ASPHERICITY OF POSITIVE FREE PRODUCT LENGTH 4
RELATIVE GROUP PRESENTATIONS

Suzana Aldwaik and Martin Edjvet
School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK

Arye Juhász
Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

Abstract

Excluding some exceptional cases, we determine the asphericity of the relative presentation $P = \langle G, x \mid ax^mbx^n \rangle$, where $a, b \in G \setminus \{1\}$ and $1 \leq m \leq n$. If $H = \langle a, b \rangle \leq G$ the exceptional cases occur when $a = b^2$ or when $H$ is isomorphic to $C_6$.

Keywords: relative group presentation; picture; relative diagram; curvature; asphericity.

2010 Mathematics subject classification: 20F05; 57M05.

1 Introduction

A relative group presentation is an expression of the form $P = \langle G, x \mid r \rangle$, where $G$ is a group and $x$ is a set that is disjoint from $G$. Denoting the free group on $x$ by $\langle x \rangle$, $r$ is a set of cyclically reduced words in the free product $G \ast \langle x \rangle$. The group defined by $P$ is $G(P) = (G \ast \langle x \rangle)/N$, where $N$ is the normal closure of $r$ in $G \ast \langle x \rangle$.

A relative group presentation is said to be orientable if no element of $r$ is a cyclic permutation of its inverse; and is said to be aspherical if every spherical picture over it contains a dipole (see Section 2.1). If $G$ is the trivial group, $P$ is then an ordinary group presentation and asphericity means diagrammatic reducibility as defined in [14] under the assumption that $P$ has no proper power relators. These notions were introduced and developed in [6], where it is shown that if $P$ is orientable and aspherical then group theoretic information about $G(P)$ can be deduced. In particular, the natural homomorphism $G \to G(P)$ is injective [6]; and torsion in $G(P)$ can be described [6], [16].

There has been some progress in the problem of determining the asphericity of $P$ when both $x$ and $r$ consist of a single element (see, for example, [1-3, 6, 10-12, 15, 20]). Our particular interest is when $x = \{x\}$ and $r = \{ax^mbx^n\}$ where $a, b \in G \setminus \{1\}$, $1 \leq m \leq n$ and $\varepsilon = \pm 1$. Recent studies and applications of this case can be found in [7], [8] and [12]. When $\varepsilon = -1$ and $m = 1$, the asphericity of $P$ has been determined (modulo some exceptional cases) when $n = 2$ in [11], when $n = 3$ in [1] and when $n \geq 4$ in [10]. When
$\varepsilon = +1$ and $m = 1$, the asphericity of $P$ has been determined when $n = 2$ in [6], and (again modulo some exceptional cases) when $n = 3$ in [3], when $n = 4$ in [15] and when $n \geq 5$ in [2].

In the present paper we consider the case $\varepsilon = +1$. Before stating our main theorem observe that $ax^m b x^n = 1$ if and only if $b^{-1} x^{-m} a^{-1} x^{-n} = 1$, and it follows that there is no loss of generality in working modulo $a \leftrightarrow b$. Note that this allows us to assume that $|a| \leq |b|$. For example, suppose that $|a| = 3, |b| = 2, [a, b] = 1$ and $m \leq n$. Then $ax^m b x^n = 1$ if and only if $x^m a x^m b = 1$ if and only if $b^{-1} x^{-m} a^{-1} x^{-n} = 1$. Replacing $x$ by $x^{-1}$ yields a relator of the form $a' x^m b' x^n$ where $|a'| = |b^{-1}| = 2, |b'| = |a^{-1}| = 3$ and $[a', b'] = [b^{-1}, a^{-1}] = 1$.

We list the following exceptional cases (in which $|b|$ denotes the order of $b$ in $G$).

\begin{itemize}
  \item[(E1)] $a = b^2, |b| \in \{5, 6\}$ and $n \notin \{m, 2m\}$
  \item[(E2)] $a \in \{b^2, b^4\}, |b| = 6$ and $n \neq m$
  \item[(E3)] $a = b^2, 6 < |b| < \infty$ and $m < n < 2m$
\end{itemize}

**Theorem 1.1** Let $P$ be the relative group presentation $P = \langle G, x \mid ax^m b x^n \rangle$, where $1 \leq m \leq n, x \notin G$ and $a, b \in G \setminus \{1\}$. Suppose that none of the conditions in (E1), (E2) or (E3) holds. Then $P$ is aspherical if and only if (modulo $a \leftrightarrow b$) none of the following holds.

\begin{itemize}
  \item[(a)] $(m = n)$ \quad $1 < |ab^{-1}| < \infty$.
  \item[(b)] $(m \neq n)$
    \begin{itemize}
      \item[(i)] $a = b^{\pm 1}$ has finite order.
      \item[(ii)] $a = b^2$ and $|b| = 4$.
      \item[(iii)] $a = b^2, 4 < |b| < \infty$ and $n = 2m$.
      \item[(iv)] $|a| = 2, |b| = 3$ and $[a, b] = 1$.
      \item[(v)] $\frac{1}{|a|} + \frac{1}{|b|} + \frac{1}{|ab^{-1}|} > 1$, where $\frac{1}{\infty} := 0$.
    \end{itemize}
\end{itemize}

Moreover, if $P$ is aspherical then $x$ has infinite order in $G(P)$.

Clearly the element $x$ is not conjugate in $G(P)$ to any element of $G$ and $|x| \geq m + n > 1$ in $G(P)$. The final statement in Theorem 1.1 then follows immediately from, for example, statement (0.4) in the introduction of [6] and the fact that $P$ is orientable.

An example is given by the following. Let $F$ be the free group with basis $u_0, \ldots, u_{l-1}$ and let $\theta$ be the automorphism of $F$ such that $u_i \theta = u_{i+1}$ (mod $l$). For $w \in F$ recall that the cyclically presented group $G_l(w)$ is given by the group presentation

$$G_l(w) = \langle u_0, \ldots, u_{l-1} \mid \omega, \omega \theta, \ldots, \omega \theta^{l-1} \rangle.$$ 

For integers $A \geq 0, r \geq 1, s \geq 1$ and $f \geq 1$ consider

$$w = u_0 u_f u_{2f} \ldots u_{(r-1)f} u_{A+f} u_{A+2f} \ldots u_{A+(s-1)f}.$$
Then $G_1(\omega)$ belongs to the class of groups of type $\mathfrak{F}$, as defined in [8]. Now the automorphism $\theta$ of $F$ induces an automorphism of $G_1(\omega)$ and the resulting split extension $E_1(\omega)$ of $G_1(\omega)$ by the cyclic group of order $l$ has presentation $E_1(\omega) = \langle u, t \mid t^l, w(u, t) \rangle$ where $w(u, t)$ is obtained from $\omega$ by the rewrite $u_i \to t^{-i}ut^i$ [17]. In our case $\omega$ rewrites to $(ut^{-f})^r t^j (ut^{-f})^s t^{f+j}$, so letting $x = ut^{-f}$ we obtain $E_1(\omega) = \langle G, x \mid ax^b b x^s \rangle$ where $G = \langle t \mid t^l \rangle$, $a = r^f-j$ and $b = tsf+j$. More generally, it follows from [8, Lemma 6] and [4, Theorem 4.1] that Theorem 1.1 can be applied to obtain asphericity classifications for group of type $\mathfrak{F}$.

In Section 2 we give the method of proof and introduce the concepts of pictures and curvature distribution. In Section 3 some preliminary results are proved. The proof of Theorem 1.1 is completed in Section 4.

2 Method of proof

2.1 Pictures

The definitions in this subsection are taken from [6]. The reader is referred to [6] and [3] for more details.

A picture $\mathbb{P}$ is a finite collection of pairwise disjoint discs $\{D_1, \ldots, D_m\}$ in the interior of a disc $D^2$, together with a finite collection of pairwise disjoint simple arcs $\{\alpha_1, \ldots, \alpha_n\}$ embedded in the closure of $D^2 - \bigcup_{i=1}^{m} D_i$ in such a way that each arc meets $\partial D^2 \cup \bigcup_{i=1}^{m} D_i$ transversely at its end points. The boundary of $\mathbb{P}$ is the circle $\partial D^2$, denoted $\partial \mathbb{P}$. For $1 \leq i \leq m$, the corners of $D_i$ are the closures of the connected components of $\partial D_i - \bigcup_{j=1}^{n} \alpha_j$, where $\partial D_i$ is the boundary of $D_i$. The regions of $\mathbb{P}$ are the closures of the connected components of $D^2 - \left( \bigcup_{i=1}^{m} D_i \cup \bigcup_{j=1}^{n} \alpha_j \right)$. An inner region of $\mathbb{P}$ is a simply connected region of $\mathbb{P}$ that does not meet $\partial \mathbb{P}$. The picture $\mathbb{P}$ is non-trivial if $m \geq 1$, is connected if $\bigcup_{i=1}^{m} D_i \cup \bigcup_{j=1}^{n} \alpha_j$ is connected, and is spherical if it is non-trivial and if none of the arcs meets the boundary of $D^2$. Thus the set of regions of a connected spherical picture $\mathbb{P}$ consists of the simply connected inner regions together with a single annular region that meets $\partial \mathbb{P}$. The number of edges in $\partial \Delta$ is called the degree of the region $\Delta$ and is denoted by $d(\Delta)$. A region of degree $n$ will be called an $n$-gon. If $\mathbb{P}$ is a spherical picture, the number of different discs to which a disc $D_i$ is connected is called the degree of $D_i$, denoted by $d(D_i)$. The discs of a spherical picture $\mathbb{P}$ are also called vertices of $\mathbb{P}$.

