DECOMPOSITION OF CARTAN MATRIX AND CONJECTURES
ON BRAUER CHARACTER DEGREES

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Abstract. Let $G$ be a finite group and $N$ be a normal subgroup of $G$. Let $J = J(F[N])$ denote the Jacobson radical of $F[N]$ and $I = \text{Ann}(J) = \{\alpha \in F[G] | J\alpha = 0\}$. We have another algebra $F[G]/I$. We study the decomposition of Cartan matrix of $F[G]$ according to $F[G/N]$ and $F[G]/I$. This decomposition establishes some connections between Cartan invariants and chief composition factors of $G$. We find that existing zero-defect $p$-block in $N$ depends on the properties of $I$ in $F[G]$ or Cartan invariants. When we consider the Cartan invariants for a block algebra $B$ of $G$, the decomposition is related to what kind of blocks in $N$ covered by $B$. We mainly consider a block $B$ of $G$ which covers a block $b$ of $N$ with $l(b) = 1$. In two cases, we prove Willems’ conjecture holds for these blocks, which covers some true cases by Holm and Willems. Furthermore We give an affirmative answer to a question by Holm and Willems in our cases. Some other results about Cartan invariants are presented in our paper.

1. Introduction

Let $A$ be a Frobenius algebra and $J(A)$ be the Jacobson radical, then the relations between the Cartan matrix of $A$ and that of $A/J(A)^i, i \geq 1$, is studied in [10]. Now suppose $A = F[G]$ is a group algebra for a finite group $G$, if $N$ is a normal subgroup of $G$, we have known some results about relations between $F[G]$ and $F[G/N]$. For example, from the Alperin, Collins and Sibley[1], we can see some connections between Cartan matrix of $F[G]$ and that of $F[G/N]$. A result of Lusztig[2, P162, Lemma 4.26] also implies their connections. For a number of relations between projective modules of $F[G]$ and $F[G/N]$, see W. Willems[14].

When $N$ is a normal subgroup of $G$, let $I$ be the annihilator of the Jacobson radical $J(F[N])$ in $F[G]$. Then $I$ will be an ideal of $F[G]$. In this paper we shall establish some relations among $F[G], F[G/N],$ and $F[G]/I$. One important result is the decomposition of Cartan matrix of $G$ in terms of Cartan matrices of $G/N$ and $F[G]/I$. As an application of this fact, we shall give connections between Cartan invariants and chief composition factors of $G$. We generalize similar results in [13].

When $B$ is a $p$-block algebra of $G$, we find that the decomposition of Cartan matrix of $B$ is heavily related to what kind of $p$-blocks of $N$ covered by $B$. There are two cases that we have more advantage to study. One is a zero-defect block in $N$. Another is a block of $N$ with only one irreducible Brauer character. So we shall discuss in what conditions $B$ will cover a zero-defect $p$-block algebra of $N$. Our

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discussions produce some conditions of existing zero-defect $p$-block algebra for $N$ in terms of $I$ or Cartan invariants.

For a block $B$ of $G$, let $k(B)$ and $l(B)$ denote the numbers of ordinary irreducible characters and irreducible Brauer characters of $B$, respectively. We mainly discuss the Cartan invariants of $B$ if it covers a block $b$ in $N$ of $l(b) = 1$.

W. Willems and T. Holm [15] present several conjectures on Brauer character degrees. Although it is not easy to prove their conjectures in general, they show that there are several affirmative answers for these conjectures. In our cases, which cover some true cases proved by them, their conjectures are proved true in this paper.

Let $p$-block $B$ of $G$ has defect group $D$ and Cartan matrix $C$, W. Willem and T. Holm present a question: is it true that

$$\text{Tr}(C) \leq l(B)|D|?$$

We give affirmative answer for their question in our cases.

As application for our results, we also consider the bounds of Cartan invariants in terms of defect groups. The motivity comes from Brauer Problem VII [2, Ch. IV, §5]. We prove the Cartan invariants are bounded by the order of defect group in our cases.

An interesting work of Cartan matrix is studying its eigenvalue, see Wada’s paper [8]. Here we present a similar result which generalize a result of [3] [2, Ch. IV, Lemma 4.26].

Cartan matrix plays an key role in modular representation of finite groups, so we can find a lot of articles about Cartan matrix, see [4] [7] [11].

This paper is organized as follows. Section 2 introduces two basic Lemmas; In section 3 we discuss relations among $F[G], F[N]$ and $F[G]/I$; Section 4 provides results about connections between block algebras of $F[G]$ and $F[G]/I$; Section 5 gives a decomposition of Cartan matrix of $G$ in terms of Cartan matrices of $F[G]/N$ and $F[G]/I$; In section 6 we study Willems’ conjecture on Brauer character degrees. In our cases we prove Willems’ conjecture and give an affirmative answer to a Willems’ question. Section 7 is the application of previous sections. We present some results on Cartan invariants.

In this paper, a $G$-module usually means a left $G$-module unless we give it under specified conditions. $F$ is always a splitting field of character $p \neq 0$. For any two $G$-modules $M, P$, we use $(M, P)^G$ denote $\text{Hom}_{F[G]}(M, P)$ in this article. We fix $A = F[G], J = J(F[N])$, the Jacobson radical of $F[N]$, and $I = \{ \alpha \in F[G] | J\alpha = 0 \}$. Some notations and basic results are refered to [10].

2. Lemmas

Let $N \trianglelefteq G$ and $J = J(F[N])$ denote the Jacobson Radical of $F[N]$. We consider $J(F[N])$ and $F[N]$ as $G$-module by $G$-conjugate action on them. For any $g \in G$, since $gJ(F[N]) = J(F[N])g$, so $F[G]J(F[N])$ is a nilpotent ideal of $F[G]$, or $F[G]J(F[N]) \leq J(F[G])$. If $S$ is a subset of $F[G]$, let

$$r(S) = \text{Ann}_r(S) = \{ a \in F[G] | Sa = 0 \}$$

denote the right annihilater ideal decided by $S$.

**Lemma 2.1.** Let $J = J(F[N])$ and let $r(J)$ denote the right annihilater ideal of $J$ in $F[G]$, then $r(J) = r(F[G]J(F[N]))$. Thus $r(J)$ is an ideal of $F[G]$.
Proof. \( r(F[G]/J(F[N])) \leq r(J) \) is easy to know. If \( a \in r(J) \), then \( Ja = 0 \). Thus \( F[G]/J(F[N])a = 0 \), which induces \( a \in r(F[G]/J(F[N])) \).

Remark 2.2. In the following sections, we use \( I \) denote \( r(J) \). According to Y. Tsushima, we can find an element \( c \) in the center of the group algebra \( F[G] \) such that \( I = F[G]c \). Hence \( F[G]/F[G]J(F[N]) \) is a symmetric algebra too.

Lemma 2.1 makes the following discussions reasonable.

