Research Article

A Legendre tau-Spectral Method for Solving Time-Fractional Heat Equation with Nonlocal Conditions

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We develop the tau-spectral method to solve the time-fractional heat equation (T-FHE) with nonlocal condition. In order to achieve highly accurate solution of this problem, the operational matrix of fractional integration (described in the Riemann-Liouville sense) for shifted Legendre polynomials is investigated in conjunction with tau-spectral scheme and the Legendre operational polynomials are used as the base function. The main advantage in using the presented scheme is that it converts the T-FHE with nonlocal condition to a system of algebraic equations that simplifies the problem. For demonstrating the validity and applicability of the developed spectral scheme, two numerical examples are presented. The logarithmic graphs of the maximum absolute errors is presented to achieve the exponential convergence of the proposed method. Comparing between our spectral method and other methods ensures that our method is more accurate than those solved similar problem.

1. Introduction

In recent years, many engineering and physical phenomena can be successfully described by models of fractional differential equations (FDEs); see, for instance, [1–7]. Thus many researchers have been interested in studying the properties of fractional calculus and finding stable and robust numerical and analytical schemes for solving FDEs such as spectral tau method [8–10], Crank-Nicolson method [11], compact finite difference approximation [12], Legendre wavelets method [13], Haar wavelet operational matrix method [14], iterative Laplace transform method [15], Lie symmetry analysis method [16], and other methods [17–20].

Recently, spectral methods [21–23] have been applied to solve ordinary FDEs (see [24, 25]) while in [26, 27] the authors introduced the operational matrices of fractional derivatives with the help of the spectral methods to solve FDEs. This is not all; the partial FDEs are also investigated by using the spectral methods. In [28–31], the tau and collocation spectral methods are implemented in combination with the operational matrices of fractional integration for approximating the solution of some classes of space-fractional differential equations.

The T-FHE is a generalization of the classical heat equation obtained by replacing the first order time derivative by a fractional derivative of order \( \nu \), \( 0 < \nu \leq 1 \). Ali and Jassim [32] used the homotopy perturbation method to solve the T-FHE, while in [33] the authors introduced a general iteration formula of variational iteration method for a solution of the T-FHE. Moreover, in [34] the differential transform method is applied to solve the T-FHE. In addition, Rostamy and Karimi [35] constructed the Bernstein operational matrix for the fractional derivatives and used it together with spectral method to solve the T-FHE.

In this paper, we consider the T-FHE with the nonlocal condition [36]:

\[
\frac{\partial^\nu u(x,t)}{\partial t^\nu} - \frac{\partial^2 u(x,t)}{\partial x^2} = q(x,t), \quad 0 < x \leq 1, \quad 0 < t \leq 1,
\]

(1)
subject to
\[ u(x,0) = u(x,1) + f(x), \quad 0 < x \leq 1, \]
\[ u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad 0 < t \leq 1, \tag{2} \]
where \( 0 < \nu \leq 1, u(x,t) \) is the temperature as a function of space \( x \) and time \( t \), and \( g(x,t) \) is known source term. Our main aim is to achieve highly accurate solution of the T-FHE with nonlocal conditions (1) and (2). The tau-spectral method is applied based on the shifted Legendre polynomials as a basis function with the help of the operational matrix of fractional integration of such polynomials. Two numerical examples are introduced and solved using the presented technique to show its accuracy and validity. Also, we introduce comparisons between our numerical results and those obtained using the implicit difference approximation (IDA).

This paper is arranged in the following way: in Section 2 we introduce some definitions and notations of fractional calculus with some properties of Legendre polynomials. In Section 3 we apply our algorithm for the solution of the T-FHE with nonlocal condition. In Section 4 two numerical examples and comparisons between our results and those obtained by the IDA are introduced. Also in Section 5, a conclusion is presented.

2. Preliminaries and Notations

2.1. Fractional Calculus Definitions. Riemann-Liouville and Caputo fractional definitions are the two most used from other definitions of fractional derivatives which have been introduced recently.