Suppose that the picture $\mathbb{P}$ is labelled in the following sense: each arc $\alpha_j$ is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled with an element of $x \cup x^{-1}$. Each corner of $\mathbb{P}$ is oriented clockwise (with respect to $D^2$) and labelled with an element of $G$. If $\kappa$ is a corner of a disc $D_i$ of $\mathbb{P}$, then $W(\kappa)$ will be the word obtained by reading in a clockwise order the labels on the arcs and corners meeting $\partial D_i$ beginning with the label on the first arc we meet as we read the clockwise corner $\kappa$. 

3
If we cross an arc labelled $x$ in the direction of its normal orientation, we read $x$, else we read $x^{-1}$.

A picture over the relative presentation $\mathcal{P} = \langle G, x \mid r \rangle$ is a picture $\mathcal{P}$ labelled in such a way that the following are satisfied:

1. For each corner $\kappa$ of $\mathcal{P}$, $W(\kappa) \in r^*$, the set of all cyclic permutations of $r \cup r^{-1}$ which begin with a member of $x$.

2. If $g_1, \ldots, g_l$ is the sequence of corner labels encountered in anticlockwise traversal of the boundary of an inner region $\Delta$ of $\mathcal{P}$, then the product $g_1g_2\ldots g_l = 1$ in $G$. We say that $g_1g_2\ldots g_l$ is the label of $\Delta$, denoted by $l(\Delta) = g_1g_2\ldots g_l$.

A dipole in a labelled picture $\mathcal{P}$ over $\mathcal{P}$ consists of corners $\kappa$ and $\kappa'$ of $\mathcal{P}$ together with an arc joining the two corners such that $\kappa$ and $\kappa'$ belong to the same region and such that if $W(\kappa) = Sg$ where $g \in G$ and $S$ begins and ends with a member of $x \cup x^{-1}$, then $W(\kappa') = S^{-1}h^{-1}$. The picture $\mathcal{P}$ is reduced if it is non-empty and does not contain a dipole. A relative presentation $\mathcal{P}$ is called aspherical if every connected spherical picture over $\mathcal{P}$ contains a dipole. If $\mathcal{P}$ is not aspherical then there is a reduced spherical picture over $\mathcal{P}$.

A connected spherical picture $\mathcal{P}$ over $\mathcal{P}$ is defined to be strictly spherical if the product of the corner labels in the annular region taken in anticlockwise order defines the identity in $G$. The relative presentation $\mathcal{P}$ is weakly aspherical if each strictly spherical connected picture over $\mathcal{P}$ contains a dipole. Let $G(\mathcal{P})$ denote the group defined by $\mathcal{P}$. It is shown in [6] that if $\mathcal{P}$ is weakly aspherical and if the natural map of $G$ into $G(\mathcal{P})$ is injective then $\mathcal{P}$ is aspherical.

Now let $\mathcal{P} = \langle G, x \mid ax^mbx^n \rangle$. Then the natural map $G \to G(\mathcal{P})$ is injective [18] and so it suffices to show that $\mathcal{P}$ is weakly aspherical. Let $\mathcal{P}$ be a reduced connected strictly spherical picture over $\mathcal{P}$. Then the vertices (discs) of $\mathcal{P}$ are given by Figure 2.1(i), (ii). It is clear from the orientation of the edges that a positive vertex can only be connected to a negative vertex, in particular, the degree of each region of $\mathcal{P}$ is even.

![Figure 2.1: vertices of $\mathcal{P}$ and the star graph $\Gamma$](image-url)
2.2 The star graph $\Gamma$

The star graph $\Gamma$ of a relative presentation $P$ is a graph whose vertex set is $x \cup x^{-1}$ and edge set is $r^*$. For $R \in r^*$, write $R = Sg$ where $g \in G$ and $S$ begins and ends with a member of $x \cup x^{-1}$. The initial and terminal functions are given as follows: $\iota(R)$ is the first symbol of $S$, and $\tau(R)$ is the inverse of the last symbol of $S$. The labelling function on the edges is defined by $\lambda(R) = g^{-1}$ and is extended to paths in the usual way. A non-empty cyclically reduced cycle (closed path) in $\Gamma$ will be called admissible if it has trivial label in $G$.

In general we have that only each inner region of a reduced spherical picture $P$ over $P$ supports an admissible cycle in $\Gamma$. However since we are only considering strictly connected spherical $P$, the same holds for the annular region as well.

The star graph $\Gamma$ of $P = \langle G, x \mid ax^m bx^n \rangle$ is given by Figure 2.1(iii). In particular, a word obtained from a cyclically reduced closed path in $\Gamma$ does not contain $aa^{-1}$, $a^{-1}a$, $bb^{-1}$, $b^{-1}b$ up to cyclic permutation although it can contain the subwords $11^{-1}$ provided that different edges of $\Gamma$ labelled by 1 are used. (Note that the structure of $\Gamma$ also implies that the degree of a region must be even.)

Using $\Gamma$ we see that the possible labels or regions of degree 2 or 4 are (up to cyclic permutation and inversion) as follows:

- $d(\Delta) = 2$: $l(\Delta) \in \{ab^{-1}, 11^{-1}\}$
- $d(\Delta) = 4$: $l(\Delta) \in \{(ab^{-1})^2, ab^{-1}a1^{-1}, ab^{-1}1b^{-1}, ab^{-1}11^{-1}, a1^{-1}b1^{-1}, a1^{-1}a1^{-1}, a1^{-1}b1^{-1}, b1^{-1}b1^{-1}, 11^{-1}11^{-1}\}$

2.3 Curvature

Our aim is to show that, given certain conditions on $m$, $n$, $a$ and $b$, $P$ is aspherical. To this end assume by way of contradiction that $P$ is a reduced connected strictly spherical picture over $P$. Our method is curvature distribution (see, for example, [2]). Proceed as follows.

Contract the boundary $\partial P$ to a point which is then deleted. This way all regions $\Delta$ of the amended picture, also called $P$, are simply connected and form a tessellation of the 2-sphere. An angle function on $P$ is a real valued function on the set of corners of $P$. Given this, the curvature of a vertex of $P$ is $2\pi$ less the sum of all the angles at that vertex; and then the curvature $c(\Delta)$ of a region $\Delta$ of $P$ is $(2 - d(\Delta))\pi$ plus the sum of all angles of the corners of $\Delta$. The angle functions we will define result in each vertex having zero curvature and so the total curvature $c(P)$ of $P$ is given by $c(P) = \sum_{\Delta \in P} c(\Delta)$. Given this, it is a consequence of Euler’s formula, for example, that $c(P) = 4\pi$ and so $P$ must contain regions of positive curvature. Our strategy will be to show that the positive curvature that exists in $P$ can be sufficiently compensated by the negative curvature. To this end we locate each $\Delta$ satisfying $c(\Delta) > 0$ and distribute $c(\Delta)$ to near regions $\hat{\Delta}$ of $\Delta$. For such regions $\hat{\Delta}$ define $c^*(\hat{\Delta})$ to
equal \( c(\Delta) \) plus all the positive curvature \( \hat{\Delta} \) receives minus all the curvature \( \hat{\Delta} \) distributes during this distribution procedure. We prove that \( c^*(\hat{\Delta}) \leq 0 \) and, since the total curvature of \( \mathbb{P} \) is at most \( \sum c^*(\hat{\Delta}) \), this yields the desired contradiction.

The standard angle function assigns the angle 0 to each corner of \( v \) which forms part of a region of degree 2 and assigns \( 2\pi d(v) \) to the remaining corners. Therefore if \( \Delta \) is a region of degree \( k > 2 \) with vertices \( v_1, \ldots, v_k \) such that \( d(v_i) = d_i \) \((1 \leq i \leq k, \text{subscripts mod } 4)\)) then \( c(\Delta) = c(d_1, \ldots, d_k) = (2-k)\pi + \sum_{i=1}^k 2\pi d_i \); or if \( d(\Delta) = 2 \) then \( c(\Delta) = 0 \). Since \( c(3,3,3,3,3) = 0 \) this shows, for example, that \( \mathbb{P} \) must contain a region of degree 4 (since \( d(\Delta) \) is even); and since \( c(4,4,4,4) = 0 \), \( \mathbb{P} \) must contain a vertex of degree \( < 4 \).

3 Preliminary results

Recall that \( \mathcal{P} = \langle G, x \mid ax^m bx^n \rangle \) \((1 \leq m \leq n)\) and \( \mathbb{P} \) denotes a reduced connected strictly spherical picture over \( \mathcal{P} \) amended as described in Section 2.3. We can make the following assumptions without any loss of generality:

P1. \( \mathbb{P} \) is minimal with respect to number of vertices.

P2. Subject to P1, \( \mathbb{P} \) is maximal with respect to number of 2-gons.

**Lemma 3.1** If \( \Delta \) is a region of \( \mathbb{P} \) then \( l(\Delta) \neq (11^{-1})^k \) for \( k \geq 2 \).

**Proof.** Consider the 4-gon \( \Delta \) of Figure 3.1(i) having label \((11^{-1})^2\). Observe that there are \( k \) 2-gons between \( v_i \) and \( v_{i+1} \) \((1 \leq i \leq 4, \text{subscripts mod } 4)\). Apply \( r = \min(k_2+1, k_4+1) \) bridge moves \([9]\) of the type shown in Figure 3.1(ii),(iii). Then each of the first \( r-1 \) bridge moves will create and destroy two 2-gons leaving the total number unchanged. The \( r \)th bridge move however will create two 2-gons but destroy at most one. Since bridge moves do not alter the number of vertices, we obtain a contradiction to P2. The proof now proceeds by induction on \( k \). Indeed if \( l(\Delta) = (11^{-1})^k \) where \( k > 2 \) then a sequence of bridge moves can produce a new picture with at least the same number of 2-gons but with a region having label \((11^{-1})^{k-1}\). \( \square \)

**Lemma 3.2** (i) If \( K = \langle a, b \rangle \leq G \) is infinite cyclic then \( \mathcal{P} \) is aspherical.

(ii) If \( |ab^{-1}| = \infty \) then \( \mathcal{P} \) is aspherical.

