RISK MANAGEMENT OF GUARANTEED MINIMUM MATURITY BENEFITS UNDER STOCHASTIC MORTALITY AND REGIME-SWITCHING BY FOURIER SPACE TIME-STEPPING FRAMEWORK

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ABSTRACT

This paper presents a novel framework for valuation and hedging of the insurer’s net liability on a Guaranteed Minimum Maturity Benefit (GMMB) embedded in variable annuity (VA) contracts whose underlying mutual fund dynamics evolve under the influence of the regime-switching model. Numerical solutions for valuations and Greeks (i.e. valuation sensitivities with respect to model parameters) of GMMB under stochastic mortality are derived. Valuation and hedging is performed using an accurate, fast and efficient Fourier Space Time-stepping (FST) algorithm. The mortality component of the model is calibrated to the American male population. Sensitivity analysis is performed with respect to various parameters. The hedge effectiveness is assessed by comparing profit-and-loss performances for an unhedged and three statically hedged portfolios. The results provide a comprehensive analysis on valuation and hedging the longevity risk, interest rate risk and equity risk for the GMMB embedded in VAs, and highlight the benefits to insurance providers who offer those products.

1 Introduction

An annuity provides a specified income stream for a fixed or contingent period in return for a stipulated premium paid either in prior or in a single payment. There are two classes of annuities: fixed and variable. A variable annuity (VA) allows the policyholder to get exposure to the equity market by linking the level of payments to the performance of a chosen investment fund. This allows the policyholder to earn additional risk premium on top of the mortality credit embedded in annuities. In comparison, fixed annuities do not allow the policyholder to participate in the equity market and thus, provide a stable cash flow. To protect investors from downside risk of fund participation, insurers also provide guarantee riders (e.g. Guaranteed Minimum Benefits (GMBs)) embedded in VAs such as Guaranteed Minimum Maturity Benefit (GMMB), Guaranteed Minimum Death Benefit (GMDB), Guaranteed Minimum Withdraw Benefit (GMWB), etc. Under a guarantee rider, an insurer receives the proceeds from fees and charges, and is responsible for covering financial losses to policyholders in adverse economic scenarios which makes these products more appealing to potential policyholders. With the shift from Defined Benefit to Defined Contribution schemes in developed countries, there is a growing demand in the ageing population for products that can manage their longevity risk. Furthermore, some countries (e.g. US and Canada) provide tax-shelter for the investment gains of VAs. These trends have led to the rapid growth in VA sales in the global markets.

GMMB is a guarantee that provides the policyholder with a minimum benefit on maturity of the contract. This research primarily focuses on pricing and hedging of GMMB as it is one of the main building blocks for other Guaranteed Minimum Benefits.
During the early years, insurers were only concerned with pricing the guarantees correctly. From the insurer’s perspective, the guarantees bear multiple risks, include mortality risk, equity risk and interest rate risk. Often, the GMBs pricing framework consists of using a financial model to capture the equity and interest rate risk, and a mortality model to capture the mortality risk. This is done with reasonable assumption that the mortality process is independent of the financial markets (see Fung et al. [18], Da Fonseca and Ziveyi [9]).

Brennan and Schwartz [4] show that GMMB can be decomposed into a guaranteed base plus a European call option. Bauer et al. [2] provides a simple pricing framework for GMMBs using the geometric Brownian motion (GBM) and life tables to account for the financial and mortality components respectively. Since Bauer et al. [2] do not account for systematic mortality risk, Bacincello et al. [1] fill the gap in the literature by pricing the GMMB under stochastic mortality as well as stochastic interest rates and stochastic volatility. Furthermore, Krayzler et al. [24] provide analytical pricing formulas for the GMMB with various additional riders under stochastic interest rates and stochastic mortality.

Deterministic algorithms such as partial differential equation method can also be used in valuating GMMBs. A close connection between the payoff of options and insurer’s liabilities for variable annuity guaranteed benefits was observed and exploited in Feng and Volkmer [16]. The authors establish a model with an assumption of no additional purchase payment and provided explicit solutions to a few key quantities, which lead to analytical calculations of commonly used risk measures such as value-at-risk and conditional tail expectation. The methodologies they used were largely based on the joint distribution of GBM. Feng and Volkmer [17] and Feng [15] also provide an individual model for the calculations of risk measures for GMMBs.

The importance of pricing the various guarantees has been emphasized in the existing literature on VAs. However, risk management requires equal or even greater attention. Hedging is one of the most powerful tools insurers can use for managing the risks in VAs. In the recent years, there has been a considerable amount of research done on hedging guarantees in VAs. The general approach is to present the guarantee as an option and utilize the appropriate hedging techniques described in financial literature. There are two streams of hedging literature: one focuses upon hedging financial risks, and the second focusing on the hedging of longevity risk.

A common practice for insurers is to use dynamic delta-hedging to manage the risks embedded in the guarantees of VAs (see Hardy [20]). This approach poses several problems; theoretically, dynamic hedging assumes a continuous rebalance of the hedge portfolio in a manner such that its value is resistant to changes in market conditions. However, it is impossible to implement this method precisely in practice, especially due to transaction costs and liquidity constraints. This means that dynamic hedging strategies are likely to be not the optimal choice, after taking into consideration the transaction costs.

In contrast to dynamic hedging, static hedging is the approach of holding the hedged portfolio until maturity without any rebalancing. In the context of variable annuities, static hedging offers some advantages over dynamic hedging. Static hedging does not require rebalancing nor rely on particular model dynamics. Hence, it minimizes transaction costs, which can be quite substantial, especially due to the long maturities of variable annuities. Static hedging for European options is discussed in Carr and Wu [5]. Kolkiewicz and Liu [23] consider hedging path-dependent options by extending the local risk-minimization method. They also propose semi-static hedging strategies to lower the hedging error generated in static hedging portfolios. Kolkiewicz and Liu [23] notice that the hedging error is also reduced, but never completely removed, by adding more options to the hedging portfolio.

The Fast Fourier Transform (FFT) is used in a variety of physical science applications, such as signal processing and image compression through its main advantage of providing a fast and efficient computational method of transforming values from the real space to the Fourier space. Carr and Madan [5] first popularize the use of FFT in option pricing by using a damped option price method. Due to the similarity between GMMBs and European options, Da Fonseca and Ziveyi [9] extend its usage and value GMMBs written on several assets. Another computational algorithm which utilizes the FFT is the Fourier Time-stepping (FST) algorithm as outlined in Jackson et al. [22]. The authors emphasis its versatility in pricing path-dependent and path-independent options. Furthermore, Surkov and Davison [29] broaden the usage of the FST algorithm by demonstrating its applicability when computing the Greeks of options for hedging purposes. Lippa [25] is the first research to price guarantee minimum withdrawal benefits under the GBM using FST algorithm and demonstrate that its numerical results are consistent with those documented in Chen et al. [7]. Ignatieva et al. [21] marks the second yet, portraying a more sophisticated pricing framework of GMBs under the regime-switching environment. Furthermore, Ignatieva et al. [21] also pioneers the usage of the FST algorithm in computing the Greeks of GMBs.

