Castelnuovo-Mumford regularity
of canonical and deficiency modules

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Abstract

We give two kinds of bounds for the Castelnuovo-Mumford regularity of the canonical module and the deficiency modules of a ring, respectively in terms of the homological degree and the Castelnuovo-Mumford regularity of the original ring.

Key words: Castelnuovo-Mumford regularity, local cohomology, canonical module, deficiency module, homological degree.

Introduction

Let $R = k[x_1, ..., x_n]$ be a standard graded polynomial ring, and let $M$ be a finitely generated graded $R$-module of dimension $d$. The canonical module $K^d(M) = \text{Ext}_R^{n-d}(M, R)(-n)$ of $M$ - originally introduced by Grothendieck - plays an important role in Commutative Algebra and Algebraic Geometry (see, e.g., [4]). It is natural to ask whether one can bound the Castelnuovo-Mumford regularity $\text{reg}(K^d(M))$ of $K^d(M)$ in terms of other invariants of $M$. Besides the canonical module we are also interested in a similar problem for all the

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deficiency modules $K^i(M) = \text{Ext}_R^{n-i}(M, R)(-n)$, $i < d$. These modules were defined in [15], Section 3.1, and can be considered as a measure for the deviation of $M$ being a Cohen-Macaulay module. Moreover, even in a rather simple case there is a close relationship between $\text{reg}(K^d(M))$ and $\text{reg}(K^i(M))$, $i < d$ (see [15], Corollary 3.1.3). One could say that the Castelnuovo-Mumford regularity $\text{reg}(M)$ controls positive components of all local cohomology modules $H^i_m(M)$ of $M$: they vanish above the level $\text{reg}(M)$. Although the negative components $H^i_m(M)_j$ do not necessarily vanish, the function $\ell(H^i_m(M)_j)$ becomes a polynomial for $j < -\text{reg}(K^i(M))$. In this sense $\text{reg}(K^i(M))$ controls the behavior of $\ell(H^i_m(M)_j)$ in negative components. Note that in a series of papers M. Brodmann and others have considered the problem when $\ell(H^i_m(M)_j)$ becomes a polynomial (see, e.g., [2], [3]). In fact, a result of [3] will play an important role in our investigation.

In this paper we will give two kinds of bounds for $\text{reg}(K^i(M))$. In Section 2 we will show that one can use the homological degree to bound $\text{reg}(K^i(M))$. The homological degree was introduced by W. Vasconcelos [19], and one can use it to bound $\text{reg}(M)$ (see [6], Theorem 2.4 and [14], Theorem 3.1). For the ring case the bound has a simple form: $\text{reg}(S) < h\text{deg}(S)$, where $S$ is a quotient ring of $R$. Our Theorem 9 says that $\text{reg}(K^i(S)) \leq d \cdot h\text{deg}(S)$ for all $i$. Thus this result complements the relationship between the Castelnuovo-Mumford regularity and the homological degree.

The core of the paper is Section 3. Here we restrict ourselves to the case of rings. We will then prove that one can bound $\text{reg}(K^i(S))$ in terms of $\text{reg}(S)$ (see Theorem 14). Although the bounds are huge numbers, they show that the Castelnuovo-Mumford regularity also controls the behavior of all local cohomology modules in negative components (in the above mentioned sense). This is a new meaning for the Castelnuovo-Mumford regularity. For example, our study gives the following consequence

**Corollary** [18] Denote by $\mathcal{H}_{n,i,r}$ the set of numerical functions $h : \mathbb{Z} \to \mathbb{Z}$ such that there exists a homogeneous ideal $I \subset R = k[x_1, ..., x_n]$ satisfying the following conditions

(i) $\text{reg } I \leq r$,
(ii) $\ell(H^i_m(R/I)_t) = h(t)$ for all $t \in \mathbb{Z}$.

Then for fixed numbers $n, i, r$ the set $\mathcal{H}_{n,i,r}$ has only finitely many elements.

In the last section 4 we will examine some cases where $\text{reg}(K^i(M))$ can be bounded by a small number. As one can expect, each of them is of a very special type. Section 1, where we collect some results on the Castelnuovo-Mumford regularity, is of preparatory character.
1 Preliminaries

In this section we recall some basis facts on the Castelnuovo-Mumford regularity. Throughout the paper let \( R = k[x_1, \ldots, x_n] \) be a standard graded polynomial ring, where \( k \) is an infinite field, and let \( \mathfrak{m} = (x_1, \ldots, x_n) \). For an arbitrary graded \( R \)-module \( N \), put

\[
\text{beg}(N) = \inf\{ i \in \mathbb{Z} \mid [N]_i \neq 0 \},
\]

and

\[
\text{end}(N) = \sup\{ i \in \mathbb{Z} \mid [N]_i \neq 0 \}.
\]

(We assume \( \text{beg}(N) = +\infty \) and \( \text{end}(N) = -\infty \) if \( N = 0 \).)

**Definition 1** Let \( M \) be a finitely generated \( R \)-module. The number

\[
\text{reg}(M) = \max\{ i + \text{end}(H^i_{\mathfrak{m}}(M)) \mid i \geq 0 \}
\]

is called the Castelnuovo-Mumford regularity of \( M \).

Note that if \( I \subset R \) is a nonzero homogeneous ideal, then

\[
\text{reg}(I) = \text{reg}(R/I) + 1.
\]

We also consider the number

\[
\text{reg}_1(M) = \max\{ i + \text{end}(H^i_{\mathfrak{m}}(M)) \mid i \geq 1 \},
\]

which is sometimes called the Castelnuovo-Mumford regularity at level one. The definition immediately gives

\[
\text{reg}(M) = \max\{ \text{reg}_1(M), \text{end}(H^0_{\mathfrak{m}}(M)) \}.
\]

(1)

The following result is the starting point for the investigation of the Castelnuovo-Mumford regularity.

**Lemma 2** ([8], Proposition 1.1 and Theorem 1.2)

\[
\text{reg}(M) = \max\{ \text{end}(\text{Tor}^R_i(k, M)) - i \mid i \geq 0 \}.
\]

The long exact sequence of local cohomology arising from a short exact sequence of modules gives:

**Lemma 3** ([7], Corollary 20.19) *Let*

\[
0 \to A \to B \to C \to 0
\]
be an exact sequence of graded $R$-modules. Then

(i) \( \text{reg}(B) \leq \max\{\text{reg}(A), \text{reg}(C)\} \),

(ii) \( \text{reg}(A) \leq \max\{\text{reg}(B), \text{reg}(C) + 1\} \).

Recall that a homogeneous element \( x \in \mathfrak{m} \) is called an \( M \)-filter regular if

\[ x \not\in p \quad \text{for all} \quad p \in (\text{Ass } M) \setminus \{\mathfrak{m}\}. \]

This is equivalent to the condition that the module \( 0 :_M x \) is of finite length. Since \( k \) is assumed to be infinite, there always exists a filter regular element with respect to a finite number of finitely generated modules.

