Long-term properties of finite-correlation time isotropic stochastic systems

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We consider finite-dimensional systems of linear stochastic differential equations
\begin{equation}
\partial_t x_k(t) = A_{kp}(t) x_p(t),
\end{equation}
where $A(t)$ is a stationary continuous real $d \times d$ matrix stochastic process with known law. It is assumed to be statistically isotropic, i.e. $A(t) O$ and $A(t)$ are identically distributed at any $t$ and have the same time correlations for any $O \in O(d)$.

The formal solution of (1) can be written as
\begin{equation}
x_k(t) = Q_{kp}(t) x_p(0),
\end{equation}
where the evolution matrix $Q$ satisfies the equation:
\begin{equation}
\partial_t Q(t) = A(t) Q(t), \quad Q(0) = 1
\end{equation}
For each continuous realization of $A(t)$ there exists a well-defined solution of (2) given by the Volterra product integral [5] (in quantum-mechanical terminology this is also called
\[ Q(t) = \prod_{\tau=0}^{t} (1 + A(\tau) \, d\tau) \] (3)

So, the solutions of (2) can be interpreted as continual products of random matrices.

In many physical problems (e.g., stochastic dynamics and turbulent transport) one is basically interested in the infinite-time asymptotic behavior of the norms of the basis multivectors [6, 7]:

\[
    E_1 = ||Q\vec{e}_1||, \quad E_2 = ||Q\vec{e}_1 \wedge Q\vec{e}_2||, \quad E_d = ||Q\vec{e}_1 \wedge ... \wedge Q\vec{e}_d||
\]

For instance, the absolute value of a passive vector advected by a random smooth flow is \( E_1 \); the density of an advected passive scalar in presence of weak molecular diffusion is \( \prod_{k=1}^{d} \min\{E_k^{-1}, 1\} \) [8, 9]; the energy of magnetic field generated by a turbulent MHD flow with strong conductivity is \( \min\{E_1^2/E_2^2, E_2^2/E_3^2\} \) [10–12].

From the multiplicative Oseledets theorem [13] it follows that almost surely, there exist the limits

\[
    \lim_{T \to \infty} \frac{1}{T} \ln E_k = \lambda_1 + \cdots + \lambda_k, \quad k = 1 \ldots d
\] (4)

The constants \( \lambda_k \) are called Lyapunov exponents (LE). Complete information about asymptotic behavior of the set \( \{E_k\} \) is contained in the generalized Lyapunov exponents (GLE) [14] defined by

\[
    w(\eta_1, ..., \eta_d) = \lim_{T \to \infty} \frac{1}{T} \ln \left\langle E_{\eta_1}^{e_1} \cdots E_{\eta_2}^{e_2} \cdots E_{\eta_d}^{e_d} \right\rangle
\] (5)

where \( \langle ... \rangle \) is the average over all realizations of \( A(t) \), \( \eta_k \in \mathbb{R} \). For example, the LE can be expressed in terms of GLE:

\[
    \lambda_k = \frac{\partial}{\partial \eta_k} w(0) \] (6)

The calculation of the values of (4) and (5) is the subject of this paper.

The most part of previous investigations considered the case of Gaussian \( A(t) \) that is delta-correlated in time. For this case, the exact expressions for LE were found in [15–17]. The delta-correlated process typically has nowhere continuous paths, so Eq. (2) requires a stochastic convention; the known results for that case refer to the Stratonovitch convention [18, 19].

However, in many physical applications random processes acting as multiplicative noise are, generally, non-Gaussian, and have finite (non-zero) correlation time. For continuous processes the solutions (3) are well defined, so there is no need in stochastic conventions. It is well known [20, 21] that for such systems, the central limit theorem 'is not valid' in the sense that all connected correlators (cumulants) of \( A(t) \) contribute equivalently to the LE and GLE. So, the result may differ essentially from the Gaussian case.
To deal with non-zero correlation time, the renovation model has been developed by different authors [22, 23]: non-stationary piecewise constant process is substituted for the stationary continuous process \( A(t) \). Alternatively, we stay in the frame of stationarity and use the large deviation principle; in quantum mechanics and quantum field theory the corresponding technics is known as the low-frequency limit. For the processes \( A_{ij}(t) \) that satisfy the large deviations principle, this allows to get exact expressions for LE and GLE: it appears that they are completely determined by the rate function of the diagonal elements \( A_{kk} \).

A. Iwasawa decomposition

To get explicit expressions for the LE and GLE, one performs the Iwasawa decomposition of the evolution matrix:

\[
Q = R D Z
\]

where \( R \) is orthogonal matrix, \( D \) is positive diagonal, and \( Z \) is upper-triangular unipotent matrix:

\[
RR^T = I, \quad D_{ij} = \delta_{ij} D_i, \quad D_i > 0, \quad Z_{i<j} = 0, \quad Z_{jj} = 1
\]

Then

\[
E_k = D_1 \ldots D_k
\]

and the LE can be expressed by

\[
\lambda_k = \lim_{T \to \infty} \frac{1}{T} \ln D_k, \quad \lambda_1 \geq \cdots \geq \lambda_k
\] (7)

The expression for the GLE takes the form

\[
w(\eta_1, \ldots, \eta_d) = \lim_{T \to \infty} \frac{1}{T} \ln \langle D_1^{\eta_1} \cdots D_d^{\eta_d} \rangle
\] (8)

The statistical properties of \( R, D \) and \( Z \) behave differently during the evolution (2); we are interested only in the statistics of \( D \).

In the next Section we consider the main strategy and the application of the large deviation theory on a simple one-dimensional example. Then we proceed to the multi-dimensional case, and make use of the isotropy to calculate the GLE. In conclusion, we reformulate the results in terms of the 'effective delta-process', which is a useful tool for physical applications.

