Abstract. Improving a result of M. Rabus we force a normal, locally compact, 0-dimensional, Frechet-Uryson, initially $\omega_1$-compact and non-compact space $X$ of size $\omega_2$ having the following property: for every open (or closed) set $A$ in $X$ we have $|A| \leq \omega_1$ or $|X \setminus A| \leq \omega_1$.

1. Introduction

E. van Douwen and, independently, A. Dow [4] have observed that under CH an initially $\omega_1$-compact $T_3$ space of countable tightness is compact. (A space $X$ is initially $\kappa$-compact if any open cover of $X$ of size $\leq \kappa$ has a finite subcover, or equivalently any subset of $X$ of size $\leq \kappa$ has a complete accumulation point). Naturally, the question arose whether CH is needed here, i.e. whether the same is provable just in ZFC. The question became even more intriguing when in [2] D. Fremlin and P. Nyikos proved the same result from PFA. Quite recently, A. V. Arhangel’skii has devoted the paper [1] to this problem, in which he has raised many related problems as well.

In [7] M. Rabus has answered the question of van Douwen and Dow in the negative. He constructed by forcing a Boolean algebra $B$ such that the Stone space $St(B)$ includes a counterexample $X$ of size $\omega_2$ to the van Douwen–Dow question, in fact $St(B)$ is the one point compactification of $X$, hence $X$ is also locally compact. The forcing used by Rabus is closely related to the one due to J. Baumgartner and S. Shelah in [3], which had been used to construct a thin very tall superatomic Boolean algebra. In particular, Rabus makes use of a so-called $\Delta$-function $f$ (which was also used and introduced in [3]) with some extra properties that are satisfied if $f$ is obtained by the original, rather sophisticated forcing argument of Shelah from [3].
In this paper we give an alternative forcing construction of counterexamples to
the van Douwen–Dow question, which we think is simpler, more direct and more
intuitive than the one in [7]. First of all, we directly force a topology \( \tau_f \) on \( \omega_2 \) that
yields an example from a \( \Delta \)-function (with no extra properties) in the ground model
which also satisfies CH. There is a wide variety of such ground models since they are
easily obtained when one forces a \( \Delta \)-function or because \( \square_{\omega_1} \) implies the existence of
a \( \Delta \)-function (cf. [3]).

Let us recall the definition of the \( \Delta \)-functions from [3].

**Definition 1.1.** Let \( f : [\omega_2]^2 \rightarrow [\omega_2]^{\leq \omega} \) be a function with \( f(\alpha, \beta) \subset \alpha \cap \beta \) for
\( \{\alpha, \beta\} \in [\omega_2]^2 \). (1) We say that two finite subsets \( x \) and \( y \) of \( \omega_2 \) are good for \( f \)
provided that for \( \alpha \in x \cap y, \beta \in x \setminus y \) and \( \gamma \in y \setminus x \) we always have
\( (a) \alpha < \beta, \gamma \Rightarrow \alpha \in f(\beta, \gamma), \)
\( (b) \alpha < \beta \Rightarrow f(\alpha, \gamma) \subset f(\beta, \gamma), \)
\( (c) \alpha < \gamma \Rightarrow f(\alpha, \beta) \subset f(\gamma, \beta). \)
(2) We say that \( f \) is a \( \Delta \)-function if every uncountable family of finite subsets of \( \omega_2 \)
takes two sets \( x \) and \( y \) which are good for \( f \).

Both in [3] and [7] the main use of the \( \Delta \)-function \( f \) is to suitably restrict the
partial order of finite approximations to a structure on \( \omega_2 \) so as to become c.c.c. This
we do as well, but in the proof of the countable compactness of \( \tau_f \) we also need the
following simple result that yields an additional property of \( \Delta \)-functions provided CH
also holds. In fact, only property 1.1.(a) is needed for this.

**Lemma 1.2.** Assume that CH holds, \( f \) is a \( \Delta \)-function, \( \{c_\alpha : \alpha < \omega_2\} \) are pairwise
disjoint finite subsets of \( \omega_2 \) and \( B \in [\omega_2]^\omega \). Then for each \( n \in \omega \) there are distinct
ordinals \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \omega_2 \) such that
\[
B \subset \bigcap \{f(\xi, \eta) : \xi \in c_{\alpha_i}, \eta \in c_{\alpha_j}, i < j < n\}.
\]

*Proof.* We can assume that \( \sup B < \min c_\alpha \) for each \( \alpha < \omega_2 \). Denote by \( S(n) \) the
statement of the lemma for \( n \). We prove \( S(n) \) by induction on \( n \). The first non-trivial
case is \( n=2 \). Assume indirectly that \( S(2) \) fails. Then for each \( \alpha < \beta < \omega_2 \) there is
\( b_{\alpha, \beta} \in B \) such that \( b_{\alpha, \beta} \notin f(\xi, \eta) \) for some \( \xi \in c_\alpha \) and \( \eta \in c_\beta \). By CH the Erdős-
Rado partition theorem [6] has the form \( \omega_2 \rightarrow (\omega_1)^2_2 \), thus there are \( I \in [\omega_2]^{\omega_1} \) and
\( b \in B \) such that for each \( \alpha \neq \beta \in I \) we have \( b \notin f(\xi, \eta) \) for some \( \xi \in c_\alpha \) and \( \eta \in c_\beta \).

Let \( d_\alpha = c_\alpha \cup \{b\} \) for \( \alpha \in I \). Since \( f \) is a \( \Delta \)-function there are \( \alpha \neq \beta \in I \) such that
\( d_\alpha \) and \( d_\beta \) are good for \( f \). But \( d_\alpha \cap d_\beta = \{b\} \) and \( b < \min c_\alpha, \min c_\beta \), so by 1.1.(a) we
have \( b \notin f(\xi, \eta) \) for each \( \xi \in c_\alpha \) and \( \eta \in c_\beta \) contradicting the choice of \( b = b_{\alpha, \beta} \). Thus
\( S(2) \) holds.

Assume now that \( S(n) \) holds for some \( n \geq 2 \) and we prove \( S(2n) \). Applying \( S(n) \)
\( \omega_2 \)-many times for \( B \) and suitable final segments of \( \{c_\alpha : \alpha < \omega_2\} \) we can obtain
$\omega_2$-many pairwise disjoint $n$-element sets $\{\alpha_0^\nu, \alpha_1^\nu, \ldots, \alpha_{n-1}^\nu\} \subset \omega_2$ such that for each $\nu < \omega_2$

$$B \subset \bigcap\{f(\xi, \eta) : \xi \in c_{\alpha_i^\nu}, \eta \in c_{\alpha_j^\nu}, i < j < n\}.$$ Let $d_\nu = \bigcup\{c_{\alpha_i^\nu} : i < n\}$ for $\nu < \omega_2$. Applying $S(2)$ for $B$ and the sequence $\{d_\nu : \nu < \omega_2\}$ we get ordinals $\nu < \mu < \omega_2$ such that $B \subset f(\xi, \eta)$ for all $\xi \in d_\nu$ and $\eta \in d_\mu$. In other words, if $i, j < n$, $\xi \in c_{\alpha_i^\nu}$ and $\eta \in c_{\alpha_j^\nu}$ then $B \subset f(\xi, \eta)$. Therefore the set $\{\alpha_0^\nu, \alpha_1^\nu, \ldots, \alpha_{n-1}^\nu, \alpha_0^\mu, \alpha_1^\mu, \ldots, \alpha_{n-1}^\mu\}$ witnesses $S(2n)$.

The following, even simpler, result about arbitrary functions $f : [\omega_2]^2 \to [\omega_2]^\omega$ with $f\{\alpha, \beta\} \subset \alpha \cap \beta$ for $\{\alpha, \beta\} \in [\omega_2]^2$ will also be needed.

**Lemma 1.3.** If $f$ is a function as above then for each $K, K' \in [\omega_2]^\omega$ there is a countable set $cl_f(K, K') \subset \omega_2$ such that

(a) $K \subset cl_f(K, K')$, sup $K = sup cl_f(K, K')$,

(b) $\forall \xi \in cl_f(K, K') \forall \eta \in cl_f(K, K') \cup K' (f\{\xi, \eta\} \subset cl_f(K, K'))$.

**Proof.** Let $K(0) = K, K(n + 1) = K(n) \cup \{f(\xi, \eta) : \xi \in K(n), \eta \in K(n) \cup K'\}$ and $cl_f(K, K') = \bigcup_{n<\omega} K(n)$.

The topology $\tau_f$ that we will construct on $\omega_2$ is right separated (in the natural order of $\omega_2$) and is also locally compact and 0-dimensional. Thus for each $\alpha \in \omega_2$ one can fix a compact (hence closed) and open neighbourhood $H(\alpha)$ of $\alpha$ such that $\max H(\alpha) = \alpha$. Conversely, if we can fix for each $\alpha \in \omega_2$ such a right-separating compact open neighbourhood $H(\alpha)$ then the family $\{H(\alpha) : \alpha < \omega_2\}$ determines the whole topology $\tau$ on $\omega_2$. In fact, using the notation $U(\alpha, b) = H(\alpha) \cup \{H(\beta) : \beta \in b\}$, it is easy to check that for each $\alpha \in \omega_2$ the family $B_\alpha = \{U(\alpha, b) : b \in [\alpha]^\omega\}$ is a $\tau$-neighbourhood base of $\alpha$. Therefore, our notion of forcing consists of finite approximations to a family $H = \{H(\alpha) : \alpha < \omega_2\}$ like above.

Now, if $H = \{H(\alpha) : \alpha < \omega_2\}$ is as required and $\beta < \alpha < \omega_2$ then either (i) $\beta \in H(\alpha)$ or (ii) $\beta \notin H(\alpha)$. If (i) holds then $H(\beta) \setminus H(\alpha)$, if (ii) holds then $H(\beta) \cap H(\alpha)$ is a compact open subset of $\beta$, hence there is a finite subset of $\beta$, call it $i\{\alpha, \beta\}$, such that this set is covered by $H[i\{\alpha, \beta\}] = \bigcup\{H(\gamma) : \gamma \in i\{\alpha, \beta\}\}$. It may come as a surprise, but the existence of such a function $i$ is also sufficient to insure that the collection $H$ be as required. More precisely, we have the following result.

**Definition 1.4.** If $H = \{H(\alpha) : \alpha \in \omega_2\}$ is a family of subsets of $\omega_2$ such that $\max H(\alpha) = \alpha$ for each $\alpha \in \omega_2$ then we denote by $\tau_H$ the topology on $\omega_2$ generated by $H \cup \{\omega_2 \setminus H : H \in H\}$ as a subbase. Clearly, $\tau_H$ is a 0-dimensional, Hausdorff and right separated topology in which the elements of $H$ are clopen.
Theorem 1.5. Assume that $H$ is as in definition 1.4 above and there is a function $i : [\omega_2]^2 \rightarrow [\omega_2]^\omega$ satisfying $i\{\alpha, \beta\} \subset \alpha \cap \beta$ for each $\{\alpha, \beta\} \in [\omega_2]^2$ such that if $\beta < \alpha$ then $\beta \in H(\alpha)$ implies $H(\beta) \setminus H(\alpha) \subset H[i\{\alpha, \beta\}]$ and $\beta \notin H(\alpha)$ implies $H(\beta) \cap H(\alpha) \subset H[i\{\alpha, \beta\}]$. Then each $H(\alpha)$ is compact in the topology $\tau_H$, hence $\tau_H$ is locally compact.

Proof. We do induction on $\alpha \in \omega_2$. Assume that for each $\beta < \alpha$ we know $H(\beta)$ is compact in $\tau_H$. By Alexander’s subbase lemma it suffices to show that any cover $K$ of $H(\alpha)$ by members of $H$ and their complements has a finite subcover. Let $K \in K$ be such that $\alpha \in K$. If $K = H(\gamma)$ then $\alpha \leq \gamma$. The case $\alpha = \gamma$ is trivial so assume $\alpha < \gamma$. But then $H(\alpha) \setminus K \subset H[i\{\alpha, \gamma\}]$ and by our inductive hypothesis $H(\beta)$ is compact for each $\beta \in i\{\alpha, \gamma\}$ hence so is $H(\beta) \cap H(\alpha) \setminus K$ being closed in $H(\beta)$. Therefore $H(\alpha) \setminus K$ is compact and so some finite $K_0 \subset K$ covers it. Then $K_0 \cup \{K\}$ covers $H(\alpha)$, hence we are done. A similar argument works if $K = \omega_2 \setminus H(\gamma)$, then using $H(\alpha) \cap H(\gamma) \subset H[i\{\alpha, \gamma\}]$ if $\alpha < \gamma$ (or the compactness of $H(\gamma)$ if $\gamma < \alpha$).