Let \( x \) be a linear \( M \)-filter regular element. Then

\[ H^i_m(M/0 :_M x) \cong H^i_m(M) \quad \text{for all} \quad i \geq 1. \]

Hence the short exact sequence induced by multiplication by \( x \)

\[ 0 \rightarrow (M/0 :_M x)(-1) \rightarrow M \rightarrow M/xM \rightarrow 0 \]

provides the exact sequence

\[ 0 \rightarrow (0 :_M x)_{j-1} \rightarrow H^0_m(M)_{j-1} \rightarrow H^0_m(M)_j \rightarrow H^0_m(M/xM)_j \rightarrow \cdots \]

\[ \cdots \rightarrow H^i_m(M)_j \rightarrow H^i_m(M/xM)_j \rightarrow H^{i+1}_m(M)_{j-1} \rightarrow H^{i+1}_m(M)_j \rightarrow \cdots \]

From this one can get (see [7], Proposition 20.20 and [12], Lemma 2):

Lemma 4 Let \( x \) be a linear \( M \)-filter regular element. Then

\[ \text{reg}_1(M) \leq \text{reg}(M/xM) \leq \text{reg } M. \]

Finally, let us recall the notion of the regularity index (of a Hilbert function). In the literature it also appears under different names like the \( a \)-invariant (see [4], Definition 4.3.6 and Theorem 4.3.5, and [20], Section B.4) or the postulation number [3].

Definition 5 Let \( H_M(t) \) and \( P_M(t) \) denote the Hilbert function and the Hilbert polynomial of \( M \), respectively. The number

\[ ri(M) = \max\{j \in \mathbb{Z} | H_M(j) \neq P_M(j)\} \]

is called the regularity index of \( M \).

Lemma 6 Let \( x \) be a linear \( M \)-filter regular element. Then

(i) (cf. [7], Proposition 20.20) \( \text{reg}(M) = \max\{\text{reg}(M/xM), \text{end}(H^0_m(M))\} \),
(ii) \( \text{reg}(M) = \max\{\text{reg}(M/xM), \text{ri}(M)\} \),

(iii) If \( M \) is a Cohen-Macaulay module of dimension \( d \), then \( \text{reg}(M) = \text{ri}(M) + d \).

**Proof.** (i) This follows from Lemma 1 and (1).

(ii) From the Grothendieck-Serre formula

\[
H_M(j) - P_M(j) = \sum_{i=0}^{d} (-1)^i \ell(H^i_{m}(M)_j),
\]

it follows that \( \text{reg} M \geq \text{ri}(M) \). By Lemma 3 we get

\[
\text{reg}(M) \geq \max\{\text{reg}(M/xM), \text{ri}(M)\}.
\]

Let \( j \geq \text{reg}(M/xM) \). Since \( \text{reg}(M) \leq j \), this yields by (2)

\[
H_M(j) - P_M(j) = \ell(H^0_{m}(M)_j).
\]

Hence

\[
\text{end}(H^0_{m}(M)) \leq \max\{\text{reg}(M/xM), \text{ri}(M)\}.
\]

Together with (i) we get

\[
\text{reg}(M) \leq \max\{\text{reg}(M/xM), \text{ri}(M)\}.
\]

(iii) This follows from (2) and the fact that \( H^i_{m}(M) = 0 \) for all \( i < d \).

2 Comparison with homological degree

From now on let \( M \) be a \( d \)-dimensional finitely generated graded \( R \)-module. The homological degree of a graded \( R \)-module \( M \) was introduced by Vasconcelos. It is defined recursively on the dimension as follows:

**Definition 7** ([19] and [20], Definition 9.4.1) The homological degree of \( M \) is the number

\[
\text{hdeg}(M) = \deg(M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \text{hdeg}(\text{Ext}^{n+i+1-d}_{R}(M, R)).
\]

Note that

(a) \( \text{hdeg}(M) \geq \deg(M) \), and the equality holds if and only if \( M \) is a Cohen-Macaulay module.
(b) \( \text{hdeg}(M) = \text{hdeg}(M/H^0_m(M)) + \ell(H^0_m(M)) \).

Let \( \text{gen}(M) \) denote the maximal degree of elements in a minimal set of homogeneous generators of \( M \). That is,

\[
\text{gen}(M) = \text{end}(M/mM).
\]

It turns out that the homological degree gives an upper bound for the Castelnuovo-Mumford regularity

**Lemma 8** ([6], Theorem 2.4 and [14], Theorem 3.1)

\[
\text{reg}(M) \leq \text{gen}(M) + \text{hdeg}(M) - 1.
\]

Let

\[
K^i(M) = \text{Ext}^{n-i}_R(M,R)(-n).
\]

The module \( K^d(M) \) is the canonical module of \( M \). Following Schenzel ([15], Section 3.1) we call the modules \( K^i(M) \), \( i < d \), as the deficiency modules of \( M \). Note that \( K^i(M) = 0 \) for \( i < 0 \) and \( i > d \). All the modules \( K^i(M) \) are finitely generated, and by [15], Section 3.1 (see Lemma 3.1.1 and page 63) we have:

\[
\dim K^i(M) \leq i \text{ for } i < d,
\]

\[
\dim K^d(M) = d, \text{ and}
\]

\[
\text{depth}(K^d(M)) \geq \min\{2, \dim M\}.
\]

By the local duality theorem (see, e.g., [4], Theorem 3.6.19), there are the following canonical isomorphisms of graded modules

\[
K^i(M) \cong \text{Hom}_k(H^i_m(M), k). \tag{3}
\]

From this and Lemma [6](ii) we obtain that

\[
\ell(H^i_m(M)_t) = P_{K^i(M)}(t) \text{ for all } t < -\text{reg}(K^i(M)).
\]

Inspired by Lemma 8 it is natural to ask whether one can use the homological degree to bound the Castelnuovo-Mumford regularity of \( K^i(M) \), too? The following theorem, which is the main result of this section, answers this question affirmatively.

**Theorem 9** For all \( i \leq d \) we have

\[
\text{reg}(K^i(M)) \leq d[\text{hdeg}(M) - \deg(M)] - \text{beg}(M) + i.
\]
Note that when \( M \) is a Cohen-Macaulay module, \( K^d(M) \) is also a Cohen-Macaulay module. It was shown in [9], Proposition 2.3 that

\[
\text{reg}(K^d(M)) = d - \text{beg}(M).
\]  

(4)

This easily follows from Lemma 6 (iii) and the Grothendieck-Serre formula (2) applied to \( K^d(M) \), or from the duality. Thus in this case we have the equality in (ii) of the above theorem.

In order to prove Theorem 9 we need some auxiliary results.

**Lemma 10** ([17], Proposition 2.4) Let \( x \) be a linear \( M \)-filter regular element. Then there are short exact sequences of graded modules

\[
0 \to (K^{i+1}(M)/xK^{i+1}(M))(1) \to K^i(M/xM) \to 0 :_{K^i(M)} x \to 0,
\]

for all integers \( i \geq 0 \).

For short, in the proof we often use the following notation

\[
K^i := K^i(M).
\]

**Lemma 11** \( \text{reg}(K^0(M)) \leq -\text{beg}(M) \).

**PROOF.** Note that \( H^0_m(M) \subseteq M \) is a submodule of finite length. Hence, by (3), we have

\[
\text{reg}(K^0) = -\text{beg}(H^0_m(M)) \leq -\text{beg}(M).
\]

\[ \square \]

In the sequel we always assume that \( x \) is a generic linear element by which we mean that \( x \) is filter regular with respect to \( M \), all the modules \( K^i(M) \) and all the iterated deficiency modules in the sense of [19], Definition 2.12. Since this is a finite collection of modules, such an element always exists.

