Bernstein-type Inequalities and Nonparametric Estimation under Near-Epoch Dependence

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Abstract

The major contributions of this paper lie in two aspects. Firstly, we focus on deriving Bernstein-type inequalities for both geometric and algebraic irregularly-spaced NED random fields, which contain time series as special case. Furthermore, by introducing the idea of "effective dimension" to the index set of random field, our results reflect that the sharpness of inequalities are only associated with this "effective dimension". Up to the best of our knowledge, our paper may be the first one reflecting this phenomenon. Hence, the first contribution of this paper can be more or less regarded as an update of the pioneering work from Xu and Lee (2018). Additionally, as a corollary of our first contribution, a Bernstein-type inequality for geometric irregularly-spaced $\alpha$-mixing random fields is also obtained. The second aspect of our contributions is that, based on the inequalities mentioned above, we show the $L_\infty$ convergence rate of the many interesting kernel-based nonparametric estimators. To do this, two deviation inequalities for the supreme of empirical process are derived under NED and $\alpha$-mixing conditions respectively. Then, for irregularly-spaced NED random fields, we prove the attainability of optimal rate for local linear estimator of nonparametric regression, which refreshes another pioneering work on this topic, Jenish (2012). Subsequently, we analyze the uniform convergence rate of uni-modal regression under the same NED conditions as well. Furthermore, by following the guide of Rigollet and Vert (2009), we also prove that the kernel-based plug-in density level set estimator could be optimal up to a logarithm factor. Meanwhile, when the data is collected from $\alpha$-mixing random fields, we also derive the uniform convergence rate of a simple local polynomial density estimator (Cattaneo, Jansson, & Ma, 2020).
1 Introduction

In time and spatial series analysis, we often encounter different kinds of dependence. A natural idea to simplify the problem is to introduce some measurement of dependence that is sufficiently general such that a large class of time or spatial series models can be included as special cases. The idea of "Near-Epoch-Dependence" (NED) was first introduced by Ibragimov (1962) to model the dependence of regularly-spaced time series. This concept was further utilized by Billingsley (2013) and McLeish (1974, 1975). The NED condition is very general, allowing for non-identically distributed sequences and it covers many important processes that are not mixing, such as, autoregressive (AR) and infinite moving average (MA(∞)) processes under very general conditions. It also includes some dynamic simultaneous equation models (cf Bierens (2012), Chapter 5 and Gallant (2009), pp 481, 502, 539). Particularly, α-mixing may fail under relatively regular conditions. Indeed, Gorodetskii (1978) demonstrated that the α-mixing property fails in the case of continuously distributed innovations when the coefficients of a linear process do not decrease quickly enough. Andrews (1985) also proved that the AR(1) process with independent Bernoulli innovations is not α-mixing. But the NED condition is satisfied in these two situations based on Proposition 1 of Jenish (2012). There are also many other interesting regularly-spaced time series models that satisfy the NED condition. Hansen (1991) proved that the GARCH(1,1) process is NED. In Sections 4.3, 4.5 and 4.6 Li, Lu, and Linton (2010) showed that AR(1)-NARCH(1,1), multivariate MA(∞) and semiparametric ARCH(∞) proposed by (Linton & Mammen, 2005) are also NED processes.

Unlike the regularly-spaced NED time series mentioned above, with the advent of spatial econometrics, irregularly-spaced NED random fields became more crucial in many non-linear spatial processes. Jenish and Prucha (2012) proved that both linear autoregressive random fields and nonlinear infinite moving average random fields are NED random fields. Jenish (2012) showed that, under mild moment and Lipschitz conditions, the nonlinear autoregressive random field is a NED random field as well, which includes many widely used spatial auto-regressive(SAR) models as special cases. More recently, Xu and Lee (2015b) studied the NED properties of SAR models with a nonlinear transformation of the dependent variable. Xu and Lee (2015a) indicates that under very mild conditions on the disturbance and weight matrix, the dependent variable in the SAR Tobit model is an $L^2$-NED random field on the disturbance.

There has been some work on nonparametric estimation and prediction
under NED dependence. Andrews (1995) established the uniform convergence rate of the kernel density estimator and Nadayara-Watson estimator under NED conditions. Lu (2001) obtained asymptotic normality of the kernel density estimator under the regularly-spaced NED process. A strong law of large number and a strong invariance principle were proven under NED conditions by Ling (2007). For local regression on regularly-spaced NED time series, Lu and Linton (2007) and Li, Lu, and Linton (2012) investigate pointwise and uniform convergence rates of local linear fitting and proved that, under some mild conditions of dependence, Stone (1980)’s optimal bound is attainable. More recently, Ren and Lu (2020) showed the asymptotic normality of the local linear estimator for quantile regression and established the Bahadur’s representation of this estimator in a weak convergence sense.

All of the work mentioned above, however, is based on regularly-spaced time series. For NED random fields, the relative results are surprisingly sparse in the literature. Hallin, Lu, and Tran (2001) analyzed marginal density estimation of regularly-spaced spatial linear processes, which are only a special example of NED processes. Later, Hallin, Lu, and Tran (2004) extended the previous research to regularly-spaced nonlinear spatial process with iid innovations. This setting is still more restrictive, however, compared with Definition 2 in Section 2. For irregularly-spaced NED random fields, the ground breaking pioneering work was that of Jenish and Prucha (2012), who thoroughly derived a uniform law of large number, central limit theorem and their applications to asymptotic behaviors of general GMM estimators. Based on this, Jenish (2012) investigated the asymptotic normality and uniform convergence rate of local linear fitting under nonstationary irregularly-spaced NED random fields.

Generally speaking, almost all of the previous research on nonparametric regression under the NED condition is kernel-based. The methods used to obtain pointwise or uniform theoretical guarantees of estimators can be categorized into two groups: The first approach is to construct an ”approximate” process whose properties are already known. Due to the definition of NED condition, the approximation error between the approximate process and original NED process converges to 0 in $L^p$ sense at a higher order. Thus, we can focus solely on the approximate process, which is assumed to be $\alpha$-mixing in this paper. (cf. Lu and Linton (2007), Li et al. (2012), Xu and Lee (2015a) and Xu and Lee (2018)). Another approach is motivated by Parzen (1962), Bierens (1983) and Andrews (1995). Because kernel is often assumed to be a probabilistic density function as well, we can use the inverse formula of the characteristic function to rewrite the kernel-based estimator.
This method is also used in Jenish (2012). But unlike Li et al. (2012), the second approach does not lead Jenish to an optimal uniform convergence rate of local linear estimators. However, neither of the two methods can be applied easily to other kinds of estimators, like sieve.

Xu and Lee (2018) have pointed out that having an exponential inequality on irregularly-spaced NED random fields is vital for proving the consistency of sieve-based estimators in many interesting spatial regression models. Indeed things would be much easier if we could discover some concentration inequalities directly about NED process, particularly irregularly-spaced NED random fields. Lemma 5.1 in Gerstenberger (2018) offers us a probabilistic inequality for NED processes under very weak dependence assumption. However, it only considers the case in which the data is collected from a regularly-spaced time series and the approximate process is $\beta$-mixing, which is not general enough to cover our setting. Xu and Lee (2018) did ground breaking work on this topic. By employing Doukhan and Louhichi (1999)’s combinatoric techniques, they proved an exponential inequality on geometric irregularly-spaced NED random fields at the order of $e$ (see Definition 2). Equipped with this result, they proved the consistency of a sieve maximum likelihood estimator for spatial auto-regressive Tobit model.

Unfortunately, there are still some possible shortcomings of their results.

S1 Their method relies heavily on the assumption that the NED condition is geometric and thus excludes many interesting time and spatial series.

S2 Their result may not be able to obtain sufficiently sharp uniform convergence rates for many widely used estimators. For example, according to Li et al. (2012), the optimality of the local linear estimator could be obtained even when the data generating process is an algebraic regularly-spaced NED time series. But Xu and Lee (2018)’s result can not provide this.

S3 The decreasing rate of their inequality is $\exp\{-\text{constant}N^{1/2d+2}\epsilon^2\}$, for $\forall \epsilon > 0$, where $d$ denotes the dimension of the index set. This directly implies that the convergence rate is affected by the dimension of the index set of the random field, which may not be true. For example, when $d = 2$ and we concentrate solely on selecting the locations along the X-axis in $\mathbb{R}^2$, similar to a time series, the index set now has linear order. Hence, the upper bound of the exponential inequality of averaged partial sums should be at the same level as in time series case.
Therefore, in order to extend the work done by Xu and Lee and important previous research about nonparametric estimation under NED condition, the contributions of this paper are twofold: Bernstein-type inequalities for both geometric and algebraic NED random fields and theoretical guarantees of some interesting nonparametric estimators beyond locally weighted least square estimator for conditional mean function. More specifically, They can be summarized as follows,

1. By adding mild assumptions (Assumption 3, 4), we use an alternative approach to derive Bernstein-type inequalities for both geometric and algebraic irregularly-spaced NED random fields. For the geometric case, our result holds for arbitrary $L^p$-NED condition, $p \geq 1$. All of the new results would be adaptive to the situation similar to the example in S3. More specifically, our result shows that, the decreasing rate of the probabilistic upper bound would only be weakly affected by the “effective dimension” (see Definition 1). Additionally, we also demonstrate a Bernstein-type inequality for geometric irregularly-spaced $\alpha$-mixing random field, which updates the earlier findings such as those in the work of Adamczak (2008), Modha and Masry (1996) and Merlevède, Peligrad, and Rio (2011).

2. By combining "chaining argument" with the Bernstein-type inequalities mentioned above, we establish two deviation inequalities for the supreme of empirical process when the data are collected from geometric NED and $\alpha$-mixing random fields respectively.(see Propositions 7 and 9)

3. To acknowledge the previous research, we revisit local linear fitting of the conditional mean function under irregularly-spaced NED random fields. We obtain the uniform convergence rate under algebraic NED conditions. Our result indicates that, under standard smoothing conditions, Stone (1982)’s optimal rate can be attained. This section takes up and extends the results of Jenish (2012), Li et al. (2012) and Lu and Linton (2007).

4. As an excellent application of the idea of local polynomial fitting, Cattaneo et al. (2020) proposed a density estimator that generically solves the problem of "boundary effects". Hence, in this paper, we investigate its uniform convergence rate under geometrical irregularly-spaced $\alpha$-mixing random fields.
5. Considering that, at a high-level, conditional modes can reveal structure that is missed by the conditional mean, we firstly obtain a non-asymptotic uniform convergence rate of kernel density estimator for geometric irregularly-spaced NED random fields. Then, based on this, we derive a uniform convergence rate for the kernel-based uni-modal regression.

6. Rigollet and Vert (2009) showed the attainability of the optimal rate for the kernel-based plug-in density level estimator for iid data. Here we prove that even when the data are collected from an irregularly-spaced geometric NED random field, the rate is at least optimal up to a logarithm factor.

The rest part of this paper is organized as follows: Section 2 introduces the definition of irregularly-spaced NED random fields and some of their useful properties. In Section 3, we thoroughly discuss contribution 1. Section 4 to 6 present the deviation inequalities of empirical process and apply them to investigate the uniform convergence rates of many interesting kernel-based nonparametric estimators which will cover the contribution 2 to 6. Lastly, the proof of all the results demonstrated in this paper is given in the Appendix.

2 Irregularly-Spaced NED Random Fields

Before we present the definition of irregularly-spaced NED random fields, we firstly introduce the following assumptions about the ”choice of locations”.

**Assumption 1** Individual units are located at $\Gamma_N \subset \Gamma \subset \mathbb{R}^d$, where $d$ is any positive integer. Here $\Gamma_N := \{s_{iN}, i = 1, 2, ..., N\}$ is the set of chosen locations.

Clearly, assumption 1 defines irregularly-spaced locations. The set $\Gamma$ here could be any region of interest on some specific time period, geographical space or more abstract economic characteristics. For example, when $d = 1$, $\Gamma$ could be a union of multiple intervals; when $d \geq 2$, $\Gamma$ might become a union of many geographical areas. In a nutshell, the reason why we introduce $\Gamma$ here is to rigorously define ”available region” for sampling and $\Gamma_N \subset \Gamma$ directly represents this idea. Until now, however, we know only that $\Gamma$ is a subset of $\mathbb{R}^d$. A natural question is whether the area of $\Gamma$ can be finite or not. In this paper, the answer is ”no” because we need the following assumption 2.
Assumption 2 For any \( s_{iN}, s_{jN} \in \Gamma_N \) and \( i \neq j \), we assume, \( \|s_{iN} - s_{jN}\| > d_0 \) for some \( d_0 > 0 \).

Here, \( \|\cdot\| \) could be any norm defined on \( \mathbb{R}^d \). Without loss of generality, in this paper it is assumed as a Euclidean norm. This assumption actually indicates that, an infinite number of selected locations could never be contained by a bounded subset of \( \mathbb{R}^d \). In the literature of spatial statistics, this is often addressed as "domain-expand asymptotics" (Lu & Tjøstheim, 2014). A formal statement of this property is given in the following proposition, which is a modification of Lemma A.1 from Jenish and Prucha (2009).

**Proposition 1** Suppose Assumptions 1 and 2 hold. Then,

(i) By denoting \( I(s, \sqrt{2d_0}/2) \) as a closed cube centered at location \( s \in \Gamma \) with the length of each edge as \( \sqrt{2d_0}/2 \), we have \( |I(s, \sqrt{2d_0}/2) \cap \Gamma_N| \leq 1 \).

(ii) There exists a constant \( C < +\infty \) such that for \( h \geq 1 \)

\[
\sup_{s \in \Gamma} |I(s, h) \cap \Gamma_N| \leq Ch^d, \ C = (2\sqrt{2}/d_0)^d.
\]

Before we exhibit Assumption 3, we introduce the definition of "effective dimension" of index set \( \Gamma_N \).

**Definition 1** (Effective Dimension) Given the set of chosen locations \( \Gamma_N \), for each \( N \), suppose \( M \)-many closed \( d \)-dimensional rectangles are needed to cover set \( \Gamma_N \), \( M \in \mathbb{N}^+ \). Denote these rectangles as \( R^d_{N, l} \), \( l = 1, \ldots, M \). For each \( l \), suppose there are \( d_1 \) out of \( d \) edges whose length are uniformly upper bounded for any \( N \) and that the length of the rest \( d - d_1 \) edges will diverge to infinity as \( N \to +\infty \). Then we define \( d_2 := \max_{1 \leq l \leq M} d_2^l \) as the "effective dimension" of set \( \Gamma_N \).

Within the rest part of this paper, with out loss of generality, suppose these \( d_1 \) edges are the \( k_i \)-th edges of \( R^d_{N, l} \), respectively, \( i = 1, \ldots, d_1 \). Furthermore, we also use \( H^N_{k_i}(l) \) to denote the length of the \( k_i \)-th edge for every \( N \) and suppose \( H^N_{k_i}(l) \leq H_0(l) \), for some \( H_0(l) > 0 \). Additionally, for each \( l \), we also denote the length of these \( d_2 \) edges as \( H^N_{k_j}(l) \). Hence \( \lim_{N \to \infty} H^N_{k_j}(l) = +\infty \).

According to Definition 1, we actually construct an array of rectangles, \( \{R^d_{N, l}\}_{N \in \mathbb{N}^+}, l = 1, 2, \ldots, M \). Please note, for any given \( l \) and any \( N \neq N' \in \mathbb{N}^+ \), \( R^d_{N, l} \) does not need to be geometrically parallel to \( R^d_{N', l} \). The reason why we introduce these array of rectangles is that, unlike regularly-spaced data, irregularity here forbids us from capturing the exact information about the
locations. Hence, the major tool of our proof, Bernstein’s Blocking technique could not be applied directly. Due to Assumption 2, by introducing the basic cubes to ”group” the locations, we can rewrite the partial sum of random variables as summation of ”groups”. This simple trick easily changes the irregularly-spaced setting into a regularly-spaced setting.

**Assumption 3** Based on Definiton 1, we assume $M = m_0$ as a fixed positive integer independent of $N$. We also assume $\max_{1 \leq j \leq d_2} H_{kj}^N(l) \leq \max_{i \neq j} ||s_i - s_j||$, $\forall s_i, s_j \in \Gamma_N \cap R^d_N$, and $\min_{1 \leq j \leq d_2} H_{kj}^N(l) \geq H_0 := \max_{1 \leq l \leq M} H_0(l)$.

The restriction $\max_j H_{kj}^N \leq \max_{i \neq j} ||s_i - s_j||$ means the volume of rectangle $R_N$ is no larger than $H_0^d (\max_{i \neq j} ||s_i - s_j||)^{d_2}$. By combining it with Assumption 4 below, we can easily obtain $\text{Vol}(R_N) = O(N)$, which builds a bridge between $R_N$ and sample size. Moreover, note $H_0$ is actually user-defined which means its only restriction $\min_{1 \leq j \leq d_2} H_{kj}^N(l) \geq H_0$ is easy to be satisfied.

Another point deserves our attention is that we assume number $M = m_0$, which is a universal positive integer independent of sample size $N$. First, this can be easily achieved. Because, for any $N$, we can always construct a d-dimensional cube whose length is equal to $\max_{i \neq j} ||s_i - s_j||$ to cover $\Gamma_N$. Moreover, please note

$$P := \mathbb{P} \left( \frac{1}{N} \sum_{i=1}^{N} Z_{s_i} > t \right) \leq \sum_{l=1}^{m_0} \mathbb{P} \left( \left| \frac{1}{N} \sum_{s_i \in R^d_N} Z_{s_i} \right| > \frac{t}{m_0} \right) := \sum_{l=1}^{m_0} P_l.$$

Since $m_0$ is a fixed number, we have $P \leq m_0 \max_{1 \leq l \leq m_0} P_l$. However, for any $l$, the cardinality of set $\{s_i \in R^d_N\}$, denoted as $|R^d_N|$, is strictly smaller than $N$ or even $|R^d_N|/N = o(1)$, which indicates that $\max_l P_l \lesssim \mathbb{P}$.

Hence, $P = O(\max_l P_l)$, which indicates that, given $N$, we only need concern one specific rectangle whose ”effective dimension” is the largest. A good example is as Figure 1. As we can see, the selected locations(black dots) are distributed along the three lines A, B and C. Suppose the length of the three lines will diverge to infinity. Then we can use three rectangles whose ”effective dimension” is one to cover all of these locations. This indicates that the ”effective dimension” of the set of these locations is 1. Together with Theorem 1 and Theorem 2, if the index set of a NED random fields are distributed as Figure 1, we know the sharpness of Bernstein-type inequalities

\footnote{A \lesssim B indicates A \leq CB for any N and some positive constant C.}
would be like time series, in which the "effective dimension" is always one. Of course, this example is only a tip of the iceberg. Many other interesting examples can be discovered.

However there are multiple methods to construct different arrays of rectangles satisfying Definition 1 and Assumption 3 to cover one group of locations. These different arrays of rectangles may lead us to different effective dimensions. But there always exists a minimum one theoretically. Our results (Theorems 1 and 2) indicate that the sharpness of the inequality will automatically coincide with the array of rectangles whose effective dimension is minimal. But unfortunately, this is only a theoretical result because we don’t know this minimum effective dimension exactly. Of course, in practice, we usually have much more information about the sampling behaviors. Since $1 \leq d_2 \leq d$, if we can directly ensure the effective dimension is equal to 1, we no longer need to worry about this problem, like Figure 1. Generally speaking, finding a general method to calculate the exact value of "effective dimension" is very helpful for us to evaluate the performance of some estimators which can be written as "arithmetic average". But this has already been beyond the scope of this paper.

**Assumption 4** Given $\Gamma_N$, we assume $\max_{i \neq j} ||s_iN - s_jN|| \leq KN^{\frac{1}{d_2}}$, for some positive constant $K$. Here, without loss of generality, for convenience in the proof, we assume $K = H_0/2$. 

