Form Factors of Exponential Operators and Exact Wave Function Renormalization Constant in the Bullough–Dodd Model

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Abstract

We compute the form factors of exponential operators $e^{kg\varphi(x)}$ in the two–dimensional integrable Bullough–Dodd model ($a_2^{(1)}$ Affine Toda Field Theory). These form factors are selected among the solutions of general nonderivative scalar operators by their asymptotic cluster property. Through analytical continuation to complex values of the coupling constant these solutions permit to compute the form factors of scaling relevant primary fields in the lightest–breather sector of integrable $\phi_{1,2}$ and $\phi_{1,5}$ deformations of conformal minimal models. We also obtain the exact wave–function renormalization constant $Z(g)$ of the model and the properly normalized form factors of the operators $\varphi(x)$ and $:\varphi^2(x):$.

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1 Introduction

In recent years much important progress has been achieved in the study of two dimensional Quantum Field Theory and related Statistical Mechanical systems. The solution of Conformal Field Theories [1, 2] has not only allowed the full characterization of fixed points in the renormalization group describing universality classes of critical models, but it has also provided the possibility of describing the renormalization group flow away from criticality by means of relevant deformations of Conformal Minimal Models [3]. In particular, Zamolodchikov showed that in some interesting cases [4] — falling in the classes of $\phi_{1,2}$, $\phi_{1,3}$ and $\phi_{2,1}$ deformations — infinitely many integrals of motion survive the deformation and the system is suitably described by an integrable relativistic scattering theory. Bootstrap techniques relying on the integrability of the model provide then a powerful tool for obtaining the exact $S$–matrix of the system which turns out to be elastic and factorizable [5, 6]. A systematic description of the $S$-matrices for all the integrable deformed minimal conformal models has been given [7, 8, 9, 10] in terms of specific reductions of the two only existing two-dimensional single-boson integrable models, namely the sinh–Gordon and the Bullough–Dodd (BD) model (the Affine Toda Field Theories $a^{(1)}_1$ and $a^{(2)}_2$) in their complex coupling constant versions which are also referred to as the sine–Gordon model and the Zhiber–Mihaïlov–Shabat (ZMS) model respectively [11, 12, 13].

It is widely believed that the knowledge of the scattering data amounts to an exhaustive solution of a quantum field theory and that, in principle, one should be able to recover from them the operator content of the theory as well as to compute the correlation functions of local operators. In order to carry out this task, the so–called form factors approach has been developed and successfully employed in many important cases [14, 15, 16, 17, 18, 19, 20, 21, 22]. The strategy of this technique relies in the reconstruction of correlation functions by means of a spectral sum

$$\langle 0|\Phi_1(x)\Phi_2(0)|0\rangle = \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \ldots \frac{d\theta_n}{2\pi} F_n^{\Phi_1}(\theta_1, \ldots, \theta_n) \left[ F_n^{\Phi_2}(\theta_1, \ldots, \theta_n) \right]^* e^{-m|x|\sum_i \cosh \theta_i},$$

(1.1)
on all the intermediate $n$–particle states of a scattering theory involving on–shell amplitudes of local operators (the so–called form factors)

$$F_n^{\Phi}(\theta_1, \ldots, \theta_n) = \langle 0|\Phi(0)|A(\theta_1)\ldots A(\theta_n)\rangle.$$  

Form factors in turn can be exactly obtained in two dimensional bootstrap systems as solutions of a system of functional equations which entail the correct analiticity and monodromy properties dictated by the $S$–matrix. The space of solutions of the system is supposed to give a faithful representation of the operatorial content of the theory [23], but the correct identification of the form factors of a specific operator within this space of solutions is in general a nontrivial problem. A useful criterion for this identification was given in [13] where it was proved that for a scaling operator $\Phi$ of scaling dimensions $2\Delta_\Phi$ the form factors divergence for large values of the rapidities is bounded by

$$\lim_{|\theta_i| \to \infty} F_n^{\Phi}(\theta_1, \ldots, \theta_n) \lesssim e^{\Delta_\Phi |\theta_i|}.$$  

\footnote{For notational convenience we assume here the spectrum to consist of a single particle $A$ and parameterize the momenta in terms of the rapidity variable $\theta$.}
More recently, it has been observed in [24] that the form factors of relevant (\(\Delta_\Phi < 1\)) scaling operators satisfy a simple factorization property given by the so-called “cluster equations”

\[
\lim_{\Delta \to \infty} F_\Phi^n(\theta_1 + \Delta, \ldots, \theta_m + \Delta, \theta_{m+1}, \ldots, \theta_n) = \frac{1}{F_0^\Phi} F_\Phi^m(\theta_1, \ldots, \theta_m) F_\Phi^{n-m}(\theta_{m+1}, \ldots, \theta_n), \tag{1.2}
\]

(\(\forall m = 1, \ldots, n - 1\)), which hold unless some internal symmetry of the theory makes some of the form factors vanish. This property had already been noticed to be satisfied in the solutions of some specific models [7, 16, 25, 26] and is believed to be a distinguishing property of exponential operators in Lagrangian theories. In particular, in ref. [25] a family of cluster solutions was found in the sinh–Gordon model and further identified with the form factors of exponential operators. These solutions were then used to compute the form factors of primary operators [27] in a class of \(\phi_{1,3}\)-deformed minimal models in which the boson of the original Lagrangian theory is still present after reduction.

In the present paper we analyze the form factors of scalar operators in the Bullough–Dodd model and focus in particular on possible solutions of cluster equations (1.2). In view of the above discussion, these solutions become particularly interesting, not only as candidate solutions for the identification of exponential operators of the model, but also because in the complex coupling constant version (ZMS model) one should be able to identify among them the scaling relevant primary fields in the reductions which describe specific deformations of minimal models.

The paper will be organized as follows. In Section 2 we review some general features of the BD model and its interpretation as a Complex Liouville Theory that allows us to map the exponential operators into the scaling primary fields of the \(\phi_{1,2}\), \(\phi_{2,1}\) and \(\phi_{1,5}\) deformed minimal models. In Section 3 we analyze the general solution of form factors equations for non–derivative scalar operators of the BD model, exhibiting the first multiparticle form factors and giving a full characterization of the dimensionality of the space of solutions. In Section 4 we study a one parameter family of cluster solutions which is shown to correspond to the form factors of exponential operators \(e^{kg\varphi(x)}\) and we determine the exact formula, eq. (4.10), which gives the dependence on \(k\) and \(g\) of these solutions. In Section 5 we check the validity of the correct interpretation of cluster solutions by comparing our results with all the known cases of form factors of scaling primary operators computed in \(\phi_{1,2}\) and \(\phi_{1,5}\) deformations of minimal models. In Section 6 we make use of the knowledge of the form factors of exponential operators to exactly compute the wave function renormalization constant of the BD model and the form factors of the fields \(\varphi(x)\) and \(\varphi^2(x)\): correctly normalized. Finally, we draw our conclusions in Section 7.

