ON THE CLASSIFICATION OF
SUBALGEBRAS OF CEND\(_N\) AND \(g_{C}\)

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ABSTRACT. The problem of classification of infinite subalgebras of Cend\(_N\) and of 
gc\(_N\) that acts irreducibly on \(\mathbb{C}[\partial]^N\) is discussed in this paper.

0. Introduction

Since the pioneering papers [BPZ] and [Bo], there has been a great deal of work
towards understanding of the algebraic structure underlying the notion of the op-
operator product expansion (OPE) of chiral fields of a conformal field theory. The
singular part of the OPE encodes the commutation relations of fields, which leads
to the notion of a Lie conformal algebra [K1-2].

In the past few years a structure theory [DK], representation theory [CK, CKW]
and cohomology theory [BKV] of finite Lie conformal algebras has been developed.
The associative conformal algebra Cend\(_N\) and the corresponding general Lie con-
formal algebra gc\(_N\) are the most important examples of simple conformal algebras
which are not finite (see Sect. 2.10 in [K1]). One of the most urgent open problems
of the theory of conformal algebras is the classification of infinite subalgebras of
Cend\(_N\) and of gc\(_N\) which act irreducibly on \(\mathbb{C}[\partial]^N\). (For a classification of such
finite algebras, in the associative case see Theorem 5.2 of the present paper, and in
the (more difficult) Lie case see [CK] and [DK].) The classical Burnside theorem states that any subalgebra of the matrix algebra Mat\(_N\mathbb{C}\) that acts irreducibly on \(\mathbb{C}^N\) is the whole algebra Mat\(_N\mathbb{C}\). This is
certainly not true for subalgebras of Cend\(_N\) (which is the “conformal” analogue
of Mat\(_N\mathbb{C}\)). There is a family of infinite subalgebras Cend\(_N, P\) of Cend\(_N\), where
\(P(x) \in \text{Mat}_N \mathbb{C}[x]\), \(\det P(x) \neq 0\), that still act irreducibly on \(\mathbb{C}[\partial]^N\). One of the
conjectures of [K2] states that there are no other infinite irreducible subalgebras of
Cend\(_N\).

One of the results of the present paper is the classification of all subalgebras of
Cend\(_1\) and determination of the ones that act irreducibly on \(\mathbb{C}[\partial]\) (Theorem 2.2).
This result proves the above-mentioned conjecture in the case \(N = 1\). For general
we can prove this conjecture only under the assumption that the subalgebra in question is unital (see Theorem 5.3). This result is closely related to a difficult theorem of A. Retakh [R] (but we avoid using it).

Next, we describe all finite irreducible modules over Cend_{N,P} (see Corollary 3.7). This is done by using the description of left ideals of the algebras Cend_{N,P} (see Proposition 1.6a). Further, we describe all extensions between non-trivial finite irreducible Cend_{N,P}-modules and between non-trivial finite irreducible and trivial finite dimensional modules (Theorem 3.10). This leads us to a complete description of finite Cend_{N}-modules (Theorem 3.28).

Next we describe all automorphisms of Cend_{N,P} (Theorems 4.2 and 4.3). We also classify all homomorphisms and anti-homomorphisms of Cend_{N,P} to Cend_{N} (Theorem 4.6). This gives, in particular, a classification of anti-involutions of Cend_{N,P}. One case of such an anti-involution \((N = 1, P = x)\) was studied by S. Bloch [B] on the level of the Lie algebra of differential operators on the circle to link representations of the corresponding subalgebra to the values of \(\zeta\)-function. Representation theory of the subalgebra corresponding to the anti-involution of Cend_{1} was developed in [KWY].

The subspace of anti-fixed points of an anti-involution of Cend_{N,P} is a Lie conformal subalgebra that still acts irreducibly on \(\mathbb{C}[\partial]^{N}\). This leads us to Conjecture 6.20 on classification of infinite Lie conformal subalgebras of gc_{N} acting irreducibly on \(\mathbb{C}[\partial]^{N}\). This conjecture agrees with the results of the papers [Z] and [DeK].

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1. LEFT AND RIGHT IDEALS OF CEND_{N,P}

First we introduce the basic definitions and notations, see [K1]. An associative conformal algebra \(R\) is defined as a \(\mathbb{C}[\partial]\)-module endowed with a \(\mathbb{C}\)-linear map,

\[ R \otimes R \longrightarrow \mathbb{C}[\lambda] \otimes R, \quad a \otimes b \mapsto a_{\lambda}b \]

called the \(\lambda\)-product, and satisfying the following axioms \((a, b, c \in R)\),

\[(A1)_{\lambda} \quad (\partial a)_{\lambda}b = -\lambda(a_{\lambda}b), \quad a_{\lambda}(\partial b) = (\lambda + \partial)(a_{\lambda}b)\]

\[(A2)_{\lambda} \quad a_{\lambda}(b_{\mu}c) = (a_{\lambda}b)_{\lambda+\mu}c\]

An associative conformal algebra is called finite if it has finite rank as \(\mathbb{C}[\partial]\)-module. The notions of homomorphism, ideal and subalgebras of an associative conformal algebra are defined in the usual way.

A module over an associative conformal algebra \(R\) is a \(\mathbb{C}[\partial]\)-module \(M\) endowed with a \(\mathbb{C}\)-linear map \(R \otimes M \longrightarrow \mathbb{C}[\lambda] \otimes M\), denoted by \(a \otimes v \mapsto a_{\lambda}^{M}v\), satisfying the
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properties:

$$(\partial a)^M\lambda v = [\partial^M, a^M\lambda]v = -\lambda (a^M\lambda v), \quad a \in R, v \in M,$$

$$a^M\lambda (b^M\mu v) = (a\lambda b)^M\lambda+\mu v, \quad a, b \in R.$$ 

An $R$-module $M$ is called trivial if $a^\lambda v = 0$ for all $a \in R, v \in M$ (but it may be non-trivial as a $\mathbb{C}[\partial]$-module).

Given two $\mathbb{C}[\partial]$-modules $U$ and $V$, a conformal linear map from $U$ to $V$ is a $\mathbb{C}$-linear map $a : U \to \mathbb{C}[\lambda] \otimes \mathbb{C} V$, denoted by $a : U \to V$, such that $[\partial, a] = -\lambda a$, that is $\partial^V a^\lambda - a^\lambda \partial^U = -\lambda a$. The vector space of all such maps, denoted by $\text{Chom}(U, V)$, is a $\mathbb{C}[\partial]$-module with

$$(\partial a)^\lambda := -\lambda a.$$ 

Now, we define $\text{Cend}V := \text{Chom}(V, V)$ and, provided that $V$ is a finite $\mathbb{C}[\partial]$-module, $\text{Cend}V$ has a canonical structure of an associative conformal algebra defined by

$$(a\lambda b)_\mu v = a\lambda (b_{\mu - \lambda} v), \quad a, b \in \text{Cend} V, \quad v \in V.$$ 

Remark 1.1. Observe that, by definition, a structure of a conformal module over an associative conformal algebra $R$ in a finite $\mathbb{C}[\partial]$-module $V$ is the same as a homomorphism of $R$ to the associative conformal algebra $\text{Cend} V$.

For a positive integer $N$, let $\text{Cend}_N = \text{Cend} \mathbb{C}[\partial]^N$. It can also be viewed as the associative conformal algebra associated to the associative algebra $\text{Diff}^N\mathbb{C}^\times$ of all $N \times N$ matrix valued regular differential operators on $\mathbb{C}^\times$, that is (see Sect. 2.10 in [K1] for more details)

$$\text{Conf}(\text{Diff}^N\mathbb{C}^\times) = \oplus_{n \in \mathbb{Z}_+} \mathbb{C}[\partial]J^n \otimes \text{Mat}_N\mathbb{C}$$

with $\lambda$-product given by $(J^k_A = J^k \otimes A)$

$$J^k_A \lambda J^l_B = \sum_{j=0}^k \binom{k}{j} (\lambda + \partial)^j J^{k+l-j}_{AB}.$$ 

Given $\alpha \in \mathbb{C}$, the natural representation of $\text{Diff}^N\mathbb{C}^\times$ on $e^{-\alpha t}\mathbb{C}^N[t, t^{-1}]$ gives rise a conformal module structure on $\mathbb{C}[\partial]^N$ over $\text{Conf}(\text{Diff}^N\mathbb{C}^\times)$, with $\lambda$-action

$$J^m_A \lambda v = (\lambda + \partial + \alpha)^m Av, \quad m \in \mathbb{Z}_+, v \in \mathbb{C}^N.$$ 

Now, using Remark 1.1, we obtain a natural homomorphism of conformal associative algebras from $\text{Conf}(\text{Diff}^N\mathbb{C}^\times)$ to $\text{Cend}_N$, which turns out to be an isomorphism (see [DK] and Proposition 2.10 in [K1]).
In order to simplify the notation, we will introduce the following bijective map, called the symbol,

\[
\text{Symb} : \text{Cend}_N \rightarrow \text{Mat}_N \mathbb{C}[\partial, x] \\
\sum_k A_k(\partial) J^k \mapsto \sum_k A_k(\partial) x^k
\]

where \( A_k(\partial) \in \text{Mat}_N(\mathbb{C}[\partial]) \). The transferred \( \lambda \)-product is

\[
A(\partial, x) \lambda B(\partial, x) = A(-\lambda, x + \lambda + \partial) B(\lambda + \partial, x).
\] (1.1)

The above \( \lambda \)-action of \( \text{Cend}_N \) on \( \mathbb{C}[\partial]^N \) is given by the following formula:

\[
A(\partial, x) \lambda v(\partial) = A(-\lambda, \lambda + \partial + \alpha) v(\lambda + \partial), \quad v(\partial) \in \mathbb{C}[\partial]^N.
\] (1.2)

Note also that under the change of basis of \( \mathbb{C}[\partial]^N \) by the invertible matrix \( C(\partial) \), the symbol \( A(\partial, x) \) changes by the formula:

\[
A(\partial, x) \mapsto C(\partial + x) A(\partial, x) C(x)^{-1}.
\] (1.3)

Observe that for any \( C(x) \in \text{Mat}_N(\mathbb{C}[x]) \), with non-zero constant determinant, the map (1.3) gives us an automorphism of \( \text{Cend}_N \).

It follows immediately from the formula for \( \lambda \)-product that

\[
\text{Cend}_{P,N} := P(x + \partial)(\text{Cend}_N) \quad \text{and} \quad \text{Cend}_{N,P} := (\text{Cend}_N)P(x),
\]

with \( P(x) \in \text{Mat}_N(\mathbb{C}[x]) \), are right and left ideals, respectively, of \( \text{Cend}_N \). Another important subalgebra is

\[
\text{Cur}_N := \text{Cur} \left( \text{Mat}_N \right) = \mathbb{C}[\partial] \left( \text{Mat}_N \mathbb{C} \right).
\] (1.4)

Remark 1.5. If \( P(x) \) is nondegenerate, i.e., \( \det P(x) \neq 0 \), then by elementary transformations over the rows (left multiplications) we can make \( P(x) \) upper triangular without changing \( \text{Cend}_{N,P} \). After that, applying to \( \text{Cend}_{N,P} \) an automorphism of \( \text{Cend}_N \) of the form (1.3), with \( \det C(x) = 1 \) (in order to multiply \( P \) on the right, which are elementary transformations over the columns), we get \( \text{Cend}_{N,P} \cong \text{Cend}_{N,D} \), with \( D = \text{diag}(p_1(x), \ldots, p_N(x)) \), where \( p_i(x) \) are monic polynomials such that \( p_i(x) \) divides \( p_{i+1}(x) \). The \( p_i(x) \) are called the elementary divisors of \( P \). So, up to conjugation, all \( \text{Cend}_{N,P} \) are parameterized by the sequence of elementary divisors of \( P \).

All left and right ideals of \( \text{Cend}_N \) were obtained by B. Bakalov. Now, we extend the classification to \( \text{Cend}_{N,P} \).
Proof. (a) By Remark 1.5, we may suppose that $a$ is isomorphic to $P$, as desired. This completes the proof of (a).

(b) All right ideals in $\text{Cend}_{N,P}$, with $\det P(x) \neq 0$, are of the form $Q(\partial + x) \text{Cend}_{N,P}$, where $Q(x) \in \text{Mat}_N(\mathbb{C}[x])$.

Proof. (a) By Remark 1.5, we may suppose that $P$ is diagonal with $\det P(x) \neq 0$. Denote by $p_1(x), \ldots, p_N(x)$ the diagonal coefficients.

Let $J \subseteq \text{Cend}_N$ be a left ideal. First, let us see that $J$ is generated over $\mathbb{C}[\partial]$ by $I := J \cap \text{Mat}_N(\mathbb{C}[x])$. If $a(\partial, x) = \sum_{i=0}^m \partial^i a_i(x) \in J$, then

$$E_{k,k}P(x)\lambda a(\partial, x) = p_k(\lambda + \partial + x)E_{k,k}a(\lambda + \partial, x)$$

$$= p_k(\lambda + \partial + x)E_{k,k}(\sum_i (\lambda + \partial)^i a_i(x)) \in \mathbb{C}[\lambda] \otimes J,$$

using that $\det P(x) \neq 0$ and considering the maximal coefficient in $\lambda$ of (1.7), we get $E_{k,k}a_m(x) \in J$ for all $k$. Hence $a_m(x) \in J$. Applying the same argument to $a(\partial, x) - \partial^m a_m(x) \in J$, and so on, we get $a_i(x) \in J$ for all $i$. Therefore, $J$ is generated over $\mathbb{C}[\partial]$ by $I := J \cap \text{Mat}_N(\mathbb{C}[x])$.

If $a(x) \in I$, then

$$E_{i,j}P(x)\lambda a(x) = p_j(\lambda + \partial + x)E_{i,j}a(x) = \lambda^{\max} E_{i,j}a(x) + \text{lower terms} \in \mathbb{C}[\lambda] \otimes J.$$ (1.8)

Therefore, $\text{Mat}_N(\mathbb{C}) \cdot I \subseteq I$.

Now, considering the next coefficient in $\lambda$ in (1.8) if $p_j$ is non-constant, or the constant term in $\lambda$ of $x E_{i,j}P(x)\lambda a(x)$ if $p_j$ is constant, we get that $xa(x) \in I$. It follows that $I$ is a left ideal of $\text{Mat}_N(\mathbb{C}[x])$. But all left ideals of $\text{Mat}_N(\mathbb{C}[x])$ are principal, i.e. of the form $\text{Mat}_N(\mathbb{C}[x])R(x)$, since $\text{Mat}_N(\mathbb{C}[x])$ and $\mathbb{C}[x]$ are Morita equivalent. This completes the proof of (a).

