Notes on functions on the unit disk

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1 Power series and their derivatives

As usual, \( \mathbb{R} \) denotes the real numbers, \( \mathbb{Z} \) denotes the integers, and \( \mathbb{C} \) denotes the complex numbers. Thus each \( \alpha \in \mathbb{C} \) can be expressed as \( a + bi \), where \( a, b \) are real numbers and \( i^2 = -1 \). We call \( a, b \) the real and imaginary parts of \( \alpha \), and denote them \( \text{Re} \alpha, \text{Im} \alpha \), respectively.

If \( z \) is a complex number, with \( z = x + yi \) where \( x, y \) are the real and imaginary parts of \( z \), then we write \( \overline{z} \) for the complex conjugate of \( z \), which is defined to be \( x - yi \). The complex conjugate of a sum or product of two complex numbers is equal to the corresponding sum or product of the complex conjugates of the two complex numbers.

The modulus of \( z \) is denoted \( |z| \) and is the nonnegative real number defined by \( |z|^2 = x^2 + y^2 \). This is equivalent to saying that \( |z|^2 = z \overline{z} \). The triangle inequality for the modulus states that \( |z + w| \leq |z| + |w| \) for all...
complex numbers $z, w$. We also have that $|zw| = |z||w|$ for all complex numbers $z, w$.

Let us write $\Delta$ for the unit disk in $\mathbb{C}$, i.e.,

$$\Delta = \{ z \in \mathbb{C} : |z| < 1 \}.$$  

(1.1)

The closed unit disk is denoted $\overline{\Delta}$,

$$\overline{\Delta} = \{ z \in \mathbb{C} : |z| \leq 1 \},$$

(1.2)

and the unit circle is denoted $\Gamma$,

$$\Gamma = \{ z \in \mathbb{C} : |z| = 1 \}.$$  

(1.3)

We shall be interested in two basic kinds of functions on $\Delta$, namely, those represented by power series of the form

$$\sum_{n=0}^{\infty} a(n) z^n$$

(1.4)

for some sequence $\{a(n)\}_{n=0}^{\infty}$ of complex coefficients, and those given as

$$\sum_{n=0}^{\infty} a(n) z^n + \sum_{n=-\infty}^{-1} a(n)z^{-n}$$

(1.5)

for some doubly-infinite sequence $\{a(n)\}_{n=-\infty}^{\infty}$ of complex numbers.

Let us briefly review some of the standard theory of power series like these. In order for the series (1.4) to converge for every $z$ in $\Delta$, it is necessary that

$$|a(n)| r^n, \ n \geq 0, \text{ be bounded for each } r \in (0, 1).$$  

(1.6)

Similarly, in order for the series in (1.5) to converge for all $z$ in $\Delta$, it is necessary that

$$|a(n)| r^n, \ n \in \mathbb{Z}, \text{ be bounded for each } r \in (0, 1).$$

(1.7)

These conditions are also sufficient for convergence, and indeed they imply that the corresponding series converge absolutely for each $z$ in $\Delta$, and uniformly on each disk of the form

$$\{ z \in \mathbb{C} : |z| \leq t \},$$

(1.8)
Moreover, the functions on \( \Delta \) given by these power series are continuously differentiable of all orders on \( \Delta \). In fact, functions of the form (1.4) are complex analytic on \( \Delta \), and functions of the form (1.5) are harmonic on \( \Delta \).

