Energy calculation of magnetohydrodynamic waves and their stability for viscous shearing flows.

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ABSTRACT

A self-consistent, thermodynamic approach is employed to derive the wave energy of a magnetohydrodynamic system within the harmonic approximation and to obtain the familiar dispersion relation from the resulting equation of motion. The evolution of the system due to an external perturbation is studied by a linear response formalism, that also gives the energy absorbed by the magnetohydrodynamic system from the external field. The calculated wave energy reveals the presence of positive and negative energy waves, that coalesce together to give rise to Kelvin - Helmholtz instability of the system. The threshold value of this instability changes only slightly in the presence of a small amount of viscosity, thus precluding the dissipative instability of the negative energy waves. The prediction of such a dissipative instability by earlier authors turns out to be the result of an erroneous choice of the viscous drag force, that violates the fundamental law of Galilean invariance.

1. Introduction

Propagation of waves in a medium is the manifestation of the system’s response to a small deviation from its local thermodynamic equilibrium. The dispersion relation \( D(k, n) = 0 \) describes the frequency \( (n) \) vs. wavenumber \( (k) \) relationship of such waves, and the form of the dispersion function \( D(k, n) \) is decided by the restoring forces and the degrees of freedom that the system possesses.

The degrees of freedom of a magnetohydrodynamic system consist of its density, pressure, velocity and its magnetic induction. The conventional method of determining the stability
of the system and also the nature of its wavemodes, is to work with the (MHD) equations connecting the above variables, thus obtaining the dispersion relation by requiring a non-trivial solution of the problem (eg. Alfven 1950; Cowling 1957; Chandrasekhar 1961). This method, though adequate to study the system’s stability, does not directly allow us to calculate the energy of the system.

In this paper, we study the response of a magnetohydrodynamic system to an external perturbation from the point of view of its energetics. The system consists of a two-dimensional slab-like inhomogeneity in the plasma and the flow parameters of an otherwise uniform magnetic medium, so that the equilibrium pressure, density, magnetic field and velocity are given by

\[
\begin{align*}
p_0(z), \rho_0(z), B_0(z), u_0(z) &= \begin{cases} 
p_{0o}, \rho_{0o}, B_{0o}, u_{0o}, & |z| \leq d, \\
p_{0e}, \rho_{0e}, B_{0e}, u_{0e}, & |z| > d, \end{cases}
\end{align*}
\]

(1)

with the magnetic field vectors and the steady flows being aligned to the axis of the slab, ie., to the X-direction. The equilibrium condition demands that the plasma inside and outside the slab are in the total (gas + magnetic) pressure balance, namely, \( p_{0o} + \frac{B_{0o}^2}{8\pi} = p_{0e} + \frac{B_{0e}^2}{8\pi} \). We here note, that Nakariakov and Roberts (1995; see also, Satya Narayanan 1991; Nakariakov, Roberts and Mann 1996) analysed the normal modes of the above equilibrium through the solutions of the slab’s dispersion relation. These authors also included plasma compressibility in their analysis. Considering the algebraic complexity of the calculations of the slab’s energetics, we here confine ourselves to an incompressible slab.

In what follows, we begin with the basic definition of the energy density of a magnetohydrodynamic system, deriving ultimately an expression for the space-averaged total energy (Hamiltonian) of the perturbed MHD slab in terms of its generalized co-ordinates, namely, its interfacial displacements \( \eta(x, z = \pm d, t) \), and their time derivatives \( \dot{\eta}(x, z = \pm d, t) \); see Section 2. In Section 3, we derive the equation of motion of the perturbed slab from the given Hamiltonian, which we solve directly to obtain the energy of excitation of the wavemodes.

The above calculation of the wave energy reveals the presence of negative energy waves (NEW), the existence of which was predicted earlier by various authors in both the ordinary hydrodynamic (eg. Benjamin 1963; Cairns 1979; Craik and Adam 1979; Ezerskii, Ostrovskii and Stepanyants 1981; Ostrovskii and Stepanyants 1982; Craik 1985; Ostrovski, Rybak and
Tsimring 1986) and the magnetohydrodynamic (e.g. Acheson 1976; Ryutova 1988; Ruderman and Goossens 1995; Ruderman et al. 1996; Joarder, Nakariakov and Roberts 1997) systems. The present paper improves upon these earlier calculations by extending the results of Cairns (1979) to the much complicated magnetohydrodynamic situations, thus providing a simple expression for the wave energy in terms of the linear dispersion function \( D(k, n) \) of the MHD slab. In his derivation of the wave energy, Cairns followed a procedure that is somewhat similar to the ones developed earlier by Stix (1962) and Witham (1974), and used the expansion \( p_{1,2} \approx D_{1,2} \left( \omega_0 + i \frac{\partial}{\partial t}, k_0 \right) A(t) \exp(ik_0x - \omega t) \) (see, Cairns 1979) for the pressures across the interface of the fluid. Such an expansion, though consistent with the dispersion relation, appears to us to be intuitively presented by Cairns. Our method, on the other hand, gives rigorous calculations of all the physical parameters and, in addition, allows one to calculate directly the energy of the system (Section 2), from which both the dispersion relation and the wave energy follows, via. the equation of motion (see Section 3).

Along with many other properties, the negative energy waves are also supposed to exhibit dissipative instability, so that the waves become overstable in the presence of any arbitrarily small amount of viscous dissipation in the medium that is in the rest frame of these negative energy waves; cf. Kikina (1967); Weissman (1970); Cairns (1979); Ezerskii et al. (1981); Ostrovskii and Stepanyants (1982); Craik (1985); Ostrovskii et al. (1986); Ruderman and Goossens (1995); Ruderman et al. (1996). In this paper, we show that this viscous overstability of the NEW is simply the result of an erroneous choice of the dissipative damping, that violates the law of Galilean invariance. In Section 4, we show that both the positive and the negative energy waves, in fact, exhibit dissipative damping under the action of a viscous drag force, that is consistent with an appropriate Galilean transformation. Concluding remarks are given in Section 5, indicating the relevance of this study to several astrophysical MHD systems.

2. Total energy of the perturbed slab

2.1. The Perturbations: Consider then, that due to the action of some external stress, the interfaces \( z = \pm d \) of the magnetic slab (see the previous section) are displaced by an amount \( \eta(x, z = \pm d, t) = \tilde{\eta}(k, z = \pm d, t) \exp(ikx) \), where \( \tilde{\eta}(k, z = \pm d, t) \) are the amplitudes of the Fourier components of the displacements \( \eta(x, \pm d, t) \) with respect to \( x \), and \( k \) is the
wavenumber. The associated z-displacements of the media internal (|z| ≤ d) and external (|z| > d) to the slab are given by

\[
\eta(x, z, t) = \begin{cases} 
\tilde{\eta}(k, z = -d, t) \exp(ikx) \exp\{k(z + d)\}, & z < -d, \\
[\alpha \cosh(kz) + \beta \sinh(kz)] \exp(ikx), & |z| \leq d, \\
\tilde{\eta}(k, z = d, t) \exp(ikx) \exp\{-k(z - d)\}, & z > d,
\end{cases}
\] (2)

thus showing that for an incompressible plasma slab, as is the case considered here, the perturbation amplitudes \( \tilde{\eta}(k, z, t) \) are evanescent in both the internal (|z| ≤ d, denoted by ‘o’) and the external (|z| > d, denoted by ‘e’) media away from the interfaces \( z = \pm d \) of the slab. Such perturbations pertain to the surface modes of the slab with the ‘cosh’ solution in equation (2) presenting a kink (even) surface mode, and the ‘sinh’ solution presenting a sausage (odd) surface mode; cf. Roberts (1981a, b), see also Edwin and Roberts (1982, 1983).

