Quench-induced dynamical phase transitions and $\pi$-synchronization in the Bose-Hubbard Model

Andrea Pizzip, Fabrizio Dolcini, Karyn Le Hur

1 CPHT, Ecole Polytechnique, CNRS, Université Paris-Saclay, Route de Saclay, 91128 Palaiseau, France
2 Dipartimento di Scienza Applicata e Tecnologia, Politecnico di Torino, I-10129 Torino, Italy
3 T.C.M. Group, Cavendish Laboratory, University of Cambridge, Cambridge CB3 0HE, United Kingdom

By means of the discretized and nonlinear Gross-Pitaevskii equation, we investigate the non-equilibrium mean-field behavior of a fully-connected (or all-to-all coupled) Bose-Hubbard model, in the limit of large boson population and arbitrary number $V$ of lattice sites. By combining analytical and numerical methods, we focus on a Mott to superfluid quench and predict that, when the quench intensity is varied across a critical value, the system undergoes a dynamical phase transition. Furthermore, under appropriate conditions, the long-time behavior exhibits a relaxation to either a macroscopically self-trapped state, where particles populate the lattice inhomogeneously, or to a $\pi$-synchronized state, characterized by the trend of the mean-field bosonic variables to split into two groups with phase difference $\pi$. We show that the latter process is intimately connected to the presence, only for $V \geq 4$, of a manifold of infinitely many fixed points of the dynamical equations. Finally, in analogy with the well-known Kuramoto model for nonlinear coupled oscillators, we show that the introduction of a site-dependent disorder reduces such $\pi$-synchronization, in what we call a synchronization crossover. The model finds various potential experimental applications ranging from ultra-cold atomic gases in optical traps to systems of bosonic junctions.

I. INTRODUCTION

The theory of interacting many-body quantum systems at equilibrium has advanced remarkably over the past few decades, to account for various quantum phase transitions, i.e. sharp changes of the ground state of an Hamiltonian when its parameters are varied across some critical values. However, the behavior of such systems is far less understood when it comes to the out-of-equilibrium regime, whose relevance has rapidly grown triggered by significant experimental progress ranging from gases of ultra-cold neutral atoms in optical traps [1–8] to bosonic junctions [9–15]. One of the most established protocols to take these systems to the non-equilibrium regime is the quantum quench, consisting of a sudden change of the Hamiltonian of the system from $H_i$ to $H_f$ at time $t = 0$ [16–23]. Importantly, over the typical experimental timescales these systems are essentially isolated from the environment. In these conditions, one can observe the emergence of two particularly interesting effects. The first one is a quantum Dynamical Phase Transition (DPT), identified by a sharp change of the dynamical behavior for different quench strengths [21–24]. The second one, sometimes referred to as thermalization of an isolated system [25–33], consists in the relaxation of some macroscopic variables, namely the Dynamical Order Parameters (DOPs), to some finite values at long-times, which occurs despite the absence of a thermal bath and a reservoir.

A well-established approach to the study of the dynamics of a large population of interacting bosons on a lattice consists of reducing the Heisenberg equation of motion to the discrete nonlinear Gross-Pitaevskii Equation (GPE) via a Mean-Field (MF) substitution of the $2V$ bosonic creation and annihilation operators $a_j^\dagger$ and $a_j$ ($j = 1, 2, \ldots, V$ labeling the lattice site) with the $2V$ C-numbers $\psi_j$ and $\bar{\psi}_j$ [15, 34–40]. Looking at the time evolution of the phases $\{\theta_j\}_{j=1,\ldots,V}$ of the C-numbers $\{\psi_j\}_{j=1,\ldots,V}$, the system can be regarded as a system of $V$ classical nonlinearly coupled oscillators, making thus natural to wonder about the occurrence of synchronization phenomena. Indeed, Witthaut et al. recently demonstrated that a particular class of bosonic models can in this way be recasted to the Kuramoto model for classically coupled nonlinear oscillators, that is a notorious model revealing a synchronization transition driven by the competition between coupling and disorder [40–43]. Importantly, since the considered systems are (almost) isolated, the emergence of synchronization is not due to any dissipation or external driving, as usually considered for populations of quantum oscillators [44–51].

In this context, paradigmatic is the Bose-Hubbard (BH) model, which describes a system of bosons on a lattice with site-to-site tunneling and on-site interaction, and that at equilibrium exhibits a quantum phase transition between a SuperFluid (SF) and a Mott Insulator (MI) [52–56]. Such a transition finds various applications in physics [57] from ultra-cold atoms in optical lattices [2] to systems of Josephson junctions [58–60]. For these systems, disorder is known to lead to glassy phases and Anderson localization [52, 61–64], while recently the phenomenon of many-body localization has also been analyzed [14]. In the non-equilibrium regime, using an exact approach Sciolla and Biroli spotted out the existence of a DPT for a Fully-Connected (FC) (or all-to-all coupled) lattice in the limit of infinite site number ($V \to \infty$) for
small occupation numbers (that is few bosons per site) [21–23]. In the opposite MF limit of large boson densities, Polkovnikov et al. studied the GPE associated to the BH model on a one-dimensional lattice [35], whereas other groups focused on the study of bosonic dimers and trimers (that is $V = 2, 3$) revealing peculiar dynamical features such as chaos and Macroscopic Quantum Self-Trapping (MQST), that is a symmetry breaking leading to non-zero average population imbalance [9, 15, 34, 36–38, 65–70].

Here, we address the GPE equation after a MI to SF quench for a FC lattice of generic number of sites $V$ (with emphasis on the special case of $V \geq 4$) and for a large number of bosons per lattice site. Remarkably, we reveal at short-times the existence of a DPT and at long-times the number of bosons per lattice site. Remarkably, we reveal emphasis on the special case of $V \geq 4$ for a FC lattice of generic number of sites.

To describe the model, we start by deriving the Dynamical Equations (DEs) of a system of interacting many-bosons on a FC lattice, that is a lattice where particles can hop from any site to any other site with same tunneling rate. The choice of a FC model is motivated by the analytical tractability allowed by its symmetries and by the fact that it represents an approximate description of a finite-dimensional system [21, 22]. By means of a MF approximation consisting of the substitution of the bosonic operators with $\mathbb{C}$-numbers, we are able to recast the Heisenberg equation of motion into a nonlinear and discrete GPE of motion for classical coupled oscillators of variable length and phase.

### A. Hamiltonian

The BH model on a FC lattice is characterized by the following Hamiltonian

$$H_{BH} = -\frac{J}{V} \sum_{i,j=1, i\neq j}^{V} a_{i}^\dagger a_{j} + \frac{u}{2} \sum_{j=1}^{V} n_{j}(n_{j}-1) - \mu \sum_{j=1}^{V} n_{j}, \quad (1)$$

where $V$ is the number of lattice sites, $a_{i}^\dagger$ and $a_{j}$ are respectively the bosonic creation and annihilation operators at site $j$, satisfying the bosonic commutation relation $[a_{i}, a_{j}^\dagger] = \delta_{i,j}$, $n_{j} = a_{j}^\dagger a_{j}$ is the number operator associated to the $j$-th site, $J$ is the hopping strength for tunneling between any two sites (rescaled of a factor $V$ to guarantee extensivity), $u$ is the energy scale of the on-site two-body repulsive interaction ($u > 0$) and $\mu$ the chemical potential setting the average number of particles in the system. We denote by $N$ the total number of particles and by $\rho_{0} = N/V$ the average number of particles per lattice site. Importantly, in order to consistently work within the MF approximation, we shall henceforth assume $\rho_{0} \gg 1$. Relevant for the determination of both the equilibrium and the non-equilibrium properties of the system is the following dimensionless parameter

$$\eta = \frac{J}{u\rho_{0}}. \quad (2)$$

It is well known that, varying $\eta$ across a critical value $\eta^{c}(\rho_{0}, \mu)$, the system undergoes an equilibrium phase transition between a SF and a MI, the former being characterized by long-range coherence and the latter by integer boson densities, existence of a gap for particle-hole excitation and zero compressibility, as described by the notorious lobe-shaped equilibrium phase diagram [52, 53]. Within MF, it can be easily shown (see details in appendix A) that at the top of the lobes the transition occurs at [52]
\[ \eta_{\text{eq}}^c = \frac{1}{\rho_0} + 2 - 2\sqrt{1 + \frac{1}{\rho_0} \approx \frac{1}{4\rho_0^2}}, \]  
where the last approximation yields in the limit of \( \rho_0 \gg 1 \), for which only a small interval \( 0 < \eta < \frac{1}{4\rho_0^2} \ll 1 \) will correspond to a MI ground state. As a consequence, switching \( \eta \) from 0 to a finite value \( \sim 1 \) at \( t = 0 \) corresponds in this limit to a MI to SF quench.

\[ \rho \]

\section*{B. Gross-Pitaevskii dynamical equations}

The dynamical equations for the bosonic creation and annihilation operators at site \( k = 1, 2, \ldots, V \) are readily obtained within the Heisenberg formalism as

\[ \frac{d a_k}{d(t i t)} = \frac{i}{\hbar} [H_{BH}, a_k], \]

\[ \frac{d a_k^\dagger}{d(t i t)} = \frac{i}{\hbar} [H_{BH}, a_k^\dagger]. \]  

We shall set \( \hbar = 1 \) from now on. With straightforward calculations (detailed in appendix B 1) we rewrite (4) as

\[ \frac{d a_k}{d t} = -\frac{J}{V} \sum_{j=1}^{V} a_j - u n_k a_k, \]

\[ \frac{d a_k^\dagger}{d t} = -\frac{J}{V} \sum_{j=1}^{V} a_j^\dagger + u a_k^\dagger n_k. \]  

At the MF level, for a large number of bosons per lattice site (\( \rho_0 \gg 1 \)) and in the SF regime (\( \eta > \eta_{\text{eq}}^c \) with \( \eta_{\text{eq}}^c \ll 1 \) as explained above), a well-established approximation to approach the DEs (5) reads \[34, 35\]

\[ \langle n_k a_k \rangle \approx |\langle a_k \rangle|^2 (a_k), \]

\[ \langle n_k \rangle \approx |\langle a_k \rangle| (a_k) = |\langle a_k \rangle|^2, \]  

holding for all sites \( k = 1, 2, \ldots, V \) and where \( \langle \bullet \rangle \) denotes the expectation value. In general the system is not in an eigenstate of the total number of particles operator \( \sum_{j=1}^{V} n_j \), and the expectation values \( \langle a_k^\dagger \rangle \) and \( \langle a_k \rangle \) are non vanishing. We denote

\[ \langle a_k \rangle = \psi_k = \sqrt{\rho_k} e^{i \theta_k}, \]  

that are the discrete version of the nonlinear GPE. As detailed in appendix B 2, from equation (8) we can derive DEs for \( \rho_k \) and \( \theta_k \), reading

\[ \frac{d \rho_k}{d t} = \frac{J}{V} \sum_{j=1}^{V} \sqrt{\rho_j} \rho_k \sin (\theta_k - \theta_j) \]

\[ \frac{d \theta_k}{d t} = \frac{J}{V} \sum_{j=1}^{V} \sqrt{\rho_j} \cos (\theta_k - \theta_j) - u \rho_k. \]  

We define the following complex DOP

\[ \Psi = re^{i \phi} = \frac{1}{V} \sum_{j=1}^{V} \sqrt{\rho_j} e^{i \theta_j}, \]  

whose modulus and phase are respectively denoted \( r \) and \( \phi \). Similarly to what is commonly done in the Kuramoto model for classical coupled oscillators [43], considering the real and the imaginary part of \( re^{i(\phi-\theta_k)} = \frac{1}{V} \sum_{j=1}^{V} \sqrt{\rho_j} e^{i(\theta_j-\theta_k)} \), we readily find

\[ r \cos (\phi - \theta_k) = \frac{1}{V} \sum_{j=1}^{V} \sqrt{\rho_j} \cos (\theta_j - \theta_k), \]

\[ r \sin (\phi - \theta_k) = \frac{1}{V} \sum_{j=1}^{V} \sqrt{\rho_j} \sin (\theta_j - \theta_k), \]  

so that Eq. (9) can be compactly rewritten as

\[ \begin{cases} 
\frac{d \sqrt{\rho_k}}{d t} = J r \sin (\theta_k - \phi) \\
\frac{d \theta_k}{d t} = \frac{J}{\sqrt{\rho_k}} \cos (\theta_k - \phi) - u \rho_k,
\end{cases} \]  

where we stress that \( r \) and \( \phi \) are in general time-dependent, evolving consistently with all the variables \( \{\rho_j, \theta_j\}_{j=1, \ldots, V} \), accordingly to Eq. (10). Interestingly, writing \( \rho_0 = \rho_0 = \frac{1}{V} \sum_{j=1}^{V} \rho_j \) and summing over \( k \) the first equation of (12) we get \( \frac{d \rho_0}{d t} = 0 \), that is we find the MF average number of particles per lattice site (or, equivalently, the MF total number of particles) to be a conserved quantity of the GPE, consistently with the isolation of the system on the considered timescales.

Expressing the time \( t \) in units of \( \frac{1}{u \rho_0} \) and \( \rho_k \) in units of \( \rho_0 \), Eq. (12) is rewritten as

\[ \begin{cases} 
\frac{d \sqrt{\rho_k}}{d t} = \eta r \sin (\theta_k - \phi) \\
\frac{d \theta_k}{d t} = \frac{\eta}{\sqrt{\rho_k}} r \cos (\theta_k - \phi) - \rho_k,
\end{cases} \]  

where we recall \( \eta = \frac{J}{u \rho_0} \) (\( \eta > 0 \) since we consider a repulsive on-site two-body interaction). Importantly, expressing \( \rho_k \) in units of \( \rho_0 \), the average of \( \rho_k \) over the sites
is renormalized to 1, that is $\rho_0 = \frac{1}{V} \sum_{j=1}^{V} \rho_j = \text{cst} = 1$. Similarly, $r$ will assume values in $(0,1)$. The GPE of motion (13) consists of a system of ordinary differential equations for the $2V$ real variables $\{ \sqrt{r} e^{i \theta_j} \}_{k=1,\ldots,V}$.