If \( M \) is a right \( G \)-module, then we let \( M^* = \text{Hom}_F(M, F) \) denotes the dual module under usual way, so it is a left \( G \)-module. Now we let \( \{P_1, P_2, \cdots, P_n\} \) denote a complete set of representatives of the isomorphism classes of principal indecomposable right \( G \)-modules. Since \( F[G] \) is a symmetric algebra, then \( \{P_1^*, P_2^*, \cdots, P_n^*\} \) is a complete set of representatives of the isomorphism classes of principal indecomposable left \( G \)-modules when we consider \( F[G] \) as a left regular \( G \)-module. We recall that the Cartan matrix \((c_{ij})\) decided by \( \{P_1, P_2, \cdots, P_n\} \) is \( c_{ij} = \) the multiplicity of \( \text{Hd}(P_j) \) as a composition factor in \( F[I] \).

Lemma 2.3. Let \( S(M^*) = \{f \in M^*|J(F[N])f = 0\} \), then \( S(M^*) = (M/MJ)^* \), where \( J = J(F[N]) \).

Proof. We know that \( S(M^*) \) is a \( G \)-submodule of \( M^* \). If \( f \in (M/MJ)^* = \text{Hom}_F(M/MJ, F) \), then \( f(MJ) = 0 \), which means \( Jf(M) = 0 \), so \( f \in S(M^*) \). Conversely, if \( f \in S(M^*) \), then \( f(MJ) = 0 \) and \( MJ \leq \text{Ker}(f) \), which induces \( f \in (M/MJ)^* \).

It is easy to know that \( \text{Soc}(M^*) \leq S(M^*) \).

Remark 2.4. As usual way, we can define \( U^\perp = \{f \in M^*|f(u) = 0, u \in U\} \), for \( U \leq M \). Then we can prove that \( S(M^*) = (MJ)^\perp \), so naturally we have \( (MJ)^\perp = (M/MJ)^* \).

3. Relations among algebra \( F[G], F[N] \) and \( F[G]/I \)

Let \( A = F[G] \) and \( I = r(J) \) as before. In this section, we study the the structure of the algebra \( A/I \), and discuss the relations among \( F[G], F[N] \), and \( A/I \).

About the algebra \( F[G]/I \), we have the following results to describe it.

Proposition 3.1. Let \( A = F[G], N \trianglelefteq G, J = J(F[N]), I = r(J) \) = \( \{\alpha \in F[G]|J\alpha = 0\} \), and as left regular module, let \( A = \bigoplus_P P \) be a decomposition of principal indecomposable modules of \( A \), then \( A/I = \bigoplus_P P/P \cap I \), where \( P/P \cap I \) is principal indecomposable \( A/I \)-module if \( P/P \cap I \neq 0 \).

Proof. Let \( \alpha \in I, \alpha = \sum_P \alpha_P \alpha_P \in P \), since \( J\alpha = 0 \), so \( J\alpha_P = 0 \), \( \alpha_P \in P \cap I \).

Thus \( I = \bigoplus_P P \cap I \), which implies \( A/I = \bigoplus_P P/P \cap I \).

In order to show that \( P/P \cap I \) is indecomposable, we consider \( \text{Hd}(P/P \cap I) \) and prove it is a simple module. First we claim that \( J(A/I) = (J(A) + I)/I \). This is because \( A/(J(A) + I) \) is semi-simple, and if \( A/M \) is simple, where \( I \leq M \), then \( J(A) + I \leq M \). Thus we obtain \( J(P/P \cap I) = (J(A)P + P \cap I)/P \cap I \), which shows \( \text{Hd}(P/P \cap I) = P/(J(A)P + P \cap I) \). Since \( P/(J(A)P + P \cap I) \) is a factor module of \( P/J(A)P \), we have \( J(A)P + P \cap I = J(A)P \) or \( J(A)P + P \cap I = P \). So \( P \cap I \leq J(A)P \) or \( P \cap I \cap P \) by Nakayama lemma. Hence we have \( \text{Hd}(P/P \cap I) = \text{Hd}(P/J(A)P) \) if \( P \cap I \neq P \).
Remark 3.2. There is a primitive idempotent $e$ such that $P = Ae$ when $P$ is left indecomposable ideal of $A$, so we can write $P \cap I = IP = Ie$ and $P/P \cap I = P/Ie$.

The proof of the result above induces the following result which suggests the relations between projective modules of $F[G]$ and $F[G]/I$.

Corollary 3.3. With the same notations as above, let $E = \text{Hd}(P/J(A)P)$ be a simple $A$-module. Then $E$ is still a simple $A/I$ module if and only if $P \cap I \neq P$. Furthermore, $E = \text{Hd}(P/P \cap I)$. □

Remark 3.4. From this result, for a simple $F[G]/I$-module $E$, if $P$ is the projective $F[G]$-cover of $E$, then $P/P \cap I$ is the projective $F[G]/I$-cover of $E$. A very natural question arises: What happens with $E$ being not simple $A/I$-module when $E$ is a simple $G$-module? We state the following result to answer this question.

Proposition 3.5. With the same notations as above, let $E = \text{Hd}(P/J(A)P)$. Suppose $E$ lie over a simple $N$-module $U$. Then $P \leq I$ if and only if $U$ is a projective simple $N$-module.

Proof. If $P \leq I$, then $J(F[N])|P = 0$, so $P_{1N}$ is a direct sum of projective simple $N$-modules. Since $E_{1N} = \text{Soc}(P)_{1N} \leq P_{1N}$, $E_{1N}\downarrow P_{1N}$, so $U \cap P_{1N}$ and $U$ is projective.

Conversely, let $T$ be the inertial group of $U$ in $G$. By Clifford theory, we have $E = F[G] \otimes_T U$, where $U = \sum_{t \in T} tU$. We can write $U = \sum_{t \in T/N} tU = \sum_{t \in T/N} tU$ and $F[T] \otimes_N U = \oplus_{t \in T/N} tU$, then we set a map $f : F[T] \otimes_N U \to U$ by $\sum_{t \in T/N} tU \mapsto \sum_{t \in T/N} tU$, which is a surjective $T$-homomorphism. Thus we get a surjective $G$-homomorphism

$$1 \otimes f : F[G] \otimes U \to F[G] \otimes U = E$$

Since $U$ is projective $N$-module and $P$ is the projective cover of $E$, then $P|F[G] \otimes U = U^{1G}$. $(U^{1G})_{1N} = \bigoplus_{g \in G/N} g \otimes U$ is the sum of projective simple $N$-modules, so is $P_{1N}$. Thus $J(F[N])|P = 0, P \leq I$.

□

The result above tells us an important fact is: When $P$ is a principal indecomposable module of $G$, then $P_{1N}$ is semi-simple if and only if $\text{Hd}(P)$ lie over a projective simple $N$-module.

According to Proposition 3.1, there exists a principal indecomposable $G$-module $P$ such that $P \leq I$ if and only if the number of classes of isomorphic simple $A/I$-modules is smaller than that of isomorphic simple $G$-modules, which is the dimension $\dim_F Z(A/J(A))$ of the center $Z(A/J(A))$ of $A/J(A)$ when $F$ is the splitting field of $A$. Hence we give the following two results, which offer conditions for existence of $p$-block of defect zero in $F[N]$.

Corollary 3.6. Suppose $A = F[G]$ and $F$ is the splitting field of $A$. Let $N$ be a normal subgroup of $G$, $J = J(F[N])$, and $I = \{ \alpha \in F[G] | J\alpha = 0 \}$. Then there not exist projective simple $N$-modules if and only if $I \leq J(A)$.