Definition 1. The integral of order \( \gamma \geq 0 \) (fractional) according to Riemann-Liouville is given by
\[ I^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) \, dt, \quad \nu > 0, \quad x > 0, \tag{3} \]
where
\[ \Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x} \, dx \tag{4} \]
is gamma function.

The operator \( I^\nu \) satisfies the following properties:
\[ I^\nu I^\mu f(x) = I^{\nu+\mu} f(x), \]
\[ I^\nu I^\mu f(x) = I^{\mu} I^\nu f(x), \tag{5} \]
\[ I^\nu x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)} x^{\beta+\nu}. \]

Definition 2. The Caputo fractional derivative of order \( \nu \) is defined by
\[ D^\nu f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} \frac{d^m f(t)}{dt^m} \, dt, \quad m-1 < \nu \leq m, \quad x > 0, \tag{6} \]
where \( m \) is the ceiling function of \( \nu \).

The operator \( D^\nu \) satisfies the following properties:
\[ D^\nu C = 0, \quad (C \text{ is constant}) \]
\[ I^\nu D^\mu f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)} (0^+) \frac{x^i}{i!}, \tag{7} \]
\[ D^\nu x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} x^{\beta-\nu}, \]
\[ D^\nu (\lambda f(x) + \mu g(x)) = \lambda D^\nu f(x) + \mu D^\nu g(x). \]

2.2. Shifted Legendre Polynomials. Assuming that the Legendre polynomial of degree \( j \) is denoted by \( P_j(z) \) (defined on the interval \((-1,1)\)), then \( P_j(z) \) may be generated using the following recurrence formulae:
\[ P_{j+1}(z) = 2j+1 \frac{j+1}{j+1} (2z-1) P_j(z) - \frac{j}{j+1} P_{j-1}(z), \quad j=1,2,\ldots, \]
\[ P_0(z) = 1, \quad P_1(z) = 2z-1. \tag{8} \]

Considering \( z = 2x-1 \), Legendre polynomials are defined on the interval \((0,1)\) that may be called shifted Legendre polynomials \( P_j^*(x) \) that were generated using the following recurrence formulae:
\[ P_{j+1}^*(x) = 2j+1 \frac{j+1}{j+1} (2x-1) P_j^*(x) - \frac{j}{j+1} P_{j-1}^*(x), \quad j=1,2,\ldots, \]
\[ P_0^*(x) = 1, \quad P_1^*(x) = 2x-1. \tag{9} \]

The orthogonality relation is
\[ \int_0^1 P_i^*(x) P_j^*(x) \, dx = \begin{cases} \frac{1}{2j+1}, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \tag{10} \]

The explicit analytical form of shifted Legendre polynomial \( P_j^*(x) \) of degree \( j \) may be written as
\[ P_j^*(x) = \sum_{k=0}^j (-1)^{j+k} (j+k)!x^k \frac{x^j}{(j-k)!((k+1)^2)}, \tag{11} \]
and this in turn enables one to get
\[ P_j^*(0) = (-1)^j, \quad P_j^*(1) = 1. \tag{12} \]
Any square integrable function \( u(x) \) defined on the interval \((0, 1)\) may be expressed in terms of shifted Legendre polynomials as

\[
 u(x) = \sum_{j=0}^{\infty} a_j P_j^* (x), \tag{13}
\]

from which the coefficients \( a_j \) are given by

\[
 a_j = (2j + 1) \int_0^1 u(x) P_j^* (x) \, dx, \quad j = 0, 1, \ldots.
\tag{14}
\]

If we approximate \( u(x) \) by the first \((N+1)\)-terms, then we can write

\[
 u_N(x) = \sum_{j=0}^{N} a_j P_j^* (x), \tag{15}
\]

which alternatively may be written in the matrix form

\[
 u_N(x) = A^T \Psi_N(x), \tag{16}
\]

with

\[
 A^T = [a_0, a_1, \ldots, a_N], \tag{17}
\]

\[
 \Psi_N(x) = [P_0^* (x), P_1^* (x), \ldots, P_N^* (x)]^T.
\]

