A Telescopic Bregmanian Proximal Gradient Method Without the Global Lipschitz Continuity Assumption

Daniel Reem · Simeon Reich · Alvaro De Pierro

Received: 19 April 2018 / Accepted: 13 March 2019 / Published online: 25 March 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
The problem of minimization of the sum of two convex functions has various theoretical and real-world applications. One of the popular methods for solving this problem is the proximal gradient method (proximal forward–backward algorithm). A very common assumption in the use of this method is that the gradient of the smooth term is globally Lipschitz continuous. However, this assumption is not always satisfied in practice, thus casting a limitation on the method. In this paper, we discuss, in a wide class of finite- and infinite-dimensional spaces, a new variant of the proximal gradient method, which does not impose the above-mentioned global Lipschitz continuity assumption. A key contribution of the method is the dependence of the iterative steps on a certain telescopic decomposition of the constraint set into subsets. Moreover, we use a Bregman divergence in the proximal forward–backward operation. Under certain practical conditions, a non-asymptotic rate of convergence (that is, in the function values) is established, as well as the weak convergence of the whole sequence to a minimizer. We also obtain a few auxiliary results of independent interest.

Keywords Bregman divergence · Lipschitz continuity · Minimization · TEPROG · Telescopic proximal gradient method · Strongly convex

Mathematics Subject Classification 90C25 · 49M27 · 47J25 · 90C30 · 54C30 · 26B25

Communicated by Hedy Attouch.
1 Introduction

1.1 Background

The problem of minimization of the sum of two convex functions appears in various areas in science and technology, including machine learning, inverse problems, image processing and signal processing [1–8]. Here, we are given an objective function, which is the sum of two convex functions, one of them is smooth (and may vanish identically), but the other may not be differentiable. The goal is to find the minimal value of the objective function over the constraint set, namely over some given constraint subset of the ambient space, and possibly also to find a minimizer (if it exists).

One of the most popular methods for solving this problem is the proximal gradient method (also called the “forward–backward algorithm” or the “proximal point algorithm”). Initial versions of this algorithm, in various settings and forms, were studied by Martinet [9], Rockafellar [10], Bruck and Reich [11], Passty [12], Brézis and Lions [13], and Nevanlinna and Reich [14], and since then many more developments have occurred and many more authors have been involved in the investigation of this algorithm. In its very basic form, this algorithm produces iterations, which are obtained by minimizing, in the iterative step, the sum of the objective function and a quadratic term.

A generalization of this basic form calls for replacing the quadratic term by a Bregman divergence (Bregman distance, Bregman measure of distance) and for replacing the objective function by an approximation of it. The Bregman divergence is a certain substitute for a distance, induced by a given well-chosen function. It has found various applications, among them in optimization, nonlinear analysis, inverse problems, machine learning and computational geometry.

A very common assumption (implicit or explicit) in the use of the various forms (Bregmanian or not) of the proximal gradient algorithm is that the gradient of the smooth term in the objective function is globally Lipschitz continuous. While this assumption holds in several important cases, it is not always satisfied (for example, in some instances of image processing algorithms [15, Subsection 5.1], [3, p. 9], [16, pp. 1063, 1065], [17, p. 262]; see also Sect. 3 and Example 7.1 below). This lack of global Lipschitz continuity casts a limitation on the method. As a matter of fact, to the best of our knowledge, only very few papers [15,18–21] do not impose this global Lipschitz condition (more details about these papers and related works can be found in Sect. 2).

1.2 Contributions

In this paper, we discuss, in a general setting, new variants of the proximal gradient method, which do not require the above-mentioned global Lipschitz continuity assumption. We consider a broad class of finite- and infinite-dimensional spaces (real reflexive Banach spaces), allow constrained minimization and backtracking, and use a Bregman divergence in the iterative step instead of just a quadratic term (this gives
more flexibility to the users). The method, assumptions and results discussed in our paper are considerably different from any other relevant work (see Sect. 2).

A key and novel contribution of our method is the dependence of the iterations on certain well-chosen subsets, which form a telescopic decomposition of the constraint set. More precisely, the forward–backward operation in a given iteration is performed over a well-chosen subset of the constraint set and not over the entire constraint set; by “telescopic decomposition” we mean that the well-chosen subset in a given iteration is contained in the corresponding subset of the next iteration and the union of all of these subsets is the entire constraint set. Hence, we regard our method as being a “telescopic proximal gradient method,” call it TEPROG, and regard the above-mentioned family of well-chosen sets as the “telescopic sequence.”

A major advantage of the above-mentioned dependence of the iterative steps on well-chosen subsets is that it gives the users a lot of flexibility and, in particular, allows them to assume that the gradient of the smooth term is Lipschitz continuous merely on the above-mentioned subsets (an assumption which frequently holds since these subsets are often bounded), rather than globally Lipschitz continuous. Based on results, which have recently been established in [22], we obtain, under certain practical conditions, explicit non-asymptotic rates of convergence to the optimal value (that is, convergence of the function values to the optimal value), as well as the weak convergence of the whole sequence to a minimizer. In fact, we show that sometimes sublinear non-asymptotic rates of convergence can be obtained, and sometimes these non-asymptotic rates can be arbitrarily close to the sublinear rate.

Another contribution of our paper is a few auxiliary results, which seem to be of independent interest. One of them is Lemma 8.5 which generalizes a key result in [2] (namely [2, Lemma 2.3]). A second result is Lemma 8.6, which presents sufficient conditions for the minimizer that appears in the proximal operation to be an interior point.

1.3 Paper Layout

In Sect. 2, we further compare our method and results to several relevant ones from the literature. In Sect. 3, we illustrate the general minimization problem that we solve in this paper using a useful model problem. In Sect. 4, we introduce some basic notation and definitions and also recall well-known facts. In Sect. 5, we present our telescopic proximal method. In Sect. 6, we present the main convergence results. In Sect. 7, we present several relevant examples. The proofs of the convergence results, as well as relevant auxiliary assertions, are presented in Sect. 8. We conclude the paper in Sect. 9. At the end of the paper, there is a short Appendix, which contains the proofs of some assertions.

2 Further Comparison to the Literature

As far as we know, our method and results are new. In particular, as far as we know the idea regarding the telescopic sequence of sets on which the gradient of the smooth
term is Lipschitz continuous (and not necessarily Lipschitz continuous on the entire constraint set) is novel. However, as mentioned in Sect. 1, there are several works which are somewhat related to our paper. In this section, we present more details about these works and elaborate more on some of the main differences between them and our own.

The relevant works are [2,15,18–21,23]. In a nutshell, in most of them (with the exception of [20]) the goal is to minimize an objective function \( F = f + g \). In addition, in most of these works (with the exception of [18,20,21]) the setting is a finite-dimensional Euclidean space. With the exception of [23], in all of these works the proximal operations do not depend on a special sequence of sets, and only in [15,19,21,23] these proximal operations depend on a Bregman divergence. With the exception of [19], both \( f \) and \( g \) are assumed to be convex. Backtracking is discussed only in [2].

Here are a few more details. In ISTA [2, Sections 2–3], the objective function \( F = f + g \) is defined on the whole space (no constrained minimization); in addition, the gradient of the smooth term \( f \) is assumed to be globally Lipschitz continuous; it is shown that under these conditions, the sequence converges non-asymptotically at a sublinear rate, namely \( O(1/k) \). For a predecessor of ISTA, see [4, Theorem 3.4].

In [20], the setting is a real reflexive Banach space which is sometimes assumed to be a real Hilbert space; each iterative step (in [20, Algorithm 2.1]) is based on a minimization of the sum of three terms: a non-smooth term, a certain convex functional (depending on the iteration), and a certain linear term; the convex functional should have a Lipschitz continuous gradient on the constraint set, and this gradient should also be strongly monotone; moreover, all of the Lipschitz constants should be uniformly bounded from above and the strong monotonicity constants should be bounded away from 0; under these and additional assumptions, among them that the gradient of the smooth term is Lipschitz continuous on a constraint subset, the main theorem [20, Theorem 2.1] shows that each weak cluster point solves the minimization problem, and that non-asymptotic convergence holds; under further assumptions, a strong convergence result is established; no rate of convergence of any kind is established. There are other theorems and algorithms in [20], but they are closely related to the ones mentioned above and the main differences between them and our paper still hold.

In the 2008 preprint [23], the space \( X \) is a real normed space which is probably either finite-dimensional (there are many similarities to [8] where \( X \) is finite-dimensional) or at least a reflexive Banach space, since otherwise it is not clear why the iterations in various places, for instance in [23, Equations (8, 12, 17, 28, 31, 45, 47, 48)], are well defined. The paper [23] is the only relevant paper of which we are aware, where there are subsets (closed and convex) \( S_k \) on which the iterations depend, but they are not almost arbitrary as in our method (see Algorithms 5.1 and 5.2): rather, they are either the whole space or they have a certain complicated form (which seems to be inspired by [24, p. 240]) in order to make sure that each of them contains an optimal solution; see [23, Equations (16, 46)]; on the other hand, they are not assumed to satisfy \( \bigcup_{k=1}^{\infty} S_k = C \) and \( S_k \subseteq S_{k+1} \) for all \( k \) as in our case (Assumption 5.2 below): actually, at least in [23, p. 8, Algorithm 3], they are required to satisfy \( S_k \supseteq S_{k+1} \) for all \( k \); in addition, [23] assumes that \( f' \) is globally Lipschitz continuous (that is, on \( C \),
which is actually the whole space in [23]) and the Bregman function $b$ is assumed to be globally strongly convex with a strong convexity parameter which is equal to 1.

In [18], the setting is a Hilbert space; the iterative schemes suggested there are based on a line search strategy; under certain assumptions, an $O(1/k)$ non-asymptotic rate of convergence is established, and under further assumptions (finite-dimensionality), an $o(1/k)$ non-asymptotic rate of convergence is proven.

In [21], the setting is a real reflexive Banach space and the iterative step is based on a Bregman divergence which is induced by a Legendre function (which should satisfy additional conditions), on a sequence of other Legendre functions and on sequences of positive parameters; a key assumption imposed there is a certain “descent condition”: in the notation of our paper (which is different from [21]), it means that if $B_b$ denotes the Bregman divergence induced by $b$ and $B_f$ denotes the Bregman divergence induced by the smooth term $f$, then it is assumed that there exists some $\beta > 0$ such that $B_b(x, y) \geq \beta B_f(x, y)$ for all $x \in \text{dom}(b)$ and $y \in \text{Int}(\text{dom}(b))$; under this condition and additional ones, a weak convergence result is established, and under further (and stronger) assumptions, a strong convergence result is also established; no rate of convergence (asymptotic or non-asymptotic) is established.

In [15], the space is a finite-dimensional Euclidean space and the iterative step is based on a Bregman divergence and on a sequence of positive parameters; the Bregman function is well chosen: for example, it should be Legendre and satisfy a descent condition similar to [21] (called “descent lemma” in [15, Condition (LC) and Lemma 1]; it seems, however, that this “descent condition” has independently been obtained in both works at around the same time); under these and additional conditions, an $O(1/k)$ non-asymptotic rate of convergence is established; under additional assumptions, the convergence of the iterative scheme to a solution of the problem is also established.

The recent paper [19] (see also some of the references therein) does not assume that $f'$ is globally Lipschitz continuous and does not assume that $f$ or $g$ are convex; since [19] is devoted to nonconvex optimization, we do not further elaborate on it, with the exception of saying that our method, the setting that we consider here, and our results are significantly different from the method, setting and results of [19], even when in [19] one restricts one’s attention to the convex case.

Finally, we note that despite the differences in the method, settings and assumptions between our paper and other papers, there are some similarities. For instance, the proof of our main convergence theorem (Theorem 6.1 below) is partly inspired by [2, Section 3] and also has some similarities to [19, Proofs of Theorem 1 and 2] (nevertheless, there are still differences between our proofs and other proofs, for instance because we use the limiting difference property of the Bregman divergence, while in the above-mentioned works this is not done).

3 A Model Problem

In this short section, we illustrate the optimization problem that we intend to solve by using a useful example involving linear inverse problems. Let $n \in \mathbb{N}$ and $p \in [2, \infty[$ be given. Assume that $0 \neq A : \mathbb{R}^n \to \mathbb{R}^m$ is a given linear operator and
\( c \in \mathbb{R}^m \) is a given vector, where \( m \in \mathbb{N} \) is given. Denote \( \|y\|_p := \left( \sum_{i=1}^m |y_i|^p \right)^{1/p} \) for all \( y = (y_i)_{i=1}^m \in \mathbb{R}^m \). Fix some \( \lambda > 0 \) (a regularization parameter) and denote \( f(x) := \frac{1}{p} \|Ax - c\|_p^p \), \( g(x) := \lambda \|x\|_1 \) and \( F := f + g \) for each \( x \in C := \mathbb{R}^n \). The minimization problem is to estimate

\[
\inf \left\{ \frac{1}{p} \|Ax - c\|_p^p + \lambda \|x\|_1 : x \in \mathbb{R}^n \right\}.
\]

In the particular case where \( p = 2 \), this is the familiar \( \ell_2 - \ell_1 \) minimization problem which is popular in machine learning, compressed sensing, and signal/image processing [1,2,7], but the choice \( p = 2 \) is somewhat arbitrary and it seems that it is driven mostly by convenience and some a priori statistical assumptions on the data which often do not hold. Perhaps another reason for the popularity of the choice \( p = 2 \) is because in this case \( f' \) is globally Lipschitz continuous, while this is no longer true when \( p > 2 \). Nevertheless, we show in Example 7.1 below, in a rather detailed manner, how one can apply TEPROG in order to solve (1) even when \( p > 2 \). TEPROG and a similar analysis can be used in closely related scenarios, for instance when the smooth term in (1) is a different proximity function, such as \( KL(c, Ax) \) or \( KL(Ax, c) \) (under some nonnegativity assumptions on \( c, x \) and \( A \)), where \( KL(w, z) := \sum_{j=1}^m \left[ w_j \log(w_j/z_j) - w_j + z_j \right] \), \( w \in [0, \infty[^m, z \in [0, \infty[^m \) is the Kullback–Leibler divergence (see [15, Subsection 5.1] and the references therein for a related discussion).

4 Notation and Definitions

This section introduces a few basic definitions, which are used later in the paper.

4.1 Basic Notation and Assumptions

Unless otherwise stated, we consider a real normed space \((X, \| \cdot \|)\), \( X \neq \{0\} \), which, frequently, will be explicitly assumed to be a real reflexive Banach space. Along the paper, we use well-known and standard notions and notations from convex analysis, such as the subdifferential of a function from \( X \) to \( ]-\infty, \infty[ \), the effective domain of such a function, the Fréchet and Gâteaux derivatives of the function (whenever they exist) and so on. See, for instance, [25–28] for some relevant sources. In particular, we let \( \langle x^*, x \rangle := x^*(x) \) for each \( x^* \) in the dual \( X^* \) of \( X \) and each \( x \in X \). We denote by \([s, t[\), \([s, t[\), \([s, t]\) and \([s, t]\) the open, right-open, left-open and closed interval, respectively, having \( s \) as its left-endpoint and \( t \) as its right-endpoint, where \(-\infty \leq s \leq t \leq \infty \).

Given \( \emptyset \neq S \subseteq X \), we denote its closure by \( \text{cl}(S) \). A function \( h : S \to X^* \) is said to be weak-to-weak* sequentially continuous at \( x \in S \) if for each sequence \( (x_i)_{i=1}^\infty \) in \( S \) which converges weakly to \( x \) and for each \( w \in X \), we have \( \lim_{i \to \infty} \langle h(x_i), w \rangle = \langle h(x), w \rangle \). If \( h \) is weak-to-weak* sequentially continuous at each \( x \in S \), then \( h \) is said to be weak-to-weak* sequentially continuous on \( S \). Of course, when \( X \) is reflexive,
then saying that \( h \) is weak-to-weak* sequential continuous is the same as saying that it is weak-to-weak sequential continuous, or, briefly, that it is weakly sequentially continuous. We say that \( h \) is coercive on \( S \) if \( \lim_{\|x\| \to \infty, x \in S} h(x) = \infty \) and supercoercive on \( S \) if \( \lim_{\|x\| \to \infty, x \in S} h(x)/\|x\| = \infty \). We say that \( b : X \to ] - \infty, \infty] \) with \( U := \text{Int}(\text{dom}(b)) \) (namely, \( U \) is the interior of the effective domain of \( b \)) is essentially smooth on \( X \), or on \( U \), if \( U \not= \emptyset \) and \( b \) is proper, convex and Gâteaux differentiable on \( U \), and also \( \lim_{t \to -\infty} \|b'(x_t)\| = \infty \) for every sequence \( (x_t)_{t=1}^{\infty} \) in \( U \) which converges to a boundary point of \( U \), where \( b' \) is the gradient of \( b \).

