A Stable Jacobi polynomials based least squares regression estimator associated with an ANOVA decomposition model.

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Abstract— In this work, we construct a stable and fairly fast estimator for solving non-parametric multi-dimensional regression problems. The proposed estimator is based on the use of multivariate Jacobi polynomials that generate a basis for a reduced size of $d$-variate finite dimensional polynomial space. An ANOVA decomposition trick has been used for building this later polynomial space. Also, by using some results from the theory of positive definite random matrices, we show that the proposed estimator is stable under the condition that the i.i.d. random sampling points for the different covariates of the regression problem, follow a $d$-dimensional Beta distribution. Also, we provide the reader with an estimate for the $L^2$-risk error of the estimator. Moreover, a more precise estimate of the quality of the approximation is provided under the condition that the regression function belongs to some weighted Sobolev space. Finally, the various theoretical results of this work are supported by numerical simulations.

Keywords: Non-parametric Regression, Jacobi polynomials, generalized polynomials chaos, ANOVA decomposition, least squares, stable regression estimator, risk error.

1 Introduction

In this work, we combine the popular technique of generalized polynomials chaos (gPC) \cite{39, 43} and a special family of $d$-variate Jacobi polynomials in order to solve a $d$-dimensional non-parametric regression problem. This last problem is one of the important as well as an active research topic from the machine learning area, see for example \cite{39, 40}. Note that a machine learning algorithm can be briefly described as an algorithm for the approximation of an unknown function $f$ that maps in general a random vectors $X \in \mathbb{R}^d$ to an observed real valued variable $Y$. An estimator or an approximation $\hat{f}$ of $f$ is constructed by the use a training data set $\{(X_i, Y_i), \ 1 \leq i \leq n\}$. Note that unlike a parametric learning algorithm, where $\hat{f}$ is given in terms of a set of fixed size of parameters, a non-parametric learning algorithm does not require any assumption about the function $f$ or its estimator $\hat{f}$. Usually, for a non-parametric (NP) model, the function $f$ lies in an infinite dimensional functional space. Consequently, the NP models have the advantage to better fit a wide range of the true functions $f$. We should mention that the multidimensional NP regression problem is frequently encountered in a wide range of scientific fields. An NP learning algorithm for solving this problem aims to provide a convenient estimate $\hat{f}$ for the true regression function $f$ associated with the regression problem,

$$Y_i = f(X_i) + \epsilon_i, \ i = 1, \ldots, n.$$  \hfill (1) 

Here, the $X_i \in \mathbb{R}^d$ are assumed to be i.i.d. random vectors and the $\{(\epsilon_i), 1 \leq i \leq n\}$ are centered i.i.d. random variables with a finite variance $\sigma^2 = \mathbb{E}[\epsilon_i^2]$ and independent from the $X_i$. The goal of a learning algorithm is to minimize the empirical risk $R_{\text{emp}}(f)$ over a given class of functional space $\mathcal{H}$. That is to solve the minimization problem

$$\hat{f} = \arg \min_{f \in \mathcal{H}} R_{\text{emp}}(f) = \arg \min_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} L(f(X_i), Y_i),$$  \hfill (2) 

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where, \( L(\cdot, \cdot) \) is a non-negative loss function. Among the frequently used loss functions from the literature, we cite the Tikhonov regularized loss \([35]\) and the weighted \( \ell_2 \) loss function \([33]\), given respectively by

\[
L_\lambda(F(X_i), Y_i) = (f(X_i) - Y_i)^2 + \lambda \|f\|_2^2, \quad L_\omega(f(X_i), Y_i) = \omega_i (f(X_i) - Y_i)^2,
\]

for some convenient regularization parameter \( \lambda > 0 \) and finite weight sequence \((\omega_i)_i\). In general, the least squares scheme is used to solve the minimization problem \([2]\).

It is well known that despite the superiority of the NP model in terms of quality of approximation of the true functional \( f \), it suffers from the curse of dimensionality for large or even moderate values of \( d \), the number of covariates. The computational load by an NP algorithm grows fast with the dimension \( d \) and its convergence rate or its associated risk rate error rate slows drastically. For instance, it has been shown in \([37]\), see also \([5, 15]\) that if \( f \) is of class \( C^p \) with \( p \)-th derivative being Hölder continuous, then the optimal convergence rate of any NP least-squares estimator \( \hat{f}_n \), solution of the minimization problem \([2]\) is given by

\[
\mathbb{E} \left[ \|f - \hat{f}_n\|_2^2 \right] = O \left( n^{-2p/(2p+d)} \right).
\]

In order to reduce the computational load required by an algorithm for solving more general models with random inputs or models from uncertainty quantification (UQ) area, a popular technique of polynomial chaos expansion (PCE) is successfully used in the literature. This technique aims to approximate the output variable \( Y \) by using a projection over a reduced size of orthogonal polynomial basis. The PCE scheme has been first introduced by N. Wiener in his pioneer work \([12]\), for the Hermite polynomials and Gaussian random variables. Recently, there is a growing interest in the study and the use for UQ applications of a generalized version of PCE, called generalized polynomial chaos (gPC).

The gPC was first introduced by \([43]\) and it aims to extend the PCE to various discrete and continuous probability distributions associated with the set of weight functions for the family of orthogonal polynomials of the Askey–scheme. For more details on PCE and gPC schemes and their associated UQ related applications, the reader is refereed to \([16, 17, 21, 22, 26, 40, 44]\). Nonetheless, the gPC based learning algorithm still has the limitation to be slow for moderate large values of the dimension \( d \). To overcome this problem, various solutions have been considered in the literature. Among the popular adopted solutions, we cite the use of sparsity and optimal sampling techniques, see for example \([20, 25, 30, 32, 36, 38]\), dimension reduction through a sensitivity analysis techniques, \([2, 8]\), as well as the use of partial functional ANOVA decomposition technique, see for example \([20, 25, 30, 32, 36, 38]\). More precisely, a partial functional ANOVA decomposition consists in the approximation of a real valued function \( f(x_1, \ldots, x_d) \) by a sum of functions with reduced number of variables of the form \( f_i(x_{i_1}, \ldots, x_{i_m}) \) where \( 1 \leq m \leq d \) and the \( i_j \in [1, d] \). Note that for the full ANOVA decomposition, that is \( m = d \), the uniqueness of the ANOVA decomposition of a function \( f \in L^1(J^d) \), \( J = [0, 1] \) has been shown in \([30]\). Also, among the popular reduced size multi-variate polynomials spaces used by a gPC scheme, we cite the total degree space of degree \( N \), given by \( \mathcal{P}^T_D = \text{Span} \{ \mathbf{x}^i = x_1^{i_1} \cdots x_d^{i_d}, \ |\mathbf{i}|_1 = \sum_{j=1}^d |i_j| \leq N \} \) and the hyperbolic cross space of degree \( N \), given by \( \mathcal{P}^{HC}_{q,N} = \text{Span} \{ \mathbf{x}^i = x_1^{i_1} \cdots x_d^{i_d}, \ |\mathbf{i}|_q = \left( \sum_{j=1}^d |i_j|^q \right)^{1/q} \leq N \}, 0 < q < 1 \). For more details on these polynomials spaces, the reader is refereed to \([7]\).

In this work, we introduce a new \( d \)-variate polynomial space constructed from uni-variate Jacobi polynomials associated with parameters \( \beta = \alpha \geq -\frac{1}{2} \) and orthonormal over \( I^d = [-1, 1]^d \). More precisely, for two positive integers \( N \geq 1 \) and \( 1 \leq m \leq \min(d,N) \), we let the ANOVA type Jacobi polynomials space

\[
\mathcal{P}_{N,m,d} = \text{Span} \left\{ \prod_{i \in \mathbf{u}} \overline{P}^{(\alpha)}_{k_i}(x_i) ; \mathbf{u} \subset [1,d] ; \ |\mathbf{u}| \leq m; \ k \in \mathbb{N}_0^d, \ |\mathbf{k}|_1 \leq N \right\}.
\]

Here, \( \overline{P}^{(\alpha)}_{k_i} \) are the orthonormal uni-variate Jacobi polynomials , associated with a parameter \( \alpha \geq -\frac{1}{2} \). The dimension of \( \mathcal{P}_{N,m,d} \) is given by \( M_{N,m,d} = \dim \mathcal{P}_{N,m,d} = \sum_{k=0}^m \binom{d}{k} \binom{N}{k} \). For the extreme case \( m = \min(d,N) \),
the space $P_{N,m,d}$ is reduced to the total degree space polynomials of degree $N$, given by $P_{N,m,d}^T$. One of the main results of this work is to prove the stability of our proposed least-squares estimator $\hat{f}_{N,n,m}^{(α)}$ which is the solution of the minimization problem (2) with the functional space $H = P_{N,m,d}$ and a loss function $L$ given by the Euclidean distance of $\mathbb{R}^n$. Note that there is a growing interest in the study of the stability issue of estimators of functions with random inputs, see for example [11, 12, 27, 28]. In the present work, we show that under the condition that the $X_i$ follow a $d$–variate Beta($\alpha + 1, \alpha + 1$) distribution with support $I^d$, the positive definite $n \times n$ random matrix involved in the construction of our proposed least-squares based estimator $\hat{f}_{N,n,m}^{(α)}$ is with high probability well conditioned in the $2$–norm. Moreover, we give an estimate for the $L^2$–risk error of a truncated version of the $\hat{f}_{N,n,m}^{(α)}$ which we denote by $\hat{F}_{N,n,m}^{(α)}$. More precisely, if $\| \cdot \|_α$ denotes the weighted $L^2$–norm associated with the weight $ω_α = \prod_{i=1}^d (1 - x_i^2)^α$, then we give an estimate of the $L^2$–risk error $E[\|f - \hat{F}_{N,n,m}^{(α)}\|_α^2]$. This latter is given in terms of the classical bias-variance decomposition. The variance term decays at a rate of $O(M_{N,m,d}^N/n)$. Here, $M_{N,m,d}$ is the dimension of our proposed polynomial space $P_{N,m,d}$. The bias term $f$ the $L^2$–risk involves the quantity $\|f - \Pi_{N,m}f\|_α$, where $\Pi_{N,m}$ denotes the orthogonal projection over $P_{N,m,d}$. An estimate of this last quantity is given under the hypothesis that the true regression function lies in a weighted Sobolev space with given Sobolev smoothness property.

This work is organized as follows. In section 2, we give some mathematical preliminaries on Matrix Chernoff eigenvalues bounds, the Gershgorin circle theorem for bounding the spectrum of a square matrix, as well as some properties of the Jacobi polynomials. In particular, we give some bounds of these polynomials that will be used for proving different results of this work. In section 3, we describe the new adopted and reduced size ANOVA type multivariate Jacobi polynomial space $P_{N,m,d}$. Moreover, we provide the reader with an estimate of the dimension of this later. Section 4 of this work is devoted to the proof of the stability of the proposed least-squares and gPC based NP regression estimator $\hat{f}_{N,n,m}^{(α)}$. In section 5, we give an estimate for the weighted $L^2$–risk error of a truncation version of the estimator $\hat{f}_{N,n,m}^{(α)}$. Moreover, in section 6, we give an estimate for the bias term of the previous weighted $L^2$–risk error, when the true regression function belongs to some weighted Sobolev space. Finally, in section 7, we give some numerical simulations that illustrate the different results of this work.

2 Mathematical Preliminaries and estimates for Jacobi polynomials

In this paragraph, we first give some mathematical preliminaries from the literature that will be used frequently in this work. Then, we give some useful estimates for the Jacobi polynomials. These estimates are needed for the proof of the stability property of our proposed special Jacobi polynomials multivariate non-parametric (NP) regression estimator.

2.1 Mathematical preliminaries

We first recall the Matrix Chernoff Theorem (see for example [11]) and the Gershgorin circle Theorem (see for example [19]) that will be useful for proving the stability of our NP regression estimator.

Matrix Chernoff Theorem: Consider a sequence of $n$ independent $D \times D$ random Hermitian matrices $\{Z_k\}$. Assume that for some $L > 0$, we have

$$0 \preccurlyeq Z_k \preccurlyeq L I_D.$$
Let
\[ A = \sum_{k=1}^{n} Z_k, \quad \mu_{\text{min}} = \lambda_{\text{min}}(\mathbb{E}(A)), \quad \mu_{\text{max}} = \lambda_{\text{max}}(\mathbb{E}(A)). \]

Then, for any \( \delta \in (0,1) \), we have
\[ \mathbb{P}(\lambda_{\text{min}}(A) \leq (1-\delta)\mu_{\text{min}}) \leq D. \exp \left( -\frac{\delta^2 \mu_{\text{min}}}{2L} \right), \quad \mathbb{P}(\lambda_{\text{max}}(A) \geq (1+\delta)\mu_{\text{max}}) \leq D. \exp \left( -\frac{\delta^2 \mu_{\text{max}}}{3L} \right) \]

**Gershgorin circle Theorem:** Let \( A = [a_{i,j}]_{1 \leq i,j \leq n} \) be a complex matrix. For \( 1 \leq i \leq n \), let \( R_i = \sum_{j \neq i} |a_{i,j}| \). Then every eigenvalue of \( A \) lies within at least one of the discs \( D(a_{ii}, R_i) \).