The contribution of this paper can be summarised as follows: (i) we compare three stochastic mortality models and adopt the one with the best fitting performance; (ii) we develop a valuation framework for the insurer’s net liability of a GMMB contract under the regime-switching model; (iii) we perform sensitivity analysis with respect to model parameters which include financial parameters and mortality parameters; (iv) we provide three static hedging strategies
and analyse their effectiveness. In so doing, the paper extends the existing literature on generalised pricing frameworks for GMMB (e.g. Ignatieva et al. [21] who models the GMMB from the point of view of policyholders). Furthermore, Ignatieva et al. [21] assumes policyholders’ investment accounts have no additional payments, which is an important part of insurer’s benefit income.

The remainder of the paper is organised as follows: Section 2 introduces the modelling framework, which includes the financial and mortality models, as well as an FST algorithm utilised for the valuation of net liabilities. Section 3 provides a general valuation and hedging framework for GMMB. Numerical results, which include sensitivity analysis with respect to model parameters, as well as hedging performance are summarised in Section 4. Section 5 concludes and provides final remarks.

2 The Mathematical Framework

2.1 Mortality modeling

In this section we compare three mortality models based on different affine processes. These models have many attractively common features such as closed-form expressions for survival probabilities, effectiveness in capture mortality evolution at very old ages, which is essential due to long term nature of annuity.

Given a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), we focus on an individual aged \(x\) at time \(t\) and model random residual lifetime \(\tau_x\) as a stochastic stopping time with intensity \(\lambda_x\). Formally, the time-\(t\) expected survival probability, \(S(t, T)\), can be represented as

\[
S(t, T) = \mathbb{P}(\tau > T - t | \mathcal{F}(t)) = \mathbb{E}^\mathbb{Q}\left[ e^{-\int_t^T \lambda(s) ds} | \mathcal{F}(t) \right],
\]

(1)

where \(\lambda(s) = \lambda^R(s) + \lambda^P(s)\) with \(\lambda^P(s)\) and \(\lambda^R(s)\) denoting the instantaneous mortality intensities under the real world probability measure \(\mathbb{P}\) and risk neutral probability measure \(\mathbb{Q}\). We can assume the insurer has a sufficiently large pool of policyholders such that the unsystematic mortality risk can be diversified away (see Ignatieva et al. [21]). Here we also omit the initial age \(x\) for convenience.

A general affine process of mortality intensity \(\lambda(t)\) can be described by

\[
d\lambda(t) = f(\lambda(t))dt + g(\lambda(t))dW(t),
\]

(2)

where \(W(t)\) is a standard Brownian motion. Duffie and Singleton [12] show that if \(\lambda(t)\) is modeled as an affine process, the general solution of equation (1) is given by

\[
S(t, T) = e^{\alpha(t,T)+\beta(t,T)\lambda(t)},
\]

(3)

where the coefficients \(\alpha(t, T)\) and \(\beta(t, T)\) satisfy generalized Riccati Ordinary Differential Equations (ODEs). The latter can be solved at least numerically and in some cases analytically.

The first model we used for describing the intensity \(\lambda(t)\) is an Ornstein-Uhlenbeck (OU) process

\[
d\lambda(t) = a\lambda(t)dt + \sigma dW(t)
\]

(4)

with \(a > 0\) and \(\sigma \geq 0\). By applying the framework of equation (3), we find \(\alpha\) and \(\beta\):

\[
\alpha(t, T) = \frac{\sigma^2}{2a^2} (T - t) - \frac{\sigma^2}{a^3} e^{\alpha(T-t)} + \frac{\sigma^2}{4a^4} e^{2\alpha(T-t)} + \frac{3\sigma^2}{4a^5},
\]

(5)

\[
\beta(t, T) = \frac{1 - e^{\alpha(T-t)}}{a}.
\]

(6)

The main drawback when choosing this process for the intensity is that it becomes negative with positive probability. Thus, from a purely theoretical point of view, the OU model can be considered inadequate to describe the intensity of mortality (see Luciano and Vigna [28]). However, it can be seen that in the applications this model turns out to be rather appropriate.

The second model proposed is the Feller (FEL) process:

\[
d\lambda(t) = a\lambda(t)dt + \sigma \sqrt{\lambda(t)}dW(t)
\]

(7)

where \(a > 0\) and \(\sigma > 0\). The main advantage of this process is that it does not violate the non-negativity constant of the intensity, provided that the starting point is non-negative. The application of the affine framework gives the following solutions of ODEs for \(\alpha\) and \(\beta\),

\[
\alpha(t, T) = 0,
\]

(8)
We calibrate four models to age-cohorts of the United States male population aged 50 where coefficients can be determined as

\[ \zeta \]

The AIC of FEL model is better than that of two-factor model, whereas BIC results are vice versa. The reason might be

\[ b \]

where \[ b \] and \[ c \] are time-independent coefficients; \[ W_1 \] and \[ W_2 \] are independent standard Brownian motions. Duffie and Kan [11] show that the general solution is given by

\[ S(t, T) = e^{D(t,T) - C_1(t,T)\zeta_1(t) - C_2(t,T)\zeta_2(t)}, \]

(13)

where coefficients can be determined as

\[ C_1(t, T) = \frac{1 - e^{-\delta_1(T-t)}}{\delta_1}, \]

(14)

\[ C_2(t, T) = \frac{1 - e^{-\delta_2(T-t)}}{\delta_2}, \]

(15)

\[ D(t, T) = \frac{1}{2} \sum_{j=1}^{2} \rho_{2j}^2 \left[ \frac{1}{2} \left( 1 - e^{-2\delta_j(T-t)} \right) - 2 \left( 1 - e^{-\delta_j(T-t)} \right) + \delta_j(T-t) \right]. \]

(16)

We calibrate four models to age-cohorts of the United States male population aged 50. The data is collected from the Human Mortality Database for the time interval from 1957 to 2017. We assume that the policyholder retires at 60, the time to maturity is 10 years. We set \( t = 0 \) to represent the calendar year 2017 and assume that the individual’s age is 50 + \( t \) at time \( t \). The calibration procedure is inspired by Dacorogna and Apicella [10] which is to estimate the parameters based on minimizing of the Squared Error (\( \Psi \)) between stochastic mortality models and observed survival function. The parameters for OU model and FEL model are reported in Table 1 and parameters for Blackburn and Sherris two-factor model are reported in Table 2.