Similarly, let \( u(x, t) \) be an infinitely differentiable function defined on \(0 < x \leq 1\) and \(0 < t \leq 1\). Then it may be expressed as

\[
 u_{M,N}(x, t) = \sum_{i=0}^{M} \sum_{j=0}^{N} u_{ij} P_i^* (t) P_j^* (x) = \Psi_M^T(t) \Psi_N(x), \tag{18}
\]

with

\[
 U = \begin{pmatrix}
  u_{00} & u_{01} & \cdots & u_{0N} \\
  u_{10} & u_{11} & \cdots & u_{1N} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{M0} & u_{M1} & \cdots & u_{MN}
 \end{pmatrix}, \tag{19}
\]

\[
 u_{ij} = (2i + 1)(2j + 1) \int_0^1 u(x, t) P_i^* (t) P_j^* (x) \, dx \, dt, \tag{20}
\]

when

\[
 \xi(i, j, k) = \sum_{l=0}^{k} (-1)^{i+j+k+l} (i+k)! (l+j)! \\
 \times (i-k)! k! \Gamma(k+n+1) (j-l)! \\
 \times ((l)!^2 (k+l+n+1))^{-1}. \tag{26}
\]

(Theorem 3. The first derivative of the shifted Legendre vector \( \Psi_N(x) \) may be expressed as

\[
 \frac{d \Psi_N(x)}{dx} = D \Psi_N(x), \tag{21}
\]

where \( D \) is the \((N+1) \times (N+1)\) operational matrix of derivative given by

\[
 D = \begin{pmatrix}
  2(2j+1), & \text{for } j = i-k, \\
  0, & \text{otherwise}.
 \end{pmatrix}
\]

Repeated use of (21) enables one to write

\[
 \frac{d^n \Psi_N(x)}{dx^n} = D^n \Psi_N(x), \tag{23}
\]

where \( q \) is a natural number and \( D^n \) means matrix power.

Theorem 4. The Riemann-Liouville fractional integral of order \( \nu \) of the shifted Legendre polynomial vector \( \Psi_M(t) \) is given by

\[
 \int_0^t \Psi_M(t) = P_v \Psi_M(t), \tag{24}
\]

where \( P_v \) is the \((M+1) \times (M+1)\) operational matrix of fractional integral of order \( \nu \) and is defined by

\[
 P_v = \begin{pmatrix}
  \sum_{k=0}^{0} \xi(0,0,k) & \sum_{k=0}^{0} \xi(0,1,k) & \cdots & \sum_{k=0}^{0} \xi(0,M,k) \\
  \sum_{k=0}^{1} \xi(1,0,k) & \sum_{k=0}^{1} \xi(1,1,k) & \cdots & \sum_{k=0}^{1} \xi(1,M,k) \\
  \vdots & \vdots & \ddots & \vdots \\
  \sum_{k=0}^{M} \xi(M,0,k) & \sum_{k=0}^{M} \xi(M,1,k) & \cdots & \sum_{k=0}^{M} \xi(M,M,k)
 \end{pmatrix}, \tag{25}
\]

where

\[
 \xi(i, j, k) = \sum_{l=0}^{k} (-1)^{i+j+k+l} (i+k)! (l+j)! \\
 \times (i-k)! k! \Gamma(k+n+1) (j-l)! \\
 \times ((l)!^2 (k+l+n+1))^{-1}. \tag{26}
\]

(See [37] for proof.)

3. Legendre tau-Spectral Method

In this section, the Legendre operational matrix of fractional integrals is applied with the help of Legendre tau-spectral method to solve the T-FHE with the nonlocal condition.