In the sequel, we let \( \Gamma(a) \) and \( B(a, b) \) respectively denote the usual Gamma and Beta functions with \( a, b > 0 \). For an integer \( k \geq 0 \) and \( \alpha \geq -\frac{1}{2} \), let \( \tilde{P}_k^{(\alpha,\alpha)} \) denote the normalized Jacobi polynomial defined on \( I = [-1,1] \) of degree \( k \) and parameters \( (\alpha, \alpha) \). We have:
\[ \tilde{P}_k^{(\alpha,\alpha)}(x) = \frac{1}{h_k^{(\alpha)}} P_k^{(\alpha)}(x), \quad h_k^{(\alpha)} = \frac{2^{2\alpha+1} \Gamma(2k+\alpha+1)}{\Gamma(k+\alpha+1) \Gamma(k+2\alpha+1)}. \]

The polynomials \( \tilde{P}_k^{(\alpha,\alpha)} \), \( k \geq 0 \) satisfy the orthonormality relation
\[ \int_I \tilde{P}_j^{(\alpha,\alpha)} \tilde{P}_k^{(\alpha,\alpha)} w_\alpha(x) dx = \delta_{j,k}, \quad w_\alpha(x) = (1-x^2)^\alpha. \]

In the sequel, in order to alleviate notations, we will use the notation \( \tilde{P}_k^{(\alpha)} \) instead of \( \tilde{P}_k^{(\alpha,\alpha)} \).

The following Lemma regroups different useful identities and inequalities that can be easily found in the literature, see for example [3][4][23].

**Lemma 1.** Let \( J_\alpha \) be the Bessel function of the first kind and order \( \alpha > -1 \). Then, we have

1. For any \( x \in \mathbb{R} \) and for any integer \( m \geq 0 \),
\[ \int_{-1}^{1} e^{ixy} \tilde{P}_m^{(\alpha)}(y) w_\alpha(y) dy = i^m \sqrt{\pi} \sqrt{2m+2\alpha+1} \frac{\Gamma(m+2\alpha+1)}{\Gamma(m+1)} \frac{J_{m+\alpha+1/2}(x)}{x^{\alpha+1/2}}. \]
2. For any \( x \in \mathbb{R} \) and any real \( \mu > -1 \), we have
\[ |J_\mu(x)| \leq \frac{|x|^\mu}{2^\mu \Gamma(\mu+1)}. \]
3. For \( x > -\frac{1}{2} \),
\[ \sqrt{2e} \left( \frac{x + \frac{1}{2}}{e} \right)^{x + \frac{1}{2}} \leq \Gamma(x+1) \leq \sqrt{2\pi} \left( \frac{x + \frac{1}{2}}{e} \right)^{x + \frac{1}{2}}. \]

**2.2 Estimates for Jacobi polynomials**

In order to provide estimates for Jacobi polynomials, we will need the following result.

**Lemma 2.** For \( \alpha \geq -\frac{1}{2} \), the function \( h_0^{(\alpha)} = 2^{2\alpha+1} \text{Beta}(\alpha+1, \alpha+1) \) is bounded as follows
\[ \frac{1}{C^2(\alpha)} \leq h_0^{(\alpha)} \leq 2\pi, \]
where \( C(\alpha) := \left[ \frac{\pi}{2} \left( \alpha + \frac{3}{4} \right) \right]^\frac{1}{4}. \)
Proof. We first establish the lower bound of $h_0^{(\alpha)}$ using (8).

$$\frac{1}{h_0^{(\alpha)}} = \frac{1}{2^{2\alpha+1}} \frac{\Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} \leq \sqrt{\pi} e^{-\frac{3}{4}} \left( \frac{\alpha + \frac{3}{4}}{2} \right)^{2\alpha+1} \left( \frac{1}{\alpha + \frac{1}{2}} \right)^{2\alpha+1} \leq \sqrt{\pi} e^{-1} \left( \alpha + \frac{3}{4} \right)^{\frac{1}{2}}.$$

For the upper bound of $h_0^{(\alpha)}$, we will use again (8). We get

$$h_0^{(\alpha)} = 2^{2\alpha+1} \frac{\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)} \leq 2^{2\alpha+2} \pi \left( \frac{\alpha + \frac{1}{2}}{e} \right)^{2\alpha+1} \frac{1}{\sqrt{2e}} \left( \frac{e}{2\alpha + \frac{3}{2}} \right)^{2\alpha+\frac{3}{2}} = 2^{2\alpha+\frac{3}{2}} \pi \left( \frac{\alpha + \frac{1}{2}}{e} \right)^{2\alpha+1} \frac{1}{\sqrt{e}} \left( \frac{e}{2\alpha + \frac{3}{2}} \right)^{2\alpha+\frac{1}{2}} = \pi \left( \frac{1}{\alpha + \frac{3}{4}} \right)^{\frac{1}{2}} \left( \frac{\alpha + \frac{1}{2}}{\alpha + \frac{3}{4}} \right)^{2\alpha+1} \leq 2\pi.$$

The following proposition provides us with some useful estimates for the normalized Jacobi polynomials $\tilde{P}_k^{(\alpha)}$.

**Proposition 1** (Bounds for $||\tilde{P}_k^{(\alpha)}||_\infty$). Under the previous notation, let $\alpha \geq -\frac{1}{2}$, then for any integer $k \geq 1$, we have

$$||\tilde{P}_k^{(\alpha)}||_\infty \leq \begin{cases} \frac{\pi^{\frac{1}{4}} e^{\alpha + \frac{1}{2}}}{\sqrt{2}} k^{\alpha + \frac{1}{2}} & \text{if } \alpha > -\frac{1}{2} \\ \frac{2}{\sqrt{\pi}} & \text{if } \alpha = -\frac{1}{2}. \end{cases}$$

Moreover, for $k = 0$,

$$||\tilde{P}_0^{(\alpha)}||_\infty \leq C(\alpha), \quad \alpha \geq -\frac{1}{2};$$

where $C(\alpha)$ is the quantity defined in Lemma 2.

Proof. Let $k \geq 1$, since

$$\frac{1}{\sqrt{h_k^{(\alpha)}}} = \frac{1}{2^\alpha \Gamma(k + \alpha + 1)} \sqrt{\Gamma(k + 1) \Gamma(k + 2\alpha + 1)} \sqrt{k + \alpha + \frac{1}{2}},$$

then

$$\frac{||P_k^{(\alpha)}||_\infty}{\sqrt{h_k^{(\alpha)} / k + \alpha + \frac{1}{2}}} = \frac{1}{2^\alpha \Gamma(\alpha + 1)} \frac{\Gamma(k + 2\alpha + 1)}{\Gamma(k + 1)}.$$

For $\alpha = -\frac{1}{2}$, one gets

$$||\tilde{P}_k^{(-\frac{1}{2})}||_\infty = \frac{||P_k^{(-\frac{1}{2})}||_\infty}{\sqrt{h_k^{(1/2)}}} = \frac{\sqrt{2}}{\Gamma(1/2)} \frac{\Gamma(k)}{\Gamma(k + 1)} \sqrt{k + 1} = \frac{\sqrt{2}}{\sqrt{\pi}} \sqrt{1 + \frac{1}{k}} \leq \frac{2}{\sqrt{\pi}}.$$
Next, for the case $\alpha > -\frac{1}{2}$ and by using the bounds of the Gamma function (8), one gets
\[
\frac{\Gamma(k + 2\alpha + 1)}{\Gamma(k + 1)} \leq \left(\frac{\pi}{e}\right)^\frac{1}{2} \left(\frac{k}{e}\right)^{2\alpha} \left(1 + \frac{2\alpha}{k + \frac{1}{2}}\right)^{k + \frac{1}{2}} \left(1 + \frac{2\alpha + \frac{1}{2}}{k}\right)^{2\alpha}.
\]
Thus,
\[
\frac{\|P_k^{(\alpha)}\|_\infty}{\sqrt{h_k^{(\alpha)}} \sqrt{k + \alpha + \frac{1}{2}}} \leq \frac{1}{2^{\alpha} \Gamma(\alpha + 1)} \left(\frac{\pi}{e}\right)^\frac{1}{2} \left(\frac{k}{e}\right)^{2\alpha} \left(\frac{2\alpha + \frac{1}{2}}{k + \frac{1}{2}}\right)^{k + \frac{1}{2}} \left(1 + \frac{2\alpha + \frac{1}{2}}{k}\right) \alpha.
\]
Since $\left(1 + \frac{2\alpha}{k + \frac{1}{2}}\right)^{k + \frac{1}{2}} \leq e^\alpha$, then we get
\[
\frac{\|P_k^{(\alpha)}\|_\infty}{\sqrt{h_k^{(\alpha)}}} \leq \frac{\Gamma(\alpha + 1)}{2^{\alpha} \Gamma(\alpha + 1)} \left(\frac{\pi}{e}\right)^\frac{1}{2} \left(\frac{k}{e}\right)^{2\alpha} \left(2\alpha + \frac{1}{2}\right)^{\alpha} \sqrt{\alpha + \frac{3}{2}} = \frac{\Gamma(\alpha + 1)}{2^{\alpha}} \left(\frac{\pi}{e}\right)^\frac{1}{2} \left(\alpha + \frac{3}{4}\right) \alpha \sqrt{\alpha + \frac{3}{2}}.
\]
Using again (8), we get $\frac{1}{\Gamma(\alpha + 1)} \leq \frac{1}{\sqrt{2\pi}} \left(\frac{e}{\alpha + \frac{1}{2}}\right)^{\alpha + \frac{1}{2}}$. Consequently, one gets
\[
\frac{\|P_k^{(\alpha)}\|_\infty}{\sqrt{h_k^{(\alpha)}}} \leq \frac{\pi^{\frac{1}{2}} k^{\alpha + \frac{3}{2}} e^{\alpha - \frac{1}{2}}}{\sqrt{2}} \left(1 + \frac{1}{\alpha + \frac{1}{2}}\right)^{\alpha + \frac{1}{2}} \leq \frac{\pi^{\frac{1}{2}} k^{\alpha + \frac{3}{2}}}{\sqrt{2} \alpha^{\frac{3}{2}}}.
\]
Finally, for $k = 0$, we have $\|\tilde{P}_0^{(\alpha)}\|_\infty^2 = \frac{1}{h_0^{(\alpha)}}$, which is, according to Lemma 2, upper bounded by $C^2(\alpha)$.

**Corollary 1.** Under the same hypothesis and notations of the previous proposition, for any $\alpha \geq -\frac{1}{2}$ and for any integer $N \geq 1$, we have
\[
\sum_{k=0}^N \|\tilde{P}_k^{(\alpha)}\|_\infty \leq \eta_\alpha \frac{(N + 1)^{\alpha + \frac{3}{2}}}{\alpha + \frac{3}{2}}, \quad \eta_\alpha = \frac{\pi^{\frac{1}{2}} e^{\alpha + \frac{3}{2}}}{\sqrt{2}}.
\]

**Proof.** From the previous proposition, we can write
\[
\sum_{k=0}^N \|\tilde{P}_k^{(\alpha)}\|_\infty = \|\tilde{P}_0^{(\alpha)}\|_\infty + \sum_{k=1}^N \|\tilde{P}_k^{(\alpha)}\|_\infty \leq \frac{\eta_\alpha}{\alpha + \frac{3}{2}} + \sum_{k=1}^N \eta_\alpha k^{\alpha + \frac{1}{2}}
\]
\[
\leq \eta_\alpha \int_0^{N+1} x^{\alpha + \frac{3}{2}} dx = \eta_\alpha \frac{(N + 1)^{\alpha + \frac{3}{2}}}{\alpha + \frac{3}{2}}.
\]

Let $L^2(I, w_\alpha)$ be the Hilbert space associated to the inner product $\langle f, g \rangle = \int_I f \cdot g \cdot w_\alpha(x) dx$. Note that the family $\{\tilde{P}_k^{(\alpha)}, k \geq 0\}$ is an orthonormal basis of $L^2(I, w_\alpha)$.