II. ONE-DIMENSIONAL CASE

A. Rate function and GLE

Consider one-dimensional differential equation with multiplicative noise:

\[
\partial_t x(t) = \xi(t) x(t), \quad x(0) = 1
\] (9)
where $\xi(t)$ is a continuous stationary random process with given law. We are interested in the moments $\langle x^n \rangle$, $\eta \in \mathbb{R}$.

For any realization of $\xi(t)$, the solution of (9) is

$$x(T) = e^{\int_0^T \xi(t) dt}$$

We assume that $\xi(t)$ satisfies the large deviation principle [24], i.e., that the probability density of its time average

$$\frac{1}{T} \int_0^T \xi(t) dt = \bar{\xi}$$

satisfies at large $T \to \infty$ the relation:

$$\rho_{\bar{\xi}}(a) \sim e^{-TJ(a)}$$

where the sign $\sim$ means that there exists the limit

$$\lim_{T \to \infty} \frac{1}{T} \ln \rho_{\bar{\xi}}(a) = -J(a)$$

Here $J(a)$ is the rate function (Cramer function). Then

$$\langle x^n(T) \rangle = \int e^{T\eta a} \rho_{\bar{\xi}}(a) da \sim e^{Tw(\eta)} \quad (10)$$

where $w(\eta)$ is the Legendre transform of the rate function,

$$w(\eta) = \sup_a (\eta a - J(a))$$

(11)

This proves the existence of the limit

$$\lim_{T \to \infty} \frac{1}{T} \ln \langle x^n(T) \rangle = w(\eta), \quad (12)$$

$w(\eta)$ is the GLE of the process $\xi(t)$. We note that, according to (10), the function $w(\eta)T$ at large $T$ coincides with the cumulant generating function of the integral $\int_0^T \xi(t) dt$.

Let the cumulants (or connected correlators)

$$\langle \xi(t_1) \ldots \xi(t_n) \rangle = W^{(n)}(t_1 - t_2, \ldots, t_1 - t_n) \quad (13)$$

be regular fast-decaying functions (i.e., let $\int W^{(n)} dt_2 \ldots dt_n$ converge). Then the cumulant-generating functional

$$W[\eta(t)] = \ln \left( e^{\int \eta(t) \xi(t)} \right)$$

(14)

can be expanded into the infinite series in $\eta(t)$:

$$W[\eta(t)] = \sum_n \frac{1}{n!} \int W^{(n)}(t_1 - t_2, \ldots, t_1 - t_n) \eta(t_1) \ldots \eta(t_n) dt_1 \ldots dt_n \quad (15)$$
In accordance with (14),
\[ \langle x^n(T) \rangle = \left\langle e^{\int_0^T \xi(t)dt} \right\rangle = e^{W_\eta[I_{[0,1]}(t/T)]} \]
(16)
where \( I_{[0,1]} \) is the indicator function of the segment \([0, 1]\). Substituting (15) we find
\[ \lim_{T \to \infty} \frac{1}{T} \ln \langle x^n(T) \rangle = \sum_n \frac{w^{(n)}}{n!} \eta^n \]
(17)
where
\[ w^{(n)} = \int W^{(n)}(t_1 - t_2, \ldots, t_1 - t_n) dt_2 \ldots dt_n \]
(18)
Comparing (10) with (17) we get
\[ w(\eta) = \sum_n \frac{w^{(n)}}{n!} \eta^n \]
(19)
From (13), (19) we see that the long-time asymptotics of the moments \( \langle x^n \rangle \), as well as the GLE, do not depend on the details of the cumulants and are determined only by their integrals (18).

**B. Low-frequency limit.**

We now consider an alternative way to find the GLE based on the Lagrangian formalism. The aim is to derive the relation between the Lagrangian density and the rate function.

So, now we start from the probability density functional \( \wp[\xi(t)] \), which can be presented in the form
\[ \wp[\xi] \sim e^{-S[\xi]}, \quad S[\xi(t)] = \int dt L(\xi, \partial_t \xi, \partial^2_t \xi, \ldots) \]
Here \( L \) is the Lagrangian density. For example, the case \( L = \frac{1}{2} (\partial_t \xi)^2 + \frac{1}{2} \xi^2 \) corresponds to Ornstein-Uhlenbeck process with the only non-zero cumulant \( W^{(2)}(t_1 - t_2) = \frac{1}{2\epsilon} e^{-|t_1 - t_2|/\epsilon} \).
Generally, \( L \) is non-local, and contains the derivatives of all orders. The generating functional can then be written in the form of the Feynman-Kac integral [25]:
\[ e^{W[\eta(t)]} \sim \int [d\xi] e^{-\int dt [L(t) - \eta(t)\xi(t)]} \]
(20)

In the previous subsection, we have seen that the logarithm of the moments \( \langle x^m \rangle \) is determined by the functions \( \eta(t) \) that are constant in the time range \([0, T]\) (16). For this reason, we are interested in the values of \( W[\eta(t)] \) on ‘slow-changing’ functions with characteristic time scale \( T: \eta_T(t) = \eta(t/T) \) (the low-frequency limit). In (20), we rescale the time \( t = \tau T \) and change accordingly the integration variable: \( \xi(t) = \xi_T(\tau) \). Then from (20) we obtain
\[ e^{W[\eta_T(t)]} \sim \int [d\xi_T] e^{-T \int d\tau [L_T(\tau) - \eta(\tau)\xi_T(\tau)]} \]
(21)
where
\[ \mathcal{L}_T(\tau) = \mathcal{L}\left(\xi_T, \frac{1}{T} \partial_\tau \xi_T, \frac{1}{T^2} \partial^2_\tau \xi_T, \ldots\right) \]

