On the structure of the ergosurface of Pomeransky-Senkov black rings

Julien Cortier
Institut de Mathématiques et de Modélisation de Montpellier
Université Montpellier 2
December 17, 2010

Abstract

We study the properties of the ergosurface of the Pomeransky-Senkov black rings, and show that it splits into an “inner” and an “outer” region. As for the singular set, the topology of the “outer ergosurface” depends upon the value of parameters.

Contents

1 Introduction 1
2 Pomeransky-Senkov black rings 2
3 The ergosurface 5
4 Acknowledgements 14

1 Introduction

There has been recently a lot of interest in studying higher-dimensional solutions of the vacuum Einstein equations, especially since the discovery, in 2001 by Emparan and Reall in [4], of five-dimensional black hole space-times in vacuum, whose sections of the event horizon are not homeomorphic to a
sphere. Those space-times are usually referred to as “black rings”, since the topology of these sections is $S^1 \times S^2$ instead.

In this work, we focus on a particular feature, the ergosurface, of a family of solutions discovered in 2006 by Pomeransky and Senkov in [5], also known as “doubly spinning black rings”. The Emparan-Reall space-times, also called “singly-spinning black rings”, can be seen as a limiting case of this family (see, e.g., the appendix of [2]). It has been shown in [4] that they possess an ergosurface homeomorphic to $\mathbb{R} \times S^1 \times S^2$. In the remainder of this paper, we first recall some definitions and facts about the Pomeransky-Senkov family of space-times (Section 2) and establish the notations. These follow the material introduced in [2]. Then we study in details the ergosurface (Section 3), which turns out to split into two subsets, the first one being the “usual” ergosurface lying outside the black hole, the second lying “under” all the Killing horizons. We finish by proving the results pointed out in [3] about the existence of two distinct regimes for the topological nature of the “upper” ergosurface.

## 2 Pomeransky-Senkov black rings

Here we follow the notations and conventions introduced in [2]. The Pomeransky-Senkov family of metrics is:

$$ds^2 = \frac{2H(x,y)k^2}{(1-\nu)^2(x-y)^2} \left( \frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} \right) - 2\frac{J(x,y)}{H(y,x)}d\varphi d\psi - \frac{H(y,x)}{H(x,y)}(dt + \Omega)^2 - \frac{F(x,y)}{H(y,x)}d\psi^2 + \frac{F(y,x)}{H(y,x)}d\varphi^2 , \quad (2.1)$$

where

$$H(x,y) = \lambda^2 + 2\nu (1-x^2) y\lambda + 2x (1-\nu^2 y^2) \lambda - \nu^2 + \nu (-\lambda^2 - \nu^2 + 1) x^2 y^2 + 1,$$

$$F(x,y) = \frac{2k^2}{(x-y)^2(1-\nu)^2} \left( (1-y^2) \left( (1-\nu)^2 - \lambda^2 \right) (\nu + 1) + y\lambda (-\lambda^2 - 3\nu^2 + 2\nu + 1) G(x) + \left( -(1-\nu)\nu (\lambda^2 + \nu^2 - 1) x^4 + \lambda (2\nu^3 - 3\nu^2 - \lambda^2 + 1) x^3 + ((1-\nu)^2 - \lambda^2) (\nu + 1)x^2 + \lambda (\lambda^2 + (1-\nu)^2) x + 2\lambda^2 \right) G(y) \right),$$

$$J(x,y) = \frac{2k^2 (1-x^2) (1-y^2) \lambda \sqrt{\nu} (\lambda^2 + 2(x+y)\nu \lambda - \nu^2 - xy\nu (-\lambda^2 - \nu^2 + 1) + 1)}{(x-y)(1-\nu)^2},$$

$$G(x) = (1-x^2) (\nu x^2 + \lambda x + 1) ,$$

$$G(y) = (1-y^2) (\nu y^2 + \lambda y + 1) .$$
and where $\Omega$ is a 1-form given by
\[ \Omega = M(x, y) d\psi + P(x, y) d\varphi , \]
with
\[
M(x, y) = \frac{2k\lambda \sqrt{(\nu + 1)^2 - \lambda^2 (y + 1)(-\lambda + \nu - 2\nu x + \nu x((\lambda + \nu - 1)x + 2y + 1))}}{(1 - \lambda + \nu)H(y, x)}
\]
\[ =: \frac{\sqrt{(\nu + 1)^2 - \lambda^2 \hat{M}(x, y)}}{1 - \lambda + \nu)H(y, x)} \]
\[ P(x, y) = \frac{2k\lambda \sqrt{\nu \sqrt{(\nu + 1)^2 - \lambda^2 (x^2 - 1)y}}}{H(y, x)} \]
\[ =: \frac{2\sqrt{\nu \hat{P}(x, y)}}{H(y, x)} , \]
where $\hat{P}$ and $\hat{M}$ are polynomials in all variables.

