The larger sieve and polynomial congruences

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Abstract
We obtain a small improvement of Gallagher’s larger sieve and we extend it to higher dimensions. We also obtain two interesting upper bounds for the number of solutions to polynomial congruences.

1 Introduction
In his paper of 1971, Gallagher introduced a new tool in number theory that is now known as the larger sieve and also as Gallagher’s larger sieve. As indicated by its name, it is a complementary inequality to the large sieve. More precisely, let $\mathcal{S}$ be a set of integers in an interval of length $M$ for which there exists a set $\mathcal{Q}$ of prime powers $q = p^{\alpha_p}$ such that each numbers $n \in \mathcal{S}$ belong to at most $\nu(q)$ congruence classes modulo $q$. Then

$$\#\mathcal{S} \leq \frac{\sum_{q \in \mathcal{Q}} \Lambda(q) - \log M}{\sum_{q \in \mathcal{Q}} \frac{\Lambda(q)}{\nu(q)} - \log M}$$

holds if the denominator is positive. Here, $\Lambda(\cdot)$ denote the classical von Mangoldt function, that is

$$\Lambda(q) = \begin{cases} \log p & \text{if } q = p^j \text{ for some prime } p \text{ and } j \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

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Inequality (1.1) has been proven to be stronger than the large sieve when most of the values of \( \nu(q) \) are small, see [5]. In the book [4], the authors propose a generalization of the larger sieve. Some related results and a discussion can be found in [3].

A seemingly unrelated subject with above is the study of polynomial congruences. Let \( f(x) := a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x] \) be a polynomial of degree \( n \geq 2 \) and \( q \geq 2 \) be an integer satisfying

\[
\gcd(a_n, \ldots, a_0, q) = 1.
\]

It has been established in [7] that the number of solutions \( N(f, q) \) to the equation

\[
(1.2) \quad f(x) \equiv 0 \pmod{q} \quad (x = 1, \ldots, q)
\]

satisfies

\[
(1.3) \quad N(f, q) \leq \left( \frac{n}{e} + O((\log n)^2) \right) q^{1-\frac{1}{n}}.
\]

It is also shown to be essentially best possible since there are infinitely many polynomials \( f(x) \) and values of \( q \) for which

\[
N(f, q) > \left( \frac{n}{e} + c_1 \log n \right) q^{1-\frac{1}{n}}
\]

for some constant \( c_1 > 0 \).

It raises the question: How many solutions \( x \) of \( f(x) \equiv 0 \pmod{q} \) can we find in an interval \( I \) of length \( q^{1/\alpha} \)? In Theorem 3 of [8], an answer has been given and it is of the shape \( \ll \log q \). By studying the argument of the demonstration of this theorem, we have been led to a small improvement in the case where \( n \) is considered as fixed. Also, our research has led us to an improvement of the inequality (1.1) as well as a generalization to higher dimensions.

Throughout the paper, we often write \( S \), with or without subscript, to denote a set of integer points in \( \mathbb{Z}^m \) for some \( m \geq 1 \). When it is the case, we often write \( \#S \) to denote \( \#S \) with the same subscript. For any integer \( q \geq 1 \), the functions \( \phi(q) \) and \( \omega(q) \) are respectively the Euler’s phi function and the number of distinct prime divisors of \( q \). For any integer \( q \geq 1 \) and prime \( p \), let’s denote by \( v_p(q) \) the unique integer \( \alpha_p \geq 0 \) for which \( p^{\alpha_p} \| q \).

For two integers \( q \) and \( \Delta \) and a real number \( \alpha \), we write \( q^{\alpha} \mid \Delta \) to signify that \( \alpha v_p(q) \leq v_p(\Delta) \) for each primes \( p \).
2 Statements of theorems

For each integer \( s \geq 2 \), let’s define

\[
c_s := \prod_{j=1}^{s} \left( \frac{(j - 1)^{2(j-1)j}}{(s + j - 2)^{s+j-2}} \right)^{\frac{1}{s(s-1)}}.
\]

**Theorem 2.1.** Let \( S \) be a set of integers in the interval \([N, N + M]\) with \( M > 0 \). Let also \( Q \) be a finite set of pairwise coprime integers. Suppose that for each \( q \in Q \) the integers \( n \in S \) belong to at most \( 1 \leq \nu(q) \leq q \) congruence classes modulo \( q \). Then, if the denominator is positive, the inequality

\[
S \leq \frac{\sum_{q \in Q} \log q - \log(c_S M)}{\sum_{q \in Q} \log q - \log(c_S M)}
\]

holds.

**Remark 2.2.** We show in Lemma 3.2 that \( c_s \) is essentially \( \frac{1}{4} + \epsilon(s) \) so that the first term in the denominator of (2.1) can be about \( \log 4 = 1.386 \ldots \) smaller than in (1.1) and still have the inequality effective. One can see directly from the proof that inequality (2.1) is at least as good as (1.1) provided \( S \geq \max_{q \in Q} \nu(q) \). Also, an inequality like (2.1) can be stated with the function \( \Lambda(\cdot) \) replacing \( \log(\cdot) \) in both sums, in which case we get an inequality that is always at least as good as (1.1).

