Novel Semi-parametric Tobit Additive Regression Models

Hailin Huang
Department of Statistics
George Washington University
Washington DC
hhlin988@gwmail.gwu.edu

Abstract—Regression method has been widely used to explore relationship between dependent and independent variables. In practice, data issues such as censoring and missing data often exist. When the response variable is (fixed) censored, Tobit regression models have been widely employed to explore the relationship between the response variable and covariates. In this paper, we extend conventional parametric Tobit models to a novel semi-parametric regression model by replacing the linear components in Tobit models with nonparametric additive components, which we refer as Tobit additive models, and propose a likelihood based estimation method for Tobit additive models. Numerical experiments are conducted to evaluate the finite sample performance. The estimation method works well in finite sample experiments, even when sample size is relatively small.

Index Terms—Semi-parametric regression; Additive models; Likelihood-based method; Tobit models; Censored data

I. INTRODUCTION

Regression analysis has been one of the most important statistical techniques in data science. To study the relationship between the response variable \( Y^* \) and a set of covariates \( X \), linear regression methods have been widely applied, which assumes that \( E(Y^*|X) = X^T \beta \) and \( \beta \) is an unknown parameter vector \([1]–[3]\). However, in practice, the response variable \( Y^* \) may not be fully observed because of data issues, such as censoring or detection limit, which is common in economic and biomedical research \([4]–[8]\). One example in medical research is that in HIV clinical studies, the HIV viral load that usually serves as the primary endpoint in HIV clinical trials, has a lower limit of detection of 50 or 20 copies/ml \([7], [9]–[14]\). And another example in economic research is that in household economics survey, household expenditures on various categories of goods vary with household income, and often censored at some lower limit \([6]\). In such situation, instead using traditional regression approach, certain statistical techniques would need to be employed to adjust for data issue. In the literature, for regression with a response variable having data issue such as censoring or detection limit, researchers proposed Tobit regression models and have been extensively studied \([6], [15]–[17]\). In Tobit models, the linearity assumption between latent true response variable \( Y^* \) and \( X \) is maintained as in linear regression models, and the censoring or detection limit problem has also been taken account \([6], [15], [16]\), i.e.

\[
Y^* = X^T \beta + \epsilon, \quad (1)
\]

\[
Y = \max(Y^*, c),
\]

where \( c \) is some fixed detection limit, \( \epsilon \) follows normal distribution with mean zero and some unknown variance \( \sigma \), and \( Y \) is the observable value of latent true response variable \( Y^* \). Likelihood approaches could be used to obtain statistical consistent estimation for \([1] [6], [15], [16]\). However, it is worthy noting that \([1] \) may be subject to linear shape restrictions. In the literature for regression analysis, additive models have been broadly acknowledged as an extension for linear models to increase model flexibility and have been widely investigated \([18], [19]\). As a natural consequence, we consider extending Tobit models to Tobit additive models by replacing the linear form \( X^T \beta \) with additive components \( \sum_{j=1}^{d} m_j(X_j) \), where \( m_j(\cdot) \) is an unspecified smooth function for a \( d \) dimensional covariate \( X \). Although there are a lot of studies in the literature for Tobit and additive models, in the literature, a direct hybrid of Tobit and additive models has not been investigated.

In this manuscript, we propose semi-parametric Tobit additive models, which is more flexible than traditional Tobit models and also takes censoring problem into consideration. The method is easy to implement with well-established optimization techniques, and can perform well even when sample size is relatively small. The manuscript is organized as follows, Section II introduces Tobit additive models and the likelihood-based estimation procedure. Section III explores the performance of the method through numerical simulation studies. Section IV states conclusions and future study directions.

II. MODEL FORMULATION AND METHODS

A. Model Formulation

We assume the following model for the latent response,

\[
Y^* = \sum_{j=1}^{d} m_j(X_j) + \epsilon, \quad (2)
\]

\[
\epsilon \sim N(0, \sigma^2),
\]
where \( X = (X_1, \ldots, X_d)^T \) is a \( d \)-dimensional covariate, \( m_j(\cdot) \)'s are unspecified smooth functions which are standardized such that \( E(m_j(X_j)) = 0 \) \([18, 19]\). The observed \( Y_i \) is such that \( Y_i = \max(Y_i^*, c) \), where \( c \) is the known lower detection limit, that we will assume to be zero W.L.O.G. \([9, 15]\). Let \( \delta_i = I(Y_i^* > 0) \), and we assume that \( \epsilon_i \)'s are independently and identically distributed (i.i.d.) from normal distribution with mean \( 0 \) and unknown finite variance \( \sigma^2 \).