### 3 An ANOVA type space based on multivariate Jacobi polynomials

In this paragraph, we describe a reduced size multidimensional polynomials space. The construction of this space is based on combining the ANOVA decomposition technique see for example [20, 25, 30, 32, 36, 38] and the total degree polynomial space, see for example [7]. For this purpose, let $D = [[1, d]] = \{1, 2, \ldots, d\}$. For this purpose, let $D = [[1, d]] = \{1, 2, \ldots, d\}$
and we will adopt the notations $u \subset D$ for subsets of $D$, $x_u = (x_i)_{i \in u}$ and $|u|$ for the length of the vector $u$.
For a given $u \subset D$, we let $F_u$ denote the subset of $\mathbb{Z}^d$ defined by

$$F_u = \{ k \in \mathbb{Z}^d / k_u = 0_{2d-|u|} ; k_j \neq 0 \forall j \in u \} .$$

First, we describe a Jacobi polynomials orthonormal basis of $L^2(I^d, w_{\alpha})$ based on the ANOVA decomposition. Here, the $d-$variate weight function $w$ is defined on $I^d$ by

$$\forall x = (x_1, \ldots, x_d) \in I^d, w_{\alpha}(x) = \prod_{i=1}^{d} w_{\alpha}(x_i) = \left( \prod_{i=1}^{d} (1 - x_i^2) \right)^{\alpha} .$$

The usual inner product associated with $L^2(I^d, w_{\alpha})$ is defined by

$$< f, g >_w = \int_{[-1,1]^d} f(x) \cdot g(x) w_{\alpha}(x) \, dx .$$

Let $\Phi_{0,\alpha}^{(\alpha)}(x) = \frac{1}{\sqrt{h_{\alpha}^{(\alpha)}}}$ and for $u \subset D$ such that $|u| \geq 1$ and $k \in F_u$, let

$$\Phi_{u,k,\alpha}^{(\alpha)}(x) := \frac{1}{\sqrt{h_{\alpha}^{(\alpha)}}} \prod_{i \in u} P_{k_i}^{(\alpha)}(x_i) .$$

**Lemma 3.** The family

$$\left\{ \Phi_{0,\alpha}^{(\alpha)} \right\} \cup \left\{ \Phi_{u,k,\alpha}^{(\alpha)} : u \subset D, \ k \in F_u \right\}$$

is an orthonormal basis of $L^2(I^d, w_{\alpha})$.

**Proof.** Since

$$\text{Span} \left\{ \Phi_{0,\alpha}^{(\alpha)} \right\} \cup \left\{ \Phi_{u,k,\alpha}^{(\alpha)} : u \subset D, \ k \in F_u \right\} = \text{Span} \left\{ \prod_{i=1}^{d} P_{k_i}^{(\alpha)}, k_i \geq 0 \right\}$$

and since this latter is dense in $L^2(I^d, w_{\alpha})$, then it suffices to establish the orthornormality of the vectors $\Phi_{0,\alpha}^{(\alpha)}$ and $\Phi_{u,k,\alpha}^{(\alpha)}$, we consider the following three cases.

1. Computation of $< \Phi_{u,k,\alpha}^{(\alpha)}, \Phi_{u',k',\alpha}^{(\alpha)} >_{\alpha}$. We first assume that $|u| \geq 1$, then we have

$$< \Phi_{u,k,\alpha}^{(\alpha)}, \Phi_{u',k',\alpha}^{(\alpha)} >_{\alpha} = \int_{[-1,1]^d} \Phi_{u,k,\alpha}^{(\alpha)}(x) \prod_{i=1}^{d} w_{\alpha}(x_i) \, dx_1 \ldots dx_d$$

$$= \int_{[-1,1]^d} \left( \prod_{i \in u} \left( P_{k_i}^{(\alpha)}(x_i) \right) \prod_{i \not\in u} w_{\alpha}(x_i) \right) \prod_{i=1}^{d} w_{\alpha}(x_i) \, dx_1 \ldots dx_d$$

$$= \left( \prod_{i \in u} \int_{-1}^{1} \left( P_{k_i}^{(\alpha)}(x_i) \right)^2 w_{\alpha}(x_i) \, dx_i \right) \left( \prod_{1 \leq j \leq |u|} \int_{-1}^{1} \left( P_{0}^{(\alpha)}(x_j) \right)^2 w_{\alpha}(x_j) \, dx_j \right) = 1 .$$

(15)

In a similar manner, we have $< \Phi_{0,\alpha}^{(\alpha)}, \Phi_{0,\alpha}^{(\alpha)} >_{\alpha} = 1$.

2. Computation of $< \Phi_{u,k,\alpha}^{(\alpha)}, \Phi_{u',l,\alpha}^{(\alpha)} >_{\alpha}$ in the case where $|u| \geq 1$ and $k \neq l$.

As $k \neq l$ then $\exists i_0 \in u$ such that $k_{i_0} \neq l_{i_0}$.
\[
\langle \Phi_{u,k}^{(\alpha)}, \Phi_{v,l}^{(\alpha)} \rangle = \int_{[-1,1]^d} \Phi_{u,k}^{(\alpha)}(x)\Phi_{v,l}^{(\alpha)}(x)w_\alpha(x)dx
\]
\[
= \left( \frac{1}{h^d_0} \right) d^{-\|u\|} \left( \prod_{j \in [N]} \Phi_{k_j}^{(\alpha)}(x_j) \left( \frac{1}{h^d_0} \right)^{\frac{d-\|u\|}{2}} \right) \prod_{s=1}^{d} w_\alpha(x_s)dx_s
\]
\[
= \left( \int_{[-1,1]^{d-1}} \Phi_{k_0}^{(\alpha)}(x_{i_0}) dx_{i_0} \right) \left( \prod_{i \in [N], j \in [l]} \Phi_{k_j}^{(\alpha)}(x_i) \left( \frac{1}{h^d_0} \right)^{\frac{2d-\|u\|-\|v\|-1}{2}} \right) \prod_{s \in [[1,d]] \setminus i_0} w_\alpha(x_s)dx_s = 0
\]

(16)

3. Computation of \( \langle \Phi_{u,k}^{(\alpha)}, \Phi_{v,l}^{(\alpha)} \rangle > \). Let us consider the case where \( u \neq v \), \( |u| \geq 1 \) and \( |v| \geq 1 \). As \( u \neq v \), we will have \( i_0 \in (u \setminus v) \) or \( v \setminus (u \setminus v) \). Without loss of generality, we will suppose that \( i_0 \in (u \setminus v) \).

\[
\langle \Phi_{u,k}^{(\alpha)}, \Phi_{v,l}^{(\alpha)} \rangle > \alpha
\]
\[
= \int_{[-1,1]^d} \Phi_{u,k}^{(\alpha)}(x)\Phi_{v,l}^{(\alpha)}(x)w_\alpha(x)dx
\]
\[
= \left( \prod_{i \in [N]} \Phi_{k_0}^{(\alpha)}(x_i) \left( \frac{1}{h^d_0} \right)^{\frac{d-\|u\|}{2}} \right) \prod_{s=1}^{d} w_\alpha(x_s)dx_s
\]
\[
= \left( \int_{[-1,1]^{d-1}} \Phi_{k_0}^{(\alpha)}(x_{i_0}) dx_{i_0} \right) \left( \prod_{i \in [N], j \in [l]} \Phi_{k_j}^{(\alpha)}(x_i) \left( \frac{1}{h^d_0} \right)^{\frac{2d-\|u\|-\|v\|-1}{2}} \right) \prod_{s \in [[1,d]] \setminus i_0} w_\alpha(x_s)dx_s = 0
\]

(17)

Similarly to (17), we have in the case where \( |v| = 0 < \Phi_{u,k}^{(\alpha)}, \Phi_{u,l}^{(\alpha)} > \alpha = 0 \).

Next, we consider the following reduced size polynomial space, defined for \( m \in [[1, N]] \) on which we will construct our estimator.

\[
\mathcal{P}_{N,m,d} = \text{Span} \left\{ \prod_{i \in [N]} \Phi_{k_i}^{(\alpha)}(x_i) ; u \subset D ; |u| \leq m; k \in F_u, ||k||_1 \leq N \right\}.
\]

(18)

We will call this space ANOVA type polynomial space. The dimension of the space \( \mathcal{P}_{N,m,d} \) as well as an estimate of its upper bound are given by the following proposition.

**Proposition 2.** For any positive integers \( 1 \leq m \leq \min(d, N) \), we have

\[
M_{N,m,d} = \dim \mathcal{P}_{N,m,d} = \sum_{k=0}^{m} \binom{d}{k} \binom{N}{k}.
\]

(19)

Moreover, we have

\[
M_{N,m,d} \leq \begin{cases} 
\frac{\pi}{e} \left( 1 + \frac{m}{\min(d,N) - m + \frac{1}{2}} \right)^{2 \left( \max(d,N) - m + \frac{1}{2} \right)} \left( \frac{\max(d,N) + \frac{1}{2}}{m + \frac{1}{2}} \right)^{2m} & \text{if } m \leq \frac{1}{2} \min(d, N), \\
2 \min(d,N) \left( 1 + \frac{m}{\max(d,N) - m} \right)^{\max(d,N) - m} \left( \frac{\max(d,N)}{m} \right)^m & \text{if } \frac{1}{2} \min(d, N) \leq m < \min(d, N)
\end{cases}
\]

(20)

and, if \( m = \min(d, N) \),

\[
M_{N,m,d} = \binom{N + d}{m} \leq \sqrt{\frac{\pi}{(2d + 2N + 1)e}} \left( 1 + \frac{N}{d + \frac{1}{2}} \right)^{d + \frac{1}{2}} \left( 1 + \frac{d}{N + \frac{1}{2}} \right)^{N + \frac{1}{2}}
\]

(21)
Proof. We first check (19). For this purpose, we first consider the two special cases $m = 2$ and $m = 3$. Then, we check the previous identity for any $m \leq \min(d, N)$. For $m = 2$, the space $\mathcal{P}_{N,2,d}$ can be rewritten as

$$\mathcal{P}_{N,2,d} = H_1^N \oplus H_2^N,$$

with

$$H_1^N = \text{Span}\{\tilde{P}_i^{(\alpha)}(x_i), \ 0 \leq l \leq N, \ 1 \leq i \leq d\}$$

and

$$H_2^N = \text{Span}\{\tilde{P}_{k_1}^{(\alpha)}(x_1)\tilde{P}_{k_2}^{(\alpha)}(x_2), \ (k_1, k_2) \in [[1, N-1]]^2, \ 2 \leq k_1 + k_2 \leq N, \ 1 \leq i < j \leq d\}.$$

It is not difficult to check that the dimensions of $H_1^N$ and $H_2^N$ are given by

$$\dim H_1^N = 1 + N d = \sum_{k=0}^{1} \binom{d}{k} \binom{N}{k}, \quad \dim H_2^N = \frac{d(d - 1)N(N-1)}{4} = \frac{d}{2} \binom{N}{2}.$$

Hence,

$$\dim \mathcal{P}_{N,2,d} = M_{N,2,d} = \sum_{k=0}^{2} \binom{d}{k} \binom{N}{k}.$$

In a similar manner; for $m = 3$, we have $\mathcal{P}_{N,3,d} = \mathcal{P}_{N,2,d} \oplus H_3^N$, with

$$H_3^N = \text{Span}\{\prod_{i=1}^{3} \tilde{P}_{k_i}^{(\alpha)}(x_{j_i}), \ (k_1, k_2, k_3) \in [[1, N-2]]^3, \ 3 \leq k_1 + k_2 + k_3 \leq N, \ 1 \leq j_1 < j_2 < j_3 \leq d\}.$$

Note that there exist $\binom{d}{3}$ different 3–tuples $(x_{j_1}, x_{j_2}, x_{j_3})$ in $H_3^N$. Moreover, for each such a tuple, there correspond $\frac{1}{2} \sum_{k=2}^{N-1} k(k-1) = \frac{N(N-1)}{6}$ different polynomials products $\prod_{i=1}^{3} \tilde{P}_{k_i}$, $3 \leq \sum_{i=1}^{3} k_i \leq N$. Consequently, we have $\dim H_3^N = \binom{d}{3} \binom{N}{3}$ and

$$\dim \mathcal{P}_{N,3,d} = \dim \mathcal{P}_{N,2,d} + \dim H_3^N = \sum_{k=0}^{3} \binom{d}{k} \binom{N}{k}.$$

Continuing in this manner, one gets the recurrence formula

$$\dim \mathcal{P}_{N,m,d} = \dim \mathcal{P}_{N,m-1,d} + \dim H_m^N = \sum_{k=0}^{m-1} \binom{d}{k} \binom{N}{k} + \binom{d}{m} \binom{N}{m} = \sum_{k=0}^{m} \binom{d}{k} \binom{N}{k}.$$

Next, to prove (20), we proceed as follows. By using (8), one gets for any integers $n \geq k \geq 0$

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \leq \sqrt{\frac{\pi}{2e^2}} \frac{(\frac{n+\frac{1}{2}}{e})^{n+\frac{1}{2}}}{(\frac{k+\frac{1}{2}}{e})^{k+\frac{1}{2}} (\frac{n-k+\frac{1}{2}}{e})^{n-k+\frac{1}{2}}} \leq \sqrt{\frac{\pi}{2e}} \frac{1}{\sqrt{k+\frac{1}{2}}} \left(1 + \frac{k}{n-k+\frac{1}{2}}\right)^{n-k+\frac{1}{2}} \left(\frac{n+\frac{1}{2}}{k+\frac{1}{2}}\right)^k. \quad (22)$$
Since for $1 \leq m \leq \frac{1}{2} \min(d,N)$, the finite sequences $\left\{(\binom{d}{i})\right\}_{0 \leq i \leq m}$, $\left\{(\binom{N}{i})\right\}_{0 \leq i \leq m}$ are increasing, then by using \cite{22}, one gets

$$M_{N,m,d} \leq (m+1)\left(\frac{d}{m}\right)\left(\frac{N}{m}\right) = (m+1) \frac{\Gamma(d+1)\cdot \Gamma(N+1)}{\Gamma(m+1)\Gamma(d-m+\frac{1}{2}) \cdot \Gamma(m+1)\Gamma(N-m+\frac{1}{2})} \leq \frac{\pi}{e} \left(1 + \frac{m}{\min(d,N) - m + \frac{1}{2}}\right)^2 \left(\frac{\max(d,N)+\frac{1}{2}}{m+\frac{1}{2}}\right)^2m \cdot \left(\frac{\max(d,N)+\frac{1}{2}}{m+\frac{1}{2}}\right)^2. \leq 2^{\min(d,N)} \max(d,N)^{\max(d,N)} \frac{m^m(\max(d,N)-m)^{\max(d,N)-m}}{\min(d,N)^{\max(d,N)-m} \cdot \min(d,N)^{\max(d,N)-m} \cdot \max(d,N)^{\max(d,N)-m}}.$$