In a similar way, but using the expression $a(\partial, x) = \sum_i \partial^i a_i(\partial + x)$, we get (b). \qed

Proposition 1.9. $\text{Cend}_{N,P} \simeq B(\partial + x)(\text{Cend}_N)A(x)$ if $P(x) = A(x)B(x)$. In particular, $\text{Cend}_{N,P} \simeq \text{Cend}_{P,N}$.

Proof. It is easy to see that the map $a(\partial, x)P(x) \to B(\partial + x)a(\partial, x)A(x)$ is an isomorphism provided that $P(x) = A(x)B(x)$. \qed

2. Classification of subalgebras of $\text{Cend}_1$

We can identify $\text{Cend}_1$ with $\mathbb{C}[\partial, x]$, then the $\lambda$–product is

$$r(\partial, x) \lambda s(\partial, x) = r(-\lambda, \lambda + \partial + x) s(\lambda + \partial, x),$$

(2.1)

where $r(\partial, x), s(\partial, x) \in \mathbb{C}[\partial, x]$.

The main result of this section is
Theorem 2.2.  
a) Any subalgebra of \( C_{end} \) is one of the following:

1. \( \mathbb{C}[^{\partial}] \);
2. \( \mathbb{C}[^{\partial}, x] \ p(x) \), with \( p(x) \in \mathbb{C}[x] \);
3. \( \mathbb{C}[^{\partial}, x] \ q(^{\partial} + x) \), with \( q(x) \in \mathbb{C}[x] \);
4. \( \mathbb{C}[^{\partial}, x] \ p(x) q(^{\partial} + x) = \mathbb{C}[^{\partial}, x] \ p(x) \cap \mathbb{C}[^{\partial}, x] \ q(^{\partial} + x) \), with \( p(x), q(x) \in \mathbb{C}[x] \).

b) The subalgebras \( \mathbb{C}[^{\partial}, x] \ p(x) \) with \( p(x) \neq 0 \), and \( \mathbb{C}[^{\partial}] \) are all the subalgebras of \( C_{end} \) that act irreducibly on \( \mathbb{C}[^{\partial}] \).

In order to prove Theorem 2.2, we first need some lemmas and the following important notation. Given \( r(^{\partial}, x) \in \mathbb{C}[^{\partial}, x] \), we denote by \( r_{i} \) and \( \tilde{r}_{j} \) the coefficients uniquely determined by

\[
 r(^{\partial}, x) = \sum_{i=0}^{n} r_{i}(x)^{\partial^{i}} = \sum_{j=0}^{m} \tilde{r}_{j}(^{\partial} + x)^{\partial^{j}} \tag{2.3}
\]

with \( r_{n}(x) \neq 0 \) and \( \tilde{r}_{m}(^{\partial} + x) \neq 0 \).

Lemma 2.4.  
Let \( S \) be a subalgebra of \( C_{end} \), let \( p(x) \) and \( q(x) \) be two non-constant polynomials, and let \( t(^{\partial}) \in \mathbb{C}[^{\partial}] \) be a non-zero polynomial.

(a) If \( t(^{\partial}) \in S \), then \( \mathbb{C}[^{\partial}] \subseteq S \).

(b) If \( t(^{\partial}), r(^{\partial}, x) \in S \) and \( r(^{\partial}, x) \) depends non-trivially on \( x \), then \( S = C_{end} \).

In particular, if \( 1 \in S \), then either \( S = \mathbb{C}[^{\partial}] \) or \( S = C_{end} \).

(c) If \( p(x) \in S \), then \( \mathbb{C}[^{\partial}, x] \ p(x) \subseteq S \).

(d) If \( q(^{\partial} + x) \in S \), then \( \mathbb{C}[^{\partial}, x] \ q(^{\partial} + x) \subseteq S \).

(e) If \( p(x) q(^{\partial} + x) \in S \), then \( \mathbb{C}[^{\partial}, x] \ p(x) q(^{\partial} + x) \subseteq S \).

Proof.  
(a) If \( t(^{\partial}) \in S \), we deduce from the maximal coefficient in \( \lambda \) of \( t(^{\partial}) \lambda \ t(^{\partial}) = t(-\lambda) \ t(\lambda + ^{\partial}) \) that \( 1 \in S \), proving (a).

(b) From (a), we have that \( 1 \in S \). Then the coefficients of \( \lambda \) in \( r(^{\partial}, x) \lambda \ 1 = r(-\lambda, \lambda + ^{\partial} + x) \) are in \( S \). Therefore, using notation (2.3), we obtain that \( \tilde{r}_{j}(^{\partial} + x) \in S \) for all \( j \). Since \( r(^{\partial}, x) \) depends non-trivially on \( x \), there exist \( j_{0} \) such that \( \tilde{r}_{j_{0}} \) is non-constant, that is \( \tilde{r}_{j_{0}}(z) = \sum_{i=0}^{l} a_{i} z^{i} \) with \( a_{i} \neq 0 \) and \( l > 0 \). Now, using that \( \mathbb{C}[^{\partial}] \subseteq S \) and

\[
 1 \lambda \tilde{r}_{j_{0}}(^{\partial} + x) = \tilde{r}_{j_{0}}(\lambda + ^{\partial} + x) = \lambda^{l} + (l a_{l}(^{\partial} + x) + a_{l-1}) \lambda^{l-1} + \text{lower powers in } \lambda
\]

we obtain that \( x \in S \). Then by induction and taking \( \lambda \)-products of type \( x_{\lambda} x^{k} \) we see that \( x^{k+1} \in S \) for all \( k \geq 1 \), proving (b).

(c) Let \( p(x) = \sum_{i=0}^{n} a_{i} x^{i} \), with \( a_{n} \neq 0 \) and \( n > 0 \). Then, considering the coefficient of \( \lambda^{n-1} \) in \( p(x) \lambda \ p(x) = p(\lambda + ^{\partial} + x)p(x) \), we get that
\( (n a_n (\partial + x) + a_{n-1}) p(x) \in S. \) Since \( S \) is a \( \mathbb{C}[\partial] \)-module, we have \( \partial p(x) \in S \), obtaining that \( x p(x) \in S. \) Applying this argument to \( x p(x) \), we get that \( x^2 p(x) \in S \), etc, and \( x^k p(x) \in S \) for all \( k > 0 \), proving (c).

(d) The proof is identical to that of (c).

(e) Assume that \( q(x + \partial)p(x) \in S. \) Then, we compute \( q(x + \partial)p(x)\lambda q(x + \partial)p(x) = q(x + \partial)p(\lambda + \partial + x)q(\lambda + x + \partial)p(x) \), and looking at the monomial of highest degree minus one, we get that \( (x + \partial)q(x + \partial)p(x) \in S \), and since by definition \( S \) is a \( \mathbb{C}[\partial] \)-module, we deduce that \( q(x + \partial)p(x) := xq(x + \partial)p(x) \in S. \) Applying this argument to \( q(x + \partial)p(x) \) we deduce that \( x^k q(x + \partial)p(x) \in S \) for any \( k \in \mathbb{Z}_+ \), and therefore \( q(x + \partial)p(x)\mathbb{C}[\partial, x] \subseteq S. \)

Lemma 2.5. Let \( S \) be a subalgebra of \( \text{Cend}_1 \) which does not contain 1.

(a) Let \( p(x) \) be of minimal degree such that \( p(x) \in S \). Then \( \mathbb{C}[\partial, x]p(x) = S. \)

(b) Let \( q(\partial + x) \) be of minimal degree such that \( q(\partial + x) \in S. \) Then \( S = \mathbb{C}[\partial, x]q(\partial + x). \)

(c) Let \( q(\partial + x)p(x) \) be of minimal degree (in \( x \)) such that \( q(\partial + x)p(x) \in S. \) Then \( S = p(x)q(\partial + x)\mathbb{C}[\partial, x]. \)

Proof. (a) From Lemma 2.4.(c), we have that \( p(x)\mathbb{C}[\partial, x] \subseteq S \) (by our assumption, \( p(x) \) is non-constant). Now, suppose that there exist \( q(\partial, x) \in S \) with \( q(\partial, x) \notin p(x)\mathbb{C}[\partial, x] \) and \( p \) as above. Then, by applying the division algorithm to each coefficient of \( q(\partial, x) = \sum_{k=0}^n q_k(x)\partial^k \), we may write \( q(\partial, x) = t(\partial, x)p(x) + r(\partial, x) \) with \( r(\partial, x) = \sum_{k=0}^n r_k(x)\partial^k = \sum_{j=0}^m \tilde{r}_j(\partial + x)\partial^k \) and \( \deg r_k < \deg p \) (cf. notation (2.3)). Using that \( p(x)\mathbb{C}[\partial, x] \subseteq S \), we obtain that \( r(\partial, x) \in S. \) Now, since

\[
\begin{align*}
\frac{r(\partial, x)}{r(\partial, x)} & = r(-\lambda, \lambda + \partial + x)r(\lambda + \partial, x), \tag{2.6}
\end{align*}
\]

looking at the coefficient of maximum degree in \( \lambda \) in (2.6), we get: \( r_n(x)\tilde{r}_m(x + \partial) \in S. \) By our assumption, one of the polynomials in this product is non-constant. If \( \tilde{r}_m(x + \partial) \) is constant, then \( r_n(x) \in S, \) but \( \deg r_n < \deg p \) which is a contradiction. If \( r_n(x) \) is constant, then \( \tilde{r}_m(x + \partial) \in S. \) Then, looking at the leading coefficient of the following polynomial in \( \lambda: p(x)\lambda \tilde{r}_m(x + \partial) = p(\lambda + \partial + x)\tilde{r}_m(x + \lambda + \partial) \) we have that \( 1 \in S, \) which contradicts our assumption.

If neither \( \tilde{r}_m(x + \partial) \) nor \( r_n(x) \) are constants, we look at \( p(x)\lambda \tilde{r}_m(x + \partial)r_n(x) = p(\lambda + \partial + x)\tilde{r}_m(\lambda + x + \partial)\tilde{r}_n(x) \in S \) and looking at the coefficient of maximum degree in \( \lambda \) we get that \( r_n(x) \in S, \) which contradicts the minimality of \( p(x). \)

(b) The proof is the same as that of (a).
(c) We may assume that \( p \) and \( q \) are non-constant polynomials, otherwise we are in the cases (a) or (b). By Lemma 2.4(e), we have \( p(x)q(x + \partial)\mathbb{C}[\partial, x] \subseteq S \). Let \( t(\partial, x) \in S \), but \( t(\partial, x) \notin \mathbb{C}[\partial, x]p(x)q(x + \partial) \). Then we may have three cases:

1. \( t(\partial, x) \in p(x)\mathbb{C}[\partial, x] \) or
2. \( t(\partial, x) \in q(\partial + x)\mathbb{C}[\partial, x] \) or
3. \( t(\partial, x) \notin p(x)\mathbb{C}[\partial, x] \) or \( t(\partial, x) \notin q(\partial + x)\mathbb{C}[\partial, x] \).

Note that these cases are mutually exclusive. Suppose we are in Case (1), so that \( t(\partial, x) = p(x)r(\partial, x) \) with \( r(\partial, x) \notin q(\partial + x)\mathbb{C}[\partial, x] \). Then we get \( r(\partial, x) = q(\partial + x)\tilde{r}(\partial, x) + s(\partial, x) \), with \( s(\partial, x) \neq 0 \), and (using notation (2.3)) \( \deg \tilde{s}_k < \deg q \) for all \( k = 0, \ldots, m \). Therefore, we have that \( t(\partial, x) = p(x)r(\partial, x) = p(x)q(\partial + x)\tilde{r}(\partial, x) + p(x)s(\partial, x) \) and then \( p(x)s(\partial, x) \in S \). Now, we can compute:

\[
p(x)s(\partial, x) = p(\lambda + \partial + x)s(-\lambda, \lambda + \partial + x)p(x)q(\lambda + \partial + x)
\]

and looking at the coefficient of maximum degree in \( \lambda \), we have (using notation (2.3)) that \( p(x)\tilde{s}_m(\partial + x) \in S \) which is a contradiction.

Similarly, Case (2) also leads to a contradiction.

In the remaining Case (3) we may assume that \( \deg p \leq \deg q \) since the case of the opposite inequality is completely analogous. We have \( t(\partial, x) \in S \), but \( \notin \mathbb{C}[\partial, x]p(x) \). Then

\[
t(\partial, x) = p(x)h(\partial, x) + r(\partial, x)
\]

with \( 0 \neq r(\partial, x) = \sum_{k=0}^{n} r_k(x)\partial^k = \sum_{j=0}^{m} \tilde{r}_j(\partial + x)\partial^k \) where \( \deg r_k < \deg p \) and \( \deg \tilde{r}_j < \deg p \).

If \( h(\partial, x) \in \mathbb{C}[\partial, x]q(\partial + x) \), then \( r(\partial, x) \in S \), but the leading coefficient of

\[
p(x)q(\partial + x) \lambda r(\partial, x) = p(\lambda + \partial + x)q(\partial + x)r(\lambda + \partial, x)
\]

is in \( S \) which is \( q(\partial + x)r_n(x) \), and this contradicts the assumption of minimality of \( p(x)q(\partial + x) \).

So, suppose that \( h(\partial, x) \notin \mathbb{C}[\partial, x]q(\partial + x) \). Then \( h(\partial, x) = \tilde{h}(\partial, x)q(\partial + x) + s(\partial, x) \) with \( 0 \neq s(\partial, x) = \sum_{k=0}^{n} s_k(x)\partial^k = \sum_{j=0}^{m} \tilde{s}_j(\partial + x)\partial^k \) and \( \deg \tilde{s}_j < \deg q \). By (2.7) we have \( p(x)s(\partial, x) + r(\partial, x) \in S \). Now, we compute:

\[
(p(x)s(\partial, x) + r(\partial, x)) \lambda p(x)q(\partial + x) \\
= (p(\lambda + \partial + x)s(-\lambda, \lambda + \partial + x) + r(-\lambda, \lambda + \partial + x))p(x)q(\lambda + \partial + x)
\]

Then the leading coefficient in \( \lambda \) is either \( p(x)\tilde{s}_m(\partial + x) \in S \), which is impossible since \( \deg \tilde{s}_m < \deg q \), or \( p(x)\tilde{r}_m(\partial + x) \in S \). But in the latter case, \( \deg \tilde{r}_m \geq \deg q \), but by construction \( \deg \tilde{r}_m < \deg p \), and this contradicts the assumption \( \deg p \leq \deg q \). \( \square \)
Proof of Theorem 2.2. (a) Let $S$ be a non-zero subalgebra of $\text{Cend}_1$. If $S \subseteq \mathbb{C}[\partial]$ then by Lemma 2.4.(a) we have that $S = \mathbb{C}[\partial]$. Therefore we may assume that there is $r(\partial, x) \in S$ which depends nontrivially on $x$. Recall that we can write

$$r(\partial, x) = \sum_{i=0}^{m} p_i(x) \partial^i = \sum_{j=0}^{n} q_j(\partial + x) \partial^j.$$ 

We have

$$r(\partial, x) \lambda r(\partial, x) = r(-\lambda, \lambda + \partial + x)r(\lambda + \partial, x) = \sum_{i=0}^{m} \sum_{j=0}^{n} q_j(\partial + x)p_i(x)(-\lambda)^j(\lambda + \partial)^i$$

Then, considering the leading coefficient of this $\lambda$-polynomial, we have $p_m(x)q_n(\partial + x) \in S$. Therefore, we may have one of the following situations:

1. $p_m(x)$ and $q_n(\partial + x)$ are constant,
2. $q_n(\partial + x)$ is constant and $p_m(x)$ is non-constant,
3. $p_m(x)$ is constant and $q_n(\partial + x)$ is non-constant, or
4. both polynomials non-constant.

