Rational components of Hilbert schemes

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Abstract

The Gröbner stratum of a monomial ideal \( j \) is an affine variety that parametrizes the family of all ideals having \( j \) as initial ideal (with respect to a fixed term ordering). The Gröbner strata can be equipped in a natural way of a structure of homogeneous variety and are in a close connection with Hilbert schemes of subvarieties in the projective space \( \mathbf{P}^n \). Using properties of the Gröbner strata we prove some sufficient conditions for the rationality of components of \( \mathcal{H}ilb_p^m(z) \). We show for instance that all the smooth, irreducible components in \( \mathcal{H}ilb_p^m(z) \) (or in its support) and the Reeves and Stillman component \( H_{RS} \) are rational.

1 Introduction

The aim of the present paper is to investigate effective methods to study the Hilbert schemes of subvarieties in the projective space \( \mathbf{P}^n \), both on the theoretical and the computational point of view, using Gröbner basis tools. Several authors have been working in this direction during last years, but our motivations mainly refer to some ideas and hints contained in a paper by Notari and Spreafico ([13]). In order to get a stratification of \( \mathcal{H}ilb_p^m(z) \), they introduce some affine varieties \( St(j) \) (here called Gröbner strata) parameterizing families of ideals in \( k[X_0, \ldots, X_n] \) having the same initial ideal \( j \) with respect to a fixed term ordering on the monomials. When only homogeneous ideals in \( k[X_0, \ldots, X_n] \) are concerned, we write \( St_h(j) \). The ideal defining a Gröbner stratum springs out of a procedure based on Buchberger’s algorithm, but involves a reduction with respect to a set of polynomials which is not a Gröbner basis.

It is not difficult to realize that the support of \( St(j) \) only depends on the initial data (the term ordering, the ideal \( j \), etc.), but one cannot be beforehand sure that different choices in the reduction steps always lead to the same ideal. In other words it is not clear if Gröbner strata are scheme-theoretically well defined. This is a crucial point: if anyone wants to deduce properties of an Hilbert scheme using a Gröbner stratum, it is necessary to consider carefully the non-reduced structures, because \( \mathcal{H}ilb_p^m(z) \) can have non-reduced components (see [7], [8] and [12]). A first achievement in this paper is the following result (see Theorem 3.6):

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Theorem A. The ideal defining $S_t(j)$ does not depend on the reduction choices.

In fact we exhibit an equivalent, but intrinsic definition for the ideal of $S_t(j)$, which by the way also allows a great simplification in the procedure for an explicit computation of this ideal.

A meaningful property that all Gröbner strata enjoy is that they are homogeneous with respect to some non-standard graduation. Homogeneous varieties are of a very special type: for instance they can be isomorphically embedded in the Zariski tangent space at the origin, which is of course the smallest affine space in which such an embedding can be done. Therefore a smooth homogeneous variety has to be isomorphic to an affine space; in fact, the variety has the same dimension as the space in which it can be embedded. For these and other properties of the homogeneous structure, we refer to [10 and 3] (for the zero-dimensional case, see also [15, Corollary 3.7]). In the quoted papers, it is explained how one can obtain directly the ideal of $S_t(j)$ in the “minimal embedding” in the Zariski tangent space, through a preliminary detection of a maximal set of “eliminable variables” (we briefly resume this method in § 4). This is a second key point in our work, because one of the main difficulties usually met studying Gröbner strata (and even more Hilbert schemes) is due to the huge number of variables that their equations involve, even in very simple cases. Besides the obvious computational gain, we would like to underline the interesting theoretical outcome of this method: most of our proofs are obtained just using the minimal embedding. In this way we can show, for instance, that there are many different monomial ideals whose homogeneous Gröbner strata are canonically isomorphic (Proposition 4.7 and Corollary 4.8), so that we can equivalently take into consideration either one of them.

Theorem B. Consider a term ordering $\leq$ in $k[X_0, \ldots, X_n]$ such that $X_n \succ \cdots \succ X_0$ and let $j \subset k[X_0, \ldots, X_n]$ be a Borel-fixed, saturated monomial ideal. If $X_1$ does not appear in any monomial of the base of $j$ of degree $> s$, then for every $m \geq s$:

$$S_t(h(j_{\geq s}) \simeq S_t(h(j_{\geq m})).$$

Especially, if $X_1$ does not appear in the base of $j$, then for every $m$:

$$S_t(h(j) \simeq S_t(h(j_{\geq m})).$$

In § 5 and § 6 we investigate more closely the natural connection between the Gröbner stratum $S_t(h(j)$ and the Hilbert scheme $\text{Hilb}^n_{p(z)}$, where $p(z)$ is the Hilbert polynomial of $k[X_0, \ldots, X_n]/j$. As $k[X_0, \ldots, X_n]/i$ and $k[X_0, \ldots, X_n]/j$ share the same Hilbert function, there is an obvious set-theoretic inclusion $S_t(h(j) \subseteq \text{Hilb}^n_{p(z)}$. However it is not a simple task to understand if this inclusion is an algebraic embedding or not. The paper [13] deals with the same question, but mainly concerns Gröbner strata of saturated ideals with respect to the term ordering $\text{DegRevLex}$: note that every subscheme $Z$ in $\mathbb{P}^n$ can be defined by the saturated ideal $I(Z)$. In this paper we prefer to consider a slightly different approach, modeled on the classical construction of the Hilbert schemes. For every $Z \in \text{Hilb}^n_{p(z)}$ we consider the ideal $I(Z)_{>r}$, where $r$ is the Gotzmann number of $p(z)$. As $r$ is the worst Castelnuovo-Mumford regularity for all $Z \in \text{Hilb}^n_{p(z)}$, the Gröbner strata (with respect to any term ordering) of monomial ideals generated in degree $r$ cover $\text{Hilb}^n_{p(z)}$.

Moreover, the subset of $\text{Hilb}^n_{p(z)}$ corresponding to $S_t(h(j_{>r})$ always contains the one corresponding to $S_t(h(j)$ and the inclusion can be strict, because ideals in $S_t(h(j)$ define subvarieties
in \( \mathbb{P}^n \) with the same Hilbert function as the subscheme \( Z = \mathcal{V}(j) \), while being in \( S_{th}(j_{\geq r}) \) only requires the same Hilbert polynomial. An interesting example of this type is that of the lexicographic saturated ideal \( \mathcal{L} \) such that \( k[\pi_0, \ldots, \pi_n]/\mathcal{L} \) has Hilbert polynomial \( p(z) \) and whose regularity is indeed the Gotzmann number \( r \) of \( p(z) \): in \( \S 7 \) we show that \( S_{th}(\mathcal{L}_{\geq r}) \) is isomorphic to an open subset of the Reeves and Stillman component \( H_{RS} \) of \( \text{Hilb}^n_{p(z)} \), while in general \( S_{th}(\mathcal{L}) \) corresponds to a locally closed subvariety of lower dimension (see [16, Remark 4.8]).

The main reason of our setting is contained in Theorem 6.3. Let \( p(z) \) be any admissible Hilbert polynomial in \( \mathbb{P}^n \) with Gotzmann number \( r \) and let \( \preceq \) be a fixed term ordering on monomials of \( k[\pi_0, \ldots, \pi_n] \). Following the classical construction, we consider \( \text{Hilb}^n_{p(z)} \) as a subvariety of a projective space through the Plücker embedding of the Grassmannian \( G(t, M) \), where \( M = \dim_k(k[\pi_0, \ldots, \pi_n], r) \) and \( t = M - p(r) \). The simple remark that the Plücker coordinates correspond to sets of \( t \) distinct monomials of degree \( r \) (that we can write in decreasing order with respect to \( \preceq \)), allows us to get a lexicographic total order on them.

**Theorem C.** If \( j \) is a monomial ideal generated in degree \( r \) such that \( k[\pi_0, \ldots, \pi_n]/j \) has Hilbert polynomial \( p(z) \) with Gotzmann number \( r \), then \( S_{th}(j) \) is naturally isomorphic to the locally closed subvariety of \( \text{Hilb}^n_{p(z)} \) given by the conditions that the Plücker coordinate corresponding to the monomial base of \( j \) does not vanish and the bigger ones vanish.

As a consequence we are able to prove that every irreducible and reduced component of \( \text{Hilb}^n_{p(z)} \) (or of its support) has an open subset which is a homogeneous affine variety (with respect to a non-standard graduation). Especially, if \( j \) is generated by the \( t \) largest degree \( r \) monomials (we call it a \((r, \preceq)\)-segment ideal), then \( S_{th}(j) \) is naturally isomorphic to an open subset of \( \text{Hilb}^n_{p(z)} \). Therefore we can easily deduce a few interesting properties of rationality for the components of Hilbert schemes (see Theorem 6.4):

**Theorem D.** Let \( H \) be an irreducible component of \( \text{Hilb}^n_{p(z)} \).

- If \( H \) is smooth, then it is rational. The same holds for its support \( \text{Supp} H \).
- If \( H \) contains a smooth point which corresponds to a \((r, \preceq)\)-segment ideal (where \( \preceq \) is any term ordering), then \( H \) is rational. The same holds for \( \text{Supp} H \).
- The Reeves and Stillman component \( H_{RS} \) of \( \text{Hilb}^n_{p(z)} \) is rational.

The last item can be obtained as a direct consequence of the previous one, because the lexicographic saturated ideal \( \mathcal{L} \) corresponds to a smooth point in \( H_{RS} \), as proved by Reeves and Stillman in [14], and \( \mathcal{L}_{\geq r} \) is a \((r, \text{Lex})\)-segment ideal. However, we can also get a new proof of this fact, not applying the quoted result by Reeves and Stillman, but proving that \( S_{th}(\mathcal{L}_{\geq r}) \) is isomorphic to an affine space using our method based on the minimal embedding (see Theorem 7.3).

It is not difficult to implement this method making use of one of the several softwares for symbolic computation. Using such a routine we are able to write equations for some Gröbner strata corresponding to the Hilbert scheme \( \text{Hilb}^n_{\mathbb{Z}_2} \), founding a computational confirmation and some improvements of the results obtained by Gotzmann in the recent preprint [5]. In fact we could find that there are two components of dimension 23 and 16, as proved by Gotzmann, and also that they are reduced, rational and intersect transversally.
The paper is organized as follows. §2 contains some general notation.

In §3 we take up the construction of Gröbner strata made in [16] and prove that they are well defined (Theorem 3.6).

In §4 we discuss the main properties of Gröbner strata as homogeneous varieties with respect to a non-standard graduation and we obtain some useful criterion in order to know when Borel-fixed monomial ideals with the same saturation define the same Gröbner stratum (Proposition 4.7 and Corollary 4.8).