As \( T \to \infty \), one can substitute the rate function
\[ J(\xi_T) = \mathcal{L}(\xi_T, 0, 0, \ldots) + C \] (22)
for \( \mathcal{L}_T \). Here \( C \) is a normalization constant. Then we estimate the integral by means of the saddle point method:
\[ \int [d\xi_T] e^{-T \int d\tau (J(\xi_T(\tau)) - \eta(\tau) \xi_T(\tau))} \sim e^{-T \int d\tau (J(\xi_s(\tau)) - \eta(\tau) \xi_s(\tau))} \]
where \( \xi_s(\tau) \) is defined by the minimum condition
\[ \frac{\partial J(\xi_s)}{\partial \xi} = \eta(\tau) \] (23)
Eventually, we get
\[ e^{W_T(\eta(t))] = e^{W_T[\eta(t/T)]} \sim e^{T \int w(\eta(\tau)) d\tau} \] (24)
where
\[ w(\eta) = \sup_{\xi} (\eta \xi - J(\xi)) \] (25)
The normalization condition \( W[0] = 0 \) results in the claim \( w(0) = 0 \). Now, substituting (24) in (16) we find
\[ \langle x^n(\tau) \rangle \sim e^{T \int_{0}^{1} w(\eta) d\tau} = e^{Tw(\eta)} \]
Comparing this with (10), (13) we see that the function \( w(\eta) \) defined in this subsection coincides with the GLE found in (11) and (19). From (11) and (25) it also follows that the function \( J(\xi) \) defined in (22) coincides with the rate function.
So, both ways to determine the rate function are equivalent. In multi-dimensional case the second way appears to be more convenient.

III. MULTI-DIMENSIONAL EQUATION

A. Equation for the Iwasawa components

Now we return to the matrix equation (2) where \( A(t) \) is a stationary continuous stochastic process with regular fast-decaying connected correlations:
\[ \langle A_{i_1 j_1}(t_1) \ldots A_{i_n j_n}(t_n) \rangle_c = W_{i_1 j_1 \ldots i_n j_n}^{(n)} (t_1 - t_2, \ldots, t_1 - t_n) \] (26)
and non-local Lagrangian density: \( \mathcal{L}_A(\mathbf{A}, \partial_1 \mathbf{A}, \partial^2_1 \mathbf{A}, \ldots) \). The well-defined solution (3) exists for any continuous realization of \( \mathbf{A}(t) \), but noncommutativity makes it difficult to
use: there is a $T$-exponential instead of a usual exponential, and it seems impossible to apply the large deviations approach for $\int A dt$. However, we will see in what follows that this is possible in the case of isotropic law of $A$ at least for the important 'diagonal' (in the sense of Iwasawa decomposition) part of the evolution matrix.

To separate the Iwasawa components, we rewrite the equation (2) in the form

$$ A = \partial_t Q Q^{-1} $$

and substitute the Iwasawa decomposition for $Q$. We obtain

$$ A = R X R^{-1}, \quad X = \xi + \zeta + \theta $$

where

$$ \xi = (\partial_t D) D^{-1}, \quad \zeta = D (\partial_t Z) Z^{-1} D^{-1}, \quad \theta = R^{-1} (\partial_t R) $$

The matrices $\xi$, $\zeta$ and $\theta$ are diagonal, nilpotent upper triangular and antisymmetric, respectively. The equation (28) can be rewritten as

$$ \partial_t D = \xi D $$
$$ \partial_t Z = D^{-1} \zeta D Z $$
$$ \partial_t R = R \theta $$

Thus, treating $\xi$, $\zeta$ and $\theta$ as independent variables, one could separate the equations for $D$ and for $R$. Moreover, the elements $D_i$ satisfy one-dimensional equations (29) same as the equation (9). So, (8) takes the form

$$ w(\eta_1, ..., \eta_d) \equiv w_\xi(\eta_1, ..., \eta_d) = \lim_{T \to \infty} \frac{1}{T} \ln \left( \int_0^T e^{\int_0^T (\xi_1 m + ... + \xi_d n_d) dt} \right) $$

To calculate this, it would be enough to know the rate function of $\xi = \{\xi_1, ..., \xi_d\}$ and make use of (25). So, in the next subsection we discuss the relation between the Lagrangian densities of $X$ and $A$.

**B. Change of variables**

One can consider (27) as a functional transformation from $A$ to $X$-variables,

$$ A = R[X]XR^{-1}[X] $$

where the dependence $R(X)$ is determined by (31). To find the probability density of $X(t)$, one has to calculate the Jacobian:

$$ J[X] = \text{Det} \left( \frac{\delta A_{ij}(t)}{\delta X_{kp}(t')} \right) $$
It was calculated, e.g., in [21]:

\[ J[X] = e \int \text{tr} (\eta_0 X(t)) \, dt \]  

(35)

where

\[ (\eta_0)_{kp} = \left( \frac{d + 1}{2} - k \right) \delta_{kp} \]  

(36)

In the Appendix A to this paper we derive this result taking the continuous limit of a stochastic difference equation.

With account of (34) and (33), from the condition \( \mathcal{P}(X) [dX] = \mathcal{P}(A) [dA] \) we get the expression for the Lagrangian density of \( X \):

\[ \mathcal{L}_X = - \text{tr} (\eta_0 X) + \mathcal{L}_A(A, \partial_t A, \partial^2_t A, \ldots) \]

where \( A \) is a function of \( X \) in accordance with (33). From (31) it follows that for any \( F(t) \),

\[ \partial_t (R(t) F(t) R^{-1}(t)) = R(t) (D_t F(t)) R^{-1}(t) \]  

(37)

where

\[ D_t F = \partial_t F + [\theta, F] \]  

(38)

where \([a, b] = ab - ba\). Substituting this for the arguments of \( \mathcal{L}_A \), we obtain

\[ \mathcal{L}_X = - \text{tr} (\eta_0 X) + \mathcal{L}_A(RXR^{-1}, R(D_t X) R^{-1}, R(D^2_t X) R^{-1}, \ldots) \]  

(39)

This expression contains \( R \), which is the Volterra product integral of the components of \( X \), \( R[X, t] = \prod_{\tau=0}^t (1 + \theta(\tau) \, d\tau) \). However, we will see below that in the case of isotropic processes \( R \) vanishes.

