theta$ is dense our experiment is boundedly complete iff the full shift experiment with $\theta \in \mathbb{R}^d$ is boundedly complete. Under $\hat{P} \neq 0$ the Wiener closure theorem implies the result, see Rudin (1991), Sect. 9.1–9.8.

(b) Boll’s convolution theorem can easily be extended to arbitrary distributions $P$ on $\mathbb{R}^d$ with $\hat{P} \neq 0$ which may not be absolutely continuous. The equality $Q * \varepsilon_\theta = K(P * \varepsilon_\theta)$ for all $\theta \in \mathbb{R}^d$ leads to $Q * N * \varepsilon_\theta = K(P * N * \varepsilon_\theta)$ for the standard normal distribution $N$ on $\mathbb{R}^d$. Then $\hat{Q} \hat{N} = \hat{P} \hat{N}$ implies the convolution theorem.

(c) Suppose that $W$ is a topological vector space and let the shift family $y \rightarrow Q * \varepsilon_y$ be weakly continuous on the closure $\overline{g(\Theta)}$ in $W$. Then the kernel representation (3.9) can be extended to enlarged parameter sets $\hat{\Theta}, \Theta \subset \hat{\Theta} \subset V_0$. A sufficient condition is

$$g(\overline{\Theta}) \subset \overline{g(\Theta)}.$$ 

The proof follows by continuity arguments. Define a new family $Q_\theta := K(P * \varepsilon_{h(\theta)})$ for all $\theta \in \hat{\Theta}$ on $W$. Since $Q'_{\theta} = Q * \varepsilon_{g(\theta)}$ holds for a dense set of parameters $g(\theta), \theta \in \Theta$, the distributions $(Q'_{\theta})_{\theta \in \Theta}$ must belong to a shift family.

**Example 16.** Let $(\Omega, \mathcal{A}, \{P_h : h \in H\})$ a Gaussian shift experiment with likelihood ratio

$$dP_h/dP_0 = \exp \left( L(h) - \frac{||h||^2}{2} \right), \ h \in H$$

see Strasser (1985), chap. 11, where $(H, \langle \cdot, \cdot \rangle)$ denotes a separable real Hilbert space and $h \mapsto L(h)$ is a centered linear Gaussian process w.r.t. $P_0$ and covariance $\text{Cov}_{P_0}(L(h), L(g)) = \langle h, g \rangle$ for all $h, g \in H$. Let $(g_i)_{i \in N}, N \subset \mathbb{N}$, denote a countable
family in $H$ and let $\mathcal{A}_0 = \sigma(L(g_i) : i \in N)$ denote the induced $\sigma$-field on $\Omega$. Then we have:

(a) If $\Theta \subset H$ is a subset of the sub-Hilbert space generated by $(g_i)_{i \in N}$, then $\mathcal{A}_0$ is sufficient for the experiment $E = (\Omega, \mathcal{A}, \{P_h : h \in \Theta\})$.

(b) Consider for each $i \in N$ a subset $\Pi_i \subset \mathbb{R}$ with $\Pi_i \neq \emptyset$. Suppose that the parameter space is rich enough in the sense that

\[
\left\{ \sum_{i \in J} \alpha_i g_i : \alpha_i \in \Pi_i, \ J \subset N, \ J \text{ finite} \right\} \subset \Theta
\]

holds. Then $\mathcal{A}_0$ is boundedly complete for $E$.

In order to give a proof of (a) observe that for each parameter $h \in H$ with Hilbert space representation $h = \sum_{i \in N} \alpha_i g_i$, we may choose densities $L(h) = \sum_{i \in N} \alpha_i L(g_i)$ such that (3.14) becomes $\mathcal{A}_0$-measurable. Thus it is easy to see that all densities (3.14) are $\mathcal{A}_0$-measurable for $h \in \Theta$. Part (b) follows from Lemma 14. Without restriction we may assume that the elements $(g_i)_{i \in N}$ are linearly independent in $H$. Otherwise we may cancel some members and $\mathcal{A}_0$ remains unchanged. For finite $J \subset N$ choose $\mathcal{A}_J = \sigma(g_i : i \in J)$ and $\Theta_J := \{ \sum_{i \in J} \alpha_i g_i : \alpha_i \in \Pi_i \}$. Obviously, $\mathcal{A}_J$ is sufficient w.r.t. $\{P_h : h \in \Theta_J\}$ by a proper choice of the densities $L(h)$. The bounded completeness of $\mathcal{A}_J$ can be proved as follows. The experiment