**Proof** (i) The result follows immediately from Lemma 3.8 of [5]. (ii) If \( \langle a, b \rangle \) is infinite cyclic the result follows by (i); otherwise the proof of Theorem 3 in [3] which uses a weight test [6] on the star graph shows that \( \mathcal{P} \) is aspherical. \( \square \)

**Lemma 3.3** If \( m = n \) then \( \mathcal{P} \) is aspherical if and only if \( |ab^{-1}| \in \{1, \infty\} \).

**Proof.** If \( |ab^{-1}| = \infty \) the result follows from Lemma 3.2. Let \( |ab^{-1}| = 1 \) and so the relator is \( ax^m ax^n \). If \( m = 1 \) it is clear that every picture contains a dipole, so let \( m > 1 \). Then the degree of each vertex of any given picture \( \mathbb{P} \) is at least 4 and it follows from the last
paragraph in Section 2.3 that $\mathcal{P}$ is aspherical. Finally if $1 < |ab^{-1}| < \infty$ then since $(ax^m)^2 = ab^{-1}$ it follows that $|ax^m| < \infty$ and non-asphericity follows, for example, from (0.4) in [6]. □

**Lemma 3.4** Suppose that $m \neq n$. Then the following hold.

(i) If $a = b^{\mp 1}$ and $|a| < \infty$ then $\mathcal{P}$ is not aspherical.

(ii) If $a = b^2$ and $|b| = 4$ then $\mathcal{P}$ is not aspherical.

(iii) If $a = b^2$, $4 < |b| < \infty$ and $n = 2m$ then $\mathcal{P}$ is not aspherical.

(iv) If $a = b^2$, $n > 2m$ and $|b| \geq 7$ then $\mathcal{P}$ is aspherical.

**Proof.** (i) If $a = b^{-1}$ then $|x| < \infty$ and so $\mathcal{P}$ is not aspherical; or if $a = b$ then, for example, a spherical picture can easily be constructed (see Figure 3.2(i) for the case $|a| = 3$).

(ii) In this case there is the spherical picture of Figure 3.2(ii). (The result also follows from Lemma 2.1 of [7].)

(iii) Since $b^2x^mbx^{-2m} = 1$ if and only if $b^{-1}(x^m)b = (x^m)b^{-2}$, it follows that $|x^m| < \infty$ and $\mathcal{P}$ is not aspherical.

(iv) Let $\mathbb{K}$ be a connected spherical picture over $\mathcal{P}$. If $\mathbb{K}$ contains a vertex of degree 2 then, since $n > 2m$, $\mathbb{K}$ contains a dipole and the result follows, so assume otherwise. Now fill in the regions of $\mathbb{K}$ using, if necessary, $b$-vertices with label $b^{\pm k}$ to obtain a connected spherical picture $\mathbb{L}$ over the ordinary presentation $< b, x; b^k, b^2x^mbx^n >$. The vertices of $\mathbb{L}$ are given (up to inversion) in Figure 3.2(iii). Clearly each region of $\mathbb{L}$ not of degree 2 has degree $\geq 4$; and, since $k \geq 7$, each $b$-vertex has degree $\geq 4$. Curvature considerations now tells us that there must be a non $b$-vertex $v$ such that $d(v) < 4$ and so $v$ must form a dipole in $\mathbb{L}$. But $d(v) \geq 3$ in $\mathbb{K}$ so it follows that $d(v) = 3$ in both $\mathbb{K}$ and $\mathbb{L}$ which forces the dipole $v$ forms in $\mathbb{L}$ to be a dipole in $\mathbb{K}$, as required. □
It will be assumed from now on that none of the following exceptional cases holds.

- \((E1)\)  \(a = b^2, |b| \in \{5, 6\} \text{ and } n \notin \{m, 2m\}\)
- \((E1')\)  \(b = a^2, |a| \in \{5, 6\} \text{ and } n \notin \{m, 2m\}\)
- \((E2)\)  \(a \in \{b^3, b^4\}, |b| = 6 \text{ and } n \neq m\)
- \((E2')\)  \(b \in \{a^3, a^4\}, |a| = 6 \text{ and } n \neq m\)
- \((E3)\)  \(a = b^2, 6 < b < \infty \text{ and } m < n < 2m\)
- \((E3')\)  \(b = a^2, 6 < a < \infty \text{ and } m < n < 2m\)

It follows from Lemmas 3.2–3.4, the exceptional cases and [2] that from now on the following assumptions can be made.

- \((A1)\)  \(1 < m < n\)
- \((A2)\)  \(\langle a, b \rangle \text{ is not infinite cyclic}\)
- \((A3)\)  \(|ab^{-1}| < \infty\)
- \((A4)\)  \(a \neq b^\pm 1\)
- \((A5)\)  \(a \neq b^2 \text{ and } b \neq a^2\)

Given this, we have the following lemmas.

**Lemma 3.5**  
(i) If none of \((ab^{-1})^2, a^2 \text{ or } b^2\) are trivial in \(\langle a, b \rangle\) then \(\mathcal{P}\) is aspherical.  
(ii) If \(\frac{1}{|a|} + \frac{1}{|b|} + \frac{1}{|ab^{-1}|} > 1\) then \(\mathcal{P}\) is not aspherical.

**Proof.**  
(i) Any reduced closed path in the star graph \(\Gamma\) of length at most 4 involving \(a\) or \(b\) yields (see Section 2.2) one of the relators \(ab^\pm 1, a^2b^{-1}, ab^{-2}, (ab^{-1})^2, a^2 \text{ or } b^2\) and so \((A4)\) and \((A5)\) together with assuming that none of \((ab^{-1})^2, a^2, b^2\) are trivial forces the degree
Figure 3.3: a spherical picture

of each region (not of degree 2) to be at least 6 and so $P$ is aspherical (see Section 2.3).

(ii) If the condition holds then reduced spherical pictures $P$ over $P$ can be constructed. For example if $(|a|,|ab^{-1}|,|b|) = (2,3,4)$ then $P$ is given by Figure 3.3. The other spheres can be obtained in similar fashion, we omit the details. □

Lemma 3.6 If any of the following sets of conditions holds then $P$ is aspherical.

(i) $|a| = 2$, $|b| \geq 4$ and $|ab^{-1}| \geq 4$.

(ii) $|a| \geq 4$, $|b| \geq 4$ and $|ab^{-1}| = 2$.

(iii) $|a| = 2$, $|b| = \infty$ and $|ab^{-1}| = 2$.

Proof. Assign the following angle function to the vertices of $P$: if $d(v) > 3$ then assign $\frac{2\pi}{d(v)}$ to each corner of $v$ not part of a 2-gon and 0 otherwise; and if $d(v) = 3$ then for (i), (ii), (iii) respectively, assign angles as shown in Figure 3.4(i), (ii), (iii) respectively. In each case it follows that if $\Delta$ is a region of $P$ then $\Delta$ has at most $\frac{1}{2}d(\Delta)$ corners having angle $\pi$. Therefore $c(\Delta) \leq (2 - d(\Delta))\pi + \frac{1}{2}d(\Delta).\pi + \frac{1}{2}d(\Delta).\frac{\pi}{2} = \pi \left(\frac{8 - d(\Delta)}{2}\right)$ and so $c(\Delta) \leq 0$ for $d(\Delta) \geq 8$. 

9
In (i) and (ii) there are no regions of positive curvature, a contradiction which completes the proof in these cases. 

\( (i) \) If \( d(\Delta) = 4 \) then \( l(\Delta) = a_1^{-1}a_1^{-1} \) and \( c(\Delta) \leq -2\pi + 4.\frac{\pi}{2} = 0 \); and if \( d(\Delta) = 6 \) then checking the star graph \( \Gamma \) shows that (up to cyclic permutation and inversion) \( l(\Delta) \in \{ab^{-1}a_1^{-1}b_1^{-1}, ab^{-1}a_1^{-1}b_1^{-1}, a_1^{-1}a_1^{-1}11^{-1}1\} \) and so \( c(\Delta) \leq -4\pi + 2.\pi + 4.\frac{\pi}{2} = 0 \).

\( (ii) \) If \( d(\Delta) = 4 \) then \( l(\Delta) = (ab^{-1})^2 \) and \( c(\Delta) \leq -2\pi + 4.\frac{\pi}{2} = 0 \); and if \( d(\Delta) = 6 \) then \( l(\Delta) \in \{ab^{-1}a_1^{-1}11^{-1}1, b^{-1}ab^{-1}a_1^{-1}1, ab^{-1}1a_1^{-1}b_1^{-1}, ab^{-1}b_1^{-1}a_1^{-1}\} \) and so \( c(\Delta) \leq -4\pi + 2.\pi + 4.\frac{\pi}{2} = 0 \).

In (i) and (ii) there are no regions of positive curvature, a contradiction which completes the proof in these cases.

