(2+1) null-plane quantum Poincaré group from a factorized universal $R$-matrix

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Abstract

The non-standard (Jordanian) quantum deformations of $so(2, 2)$ and (2+1) Poincaré algebras are constructed by starting from a quantum $sl(2, \mathbb{R})$ basis such that simple factorized expressions for their corresponding universal $R$-matrices are obtained. As an application, the null-plane quantum (2+1) Poincaré Poisson-Lie group is quantized by following the FRT prescription. Matrix and differential representations of this null-plane deformation are presented, and the influence of the choice of the basis in the resultant $q$-Schrödinger equation governing the deformed null plane evolution is commented.
1 Introduction

Among quantum deformations of Poincaré algebra we find three remarkable Hopf structures. Two of them are obtained in a natural way within a purely kinematical framework encoded within the usual Poincaré basis. They are the well-known $\kappa$-Poincaré algebra [1, 2] where the deformation parameter can be interpreted as a fundamental time scale and a $q$-Poincaré algebra [3] where the quantum parameter is a fundamental length. On the other hand, the remaining structure (the null-plane quantum Poincaré algebra recently introduced in [4, 5]) strongly differs from the previous ones: firstly, it is constructed in a null-plane context where the Poincaré invariance splits into a kinematical and dynamical part [6] and, secondly, this case is a quantization of a non-standard (triangular) coboundary Lie bialgebra.

We also recall that the related problem of obtaining universal $R$-matrices for standard Poincaré deformations has been only solved in (2+1) dimensions [7, 8]. In the non-standard case, relevant successes have been recently obtained: the universal $R$-matrix for the (1+1) case has been deduced in [9, 10, 11] and a (2+1) solution has been recently given in [12].

The aim of this letter is twofold: on one hand, to present a simplified construction of the universal $R$-matrices for non-standard quantum $so(2, 2)$ and (2+1) Poincaré algebras, which is based on the $sl(2, \mathbb{R})$ factorized $R$-matrix introduced in [13] (sections 2 and 3). Within this construction a (non-linear) change of basis (whose origin lies in a $T$-matrix construction [14]) is essential, and will lead to rather new expressions for all these quantizations. From a physical point of view, it is important to stress that $so(2, 2)$ is interpreted in a conformal context: i.e., its classical counterpart acts as the group of conformal transformations on the (1+1) Minkowskian space-time. The second objective is to get a deeper insight in the (2+1) null-plane quantization by constructing the associated quantum group and by exploring its representation theory. All these results are presented in Section 4.

2 Non-standard quantum $so(2, 2)$ revisited

Let us consider the coproduct and the commutation relations of the non-standard quantum $sl(2, \mathbb{R})$, denoted by $U_z sl(2, \mathbb{R}) = \langle A, A_+, A_- \rangle$, as [13]:

$$\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1,$$
$$\Delta(A) = 1 \otimes A + A \otimes e^{2zA_+},$$
$$\Delta(A_-) = 1 \otimes A_- + A_- \otimes e^{2zA_+},$$

(2.1)

$$[A, A_+] = \frac{e^{2zA_+} - 1}{z}, \quad [A, A_-] = -2A_- + zA^2, \quad [A_+, A_-] = A,$$

(2.2)

and the quantum Casimir belonging to the centre of $U_z sl(2, \mathbb{R})$ given by:

$$C_z = \frac{1}{2} A e^{-2zA_+} A + \frac{1 - e^{-2zA_+}}{2z} A_- A_+ - 1,$$

(2.3)
These relations are obtained from the ones given in [4, 15] in terms of \( \{ J_3, J_+, J_- \} \), by means of the change of basis [13]

\[
A_+ = J_+, \quad A = e^{zJ_+} J_3, \\
A_- = e^{zJ_+} J_- - \frac{z}{4} e^{zJ_+} \sinh(zJ_+). \tag{2.4}
\]

In [13] it is shown that the universal element

\[
R_z = \exp\{-zA_+ \otimes A\} \exp\{zA \otimes A_+\} \tag{2.5}
\]

is a solution of the quantum Yang–Baxter equation and also verifies the property

\[
\sigma \circ \Delta(X) = R_z \Delta(X) R_{z-1}, \quad \forall X \in U_z sl(2, \mathbb{R}), \tag{2.6}
\]

being \( \sigma \) the flip operator \( \sigma(a \otimes b) = b \otimes a \). Hence, \( R_z \) is the quantum universal \( R \)-matrix for \( U_z sl(2, \mathbb{R}) \).