Consider the T-FHE with the nonlocal condition

\[
 \frac{\partial^\nu u(x,t)}{\partial t^\nu} - \frac{\partial^2 u(x,t)}{\partial x^2} = q(x,t),
\]

\[
 0 < x \leq 1, \quad 0 < t \leq 1, \quad 0 < \nu \leq 1,
\]

\[
 u(x,0) = u(x,1) + f(x), \quad 0 < x \leq 1,
\]

\[
 u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad 0 < t \leq 1.
\]

\[
 u(x,0) = u(x,1) + f(x), \quad 0 < x \leq 1,
\]

\[
 u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad 0 < t \leq 1.
\]
We integrate (27) of order $\nu$ and making use of (7), we have
\[ u(x,t) - (u(x,1) + f(x)) - I_t^\nu \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) = I_t^\nu q(x,t), \quad 0 < \nu \leq 1, \]
\[ u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad 0 < t \leq 1. \]

In order to use tau-spectral method based on the shifted Legendre operational matrix for fractional integrals to solve the fully integrated problem (28), we approximate $(x,t)$, $f(x)$, and $q(x,t)$ by the shifted Legendre polynomials as
\[ u_{M,N}(x,t) = \Psi_M^T(t) U \psi_N(x), \]
\[ f_M(x) = \Psi_M^T(t) F \psi_N(x), \]
\[ q_{M,N}(x,t) = \Psi_M^T(t) Q \psi_N(x), \] (29)
where $U$ is the unknown coefficients $(M+1) \times (N+1)$ matrix and $F$ and $Q$ are known matrices that can be written as
\[ F = \begin{pmatrix} f_0 & f_1 & \cdots & f_N \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \]
\[ Q = \begin{pmatrix} q_{00} & q_{01} & \cdots & q_{0N} \\ q_{10} & q_{11} & \cdots & q_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ q_{M0} & q_{M1} & \cdots & q_{MN} \end{pmatrix}, \] (30)
where $f_j$ and $q_{ij}$ are given as in (14) and (20), respectively. Using (29), it is easy to write
\[ u_{M,N}(x,1) = \Psi_M^T(1) U \psi_N(x), \]
\[ = \sum_{k=0}^{M} u_{k0} \sum_{k=0}^{M} u_{k1} \cdots \sum_{k=0}^{M} u_{kN} \psi_N(x), \]
\[ = \Psi_M^T(t) V \psi_N(x), \] (31)
where $V$ is a $(M+1) \times (N+1)$ matrix that can be written as
\[ V = \begin{pmatrix} \sum_{k=0}^{M} u_{k0} & \sum_{k=0}^{M} u_{k1} & \cdots & \sum_{k=0}^{M} u_{kN} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \] (32)
Making use of (23), (24), and (29) enables one to write
\[ I_t^\nu \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) = I_t^\nu \Psi_M^T(t) \left( D^2 \psi_N(x) \right), \]
\[ = \Psi_M^T(t) P^x \Psi_N^T(x). \] (33)

In addition, if we use (24) and (29), we obtain
\[ I_t^\nu q_{M,N}(x,t) = \Psi_M^T(t) P^x \Psi_N^T(x). \] (34)

Equations (31) and (34) enable one to write the residual $R_{M,N}(x,t)$ for (28) in the form
\[ R_{M,N}(x,t) = \Psi_M^T(t) \left[ U - V - F \cdot D^2 - P^x \cdot Q \right] \psi_N(x). \] (35)

As in a typical tau method (see [22, 38, 39]) we generate $(M + 1)(N + 1)$ linear algebraic equations in the unknown expansion coefficients, $u_{ij}, i = 0,1,\ldots,M; \ j = 0,1,\ldots,N-2$, namely;
\[ \int_0^1 \int_0^1 R_{M,N}(x,t) P_i^x(t) P_j^y(x) dx \ dt = 0, \]
\[ i = 0,1,\ldots,M, \ j = 0,1,\ldots,N-2, \] (36)
and the rest of linear algebraic equations are obtained from the boundary conditions, as
\[ \Psi_M^T(t) U \psi_N(0) = g_0(t_i), \]
\[ \Psi_M^T(t) U \psi_N(1) = g_1(t_i), \] (37)
where $t_i, i = 0,1,\ldots,M$ are the roots of $P_{M+1}(t)$. The number of the unknown coefficients $u_{ij}$ is equal to $(M+1)(N+1)$ and can be obtained from (36) and (37). Consequently $u_{M,N}(x,t)$ given in (29) can be calculated.