It is now very natural to try to force a generic 0-dimensional, locally compact and right separated topology on $\omega_2$ by finite approximations (or pieces of information) of $H$ and $i$. As was already mentioned, the $\Delta$-function $f$ comes into the picture when one wants to make this forcing c.c.c. The technical details of this are done in section 2.

We call the family $H$ coherent if $\beta \in H(\alpha)$ implies $H(\beta) \subset H(\alpha)$. Clearly, this makes things easier because then $H(\beta) \setminus H(\alpha) = \emptyset$, hence there is no problem covering it, the requirement on $i$ is only that if $\beta \notin H(\alpha)$ and $\beta < \alpha$ then $H(\beta) \cap H(\alpha) \subset H[i\{\alpha, \beta\}]$. The original forcing of Baumgartner and Shelah from [3] (when translated to scattered, right separated, locally compact spaces rather than superatomic Boolean algebras) actually produced such a coherent family $H$. This is interesting because if $H$ is coherent and $\tau_H$ is separable, which we have almost automatically if $H$ is obtained generically, then $\tau_H$ is also countably tight!

Theorem 1.6. If there is a coherent family $H$ of right separating compact open sets for a separable topology $\tau$ on $\omega_2$ then $t(\omega_2, \tau) = \omega$.

Proof. Let $X = (\omega_2, \tau)$. Then for each $\alpha \in \omega_2$ we have $t(\alpha, X) = t(\alpha, H(\alpha))$, hence it suffices to prove $t(\alpha, H(\alpha)) = \omega$. We do this by induction on $\alpha$. So assume it for each $\beta < \alpha$. If we had $t(\alpha, H(\alpha)) = \omega_1$ then $H(\alpha)$ would contain a free sequence $S = \{x_\nu : \nu < \omega_1\}$ of length $\omega_1$. $S$ must converge to $\alpha$ since for each $\beta < \alpha$ we have, by the inductive hypothesis, $t(H(\beta)) \leq \omega$, hence $|H(\beta) \cap S| = \omega$. Let $F_\nu = \{x_\mu : \mu < \nu\}$ for each $\nu < \omega_1$. Since $S$ is free we have $\alpha \notin F_\nu$ for all $\nu < \omega_1$ and the sequence $<F_\nu : \nu < \omega_1>$ is (strictly) increasing. For each $\nu < \omega_1$ there is a finite subset $b_\nu$ of $F_\nu$ such that $F_\nu \subset H[b_\nu]$. Now, if $\mu < \nu$ then $b_\mu \subset F_\mu \subset F_\nu \subset H[b_\nu]$ implies that for each $\beta \in b_\mu$ we have $H(\beta) \subset H[b_\nu]$ by the coherence of $H$, hence $H[b_\nu] \subset H[b_\nu]$. But we have seen above that $|H[b_\nu] \cap S| \leq \omega$ for each $\nu \in \omega_1$, while
of course $S \subset \bigcup \{ H[b_\nu] : \nu < \omega_1 \}$, hence the $H[b_\nu]$'s yield a strictly increasing $\omega_1$-sequence of clopen sets in a separable space, which is a contradiction completing the proof. □

Ironically, this general result that gives countable tightness so easily cannot be used in our construction because we had to abandon the coherency of $H$ in our effort to insure countable compactness (implied by the initial $\omega_1$-compactness) of $\tau_H$.

We mentioned above that our examples, by genericity, are separable. But this is not a coincidence. It is well-known and very easy to prove that if $X$ is an initially $\omega_1$-compact space then $t(X) \leq \omega$ implies that $X$ has no uncountable free sequence. (Moreover, if $X$ is $T_3$ the converse of this is also true.) Hence the following easy, but perhaps not widely known, result immediately implies that any non-compact, initially $\omega_1$-compact space of countable tightness contains a countable subset whose closure is not compact. Thus if there is a counterexample to the van Douwen–Dow question then there is also a separable one.

**Lemma 1.7.** If $Y$ is a non-compact topological space, then for some ordinal $\mu$ the space $Y$ contains a free sequence $\{ y_\xi : \xi < \mu \} \subset Y$ with non-compact closure.

**Proof.** If $Y$ is non-compact, then $Y$ has an strictly increasing open cover $\{ U_\alpha : \alpha < \kappa \}$ for some regular cardinal $\kappa$. We pick points $y_\xi \in X$ and ordinals $\alpha_\xi < \kappa$ by recursion as follows. If the closure of the set $Y_\xi = \{ y_\eta : \eta < \xi \}$ in $Y$ is compact, then pick $\alpha_\xi \in \kappa$ such that $\overline{Y_\xi} \subset U_{\alpha_\xi}$ and let $y_\xi \in Y \setminus U_{\alpha_\xi}$.

The sequence $\alpha_\xi$ is strictly increasing because $y_\eta \in U_{\alpha_\xi} \setminus U_{\alpha_\eta}$ for $\eta < \xi$. So for some $\xi \leq \kappa$ the closure of $Y_\xi$ is non-compact. But $Y_\xi$ is also free because for each $\eta < \xi$ we have $\overline{Y_\eta} \subset U_{\alpha_\eta}$ and $(Y_\xi \setminus Y_\eta) \cap U_{\alpha_\eta} = \emptyset$. So we are done. □

Note that under CH the weight of a separable $T_3$ space is $\leq \omega_1$, and an initially $\omega_1$-compact space of weight $\leq \omega_1$ is compact, hence the CH result of van Douwen and Dow is a trivial consequence of 1.7. Arhangel’skii asked the question, [1, problem 3], whether in this CH can be weakened to $2^\omega < 2^{\omega_1}$? We shall answer this question in the negative: theorem 3.9 implies that the existence of a counterexample to the van Douwen–Dow question is consistent with practically any cardinal arithmetic that violates CH.

In [1, problem 17] Arhangel’skii asked if it is provable in ZFC that an initially $\omega_1$-compact subspace of a $T_3$ space of countable tightness is always closed. (Clearly this is so under CH or PFA, or in general if the answer to the van Douwen–Dow question is “yes”.) In view of our next result both Rabus’ and our spaces give a negative answer to this question. More generally we have the following result.

**Theorem 1.8.** If $X$ is a locally compact counterexample to van Douwen–Dow then the one-point compactification $\alpha X = X \cup \{ p \}$ of $X$ also has countable tightness. On the other hand, $X$ is an initially $\omega_1$-compact non-closed subset of $\alpha X$. 
Let \( A \subseteq X \) be such that \( p \in \overline{A} \) \ltx \ie \ \overline{A}^X \) is not compact\r
. By lemma 1.7 and our preceding remark then there is a countable set \( S \subseteq \overline{A}^X \) such that \( S^X \) is not compact. But by \( t(X) = \omega \) then there is a countable \( T \subseteq A \) for which \( S \subseteq T^X \), hence \( T^X \) is non-compact as well, so \( p \in T \). Consequently we have \( t(p, \alpha X) = \omega \) and so \( t(\alpha X) = \omega \). \( \square \)

2. THE FORCING CONSTRUCTION

The following notation will be used in the definition of the poset \( P_f \). Given a function \( h \) and \( a \subseteq \text{dom}(h) \) we write \( h[a] = \cup \{ h(\xi) : \xi \in a \} \). Given non-empty sets \( x \) and \( y \) of ordinals with \( x \neq \text{sup} \ y \) let

\[
x \ast y = \begin{cases} 
    x \cap y & \text{if } x \notin y \text{ and } y \notin x, \\
    x \setminus y & \text{if } x \in y, \\
    y \setminus x & \text{if } y \in x.
\end{cases}
\]

**Definition 2.1.** For each function \( f : [\omega_2]^2 \rightarrow [\omega_2]^{\leq \omega} \) satisfying \( f(\alpha, \beta) \subset \alpha \cap \beta \) for any \( \{ \alpha, \beta \} \in [\omega_2]^2 \) we define a poset \( P_f = (P_f, \leq) \) as follows. The underlying set of \( P_f \) is the family of triples \( p = (a, h, i) \) for which

(i) \( a \in [\omega_2]^{<\omega} \), \( h : a \rightarrow \mathcal{P}(a) \) and \( i : [a]^2 \rightarrow \mathcal{P}(a) \) are functions,

(ii) \( \max h(\xi) = \xi \) for each \( \xi \in a \),

(iii) \( i(\xi, \eta) \subset f(\xi, \eta) \) for each \( \{ \xi, \eta \} \in [a]^2 \),

(iv) \( h(\xi) \ast h(\eta) \subset h(i(\xi, \eta)) \) for each \( \{ \xi, \eta \} \in [a]^2 \).

We will often write \( p = (a^p, h^p, i^p) \) for \( p \in P_f \).

For \( p, q \in P_f \) let \( p \leq q \) if \( a^p \supseteq a^q \), \( h^p(\xi) \cap a^q = h^q(\xi) \) for \( \xi \in a^q \), and \( i^p \supseteq i^q \). If \( p \in P_f \), \( \alpha \in a^p \), \( b \subseteq a^p \cap \alpha \), let us write \( a^p(\alpha, b) = h^p(\alpha) \setminus h^p[b] \).

**Lemma 2.2.** For each \( \alpha < \omega_2 \) the set \( D_\alpha = \{ p \in P_f : \alpha \in a^p \} \) is dense in \( P_f \).

**Proof.** Let \( q \in P_f \) with \( \alpha \not\in a^q \). Define the condition \( p \leq q \) by the stipulations \( a^p = a^q \cup \{ \alpha \} \), \( h^p(\alpha) = \{ \alpha \} \), \( h^p(\xi) = h^q(\xi) \) and \( i^p(\alpha, \xi) = \emptyset \) for \( \xi \in a^q \). Then clearly \( p \in D_\alpha \). \( \square \)

**Definition 2.3.** If \( \mathcal{G} \) is a \( P_f \)-generic filter over \( V \), in \( V[\mathcal{G}] \) we can define the topological space \( X_f[\mathcal{G}] = X_f = \langle \omega_2, \tau_f \rangle \) as follows. For \( \alpha \in \omega_2 \) put \( H(\alpha) = \cup \{ h^p(\alpha) : p \in \mathcal{G} \wedge \alpha \in a^p \} \), let \( \mathcal{H} = \{ H(\alpha) : \alpha < \omega_2 \} \) and let \( \tau_f = \tau_{\mathcal{H}} \) as defined in 1.4, that is, \( \tau_f \) is the topology on \( \omega_2 \) generated by \( \mathcal{H} \cup \{ \omega_2 \setminus H : H \in \mathcal{H} \} \) as a subbase.

If \( \mathcal{G} \) is a \( P_f \)-generic filter over \( V \) then by lemma 2.2 we have \( \cup \{ a^p : p \in \mathcal{G} \} = \omega_2 \), and for each \( \alpha < \omega_2 \) \( \max H(\alpha) = \alpha \) and \( H(\alpha) \) is clopen in \( X_f \). Thus \( X_f \) is 0-dimensional and right separated. Of course, neither \( f \) nor \( i \) is needed for this. As was explained in
section 1, we need \( f \) to be a \( \Delta \)-function in order to make \( P_f \) c.c.c (which insures that no cardinal is collapsed), and the function \( i \) is used to make \( X_f \) also locally compact.

**Theorem 2.4.** If \( CH \) holds and \( f \) is a \( \Delta \)-function, then \( P_f \) satisfies the c.c.c and \( V^{P_f} \models \text{"}X_f = \langle \omega_2, \tau_f \rangle \text{"} \) is a 0-dimensional, right separated, locally compact space having the following properties:

(i) \( t(X_f) = \omega \),

(ii) \( \forall A \in [\omega_2]^\omega \exists \alpha \in \omega_2 \ | A \cap H(\alpha) | = \omega_1 \)

(iii) \( \forall A \in [\omega_2]^\omega \ (\overline{A} \text{ is compact or } |\omega_2 \setminus \overline{A} | \leq \omega_1). \)

Consequently, in \( V^{P_f} \), \( X_f \) is a locally compact, normal, countably tight, initially \( \omega_1 \)-compact but non-compact space.

**Proof of theorem 2.4.** To show that \( P_f \) satisfies c.c.c we will proceed in the following way. We first formulate when two conditions \( p \) and \( p' \) from \( P_f \) are called good twins (definition 2.5), then we construct the amalgamation \( r = p + p' \) of \( p \) and \( p' \) (definition 2.6) and show that \( r \) is a common extension of \( p \) and \( p' \) in \( P_f \). Finally we prove in lemma 2.8 that every uncountable family of conditions contains a couple of elements which are good twins.