In a similar manner, for $\frac{1}{2} \min(d,N) \leq m < \min(d,N)$, we have

$$M_{N,m,d} \leq \sum_{k=0}^{m} \left(\binom{d}{k}\right)\left(\binom{N}{k}\right) \leq \sum_{k=0}^{\min(d,N)} \left(\binom{\min(d,N)}{k}\right) \cdot \sum_{k=0}^{\max(d,N)} \left(\binom{\max(d,N)}{k}\right) \leq 2^{\min(d,N)} \max(d,N)^{\max(d,N)} \frac{m^m(\max(d,N)-m)^{\max(d,N)-m}}{\min(d,N)^{\max(d,N)-m} \cdot \min(d,N)^{\max(d,N)-m} \cdot \max(d,N)^{\max(d,N)-m}}.$$

This last inequality is a consequence of the inequality,

$$\sum_{k=0}^{m} \left(\binom{n}{k}\right) \leq \frac{n^n}{m^m(n-m)^{n-m}}, \quad \forall \ 0 \leq m < n.$$

The previous inequality is a direct consequence of the following upper bound for the binomial coefficient, see \cite{13}, p.353

$$\left(\frac{n}{k}\right) \leq 2^n H(k/n),$$

where for $0 < p < 1$, $H(p)$ is the binary entropy function, given by $H(p) = -p \log_2(p) - (1 - p) \log_2(1 - p)$. For the last case where $m = \min(d,N)$, using Chu-Vandermonde’s identity, see for example \cite{31}, we get

$$\sum_{k=0}^{m} \left(\binom{d}{k}\right)\left(\binom{N}{k}\right) = \sum_{k=0}^{\min(d,N)} \left(\binom{\min(d,N)}{k}\right) \cdot \sum_{k=0}^{\max(d,N)} \left(\binom{\max(d,N)}{k}\right) = \sum_{k=0}^{\min(d,N)} \left(\binom{\min(d,N)}{k}\right) \cdot \sum_{k=0}^{\max(d,N)} \left(\binom{\max(d,N)}{k}\right) = \left(\frac{N+d}{m}\right).$$

Using \cite{3} we get

$$\left(\frac{N+d}{m}\right) = \frac{\Gamma(N+d+1)}{\Gamma(d+1)\Gamma(N+1)} \leq \sqrt{\frac{\pi}{2e}} \frac{1}{\sqrt{N+d+\frac{1}{2}}} \left(\frac{N+d+\frac{1}{2}}{d+\frac{1}{2}}\right)^{d+\frac{1}{2}} \left(\frac{N+d+\frac{1}{2}}{N+\frac{1}{2}}\right)^{N+\frac{1}{2}}. \leq 2^{\min(d,N)} \max(d,N)^{\max(d,N)} \frac{m^m(\max(d,N)-m)^{\max(d,N)-m}}{\min(d,N)^{\max(d,N)-m} \cdot \min(d,N)^{\max(d,N)-m} \cdot \max(d,N)^{\max(d,N)-m}}.$$

Remark 1. For the particular case $m = d$, the space $\mathcal{P}_{N,d,d}$ is the usual total degree polynomial of order $N$. From Proposition \cite{3} we have $M_{N,m,d} = \left(\frac{N+d}{d}\right)$ which recovers the well known dimension of the total degree polynomial space.

4 Stability of the NP estimator

In this section, we first describe our proposed Least squares NP regression estimator based on the use of the orthonormal $d$–variate polynomials $\Phi_{u,k}^{(\alpha)}$. Then, we prove the stability of the proposed estimator under the condition that the random sampling covariates follow a $d$–dimensional Beta distribution.

Recall that the regression system at hand is given by

$$Y_i = f(X_i) + \varepsilon_i, \quad i = 1, \ldots, n. \quad (24)$$
Assuming that $\hat{K} \subset D$ the regression function $f$ its condition number denoted by $g$. Let $\beta(\alpha)$, we obtain $g$.

Consider the positive definite random matrix (to be checked later on), $\sigma$ to be i.i.d. random vectors and the $\varepsilon$ to be i.i.d. random variables with a finite variance $\sigma^2 = \mathbb{E}[\varepsilon^2]$. Also we assume that the $X_i$ are independent from the $\varepsilon_j$. In the sequel, we assume that for a real $\alpha \geq -\frac{1}{2}$, the set $\{X_i, i = 1, \ldots, n\}$ is a random sampling set with the $X_i$ following a $d$-variates Beta($\alpha + 1, \alpha + 1$) distribution. For two positive integers $m \leq d$ and $N$, we build an estimator $\hat{f}_{N,m}(\alpha)$ of the regression function $f$ which is obtained using the approximation of $f$ by its projection over $\mathcal{P}_{N,m,d}$. For $\mathbf{u} \in D$ with $|u| \geq 1$ and $\mathbf{k} \in F_{\mathbf{u}}$, let $\Phi_{\mathbf{k},\mathbf{u}} := \Phi_{\mathbf{k},\mathbf{u},\mathbf{u}}$. Let $K_{N,0} := \{(0, \ldots, 0) \in \mathbb{R}^d\}$ and for $p \in [[1, m]]$, let $K_{N,p}$ be the subsets defined as follows

$$K_{N,p} := \{\mathbf{k} \in F_{\mathbf{u}} : |\mathbf{k}| = p : |k_{\mathbf{u}}| \leq N\}, \quad K_{N,m,d} := \bigcup_{p=0}^{m} K_{N,p}.$$ 

Let $g : K_{N,m,d} \mapsto [[1, M_{N,m,d}]]$ be a correspondence (order) defined on the indexes $(\mathbf{k}, \mathbf{u})$ of our basis : $g(\mathbf{k}, \mathbf{u}) \in [[1, M_{N,m,d}]]$. Then, we can introduce the notation

$$\Psi_j^{(\alpha)} := \Phi_{g^{-1}(j)}, \quad \text{with } i \in [[1, M_{N,m,d}]].$$

Using these definitions, our NP regression estimator is given by

$$\hat{f}_{N,m}(\alpha, x) = \sum_{\mathbf{k} \in K_{N,m,d}} \hat{C}_{\mathbf{k}}(f) \Phi_{\mathbf{k},\mathbf{u}}(x) = \sum_{j=1}^{M_{N,m,d}} \hat{C}_j(f) \Psi_j(x), \quad x \in \mathbb{R}^d.$$

(25)

Assuming that $\hat{f}_{N,m}(\alpha)$ satisfies $\hat{f}_{N,m}(\alpha, X_i) = Y_i$ and multiplying the previous equation by $\frac{[2^{\alpha+1}B(\alpha+1,\alpha+1)]^{\frac{1}{2}}}{n^{\frac{1}{2}}}$, we obtain

$$\sum_{j=1}^{M_{N,m,d}} \hat{C}_j(f) \frac{[2^{\alpha+1}B(\alpha+1,\alpha+1)]^{\frac{1}{2}}}{n^{\frac{1}{2}}} \Psi_j(X_i) = \frac{[2^{\alpha+1}B(\alpha+1,\alpha+1)]^{\frac{1}{2}}}{n^{\frac{1}{2}}} Y_i.$$

(26)

This system can be rewritten as a system of linear equations where the unknown is the expansion coefficients vector $\hat{C}_{N,m} = \left(\hat{C}_j\right)_{1 \leq j \leq M_{N,m,d}}$ :

$$F_{N,m}^{(\alpha)} \hat{C}_{N,m}^{(\alpha)} = \frac{[2^{\alpha+1}B(\alpha+1,\alpha+1)]^{\frac{1}{2}}}{n^{\frac{1}{2}}} Y.$$

Consider the positive definite random matrix (to be checked later on),

$$G_{N,m}^{(\alpha)} = \left(F_{N,m}^{(\alpha)}\right)^T \cdot F_{N,m}^{(\alpha)},$$

(27)

Then, we have

$$\hat{C}_{N,m}^{(\alpha)} = \left(G_{N,m}^{(\alpha)}\right)^{-1} \left(F_{N,m}^{(\alpha)}\right)^T \frac{[2^{\alpha+1}B(\alpha+1,\alpha+1)]^{\frac{1}{2}}}{n^{\frac{1}{2}}} Y,$$

(28)

with

$$F_{N,m}^{(\alpha)}(i, j) = \frac{[2^{\alpha+1}B(\alpha+1,\alpha+1)]^{\frac{1}{2}}}{n^{\frac{1}{2}}} \Psi_j^{(\alpha)}(X_i).$$

(29)

Next, we show that our proposed NP regression estimator is stable in the sense that the random matrix $G_{N,m}^{(\alpha)}$ is well conditioned with respect to the 2–norm. As the random matrix $G_{N,m}^{(\alpha)}$ is positive definite, its condition number denoted by $\kappa_2 \left(G_{N,m}^{(\alpha)}\right)$ is given by

$$\kappa_2 \left(G_{N,m}^{(\alpha)}\right) = \frac{\lambda_{\max} \left(G_{N,m}^{(\alpha)}\right)}{\lambda_{\min} \left(G_{N,m}^{(\alpha)}\right)}.$$

(30)
Recall that the i.i.d. random samples \((X_i)_{1 \leq i \leq n}\) follow the \(d\)-variate Beta distribution on \(I^d\) with the density function \(g_\alpha\) given by
\[
g_\alpha(x) = \frac{1}{[2^{2\alpha+1}B(\alpha+1,\alpha+1)]^d} w_\alpha(x) \cdot I_{I^d}(x) = \frac{1}{(h_0^{(\alpha)})^d} w_\alpha(x) \cdot I_{I^d}(x). \tag{31}
\]
Before stating the theorem about the condition number upper bound (Theorem 1), we need the following technical lemma.

**Lemma 4.** For \(\alpha \geq -\frac{1}{2}\), \(N \in \mathbb{N}\) and a positive integer \(m \leq d\), let
\[
D(N, \alpha) := \begin{cases} \frac{\pi^{\frac{d}{2}} N^{(\alpha + \frac{1}{2})} e^{(\alpha + \frac{1}{2})}}{2} & \text{if } \alpha > -\frac{1}{2} \\ 2 & \text{if } \alpha = -\frac{1}{2}. \end{cases} \tag{32}
\]
Then, we have
\[
\left\| \Phi^{(\alpha)}_0 \right\|_\infty = \left( \frac{1}{h_0^{(\alpha)}} \right)^\frac{d}{2}. \tag{33}
\]
Moreover, for \(1 \leq |u| \leq m\) and \(\|k_u\|_1 \leq N\), we have
\[
\left\| \Phi^{(\alpha)}_{k_u} \right\|_\infty \leq \left( \frac{1}{h_0^{(\alpha)}} \right)^\frac{d}{2} \cdot |D(N, \alpha)|^m. \tag{34}
\]

**Proof.** Equality (33) is immediate since \(\Phi^{(\alpha)}_0 = \left[ \tilde{P}^{(\alpha)}_0 \right]^d = \left( \frac{1}{h_0^{(\alpha)}} \right)^\frac{d}{2}. \) Also from the definition of \(\Phi^{(\alpha)}_{k_u}\), we have
\[
\left\| \Phi^{(\alpha)}_{k_u} \right\|_\infty \leq \left\| \tilde{P}^{(\alpha)}_0 \right\|_\infty^{|d-u|} \prod_{i \in u} \left\| \tilde{P}^{(\alpha)}_{k_i} \right\|_\infty = \left( \frac{1}{h_0^{(\alpha)}} \right)^\frac{d-|u|}{2} \prod_{i \in u} \left\| \tilde{P}^{(\alpha)}_{k_i} \right\|_\infty.
\]
Next, for the case where \(\alpha > -\frac{1}{2}\) and by using Proposition 4 and Lemma 2 one gets:
\[
\left\| \Phi^{(\alpha)}_{k_u} \right\|_\infty \leq \left( \frac{1}{h_0^{(\alpha)}} \right)^\frac{d}{2} \prod_{i \in u} \left[ \frac{\pi^{\frac{d}{2}} N^{\alpha + \frac{1}{2}} e^{\alpha + \frac{1}{2}}}{\sqrt{2}} \right]^{\frac{|u|}{2}} \cdot \left[ \frac{\pi^{\frac{d}{2}} N^{(\alpha + \frac{1}{2})} |u| e^{(\alpha + \frac{1}{2})} |u|}{\sqrt{2} |u|} \right]^{\frac{|u|}{2}} \tag{35}
\]
\[
\leq \left( \frac{1}{h_0^{(\alpha)}} \right)^\frac{d}{2} \left[ \pi^{\frac{d}{2}} N^{(\alpha + \frac{1}{2})} |u| e^{(\alpha + \frac{1}{2})} |u| \right] \cdot \left[ \frac{\pi^{\frac{d}{2}} N^{(\alpha + \frac{1}{2})} e^{(\alpha + \frac{1}{2})}}{\sqrt{2} |u|} \right]^{m}.
\]
Finally, for the case \(\alpha = -\frac{1}{2}\) and by using again Proposition 4 and Lemma 2 one gets
\[
\left\| \Phi^{(\alpha)}_{k_u} \right\|_\infty \leq \left( \frac{1}{h_0^{(\alpha)}} \right)^\frac{d}{2} \left( h_0^{(-\frac{1}{2})} \right)^\frac{|u|}{2} \prod_{i \in u} \left[ \frac{2}{\sqrt{\pi}} \right]^{\frac{|u|}{2}} \leq \left( \frac{1}{h_0^{(\alpha)}} \right)^\frac{d}{2} 2^m. \tag{36}
\]
In the following theorem, we show that with high probability, the 2–norm condition number of the positive definite random matrix $G_{N,n,m}^{\alpha}$ is bounded by a convenient constant depending on a parameter $0 < \delta < 1$. The results of this theorem can be considered as a generalization of a similar result given in [6] for tensor product multivariate Jacobi polynomials basis.