Let us be more precise. If we write the complex variable \( z \) as \( x + iy \), where \( x, y \) are real, then we can define the first-order complex derivatives \( \partial/\partial z \) and \( \partial/\partial \bar{z} \) in terms of the usual partial derivatives \( \partial/\partial x \) and \( \partial/\partial y \) by

\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

It is easy to see that

\[
\frac{\partial}{\partial z} (z) = \frac{\partial}{\partial \bar{z}} (\bar{z}) = 1, \quad \frac{\partial}{\partial z} (\bar{z}) = \frac{\partial}{\partial \bar{z}} (z) = 0.
\]

The usual Leibnitz rule for first-order derivatives of a product of functions works for \( \partial/\partial z \) and \( \partial/\partial \bar{z} \) just as it does for \( \partial/\partial x \) and \( \partial/\partial y \), and thus

\[
\frac{\partial}{\partial z} (z^n) = nz^{n-1}, \quad \frac{\partial}{\partial \bar{z}} (\bar{z}^n) = n \bar{z}^{n-1}, \quad \frac{\partial}{\partial z} (\bar{z}^n) = \frac{\partial}{\partial \bar{z}} (z^n) = 0
\]

for all positive integers \( n \). Of course \( \partial/\partial z \) and \( \partial/\partial \bar{z} \) applied to constant functions is equal to 0, which corresponds to the case \( n = 0 \).

Notice that

\[
\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),
\]

i.e., \( \partial/\partial z \partial/\partial \bar{z} \) is equal to 1/4 times the usual two-dimensional Laplace operator. A continuously differentiable function \( f(z) \) defined on some open subset of \( \mathbb{C} \) is complex-analytic or holomorphic if and only if \( \partial/\partial z f(z) = 0 \) on that open set, and similarly a twice-continuously differentiable function \( h(z) \) defined on some open subset of \( \mathbb{C} \) is harmonic if and only if \((\partial^2/\partial x^2 + \partial^2/\partial y^2)h = 0 \) on that open set, which is the same as saying that \((\partial/\partial z)(\partial/\partial \bar{z})h(z) = 0 \) on that open set.

If \( f(z) \) is defined on \( \Delta \) by a convergent power series (1.4), then \( f(z) \) is holomorphic on \( \Delta \) and

\[
\frac{\partial}{\partial z} f(z) = \sum_{n=1}^{\infty} n a(n) z^{n-1}.
\]
This follows from standard results about differentiating power series. Note that the series on the right also converges for all \( z \) in \( \Delta \). Similarly, if \( h(z) \) is defined on \( \Delta \) by convergent series as in (1.5), then \( h(z) \) is harmonic and

\[
\frac{\partial}{\partial z} h(z) = \sum_{n=1}^{\infty} n a(n) z^{n-1}, \quad \frac{\partial}{\partial \overline{z}} h(z) = \sum_{n=-\infty}^{-1} (-n) a(n) \overline{z}^{-n-1}. \tag{1.14}
\]

Again these two series converge for all \( z \) in \( \Delta \).

2 Continuous functions and the Poisson kernel

Let \( h(z) \) be a continuous function on the closed unit disk \( \overline{\Delta} \) which can be represented by a series expansion as in (1.5) on the open unit disk \( \Delta \), i.e.,

\[
h(z) = \sum_{n=0}^{\infty} a(n) z^n + \sum_{n=-\infty}^{-1} a(n) \overline{z}^{-n} \quad \text{for } z \in \Delta. \tag{2.1}
\]

In particular it is assumed that the series in (2.1) converges for all \( z \) in \( \Delta \).

We would like to determine the coefficients \( a(n), n \in \mathbb{Z} \), in (2.1) from the knowledge of \( h \) on the unit circle \( \Gamma \). Recall that if \( \phi(z) \) is a continuous function on \( \Gamma \), then the integral

\[
\int_{\Gamma} \phi(z) |dz|
\]

is defined, where \(|dz|\) is the element of integration by arclength. We would like to show that

\[
a(n) = \frac{1}{2\pi} \int_{\Gamma} h(z) z^n |dz|, \tag{2.3}
\]

for all integers \( n \), which is equivalent to

\[
a(n) = \frac{1}{2\pi} \int_{\Gamma} h(z) \overline{z}^{-n} |dz|, \tag{2.4}
\]

since \( z^{-1} = \overline{z} \) when \(|z| = 1\).