### 4.2 Relative Uniform Convexity

The next definition presents the central concepts of uniform convexity, relative uniform convexity and (relative) strong convexity.

**Definition 4.1** Assume that \( b : X \to ] - \infty, \infty] \) is convex and proper. Suppose that \( S_1 \) and \( S_2 \) are two nonempty subsets (not necessarily convex) of \( \text{dom}(b) \).

(I) The function \( b \) is called uniformly convex relative to \((S_1, S_2)\) (or relatively uniformly convex on \((S_1, S_2)\)) if there exists \( \psi : [0, \infty[ \to [0, \infty] \), called a relative gauge, such that \( \psi(t \in ]0, \infty[ \) whenever \( t > 0 \) and for each \( \lambda \in ]0, 1[ \) and each \((x, y) \in S_1 \times S_2\),

\[
b(\lambda x + (1 - \lambda)y) + \lambda (1 - \lambda) \psi(\|x - y\|) \leq \lambda b(x) + (1 - \lambda)b(y). \tag{2}\]

If \( S := S_1 = S_2 \) and \( b \) is uniformly convex relative to \((S_1, S_2)\), then \( b \) is said to be uniformly convex on \( S \).

(II) The optimal gauge of \( b \) relative to \((S_1, S_2)\) is the function defined for each \( t \in ]0, \infty[ \) by:

\[
\psi_{b, S_1, S_2}(t) := \inf \left\{ \frac{\lambda b(x) + (1 - \lambda)b(y) - b(\lambda x + (1 - \lambda)y)}{\lambda (1 - \lambda)} : (x, y) \in S_1 \times S_2, \|x - y\| = t, \lambda \in ]0, 1[ \right\}, \tag{3}
\]

where we use the standard convention that \( \inf \emptyset := \infty \), namely if there does not exist \((x, y) \in S_1 \times S_2\) such that \( \|x - y\| = t \), then \( \psi_{b, S_1, S_2}(t) := \infty \). The optimal gauge is also called the modulus of relative uniform convexity of \( b \) on \((S_1, S_2)\) or simply the optimal relative gauge, and \( b \) is uniformly convex on \((S_1, S_2)\) if and only if \( \psi_{b, S_1, S_2}(t) > 0 \) for every \( t \in ]0, \infty[ \). If \( S := S_1 = S_2 \), then we denote \( \psi_{b, S} := \psi_{b, S_1, S_2} \) and call \( \psi_{b, S} \) the modulus of uniform convexity of \( b \) on \( S \).

(III) The function \( b \) is said to be uniformly convex on closed, convex and bounded subsets of \( \text{dom}(b) \) if \( b \) is uniformly convex on each nonempty subset \( S \subseteq \text{dom}(b) \) which is closed, convex and bounded.

(IV) The function \( b \) is said to be strongly convex relative to \((S_1, S_2)\) if there exists \( \mu > 0 \) (which depends on \( S_1 \) and \( S_2 \)), called a parameter of strong convexity of \( b \) on \((S_1, S_2)\), such that \( b \) is uniformly convex relative to \((S_1, S_2)\) with \( \psi(t) := \)
\[ \frac{1}{2} \mu t^2, \ t \in [0, \infty[ , \text{ as a relative gauge. If } S := S_1 = S_2 \text{ and } b \text{ is strongly convex relative to } (S_1, S_2), \text{ then } b \text{ is said to be strongly convex on } S, \text{ namely for each } \lambda \in ]0, 1[ \text{ and each } x, y \in S, \text{ we have} \]

\[ b(\lambda x + (1 - \lambda) y) \leq \lambda b(x) + (1 - \lambda) b(y) - \frac{1}{2} \mu \lambda (1 - \lambda) \|x - y\|^2. \tag{4} \]

**Remark 4.1** The notion of relative uniform convexity was introduced and investigated in [22], where various examples, results and other relevant details can be found. Of course, uniformly convex and strongly convex functions (not relatively uniformly convex) are well-known concepts: see, for instance, [29, pp. 63–66] and [28, pp. 203–221].

### 4.3 Bregman Divergences

We now discuss (semi) Bregman functions and divergences.

**Definition 4.2** Suppose that \( b : X \to ] - \infty, \infty[ \). Let \( \emptyset \neq U \subseteq X \).

1. We say that \( b \) is a *semi-Bregman function* with respect to \( U \) (the zone of \( b \)) if the following conditions hold:
   - (i) \( U = \text{Int}(\text{dom}(b)) \) (in particular, \( \text{Int}(\text{dom}(b)) \neq \emptyset \)) and \( b \) is Gâteaux differentiable in \( U \).
   - (ii) \( b \) is convex and lower semicontinuous on \( X \) and strictly convex on \( \text{dom}(b) \).

2. We say that \( B \) is the *Bregman divergence* (or the *Bregman distance*, or the *Bregman measure of distance*) associated with \( b \), if \( B \) is defined by

\[
B(x, y) := \begin{cases} 
   b(x) - b(y) - (b'(y), x - y), & \forall (x, y) \in \text{dom}(b) \times \text{Int}(\text{dom}(b)), \\
   \infty, & \text{otherwise.}
\end{cases}
\tag{5}
\]

3. We say that \( b \) (or \( B \)) has the *limiting difference property* if for each \( x \in \text{dom}(b) \) and each weakly convergent sequence \( (y_i)_{i=1}^\infty \) in \( U \), if the weak limit of \( (y_i)_{i=1}^\infty \) is some \( y \in U \), then \( B(x, y) = \lim_{i \to \infty} (B(x, y_i) - B(y, y_i)) \).

4. We say that \( B \) has bounded level-sets of the first type if for each \( \gamma \in [0, \infty[ \) and each \( x \in \text{dom}(b) \), the level-set \( L_1(x, \gamma) := \{ y \in U : B(x, y) \leq \gamma \} \) is bounded.

Here are a few comments regarding Definition 4.2.

**Remark 4.2** (i) The Bregman divergence is, of course, a classical notion. It was introduced by Bregman [30] in 1967 and since then it has found applications in various fields, among them in optimization, nonlinear analysis, inverse problems, machine learning, and computational geometry. This divergence is not a true metric (for example, because it does not satisfy the triangle inequality), but it still enjoys various properties which make it a useful substitute for a distance (for example, \( B(x, y) \geq 0 \) for all \( x \in \text{dom}(b) \) and \( y \in \text{Int}(\text{dom}(b)) \) and \( B(x, y) = 0 \) if and only if \( x = y \in U \); see, for instance, [22, Proposition 4.13(III)]). Many more details about this notion, including historical details, a long list of relevant
references, various mathematical properties, and a thorough re-examination, can be found in the recent paper [22].

(ii) A semi-Bregman function generalizes the notion of “a Bregman function” (the term “semi-Bregman function” seems to be new, although it has been used here and there without this explicit name). In finite-dimensional Euclidean spaces, Bregman functions should satisfy more conditions in addition to the ones mentioned in Definition 4.2. See, for example, [31, Definition 2.1], [32, Definition 2.1], [33, Definition 2.1] (in this finite-dimensional setting, the limiting difference property is just a consequence of the classical definition and it is not mentioned explicitly).

(iii) A sufficient condition which ensures that $b$ has the limiting difference property is that $b'$ is weak-to-weak* sequentially continuous: see [22, Proposition 4.13(XIX)]. Of course, if $X$ is finite-dimensional, then $b'$ is continuous (this is a consequence of [26, Corollary 25.5.1, p. 246]) and hence weak-to-weak* sequentially continuous; see [22, Proposition 5.6 and Remark 5.7] for sufficient conditions which ensure that $b'$ is weak-to-weak* sequentially continuous in infinite-dimensional settings.

(iv) An immediate sufficient condition which ensures that all the first-type level-sets of $B$ will be bounded is that $U$ is bounded. A less immediate such sufficient condition is that for each $x \in \text{dom}(b)$, there exists $r_x \geq 0$ such that the subset $\{ w \in U : \|w\| \geq r_x \}$ is nonempty and $b$ is uniformly convex relative to $\{ x, \{ w \in U : \|w\| \geq r_x \} \}$ with a gauge $\psi_x$ which satisfies $\lim_{t \to \infty} \psi_x(t) = \infty$: see [22, Proposition 4.13(XV)]. This latter condition holds, in particular, if $b$ is uniformly convex on $\text{dom}(b)$, as follows from [22, Lemma 3.3].

### 4.4 The Proximal Gradient Method

Here, we briefly recall very well-known versions of the proximal gradient method (the proximal point algorithm). In its very basic form, this algorithm can be written as follows:

$$
x_k := \arg\min_{x \in C} (F(x) + c_k \|x - x_{k-1}\|^2), \quad k \geq 2,
$$

where $F$ is the objective function to be minimized (it should satisfy certain assumptions, for example, to be proper, convex and lower semicontinuous), $C$ is the constraint subset over which the minimal value of $F$ is sought ($C$ is assumed to be a nonempty, closed and convex subset of the ambient space $X$), $x_1 \in X$ is some initial point, and $c_k$ is some positive parameter which may or may not depend on the iteration, for example, $c_k = 0.5L$ for every $k \geq 2$, where $L$ is a fixed positive number. It is well known that $(x_k)_{k=1}^\infty$ is well defined (that is, the minimizer in (6) exists and is unique) and converges weakly, at least under certain assumptions. See, for example, [4, Theorem 3.4] and [10, p. 878 and Theorem 1 (p. 883)].

A further generalization of (6) is to replace the regularization term $\|x - x_k\|^2$ by a Bregman divergence $B(x, x_{k-1})$ and to replace the function $F$ by an approximation function $F_k$ so that the iterative scheme becomes
\[ x_k := \arg\min_{x \in C} (F_k(x) + c_k B(x, x_{k-1})), \quad k \geq 2. \]  

(7)

One possible choice for \( F_k \) is simply \( F_k := F \), as done in [34] (the paper which started the investigation of the proximal gradient method in the context of Bregman divergences). However, when \( F = f + g \) and \( f' \) exists, it is common to take \( F_k(x) := f(x_{k-1}) + \langle f'(x_{k-1}), x - x_{k-1} \rangle + g(x), \quad x \in C \), namely \( F_k \) is the sum of \( g \) with a linear term which approximates the smooth term \( f \). A very partial list of references which consider, in various settings, the proximal gradient method with a Bregman divergence, is [8,23,35–40].

5 TEPROG: A Telescopic Proximal Bregmanian Method

In this section, we present TEPROG, namely our new variants of the proximal gradient method with a Bregman divergence. We impose the following assumptions: The ambient space \((X, \| \cdot \|)\) is a real reflexive Banach space; \( b : X \to ]-\infty, \infty] \) is a semi-Bregman function (see Definition 4.2); in particular, its zone is \( U := \text{Int}(\text{dom}(b)) \neq \emptyset \); we denote by \( B \) the associated Bregman divergence of \( b \) [defined in (5)]; we assume that \( C \subseteq \text{dom}(b) \), which is the constraint subset over which we want to perform the minimization process, is a nonempty, closed and convex subset of \( X \) which satisfies \( C \cap U \neq \emptyset \); we are given a function \( f : \text{dom}(b) \to \mathbb{R} \) which is convex on \( \text{dom}(b) \) and Gâteaux differentiable in \( U \); we assume that the restriction of \( f \) to \( C \) is lower semicontinuous; we are also given a function \( g : C \to ]-\infty, \infty] \) which is convex, proper and lower semicontinuous. Our goal is to solve the minimization problem

\[
\inf \{ F(x), \ x \in C \},
\]

where \( F : X \to ]-\infty, \infty] \) is the function defined by

\[
F(x) := \begin{cases} 
  f(x) + g(x), & x \in C, \\
  \infty, & x \notin C.
\end{cases}
\]  

(9)

We assume from now on that \( C \) contains more than one point, since otherwise (8) is trivial. The assumptions on \( f, g \) and \( C \) imply that \( F \) is convex, proper and lower semicontinuous. The values of \( g \) outside \( C \) and of \( f \) outside \( \text{dom}(b) \) are not very important for us, but, as is common in optimization theory, we may assume that they are equal to \( +\infty \).

We define as follows a proximal function \( p_{L,\mu,S} \) which depends on three parameters. The first two parameters are arbitrary \( L > 0 \) and \( \mu > 0 \). The third parameter is an arbitrary closed and convex subset \( S \subseteq C \) which has the following properties: first, \( b \) is strongly convex on \( S \) with a strong convexity parameter \( \mu > 0 \); second, \( g \) is proper on \( S \); third, \( S \cap U \neq \emptyset \) (Assumption 5.2 below ensures the existence of such a subset \( S \)). Given \( y \in S \cap U \), let

\[
p_{L,\mu,S}(y) := \arg\min\{ Q_{L,\mu,S}(x, y) : \ x \in S \}.
\]  

(10)
where \( Q_{L,\mu,S}(x, y) \) is defined as follows:

\[
Q_{L,\mu,S}(x, y) := f(y) + \langle f'(y), x - y \rangle + \frac{L}{\mu} B(x, y) + g(x), \quad \forall x \in S, \quad \forall y \in U.
\]

(11)

Our assumptions and Lemma 8.3 below ensure that the function \( u_y(x) := Q_{L,\mu,S}(x, y), x \in S \), has a unique minimizer in \( S \). Thus \( p_{L,\mu,S}(y) \) is well defined. We impose the following additional assumptions:

**Assumption 5.1** The set of minimizers of \( F \), namely

\[
\text{MIN}(F) := \left\{ x_{\min} \in C : F(x_{\min}) = \inf \{ F(x) : x \in C \} \right\},
\]

is nonempty and contained in \( U \).

**Assumption 5.2** There is a sequence of subsets \( (S_k)_{k=1}^{\infty} \) in \( C \), which, for the sake of convenience, we refer to as “a telescopic sequence,” that has the following properties for each \( k \in \mathbb{N} \): \( S_k \) is closed and convex; \( S_k \cap U \) contains more than one point; \( S_k \subseteq S_{k+1} \subseteq C \); the function \( b \) is strongly convex on each \( S_k \) with a parameter \( \mu_k \) such that \( \mu_{k+1} \leq \mu_k \); finally, \( \bigcup_{k=1}^{\infty} S_k = C \). In addition, \( g \) is proper on \( S_1 \) (and hence on each \( S_k, k \in \mathbb{N} \)).

**Assumption 5.3** \( f' \) is Lipschitz continuous on \( S_k \cap U \) for each \( k \in \mathbb{N} \). We denote by \( L(f', S_k \cap U) := \sup \{ \| f'(x) - f'(y) \| / \| x - y \| : x \in S_k \cap U, y \in S_k \cap U, x \neq y \} \) the best (smallest) Lipschitz constant of \( f' \) on \( S_k \cap U \).

**Assumption 5.4** For all \( L > 0, k \in \mathbb{N} \) and \( y \in S_k \cap U \), we have \( p_{L,\mu_k,S_k}(y) \in S_k \cap U \), where \( \mu_k \) is the strong convexity parameter of \( b \) over \( S_k \).