\[
(\mathbb{R}^J, \mathcal{B}^J, \{ \mathcal{L}((L(g_i))_{i \in J} | h) : h \in \Theta_J \})
\]

is an exponential family of normal distributions with non-singular covariance matrix on $\mathbb{R}^J$. The family is boundedly complete since $\times_{i \in J} \Pi_i$ has an inner point.

**Example 17.** (a) Let $P_\vartheta := \mathcal{L}((X_t(t))_{0 \leq t \leq 1} | \vartheta)$ denote the distribution of the signal plus noise model (1.17) on $C[0, 1]$ and consider the Hilbert space $L_2[0, 1]$ as parameter space. Via the injection $\vartheta \mapsto h(\vartheta)$ given by (1.18) the family is a shift family

\[
P_\vartheta = P_0 * \varepsilon_{h(\vartheta)}, \ \vartheta \in L_2[0, 1],
\]

where $P_0$ denotes the Wiener measure on $C[0, 1]$. The present family (3.16) is a Gaussian shift experiment where the densities (3.14) are determined by the stochastic integral

\[
L(\vartheta) = \int_0^1 \vartheta(s) B_1(ds)
\]

w.r.t. Brownian motion. Observe, that the form of the densities follows from the well-known Girsanov formula. Example 14 provides conditions for the completeness of $\mathcal{B}(C[0, 1])$ w.r.t. $E = \{P_\vartheta : \vartheta \in \Theta\}$. We will only give a simple example. Let

\[
\Theta_1 \subset \Theta
\]

with non-negative rational coefficients. If

\[
\Theta_1 \subset \Theta
\]

holds, then $\mathcal{B}(C[0, 1])$ is boundedly complete for $E$ and the assumptions of Theorem 12 are fulfilled since $\Theta_1 + \Theta_1 = \Theta_1$ holds. Observe that $\Theta_1$ generates the whole Hilbert space $L_2[0, 1]$ and thus the projections $f \mapsto f(t)$ on $C[0, 1]$ are $P_0$ a.e. measurable w.r.t. to $\mathcal{A}_0$. To see this consider $\vartheta = 1_{[0,1]}$. 

Let now $T : C[0, 1] \to W$ be an estimator of a function $g : \Theta \to W, \ g = f \circ h_{|_{\Theta}}$ defined in (3.2) which factorizes via our signal $h$. Suppose that $T$ is an equivariant estimator in law, i.e.

$$L(T(X(\cdot))|\vartheta) = L(T(B(\cdot))) * \varepsilon_{g(\vartheta)} =: Q * \varepsilon_{g(\vartheta)}$$

for each $\vartheta \in \Theta$. Then the estimator $T_0 := f(X(\cdot))$ is superior in the sense that $Q = \nu * L(T_0|\vartheta = 0)$ holds.

(b) As application the convolution theorem (1.23) will be established for arbitrary sample size $n$. Assume for instance that $\vartheta_0 + \Theta_1 \subset \Theta$ holds where $\Theta_1$ denotes again the polynomials of (3.18). Due to the translation invariance of the problem we may assume that $\vartheta_0 = 0$ holds. In a first step the sample size $n$ will be reduced. By (3.17) the product densities are given by the stochastic integral w.r.t. to the processes (3.20)

$$\frac{dP^n_0}{dP^n_\eta} = \exp \left( \int_0^1 \vartheta(s) \left( \sum_{i=1}^n X_i \right) ds + n \|\vartheta\|^2 \right).$$