Notice first that

\[
\int_{\Gamma} |dz| = 2\pi, \quad \int_{\Gamma} z^m |dz| = 0 \quad \text{when } m \in \mathbb{Z}, \ m \neq 0. \tag{2.5}
\]
If \( r \in (0, 1) \), then
\[
a(n) r^{|n|} = \frac{1}{2\pi} \int_{\Gamma} h(rz) \overline{z}^{|n|} |dz|
\]
for all integers \( n \), which is equivalent to
\[
a(n) r^{|n|} = \frac{1}{2\pi} \int_{\Gamma} h(rz) z^{-n} |dz|.
\]

This follows by substituting the series expansion for \( h(rz) \) into the integral and interchanging the order of integration and summation, which is permissible because of uniform convergence. One can then take the limit as \( r \to 1 \) using the fact that \( h(rz) \) tends to \( h(z) \) uniformly for \( |z| = 1 \) since \( h \) is continuous and therefore uniformly continuous on the closed unit disk, because the latter is compact.

If \( \zeta \) lies in \( \Delta \), then it follows from the series expansion for \( h(\zeta) \) that
\[
h(\zeta) = \int_{\Gamma} h(z) P(z, \zeta) |dz|,
\]
where
\[
P(z, \zeta) = \frac{1}{2\pi} \left( \sum_{n=0}^{\infty} \overline{z}^n \zeta^n + \sum_{n=-\infty}^{-1} z^{-n} \overline{\zeta}^{-n} \right).
\]

As before, it is permissible to interchange the order of integration and summation to obtain (2.8) because of uniform convergence. The function \( P(z, \zeta) \), \( z \in \Gamma, \zeta \in \Delta \), is called the Poisson kernel.

We can rewrite (2.9) as
\[
P(z, \zeta) = \frac{1}{2\pi} \left( -1 + 2 \text{Re} \sum_{n=0}^{\infty} \overline{z}^n \zeta^n \right)
= \frac{1}{2\pi} \left( -1 + 2 \text{Re} \frac{1}{1 - z\overline{\zeta}} \right).
\]

This can be simplified further as
\[
P(z, \zeta) = \frac{1}{2\pi} \left( \frac{-|1 - \overline{z}\zeta|^2}{|1 - \overline{z}\zeta|^2} + \frac{2 \text{Re}(1 - z\overline{\zeta})}{|1 - \overline{z}\zeta|^2} \right)
= \frac{1}{2\pi} \frac{1 - |\zeta|^2}{|z - \zeta|^2}
\]
for \( z \in \Gamma, \zeta \in \Delta \).
Notice that
\[ P(z, \zeta) \geq 0 \quad \text{(2.12)} \]
in particular. For each \( \zeta \) in \( \Delta \), we have that
\[ \int_{\Gamma} P(z, \zeta) |dz| = 1. \quad \text{(2.13)} \]
This is an easy consequence of the series expansion \( \text{(2.9)} \).

Suppose that we start with an arbitrary real polynomial on \( C \), which is to say a polynomial in the real variables \( x, y \) which are the real and imaginary parts of the complex variable \( z \). This is equivalent to saying that we have a general polynomial in \( z \) and \( \overline{z} \). The harmonic polynomials are the ones which can be written as a sum of a polynomial in \( z \) alone and a polynomial in \( \overline{z} \) alone. Harmonic polynomials define functions on the unit disk of the sort under consideration, with only finitely many terms in the expansion \( \text{(2.1)} \).

Given an arbitrary real polynomial on \( C \), we can replace it with a harmonic polynomial such that the two polynomials are equal on the unit circle. Indeed, to do this one simply replaces each monomial \( z^j \overline{z}^k \) with \( z^{j-k} \) when \( j \geq k \) and with \( \overline{z}^{k-j} \) when \( j \leq k \). The harmonic polynomial is uniquely determined by its restriction to the unit circle, as in the preceding computations, and hence is determined by the initial real polynomial.