![Figure 1: 1](image)
Assumption 4 is dedicated to ensuring that the distribution of the chosen locations is not too sparse. Therefore, as we mentioned before, we can build a connection between the volume of the rectangle $R^l_N$ defined in Definition 1 and sample size $N$. Note that, the $K$ here could be any fixed positive constant. Changing $K$ does not affect the sharpness of the inequalities in Theorems 1 and 2 but only the constant factors.

For any random vector $v \in \mathbb{R}^D$, $||v||_p = [E||v||^p]^\frac{1}{p}$ denotes the $L^p$-norm of Euclidean norm of vector $v$ with respect to a probabilistic measure. For any subset $E \subset \mathbb{R}^d$, denote its cardinality as $|E|$. Based on assumptions 1, We introduce the definitions and properties of $\mathbb{R}^D$-valued ($D \geq 1$) NED random fields as follow.

**Definition 2** Let $Z = \{Z_{s_{iN}}, s_{iN} \in \Gamma_N\}$ be an $\mathbb{R}^D$-valued random field and $\epsilon = \{\epsilon_{s_{iN}}, s_{iN} \in \Gamma_N\}$ also an random field, where $\Gamma_N$ satisfies Assumption 1. $\mathcal{F}_{i,N}(r)$ is denoted as a $\sigma$-field generated by the random vectors $\epsilon_{s_{jN}}$ satisfying that $||s_{jN} - s_{iN}|| \leq r$. $Z$ is said to be $L^p$-near-epoch dependent ($L^p$-NED) on $\epsilon$ if, for some $p > 0$, we have

$$||Z_{s_{iN}} - E(Z_{s_{iN}}|\mathcal{F}_{i,N}(r))||_p \leq \alpha_{iN}\psi_Z(r).$$

(1)

Here $\{\alpha_{iN}\}$ is some array of finite positive constants. Function $\psi_Z(r)$ is a positive-valued non-increasing function that satisfies $\lim_{r \to +\infty} \psi_Z(r) = 0$. We call $Z$ a geometric or algebraic irregularly-spaced $L^p$ NED random field on $\epsilon$ at the order of $\nu$, when $\psi_Z(r) \leq \nu^{-br^\gamma} (\nu > 1, b, \gamma > 0)$ or $\psi_Z(r) \leq r^{-\nu} (\nu > 0)$.

Definition 2 is directly borrowed from Jenish and Prucha (2012). As we mentioned in Section 1, a widely used setting is to assume that random field $\epsilon$ is a strong mixing random field. In this paper, this setting will also be adopted. Thus, in order to prepare the next sections, we first review the definition of a strong mixing random field first.

**Definition 3** Given $\Gamma_N$ satisfying Assumption 1, we say that a process $\epsilon := \{\epsilon_{s_{iN}}, s_{iN} \in \Gamma_N\}$ is an $\alpha$-mixing random field, if $\forall \ U, V \subset \Gamma_N$, such that $\max\{|U|, |V|\} < +\infty$, the following condition is satisfied,

$$\alpha(U, V) := \sup\{|P(AB) - P(A)P(B)|, A \in \sigma(U), B \in \sigma(V)\} \leq \phi(|U|, |V|)\psi_{\epsilon}(\rho(U, V)), $$

where $\sigma(U)$ denotes the sigma-algebra generated by $\epsilon_{s_{iN}}$s whose coordinates belong to set $U$. Function $\phi : \mathbb{N}^2 \to \mathbb{R}^+$ is symmetric and increasing in each
of its two arguments and $\psi_\epsilon(\cdot)$ shares the same property as function $\psi_\infty(\cdot)$ defined in Definition 2. Furthermore, $\rho(U, V) = \inf\{||u - v||, u \in U, v \in V\}$, where $|| \cdot ||$ denotes the Euclidean norm.

Based on the two definitions above, we can obtain the following propositions which will be very useful in deriving concentration inequalities for NED random fields later.

**Proposition 2** Given random fields $Z = \{Z_{s_iN}, s_iN \in \Gamma_N\}$ and $\epsilon = \{\epsilon_{s_iN}, s_iN \in \Gamma_N\}$, where $\Gamma_N$ satisfies Assumption 1 and $Z_{s_iN} = (Z_{s_iN}^1, ..., Z_{s_iN}^D) \in \mathbb{R}^D$, $Z$ is $L^p$-NED on $\epsilon$ if and only if for every $k = 1, ..., D$, $\{Z_{s_iN}^k, s_iN \in \Gamma_N\}$ is also $L^p$-NED on $\epsilon$.

**Proposition 3** Suppose random field $Z$ is $L^p$-NED on random field $\epsilon$ and for any $s_iN \in \Gamma_N$, $Z_{s_iN} \in Z \subset \mathbb{R}^D$. Let $f$ be any real-valued measurable function whose domain is $Z$. If $f$ is Lipschitz continuous on $Z$ with respect to Euclidean norm and $\text{Lip}(f)$ as the Lipschitz constant, random field $f(Z) := \{f(Z_{s_iN}), s_iN \in \Gamma_N\}$ is also $L^p$-NED on $\epsilon$. Furthermore, we have

$$||f(Z_{s_iN}) - E[f(z_{s_iN})|\mathcal{F}_{iN}(r)]||_p \leq 2\text{Lip}(f)\alpha_iN\psi_\infty(r).$$

(2)

**Proposition 4** Suppose random field $Z$ is $L^p$-NED on strong mixing random field $\epsilon$ satisfying Definition 3 and for any $s_iN \in \Gamma_N$, $Z_{s_iN} \in Z \subset \mathbb{R}$. For all $U, V \subset \Gamma_N$, such that $\max\{|U|, |V|\} < +\infty$, denote $U = \{s_iN, i = 1, ..., |U|\}$, $V = \{t_jN, j = 1, ..., |V|\}$. Let $g: Z^{|U|} \to R$, $h: Z^{|V|} \to R$ be two coordinate wise Lipschitz functions with $\text{Lip}(g)$ and $\text{Lip}(h)$ as Lipschitz constants. When $p \geq 2$, for some $\delta > 0$ and $r < \rho(U, V)/2$, we have

$$\left|\text{Cov}(g(Z_{s_iN1}, ..., Z_{s(i|U|N)}, h(Z_{t1N}, ..., Z_{t(|V|N)}))\right|
\leq 4\text{Lip}(f)\text{Lip}(g)(\sum_{i=1}^{|U|}\alpha_{s_iN})(\sum_{j=1}^{|V|}\alpha_{t_jN})\psi_\infty^2(r)
+ 4\text{Lip}(g)(\sum_{i=1}^{|U|}\alpha_{s_iN})\psi_\infty(r)||h||_p
+ 4\text{Lip}(f)(\sum_{j=1}^{|V|}\alpha_{t_jN})\psi_\infty(r)||g||_p
+ \phi(|U|, |V|)\psi_\epsilon^p\phi^p + (\rho(U, V) - 2r)||g||_{p+\delta}||h||_{p+\delta}.$$  

(3)

Proposition 2 is obvious and easy to be proved only using the basic property of conditional expectations and the relationship between $L^1$ and Euclidean norm in $\mathbb{R}^D$. Proposition 3 works as a foundation of Proposition 4 and allows us to concentrate solely on real-valued NED processes when the statistic can be described as an empirical process whose index set is
Lipschitz class. Proposition 4 is actually an updated result of Lemma A.2 from Xu and Lee (2018). The only difference is that our (3) relies on $L^p$-norm whereas their result depends on $L^\infty$-norm. Proposition 2 allows us to apply the concentration inequalities displayed in Section 3 directly to many interesting estimators of regression functions. After all, under mild conditions, most of them can be regarded as a Lipschitz function on the data, like kernel or kNN smoother, Hermite polynomial or many other orthogonal series estimators.

3 Bernstein-type Inequalities for NED

This section introduces Bernstein-type inequalities for geometric and algebraic irregularly-spaced NED random fields. Compared to a Hoeffding-type inequality, a significant advantage of a Bernstein-type inequality is that it allows for localization, due to its specific dependence on the variance term of random variables. Thus, in the asymptotic or non-asymptotic analysis of many estimators, optimal rate is often achieved by applying Bernstein-type inequalities.

Since the work of Carbon (1983) and Collomb (1984) in the mid-1980s, there have been various generalizations of Bernstein-type inequalities to different types of dependent stochastic processes. Because only NED random fields with respect to $\alpha$-mixing processes are discussed in this paper, we review here only the previous research about Bernstein-type inequalities on $\alpha$-mixing processes. Based on coupling and blocking techniques, Rio (1995) and Liebscher (1996) derived Bernstein-type inequalities for stationary real-valued $\alpha$-mixing processes under a very general assumption of decay of dependence. Bryc and Dembo (1996) investigated the large deviation phenomenon for $\alpha$-mixing processes under exponential decay. Using a combination of Cantor set and Bernstein Blocking technique, Merlevède, Peligrad, and Rio (2009) explored a Bernstein-type inequality under geometric $\alpha$-mixing conditions. More recently, Valenzuela-Domínguez, Krebs, and Franke (2017) and Krebs (2018) extended this work to geometric strong mixing spatial lattice process and exponential graph. However, all of the previous results only focus on the regularly-spaced case. Unfortunately, there is surprisingly little research on Bernstein-type inequalities or more general concentration inequalities for irregularly-spaced data. Based on a martingale approach, Delyon (2009) developed an exponential inequality for irregularly-spaced mixing random fields. The preconditions of this result are difficult to specify and fail in including the NED condition. Then, as
mentioned in Section 1, Xu and Lee (2018) delivered the first direct result regarding exponential inequality for geometric irregularly-spaced NED random fields.

Our inequalities generally share the same form as those of Rio (1995) and Liebscher (1996), with the upper bound being composed of an exponential part and a remainder. Together with the blocking technique, the major idea of our proof of Theorem 1 and 2 is to use of the definition of NED. This is quite similar to the tricks demonstrated in Rio (1995). More specifically, by following Definition 2, suppose $Z$ is an irregularly-spaced $L^p$ NED random field on an $\alpha$-mixing random field $\epsilon$. Our result indicates that the probabilistic upper bound of the averaged partial sum process is actually governed by the concentration phenomenon of the random field $\epsilon$. Indeed this is intuitive according to Definition 2. Note, for each $Z_{s_iN} \in Z$, the projection error, i.e. $Z_{s_iN}$’s projection on the orthogonal complete space of $\mathcal{M}(\mathcal{F}_iN(r))$, will converge to 0 with respect to $L^p$ norm, as $r \to +\infty$, where $\mathcal{M}(\mathcal{F}_iN(r))$ denotes the collection of all real-valued $\mathcal{F}_iN(r)$ measurable functions. If we let $r$ be non-decreasing with respect to sample size $N$, we ensure that random field(array) $\epsilon$ is an approximation of field(array) $Z$ under the $L^p$-norm.

Therefore, the assumption of the quantity of $p$ becomes crucial. Many previous literature assumes $p \geq 2$, e.g. Lu and Linton (2007), Li et al. (2012) and Xu and Lee (2018). Recently, Ren and Lu (2020) investigates the asymptotic properties of local linear quantile regression only under the condition $p = 1$ only. In this paper, except for the theorems in Sections 3.2 and 4.1, we always assume $p \geq 1$, which is in concordance with most of the previous research.

### 3.1 Geometric NED

We now present the Bernstein-type inequality for irregularly-spaced geometric NED random fields. Our method can be regarded as a modification of the proof of Theorem 3.1 in Krebs (2018).

**Theorem 1 (Geometric NED)** According to Definition 2, suppose $Z := \{Z_{s_iN}\}_{s_iN \in \Gamma_N}$ is a real-valued geometric irregularly-spaced $L^p (p \geq 1)$ NED random field on $\epsilon := \{\epsilon_{s_iN}\}_{s_iN \in \Gamma_N}$ at the order of $\nu$. For any $i$ and $N$, assume $\|Z_{s_iN}\|_\infty \leq A$, $E[Z_{s_iN}^2] \leq \sigma^2$ and $E[Z_{s_iN}] = 0$. $\epsilon$ is defined as Definition 3. Furthermore, assume max$_{iN} \alpha_iN \leq N^\kappa$, for some $\kappa > 0$, $\phi(x,y) = (x+y)^\tau$, $\tau > 0$ and $\psi(t) = \psi(t) = \nu^{-b}\tau$. Then, under Assump-
tions 1 to 4, for ∀ \( N \geq N_0 \in \mathbb{N}^+ \) and ∀ \( \beta \geq 0 \), we have,

\[
P \left( \left| \sum_{i=1}^{N} Z_{s_iN} \right| \geq t \right) \leq 2K_1 \left( \frac{1}{N^{\kappa + 2\beta + \tau + 1}t} \right)^p \\
+ 4 \exp \left( -\frac{Ni^2}{K_2 (\log N)^{d_2} \left( (C_0 \sigma)^2 + \frac{4C_{0A}A}{3} \right)} \right) \tag{4}
\]

where positive integer \( N_0 \) is specified in the proof and

\[
K_1 = \left( \frac{C_0}{H_0^d} \right)^{2(\kappa + \beta) + \tau + 1}, \\
K_2 = 2^{8 + d_2} \left( C_0 \left( \frac{2(\kappa + \beta) + \tau + 1}{b} \right) \right)^{\frac{1}{2}}, \\
C_0 = \left( \left[ \frac{H_0}{\sqrt{2d_0/2}} \right] + 1 \right)^d.
\]

**Remark 1** Note that even though the first term of the upper bound seems to be a polynomial term, it actually plays a role as a higher order term in deriving the convergence rate of many estimators. The reason for this is that Theorem 1 is valid for arbitrary \( \beta \geq 0 \). When we apply the Bernstein-inequality to investigate the convergence rate of estimators, say \( aN \), a common step is to replace \( t \) with \( aN \). Until now, almost all the estimators converge no faster than \( N^{-\eta} \), for some \( \eta > 0 \). In fact, according to to the second term, \( \eta \) can not be set as some real number larger than \( 1/2 \), which has already achieved the parametric rate. Thus, we can easily find a sufficiently large \( \beta \) such that, after setting \( t = aN \), the first term is at a higher order rate compared with the second term. However, no matter how larger the \( \beta \) we select, it only enlarges the constant \( K_2 \) in the second term and does not change the sharpness of the second term asymptotically.

**Remark 2** Unlike the usually defined uniform NED condition (see Definition 1, Jenish and Prucha (2012)), the assumption \( \max_i N \alpha_i \leq N^\kappa \) is more general and could be more useful in some applications. For example, suppose \( Z = \{Z_{s_iN}\}_{s_iN} \in \Gamma_N \) is a \( Z \)-valued \( L^p \) irregularly-spaced NED random field on \( \epsilon = \{\epsilon_{s_iN}\}_{s_iN} \in \Gamma_N, Z \subset \mathbb{R}^D \). Let \( f : \mathbb{Z} \to \mathbb{R} \) be a Lipschitz continuous function with module Lip\( (f) \). Then according to Proposition 2, we know \( \{f(Z_{s_iN})\}_{s_iN} \in \Gamma_N \) can be regarded as a real-valued NED process on \( \epsilon \) and \( \|f(Z_{s_iN}) - E[f(Z_{s_iN}) | \mathcal{F}_{iN}(r)] \|_p \leq 2\text{Lip}(f)\psi_Z(r) \). But sometimes Lip\( (f) \) may diverge to infinity as \( N \to +\infty \). A typical example is a smoothing kernel function from Lipschitz class, \( K(Z_{s_iN} - z/h) \), whose Lipschitz module is \( O(1/h) \).
Actually the first term on the right side of (4) denotes the approximation error between process \( Z \) and \( \epsilon \) whereas the second term is an upper bound of the concentration phenomenon of the projection process \( \{ E[Z_{s_iN}|F_{iN}(r)]\}_{s_iN \in \Gamma_N} \), which is an \( \alpha \)-mixing sequence. Therefore, a natural corollary of Theorem 1 is as follow,

**Corollary 1 (geometric \( \alpha \)-Mixing)** Suppose \( \epsilon := \{ \epsilon_{s_iN} \}_{s_iN \in \Gamma_N} \) is a mean-zero and real-valued \( \alpha \)-mixing random field that satisfies, for any \( i \) and \( N \), \( ||\epsilon_{s_iN}||_\infty \leq A \) and \( E[\epsilon_{s_iN}^2] \leq \sigma^2 \). Assume \( \phi(x,y) = (x + y)^\tau \) and \( \psi_\epsilon(t) = \nu^{-bt^\gamma} \), for some \( \tau, \gamma, b > 0 \). Then, under Assumptions 1 to 4, we have

\[
P \left( \left| N^{-1} \sum_{i=1}^{N} \epsilon_{s_iN} \right| \geq t \right) \leq 4 \exp \left( -\frac{Nt^2}{K_2(\log_\nu N)^{\frac{d_2}{4}} ((C_0\sigma)^2 + 4C_0At^3)} \right),
\]

where \( K_2 \) and \( C_0 \) are inherited from Theorem 1 directly.

In order to facilitate comparison with the pioneering work of Xu and Lee, based on our notation, we first brief their result here.

**Theorem A.1 Xu and Lee (2018)** Under Assumptions 1 and 2, suppose random field \( \{ Z_{s_iN} \}_{s_iN \in \Gamma_N} \) is a real-valued random field satisfying \( E[Z_{s_iN}] = 0 \) and \( \sup_{s_iN} ||Z_{s_iN}||_\infty \leq A \). Also assume for any \( U = \{ s_iN, i = 1, ..., |U| \} \), \( V = \{ t_jN, j = 1, ..., |V| \} \subset \Gamma_N \),

\[
\text{Cov} \left( \prod_{i=1}^{[U]} Z_{s_iN}, \prod_{i=1}^{[V]} Z_{t_jN} \right) \leq C^* \epsilon^{[U]+[V]} \exp(-a\rho(U,V))A^{[U]+[V]},
\]

for some \( C^* \) and \( a > 0 \). Then we have

\[
P \left( \left| N^{-1} \sum_{i=1}^{N} Z_{s_iN} \right| \geq t \right) \leq C_1^* \exp \left( -C_2^* N^{\frac{1}{2(\tau+\gamma)}} t^2 \right),
\]

where both \( C_1^* \) and \( C_2^* \) are positive constants but \( C_2^* \) has nothing to do with variance information.

The advantage of (4) and (5) are relatively obvious. First, they are adaptive to the counterexamples mentioned in the third limitation of (7) in Section 1. More specifically, only the "effective dimension" would really affect the sharpness of inequality up to a \( \log_\nu N \) level, which is clearly sharper than (7). Secondly, Theorem 1 holds for the NED condition under the \( L^p \)-norm for any \( p \geq 1 \), whereas (7) is only valid for the case \( p \geq 2 \). Meanwhile,
as shown in the applications, (4) and (5) can be easily used to investigate of the attainability of optimality for many widely used estimators. Furthermore, compared with (7), our results still lead us to obtain sufficiently sharp convergence rate of estimators whose variance becomes arbitrary small asymptotically, like kernel density estimator. Lastly, (4) and (5) are valid for any $\nu > 0$, while according to the proof of (7), $\nu = e$ is a necessary setting. Generally speaking, Theorem 1 and Corollary 1 almost solve the above mentioned shortcomings of the result from Xu and Lee (2018)(see S1-S3).

Unfortunately, most of time, there is no "free lunch". Xu and Lee (2018) directly obtained an exponential inequality on a irregularly-spaced weakly-dependent random field with zero mean and bounded support, which contains both NED and $\alpha$-mixing random fields as special cases. The only preconditions about the sampling of locations are Assumptions 1 and 2. Therefore, compared with (4) and (5), (7) coincides with more general setting.