B. Likelihood-Based Estimation Approach

The likelihood based estimation method combines the idea in estimation of additive models \([18, 20]\) and maximum likelihood estimation methods for Tobit models in the literature \([6, 15]\). In the estimation of Tobit models \([1]\), \( \beta \) could be obtained by maximizing

\[
L_n(\gamma) = \prod_{i=1}^{n} \left\{ \frac{\phi(Y_i - X^T \beta)}{\sigma} \right\}^{\delta_i} \left\{ 1 - \Phi\left( \frac{X^T \beta}{\sigma} \right) \right\}^{(1-\delta_i)},
\]

where \( \phi \) and \( \Phi \) are the standard normal density and distribution functions, respectively. Correspondingly, it is also equivalent to maximize the log-likelihood function given below

\[
\log(L_n(\theta_\kappa, \sigma)) = \sum_{i=1}^{n} \delta_i \left\{ -\log \sigma - \frac{(Y_i - B_{\kappa}(X)^T \theta_\kappa)^2}{2\sigma^2} \right\} + \sum_{i=1}^{n} (1 - \delta_i) \log \left\{ 1 - \Phi\left( \frac{B_{\kappa}(X)^T \theta_\kappa}{\sigma} \right) \right\}.
\]

Numerically, well-established optimization methods like Newton-Raphson or BFGS algorithm can be used to solve \([5]\) and \([6]\). In addition, to select the tuning parameter \( \kappa \) that used in the B-spline basis functions, we borrow the idea of cross-validation, and propose a 5-fold cross-validation likelihood score function (for parameter tuning) as follows:

\[
CV(\kappa) = \sum_{l=1}^{5} \log(L_n(\theta_\kappa, \sigma))^{-l},
\]

where \( \log(L_n(\theta_\kappa, \sigma))^{-l} \) is the log likelihood with the \( l \)th folder removed and using tuning parameter \( \kappa \). We maximize this function over a grid of \( \kappa \) values and choose the \( \kappa \) which yields the maximum cross-validation score.

III. NUMERICAL EXPERIMENTS

In this section, we investigate the finite sample performance of our proposed model and methods by Monte Carlo simulations. We generate 50 replicates, each consisting of \( n \) observations from the following model:

\[
Y^* = \sum_{j=1}^{2} m_j(X_j) + 0.2\epsilon,
\]

where \( n \) is chosen to be 80 or 160, and \( \epsilon \) follows a standard normal distribution. Both \( X_1 \) and \( X_2 \) are generated independently and following an uniform distribution on \([0, 1]\). The components of the additive model \((7)\) are: \( m_1(\cdot) = (s - 0.5)^2 - 1/12 \), where one component is linear and the other one is nonlinear. The observed response variable \( Y_i = \max(Y_i^*, c) \), where \( c \) is determined by censoring proportions (Cens): Cen=5%, Cen=15% and Cen=30%.

To implement our likelihood-based estimator, the basis functions used in the estimation are chosen to be cubic B-splines with 1 inner knot decided by cross-validation, the selection of cubic B-spline is based on some previous numerical studies and its desired theoretical property \([21, 22]\). Since the Tobit model can only work with linear relationship, it is by theory can not work with nonlinear relationship, we compare the performance of our method with nonparametric estimation method (NP) for additive models \([23]\). Table 1 summarizes the IMSE (integrated mean square error) of \( \hat{m}_1(\cdot) \) and \( \hat{m}_2(\cdot) \) at 50 equally-spaced grid points over \([0, 1]\). From Table 1, it can be seen that in general, the IMSE’s of Tobit Additive method are much smaller than those of the NP method. For
TABLE I

$10^4 \times \text{IMSE of } \hat{m}_1(\cdot) \text{ and } \hat{m}_2(\cdot) \text{ at 50 equally-spaced grid points over } [0, 1]$.