Two comments concerning this universal \( R \)-matrix are in order: firstly, the significant simplification obtained (as far as the commutation rules (2.2) are concerned) with respect to the original formulation of this non-standard \( sl(2, \mathbb{R}) \) deformation. Secondly, we recall that the factorized expression (2.5) comes from a universal \( T \)-matrix formalism [11, 14]. From this point of view, the interest of finding such a kind of factorized expressions is directly related to the interpretation of the transfer matrices as quantum monodromies and the obtention of more manageable algebraic models in quantum field theory (see [13]).

Let us now consider two copies of the non-standard quantum \( sl(2, \mathbb{R}) \) algebra, the former with \( z \) and the latter with \( -z \) as deformation parameters: \( U_z^{(1)} sl(2, \mathbb{R}) = \langle A^1, A_+^1, A_-^1 \rangle \) and \( U_{-z}^{(2)} sl(2, \mathbb{R}) = \langle A^2, A_+^2, A_-^2 \rangle \). The generators defined by

\[
K = \frac{1}{2}(A_1 - A_2), \quad D = \frac{1}{2}(A_1 + A_2), \\
H = A_+^1 + A_-^1, \quad P = A_+^2 - A_-^2, \\
C_1 = -A_-^1 - A_+^2, \quad C_2 = A_1^1 - A_2^1, \tag{2.7}
\]

give rise to a non-standard quantum deformation of \( so(2, 2) \) [1]:

\[
U_z so(2, 2) \simeq U_z^{(1)} sl(2, \mathbb{R}) \oplus U_{-z}^{(2)} sl(2, \mathbb{R}).
\]

At a purely classical level, \( SO(2, 2) \) can be regarded in this basis as the group of conformal transformations of the (1+1) Minkowskian space-time, where \( K \) generates the boosts, \( H \) the time translations, \( P \) the space translations, \( D \) is a dilation generator and \( C_1, C_2 \) generate specific conformal transformations. The Hopf algebra structure of \( U_z so(2, 2) \) obtained in this way is given by the following coproduct (\( \Delta \)), counit (\( \epsilon \)), antipode (\( \gamma \)) and commutation relations:

\[
\Delta(H) = 1 \otimes H + H \otimes 1, \quad \Delta(P) = 1 \otimes P + P \otimes 1, \\
\Delta(K) = 1 \otimes K + K \otimes e^{zP} \cosh zH + D \otimes e^{zP} \sinh zH,
\]

3
\[ \Delta(D) = 1 \otimes D + D \otimes e^{zP} \cosh zH + K \otimes e^{zP} \sinh zH, \quad (2.8) \]
\[ \Delta(C_1) = 1 \otimes C_1 + C_1 \otimes e^{zP} \cosh zH - C_2 \otimes e^{zP} \sinh zH, \]
\[ \Delta(C_2) = 1 \otimes C_2 + C_2 \otimes e^{zP} \cosh zH - C_1 \otimes e^{zP} \sinh zH, \]
\[ \epsilon(X) = 0, \quad \text{for } X \in \{K, H, P, C_1, C_2, D\}, \quad (2.9) \]
\[ \gamma(H) = -H, \quad \gamma(P) = -P, \]
\[ \gamma(K) = -K e^{-zP} \cosh zH + D e^{-zP} \sinh zH, \]
\[ \gamma(D) = -D e^{-zP} \cosh zH + K e^{-zP} \sinh zH, \quad (2.10) \]
\[ [K, H] = \frac{1}{2} (e^{zP} \cosh zH - 1), \quad [K, P] = \frac{1}{2} e^{zP} \sinh zH, \]
\[ [K, C_1] = C_2 - z(K^2 + D^2), \quad [K, C_2] = C_1 + 2zKD, \]
\[ [D, H] = \frac{1}{2} e^{zP} \sinh zH, \quad [D, P] = \frac{1}{2} (e^{zP} \cosh zH - 1), \quad (2.11) \]
\[ [D, C_1] = -C_1 - 2zKD, \quad [D, C_2] = -C_2 + z(K^2 + D^2), \]
\[ [H, C_1] = -2D, \quad [H, C_2] = 2K, \quad [P, C_1] = -2K, \quad [P, C_2] = 2D, \]
\[ [K, D] = 0, \quad [H, P] = 0, \quad [C_1, C_2] = 0. \]