**Definition 2.5.** Let \( p = \langle a, h, i \rangle \) and \( p' = \langle a', h', i' \rangle \) be from \( P_f \). We say that \( p \) and \( p' \) are good twins provided

1. \( p \) and \( p' \) are twins, i.e., \( |a| = |a'| \) and the natural order-preserving bijection \( e = e_{p,p'} \) between \( a \) and \( a' \) is an isomorphism between \( p \) and \( p' \):
   i. \( h'(e(\xi)) = e''h(\xi) \) for each \( \xi \in a \),
   ii. \( i'(e(\xi), e(\eta)) = e''i(\xi, \eta) \) for each \( \{\xi, \eta\} \in [a]^2 \),
   iii. \( e(\xi) = \xi \) for each \( \xi \in a \cap a' \),
2. \( i(\xi, \eta) = i'(\xi, \eta) \) for each \( \{\xi, \eta\} \in [a \cap a']^2 \),
3. \( a \) and \( a' \) are good for \( f \).

Let us remark that, in view of (ii) and (iii), condition (2) can be replaced by “\( i(\xi, \eta) \subset a \cap a' \) for each \( \{\xi, \eta\} \in [a \cap a']^2 \).”

**Definition 2.6.** If \( p = \langle a, h, i \rangle \) and \( p' = \langle a', h', i' \rangle \) are good twins we define the amalgamation \( r = \langle b, g, j \rangle \) of \( p \) and \( p' \) as follows:

Let \( b = a \cup a' \). For \( \xi \in h[a \cap a'] \cup h'[a \cap a'] \) define

\[
\delta \xi = \min \{ \delta \in a \cap a' : \xi \in h(\delta) \cup h'(\delta) \}.
\]

Now, for any \( \xi \in b \) let

\[
(\bullet) \quad g(\xi) = \begin{cases} 
    h(\xi) \cup h'(\xi) & \text{if } \xi \in a \cap a', \\
    h(\xi) \cup \{ \eta \in a' \setminus a : \delta_\eta \in h(\xi) \} & \text{if } \xi \in a \setminus a', \\
    h'(\xi) \cup \{ \eta \in a \setminus a' : \delta_\eta \in h'(\xi) \} & \text{if } \xi \in a' \setminus a.
\end{cases}
\]
Finally for \( \{\xi, \eta\} \in [b]^2 \) let

\[
(j) \quad j(\xi, \eta) = \begin{cases} 
i(\xi, \eta) & \text{if } \xi, \eta \in a, \\
i'(\xi, \eta) & \text{if } \xi, \eta \in a', \\
f(\xi, \eta) \cap b & \text{otherwise.}
\end{cases}
\]

(Observable that \( j \) is well-defined because 2.5.(2) holds.) We will write \( r = p + p' \) for the amalgamation of \( p \) and \( p' \).

**Lemma 2.7.** If \( p \) and \( p' \) are good twins then their amalgamation, \( r = p + p' \), is a common extension of \( p \) and \( p' \) in \( P_f \).

**Proof.** First we prove two claims.

**Claim 2.7.1.** Let \( \eta \in a \) and \( \delta \in a \cap a' \). Then \( \eta \in h(\delta) \) if and only if \( \delta_\eta \) is defined and \( \delta_\eta \in h(\delta) \). (Clearly, we also have a symmetric version of this statement for \( \eta \in a' \).)

**Proof of claim 2.7.1.** Assume first \( \eta \in h(\delta) \). Then \( \delta_\eta \) is defined and clearly \( \delta_\eta \in h(\delta) \) if \( \delta_\eta = \delta \). So assume \( \delta_\eta \neq \delta \). Since \( i(\delta_\eta, \delta) \subset a \cap a' \) and \( \max i(\delta_\eta, \delta) < \delta_\eta \) we have \( \eta \notin h[i(\delta_\eta, \delta)] \) by the choice of \( \delta_\eta \). Thus from \( p \in P_f \) we have

\[
(i) \quad \eta \notin h(\delta_\eta) \ast h(\delta).
\]

Then \( h(\delta_\eta) \ast h(\delta) \neq h(\delta_\eta) \cap h(\delta) \) by (\( i \)). Since \( \eta \in h(\delta) \) implies \( \delta_\eta < \delta \), we actually have \( h(\delta_\eta) \ast h(\delta) = h(\delta_\eta) \setminus h(\delta) \). Thus \( \delta_\eta \in h(\delta) \) by the definition of the operation \( \ast \).

On the other hand, if \( \delta_\eta \in h(\delta) \), then either \( \delta_\eta = \delta \) or \( h(\delta_\eta) \ast h(\delta) = h(\delta_\eta) \setminus h(\delta) \). Thus \( \eta \in h(\delta) \) because in the latter case again \( \eta \notin h[i(\delta_\eta, \delta)] \), hence (\( i \)) holds. □

**Claim 2.7.2.** If \( \xi \in a \cap a' \) then

\[
g(\xi) = h(\xi) \cup \{\eta \in a' \setminus a : \delta_\eta \in h(\xi)\} = h'(\xi) \cup \{\eta \in a \setminus a' : \delta_\eta \in h'(\xi)\}.
\]

**Proof of claim 2.7.2.** Conditions 2.5.1(i) and (iii) imply \( h(\xi) \cap a \cap a' = h'(\xi) \cap a \cap a' \) and so

\[
g(\xi) = h(\xi) \cup h'(\xi) = h(\xi) \cup ((a' \setminus a) \cap h'(\xi)).
\]

By claim 2.7.1 we have

\[
(a' \setminus a) \cap h'(\xi) = \{\eta \in a' \setminus a : \delta_\eta \in h'(\xi)\}.
\]

But by 2.5.(1) we have \( \delta_\eta \in h(\xi) \) iff \( \delta_\eta \in h'(\xi) \), hence it follows that

\[
g(\xi) = h(\xi) \cup \{\eta \in a' \setminus a : \delta_\eta \in h(\xi)\}.
\]

The second equality follows analogously. □

Next we check \( r \in P_f \). Conditions 2.1.(i)–(iii) for \( r \) are clear by the construction. So we should verify 2.1.(iv).

Let \( \xi \neq \eta \in b \) and \( \alpha \in g(\xi) \ast g(\eta) \). We need to show that \( \alpha \in g[j(\xi, \eta)] \). We will distinguish several cases.


Case 1. \( \xi, \eta \in a \) ( or \( \xi, \eta \in a' \)).

Since \( g(\xi) \cap a = h(\xi) \) and \( g(\eta) \cap a = h(\eta) \) we have \((g(\xi) * g(\eta)) \cap a = h(\xi) * h(\eta)\) by the definition of operation \(*\). Thus \((g(\xi) * g(\eta)) \cap a \subset h[i(\xi, \eta)] = g[j(\xi, \eta)] \cap a \subset g[j(\xi, \eta)]\). So we can assume that \( \alpha \in a' \setminus a \). We know that \( \delta_\alpha \) is defined because \( \alpha \in g(\xi) \cup g(\eta) \) is also satisfied. Since \( \alpha \in g(\xi) \) iff \( \delta_\alpha \in h(\xi) \) and \( \alpha \in g(\eta) \) iff \( \delta_\alpha \in h(\eta) \) by (\(\bullet\)) it follows that \( \delta_\alpha \in h(\xi) * h(\eta) \). Thus there is \( \nu \in i(\xi, \eta) \) such that \( \delta_\alpha \in h(\nu) \). But \( i(\xi, \eta) = j(\xi, \eta) \) and \( \alpha \in g(\nu) \) by (\(\bullet\)). Thus \( \alpha \in g[j(\xi, \eta)]\).

Case 2. \( \xi \in a \setminus a' \) and \( \eta \in a' \setminus a \).

We can assume that \( \alpha \in a \), since the \( \alpha \in a' \) case is done symmetrically.

Subcase 2.1. \( g(\xi) * g(\eta) = g(\eta) \setminus g(\xi) \).

Then \( \alpha \in g(\eta) \) and \( \eta \in g(\xi) \) so \( \delta_\alpha \) and \( \delta_\eta \) are both defined and \( \delta_\alpha \in h'(\eta) \), \( \delta_\eta \in h(\xi) \) hold, hence \( \alpha \leq \delta_\alpha < \eta \leq \delta_\eta < \xi \). But \( \alpha \) and \( \alpha' \) are good for \( f \), so by 1.1(a) we have \( \delta_\alpha \in f(\eta, \xi) \cap b = j(\xi, \eta) \). Thus \( \alpha \in h(\delta_\alpha) \subset g(\delta_\alpha) \subset g[j(\xi, \eta)] \) which was to be proved.

Subcase 2.2. \( g(\xi) * g(\eta) = g(\xi) \cap g(\eta) \) or \( g(\xi) * g(\eta) = g(\xi) \setminus g(\eta) \).

Since now \( \alpha \in g(\xi) * g(\eta) \subset g(\xi) \), by the definition of the operation \(*\) we have

\[
(1) \quad |\{\alpha, \xi\} \cap g(\eta)| = 1.
\]

Thus, by the definition of \( g(\eta) \), \( \delta^* = \min\{\delta \in a \cap a' : \alpha \in h(\delta) \vee \xi \in h(\delta)\} \) is well-defined and \( \delta^* < \eta \). If \( \delta^* < \xi \) then \( \alpha \in h(\delta^*) \) and by 1.1(a) we have \( \delta^* \in f(\xi, \eta) \cap b = j(\xi, \eta) \) for \( a \) and \( a' \) are good for \( f \), and so \( \alpha \in g[j(\xi, \eta)] \).

Thus we can assume \( \xi < \delta^* \). We know that \( \delta^* = \delta_\alpha \) or \( \delta^* = \delta_\xi \) by the choice of \( \delta^* \), but \( \delta_\alpha = \delta_\xi \) is impossible by (1). Thus

\[
(2) \quad |\{\alpha, \xi\} \cap h(\delta^*)| = 1.
\]

Since \( \alpha \in g(\xi) \) implies \( \alpha \in h(\xi) \) and we have \( \xi < \delta^* \), (2) implies \( \alpha \in h(\xi) * h(\delta^*) \) and so \( \alpha \in h[i(\xi, \delta^*)] \). But \( i(\xi, \delta^*) \subset f(\xi, \delta^*) \subset f(\xi, \eta) \) because \( a \) and \( a' \) are good for \( f \), so 1.1(b) or (c) may be applied. Consequently, we have \( i(\xi, \delta^*) \subset j(\xi, \eta) \) by (\(\bullet\)). Hence \( \alpha \in g[j(\xi, \eta)] \) which was to be proved.

Since we investigated all the cases it follows that \( r \) satisfies 2.1.(iv), that is, \( r \in P_f \).

Since \( r \leq p \) or \( q \) are clear from the construction, the lemma is proved. \( \square \)

Lemma 2.8. Every uncountable family \( F \) of conditions in \( P_f \) contains a couple of elements which are good twins. Consequently, \( P_f \) satisfies c.c.c.
Proof. By standard counting arguments $\mathcal{F}$ contains an uncountable subfamily $\mathcal{F}'$ such that every pair $p \neq p' \in \mathcal{F}'$ satisfies 2.5.(1)-(2). But $f$ is a $\Delta$-function, so there are $p \neq p' \in \mathcal{F}'$ such that $a^p$ and $a^{p'}$ are good for $f$, i.e. $p$ and $p'$ satisfies 2.5.(3), too. In other words, $p$ and $p'$ are good twins and so $r = p + p'$ is a common extension of $p$ and $p'$ in $P_f$. □

Let $\mathcal{G}$ be a $P_f$-generic filter over $V$. As in definition 2.3, let $H(\alpha) = \bigcup\{h^p(\alpha) : p \in \mathcal{G} \land \alpha \in a^p\}$ for $\alpha \in \omega_2$, and let $\tau_f$ be the topology on $\omega_2$ generated by $\{H(\alpha) : \alpha \in \omega_2\} \cup \{\omega_2 \setminus H(\alpha) : \alpha \in \omega_2\}$ as a subbase. Put $i = \bigcup\{i^p : p \in \mathcal{G}\}$.

Since $X_f$ is generated by a clopen subbase and $\max(H(\alpha)) = \alpha$ for each $\alpha \in \omega_2$ by 2.1(ii), it follows that $X_f$ is 0-dimensional and right separated in its natural well-order.

The following proposition is clear by 2.1.(iv) and by the definition of $H$ and $i$.

**Proposition.** $H(\alpha) \ast H(\beta) \subset H[i(\alpha, \beta)]$ for $\alpha < \beta < \omega_2$. So by 1.5 every $H(\alpha)$ is a compact open set in $X_f$.

**Definition 2.9.** For $\alpha \in \omega_2$ and $b \in [\alpha]^{<\omega}$ let

$$U(\alpha, b) = H(\alpha) \setminus H[b]$$

and let

$$B_\alpha = \{U(\alpha, b) : b \in [\alpha]^{<\omega}\}.$$ 

By theorem 1.5 every $H(\alpha)$ is compact and $B_\alpha$ is a neighborhood base of $\alpha$ in $X_f$. Thus $X_f$ is locally compact and the neighbourhood base $B_\alpha$ of $\alpha$ consists of compact open sets.