However, an important criterion for evaluating a "trade-off" is to check whether the additional restriction is sufficiently mild while the gain is significant. In this paper, these additional restrictions come from two directions. First, compared with (4), (7) does not rely on Assumptions 3 and 4. But as we argued in Section 2, however, Assumption 3 may be seen as a natural consequence of Assumption 2 and Assumption 4 holds for many practical examples. Second, Theorem 1 and Corollary 1 concentrate solely on NED and $\alpha$-mixing conditions whereas Xu and Lee’s result is valid for any weakly-dependent random field satisfying (6). Note, however, that the motivation for both (4) and (7) is to derive a useful concentration inequality for the geometric irregularly-spaced NED random field. Hence our result automatically fits in well with all the applications mentioned in Xu and Lee (2018), particularly spatial autoregressive Tobit model(SART). This, to some degree, reflects the generality of our results (4) and (5) as well.

As for Corollary 1, compared with Adamczak (2008), Modha and Masry (1996) and Merlevède et al. (2011), our result is either sharper or valid for much more general settings. It seems that Delyon (2009)'s result could be as sharp as the iid setting. But Corollary 1 is derived solely based the covariance inequality in Lemma 2, which is entirely different from the preconditions of Delyon’s work.

### 3.2 Algebraic NED

In addition to geometric NED, according to Proposition 1 from Xu and Lee (2015a), SART may satisfy the algebraic NED condition only sometimes.
Thus to seek a Bernstein-type inequality for an algebraic irregularly-spaced NED random field is crucial. Before presenting Theorem 2, we introduce the following assumptions and proposition.

**Assumption 5** We assume the $H_0$ defined in Assumption 3 satisfies $H_0 \geq 4C_0$.

**Proposition 5** Based on Assumption 3, suppose the edges that are going to be infinitely long are the $k_j$-th edges, $j = 1, 2, ..., d_2$. Let $\hat{N} = \prod_{j=1}^{d_2} H_{k_j}^N$, where $H_{k_j}^N$ denotes the length of the $k_j$-th edge of $R_N$. Then, under Assumptions 2 and 4, we have,

$$\frac{H_{0}^{d_2}N}{C_0} \leq \hat{N}, \quad \hat{N} \leq \frac{H_{0}^{d_2-d_1}}{2^{d_2}} N.$$  

Assumption 5 is simply an assumption without loss of generality designed to make the result more compact. Proposition 5 is nothing other than a direct result of Assumptions 2 and 4.

**Theorem 2 (Algebraic NED)** Suppose $Z := \{Z_{s_iN}\}_{s_iN \in \Gamma_N}$ is a real-valued algebraic irregularly-spaced $L^s (s \geq 2)$ NED random field on $\epsilon := \{\epsilon_{s_iN}\}_{s_iN \in \Gamma_N}$ at the order of $\nu_1$, which indicates $\psi_Z(t) = t^{-\nu_1}$, for some $\nu_1 > 0$. For any $i$ and $N$, $\|Z_{s_iN}\|_{\infty} \leq A$, $\|Z_{s_iN}\|_2 \leq \sigma$, $\|Z_{s_iN}\|_{2+\delta} \leq \sigma_{2+\delta}$, and $E[Z_{s_iN}] = 0$. Denote $\max_{ij} |E[Z_{s_iN}Z_{s_jN}]| := \sigma_{ij}$. $\epsilon$ is defined as Definition 3. Assume $\max_{i,N} \alpha_{iN} \leq \alpha(N)$, $\phi(x,y) = (x+y)^\tau$ and $\psi_\epsilon(t) = t^{-\nu_2}$, for some $\tau$ and $\nu_2 > 0$. $\alpha(N)$ is non-decreasing with respect to $N$. Set $q \in \mathbb{N}$, where $H_{k_j}^N$’s are defined as Proposition 5. Then, under Assumptions 1-5, for some $C > 0$, $\forall \ t > 0$, we have,

$$P \left( \left| \sum_{i=1}^{N} \frac{Z_{s_iN}}{N} \right| \geq t \right) \leq 2 \exp \left( -\frac{H_{0}^{d_1}N t^2}{32C_0 \left( 2^{d_2-1}v(q) + \frac{AH_{0}^{d_1}N t}{12(2^{d_2})C_0} \right)} \right)$$  

$$+ 11K_3 \left( 1 + \frac{8AH_{0}^{d_2-d_1}}{t} \right)^{\frac{1}{2}} q^{d_2} \left( \frac{N}{q^{d_2}} \right)^{\frac{\nu_2}{\alpha(N)} - \frac{\nu_1}{\alpha(N)}} + K_4 \left( \frac{N}{q^{d_2}} \right)^{\frac{\nu_2}{\alpha(N)}} \left( \frac{\alpha(N)}{t} \right)^s,$$

(8)
where
\[ v(q) = \max \left\{ \sigma^2 + \sigma_{ij}^2 B(N, q), 3 \left( C_0 C^* \left( \frac{2}{3H_0} \right) \frac{N}{q^{d_2}} (\sigma_{ij} \vee \sigma_{ij}^2 \vee \frac{2}{3d_2} \alpha(N) \left( \frac{N}{q^{d_2}} \right)^{-\frac{\nu_1}{d_2}}) \right) \right\}, \]
\[ B(N, q) = C^* \left( \frac{N}{q^{d_2}} \right)^{-\frac{\nu_2 \delta}{2(2+\nu_2)}} + 1, \quad C_p \leq \frac{C_0}{2d_2 H_0^{d_1}} \left( \frac{N}{q^{d_2}} \right) \text{ and } C_p = O \left( \frac{N}{q^{d_2}} \right). \]

\( K_3, K_4, C^*, C^{**} \in \mathbb{R}^+ \) are specified in the proof.

The core idea for obtaining the Bernstein-type inequality under an algebraic NED condition is motivated by Q. Yao (2003), in which, a blocking technique and Bradley’s coupling lemma are combined to obtain an exponential inequality for regularly-spaced \( \alpha \)-mixing random field. Unfortunately, Theorem 2 can not generally help us to attain the optimality of the \( L^\infty \) convergence rate for an arbitrary estimator under NED condition. The major reason is that \( v(q) \) may not be sharp enough. We suppose this cannot be improved based on our approach. However, the situation is much better if we focus on kernel smoothing. The term \( \sigma_{ij} := \max_{i,j} |E[Z_{s_iN} Z_{s_jN}]| \) in \( v(p) \) could explain the reason for this. Indeed, according to lemmas A.2 and A.3 in Lu and Linton (2007), as well as lemma A.5 and Theorem 3.1 in Li et al. (2012), by letting \( Z_{s_{iN}} = K(X_{s_{iN}} - x/h)Y_{s_{iN}} \), we have \( E[Z_{s_{iN}} Z_{s_{jN}}] = O(h^{2D}) \), provided \( X \in \mathbb{R}^D \). This result relies heavily on the convolution property of the kernel smoother, which may not be satisfied by many other estimators.

Due to the second and third terms on the right side of (8), another significant phenomenon displayed by (8) is the effect of \( d_2 \), which is defined as ”effective dimension” (see. Assumption 3). Meanwhile, note that rate of the first term within the upper bound is independent of \( d_2 \). Therefore, under algebraic NED condition, inequality (8) is also adaptive to the effective dimension \( d_2 \). Moreover, as shown later in Section 4.1, Theorem 2 helps us to obtain the optimality of local linear fitting for algebraic irregularly-spaced NED random fields, which to some degree extends the result from Jenish (2012).

4 Locally Polynomial Regression

In this section, we will revisit local polynomial fitting for irregularly-spaced dependent data. Section 4.1 discusses the uniform convergence rate of local linear fitting of conditional mean function under the algebraic NED condition, which completes or refreshes the results of Li et al. (2012) and Jenish
In Section 4.2, Cattaneo et al. (2020) recently proposed a generically unbiased density estimator by using local polynomial regression to smooth the empirical cumulative distribution function of a random variable. We will investigate its uniform convergence rate for the geometric irregularly-spaced $\alpha$-mixing random field.

### 4.1 Conditional Mean Function Estimation

Nonparametric modeling for spatial data has attracted much attention over the past two decades, particularly that using kernel-based local estimator. In this section, we will investigate the uniform convergence rate of local linear fitting for an algebraic irregularly-spaced NED random field. More specifically, suppose we have the following regression model,

$$Y_{s_{iN}} = m(X_{s_{iN}}) + e_{s_{iN}},$$

$$E[e_{s_{iN}}|X_{s_{iN}}] = 0.$$  

According to Lu and Linton (2007), Li et al. (2012) and Jenish (2012), given $X_{s_{iN}} = x$, local linear estimator of $m(x)$ is defined as,

$$\hat{m}(x) = (1, 0)\hat{\theta}(x), \quad \hat{\theta}(x) = S_N(x)^{-1}T_N(x),$$

$$S_N(x) = \frac{1}{N} \sum_{i=1}^{N} K_{ih}(x)U_i(x)U_i(x)^T; \quad T_N(x) = \frac{1}{N} \sum_{i=1}^{N} K_{ih}(x)U_i(x)Y_{s_{iN}};$$

$$K_{ih}(x) = \frac{1}{h}K\left(\frac{X_{s_{iN}} - x}{h}\right); \quad U_i(x) = \left(1, \left(\frac{X_{s_{iN}} - x}{h}\right)^T\right)^T,$$

where 0 is a D-dimensional zero vector.

**Assumption 6** $Z := \{X_{s_{iN}}, Y_{s_{iN}}\}_{s_{iN} \in \Gamma_N}$ is an algebraic and irregularly-spaced $L^p$-NED ($p \geq 2$) process on process $\epsilon := \{e_{s_{iN}}\}_{s_{iN} \in \Gamma_N}$. Here $\epsilon$ is an $\alpha$-mixing random field defined as Definition 3. Here $\Gamma_N \subset \mathbb{R}^d$ and max $\alpha_{iN} \leq 1$.

**Assumption 7** Let $\psi_Z(t) = t^{-\nu_1}$ and $\psi_\epsilon(t) = t^{-\nu_2}$, for some $\nu_1$ and $\nu_2 > 0$. For any finite subset $U$ and $V$ in $\Gamma_N$, $\phi(|U|, |V|) = (|U| + |V|)^\tau$, where $\tau > 0$.

**Assumption 8** $(X_{s_{iN}}, Y_{s_{iN}})$ is identical to random vector $(X, Y) \in \mathcal{X} \times \mathbb{R}$, where $\mathcal{X}$ is compact in $\mathbb{R}^D$. Assume $E[Y^2|X = x] \leq \sigma_Y^2$, $E[Y^p_0|X = x] < +\infty$, for some $p_0 > 2$ and $\forall \ x \in \mathcal{X}$. Assume that the conditional mean function $m(x)$ is H"older continuous on $\mathcal{X}$ with exponent $\beta > 0$. 

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Assumption 9 The kernel function $K(u)$ is Lipschitz continuous with compact support. Without loss of generality, we assume $K(u) \in \text{Lip}([-1,1]^D)$.

Assumption 10 We assume $p_0$ is large enough such that

(1) There exists some $\alpha > 0$, such that, $\log N/N^{1-2/p_0} h^D = O(h^\alpha)$.

(2) $N^{1-2/p_0} h^D + 1 = O(1)$.

Assumption 11 Based on the $p_0$ and $\alpha$ given in Assumption 10, $\nu_1$ is sufficiently large such that

(1) $N^{1-2/p_0} h^{\alpha \nu_1 + D-1} = O(h^D)$, (2) $h^{\alpha \nu_1} N^2 \log N = o(1)$.

Assumption 12 Based on the $p_0$ and $\alpha$ given in Assumption 10, $\nu_2$ is sufficiently large such that

(1) $\left(\frac{\alpha \nu_2 (p_0-2)}{2d_2} + 2D\right) \frac{1}{p_0} > D + \frac{1}{2}$, (2) $h^{\alpha \nu_2} (\frac{N^2 \log N}{h^D} + 1) = o(1)$.

Theorem 3 Under Assumptions 1-4 and 6-11, we have

$$\sup_{x \in X} |\hat{m}(x) - m(x)| = O_p(\sqrt{\frac{\log N}{Nh^D}} + h^\beta).$$

Assumption 10 requires $p_0$ to be sufficiently large. More specifically, (1) in Assumption 10 is easy to achieve. For example, by setting $h = O(\log N/N)^{1/D}$ and conditional mean function as Lipschitz continuous function ($\beta = 1$), we have

$$\frac{\log N}{N^{1-2/p_0} h^D} = \frac{(\log N)^{1-\frac{1}{2+D}}}{N^{1-\frac{1}{2+D}}},$$

Since $p_0 > 2$ and $h = O(\log N/N)^{1/D}$, we instantly know that there exists an $\alpha > 0$ such that (1) of Assumption 10 is satisfied. Under these similar settings about $h$ and $\beta$, (2) in Assumption 10 can be easily satisfied as well. These settings are widely accepted when we try to obtain the optimal uniform convergence rate. Furthermore, to check whether $h = O(\log N/N)^{1/D}$ is feasible, we only need to check whether it can satisfy Assumption 11 and Assumption 12 (2). Actually, for any given $\alpha$, $D$, $d_2$, $\tau$ and $p_0$, there always
exist $\nu_1$ and $\nu_2$ satisfying Assumptions 11 and 12. Therefore, according to Theorem 3, after some simple calculation, we know that the local linear estimator of the conditional mean function for algebraic irregularly-spaced NED random fields could achieve the minimax optimal rate proved by Stone (1982).

Compared with Theorem 3 in Jenish (2012), our result is obviously sharper but seems to cover only the stationary case. Our result can be easily extended, however, to the nonstationary case by employing Assumption 5 from Jenish (2012), because this assumption is not involved in deriving the uniform convergence rate. Moreover, Theorem 3 also extends Li et al. (2012)’s result to irregularly-spaced setting.

### 4.2 Density Estimation

The kernel density estimator (KDE) is one of the most widely used density estimators and has been thoroughly investigated. It is well known that, KDE suffers from the boundary effect, which means KDE is inconsistent when the evaluation point is close to the boundary of the support set. Many good modification methods have been proposed to solve this problem. Karunamuni and Alberts (2005) conducted a useful literature review on this topic. Cattaneo et al. (2020) proposed a method called simple local polynomial density estimator (SLPDE), which generically solve this problem. In particular, compared with the methods mentioned in Karunamuni and Alberts (2005), except for being adaptive to the boundaries of the support of the density without specific data modification or additional tuning parameter, SLPDE inherits all the advantages of local polynomial fitting. Thus, in this section, we will focus on the uniform convergence rate for SLPDE for a real-valued geometric irregularly-spaced $\alpha$-mixing random field.

First, we brief the main idea of SLPDE. It is obvious that, for any point $x$ in the support set, by denoting $F(x)$ as the cumulative distribution function (CDF), we have

$$F(x) = E[F(X)|X = x] := E[U|X = x].$$

Hence, if we consider $F(X)$ to be another random variable $U$ (uniformly distributed over $[0,1]$), $F(x)$ becomes a conditional mean function $E[U|X = x]$ and density function $f(x)$ is the derivative of the conditional mean function. Obviously, local polynomial regression can be used to fit these functions. However, this is not a feasible estimator because we do not know the true CDF. A natural idea is to use the empirical distribution function $F_N$ to
where

\[ \hat{\theta}(x) = U_N(x)^{-1}V_N(x), \]

and

\[ U_N(x) = \frac{1}{N} \sum_{i=1}^{N} K_{ih} U \left( \frac{X_i - x}{h} \right) U \left( \frac{X_i - x}{h} \right)^T \]

\[ V_N(x) = \frac{1}{N} \sum_{i=1}^{N} K_{ih} U \left( \frac{X_i - x}{h} \right) F_N(X_i), \]

\[ X_i = X_{s_iN}, \quad K_{ih} = \frac{1}{h} K \left( \frac{X_i - x}{h} \right), \quad U(x) = (1, x, x^2, \ldots, x^p)^T, \]

\[ \hat{\theta}(x) = (\hat{F}(x), \hat{F}'(x)h, \ldots, \frac{1}{p!} \hat{F}^{(p)}(x)h^p)^T. \]

Denote \( \theta(x) = (F(x), F'(x)h, \ldots, \frac{1}{p!} F^{(p)}(x)h^p)^T \) and \( v_k \) as the k-th element of vector \( v \). A natural inequality is

\[ ||\hat{F}^{(k-1)}(x) - F^{(k-1)}(x)||_\infty \leq h^{-(k-1)} ||\hat{\theta}(x)_k - \theta(x)_k||_\infty. \]

Therefore, to obtain the uniform convergence rate of SLPDE, it suffice to prove the convergence rate of \( ||\hat{\theta}(x)_2 - \theta(x)_2||_\infty \). Additionally, before demonstrating our result, we exhibit the following propositions which are crucial to our proof and may be of independent interest.

**Proposition 6** Based on Definition 3 and Assumptions 1-4, suppose \( X := \{X_{s_iN}\}_{s_iN \in G_N} \) is an \( \mathbb{R}^d \)-valued irregularly-spaced \( \alpha \)-mixing random field. Assume \( \phi(x, y) = (x+y)^\gamma \) and \( \psi(t) = e^{-bt}\gamma \), for some \( \tau, \gamma, b > 0 \). Suppose \( G \) is a class of measurable functions and the lower graph set of each function has a finite VC-dimension, say \( V_{G+} \). Furthermore, assume for \( \forall g \in G \), we have \( ||g||_\infty \leq A \), for some \( A > 0 \). Then, under Assumptions 1 to 4, if \( V_G \geq 2 \), \( \exists N^* \in \mathbb{N} \), such that for \( \forall N \geq N^* \), we have

\[ P \left( \sup_{g \in G} \left| N^{-1} \sum_{i=1}^{N} (g(X_{s_iN}) - E[g(X_{s_iN})]) \right| > t \right) \leq 40 \exp \left( -\frac{Nt^2}{(48A)^2} \right), \quad (9) \]

where \( N^* = N_0 \lor N_1 \lor N_2 \), \( N_1 = \min\{N \in \mathbb{N} : \sqrt{N}t \geq 36A\} \) and

\[ N_0 = \min \left\{ N \in \mathbb{N} : \sqrt{N}t \geq 48A \int_{16A}^{\infty} V_{G+}^+ \log 3 \left( \frac{8e}{u^2} \log \frac{12e}{u^2} \right) du \right\}, \]

\[ N_2 = \min \left\{ N \in \mathbb{N} : \frac{N}{(\log N)^{d_2/\gamma}} \geq 4K_2 \log 8C_0A((C_0A) + 8t/3)/t^2 \right\}. \]

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A natural corollary of this proposition is the following weak-version of Dvoretzky–Kiefer–Wolfowitz (DKW) inequality for geometric irregularly-spaced $\alpha$-mixing random field, which will be very useful later in the proof of Theorem 4.

**Corollary 2** Based on Definition 3, suppose $X := \{X_{s_i,N}\}_{s_i,N \in \Gamma_N}$ is a strict stationary $\mathbb{R}^D$-valued irregularly-spaced $\alpha$-mixing random field. Assume $\phi(x, y) = (x + y)^\tau$ and $\psi(t) = e^{-bt^\gamma}$, for some $\tau, \gamma, b > 0$. Let $A$ be the collection of all rectangles in $\mathbb{R}^D$. Let $P_N$ and $P$ denote the empirical and true probability measure respectively. Then, for sufficiently large $N$, we have

$$P \left( \sup_{A \in A} |P_N(A) - P(A)| \geq t \right) \leq 40 \exp \left( -\frac{Nt^2}{2304} \right). \quad (10)$$

What is surprising here is that, both (9) and (10) share the same decreasing rate as Hoeffding inequality for the iid case. Unfortunately, this still cannot help us to obtain the optimal rate of estimators if the variance were to become sufficiently small as the sample size diverges to infinity. Therefore, in order to solve this problem, we take one further step and deliver the following proposition.