### IV. ISOTROPIC SYSTEMS

We now make use of the claim that \( A \) is isotropic, in particular, \( A(t) \) has the same probability density as \( OA(t)O^T \) for any \( O \in O(d) \). For such processes, the Lagrangian density \( \mathcal{L}_A \) can be presented as a sum of different combinations of traces with arguments containing products of \( A, A^T \), and their derivatives. As one substitutes \( RXR^{-1} \) for \( A \), the ‘plates’ \( R, R^{-1} \) in the expressions vanish; but, in accordance with (37) and (39), all time derivatives in the expression for the Lagrangian density change to ‘long derivatives’ (38),

\[ A \mapsto X, \quad \partial^p_t A \mapsto D^p_t X \]

As we proceed from \( \mathcal{L}_X \) to the rate function as in (22), we have to set all the time derivatives of \( X \) zero. But the commutators stay in their places, so the rate function takes the form

\[ J_X(X) = \mathcal{L}_X(X, 0, 0, \ldots) + C = - \text{tr} (\eta_0 X) + \mathcal{L}_A(X, [\theta, X], [\theta, [\theta, X]], \ldots) + C \]  

(40)
One can separate the rate function of the matrix $A$:

$$J_A(A) = \mathcal{L}_A(A, 0, \ldots)$$

and present $J_X$ in the form:

$$J_X(X) = -\text{tr}(\eta_0 X) + J_A(X) + \delta J(X) + C,$$  \hspace{1cm} (41)

$$\delta J(X) = \mathcal{L}_A(X, [\theta, X], [\theta, [\theta, X]], \ldots) - \mathcal{L}_A(X, 0, \ldots)$$

We define $w_X(\mu)$ for the matrix process $X$ in the same way as in (11), (25),

$$w_X(\mu) = \sup_X \left( \text{tr}(\mu X) - J_X(X) \right) = \lim_{T \to \infty} \frac{1}{T} \ln \left( \frac{1}{T} \int_0^T \text{tr}(\mu X) dt \right)$$  \hspace{1cm} (42)

To this purpose, we have to find the extremum point $X_s$ analogous to $\xi_s$ in (23):

$$\frac{\partial J_X(X_s)}{\partial X_{qr}} = \mu_{qr}$$  \hspace{1cm} (43)

This equation system is very complicated. However, it can be simplified significantly as we are interested only in the GLE of the diagonal elements $X_{kk} = \xi_k$. From (32) it follows that

$$w_\xi(\eta_1, \ldots, \eta_d) = w_X(\eta)$$  \hspace{1cm} (44)

where $\eta = \text{diag}(\eta_1, \eta_2, \ldots, \eta_d)$. So, to find $w_\xi$ we can restrict ourselves to the diagonal matrices $\mu = \eta$ in (43), so the extremum condition takes the form:

$$\frac{\partial J_X(X_s)}{\partial X_{qr}} = 0, \quad q \neq r; \quad \frac{\partial J_X(X_s)}{\partial X_{qq}} = \eta_q, \quad q = 1..d$$  \hspace{1cm} (45)

This system has the diagonal solution:

$$X_s = \xi_s$$  \hspace{1cm} (46)

where $\xi_s$ satisfies the relation

$$\frac{\partial J_A(\xi_s)}{\partial X_{qq}} - (\eta_0)_{qq} = \eta_q$$  \hspace{1cm} (47)

Indeed, we recall that $X = \xi + \theta + \zeta$, and $J_X(X)$ is a combination of traces $\text{tr} f(\xi, \theta, \zeta)$ where $f$ is some product of the matrices. Since $\xi$ is diagonal and $\theta, \zeta$ have zero diagonal elements, each summand in $J_X$ contains either zero or more than one of the matrices $\theta, \zeta$. Taking the derivative with respect to $X_{qr}, q \neq r$ leaves the summands with at least one multiplier $\theta$ or $\zeta$, and subsequent setting $X = \xi_s$ makes them zero; so, the first equation in (45) holds automatically. On the other hand, each summand in $\delta J(X)$ contains at least one $\theta$ as a multiplier, so $\partial\delta J/\partial X_{qq} = 0$.

We note that $\theta_s = 0$ and thus, $\delta J(X_s) = 0$. From (44), (42), (41) it then follows

$$w_\xi(\eta_1, \eta_2, \ldots, \eta_d) = \text{tr}((\eta + \eta_0)\xi_s) - J_A(\xi_s) - C$$  \hspace{1cm} (48)
The statistical isotropy of $A$ implies $\langle A_{i\neq j} \rangle = 0$. Thus, $\partial J_A/\partial A_{qr}(\xi)$ = 0 for $q \neq r$, and $J_A(\xi)$ coincides with the rate function of the diagonal elements $\alpha_q = A_{qq}$:

$$J_\alpha(\alpha_1, \ldots, \alpha_d) = J_A(\alpha),$$

$$\alpha = \text{diag}(\alpha_1, \ldots, \alpha_d)$$

So, (48) proves that the GLE and, as a consequence, also the Lyapunov exponents are completely determined by the rate function of the diagonal elements of $A$.