Let us see what happens in each case:

1. By Lemma 2.4.(b), we have that $S = \text{Cend}_1$.
2. In this case, we may take $p(x) \in S$ of minimal degree, then using Lemma 2.5.(a) we have $S = \mathbb{C}[\partial, x]p(x)$.
3. It is completely analogous to (2).
4. Here, we have that $p(x)q(x + \partial) \in S$ and, again we may assume that it has minimal degree. Now, by Lemma 2.5.(c), we finish the proof of (a).

The proof of (b) is straightforward. \(\square\)

3. Finite modules over $\text{Cend}_{N,P}$

Given $R$ an associative conformal algebra (not necessarily finite), we will establish a correspondence between the set of maximal left ideals of $R$ and the set of irreducible $R$-modules. Then we will apply it to the subalgebras $\text{Cend}_{N,P}$.

First recall that the following property holds in an $R$-module $M$ (cf. Remark 3.3 [DK]):

$$a_\lambda(b_\partial-\mu v) = (a_\lambda b)_\partial-\mu v \quad a, b \in R, \ v \in M. \quad (3.1)$$

Remark 3.2. (a) Let $v \in M$ and fix $\mu \in \mathbb{C}$, then due to (3.1) we have that $R_\partial-\mu v$ is an $R$-submodule of $M$.

(b) $\text{Tor} M$ is a trivial $R$-submodule of $M$ (Lemma 8.2, [DK]).

(c) If $M$ is irreducible and $M = \text{Tor} M$, then $M \simeq \mathbb{C}$.

(d) If $M$ is a non-trivial finite irreducible $R$-module, then $M$ is free as a $\mathbb{C}[\partial]$-module.
Lemma 3.3. Let $M$ be a non-trivial irreducible $R$-module. Then there exists $v \in M$ and $\mu \in \mathbb{C}$ such that $R_{-\partial - \mu}v \neq 0$. In particular, $R_{-\partial - \mu}v = M$ if $M$ is irreducible.

Proof. Suppose that $R_{-\partial - \mu}v = 0$ for all $v \in M$ and $\mu \in \mathbb{C}$, then we have that $r_{-\partial - \mu}v = 0$ in $\mathbb{C}[\mu] \otimes M$ for all $r \in R$ and $v \in M$. Thus writing down $r_{-\partial - \mu}v$ as a polynomial in $\mu$ and looking at the $n$-products that are going to appear in this expansion, we conclude that $r_{\lambda}v = 0$ for all $v \in M$ and $r \in R$. Hence $M$ is a trivial $R$-module, a contradiction. \hfill \Box

By Lemma 3.3, given a non-trivial irreducible $R$-module $M$ we can fix $v \in M$ and $\mu \in \mathbb{C}$ such that $R_{-\partial - \mu}v = M$ and consider the following map

$$\phi : R \rightarrow M, \quad r \mapsto r_{-\partial - \mu}v.$$ 

Observe that $\phi(\partial r) = (\partial + \mu) \phi(r)$ and using (3.1) we also have $\phi(r_{\lambda}s) = r_{\lambda}\phi(s)$. Therefore, the map $\phi$ is a homomorphism of $R$-modules into $M_{-\mu}$, where $M_{-\mu}$ is the $\mu$-twisted module of $M$ obtained by replacing $\partial$ by $\partial + \mu$ in the formulas for the action of $R$ on $M$, and $\text{Ker}(\phi)$ is a maximal left ideal of $R$. Clearly this map is onto $M_{-\mu}$.

Therefore we have that $M_{-\mu} \simeq (R/\text{Ker} \phi)$ as $R$-modules, or equivalently,

$$M \simeq (R/\text{Ker} \phi)_{\mu}. \quad (3.4)$$

On the other hand, it is immediate that given any maximal left ideal $I$ of $R$, we have that $(R/I)_{\mu}$ is an irreducible $R$-module. Therefore we have proved the following

Theorem 3.5. Formula (3.4) defines a surjective map from the set of maximal left ideals of $R$ to the set of equivalence classes of non-trivial irreducible $R$-modules.

Remark 3.6. (a) Observe that given an $R$-module $M$ and $v \in M$, the set $I = \{a \in R \mid a_{\lambda}v = 0\}$ is a left ideal, but not necessarily $M \simeq R/I$. For example, consider $\mathbb{C}[\partial]$ as a $\text{Cend}_1$-module, then the kernel of $a \mapsto a_{\lambda}v$ is $\{0\}$.

(b) If we fix $\mu \in \mathbb{C}$, there are examples of irreducible modules where $R_{-\partial - \mu}v = 0$ for all $v \in M$ (cf. Lemma 3.3). Indeed, consider $\mathbb{C}[\partial]$ as a $\text{Cend}_{1,(x+\mu)}$-module.

Using Remark 3.2, Proposition 1.6 and Theorem 3.5, we have

Corollary 3.7. The $\text{Cend}_{N,P}$-module $\mathbb{C}[\partial]^N$ defined by (1.2) is irreducible if and only if $\text{det} P(x) \neq 0$. These are all non-trivial irreducible $\text{Cend}_{N,P}$-modules up to equivalence, provided that $\text{det} P(x) \neq 0$.

Note that Corollary 3.7 in the case $P(x) = I$, have been established earlier in [K2], by a completely different method (developed in [KR]).

A subalgebra $S$ of $\text{Cend}_N$ is called irreducible if $S$ acts irreducibly in $\mathbb{C}[\partial]^N$. 

Corollary 3.8. The following subalgebras of $\text{Cend}_N$ are irreducible: $\text{Cend}_{N,P}$ with $\det P(x) \neq 0$, and $\text{Cur}_N$ or conjugates of it by automorphisms (1.3).

Remark 3.9. It is easy to show that every non-trivial irreducible representation of $\text{Cur}_N$ is equivalent to the standard module $\mathbb{C}[\partial]^N$, and that every finite module over $\text{Cur}_N$ is completely reducible.

We will finish this section with the classification of all extensions of $\text{Cend}_{N,P}$-modules involving the standard module $\mathbb{C}[\partial]^N$ and finite dimensional trivial modules, and the classification of all finite modules over $\text{Cend}_N$.

We shall work with the standard irreducible $\text{Cend}_{N,P}$-module $\mathbb{C}[\partial]^N$ with $\lambda$-action (see (1.2))

\[ a(\partial, x)P(x)\lambda v(\partial) = a(-\lambda, \lambda + \partial + \alpha)P(\lambda + \partial)v(\lambda + \partial). \]

Consider the trivial $\text{Cend}_{N,P}$-module over the finite dimensional vector space $V_T$, whose $\mathbb{C}[\partial]$-module structure is given by the linear operator $T$, that is: $\partial \cdot v = T(v)$, $v \in V_T$. As usual, we may assume that $P(x) = \text{diag}\{p_1(x), \cdots, p_N(x)\}$. We shall assume that $\det P \neq 0$.

Theorem 3.10. a) There are no non-trivial extensions of $\text{Cend}_{N,P}$-modules of the form:

\[ 0 \to V_T \to E \to \mathbb{C}[\partial]^N \to 0. \]

b) If there exists a non-trivial extension of $\text{Cend}_{N,P}$-modules of the form

\[ 0 \to \mathbb{C}[\partial]^N \to E \to V_T \to 0, \tag{3.11} \]

then $\det P(\alpha + c) = 0$ for some eigenvalue $c$ of $T$. In this case, all torsionless extensions of $\mathbb{C}[\partial]^N$ by finite dimensional vector spaces, are parameterized by decompositions $P(x + \alpha) = R(x)S(x)$ and can be realized as follows. Consider the following isomorphism of conformal algebras:

\[ \text{Cend}_{N,P} \to S(\partial + x)\text{Cend}_N R(x), \quad a(\partial, x)P(x) \mapsto S(\partial + x)a(\partial, x)R(x), \]

where $P(x+\alpha) = R(x)S(x)$, (this is the isomorphism between $\text{Cend}_{N,S}$ and $\text{Cend}_{S,N}$ (Proposition 1.9), restricted to $\text{Cend}_{N,R}S(x)$). Using this isomorphism, we get an action of $\text{Cend}_{N,P}$ on $\mathbb{C}[\partial]^N$:

\[ a(\partial, x)P(x)\lambda v(\partial) = S(\partial)a(-\lambda, \lambda + \partial + \alpha)R(\lambda + \partial)v(\lambda + \partial). \]

Then $S(\partial)\mathbb{C}[\partial]^N$ is a submodule isomorphic to the standard module, of finite codimension in $\mathbb{C}[\partial]^N$. 

c) If $E$ is a non-trivial extension of $\text{Cend}_{N,P}$-modules of the form:

$$0 \to \mathbb{C}[\partial]^N \to E \to \mathbb{C}[\partial]^N \to 0,$$

then $E = \mathbb{C}[\partial]^N \otimes \mathbb{C}^2$ as a $\mathbb{C}[\partial]$-module (with trivial action of $\partial$ on $\mathbb{C}^2$) and $\text{Cend}_{N,P}$ acts by

$$a(\partial, x) = a(-\lambda, \lambda + \partial \otimes 1 + 1 \otimes J)c(\lambda + \partial)(1 \otimes u),$$

(3.12)

where $J$ is a $2 \times 2$ Jordan block matrix.

Proof. a) Consider a short exact sequence of $R = \text{Cend}_{N,P}$-modules

$$0 \to T \to E \to V \to 0,$$

(3.13)

where $V$ is irreducible finite, and $T$ is trivial (finite dimensional vector space). Take $v \in E$ with $v \notin T$, and let $\mu \in \mathbb{C}$ be such that $A := R_{-\partial-\mu}v \neq 0$. Then we have three possibilities:

1) The image of $A$ in $V$ is 0, then $A = T$, which is impossible since $A$ corresponds to a left ideal of $\text{Cend}_{N,P}$.

2) The image of $A$ in $V$ is $V$ and $A \cap T = 0$, then $A$ is isomorphic to $V$, hence the exact sequence splits.

3) The image of $A$ in $V$ is $V$ and $T = A \cap T \neq 0$. Now, if $T' = T$ then $A = E$ and $E$ is a cyclic module, which is impossible since it has torsion. If $T' \neq T$, we consider the exact sequence $0 \to T' \to A \to V \to 0$, by an inductive argument on the dimension of the trivial module, the last sequence split, i.e. $A = T' \oplus V' \subset E$ with $V' \simeq V$, hence $E = T \oplus V'$ as $\text{Cend}_{N,P}$-modules, proving (a).

b) We may assume without loss of generality that $\alpha = 0$. Consider an extension of $\text{Cend}_{N,P}$-modules of the form (3.11). As a vector space $E = \mathbb{C}[\partial]^N \oplus V_T$. We have, for $v \in V_T$:

$$\partial v = T(v) + g_v(\partial), \quad \text{where} \quad g_v(\partial) \in \mathbb{C}[\partial]^N,$$

$$x^jBP(x)\lambda v = f_{i,v}^B(\lambda, \partial), \quad \text{where} \quad f_{i,v}^B(\lambda, \partial) \in (\mathbb{C}[\partial]^N)[\lambda], \quad B \in \text{Mat}_N \mathbb{C}.$$

(3.14)

Let $P(x) = \sum_{i=0}^m Q_i x^i$. Since

$$(x^kAP(x)\lambda x^jBP(x))_{\lambda+\mu} v = (\lambda + \partial + x)^kAP(\lambda + \partial + x)x^jBP(x)_{\lambda+\mu} v$$

$$= \sum_{i=0}^m \sum_{j=0}^{i+k} \binom{i+k}{j} (\lambda + \partial)^{i+k-j} x^{i+l} AQ_iBP(x)_{\lambda+\mu} v$$

$$= \sum_{i=0}^m \sum_{j=0}^{i+k} \binom{i+k}{j} (-\mu)^{i+k-j} f_{i+j+l,v}^A Q_i B(\lambda + \mu, \partial)$$
and
\[ x^k AP(x)_{\lambda} (x^l BP(x)_{\mu} v) = x^k AP(x)_{\lambda} (f^v,B_{l} (\mu, \partial)) = (\lambda + \partial)^k AP(\lambda + \partial) f^v,B_{l} (\mu, \lambda + \partial) \]

must be equal by (A2)\(\lambda\), we have the functional equation
\[ (\lambda + \partial)^k AP(\lambda + \partial) f^v,B_{l} (\mu, \lambda + \partial) = \sum_{i=0}^{m} \sum_{j=0}^{i+k} \binom{i+k}{j} (-\mu)^{i+k-j} f^{v,AQ_{i}}_{j+l} B(\lambda + \mu, \partial). \]

If we put \( \mu = 0 \) in (3.15), we get
\[ (\lambda + \partial)^k AP(\lambda + \partial) f^v,B_{0} (0, \lambda + \partial) = \sum_{i=0}^{m} f^{v,AQ_{i}}_{i+k+l} B(\lambda, \partial). \]

Since the right-hand side of (3.16) is symmetric in \( k \) and \( l \), so is the left-hand side, hence, in particular, we have
\[ (\lambda + \partial)^k AP(\lambda + \partial) f^v,B_{0} (0, \lambda + \partial) = AP(\lambda + \partial) f^v,B_{0} (0, \lambda + \partial). \]

Taking \( A = I \) and using that \( \det P \neq 0 \), we get
\[ f^v,B_{k} (0, \lambda + \partial) = (\lambda + \partial)^k f^v,B_{0} (0, \lambda + \partial). \]

Furthermore, by (A1)\(\lambda\), we have \([\partial, x^k AP(x)_{\lambda}] v = -\lambda x^k AP(x)_{\lambda} v\), which gives us the next condition:
\[ (\lambda + \partial) f^v,A_{k} (\lambda, \partial) = f^{T(v),A}_{k} (\lambda, \partial) + (\lambda + \partial)^k AP(\lambda + \partial) g_v(\lambda + \partial). \]

We shall prove that if \( c \) is an eigenvalue of \( T \) and \( p_j(c) \neq 0 \) for all \( 1 \leq j \leq N \), then (after a change of complement) the generalized eigenspace of \( T \) corresponding to the eigenvalue \( c \) is a trivial submodule of \( E \) (hence is a non-zero torsion submodule). Indeed, let \( \{v_1, \cdots, v_s\} \) be vectors corresponding to one Jordan block of \( T \) associated to \( c \), that is \( T(v_1) = cv_1 \) and \( T(v_{i+1}) = cv_{i+1} + v_i \) for \( i \geq 1 \). Then (3.18) with \( v = v_1 \) becomes
\[ (\lambda + \partial - c) f^{v_1,A}_{k} (\lambda, \partial) = (\lambda + \partial)^k AP(\lambda + \partial) g_v(\lambda + \partial) \]

Observe that the right-hand side of (3.19) depends on \( \lambda + \partial \), so \( f^{v_1,A}_{k} (\lambda, \partial) = f^{v_1,A}_{k} (0, \lambda + \partial) \). Then using (3.17), we have
\[ f^{v_1,A}_{k} (\lambda, \partial) = f^{v_1,A}_{k} (0, \lambda + \partial) = (\lambda + \partial)^k f^{v_1,A}_{k} (0, \lambda + \partial) = (\lambda + \partial)^k f^{v_1,A}_{k} (\lambda, \partial) \]
Similarly, considering (3.18) with \( v = v_{i+1} \) \((i \geq 1)\), we get
\[
(\lambda + \partial - c) f^{v_{i+1},A}_{k}(\lambda, \partial) = f^{v_{i},A}_{k}(\lambda, \partial) + (\lambda + \partial)^{k} AP(\lambda + \partial)g_{v_{i+1}}(\lambda + \partial)
\]
\[
= (\lambda + \partial)^{k} \left[ f^{v_{i},A}_{0}(0, \lambda + \partial) + AP(\lambda + \partial)g_{v_{i+1}}(\lambda + \partial) \right]
\]
(3.21)

Again, since the right hand side of (3.21) depends only on \( \lambda + \partial \), we have that (3.20) also holds for any \( v_{i} \).