2 The Bullough–Dodd model and its interpretation as a Complex Liouville Theory

The so–called Bullough–Dodd (BD) model [11] is a two–dimensional integrable Lagrangian QFT, namely the \(a_2^{(2)}\) Affine Toda Field Theory [13], defined by the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{m_0^2}{6g^2} \left(2 :e^{g\varphi}:+:e^{-2g\varphi}:\right), \tag{2.1}
\]
where the exponentials are normal ordered. This model is the only 2D integrable QFT involving a single bosonic field which exhibits the $\phi^3$ property (i.e. the elementary particle appears as a bound state of itself). The spectrum of the theory consists of a single bosonic massive particle $A$ of mass $m$ and the exact $S$–matrix of the model factorises into two–particle amplitudes given by the following function of the relative rapidity variable $\theta$ 

$$S(\theta) = f_{\frac{\phi}{3}}(\theta) f_{\frac{A}{3}}(\theta) f_{\frac{-\phi}{3}}(\theta),$$

where we have used the building block function

$$f_x(\theta) \equiv \frac{\tanh \frac{\theta + i\pi x}{2}}{\tanh \frac{\theta - i\pi x}{2}}.$$

The renormalized coupling constant $B$ is given by

$$B(g) = \frac{g^2/2\pi}{1+g^2/4\pi},$$

and ranges from 0 to 2 for real values of $g$. The $S$–matrix exhibits a weak–strong coupling duality under the transformation $g \leftrightarrow 4\pi/g$ or equivalently $B \leftrightarrow 2 - B$. For later use we also define the following duality–invariant function of the coupling constant

$$c = \cos \left( \frac{B + 2}{3} \pi \right).$$

The $S$–matrix has a simple pole at $\theta = 2\pi i/3$ corresponding to the bound state represented by the particle $A$ itself. The on–shell three–point coupling constant is given by

$$\Gamma^2 = -i \lim_{\theta \to 2\pi i/3} (\theta - \frac{2\pi i}{3}) S(\theta) = 2\sqrt{3} \frac{(c + 1)(1 + 2c)}{(c - 1)(1 - 2c)},$$

and vanishes both at the free field limiting values $B = 0, 2$ and at the self–dual point $B = 1$.

For imaginary values of the coupling constant $g$ (i.e. $B < 0$), the BD model — which is then referred to as the Zhiber–Mihailov–Shabat (ZMS) model [12] — permits the description of $\phi_{1,2}$ and $\phi_{2,1}$ deformations of conformal minimal models. Indeed, starting from the observation that the ZMS has a non–unitary $S$–matrix related to the Izergin–Korepin $R$–matrix, Smirnov exploited the quantum group $SL(2)_q$ invariance of the $S$–matrix in order to recover unitarity in specific reductions of the model. The $S$–matrices of the above–mentioned deformed minimal models were in this way obtained from RSOS restrictions of the Izergin–Korepin $R$–matrix at specific values of the coupling constant at which $q$ is a root of unity [10]. More recently the possibility of studying some relevant $\phi_{1,5}$ deformations of specific non–unitary minimal models has been considered as well which relies again on quantum group reductions of the ZMS model [30].

Some insight can be obtained if one considers the ZMS model by interpreting one of the exponential operators in the Lagrangian as a deformation of a Complex Liouville Theory (CLT) [31]. The other exponential then plays the role of a screening operator. If we require the CLT to describe the minimal model $\mathcal{M}_{r,s}$ with central charge

$$C = 1 - \frac{6(r-s)^2}{r s}, \quad s > r \quad \text{relative primes},$$
and primary fields $\phi_{m,n}$ of conformal dimensions

$$\Delta_{m,n} = \frac{(m r - m s)^2 - (r - s)^2}{4 r s} \quad m = 1, \ldots r - 1; \quad n = 1, \ldots s - 1,$$

the above interpretation leads to a four–fold choice: in fact, after choosing one of the two exponentials in eq. (2.1) as the screening operator, one can still choose two possible values of $g$ as a function of $r$ and $s$ in order to correctly set its conformal dimensions to be $\Delta = 1$. In the Complex Liouville Theory, the primary operators will be given by the following exponential operators

$$\phi_{m,n} = e^{k_{m,n} g \varphi} \quad m = 1, \ldots r - 1; \quad n = 1, \ldots s - 1;$$

with the identification

$$\phi_{m,n} \equiv \phi_{r-m,s-n},$$

which entails the correct symmetry of the Kac Table of the model. In eq. (2.3), the dependence of $k_{m,n}$ and $B(g)$ from the integers $r, s, m, n$ depends on the choice made for the screening operator, as summarized in Table 1 and the deforming exponential can be easily shown to correspond to one of the primary fields $\phi_{1,2}, \phi_{2,1}, \phi_{1,5}$ or $\phi_{5,1}$. However, one can easily check that, while the primary field $\phi_{1,2}$ is relevant in any minimal model, the field $\phi_{5,1}$ is on the contrary always irrelevant and therefore does not yield renormalizable deformations. As for the fields $\phi_{2,1}$ and $\phi_{1,5}$, they can be shown to be relevant only in disjoint sets of models: the field $\phi_{2,1}$ is relevant for the class of minimal models $\mathcal{M}_{r,s}$ with $s < 2r$ which includes all the unitary cases $\mathcal{M}_{r,r+1}$, while $\phi_{1,5}$ is relevant for the complementary class of non–unitary models $s > 2r$. Notice that in order to decide whether the deformation is relevant or not it is sufficient to require that the coupling constant $g$ be imaginary, namely that $B < 0$ (see Table 1).

The spectrum of the reduced ZMS model in general consists of kinks together with a cascade of their possible bound states [10]. This spectrum does not always contain the original BD boson: while this particle is always present in the $\phi_{1,2}$ deformations (where it appears as the lightest breather of two fundamental kinks), it is on the contrary never present in the spectrum of $\phi_{2,1}$ deformations. In the relevant $\phi_{1,5}$ deformations the presence of the BD boson depends on the specific model (see [30]).

We will not enter in further details on the reductions of the ZMS model which can be found in the original literature [10, 30]. The information collected in this chapter is all we need for establishing the correct mapping between exponential operators of the BD model and primary fields of the reduced models.

## 3 Form Factor Equations in the BD model

We now turn to the problem of determining the on–shell matrix elements (form factors) of a local operator $\Phi(x)$ in the BD model. In the framework of two–dimensional integrable QFT, this problem is reduced to the

\[\text{In the following we will always consider normal ordered exponential operators omitting the notation } e^{\alpha \varphi(x)} .\]
problem of studying a set of coupled functional equations \[14, 15\] namely the Watson monodromy equations

\[
F_n(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = S(\theta_i - \theta_{i+1}) F_n(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n),
\]
\[
F_n(\theta_1 + 2\pi i, \theta_2, \ldots, \theta_n) = F_n(\theta_2, \ldots, \theta_n, \theta_1),
\]
(3.1)

as well as the recursive residue equations on annihilation poles (kinematical residue equations)

\[
\lim_{\theta \to \theta'} F_{n+2}(\theta' + \pi, \theta, \theta_1, \ldots, \theta_n) = i \left(1 - \prod_{i=1}^{n} S(\theta - \theta_i)\right) F_n(\theta_1, \ldots, \theta_n),
\]
(3.2)

and bound state poles (dynamical residue equations)

\[
\lim_{\alpha \to \pm \frac{2\pi}{3i}} \left(\alpha - \frac{2\pi i}{3}\right) F_{n+2}(\theta + \alpha/2, \theta - \alpha/2, \theta_1, \ldots, \theta_n) = i \Gamma F_{n+1}(\theta, \theta_1, \ldots, \theta_n).
\]
(3.3)