Assumptions 5.1–5.4 occur frequently in applications. Indeed, the assumption that \( \text{MIN}(F) \) is nonempty (in Assumption 5.1) holds if, for example, \( C \) is compact (in particular, when \( X \) is finite-dimensional and \( C \) is closed and bounded) or when both \( g \) and the restriction of \( f \) to \( C \) are coercive, since then \( F \) is proper, coercive, convex, and lower semicontinuous on the closed and convex subset \( C \) and hence, by a well-known and classical result [41, Corollary 3.23, p. 71], it has a minimizer in \( C \); the assumption that \( \text{MIN}(F) \subseteq U \) trivially holds if \( C \subseteq U \), but it may hold in other cases as well (see, for instance, Examples 7.2 and 7.3 below). Assumption 5.2 holds frequently: for instance, if \( b \) is strongly convex on bounded and convex subsets of \( \text{dom}(b) \), then we can take each \( S_k \) to be the intersection of \( C \) with a closed ball with center at some \( y_0 \in C \cap U \), where the radii of these balls are increasing, or, if \( b \) is strongly convex on \( C \), then we may take \( S_k := C, k \in \mathbb{N} \) (with the hope that the other assumptions will be satisfied with this choice). Assumption 5.3 holds frequently, for instance if \( f' \) is continuous on \( U \) and \( S_k \) is bounded for each \( k \) and \( f'' \) exists and is bounded on bounded subsets of \( C \cap U \) (as follows from the generalized Mean Value Theorem [42, Theorem 1.8, pp. 13, 23]). Assumption 5.4 holds, for instance, when \( C \subseteq U \) or (as follows from Lemma 8.6 and Remark 8.1 below) when \( X \) is finite-dimensional, \( b \) is essentially smooth on \( U \), and either \( b \) is continuous on \( \text{dom}(b) \) or \( g \) is continuous on...
C or there exists a point \( x \in S_1 \cap U \) such that \( g(x) \in \mathbb{R} \). Anyhow, Assumption 5.4 ensures that \( x_k \) defined below (in either (13) or (15)) satisfies \( x_k \in S_k \cap U \) for all \( k \in \mathbb{N} \).

Our telescopic proximal gradient method is presented below. We consider two versions of it: one with a Lipschitz step size rule and the other with a backtracking step size rule.

**Algorithm 5.1 (TEPROG with a Lipschitz step size rule):**

*Input:* A positive number \( L_1 \geq L(f', S_1 \cap U) \).

*Step 1 (Initialization)* an arbitrary point \( x_1 \in S_1 \cap U \).

*Step \( k, k \geq 2 \): \( L_k \) is arbitrary such that \( L_k \geq \max\{L_{k-1}, L(f', S_k \cap U)\} \). Given \( \mu_k > 0 \), which is a parameter of strong convexity of \( b \) on \( S_k \) (see Assumption 5.2), let

\[
x_k := p_{L_k, \mu_k, S_k}(x_{k-1}).
\]  

(13)

**Algorithm 5.2 (TEPROG with a backtracking step size rule):**

*Input* \( \eta > 1 \); if \( S_k = C \) for all \( k \in \mathbb{N} \), then another input is an arbitrary positive number \( L_1 \); otherwise, another input is any \( L_1 \) satisfying \( 0 < L_1 \leq \eta L(f', S_1 \cap U) \).

*Step 1 (Initialization)* an arbitrary point \( x_1 \in S_1 \).

*Step \( k, k \geq 2 \): Let \( \mu_k > 0 \) be a parameter of strong convexity of \( b \) on \( S_k \) (the existence of \( \mu_k \) is ensured by Assumption 5.2). Find the smallest nonnegative integer \( i_k \) such that with \( L_k := \eta^{i_k} L_{k-1} \), we have

\[
F(p_{L_k, \mu_k, S_k}(x_{k-1})) \leq Q_{L_k, \mu_k, S_k}(p_{L_k, \mu_k, S_k}(x_{k-1}), x_{k-1}).
\]  

(14)

Now let

\[
x_k := p_{L_k, \mu_k, S_k}(x_{k-1}).
\]  

(15)

**Remark 5.1** The backtracking step size rule is well defined in the sense that (14) does occur for some \( i_k \). Indeed, given \( 2 \leq k \in \mathbb{N} \), we first observe that [22, Proposition 4.13(i)], when applied to \( \psi(t) := 0.5 \mu_k t^2 \), \( t \in [0, \infty[ \), \( y := x_{k-1} \in S_{k-1} \cap U \subseteq S_k \cap U \) and \( x := p_{L_k, \mu_k, S_k}(x_{k-1}) \in S_k \cap U \), implies that the following inequality holds: \( 0.5 \mu_k \| p_{L_k, \mu_k, S_k}(x_{k-1}) - x_{k-1} \|^2 \leq B(p_{L_k, \mu_k, S_k}(x_{k-1}), x_{k-1}) \). By combining this inequality with Lemma 8.7 below (in which we take any \( L \geq L(f', S_k \cap U) \)) and with (11), we see that

\[
F(p_{L_k, \mu_k, S_k}(x_{k-1})) = f(p_{L_k, \mu_k, S_k}(x_{k-1})) + g(p_{L_k, \mu_k, S_k}(x_{k-1}))
\]

\[
\leq g(p_{L_k, \mu_k, S_k}(x_{k-1})) + f(x_{k-1}) + (f'(x_{k-1}) - f'(x_{k-1})) + \frac{1}{2}L\|p_{L_k, \mu_k, S_k}(x_{k-1}) - x_{k-1}\|^2
\]

\[
\leq g(p_{L_k, \mu_k, S_k}(x_{k-1})) + f(x_{k-1}) + (f'(x_{k-1}) - f'(x_{k-1})) + \frac{L}{\mu_k}B(p_{L_k, \mu_k, S_k}(x_{k-1}), x_{k-1})
\]

\[
= Q_{L_k, \mu_k, S_k}(p_{L_k, \mu_k, S_k}(x_{k-1}), x_{k-1}).
\]  

(16)

Since, by taking \( i_k \) large enough, we can ensure that \( L_k \geq L(f', S_k \cap U) \), if we let \( L := L_k \), then (16) implies that (14) holds. It may happen, however, that (14) holds
even when $L_k < L(f', S_k \cap U)$. In the Lipschitz step size rule $L_k \geq L(f', S_k \cap U)$ for all $k \geq 2$ by the definition of $L_k$, and hence (14) holds in this case too, as a result of (16). Finally, since $Q_{L_k, \mu_k, S_k}(p_{L_k, \mu_k, S_k}(x_{k-1}), x_{k-1}) \in \mathbb{R}$ according to Lemma 8.3 below, we can conclude from (16) that $F(x_k) \in \mathbb{R}$ for all $k \geq 2$.

**Remark 5.2** The constructions of $L_k$ in both the Lipschitz and the backtracking step size rules imply immediately that $(L_k)_{k=1}^{\infty}$ is increasing. Actually, by using simple induction it can be shown that in the backtracking step size rule, if $S_j \neq C$ for some $j$, then $L_k \leq \eta L(f', S_k \cap U)$ for each $k \in \mathbb{N}$: see the Appendix for the details.

The above discussion implies that we can construct an increasing sequence $(\tau_k)_{k=1}^{\infty}$ of positive numbers which always satisfies

$$L_k \leq \tau_k, \quad \forall k \in \mathbb{N}, \quad (17)$$

and sometimes also satisfies

$$\tau_k \leq \eta L(f', S_k \cap U), \quad \forall k \in \mathbb{N}. \quad (18)$$

If we are interested only in (17), as in the Lipschitz step size rule, then, of course, we can simply take $\tau_k := L_k$ for all $k \in \mathbb{N}$, but we are free to select the $\tau_k$ parameters in any other way which guarantees (17). Suppose that we are in the backtracking step size rule and we are interested in both (17) and (18). If $S_j \neq C$ for some $j \in \mathbb{N}$, then $L_1 \leq \eta L(f', S_1 \cap U)$ and from the above discussion we can take $\tau_k := \eta L(f', S_k \cap U)$ for all $k \in \mathbb{N}$; this choice ensures that both (17) and (18) hold. Otherwise, $S_k = C$ for all $k \in \mathbb{N}$. In order to make sure that both (17) and (18) hold, the previous discussion implies that we can take an arbitrary $L_1 \leq \eta L(f', S_1 \cap U)$ and define $\tau_k := \eta L(f', S_k \cap U)$ for all $k \in \mathbb{N}$. If we are in the backtracking step size rule but select some $L_1 > \eta L(f', S_1 \cap U)$, then a simple analysis shows that actually $L_{k+1} = L_k$ for each $k \in \mathbb{N}$ and hence we can let $\tau_k := L_k$ for each $k \in \mathbb{N}$; see the Appendix for the simple details.

### 6 The Convergence Theorem and Several Corollaries

In this section, we present Theorem 6.1 below. It asserts that under some assumptions, the iterative scheme produced by TEPROG (Algorithms 5.1 or 5.2) converges at a certain non-asymptotic rate of convergence, and under further assumptions, also converges weakly. A few corollaries are presented after the theorem itself.

**Theorem 6.1** In the framework of Sect. 5, for each minimizer $x_{\min} \in \text{MIN}(F)$, there exists $k_0 \in \mathbb{N}$ (which satisfies $k_0 \geq 2$ if $F(x_1) = \infty$) such that for each $k \geq k_0$, we have

$$F(x_{k+1}) - F(x_{\min}) \leq \frac{\tau_{k+1}B(x_{\min}, x_{k_0})}{(k + 1 - k_0)\mu_{k+1}}. \quad (19)$$

In addition, if

$$\lim_{k \to \infty} \frac{\tau_k}{k \mu_k} = 0, \quad (20)$$

 Springer
then \((x_k)_{k=1}^\infty\) converges non-asymptotically to the minimal value of \(F\). Moreover, if (20) holds, \(B\) has the limiting difference property and all of its first type level-sets are bounded, then there exists a point \(z_\infty \in \text{MIN}(F)\) such that \(z_\infty = \lim_{k \to \infty} x_k\) weakly. In particular, if (20) holds, \(b'\) is weak-to-weak* sequentially continuous on \(U\), and either \(U\) is bounded or for each \(x \in C\), there exists \(r_x \geq 0\) such that \(\{y \in U : \|y\| \geq r_x\} \neq \emptyset\) and \(b\) is uniformly convex relative to \(\{|x|, \{y \in U : \|y\| \geq r_x\}\}\) with a gauge \(\psi_x\) satisfying \(\lim_{t \to \infty} \psi_x(t) = \infty\) (this latter condition holds, in particular, if \(b\) is uniformly convex on \(U\)), then \((x_k)_{k=1}^\infty\) converges weakly to some point in \(\text{MIN}(F)\).

**Corollary 6.1** Consider the framework of Sect. 5, and assume further that \(f''\) exists, is bounded and uniformly continuous on bounded subsets of \(C \cap U\), and that \(b\) is strongly convex on \(C\) with a strong convexity parameter \(\mu > 0\). Then, we can construct a sequence \((x_k)_{k=1}^\infty\), by either Algorithm 5.1 or 5.2, which converges in the function values to a solution of (8), at a rate of convergence which can be arbitrarily close to \(O(1/k)\). In particular, for all \(x_{\text{min}} \in \text{MIN}(F), q \in ]0, 1[ , y_0 \in C \cap U\) and \(\alpha > \|f''(y_0)\|\), there is a sequence \((x_k)_{k=1}^\infty\) and an index \(k_0 \in \mathbb{N}\) such that

\[
F(x_{k+1}) - F(x_{\text{min}}) \leq \frac{k + 1}{k + 1 - k_0} \cdot \frac{\alpha B(x_{\text{min}}, x_{k_0})}{\mu(k + 1)^{1-q}}, \quad \forall k \geq k_0.
\]

Moreover, if for each \(x \in C\), there exists \(r_x \geq 0\) such that \(\{y \in U : \|y\| \geq r_x\} \neq \emptyset\) and \(b\) is uniformly convex relative to \(\{|x|, \{y \in U : \|y\| \geq r_x\}\}\) with a gauge \(\psi_x\) satisfying \(\lim_{t \to \infty} \psi_x(t) = \infty\) (a condition which holds, in particular, when \(b\) is uniformly convex on \(U\)), and if \(b'\) is weak-to-weak* sequentially continuous on \(U\), then \((x_k)_{k=1}^\infty\) converges weakly to some point in \(\text{MIN}(F)\).

**Corollary 6.2** In the framework of Sect. 5, suppose that \(f'\) is Lipschitz continuous on \(C \cap U\) and \(b\) is strongly convex on \(C\) with a strong convexity parameter \(\mu > 0\). Then, by denoting \(S_k := C\) and \(\mu_k := \mu\) for each \(k \in \mathbb{N}\), the sequence \((x_k)_{k=1}^\infty\), which is obtained by either Algorithm 5.1 or 5.2, converges in the function values to the minimal value of \(F\) at a rate of \(O(1/k)\). Furthermore, \(k_0\) (that is, the index which is guaranteed in the formulation of Theorem 6.1) satisfies \(k_0 = 1\), unless \(F(x_1) = \infty\) and then \(k_0 = 2\). Moreover, if, in addition, \(b'\) is weak-to-weak* sequentially continuous on \(U\) and either \(C\) is bounded or \(b\) is uniformly convex on \(\text{dom}(b)\), then the above-mentioned sequence \((x_k)_{k=1}^\infty\) converges weakly to a solution of (8).

**Remark 6.1** It is possible to weaken a bit some of the assumptions needed for the definition of TEPROG and for the non-asymptotic convergence. Indeed, if \(b\) is semi-Bregman with the exception of being strictly convex on \(\text{dom}(b)\), then \(p_{L_k, \mu_k, S_k}(x_{k-1})\) will be a nonempty subset of \(S_k\) (this is a consequence of Lemma 8.3) and hence (by Assumption 5.4) of \(S_k \cap U\), and \(x_k\) from either Algorithms 5.1 or 5.2 can be taken to be any point in \(p_{L_k, \mu_k, S_k}(x_{k-1})\). The proof of the non-asymptotic convergence in Theorem 6.1 remains as it is. However, it is an open problem whether the weak convergence result holds when \(b\) is no longer strictly convex, since the derivation of this convergence result is based on Lemma 8.1, which is based on the assumption that \(b\) is strictly convex.
7 Examples

In this section, we present a few examples which illustrate our convergence results.

Example 7.1 \((\ell_p - \ell_1\) optimization) We use the notation of Section 3 and show how to solve, using the Lipschitz step size rule version of TEPROG, the \(\ell_p - \ell_1\) optimization problem (1) mentioned there. For doing this, we need to choose the sets \(S_k\) and the Bregman function \(b\), to estimate \(\mu_k\), \(L_k\) and \(\tau_k\), and also to show how to compute \(x_k\) from (13).

Let \(X:=\mathbb{R}^n\) and \(Y:=\mathbb{R}^m\). Denote \(C:=X\). Fix some \(r \in [2, \infty[\) and endow \(X\) with the norm \(\|x\|_r = (\sum_{j=1}^n |x_j|^r)^{1/r}\). For each \(k \in \mathbb{N}\), let \(S_k:=[-\rho_k, \rho_k]^n\), where \((\rho_k)_{k=1}^\infty\) is an increasing sequence of positive numbers which satisfies both \(\lim_{k \to \infty} \rho_k = \infty\) and \(\lim_{k \to \infty} \rho_k^{p-2}/k = 0\) (for instance, we can take \(\rho_k := k^\sigma\), where \(\sigma\) is a fixed number which satisfies \(\sigma \in]0, 1/(p-2)[\) if \(p > 2\) and can be an arbitrary positive number if \(p = 2\)). Then \(S_k \subset S_{k+1}\) for all \(k \in \mathbb{N}\) and \(\cup_{k=1}^\infty S_k = C = \mathbb{R}^n\).

Denote \(b(x):=\frac{1}{2}\|x\|_2^2\) for each \(x \in X\). It is well known and can easily be proved that \(b\) is strongly convex on \(\mathbb{R}^n\) with the Euclidean norm, where the strong convexity parameter is 1: see, for instance, [22, Subsection 11.4]. Since \(r \geq 2\), it follows that \(\|x\|_2 \geq \|x\|_r\) for every \(x \in \mathbb{R}^n\). This inequality and simple calculations show that \(b\) is strongly convex on \((X, \| \cdot \|_r)\), again with 1 as a strong convexity parameter (for a more general statement, see [22, Proposition 5.3]). Thus, \(b\) is strongly convex on \(S_k\) (for each \(k \in \mathbb{N}\)) with \(\mu_k := 1\) as a strong convexity parameter.