Note that (2.8) presents an interesting feature: both generators \( H \) and \( P \) are primitive ones. Therefore, this conformal approach to \( \text{so}(2, 2) \) leads to a quantum structure that can be interpreted as an attempt in order to deform the \((1+1)\) Minkowskian space and time in a rather symmetrical way.

Two elements of the centre of \( U_z \text{so}(2, 2) \) are constructed from the quantum Casimirs (2.3) of \( U_z^{(1)} \text{sl}(2, \mathbb{R}) \) and \( U_z^{(2)} \text{sl}(2, \mathbb{R}) \) as:
\[ C_1^q = C_1^{(1)} + C_1^{(2)} - C_2^{(2)}, \quad C_2^q = C_2^{(1)} - C_2^{(2)}. \quad (2.12) \]

After a straightforward computation we get:
\[ C_1^q = Ke^{-zP} \cosh(zH)K + De^{-zP} \cosh(zH)D \]
\[ -K e^{-zP} \sinh(zH)D - D e^{-zP} \sinh(zH)K \]
\[ + \frac{1}{2z}(1 - e^{-zP} \cosh zH)C_2 + C_2 \frac{1}{2z}(1 - e^{-zP} \cosh zH) \]
\[ - e^{-zP} \sinh zH C_1 - C_1 e^{-zP} \sinh zH \]
\[ = 2z (1 - e^{-zP} \cosh(zH) - 2z e^{-zP} \cosh zH - 1), \quad (2.13) \]
\[ C_2^q = Ke^{-zP} \cosh(zH)D + De^{-zP} \cosh(zH)K \]
\[ - K e^{-zP} \sinh(zH)K - D e^{-zP} \sinh(zH)D \]
\[ - \frac{1}{2z}(1 - e^{-zP} \cosh zH)C_1 - C_1 \frac{1}{2z}(1 - e^{-zP} \cosh zH) \]
\[ + e^{-zP} \sinh zH C_2 + C_2 e^{-zP} \sinh zH \]
\[ = 2z (1 - e^{-zP} \cosh(zH) - 2z e^{-zP} \sinh zH). \quad (2.14) \]
Likewise, the universal $R$-matrix for $U_z\text{so}(2, 2)$ can be easily deduced as a product of those corresponding to the two copies of $U_z\text{sl}(2, \mathbb{R})$ \cite{2,3}:

$$R_z = R_z^{(1)} R_z^{(2)} = \exp\{-zA_1^+ \otimes A_1^-\} \exp\{zA_1^+ \otimes A_1^-\} \exp\{-zA_2^+ \otimes A_2^-\} \exp\{zA_2^+ \otimes A_2^-\}$$

$$= \exp\{-zA_1^+ A_1^- + zA_2^+ A_2^-\} \exp\{zA_1^+ A_1^- - zA_2^+ A_2^-\}$$

$$= \exp\{-z(H \otimes K + P \otimes D)\} \exp\{z(K \otimes H + D \otimes P)\},$$

which can be finally written in a complete “factorized” form as:

$$R_z = \exp\{-zH \otimes K\} \exp\{-zP \otimes D\} \exp\{zD \otimes P\} \exp\{zK \otimes H\}. \quad (2.15)$$

The first order in $z$ gives the classical $r$-matrix

$$r = z(K \wedge H + D \wedge P), \quad (2.16)$$

which, as expected, is a solution of the classical Yang–Baxter equation.

