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Abstract. The fractal structure of the wake behind a 6:1 prolate spheroid has been analysed using data from a direct numerical simulation. The turbulent/non-turbulent interface was investigated by studying interfaces of turbulence related quantities, such as e.g. vorticity, helicity and dissipation. A box-counting dimension of around 2.7 was found for the turbulent/non-turbulent interface, which is higher than reported in most other studies. This value is close to 8/3, the expected dimension of a fractional Brownian surface with a \(-5/3\) power law in the power spectrum. Contrary to earlier studies, the fractal scaling only persists down to around 20 times the Kolmogorov length scale. The helicity interface in particular showed a high consistency for the box-counting dimension, and a higher dimension at around 2.8.

1. Introduction

Although the interface between turbulent and non-turbulent flow is often for simplicity treated as a smooth surface, it is actually highly irregular and difficult to characterise. Such an interface can be found at the outer boundary of a turbulent wake flow. This boundary defines the largest length scale of the turbulence, and contains regions of high vorticity and dissipation.

A fractal description of the interface between turbulent and non-turbulent flow has been proposed by several authors, and the dimensions of various interfaces have been determined (see e.g. Sreenivasan & Meneveau, 1986; Sreenivasan et al., 1989; Sreenivasan & Prasad, 1989). Newly available data from direct numerical simulations (DNS) have made it possible to study the turbulent/non-turbulent interface and determine the box-counting dimension, using the instantaneous velocity field.

The purpose of this work has been to study the interface between turbulent and non-turbulent flow, at the outer boundary of a turbulent bluff body wake. Velocity data from DNS have been used to calculate various turbulence related properties to distinguish between turbulent and non-turbulent flow. The fractal properties of the resulting surfaces have been investigated by computing the box-counting dimension.

2. Methodology

DNS of the flow behind a 6:1 prolate spheroid, with the major axis orthogonally oriented to the oncoming flow (El Khoury et al., 2010), provided a substantial amount of velocity data that was used for this study. The wake flow was characterised at a Reynolds number \(Re = \frac{U_0 D}{\nu} = 10000\), where \(D\) is the equatorial diameter, \(U_0\) the free-stream velocity and \(\nu\) the kinematic viscosity. The incompressible Navier-Stokes equations were solved using the
Table 1: Lower, middle and upper threshold values for turbulent interfaces.

|     | $I$ | $k/U_0^2$ | $\varepsilon/(u_0/D)^2$ | $\omega/(U_0/D)$ | $|h|/(U_0^2/D)$ |
|-----|-----|-----------|-----------------|-----------------|----------------|
| Lower threshold | 0.02 | 0.01 | 0.0005 | 0.5 | 0.2 |
| Middle threshold | 0.06 | 0.03 | 0.0015 | 1.5 | 0.6 |
| Upper threshold  | 0.10 | 0.05 | 0.0025 | 2.5 | 1.0 |

MGLET solver (Manhart, 2004) on a non-equidistant grid. These results were then interpolated on an equidistant grid for further analysis.

The available data consisted of all three velocity components in both the equatorial and the meridional plane, and adjacent planes, for several time samples. The time samples were equally spaced, each 100 time steps apart.

In order to study the geometry of the turbulent/non-turbulent interface, this interface had to be defined. To distinguish between turbulent and non-turbulent regions of the flow, different turbulence-related quantities were computed from the velocity field. These quantities were turbulence intensity ($I$), turbulence energy ($k$), dissipation ($\varepsilon$), vorticity ($\omega$) and helicity ($h$). A set of threshold values were then prescribed to provide a measure of how large the different turbulence quantities had to be for the region to be defined as turbulent and thus inside the wake. All of these quantities are characteristic for turbulent flow and can be considered as different ways of defining a turbulent/non-turbulent interface. The three different threshold values chosen are listed in table 1 for each interface type.

The boundary of the wake was defined by applying the threshold value as a filter and describing points with values higher than this threshold as inside the wake. Figure 1 shows the isolated boundary of the wake in the meridional and equatorial planes for the vorticity interface. This highly fragmented geometry of the wake boundary is characteristic for all the interface types and threshold values.

