Thermodynamics of strange baryon system from coupled-channel analysis and missing states

César Fernández-Ramírez, Pok Man Lo, and Peter Petreczky

1 Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, 04510 Mexico City, Mexico
2 Institute of Theoretical Physics, University of Wrocław, PL-50204 Wrocław, Poland
3 Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA

(Dated: June 7, 2018)

We study the thermodynamics of the strange baryon system using an S-matrix formulation of statistical mechanics. For this purpose, we employ an existing coupled-channel study involving \( \bar{K}N, \pi\Lambda, \pi\Sigma \) interactions in the \( S = -1 \) sector. A novel method is proposed to extract an effective phase shift due to the interaction, which can subsequently be used to compute various thermal observables via a relativistic virial expansion. As an application of the calculation scheme, we compute the correlation of the net baryon number with strangeness \( \langle \chi_{BS} \rangle \) for an interacting hadron gas. We show that the S-matrix approach, which entails a consistent treatment of resonances and naturally incorporates the additional hyperon states which are not listed by the Particle Data Group, leads to an improved description of the lattice data over the Hadron Resonance Gas model.

PACS numbers: 25.75.-q; 25.75.Ld; 12.38.Mh; 24.10.Nz

I. INTRODUCTION

It was conjectured long time ago that thermodynamics of hadrons can be understood in terms of the Hadron Resonance Gas (HRG) model [1]. The essence of this model is that interacting hadron gas can be replaced by an uncorrelated gas of hadrons and hadronic resonances, and as a first approximation, resonances are treated as zero width states. Lattice QCD (LQCD) calculations confirm that indeed the thermodynamics below the chiral transition temperature \( T < T_c \simeq 155 \text{ MeV} \) can be understood in terms of the HRG model. The model appears to describe the pressure and the trace anomaly calculated on the lattice [2] [8]. Fluctuations, \( \chi_n \) and correlation \( \chi_{XY} \) of conserved charges defined as the derivatives of the pressure with respect to chemical potentials \( \mu_X, \mu_Y \),

\[
\chi_n^X = T^n \frac{\partial^n (p(T, \mu_X)/T^4)}{\partial \mu_X^n} \bigg|_{\mu_X=0},
\]

\[
\chi_{XY} = T^{n+m} \frac{\partial^{n+m} (p(T, \mu_X, \mu_Y)/T^4)}{\partial \mu_X^n \partial \mu_Y^m} \bigg|_{\mu_X=0, \mu_Y=0},
\]

are also reasonably well described by the HRG model [0] [9] [11]. Here the conserved charges \( X, Y = B, Q, S \) correspond to baryon number, electric charge and strangeness.

Around the same time a more systematic approach to study thermodynamics of hadrons was proposed by Dashen, Ma, and Bernstein– the S-matrix formulation of statistical mechanics [12]. It is an extension of the usual virial expansion to the relativistic case. When used in conjunction with the empirical phase shifts from scattering experiments, this approach offers a model independent way to consistently incorporate the effects of hadronic interactions, including the appearance of broad resonances and purely repulsive channels. The analysis of Venugopalan and Prakash [13] along these lines showed that for the pressure there is a large cancellation of different non-resonant repulsive and attractive contribution and therefore the pressure to a fairly good approximation can be understood solely in terms of resonances alone, thus justifying the use of the HRG model [13]. For a more recent analysis, see Ref. [14]. To what extends this simplifications can be justified for the fluctuations and correlations of conserved charges remains to be seen. Recently the S-matrix approach has been applied to analyze the LQCD result on the baryon electric charge correlation \( H_{BQ} \) [13] and higher order fluctuations [16] of the baryon charge such as \( \chi^4 \). The former observable is particularly sensitive to the interaction between pions and nucleons. It was found that the use of the effective density of states constructed from the phase shifts of a partial wave analysis (PWA) leads to an improved description of the LQCD result up to a temperature \( T \approx 160 \text{ MeV} \) over that of the HRG model. The source of the improvement in the quantitative description of the LQCD result within the S-matrix approach is twofold. First, the inclusion of non-resonant, often purely repulsive, channels yields an important contribution at low invariant masses. Second, a consistent treatment of the interactions is pivotal in channels with broad resonances. For such a resonance, the thermal contribution can be significantly reduced relative to the HRG prediction owing to the fact that a substantial part of the effective density of states is found at large masses, which are suppressed by the Boltzmann factors.