Using that \( p_{j}(c) \neq 0 \) \((j = 1, \ldots, N)\) (recall that \( P \) is diagonal), and taking \( A = E_{i,l} \), we obtain from (3.19) with \( k = 0 \) that
\[
f^{v_{1},A}_{0}(\lambda, \partial) = AP(\lambda + \partial)h_{v_{1}}(\lambda + \partial)
\]
(3.22.a)

where \( g_{v_{1}}(\partial) = (\partial - c)h_{v_{1}}(\partial) \). Now, (3.21) with \( k = 0 \) and \( i = 1 \) becomes (by (3.22.a))
\[
(\lambda + \partial - c) f^{v_{2},A}_{0}(\lambda, \partial) = f^{v_{1},A}_{0}(\lambda, \partial) + AP(\lambda + \partial)g_{v_{2}}(\lambda + \partial)
\]
\[
= AP(\lambda + \partial) ( h_{v_{1}}(\lambda + \partial) + g_{v_{2}}(\lambda + \partial) )
\]

As in (3.22.a), we get
\[
f^{v_{2},A}_{0}(\lambda, \partial) = AP(\lambda + \partial)h_{v_{2}}(\lambda + \partial)
\]

where \( g_{v_{2}}(\partial) + h_{v_{1}}(\partial) = (\partial - c)h_{v_{2}}(\partial) \). Similarly, we obtain for all \( i \geq 1 \),
\[
f^{v_{i+1},A}_{0}(\lambda, \partial) = AP(\lambda + \partial)h_{v_{i+1}}(\lambda + \partial)
\]
(3.22.b)

where \( g_{v_{i+1}}(\partial) + h_{v_{i}}(\partial) = (\partial - c)h_{v_{i+1}}(\partial) \). Changing the basis to \( v'_{i} = v_{i} - h_{v_{i}}(\partial) \), we have from (3.22.a-b) that \( x^{k} AP(x) \chi v'_{i} = 0 \) and
\[
\partial v'_{i} = T(v_{i}) + g_{v_{i}}(\partial) - \partial h_{v_{i}}(\partial)
\]
\[
= cv_{1} + (\partial - c)h_{v_{1}}(\partial) - \partial h_{v_{1}}(\partial) = cv'_{1},
\]

\[
\partial v'_{i+1} = T(v_{i+1}) + g_{v_{i+1}}(\partial) - \partial h_{v_{i+1}}(\partial)
\]
\[
= cv_{i+1} + v_{i} + (\partial - c)h_{v_{i+1}}(\partial) - \partial h_{v_{i+1}}(\partial) - h_{v_{i}}(\partial) - h_{v_{i}}(\partial)
\]
\[
= cv'_{i+1} + v'_{i}.
\]

Hence, the \( T \)-invariant subspace spanned by \( \{ v'_{i} \} \) is a trivial submodule of \( E \). Therefore, if \( p_{j}(c) \neq 0 \) for all \( j \) and all eigenvalues \( c \) of \( T \), then \( E \) is a trivial extension. This proves the first part of (b).
Now suppose that the extension $E$ of $\mathbb{C}[\partial]^N$ by a finite dimensional vector space have no non-zero trivial submodule (equivalently, $E$ is torsionless). By Remark 3.2.(b), $E$ must be a free $\mathbb{C}[\partial]$-module of rank $N$.

Then, the problem reduces to the study of a $\text{Cend}_N$-module structure on $E = \mathbb{C}[\partial]^N$, but using Remark 1.1, this is the same as a non-zero homomorphism from $\text{Cend}_{N,P}$ to $\text{Cend}_N$. So, the end of this proof also gives us the classification of all these homomorphisms.

Denote by $\phi : \text{Cend}_{N,P} \to \text{Cend}_N$ the (non-zero) homomorphism associated to $E$. It is an embedding (due to irreducibility) of free $\mathbb{C}[\partial]$-modules $\mathbb{C}[\partial]^N \to \mathbb{C}[\partial]^N$, hence it is given by a non-degenerate matrix $S(\partial) \in \text{Mat}_N \mathbb{C}[\partial]$. Hence the action on $E$ of $\text{Cend}_{N,P}$ is given by the formula:

$$\phi(a(\partial, x)P(x))_\lambda (S(\partial)v) = S(\partial) a(-\lambda, \lambda + \partial + \alpha) P(\lambda + \partial + \alpha) v \quad \text{for all } v \in \mathbb{C}^N.$$\[1\]

Furthermore, we have:

$$\phi(a(\partial, x)P(x)) S(x))_\lambda v = \phi(a(\partial, x)P(x))_\lambda (S(\partial)v)$$

$$= (S(\partial + x)a(\partial, x + \alpha)P(x + \alpha))_\lambda v \quad \text{for all } v \in \mathbb{C}^N.$$\[2\]

Hence $\phi(a(\partial, x)P(x)) = S(\partial + x)a(\partial, x + \alpha)P(x + \alpha)S^{-1}(x)$, and this is in $\text{Cend}_N$ if and only if $R(x) := P(x + \alpha)S^{-1}(x) \in \text{Mat}_N \mathbb{C}[x]$, proving b).

c) Consider a short exact sequence of $R = \text{Cend}_{N,P}$-modules

$$0 \to V \to E \to V' \to 0,$$

where $V$ and $V'$ are irreducible finite. Take $v \in E$ with $v \notin V$, and let $\mu \in \mathbb{C}$ be such that $A := R_{-\partial - \mu}v \neq 0$. Then we have three possibilities:

1) The image of $A$ in $V'$ is 0, then $A = V$, which is impossible because $v \notin V$.

2) The image of $A$ in $V'$ is $V'$ and $A \cap V = 0$, then $A$ is isomorphic to $V'$, hence the exact sequence splits.

3) The image of $A$ in $V'$ is $V'$ and $A \cap V = V$, hence $A = E$ and $E$ is a cyclic module, hence corresponds to a left ideal which is contained in a unique max ideal (otherwise the sequence splits). It is easy to see then that $E$ is the indecomposable module given in (3.12), where $J$ is the $2 \times 2$ Jordan block. \[Q.E.D.\]

**Corollary 3.27.** There are no non-trivial extensions of $\text{Cend}_N$-modules of the form:

$$0 \to V_T \to E \to \mathbb{C}[\partial]^N \to 0 \quad \text{or} \quad 0 \to \mathbb{C}[\partial]^N \to E \to V_T \to 0$$
Theorem 3.28. Every finite $\text{Cend}_N$-module is isomorphic to a direct sum of its (finite dimensional) trivial torsion submodule and a free finite $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial]^N \otimes T$ on which the $\lambda$-action is given by
\[
a(\partial, x)\lambda(c(\partial) \otimes u) = a(-\lambda, \lambda + \partial \otimes 1 + 1 \otimes \alpha)c(\lambda + \partial)(1 \otimes u),
\] (3.29)
where $\alpha$ is an arbitrary operator on $T$.

Proof. Consider a short exact sequence of $R = \text{Cend}_N$-modules
\[
0 \to V \to E \to V' \to 0,
\]
where $V$ and $V'$ are irreducible finite. By Theorem 3.10(c), the exact sequence split or $E$ is the indecomposable module that corresponds to a $2 \times 2$ Jordan block $J$, i.e., $E = \mathbb{C}[\partial]^N \otimes \mathbb{C}^2$, and $R$ acts via (3.29), where $S = 0$, $\beta = 0$ and $\alpha = J$.

Next, using Corollary 3.27, the short exact sequences of $R$-modules $0 \to V \to E \to C \to 0$ and $0 \to C \to E \to V \to 0$, where $C$ is a trivial 1-dimensional $R$-module, and $V$ is a standard $R$-module (1.2), split.

Recall [K1] that an $R$-module is the same as a module over the associated extended annihilation algebra $(\text{Alg}R)^- = \mathbb{C}\partial \ltimes (\text{Alg}R)_-$, where $(\text{Alg}R)_-$ is the annihilation algebra. For $R = \text{Cend}_N$ one has:
\[
(\text{Alg}R)_- = (\text{Diff}^N \mathbb{C}), (\text{Alg}R)^- = \mathbb{C}\partial \ltimes (\text{Alg}R)_-,
\]
where $\partial$ acts on $(\text{Alg}R)_-$ via $-\text{ad}\partial t$. Furthermore, viewed as an $(\text{Alg}R)_-$-module, all modules (1.2) are equivalent to the module $F = \mathbb{C}[t, t^{-1}]^N / \mathbb{C}[t]^N$, and the modules (1.2) are obtained by letting $\partial$ act as $-\partial t + \alpha$.

Let $M$ be a finite $R$-module. Then it has finite length and, by Corollary 3.7, all its irreducible subquotients are either trivial 1-dimensional or are isomorphic to a standard $R$-module (1.2). Since in the case b) above, the exact sequence splits when restricted to $(\text{Alg}R)_-$, we conclude that, viewed as an $(\text{Alg}R)_-$-module, $M$ is a finite direct sum of modules equivalent to $F$ or trivial 1-dimensional. Thus, viewed as an $(\text{Alg}R)_-$-module, $M = S \oplus (F \otimes T)$, where $S$ and $T$ are trivial $(\text{Alg}R)_-$-modules.

The only way to extend this $M$ to an $(\text{Alg}R)^-$-module is to let $\partial$ act as operators $\alpha$ and $\beta$ on $T$ and $S$, respectively, and as $-\partial_t$ on $F$, which gives (3.29). □

4. Automorphisms and anti-automorphisms of $\text{Cend}_N, P$

A $\mathbb{C}[\partial]$-linear map $\sigma : R \to S$ between two associative conformal algebras is called a homomorphism (resp. anti-homomorphism) if
\[
\sigma(a_\lambda b) = \sigma(a)_\lambda \sigma(b) \quad (\text{resp } \sigma(a_\lambda b) = \sigma(b)_-\lambda-\partial \sigma(a)).
\]
An anti-automorphism \( \sigma \) is an anti-involution if \( \sigma^2 = 1 \).

An important example of an anti-involution of \( \text{Cend}_N \) is:

\[
\sigma(A(\partial, x)) = A^t(\partial, -x - \partial)
\]

where the superscript \( t \) stands for the transpose of a matrix.

By Corollary 3.7 we know that all irreducible finite \( \text{Cend}_N \)-modules are of the form \((\alpha \in \mathbb{C})\):

\[
A(\partial, x)v(\partial) = A(-\lambda, \lambda + \partial + \alpha)v(\lambda + \partial).
\]

Hence, twisting one of these modules by an automorphism of \( \text{Cend}_N \) gives again one of these modules, and we get the following

**Theorem 4.2.** All automorphisms of \( \text{Cend}_N \) are of the form:

\[
A(\partial, x) \mapsto C(\partial + x)A(\partial, x + \alpha)C(x)^{-1},
\]

where \( \alpha \in \mathbb{C} \) and \( C(x) \) is a matrix with a non-zero constant determinant.

This result can be generalized as follows.