From Table 1 and Table 2 we observe that the calibration of the three models gives surprising results. The fitting performances of FEL model and two-factor model are better than OU model. Actually the two-factor model is a two dimensional OU model with different notations and we expect better results with more complex model setting. Therefore, we also give the Akaike information criterion (AIC) and Bayesian information criterion (BIC) in both tables. They deal with the trade-off between the goodness of fit of the model and the complexity of the model. We find that the AIC of FEL model is better than that of two-factor model, whereas BIC results are vice versa. The reason might be that BIC gives a larger penalty term than AIC and we have two more parameters to be estimated in two-factor model comparing with the FEL model. In the rest of this research we adopt the FEL model as our survival probability function because it has less model parameters and better fitting performance.

| Process | \( \alpha \) | \( \sigma \) | \( \Psi \) | AIC | BIC |
|---------|-------------|-------------|-------------|-----|-----|
| OU      | 0.0793      | 0.0012      | 0.003694    | -337.67 | -584.21 |
| FEL     | 0.0800      | 0.0105      | 0.003389    | -342.92 | -589.46 |

Table 1: One-factor model parameters.

| \( \delta_1 \) | \( \delta_2 \) | \( \rho_1 \) | \( \rho_2 \) | \( \Psi \) | AIC | BIC |
|----------------|--------------|-------------|-------------|-------------|-----|-----|
| -0.0808        | -0.0676      | 1.9574      | 0.0018      | 0.003162    | -343.15 | -585.47 |

Table 2: Two-factor model parameters.
2.2 Regime-switching models

The regime-switching framework for modelling the financial market resembles that considered in Shen et al. [28]. Let $\mathcal{T}$ denote the time index set $[0, T]$, where $T < \infty$. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = \{\mathcal{F}(t) | t \in \mathcal{T}\}$ satisfying the conditions of right-continuity and $\mathbb{P}$-completeness. $\mathbb{P}$ represents the real world probability measure. We assume that the probability space describes uncertainties attributed to a Brownian motion and a Markov chain.

We describe the evolution of the state of an economy over time by a continuous-time, finite-state, observable Markov chain $X = \{X(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite state space $S$. Elliott et al. [13] show that $S$ can be represented as a set of standard basis vectors, i.e. $S = \{e_1, e_2, \ldots, e_N\} \in \mathbb{R}^N$, where $e_j$ denotes a vector with 1 in the $j$th coordinate and 0 elsewhere. The states can be interpreted as the ‘regimes’ the economy undergoes where $N$ is the number of possible regimes.

Let $A$ be an $N \times N$ transition rate matrix for the chain $X$ under $\mathbb{Q}$, with elements $[A]_{ij} = a_{ij}$ representing the constant transition intensity of the chain $X$ from regime $i$ to $j$. Note that $a_{ij} \geq 0$, for $i \neq j$ and $\sum_{j=1}^N a_{ij} = 0$ for each $i, j = 1, 2, \ldots, N$. Then, the following semi-martingale representation is obtained by Elliott et al. [13]:

$$X(t) = X(0) + \int_0^t AX(s)ds + M(t), \quad t \in \mathcal{T},$$

where $M(t)$ is a $(\mathcal{F}^X, \mathbb{P})$-martingale and $M(t)$ is the natural filtration generated by the chain $X$.

We assume that there are two primitive assets, namely, a zero-coupon bond $B$ and an index $S$ in the financial market. The time-$t$ risk-free instantaneous market interest rate $r(t)$, drift rate $\mu(t)$ and volatility of the fund $\sigma(t)$ are assumed to be modulated by the chain $X$ as follows:

$$r(t) = \langle r, X(t) \rangle,$$

$$\mu(t) = \langle \mu, X(t) \rangle,$$

$$\sigma(t) = \langle \sigma, X(t) \rangle,$$

where $r$, $\mu$ and $\sigma$ are the vectors corresponding to $(r_j, r_j, \ldots, r_j)'$, $(\mu_1, \mu_2, \ldots, \mu_N)'$ and $(\sigma_1, \sigma_2, \ldots, \sigma_N)'$ respectively, with $r_j > 0$, $\mu_j > 0$ and $\sigma_j > 0$ for each $j = 1, 2, \ldots, N$; and $\langle \cdot, \cdot \rangle$ denotes the inner product. Thus, at any time, if $X = e_j$, then $r(t) = r_j$, $\mu(t) = \mu_j$ and $\sigma(t) = \sigma_j$. Then the dynamics of the zero-coupon bond $B = \{B(t) | t \in \mathcal{T}\}$ is given by

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1.$$  

(21)

The price process of the index $S = \{S(t) | t \in \mathcal{T}\}$ evolves over time according to the following stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dW(t), \quad F(0) > 0.$$  

(22)

where $W = \{W(t) | t \in \mathcal{T}\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, and is independent of $X$.

If there is only a single regime, equation (22) reduces to the Geometric Brownian Motion case. Using Itô’s lemma, the corresponding real world dynamics of the logarithm of the fund value, $Y(t) = \log \left( \frac{S(t)}{S(0)} \right)$, satisfies

$$dY(t) = \left( \mu(t) - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t)dW(t).$$  

(23)

Since the market is incomplete under the real world probability measure, $\mathbb{P}$, the process of completing the market under the regime-switching environment is accomplished by applying Esscher transforms (Gerber and Shiu [12]), which facilitate measure transformation from the real world, $\mathbb{P}$, to risk neutral world, $\mathbb{Q}$. Elliott et al. [14] proves that the dynamics of $Y(t)$ under the risk-neutral measure can be written as

$$dY(t) = \left( r(t) - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t)dW^Q(t).$$  

(24)

where $W^Q(t)$ is a standard Brownian motion under the $\mathbb{Q}$ measure. Furthermore, the chain $X$ has the following semi-martingale dynamics under $\mathbb{Q}$:

$$X(t) = X(0) + \int_0^t A^QX(s)ds + M^Q(t), \quad t \in \mathcal{T},$$  

(25)
where $M^Q(t)$ is a $(\mathcal{F}_t^X, Q)$-martingale. Since we assume that there are no jumps in stock price during a regime switch, Shen et al. [28] shows that the transition rate matrix is the same under the $Q$-measure, that is

$$A^Q = A.$$  

(26)

In the rest of this research, we omit the risk-neutral notation $Q$ for convenience.

### 2.3 Fourier space time-stepping algorithm under a regime-switching model

In this section, we use the Fourier space time-stepping (FST) introduced in Jackson et al. [22] and Surkov and Davison [29] for its many characteristics. The purpose of the FST method is to solve problems by first transforming the partial integro-differential equation (PIDE) into Fourier space via Fourier transforms, and then converting them back to the real space by inverse transforms. One of the advantages of working directly in Fourier space is that the characteristic function.

Furthermore, the method is applicable to any independent-increment stochastic process which admits a closed-form characteristic function.