It is natural to expand these studies to include the strange baryons. This problem demands a coupled-channel treatment as inelastic interaction among \( (\bar{K}N, \pi\Lambda, \pi\Sigma) \) sets in at rather low momentum. Moreover, the available experimental data are insufficient to constrain individual scattering process and model extrapolations for poorly (if at all) measured amplitudes are hard to avoid. A channel-by-channel analysis to describing the thermal system, as is done previously, would be
rather inefficient. In the hadron physics community, a multi-channel S-matrix is usually constructed through a model of the amplitude that attempts to incorporate unitarity, analyticity, crossing symmetry and any underlying symmetry of the given reactions to describe the available data. The model S-matrix thus obtained can be used to study the thermal properties of the hadronic medium. In this paper we study the pressure of strange baryons in a given partial wave, that the description of the lattice data on \( \chi^2 \) of Particle Data Group (PDG) [17]. Thus, the S-matrix approach with the list of three and four star resonances in our analysis is that the PWA has more resonances than lattice data over the HRG model. An important feature acting hadron gas.

We show that the proper treatment of resonances in the S-matrix approach lead to an improved description of the lattice data over the HRG model. An important feature of our analysis is that the PWA has more resonances than the list of three and four star resonances in Particle Data Group (PDG) [17]. Thus, the S-matrix approach with the state of the art PWA confirms earlier an conjecture that the description of the lattice data on \( \chi^2 \) requires additional baryon resonances not listed in the PDG [18] [19].

II. EFFECTIVE PHASE SHIFT FOR COUPLED-CHANNEL SYSTEM

Before getting into the details of extracting an effective phase shift from a full-fledged coupled-channel PWA for the \( S = -1 \) baryons, we begin by introducing the method in a simpler setting: an elementary two-channel resonance decay model.

Consider the following fact of a unitary S-matrix \( S \) (in a given partial wave),

\[
SS^\dagger = \mathbb{1} \\
\Rightarrow \det S \times \det S^\dagger = 1 \\
\Rightarrow \ln \det S + \ln \det S^\dagger = 0.
\]

Using \( \ln \det S^\dagger = (\ln \det S)^* \), we see that unitarity dictates the quantity \( (\ln \det S) \) to be purely imaginary. This motivates the definition of a generalized phase shift function [20] [21] \( Q \)

\[
Q \equiv \frac{1}{2} \text{Im} (\text{tr} \ln S) \\
= \frac{1}{2} \text{Im} (\ln \det S).
\]

The determinant operation makes this quantity invariant under any unitary rotation \( U \) of the S-matrix

\[
S \to U^\dagger SU.
\]

In the single channel case \( Q \) reduces to the standard expression of a scattering phase shift. The physical meaning of this quantity in the general N-channel case can be clarified by studying a simple example. Consider a single relativistic Briet-Wigner resonance of mass \( m_{\text{res}} \) decaying via two channels. The S-matrix can be parametrized as [22] [24]

\[
S(s) = \mathbb{1} + i \hat{T}(s),
\]

where, e.g. for the \( l = 0 \) partial wave,

\[
\hat{T}(s) = \frac{-2 \sqrt{s} \gamma_{\text{res}}}{s - m_{\text{res}}^2 + i \sqrt{s} \gamma_{\text{res}}} \times \hat{i} \]

\[
\hat{i} = \frac{1}{g_a^2 \phi_a + g_b^2 \phi_b} \left( g_a g_b \sqrt{\phi_a \phi_b} - g_b \phi_b \right).
\]

In this parametrization, \( \sqrt{s} \) is the invariant mass, \( g_{a,b} \)'s are coupling constants (with \( g_a^2 + g_b^2 = 1 \)), and \( \phi_{a,b}(s) \)'s are the relevant Lorentz invariant phase space [20]. For the two-body case it takes the generic form

\[
\phi_2(s) = \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^6} \frac{1}{4E_1 E_2} \\
\times (2\pi)^4 \delta(\sqrt{s} - E_1 + E_2) \delta^{(3)}(\vec{p}_1 + \vec{p}_2) \\
= \frac{q}{4\pi s},
\]

where

\[
q = \frac{1}{2} \sqrt{s} \sqrt{1 - \frac{(m_1 + m_2)^2}{s}} \sqrt{1 - \frac{(m_1 - m_2)^2}{s}},
\]

and \( m_1, m_2 \) are the masses of the particles making up the channel. The (energy-dependent) total width of the resonance can be computed by

\[
\gamma_{\text{res}}(s) = \gamma_0 \times \left( g_a^2 \phi_a + g_b^2 \phi_b \right),
\]

with a width parameter \( \gamma_0 \).