### 3 (2+1) null-plane quantum Poincaré algebra

#### 3.1 Null-plane classical Poincaré algebra

We briefly describe the classical structure of the (2+1) Poincaré algebra $\mathcal{P}(2+1)$ in relation with the null-plane evolution scheme \cite{17}, in which the initial state of a quantum relativistic system can be defined on a light-like plane $\Pi^\tau_n$ defined by $n \cdot x = \tau$, where $n$ is a light-like vector and $\tau$ a real constant. In particular, if $n = (\frac{1}{2}, 0, \frac{1}{2})$ and the coordinates

$$x^- = n \cdot x = \frac{1}{2}(x^0 - x^2), \quad x^+ = x^0 + x^2, \quad (3.1)$$

are considered, a point $x \in \Pi^\tau_n$ will be labelled by $(x^+, x^1)$ while the remaining one $(x^-)$ plays the role of a time parameter $\tau$. A basis $\{P_+, P_-, P_1, E_1, F_1, K_2\}$ of the (2+1) Poincaré algebra consistent with these coordinates is provided by the generators $P_+, P_-, E_1$ and $F_1$ which are defined in the terms of the usual kinematical ones $\{P_0, P_1, P_2, K_1, K_2, J\}$ by:

$$P_+ = \frac{1}{2}(P_0 + P_2), \quad P_- = P_0 - P_2, \quad E_1 = \frac{1}{2}(K_1 + J), \quad F_1 = K_1 - J. \quad (3.2)$$

This “null-plane” basis has the following non-vanishing commutation rules:

$$[K_2, P_\pm] = \pm P_\pm, \quad [K_2, E_1] = E_1, \quad [K_2, F_1] = -F_1,$$

$$[E_1, P_\pm] = P_\pm, \quad [F_1, P_\pm] = P_\mp, \quad [E_1, F_1] = K_2,$$

$$[P_+, F_1] = -P_1, \quad [P_-, E_1] = -P_1. \quad (3.3)$$

The operators $\{P_+, P_1, E_1, K_2\}$ are the infinitesimal generators of the stability group $S_+$ of the null-plane $\Pi^\tau_n$ $(\tau = 0)$. The remaining generators have a dynamical
This null-plane Poincaré algebra is naturally linked by a contraction procedure to so(2, 2) when the latter is written in a basis of the kind (2.7). The explicit form of that contraction mapping is as follows:

\[ M^2 = 2P_-P_+ - P_+^2, \]  
\[ L = K_2P_1 + E_1P_- - F_1P_+. \]  

### 3.2 Null-plane quantum Poincaré algebra

This null-plane Poincaré algebra is naturally linked by a contraction procedure to so(2, 2) when the latter is written in a basis of the kind (2.7). The explicit form of that contraction mapping is as follows:

\[ P_+ = \varepsilon \frac{1}{\sqrt{2}} P, \quad P_1 = \varepsilon K, \quad P_- = -\varepsilon \frac{1}{\sqrt{2}} C_2, \]  
\[ E_1 = -\frac{1}{\sqrt{2}} H, \quad F_1 = \frac{1}{\sqrt{2}} C_1, \quad K_2 = D. \]  

In the quantum case, the deformation parameter has also to be transformed as \( w = \frac{1}{\epsilon \sqrt{2}} z \). Therefore, by applying (3.6) in the Hopf algebra of \( U_+so(2, 2) \) (2.8–2.11) and then by making the limit \( \varepsilon \to 0 \) we get the resulting Hopf structure for the quantum (2+1) null plane Poincaré algebra \( U_w\mathcal{P}(2 + 1) \):

