We work over a fixed algebraically closed field $k$ of characteristic zero.

1.1. Exceptional collections. Let $X$ be a smooth projective variety over the field $k$ and let $\mathbf{D}^b(X)$ be the bounded derived category of coherent sheaves on $X$. An object $E$ of $\mathbf{D}^b(X)$ is called exceptional, if we have

$$\operatorname{Hom}(E, E) = \text{id}_E \text{ and } \operatorname{Ext}^i(E, E) = 0 \text{ for all } i \neq 0.$$ 

A sequence of exceptional objects $E_1, \ldots, E_n$ is called an exceptional collection, if we have

$$\operatorname{Ext}^k(E_i, E_j) = 0 \text{ for } i > j \text{ and } \forall k.$$ 

We denote by $\langle E_1, \ldots, E_n \rangle$ the smallest full triangulated subcategory of $\mathbf{D}^b(X)$ containing the objects $E_1, \ldots, E_n$. If we have

$$\mathbf{D}^b(X) = \langle E_1, \ldots, E_n \rangle,$$

then the collection $E_1, \ldots, E_n$ is called full.

Example 1.1.

(1) On the projective space $\mathbb{P}^n$ we have the famous Beilinson collection [1]

$$\mathbf{D}^b(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle. \quad (1.1)$$

(2) For Grassmannians $\text{Gr}(k, n)$ and quadrics $Q_n$ full exceptional collections were constructed by Kapranov [11]. In the case of $\text{Gr}(2, 4)$ Kapranov’s collection takes the form

$$\mathbf{D}^b(\text{Gr}(2, 4)) = \langle \mathcal{O}, \mathcal{U}, S^2 \mathcal{U}, \mathcal{O}(1), \mathcal{U}(1), \mathcal{O}(2) \rangle, \quad (1.2)$$

where $\mathcal{U}$ is the tautological subbundle on $\text{Gr}(2, 4)$.

1.2. Lefschetz exceptional collections. In this talk we are interested in a particular class of exceptional collections, called Lefschetz collections introduced by Alexander Kuznetsov in [13] in the context of homological projective duality. The main goal of the talk is to explain that Lefschetz collections also have close connections with quantum cohomology and mirror symmetry.

Definition 1.2. Let $X$ be a smooth projective variety over $k$ and let $\mathcal{O}(1)$ be a line bundle on $X$. For an object $F \in \mathbf{D}^b(X)$ we denote $F(1) := F \otimes \mathcal{O}(1)$.
(i) A **Lefschetz collection** with respect to \( \mathcal{O}(1) \) is an exceptional collection, which has a block structure

\[
\begin{align*}
E_1, E_2, \ldots, E_{\sigma_0}; & \quad E_1(1), E_2(1), \ldots, E_{\sigma_1}(1); \ldots; E_1(m-1), E_2(m-1), \ldots, E_{\sigma_{m-1}}(m-1)
\end{align*}
\]

where \( \sigma = (\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{m-1} \geq 0) \) is a non-increasing sequence of non-negative integers called the **support partition** of the collection. In the above notation we use semicolons to separate the blocks. The block \((E_1, E_2, \ldots, E_{\sigma_0})\) is called the starting block. We use notation \((E_\bullet, \sigma)\) for a Lefschetz collection with support partition \(\sigma\).

(ii) If \(\sigma_0 = \sigma_1 = \cdots = \sigma_{m-1} \), then the Lefschetz collection is called **rectangular**. Otherwise, its rectangular part is defined to be the subcollection

\[
E_1, E_2, \ldots, E_{\sigma_{m-1}}; E_1(1), E_2(1), \ldots, E_{\sigma_{m-1}}(1); \ldots; E_1(m-1), E_2(m-1), \ldots, E_{\sigma_{m-1}}(m-1).
\]

(iii) The **residual category** of a Lefschetz collection is defined as the orthogonal of its rectangular part

\[
\mathcal{R} = \left< E_1, E_2, \ldots, E_{\sigma_{m-1}}; \ldots; E_1(m-1), E_2(m-1), \ldots, E_{\sigma_{m-1}}(m-1) \right>^\perp.
\]