Thus $\sqrt{n}S$ is sufficient where $S$ is the mean statistic (1.20). If we turn to the local parameterization (1.21) we have

$$L \left( \sqrt{n}S \big| P^n_{\eta/\sqrt{n}} \right) = L(X_1(\cdot)|P_\eta)$$

on $C[0, 1]$. Since $\sqrt{n}S$ is sufficient there exists a kernel

$$C : C[0, 1] \times B(C[0, 1]^n) \to [0, 1]$$

which reproduces the product measures from the image distributions

$$P^n_{\eta/\sqrt{n}}(\cdot) = \int C(x, \cdot) dP_\eta(x),$$

see Pfanzagl (1994), Prop. 1.3.1 for instance. For $g = h$ we now find the kernel required in (3.0), namely

$$Q_h(\cdot) := L \left( T \big| P^n_{\eta/\sqrt{n}} \right) (\cdot) = \int C(x, T^{-1}(\cdot)) P_0 * \varepsilon_{h(\vartheta)}(dx)$$

Thus Theorem 12 implies the convolution theorem (1.23) and $S$ is the best equivariant estimator, see Pötzelsberger et al. (2000) for more details about equivariant estimation.

**Example 18.** The treatment of the Levy processes of Example 5 is based on a completeness result about countable product experiments

$$\times_{i=1}^\infty \Omega_i, \otimes_{i=1}^\infty A_i, \left\{ \otimes_{i=1}^\infty P_{s_i}^{(i)} : (s_i)_i \in \Theta \right\}$$

where $\Theta \subset \times_{i=1}^\infty \Theta_i$ is a suitable parameter space. Suppose that each factor

$$\left( \Omega_i, A_i, \left\{ P_{s_i}^{(i)} : s_i \in \Theta_i \right\} \right)$$

is boundedly complete. Suppose that $s_0 = (s_0^i)_{i \in \mathbb{N}} \in \Theta$ denotes a fixed parameter. Then Lemma 14 immediately implies that (3.26) is boundedly complete for

$$\Theta = \left\{ (s_i)_{i \in \mathbb{N}} \in \times_{i=1}^\infty \Theta_i : s_i = s_i^{(0)} \text{ finally} \right\},$$
see also Landers and Rogge (1976) for related argument for the completeness of product experiments. Let us now turn to Example 5. Define next $t_0 = 0$, $\vartheta_0^{(0)} = 0$ and $\vartheta_0 = 0$. Then the transformation

\begin{equation}
(3.27) \quad X \to (Y_i + s_i)_{i \in \mathbb{N}}
\end{equation}

given by $Y_i := Z_{i_1} - Z_{i_{i-1}}$, $s_i = \vartheta_i - \vartheta_{i-1}$, denotes a one to one transformation of the shifted Levy process \((1.24)\) to the independent increment processes of the right hand side of \((3.27)\). Observe that by Wiener’s closure theorem the family \((\mathcal{L}(Y_i + s_i))_{i \in \mathbb{N}}\) is boundedly complete for each $i$ since the Fourier transforms of absolutely continuous distributions do not vanish, see Remark 13. Thus the Levy process model is boundedly complete iff the product model of the right hand side of \((3.27)\) is. Our results above imply that this results is true if we take $s_i^{(0)} := \vartheta_i^{(0)} - \vartheta_{i-1}^{(0)}$ and if we restrict ourselves to shifts $(\vartheta_i)_{i \in \mathbb{N}}$ of the Levy process with $\vartheta_0 = \vartheta_0^{(0)}$ finally. The convolution theorem \((1.20)\) follows again from Theorem 12.

Note that we may assume without restrictions (after applying a shift to $Y_i$) that $\vartheta^0 = 0 \in \mathbb{R}^N$ holds. Then the choice $\Theta_1 = \{(\vartheta_i)_{i \in \mathbb{N}} \in \mathbb{Q}^N : \vartheta_i = 0 \text{ finally}\}$ is appropriate.

Remark 19. The present convolution theorem can be extended to differentiable statistical functionals $\kappa : \Theta \to \mathbb{R}$ in the sense of van der Vaart (1988,1989,1991). For LAN families the linearization of $\kappa$ at $\vartheta_0$ via the canonical gradient implies the result.