Proof. Notice that $J(A/I) = (I + J(A))/I$ and let $\overline{A} = A/I$, then

$$\dim_F Z(A/(I + J(A))) = \dim_F Z(\overline{A}/J(\overline{A})),$$

which is the number of classes of isomorphic simple $A/I$-modules. Since

$$\dim_F Z(A/J(A)) = l(G)$$

when $A = F[G]$, the assertion holds. □
4. RELATIONS BETWEEN BLOCKS OF \( G \) AND \( F[G]/I \)

Now we are going to prove some results for block algebras of \( G \) and \( F[G]/I \).

**Theorem 4.1.** Let \( P_1, P_2 \) be any two principal indecomposable modules belonging to the same \( p \)-block of \( G \). Then \( P_1 \leq I \) if and only if \( P_2 \leq I \).

**Proof.** Since any two principal indecomposable modules are linked \( \mathbb{S} \) if they are in the same block, we only need to suppose \( \text{Hom}_{F[G]}(P_1, P_2) = (P_1, P_2)^G \neq 0 \).

Let \( E_i = \text{Hd}(P_i), E_j = \text{Hd}(P_j) \) and they are supposed to lie over simple \( N \)-modules \( U_i \) and \( U_j \), respectively. By Proposition 3.5, it is sufficient to prove: if \( U_i \) is projective, then \( U_j \) is projective too.

\( 0 \neq (P_i, P_j)^G \leq (P_i, P_j)^N \) yields \( (P_i, P_j)^N \neq 0 \). In the proof of Proposition 3.5, we have proved \( P_i | (U_i)^G \), and consequently \( (P_i)_{1_N} = \oplus (g \otimes U_i) \), a direct sum of projective simple \( N \)-modules. Then

\[
(P_i, P_j)^N = (\oplus (g \otimes U_i), \text{Soc}((P_j)_{1_N}))^N.
\]

Next we claim that:

- For a simple \( N \)-module \( U, U | (E_j)_{1_N} \) if and only if \( U | \text{Soc}((P_j)_{1_N}) \).

As \( E_j = \text{Soc}(P_j) \leq P_j, (E_j)_{1_N} \leq \text{Soc}((P_j)_{1_N}) \). If \( E \) is a simple \( G \)-module lie over a simple \( N \)-module \( U \) such that \( U | \text{Soc}((P_j)_{1_N}) \), then \( E = \sum_{g \in G} gU \leq P_j \) and so \( E = E_j \). Hence \( U | (E_j)_{1_N} \).

Hence \( \text{Soc}(P_j) \) and \( \text{Soc}((P_j)_{1_N}) \) are decomposed a sum of simple \( N \)-modules which are conjugate to the same simple \( N \)-module. Finally, since

\[
0 \neq (P_i, P_j)^N = (\oplus (g \otimes U_i), \text{Soc}((P_j)_{1_N}))^N
\]

so

\[
(\oplus (g \otimes U_i), (E_j)_{1_N})^N \neq 0
\]

by the argument we claim above. Therefore there exist some \( g, x \in G, g \otimes U_i \cong x \otimes U_j \). So \( U_j \) is projective as \( U_i \) is projective. \( \square \)

**Remark 4.2.** By the result above, if a principal indecomposable module \( P \) is in a \( p \)-block algebra \( B \), then \( P \leq I \) if and only if \( B \leq I \).

We consider relations between \( I \) and a block \( B \) of \( G \), then the previous results Proposition 3.1 and Theorem 4.1 tell us that \( B \leq I \) or \( B \cap I = Ie_B \leq J(B) = BJ(F[G]) = J(F[G])e_B \), where \( e_B \) denote the block idempotent of \( B \). For a block \( b \) of \( N \), let \( \text{Bl}(G|b) \) denote all blocks of \( G \) which lie over \( b \). Let \( \text{Bl}(N|0) \) denote all blocks of \( N \) of defect zero. Thus we have the following result to describe \( I \).

**Corollary 4.3.** Let \( N \leq G \) and \( N_p \) denote the set of all \( p \)-elements of \( N \). Let \( c = \sum_{x \in N_p} x \). Then

\[
I = F[G]c^2 \bigoplus_{e_B \notin I} (\bigoplus_{e_B \notin I} Ie_B).
\]

In particular, \( I \leq J(F[G]) \) if and only if \( c^2 = 0 \).

**Proof.** By the previous results, we can write

\[
I = (\bigoplus_{e_B \in I} F[G]e_B) \bigoplus (\bigoplus_{e_B \notin I} Ie_B),
\]

so we just need to prove \( \bigoplus_{e_B \notin I} F[G]e_B = F[G]c^2 \).
Notice $e_B \in I$ if and only if $B$ lie over a block $b$ of $N$ with defect zero. By Tsushima\[12], $c^2$ is the sum of all block idempotents $e_b$ in $F[N]$ with defect zero. Hence we have
\[
c^2 = \sum_{\substack{\text{B}\in \text{Bl}(N[N])}} e_b = \sum_{\substack{\text{B}\in \text{Bl}(N[N])}} \sum_{b'} e_{b'} (\text{where } b_i \text{ denote the representative of } G\text{-orbit of } \text{Bl}(N[N]))
\]
\[
= \sum_{b_i} \text{Tr}_{T(b_i)}(e_{b_i})
\]
\[
(\text{sum of some central idempotents in } G)
\]
\[
= \sum_{b_i} \sum_{B\in \text{Bl}(G[b_i])} e_B^b
\]
\[
= \sum_{B \leq I} e_B
\]
which implies our result.

If $N$ is a $p$-solvable group, then we can prove $I = F[G]e$ from result in \[2\] Chap X, P422. So $I \leq J(F[G])$ if and only if $c \in J(F[G])$. If $N$ is not $p$-solvable group, the Corollary 4.3 still implies $I \leq J(F[G])$ if and only if $c \in J(F[G])$, as $c^2$ is an idempotent. Thus Corollary 4.3 is conformed with Corollary 3.6.

An immediate consequence of the results above is

**Theorem 4.4.** Let $F[G] = A = \bigoplus_i B_i$ be a decomposition of $p$-block algebra of $A$. $\overline{B}_i$ denotes the image of $B_i$ under the natural map $A \rightarrow A/I$. Then

1. $\overline{B}_i = 0$ if and only if $(B_i)_N$ is the sum of all blocks of defect zero in $F[N]$.
2. if $\overline{B}_i \neq 0$, then the simple $G$-modules of $B_i$ are also the simple $A/I$-modules of $\overline{B}_i$.
3. $A/I = \bigoplus_i \overline{B}_i$, where $\overline{B}_i = 0$ or $B_i/B_i \cap I$ with $B_i \cap I \leq J(B_i)$. □

5. AN ALGEBRAIC DECOMPOSITION OF CARTAN MATRIX

In the following, we are going to give a relation among the Cartan matrices of $F[G]$, $F[G/N]$ and $F[G/r(J(F[N]))]$. Now let $C_G = (c_{ij})$ denote the Cartan matrix of $G$ and $C_G^{\overline{N}} = (c_{ij}^{\overline{N}})$ denote the Cartan matrix of $F[G/N]$. For simple $F[G/N]$-module $E_i$, let $\overline{E}_i$ and $\overline{P}_i$ denote the projective $F[G/N]$-cover and $F[G]$-cover of $E$, respectively. Then by \[8\] Chapter 2, §11, we have $\overline{E}_i \cong P_i/P,J$ as $F[G]$-module, where $J = J(F[N])$.