Unfortunately, the family $\mathcal{H} = \{H(\alpha) : \alpha < \omega_2\}$ is not coherent, so we can’t apply theorem 1.6 to prove that $X_f$ is countably tight. It will however follow from the following result.

**Lemma 2.10.** In $V^{P_f}$, if a sequence $\{z_\zeta : \zeta < \omega_1\} \subset H(\beta)$ converges to $\beta$, then there is some $\xi < \omega_1$ such that $\beta \in \{z_\zeta : \zeta < \xi\}$.

**Proof.** Assume on the contrary that for each $\xi < \omega_1$ we can find a finite subset $b_\xi \subset \beta$ such that $\{z_\zeta : \zeta < \xi\} \cap U(\beta, b_\xi) = \emptyset$, that is, $\{z_\zeta : \zeta < \xi\} \subset H[b_\xi]$.

Fix now a condition $p \in P_f$ which forces the above described situation and decides the value of $\beta$. Then, for each $\xi < \omega_1$ we can choose a condition $p_\xi \leq p$ which decides the value of $z_\zeta$ and $b_\xi$. We can assume that $\{a^p_\zeta : \zeta < \omega_1\}$ forms a $\Delta$-system with kernel $D$, $z_\zeta \in a^p_\zeta \setminus D$ and that $z_\xi < z_\eta$ for $\xi < \eta < \omega_1$.

**Claim.** Assume that $\xi < \eta < \omega_1$, $p_\xi$ and $p_\eta$ are good twins and $r = p_\xi + p_\eta$. Then $r \vdash "z_\xi \in H[D \cap \beta]^\theta."$
Proof of the claim. Indeed, \( z_\xi \in h^r[b_\eta] \) because \( r \leq p_\xi, p_\eta \). Since \( z_\xi \in a^{p_\xi} \setminus a^{p_\eta} \), (●) and claim 2.7.2 imply that \( z_\xi \in h^r[b_\eta] \) holds if and only if \( \delta_z \xi \in D = a^{p_\xi} \cap a^{p_\eta} \) is defined and \( \delta_z \xi \in h^{p_\eta}[b_\eta] \). Since \( b_\eta \subset \beta \), we also have \( \delta_z \xi < \beta \) and so \( z_\xi \in h^r[D \cap \beta] \). □

Applying lemma 2.8 to appropriate final segments of \( \{p_\xi : \xi < \omega_1 \} \) we can choose, by induction on \( \mu < \omega_1 \), pairwise different ordinals \( \mu < \xi_\mu < \eta_\mu < \omega_1 \) with \( \eta_\mu < \xi_\mu \) if \( \mu < \nu \) such that \( p_\xi \mu, p_\eta \mu \) are good twins. Let \( r_\mu = p_\xi \mu + p_\eta \mu \). Since \( P_f \) satisfies c.c.c there is a condition \( q \models \{\mu \in \omega_1 : r_\mu \in G\} = \omega_1 \). Thus, by the claim \( q \models \{\{\xi : \xi < \omega_1\} \cap H[D \cap \beta] = \omega_1\} \). By lemma 2.10 there is some \( \xi < \omega_1 \) with \( \beta \in \{\{\xi : \xi < \omega_1\} \subset Y \) contradicting \( \beta \in Y \). □

Corollary 2.11. \( t(X_\xi) = \omega \).

Proof. Assume on the contrary that \( \alpha \in \omega_2 \) and \( t(\alpha, X_\xi) = t(\alpha, H(\alpha)) = \omega_1 \). Then there is an \( \omega \)-closed set \( Y \subset H(\alpha) \) that is not closed. Since the subspace \( H(\alpha) \) is compact and right separated and so it is pseudo-radial, for some regular cardinal \( \kappa \) there is a sequence \( \{z_\xi : \xi < \kappa\} \subset Y \) which converges to some point \( \beta \in H(\alpha) \setminus Y \). Since \( Y \) is \( \omega \)-closed and \( |Y| \leq |H(\alpha)| = \omega_1 \) we have \( \kappa = \omega_1 \). By lemma 2.10 there is some \( \xi < \omega_1 \) with \( \beta \in \{\{\xi : \xi < \omega_1\} \subset Y \) contradicting \( \beta \notin Y \). □

Lemma 2.12. In \( V^{P_f} \), for each uncountable \( A \subset X_\xi \) there is \( \beta \in \omega_2 \) such that \( |A \cap H(\beta)| = \omega_1 \).

Proof. Assume that \( p \models \text{“}A = \{\alpha_\xi : \xi < \omega_1\} \in [\omega_1]^{\omega_1}\text{”} \). For each \( \xi < \omega_1 \) pick \( p_\xi \leq p \) and \( \alpha_\xi \in \omega_2 \) such that \( p_\xi \models \alpha_\xi = \alpha_\xi \). Since \( P_f \) satisfies c.c.c we can assume that the \( \alpha_\xi \) are pairwise different. Let \( \text{sup}\{\alpha_\xi : \xi < \omega_1\} < \beta < \omega_2 \). Now for each \( \xi < \omega_1 \) define the condition \( q_\xi \leq p_\xi \) by the stipulations \( a^{q_\xi} = a^{p_\xi} \cup \{\beta\}, h^{q_\xi}(\beta) = a^{p_\xi} \) and \( i^{q_\xi}(\nu, \beta) = \emptyset \) for \( \nu \in a^{p_\xi} \). Then \( q_\xi \in P_f \) and \( q_\xi \models \alpha_\xi \in H(\beta) \). But \( P_f \) satisfies c.c.c, so there is \( q \leq p \) such that \( q \models \text{“}|\{\xi \in \omega_1 : q_\xi \in G\}| = \omega_1\text{”} \). Thus \( q \models \text{“}|A \cap H(\beta)| = \omega_1\text{”} \). □

Since every \( H(\beta) \) is compact, lemma 2.12 above clearly implies that \( X_\xi \) is \( \omega_1 \)-compact, i.e. every subset \( S \subset X_\xi \) of size \( \omega_1 \) has a complete accumulation point.

Now we start to work on (iii): in \( V^{P_f} \) the closure of any countable subset \( Y \) of \( X_\xi \) is either compact or it contains a final segment of \( \omega_2 \). If \( Y \) is also in the ground model, then actually the second alternative occurs and this follows easily from the next lemma.

Lemma 2.13. If \( p \in P_f, \beta \in a^p, b \subset a^p \cap \beta, \alpha \in \beta \setminus a^p \), then there is a condition \( q \leq p \) such that \( \alpha \in u^p(\beta, b) \).

Proof. Define the condition \( q \leq p \) by the following stipulations: \( a^q = a^p \cup \{\alpha\} \), \( h^q(\alpha) = \{\alpha\} \),

\[
h^q(\nu) = \begin{cases} h^p(\nu) \cup \{\alpha\} & \text{if } \beta \in h^p(\nu) \\ h^p(\nu) & \text{if } \beta \notin h^p(\nu) \end{cases}
\]
for \( \nu \in a^p \), and let \( i^q \supseteq i^p \) and \( i^q(\alpha, \nu) = \emptyset \) for \( \nu \in a^p \).

To show \( q \in P_f \) we need to check only 2.1(iv). Assume that \( \alpha \in h^p(\nu) \ast h^p(\mu) \). Then by the construction of \( q \) we have \( \beta \in h^p(\nu) \ast h^p(\mu) \). Thus there is \( \xi \in i^p(\nu, \mu) \) with \( \beta \in h^p(\xi) \). But then \( \alpha \in h^p(\xi) \), so by \( i^q(\nu, \mu) = i^p(\nu, \mu) \) we have \( \alpha \in h^q[i^q(\nu, \mu)] \). In view of \( h^p(\nu) \ast h^p(\mu) \subset h^p[i^p(\nu, \mu)] \) we are done. Thus \( q \in P_f \), \( q \leq p \) and clearly \( \alpha \in u^q(\beta, b) \), so we are done. \( \square \)

This lemma yields the following corollary.

**Corollary 2.14.** If \( Z \in \big[ \omega_2 \big]^\omega \cap V \) and \( \beta \in \omega_2 \setminus \sup Z \) then \( \beta \in Z \).

**Proof.** Let \( U(\beta, b) \) be a neighbourhood of \( \beta \), \( b \in \big[ \beta \big]^{<\omega} \). Since \( p \models \text{“} U(\beta, b) \supset u^p(\beta, b) \text{”} \) for each \( p \in P_f \) and the set
\[
D_{\beta, b, Z} = \{ q \in P_f : u^q(\beta, b) \cap Z \neq \emptyset \}
\]
is dense in \( P_f \) by the previous lemma, it follows that \( U(\beta, b) \) intersects \( Z \). Consequently \( \beta \in Z \). \( \square \)

The space \( X_f \) is right separated, i.e. scattered, so we can consider its Cantor-Bendixon hierarchy. According to corollary 2.14 for each \( \alpha < \omega_2 \) the set \( A_\alpha = [\omega_\alpha, \omega_\alpha + \omega] \) is a dense set of isolated points in \( X_f [\omega_2 \setminus \omega_\alpha] \). Thus the \( \alpha \)-th Cantor-Bendixon level of \( X_f \) is just \( A_\alpha \). Therefore \( X_f \) is a thin very tall, locally compact scattered space in the sense of \( [8] \). Let us emphasize that CH was not needed to get this result, hence we have also given an alternative proof of the main result of \( [3] \).

Now we continue to work on proving property 2.4(iii) of \( X_f \).

Given \( p \in P_f \) and \( b \subset a^p \) let \( p[b] = \left\langle b, h, i^p \big[ \big[ b \big] \big] \right\rangle \) where \( h \) is the function with \( \text{dom}(h) = b \) and \( h(\xi) = h^p(\xi) \cap b \) for \( \xi \in b \). Let us remark that \( p[b] \) in not necessarily in \( P_f \). In fact, \( p[b] \in P_f \) if and only if \( i^p(\xi, \eta) \subset b \) for each \( \xi \neq \eta \in b \). Especially, if \( b \) is an initial segment of \( a^p \), then \( p[b] \in P_f \). The order \( \leq \) of \( P_f \) can be extended in a natural way to the restrictions of conditions: if \( p \) and \( q \) are in \( P_f \), \( b \subset a^p \), \( c \subset a^q \), define \( p[b] \leq q[c] \) iff \( b \supseteq c \), \( h^p(\xi) \cap c = h^q(\xi) \cap c \) for each \( \xi \in c \), and \( i^p[c] = i^q[c] \).

Clearly if \( p[b] \in P_f \) and \( q[c] \in P_f \) then the two definitions of \( \leq \) coincide.

**Definition 2.15.** Let \( p, p' \in P_f \) with \( a^p = a^{p'} \). We write \( p \prec p' \) if for each \( \alpha \in a^p \) and \( b \subset a^p \cap \alpha \) we have \( u^p(\alpha, b) \subset u^{p'}(\alpha, b) \).

The following technical result will play a crucial role in the proof of 2.4(iii). Part (c) in it will enable us to “insert” certain things in \( H(\gamma_0) \) in a non-trivial way. But there is a price we have to pay for this: this is the point where the coherency of the \( H(\alpha) \) has to be abandoned. Part (d) will be needed in section 3.

**Lemma 2.16.** Assume that \( s = \langle a^s, h^s, s^s \rangle \in P_f \), \( a^s = S \cup E \cup F \), \( Q \subset S \), \( S <_{on} E <_{on} F \), \( E = \{ \gamma_i : i < k \} \), \( \gamma_0 < \gamma_1 < \cdots < \gamma_{k-1} \), \( F = \{ \gamma_{i,0}, \gamma_{i,1} : i < k \} \), moreover
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(1) \( \forall i < k \ h^s(\gamma_{i,0}) \cap h^s(\gamma_{i,1}) = h^s[Q \cup E] \),
(2) \( \forall i < k \ \forall \zeta \in S \ f(\zeta, \gamma_{i}) = f(\zeta, \gamma_{i,0}) = f(\zeta, \gamma_{i,1}) \).

Then there is a condition \( r = \langle a^r, h^r, i^r \rangle \) with \( a^r = S \cup E \) such that

(a) \( r \leq s[S] \),
(b) \( r \leq s[(Q \cup E)] \),
(c) \( S \setminus h^s[Q \cup E] \subseteq h^r(\gamma_0) \),
(d) \( s[(S \cup E)] < r \).