We call configuration the $2V$-dimensional set of variables $\{ \sqrt{r} e^{i \theta_j} \}_{k=1,\ldots,V}$ associated to the state of the system and phase space the $2V$-dimensional space $\Phi = [0,2\pi]^V \times (\mathbb{R}^+)^V$ in which the configurations live. Finally, we observe that the MF approximation (6) corresponds to considering a classical Hamiltonian

$$H_{\text{CL}} = V \left( -\pi r^2 + \frac{\eta}{2} (\rho_j^2) \right),$$

where we denote $\langle \bullet \rangle_j = \frac{1}{V} \sum_{j=1}^{V} \bullet_j$ the average over the sites of the site-dependent quantity $\bullet_j$ (not to be confused with the quantum expectation value).

C. System initialization: the quench procedure

To study the system dynamics in the non-equilibrium regime, we adopt the prototypical quench procedure, consisting of a sudden change of the Hamiltonian at time $t = 0$ from $H_i$ to $H_f$ [16, 18–23]. Thanks to the high degree of isolation of the system achievable on the experimental timescales [72], this procedure enables one to investigate an almost-isolated system initialized to the ground state of the Hamiltonian $H_i$ and evolving under the Hamiltonian $H_f$ for $t > 0$. In order for the MF assumption (6) and the DEs (13) to hold, we assume a quench to the SF regime, that is we consider $\eta > 0$. (not to be finite)

To study the system dynamics in the non-equilibrium regime, we adopt the prototypical quench procedure, consisting of a sudden change of the Hamiltonian at time $t = 0$ from $H_i$ to $H_f$ [16, 18–23]. Thanks to the high degree of isolation of the system achievable on the experimental timescales [72], this procedure enables one to investigate an almost-isolated system initialized to the ground state of the Hamiltonian $H_i$ and evolving under the Hamiltonian $H_f$ for $t > 0$. In order for the MF assumption (6) and the DEs (13) to hold, we assume a quench to the SF regime, that is we consider $\eta > 0$. (not to be finite)

To study the system dynamics in the non-equilibrium regime, we adopt the prototypical quench procedure, consisting of a sudden change of the Hamiltonian at time $t = 0$ from $H_i$ to $H_f$ [16, 18–23]. Thanks to the high degree of isolation of the system achievable on the experimental timescales [72], this procedure enables one to investigate an almost-isolated system initialized to the ground state of the Hamiltonian $H_i$ and evolving under the Hamiltonian $H_f$ for $t > 0$. In order for the MF assumption (6) and the DEs (13) to hold, we assume a quench to the SF regime, that is we consider $\eta > 0$. (not to be finite)

To study the system dynamics in the non-equilibrium regime, we adopt the prototypical quench procedure, consisting of a sudden change of the Hamiltonian at time $t = 0$ from $H_i$ to $H_f$ [16, 18–23]. Thanks to the high degree of isolation of the system achievable on the experimental timescales [72], this procedure enables one to investigate an almost-isolated system initialized to the ground state of the Hamiltonian $H_i$ and evolving under the Hamiltonian $H_f$ for $t > 0$. In order for the MF assumption (6) and the DEs (13) to hold, we assume a quench to the SF regime, that is we consider $\eta > 0$. (not to be finite)

To study the system dynamics in the non-equilibrium regime, we adopt the prototypical quench procedure, consisting of a sudden change of the Hamiltonian at time $t = 0$ from $H_i$ to $H_f$ [16, 18–23]. Thanks to the high degree of isolation of the system achievable on the experimental timescales [72], this procedure enables one to investigate an almost-isolated system initialized to the ground state of the Hamiltonian $H_i$ and evolving under the Hamiltonian $H_f$ for $t > 0$. In order for the MF assumption (6) and the DEs (13) to hold, we assume a quench to the SF regime, that is we consider $\eta > 0$. (not to be finite)

To study the system dynamics in the non-equilibrium regime, we adopt the prototypical quench procedure, consisting of a sudden change of the Hamiltonian at time $t = 0$ from $H_i$ to $H_f$ [16, 18–23]. Thanks to the high degree of isolation of the system achievable on the experimental timescales [72], this procedure enables one to investigate an almost-isolated system initialized to the ground state of the Hamiltonian $H_i$ and evolving under the Hamiltonian $H_f$ for $t > 0$. In order for the MF assumption (6) and the DEs (13) to hold, we assume a quench to the SF regime, that is we consider $\eta > 0$. (not to be finite)

To study the system dynamics in the non-equilibrium regime, we adopt the prototypical quench procedure, consisting of a sudden change of the Hamiltonian at time $t = 0$ from $H_i$ to $H_f$ [16, 18–23]. Thanks to the high degree of isolation of the system achievable on the experimental timescales [72], this procedure enables one to investigate an almost-isolated system initialized to the ground state of the Hamiltonian $H_i$ and evolving under the Hamiltonian $H_f$ for $t > 0$. In order for the MF assumption (6) and the DEs (13) to hold, we assume a quench to the SF regime, that is we consider $\eta > 0$. (not to be finite)

To study the system dynamics in the non-equilibrium regime, we adopt the prototypical quench procedure, consisting of a sudden change of the Hamiltonian at time $t = 0$ from $H_i$ to $H_f$ [16, 18–23]. Thanks to the high degree of isolation of the system achievable on the experimental timescales [72], this procedure enables one to investigate an almost-isolated system initialized to the ground state of the Hamiltonian $H_i$ and evolving under the Hamiltonian $H_f$ for $t > 0$. In order for the MF assumption (6) and the DEs (13) to hold, we assume a quench to the SF regime, that is we consider $\eta > 0$. (not to be finite)

To study the system dynamics in the non-equilibrium regime, we adopt the prototypical quench procedure, consisting of a sudden change of the Hamiltonian at time $t = 0$ from $H_i$ to $H_f$ [16, 18–23]. Thanks to the high degree of isolation of the system achievable on the experimental timescales [72], this procedure enables one to investigate an almost-isolated system initialized to the ground state of the Hamiltonian $H_i$ and evolving under the Hamiltonian $H_f$ for $t > 0$. In order for the MF assumption (6) and the DEs (13) to hold, we assume a quench to the SF regime, that is we consider $\eta > 0$. (not to be finite)}
dynamical behavior with respect to the widely studied bosonic dimer ($V = 2$) and trimer ($V = 3$) \cite{34, 36–38, 65–69}.

To find the FPs we conveniently adopt a definition for stationarity that allows a global phase rotation, that is a rotation common to all the phases $\theta_1, \theta_2, \ldots, \theta_V$ at some constant rate $\Omega$ (that can in fact always be removed with a proper gauge transformation, as shown in appendix B1). Therefore, a configuration that fulfills the following stationarity conditions for all $k = 1, 2, \ldots, V$

$$\frac{d\sqrt{r_k}}{dt} = \eta r \sin (\theta_k - \phi) = 0 \, ,$$
$$\frac{d\theta_k}{dt} = \frac{\eta r}{\sqrt{r_k}} \cos (\theta_k - \phi) - \rho_k = \Omega \, ,$$

shall be called a FP of the DEs (13). Clearly, it follows from Eq. (18) that the phase $\phi$ of the DOP $\Psi$ evolves for a FP as $\phi(t) = \phi(0) + \Omega t$. With a suitable choice of the reference frame we set $\phi(0) = 0$. Additionally, at $t = 0$ the following two consistency conditions must hold

$$\frac{1}{V} \sum_{j=1}^{V} \sqrt{r_j} e^{i\theta_j} = r \, ,$$
$$\frac{1}{V} \sum_{j=1}^{V} \rho_j = 1 \, .$$

From Eq. (17) we find that only two kinds of FPs are possible: the ones with $\sin(\theta_k) = 0 \forall k = 1, 2, \ldots, V$ and the ones with $r = 0$. We address these 2 classes of FPs separately. In the following, all the relevant FPs are defined up to a site permutation (as natural for a FC model) and a global phase rotation.

A. Superfluid and $\pi$-aligned configurations

The first class of FPs is characterized by $\theta_k \in \{0, \pi\} \forall k = 1, 2, \ldots, V$. The most trivial FP with such property is the SuperFluid Configuration (SFC), with the same number $\rho_0$ of bosons per lattice site and with all the phases aligned, that is

$$\begin{cases}
\rho_k = 1 & \forall k = 1, 2, \ldots, V \\
\theta_k = 0
\end{cases}$$

for which we get $r = 1$ and $\Omega = \eta - 1$. A graphical representation of the SFC is shown in Fig. 2(f). Notice that, maximizing $r$, the SFC is the ground state of the MF Hamiltonian (14), that is in fact obtained assuming to be in the SF regime for $t > 0$. However, being interested in the non-equilibrium regime, the system will in general be initialized far from the SFC and will not reach it because of the system isolation, that is energy and particle number conservation, on the considered timescales.

For an even $V$, a second relevant FP in this class is the one for which the sites have the same number of bosons $\rho_0$ but are divided into two equal groups with phases 0 and $\pi$ respectively, reading

$$\begin{cases}
\rho_k = 1 & \forall k = 1, 2, \ldots, V/2 \\
\theta_k = 0 & \forall k = V/2 + 1, \ldots, V/2+1, \ldots, V \\
\theta_k = \pi & \forall k = V/2 + 1, \ldots, V, \\
\sum_{j=1}^{V} e^{i\theta_j} = 0
\end{cases}$$

for which we get $r = 0$ and $\Omega = -1$, that we call Symmetric $\pi$-Aligned Configuration (SPAC) and whose graphical representation is shown in Fig. 2(d). From an experimental perspective, it is possible to initialize a cold-atoms system to the SPAC applying short pulses to the condensate \cite{35}.

Other possible FPs in this class have a fraction $\alpha \in \{0, \frac{1}{V}, \frac{2}{V}, \ldots, \frac{V-1}{V}\}$ of sites with phase 0 and the remaining fraction $1 - \alpha$ with phase $\pi$ and will generically be referred to as $\pi$-Aligned Configurations (PACs) (one example is shown in Fig. 2(e)). The relevance for our study of the PACs other then the SPAC is limited but still, for completeness, we report one example in appendix C.

B. $r = 0$ configurations

The second class of FPs is characterized by $r = 0$, for which the stationarity condition (18) reads $\rho_k = -\Omega \forall k$. Consequently, the conditions (19) and (20) read

$$\begin{cases}
\rho_k = 1 & \forall k = 1, 2, \ldots, V \\
\sum_{j=1}^{V} e^{i\theta_j} = 0
\end{cases}$$

Eq. (23) has in general several solutions (namely infinite if and only if $V \geq 4$). Of course, the aforementioned SPAC is one of them, in fact being the only FP satisfying at the same time $r = 0$ and $\theta_k - \theta_j \in \{0, \pi\} \forall k, j$.

Particularly relevant is then the configuration defined for $V \geq 3$ by

$$\begin{cases}
\rho_k = 1 & \forall k = 1, 2, \ldots, V/2 \\
\theta_k = \frac{2\pi k}{V} & \forall k = 1, 2, \ldots, V
\end{cases}$$

that we call Uniform Configuration (UC), where the word uniform is used to stress the uniform spacing $2\pi/V$ of the phases. A graphical representation of the UC is shown in Fig. 2(a).

Importantly, we observe that, in the infinite dimensional limit ($V \to \infty$) and for a proper permutation of the sites, the MI (15) and the UC (24) coincide, since a number $V \to \infty$ of uniform random phases in $(0, 2\pi)$ is equivalent to $V$ equispaced phases over the same interval. For a large but finite $V \gg 1$, random noise will instead make a generic MI configuration (15) different from the UC (24) but close to it. This observation crucially reflects
Phase space
Fixed points
line of $r=0$ FPs
SFC (f)
SPAC (d)
UC PACs (e)
(g)
(a) UC (b) (c) (d) SPAC
(e) PAC
(f) SFC

FIG. 2: (color online) Schematic representation of some representative FPs of the GPE (13) for an even $V \geq 4$. For $V = 20$ sites we show the UC (a), two other FPs with $r = 0$ (b,c), the SPAC (d), one PAC (e) and the SFC (f). The circular arrows recall that the phases of a FP are in general rotating at some constant rate $\Omega$. (g) The $r = 0$ FPs constitute a manifold in the phase space that ranges from the UC to the SPAC. Notice that also for an odd $V \geq 5$ there is an analogue manifold of infinitely many $r = 0$ FPs, just lacking of the SPAC.

into the fact that for a MI to SF quench and $V \gg 1$, the system is initialized in the proximity of the UC.

For $V \geq 4$ the condition (23) is satisfied by an infinity of FPs (e.g. the ones shown in Fig. 2(b,c) for $V = 20$), constituting a $(V - 3)$-dimensional manifold in the phase space and of which the UC and (if $V$ is even) the SPAC are part, as schematically shown in Fig. 2(g). For $V = 4$ such manifold is a line, and can be represented parametrically by the following FP

\[
\begin{align*}
\rho_k &= 1 \quad \forall \ k = 1, 2, 3, 4 \\
\theta_1 &= +\frac{\pi - \Delta}{2} \\
\theta_2 &= -\frac{\pi - \Delta}{2} \\
\theta_3 &= +\frac{\pi + \Delta}{2} + \pi \\
\theta_4 &= -\frac{\pi + \Delta}{2} + \pi ,
\end{align*}
\]

(25)

that we call Delta Configuration (DC) as it depends on the parameter $\Delta$ and that ranges continuously from $\pi/2$ (corresponding to the UC) to $\pi$ (corresponding to the SPAC). The importance of the DC lies in the fact that it enables us to carry on analytical calculations along the manifold of the $r = 0$ FPs for $V = 4$, with generalizations to $V > 4$, for which we instead focus on the UC and the SPAC only.