**Theorem 4.3.** Let \( P(x) \in \text{Mat}_N \mathbb{C}[x] \) with \( \det P(x) \neq 0 \). Then all automorphisms of \( \text{Cend}_{N,P} \) are those that come from \( \text{Cend}_N \) by restriction. More precisely, any automorphism is of the form:

\[
A(\partial, x)P(x) \mapsto C(\partial + x)A(\partial, x + \alpha)B(x)P(x),
\]

where \( \alpha \in \mathbb{C} \), and \( B(x) \) and \( C(x) \) are invertible matrices in \( \text{Mat}_N \mathbb{C}[x] \) such that

\[
P(x + \alpha) = B(x)P(x)C(x).
\]

**Proof.** Let \( \pi'(a) = \pi(s(a)) \), where \( \pi \) is the standard representation and \( s \) is an automorphism of \( \text{Cend}_{N,P} \). Since it is equivalent to the standard representation due to Corollary 3.7, we deduce that \( s(a(\partial, x)) = C(\partial + x)a(\partial, x + \alpha)C(x)^{-1} \) for some invertible (in \( \text{Mat}_N \mathbb{C}[x] \)) matrix \( C(x) \). But \( C(\partial + x)\text{Cend}_{N,P}C(x)^{-1} = \text{Cend}_{N,P} \) if and only if (4.5) holds. Indeed, we have: \( C(\partial + x)P(x + \alpha)C(x)^{-1} = A(\partial, x)P(x) \) for some \( A(\partial, x) \in \text{Cend}_N \). Taking determinants of both sides of this equality, we see that \( \det A(\partial, x) \) is a non-zero constant. Hence \( B(x) := P(x + \alpha)C(x)^{-1}P(x)^{-1} \) is invertible in \( \text{Mat}_N \mathbb{C}[x] \), finishing the proof. \( \square \)
Theorem 4.6. Let $P(x) \in \text{Mat}_N\mathbb{C}[x]$ with $\det P(x) \neq 0$. Then we have,

a) All non-zero homomorphisms from $\text{Cend}_N,P$ to $\text{Cend}_N$ are of the form:

$$a(\partial, x)P(x) \mapsto S(\partial + x)a(\partial, x + \alpha)R(x),$$

where $\alpha \in \mathbb{C}$, and $R(x)$ and $S(x)$ are matrices in $\text{Mat}_N\mathbb{C}[x]$ such that

$$P(x + \alpha) = R(x)S(x).$$

(b) All non-trivial anti-homomorphisms from $\text{Cend}_N,P$ to $\text{Cend}_N$ are of the form:

$$a(\partial, x)P(x) \mapsto A(\partial + x)a^t(\partial, -\partial - x + \alpha)B(x),$$

where $\alpha \in \mathbb{C}$, and $A(x)$ and $B(x)$ are matrices in $\text{Mat}_N\mathbb{C}[x]$ such that

$$P^t(-x + \alpha) = B(x)A(x).$$

(c) The conformal algebra $\text{Cend}_N,P$ has an anti-automorphism (i.e. it is isomorphic to its opposite conformal algebra) if and only if the matrices $P^t(-x + \alpha)$ and $P(x)$ have the same elementary divisors for some $\alpha \in \mathbb{C}$. In this case, all anti-automorphisms of $\text{Cend}_N,P$ are of the form:

$$a(\partial, x)P(x) \mapsto Y(\partial + x)a^t(\partial, -\partial - x + \alpha)W(x)P(x),$$

where $Y(x)$ and $W(x)$ are invertible matrices in $\text{Mat}_N\mathbb{C}[x]$ such that

$$P^t(-x + \alpha) = W(x)P(x)Y(x).$$

(d) The conformal algebra $\text{Cend}_N,P$ has an anti-involution if and only if there exist an invertible in $\text{Mat}_N\mathbb{C}[x]$ matrix $Y(x)$ such that

$$Y^t(-x + \alpha)P^t(-x + \alpha) = \epsilon P(x)Y(x)$$

for $\epsilon = 1$ or $-1$. In this case all anti-involutions are given by

$$\sigma_{P,Y,\epsilon,\alpha}(a(\partial, x)P(x)) = \epsilon Y(\partial + x)a^t(\partial, -\partial - x + \alpha)Y^t(-x + \alpha)^{-1}P(x),$$

where $Y(x)$ is an invertible in $\text{Mat}_N\mathbb{C}[x]$ matrix satisfying (4.13).

Proof. a) Follows by the end of proof of Theorem 3.10(b).

b) Since composition of two anti-homomorphisms is a homomorphism, using the anti-involution (4.1) we see that any anti-homomorphism must be of the form

$$a(\partial, x)P(x) \rightarrow R^t(-\partial - x)a^t(\partial, -\partial - x + \alpha)S^t(-x)$$
with \( P(x + \alpha) = R(x)S(x) \). Then, (4.9) and (4.10) follows by taking \( A(x) = S^t(-x) \) and \( B(x) = R^t(-\partial - x) \).

c) Let \( \phi \) be an anti-automorphism of \( \text{Cend}_{N,P} \). In particular, it is an anti-homomorphism as in part b), whose image is \( \text{Cend}_{N,P} \). Then, for all \( a(\partial, x)P(x) \in \text{Cend}_{N,P} \), we have that \( \phi(a(\partial, x)p(x)) = A(\partial + x)a^t(\partial, -\partial - x + \alpha) B(x) \in \text{Cend}_{N,P} \). Then taking \( a(\partial, x) \) the identity matrix we have that

\[
A(\partial + x)B(x) = b(\partial, x)P(x), \quad \text{for some } b(\partial, x) \in \text{Cend}_{N,P}.
\]

Recall that \( P^t(-x + \alpha) = B(x)A(x) \). Taking determinant of both sides of (4.16), and comparing its highest degrees in \( x \), we deduce that \( \det b(\partial, x) \) and \( \det A(x) \) are both (non-zero) constants. Now, from (4.16), we see that \( A^{-1}(\partial + x)b(\partial, x) \) does not depend on \( \partial \). Then we have \( B(x) = W(x)P(x) \), where \( W(x) = A^{-1}(\partial + x)b(\partial, x) \) is an invertible matrix. Therefore,

\[
\phi(a(\partial, x)P(x)) = A(\partial + x)a^t(\partial, -\partial - x + \alpha) W(x)P(x), \tag{4.17}
\]

with \( A, W \) invertible matrices such that

\[
W(x)P(x)A(x) = P^t(-x + \alpha). \tag{4.18}
\]

d) Now suppose that \( \phi \) is an anti-involution. Then it is as in (4.11), and it also satisfies \( \phi^2 = \text{id} \). This condition implies that

\[
a(\partial, x)P(x) = Y(\partial + x)W^t(-\partial - x + \alpha)a(\partial, x)Y^t(-x + \alpha)W(x)P(x) \tag{4.19}
\]

for all \( a(\partial, x) \in \text{Cend}_{N,P} \). Denote \( Z(x) = Y^t(-x + \alpha)W(x) \). Taking \( a(\partial, x) = \text{Id} \) in (4.19) and using that \( \det P(x) \neq 0 \), we have \( Y(\partial + x)W^t(-\partial - x + \alpha) = Z^{-1}(x) \). Now, (4.19) becomes \( a(\partial, x)P(x) = Z^{-1}(x)a(\partial, x)Z(x)P(x) \). Hence, we obtain \( Z(x) = \varepsilon \text{id} \), with \( \varepsilon = 1 \) or \(-1\). Thus, \( Y^{-1}(X) = \varepsilon W^t(-x + \alpha) \). From (4.12) we deduce that

\[
P(x)Y(x) = \varepsilon(P(-x + \alpha)Y(-x + \alpha))^t. \tag{4.20}
\]

This condition is also sufficient. There exists an anti-involution if (4.20) holds for some invertible matrix \( Y \), and it is given by

\[
\phi(a(\partial, x)P(x)) = \varepsilon Y(\partial + x)a^t(\partial, -\partial - x + \alpha)Y^t(-x + \alpha)^{-1}P(x). \quad \square
\]

Two anti-involutions \( \sigma, \tau \) of an associative conformal algebra \( R \) are called conjugate if \( \sigma = \varphi \circ \tau \circ \varphi^{-1} \) for some automorphism \( \varphi \) of \( R \). Recall that two matrices \( a \) and \( b \) in \( \text{Mat}_{N}[x] \) are called \( \alpha \)-congruent if \( b = c^*ac \) for some invertible in \( \text{Mat}_{N}[x] \) matrix \( c \), where \( c(x)^* := c(-x + \alpha)^t \). We shall simply call them congruent if \( \alpha = 0 \). The following proposition gives us a characterization of equivalent anti-involutions \( \sigma_{P,Y,\varepsilon,\alpha} \) in \( \text{Cend}_{N,P} \) (defined in (4.14)) and relates anti-involutions for different \( P \).
Proposition 4.21. (a) The anti-involutions \( \sigma_{P,Y_1,\epsilon_1,\alpha} \) and \( \sigma_{P,Y_2,\epsilon_2,\gamma} \) of \( \text{Cend}_N \) are conjugate if and only if \( \epsilon_1 = \epsilon_2 \) and \( P(x + \frac{\gamma - \alpha}{2})Y_2(x + \frac{\gamma - \alpha}{2}) \) is \( \alpha \)-congruent to \( P(x)Y_1(x) \).

(b) Let \( \varphi_Y \) be the automorphism of \( \text{Cend}_N \) given by
\[
\varphi_Y(a(\partial, x)) = Y(\partial + x)^{-1}a(\partial, x)Y(x),
\]
where \( Y \) is an invertible matrix in \( \text{Mat}_N\mathbb{C}[x] \), and let \( P \) and \( Y \) satisfying (4.13). Then
\[
\sigma_{P,Y,\epsilon,\alpha} = \varphi_Y^{-1} \circ \sigma_{P,Y,I,\epsilon,\alpha} \circ \varphi_Y.
\]

(c) Let \( c_\alpha \) be the automorphism of \( \text{Cend}_N \) given by \( c_\alpha(a(\partial, x)) = a(\partial, x + \alpha) \), where \( \alpha \in \mathbb{C} \). Suppose that \( P^t(-x + \alpha) = \epsilon P(x) \), for \( \epsilon = 1 \) or \( -1 \), then \( Q(x) := P(x + \frac{\alpha}{2}) \) satisfies \( Q^t(-x) = \epsilon Q(x) \) and
\[
\sigma_{P,I,\epsilon,\alpha} = c_1^{-1} \circ \sigma_{P,I,\epsilon,0} \circ c_2\).
\]

Proof. (a) Let \( \varphi_{B,C,\alpha} \) be the automorphism of \( \text{Cend}_N \) given by in (4.4) and (4.5). A straightforward computation shows that \( \varphi_{B,C,\beta}^{-1} \circ \sigma_{P,Y,\epsilon,\alpha} \circ \varphi_{B,C,\beta} = \sigma_{P,Y,\epsilon,\alpha} \), where \( \tilde{P}(x) = C^{-1}(x - \beta)Y(x - \beta)B^t(-x + \alpha + \beta) \) and \( P(x + \beta) = B(x)P(x)C(x) \). Hence, if \( \sigma_{P,Y_1,\epsilon_1,\alpha} \) and \( \sigma_{P,Y_2,\epsilon_2,\gamma} \) are conjugate, then \( \epsilon_1 = \epsilon_2 \) and \( Y_2(x) = C^{-1}(x - \beta)Y(x - \beta)B^t(-x + \alpha + \beta) \), with \( \beta = \gamma - \alpha/2 \). Therefore, \( P(x + \beta)Y_2(x + \beta) = B(x)P(x)Y_1(x)B^t(-x + \alpha) \), that is \( P(x + \frac{\gamma - \alpha}{2})Y_2(x + \frac{\gamma - \alpha}{2}) \) is \( \alpha \)-congruent to \( P(x)Y_1(x) \).

Conversely, suppose that \( P(x + \frac{\gamma - \alpha}{2})Y_2(x + \frac{\gamma - \alpha}{2}) = B(x)P(x)Y_1(x)B^t(-x + \alpha) \) for some \( B(x) \) invertible matrix in \( \text{Mat}_N\mathbb{C}[x] \). Recall that \( Y_1 \) and \( Y_2 \) are invertibles. Then \( C(x) := Y_1(x)B^t(-x + \alpha)Y_2(x + \frac{\gamma - \alpha}{2})^{-1} \) is an invertible matrix in \( \text{Mat}_N\mathbb{C}[x] \), satisfies \( P(x + \frac{\gamma - \alpha}{2}) = B(x)P(x)C(x) \), and it is easy to check that the anti-involutions are conjugated by the automorphism \( \varphi_{B,C,\frac{\gamma - \alpha}{2}} \), proving (a). Parts (b) and (c) are straightforward computations. \( \square \)

Theorem 4.24. Any anti-involution of \( \text{Cend}_N \) is, up to conjugation by an automorphism of \( \text{Cend}_N \):

\[
a(\partial, x) \mapsto a^*(\partial, -\partial - x),
\]
where \( * \) is the adjoint with respect to a non-degenerate symmetric or skew-symmetric bilinear form over \( \mathbb{C} \).

Proof. Using Theorem 4.6(d), we have that any anti-involution of \( \text{Cend}_N \) has the form \( \sigma(a(\partial, x)) = c(\partial + x)a(\partial, -\partial - x + \alpha)^t c(x)^{-1} \), where \( c(x) \) is an invertible matrix such that \( c(x)^t = \varepsilon c(-x + \alpha) \), with \( \varepsilon = 1 \) or \( -1 \). By Proposition 4.21(c), we may suppose that \( \alpha = 0 \). Now, the proof follows because \( c(x) \) is congruent to a constant symmetric or skew-symmetric matrix, by the following general theorem of Djokovic.
**Theorem 4.25.** (Djokovic, [D1-2]) If $A$ is invertible in $\text{Mat}_N(\mathbb{C}[x])$ and $A^* = A$ (resp. $A^* = -A$) where $A(x)^* = A(-x)$, then $A$ is congruent to a symmetric (resp. skew-symmetric) matrix over $\mathbb{C}$.

**Proof.** The symmetric case follows by Proposition 5 in [D1]. The skew-symmetric case was communicated to us by D. Djokovic and we will give the details here. Suppose $A^* = -A$. By Theorem (2.2.1), Ch. 7 in [Kn] it follows that $A$ has to be isotropic, i.e. there exists a non-zero vector $v$ in $\mathbb{C}[x]^N$ such that $v^*Av = 0$. We can assume that $v$ is primitive (i.e., the greatest common divisor of its coordinates is 1).

But then $\mathbb{C}[x]v$ is a direct summand: $\mathbb{C}[x]^N = \mathbb{C}[x]v \oplus M$, for some $\mathbb{C}[x]$-submodule $M$ of $\mathbb{C}[x]^N$. Then we have $\mathbb{C}[x]^N = (\mathbb{C}[x]v)^\perp \oplus M^\perp$ and $M^\perp$ is a free rank one $\mathbb{C}[x]$-module, that is $M^\perp = \mathbb{C}[x]w$ for some $w \in \mathbb{C}[x]^N$. Since $\mathbb{C}[x]v \subseteq (\mathbb{C}[x]v)^\perp$, the submodule $P = \mathbb{C}[x]v + \mathbb{C}[x]w$ is free of rank two. If $Q = M \cap (\mathbb{C}[x]v)^\perp$, then since $\mathbb{C}[x]v \subseteq (\mathbb{C}[x]v)^\perp$ we have $(\mathbb{C}[x]v)^\perp = \mathbb{C}[x]v \oplus Q$ and

$$\mathbb{C}[x]^N = (\mathbb{C}[x]v)^\perp \oplus \mathbb{C}[x]w = P \oplus Q.$$ 

Since $Q = P^\perp$, the submodule generated by $v$ and $w$ is a direct summand. Choose $w' \in P$ such that $v^*Aw' = 1$. Then $v, w'$ must be a free basis of $P$ and the corresponding $2 \times 2$ block is of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & f \end{pmatrix}$$

for some skew element $f = g - g^*$ (cf. Proposition 5 [D1]). One can now replace $f$ by 0, by taking the basis $v, w'$, and use induction to finish the proof. \quad \Box

**Remark 4.26.** We do not know any counter-examples to the following generalization of Djokovic’s theorem: If $A \in \text{Mat}_N(\mathbb{C}[x])$ and $A^* = A$ (resp. $A^* = -A$) where $A(x)^* = A(-x)$, then $A$ is congruent to a direct sum of $1 \times 1$ matrices of the form $(p(x))$ where $p$ is an even (resp. odd) polynomial and $2 \times 2$ matrices of the form

$$\begin{pmatrix} 0 & q(x) \\ -q(x) & 0 \end{pmatrix}$$

where $q(x)$ is an odd (resp. even) polynomial.

As a consequence of Theorem 4.6, we have the following result.