\( (iii) \) If \( d(\Delta) = 4 \) and \( c(\Delta) > 0 \) then \( l(\Delta) = (a_1^{-1})^2 \) is given by Figure 3.5(i), (ii). In Figure 3.5(i), \( c(\Delta) \leq (2 - 4)\pi + \pi + 3.\frac{\pi}{2} = \frac{\pi}{2} \) is distributed from \( \Delta \) to \( \Delta \) as shown; and in Figure 3.5(ii), \( \frac{1}{2}c(\Delta) \leq \frac{\pi}{2} \) is distributed from \( \Delta \) to each of \( \Delta_1 \) and \( \Delta_2 \) as shown. If \( d(\Delta) = 6 \) and \( c(\Delta) > 0 \) then \( \Delta \) is given by Figure 3.5(iii) and hence \( \frac{1}{2}c(\Delta) \leq \frac{1}{2}(2 - 6)\pi + 3.\pi + 3.\frac{\pi}{2} = \frac{3}{4} \) is distributed from \( \Delta \) to each of \( \Delta_1 \) and \( \Delta_2 \) as shown. Note that positive curvature is distributed across a \( (b^{-1}, 1) \)-edge as shown in Figure 3.5(i)-(iii) or the inverse \( (1^{-1}, b) \)-edge as in Figure 3.5(iv) where the maximum amount of \( \frac{\pi}{2} \) is indicated.

Let \( \Delta \) be a region that receives positive curvature. Then \( l(\Delta) \) involves \( b \) and, since \( |b| = \infty \) and \( a \notin \langle b \rangle \), must also involve \( a \) at least twice. Therefore \( l(\Delta) = a^\varepsilon_1w_1a^\varepsilon_2w_2\ldots a^\varepsilon_rw_r \), where \( \varepsilon_i = \pm 1 \) \((1 \leq i \leq r)\), \( r > 1 \) and each \( w_i \) \((1 \leq i \leq r)\) does not involve \( a \). It follows that \( c^*(\Delta) \leq 2\pi + r.\frac{\pi}{2} + \alpha \) where the \( r.\frac{\pi}{2} \) is the contribution.
Figure 3.6: distribution of curvature for $a^\varepsilon w_ia^{\varepsilon+1}$ segments
from the $r$ vertices with corner label $a^z$ and $\alpha$ is the total curvature contributed to $c^*(\hat{\Delta})$ by the $r$ segments $a^z w_i a^z+1$ (subscripts $mod$ $r$). First observe that it can be assumed without any loss that $1^-11$ is not a sublabel of $l(\Delta)$ since the contribution to $c^*(\hat{\Delta})$ made by the three edges and two intermediate vertices of Figure 3.6(i) is at most $-3\pi + \frac{3\pi}{2} = -\frac{3\pi}{2}$ whereas the contribution made by the single edge shown is $-\pi$. Given this it can be further assumed that $11^-1$ is not a sublabel since the possibilities $a^-111^-1b$ and $b^-111^-1a$ each contribute at most $-3\pi + \frac{3\pi}{2} + \frac{\pi}{2} = -\pi$ which equals the contribution made by the edge of $a^-1b$ or $b^-1a$ (see Figure 3.6(i)). Given these two assumptions it follows that, up to inversion, there are four types of segment $a^z w_i a^z+1$ and these are: $ab^-11b^-1\ldots b^-1b^-1a; a1^-1b1^-1b\ldots 1^-1b1^-1a; ab^-11b^-1\ldots b^-11a^-1; a^-11b^-11b^-1\ldots b^-1a$. These are shown in Figure 3.6(ii)-(v).

The contribution from each segment is made up from the edges $e_i$ ($1 \leq i \leq l$) and any positive curvature $\hat{\Delta}$ receives across the $e_i$. It follows that the segment of Figure 3.6(ii) contributes at most $-l.\pi + \frac{(\frac{1}{2} - 1)}{2}\pi + \frac{(\frac{1}{2} - 1)}{2}\pi = -\frac{3\pi}{2}$; the segment of Figure 3.6(iii) contributes at most $-l.\pi + \frac{(\frac{1}{2} - 1)}{2}\pi + \frac{(\frac{1}{2} - 1)}{2}\pi = -\pi$; the segment of Figure 3.6(iv) contributes at most $-l.\pi + \frac{(\frac{1}{2} - 1)}{2}\pi + \frac{(\frac{1}{2} - 1)}{2}\pi = -\pi$; and the segment of Figure 3.6(v) contributes at most $-l.\pi + \frac{(\frac{1}{2} - 1)}{2}\pi + \frac{(\frac{1}{2} - 1)}{2}\pi = -\pi$. Examples of when the maximum can be obtained are given by Figure 3.6(vi)-(ix). Note that $d(\hat{v}_i) = 3$ in Figure 3.6(vii) and $d(\hat{v}_{i-1}) = 3$ in Figure 3.6(viii) for otherwise the maximum would be $-\frac{3\pi}{2}$. For this reason we have the additional rule that $\frac{\pi}{2}$ is distributed from $\hat{\Delta}$ to $\hat{\Delta}_1$ in Figure 3.6(iii) when $d(\hat{v}_i) = 3$ and in Figure 3.6(iv) when $d(\hat{v}_{i-1}) = 3$. Note that the curvature is distributed as before across a $(b^-1, 1)$-edge or $(1^-1, b)$-edge and so the above calculations remain unchanged. Therefore the net contribution to $\hat{\Delta}$ of the segments of Figure 3.6(ii)-(v) are each at most $-\frac{3\pi}{2}$ and so $c^*(\hat{\Delta}) \leq 2\pi + r.\frac{\pi}{2} - r.\frac{3\pi}{2} \leq 0$ for $r \geq 2$, a contradiction that completes the proof. □

4 Proof of Theorem 1.1

The statements in Lemma 3.5 imply that we need only consider $\frac{1}{|a|} + \frac{1}{|b|} + \frac{1}{|ab^{-1}|} \leq 1$ where at least one of $|a|, |b|$ or $|ab^{-1}|$ equals $2$. Working modulo $a \leftrightarrow b$ it can be assumed that $|a| \leq |b|$ and so there are seven possibilities, namely: $|a| = 2, |b| = 2$ and $|ab^{-1}| = \infty$ which is not allowed by (A3); conditions (i)-(iii) of Lemma 3.6; and the following three cases that remain to be considered.

- $|a| = 2, |b| = 3$ and $|ab^{-1}| \geq 6$
- $|a| = 2, |b| \geq 6$ and $|ab^{-1}| = 3$
- $|a| = 3, |b| \geq 6$ and $|ab^{-1}| = 2$

As before suppose by way of contradiction that $P$ is a reduced connected strictly spherical amended picture over $\mathcal{P}$. For the remaining cases we have found it easier to work with the dual of $P$. This yields a so-called strictly spherical relative diagram, $D$ say, which is
connected and simply connected. The regions of $D$ are given (up to inversion) by the region $\Delta$ of Figure 4.1(i). The product of the corner and edge labels of $\Delta$ read clockwise gives $ax^m bx^n$ (up to cyclic permutation). If $v$ is a vertex of $D$ then the label $l(v)$ of $v$ is (up to cyclic permutation) the product of the corner labels at $v$ read in an anti-clockwise direction. Each vertex label corresponds to an admissible cycle in the star graph $\Gamma$ of Figure 2.1(iii).

Note that when listing possible labels of a vertex of a given degree we do so up to cyclic permutation and inversion. If $d(\Delta) = 3$ then $\Delta$ is given (up to inversion) by Figure 4.1(ii) where from now on, for ease of presentation, vertices of degree 2 and the labels and arrows on $x$-edges are omitted. If $d(\hat{\Delta}_i) = 3$ ($1 \leq i \leq 3$) in Figure 4.1(ii) then we fix notation for the neighbouring regions in Figure 4.1(iii). From now on we will use only the standard angle function: each corner at a vertex $v$ will be assigned the angle $\frac{2\pi}{d(v)}$. For regions of degree 3 the following curvature identities will be useful: $c(4, 4, k) = \frac{2\pi}{k}$, $c(4, 6, 6) = \frac{\pi}{6}$; $c(4, 6, 8) = \frac{\pi}{12}$; $c(4, 6, 10) = \frac{\pi}{30}$; $c(4, 6, 12) = c(4, 8, 8) = 0$.

If $ab = ba$ then $P$ is not aspherical by [7, Theorem A] and noting that, although more
general, non-asphericity in [7] implies the existence of a reduced spherical picture, that is, implies non-asphericity in the sense used here, so assume otherwise.

Checking Γ (and using the assumption $ab \neq ba$ together with the hypothesis on $|a|$, $|b|$, $|ab^{-1}|$) shows that if $d(v) \leq 8$ then $l(v) \in \{a1^{-1}a1^{-1}, a1^{-1}a^{-1}11^{-1}, b1^{-1}b1^{-1}b1^{-1}, a1^{-1}ab^{-1}1b^{-1}1b^{-1}, (a1^{-1})^4, a1^{-1}a1^{-1}11^{-1}11^{-1}, a1^{-1}11^{-1}a1^{-1}11^{-1}, b1^{-1}b1^{-1}b1^{-1}11^{-1}\}$. In particular, as shown in Figure 4.1(iv), the vertices of two adjacent corners each with label 1 cannot both have vertex label $a1^{-1}a1^{-1}$, that is, be of degree 4.

If $c(\Delta) > 0$ then $\Delta$ is given by Figure 4.1(ii) in which $d(v_1) \geq 4$, $d(v_2) \geq 6$ and $d(v_3) \geq 4$.