Let \(h(y):=\frac{1}{p}\|y\|_p^p\) for all \(y \in Y\). Given a nonempty and bounded subset \(T\) of \(Y\), since we have \(p \in [2, \infty[\), it essentially follows from [36, pp. 48–49] that \(h'\) is Lipschitz continuous over \(T\) with a Lipschitz constant \((p-1)(2M_T)^{p-2}\), where \(M_T\) is an upper bound on the norms of the elements of \(T\) (but we note that there is a small mistake in [36, Expression (26)]: instead of the inequality \(\|h'(x) - h'(y)\| \leq (p-1)(\|x\|_p + \|y\|_p)^{(p-2)/p}\|x - y\|_p\) written there, the following expression should be written: \(\|h'(x) - h'(y)\| \leq (p-1)(\|x\|_p + \|y\|_p)^{p-2}\|x - y\|_p\). Since \(f(x) = h(Ax - c)\) for every \(x \in X\), a direct calculation shows that \(f'\) is Lipschitz continuous on \(S\) with \((p-1)(2M_{AS-c})^{p-2}\|A\|\) as a Lipschitz constant, where \(M_{AS-c}\) is an upper bound on the norms of the elements of the set \(AS-c\) and \(\|A\| := \sup\{\|Ax\|_p : \|x\|_r = 1\}\) is the operator norm of \(A\). Since the norm of any element in \(AS-c\) is bounded by \(\|A\|M_S + c\|\|_p\), where \(M_S\) is an upper bound on the norms of the elements of \(S\), and since \(\|x\|_r \leq n^{1/r}\rho_k\) for all \(x \in S_k\), we conclude that for all \(k \in \mathbb{N}\), the function \(f'\) is Lipschitz continuous on \(S_k\) with \(L_k := (p-1)2^{p-2}\|A\|\|\|\|_{1/r}\rho_k + \|c\|_p\|^p\) as a Lipschitz constant. Then, \((L_k)_{k=1}^\infty\) is an increasing sequence and Remark 5.2 ensures that we can take \(\tau_k := L_k\) for all \(k \in \mathbb{N}\).

Fix \(2 \leq k \in \mathbb{N}\). We need to compute \(x_k\). It follows from (13), (10) and (11) that

\[
x_k = \arg\min_{w \in S_k} \left\{ f(x_{k-1}) + (f'(x_{k-1}), w - x_{k-1}) + \frac{L_k}{2\mu_k} \|w - x_{k-1}\|_2^2 + \lambda\|w\|_1 \right\}.
\]

Denote by \(\phi_j\) the \(j\)-th component of \(f'(x_{k-1})\), \(j \in \{1, \ldots, n\}\). Direct calculations imply the equality \(\phi_j = \sum_{i=1}^m (Ax_{k-1})_i - c_i\|\|^p\|c\|^2\). Since \((Ax_{k-1})_i\) is the \(i\)th component of the vector \(Ax_{k-1} \in \mathbb{R}^m\) and \(A_{ij}\) is the \((i, j)\)-entry of the...
matrix representation of $A$ for each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Denote $\alpha_{k-1} := f(x_{k-1}) - f'(x_{k-1})(x_{k-1})$. In addition, for each $j \in \{1, \ldots, n\}$ let $H_j : [-\rho_k, \rho_k] \to \mathbb{R}$ be defined by $H_j(t) := \phi_j t + 0.5(L_k/\mu_k)(t - x_{k-1,j})^2 + \lambda |t|$, $t \in [-\rho_k, \rho_k]$, where $x_{k-1,j}$ is the $j$th component of $x_{k-1}$. These notations and (22) imply that

$$x_k = \text{argmin} \left\{ \alpha_{k-1} + \sum_{j=1}^{n} H_j(w_j) : (w_j)_{j=1}^{n} \in [-\rho_k, \rho_k]^{n} \right\}. \quad (23)$$

Thus, in order to compute $x_k$ it is sufficient to find, for each $j \in \{1, \ldots, n\}$, a minimizer of $H_j$ on $[-\rho_k, \rho_k]$. Since $H_j$ is differentiable on the set $[-\rho_k, \rho_k] \setminus \{0, -\rho_k, \rho_k\}$ and since $H_j'$ satisfies the equality $H_j'(t) = \phi_j + (L_k/\mu_k)(t - x_{k-1,j}) + \lambda \cdot \text{sign}(t)$ for each $t \in [-\rho_k, \rho_k] \setminus \{0, -\rho_k, \rho_k\}$, considerations from elementary calculus show that the minimal value of $H_j$ is attained, and, moreover, this minimal value can be attained only at one of the following (at most) five points: $t_j(1) := -\rho_k$, $t_j(2) := \rho_k$, $t_j(3) := 0$, $t_j(4) := (\mu_k/L_k)(-\phi_j - \lambda) + x_{k-1,j}$ (only if the expression which defines $t_j(4)$ is in $[0, \rho_k]$), and $t_j(5) := (\mu_k/L_k)(-\phi_j + \lambda) + x_{k-1,j}$ (only if the expression which defines $t_j(5)$ is in $[\rho_k, 0]$).

Now, we merely need to compute $H_j(t_j(1)), \ldots, H_j(t_j(5))$ and to find the minimal value among them. Then, we let $\tilde{w}_j$ be the corresponding argument among the $t_j(1), \ldots, t_j(5)$ which leads to this minimal value ($\tilde{w}_j$ is unique since $H_j$ is strictly convex). By repeating this process for all $j \in \{1, \ldots, n\}$ and using (23) we see that $x_k = (\tilde{w}_j)_{j=1}^{n}$.

Finally, since $\lim_{\|x\| \to \infty} F(x) = \infty$ and $F$ is continuous over $X$, a well-known result in classical analysis implies that $\text{MIN}(F) \neq \emptyset$. We conclude that Assumptions 5.1–5.4 hold. Since we assume that $\lim \alpha_k = 0$, Theorem 6.1 implies that the proximal sequence $(x_k)_{k=1}^{\infty}$, which is obtained from Algorithm 5.1 converges to a point in $\text{MIN}(F)$, and (19) implies that the non-asymptotic rate of convergence is $O(\rho_k^{-2}/k)$. In particular, if $\rho_k := k^\sigma$ for all $k \in \mathbb{N}$, where $\sigma \in [0, 1/(p-2))$, if $p > 2$ and can be an arbitrary positive number if $p = 2$, then the non-asymptotic rate of convergence is $O(1/k^{1-\sigma(p-2)})$, that is, by letting $\sigma$ be arbitrarily close to 0, the non-asymptotic rate of convergence can be arbitrarily close to $O(1/k)$.

**Example 7.2** Fix some $p \in [1, \infty]$ and let $(X, \| \cdot \|)$ be $\mathbb{R}^3$ with the $\ell_p$ norm. Let $C := C_0$ where $C_0 := \{w = (w_1, w_2, w_3) \in [0, 1]^3 : \sum_{j=1}^{3} w_j = 1\}$ is the probability simplex. Let $U := \{0, \infty\}^3$ and $b$ be the negative Boltzmann–Gibbs–Shannon entropy function defined by $b(w) := \sum_{j=1}^{3} w_j \log(w_j)$ for $w = (w_j)_{j=1}^{3} \in U$ and $b(w) := 0$ for $w$ on the boundary of $U$ and $b(w) := \infty$ for $w \notin \text{cl}(U)$. Let $f(w) := \frac{4}{15} \left( (w_1 + w_2)^{5/2} + (w_2 + w_3)^{5/2} + (w_3 + w_1)^{5/2} \right)$ for all $w \in \text{cl}(U) = \text{dom}(b)$. Let $S_k := C$ for all $k \in \mathbb{N}$. Let $I$ be a nonempty finite set and for each $i \in I$, let $g_i : X \to \mathbb{R}$ be the linear function defined for each $w \in X$ by $g_i(w) := \sum_{j=1}^{3} a_{ij} w_j$, where $a_{ij} \in \mathbb{R}$ for each $i \in I$ and $j \in \{1, 2, 3\}$, and $\sum_{j=1}^{3} a_{ij} \leq 1$ for each $i \in I$. Assume also that $\min \{a_{i1}, a_{i2}, a_{i3} : i \in I\} \geq 0.27$. This condition holds, for instance, when $\min \{a_{i1}, a_{i2}, a_{i3}\} \geq 0.27$ for all $i \in I$. Let $g(w) := \max \{g_i(w) : i \in I\}$ for all $w \in C$. Then, $g$ is convex and continuous, but usually non-smooth.

Since $C$ is bounded, it follows that $b$ is strongly convex on $C$ (see [22, Subsection 6.3]). A somewhat technical but simple verification (the argument is based on
the component-wise monotonicity of our norm and its dual, the fact that \(|w_i| \leq \| (w_1, w_2, w_3) \|_i\) for all \(i \in \{1, 2, 3\}\), and also on the mean value theorem applied to the function \(t \mapsto (2/3)t^{3/2}\) on a bounded interval contained in \([0, \infty)\) shows that \(f'\) is Lipschitz continuous on any bounded and convex subset of \(U \cap C\) with \(4\sqrt{2}\| (1, 1, 1) \|_q \sqrt{MC} = 1\) as a (not necessarily optimal) global Lipschitz constant; here \(\| \cdot \|_q\) is the dual norm (namely \((1/p) + (1/q) = 1\)) and \(MC\) is the radius of a ball which contains \(C\) and has 0 as its center (any such ball is fine). In addition, \(F := f + g\) is convex and continuous on the compact subset \(C\) and hence \(\text{MIN}(F) \neq \emptyset\). Moreover, by considerations from elementary calculation we have \(F(w) \geq (4/15)(1 + 2^{-3/2}) + 0.27 > 0.63\) for each \(w\) in the intersection of \(C\) with the boundary of \(U\) (that is, any \(w\) which belongs to the union of the segments \(\{(y_1, y_2, 0) \in [0, 1]: y_1 + y_2 = 1\}, \{(0, y_2, y_3) \in [0, 1]: y_2 + y_3 = 1\}\) and also that \(F(c) < 0.63\) for \(c = (1/3, 1/3, 1/3) \in C\). Since obviously \(F(x) \leq F(c)\) for every \(x \in \text{MIN}(F)\), it follows that no point in \(\text{MIN}(F)\) can belong to the intersection of \(C\) with the boundary of \(U\). Hence, \(\text{MIN}(F) \subset U\). Since \(X\) is finite-dimensional, \(b'\) is weak-to-weak*. Finally, direct calculation shows that \(b\) is essentially smooth and hence Assumption 5.4 holds (see the discussion after Assumption 5.4). Thus, all the conditions needed in Corollary 6.2 are satisfied. Therefore, the proximal sequence \((x_k)_{k=1}^\infty\) which is obtained from either Algorithms 5.1 or 5.2 converges to a point in \(\text{MIN}(F)\), and the function values rate of convergence is \(O(1/k)\).

**Example 7.3** Consider the setting of Example 7.2, where we re-define \(C\) to be the prism which is obtained from the intersection of the following halfspaces: \(\{w \in \mathbb{R}^3 : w_1 + w_2 + w_3 \geq 1\}\) (on the boundary of this halfspace the probability simplex \(C_0\) is located), \(\{w \in \mathbb{R}^3 : -2w_1 + w_2 + w_3 \leq 1\}, \{w \in \mathbb{R}^3 : w_1 - 2w_2 + w_3 \leq 1\}\), and \(\{w \in \mathbb{R}^3 : w_1 + w_2 - 2w_3 \leq 1\}\). Now \(C\) is unbounded, but since \(F\) is continuous on \(C\) and \(\lim_{\|w\| \to \infty, w \in C} F(w) = \infty\), it follows that \(\text{MIN}(F) \neq \emptyset\). Since the intersection of \(C\) with the boundary of \(U\) is as in Example 7.2 (namely, the boundary of \(C_0\)), we have \(\text{MIN}(F) \subset U\), that is, Assumption 5.1 holds. For each \(k \in \mathbb{N}\), let \(V_k\) be the closed ball of radius \(r_k := \sqrt{k}\) and center at the origin and let \(S_k := C \cap V_k\). It follows from [22, Subsection 6.3] that \(b\) is essentially smooth on \(S_k\) with \(\mu_k := \beta/r_k\) as a strong convexity parameter, where \(\beta\) is some positive constant not depending on \(k\) (note that one should not expect \(b\) to be globally strongly convex, since [22, Subsection 6.5] shows that \(b\) is not even uniformly convex on \(U\)). Thus, Assumption 5.2 holds. As we saw in Example 7.2, \(b\) is essentially smooth and hence Assumption 5.4 holds (see the discussion after Assumption 5.4). The same reasoning as the one used in Example 7.2 can be used to show that \(f'\) is Lipschitz continuous on \(S_k \cap U\) with a Lipschitz constant \(L_k := 4\sqrt{2}\| (1, 1, 1) \|_q \sqrt{r_k} = O(k^{0.25})\). Hence \((L_k)_{k=1}^\infty\) is increasing and Assumption 5.3 holds. Finally, [22, Subsection 6.4] shows that for each \(x \in \text{cl}(U)\) (in particular, for each \(x \in C\)), there exists \(r_x \geq 0\) such that \(b\) is uniformly convex relative to \(\{x\}, \{y \in U : \|x\| \geq r_x\}\), with some gauge \(\psi_x\) satisfying \(\lim_{t \to \infty} \psi_x(t) = \infty\) (namely, \(r_x := 2\|x\|\) and \(\psi_x(t) = \gamma t, t \in [0, \infty]\), for some \(\gamma > 0\) independent of \(t\)). Thus, if we denote \(\tau_k := L_k\) for every \(k \in \mathbb{N}\), then Theorem 6.1 and Remark 5.2 imply that the proximal sequence \((x_k)_{k=1}^\infty\) generated by Algorithm 5.1 converges to a point in \(\text{MIN}(F)\), and (19) implies that the non-asymptotic rate of convergence is \(O(1/k^{0.25})\).
Example 7.4 Let $X := \ell_2$. Let $\| (x_i)_{i=1}^\infty \| := \sum_{i=1}^{2n} |x_i| + \sqrt{\sum_{i=2n+1}^{\infty} x_i^2}$ for all $x = (x_i)_{i=1}^\infty \in X$, where $n \in \mathbb{N} \cup \{0\}$ and $\sum_{i=1}^{2n} |x_i| := 0$ if $n = 0$. A simple verification shows that $(X, \| \cdot \|)$ is isomorphic to $(X, \| \cdot \|_{\ell_2})$. Let $C := \{ x \in X : x_i \geq 0 \ \forall i \in \mathbb{N} \}$ be the nonnegative orthant and consider the function

$$b(x) := \left\{ \begin{array}{ll} \sum_{i=1}^{\infty} \left( e^{(x_{2i-1}+x_{2i})^2} + e^{(x_{2i-1}-x_{2i})^2} - 2 \right), & x \in C, \\ \infty, & \text{otherwise.} \end{array} \right.$$ 

Considerations similar to the ones presented in [22, Section 10] show that this function is a well-defined semi-Bregman function which satisfies various additional properties, including the limiting difference property. Moreover, these considerations show that if $n = 0$, then $b$ is strongly convex on $C$ with $\mu = 4$ as a strong convexity parameter, and if $n > 0$, then $b$ is strongly convex on $C$ with $\mu = 1/n$ as a strong convexity parameter. Fix some $\beta \geq 2$ and a sequence $(p_i)_{i=1}^\infty$ of real numbers in $[2, \beta]$, and let $f(x) := \sum_{i=1}^{\infty} x_i^{p_i}$ for every $x \in C$. Then, $f$ is well defined, convex and continuous on $C$ and differentiable in $U := \text{Int}(\text{dom}(b)) = \{ x \in X : x_i > 0 \ \forall i \in \mathbb{N} \}$. Fix some $\lambda > 0$ (a regularization parameter) and define $g(x) := \lambda \| x \|$ for each $x \in C$. Let $F := f + g$. For each $2 \leq k \in \mathbb{N}$, let $S_k$ be the intersection of $C$ with the ball of radius $r_k := k^\sigma$ and center 0, where $\sigma \in [0, 1/(\beta - 2)]$ is fixed in advance (if $\beta = 2$, then $\sigma$ can be an arbitrary positive number). In addition, let $S_1 := S_2$ and $r_1 := r_2$. Since $b$ is strongly convex on $C$ with a strong convexity parameter $\mu > 0$, it is strongly convex on $S_k$ for each $k \in \mathbb{N}$ with $\mu_k := \mu$ as its strong convexity parameter.