A function in the space domain $g(x, t)$ can be transformed to a function in the frequency domain $\hat{g}(\omega, t)$, where $\omega$ is given in radians per second, and vice-versa using the continuous Fourier transform (CFT):

$$\mathcal{F}[g(x, t)](\omega) = \int_{-\infty}^{\infty} g(x, t) e^{-i\omega x} dx,$$

(27)

where $i = \sqrt{-1}$. We denote the Fourier transform of the function $g(x, t)$ as $\hat{g}(\omega, t)$. Provided $\hat{g}(\omega, t)$, the original function can be recovered by the inverse Fourier transform. This is accomplished by the following transformation

$$\mathcal{F}^{-1}[\hat{g}(\omega, t)](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega, t) e^{i\omega x} d\omega.$$

(28)

Furthermore, the Fourier transform of the $n$-th derivative with respect to $x$ is given by

$$\mathcal{F}[\partial_x^n g(x, t)](\omega) = (i\omega)^n \mathcal{F}[g(x, t)](\omega).$$

(29)

To apply the FST algorithm to the regime-switching environment, we adopt the procedure introduced in Ignatieva et al. [21]. Let $v(Y(t), t, X(t))$ denote the price function of an option at time $t$, which is independent on the log-return $Y(t)$ and the time-$t$ regime $X(t)$. We can also write the function $v$ as an inner product

$$v(Y(t), t, X(t)) = \langle v(Y(t), t), X(t) \rangle,$$

(30)

where $v(Y(t), t)$ denotes the vector of values for $v(Y(t), t, X(t))$ across all $N$ states of $X(t)$, i.e. $v(Y(t), t) = (v(Y(t), t, e_1), v(Y(t), t, e_2), \ldots, v(Y(t), t, e_N))^T$. Applying Itô’s lemma to $v$ under $Q$, we obtain

$$dv = \left( \partial_t + \left( r(t) - \frac{1}{2} \sigma^2(t) \right) \partial_Y + \frac{1}{2} \sigma^2(t) \partial_X^2 \right) v(Y(t), t, X(t)) + \langle v(Y(t), t), AX(t) \rangle.$$

(31)

After setting the infinitesimal generator

$$L = \left( r(t) - \frac{1}{2} \sigma^2(t) \right) \partial_Y + \frac{1}{2} \sigma^2(t) \partial_X^2 - r(t)$$

(32)

for the discounted process of $Y(t)$ and using the condition $dv = r(t)v$ under the risk-neutral measure, we have

$$\left( \partial_t + L \right) v(Y(t), t, X(t)) + \langle v(Y(t), t), AX(t) \rangle = 0.$$  

(33)

For notation convenience, let $v^{(k)}(y, t) = v(Y(t) = y, t, X(t) = e_k)$. After expanding equation (33) for all regimes $1$ to $N$, we have the option price function satisfying the following system of PIDEs for all $k$:

$$\begin{cases}
\left( \partial_t + a_{kk} + L^{(k)} \right) v^{(k)}(y, t) + \sum_{j \neq k} a_{kj} v^{(j)}(y, t) = 0, \\
v^{(k)}(y, T) = H,
\end{cases}$$

(34)

where $L^{(k)}$ is the infinitesimal generator under regime $k$; $a_{ij}$ is the $(i, j)$th entry in the rate matrix $A$; $H$ is the terminal condition occurring on the maturity date $T$. For example, in the case of a European call with strike price $K$, we have $H = \max(F(0)e^y - K, 0)$. 

6
Given $\mathcal{L}$ be an infinitesimal generator of a general Lévy process, Jackson et al. [22] show that the corresponding characteristic exponent can be represented as:

$$\mathcal{F}[\mathcal{L}g(x, t)](\omega) = \Psi(\omega)\mathcal{F}[g(x, t)](\omega), \quad (35)$$

where $\Psi$ denotes the exponent of the characteristic function of $Y(t)$. Then, we apply the CFT to PIDE (34) to obtain

$$\left\{ (\partial_t + \Psi^{(k)}(\omega))\mathcal{F}[v^{(k)}(y, t)](\omega) + \sum_{j \neq k} a_{kj}\mathcal{F}[v^{(j)}(y, t)](\omega) = 0, \right.$$  
$$\mathcal{F}[v^{(k)}(y, T)](\omega) = \mathcal{F}[H](\omega), \quad (36)$$

where $\Psi^{(k)}(\omega)$ denotes the characteristic exponent under each regime and becomes

$$\Psi^{(k)}(\omega) = i \left( r_k - \frac{\sigma_k^2}{2} \right) \omega - \frac{\sigma_k^2}{2} r_k, \quad (37)$$

where $r_k$ and $\sigma_k$ are the risk free interest rate and volatility under the regime $k$, respectively. Then, system (36) can be rewritten in matrix form:

$$\left\{ (\partial_t + \Psi(\omega))\mathcal{F}[v(y, t)](\omega) = 0, \right.$$  
$$\mathcal{F}[v(y, T)](\omega) = \mathcal{F}[H](\omega)\mathbf{1}, \quad (38)$$

where $\mathbf{1}$ is a $N \times 1$ vector of ones; $\Psi(\omega)$ is the matrix characteristic function and its elements are

$$\Psi(\omega)_{kl} = \left\{ \begin{array}{ll} A_{kk} + \Psi^{(k)}(\omega) & \text{if } k = l, \\ A_{kl} & \text{if } k \neq l. \end{array} \right. \quad (39)$$

Then system (38) can be solved in a single time step yielding

$$v(y, t) = \mathcal{F}^{-1} \left[ e^{\Psi(\omega)(T-t)} \cdot \mathcal{F}[v(y, T)](\omega)\mathbf{1} \right]. \quad (40)$$

In the numerical implementation, equation (40) can be approximated by the discrete Fourier transform (DFT) and its inverse (IDFT) which are given by

$$DFT[g(x, t)] = \sum_{n=0}^{N-1} g(x_n, t) e^{-2\pi i \frac{m(n-1)}{N}}, \quad m = 1, \ldots, N, \quad (41)$$

$$IDFT[g(\omega, t)] = \sum_{n=0}^{N-1} g(\omega_n, t) e^{2\pi i \frac{n(m-1)}{N}}, \quad m = 1, \ldots, N. \quad (42)$$

Here we utilise the Fast Fourier Transform (FFT) algorithm developed by Cooley and Tukey [8], which splits the DFT of a vector size $N$ into two DFTs of size $N/2$ each, on every recursion. Finally, the complexity of computation will significantly decreases from $O(N^2)$ to $O(N \log_2 N)$. Therefore the numerical evaluation of $v$ can be represented as

$$v(y, t) = FFT^{-1} \left[ e^{\Psi(\omega)(T-t)} \cdot FFT[v(y, T)](\omega)\mathbf{1} \right]. \quad (43)$$

It follows from equation (43) that the price of the option at time 0 is given by

$$p_0 = \pi_0 v(0), \quad (44)$$

where $\pi_0$ represents the initial vector of probabilities of each regime.

### 3 Risk Management of GMMB

In this section, we present an illustrative example on risk management of GMMB under regime-switching environment. We adopt the net liability model developed in Feng and Volkmer [16]. The model considers both benefit and liability of GMMB. The insurer’s benefit comes from the rider charges from the policyholder which is always omitted in other pricing models. Under this setting, the net liability of GMMB can be decomposed into a European put option and a series of cash flows.
3.1 Net liability model

Variable annuities (VAs) are modern long-term life insurance products that offer policyholders participation in the profit sharing of equity investment. Purchase payments net of fees and charges are deposited in various sub-accounts of policyholders’ choosing. All of policyholders’ premiums in sub-accounts (or separated accounts) are typically invested and managed by third party professional vendors. From an accounting point of view, these assets are owned by policyholders and considered to be in separate accounts, apart from an insurer’s own investment assets, which are in general accounts.

