DOES NONCOMMUTATIVE GEOMETRY ENCOMPASS LATTICE GAUGE THEORY?

Meinulf GÖCKELER, 1
Institut für Theoretische Physik,
Universität Regensburg,
D–93040 Regensburg

Thomas SCHÜCKER, 2
Centre de Physique Théorique,
CNRS - Luminy, Case 907
F–13288 Marseille Cedex 9

Abstract

We are unable to formulate lattice gauge theories in the framework of Connes’ spectral triples.

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1 also at Université de Provence meinulf.goeckeler@physik.uni-regensburg.de
2 also at Université de Provence schucker@cpt.univ-mrs.fr
1 Introduction

Our fascination for Connes’ noncommutative geometry [1] has two sources: 1) It is general enough to treat continuous and discrete spaces on equal footing. 2) It has enough structure to include the Yang–Mills action. A natural question then is whether noncommutative geometry is compatible with standard formulations of lattice field theories [2]. In a more general frame of noncommutative geometry starting from differential algebras Dimakis, Müller–Hoissen & Striker [3] gave an affirmative answer. Since then Connes [4] completed the axiomatic foundation of noncommutative (Riemannian) geometry in terms of spectral triples. The triple consists of an associative involution algebra \( \mathcal{A} \), a faithful representation \( \rho \) on a Hilbert space \( \mathcal{H} \) and a selfadjoint operator \( \mathcal{D} \) on \( \mathcal{H} \), ‘the Dirac operator’. In even dimensions one also requires the existence of a ‘chirality’, a unitary operator \( \chi \) on \( \mathcal{H} \) of square one, that anticommutes with \( \mathcal{D} \). A real spectral triple has furthermore a ‘real structure’, an antiunitary operator \( \mathcal{J} \) on \( \mathcal{H} \) of square plus or minus one. These five items are to satisfy axioms. These axioms generalize properties of the commutative spectral triple of Riemannian spin manifolds \( M \). There \( \mathcal{A} = \mathcal{C}^\infty(M) \) is the commutative algebra of functions on spacetime, \( \mathcal{H} \) is the space of square integrable spinors, \( \mathcal{D} = \partial \), \( \chi = \gamma_5 \) and the real structure is given by charge conjugation. These axioms are tailored such that there is a one–to–one correspondence between commutative spectral triples and Riemannian spin manifolds. The items of the spectral triple allow to construct a differential algebra which is no longer chosen by hand. For the commutative spectral triple of a spacetime \( M \) this differential algebra is isomorphic to de Rham’s algebra of differential forms.

2 Any spectral triple for a lattice?

We tried to construct a lattice action in terms of spectral triples starting from a lattice version of the Dirac operator. We failed already at the level of the axioms. More precisely, it is the first order axiom,

\[
[[\mathcal{D}, \rho(a)], J \rho(\tilde{a}) J^{-1}] = 0, \quad \text{for all } a, \tilde{a} \in \mathcal{A},
\]

that puts us out of business. On a smooth spacetime \( M \) this equation just says that the Dirac operator \( \partial \) is a first order differential operator.

Consider a finite hypercubic lattice of \( N^4 \) points labeled by discrete 4-vectors \( x \) or \( y \). We take for the algebra

\[
\mathcal{A} = \bigoplus_{i=1}^{N^4} \mathbb{C} \ni f,
\]