A direct calculation shows that the phase shift function \( Q \) defined in Eq. 4 is given by

\[
Q(s) = \delta_{\text{res}}(s) = \tan^{-1} \left( \sqrt{s} \gamma_{\text{res}} \right) \\
\frac{1}{s - m_{\text{res}}^2}.
\]

We see that \( Q \) correctly recovers the phase shift of a relativistic Briet-Wigner (B-W) resonance [23]. This result is independent of the basis being used. In fact one can rewrite the model S-matrix in Eqs. [6] and [7] as
This means that the “observed” S-matrix is related to a diagonal eigenmatrix $S_d$ (made up of eigenphases [26,27]) by a rotation matrix whose elements can be related to the energy-dependent branching fractions:

$$S = U^\dagger S_d U \quad \text{(12)}$$

with

$$S_d = \begin{pmatrix} e^{2i\delta_{\text{res}}(s)} & 0 \\ 0 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \text{(13)}$$

Note that $Q$ is invariant under a change of basis and is hence independent of these branching fractions.

To obtain other channel-specific quantities, we compare the model S-matrix with the general two-channel parametrization

$$S = \begin{pmatrix} \eta e^{2i\varphi_a} & i \sqrt{1-\eta^2} e^{i(Q_a+Q_b)} \\ i \sqrt{1-\eta^2} e^{i(Q_a+Q_b)} & \eta e^{2i\varphi_b} \end{pmatrix} \quad \text{(15)}.$$
The spectral functions within a channel can be computed using the branching fractions in Eq. (14)

\[ B_i(s) = BR_i \times B(s) \]

and

\[ A_i(s) = -2\sqrt{s} \sin 2Q_i \]

The equivalence between all the expressions in Eq. (19) are numerically checked.

We briefly summarize the key features of the effective spectral function $B(s)$ and its differences from the standard spectral function $A(s)$, see Fig. 2.

(i) One observes irregularities in the $B(s)$ function at $\sqrt{s} = m_\pi + m_\eta$ and $\sqrt{s} = 2m_\pi$. These are integrable divergences associated with the appearance of the $S$-wave two-body decay channel. It can be shown that these threshold effects give a finite contribution to the physical observables, with a strength that depends on the scattering length of the channel;

(ii) The apparent “shift” of the strength towards lower invariant masses of $B(s)$ compared to $A(s)$ is a familiar feature. This effect originates from the non-resonant scattering effect, which is not properly accounted for by $A(s)$. The enhancement near threshold can give a substantial contribution to the soft momentum of the decay particles. This feature has been discussed in details in Ref. [28].

(iii) The parametrization for the resonance decay presented in Eqs. (16) and (17) is quite robust. It goes beyond the usual assumption of a narrow resonance, where one replaces $\sqrt{s} \rightarrow m_{\text{res}}$ in the phase spaces $\phi_i$, and sometimes also in prefactor multiplying $\gamma_{\text{res}}$ in Eq. (7). To describe higher partial waves, one needs to incorporate the right angular momentum barrier to the phase space $\phi_i$. It is a basic approximation to the integral involved in computing the imaginary part of the self energy of the resonance [20, 22, 25],

\[ g_{\alpha}^2 \phi_a \leftrightarrow \int d\phi_a |\Gamma_{\text{res} \rightarrow a}|^2. \]

Note that the definition for energy dependent branching fractions in Eq. (14) remains unchanged.

In an actual PWA, the S-matrix would include multiple resonances and the effects from the non-resonant background. These models are constructed to describe a wide range of experimental scattering data and their parameters fitted to data. The S-matrix obtained is usually employed to assess the existence of a resonance, and to extract its parameters. Here we have proposed an additional use of the S-matrix – the determination of an effective level density.