Example 20. This example shows that generally the convolution factors are not unique on $\mathbb{R}$ when the assumptions of Wiener’s closure theorem do not hold. In particular we present different probability measures $\mu, \nu$ such that the equality $\mu * \eta = \nu * \eta$ holds for each probability measure $\eta$ with Fourier transform vanishing outside of the interval $[-1,1]$. Let $\nu$ be a probability measure on $(\mathbb{R},\mathcal{B})$ with Lebesgue density $f : x \to \frac{2 \sin \pi x^2}{\pi x^2}$. It is easy to show that $\nu$ possesses the Fourier transform $\widehat{\nu}(t) = (1 - |t|)1_{[-1,1]}(t)$, see Gnedenko (1968), p. 236. Next we regard the probability measure

$$
\mu_0 = \frac{1}{2} \varepsilon_0 + \sum_{k=1}^{\infty} \frac{2}{\pi^2 (2k + 1)^2} \left( \varepsilon_{4k+2} + \varepsilon_{-(4k+2)} \right),
$$

which $\pi$-periodical Fourier transform is $\widehat{\mu}_0(t) = 1 - \frac{2}{\pi} |t|$ for $|t| \leq \frac{\pi}{2}$, see Renyi (1966), p. 271. The probability measure $\mu$ is defined as distribution of $\mu_0$ which is scaled with the factor $\frac{\pi}{2}$. The Fourier transform of $\mu$ is then $\widehat{\mu}(t) = 1 - |t|$ for $|t| \leq 1$. Thus we obtain $\mu * \eta = \nu * \eta$ for each probability measure $\eta$, which Fourier transform have a support inside of the interval $[-1,1]$.

4. Convergence to Limit Experiments

In this section it is pointed out how the convergence of statistics and the convergence of experiments are related. We present an elementary proof of the main Theorem 21 for distributions $Q_0$ on standard Borel spaces. This theorem is a special case of more general results of Le Cam, see Le Cam and Yang (2000), Le Cam (1994), Th. 1 and also Strasser (1985) or Torgersen (1991). Let $E_n = (\Omega_n, \mathcal{A}_n, \{P_{n,\vartheta} : \vartheta \in \Theta_n\})$ denote as in \((2.3)\) a statistical experiment and let $E = (\Omega, \mathcal{A}, \{P_{\vartheta} : \vartheta \in \Theta\})$ be another experiment with $\Theta_n \uparrow \Theta$. Recall from Strasser (1985), Sect. 60, that $E_n$ is
said to be weakly convergent to $E$, if all finite dimensional marginals distributions of the loglikelihood processes

$$L \left( \left( \log \frac{dP_{n,t}}{dP_{n,s}} \right)_{t \in I} \middle| P_{n,s} \right) \to L \left( \left( \log \frac{dP_t}{dP_s} \right)_{t \in I} \middle| P_s \right)$$

weakly converge for all $s \in \Theta$ and all $I \subset \Theta$, $|I| < \infty$.

Let $D$ denote a standard Borel space and $\mathcal{D}$ its Borel $\sigma$-field on $D$.

**Theorem 21.** Suppose that the sequence of experiments $E_n$ converges weakly to a limit experiment $E$. Let

$$(4.1) \quad T_n : \Omega_n \to D$$

be a sequence of statistics with values in a standard Borel space $D$ such that

$$(4.2) \quad L(T_n \mid P_{n,\vartheta}) \to Q_\vartheta$$

weakly converges to the same distribution $Q_\vartheta$ on $D$ for all $\vartheta \in \Theta$. Then there exists a kernel $K : \Omega \times \mathcal{D} \to [0, 1]$ with

$$(4.3) \quad Q_\vartheta = \int K(x, \cdot) \, dP_\vartheta$$

for all $\vartheta \in \Theta$. In particular, $E$ is more informative than $(D, \mathcal{D}, \{Q_\vartheta : \vartheta \in \Theta\})$.