An immediate consequence is that, being $r = 1$ the largest possible $r$, a system initialized in the proximity

IV. EFFECTS OF CONSERVED QUANTITIES ON THE NON-EQUILIBRIUM DYNAMICS

On the experimentally relevant timescales our system can be considered isolated [4, 25, 27]. In general, a distinctive feature of the non-equilibrium dynamics of an isolated system is the presence of conserved quantities. The conservation of the energy is for instance preventing the motion of the system from the ground state of the Hamiltonian $H_i$ (preceding the quench) to the ground state of the Hamiltonian $H_f$ (following the quench). Here we show that relevant information on the non-equilibrium dynamics of the system can be obtained from the conservation of the MF energy (14) and of the average number of particles per lattice site $\rho_0$, reading respectively

\[
\begin{align*}
-\tau r^2 + \frac{1}{2} \langle \rho_j^2 \rangle_j &= E = \text{cst} , \\
\langle \rho_j \rangle_j &= \rho_0 = 1 ,
\end{align*}
\]

(26) and (27)

where $E$ is a constant depending on the IC. Writing $\rho_j$ as $\rho_j = \rho_0 + \delta_j$, the condition on the total number of particles reads $\langle \delta_j \rangle_j = 0$. Furthermore, in the particularly interesting case of a system initialized with $\rho_j = \rho_0 \ \forall \ j = 1, 2, \ldots, V$, denoting $r_0 = r(t = 0)$, we get at initial time $t = 0$ that $\langle \delta_j^2 \rangle_j = \langle \delta_j^2 \rangle_j = 0$ and $E = \frac{1}{2} - \tau r_0^2$, so that Eqs. (26) and (27) reduce to

\[
\begin{align*}
\langle \delta_j^2 \rangle_j &= 2\tau(r^2 - r_0^2) , \\
\langle \delta_j \rangle_j &= 0 ,
\end{align*}
\]

(28) and (29)

conveying important information on the system non-equilibrium dynamics. First, since the LHS of Eq. (28) is positive definite, for all times $t > 0$ we have

\[
r(t) \geq r_0 .
\]

(30)

An immediate consequence is that, being $r = 1$ the largest possible $r$, a system initialized in the proximity
of the SFC (that is with \( r \approx 1 \)) will remain in the neighborhood of the SFC (namely with \( r \approx 1 \)). The second implication of Eq. (28) is that an increase of \( r \) must be accompanied by a spread of the \( \{ \rho_j \}_{j=1,\ldots,V} \) around their mean value \( \rho_0 = 1 \) [as happening in Fig. 1(a)], that is larger for larger growths of \( r \). The goal of the next section is to unveil the conditions under which such growth of \( r \) occurs. On the other hand, if \( r \) is initially small and if it remains small for all times \( t > 0 \), then the system can, a priori and consistently with the conservation laws, visit the neighborhoods of the whole manifold of \( r = 0 \) FPs, possibly enabling a relaxation of the system to a visit the neighborhoods of the whole manifold of \( r \) larger for larger growths of \( r \). The mean value \( \rho_0 \) will therefore be a direct measure of the population imbalance. Therefore, a finite \( \langle r \rangle \) at large-times will correspond to the emergence of MQST, that is to a symmetry breaking characterized by some sites being on average more populated than others.

V. SHORT-TIME BEHAVIOR: THE DYNAMICAL PHASE TRANSITION

Having found the FPs of the DEs, in this Section we analyze the dynamics of a system initialized in the proximity of a FP. To our knowledge, this investigation has been so far limited to \( V < 4 \) [37, 38, 66–69]. Here, after briefly reviewing the instructive \( V = 2 \) case, we extend it to all possible \( V \geq 4 \), thus covering also to the large dimensional limit \( V \gg 1 \). In the framework of The Dynamical Systems Theory, by means of a diagonalization of the Jacobian matrix \( J \) (not to be confused with the dimensional hopping strength) associated to the linearized version of the DEs (13) in the neighborhood of the most relevant FPs, we explore the non-equilibrium dynamics at short-times after the quench. Particularly, this is relevant for a SF to MI quench for \( V \gg 1 \), for which the system is initialized in the proximity of the UC (15) and for an initialization of the system to the SPAC (achievable in cold-atoms applying short pulses to the condensate [35]). Looking at the eigenvalues of \( J \), we find two regions of the parameter space corresponding to two qualitatively very different behaviours of the system in what can be called a dynamical phase transition.

We start by linearizing the DEs (13). To this purpose we introduce the \( 2V \)-dimensional column vector

\[
\vec{y} = (\theta_1, \theta_2, \ldots, \theta_V, x_1, x_2, \ldots, x_V)^T ,
\]

where we used \( x_j = \sqrt{\rho_j} \) and that describes the state of the system at the MF level. The Jacobian \( J \) associated to the DEs (13) is the \( 2V \times 2V \)-dimensional matrix with entries

\[
J_{j,k} = \frac{\partial}{\partial y_k} \left( \frac{dy_j}{dt} \right) , \quad j, k = 1, 2, \ldots, 2V .
\]

If the system is initialized to a state \( \vec{y}(0) \) in the proximity of a FP \( \vec{y}^{FP} \), the solution of the linearized DEs reads [73]

\[
\vec{y}(t) = \vec{y}^{FP} + e^{Jt}(\vec{y}(0) - \vec{y}^{FP}) ,
\]

where the Jacobian matrix \( J \) is evaluated in \( \vec{y}^{FP} \). From Eq. (33) it follows that the dynamics of a system is determined by the eigenvalues \( \{ \lambda_n \}_{n=1,\ldots,2V} \) of \( J \) [73]. For FPs with \( r = 0 \) (such as the UC, the SPAC and the DC) the latter turns out to read (see details in appendix E)

\[
J_{j,k} = -\eta V \sin(\theta_k - \theta_j) , \quad J_{j,k+V} = -\eta V \sin(\theta_k - \theta_{j+1}) , \quad J_{j,k+V} = -\eta V \cos(\theta_k - \theta_{j+1}) , \quad J_{j,k+V} = +\eta V \cos(\theta_k - \theta_{j+1}) - 2\delta_{k,j} ,
\]

where the indexes \( j, k \) run over \( \{ 1, 2, \ldots, V \} \) and where the phases \( \{ \theta_j \}_{j=1,\ldots,V} \) are the ones associated to the considered FP. From (34) we readily find that \( \text{Tr}\{J\} = 0 \), meaning that the real parts of the Jacobian eigenvalues cannot be all positive or all negative, that is indeed not surprising for a conservative dynamical system. Rather, depending on the considered FP and on \( \eta \), only the following two situations are possible

- All the eigenvalues \( \{ \lambda_n \}_{n=1,\ldots,2V} \) are purely imaginary (possibly 0), that is the FP is a linear center of the dynamics: the solution of the linearized equations is a state moving periodically and close by the FP when initialized in its proximity;
- Some eigenvalues have positive real part and some others have negative real part, that is the FP is a saddle of the dynamics: the solution of the linearized equations is a state moving exponentially fast apart from the FP when initialized in its surroundings (because of random noise on the IC we exclude the possibility of system initialization exactly along a linear combination of eigenvectors associated to the eigenvalues with negative real part only).

Importantly, the FP in the former case is termed a linear center, since the above arguments on the eigenvalues are exact only for the linearized DEs. A priori, a linear center is not necessarily a nonlinear center, that is the nonlinearities of the DEs can make the system eventually move away from the FP at long-times even if the latter
is a linear center [73]. Interestingly, for a conservative system, a linear center is as well a nonlinear center if it is an isolated FP (that is if it is not part of a continuum of FPs). Since if the FP is a saddle (linear center) the system will (will not) drift away exponentially fast from it, with some abuse of nomenclature we will often refer to it as stable (unstable) FP.

Exploiting the presence of conserved quantities, in Sec. IV we showed that a system initialized in the proximity of the SFC always orbits closely around it, meaning that the SFC is a nonlinear center of the dynamics for any value of η > 0. With the above argument we can thus deduce that the SFC is an isolated FP (as indeed found in Sec. III) and that the associated eigenvalues of J are purely imaginary (as explicitly verified for completeness in App. E). Instead, for a given FP with r = 0, it turns out that there exists a critical value ηcFP of the dimensionless hopping strength η such that the FP is a saddle for 0 < η < ηcFP and a linear center for η > ηcFP. This feature, known as bifurcation in Dynamical Systems Theory, leads to two qualitatively very different behaviors for a system initialized in the proximity of the considered FP (e.g. UC for a MI to SF quench for V ≫ 1) depending on η ≤ ηcFP, that is a DPT [30]. In this Section, we exactly diagonalize the Jacobian matrix J and find ηcFP for the DC (V = 4), the UC (∀ V ≥ 3) and the SPAC (∀ even V ≥ 2), thus locating the DPT.

A. A short review of the V = 2 case

Before addressing the higher V case, it is useful to recall the results of the two-sites system (that is a bosonic dimer) [9, 15, 34, 37, 38, 65–67], with further details given in Appendix D. Exploiting the constraint of conservation of the total number of particles (ρ1 + ρ2 = 2ρ0 = 2), one can reduce the DEs (13) to

\[
\begin{align*}
\frac{\partial \theta}{\partial t} &= -\delta - \frac{\eta}{2} \frac{\delta}{\sqrt{1 - \frac{\delta^2}{4}}} \cos \theta \\
\frac{\partial \delta}{\partial t} &= 2\eta \sqrt{1 - \frac{\delta^2}{4}} \sin \theta ,
\end{align*}
\]

(35)

where δ = ρ1 − ρ2 is the population imbalance and θ = θ1 − θ2 is the phase difference between the two sites. The Jacobian eigenvalues associated to the DEs (35) for the various FPs are

\[
\begin{align*}
\lambda_{1,2}^{S\text{FC}} &= \pm 2i \sqrt{\frac{\eta}{2} \left( \frac{\eta}{2} + 1 \right)} , \\
\lambda_{1,2}^{S\text{PAC}} &= \pm 2 \sqrt{\frac{\eta}{2} \left( 1 - \frac{\eta}{2} \right)} , \\
\lambda_{1,2}^{P\text{AC}} &= \pm 2i \sqrt{\frac{\eta}{2} \left( 1 + \frac{4}{\eta} \right)} ,
\end{align*}
\]

(36)

where PAC+ and PAC− are two possible PACs existing only for η < ηc = 2 and with r ≠ 0. Being λ1,2S\text{FC} and λ1,2S\text{PAC} purely imaginary for any η > 0, the SFC and (when existing) the PAC± are linear centers of the dynamics. Conversely, the SPAC presents a double nature depending on the value of η; for 0 < η < ηc = 2 it is a saddle of the dynamics (λ1,2S\text{PAC} > 0 and λ1,2S\text{PAC} < 0) whereas for η > ηc = 2 it is a linear center of the dynamics (λ1,2S\text{PAC} are both purely imaginary). The nature of the FPs is intimately related to the shape of the MF energy landscape [Fig. 4(a,c)], and heavily impacts on the features of the trajectories of the system in the phase space [Fig. 4(b,d)]. Trajectories starting in the proximity of the SPAC will closely orbit around it for η > ηc, and instead drift away from it exponentially rapidly (and eventually come back at later times) if 0 < η < ηc. The instability of δ = 0 in the latter case is at the origin of the MQST, that is an average non-zero population imbalance [15]. A system initialized in the surroundings of the SFC will instead closely orbit around it for any η > 0.

Importantly, being the system conservative and being all the FPs isolated, linear centers of the dynamics will always be nonlinear centers as well. This powerful information, extendible to V = 3 but in stark contrast with V ≥ 4, guarantees that the solution of the linearized DEs is accurate even at long-times and for the whole nonlinear DEs (35) when close to a FP, no matter if this is a saddle or a linear center. This can be understood clearly looking at Fig. 4(c), where we show the energy landscape in the surroundings of the SPAC for η > ηc, and one possible trajectory (in blue). Since energy is conserved and the SPAC is isolated, the trajectory must necessarily be a cycle around the SPAC, even at long-times.

B. Stability of the uniform configuration (V ≥ 3) and macroscopic quantum self-trapping

We now consider the case of the UC, that is we evaluate the Jacobian matrix (34) for the configuration (24). This case is particularly relevant since for a MI to SF quench and V ≫ 1 the system is initialized in the proximity of the UC, so that we observe a DPT at ηUC.

With a proper Fourier transform of the eigenvalue problem associated to the Jacobian matrix J (details in appendix E), we obtain the following eigenvalues λ and respective algebraic multiplicities ma for a generic number of lattice sites V ≥ 3

\[
\begin{align*}
\lambda_0 &= 0 , \\
ma &= 2V - 4 , \\
\lambda_1^± &= \frac{i\eta \pm \sqrt{4\eta - \eta^2}}{2} , \\
ma &= 1 , \\
\lambda_{±1} &= \frac{-i\eta \pm \sqrt{4\eta - \eta^2}}{2} , \\
ma &= 1 .
\end{align*}
\]

(37)–(39)

For completeness we report the corresponding characteristic polynomial P(λ) = \prod_{n=1}^{V}(λ - \lambda_n) of the Jacobian
with characteristic timescale (dashed blue line in Fig. 5 (b))

$$\tau^{UC} = \frac{2}{\sqrt{4\eta - \eta^2}} \quad (42)$$

and where we used the symbol $\sim$ meaning that the exponential divergence will occur after a possible very short transient in which the system aligns with the unstable eigenvector. Such short transient and the exponential growth of $r$ at short-times for $\eta < \eta_{UC}^{\nu}$ are correctly observed for $\eta = 2$ in the inset of Fig. 6(d), with logarithmic ordinate axis. Close to the DPT we have $\tau^{UC} \sim \sqrt{1 - \frac{a}{\eta_0} - \eta}$ with critical exponent $\beta = 1/2$. For $\eta < \eta_{UC}^{\nu}$, the increase of $r$ corresponds to an increase of the spread of the boson numbers at each site $\{\rho_j\}_{j=1,...,V}$ around their mean value $\rho_0$ [see Sec. IV and Fig. 6(b)], that is to a symmetry breaking and the emergence of MQST. In particular, from equation (28) we get that the variance over the sites of the number of bosons $\{\rho_j\}_{j=1,...,V}$ at each site reads $\text{Var}[\rho_j] = \langle (\rho_j - \rho_0)^2 \rangle_j = 2\eta(t^2 - \rho_0^2)$ and thus grows as $\sim e^{2H/\tau^{UC}}$ at short-times. Noticeably, the emergence MQST is a spontaneous process, since on the timescales of our study the system is considered to be isolated.

For $\eta > \eta_{UC}^{\nu} = 4$ instead $\lambda^+ = \lambda_{1}^+, \lambda^- = \lambda_{1}^-, \lambda^+_1, \lambda^-_1$ are all purely imaginary, and the UC is a linear center of the dynamics. At least at short-times and the system will cycle around the UC. For a MI to SF quench and large $V$, $r$ will correspondingly remain small ($\sim 1/\sqrt{V}$) and fluctuate in time.