**Proposition 7** Based on Definition 2 and Assumptions 1-4, suppose that $\{X_{s_i,N}, Y_{s_i,N}\}_{s_i,N \in \Gamma_N}$ is an $\mathcal{X} \times [-L, L]$-valued irregularly-spaced $\alpha$-mixing random field, for some $L > 0$ and $\mathcal{X} \times [-L, L] \subset \mathbb{R}^{D+1}$. Assume $\phi(x, y) = (x + y)^\tau$ and $\psi(t) = e^{-bt^\gamma}$, for some $\tau, \gamma, b > 0$. Let $\mathcal{G}$ be a class of real-valued measurable functions $g : \mathcal{X} \to \mathbb{R}$. Assume for any $g \in \mathcal{G}$, $\|g\|_{\infty} \leq A$ and $\|g\|_2 \leq \sigma$. Then, for any $N, t$ that satisfy

\begin{align*}
\text{C1.} \quad & tA \leq \frac{3A^2 L}{40}, \\
\text{C2.} \quad & \sqrt{\frac{Nt^2}{(\log N)^{\frac{d_j}{\tau}}}} \geq 2\sqrt{10K_2\sigma}C_0L \int_{\frac{2}{\frac{1}{\tau}}}^{\sigma^{1/4}} \sqrt{\log N(u^2, \mathcal{G}, \|\cdot\|_{\infty})} du, \\
\text{C3.} \quad & \sqrt{\frac{Nt^2}{(\log N)^{\frac{d_j}{\tau}}}} \geq 24\sqrt{10K_2C_0}L\sigma,
\end{align*}

we have

$$P \left( \sup_{g \in \mathcal{G}} |P_N(gY) - P(gY)| \geq t \right) \leq 8 \exp \left( -\frac{Nt^2}{1440C_0^2K_2(\log N)^{\frac{d_j}{\tau}} L^2\sigma^2} \right), \quad (11)$$

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where \( P_N(gY) = N^{-1} \sum_{i=1}^{N} g(X_{s_iN})Y_{s_iN} \) and \( P(gY) = E[N^{-1} \sum_{i=1}^{N} g(X_{s_iN})Y_{s_iN}] \).

Condition C1 is the reason why only the variance term is involved in the upper bound. For the case where the variance is asymptotically larger than 0, C1 is trivial. When the variance term convergence to 0 as sample size grows, this condition C1 is also suitable for many situations, such as kernel-based estimators. This can be checked in the proof of Proposition 8 and Theorem 4.

**Assumption 13** The kernel function \( K(u) \) is a Lipschitz continuous, non-negative and symmetric function with compact support \([-1,1]\).

**Assumption 14** Assume \( \{X_{s_iN}\}_{s_iN} \in \Gamma_N \) is a real-valued strict stationary geometric \( \alpha \)-mixing random field sharing the same marginal distribution as random variable \( X \). For any subsets \( U \) and \( V \) of \( \Gamma_N \) such that \(|U| \vee |V| < +\infty\), \( \alpha(\sigma(U),\sigma(V)) \leq (|U| \vee |V|)^{\tau} \exp(-b\rho(U,V)^{\gamma}) \), for some \( b, \tau > 0 \) and \( \gamma > d_2 \). \( d_2 \) here is the "effective dimension" defined in Assumption 3.

**Assumption 15** Assume \( X \) has bounded support \([a,b]\). The CDF of \( X \), say \( F(x) \), is \( p+1 \) times continuously differentiable for some \( p \geq 1 \) and its probability density function, \( f(x) \), satisfies \( \inf_{x \in [a,b]} f(x) > 0 \).

**Assumption 16** For sample size \( N \) and bandwith \( h \), we assume, for any \( l = 0, 1, \ldots, p, \)

\[
\lim_{N \to +\infty} h = 0, \quad \lim_{N \to +\infty} \frac{\log N}{Nh^{2l+1}} = o(1).
\]

**Proposition 8** Under Assumptions 1-4 and 13-16, we have,

\[
\sup_{x \in [a,b]} \left| \frac{1}{N} \sum_{i=1}^{N} g_h^l(X_{s_iN}) F_N(X_{s_iN}) - E \left[ g_h^l(X) F_N(X) \right] \right| = O_p \left( \sqrt{\frac{\log N}{Nh^{l+1}}} \right),
\]

where

\[
g_h^l(X) = \left( \frac{X-x}{h} \right)^l K_h \left( \frac{X-x}{h} \right), \quad K_h \left( \frac{u}{h} \right) = \frac{1}{h} K \left( \frac{u}{h} \right).
\]

**Theorem 4** Under Assumptions 1-4 and 13-16, we have, for any \( v = 0, 1, 2, \ldots, p \) and \( \forall \ x \in [a,b] \)

\[
\sup_{x \in [a,b]} |\hat{F}^{(v)}(x) - F^{(v)}(x)| = O_p \left( \sqrt{\frac{\log N}{Nh^{2v+1}}} + h^{p+1-v} \right).
\]
More specifically, for density function,

$$\sup_{x \in [a,b]} |\hat{f}(x) - f(x)| = \sup_{x \in [a,b]} |\hat{F}'(x) - F'(x)| = O\left(\sqrt{\frac{\log N}{Nh^3}} + h^p\right).$$

Assumptions 13, 15 and 16 are borrowed directly from Cattaneo et al. (2020). Note that Assumption 15 actually indicates that the density function of $X$ belongs to the $p-th$ order of Hölder class. Theorem 4 reflects that, unlike kernel density estimator, SLPDE may not be able to achieve the optimal uniform convergence rate. However, as shown by Theorem 4 and the simulation part of Cattaneo et al. (2020), SLPDE is asymptotically unbiased for any $x \in [a,b]$ which significantly outperforms the kernel density estimator near the boundary points.

5 Modal Regression

In standard prediction problems, we are interested in predicting the value of output $Y$ given $X = x$. Conditional mean regression is a standard method that has been thoroughly studied. This method may fail, however, when the conditional density function $f_{Y|X}(y|x)$ is significantly asymmetric. After all, prediction via conditional mean function only tells us an “average-value” of random variable $Y$ given $X = x$, whereas what we are really interested in is to investigate, the value of output $Y$ on condition of $X = x$. Under this circumstance, global or local modal regression would seem to be a more natural approach.

On this topic, there has been much outstanding previous research. Here we only name a few. For global modal regression, Lee (1989), Sager and Thisted (1982), W. Yao and Li (2014) assume a parametric linear model, 

$$\text{Mode}(Y|X = x) = \beta_0 + \beta^T x,$$

where $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^D$ are unknown coefficients and Mode$(Y|X = x)$ denotes the global mode of $Y$ given $X = x$. Nonparametric global modal regression is flexible because the linearity of conditional mode function is not needed. Collomb, Härdle, and Hassani (1986) investigate the uniform convergence rate of conditional mode function estimator based on kernel density estimation under $\phi$-mixing process. More recently, W. Yao, Lindsay, and Li (2012) propose a mode estimation based on local polynomial smoothing. Unlike global modal regression which seeks to detect the global maximum point of conditional density $f_{Y|X}(y|x)$ (or equally, joint density, $f(x,y)$) given $X = x$, local modal regression tends to detect the local extreme points satisfying, $\partial f(x,y)/\partial y = 0$ and $\partial^2 f(x,y)/\partial y^2 < 0$. Scott (2015) proposed an estimator of this according
to commonly used kernel-based joint density estimates. Motivated by this, Chen, Genovese, Tibshirani, and Wasserman (2016) thoroughly investigated the uniform convergence rate and confidence set of local modal regression based on the kernel density estimator. Here, we focus on nonparametric global modal regression in a setting like that of Collomb et al.

More specifically, our object is to detect

$$\arg \max_{y \in \mathcal{Y}_x} f(y|x) := y_x,$$

where $\mathcal{Y}_x$ is the support of $Y$ given $X = x$. Hence a natural estimator is

$$\hat{y}_x.$$
the $L^p$ distance of some probabilistic measure. The sub-Gaussianity here actually offers us exponential level tail probability to control the chaining (Pollard, 2015). This point can also be shown by the proof of Theorem 19.1 in Györfi, Kohler, Krzyżak, and Walk (2002) and Proposition 6 and 7 in this paper. Thus it seems that Theorem 1 may not be suitable for this argument. However, things are actually not that severe for geometric NED. Because, as we mentioned in Remark 1, a key advantage of Theorem 1 is that it is valid for any $\beta > 0$. That is to say, setting $\beta$ as an arbitrarily large positive number does not change the decreasing rate of the exponential part of (4) while makes the polynomial part arbitrarily sharper. This flexibility allows us to continue using a chaining argument. More specifically, similar to Proposition 7, we have Proposition 9 below.

**Assumption 20** Let $\mathcal{G}$ be a class of real-valued functions defined on set $\mathcal{X} \times \Theta$, where $\mathcal{X}$ is defined as Assumption 19. Here $\Theta$ is compact sets in $\mathbb{R}^m$ respectively. Furthermore, for any given $x \in \mathcal{X}$, we assume

$$||g(x, \theta) - g(x, \theta')||_\infty \leq C_{\text{Lip}}||\theta - \theta'||_E,$$

where $||\cdot||_E$ is any given norm on $\mathbb{R}^m$. Similarly, for any given $\theta \in \Theta$, we assume

$$|g(x, \theta) - g(x', \theta)| \leq D_{\text{Lip}}||x - x'||_F,$$

where $||\cdot||_F$ is any given norm on $\mathbb{R}^D$. Here both $C_{\text{Lip}}$ and $D_{\text{Lip}}$ are positive constants allowed to diverge to infinitely large as sample size $N \to +\infty$. More specifically, without loss of generality, we assume $D_{\text{Lip}} = N^\alpha$, for some $\alpha > 0$. Furthermore, we also assume for any $g \in \mathcal{G}$, $||g||_\infty \leq A$ and $\sup_{\theta \in \Theta}||g||_2 \leq \sigma$.

**Proposition 9** Based on Assumptions 18-20, without loss of generality, we further assume that $\max_{i,N} \alpha_i = 1$ and there exists some positive constant $L$ such that $\mathcal{X} \subset [-L, L]^D$, $\Theta \subset [-L, L]^m$ and $Y \subset [-L, L]$ respectively. Suppose $N$ and $t$ satisfy conditions C1, C3 in Proposition 7 and the following conditions C4 and C5,
\textbf{C4.} \( \exists \beta_0 > 0 \) and \( C_{\beta_0} > 0 \), such that

\[
\max\{A, B, C\} \leq C_{\beta_0} N^{\beta_0},
\]

\[
A = t^2 \left( \log_2 \frac{4L\sigma}{t} \right)^{\frac{3}{2}},
\]

\[
B = \frac{C_2^2 \log_2 \frac{16L_2}{t}}{32\sigma} \int_{t/16L}^{\sigma/4} \left( \frac{C_{Lip}}{u} \right)^{2m} \, du,
\]

\[
C = 160C_1^3 C_0^2 \sigma \left( \log N \right)^{\frac{d_2}{d}} \int_{t/16L}^{\sigma/4} \left( \frac{C_{Lip}}{u} \right)^{3m} \, du.
\]

\textbf{C5.} \( N^\alpha \geq 2A \).

\textbf{C6.}

\[
\sqrt{\frac{N t^2}{(\log N)^{\frac{d_2}{d}}} \geq 2 \sqrt{10K_2\sigma L} \int_{\frac{1}{2}(\frac{1}{L})^{1/4}} \sqrt{m \log \left( \frac{C_1 C_{Lip}}{u} \right)} \, du.}
\]

For any \( \beta > \beta_0 \), we have

\[
P \left( \sup_{g \in G} |P_N(gY) - P(gY)| \geq t \right) 
\leq 8 \exp \left( - \frac{N t^2}{1440C_0^2 K_2 (\log N)^{\frac{d_2}{d}} L^2 \sigma^2} \right) + \frac{1}{N^{\alpha + \tau + 1}}
\]

Here \( \tau, K_2 \) and \( C_0 \) are constants inherited from Theorem 1. \( C_1 \) is a positive constant specified in the proof.

Assumption 20 indicates that the functions in set \( G \) can be parametrized by \( \theta \) and is Lipschitz continuous on the parameter space \( \Theta \) and support set \( \mathcal{X} \) respectively. This is very crucial to prove Proposition 9. Because based on the parametrization, the covering number of \( G \) becomes much smaller (see Lemma 19 and 21 in \textit{Hang, Steinwart, Feng, and Suykens (2018)}), which allows us to combine the “chaining argument” with the polynomial part in (4). Additionally, in order to apply Theorem 1, we need to ensure that random field \( \{W_{iN}\}_{s_{iN} \in \Gamma_N}, W_{iN} := g(X_{iN}, \theta)Y_{iN} \), is still a geometrical NED random field and the NED coefficient can be bounded exponentially. This can be guaranteed by Assumption 20 and Proposition 3. As a corollary of Proposition 9, we manage to obtain a non-asymptotic uniform convergence rate of kernel density estimator under geometric NED conditions.
Condition C4 seems to be a little bit complex. But it’s easy to be achieved when the growth condition of the covering number of function class $G$ is at level $O(N^\varphi)$, $\varphi > 0$, while the parameter $\beta$ in Theorem 1 is purely user-defined and allowed to be set as arbitrary positive real number. Actually, this point can be reflected by the proof of Theorem 5, in which condition C4 is easily satisfied if we set the kernel function used in density estimation as Lipschitz or Hölder continuous function like Assumption 20.

**Theorem 5** Let $Z_i := (X_{s_iN}, Y_{s_iN})$, $s_iN \in \Gamma_N$ be a sequence of $Z := \mathcal{X} \times \mathcal{Y}$-valued random vectors satisfying Assumptions 18 and 19. Denote $Z = (X,Y)$ and $G(u)$ is a non-negative Lipschitz continuous function on $\mathbb{R}^{D+1}$ and is uniformly upper bounded by $A > 0$. Then, together with Assumptions 1-4,

$$\sup_{z \in Z} \left| \hat{f}(z) - f(z) \right| \lesssim \sqrt{\log N} Nh^{D+1} + h^\alpha,$$

$$\hat{f}(z) = \frac{1}{Nh^{D+1}} \sum_{i=1}^{N} G \left( \frac{Z_i - z}{h} \right).$$

Then as a natural corollary of Theorem 5, we instantly obtain the uniform convergence rate of the uni-modal regression as follow,

**Assumption 21** (Uniqueness) For the conditional density $f(y|x)$, we assume for any fixed $x \in \mathcal{X}$, for $\forall \xi > 0$, $\exists a > 0$, such that, for any mapping $t : x \rightarrow y$, $\sup_{x \in \mathcal{X}} |y_x - t(x)| > \xi$ always implies $\sup_{x \in \mathcal{X}} |f(y_x|x) - f(t(x)|x)| > a$.

**Theorem 6** Under Assumption 1-4, 13 and 17-21, by assuming $\mathcal{X}$ and $\mathcal{Y}$ as compact sets, we have

$$\sup_{x \in \mathcal{X}} |\hat{y}_x - y_x| = O_p \left( \sqrt{\frac{\log N}{Nh^{D+1}}} + h^\alpha \right).$$

Moreover, by choosing $h = (\log N/N)^{\frac{1}{2\alpha+D+1}}$,

$$\sup_{x \in \mathcal{X}} |\hat{y}_x - y_x| = O_p \left( \frac{\log N}{N} \right)^{\frac{\alpha}{2\alpha+D+1}}.$$

Assumption 21 is directly borrowed from Collomb et al. (1986) and can be easily satisfied. For example, assume for each $x \in \mathcal{X}$, $\sup_{y \in \mathcal{Y}_x} |f(y|x)|$
uniquely exists. Without loss of generality, suppose there exists $x^* \in X$ such that, $|y_{x^*} - t(x^*)| = \sup_{x \in X} |y_x - t(x)|$. Then, according to Assumption 19, when $|y_{x^*} - t(x)| > \xi$, we know, there always exists $a > 0$, such that, $|f(y|x^*) - f(y|t(x^*))| > a$. Otherwise, the uniqueness assumption is violated. As for the compactness of $X$ and $Y$, this condition can be released to unbounded support by using the truncation method from Hansen (2008). But here we only consider the simpler case.

Theorem 5 shows that the optimal rate of estimator $\tilde{y}_x$ is $(\log N/N)^{\alpha/(2\alpha + D + 1)}$ and cannot be promoted. This is because, due to the objective function, in order to detect $y_x$, a step we cannot avoid is to estimate the joint density $f(x,y)$, whereas, for kernel density estimator, $(\log N/N)^{\alpha/(2\alpha + D + 1)}$ has already achieved the mini-max lower bound for the $L^\infty$ distance, as shown by Stone (1982).

6 Density Level Set Estimation

Set $\mu$ as a sigma finite measure on $Z \subset \mathbb{R}^D$. Consider $(Z_{s,N})$ as a strictly stationary $Z$-valued geometric $L^p$ NED process on random field $\epsilon$ at order of $e$. Here $\epsilon$ is defined as Definition 3. For each $Z_{s,N}$, assume there is an unknown marginal probability density $f$ with respect to measure $\mu$. For fixed $\lambda > 0$, our goal is to estimate the $\lambda$-level set of density $f$ defined by

$$\Gamma_f(\lambda) := \{ z \in Z : f(z) > \lambda \}. \quad (12)$$

Another widely used definition of $\lambda$-level set is as follow,

$$\tilde{\Gamma}_f(\lambda) := \{ z \in Z : f(z) \geq \lambda \}. \quad (13)$$

Throughout this subsection, we always consider $\lambda$ to be fixed. When no confusion is possible, we also use notations $\Gamma(\lambda)$ or $\Gamma(\tilde{\Gamma}_f(\lambda))$ to represent $\Gamma_f(\lambda)$ or $\tilde{\Gamma}_f(\lambda))$. If we allow some part of the density function to be "flat", then the two definitions are different from each other. But we only focus on estimating (12) here.

Level set estimation is a core tool in many branches of machine learning and statistics. A typical example is anomaly detection. In this research area, as pointed out by Steinwart, Hush, and Scovel (2005) and references therein, an approach is to assume that observations are abnormal if they are observed outside of "high-probability area". This "high-probability area" is the region in which most of the probability mass lies in. Obviously, to detect this "high-probability area", a straightforward idea is to use a $\lambda$-level set of
a density function, provided that the density function of data has been well defined. When \( \lambda = 0 \), this question corresponds to support estimation which has been examined by Devroye and Wise (1980).

To estimate the \( \lambda \)-level set of \( f \), we here introduce a plug-in estimator with offset \( l \). That is to say, based on sample, \( (Z_{siN})_{i=1}^{N} \), given a product kernel density estimator \( \hat{f} \), we estimate \( (12) \) by

\[
\hat{\Gamma} := \{ z \in Z : \hat{f}(z) \geq \lambda + l_n \},
\]

where \( \{l_n\}_{n \in \mathbb{N}} \) is a positive and monotonically non-increasing number sequence such that \( \lim l_n = 0 \). For estimator \( (14) \), Rigollet and Vert (2009) had thoroughly investigated its convergence rates under two different measures of performance. By introducing some local smoothness assumptions for the density function within the neighborhood of "level \( \lambda \)" they showed their rates are minimax optimal. Hence, we follow their lead here and extend the dependence condition of their discussion from iid to NED.