In §5, we focus our attention on ideals generated in degree \( r \), where \( r \) is the Gotzmann number of their Hilbert polynomials, and prove that their Gröbner strata can be defined by minors of suitable matrices (Theorem 5.5).

§6 represents the heart of the work. We show that there is a close connection between the above quoted matrices defining Gröbner strata and those appearing in the classical construction of Hilbert schemes and obtain as a consequence the main results of the paper about rational components (Theorem 6.4).

Finally, in §7 we prove the rationality of the Reeves-Stillman component \( H_{RS} \) of \( \text{Hilb}^{\alpha}_p(z) \) using our method based on the minimal embedding and in §8 we apply this same method in order to perform some explicit computations about \( \text{Hilb}^{\frac{3}{2}}_4 \).

2 Notation

Throughout the paper, we will consider the following general notation.

1. During the construction of Gröbner strata, we work on a field \( k \) of any characteristic, whereas when we study the Hilbert scheme, we will suppose that \( k \) is algebraically closed.

2. \( k[X_0, \ldots, X_n] \) is the polynomial ring in the set of variables \( X_0, \ldots, X_n \) that we will often denote by the compact notation \( X \), so that \( k[X] := k[X_0, \ldots, X_n] \); we will denote by \( X^\alpha \) the generic monomial in \( k[X] \), where \( \alpha \) represents a multi-index \( (\alpha_0, \ldots, \alpha_n) \), that is \( X^\alpha := X_0^{\alpha_0} \cdots X_n^{\alpha_n} \). \( j \) will be a monomial ideal in \( k[X] \) with base \( \{ X^\gamma_1, \ldots, X^\gamma_t \} \) and \( \text{Syz}(j) \) its \( k[X] \)-module of syzygies.

\( \preceq \) will be a fixed term ordering on the set \( T_X \) of monomials in \( k[X] \) and we always assume that \( X_n \succ \cdots \succ X_0 \). We know that it is not the usual assumption, but this choice will result more convenient, mainly in last sections. As the term order \( \preceq \) is fixed, we often omit to indicate it; for instance the stratum of an ideal \( j \) also depends on the term ordering, but in general we will omit it in the symbol \( \text{St}(j, T) \), that more precisely should be \( \text{St}(j, T, \preceq) \).

For every polynomial \( F \) in \( k[X] \) (or \( k[X, C], k[C] \)), \( \text{LT}(F) \) is its leading term with respect to the fixed term ordering; in the same way, if \( a \) is an ideal, \( \text{LT}(a) \) is its initial ideal.

3. We will introduce a second set of variables \( C_\alpha \) that we will denote with \( C \). So \( k[X, C] \) will be the polynomial ring in the variables \( X \) and \( C \) and \( T_{X,C} \) the corresponding set of monomials. We will consider on \( T_{X,C} \) any term ordering which is an elimination ordering of the variables \( X \) and coincides with \( \preceq \) on \( T_X \); we will denote by the same symbol \( \preceq \) also this term ordering on \( T_{X,C} \) and the induced term ordering on \( T_C \).

\( \mathbb{T}_{X,C}, \mathbb{T}_X, \mathbb{T}_C \) will denote the multiplicative groups of Laurent monomials.
4. Let $G$ be any polynomial in $k[C, X]$. An $X$-monomial of $G$ is a monomial of $T_X$ that appears in $G$ considered as a polynomial in the variables $X$ with coefficients in the ring $k[C]$; the $X$-coefficients of $G$ are the elements of $k[C]$ that are coefficients of an $X$-monomial. Note that the $X$-coefficients are polynomials, but not necessary monomials.

5. $X^{\alpha} \mid X^{\gamma}$ means that $X^{\alpha}$ divides $X^{\gamma}$, that is, there exists a monomial $X^{\beta}$ such that $X^{\alpha} \cdot X^{\beta} = X^{\gamma}$. If such monomial does not exist, we will write $X^{\alpha} \nmid X^{\gamma}$.

6. Given any subscheme $Z$ in $\mathbb{P}^n$, we will denote by $\text{Supp} Z$ its support and by $I(Z)$ the saturated ideal in $k[X]$ that defines $Z$. Given any ideal $a$, we will denote by $\mathcal{V}(a)$ the zero set of $a$.

7. $\mathcal{H}_{\text{hilb}}^{p(z)}$ will denote the Hilbert scheme parametrizing all subschemes $Z$ in $\mathbb{P}^n$ with Hilbert polynomial $p(z)$. $r$ will be the Gotzmann number of $p(z)$; that is, the worst Castelnuovo-Mumford regularity among subschemes parametrized by $\mathcal{H}_{\text{hilb}}^{p(z)}$. When we write that an ideal $i \subset k[X]$ belongs to $\mathcal{H}_{\text{hilb}}^{p(z)}$, we will mean that $i$ is generated in degree $r$ and that the Hilbert polynomial of $\text{Proj} k[X]/i$ is $p(z)$. By abuse of notation we will say that any such ideal $i$ has Hilbert polynomial $p(z)$ referring to the Hilbert polynomial of the quotient, even if the real Hilbert polynomial of $i$ is $(\binom{p+n}{n} - p(z))$.

## 3 The ideal of a Gröbner Stratum

Now we introduce the Gröbner strata and prove some properties, generalizing definitions and results of the paper [16].

**Definition 3.1.** The tail of $X^{\gamma}$ with respect to $j$ (and to the fixed term ordering $\preceq$) is the set of monomials:
\[
T_j = \{ X^{\alpha} \in T_X \mid X^{\alpha} \prec X^{\gamma}, \ X^{\alpha} \notin j \} \tag{1}
\]

Every ideal $i$ such that $\text{LT}(i) = j = (X^{\gamma_1}, \ldots, X^{\gamma_l})$ has a reduced Gröbner base of the type \{ $f_1, \ldots, f_t$ \} where:
\[
f_i = X^{\gamma_i} + \sum_{X^{\alpha} \in T_{\gamma_i}} c_{i\alpha} X^{\alpha} \tag{2}
\]

and $c_{i\alpha} \in k$, $c_{i\alpha} = 0$ except finitely many of them. It is very natural to parametrize the family of all the ideals $i$ by the coefficients $c_{i\alpha}$; in this way it corresponds to a subset of $k^T$, where $T = T_{\gamma_1} \times \cdots \times T_{\gamma_l}$.

In many interesting cases, $T_{\gamma_i}$ are finite sets and so $k^T$ is an affine space: this happens for instance if $j$ is a zero-dimensional ideal or if $\preceq$ is a suitable term ordering; in other cases, for instance when only homogeneous ideals are concerned, $T$ can be infinite, but we can restrict our interest to a suitable finite subset. The following definition extends and includes all the previous cases.

**Definition 3.2.** Let us fix $T = \{T_1, \ldots, T_l\}$ where $T_i$ is a finite subset of the tail of $X^{\gamma_i}$ with respect to $j$. We will denote by $\text{St}(j, T)$ the family of all ideals $i$ in $k[X]$ such that $\text{LT}(i) = j$ and whose reduced Gröbner basis $f_1, \ldots, f_t$ is of the type:
\[
f_i = X^{\gamma_i} + \sum_{X^{\alpha} \in T_i} c_{i\alpha} X^{\alpha}. \tag{3}
\]
Moreover we will use the following special notation:
i) $St(i)$, if $T_i = T_{\gamma_i}$ (of course only if $T_{\gamma_i}$ are finite sets): $St(i)$ parametrizes all the ideals $i$ such that LT($i$) = $j$.

ii) $St_h(i)$, if $T_i$ is the subset of $T_{\gamma_i}$ of the monomials with the same degree as $X^{\gamma_i}$: $St_h(i)$ parametrizes all the homogeneous ideals $i$ such that LT($i$) = $j$.

**Remark 3.3.** It will be clear later that the term ordering affects the construction of a Gröbner stratum only because it states which monomials can belong to the tails; in fact two different term ordering giving the same tails will lead to the same Gröbner strata.

Every ideal $i$ in the family $St(j, T)$ is uniquely determined by a point in the affine space $\mathbb{A}^N$ ($N = \sum_i |T_i|$) where we fix coordinates $C_{i,\alpha}$ corresponding to the coefficients $c_{i\alpha}$ that appear in (3). The subset of $\mathbb{A}^N$ corresponding to $St(j, T)$ turns out to be a closed algebraic set. More precisely, we will see how it can be endowed in a very natural way with a structure of affine subvariety, possibly reducible or non reduced, that is we will see that it can be obtained as the variety defined by an ideal $h(j, T)$ in $k[C]$, where $C$ is the set of variables $C_{i\alpha}$.

In the following, we refer to the terminology introduced in Notation 4 for what concerns the polynomials in $k[X, C]$.

**Definition 3.4.** We will denote by $h(j, T)$ and $L(j, T)$ respectively any ideal in $k[C]$ that can be obtained in the following way.

- Let $B = \{F_1, \ldots, F_t\}$ be the set of polynomials in $k[X, C]$ given by:

  $$F_i = X^{\gamma_i} + \sum_{X^\alpha \in T_i} C_{i\alpha} X^\alpha. \quad (4)$$

- Consider any term order in $k[X, C]$ which is an elimination order for the variables $X$ and that coincides with $\preceq$ for monomials in $T_X$; there will be no confusion if we denote it by the same symbol $\preceq$. With respect to such a term order, the leading term of $F_i$ is $X^{\gamma_i}$.

- Fix the subset $P$ of $\{(i, j) \mid 1 \leq i < j \leq m\}$ corresponding to any set of generators for Syz($j$);

- For every $(i, j) \in P$, let $R_{ij}$ be a complete reduction of the $S$-polynomial $S(F_i, F_j)$ with respect to $B$.

- For every $(i, j) \in P$, let $M_{ij}$ be a complete reduction of $S(F_i, F_j)$ with respect to $j$.

- $h(j, T)$ is the ideal in $k[C]$ generated by the $X$-coefficients of the polynomials $R_{ij}$, $(i, j) \in P$.

- $L(j, T)$ is the $k$-vector space in $(C)$ generated by the $X$-coefficients of $M_{ij}$, $(i, j) \in P$.

It is almost evident, that the definition of $h(j, T)$ is nothing else than Buchberger’s characterization of Gröbner bases if we think to the $C_{i\alpha}$’s as constant in $k$ instead of variables. In fact the variables $C$ do not appear in the leading terms of $F_i$ and so their specialization in $k$ commutes with reduction with respect to $B$. Thus $(\ldots, c_{i\alpha}, \ldots)$ is a point in the algebraic set $\mathcal{V}(h(j, T))$ in $\mathbb{A}^N$ if and only if it corresponds to polynomials $f_1, \ldots, f_m$ in $k[X]$ that are
a Gröbner base. Then \( V(\mathfrak{h}(j, T)) \) is uniquely defined; however a priori the ideal \( \mathfrak{h}(i, T) \) could depend on the choices we perform computing it, that is on the choice of the set \( P \) of generators for \( \text{Syz}(j) \) and on the choice of a reduction for the \( S \)-polynomials \( S(F_i, F_j) \) with respect to \( B \) (which in general is not uniquely determined).