The most general solution to the monodromy equations \[14, 15\] can be written in the following form \[14\]

\[
F^\Phi_n(\theta_1, \ldots, \theta_n) = R^\Phi(\theta_1, \ldots, \theta_n) \prod_{i < j} F_{n}^{\text{min}}(\theta_i - \theta_j),
\]

where \(R^\Phi(\theta_1, \ldots, \theta_n)\) is any symmetric \(2\pi i\)-periodic function in the variables \(\theta_i\) and the “minimal” two–particle form factor \(F^{\text{min}}(\theta)\) is given by the following function

\[
F^{\text{min}}(\theta) = \mathcal{N}(B) \frac{g_0(\theta) g_2(\theta)}{g_{-\alpha}(\theta) g_{\alpha}(\theta)},
\]
(3.4)

where \(g_\alpha(\theta)\) is defined by

\[
g_\alpha(\theta) = \exp \left[2 \int_0^{\infty} \frac{dt}{t} \frac{\cosh((\alpha - 1/2)t)}{\cosh(t/2)} \sin^2((i\pi - \theta)t/2)\right].
\]

In eq. (3.4), \(\mathcal{N}(B)\) is the following normalization constant

\[
\mathcal{N}(B) = \exp \left[-4 \int \frac{dt}{t} \frac{\sinh(t/2) \cosh(t/6)}{\sinh^2 t} \left(\cosh(t/3) - \cosh((B - 1)t/3)\right)\right],
\]
(3.5)

chosen such that \(F^{\text{min}}(\infty) = 1\). For real values of the coupling constant, namely for \(B \in (0, 2)\), \(F^{\text{min}}(\theta)\) has neither poles nor zeros in the physical strip \(\text{Im} \theta \in (0, \pi)\), since the same property is shared by \(g_\alpha(\theta)\) when \(\alpha \in (0, 1)\). The analytical continuation of \(F^{\text{min}}(\theta)\) for imaginary values of the coupling constant \(g\) \((B < 0)\) develops poles in \(\theta\) which can be explicitly exhibited by using the following functional relations

\[
g_{1+\alpha}(\theta) = g_{-\alpha}(\theta),
\]
\[
g_\alpha(\theta) g_{-\alpha}(\theta) = \mathcal{P}_\alpha(\theta) = \frac{\cos \pi \alpha - \cosh \theta}{2 \sinh^2 \frac{\pi \alpha}{2}},
\]

satisfied by the functions \(g_\alpha(\theta)\).

Notice that we have not mentioned yet the dependence of the form factors \(F^\Phi_n\) on the operator \(\Phi(x)\). Indeed, in the system of equations (3.1), (3.2) and (3.3) this dependence is not explicit and further physical requirements are necessary to identify in the space of solutions the form factors of a specific operator.
3.1 General solutions for scalar non–derivative operators

In this work we are mainly concerned with the analysis of form factors of scalar operators which are local nonderivative functions of the field \( \varphi(x) \). This infinite dimensional operatorial space can be spanned for instance by the basis of polynomials in \( \varphi(x) \) or by the basis of exponentials \( e^{\alpha \varphi(x)} \). A suitable parameterization of the form factors for this class of operators is the following

\[
F^\Phi_n(\theta_1, \ldots, \theta_n) = H^\Phi_n Q^\Phi_n(x_1, \ldots, x_n) \prod_{i<j} \frac{F_{\text{min}}(\theta_i - \theta_j)}{(x_i + x_j)(x_i^2 + x_ix_j + x_j^2)},
\]

where \( x_i = e^{\theta_i} \). The pole structure expected to reflect the correct analyticity properties is explicitly shown in the denominator of (3.6), where annihilation and bound state simple poles are present at relative rapidities \( \theta_{ij} = i\pi \) and \( \theta_{ij} = 2\pi i/3 \), respectively. \( Q^\Phi_n \) is a homogeneous symmetrical polynomial in the variables \( x_i \) whose total degree is determined by Lorentz invariance to be \( d_n = \frac{3n(n-1)}{2} \). The constants \( H^\Phi_n \) are conveniently chosen to be

\[
H^\Phi_n = t \mu^n(B),
\]

in order to obtain a simplified version of recursive equations on the polynomials \( Q^\Phi_n \). In eq. (3.7), \( t \) is a free parameter which will have an important role in the discussion of cluster solutions whereas

\[
\mu(B) = \frac{\sqrt{3} \Gamma(B)}{F_{\text{min}}(\frac{2\pi i}{3})}.
\]

With the above choice of \( H^\Phi_n \), the dynamical recursive equations (3.3) read

\[
Q_n(\omega x, \omega^{-1} x, x_1, \ldots, x_{n-2}) = -x^3 D_{n-2}(x|x_1, \ldots, x_{n-2}) Q_{n-1}(x, x_1, \ldots, x_{n-2}),
\]

where \( \omega = e^{i\pi/3} \) and the polynomial \( D_n \) is given by

\[
D_n(x|x_1, \ldots, x_n) = \sum_{k_1, k_2, k_3=0}^{n} x^{3n-k_1-k_2-k_3} \sigma_{k_1}^{(n)} \sigma_{k_2}^{(n)} \sigma_{k_3}^{(n)} \cos((k_2 - k_3)(B + 2\pi i/3)).
\]

The last expression is written in the usual basis of symmetrical polynomials \( \sigma_k^{(n)} \) which are defined by the generating function

\[
\sum_{k=0}^{n} x^{n-k} \sigma_k^{(n)} = \prod_{i=1}^{n} (x + x_i).
\]

In expression (3.9) we can get rid of the trigonometrical dependence on the coupling constant \( B \) by exploiting the following recursive relation

\[
\cos((n+1)\alpha) = 2 \cos(n\alpha) \cos\alpha - \cos((n-1)\alpha),
\]

which allows us to express cosines of multiple angles as polynomials of \( \cos\alpha \). In this way we can cast the dependence of eq. (3.8) on the coupling constant into a rational dependence on the variable \( c \) defined in eq. (2.2).

The kinematical residue equations on annihilation poles (3.2) can be written as

\[
Q_n(-x, x, x_1, \ldots, x_{n-2}) = (-)^n K x^3 U_{n-2}(x|x_1, \ldots, x_{n-2}) Q_{n-2}(x_1, \ldots, x_{n-2}),
\]

where
with

\[ U_n(x|x_1, \ldots, x_n) = 2 \sum_{k_1, \ldots, k_6=0}^n (-)^{k_2+k_3+k_5} x^{6n-(k_1+\cdots+k_6)} \sigma_{k_1}^{(n)} \cdots \sigma_{k_6}^{(n)} \cdot \sin \left( (2(k_2 + k_4 - k_1 - k_3) + B(k_3 + k_6 - k_4 - k_5)) \pi/3 \right), \]

(3.12)

and

\[ K = \frac{(2c-1)}{4 \sqrt{3}(1+c)(2c+1)}. \]

Before solving the system of recursive equations, let us derive some important properties on the space of solutions from a direct analysis of the equations (3.8) and (3.11).