**Corollary 2.3.** Assume that we are in the situation of Theorem 2.1. We either have \( S \leq 1243 \) or we have that

\[
S \leq \frac{\sum_{q \in Q} \Lambda(q) - \log M + 1.38}{\sum_{q \in Q} \nu(q) - \log M + 1.38}
\]

holds if the denominator is positive.

**Remark 2.4.** It is possible to show that an inequality like (2.2) cannot hold if the constant is too large. In fact, using the polynomial \( P(x) = x^2 + x \), one can show that the optimal constant has to be less than

\[
2 - \log(2) + 2\gamma + 4 \sum_{p \geq 3} \frac{\log(p)}{p^2 - 1} \leq 3.817.
\]

Let \( v_1, \ldots, v_{m+1} \) be points in \( \mathbb{R}^m \). We define the quantity

\[
D(v_1, \ldots, v_{m+1}) := \left\| \begin{array}{cccc}
1 & 1 & \cdots & 1 \\
v_1 & v_2 & \cdots & v_{m+1}
\end{array} \right\|.
\]
The points \(v_1, \ldots, v_{m+1}\) are in the same (affine) hyperplane if and only if
\[
D(v_1, \ldots, v_{m+1}) = 0.
\]

Let \(\Gamma \subseteq \mathbb{Z}^m\) be a lattice in \(\mathbb{R}^m\). We denote by \(|\Gamma|\) the \(m\)-dimensional volume of the fundamental parallelepiped of the lattice \(\Gamma\). For a fixed set \(\Omega \in \mathbb{R}^m\), we write
\[
t(\Omega) := \sup_{v_1, \ldots, v_{m+1} \in \Omega} D(v_1, \ldots, v_{m+1}).
\]

Theorem 2.5. Let \(S\) a set of integer points included in a set \(\Omega \in \mathbb{R}^m\) \((m \geq 2)\) of nonzero \(m\)-dimensional volume. Let also \(\mathcal{L}\) be a set of lattices \(\Gamma \subset \mathbb{Z}^m\). Suppose that for each lattices \(\Gamma \in \mathcal{L}\), the points of \(S\) belong to at most \(v(\Gamma)\) equivalence classes of \(\mathbb{Z}^m / \Gamma\) and that
\[
\min_{v_1, \ldots, v_{m+1} \in S, \ v_i \neq v_j \text{ for } i \neq j} D(v_1, \ldots, v_{m+1}) > 0. \tag{2.3}
\]
Suppose also that the values of \(|\Gamma|\) are pairwise coprime. Then,
\[
S < \max \left( \gamma_m \max_{1 \leq s \leq m} \left( \frac{\sum_{\Gamma \in \mathcal{L}} \log \frac{|\Gamma|}{v(\Gamma)^s}}{\sum_{\Gamma \in \mathcal{L}} \log \frac{|\Gamma|}{v(\Gamma)^s} - \log t(\Omega)} \right)^{1/s}, (m + 1) \max_{\Gamma \in \mathcal{L}} v(\Gamma) \right) \tag{2.4}
\]
if
\[
\sum_{\Gamma \in \mathcal{L}} \log \frac{|\Gamma|}{v(\Gamma)^m} - \log t(\Omega) > 0.
\]
We have set \(\gamma_m := \left\lfloor \frac{m+1}{2} \right\rfloor \frac{m(m+1)}{2} \).

Remark 2.6. The hypothesis (2.3) is really strong and seems difficult to deal with in practice. For this reason, we have included Lemma 3.7. We have also included in Lemma 3.5 an estimate for the value of \(t(\Omega)\) in the case where \(\Omega\) is a \(m\)-dimensional parallelepiped.

Finally, our considerations of the initial problem have led us to the following theorem. It is an improvement of Theorem 3 of [8].

Theorem 2.7. Consider the polynomial \(P(x) := a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]\) of degree \(n\) and \(q \geq 2\) be an integer satisfying \(\gcd(a_n, \ldots, a_0, q) = 1\). Let \(I\) be an interval of length at most \(q^{1/n}\). The number \(W\) of solutions to the congruence
\[
P(x) \equiv 0 \pmod{q} \quad (x \in I) \tag{2.5}
\]
satisfies
\[
W \leq 2(n - 1)^2 \omega(q). \tag{2.6}
\]
Corollary 2.8. Consider the polynomial $P(x) := a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ of degree $n$ and $q \geq 2$ be an integer satisfying $\gcd(a_n, \ldots, a_0, q) = 1$. Let $\mathcal{I}$ be an interval of length $L$. The number $W$ of solutions to the congruence

\begin{equation}
(2.7) \quad P(x) \equiv 0 \pmod{q} \quad (x \in \mathcal{I})
\end{equation}

satisfies

\begin{equation}
(2.8) \quad W \leq 2(n - 1)^2 \omega(q) \left( \frac{L}{q^{1/n}} + 1 \right).
\end{equation}

We also have a modest improvement of Theorem 2.7 in a very particular case.