Thanks again to Buchberger’s criterion, we can prove that in fact \( \mathfrak{h}(j, T) \) only depends on \( j, T \) and of course \( \preceq \) because it can be defined in an equivalent intrinsic way.

**Proposition 3.5.** Let \( j \subseteq k[X], \ B = \{F_1, \ldots, F_i\} \subseteq k[X, C] \) and \( \preceq \) be as above and consider an ideal \( \mathfrak{a} \) in \( k[C] \) with Gröbner base \( A \). The following are equivalent:

1) \( B \cup A \) is a Gröbner base in \( k[X, C] \);

2) \( \mathfrak{a} \) contains all the \( X \)-coefficients of every complete reduction of \( S(F_i, F_j) \) with respect to \( B \) for every \( i, j \);

3) \( \mathfrak{a} \) contains all the \( X \)-coefficients of some (even partial) reduction with respect to \( B \) of \( S(F_i, F_j) \) for every \( i, j \);

4) \( \mathfrak{a} \) contains all the \( X \)-coefficients of some (even partial) reduction with respect to \( B \) of \( S(F_i, F_j) \), for every \( (i, j) \) corresponding to a set of generators of \( \text{Syz}(j) \).

**Proof.** \( 1 \Rightarrow 2 \) let \( G \) be a complete reduction of \( S(F_i, F_j) \) with respect to \( B \). By hypothesis, \( G \) must be reducible to \( 0 \) through \( B \cup A \), so that the next step of reduction have to be performed just using \( A \). But any step of reduction through \( A \) does not change the \( X \)-monomials and only modifies the \( X \)-coefficients; then \( G \overset{A}{\rightarrow} 0 \), that is every \( X \)-coefficient in \( G \) can be reduced to \( 0 \) using \( A \): this shows that all the \( X \)-coefficients in \( G \) belong to \( \mathfrak{a} \).

\( 2 \Rightarrow 3 \) and \( 3 \Rightarrow 2 \) are obvious.

\( 4 \Rightarrow 1 \) we can check that \( B \cup A \) is a Gröbner base using the refined Buchberger criterion (see for instance [2, Theorem 9, page 104]). If \( A = \{a_1, \ldots, a_r\} \), a set of generators for \( \text{Syz}(X^{r_1}, \ldots, X^{r_n}, \text{LT}(a_1), \ldots, \text{LT}(a_r)) \) can be obtained as the union of a set of generators for \( \text{Syz}(X^{r_1}, \ldots, X^{r_n}) \), a set of generators for \( \text{Syz} \)\((\text{LT}(a_1), \ldots, \text{LT}(a_r)) \) and the obvious syzygies of \( (X^{r_1}, \text{LT}(a_i)) \). Then:

- \( S(a_i, a_j) \rightarrow_{B \cup A} 0 \), since \( A \) is a Gröbner base and \( A \subseteq B \cup A \);
- \( S(a_i, F_j) \rightarrow_{B \cup A} 0 \), since the leading terms of \( a_i \) and \( F_j \) are coprime and \( a_i, F_j \in B \cup A \);
- \( S(F_i, F_j) \rightarrow_{B \cup A} 0 \) in at least one way, by hypothesis.

There are many ideals \( \mathfrak{a} \) fulfilling the equivalent conditions of Proposition 3.5 for instance we can consider the irrelevant maximal ideal in \( k[C] \) or any ideal obtained accordingly with condition 3. Moreover, if \( \mathfrak{a} \) satisfies those conditions and \( \mathfrak{a}' \supseteq \mathfrak{a} \), then also \( \mathfrak{a}' \) does, and if the ideals \( \mathfrak{a}_i \) satisfy the conditions, then also their intersection \( \bigcap \mathfrak{a}_i \) does. As a consequence of these remarks we obtain the proof of the unicity of the ideal \( \mathfrak{h}(j, T) \) given by Definition 3.4

**Theorem 3.6.** Let \( j \) and \( T \) as above. Then:

i) \( \mathfrak{h}(j, T) \) is uniquely defined; in fact \( \mathfrak{h}(j, T) = \bigcap \mathfrak{a}, \mathfrak{a} \) satisfying the equivalent conditions of Proposition 3.5.

ii) \( L(j, T) \) is uniquely defined.
Proof. If \( \mathfrak{h} \) is one of the ideals \( \mathfrak{a} \), because it satisfies condition \( [1] \); on the other hand, if \( \mathfrak{a} \) satisfies condition \( [2] \), then clearly \( \mathfrak{a} \supseteq \mathfrak{h} \).

For \( [3] \) it is sufficient to observe that the generators for \( L(i, T) \) are the degree 1 homogeneous components (here “homogeneous” is related to the usual graduation of \( k[C] \) that is the \( \mathbb{Z} \)-graduation with variables of degree 1) of the generators of \( \mathfrak{h}(i, T) \) given in its construction (Definition 3.4).

By abuse of notation we will denote by the same symbol \( St(j, T) \) the family of ideals and the subvariety in \( \mathbb{A}^N \) given by the ideal \( \mathfrak{h}(i, T) \). Note that \( \mathfrak{h}(i, T) \) is not always a prime ideal and so \( St(j, T) \) is not necessarily irreducible nor reduced, as shows the following trivial example.

**Example 3.7.** Let \( j = (x^2, xy) \subset k[x, y] \) and \( \preceq \) be any term ordering. Let us choose \( T = (T_{x^2} = \emptyset, T_{xy} = \{y\}) \) and construct the ideal of the Gröbner stratum \( St(j, T) \) according to Definition 3.4:

\[
\{F_1 = x^2, F_2 = xy + Cy\}, \quad S_{12} = yF_1 - xF_2 = -Cxy \frac{F_1}{F_2}, \quad R_{12} = -Cxy + CF_2 = C^2y.
\]

Then \( \mathfrak{h}(j, T) = (C^2) \) that is \( St(j, T) \) is a double point in the affine space \( \mathbb{A}^1 \).

4 Gröbner strata and homogeneous varieties

In this section we will see how every Gröbner stratum \( St(j, T) \) is in a very natural way a homogeneous variety with respect to a suitable non-standard graduation on \( k[C] \), so that we can apply the nice properties typical of this kind of varieties and especially those obtained in [10] and in [4].

For the meaning of \( j \), \( k[X] \), \( \{X^{\gamma_1}, \ldots, X^{\gamma_t}\} \) and \( \preceq \) we refer to Notation 2 and for \( k[X, C] \), \( \{F_1, \ldots, F_t\} \), \( St(j, T) \), \( \mathfrak{h}(i, T) \) to the previous section.

First of all, we recall the definitions and properties that we will use more often.

**Definition 4.1.** (See [4], Definition 2.1) or [10], Definition 2.7) Let \((G, +)\) be an abelian group and let \( g_1, \ldots, g_s \) be elements in \( G \) (not necessarily distinct).

The group homomorphism:

\[
\lambda : \mathbb{T}_Y \rightarrow G \text{ given by } Y_i \mapsto g_i
\]

induces a graduation on the polynomial ring \( k[Y] := k[Y_1, \ldots, Y_\gamma] \).

A polynomial \( F \in k[Y] \) is \( \lambda \)-homogeneous of \( \lambda \)-degree \( g \) if \( \lambda(Y^\beta) = g \) for every monomial \( Y^\beta \) that appears in \( F \). A \( \lambda \)-homogeneous ideal \( \mathfrak{a} \) is a proper ideal \( (\mathfrak{a} \neq k[X, C]) \) generated by \( \lambda \)-homogeneous polynomials. A \( \lambda \)-cone is the subvariety \( V \) defined by a \( \lambda \)-homogeneous ideal.

We always assume that \( \lambda \) is positive (see [4], Chapter 4)), that is that the following equivalent conditions are satisfied:

i) \( k[Y]_{0_G} = k \);

ii) \( \sum n_i g_i = 0_G \) with \( n_i \geq 0 \iff n_i = 0 \) for every \( i \);

iii) \( G \) can be endowed of a structure of totally ordered group \((G, +, \preceq)\) in such a way that \( \lambda(Y_i) > 0_G \) for every \( i = 1, \ldots, s \).
(for the equivalence of these three conditions see [4, Lemma 2.2]).

**Proposition 4.2.** (See [16, Lemma 2.8]) Let $G$ be the free abelian group $\mathbb{Z}^{n+1}$. The group homomorphism:

$$
\lambda : \overline{T}_C \longrightarrow G \quad \text{given by} \quad C_{i\alpha} \mapsto \gamma_i - \alpha
$$

induces a positive graduation on $k[C]$ with respect to which $h(j, T)$ is a $\lambda$-homogeneous ideal.

**Proof.** First of all we remark that condition (ii) of Definition 4.1 holds: in fact we obtain a total ordering on $G$ using the term order $\preceq (\alpha \preceq \beta \iff X^\alpha \preceq X^\beta)$ and that $\lambda(C_{i\alpha}) > 0_G$ because $X^{\gamma_i} \succ X^\alpha$.

We can consider $\lambda$ as the restriction to $\overline{T}_C$ of a group homomorphism $\overline{T}_{X,C} \rightarrow G$ (that we can denote again by $\lambda$) given by $\lambda(X^\alpha) = \alpha$. Every monomial that appears in $F_i$ is of the type $C_{i\alpha}X^\alpha$ and so its $\lambda$-degree is $\lambda(C_{i\alpha}X^\alpha) = \lambda(C_{i\alpha}) + \lambda(X^\alpha) = \gamma_i$. Thus all the polynomials $F_i$ are $\lambda$-homogeneous and then also the $S$-polynomials $S(F_i, F_j)$ and their reductions are $\lambda$-homogeneous. Finally, the $X$-coefficients in any $\lambda$-homogeneous polynomial (which are polynomials in $k[C]$) are $\lambda$-homogeneous. \qed

We now recall some properties of $L(j, T)$ proved in [16, Proposition 2.4] and [4, Theorem 3.2].

**Theorem 4.3.** The linear space $V(L(j, T))$ can be naturally identified with the Zariski tangent space to $St(j, T)$ at the origin.

If $C'' \subset C$ is any subset of $ed := \dim V(L(j, T))$ variables such that $L(j, T) \oplus \langle C'' \rangle = \langle C \rangle$, then $h(j, T) \cap k[C'']$ defines a $\lambda$-homogeneous subvariety in $\mathbb{A}^{ed}$ isomorphic to $St(h, T)$.

We may summarize the previous result saying that $St(h, T)$ can be embedded in its Zariski tangent space at the origin. This explains the following terminology.