The coefficients \( \alpha \) and \( \beta \) in equation (2) can be evaluated from the condition of continuity of the z-displacements across the interfaces \( z = \pm d \) of the slab. Retaining the above terminology to reperesent the perturbations that are of even or of odd symmetry with respect to the axis (z = 0) of the magnetic slab, we thus obtain

\[
\alpha = \frac{\tilde{\eta}_{\text{kink}}(k, t)}{\cosh(kd)}, \quad \tilde{\eta}_{\text{kink}}(k, t) = \frac{1}{2} \{\tilde{\eta}(k, z = -d, t) + \tilde{\eta}(k, z = d, t)\}, \quad \beta = \frac{\tilde{\eta}_{\text{sausage}}(k, t)}{\sinh(kd)}, \quad \tilde{\eta}_{\text{sausage}}(k, t) = \frac{1}{2} \{\tilde{\eta}(k, z = -d, t) - \tilde{\eta}(k, z = d, t)\}. \] (3)

So far, we considred only the z-displacements of the magnetic plasma. The perturbations in other magnetohydrodynamic variables can be obtained by applying equations (2) and (3) to the linearized MHD equations (eg. Chandrasekhar 1961). In medium ‘o’ internal to the slab, these fluctiations are written as

\[
\delta u_{x0} = -(n + ku_{00}) \left\{ \tilde{\eta}_{\text{kink}}(k, t) \frac{\sinh(kz)}{\cosh(kd)} + \tilde{\eta}_{\text{sausage}}(k, t) \frac{\cosh(kz)}{\sinh(kd)} \right\} \exp(ikx), \quad (4 \, a)
\]

\[
\delta u_{z0} = \left( \frac{n + ku_{00}}{n} \right) \left\{ \tilde{\eta}_{\text{kink}}(k, t) \frac{\cosh(kz)}{\cosh(kd)} + \tilde{\eta}_{\text{sausage}}(k, t) \frac{\sinh(kz)}{\sinh(kd)} \right\} \exp(ikx), \quad (4 \, b)
\]

\[
\delta B_{x0} = -kB_{00} \left\{ \tilde{\eta}_{\text{kink}}(k, t) \frac{\sinh(kz)}{\cosh(kd)} + \tilde{\eta}_{\text{sausage}}(k, t) \frac{\cosh(kz)}{\sinh(kd)} \right\} \exp(ikx), \quad (4 \, c)
\]
\[ \delta B_z = \left( \frac{kB_0}{n} \right) \left\{ \hat{\eta}_\text{kink}(k, t) \frac{\cosh(kz)}{\cosh(kd)} + \hat{\eta}_\text{sausage}(k, t) \frac{\sinh(kz)}{\sinh(kd)} \right\} \exp(ikx). \] (4 d)

Here, \( \delta u_x \) and \( \delta u_z \) are the perturbations in the x- and the z- components of the velocity, whereas, \( \delta B_x \) and \( \delta B_z \) are the perturbations in the x- and the z- components of the magnetic field, respectively. In writing equation (4), we have used only the \( n \)-th temporal Fourier mode, so that \( \hat{\eta}(k, t) = in\tilde{\eta}(k, t) \). For any arbitrary \( \tilde{\eta}(k, t) \), equation (4) can be generalized by appropriately summing over all possible \( n \)s; such an exercise, however, does not alter the results that follow, as can be verified by using the condition of independence of the Fourier components.

The fluctuations of the MHD variables in the external (\( |z| > d \), denoted by ‘e’ ) media can similarly be derived. They are

\[ \delta u_{xe} = \mp(n + ku_0)\tilde{\eta}(k, z = \mp d, t) \exp(ikx) \exp\{\pm k(z \pm d)\}, \] (5 a)

\[ \delta u_{ze} = \left( \frac{n + ku_0}{n} \right) \tilde{\eta}(k, z = \mp d, t) \exp(ikx) \exp\{\pm k(z \pm d)\}, \] (5 b)

\[ \delta B_{xe} = \mp(kB_0)\tilde{\eta}(k, z = \mp d, t) \exp(ikx) \exp\{\pm k(z \pm d)\}, \] (5 c)

and

\[ \delta B_{ze} = \left( \frac{kB_0}{n} \right) \tilde{\eta}(k, z = \mp d, t) \exp(ikx) \exp\{\pm k(z \pm d)\}. \] (5 d)

2.2. The Energy Densities: To calculate the total energy (or, the Hamiltonian) of the magnetic slab, we first note that, the energy density of an incompressible magnetohydrodynamic system is given by (see the Appendix)

\[ \varepsilon = p + \frac{B^2}{8\pi} + \frac{\rho_0}{2}u^2, \] (6)

where, \( p = p_0 + \delta p \), \( B = |B| = |B_0 + \delta B| \), and \( u = |u| = |u_0 + \delta u| \) are the total pressure, the total magnetic field strength and the total velocity of the medium, respectively. For the specific situation considered here (see Section 1), we can now use equation (6) to write the magnetohydrodynamic equations of motion in the following particular form:
∇\varepsilon = -i\rho_0 (n + ku_0) \delta \mathbf{u} + \frac{ikB_0}{4\pi} \delta \mathbf{B} + \frac{1}{8\pi} \nabla |\delta \mathbf{B}|^2 + \rho_0 u_0 \nabla (\delta u_x), \quad (7)

that must be integrated, with the help of equations (4) and (5), to obtain the expressions for the energy densities separately for each of the media internal (‘\(o\)’) and external (‘\(e\)’) to the slab. Consider first the internal (‘\(o\), \(|z| \leq d\)’) medium. Equation (4) then allows us to evaluate the various terms on the left hand side of equation (7) up to the second order of smallness in \(\tilde{\eta}(k,t)\). Writing \(\eta(x,t) = \tilde{\eta}(k,t) \exp(ikx)\), we thus get,

\[
\nabla \varepsilon_o = \nabla \left[ \frac{k^2B_{0o}^2}{8\pi} \left\{ \eta_{\text{kink}}(x,t) \frac{\sinh(kz)}{\cosh(kd)} + \eta_{\text{sausage}}(x,t) \frac{\cosh(kz)}{\sinh(kd)} \right\}^2 + \frac{1}{n^2} \left[ \dot{\eta}_{\text{kink}}(x,t) \frac{\cosh(kz)}{\cosh(kd)} + \dot{\eta}_{\text{sausage}}(x,t) \frac{\sinh(kz)}{\sinh(kd)} \right]^2 \right] + \left\{ \rho_{0o} \frac{(n + ku_{0o})^2}{k} - \frac{kB_{0o}^2}{4\pi} \right\} \left\{ \eta_{\text{kink}}(x,t) \frac{\sinh(kz)}{\cosh(kd)} + \eta_{\text{sausage}}(x,t) \frac{\cosh(kz)}{\sinh(kd)} \right\} + \rho_{0o} u_{0o} (\delta u_{zo}), \quad (8)
\]

which, after integration, gives the following expression for the energy density \(\varepsilon_o\) of the medium internal to the slab:

\[
\varepsilon_o = \left[ \rho_{0o} \frac{(n + ku_{0o})^2}{k} - \frac{kB_{0o}^2}{4\pi} \right] \left\{ \eta_{\text{kink}}(x,t) \frac{\sinh(kz)}{\cosh(kd)} + \eta_{\text{sausage}}(x,t) \frac{\cosh(kz)}{\sinh(kd)} \right\} + \frac{k^2B_{0o}^2}{8\pi} \left\{ \eta_{\text{kink}}(x,t) \frac{\sinh(kz)}{\cosh(kd)} + \eta_{\text{sausage}}(x,t) \frac{\cosh(kz)}{\sinh(kd)} \right\}^2 + \frac{1}{n^2} \left[ \dot{\eta}_{\text{kink}}(x,t) \frac{\cosh(kz)}{\cosh(kd)} + \dot{\eta}_{\text{sausage}}(x,t) \frac{\sinh(kz)}{\sinh(kd)} \right]^2 \right\} + \rho_{0o} u_{0o} (\delta u_{zo}) + \varepsilon_{0o} + \delta \varepsilon_{0o}(x,z,t) + K_o. \quad (9)
\]

Here, the quantities \(\varepsilon_{0o}\) and \(K_o\) are constants of integration, that are independent of \(x\) and \(z\). The function \(\delta \varepsilon_{0o}(x,z,t)\) in equation (9) represents some initial fluctuations in the equilibrium energy density \(\varepsilon_{0o} = p_{0o} + B_{0o}^2/8\pi + (\rho_{0o}/2) u_{0o}^2\) of the medium ‘\(o\)’, that are the manifestations of some externally applied initial stress on the system. Such a stress does not appear explicitly in equation (7), but is incorporated phenomenologically in equation (9) to facilitate the study of the system’s response \(\eta(x,z,t)\) to such external perturbations \(\delta \varepsilon_0\) in Section 3.2 of this paper.
In what follows, we further consider the fluctuations $\delta \varepsilon_0(x, z, t)$ to have only a piece-wise dependence on the $z$-locations; i.e., for any particular value of $x$, the magnitudes of $\delta \varepsilon_0(x, z, t)$ may change abruptly across the interfaces $z = \pm d$ of the slab, but are constant in each of the three media, namely, the medium ‘o’ inside ($|z| \leq d$) the slab, and the medium ‘e’ on either ($z < d$ and $z > d$) side of the slab. For the medium ‘o’, we can write $\delta \varepsilon_{0o}(x, z, t) = \delta \varepsilon_{0o}(x, z = 0, t)$ in such a case.

