THE ANTIPODE OF A DUAL QUASI-HOPF ALGEBRA WITH NONZERO INTEGRALS IS BIJECTIVE

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Dedicated to Fred Van Oystaeyen for his sixtieth birthday

Abstract. For $A$ a Hopf algebra of arbitrary dimension over a field $K$, it is well-known that if $A$ has nonzero integrals, or, in other words, if the coalgebra $A$ is co-Frobenius, then the space of integrals is one-dimensional and the antipode of $A$ is bijective. Bulacu and Caenepeel recently showed that if $H$ is a dual quasi-Hopf algebra with nonzero integrals, then the space of integrals is one-dimensional, and the antipode is injective. In this short note we show that the antipode is bijective.

1. Introduction

The definition of quasi-Hopf algebras and the dual notion of dual quasi-Hopf algebras is motivated by quantum physics and dates back to work of Drinfeld [4]. The theory of integrals for quasi-Hopf algebras was studied in [9] [3] [2]. In [2], Bulacu and Caenepeel showed that a dual quasi-Hopf algebra is co-Frobenius as a coalgebra if and only if it has a nonzero integral. In this case, the space of integrals is one-dimensional and the antipode is injective, so that for finite dimensional dual quasi-Hopf algebras the antipode is bijective. In this note, we use the ideas from a new short proof of the bijectivity of the antipode for Hopf algebras by the second author [7] to show that the antipode of a dual quasi-Hopf algebra with integrals is bijective, thus extending the classical result of Radford [10] for Hopf algebras.

In this paper we prove

Theorem 1.1. Let $H$ be a co-Frobenius dual quasi-Hopf algebra, equivalently, a dual quasi-Hopf algebra having nonzero integrals. Then the antipode of $H$ is bijective.

2. Preliminaries

In this section we briefly review the definition of a dual quasi-Hopf algebra over a field $K$. We refer the reader to [1] [3] [11] for the basic definitions and properties of coalgebras and their comodules and of Hopf algebras. For the definition of dual quasi-Hopf algebra we follow [8] Section 2.4].

Definition 2.1. A dual quasi-bialgebra $H$ over $K$ is a coassociative coalgebra $(H, \Delta, \varepsilon)$ together with a unit $u : K \to H$, $u(1) = 1$, and a not necessarily associative multiplication $M : H \otimes H \to H$. The maps $u$ and $M$ are coalgebra maps. We write $ab$ for $M(a \otimes b)$.

The first author’s research was supported by an NSERC Discovery Grant.

The second author was partially supported by the contract nr. 24/28.09.07 with UEFISCU “Groups, quantum groups, corings and representation theory” of CNCIS, PN II (ID 1002).
As well, there is an element \( \varphi \in (H \otimes H \otimes H)^* \) called the reassociator, which is invertible with respect to the convolution algebra structure of \((H \otimes H \otimes H)^*\). The following relations must hold for all \( h, g, f, e \in H \):

1. \[ h_1(g_1f_1)\varphi(h_2g_2f_2) = \varphi(h_1g_1f_1)(h_2g_2)f_2 \]
2. \[ 1h = h1 = h \]
3. \[ \varphi(h_1g_1f_1\epsilon_1)\varphi(h_2g_2f_2\epsilon_2) = \varphi(g_1f_1\epsilon_1)\varphi(h_1g_2f_2\epsilon_2)\varphi(h_2g_3f_3) \]
4. \[ \varphi(h, 1, g) = \varepsilon(h)\varepsilon(g) \]

Here we use Sweedler’s sigma notation with the summation symbol omitted.