A – It is easy to prove that in the space of symmetrical polynomials of degree \( d_n = \frac{3n(n-1)}{2} \), the only polynomials which have zeros both at \( x_i/x_j = e^{2\pi i/3} \) and at \( x_i/x_j = -1 \) are given by

\[ K^{(n)}(\{x_i\}) = \prod_{1 \leq i < j \leq n} (x_i + x_j)(x_i^2 + x_i x_j + x_j^2) \]

\[ = \det \left| \sigma_{2j-i}^{(n)} \right|_{1 \leq i,j \leq n-1} \det \left| \sigma_{2j/2-i+1+(-)i+1}^{(n)} \right|_{1 \leq i,j \leq 2n-2}, \]

up to a multiplicative constant. This is therefore the only possible kernel for the whole system of recursive equations. Hence, after fixing all the polynomials \( Q_i \) for \( i = 1, \ldots, n-1 \), the most general solution \( Q_n \) of the system of equations (3.8) and (3.11) will be then given by

\[ Q_n = Q_n^* + \lambda_n K^{(n)}(\{x_i\}), \]

(3.13)

where \( Q_n^* \) is a specific solution and \( \lambda_n \) is a free parameter. The space of solutions will be organized correspondingly, namely every operator will be identified by a succession of parameters \( \lambda_i, i = 1, \ldots, \infty \) and the general solution for a \( n \)-particle form factor will be described by an \( n \)-dimensional vector space of solutions \( Q_n \) spanned by the parameters \( \lambda_1, \ldots, \lambda_n \).

B – The partial degree of the general polynomial \( Q_n \) with respect to any of the variables \( x_i \) is exactly \( d_n^{(i)} = 3(n-1) \). This can be easily shown by induction observing that \( Q_1 \) must be a constant for Lorentz invariance and making use of equations (3.8), (3.11) and (3.13). This implies in particular that the form factors of this class of scalar operators of the theory have bounded asymptotic behavior for large values of the rapidities,

\[ \lim_{\Lambda \to \infty} F_n^\Phi(\theta_1 + \Lambda, \ldots, \theta_k + \Lambda, \theta_{k+1}, \ldots, \theta_n) < \infty \quad \forall k = 1, \ldots, n-1. \]

This observation enables us to look for cluster solutions of form factors equations within this general class of solutions (see eq. (1.2)).

We now turn to the actual computation of the first multiparticle general solutions to the system of recursive equations (3.8) and (3.11). The most direct way of computing these solutions consists in parameterizing any polynomial \( Q_n \) as the most general polynomial of degree \( d_n = \frac{3n(n-1)}{2} \) in the basis of symmetrical polynomials
\( \sigma_k^{(n)} \) and to impose on the coefficients of the expansion the constraints coming from the recursive equations. We report here the result of the first general multiparticle form factors in the space of scalar non–derivative operators. Lorentz invariance requires \( Q_1 \) to be a constant

\[ Q_1 = \lambda_1 , \]

hence in order not to have two different overall normalization constants we can set for the time being \( t = 1 \) in eq. (3.7). The next most general solutions are given by

\[
Q_2(x_1, x_2) = -\lambda_1 \sigma_1^3 - \lambda_2 \mathcal{K}^{(2)},
\]

\[
Q_3(x_1, x_2, x_3) = \lambda_1 \left( \sigma_1 \sigma_2^4 + \sigma_1^4 \sigma_2 \sigma_3 + \frac{4 c^2 - 1}{2 (1 + c)} \sigma_1^2 \sigma_2^2 \sigma_3 - \frac{3}{2 (1 + c)} \left( \sigma_2^3 \sigma_3 + \sigma_1^3 \sigma_3^2 \right) \right)
\]

\[
+ \lambda_2 \left( \sigma_1 \sigma_2^4 + \sigma_1^4 \sigma_2 \sigma_3 - 2 (1 - c) \left( \sigma_1^2 \sigma_2^2 \sigma_3 - \sigma_1 \sigma_2 \sigma_3^2 \right) - \sigma_2^3 \sigma_3 - \sigma_1^3 \sigma_3^2 \right)
\]

\[
+ \lambda_3 \mathcal{K}^{(3)},
\]

where the residual kernel freedom of each solution has been explicated\(^3\). Notice that in the above solutions the trigonometrical dependence on the coupling constant has been hidden in a simple rational dependence on the self–dual variable \( c \) defined in eq. (2.2). This major simplification has been made possible by noticing that the systematic solution of the dynamical recursive equations alone (3.8) yields polynomials \( Q_n \) which already have the correct single–parameter kernel ambiguity (3.13) expected for the whole system. It therefore means that, actually the dynamical recursive equations (3.8) are equivalent to the system of the two coupled equations (3.8) and (3.11).

The general solutions that we have found must include in particular the form factors of the elementary field \( \varphi(x) \) which were first studied in \([28]\). One can prove that they can in fact be selected by imposing either the asymptotic vanishing of the form factors for large values of the rapidities (i.e. imposing the cancellation of the highest partial degree terms in \( Q_n \)) or the proportionality \( Q_n \sim \sigma_n \) \([28]\). The \( \lambda_i \) are then determined to be in this case

\[ \lambda^\varphi_2 = -\lambda^\varphi_1 , \]

\[ \lambda^\varphi_i = 0 \quad \forall i > 2 . \]

Finally the overall normalization is fixed by\(^4\)

\[ \langle \varphi(0) | A \rangle = \frac{Z^{1/2}}{\sqrt{2}} , \]

which sets \( \lambda_1^\varphi = \mu^{-1} Z^{1/2} / \sqrt{2} \). In the above expression \( Z \) is the wave function renormalization constant of the theory which will be exactly computed in Section 6.

\(^3\)We do not report here the general solution of \( Q_4 \) which already contains an extremely large number of terms and is not particularly useful for the purposes of this work.

\(^4\)Our convention on the normalization of states is \( \langle A(\theta_1) | A(\theta_2) \rangle = 2 \pi \delta(\theta_1 - \theta_2) = 2 \pi E_1 \delta(p_1 - p_2) \).
By using the above general solutions we can also identify the 1–parameter family of the trace $\Theta(x)$ of the Stress–Energy tensor for different values of the background charge. This family of operators was studied in ref. [26] where the authors showed that different choices of $\Theta(x)$ select different possible ultraviolet limits of the theory. In order to identify these form factors it is sufficient to impose the proportionality $Q_n \sim \sigma_1 \sigma_{n-1}$ for $n \geq 3$, as it can be shown from the conservation of the Stress–Energy–Tensor. In this way one determines all the free kernel parameters $\lambda_i$ but the first two. The parameter $\lambda_3^\Theta$ is found to be for example

$$\lambda_3^\Theta = \lambda_2^\Theta + \frac{3\lambda_1^\Theta}{2c+2}.$$  

Finally, imposing the overall normalization

$$F_2^\Theta(i\pi) = 2\pi m^2,$$

one determines

$$\lambda_2^\Theta = \frac{\pi m^2}{(c-1)\Gamma^2},$$

and obtains a one–parameter family of independent operators for arbitrary $\lambda_1^\Theta$ which coincides with the one analyzed in ref. [26].

In order to identify different operators in this general space of solutions one must resort to more powerful techniques. We will see in the following section how the imposition of the cluster equations (1.2) enables us to extract the form factors of a whole basis in the space of non–derivative scalar operators.

4 Form Factors of Exponential Operators

In this chapter we study the existence of solutions of the form factor equations which also satisfy the further requirement given by the so–called cluster equations (1.2) imposed on a multiparticle form factor $F_n$. This restrictive set of non–linear equations is believed to select out the exponential operators in a Lagrangian theory [15, 25]. More recently it has been shown in ref. [24] that these equations are the distinguishing property of scaling operators in the conformal limit of a two–dimensional field theory at least in the cases where there is no symmetry preventing the form factors from being non–vanishing. This observation has been confirmed and successfully employed in ref.’s [21, 33] for identifying the complete set of scaling primary fields in some massive deformations of minimal models. Cluster solutions become therefore objects of utmost interest in the BD model because the two ways of looking at them either as exponential operators or as scaling fields, converge in this theory where specific exponentials are identified with primary operators in the reduced models describing deformations of conformal field theories.