**Proof.** Recall that on standard Borel spaces the set of probability measures is a separable metric space w.r.t. the topology of weak convergence. Thus we may choose a countable dense subset $\{Q_{\vartheta_j} : j \in \mathbb{N}\}$ of $\{Q_\vartheta : \vartheta \in \Theta\}$. We will identify $Q_{\vartheta_j} =: Q_j$ and $P_{\vartheta_j} =: P_j$. Introduce the additional distributions

$$(4.4) \quad Q_0 = \sum_{j=1}^{\infty} \frac{Q_j}{2^j} \quad \text{and} \quad P_0 = \sum_{j=1}^{\infty} \frac{P_j}{2^j}.$$  

Write shortly $P_{n,j} := P_{n,\vartheta_j}$. Define

$$(4.5) \quad P_{n,0} := a_n \sum_{j=1}^{\infty} \frac{P_{n,j}}{2^j} \mathbf{1}_{\Theta}(\vartheta_j),$$

where $a_n$ are normalizing constants. The proof of Theorem 21 relies on two lemmas. First we add $P_{n,0}$ and $P_0$ to our experiments.

**Lemma 22.** If $E_n \to E$ converges weakly then also

$$(\Omega_n, \mathcal{A}_n, \{P_{n,j} : j \in \mathbb{N}_0\}) \to (\Omega, \mathcal{A}, \{P_j : j \in \mathbb{N}_0\})$$

converges weakly.

**Proof.** Consider the subexperiments for $I = \{1, \ldots, k\}$ and add $P_{n,0}' := \sum_{j=1}^{m} c_j P_{n,j}$ and $P_0' := \sum_{j=1}^{m} c_j P_j$, where $0 < c_j < 1$, $\sum_{j=1}^{m} c_j = 1$, $m \in \mathbb{N}$ for $n$ large enough. It is easy to see that the experiments $\{P_{n,0}', P_{n,1}', \ldots, P_{n,k}'\}$ weakly converge to $\{P_0', P_1', \ldots, P_k'\}$, since their likelihood processes can be expressed by linear dependence by the likelihood processes of $E_n$ and $E$. If $m$ tends to infinity we find coefficients $c_j$ such that $P_{n,0}'$ tends to $P_{n,0}$ uniformly in $n$ w.r.t. the norm of total variation. Thus $P_{n,0}'$ and $P_0'$ may be substituted by $P_{n,0}$ and $P_0$ and the convergence of experiments carries over. \(\square\)
In the case of Theorem 21 we have

**Lemma 23.** (a) There exist a subsequence \( \{m\} \subset \mathbb{N} \) and a probability measure \( \mu_0 \) on \( \mathbb{R}^N \times D \) such that

\[
\mathcal{L} \left( \left( \left( \log \frac{dP_{m,j}}{dP_{m,0}} \right)_{j \in \mathbb{N}} , T_m \right) \right) \bigg| P_{m,0} \to \mu_0
\]

weakly converges as \( m \to \infty \).

(b) Under \( P_{m,j} \) the limit law \( \mu_j \) of \( \{4.6\} \) exists. We have \( \mu_j \ll \mu_0 \) with density

\[
\frac{d\mu_j}{d\mu_0} (x, d) = \exp \left( p_j (x) \right)
\]
on \( \mathbb{R}^N \times D \) where \( p_j (x) \) denote the \( j \)-th coordinate of \( x \in \mathbb{R}^N \).

**Proof.** Part (a) follows from the tightness of the marginals. Part (b) is a consequence of the third Lemma of Le Cam, see van der Vaart (1988), Appendix A1, Janssen (1998), Sect. 14 or Jacod and Shiryaev (2003). Observe, that Le Cam’s third Lemma also holds for non-Gaussian limit experiments. Only contiguity of \( (P_{m,j})_m \) w.r.t \( (P_{m,0})_m \) is required. This condition follows from Lemma 22 and the first Lemma of Le Cam. \( \square \)