The parameter $k$ is assumed to be in $\mathbb{R}^*$, while the parameters $\lambda$ and $\nu$ have been restricted in \[5\] to belong to the set \[ \mathcal{U} := \{ (\nu, \lambda) : \nu \in (0, 1), 2\sqrt{\nu} \leq \lambda < 1 + \nu \} . \]

The coordinates $x, y, \phi, \psi, t$ vary within the ranges $-1 \leq x \leq 1, -\infty < y < -1, 0 \leq \phi \leq 2\pi, 0 \leq \psi \leq 2\pi$ and $-\infty < t < \infty$.

We also introduce the quantities
\[ y_h := -\lambda - \frac{\sqrt{\lambda^2 - 4\nu}{2\nu}}, \quad y_c := -\lambda + \frac{\sqrt{\lambda^2 - 4\nu}}{2\nu} , \]

In \[2\], for every $(\nu, \lambda) \in \mathcal{U}$, the submanifold delimited by the coordinates above, with the additional condition $y_h < y < -1$, and equipped with the metric \[2.1\], has been shown to be a regular asymptotically flat space-time, with the asymptotically flat region located at $(x, y) = (-1, -1)$. In fact, the set $\{y = y_h\}$ corresponds to an event horizon, $\{y = y_c\}$ to another Killing horizon, and the space-time has been shown to extend smoothly through these horizons up to the singular set \[ \text{Sing} := \{ H(x, y) = 0 \} , \]
where the

\[ ^1 \text{Strictly speaking, } \nu = 0 \text{ is allowed in } \mathcal{U}. \text{ It is shown there that this corresponds to Emparan-Reall metrics (compare Appendix in } \mathcal{U}) \text{, which have already been analysed elsewhere } \[1\], \text{ and so we only consider } \nu > 0. \]
curvature tensor blows up at least for some values of the parameters. At this stage, it is convenient to work with a new coordinate $Y$, defined as

$$Y = -\frac{1}{y},$$

because the singular set has components which blow up to infinity in $(x, y)$-coordinates. Since $H(x, y)$ is a second-order polynomial in $y$, we can now write Sing as the union of the graphs of the functions $y_+(x)$ and $y_-(x)$, or, better, we write Sing in $(x, Y)$-coordinates as the union of the graphs of $Y_+(x)$ and $Y_-(x)$, both defined on a subset of $[-1, 1]$ such that the discriminant of the polynomial $H(x, y)$ is non-negative. Equivalently, if we define $\tilde{H}(x, Y) := Y^2 H(x, -1/Y)$, then $Y_\pm(x)$ are the roots of the second order polynomial $\tilde{H}(x, Y)$ in $Y$.

In fact, we can observe in these coordinates that the analytic extension done so far up to Sing can be performed further across the set $\{Y = 0\}$, since the branches $x \mapsto Y_\pm(x)$ are not defined on the whole set $[-1, 1]$, but only on a subinterval of $(-1, 1]$. Following the notations of [2], we have the inequalities (wherever $Y_+$ and $Y_-$ are defined):

$$Y_+(x) \leq Y_-(x) < Y_c < Y_h.$$

These confirm that the singular set Sing lies “under” both Killing horizons. On the other hand, the singular set has the remarkable property to have a varying topology, depending on the values of $(\nu, \lambda) \in \mathcal{U}$. Indeed, the topology is $\mathbb{R} \times T^3$ when $\nu + \lambda < 1$, $\mathbb{R} \times S^1 \times S^2$ when $\nu + \lambda > 1$, and a “pinched” $\mathbb{R} \times S^1 \times S^2$ when $\nu + \lambda = 1$. These results are illustrated on Figure 1.3 of [2]. Some other properties of the singular set have been derived (see Theorem 5.6 in [2]). Most of them come from the study of the polynomial $\tilde{W}$, which is, up to a factor $\frac{1}{4}$, the discriminant of the second-order polynomial in $x$, $H(x, y)$:

$$\tilde{W} := \nu \left( (x^2 - 1)^2 \nu \lambda^2 + x (\lambda^2 + 2x \lambda - \nu^2 + 1) (2\lambda \nu + x (\lambda^2 + \nu^2 - 1)) \right).$$

(2.3)

In particular we recall the following result, which turns out to be useful for the next section:

**Lemma 2.1** In the region $\{-1 \leq x \leq 0, \ y < y_c\}$, the two branches $y_\pm$ which exist for small negative values of $x$ meet smoothly at some $\bar{x} \in (-1, 0)$, where $\bar{x}$ is a simple root of $\tilde{W}$. 