First, we introduce the two measures for performance of estimator \( (14) \). Recall that \( \mu \) is a sigma finite measure on \( Z \) and define \( \nu(G) = \int_{G} |f(x) - \lambda| \mu(dx) \). To measure the performance of \( (14) \), we use the following two pseudo-distances between \( G_1 \) and \( G_2 \in Z \cap B_{\mathbb{R}^D} \), where \( B_{\mathbb{R}^D} \) denotes the Borel field on \( \mathbb{R}^D \):

\[
d_{\Delta}(G_1, G_2) = \mu(G_1 \Delta G_2),
\]

\[
d_{H}(G_1, G_2) = \nu(G_1 \Delta G_2) = \int_{G_1 \Delta G_2} |f(x) - \lambda| \mu(dx).
\]

Obviously, compared with with \( d_{\Delta}(G_1, G_2) \), an advantage of \( d_{H}(G_1, G_2) \) is that, it is invariant if we replace \( \Gamma \) with \( \bar{\Gamma} \) (see Rigollet and Vert (2009)). However, \( d_{\Delta}(\cdot, \cdot) \) is quite sensitive to the difference between \( \Gamma \) and \( \bar{\Gamma} \), that is to say, \( \{f = \lambda\} \). Actually, by using idea of "\( \rho \)-exponent at level \( \lambda \) with respect to measure \( \mu \)" defined as follows, we can build a link between the two measurements.

**Definition 4** For any given \( \rho, \lambda \geq 0 \), a function \( f : Z \to \mathbb{R} \) is said to be \( \rho \)-exponent at level \( \lambda \) with respect to \( \mu \) if there exists some positive constants \( c_0 \) and \( \epsilon_0 \), for \( \forall 0 < \epsilon \leq \epsilon_0 \),

\[
\mu\{z \in Z : 0 < |f(z) - \lambda| \leq \epsilon\} \leq c_0 \epsilon^\rho.
\]

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This definition is introduced by Polonik (1995). For $\rho = 0$, the existence of $c_0$ and $\epsilon_0$ is trivial. When $\rho > 0$, it measures the concentration of the value of function. Based on Definition 4, Proposition 2.1 in Rigollet and Vert (2009) shows that for any fixed $\lambda, \rho \geq 0$, if the density function $f$ satisfies Definition 4, then for any $L_{\mu} > 0$, there exists a $C > 0$ such that for any $G_1$ and $G_2$ satisfying $\mu(G_1 \Delta G_2) \leq L_{\mu}$, we have

$$d_{\Delta}(G_1, G_2) \leq \mu(G_1 \Delta G_2 \cap \{f = \lambda\}) + C(d_H(G_1, G_2))^{\frac{p}{1+\rho}}. \quad (16)$$

Meanwhile, an assumption tightly connected with the parameter “exponent-$\gamma$” is the ”$L^p$-NED condition”. More specifically, we need the following assumption.

**Assumption 22** Based on Definitions 2 and 3, assume $1 \leq d_2 \leq d$, $p > 1 + \rho$, $\max \alpha_i N \leq 1$, $\psi_Z(r) = \psi_{\epsilon}(r) = \exp(-br^\gamma)$ and $\gamma > d_2$.

Note that even if we assume $p > 1 + \rho$, because our result holds for any $\rho > 0$, when $0 < \rho < 1$, $L^2$-NED random field will be sufficiently general to cover this requirement. Additionally, the requirement $1 < \rho < 1$ can be easily fulfilled when (4) in Assumption 23 is satisfied, provided that $\beta \geq 1$. According to some examples mentioned by Xu and Lee (2015a), Xu and Lee (2018) and Jenish and Prucha (2012), $L^2$-NED process has been general enough to cover many interesting processes that are widely used in time(spatial) series analysis.

Except for the choice of measure of performance, another key characteristic influencing the convergence rate of the level set estimator is the smoothness of the density function. Rigollet and Vert (2009) showed that the fast rate of plug-in estimator of density level set with respect to (12) and (13) are only associated with the convergence rate of density estimator, $\hat{f}$, of the neighborhood around level-$\lambda$, which is defined as

$$D(\eta) = \{z \in Z : f(z) \in (\lambda - \eta, \lambda + \eta)\}.$$ 

Because it is widely known that smoothing conditions are crucial to the convergence rate of the density estimator, Rigollet and Vert’s discovery allows us to ensure the consistency of plug-in level set estimator by only depending on the assumptions about smoothness of the neighborhood and for the points belonging to $D(\eta)^c$, $\hat{f}$ converges at least $O((\log n)^{-1/2})$-fast. In this section, since we aim to discuss the attainability of optimality of convergence rates of $\hat{f}(\lambda)$ with respect to $d_{\Delta}$ and $d_H$ respectively, we deliver the following assumption about the smoothness of density $f$ on $D(\eta)$ and $D(\eta)^c$. 

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**Definition 5 (Locally Hölder Class)** Fix $L > 0$, $r > 0$ and denote by $\Sigma(\beta, L, r, x_0)$ the set of functions $g : Z \to \mathbb{R}$ that are $[\beta]$-times continuously differentiable at point $x_0$ and satisfy, for any $s = (s_1, \ldots, s_D) \in \mathbb{N}^D$,

$$|g(z) - g^{(\beta)}_{z_0}(z)| \leq L||z - z_0||^\beta, \quad \forall \ z \in B(z_0, r),$$

$$g^{(\beta)}_{z_0}(z) = \sum_{|s| \leq [\beta]} \frac{(z - z_0)^s}{s!} \partial_{z_1}^{s_1} \cdots \partial_{z_D}^{s_D}g(z_0),$$

where $|s| = s_1 + \cdots + s_D$, $s! = s_1! \cdots s_D!$, $z^s = z_1^{s_1} \cdots z_D^{s_D}$.

The set $\Sigma(\beta, L, r, x_0)$ is called locally Hölder class of functions.

**Assumption 23** Fix measure $\mu$ to be Lebesgue measure on $\mathbb{R}^D$. Fix $\beta > 0$, $L > 0$, $r > 0$, $\lambda > 0$ and $\gamma > 0$. Let $P_{\Sigma}(\beta, L, r, \lambda, \gamma, \beta', L^*) := P_{\Sigma}$ denote the class of probability density functions $f$ on $Z$ for which $\exists \eta > 0$ such that:

1. $f \in \Sigma(\beta, L, r, x_0)$ almost surely for all $x_0 \in D(\eta)$. $\exists \beta' > 0$ such that $f \in \Sigma(\beta', L, r, x_0)$ almost surely for all $x_0 \in D(\eta)^c$.
2. $f$ has $p$-exponent at level $\lambda$ with respect to the measure $\mu$.
3. $f$ is uniformly bounded by a constant $L^*$.
4. $\gamma(\beta \wedge 1) \leq 1$.

Assumption 23 is directly inherited from Definition 4.1 in Rigollet and Vert (2009). According to (1) in Assumption 23, we know that the smoothness of density function is allowed to be different. It is widely known that parameters $\beta$ and $\beta'$ here are usually expected to control the convergence rate of density estimator. Actually, as we will show later, $\beta'$ could be allowed to be arbitrarily close to 0 while this won’t affect the convergence rate of level set estimator. (4) indicates that there exits densities such that both $\Gamma$ and $\Gamma^c$ have non-empty interior (Audibert & Tsybakov, 2005).

Additionally, we also admit the following assumption on the kernel function used in (15).

**Assumption 24** Let $K$ be a real-valued function on $\mathbb{R}$ with support $[-1, 1]$. For fixed $\alpha > 0$, the function $K$ satisfies the following conditions:

1. $\int K = 1$, $||K||_p < \infty$ for any $1 \leq p \leq \infty$.
2. $\int |t|^{\alpha}|K(t)|dt < \infty$, for any $\alpha > 0$. In the case $[\alpha] \geq 1$, we instantly have, for any $s > 0$ such that $s \leq [\alpha]$, $\int t^sK(t)dt = 0$. 

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The first-order derivative of $K$ is uniformly bounded on $[-1, 1]$.

Assumption 24 coincides the definition of "α-valid" kernel from Tsybakov (2004b). Section 1.22 in Tsybakov (2004a) clearly demonstrates how to construct such kind of kernel by using the orthonormal property of Legendre polynomial.

Generally speaking, we follow the method of Rigollet and Vert (2009) to prove Theorem 7. Therefore, we demonstrate the following result about the "fast" rate of kernel-based plug-in estimator. This can be regarded as a direct extension of Theorem 3.1 in Rigollet and Vert.

**Definition 6** Let $\mathcal{P}$ be a family of probability densities of $Z \subset \mathbb{R}^D$ and fix an $\triangle > 0$. Let $\varphi = (\varphi_N)$ and $\phi = (\phi_N)$ be two monotonically non-increasing number sequences.

1. We say density estimator $\hat{f}$ is "pointwise convergent at a rate $\phi_N$ with remainder $N^{\theta p}$ uniformly over $\mathcal{P}$" if there exist positive constants $c_1$, $c_2$, $c_\phi$ such that, for almost all $z \in Z$ with respect to measure $\mu$, we have, for $\forall \theta > 0$, $c_\phi \phi_N < \delta < \triangle$ and some constant $B > 0$

$$
\sup_{f \in \mathcal{P}} \mathbb{P} \left( \left| \hat{f}(z) - f(z) \right| \geq \delta \right) \leq c_1 \exp \left( -c_2 (\delta/\phi_N)^2 \right) + \left( \frac{B}{N^{\theta p} \delta} \right)^p. \quad (17)
$$

2. We say density estimator $\hat{f}$ is "$(\varphi, \phi)$-locally pointwise convergent with remainder $N^{\theta p}$ in $D(\eta)$ uniformly over $\mathcal{P}$" if it is "pointwise convergent at a rate $\phi_N$ with remainder $N^{\theta p}$ uniformly over $\mathcal{P}$" and $\exists c_3$, $c_4$, $c_\varphi > 0$, for $\mu$-almost all $z \in D(\eta)$ we have

$$
\sup_{f \in \mathcal{P}} \mathbb{P} \left( \left| \hat{f}(z) - f(z) \right| \geq \delta \right) \leq c_3 \exp \left( -c_4 (\delta/\varphi_N)^2 \right) + \left( \frac{B}{N^{\theta p} \delta} \right)^p. \quad (18)
$$

$p$ here is the $L^p$ condition of NED.(see Definition 2).

Definition 6 is a direct variation of Definition 3.1 in Rigollet and Vert (2009) which plays a key role in obtaining the fast rate of kernel-based plug-in density level set estimator, $\hat{\Gamma}$, under near epoch dependence. Compared with Rigollet’s definition, a core difference here is the additional polynomial part $(1/N^{\theta p})^p$. Fortunately, similar to the proof of Theorem 5 in Appendix D, by making use of the property that $\theta$ defined in Definition 6 is allowed to be arbitrary large, we can still obtain a result which is nearly the same as Theorem 3.1 in Rigollet and Vert (2009).
Theorem 7 Fix $\lambda > 0$, $\Delta > 0$ and let $\mathcal{P}$ be a class of densities on $\mathbb{Z}$. Let $\varphi_N \lor \phi_N = o(1/\sqrt{\log N})$, $l_N \land \phi_N \land \varphi_N \geq CN^{-\nu}$, for some $C, \nu > 0$. Let $f$ be a density estimator of $f \in \mathcal{P}$ such that $\mu(\hat{f} \geq \lambda) \leq M$ holds for some $M > 0$ almost surely with respect to $\mathcal{P}$. Assume $\hat{f}$ is $(\varphi, \phi)$-locally convergent with remainder $N^{\theta \rho}$ in $D(\eta)$ uniformly over $\mathcal{P}$. Then if $f$ is $\rho$-exponent at level $\lambda$. Then we have

$$\sup_{f \in \mathcal{P}} E \left[ d_H(\hat{\Gamma}, \Gamma) \right] \lesssim \varphi_N^{1+\rho}, l_N \lesssim \varphi_N. \quad (19)$$

$$\sup_{f \in \mathcal{P}} E \left[ d_{\triangle}(\hat{\Gamma}, \Gamma) \right] \lesssim (\varphi_N \sqrt{\log N})^\rho, l_N = O(\varphi_N \sqrt{\log N}). \quad (20)$$

$(19)$ and $(20)$ show that, under two different measures of performance, the convergence rates have nothing to do with $\phi_N$. Considering that the smoothness of density function significantly affects the local convergence rate of density estimator, Theorem 7 to some degree indicates that, to obtain the convergence rate of level set estimator, only assumptions about local smoothness are in demand. Thus, by combining Definition 6 and Theorem 7, we immediately have the following theorem which indicates that, estimator $\hat{\Gamma}$ is optimal with respect to $"d_{\triangle}"$ and $"d_H"$ when the density function $f$ satisfies Assumption 23.

Theorem 8 Suppose Assumptions 1-4, 22-24 have been satisfied. Define $\beta^* = \beta \lor \beta'$ and let density estimator $\hat{f}$ has form $(15)$ whose kernel function satisfies Assumption 23 with $\alpha = \beta^*$. Then we have

1. When $l_N \lesssim N^{-\frac{2\beta}{2\beta+D}}$ and $h = O(N^{-\frac{1}{2\beta+D}})$,

$$\sup_{f \in \mathcal{P}} E \left[ d_H(\hat{\Gamma}, \Gamma) \right] \lesssim (N^{-\frac{\beta}{2\beta+D}} \log N)^{\frac{\beta}{2\gamma}}. \quad (21)$$

2. When $l_N = O(N^{-\frac{\beta}{2\beta+D} \sqrt{\log N}})$ and $h = O(N/\log N)^{-\frac{1}{2\beta+D}}$,

$$\sup_{f \in \mathcal{P}} E \left[ d_{\triangle}(\hat{\Gamma}, \Gamma) \right] \lesssim N^{-\frac{\beta}{2\beta+D} \log N} \rho^{\frac{\beta}{2\beta+D} + \frac{d_\gamma}{2\gamma}}. \quad (22)$$

Compared with Theorem 5.1 in Rigollet and Vert (2009), we can instantly know that $(21)$ and $(22)$ are optimal up to a logarithm factor. More specifically, when the data is iid, that is to say, $\gamma = +\infty$, Theorem 8 coincides with Corollary 4.1 in Rigollet and Vert (2009) under nearly the same conditions.
7 Conclusion and Discussion

This paper is dedicated to two topics under different near-epoch dependent conditions. First, we focus on deriving Bernstein-type inequalities for geometric and algebraic irregularly-spaced NED processes. As a corollary, we also obtained a new Bernstein-type inequality for a geometric irregularly-spaced $\alpha$-mixing process. All of our inequalities are adaptive to the "effective dimension" of the index set, which allows us to obtain a much sharper upper bound when the "effective dimension" is significantly smaller than the dimension of the index set. To our best knowledge, this may be the first result reflecting this phenomenon. Our major tools for proving the results are the Bernstein blocking technique and a definition of NED that sees the NED process as a functional version of a given process, with the orthogonal complement of its projection on the given process vanishing with respect to the $L^p$-norm, $p \geq 1$. However, an inevitable disadvantage of the Bernstein blocking technique is that the sharpness of inequality is determined by the design of the blocks. We suppose this is purely artificial. It is widely known that NED process is generically connected with martingale process. Hence, a possible alternative approach may be martingale method (e.g. Kontorovich and Raginsky (2017)). We leave this topic for our future research. Additionally, even though we have too some degree obtained Bernstein-type inequalities for irregularly-spaced NED random fields, our results rely heavily on pure domain-expand asymptotics. Thus, another important question is what would happen if things become worse, such as the "domain-expand and infill asymptotics" (Lu & Tjøstheim, 2014) or "stochastic sampling with mixed asymptotics" (Lahiri, 2003)?

For applications, based on these inequalities, we firstly derive two deviation inequalities for geometric $\alpha$-mixing and $L^p$-NED random fields with irregularly-spaced locations. Based on this, we further investigate the uniform convergence of local linear and uni-modal regression estimators for algebraic and geometric NED processes respectively. Additionally, we investigate the uniform convergence rate of a simple local polynomial density estimator for a geometric $\alpha$-mixing random field. Our result shows that the local linear estimator could achieve the optimal rate demonstrated by Stone (1982), which refreshes the result from Jenish (2012). We also obtain the convergence rate of the kernel-based uni-modal regression and density level set estimators. Beyond the applications showed in this paper, another potentially interesting and direct application is to prove the consistency of sieve maximum likelihood of the spatial auto-regressive Tobit model when SART satisfies only the algebraic NED condition. This could significantly
extend the generality of the result of Xu and Lee (2018).

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A Proof Related to Section 2

Proof of Proposition 1
For (i), we prove it by contradiction. Set $B(s, d_0/2)$ as an open ball in $\mathbb{R}^d$ whose center is $s \in \Gamma$ and radius is $d_0/2$. Obviously, $I(s, \sqrt{2}d_0/2) \subset B(s, d_0/2)$. If $B(s, d_0/2)$ contains two elements in $\Gamma_N$, say $s_{iN}$ and $s_{jN}$, a direct consequence is $|s_{jN} - s| \vee |s_{iN} - s| \leq d_0/2$. Thus, due to triangle inequality, $|s_{jN} - s_{iN}| \leq |s_{jN} - s| + |s_{iN} - s| \leq d_0$, which violates Assumption 2. Since $I(s, \sqrt{2}d_0/2) \subset B(s, d_0/2)$, we finish the proof.

For (ii), note that for $\forall h > 0$, $\sqrt{2}h/d_0 \leq \lceil \sqrt{2}h/d_0 \rceil + 1$. Thus $I(s, h) \subset I(s, \lceil \sqrt{2}h/d_0 \rceil \sqrt{2}d_0/2)$. Clearly, $I(s, \lceil \sqrt{2}h/d_0 \rceil \sqrt{2}d_0/2)$ can be partitioned into $\lceil \sqrt{2}h/d_0 \rceil + 1$ cubes whose length of each is $\sqrt{2}d_0/2$. Then in light of Proposition 1 (i), $|I(s, h) \cap \Gamma_N| \leq (\lceil \sqrt{2}h/d_0 \rceil + 1)d \leq (2\sqrt{2}h/d_0)^d$.

Proof of Proposition 2
Given a $s_{iN} \in \Gamma_N$, for $\forall k = 1, ..., D$, let $Z^{\text{th}}_{s_{iN}}$ be the $k$-th element of random vector $Z_{s_{iN}}$. Due to some geometrical property of Euclidean space, we have $|Z_{s_{iN}}^k - E[Z_{s_{iN}}^k | F_{iN}(r)]| \leq |Z_{s_{iN}} - E[Z_{s_{iN}} | F_{iN}(r)]| \leq \sum_{k=1}^{D} |Z_{s_{iN}}^k - E[Z_{s_{iN}}^k | F_{iN}(r)]|$. Together with the monotonicity of $L^p$-norm and Minkovski inequality, we finish the proof of Proposition 2.