The box-counting dimension is a property that describes the degree of irregularity and fragmentation of the surface, and it is useful when studying geometries with some degree of self-similarity. It is quite common to use the term fractal dimension, but this is avoided here as the term more often refers to the Hausdorff dimension (see e.g Falconer, 2003; Vassilicos, 1989). In order to determine the box-counting dimension, a box-counting algorithm was established, and the dimension was determined as the negative slope in a log-log box-counting plot. The box-counting plot displays how many boxes of size $\delta$ are required to cover the geometry, for various values of $\delta$.

The available data made it possible to determine the box-counting dimension in two orthogonal planes for the same time samples. While the actual geometry of the interface is a surface in three-dimensional Euclidian space, the geometry studied using the box-counting algorithm is a line in two-dimensional space. To estimate the box-counting dimension for a surface, using the intersection of a plane and that surface, one uses the additive law for intersections, i.e. the dimension of the surface is equal to the dimension of the interface plus 1 (Sreenivasan, 1991). This only applies when the result is independent of the orientation of the intersection, and will be invalidated if there are considerable differences between the dimensions in the equatorial and meridional planes.

To identify any possible temporal or rotational dependence, the box-counting dimension was determined in both orthogonal planes, and for several different time samples. This was done for all the interface types.
Figure 1: The vorticity interfaces for the middle threshold in the meridional (a) and equatorial (b) planes represent highly fragmented geometries. Regions of low-vorticity flow are entrained in the high-vorticity interior of the wake.

3. Results

Box-counting results for the vorticity interface with the middle threshold in the equatorial plane are shown in figure 2. The number of boxes needed to cover the boundary of the wake when defined using a vorticity threshold, is plotted against the size of the boxes. It is observed that the box-counting plot can be divided into different ranges, an initial linear range, that transitions smoothly to the next steeper linear range called the fractal range, and lastly a more scattered range. The initial linear range has a slope equal to $-1$ corresponding to a box-counting dimension of 1, which is equal to the topological dimension of a smooth line. Thus for the smallest scales, the interface is smooth and non-fractal. This is consistent with what one would expect, considering that turbulence is smooth and viscous at the smallest scales. The slope of $-1$ persists to around
Figure 2: Box-counting results (○) for the middle threshold of vorticity interface in equatorial plane exhibit a box-counting dimension of 1.75, as indicated by the power-law regression line $\propto (\delta/D)^{-1.75}$ (—). 

$\delta/D = 0.07$, or slightly more than ten times the Kolmogorov microscale. Up to this value the vorticity interface can be considered as smooth.

In the fractal range, the slope is approximately constant and equal to 1.75. The line defined by the vorticity interface can therefore be described as approximately fractal with a box-counting dimension of 1.75 in this range. The fractal range is roughly limited to length scales between $0.15D$ and $3D$. For the highest values of $\delta/D$, it is difficult to determine the slope or whether it is linear at all. This is due to what is called a staircasing effect (Kruger, 1996), which causes the scattering.

The box-counting dimension has been estimated for the three different thresholds and for different time samples for the vorticity interface. The results are plotted in figure 3 as box-counting dimension against time. Figure 3a shows the results in the meridional plane and figure 3b shows the results in the equatorial plane, for time samples from $tU_0/D = 0$ to $tU_0/D = 12.8$, spanning approximately two vortex shedding periods in the equatorial plane. The highest threshold consistently gives the highest dimension value and the lowest threshold gives the lowest value. The difference between the dimension for the lowest threshold and for the two others is consistent and significant for all time samples. There is no time dependency for the box-counting dimension, and the small differences for the different time samples can most likely be attributed to statistical uncertainty.

Box-counting and dimension plots were studied for all the different interface types, turbulence intensity, turbulence energy, dissipation, vorticity and helicity. Similar smoothness was found for the smallest scales for all interfaces, and all exhibited fractal-like behaviour in the middle range. The slope, and thus box-counting dimension, in the box-counting plots differed between the interface types, as did the width of this range.