Unlike $\varepsilon_{0o}$ and $\delta \varepsilon_{0o}(x, 0, t)$, the quantity $K_o$ in equation (9) contains terms, that are of second order smallness in the amplitudes of the interfacial displacements $\tilde{\eta}(x, \pm d, t)$. Such second order terms do not follow from the linearized equation (8), but are ought to be introduced, as we have to determine all the harmonic terms in the system. A general expression for $K_o$ can therefore be written as

$$K_o = a \tilde{\eta}_{\text{kink}}^2(k, t) + b \tilde{\eta}_{\text{sausage}}^2(k, t) + c \tilde{\eta}_{\text{kink}}(k, t)\tilde{\eta}_{\text{sausage}}(k, t),$$  \hspace{1cm} (10)

where, the coefficients $a$, $b$ and $c$ must be determined separately from some physical considerations. In order to determine these coefficients, it is convenient to first drop the term $\rho_{0o}u_{0o}(\delta u_{xo})$ from equation (9). Such terms do not contribute to the total Hamiltonian $H(\tilde{\eta}(k, \pm d, t), \tilde{\eta}(k, \pm d, t))$ of the system, as the integrals like $\rho_{0o} \int \int (\delta u_x) \, dz \, dx$ represent the velocity of the centre of mass of the respective layers, and hence are identically zero owing to wave propagation. Unfortunately, this property of the integrals has to be imposed and cannot be demonstrated here, as the linearization limit that we follow, introduces some unphysical $\tilde{\eta}^2(k, \pm d, t)$ -type terms in the Hamiltonian of the system.

In order to identify the constants $a$, $b$, $c$ in equation (10), we use now the condition of equipartition. After dropping the term involving $\delta u_{xo}$ in equation (9), we next consider an averaging of the energy density over a time-scale that is much longer than any periodicity present in the system. Such an averaging retains only the second order terms on the right hand side of equation (9), while all the terms linear in $\tilde{\eta}(k, \pm d, t)$ drop out. By denoting time averages by overbars, and by using the Parseval formula for Fourier transforms, we thus obtain
The energy density is given by

\[ 2\varepsilon_o(x, z, t) - \varepsilon_0 - 2\varepsilon_0(x, 0, t) = \left\{ \begin{array}{l}
\left[ \frac{k^2 B^2_{0o} \sinh^2(kz)}{8\pi} \cosh^2(kd) + a \right] \eta^2_{kink}(k, t) \\
+ \left[ \frac{k^2 B^2_{0o} \sinh^2(kz)}{8\pi} \cosh^2(kd) + b \right] \eta^2_{sausage}(k, t) + \frac{k^2 B^2_{0o}}{8\pi n^2} \left[ \frac{\cosh^2(kz)}{\cosh^2(kd)} - \eta^2_{kink}(k, t) \right] \\
+ \frac{\sinh^2(kz)\eta^2_{kink}(k, t)}{\sinh^2(kd) \eta^2_{sausage}(k, t)} + \frac{k^2 B^2_{0o} \sinh(2kz)}{4\pi} \frac{\sinh(2kd)}{2 \eta^2_{sausage}(k, t)} + c \right\} \eta_{kink}(k, t) \eta_{sausage}(k, t) \]

(11)

On the basis of the principle of equipartition of energy in harmonic oscillators, we may now argue, that the contributions from the terms containing \( \eta^2_{kink/sausage}(k, t) \) must be equal to the contributions from the terms containing \( \eta^2_{kink/sausage}(k, t) \). Applying similar arguments for the cross terms (containing \( \eta_{kink}(k, t) \eta_{sausage}(k, t) \) etc.), we ultimately get

\[ a = \frac{k^2 B^2_{0o}}{8\pi} \frac{1}{\cosh^2(kd)}, \quad b = \frac{k^2 B^2_{0o}}{8\pi} \frac{1}{\sinh^2(kd)}, \quad c = 0. \]

(12)

Using equations (9-12), we finally arrive at the expression for the energy density of the medium ‘o’ internal to the slab. This expression reads

\[ \varepsilon_o(x, z, t) = \varepsilon_0 + \delta \varepsilon_{0o}(x, 0, t) + \frac{k^2 B^2_{0o}}{8\pi} \left\{ \left[ \frac{\eta^2_{kink}(k, t)}{2 \cosh^2(kd)} \right] \cosh^2(kz) \right\} + \frac{k^2 B^2_{0o}}{8\pi n^2} \left[ \frac{\cosh^2(kz)}{2 \cosh^2(kd)} \right] \]

(13)

The energy density of the external media can be derived by following a similar procedure.

This energy density is given by

\[ \varepsilon_o(x, z, t) = \frac{k^2 B^2_{0o}}{8\pi} \left[ \eta^2(x, \mp d, t) + \frac{1}{n^2} \eta^2(x, \mp d, t) \right] \exp \{ \pm 2k (z \pm d) \}
\]

(14)
where, the ‘-’ and the ‘+’ signs denote the regions $z < -d$ and $z > d$, respectively. In writing equation (14), we have assumed that some initial fluctuations, in the form of $\delta \varepsilon_{0e}(x, \mp d, t)$, is imposed on the equilibrium state $\varepsilon_{0e}$ of the external plasma on both the sides of the magnetic slab, that are the manifestations of some external stress applied on the system; see the discussions following equation (9) above.

2.3. The Total Hamiltonian: The total energy, or the Hamiltonian $H(\tilde{\eta}, \dot{\tilde{\eta}})$ of the given MHD system is derived by integrating its energy density over the entire volume of the system consisting of the magnetic slab in its magnetic environment. Thus

$$H(\tilde{\eta}(k, z = \pm d, t), \dot{\tilde{\eta}}(k, z = \pm d, t)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon(x, z, t) \, dz \, dx = \int_{-\infty}^{\infty} \left\{ \int_{-d}^{d} \varepsilon_{0e}(x, z, t) \, dz + \int_{-\infty}^{\infty} \varepsilon_{0e}(x, z, t) \, dz \right\} dx, \tag{15}$$

where, the quantities $\varepsilon_{0e}(x, z, t)$ and $\varepsilon_{e}(x, z, t)$ are as given in equations (13) and (14), respectively.

After performing the integrations as indicated in equation (15), while retaining terms, that are of second order of smallness in the displacements $\eta(x, z, t)$, we finally arrive at the required expression for the total energy of the perturbed magnetic slab. This expression reads

$$H(\eta_{\text{kink}}, \eta_{\text{sausage}}, \dot{\eta}_{\text{kink}}, \dot{\eta}_{\text{sausage}}) = -\tilde{\sigma}_{\text{kink}}(-k, t) \tilde{\eta}_{\text{kink}}(k, t) - \tilde{\sigma}_{\text{sausage}}(-k, t) \tilde{\eta}_{\text{sausage}}(k, t)$$

$$+ \eta_{\text{kink}}^{2}(k, t) \left\{ \left[ \rho_{0e} \frac{(n + k u_{0e})^{2}}{k} - \frac{3 k B_{0e}^{2}}{16 \pi} \right] + \left[ \rho_{0o} \frac{(n + k u_{0o})^{2}}{k} - \frac{k B_{0o}^{2}}{16 \pi} \right] \tan(k d) \right\}$$

$$+ \left[ \rho_{0o} \frac{(n + k u_{0o})^{2}}{k} - \frac{k B_{0o}^{2}}{4 \pi} \right] \frac{2 \sinh(2kd) + 4(kd)}{8 k \cosh^{2}(kd)} \right\} + \dot{\eta}_{\text{sausage}}^{2}(k, t) \left\{ \left[ \rho_{0e} \frac{(n + k u_{0e})^{2}}{k} - \frac{3 k B_{0e}^{2}}{16 \pi} \right] \right.$$ 

$$+ \left[ \rho_{0o} \frac{(n + k u_{0o})^{2}}{k} - \frac{k B_{0o}^{2}}{4 \pi} \right] \coth(kd) + \frac{k B_{0o}^{2}}{8 \pi} \left[ \frac{2 \sinh(2kd) - 4(kd)}{8 k \sinh^{2}(kd)} \right] \right\}$$

$$+ \eta_{\text{kink}}^{2}(k, t) \left\{ \frac{k B_{0e}^{2}}{16 \pi n^{2}} + \frac{k B_{0o}^{2}}{8 \pi n^{2}} \left[ \frac{2 \sinh(2kd) + 4(kd)}{8 k \cosh^{2}(kd)} \right] \right\}$$

$$+ \dot{\eta}_{\text{sausage}}^{2}(k, t) \left\{ \frac{k B_{0o}^{2}}{16 \pi n^{2}} + \frac{k B_{0o}^{2}}{8 \pi n^{2}} \left[ \frac{2 \sinh(2kd) - 4(kd)}{8 k \sinh^{2}(kd)} \right] \right\}, \tag{16}$$