To understand our proof in the following result, we need to know the basic fact: as Frobenius algebra, $A \cong (A_N)^*$, so if $\{P\}$ is right principle indecomposable module, then $\{P^*\}$ is left principle indecomposable module.

In order to make our discussion more clearly, We divide the set of all right principal indecomposable modules(up to isomorphisms) of the group $G$ into two subsets: let
\[
S_1 = \{P \mid \text{Hd}(P) \text{ a simple module of } G/N\}
\]
\[
S_2 = \{P \mid \text{Hd}(P) \text{ a simple module } \not\in G/N\}.
\]
We describe the difference between $S_1$ and $S_2$ as follows:

**Proposition 5.1.** Let $J = J(F[N])$ be the Jacobson radical of $F[N]$ and $A_N$ the augmentation ideal of $F[N]$. 1) If $P \in S_1$, then $PJ = PA_N$; 2) If $P \in S_2$, then $PA_N = P$.

**Proof.** (1) In fact it was proved in \[8\] Chapter 2, §11.
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(2) Let $E = \text{Hd}(P)$ for $P \in S_2$, then $E \not\in G/N$. If $PA_N \neq P$, then $PA_N \leq PJ(F[G])$. Let $P = P/PA_N$, then $P$ is a $G/N$-module which has $\text{Hd}(P)$ in $G/N$, a contradiction. \hfill \Box

We let the Cartan matrix $C = (c_{ij})$ of $F[G]$ be decided by the set of right principal indecomposable modules: $P_1, P_2, \ldots, P_l$. Then

$$c_{ij} = \dim_F(P_j, P_i)^G,$$

that is to say $c_{ij} = c_{ji} = c_{ij}^* = c_{ij}^*_{ij}. \text{[16]}$. Let $(\overline{c_{ij}})$ be the corresponding Cartan matrix of $F[G]/F[G]J$ decided by $\{P_i/P_j, J\}$, where $J = J(F[N])$. Then $\overline{c_{ij}} = \dim_F(P_j/P_i, P_j/P_j)^G$. Since $F[G]/F[G]J$ is also a symmetric algebra, so $\overline{c_{ij}} = \overline{c_{ij}}^* = \overline{c_{ij}}^*$. The following result helps us to calculate $\overline{c_{ij}}^*$.\hfill \Box

Lemma 5.2. With the same notations as above, $N$ is a normal subgroup of $G$. Let $P$ is the right principal indecomposable module with head $\text{Hd}(P)$ in $F[G]/F[G]J$, where $J = J(F[N])$. Then $(P/PJ)^* = P^*/JP^*$. \hfill \Box

Proof. We should notice both $(P/PJ)^*$ and $P^*/JP^*$ are left principal indecomposable modules of $F[G]/F[G]J$. If we prove they have the same head, they are equal.

First we have $(\text{Hd}(M))^* = \text{soc}(M^*)$, $(\text{Hd}(M)) = (\text{soc}(M))^*$ for any $G$-module $M$ by duality. Hence $\text{Hd}((P/PJ)^*) = (\text{soc}(P/PJ))^* = (\text{Hd}(P))^*$. On the other hand, we have $\text{Hd}(P^*/JP^*) = \text{Hd}(P^*) = (\text{soc}(P))^* = (\text{Hd}(P))^*$. So they have the same head. \hfill \Box

Theorem 5.3. With the same notations as above, let $J = J(F[N])$ and $I = r(J(F[N]))$. We use $\overline{c} = (\overline{c_{ij}})$ and $C_J = (x_{ij})$ to denote the Cartan matrix of $F[G]/F[G]N$ and $F[G]/I$, respectively. Then

1. If $P_i, P_j \in S_1$, then $c_{ij} = \overline{c_{ij}} + x_{ij}$
2. If $P_i \in S_1$ and $P_j \in S_2$, then $c_{ij} = x_{ij}$
3. If $P_i, P_j \in S_2$, then $c_{ij} = a_{ij} + x_{ij}$, where $a_{ij}$ is the Cartan number of Algebra $F[G]/F[G]J$.

Proof. \hfill \Box

Case One: For projective module $P_j$, we have

\begin{equation}
0 \rightarrow P_jJ \rightarrow P_j \rightarrow P_j/P_jJ \rightarrow 0,
\end{equation}

so we have

\begin{equation}
0 \rightarrow (P_i, P_j J)^G \rightarrow (P_i, P_j)^G \rightarrow (P_i, P_j/P_j J)^G \rightarrow 0
\end{equation}

as $P_i$ is $G$-projective module. Thus

$$\dim_F(P_i, P_j)^G = \dim_F(P_i, P_j J)^G + \dim_F(P_i, P_j/P_j J)^G.$$

Since $\overline{P_i} \cong P_i/P_jJ$, so we got

$$c_{ij} = \dim_F(P_i, P_j J)^G + \overline{c_{ij}}.$$ 

Now for $\dim_F(P_i, P_j J)^G$, first we know by \text{[16]}

\begin{equation}
\dim_F(P_i, P_j J)^G = \dim_F((P_j J)^*, P_i^*)^G.
\end{equation}
On the other hand, we have the following from (4.1)

\[(5.3)\quad 0 \longrightarrow (P_j/P_j J)^* \longrightarrow P_j^* \longrightarrow (P_j J)^* \longrightarrow 0.\]

By Lemma 2.3, we have \(S(P_j^*) = (P_j/P_j J)^*\), then by (4.3) we get

\[(P_j J)^* \cong P_j^*/S(P_j^*).\]

Thus

\[
\dim_F((P_j J)^*, P_i^*)^G = \dim_F(P_i^*/S(P_i^*), P_i^*)^G
\]

= Multiplicity of \(\text{Soc}(P_i^*)\) as the composition factor in \(P_j^*/S(P_j^*)\)

= Multiplicity of \(\text{Hd}(P_i^*)\) as the composition factor in \(P_j^*/S(P_j^*)\)

= \(x_{ji}\)

Notice \(P_j^* \cap I = S(P_j^*)\), then by (4.2), we have

\[(5.4)\quad c_{ij} = \overline{c_{ij}} + x_{ji}^*,\]

where \((x_{ji}^*)\) is the left Cartan matrix of \(F[G]/r(J)\), decided by

\[\{P_i^*/S(P_i^*), P_2^*/S(P_2^*), \ldots, P_n^*/S(P_n^*)\}.\]

Similarly we can get

\[
c_{ij}^* = \dim_F(P_i^*, J P_j^*)^G + \dim_F(P_i^*, P_j^*/J P_j^*)^G
\]

= \(\dim_F(J P_j^*, P_j^*)^G + \dim_F(J P_j^*/J P_j^*, P_j^*/J P_j^*)^G\)

= \(\dim_F((J P_j^*)^*, P_j^*)^G + \overline{c_{ij}^*}\)

by Lemma 4.2. Since \((J P_j^*)^* = P/I \cap P\) by the same arguments as above,

\[\dim_F((J P_j^*)^*, P_j^*)^G = \dim_F(P_j/I \cap P_j, P_j)^G\]

= the number of \(\text{Soc}(P_j)\)

= the number of \(\text{Hd}(P_j)\)

= the number of \(\text{Hd}(P_j/I \cap P_j)\)

= the composition factors of \(P_j/I \cap P_j\)

= \(x_{ji}\).