Theorem 2.9. Consider the polynomial

\begin{equation}
(2.9) \quad P(x) := x^n + d
\end{equation}

of degree $n \geq 2$ with $d \in \mathbb{Z}$. Let $q \geq 2$ be an integer and $\mathcal{I}$ be an interval of length at most $q^{1/n}$. The number $W$ of solutions to the congruence

\begin{equation}
(2.10) \quad P(x) \equiv 0 \pmod{q} \quad (x \in \mathcal{I})
\end{equation}

satisfies

\begin{equation}
(2.11) \quad W \leq n \omega(q).
\end{equation}

3 Preliminary lemmas

Lemma 3.1. For each $s \geq 2$, we have

$$
\max_{0 \leq \xi_1 \leq \cdots \leq \xi_s \leq 1} \prod_{1 \leq i < j \leq s} (\xi_j - \xi_i) = c_s^{(s)}
$$

Proof. This is a restatement of Theorem 8.5.2 of \cite{1} with $p = q = 0$. \qed

Lemma 3.2. For each $s \geq 2$, the inequality

$$
c_s < \frac{1}{4} \exp \left( \frac{s \log(2s) + \frac{1}{7} \log(s)}{s(s - 1)} \right)
$$

holds.
Proof. We proceed by induction. We start by checking that the result is true for \( 2 \leq s \leq 199 \). Now, we write

\[ a_s := s(s - 1) \log c_s \]

and

\[ f(s) := -s(s - 1) \log 4 + s \log 2s + \frac{\log s}{4} - \frac{1}{s}. \]

We verify that \( a_{200} \leq f(200) \). For \( s \geq 200 \), we suppose that \( a_s \leq f(s) \) and we want to establish that \( a_{s+1} \leq f(s+1) \). It is enough to establish that

\[ a_{s+1} - a_s \leq f(s+1) - f(s). \]

We have

\[ a_{s+1} - a_s = (s+1) \log(s+1) + (s-1) \log(s-1) - (2s-1) \log(2s-1) - 2s \log 2 \]

and

\[ f(s+1) - f(s) = -2s \log 4 + (s+1) \log(s+1) - s \log s + \frac{1}{4} \log(1 + \frac{1}{s}) + \frac{1}{s(s+1)}. \]

Comparing (3.2) with (3.3), we observe that (3.1) holds if and only if

\[ g(2s-1) - g(s-1) \leq \frac{1}{4} \log \left( 1 + \frac{1}{s} \right) + \frac{1}{s(s+1)} \]

holds, where we have written \( g(x) := x \log \left( 1 + \frac{1}{x} \right) \). Now, we make use of the inequality

\[ \frac{1}{x} - \frac{1}{2x^2} \leq \log \left( 1 + \frac{1}{x} \right) \leq \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} \quad (x > 1), \]

to establish that the inequality (3.4) holds if

\[ 0 \leq \frac{1}{4s} + \frac{1}{2(2s-1)} - \frac{1}{2(s-1)} + \frac{1}{s(s+1)} - \frac{1}{8s^2} - \frac{1}{3(2s-1)^2} \]

holds. We verify that this is the case for \( s \geq 200 \), which completes the induction step.

For a fixed \( n \geq 2 \), we consider the multiplicative function \( g(n, q) \), i.e.

\[ g(n, q) = \prod_{p^j || q} g(n, p^j), \]

defined by

\[ g(n, p^j) := \begin{cases} 
\gcd(n, \phi(p^j)) & \text{if } p \geq 3, \\
1 & \text{if } p^j = 2, \\
\gcd(n, 2) & \text{if } p^j = 4, \\
\gcd(n, 2) \cdot \gcd(n, \phi(2^{j-1})) & \text{if } p = 2 \text{ and } j \geq 3. 
\end{cases} \]

In particular, \( g(n, q) \leq 2^{\omega(q)}. \)
Lemma 3.3. Let $P(x)$ be the polynomial (2.9) with $n \geq 2$. Let also $q \geq 2$ be an integer satisfying $\gcd(d, q) = 1$. Then the total number of solutions (mod $q$) to the congruence

$$P(x) \equiv 0 \pmod{q}$$

is of at most $g(n, q)$.

Proof. The proof is an easy exercise that uses a primitive root of $(\mathbb{Z}/p^\alpha \mathbb{Z})^*$, for any odd prime $p$ and $\alpha \geq 1$, together with the fact that any element of $(\mathbb{Z}/2^\alpha \mathbb{Z})^*$, with $\alpha \geq 2$, has a unique representation as $(-1)^a 5^b$ with $a \in \{0, 1\}$ and $b \in \{1, \ldots, 2^{\alpha-2}\}$. The multiplicativity follows from the Chinese remainder theorem. \(\square\)