4
We observed in [2], (see Remark 5.3) that the root $\bar{x}$ is simple, and is the largest negative one $\tilde{W}$, and therefore the function $(\nu, \lambda) \mapsto \bar{x}(\nu, \lambda)$ defined in $\mathcal{W}$ is continuous.

3 The ergosurface

The ergosurface of the Pomeransky-Senkov space-time is the set of zeros of

$$g_{tt} = -\frac{H(y, x)}{H(x, y)}.$$ 

Since this function cannot vanish on the singular set $\text{Sing}$ for allowed values of the parameters and variables (see Section 3 of [2]), the ergosurface coincides with the set of zeros of $H(y, x)$. In this section we derive a more precise version of the results of [3] about the topology of the ergosurface, introduced below. Similarly to the singular set $\{H(x, y) = 0\}$, this topology depends upon the values of the parameters $\nu$ and $\lambda$. In what follows, we prove that the ergosurface exists and splits into two parts which extend over all the region $x \in [-1, 1]$ (Proposition 3.1). Next, Proposition 3.5 shows that one of these parts, referred to as the outer ergosurface, lies above the event horizon $\{y = y_h\}$, while Proposition 3.6 shows that the other part, referred to as the inner ergosurface, lies under both Killing horizons. Finally Theorem 3.7 discusses the topology of the ergosurface for the allowed values of $\lambda$ and $\nu$.

We start by showing that the ergosurface is globally defined:

**Proposition 3.1** The equation $H(y, x) = 0$ has solutions $y_{e, \pm}(x)$ in $\mathbb{R} \cup \{\infty\}$ for all $x$ in $[-1, 1]$.

**Proof:** In order to see this, we begin by noting the identity

$$H(y, x) = H\left(\nu x, \frac{y}{\nu}\right),$$

holding for any $\nu \neq 0$, and which enables us to use some of the properties of the singular set $\text{Sing}$ recalled in Section 2. Hence, it is sufficient to show that the polynomial $\tilde{W}(x)$, defined by the formula (2.3) above, is nonnegative for $x \in [-\nu, \nu]$. To do so, we write

$$\tilde{W}(x = \nu) = \nu^2(\lambda^2 + 2\nu \lambda + \nu^3 - \nu)(\lambda^2 \nu + 1 + 2\nu \lambda - \nu^2),$$

$$\tilde{W}(x = -\nu) = \nu^2(\lambda^2 - 2\nu \lambda + \nu^3 - \nu)(\lambda^2 \nu + 1 - 2\nu \lambda - \nu^2).$$

We need two lemmata:
Lemma 3.2 \( \bar{W}(x = \nu) \) and \( \bar{W}(x = -\nu) \) are positive.

Proof of the lemma 3.2. For (3.2), the second factor \((\lambda^2 + 2\nu \lambda + \nu^3 - \nu)\) is positive since we have \(\lambda^2 \geq 4\nu > \nu\) for admissible \(\nu\) and \(\lambda\), and the third factor \((\lambda^2 \nu + 1 + 2\nu \lambda - \nu^2)\) is obviously positive as well there. The proof of the positivity in (3.3) requires some more effort. Let us begin with its second factor,

\[
h_1(\nu, \lambda) = \lambda^2 - 2\nu \lambda + \nu^3 - \nu.
\]

We have \(\partial_\lambda h_1 = 2(\lambda - \nu) > 0\), hence \(\lambda \mapsto h_1(\nu, \lambda)\) is increasing on \([2\sqrt{\nu}, 1 + \nu)\), and since

\[
h_1(\nu, 2\sqrt{\nu}) = \nu(3 - 4\sqrt{\nu} + \nu^2) = \nu(1 - \sqrt{\nu})^2(3 + 2\sqrt{\nu} + \nu),
\]

positive for all \(\nu\) in \((0, 1)\), therefore \(h_1\) is positive on \(\mathcal{U}\). Then, for the third factor of (3.3):

\[
h_2(\nu, \lambda) = \lambda^2 \nu - 2\nu \lambda + 1 - \nu^2,
\]

The first derivative reads \(\partial_\lambda h_2 = 2\nu(\lambda - 1)\), hence the minimum on the interval \([2\sqrt{\nu}, 1 + \nu)\) of \(h_2(\nu, \cdot)\) is reached at \(\lambda = \lambda_m := \max(1, 2\sqrt{\nu})\). From the definition above, \(\lambda_m = 1\) for \(\nu \in (0, 1/4]\) (critical point), and \(\lambda_m = 2\sqrt{\nu}\) for \(\nu \in [1/4, 1)\) (\(h_2(\nu, \cdot)\) is increasing in the whole interval \([2\sqrt{\nu}, 1 + \nu)\) in this case). But on the one hand, we have \(h_2(\nu, 1) = 1 - \nu - \nu^2\) positive for all \(\nu\) in \((0, 1/4]\), and on the other hand,

\[
h_2(\nu, 2\sqrt{\nu}) = 3\nu^2 - 4\nu\sqrt{\nu} + 1 = (1 - \sqrt{\nu})^2(3\nu + 2\sqrt{\nu} + 1),
\]

positive in particular on \([1/4, 1)\). This finishes to show that all the factors of (3.3) are positive, and hence to prove the lemma. \(\square\)