Proof of Proposition 3
Since $f$ is Lipschitz on $Z$, we have

$$|f(Z_{s_{iN}}) - E[f(Z_{s_{iN}}) | F_{iN}(r)]|$$
$$\leq |f(Z_{s_{iN}}) - f(E[Z_{s_{iN}} | F_{iN}(r)])| + |f(E[Z_{s_{iN}} | F_{iN}(r)]) - E[f(Z_{s_{iN}}) | F_{iN}(r)]|$$
$$\leq Lip(f)||Z_{s_{iN}} - E[Z_{s_{iN}} | F_{iN}(r)]|| + |E \{f(E[Z_{s_{iN}} | F_{iN}(r)]) - f(Z_{s_{iN}}) | F_{iN}(r)\}|$$
$$:= A_1 + A_2$$

Apparently, due to the Lipschitz property of $f$,

$$||A_1||_p \leq Lip(f)||Z_{s_{iN}} - E[Z_{s_{iN}} | F_{iN}(r)]||_p.$$

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Similarly, based on basic properties of conditional expectation,
\[
|A_2|_p = \|E \{f(E[Z_{s,N}|\mathcal{F}_I(r)]) - f(Z_{s,N})|\mathcal{F}_I(r)\}\|_p
= \{E|E \{f(E[Z_{s,N}|\mathcal{F}_I(r)]) - f(Z_{s,N})|\mathcal{F}_I(r)\}|^p\}^{1/p}
\leq \{E \{E \{|f(E[Z_{s,N}|\mathcal{F}_I(r)]) - f(Z_{s,N})|^p|\mathcal{F}_I(r)\}\}^p\}^{1/p}
\leq \text{Lip}(f) \{E \{E \{|E[Z_{s,N}|\mathcal{F}_I(r)] - Z_{s,N}|p|\mathcal{F}_I(r)\}\}^{1/p}
= \text{Lip}(f)\|E[Z_{s,N}|\mathcal{F}_I(r)] - Z_{s,N}\|_p
\]

Finally, by applying Minkowski inequality and definition of NED, we obtain
\[
\|f(Z_{s,N}) - E[f(Z_{s,N})|\mathcal{F}_I(r)]\|_p \leq \|A_1\|_p + \|A_2\|_p \leq 2\text{Lip}(f)\alpha_i N\psi_Z(r).
\]

**Proof of Proposition 4**

Define \(g^r = E[g(Z_{s1,N},...,Z_{s|U|N})|\mathcal{F}_U(r)]\) and \(h^r = E[h(Z_{t1,N},...,Z_{t|V|N})|\mathcal{F}_V(r)],\)
where \(\mathcal{F}_U(r) = \sigma(\bigcup_{j=1}^{V} \mathcal{F}_j(r)), \mathcal{F}_V(r) = \sigma(\bigcup_{j=1}^{V} \mathcal{F}_j(r))\). Then, by denoting \(\Delta g = g - g^r, \Delta h = h - h^r\), we have
\[
\text{Cov}(g(Z_{s1,N},...,Z_{s|U|N}), h(Z_{t1,N},...,Z_{t|V|N}))
= \text{Cov}(g, h) = \text{Cov}(\Delta g + g^r, \Delta h + h^r)
= \text{Cov}(\Delta g, \Delta h) + \text{Cov}(g^r, \Delta h) + \text{Cov}(\Delta g, h^r) + \text{Cov}(g^r, h^r).
\]

Note \(\text{Cov}(\Delta g, \Delta h) = E(\Delta g \Delta h), \text{since } E\Delta h = E\Delta g = 0\). Meanwhile,
\[
\text{Cov}(\Delta g, h^r) = E[(\Delta g - E\Delta g)(h^r - Eh^r)] = E[\Delta g(h^r - Eh^r)]
= E[\Delta g(h^r + \Delta h - \Delta h + E\Delta h - E\Delta h - Eh^r)]
= E[\Delta g(h - Eh)] - E[\Delta g \Delta h]
\]

Similarly, \(\text{Cov}(g^r, \Delta h) = E[(g - Eg)\Delta h] - E[\Delta g \Delta h].\)

Therefore, for \(\forall p \geq 2\) and \(q = \frac{p}{p-1},\)
\[
|\text{Cov}(g, h)| \leq E|\Delta g \Delta h| + E|\Delta h(g - Eg)| + E|\Delta g(h - Eh)| + |\text{Cov}(g^r, h^r)|
\leq ||\Delta g||_q ||\Delta h||_p + ||\Delta h||_q ||g - Eg||_p + ||\Delta g||_q ||h - Eh||_p + |\text{Cov}(g^r, h^r)|
\leq I_1 + I_2 + I_3 + I_4
\]

For \(I_4, \text{since } g^r \text{ and } h^r \text{ are } \mathcal{F}_U(r) \text{ and } \mathcal{F}_V(r)\) measurable, according to Davydov (1968), for \(\forall \delta > 0, \text{we have}\)
\[
I_4 \leq \phi(|U|, |V|)\psi_{\epsilon^{p+\delta}}(\rho(U, V))(|g^r||p+\delta||h^r||p+\delta).
\]

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Also note that,

$$||g||_p = (E|E[g|F_U(r)]|^p)^{1/p} \leq (E[E[|g|^p|F_U(r)]])^{1/p} = ||g||_p \leq ||g||_{p+\delta}.$$ 

$$I_4 \leq \phi(|U|, |V|)\psi\epsilon^{p/2} (\rho(U, V))||g||_{p+\delta}||h||_{p+\delta}.$$ 

Please remark \( ||g - Eg||_p \leq ||g||_p + \int |Eg|^p dP \leq 2 ||g||_p \), where the second inequality is due to Jensen inequality. Similarly, we have \( ||h - Eh||_p \leq 2 ||h||_p \). Since \( p \geq 2, q = p/p - 1 \leq p \). Thus \( ||\Delta g||_q \leq ||\Delta g||_p \) and \( ||\Delta h||_q \leq ||\Delta h||_p \).

Moreover, to bound \( ||\Delta g||_p \), by applying Minkowski inequality, we have

$$||\Delta g||_p \leq ||\bar{g} - g\bar{r}||_p + ||\bar{g} - g^\prime||_p := G_1 + G_2,$$

where \( \bar{g} = g(E[Z_{s_iN}|F_{iN}(r)], i = 1, \ldots, |U|) \). For \( G_1 \), since \( g \) is coordinate wise Lipschitz continuous on \( \mathcal{Z}^{|U|} \), we have

$$G_1 \leq Lip(g)\sum_{i=1}^{|U|}||Z_{s_iN} - E[Z_{s_iN}|F_{iN}(r)]||_p \leq Lip(g)(\sum_{i=1}^{|U|}\alpha_{iN})\psi_Z(r).$$

As for \( G_2 \), similarly to bounding "||A_2||_p" in Proposition 3,

$$\therefore G_2 \leq Lip(g)||\sum_{i=1}^{|U|}E[|Z_{s_iN} - E[Z_{s_iN}|F_{iN}(r)]||F_U(r)||_p$$

$$\leq Lip(g)\sum_{i=1}^{|U|}E[|Z_{s_iN} - E[Z_{s_iN}|F_{iN}(r)]||F_U(r)||_{F_U(r)}]$$

$$\leq Lip(g)\sum_{i=1}^{|U|}E(E[|Z_{s_iN} - E[Z_{s_iN}|F_{iN}(r)]||F_U(r)])^{1/2}$$

$$= Lip(g)\sum_{i=1}^{|U|}||Z_{s_iN} - E[Z_{s_iN}|F_{iN}(r)]||_p \leq Lip(g)(\sum_{i=1}^{|U|}\alpha_{iN})\psi_Z(r).$$

Therefore, we prove that \( ||\Delta g||_p \leq 2 Lip(g)(\sum_{j=1}^{|U|}\alpha_{jN})\psi_Z(r) \). Similarly, we can obtain \( ||\Delta h||_p \leq 2 Lip(h)(\sum_{j=1}^{|U|}\alpha_{jN})\psi_Z(r) \) as well. Together with all the results above, we obtain

$$I_1 \leq ||\Delta g||_p||\Delta h||_p \leq 4 Lip(g)Lip(h)(\sum_{i=1}^{|U|}\alpha_{iN})(\sum_{j=1}^{|U|}\alpha_{jN})\psi_Z^2(r),$$

$$I_2 \leq ||\Delta g||_p||h - Eh||_p \leq 2 ||\Delta g||_p||h||_p \leq 4 Lip(g)(\sum_{i=1}^{|U|})\psi_Z(r),$$

$$I_3 \leq 4 Lip(h)(\sum_{j=1}^{|U|}\alpha_{jN})\psi_Z(r).$$

Q.D.E
B Proof Related to Section 3

Lemma 1 Suppose process $Z$ is an $L^p$-NED random field on process $\epsilon$. Provided that for $\forall s_{iN} \in \Gamma_N$, $||Z_{s_{iN}}||_\infty \leq A$ and $||Z_{s_{iN}}||_2 \leq \sigma$, for some $A$ and $\sigma$ strictly larger 0, we have, for $\forall r > 0$,

$$||E[Z_{s_{iN}} | \mathcal{F}_{iN}(r)]||_2 \leq \sigma, \quad ||E[Z_{s_{iN}} | \mathcal{F}_{iN}(r)]||_\infty \leq A.$$  

Proof: Note for any $p \geq 1$, since Jensen inequality still holds for conditional expectation, we have

$$||E[Z_{s_{iN}} | \mathcal{F}_{iN}(r)]||_p = \{E[|E[Z_{s_{iN}} | \mathcal{F}_{iN}(r)]|^p]\}^{\frac{1}{p}} \leq \{E[|E[Z_{s_{iN}} | \mathcal{F}_{iN}(r)]|^p]\}^{\frac{1}{p}} = ||Z_{s_{iN}}||_p.$$  

Thus, the results when $p = 2$ and $\infty$ respectively.

Lemma 2 (Davydov (1968)) Suppose $\epsilon$ is an $\alpha$-mixing random field defined as Definition 3. For any finite subset $U, V \subset \Gamma_N$, let $\xi$ and $\eta$ be $\sigma(U)$ and $\sigma(V)$ measurable respectively. Then, for any $p_1, p_2 \geq 1$ and $\frac{1}{p_2} + \frac{1}{p_1} + \frac{1}{p_3} = 1$, we have the following covariance inequality,

$$|\text{Cov}(\xi, \eta)| \leq \phi(|U|, |V|)^{\frac{1}{p_1}}||\xi||_{p_2}||\eta||_{p_3}.$$  

Proof of Theorem 1

The whole proof will be generally decomposed into three steps. The first step is to introduce the blocking technique which is essentially the same as Hang and Steinwart (2017). The second step is to bound the sum of the difference between $Z_{s_{iN}}$ and its projection on sigma field $\mathcal{F}_{iN}(r)$ of any given $r > 0$. The last step actually aims at deriving a concentration inequality for $\alpha$-mixing random field. The method used in this step is strongly motivated by Valenzuela-Domínguez et al. (2017).

Step 1 (Blocking)

Firstly, without loss of generality, we assume $R_N = \prod_{k=1}^{d}(0, H_k)$ and $H_1^n = H_2^n = \cdots = H_{d_1}^n = H_0$, where $0 \leq d_1 < d$. Thus, according to Assumption 3, we have , for $\forall k = d_1 + 1, \ldots, d$, we have $H_k^N \nearrow +\infty$, as $N \nearrow \infty$. By denoting $\bar{N} = \prod_{k=d_1+1}^{d}H_k$, Assumption 4 implies $\text{Vol}(R_N) \leq K^{d_2}N$, where $\text{Vol}(R_N) = H_0^{d_1} \bar{N}$. Now, for $k = d_1 + 1, \ldots, d$, let

$$m_k = \begin{cases} \frac{H_k^N}{H_0} & \text{if } \frac{H_k^N}{H_0} \in \mathbb{N} \\ \left[\frac{H_k^N}{H_0}\right] + 1 & \text{else} \end{cases}.$$  

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We thus partition $R_N$ into $\prod_{k=d_1+1}^d m_k$ many smaller cubes whose length of each edge is $H_0$. Here we name these cubes as $B_l$, $l = 1, \ldots, M$, $M = \prod_{k=d_1+1}^d m_k$. Obviously, based on Proposition 1, for each $l$, $|B_l \cap \Gamma_N| \leq \left(\frac{H_0}{\sqrt{2d_0/2}}\right) + 1)^d := C_0$. By defining $G_{lN} = \sum_{i=1}^N 1[s_{iN} \in B_l]Z_{s_{iN}}$, conditions of $Z_{s_{iN}}$ indicate that, for $\forall l = 1, \ldots, M$, $E[G_{lN}] = 0$, $\|G_{lN}\|_{\infty} \leq A$ and $E[G_{lN}^2] \leq C_0^2\sigma^2$.

Secondly, note that $\{G_{lN}\}$ can be regarded as "area-located" spatial data. Hence, index $l$ can be replaced by vector $j \in \mathbb{Z}^{d_2+}$, $j = (j_{d_1+1}, \ldots, j_d)$, in which, for $\forall k = d_1+1, \ldots, d$, $1 \leq j_k \leq m_k$. Now, suppose $\{P_{\hat{N}}\}$ is a positive-integer valued number sequence such that $P_{\hat{N}} \nearrow +\infty$, as $\hat{N} \nearrow +\infty$. Denote $[m_k/P_{\hat{N}}]$ as $L_N^k$ and let $r_k = m_k - P_{\hat{N}} L_N^k$. Furthermore, define

$$I_{jk} = \begin{cases} j_k, & j_k + P_{\hat{N}}, \ldots, j_k + L_N^k P_{\hat{N}} \ 1 \leq j_k \leq r_k \\ j_k, & j_k + P_{\hat{N}}, \ldots, j_k + (L_N^k - 1) P_{\hat{N}} \ r_k < j_k \leq P_{\hat{N}}, \end{cases}$$

and $I_j = \prod_{k=d_1+1}^d I_{jk}$. Therefore $j \in [1, P_{\hat{N}}]^{d_2} \cap \mathbb{Z}^{d_2+}$. Additionally, for each vector $j$, equip it with a scalar $j$ for later notation, $j = 1, 2, \ldots, P_{\hat{N}}^{d_2}$.

Above all, we rewrite the partial sum of random variables as follow,

$$S_{N} = \frac{\sum_{i=1}^N Z_{s_{iN}}}{N} = \frac{\sum_{l=1}^M \sum_{i=1}^N 1[s_{iN} \in B_l]Z_{s_{iN}}}{N} = \frac{\sum_{j \in \lfloor 1, P_{\hat{N}} \rfloor^{d_2} \cap \mathbb{Z}^{d_2+}} \sum_{j' \in I_j} G_{j'N}}{N} = \frac{\sum_{j \in \lfloor 1, P_{\hat{N}} \rfloor^{d_2} \cap \mathbb{Z}^{d_2+}} \sum_{i=1}^{|I_j|} G_{iN}}{N} = \frac{\left(\sum_{j \in \lfloor 1, P_{\hat{N}} \rfloor^{d_2} \cap \mathbb{Z}^{d_2+}} \sum_{i=1}^{|I_j|} G_{iN}\right)|I_j|}{N} = \sum_{j \in \lfloor 1, P_{\hat{N}} \rfloor^{d_2} \cap \mathbb{Z}^{d_2+}} \frac{|I_j|}{|I_j|}.$$ 

Since $\prod_{k=d_1+1}^d L_N^k \leq |I_j| \leq \prod_{k=d_1+1}^d L_N^k$, according to the definition of $L_N^k$, it can be easily specified that

$$p_j = \left(\frac{2}{P_{\hat{N}} H_0}\right)^{d_2} \left(\frac{\hat{N}}{N}\right)^{d_2} \left(\frac{2}{P_{\hat{N}} H_0}\right)^{d_2} \frac{Vol(R_N)}{N} \leq \left(\frac{2}{P_{\hat{N}} H_0}\right)^{d_2} \left(\frac{KN}{N}\right)^{d_2} = \frac{1}{P_{\hat{N}}^{d_2}},$$

which implies $\sum p_j \leq 1$. Now we have

$$P \left(\frac{S_{N}}{N} \geq t\right) = P \left(\left|\sum_{j \in \lfloor 1, P_{\hat{N}} \rfloor^{d_2} \cap \mathbb{Z}^{d_2+}} \frac{|I_j|}{|I_j|} \right| \geq t\right). \quad (B.1)$$

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Let $G_{iN}^r = \sum_{i=1}^{N} 1_{[s_iN \in B_i]} E[Z_{s_iN} | F_{iN}(r)]$ for any $0 < r < P_N/2$. Then we bound (B.1) as follow,

\[
P \left( \left| \frac{S_N}{N} \right| \geq t \right) \leq P \left( \left| \sum_{j=1}^{P^d_N} p_j \sum_{l=1}^{r} \frac{|G_{iN} - G_{iN}^r|}{|I_l|} \right| \geq \frac{t}{2} \right) \\
+ P \left( \left| \sum_{j=1}^{P^d_N} p_j \sum_{l=1}^{r} |G_{iN}^r| \right| \geq \frac{t}{2} \right) \\
:= Q_1 + Q_2. 
\]

Therefore, according to (B.2), it suffice to give upper bounds to $Q_1$ and $Q_2$ respectively. As we mentioned before, $Q_1$ essentially means the summation of projection error between $Z_{s_iN}$'s and $E[Z_{s_iN} | F_{iN}(r)]$. To bound $Q_2$ is actually equal to bound the partial sums of some measurable functions of $\alpha$-mixing random field.

**Step 2 (Projection Error)**

Proof of this step is very simple since we only make use of Markov inequality the definition of NED condition. More particularly, we have,

\[
Q_1 \leq \frac{2^p E \left| \sum_{j=1}^{P^d_N} p_j \sum_{l=1}^{r} \frac{|G_{iN} - G_{iN}^r|}{|I_l|} \right|^p}{t^p} \\
\leq \frac{\left( 2|| \sum_{j=1}^{P^d_N} p_j \sum_{l=1}^{r} \frac{|G_{iN} - G_{iN}^r|}{|I_l|} ||_p \right)^p}{t^p} \\
\leq \frac{\left( 2 \sum_{j=1}^{P^d_N} p_j \sum_{l=1}^{r} \frac{|G_{iN}^r|}{|I_l|} \right)^p}{t^p} \\
\leq \frac{\left( 2 \sum_{j=1}^{P^d_N} p_j \max_{1 \leq l \leq |I_j|} ||G_{iN} - G_{iN}^r||_p \right)^p}{t^p} \\
\leq \left( \frac{2 (N^\kappa \nu^{-br\gamma})}{t} \right)^p, \ \forall \ p \geq 1, \ 0 \leq r < \frac{P_N}{2}.
\]

The third and the fifth inequalities are due to Minkowski inequality of $L^p$-norm and definition of $L^p$ NED condition.
Step 3 (Bernstein-type Inequality for Process \( \{G_{IN}^r\} \))

Due to Chernoff’s approach, we have, for \( \forall \; \lambda > 0 \),

\[
P\left( \sum_{j=1}^{p^d_2} p_j \sum_{l=1}^{|I_j|} \frac{G_{IN}^r}{|I_j|} \geq \frac{t}{2} \right) \leq \exp \left( -\lambda \frac{t}{2} \right) E \left[ \exp \left( \lambda \sum_{j=1}^{p^d_2} p_j \sum_{l=1}^{|I_j|} \frac{G_{IN}^r}{|I_j|} \right) \right].
\]

Recall for each \( j \), \( p_j \leq \frac{1}{p^d_2} \). Due to the monotonicity of exponential function and Jensen inequality,

\[
E \left[ \exp \left( \lambda \sum_{j=1}^{p^d_2} p_j \sum_{l=1}^{|I_j|} \frac{G_{IN}^r}{|I_j|} \right) \right] \leq \sum_{j=1}^{p^d_2} \frac{1}{p^d_2} E \left[ \exp \left( \lambda \sum_{l=1}^{|I_j|} \frac{G_{IN}^r}{|I_j|} \right) \right].
\]

Thus, it suffice to obtain an upper bound of the Laplace transformation of \( \sum_{l=1}^{|I_j|} \frac{G_{IN}^r}{|I_j|} \) uniformly over \( j \).

Given \( 0 \leq r < P_N^r/2 \), define \( U_l = \bigcup_{s_i \in B(s_iN, r)} B(s_iN, r) \) and \( \sigma(U_l) = \sigma(\epsilon_{s_iN}, s_iN \in U_l) \). Then \( G_{IN}^r \) is \( \sigma(U_l) \) measurable and

\[
|\{s_j \in U_l\}| \leq (\sqrt{2(H_0 + 2r)/d_0}) + 1)^d := C(r).
\]

According to Lemma 1 and definition of \( G_{IN}^r \), it’s easy to obtain, for any \( l = 1, \ldots, |I_j| \), \( ||G_{IN}^r||_\infty \leq C_0 A \), \( ||G_{IN}^r||_2 \leq \sigma \). Furthermore, \( G_{IN}^r \) can be regarded as a measurable function of \( \epsilon_{s_iN} \)’s whose \( s_i \in U_l \). Since random field \( \epsilon \) is an \( \alpha \)-mixing random field(array), by applying Lemma 2, we can do the following calculation.