Lemma 3.4. Let $P(x) := a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ be a polynomial of degree $n \geq 1$ and $q \geq 2$ be an integer satisfying $\gcd(a_n, \ldots, a_0, q) = 1$. Let also $x_1 < x_2 < \cdots < x_s$ be a sequence of solutions to the congruence

$$P(x) \equiv 0 \pmod{q}.$$

Consider the product

$$\Delta := \prod_{1 \leq i < j \leq s} (x_j - x_i).$$

If $s \geq n + 1$ then

$$q^{\frac{s^2}{2n}} | \Delta.$$

Proof. This result is proved in Lemma 2.5 of [9] for $n \geq 2$ and it is clear for $n = 1$. \(\square\)

Lemma 3.5. Let $\Omega \in \mathbb{R}^m$ be a closed parallelepiped of nonzero $m$-dimensional volume. Then

$$(3.5) \hspace{1cm} t(\Omega) \leq \frac{(m + 2)^{\frac{m+1}{2}}}{2^m} \text{Vol}(\Omega).$$

Also, we have that $t(1) = t(2) = \text{Vol}(\Omega)$, $t(3) = 2\text{Vol}(\Omega)$ and that $t(4) = 3\text{Vol}(\Omega)$.

Proof. It is enough to prove the result for the cube $[0, 1]^m$. This is a situation that is similar to a famous problem, see [2]. Let $A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_{m+1} \end{pmatrix}$ be a matrix that realizes an extremum of the function $\det A$. Suppose at first that one of the vectors $v_j = (a_{2,j}, \ldots, a_{m+1,j})^t$ has a coordinate $0 < a_{i,j} < 1$. We then deduce that

$$0 = \frac{d}{dx_{i,j}} \det A \bigg|_{x_{i,j} = a_{i,j}} = (-1)^{i+j} \det A_{i,j}.$$
where \( x_{i,j} \) is a variable in position \((i, j)\) in \( A \), where the last equality follows by expanding using the \( j \)-th column and where \( A_{i,j} \) is the submatrix \( m \times m \) obtained by removing the \( i \)-th row and the \( j \)-th column. We deduce that \( \det A_{i,j} = 0 \) so that \( \det A \) remains invariant by a modification of the entry \( a_{i,j} \). We therefore consider the new matrix \( A_1 \) for which \( a_{i,j} = 0 \) and all the other entries are the same as in the matrix \( A \). We repeat this process until we get to a matrix \( A' \) composed only of 0 and 1.

Now, to obtain inequality (3.5), we consider the matrix

\[
B := \begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 \\
x & v_1 & \cdots & v_{m+1}
\end{pmatrix}
\]

where \( x = (\frac{1}{2}, \ldots, \frac{1}{2})' \). We observe that \( \det A = \det B \) and the result follows by subtracting the first column from the others and by using Hadamard’s inequality on the rows. The other statements can be verified directly with a computer. The proof is completed.

**Lemma 3.6.** Let \( P(x) := x(x - 1) \cdots (x - d + 1) \) be a polynomial of degree \( d \geq 3 \). Let also \( x_1, \ldots, x_n, X \) be positive real numbers satisfying \( x_1 + \cdots + x_n = X \) and \( X \geq dn \). Then,

\[
(3.6) \quad P(x_1) + \cdots + P(x_n) \geq nP\left(\frac{X}{n}\right).
\]

**Proof.** Clearly \( 0 \leq x_i \leq X \) for each \( i = 1, \ldots, n \) and we can assume that \( x_1 \geq x_2 \geq \cdots \geq x_n \). Let \( j + 1 \) be the number of nonzero values of \( x_i \). Suppose that \( j \geq 1 \) and consider the function

\[
F(z_1, z_2, \ldots, z_j) := P(z_1) + \cdots + P(z_j) + P(X - z_1 - \cdots - z_j).
\]

If \( F \) reaches a local extremum at \((x_1, \ldots, x_j)\), then

\[
0 = \frac{d}{dx_i} F(z_1, \ldots, z_j) \bigg|_{z_1 = x_1, \ldots, z_j = x_j} = P'(x_i) - P'(X - x_1 - \cdots - x_j) \quad (i = 1, \ldots, j).
\]

We deduce that

\[
(3.7) \quad P'(x_1) = \cdots = P'(x_{j+1}).
\]

One can establish the inequality

\[
\max_{x \in [0, d-1]} \prod_{i=0}^{d-1} |x - i| \leq (d - 1)! \quad (k = 0, \ldots, d - 1).
\]
We deduce that $\max_{x \in [0, d-1]} |P'(x)| \leq d!$ and consequently $|P'(x)| < P'(d)$ for each $x \in [0, d)$. For $x \geq d$, the function $P'(x)$ is strictly increasing. Now, since $x_1 + \cdots + x_{j+1} = X$, we must have $\max_i x_i \geq \frac{X}{j+1}$. It follows that if $\frac{X}{j+1} \geq d$, then (3.7) implies that $x_1 = \cdots = x_{j+1} = \frac{X}{j+1}$. We have therefore shown that the minimum of the left expression in (3.6) is of the form $(j+1)P\left(\frac{X}{j+1}\right)$ for a value of $j = 0, \ldots, n - 1$. We then notice that

$$\frac{d}{dt} t P\left(\frac{X}{t}\right) = P\left(\frac{X}{t}\right) - \frac{X}{t} P'\left(\frac{X}{t}\right) < 0$$

if $\frac{X}{t} > d - 1$. The proof is thus completed.