In order to impose the cluster equations (1.2) we fix the overall normalization of the form factors by adopting the convenient choice $F_0 = 1$ and choose

$$Q_1 = 1.$$
Equations (1.2) then amount to requiring the following property on the polynomial $Q_n$

$$\lim_{\Lambda \to \infty} \frac{Q_n}{K^{(n)}}(\Lambda x_1, \ldots, \Lambda x_m, x_{m+1}, \ldots, x_n) = \frac{Q_m}{K^{(m)}}(\{x_i\}_{i=1}^{m}) \frac{Q_{n-m}}{K^{(n-m)}}(\{x_i\}_{i=m+1}^{n}),$$

(4.1)

where $t$ — the variable introduced in eq. (2.2) — is now switched on and treated as a free parameter. These further restrictions imposed on the general solutions of residue equations determine level by level all the $\lambda_n$ parameters as functions of $t$. At any given level $n$, the number of equations which determine the only free parameter left $\lambda_n$, grows rapidly with $n$, therefore the very existence of a cluster solution is not at all obvious. For the first computed solutions however, all the equations on a given $\lambda_n$ turn out to be identical and we believe that this should be the case at any level. In this way we obtain a one–parameter family of solutions for $t$ arbitrary, of which we report the first multiparticle representatives in Appendix A. Notice that $t$ is not an overall normalization factor since the normalization of the form factors has been fixed by $F_0 = 1$ and indeed, due to the nonlinearity of (1.1), the solutions $Q_n(t)$ turn out to be polynomials in $t$ of degree $n - 1$. This means that $t$ defines through the polynomials of Appendix A and eqs. (3.6) and (3.7) a one–parameter family of solutions $F_n^{(t)}$ corresponding to independent operators. If we make then the hypothesis that these solutions actually correspond to the form factors of the exponential operators $e^{kg\varphi(x)}$,

$$F_n^{(t)}(\theta_1, \ldots, \theta_n) = \frac{\langle 0|e^{kg\varphi(0)}|\Lambda(x_1) \cdots A(x_n)\rangle}{\langle 0|e^{kg\varphi(0)}|0\rangle},$$

(4.2)

we are forced to consider $t$ as a well–defined function $t(k,B)$ of $k$ and $B$ rather than a free parameter. In particular, in order to establish the one–to–one correspondence between cluster solutions and exponential operators it is of particular interest to compute the normalization–invariant quantity

$$F_1^{(t)} = \frac{\langle 0|e^{kg\varphi(0)}|A\rangle}{\langle 0|e^{kg\varphi(0)}|0\rangle} = \mu(B)t(k,B).$$

(4.3)

In the following we consider in detail some conditions that we can impose on the function $t(k,B)$ in order to find its exact form.

### 4.1 The Function $t(k,B)$

The first information on $t(k,B)$ can be obtained from the computation of the conformal dimensions $\Delta = -g^2 k^2/8 \pi$ of the operators $e^{kg\varphi(x)}$ in the free–boson ultraviolet limit at lowest order in $g^2$. These can be easily obtained from the analysis of the short distance behavior of the correlator $\langle 0|e^{kg\varphi(x)}e^{kg\varphi(0)}|0\rangle$ by means of eq. (1.1) and the cluster solutions $F_n^{(t)}$. We obtain

$$\Delta = -g^2 \lim_{g \to 0} \mu(B)^2 t(k,B)^2 \frac{t(k,B)^2}{4 \pi g^2} = -\frac{g^2 t(k,0)^2}{8 \pi},$$

from which one obtains the important relation

$$\lim_{B \to 0} t(k,B) = k.$$

(4.4)

Notice that from a computational point of view there is no difficulty in obtaining the next multiparticle solutions since the dynamical recursive equations (4.1) are linear equations in the unknown coefficients of independent monomials in the $\varphi$’s and the dependence on the coupling constant is simply a rational dependence on $c$. 

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Furthermore, from the expressions (A.3) and (A.4), by imposing the proportionality 
\[ Q_n \sim \sigma_1 \sigma_{n-1} \]
one can easily verify that the only cluster solutions which also belong to the class of possible traces of the stress–energy tensor are defined by the solutions of

\[ -1 + 2c + 2t + 2ct + 2t^2 + 2ct^2 = 0 , \]

namely

\[ t^\pm = \begin{cases} 
\frac{\sin((B + 1)\pi/6)}{\cos((B + 2)\pi/6)} \\
\frac{\sin((B - 3)\pi/6)}{\cos((B + 2)\pi/6)} 
\end{cases} . \quad (4.5) \]

These two solutions correspond to the ones found in ref. [26] and identified with the form factors of the fundamental vertex operators \( e^{g\varphi} \) and \( e^{-2g\varphi} \) which appear in the Lagrangian density. This can also be obtained immediately by taking the limit \( B \to 0 \) in eq. (4.5) which gives respectively \( k = 1, -2 \) in virtue of (4.4).

Therefore we have also the two following important requirements on \( t(k, B) \):

\[ t(1, B) = \frac{\sin((B + 1)\pi/6)}{\cos((B + 2)\pi/6)} , \quad (4.6) \]

\[ t(-2, B) = \frac{\sin((B - 3)\pi/6)}{\cos((B + 2)\pi/6)} . \quad (4.7) \]

As a limiting case of the cluster solutions we can also recover the form factors of the fundamental field \( \varphi(x) \) which is naturally obtained from the vertex operators in the limit \( k \to 0 \). These form factors of course satisfy a trivial cluster property because they vanish for large rapidities and therefore satisfy eq. (4.1) with \( t = 0 \).

Hence we get one more information

\[ \lim_{k \to 0} t(k, B) = 0 . \quad (4.8) \]

Indeed one can easily check that the form factors we had obtained in the previous section for the field \( \varphi(x) \) from the most general solutions of residue equations satisfy,

\[ F^\varphi_n = \lambda^\varphi \lim_{t \to 0} \frac{F_n^{(t)}}{t} . \]

A remarkable check on the correct identification of these operators is obtained studying the quantum equations of motion of the model

\[ \Box \varphi + \frac{m^2}{3g} \left( e^{g\varphi} - e^{-2g\varphi} \right) = 0 . \]

If our identification is correct we should find\[6\]

\[ m^2 \sigma^{(n)}_{\varphi} \left[ \sigma^{(n-1)}_{\varphi} \right] \left( F_n^\varphi + \tau \left( F_n^{(t^+)} - F_n^{(t^-)} \right) \right) = 0 , \]

with some constant \( \tau \), or equivalently

\[ \lambda^\varphi m^2 \sigma^{(n)}_{\varphi} \sigma^{(n-1)}_{\varphi} Q_n(0) + \tau \left( t^+ Q_n(t^+) - t^- Q_n(t^-) \right) = 0 . \]

\[ ^{6} \text{In general} F^\Box \Phi = -m^2 \sigma^{(n-1)}_{\Phi} F_n^\Phi \text{ for any field } \Phi(x). \]
Indeed this last equation can be verified to hold on the solutions given in Appendix A with
\[ \tau = -\frac{\lambda^2 m^2}{\sqrt{3}} \tan((B + 2)\pi/6). \]

The non–perturbative nature of this last check shows that the identification of cluster solutions as vertex operators is far beyond a semiclassical one for small coupling constant.