**Remark 1.3.**

1. The residual category \(\mathcal{R}\) of a Lefschetz collection vanishes if and only if the Lefschetz collection is full and rectangular.
2. The definitions imply that we have a semiorthogonal decomposition

\[
D^b(X) = \langle \mathcal{R}; E_1, E_2, \ldots, E_{\sigma_{m-1}}; \ldots; E_1(m-1), E_2(m-1), \ldots, E_{\sigma_{m-1}}(m-1) \rangle.
\]

3. An important feature of the residual category is the existence of a natural autoequivalence \(\tau_\mathcal{R}: \mathcal{R} \to \mathcal{R}\) called the **induced polarization** such that \(\tau_\mathcal{R}^m \cong S^{-1}_\mathcal{R}[\text{dim} X]\), where \(S_\mathcal{R}\) is the Serre functor of \(\mathcal{R}\). It is useful to think of \(\tau_\mathcal{R}\) as the analogue of the twist by \(\mathcal{O}(1)\) in \(D^b(X)\). For more details on \(\tau_\mathcal{R}\) we refer to [15, 16].

**Example 1.4.**

1. Collection (1.1) is a Lefschetz collection with

\[
E_\bullet = (\mathcal{O}) \quad \text{and} \quad \sigma = (1^n) := (1, \ldots, 1)_n.
\]

2. Collection (1.2) is a Lefschetz collection with

\[
E_\bullet = (\mathcal{O}, \mathcal{U}^\vee, \mathcal{S}_2^2 \mathcal{U}^\vee) \quad \text{and} \quad \sigma = (3, 2, 1).
\]

3. On \(\text{Gr}(2, 4)\) there exist a Lefschetz collection with a smaller starting block than (1.2). Namely, we have

\[
D^b(\text{Gr}(2, 4)) = \langle \mathcal{O}, \mathcal{U}^\vee; \mathcal{O}(1), \mathcal{U}^\vee(1); \mathcal{O}(2); \mathcal{O}(3) \rangle,
\]

For this collection we have

\[
E_\bullet = (\mathcal{O}, \mathcal{U}^\vee) \quad \text{and} \quad \sigma = (2, 2, 1, 1).
\]
1.3. Lefschetz collections and quantum cohomology. Let $X$ now be a Fano variety over $k$. Roughly speaking the main conjectures of [15,16] say that there is a deep relation between the small quantum cohomology of $X$ and the structure of Lefschetz collections and their residual categories. Let us now be more precise.

Let $X$ be a Fano variety with vanishing odd cohomology. We denote by $\text{QH}_\text{can}(X)$ its small quantum cohomology specialized at the canonical class. If the Picard rank of $X$ is one, and consequently there is only one deformation parameter $q$ in the small quantum cohomology, $\text{QH}_\text{can}(X)$ is the small quantum cohomology at $q = 1$; this is the case for almost all examples appearing below. Thus, $\text{QH}_\text{can}(X)$ is a finite dimensional commutative $\mathbb{C}$-algebra, whose underlying vector space is canonically isomorphic to $H^*(X, \mathbb{C})$.

Now we define the quantum spectrum of $X$ as

$$\text{QS}_X := \text{Spec}(\text{QH}_\text{can}(X)),$$

which is a finite scheme endowed with an action of the group $\mu_m$, where $m$ is the Fano index of $X$. The anticanonical class $-K_X$ defines a morphism

$$\kappa: \text{QS}_X \to \mathbb{A}^1,$$

which is equivariant with respect to the standard action of $\mu_m$ on $\mathbb{A}^1$. Finally, we define

$$\text{QS}_X^\circ := \kappa^{-1}(\mathbb{A}^1 \setminus \{0\}) \quad \text{and} \quad \text{QS}_X^\circ := \text{QS}_X \setminus \text{QS}_X^\circ.$$

The action of $\mu_m$ on $\text{QS}_X^\circ$ is free, as it is free on $\mathbb{A}^1 \setminus \{0\}$. We refer to [16, Introduction] for more details on the setup.