**Lemma 3.7.** Let $\mathcal{N}$ be a finite set of points in $\mathbb{R}^m$. Let $\mathcal{S} \subseteq \mathcal{N}$ a subset of maximal cardinality for which

$$\min_{v_1, \ldots, v_{m+1} \in \mathcal{S} \atop v_i \neq v_j \text{ for } i \neq j} D(v_1, \ldots, v_{m+1}) > 0.$$

Let also $K$ be the maximal number of points in $\mathcal{N}$ that are all included in an hyperplane. Then

$$\#\mathcal{N} \leq K \max\left(1, \binom{S}{m}\right).$$

**Proof.** If such a set $\mathcal{S}$ does not exist, then we have $\#\mathcal{N} \leq K$. Otherwise, since $\mathcal{S}$ is a set of maximal cardinality, it follows that each point $v \in \mathcal{N} \setminus \mathcal{S}$ is included in an hyperplane defined by at least one set of $m$ points of $\mathcal{S}$. There are $\binom{S}{m}$ such sets of $m$ points. By hypothesis, each of these sets defines a distinct hyperplane and then each such hyperplane contains at most $K$ points of $\mathcal{N}$. The result follows.

### 4 Proof of Theorem 2.1

Let’s denote by $x_1, \ldots, x_S$ the ordered list of numbers in $\mathcal{S}$. We then consider the product

$$\Delta := \prod_{1 \leq i < j \leq S} (x_j - x_i).$$

On the one hand, using Lemma 3.1, we have

$$\Delta = M^{(s)} \prod_{1 \leq i < j \leq S} \left(\frac{x_j - x_i}{M}\right) \leq M^{(s)} \max_{0 \leq \xi_1 \leq \cdots \leq \xi_S \leq 1} \prod_{1 \leq i < j \leq S} (\xi_j - \xi_i) = \left(c_S M\right)^{\binom{S}{2}}.$$
On the other hand, let’s fix an integer $q \in \mathbb{Q}$ and partition the set $S$ into the $\nu(q)$ disjoint subsets $S_r$ that contain the ordered set of numbers $x_{r,1}, \ldots, x_{r,S_r}$ from $S$ that belong to the same congruence class modulo $q$. We then write
\[
\Delta_r := \prod_{1 \leq i < j \leq S_r} (x_{r,j} - x_{r,i})
\]
and notice that
\[
q^{\binom{S_r}{2} + \cdots + \binom{S_r}{2}} \mid \Delta_1 \cdot \cdots \cdot \Delta_{\nu(q)} \mid \Delta.
\]
Now, we find
\[
\sum_{r=1}^{\nu(q)} \binom{S_r}{2} = \frac{1}{2} \sum_{r=1}^{\nu(q)} S_r^2 - \frac{S}{2}
\geq \frac{S^2}{2\nu(q)} - \frac{S}{2}
\]
using Cauchy-Schwarz’s inequality. Since the values of $q \in \mathbb{Q}$ are pairwise coprime, we get to
\[
\prod_{q \in \mathbb{Q}} q^{\frac{S^2}{2\nu(q)} - \frac{S}{2}} \mid \Delta \leq (c_SM)\binom{S}{2}.
\]
The result easily follows.

5 Proof of Theorem 2.5

For a fixed $m \geq 2$, we write the sequence of integer points in $S$ as $v_1, \ldots, v_S$ and consider the product
\[
\Delta := \prod_{1 \leq i_1 < \cdots < i_{m+1} \leq S} D(v_{i_1}, \ldots, v_{i_{m+1}}).
\]
Clearly,
\[
\Delta \leq t(\Omega)\binom{S}{m+1}.
\]
Now, let’s fix a lattice $\Gamma \in \mathcal{L}$ and partition the set $S$ into the $\nu(\Gamma)$ disjoint subsets $S_r$ that contain the set of integer points $v_{r,1}, \ldots, v_{r,S_r}$ from $S$ that belong to the same equivalence class of $\mathbb{Z}^m/\Gamma$. We then define
\[
\Delta_r := \prod_{1 \leq i_1 < \cdots < i_{m+1} \leq S_r} D(v_{r,i_1}, \ldots, v_{r,i_{m+1}}).
\]
and notice that
\[
|\Gamma|\binom{S_r}{m+1} + \cdots + \binom{S_r}{m+1} \mid \Delta_1 \cdot \cdots \cdot \Delta_{\nu(\Gamma)} \mid \Delta.
\]
From Lemma 3.6 and the hypothesis $S \geq (m + 1)v(\Gamma)$ (otherwise (2.4) is trivial), we get

$$|\Gamma|^{v(\Gamma)\left(\frac{S}{m+1}\right)} | \Delta.$$  

By assumption the values of $|\Gamma|$ are pairwise coprime and the inequality $S \geq (m + 1)v(\Gamma)$ holds for each $\Gamma$. We deduce that

$$\prod_{\Gamma} |\Gamma|^{v(\Gamma)\left(\frac{S}{m+1}\right)} \leq t(\Omega)^{\left(\frac{S}{m+1}\right)}.$$  