Now suppose that we start with a continuous function \( h(z) \) on \( \Gamma \), and we extend \( h \) to a function on the closed unit disk through the formula \( \text{(2.8)} \) for \( \zeta \in \Delta \). By construction, \( h \) is represented by the series expansion \( \text{(2.1)} \) on \( \Delta \), with the coefficients \( a(n) \) as in \( \text{(2.3)} \) and \( \text{(2.4)} \). More precisely, the \( a(n) \)'s are bounded in this case, which ensures that the relevant series converge on \( \Delta \). In fact the extended function is continuous on all of \( \Delta \).

Continuity on \( \Delta \) can be viewed as a consequence of the series expansion, or of continuity properties of \( P(z, \zeta) \). Let \( w \) be an element of \( \Gamma \), and let us consider continuity of the extended function at \( w \). We only need to be concerned about nearby points \( \zeta \) in \( \Delta \), because our original function is continuous on \( \Gamma \) by assumption. In order to deal with these points \( \zeta \), observe that for each \( \rho > 0 \) we have
\[ \lim_{\zeta \to w} P(z, \zeta) = 0 \quad \text{(2.14)} \]
uniformly on \( \{ \zeta \in \Gamma : |\zeta - w| \geq \rho \} \). This is not hard to check, and once one has this, one can also verify the desired continuity at \( w \).
Of course a continuous function on the unit circle can be approximated uniformly by a sequence of functions which are restrictions of real polynomials on the complex plane to the unit circle. For each of these approximations we can get a harmonic extension which is a polynomial, as discussed previously. The resulting sequence of harmonic polynomials converges uniformly on the closed unit disk to a continuous function on the closed unit disk which is the extension of the initial continuous function on the unit circle to a continuous function on the closed unit disk which is harmonic on the open unit disk.

3 Normal families

Let $\mathcal{H}$ be a collection of functions on $\Delta$ represented by power series as in (1.5). We say that $\mathcal{H}$ is a normal family if for every $r \in (0, 1)$ there is a real number $M(r)$ such that

$$|h(z)| \leq M(r) \text{ when } h \in \mathcal{H} \text{ and } |z| \leq r.$$  

(3.1)

Similarly, a collection $\mathcal{C}$ of functions on $\mathcal{Z}$ is said to be a normal family if for every $r \in (0, 1)$ there is a real number $C(r)$ such that

$$|a(n)| r^{|n|} \leq C(r) \text{ when } a \in \mathcal{C} \text{ and } n \in \mathcal{Z}.$$  

(3.2)

If $\mathcal{H}$ is a normal family of functions on $\Delta$ represented by power series as in (1.5), then from $\mathcal{H}$ we get a collection of $\mathcal{C}$ of functions on $\mathcal{Z}$, namely, the power series coefficients of the functions in $\mathcal{H}$, and $\mathcal{C}$ is a normal family. This can be derived from the formulas (2.6), (2.7) for the coefficients of a given function. Conversely, if $\mathcal{C}$ is a normal family of functions on $\mathcal{Z}$, then we get a family $\mathcal{H}$ of functions on $\Delta$ which can be represented by power series in (1.5), namely, the series whose coefficients are in $\mathcal{C}$. The conditions on the elements of $\mathcal{C}$ are strong enough to ensure that the corresponding series converge on all of $\Delta$, and in fact that the family of functions that results is a normal family.