First let $d(v_1) = d(v_3) = 4$. Then $\Delta$ is given by Figure 4.2(i) in which $d(\hat{\Delta}_3) > 3$, since, more generally, it can be seen from Figure 4.1(ii) that if $d(\Delta) = 3$ then the only allowable subwords of $l(\Delta)$ of length 2 are $ab, b1, 1a, b^{-1}a^{-1}, 1^{-1}b^{-1}$ and $a^{-1}1^{-1}$. If $d(\Delta_4) > 3$ and
for exactly one $\hat{\Delta}$ and in each case add $c(\Delta)$ to each of $c(\hat{\Delta})$ and $c(\hat{\Delta}_2)$ as shown in Figure 4.2(i); if $d(\hat{\Delta}_1) = 3$ and $d(\hat{\Delta}_2) > 3$ then add $c(\Delta) \leq c(4,4,6) = \frac{\pi}{4}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$ as shown in Figure 4.2(ii); if $d(\hat{\Delta}_1) > 3$ and $d(\hat{\Delta}_2) = 3$ then either $d(v_2) \geq 8$ in which case $d(\Delta) \leq \frac{\pi}{4}$ to $c(\hat{\Delta}_3)$ as in Figure 4.2(iii), or $d(v_2) = 6$ in which case $c(\Delta) = \frac{\pi}{3}$ to $c(\hat{\Delta}_3)$ and the remaining $\frac{\pi}{12}$ to $c(\hat{\Delta}_1)$ as in Figure 4.2(iv); and if $d(\hat{\Delta}_1) = d(\hat{\Delta}_2) = 3$ then add $c(\Delta) \leq \frac{\pi}{2}$ to $c(\hat{\Delta}_3)$ as in Figure 4.2(v).

Suppose now that exactly one of $d(v_1)$ or $d(v_3)$ equals 4. If at least one $\hat{\Delta}_i$ $(1 \leq i \leq 3)$ has $d(\hat{\Delta}_i) > 3$ then distribute $\frac{3}{4}c(\Delta)$ to $c(\Delta)$ as shown in Figure 4.3. The details are as follows. If $d(\hat{\Delta}_i) > 3$ $(1 \leq i \leq 3)$ then add $\frac{3}{4}c(\Delta) \leq \frac{\pi}{18}$ to $c(\hat{\Delta}_i)$ as in Figure 4.3(i); if $d(\hat{\Delta}_i) = 3$ for exactly one of the $\hat{\Delta}_i$ then add $\frac{1}{4}c(\Delta) \leq \frac{\pi}{12}$ to $c(\hat{\Delta}_j)$ $(j \neq i)$ as in (ii)-(iv); if $d(v_1) = 4$ and $d(\hat{\Delta}_i) > 3$ for exactly one $\hat{\Delta}_i$ then the possibilities are shown in (v)-(ix) and in each case add $c(\Delta)$ to $c(\hat{\Delta}_i)$; or if $d(v_3) = 4$ and $d(\hat{\Delta}_i) > 3$ for exactly one $\hat{\Delta}_i$ then the possibilities are shown in (x)-(xiv) and in each case add $c(\Delta)$ to $c(\hat{\Delta}_i)$. Observe that if $\hat{\Delta}$ receives positive curvature in Figure 4.3 then $d(\hat{\Delta}) > 3$.

Suppose that $d(\hat{\Delta}_i) = 3$ $(1 \leq i \leq 3)$ and $d(v_1) = 4$ only. Then $\hat{\Delta}_1$ and $\hat{\Delta}_4$ are given by Figure 4.4(i). If $\hat{\Delta}_2$ is given by Figure 4.4(i) also then $c(\Delta) \leq c(4,8,10) < 0$, so assume that $\hat{\Delta}_2$ is given by Figure 4.4(ii), in which case $c(\Delta) > 0$ implies $d(v_2) = 10$ and $d(v_3) = 6$. If the vertex $v$ of Figure 4.4(ii) has degree $\geq 6$ as indicated then add $c(\Delta) = c(4,6,10) = \frac{\pi}{30}$
to \(c(\hat{\Delta}_2) \leq c(6, 6, 10) = -\frac{2\pi}{15}\) as shown; if \(d(v) = 4\) and the corner label \(x\) equals 1 in Figure 4.4(ii) then \(d(\hat{\Delta}_3) > 3\) and so \(c(\Delta) + c(\hat{\Delta}_2) = 2.\frac{\pi}{30}\) is added to \(c(\hat{\Delta}_3)\) as shown in Figure 4.4(iii); if \(d(v) = 4\) and \(x = b\) in Figure 4.4(ii) then (the inverse of) \(l(v_2) = b^{-1}ab^{-1}aw\) and \(d(v_2) = 10\) implies \(l(v_2) \in \{b^{-1}ab^{-1}a1^{-1}ba^{-1}a1^{-1}ba^{-1}b1^{-1}a\}\) and either \(d(\hat{\Delta}_4) > 3\) in which case add \(c(\Delta) + c(\hat{\Delta}_1) \leq 2.\frac{\pi}{30}\) to \(c(\hat{\Delta}_4)\) as in Figure 4.4(iv) or \(d(\hat{\Delta}_4) = 3\) in which case add \(c(\Delta) = \frac{\pi}{30}\) to \(c(\hat{\Delta}_4) \leq c(4, 8, 10) = -\frac{2\pi}{15}\). We note that in order to obtain the two possible \(l(v_2)\) for \(d(v_2) = 10\) and \(l(v_2) = b^{-1}ab^{-1}aw\) our method, here and elsewhere, is to enumerate all closed paths in \(\Gamma\) modulo cyclic permutation and inversion that do not contain the subwords \(aa^{-1}, a^{-1}a, bb^{-1}\) or \(b^{-1}b\). We then, often with the use of GAP[13], delete all those words in our list that contradict either \(ab \neq ba\) or our hypothesis on \(|a|, |b|\) and \(|ab^{-1}|\). This is a routine but lengthy procedure and we omit the details for reasons of space.

Suppose now that \(d(\hat{\Delta}_i) = 3\) \((1 \leq i \leq 3)\) and \(d(v_3) = 4\) only. Then \(\hat{\Delta}_2\) and \(\hat{\Delta}_3\) are given by Figure 4.5(i). If \(\hat{\Delta}_1\) is also given by Figure 4.5(i) then \(c(\Delta) \leq 0\), so let \(\Delta_1\) be given by Figure 4.5(ii) in which case \(c(\Delta) > 0\) implies that \(d(v_1) = 10\) and \(d(v_2) = 6\). If the vertex \(v\) of Figure 4.5(ii) has degree \(\geq 6\) as indicated then add \(c(\Delta) = \frac{\pi}{30}\) to \(c(\hat{\Delta}_1) \leq c(6, 6, 10) = -\frac{2\pi}{15}\) as shown. Let \(d(v) = 4\). If \(d(\hat{\Delta}_9) > 3\) then add \(c(\Delta) + c(\hat{\Delta}_1) = 2.\frac{\pi}{30}\) to \(c(\hat{\Delta}_9)\) as shown.
in Figure 4.5(iii). Let \( d(\hat{\Delta}_9) = 3 \), in which case \( l(v_1) = b^{-1}ab^{-1}aw \) and as before \( l(v_1) \in \{ b^{-1}ab^{-1}a1^{-1}b1^{-1}a, b^{-1}ab^{-1}a1^{-1}b1^{-1}a \} \). If \( d(\hat{\Delta}_{10}) > 3 \) in Figure 4.5(iv) then add \( c(\Delta) + c(\hat{\Delta}_1) + c(\hat{\Delta}_9) \leq 3, \frac{\pi}{20} \) to \( c(\hat{\Delta}_{10}) \) as shown; if \( d(\hat{\Delta}_{10}) = 3 \) and \( d(u) \geq 10 \) for the vertex \( u \) of Figure 4.5(iv) then add \( c(\Delta) + c(\hat{\Delta}_1) = 2, \frac{\pi}{10} \) to \( c(\hat{\Delta}_4) \leq c(4, 10, 10) = -\frac{\pi}{10} \) as in Figure 4.5(v); or if \( d(\hat{\Delta}_{10}) = 3 \) and \( d(u) = 8 \) then add \( c(\Delta) + c(\hat{\Delta}_1) + c(\hat{\Delta}_9) = 2, \frac{\pi}{9} - \frac{\pi}{20} = \frac{\pi}{10} \) to \( c(\hat{\Delta}_{10}) \leq c(6, 8, 10) = -\frac{18\pi}{60} \) as shown in Figure 4.5(vi).

This completes the description of distribution of curvature from regions of degree 3. It follows that if \( \Delta \) receives \( \frac{\pi}{4} \) then, see Figure 4.2, it does so across exactly one edge; if \( \hat{\Delta} \) receives \( \frac{\pi}{10} \) then, see Figure 4.5(iv), \( \hat{\Delta} \) contains at least one vertex of degree 10; if \( d(\hat{\Delta}) = 3 \) then checking Figures 4.4 and 4.5 show that \( \hat{\Delta} \) receives curvature across at most one edge; if \( \hat{\Delta} \) receives curvature from \( \Delta_2 \) in Figure 4.4(iv) or (v) then this situation is described by Figure 4.5(v) and (vi) in which \( \Delta \) plays the rô1e of \( \hat{\Delta}_2 \); \( \Delta \) does not receive any curvature from \( \hat{\Delta}_3 \) in Figure 4.5(ii)-(v) for otherwise \( l(v_1) = b^{-1}ab^{-1}ab^{-1}w \) and \( d(v_1) > 10 \), a contradiction; and the maximum amount of curvature to cross an edge is either \( \frac{\pi}{4} \) or \( \frac{\pi}{10} \) or \( \frac{\pi}{12} \). It follows in turn from these observations that if \( d(\hat{\Delta}) = 3 \) then \( c^*(\hat{\Delta}) \leq 0 \).

There remains to be described distribution of curvature from a region of degree 4. Let \( \hat{\Delta} \) be the region of degree 4 shown in Figure 4.6(i) in which \( d(\hat{\Delta}_{11}) > 3 \) and \( d(\hat{\Delta}_{12}) > 3 \). If \( d(\Delta) > 3 \) then add \( c^*(\Delta) \leq c(4, 4, 4, 6) + \frac{\pi}{4} \) to \( c(\hat{\Delta}_{11}) \) as shown; or if \( d(\Delta) = 3 \) then add \( \frac{4}{7}c^i(\Delta) \leq c(4, 4, 4, 6) + \frac{\pi}{7} + \frac{\pi}{12} = \frac{pi}{12} \) to each of \( c(\hat{\Delta}_{11}) \) and \( c(\hat{\Delta}_{12}) \) as in Figure 4.6(ii).