\[ \Delta(P_+) = 1 \otimes P_+ + P_+ \otimes 1, \quad \Delta(E_1) = 1 \otimes E_1 + E_1 \otimes 1, \]  
\[ \Delta(P_-) = 1 \otimes P_- + P_- \otimes e^{2wP_+}, \quad \Delta(P_1) = 1 \otimes P_1 + P_1 \otimes e^{2wP_+}, \]  
\[ \Delta(F_1) = 1 \otimes F_1 + F_1 \otimes e^{2wP_+} - 2wP_- \otimes e^{2wP_+} E_1, \]  
\[ \Delta(K_2) = 1 \otimes K_2 + K_2 \otimes e^{2wP_+} - 2wP_1 \otimes e^{2wP_+} E_1, \]  
\[ \epsilon(X) = 0, \quad \text{for } X \in \{ K_2, P_+, P_-, P_1, E_1, F_1 \}, \]  
\[ \gamma(P_+) = -P_+, \quad \gamma(E_1) = -E_1, \]  
\[ \gamma(P_-) = -P_- e^{-2wP_+}, \quad \gamma(P_1) = -P_1 e^{-2wP_+}, \]  
\[ \gamma(F_1) = -F_1 e^{-2wP_+} - 2wP_- e^{-2wP_+} E_1, \]  
\[ \gamma(K_2) = -K_2 e^{-2wP_+} - 2wP_1 e^{-2wP_+} E_1, \]  
\[ [K_2, P_+] = \frac{1}{2w} (e^{2wP_+} - 1), \quad [K_2, P_-] = -P_- - wP_1^2, \]  
\[ [K_2, E_1] = E_1 e^{2wP_+}, \quad [K_2, F_1] = -F_1 - 2wP_1 K_2, \]  
\[ [E_1, P_1] = \frac{1}{2w} (e^{2wP_+} - 1), \quad [F_1, P_1] = P_- + wP_1^2, \]  
\[ [E_1, F_1] = K_2, \quad [P_+, F_1] = -P_1, \quad [P_-, E_1] = -P_1, \]  

where the remaining commutators are zero. Note that the generators of the null-plane stability group close a Hopf subalgebra \( U_w S_+ \). As a byproduct of the original
change of basis within $sl(2, \mathbb{R})$, these commutation rules are simpler than the ones given in [3].

The quantum Casimirs belonging to the centre of $U_\omega \mathcal{P}(2 + 1)$ are deduced from (2.13) and (2.14) by means of the limits:

\[
M_q^2 = \lim_{\varepsilon \to 0} (\varepsilon^2 C_1^q), \quad L_q = \frac{1}{2} \lim_{\varepsilon \to 0} (\varepsilon C_2^q), \tag{3.11}
\]

explicitly,

\[
M_q^2 = P_+ \frac{1 - e^{-2wP_+}}{w} - P_1^2 e^{-2wP_+}, \tag{3.12}
\]

\[
L_q = K_2 P_1 e^{-2wP_+} + E_1 (P_+ + w P_1^2) e^{-2wP_+} - F_1 \frac{1 - e^{-2wP_+}}{2w}. \tag{3.13}
\]

The universal $R$-matrix $U_\omega \mathcal{P}(2 + 1)$ is also directly obtained from (2.15) and reads:

\[
R_\omega = \exp \{2w E_1 \otimes P_1\} \exp \{-2wP_+ \otimes K_2\} \exp \{2wK_2 \otimes P_+\} \exp \{-2wP_1 \otimes E_1\}. \tag{3.14}
\]

A differential representation of $U_\omega S_+$ with coordinates $(p_+, p_1)$ can be given as follows:

\[
P_+ = p_+, \quad P_1 = p_1, \quad K_2 = \frac{e^{2wp_+} - 1}{2w} \partial_+, \quad E_1 = \frac{e^{2wp_+} - 1}{2w} \partial_1, \tag{3.15}
\]

where $\partial_+ = \frac{\partial}{\partial p_+}$ and $\partial_1 = \frac{\partial}{\partial p_1}$. With the aid of the quantum Casimirs a spin-zero differential representation ($L_q = 0$) for the two remaining generators of $U_\omega \mathcal{P}(2 + 1)$ can be deduced:

\[
P_- = \frac{w(m_q^2 + p_1^2 e^{-2wp_+})}{1 - e^{-2wp_+}}, \quad F_1 = p_1 \partial_+ + \frac{w(m_q^2 + p_1^2)}{1 - e^{-2wp_+}} \partial_1, \tag{3.16}
\]