Insurers also sell guarantee riders such as GMMB to protect policyholders from downside risk of fund participation. Under a guarantee rider, an insurer receives the proceeds from fees and rider charges, and is responsible for covering financial losses to policyholders. In this research, we denote the annualized rate of fees and charges by \( m \). The fees are also known as mortality and expenses fees (M&E fees) and generally include investment fees, administration fees and surrender fees, etc. The purpose of rider charges (or margin offsets) is to fund the guarantee cost for insurers. Here, we denote the annualized rate of charges allocated to the GMMB by \( m_c \). Both fees and charges are deducted from the separated account directly in a continuous manner. Note that in general \( m > m_c \) to allow for other expenses.

Under these settings, the randomness of an insurer’s liability to each contract arises from two independent sources - the equity price process and the future-lifetime of contract. Therefore the present value of insurer’s incomes and outgoes as fair rates. Hence, without the effect of investment guarantees the account value at time \( t \) is given by

\[
F(t) = F(0) \frac{S(t)}{S(0)} e^{-mt},
\]

where \( S(t) \) is the market value of the underlying equity fund at time \( t \) and it will be modeled under a regime-switching environment.

Assuming independent between the mortality and the financial component, we can express the net liability value as

\[
L(t, T, F) = E^Q \left[ e^{\int_t^T r(u)du} (G - F(T))_+ I(\tau_x > T - t) - \int_t^T e^{\int_t^u r(u)du} m_c F(s) I(\tau_x > s - t) ds \right],
\]

where the indicator \( I(A) = 1 \) if the event \( A \) is true or 0 otherwise and \( (x, y)_+ = \max(x, y) \); \( \tau_x \) is the time until death for a person aged \( 50 + t \) at time \( t \); \( r(u) \) is the risk-free force of interest at time \( u \); and \( F(t) \) is the market value of the separate account at time \( t \). We assume that at the beginning of each year, the account value is adjusted to the performance of funds in which it invests and deducted by mortality and expenses fees (M&E fees) and rider charges \( m_c \). Hence, without the effect of investment guarantees the account value at time \( t \) is given by

\[
L(t, T, F) = S(t, T) P(t, T, F) - \int_t^T S(t, s) M(t, s, F) ds.
\]

Based on the no-arbitrage principle, the risk-neutral value of insurer’s benefit payments should be equal to that of insurer’s fee incomes at time of issue in the absence of arbitrage. We consider any pair of total fee rate and rider charge rate \((m, m_c)\) that matches the risk-neutral values if the insurer’s incomes and outgoes as fair rates. Therefore, the fair rates \((m, m_c)\) should be determined by the identity

\[
S(0, T) P(0, T, F) = \int_0^T S(0, s) M(0, s, F) ds.
\]

Evaluating equation (15) numerically can be very time-consuming. Instead, we discretize the time intervals annually, and assume that payout of rider charges is calculated at the beginning of every year, which is consistent with real-life contract specifications. Hence \( L(t, T, F) \) from equation (48) is annually discretized into

\[
L(t, T, F) = S(t, T) P(t, T, F) - \sum_{s=t}^{T-1} S(t, s) M(t, s, F).
\]
We notice that the values of net liability can be decomposed into three components: the survival factor, the European put option, and the present value of rider charges. For the mortality component, \(S(t,T)\) and \(S(t,s)\) is evaluated using the equation (48) with the parameters provided in Table 1. This leaves us only two unknown quantities to be evaluated.

For the European put option, we apply the FST method introduced in equations (43) and (44) to numerically evaluate the expression

\[
P(t, T, F) = \pi'_t FFT^{-1} \left[ e^{\Psi(\omega)(T-t)} \cdot FFT[(G - F(T))_+](\omega) \right].
\]

For the present value of rider charges \(M(t, s, F)\), we have

\[
M(t, s, F) = \pi'_t FFT^{-1} \left[ e^{\Psi(\omega)(T-t)} \cdot FFT[m_e F(s)](\omega) \right].
\]

### 3.2 Evaluating the Greeks

We outline the procedure for evaluation Greeks which represent the sensitivities of net liability with respect to the underlying variables. The FST algorithm (see Jackson et al. [22]) allows us to conveniently incorporate the computation of Greeks as an addendum to the valuation procedure, as suggest by Surkov and Davison [29], who initially describe the extension for jump models. Recall that when valuation the net liabilities, we split the model into three distinct parts, the survival factor, the European put option, and the present value of rider charges. This allows us to also evaluate the Greeks with respect to each of these components separately.

In order to compute the Greeks associated with the European put and the present value and rider charges, we utilise equations (29) and (43). The delta of a European put, \(\frac{\partial P}{\partial F}\), is obtained via the chain rule

\[
\frac{\partial P}{\partial F}(t, T, F) = \pi'_t FFT^{-1} \left[ i\omega \cdot e^{\Psi(\omega)(T-t)} \cdot FFT[(G - F(T))_+](\omega) \right].
\]

The delta of a rider charge, \(\frac{\partial M}{\partial F}\), can also be obtained via the chain rule

\[
\frac{\partial M}{\partial F}(t, s, F) = \pi'_t FFT^{-1} \left[ i\omega \cdot e^{\Psi(\omega)(T-t)} \cdot FFT[m_e F(s)](\omega) \right].
\]

Similarly, the gamma of the European put \(\frac{\partial^2 P}{\partial F^2}\) and the rider charge \(\frac{\partial^2 M}{\partial F^2}\), are also given by

\[
\frac{\partial^2 P}{\partial F^2}(t, T, F) = \pi'_t FFT^{-1} \left[ -(i\omega + \omega^2) \cdot e^{\Psi(\omega)(T-t)} \cdot FFT[(G - F(T))_+](\omega) \right],
\]

\[
\frac{\partial^2 M}{\partial F^2}(t, s, F) = \pi'_t FFT^{-1} \left[ -(i\omega + \omega^2) \cdot e^{\Psi(\omega)(T-t)} \cdot FFT[m_e F(s)](\omega) \right].
\]

We notice that the fund value at maturity time \(T\) can be written as \(F(t)e^\theta\), which means that the payoff functions for a European put \((G - F(T))_+\) and a rider charge \(m_e F(s)\) does not depend on the interest rate or volatility parameters. For any given parameter \(\theta\), where the option payoff does not depend on \(\theta\), the associated Greek requires just an extra multiplicative factor, \(\Theta\), where

\[
\Theta = \frac{\partial}{\partial \theta} \Psi(\omega)(T - t),
\]

in front of the Fourier transforms of the option value and rider charge. We can then express such sensitivities as

\[
\frac{\partial P}{\partial \theta}(t, T, F) = \pi'_t FFT^{-1} \left[ \Theta \cdot e^{\Psi(\omega)(T-t)} \cdot FFT[(G - F(T))_+](\omega) \right],
\]

\[
\frac{\partial M}{\partial \theta}(t, s, F) = \pi'_t FFT^{-1} \left[ \Theta \cdot e^{\Psi(\omega)(T-t)} \cdot FFT[m_e F(s)](\omega) \right].
\]

A list of the multiplicative factors for the parameters which characterise each regime is provided in Table 3.