**Lemma 12** Assume \( \text{depth}(M) > 0 \) and \( 1 \leq i < d \). Then

\[
\text{reg}(K^i(M)) \leq -\text{beg}(M) + \sum_{j=1}^{i} \left( \begin{array}{c} d \\ j \end{array} \right) \text{hdeg}(K^{ij}(M)) + i.
\]

**PROOF.** Let \( x \in R \) be a generic linear element and \( j \geq 0 \). By Lemma 11 there is an exact sequence

\[
0 \to (K^{j+1}/xK^{j+1})(1) \to K^j(M/xM) \to 0 :_{K^j} x \to 0.
\]
Taking the tensor product with $k$ we get the exact sequence

$$K^j(M/xM)/mK^j(M/xM) \leftarrow (K^{j+1}/mK^{j+1})(1) \leftarrow \text{Tor}_1^R(k, 0 :_{K^j} x).$$

This implies that

$$\text{gen}(K^{j+1}) = \text{end}(K^{j+1}/mK^{j+1})$$

$$\leq \max\{\text{gen}(K^j(M/xM)), \text{end}(\text{Tor}_1^R(k, 0 :_{K^j} x))\} + 1.$$

Since $0 :_{K^j} x$ is of finite length,

$$0 :_{K^j} x \subseteq H^0_m(K^j).$$

Hence, by Lemma 2,

$$\text{end}(\text{Tor}_1^R(k, 0 :_{K^j} x)) - 1 \leq \text{reg}(0 :_{K^j} x) \leq \text{end}(H^0_m(K^j)) \leq \text{reg}(K^j).$$

Combining this with the fact that

$$\text{gen}(K^j(M/xM)) \leq \text{reg}(K^j(M/xM))$$

(look again at Lemma 2), we get

$$\text{gen}(K^{j+1}) \leq \max\{\text{reg}(K^j(M/xM)), \text{reg}(K^j) + 1\} + 1$$

$$\leq \max\{\text{reg}(K^j(M/xM)) + 1, \text{reg}(K^j) + 2\}. \quad (5)$$

Note that

$$\text{beg}(M/xM) \geq \text{beg}(M).$$

We now prove the claim by induction on $i$. Let $i = 1$. An application of (5) to the case $j = 0$ together with Lemma 11 yields

$$\text{gen}(K^1) \leq \max\{-\text{beg}(M/xM), -\text{beg}(M)\} + 2 = -\text{beg}(M) + 2.$$

By Lemma 8 we then get

$$\text{reg}(K^1) \leq \text{gen}(K^1) + \text{hdeg}(K^1) - 1 \leq \text{hdeg}(K^1) - \text{beg}(M) + 1$$

$$\leq d \cdot \text{hdeg}(K^1) - \text{beg}(M) + 1.$$

Thus the claim holds for $K^1$.

Let $2 \leq i \leq d - 1$. By the induction hypothesis we have

$$\text{reg}(K^{i-1}) \leq -\text{beg}(M) + \sum_{j=1}^{i-1} \binom{d}{j} \text{hdeg}(K^j) + i - 1. \quad (6)$$
For a Noetherian graded module $N$ over $S$, let $\overline{N}$ denote the module $N/H^0_m(N)$. Note that $\text{depth}(\overline{N}) > 0$ if $\dim N > 0$, and for all $j > 0$ we have

$$K^j(N) \cong K^j(\overline{N}).$$

(7)

Since $\dim M/xM = d - 1$ and $0 < i - 1 < d - 1$, again by the induction hypothesis applied to $M/xM$, the following holds

$$\text{reg}(K^{i-1}(M/xM)) = \text{reg}(K^{i-1}(\overline{M/xM}))$$

$$\leq - \text{beg}(M/xM) + \sum_{j=1}^{i-1} \binom{d-1}{j} \text{hdeg}(K^j(\overline{M/xM})) + i - 1$$

$$\leq - \text{beg}(M/xM) + \sum_{j=1}^{i-1} \binom{d-1}{j} \text{hdeg}(K^j(M/xM)) + i - 1.$$

Since $\text{depth}(M) > 0$, we have by the inequality (10) in [19]

$$\text{hdeg}(K^j(M/xM)) \leq \text{hdeg} K^j + \text{hdeg} K^{j+1}.$$

So

$$\text{reg}(K^{i-1}(M/xM)) \leq$$

$$\leq - \text{beg}(M) + \sum_{j=1}^{i-1} \binom{d}{j} \text{hdeg}(K^j) + \binom{d-1}{i-1} \text{hdeg}(K^i) + i + 1.$$  (\*)

By (5) and (6) this yields

$$\text{gen}(K^i) \leq - \text{beg}(M) + \sum_{j=1}^{i-1} \binom{d}{j} \text{hdeg}(K^j) + \binom{d-1}{i-1} \text{hdeg}(K^i) + i + 1.$$  (**)

Hence, by Lemma 8, we then get

$$\text{reg}(K^i) \leq \text{gen}(K^i) + \text{hdeg}(K^i) - 1$$

$$\leq - \text{beg}(M) + \sum_{j=1}^{i-1} \binom{d}{j} \text{hdeg}(K^j) + \binom{d-1}{i-1} \text{hdeg}(K^i) + i + 1$$

$$+ \text{hdeg}(K^i) - 1$$

$$\leq - \text{beg}(M) + \sum_{j=1}^{i-1} \binom{d}{j} \text{hdeg}(K^j) + \binom{d-1}{i-1} \text{hdeg}(K^i) + i$$

$$+ \binom{d-1}{i} \text{hdeg}(K^i)$$

$$= \sum_{j=1}^{i} \binom{d}{j} \text{hdeg}(K^j) - \text{beg}(M) + i.$$
Lemma 12 is thus completely proved. □

Remark. Let us take an extra look at the case $i = d - 1$, where $d \geq 2$. If the equality holds, i.e.

$$
\text{reg} (K^{d-1}(M)) = - \text{beg}(M) + \sum_{j=1}^{d-1} \binom{d}{j} \text{hdeg} (K^j(M)) + d - 1,
$$

then we must have the equality in (**), too. Using (*), (6) and (5), this yields $\text{hdeg} (K^{d-1}(M)) = 0$, or equivalently $K^{d-1}(M) = 0$.

**Proof of Theorem 9.** Since $\text{hdeg}(M) \geq \text{deg}(M)$, by Lemma 11 we may assume that $d \geq 1$ and $i \geq 1$. Let $\overline{M} = M/H^0_m(M)$.

(i) First consider the case $1 \leq i < d$. Since $\text{depth}(\overline{M}) > 0$, the formula of $\text{hdeg}(\overline{M})$ in Definition 7 can be rewritten as follows

$$
\text{hdeg}(\overline{M}) = \text{deg}(\overline{M}) + \sum_{j=1}^{d-1} \binom{d-1}{j} \text{hdeg}(K^{d-j-1}(\overline{M}))
$$

$$
= \text{deg}(\overline{M}) + \sum_{j=1}^{d-1} \binom{d-1}{j} \text{hdeg}(K^j(\overline{M}))
$$

$$
\geq \text{deg}(\overline{M}) + \sum_{j=1}^{i} \binom{d-1}{j} \text{hdeg}(K^j(\overline{M}))
$$

$$
\geq \text{deg}(\overline{M}) + \frac{1}{d} \sum_{j=1}^{i} \binom{d}{j} \text{hdeg}(K^j(\overline{M})).
$$

Consequently, Lemma 12 and (7) give

$$
\text{reg}(K^i) = \text{reg}(K^i(\overline{M})) \leq d(\text{hdeg}(\overline{M}) - \text{deg}(\overline{M})) - \text{beg}(\overline{M}) + i.
$$

Since $\text{hdeg}(\overline{M}) \leq \text{hdeg}(M)$, $\text{deg}(\overline{M}) = \text{deg}(M)$ and $\text{beg}(\overline{M}) \geq \text{beg}(M)$, the above inequality yields

$$
\text{reg}(K^i) \leq d(\text{hdeg}(M) - \text{deg} M) - \text{beg}(M) + i.
$$

Thus (i) is proved.