**Theorem 1.** Under the previous notation and hypotheses on $N, m, \alpha$, let $0 < \delta < 1$ and let $G_{N,n,m}^{\alpha}$ be the random matrix given by (27). Then, we have

$$
\mathbb{P} \left( \kappa_2 \left( G_{N,n,m}^{\alpha} \right) \leq \frac{1 + \delta}{1 - \delta} \right) \geq 1 - 2M_{N,m,d} \exp \left( -\delta^2 \frac{n}{3D^2m(N,\alpha)M_{N,m,d}} \right).
$$

(37)

Here, the quantity $D(N, \alpha)$ is as defined by Lemma 2 and $M_{N,m,d}$ is as given by Proposition 2.

**Proof.** From (29), we have $F_{N,n,m}^{\alpha} F(p, i) = \frac{[2^{\alpha+1}B(\alpha+1,\alpha+1)]^\frac{\delta}{2}}{n^\frac{\delta}{2}} \Phi_i^{\alpha}(X_p)$ and $G_{N,n,m}^{\alpha} = \left( F_{N,n,m}^{\alpha} \right)^T F_{N,n,m}^{\alpha}$, one gets

$$
G_{N,n,m}^{\alpha}(i,j) = \sum_{p=1}^{n} \left[ \frac{2^{\alpha+1}B(\alpha+1,\alpha+1)}{n^\frac{\delta}{2}} \Phi_i^{\alpha}(X_p) \Phi_j^{\alpha}(X_p) \right] = \sum_{p=1}^{n} \frac{\left( h_0^{\alpha} \right)^d}{n} \Phi_i^{\alpha}(X_p) \Phi_j^{\alpha}(X_p).
$$

Consequently, one gets

$$
\mathbb{E}[G_{N,n,m}^{\alpha}(i,j)] = \frac{\left( h_0^{\alpha} \right)^d}{n} \sum_{p=1}^{n} \mathbb{E} \left[ \Phi_i^{\alpha}(X_p) \Phi_j^{\alpha}(X_p) \right].
$$

Now to compute $\mathbb{E} \left[ \Phi_i^{\alpha}(X_p) \Phi_j^{\alpha}(X_p) \right]$, we let $i = g(k,u)$ and $j = g(l,v)$, then by using Lemma 3, we obtain

$$
\mathbb{E} \left[ \Phi_i^{\alpha}(X_p) \Phi_j^{\alpha}(X_p) \right] = \mathbb{E} \left[ \Phi_i^{\alpha}(X_p) \Phi_j^{\alpha}(X_p) \right] = \int_{[-1,1]^d} \Phi_i^{\alpha}(x) \Phi_j^{\alpha}(x) \prod_{a=1}^{d} w_a(x_a) dx_a
$$

\[
\frac{\left( h_0^{\alpha} \right)^d}{n} \Phi_i^{\alpha}(x) \Phi_j^{\alpha}(x) \prod_{a=1}^{d} w_a(x_a) dx_a = \frac{\left( h_0^{\alpha} \right)^d}{n} \delta_{i,j}.
\]

(38)

Hence, we have $\mathbb{E} \left[ \Phi_i^{\alpha}(X_p) \Phi_j^{\alpha}(X_p) \right] = \frac{1}{\left( h_0^{\alpha} \right)^d} \delta_{i,j}$. Therefore $\mathbb{E}[G_{N,n,m}^{\alpha}]$ is the identity matrix of dimension $M_{N,m,d}$. Also, note that

$$
G_{N,n,m}^{\alpha} = \sum_{p=1}^{n} H_p,
$$

where $H_p(i,j) = \frac{(h_0^{\alpha})^d}{n} \Phi_i^{\alpha}(X_p) \Phi_j^{\alpha}(X_p)$. We show that, for all $p \in [1,n]$, all eigenvalues of $H_p$ are non-negative. Moreover, we give an estimate for an upper bound for the eigenvalues of $H_p$. The matrix $H_p$ can be written as $H_p = A_p^T A_p$ where $A_p := \frac{(h_0)^d}{\sqrt{n}} \left( \Phi_1^{\alpha}(X_p), \ldots, \Phi_{M_{N,m,d}}^{\alpha}(X_p) \right)$. The relation $H_p = A_p^T A_p$ implies that the eigenvalues of all the matrices $H_p$ are non-negative. Applying Gershgorin Theorem to $H_p$, one gets

$$
\lambda_{\text{max}}(H_p) \leq \max_{1 \leq i \leq M_{n,m,d}} \left( \frac{(h_0)^d}{\sqrt{n}} \left( |\Phi_i^{\alpha}(X_p)|^2 + \sum_{j \neq i} |\Phi_j^{\alpha}(X_p)\Phi_j^{\alpha}(X_p)| \right) \right).
$$
Therefore \( \lambda_{\text{max}}(H_p) \leq \frac{D(N, \alpha)^{2n}}{n} \cdot M_{N,m,d} \) for all \( l \in [1, n] \). Next, by applying the matrix Chernoff Theorem to the matrix \( G_{N,n,m}^{(\alpha)} \) as a sum of positive semi definite random matrices \( G_{N,n,m}^{(\alpha)} \) and satisfying \( 0 \leq H_p \leq L \cdot I_{M_{N,m,d}} \), let \( L := \frac{(D(N, \alpha))^{2n}}{n} \cdot M_{N,m,d} \) and as \( \mu_{\min} = \lambda_{\min} \mathbb{E} \left( G_{N,n,m}^{(\alpha)} \right) = 1 \), \( \mu_{\max} = \lambda_{\max} \mathbb{E} \left( G_{N,n,m}^{(\alpha)} \right) = 1 \), we get:

\[
\mathbb{P} \left( \lambda_{\min} \left( G_{N,n,m}^{(\alpha)} \right) \geq 1 - \delta \right) \geq 1 - M_{N,m,d} \cdot \exp \left( - \frac{\delta^2 n}{2(D(N, \alpha))^{2m} \cdot M_{N,m,d}} \right)
\]

and

\[
\mathbb{P} \left( \lambda_{\max} \left( G_{N,n,m}^{(\alpha)} \right) \leq 1 + \delta \right) \geq 1 - M_{N,m,d} \cdot \exp \left( - \frac{\delta^2 n}{3(D(N, \alpha))^{2m} \cdot M_{N,m,d}} \right)
\]

for \( \delta \in (0, 1) \). Finally, consider the events \( A_1 := \left( \kappa_2 \left( G_{N,n,m}^{(\alpha)} \right) \leq \frac{1+\delta}{1-\delta} \right) \), \( A_2 = \left( \lambda_{\min} \left( G_{N,n,m}^{(\alpha)} \right) \geq 1 - \delta \right) \) and \( A_3 := \left( \lambda_{\max} \left( G_{N,n,m}^{(\alpha)} \right) \leq 1 + \delta \right) \). Using the fact that \( \mathbb{P}(A_1) \leq \mathbb{P}(A_2) + \mathbb{P}(A_3^c) \), we get

\[
\mathbb{P}(A_1) \geq 1 - 2M_{N,m,d} \cdot \exp \left( - \frac{\delta^2 n}{3(D(N, \alpha))^{2m} \cdot M_{N,m,d}} \right).
\]

**Remark 2.** By using (37) and (38), one concludes that a convenient choice for the value of the parameter \( \alpha \) is given by \( \alpha = -\frac{1}{2} \). For this value, a given upper bound for 2–norm condition number of the random matrix \( G_{N,n,m}^{(\alpha)} \) is obtained with fewer number of \( n \), the number of random sampling points \( X_i \). This behaviour is illustrated by the numerical simulations given by Table 1 of the numerical examples section.

### 5 \( L^2 \)– Risk error of the NP regression estimator

In this section, we give an estimate for the \( L^2 \)–risk error of our proposed NP regression estimator. For this purpose and it is done in [12], we assume that there exists a constant \( K_f \) such that

\[
\|f(x)\| \leq K_f, \quad \forall x \in I^d.
\]

We let \( \tilde{F}_{N,n,m}^{(\alpha)} \) denote the truncated version of the estimator \( \tilde{f}_{N,n,m}^{(\alpha)} \), given by

\[
\tilde{F}_{N,n,m}^{(\alpha)}(x) := \text{Sign} \left( \tilde{f}_{N,n,m}^{(\alpha)}(x) \right) \min \left\{ K_f, \left| \tilde{f}_{N,n,m}^{(\alpha)}(x) \right| \right\}.
\]

**Theorem 2.** Let \( f \) be a function satisfying the hypothesis (40) and let \( \tilde{F}_{N,n,m}^{(\alpha)} \) be the truncated version of the estimator \( \tilde{f}_{N,n,m}^{(\alpha)} \), given by (42). For \( \alpha \geq -\frac{1}{2} \) and \( 0 < \delta < 1 \), we have

\[
\mathbb{E} \left[ \|f - \tilde{F}_{N,n,m}^{(\alpha)}\|_2^2 \right] \leq 4K_f^2 \left( h_0^{(\alpha)} \right)^d M_{N,m,d} \cdot \exp \left( - \frac{\delta^2 n}{2(D(N, \alpha))^{2m} \cdot M_{N,m,d}} \right) + \frac{M_{N,m,d}}{n(1-\delta)^2} \left( |D(N, \alpha)|^{2m} \|f - \Pi_{N,m}f\|_2^2 + \sigma^2 \left( h_0^{(\alpha)} \right)^d \right) + \|f - \Pi_{N,m}f\|_2^2,
\]

where the quantity \( D(N, \alpha) \) is as defined by Lemma 3.