By analogy with (42), we define the local cumulant-generating function of these diagonal elements:

$$w_\alpha(\eta) = \sup_{\alpha} (\text{tr} (\eta \alpha) - J_\alpha(\alpha_1, \ldots, \alpha_d)) = \lim_{T \to \infty} \frac{1}{T} \ln \left\langle e^{\int_0^T \text{tr} (\eta \alpha) dt} \right\rangle$$

It is related to the cumulants of $\alpha_i$ by

$$w_\alpha(\eta_1, \ldots, \eta_d) = \sum_{n=1}^{\infty} \sum_{i_1 \ldots i_n} \frac{w^{(n)}_{i_1 \ldots i_n}}{n!} \eta_{i_1} \ldots \eta_{i_n}$$

where

$$w^{(n)}_{i_1 \ldots i_n} = \int dt_1 \ldots dt_n \langle \alpha_{i_1}(t_1) \ldots \alpha_{i_n}(t_n) \rangle_c$$

From (48) we see that $w_\xi(\eta_1, \ldots, \eta_d)$ can be reduced to $w_\alpha(\eta + \eta_0)$; with account of the normalization condition $w_\xi(0) = \ln(1) = 0$, we obtain GLE and the Lyapunov exponents:

$$w_\xi(\eta_1, \eta_2, \ldots, \eta_d) = w_\alpha(\eta + \eta_0) - w_\alpha(\eta_0)$$

$$\lambda_k = \frac{\partial}{\partial \eta_k} w_\alpha(\eta_0)$$

$$(\eta_0)_{kp} = \left( \frac{d + 1}{2} - k \right) \delta_{kp}$$

A. Gaussian process

Consider now the important particular case: let $A(t)$ be Gaussian continuous process with zero mean and given second-order correlator:

$$\langle A_{ij}(t) \rangle = 0, \quad \langle A_{ij}(t_1) A_{kp}(t_2) \rangle_c = K_{(ij)(kp)}(t_1 - t_2)$$

where

$$K_{(ij)(kp)} = -a \delta_{ij} \delta_{kp} + b \delta_{ik} \delta_{jp} + c \delta_{ip} \delta_{jk}$$

where $a, b, c$ are some constants and $\Phi(t)$ is some regular even fast decaying function, $\int \Phi(t) dt = 1$. The form (52) of $K_{(ij)(kp)}$ is determined by the isotropy.
From (49) it follows the expression for the cumulant generation function of the diagonal elements of $A$:

$$w_\alpha(\eta) = -\frac{a}{2}(\text{tr} \, \eta)^2 + \frac{b + c}{2} \text{tr} \, \eta^2$$

From (50) we get

$$w_\xi(\eta_1, ..., \eta_d) = (b + c) \sum_{k=1}^{d} \left( \frac{d + 1}{2} - k \right) \eta_k + \frac{1}{2} \sum_{k,p=1}^{d} ((b + c) \delta_{kp} - a) \eta_k \eta_p, \quad (53)$$

$$\lambda_k = (b + c) (d + 1 - 2k) \quad (54)$$

In many applications one considers traceless matrices $\text{tr} \, A = 0$. With this additional requirement, the coefficients $a, b, c$ are associated by the relation $b + c - ad = 0$, which can be taken into account in (53).

V. CONCLUSION. EFFECTIVE $\delta$-PROCESS

So, in the paper we consider linear stochastic equations systems (1) with statistically isotropic matrix random process $A(t)$ that have regular fast-decaying connected correlations (26). We find the explicit expressions (50) for the generalized Lyapunov exponents in terms of rate functions of the diagonal elements of $A$.

Now we reformulate the results in the form that is useful for physical applications.

We find that the correlations of the diagonal elements of $A$ contribute to GLE only via their integrals:

$$w_{i_1...i_n}^{(n)} = \int dt_2...dt_n W_{i_1j_1...i_nj_n}^{(n)}(t_1 - t_2, ..., t_1 - t_n) \quad \text{(no summation)}$$

In (50), the GLE are expressed in terms of the cumulant-generating function of the diagonal elements of the matrix $A$,

$$w_\alpha(\eta_1, ..., \eta_d) = \sum_{n=1}^{\infty} \sum_{i_1...i_n} w_{i_1...i_n}^{(n)} \eta_{i_1}...\eta_{i_n} / n!$$

So, there exists the sequence of formal random processes $A_\epsilon$ with connected correlations

$$\frac{1}{\epsilon^{n-1}} W_{i_1j_1...i_nj_n}^{(n)} \left( \frac{t_1 - t_2}{\epsilon}, ..., \frac{t_1 - t_n}{\epsilon} \right)$$

which produce identical GLE. Going to the formal limit $\epsilon \to 0$ one can define the 'effective $\delta$-process' $A_0$ with singular correlation functions

$$\Delta_{i_1...k_p}^{(n)} = w_{i_1...k_p}^{(n)} \delta(t_1 - t_2) ... \delta(t_1 - t_n) \quad (55)$$

This formal process provides the same GLE and allows to split correlations and get closed equations for different averages (see Appendix B). Despite its formal nature, it is a handy
instrument for calculations for the problems that appear in theory of turbulence, turbulent transport, kinematic dynamo in turbulent flows etc [26–28].

In the particular case of the Gaussian isotropic processes, the result (54) coincides with the well-known expressions [18, 19] obtained for the differential equation $Q = dW(t)/dtQ$ in the frame of Stratonovich stochastic convention, $dQ(t) = dW(t) \circ Q(t)$. Thus, for these processes the effective $\delta$-process has not only formal sense but can also be expressed in terms of the Wiener process’ derivative. This corresponds to the Wong-Zakai theorem [29].

We also make some notes on the relation between our approach and the renovation model [23]. In our approach, the noise is stationary for any correlation time while in the renovation model it becomes stationary only as $\tau \to 0$. Actually, the results of both approaches coincide as $\tau \to 0$. Possibly, the non-stationarity can be taken into account in our approach by means of corrections to $w_{i_1\cdots i_n}^{(n)}$. However, this is a subject for separate issue.

It is also important to note that even in the isotropic case, one can substitute the effective $\delta$-process for the real matrix process only when calculating the long-term asymptotics of $E_k$ and their combinations. For the quantities that depend on the non-diagonal elements $\theta_{ij}$ and $\zeta_{ij}$ (e.g., the coordinates $x_k$), the asymptotic behavior is determined not only by the rate function of the matrix elements $A_{ij}$ but also by the shape of their correlation functions. This illustrates the fact that the possibility to introduce the effective $\delta$-process is a non-trivial feature of multi-dimensional isotropic stochastic systems with multiplicative noise.