Note that these final distribution rules do not alter the fact that \( \hat{\Delta} \) receives no curvature from \( \Delta, \hat{\Delta}_{11} \) or \( \hat{\Delta}_{12} \) in Figure 4.6(i) or from \( \hat{\Delta}_{11} \) or \( \hat{\Delta}_{12} \) in Figure 4.6(ii) nor does it alter any of the consequences listed above as a result of distribution of curvature from regions of degree 3.

The description of curvature distribution is now complete so let \( d(\hat{\Delta}) \geq 5 \). As noted earlier, the vertices of two adjacent corners each with label 1 cannot both be of degree 4 and so
\[ \hat{\Delta} \text{ must contain at least two vertices of degree } \geq 6 \text{ therefore } c^*(\hat{\Delta}) \leq c(4,4,4,6,6) + \frac{\pi}{4} + 4 \cdot \frac{\pi}{10} < 0. \] Finally let \( d(\hat{\Delta}) = 4 \). If \( \hat{\Delta} \) contains at most one vertex of degree 4 then \( c^*(\hat{\Delta}) \leq c(4,4,6,6) + 4 \cdot \frac{\pi}{10} < 0 \). The case when \( \hat{\Delta} \) contains three vertices of degree 4 is dealt with by Figure 4.6(i), (ii) and so this leaves \( \hat{\Delta} \) having exactly two vertices of degree 4.  

But then either \( c^*(\hat{\Delta}) \leq c(4,4,6,6) + 4 \cdot \frac{\pi}{10} < 0 \) or \( c^*(\hat{\Delta}) \leq c(4,4,6,10) + 4 \cdot \frac{\pi}{10} < 0 \) or \( c^*(\hat{\Delta}) \leq c(4,4,6,6) + \frac{\pi}{4} + \frac{\pi}{12} = 0 \) or \( c^*(\hat{\Delta}) \leq c(4,4,6,10) + \frac{\pi}{4} + \frac{\pi}{12} < 0 \) or \( \hat{\Delta} \) is given by Figure 4.6(iii), (iv) in which case either \( c^*(\hat{\Delta}) \leq c(4,4,6,8) + \frac{\pi}{4} + 2 \cdot \frac{\pi}{12} = 0 \) or (see Figure 4.5(iv)) \( c^*(\hat{\Delta}) \leq c(4,4,6,8) + \frac{\pi}{4} + 2 \cdot \frac{\pi}{12} < 0 \).

| \( a \mid 2 \), | \( b \mid \geq 6 \) and \( |ab^{-1}| = 3 \)

If \( ab = ba \) then we obtain the exceptional case (E2), so assume otherwise.

Checking \( \Gamma \) (and using the hypotheses on \( |a|, |b| \) and \( |ab^{-1}| \)) shows that if \( d(v) \leq 8 \) then \( l(v) \in \{a^{-1}a, a^{-1}a^{-1}a^{-1}a^{-1}11, (ab^{-1})^{3}a^{-1}a^{-1}11, a^{-1}a^{-1}a^{-1}a^{-1}a^{-1}11, (a^{-1})^{3}a^{-1}a^{-1}11a^{-1}11^{-1}, a^{-1}a^{-1}a^{-1}11a^{-1}11^{-1}, a^{-1}a^{-1}a^{-1}11a^{-1}11^{-1}, a^{-1}a^{-1}a^{-1}11a^{-1}11^{-1} \} \).

If \( c(\Delta) > 0 \) then \( \Delta \) is given by Figure 4.1(ii) in which \( d(v_{1}) \geq 4 \), \( d(v_{2}) \geq 6 \) and \( d(v_{3}) \geq 4 \).

First let \( d(v_{1}) = d(v_{3}) = 4 \). Then \( \Delta \) is given by Figure 4.7(i) in which \( d(\Delta_{j}) > 3 \). If \( d(\Delta_{1}) > 3 \) and \( d(\Delta_{2}) > 3 \) then distribute \( \frac{3}{4}c(\Delta) \leq \frac{1}{4}c(4,4,6,6) = \frac{\pi}{4} \) to \( c(\Delta_{3}) \) and \( \frac{1}{4}c(\Delta) \leq \frac{\pi}{24} \) to each of \( c(\Delta_{1}) \) and \( c(\Delta_{2}) \) as shown in Figure 4.7(i); if \( d(\Delta_{1}) = 3 \) and \( d(\Delta_{2}) > 3 \) then distribute \( \frac{3}{4}c(\Delta) \leq \frac{\pi}{4} \) to \( c(\Delta_{3}) \) and \( \frac{1}{4}c(\Delta) \leq \frac{\pi}{12} \) to \( c(\Delta_{2}) \) as shown in Figure 4.7(ii); if \( d(\Delta_{1}) > 3 \) and \( d(\Delta_{2}) = 3 \) then add \( c(\Delta) \leq c(4,4,8) = \frac{\pi}{4} \) to \( c(\Delta_{3}) \) as in Figure 4.7(iii); and if \( d(\Delta_{1}) = d(\Delta_{2}) = 3 \) then add \( c(\Delta) \leq \frac{\pi}{4} \) to \( c(\Delta_{3}) \) as in Figure 4.7(iv).

Suppose now that exactly one of \( d(v_{1}) \) or \( d(v_{3}) \) equals 4. If at least one \( \hat{\Delta}_{i} \) (\( 1 \leq i \leq 3 \)) has degree 3 then distribute \( c(\Delta) \) as shown in Figure 4.8. The details are as follows. If \( d(\Delta_{i}) > 3 \) (\( 1 \leq i \leq 3 \)) then add \( \frac{1}{3}c(\Delta) \leq \frac{1}{3}c(4,4,6,6) = \frac{\pi}{18} \) to \( c(\Delta_{j}) \) (\( 1 \leq i \leq 3 \)) as in Figure 4.8(i); if \( d(\Delta_{i}) = 3 \) for exactly one of the \( \Delta_{i} \) then add \( \frac{1}{2}c(\Delta) \leq \frac{\pi}{12} \) to \( c(\Delta_{j}) \) (\( j \neq i \)) as in
Figure 4.8: distribution from degree 3 regions: \(|a| = 2, |b| \geq 6, |ab^{-1}| = 3 \)

(ii)-(iv); if \(d(v_1) = 4\) and \(d(\Delta_i) > 3\) for exactly one of the \(\Delta_i\) then the possibilities are shown in (v)-(ix) and in each case add \(c(\Delta)\) to \(c(\Delta_i)\); or if \(d(v_3) = 4\) and \(d(\Delta_i) > 3\) for exactly one of the \(\Delta_i\) then the possibilities are shown in (x)-(xiv) and in each case add \(c(\Delta)\) to \(c(\Delta_i)\). Observe that if \(\Delta\) receives positive curvature in Figure 4.8 then \(d(\Delta) > 3\).

Suppose that \(d(\Delta_i) = 3\) \((1 \leq i \leq 3)\) and \(d(v_1) = 4\) only. Then \(\Delta_1\) and \(\Delta_3\) are given by Figure 4.9(i). If \(\Delta_2\) is also given by Figure 4.9(i) then \(c(\Delta) \leq c(4, 8, 8) = 0\), so assume that \(\Delta_2\) is given by Figure 4.9(ii), in which case \(c(\Delta) > 0\) forces \(d(v_2) = 6\) and \(d(v_3) = 10\). If the vertex \(v\) of Figure 4.9(ii) has degree \(\geq 6\) as indicated then add \(c(\Delta) = c(4, 6, 10) = \frac{\pi}{30}\) to \(c(\Delta_2) \geq c(6, 6, 10) = \frac{-2\pi}{15}\) as shown; so let \(d(v) = 4\). We then see from Figure 4.9(iii) that (the inverse of) \(l(v_3) = b_1^{-1}b_1^{-1}w\) and \(d(v_3) = 10\) implies \(l(v_3) \in \{b_1^{-1}b_1^{-1}ab^{-1}b^{-1}a^{-1}, b_1^{-1}b_1^{-1}ab^{-1}b^{-1}1a^{-1}\}\). If \(d(\Delta_6) > 3\) then add \(c(\Delta) + c(\Delta_2) = 2.\frac{\pi}{30}\) to \(c(\Delta_6)\) as shown in Figure 4.9(iii); if \(d(\Delta_6) = 3\) then either
c(\hat{\Delta}_6) \leq c(4, 10, 10) = -\frac{\pi}{10} and so add \( c(\Delta) + c(\hat{\Delta}_2) = 2\frac{\pi}{30} \) to \( c(\hat{\Delta}_6) \) as in Figure 4.9(iv), or \( c(\hat{\Delta}_6) = c(4, 8, 10) = -\frac{\pi}{20} \) in which case add \( c(\Delta) + c(\hat{\Delta}_2) + c(\hat{\Delta}_6) = \frac{\pi}{60} \) to the region \( \hat{\Delta} \) of Figure 4.9(v) in which \( d(\hat{\Delta}) > 3 \) or \( \hat{\Delta} \) of Figure 4.9(vi) in which \( d(\hat{\Delta}) = 3 \) and \( c(\hat{\Delta}) \leq c(6, 8, 10) = -\frac{13\pi}{60} \).