Proof. Let \( a^r = S \cup E \) and write \( C = S \setminus h^s[Q \cup E] \). For \( \xi \in a^r \) we set

\[
h^r(\xi) = \begin{cases} 
    h^s(\xi) \cup C & \text{if } \xi = \gamma_i \text{ and } \gamma_0 \in h^s(\gamma_i), \\
    h^s(\xi) & \text{otherwise}.
\end{cases}
\]

For \( \xi \neq \eta \in a^r \) we let

\[
i^r(\xi, \eta) = \begin{cases} 
i^s(\xi, \eta) & \text{if } \xi, \eta \in Q \cup E \text{ or } \xi, \eta \in S, \\
f(\xi, \eta) \cap a^r & \text{otherwise}.
\end{cases}
\]

Finally let \( r = \langle a^r, h^r, i^r \rangle \).

We claim that \( r \) satisfies the requirements of the lemma. (a), (b) and (c) are clear from the definition of \( r \), once we establish that \( r \in P_f \). To see that it suffices to check only 2.1.(iv) because the other requirements are clear from the construction of \( r \). So let \( \xi < \eta \in a^r \). We have to show \( h^r(\xi) * h^r(\eta) \subset h^r[i^r(\xi, \gamma_i)] \).

If \( \xi, \eta \in S \), then \( h^r(\xi) * h^r(\eta) \subset h^r[i^r(\xi, \eta)] \) holds because \( r[S] = s[S] \in P_f \).

So we can assume that \( \eta = \gamma_i \) for some \( i < k \).

Case 1. \( \xi \in S \).

Subcase 1.1. \( \xi \notin h^r(\gamma_i) \), hence \( h^r(\xi) * h^r(\gamma_i) = h^r(\xi) \cap h^r(\gamma_i) \).

In this case we also have \( h^s(\xi) * h^s(\gamma_i) = h^s(\xi) \cap h^s(\gamma_i) \) and so

(3) \( h^s(\xi) \cap h^s(\gamma_i) \subseteq h^s[i^s(\xi, \gamma_i)] \subseteq h^r[i^r(\xi, \gamma_i)] \)

for \( i^r(\xi, \gamma_i) \supseteq i^s(\xi, \gamma_i) \). If \( \gamma_0 \notin h^s(\gamma_i) \), then \( h^r(\gamma_i) = h^s(\gamma_i) \) and since \( h^r(\xi) = h^s(\xi) \) we have \( h^r(\xi) * h^r(\gamma_i) = h^s(\xi) \cap h^s(\gamma_i) \subseteq h^r[i^r(\xi, \gamma_i)] \) by (3). Assume now that \( \gamma_0 \in h^s(\gamma_i) \). Thus \( h^r(\gamma_i) = h^s(\gamma_i) \cup C \), and so \( \xi \notin C \), that is \( \xi \in h^s[Q \cup E] \). Then

(4) \( h^r(\xi) * h^r(\gamma_i) = h^r(\xi) \cap h^r(\gamma_i) = (h^s(\xi) \cap h^s(\gamma_i)) \cup (h^s(\xi) \cap C) \).

By (3) above it is enough to show that \( h^s(\xi) \cap C \subset h^r[i^r(\xi, \gamma_i)] \). Since \( h^s(\xi) \cap C = \emptyset \) for \( \xi \in Q \) we can assume that \( \xi \notin Q \). By (1) \( h^s[Q \cup E] = h^s(\gamma_{i,0}) \cap h^s(\gamma_{i,1}) \) and so

(5) \( h^s(\xi) \cap C = h^s(\xi) \setminus h^s[Q \cup E] = (h^s(\xi) \setminus h^s(\gamma_{i,0})) \cup (h^s(\xi) \setminus h^s(\gamma_{i,1})) \).

Since \( \xi \in h^s[Q \cup E] \subset h^s(\gamma_{i,j}) \) for \( j = 0, 1 \) we have

(6) \( h^s(\xi) \setminus h^s(\gamma_{i,j}) = h^s(\xi) * h^s(\gamma_{i,j}) \subset h^s[i^s(\xi, \gamma_{i,j})] \).
By (ii), \( i^* (\xi, \gamma_{i,j}) \subset f (\xi, \gamma_{i,j}) \cap a^s = f (\xi, \gamma_i) \cap a^s \). Since \( \xi \notin Q \) it follows that 
\( f (\xi, \gamma_i) \cap a^s = i^* (\xi, \gamma_i) \). Thus from (5) we obtain
\[
(6) \quad h^s (\xi) \setminus h^s (\gamma_{i,j}) \subset h^s [i^* (\xi, \gamma_i)] = h^r [i^* (\xi, \gamma_i)].
\]
Putting (4) and (6) together we get 
\( h^s (\xi) \cap C \subset h^r [i^* (\xi, \gamma_i)] \) which was to be proved.

**Subcase 1.2.** \( \xi \in h^r (\gamma_i) \), hence \( h^r (\xi) \ast h^r (\gamma_i) = h^r (\xi) \setminus h^r (\gamma_i) \).

If \( \xi \in h^s (\gamma_i) \) then since \( h^s (\xi) = h^r (\xi) \) we have 
\( h^r (\xi) \ast h^r (\gamma_i) = h^r (\xi) \setminus h^r (\gamma_i) \subset h^s (\xi) \setminus h^s (\gamma_i) \subset h^s [i^* (\xi, \gamma_i)] \subset h^r [i^* (\xi, \gamma_i)] \) and we are done.

So we can assume that \( \xi \notin h^s (\gamma_i) \) and so \( h^r (\gamma_i) \neq h^s (\gamma_i) \). By the construction of \( r \), we have \( \gamma_0 = h^s (\gamma_i) \), \( h^r (\gamma_i) = h^s (\gamma_i) \cup C \) and so \( \xi \in C \), i.e. \( \xi \notin h^s [Q \cup E] \). By (i) we can assume that \( \xi \notin h^s (\gamma_{i,0}) \). So \( s \in P^r \) implies
\[
(7) \quad h^s (\xi) \cap h^s (\gamma_{i,0}) = h^s (\xi) \ast h^s (\gamma_{i,0}) = h^s [i^* (\xi, \gamma_{i,0})]
\]
We have
\[
(8) \quad h^r (\xi) \ast h^r (\gamma_i) = h^s (\xi) \setminus (h^s (\gamma_i) \cup C) \subset h^s (\xi) \setminus C
\]
and applying (i) again
\[
(9) \quad h^s (\xi) \setminus C = h^s (\xi) \cap h^s [Q \cup E] \subset h^s (\xi) \setminus h^s (\gamma_{i,0}).
\]
By (ii), \( i^* (\xi, \gamma_{i,0}) \subset f (\xi, \gamma_{i,0}) \cap a^s = f (\xi, \gamma_i) \cap a^s \). Since \( \xi \notin Q \subset h^s [Q \cup E] \) it follows that 
\( f (\xi, \gamma_i) \cap a^s = i^* (\xi, \gamma_i) \) and so (7)–(9) together yield
\[
(10) \quad h^r (\xi) \ast h^r (\gamma_i) \subset h^r [i^* (\xi, \gamma_i)]
\]
which was to be proved.

**Case 2.** \( \xi = \gamma_j \) for some \( j < i \).

Since \( i^* (\gamma_j, \gamma_i) = i^* (\gamma_j, \gamma_i) \) we have
\[
(11) \quad h^s (\gamma_j) \ast h^s (\gamma_i) \subset h^r [i^* (\gamma_j, \gamma_i)]
\]
It is easy to check that
\[
(12) \quad h^r (\gamma_j) \ast h^r (\gamma_i) = \begin{cases} h^s (\gamma_j) \ast h^s (\gamma_i) & \text{if } \gamma_0 \notin h^s (\gamma_j) \ast h^s (\gamma_i), \\ h^s (\gamma_j) \ast h^s (\gamma_i) \cup C & \text{if } \gamma_0 \in h^s (\gamma_j) \ast h^s (\gamma_i). \end{cases}
\]
So we are done if \( \gamma_0 \notin h^s (\gamma_j) \ast h^s (\gamma_i) \). Assume \( \gamma_0 \in h^s (\gamma_j) \ast h^s (\gamma_i) \). Then there is
\( \gamma_l \in i^* (\gamma_j, \gamma_i) \) with \( \gamma_0 \in h^s (\gamma_l) \). Thus, by the construction of \( r \) we have
\[
(13) \quad C \subset h^r (\gamma_l) \subset h^r [i^* (\gamma_j, \gamma_i)]
\]
But (12) and (13) together imply what we wanted.

Thus we proved \( r \in P^r \).

Clearly \( r \) satisfies 2.16.(a)–(c). To check 2.16.(d) write \( s^r = s \mid (S \cup E) \) and let \( \alpha \in S \cup E \) and \( b \subset (S \cup E) \cap \alpha \). We need to show that \( u^r (\alpha, b) \subset u^r (\alpha, b) \). Since
$S \cup E$ is an initial segment of $a^*$, we have $u^*(\alpha, b) = u^*(\alpha, b)$. If $\alpha \in S$, then also $u^*(\alpha, b) = u^*(\alpha, b)$, so we can assume $\alpha \in E$.

Let $\xi \in u^*(\alpha, b) = u^*(\alpha, b)$. Then $\xi \in h^*(\alpha) \subset h^*[E]$, and hence $\xi \notin C$. But $h^*[b] \setminus h^*[b] \subset C$, more precisely, it is empty or just $C$. Since $\xi \in u^*(\alpha, b)$, it follows that $\xi \notin h^*[b]$ and so $\xi \notin h^*[b]$ because $\xi \notin C$. Thus $\xi \in u^*(\alpha, b)$. Hence $r$ satisfies (d).

The lemma is proved. \(\square\)

**Lemma 2.17.** In $V^{P_f}$, if $Y \subset \omega_2$ is countable, then either $\overline{Y}$ is compact or $|\omega_2 \setminus \overline{Y}| \leq \omega_1$.

**Proof.** Assume that $1_{P_f} \models " \overline{Y} = \{ \hat{y}_n : n \in \omega \} \subset \omega_2"$. For each $n \in \omega$ fix a maximal antichain $C_n \subset P_f$ such that for each $p \in C_n$ there is $\alpha \in a^p$ with $p \models " \hat{y}_n = \check{\alpha}"$. Let $A = \bigcup \{ a^p : p \in \bigcup_{n<\omega} C_n \}$. Since every $C_n$ is countable by c.c.c we have $|A| = \omega$.

Assume also that $1_{P_f} \models " \overline{\omega_2} is not compact", that is, $Y$ cannot be covered by finitely many $H(\delta)$ in $V^{P_f}$.

Let

$$I = \{ \delta < \omega_2 : \exists p \in P_f \ p \models " \delta \notin \overline{Y}" \}.$$ 

Clearly $1_{P_f} \models \omega_2 \setminus I \subset \overline{Y}$. Since $\omega_2 \setminus I$ is in the ground model, by corollary 2.14 it is enough to show that $\omega_2 \setminus I$ is infinite. Actually we will prove much more:

**Claim.** $I$ is not stationary in $\omega_2$.

Assume on the contrary that $I$ is stationary. Let us fix, for each $\delta \in I$, a condition $p_\delta \in P_f$ and a finite set $D_\delta \in [\delta]^{<\omega}$ such that $p_\delta \models " \hat{Y} \cap U(\delta, D_\delta) = \emptyset"$. For each $\delta \in I$ let $Q_\delta = a^{p_\delta} \cap \delta$ and $E_\delta = a^{p_\delta} \setminus \delta$. We can assume that $D_\delta \subset Q_\delta$ and sup $A < \delta$ for each $\delta \in I$.

Let $B_\delta = \text{cl}_f (A \cup Q_\delta, E_\delta)$ for $\delta \in I$ (see 1.3). For each $\delta \in I$ the set $B_\delta$ is countable with sup $B_\delta = \sup (A \cup Q_\delta)$, so we can apply Fodor’s pressing down lemma and CH to get a stationary set $J \subset I$ and a countable set $B \subset \omega_2$ such that $B_\delta = B$ for each $\delta \in J$.

By thinning out $J$ and with a further use of CH we can assume that for a fixed $k \in \omega$ we have

1. $E^\delta = \{ \gamma_i^\delta : i < k \}$ for $\delta \in J$, $\gamma_0^\delta < \gamma_1^\delta < \cdots < \gamma_{k-1}^\delta$,

2. $f(\xi, \gamma_i^\delta) = f(\xi, \gamma_i^{\delta'})$ for each $\xi \in B$, $\delta, \delta' \in J$ and $i < k$.