C. Stability of the symmetric $\pi$-aligned configuration (even $V$)

To obtain information on the stability of the SPAC, assuming an even number of lattice sites $V$, we diagonalize exactly the Jacobian matrix $J$ evaluated in the configuration (22), finding (see appendix E) the following eigenvalues $\lambda$ and respective algebraic multiplicities $m_a$

$$\lambda_0 = 0 \quad m_a = 2V - 2 \quad (43)$$

$$\lambda^+ = +\sqrt{\eta} \quad m_a = 1 \quad (44)$$

$$\lambda^- = -\sqrt{\eta} \quad m_a = 1 \quad (45)$$

corresponding to the roots of the characteristic polynomial $P(\lambda)$ of the Jacobian matrix $J$, that we report for completeness

$$P(\lambda) = \lambda^{2V-4} (\lambda^2 + \eta(\eta - 2)) \quad (46)$$

For a system initialized in the proximity of the SPAC the DPT is thus located at $\eta_{SPAC}^{\nu} = 2$. Indeed for $\eta < \eta^{\nu}_{SPAC}$ we get...
\( \eta_{c}^{SPAC} \) we get \( \lambda^{+} > 0 \) and \( \lambda^{-} < 0 \), meaning the SPAC to be a saddle of the dynamics with divergence timescale \( \tau_{SPAC} = 1/\lambda^{+} = (\eta(2 - \eta))^{-1/2} \) (continuous red line in Fig. 5 (b)). Again, for \( \eta < \eta_{c}^{SPAC} \) and initialization of the system in the proximity of the SPAC, the instability reflects at short-times into the exponential growth of \( \rho \) with correspondent growth of the spread of the number of bosons per each lattice site \( \text{Var}[\rho_j] = \langle (\rho_j - \rho_0)^2 \rangle_j = 2\eta(t^2 - t_0^2) \). Conversely, for \( \eta > \eta_{c}^{SPAC} \) the non-zero Jacobian eigenvalues are purely imaginary and the SPAC is a linear center.

D. Stability of the delta configuration \((V = 4)\)

We now aim, for \( V = 4 \), to study the linear stability of the DC, that runs parametrically over the whole manifold of \( r = 0 \) FPs, ranging from the UC to the SPAC. With the help of a symbolic manipulation software we plug the configuration (25) into the Jacobian matrix (34), exactly finding its associated characteristic polynomial

\[
P(\lambda) = \lambda^4 + \eta(\eta - 2)\lambda^2 + \eta^2 \sin(\Delta)
\]

whose roots are the eigenvalues. Setting \( \lambda = i\alpha \) and excluding the zero eigenvalues, we can equivalently look at the roots of

\[
p(\alpha) = \alpha^4 - \eta(\eta - 2)\alpha^2 + \eta^2 \sin(\Delta).
\]

(Not) real roots of \( p(\alpha) \) correspond to (non) purely imaginary eigenvalues of \( J \), that is to the DC being a (saddle) linear center of the dynamics. For \( \eta < 2 \) the polynomial \( p(\alpha) \) has no real roots, meaning that the DC is a saddle of the dynamics for any \( \Delta \). For \( \eta \geq 2 \) the roots of (48) are given by \( \alpha^2 = a\alpha^2 + b = 0 \) with \( a, b \geq 0 \) and are real if and only if \( b \leq \frac{a^2}{2} \). The critical condition reads therefore \( b = \frac{a^2}{2} \), giving

\[
\eta_{c}^{DC} = 2(1 + \sin(\Delta)),
\]

such that the DC corresponding to a given \( \Delta \) is a saddle (linear center) of the dynamics if \( \eta < \eta_{c}^{DC} \) \((\eta > \eta_{c}^{DC})\), as shown in the dynamical phase diagram of Fig. 5 (a). In particular, we find that \( \eta_{c} = 2 \) for the SPAC \((\Delta = \pi)\) and that \( \eta_{c} = 4 \) for the UC \((\Delta = \pi/2)\), in agreement with the previous results.

As a final remark, we stress that the validity of the present linear stability analysis is limited to short-times only. Indeed, in the long-time regime the nonlinearities of the DEs (13) crucially impact on the system dynamics. For instance, for \( V \geq 4 \) and \( \eta > \eta_{c}^{UC} \), the UC is a linear center but not necessarily a nonlinear center, since it is non isolated (it is in fact part of the manifold of the \( r = 0 \) FPs). This means that in the long-time and nonlinear regime a system initialized in the proximity of the UC (as for the MI to SF quench for \( V \gg 1 \)) can a priori still drift away from it, even for \( \eta > \eta_{c}^{UC} \). This reasoning is peculiar of the \( V \geq 4 \) case and at the basis of the possible emergence of the \( \pi \)-synchronization of the bosonic phases \( \{\theta_j\}_{j=1,...,V} \) that is addressed in the next section.

### VI. LONG-TIME DYNAMICS AND \( \pi \)-SYNCHRONIZATION

In this section we go beyond the linear analysis presented above and investigate the long-time \((t \gg 1/|\lambda^{\pm\pm}_c|)\) nonlinear dynamics. To this purpose we solved numerically the DEs (8) that are analogue to Eq. (13) with the MATLAB built-in adaptive ordinary differential equations solver ode45. Focusing on the MI to SF quench for \( V \gg 1 \), we initialize the system as in (15), and thus in the proximity of the UC, so that the DPT is located at a critical dimensionless hopping strength \( \eta_{c}^{UC} = 4 \) and that \( r_0 = r(t = 0) \sim \frac{1}{\sqrt{V}} \ll 1 \). Averaging the MF observables over a large number of simulations (each one with different random initial phases \( \{\theta_j\}_{j=1,...,V} \)) we can approximately compute the dynamics of the expectation value of physical observables at any time \( t \geq 0 \) according to Eq. (16). In this way, we find some macroscopical DOPs to asymptotically relax to finite values. In particular, \( \langle r \rangle \) tends to a finite value for \( \eta < \eta_{c}^{UC} \), whereas for \( \eta > \eta_{c}^{UC} \) the system relaxes to a \( \pi \)-synchronized state, that is characterized by the tendency of the phases \( \{\theta_j\}_{j=1,...,V} \) to split into two groups with phase difference \( \pi \).
A. Long-time dynamics for $\eta < 4$

As emerged from the linear stability analysis of Sec. V, if $\eta < \eta^{UC}_{c}$, at short-times the system drifts away from the UC with the modulus $r$ of the DOP $\Psi$ growing exponentially as in Eq. (41), capturing the emergence of the MQST. The effects of the nonlinearities of the DEs (13) emerge instead in the long-time regime. If we look at the dynamics of $r$ for a given IC (e.g. Fig. 6(a,b) for $\eta = 2$ at $t = 0$ and $t = 1000$ respectively), we find that, after the initial growth, $r$ fluctuates in time around a finite value. When considering the average $\langle r(t) \rangle$ over a large number of simulations [in the spirit of Eq. (16)], these long-time fluctuations vanish, revealing an asymptotic relaxation to a finite value (e.g. $\langle r \rangle \rightarrow 0.38$ for $\eta = 2$), as showed in Fig. 6(d). As explained in Sec. IV, the finiteness of $\langle r \rangle$ at long-times corresponds to a well-defined spread of the $\{\rho_j\}_{j=1,...,V}$ around their mean value $\rho_0$ (that is MQST), that is displayed at $t = 1000$ for one specific IC in Fig. 6(b). Notice that the growth of $\langle r \rangle$ does not indicate at all a tendency of the system to reach the SFC (for which $r = 1$ and $\rho_j = \rho_0$ for $j = 1, 2, \ldots, V$). In Sec. IV we have in fact shown this to be forbidden by the presence of conserved quantities in the non-equilibrium regime. The relaxation of the system to the SFC will possibly happen on much longer timescales thanks to the interaction with the environment, that goes beyond the interests of our study.

B. Long-time dynamics for $\eta > 4$

As shown in Sec. V by solving the linearized DEs, if $\eta > \eta^{UC}_{c}$ at short-times the system orbits in the phase space around the UC, that is in fact a linear center of the dynamics. Correspondingly, $r$ remains small (in the same order of $\rho_0$, meaning that no MQST occurs) and fluctuates, eventually relaxing at long-times (dashed red line for $\eta = 5$ in Fig. 6(d)). However, in striking contrast with the $V = 2, 3$ cases, the UC is a non-isolated FP (it is in fact part of the continuous manifold of $r = 0$ FPs), and in general is thus not a nonlinear center of the dynamics, despite the system being conservative. That is, when considering the whole nonlinear DEs (13), at long-times the system can actually drift away from the IC, moving in the proximity of the manifold of the $r = 0$ FPs and along it, still conserving energy and total number of particles. For instance, considering the particular initialization at $t = 0$ of Fig. 6(a), the system at $t = 1000$ for $\eta = 5$ looks considerably differently but still with $r \ll 1$, [Fig. 6(c)].

To track the position of the system in the phase space with respect to the manifold of $r = 0$ FPs we introduce therefore a $\pi$-synchronization DOP $S$ defined as

$$S(t) = \langle \psi_j \psi_k \phi_j^\dagger \phi_k^\dagger \rangle_{j,k},$$

where $\langle \cdot, \cdot \rangle_{j,k} = [V^2]^{-1} \sum_{j,k}^V \cdot, \cdot$ denotes the average over all the sites $j, k = 1, 2, \ldots, V$. An interpretation of $S$ is easily accessible expressing it as $S = \langle \rho_j \rho_k [2 \cos(\theta_j - \theta_k)^2 - 1] \rangle_{j,k}$. This quantity, which resembles the order parameter typically considered in the study of liquid crystals at equilibrium [71], provides a measure of the tendency of the MF phases $\{\theta_j\}_{j=1,...,V}$ to $\pi$-synchronize. In the sense of Eq. (16), averaging $S$ over a large number of different MF evolutions we approximate the expectation value of the corresponding quantum operator $\frac{1}{2} \{a_j a_j a_k a_k + h.c.\}_{j,k}$. On the manifold of the $r = 0$ FPs, $S$ ranges from 0 (for the UC) to 1 (for the SPAC). Looking at the evolution of $S$ we are therefore able to quantify the position of the system with respect to such manifold in time. This can be clearly seen in the case of $V = 4$, for which the parametric DC is characterized by $S(\Delta) = \cos^2(\Delta)$, and works analogously for larger $V$. In Fig. 6(c) we plot $\langle S \rangle$ against $t$ for $V = 500$. For $\eta = 5 > \eta^{UC}_{c}$ (red dashed line) $\langle S \rangle$ increases from 0, corresponding to the initial MI, up to a finite value 0.35, underlying respectively complete uncorrelation and $\pi$-synchronization of the bosonic phases $\{\theta_j\}_{j=1,...,V}$. Importantly, the growth at short-times is not exponential, confirming once more to be intimately connected to the nonlinearities of the DEs. Once more, we stress that this phenomenon is enabled by the manifold of isoenergetic $r = 0$ FPs that, only for $V \geq 4$, opens a channel for the non-equilibrium dynamics connecting the UC to the SPAC, towards which the system shifts robustly. Notice that, since the system is isolated, the stabilization of $\langle S \rangle$ for large $V$ is an intrinsic property and is not due to the presence of driving and dissipation, as typically considered in the literature [44-51].

In Fig. 7 we schematically show some possible trajectories of the system (in blue) in the phase space for a large and even $V$ (the schematic for a odd $V$ would be very similar though, just without the SPAC), together with the graphical representation of the most relevant configurations. For a MI to SF quench and $\eta = 2 < \eta^{UC}_{c} = 4$ (a), the system drifts away from the manifold of $r = 0$ FPs because of the linear instability, eventually relaxing to a state characterized by $\langle r \rangle \approx 0.38$ and by consequent spread of the number of bosons per site $\langle \rho_j \rangle_{j=1,...,V}$. Conversely, for $\eta = 5$, the system closely orbits around the aforementioned manifold (since the latter is made of linear centers of the dynamics), while progressively shifting towards the SPAC due to the nonlinearities of the DEs (b).

VII. DISORDER-INDUCED SYNCHRONIZATION Crossover

Having discovered in the previous section that the dynamics generated by the nonlinear GPE can lead at long-times to a partial $\pi$-synchronization of the MF bosonic phases $\{\theta_j\}_{j=1,...,V}$, we wonder now about the robustness of such synchronization against the introduction of some site-dependent disorder. We consider thus the following disordered version of the BH Hamiltonian

$$H = \sum_j (D_j - \Delta) \phi_j^\dagger \phi_j + \Delta \sum_j \phi_j^\dagger \phi_j + \sum_{j<k} V_{jk} \phi_j^\dagger \phi_k + \sum_j \phi_j^\dagger \phi_j,$$

where $D_j$ is the disorder potential on the site $j$ and $V_{jk}$ is the coupling constant between the sites $j$ and $k$. The disorder is assumed to be random and independent on the sites, with a distribution $P(D_j)$.

The disorder introduces fluctuations in the energy levels of the system, which can lead to a breaking of the $\pi$-synchronization. We study this effect by numerically solving the disordered GPE and comparing the results with the undisturbed case. We find that the presence of disorder can disrupt the $\pi$-synchronization, depending on the strength and distribution of the disorder. This is demonstrated through the behavior of the DOP $S(t)$, which shows a decrease in the value of $\langle S \rangle$ as the disorder increases.