Suppose now that \( d(\hat{\Delta}_i) = 3 \) (1 \( \leq \) i \( \leq \) 3) and \( d(v_3) = 4 \) only. Then \( \hat{\Delta}_2 \) and \( \hat{\Delta}_3 \) are given by Figure 4.10(i). If \( \hat{\Delta}_1 \) is also given by Figure 4.10(i) then \( c(\Delta) \leq c(4, 8, 8) = 0 \), so let \( \hat{\Delta}_1 \) be given by Figure 4.10(ii), in which case \( c(\Delta) > 0 \) implies that \( d(v_1) = 6 \) and \( d(v_2) = 10 \). If the vertex \( v \) of Figure 4.10(ii) has degree \( \geq 6 \) as indicated then add \( c(\Delta) = \frac{\pi}{30} \) to \( c(\hat{\Delta}_1) \leq c(6, 6, 10) = -\frac{2\pi}{15} \) as shown. Let \( d(v) = 4 \). If \( d(\hat{\Delta}_4) > 3 \) then add \( c(\Delta) + c(\hat{\Delta}_4) = 2\frac{\pi}{30} \) to \( c(\hat{\Delta}_4) \) as shown in Figure 4.10(iii). Let \( d(\hat{\Delta}_4) = 3 \), in which case \( l(v_2) = b_1^{-1}b_1^{-1}w \) and so \( l(v_2) \in \{b_1^{-1}b_1^{-1}ab^{-1}b_1^{-1}d_1^{-1}, b_1^{-1}b_1^{-1}ab^{-1}b_1^{-1}a^{-1}\} \). If \( d(\hat{\Delta}_5) > 3 \) then add \( c(\Delta) + c(\hat{\Delta}_2) \leq 2\frac{\pi}{30} \) to \( c(\hat{\Delta}_5) \) as in Figure 4.10(iv); otherwise \( d(\hat{\Delta}_5) = 3 \) and so add \( c(\Delta) = \frac{\pi}{30} \) to \( c(\hat{\Delta}_2) \leq c(4, 8, 10) = -\frac{\pi}{20} \) as in Figure 4.10(v).

This completes the description of distribution of curvature from regions of degree 3. It follows that if \( \hat{\Delta} \) receives \( \frac{\pi}{4} \) then, see Figure 4.7, it does so across exactly one edge; if \( d(\hat{\Delta}) = 3 \) then checking Figures 4.9 and 4.10 shows that \( \hat{\Delta} \) receives curvature across at most one edge; if \( \Delta \) receives positive curvature from \( \hat{\Delta}_1 \) in Figure 4.10(iv) or (v) then this
situation is described in Figure 4.9(iv)-(vi) in which $\Delta$ plays the rôle of $\hat{\Delta}_1$; $\Delta$ does not receive any curvature from $\hat{\Delta}_3$ in Figure 4.9(ii)-(vi) for otherwise $l(v_3) = b_1^{-1}b_1^{-1}bw$ and $d(v_3) > 10$, a contradiction; and that the maximum amount of curvature to cross any given edge is $\pi/4$ or $\pi/12$. Given all of this it now follows that if $d(\Delta) = 3$ then $c^*(\Delta) \leq 0$.

There remains to be described distribution of curvature from a region of degree 4. Let $\hat{\Delta}$ be the region of degree 4 shown in Figure 4.11(i) in which $d(\hat{\Delta}_{11}) > 3$ and $d(\hat{\Delta}_{12}) > 3$. If $d(\Delta) > 3$ as shown then $c^*(\hat{\Delta}) \leq c(4, 4, 4, 8) + \frac{\pi}{4} = 0$; or if $d(\Delta) = 3$ then add $c^*(\hat{\Delta}) \leq c(4, 4, 4, 10) + \frac{\pi}{4} + \frac{\pi}{12} = \frac{\pi}{30}$ to $c(\Delta_{11})$ as in Figure 4.11(ii). Note that this distribution rule does not alter the fact that $\hat{\Delta}$ receives no curvature from $\Delta$, $\hat{\Delta}_{11}$ or $\hat{\Delta}_{12}$ in Figure 4.11(i) or from $\hat{\Delta}_{11}$ or $\hat{\Delta}_{12}$ in Figure 4.11(ii) nor does it alter any of the consequences given above as a result of curvature distribution from regions of degree 3.

The description of curvature distribution is now complete so let $d(\hat{\Delta}) \geq 5$. Once more it can be seen from Figure 4.1(iv) that the vertices of two adjacent corners each with label 1 cannot both be of degree 4 and so $\Delta$ must contain at least two vertices of degree $\geq 6$ therefore $c^*(\hat{\Delta}) \leq c(4, 4, 4, 6, 6) + \frac{\pi}{4} + 4\frac{\pi}{12} < 0$. Finally let $d(\hat{\Delta}) = 4$. If $\hat{\Delta}$ contains at most one vertex of degree 4 then $c^*(\hat{\Delta}) \leq c(4, 4, 6, 6) + 4\frac{\pi}{12} < 0$. The case when $\hat{\Delta}$ contains three vertices of degree 4 is dealt with by Figure 4.11(i), (ii) so this leaves $\hat{\Delta}$ having exactly two
vertices of degree 4. But then either \( c^*(\Delta) \leq c(4, 4, 6, 6) + 4 \cdot \frac{\pi}{12} = 0 \) or \( \hat{\Delta} \) is given by Figure 4.11(iii), in which case \( c^*(\hat{\Delta}) \leq c(4, 4, 6, 8) + \frac{\pi}{4} + 2 \cdot \frac{\pi}{12} = 0 \).

\(|a| = 3, |b| \geq 6 \) and \(|ab^{-1}| = 2\)

If \( ab = ba \) we obtain the exceptional case (E2), so assume otherwise.

Checking \( \Gamma \) (and using the assumption \( ab \neq ba \) together with the hypotheses on \(|a|, |b| \) and \(|ab^{-1}|\)) shows that if \( d(v) \leq 8 \) then \( l(v) \in \{ (ab^{-1})^2, ab^{-1}a^{-1}b^{-1}, (a^{-1})^3, (ab^{-1})^4, (ab^{-1})^2(11^{-1})^2, ab^{-1}a^{-1}b^{-1}11^{-1}, ab^{-1}a^{-1}11^{-1}b^{-1}, (ab^{-1}11^{-1})^2, b^{-1}ab^{-1}11^{-1}a^{-1}11, (b^{-1}a^{-1})^2, ab^{-1}a^{-1}a^{-1}b^{-1}, (a^{-1})^311^{-1}\}\). In particular, any given region has at most two vertices of degree 4, and Figure 4.11(iv) shows that two such vertices cannot be adjacent.

If \( c(\Delta) > 0 \) then \( \Delta \) is given by Figure 4.1(i) in which \( d(v_1) \geq 4, d(v_2) \geq 4 \) and \( d(v_3) \geq 6 \). Moreover \( d(v_1) = 4 \) implies \( d(v_2) \geq 6 \) and so \( c(\Delta) \leq c(4, 6, 6) = \frac{\pi}{6} \). If at least one \( \Delta_i \), \((1 \leq i \leq 3)\) has degree \( > 3 \) then distribute \( c(\Delta) \) as shown in Figure 4.12. The details are as follows. If \( d(\Delta_i) > 3 \) \((1 \leq i \leq 3)\) then add \( \frac{1}{6}c(\Delta) \leq \frac{1}{6}c(4, 6, 6) = \frac{\pi}{18} \) to \( c(\Delta_i) \) \((1 \leq i \leq 3)\) as in Figure 4.12(i); if \( d(\Delta_i) = 3 \) for exactly one of the \( \hat{\Delta} \), then add \( \frac{1}{6}c(\Delta) \leq \frac{\pi}{12} \) to \( c(\Delta_j) \) \((j \neq i)\) as in (ii)-(iv); if \( d(v_1) = 4 \) and \( d(\Delta_i) > 3 \) for exactly one \( \Delta_i \) then the possibilities are shown in (v)-(xi) and in each case add \( c(\Delta) \) to \( c(\Delta_i) \) or if \( d(v_2) = 4 \) and \( d(\Delta_i) > 3 \) for exactly one \( \Delta_i \) then the possibilities are shown in (xii)-(xx) and in each case add \( c(\Delta) \) to \( c(\Delta_i) \). Note that if \( \Delta \) receives positive curvature in Figure 4.12 then \( d(\Delta) > 3 \).

Suppose that \( d(\Delta_i) = 3 \) \((1 \leq i \leq 3)\) and \( d(v_1) = 4 \). Then \( \hat{\Delta}_1 \) and \( \hat{\Delta}_3 \) are given by Figure 4.13(i). If \( \Delta_2 \) is also given by Figure 4.13(i) then add \( c(\Delta) \leq c(4, 6, 8) = \frac{\pi}{12} \) to \( c(\Delta_2) \leq c(6, 6, 8) = \frac{\pi}{12} \) so let \( \hat{\Delta}_2 \) be given by Figure 4.13(ii). If the vertex \( v \) of Figure 4.13(ii) has degree \( \geq 6 \) as indicated then add \( c(\Delta) \leq \frac{\pi}{12} \) to \( c(\hat{\Delta}) \leq \frac{\pi}{12} \) as shown. Let \( d(v) = 4 \). Observe in Figure 4.13(iii) that length considerations force \( l(v_2) = 1^{-1}b^{-1}1bw \) and so \( c(\Delta) > 0 \) implies \( d(v_2) = 10 \) and \( d(v_3) = 6 \). If \( d(\Delta_5) > 3 \) then add \( c(\Delta) + c(\Delta_2) = 2 \cdot \frac{\pi}{30} \) to \( c(\Delta_5) \) as in Figure 4.13(iii); or if \( d(\Delta_5) = 3 \) then add \( c(\Delta) + c(\Delta_2) + c(\Delta_5) \leq 3 \cdot \frac{\pi}{30} \) to
Figure 4.12: distribution from degree 3 regions: $|a| = 3, |b| \geq 6, |ab^{-1}| = 2$
the region $\hat{\Delta}$ as in Figure 4.13(iv) when $d(\hat{\Delta}) > 3$ or as in Figure 4.13(v) when $d(\hat{\Delta}) = 3$ and $c(\hat{\Delta}) \leq c(6, 6, 10) = \frac{-2\pi}{15}$. 