4. Numerical Experiments

In order to highlight the accuracy of the presented scheme, we implement it to solve two numerical examples, and also comparisons between their exact solutions with the approximate solutions achieved using the presented scheme and with those achieved using other methods are made.

**Example 1.** We consider the following problem [36]:
\[ \frac{\partial^5 u(x,t)}{\partial t^{0.5}} - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{2 t^{1.5} \sin(2 \pi x)}{\Gamma(1.5)} + 4 \pi^2 t^2 \sin(2 \pi x), \]
\[ u(x,0) = u(x,1) - \sin(2 \pi x), \quad 0 < x \leq 1, \]
\[ u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \leq 1, \] (38)
with exact solution $u(x,t) = t^2 \sin(2 \pi x)$.

Karataş et al. [36] introduced this problem and applied the IDA method to approximate its solution at various choices of time and space nodes $M$ and $N$.

We apply our numerical scheme for this problem. In order to show that our scheme is more accurate than the IDA method, in Table 1, we compare the maximum absolute errors (MAEs) achieved using our scheme with those obtained
Table 1: Comparison of our scheme with the IDA [36] at various choices of \( N, (N = M) \) for Example 1.

| \( N = M \) | Our scheme MAEs | IDA [36] MAEs |
|-------------|----------------|--------------|
| 6           | 2.541507 \( \times 10^{-2} \) | 2.297695 \( \times 10^{-1} \) |
| 8           | 8.774320 \( \times 10^{-4} \) | 5.383793 \( \times 10^{-2} \) |
| 10          | 2.118341 \( \times 10^{-5} \) | 1.391800 \( \times 10^{-2} \) |
| 12          | 1.832678 \( \times 10^{-6} \) | 3.843610 \( \times 10^{-3} \) |
| 14          | 1.037991 \( \times 10^{-6} \) | 1.152111 \( \times 10^{-3} \) |
| 16          | 6.590323 \( \times 10^{-7} \) | 3.844224 \( \times 10^{-4} \) |
| 18          | 4.383403 \( \times 10^{-7} \) | 1.447756 \( \times 10^{-4} \) |

Using the IDA [36] method at different values of \( N, (N = M) \). Moreover, Figure 1 plots the absolute error function at \( M = N = 18 \), while Figure 2 plots the absolute error function for \( t = 0.5 \) at \( M = N = 18 \).

Example 2. Consider the following problem:

\[
\frac{\partial^\nu u(x,t)}{\partial t^\nu} - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{2^\nu \ln (1 + x (1 - x))}{\Gamma (3 - \nu)} + \frac{t^\nu (2x^2 - 2x - 1)}{(x^2 - x + 1)^2}, \quad 0 < \nu \leq 1,
\]

\[
u (x,0) = u(x,1) - \ln (1 + x (1 - x)), \quad 0 < x \leq 1,
\]

\[
u (0,t) = 0, \quad u (1,t) = 0, \quad 0 < t \leq 1,
\]

with exact solution \( u(x,t) = t^\nu \ln (1 + x (1 - x)) \).

Karatay et al. [36] introduced this problem and solved it for two choices of \( \nu, \nu = 0.45, 0.95 \) at different values of \( N \) and \( M \). Table 2 lists the MAEs for \( \nu = 0.45, 0.95 \) using our scheme at \( N = M = 4, 6, 8, 10, 12, 14, 16, 18 \) and a comparison with those obtained in [36] at \( M = 16, N = 2, 4, 8, 16, 32, 64, 128, 256 \) and \( N = 16, M = 2, 4, 8, 16, 32, 64, 128, 256 \). Figures 3 and 4 plot the absolute error functions at \( M = N = 20 \) with \( \nu_1 = 0.45 \) and \( \nu = 0.95 \), respectively.
Table 2: Comparison of our scheme with the IDA [36] at various choices of \( N \) and \( M \) for Example 2.