(ii) We now prove for the claim for $\text{reg}(K^d(M))$. We do induction on $d$.

If $d = 1$, then $\overline{M}$ is a Cohen-Macaulay module. By (4) we have

$$
\text{reg}(K^1) = \text{reg}(K^1(\overline{M})) = 1 - \text{beg}(\overline{M}) \leq 1 - \text{beg}(M)
$$

$$
\leq 1 + \text{hdeg}(M) - \text{deg} M - \text{beg}(M).
$$

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Let $d \geq 2$. Let $x$ be a generic linear element. Since depth $K^d > 0$, one has by Lemma 3 (i)
\[ \text{reg}(K^d) = \text{reg}(K^d/xK^d). \]
Using the short exact sequence
\[ 0 \to (K^d/xK^d)(1) \to K^{d-1}(M/xM) \to 0 : K^{d-1} x \to 0, \]
and Lemma 3 (ii) we then get
\[ \text{reg}(K^d) \leq \max\{\text{reg}(K^{d-1}(M/xM)), \text{reg}(K^{d-1}) + 1\} + 1. \tag{8} \]
If $K^{d-1} \neq 0$, then by Part (i) and the remark after Lemma 12 it already holds that $\text{reg}(K^{d-1}) + 2 < d(\text{hdeg}(M) - \text{deg}(M)) - \text{beg}(M) + d + 1$. Hence
\[ \text{reg}(K^{d-1}) + 2 \leq d(\text{hdeg}(M) - \text{deg}(M)) - \text{beg}(M) + d. \]
This inequality trivially holds if $K^{d-1} = 0$. On the other hand, by the induction hypothesis
\[ \text{reg}(K^{d-1}(M/xM)) \leq (d - 1)(\text{hdeg}(M/xM) - \text{deg}(M/xM)) - \text{beg}(M/xM) + d - 1 \]
\[ \leq d(\text{hdeg}(M/xM) - \text{deg}(M)) - \text{beg}(M) + d - 1. \]
We now distinguish two cases:

- Assume depth $M > 0$. By [19], Theorem 2.13, we have
  \[ \text{hdeg}(M/xM) \leq \text{hdeg}(M). \]
  Hence
  \[ \text{reg}(K^{d-1}(M/xM)) + 1 \leq d(\text{hdeg}(M) - \text{deg}(M)) - \text{beg}(M) + d. \]
  Summing up we obtain
  \[ \text{reg}(K^d) \leq d(\text{hdeg}(M) - \text{deg}(M)) - \text{beg}(M) + d. \]

- We now consider the case depth$(M) = 0$. Since $\text{hdeg}(M) \leq \text{hdeg}(M)$, $\text{deg}(M)$ and $\text{beg}(M) \geq \text{beg}(M)$, by (7) we get
  \[ \text{reg}(K^d) = \text{reg}(K^d(M)) \leq d(\text{hdeg}(M) - \text{deg}(M)) - \text{beg}(M) + d \]
  \[ \leq d(\text{hdeg}(M) - \text{deg}(M)) - \text{beg}(M) + d. \]

The proof of Theorem 9 is completed. \qed

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Remark 13 Inspired by the homological degree Vasconcelos also introduced a class of functions, called extended degree $\text{Deg}(M)$ (see [19] and [20], p. 263). This class contains $\text{hdeg}(M)$. In fact, Theorem 3.1 in [14] (see also [6], Theorem 2.4) states that

$$\text{reg}(M) \leq \text{gen}(M) + \text{Deg}(M) - 1.$$ 

It is interesting to ask whether one can replace $\text{hdeg}(M)$ in Theorem 9 by $\text{Deg}(M)$. Our method is not applicable in this case, because the definition of an arbitrary extended degree does not explicitly contain the information on $K_i^i(M)$.

3 Castelnuovo-Mumford of a ring and its deficiency modules

In this section we will consider a quotient ring $S = R/I$, and give a bound for $\text{reg}(K^i(S))$, $i \leq d$, in terms of $\text{reg}(S)$. We always assume that $I$ is a non-zero homogeneous ideal containing no linear form. Note that it is unclear whether one can bound $\text{hdeg}(S)$ in terms of $\text{reg}(S)$. Therefore the following bound is independent from that of Theorem 9.

**Theorem 14** Let $S = R/I$ be a quotient ring of a polynomial ring $R = k[x_1, ..., x_n]$ ($n \geq 2$) modulo a homogeneous ideal $I \subseteq R$ as above. Then

$$\text{reg}(K^i(S)) < \begin{cases} 4(\text{reg } I)^{n-1} - 4(\text{reg } I)^{n-2} & \text{if } i = 1, \\ (2 \text{reg } I)^{n-(n+i-1)} & \text{if } i \geq 2. \end{cases}$$

In order to prove this theorem we need a result of M. Brodmann, C. Matteotti and N. D. Minh [3]. Following the notation there, we set

$$h^i_S(t) = \ell(H^i_m(S)_t) = H^i_{K^i(S)}(-t),$$

$$d^0_S(t) = H_S(t) - h^0_S(t) + h^1_S(t),$$

$$d^i_S(t) = h^{i+1}_S(t), \quad i \geq 1.$$ 

Since $K^i(S)$ is a finitely generated $R$-module, there is a polynomial $q^i_S(t)$ such that

$$d^i_S(t) = q^i_S(t) \quad \text{for } t \ll 0.$$ 

For $i \geq 0$, let

$$\Delta_i = \sum_{j=0}^{i} \binom{i}{j} (d^j_S(-j) + |q^j_S(-j)|).$$

Then Proposition 3.22 (c) of [3] can be reformulated as follows.
Lemma 15 For all $i \geq 1$ we have

$$ri(K^i(S)) \leq [2(1 + \Delta_{i-1})]^{2^{i-1}} - 2.$$  

PROOF. Set

$$\nu_S^i = \inf\{t \in \mathbb{Z} | d_S^i(t) \neq q_S^i(t)\}.$$  

Proposition 3.22 (c) of [3] states that

$$\nu_S^i \geq -(2(1 + \Delta_i))^{2^i} + 2,$$

for all $i \geq 0$. Since $ri(K^i(S)) = -\nu_S^i$ for $i \geq 2$, the assertion holds for $i \geq 2$. From the definition of $d_S^i(t)$ we also have

$$ri(K^1(S)) \leq \max\{0, -\nu_S^0\} \leq 2\Delta_0.$$

□

Lemma 16 For $0 \leq i < d = \dim S$ and all $t \in \mathbb{Z}$ we have

$$h_S^i(t) < (\text{reg}(I))^{n-i-1}\binom{\text{reg}(S) - t}{i}.$$  

PROOF. This follows from [11], Theorem 3.4 and Remark 3.5. □

PROOF OF THEOREM 14 We divide the proof of Theorem 14 into proving several claims. In the proof we simply write

$$K^i := K^i(S) \text{ and } r := \text{reg}(I) = \text{reg}(S) + 1.$$  

CLAIM 1. Let

$$\overline{K^{i+1}} = K^{i+1}/H^0_m(K^{i+1}),$$

and let $\alpha$ be an integer such that

$$\alpha \geq \max\{\text{reg}(K^i), \text{reg}(K^i(S/xS))\},$$

where $x$ is a generic linear element. Then

$$\text{reg}(\overline{K^{i+1}}) \leq \text{reg}(K^{i+1}/xK^{i+1}) \leq \alpha + 2.$$  