where $m_q^2$ is the eigenvalue of the $q$-Casimir (3.12). Similarly to the classical case, we can take the coordinate $x^-$ as an evolution parameter ($\tau$) and thus we can consider a wave function $\psi(p_+, p_1, \tau)$ whose evolution is determined by the $q$-Scrödinger equation provided by the Hamiltonian $P_-$: $i \partial_\tau \psi = P_- \psi$. In terms of the representation (3.16) we get:

\[
i \partial_\tau \psi(p_+, p_1, \tau) = \frac{w(m_q^2 + p_1^2 e^{-2wp_+})}{1 - e^{-2wp_+}} \psi(p_+, p_1, \tau), \tag{3.17}
\]

which is different from the one given in [3] for the (3+1) case. This fact can be more clearly appreciated by writing the power series expansion in $w$ of $P_-:

\[
P_- = \frac{w(m_q^2 + p_1^2 e^{-2wp_+})}{1 - e^{-2wp_+}} = \frac{w(m_q^2 e^{wp_+} + p_1^2 e^{-wp_+})}{2 \sinh wp_+} \frac{2 \cosh wp_+}{2p_+} + w^2 \frac{m_q^2 + p_1^2}{2} + o(w^3). \tag{3.18}
\]
The zero-term in $w$ can be identified with a kinetic term of the null-plane bound state equation in quantum chromodynamics [18, 19] while all remaining terms in $w$ constitute a dynamical part, now including a first order term in $w$ (that is absent in [5]). Therefore, this deformation of the null-plane symmetry has some intrinsic dynamical content whose explicit description depends on the way in which the deformation is constructed.

4 (2+1) Null-plane quantum Poincaré group

The Lie bialgebra underlying the quantum Hopf algebra of $U_w \mathcal{P}(2+1)$ is generated by the non-standard classical $r$-matrix (first order in $w$ of (3.14)):

$$r = 2(K_2 \wedge P_+ + E_1 \wedge P_1), \quad (4.1)$$

which provides the cocommutators $\delta(X) = [1 \otimes X + X \otimes 1, r]$:

$$
\begin{align*}
\delta(P_+) &= 0, & \delta(E_1) &= 0, \\
\delta(P_1) &= 2P_1 \wedge P_+, & \delta(P_-) &= 2P_- \wedge P_+, \\
\delta(F_1) &= 2(F_1 \wedge P_+ + E_1 \wedge P_-), \\
\delta(K_2) &= 2(K_2 \wedge P_+ + E_1 \wedge P_1).
\end{align*}
\quad (4.2)
$$

They are related to the first order term in the deformation parameter of the coproduct (3.7) by means of $\delta = \Delta^{(1)} - \sigma \circ \Delta^{(1)}$.

The $r$-matrix (4.1) also allows to deduce the associated Poisson structure to the Poincaré algebra. Let the four-dimensional matrix representation of $\mathcal{P}(2+1)$ given by:

$$
\begin{align*}
D(P_+) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & D(P_-) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & D(P_1) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
D(E_1) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & D(F_1) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & D(K_2) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{align*}
\quad (4.3)
$$

Then a $4 \times 4$ representation of the element $g = e^{a^+ P_+} e^{a^- P_-} e^{a^1 P_1} e^{f^1 E_1} e^{f^2 K_2}$ belonging to the (2+1) Poincaré group is

$$
D(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{a^+}{2} + a^- & \Lambda_0 & \Lambda_1 & \Lambda_0 \\ a^1 & \Lambda_1 & \Lambda_1 & \Lambda_1 \\ \frac{a^+}{2} - a^- & \Lambda_2 & \Lambda_2 & \Lambda_2 \end{pmatrix},
\quad (4.4)
$$
where the $\Lambda^\mu_\nu$ are the matrix elements of the Lorentz subgroup (whose generators are $E_1$, $F_1$ and $K_2$) satisfying the pseudo-orthogonality condition:

$$\Lambda^\mu_\nu \Lambda^\nu_\rho \eta^{\rho\sigma} = \eta^{\mu\sigma}, \quad (\eta^{\mu\sigma}) = \text{diag} (1, -1, -1). \quad (4.5)$$