**Definition 2.2.** A dual quasi-bialgebra \( H \) is called a dual quasi-Hopf algebra if there exists an antilinear isomorphism \( S \) of the coalgebra \( H \) and elements \( \alpha, \beta \in H^* \) such that for all \( h \in H \):

5. \[ S(h_1)\alpha(h_2)h_3 = \alpha(h), \quad h_1\beta(h_2)S(h_3) = \beta(h)1 \]
6. \[ \varphi(h_1\beta(h_2), S(h_3), \alpha(h_4)h_5) = \varphi^{-1}(S(h_1), \alpha(h_2)h_3, \beta(h_4)S(h_5)) = \varepsilon(h). \]

Let \( H \) be a dual quasi-Hopf algebra. As in the Hopf algebra case, a left integral on \( H \) is an element \( T \in H^* \) such that \( h^*T = h^*(1)T \) for all \( h^* \in H^* \); the space of left integrals is denoted by \( \int_l \) and by \([2, \text{Proposition 4.7}]\) has dimension 0 or 1. Right integrals are defined analogously with space of right integrals denoted by \( \int_r \). Suppose \( 0 \neq T \in \int_l \). It is easily seen that \( \int_l \) is a two sided ideal of the algebra \( H^* \), and \( KT \subseteq \text{Rat}(H^*) \) with right comultiplication given by \( T \mapsto T \otimes 1 \). Since for co-Frobenius coalgebras \( \text{Rat}(H^*) = \text{Rat}(H^*_{\text{HF}}) = \text{Rat}(H^*_{\text{HF}}) \), \( KT \) must have left comultiplication \( T \mapsto a \otimes T \). By coassociativity, \( a \) is a grouplike element, called the distinguished grouplike of \( H \). Then, for all \( h^* \in H^* \),

\[ T h^* = h^*(a)T. \]

From \([2, \text{Proposition 4.2}]\), the function \( \theta^* : \int_l \otimes H \to \text{Rat}(H^*) \)

\[ \theta^*(T \otimes h) = \sigma(S(h_3) \otimes \alpha(h_4)h_5)(S(h_4) \to T)\sigma^{-1}(S(h_3) \otimes \beta(S(h_2))S^2(h_1)) \]

is an isomorphism of right \( H \)-comodules, where \( \sigma : H \otimes H \to H^* \) is defined by \( \sigma(h \otimes g)(f) = \varphi(f, h, g) \), \( \sigma^{-1} \) is the convolution inverse of \( \sigma \), and, as usual, \( (h \to T)(g) = T(gh) \).

### 3. Proof of the theorem

Let \( H \) be a quasi-Hopf algebra with \( 0 \neq T \in \int_l \). As in \([7]\), for each right \( H \)-comodule \((M, \rho)\), we denote by \( ^aM \) the left \( H \)-comodule structure on \( M \) defined by \( m^a_{\rho} \otimes m_0 = aS(m_1) \otimes m_0 \), where \( \rho(m) = m_0 \otimes m_1 \). Denote the induced right \( H^* \)-module structure on \(^aM \) by \( m^a \), \( h^* = h^*(m^a_{\rho})m_0 = h^*(aS(m_1))m_0 \). By \([2, \text{Corollary 4.4}]\) the antipode \( S \) of \( H \) is injective, and therefore has a left inverse \( S^l \). Then, for \( \sigma \) as above, we have the following analogue of \([7, \text{Proposition 2.5}]\):
**Proposition 3.1.** The map \( p : {aH} \rightarrow \text{Rat}(H^*) \) defined by

\[
(9) \quad p(h) = \sigma(S(S^l(h_3)) \otimes \alpha(S^l(h_2)S^l(h_1)) \ast (h_4 \rightarrow T) \ast \sigma^{-1}(h_5 \beta(h_6) \otimes S(h_7))
\]

is a surjective morphism of left \( H \)-comodules.