PROOF. Since $\text{reg}(\overline{K^{i+1}}) = \text{reg}_1(K^{i+1})$, the first inequality holds by Lemma 4. For the second inequality, by applying Lemma 3 to the exact sequence of
Lemma 10

\[ 0 \to (K^{i+1}/xK^{i+1})(1) \to K^i(S/xS) \to 0 : K^i, x \to 0, \]

we get

\[
\text{reg}(K^{i+1}/xK^{i+1}) \leq 1 + \max\{\text{reg}(K^i(S/xS)), \text{reg}(0 : K^i, x) + 1\} \\
\leq 1 + \max\{\text{reg}(K^i(S/xS)), \text{end}(H^0_m(K^i)) + 1\} \\
\leq 1 + \max\{\text{reg}(K^i(S/xS)), \text{reg}(K^i) + 1\} \\
\leq \alpha + 2.
\]

□

CLAIM 2. \(\text{reg}(K^1) < 4(\text{reg } I)^{n-1} - 4(\text{reg } I)^{n-2}.\)

PROOF. Let \(d = 1.\) By (7), \(\text{reg}(K^1) = \text{reg}(K^1(S)).\) Since \(S\) is a Cohen-Macaulay ring, by (4) we have \(\text{reg}(K^1) = 1.\) Since \(I\) is a non-zero ideal and contains no linear form, \(\text{reg } I \geq 2.\) Hence the claim obviously holds in this case.

Let \(d \geq 2.\) Since \(H^0_m(S)\) is a submodule of finite length of \(S,\) \(\text{reg}(K^0) < 0\) for all \(S.\) Let \(x\) be a generic linear element. By Claim 3 we get

\[
\text{reg}(K^1) \leq \text{reg}(K^1/xK^1) \leq 2 + \max\{\text{reg}(K^0(S/xS)), \text{reg}(K^0)\} \leq 1. \quad (10)
\]

Now we estimate \(\Delta_0.\) By Lemma 16

\[
d_S^0(0) = H_S(0) - h_S^0(0) + h_S^1(0) = 1 + h_S^1(0) \\
< 1 + r^{n-2} \text{reg}(S) = 1 + r^{n-1} - r^{n-2}.
\]

Hence

\[
d_S^0(0) \leq r^{n-1} - r^{n-2}.
\]

Note that

\[
q_S^0(-t) = P_{K^1}(t) = P_{\overline{K^1}}(t).
\]

Since \(\text{reg}(K^1) \leq 1,\)

\[
h_{\overline{K^1}}^1(1) = 0.
\]

As \(\dim \overline{K^1} \leq 1,\) applying the Grothendieck-Serre formula (see 2) to \(\overline{K^1}\) we get

\[
\overline{P_{K^1}}(1) = \overline{H_{K^1}}(1).
\]
Moreover, in this case $P_{K^r}(t)$ is a constant. By Lemma 16 this gives

$$q^0_S(0) = P_{K^r}(0) = P_{K^r}(1) = H_{K^r}(1) \leq H_{K^1}(1) = h^1_S(-1) < r^{n-2} \text{reg}(S) = r^{n-1} - r^{n-2}.$$ 

Putting all together we get

$$\Delta_0 = d^0_S(0) + |q^0_S(0)| < 2r^{n-1} - 2r^{n-2}. \quad (11)$$

From (10) and Lemma 15 we can now conclude by Lemma 6 (ii) that

$$\text{reg}(K^1) \leq \max\{1, 2\Delta_0\} < 4r^{n-1} - 4r^{n-2}. \quad \square$$

**CLAIM 3.** For $i \geq 1$ we have

$$\Delta_i < i\Delta_{i-1} + (\text{reg } I)^{n-1} + |P_{K^{i+1}}(-i)|.$$

**PROOF.** By (9) we have

$$\Delta_i = \sum_{j=0}^{i-1} \binom{i}{j} (d^j_S(-j) + |q^j_S(-j)|)$$

$$= \sum_{j=0}^{i-1} \binom{i}{j} (d^j_S(-j) + |q^j_S(-j)|) + h^{i+1}_S(-i) + |P_{K^{i+1}}(-i)|$$

$$\leq i\Delta_{i-1} + h^{i+1}_S(-i) + |P_{K^{i+1}}(-i)|.$$

By Lemma 16 we know that

$$h^{i+1}_S(-i) \leq r^{n-i-2} \binom{r + i - 1}{i + 1} < r^{n-i-2}r^{i+1} = r^{n-1}.$$

Since $P_{K^{i+1}}(t) = P_{K^{i+1}}(t)$, the claim follows. \quad \square

**CLAIM 4.** Keep the notation and assumptions of Claim 1 with the additional assumption that $\alpha \geq 0$. For all $1 \leq i < d - 1$,

$$|P_{K^{i+1}}(-i)| < \frac{1}{2} (\text{reg } I)^{n-i-2} (\text{reg }(I) + \alpha + 2i + 1)^{2i+2}.$$
PROOF. By Claim 3, \( \text{reg}(K^{i+1}) \leq \alpha + 2 \). Since \( \text{depth}(K^{i+1}) > 0 \) (if \( K^{i+1} \neq 0 \)), this implies that

\[
P_{K^{i+1}}(t) = H_{K^{i+1}}(t) \text{ for all } t \geq \alpha + 2.
\]

In the case \( P_{K^{i+1}}(t) = 0 \) there is nothing to prove. Assume that \( \text{deg}(P_{K^{i+1}}(t)) = p \geq 0 \). Using the Lagrange’s interpolation formula

\[
P_{K^{i+1}}(t) = \sum_{j=0}^{p} \frac{[t - (\alpha + 2)] \cdots [t - (\alpha + 2 + j)] \cdots [t - (\alpha + 2 + p)]}{(j - 0)(j - 1) \cdots (j - j) \cdots (j - p)} \times P_{K^{i+1}}(\alpha + 2 + j),
\]

where \( \ast \) means that the corresponding term is omitted, we get

\[
P_{K^{i+1}}(-i) = \sum_{j=0}^{p} (-1)^j \frac{(i + \alpha + 2) \cdots (i + \alpha + 2 + j) \cdots (i + \alpha + 2 + p)}{|(j - 0)(j - 1) \cdots (j - j) \cdots (j - p)|} \times H_{K^{i+1}}(\alpha + 2 + j).
\] (12)