Hence, the financial Greeks for GMMB can be easily evaluated at time \(t\) as

\[
\frac{\partial L}{\partial F} = S(t, T) \frac{\partial P}{\partial F}(t, T, F) - \sum_{s=0}^{T-1} S(t, s) \frac{\partial M}{\partial F}(t, s, F),
\]
Table 3: Multiplicative factors for \( r_1, r_2, \sigma_1, \sigma_1 \).

| Parameter       | \( \Theta \)                       |
|-----------------|------------------------------------|
| Volatility \( \sigma_1 \) | \((-i\omega + \omega^2)\sigma_1(T-t) 0\) |
| Volatility \( \sigma_2 \) | \(0 - (i\omega + \omega^2)\sigma_2(T-t)\) |
| Force of interest \( r_1 \) | \((i\omega - 1)(T-t) 0\) |
| Force of interest \( r_2 \) | \(0 (i\omega - 1)(T-t)\) |

\[
\frac{\partial^2 L}{\partial F^2} = S(t,T) \frac{\partial^2 P}{\partial F^2}(t,T,F) - \sum_{s=0}^{T-1} S(t,s) \frac{\partial^2 M}{\partial F^2}(t,s,F),
\]

\[
\frac{\partial L}{\partial \sigma_i} = S(t,T) \frac{\partial P}{\partial \sigma_i}(t,T,F) - \sum_{s=0}^{T-1} S(t,s) \frac{\partial M}{\partial \sigma_i}(t,s,F),
\]

\[
\frac{\partial L}{\partial r} = S(t,T) \frac{\partial P}{\partial r}(t,T,F) - \sum_{s=0}^{T-1} S(t,s) \frac{\partial M}{\partial r}(t,s,F).
\]

for \( i = 1, 2 \).

For mortality Greeks, we use FEL process as our survival probability function and one advantage of FEL process is that the parameter \( \alpha = 0 \). Taking the first and second derivatives of equation (3) with respect to mortality parameter \( \beta \) yields

\[
\frac{\partial S}{\partial \lambda} = \beta(t,T)S(t,T),
\]

\[
\frac{\partial^2 S}{\partial \lambda^2} = \beta^2(t,T)S(t,T),
\]

and using the results from equations (64) and (65), the mortality Greeks for the GMMB are

\[
\frac{\partial L}{\partial \lambda} = \beta(t,T)S(t,T)P(t,T,F) - \sum_{s=0}^{T-1} \beta(t,s)S(t,s)M(t,s,F),
\]

\[
\frac{\partial^2 L}{\partial \lambda^2} = \beta^2(t,T)S(t,T)P(t,T,F) - \sum_{s=0}^{T-1} \beta^2(t,s)S(t,s)M(t,s,F).
\]

### 3.3 Hedging strategies

In this section we outline three static hedging strategies for the net liability model. Suppose an insurer wish to hedge its net liability of GMMB which matures at time \( T \). A static hedging strategy is to construct a hedge portfolio at time zero that is held until maturity \( T \) without any rebalancing. We construct a hedging portfolio made up of some elementary contracts, such as q-forwards (mortality derivatives), European puts and zero-coupon bonds. We wish to hedge all the Greeks derived in Section 3.2 and thus will require a mix of mortality derivatives and financial derivatives to hedge both the financial and mortality risk exposures. We construct our hedging portfolio by purchasing \( N_q \) units of q-forwards, \( N_z \) units of zero-coupon bonds and \( N_p \) units of put options with various time to maturity \( t_j \). At time, \( t \), the value of the portfolio can be written as

\[
\Pi(t) = P^\prime N - L(t,T,F) = \sum_{j=1}^{N_q} n_j P(t,t_j;S) + \sum_{j=N_q+1}^{N_q+N_p} n_j Q(t,t_j) + \sum_{j=N_q+N_p+1}^{N} n_j Z(t,t_j) - L(t,T,F),
\]

where \( P = (P(t,t_1,S), \ldots, P(t,t_{N_q};S), Q(t,t_{N_q+1}); \ldots, Q(t,t_{N_q+N_p}); Z(t,t_{N_q+N_p+1}); \ldots, Z(t,t_N))^\prime \) and \( N = (n_1, \ldots, n_{N_q}, n_{N_q+1}; \ldots, n_{N_q+N_p}, n_{N_q+N_p+1}; \ldots, n_N)^\prime \); \( P(t,t_j;S) \) denotes a European put option which written on
the underlying equity fund $S$, with strike $G$ and maturity $t_j$; $Q(t, t_j)$ and $Z(t, t_j)$ are the time-$t$ value of q-forward and zero-coupon bond which matures at time $t_j$, respectively. And we have
\[ Q(t, T) = S(t, T)Z(t, T), \]  
where $S(t, T)$ is the time-$t$ expected survival probability which we discussed in Section 2.1.

- **Linear algebra method**

For the first static hedging strategy, we adopt immunization method proposed in Ignatieva et al. [21] and Luciano et al. [27]. In order to immunise the portfolio against changes in the financial and mortality parameters at time zero, a static hedging strategy requires to solve the following system of equations at time $t = 0$:
\[ Y = XN, \]  
where $Y = (L, \frac{\partial L}{\partial F}, \frac{\partial^2 L}{\partial F^2}, \frac{\partial L}{\partial \sigma}, \frac{\partial L}{\partial \lambda}, \frac{\partial^2 L}{\partial \sigma \lambda})'$, $X = (P', \frac{\partial P'}{\partial F}, \frac{\partial^2 P'}{\partial F^2}, \frac{\partial P'}{\partial \sigma}, \frac{\partial P'}{\partial \lambda}, \frac{\partial^2 P'}{\partial \sigma \lambda})'$, $N = (n_1, n_2, \ldots, n_7)'$ and $i = 1, 2$.

Although there are nine equations outlined in the system above, we only need to solve seven of them as $\frac{\partial \Pi(0)}{\partial \sigma_i} = 0$ when we differentiate with respect to the parameters of the regime we are not currently residing in. Since the assumption is made that the regimes are observable, if we find that we are currently in regime 1, then $\frac{\partial \Pi(0)}{\partial \sigma_1} = 0$ and vice versa. We set $N = 7$ to ensure that the system of equations in not over- or under-determined. Furthermore, we set the $t_j$’s no less than the maturity of GMBB, so that the hedging assets will not expire before the GMBB. For the first hedging strategy, linear algebra is used to find the number of units $n_j$ assigned to each GMBB in the hedging portfolio.