In conclusion, we have studied the long-time dynamics of the Bose-Hubbard model with site-dependent disorder, focusing on the emergence of $\pi$-synchronization. We have shown that, in the absence of disorder, the system can reach a stable state characterized by $\langle S \rangle = 0.35$, corresponding to $\pi$-synchronization of the MF phases. However, the introduction of disorder disrupts this synchronization, leading to a decrease in $\langle S \rangle$. This result highlights the sensitivity of the synchronization phenomenon to the presence of disorder, which is an important aspect in the study of complex systems.
t = 0; IC = MI

\[ S = 0 \]
\[ r = 0 \]

FIG. 6: (color online) Exact numerical solution of the nonlinear dynamics (13) up to long-times for \( V = 500 \) sites and a MI to SF quench. (a-c) Graphical representation of the MF variables for a single simulation with IC given by Eq. (15). For graphical clarity, only the blue markers of 300 out of the \( V = 500 \) bosonic variables are represented. At time \( t = 0 \) the phases are randomly distributed (a) whereas at \( t = 1000 \) either the \( \{ \rho_j \} \) (b) or the \( \{ \theta_j \} \) (c) are \( \pi \)-synchronized. (d) Dynamics of the expected value \( \langle r \rangle \) of the DOP \( r = | \langle \Psi | \rangle |^2 \) obtained according to Eq. (16) as an average over 3000 simulations (each one for a different random IC (15)). For \( \eta = 2 < \eta_{UC}^{V} = 4 \) (blue continue line), the DOP \( \langle r \rangle \) grows exponentially at short-times \( \langle r \rangle \sim \exp \{ \eta \tau \} \), see inset with logarithmic ordinate axis) and relaxes to a finite value \( 0.38 \) at long-times, indicating MQST and reflected in the spread of the \( \{ \rho_j \} \) in (b). For \( \eta > \eta_{UC}^{V} \) (red dashed line) \( \langle r \rangle \) remains instead small. (b) Dynamics of the expected value \( \langle S \rangle \) of the \( \pi \)-synchronization parameter \( S \) obtained as an average over 3000 simulations. For \( \eta = 5 > \eta_{UC}^{V} \), \( \langle S \rangle \) asymptotically relaxes to a finite value \( 0.35 \), indicating a robust shift of the system towards the SPAC.

\[ \rho = H_{BH} + \sum_{j=1}^{V} \Omega_j n_j, \tag{51} \]

where \( \{ \Omega_j \} \) denotes a set of independent and identically distributed Gaussian random numbers of zero mean and standard deviation \( \Sigma \). Applying a MF approximation analogue to the one used to obtain (13), we find the following GPE associated to the Hamiltonian (51)

\[ \begin{align*}
    \frac{d\sqrt{\rho_k}}{dt} &= \eta r \sin (\theta_k - \phi), \\
    \frac{d\theta_k}{dt} &= \frac{\eta r}{\sqrt{\rho_k}} \cos (\theta_k - \phi) - \rho_k + \omega_k,
\end{align*} \tag{52} \]

where \( \omega_k = \Omega_k/\rho_k \). We call disorder strength the dimensionless parameter \( \sigma = \frac{\omega}{\omega_{max}} \), that is the standard deviation of the random numbers \( \{ \omega_j \} \). Interpreting the variables \( \{ \theta_j \} \) as the phases of a population of classical oscillators (one per lattice site), in Eq. (52) the disorder can be regarded as affecting the oscillators natural frequencies \( \{ \omega_j \} \), thus competing against the tendency of the oscillators to \( \pi \)-synchronize. This is reminiscent of the Kuramoto model for classically coupled nonlinear oscillators [41–43].

To investigate such competition, we solved numerically the DEs (52) for \( V = 300 \) lattice sites and a MI to SF quench. Looking at the MF dynamics obtained for \( \eta = 5 \) for one given IC [shown in Fig. 8(a)], at time \( t = 200 \) we observe a clear reduction of the \( \pi \)-synchronization in the disordered case \( \sigma = 0.045 \), Fig. 8(c)] with respect to the non-disordered one \( \sigma = 0 \). In Fig. 8(d) we show \( \langle S \rangle \) to decrease for an increasing disorder strength \( \sigma = 0, 15, 30, 45, 60 \times 10^{-3} \). In Fig.
We numerically solve the DEs (52) for \( \eta = 200 \) at time \( t = 0 \) and at time \( t = 200 \) for \( \sigma = 0 \) (b) and \( \sigma = 45 \times 10^{-3} \) (c). The polar histogram of the phases \( \{ \theta_j \} \) helps to visualize the reduction of \( \pi \)-synchronization due to the disorder. (d) Time dynamics of the expected value \( \langle S \rangle \) versus the disorder strength \( \sigma \). For increasing disorder, the \( \pi \)-synchronization is progressively broken in a synchronization crossover with onset decreasing with \( \eta \).

\[ \langle S \rangle(t \to \infty) \]

\[ \sigma = 0, 15, 30, 45, 60 \times 10^{-3} \]

8(e) we plot the asymptotic value of \( \langle S \rangle \) against the disorder strength for \( \eta = 5 \) (continue blue line) and \( \eta = 10 \) (dashed red line), showing a synchronization crossover with onset decreasing with \( \eta \). For small (large) disorder \( \sigma \), at long-time the MF phases \( \{ \theta_j \} \) are \( \pi \)-synchronized (uncorrelated).

**VIII. CONCLUSIONS**

In conclusion, we studied the non-equilibrium dynamics induced by a quantum quench to the SF regime in the BH model (1) on a FC (or all-to-all coupled) \( V \)-dimensional lattice, with potential experimental applications ranging from cold-atoms to systems of bosonic junctions. Assuming large bosonic populations and performing a MF approximation, we derived the discrete and nonlinear GPE of motion (13), which amounts to a problem of \( V \) classical and nonlinearly coupled oscillators with variable phase and length \( \{ \theta_j, \sqrt{\rho_j \tau_j} \} \). Having compacted the GPE thanks to the introduction of a complex DOP \( r \), going beyond previous studies on the bosonic dimers and trimers (that is \( V = 2 \) and \( V = 3 \) respectively) [34, 36–38, 65–69] and on a one-dimensional lattice [35], we showed that for \( V \geq 4 \) there exists a peculiar class of infinitely many FPs of the GPE (namely the ones with \( r = 0 \) and \( \rho_j = \rho_0 \forall j \) forming a manifold in the phase space. Among the FPs of such manifold, particularly relevant are the SPAC (22) and the UC (24), the latter being in the proximity of the IC in case of a MI to SF quench for \( V \gg 1 \). Linearizing the GPE and diagonalizing the respective Jacobian matrix, we studied the short-time dynamics for a system initialized in the proximity of the most relevant FPs, that is the UC (\( V \geq 3 \)), the SPAC (\( V \) even \( V \)) and the DC (a parametric FP spanning the \( r = 0 \) manifold for \( V = 4 \)). We found that, depending on the considered FP, there exists a certain critical hopping strength \( \eta_{c, \mathrm{FP}} \) such that at short-times if \( \eta > \eta_{c, \mathrm{FP}} \) (i.e. small interactions) the system remains close to the FP (and thus to the IC), whereas if \( \eta < \eta_{c, \mathrm{FP}} \) (i.e. large interactions) it drifts away from the FP exponentially fast \( (r) \sim e^{\eta/\eta_{c, \mathrm{FP}}} \), indicating MQST. Such sharp change of dynamic behavior when varying \( \eta \) across a critical value is a DPT, that we located at \( \eta_{c, \mathrm{UC}} = 4 \), \( \eta_{c, \mathrm{SPAC}} = 2 \) and at \( \eta_{c, \mathrm{DC}} = 2(1+\sin(\Delta)) \). Furthermore, for a MI to SF quench with \( V \gg 1 \), we investigated numerically the long-time dynamics generated by the whole, nonlinear GPE, computing expectation values as averages over the random IC. For \( \eta < \eta_{c, \mathrm{UC}} = 4 \) we proved the emergence of MQST whereas for \( \eta > \eta_{c, \mathrm{UC}} = 4 \), we showed that the system exhibits a slow drift in the phase space from the proximity of the UC towards the SPAC, eventually relaxing to a state where the MF bosonic phases \( \{ \theta_j \} \) are \( \pi \)-synchronized. We argued this phenomenon, which we quantified with a suitable DOP \( S \), to be intimately connected to the presence of the manifold of \( r = 0 \) FPs. We finally studied the robustness of the long-time \( \pi \)-synchronization against the introduction in the model of site-dependent disorder, finding that \( \langle S \rangle \) vanishes in a synchronization crossover for an increasing disorder strength.

**Future developments.** We conclude by outlining possible developments of the present investigation. A fascinating challenge is to work out analytical approaches that enable to understand at a deeper level the emergence of \( \pi \)-synchronization, that is intimately connected to the nonlinear terms of the GPE. One possibility is represented by the continuum limit for \( V \to \infty \) of the nonlinear GPE, that is addressed in appendix F. Another interesting aspect to be analyzed is the effect of the lat-
tice dimensionality on the structure of the FPs in the phase space and consequently on the \( \pi \)-synchronization, for instance considering 2D or 3D lattices. Furthermore, we notice that the discrete and nonlinear GPE is potentially a powerful tool to address, at the MF level, the emergence of spontaneous synchronization in generic (almost) isolated many-bosons quantum systems different from the one considered in the present work or in Refs. \cite{35, 40}. As seen, under particular circumstance, such systems can in fact be turned at the MF level into systems of nonlinearly coupled classical oscillators (in general of variable phase and length), for which synchronization is a universal and fundamental concept \cite{74}. Finally, while the purpose of the present article was the investigation of the model within the MF approximation, the analysis of quantum fluctuations is also an interesting question that deserves a separate and detailed investigation.

Acknowledgments

A.P. acknowledges financial support from the PALM Labex, Paris-Saclay, Grant No. ANR-10-LABX-0039, and from the Royal Society and the EPSRC. K.L.H acknowledges funding from the ANR BOCA and the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) via Research Unit FOR 2414 under project number 277974659. We thank Marco Schiro for his fruitful comments and we also acknowledge discussions at CIFAR meetings in Canada and at the Centre de Recherches Mathematiques in Montreal.

Appendix A: Mean-field equilibrium phase diagram

At MF, the critical line separating the MI and the SF phases (originating the notorious lobe-shaped phase diagram) is given by \cite{52}

\[
1 + J \left( \frac{\rho_0}{u(\rho_0 - 1) - \mu} + \frac{\rho_0 + 1}{\mu - u\rho_0} \right) = 0 ,
\]

that is

\[
J/u = \frac{\rho_0 - 1 - \mu/u)(\mu/u - \rho_0)}{1 + \frac{\rho_0}{\mu}} .
\]

(A2)

Differentiating (A2) we get

\[
\frac{dJ}{d\mu} = \frac{\rho_0}{\mu} - 2\mu - 1
\]

(A3)

Enforcing \( dJ/d\mu = 0 \) we obtain the position \( \mu_T \) of the top of the lobes

\[
\mu_T = -1 + \sqrt{\rho_0^2 + \rho_0}
\]

(A4)

and plugging it into (A2) we get the value \( \eta_T \) of the dimensionless hopping strength \( \eta \) at the top of the lobes

\[
\eta_T = \frac{J_T}{u\rho_0} = \frac{1}{\rho_0} + 2 - 2\sqrt{1 + \frac{1}{\rho_0}} \approx \frac{1}{4\rho_0^2} ,
\]

(A5)

where the last approximation holds for \( \rho_0 \gg 1 \) and is obtained Taylor expanding the square root at second order in \( 1/\rho_0 \).

Appendix B: Dynamical equations

1. Explicitation of the Heisenberg equation of motion

To compute the commutators of Eq. (4) we evaluate the following terms

\[
[n_j, a_k] = -\delta_{j,k}a_j ,
\]

\[
[n_j, a_k^\dagger] = \delta_{j,k}a_j^\dagger ,
\]

\[
n_j(n_j - 1), a_k = -2n_ja_j\delta_{j,k} ,
\]

\[
n_j(n_j - 1), a_k^\dagger = +2a_j^\dagger n_j\delta_{j,k} ,
\]

\[
[a_j^\dagger a_j + a_j^\dagger a_j^\dagger] = -\delta_{i,k}a_{j,k} - \delta_{j,k}a_k^\dagger ,
\]

\[
[a_j^\dagger a_j + a_j^\dagger a_j^\dagger] = \delta_{i,k}a_k^\dagger + \delta_{j,k}a_k^\dagger ,
\]

so that Eq. (4) is explicitly rewritten as

\[
\frac{da_j}{d\Omega_G} = \frac{J}{V} \sum_{j \neq k} a_j - u n_k a_k + \mu a_k ,
\]

\[
\frac{da_j^\dagger}{d\Omega_G} = -\frac{J}{V} \sum_{j \neq k} a_j^\dagger + u a_k^\dagger n_k - \mu a_k^\dagger .
\]

(B2)

Obviously, the two equations of (B2) are one the Hermitian conjugate of the other. Exploiting the gauge freedom we can safely operate the following substitution for the bosonic creation and annihilation operators

\[
a_j^\dagger \rightarrow a_j^\dagger e^{-i\Omega_G t} ,
\]

\[
a_j \rightarrow a_j e^{i\Omega_G t} ,
\]

(B3)

where \( \Omega_G \) is an arbitrary real and time-independent number. Under the transformation (B3), the bosonic commutation relations are in fact preserved, that is

\[
[a_k e^{i\Omega_G t}, a_j^\dagger e^{-i\Omega_G t}] = [a_k, a_j^\dagger] = \delta_{k,j} ,
\]

meaning that \( a_j e^{i\Omega_G t} \) and \( a_j^\dagger e^{-i\Omega_G t} \) are still respectively annihilation and creation bosonic operators associated to the \( j \)-th site. Under the gauge transformation (B3), (B2) transforms into
that is the system of DEs (9).

Appendix C: π-aligned stationary configurations

We complement Sec. III showing one example of stationary PAC, that is the one with \( \alpha = 1/2 \) for an even \( V \), for which at time \( t = 0 \)

\[
\psi_j = \begin{cases} 
+\sqrt{1 + \delta} & \text{if } j = 1, 2, \ldots, V/2 \\
-\sqrt{1 - \delta} & \text{if } j = V/2 + 1, \ldots, V ,
\end{cases}
\]

where \( \delta \) is a parameter to be determined. Without loss of generality we assume \( \delta \in (0, 1) \), since a negative \( \delta \) would actually be reabsorbed in our study just with an initial \( \pi \) rotation of the reference frame. For the configuration (C1), the DOP (10) at time \( t = 0 \) reads

\[
r e^{i \phi} = \frac{1}{V} \sum_{j=1}^{V} \sqrt{\rho_j} e^{i \phi_j} = \frac{\sqrt{1 + \delta} - \sqrt{1 - \delta}}{2},
\]

that is

\[
r(0) = \frac{\sqrt{1 + \delta} - \sqrt{1 - \delta}}{2}, \quad \phi(0) = 0 ,
\]

where \( r > 0 \) consistently with the assumption \( \delta > 0 \). Enforcing the two groups of phases to rotate at the same rate, the stationarity condition (18) reads

\[
-(1 + \delta) + \eta \frac{r}{\sqrt{1 + \delta}} = -(1 - \delta) - \eta \frac{r}{\sqrt{1 - \delta}} ,
\]

from which in few easy steps we get

\[
2\delta = \frac{\eta \delta}{\sqrt{1 - \delta^2}} ,
\]

that is solved either by \( \delta = 0 \) (corresponding to the SPAC) or by \( \delta = \sqrt{1 - \left(\frac{2}{V}\right)^2} \). The former solution is present for any \( \eta \), the latter only if \( \eta < \eta_{SPAC} = 2 \).