The proof of Theorem 21 can be completed by the following arguments. Consider the canonical projections \( \pi_1 : \mathbb{R}^N \times D \to \mathbb{R}^N \) and \( \pi_2 : \mathbb{R}^N \times D \to D \). Obviously, \( \pi_1 \) is sufficient for \( \{\mu_j : j \in \mathbb{N}_0\} \). Thus there exists a version of the conditional distribution of \( (\pi_1, \pi_2) \) given \( \pi_1 \)

\[
C : \mathbb{R}^N \times (\mathcal{B}^N \otimes D) \to [0,1]
\]

which is independent of \( j \), i.e. \( \mu_j = \int C (x, \cdot) \, d\mu_j^{\pi_1} (x) \). Now we may choose

\[
K (\omega, A) = C \left( \left( \left( \log \frac{dP_{j}}{dP_0} (\omega) \right)_{j \in \mathbb{N}} , \pi_2^{-1} (A) \right) \right)
\]

for \( A \in D \) and \( \{4.3\} \) holds for all \( \vartheta \in \{\vartheta_j : j \in \mathbb{N}\} \). That equality can be extended to all \( \vartheta \in \Theta \) by the following continuity arguments. Define new distributions \( Q_{\vartheta} = K (P_{\vartheta}) \) for all \( \vartheta \in \Theta \) via \( \{3.6\} \). Since \( Q_{\vartheta_j} = Q_{\vartheta} \) holds and \( \{Q_{\vartheta_j} : j \in \mathbb{N}\} \) is dense we have \( Q_{\vartheta} = Q_{\vartheta} \) for all \( \vartheta \in \Theta \) and \( \{4.3\} \) holds. These arguments finish the proof of Theorem 21. \( \square \)

Observe that \( \{P_{\vartheta_j} : j \in \mathbb{N}\} \) and \( \{Q_{\vartheta_j} : j \in \mathbb{N}\} \) are dense in \( E \) and \( \{Q_{\vartheta} : \vartheta \in \Theta \} \), respectively, the norm of total variation. By continuity arguments equation \( \{4.3\} \) carries over for all \( \vartheta \in \Theta \). We like to present another example where the convolution theorem holds for a non Gaussian limit experiment.

**Example 24.** [Estimating the endpoints of a distribution] Let \( f \) be a continuous smooth density on some real interval \( [a, b] \) with \( f(a) > 0 \) and \( f(b) > 0 \) where \( f \) vanishes outside of the interval. We want to estimate the endpoints \( a, b \) based on i.i.d. replications of sample size \( n \). A local model is now given by the parameterization of the endpoints

\[
a + \frac{\vartheta_1}{n} \quad \text{and} \quad b + \frac{\vartheta_2}{n}
\]
for a suitable pair of local parameters \( \vartheta = (\vartheta_1, \vartheta_2) \in \Theta_n \subset \mathbb{R}^2 \). We will restrict ourselves to the following shift and scale submodel (4.11). Let \( Y_1, Y_2, \ldots, Y_n \) be i.i.d. random variables with common density \( f \). Our observations for \( 1 \leq i \leq n \) are

\[
Z_{n,i} := \left( 1 + \frac{\vartheta_2 - \vartheta_1}{n(b-a)} \right) (Y_i - a) + a + \frac{\vartheta_1}{n}
\]

which have just the endpoints (4.10). For suitable parameters \( \vartheta = (\vartheta_1, \vartheta_2) \) introduce

\[
P_{n,\vartheta} := \mathcal{L}\left( (Z_1, \ldots, Z_n) \mid \vartheta \right)
\]

and let \( Z_{1:n} \leq Z_{2:n} \leq \ldots \leq Z_{n:n} \) denote the order statistics of (4.11). We will only sketch the general results and indicate how to prove convergence of the experiments.

A rigorous proof is only given for the uniform distribution. Let \( X_1 \) and \( X_2 \) denote two independent standard exponential random variables with \( E(X_i) = 1 \) for \( i = 1, 2 \).