We take the logarithm and send everything to the left hand side. We get the inequality

$$a_mb_mS^m - a_{m-1}b_{m-1}S^{m-1} + \cdots + (-1)^ma_0b_0 \leq 0$$

where

$$x(x - 1) \cdots (x - m) = a_mx^{m+1} - a_{m-1}x^m + \cdots (-1)^ma_0x$$

and where

$$b_i := \sum_{t \in \mathcal{L}} \frac{\log |\Gamma|}{v(\Gamma)^i} - \log t(\Omega).$$

The hypothesis $b_m > 0$ implies that $b_i > 0$ for each $i = 0, \ldots, m$. We deduce that

$$S \leq \left\lfloor \frac{m + 1}{2} \right\rfloor \max_{1 \leq s \leq m} \frac{a_{m-s}b_{m-s}}{b_mS^{s-1}}$$

from which the result follows after a simple computation.

6 Proof of Theorem 2.7

From the proof of Lemma 2.5 of [9], we know that we can assume that $P(x) = \prod_{j=1}^n (x - a_j)$. Also, we can assume that $a_1 = 0$.

Step 1: We have an integer $q \geq 2$, a polynomial $P(x)$ of degree $n$ and an interval $I$ of length $\leq q^{1/n}$. We want to find an upper bound for the number of solutions $W$ to the system (2.5). Let’s fix a prime power $q_1 = p^\alpha \| q$ for which $q_1 \geq q^{1/\omega(q)}$. We consider two cases.

Case 1: The solutions $W$ to (2.5) are in exactly $2 \leq t \leq n$ congruence classes modulo $p$. Consider a congruence class, say $\ell$ (mod $p$), that has the most solutions, a set we denote by $W'$. We have $\#W \leq ts$ where $\#W' = s$. We can assume that $s \geq 2$ since otherwise (2.6) holds. Let’s define the polynomial

$$P_\ell(x) := \prod_{1 \leq j \leq n, a_j \equiv \ell \pmod p} (x - a_j).$$
We remark that $P_{t}(x)$ is of degree at most $n + 1 - t$. Now, we write the solutions in $W'$ as $x_{1} < \cdots < x_{s}$ and define
\[
\Delta := \prod_{1 \leq i < j \leq s} (x_{j} - x_{i}).
\]
Clearly, $\Delta \leq q^{s^{2}/n}$. Also, using Lemma 3.4 for the polynomials $P(x)$ and $P_{t}(x)$, we get
\[
q^{2s^{2}/(n+1-t)} q^{s^{2}/n} \quad \text{where} \quad q^{2s^{2}/(n+1-t)} q^{s^{2}/n} \leq q^{2s^{2}/n}.
\]
We deduce that
\[
W \leq ts \leq \frac{t}{t-1} \left(1 - \frac{1}{n}\right) n(n+1-t) \omega(q)
\]
and the result follows.

Case 2: The solutions $W$ to (2.5) are in only one congruence class modulo $p$. In this case, since $P(0) \equiv 0 \pmod{q}$, we have that this class is $0 \pmod{p}$. Also, we must have $p \mid a_{i}$ for $i = 1, \ldots, n$. Writing $x = pz$, we get
\[
P(x) \equiv 0 \pmod{q} \quad \Longrightarrow \quad P_{1}(z) \equiv 0 \pmod{\frac{q}{p_{\min(a,n)}}}
\]
where $P_{1}(z) = \prod_{j=1}^{n} (z - a_{j,1})$ and $a_{j,1} = \frac{a_{j}}{p}$. We have thus transformed our problem into another one with the integer $q' = \frac{q}{p_{\min(a,n)}}$, the polynomial $P_{1}(x)$ and an interval of length $\frac{q'^{1/n}}{p} \leq q'^{1/n}$.

Step 2: If $q' \geq 2$ we return to Step 1 with $q'$ instead of $q$, $P_{1}(x)$ instead of $P(x)$ and $\mathcal{I}_{1}$ of length $\leq q'^{1/n}$ instead of $\mathcal{I}$. If we are not in Case 1 at some stage, we will get to $q' = 1$ and $\mathcal{I}_{1}$ of length at most 1 so that $W \leq 2$. The proof is completed.