**Definition 4.4.** The number $ed$ is the embedding dimension of $St(j, T)$. The complement $C' := C \setminus C''$ is a maximal set of eliminable variables for $h(j, T)$.

**Corollary 4.5.** In the above notation, the following statements are equivalent:

1. $St(j, T) \simeq \mathbb{A}^{ed}$;
2. $St(j, T)$ is smooth;
3. the origin is a smooth point for $St(j, T)$;
4. $ed \leq \dim St(j, T)$.

Note that in general a maximal set of eliminable variables (and so its complementary) is not uniquely determined. However, if $C_{i\alpha} \in L(j, T)$, then $C_{i\alpha}$ belongs to any set of eliminable variables; on the other hand, if $C_{i\alpha}$ does not appear in any element of $L(j, T)$, then $C_{i\alpha}$ does not belong to any set of eliminable variables.

There is an easy criterion that allows us to decide if a variable is eliminable or not.

**Criterion 4.6.** Let $LT(F_i) = X^{\gamma_i}$, $LT(F_j) = X^{\gamma_j}$ and let $C_{i\beta}$ be a variable appearing in the tail of $F_i$. Using the reduction with respect to $j$ of a $\lambda$-homogeneous polynomial $X^{\delta}F_i - X^{\eta}F_j$ we can see that:
i) if $X^{\delta+\beta} \notin j$ and $X^{\delta+\beta-\eta}$ is not a monomial that appears in $F_j$, then $C_{i\beta} \in L(j,T)$;

ii) if $X^{\delta+\beta} \notin j$ and $X^{\beta'} = X^{\delta+\beta-\eta}$ is a monomial that appears in $F_j$, then $C_{i\beta} - C_{j\beta'} \in L(j,T)$

Moreover if $C_{i\beta} - C_{j\beta'} \in L(j,T)$, then every maximal set of eliminable variables must contain at least either one of them.

In most cases the number $N = |C|$ is very big and $\mathfrak{h}(j,T)$ needs a lot of generators so that finding it explicitly is a very heavy computation. On the contrary $L(j,T)$ is very fast to compute and so we can easily obtain a set of eliminable variables $C'$; a foregoing knowledge of $C'$ allows a simpler computation of the ideal $\mathfrak{h}(j,T) \cap k[C']$ that gives $St(j,T)$ embedded in the affine space of minimal dimension $\mathbb{A}^d$. Furthermore, in many interesting cases we can greatly bring down the number of involved variables thanks to another kind of argument.

Let $X^\alpha, X^{\gamma_1}, \ldots, X^{\gamma_t}$ be monomials in $k[X]$ such that $|\gamma_i| > m := |\alpha|$ for every $i = 1, \ldots, t$ and consider the monomial ideals $j_0 = (X^\alpha, X^{\gamma_1}, \ldots, X^{\gamma_t})$ and $j = (j_0)_{m+1}$, that is $j = (X_nX^\alpha, \ldots, X_0X^\alpha, X^{\gamma_1}, \ldots, X^{\gamma_t})$. We can obtain the homogeneous stratum $\mathcal{S}t_h(j_0)$ starting from polynomials $F_j \in k[X,C]$ with $LT(F_j) = X^{\gamma_j}$ for $j > 0$ and $LT(F_0) = X^\alpha$; moreover we can obtain $\mathcal{S}t_h(j)$ starting from the same polynomials $F_j$ if $j = 1, \ldots, t$ (note that the reduced homogeneous tail of $X^{\gamma_j}$ with respect to $j$ and that with respect to $j_0$ coincide) and new polynomials $G_i = X_iX^\alpha + \sum C'_{i\eta}X^\eta$, for every $i > 0$ and $G_0 = X_0(X^\alpha + \sum C_{0\beta}X^\beta) + \sum D_\delta X^\delta$. Writing $G_i$ ($i > 0$) we use new variables $C'_{i\eta} \in C'$, while writing $G_0$ we use the variables $C_{0\beta} \in C$ that appear in $F_0$ for the coefficients of monomials $X_0X^\beta$ and new variables $D$ for monomials $X^\delta$ such that $X_0 \nmid X^\delta$

In this way we obtain the ideal $\mathfrak{h}(j_0)$ of $\mathcal{S}t_h(j_0)$ in the polynomial ring $k[C]$ and the ideal $\mathfrak{h}(j)$ of $\mathcal{S}t_h(j)$ in $k[C,C',D]$.

**Proposition 4.7.** In the above described situation, let $T_i$ be the homogeneous tail of $X_iX^\alpha$ reduced with respect to $j$.

i) If we assume that $X_0X^{\gamma} \notin j$ for every $X^{\gamma} \in T_i$, then there is a set of eliminable variables containing $C'$, and so $\mathcal{S}t_h(j)$ can be defined by an ideal $\mathfrak{h}_1 \subseteq k[C,D]$.

ii) If moreover we assume that $X_1X^{\gamma} \notin j$ for every $X^{\gamma} \in T_0$ such that $X_0 \nmid X^{\gamma}$, then also the variables $D$ can be eliminated; furthermore $\mathfrak{h}(j) \cap k[C] = \mathfrak{h}(j_0)$, so that in a natural way:

$$\mathcal{S}t_h(j) \simeq \mathcal{S}t_h(j_0).$$

**Proof.** Let $C'_{i\eta}$ be any variable in $C'$. By the hypothesis, $X_0X^{\eta} \notin j$. Then, using Criterion 4.6 we can easily see that the $X$-coefficient of $X_0X^{\eta}$ in the reduction of $S(G_i, G_0) = X_0G_i - X_iG_0$ with respect to $j$ is $C'_{i\eta}$, if $X_i \nmid X^\eta$, and $C'_{i\eta} - C_{0\beta}$, if $X_0X^{\eta} = X_1X^\beta$. Then there is a set of eliminable variables for $\mathfrak{h}(j)$ containing $C'$.

First of all, we will prove that under the new stronger hypothesis, we also have $D_\delta \in \mathfrak{h}(j)$ for every $D_\delta \in D$ that is such that $X^\delta \in T_0$ and $X_0 \nmid X^\delta$. Using the above introduced notation we have $S(G_0, G_1) = X_1G_0 - X_0G_1 = (X_1X \sum C_{0\beta}X^\beta) + (X_1 \sum D_\delta X^\delta) - (X_0 \sum C_{i\eta}X^{\eta})$

Thanks to the hypothesis posed in 6, no monomial in the third bracket can be reduced; the same holds for the second bracket thanks to the new hypothesis posed in 7.
If $R$ is a complete reduction of $X_1 \sum C_{0\beta}X^\beta$ with respect to \{G_0, \ldots, G_n, F_1, \ldots, F_t\}, then $X_0R$ is a complete reduction of the first bracket, because all the monomials in $R$ belong to $T_1$ and so for them the condition given in (3) holds.

If we extract the $X$-coefficients in $X_0R + (X_1 \sum D_\delta X^\delta) - (X_0 \sum C_{1\eta}X^\eta)$ we see that $D_\delta \in \mathfrak{h}(j)$ because $X_0 \nmid X^\delta$.

As we have proved in Theorem 3.6, \(\mathfrak{h}(j)\) is the smallest ideal in \(k[C, C', D]\) such that \(\{\ldots, G_i, \ldots, F_j \ldots \} \cup \mathfrak{h}(j)\) is a \(\text{Gröbner} \) base. This is equivalent to say that \(\mathfrak{h}_1 := \mathfrak{h}(j) \cap k[C, C']\) is the smallest ideal such that \(X_0F_0, G_1 \ldots, G_n, F_1, \ldots, F_t\} \cup \mathfrak{h}_1\) is a \(\text{Gröbner} \) base. In fact, \(G_0\) is equivalent modulo \(\mathfrak{h}(j)\) to \(X_0F_0\) because \(D \subset \mathfrak{h}(j)\) and the variables \(D\) do not appear in \(\{X_0F_0, G_1 \ldots, G_n, F_1, \ldots, F_t\}. \) Note that \((\text{LT}(X_0F_0), \text{LT}(G_1), \ldots, \text{LT}(F_t)) = j\).

Thanks to the condition posed in (3), for every integer \(s \geq 1\), the \(S\)-polynomial \(S(G_s, X_0F_0) = X_0(G_s - X_0F_0)\) reduce to 0 if and only if \(G_s - X_0F_0\) reduces to 0. Then we can eliminate all the variables \(C'\) and substitute \(G_s\) with \(X_0F_0\), obtaining that \(\mathfrak{h}_2 := \mathfrak{h}_1 \cap k[C]\) is the smallest ideal in \(k[C]\) such that \(\{X_nF_0, \ldots, X_0F_0, F_1, \ldots, F_t\} \cup \mathfrak{h}_2\) is a \(\text{Gröbner} \) base.

Furthermore, the set of \(S\)-polynomials among elements of \(B = \{X_0F_0, \ldots, X_nF_0, F_1, \ldots, F_t\}\) is also the set of \(S\)-polynomials among elements of \(B_1 = \{F_0, F_1, \ldots, F_t\}; \) then \(\mathfrak{h}_2\) is the minimal ideal in \(k[C]\) such that \(B_1 \cup \mathfrak{h}_2\) is a \(\text{Gröbner} \) base that is \(\mathfrak{h}_2 = \mathfrak{h}(j_0), \) thanks again to Theorem 3.6.

**Corollary 4.8.** Let \(j_0 = (X^{\alpha_1}, \ldots, X^{\alpha_t})\) be a \(\text{Borel-fixed} \) saturated monomial ideal generated in degree \(\leq r\) and let \(j := (j_0)_{\geq r}. \)

i) There is a set of eliminable variables of \(\mathfrak{h}(j)\) that contains all variables except at most the ones appearing in polynomials \(F_i\) such that \(\text{LT}(F_i) = X^{\alpha_i}X_0^{r-|\alpha_i|}. \)

ii) If \(s\) is the maximal degree of monomials in the base of \(j\) in which \(X_1\) appears, then \(\text{St}(j_{\geq m}) \simeq \text{St}(j_{\geq s})\) for every \(m \geq s. \)

iii) Especially, if \(X_1\) does not appears in the monomial base of \(j, \) then \(\text{St}(j_{\geq m}) \simeq \text{St}(j)\) for every \(m. \)

**Proof.** It is sufficient to apply Proposition 4.7 by induction on the number of monomials of the base of \(j. \)

5 \(\text{Gröbner strata and regularity}\)

In the present and following sections \(k[X], \preceq\) and \(j = (X^{\gamma_1}, \ldots, X^{\gamma_t})\) will be as in the previous, but from now on we will consider only homogeneous ideals (with respect to the usual graduation) and \(T_i\) will be the complete homogeneous tail of \(X^{\gamma_i}\) so that the only involved strata will be the homogeneous strata \(\text{St}_h(j)\) introduced in Definition 3.2. Since every tail is fixed by \(\preceq, \) we will simply denote ideals defining \(\text{Gröbner} \) strata by \(\mathfrak{h}(j). \)

Let \(p(z)\) be any admissible Hilbert polynomial for subschemes in \(P^n. \) Our goal is to show that the Hilbert scheme \(\mathcal{H}_{\text{hilb}}^{n}(p(z))\) can be covered by homogeneous strata of the type \(\text{St}_h(j)\). In order to prove that, it is convenient to think of \(\mathcal{H}_{\text{hilb}}^{n}(p(z))\) and \(\text{St}_h(j)\) as schemes parameterizing the same kind of objects, namely homogeneous ideals in \(k[X];\) as many ideals define the same subscheme \(Z \subset P^n,\) the problem is to select a unique ideal in \(k[X]\) for every subscheme \(Z.\) The most common choice is to associate to \(Z\) the only homogeneous saturated ideal \(I(Z)\)
such that $Z = \text{Proj} (k[X]/I(Z))$; this point of view is that assumed for instance in [13] and in [16], where homogeneous strata of saturated ideals are considered.