| \( N \) | \( \gamma = 0.45 \) | \( \gamma = 0.95 \) | \( M = 16 \) | \( \gamma = 0.45 \) | \( \gamma = 0.95 \) | \( M = 16 \) | \( \gamma = 0.45 \) | \( \gamma = 0.95 \) |
|---|---|---|---|---|---|---|---|---|
| 4 | \(1.08 \cdot 10^{-3}\) | \(1.10 \cdot 10^{-3}\) | 2 | \(1.46 \cdot 10^{-2}\) | \(7.45 \cdot 10^{-3}\) | 2 | \(3.88 \cdot 10^{-2}\) | \(4.30 \cdot 10^{-2}\) |
| 6 | \(5.84 \cdot 10^{-5}\) | \(5.86 \cdot 10^{-5}\) | 4 | \(7.59 \cdot 10^{-3}\) | \(4.06 \cdot 10^{-3}\) | 4 | \(9.51 \cdot 10^{-3}\) | \(9.47 \cdot 10^{-3}\) |
| 8 | \(6.33 \cdot 10^{-6}\) | \(5.01 \cdot 10^{-6}\) | 8 | \(4.03 \cdot 10^{-3}\) | \(2.30 \cdot 10^{-3}\) | 8 | \(3.65 \cdot 10^{-3}\) | \(2.96 \cdot 10^{-3}\) |
| 10 | \(1.86 \cdot 10^{-6}\) | \(1.15 \cdot 10^{-6}\) | 16 | \(2.24 \cdot 10^{-3}\) | \(1.41 \cdot 10^{-3}\) | 16 | \(2.24 \cdot 10^{-3}\) | \(1.41 \cdot 10^{-3}\) |
| 12 | \(9.03 \cdot 10^{-7}\) | \(5.12 \cdot 10^{-7}\) | 32 | \(1.35 \cdot 10^{-3}\) | \(9.59 \cdot 10^{-4}\) | 32 | \(1.89 \cdot 10^{-3}\) | \(1.02 \cdot 10^{-3}\) |
| 14 | \(5.07 \cdot 10^{-7}\) | \(2.81 \cdot 10^{-7}\) | 64 | \(9.02 \cdot 10^{-4}\) | \(7.33 \cdot 10^{-4}\) | 64 | \(1.81 \cdot 10^{-3}\) | \(9.31 \cdot 10^{-4}\) |
| 16 | \(3.06 \cdot 10^{-7}\) | \(1.69 \cdot 10^{-7}\) | 128 | \(6.78 \cdot 10^{-4}\) | \(6.19 \cdot 10^{-4}\) | 128 | \(1.78 \cdot 10^{-3}\) | \(9.07 \cdot 10^{-4}\) |
| 18 | \(1.95 \cdot 10^{-7}\) | \(1.69 \cdot 10^{-7}\) | 256 | \(5.66 \cdot 10^{-4}\) | \(5.62 \cdot 10^{-4}\) | 256 | \(1.78 \cdot 10^{-3}\) | \(9.01 \cdot 10^{-4}\) |

From Tables 1 and 2 and Figures 1 and 2 introduced above, it is shown that the proposed scheme is more accurate than the IDA method introduced by Karatay et al. [36].

5. Conclusion

An effective and accurate numerical scheme was developed to approximate the solution of the T-FHE with the nonlocal condition. The developed approach is based on the Legendre tau-spectral method combined with the operational matrix of fractional integration (described in the Riemann-Liouville sense) for orthogonal polynomials. A good approximation of the exact solution was achieved by using a limited number of the basis function.