**Remark 6.1.** We can also proceed as in the proof of Theorem 2.1 to find an upper bound for $W$. We write the solutions of (2.6) as $x_{1} < \cdots < x_{W}$ and define
\[
\Delta := \prod_{1 \leq i < j \leq W} (x_{j} - x_{i}).
\]
Proceeding as usual and using Lemma 3.4, we get to
\[
q^{w^{2}/2n} w^{2} \quad | \quad \Delta \leq (c_{W} q^{\frac{1}{n}})^{w^{2}-w}.\]
Thus we have
\[
\left( \frac{1}{c_W} \right)^{W-1} \leq q^{1 - \frac{1}{n}}
\]
and we deduce from Lemma 3.2 that
\[
W - \frac{\log W}{\log 4} - \frac{\log W}{4W \log 4} \leq \left( 1 - \frac{1}{n} \right) \frac{\log q}{\log 4} + \frac{3}{2}.
\]
Now, we write \(F(x) := x - \frac{\log x}{\log 4} - \frac{\log x}{4x \log 4} \) and show that \(F'(x) > 0\) for \(x \geq 1\) and that
\[
F \left( x + \frac{\log x}{\log 4} + \frac{2}{3} \right) \geq x \quad (x \geq \frac{7}{4}).
\]
From there we get
\[
W < \left( 1 - \frac{1}{n} \right) \frac{\log q}{\log 4} + \frac{\log \log q}{\log 4} + 3.
\]

7 Proof of Theorem 2.9

We can assume that \(d \in \{1, \ldots, q\}\). We first show that it is enough to prove the theorem with the supplementary assumption \(\gcd(d, q) = 1\). Indeed, assume that \(\gcd(d, q) = r\). Let’s define the function
\[
\gamma_n(r) := \prod_{p^\alpha \parallel r} p^{\left\lfloor \frac{\alpha}{n} \right\rfloor}.
\]
Each solutions \(x \in I\) of \((2.10)\) must also satisfy \(\gamma_n(r) \mid x\), Thus, by writing \(x = \gamma_n(r)z\), we get to the congruence
\[
\gamma_n(r)^n z^n + d \equiv 0 \pmod{q} \quad \implies \quad \frac{\gamma_n(r)^n}{r} z^n + \frac{d}{r} \equiv 0 \pmod{\frac{q}{r}}.
\]

Case 1: \(\frac{q}{r} > 1\). If \(\gcd\left(\frac{\gamma_n(r)^n}{r}, \frac{q}{r}\right) > 1\) then we have \(W = 0\) and otherwise we multiply the above equation by the multiplicative inverse of \(\frac{\gamma_n(r)^n}{r} \pmod{\frac{q}{r}}\) and retrieve a polynomial of the shape \((2.9)\). We remark that \(z\) is in an interval of length at most \(\frac{q^{1/n}}{\gamma_n(r)} \leq \left( \frac{q}{r}\right)^{1/n} = 1\), so that we have transformed the original problem into a problem that has the desired property.

Case 2: \(\frac{q}{r} = 1\). In this case, since \(z\) is in an interval of length at most \(\frac{q^{1/n}}{\gamma_n(r)} \leq \left( \frac{q}{r}\right)^{1/n} = 1\), we have at most two solutions and \((2.11)\) holds.

We are now ready to prove \((2.11)\) under the hypothesis \(\gcd(d, q) = 1\). We begin with the case \(\omega(q) \geq 2\). Let \(W\) be the set of solutions to the equation \((2.10)\). For each prime \(p\) with \(p^a || q\) we denote by \(v_p\) the number of solutions to the equation \(P(x) \equiv 0 \pmod{p^a}\). Suppose at first that there is a prime
number \( p \) for which \( p^\alpha \| q \) and \( p^\alpha > q^{-\omega(q)+1} \) \( (p^\alpha = q^{-\omega(q)+1} \) is impossible). From Lemma \ref{lemma3.3} the numbers \( x \in \mathcal{W} \) are in at most \( v_p \leq n \) congruence classes modulo \( p^\alpha \). Let’s denote by \( \mathcal{W}' \) the set of solutions \( x \in \mathcal{W} \) that are in one of the most popular congruence classes modulo \( p^\alpha \). We write \( s := \# \mathcal{W}' \), so that \( W \leq ns \). Now, set \( q_1 := p^\alpha \) and \( q_2 := \frac{q}{p^\alpha} \) and consider the product

\[
\Delta := \prod_{\substack{x_1 < x_2 \\ x_1, x_2 \in \mathcal{W}'}} (x_2 - x_1)
\]

From Lemma \ref{lemma3.4} if \( s \geq 2 \), then \( q_2^{\frac{2}{s-1}} \) | \( \Delta \). Also, since \( q_1^{-\frac{1}{s^2}} \) | \( \Delta \), we must have

\[
q_1^{\binom{s}{2}} q_2^{\frac{2}{s-1}} \leq q_2^{\frac{2}{s-1}} = (q_1 q_2)^{\binom{s}{2}}
\]

thus \( q_1 \leq q_2^{\frac{1}{s-1}} \). However, \( q_1 > q^{-\omega(q)+1} \Rightarrow q_1 > q_2^{\frac{1}{\omega(q)}} \). We deduce that

\[
q_2^{-\omega(q)} < q_1 \leq q_2^{\frac{1}{s-1}}
\]

so that \( s \leq \omega(q) \) and \( W \leq n \omega(q) \) if \( s \geq 2 \) and \( W \leq n \) otherwise.