Suppose that $\mathcal{H}$ is a normal family of functions on $\Delta$ represented by power series as in (1.5), and that $\mathcal{C}$ is the corresponding normal family of functions on $\mathcal{Z}$. If $\{h_j\}_{j=1}^\infty$ is a sequence of functions in $\mathcal{H}$ and $h$ is another function in $\mathcal{H}$, then a natural form of convergence for $\{h_j\}_{j=1}^\infty$ to $h$ is uniform convergence on every closed disk $\{z \in \mathcal{C} : |z| \leq r\}$ for $0 < r < 1$. Similarly, if $\{a_j\}_{j=1}^\infty$ is a sequence of functions in $\mathcal{C}$, and $a$ is another function in $\mathcal{C}$,
then a natural kind of convergence for \( \{a_j\}_{j=1}^{\infty} \) to \( a \) is convergence at each element of \( \mathbb{Z} \), i.e., to have \( \lim_{j \to \infty} a_j(n) = a(n) \) for all \( n \in \mathbb{Z} \). If \( a_j(n) \), \( n \in \mathbb{Z} \), are the power series coefficients for \( h_j \) for each \( j \), and if \( a(n) \), \( n \in \mathbb{Z} \) are the power series coefficients of \( h \), then convergence of \( \{h_j\}_{j=1}^{\infty} \) to \( h \) in the sense just described is equivalent to convergence of \( \{a_j\}_{j=1}^{\infty} \) to \( a \) in the sense just described for it. Indeed, to go from convergence of the \( h_j \)'s to convergence of the \( a_j \)'s, one can use the formulae \( (2.6) \), \( (2.7) \) for the \( a_j(n) \)'s and \( a(n) \)'s in terms of the \( h_j \)'s and \( h \). For the converse one can basically just sum the series. For a fixed \( r \), the normality condition for \( C \) can be used to show that the total contribution of the coefficients corresponding to large \( |n| \) is small uniformly on the disk \( \{ z \in \mathbb{C} : |z| \leq r \} \). Thus the convergence is largely affected by what happens for finite ranges of \( n \), for which the convergence of the \( a_j \)'s can be used.

One also has compactness results for these normal families, i.e., every sequence in the family has a subsequence which converges in the sense described in the preceding paragraph. This is a version of the well-known Arzela–Ascoli theorem. Because of the previous remarks, it is enough to establish this compactness property for \( C \), where it can be derived using compactness of closed disks in \( \mathbb{C} \).

### 4 Inner products and shift operators

Let us define an inner product on the space of continuous functions on \( \Gamma \) by

\[
\langle \phi_1, \phi_2 \rangle = \frac{1}{2\pi} \int_{\Gamma} \phi_1(z) \overline{\phi_2(z)} |dz|.
\]

With respect to this basis, the functions \( z^m, \overline{z}^n \), and the constant function \( 1 \) are orthonormal, where \( m \) and \( n \) run through the set of positive integers. These functions are the eigenfunctions for differentiation on the unit circle.

As in Section 2 for each continuous function \( \phi \) on the unit circle there is a unique continuous function \( \Phi \) on the closed unit disk such that \( \Phi \) and \( \phi \) are equal on the unit circle, and \( \Phi \) can be represented as

\[
\Phi(z) = \sum_{n=0}^{\infty} a(n, \phi) z^n + \sum_{n=-\infty}^{-1} a(n, \phi) \overline{z}^{-n}
\]

for all \( z \) in the open unit disk. For each \( r \in (0, 1) \), we have that

\[
\frac{1}{2\pi} \int_{\Gamma} |\Phi(rz)|^2 |dz| = \sum_{n=-\infty}^{\infty} |a(n, \phi)|^2 r^{2|n|}.
\]
This can be checked by substituting the series expansion for $\Phi$ and computing in a simple way. Everything converges nicely, because $r \in (0, 1)$.