Therefore we have

\[(5.5)\quad c_{ij}^* = \overline{c_{ij}^*} + x_{ji}.\]

Since the Cartan matrices of \(F[G]\) and \(F[G/N]\) are dual and symmetric by 10, so \(c_{ij} = \overline{c_{ij}} + x_{ij}\) by equality (4.4) and (4.5).

**Case two:** Let \(P_i \in S_2, P_j \in S_1\). Then

\[(P_i, P_j)^G = (P_i A_N, P_j)^G = (P_i, P_j J)^G\]

By the same arguments as in Case one, we have \(c_{ij} = x_{ij}^*\). Thus \(c_{ij} = x_{ij}\) as \(c_{ij}\) is dual and symmetric.

**Case three:** Let \(P_i, P_j \in S_2\). Then by the same reason as in Case one, we have

\[\dim_F(P_i, P_j)^G = \dim_F(P_i, P_j J)^G + \dim_F(P_i, P_j/J P_j)^G.\]
Let \( a_{ij} = \dim_F (P_i, P_j / P_j J)^G = \dim_F (P_i / P_i J, P_j / P_j J)^G \), which is Cartan number from the symmetric algebra \( F[G] / F[G] J \), so \( a_{ij} \) is dual and symmetric. The same arguments as before induces \( c_{ij} = a_{ij} + x_{ij} \).

Then the assertion follows.

Consider two extreme examples for Theorem 4.3. If \( N \) is a \( p' \)-prime group, then all \( x_{ij} = 0 \). The Cartan numbers from \( S_2 \) tell nothing new by Theorem 4.3. If \( N \) is a \( p \)-group, then set \( S_2 \) is an empty set. Only Case one could happen.

We apply the results above to consider the Cartan matrix of a \( p \)-block algebra of \( G \). If \( B \) is a \( p \)-block algebra of \( G \) covering a \( p \)-block algebra of \( N \) of defect zero, then \( B \) has no simple modules in \( F[G] / I \) by Theorem 3.10. Hence all Cartan numbers \( x_{ij} \) of \( F[G] / I \) related to the principal indecomposable modules in \( B \) will be zero. Furthermore, we know the Cartan matrix of a block algebra can not split into two unconnected parts, which shows that all principal indecomposable modules of \( B \) are in \( S_1 \) or all in \( S_2 \). So we have the following

**Corollary 5.4.** Let \( N \) be a normal subgroup of group \( G \). Then \( B \) is a \( p \)-block algebra of \( G \) covering a zero-defect \( p \)-block of \( N \) if and only if the Cartan matrix of \( B \) is equal to that of image of \( B \) in \( F[G/N] \) or that of image of \( B \) in \( F[G] / F[G] J, \) where \( J = J(F[N]) \).

**Proof.** The assumption must holds by the arguments above if \( B \) covers a zero-defect \( p \)-block of \( N \). Conversely, if the Cartan matrix of \( B \) is equal to that of image of \( B \) in \( F[G/N] \) or that of image of \( B \) in \( F[G] / F[G] J, \) we can find some principal indecomposable module \( P_i \) in \( B \) such that Cartan number \( x_{ii} = 0 \) of \( F[G] / I \) corresponding to \( P_i \). Thus \( P_i \leq I \) and \( B \) covers a zero-defect \( p \)-block of \( N \) from Theorem 3.10.

**Remark 5.5.** If \( N \) is \( p' \)-group and the image of \( B \) in \( F[G/N] \) is not zero, it is well known that \( B \) and its image have the same irreducible characters. We generalize these results.

Notation: In the following, when we say a Cartan number \( c \) is corresponding to a simple module \( E \), it means that \( c \) is the multiplicity of \( E \) as a composition factors in the projective cover of \( E \).

**Corollary 5.6.** Let \( N \) be a normal subgroup of \( G \) and \( G/N \) is a \( p \)-group. Number \( c_{11} \) denote the Cartan number corresponding to the trivial \( G \)-module \( F \). Then \( c_{11} \geq |G/N| \) and the equality holds if and only if \( G \) is \( p \)-nilpotent and \( N = O_{p'}(G) \).

**Proof.** Since \( G/N \) is a \( p \)-group, the set \( S_1 \) contains only one element \( P \) which is the projective cover of trivial \( G \)-module \( F \). The Cartan number \( c_{11} \) in \( G/N \) corresponding to the trivial \( G/N \)-module is \( |G/N| \), so \( c_{11} \geq |G/N| \). The equality holds if and only if the principal \( p \)-block covers a defect-zero block in \( N \), but this is equal to the principal \( p \)-block having only one irreducible Brauer character.

In the following we discuss the relations between the composition factors and Cartan numbers for a group \( G \). We will generalize a result in [13] about a lower bound for the first Cartan invariant \( c_{11} \) in terms of the chief composition factors of \( G \).

Let the group \( G \) has a series of normal subgroups as following

\[
1 = G_0 \leq G_1 \leq \cdots \leq G_n \leq G_{n+1} = G.
\]
Let $k$ be the number of factors which are $p$-groups (non-trivial) in $G_i/G_{i-1}$, $i = 1, 2, ..., n + 1$.

**Theorem 5.7.** With the same notation as above, if $E_j$ is a simple module of $G/G_n$, then the Cartan number $c_{jj} \geq k + 1$, where $c_{jj}$ is the Cartan number of $G$ corresponding to $E_j$. In particular, if $G/G_n$ is a $p$-group, then $c_{11} \geq |G/G_n| + k - 1$.

**Proof.** First, we should know $E_j$ is also a simple $G/G_i$-module for $i = 0, 1, ..., n - 1$, as $G_i$ acts trivially on $E_j$.

When $i = 1$, we have $c_{jj} = c_{jj}^{(1)} + x_{jj}^{(1)}$ by Theorem 4.3, where $c_{jj}^{(1)}$ is the Cartan number of $F[G/G_1]$ corresponding to $E_j$ and $x_{jj}^{(1)}$ is the Cartan number of $F[G]/I$ corresponding to $E_j$, $I = \{ \alpha \in F[G] | J(F[G_1]) | \alpha = 0 \}$.

Considering $c_{jj}^{(1)}$ in $F[G/G_1]$ and $G_2/G_1$ as normal subgroup of $G/G_1$, we have $c_{jj}^{(1)} = c_{jj}^{(2)} + x_{jj}^{(2)}$, where $c_{jj}^{(2)}$ is the Cartan number of $F[G/G_2]$ corresponding to $E_j$ and $x_{jj}^{(2)}$ is the Cartan number of $F[G/G_1]/I_1, I_1 = \text{Ann}_* J(F[G_2/G_1])$.