Next, we prove a second lemma:

Lemma 3.3 For the particular values \((\nu, \lambda = 1 - \nu)\) of \((\nu, \lambda)\) in \(\mathcal{U}\), the whole interval \([-\nu, \nu]\) is included in the set \(\bar{\Omega}_{\nu, \lambda} = \{x : \bar{W}(x) \geq 0\}\), and \(\bar{W}(x)\) has four real valued roots.

Proof of the lemma 3.3. In this case (which corresponds to \(\nu \in (0, 3 - 2\sqrt{2})\)), \(\bar{W}(x)\) takes the form

\[
\bar{W}(x) = \nu^2(1 - \nu)^2(x - 1)(x + 1)(x - (\sqrt{5} + 2))(x + \sqrt{5} - 2).
\]
Here, all the factors but the last one of $\tilde{W}$ are positive for $x$ in $[-\nu_0, \nu_0]$, where $\nu_0 := 3 - 2\sqrt{2} < 1$. In order to determine the sign of the last factor $x + \sqrt{5} - 2$, we see that for all $x$ in $[-\nu_0, \nu_0]$, we have

$$x + \sqrt{5} - 2 \geq -\nu_0 + \sqrt{5} - 2 = \sqrt{5} + 2\sqrt{2} - 5,$$

the last term being positive, as can be seen from the inequality

$$(\sqrt{5} + 2\sqrt{2})^2 = 13 + 4\sqrt{10} > 13 + 4\sqrt{9} = 5^2,$$

and Lemma 3.3 is proved. □

These two lemmata enable us to conclude the proof of Proposition 3.1: Indeed, recall, from the Lemma 2.1 and the remark that follows at the end of Section 2, that the map $\bar{x}$ is continuous on $U$, and so is the map $(\nu, \lambda) \mapsto \bar{x}(\nu, \lambda) + \nu$. The connectedness of $U$ implies that its image by the map $\bar{x} + \nu$ is a connected subset of $\mathbb{R}$ and hence an interval. Clearly 0 cannot belong to this interval because $\tilde{W}(x = -\nu) > 0$ as shown in Lemma 3.2 and thus, either

$$\bar{x}(\nu, \lambda) < -\nu, \ \forall (\nu, \lambda) \in U,$$

or

$$\bar{x}(\nu, \lambda) > -\nu, \ \forall (\nu, \lambda) \in U.$$

But for $\lambda = 1 - \nu$, since $\bar{x}$ is negative by definition, Lemma 3.3 entails $\bar{x}(\nu, 1-\nu) < -\nu$ and hence only the possibility $\bar{x}(\nu, \lambda) < -\nu$ for all $(\nu, \lambda) \in U$ remains.

Then, assume by contradiction that there exists $(\nu_q, \lambda_q)$ in $U$ and $x_q$ in $[-\nu_q, \nu_q]$ such that $\tilde{W}(x_q)$ is negative. By Lemma 3.2, $\tilde{W}(x = \pm\nu_q)$ are both positive and therefore there is an even number of roots (counting multiplicity) of $\tilde{W}(x)$ inside the interval $[-\nu_q, \nu_q]$, and at least two of them or distinct. Since there cannot be four roots in $[-\nu_q, \nu_q]$ (as $\bar{x}$ is outside this interval), there are exactly two distinct ones in $[-\nu_q, \nu_q]$. This implies that, for $(\nu, \lambda) = (\nu_q, \lambda_q)$, the polynomial $\tilde{W}(x)$ has four distinct roots; two are in the interval $[-\nu_q, \nu_q]$, one in the interval $(\infty, 0)$ and, given that $\tilde{W}(x)$ is positive for $|x|$ large enough, the remaining root must lie also in the interval $(-\infty, -\nu_q)$. Therefore $(\nu_q, \lambda_q)$ belongs to the connected subset $\mathcal{U}_1 = U \cap \{\lambda < \chi(\nu)\}$, which was defined in Lemma 5.1 of [2] as the domain where $\tilde{W}$ has four distinct roots, and so does $(\nu, 1-\nu)$ according to Lemma 3.3. The roots of $\tilde{W}(x)$, for values of $(\nu, \lambda)$ in $\mathcal{U}_1$, are simple and thus they are smooth functions of $(\nu, \lambda)$ in $\mathcal{U}_1$. 7
Let us denote by $x_4(\nu, \lambda)$ the largest of these roots. Again, $x_4(\mathcal{W}_1) \subset \mathbb{R}$ is a real interval as $\mathcal{W}_1$ is connected and $x_4$ is continuous.