We can take the limit as $r \to 1$ to obtain that

$$
\frac{1}{2\pi} \int_{\Gamma} |\phi(z)|^2 |dz| = \sum_{n=-\infty}^{\infty} |a(n, \phi)|^2.
$$

(4.4)

In particular, the series on the right side converges, which is to say that $a(n, \phi)$ lies in $\ell^2(\mathbb{Z})$. Similarly, for a pair of continuous functions $\phi_1, \phi_2$ on the unit circle, $\Phi_1, \Phi_2, a(n, \phi_1)$, and $a(n, \phi_2)$ can be defined just as for $\phi$, and we have that

$$
\frac{1}{2\pi} \int_{\Gamma} \Phi_1(rz) \Phi_2(rz) |dz| = \sum_{n=-\infty}^{\infty} a(n, \phi_1) a(n, \phi_2) r^{2|n|}
$$

(4.5)

for $r \in (0, 1)$, and hence

$$
\frac{1}{2\pi} \int_{\Gamma} \phi_1(z) \phi_2(z) |dz| = \sum_{n=-\infty}^{\infty} a(n, \phi_1) a(n, \phi_2).
$$

(4.6)

Consider the linear operator on $\ell^2(\mathbb{Z})$ which sends a function $a(n)$ in $\ell^2(\mathbb{Z})$ to the function $a(n - 1)$. This is called the forward shift operator on $\ell^2(\mathbb{Z})$. This operator maps $\ell^2(\mathbb{Z})$ onto itself and preserves the inner product

$$
\sum_{n=-\infty}^{\infty} a_1(n) \overline{a_2(n)}
$$

(4.7)

on $\ell^2(\mathbb{Z})$. We can use the correspondence between functions on the unit circle and functions on $\mathbb{Z}$ to convert this to an operator acting on functions on the unit circle, and in fact it corresponds to the operator

$$
\phi(z) \mapsto z \phi(z).
$$

(4.8)

It is easy to check directly that this operator preserves the inner product of two continuous functions on the unit circle.

Let $\mathbb{Z}_{+,0}$ denote the set of nonnegative integers, i.e., $\mathbb{Z}_{+,0} = \mathbb{Z}_+ \cup \{0\}$. Consider the linear operator on $\ell^2(\mathbb{Z}_{+,0})$ which sends a function $a(n)$ to the function equal to $a(n - 1)$ when $n \geq 1$ and equal to 0 for $n = 0$. Like the previous operator, this one preserves the standard inner product on $\ell^2(\mathbb{Z}_{+,0})$. This operator does not map $\ell^2(\mathbb{Z}_{+,0})$ onto itself, but onto a proper subspace.
of codimension 1. In terms of the correspondence with functions, one should now restrict one’s attention to functions whose series expansions on the unit disk are of the form \( \sum_{n=0}^{\infty} a(n) z^n \). The operator again corresponds to multiplication by \( z \) at the level of functions, and this works on the closed unit disk.

A bounded linear operator \( T \) on a Hilbert space \( \mathcal{H} \) is said to be a contraction if its operator norm is less than or equal to 1. If \( p(z) \) is a polynomial on the complex plane, so that

\[
p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0,
\]

where \( c_0, \ldots, c_{n-1}, c_n \) are complex numbers, then we can define a bounded linear operator \( p(T) \) on \( \mathcal{H} \) by

\[
p(T) = c_n T^n + c_{n-1} T^{n-1} + \cdots + c_0 I,
\]

where \( I \) denotes the identity operator on \( \mathcal{H} \). A remarkable result of von Neumann and Heinz states that the operator norm of \( p(T) \) is less than or equal to the supremum of \( |p(z)| \) over \( z \) in the closed unit disk. For the shift operators described before, this can be derived from the representation of the operators in terms of multiplication by \( z \) on functions on the unit circle, in which case \( p \) of the operator corresponds to multiplication by \( p(z) \) on the unit circle.

5 Convex means

Let \( \phi(s) \) be a continuous real-valued function on \( [0, \infty) \) which is monotone increasing and convex. For the purposes of this section, one may as well assume that \( \phi(0) = 0 \).