This example, holding for whatever even \( V \), has strong analogies with the \( V = 2 \) case treated in Sec. VA.

Appendix D: Two-sites model

In this section we review some results on the simple case of \( V = 2 \) (a bosonic dimer) [9, 15, 34, 37, 38, 65–67]. The DEs (13) for \( V = 2 \) read

\[
\begin{aligned}
\frac{d \rho_k}{d t} &= 2 \sqrt{\frac{V}{V_k}} \sqrt{\rho_j \rho_k} \sin (\theta_k - \theta_j) \\
\frac{d \theta_k}{d t} &= V \sqrt{\frac{V}{V_k}} \cos (\theta_k - \theta_j) - u \rho_k ,
\end{aligned}
\]
and the nature of the FPs generally depends on the sign
of the on-site interaction $u$. In the present work we only focus on repulsive in-situ interaction ($u, \eta > 0$).

To study the stability of the FPs, we diagonalize the 2 dimensional Jacobian matrix $J$ associated to the DEs (D2). For the various FPs we find the following Jacobian matrices and associated eigenvalues $\lambda_{1,2}$

$$J^{SFC} = \begin{pmatrix} 0 & -\frac{\eta}{2} - 1 \\ 2\eta & 0 \end{pmatrix} \rightarrow \lambda_{1,2}^{SFC} = \pm 2i\sqrt{\frac{\eta}{2} + \frac{\eta^2}{4}}, \quad (D7)$$

$$J^{SPAC} = \begin{pmatrix} 0 & -\frac{\eta}{2} - 1 \\ -2\eta & 0 \end{pmatrix} \rightarrow \lambda_{1,2}^{SPAC} = \pm 2\sqrt{\frac{\eta}{2} - \frac{\eta^2}{4}}, \quad (D8)$$

$$J^{PAC_\pm} = \begin{pmatrix} 0 & -\frac{\eta}{2} - 1 \\ -2\eta & 0 \end{pmatrix} \rightarrow \lambda_{1,2}^{PAC_\pm} = \pm 2i\sqrt{\frac{\eta}{2} + \frac{\eta^2}{4}}. \quad (D9)$$

Notice that, as shown in Sec. V C, considering the 4 × 4-dimensional Jacobian (32) would just have generated 2 additional zero eigenvalues, that are nevertheless not relevant for the determination of the stability of the FPs.

**Appendix E: Linear stability analysis**

In this section we derive the Jacobian matrix associated to the linearized DEs and diagonalize it exactly, finding its eigenvalues (and eventually eigenvectors), for any $V$ and for the various relevant FPs, that are the UC, the SPAC, and the SFC. Since the stationarity condition (18) allows a global rotation of the phases at rate $\Omega$, we move to a frame rotating exactly at the angular speed $\Omega$ associated to the considered FP, where the equations of motion (13) read

$$\begin{cases}
\frac{\partial \phi_j}{\partial t} = -x_j + \eta \frac{1}{x_j} \cos (\phi - \theta_j) - \Omega \\
\frac{\partial \theta_j}{\partial t} = -\eta \sin (\phi - \theta_j)
\end{cases}, \quad (E1)$$

with $x_j = \sqrt{\rho_j}$. In the new frame we recover the familiar definition of FP, reading $\frac{\partial \theta_j}{\partial t} = 0$ (i.e. the phases of a FP are not rotating). Since the first and the second halves of the state vector $\vec{y}$ refer respectively to the phases and to the moduli of the MF bosonic variables, it is convenient to distinguish the following 4 terms of the Jacobian (32)
\[ J_{j,k} = \frac{\partial}{\partial t_k} \left( \frac{d\theta_j}{dt} \right), \]
\[ J_{j,V,k+V} = \frac{\partial}{\partial x_k} \left( \frac{dx_j}{dt} \right), \]
\[ J_{j+V,k+V} = \frac{\partial}{\partial x_k} \left( \frac{dx_j}{dt} \right), \]
\[ J_{j,k+V} = \frac{\partial}{\partial x_k} \left( \frac{d\theta_j}{dt} \right), \] (E2)

where \( j, k = 1, 2, \ldots, V \), as we implicitly assume also in the following. To build the Jacobian matrix we evaluate the following partial derivatives

\[ \frac{\partial}{\partial \theta_k} r \cos(\phi - \theta_j) = -\frac{x_k}{V} \sin(\theta_k - \theta_j) + \delta_{k,j} r \sin(\phi - \theta_j), \]
\[ \frac{\partial}{\partial \theta_k} \sin(\phi - \theta_j) = \frac{x_k}{V} \cos(\theta_k - \theta_j) - \delta_{k,j} r \cos(\phi - \theta_j), \]
\[ \frac{\partial}{\partial x_k} r \cos(\phi - \theta_j) = \frac{1}{V} \cos(\theta_k - \theta_j), \]
\[ \frac{\partial}{\partial x_k} \sin(\phi - \theta_j) = \frac{1}{V} \sin(\theta_k - \theta_j), \] (E3)

\[ \delta_{k,j} \text{ being the Kronecker delta (} \delta_{k,j} = 1 \text{ if } k = j, \delta_{k,j} = 0 \text{ else). Using (E3), the Jacobian (32) reads} \]
\[ J_{j,k} = \frac{\eta}{V} \left[ \frac{x_k}{x_j} \sin(\theta_j - \theta_k) + \delta_{k,j} \frac{rV}{x_j} \sin(\phi - \theta_j) \right], \]
\[ J_{j+V,k+V} = -\frac{\eta}{V} \sin(\theta_k - \theta_j), \]
\[ J_{j+V,k} = -\frac{\eta}{V} \left[ x_k \cos(\theta_k - \theta_j) - \delta_{k,j} rV \cos(\phi - \theta_j) \right], \]
\[ J_{j,k+V} = -2\delta_{k,j} x_k + \eta \frac{1}{V} x_j \cos(\theta_k - \theta_j) + \]
\[ - \delta_{k,j} r \frac{1}{x_k^2} \cos(\phi - \theta_k). \] (E4)

In particular, in the case of FPs with \( r = 0 \), like the UC, the SPAC and the DC, (E4) simplifies to
\[ J_{j,k} = -\frac{\eta}{V} \sin(\theta_k - \theta_j), \]
\[ J_{j+V,k+V} = -\frac{\eta}{V} \sin(\theta_k - \theta_j), \]
\[ J_{j+V,k} = -\frac{\eta}{V} \cos(\theta_k - \theta_j), \]
\[ J_{j,k+V} = +\frac{\eta}{V} \cos(\theta_k - \theta_j) - 2\delta_{k,j}, \] (E5)

that is Eq. (34). Having written explicitly \( J \), we now aim to diagonalize it exactly, that is to solve the following eigenvalue problem
\[ J \vec{g} = \lambda \vec{g}. \] (E6)

1. Diagonalization of \( J \) for the UC

Since the first and the second halves of the state vector \( \vec{y} \) (31) refer respectively to the phases and to the moduli of the MF bosonic variables, it is handy to write \( \vec{y} \) as
\[ \vec{y} = \begin{pmatrix} \vec{y}^{(1)} \\ \vec{y}^{(2)} \end{pmatrix}, \] (E7)

with \( \vec{y}^{(1)} \) and \( \vec{y}^{(2)} \) \( V \)-dimensional column vectors defined by \( y_j^{(1)} = \theta_j \) and \( y_j^{(2)} = x_j \). Performing the multiplication of the matrix \( J \) times the column vector \( \vec{y} \) we can thus write
\[ (J \vec{y})_j = -\eta \sum_{k=1}^{V} \sin(\theta_k - \theta_j) y_j^{(1)} \]
\[ + \frac{\eta}{V} \sum_{k=1}^{V} \cos(\theta_k - \theta_j) y_j^{(2)} - 2y_j^{(2)} \]
\[ , \]
\[ (J \vec{y})_{j+V} = -\eta \sum_{k=1}^{V} \sin(\theta_k - \theta_j) y_j^{(2)} \]
\[ - \eta \sum_{k=1}^{V} \cos(\theta_k - \theta_j) y_j^{(1)} \] (E8)

Since the sine and the cosine can be written in terms of exponentials and since for the UC \( \theta_k = \frac{2\pi}{V} k \), the form of (E8) suggests us to introduce the following Discrete Fourier Transform (DFT)
\[ \vec{v}_q = \frac{1}{V} \sum_{k=1}^{V} e^{iq\frac{2\pi}{V} k} v_k \quad q \in \mathbb{Z}, \] (E9)

where \( \vec{v} \) is a \( V \)-dimensional vector and where we denoted \( q \) the Fourier wavenumber. It is easy to verify that \( \vec{v}_q = \vec{v}_{q'} \) if \( q - q' \in \mathbb{Z} \), so that it is possible to restrict, without loss of generality, \( q \in \{ 0, 1, 2, \ldots, V - 1 \} \) and to refer to \( q = V - 1 \) as to \( q = -1 \). Looking at (E8) we are thus interested in the evaluation of the following terms

\[ \frac{1}{V} \sum_{k=1}^{V} \sin(\theta_k - \theta_j) v_k = \frac{1}{V} \sum_{k=1}^{V} e^{i(\theta_k - \theta_j)} - c.c. v_k \]
\[ = \vec{v}_1 e^{-i\theta_j} - c.c \quad \text{Im} \{ \vec{v}_1 e^{-i\theta_j} \}, \]
\[ \frac{1}{V} \sum_{k=1}^{V} \cos(\theta_k - \theta_j) v_k = \frac{1}{V} \sum_{k=1}^{V} e^{i(\theta_k - \theta_j)} + c.c. v_k \]
\[ = \vec{v}_1 e^{-i\theta_j} + c.c \quad \text{Re} \{ \vec{v}_1 e^{-i\theta_j} \}. \] (E10)

where Re and Im denote respectively the real and the imaginary part. Having introduced the DFT (E9) and
performing a DFT on it. To do it, we evaluate the eigenvalues \( \{ \lambda_y \} \), having evaluated the terms of \( \text{(E10)} \), we can write Eq. \( \text{(E8)} \) in the following compact form

\[
\begin{align*}
(J \vec{y})_j &= -\eta \text{Im}\left\{ \hat{y}_1(1)e^{-i\theta_j} \right\} + \eta \text{Re}\left\{ \hat{y}_1(2)e^{-i\theta_j} \right\} - 2y_j(2), \\
(J \vec{y})_{j+V} &= -\eta \text{Im}\left\{ \hat{y}_1(2)e^{-i\theta_j} \right\} + \eta \text{Re}\left\{ \hat{y}_1(1)e^{-i\theta_j} \right\},
\end{align*}
\] (E11)

that allows us to write the eigenvalue problem \( \text{(E6)} \) as

\[
\begin{align*}
\lambda y_1^{(1)} &= -\eta \text{Im}\left\{ \hat{y}_1(1)e^{-i\theta_j} \right\} + \eta \text{Re}\left\{ \hat{y}_1(2)e^{-i\theta_j} \right\} - 2y_j(2), \\
\lambda y_2^{(2)} &= -\eta \text{Im}\left\{ \hat{y}_1(2)e^{-i\theta_j} \right\} - \eta \text{Re}\left\{ \hat{y}_1(1)e^{-i\theta_j} \right\}.
\end{align*}
\] (E12)

The solution of \( \text{(E12)} \) will provide us with the Jacobian eigenvalues \( \{ \lambda_n \} \) that can be approached performing a DFT on it. To do it, we evaluate the following terms

\[
\begin{align*}
\text{Im}\{Ae^{-i\theta_j}\}_1 &= \frac{1}{V} \sum_{j=1}^{V} \frac{A - A^*e^{2i\theta_j}}{2i} = -\frac{A}{2}, \\
\text{Re}\{Ae^{-i\theta_j}\}_1 &= \frac{1}{V} \sum_{j=1}^{V} \frac{A + A^*e^{2i\theta_j}}{2} = \frac{A}{2}, \\
\text{Im}\{Ae^{-i\theta_j}\}_{1-} &= \frac{1}{V} \sum_{j=1}^{V} \frac{Ae^{-2i\theta_j} - A^*}{2i} = \frac{A^*}{2}, \\
\text{Re}\{Ae^{-i\theta_j}\}_{1-} &= \frac{1}{V} \sum_{j=1}^{V} \frac{Ae^{-2i\theta_j} + A^*}{2} = \frac{A}{2}, \\
\text{Im}\{Ae^{-i\theta_j}\}_q &= \frac{1}{V} \sum_{j=1}^{V} \frac{Ae^{i(q-1)\theta_j} - A^*e^{i(q+1)\theta_j}}{2i} = 0, \\
\text{Re}\{Ae^{-i\theta_j}\}_q &= \frac{1}{V} \sum_{j=1}^{V} \frac{Ae^{i(q-1)\theta_j} + A^*e^{i(q+1)\theta_j}}{2} = 0.
\end{align*}
\] (E13)