The simple example just presented motivates a robust way to extract, for a coupled-channel system, an effective phase shift function $Q(s) = \frac{1}{2} \Im \ln S$, which is the suitable generalization of a single-channel phase shift. The phase shift function $Q$ thus defined is invariant under unitary rotations of the basis states, while the S-matrix element, which describes individual scattering process, depends on the choice of basis used in the coupled-channel study.

According to the S-matrix formulation of the statistical mechanics, the effective spectral function $B(s) = 2\frac{d}{d\sqrt{s}} Q(s)$, plays the role of an effective level density due to the interaction, which enters the thermodynamical description of an interacting system in the form of a virial expansion. In the next section we apply this formulation to study the thermal system of $|S| = 1$ strange baryons.

### III. THE PRESSURE OF STRANGE BARYONS

In the S-matrix approach to statistical mechanics, the thermodynamic pressure can be written as a sum of two pieces

\[ P = P_0 + \Delta P_{\text{int}}. \]
$P_0$ is the pressure of an uncorrelated gas of particles that do not decay under the strong interaction (i.e. ground-state particles), such as pions, kaons, and nucleons:

$$P_0 = \sum_{a \in \text{gs}} d_a \int \frac{d^3k}{(2\pi)^3} \times \left[ \pm \ln \left( 1 \pm e^{-\beta(\sqrt{p^2 + m_a^2} - \mu_a)} \right) \right],$$

(22)

where $\mu_a = B_a \mu_B + Q_a \mu_Q + S_a \mu_S$ with $(B_a, Q_a, S_a)$ being the baryon number, electric charge and strangeness of the particle species $a$ and $(\mu_B, \mu_Q, \mu_S)$ are the relevant chemical potentials. The choice of Fermi-Dirac or Bose-Einstein statistics depends on the quantum numbers of the species.

The interaction contribution $\Delta P_{\text{int}}$ due to two-body scatterings involves an integral over the invariant mass $\sqrt{s}$

$$\Delta P_{\text{int}} = \frac{T}{V} \langle \ln Z \rangle_{\text{int.}} \approx \sum_{ab} \sum_f T \int_{m_{ab}^\text{th}}^{\infty} d\sqrt{s} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4i\pi} \text{tr} \left[ S^{-1} \partial S - (\partial S^{-1}) S \right] \left[ \pm \ln \left( 1 \pm e^{-\beta(\sqrt{p^2 + s} - \mu_a)} \right) \right].$$

(23)

Here $S$ is the S-matrix of the scattering process $ab \to f$ with threshold $m_{ab}^\text{th}$ and $\partial$ stands for the derivative with respect to $\sqrt{s}$. The sum over $f$ implies summation over all allowed final states, while the sum over $ab$ should encompass all possible pairs of ground state hadrons: $\pi \pi$, $\pi K$, $\pi N$, $KN$, $NN$, etc.

In this paper we are interested in the pressure of strange baryons. The dominant contribution to the strange baryon pressure comes from $|S| = 1$ sector. The contribution of the $|S| = 2$ and $|S| = 3$ sectors is significantly smaller. For example, at $T = 155$ MeV $|S| = 2$ and $|S| = 3$ baryons contribute 20% and 1.4% respectively to the total strange baryonic pressure. For the calculation of the $|S| = 1$ strange baryonic pressure the most important interactions are the kaon-nucleon ($KN$) interactions, antikaon-nucleon interactions ($\bar{K}N$), as well as the interactions of non-strange pseudo-scalar mesons with hyperons, i.e. the $\pi \Lambda$, $\pi \Sigma$, $\eta \Lambda$ and $\eta \Sigma$ interactions. The hyperon-nucleon interactions are suppressed due to the large mass of the hyperon and the nucleon.

The $KN$ scattering is known to have a lot of resonances. As mentioned before a coupled channel of these resonances is needed. Many of the resonances that are present in $KN$ scattering also couple to $\pi \Lambda$, $\pi \Sigma$, $\eta \Lambda$ and $\eta \Sigma$. Furthermore, some of these resonances can also decay into quasi two-body final states like $K^* N$, $\bar{K} \Delta$, $\pi \Sigma^*(1385)$ and $\pi \Lambda^*(1520)$, i.e. final states that contain resonances $K^*$, $\Delta$, $\Sigma^*(1385)$ and $\Lambda^*(1520)$. Based on the principle of effective elementarity [29], narrow resonances such as $K^*$, $\Sigma^*(1385)$ and $\Lambda^*(1520)$ (with width of 51 MeV, 38 MeV and 16 MeV, respectively), can be approximately treated as stable states when calculating their contributions to the thermodynamics. In addition, due to the long lifetime they can interact multiple times with other stable particles in the medium before decaying. Such interaction may be treated as effectively two-body under the same principle. This is in line with the framework of isobar decomposition and is compatible with the current PWA.