Let $\delta = \min J$, $D = D_\delta$, $E = E_\delta$, $p = p_\delta$, $Q = Q_\delta$. By lemma 1.2 there are ordinals $\delta_j \in J$ with $\delta < \delta_0 < \delta_1 < \cdots < \delta_{2k-1}$ such that

$$(\ast) \quad B \cup E \subset \bigcap \{ f(\xi, \eta) : \xi \in E_{\delta_i}, \eta \in E_{\delta_j}, i < j < 2k \}. $$

For $i < k$ and $j < 2$ let $\gamma_i = \gamma_i^\delta$ and $\gamma_{i,j} = \gamma_{2i+j}^\delta$. Let $F = \{ \gamma_{i,j} : i < k, j < 2 \}$.

We know that $a^p = Q \cup E$. Define the condition $q \in P_f$ by the following stipulations:
Since \( a^q = a^p \cup F \) and \( q \leq p \),

(i) \( h^q(\gamma_{ij}) = \{\gamma_{ij}\} \cup a^p \) for \( \langle i, j \rangle \in k \times 2 \).

(ii) \( i^q(\gamma_{i0}) = a^p \) for \( \langle i_0, j_0 \rangle \neq \langle i_1, j_1 \rangle \in k \times 2 \).

(iii) \( r \ni (\gamma_{i0}, \gamma_{i1}) = 0 \) if \( \langle i, j \rangle \in k \times 2 \).

Since \( a^p \subseteq B \cup E \), \((*)\) implies that \( q \in P_f \).

Since \( 1_{P_f} \vdash \cdot \ H[Q \cup E] \not\vdash \cdot \ Y \cdot \), there is a condition \( t \leq q \), a natural number \( n \) and an ordinal \( \alpha \) such that \( t \vdash \cdot \alpha = \dot{y}_n \cdot \) but \( \alpha \in a^s \setminus h^t[Q \cup E] \). Since \( C_n \) is a maximal antichain we can assume that \( t \leq v \) for some \( v \in C_n \).

Let \( s = t \big| (B \cup E \cup F) \). Then \( s \in P_f \) because for each pair \( \xi < \eta \in a^s \) if \( \xi \in B \) then \( i^q(\xi, \eta) \subset f(\xi, \eta) \subset B \) and so \( i^q(\xi, \eta) \subset a^s \), and if \( \xi, \eta \in E \cup F \) then \( i^q(\xi, \eta) = i^q(\xi, \eta) \subset Q \cup E \subset a^s \). Moreover \( s \leq v \) because \( a^s \subset B \). Thus \( s \vdash \cdot \alpha = \dot{y}_n \cdot \) and \( \alpha \notin h^s[Q \cup E] \). Let \( S = a^s \cap B \).

Since \( i^q(\gamma_{i0}, \gamma_{i1}) = i^q(\gamma_{i0}, \gamma_{i1}) = Q \cup E \), and \( \gamma_{i,j} \notin h^s(\gamma_{i,j-1}) \) we have

\[
(14) \quad h^s(\gamma_{i0}) \cap h^s(\gamma_{i1}) = h^s(\gamma_{i0}) * h^s(\gamma_{i1}) \subset h^s[Q \cup E].
\]

Moreover, if \( \eta \in Q \cup E \) and \( j < 2 \) then \( \xi \in h^s(\gamma_{i,j}) \subset h^s(\gamma_{i,j}) \) and \( i^q(\xi, \gamma_{i,j}) = 0 \), consequently

\[
(15) \quad h^s(\xi) \setminus h^s(\gamma_{i,j}) = h^s(\xi) * h^s(\gamma_{i,j}) \subset h^s[i^q(\xi, \gamma_{i,j})] = 0.
\]

Putting (14) and (15) together it follows that \( h^s(\gamma_{i0}) \cap h^s(\gamma_{i1}) = h^s[Q \cup E] \).

Thus we can apply 2.16 to get a condition \( r \) such that \( r \leq s \big| (Q \cup E) = p \), \( r \leq s[S \leq v \) and \( \alpha \in S \setminus h^s[Q \cup E] \subset h^s(\gamma_0) = h^s(\delta) \). Since \( D \subset Q \), we have \( \alpha \in h^s(\delta) \setminus h^s[D] \).

Thus

\[
r \vdash \cdot \alpha \in \dot{Y} \cap (H(\delta) \setminus H[D]).
\]

On the other hand \( r \vdash \cdot \dot{Y} \cap (H(\delta) \setminus H[D]) = 0 \) because \( r \leq p \). With this contradiction the claim is proved and this completes the proof of the lemma.

Clearly, lemma 2.17 implies that \( X_f \) is countably compact.

**Corollary 2.18.** If \( F \subset X \) is closed (or open), then either \( |F| \leq \omega_1 \) or \( |X \setminus F| \leq \omega_1 \).

**Proof.** If \( |F| = \omega_2 \) then \( F \) is not compact, so by lemma 1.7 \( F \) contains a free sequence \( Y \) with non-compact closure. But \( F \) is initially \( \omega_1 \)-compact and countably tight, so \( Y \) is countable. Consequently, we have \( |\omega_2 \setminus \dot{Y}| \leq \omega_1 \) by lemma 2.17 and so \( |X \setminus F| \leq \omega_1 \).

**Corollary 2.19.** \( X_f \) is normal and \( z(X_f) = \text{hd}(X_f) \leq \omega_1 \).

**Proof.** To show that \( X_f \) is normal let \( F_0 \) and \( F_1 \) be disjoint closed subsets of \( X_f \). Since at least one of them is compact by lemma 2.17 they can be separated by open subsets of \( X_f \) because \( X_f \) is \( T_3 \).
Concerning the hereditarily density of $X_f$ it follows easily from corollary 2.18 that $X_f$ does not contain a discrete subspace of size $\omega_2$. But $X_f$ is right separated, so all the left separated subspaces of $X_f$ are of size $\leq \omega_1$, that is, $\chi(X) \leq \omega_1$. □

Thus theorem 2.4 is proved. □

We know that the space $X_f$ is not automatically hereditarily separable, so the following question of Arhangel’skiǐ, [1, problem 5], remains unanswered: Is it true in ZFC that every hereditarily separable, initially $\omega_1$-compact space is compact?

As we have seen our space $X_f$ is normal. However, we don’t know whether $X_f$ is or can be made hereditarily normal, i.e. $T_5$. This raises the following problem.

**Problem 1.** Is it provable in ZFC that every $T_5$, countably tight, initially $\omega_1$-compact space is compact?

### 3. Making $X_f$ Frechet-Uryson

In [1, problem 12] Arhangel’skiǐ asks if it is provable in ZFC that a normal, first countable initially $\omega_1$-compact space is necessarily compact. We could not completely answer this question, but in this section we show that the Frechet-Uryson property (which is sort of half-way between countable tightness and first countability) in not enough to get compactness.

To achieve that we want to find a further extension of the model $V^{P_f}$ in which $X_f$ becomes Frechet-Uryson but its other properties are preserved, for example, $X_f$ remains initially $\omega_1$-compact and normal. Since $X_f$ is countably tight and $\chi(X_f) \leq \omega_1$ it is a natural idea to make $X_f$ Frechet-Uryson by constructing a generic extension of $V^{P_f}$ in which $X_f$ remains countably tight and $p > \omega_1$, i.e. $MA_{\omega_1}(\sigma$-centered) holds (see [9, theorem 8]).

The standard c.c.c poset $P$ which forces $p > \omega_1$ is obtained by a suitable finite support iteration of length $2^{\omega_1}$. During this iteration in the $\alpha^{th}$ step we choose a non-principal filter $F \subset \mathcal{P}(\omega)$ generated by at most $\omega_1$ elements and we add a new subset $A$ of $\omega$ to the $\alpha^{th}$ intermediate model so that $A$ is almost contained in every element of $\mathcal{F}$, i.e. $A \setminus F$ is finite for each $F \in \mathcal{F}$. It is well-known and easy to see that $P$ has property $K$. Thus, by theorem 3.1 below, $X_f$ remains countably tight in $V^{P_f+R}$ and so indeed $X_f$ becomes Frechet-Uryson in that model. Moreover, theorem 3.2 implies that the $\omega_1$-compactness of $X_f$ is also preserved. Unfortunately, we could not prove that forcing with $P$ preserves the countable compactness of $X_f$.

So, instead of aiming at $p > \omega_1$ we will consider only those filters during the iteration which are needed in proving the Frechet-Uryson property of $X_f$. As we will see, we can handle these filters in such a way that our iterated forcing $R$ preserves not only the countable compactness of $X_f$ but also property (iii) from theorem 2.4: in $V^{P_f+R}$ the closure of any countable subset of $X_f$ is either compact or contains a
Theorem 3.1. If the topological space $X$ is right separated, compact, countably tight and the poset $R$ has property $K$ then forcing with $R$ preserves the countable tightness of $X$.

Proof. First we recall that $X$ remains compact (and clearly right separated) in any extension of the ground model by $[5,\text{lemma 7}].$ Since $F(X) = t(X)$ for compact spaces, assume indirectly that $1 \Vdash "\{\dot{z}_\xi : \xi < \omega_1\} \subset X \text{ is a free sequence}".$ For every $\xi < \omega_1$ we have that $1 \Vdash "\{\check{z}_\xi : \zeta < \xi\} \text{ and } \{\check{z}_\xi : \xi \leq \zeta < \omega_1\} \text{ are disjoint compact sets}"$ and $X$ is $T_3,$ so we can fix a condition $p_\xi \in P,$ open sets $U_\xi$ and $V_\xi$ from the ground model and a point $z_\xi \in X$ such that $\bigcup_{\xi} \bigcap V_\xi = \emptyset$ and

$$p_\xi \Vdash "\{\check{z}_\xi : \zeta < \xi\} \subset U_\xi, \{\check{z}_\xi : \xi \leq \zeta < \omega_1\} \subset V_\xi \text{ and } \dot{z}_\xi = z_\xi."$$

Since $R$ has property $K,$ there is an uncountable set $I \subset \omega_1$ such that the conditions $\{p_\xi : \xi \in I\}$ are pairwise compatible.

We claim that the sequence $\{z_\xi : \xi \in I\}$ is an uncountable free sequence in the ground model which contradicts $F(X) = t(X) = \omega.$ Indeed let $\xi \in I.$ If $\zeta \in I \cap \xi,$ then $p_\xi$ and $p_\zeta$ has a common extension $q$ in $P$ and we have

$$q \Vdash "\dot{z}_\zeta = z_\zeta \text{ and } \{\check{z}_\eta : \eta < \xi\} \subset \dot{U}_\xi."$$

Hence $z_\zeta \in U_\zeta.$ Similarly for $\zeta \in I \setminus \xi$ we have $z_\zeta \in V_\zeta.$ Therefore $\overline{U}_\xi$ and $\overline{V}_\xi$ separate $\{z_\xi : \xi \in I \cap \xi\}$ and $\{z_\xi : \xi \in I \setminus \xi\}$ which implies that $\{z_\xi : \xi \in I\}$ is really free. \qed

Theorem 3.2. Forcing with a c.c.c poset $R$ over $V^{P_f}$ preserves property 2.4.(ii) of the space $X_f,$ i.e. for each uncountable $A \subset X_f$ there is $\beta \in \omega_2$ such that $A \cap H(\beta)$ is uncountable.

Proof. We work in $V^{P_f}.$ Assume that $r \Vdash R "$\hat{A} = \{\hat{\alpha}_\xi : \xi < \omega_1\} \in \left[ X_f \right]^{\omega_1}\".$ For each $\xi < \omega_1$ pick a condition $r_\xi \leq r$ from $R$ which decides the value of $\hat{\alpha}_\xi,$ $r_\xi \Vdash R "$\hat{\alpha}_\xi = \alpha_\xi\".$ Since $R$ satisfies c.c.c, $\{\alpha_\xi : \xi \in \omega_1\}$ is uncountable, hence as $X_f$ has property (ii) in $V^{P_f},$ for some $\beta \in \omega_2$ the set $I = H(\beta) \cap \{\alpha_\xi : \xi < \omega_1\}$ is also uncountable. Since $R$ satisfies c.c.c there is a condition $q \leq r$ in $R$ such that $q \Vdash R "$\|\{\xi \in I : r_\xi \in G\}\| = \omega_1\"$, where $G$ is the $R$-generic filter over $V^{P_f}.$ Thus $q \Vdash R "$\|A \cap H(\beta)\| = \omega_1\"$ which was to be proved. \qed

Of course, theorem 3.2 implies that forcing with any c.c.c poset $R$ preserves the $\omega_1$-compactness of $X_f.$ It is much harder to find a property of a poset $R$ which guarantees that forcing with $R$ over $V^{P_f}$ preserves the countable compactness of $X_f.$ We will proceed in the following way. In definition 3.3 we formulate when a poset
\( \hat{R} \) is called \textit{nice (over} \( P_f \), and then in theorem 3.4 we show that forcing with a nice poset preserves not only the countable compactness of \( X_f \), but also property 2.4(iii): the closure of any countable subset of \( X_f \) is either compact or contains a final segment of \( \omega_2 \). Finally, in definitions 3.5 and 3.6 we describe a class of finite support iterated forcings, which by theorem 3.7 are nice and have property K , and then in theorem 3.9 we show that forcing with a suitable member of this class makes \( X_f \) Frechet-Uryson.