A being an arbitrary complex number and \( (\bullet)_j \) being an alternative notation for the DFT of the function \( \bullet \) with respect to the Fourier wavenumber \( q = 0, 1, 2, \ldots, V - 1 \). Importantly, we notice that expressions \( \text{(E13)} \) for \( q = \pm 1 \) are valid if and only if \( V \geq 3 \), since \( \sum_{j=1}^{V} A e^{\pm 2i\theta_j} \neq 0 \) for \( V = 2 \). We therefore assume for the following treatment that \( V \geq 3 \). Performing the DFT of \( \text{(E12)} \) for \( q = \pm 1 \) and exploiting the expressions \( \text{(E13)} \), we get

\[
\begin{align*}
\lambda y_1^{(1)} &= \pm i \frac{q}{2} y_1^{(1)} + \left( \frac{q}{2} - 2 \right) y_1^{(2)}, \\
\lambda y_1^{(2)} &= \pm \frac{q}{2} y_1^{(2)} - \frac{q}{2} y_1^{(1)}, \\
\lambda y_{-1}^{(1)} &= -\pm i \frac{q}{2} y_{-1}^{(1)} + \left( \frac{q}{2} - 2 \right) y_{-1}^{(2)}, \\
\lambda y_{-1}^{(2)} &= -\pm \frac{q}{2} y_{-1}^{(2)} - \frac{q}{2} y_{-1}^{(1)},
\end{align*}
\] (E14)

that are 2-dimensional eigenvalue problems for \( y_1^{(1)}, y_1^{(2)} \) and \( y_{-1}^{(1)}, y_{-1}^{(2)} \) respectively and where we recall the subscripts \( \pm 1 \) to refer to the Fourier wavenumber \( q \) and the superscripts \( 1, 2 \) to refer to the bipartition of \( \vec{y} \) in first and second halves. We rewrite the problems \( \text{(E14)} \) and \( \text{(E15)} \) in matricial form as

\[
\begin{align*}
\left( \begin{array}{c}
\frac{i q}{2} & \left( \frac{q}{2} - 2 \right) \\
-\frac{q}{2} & \frac{q}{2}
\end{array} \right)
\begin{pmatrix}
y_1^{(1)} \\
y_1^{(2)}
\end{pmatrix} &= \lambda
\begin{pmatrix}
y_1^{(1)} \\
y_1^{(2)}
\end{pmatrix}, \\
\left( \begin{array}{c}
-\frac{q}{2} & \left( \frac{q}{2} - 2 \right) \\
\frac{q}{2} & -\frac{q}{2}
\end{array} \right)
\begin{pmatrix}
y_{-1}^{(1)} \\
y_{-1}^{(2)}
\end{pmatrix} &= \lambda
\begin{pmatrix}
y_{-1}^{(1)} \\
y_{-1}^{(2)}
\end{pmatrix},
\end{align*}
\] (E16)

and find the respective characteristic polynomials \( P_{\pm 1}(\lambda) \), eigenvalues \( \lambda_{\pm 1} \) and eigenvectors \( v_{\pm 1} \) and \( v_{\pm 2} \)

\[
\begin{align*}
P_1(\lambda) &= \lambda^2 - i\eta \lambda - \eta, \\
P_{-1}(\lambda) &= \lambda^2 + i\eta \lambda - \eta, \\
\lambda_{1}^\pm &= \pm \frac{i\eta \pm \sqrt{4\eta - \eta^2}}{2}, \\
\lambda_{-1}^\pm &= \pm \frac{i\eta \pm \sqrt{4\eta - \eta^2}}{2}, \\
v_1^\pm &= (\mp \sqrt{\eta(4 - \eta^2)} \eta)^T, \\
v_{-1}^\pm &= v_1^\pm.
\end{align*}
\] (E17)

We proceed looking for other non-zero eigenvalues, that is for \( \lambda \notin \{ 0, \lambda_1^+, \lambda_1^-, \lambda_{-1}^+, \lambda_{-1}^- \} \). Performing the DFT of equation \( \text{(E12)} \) for \( q \neq \pm 1 \) we get

\[
\begin{align*}
\lambda y_q^{(1)} &= -2y_q^{(2)} \quad \text{for} \quad q = 0, 2, 3, \ldots, V - 2, \quad \text{(E21)}
\end{align*}
\]

that, assuming \( \lambda \neq 0 \), is solved by \( y_q^{(1)} = y_q^{(2)} = 0 \) for \( q = 0, 2, 3, \ldots, V - 2 \). We observe that, if \( \lambda \notin \{ \lambda_1^+, \lambda_1^-, \lambda_{-1}^+, \lambda_{-1}^- \} \), then \( y_q^{(1)} = y_q^{(2)} = y_{-q}^{(1)} = y_{-q}^{(2)} = 0 \), since Eq. \( \text{(E16)} \) and \( \text{(E17)} \) still need to be satisfied. This implies that \( \vec{y} = 0 \), being all its Fourier components equal to 0. Thus, we conclude that the only non-zero eigenvalues are \( \lambda_1^+, \lambda_1^-, \lambda_{-1}^+, \lambda_{-1}^- \), and that \( \lambda_0 = 0 \) is an eigenvalue with algebraic multiplicity \( m_0 = 2V - 4 \).

Summing up, the eigenvalues of the Jacobian matrix and their respective algebraic multiplicities for the UC for a number of sites \( V \geq 3 \) are
\[ \lambda_0 = 0 , \quad m_a = 2V - 4 , \quad (E22) \]
\[ \lambda_1^+ = \frac{i\eta + \sqrt{4\eta^2 - \eta^2}}{2} , \quad m_a = 1 , \quad (E23) \]
\[ \lambda_1^- = \frac{i\eta - \sqrt{4\eta^2 - \eta^2}}{2} , \quad m_a = 1 , \quad (E24) \]
\[ \lambda_2^+ = \frac{-i\eta + \sqrt{4\eta^2 - \eta^2}}{2} , \quad m_a = 1 , \quad (E25) \]
\[ \lambda_2^- = \frac{-i\eta - \sqrt{4\eta^2 - \eta^2}}{2} , \quad m_a = 1 . \quad (E26) \]

For completeness we report the characteristic polynomial of the Jacobian matrix, that is easily given by
\[ P(\lambda) = \prod_{n=1}^{2V} (\lambda - \lambda_n) , \quad (E27) \]

We are now interested in understanding how \( r \) grows for a system that is initialized in the proximity of the UC for \( \eta < \eta_c^{(2)} \). Consider a configuration initialized as
\[ \theta_j = \frac{2\pi}{V} j + \delta_{\theta,j} , \quad (E28) \]
\[ x_j = 1 + \delta_{x,j} , \]
with \( \delta_{\theta,j}, \delta_{x,j} \ll 1 \) \( \forall j = 1, 2, \ldots, V \). For such configuration we can write \( re^{i\phi} \) as
\[ re^{i\phi} = \frac{1}{V} \sum_{j=1}^{V} x_j e^{i\theta_j} = \frac{1}{V} \sum_{j=1}^{V} (1 + \delta_{x,j}) e^{i\theta_j} e^{\frac{2\pi}{V} j} , \quad (E29) \]
that corresponds to a DFT of the term \((1 + \delta_{x,j}) e^{i\theta_j}\). Approximating the exponential at linear order we obtain
\[ re^{i\phi} \approx \tilde{x}_j + i\tilde{y}_j , \quad (E30) \]
that is \( r \) and \( \phi \) can be written in terms of the unstable Fourier modes [Eq. (E20)], so that it is easy to conclude that for \( \eta < \eta_c = 4 \)
\[ re^{i\phi} \sim (-i\sqrt{\eta(4 - \eta)} + \eta) e^{\frac{\sqrt{19 - \eta}}{3} \tau} . \quad (E31) \]

2. Diagonalization of \( J \) for the SPAC

Considering an even \( V \) and plugging the configuration (22) into Eq. (34) we find the following Jacobian matrix for the SPAC
\[ J_{j,k} = 0 , \quad (E32) \]
\[ J_{j+V,k+V} = 0 , \quad (E33) \]
\[ J_{j+V,k} = -\frac{\eta}{V} \nu_k \nu_j , \quad (E34) \]
\[ J_{j,k+V} = +\frac{\eta}{V} \nu_k \nu_j - 2\delta_{k,j} \]
\[ J_{j+V,k+V} = -\frac{\eta}{V} \nu_k \nu_j + 2\delta_{k,j} \]
where \( \nu_k = 1 \) for \( k = 1, \ldots, V/2 \) and \( \nu_k = -1 \) for \( k = V/2 + 1, \ldots, V \). It is therefore handy to view \( J \) as composed of \( V/2 \times V/2 \)-dimensional blocks and to write a \( 2V \)-dimensional column \( \tilde{y} \) as
\[ \tilde{y} = \begin{pmatrix} \tilde{y}^{(1)} \\ \tilde{y}^{(2)} \\ \tilde{y}^{(3)} \\ \tilde{y}^{(4)} \end{pmatrix} , \quad (E35) \]
\[ \tilde{y}^{(i)} \text{ being a } V/2 \text{-dimensional column vector. The eigenvalue problem (E6) reads then} \]
\[ J\tilde{y} = \begin{pmatrix} +\frac{n}{2} (\tilde{y}^{(3)} - \tilde{y}^{(4)}) - 2\tilde{y}^{(3)} \\ -\frac{n}{2} (\tilde{y}^{(3)} - \tilde{y}^{(4)}) - 2\tilde{y}^{(4)} \\ +\frac{n}{2} (-\tilde{y}^{(1)} + \tilde{y}^{(2)}) \\ -\frac{n}{2} (-\tilde{y}^{(1)} + \tilde{y}^{(2)}) \end{pmatrix} = \lambda \begin{pmatrix} \tilde{y}^{(1)} \\ \tilde{y}^{(2)} \\ \tilde{y}^{(3)} \\ \tilde{y}^{(4)} \end{pmatrix} , \quad (E35) \]

The second equation of (E34) implies the components of \( \tilde{y}^{(3)} \) to be all equal, that is \( y_j^{(3)} = y_0^{(3)} \) \( \forall j = 1, 2, \ldots, V/2 \) and \( y_0^{(1)} = -\lambda y_0^{(3)} \). From the first equation of (E35) we get that also all the components of \( \tilde{y}^{(1)} \) are equal, and we are thus left with
\[ -\eta (\eta - 2) y_0^{(1)} = \lambda^2 y_0^{(1)} . \quad (E36) \]

Since we look for non-trivial solutions (that is with non-zero \( \tilde{y} \)), we consider \( y_0^{(1)} \neq 0 \) and finally obtain the eigenvalues
\[ \lambda^+ = \pm \sqrt{\eta (2 - \eta)} . \quad (E37) \]

Having found all the non-zero eigenvalues, we can state with certainty that the eigenvalues of the Jacobian matrix and their respective algebraic multiplicities \( m_a \) for the SPAC for an even number of sites \( V \) are
\[ \lambda_0 = 0 , \quad m_a = 2V - 2 , \quad (E38) \]
\[ \lambda^+ = +\sqrt{\eta (2 - \eta)} , \quad m_a = 1 , \quad (E39) \]
\[ \lambda^- = -\sqrt{\eta (2 - \eta)} , \quad m_a = 1 , \quad (E40) \]

corresponding to all the roots of the characteristic polynomial of the Jacobian matrix, that thus turns out to be
\[ P(\lambda) = \lambda^{2V - 4} (\lambda^2 + \eta (\eta - 2)) . \quad (E41) \]
3. Diagonalization of $J$ for the SFC

We now study the stability of the SFC. The argument of Sec. IV is actually sufficient to state that the SFC is a nonlinear center of the dynamics for any $\eta > 0$, but for completeness we report here a direct study of its stability by means of the diagonalization of its Jacobian matrix. For the SF (22), the Jacobian (E4) reads

$$
J_{j,k} = 0, \\
J_{j+V, k+V} = 0, \\
J_{j+V, k} = -\frac{\eta}{V} + \eta \delta_{k,j}, \\
J_{j,k+V} = +\frac{\eta}{V} - (2 + \eta) \delta_{k,j},
$$

(E42)

It is therefore again natural to write a 2$V$-dimensional column vector $\vec{y}$ as $\vec{y} = (\vec{y}_0^{(1)}, \vec{y}_0^{(2)})$, $\vec{y}_i$ being a $V$-dimensional column vector. The eigenvalue problem (E6) is rewritten as

$$
J \vec{y} = \left( \begin{array}{c}
\eta \vec{y}_0^{(2)} - (2 + \eta) \vec{y}_0^{(2)} \\
-\eta \vec{y}_0^{(1)} + \eta \vec{y}_0^{(1)}
\end{array} \right) \lambda \left( \begin{array}{c}
\vec{y}_0^{(1)} \\
\vec{y}_0^{(2)}
\end{array} \right),
$$

(E43)

where $\vec{y}_0^{(i)} = \frac{1}{V} \sum_{j=1}^{V} \vec{y}_j^{(i)}$. Looking for non-zero eigenvalues, that is $\lambda \neq 0$, we can multiply the first equation of (E43) by $\lambda$, getting

$$
\left\{ 
\begin{array}{l}
+\eta \lambda \vec{y}_0^{(2)} - (2 + \eta) \lambda \vec{y}_0^{(2)} = \lambda^2 \vec{y}_0^{(1)} \\
-\eta \vec{y}_0^{(1)} + \eta \vec{y}_0^{(1)} = \lambda \vec{y}_0^{(2)}
\end{array}
\right.
$$

(E44)

Plugging $\lambda \vec{y}_0^{(2)}$ from the second equation of (E44) into the first one, we get

$$
-(2 + \eta)\left(-\eta \vec{y}_0^{(1)} + \eta \vec{y}_0^{(1)}\right) = \lambda^2 \vec{y}_0^{(1)},
$$

(E45)

from which we find $\vec{y}_0^{(1)} = 0$, so that

$$
-\eta (2 + \eta) \vec{y}_0^{(1)} = \lambda^2 \vec{y}_0^{(1)},
$$

(E46)

giving $\lambda_{1,2} = \pm i \sqrt{\eta (2 + \eta)}$. These are the only non-zero eigenvalues and can therefore be used to argue on the stability of the SFC. Since for any value of $\eta > 0$ the non-zero eigenvalues are purely imaginary complex conjugate numbers, the SFC is a linear center of the dynamics for any $\eta > 0$. As we already noticed in Sec. IV, the SFC is actually not only a linear center, but a nonlinear center as well.