Here we prefer a different approach, that directly calls back to the explicit construction of Hilbert schemes.

**Definition 5.1.** Given an admissible Hilbert polynomial $p(z)$ for subschemes in $\mathbb{P}^n$, we will denote by $r$ the Gotzmann number of $p(z)$, that is the worst regularity of saturated ideals defining subschemes in $\text{Hilb}^n_{p(z)}$. Moreover we set: $M := \binom{n+r}{n}$, $t := M - p(r)$, $M_1 := \binom{n+r+1}{n}$ and $t_1 := M_1 - p(r+1)$.

Macaulay’s Theorem states that $r$ is the regularity of the lexsegment ideal with Hilbert polynomial $p(z)$ (for the definition and the main properties of regularity and for some consequences, we refer to [6]).

As $Z = \text{Proj} (k[X]/I(Z)) = \text{Proj} (k[X]/I(Z)_{\geq r})$, $Z$ can be uniquely identified by the ideal $I(Z)_{\geq r}$, which is generated by $t$ linearly independent degree $r$ homogeneous polynomials $F_1, \ldots, F_t$ or, more precisely, by the $t$-dimensional $k$-vector space $I(Z)_r$: $\text{Hilb}^n_{p(z)}$ can be realized as a closed subscheme in the grassmannian of the $t$-dimensional vector spaces in $k[X]_r$. A $t$-dimensional vector space in $k[X]_r$ gives a point in $\text{Hilb}^n_{p(z)}$ if and only if it generates an ideal $i$ having $p(z)$ as Hilbert polynomial.

**Notation 5.2.** From now on, $i \in \text{Hilb}^n_{p(z)}$ will mean that $i = I(Z)_{\geq r}$ for some closed subscheme $Z$ in $\mathbb{P}^n$ with Hilbert polynomial $p(z)$. Equivalently we can say that $i \in \text{Hilb}^n_{p(z)}$ if and only if $i$ is an homogeneous ideal in $k[X]$ with Hilbert polynomial $p(z)$ (for the meaning of “Hilbert polynomial of $i$” see Notation [7] which is generated in degree $r$, where $r$ is the Gotzmann number of $p(z)$.

**Remark 5.3.** If $i \in \text{Hilb}^n_{p(z)}$, then $i$ is $r$-regular and it has a free resolution of the type:

$$0 \rightarrow k[X](-r-\lambda)^{n\lambda} \rightarrow \cdots \rightarrow k[X](-r-1)^{n_1} \rightarrow k[X](-r)^{n_0} \rightarrow i \rightarrow 0 \quad (7)$$

([3 Theorem 1.2]). Then we can find a set of generators for the first syzygies $\text{Syz}(i)$ in degree $r + 1$.

If we take into consideration the homogeneous Gröbner strata $\mathcal{S}_{t,h}(j)$ and select the monomial ideal $j$ in $\text{Hilb}^n_{p(z)}$, we obtain the intended direct relation between Gröbner strata and Hilbert schemes.

**Lemma 5.4.** If $j \in \text{Hilb}^n_{p(z)}$, then (at least set-theoretically) $\mathcal{S}_{t,h}(j) \subseteq \text{Hilb}^n_{p(z)}$.

**Proof.** Let $i$ be any ideal in $\mathcal{S}_{t,h}(j)$. By hypothesis $\text{LT}(i) = j$ and then $i$ and $j$ share the same Hilbert function. Therefore $i$ is generated in degree $r$ and has Hilbert polynomial $p(z)$ and then $i \in \text{Hilb}^n_{p(z)}$. $lacksquare$

Now we will see that the set-theoretic inclusions are in fact algebraic maps and that for some ideals they are open injections. It is a little surprising that $\mathcal{S}_{t}(j)$ directly depends on the ideal $j$ and not only on the subscheme $Z = \text{Proj} (k[X]/j)$ it defines. For instance the saturated lexicographic ideal $\mathcal{L}$ with Hilbert polynomial $p(z)$ and its truncation $\mathcal{L}' = \mathcal{L}_{\geq r}$ define the same subscheme in $\mathbb{P}^n$; however, as we will see, $\mathcal{S}_{t,h}(\mathcal{L}')$ is an open subset of the Reeves-Stillman component of $\text{Hilb}^n_{p(z)}$, whereas $\mathcal{S}_{t,h}(j)$ can be a locally closed subset of lower dimension in the same component.
Let \( j \) be a monomial ideal in \( \mathcal{H} \text{ilb}^n_{p(2)} \). As seen in §3 every ideal \( i \) such that \( \text{LT}(i) = j \) has a (unique) reduced Gröbner basis \( \{ f_1, \ldots, f_t \} \) where \( f_i \) is as in Definition 3.2(3). Not every ideal generated by \( t \) polynomials of such a type has \( j \) as initial ideal. In order to obtain equations for \( \mathcal{S} \text{th}(j) \) we consider the coefficients \( c_{i\alpha} \) appearing in the \( f_i \) as new variables; more precisely let \( C = \{ C_{i\alpha}, i = 1, \ldots, t, \ X^\alpha \in k[X]_r \setminus j \_r \text{ and } X^\alpha < X^\gamma \} \) be new variables and consider \( t \) polynomials in \( k[X, C] \) of the following type:

\[
F_i = X^\gamma + \sum_{X^\alpha \in T_i} C_{i\alpha} X^\alpha
\]

where \( T_i = T_{\gamma i} \cap k[X]_r \) (Definition 3.4). We obtain the ideal \( \mathfrak{h}(j) \) of \( \mathcal{S} \text{th}(j) \) collecting the \( X \)-coefficients of some complete reduction with respect to \( F_1, \ldots, F_t \) of all the \( S \)-polynomials \( S(F_i, F_j) \), corresponding to a set of generators for \( \text{Syz}(j) \) (see Theorem 3.6 and Proposition 3.5[3]).

**Theorem 5.5.** In the above notation, let \( j \in \mathcal{H} \text{ilb}^n_{p(2)} \) and let \( A \) be the \( t(n + 1) \times M_1 \) matrix whose entries are the \( X \)-coefficients of \( X_j F_i \), for all \( j = 0, \ldots, n \) and \( i = 1, \ldots, t \).

Then the ideal \( \mathfrak{h}(j) \) of the homogeneous stratum \( \mathcal{S} \text{th}(j) \) is generated by the \( (t_1 + 1) \times (t_1 + 1) \) minors of \( A \).

**Proof.** By abuse of notation we write in the same way a polynomial and the rows of its \( X \)-coefficients. As in Definition 3.3 we consider a term order on \( \mathbb{T}_{X, C} \) which is an elimination order of the variables \( X \) and coincides with the fixed term ordering \( \preceq \) on \( \mathbb{T}_X \). It is quite evident by elementary arguments of linear algebra, that the ideal \( a \subseteq k[C] \), generated by all \( (t_1 + 1) \times (t_1 + 1) \) minors, does not change if we perform some row reduction on \( A \). Let \( \mathcal{P} \) be a set of \( t_1 \) rows whose leading terms are a base of \( j_{r+1} \). If \( X_h F_i \notin \mathcal{P} \), then it has the same leading term than one in \( \mathcal{P} \), say \( X_k F_j \); we can substitute \( X_h F_i \) with \( X_h F_i - X_k F_j \). In this way the rows not in \( \mathcal{P} \) become precisely all the \( S \)-polynomials \( S(F_i, F_j) \) that have \( X \)-degree \( r + 1 \).

At the end of this sequence of rows reductions, we can write the matrix as follows:

\[
\begin{pmatrix}
D & E \\
S & L
\end{pmatrix}
\]

where \( D \) is a \( t_1 \times t_1 \) upper-triangular matrix with 1’s along the main diagonal, whose rows correspond to \( \mathcal{P} \) and whose columns correspond to monomials in \( j_{r+1} \).

Using rows in \( \mathcal{P} \), we now perform a sequence of rows reductions on the following ones, in order to annihilate all the coefficients of monomials in \( j_{r+1} \), that is the entries of the submatrix \( S \): if \( a(C) \) is the first non-zero entry in a row not in \( \mathcal{P} \) and its column corresponds to the monomial \( X^\gamma \in j_{r+1} \), we add to this row \(-a(C)X_k F_j \), where \( X_k F_j \in \mathcal{P} \) and \( \text{LT}(X_k F_j) = X^\gamma \). This is nothing else than a step of reduction with respect to \( \{ F_1, \ldots, F_t \} \). At the end of this second turn of rows reductions, we can write the matrix as follows:

\[
\begin{pmatrix}
D & E \\
0 & R
\end{pmatrix}
\]

where the rows in \((D \mid E)\) are unchanged whereas the rows in \((0 \mid R)\) are the \( X \)-coefficients of complete reductions of \( S \)-polynomials in \( X \)-degree \( r + 1 \). Then \( a \) is generated by the entries of \( R \) and so \( a \subset \mathfrak{h}(j) \).
We can see that this inclusion is in fact an equality taking in mind Remark \ref{rem:5.3} and Proposition \ref{prop:3.5}: the first one says that $\text{Syz}(j)$ is generated in degree $r+1$ and the second one that in this case $h(j)$ is generated by the $X$-coefficients of complete reductions of the $S$-polynomials $S(F_i, F_j)$ of $X$-degree $r+1$.

The following corollary just express in an explicit way two properties contained in the proof of Theorem \ref{thm:5.5}

**Corollary 5.6.** In the above notation:

- the ideal $h(j)$ is generated by the entries of the submatrix $R$ in \eqref{eq:10};
- the vector space $L(j)$ is generated by the entries of the submatrix $L$ in \eqref{eq:9}.