We continue in the same way as above, then we end in $n$th step:

$$c_{jj} = c_{jj}^{(n)} + x_{jj}^{(n)} + \cdots + x_{jj}^{(2)} + x_{jj}^{(1)},$$

where $c_{jj}^{(n)}$ is the Cartan number of $G/G_n$ corresponding to $E_j$. Notice $x_{jj}^{(t)} \neq 0$ if $G_i/G_{i-1}$ is a $p$-group. If $G/G_n$ is a $p$-group, $c_{jj}^{(n)} \geq |G/G_n|$ by Corollary 4.6. This completes our proof. \(\square\)

There is a block version to state Theorem 5.7. With the same series of normal subgroups of $G$ as above, for a simple module $E$ in $G/G_n$, we suppose $B^{(i)}$ is the $p$-block containing $E$ in $G/G_i$. Assume $p$-block $B$ of $G$ contains the simple module $E$, then there are series of natural epimorphisms as follows:

$$B \xrightarrow{f_1} B^{(1)} \xrightarrow{f_2} B^{(2)} \rightarrow \cdots \rightarrow B^{(n)}$$

where $f_i$ is the composition of natural map from $G/G_{i-1}$ to $G/G_i$ and the projective to $B^{(i)}$ from the image of $B^{(i-1)}$ in $G/G_i$. By Corollary 4.4, we know $B^{(i)}$ and $B^{(i+1)}$ (Assume $B^{(0)} = B$, $i = 0, 1, ..., n - 1$) have the same Cartan matrix if and only if $B^{(i)}$ covers a zero-defect $p$-block in $F[G_{i+1}/G_i]$. Let $k$ be the number of $p$-blocks $B^{(i)}$ which covers non-zero-defect $p$-blocks, $i = 0, 1, ..., n - 1$, then we have

**Theorem 5.8.** Under the same notation as above, let $c$ be the Cartan number corresponding to simple module $E$ in $G$. If $E$ belongs to a $p$-block $B^{(n)}$, then $c \geq k + 1$.

**Proof.** By the arguments above. \(\square\)

6. CARTAN INVARIANTS AND DIMENSIONS OF BRAUER CHARACTERS

For $N \leq G$ and a $p$-block $b$ in $N$, we let $\text{Bl}(G|b)$ denote all blocks of $G$ covering $b$, it is an important to find connections between them. In general case, it is not easy. If there exists defect zero blocks in $N$, we will have more advantages to study those blocks in $G$ covering defect zero blocks in $N$. In this section we consider some $p$-blocks of $G$ which cover a $p$-block $b$ with only one Brauer irreducible character in $N$ for $N \leq G$. We obtain some results about Cartan invariants and degrees of Brauer characters of $B$. In [15] and [6], T. Holmes and W. Willems present several conjectures on Brauer character degrees. Some true cases are proved by them. Here
we show some more general true cases covering their special cases. Furthermore in some cases, we give affirmative answers to a question presented by T. Holms and W. Willems in [6].

If $E_i$ denote a simple $G$-module covering a simple $N$-module $V$, by Clifford Theorem we have as $N$-module:

$$E_i = e_i \bigoplus_{t \in G/T_V} V^t,$$

where $e_i$ is the ramification coefficient of $E_i$, $T_V$ is the initial group of $V$, and $V^t$ is a conjugate module of $V$.

Now let $P_i$ is the projective cover of $E_i$. As $N$-module, $P_i$ or its character form has a similar decomposition. See W. Feit [2, Lemma 1.3, Ch. VI], H. Nagao and Y. Tsushima [9, P389, Problem 10], and Navarro [10, Corollary 8.8, VIII]. Here we will improve their result with a new proof. Let $P^t_V$ denote the projective cover of $V \in \mathbb{IBr}(N)$, then $P^t_V$ is a conjugate module of $P_V$ for $t \in G$ and projective cover of $V^t$.

**Theorem 6.1.** With the same notation as above, if

$$E_i = e_i \bigoplus_{t \in G/T_V} V^t,$$

then

$$P_i = a_i \bigoplus_{t \in G/T_V} P^t_V,$$

where $a_i \geq e_i$, the equality holds if and only if $E_i$ is $N$-projective.

**Proof.** By Frobenius-Nakayama-reciprocity theorem, there is

$$P^G_V = \bigoplus_i e_i P_i,$$

so $(P_i)_N (P^G_V)_N = \bigoplus_{t \in G/N} P^t_V$. Since all $P^t_V, t \in G$ are projective indecomposable (as projective cover of $V^t$), we have $(P_i)_N = \bigoplus_i P^t_V$ and $\Hd((P_i)_N) = \bigoplus_i \Hd(P^t_V)$.

Now as $G$-module, it holds $P_i J(F[N]) \leq P_i J(FG)$, so we have exact series:

$$0 \longrightarrow P_i J(FG) / P_i J(F[N]) \longrightarrow P_i / P_i J(F[N]) \longrightarrow P_i / P_i J(FG) \longrightarrow 0.$$

As $N$-modules, the exact series above is split since $P_i / P_i J(F[N])$ is completely reducible. Thus

$$(\Hd(P_i))_N = (P_i / P_i J(FG))_N (P_i / P_i J(F[N]))_N = \Hd((P_i)_N).$$

As $E_i = \Hd(P_i)$, It follows that

$$(P_i)_N = \bigoplus_{t \in G/T_V} a_t P^t_V, a_t \geq e_i.$$

Since

$$a_t = \dim_F((P_t)_N, V^t)_N$$

$$= \dim_F(P_t, (V^t)^G_G)$$

$$= \dim_F(P_t, V^G_G)$$

$$= \dim_F((P_t)_N, V)_N$$

$$= a_1$$
so let \( a_i = a_1 \) which only depends \( P_i \). Comparing \( e_i \) and \( a_i \), the arguments above tell us that \( a_i = e_i \) if and only if \( P_i J(F[G]) = P_i J(F[N]) \), which is equivalent to that \( E_i \) is \( N \)-projective\(^2\) Lemma 2.1, Ch. VI]. Our proof is finished. □

In order to understand Willems’ conjecture, let \( \{ \phi_i | i = 1, 2, ..., l(B) \} \) = IBr\((B)\) and \( \Phi_i, i = 1, 2, ..., l(B) \) is the Brauer character of principal indecomposable module corresponding to \( \phi_i \). We denote defect group of block \( B \) by \( D_B \). The number \( e_i \) and \( a_i \), appeared in Theorem 6.1, are the ramification coefficient of \( \phi_i \) and \( \Phi_i \), respectively.