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Appendix A: Calculation of the Jacobian (34)

The Jacobian was calculated in [21] by means of operations with continuous stochastic processes and functional integrals. The notion of Jacobian is difficult to define for a continuous process. It is natural to define it for a discrete process and then take the continuous limit. The result must not depend on the way of discretization. Here we calculate the Jacobian for one particular discretization of the random process $Q$ and the corresponding stochastic difference equation. Then we show that for another choice of the discretization with the same continuous limit, the result is the same. In analogous way, one can check that any difference equation with the same continuous limit would lead to the same result.
Discretization à la Stratonovich

So, again we start with the equation (2),

$$\partial_t Q(t) = A(t)Q(t), \quad Q(0) = 1$$

Although $A(t)$ is a finite - correlation time process and thus the differential equation is well defined, we still consider its discrete analog in order to substitute ordinary derivatives for variational derivatives. Thus, we split $T$ into $N$ discrete intervals, $\Delta t$ being much smaller than the correlation time of $A$, and consider the discrete equation:

$$\Delta Q_t \equiv Q_{t+1} - Q_t = A_t \frac{Q_t + Q_{t+1}}{2} \Delta t\quad (56)$$

We use the Stratonovich-type discretization because to calculate the Jacobian, we need the second order accuracy in $\Delta t$, and the Stratonovich choice allows to get this accuracy for $\Delta Q$ without writing the second order derivative.

Multiplying (56) by $Q_t^{-1}$ and by $Q_{t+1}^{-1}$ and taking the sum, we get:

$$A_t \Delta t = \frac{1}{2} (Q_{t+1}Q_t^{-1} - Q_tQ_{t+1}^{-1}) + O(\Delta t^3)$$

Now we make use of the Iwasawa decomposition and substitute $Q_t = R_tD_tZ_t$, $Q_{t+1} = R_{t+1}D_{t+1}Z_{t+1}$. With unified notation $\Delta F_t = F_{t+1} - F_t$ and taking into account $\Delta (F^{-1})_t = -F_t^{-1}\Delta F_tF_t^{-1} + F_t^{-1}\Delta F_tF_t^{-1} + O(\Delta t^3)$ we obtain

$$\begin{align*}
A \Delta t &= \frac{1}{2} (\Delta R_tD_{t+1}Z_{t+1} + R_t\Delta D_tZ_{t+1} + R_tD_t\Delta Z_t) Z_t^{-1}D_t^{-1}R_t^{-1} - \\
&\quad -\frac{1}{2} R_tD_tZ_t(\Delta (Z_t^{-1})D_t^{-1}R_t^{-1} + Z_t^{-1}\Delta (D_t^{-1})R_t^{-1} + Z_t^{-1}D_t^{-1}\Delta (R_t^{-1})) = \\
&= R_t \left[ \frac{1}{2} (R_t^{-1}R_{t+1} - R_{t+1}^{-1}R_t) + \frac{1}{2} (R_t^{-1}R_{t+1}\Delta D_tD_t^{-1} + \Delta D_tD_t^{-1}R_t^{-1}R_t - (\Delta D_tD_t^{-1})^2) + \\
&\quad + \frac{1}{2} (R_t^{-1}R_{t+1}\Delta Z_tZ_t^{-1}D_t^{-1} + D_t\Delta Z_tZ_t^{-1}D_t^{-1}R_t^{-1}R_t - (D_t\Delta Z_tZ_t^{-1}D_t^{-1})^2) \right] R_t^{-1} + O(\Delta t^3)\quad (57)
\end{align*}$$

We denote

$$\theta_t \Delta t = \frac{1}{2} (R_t^{-1}R_{t+1} - R_{t+1}^{-1}R_t) = \frac{1}{2} (R_t^{-1} + R_{t+1}^{-1}) \Delta R_t\quad (58)$$

This is an antisymmetric matrix; from (58) it follows that, with accuracy $O(\Delta t^3)$,

$$\Delta R_t = \frac{R_t + R_{t+1}}{2} \theta_t \Delta t\quad (59)$$

which is the discrete analog to (31) in accord with the Stratonovich approach. Analogously, we claim that

$$\Delta D_t = \xi_t \frac{D_t + D_{t+1}}{2} \Delta t$$

and arrive at

$$\xi_t \Delta t = \Delta D_tD_t^{-1} - \frac{1}{2} (\Delta D_tD_t^{-1})^2 + O(\Delta t^3)$$
Demanding
\[ \Delta Z_t = \left( \frac{D_t + D_{t+1}}{2} \right)^{-1} \zeta_t \left( \frac{D_t + D_{t+1}}{2} \right) \left( \frac{Z_t + Z_{t+1}}{2} \right) \Delta t \]
we also get (up to \( O(\Delta t^3) \) accuracy):
\[ \zeta_t \Delta t = (D + \frac{\Delta D}{2}) \Delta Z \left( \frac{Z_t + Z_{t+1}}{2} \right)^{-1} (D + \frac{\Delta D}{2})^{-1} \]
\[ = \left( 1 + \frac{\Delta D \cdot D^{-1}}{2} \right) D_t \Delta Z_t D^{-1} \left( 1 - \frac{D_t \cdot D_t^{-1}}{2} \right) \left( 1 - \frac{D_t \cdot D_t^{-1}}{2} \right) + O(\Delta t^3) \]
The first summand in the round brackets in (57) is \( \theta_t \); now, we note that the summand in the second brackets is
\[ R_t^{-1} R_{t+1} \xi_t \Delta t + \xi_t \Delta t R_t^{-1} R_{t+1} + O(\Delta t^3) = (1 + \theta_t \Delta t) \xi_t \Delta t + \xi_t \Delta t (1 + \theta_t \Delta t) + O(\Delta t^3) \]
Finally, the third bracket can be written as
\[ R_t^{-1} R_{t+1} \zeta_t \Delta t + \zeta_t \Delta t R_t^{-1} R_{t+1} + O(\Delta t^3) \]
Summarizing, we rewrite (57) as
\[ A_t = R_t \left( X_t + \frac{1}{2} [\theta_t, X_t] \Delta t \right) R_t^{-1} + O(\Delta t^2) \]
where
\[ X_t = \theta_t + \xi_t + \zeta_t \]
Eq. (60) is the discrete analog to Eq. (27).