**Proof.** As it is done in [12], we write \( \Omega \) as \( \Omega_+ \cup \Omega_- \), where

\[
\Omega_- := \{(X_1, \ldots, X_n) : \lambda_{\min}(G) < 1 - \delta\} \quad \text{and} \quad \Omega_+ := \{(X_1, \ldots, X_n) : \lambda_{\min}(G) \geq 1 - \delta\}.
\]
Then
\[
\mathbb{E}\left[\|f - \hat{F}_{N,n,m}^{(\alpha)}\|_\alpha^2\right] = \int_{\Omega_-} \|f - \hat{F}_{N,n,m}^{(\alpha)}\|_\alpha^2 d\rho_n + \int_{\Omega_+} \|f - \hat{F}_{N,n,m}^{(\alpha)}\|_\alpha^2 d\rho_n,
\]
where \(d\rho_n\) is the probability measure on \((I^d)^n\) given by the tensor product
\[
d\rho_n(u_1, \ldots, u_n) = \prod_{k=1}^n d\rho(u_k),
\]
with \(\rho\) given in [31]. To get an upper bound for \(\int_{\Omega_-} \|f - \hat{F}_{N,n,m}^{(\alpha)}\|_\alpha^2 d\rho_n\), we proceed as follows. From (39), we have
\[
P\left(\lambda_{\min}\left(G_{N,n,m}^{(\alpha)}\right) < 1 - \delta\right) \leq M_{N,m,d} \cdot \exp\left(-\frac{\delta^2 n}{2[D(N,\alpha)]^{2m} \cdot M_{N,m,d}}\right).
\]
Then
\[
\int_{\Omega_-} \|f - \hat{F}_{N,n,m}^{(\alpha)}\|_\alpha^2 d\rho_n \leq 4K_2 \int_{\Omega_-} d\rho_n = 4K_2^2 \left(h_0^{(\alpha)}\right)^d \mathbb{P}\left(\lambda_{\min}\left(G_{N,n,m}^{(\alpha)}\right) < 1 - \delta\right) \leq 4K_2^2 \left(h_0^{(\alpha)}\right)^d M_{N,m,d} \cdot \exp\left(-\frac{\delta^2 n}{2[D(N,\alpha)]^{2m} \cdot M_{N,m,d}}\right).
\]
For an estimate of an upper bound of \(\int_{\Omega_+} \|f - \hat{F}_{N,n,m}^{(\alpha)}\|_\alpha^2 d\rho_n\), we use the fact that \(|f - \hat{F}_{N,n,m}^{(\alpha)}|\) is upper bounded by \(|f - \hat{F}_{N,n,m}^{(\alpha)}|\) and the fact that \((f - \Pi_{N,m} f)\) and \((\Pi_{N,m} f - \hat{F}_{N,n,m}^{(\alpha)})\) are orthogonal. Hence, we get
\[
\int_{\Omega_+} \|f - \hat{F}_{N,n,m}^{(\alpha)}\|_\alpha^2 d\rho_n \leq \int_{\Omega_+} \|f - \hat{F}_{N,n,m}^{(\alpha)}\|_\alpha^2 d\rho_n + \int_{\Omega_+} \|\Pi_{N,m} f - \hat{F}_{N,n,m}^{(\alpha)}\|_\alpha^2 d\rho_n.
\]
On the other hand, let \(C_{N,n,m}^{(\alpha)}\), the vector containing the coefficients of \(\Pi_{N,m} f\) on the basis \(\{\psi_1^{(\alpha)}, \ldots, \psi_{M_{N,m,d}}^{(\alpha)}\}\), i.e.,
\[
\Pi_{N,m} f(X_i) = \sum_{j=1}^{M_{N,m,d}} C_{N,n,m}^{(\alpha)}(\psi_j^{(\alpha)})(X_i)
\]
for \(i = 1, \ldots, n\). This leads to
\[
C_{N,n,m}^{(\alpha)} = \left(G_{N,n,m}^{(\alpha)}\right)^{-1}\left(F_{N,n,m}^{(\alpha)}\right)^T \cdot \sqrt{\left(h_0^{(\alpha)}\right)^d \frac{1}{n}} \cdot (\Pi_{N,m} f(X_i))_{1 \leq i \leq n}.
\]
The identity (28) used for the computation of \(\hat{\gamma}_{N,n,m}^{(\alpha)}\) can be rewritten as:
\[
\hat{\gamma}_{N,n,m}^{(\alpha)} = \left(G_{N,n,m}^{(\alpha)}\right)^{-1}\left(F_{N,n,m}^{(\alpha)}\right)^T \cdot \sqrt{\left(h_0^{(\alpha)}\right)^d \frac{1}{n}} \cdot (f(X_i) + \epsilon_i)_{1 \leq i \leq n}.
\]
Combining the previous two equations, we get
\[
\hat{\gamma}_{N,n,m}^{(\alpha)} - C_{N,n,m}^{(\alpha)} = \left(G_{N,n,m}^{(\alpha)}\right)^{-1}\left(F_{N,n,m}^{(\alpha)}\right)^T \cdot \sqrt{\left(h_0^{(\alpha)}\right)^d \frac{1}{n}} \cdot (g(X_i) + \epsilon_i)_{1 \leq i \leq n}.
\]
where \( g(X_i) := (f - \Pi_{N,m}(f))(X_i) \). From Parseval’s equality, we have on \( \Omega_+ \)
\[
\|\Pi_{N,m}(f) - \hat{f}^{(\alpha)}_{N,m}\|_2^2 = \|\hat{G}^{(\alpha)}_{N,m} - G^{(\alpha)}_{N,m}\|_2^2 \leq \frac{\left(\frac{h_0^{(\alpha)}}{n}\right)^d}{n(1-\delta)^2} \left\| \left( G^{(\alpha)}_{N,m} \right)^{-1} \right\|_2^2 \left\| \left( G^{(\alpha)}_{N,m} \right)^T \cdot (g(X_i) + \epsilon_i)_{1 \leq i \leq n} \right\|_2^2 \leq \frac{\left(\frac{h_0^{(\alpha)}}{n}\right)^d}{n(1-\delta)^2} \left\| \left( G^{(\alpha)}_{N,m} \right)^T \cdot (g(X_i) + \epsilon_i)_{1 \leq i \leq n} \right\|_2^2 \leq \frac{\left(\frac{h_0^{(\alpha)}}{n}\right)^d}{n^2(1-\delta)^2} \sum_{j=1}^{M_{N,m,d}} \sum_{k=1}^{n} \left( (\Psi_j(X_k)(g(X_k) + \epsilon_k)) \cdot (\Psi_j(X_i)(g(X_i) + \epsilon_i)) \right).
\]

Considering the expectations of both sides of the previous inequality and taking into account that
\(<\Psi_j^{(\alpha)}, g>_{\alpha} = 0 \) and that the \( \epsilon_k \) are independent from the \( X_k \) with \( E(\epsilon_k) = 0 \) and \( E(\epsilon_k^2) = \sigma^2 \), we get
\[
E \left[ \|\Pi_{N,m}(f) - \hat{f}^{(\alpha)}_{N,m}\|_2^2 \right] \leq \frac{\left(\frac{h_0^{(\alpha)}}{n}\right)^d}{n^2(1-\delta)^2} \sum_{j=1}^{M_{N,m,d}} \sum_{k=1}^{n} \left( E[\Psi_j^2(X_k)g^2(X_k)] + \sigma^2 \cdot E[\Psi_j^2(X_k)] \right) \leq \frac{\left(\frac{h_0^{(\alpha)}}{n}\right)^d}{n^2(1-\delta)^2} \sum_{j=1}^{M_{N,m,d}} \sum_{k=1}^{n} E[\Psi_j^2(X_k)g^2(X_k)] + \frac{\left(\frac{h_0^{(\alpha)}}{n}\right)^d}{n(1-\delta)^2} \cdot \sigma^2 M_{N,m,d}.
\]

Using Lemma \[\[\]\] we have :
\[
E \left[ \Psi_j^2(X_k)g^2(X_k) \right] \leq \|\Psi_j\|_\infty^2 \cdot \|g^2(X_k)\| \leq (D(N, \alpha))^{2m} \cdot \frac{\|f - \Pi_{N,m,f}\|_\alpha^2}{\left(\frac{h_0^{(\alpha)}}{n}\right)^{2d}}.
\]

This implies that
\[
\int_{\Omega_+} \|\Pi_{N,m}(f) - \hat{f}^{(\alpha)}_{N,m}\|_2^2 d\nu_n \leq E \left[ \|\Pi_{N,m}(f) - \hat{f}^{(\alpha)}_{N,m}\|_\alpha^2 \right] \leq \frac{M_{N,m,d}}{n(1-\delta)^2} \left( [D(N, \alpha)]^{2m} \cdot \|f - \Pi_{N,m,f}\|_\alpha^2 + \sigma^2 \left(\frac{h_0^{(\alpha)}}{n}\right)^d \right).
\]

\[\square\]

### 6 Quality of the estimation in a weighted Sobolev space

In this paragraph, we give a precise rate of convergence of our estimator in the case where the \( d \)-variate regression functions belongs to a weighted Sobolev space. More precisely, we give an estimate for the term \( \|f - \Pi_{N,m,f}\|_\alpha^2 \) in the \( L^2 \)-risk error given by Theorem 2. For this purpose, we recall some definitions and results mainly borrowed from \[\[\]\]. Let \( T \) be a unit torus and let \( f \in L^2(T^d) \). The Fourier series expansion of \( f \) is given by
\[
f(x) = \sum_{p \in \mathbb{Z}^d} \hat{f}_p \cdot e^{2\pi i p \cdot x}, \quad x \in T^d, \quad (or \mathbb{R}^d).
\]

We have the following partition of \( \mathbb{Z}^d : \mathbb{Z}^d = \bigcup_{u \in D} F_u \). This gives the analysis of variance (ANOVA) decomposition of \( f \) :
\[
f(x) = f_0 + \sum_{i=1}^{d} f_i(x_i) + \sum_{i=1}^{d} \sum_{j=i+1}^{d} f_{i,j}(x_i, x_j) + \ldots + f_D(x) = \sum_{u \in D} f_u(x_u),
\]

where \( f_0 \) is the constant term, \( f_i \) are the \( i \)-dimensional unknown functions, \( f_{i,j} \) are the \( i,j \)-dimensional unknown functions, and so on.
where the functions \( f_u \) are called ANOVA terms, see [30]. Let \( U \subset P(D) \). The truncated ANOVA decomposition over \( U \) is defined as:

\[
T_U(f) := \sum_{u \in U} f_u .
\]

In particular, for \( 1 \leq s \leq d \), we define

\[
T_s(f) := \sum_{|u| \leq s} f_u .
\]

**Definition 1.** Let \( s > 0 \). Let \( w^{(s)} : \mathbb{Z}^d \to [1, \infty) \) be the weight function defined by

\[
w^{(s)}(p) := \prod_{j=1}^d (1 + |p_j|)^s , \quad \forall \ p \in \mathbb{Z}^d .
\]

We associate to this weight function the Sobolev space

\[
H^s(I^d) := \left\{ f \in L^2(I^d); \| f \|_{H^s(I^d)} := \sum_{p \in \mathbb{Z}^d} (w^{(s)}(p))^2 \cdot |a_p(f)|^2 < +\infty \right\} ,
\]

and the weighted Wiener algebra

\[
A^s(I^d) := \left\{ f \in L^1(I^d); \| f \|_{A^s(I^d)} = \sum_{p \in \mathbb{Z}^d} w^{(s)}(p) \cdot |a_p(f)| < +\infty \right\}.
\]

The following lemma provides us with an estimate for the decay rate of the expansion series expansion coefficients of the \( f_u \) when \( f \in H^{s+\frac{d}{2}}(I^d) \) and with respect to the basis functions \( \Phi^{(\alpha)}_{k,v} \).

**Lemma 5.** Let \( 0 < \xi < \frac{1}{2} \), \( N \in \mathbb{N} \) and \( \mathbf{k} \in F_u \) such that \( |\mathbf{k}|_1 \geq N + 1 \) with

\[
\frac{N+1}{|\mathbf{k}|_1} \geq -\ln (\xi) \left( 1 + s + \frac{d}{2} \right).
\]

For any \( f \in H^{s+\frac{d}{2}}(I^d) \)

\[
|C_{\mathbf{k},u}(f_u)| = \langle f_u, \Phi^{(\alpha)}_{\mathbf{k},u} >_{\alpha} | \lesssim_{\alpha,d,s} |\mathbf{k}|^{-s-\frac{d}{2}} \left( \| f_u \|_2 + \| f_u \|_{H^{s+\frac{d}{2}}(I^d)} \right) .
\]

**Proof.** We have:

\[
f_u(x) = \sum_{p \in \mathbb{Z}^d} a_p(f_u) e^{2\pi i p x}, \quad x \in I^d .
\]

For \( \mathbf{v} \in \mathbb{Z}^d \) and \( \mathbf{k} \in \mathbb{Z}^{|\mathbf{v}|} \), we have :

\[
C_{\mathbf{k},\mathbf{v}}(f_u) = \langle f_u, \Phi^{(\alpha)}_{\mathbf{k},\mathbf{v}} >_{\alpha} = \sum_{p \in \mathbb{Z}^d} a_p(f_u) e^{2\pi i p \mathbf{v}} \cdot \langle \Phi^{(\alpha)}_{\mathbf{k},\mathbf{v}}(\mathbf{x}) >_{\alpha} = \sum_{p \in \mathbb{Z}^d} a_p(f_u) d_{\mathbf{k},\mathbf{v},p} ,
\]

where \( d_{\mathbf{k},\mathbf{v},p} = \langle e^{2\pi i p \mathbf{v}} , \Phi^{(\alpha)}_{\mathbf{k},\mathbf{v}}(\mathbf{x}) >_{\alpha} \). Note that if \( \mathbf{v} \neq \mathbf{u} \), then

\[
C_{\mathbf{k},\mathbf{v}}(f_u) = \langle f_u, \Phi^{(\alpha)}_{\mathbf{k},\mathbf{v}} >_{\alpha} = 0 .
\]
Consequently, we need only to estimate \( C_{k, \mathbf{u}}(f_{\mathbf{u}}) \). In the special case \( |\mathbf{u}| = 0 \), it is easy to show that \( |d_p := |e^{2i \mathbf{p} \cdot \mathbf{x}} \Phi^{(\alpha)}_0(\mathbf{x})| > \alpha \| \leq 1 \). Next, for \( |\mathbf{u}| > 0 \), we have