The previous considerations imply that the greatest root of $\tilde{W}(x)$ for $(\nu, \lambda) = (\nu_q, \lambda_q)$ fulfills $x_4(\nu_q, \lambda_q) < \nu_q$ and from Lemma 3.3 we deduce $x_4(\nu, 1 - \nu) = 2 + \sqrt{5} > \nu$, $\forall \nu \in (0, 1)$. Hence, the continuous map $(\nu, \lambda) \mapsto x_4(\nu, \lambda) - \nu$ changes sign on $\mathcal{W}_1$. Therefore, by the intermediate value theorem, there exits some $(\nu, \lambda) \in \mathcal{W}_1$ such that $x_4(\nu, \lambda) = \nu$. But this is impossible since $\tilde{W}(x_4(\nu, \lambda)) = 0$ by definition of $x_4$, whereas $\tilde{W}(\nu) > 0$ as shown in Lemma 3.2.

This shows that $\tilde{W}(x) \geq 0$, for $x$ in $[-\nu, \nu]$, and the corresponding solutions $y_\pm(x)$ are therefore defined on $[-\nu, \nu]$, except $y_+^-(x)$ which diverges at $x = 0$. However, $Y_\pm$ are both well defined on $[-\nu, \nu]$. The relation (3.1) finishes the proof. □

We have just shown that $H(y, x) = 0$ has two solutions $y_-^-(x)$, $y_+^+(x)$ which are defined $\forall x \in [0, 1]$ (allowing an infinite value for $y_+^+(x)$ at $x = 0$). Note that, from (3.1), they are related to the solutions $y_\pm(x)$ of $H(x, y)$ by

$$y_\pm(x) = \nu y_\pm(\nu x),$$

(3.4)

for all $x$ in $[-1, 1]$. Equivalently, we have in $(x, Y)$-coordinates:

$$Y_\pm(x) = \frac{1}{\nu} Y_\pm(\nu x),$$

(3.5)

which means that, in the $(x, Y)$-coordinates, the graphs of $Y_\pm$ are obtained from the graphs of $Y_\pm$ by an homothety of center $(0, 0)$, and of ratio $1/\nu$. In particular, we have $Y_\pm^-(x) > Y_\pm^+(x)$, and we define the outer ergosurface as the set $\{y = y_-^-(x) : x \in [-1, 1]\}$, or, in $(x, Y)$-coordinates, $\{Y = Y_-^-(x) : x \in [-1, 1]\}$. We now prove that this last set lies above the event horizon.

To that end we need a lemma:

**Lemma 3.4** $x \mapsto y_-^-(x)$ reaches its minimum at one of the ends $\pm 1$.

**Proof:** From the equality $y_-^-(x) = \nu y_-^-(\nu x)$, it suffices to prove that the function $x \mapsto y_-^-(x)$ reaches its minimum at one of the ends $-\nu$ or $\nu$ in the interval $[-\nu, \nu]$. If not, there would exist a strict local minimum of $y_-$ located at $x = x_m \in (-\nu, \nu)$. Let $y_m := y_-(x_m)$. Then, for $\varepsilon > 0$ small enough, there exist two values $x_1$ and $x_2$ such that $y_-(x_i) = y_m + \varepsilon$, with $-\nu < x_1 < x_m < x_2 < \nu$, and such that $y_-$ is strictly decreasing in a
neighbourhood of \( x_1 \) (see figure 3.1). Furthermore, recall that the graphs of \( y_+ \) and \( y_- \) join smoothly at \( x = \bar{x} \), where \( \bar{x} \) (introduced in Lemma 2.1) is the greatest negative root of \( \bar{\nu} \) (in particular we have \(-1 < \bar{x} < -\nu\)). Hence, in the interval \([\bar{x}, x_1]\), the graphs of \( y_+ \) and \( y_- \) form a smooth connected curve \( C \) (if \( x_1 > 0 \), there is another connected component of the graph of \( y_+ \) which is not of interest here). Therefore the points of \( C \) in the interval \([\bar{x}, x_1]\) are \( \bar{\nu} \) equivalent to the curve \( y \). Hence we have the inequalities \( y(p) < y(x_1) = y(0) \) and \( p \ll \bar{s} \). On the other hand the property \( \partial_y y_-(x_1) < 0 \) implies that there exists a real number \( q \ll (0, 1) \) such that \( y(q) \ll y(x(q)) \ll y_-(x_1) \) which according to (3.6) is equivalent to \( y(q) > y(0) \). Hence we have the inequalities \( y(p) < y(0) \ll y(q) \ll y(q) \) with \( q \ll p \) and given that \( y(s) \) is continuous in the interval \([\mu, \nu]\) the intermediate value theorem tells us that there exists a real number \( s^* \ll [\mu, \nu] \) such that \( y(s^*) = y(0) \). If we define \( x^* \ll x(s^*) \), then eq. (3.6) implies that, either \( y_-(x^*) = y(0) = y_m + \varepsilon \) if \( s^* \ll \bar{s} \) or \( y_+(x^*) = y(0) = y_m + \varepsilon \) if \( s^* > \bar{s} \). In any case the conclusion is that the numbers \( x^* \ll x_1 \ll x_2 \) are three distinct roots of the polynomial \( H(x, y_m + \varepsilon) \) of order two in \( x \), which is impossible.