**Theorem 4.27.** Let $P(x), Q(x) \in \text{Mat}_N[\mathbb{C}[x]]$ be two non-degenerate matrices. Then $\text{Cend}_{N,P}$ is isomorphic to $\text{Cend}_{N,Q}$ if and only if there exist $\alpha \in \mathbb{C}$ such that $Q(x)$ and $P(x + \alpha)$ have the same elementary divisors.

**Proof.** We may assume that $P$ is diagonal. Let $\phi : \text{Cend}_{N,P} \longrightarrow \text{Cend}_{N,Q}$ be an isomorphism. In particular it is a homomorphism from $\text{Cend}_{N,P}$ to $\text{Cend}_{N}$ whose
image is $\text{Cend}_{N,Q}$. Then, by Theorem 4.6(a), we have that $\phi(a(\partial, X)P(X)) = A(\partial + x)a(\partial, x + \alpha)B(x)$, with $P(x + \alpha) = B(x)A(x)$. In particular

$$A(\partial + x)a(\partial, x + \alpha)B(x) = Q(x)$$  \hfill (4.28)

for some $a(\partial, x)P(x) \in \text{Cend}_{N,P}$.

Taking determinant in both sides of (4.28), and comparing its highest degrees in $\partial$, we can deduce that $\det A(x)$ is constant. Now, define the isomorphism $\phi_2 = \chi_A \circ \phi : \text{Cend}_{N,P} \to \text{Cend}_{N,Q_A}$, where $\chi_A(a(\partial, x)) = A^{-1}(\partial + x)a(\partial, x)A(x)$. Hence $\phi_2(a(\partial, x)P(x)) = a(\partial, x + \alpha)B(x)A(x)$. Since $\phi_2$ is an isomorphism, we have that

$$B(x)A(x) = D(x)Q(x)A(x) \quad \text{and} \quad C(x)B(x)A(x) = Q(x)A(x)$$

for some $C(x)$ and $D(x)$ (obviously $C$ and $D$ does not depend on $\partial$). Comparing these two formulas, we have that $C(x)D(x) = Id$. Then both are invertible matrices, and $Q(x)A(x) = C(x)B(x)A(x) = C(x)P(x + \alpha)$ for some invertible matrices $A$ and $C$. \hfill \Box

5. On irreducible subalgebras of $\text{Cend}_N$

In this section we study the conformal analog of the Burnside Theorem. A subalgebra of $\text{Cend}_N$ is called irreducible if it acts irreducibly on $\mathbb{C}[\partial]^N$. The following is the conjecture from [K2] on the classification of such subalgebras:

**Conjecture 5.1.** Any irreducible subalgebra of $\text{Cend}_N$ is either $\text{Cend}_{N,P}$ with $\det P(x) \neq 0$ or $C(x + \partial) \text{ Cur}_N C(x)^{-1}$ (i.e. is a conjugate of $\text{Cur}_N$), where $\det C(x) = 1$.

The classification of finite irreducible subalgebras follows from the classification in [DK] at the Lie algebra level:

**Theorem 5.2.** Any finite irreducible subalgebra of $\text{Cend}_N$ is a conjugate of $\text{Cur}_N$.

*Proof.* Let $R$ be a finite irreducible subalgebra of $\text{Cend}_N$. Then the Lie conformal algebra $R_-$ (with the bracket $[a\lambda b] = a\lambda b - b_{-\partial - \lambda}a$), of course, still acts irreducibly on $\mathbb{C}[\partial]^N$. By the conformal analogue of the Cartan-Jacobson theorem [DK] applied to $R_-$, a conjugate $R_1$ of $R$ either contains the element $xI$, or is contained in $\text{Mat}_N \mathbb{C}[\partial]$. The first case is ruled out since then $R_1$ is infinite. In the second case, by the same theorem, $R_1$ contains $\text{Cur} \mathfrak{g}$, where $\mathfrak{g} \subseteq \text{Mat}_N \mathbb{C}$ is a simple Lie algebra acting irreducibly on $\mathbb{C}^N$, provided that $N > 1$.

By the classical Burnside theorem, we conclude that $R_1 = \text{Mat}_N \mathbb{C}[\partial]$ in the case $N > 1$. It is immediate to see that the same is true if $N = 1$ (or we may apply Theorem 2.2). \hfill \Box
Theorem 5.3. If \( S \subseteq \text{Cend}_N \) is an irreducible subalgebra such that \( S \) contains the identity matrix \( \text{Id} \), then \( S = \text{Cur}_N \) or \( S = \text{Cend}_N \).

Proof. Since \( \text{Id} \in S \), and using the idea of (1.7), we have that \( S = \mathbb{C}[\partial]A \), where \( A = S \cap \text{Mat}_N \mathbb{C}[x] \). Observe that \( A \) is a subalgebra of \( \text{Mat}_N \mathbb{C}[x] \). Indeed,

\[
P(x)Q(x) = P(x)\lambda Q(x)_{|\lambda=-\partial} \in S \quad \text{for all } P, Q \in A.
\]

In order to finish the proof, we should show that \( A = \text{Mat}_N \mathbb{C} \) or \( A = \text{Mat}_N \mathbb{C}[x] \). Observe that \( A \) is invariant with respect to \( d/dx \). Because \( P(x)\lambda(\text{Id}) = P(\lambda+\partial+x) \in \mathbb{C}[\lambda] \otimes S \).

Let \( A_0 \subset \text{Mat}_N \mathbb{C} \) be the set of leading coefficients of matrices from \( A \). This is obviously a subalgebra of \( \text{Mat}_N \mathbb{C} \) that acts irreducibly on \( \mathbb{C}^N \). Otherwise we would have a non-trivial \( A_0 \)-invariant subspace \( u \subset \mathbb{C}^N \). Let \( U \) denote the space of vectors in \( \mathbb{C}[\partial]^N \) whose leading coefficients lie in \( u \); this is a \( \mathbb{C}[\partial] \)-submodule. But we have:

\[
a(x)\lambda u(\partial) = a(\lambda+\partial)u(\lambda+\partial) = \sum_{j \geq 0} \frac{\lambda^j}{j!} (a(\lambda+\partial)u(\lambda+\partial))^{(j)}|_{\lambda=0},
\]

where \( (j) \) stands for \( j \)-th derivative. Since both \( A \) and \( U \) are invariant with respect to the derivative by the indeterminate, we conclude that \( U \) is invariant with respect to \( A \), hence with respect to \( S = \mathbb{C}[\partial]A \).

Thus, \( A_0 = \text{Mat}_N \mathbb{C} \). Therefore \( A \) is a subalgebra of \( \text{Mat}_N \mathbb{C}[x] \) that contains \( \text{Mat}_N \mathbb{C} \) and is \( d/dx \)-invariant. If \( A \) is larger than \( \text{Mat}_N \mathbb{C} \), applying \( d/dx \) a suitable number of times, we get that \( A \) contains a matrix of the form \( xa \), where \( a \) is a non-zero constant matrix (we can always subtract the constant term). Hence \( A \supseteq x(\text{Mat}_N \mathbb{C})a(\text{Mat}_N \mathbb{C}) = x\text{Mat}_N \mathbb{C} \), hence \( A \) contains \( x^k \text{Mat}_N \mathbb{C} \) for all \( k \in \mathbb{Z}_+ \). □

6. Lie conformal algebras \( gc_N \), \( oc_{N,P} \) and \( spc_{N,P} \)

A Lie conformal algebra \( R \) is a \( \mathbb{C}[\partial] \)-module endowed with a \( \mathbb{C} \)-linear map \( R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes R, \ a \otimes b \mapsto [a_\lambda b] \), called the \( \lambda \)-bracket, satisfying the following axioms \( (a, b, c \in R) \),

\[
(C1)_{\lambda} \quad [(\partial a)_{\lambda} b] = -\lambda [a_\lambda b], \quad [a_\lambda (\partial b)] = (\lambda + \partial)[a_\lambda b]
\]

\[
(C2)_{\lambda} \quad [a_\lambda b] = -[a_{-\partial-\lambda} b]
\]

\[
(C3)_{\lambda} \quad [a_\lambda [b_\mu c] = [a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]].
\]

A module \( M \) over a conformal algebra \( R \) is a \( \mathbb{C}[\partial] \)-module endowed with a \( \mathbb{C} \)-linear map \( R \otimes M \rightarrow \mathbb{C}[\lambda] \otimes M, \ a \otimes v \mapsto a_\lambda v \), satisfying the following axioms \( (a, b \in R), \ v \in M \),
(M1)\( \lambda \)  \( (\partial a)^M_\lambda v = [\partial^M, a^M_\lambda]v = -\lambda a^M_\lambda v, \)

(M2)\( \lambda \)  \( [a^M_\lambda, b^M_\mu]v = [a_\lambda b^M_{\lambda+\mu}, v]. \)

Let \( U \) and \( V \) be modules over a conformal algebra \( R \). Then, the \( \mathbb{C}[\partial] \)-module \( N := \text{Chom}(U, V) \) has an \( R \)-module structure defined by

\[
(a^N_\lambda \varphi)_\mu u = a^V_\lambda (\varphi_{\mu-\lambda} u) - \varphi_{\mu-\lambda} (a^U_\mu u),
\]

where \( a \in R, \varphi \in N \) and \( u \in U \). Therefore, one can define the contragradient conformal \( R \)-module \( U^* = \text{Chom}(U, \mathbb{C}) \), where \( \mathbb{C} \) is viewed as the trivial \( R \)-module and \( \mathbb{C}[\partial] \)-module. We also define the tensor product \( U \otimes V \) of \( R \)-modules as the ordinary tensor product with \( \mathbb{C}[\partial] \)-module structure \( (u \in U, v \in V) \):

\[
\partial (u \otimes v) = \partial u \otimes v + u \otimes \partial v
\]

and \( \lambda \)-action defined by \( (r \in R) \):

\[
r_\lambda (u \otimes v) = r_\lambda u \otimes v + u \otimes r_\lambda v.
\]

**Proposition 6.2.** Let \( U \) and \( V \) be two \( R \)-modules. Suppose that \( U \) has finite rank as a \( \mathbb{C}[\partial] \)-module. Then \( U^* \otimes V \simeq \text{Chom}(U, V) \) as \( R \)-modules, with the identification \( (f \otimes v)_\lambda(u) = f_{\lambda+\partial v}(u) v, f \in U^*, u \in U \) and \( v \in V \).

**Proof.** Define \( \varphi : U^* \otimes V \to \text{Chom}(U, V) \) by \( \varphi(f \otimes v)_\lambda(u) = f_{\lambda+\partial v}(u) v. \) Observe that \( \varphi \) is \( \mathbb{C}[\partial] \)-linear, since

\[
\varphi(\partial(f \otimes v))_\lambda(u) = \varphi(\partial f \otimes v + f \otimes \partial v)_\lambda(u) = (\partial f)_{\lambda+\partial v}(u) v + f_{\lambda+\partial v}(u) \partial v
\]

\[
= -(\lambda + \partial V)f_{\lambda+\partial v}(u) v + f_{\lambda+\partial v}(u) \partial v = -\lambda f_{\lambda+\partial v}(u) v
\]

\[
= -\lambda \varphi(f \otimes v)_\lambda(u) = \partial(\varphi(f \otimes v))_\lambda(u)
\]

and \( \varphi \) is a homomorphism, since

\[
\varphi\left(r_\lambda(f \otimes v)\right)_\mu(u) = \varphi\left(r_\lambda f \otimes v + f \otimes r_\lambda v\right)_\mu(u)
\]

\[
= (r_\lambda f)_{\mu+\partial v}(u) v + f_{\mu+\partial v}(u) (r_\lambda v)
\]

\[
= -f_{\mu-\lambda+\partial v}(r_\lambda u) v + f_{\mu+\partial v}(u) (r_\lambda v)
\]

and

\[
\left(r_\lambda(\varphi(f \otimes v))\right)_\mu(u) = r_\lambda\left(\varphi(f \otimes v)_{\mu-\lambda}(u)\right) - \varphi(f \otimes v)_{\mu-\lambda}(r_\lambda u)
\]

\[
= r_\lambda(f_{\mu-\lambda+\partial v}(u) v) - f_{\mu-\lambda+\partial v}(r_\lambda u) v
\]

\[
= f_{\mu+\partial v}(u) (r_\lambda v) - f_{\mu-\lambda+\partial v}(r_\lambda u) v.
\]
The homomorphism \( \varphi \) is always injective. Indeed, if \( \varphi(f \otimes v) = 0 \), then \( f_{\mu+\partial}v = 0 \) for all \( u \in U \). Suppose that \( v \neq 0 \), then \( f_{\lambda+\partial}v = 0 \), that is \( f = 0 \).

It remains to prove that \( \varphi \) is surjective provided that \( U \) has finite rank as a \( \mathbb{C}[\partial] \)-module. Let \( g \in \text{Chom}(U, V) \), and \( U = \mathbb{C}[\partial]\{u_1, \cdots, u_n\} \). Then, there exist \( v_{ik} \in U \) such that

\[
g_{ij}(u_i) = \sum_{k=0}^{m_i} (\lambda + \partial V)^k v_{ik} = \sum_{k=0}^{m_i} \varphi(f_{ik} \otimes v_{ik}) \lambda(u_i),
\]

where \( f_{ik} \in U^* \) is defined (on generators) by \( f_{ik}(u_j) = \delta_{i,j} \lambda^k \). Therefore, \( g = \varphi(\sum_{i=0}^{n} \sum_{k=0}^{m_i} f_{ik} \otimes v_{ik}) \), finishing the proof. \( \square \)

In general, given any associative conformal algebra \( R \) with \( \lambda \)-product \( a \lambda b \), the \( \lambda \)-bracket defined by

\[
[a \lambda b] := a \lambda b - b \lambda - \lambda a
\]

(6.3)

makes \( R \) a Lie conformal algebra.

Let \( V \) be a finite \( \mathbb{C}[\partial] \)-module. The \( \lambda \)-bracket (6.3) on \( \text{Cend} V \), makes it a Lie conformal algebra denoted by \( \text{gc} \) \( V \) and called the general conformal algebra (see [DK], [K1] and [K2]). For any positive integer \( N \), we define \( \text{gc}_N := \text{gc} \mathbb{C}[\partial] \mathbb{C}[\partial, x] \), and the \( \lambda \)-bracket (6.3) is by (1.1):

\[
[A(\partial, x) \lambda B(\partial, x)] = A(-\lambda, x + \lambda + \partial)B(\lambda + \partial, x) - B(\lambda + \partial, -\lambda + x)A(-\lambda, x).
\]

Recall that, by Theorem 4.24, any anti-involution in \( \text{Cend}_N \) is, up to conjugation

\[
\sigma_*(A(\partial, x)) = A^*(\partial, -\partial - x),
\]

(6.4)

where \( * \) stands for the adjoint with respect to a non-degenerate symmetric or skew-symmetric bilinear form over \( \mathbb{C} \). These anti-involutions give us two important subalgebras of \( \text{gc}_N \): the set of \( \sigma_x \) fixed points is the orthogonal conformal algebra \( \text{oc}_N \) (resp. the symplectic conformal algebra \( \text{spc}_N \)), in the symmetric (resp. skew-symmetric) case.