**Definition 3.3.** Let \( \hat{R} \) be a name for a poset in \( V^{P_f} \). We say that \( \hat{R} \) is \textit{nice (over} \( P_f \) if there is a dense subset \( \mathcal{D} \) of the iteration \( P_f * \hat{R} \) with the following property: If \( \langle p_0, r_0 \rangle, \langle p_1, r_1 \rangle \in \mathcal{D}, p, p' \in P_f \) are such that \( p \leq p_0, p_1, p' \leq p_0, p_1 \), and

\[
p \Vdash \text{“} r_0 \text{ and } r_1 \text{ are compatible in } \hat{R} \text{“},
\]

moreover we have either \( p \leq p' \) or \( p \nleq p' \) (see definition 2.15) then we also have

\[
p' \Vdash \text{“} r_0 \text{ and } r_1 \text{ are compatible in } \hat{R} \text{“}.
\]

**Theorem 3.4.** If CH holds in \( V \), \( f \) is a \( \Delta \)-function and \( \hat{R} \) is a \( P_f \)-name for a c.c.c poset which is nice over \( P_f \), then 2.4.(iii) is preserved by forcing with \( \hat{R} \), i.e.

\[
V^{P_f * \hat{R}} \models \forall Y \in [X_f]^{\omega} \left( Y \text{ is compact or } |X_f \setminus Y| \leq \omega_1 \right).
\]

**Proof.** Let \( \mathcal{D} \subset P_f * \hat{R} \) witness that \( \hat{R} \) is nice.

Assume that \( 1_{P_f * \hat{R}} \Vdash \text{“} \hat{Y} = \{ \hat{y}_n : n \in \omega \} \subset \omega_2 \text{“} \). For each \( n \in \omega \) fix a maximal antichain \( C_n \subset \mathcal{D} \) such that for each \( \langle p, r \rangle \in C_n \) there is \( \alpha \in a^p \) with \( \langle p, r \rangle \Vdash \text{“} \hat{y}_n = \hat{\alpha} \text{“} \). Let \( A = \bigcup \{ a^p : \langle p, r \rangle \in \bigcup_{n<\omega} C_n \} \). Since every \( C_n \) is countable by c.c.c we have \( |A| = \omega \).

Assume that \( 1_{P_f * \hat{R}} \Vdash \text{“} \bar{Y} \text{ is not compact} \text{“} \), that is, \( \bar{Y} \) can not be covered by finitely many \( H(\delta) \) in \( V^{P_f * \hat{R}} \).

Let

\[
I = \{ \gamma < \omega_2 : \exists \langle p, r \rangle \in P_f * \hat{R} \langle p, r \rangle \Vdash \text{“} \gamma \notin \bar{Y} \text{“} \}.
\]

Since \( \omega_2 \setminus I \) is in the ground model and \( 1_{P_f * \hat{R}} \Vdash \omega_2 \setminus I \subset \bar{Y} \), it is enough to show that \( \omega_2 \setminus I \) is infinite. Indeed, in this case \( \omega_2 \setminus I \) contains a final segment of \( \omega_2 \) by corollary 2.14. (Note that the closure of a ground model set does not change under any further forcing.) Thus the next claim completes the proof of this theorem.

**Claim.** \( I \) is not stationary.

Assume on the contrary that \( I \) is stationary. For each \( \delta \in I \) fix a condition \( \langle p_\delta, r_\delta \rangle \in P_f * \hat{R} \) and a finite set \( D_\delta \in [\delta]^{<\omega} \) such that \( \langle p_\delta, r_\delta \rangle \Vdash \text{“} Y \cap U(\delta, D_\delta) = \emptyset \text{“} \).

For each \( \delta \in I \) write \( Q_\delta = a^{p_\delta} \cap \delta \) and \( E_\delta = a^{p_\delta} \setminus \delta \). We can assume that \( D_\delta \subset Q_\delta \), \( \delta \in E_\delta \) and \( \sup A < \delta \) for each \( \delta \in I \).
Let $B_{\delta} = \text{cl}_f(A \cup Q_{\delta}, E_{\delta})$ for $\delta \in I$ (see 1.3). Since $B_{\delta}$ is a countable set with $\sup(B_{\delta}) = \sup(A \cup Q_{\delta})$ and so $\sup(B_{\delta}) < \delta$, we can apply Fodor’s pressing down lemma and CH to get a stationary set $J \subset I$ and a countable set $B \subset \omega_2$ such that $B_{\delta} = B$ for each $\delta \in J$.

By thinning out $J$ and with another use of CH we can assume that for some fixed $k \in \omega$ we have

(i) $E^\delta = \{i : i < k\}$ for $\delta \in J$, $\gamma_0^\delta < \gamma_1^\delta < \cdots < \gamma_{k-1}^\delta$,
(ii) $f(\xi, \gamma_i^\delta) = f(\xi, \gamma_j^\delta)$ for each $\xi \in B$, $\gamma_i^\delta, \gamma_j^\delta \in J$, and $i < j < k$.

Let $\delta = \min J$, $D = D_{\delta}$, $E = E_{\delta}$, $p = p_{\delta}$, $r = r_{\delta}$, $Q = Q_{\delta}$. By lemma 1.2 there are $2k$ ordinals $\delta_j \in J$ with $\delta < \delta_0 < \delta_1 < \cdots < \delta_{2k-1}$ from $J$ such that

(*) $B \cup E \subset \bigcap\{f(\xi, \eta) : \xi \in E_{\delta_j}, \eta \in E_{\delta_j}, i < j < 2k\}$.

For $i < k$ and $j < 2$, let $\gamma_i = \gamma_i^\delta$, $\gamma_{i,j} = \gamma_{i,j}^\delta$. Let $F = \{i,j : i < k, j < 2\}$.

We know that $a^p = Q \cup E$. Define the condition $q \in P_f$ by the following stipulations:

(i) $a^q = a^p \cup F$ and $q \leq p$,
(ii) $h^q(\gamma_{i,j}) = \{\gamma_{i,j}\} \cup a^p$ for $\langle i,j \rangle \in k \times 2$,
(iii) $i^q(\gamma_{i_0,j_0}, \gamma_{i_1,j_1}) = a^p$ for $\langle i_0,j_0 \rangle \neq \langle i_1,j_1 \rangle \in k \times 2$,
(iv) $i^q(\eta_{i,j}) = \emptyset$ for $\eta \in a^p$ and $\langle i,j \rangle \in k \times 2$.

Since $a^p \subset B \cup E$, (*) implies that $q \in P_f$ and so $\langle q, r \rangle \in P_\ast \hat{R}$.

Since $1_{P_\ast \hat{R}} \vDash "H[Q \cup E] does not cover Y"$, there is a condition $t, u \leq \langle c, d \rangle$ for some $\langle c, d \rangle \in C_n$, so

(*') $t_{\ast} \vDash "r \text{ and } d \text{ are compatible in } \hat{R}"$.

Since $t \leq w \leq c$ and $\hat{R}$ is nice for $P_f$ we have

(**) $w_{\ast} \vDash "r \text{ and } d \text{ are compatible in } \hat{R}"$.

We know $s \leq c$ and so $s_{\ast} \vDash "\alpha = \gamma_n"$ and $\alpha \in a^s \setminus h^s[Q \cup E]$.

Since $i^s(\gamma_{0,0}, \gamma_{i,j}) = i^s(\gamma_{0,0}, \gamma_{i,j}) = Q \cup E$, and $\gamma_{i,j} \notin h^s(\gamma_{i,j} - j)$ we have

(16) $h^s(\gamma_{i,0}) \cap h^s(\gamma_{i,j}) = h^s(\gamma_{i,0}) \ast h^s(\gamma_{i,j}) \subset h^s[Q \cup E]$.

If $\xi \in Q \cup E$, then $\xi \in h^s(\gamma_{i,j})$ and $i^s(\xi, \gamma_{i,j}) = \emptyset$ and so

(17) $h^s(\xi) \setminus h^s(\gamma_{i,j}) = h^s(\xi) \ast h^s(\gamma_{i,j}) \subset h^s[i^s(\xi, \gamma_{i,j})] = \emptyset$. 
Putting (16) and (17) together it follows that \( h^*(\gamma_{\iota,0}) \cap h^*(\gamma_{\iota,1}) = h^*[Q \cup E] \). Thus we can apply 2.16 to get a condition \( p^* \in P_f \) such that \( p^* \leq s[Q \cup E] = p, p^* \leq s[S \leq c, w \prec p^* \text{ and } \alpha \in S \setminus h^*[Q \cup E] \subseteq h^{p'}(\gamma_0) = h^{p'}(\delta) \). But \( D \subset Q \), so \( \alpha \in h^{p'}(\delta) \setminus h^{p'}[D] \).

Hence \( p^* \Vdash " \alpha \in U(\delta, D)". \)

Since \( p^* \leq p, c \) and \( w \prec p^* \) and \( \dot{R} \) is nice, it follows from (**) that

\[
(**) \quad p^* \Vdash " r \text{ and } d \text{ are compatible in } \dot{R}".
\]

Thus \( \langle p, r \rangle \) and \( \langle c, d \rangle \) have a common extension \( \langle p^*, r^* \rangle \) in \( P \ast \dot{R} \).

But then

\[
\langle p^*, r^* \rangle \Vdash \dot{y}_n = \alpha \in \dot{Y} \cap U(\delta, D)
\]

because \( \langle p^*, r^* \rangle \leq \langle c, d \rangle \). On the other hand \( \langle p^*, r^* \rangle \Vdash \" \dot{Y} \cap U(\delta, D) = \emptyset \" \) because \( \langle p^*, r^* \rangle \leq \langle p, r \rangle \). Contradiction, the claim is proved, which completes the proof of the theorem. \( \square \)

**Definition 3.5.** Assume that \( A \in \left[X_f \right]^\omega \) and \( \alpha \in \overline{A} \). We define the poset \( Q(A, \alpha) \) as follows. Its underlying set is \( \left[ A \right]^{\leq \omega} \times \left[ \alpha \right]^{\omega} \). If \( \langle s, C \rangle \) and \( \langle s', C' \rangle \) are conditions, let \( \langle s, C \rangle \leq \langle s', C' \rangle \) if and only if \( s \supseteq s', C \supseteq C' \) and \( s \setminus s' \subset U(\alpha, C') \). For \( q \in Q(A, \alpha) \) write \( q = \langle s', C' \rangle \) and \( \text{supp}(q) = s' \cup C' \).

It is well-known and easy to see that if \( G \) is a \( Q(A, \alpha) \)-generic filter, then \( S = \cup \{ s^\beta : q \in G \} \) is a sequence from \( A \) which converges to \( \alpha \), i.e. every open neighbourhood of \( \alpha \) contains all but finitely many points of \( S \).

Clearly \( Q(A, \alpha) \) is \( \sigma \)-centered and well-met. In fact, if \( p_0 = \langle s_0, C_0 \rangle \) and \( p_1 = \langle s_1, C_1 \rangle \) are compatible, then \( p_0 \land p_1 = \langle s_0 \cup s_1, C_0 \cup C_1 \rangle \).

**Definition 3.6.** A finite support iterated forcing \( \langle R_\xi : \xi \leq \kappa \rangle \) over \( V^{P_f} \) is called an FU-iteration if for each \( \xi < \kappa \) we have

\[
1_{P_f \ast R_\xi} \Vdash R_{\xi+1} = R_\xi \ast Q(\dot{A}, \dot{\alpha}) \text{ for some } \dot{A} \in \left[X_f \right]^\omega \text{ and } \dot{\alpha} \in \overline{A}.
\]

Since every \( Q(A, \alpha) \) is \( \sigma \)-centered, it is clear that any FU-iteration is c.c.c, in fact it even has property K. The really important, and much less trivial, property of them is given in our next result.

**Theorem 3.7.** Any FU-iteration is nice over \( P_f \).

**Proof.** Assume that \( \langle R_\xi : \xi \leq \kappa \rangle \) is an FU-iteration, \( R_{\xi+1} = R_\xi \ast Q(\dot{A}_\xi, \dot{\alpha}_\xi) \). Write \( Q^* = \left[ \omega_2 \right]^{\omega} \times \left[ \omega_2 \right]^{\omega} \). Clearly \( 1_{P_f \ast R_\xi} \Vdash " Q(\dot{A}_\xi, \dot{\alpha}_\xi) \subset Q^* " \).