The logarithmic graphs of the maximum absolute errors were presented to achieve the exponential convergence of the proposed method. Comparisons between our approximate solutions of test problems with their exact solutions and the approximate solutions achieved by the IDA method were introduced to confirm the validity and accuracy of our scheme.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] H. G. Sun, W. Chen, H. Wei, and Y. Q. Chen, "A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems," European Physical Journal: Special Topics, vol. 193, no. 1, pp. 185–192, 2011.
[2] H. G. Sun, W. Chen, C. Li, and Y. Q. Chen, "Fractional differential models for anomalous diffusion," Physica A: Statistical Mechanics and Its Applications, vol. 389, no. 14, pp. 2719–2724, 2010.
[3] S. Chen, F. Liu, P. Zhuang, and V. Anh, "Finite difference approximations for the fractional Fokker-Planck equation," Applied Mathematical Modelling. Simulation and Computation for Engineering and Environmental Systems, vol. 33, no. 1, pp. 256–273, 2009.
[4] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, Fractional Calculus Models and Numerical Methods, vol. 3 of Series on Complexity, Nonlinearity and Chaos, World Scientific, 2012.
[5] X.-J. Yang, Local Fractional Functional Analysis & Its Applications, Asian Academic, 2011.
[6] S. H. Yan, X. H. Chen, G. N. Xie, C. Cattani, and X. J. Yang, "Solving fokker-planck equations on cantor sets using local FRActional decomposition method," Abstract and Applied Analysis, vol. 2014, Article ID 396469, 6 pages, 2014.
[7] S. Das, Functional Fractional Calculus for System Identification and Controls, Springer, New York, NY, USA, 2008.
A.-M. Yang, A. S. Alofi, “The operational matrix of fractional integration for shifted Chebyshev polynomials,” Applied Mathematics Letters, vol. 26, no. 1, pp. 25–31, 2013.

W. M. Abd-Elhameed and Y. H. Youssri, “New ultraspherical wavelets spectral solutions for fractional Riccati differential equations,” Abstract and Applied Analysis, vol. 2014, Article ID 626275, 8 pages, 2014.

E. H. Doha, A. H. Bhrawy, D. Baleanu, and S. S. Ezz-Eldien, “On Jacobi approximations for solving fractional differential equations,” Applied Mathematics and Computation, vol. 219, no. 15, pp. 8042–8056, 2013.

C. C. Celik and M. Duman, “Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative,” Journal of Computational Physics, vol. 231, no. 4, pp. 1743–1750, 2012.

H. Zhou, W. Y. Tian, and W. H. Deng, “Compact finite difference approximations for space fractional diffusion equations,” http://arxiv.org/abs/1204.4870.

M. H. Heydari, M. R. Hooshmandasl, and F. Mohammadi, “Legendre wavelets method for solving fractional partial differential equations with Dirichlet boundary conditions,” Applied Mathematics and Computation, vol. 234, pp. 267–276, 2014.

S. S. Ray, “On Haar wavelet operational matrix of general order and its application for the numerical solution of fractional Bagley Torvik equation,” Applied Mathematics and Computation, vol. 218, no. 9, pp. 5239–5248, 2012.

H. Jafari, M. Nazari, D. Baleanu, and C. M. Khalique, “A new approach for solving a system of fractional partial differential equations,” Computers & Mathematics with Applications, vol. 66, no. 5, pp. 838–843, 2013.

G. W. Wang, T. Z. Xu, and T. Feng, “Lie symmetry analysis and explicit solutions of the time fractional fifth-order KdV equation,” PLoS ONE, vol. 9, no. 2, Article ID e88336, 2014.

A. Atangana and S. B. Belhaouari, “Solving partial differential equation with space- and time-fractional derivatives via homotopy decomposition method,” Mathematical Problems in Engineering, vol. 2013, Article ID 318590, 9 pages, 2013.

R. Darzi, B. Mohammadzade, S. Mousavi, and R. Beheshhti, “Sumudu transform method for solving fractional differential equations and fractional diffusion-wave equation,” The Journal of Mathematics and Computer Science, vol. 6, pp. 79–84, 2013.

A. Ansari, A. R. Sheikhani, and H. S. Najafi, “Solution to system of partial fractional differential equations using the fractional exponential operators,” Mathematical Methods in the Applied Sciences, vol. 35, no. 1, pp. 119–123, 2012.