| n   | Cen | Tobit Additive | NP  |
|-----|-----|----------------|-----|
|     |     | IMSE($m_1$) | IMSE($m_2$) | IMSE($m_1$) | IMSE($m_2$) |
| 80  | 5%  | 26.9         | 26.0         | 60.9         | 41.9         |
|     | 15% | 35.6         | 27.5         | 79.6         | 42.2         |
|     | 30% | 67.9         | 31.9         | 107.7        | 44.3         |
| 160 | 5%  | 12.5         | 10.5         | 36.5         | 18.4         |
|     | 15% | 13.3         | 10.8         | 42.6         | 19.1         |
|     | 30% | 19.0         | 11.9         | 72.7         | 20.9         |

Fig. 1. For linear function: the true curves (solid lines), estimated median curves (dashed lines), and associated 95% confidence bands (dotted lines) for $m_1(\cdot)$, using the Tobit Additive method, the NP method (from bottom to upper) when cen=30%.

Fig. 2. For quadratic function: the true curves (solid lines), estimated median curves (dashed lines), and associated 95% confidence bands (dotted lines) for $m_2(\cdot)$, using the Tobit Additive method, the NP method (from bottom to upper) when cen=30%.

Figures (1) and (2) present the point-wise median curves of the estimated component functions $\hat{m}_1(\cdot)$ and $\hat{m}_2(\cdot)$ at the selected grid points when $n = 80$ and Cens = 0.3, as well as the 95% confidence bands. The discrepancy between the true curves and fitted median curves provide a measure for the bias of the estimators, and the 2.5% and 97.5% lines demonstrate the variability of the estimators. From Figures 1, it can be seen that the fitted values for both methods are close to the true values, and the confidence bands in general covers the true curves, while the Tobit Additive method can produce a better point estimation and narrow confidence bands compared both methods, the IMSE’s increase when Cen’s increase, and the IMSE’s will decrease with the increase of the sample size.
with the NP approach.

IV. CONCLUSIONS

In this research article, a novel semi-parametric Tobit additive regression model was proposed to account for censoring or limit detection in regression analysis and to extend flexibility of Tobit models. The method is straightforward and easy to implement, and perform well in numerical experiments. In the future, the research direction would be in theoretical perspective, in which asymptotic properties for the estimators could be further investigated.