**Corollary 6.2.** With the same notations as above, let \( N \) be a normal subgroup of \( G \) and \( B \in \text{Bl}(G|b) \) for \( b \in \text{Bl}(N) \). Let \( l(b) = 1 \), then

\[
\frac{\text{Dim}_F B}{|D_B|} \geq \sum_{i=1}^{l(B)} \phi_i(1)^2,
\]

the equality holds if and only if \( D_B \leq N \). In particular, if \( B \) has a normal defect subgroup of \( G \), then

\[
\frac{\text{Dim}_F B}{|D_B|} = \sum_{i=1}^{l(B)} \phi_i(1)^2,
\]

**Proof.** By our condition, we suppose that IBr\((b) = \theta \) and \( \Psi \) is the only Brauer character offered by the principal indecomposable module of \( b \). Since

\[
\text{Dim}_F(B) = \sum_{i=1}^{l(B)} \phi_i(1)\Phi_i(1) = \sum_{i=1}^{l(B)} \phi_i(1)a_i|G/T_b|\Psi(1) = \sum_{i=1}^{l(B)} \phi_i(1)a_i|G/T_b||D_B|\theta(1) \geq \sum_{i=1}^{l(B)} \phi_i(1)^2|D_b|.
\]

The equality above holds if and only if all \( a_i = e_i, i = 1, 2, ..., l(B) \), which is equivalent to \( D_B \leq N \) by \([2]\) Theorem 2.3, Ch. VI], or \( D_B = D_b \). □

By Corollary 6.2, if \( B \) covers a block \( b \) of \( N \) with \( l(b) = 1 \) and \( D_B \leq N \), then Willems’ local conjecture holds. In particular, his conjecture holds for a block \( B \) with a normal defect subgroup.

Without the condition \( l(b) = 1 \), the following result present another true case for Willems’ local conjecture. In fact, Willems’ conjecture holds for \( p \)-solvable groups by Holm and Willems\([6]\). We prove the conjecture in a weaker condition: there exists a normal and \( p \)-solvable subgroup \( N \) containing the defect group \( D_B \) of block \( B \). We suppose that \( \phi_i \) lie over \( \theta_i \in \text{IBr}(b) \) and that \( \Psi_i \) denotes the projective Brauer character corresponding to \( \theta_i \).

**Corollary 6.3.** With the same notations as above, for a normal subgroup \( N \) and \( B \in \text{Bl}(G|b), b \in \text{Bl}(N) \), if defect group \( D_B \leq N \) and \( N \) is \( p \)-solvable group, then

\[
\frac{\text{Dim}_F(B)}{|D_B|} \leq \sum_{i=1}^{l(B)} \phi_i(1)^2,
\]

so Willems’ local conjecture holds for these blocks \( B \).
First notice $D_b = D_B$ from $D_B \leq N$. By our condition, all $e_i = a_i$, $i = 1, 2, \ldots, l(B)$. Hence we have

$$
\text{Dim}_F(B) = \sum_{i=1}^{l(B)} \varphi_i(1)\Phi_i(1)
= \sum_{i=1}^{l(B)} \varphi_i(1)a_i|G/T_\theta_i|\Psi_i(1)
\leq \sum_{i=1}^{l(B)} \varphi_i(1)a_i|G/T_\theta_i||N|\theta_i(1)_p
\leq \sum_{i=1}^{l(B)} \varphi_i(1)|D_B\theta_i(1)|
(\text{as } |N|_p \leq |D_B\theta_i(1)_p|)
= \sum_{i=1}^{l(B)} \varphi_i(1)^2|D_B|
$$

which induces our result. □

Let $C$ denote the Cartan matrix of a $p$-block $B$, with defect group $D$. There is a question from T. Holm and W. Willems[6]:

$$\text{Tr}(C) \leq l(B)|D|?$$

The question is important as it implies Willems' conjecture. We give affirmative answer in some cases for their question. We keep to use the notations as before.

**Lemma 6.4.** For a normal subgroup $N$ of $G$ and $b \in \text{Bl}(N)$, suppose $B \in \text{Bl}(G|b)$, with Cartan matrix $C = (c_{ij})$. Let $D_B$ and $D_b$ denote the defect groups of $B$ and $b$, respectively. If $l(b) = 1$, then

1. $c_{ii} \leq \frac{a_i}{e_i}|D_b|$, $c_{ij} \leq \frac{a_i}{e_j}|D_b|$.

2. Furthermore if the simple module $E_i$ affording $\varphi_i$ is $N$-projective, then $c_{ii} \leq |D_b|$.

**Proof.** Let $\text{IBr}(b) = \{\theta\}$ and $\Psi$ denote the projective Brauer character corresponding to $\theta$. Since

$$
\Phi_i(1) = \sum_{j=1}^{l(B)} c_{ij}\varphi_j(1)
= \sum_{j=1}^{l(B)} c_{ij}e_j|G/T_\theta_i|\theta(1)
= a_i|G/T_\theta_i|\Psi(1)
= a_i|G/T_\theta_i||D_b\theta(1)|,
$$

then we get

$$|D_b|a_i = \sum_{j=1}^{l(B)} c_{ij}e_j \geq c_{ij}e_j,$$

which induces (1).

When $E_i$ is $N$-projective, then $a_i = e_i$. So we get (2). □

The following result gives affirmative answers to Wellems’ question in some cases.

**Theorem 6.5.** For $N \lhd G$, suppose $p$-block $B$ has a defect group $D_B$ and covers a $p$-block $b$ in $N$. If $l(b) = 1$ and $D_B \leq N$, then we have

$$\text{Tr}(C) \leq l(B)|D_B|.$$ 

In particular, if $B$ has a normal defect group $D_B$, then

$$\text{Tr}(C) \leq l(B)|D_B|.$$
Proof. By our condition, all $a_i = e_i$, $i = 1, 2, ..., l(B)$ and $D_B = D_b$, defect group of $b$, so our result follows from Lemma 6.4.

Since T. Holm and W. Willems has proved that

$$\frac{\text{Dim}_F(B)}{\text{Tr}(C)} \leq \sum_{i=1}^{l(B)} \varphi_i(1).$$

Thus by Theorem 6.5 we show again that their conjecture holds for a block $B$, if it covers a block $b$ in $N$, with $l(b) = 1$ and $D_B \leq N$, or if $B$ has a normal defect group, as proved by Corollary 6.2.

7. SOME APPLICATIONS FOR OUR RESULTS

In this section, we further consider applications for our results in the Sections before. Specially when $N$ is a normal $p$-subgroup of $G$. For example, Brauer Problem VII [2] Ch. IV, §5 is about bounds of Cartan invariants in terms of defect groups. If $G$ is $p$-solvable, the upper bound is the order of defect group by Fong [2] Ch. X, §4. Under our condition, we will give some bounds for Cartan invariants. These results can help us understand the connection between the group invariants of $G$ and algebraic invariants of $F[G]$. We keep the same notation as before.

The proposition 3.1 will be improved with the new condition. The part 1 in the following result can be obtained directly from [13], but we prove it here in a simple way.

**Proposition 7.1.** Let $A = F[G]$ and $N$ be a normal $p$-subgroup of $G$. Then:

1. Let $\hat{N} = \sum_{x \in N} x \in Z(A)$, the center of $A$, then
   $$I = \{ \alpha = \sum_{g \in G/N} a_y \hat{N} g \in F \} = A\hat{N},$$
   an ideal of $A$ generated by $\hat{N}$.
2. $I^2 = 0$, and $\text{Soc}(G) \leq I \leq J(A)$, where $\text{Soc}(G)$ is the socle of $A$
3. If $A = \bigoplus P_i$, a decomposition of principal indecomposable module of $A$, then $A/I = \bigoplus P_i / P_i \cap I$, a similar decomposition of $A/I$ with the same number of summands as in $A$.