**Remark 5.7.** An interesting consequence of this theorem is that Gröbner strata equations are substantially independent of the term ordering. The term ordering sets which monomials can appear in the tails $T_i$, but if two different term orderings determine the same tail for each monomials in $j$, then they lead to the same Gröbner stratum structure. We will take up this remark in the next paragraph concerning Hilbert schemes.

**Remark 5.8.** The construction of the Gröbner stratum of a saturated ideal requires a number of coefficients $C$ lower than that our approach needs, because a saturated monomial ideal $j$ has less generators of lower degree than its truncation $j_{\geq r}$.

Using our approach the number of coefficients $C$ suffers a great increase, due both to the higher number of generators and to the to the fact that homogeneous tails in higher degree are “stretched out”. One could think that this involves an unnecessary lost of computational efficiency; however a large number of the coefficients can be easily eliminated during the computation (see for instance Proposition \ref{prop:4.7}) and moreover some of the “new” coefficients give tangible improvement.

Moreover, the construction made in this paragraph shows that we have to consider only the “linear” $S$-polynomials and that we can use the very-well-studied and improved tools of linear algebra. In conclusion, we think that the complication of computational aspects is a small penalty comparing to the possibilities that this technique opens.

## 6 Gröbner strata that are open subset of an Hilbert scheme

In the present section we will prove that every homogeneous Gröbner stratum $S_{t,h}(j)$, where $j \in H_{\text{Hilb}}^n_{p(z)}$, can be naturally identified with a locally closed subvariety of $H_{\text{Hilb}}^n_{p(z)}$ and that it is an open subset of $H_{\text{Hilb}}^n_{p(z)}$ if $j$ is generated by the first $t$ monomials in $k[X]$ with respect to the fixed term ordering $\preceq$. As a consequence we obtain the main results of the paper about the rationality of some components of $H_{\text{Hilb}}^n_{p(z)}$.

For the meaning of $p(z)$, $r$, $t$, $M, t_1, M_1$ and $i \in H_{\text{Hilb}}^n_{p(z)}$ we refer to Definition \ref{def:5.1} and Notation \ref{not:5.2}.

First of all we recall how equations defining $H_{\text{Hilb}}^n_{p(z)}$ are usually obtained. Every ideal $i \in H_{\text{Hilb}}^n_{p(z)}$ is generated by the $t$-dimensional vector space $i$. On the other hand thanks to Gotzmann’s Persistence Theorem (see for instance \cite[Theorem 3.8]{gotzmann}) a $t$-dimensional vector space $V \subset k[X]$ generates an ideal $i \in H_{\text{Hilb}}^n_{p(z)}$ if and only if $\dim_k \langle X_0 V, \ldots, X_n V \rangle = t_1$.

Therefore $H_{\text{Hilb}}^n_{p(z)}$ can be thought as the subscheme of the grassmannian $G(t, M)$ defined by the previous condition; moreover by the Plücker embedding of the grassmannian in a
projective space $\mathbb{P}^q$, $\mathcal{Hilb}^n_{p(z)}$ becomes a closed subscheme (not necessarily irreducible and reduced) of $\mathbb{P}^q$.

Here we are not interested in finding explicit equations for $\mathcal{Hilb}^n_{p(z)}$ in $\mathbb{P}^q$, but only equations defining each open subset $U \cap \mathcal{Hilb}^n_{p(z)}$, where $U$ is the open subset of $G(t, M)$ given by a non-vanishing Plücker coordinate.

**Definition 6.1.** Thinking of $k[X], r$ as the vector space generated by its monomials, we can identify every Plücker coordinate with a suitable monomial ideal $j$ generated by $t$ monomials of degree $r$. We will denote by $U_j$ and $\mathcal{H}_j$ respectively the open subsets of $G(t, M)$ and of $\mathcal{Hilb}^n_{p(z)}$ where the Plücker coordinate corresponding to $j$ does not vanish.

In a natural way $U_j$ is isomorphic to the affine space $\mathbb{A}^{t(M-t)}$. In fact, if $j = (X^{\gamma_1}, \ldots, X^{\gamma_n})$, every point in $U_j$ is uniquely identified by the reduced, ordered set of generators $\langle g_1, \ldots, g_t \rangle$ of the type $g_i = X^{\gamma_i} + \sum c_{i\alpha} X^\alpha$, where $c_{i\alpha} \in k$ and $X^\alpha$ is any monomial in $k[X]_r \setminus j$. Then we consider on $\mathbb{A}^{t(M-t)}$ the coordinates $C_{i\alpha}$. Note that each $C_{i\alpha}$ naturally corresponds to the Plücker coordinate $j' = (X^{\gamma_1}, \ldots, X^{\gamma_{n-1}}, X^{\alpha}, X^{\gamma_{n+1}}, \ldots X^{\gamma_t})$ (but of course not all the Plücker coordinates are of this type).

Now we can mimic the construction of Gröbner strata and obtain the defining ideal of $\mathcal{H}_j$ as a subscheme of $\mathbb{A}^{t(M-t)}$. Let us consider the set of variables $\overline{C} = \{C_{i\alpha}, i = 1, \ldots, t, \ X^\alpha \in k[X]_r \setminus j \}$ and $t$ polynomials $G_1, \ldots, G_t$ in $k[X,\overline{C}]$ of the type:

$$G_i = X^{\gamma_i} + \sum C_{i\alpha} X^\alpha$$

(11)

and let $B$ be the $(n+1)t \times M_1$ matrix whose entries are the $X$-coefficients of the polynomials $X_j G_i$. Then consider the ideal $\mathfrak{b}(j) \subset k[\overline{C}]$ generated by the $(t_1+1) \times (t_1+1)$ minors in $B$.

**Proposition 6.2.** $\mathfrak{b}(j)$ is the ideal of $\mathcal{H}_j$ as a closed subscheme of $\mathbb{A}^{t(M-t)}$.

**Proof.** Every ideal $i \in U_j$ can be obtained from $(G_1, \ldots, G_t)$ specializing (in a unique way) the variables $C_{i\alpha}$ to $c_{i\alpha} \in k$. Obviously not all the specializations give ideals $i \in \mathcal{H}_j$, that is with Hilbert polynomial $p(z)$ (more precisely, such that $k[X_0, \ldots, X_n]/i$ has Hilbert polynomial $p(z)$), because we have to ask both $\dim_k(i_r) = t$ and $\dim_k(i_{r+1}) = t_1$: thanks to Gotzmann’s persistence we know that these two necessary conditions are also sufficient.

In the open subset $U_j$ the first condition always holds and the rank of every specialization of $B$ is $\geq t_1$ by Macaulay estimate of the growth of ideals (see [5] § 3 or [9] Corollary 5.5.28). Therefore $\mathcal{H}_j$ is given by the condition $rk(B) \leq t_1$.

We can order the set of Plücker coordinates in the following way. We write the $t$ monomials corresponding to each Plücker coordinate in decreasing order with respect to $\preceq$; if $j_1 = (X^{\alpha_1} \succ \cdots \succ X^{\alpha_t})$ and $j_2 = (X^{\beta_1} \succ \cdots \succ X^{\beta_t})$, then $j_1 \succ j_2$ if $X^{\alpha_i} = X^{\beta_i}$ for every $i$ lower than some $s$ and $X^{\alpha_s} \succ X^{\beta_s}$.

It is now easy to compare, for a same monomial ideal $j \in \mathcal{Hilb}^n_{p(z)}$, the Gröbner stratum $\mathcal{S}_{\mathfrak{b}_h}(j)$ and the open subset $\mathcal{H}_j$. We underline that for our purpose it will be sufficient to consider the open subsets $\mathcal{H}_j$ corresponding to monomial ideals $j \in \mathcal{Hilb}^n_{p(z)}$, because (at least set-theoretically) they cover $\mathcal{Hilb}^n_{p(z)}$. In fact, if $i \in \mathcal{Hilb}^n_{p(z)}$, then also $LT(i) \in \mathcal{Hilb}^n_{p(z)}$ and we have at least $i \in \mathcal{H}_{LT(i)}$.

**Theorem 6.3.** Let $p(z)$ be any admissible Hilbert polynomial in $\mathbb{P}^n$ with Gotzmann number $r$. Let us fix any term ordering $\preceq$ on $\mathbb{T}_X$.
i) If $j$ is a monomial ideal in $\text{Hilb}_{p(z)}^n$, then $\text{St}_h(j)$ is naturally isomorphic to the locally closed subvariety of $\text{Hilb}_{p(z)}^n$ given by the conditions that the Plücker coordinate corresponding to $j$ does not vanish and the preceding ones vanish.

ii) For every isolated, irreducible component $H$ of $\text{Hilb}_{p(z)}^n$, there is a monomial ideal $j \in \text{Hilb}_{p(z)}^n$ such that an irreducible component of $\text{Supp} \text{St}_h(j)$ is an open subset of $\text{Supp} H$. Then $\text{Supp} H$ has an open subset which is a homogeneous affine variety with respect to a non-standard positive graduation.

Proof. i) We obtain the two affine varieties $\text{St}_h(j)$ and $\mathcal{H}_i$ in a quite similar way (for $\text{St}_h(j)$ see Theorem 6.3 and for $\mathcal{H}_i$ see Proposition 6.2). The only difference comes from the definition of the set of polynomials $F_1, \ldots, F_r$ given in (8), leading to equations for $\text{St}_h(j)$, and the set of polynomials $G_1, \ldots, G_r$ given in (11), leading to equations for $\mathcal{H}_i$; in $G_i$ the sum is over all the degree $r$ monomials $X^\alpha \not\in j$ whereas in $F_i$ we also assume that $X^\alpha < \text{LT}(F_i)$. Therefore we can think of $\text{St}_h(j)$ as the affine variety defined by the ideal $\overline{b(j)}$ in the ring $k[X, \overline{C}]$, where $\overline{C} = \{C_i \mid i = 0, \ldots, n, \ X^\alpha \in k[X] \setminus \{j\} \}$ generated by $b(j)$ and by $(C_i \mid X^\alpha \prec \text{LT}(F_i))$, namely $\overline{b(j)} = b(j)k[\overline{C}] + (\overline{C} \setminus C)$. Now we can conclude because the Plücker coordinates higher than $j$ vanish if and only if all the $C_i$ such that $X^\alpha \prec \text{LT}(F_i)$ vanish.

As $j$ varies among the finite set of the monomial ideals in $\text{Hilb}_{p(z)}^n$, the Gröbner strata $\text{St}_h(j)$ give a set theoretical covering of $\text{Hilb}_{p(z)}^n$ by locally closed subschemes. Then there is a suitable ideal $j$ such that an irreducible component of $\text{Supp} \text{St}_h(j)$ is an open subset of $H$. We have seen in the previous sections that $\text{St}_h(j)$ has a structure of homogeneous affine scheme with respect to a non-standard positive graduation $\lambda$. Then also its support and the irreducible components of the support are homogeneous (see [4, Corollary 2.7]).