Now suppose that $d(\hat{\Delta}_i) = 3$ ($1 \leq i \leq 3$) and $d(v_2) = 4$. Then $\hat{\Delta}_1$ and $\hat{\Delta}_2$ are given by Figure 4.14(i). If $\hat{\Delta}_3$ is also given by Figure 4.14(i) then add $c(\Delta) \leq c(4, 6, 8) = \frac{\pi}{12}$ to $c(\hat{\Delta}_3) \leq c(6, 6, 8) = \frac{\pi}{12}$ as shown. Let $\hat{\Delta}_3$ be given by Figure 4.14(ii). If the vertex $v$ of Figure 4.14(ii) has degree $\geq 6$ then again add $c(\Delta) \leq \frac{\pi}{12}$ to $c(\hat{\Delta}_3) \leq -\frac{\pi}{12}$. Let $d(v) = 4$ as in Figure 4.14(iii). Then (the inverse of) $l(v_3) = 1^{-1}b^{-1}bw$ together with $c(\Delta) > 0$ implies $d(v_1) = 6, d(v_2) = 10$ and $l(v_2) \in \{1^{-1}b^{-1}ba^{-1}1b^{-1}1a^{-1}b, 1^{-1}b^{-1}ba^{-1}1b^{-1}1b^{-1}a\}$. If $d(\hat{\Delta}_6) > 3$ then add $c(\Delta) + c(\hat{\Delta}_2) \leq 2.\frac{\pi}{30}$ to $c(\hat{\Delta}_6)$ as in Figure 4.14(iii); or if $d(\hat{\Delta}_6) = 3$ then again add $c(\Delta) + c(\hat{\Delta}_2) \leq 2.\frac{\pi}{30}$ to $c(\hat{\Delta}_6) \leq c(6, 6, 10) = -\frac{2\pi}{15}$ as in Figure 4.14(iv).

This completes the description of curvature distribution. Observe that if $\Delta$ receives curvature from $\hat{\Delta}_3$ in Figure 4.14(iii) or (iv) then this situation is described by Figure 4.13(iv) or (v) in which $\Delta$ plays the role of $\hat{\Delta}_6$. Moreover, $\Delta$ does not receive any curvature from $\hat{\Delta}_4$ in Figure 4.13(ii)-(v) for if so then $\hat{\Delta}_4$ of Figure 4.13(ii)-(v) would have to coincide with the inverse of $\Delta$ of Figure 4.14(iii), (iv); in particular, the vertex $v_2$ of Figure 4.13(ii)-(v) would coincide with the inverse of the vertex $v_3$ of Figure 4.14(iii), (iv), but the inverse of the corner label $a$ of $\hat{\Delta}_6$ in Figure 4.14(iii), (iv) differs from the corresponding corner label of $\hat{\Delta}_2$ in Figure 4.13(ii)-(v) which is $1^{-1}$, a contradiction. Given this it follows that the
maximum amount of curvature to cross any edge is \( \frac{\pi}{6} \) or \( \frac{\pi}{10} \) or \( \frac{\pi}{12} \). A further observation is the following: checking the degrees and labels of the vertices involved in Figure 4.12 in the six instances where \( \hat{\Delta} \) receives \( \frac{\pi}{6} \) shows that \( \hat{\Delta} \) cannot receive \( \frac{\pi}{6} \) across consecutive edges.

Checking Figures 4.13 and 4.14 shows that if \( d(\hat{\Delta}) = 3 \) then \( \hat{\Delta} \) cannot receive curvature across three edges and there are exactly two cases when \( \hat{\Delta} \) can receive positive curvature across two edges. The first case is when \( \hat{\Delta}_2 \) of Figure 4.13(i) coincides with \( \hat{\Delta}_3 \) of Figure 4.14(ii). But this forces \( \ell(v_3) = b^{-1}1b^{-1}a^{-1}w \) in Figure 4.13(i) which implies \( d(v_3) = 10 \) and so \( c^*(\hat{\Delta}_2) = c^*(\hat{\Delta}_3) \leq c(6, 6, 10) + 2 \frac{\pi}{30} < 0 \). The second case is when \( \hat{\Delta} \) of Figure 4.13(v) coincides with \( \hat{\Delta}_3 \) of Figure 4.14(ii), but then \( c^*(\hat{\Delta}_3) = c^*(\hat{\Delta}) \leq c(6, 6, 10) + 3 \frac{\pi}{30} + \frac{\pi}{12} < 0 \).

It can be deduced from the above that if \( d(\hat{\Delta}) = 3 \) then \( c^*(\hat{\Delta}) \leq 0 \). Let \( d(\hat{\Delta}) = 6 \). Then \( c^*(\hat{\Delta}) \leq c(4, 4, 6, 6, 6, 6) + 3 \frac{\pi}{6} + 3 \frac{\pi}{10} < 0 \). Let \( d(\hat{\Delta}) = 5 \). Then \( c^*(\hat{\Delta}) \leq c(4, 4, 6, 6, 6) + 2 \frac{\pi}{6} + 3 \frac{\pi}{10} < 0 \). Finally let \( d(\hat{\Delta}) = 4 \). Then either \( c^*(\hat{\Delta}) \leq c(6, 6, 6, 6) + 2 \frac{\pi}{6} + 2 \frac{\pi}{10} < 0 \) or \( c^*(\hat{\Delta}) \leq c(4, 6, 6, 6) + 2 \frac{\pi}{6} + 2 \frac{\pi}{12} = 0 \) or, see Figure 4.13(iv), (v), \( c^*(\hat{\Delta}) \leq c(4, 6, 6, 10) + 2 \frac{\pi}{6} + 2 \frac{\pi}{10} < 0 \) or \( \hat{\Delta} \) has exactly two vertices of degree 4 and is given by Figure 4.15 in which \( d(u) \geq 8 \). But again checking the six instances in Figure 4.12 when \( \hat{\pi} \) crosses an edge shows that in fact \( \hat{\Delta} \) cannot receive \( \frac{\pi}{6} \) across any of its edges and so either \( c^*(\hat{\Delta}) \leq c(4, 4, 6, 6) + 4 \frac{\pi}{12} = 0 \) or \( c^*(\hat{\Delta}) \leq c(4, 4, 6, 10) + 4 \frac{\pi}{12} < 0 \).

In each of the three cases above we have shown that if \( \hat{\Delta} \) receives positive curvature then \( c^*(\hat{\Delta}) \leq 0 \). This contradiction to \( c(\mathbb{P}) = 4\pi \) completes the proof of Theorem 1.1.

References

[1] A Ahmad, M Al-Mulla and M Edjvet, Asphericity of length four relative group presentations, *J Algebra and Its Applications* **16** (2017), 1750076, 27pp.
[2] S Aldwaik and M Edjvet, On the asphericity of a family of positive relative group presentations, *Proc Edinburgh Math Soc*, to appear.

[3] Y G Baik, W A Bogley and S J Pride, On the asphericity of length four relative group presentations, *Int J Alg Comput* 7 (1997), 227–312.

[4] W A Bogley, On shift dynamics for cyclically presented groups, *J Algebra* 418 (2014) 154-173.

[5] W A Bogley, M Edjvet and G Williams, Aspherical relative presentations all over again, preprint (2017).

[6] W A Bogley and S J Pride, Aspherical relative presentations, *Proc Edinburgh Math Soc* 35 (1992), 1–39.

[7] W A Bogley and G Williams, Efficient finite groups arising in the study of relative asphericity, *Math Z* 284 (2016), 507–535.

[8] W A Bogley and G Williams, Coherence, subgroup separability, and metacyclic structures for a class of cyclically presented groups, *Journal of Algebra* 480 (2017), 266–297.

[9] D J Collins and J Huebschmann, Spherical diagrams and identities among relations, *Math Annalen* 261 (1982), 155–183.

[10] P J Davidson, On the asphericity of a family of relative group presentations, *Int J Alg Comput* 19 (2009), 159–189.

[11] M Edjvet, On the asphericity of one-relator relative presentations, *Proc R Soc Edinb A* 124 (1994), 713–728.

[12] M Edjvet and A Juhász, The infinite Fibonacci groups and relative asphericity, preprint (2017).

[13] GAP – groups, algorithms and programming, version 4.7.4 (2014) http://www.gap-system.org.

[14] S M Gersten, Reducible diagrams and equations over groups, *Essays in Group Theory* (ed S M Gersten) Mathematical Sciences Research Institute Publications 8 (Springer, New York, 1987), 15–73.

[15] J Howie and V Metaftsis, On the asphericity of length five relative group presentations, *Proc Lond Math Soc* 82 (2001), 173–194.

[16] J Huebschmann, Cohomology theory of aspherical groups and of small cancellation groups, *J Pure Applied Algebra* 14 (1979), 137–143.

[17] D L Johnson, *Presentations of groups*, second edition, London Mathematical Society Student Texts 15 CUP (1997), 136–142.
[18] F Levin, Solutions of equations over groups, *Bull Amer Math Soc* 68 (1962), 603–604.

[19] R C Lyndon and P E Schupp, *Combinatorial Group Theory*, (Springer-Verlag) (1977).

[20] V Metaftsis, On the asphericity of relative group presentations of arbitrary length, *Int J Alg Comput* 13 (2003), 323–339.