For \( \forall \; p_2, p_3 \geq 1 \) and \( \sum_{i=1}^3 1/p_i = 1 \), by letting \( \sum_{l=1}^n G_{IN}^r/|I_j| = S_{nN}^r \),
Let $p \in (0, 1)$ and $|I_j| = n$, for $n = 2, \ldots, |I_j|$, 

$$E \left[ \exp \left( \frac{\lambda \sum_{l=1}^{|I_j|} G_{lN}}{|I_j|} \right) \right] = E \left[ \exp \left( \lambda S_{|I_j|-1,N}^r \right) \exp \left( \frac{\lambda G_{|I_j|N}^r}{|I_j|} \right) \right]$$

$$\leq Cov \left( \exp \left( \lambda S_{|I_j|-1,N}^r \right), \exp \left( \frac{\lambda G_{|I_j|N}^r}{|I_j|} \right) \right) + E \left[ \exp \left( \lambda S_{|I_j|-1,N}^r \right) \right] E \left[ \exp \left( \frac{\lambda G_{|I_j|N}^r}{|I_j|} \right) \right]$$

$$\leq \left\{ \left( |I_j| C(r) \right)^{\nu - b(P_N - 2r)^\gamma} \right\} \left[ \exp \left( \frac{\lambda G_{|I_j|N}^r}{|I_j|} \right) \right] \left\| \exp \left( \frac{\lambda G_{|I_j|N}^r}{|I_j|} \right) \right\|_{p_1} + E \left[ \exp \left( \frac{\lambda G_{|I_j|N}^r}{|I_j|} \right) \right] \left\| \exp \left( \lambda S_{|I_j|-1,N}^r \right) \right\|_{p_2},$$

let $p_3 = \infty$, thus $\frac{1}{p_1} + \frac{1}{p_2} = 1,$

$$\leq \left\{ \left( |I_j| C(r) \right)^{\nu - b(P_N - 2r)^\gamma} \right\} \left[ \exp \left( \frac{\lambda G_{|I_j|N}^r}{|I_j|} \right) \right] \left\| \exp \left( \frac{\lambda G_{|I_j|N}^r}{|I_j|} \right) \right\|_{\infty} + E \left[ \exp \left( \frac{\lambda G_{|I_j|N}^r}{|I_j|} \right) \right] \left\| \exp \left( \lambda S_{|I_j|-1,N}^r \right) \right\|_{p_2},$$

repeat the calculation above $|I_j| - 1$ times,

$$\leq \prod_{l=0}^{\frac{|I_j|-2}{|I_j|-1}} \left\{ \left( |I_j| - l \right)^{\nu - b(P_N - 2r)^\gamma} \right\} \left[ \exp \left( \frac{\lambda p_1^l G_{|I_j|-l,N}^r}{|I_j|} \right) \right] \left\| \exp \left( \frac{\lambda p_1^l G_{|I_j|-l,N}^r}{|I_j|} \right) \right\|_{\infty} + E \left[ \exp \left( \frac{\lambda p_2^l G_{|I_j|-l,N}^r}{|I_j|} \right) \right] \left\| \exp \left( \lambda S_{|I_j|-1,N}^r \right) \right\|_{p_2},$$

Let $p_2 = \frac{1}{p_1}$ and $\frac{1}{p_1} = 1 - \frac{1}{p_2} = 1 - 2^{-\frac{1}{|I_j|-1}}, \ldots, 1 < p_2 \leq 2,$

$\frac{1}{2} \leq \frac{1}{p_1} < 1$ and for all $0, \ldots, |I_j| - 2, p_2^l \leq 2$. Now we bound $E_k$ uniformly over $l, k = 0, 1, 2, 3$. For $E_0$, since $\nu > 1$ and $\frac{1}{p_1} < 1$, we have

$$E_0 \leq (C(r)|I_j|)^{\nu - b(P_N - 2r)^\gamma}$$

For $E_1$, by assuming $\lambda < \frac{3|I_j|}{4C_N^{\nu - 2r}}$, we obtain $E_1 \leq e^2$. Considering probabilistic measure is a finite measure, $E_3$ can be bounded by $e$. At last, we focus on
deriving an upper bound of $E_2$. Since $\max_{1 \leq |I_j| < P_2^l} p_2^{l-1} \leq 4$,

\[
\exp \left( \frac{\lambda_p^{2l} G_{|I_j|} - t_n}{|I_j|} \right) \leq \exp \left( 4 \lambda G_{|I_j| - t_n} \right)
\]

$$(\delta := 4\lambda) \leq 1 + \frac{\delta G_{|I_j| - t_n}}{|I_j|} + \frac{(\delta G_{|I_j| - t_n})^2}{2(|I_j|)^2} + \sum_{q=3}^{\infty} 1 \left( \frac{\delta G_{|I_j| - t_n}}{|I_j|} \right)^q$$

\[
\leq 1 + \frac{\delta G_{|I_j| - t_n}}{|I_j|} + \frac{(\delta G_{|I_j| - t_n})^2}{2(|I_j|)^2} \left( 1 + \sum_{q=1}^{\infty} \left( \frac{\delta C_0 A}{3|I_j|} \right)^q \right)
\]

\[
\leq 1 + \frac{\delta G_{|I_j| - t_n}}{|I_j|} + \frac{(\delta G_{|I_j| - t_n})^2}{2(|I_j|)^2} \left( \frac{1}{1 - \frac{\delta C_0 A}{3|I_j|}} \right).
\]

Therefore, we obtain that,

\[
E_2 \leq 1 + \frac{1}{2} \left( \frac{\delta \sigma}{|I_j|} \right)^2 \left( \frac{1}{1 - \frac{\delta C_0 A}{3|I_j|}} \right) \leq \exp \left( \frac{8\lambda^2 (C_0 \sigma)^2}{|I_j| (|I_j| - \frac{4\delta C_0 A}{3})} \right).
\]

Above all,

\[
E \left[ \exp \left( \frac{\lambda \sum_{l=1}^{|I_j|} G_{|I_j|}}{|I_j|} \right) \right]
\]

\[
\leq \prod_{l=0}^{|I_j|-2} \left\{ \min(C(|I_j|)) \right\}^\tau \nu^{-\frac{(P_N - 2r)^\gamma}{2}} + \exp \left( \frac{8\lambda^2 (C_0 \sigma)^2}{|I_j| (|I_j| - \frac{4\delta C_0 A}{3})} \right) \frac{1}{P_2^{\frac{1}{2}}} \cdot e
\]

\[
\leq e \left( e^{\frac{3}{2}} C(r) |I_j|^\tau \nu^{-\frac{(P_N - 2r)^\gamma}{2}} + \exp \left( \frac{8\lambda^2 (C_0 \sigma)^2}{|I_j| (|I_j| - \frac{4\delta C_0 A}{3})} \right) \right)^{|I_j|}
\]

\[
\leq e \exp \left( e^{\frac{3}{2}} C(r) |I_j|^{\tau + 1} \nu^{-\frac{(P_N - 2r)^\gamma}{2}} \exp \left( \frac{8\lambda^2 (C_0 \sigma)^2}{|I_j| (|I_j| - \frac{4\delta C_0 A}{3})} \right) \right).
\]

The third inequality holds because for each $l$, $1/p_2^l \leq 1$. The fourth inequality holds since $\lambda < 3|I_j|/4C_0 A$, which indicates $|I_j| - 3|I_j|/4C_0 A > 0$. Furthermore, note that

\[
C(r) = \left( \left[ \frac{H_0 + 2r}{\sqrt{2d_0 / 2}} + 1 \right] \right)^d \leq \left( \frac{2^{\frac{3}{2}} \gamma}{d_0} \right)^d (H_0 + 2r)^d,
\]

\[
|I_j| \leq \prod_{k=d_1+1}^d (L_N^k + 1) \leq 2^{d_2} \prod_{k=d_1+1}^d L_N^k \leq 2^{d_2} \prod_{k=d_1+1}^d \frac{m_k}{P_N} = \left( \frac{2}{P_N} \gamma \right)^{d_2} \tilde{N}.
\]
Hence, by setting \( r = \frac{P_{\hat{N}}}{3} \), by setting \( H_0 \leq \frac{P_{\hat{N}}}{3} \) and some simple algebra,

\[
E \left[ \exp \left( \frac{\lambda \sum_{l=1}^{p^d} G_{lN}^r}{|I_j|} \right) \right] \\
\leq e \exp \left\{ e^{\frac{3}{2} \frac{\tau d - d_2 (\tau + 1)}{H_0} \left( \frac{\log N}{\nu} \right) \gamma} \right\} \exp \left( \frac{8 \lambda^2 (C_0 \sigma)^2}{(\|I_j| - \frac{4M_0 A}{3})} \right).
\]

Now, we let \( P_{\hat{N}} = 3 \left( \frac{(2(\kappa + \beta) + \tau + 1) \log_\nu \hat{N}}{b} \right)^{\frac{1}{\gamma}} \) for any given \( \beta \geq 0 \), we obtain

\[
E \left[ \exp \left( \frac{\lambda \sum_{l=1}^{p^d} G_{lN}^r}{|I_j|} \right) \right] \\
\leq e \exp \left\{ e^{\frac{3}{2} \frac{\tau d - d_2 (\tau + 1)}{H_0} \left( \frac{\log N}{\nu} \right) \gamma} \right\} \exp \left( \frac{8 \lambda^2 (C_0 \sigma)^2}{(\|I_j| - \frac{4M_0 A}{3})} \right).
\]

Let \( \hat{N} \geq \hat{N}_0 \), where

\[
\hat{N}_0 := \min \left\{ \hat{N} \in \mathbb{R}^+, \frac{(\log_\nu \hat{N})^{\frac{\tau d - d_2 (\tau + 1)}{N^{\kappa + \beta}}} \leq \frac{1}{C_1} \right\}. \tag{B.3}
\]

Then,

\[
E \left[ \exp \left( \frac{\lambda \sum_{l=1}^{p^d} G_{lN}^r}{|I_j|} \right) \right] \leq 2 \exp \left( \frac{8 \lambda^2 (C_0 \sigma)^2}{(\|I_j| - \frac{4M_0 A}{3})} \right).
\]

Define \( h(\lambda) = \frac{8 \lambda^2 (C_0 \sigma)^2}{(\|I_j| - \frac{4M_0 A}{3})} - \frac{\lambda}{2} \). We thus obtain Chernoff-type upper bound as follow,

\[
P \left( \sum_{j=1}^{p^d} p_j |I_j| G_{lN}^r \geq t \right) \leq 2 \exp \{ h(\lambda) \}.
\]

Here, let \( \lambda^* = \frac{|I_j|}{\xi (C_0 \sigma)^2 + \frac{4M_0 A}{3} t} \) for some \( \xi \geq 1 \). Thus \( \lambda^* \leq \frac{3|I_j|}{4C_0 A t} < \frac{3|I_j|}{4C_0 A} \),

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which coincides with the previous requirement. Then

\[
h(\lambda^*) = \frac{8(C_0\sigma|I_j|t)^2}{\xi^2(|I_j| - \frac{4C_0A|I_j|t}{3\xi (C_0\sigma)^2} + \frac{4C_0A^2t}{3}) - \frac{|I_j|t^2}{2\xi((C_0\sigma)^2 + \frac{4C_0A}{3}t)}} - \frac{|I_j|t^2}{2\xi((C_0\sigma)^2 + \frac{4C_0A}{3}t)}
\]

\[
\left(\therefore \xi \geq 1 \text{ and } C_0A > 0, \therefore 1 - \frac{1}{\xi} \geq 0, \frac{4}{3} C_0A|I_j|t(1 - \frac{1}{\xi}) \geq 0\right)
\]

\[
\leq \frac{(8/\xi^2)(C_0\sigma)^2|I_j|^2}{(C_0\sigma)^2(\xi^2 + \frac{4C_0A}{3}t)} - \frac{(1/2\xi)|I_j|^2}{(C_0\sigma)^2(\xi^2 + \frac{4C_0A}{3}t)}
\]

\[
= \frac{(8/\xi^2 - 1/2\xi)|I_j|^2}{(C_0\sigma)^2 + \frac{4C_0A}{3}t}
\]

(\text{let } \xi = 32)

\[
= 2\exp\left(-\frac{|I_j|t}{128(C_0\sigma)^2 + \frac{4C_0A}{3}t}\right).
\]

Due to sub-Gaussianity and definition of \(P_{\hat{N}}\), we manage to prove, for \(\forall \ t > 0\),

\[
Q_2 \leq 4\exp\left(-\frac{|I_j|t}{128(C_0\sigma)^2 + \frac{4C_0A}{3}t}\right), \quad (B.4)
\]

\[
Q_1 \leq 2\left(\frac{\hat{N}}{\hat{N}^2(\kappa + \beta) + \tau + 1}\right)^p. \quad (B.5)
\]

The last job is to specify the relationship between \(N\) and \(\hat{N}\). Recall Assumptions 3 actually indicates that \(\Gamma_N \in \hat{R}_N\) while \(R_N\) can be decomposed into \(\prod_{k=d_1+1}^d m_k\)-many cubes. Each cube contains at most \(C_0\) locations. Thus, we have

\[
N \leq C_0\prod_{k=d_1+1}^d m_k = \frac{C_0}{H_0^{d_2}}\prod_{k=d_1+1}^d H_k^N = \frac{C_0\hat{N}}{H_0^{d_2}}.
\]

\[
\Rightarrow \frac{H_0^{d_2} \hat{N}}{C_0} \leq \hat{N}. \quad (B.6)
\]

Conversely, according to Assumption 4,

\[
Vol(R_N) = H_0^{d_1}\hat{N} \leq (KN^{\frac{1}{d_2}})^{d_2}
\]

\[
\Rightarrow \hat{N} \leq \frac{K^{d_2}}{H_0^{d_1}}N = \frac{H_0^{d_2-d_1}}{2^{d_2}}N. \quad (B.7)
\]

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Therefore, we obtain

\[ |J_j| \geq \prod_{k=d_1+1}^{d} L_{N}^{k} = \prod_{k=d_1+1}^{d} \left\lceil \frac{m_k}{P_N} \right\rceil \geq \left( \frac{1}{P_N} \right)^{d_2} \prod_{k=d_1+1}^{d} m_k \]

\[ = \left( \frac{1}{2P_N H_0} \right)^{d_2} \tilde{N} = \frac{\tilde{N}}{(2H_0 \left[ \frac{(2(\kappa+\beta)+\tau+1)}{b} \right]^{\frac{1}{7}})^{d_2} (\log_\nu \tilde{N})^{\frac{d_2}{\gamma}}} \]

(\text{let } N \geq \frac{H_0^{d_2-d_1}}{2^{d_2}}) \quad \text{(B.8)}

\[ \geq \frac{N}{2 \left( 2C_0 \left[ \frac{(2(\kappa+\beta)+\tau+1)}{b} \right]^{\frac{1}{7}} \right)^{d_2} (\log_\nu N)^{d_2}} \quad \text{(B.9)} \]

and

\[ \left( \frac{1}{N} \right)^{2(\kappa+\beta)+\tau+1} \leq \left( \frac{C_0 / H_0^{d_2}}{N} \right)^{2(\kappa+\beta)+\tau+1} \quad \text{(B.10)} \]

Hence, by combining (B.9) and (B.10) with (B.4) and (B.5) respectively, we obtain the result shown in Theorem 1. To specify the \( N_0 \), we firstly apply (B.6) and (B.7) to \( \hat{N}_0 \). We thus obtain a sufficient condition for (B.3) as follow,

\[ N \geq N_1 := \min \left\{ \begin{array}{l} N \in \mathbb{N} : \left( \frac{\log_\nu \frac{H_0^{d_2-d_1} N}{2^{d_2}}}{N^{\kappa+\beta}} \right)^{\frac{r d_2 - d_2(r+1)}{\gamma}} \end{array} \right\} . \]

Recall we also require \( P_\hat{N} \geq H_0 \), which, due to (B.6), yields

\[ N \geq \frac{C_0}{H_0^{d_2}} \nu^{2(\kappa+\beta)+\tau+1} := N_2. \]

Together (B.8), we specify \( N_0 \) as \( N_0 := \max \{ N_1, N_2, \frac{H_0^{d_2-d_1}}{2^{d_2}} \} \).
Proof of Corollary 1

Similar to Step 1 of the proof of Theorem 1, we can rewrite the averaged partial sum as follow,
\[
\sum_{i=1}^{N} \epsilon_{s_i N} / N^{-1} = \sum_{j=1}^{p_{d_2}} p_j \sum_{i=1}^{|I_j|} G_{lN} / |I_j|.
\]
Here \( G_{lN} = \sum_{i=1}^{N} 1[ s_i N \in B_j ] \epsilon_{s_i N} \). \( p_j \) and \(|I_j|\) are directly inherited from proof of Theorem 1. Then, by using Chernoff approach, we obtain
\[
P \left\{ \left| \frac{\sum_{i=1}^{N} \epsilon_{s_i N}}{N} \right| \geq t \right\} \leq \exp (-\lambda t) \sum_{j=1}^{p_{d_2}} 1 \frac{1}{P_{d_2}} \exp \left( \lambda \sum_{i=1}^{|I_j|} G_{lN} / |I_j| \right)
\]
Since process \( \{ \epsilon_{s_i N} \}_{s_i N \in \Gamma_N} \) is an \( \alpha \)-mixing random field with geometric decay, from now on, we only need to follow exactly the same procedures demonstrated in Step 3 of proof of Theorem 1.

**Lemma 3** (Bradley’s Lemma) Let \((X, Y)\) be a \( R^n \times R \)-valued random vector such that \( Y \in L^p(P) \) for some \( p \in [1, +\infty) \). Let \( c \) be a real number such that \( ||Y + c||_p > 0 \), and \( \xi \in [0, ||Y + c||_p] \). Then, there exists a random variable \( Y^* \) such that

1. \( P_{Y^*} = P_Y \) and \( Y^* \) is independent of \( X \).
2. \( P(||Y - Y^*|| > \xi) \leq 11 \left( \frac{||Y + c||_p}{\xi} \right)^{2d_2+1} \cdot \alpha(\sigma(X), \sigma(Y))^{2d_2+1} \).