Proof. Notice $I = r(J(F[N])) = \{ \alpha \in F[G] | J(F[N]) \alpha = 0 \}$, and $J(F[N]) = \langle x - 1 | g \in N \rangle$, an ideal of $F[N]$ generated by $x - 1, x \in N$. Let $\alpha = \sum_{g \in G} a_g g \in I$, then for any $x \in N$,

$$(x - 1) \alpha = (x - 1) \sum_{g} a_g g = \sum_{g} a_g x g - \sum_{g} a_g g = \sum_{g} (a_{x^{-1} g} - a_g) g = 0.$$

Thus $a_{x^{-1} g} = a_g$, for any $x \in N, g \in G$. Assertion 1 follows.

Notice $\hat{N}^2 = |N| \hat{N} = 0$, Hence $I^2 = 0$, $I \leq J(A)$ by assertion 1. Since $J(F[N]) \text{Soc}(G) \leq J(A) \text{Soc}(G) = 0$, $\text{Soc}(G) \leq I$. Assertions 2 follows.

Since we have $J(A/I) = J(A)/I$ by Proposition 3.1 and assertion 2, thus

$$\text{Hd}(A/I) = \text{Hd}(A) = A/J(A),$$

so there is no $P_i$ such that $P_i \cap I = P_i$. Assertion 3 follows. □
On the bounds of Cartan invariants, we have following results:

**Corollary 7.2.** If simple module \( E_i \) has a normal vertex \( N \), then
\[
c_{ii} \leq |N|
\]

**Proof.** Easy to see by Lemma 6.4. \( \square \)

**Theorem 7.3.** For \( N \trianglelefteq G \) and \( b \in \text{Bl}(N) \), suppose \( B \in \text{Bl}(G|b) \), with a defect group \( D_B \). If \( l(b) = 1 \) and \( D_B \leq N \), then
\[
c_{ij} \leq |D_B|, \quad i, j = 1, 2, \ldots, l(B).
\]
In particular, if \( D_B \) is a normal defect group of \( G \), then
\[
c_{ij} \leq |D_B|, \quad i, j = 1, 2, \ldots, l(B).
\]

**Proof.** According to Lemma 6.4, we have
\[
c_{ii} \leq |D_B|, \quad i = 1, 2, \ldots, l(B).
\]
Hence
\[
c_{ij} \leq \sqrt{c_{ii}c_{jj}} \leq |D_B|,
\]
for any \( 1 \leq i, j \leq l(B) \). \( \square \)

Applying Theorem 5.7 and Corollary 7.2, we give a result about a series of normal subgroups of \( G \).

**Corollary 7.4.** Suppose we have a series of normal subgroup of \( G \):
\[
1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n \trianglelefteq G_{n+1} = G.
\]
Let \( k \) be the number of factors which are \( p \)-groups(non-trivial) in \( G_i/G_{i-1}, i = 1, 2, \ldots, n+1 \). If \( E_i \) is a simple \( G/G_n \)-module and has a normal vertex \( N \), then
\[
k + 1 \leq c_{ii} \leq |N|.
\]

**Proof.** Easy to see by Theorem 5.7 and Corollary 7.2. \( \square \)

We obtain a result about eigenvalues of Cartan matrix \( C = (c_{ij}) \) of \( B \), which generalize a result when \( D_B \) is a normal defect group. [3][2, Ch. IV, §4.26].

**Theorem 7.5.** For \( N \trianglelefteq G \) and \( b \in \text{Bl}(N) \), suppose \( B \in \text{Bl}(G|b) \), with defect group \( D_B \). If \( l(b) = 1 \) and \( D_B \leq N \), then \( |D_B| \) is an eigenvalue of Cartan matrix \( C = (c_{ij}) \) of \( B \) with an eigenvector
\[
(e_1, e_2, \ldots, e_{l(B)}).
\]
In particular, the conclusion holds for a \( p \)-block \( B \) with a normal defect group and the eigenvector is
\[
(\varphi_1(1), \varphi_2(1), \ldots, \varphi_{l(B)}(1)).
\]

**Proof.** As proof in Lemma 6.4, we get
\[
|D_b|a_i = \sum_{j=1}^{l(B)} c_{ij} e_j.
\]
Notice all $a_i = e_i, i = 1, 2, ..., l(B)$, and $D_b = D_B$ under our condition. So
\[
\begin{pmatrix}
e_1 \\
e_2 \\
\vdots \\
e_{l(B)}
\end{pmatrix}
\begin{vmatrix}
D_B
\end{vmatrix}
= \begin{pmatrix}
e_1 \\
e_2 \\
\vdots \\
e_{l(B)}
\end{pmatrix}
\begin{pmatrix}
c_{ij} \\
c_{ij} \\
\vdots \\
c_{ij}
\end{pmatrix}
\] by which the assertion follows. When $N = D_B \triangleleft G$, all $e_i = \varphi_i(1), i = 1, 2, ..., l(B)$.

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\section*{References}

1. J. L. Alperin, M. Collins and D. Sibley, \textit{Projective modules, filtrations, and Cartan invariants}, Bull. L. M. S., Vol16(1984), 416-420
2. W. Feit, \textit{The representation theory of finite groups}, North-Holland Publishing Company, Amsterdam, 1980
3. M. Kiyota, M. Murai and T. Wada, \textit{Rationality of eigenvalues of Cartan matrices in finite groups}, J. Algebra, 249(2002), 110-119
4. Laszlo Hethelyi, Erzsebet Horvath, Burkhard Kulshammer, John Murray, \textit{Central ideals and Cartan invariants of symmetric algebras}, J. Algebra, 296(2006), 177-195
5. Burkhard Kulshammer, \textit{Group-theoretical descriptions of ring-theoretical invariants of group algebras}, Progress in mathematics, Vol.95(1991), 425-442
6. Thorsten Holm and Wolfgang Willems, \textit{A local conjecture on Brauer character degrees of finite groups}, Trans. Am. Math. Soc., 359(2007), 591-603
7. R. Knorr and G. Robinson, \textit{Some remarks on a conjecture of Alperin}, J. London Math. Soc. (2)39(1989), 48-60
8. P. Landrock, \textit{Finite group algebras and their modules}, Cambridge Univ. Press, Cambridge, 1983
9. Hirosi Nagao and Yukio Tsushima, \textit{Representations of finite groups}, Academic Press, INC., 1989
10. G. Navarro, \textit{Characters and blocks of finite groups}, L. M. S. Lecture Note Series 250, Cambridge University Press, 1998
11. Shigeo Koshitani, \textit{Cartan invariants of group algebra of finite groups}, Proceedings of the AMS, 124(8)(1996), 2319-2323
12. Y. Tsushima, \textit{On the block of defect zero}, Nagoya Math. J. 44(1971), 57-59
13. Y. Tsushima, \textit{On the annihilator ideals of the radical of a group algebra}, Osaka J. Math. 8(1971), 91-97
14. W. Von Willems, \textit{On the projectives of a group algebra}, Math. Z.171(1980), 163-174
15. W. Willems, \textit{On degrees of irreducible Brauer characters}, Trans. Am. Math. Soc., Vol. 357(2005), 2379-2387
16. Zeng Jiwen, \textit{Some results on the Cartan matrix of a Frobenius Algebra}, Communications in algebra, 24(14)(1996), 4385-4396

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