9
where, we have used the definitions

\[
\delta \varepsilon_{0e}(x, -d, t) - \delta \varepsilon_{0e}(x, d, t) = \sum_{k'} \bar{\sigma}_{\text{kink}} (k', t) \exp (ik'x),
\]

(17 a)

and

\[
\delta \varepsilon_{0e}(x, -d, t) + \delta \varepsilon_{0e}(x, d, t) - 2\delta \varepsilon_{0o}(x, 0, t) = \sum_{k'} \bar{\sigma}_{\text{sausage}} (k', t) \exp (ik'x),
\]

(17 b)

with \( \bar{\sigma}_{\text{kink}} (k', t) \) and \( \bar{\sigma}_{\text{sausage}} (k', t) \) being the \( k' \)-th Fourier amplitudes of the externally applied stresses, that excite the kink (even) and the sausage (odd) perturbations of the slab, respectively.

### 3. Excitation energy of the wavemodes.

#### 3.1. Equation of Motion and the Dispersion Relations:

The Hamiltonian, that we derive in the previous section (see equation (16)), immediately allows us to write down the equation of motion for the perturbed magnetic slab. This equation reads

\[
\begin{align*}
\left\{ \frac{kB_{0e}^2}{8\pi n^2} + \frac{k^2 B_{0o}^2}{8\pi n^2} \left[ \frac{2 \sinh (kd) \pm 4 (kd)}{4k \cosh^2 (kd)} \right] \right\} \ddot{\bar{\eta}}_{\text{(kink/ sausage)}} (k, t) \\
+ 2 \left\{ \rho_{0e} \left( \frac{n + ku_{0e}}{k} \right)^2 - \frac{3kB_{0e}^2}{16\pi} \right\} + \left[ \rho_{0o} \left( \frac{n + ku_{0o}}{k} \right)^2 - \frac{kB_{0o}^2}{4\pi} \right] f(k) \\
+ \frac{k^2 B_{0o}^2}{8\pi} \left[ \frac{2 \sinh (2kd) \pm 4kd}{8k \cosh^2 (kd)} \right] \right\} \bar{\eta}_{\text{(kink/ sausage)}} (k, t) \\
= \bar{\sigma}_{\text{(kink/ sausage)}} (-k, t),
\end{align*}
\]

(18)

where, the ‘+’ and the ‘-’ sign applies to the cases of the kink (with \( f(k) = \tanh (kd) \)) and the sausage (with \( f(k) = \coth (kd) \)) perturbations, respectively.

To check the correctness of equations (16) and (18), we consider, for the time being, that there is no external stress applied on the system, ie., \( \bar{\sigma}(-k, t) = 0 \). The motion of the plasma slab then consists of its various eigenmodes, as revealed by its dispersion relations \( D(k, n) = 0 \), connecting the frequency \( (n) \) and the wavenumber \( (k) \) of any temporal Fourier component \( \tilde{\eta}(k, n) \exp (int) \) of the displacements \( \tilde{\eta}(k, t) \) of the slab. For the kink and the sausage disturbances, these dispersion relations are obtained by substituting \( \partial/\partial t = in \) in equation (18). For arbitrary \( \tilde{\eta}(k, t) \), we thus obtain
\[ D_{\text{kink/sausage}}(k, n) = \rho_{0e} \left[ (n + k u_{0e})^2 - k^2 c_{Ae}^2 \right] + \rho_{0o} \left[ (n + k u_{0o})^2 - k^2 c_{Ao}^2 \right] f(k) = 0, \quad (19) \]

where, we have introduced the Alfvén speeds \( c_{Ao} = B_{0o}/(4\pi\rho_{0o})^{1/2} \) and \( c_{Ae} = B_{0e}/(4\pi\rho_{0e})^{1/2} \) in the two media ‘o’ and ‘e’, respectively.

We here note, that equation (19) is essentially the same as the dispersion relations obtained by Nakariakov and Roberts (1995), for the normal modes of an incompressible magnetic slab embedded in an incompresible MHD medium, where there is a relative tangential velocity between the slab and its environment. This equation admits solutions

\[ n_{\pm} = -k \left[ \{\alpha_{ef} u_{0e} + \alpha_{of} u_{0o}\} \pm \left\{ c_{kf}^2 - \alpha_{of} \alpha_{ef} (u_{0e} - u_{0o})^2 \right\}^{1/2} \right], \quad (20 \ a) \]

with

\[ \alpha_{ef} = \frac{\rho_{0e}}{\rho_{0e} + \rho_{0o} f(k)}, \quad \alpha_{of} = \frac{\rho_{0o} f(k)}{\rho_{0e} + \rho_{0o} f(k)}, \quad \text{and} \quad c_{kf}^2 = \frac{\rho_{0e} c_{Ae}^2 + \rho_{0o} f(k) c_{Ao}^2}{\rho_{0e} + \rho_{0o} f(k)}. \quad (20 \ b) \]

Each of the \textit{kink} \( f(k) = \tanh (kd) \) and the \textit{sausage} \( f(k) = \coth (kd) \) solutions then allows two distinct eigenmodes of the slab, that are represented by the ‘+’ and the ‘-’ signs in equation (20). These modes are purely oscillatory (giving surface waves) when the discriminant of equation (20) is real, but one of them (the ‘+’ mode) becomes a growing mode (giving Kelvin-Helmholtz (K-H) instability) when this discriminant is imaginary. The instability thus sets in for relative velocities

\[ |u_{0e} - u_{0o}| \geq c_{kf}/(\alpha_{of}\alpha_{ef})^{1/2}, \quad (21) \]

ie., for values at which the ‘+’ and the ‘-’ surface modes of either the kink or the sausage-type \textit{coalesce} together in the real \( n \) vs. \( k \) plane to produce an unsatable region of complex frequency; see Cairns (1979) for some examples of such \textit{coalescence instabilities} drawn from hydrodynamics. Such a coalescence instability occurs only when the modes involved have energies of opposite sign; see the discussions in the next section. We also note, that equation (21) reduces to the instability criterion given by Singh and Talwar (1994) and Nakariakov and Roberts (1995) in a situation, where the plasma slab moves in a static environment, ie., when \( u_{0e} = 0 \).
Equation (21) further shows, that the kink and the sausage modes merge together to give only two (‘+’ and ‘−’) surface modes in the case of an infinitely wide \((kd \to \infty, f(k) \to 1)\) slab, so that the quantity \(c_{kf}\) coincides with the phase speed \(c_k = (\rho_0 \omega c_{A0}^2 + \rho_0 e c_{Ae}^2)^{1/2} / (\rho_0 + \rho_0 e)^{1/2}\) of the hydromagnetic surface waves (Roberts 1981a,b) in a single surface of discontinuity separating two uniform magnetic plasma media. The instability criterion reduces to the classical threshold for the K-H instability of a magnetic tangential discontinuity (Chandrasekhar 1961) in this limit of an infinitely thick plasma slab.