Taking into account that

$$\frac{1}{4i\pi} \text{tr} \left( S^{-1} \partial S - (\partial S^{-1}) S \right) = \frac{1}{\pi} \frac{dQ}{d\sqrt{s}} = \frac{1}{2\pi} B(s),$$

(24)

and in particular at $\mu_Q = 0$, the pressure of $|S| = 1$ baryons can be written as

$$P_{|S|=1}^{\text{res}}(T, \mu_B, \mu_S) = \sum_{a=\Lambda, \Sigma, \Sigma^{1385}} d_{IJ} p_0(M_a, T) \cosh(\beta(\mu_B - \mu_S)) + P_{\text{int}}^{\text{res}}(T) \cosh(\beta(\mu_B - \mu_S)) + P_{\text{int}}^{\text{KN}}(T) \cosh(\beta(\mu_B + \mu_S)), $$

(25)

$$P_{\text{int}}^{\text{res}}(T) = T \int d\sqrt{s} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi} B(s) e^{-\beta \sqrt{p^2 + s}} = \frac{1}{2\pi} \int d\sqrt{s} p_0(\sqrt{s}, T) B(s),$$

(26)

$$p_0(x, T) = \frac{x^2 T^2}{\pi^2} K_2(\beta x).$$

(27)

Here $d_{IJ} = (2J + 1)(2I + 1)$ is the spin and isospin degeneracy factor and $K_2(x)$ is the Bessel function of the sec-
ond kind. The thresholds of different scattering processes are implicitly encoded in $B(s)$. We used the Boltzmann approximation in the above equation since $\beta m_{th}^{ab} \gg 1$. We also performed the calculations without the Boltzmann approximation and found that the difference is tiny.

The first term in Eq. (25) corresponds to the free gas pressure. In addition to the ground state hyperons $\Lambda$ and $\Sigma$, we also include the narrow resonance $\Sigma^* (1385)$ in this term, since it is not reconstructed in the current PWA. The $\Lambda^* (1520)$ state, on the other hand, is dynamically generated, with parameters close to the ones in the PDG [17]. It is included in the $I = 0$ ($D_{03}$) component of $p_{int}^{res}$ (see below). The last term in Eq. (25) corresponds to $KN$ interactions and will be discussed later. Here we only note that it has a different dependence on the chemical potentials.

To evaluate the effective phase shift function $Q(\sqrt{s})$ for the coupled-channel system, we use a coupled PWA [30] by Joint Physics Analysis Center (JPAC) collaboration which computes the scattering amplitude ($T$-matrix) for the following partial waves: $S_{01}$, $P_{01}$, $P_{03}$, $D_{03}$, $D_{05}$, $F_{05}$, $F_{07}$ and $G_{07}$ for isospin zero ($I = 0$) and $S_{11}$, $P_{11}$, $P_{13}$, $D_{13}$, $D_{15}$, $F_{15}$, $F_{17}$ and $G_{17}$ for isospin one ($I = 1$) cases. Here the second subscripts in the partial wave labels stands for the spin ($2 \times J$). The $T$-matrix, and hence the $S$-matrix derived, describes the coupled-channel interaction of a system of 16 basis Fock states. It includes such major channels as $KN$, $\pi\Lambda$, $\pi\Sigma$, $\eta\Lambda$ and $\eta\Sigma$. Furthermore, for $I = 0$ the quasi two-body states like $K^*N$, $K\Delta$, $\pi\Sigma^*(1385)$, i.e. final states that contain resonances $K^*$, $\Delta$ and $\Sigma^*(1385)$ are also included. As discussed above these quasi two body channels are important for thermodynamics. Moreover, the JPAC analysis includes dummy channels labeled as $\sigma\Lambda$ and $\sigma\Sigma$ to account for remaining inelasticities not taken into account by the channels discussed above [30]. Here $\sigma$ is a fictitious meson with the mass of two pion masses. In principle many of these channels should be treated as genuine three-body final states. This, at the moment, remains a challenging task. Nevertheless, three-body unitarity studies are currently under development in the PWA field, particularly related to LQCD calculations [31,32].