\[ \square \]

**Proposition 3.5** We have the inequality \( y_{e-}(x) > y_h \), for all \( x \) in \([-1, 1]\).

**Proof:** We first write the expressions:

\[
H(y_h, -1) = \frac{\lambda(1 + \nu - \lambda)}{2\nu} \left( (\lambda - \sqrt{\lambda^2 - 4\nu}) (\lambda + \nu - 1) - 4\nu \right),
\]

\[
H(y_h, 1) = -\frac{\lambda(1 + \nu + \lambda)}{2\nu} \left( (\lambda - \sqrt{\lambda^2 - 4\nu}) (\lambda + 1 - \nu) - 4\nu \right).
\]

We will check in a moment that they are both negative. This will imply the proposition 3.5. In order to see this, note first that the negativity of these expressions, together with the positivity of both \( H(-1, \pm 1) \) (see Section 3.
of [2) show that both (continuous) functions $y \mapsto H(y, \pm 1)$ vanish on the interval $(y_h, -1)$. And since

$$\lim_{y \to -\infty} H(y, \pm 1) = +\infty,$$

these two functions vanish once again in the interval $(-\infty, y_h)$. Hence, we obtain that $y_{e-}(x = \pm 1)$ belongs to the interval $(y_h, -1)$, and $y_{e+}(x = \pm 1)$ belongs to the interval $(-\infty, y_h)$. This entails $y_h < y_{e-}(x = \pm 1)$ and from Lemma [3.4] we conclude that $y_h < y_{e-}(x)$, $\forall x \in [-1, 1]$ as desired. We now turn to the proof of the negativity of the quantities $H(y_h, 1)$ written above: $H(y_h, -1)$ has the sign of the second factor, $$(\lambda - \sqrt{\lambda^2 - 4\nu})(\lambda + \nu - 1) - 4\nu.$$ But we have the inequalities $0 < \lambda - \sqrt{\lambda^2 - 4\nu} < \lambda < 2$ and $\lambda + \nu - 1 < 2\nu$ for admissible $\nu$ and $\lambda$, so the negativity of $H(y_h, -1)$ follows. Then, $H(y_h, 1)$ has the sign of $-u(\nu, \lambda)$, where

$$u(\nu, \lambda) = (\lambda - \sqrt{\lambda^2 - 4\nu})(\lambda + 1 - \nu) - 4\nu.$$
With the change of variables \( \lambda = 2\sqrt{\nu}\cosh(\eta) \), we get

\[
u(\nu, \lambda) = 2\sqrt{\nu}(\sqrt{\nu}e^{-2\eta} + (1 - \nu)e^{-\eta} - \sqrt{\nu}) \;
\]

the roots of the right-hand term, which is an order-two polynomial in \( e^{-\eta} \), are \(-1/\sqrt{\nu}\) and \(\sqrt{\nu}\). But the following equivalence holds:

\[
e^{-\eta} > \sqrt{\nu} \iff \lambda < 1 + \nu,
\]

thus \(\nu(\nu, \lambda)\) is positive for any admissible \((\nu, \lambda)\).

□

We now turn to the inner part of the ergosurface

\[\{y = y_{e+}(x) : x \in [-1, 1]\},\]

or, in \((x, Y)\)-coordinates,

\[\{Y = Y_{e+}(x) : x \in [-1, 1]\}.
\]

We wish to prove that the inner ergosurface lies beyond the Killing horizons, and even beyond the singular set \(\{H(x, y) = 0\}\), in the sense that

\[Y_{e+} \leq Y_+ \leq Y_- < Y_c < Y_h\]

wherever these functions are defined. This is illustrated in Figures 1.3 and 1.4 of [2]. This inner part reaches and crosses the set \(\{y = \pm \infty\} = \{Y = 0\}\) at \(x = 0\) only, we therefore make use of the coordinates \((x, Y)\) in the following:

**PROPOSITION 3.6** We have the inequality \(Y_{e+}(x) < Y_c\) for all \(x\) in \([-1, 1]\).
Moreover, we have \(Y_{e+}(x) \leq Y_+(x)\) for all \(x \in \tilde{\Omega}_{\nu, \lambda}\), with equality only at \(x = 0\).