We consider the elements of \( P_f \ast \dot{R}_\kappa \) as pairs \( \langle p, \dot{r} \rangle \), where \( p \in P_f \) and \( p \Vdash " \dot{r} \text{ is a finite function, } \text{dom}(\dot{r}) \subset \kappa \text{ and } \dot{r}[\xi] \Vdash \dot{r}(\xi) \in Q(\dot{A}_\xi, \dot{\alpha}_\xi) \text{ for each } \xi \in \text{dom}(\dot{r})". \)
Definition 3.8. A condition \( \langle p, r \rangle \in P_f \times R_\kappa \) is called determined if \( r \in \text{Fn}(\kappa, Q^*, \omega) \), (i.e. \( r \) is a finite function, not only a \( P_f \)-name of a finite function) moreover for each \( \xi \in \text{dom}(r) \) we have \( \text{supp}(r(\xi)) \subseteq a^p \) and there is an ordinal \( \alpha \in a^p \) such that \( \langle p, r(\xi) \rangle \models \langle \check{\alpha} = \check{\alpha} \rangle \).

Clearly the family \( D \) of the determined conditions is dense in \( P_f \times R_\kappa \). We claim that \( D \) witnesses that \( R_\kappa \) is nice. Indeed let \( \langle p_0, r_0 \rangle, \langle p_1, r_1 \rangle \in D \), \( p, p' \in P_f \) be such that \( p \leq p_0, p_1, p' \leq p_0, p_1 \),

\[ (+) \quad p \vdash \langle \check{r_0} \text{ and } \check{r_1} \rangle \text{ are compatible in } \hat{R} \]

and either \( p \leq p' \) or \( p < p' \).

Write \( r_i(\xi) = \langle s_i(\xi), C_i(\xi) \rangle \) for \( \xi \in \text{dom}(r_i) \). Let \( D = \text{dom}(r_0) \cup \text{dom}(r_1) \). For \( i < 2 \) and \( \xi \in D \) let

\[
s_i^*(\xi) = \begin{cases} s_i(\xi) & \text{if } \xi \in \text{dom}(r_i), \\ \emptyset & \text{otherwise}, \end{cases}
\]

and

\[
C_i^*(\xi) = \begin{cases} C_i(\xi) & \text{if } \xi \in \text{dom}(r_i), \\ \emptyset & \text{otherwise}. \end{cases}
\]

Consider the condition \( r^* \in \text{Fn}(\kappa, Q^*, \omega) \) defined by the stipulations \( \text{dom}(r^*) = D \) and \( r^*(\xi) = \langle s_0^*(\xi) \cup s_1^*(\xi), C_0^*(\xi) \cup C_1^*(\xi) \rangle \) for \( \xi \in D \). We show that \( p' \models \langle \check{r^*} \text{ is a common extension of } \check{r_0} \text{ and } \check{r_1} \text{ in } R_\kappa \rangle \). We prove a bit more: we show, by a finite induction, that for each \( \xi \in D \)

\[ p' \models \langle \check{r^*}(\xi + 1) \in R_{\xi+1} \rangle \text{ and } \check{r^*}(\xi + 1) \leq \check{r_0}(\xi + 1, \check{r_1}(\xi + 1) \rangle. \]

The non-trivial step is when \( \xi \in \text{dom}(r_0) \cap \text{dom}(r_1) \).

Since \( \langle p', r^*(\xi) \rangle \leq \langle p', r_i(\xi) \rangle \) the induction hypothesis, it follows that \( \langle p', r^*(\xi) \rangle \models \langle \check{A}_\xi \rangle \subseteq s_0(\xi) \) and so \( \langle p', r^*(\xi) \rangle \models \langle \check{A}_\xi, \check{\alpha}_\xi \rangle \), that is, \( p' \models \langle \check{r^*}(\xi + 1) \in R_{\xi+1} \rangle \).

By \((+\rangle\) for some \( P_f \)-name \( \check{r} \) and \( P_f \times R_\xi \)-name \( \check{q} \) we have \( \langle p, \check{r} \rangle \in P_f \times R_\xi \) and \( \langle p, \check{q} \rangle \models \langle \check{q} \in Q(\check{A}_\xi, \check{\alpha}_\xi) \text{ is a common extension of } \check{r_0}(\xi) \text{ and } \check{r_1}(\xi) \rangle \). Hence \( \langle p, \check{r} \rangle \models \langle \check{s_\xi \setminus s_0(\xi)} \subseteq U(\alpha, C_0(\xi)) \rangle \) and \( \check{s_\xi \setminus s_0(\xi)} \subseteq U(\alpha, C_0(\xi)) \). Therefore \( \langle p, \check{r} \rangle \models \langle \check{s_1(\xi) \setminus s_0(\xi)} \subseteq U(\alpha, C_0(\xi)) \rangle \). By \( \text{supp}(r_i(\xi)) \subseteq a^p \), this can only happen if

\[ (\dagger) \quad s_1(\xi) \setminus s_0(\xi) \subseteq u^p(\alpha, C_0(\xi)). \]

But both \( p \leq p' \) and \( p < p' \) together with \((\dagger)\) imply

\[ (\ddagger) \quad s_1(\xi) \setminus s_0(\xi) \subseteq u^{p'}(\alpha, C_0(\xi)). \]

Thus \( p' \models \langle \check{s_1(\xi) \setminus s_0(\xi)} \subseteq U(\alpha, C_0(\xi)) \rangle \), i.e. \( \langle p', r^*(\xi) \rangle \models \langle \check{r^*(\xi)} \leq \check{r_0(\xi)} \rangle \). Similarly, it can be proved that \( \langle p', r^*(\xi) \rangle \models \langle \check{r^*(\xi)} \leq \check{r_1(\xi)} \rangle \). Thus we have carried out the induction step and the theorem is proved. \( \square \)
Now we are ready to prove the main result of this section.

**Theorem 3.9.** Assume that CH holds in the ground model \( V \), there is a \( \Delta \)-function \( f \) and \( \lambda \) is a cardinal such that \( \omega_1 < \lambda = \lambda^\omega \). Then in \( V^P_f \) there is an FU-iteration \( \hat{R}_\lambda \) of length \( \lambda \) such that in \( V^{P_f \ast \hat{R}_\lambda} \) the space \( X_f \) is Frechet-Uryson and satisfies 2.4(i)–(iii), moreover \( V^{P_f \ast \hat{R}_\lambda} \models \forall \kappa \exists \lambda (\kappa^\lambda) \) for each cardinal \( \kappa \geq \omega \).

**Proof.** Since \( |P_f| = \omega_2 \) and \( P_f \) satisfies c.c.c we have \( (\lambda^\omega)^{V^{P_f}} \leq (|P_f| \lambda^\omega)^V = (\lambda^\omega)^V = \lambda \). Therefore, using a suitable book-keeping procedure in \( V^{P_f} \) (see [6, Ch VIII. 6.3] for this technique) we can construct an FU-iteration \( \langle R_\xi : \xi \leq \lambda \rangle \), \( R_{\xi+1} = R_\xi \ast Q(A_\xi, \alpha_\xi) \), having the following property: for every pair \( \langle A, \alpha \rangle \) if \( A \) is a countable subset of \( X_f \) in \( V^{P_f \ast \hat{R}_\lambda} \) and \( \alpha \in A \) then for some \( \xi < \lambda \) we have \( V^{P_f \ast R_\xi} \models \langle A, \alpha \rangle \in Q(A, \alpha) \). Thus

\[
(*) \quad V^{P_f \ast \hat{R}_\lambda} \models \text{“if } \alpha \in \omega_2 \text{ is in the closure of a countable set } A \subseteq X_f \text{ then there is a sequence } \{ s_n : n \in \omega \} \subseteq A \text{ which converges to } \alpha”.
\]

Since \( R_\lambda \) has property K in \( V^{P_f} \), by theorem 3.1 the space \( X_f \) remains countably tight in \( V^{P_f \ast \hat{R}_\lambda} \). Putting together this observation with (\( * \)) it follows that \( V^{P_f \ast \hat{R}_\lambda} \models \langle X_f \rangle \text{ is Frechet-Uryson}”. \)

An FU-iteration is nice by theorem 3.7, and so \( X_f \) has property 2.4(iii) in \( V^{P_f \ast \hat{R}_\lambda} \) by theorem 3.4. Since \( X_f \) is right separated it remains locally compact in any extension of \( V^{P_f} \). Since \( \hat{R}_\lambda \) satisfies c.c.c the space \( X_f \) has property 2.4(ii) in \( V^{P_f \ast \hat{R}_\lambda} \) by theorem 3.2. All this implies that \( X_f \) remains initially \( \omega_1 \)-compact and normal.

Finally we investigate the cardinal exponents in \( V^{P_f \ast \hat{R}_\lambda} \). Since \( |P_f| = \omega_2 \) and \( 1_{P_f} \models \langle |R_\lambda| = \lambda \rangle \), the iteration \( P_f \ast R_\lambda \) contains a dense subset \( D \) of cardinality \( \leq \lambda \). Since \( P_f \ast R_\lambda \) satisfies c.c.c it follows that for each \( \kappa \geq \omega \) we have \( (2^\kappa)^{V^{P_f \ast \hat{R}_\lambda}} \leq (|D|^{\kappa})^V = (\lambda^\kappa)^V \). On the other hand every successor step of an FU-iteration introduces a new subset of a countable set, and so \( (2^\omega)^{V^{P_f \ast \hat{R}_\lambda}} \geq \lambda \). Consequently, \( V^{P_f \ast \hat{R}_\lambda} \models \forall \kappa \geq \omega_1 \exists \lambda (\kappa^\lambda) \geq \lambda^\omega \), which proves what we wanted. \( \square \)

Theorem 3.9 answers a question raised by Arhangel’skii [1, problem 3], in the negative: CH can not be weakened to \( 2^\omega < 2^{\omega_1} \) in the theorem of van Douwen and Dow. In fact we proved much more: the existence of a Frechet-Uryson, initially \( \omega_1 \)-compact and non-compact space is consistent with practically any cardinal arithmetic that violates CH. More precisely, if we have a ZFC model \( V \) in which CH holds and \( \lambda = \lambda^\omega \leq 2^{\omega_1} \), then we can find a cardinal preserving generic extension \( W \) of \( V \) which contains a Frechet-Uryson and normal counterexample to the van Douwen–Dow question, \( (2^\omega)^W = \lambda \), moreover \( (2^\kappa)^W = (2^\kappa)^V \) for each \( \kappa \geq \omega_1 \). We can obtain \( W \) as follows. First we force the \( \sigma \)-complete poset \( P \) of Shelah (see [3]) which introduces a \( \Delta \)-function \( f \) in \( V^P \). Since \( P \) is \( \sigma \)-complete and \( |P| = \omega_2 \), forcing with \( P \) does not change \( 2^\kappa \) for any \( \kappa \geq \omega \). Now forcing with \( P_f \) over \( V^P \) introduces
the counterexample $X_f$ to the van Douwen–Dow question. Since $|P_f| = \omega_2$, the cardinal exponents are the same in $V^P$ and in $V^{P_f}$ for uncountable cardinals and $(2^{\omega})^{V^{P_f}} = \omega_2$. Finally we can apply theorem 3.9 to get the desired final model $W = V^{P_f*P_f+Rs}$. Let us remark that we have $(2^\kappa)^W = (2^\kappa)^V$ for $\kappa \geq \omega_1$ because $(2^\kappa)^V = (\lambda^\kappa)^V$ by $\lambda \leq 2^{\omega_1}$.

Let us remark that for any cardinal $\kappa$ the poset $Fn(\kappa, 2, \omega)$, i.e. the forcing notion that adds $\kappa$ many Cohen reals, is clearly nice over $P_f$, as is witnessed by the dense set of the determined conditions. Thus, by theorems 3.1, 3.2 and 3.4 , adding Cohen reals will preserve properties 2.4(i)–(iii) of $X_f$, especially $X_f$ remains countably tight and initially $\omega_1$-compact. It is worthwhile to mention that, in contrast with this, Alan Dow proved in [4] that if CH holds in the ground model $V$ then adding Cohen reals can not introduce a countably tight, initially $\omega_1$-compact and non-compact $T_3$ space.

Let us finish by formulating the following higher cardinal version of the van Douwen–Dow problem:

**Problem 2.** Is it provable in ZFC that an initially $\omega_2$-compact $T_3$ space of countable tightness is compact ?

The main problem is trying to use out approach that worked for $\omega_2$ (instead of $\omega_3$) here is that no $\Delta$-function may exist for $\omega_3$! This problem has come up already in the efforts trying to lift the result of [3] from $\omega_2$ to $\omega_3$.

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