\[
|d_{k, \mathbf{u}, p}| = \left| e^{2i \mathbf{p} \cdot \mathbf{x}} \left( \frac{1}{h_0(\alpha)} \right)^{d-|\mathbf{u}|} \prod_{j \in \mathbf{u}} \tilde{P}^{(\alpha)}_{k_j}(x_j) \right| \\
= \left| \int_{[-1, 1]^d} \prod_{l=1}^{d} e^{2i \pi p_l x_l} \left( \frac{1}{h_0(\alpha)} \right)^{d-|\mathbf{u}|} \prod_{j \in \mathbf{u}} \tilde{P}^{(\alpha)}_{k_j}(x_j) \mathbf{u}(x) \, dx \right| \\
= \left| \prod_{l \in \mathcal{D}} \int_{-1}^{1} e^{2i \pi p_l x_l} \left( \frac{1}{h_0(\alpha)} \right)^{d-|\mathbf{u}|} \prod_{j \in \mathbf{u}} \tilde{P}^{(\alpha)}_{k_j}(x_j) w_\alpha(x_l) \, dx_l \right| \times \left| \prod_{j \in \mathbf{u}} \int_{-1}^{1} e^{2i \pi p_j y} \tilde{P}^{(\alpha)}_{k_j}(y) w_\alpha(y) \, dy \right| \\
\leq \prod_{j \in \mathbf{u}} \int_{-1}^{1} e^{2i \pi p_j y} \tilde{P}^{(\alpha)}_{k_j}(y) w_\alpha(y) \, dy = \prod_{j \in \mathbf{u}} |d_{p_j, k_j}|,
\]

where, for \( l \in \mathbb{Z} \) and \( r \in \mathbb{N} \), \( d_{l, r} := \int_{-1}^{1} e^{2i \pi p_j y} \tilde{P}^{(\alpha)}_{k_j}(y) w_\alpha(y) \, dy \). Using (6), we get

\[
d_{l, r} = i^r \sqrt{\pi} \sqrt{2r + 2\alpha + 1} \sqrt{\frac{\Gamma(r + 2\alpha + 1)}{\Gamma(r + 1)}} \frac{J_{r+\alpha+\frac{1}{2}}(2\pi l)}{(2\pi l)^{\alpha+\frac{1}{2}}},
\]

where \( J_\alpha \) is the Bessel function of the first kind and order \( \alpha > -1 \). Note that since \( d_{l, r} = (-1)^r d_{-l, r} \) then \( |d_{l, r}| = |d_{-l, r}| \) which allow us to consider only the case where \( l \geq 0 \). Now, we apply the inequality (7) to the Bessel function \( J_\alpha \) of the previous equation. We obtain:

\[
|d_{l, r}| \leq \sqrt{\pi} \sqrt{2r + 2\alpha + 1} \sqrt{\frac{\Gamma(r + 2\alpha + 1)}{\Gamma(r + 1)}} \frac{1}{\Gamma(r + \alpha + \frac{3}{2})} \frac{(\pi |l|)^r}{(2\pi l)^{\alpha+\frac{1}{2}}},
\]

Using (8), we get

\[
|d_{l, r}| \leq \sqrt{r} (e \pi |l|)^{\frac{1}{2}} \sqrt{\alpha + \frac{3}{2}} \frac{1}{2^{\alpha+\frac{1}{2}}} \frac{(r + 2\alpha + \frac{1}{2})^{\frac{r+\alpha+\frac{1}{2}}{2}}}{(r + \frac{1}{2})^{2\alpha+\frac{1}{2}}}(r + \alpha + 1)^{r+\alpha+1}.
\]

\[
\leq K(\alpha) \sqrt{r} \left( \frac{e \pi |l|}{r + \alpha} \right)^r,
\]

with \( K(\alpha) := \frac{\pi e^{\frac{1}{2}}}{\sqrt{2}} \sqrt{\alpha + \frac{3}{2}} (\alpha + 1)^{\alpha} \). We also have, from Cauchy-Schwarz inequality that \( |d_{p_j, k_j}| \leq 1 \) for all \( k_j \in \mathbb{N} \) and \( p_j \in \mathbb{Z} \). Injecting this in (53), we get

\[
|d_{k, \mathbf{u}, p}| \leq |d_{p_{j_0}, k_{j_0}}|
\]

where \( j_0 \in [[1, d]] \) is such that \( |||k||| = |k_{j_0}| \). Let \( 0 < \xi < 1 \) such that \( ||p||| \leq \xi |||k||| \). This implies that

\[
|d_{k, \mathbf{u}, p}| \leq \sqrt{|||k|||} \xi |||k|||, \quad \text{if} \quad ||p||| \leq \xi |||k|||.
\]

where the notation \( \leq \gamma \) means in general that the inequality is true up to constant depending only on a variable \( \gamma \). Let us now re-write the expression of \( C_{k, \mathbf{u}}(f_{\mathbf{u}}) \) in the following manner:

\[
C_{k, \mathbf{u}}(f_{\mathbf{u}}) = \sum_{||p||| \leq \xi |||k|||} a_p(f_{\mathbf{u}}) d_{k, \mathbf{u}, p} + \sum_{||p||| > \xi |||k|||} a_p(f_{\mathbf{u}}) d_{k, \mathbf{u}, p}.
\]
To get an upper bound for $|S_1|$, we let $\mathcal{A} := \{p \in \mathbb{Z}^d : ||p||_\infty \leq \frac{\xi}{e\pi} \}$. This set contains at most $\left( 2 \left[ \frac{\xi}{e\pi} ||k||_\infty \right] + 1 \right)^d$ elements where $[x]$ denotes the integer part of $x$. Hence, by using Bessel’s and Cauchy-Schwartz inequalities, we obtain

\[
|S_1| = \sum_{p \in \mathcal{A}} a_p(f_u) d_{k,u,p} \leq \left( \sum_{p \in \mathcal{A}} |a_p(f_u)|^2 \right) \cdot \left( \sum_{p \in \mathcal{A}} |d_{k,u,p}|^2 \right)^{\frac{1}{2}} \\
\leq \sum_{p \in \mathcal{A}} \left( ||k||_\infty \xi^2 ||k||_\infty \right)^{\frac{1}{2}} \cdot ||f_u||_2 \\
\leq \alpha \sqrt{||k||_\infty ||k||_\infty \left( 2 \left[ \frac{\xi}{e\pi} ||k||_\infty \right] + 1 \right)^{\frac{1}{2}}} \cdot ||f_u||_2 \\
\leq \alpha \sqrt{2 ||k||_\infty ||k||_\infty ||f_u||_2}.
\]

(55)

Next, we get an upper bound for $|S_2|$. For this purpose, we suppose that $f \in H^{s+\frac{d}{2}}(I^d)$. It follows, using Lemma 3.9 of [30], that $f_u \in H^{s+\frac{d}{2}}(I^d) \forall |u| \leq d$. Applying Cauchy-Schwartz inequality to $|S_2|^2$ followed by Bessel’s inequality, we obtain

\[
|S_2|^2 \leq \sum_{p \in \mathcal{A}^c} |a_p(f_u)|^2 \cdot \left( \sum_{p \in \mathcal{A}^c} |d_{k,u,p}|^2 \right) \\
\leq \sum_{p \in \mathcal{A}^c} |a_p(f_u)|^2 \cdot \left( \sum_{p \in \mathcal{A}^c} \|\Phi_{u,k}^{(\alpha)} u^{(\alpha)}\|_2^2 \right) \\
\leq \sum_{p \in \mathcal{A}^c} |a_p(f_u)|^2 \cdot \|\Phi_{u,k}^{(\alpha)} u^{(\alpha)}\|_\infty \leq \sum_{p \in \mathcal{A}^c} |a_p(f_u)|^2.
\]

(56)

Note that for $p \in \mathcal{A}^c$, we have $\left[ \prod_{j=1}^d (1 + |p_j|)^{s+\frac{d}{2}} \right]^2 \geq \|p\|_{2s+d} \geq \left[ \frac{\xi}{e\pi} ||k||_\infty \right]^{2s+d}$. Consequently, for $f_u \in H^{s+\frac{d}{2}}(I^d)$, we have:

\[
|S_2|^2 \leq \sum_{p \in \mathcal{A}^c} |a_p(f_u)|^2 \leq \left( \frac{e\pi}{\xi ||k||_\infty} \right)^{2s+d} \sum_{p \in \mathcal{A}^c} \left( 1 + \sum_{j=1}^d |p_j|^2 \right)^{s+\frac{d}{2}} |a_p(f_u)|^2 \leq \left( \frac{e\pi}{\xi ||k||_\infty} \right)^{2s+d} \||f_u||_{H^{s+\frac{d}{2}}(I^d)}^2.
\]

Hence,

\[
|S_2| \leq \alpha, d, s ||k||_{\infty}^{-s-\frac{d}{2}} \|f_u\|_{H^{s+\frac{d}{2}}(I^d)}.
\]

(57)

Combining (55) and (57), we get

\[
|C_{k,u}(f_u)| \leq \alpha, d, s \left( ||k||_{\infty}^{d+1} \xi ||k||_\infty \|f_u\|_2 + ||k||_{\infty}^{-s-\frac{d}{2}} \|f_u\|_{H^{s+\frac{d}{2}}(I^d)} \right) \\
\leq \left( ||k||_{\infty}^{d+1} \xi ||k||_\infty + ||k||_{\infty}^{-s-\frac{d}{2}} \right) \left( \|f_u\|_2 + \|f_u\|_{H^{s+\frac{d}{2}}(I^d)} \right)
\]

(58)

Finally, if $||k||_1 \geq (N + 1)$ and $N \in \mathbb{N}$ is such that $\frac{N+1}{|u| (\ln(N+1) - \ln(|u|))} = \frac{1}{\ln(\frac{1}{2})} (1 + s + \frac{d}{2})$, then $||k||_{\infty}^{d+1} \xi ||k||_\infty \leq ||k||_{\infty}^{-s-\frac{d}{2}}$. Consequently

\[
|C_{k,u}(f_u)| \leq \alpha, d, s ||k||_{\infty}^{-s-\frac{d}{2}} \left( \|f_u\|_2 + \|f_u\|_{H^{s+\frac{d}{2}}(I^d)} \right).
\]
Theorem 3. Let $f \in H^{s + \frac{d}{2}}(I^d)$ and let $1 \leq m \leq \min\{N, d\}$ with $N > e.m - 1$ and such that 

$$\frac{N+1}{m} \leq \ln \left( \frac{N+1}{m} \right) \leq -\ln \left( \xi \right) \left( 1 + s + \frac{d}{2} \right),$$

where $0 < \xi < \frac{1}{2}$. Suppose that $s + \frac{d}{2} > m + 1$, then 

$$\|f - \Pi_{N,m}(f)\|_\alpha \lesssim_{\alpha, d, s} \frac{1}{2^{(s+d/2)(m+1)}} \|f\|_{H^{s + \frac{d}{2}}(I^d)} + \frac{m \|f\|}{s + \frac{d}{2} - m} \left( 2 + \frac{m \|f\|}{N+1} \right)^d \left( \frac{m \|f\|}{N+1} \right)^{s + \frac{d}{2} - m}. \quad (59)$$

and 

$$\|f - \Pi_{N,m}(f)\|_\infty \lesssim_{\alpha, d, s} \frac{1}{2^{(s+d/2)(m+1)}} \|f\|_{\mathcal{A}^s(I^d)} + \frac{\|D^m(N,\alpha) m\|}{s + \frac{d}{2} - m} \left( 2 + \frac{m \|f\|}{N+1} \right)^d \left( \frac{m \|f\|}{N+1} \right)^{s + \frac{d}{2} - m}. \quad (60)$$

Here, $\|f\| = \|f\|_2 + \|f\|_{H^{s + \frac{d}{2}}}$ and $D(N,\alpha)$ is as given by Lemma [4].