The Poisson brackets of the coordinate functions on the Poincaré group are obtained by calculating the Poisson bivector

$$\{D(g) \otimes D(g)\} = [r, D(g) \otimes D(g)], \quad (4.6)$$

writing the $r$-matrix \((4.1)\) in terms of the matrix representation \((4.3)\). The final result can be summarized as follows:

$$\{a^+, a^1\} = -2w a^1, \quad \{a^+, a^-\} = -2w a^-, \quad \{a^1, a^-\} = 0,$$
$$\{\Lambda^\mu_\nu, \Lambda^\nu_\rho\} = 0, \quad \nu, \mu, \rho, \sigma = 0, 1, 2;$$
$$\{\Lambda^\mu_\nu, a^+\} = -2\delta_{\mu 0} \Lambda^\nu_0 - 2\delta_{\nu 0} \Lambda^\mu_0 - (\mu - 1)(\nu - 1) + (\Lambda^\mu_0 + \Lambda^\nu_0)(\Lambda^\nu_0 + \Lambda^\mu_0) + \Lambda^\nu_\nu(\Lambda^\mu_\nu + \Lambda^\nu_\mu - 1),$$
$$\{\Lambda^\mu_\nu, a^-\} = \frac{1}{2}(\mu - 1)(\nu - 1) + \frac{1}{2}(\Lambda^\mu_\nu + \Lambda^\nu_\mu)(\Lambda^\nu_\nu - \Lambda^\mu_\mu). \quad (4.7)$$

It is worth comparing these expressions with the results related to the classical $r$-matrix of the $\kappa$-Poincaré algebra \([20, 21, 22]\).

The classical matrix representation \((4.3)\) is also valid for $U_w \mathcal{P}(2+1)$ since $D(P_+)^2$ vanishes. This fact can be used to get an explicit expression for $\mathcal{R}_w$:

$$D(\mathcal{R}_w) = I \otimes I + 2w(D(K_2) \wedge D(P_+) + D(E_1) \wedge D(P_1)), \quad (4.8)$$

where $I$ is the $4 \times 4$ identity matrix. The fulfillment of property \((2.6)\) allows to apply the FRT method \([23]\):

$$R T_1 T_2 = T_2 T_1 R, \quad (4.9)$$

where $R$ is \((4.8)\), $T_1 = T \otimes I$, $T_2 = I \otimes T$, being $T$ the group element \((4.4)\) but now with non-commutative entries: $\hat{\Lambda}^\mu_\nu$ and $\hat{\dot{a}}^i$. The commutation relations of the quantum Poincaré group read

$$[\hat{a}^+, \hat{a}^1] = -2w \hat{a}^1, \quad [\hat{a}^+, \hat{a}^-] = -2w \hat{a}^-, \quad [\hat{a}^1, \hat{a}^-] = 0,$$
$$[\hat{\Lambda}^\mu_\nu, \hat{\Lambda}^\nu_\rho\] = 0, \quad \nu, \mu, \rho, \sigma = 0, 1, 2;$$
$$[\hat{\Lambda}^\mu_\nu, \hat{\dot{a}}^+\] = -2\delta_{\mu 0} \hat{\Lambda}^\nu_0 - 2\delta_{\nu 0} \hat{\Lambda}^\mu_0 - (\mu - 1)(\nu - 1) + (\hat{\Lambda}^\mu_0 + \hat{\Lambda}^\nu_0)(\hat{\Lambda}^\nu_0 + \hat{\Lambda}^\mu_0) + \hat{\Lambda}^\nu_\nu(\hat{\Lambda}^\mu_\nu + \hat{\Lambda}^\nu_\mu - 1),$$
$$[\hat{\Lambda}^\mu_\nu, \hat{\dot{a}}^-\] = \frac{1}{2}(\mu - 1)(\nu - 1) + \frac{1}{2}(\hat{\Lambda}^\mu_\nu + \hat{\Lambda}^\nu_\mu)(\hat{\Lambda}^\nu_\nu - \hat{\Lambda}^\mu_\mu). \quad (4.10)$$

with the additional relations:

$$\hat{\Lambda}^\mu_\nu \hat{\Lambda}^\nu_\rho \eta^{\rho\sigma} = \eta^{\mu\sigma}, \quad (\eta^{\mu\sigma}) = \text{diag} (1, -1, -1). \quad (4.11)$$