We claim that $f'$, which exists in $U$, is Lipschitz continuous on $S_k \cap U$ for each $k \in \mathbb{N}$. Indeed, we first observe that if, given $i \in \mathbb{N}$, we define $h_i(t) := t^{p_i-1}$ for every $t \in [0, r_k]$, then the mean value theorem implies that for each $t, s \in [0, r_k]$, there exists some $\theta$ between $t$ and $s$ such that $h_i(t) - h_i(s) = h'(\theta)(t - s)$. Hence,

$$|h_i(t) - h_i(s)| = (p_i - 1)\theta^{p_i-2}|t - s| \leq (\beta - 1)r_k^{p_i-2}|t - s| \leq (\beta - 1)r_k^{p_i-2}|t - s|,$$

where we used in the last inequality the assumptions that $\beta \geq 2$ and $r_k \geq 1$ for all $k \in \mathbb{N}$. Since $f'(x)(w) = \sum_{i=1}^{\infty} p_i x_i^{p_i-1} w_i$ for every $x \in U$ and $w \in X$, the above inequality, the Cauchy–Schwarz inequality and simple calculations imply that for all $x, y \in S_k \cap U$,

$$\| f'(x) - f'(y) \| = \sup_{w \in X, \| w \|=1} | f'(x)(w) - f'(y)(w) |$$

$$= \sup_{w \in X, \| w \|=1} \left| \sum_{i=1}^{\infty} p_i w_i (x_i^{p_i-1} - y_i^{p_i-1}) \right|$$

$$\leq \beta \sup_{w \in X, \| w \|=1} \left\{ \sum_{i=1}^{\infty} w_i^2 \right\} \left\{ \sum_{i=1}^{\infty} |x_i^{p_i-1} - y_i^{p_i-1}|^2 \right\}^{1/2}.$$
Assumption 5.1 holds. Direct calculations show that nonempty, closed and convex subset $\tau$ implies that the non-asymptotic rate of convergence is $\sigma$ that Assumptions 5.2 and 5.4 hold. To see that also Assumption 5.3 holds, we need to connect, we note that the fact that $\mu$ is Lipschitz continuous on $S_k \cap U$ with $L_k := \beta(\beta - 1)r_k^{\beta - 2}$ as a Lipschitz constant. Since $\lim_{\|x\| \to \infty, x \in C} F(x) = \infty$, and $F$ is convex and continuous over the nonempty, closed and convex subset $C$, a well-known result [41, Corollary 3.23, p. 71] implies that MIN($F) \neq \emptyset$. We conclude that Assumptions 5.1–5.4 hold. Hence, if we let $\tau_k := L_k$ for all $k \in \mathbb{N}$, then Theorem 6.1 implies that the proximal sequence $(x_k)_{k=1}^{\infty}$ generated by Algorithm 5.1 converges weakly to a point in MIN($F$), and (19) implies that the non-asymptotic rate of convergence is $O(1/k^{1-\sigma(\beta - 2)}$).

**Example 7.5** Let $p \in [1, 2]$ be given. Let $X$ be $\ell_p$ with the $\| \cdot \|_p$ norm. This is a reflexive Banach space which is not isomorphic to a Hilbert space unless $p = 2$. Let $C := X$ and $b : X \to \mathbb{R}$ be defined by $b(y) := \frac{1}{2}\|y\|_p^2$, $y \in X$. Then, $U := \text{Int}(\text{dom}(b)) = X$. Define $h : X \to \mathbb{R}$ by $h(y) := \sum_{i=1}^{\infty} |y_i|^{p+2}$. Suppose that $0 \neq A : X \to X$ is a given bounded linear operator and that $c \in X$ is a given vector. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be defined by $f(y) := h(Ay - c)$ and $g(y) := \max\{\lambda \|y\|_p, \sup\{\lambda_i |y_i| : i \in \mathbb{N}\}\}$ for all $y = (y_i)_{i=1}^{\infty} \in X$, respectively, where $\lambda > 0$ is given and $(\lambda_i)_{i=1}^{\infty}$ is a given bounded sequence of positive parameters. The facts that $p \in [1, 2]$ and $\lim_{i \to \infty} y_i = 0$ for each $y \in X$ imply that $h$ and hence $f$ are well defined and smooth. In addition, $f$ is convex. Let $F := f + g$. Then, $g$ and $F$ are convex, proper and lower semicontinuous.

We want to solve (8). Since $F$ is also coercive, it has a minimizer and hence Assumption 5.1 holds. Direct calculations show that $b^* : X \to X^* \cong \ell_q$ (where $q = p/(p - 1)$) exists and satisfies $b'(y) = \|y\|_p^{2-p}(|y_i|^{p-1}\text{sign}(y_i))_{i=1}^{\infty}$ for all $y \in X$. In addition, $b$ is strongly convex on $X$ as follows, for instance, from [43, Example 6.7] (since there $\rho = \max\{2, p\} = 2$, and [43, Inequality (6.8)] is equivalent to strong convexity as follows from [28, Corollary 3.5.11(i)–(v), pp. 217–218]; in this connection, we note that the fact that $b$ is strongly convex has been observed long before, for example, in [44, p. 314]). Denote by $\mu$ the strong convexity parameter of $b$ and let $\mu_k := \mu$ for all $k \in \mathbb{N}$. It follows from Remark 4.2(iv) that all the first-type level-sets of $B$ are bounded.

For each $k \in \mathbb{N}$, let $S_k$ be the closed ball of radius $\rho_k$ about the origin, where $\lim_{k \to \infty} \rho_k^p/k = 0$, $\lim_{k \to \infty} \rho_k = \infty$ and $\rho_{k+1} \geq \rho_k$ for all $k \in \mathbb{N}$ (for example, we can fix $\sigma \in ]0, 1/p[$ and take $\rho_k := k^{\sigma}$). The previous lines immediately imply that Assumptions 5.2 and 5.4 hold. To see that also Assumption 5.3 holds, we need to show that $f'$ is Lipschitz continuous. In order to show this, we first observe that since $f'(y) = h'(Ay - c)A$ for every $y \in X$, we have

\[
\|f'(u) - f'(v)\|_q = \sup_{\|u\|_p = 1} |(h'(Au - c) - h'(Av - c))(Aw)| \\
\leq \|h'(Au - c) - h'(Av - c)\|_q \|A\|
\]
for all $u$ and $v$ in $X$. Since $\|Ay - c\|_p \leq \|A\|\rho_k + \|c\|_p$ for all $y \in S_k$, it is sufficient to show that $h'$ is Lipschitz continuous on the ball of radius $\|A\|\rho_k + \|c\|_p$ about the origin.

Indeed, direct calculations show that $h'(y) := (\phi(\gamma_i))_{i=1}^{\infty}$, $y \in X$, where $\phi : \mathbb{R} \to \mathbb{R}$ is defined by $\phi(s) := (p + 2)|s|^{p+1}\text{sign}(s)$, $s \in \mathbb{R}$. Elementary considerations (based on the Taylor expansion with remainder in Lagrange’s form and the fact that $\phi''$ exists and is continuous and bounded on any compact interval) imply that $h''$ exists. Moreover, these considerations imply that $h''$ satisfies the equality $h''(y)(w, \tilde{w}) = (p+2)(p+1)\sum_{i=1}^{\infty} |yi|^p w_i \tilde{w}_i$ for all $y \in X$ and $w, \tilde{w} \in X$ which satisfy the condition $\|w\|_p = 1 = \|\tilde{w}\|_p$. In particular, $h'$ is continuous. Since $|w_i| \leq \|w\|_p = 1$ for all $i \in \mathbb{N}$, one has $|h''(y)(w, \tilde{w})| \leq (p+2)(p+1)\sum_{i=1}^{\infty} |y_i|^p = (p+2)(p+1)\|y\|_p^p$.

Since the definition of $h''$ implies that $\|h''(y)\| = \sup_{\|w\|_p=1,\|\tilde{w}\|_p=1} |h''(y)(w, \tilde{w})|$, the previous lines imply that $\|h''\|$ is bounded by $(p+2)(p+1)M^p$ on the ball of radius $M > 0$ about the origin. Therefore, the (generalized) Mean Value Theorem applied to $h'$ (see [42, Theorem 1.8, p. 13, and also p. 23]) implies that $h'$ is Lipschitz continuous on this ball with $(p+2)(p+1)M^p$ as a Lipschitz constant. This is true, in particular, for $M := \|A\|\rho_k + \|c\|_p$, and so the previous lines imply that $f'$ is Lipschitz continuous on $S_k$ with a Lipschitz constant $L_k := (p+2)(p+1)(\|A\|\rho_k + \|c\|_p)\|A\|^2$.

Since $L_k = O(\rho_k^p)$, by letting $\tau_k := L_k$ and using our assumption that $\lim_{k \to \infty} \rho_k^p/k = 0$, we have $\lim_{k \to \infty} \tau_k/(k\mu_k) = 0$. As a result, if $(x_k)_{k=1}^{\infty}$ is the sequence defined by Algorithm 5.1, then Theorem 6.1 implies that it converges non-asymptotically to the minimal value of $F$ and the rate of non-asymptotic convergence is $O(\rho_k^p/k)$. We note that it is not clear whether $(x_k)_{k=1}^{\infty}$ converges weakly to a minimizer of $F$, since $b'$ is not weak-to-weak* sequentially continuous unless $p = 2$ (for instance, if $(\epsilon_k)_{k \in \mathbb{N}}$ is the canonical basis in $X$, then $(\epsilon_k + e_1)_{k=1}^{\infty}$ converges weakly to $e_1$ but $\lim_{k \to \infty} \langle b'(\epsilon_k + e_1), e_1 \rangle = 2(2-p)/p \neq 1 = \langle b'(e_1), e_1 \rangle$). The issue of weak convergence of $(x_k)_{k=1}^{\infty}$ is left as an open problem for the future.

### 8 Proofs

This section contains the proofs of Theorem 6.1 and the corollaries which follow it. The proofs are based on several auxiliary assertions. Since some of the proofs are either standard or follow directly from well-known results, we either omit them completely or only sketch them briefly. Full proofs of these omitted proofs can be found in [45, Sections 8, 10].

We start with the following general lemma which slightly modifies [46, Lemma 3.4]. Its proof can be found in the Appendix at the end of this paper.

**Lemma 8.1** Suppose that $X \neq \{0\}$ is a real reflective Banach space and $b : X \to \mathbb{R}$ is a semi-Bregman function with a zone $U$. Let $B : X^2 \to \mathbb{R}$ be the Bregman divergence associated with $b$ and defined in (5), and suppose further that $B$ satisfies the limiting difference property. If $(x_k)_{k=1}^{\infty} \subset U$ is a bounded sequence in $U$ having the properties that all of its weak cluster points are in $U$ and $\lim_{k \to \infty} B(q, x_k)$ exists and is finite for each weak cluster point $q \in X$ of $(x_k)_{k=1}^{\infty}$, then $(x_k)_{k=1}^{\infty}$ converges weakly to a point in $U$. 

© Springer
The next lemma, which is probably known, generalizes [25, Proposition 11.14, p. 193] from real Hilbert spaces to real normed spaces. The proofs in both cases are fairly similar.

**Lemma 8.2** Suppose that $X \neq \{0\}$ is a real normed space and let $\emptyset \neq S \subseteq X$ be closed, convex and unbounded. Let $u : S \to ]-\infty, \infty]$ be defined by $u(x) := v(x) + w(x)$ for each $x \in S$, where $v : S \to ]-\infty, \infty]$ is convex, proper, and lower semicontinuous, and $w : S \to ]-\infty, \infty]$ is supercoercive on $S$. Then, $u$ is supercoercive on $S$.

The next lemma is needed for the formulation of the proximal forward–backward algorithms presented in Sect. 5.

**Lemma 8.3** Let $X \neq \{0\}$ be a real reflexive Banach space. Suppose that $b : X \to ]-\infty, \infty]$ is lower semicontinuous, convex and proper on $X$, that $U := \text{Int(dom}(b)) \neq \emptyset$, and that $b$ is Gâteaux differentiable in $U$. Suppose that $S \subseteq \text{dom}(b)$ is nonempty, closed and convex, and that $b$ is strictly convex on $S$. Suppose also that $f : \text{dom}(b) \to \mathbb{R}$ is Gâteaux differentiable in $U$ and that $g : S \to ]-\infty, \infty]$ is proper, lower semicontinuous and convex. Fix arbitrary $L > 0$ and $\mu > 0$, let $B$ be the associated Bregman divergence of $b$, and let $Q_{L, \mu, S}$ be defined in (11). Fix some $y \in U$ and assume that at least one of the following conditions holds:

(i) $S$ is bounded;
(ii) $S$ is unbounded and $b$ is uniformly convex relative to $(S, \{y\})$ with a relative gauge $\psi$ which satisfies $\lim_{t \to \infty} \psi(t)/t = \infty$.

Then, the function $u_y : S \to ]-\infty, \infty]$ defined by

\[
u_y(x) := Q_{L, \mu, S}(x, y), \quad x \in S, \tag{24}\]

has a unique minimizer $z \in S$ (and $u_y(z)$ is a real number). In particular, if $S \cap U \neq \emptyset$ and $b$ is uniformly convex on $S$, then $u_y$ has a unique minimizer $z \in S$ for each $y \in S \cap U$.

**Proof** Suppose first that Condition (i) holds. Then, the existence of a minimizer is just a consequence of [41, p. 11] since $S$ is compact with respect to the weak topology (because $X$ is reflexive) and $u_y$ is lower semicontinuous with respect to the weak topology.

Consider now the case of Condition (ii). Since $S \subseteq \text{dom}(b)$ and $y \in U$, the assumed uniform convexity of $b$ relative to $(S, \{y\})$ with a relative gauge $\psi$ implies, according to [22, Proposition 4.13(i)], that $B(x, y) \geq \psi(\|x - y\|)$ for all $x \in S$. Since we assume that $\lim_{t \to \infty} \psi(t)/t = \infty$ and that $S$ is unbounded, we have

\[rac{B(x, y)}{\|x\|} \geq \frac{\psi(\|x - y\|)}{\|x - y\|} \cdot \frac{\|x - y\|}{\|x\|} \xrightarrow[\|x\| \to \infty, x \in S]{} \infty.
\]

This fact, (11), (24) and Lemma 8.2 imply that $\lim_{\|x\| \to \infty, x \in S} u_y(x)/\|x\| = \infty$ and hence, in particular, $\lim_{\|x\| \to \infty, x \in S} u_y(x) = \infty$. Therefore, we can use [41, Corollary 3.23, p. 71] to conclude that $u_y$ has a minimizer $z \in S$. 

\[\square\] Springer
The fact that $u_y(z)$ is a real number is an immediate consequence of the definition of $u_y$, the assumption that $g$ is proper on $S \subseteq \text{dom}(b)$, and the fact that $z$ is a minimizer of $u_y$ on $S$. The uniqueness of the minimizer is just a well-known consequence of the fact that $u_y$ is strictly convex on $S$ (since $u_y$ is the sum of $b$ and another convex function, and we assume that $b$ is strictly convex on $S$).

Finally, assume that $S \cap U \neq \emptyset$ and that $b$ is uniformly convex on $S$. If $S$ is bounded, then we are in the case of Condition (i) and the assertion follows from previous paragraphs. Assume now that $S$ is unbounded. We observe that the modulus of convexity $\psi_{b,S}$ of $b$ is a relative gauge on $(S, \{y\})$ for all $y \in S \cap U$ (this is true even if $S$ is bounded). Now, fix some $y \in S \cap U$. Since $S$ is unbounded, [22, Lemma 3.3] implies that $\lim_{t \to \infty} \psi_{b,S}(t)/t = \infty$. Thus, previous paragraphs imply that $u_y$ has a unique minimizer on $S$. \qed

The next lemma generalizes [47, Lemma 5.2] to the setting of Bregman divergences ([47, Lemma 5.2] by itself extends [2, Lemma 2.2]). Its proof is standard and uses Lemma 8.3, Fermat’s rule (namely the very simple necessary and sufficient condition for minimality [27, p. 96]) and the sum rule for subdifferentials [27, Theorem 5.38, pp. 77–79] (we use the existence of a point in $\text{dom}(g)$ at which either $b$ or $g$ are continuous in the application of this rule).