The results show that the box-counting dimension is not time dependent, even though the wake clearly exhibits periodic vortex shedding. Despite the fact that the geometrical shape of the wake obviously changes with time, the fractal properties do not. This is because the box-counting dimension is related to the wiggliness, roughness or space-filling properties of the wake, and not the actual macroscopic shape.

While neither of the interface types revealed any significant temporal dependence for the box-
counting dimension, there were differences in how much the dimension changed for the different threshold values. Figure 4 shows the dimensions of the surfaces in three-dimensional space for the different interface types and for the meridional (4a) and equatorial (4b) planes, averaged over all time samples and threshold values. The error-bars represent the maximum and minimum values of the given dimension. Rotational invariance is assumed and makes it possible to use the additive law for intersections, and thus adding 1 to the dimension in the two-dimensional plane yields the dimension of the surface in three-dimensional space.

The two figures 4a and 4b also make it possible to determine if rotational invariance is a reasonable assumption. There does not seem to be much difference between the values in the two different planes. However, the values are less scattered in the meridional plane, and the
Figure 4: The dimensions for the five different interface types are plotted as error bars for the meridional and equatorial planes. The centre points are the average over all time samples and the limits are the maximum and minimum values.

The box-counting dimension of the intensity interface is higher in the meridional plane. It is difficult to say if these are important differences or just random deviations. There does not seem to be any consistent differences between the two planes, and the assumption of rotational invariance is not unreasonable.

Due to the different values estimated by the various interface types, a general box-counting dimension for the turbulent/non-turbulent interface is difficult to determine. What can be stated is that the interface bounding the wake is approximately fractal in the range between around 0.1D to 1D, and that the dimension has a value that is likely between 2.6 and 2.8, using the additive law for intersections. The implication of this is that between 0.1D and 1D the turbulence scales somewhat similarly. This range is very narrow due to the relatively low Reynolds number, and it is expected that the range would widen correspondingly when increased
Reynolds number increases the scale range.

A small scale cut-off of 0.1D corresponds to approximately twenty times the Kolmogorov microscale. The Kolmogorov microscale is estimated using the middle dissipation threshold at 0.0051D. This value is used as a global microscale for all the interfaces, despite the fact that the Kolmogorov microscale is only going to be constant and equal to this value for the middle threshold dissipation interface. Earlier studies (Sreenivasan & Meneveau, 1986; Sreenivasan et al., 1989; Sreenivasan & Prasad, 1989) have shown fractal scaling for scales down to the Kolmogorov microscale. This could be caused by numerical diffusion for the simulation as sufficiently large numerical diffusion can create the effect of a low Schmidt number and increase the cut-off scale.

The different interface types yield different box-counting dimensions, suggesting that the different turbulence quantities are distributed on surfaces of varying degrees of roughness. Most notably, of the difference between the results for the various interface types, is that the helicity interface exhibits a higher box-counting dimension and a slightly wider range of scaling, than observations made for the other interfaces. This suggests that the helicity interface has a higher roughness, and is more space-filling than the other interfaces.

The results are also much less scattered for the helicity interface than for the other interfaces, suggesting that the dimension of the helicity interface is less dependent on the threshold value. In order to further investigate this effect, threshold values ranging from 0.01U_0^2/D to 4.0U_0^2/D were tested for the helicity interface, with little or no change in box-counting dimension. Thus the dimension of the helicity interface remains almost constant for a very large range of threshold values, suggesting that the geometrical roughness of the interface remains almost constant. This could imply that of the interfaces tested here, the helicity interface is the best suited for a fractal description.