3.2. Evolution of the System under External Stress: With this brief discussion on the nature of the normal modes of the slab, we now turn to the calculations for the work done by an external stress \(\tilde{\sigma}(-k, t)\) in exciting each of these modes, which, in turn, is stored as the energy of that particular wavemode of the system. To find this wave energy, we consider that the external stresses begin to act on the system at a time \(t = 0\), so that

\[
\tilde{\sigma}_{(\text{kink/sausage})}(-k, t) = \tilde{\sigma}_{(\text{kink/sausage})}(-k) \Theta(t) F(t), \tag{22}
\]

with \(F(t)\) being an arbitrary function of time \(t\), and \(\Theta(t)\) being a Heaviside unit step function. The causal response of the system to this external stress is given in terms of a response function \(G(k, t)\). Thus

\[
\tilde{\eta}_{(\text{kink/sausage})}(k, t) = \begin{cases} 
0 & \text{if } t < 0, \\
\int_0^t G_{(\text{kink/sausage})}(k, t-t') F(t') \, dt' & \text{if } t \geq 0,
\end{cases} \tag{23}
\]

the Laplace transform of which is given by

\[
\tilde{\eta}_{(\text{kink/sausage})}^L(k, s) = G_{(\text{kink/sausage})}^L(k, s) F^L(s), \tag{24}
\]

with the superscript ‘\(L\)’ denoting a Laplace transform. The one-sided Fourier transform \(\tilde{\eta}(k, \omega)\) of \(\tilde{\eta}(k, t > 0)\) can now be found from equation (24), by analytically continuing \(s\) to \(i\omega\), so that

\[
\tilde{\eta}_{(\text{kink/sausage})}(k, \omega) = \tilde{\eta}_{(\text{kink/sausage})}^L(k, i\omega) = G_{(\text{kink/sausage})}^L(k, i\omega) F^L(i\omega). \tag{25}
\]
With the help of equations (19) and (25), equation (18) yields

\[ D_{\text{kink/sausage}}(k, \omega)G_{\text{kink/sausage}}^L(k, i\omega)F^L(i\omega) = k\tilde{\sigma}_{\text{kink/sausage}}(-k)F^L(i\omega), \]  

(26)

whence we obtain

\[ G_{\text{kink/sausage}}^L(k, i\omega) = k\tilde{\sigma}_{\text{kink/sausage}}(-k)/D_{\text{kink/sausage}}(k, \omega). \]  

(27)

Equation (27) helps us to find \( G^L(k, s) \) for any complex value of \( s \), by means of analytic continuation. Thus

\[ G_{\text{kink/sausage}}^L(k, s) = k\tilde{\sigma}_{\text{kink/sausage}}(-k)/D_{\text{kink/sausage}}(k, -is), \]  

(28)

which, combined with equation (24), ultimately gives the expression for the Laplace transform \( \tilde{\eta}^L_{\text{kink/sausage}}(k, s) \) of the displacements \( \tilde{\eta}_{\text{kink/sausage}}(k, t) \) for any complex value of \( s \). This expression is given as

\[ \tilde{\eta}^L_{\text{kink/sausage}}(k, s) = k\tilde{\sigma}_{\text{kink/sausage}}(-k)F^L(s)/D_{\text{kink/sausage}}(k, -is), \]  

(29)

from which we can find the time evolution of the displacements \( \tilde{\eta}_{\text{kink/sausage}}(k, t) \) by means of Bromwich’s integral formula. Thus,

\[ \tilde{\eta}_{\text{kink/sausage}}(k, t) = \frac{k\tilde{\sigma}_{\text{kink/sausage}}(-k)}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F^L(s) \exp(st) \, ds/D_{\text{kink/sausage}}(k, -is), \]  

(30)

where, the real constant \( \gamma \) is so chosen that the singularities of the integrand lie to the left of the line \( s = \gamma \), the singularities themselves being of the nature of simple poles, that are the solutions of the dispersion relations \( D_{\text{kink/sausage}}(k, n) = 0 \). In the present case, these singularities are \( s_{\pm} = in_{\pm} \), with \( n_{\pm} \) being simply the frequencies given in equation (20) above.

To evaluate the integral given in equation (30), we need now to consider some particular form of the function \( F(t) \) for the time dependent part of the external stress \( \tilde{\sigma}_{\text{kink/sausage}}(-k, t) \); see, equation (22). A common model to choose is an exponential one, so that
\[ F(t) = A \exp(-\lambda t), \] (31)

with \( A \) being a constant, and the exponent \( \lambda \) being positive definite. For this particular functional form of the external stress \( \tilde{\sigma}(-k, t) \), equation (30) yields

\[
\tilde{\eta}(k, t) = \frac{k \tilde{\sigma}(k, t)}{2\pi i} A \int_{\gamma}^{\gamma+\infty} \exp(st) ds \frac{1}{s + \lambda} D(k, -is) \quad \text{with the second expression following from the residue theorem.} \]

The first term in this expression is a transient term, that decays with the decay of the external stress, whereas, the last two terms are due to disturbances that live on even after the withdrawal of the external force and, therefore, pertain to the eigenmode oscillations of the slab. For the sake of illustrations, we may consider a \textit{delta function} type external stressing, so that \( \lambda \to \infty \), and \( A/\lambda \to 1 \). The displacements \( \tilde{\eta}(k, t) \) then evolve as

\[
\tilde{\eta}(k, t) = i k \tilde{\sigma}(k, t) \frac{\exp(in_+ t)}{[\partial D(k, n)/\partial n]_{n_+}} \frac{\exp(in_- t)}{[\partial D(k, n)/\partial n]_{n_-}} \left[ \exp(in_+ t) - \exp(in_- t) \right], \quad \text{(33)}
\]

giving us the familiar result, that the application of an instantaneous external stress \( \tilde{\sigma}(-k) \delta(t) \) on the MHD slab creates long lived excitations pertaining to the normal surface modes of the slab, that have frequencies \( n_+ \) and \( n_- \) as given by the dispersion relations (20). Equation (33) further shows that the two surface modes, denoted here by a ‘+’ and a ‘-’ sign, are out of phase with the external stress \( \tilde{\sigma}(-k, t) \) by \( \pi/2 \) and \( 3\pi/2 \), respectively. The modes
are of same amplitude, i.e., \( |\tilde{\eta}^+_{(\text{kink/sausage})}(k, t)| = |\tilde{\eta}^-_{(\text{kink/sausage})}(k, t)| \), but their different phase relations with respect to \( \tilde{\sigma}(-k, t) \) lead to a difference in their respective energy absorption rates, as will be presented in the next section.

3.3. Energy Absorbed by the Modes: Equation (33) shows that the rate of absorption of energy per unit area by each of the \( \tilde{\eta}^\pm_{(\text{kink/sausage})}(k, t) \) mode, from the external stress \( \tilde{\sigma}_{(\text{kink/sausage})}(-k, t) \) is given by

\[
\dot{h}^\pm(\tilde{\eta}^\pm_{(\text{kink/sausage})}(k, t), \dot{\tilde{\eta}}^\pm_{(\text{kink/sausage})}(k, t)) = \tilde{\sigma}^\pm_{(\text{kink/sausage})}(-k, t)\tilde{\eta}^\pm_{(\text{kink/sausage})}(k, t)
\]

whence we calculate the total energy (per unit area) absorbed by the mode, by integrating equation (34) over a time \( t \) that is much longer than the decay time of the stress, i.e. \( t \gg 1/\lambda \).