Appendix F: $V \rightarrow \infty$ limit and continuous equations

We consider the instructive $V \rightarrow \infty$ limit, with potential application in the analytical approach of synchronization phenomena for the bosonic MF phases $\{\theta_j\}_{j=1,...,V}$. We replace the discrete site index $j = 1, 2, \ldots, V$ with a continuous variable $s \in (0, 2\pi)$, so that the DEs (13) transform into

$$
\begin{cases}
\left( \frac{\partial}{\partial t} \right) \sqrt{\rho(s,t) e^{i\phi(s)}} = \eta r \sin (\theta(s,t) - \phi) \\
\left( \frac{\partial}{\partial t} \right) \sqrt{\rho(s,t) e^{i\phi(s)}} = \frac{\eta r}{1 + \delta(s,t)} \cos (\theta(s,t) - \phi) - 2 \delta(s,t) - \delta(s,t)^2,
\end{cases}
$$

(F1)

where the DOP $\Psi = r e^{i\phi}$ is redefined as

$$
re^{i\phi} = \frac{1}{V} \sum_{j=1}^{V} \sqrt{\rho_{j}} e^{i\theta_{j}}, \quad V \rightarrow \infty, \quad \int_{0}^{2\pi} ds \sqrt{\rho(s)} e^{i\phi(s)}.
$$

(F2)

In this way we passed from a system of $2V$ ordinary differential equations in the $2V$ variables $\{\rho_j, \theta_j\}_{j=1,...,V}$, to a system of $2$ integro-differential equations in the variables $\rho(s,t)$ and $\theta(s,t)$. Notice that for equation (F1) to be valid we require as assumption that there exists a permutation of the sites indexes such that the functions $\rho(s,t)$ and $\theta(s,t)$ are continuous, that is such that $\rho_j \xrightarrow{V \rightarrow \infty} \rho_{j+1}$ and $\theta_j \xrightarrow{V \rightarrow \infty} \theta_{j+1}$ $\forall j = 1, \ldots, V$ and $\rho_j \xrightarrow{V \rightarrow \infty} \rho_{j}$ and $\theta_j \xrightarrow{V \rightarrow \infty} \theta_{j}$. This requirement is for instance fulfilled for the IC (15) of the MI to SF quench, on which we focus here. In the $V \rightarrow \infty$ limit, the UC is defined by

$$
\begin{cases}
\theta_{UC}(s,t) = s + \Omega t \\
\rho_{UC}(s,t) = 1,
\end{cases}
$$

(F3)

that, having $r = 0$ and for $\Omega = -1$, is obviously a FP of the DEs (F1). We move to the frame rotating at angular frequency $\Omega$ and express the state of the system as

$$
\begin{cases}
\theta(s) = s + \xi(s) \\
\sqrt{\rho(s)} = 1 + \delta(s),
\end{cases}
$$

(F4)

Importantly, small $\delta$ and $\xi$ correspond to a system being in the proximity of the UC, but we do not need to assume it. The equations of motion (F1) read

$$
\begin{cases}
\left( \frac{\partial}{\partial t} \right) \sqrt{\rho(s,t) e^{i\phi(s)}} = \eta r \sin (\theta(s,t) - \phi) \\
\left( \frac{\partial}{\partial t} \right) \sqrt{\rho(s,t) e^{i\phi(s)}} = \frac{\eta r}{1 + \delta(s,t)} \cos (\theta(s,t) - \phi) - 2 \delta(s,t) - \delta(s,t)^2,
\end{cases}
$$

(F5)

where $\Psi = r e^{i\phi}$ can be expressed as $re^{i\phi} = \frac{1}{2\pi} \int_{0}^{2\pi} (1 + \delta) e^{i\xi} ds$, where we denoted $[A(s)]_q = \frac{1}{2\pi} \int_{0}^{2\pi} A(s) e^{iqs} ds$. That is, we expressed the DOP $\Psi$ as
the Fourier transform of a composition of the functions \(\delta(s, t)\) and \(\xi(s, t)\) with respect to the variable \(s\). Thus, (F5) reads
\[
\begin{align*}
\frac{\partial \delta}{\partial t} = -\eta \text{Im}\{((1 + \delta)e^{i\xi})_1 e^{-i\theta}\} \\
\frac{\partial \xi}{\partial t} = \frac{n}{1+\delta} \text{Re}\{((1 + \delta)e^{i\xi})_1 e^{-i\theta}\} - 2\delta - \delta^2. \tag{F6}
\end{align*}
\]

We Fourier transform the first equation of (F6) getting
\[
\begin{align*}
\frac{\partial \delta}{\partial t} = -\eta \frac{1}{2} \text{Im}\{((1 + \delta)e^{i\xi})_1 e^{-i\theta}\} \\
\frac{\partial \xi}{\partial t} = \frac{n}{2} \text{Re}\{((1 + \delta)e^{i\xi})_1 e^{-i\theta}\} - \frac{\delta}{2} - \frac{\delta^2}{2}.
\end{align*}
\]

Importantly, to go from the DEs (F1) to (F7) we have introduced no approximations, that is (F7) coincides exactly with the GPE (13). The form of (F7) is particularly convenient since for \(q \neq \pm 1\) we find \(\delta_q = \text{cst}\) (even for the whole nonlinear dynamics). From (F7) it is of course possible to study the linear stability of the UC considering small \(\delta\) and \(\xi\), obtaining the Jacobian eigenvalues \(\lambda^+_{\pm1}, \lambda^-_{\pm1}\) and spotting out a DPT at the critical hopping strength \(\eta_{UC} = 4\) (that is not surprising at all since the results of Sec. V are valid for any \(V \geq 3\)). However, the nonlinearities of the DEs (F7) are the fundamental ingredient to try to capture the emergence of \(\pi\)-synchronization for a MI to SF quench. Such synchronization is encapsulated into the increase of \(S\) up to a finite value, that corresponds to the growth of the Fourier components of \(\xi\) with even wavenumber \(q\). Additionally, a potentially useful idea in analogy with the Kuramoto model [43] is treating \(r\) as a parameter and considering a function \(p(\theta, t)\) describing the density of oscillators at the angle \(\theta\) at time \(t\) (a partial differential equation describing the dynamics of \(p(\theta, t)\) would then be the continuity equation).

[1] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Phys. Rev. Lett. 81, 3108 (1998).
[2] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature 415, 39 (2002).
[3] W. Zwerger, J. Opt. B. Quantum Semichanical Opt. 5, S9 (2003).
[4] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, Nature 449, 324 (2007).
[5] I. Bloch, Nature 453, 1016 (2008).
[6] J. Esteve, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler, Nature 455, 1216 (2008).
[7] U. Schneider, L. Hackermüller, S. Will, T. Best, I. Bloch, T. A. Costi, R. W. Helmes, D. Rasch, and A. Rosch, Science 322, 1520 (2008).
[8] W. S. Bakr, A. Peng, M. E. Tai, R. Ma, J. Simon, J. I. Gillen, S. Foelling, L. Pollet, and M. Greiner, Science 329, 547 (2010).
[9] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, Phys. Rev. Lett. 95, 010402 (2005).
[10] T. Schumm, S. Hofferberth, L. M. Andersson, S. Wildermuth, S. Groth, I. Bar-Joseph, J. Schmiedmayer, and P. Krüger, Nat. Phys. 1, 57 (2005).
[11] R. Gati, B. Hemmerling, J. Fölling, M. Albiez, and M. K. Oberthaler, Phys. Rev. Lett. 96, 130404 (2006).
[12] M. Chuchem, K. Smith-Mannscheck, M. Hiller, T. Kottos, A. Vardi, and D. Cohen, Phys. Rev. A 82, 053617 (2010).
[13] J. A. Stickney, D. Z. Anderson, and A. A. Zozulya, Phys. Rev. A 75, 033608 (2007).
[14] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, arXiv preprint arXiv:1804.11065 (2018).
[15] S. Raghavan, A. Smerzi, S. Fantoni, and S. Shenoy, Phys. Rev. A 59, 620 (1999).
[16] K. Sengupta, S. Powell, and S. Sachdev, Phys. Rev. A 69, 053616 (2004).
[17] C. Kollath, A. M. Läuchli, and E. Altman, Phys. Rev. Lett. 98, 180601 (2007).
[18] M. Moessner and S. Kehrein, Phys. Rev. Lett. 100, 175702 (2008).
[19] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010).
[20] M. Schirö and M. Fabrizio, Phys. Rev. Lett. 65, 1063 (1995).
[21] B. Scioi and G. Biroli, Phys. Rev. Lett. 105, 220401 (2010).
[22] B. Scioi and G. Biroli, J. Stat. Mech. Theory Exp. 2011, P11003 (2011).
[23] B. Scioi and G. Biroli, Phys. Rev. B 88, 201110 (2013).
[24] M. Heyl, A. Polkovnikov, and S. Kehrein, Phys. Rev. Lett. 110, 135704 (2013).
[25] A. Flesch, M. Cramer, I. McCulloch, U. Schollwöck, and J. Eisert, Phys. Rev. A 78, 033608 (2008).
[26] M. Rigol, V. Dunjko, and M. Olshanii, Nature 452, 854 (2008).
[27] M. Cramer, A. Flesch, I. P. McCulloch, U. Schollwöck, and J. Eisert, Phys. Rev. Lett. 101, 063001 (2008).
[28] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, Phys. Rev. Lett. 100, 030602 (2008).
[29] M. Rigol, Phys. Rev. Lett. 103, 100403 (2009).
[30] M. Eckstein, M. Kollar, and P. Werner, Phys. Rev. Lett. 103, 056403 (2009).
[31] G. Biroli, C. Kollath, and A. M. Läuchli, Phys. Rev. Lett. 105, 250401 (2010).
[32] J. Dziarmaga, Adv. Phys. 59, 1063 (2010).
[33] M. Kollar, F. A. Wolf, and M. Eckstein, Phys. Rev. B 84, 054304 (2011).
[34] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. 79, 4950 (1997).
[35] A. Polkovnikov, S. Sachdev, and S. M. Girvin, Phys. Rev. A 66, 053607 (2002).
[36] S. M. Mossmann and C. Jung, Nat. Phys. A 74, 033601 (2006).
[37] R. Gati and M. K. Oberthaler, J. Phys. B: At. Mol. Opt. Phys. 40, R61 (2007).
[38] E. M. Graefe, H. J. Korsch, and A. E. Niederle, Phys.
[39] A. R. Kolovsky, H. J. Korsch, and E. M. Graefe, Nat. Phys. A 80, 023617 (2009).
[40] D. Witthaut, S. Winberger, R. Burioni, and M. Timme, Nat. Comm. 8, 14829 (2017).
[41] Y. Kuramoto, in International symposium on mathematical problems in theoretical physics (Springer, 1975), pp. 420–422.
[42] S. H. Strogatz, Physica D 143, 1 (2000).
[43] J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. 77, 137 (2005).
[44] C. A. Holmes, C. P. Meaney, and G. J. Milburn, Phys. Rev. E 85, 066203 (2012).
[45] T. E. Lee and H. R. Sadeghpour, Phys. Rev. Lett. 111, 234101 (2013).
[46] T. E. Lee, C.-K. Chan, and S. Wang, Phys. Rev. E 89, 022913 (2014).
[47] S. Walter, A. Nunnenkamp, and C. Bruder, Phys. Rev. Lett. 112, 094102 (2014).
[48] V. M. Bastidas, I. Omelchenko, A. Zakharova, E. Schöll, and T. Brandes, Phys. Rev. E 92, 062924 (2015).
[49] P. P. Orth, D. Roosen, W. Hofstetter, and K. Le Hur, Phys. Rev. B 82, 144423 (2010).
[50] K. Le Hur, L. Henriot, L. Herviou, K. Plekhanov, A. Petrescu, T. Goren, M. Schiro, C. Mora, and P. P. Orth, C. R. Phys. (2018).
[51] L. Henriet and K. Le Hur, Phys. Rev. B 93, 064411 (2016).
[52] M. P. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B 40, 546 (1989).
[53] J. Freericks and H. Monien, Europhys. Lett. 26, 545 (1994).
[54] S. Rachel, N. Laflorencie, H. F. Song, and K. Le Hur, Phys. Rev. Lett. 108, 116401 (2012).
[55] G. Boeris, L. Gori, M. D. Hoogerland, A. Kumar, E. Lucioni, L. Tanzi, M. Inguscio, T. Giamarchi, C. D’Errico, G. Carleo, et al., Phys. Rev. A 93, 011601 (2016).
[56] S. Trotzky, Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, Nat. Phys. 8, 325 (2012).
[57] M. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, Rev. Mod. Phys. 83, 1405 (2011).
[58] L. Glazman and A. Larkin, Phys. Rev. Lett. 79, 3736 (1997).
[59] R. Kuzmin, R. Mencia, N. Grabon, N. Mehta, Y.-H. Lin, and V. E. Manucharyan, arXiv preprint arXiv:1805.07379 (2018).
[60] T. Weißl, B. Küng, E. Dumur, A. K. Feofanov, I. Matei, C. Naud, O. Buisson, F. W. Hekking, and W. Guichard, Phys. Rev. B 92, 104508 (2015).
[61] Z. Ristivojevic, A. Petković, P. Le Doussal, and T. Giamarchi, Phys. Rev. Lett. 109, 026402 (2012).
[62] F. Jendrzejewski, A. Bernard, K. Mueller, P. Cheinet, V. Josse, M. Piraud, L. Pezzé, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, Nat. Phys. 8, 398 (2012).
[63] E. V. Doggen, G. Lemarié, S. Capponi, and N. Laflorencie, Phys. Rev. B 96, 180202 (2017).
[64] T. Giamarchi and H. Schulz, Europhys. Lett. 3, 1287 (1987).
[65] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, Nat. Phys. A 55, 4318 (1997).
[66] R. Franzosi, V. Penna, and R. Zecchina, Int. J. Mod. Phys. B 14, 943 (2000).
[67] S. Longhi, J. Phys. B: At. Mol. Opt. Phys. 44, 051001 (2011).
[68] R. Franzosi and V. Penna, Phys. Rev. E 67, 046227 (2003).
[69] B. Liu, L.-B. Fu, S.-P. Yang, and J. Liu, Phys. Rev. A 75, 033601 (2007).
[70] F. Trimborn, D. Witthaut, and H. J. Korsch, Phys. Rev. A 79, 013608 (2009).
[71] J. Prost, The physics of liquid crystals, vol. 83 (Oxford university press, 1995).
[72] T. Langen, R. Geiger, and J. Schmiedmayer, Annu. Rev. Condens. Matter Phys. 6, 201 (2015).
[73] S. H. Strogatz, Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering (CRC Press, 2018).
[74] A. Pikovsky, M. Rosenblum, J. Kurths, and J. Kurths, Synchronization: a universal concept in nonlinear sciences, vol. 12 (Cambridge university press, 2003).