**Lemma 8.4** Consider the setting of Lemma 8.3 and let $\tilde{g} : X \to (-\infty, \infty]$ be defined by $\tilde{g}(x) := g(x)$ whenever $x \in S$ and by $\tilde{g}(x) := \infty$ whenever $x \notin S$. Assume that $S \cap U \neq \emptyset$ and that there exists a point in $\text{dom}(g)$ at which either $b$ or $g$ are continuous. Then, an element $z \in S \cap U$ is a minimizer of $u_y$ in $S$ if and only if there exists $\gamma \in \partial \tilde{g}(z)$ such that

$$f'(y) + \gamma = \frac{L}{\mu} (b'(y) - b'(z)). \quad (25)$$

**Remark 8.1** A crucial assumption in Lemma 8.4 is the existence of a point in $\text{dom}(g)$ at which either $b$ or $g$ are continuous. Hence, it is of interest to present several rather practical sufficient conditions which ensure the existence of such a point. As we show below, each one of the following conditions achieves this goal:

(i) $g$ is continuous on $\text{dom}(g)$,
(ii) $b$ is continuous on $\text{dom}(g)$,
(iii) $b$ is continuous on $\text{dom}(b)$,
(iv) $\text{dom}(g) \cap U \neq \emptyset$,
(v) $S \subseteq U$.

Indeed, Conditions (i)–(ii) immediately imply the required assumption and Condition (iii) is a particular case of Condition (ii) because $\text{dom}(g) \subseteq S \subseteq \text{dom}(b)$. Assume now that Condition (iv) holds. Since we assume that $b$ is lower semicontinuous and convex, that $X$ is a Banach space, and that $U := \text{Int}(\text{dom}(b)) \neq \emptyset$, we can apply [48, Proposition 3.3, p. 39] to conclude that $b$ is continuous on $U$. In particular, $b$ is continuous at any point in $U$ which belongs to $\text{dom}(g)$, namely the assertion follows. Condition (v) is just a particular case of Condition (iv) because $\text{dom}(g) \subseteq S$ and $g$ is proper on $S$. \hfill \qed

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{A figure related to the text content.}
\end{figure}
The following lemma generalizes [2, Lemma 2.3] (finite-dimensional Euclidean spaces) and [47, Lemma 3.1] (real Hilbert spaces) to our setting of real reflexive Banach spaces and Bregman divergences.

**Lemma 8.5** Consider the setting of Lemma 8.4 and let $F(x) := f(x) + g(x)$ for each $x \in S$. Suppose that for some $y \in U$, the minimizer $z$ of $u_y$ (from (24)) satisfies $z \in S \cap U$ and

$$F(z) \leq Q_{L,\mu,S}(z, y).$$

Then for all $x \in S$,

$$F(x) - F(z) \geq \frac{L}{\mu} (B(x, z) - B(x, y)).$$  \hspace{1cm} (27)

**Proof** Since $u_y(z) = Q_{L,\mu,S}(z, y) \in \mathbb{R}$ (according to Lemma 8.4), since the ranges of $f$ and $g$ do not include $-\infty$ and since (26) holds, we have $F(z) \in \mathbb{R}$. Let $\tilde{f} : X \to [-\infty, \infty]$ be the function defined by $\tilde{f}(x) := f(x)$ if $x \in \text{dom}(b)$ and $\tilde{f}(x) := \infty$ otherwise. The convexity of $\text{dom}(b)$ and of $f$ on $\text{dom}(b)$ imply that $\tilde{f}$ is convex on $X$. Since $f$ is Gâteaux differentiable in $U$ and since $\tilde{f}(x) = f(x)$ for each $x \in U$, the assumption that $y \in U$ and [27, Theorem 5.37, p. 77] imply that $\{f'(y)\} = \{\tilde{f}'(y)\} = \partial \tilde{f}(y)$. Let $\gamma$ be a vector in $\partial \tilde{g}(z)$ which satisfies (25), the existence of which is ensured by Lemma 8.4. Since $S \subseteq \text{dom}(b)$, the subgradient inequality and the equalities $\tilde{f}(x) = f(x)$ and $\tilde{g}(x) = g(x)$ for all $x \in S$, imply that

$$f(x) \geq f(y) + \langle f'(y), x - y \rangle, \quad x \in S,$$

and

$$g(x) \geq g(z) + \langle \gamma, x - z \rangle, \quad x \in S.$$ \hspace{1cm} (29)

It follows from the equality $F(x) = f(x) + g(x)$ for every $x \in S$, from (26), (28), (29), from (11) with $z$ instead of $x$, from Lemma 8.4, and from (5), that for all $x \in S$, we have

$$F(x) - F(z) \geq (f(x) + g(x)) - Q_{L,\mu,S}(z, y)$$

$$\geq f(y) + \langle f'(y), x - y \rangle + g(z) + \langle \gamma, x - z \rangle - Q_{L,\mu,S}(z, y)$$

$$= (f'(y) + \gamma, x - z) - (L/\mu)B(z, y)$$

$$= (\langle L/\mu \rangle b(y) - b'(z)), x - z) - (L/\mu)B(z, y)$$

$$= (\langle L/\mu \rangle b(y) - b'(z)), x - z) - (L/\mu)B(z, y)$$

$$= (L/\mu) (b(x) - B(x, y)) - (L/\mu) (b(z) - B(x, z))$$

$$= (L/\mu) (b(x) - B(x, y)),$$

as claimed. \hspace{1cm} (27)

\(\Box\)
The next lemma is needed in order to ensure simple sufficient conditions for the minimizer of \( u_y \) from (24) to belong to \( S \cap U \). These conditions are useful for Assumption 5.4.

**Lemma 8.6** Consider the setting of Lemma 8.4 and suppose in addition that at least one of the following conditions holds:

(A) \( S \subseteq U \),
(B) For each \( x \in S \cap \partial U \), either \( \partial b(x) = \emptyset \) or \( \partial \tilde{g}(x) = \emptyset \), where \( \partial U \) is the boundary of \( U \) and the function \( \tilde{g} : X \to ]-\infty, \infty[ \) is defined as \( \tilde{g}(x) := g(x) \) if \( x \in S \) and \( \tilde{g}(x) := \infty \) otherwise.

Then, the minimizer \( z \) of \( u_y \) (from (24)) satisfies \( z \in S \cap U \). In particular, if the assumptions mentioned in Lemma 8.4 hold, \( X \) is finite-dimensional and \( b \) is essentially smooth on \( U \), then \( z \in S \cap U \).

**Proof** The case of Condition (A) is obvious, and so from now on assume that Condition (B) holds. In what follows, we suppose to the contrary that \( z \notin S \cap \partial U \).

Let \( \tilde{u}_y : X \to ]-\infty, \infty[ \) be defined by \( \tilde{u}_y(x) := u_y(x) \) when \( x \in S \) (where \( u_y \) is defined in (24)), and \( \tilde{u}_y(x) := \infty \) otherwise. We can write

\[
\tilde{u}_y(x) := f(y) + \langle f'(y), x - y \rangle + \frac{L}{\mu} b(x, y) + \tilde{g}(x), \quad \forall x \in X.
\]

Then, \( \tilde{u}_y \) is convex, proper and lower semicontinuous. We know from Lemma 8.3 that \( z \) is the unique global minimizer of \( u_y \) on \( S \). Since \( \tilde{u}_y(x) = \infty \) when \( x \notin S \), it follows that \( z \) is also the global minimizer of \( \tilde{u}_y \) on \( X \). Thus, the simple necessary and sufficient condition for minimizers [27, p. 96] (Fermat’s rule) implies that \( 0 \in \partial \tilde{u}_y(z) \). Since for all \( y \in U \) and all \( x \in X \), we have

\[
\tilde{u}_y(x) = f(y) + \langle f'(y), x - y \rangle + \tilde{g}(x) + (L/\mu)(b(x) - b(y) - \langle b'(y), x - y \rangle),
\]

we see that \( \tilde{u}_y \) is the sum of the following two convex and proper functions which are defined by \( \tilde{u}_{1,y}(x) := f(y) + \langle f'(y), x - y \rangle - (L/\mu)(b(y) + \langle b'(y), x - y \rangle) + \tilde{g}(x) \) and \( \tilde{u}_{2,y}(x) := (L/\mu)b(x) \) for each \( x \in X \). We have \( \text{dom}(\tilde{u}_{1,y}) = \text{dom}(g) \) and \( \text{dom}(\tilde{u}_{2,y}) = \text{dom}(b) \). Moreover, the set of points at which \( \tilde{u}_{1,y} \) is finite and continuous coincides with the set of points at which \( \tilde{g} \) is finite and continuous, namely with the set of points at which \( g \) is continuous. In addition, the set of points at which \( \tilde{u}_{2,y} \) is finite and continuous coincides with the set of points at which \( b \) is finite and continuous. Since we assume that the setting of Lemma 8.4 holds, there exists some point in \( \text{dom}(b) \cap \text{dom}(g) \) at which either \( b \) or \( g \) are continuous. Hence, we can apply the sum rule [27, Theorem 5.38, p. 77] and its proof [27, Theorem 5.38, pp. 78–79] to conclude that \( \partial \tilde{u}_y(x) = \partial \tilde{u}_{1,y}(x) + \partial \tilde{u}_{2,y}(x) \) for every \( x \in X \) and, furthermore, that the subsets \( \partial \tilde{u}_{1,y}(x) \) and \( \partial \tilde{u}_{2,y}(x) \) are nonempty whenever \( \partial \tilde{u}_y(x) \) is nonempty.

Since we already know that \( 0 \in \partial \tilde{u}_y(z) \), it follows that both \( \partial \tilde{u}_{1,y}(z) \neq \emptyset \) and \( \partial \tilde{u}_{2,y}(z) \neq \emptyset \). Therefore, by direct computation (or by the sum rule), \( \partial \tilde{g}(z) = -f'(y) + (L/\mu) b'(y) \neq \emptyset \) and \( \partial b(z) = (\mu/L) \partial \tilde{u}_{2,y}(z) \neq \emptyset \). These relations contradict Condition (B).
The preceding discussion proves that \( z \) cannot be in \( S \cap \partial U \). Since obviously \( z \), which is the minimizer of \( u_y \) on \( S \), belongs to \( S \), it follows that \( z \notin \partial U \). But \( S \subseteq \text{dom}(b) \subseteq \text{cl} (\text{dom}(b)) = \text{cl}(U) \) (the last equality is just the well-known fact that says that the closure of the nonempty interior of a convex subset is equal to the closure of the subset itself [27, Theorem 2.27(b), p. 29]), and hence \( z \in S \subseteq \text{cl}(U) \). This fact and the equality \( \text{cl}(U) = U \cup \partial U \) imply that \( z \in U \). Hence, \( z \in S \cap U \), as required.

It remains to show that \( z \in S \cap U \) in the particular case where \( X \) is finite-dimensional and \( b \) is essentially smooth. Under these assumptions, we can apply [26, Theorem 26.1, pp. 251–252] which implies that \( \partial b(x) = \emptyset \) for each boundary point \( x \) of \( U \). Thus, Condition (B) holds and the assertion follows from previous paragraphs.

\begin{remark}
In view of the results of [49, Section 5], mainly the ones related to essentially smooth functions defined on reflexive Banach spaces, it might be that the finite-dimensional sufficient condition mentioned in Lemma 8.6 can be extended in one way or another to a class of functions defined on some infinite-dimensional spaces.
\end{remark}

The following lemma is needed for proving that the proximal forward–backwards algorithms in Sect. 5 are well defined (see Remark 5.1). Versions of it are known in more restricted settings [29, Lemma 1.2.3, pp. 22–23], [47, Lemma 5.1], [20, Lemma 2.1]. Its proof follows immediately from the definition \( \phi(t) := f(y + t(x - y)), t \in [0, 1] \), the fact that \( f(x) = \phi(1) = \phi(0) + \int_{0}^{1} \phi'(t)dt \), the assumption that \( f' \) is Lipschitz on \([x, y] \), the triangle inequality for integrals, and the inequality \(|x^*(x)| \leq \|x^*\|\|x\|\) for all \( x^* \in X^* \) and \( x \in X \).

\begin{lemma}
Let \((X, \| \cdot \|)\) be a real normed space and let \( U \) be an open subset of \( X \). Suppose that \( x, y \in U \) are given and that the line segment \([x, y]\) is contained in \( U \). Let \( f : U \to \mathbb{R} \) be a continuously Fréchet differentiable function the derivative \( f' \) of which is Lipschitz continuous along the line segment \([x, y]\) with a Lipschitz constant \( L(f', [x, y]) \geq 0 \). Then, the following inequality is satisfied for all \( L \geq L(f', [x, y]) \):

\[ f(x) \leq f(y) + (f'(y), x - y) + \frac{1}{2} L\|x - y\|^2. \]  

(31)
\end{lemma}

The following proposition is needed for the proof of Corollary 6.1. Its proof can be found in [50].

\begin{proposition}
Suppose that \( f : U \to \mathbb{R} \) is a twice continuously (Fréchet) differentiable function defined on an open and convex subset \( U \) of some real normed space \((X, \| \cdot \|), X \neq \{0\}\). Suppose that \( C \) is a convex subset of \( X \) which has the property that \( C \cap U \neq \emptyset \). Assume that \( f'' \) is bounded and uniformly continuous on bounded subsets of \( C \cap U \). Fix an arbitrary \( y_0 \in C \cap U \), and let \( s_0 := \|f''(y_0)\| \) and \( s := \sup\{\|f''(x)\| : x \in C \cap U \} \). If \( s = \infty \), then for each strictly increasing sequence \((\lambda_k)_{k=1}^\infty \) of positive numbers which satisfies \( \lambda_1 > s_0 \) and \( \lim_{k \to \infty} \lambda_k = \infty \), there exists an increasing sequence \((S_k)_{k=1}^\infty \) of bounded and convex subsets of \( C \) (and also closed if \( C \) is closed) such that \( S_k \cap U \neq \emptyset \) for all \( k \in \mathbb{N} \), that \( \bigcup_{k=1}^\infty S_k = C \), and that for each \( k \in \mathbb{N} \), the function \( f' \) is Lipschitz continuous on \( S_k \cap U \) with \( \lambda_k \) as
\end{proposition}
a Lipschitz constant; moreover, if $C$ contains more than one point, then also $S_k \cap U$ contains more than one point for each $k \in \mathbb{N}$. Finally, if $s < \infty$, then $f'$ is Lipschitz continuous on $C \cap U$ with $s$ as a Lipschitz constant.

Now, it is possible to prove Theorem 6.1 and the corollaries which follow it. The proofs are based on the previous assertions and also on the notation and assumptions of Sect. 5.

**Proof of Theorem 6.1** Since $\cup_{k=1}^{\infty} S_k = C$ and since $x_{\min} \in \text{MIN}(F) \subseteq C$, there exists an index $k_0 \in \mathbb{N}$ such that $x_{\min} \in S_{k_0}$. Since $S_k \subseteq S_{k+1}$ for all $k \in \mathbb{N}$ (Assumption 5.2), one has $x_{\min} \in S_k$ for all $k \geq k_0$. For a technical reason [see the discussion after (36)], if $F(x_1) = \infty$, then we take $k_0$ to be at least 2. This is possible since by definition, $k_0$ is just an index (not necessarily the first) for which $x_{\min} \in S_{k_0}$, and hence, if $x_{\min} \in S_1$, then also $x_{\min} \in S_2$ by the inclusion $S_1 \subseteq S_2$.