Proposition 6.5. The subalgebras \( \text{oc}_N \) and \( \text{spc}_N \) are simple.

Proof. We will prove that \( \text{oc}_N \) is simple. The proof for \( \text{spc}_N \) is similar. Let \( I \) be a non-zero ideal of \( \text{oc}_N \). Let \( 0 \neq A(\partial, x) \in I \), then \( A(\partial, x) = \sum_{i=0}^{m} \partial^i a_i(x) = \sum_{i=0}^{m} \partial^i \tilde{a}_i(\partial + x) \), with \( a_i(x), \tilde{a}_i(x) \in \text{Mat}_N \mathbb{C}[x] \). Now, using that \( A(\partial, x) = -A^t(\partial, -\partial - x) \), we obtain that \( n = m \) and \( a_i(x) = -\tilde{a}_i^t(-x) \). Computing the \( \lambda \)-bracket

\[
x E_{ij} - (-\partial - x) E_{ji} \lambda A(\partial, x)] = \lambda^{m+1}(E_{ij} a_m(x) - a_m^t(-\partial - x) E_{ji}) + \lambda^m \ldots
\]
we deduce that \( E_{ij} a_m(x) - a'_m(-\partial - x) E_{ji} \in I \), with \( a_m \neq 0 \). By taking appropriate \( i \) and \( j \), we have that there exist polynomials \( b_k(x) \) such that
\[
\sum_{k=1}^{N} (b_k(x) E_{ik} - b_k(-\partial - x) E_{ki}) \in I,
\]
with \( b_r \neq 0 \) for some \( r \neq i \). Now by computing \([ (2x + \partial) E_{rr} \lambda \sum_{k=1}^{N} (b_k(x) E_{ik} - b_k(-\partial - x) E_{ki}) ]\) and looking at its leading coefficient in \( \lambda \), we show that \( E_{ri} - E_{ir} \in I \), with \( r \neq i \). Taking brackets with elements in \( o_N \), we have \( E_{ij} - E_{ji} \in I \) for all \( j \neq i \). Now, we can see from the \( \lambda \)-brackets \( [ x E_{ri} - (-\partial - x) E_{ir} \lambda E_{ir} - E_{ri} ] = (2x + \partial)(E_{ii} - E_{rr}) \) and \([ (2x + \partial) E_{ii} \lambda (2x + \partial)(E_{ii} - E_{rr}) ] \) that \( (2x + \partial) E_{ii} \in I \) for all \( i \). The other generators are obtained by \( (k \neq i, j) \)
\[
[(-x)^k E_{ik} - (\partial + x)^k E_{ki} \lambda E_{jk} - E_{kj}]_{\lambda=0} = x^k E_{ij} - (\partial - x)^k E_{ji}.
\]
Similarly, we can see that \( (x^k - (\partial - x)^k) E_{ii} \in I \), finishing the proof. \( \square \)

The conformal subalgebras \( o_N \) and \( spc_N \), as well as the anti-involutions given by (6.4), and their generalizations can be described in terms of conformal bilinear forms. Let \( V \) be a \( \mathbb{C}[\partial] \)-module. A conformal bilinear form on \( V \) is a \( \mathbb{C} \)-bilinear map \( \langle \cdot, \cdot \rangle_\lambda : V \times V \to \mathbb{C}[\lambda] \) such that
\[
\langle \partial v, w \rangle_\lambda = -\lambda \langle v, w \rangle_\lambda = -\langle v, \partial w \rangle_\lambda, \quad \text{for all } v, w \in V.
\]
The conformal bilinear form is non-degenerate if \( \langle v, w \rangle_\lambda = 0 \) for all \( w \in V \), implies \( v = 0 \). The conformal bilinear form is symmetric (resp. skew-symmetric) if \( \langle v, w \rangle_\lambda = \epsilon \langle w, v \rangle_{-\lambda} \) for all \( v, w \in V \), with \( \epsilon = 1 \) (resp. \( \epsilon = -1 \)).

Given a conformal bilinear form on a \( \mathbb{C}[\partial] \)-module \( V \), we have a homomorphism of \( \mathbb{C}[\partial] \)-modules, \( L : V \to V^* \), \( v \mapsto L_v \), given as usual by
\[
(L_v)_\lambda w = \langle v, w \rangle_\lambda, \quad v \in V. \quad (6.6)
\]
Let \( V \) be a free finite rank \( \mathbb{C}[\partial] \)-module and fix \( \beta = \{ e_1, \cdots, e_N \} \) a \( \mathbb{C}[\partial] \)-basis of \( V \). Then the matrix of \( \langle \cdot, \cdot \rangle_\lambda \) with respect to \( \beta \) is defined as \( P_{i,j}(\lambda) = \langle e_i, e_j \rangle_\lambda \). Hence, identifying \( V \) with \( \mathbb{C}[\partial]^N \), we have
\[
\langle v(\partial), w(\partial) \rangle_\lambda = v^t(-\lambda) P(\lambda) w(\lambda). \quad (6.7)
\]
Observe that \( P^t(-x) = \epsilon P(x) \) with \( \epsilon = 1 \) (resp. \( \epsilon = -1 \)) if the conformal bilinear form is symmetric (resp. skew-symmetric). We also have that \( \text{Im} \ L = P(-\partial)V^* \), where \( L \) is defined in (6.6). Indeed, given \( v(\partial) \in V \), consider \( g_\lambda \in V^* \) defined by \( g_\lambda(w(\partial)) = v^t(-\lambda)w(\lambda) \), then by (6.7)
\[
(L_{v(\partial)})_\lambda w(\partial) = v^t(-\lambda) P(\lambda) w(\lambda) = g_\lambda(P(\partial)w(\partial)) = (P(-\partial)g)_\lambda(w(\partial)),
\]
where in the last equality we are identifying $V^*$ with $\mathbb{C}[\partial]^N$ in the natural way, that is $f \in V^*$ corresponds to $(f_{-\partial e_1}, \cdots, f_{-\partial e_N}) \in \mathbb{C}[\partial]^N$. Therefore, if the conformal bilinear form is non-degenerate, then $L$ gives an isomorphism between $V$ and $P(-\partial)V^*$, with det $P \neq 0$.

Suppose that we have a non-degenerate conformal bilinear form on $V = \mathbb{C}[\partial]^N$ which is also symmetric or skew-symmetric. Denote by $P(\lambda)$ the matrix of this bilinear form with respect to the standard basis of $\mathbb{C}[\partial]^N$. Then for each $a \in \text{Cend}_N$ and $w \in V$, the map $f^{a,w}_\lambda(v) := \langle w, a \mu \rangle_{\lambda - \mu}$ is in $\mathbb{C}[\mu] \otimes V^*$, that is $f^{a,w}_\lambda$ is a $\mathbb{C}$-linear map, $f^{a,w}_\lambda(\partial v) = \lambda f^{a,w}_\lambda(v)$ and depends polynomially on $\mu$, because $\deg_\mu f^{a,w}_\lambda(v) \leq \max\{\deg_\mu f^{a,w}_\lambda(e_i) : i = 1, \cdots, N\}$. Observe that if we restrict to $\text{Cend}_{N,P}$, then $f^{a,P,w}_\lambda = (P(-\partial)f^{a,w}_\lambda) \in \text{Im} \, L$. Therefore, since $\langle , \rangle_\lambda$ is non-degenerate, there exists a unique $(aP)_{\mu}^*w \in \mathbb{C}[\mu] \otimes V$ such that $f^{aP,w}_\lambda(v) = \langle w, aP_{\mu}^*w \rangle_{\lambda - \mu} = \langle (aP)_{\mu}^*w, v \rangle_{\lambda}$. Thus, we have attached to each $aP \in \text{Cend}_{N,P}$ a map $(aP)^*: V \rightarrow \mathbb{C}[\mu] \otimes V, w \mapsto (aP)^*_\mu w$, where the vector $(aP)^*_\mu w$ is determined by the identity

$$\langle aP_{\mu}^*w, v \rangle_{\lambda} = \langle v, (aP)^*_\mu w \rangle_{\lambda - \mu}.$$ 

Observe that $(aP)^*_\mu(\partial w) = (\partial + \mu)(aP)^*_\mu w$, that is $(aP)^* \in \text{Cend}_N$. Indeed,

$$\langle v, (aP)^*_\mu(\partial w) \rangle_{\lambda - \mu} = \langle aP_{\mu}^*w, \partial w \rangle_{\lambda} = \lambda \langle aP_{\mu}^*w, w \rangle_{\lambda}$$

$$= -(\langle (aP)^*_\mu w, \partial v \rangle_{\lambda} - \langle aP_{\mu}^*w, \partial v \rangle_{\lambda})$$

$$= \mu \langle v, (aP)^*_\mu w \rangle_{\lambda - \mu} + (\partial v, (aP)^*_\mu w)_{\lambda - \mu}$$

Moreover we have the following result:

**Proposition 6.8.** (a) Let $\langle , \rangle_\lambda$ be a non-degenerate symmetric or skew-symmetric conformal bilinear form on $\mathbb{C}[\partial]^N$, and denote by $P(\lambda)$ the matrix of $\langle , \rangle_\lambda$ with respect to the standard basis of $\mathbb{C}[\partial]^N$ over $\mathbb{C}[\partial]$. Then the map $aP \mapsto (aP)^*$ from $\text{Cend}_{N,P}$ to $\text{Cend}_N$ defined by

$$\langle a_{\mu}v, w \rangle_{\lambda} = \langle v, a_{\mu}^*w \rangle_{\lambda - \mu}. \quad (6.9)$$

is the anti-involution of $\text{Cend}_{N,P}$ given by

$$(a(\partial, x)P(x))^* = \epsilon a^4(\partial, -\partial - x)P(x), \quad (6.10)$$

where $P^*(-x) = \epsilon P(x)$ with $\epsilon = 1$ or $-1$, depending on whether the conformal bilinear form is symmetric or skew-symmetric.

(b) Consider the Lie conformal subalgebra of $g_{C_N}$ defined by

$$g_\ast = \{ a \in \text{Cend}_{N,P} : a^* = -a \}$$

$$= \{ a \in \text{Cend}_{N,P} : \langle a_{\mu}v, w \rangle_{\lambda} + \langle v, a_{\mu}w \rangle_{\lambda - \mu} = 0, \text{ for all } v, w \in \mathbb{C}[\partial]^N \},$$
where * is defined by (6.10). Then under the pairing (6.6) we have \( \mathbb{C}[\partial]^N \simeq P(-\partial)(\mathbb{C}[\partial]^N)^* \) as \( g_\ast \)-modules.

**Proof.** (a) First let us check that \( \varphi(aP) = (aP)^* \) defines an anti-homomorphism from \( \text{Cend}_{N,P} \) to \( \text{Cend}_N \). Since \( (a, b \in \text{Cend}_{N,P}) \)

\[
(v, (a_\mu b_\gamma^* w)\lambda - \gamma) = \langle (a_\mu b_\gamma^* w)\lambda - \gamma \rangle_v = \langle a_\mu b_\gamma^* (w)\lambda - \gamma \rangle_v = \langle b_\gamma^* (w)\lambda - \mu \rangle_v = \langle v, b_\gamma^* (a_\mu w)\lambda - \mu \rangle_v
\]

we have that \( \varphi(a_\mu b_\gamma) = (\varphi(b_\gamma^* \varphi(a_\mu)\gamma) = (\varphi(b_\gamma^* \varphi(a_\mu)\gamma).

Now, using Theorem 4.6(b), we have that\( (a_\mu b_\gamma) = (\varphi(b_\gamma^* \varphi(a_\mu)\gamma), \varphi(b_\gamma^* \varphi(a_\mu)\gamma). \)

with \( \alpha \in \mathbb{C} \) and \( P^t(-x + \alpha) = B(x)A(x) \). Replacing \( \varphi(aP) \) in (6.9) and using (6.7), we obtain\n
\[
P(\lambda - \mu) a^t(-\mu, \mu - \lambda) P(\lambda) = P(\lambda - \mu) A(\lambda - \mu) a^t(-\mu, \mu - \lambda + \alpha) B(\lambda), \text{ for all } a(\partial, x).
\]

(6.11)

Taking \( a(\partial, x) = I \) and using that \( \det P \neq 0 \), we have \( P(\lambda) = A(\lambda - \mu) B(\lambda) \). Since the left hand side does not depend on \( \mu \), we get \( A = A(x) \in \text{Mat}_N \mathbb{C} \), with \( \det A \neq 0 \).

Using that \( \epsilon P(x - \alpha) = P^t(-x + \alpha) = B(x)A(x) \), then (6.11) become\n
\[
a^t(-\mu, \mu - \lambda) \epsilon B(\lambda + \alpha) A = A a^t(-\mu, \mu - \lambda + \alpha) B(\lambda), \text{ for all } a(\partial, x).
\]

In particular, we have \( \epsilon B(\lambda + \alpha) A = AB(\lambda) \). Hence \( a^t(-\mu, \mu - \lambda) A = A a^t(-\mu, \mu - \lambda + \alpha) B(\lambda) \) for all \( a(\partial, x) \), getting \( \alpha = 0 \) and \( A = cI \). Therefore,

\[
\varphi(a(\partial, x) P(x)) = \epsilon a^t(\partial, -\partial - x) P(x),
\]

with \( P^t(-x) = \epsilon P(x) \) with \( \epsilon = 1 \) or \( -1 \), depending on whether the conformal bilinear form is symmetric or skew-symmetric, getting (a).

(b) Using (6.6), we obtain for all \( a \in g_\ast \) and \( v, w \in \mathbb{C}[\partial]^N \) that

\[
(L_{a_\mu v})_\lambda \lambda = \langle a_\mu v, w\rangle_\lambda = -\langle v, a_\mu w\rangle_\lambda - (L_v)_\lambda - (L_v)_\lambda (a_\mu w) = (a_\mu (L_v))_\lambda (w).
\]

finishing the proof. \( \square \)

Observe that \( \text{oc}_N \) (resp. \( \text{spc}_N \)), can be described as the subalgebra \( g_\ast \) of \( g_\ast \) in Theorem 6.8(b), with respect to the conformal bilinear form

\[
\langle p(\partial) v, q(\partial) w\rangle_\lambda = p(-\lambda) q(\lambda) \langle v, w\rangle \text{ for all } v, w \in \mathbb{C}^N,
\]
where \((\cdot, \cdot)\) is a non-degenerate symmetric (resp. skew-symmetric) bilinear form on \(\mathbb{C}^N\). For general \(P\), see (6.16) below.