This absorption is given by

\[
\Delta h^\pm_{(\text{kink/sausage})}(n^\pm) = \int_0^\infty \dot{h}^\pm(\tilde{\eta}^\pm_{(\text{kink/sausage})}(k, t), \dot{\tilde{\eta}}^\pm_{(\text{kink/sausage})}(k, t)) \, dt
\]

\[
= \frac{k A^2 |\tilde{\sigma}_{(\text{kink/sausage})}(-k)|^2 n^\pm}{\left[ \frac{\partial D_{(\text{kink/sausage})}(k, n)}{\partial n} \right]_{n^\pm}} n^\pm + \lambda \exp \left\{ (in^\pm - \lambda) t \right\},
\]

which, after substitution in favour of the modal amplitude \( \tilde{\eta}^\pm_{(\text{kink/sausage})}(k, t) \) in equation (32), ultimately yields

\[
\Delta h^\pm_{(\text{kink/sausage})}(n^\pm) = \left( n \frac{\partial D_{(\text{kink/sausage})}(k, n)}{\partial n} \right)_{n^\pm} \frac{1}{k} |\tilde{\eta}^\pm_{(\text{kink/sausage})}(k, t)|^2.
\]

We here note that, inspite of the complexities presented by the magnetic field, the expression of wave energy presented in equation (36) is essentially the same as the one given in Cairns (1979) in the case of a purely hydrodynamic system. Unlike Cairns (1979), whose method was intuitive (see Section 1 for details), we here derive our results directly from the equation of motion (18) of the system. The particular models of external stress that we assume for the purpose of demonstrations, do not have any bearing on our final result in equation (36), thus implying that this expression for the wave energy is truly a generalized expression for any hydrodynamic or magnetohydrodynamic system.
3.4. Waves of Negative Energy (NEW): Consider, for simplicity’s sake, a frame of reference co-moving with the external medium, so that $u_{0e} = 0$. Consider further, that the velocity of the slab $u_{0o}$ is increased gradually from zero through positive values in this frame of reference. Equation (20) in Section 3.1 then shows that, as long as $u_{0o} < c_k f / (\alpha_{of})^{1/2}$, both the ‘+’ and the ‘-’ modes present oscillatory surface waves with $n_+ < 0$ and $n_- > 0$, thus implying that the ‘+’ wave propagates in the positive X-direction, whereas, the ‘-’ wave propagates in the negative X-direction. As the value of $u_{0o}$ is increased through $c_k f / (\alpha_{of})^{1/2}$, oscillatory surface modes still pertain, but now with $n_\pm < 0$, thus implying that both the ‘+’ and the ‘-’ waves now propagate in the positive X-direction. Thus, with the increase of the slab speed past its critical value $c_k f / (\alpha_{of})^{1/2}$, the ‘-’ surface wave reverses the direction of its phase propagation to be simply carried by the flow. In other words, the ‘-’ surface wave changes its character from a forward wave to a backward wave (eg. Ostrovskii et al. 1986). To examine the energetics of these surface waves, we note from equation (36) that,

$$\Delta h_{(kink/sausage)}(n_\pm) = \pm n_\pm (n_+ - n_-) \frac{1}{k} |\tilde{\eta}_{(even/odd)}(k, t)|^2$$

(37)

so that,

$$\Delta h_{(kink/sausage)}(n_\pm) > 0, \text{ when } u_{0o} < c_k f / (\alpha_{of})^{1/2},$$

(38 a)

and

$$\Delta h_{(kink/sausage)}(n_+) > 0 \text{ and } \Delta h_{(kink/sausage)}(n_-) < 0,$$

$$\text{ when } c_k f / (\alpha_{of}\alpha_{ef})^{1/2} > u_{0o} > c_k f / (\alpha_{of})^{1/2}.$$  

(38 b)

The backward ‘-’ surface wave is then also a negative energy wave in this particular reference frame - a result, that is in agreement with Cairns (1979), Ostrovskii et al. (1986) and Ryutova (1988). As the velocity $u_{0o}$ of the slab passes through its threshold value $c_k f / (\alpha_{of}\alpha_{ef})^{1/2}$ for K-H instability, an unstable region is produced by a coalescence of the positive and the negative energy modes. We may note that, although the sign of energy of the modes depends on the choice of the co-ordinate frame, the existence criterion of the unstable branch (see
equation (21)) is independent of such a choice. Also invariant is the total energy absorbed by the system from the external perturbation. For a $\delta$-function type perturbation, this energy is given by (see equation 35)

\[
\Delta h_{\text{(kink/sausage)}} = \Delta h_{\text{(kink/sausage)}}(n_+) + \Delta h_{\text{(kink/sausage)}}(n_-) = k|\tilde{\sigma}_{\text{(kink/sausage)}}(-k)|^2, \quad (39)
\]

which is less than the excitation energy $\Delta h_{\text{(kink/sausage)}}(n_+)$ of the ‘+’ mode alone in the above example. This extra energy is released during the process of excitation of a negative energy wave, thus exciting simultaneously a positive energy wave through the mode interactions in the presence of the external stress $\tilde{\sigma}(-k, t)$.

4. Effects of viscosity on the wavemodes

To examine the effects of *viscous dissipation* on the surface modes of the slab, we first observe that a canonical form of the stress-free equation of motion of the slab can be obtained by substituting $\partial/\partial t$ for $\dot{\eta}$ in equation (20). This equation is

\[
\ddot{\eta}_{s}(k, t) + 2i k \bar{U} \dot{\eta}_{s}(k, t) + k^2 (\delta n)^2 \eta_{s}(k, t) = 0, \quad (40 \ a)
\]

with

\[
\bar{U} = (\alpha_{ef} u_{0e} + \alpha_{of} u_{0o}), \quad (\delta n)^2 = c_{kf}^2 - \alpha_{of} \alpha_{ef} (u_{0e} - u_{0o})^2, \quad (40 \ b)
\]

in a frame ‘s’(say), in which the two fluids ‘o’ and ‘e’ are seen to move with velocities $u_{0o}$ and $u_{0e}$, respectively. In this frame of reference ‘s’, a wave profile $\eta^s(x, z', t)$ at any point $z = z'$ inside the slab is seen to have a dependence given by $\eta^s(x, z', t) = \eta^s(t = 0) \exp \left\{ i \left( n_{z}^s \cdot t + kx \right) \right\}$, in which the frequencies $n_{z}^s$ have a nett drift term $k\bar{U}$, so that $n_{z}^s \neq -n_{z}^s$, signifying that the reflection symmetry is lost.

Consider now a frame of reference ‘r’, that moves with a relative velocity $\bar{U}$ with respect to the ‘s’ frame, so that the transformation r→s is given by

\[
\tilde{\eta}_{s}(k, t) = \tilde{\eta}_{s}(k, t) \exp \left( -ik\bar{U}t \right), \quad (41)
\]
with $\tilde{\eta}_{r(kink/sausage)}(k, t)$ satisfying the equation of a simple harmonic oscillator

$$\ddot{\tilde{\eta}}_{r(kink/sausage)}(k, t) + k^2(\delta n)^2 \tilde{\eta}_{r(kink/sausage)}(k, t) = 0. \tag{42}$$

A wave profile has a dependence given by $\eta^r(x, z', t) = \eta^r(t = 0) \exp\{i (n^r \pm t + kx)\}$ in this reference frame ‘r’, which yields $n^r_+ = -n^r_- = \delta n$ (see equations (20) and (40), with $\bar{U} = 0$). The slab waves in this ‘r’ frame thus possess a reflection symmetry, since $\delta n$ is an invariant that depends only on the relative velocity $|u_{0e} - u_{0o}|$, and not on the drift velocity $\bar{U}$.

While examining the effect of viscosity on the surface modes of the slab, we must begin our investigations by calculating the viscous dissipation as is seen in the reference frame ‘r’. This approach is in agreement with Rayleigh (1883), who argued that in a moving stream flowing with a velocity $U$, the nett pressure fluctuation due to viscous drag must be $\delta p = \chi_d (\phi - \bar{U} x)$, as measured in the frame of reflection symmetry of the perturbations, with $\chi_d$ being a viscous drag coefficient and $\phi(x, z, t)$ being the velocity potential. In the present case of the magnetic modes of the slab, this requires that the rate of mechanical energy dissipation of the slab due to viscous damping (eg. Landau and Lifshitz 1959a) is given by

$$\hat{\mathcal{E}}^r_\nu = - \int \left[ \int_{-\infty}^{-d} \Psi \, dz + \int_{d}^{\infty} \Psi \, dz \right] dx, \tag{43 a}$$

where,

$$\Psi = 2\rho_{0e}\nu_e \left[ \left( \frac{\partial}{\partial x} (\delta u_{xe})^r \right)^2 + \left( \frac{\partial}{\partial z} (\delta u_{ze})^r \right)^2 + \left( \frac{\partial}{\partial z} (\delta u_{ze})^r \right)^2 \right], \tag{43 b}$$

with $\nu_e$ being the kinematic viscosity in medium ‘e’, whereas, medium ‘o’ is taken to be inviscid. It is possible to use the classical gas dynamical formula for viscous dissipation (as given in equation (43)), while retaining the velocity discontinuities at the slab interfaces, only in such a situation, where either the internal or the external medium alone has viscous dissipation and the other medium is inviscid, i.e., the details arising due to boundary layer may be ignored and the tangential discontinuity of the velocities at the interfaces still remains a valid condition. We however note that, ideally one should consider an anisotropic viscous stress tensor in the presence of a magnetic field (cf. Braginskii 1965), as in the situation
considered here. Whatever the case may be, the specific choice of the viscous stress tensor is not expected to change the overall stability properties of the modes about which we are mainly concerned in this paper.