**Proof.** Let \( \Psi := \sigma(S(S^l(h_3)) \otimes \alpha(S^l(h_2)S^l(h_1)) \). Then for \( c^* \in H^* \) and \( g \in H \):

\[
(p(h) \ast c^*)(g) = p(h)(g_1)c^*(g_2)
\]

\[
= \Psi(g_1)T(g_2h_4)\sigma^{-1}(h_5\beta(h_6) \otimes S(h_7)))(g_3)c^*(g_4)
\]

\[
= \Psi(g_1)T(g_2h_4)\varphi^{-1}(g_3, h_5\beta(h_6), S(h_7))c^*(g_4)
\]

\[
= \Psi(g_1)T(g_2h_4)\varphi^{-1}(g_3, h_5, S(h_7))c^*(g_4\beta(h_6))
\]

\[
= \Psi(g_1)T(g_2h_4)\beta(h_7)c^*(\varphi^{-1}(g_3, h_5, S(h_9))g_4\beta(h_6)(S(h_8)))
\]

\[
= \Psi(g_1)T(g_2h_4)\beta(h_7)c^*((g_3h_5S(h_9))\varphi^{-1}(g_4, h_6, S(h_8)))
\]

\[
= \Psi(g_1)T(g_2h_4)\beta(h_7)c^*((g_3h_5S(h_9))\varphi^{-1}(g_4, h_6, S(h_8)))
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\]

Thus \( p \) is left \( H \)-colinear. Finally, we note that \( p \circ S = \theta^*(T \otimes -) \) where \( \theta^* \) is the isomorphism from \([5]\) so that \( p \) is surjective. \( \square \)

Let \( c \) be a grouplike element of \( H \). From \([2]\) p.580, \( c \) is invertible with inverse \( S(c) \). We will show that left multiplication by \( c \) has an inverse too.

Let \( \theta_c \in \text{End}(H) \) be defined by \( \theta_c(h) = ch \) and define the coinner automorphisms \( q_c \) and \( r_c = q_c^{-1} \in \text{End}(H) \) by:

\[
q_c(h) = \varphi^{-1}(c, S(c), h_1)h_2\varphi(c, S(c), h_3) \text{ and } r_c(h) = \varphi(c, S(c), h_1)h_2\varphi^{-1}(c, S(c), h_3).
\]

**Lemma 3.2.** For any grouplike element \( c \) and \( \theta_c, r_c, q_c \) as above, \( \theta_c \circ \theta_c^{-1} = r_c \) and thus \( \theta_c \) is bijective with inverse \( \theta_c^{-1} = \theta_c^{-1} \circ q_c = q_c^{-1} \circ \theta_c^{-1} \).

**Proof.** Using \([1]\) and the fact that \( c^{-1} = S(c) \), we see that

\[
\theta_c \circ \theta_c^{-1}(h) = c(c^{-1}h) = \varphi(c, S(c), h_1)(cS(c))h_2\varphi^{-1}(c, S(c), h_3) = r_c(h).
\]

The same formula for \( c^{-1} = S(c) \) yields \( \theta_{c^{-1}} \circ \theta_c = r_{c^{-1}} \) and the statement then follows directly. \( \square \)
We can now prove our main result.

Proof of Theorem 1.1.
We only need to prove the surjectivity. The proof goes along the lines of the proof of [7, Theorem 2.6], but with the difference that here the antipode is not necessarily an anti-morphism of algebras.

Let \( \pi \) be the composition map \( aH \xrightarrow{p} \text{Rat}(H^*_H) \xrightarrow{\sim} H \otimes \int_r \simeq H \), where the last two isomorphisms follow by left-right symmetry of the results of [2]. Since \( H \) is a co-Frobenius coalgebra, \( HH \) is projective by [5, Theorem 1.3] or [2, Theorem 4.5, (x)], and as \( \pi \) is surjective, there is a morphism of left \( H \)-comodules \( \lambda : H \to aH \) such that \( \pi \lambda = \text{Id}_H \). We then have

\[ aS(\lambda(h)_2) \otimes \lambda(h)_1 = \lambda(h)^a_{-1} \otimes \lambda(h)^a_0 = h_1 \otimes \lambda(h_2). \]

Applying \( \text{Id} \otimes \varepsilon \pi \), we get \( aS(\varepsilon \pi(\lambda(h)_1)) \lambda(h)_2) = h \) for any \( h \in H \). Thus \( \theta_0 \circ S \) is surjective and since \( \theta_0 \) is bijective by Lemma 3.2, \( S \) is surjective also. \( \square \)

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