Now, if such a prime number does not exist, it is because \( 2^\alpha \| q \) with \( 2^\alpha > q^{-\omega(q)+1} \). The rest of the argument is similar except that \( v_2 \leq 2n \), so that \( W \leq 2ns \). If \( s \geq 2 \), we still come to the conclusion \( q_1 \leq q_2^{\frac{1}{s-1}} \) except that now \( q_1 > q^{-\omega(q)+1} \Rightarrow q_1 > q_2^{\frac{2}{s-1}} \). We deduce that \( s \leq \omega(q) \), so that \( W \leq n \omega(q) \) if \( s \geq 2 \) and \( W \leq 2n \) otherwise. We have established \eqref{2.11} in the case \( \omega(q) \geq 2 \).

We now assume that \( \omega(q) = 1 \). Since \( q^{1/n} \leq q \), we deduce from Lemma \ref{lemma3.3} that \( q = 2^\alpha \) for some \( \alpha \geq 3 \). Then, again from Lemma \ref{lemma3.3} we deduce that \( n = 2^k \) for some \( k \geq 1 \).

We first consider the case \( n = 2 \). One can show with the help of the representation \( x \equiv (-1)^a 5^b \pmod{2^\alpha} \) (see the proof of Lemma \ref{lemma3.3}) that if the equation \eqref{2.11} has a solution, then it has 4 solutions and they are of the form \( x \equiv \pm (2^\alpha - 1)z \pmod{2^\alpha} \) for some \( z \in \{1, \ldots, 2^{\alpha-2} - 1\} \pmod{2^\alpha} \). The result \eqref{2.11} follows from \( [2^{\alpha/2}] \leq 2^{\alpha-2} \) for \( \alpha \geq 3 \).

We now turn to the case \( n = 2^k \) for some \( k \geq 2 \). Let’s write

\[
\mathcal{T}_1 := \{1, \ldots, 2^{\alpha-1} - 1\} \pmod{2^\alpha}
\]

\[
\mathcal{T}_2 := \{2^{\alpha-1} + 1, \ldots, 2^\alpha - 1\} \pmod{2^\alpha}.
\]

Again, since every solution \( x \in \mathcal{T}_1 \) to \eqref{2.11} has its associated solution \( -x \in \mathcal{T}_2 \), we deduce that \eqref{2.11} holds if all the odd numbers in \( \mathcal{I} \) are included in one of \( \mathcal{T}_1 \) or \( \mathcal{T}_2 \). If it is not the case, then since

\[
(x + 2^{\alpha-2})2^k \equiv x^{2^k} \pmod{2^\alpha} \quad (k \geq 2, \alpha \geq 3),
\]
we deduce that the number of solutions to (2.10) is the same with \( I \) replaced by \( I' := I + 2^{\alpha-2} \). Now, since \( 2^{\alpha/4} < 2^{\alpha-2} \) for \( \alpha \geq 3 \), we must have that all the odd numbers in \( I' \) are included in one of \( T_1 \) or \( T_2 \). The proof is completed.

8 Concluding remarks

It is interesting to consider Theorem 2.5 with \( m = 2 \). Let \( \alpha := a + bi \) with \( a, b \in \mathbb{Z} \). The ideal \( (\alpha) \subseteq \mathbb{Z}[i] \) can also be seen as the lattice \( \Gamma \) generated by \( v_1 := (a, b) \) and \( v_2 := (-b, a) \) in \( \mathbb{Z}^2 \). The fundamental domain of \( \Gamma \) is a square of area \( N(\alpha) \) with the base \( v_1, v_2 \), precisely \( \{ \lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in [0, 1) \} \).

Here and throughout, the norm is \( N(z_1 + z_2 i) = z_1^2 + z_2^2 = \|z_1 + z_2 i\|^2 \) \((z_1, z_2 \in \mathbb{R})\) as usual.

Assume that we have a bounded set \( \Omega \in \mathbb{R}[i] \) of diameter, defined by

\[
d(\Omega) := \sup_{\alpha_1, \alpha_2 \in \Omega} (N(\alpha_2 - \alpha_1))^{1/2},
\]

nonzero. Assume also that we have a set \( \mathcal{S} \) that contains \( x_1, \ldots, x_S \) elements of \( \mathbb{Z}[i] \) and that we have a set \( \mathcal{Q} \) of pairwise coprime elements of \( \mathbb{Z}[i] \) such that for each \( q \in \mathcal{Q} \) the elements of \( \mathcal{S} \) are in at most \( \nu(q) \) of the \( N(\alpha) \) equivalence classes of \( \mathbb{Z}[i]/(\alpha) \). Then, considering

\[
\Delta := \prod_{1 \leq i < j \leq S} N(x_j - x_i)
\]

and arguing as in the proof of Theorem 2.1 we get to

\[
S \leq \frac{\sum_{q \in \mathcal{Q}} \log N(q) - 2 \log d(\Omega)}{\sum_{q \in \mathcal{Q}} \frac{\log N(q)}{\nu(q)} - 2 \log d(\Omega)}
\]

provided that the denominator is positive.

It is a refinement of Theorem 2.5 in a very special case. We are not aware of this kind of generalization in \( \mathbb{R}^m \) for any \( m \geq 3 \).

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