From Remark 5.1, it follows that for all $i \in \mathbb{N}$, inequality (26) holds with $S := S_i + 1$, $L := L_{i+1}$, $y := x_i$, $\mu := \mu_{i+1}$, and $z := x_{i+1}$, and also that $F(x_{i+1})$ is finite for all $i \in \mathbb{N}$. According to our assumption, $b$ is strongly convex on $S_k$ for all $k \in \mathbb{N}$. Moreover, $x_{\min} \in S_k$ for all $k \geq k_0$ and $z_k \in S_k \cap U$ for each $k \in \mathbb{N}$ (as follows from Assumption 5.4 and an induction argument). Thus, we can use Lemma 8.5, where in (27) we substitute $x := x_{\min}$, $y := x_i$, $S := S_i + 1$, $\mu := \mu_{i+1}$, $L := L_{i+1}$ and $z := x_{i+1}$, where $k_0 \leq i \leq k$. This yields

$$F(x_{\min}) - F(x_{i+1}) \geq (L_{i+1}/\mu_{i+1})(B(x_{\min}, x_{i+1}) - B(x_{\min}, x_i)).$$

(32)

Since $F(x_{\min}) - F(x_{i+1}) \leq 0$, it follows from (32) that $B(x_{\min}, x_{i+1}) \leq B(x_{\min}, x_i) \leq 0$. From (17), (32), the inequality $F(x_{\min}) - F(x_{i+1}) \leq 0$, and the inequalities $\mu_{i+1} \geq \mu_{k+1}$ (Assumption 5.2) and $\tau_{i+1} \leq \tau_{k+1}$ (Remark 5.2) for all $i \in \{k_0, \ldots, k\}$, it follows that

$$\frac{F(x_{\min}) - F(x_{i+1})}{\tau_{k+1}} \geq \frac{F(x_{\min}) - F(x_{i+1})}{\tau_{i+1}} \geq \frac{F(x_{\min}) - F(x_{i+1})}{L_{i+1}} \geq \frac{B(x_{\min}, x_{i+1}) - B(x_{\min}, x_i)}{\mu_{i+1}} \geq \frac{B(x_{\min}, x_{i+1}) - B(x_{\min}, x_i)}{\mu_{k+1}}.$$

(33)

By summing (33) from $i := k_0$ to $i := k$, we obtain

$$(k + 1 - k_0)F(x_{\min}) - \sum_{i=k_0}^{k} F(x_{i+1}) \geq (\tau_{k+1}/\mu_{k+1})(B(x_{\min}, x_{k+1}) - B(x_{\min}, x_{k_0})).$$

(34)

Using Lemma 8.5, where in (27) we substitute $x := x_i$, $y := x_i$, $S := S_i + 1$, $L := L_{i+1}$, $i \geq k_0$, $z := x_{i+1}$, and $\mu := \mu_{i+1}$, using (17) and the nonnegativity of $B$, and the fact that $B(x_i, x_i) = 0$, and using the inequality $\mu_{k_0} \geq \mu_{i+1}$ for all $i \in \{k_0, \ldots, k\}$ (as a result of Assumption 5.2), we obtain

$$F(x_i) - F(x_{i+1}) \geq (L_{i+1}/\mu_{i+1})B(x_i, x_{i+1}) \geq (L_{i+1}/\mu_{k_0})B(x_i, x_{i+1}).$$

(35)
After multiplying (35) by $i - k_0$, summing from $i := k_0$ to $i := k$, and performing simple manipulations, we arrive at

$$-(k + 1 - k_0)F(x_{k+1}) + \sum_{i=k_0}^{k} F(x_{i+1})$$

$$= \sum_{i=k_0}^{k} ((i - k_0)F(x_i) - (i + 1 - k_0)F(x_{i+1}) + F(x_{i+1}))$$

$$\geq \sum_{i=k_0}^{k} (L_{i+1}/\mu_{k_0})(i - k_0)B(x_i, x_{i+1}). \quad (36)$$

There is a minor issue related to (36) which should be noted: if $k_0 = 1$ and $F(x_1) = \infty$, then one of the terms in (36) is $0 \cdot \infty$, and therefore, it is not defined. In order to avoid this possibility, we re-defined $k_0$ in advance (see the beginning of the proof) to be 2 (the only case where $F(x_k) = \infty$ is when $k = 1$, since Remark 5.1 ensures that $F(x_k) \in \mathbb{R}$ for all $k \geq 2$).

After summing (34) and (36) and using the nonnegativity of some terms, it follows that

$$(k + 1 - k_0)(F(x_{\min}) - F(x_{k+1}))$$

$$\geq (\tau_{k+1}/\mu_{k+1})(B(x_{\min}, x_{k+1}) - B(x_{\min}, x_{k_0})) + \sum_{i=k_0}^{k} (i - k_0)(L_{i+1}/\mu_{k_0})B(x_i, x_{i+1})$$

$$\geq -(\tau_{k+1}/\mu_{k+1})B(x_{\min}, x_{k_0}). \quad (37)$$

This inequality implies (19), as claimed.

Now, if (20) holds, then obviously $\lim_{k \to \infty} F(x_k) = F(x_{\min})$. From now on we assume that (20) holds, that $B$ has the limiting difference property and that all of its first-type level-sets are bounded. Our goal is to prove that the weak limit $\lim_{k \to \infty} x_k$ exists and belongs to $\text{MIN}(F)$. Since $F(x_{\min}) - F(x_k) \leq 0$ for all $k \in \mathbb{N}$, it follows from (33) that

$$B(x_{\min}, x_{k+1}) \leq B(x_{\min}, x_k), \quad \forall k \geq k_0, k \in \mathbb{N}, \quad (38)$$

namely, the sequence of nonnegative numbers $(B(x_{\min}, x_k))_{k=k_0}^{\infty}$ is decreasing. In particular, one has $B(x_{\min}, x_k) \leq B(x_{\min}, x_{k_0})$ for every $k \geq k_0$. This inequality and the nonnegativity of $B$ imply that $(B(x_{\min}, x_k))_{k=k_0}^{\infty}$ is bounded. By our assumption on the boundedness of the first-type level-sets of $B$, the set $\{y \in U : B(x_{\min}, y) \leq B(x_{\min}, x_{k_0})\}$ is bounded. Since $(x_k)_{k=k_0}^{\infty}$ is contained in this level-set, $(x_k)_{k=k_0}^{\infty}$ and hence $(x_k)_{k=k_0}^{\infty}$ are bounded.

Let $x_\infty$ be an arbitrary weak cluster point of $(x_k)_{k \in \mathbb{N}}$. Such a cluster point exists because $X$ is reflexive and $(x_k)_{k \in \mathbb{N}}$ is bounded. Since $C$ is closed and convex, and $X$ is reflexive, $C$ is weakly closed and hence $x_\infty \in C$. Now, we observe that (20) implies that
next we go to a subsequence which converges weakly to \( x_\infty \), and then, we recall that \( F \) is convex and lower semicontinuous (with respect to the norm topology), and hence weakly lower semicontinuous \([41, \text{Corollary 3.9, p. 61}]\). These observations and (19) imply that \( F(x_\infty) \leq F(x_{\text{min}}) \). This proves that \( F(x_\infty) = F(x_{\text{min}}) \) because \( F(x_{\text{min}}) \) is the minimal value of \( F \), and hence, \( F(x_{\text{min}}) \leq F(x_\infty) \) holds trivially.

From (38), it follows that \( \lim_{k \to \infty} B(x_{\text{min}}, x_k) \) exists (and is finite). Since the previous paragraph shows that \( x_\infty \in \text{MIN}(F) \), it follows from Assumption 5.1 that \( x_\infty \in U \). Moreover, since \( x_\infty \in C \) and \( C = \bigcup_{i=1}^{\infty} S_i \), there exists an index \( i_0 \in \mathbb{N} \) such that \( x_\infty \in S_{i_0} \). Assumption 5.2 implies that \( x_\infty \in S_i \) for all \( i \geq i_0 \). By using Lemma 8.5, where in (27) we take any \( i_0 \leq i \in \mathbb{N} \) and substitute \( S := S_{i+1}, \mu := \mu_{i+1}, L := L_{i+1}, z := x_{i+1}, x := x_i \), we have

\[
F(x_\infty) - F(x_{i+1}) \geq \left( L_{i+1}/\mu_{i+1} \right) (B(x_\infty, x_{i+1}) - B(x_\infty, x_i)).
\]

Since \( F(x_\infty) = F(x_{\text{min}}) \leq F(x_{i+1}) \) as shown a few lines above, one has \( B(x_{\infty}, x_{i+1}) \leq B(x_{\infty}, x_i) \) for each \( i \geq i_0 \). Therefore, \( (B(x_\infty, x_i))_{i=0}^{\infty} \) is a decreasing sequence of nonnegative numbers and, as a result, \( \lim_{i \to \infty} B(x_\infty, x_i) \) exists. Since \( x_\infty \) was an arbitrary weak cluster point of \( (x_k)_{k=1}^{\infty} \), since we assume that \( B \) has the limiting difference property and since \( (x_k)_{k=1}^{\infty} \) is a bounded sequence in \( U \), Lemma 8.1 ensures that \( (x_k)_{k=1}^{\infty} \) converges weakly to a point \( z_\infty \in U \). From previous paragraphs, we have \( z_\infty \in \text{MIN}(F) \).

Finally, if \( b' \) is weak-to-weak* sequentially continuous and either \( U \) is bounded or for each \( x \in C \), there exists \( r_x \geq 0 \) such that \( \{ y \in U : \| y \| \geq r_x \} \neq \emptyset \) and \( b \) is uniformly convex relative to \( (\{ x \}, \{ y \in U : \| y \| \geq r_x \}) \) with a gauge \( \psi_x \) which satisfies \( \lim_{r \to \infty} \psi_x(r) = \infty \), then Remarks 4.2(iii) and 4.2(iv) imply that \( B \) has the limiting difference property and its first-type level-sets are bounded. Hence, the previous paragraph implies that \( (x_k)_{k=1}^{\infty} \) converges weakly to a point \( z_\infty \in \text{MIN}(F) \).

\[\square\]

**Proof of Corollary 6.1** Suppose first that \( \sup \{ \| f''(x) \| : x \in C \cap U \} = \infty \). Fix some \( y_0 \in C \cap U \) and let \( (\lambda_k)_{k=1}^{\infty} \) be any strictly increasing sequence of positive numbers which satisfies \( \lambda_1 > \| f''(y_0) \|, \lim_{k \to \infty} \lambda_k/k = 0 \) and \( \lim_{k \to \infty} \lambda_k = \infty \) (say, \( \lambda_k = \alpha k^q \) for all \( k \in \mathbb{N} \), where \( \alpha > \| f''(y_0) \| \) and \( q \in (0, 1) \) are fixed). Since \( \lim_{k \to \infty} \lambda_k = \infty \), \( \lambda_1 > \| f''(y_0) \| \) and \( C \) contains more than one point, Proposition 8.1 implies that there is an increasing sequence of nonempty closed and convex subsets \( S_k \subseteq C \) such that \( S_k \cap U \) contains more than one point and \( L(f', S_k \cap U) \leq \lambda_k \) for every \( k \in \mathbb{N} \), and such that \( \bigcup_{k=1}^{\infty} S_k = C \).

Let \( \mu_k := \mu \) for all \( k \in \mathbb{N} \) and consider two cases. In the first case, we are in the Lipschitz step size rule (Algorithm 5.1), and in the second case we are in the backtracking step size rule (Algorithm 5.2). If the first case holds, then let \( L_1 \) be any positive number satisfying \( L_1 \geq L(f', S_1) \) and let \( L_k := \lambda_k \) for each \( k \geq 2 \); in addition, let \( \tau_k := L_k \) for every \( k \in \mathbb{N} \). If the second case holds, then let \( L_1 \) be any positive number satisfying \( L_1 \leq \eta L(f', S_1) \) and let \( L_k := \eta \lambda_k L_k-1 \) whenever \( k \geq 2 \), where \( i_k \) is defined in Algorithm 5.2; in addition, let \( \tau_k := \eta \lambda_k \) for all \( k \in \mathbb{N} \).
These choices ensure that for each $k \in \mathbb{N}$, we have $\tau_{k+1} \geq \tau_k \geq L_k$. Indeed, in the Lipschitz step size rule this assertion follows immediately from the definition of $\tau_k$ and the assumption that $(\lambda_j)_{j=1}^{\infty}$ is increasing; in the backtracking step size rule the assertion follows from the definition of $\tau_k$ and the inequality $L_k \leq \eta L(f', S_k)$ (as shown in Remark 5.2; note that the proof there holds whenever $L_1 \leq \eta L(f', S_1 \cap U)$, no matter whether $S_j \neq C$ for some $j \in \mathbb{N}$ or not) and hence, from the choice of $\lambda_k$, we have $L_k \leq \eta \lambda_k = \tau_k$ for each $k \in \mathbb{N}$. In addition, in the first case we also have $L_k = \lambda_k \geq L(f', S_k \cap U)$, as follows from the previous paragraph. Hence, we can use Theorem 6.1 which implies that (19) holds (and hence also (21) when $\lambda_k = \alpha k^q$).

It remains to consider the case where $\sup\{\|f''(x)\| : x \in C \cap U\} < \infty$. In this case, $f'$ is Lipschitz continuous on $C \cap U$ (see Proposition 8.1). Denote $S_k := C$ for each $k \in \mathbb{N}$. In the case of Algorithm 5.1, take any positive number $L$ which satisfies $L \geq L(f', C \cap U)$ and denote $L_k := L := \tau_k$ for every $k \in \mathbb{N}$. In the case of Algorithm 5.2, select the $L_k$ parameters according to the rule mentioned there with $L_1 > \eta L(f', C \cap U)$. Remark 5.2 ensures that $L_k = L_1$ for each $k \in \mathbb{N}$. Let $\tau_k := L_1$ for all $k \in \mathbb{N}$. Now, we can use Theorem 6.1 which implies that an $O(1/k)$ rate of convergence in the function values holds [see (19)].

Finally, suppose that for each $x \in C$, there exists $r_x \geq 0$ such that $\{y \in U : \|y\| \geq r_x\} \neq \emptyset$ and $b$ is uniformly convex relative to $\{(x), \{y \in U : \|y\| \geq r_x\}\}$ with a gauge $\psi_x$ satisfying $\lim_{t \to \infty} \psi_x(t) = \infty$, and that $b'$ is weak-to-weak* sequentially continuous on $U$. Since (20) holds by our choice of $\tau_k$ and $L_k$, Theorem 6.1 ensures that $(x_k)_{k=1}^{\infty}$ converges weakly to a solution of (8). $\square$

Proof of Corollary 6.2 Denote, as stated, $S_k := C$ and $\mu_k := \mu$ for each $k \in \mathbb{N}$. The assumption on $b$ implies that $(S_k)_{k=1}^{\infty}$ is a telescopic sequence in $C$. The assumption on $f'$ implies that $f'$ is Lipschitz continuous on each $S_k$ with a Lipschitz constant $L(f', C \cap U)$. In the case of Algorithm 5.1, take any positive number $L$ which satisfies $L \geq L(f', C \cap U)$ and denote $L_k := L := \tau_k$ for each $k \in \mathbb{N}$. In the case of Algorithm 5.2, select the $L_k$ parameters according to the rule mentioned there with $L_1 > \eta L(f', C \cap U)$. Remark 5.2 ensures that $L_k = L_1$ for each $k \in \mathbb{N}$. Let $\tau_k := L_1$ for all $k \in \mathbb{N}$. In both cases, we can use Theorem 6.1 which implies that an $O(1/k)$ rate of convergence in the function values holds [see (19)], namely that $F(x_k) - F^* = O(1/k)$ for all $k \geq k_0$. The choice of $k_0$ (see the first lines of proof of Theorem 6.1) shows that we can take $k_0 = 1$, unless $F(x_1) = \infty$, where in this latter case we can take $k_0 = 2$.

As for the convergence of $(x_k)_{k=1}^{\infty}$ to a solution of (8), we separate the proof into two cases, according to either the assumption that $C$ is bounded or the assumption that $b$ is uniformly convex on dom$(b)$. In the first case, the proof is similar to the proof of Theorem 6.1, where the only difference is that we do not need to invoke any assumption on the boundedness of the first-type level-sets of $B$ since this assumption was needed there only to ensure the boundedness of $(x_k)_{k=1}^{\infty}$ [see the lines after (38)], and here we assume in advance that $C$ is bounded and, according to Assumption 5.4, we know that $(x_k)_{k=1}^{\infty}$ is contained in $C$.