The constraints obtained for the function \( t(k, B) \), eqs. (4.4), (4.6), (4.7) and (4.8) are not sufficient to determine its form and, in particular, little information is given on the dependence on \( k \). We will see however in the next section that some additional requirements coming from the reductions of the ZMS model impose a periodicity condition in \( k \) for the function \( t(k, B) \)

\[ t(k, B) = t(k + 6/B, B), \quad (4.9) \]

which suggests the following conjecture:

\[ t(k, B) = \frac{\sin(kB \pi/6) \sin((kB + B + 2)\pi/6)}{2 \sin(B \pi/6) \sin((2 - B)\pi/6) \cos((B + 2)\pi/6)}. \quad (4.10) \]

This function satisfy all the aforementioned requirements. A decisive check of the validity of this expression will be obtained in the following chapter by the comparison with explicit computations of form factors of primary operators in specific reductions of the ZMS model. This formula may be regarded as one of the main results of the paper. In fact it allows us to explicitly assign to every vertex operator \( e^{kg\varphi} \) in the BD model its form factors \( F_{n}^{(k)} \) which are obtained from the cluster solutions \( Q_{n}(t) \) of Appendix A through the parameterization (3.6) and eq. (3.7) by replacing \( t = t(k, B) \).

5 Form Factors in the reductions of the ZMS model

We now turn our attention to the analitical continuation of the model to imaginary values of the coupling constant \( g \), namely to possible reductions of the ZMS model. In these models the spectrum is no more a single–particle one as in the real coupling BD model, but it has a richer structure that depends on the model analyzed. We consider here only those restrictions whose spectrum still contains the elementary boson excitation of the BD model, namely \( \phi_{1,2} \) and some cases of \( \phi_{1,5} \) deformations.\(^7\) If we assume that the identification obtained between cluster solutions and vertex operators of the model is exact, we are then led to establish a correspondence between the form factors of exponential operators \( e^{kg\varphi(x)} \) in the BD model and the form factors\(^8\) of scaling primary operators in the deformations according to the correspondence given by eq. (2.3) and Table 1. An immediate consistency requirement for this procedure is obtained by imposing that the form factors respect the symmetry (2.4) of the Kac table of minimal models. For example, the quantity \( F_{1}^{(t)} \) of eq. (4.3) should have the same value if evaluated at \( k = k_{m,n} \) and \( k = k_{r-m,s-n} \). Imposing this condition both in the \( \phi_{1,2} \)

\(^7\)To avoid confusion we stress that the elementary BD scalar boson, which is created from the field \( \varphi(x) \), is not the fundamental particle in the bootstrap of the reductions of the ZMS model which is instead a three–component kink.

\(^8\)The form factors in the reduced model must be intended as the matrix elements on the BD breather sector.
deformations and in the $\phi_{1,5}$ relevant ones we obtain respectively that the following two symmetries of the function $t(k, B)$ must hold

$$
\begin{align*}
t(k, B) &= t(-k - 1 - 2/B, B), \\
t(k, B) &= t(-k - 1 + 4/B, B),
\end{align*}
$$

which in particular entail the above mentioned periodicity in $k$, equation (4.9). Both these symmetries are indeed separately satisfied by the function (4.10).

A precise check on the validity of equation (4.10) is provided by comparing its predictions with the form factors of scaling primary operators in $\phi_{1,2}$ and $\phi_{1,5}$ deformations which can be found in literature. We have indeed computed the normalization invariant ratio $F_1/F_0$ using eq. (4.3) and the assignments of Table 1, for all the known cases of primary form factors which have been analyzed in literature [16, 19, 21, 20, 33] (see Table 2) and a perfect agreement has been found with all the values reported in the references. We stress here the fact that in the references considered, the form factors of primary operators have been identified by different techniques: in ref.’s [16, 19, 20] the identification has been obtained by using the correspondence between the deforming field and the trace of the stress–energy tensor whereas in ref.’s [21, 33] the form factors of the primary fields have been identified with the finite number of solutions of a non–linear system of cluster equations involving the form factors relative to the whole particle spectrum of the reduced models.

6 The Wave Function Renormalization Constant of the BD Model and the Form Factors of $\varphi(x)$ and $\varphi^2(x)$:

The form factors $F_n^{(k)}(\theta_1, \ldots, \theta_n)$ that we have computed have been so far conveniently normalized putting $F_0 = 1$. From equation (4.12) one immediately observes that these form factors are invariant under an additive redefinition of the field $\varphi(x) \to \varphi(x) + \text{const.}$. We remove this ambiguity on the definition of the field $\varphi(x)$ by imposing that its vacuum expectation value $\langle 0 | \varphi(x) | 0 \rangle$ be zero, namely subtracting from the original Lagrangian field the value of the one point tadpole function. Consider now the following expansion of the form factors of exponential operators:

$$
\langle 0 | e^{kg\varphi(0)} | A(\theta_1) \cdots A(\theta_n) \rangle = \sum_{j=1}^{\infty} \frac{k^j g^j}{j!} \langle 0 : \varphi^j(0) : | A(\theta_1) \cdots A(\theta_n) \rangle,
$$

and of the vacuum expectation value

$$
\langle 0 | e^{kg\varphi(0)} | 0 \rangle = \sum_{j=0}^{\infty} \frac{k^j g^j}{j!} \langle 0 : \varphi^j(0) : | 0 \rangle = 1 + o(k^2).
$$
If we now expand the form factors $F_n^{(k)}$ that we have obtained in series of $k$ we can identify the form factors of the fields $\varphi(x)$ and $:\varphi^2(x)$: as the coefficients of order $k$ and $k^2$ respectively.

$$F_n^{(k)}(\theta_1, \ldots, \theta_n) = \frac{(0|e^{ik\varphi(0)}|A(\theta_1) \cdots A(\theta_n))}{(0|e^{ig\varphi(0)}|0)} = k \left\langle 0|\varphi(x)\rangle_{(0)}|A(\theta_1) \cdots A(\theta_n)) + \frac{k^2 g^2}{2} \left\langle 0|:\varphi^2(x):\rangle_{(0)}|A(\theta_1) \cdots A(\theta_n)) + o(k^3). \right.$$  \hfill (6.1)

This procedure gives the form factors of $\varphi(x)$ and $:\varphi^2(x)$: with the correct overall normalization of the fields. This observation in particular allows the exact determination of the wave function renormalization constant $Z(B)$ of the BD model. In fact, considering the first order expansion in $k$ of $F_1^{(k)}$

$$F_1^{(k)} = \mu(B) t(k, B) = \mu(B) \frac{k B \pi \tan((B + 2)\pi/6)}{12 \sin(B\pi/6) \sin((2 - B)\pi/6)} + o(k^2)$$

$$= k g \langle 0|\varphi(0)\rangle_{(0)} + o(k^2)$$

$$= \frac{k g Z^{1/2}}{\sqrt{2}} + o(k^2),$$

one easily obtains the following expression for $Z(B)$

$$Z(B) = \mu(B)^2 B (2 - B) \frac{\pi}{288} \left(\frac{\tan((B + 2)\pi/6)}{\sin(B\pi/6) \sin((2 - B)\pi/6)}\right)^2$$

$$= \frac{2 \pi}{3 \sqrt{3}} \frac{B (2 - B)}{\mathcal{N}(B)} \frac{(c - 1)}{(1 + 2 c)(1 - 2 c)},$$

where $\mathcal{N}(B)$ is defined in eq. (3.3). The function $Z(B)$ is manifestly dual with respect to the weak–strong coupling transformation $B \leftrightarrow 2 - B$ and can be easily shown to coincide at lowest order in $g^2$ with the correct perturbative result coming from the one–loop self energy diagram

$$Z = 1 - \frac{g^2}{12} \left(\frac{1}{\pi} - \frac{1}{3\sqrt{3}}\right) + o(g^4).$$

A plot of the function $Z(B)$ is given in Figure 2. Notice the tiny deviation of the constant from the free field value $Z = 1$ on the entire range of the coupling constant $B \in [0, 2]$.