Calculation of the determinant

So, now we have to calculate the Jacobian \( J = \left| \partial A_{ij, t} / \partial X_{km, t'} \right| \).
First, we note that from (60) and (59) it follows
\[ \partial A_{ij, t} / \partial X_{km, t'} = 0 \quad \text{for any} \quad t < t' \]
(This is the manifestation of causality) Thus, the matrix \( (\partial A / \partial X)_{ij, t; km, t'} \) is a block triangular matrix, and its determinant is equal to the product of the determinants of the diagonal blocks, \( t = t' \):
\[ J = \prod_t \left| \frac{\partial A_{ij, t}}{\partial X_{km, t}} \right| \] \quad (61)
Second, we note that, in accordance with the causality principle, from (59) it follows that the value of the rotation matrix \( R_t \) depends only on the ‘previous-time’ values of \( \theta_{t' < t} \) and does not depend on the ‘simultaneous’ value \( \theta_t \),
\[ \partial R_t / \partial \theta_{t' \geq t} = 0 \]
Thus, in (60) the derivative must be taken only over the multiplier in the square brackets.
We now introduce the multiindices \( \alpha = \{ij\} \) and the \( d^2 \times d^2 \) matrix
\[ R_{ij, km} = R_{ik} R_{jm} \]
Then (60) can be written in the form:

\[ A_\alpha = R_{\alpha\beta} (X_\beta + M_\beta \Delta t) \]

where

\[ M = \frac{1}{2} [\theta, X] \]

So,

\[ \left| \frac{\partial A_\alpha}{\partial X_\gamma} \right| = |R_{\alpha\beta}| \cdot \left| \delta_{\beta\gamma} + \frac{\partial M_\beta}{\partial X_\gamma} \Delta t \right| \]

Since the matrices \( R \) are orthogonal, \( R \) is also orthogonal, i.e., \( RR^T = 1 \). Thus, \( \text{det } R = 1 \).

The second determinant, to an accuracy of \( \sim O(\Delta t) \), can be reduced to the trace of \( \partial M / \partial X \):

\[ \left| \frac{\partial A_\alpha}{\partial X_\gamma} \right| = 1 + \left( \frac{\partial M_\beta}{\partial X_\gamma} \right) \delta_{\beta\gamma} \Delta t + O(\Delta t^2) \] (62)

Finally, we make use of the fact that only the lower triangular part of \( X \) determines the values of \( \theta \), while the diagonal and upper triangular components are 'responsible' for \( \xi \) and \( \zeta \), correspondingly. So,

\[ \theta_{ij} = \begin{cases} X_{ij} & \text{if } i > j, \\ 0 & \text{if } i = j, \\ -X_{ji} & \text{if } i < j \end{cases} \]

In particular, \( \partial \theta_{ij} / \partial X_{km} = 0 \) if \( k \leq m \). Thus, taking the derivative of \( M \), we obtain

\[ \text{tr} \left( \frac{\partial M}{\partial X} \right) = \sum_{i,j} \left( \frac{\partial M_{ij}}{\partial X_{ij}} \right) = \frac{1}{2} \sum_{i,j} (\theta_{ii} - \theta_{jj}) + \sum_{i>j} (X_{jj} - X_{ii}) = \sum_{j=1}^{d} (d - 2j + 1)X_{jj} \]

Combining this with (62) and (61), we eventually get

\[ J = \prod_{t} \left( 1 + \sum_{j=1}^{d} \eta_j X_{jj} \Delta t \right) + O(\Delta t^2) = e^{\sum_{j=1}^{d} \frac{\eta_j X_{jj} \Delta t}{2}} \]

Taking the continuous limit \( \Delta t \to 0 \) we arrive at the integral

\[ J = \int \sum_{j=1}^{d} \eta_j X_{jj} dt \]

which coincides with Eq-s (35), (36).

**Discretization à la Ito**

The same result can be obtained in other discretization settings; However, in general case one has to keep the terms up to the second order in \( \Delta t \) in the initial difference equation for \( Q \). Here we consider the Itô-type discretization.
We rewrite the initial continuous differential equation (2), \( \partial_t Q = AQ \), in the integral form:

\[
Q(t) = Q_0 + \int_{t_0}^{t} AQ\,dt
\]

Applying this equation to the time range from \( t \) to \( t + \Delta t \) and solving it by means of iterations, after two iterations we get:

\[
Q(t + \Delta t) = Q(t) + \bar{A}Q(t)\Delta t + \frac{1}{2}\bar{A}^2Q(t)\Delta t^2
\]

where \( \bar{A} \) is the time average of \( A(t) \) over the range \( \Delta t \). Basing on this equation, we write the difference equation:

\[
Q_{t+1} = Q_t + \left( A_t\Delta t + \frac{1}{2}A_t^2\Delta t^2 \right) Q_t
\] (63)

Multiplying (63) by \( Q_t^{-1} \) and making use of the Iwasawa decomposition, we present the left-hand side in the form:

\[
Q_{t+1}Q_t^{-1} = R_t \left( 1 + R_t^{-1}\Delta R_t \right) \left( 1 + \Delta(DZ)_t(DZ)_t^{-1} \right) R_t^{-1}
\] (64)

where \( \Delta F_t \equiv F_{t+1} - F_t \).