**Conjecture 1.5 ([16, Conjecture 1.3]).** Let $X$ be a Fano variety of index $m$ over an algebraically closed field $k$ of characteristic zero and assume that the big quantum cohomology $\text{BQH}(X)$ is generically semisimple.

1. There is an $\text{Aut}(X)$-invariant exceptional collection $E_1, \ldots, E_k$ in $\text{D}^b(X)$, where $k$ is the length of $\text{QS}_X^\circ$ divided by $m$. This collection extends to a rectangular Lefschetz collection

$$E_1, E_2, \ldots, E_k; E_1(1), E_2(1), \ldots, E_k(1); \ldots; E_1(m - 1), E_2(m - 1), \ldots, E_k(m - 1). \quad (1.5)$$

in $\text{D}^b(X)$.

2. The residual category $\mathcal{R}$ of (1.5) has a completely orthogonal $\text{Aut}(X)$-invariant decomposition

$$\mathcal{R} = \bigoplus_{\xi \in \text{QS}_X^\circ} \mathcal{R}_\xi$$

with components indexed by closed points $\xi \in \text{QS}_X^\circ$. Moreover, the component $\mathcal{R}_\xi$ of $\mathcal{R}$ is generated by an exceptional collection of length equal to the length of the localization $(\text{QS}_X^\circ)_\xi$ at $\xi$.

3. The induced polarization $\tau_\mathcal{R}$ permutes the components $\mathcal{R}_\xi$. More precisely, for each point $\xi \in \text{QS}_X^\circ$ it induces an equivalence

$$\tau_\mathcal{R}: \mathcal{R}_\xi \overset{\sim}{\longrightarrow} \mathcal{R}_{g(\xi)},$$

where $g$ is a generator of $\mu_m$. 


Thus, intuitively the points of $QS_X^X$ correspond to the rectangular part (1.5) and the twist by $\mathcal{O}(1)$ corresponds to the action of $\mu_m$ on $QS_X^X$; points of $QS_X^0$ give rise to an exceptional collection in $\mathcal{R}$.

Below we discuss two particular instances of the above conjecture. In the first case we assume that the small quantum cohomology, or rather $\text{QH}_{\text{can}}(X)$, is semisimple, and in the second case we consider a particular class of homogeneous varieties called coadjoint varieties, whose $\text{QH}_{\text{can}}(X)$ is almost never semisimple.

1.4. Cases with semisimple $\text{QH}_{\text{can}}(X)$. The simplest example, where Conjecture 1.5 holds, is provided by $\mathbb{P}^n$. Indeed, it is well-known that we have

$$\text{QH}_{\text{can}}(\mathbb{P}^n) = \mathbb{C}[h]/(h^{n+1} - 1),$$

and, therefore, the quantum spectrum $QS_{\mathbb{P}^n}$ is a reduced subscheme of $\mathbb{A}^1$ supported at the points $\zeta^i$ with $i \in [0, n]$, where $\zeta$ is a primitive $(n+1)$-st root of unity. The action of $\mu_{n+1}$ on $QS_{\mathbb{P}^n} = QS_{\mathbb{P}^n}^X$ is the usual action of $\mu_{n+1}$ on $(n+1)$-st roots of unity. Therefore, this action has only one orbit and, according to Conjecture 1.5, in $D^b(\mathbb{P}^n)$ we should expect to have a Lefschetz collection, whose starting block $E_0$ consists of one object and whose support partition is of the form $\sigma = (1^{n+1})$. Since $QS_{\mathbb{P}^n}^0 = \emptyset$, the residual category vanishes. The Beilinson collection (1.1) satisfies all these requirements.

More generally, if $\text{QH}_{\text{can}}(X)$ is semisimple, then Conjecture 1.5 gives a full description of the residual category. Indeed, since each component $\mathcal{R}_\xi$ is generated by one exceptional object Conjecture 1.5(ii) says that the residual category $\mathcal{R}$ is generated by a completely orthogonal exceptional collection (cf. [15, Conjecture 1.12]).