The correctly normalized form factors of the field $\varphi(x)$ are given by

$$F_n^{\varphi} = g^{-1} \frac{d}{dk} F_n^{(k)} \bigg|_{k=0} = \frac{Z^{1/2}}{\mu\sqrt{2}} \frac{F_n^{(t)}}{t} \bigg|_{t=0},$$

while the exact form factors of the field $:\varphi^2(x)$: are simply obtained by

$$F_n^{\varphi^2} = g^{-2} \frac{d^2}{dk^2} F_n^{(k)} \bigg|_{k=0},$$

For example we can compute

$$F_1^{\varphi^2} = \langle 0|:\varphi^2(0):\rangle_{(0)}|A\rangle = \mu(B) g^{-2} \frac{d^2}{dk^2} t(k, B) \bigg|_{k=0}$$

$$= \mu(B) B (2 - B) \frac{\pi}{144} \frac{1}{\sin(B\pi/6) \sin((2 - B)\pi/6)},$$

\hfill 9

Here and in the following we will adopt the notation $F_n^{(k)}$ instead of $F_n^{(t)}$ to stress the dependence on $k$. The relation between the two expressions is obviously given by $t = t(k, B)$ eq. (4.11).
which exactly matches at lowest order in $g$ with the one loop calculation

$$
(0|:\varphi^2(0):|A\rangle = \frac{g}{6\sqrt{6}} + o(g^3).
$$

In a similar way we get

$$
F_2^{\varphi^2}(\theta_1 - \theta_2) = (0|:\varphi^2(0):|A(\theta_1) A(\theta_2)) = \mu^2(B) B (2 - B) \frac{\pi}{288} \frac{1}{\sin(B\pi/6) \sin((2 - B)\pi/6))^2}
\cdot \left( \sigma_1^2 \tan^2((B + 2)\pi/6) - \sigma_1 \sigma_2 (2 \sin(B\pi/6) \sin((2 - B)\pi/6) + \tan^2((B + 2)\pi/6)) \right)
\cdot \frac{F_{\min}(\theta_1 - \theta_2)}{(x_2 + x_2)(x_1^2 + x_1 x_2 + x_2^2)}.
$$

Notice that in order to obtain the form factors of arbitrary operators $:\varphi^n(x)$: one should exactly compute the vacuum expectation value $(0|e^{kg\varphi(0)}|0)$ of the exponential operators and make use of expansion (6.1) (for the sine–Gordon model the vacuum expectation value of the exponential operators has been recently obtained in ref. [34]).

7 Conclusions

In this paper we have computed, in the framework of the bootstrap approach to integrable models, the first multiparticle solutions of form factor equations for general non–derivative scalar operators in the BD model. Among these solutions we have selected a one–parameter family of cluster solutions which have been identified by means of the central result eq. (4.10) with the form factors of exponential operators $e^{kg\varphi}$. In the complex coupling constant version of the model, the form factors of exponential operators allow to identify the form factors of relevant primary operators in the sector of the lightest breather of $\phi_{1,2}$ and $\phi_{1,5}$ deformations of minimal models and perfect agreement has been found with all the examples that we have found in literature. Finally, by using the cluster solutions, we have computed the form factors of the fields $\varphi(x)$ and $:\varphi^2(x)$: with the correct overall normalization and determined in this way the non–perturbative exact wave function renormalization constant of the model.

We have therefore obtained the characterization of form factors for a whole basis in the space of scalar non–derivative operators and we have found complete consistency, in a non–perturbative setting, between the axiomatic $S$–matrix approach to bootstrap systems and the Lagrangian approach to quantum field theories.

This work also yields an efficient tool for the identification of relevant primary fields among the cluster solutions of massive $\phi_{1,2}$ and $\phi_{1,5}$ deformations of minimal models.
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Appendix A

In this Appendix we list the first solutions of the one–parameter family of $Q_n$ polynomials of cluster solutions in the BD model. In the following expressions, the variable $c$ is the dual–invariant function of the coupling constant defined in eq. (2.2) and $t$ is a free parameter. The solutions are identified with those of the basis of operators $e^{kg\varphi}$ by means of eq. (4.10) which determines $t$ as a function of $k$ and $g$.

$$Q_1(t) = 1,$$  \hfill (A.1)

$$Q_2(t) = t \sigma_1^3$$

\[ - (1 + t) \sigma_1 \sigma_2, \]  \hfill (A.2)

$$Q_3(t) 2 (1 + c) = 2 (1 + c) t^2 \sigma_1^3 \sigma_2^3$$

\[ -2 (1 + c) t (1 + t) \sigma_1 \sigma_2 \sigma_3 \]  \hfill (A.3)

\[ -2 (1 + c) t (1 + t) \sigma_1 \sigma_2^2 \sigma_3 \]

\[ + (3 + 4 t - 4 c^2 t - 2 t^2 - 2 c t^2) \sigma_1^2 \sigma_2 \sigma_3 \]

\[ + (-1 + 2 c + 2 t + 2 c t + 2 t^2 + 2 c t^2) \sigma_2^3 \sigma_3 \]

\[ + (-1 + 2 c + 2 t + 2 c t + 2 t^2 + 2 c t^2) \sigma_1 \sigma_3 \]

\[ + 4 (-1 + c) (1 + c) (1 + t) \sigma_1 \sigma_2 \sigma_3^2, \]

$$Q_4(t) 2 (1 + c) =$$

\[ 2 (1 + c) t^3 \sigma_1^3 \sigma_2^3 \sigma_3 \]

\[ -2 (1 + c) t^2 (1 + t) \sigma_1 \sigma_2^4 \sigma_3 \]

\[ -2 (1 + c) t^2 (1 + t) \sigma_1^4 \sigma_2 \sigma_3 \]

\[ + t (3 + 4 t - 4 c^2 t - 2 t^2 - 2 c t^2) \sigma_1^2 \sigma_2^2 \sigma_3 \]

\[ + t (-1 + 2 c + 2 t + 2 c t + 2 t^2 + 2 c t^2) \sigma_2^3 \sigma_3 \]

\[ + t (-1 + 2 c + 2 t + 2 c t + 2 t^2 + 2 c t^2) \sigma_1^3 \sigma_3 \]

\[ + 4 (-1 + c) (1 + c) t (1 + t) \sigma_1 \sigma_2 \sigma_3^5 \]

\[ -2 (1 + c) t^2 (1 + t) \sigma_1^3 \sigma_2^4 \sigma_4 \]

\[ + 2 (1 + c) t (1 + t)^2 \sigma_1 \sigma_2^5 \sigma_3 \sigma_4 \]

\[ + t (3 + 4 t - 4 c^2 t - 2 t^2 - 2 c t^2) \sigma_1^4 \sigma_2^2 \sigma_3 \]

\[ + 2 (1 + t) (-2 + c - 2 t + 2 c t + 4 c^2 t + 3 t^2 + 3 c t^2) \sigma_1^2 \sigma_2 \sigma_3 \]

\[ + (1 + t) (-2 - c - 2 t - 2 c t - 2 t^2 - 2 c t^2) \sigma_2^2 \sigma_3 \]

\[ + t (-1 + 2 c + 2 t + 2 c t + 2 t^2 + 2 c t^2) \sigma_1^5 \sigma_3 \sigma_4] \]