In terms of PWA one can write

$$Q = \sum_i d_{ij} Q^i$$

with index $l$ labeling different partial waves, $S_{01}$, $S_{11}$, $P_{03}$ etc., and $Q^i$ is obtained by numerically calculating the determinant of the $16 \times 16$ S-matrix $S^i$, for each partial wave. The corresponding numerical results for the generalized phase shifts are shown in Fig. 3. One can imagine a simple scenario where the phase shift is dominated by the sum of step functions as asserted by the HRG model. However, Fig. 3 shows that this simple scenario is not realized in general. In Fig. 4 we show the effective spectral functions for the five lowest partial waves with $I = 0$ and $I = 1$. We also compare the effective spectral functions with the sum of B-W parametrization of the resonances in each channel. For partial waves dominated by narrow resonances like $P_{03}$, $D_{03}$, $P_{13}$, $D_{13}$ the B-W parametrization gives a fair description of the effective spectral function although not accurate at the quantitative level. The B-W parametrization also does a fair job for $G_{17}$ partial wave (not shown). However, for all other cases the B-W parametrization does not describe the effective spectral function. Furthermore, the simple K-matrix parametrization advocated in Ref. [35] also does not provide a good description of the spectral densities. In fact, it is known that a more sophisticated treatment of the K-matrix, e.g. maintaining the exchange symmetry and unitarity, is required to produce reliable results on phase shifts and scattering amplitudes [30,37].

With the extracted effective spectral functions $B(s)$ it is straightforward to calculate numerically the contribution of the coupled channel interactions to the partial pressure of $|S| = 1$ baryons. The result is shown in Fig. 5. In Table I we give the individual contributions to the $|S| = 1$ baryonic pressure from different partial waves. We also performed calculations using simplified spectral functions that are the sum of delta functions with peak positions corresponding to resonance location for each partial wave. This corresponds to the HRG approxima-
FIG. 4. The effective spectral functions \((B)\) for \(I = 0\) (left) and \(I = 1\) (right) and different partial waves. The dashed lines correspond to B-W parametrization of the resonances.

FIG. 5. The pressure of \(|S| = 1\) baryons calculated in the S-matrix based relativistic virial expansion and using HRG model with various particle content (PWA,QM,PDG). Also shown in the figure are the lattice results for the \(|S| = 1\) pressure \([19]\).

| \(I = 0\) | \(I = 1\) |
|--------------|--------------|
| S-mat. HRG   | S-mat. HRG   |
| \(S_{01}\)   | 0.916        | 1.139        |
| \(P_{01}\)   | 0.539        | 0.607        |
| \(P_{03}\)   | 0.426        | 0.403        |
| \(D_{03}\)   | 1.091        | 1.127        |
| \(D_{05}\)   | 0.363        | 0.221        |
| \(F_{05}\)   | 0.261        | 0.308        |
| \(F_{07}\)   | 0.160        | 0.085        |
| \(G_{07}\)   | 0.173        | 0.057        |

TABLE I. The contributions to \(|S| = 1\) baryonic pressure from different the partial waves in the S-matrix approach and in HRG approximation in units of \(10^{-3}T^4\) at \(T = 150\) MeV.

The contributions from the S-matrix virial expansion are either significantly smaller or significantly larger than the HRG result. Qualitatively the situation is similar to the study of baryon number electric charge correlations, where it was also found that relativistic virial expansion and HRG model give quite different results \([15]\). Very interestingly, however, after adding the contribution from all partial waves, the two approaches give very similar results as shown in Fig. 5. This illustrates that spectra with drastically different shape may produce similar temperature dependence in a given thermal observable, i.e. a given thermal quantity does not uniquely fix the spectrum. In particular, the excellent agreement of the HRG model with various lattice results should not be taken as a justification of the zero-width approximation in treating resonances. Such an assumption, in many cases, is not supported by empirical findings. Instead, the current approach suggests multiple mechanisms are at work in the thermodynamic quantities: threshold effects, repulsive channels, coupled-channel effects, and the effect of averaging over many channels. A momentum-differential observable such as the momentum spectrum \([38, 39]\) would allow one to differentiate between different models of the effective spectral functions.