We now substitute the expressions

\[
\delta u^r_{xe}(x, z, t) = \mp k \dot{\eta}^r(k, \mp d, t) \exp \{ \pm k (z \pm d) \} \exp (ikx) \\
= \mp (\delta n) \dot{\eta}(k, \mp d, t) \exp \{ \pm k (z \pm d) \} \exp (ikx),
\]

(44 a)

and

\[
\delta u^r_{xe}(x, z, t) = \dot{\eta}^r(k, \mp d, t) \exp \{ \pm (z \pm d) \} \exp (ikx) \\
= (\delta n) \dot{\eta}(k, \mp d, t) \exp \{ \pm k (z \pm d) \} \exp (ikx),
\]

(44 b)

for the various perturbations in equation (43). Using the definitions given in equation (3, see Section 2.1), we thus obtain

\[
\hat{E}^r_{\nu} \left( \dot{\eta}_{(kink/sausage)}^r(k, t), \dot{\eta}_{(kink/sausage)}^s(k, t) \right) = -2i \rho_0 e \nu e k(\delta n) \left[ \dot{\eta}_{(kink)}^r(k, t) \dot{\eta}_{(kink)}^s(k, t) \\
+ \dot{\eta}_{(sausage)}^r(k, t) \dot{\eta}_{(sausage)}^s(k, t) \right],
\]

(45)

for the rate of viscous dissipation in frame ‘r’.

The energy thus dissipated in frame ‘r’ gives rise to an increment \( \delta S_{0e} \) in the entropy of the system, that must be invariant in all frames. Noting that, \( \delta n \) in equation (45) is an invariant (see, equation (40b)), we obtain the rate of increase of entropy (or, the heating rate) in terms of the quantities defined in the frame ‘s’. Thus, substituting \( \delta n = n + k \bar{U} \), we have

\[
T_{0e} \frac{\delta S_{0e}}{\delta t} \left( \dot{\eta}_{(kink/sausage)}^r(k, t), \dot{\eta}_{(kink/sausage)}^s(k, t) \right) = 2t \rho_0 e \nu e k(n + k \bar{U}) \left[ \dot{\eta}_{(kink)}^r(k, t) \dot{\eta}_{(kink)}^s(k, t) \\
+ \dot{\eta}_{(sausage)}^r(k, t) \dot{\eta}_{(sausage)}^s(k, t) \right],
\]

(46)

where, \( T_{0e} \) is the equilibrium temperature of the medium ‘e’.

The heating rate being thus known, we demand that, the thermodynamic potential \( \Phi_0 \) (\( = H - T_{0e} S_{0e} \)) must be minimum at all instants for the wave propagation to be a manifestation
of the system’s response to its departure from equilibrium (cf. Landau and Lifshitz 1959b; Glansdorff and Prigogine 1971). This shows,

\[
\delta \Phi_0(t) \equiv \Phi_0(t + \delta t) - \Phi_0(t) = \left\{ \frac{\partial H}{\partial \tilde{\eta}(\text{kink/sausage})} \tilde{\eta}(\text{kink/sausage}) + \frac{\partial H}{\partial \tilde{\eta}(\text{kink/sausage})} \dot{\tilde{\eta}}(\text{kink/sausage}) - T_0 \frac{\delta S_0}{\delta t} \right\} \delta t = 0,
\]

(47)

for any infinitesimal \( \delta t \). In equation (47), we have dropped the superscript ‘s’, still indicating the observer’s frame. With the help of equation (18), we then obtain (after substituting \( \tilde{\sigma}(\text{kink/sausage})(-k, t) = 0 \)),

\[
\left\{ \frac{k^2 B_{0e}^2}{16\pi n^2} + \frac{k^2 B_{0o}^2}{8\pi n^2} \left[ \frac{2 \sinh (2kd) \pm 4(kd)}{8 \cosh^2 (kd)} \right] \right\} \tilde{\eta}(\text{kink/sausage})(k, t) \dot{\tilde{\eta}}(\text{kink/sausage})(k, t) + \left\{ \rho_{0e} (n + k u_{0e})^2 - \frac{3k^2 B_{0e}^2}{16\pi} \right\} \tilde{\eta}(\text{kink/sausage})(k, t) \dot{\tilde{\eta}}(\text{kink/sausage})(k, t) + \frac{k^2 B_{0o}^2}{8\pi} \left[ \frac{2 \sinh (2kd) \pm 4(kd)}{8 \cosh^2 (kd)} \right] \tilde{\eta}(\text{kink/sausage})(k, t) \dot{\tilde{\eta}}(\text{kink/sausage})(k, t) - \rho_{0e} \nu_e k^2 \left( \frac{n + k \bar{U}}{n} \right) \tilde{\eta}^2(\text{kink/sausage})(k, t) = 0,
\]

(48)

which is true for all \( \tilde{\eta}(\text{kink/sausage})(k, t) \). Substituting \( \partial/\partial t = i \nu \), and also requiring a non-trivial solution, we then obtain the dispersion relation

\[
\rho_{0e} \left( (n + k u_{0e})^2 - k^2 c_{Ae} \right) + \rho_{0o} \left( (n + k u_{0o})^2 - k^2 c_{Ao} \right) f(k) = i \rho_{0e} \nu_e k^2 (n + k \bar{U}),
\]

(49)

for the viscous surface modes of the slab. Note the factor \( (n + k \bar{U}) \) in the damping term of equation (49). This factor differentiates equation (49) from the earlier results (cf. Kikina 1967; Weissman 1970; Cairns 1979; Ezerskii et al. 1981; Ostrovskii and Stepanyants 1982; Ostrovskii et al. 1986; Ruderman and Goossens 1995), in which the viscous drag force was proportional to the frequency \( n \) of the waves in the observer’s frame ‘s’, thus depending on the velocity \( \bar{U} \) of the material. In view of this important difference, it is here pertinent, that we discuss the significance of equation (49) in some detail.
The flow of the two fluids creates a momentum flux $\rho_0 u_0 + \rho_0 u_0$ per unit volume in the observer’s (‘s’) frame, that is equivalent to imposing a velocity $\bar{U} = (\alpha_{of} u_0 + \alpha_{ef} u_0)$ on all matter in the wave profile. Moving to any other frame, where velocities of the fluids are $u'_{0o} = u_{0o} + u$ and $u'_{0e} = u_{0e} + u$, we have $\bar{U}' = \alpha_{of} u'_{0o} + \alpha_{ef} u'_{0e} = \bar{U} + u$. This nett velocity of the wave profile appears purely due to Galilean transformation, and should not contribute to any process of exchange of energy or momentum within the system, and thus cannot contribute to dissipation. Contrary to the earlier results, the expression for the viscous drag must, therefore, have no explicit dependence on the drift velocity $\bar{U}$, as is evident by the appearance of the invariant factor $\delta n = n + k\bar{U}$ in the damping force in equation (49).

Returning to the modes of oscillations of the magnetic slab, equation (49) yields solutions

$$n_\pm = \frac{i}{2} \alpha_{ef} \nu_e k^2 - k \left[ \bar{U} \pm \left[ \left( \frac{\nu_e k^2}{\alpha_{ef} \alpha_{of}} - \frac{1}{4} \frac{\alpha_{ef} \nu_e^2 k^2}{\alpha_{of} \alpha_{ef}} \right) \right]^{1/2} \right],$$

(50)

thus showing that, for flow velocities below the threshold for the K-H instability, the principal effect of viscous dissipation is to introduce a damping for both the positive and the negative energy modes of the magnetic slab - a result, that is in contradiction to the earlier results (see the references above), which predict a dissipative instability for the negative energy waves. The main consequence of our considering the correct Galilean transformation, while examining the viscous effects on the slab waves, is then the establishment of the fact that, the stability property of the modes remains non-singular in the presence of a small dissipation, that changes only slightly the threshold for the K-H instability of the slab, with the modified instability criterion given by

$$|u_{0e} - u_{0o}| \geq \left[ \frac{\nu_e^2 k^2}{\alpha_{ef} \alpha_{of}} - \frac{1}{4} \left( \frac{\alpha_{ef}}{\alpha_{of}} \right) \nu_e^2 k^2 \right]^{1/2},$$

(51)

that smoothly approaches the adiabatic criterion in equation (21) for a vanishingly small kinematic viscosity $\nu_e \to 0$. 