Corollary 6.4. Every smooth irreducible component $H$ of $\text{Hilb}_{p(z)}^n$ is rational.

The same holds for every smooth, irreducible component of $\text{Supp} \text{Hilb}_{p(z)}^n$.

Proof. If $H$ is a smooth, irreducible component of $\text{Hilb}_{p(z)}^n$, then it is also reduced. Thanks to Theorem 6.2 we know that an open subset of $H$ is an affine homogeneous variety with respect to a positive graduation. Moreover this open subset is also smooth and so it is isomorphic to an affine space, by Corollary 4.5.

Remark 6.5. The ideal $b(j)$ of the Gröbner stratum $\text{St}_h(j)$ is homogeneous with respect to a positive graduation $\lambda$ on $k[C]$ given by $\lambda(C_{i\alpha}) = \gamma_i - \alpha$, where $C_{i\alpha}$ appears in the polynomial $F_i$ as the coefficient of the degree $r$ monomial $X^\alpha$ such that $X^\alpha \not\in j$ and $X^\alpha < X^\gamma = \text{LT}(F_i)$ (see Definitions 4.7 and 4.2). We could also consider a graduation $\lambda'$ on $k[C]$ such that the ideal $b(j)$ of $\mathcal{H}_i$ is homogeneous by the analogous definition: $\lambda'(C_{i\alpha}) = \gamma_i - \alpha$ if $C_{i\alpha}$ appears in $G_i$. However this graduation $\lambda'$ is not necessarily positive and so it gives less interesting consequences.

If an irreducible component $H$ of $\text{Hilb}_{p(z)}^n$ is also reduced, Theorem 6.2 insures that there is an open subset of $H$ which has the structure of homogeneous variety with respect to a positive graduation induced from that of a suitable Gröbner stratum $\text{St}_h(j)$.

On the other hand, in the case of a non-reduced component we only know that the support of a suitable open subset is homogeneous with respect to a positive graduation, but this not implies that the open subset itself is homogeneous.

Now we consider a special case in which we obtain a positive graduation on an open subset of an irreducible component of $\text{Hilb}_{p(z)}^n$, even if not reduced.
**Definition 6.6.** Given any term order $\preceq$ in $\mathbb{T}_X$, a $(m, \preceq)$-segment is a subset $S$ of $k[X]_m$ containing the first $|S|$ monomials of degree $m$ with respect to $\preceq$, namely such that:

$$\forall X^\beta \in k[X]_m, \forall X^\gamma \in S: X^\beta \preceq X^\gamma \Rightarrow X^\beta \in S.$$ 

An $(m, \preceq)$-segment ideal is a monomial ideal $j$ which is generated by a $(m, \preceq)$-segment.

If $\mathcal{L}$ is the saturated lexsegment ideal, then for every $m \geq r$ (that is for every $m$ higher than the regularity of $\mathcal{L}$), the ideal $\mathcal{L}_{\geq m}$ is a $(m, \text{Lex})$-segment ideal. This property does not hold in general if the term ordering in not $\text{Lex}$, so that $j_{\geq m}$ can be a $(m, \preceq)$-segment ideal, while $j_{\geq m+1}$ is not a $(m+1, \preceq)$-segment ideal. A trivial case is for instance that of the ideal $(x)$ in $k[x, y, z]$ which is $(1, \text{DegRevLex})$-segment ideal, whereas $(x)_{\geq 2} = (x^2, xy, xz)$ is not $(2, \text{DegRevLex})$-segment ideal, because it contains $xz$ and does not contain $y^2$.

The definition of $(m, \preceq)$-segment ideal is not equivalent, but it is very close to that of extremal ideal given in [17].

**Corollary 6.7.** Let $p(z)$ any admissible Hilbert polynomial in $\mathbb{P}^n$ with Gotzmann number $r$ and let $H$ be an isolated, irreducible component of $\mathcal{H}_{p(z)}^n$.

If $H$ contains a point corresponding to an $(r, \preceq)$-segment ideal $j \in \mathcal{H}_{p(z)}^n$ with respect to some term ordering $\preceq$ in $\mathbb{T}_X$, then $\text{St}_h(j)$ is an open subset of $H$, so that $H$ has an open subset which is an homogeneous affine variety with respect to a non-standard positive graduation.

**Proof.** If $j$ is a $(r, \preceq)$-segment ideal, then there are no Plücker coordinates preceding that corresponding to $j$. Thus $\text{St}_h(j) \cong \mathcal{H}_j$ (see Theorem 6.3) and so $\mathcal{H}_j$ is an affine homogeneous scheme with respect to a positive graduation. ■

**Corollary 6.8.** Let $H$ be an irreducible component of $\mathcal{H}_{p(z)}^n$ and let $j$ be a $(r, \preceq)$-segment ideal. If $j \in H$ and either of the following condition holds:

i) $\text{St}_h(j)$ is an affine space,

ii) $j$ is a smooth point of $\text{St}_h(j)$,

iii) $j$ is a smooth point of $\mathcal{H}_{p(z)}^n$,

then $H$ is rational.

**Proof.** It is a straightforward consequence of the previous result and of Corollary 4.5. ■

## 7 Gröbner stratum of the lexsegment ideal

In this paragraph the term ordering $\preceq$ will be the lexicographic term ordering $\text{Lex}$.

As a first application to the results obtained in §6 we take into consideration the lexicographic ideal $\mathcal{L}$. For every admissible Hilbert function $p(z)$ on $\mathbb{P}^n$, $\mathcal{H}_{p(z)}^n$ contains the ideal generated by the first $t$ monomials in degree $r$ with respect to the term ordering $\text{Lex}$. In the paper [14] it is proved that the point of $\mathcal{H}_{p(z)}^n$ (usually called lexicographic point) corresponding to the subscheme $	ext{Proj} k[X]/\mathcal{L}$ is smooth and Reeves and Stillman get the proof by a computation of the Zariski tangent space dimension. The only component of $\mathcal{H}_{p(z)}^n$ containing the lexicographic point is usually denoted by $H_{RS}$. As a consequence of the quoted result by Reeves and Stillman and of Corollary 6.8, we then obtain:
Corollary 7.1. The Reeves and Stillman component $H_{RS}$ of $\mathcal{Hilb}^n_{p(z)}$ is rational.

However we prefer to present here a new, self-included proof, in order to explain how our technique can be used as a theoretical, as well as a computational, tool.

First of all, we recall briefly the notation used in [14]. Moving from [10], Reeves and Stillman work with lexicographic saturated ideals; they suppose $X_0 > X_1 > \ldots > X_n$ and identify the ideals with arrays composed by terms ordered: $X_{i_1} \succ \ldots \succ X_{i_k}$, where $i_1 > \ldots > i_k$. Since we are going to prove the same result, we will assume a quite similar notation, but not exactly the same because in this paper the variables $X$ are ordered in the reverse way with respect to [14] and moreover we consider ideals generated in degree $r$ instead of saturated ideals.

Note that $r$ is precisely the Gotzmann number of the Hilbert polynomial of $\mathcal{L}$ (more precisely of $k[X]/\mathcal{L}$).

Theorem 7.3. The homogeneous Gröbner stratum $\mathcal{S}_h(\mathcal{L}(a_n, \ldots, a_1))$ of the lexicographic ideal $\mathcal{L}(a_n, \ldots, a_1) \in \mathcal{Hilb}^n_{p(z)}$ is isomorphic to an affine space. Therefore the component $H_{RS}$ of $\mathcal{Hilb}^n_{p(z)}$ is rational.

Proof. Thanks to Corollary 4.8 we obtain the complete statement proving that the homogeneous Gröbner stratum $\mathcal{S}_h(\mathcal{L}(a_n, \ldots, a_1))$ is an affine space, that is showing that a same number is both a lower-bound for its dimension and an upper-bound for its embedding dimension; the first part corresponds to Theorem 4.1 of [14] (here in terms of initial ideals) and the second one corresponds to Theorem 3.3 of [14].

We proceed by induction on the number $n$ of variables and on the Gotzmann number $r$.

In order to obtain an upper-bound for the embedding dimension we look for a maximal set of eliminable variables $C' \subset C$, using Criterion 4.6. If $X_{a_1} > \ldots > X_{a_m}$ is the monomial base of the saturation $j$ of $\mathcal{L}(a_n, \ldots, a_1)$, then we can assume that the polynomials $F_1, \ldots, F_l \in k[X,C]$ (that we use in order to construct $\mathcal{S}_h(\mathcal{L}(a_n, \ldots, a_1))$: see Definition 3.4) are ordered so that $\text{LT}(F_i) = X_{a_i}^r X_0^{|a_i|}$ for $i = 1, \ldots, m$. Thanks to Corollary 4.8 we can start the construction of $C'$, putting inside all the variables appearing in $F_j$ for every $j > m$.

We divide the proof in 3 steps.

Step 1 The zero-dimensional case: $\mathcal{S}_h(\mathcal{L}(0, \ldots, 0, a_1)) \simeq \mathbb{A}^{na_1}$.

Claim 1i: $\dim \mathcal{S}_h(\mathcal{L}(0, \ldots, 0, a_1)) \geq na_1$.

Let us denote $\mathcal{L}(0, \ldots, 0, a_1)$ by $\mathcal{L}$. The zero-dimensional scheme $Z$ of $a_1$ general points in $\mathbb{P}^n$ has Gotzmann number $a_1$ and Hilbert polynomial $p(z) = a_1$. Moreover $\text{LT}(I(Z)_{\geq a_1}) \supseteq \mathcal{L}$, because for every monomial $X^{\gamma} \supseteq X_{a_1}^{a_1}$ we can find some homogeneous polynomial of the type $X^{\gamma} - \sum_{j=1}^{a_1} b_j X_1^{a_1-j} X_0^j$ vanishing in the $a_1$ points of $Z$: we can find the $b_j$'s solving a $a_1 \times a_1$ linear system with a Vandermonde associated matrix. As both $\text{LT}(I(Z)_{\geq a_1})$ and $\mathcal{L}$ are generated in degree $a_1$, they coincide; so $I(Z)_{\geq r} \in \mathcal{S}_h(\mathcal{L})$ and we conclude since we can choose $Z$ in a family of dimension $na_1$. 

Claim 1ii: \( \text{ed } S_{th}(\mathcal{L}(0, \ldots, 0, a_1)) \leq na_1 \).

The saturation of \( \mathcal{L} \) is the ideal \((X_n, X_{n-1}, \ldots, X_2, X_1^{a_1}) \), which is generated by \( n \) monomials; moreover there are only \( a_1 \) monomials of degree \( a_1 \) not contained in \( \mathcal{L} \): Corollary 4.8 leads to the conclusion.