Since \( \dim K^{i+1} \leq i + 1, p \leq i + 1 \). By Lemma 16 one has

\[
H_{K^{i+1}}(\alpha + 2 + j) \leq H_{K^{i+1}}(\alpha + 2 + j) = h_{S}^{i+1}(\alpha + 2 + j) \\
\leq \frac{r^{n-i-2}(r - 1 + \alpha + 2 + j)}{i + 1} \\
< \frac{r^{n-i-2}(r + \alpha + 1 + p)^{i+1}}{(i + 1)!} \\
\leq \frac{1}{2} r^{n-i-2}(r + \alpha + i + 2)^{i+1}.
\]

for all \( j \leq p \). Obviously

\[
(i + \alpha + 2) \cdots (i + \alpha + 2 + j) \cdots (i + \alpha + 2 + p) \leq (\alpha + i + 2 + p)^p \leq (\alpha + 2i + 3)^{i+1}.
\]

Since \( r \geq 2 \), the above estimations imply that all numerators in (12) are strictly less than

\[
A := \frac{1}{2} r^{n-i-2}(r + \alpha + 2i + 1)^{2i+2}.
\]

All the denominators in the alternating sum (12) are bigger or equal to \( (\lfloor \frac{p}{2} \rfloor)^2 \). There are at most \( \lfloor \frac{p}{2} \rfloor + 1 \) terms with the same sign. This implies that the subsum of all terms with the same sign in (12) has the absolute value less than \( A \) if \( p \geq 4 \). The same holds for \( p \leq 3 \) by a direct checking. Hence \( |P_{K^{i+1}}(-i)| < A \).
CLAIM 5. Assume that \( d \geq 3 \). Then

\[
\Delta_1 < \frac{1}{2} (2 \text{reg}(I))^{n(n+1)} - (\text{reg} I)_n - n,
\]

and

\[
\text{reg}(K^2) < (2 \text{reg}(I))^{2n(n+1)} - 2(\text{reg} I)_n - 2n.
\]

PROOF. By Claim 2

\[
\text{reg}(K^1) < 4r^{n-1} - 4r^{n-2} =: \alpha.
\]

Let \( x \) be a generic linear element. By Lemma \( \text{iii} \) \( \text{reg}(S/xS) \leq \text{reg}(S) = r-1 \). Again by Claim 2 this yields

\[
\text{reg}(K^1(S/xS)) < 4(\text{reg}(S/xS))^{n-1} - 4(\text{reg}(S/xS))^{n-2} \leq 4r^{n-1} - 4r^{n-2} = \alpha.
\]

Hence we can apply Claim 4 with \( i = 1 \) and \( \alpha > 0 \) to get

\[
|P_{K^2}(-1)| < \frac{1}{2} r^{n-3}(r + 4r^{n-1} - 4r^{n-2} + 3)^4.
\]

By Claim 3 and (11) we obtain

\[
\Delta_1 < \Delta_0 + r^{n-1} + \frac{1}{2} r^{n-3}(r + 4r^{n-1} - 4r^{n-2} + 3)^4
\]

\[
< 3r^{n-1} - 2r^{n-2} + \frac{1}{2} r^{n-3}(4r^{n-1} - 4r)^4 \quad \text{ (since } n \geq 4, r \geq 2) \]

\[
< 3r^{n-1} - 2r^{n-2} + \frac{1}{2} r^{n-1}4^4(r^{4(n-2)} - 4) \]

\[
< \frac{1}{2} (2r)^{n(n+1)} - r^n - n \quad \text{ (since } n \geq 4).\]

Thus the first inequality is proven.

Furthermore, by the inequalities at the beginning of the proof, we can use Claim 1 to get

\[
\text{reg}(K^2/xK^2) \leq \alpha + 2 = 4r^{n-1} - 4r^{n-2} + 2.
\]

Hence, by Lemma \( \text{I.5} \) and Lemma \( \text{iii} \) (ii), this implies

\[
\text{reg}(K^2) \leq \max\{\text{reg}(K^2/xK^2), [2(1 + \Delta_1)]^2 - 2\}
\]

\[
\leq \max\{4r^{n-1} - 4r^{n-2} + 2, [2(1 + \frac{1}{2}(2r)^{n(n+1)} - r^n - n)]^2 - 2\}
\]

\[
\leq (2r)^{2n(n+1)} - 2r^n - 2n.
\]

This is the second inequality of the claim. \( \square \)
CLAIM 6. Assume that $1 \leq i < d - 1$. Then
\[
\Delta_i < \frac{1}{2} \left( \text{reg}(I) \right)^{n-(n+i)2^i} - \text{reg}(I)^n - n,
\]
and
\[
\text{reg}(K^{i+1}) < \left( \text{reg}(I) \right)^{n-(n+i)2^i} - 2(\text{reg}(I))^n - 2n.
\]

PROOF. We do induction on $i$. The case $i = 1$ is Claim 5. Let $i \geq 2$ and let $x$ be a generic linear element. By the induction hypothesis we have
\[
\text{reg}(K^i) < \beta - 2r^n - 2n,
\]
where
\[
\beta := (2r)^{n-(n+i-1)2^i}.
\]
Since $\text{reg}(S/xS) \leq \text{reg}(S) = r - 1$, the induction hypothesis also gives
\[
\text{reg}(S/xS) < [2(\text{reg}(S/xS) + 1)]^{n-(n+i-1)2^i} - 2(\text{reg}(S/xS + 1))^n - 2n
\]
\[
\leq (2r)^{n-(n+i-1)2^i} - 2r^n - 2n = \beta - 2r^n - 2n.
\]
Applying Claim 3, Claim 4 (with $\alpha := \beta - 2r^n - 2n$) and the induction hypothesis on $\Delta_{i-1}$, we get
\[
\Delta_i < i\Delta_{i-1} + (\text{reg}(I))^{n-1} + |P_{K^{i+1}}(-i)|
\]
\[
\leq i(\frac{1}{2}\beta - r^n - n) + r^{n-1} + \frac{1}{2}r^{n-i-2}(r + \beta - 2r^n - 2n + 2i + 1)^{2i+2}
\]
\[
< \frac{1}{2}\beta^{2i+3} - r^n - n
\]
\[
< \frac{1}{2}\beta^{n+i} - r^n - n \quad \text{(since $n \geq i + 3$)}
\]
\[
= \frac{1}{2}(2r)^{n-(n+i)2^i} - r^n - n.
\]

Furthermore, by recalling the inequalities at the beginning of the induction step, we can use Claim 1 to have
\[
\text{reg}(K^{i+1}/xK^{i+1}) \leq \beta - 2r^n - 2n + 2.
\]
By Lemma 15 and Lemma 6 (ii), this now implies that
\[
\text{reg}(K^{i+1}) \leq \max\{\text{reg}(K^{i+1}/xK^{i+1}), [2(1 + \Delta_i)]^{2^i} - 2\}
\]
\[
\leq \max\{\beta - 2r^n - 2n + 2, [2(1 + \frac{1}{2}\beta^{n+i} - r^n - n)]^{2^i} - 2\}
\]
\[
< \beta^{(n+i)2^i} - 2r^n - 2n
\]
\[
= (2r)^{n-(n+i)2^i} - 2r^n - 2n.
\]
Claim 6 is thus completely proven. □

The cases \( i = 1 \) and \( 2 \leq i \leq d-1 \) of Theorem 14 were proved in Claim 2 and Claim 6, respectively. To finish the proof of Theorem 14, we only have to show the following stronger bound

**CLAIM 7.** Let \( d \geq 2 \). Then

\[
\text{reg}(K^d) < (2 \text{reg}(I))^{n-(n+d-2)2^{(d-1)(d-2)}} - 2(\text{reg} I)^n - 2n + 2.
\]

**PROOF.** Let

\[
\beta(d) = (2r)^{n-(n+d-2)2^{(d-1)(d-2)}} - 2r^n - 2n.
\]

We will prove by induction on \( d \) that \( \text{reg}(K^d) < \beta(d) + 2 \).