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5. Concluding remarks

Occurrence of magnetic structures is abundant in various astrophysical situations, such as in the solar photospheric flux tubes, or in the solar coronal plasma loops. Such magnetic structures are often associated with field-aligned plasma flows, with the velocities of these flows being different inside the structures than those outside, thus producing shearing motions in the plasma medium. Detailed understandings of the complex interactions of such shearing flows with the oscillatory motions of the structures are necessary to study accurately the energy transport processes in astrophysics, such as the mechanisms of non-thermal energy transport from the solar sub-surface layers to the upper solar atmosphere. Certain investigations have been carried out (e.g. Ryutova 1988; Nakariakov and Roberts 1995; Nakariakov et al. 1996; Ruderman and Goossens 1995; Ruderman et al. 1996; Joarder et al. 1997) in this direction, that highlighted the role of negative energy waves in such processes. As a further contribution to such investigations, we here examined in detail certain specific aspects of the interactions of magnetohydrodynamic waves with shearing flows, and particularly of the negative energy waves, by using a self-consistent thermodynamic approach. This approach helped us to generalize the expression for the hydrodynamic wave energy given in Cairns (1979) to magnetohydrodynamics (see, equation (36) in Section 3.3), thus enabling us to calculate the energy of the hydromagnetic waves (of course in the harmonic approximation), when the linear dispersion relations of such waves are known along with the observationally obtained informations regarding the wave amplitudes. Once the wave energy is thus calculated, equation (35) then guides us to obtain a rough estimate of the generating stresses $\tilde{\sigma}(-k, t)$ of the waves. Such estimates of the stresses may be of great importance in several astrophysical situations, particularly in solar MHD cases, where such estimates may provide us with some clues regarding the physical processes that may be taking place in the sub-surface layers of the Sun, or in the regions of complex magnetic topology in the solar atmosphere, about which we have very little direct observational evidence. Finally, by incorporating viscosity, we obtain the dispersion relations (equation 49 in Section 4) which, while precluding the possibility of dissipative instability, sets the correct conditions for the stability of the system (equation (51)) and also yields the time scales for the decay of the disturbances in the surface modes of MHD systems. It is to be hoped, that the present study
would provide us with some guidance in gaining further physical insights into the complex
nature of the interactions between the magnetic field and the fluid flows in various astro-
physical systems,—both for estimates in terms of energetics as also in the study of evolutions
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References.

Acheson, D. J. 1976 J. Fluid Mech. 77, 433.

Alfven, H. 1950 *Cosmical Electrodynamics*. Clarendon Press, Oxford.

Benjamin, T. B. 1963 *J. Fluid Mech.* 16, 436.

Braginskii, S. I. 1965 in *Rev. Plasma Phys.* (ed. M. A. Leontovich), I, p. 205.

Cairns, R. A. 1979 *J. Fluid Mech.* 92, 1.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Clarendon Press, Oxford.

Cowling, T. G. 1957 *Magnetohydrodynamics*. Wiley-Interscience, New York.

Craik, A. D. D. 1985 *Wave Interaction and Fluid Flows*. Cambridge Univ. Press, Cambridge.

Craik, A. D. D. and Adam, J. A. 1979 *J. Fluid Mech.* 92, 15.

Edwin, P. M. and Roberts, B. 1982 *Solar Phys.* 76, 239.

Edwin, P. M. and Roberts, B. 1983 *Solar Phys.* 88, 179.

Ezerskii, A. B., Ostrovskii, L. A. and Stepanyants, Yu. A. 1981 *Izv. Atmos. Ocean. Phys.* 17, 890.

Glansdorff, P. and Prigogine, I. 1971 *Thermodynamic Theory of Structure, Stability and Fluctuations*. Wiely-Interscience, New York.

Joarder, P. S., Nakariakov, V. M. and Roberts, B. 1997 *Solar Phys.* (in Press).

Kikina, N. G. 1967 *Sov. Phys. Akust.* 13, 184.

Landau, L. D. and Lifshitz, E. M. 1959a *Fluid Mechanics*. Pergamon Press, Oxford.

Landau, L. D. and Lifshitz, E. M. 1959b *Statistical Physics*, Part I. Pergamon Press, Oxford.

Nakariakov, V. M. and Roberts, B. 1995 *Solar Phys.* 159, 213.

Nakariakov, V. M., Roberts, B. and Mann, G. 1996 *Astron. Astrophys.*, 311, 311.

Ostrovskii, L. A. and Stepanyants, Yu. A. 1982 *Izv. Akad. Nauk SSSR. Ser Mekh. Zheidk. Gaza* No. 4, 63.

Ostrovskii, L. A., Rybak, S. A. and Tsimring, L. Sh. 1986 *Sov. Phys. Usp.* 29, 1040.

Rayleigh, Lord. 1883 *Proc. Lon. Math. Soc.* XV, 69.

Roberts, B. 1981a *Solar. Phys.* 69, 27.

Roberts, B. 1981b *Solar. Phys.* 69, 39.

Ruderman, M. S. and Goossens, M. 1995 *J. Plasma Phys.* 54, 149.
Ruderman, M. S., Verwichte, E., Erdelyi, R. and Goossens, M. 1996 *J. Plasma Phys.* **56**, 285.

Ryutova, M. P. 1988 *Sov. Phys. JETP.* **67**, 1594.

Satya Narayanan, A. 1991 *Plasma Phys. Control. Fusion* **33**, 333.

Singh, A. P. and Talwar, S. P. 1994 *Solar Phys.* **149**, 331.

Sommerfeld, A. 1950 *Mechanics of Deformable bodies.* Academic Press, New York.

Stix, T. H. 1962 *The Physics of Plasma Waves.* Mc.Graw-Hill, New York.

Weissman, M. A. 1970 *Notes on Summer Study Prog. Geophys. Fluid Dyn.* Woods Hole Oceanog. Inst. no. 70-50.

Witham, G. B. 1974 *Linear and Non-Linear Waves.* Wiley-Interscience, New-York.
Appendix. Energy density of an incompressible MHD plasma.

The *energy density* of an incompressible magnetohydrodynamic system is defined as

\[ \varepsilon(r, t) = \varepsilon_i(r, t) + \frac{B^2(r, t)}{8\pi} + \frac{\rho_0}{2} u^2(r, t), \]  

(A.1)

where, \( \varepsilon_i \) is the *thermodynamic internal energy* of the plasma, and the quantities \( \rho_0 \), \( u \) and \( B \) are as defined in equation (2) of the main text.

In the present case, we consider the fluid to be *incompressible*, and the hydrodynamic processes to be *adiabatic*. In that case, we write (cf. Landau and Lifshitz, 1959b)

\[ \varepsilon_i(r, t) = \mu(r, t) / \upsilon, \]  

(A.2)

where, \( \mu(r, t) \) is the local *chemical potential* of the system, and \( \upsilon \) is its *specific volume*. Thus, for any fluctuation in the thermodynamic state of the system, we have

\[ \delta\varepsilon_i(r, t) = \delta\mu(r, t) / \upsilon, \]  

(A.3)

where, \( \upsilon \) is a constant in an incompressible fluid. Further,

\[ \delta\mu = -s \, dT + \upsilon \, dp, \]  

(A.4)

where, \( s \) is the *specific entropy* of the system.

If we now assume the *electrical* and the *thermal conductivities* of the fluid to be infinite, and its *viscosity coefficient* to be zero, then the system cannot support any thermal gradients, and, therefore, \( \delta T \) is zero at all points. Combining equations (A.2-A.4), we then find that the fluctuations in the thermodynamic energy density to be (see, Sommerfeld 1950 for an alternative interpretation)

\[ \delta\varepsilon_i = \delta p, \]  

(A.5)
so that, neglecting the integration constant, we ultimately obtain equation (6, Section 2.2) of the main text, i.e.,

\[
\varepsilon (r, t) = p (r, t) + \frac{B^2 (r, t)}{8\pi} + \frac{\rho_0}{2} u^2 (r, t) \tag{A.6}
\]

for an \textit{incompressible} fluid.