There is a number of cases with semisimple $\text{QH}_{\text{can}}(X)$, where our conjecture is known to hold:

1. for $\text{Gr}(k, n)$ with for $k = p$ a prime number (see [15]);
2. for quadrics $Q_n$ this follows from Kapranov’s work [11] (see [15, Example 1.6]);
3. for $\text{OG}(2, 2n + 1)$ this follows from Kuznetsov’s work [14] (see [15, Example 1.9]);
4. for $\text{IG}(3, 8)$ and $\text{IG}(3, 10)$ this holds by [10, 18];
5. for $\text{IG}(4, 8)$ and $\text{IG}(5, 10)$ this should follow from [6, 21];
6. for the Cayley plane $E_6/P_1$ this holds by [2, 5, 17];
7. for the Cayley Grassmannian this holds by [3, 9];
8. for the $G_2$-Grassmannian $G_2/P_2$ this holds by [12];
9. for some horospherical varieties of Picard rank one by [7, 8, 14, 19];

In the above list $\text{IG}(k, 2n)$ is the variety parametrizing $k$-dimensional isotropic subspaces in a $2n$-dimensional vector space with a symplectic form; this is a homogeneous space for the symplectic group $\text{Sp}_{2n}$. The variety $\text{OG}(2, 2n + 1)$ parametrizes 2-dimensional isotropic subspaces in a $(2n+1)$-dimensional vector space with a symmetric non-degenerate form; this is a homogeneous space for the special orthogonal group $\text{SO}_{2n+1}$. We refrain from recalling the definitions of all the other varieties in the above list.

1.5. Cases with non-semisimple $\text{QH}_{\text{can}}(X)$. If the algebra $\text{QH}_{\text{can}}(X)$ is not semisimple, then Conjecture 1.5 does not give a full description of the orthogonal components $\mathcal{R}_\xi$ of the residual category. However, under mirror symmetry, the locus $QS_X^0$ corresponds to the critical points of the mirror Landau–Ginzburg model $f$ of the Fano variety $X$ in the fiber over zero $f^{-1}(0)$. We expect that for each $\xi \in QS_X^0$ the component $\mathcal{R}_\xi$ is equivalent to the
Fukaya–Seidel category of the corresponding critical point in $f^{-1}(0)$. Below we illustrate this phenomenon.

**Symplectic isotropic Grassmannians** $IG(2, 2n)$. The variety $IG(2, 2n)$ parametrizes 2-dimensional isotropic subspaces in a $2n$-dimensional vector space with a symplectic form. It is embedded into the usual Grassmannian $IG(2, 2n) \subset Gr(2, 2n)$ and is, in fact, a hyperplane section of $Gr(2, 2n)$.

The Fano index of $IG(2, 2n)$ is equal to $2n - 1$ and $\dim \mathbb{C}(H^*(IG(2, 2n), \mathbb{C})) = 2n(n - 1)$. We know by [4, Proposition 4.3] that $QS^\times_{IG(2, 2n)}$ is a disjoint union of $n - 1$ orbits of $\mu_{2n-1}$, each of which consists of $2n - 1$ reduced points, and $QS^\circ_{IG(2, 2n)}$ consists of one non-reduced point $\xi_0$ with local algebra $\mathbb{C}[t]/t^{n-1}$.

Since $\mathbb{C}[t]/t^{n-1}$ is isomorphic to the Jacobi algebra of an isolated hypersurface singularity of type $A_{n-1}$, the above description of $QS^\circ_{IG(2, 2n)}$ suggests that in Conjecture 1.5 we should expect the residual category $\mathcal{R} = R_{\xi_0}$ to be equivalent to the Fukaya–Seidel category of an isolated hypersurface singularity of type $A_{n-1}$, which by [22] is equivalent to the derived category of representations of a quiver of type $A_{n-1}$.