**PROOF:** Let us first write:

\[
H(y_c, -1) = \frac{\lambda(1 + \nu - \lambda)}{2\nu} \left( (\lambda + \sqrt{\lambda^2 - 4\nu}) (\lambda + \nu - 1) - 4\nu \right),
\]

\[
H(y_c, 1) = -\frac{\lambda(1 + \nu + \lambda)}{2\nu} \left( (\lambda + \sqrt{\lambda^2 - 4\nu}) (\lambda + 1 - \nu) - 4\nu \right).
\]

Again, these expressions are both negative. Indeed, we have the inequalities \(\lambda + \sqrt{\lambda^2 - 4\nu} < 1 + \nu + (1 - \nu) = 2\) for allowed values of \(\lambda\) and \(\nu\). Since
\[ \lambda + \nu - 1 < 2\nu, \] the whole factor \( (\lambda + \sqrt{\lambda^2 - 4\nu}) (\lambda + 1 - \nu) - 4\nu \) is negative, thus \( H(y_c, -1) \) is negative. Then, it is straightforward to see that \( H(y_c, 1) < H(y_h, 1) \), itself negative, when comparing both expressions. Since we have the limits \( H(y, \pm 1) \to +\infty \) as \( y \to -\infty \), the function \( H(y, \pm 1) \) must have a zero in the interval \((-\infty, y_c)\) and therefore we are led to the inequality \( y_{e+}(x = \pm 1) < y_c \)\(^r\) which in turn entails \( Y_{e+}(\pm 1) < Y_c \). Now, we have that the function \( Y_{e+}(x) \) does not admit any strict maximum on the interval \((-1, 1)\) (the proof of this fact is similar to the proof of Lemma 3.4) from which we conclude that \( Y_{e+}(x) < Y_c, \forall x \in [-1, 1] \).

In order to show the second part, we first compare \( Y_{e+} \) and \( Y_+ \) in a neighbourhood of \( x = 0 \). To do so, we write the asymptotic expansion:

\[
Y_+(x) = -\nu x \left( 1 - \frac{1-\nu}{2\nu\lambda}((1+\nu)^2 - \lambda^2)x + o(x) \right);
\]

so that, using the fact that \( Y_{e+}(x) = Y_+(\nu x)/\nu \), we obtain

\[
Y_{e+}(x) - Y_+(x) = -\frac{(1-\nu)^2((1+\nu)^2 - \lambda^2)}{2\nu\lambda}x^2 + o(x^2).
\]

The order-two term in the above expansion is negative for allowed values of the parameters \( \nu \) and \( \lambda \). Therefore, we get \( Y_{e+} \leq Y_+ \) in a neighbourhood of \( x = 0 \), with equality at \( x = 0 \) only: \( Y_{e+}(0) = Y_+(0) = 0 \). Furthermore, this inequality extends globally to the set \( \tilde{\Omega}_{\nu, \lambda} = \{ x : \tilde{W}(x) \geq 0 \} \); otherwise, either there would exist some \((x, y)\), with \( x \in \tilde{\Omega}_{\nu, \lambda} \) and \( |y| > 1 \) such that \( H(x, y) = H(y, x) = 0 \), which is impossible from the analysis subsequent to Equation (3.1) in Section 3 of \( [2] \), or there would be an equality \( Y_{e+}(x) = Y_+(x) = 0 \) at some \( x \in [-1, 1] \setminus \{0\} \), which is again impossible. Indeed, the only candidate for such a zero of \( Y_+ \) is \( x = x_* \), which exists if \( \nu^2 + \lambda^2 \neq 1 \), and reads (see subsection 5.4 of \( [2] \)):

\[
x_* = \frac{-2\nu\lambda}{\nu^2 + \lambda^2 - 1}.
\]

But then, it is easy to check that \( Y_{e+}(x_*) = Y_+(\nu x_*)/\nu \neq 0 \), from the expression of \( y_\pm \) written in subsection 5.4 of \( [2] \).

\[ \square \]

We now have the main statement:

\(^2\)Recall from the proof of proposition 3.5 the inequality \( y_h < y_{e-}(x = \pm 1) \) which rules out the possibility \( y_{e-}(x = \pm 1) < y_c \).
**Theorem 3.7** The ergosurface always has a connected component, diffeomorphic to $\mathbb{R} \times S^1 \times S^2$, lying beyond all Killing horizons, and an “outer part”, lying above the event horizon, such that:

- For $(\nu, \lambda) \in \mathcal{U}$ such that $\nu + \lambda < 1$, the outer ergosurface is diffeomorphic to $\mathbb{R} \times S^1 \times S^2$.