Note that the HRG approximation discussed above and labeled as HRG PWA is different from standard HRG model which uses only well established, i.e. four and three star resonances from PDG \([17]\) and labeled as HRG PDG. This is due to the fact that the number of hyperon resonances identified in JPAC PWA analysis is larger than the list of three star or four star resonances listed by PDG \([30]\). The JPAC analysis identifies \(A^*\) and \(\Sigma^*\) resonances that do not appear in PDG as well established, i.e. three star or four star resonances. The analysis confirms some two and one star resonances that appear in PDG. At the same time other two and one star resonances are not confirmed by JPAC, instead new hyperon resonances are identified \([30]\). Furthermore, in Fig. 5 we show the HRG result, which in addition to the PDG states also includes \(|S| = 1\) baryons predicted in the quark model (QM) \([40]\) and therefore labeled as
HRG QM. Interestingly enough, HRG PWA and HRG QM results agree reasonably well despite differences in the particle content. Finally, the calculations are compared to the lattice result for the $|S| = 1$ baryon pressure extracted from strangeness fluctuations and baryon-strangeness correlations using the HRG Ansatz for the strange pressure [19]. These lattice results are significantly higher than the HRG result with PDG states and agree better with the calculations that include the missing states.

So far we did not discuss the contribution of $KN$ interactions to the strange baryonic pressure, which is given by the third term in Eq. (25). These interactions do not have known resonances and have been analyzed using the PWA in [41–43]. The SAID interface gives the phase shift of elastic $KN$ scattering [44]. The inelastic channels are included explicitly through the inelastic parameter. To estimate the contribution of $KN$ we write

$$P_{\text{int}}^{KN}(T) = \sum I \int d\sqrt{s} \rho_0(\sqrt{s}, T) \frac{1}{\pi} \frac{d\delta^I}{d\sqrt{s}}.$$  (29)

Here it was assumed that $S = \sum_i \exp(2i\delta^I)$, with $\delta^I$ the elastic $KN$ scattering phase shifts. We use the results of Ref. [43] for $\delta^I$ and consider partial waves up to $G_{09}$ and $G_{19}$. The numerical results for different partial wave contributions to $P_{\text{int}}^{KN}$ are shown in Fig. 6 separately for $I = 0$ and $I = 1$ channels. It is obvious that the largest contribution to $P_{\text{int}}^{KN}$ comes from $I = 1$ partial waves. The absolute value of the contribution from $KN$ interactions is always smaller than the total resonance contribution in each partial waves. The typical strength of the $KN$ contribution relative to resonant ($KN$ etc) contribution varies between 5% and 25%. Furthermore, we observe large cancellations between the contributions from different partial waves, c.f. Fig. 6. Thus, the total contribution from $KN$ interactions turns out to be very small and can be neglected.

IV. BARYON-STRANGENESS CORRELATIONS

In the previous section we compared the pressure of $|S| = 1$ baryons obtained in the S-matrix approach with the LQCD estimate [19]. The LQCD estimate of the $|S| = 1$ baryonic pressure was obtained from the baryon strangeness correlations and strangeness fluctuations and the HRG Ansatz for the pressure, and thus is model dependent. For a more direct comparison with LQCD we consider the second order baryon strangeness correlation, $\chi_{BS}^{S=1}$. This quantity, however, receives significant contribution from $|S| = 2$ and $|S| = 3$ baryons. Unlike for $|S| = 1$ sector no PWA is available here. Furthermore, the number of well established (four or three star) baryons is much smaller. There are only five well established $|S| = 2$ baryons and only one well established $|S| = 3$ baryon ($\Omega(1672)$) [17]. On the other hand all these states are either stable under strong interactions or quite narrow, $\Gamma \lesssim 30$ MeV. Therefore, all the well established $|S| = 2$ and $|S| = 3$ baryons can be treated as “elementary” states to a good approximation, i.e. their contribution can be calculated using the ideal gas expression [29]. In Fig. 7 we show $\chi_{BS}^{S=1}$ obtained in LQCD [10] together with the ideal gas result that contains all the elementary states ($\Lambda$, $\Sigma$, $\Sigma^*(1385)$ and the well established $|S| = 2,3$ baryons). The elementary states account for about 60% of $\chi_{BS}^{S=1}$. The remainder has to come from the interactions. If we add the interactions from the $|S| = 1$ sector based on PWA, we see a substantial improvement in the description of the LQCD result by the S-matrix approach, over the standard HRG result that contains only the well established states. This demonstrates the importance of the consistent treatment of resonances and the need to incorporate additional hyperon states.