Then, \(oc_N\) (resp. \(spc_N\)) is the \(\mathbb{C}[\partial]\)-span of \(\{y_A^n := x^n A - (-\partial - x)^n A^* : A \in \text{Mat}_N\mathbb{C}\}\), where * stands for the adjoint with respect to a non-degenerate symmetric (resp. skew-symmetric) bilinear form over \(\mathbb{C}\). Therefore we have that \(gc_N = oc_N \oplus M_N\) (resp. \(gc_N = spc_N \oplus M_N\)), where \(M_N\) is the set of \(\sigma\)-fixed points, i.e.

\[
M_N = \mathbb{C}[\partial]\text{-span of } \{w_A^n := x^n A + (-\partial - x)^n A^* : A \in \text{Mat}_N\mathbb{C}\}. \tag{6.12}
\]

We are using the same notation \(M_N\) in the symmetric and skew-symmetric case. Observe that \(M_N\) is an \(oc_N\)-module (resp. \(spc_N\)-module) with the action given by

\[
y_A^n \cdot w_B^m = (\lambda + \partial + w_{AB})^n w_{AB}^m - (-\partial - w_{A\cdot B})^n w_{A\cdot B}^m
+ (-1)^n(-\lambda - \partial - w_{AB})^{m+n} - (-\lambda + w_{BA})^m w_{BA}^n \tag{6.13}
\]

Let us give a more conceptual understanding of the module \(M_N\). Let \(V = \mathbb{C}[\partial]^N\). By definition, \(V^* = \text{Chom}(V, \mathbb{C}) = \{\alpha : \mathbb{C}[\partial]^N \rightarrow \mathbb{C}[\lambda] : \alpha \lambda \partial = \lambda \alpha \lambda\} \) and given \(\alpha \in V^*\) it is completely determined by the values in the canonical basis \(\{e_i\}\) of \(\mathbb{C}^N\), this is \(p_{\alpha}(\lambda) := (\alpha e_1, \cdots, \alpha e_N) \in \mathbb{C}[\lambda]^N\). Thus, we may identify \(V^* \simeq \mathbb{C}[\lambda]^N\) and \(\mathbb{C}[\partial]\)-module structure given by \((\partial p)(\lambda) = -\lambda p(\lambda)\).

We have that \(gc_N\) acts on \(V\) by the \(\lambda\)-action

\[
A(\partial, x) \lambda v(\partial) = A(-\lambda, \lambda + \partial)v(\lambda + \partial), \quad v(\partial) \in \mathbb{C}[\partial]^N,
\]

and on \(V^*\) by the contragradient action, given by

\[
A(\partial, x) \lambda v(\partial) = -^tA(-\lambda, -\partial)v(\lambda + \partial), \quad v(\partial) \in \mathbb{C}[\partial]^N.
\]

It is easy to check that \((V^*)^* \simeq V\) as \(gc_N\)-modules. Observe that by Proposition 6.8(b), \(V \simeq V^*\) as \(oc_N\)-modules and \(spc_N\)-modules.

We define the 2nd exterior power \(\Lambda^2(V)\) and the 2nd symmetric power \(S^2(V)\) in the usual way with the induced \(\mathbb{C}[\partial]\)-module and \(gc_N\)-module structures.

**Proposition 6.14.** (a) \(V \otimes V = S^2(V) \oplus \Lambda^2(V)\) is the decomposition of \(V \otimes V\) into a direct sum of irreducible \(gc_N\)-modules. And \(V^* \otimes V\) is isomorphic to the adjoint representation of \(gc_N\).

(b) \(gc_N \simeq V \otimes V = S^2(V) \oplus \Lambda^2(V)\) is the decomposition of \(gc_N\) into a direct sum of irreducible \(oc_N\)-modules, where \(\Lambda^2(V)\) is isomorphic to the adjoint representation of \(oc_N\), and \(M_N \simeq S^2(V)\) as \(oc_N\)-modules.

(c) \(gc_N \simeq V \otimes V = S^2(V) \oplus \Lambda^2(V)\) is the decomposition of \(gc_N\) into a direct sum of irreducible \(spc_N\)-modules, where \(S^2(V)\) is isomorphic to the adjoint representation of \(spc_N\), and \(M_N \simeq \Lambda^2(V)\) as \(spc_N\)-modules.

**Proof.** (a) Follows from Proposition 6.2 and part (b).
(b) Define \( \varphi : V \otimes V \to gc_N \) by

\[
\varphi(p(\partial)e_i \otimes q(\partial)e_j) = p(-x)q(x + \partial)E_{ji}
\]

It is easy to check that this is an \( oc_N \)-module isomorphism. Note that \( \sigma_* \) defined in (6.4) corresponds via \( \varphi \) to \( \sigma(p(\partial)e_i \otimes q(\partial)e_j) = q(\partial)e_j \otimes p(\partial)e_i \). Therefore it is immediate that \( M_N \simeq S^2(V) \) and \( \Lambda^2(V) \simeq oc_N \). It remains to see that \( M_N \) is an irreducible \( oc_N \)-module. Let \( W \neq 0 \) be a \( oc_N \)-submodule of \( M_N \) and \( 0 \neq w(\partial, x) = \sum_{i,j} q_{ij}(\partial, x)E_{ij} \in W \). We may suppose that \( q_{11} \neq 0 \). Computing \( [y_{E_{11}}^1 \lambda w(\partial, x)] \) and looking at the highest degree of \( \lambda \) that appears in the component \( E_{11} \), we deduce that there exists in \( W \) an element of the form \( w' = \sum_i (p_i(\partial, x)E_{1i} + q_i(\partial, x)E_{i1}) \), with \( p_1 = q_1 = 1 \). Now, computing \( [y_{E_{12}}^1 \lambda w''(\partial, x)] \) we have that \( w'' = r(\partial, x)E_{11} + w_{E_{12}}^1 + \text{terms out of the first column and row} \in W \). And from \( [y_{E_{11}}^1 \lambda w''(\partial, x)] \) and looking at the highest degree in \( \lambda \), we have that if \( r(\partial, x) \) is non-constant, \( w_{E_{11}}^0 \in W \), and if \( r(\partial, x) \) is constant, \( w_{E_{11}}^0 + w_{E_{12}}^1 \in W \). In both cases, by (6.6) we have that \( w_I^I \in W \). Now, looking at \( (n >> 0 \text{ and } A \text{ arbitrary}) \)

\[
y_{A}^n w_I^0 = \lambda^n 2w_A^0 + \lambda^{n-1}2n(\partial w_A^0 + w_A^1) + \lambda^{n-2}2\left(\frac{n}{2}\right)(\partial^2 w_A^0 + 2\partial w_A^1 + w_A^2) + \cdots
\]

we get \( W = M_N \), finishing part (b).

(c) The proof is similar to (b), with \( \varphi : V \otimes V \to gc_N \) defined by \( \varphi(p(\partial)e_i \otimes q(\partial)e_j) = p(-x)q(x + \partial)E_{ij}^\dagger \), where \( E_{ij}^\dagger = -E_{j,i}^{N+i,j} \). \( E_{ij}^{N+i,j}, E_{i,j}^{N+j,i} = E_{i,j}^\dagger, E_{i,j}^\dagger = -E_{j,i}^{N+j,i} \) and \( E_{i,j}^\dagger = -E_{j,i} \), for all \( 1 \leq i, j \leq N \).

Observe that \( gc_{N,P} := gc_NP(x) \) is a conformal subalgebra of \( gc_N \), for any \( P(x) \in \text{Mat}_N\mathbb{C}[x] \).

A matrix \( Q(x) \in \text{Mat}_N\mathbb{C}[x] \) will be called hermitian (resp. skew-hermitian) if

\[
Q^\dagger(-x) = \varepsilon Q(x) \quad \text{with } \varepsilon = 1 \quad (\text{resp. } \varepsilon = -1).
\]

Denote by \( o_{P,Y,\varepsilon,\alpha} \) the subalgebra of \( gc_{N,P} \) of \( -\sigma_{P,Y,\varepsilon,\alpha} \)-fixed points. By Proposition 4.21 (b)-(c), we have the following isomorphisms, obtained by conjugating by automorphisms of \( Cend_N \)

\[
o_{P,Y,\varepsilon,\alpha} \simeq o_{P,Y,\varepsilon,\alpha} \simeq o_{Q,Y,\varepsilon,\alpha}, \quad (6.15)
\]

where \( Q(x) = (PY)(x+\alpha/2) \) is hermitian or skew-hermitian, depending on whether \( \varepsilon = 1 \) or \(-1 \). Therefore, up to conjugacy, we may restrict our attention to the family of subalgebras (6.15), that is it suffices to consider the anti-involutions

\[
\sigma_{P,Y,\varepsilon,\alpha}(a(\partial, x)P(x)) = \varepsilon a^\dagger(\partial, -\partial - x)P(x)
\]
where $P$ is non-degenerate hermitian or skew-hermitian, depending on whether $\varepsilon = 1$ or $-1$. From now on we shall use the following notation
\[
\begin{align*}
oc_{N,P} & := o_{P,I,1,0} \quad \text{if } P \text{ is hermitian} \\
\spc_{N,P} & := o_{P,I,-1,0} \quad \text{if } P \text{ is skew-hermitian.}
\end{align*}
\tag{6.16}
\]

These subalgebras are those obtained in Theorem 6.8(b) in a more invariant form. In the special case $N = 1$ and $P(x) = x$, the involution $\sigma_{x,I,-1,0}$ is the conformal version of the involution given by Bloch in [B].

Note that $gc_{N,P} \simeq \noc_{N} \cdot P(x) \oplus M_{N} \cdot \bar{P}(x)$. If $P$ is hermitian, then $\noc_{N,P} = \noc_{N} \cdot P(x)$ and $M_{N} \cdot \bar{P}(x)$ is an $\noc_{N,P}$-module. If $P$ is skew-hermitian, then $\spc_{N,P} = M_{N} \cdot \bar{P}(x)$, and $\noc_{N} \cdot P(x)$ is a $\spc_{N,P}$-module.

Remark 6.17. (a) The subalgebras $gc_{N}$, $gc_{N,xI}$, $\noc_{N}$ and $\spc_{N,xI}$ contain the conformal Virasoro subalgebra $\C[\partial](x + \alpha\partial)I$, for $\alpha$ arbitrary, $\alpha = 0$, $\alpha = \frac{1}{2}$ and $\alpha = 0$ respectively.

(b) Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, then by (6.15) we obtain
\[
\spc_{N} = o_{I,J,-1,0} \simeq o_{J,I,-1,0} = \spc_{N,J}.
\]

(c) Using the proof of Proposition 6.5, one can prove that $\noc_{N,P}$ and $\spc_{N,P}$, with $\det P(x) \neq 0$, are simple if $P(x)$ satisfies the property that for each $i$ there exists $j$ such that $\deg P_{ij}(x) > \deg P_{ik}(x)$ for all $k \neq j$.

Proposition 6.18. The subalgebras $\noc_{N,P}$ and $\spc_{N,P}$, with $\det P(x) \neq 0$, acts irreducibly on $\C[\partial]^{N}$.

Proof. Let $M$ be a non-zero $\noc_{N,P}$-submodule of $\C[\partial]^{N}$ and take $0 \neq v(\partial) \in M$. Since $\det P(x) \neq 0$, there exists $i$ such that $P(y)v(y)$ has non-zero $i$th-coordinate that we shall denote by $b(y)$. Recall that $\{(x^{k}A - (\partial - x)^{k}A^{t})P(x) \mid A \in \text{Mat}_{N}\C\}$ generates $\noc_{N,P}$. Now, looking at the highest degree in $\lambda$ in
\[
(2x + \partial)E_{ii}P(x) \lambda v(\partial) = (\lambda + 2\partial)b(\partial + \lambda)\lambda i
\]
we deduce that $e_{i} \in M$. Now, since the $i$th-column of $P = (P_{r,j})$ is non-zero, we can take $k$ such that $P_{k,i}(x) \neq 0$ has maximal degree in $x$, in the $i$th-column. Then, considering the $\lambda$ action of $(xE_{jk} - (\partial - x)E_{kj})P(x)$ on $e_{i}$, for $j = 1, \ldots, N$, and looking at the highest degree in $\lambda$, we have that $e_{j} \in M$ for all $j = 1, \ldots, N$. Therefore $M = \C[\partial]^{N}$. A similar argument also works for $\spc_{N,P}$. \qed

Proposition 6.19. (a) The subalgebras $\noc_{N,P}$ and $\noc_{N,Q}$ (resp. $\spc_{N,P}$ and $\spc_{N,Q}$) are conjugated by an automorphism of $\text{Cend}_{N}$ if and only if $P$ and $Q$ are congruent hermitian (resp. skew-hermitian) matrices.
(b) The subalgebras $oc_{N,P}$ and $spc_{N,Q}$ are not conjugated by any automorphism of $Cend_N$.

Proof. By Theorem 4.2, any automorphism of $Cend_N$ has the form $\varphi_A(a(\partial,x)) = A(\partial + x)a(\partial, x + \alpha)A(x)^{-1}$, with $A(x)$ an invertible matrix in $\text{Mat}_N\mathbb{C}[x]$. Suppose that the restriction of $\varphi_A$ to $oc_{N,P}$ gives us an isomorphism between $oc_{N,P}$ and $oc_{N,Q}$. Then $\varphi_A(a(\partial,x)P(x)) = A(\partial + x)a(\partial, x + \alpha)D(x)Q(x)$ for all $a(\partial,x) \in oc_{N}$, where $D$ is an invertible matrix in $\text{Mat}_N\mathbb{C}[x]$ and $P(x + \alpha) = D(x)Q(x)A(x)$. But the image is in $oc_{N,Q}$ if and only if (applying $\sigma_{Q,I,1,0}$)

$$a(\partial,x - \alpha)R(x) = R^t(-\partial - x)a(\partial,x + \alpha)$$

for all $a(\partial,x) \in oc_{N}$,

where $R(x) = A^t(-x)D(x)^{-1}$. Therefore, we must have $\alpha = 0$ and $R = c \text{Id}$ ($c \in \mathbb{C}$), that is $D(x) = cA^t(-x)$. Hence $P(x) = cA^t(-x)Q(x)A(x)$, proving (a). Part (b) follows by similar arguments. □

A classification of finite irreducible subalgebras of $gc_N$ was given in [DK]. In view of the discussion of this section, it is natural to propose the following conjecture.

**Conjecture 6.20.** Any infinite Lie conformal subalgebra of $gc_N$ acting irreducibly on $\mathbb{C}[\partial]^N$ is conjugate by an automorphism of $Cend_N$ to one of the following subalgebras:

(a) $gc_{N,P}$, where $\det P \neq 0$,

(b) $oc_{N,P}$, where $\det P \neq 0$ and $P(-x) = P^t(x)$,

(c) $spc_{N,P}$, where $\det P \neq 0$ and $P(-x) = -P^t(x)$.

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