Now we formally define \( \xi_t \) by

\[
\xi_t \Delta t = \Delta D_tD_t^{-1} - \frac{1}{2}(\Delta D_tD_t^{-1})^2
\]

which coincides up to the second order in \( \Delta t \) with

\[
\Delta D_t = D_{t+1} - D_t = (\xi_t\Delta t + \frac{1}{2}\xi_t^2\Delta t^2)D_t
\]

In accordance with the chosen prescription, this difference equation corresponds to the differential equation (29). Accordingly, (31) corresponds to

\[
\Delta R_t = R_{t+1} - R_t = R_t(\theta_t\Delta t + \frac{1}{2}\theta_t^2\Delta t^2)
\]

What about the equation (30) for the upper-triangular part, with account of (29) we rewrite it in the form \( \partial_t(DZ) = (\xi + \zeta)DZ \). Then, the corresponding difference equation takes the form:

\[
\Delta(DZ)_t = D_{t+1}Z_{t+1} - D_tZ_t = \left( (\xi_t + \zeta_t)\Delta t + \frac{1}{2}(\xi_t + \zeta_t)^2\Delta t^2 \right)D_tZ_t
\]

Substituting these expressions in (64) and keeping the terms of the order of \( \Delta t^2 \) we obtain:

\[
Q_{t+1}Q_t^{-1} = 1 + R_t \left( (\theta_t + \xi_t + \zeta_t)\Delta t + (\theta_t^2 + (\xi_t + \zeta_t)^2)\Delta t^2/2 + \theta_t(\xi_t + \zeta_t)\Delta t^2 \right) R_t^{-1}
\]

Combining this with (63), we get

\[
A_t = R_t \left( X_t + \frac{1}{2}[\theta_t, X_t]\Delta t \right) R_t^{-1}, \quad X_t = \xi_t + \zeta_t + \theta_t
\]

This result coincides with (60) obtained in the Stratonovich convention. The rest of the derivation is the same as in the Stratonovich case.
Appendix B. Correlation splitting for $\delta$-process

In this appendix, we derive the analog of the Furutsu-Novikov formula for the $\delta$-processes. The Furutsu-Novikov relation for regular processes has the form [30, 31]:

$$
\langle A_{ij}(t)g[A]\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} \langle A_{ij}(t)A_{i_1j_1}(t_1) \ldots A_{i_nj_n}(t_n)\rangle \left\langle \frac{\delta^n g[A]}{\delta A_{i_1j_1}(t_1) \ldots \delta A_{i_nj_n}(t_n)} \right\rangle dt_1 \ldots dt_n
$$

(65)

for any regular functional $g[A]$. For the $\delta$-process with correlation functions (55),

$$
\Delta_{ij\ldots kp}^{(n)} = w_{ij\ldots kp}^{(n)} \delta(t_1 - t_2) \ldots \delta(t_1 - t_n) ,
$$

(66)

it takes the form:

$$
\langle A_{ij}(t)g[A]\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} w_{ij\ldots kp}^{(n+1)} \left\langle \frac{\delta^n g[A]}{\delta A_{i_1j_1}(t_1) \ldots \delta A_{i_nj_n}(t_n)} \right\rangle
$$

(67)

Here all the variational derivatives are taken at the same moment $t$.

However, in physical applications one often has to deal with 'causal' functionals, i.e., the functionals that depend explicitly on time and satisfy the causality principle:

$$
\frac{\delta g[t, A]}{\delta A_{ij}(t')} = 0 \quad \text{if} \quad t' > t
$$

For these functionals,

$$
\frac{\delta^n g[t, A]}{\delta A_{i_1j_1}(t_1) \ldots \delta A_{i_nj_n}(t_n)} = I_{[0, \infty)}(t-t_1) \ldots I_{[0, \infty)}(t-t_n)G_{ij\ldots jn}^{(n)}[t, t_1, \ldots, t_n; A] ,
$$

and

$$
\langle A_{ij}(t')g[t, A]\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} w_{ij\ldots jn}^{(n+1)} \left\langle G_{ij\ldots jn}^{(n)}[t, t', \ldots, t'; A] \right\rangle , \quad t' < t
$$

(68)

$$
\langle A_{ij}(t')g[t, A]\rangle = \langle A_{ij}(t')\rangle \langle g[t, A] \rangle , \quad t' > t
$$

(69)

But (67) is inapplicable for $t = t'$ because it contains undefined values $I_{[0, \infty)}(0)$. So, to calculate the simultaneous correlator, we have to return to (65) and consider some sequence of (formal) processes with cumulants converging to (66). Thus, we choose the sequence of cumulants

$$
\frac{w_{ij\ldots jn}^{(n)}}{n} \sum_{k=1}^{n} \prod_{t_1=1}^{n} \delta_c(t_k - t_1),
$$

where $\delta_c(t)$ are even regular functions, $\delta_c(t) \xrightarrow{t \to 0} \delta(t)$, $\int \delta_c dt = 1$. With account of

$$
\int_{t}^{t} dt_1 \ldots \int_{t}^{t} dt_n \prod_{l=1}^{n} \delta_c(t_l - t_l) = \frac{1}{2^n}.
$$
\[ \int^t dt_1 \cdots \int^t dt_n \delta_\epsilon(t - t_k) \prod_{l=1, l \neq k}^n \delta_\epsilon(t_k - t_l) = \frac{1}{n} \left( 1 - \frac{1}{2^n} \right), \]

we arrive at

\[ \langle A_{ij}(t) g[t, A] \rangle = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} w^{(n+1)}_{i_1 j_1 \ldots i_n j_n} \langle G_{i_1 j_1 \ldots i_n j_n}^{(n)}[t, \ldots, t; A] \rangle \]  

(70)

This relation was presented without derivation in [32]; here we derive it taking the formal limit.

Comparing (70) with (68) we see that the correlations of 'causal' functionals are discontinuous at \( t' = t \) for \( \delta \)-processes, and the 'naive' convention \( I_{[0, \infty)}(0) = 1/2 \) is valid only for the Gaussian processes.

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