Proof. The orthogonal projection of $f$ on $\mathcal{P}_{N,m,d}$, $\Pi_{N,m,f}$, verifies 

$$\|f - \Pi_{N,m}(f)\|_\alpha = \left\| \left[ f - T_m(f) \right] + \left[ T_m(f) - \Pi_{N,m}(T_m(f)) \right] + \left[ \Pi_{N,m}(T_m(f)) - \Pi_{N,m}(f) \right]\right\|_\alpha.$$

Note that for $N \geq m$, $\Pi_{N,m}(T_m(f)) = \Pi_{N,m}(f)$ and consequently, we have 

$$\|f - \Pi_{N,m}(f)\|_\alpha \leq \|f - T_m(f)\|_\alpha + \|T_m(f) - \Pi_{N,m}(T_m(f))\|_\alpha.$$

From [33], we have for $f \in H^{s + \frac{d}{2}}(I^d)$ 

$$\|f - T_m(f)\|_{L^2(I^d)} \leq \frac{1}{2^{(s+d/2)(m+1)}} \|f\|_{H^{s + \frac{d}{2}}(I^d)}.$$

To get an upper bound for $\|T_m(f) - \Pi_{N,m}(T_m(f))\|_\alpha$, we proceed as follows. The function $\Pi_{N,m}f$ writes as $\Pi_{N,m}f = \sum_{|u| \leq m; |k|_1 \leq N} C_{k,u}(f)\Phi_{k,u}(x)$, hence,

$$\|T_m(f) - \Pi_{N,m}(T_m(f))\|_\alpha = \left\| \sum_{|u| \leq m} f_u - \Pi_{N,m} \left( \sum_{|u| \leq m} f_u \right) \right\|_\alpha = \left\| \sum_{|u| \leq m} \left[ f_u - \Pi_{N,m}(f_u) \right] \right\|_\alpha = \sum_{|u| \leq m; |k|_1 \geq N+1} C_{k,u}(f_u) \Phi_{k,u}^{(\alpha)} \right\|_\alpha \quad (61)$$

Consequently, we get 

$$\|T_m(f) - \Pi_{N,m}(T_m(f))\|_\alpha \leq \sum_{|u| \leq m; |k|_1 \geq N+1} |C_{k,u}(f_u)| \cdot \left\| \Phi_{k,u}^{(\alpha)} \right\|_\alpha = \sum_{|u| \leq m; |k|_1 \geq N+1} |C_{k,u}(f_u)|$$

$$= \sum_{|u| = 1} \sum_{|k|_1 \geq N+1} |C_{k,u}(f_u)| \quad (62)$$

$$\leq \sum_{|u| = 1} \sum_{|k|_1 \geq N+1} |C_{k,u}(f_u)|.$$
Let $1 \leq |u| \leq m$ and $k \in F_u$. Then for $1 \leq j \leq m$ and $l \in \mathbb{N}$, the number of elements $(u, k)$ such that $|u| = j$ and $\|k\|_{\infty} = l$ is \((j)\)$(l + 1)^{j-1}$. Let $\|f_u\| := \|f_u\|_2 + \|f_u\|_{H^{r+\frac{d}{2}}}$. Using the result of Lemma 3.9 in [30] and adapting its proof for the $||.||_{H^r}$-norm, we conclude that for all $u \in D$ with $|u| \geq 1 \|f_u\| \leq \|f\|$. Moreover, by using the result of Lemma 3.9 we get

$$
\|T_m(f) - \Pi_{N,m}(T_m(f))\|_\alpha \lesssim_{\alpha,d,s} \|f\| \cdot \sum_{j=1}^{m} \left( \begin{array}{c} d \\ j \end{array} \right) j \sum_{l \geq \frac{N+1}{l}} (l + 1)^{j-1} \frac{1}{l^{s+\frac{d}{2}}}
\leq \|f\| \cdot \sum_{j=1}^{m} \left( \begin{array}{c} d \\ j \end{array} \right) j \left( 1 + \frac{m}{N+1} \right)^{j-1} \sum_{l \geq \frac{N+1}{l}} l^{j-1-s-\frac{d}{2}}
\leq \frac{m\|f\|}{s + \frac{d}{2} - m} \left( 2 + \frac{m}{N+1} \right)^d \left( \frac{m}{N+1} \right)^{s+\frac{d}{2} - m}.
$$

(63)

In a similar manner, we have

$$
\|f - \Pi_{N,m,f}\|_\infty \leq \|f - T_m(f)\|_\infty + \|T_m(f) - \Pi_{N,m}(T_m(f))\|_\infty.
$$

In [30], it has been shown that

$$
\|f - T_m(f)\|_\infty \leq \frac{1}{2}\|f\|_{A^r(t^*)}.
$$

On the other hand and by using Lemma 4 and Lemma 5, one gets

$$
\|T_m(f) - \Pi_{N,m}(T_m(f))\|_\infty \leq \sum_{|u|=1}^{m} \sum_{\|k\|_{\infty} \geq \frac{N+1}{|u|}} |C_{k,u}(f_u)| \cdot \|\Phi_{k,u}\|_{\infty}
\leq \sum_{|u|=1}^{m} \sum_{\|k\|_{\infty} \geq \frac{N+1}{|u|}} |C_{k,u}(f_u)| \cdot \frac{D^m(N,\alpha)}{\left( b_0^{(\alpha)} \right)^{\frac{d}{2}}}
\lesssim_{\alpha,d,s} \frac{D^m(N,\alpha)}{\left( b_0^{(\alpha)} \right)^{\frac{d}{2}}} \cdot \|f\| \sum_{|u|=1}^{m} \sum_{\|k\|_{\infty} \geq \frac{N+1}{|u|}} \|k\|_{\infty}^{-s-d}
\leq \frac{D^m(N,\alpha)}{\left( b_0^{(\alpha)} \right)^{\frac{d}{2}}} \|f\| \sum_{j=1}^{m} \left( \begin{array}{c} d \\ j \end{array} \right) j \sum_{l \geq \frac{N+1}{l}} l^{j-1-s-\frac{d}{2}} \left( 1 + \frac{1}{l} \right)^{j-1}
\leq \frac{m\|f\|D^m(N,\alpha)}{s + \frac{d}{2} - m} \left( 2 + \frac{m}{N+1} \right)^d \left( \frac{m}{N+1} \right)^{s+\frac{d}{2} - m}.
$$

Remark 3. From (69), one concludes that under the conditions that $\gamma_{m,N} = \frac{m}{N+1} \leq 1$ and $s + \frac{d}{2} - m > 0$ with $2(s+\frac{d}{2})(m+1) > 3d\gamma_{m,N}^{-\frac{d}{2}-s}$, we have

$$
\|f - \Pi_{N,m}(f)\|_\alpha = O\left( 3^{d_s} \gamma_{m,N}^{-\frac{d}{2}} \right).
$$

The previous estimate together with the estimate given by Theorem 2, provide us with a precise estimate for the $L^2$-risk error of our estimator $F_{N,n,m}^{(\alpha)}$ when the regression function belongs to the weighted Sobolev space [50].
7 Numerical Examples

In this paragraph, we give three numerical examples that illustrate the different results of this work.

Example 1: In this first example, we illustrate the results of Proposition 2 and Theorem 1. For this purpose, we have considered the following parameter values: \( d = 4 \) and \( d = 6 \), for the dimension with the values of \( 2 \leq N \leq 5 \), for the parameter relative to the total degree \( d \)-variate Jacobi polynomials space. These polynomials are associated with the two special values of \( \alpha = -0.5 \) and \( \alpha = 0.5 \). Moreover, we have considered the value of \( m = 2 \), that we restrict ourselves to the ANOVA decomposition with \( m = 2 \) interactions between the covariables. Also, for the case of \( d = 4 \), (respectively \( d = 6 \)), we have considered a random sampling set with size \( n = 900 \) (respectively \( n = 1600 \)) and following a multivariate Beta\((\alpha+1, \alpha+1)\) distribution. Then, we have computed the \( \kappa_2 \left( G_{N,n,m}^{(\alpha)} \right) \), the \( 2 \)-condition number of the random projection matrix \( G_{N,n,m}^{(\alpha)} \), given by (29). Also, for each values of the couple \((d,N)\), we have provided the dimension \( M_{N,2,d} \) of the considered \( d \)-variate polynomial space. The obtained numerical results are given by the following Table 1.

| \( d = 4 \) | \( M_{N,2,d} \) | \( \kappa_2 \left( G_{N,n,2}^{(-0.5)} \right) \) | \( \kappa_2 \left( G_{N,n,2}^{(0.5)} \right) \) | \( d = 6 \) | \( M_{N,2,d} \) | \( \kappa_2 \left( G_{N,n,2}^{(-0.5)} \right) \) | \( \kappa_2 \left( G_{N,n,2}^{(0.5)} \right) \) |
|---|---|---|---|---|---|---|---|
| 2 | 15 | 8.80 | 6.70 | 2 | 28 | 16.90 | 11.75 |
| 3 | 31 | 20.32 | 32.50 | 3 | 64 | 24.35 | 47.80 |
| 4 | 53 | 41.16 | 48.25 | 4 | 115 | 54.20 | 87.75 |
| 5 | 81 | 65.19 | 135.45 | 5 | 181 | 223.22 | 312.77 |

Table 1: The Jacobi polynomial space dimension and the \( 2 \)-condition number of the random projection matrix for \( m = 2 \), \((d,n) = (4,900), (6,1600), \alpha = \frac{1}{2}, -\frac{1}{2} \) and \( 2 \leq N \leq 5 \).

Also, in Figure 1, we give the plots of the spectrum of \( G_{N,n,2}^{(-0.5)} \), for \((d,n) = (4,900), (6,1600)\) and the different values of \( 2 \leq N \leq 5 \). Note that these plots are fairly coherent with the predicted theoretical behaviour of the spectrum of the random matrix \( G_{N,n,2}^{(\alpha)} \), given by Theorem 1 and in particular by the lower and upper bounds (39) and (40).

Example 2: In this second example, we illustrate the performance of \( \hat{f}_{N,n,m}^{(\alpha)} \), our proposed stable NP regression estimator, that is based on least squares by means of multivariate Jacobi polynomials. For this purpose, we have considered the NP regression problem (24) for the special case of the dimension \( d = 4 \) with a synthetic test true regression function \( f \), given by

\[
 f(x,y,z,t) = x + (2y - 1)^2 + \frac{\sin(2\pi z)}{2 - \sin(2\pi z)} + 0.1 \sin(2\pi t) + 0.2 \cos(2\pi t) \\
 + 0.3 (\sin(2\pi t))^2 + 0.4 (\cos(2\pi t))^3 + 0.5 (\sin(2\pi t))^3 
\]

Note that this test regression function corresponds to an additive multidimensional regression model. Hence, \( m = 1 \) is the appropriate value of this parameter. Then, we have constructed our estimator \( \hat{f}_{N,n,1}^{(\alpha)} \), with \( \alpha = -\frac{1}{2}, n = 900 \) i.i.d. random sampling points following a 4-D Beta\((\alpha + 1, \alpha + 1)\) distribution. Also, we have considered the different values of \( N = 4, 6, 8, 10 \) together with a noise free model as well as noise\(^\_\)
models associated to two different values of $\sigma = 0.1, 0.5$. We have computed the empirical mean squared error over a test random set of size $n$ with i.i.d. random points $X_i$ following also a 4–D Beta$(\alpha + 1, \alpha + 1)$ distribution. This empirical mean squared error is given by

$$MSE = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{f}_{N,n,1}^{(\alpha)}(X_i) - f(X_i) \right)^2.$$ 

The obtained numerical results are given by the following Table 2.

| $N$ | $\sigma = 0$  | $\sigma = 0.1$ | $\sigma = 0.5$ |
|-----|----------------|----------------|----------------|
| 4   | $2.69e-2$      | $3.99e-2$      | $4.42e-2$      |
| 6   | $1.60e-2$      | $2.11e-2$      | $2.53e-2$      |
| 8   | $5.89e-3$      | $1.05e-2$      | $1.79e-2$      |
| 10  | $9.29e-4$      | $2.63e-3$      | $6.20e-2$      |

Table 2: Numerical simulations for test function (64).

Note that the numerical results given by Table 2 are coherent with the theoretical $L^2$–risk and the approximation error of the proposed estimator $\hat{f}_{N,n,1}^{(\alpha)}$, given by Theorems 2 and 3. Also, note that the loss of accuracy we have observed for the special values of $N = 10$ and $\sigma = 0.5$ is due to the fact that for the given couple $(N, n) = (10, 900)$, the 2–condition number of the random matrix $G^{(\alpha)}_{N,n,m}$ is relatively large to handle noised data with relatively large $\sigma = 0.5$ Nonetheless, for $n = 1600$ and the same values of the parameters,
we have obtained an $MSE \approx 1.16e - 2$.

**Example 3:** In this last example, we consider the Kriging model test function borrowed from [10] and given, for $x_1, x_2, x_3, x_4 \in [0, 1]$, by

$$f(x_1, x_2, x_3, x_4) = 1 + \exp \left[ -2((x_1 - 1)^2 + x_2^2) - 0.5(x_3^2 + x_4^2) \right] + \exp \left[ -2(x_1^2 + (x_2 - 1)^2) - 0.5(x_3^2 + x_4^2) \right].$$

(65)

Then, we have considered the values of the parameters $m = 2$, $\alpha = -\frac{1}{2}$, $n = 1600$ and constructed our estimator $\hat{f}^{(N)}_{\alpha}$, with $N = 4, 5, 6$. As in the previous example, we have computed the different $MSE$: the empirical mean squared errors for the different values of the Gaussian noise standard deviation $\sigma = 0, 0.1, 0.5$. These $MSE$ are computed by the use of a new set of $n_1 = 400$ i.i.d. random sampling points following a multivariate Beta($\alpha + 1, \alpha + 1$) distribution. The obtained numerical results are given by Table 3.

| $N$ | $\sigma = 0$ | $\sigma = 0.1$ | $\sigma = 0.5$ |
|-----|--------------|---------------|---------------|
| 4   | 5.35e - 3    | 5.94e - 3     | 1.38e - 2     |
| 5   | 1.57e - 3    | 1.52e - 2     | 2.57e - 2     |
| 6   | 1.06e - 3    | 1.07e - 2     | 2.69e - 2     |

Table 3: Numerical simulations for test function (65).

Note that these numerical results are also coherent with the theoretical $L^2$–risk and the approximation error given by Theorems 2 and 3. For the noise free model, that is $\sigma = 0$, the larger $N$, the smallest is the empirical mean squared error. For the noised versions of the model, that is for $\sigma = 0.1$ or $\sigma = 0.5$, the situation is slightly reversed. This is due to the contribution of the variance term. According to Theorem 2, this last quantity is affected by larger values of the parameter $N$.

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