- For $(\nu, \lambda) \in \mathcal{U}$ such that $\nu + \lambda > 1$, the outer ergosurface is diffeomorphic to $\mathbb{R} \times (S^3 \cup S^3)$, that is to say the space cross-sections have the topology of two disjoint copies of a 3-sphere.

- In the limiting case $\nu + \lambda = 1$, the outer ergosurface is a “pinched” $\mathbb{R} \times S^1 \times S^2$.

**Proof:** From the previous results in this section, in particular from Proposition 3.6, the inner part of the ergosurface is defined for all allowed values of the parameters, lies under the Killing horizon $\{y = y_c\}$, “under” the singular set, and meets the latter only at $(x, Y) = (0, 0)$. In particular, it is smooth, connected, reaches the axis of rotation $x = \pm 1$ related to the coordinate $\varphi$, and is therefore diffeomorphic to $\mathbb{R} \times S^1 \times S^2$.

For the outer part, we study its intersection with the set $\{y = -1\}$. To do so, recall the formula:

$$H(-1, x) = (1 + \nu - \lambda)\left(\nu(1 + \lambda - \nu)x^2 + 1 - \nu - \lambda\right).$$

In particular, we have $H(-1, \pm 1) = (1-\nu)(1+\nu-\lambda)^2 > 0$. Since $H(y_h, \pm 1) < 0$, we obtain that $y_{e_{-}}(\pm 1)$ are both in $(y_h, -1)$.

Then, the roots $x$ of $H(-1, x) = 0$ exist if and only if $\nu + \lambda \geq 1$, and read

$$x \in \left\{ \pm \sqrt[\nu(1 + \lambda - \nu)]{\frac{\nu + \lambda - 1}{\nu(1 + \lambda - \nu)}} \right\}.$$

Those values of the roots are in $(-1, 1)$ for all allowed values of $\nu$ and $\lambda$ such that $\nu + \lambda \geq 1$, as shown in the proof of Lemma 3.3 in Section 3 of [2].

From this, we conclude that if $\nu + \lambda < 1$, the whole graph of $y_{e_{-}}$ forms a connected set in the spacetime, above the event horizon, and since it is smooth and defined for all allowed values of the parameter $x$, it is again diffeomorphic to $\mathbb{R} \times S^1 \times S^2$. 
If $\nu + \lambda > 1$, the intersection of the graph of $y_{e-}$ with the spacetime has two connected components, $\{y_{e-}(x) : x \in [-1, -\sqrt{\frac{\nu+\lambda-1}{\nu(1+\lambda-\nu)}}] \}$ and $\{y_{e-}(x) : x \in [\sqrt{\frac{\nu+\lambda-1}{\nu(1+\lambda-\nu)}}, 1]\}$. Therefore, after taking into account the rotations along the coordinates $\varphi$ and $\psi$, we obtain that the part of the outer ergosurface which lies in the spacetime is diffeomorphic to $\mathbb{R} \times (S^3 \cup S^3)$ (the product of the real line with two disjoint 3-spheres). In particular, it has two connected components.

In the limiting case $\nu + \lambda = 1$, the graph of $y_{e-}$ intersects the set $\{y = -1\}$ exactly once, at $x = 0$. Otherwise, it is located in the spacetime. The ergosurface in this case is therefore diffeomorphic to a “pinched” $\mathbb{R} \times S^1 \times S^2$. \qed

4 Acknowledgements

The author is grateful to P.T. Chruściel and to A. García-Parrado for many valuable suggestions and comments.

References

[1] P.T. Chruściel and J. Cortier, Maximal analytic extensions of the Emparan-Reall black ring, J. Diff. Geom. 85 (2010) 425-459.

[2] P.T. Chruściel, J. Cortier and A. García-Parrado Gómez-Lobo, On the global structure of the Pomeransky-Senkov black holes, (2009), arXiv:0911.0802 [gr-qc].

[3] M. Durkee, Geodesics and symmetries of doubly spinning black rings, Class. Quantum Grav. 26 (2009), 085016, 33.

[4] R. Emparan and H.S. Reall, A rotating black ring in five dimensions, Phys. Rev. Lett. 88 (2002), 101101, arXiv:hep-th/0110260.

[5] A.A. Pomeransky and R.A. Sen’kov, Black ring with two angular momenta, (2006), hep-th/0612005.