Let \( d = 2 \). This case was considered in [9], Theorem 2.9 and the bound there is much smaller. For the convenience of the reader we give here a direct proof of the weaker bound: \( \text{reg}(K^2) < \beta(2) + 2 \). Let \( x \) be, as usual, a generic linear element. Since \( \text{reg}(S/xS) \leq \text{reg}(S) = r - 1 \) by Claim 2, both \( \text{reg}(K^1) \) and \( \text{reg}(K^1(S/xS)) \) are less than \( 4r^{n-1} - 4r^{n-2} \). Since \( n \geq 3 \),

\[
4r^{n-1} - 4r^{n-2} < \beta(2) = (2r)^n - 2r^n - 2n.
\]

By (8) we then get \( \text{reg}(K^2) < \beta(2) + 2 \).

Let \( d \geq 3 \). By Claim 6,

\[ \text{reg}(K^{d-1}) < \beta(d). \]

Since \( \dim S/xS = d - 1 \), by the induction hypothesis the following holds

\[
\text{reg}(K^{d-1}(S/xS)) < [2(\text{reg}(S/xS) + 1)]^{n-(n+d-2)2^{(d-1)(d-3)}} - 2(\text{reg}(S/xS) + 1)^n - 2n + 2 < (2r)^{n-(n+d-2)2^{(d-1)(d-2)}} - 2r^n - 2n = \beta(d).
\]

Hence, again by (8), we get \( \text{reg}(K^d) < \beta(d) + 2 \), as required. □

**Remark 17** Assume that \( S \) is a generalized Cohen-Macaulay ring, i.e. all modules \( K^i(S), \ i < d \), are of finite length. In this case \( q_S(t) = 0 \) for all \( i \leq d - 2 \), and the proof of Theorem 14 will be substantially simplified. It gives

\[
\text{reg}(K^i(S)) < [2^{i+1}((\text{reg} I)^{n-1} - (\text{reg} I)^{n-2})]^{2^{i-1}}.
\]
It is still a huge number. We don’t know whether one can give a linear bound even in this case.

The bounds in Theorem 14 are huge. However, this theorem demonstrates that the Castelnuovo-Mumford regularity $\text{reg}(S)$ also controls the behavior of local cohomology modules in negative components. To understand better this phenomenon, let us state some consequences. The first corollary is formulated in the spirit of [3], Theorem 4.8.

**Corollary 18** Denote by $\mathcal{H}_{n,i,r}$ the set of numerical functions $h : \mathbb{Z} \to \mathbb{Z}$ such that there exists a homogeneous ideal $I \subset R = k[x_1, \ldots, x_n]$ satisfying the following conditions

(i) $\text{reg} I \leq r$,
(ii) $\ell(H^i_{m}(R/I)_t) = h(t)$ for all $t \in \mathbb{Z}$.

Then for fixed numbers $n, i, r$ the set $\mathcal{H}_{n,i,r}$ has only finitely many elements.

**Proof.** Note that $h(t) = 0$ for all $t \geq r$. By Theorem 14, $\text{reg}(K^i(S))$ is bounded by a number $f(n, r)$ depending on $n$ and $r$. By Lemma 16 for each $t$ with $-(f(n, r) + n) \leq t \leq r$, the value $h(t) = \ell(H^i_{m}(R/I)_t)$ is also bounded by a function $g(n, r)$. This implies that there are only finitely many choices of the initial values of $h(t)$. Since $P_K^i(t) = \ell(H^i_{m}(R/I)_{-t})$ in $n$ points $t = f(n, r) + 1, \ldots, f(n, r) + n$, and the degree of $P_K^i(t)$ is less than $n$, the number of possible polynomials $P_K^i(t)$ is finite. Moreover $h(t) = P_K^i(-t)$ for all $t < -f(n, r)$. These statements together imply the finiteness of the set $\mathcal{H}_{n,i,r}$. □

Assume that $k$ is an algebraically closed field. A famous result of Kleiman states that there exists only a finite number of Hilbert functions associated to reduced and equi-dimensional $k$-algebras $S$ such that $\text{deg}(S) \leq e$ and $\text{dim}(S) = d$. In a recent paper [11] the first author was able to extend this result to all reduced algebras. Recall that

$$\text{adeg} S = \sum_{p \in \text{Ass}(S)} \ell(H^0_{m_p}(S_p))e(S/p)$$

is called arithmetic degree of $S$ (see [1], Definition 3.4 or [20], Definition 9.1.3). The arithmetic degree agrees with $\text{deg}(S)$ if and only if $S$ is equi-dimensional. Inspired by Kleiman’s result we formulate the following two corollaries.

**Corollary 19** Denote by $\mathcal{H}_{d,i,a}$ the set of all numerical functions $h : \mathbb{Z} \to \mathbb{Z}$ for which there exists a reduced $k$-algebra $S$ such that $\text{adeg}(S) \leq a$, $\text{dim}(S) = d$ and $\ell(H^i_{m}(S)_t) = h(t)$ for all $t \in \mathbb{Z}$. Assume that $k$ is an algebraically closed field. Then for fixed numbers $d, i, a$ the set $\mathcal{H}_{d,i,a}$ is finite.
**Corollary 20** Denote by $H'_{n,i,\delta}$ the set of all numerical functions $h : \mathbb{Z} \to \mathbb{Z}$ such that there exists a homogeneous ideal $I \subset R = k[x_1, ..., x_n]$ satisfying the following conditions

(i) $I$ is generated by forms of degrees at most $\delta$,
(ii) $\ell(H^i_m(R/I)_t) = h(t)$ for all $t \in \mathbb{Z}$.

Then for fixed numbers $n, i, \delta$ the set $H'_{n,i,\delta}$ has only finitely many elements.

**PROOF.** Under the assumption, by [11], Theorem 2.1 (see also [1], Proposition 3.8), $\text{reg}(S)$ is bounded by $f(n, \delta)$. Hence the assertion follows from Corollary 18. □

4 Examples

We believe that there should be a much better bound for $\text{reg}(K^i(S))$ in terms of $\text{reg}(S)$ than the one given in the previous section. In this section we show this for some particular cases.

1. If $S = R/I$ is the coordinate ring of a smooth projective variety over a field of characteristic zero, then M. Chardin and B. Ulrich ([5], Theorem 1.3) showed that

\[ \text{reg}(K^d(S)) = d. \]

This is a consequence of Kodaira’s vanishing theorem.

2. Assume that $M$ is a generalized Cohen-Macaulay module, i.e. all modules $K^i(M)$, $i < d$, are of finite length. In the general case there is no known good bound for $\text{reg}(K^i(M))$ (see Remark 17). However, there is a good one in terms of the annihilators of $K^i(M)$ (see [13], Proposition 2.4 and Corollary 2.5). We recall here a nice case.