These box-counting dimension values are higher than most earlier published results for scalar interfaces (see e.g. Sreenivasan & Meneveau, 1986; Sreenivasan et al., 1989; Sreenivasan & Prasad, 1989; Lane-Serff, 1993). It is likely that the difference between the results obtained from DNS and earlier reported results, is largely attributed to the added fragmentation of the surfaces described by instantaneous turbulence quantities, compared to surfaces described by scalar fields. The line in figure 1b is not just highly irregular, it is also very fragmented. This line is more fragmented than the boundary seen from the typical experimental flow visualization of e.g. Sreenivasan et al. (1989). The smoke or dye in an experimental flow visualisation will tend to create more contiguous surfaces than the turbulence quantity interfaces studied here, because the smoke or dye does not vanish in regions of e.g. low vorticity.

The box-counting dimensions found for most of the interfaces come close to 8/3 ≈ 2.67, which is the dimension of a fractional Brownian surface with a Hurst index of 1/3. A Hurst index equal to 1/3 implies a −5/3 power law in the power spectrum and a second order structure function S_2(r) = \[u(x + r) - u(x)\]^2 proportional to r^{2/3} (Falconer, 2003, pp. 267–271). Mandelbrot (1975) found a box-counting dimension of 8/3 for simulated scalar iso-surfaces in k^{-5/3} homogeneous turbulence. A similar result was found by Scotti et al. (Scotti et al., 1995) for velocity signals in hydrodynamic turbulence.

It is possible to estimate the area of the surface for a geometry with a finite fractal range. If the upper cut-off length in the fractal range is called L_0 and the lower cut-off length is called \(\eta_c\), then the area ratio for the two different scales is expressed as \(A/A_0 = (\eta_c/L_0)^{2-D_{bc}}\). Using averaged box-counting dimensions and averaged values for \(\eta_c/L_0\), the area ratios are computed for the five different interface types and presented in table 2. Also presented in table 2 are average values of \(\eta_c/L_0\) and \(D_{bc}\) for the different interfaces.

The results in table 2 show the importance of the roughness or wiggliness of a surface, as represented in the box-counting dimension, when estimating the area. There is a large difference in area for the various interface types, due to the differences in box-counting dimension. The
Table 2: Area of irregular interfaces for different interface types.

| I  | $k/U_0^2$ | $\varepsilon/(u_0/D)^2$ | $\omega/(U_0/D)$ | $|h|/(U_0^2/D)$ |
|----|-----------|------------------------|----------------|----------------|
| $\eta_c/L_0$ | 0.054 | 0.047 | 0.042 | 0.058 | 0.027 |
| $D_{bc}$ | 2.72 | 2.64 | 2.74 | 2.73 | 2.83 |
| $A/A_0$ | 8.18 | 7.08 | 10.40 | 7.99 | 20.04 |

surface areas for all interface types are much larger than an assumed smooth area. For this particular example, the fractal range is quite narrow. For flows at considerably higher Reynolds numbers, the corresponding interfacial areas will be much larger. If one is to determine the transport of some property across the interface, a reasonably accurate estimate of the surface of the interface is important.

4. Conclusion

The fractal structure of the wake behind a 6:1 prolate spheroid has been analysed using data from a direct numerical simulation. These data have been applied to estimate the value of the box-counting dimension of interfaces defined by various turbulence-related quantities in the wake. All of these quantities are characteristic for turbulent flow and can be considered as different types of turbulent/non-turbulent interfaces.

The box-counting dimension for these turbulent/non-turbulent interfaces appears to be around 2.7 for a variety of interface types, threshold values, time samples and for both equatorial and meridional planes. This value is close to the expected value for a fractional Brownian surface with a $-5/3$ energy spectrum. There is no time dependency of the box-counting dimension, and no consistent difference between the equatorial and the meridional plane.

The area of the boundary of the wake ranges from factors 7 to 20, to that of a smooth area. This is important if the transport across this surface is to be estimated. Since the area of the interface will be much larger for higher Reynolds numbers, this effect will have a greater consequence for the prediction of interfacial transport under such flow conditions.

The helicity interface, in particular, shows very little variation in box-counting dimension, also for an extended range of threshold values. The box-counting dimension is somewhat higher than 2.8, and remains almost constant for all tested threshold values, time samples and for both planes. A possible implication of this is that the helicity interface is particularly amenable to a fractal description.

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