A.-M. Yang, Y.-Z. Zhang, C. Cattani et al., “Application of local fractional series expansion method to solve Klein-Gordon equations on Cantor sets,” Abstract and Applied Analysis, vol. 2014, Article ID 372741, 6 pages, 2014.

E. H. Doha, A. H. Bhrawy, “A Jacobi-Gauss-Lobatto collocation method for solving generalized Fitzhugh-Nagumo equation with time-dependent coefficients,” Applied Mathematics and Computation, vol. 222, pp. 255–264, 2013.

C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, Spectral Methods in Fluid Dynamics, Springer, New York, NY, USA, 1988.

A. Saadatmandi and M. Dehghan, “A new operational matrix for solving fractional-order differential equations,” Computers & Mathematics with Applications, vol. 59, no. 3, pp. 1326–1336, 2010.

E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, “Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations,” Applied Mathematical Modelling, Simulation and Computation for Engineering and Environmental Systems, vol. 35, no. 12, pp. 5662–5672, 2011.

A. H. Bhrawy and M. M. Al-Shomrani, “A shifted Legendre spectral method for fractional-order multi-point boundary value problems,” Advances in Difference Equations, vol. 2012, article 8, 2012.

E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, “Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order,” Computers & Mathematics with Applications, vol. 62, no. 5, pp. 2364–2373, 2011.

E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, “A new Jacobi operational matrix: an application for solving fractional differential equations,” Applied Mathematical Modelling, vol. 36, no. 10, pp. 4931–4943, 2012.

A. Saadatmandi and M. Dehghan, “A tau approach for solution of the space fractional diffusion equation,” Computers & Mathematics with Applications, vol. 63, no. 3, pp. 1135–1142, 2011.

E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, “Numerical approximations for fractional diffusion equations via a Chebyshev spectral-tau method,” Central European Journal of Physics, vol. 11, pp. 1494–1503, 2013.

A. H. Bhrawy, “A new numerical algorithm for solving a class of fractional advection-dispersion equation with variable coefficients using Jacobi polynomials,” Abstract and Applied Analysis, vol. 2013, Article ID 954983, 9 pages, 2013.

R. Ren, H. Li, W. Jiang, and M. Song, “An efficient Chebyshev tau method for solving the space fractional diffusion equations,” Applied Mathematics and Computation, vol. 224, pp. 259–267, 2013.

E. J. Ali and A. M. Jassim, “Development treatment of initial boundary value problems for one dimensional heat-like and wave-like equations using homotopy perturbation method,” Journal of Basrah Researches, vol. 39, no. 1, 2013.

F. Yin, J. Song, and X. Cao, “A general iteration formula of VIM for fractional heat- and wave-like equations,” Journal of Applied Mathematics, vol. 2013, 9 pages, 2013.

A. Secer, “Approximate analytic solution of fractional heat-like and wave-like equations with variable coefficients using the differential transforms method,” Advances in Difference Equations, vol. 2012, 10 pages, 2012.

D. Rostamy and K. Karimi, “Bernstein polynomials for solving fractional heat- and wave-like equations,” Fractional Calculus and Applied Analysis, vol. 15, no. 4, pp. 556–571, 2012.

I. Karatay, S. R. Bayramoglu, and A. Sahin, “Implicit difference approximation for the time fractional heat equation with the nonlocal condition,” Applied Numerical Mathematics, vol. 61, no. 12, pp. 1281–1288, 2011.

M. H. Akram, M. H. Atabakzadeh, and G. H. Erjaei, “The operational matrix of fractional integration for shifted Legendre polynomials,” Iranian Journal of Science and Technology, vol. 37, pp. 439–444, 2013.

A. H. Bhrawy, M. M. Alghamdi, and T. M. Taha, “A new modified generalized Laguerre operational matrix of fractional integration for solving fractional differential equations on the half line,” Advances in Difference Equations, vol. 2012, article 179, 2012.

D. Baleanu, A. H. Bhrawy, and T. M. Taha, “Two efficient generalized Laguerre spectral algorithms for fractional initial value problems,” Abstract and Applied Analysis, vol. 2013, Article ID 546502, 10 pages, 2013.