However the agreement remains incomplete, and we see that further interaction strength in the strange baryon system is needed to reproduce the LQCD results. This may come from an improved analysis of the $|S| = 1$ hyperon system, e.g. more realistic interaction vertices and the systematic inclusion of the multi-hadron scatterings.
For this one needs more precise experimental information on the hyperons.

The enhancement may also come from an improved treatment of the $|S| = 2$ and $|S| = 3$ sectors. Based on the quark model calculations [40, 45], and LQCD [46] we expect many more $|S| = 2$ and $|S| = 3$ resonances should exist than in the list of well established states in PDG. This also suggests that interactions in the $|S| = 2$ and $|S| = 3$ is mostly resonant. To investigate the possible impact of these states we included the missing $|S| = 2$ and $|S| = 3$ baryon resonances to $\chi^{BS}_{11}$ using the HRG approximation. As one can see from Fig. 7 including these states further improve the agreement with LQCD results. Of course, a more definitive assessment can be made when PWA in $|S| = 2$ and $|S| = 3$ sectors becomes available.

Previously [47] it was shown that the HRG model, supplemented with additional hyperon states (one and two star resonances), can also yield a reasonable description of the lattice results, despite essential differences in the distribution of strength in the effective spectral function (as a function of center-of-mass energy) between the two approaches. This underlines the fact that spectral functions with drastically different shapes may produce similar temperature dependence in a given thermal observable. Nevertheless, the S-matrix analysis presented in this work is expected to yield a more reliable description since it is consistent with many known facts of the hadrons, e.g. resonance widths and inelasticities.

It would be interesting to study higher order baryon strangeness correlations, e.g. $\chi^{BS}_{22}$ and $\chi^{BS}_{33}$. However, as it was pointed out in Ref. [16] these will be sensitive to the repulsive baryon-baryon interactions, which are not very well known in the case of strange baryons. The repulsive baryon-baryon interactions need to be studied and taken into account before applying the virial expansion to higher order baryon strangeness correlation. These interactions may also explain why HRG with additional QM states seems to be disfavored by the lattice results on the ratio $\chi^S_3/\chi^S_2$ [19].

V. CONCLUSIONS

In this paper the partial pressure of strange baryons and baryon-strangeness correlations have been discussed within the S-matrix approach, based on the state of the art coupled-channel analysis by JPAC. It was found that the proper treatment of resonances, and the natural incorporation of additional hyperon states which are not listed in the Particle Data Group in the S-matrix approach, lead to an improved description of the lattice data over the standard Hadron Resonance Gas model. Thus, the presented analysis supports the earlier claim that the incorporation of extra hyperon states is required to explain the lattice results of the BS correlations. It was pointed out, however, that baryon-baryon interactions may be important for the analysis of higher order baryon-strangeness correlations.

ACKNOWLEDGMENTS

CFR was partly supported by PAPIIT-DGAPA (UNAM, Mexico) grant No. IA101717 and CONACYT (Mexico) grant No. 251817. PML was partly supported by the Polish National Science Center (NCN), under Maestro Grant No. DEC-2013/10/A/ST2/00106 and by the ExtrMe Matter Institute EMMI at the GSI Helmholtzzentrum fuer Schwerionenphysik, Darmstadt, Germany. PP was supported by U.S. Department of Energy under Contract No. de-sc0012704. We used the web interface of the DATA analysis Center at George Washington University (http://gwdata.phys.gwu.edu/) and Joint Physics Analysis Center (http://www.indiana.edu/~jpac/index.html) for the partial wave analysis. PML is grateful for the fruitful discussions with Bengt Friman, Krzysztof Redlich, Ted Barnes, Michael Döring, and Jose R. Peláez. He also thanks Olaf Kaczmarek, Christian Schmidt and Frithjof Karsch for constructive comments and the warm hospitality at the Bielefeld University. PP thanks Igor Strakovsky for useful correspondence and Claudia Ratti for useful correspondence and the lattice data.

[1] R. Hagedorn, Nuovo Cim. Suppl. 3, 147 (1965).

[2] S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo (Wuppertal-Budapest), JHEP