**Step 2:** If \( S_{th}(\mathcal{L}(a_n, \ldots, a_2, 0)) \simeq \Lambda^K \) then \( S_{th}(\mathcal{L}(a_n, \ldots, a_2, a_1)) \simeq \Lambda^{K+na_1} \).

Claim 2i: \( \dim S_{th}(\mathcal{L}(a_n, \ldots, a_2, a_1)) \geq \dim S_{th}(\mathcal{L}(a_n, \ldots, a_2, 0)) + na_1 = K + na_1 \).

Let us denote \( \mathcal{L}(a_n, \ldots, a_2, a_1) \) by \( \mathcal{L} \) and \( \mathcal{L}(a_n, \ldots, a_2, 0) \) by \( \mathcal{L}_1 \). Let \( Y \) be any closed subscheme in \( \mathbb{P}^n \) such that \( I(Y)_{\geq r} \in \mathcal{I}_h(\mathcal{L}_1) \) and consider the set \( Z \) of \( a_1 \) points in \( \mathbb{P}^n \). If we choose the \( a_1 \) points in \( Z \) general enough, then \( I(Z \cup Y) = I(Z) \cdot I(Y) \). Then we conclude thanks to the previous step, as \( L(I(Z)) = \mathcal{L}(0, \ldots, 0, a_1) \) and \( \mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}(0, \ldots, 0, a_1) \).

Claim 2ii: \( \text{ed } S_{th}(\mathcal{L}(a_n, \ldots, a_2, a_1)) \leq \text{ed } S_{th}(\mathcal{L}(a_n, \ldots, a_2, 0)) + na_1 = K + na_1 \).

First of all, let us consider all the polynomials \( F_i \) such that \( X_0^{r-a_1} \mid \text{LT}(F_i) \) and the set of variables \( C_{i, \beta} \) appearing in them such that \( X^\beta = X^{\beta_1}X_0^{r-a_1} \) for some monomial \( X^{\beta_1} \notin \mathcal{L}_1 \): a multiple of \( X^\beta \) belongs to \( \mathcal{L} \) if and only the corresponding multiple of \( X^{\beta_1} \) belongs to \( \mathcal{L}_1 \). Then \( F_i = X_0^{r-a_1}F_{i_1} + \ldots \), where the \( F_{i_1} \)'s are the polynomials that appear in the definition of \( S_{th}(\mathcal{L}_1) \). Using the \( S \)-polynomials involving couples of such polynomials we see that \( L(\mathcal{L}_1) \subseteq L(\mathcal{L}) \); thus all the variables \( C_{i, \beta} \) of this type are eliminable, except at most \( K = ed \mathcal{L}_1 \) of them.

Moreover, for every \( i \leq n \) there are \( a_1 \) variables \( C_{i, \beta} \) such that \( X^\beta \notin \mathcal{L}, X^\beta \in \mathcal{L}_1 \): the are \( X_n^{a_1} \cdot X_2^{a_2}X_1^{a_1-j}X_0^{r-a_1}, \ j = 1, \ldots, a_1 \).

If we specialize to \( 0 \) all the variables of the two above considered types, the embedding dimension drops at most by \( \text{ed } S_{th}(\mathcal{L}_1) + na_1 = K + na_1 \).

Now it will be sufficient to verify that all the remaining variables \( C_{i, \beta} \) are eliminable, using Criterion 4.6.

Assume that \( X^\beta \prec X_n^{a_1} \cdot X_2^{a_2} \) and \( X_0^{r-a_1} \upharpoonright X^\beta \).

- If \( i > n \), all the variables are eliminable using those appearing in \( F_1, \ldots, F_n \), thanks to Corollary 4.8.
- If \( i < n \), using \( S(F_i, F_j) \), where \( \text{LT}(F_j) = X^{a_1}X_1^{r-a_1-j} \), we see that \( C_{i, \beta} \in L(\mathcal{L}) \).
- If \( i = n \), using \( S(F_n, F_{n-1}) = X_2X_0^{a_1-1} - X_1^{a_1}F_{n-1} \), we see that \( C_{n, \beta} \in L(\mathcal{L}) \) (note that by the previous idem \( C_{n-1, \beta_{r-1}} \in L(\mathcal{L}) \)).

**Step 3:** If \( \mathcal{L}(a_n, \ldots, a_2, 0) \simeq \Lambda^{K_1} \) then \( \mathcal{L}(a_d, \ldots, a_2, 0) \simeq \Lambda^{K_2} \) where \( d \) is the maximal index \( < n \) such that \( a_d \neq 0 \) and \( K_2 = K_1 + (n-d)(d+1) + (a_n+a) - 1 \) (or \( K_2 = K_1 + (a_n+a) - 1 \) if \( d \) does not exist).

Here we compare the ideal \( \mathcal{L} = \mathcal{L}(a_n, \ldots, a_2, 0) \) in \( k[X] \) and the ideal \( \mathcal{L}_1 = \mathcal{L}(a_d, \ldots, a_2, 0) \) in \( k[X_0, \ldots, X_d] \). Observe that both \( j := \text{sat}(\mathcal{L}) \) and \( j_1 := \text{sat}(\mathcal{L}_1) \) fulfill the hypothesis of Corollary 4.8; then it holds \( S_{th}(\mathcal{L}) \simeq S_{th}(j) \) and \( S_{th}(\mathcal{L}_1) \simeq S_{th}(j_1) \). The statement for the saturated ideals \( j \) and \( j_1 \) is proved using the same technique as above in [16, Proposition 4.5].
8 Examples

In this final section we apply our technique to the Hilbert scheme \( \mathcal{H} \text{ilb}^3_{12} \), studied by Gotzmann in [5]. In that paper a complete list is presented of the Borel-fixed, saturated, monomial ideals of \( k[X_0, X_1, X_2, X_3] \) corresponding to points of \( \mathcal{H} \text{ilb}^3_{12} \). They are:

\[
\begin{align*}
\mathfrak{b}_3 &= (X_3^2, X_3X_2, X_2^3), \\
\mathfrak{b}_4 &= (X_2^3, X_3X_2, X_3X_1^2, X_2^4), \\
\mathfrak{b}_5 &= (X_2^3, X_3X_2, X_3X_1, X_2^5, X_4X_1), \\
\mathfrak{b}_6 &= (X_3, X_2^5, X_2^4X_1^2),
\end{align*}
\]

(recall that for us \( X_3 \triangleright X_2 \triangleright X_1 \triangleright X_0 \)). The index \( s \) in \( \mathfrak{b}_s \) is the regularity of the ideal. Moving from this point, Gotzmann proves that there are two irreducible components: the first containing \( \mathfrak{b}_3 \) with dimension 16 and the second (the Reeves-Stillman one) containing \( \mathfrak{b}_6 \) with dimension 23. Now we obtain a computational confirmation of this result and moreover we prove that the two components are reduced, rational and that they have a transversal intersection.

Since the Gotzmann number of the Hilbert polynomial \( p(z) = 4z \) is 6, we consider the truncated ideals \( j_s = (\mathfrak{b}_s)_{\geq 6} \).

First of all, we consider \( j_6 \): it is the lexsegment ideal \( \mathcal{L}(0, 4, 2) \). In the previous section we proved that the Gröbner stratum \( S_{\mathfrak{L}}(\mathcal{L}(0, 4, 2), \text{Lex}) \) is an open subset of \( \mathcal{H} \text{ilb}^3_{12} \) and that it is an affine space; an easy computation gives the dimension 23.

Now we will show that the ideals \( j_s \) for \( s = 3 \) and \( s = 4 \) (it will not be necessary to examine \( j_5 \)) are generated by a \( (6, \preceq_s) \)-segment according to a suitable term ordering \( \preceq_s \) (depending on \( j_s \)), that we will look for in the following way.

We consider the partial order on the set \( W \) of the degree 6 monomials of \( k[X_0, X_1, X_2, X_3] \) given by the term ordering condition of compatibility with the multiplication (as introduced in [11]), that is the transitive closure of the partial order:

\[
X^\gamma \succeq X^\beta \quad \iff \quad X^\gamma = X_i X^\beta / X_{i-1} \text{ for some } i > 0.
\]

As shown in [18], the set of monomials of a fixed degree in a Borel-fixed monomial ideal is a filter for this partial ordering. For every ideal \( j_s \) we determine the minimal monomials in \( W \cap j_s \) (with respect to \( \succeq \)) and the maximal monomials in its complementary \( W \setminus j_s \). Then we look for a term ordering \( \preceq_s \) given by a matrix of the type:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
a & b & c & d \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

where \( a, b, c, d \) should satisfy two conditions:

i) \( a > b \geq c \geq d \) (to respect the order on variables \( X_3 \triangleright X_2 \triangleright X_1 \triangleright X_0 \));

ii) \( (a, b, c, d) \cdot \beta > (a, b, c, d) \cdot \alpha \) for every minimal monomial \( X^\beta \) in \( W \cap j_s \) and maximal monomial \( X^\alpha \) in \( W \setminus j_s \).
Of course in general no such a matrix exists, but if we can find it, then we can apply Corollary 6.7 and say that the Gröbner stratum $St_h(j_3, \preceq)$ is an open subset of $\text{Hilb}^4_{12}$.

$j_3$ There are two minimal degree 6 monomials in $j_3$: $X_3X_2X_0^3$ and $X_3^2X_0^3$ and three maximal elements in the complementary: $X_3X_1^5$, $X_2X_1^4$, and $X_2^2X_1^2X_0^2$. The vector $(3, 2, 1, 1)$ satisfies the required conditions. Thanks to Corollary 4.8, we know that $St_h(j_3, \preceq_3) \simeq St_h(b_3, \preceq_3)$. Using a computer routine based on Definition 3.4 we obtain explicit equations for $St_h(b_3, \preceq_3)$ founding that it is an affine space of dimension 16.

$j_4$ The minimal degree 6 monomials in $j_4$ are $X_3X_2X_0^3$, $X_3X_1^2X_0^2$, $X_2X_0^4$, the maximal in the complementary are $X_3X_1X_0^5$, $X_2^3X_1^2$ and the weight vector $(15, 5, 2, 1)$ satisfies the required conditions. By Corollary 4.8, we can see that $St_h(j_4, \preceq_4) \simeq St_h((b_4)_{\geq 3}, \preceq_4)$ and by the computer routine for an explicit computation of $S = St_h((b_4)_{\geq 3}, \preceq_4)$.

Looking at the ideals $(K)$ of $S_1$ and $(\mathfrak{h} : K)$ of $S_2$, it is easy to control that they have a transversal intersection. Finally the ideal $(\mathfrak{h} : K)$ of $S_2$ presents some more eliminable variables. If we consider the minimal embedding of $S_2$ we see that $S_2 \simeq \mathbb{A}_{16}$.

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