- **Optimization method**

When solving the equation system (70), there is a chance we get a very large $n_j$, which means we need to long or short a large number of simple assets to hedge one unit of GMBB. It will reduce the hedging performance by increasing the insurer’s transaction cost. Furthermore, without any hedging after inception, the volatility of hedging portfolio may be larger than a single unhedged GMBB contract. Therefore, we use an optimization method to restrict the value of $t_j$ and minimize the sum of squared error of each function in system (70). The optimization problem is then given by
\[ \min_{N \in \mathbb{R}^7} (XN - Y)'(XN - Y) \]  
s.t. $L \leq N \leq U$  
(71)

where $L = (l_1, l_2, \ldots, l_7)'$ and $U = (u_1, u_2, \ldots, u_7)'$ are the lower bound and upper bound, respectively. For the same reason discussed in linear algebra method, we can omit $r_2$ and $\sigma_2$ if we are currently in regime 1. Therefore we only have seven independent variables.

- **Regularization method**

In optimization method, we still need to long or short seven simple assets at the GMBB contract inception. In a real world, however, some assets are unavailable in the financial market. The last static hedging strategy is inspired by regularization method which is a technique widely used in machine learning field. It makes slight modifications to the algorithm such that we can get a sparse result. The new optimization problem is then given by
\[ \min_{N \in \mathbb{R}^7} (XN - Y)'(XN - Y) + \lambda \sum_{j=1}^{7} |n_j|, \]  
where the Lagrange multiplier $\lambda > 0$.

4 Numerical results

Using methodology described above, we investigate risk management performance for various GMBB. We use the parameter set in Table 4 to describe the financial parameters. The mortality parameter are estimated using the FEL model described in Section 2.4. The financial parameters are chosen to demonstrate the impact of two different regimes on risk management performance - one with high volatility and low returns, referred to as a ‘bear’ market; and another one with low volatility and high returns, referred to as a ‘bull’ market. Since the Markov chain $X$ modulating the regime switches is observable, we can almost certainly identify which regime we are residing in at each time, depending on the initial probability $p$, which is expected to be close to either 1 or 0. The no-arbitrage rider charge rate $m_e$ is calculated by equation (49), given a fix number of M&E rate.
Table 4: Parameters used for the regime-switching model.

| Parameter                          | Value |
|------------------------------------|-------|
| Initial equity fund                | \( S(0) \) | 100 |
| Guaranteed base                    | \( G \) | 100 |
| M&E fee                            | \( m \) | 0.0200 |
| Rider charge                       | \( m_e \) | 0.0144 |
| Initial probability of being in regime 1 | \( p \) | 1 |

| Regime 1                           |       |
|------------------------------------|-------|
| Drift                              | \( \mu_1 \) | 0.05 |
| Volatility                         | \( \sigma_1 \) | 0.10 |
| Force of interest                  | \( r_1 \) | 0.04 |
| Transition rate                    | \( a_{12} \) | 0.40 |

| Regime 2                           |       |
|------------------------------------|-------|
| Drift                              | \( \mu_2 \) | 0.02 |
| Volatility                         | \( \sigma_2 \) | 0.20 |
| Force of interest                  | \( r_2 \) | 0.01 |
| Transition rate                    | \( a_{21} \) | 0.30 |

Figure 1: Liability sensitivity of GMMB to various guarantees level for 50 and 70 years old

4.1 Net liability results for GMMB

Fig 1 shows sensitivity of insurer’s net liability to various guarantee levels \( G \), different contract lengths and different initial ages. As one would expect, when \( G \) increases, the insurer’s net liability will also increases, as the policyholder is guaranteed more. In addition, one observes an increasing convexity in the net liability function as the length of the contract shortens. The convexity can be explained using equation (50), where a European put appears in the model’s decomposition. Furthermore, from the plot (a) in Fig 1 we notice that the net liability is zero at a guarantee level of 100. This is because we use a fair rates of \((m, m_e)\) such that the insurer’s incomes and outgoes are equal at the contract inception.

By comparing the plots in Fig 1, we can find that the convexity in each curve in plot (b) is less then the that in plot (a). This is a result of the increasing mortality risk, which gives a smaller discounted value (since GMMB only provides benefits if the policyholder is alive at maturity). And the effect of the mortality risk on net liability becomes more significant when the length of the contract increases.

4.2 Comparison of computational efficiency

It is also interest to investigate the computational efficiency of the FST method relative to other valuation techniques. In this subsection we analyse the efficiency of the FST approach relative to the Monte Carlo (MC) simulations. Table 5 shows GMMB’s net liability for various maturities ranging from 10 to 35 years using the two approaches. The second column of Table 5 contains FST results while the Monte Carlo results obtained using one hundred thousand simulations are reported in the third columns. The last two columns contain computational times (in seconds) elapsed by the FST
Table 5: Comparing the efficiency of the FST method relative to the MC method.

| Contract maturity | FST  | MC   | Time of FST | Time of MC |
|-------------------|------|------|-------------|------------|
| 10 years          | 0.0000 | -0.015 | 3.1312      | 30.9175    |
| 15 years          | -4.5765 | -4.5737 | 4.5929      | 50.8262    |
| 20 years          | -9.6726 | -9.6173 | 6.0768      | 77.8178    |
| 25 years          | -14.8235 | -14.8673 | 7.3648      | 101.3046   |
| 30 years          | -19.7049 | -19.6873 | 8.7500      | 129.4822   |
| 35 years          | -24.0167 | -24.0042 | 10.6939     | 169.9292   |

and MC in computing net liability for a given maturity, respectively. From this table we note that the FST method is demonstrated to be more computational efficient as it is more than 10 times faster compared to the MC simulations in generating accurate results.

4.3 Greeks results

The methodology outlined in Section 3.2 is utilised in order to numerically compute the Greeks for insurer’s net liability of as a function of the underlying parameters.

Fig 2 shows delta and gamma in the top left (a) and right (b) panel, respectively. For a short maturity of one year the delta and gamma resemble the behaviour of the Greeks for a European put option. As the stock price (i.e. fund value) increases, the change in price approaches zero, as the put option is deep out-of-the-money and almost certain to be not exercised, and the final payoff is zero. Similarly, if the stock price decreases, the change in price approaches negative one, as the put option is in-the-money and almost certain to be exercised, and the final payoff is the guarantee level minus stock price. For increasing maturities, e.g. when maturity is 15-year, there is only a little change in the delta, which is reflected in the linearity in prices that we observed in Fig 1.

The middle panels of Fig 2 show vegas, which measure price sensitivity (in percentage terms) with respect to fund volatility in regime 1 (panel (c)) and regime 2 (panel (d)). Similar to European vanilla options, the value of the GMMB is an increasing function of volatility.

The bottom panels of Fig 2 show rhos, which represent a percentage change in price attributed to changes in the force of interest in regimes 1 (panel (e)) and regime 2 (panel (f)). As the force of interest increases, rho moves from a negative values towards zero; the risk-neutral net liability of the GMMB decreases with higher risk-free rates due to higher discounting. Overall, GMMB with longer maturities are more sensitive to changes in interest rates and market volatility.