for the algebra
represented on
\[ \mathcal{H} = \bigoplus_{k=1}^{N^4} \mathbb{C}^k \ni \psi, \]  
by
\[ (\rho(f)\psi)(x) := f(x)\psi(x). \]  
For later purposes we include \( k \) additional degrees of freedom that will be spin or colour. As a matrix the representation is \( \rho(f)(x, y) = f(x)\delta_{xy} \otimes 1_k \). We take the real structure \( J \) to be such that \( J\rho(f)J^{-1} = \rho(\bar{f}) \) with \( z \mapsto \bar{z} \) meaning complex conjugation. (In the concrete examples discussed below this will be fulfilled.) For the Dirac operator, we keep a general matrix \( D(x, y) \), where the additional indices running from 1 to \( k \) are suppressed. Then the double commutator in the first order axiom becomes,
\[ [[D, \rho(f)], \rho(g)](x, y) = (f(y) - f(x))(g(y) - g(x))D(x, y). \]  
The double commutator vanishes if and only if the Dirac operator is diagonal in \((x, y)\). This excludes any kind of difference operator on the lattice as for instance the Dirac–Kähler operator,
\[ D(x, y) = \sum_{\mu=1}^{4} \frac{i}{2} \eta_{\mu}(x) \left[ \delta_{y,x+\hat{\mu}} - \delta_{y,x-\hat{\mu}} \right]. \]  
Here
\[ \eta_{\mu}(x) := (-1)^{\sum_{\nu=1}^{4} x_{\nu}}, \]  
and \( \hat{\mu} \) denotes the lattice unit vector in \( \mu \)-direction . We define the chirality by the matrix
\[ \chi(x, y) := \epsilon(x)\delta_{xy}, \quad \epsilon(x) := (-1)^{x_1+x_2+x_3+x_4}. \]  
Then, besides the first order axiom, all other axioms by Connes are satisfied, in particular Poincaré duality. Indeed, if \( \{ p_x \}_{x \in \mathbb{N}^4} \) is the set of minimal projectors in \( A \) defined by \( p_x(y) = \delta_{xy} \), then the intersection form,
\[ \cap_{xy} := \text{tr} \left[ \chi \rho(p_x)J\rho(p_y)J^{-1} \right] = \epsilon(x)\delta_{xy}, \]  
is non–degenerate.

On the other hand Poincaré duality fails for the naive lattice Dirac operator acting on \( k = 4 \) component spinors \( \psi(x) \),
\[ D(x, y) = \frac{i}{2} \sum_{\mu=1}^{4} \left[ \delta_{y,x+\hat{\mu}} - \delta_{y,x-\hat{\mu}} \right] \gamma^\mu. \]
Here the $\gamma^\mu$ are the four Euclidean, Hermitian Dirac matrices. The chirality is $\chi = 1_{N^4} \otimes \gamma_5$ and the real structure $J$ is charge conjugation on each lattice point. Then the intersection form vanishes identically because $\text{tr} \gamma_5 = 0$.

At this point we recall that Connes’ geometric formulation of the standard model of electro–weak and strong interaction grew out of the two point space with Dirac operator

$$D = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix},$$  \hspace{1cm} (11)$$

with $m$ being the inverse distance between the two points. The two point space is a one dimensional lattice and this $D$ is a kind of lattice Dirac operator. Still the standard model satisfies the first order condition and does so by adding antiparticles and strong interactions. We try to copy this trick:

$$A = (\mathbb{C} \oplus \mathbb{C}) \oplus M_k(\mathbb{C}) \ni (a, b, c),$$ \hspace{1cm} (12)$$

$$\mathcal{H} = (\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^k \oplus (\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^k,$$ \hspace{1cm} (13)$$

$$\rho(a, b, c) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \otimes 1_k \begin{pmatrix} 0 & 0 \\ 1_2 \otimes \bar{c} \end{pmatrix},$$ \hspace{1cm} (14)$$

$$D = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \otimes 1_k \begin{pmatrix} 0 & 0 \\ 1_2 \otimes \bar{c} \end{pmatrix},$$ \hspace{1cm} (15)$$

$$\chi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes 1_k \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \otimes 1_k,$$ \hspace{1cm} (16)$$

$$J = \begin{pmatrix} 0 & 1_2 \otimes 1_k \\ 1_2 \otimes 1_k \end{pmatrix} \circ \text{complex conjugation.}$$ \hspace{1cm} (17)$$

Now the first order axiom works because the ‘colour’ $M_k(\mathbb{C})$ is vectorlike. However the Poincaré duality fails, the intersection form is degenerate. Looking at the standard model, this is not that astonishing. Our model is necessarily closer to the standard model with right–handed neutrinos for which Poincaré duality fails as well.

\section{Conclusion}

Our conclusion is short and disappointing. We were looking for an overlap of lattice field theory and noncommutative geometry in the hope it would shed light on the definition of the functional integral within Connes’ geometry. But so far this overlap seems to be empty in Connes’ strict sense.
References

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