**Proposition 5.1** Let \( h(z) \) be a continuous complex-valued function on the closed unit disk \( \Delta \) which can be represented by a power series

\[
\sum_{n=0}^{\infty} a(n) z^n + \sum_{n=-\infty}^{-1} a(n) z^{-n}
\]

on the open unit disk \( \Delta \). For each real number \( r \in (0, 1) \) we have that

\[
\frac{1}{2\pi} \int_{\Gamma} \phi(|h(rz)|) |dz| \leq \frac{1}{2\pi} \int_{\Gamma} \phi(h(z)) |dz|.
\]
Also,
\[(5.4) \quad \sup_{z \in \Gamma} |h(rz)| \leq \sup_{z \in \Gamma} |h(z)|.\]

To prove this, we use the Poisson integral formula
\[(5.5) \quad h(\zeta) = \int_{\Gamma} h(z) P(z, \zeta) |dz|,\]
for $\zeta$ in $\Delta$, as in (2.8). Because $P(z, \zeta)$ is nonnegative, we have that
\[(5.6) \quad |h(\zeta)| \leq \int_{\Gamma} |h(z)| P(z, \zeta) |dz|\]
for all $\zeta$ in $\Delta$. The monotonicity of $\phi(s)$ implies that
\[(5.7) \quad \phi(|h(\zeta)|) \leq \phi \left( \int_{\Gamma} |h(z)| P(z, \zeta) |dz| \right).\]

Now we apply the convexity of $\phi(s)$ to obtain
\[(5.8) \quad \phi(|h(\zeta)|) \leq \int_{\Gamma} \phi(|h(z)|) P(z, \zeta) |dz|\]
This also employs the nonnegativity of $P(z, \zeta)$ and (2.13). In other words, the value of $\phi$ at an average of numbers is less than or equal to the average of the values of $\phi$ at those same numbers.

For $r \in (0, 1)$ we obtain that
\[(5.9) \quad \frac{1}{2\pi} \int_{\Gamma} \phi(|h(rw)|) |dw| \leq \frac{1}{2\pi} \int_{\Gamma} \int_{\Gamma} \phi(|h(z)|) P(z, rw) |dz| |dw|.\]
Using (2.13) we obtain that
\[(5.10) \quad \frac{1}{2\pi} \int_{\Gamma} \phi(|h(rw)|) |dw| \leq \frac{1}{2\pi} \int_{\Gamma} \phi(|h(z)|) |dz|,\]
which is exactly what we wanted.

It is easy to see that
\[(5.11) \quad |h(\zeta)| \leq \sup_{a \in \Gamma} |h(a)| \int_{\Gamma} P(z, \zeta) |dz| = \sup_{a \in \Gamma} |h(a)|\]
for all $\zeta$ in $\Delta$, by (5.6) and (2.13). This completes the proof of Proposition 5.1.
Corollary 5.12 Let $h(z)$ be a function on the open unit disk $\Delta$ which can be represented by a power series

$$\sum_{n=0}^{\infty} a(n) z^n + \sum_{n=-\infty}^{-1} a(n) z^{-n}$$

there. For each pair of real numbers $r, s \in (0, 1)$ with $r \leq s$ we have that

$$\frac{1}{2\pi} \int_{\Gamma} \phi(|h(rz)|) |dz| \leq \frac{1}{2\pi} \int_{\Gamma} \phi(h(sz)) |dz|.$$  

(5.14)

Also,

$$\sup_{z \in \Gamma} |h(rz)| \leq \sup_{z \in \Gamma} |h(sz)|.$$  

(5.15)

This follows from Proposition 5.1 applied to the function $h(sz)$, which is in fact defined on a neighborhood of the closed unit disk. If $h(z)$ is complex analytic function on the open unit disk, which is to say that $h(z)$ can be represented by a series of the form

$$\sum_{n=0}^{\infty} a(n) z^n,$$

then there are remarkable extensions of these results to larger classes of monotone functions $\phi(t)$, namely to functions which can be expressed as a convex function of $\log t$. In particular, a basic class of examples of $\phi(t)$ for harmonic functions are the functions $\phi(t) = t^p$ with $p \geq 1$, while for holomorphic functions one can allow $p > 0$.

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