REFERENCES

[1] G. A. Seber and A. J. Lee, Linear regression analysis. John Wiley & Sons, 2012, vol. 329.
[2] J. Neter, M. H. Kutner, C. J. Nachtsheim, and W. Wasserman, Applied linear statistical models. Irwin Chicago, 1996, vol. 4.
[3] H. Huang, J. Shangguan, Y. Li, and H. Liang, “Bi-level variable selection in high-dimensional tobit models,” Statistics and Its Interface, vol. 13, no. 2, pp. 151–156, 2020.
[4] H. Huang, J. Shangguan, P. Ruan, and H. Liang, “Bi-level feature selection in high dimensional aft models with applications to a genomics study,” Statistical applications in genetics and molecular biology, vol. 18, no. 5, 2019.
[5] H. Huang, Y. Li, H. Liang, and C. O. Wu, “Decomposition feature selection with applications in detecting correlated biomarkers of bipolar disorders,” Statistics in medicine, vol. 38, no. 23, pp. 4574–4582, 2019.
[6] J. Tobin, “Estimation of relationships for limited dependent variables,” Econometrica, vol. 26, pp. 24–36, 1958.
[7] C. Orkin, E. Delescu, P. E. Sax, J. R. Arribas, S. K. Gupta, C. Martorell, J. L. Stephens, H.-J. Stellbrink, D. Wohl, F. Maggioni, M. A. Thompson, D. Podzamczer, D. Hagins, J. A. Flamm, C. Brinson, A. Clarke, H. Huang, R. Acosta, D. M. Brainard, S. E. Collins, and H. Martin, “Fixed-dose combination bictegravir, emtricitabine, and tenofovir alafenamide versus dulotegravir-containing regimens for initial treatment of hiv-1 infection: week 144 results from two randomised, double-blind, multicentre, phase 3, non-inferiority trials,” The Lancet HIV, vol. 7, no. 6, pp. e389–e400, 2020. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S2352301820300990
[8] H. Huang, J. Shangguan, X. Li, and H. Liang, “High-dimensional single-index models with censored responses,” Statistics in medicine, vol. 39, no. 21, pp. 2743–2754, 2020.
[9] D. A. Wohl, Y. Yazdanpanah, A. Baumgarten, A. Clarke, M. A. Thompson, C. Brinson, D. Hagins, M. N. Ramgopal, A. Antinori, X. Wei et al., “Bictegravir combined with emtricitabine and tenofovir alafenamide versus dulotegravir, abacavir, and lamivudine for initial treatment of hiv-1 infection: week 96 results from a randomised, double-blind, multicentre, phase 3, non-inferiority trial,” The Lancet HIV, 2019.
[10] K. M. Erlandson, C. C. Carter, K. Melbourne, T. T. Brown, C. Cohen, M. Das, S. Esser, H. Huang, J. R. Koethe, H. Martin et al., “Weight change following antiretroviral therapy switch in people with viral suppression: Pooled data from randomized clinical trials,” Clinical Infectious Diseases, 2021.
[11] R. K. Acosta, G. Q. Chen, S. Chang, R. Martin, X. Wang, H. Huang, D. Brainard, S. E. Collins, H. Martin, and K. L. White, “Three-year study of pre-existing drug resistance substitutions and efficacy of bictegravir/emtricitabine/tenofovir alafenamide in hiv-1 treatment-naive participants,” Journal of Antimicrobial Chemotherapy, 2021.
[12] C. Orkin, P. Sax, J. Arribas, S. Gupta, C. Martorell, J. Stephens, H. Stellbrink, E. Dejesus, F. Maggioni, H. Huang et al., “Long-term efficacy and safety of bictegravir/emtricitabine/tenofovir alafenamide (b/f/t) in art-naive adults,” in HIV MEDICINE, vol. 20. WILEY 111 RIVER ST, HOBOKEN 07030-5774, NJ USA, 2019, pp. 94–95.
[13] I. Brar, P. Ruane, D. Ward, J.-m. Molina, A. Mills, M. Berhe, C. Brinson, M. Ramgopal, P. Benson, K. Henry et al., “1028. Long-term follow-up after a switch to bictegravir, emtricitabine, tenofovir alafenamide from dulotegravir, abacavir, lamivudine,” in Open Forum Infectious Diseases, vol. 7, no. Supplement_1. Oxford University Press US, 2020, pp. S543–S544.
[14] A. Castagna, D. S. C. Hui, K. M. Mullane, K. M. Mullane, M. Jain, M. Galli, S.-C. Chang, R. H. Hyland, D. SenGupta, H. Cao et al., “548. baseline characteristics associated with clinical improvement and mortality in hospitalized patients with moderate covid-19,” in Open Forum Infectious Diseases, vol. 7, no. Supplement_1. Oxford University Press US, 2020, pp. S340–S340.
[15] T. Amemiya, “Regression analysis when the dependent variable is truncated normal,” Econometrica: Journal of the Econometric Society, pp. 997–1016, 1973.
[16] ——, “Tobit models: a survey,” J. Econometrics, vol. 24, no. 1-2, pp. 3–61, 1984. [Online]. Available: http://dx.doi.org/10.1016/0304-4076(84)90074-5
[17] H. Huang, Y. Li, H. Liang, and Y. Tang, “Estimation of single-index models with fixed censored responses,” Statistica Sinica, vol. 30, no. 2, pp. 829–843, 2020.
[18] T. Hastie and R. Tibshirani, “Generalized additive models: some applications,” Journal of the American Statistical Association, vol. 82, no. 398, pp. 371–386, 1987.
[19] T. J. Hastie and R. J. Tibshirani, Generalized additive models. CRC press, 1990, vol. 43.
[20] A. Buja, T. Hastie, and R. Tibshirani, “Linear smoothers and additive models,” The Annals of Statistics, pp. 453–510, 1989.
[21] J. L. Horowitz, E. Mannen et al., “Nonparametric estimation of an additive model with a link function,” The Annals of Statistics, vol. 32, no. 6, pp. 2412–2443, 2004.
[22] J. L. Horowitz and E. Mannen, “Oracle-efficient nonparametric estimation of an additive model with an unknown link function,” Econometric Theory, vol. 27, no. 03, pp. 582–608, 2011.
[23] H. Huang, Y. Tang, Y. Li, and H. Liang, “Estimation in additive models with fixed censored responses,” Journal of Nonparametric Statistics, vol. 31, no. 1, pp. 131–143, 2019.