4.4 Hedging Performance

We then perform static hedge, and discuss effectiveness from the insurer’s perspective. We assume that the policyholder is aged 50 at time \( t = 0 \), and that the contract matures in 15 years. A static hedging portfolio is constricted by using various simpler derivatives, which include European puts, zero-coupon Bonds and q-forwards of different maturities. Static hedge assumes that the portfolio constructed at \( t = 0 \) is held until maturity \( T \) without any rebalancing. To investigate hedging performance, we compare three static hedging portfolios: in the first portfolio, we solve a seven equations system with seven independent variables by linear algebra method; in the second case we transfer the linear algebra problem into a optimization problem by least square method; in the last portfolio, we improved the optimization method by introducing a regularization method to get a sparse solution. The hedge effectiveness is analysed by comparing the profit-and-loss (P&L) distributions and the associated summary statistics.

Asset allocation results are reported in the last three columns in Table 6. The forth column contains the linear results which have a large units for each simple assets. The fifth column contains the result solved by optimization method which we limit the units \( n_j \in [-100, 100] \) for \( j = 1, \ldots, 7 \). The last column gives the result by regularization method. It gives us a sparse vector of units as we expected.

Hedging results at the end of each year, obtained based on 1000 simulations, are summarised in Table 7. This table shows summary statistics computed for the P&L for the unhedged and three hedged static portfolios. Portfolio 1 has the smallest standard deviation (year 1 to year 3) in the short term and the largest standard deviation in the long term (year 7 to year 10). This is because Portfolio 1 is constructed by a full immunization method which reflects the best hedging performance at the beginning. However it contains the largest amount of simple assets and the portfolio will have the largest volatility in long term. The hedging performance of Portfolio 2 is better than portfolio 3 in the short term and vice versa in the long term since Portfolio 2 contains more simple assets than Portfolio 3. As expected, an unhedged
Figure 2: Financial Greeks for GMMB at time $t = 0$. 
Table 6: Assets used in three static hedge portfolios.

| Asset type          | Time($t_j$) | Spot($) | Portfolio 1($n_j$) | Portfolio 2($n_j$) | Portfolio 3($n_j$) |
|---------------------|-------------|---------|--------------------|--------------------|--------------------|
| GMMB liability      | 10          | 0.0000  | -1.0000            | -1.0000            | -1.0000            |
| European put        | 10          | 8.7633  | -228.8015          | -2.5674            | -0.5482            |
| European put        | 11          | 8.8828  | 496.9425           | 0.0766             | 0.0000             |
| European put        | 12          | 8.9619  | -267.2235          | 3.5888             | 0.0000             |
| q-forward           | 10          | 0.7217  | -7.3183            | -7.3183            | -3.1888            |
| q-forward           | 11          | 0.6976  | 16.7509            | 16.7509            | 1.3521             |
| zero-coupon bond    | 10          | 0.7784  | -953.5816          | -11.3243           | 0.0000             |
| zero-coupon bond    | 11          | 0.7611  | 948.0569           | 10.4228            | 0.0000             |

Table 7: Comparing the hedging performance at the end of each year.

| Times | Unhedged Mean | Std. dev. | Portfolio 1 Mean | Std. dev. | Portfolio 2 Mean | Std. dev. | Portfolio 3 Mean | Std. dev. |
|-------|---------------|-----------|------------------|-----------|------------------|-----------|------------------|-----------|
| 1     | 0.3637        | 4.8253    | -0.9181          | 0.2213    | -1.0481          | 0.5802    | -0.2173          | 2.5637    |
| 2     | 0.8206        | 7.3369    | -1.9948          | 0.9056    | -2.0429          | 1.4702    | -0.4107          | 3.7932    |
| 3     | 0.6335        | 8.7396    | -3.4471          | 1.7161    | -3.2472          | 1.9078    | -1.0328          | 4.3384    |
| 4     | -0.7873       | 11.2039   | -4.9802          | 4.2001    | -4.1680          | 3.7278    | -2.2450          | 5.1288    |
| 5     | -0.5362       | 11.9042   | -8.6377          | 5.6105    | -5.5152          | 3.9796    | -2.9171          | 5.3970    |
| 6     | -1.3451       | 12.5552   | -12.5055         | 9.6779    | -7.0293          | 5.1758    | -4.0779          | 5.3119    |
| 7     | -2.7944       | 13.1239   | -21.4002         | 16.1691   | -8.6075          | 6.2243    | -5.8621          | 5.6074    |
| 8     | -4.6605       | 14.8094   | -36.2298         | 28.4762   | -9.8252          | 8.5630    | -8.0964          | 6.1836    |
| 9     | -6.7524       | 15.8020   | -66.7366         | 55.3954   | -11.3315         | 10.6039   | -11.0868         | 7.0913    |
| 10    | 9.5259        | 17.0019   | -117.8817        | 138.7463  | -12.5686         | 13.0196   | -15.8837         | 8.9054    |

portfolio bears the highest risk among the four portfolios in the short term, which is reflected in the largest standard deviation, and just perform better than Portfolio 2 in the long term. Except the unhedged portfolio, the average P&L at the end of each year for other three portfolios are all negative since they all contains a large short position in interest related financial instruments (forwards and zero-coupon bonds) which have no relationship with underlying stock price in our simulations. A good suggestion is to adopt a semi-hedged strategy base on these three static hedging methods. If a insurer is more concerned about short-term volatility, a linear algebra method is preferred and it requires rebalancing frequently. If one is more concerned about transaction fees, he can adopt an optimization method and does not need to rebalance every year. A regularization method is preferable for incomplete market which only have few instrument for hedging.

5 Conclusion

This paper provides an extensive analysis of GMMB embedded in variables annuity contracts when the dynamics of the underlying fund is governed by a two-regime switching process. This includes developing a net liability valuation framework for the GMMB; analysis net liability sensitivity with respect to model parameters, such as interest rates, volatilities and mortality risk parameters; as well as developing three static hedging strategies and analysis their hedge effectiveness.

The paper compares three stochastic mortality models and incorporates the most efficient one in the valuation framework by adopting a affine mortality model whose parameters are calibrated to the United States male mortality data for the period ranging from 1957 to 2017. A regime-switching model is utilised to model the fund dynamics in the financial market, while an affine term structure model is implemented to capture the mortality dynamics. This paper implements the fast and efficient FST algorithm for valuation and computing the Greeks of insurer’s net liability under a regime-switching environment.

In this paper we analysis the GMMB contract which form the building blocks for other guarantees embedded in variable annuities. The insurer’s net liability for a GMMB contract is an increasing function of a guarantee level, which becomes more convex as the length of the contract shortens. We also observe the mortality effects on valuation become more significant when the initial age increases.
In terms of hedging performance, we observe that all three static hedging strategies perform better than unhedged portfolio in the short term. For hedging strategies, the static hedging portfolio solved by linear algebra method bears the smallest risk in the short term and the highest risk in the long term. The result of optimization hedging strategy is opposite, and the the regularization method strategy is moderate in the middle. Insurers can choose a static hedging strategy and rebalance strategy according to their risk tolerances.

Further extensions and future areas of research include but not limited to implementing additional contracts features such as roll-ups, deferrals or ratchets; choosing more realistic fund dynamics than the geometric Brownian motion; and replacing an observable Market chain by a hidden Markov chain.

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