A Lefschetz collection on $IG(2, 2n)$ satisfying these conjectures was constructed in [14]. Indeed, let us define

$$E_i = S^{i-1}U^\vee \quad \text{for} \quad 1 \leq i \leq n,$$

$$\sigma = (n^{n-1}, (n - 1)^n).$$

Then $(E^\bullet, \sigma)$ is a full Lefschetz collection by [14, Theorem 5.1]. Moreover, by [4, Theorem 9.6], its residual category is equivalent to the derived category of representations of $A_{n-1}$ quiver.

**Coadjoint varieties.** The variety $IG(2, 2n)$ considered above fits naturally into a series of examples. Namely, $IG(2, 2n)$ is the coadjoint variety in Dynkin type $C_n$. In general, the coadjoint variety of a simple algebraic group $G$ is the highest weight vector orbit in the projectivization of the irreducible $G$-representation, whose highest weight is the highest short root. Therefore, a coadjoint variety is uniquely determined by the Dynkin type of $G$, which can be $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. Coadjoint varieties are of Picard rank one except for type $A_n$, where the Picard rank is two.

The quantum spectrum of a coadjoint variety is described in [20]. To state the result we denote by $T(G)$ the Dynkin diagram of $G$ and by $T_{\text{short}}(G)$ the subdiagram of $T(G)$ consisting of vertices corresponding to short roots. Thus, we have the table

| $T$ | $A_n$ | $B_n$ | $C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $T_{\text{short}}$ | $A_n$ | $A_1$ | $A_{n-1}$ | $D_n$ | $E_6$ | $A_2$ | $A_1$ |

Now we are ready to describe the quantum spectrum of coadjoint varieties.

**Lemma 1.6** ([20]). Let $X$ be the coadjoint variety of a simple algebraic group $G$. Then

1. $QS^\times(X)$ consists of reduced points;
2. if $T(G) = A_n$ and $n$ is even, then $QS^\circ(X) = \emptyset$;
3. otherwise, $QS^\circ(X)$ has a unique point and the local algebra at this point is isomorphic to the Jacobi ring of an isolated hypersurface singularity of Dynkin type $T_{\text{short}}(G)$.

Similarly to the case of $IG(2, 2n)$ the above description suggests the following.
Conjecture 1.7 ([16, Conjecture 1.8]). Let $X$ be the coadjoint variety of a simple algebraic group $G$ over an algebraically closed field of characteristic zero. Then $D^b(X)$ has an $\text{Aut}(X)$-invariant rectangular Lefschetz exceptional collection with residual category $\mathcal{R}$ and

1. if $T(G) = A_n$ and $n$ is even, then $\mathcal{R} = 0$;
2. otherwise, $\mathcal{R}$ is equivalent to the derived category of representations of a quiver of Dynkin type $T_{\text{short}}(G)$.

This conjecture is by now known in all Dynkin types except for $E_6, E_7, E_8$. Indeed, this is [4, Theorem 9.6] for type $C_n$, [15, Example 1.6] for types $B_n$ and $G_2$, [16, Theorem 1.9] for types $A_n$ and $D_n$, [2, Theorem 1.4] for type $F_4$.

Remark 1.8. Above we have discussed only coadjoint varieties, but one can also define adjoint varieties. The adjoint variety of a simple algebraic group $G$ is the highest weight vector orbit in the projectivization of the irreducible $G$-representation, whose highest weight is the highest root. If $T(G)$ is simply laced, then all roots have the same length and, therefore, adjoint and coadjoint varieties coincide.

Let $X$ be the adjoint variety of a simple algebraic group $G$ whose Dynkin type is not simply laced, i.e. $B_n, C_n, F_4, G_2$. Then by [20, Theorem 9.1] we know that $QS^g(X) = \emptyset$ and by Conjecture 1.5 we should expect a rectangular Lefschetz collection in $D^b(X)$. This is by now known in all cases: [14, Theorem 7.1] for type $B_n$, [15, Example 1.4] for type $C_n$, [23, Theorem 1.1] for type $F_4$, and [12, §6.4] for type $G_2$.

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Universität Augsburg, Institut für Mathematik, Universitätstr. 14, 86159 Augsburg, Germany
Email address: maxim.smirnov@math.uni-augsburg.de
Email address: maxim.n.smirnov@gmail.com