In the second case, if dom$(b)$ is bounded, then we continue as in the first case above since $C \subseteq$ dom$(b)$. Assume now that dom$(b)$ is not bounded. Then, $U$ is also not bounded, because otherwise cl$(U)$ is bounded too and then the equality cl$(\text{dom}(b)) =$
\( \text{cl}(U) \) implies that \( \text{dom}(b) \) is bounded, a contradiction. Now, let \( x \in C \) be given and let \( r_x \) be an arbitrary nonnegative number. Then, the set \( \{ y \in U : \| y \| \geq r_x \} \) is not empty (otherwise \( U \) is bounded). Since \( b \) is uniformly convex on \( \text{dom}(b) \), it is obviously uniformly convex relative to the pair \((\{x\}, \{ y \in U : \| y \| \geq r_x \})\), where the relative gauge \( \psi \) is simply the modulus of uniform convexity of \( b \) on \( \text{dom}(b) \) (see Definition 4.1). This gauge satisfies \( \lim_{t \to \infty} \psi(t) = \infty \), as follows from [22, Lemma 3.3], because \( \text{dom}(b) \) is unbounded (actually, [22, Lemma 3.3] implies that \( \psi \) is even supercoercive). Since \( b' \) is weak-to-weak* sequentially continuous, we conclude from Theorem 6.1 that \( (x_k)_{k=1}^\infty \) converges weakly to a solution of (8).

\[ \square \]

9 Conclusions

In this paper, we presented, in a rather general setting, new Bregmanian variants of the proximal gradient method for solving the widely useful minimization problem of the sum of two convex functions over a constraint set. A major advantage of our method (TEPROG) is that it does not require the smooth term in the objective function to have a Lipschitz continuous gradient, an assumption which restricts the scope of applications of the proximal gradient method, but nonetheless is imposed in almost all of the many works devoted to this method. We were able to do so by decomposing the constraint set into a certain telescopic union of subsets, and performing the minimization needed in each of the iterative steps over one subset from this union, instead of over the entire constraint set. Moreover, under practical assumptions, we were able to prove a sublinear non-asymptotic convergence of TEPREG (or, sometimes, a rate of convergence which is arbitrarily close to sublinear) to the optimal value of the objective function, as well as the weak convergence of the iterative sequence to a minimizer. We have also obtained a few results which, we feel, are of independent interest, such as Lemmas 8.5 and 8.6.

We believe that TEPREG, as well as the “telescopic” idea (regarding the telescopic sequence), holds a promising potential to be applied in other theoretical and practical scenarios. It will be interesting to test TEPREG numerically and to compare it with other methods, and also to check whether suitable inexact versions of TEPREG, namely ones which allow errors to appear during the iterative process, exhibit similar convergence properties (in this connection, see [43,47] and the references therein).

Acknowledgements Part of the work of Daniel Reem was done when he was at the Institute of Mathematical and Computer Sciences (ICMC), University of São Paulo, São Carlos, Brazil (2014–2016), and was supported by FAPESP 2013/19504-9. It is a pleasure for him to thank Alfredo Iusem and Jose Yunier Bello Cruz for helpful discussions regarding some of the references. Simeon Reich was partially supported by the Israel Science Foundation (Grants 389/12 and 820/17), by the Fund for the Promotion of Research at the Technion and by the Technion General Research Fund. Alvaro De Pierro thanks CNPq Grant 306030/2014-4 and FAPESP 2013/19504-9. All the authors wish to express their thanks to three referees for their feedback which helped to improve the presentation of the paper.
Appendix: Some Proofs

Here, we provide the proofs of some claims mentioned earlier.

**Proof of some claims mentioned in Remark 5.2** We first show that in the backtracking step size rule, if $S_j \neq C$ for some $j$, then $L_k \leq \eta L(f', S_k \cap U)$ for each $k \in \mathbb{N}$. The case $k = 1$ holds by our assumption on $L_1$ since $S_j \neq C$ for some $j \in \mathbb{N}$. Let $k = 2$ and suppose, by induction, that the claim holds for all natural numbers between 1 to $k - 1$. If, to the contrary, we have $L_k > \eta L(f', S_k \cap U)$, then $\eta^{k-1} L_{k-1} > L(f', S_k \cap U)$ because $L_k/\eta = \eta^{k-1} L_{k-1}$. Hence, by using (16) with $L := \eta^{k-1} L_{k-1}$, we conclude that (14) holds with $L$ instead of $L_k$, a contradiction to the minimality of $i_k$ unless $i_k = 0$. But when $i_k = 0$, we have $L_k = L_{k-1}$, hence, using the induction hypothesis and the fact that $L(f', S_{k-1} \cap U) \leq L(f', S_k \cap U)$ (this latter fact follows immediately from the equality $L(f', S_k \cap U) := \sup\{\|f'(x) - f'(y)\|/\|x - y\| : x, y \in S_k \cap U, x \neq y\}$ and the assumption $S_{k-1} \subseteq S_k$), we have $L_k = L_{k-1} \leq \eta L(f', S_{k-1} \cap U) \leq \eta L(f', S_k \cap U)$, a contradiction to the assumption on $L_k$.

Now, we show that if we are in the backtracking step size rule, and we also have $S_k = C$ for all $k \in \mathbb{N}$ and $L_1 > \eta L(f', S_1 \cap U)$, then $L_{k+1} = L_k$ for each $k \in \mathbb{N}$. Indeed, since $S_1 = C$ and $L_{k+1} \geq L_k$ for each $k \in \mathbb{N}$ (as shown in Remark 5.2), we have $L_k > \eta L(f', C \cap U)$ for all $k \in \mathbb{N}$. If we do not have $L_{k+1} = L_k$ for each $k \in \mathbb{N}$, then $i_{k+1} > 0$ for some $k \in \mathbb{N}$ and for this $k$ we have $\eta^{k+1} L_k = L_{k+1}/\eta > L(f', C \cap U)$. Thus, (16) with $L := \eta^{k+1} L_k$ implies that (14) holds with $L$ instead of $L_k$, a contradiction to the minimality of $i_{k+1}$. 

**Proof of Lemma 8.1** Since $(x_k)_{k=1}^{\infty}$ is a bounded sequence in a reflexive Banach space, a well-known classical result implies that $(x_k)_{k=1}^{\infty}$ has at least one weak cluster point $q \in X$, which, by our assumption, is in $U$. Suppose to the contrary that there are at least two different weak cluster points $q_1 := w\lim_{k \to \infty, k \in N_1} x_k$ and $q_2 := w\lim_{k \to \infty, k \in N_2} x_k$ in $X$, where $N_1$ and $N_2$ are two infinite subsets of $\mathbb{N}$. By our assumption, $q_1, q_2 \in U$, and hence, since $b$ satisfies the limiting difference property, we have

$$B(q_2, q_1) = \lim_{k \to \infty, k \in N_1} (B(q_2, x_k) - B(q_1, x_k))$$

(40a)

and

$$B(q_1, q_2) = \lim_{k \to \infty, k \in N_2} (B(q_1, x_k) - B(q_2, x_k)).$$

(40b)

Since we assume that $L_1 := \lim_{k \to \infty} B(q_1, x_k)$ and $L_2 := \lim_{k \to \infty} B(q_2, x_k)$ exist and are finite, we conclude from (40) that $B(q_2, q_1) = L_2 - L_1 = -(L_1 - L_2) = -B(q_1, q_2)$. The assumptions on $b$ imply that $B$ is nonnegative (see, for example, [22, Proposition 4.13(III)]), and hence $0 \leq B(q_1, q_2) = -B(q_1, q_2) \leq 0$. Thus, $B(q_1, q_2) = B(q_2, q_1) = 0$. Since $b$ is strictly convex on $U$, for all $z_1 \in \text{dom}(b)$ and $z_2 \in U$, we have $B(z_1, z_2) = 0$ if and only if $z_1 = z_2$ (see, for example, [22, Proposition 4.13(III)]). Hence $q_1 = q_2$, a contradiction to the initial assumption. Thus, all the weak cluster points of $(x_k)_{k=1}^{\infty}$ coincide.

We claim that $(x_k)_{k=1}^{\infty}$ converges weakly to the unique cluster point $q$. Indeed, otherwise there are a weak neighborhood $V$ of $q$ and a subsequence $(x_{k_j})_{j=1}^{\infty}$ of
\((x_k)_{k=1}^{\infty}\) which is located outside \(V\). But this subsequence is a bounded sequence in a reflexive Banach space (since \((x_k)_{k=1}^{\infty}\) is bounded), and hence, it has a subsequence which converges weakly, as proved above, to \(q\). Thus, infinitely many elements of \((x_j)_{j=1}^{\infty}\) are in \(V\), a contradiction. \(\square\)

**References**

1. Bach, F., Jenatton, R., Mairal, J., Obozinski, G.: Optimization with sparsity-inducing penalties. Found. Trends Mach. Learn. 4, 1–106 (2012)
2. Beck, A., Teboulle, M.: A fast iterative shrinkage-thresholding algorithm for linear inverse problems. SIAM J. Imag. Sci. 2, 183–202 (2009)
3. Bertero, M., Boccacci, P., Desiderà, G., Vicidomini, G.: Image deblurring with Poisson data: from cells to galaxies. Inverse Prob. 25, 123006 (2009)
4. Combettes, P.L., Wajs, V.R.: Signal recovery by proximal forward-backward splitting. Multiscale Model. Simul. 4, 1168–1200 (2005)
5. De Mol, C., De Vito, E., Rosasco, L.: Elastic-net regularization in learning theory. J. Complex. 25, 201–230 (2009)
6. Figueiredo, M.A.T., Bioucas-Dias, J.M., Nowak, R.D.: Majorization-minimization algorithms for wavelet-based image restoration. IEEE Trans. Image Process. 16, 2980–2991 (2007)
7. Parikh, N., Boyd, S.: Proximal algorithms. Found. Trends Optim. 1, 127–239 (2014)
8. Tseng, P.: Approximation accuracy, gradient methods, and error bound for structured convex optimization. Math. Program. Ser. B. 125, 263–295 (2010)
9. Martinet, B.: Régularisation d’inéquations variationnelles par approximations successives. Rev. Française Inf. Rech. Oper. 4, 154–158 (1970)
10. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. 14, 877–898 (1976)
11. Bruck, R.E., Reich, S.: Nonexpansive projections and resolvents of accretive operators in Banach spaces. Houston J. Math. 32, 459–470 (1977)
12. Passy, G.B.: Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. J. Math. Anal. Appl. 72, 383–390 (1979)
13. Brézis, H., Lions, P.L.: Produits infinis de résolvantes. Israel J. Math. 29, 329–345 (1978)
14. Nevanlinna, O., Reich, S.: Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces. Israel J. Math. 32, 44–58 (1979)
15. Bauschke, H.H., Bolte, J., Teboulle, M.: A descent lemma beyond Lipschitz gradient continuity: first-order methods revisited and applications. Math. Oper. Res. 42, 330–348 (2017)
16. Markham, J., Conchello, J.A.: Fast maximum-likelihood image-restoration algorithms for three-dimensional fluorescence microscopy. J. Opt. Soc. Am. A 18, 1062–1071 (2001)
17. Dey, N., Blanc-Feraud, L., Zimmer, C., Roux, P., Kam, Z., Olivo-Marin, J.C., Zerubia, J.: Richardson–Lucy algorithm with total variation regularization for 3D confocal microscope deconvolution. Microsc. Res. Tech. 69, 260–266 (2006)
18. Cruz, J.Y.B., Nghia, T.T.A.: On the convergence of the forward–backward splitting method with line-searches. Methods Softw. Optim. 31, 1209–1238 (2016)
19. Bolte, J., Sabach, S., Teboulle, M., Vaisbourd, Y.: First order methods beyond convexity and Lipschitz gradient continuity with applications to quadratic inverse problems. SIAM J. Optim. 28, 2131–2151 (2018)
20. Cohen, G.: Auxiliary problem principle and decomposition of optimization problems. J. Optim. Theory Appl. 32, 277–305 (1980)
21. Nguyen, Q.V.: Forward-backward splitting with Bregman distances. Vietnam J. Math. 45, 519–539 (2017)
22. Reem, D., Reich, S., De Pierro, A.: Re-examination of Bregman functions and new properties of their divergences. Optimization 68, 279–348 (2019)
23. Tseng, P.: On accelerated proximal gradient methods for convex-concave optimization (2008). Preprint. https://www.mit.edu/~dimitrib/PTseng/papers/apgm.pdf. Accessed 15 Oct 2018
24. Nemirovski, A.: Prox-method with rate of convergence $O(1/t)$ for variational inequalities with Lipschitz continuous monotone operators and smooth convex-concave saddle point problems. SIAM J. Optim. 15, 229–251 (2004)

25. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. CMS Books in Mathematics, 2nd edn. Springer, Cham (2017)

26. Rockafellar, R.T.: Convex Analysis. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton (1970)

27. van Tiel, J.: Convex Analysis: An Introductory Text. Wiley, Belfast (1984)

28. Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific Publishing, River Edge (2002)

29. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. CMS Books in Mathematics, 2nd edn. Springer, Cham (2017)

30. Rockafellar, R.T.: Convex Analysis. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton (1970)

31. Censor, Y., Lent, A.: An iterative row-action method for interval convex programming. J. Optim. Theory Appl. 34, 321–353 (1981)

32. Censor, Y., Reich, S.: Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization. Optimization 37, 323–339 (1996)

33. De Pierro, A.R., Iusem, A.N.: A relaxed version of Bregman’s method for convex programming. J. Optim. Theory Appl. 51, 421–440 (1986)

34. Censor, Y., Zenios, A.: Proximal minimization algorithm with $D$-functions. J. Optim. Theory Appl. 73, 451–464 (1992)

35. Beck, A., Teboulle, M.: Mirror descent and nonlinear projected subgradient methods for convex optimization. Oper. Res. Lett. 31, 167–175 (2003)

36. Butnariu, D., Iusem, A.N., Zălinescu, C.: On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces. J. Convex. Anal. 10, 35–61 (2003)

37. Chen, G., Teboulle, M.: Convergence analysis of a proximal-like minimization algorithm using Bregman functions. SIAM J. Optim. 3, 538–543 (1993)

38. Osher, S., Burger, M., Goldfarb, D., Xu, J., Yin, W.: An iterative regularization method for total variation-based image restoration. Multiscale Model. Simul. 4, 460–489 (2005)

39. Yin, W., Osher, S., Goldfarb, D., Darbon, J.: Bregman iterative algorithms for $\ell_1$-minimization with applications to compressed sensing. SIAM J. Imaging Sci. 1, 143–168 (2008)

40. Zaslavski, A.J.: Convergence of a proximal point method in the presence of computational errors in Hilbert spaces. SIAM J. Optim. 20, 2413–2421 (2010)

41. Brezis, H.: Functional Analysis. Sobolev Spaces and Partial Differential Equations. Springer, New York (2011)

42. Ambrosetti, A., Prodi, G.: A Primer of Nonlinear Analysis. Cambridge University Press, New York, USA (1993)

43. Reem, D., Reich, S.: Solutions to inexact resolvent inclusion problems with applications to nonlinear analysis and optimization. Rend. Circ. Mat. Palermo 2(67), 337–371 (2018)

44. Reich, S.: Nonlinear semigroups, holomorphic mappings, and integral equations. In: Proceedings of Symposia Pure Mathematics Part 2. Nonlinear functional analysis and its applications, Berkeley, California, 1983, vol. 45, pp. 307–324. American Mathematical Society, Providence (1986)

45. Reem, D., Reich, S., De Pierro, A.: A telescopic Bregmanian proximal gradient method without the global Lipschitz continuity assumption (2019). arXiv:1804.10273 [math.OC] ([v4], 19 Mar 2019)

46. Reem, D.: The Bregman distance without the Bregman function II. In: Reich, S., Zaslavski, A.J. (eds.) Optimization Theory and Related Topics, Contemporary Mathematics, vol. 568, pp. 213–223. American Mathematical Society, Providence (2012)

47. Reem, D., Pierro, A.D.: A new convergence analysis and perturbation resilience of some accelerated proximal forward–backward algorithms with errors. Inverse Prob. 33, 044001 (2017)

48. Phelps, R.R.: Convex Functions, Monotone Operators and Differentiability, vol. 1364, 2nd edn. Springer, Berlin (1993). Closely related material can be found in “Lectures on maximal monotone operators”

49. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. Commun. Contemp. Math. 3, 615–647 (2001)
50. Reem, D., Reich, S., De Pierro, A.: Stability of the optimal values under small perturbations of the constraint set. arXiv:1902.02363 [math.OC][v1], 6 Feb 2019

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.