Well-known results from extreme value theory prove that (4.12)

\[
n (Z_{1:n} - a) \to \frac{X_1 + \vartheta_1}{f(a)}
\]

and

\[
n (Z_{n:n} - b) \to \frac{-X_2 + \vartheta_2}{f(b)}
\]

are convergent in distribution under \( \vartheta \). The convergence of these distributions can also be shown w.r.t. the norm \( \| \cdot \| \) of total variation. Under regularity assumptions concerning the smoothness of the density \( f \) the extreme order statistics \( (Z_{1:n}, Z_{n:n}) \) are asymptotically sufficient and weak convergence of the experiments

\[
(\mathbb{R}^n, \mathcal{B}^n, \{P_{n,\vartheta} : \vartheta = (\vartheta_1, \vartheta_2) \in \Theta_n\})
\]

\[
\to \left( (\mathbb{R}^2, \mathcal{B}^2, \mathcal{L}\left( \frac{X_1 + \vartheta_1}{f(a)}, \frac{-X_2 + \vartheta_2}{f(b)} \right) : \vartheta = (\vartheta_1, \vartheta_2) \in \Theta \right) \]

holds provided \( \Theta_n \uparrow \Theta \). The details are figured out for the uniform distribution only.

**Lemma 25.** The convergence of experiments (4.13) holds for the uniform density \( f = 1_{[0,1]} \).

**Proof.** In this case the extreme order statistics \( (Z_{1:n}, Z_{n:n}) \) are finite sample sufficient and the experiments

\[
\{P_{n,\vartheta}\} \text{ and } \{\mathcal{L}(nZ_{1:n}, n(Z_{n:n} - 1) \mid \vartheta)\}
\]

are equivalent in Le Cam’s sense. Thus it is sufficient to prove the weak convergence of the latter experiment. Consider first \( \vartheta = 0 \). It is well-known, see Reiss (1989), Section 5.1 and p. 121, that

\[
\| \mathcal{L}(nZ_{1:n}, n(Z_{n:n} - 1) \mid 0) - \mathcal{L}(X_1, -X_2) \| \to 0
\]

holds w.r.t. the norm of total variation. This assertion is due to the fact that lower and upper extreme become asymptotically independent. The same result holds under \( \vartheta \) where \( \mathcal{L}(X_1, -X_2) \) is replaced by \( \mathcal{L}(X_1 + \vartheta_1, -X_2 + \vartheta_2) \), see also (4.13) and (4.11). Note that (4.17) is equivalent to the \( L_1 \)-convergence of the densities which are easy to handle under the present shift and scale family. It is well-known that the convergence of distributions w.r.t. total variation implies the convergence of the underlying experiments.

\( \square \)
Let now $\vartheta \in \Theta$ belong to a dense set of $\mathbb{R}^2$. Then the assumptions of the convolution theorem hold. Observe that the limit experiment of (4.15) is boundedly complete by Wiener’s closure theorem, see Remark 15. Let $T_n : \mathbb{R}^n \to \mathbb{R}^2$ be a sequence of $\Theta$ regular estimators of the endpoints for the uniform distribution at $(a, b) = (0, 1)$ with

$$L \left( n \left( T_n - \left( \frac{\vartheta_1}{n}, 1 + \frac{\vartheta_2}{n} \right) \right) \right) \rightarrow L(Q)$$

in distribution for all $\vartheta \in \Theta$. Then $L(Q)$ is given by a convolution product

$$L(Q) = \nu * L(X_1, -X_2).$$

Under regularity assumption the same holds for densities $f$ where the convolution bound arises from the limit experiment (4.15).

**Remark 26.** (a) Limit experiments for densities with jumps were treated by Ibragimov and Has’minskii (1981), Chap. V, Pflug (1983), Strasser (1985b), Sect. 19 and in the appendix of Janssen and Mason (1990), p. 215. In these references the reader will find regularity conditions concerning the density $f$ which ensure weak convergence of the experiments in (4.15).

(b) If the density has only a single jump at the lower endpoint then the limit experiment (4.15) has to be modified and it is just $\{ L \left( \frac{X_2}{f(a)} : \vartheta_1 \right) \}$. For this kind of limit experiment Millar (1983), p. 157 already obtained a convolution theorem which is similar to (4.19).

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