A module $M$ is called Buchsbaum module if the difference $\ell(M/qM) - e(q, M)$ between the length and the multiplicity is a constant, when $q$ runs over all homogeneous parameter ideals of $M$. In this case $mK^i(M) = 0$ for all $i < d$. Proposition 2.4 (i) in [13] states that if $M$ is a Buchsbaum module, then

\[ \text{reg}(K^i(M)) \leq i - \text{beg}(M), \quad i \leq d. \]
3. Assume that $I$ is a monomial ideal and that $S = R/I$ is a generalized Cohen-Macaulay ring. Then by [18], Proposition 1 we have

$$\text{reg}(K^i(S)) = \text{end}(K^i(S)) \leq 0 \text{ for } i < d.$$  \hfill (14)\]

We also get

**Proposition 21** Assume that $I$ is a monomial ideal and that $S = R/I$ is a generalized Cohen-Macaulay ring. Then

$$\text{reg}(K^d(S)) \leq d.$$

**PROOF.** Since $S$ is a generalized Cohen-Macaulay ring, we have by [15], Corollary 3.1.3 the following isomorphisms

$$H^{d+1-i}_m(K^d) \cong K^i,$$

for all $2 \leq i < d$, and there is an exact sequence of graded modules

$$0 \to K^1 \to H^d_m(K^d) \to \text{Hom}(S,k) \to K^0 \to 0.$$

We also have $\text{depth}(K^d) \geq \min\{2, \dim(S)\}$. Combining this with (14) implies the assertion.

4. In some cases when $M$ is not necessarily a generalized Cohen-Macaulay module, good bounds can still be found for $\text{reg}(K^d(M))$ (see [9]). In order to extend these results to all $\text{reg}(K^i(M))$, let us recall some definitions.

For an integer $0 \leq i \leq d$, let $M^i$ denote the largest graded submodule of $M$ such that $\dim M^i \leq i$. Let $M^{-1} = 0$. The increasing filtration

$$0 = M^{-1} \subset M^0 \subset \cdots \subset M^d = M$$

is called the dimension filtration of $M$. This filtration is well-defined and unique. We put

$$\mathcal{M}^i = M^i/M^{i-1} \text{ for all } 0 \leq i \leq d.$$

Note that $\mathcal{M}^i$ is either zero or of dimension $i$. A module $M$ is called a sequentially Cohen-Macaulay (sequentially Buchsbaum) module if each module $\mathcal{M}^i$ is either zero or a Cohen-Macaulay (Buchsbaum, respectively). The notion of a sequentially Cohen-Macaulay module was introduced by R. Stanley (see, e.g., [10]).

**Proposition 22** (i) If $M$ is a sequentially Cohen-Macaulay module, then for all $i \leq d$ we have

$$\text{reg}(K^i(M)) \leq i - \text{beg}(M).$$
(ii) If $M$ is a sequentially Buchsbaum module, then for all $i \leq d$ we have

$$\text{reg}(K^i(M)) \leq i + 1 - \text{beg}(M).$$

**Proof.** (i) Under the assumption, $K^i(M) \cong K^i(M^i)$ by [16], Lemma 5.2. If $M^i = 0$, then there is nothing to prove. Otherwise, $M^i$ is a Cohen-Macaulay module of dimension $i$. By (4) we have

$$\text{reg}(K^i(M)) \leq i - \text{beg}(M).$$

(ii) We do induction on $d$. If $d = 1$ then $M$ is a sequentially Cohen-Macaulay module. Hence the assertion holds true by (i). Let $d \geq 2$. By Lemma [11] we may assume that $i > 0$. The case $i = d$ is [9], Proposition 2.2. Let $1 \leq i < d$. The exact sequence

$$0 \rightarrow M^{d-1} \rightarrow M \rightarrow M^d \rightarrow 0$$

gives the long exact sequence of cohomology

$$K^i(M^d) \xrightarrow{\varphi} K^i(M) \xrightarrow{\psi} K^i(M^{d-1}) \xrightarrow{\chi} K^{i-1}(M^d).$$

This breaks up into two short exact sequences

$$0 \rightarrow \text{Im} \varphi \rightarrow K^i(M) \rightarrow \text{Im} \psi \rightarrow 0,$$

$$0 \rightarrow \text{Im} \psi \rightarrow K^i(M^{d-1}) \rightarrow \text{Im} \chi \rightarrow 0.$$

Note that $\dim(M^d) = d$, and that by the assumption $M^d$ is a Buchsbaum module. Hence $K^i(M^d)$ and $K^{i-1}(M^d)$ are modules of finite length. Since $\text{beg}(M^d) \geq \text{beg}(M)$, we have by [13]

$$\text{reg}(\text{Im} \varphi) \leq \text{reg}(K^i(M^d)) \leq i - \text{beg}(M^d) \leq i - \text{beg}(M),$$

and

$$\text{reg}(\text{Im} \chi) \leq \text{reg}(K^{i-1}(M^d)) \leq i - 1 - \text{beg}(M).$$

Using Lemma [3] and the above two short exact sequences we obtain

$$\text{reg}(K^i(M)) \leq \max\{\text{reg}(\text{Im} \varphi), \text{reg}(\text{Im} \psi)\} \leq \max\{i - \text{beg}(M), \text{reg}(K^i(M^{d-1})), \text{reg}(\text{Im} \chi) + 1\} = \max\{i - \text{beg}(M), \text{reg}(K^i(M^{d-1}))\}.$$

Since $M^{d-1}$ is also a sequentially Buchsbaum module (of dimension at most $d - 1$), we know by the induction hypothesis that

$$\text{reg}(K^i(M^{d-1})) \leq i - \text{beg}(M^{d-1}) \leq i - \text{beg}(M).$$
Consequently, \( \text{reg}(K^i(M)) \leq i - \text{beg}(M) \). \( \square \)

Let \( \text{gin}(I) \) denote the generic ideal of \( I \) with respect to a term order. It is a so-called Borel-fixed ideal, and by [10], Theorem 2.2, \( \frac{R}{\text{gin}(I)} \) is a sequentially Cohen-Macaulay ring. Hence we get:

**Corollary 23** For an arbitrary homogeneous ideal \( I \subset R \) we have

\[
\text{reg}(K^i(\frac{R}{\text{gin}(I)})) \leq i.
\]

Unfortunately we cannot use this result to bound \( \text{reg}(K^i(R/I)) \). The reason is the following. We always have

\[
\ell(K^i(\frac{R}{\text{gin}(I)}), j) \geq \ell(K^i(\frac{R}{I}), j) \quad \text{for all } j \in \mathbb{Z}.
\]

It is well-known that many invariants increase by passing from \( I \) to \( \text{gin}(I) \), but remain unchanged if one takes the generic initial ideal \( \text{Gin}(I) \) with respect to the reverse lexicographic order. So, if the equality

\[
\ell(K^i(\frac{R}{\text{Gin}(I)}), j) = \ell(K^i(\frac{R}{I}), j)
\]

would hold for all \( j \in \mathbb{Z} \), then using Corollary 23 one would get a good bound for \( ri(K^i(R/I)) \). From that, by the method of Section 3 one would get a good bound for \( \text{reg}(K^i(R/I)) \). Unfortunately, this is almost impossible. Namely, J. Herzog and E. Sbarra ([10], Theorem 3.1) proved that

\[
\ell(K^i(\frac{R}{\text{Gin}(I)}), j) = \ell(K^i(\frac{R}{I}), j),
\]

for all \( i \leq d \) and all \( j \in \mathbb{Z} \) if and only if \( \frac{R}{I} \) is itself a sequentially Cohen-Macaulay ring. But the latter case was settled in Proposition 22 (i).

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