As it happened with the $\kappa$-Poincaré group \([21, 22]\) these commutation relations are also a Weyl quantization $[,] \rightarrow w^{-1}[,]$. of the Poisson brackets of the coordinate functions on the Poincaré group \((4.7)\), and moreover, all the Lorentz coordinates $\hat{\Lambda}^\mu_\nu$
commute among themselves so that there is no ordering ambiguity. The associated coproduct, counit and antipode can be deduced from relations \( \Delta(T) = T \otimes T \), \( \epsilon(T) = I \) and \( \gamma(T) = T^{-1} \), respectively. In particular, the coproduct is:

\[
\begin{align*}
\Delta(\hat{a}^+) &= \hat{a}^+ \otimes 1 + \frac{1}{2}(\hat{\Lambda}_0^0 + \hat{\Lambda}_2^2 + \hat{\Lambda}_2^0 + \hat{\Lambda}_0^2) \otimes \hat{a}^+ \\
&\quad + (\hat{\Lambda}_1^0 + \hat{\Lambda}_2^1) \otimes \hat{a}^1 + (\hat{\Lambda}_0^0 + \hat{\Lambda}_2^2 - \hat{\Lambda}_0^2 - \hat{\Lambda}_2^2) \otimes \hat{a}^- \\
\Delta(\hat{a}^1) &= \hat{a}^1 \otimes 1 + \frac{1}{2}(\hat{\Lambda}_1^1 + \hat{\Lambda}_2^1) \otimes \hat{a}^+ + \hat{\Lambda}_1^1 \otimes \hat{a}^1 + (\hat{\Lambda}_1^0 - \hat{\Lambda}_1^2) \otimes \hat{a}^- , \\
\Delta(\hat{a}^-) &= \hat{a}^- \otimes 1 + \frac{1}{4}(\hat{\Lambda}_0^0 - \hat{\Lambda}_2^0 + \hat{\Lambda}_2^2 - \hat{\Lambda}_0^2) \otimes \hat{a}^+ \\
&\quad + \frac{1}{2}(\hat{\Lambda}_1^0 - \hat{\Lambda}_1^2) \otimes \hat{a}^1 + \frac{1}{2}(\hat{\Lambda}_0^0 - \hat{\Lambda}_0^2 - \hat{\Lambda}_2^0 + \hat{\Lambda}_2^2) \otimes \hat{a}^- , \\
\Delta(\hat{\Lambda}_\mu^\sigma) &= \hat{\Lambda}_\mu^\sigma \otimes \hat{\Lambda}_\sigma^\nu .
\end{align*}
\]

The quantum (2+1) Poincaré plane of coordinates \((\hat{x}^+, \hat{x}^1, \hat{x}^-)\) characterized by

\[
[\hat{x}^+, \hat{x}^1] = -2w \hat{x}^1, \quad [\hat{x}^+, \hat{x}^-] = -2w \hat{x}^-, \quad [\hat{x}^1, \hat{x}^-] = 0,
\]

is easily derived from the first three commutators of (4.10). Note that it includes, as a particular case, the quantum (1+1) Poincaré plane \([\hat{x}^+, \hat{x}^-] = -2w \hat{x}^-\) \([11]\). The coordinates \((\hat{x}^+, \hat{x}^1)\) could be interpreted as the parameters of a quantum light-like plane while the remaining one \(\hat{x}^-\) would be a quantum time.

**Acknowledgements**

This work has been partially supported by DGICYT (Project PB94–1115) from the Ministerio de Educación y Ciencia de España.

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