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\[
+ (1 - 2c - 4t + 4c^2 t - 2t^2 + 14c t^2 + 8c^2 t^2 - 8c^3 t^2 + 6t^3 + 6ct^3) \sigma_1^3 \sigma_2 \sigma_3 \sigma_4 \\
+ (7 - 8c + 9t - 14ct - 12c^2 t + 8c^3 t - 2t^2 - 6ct^2 - 4c^2 t^2 - 2t^3 - 2ct^3) \sigma_1^3 \sigma_2 \sigma_3 \sigma_4 \\
+ t (3 - 14c + 8c^3 - 6t - 14ct + 8c^3 t - 6t^2 - 6ct^2) \sigma_1^2 \sigma_3 \sigma_4 \\
+ 2(-1 + c) (1 - 2 + 2c - 2ct - 2t^2 - 2ct^2) \sigma_2 \sigma_3 \sigma_4 \\
+ t (-1 + 2c + 2t + 2ct + 2t^2 + 2ct^2) \sigma_1^4 \sigma_2^3 \sigma_4^2 \\
+ (1 + t) (1 - 2c - 2t - 2ct - 2t^2 - 2ct^2) \sigma_1 \sigma_2 \sigma_3 \sigma_4^2 \\
+ 4 (-1 + c) (1 + c) t (1 + t) \sigma_1^5 \sigma_2 \sigma_3 \sigma_4^2 \\
+ (7 - 8c + 9t - 14ct - 12c^2 t + 8c^3 t - 2t^2 - 6ct^2 - 4c^2 t^2 - 2t^3 - 2ct^3) \sigma_1^3 \sigma_2^2 \sigma_3 \sigma_4^2 \\
+ (-5 + 14c - 8c^2 - 6t - 2ct + 4c^2 t - 6t^2 - 14ct^2 + 8c^3 t^2 - 4t^3 - 4ct^3) \sigma_1 \sigma_2^3 \sigma_3 \sigma_4^2 \\
+ t (3 - 14c + 8c^3 - 6t - 14ct + 8c^3 t - 6t^2 - 6ct^2) \sigma_1^4 \sigma_3^2 \sigma_4^2 \\
+ 2(-5 + 14c - 8c^2 - 2t + 8ct - 6c^2 t - 8c^3 t + 8c^4 t + t^2 - 11ct^2 - 4c^2 t^2 \\
\quad + 8c^3 t^2 - 3t^3 - 3ct^3) \sigma_1 \sigma_2 \sigma_3 \sigma_4^2 \\
+ (1 + 4c - 4c^2 + 2t) (-1 + 2c + 2t + 2ct + 2t^2 + 2ct^2) \sigma_2 \sigma_3^2 \sigma_4^2 \\
+ (-4c + 12c^2 - 8c^3 - 3t + 22ct - 16c^3 t + 6t^2 + 22ct^2 - 16c^3 t^2 + 6t^3 + 6ct^3) \sigma_1 \sigma_3 \sigma_4^2 \\
+ 2 (-1 + c) (1 - 2c - 2t - 2ct - 2t^2 - 2ct^2) \sigma_1 \sigma_2 \sigma_4^3 \\
+ (1 + 4c - 4c^2 + 2t) (-1 + 2c + 2t + 2ct + 2t^2 + 2ct^2) \sigma_1^2 \sigma_2 \sigma_4^3 \\
+ (-4c + 12c^2 - 8c^3 - 3t + 22ct - 16c^3 t + 6t^2 + 22ct^2 - 16c^3 t^2 + 6t^3 + 6ct^3) \sigma_1 \sigma_3 \sigma_4^3 \\
+ (9 - 30c + 20c^2 + 16c^3 - 16c^4 + 8t - 16ct + 8c^2 t + 16c^3 t - 16c^4 t \\
\quad + 2t^2 + 10ct^2 - 8c^3 t^2 + 2t^3 + 2ct^3) \sigma_1 \sigma_2 \sigma_3 \sigma_4^3 \\
+ (4c - 4c^2 + t) (1 - 2c - 2t - 2ct - 2t^2 - 2ct^2) \sigma_3 \sigma_4^3 \\
+ (4c - 4c^2 + t) (1 - 2c - 2t - 2ct - 2t^2 - 2ct^2) \sigma_1 \sigma_4^4.
\]
Table Captions

**Table 1** Complex Liouville Theory assignments between exponential operators and primary fields for different choices of the screening operator.

**Table 2** Primary operators in ZMS reduced models for which the form factors have been computed in literature.

| Screening operator | Deformation | $B$ | $k_{m,n}$ |
|--------------------|-------------|-----|-----------|
| $e^{-2g\phi}$     | $e^{g\phi} = \phi_{1,2}$ | $2\frac{r}{r-2s}$ | $(n-1)-(m-1)\frac{s}{r}$ |
| $e^{-2g\phi}$     | $e^{g\phi} = \phi_{2,1}$ | $2s-2r$ | $(m-1)-(n-1)\frac{s}{r}$ |
| $e^{g\phi}$       | $e^{-2g\phi} = \phi_{1,5}$ | $\frac{4s}{2s-1}$ | $\frac{1}{2}((1-n)-(1-m)\frac{s}{r})$ |
| $e^{g\phi}$       | $e^{-2g\phi} = \phi_{5,1}$ | $\frac{4s}{2s-1}$ | $\frac{1}{2}((1-m)-(1-n)\frac{s}{r})$ |

**Table 1**

| Model | Deformation | Primaries analyzed | $F_1/F_0$ | Reference |
|-------|-------------|--------------------|----------|-----------|
| $\mathcal{M}_{2,5}$ | $\phi_{1,2}$ | $\phi_{1,2}$ | 0.8372182$i$ | [16] |
| $\mathcal{M}_{2,7}$ | $\phi_{1,2}$ | $\phi_{1,2}$ | 0.8129447$i$ | [33] |
|       |             | $\phi_{1,3}$ | 1.245504$i$ | [33] |
| $\mathcal{M}_{2,9}$ | $\phi_{1,2}$ | $\phi_{1,2}$ | 0.7548302$i$ | [33] |
|       |             | $\phi_{1,3}$ | 1.288576$i$ | [33] |
|       |             | $\phi_{1,4}$ | 1.564863$i$ | [33] |
| $\mathcal{M}_{3,4}$ | $\phi_{1,2}$ | $\phi_{1,2}$ | $-0.6409021$ | [19] [21] |
|       |             | $\phi_{2,1}$ | $-3.706584$ | [21] |
| $\mathcal{M}_{4,5}$ | $\phi_{1,2}$ | $\phi_{1,2}$ | $-0.8113145$ | [20] |
| $\mathcal{M}_{6,7}$ | $\phi_{1,2}$ | $\phi_{1,2}$ | $-0.9499626$ | [20] |
| $\mathcal{M}_{2,9}$ | $\phi_{1,4} \equiv \phi_{1,5}$ | $\phi_{1,2}$ | $-0.5483649$ | [33] |
|       |             | $\phi_{1,3}$ | $-1.476188$ | [33] |
|       |             | $\phi_{1,4}$ | $-2.169493$ | [33] |

**Table 2**
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Figure 1: Plot of the wave function renormalization constant $Z(B)$ of the Bullough–Dodd model.