Proof of Theorem 2

**Step 1** (Blocking)

Similar to the proof of Theorem 1, we let \( R_N = \prod_{k=d_1+1}^{d}(0, H_k^N] \otimes (0, H_0^{d_1}] \), where \( \otimes \) denotes the Cartesian product among intervals. Without loss of generality, set \( \forall d_1+1 \leq k \leq d, \min_{d_1+1 \leq k \leq d} H_k^N \geq H_0 \). For \( \forall d_1+1 \leq k \leq d, 1 \leq q \leq N/2 \), let \( p_k = H_k^N / 2q \). Then \( R_N \) is partitioned into \( 2^{d_2} \) groups of rectangles. Within each group, the quantity of rectangle is \( q^{d_2} \) and the volume of each of these rectangles is \( \prod_{k=d_1+1}^{d} p_k := p \). Here we name the \( j \)-th rectangle belonging to the \( m \)-th group as \( A_{jm} \). Hence for \( \forall j, m \), \( Vol(A_{jm}) = H_0^{d_1} \prod_{k=d_1+1}^{d} p_k := H_0^{d_1} p \). Additionally, due to Proposition 1, we know that \( |\Gamma_N \cap A_{jm}| \leq C_0 \prod_{k=d_1+1}^{d} ([p_k / H_0] + 1) := C_p \), where \( C_0 \) is inherited from Theorem 1. Thus, we can write the partial sum process as follow,
\[
S_N = \sum_{i=1}^{N} Z_{s_i N} = \sum_{j=1}^{q^{d_2}} \sum_{m=1}^{2^{d_2}} \left( \sum_{i=1}^{N} 1[ s_i N \in A_{jm} ] Z_{s_i N} \right)
:= \sum_{m=1}^{2^{d_2}} \sum_{j=1}^{q^{d_2}} G_{jm}.
\]
Define $G_{jm}^r = \sum_{i=1}^N 1[s_i \in A_{jm}]E[Z_{siN}|\mathcal{F}_{iN}(r)]$. Then, for any $t > 0$,

$$P(|S_N| \geq Nt) \leq P \left( \sum_{m=1}^{2^{d_2}} \left| \sum_{j=1}^{q^{d_2}} G_{jm} \right| \geq Nt \right) \leq \sum_{m=1}^{2^{d_2}} P \left( \sum_{j=1}^{q^{d_2}} G_{jm} \geq \frac{Nt}{2^{d_2}} \right) \leq \sum_{m=1}^{2^{d_2}} P \left( \sum_{j=1}^{q^{d_2}} (G_{jm} - G_{jm}^r) \geq \frac{Nt}{2^{d_2+1}} \right) + \sum_{m=1}^{2^{d_2}} P \left( \sum_{j=1}^{q^{d_2}} G_{jm}^r \geq \frac{Nt}{2^{d_2+1}} \right) =: Q_3 + Q_4$$

Hereafter, denote $t/2^{d_2+1}$ as $\eta$. Hence the result can be obtained if we find probabilistic upper bounds for $Q_3$ and $Q_4$ uniformly over $m$.

**Step 2 (Projection Error)**

Firstly, we have the following argument existed.

$$Q_3 \leq \sum_{m=1}^{2^{d_2}} P \left( \sum_{j=1}^{q^{d_2}} (G_{jm} - G_{jm}^r) \geq \tilde{\eta} \frac{N^2}{N} \right) \leq \sum_{m=1}^{2^{d_2}} P \left( \sum_{j=1}^{q^{d_2}} (G_{jm} - G_{jm}^r) \geq \frac{2^{d_2} \tilde{\eta} N^2}{H_0^{d_2-d_1}} \right)$$

Let $\tilde{\eta} = \frac{2^{d_2} \eta}{H_0^{d_2-d_1}} = \frac{t}{2H_0^{d_2-d_1}}$. According to the same procedures as Step 2 in the proof of Theorem 1, we obtain, for $t > 0$,

$$Q_3 \leq \left( \frac{C_02^{d_2+1}}{H_0^{d_1}} \right)^s \left( \frac{\alpha(N)t^{-d_1}}{t} \right)^s$$

**Step 3 (Moment Upper Bound for r.v $G_{jm}^r$)**

Due to the definition of $G_{jm}^r$, it’s obvious that, for any $j$ and $m$, $E[G_{jm}^r] = 0$, $\|G_{jm}^r\|_{+ \infty} \leq C_p A$. Now we focus on deriving an upper bound for $Var(G_{jm}^r)$.

Firstly, according to the basic property of variance operator, for some $r > 0$,

$$Var(G_{jm}^r) = Var \left( \sum_{i=1}^N 1[s_i \in A_{jm}]E[Z_{siN}|\mathcal{F}_{iN}(r)] \right) = Var \left( \sum_{i=1}^N 1[s_i \in A_{jm}]Z_{iN}^r \right)$$

$$\leq \sum_{s_i \in A_{jm}} Var(Z_{iN}^r) + \sum_{i \neq j} |Cov(Z_{iN}^r, Z_{jN}^r)|$$

$$=: \Xi_1 + \Xi_2.$$

Obviously, $\Xi_1 \leq C_p \sigma^2$. We only need to concentrate on $\Xi_2$, which can be decomposed as follow,

$$\Xi_2 = \sum_{s_i \in A_{jm}} \sum_{s_j \notin A_{jm}, \|s_i - s_j\| \leq 2r} |Cov(Z_{iN}^r, Z_{jN}^r)| + \sum_{s_i \in A_{jm}} \sum_{s_j \notin A_{jm}, \|s_i - s_j\| > 2r} |Cov(Z_{iN}^r, Z_{jN}^r)|$$

$$=: \Xi_{21} + \Xi_{22}.$$
For $\Xi_{21}$, since $Z_{iN}^r = Z_{s_iN}^r - Z_{s_iN} = \Delta_{iN}^r + Z_{s_iN}$, by using definition of NED and Cauchy-Schwarz inequality, we obtain
\[
|\text{Cov}(Z_{iN}^r, Z_{jN}^r)| \leq |E[Z_{s_iN} Z_{s_jN}] + ||\Delta_{iN}^r||_2||\Delta_{jN}^r||_2 + 2\sigma \max_i ||\Delta_{iN}||_2
\]

\[
\leq |E[Z_{s_iN} Z_{s_jN}] + (\alpha(N)r^{-\nu_1})^2 + 2\sigma\alpha(N)r^{-\nu_1}
\]

\[
\leq \sigma_{ij} + (\alpha(N)r^{-\nu_1})^2 + 2\sigma\alpha(N)r^{-\nu_1}.
\]

Here $\sigma_{ij} = \max_{i,j} |E[Z_{s_iN} Z_{s_jN}]|$. Therefore, together with Proposition 1, provided that $\alpha(N)r^{-\nu_1} \leq \sigma_{ij}$, we obtain
\[
\Xi_{21} \leq C_pC_0 \left( \left[ \frac{r}{H_0} \right] + 1 \right) \max_{i,j} |\text{Cov}(Z_{iN}^r, Z_{jN}^r)| \leq 3C_p \left( C_0 \left( \frac{2}{3H_0} \right)^{d_2} \right) r^{d_2}(\sigma_{ij} \vee \sigma_{ij}^3 \vee 2\sigma\sigma_{ij}).
\]

For $\Xi_{22}$, recall $Z_{s_iN}^r$ is a measurable function of random variable $\epsilon_{s_iN}$ that $s_{jN} \in B(s_{jN}, r)$ and $\{\epsilon_{s_iN}\}_{s_iN \in \Gamma N}$ is an $\alpha$-mixing random field. Then, based on Lemma 2, by denoting $\max_{d_1 + 1 \leq k \leq d} p_k = \bar{p}$, $\min_{d_1 + 1 \leq k \leq d} p_k = \underline{p}$ and letting $r = \frac{p}{3}$, we have,
\[
\Xi_{22} = \sum_{s_{iN} \in A_j \cap s_{jN} \in B(s_{jN}, \epsilon_{s_{iN}}) \in (2r, \bar{p})} \sum_{\epsilon_{s_{iN}} \in \Gamma N} |\text{Cov}(Z_{iN}^r, Z_{jN}^r)|
\]

\[
\leq \sum_{s_{iN} \in A_j \cap s_{jN} \in B(s_{jN}, \epsilon_{s_{iN}}) \in (2r, \bar{p})} \sum_{\epsilon_{s_{iN}} \in \Gamma N} 2^\tau \left( \frac{2p}{3} \right)^{\frac{\nu_1\delta_1}{2+\delta}} \sigma_{ij}^2
\]

\[
\leq C_p\sigma_{ij}^{2+\delta} 2^\tau \left( \frac{2}{3} \right)^{\frac{\nu_1\delta_1}{2+\delta}} (C_p\bar{p}^{-\frac{\nu_1\delta_1}{2+\delta}})
\]

\[
\leq C_p\sigma_{ij}^{2+\delta} 2^\tau \left( \frac{2}{3} \right)^{\frac{\nu_1\delta_1}{2+\delta}} \left( \frac{C_0}{2d_2H_0^d q^{d_2}\bar{p}^{-\frac{\nu_1\delta_1}{2+\delta}}} \right).
\]

According to Assumption 4, $\min_{d_1 + 1 \leq k \leq d} H_{N}^k \leq \max_{d_1 + 1 \leq k \leq d} H_{N}^k \leq KN^{1/d_2}$. Furthermore, according to geometric-mean inequality, we know $\min_{d_1 + 1 \leq k \leq d} H_{N}^k = O(\max_{d_1 + 1 \leq k \leq d} H_{N}^k)$ and $p = O(p^{1/d_2})$. This implies, $\exists C^*> 0$ such that,
\[
\Xi_{22}^2 \leq C_p\sigma_{ij}^{2+\delta} C^\tau \left( \frac{N}{q^{d_2}} \right)^{\frac{\nu_1\delta_1}{2d_2(2+\delta)+1}} := B(N, q).
\]

Therefore,
\[
\Xi_2 \leq 3C_p \left( C_0 \left( \frac{2}{H_0} \right)^{d_2} \right) r^{d_2}(\sigma_{ij} \vee \sigma_{ij}^3 \vee 2\sigma\sigma_{ij}) + C_p\sigma_{ij}^{2+\delta} B(N, q).
\]

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Based on some simple algebra,

\[
\text{Var}(G_{\text{jm}}) \leq C_p \sigma^2 + 3C_p \left( C_0 \left( \frac{2}{H_0} \right)^d \right) r^{d_2} (\sigma_{ij} \lor \sigma_{ij}^2 \lor 2\sigma_{ij}) + C_p \sigma_{2+\delta}^2 B(N, q)
\]

\[
\leq 2C_p(\sigma^2 + \sigma_{2+\delta}^2 B(N, q) \lor 3 \left( C_0 \left( \frac{2}{H_0} \right)^d \right) r^{d_2} (\sigma_{ij} \lor \sigma_{ij}^2 \lor 2\sigma_{ij})).
\]

Recall \( r = \frac{p}{3} \), we obtain

\[
\text{Var}(G_{\text{jm}}) \leq 2C_p(\sigma^2 + \sigma_{2+\delta}^2 B(N, q) \lor 3 \left( C_0 \left( \frac{2}{3H_0} \right)^d \right) \frac{p}{r^{d_2}} (\sigma_{ij} \lor \sigma_{ij}^2 \lor 2\sigma_{ij})).
\]

We thus get, there exists some \( C^{**} > 0 \), such that

\[
\text{Var}(G_{\text{jm}}) \leq 2C_p \max \left\{ \sigma^2 + 2\sigma_{2+\delta}^2 B(N, q), 3 \left( C_0 C^{**} \left( \frac{2}{3H_0} \right)^d \right) \frac{N}{q^{d_2}} (\sigma_{ij} \lor \sigma_{ij}^2 \lor 2\sigma_{ij}) \right\}
\]

\[
:= 2C_p \nu(q).
\] (B.11)

**Step 4** (Bernstein-type Inequality for \( \{G_{\text{jm}}\} \))

Define \( O_{\text{jm}} = \{ s_jN : s_jN \in \cap_{s_iN \in A_{jm}} B(s_iN, r) \} \), where \( B(s_iN, r) \) denotes the closed ball centered at \( s_iN \) at the radius of \( r \). For any given \( m \), according the Step 1, we know \( G_{\text{jm}}^r \) is a real-valued measurable function of \( \epsilon_{s_iN}'s \) whose \( s_iN \in O_{\text{jm}}^r \), hence \( \sigma(O_{\text{jm}}^r) \)-measurable. Moreover, the distance between any \( O_{\text{jm}}^r \) and \( O_{\text{im}}^r \) is at least \( \min_{d_1+1 \leq k \leq d} p_k - 2r \), provided that \( 0 < r < \min_{d_1+1 \leq k \leq d} p_k / 2 \). Then, for any \( i \neq j \), according to Definition 3, the \( \alpha \)-mixing coefficient between sigma fields \( \sigma(G_{\text{im}}^r) \) and \( \sigma(G_{\text{jm}}^r) \) can bounded by inequality,

\[
\alpha(\sigma(G_{\text{im}}^r), \sigma(G_{\text{jm}}^r)) \leq \alpha(\sigma(O_{\text{im}}^r), \sigma(O_{\text{jm}}^r)) \leq \phi(|O_{\text{im}}^r|, |O_{\text{jm}}^r|) \psi_r(\rho(O_{\text{im}}^r, O_{\text{jm}}^r)).
\]

Based on Lemma 3, for any given \( m \), there exists a sequence of independent random variables, say \( \{W_{jm}\}_{j=1}^{d_2} \), such that, for each \( j \), \( W_{jm} \) is
identically distributed as $G_{jm}^r$. Furthermore, for any $s \geq 1$,

$$P(|W_{jm} - G_{jm}^r| > \xi) \leq 11 \left( \frac{||G_{jm}^r + c||_s}{\xi} \right)^{\frac{s}{2s+1}} \left[ \alpha(\sigma(G_{im}^r), \sigma(G_{jm}^r)) \right]^{\frac{2s}{2s+1}}$$

$$\leq 11 \left( \frac{||G_{jm}^r + c||_s}{\xi} \right)^{\frac{s}{2s+1}} \left[ \phi(||O_{im}^r||, ||O_{jm}^r||) \psi(\rho(O_{im}^r, O_{jm}^r)) \right]^{\frac{2s}{2s+1}}$$

$(s = \infty) \leq 11 \left( \frac{||G_{jm}^r + c||_{\infty}}{\xi} \right)^{\frac{1}{2}} \phi(||O_{im}^r||, ||O_{jm}^r||) \left( \frac{\rho}{3} \right)^{-\nu_2}$

$$\leq 11 \left( \frac{||G_{jm}^r + c||_{\infty}}{\xi} \right)^{\frac{1}{2}} \left( C_0 \left( \frac{4}{H_0} \right)^{d_2} p \right)^{p \left( \frac{\rho}{3} \right)^{-\nu_2}}.$$ 

Now let $p = \infty$, $c = \delta C_p A$, $\delta = 1 + \frac{\eta}{2A}$, and $\xi = \min\{\hat{N}\hat{\eta}/2q^{d_2}, (\delta - 1)C_p A\}$. Since $||G_{jm}^r + c||_{\infty} \geq \delta C_p A - ||G_{jm}^r||_{\infty} \geq (\delta - 1)C_p A \geq \xi$, our setting coincides with the preconditions of Lemma 3. Therefore, we obtain

$$P(|W_{jm} - G_{jm}^r| > \xi) \leq 11 \left( \frac{(\delta + 1)C_p A}{\min\{\hat{N}\hat{\eta}/2q^{d_2}, (\delta - 1)C_p A\}} \right)^{\frac{1}{2}} \left( C_0 \left( \frac{4}{H_0} \right)^{d_2} p \right)^{p \left( \frac{\rho}{3} \right)^{-\nu_2}}$$

$$\leq 11 \left( C_0 \left( \frac{2}{H_0} \right)^{d_2} \sqrt{1 + \frac{4A'}{\hat{\eta}}} \right)^{\frac{1}{2}} \left( C_0 \left( \frac{4}{H_0} \right)^{d_2} \right)^{p \left( \frac{\rho}{3} \right)^{-\nu_2}}$$

$$\leq 11 \left( 1 + \frac{4A'}{\hat{\eta}} \right)^{\frac{1}{2}} \left( \frac{p}{3} \right)^{-\nu_2}. \quad (B.12)$$

The last inequality is due to Assumption 5. Furthermore, similar to the
proof of Theorem 1.3 in Bosq (2012), we have

\[ Q_4 \leq P \left( \sum_{j=1}^{q^d_2} G_{jm}^r \geq \hat{N}\bar{\eta} \right) \]

\[ \leq P \left( \sum_{j=1}^{q^d_2} G_{jm}^r \geq \hat{N}\bar{\eta}; |W_{jm} - G_{jm}^r| \leq \xi \right) + \sum_{j=1}^{q^d_2} P \left( |W_{jm} - G_{jm}^r| > \xi \right) \]

\[ \leq P \left( \sum_{j=1}^{q^d_2} W_{jm} \geq \hat{N}\bar{\eta} - q^d_2 \xi \right) + \sum_{j=1}^{q^d_2} P \left( |W_{jm} - G_{jm}^r| > \xi \right) \]

:= Q_{41} + Q_{42}.

By applying (B.12), we instantly obtain

\[ Q_{42} \leq 11q^d_2 \left( 1 + \frac{4A}{\bar{\eta}} \right) \frac{1}{2} p\left( \frac{p}{3} \right)^{-\nu_2} = 11q^d_2 \left( 1 + \frac{8AH^d_2 - d_1}{t} \right) \frac{1}{2} p\left( \frac{p}{3} \right)^{-\nu_2}. \]

For \( Q_{41} \), note \( \{W_{jm}\}_{j=1}^{q^d_2} \) is a mutually independent sequence of real-valued random variables. Thus, Bernstein inequality for sum of independent random variables could be applied, which yields

\[ Q_{41} \leq 2 \exp \left( -\frac{(\hat{N}\bar{\eta}/2)^2}{2(\sum_{j=1}^{q^d_2} \max_j \text{Var}(W_{jm}) + \|W_{jm}\|_{\infty}\hat{N}\bar{\eta})} \right) \]

\[ \leq 2 \exp \left( -\frac{(\hat{N}\bar{\eta}/2)^2}{2(2q^d_2 C_p v(q) + \frac{C_p A\hat{N}\bar{\eta}}{6})} \right) \]

\[ \leq 2 \exp \left( -\frac{(\hat{N}\bar{\eta}/2)^2}{2(2q^d_2 pv(q) + \frac{p A\hat{N}\bar{\eta}}{6})} \right) \]

\[ = 2 \exp \left( -\frac{\hat{N}t^2}{32H^d_0 A_0^{-d_1} \left( 2d_2 - 1 v(q) + \frac{A pt}{12H_0^2 d_2 - d_1} \right)} \right) \]

The second and third inequalities are due to (B.11) and \( C_p \leq C_0(2/H_0)^{d_2} p \leq p \) respectively. The last equality is according to \( \hat{N} = (2q)^{d_2} p \) and \( \bar{\eta} = \)
\[
\frac{t}{2H_0^{d_2-d_1}}. \text{ Together all the results above, we get}
\]

\[
P\left(\left| \frac{\sum_{i=1}^{N} Z_{s_i, N}}{N} \right| \geq t \right) \leq 2 \exp \left( - \frac{\hat{N} t^2}{32 H_0^{d_2-d_1} \left( 2^{d_2-1} v(q) + \frac{A p t}{12 H_0^{d_2-d_1}} \right)} \right)
\]

\[
+11 q^{d_2} \left( 1 + \frac{8 A H_0^{d_2-d_1}}{t} \right)^{\frac{1}{2}} p^{\tau} \left( \frac{p}{3} \right)^{-\nu_2} + \left( \frac{C_0 2^{d_2+1}}{H_0^{d_1}} \right)^{s} \left( \frac{\alpha(N)p^{-\nu_1}}{3t} \right)^{s}
\]

\[ := U_1 + U_2 + U_3. \]

Together with Proposition 5, we have

\[
U_1 \leq 2 \exp \left( - \frac{H_0^{d_1} N t^2}{32 C_0 \left( 2^{d_2-1} v(q) + \frac{A H_0^{d_1} N t}{12(2q)^{d_2} C_0} \right)} \right).
\]

For \( U_2 \) and \( U_3 \), the key is to specify \( p \) and \( q \). According to Assumption 3 and Assumption 4, we have \( \min_{d_1+1 \leq k \leq d} H_k^N \leq \max_{d_1+1 \leq k \leq d} H_k^N \leq K N^{1/d_2} \).

Then, according to geometric-mean inequality, we know \( \min_{d_1+1 \leq k \leq d} H_k^N = O(\max_{d_1+1 \leq k \leq d} H_k^N) \) and \( q \leq p^{1/d_2} \). Thus, there exists some constant \( K_3 \) and \( K_4 > 0 \), such that,

\[
U_1 + U_2 \leq 11 K_3 \left( 1 + \frac{8 A H_0^{d_2-d_1}}{t} \right)^{\frac{1}{2}} q^{d_2} \left( \frac{N}{q^{d_2}} \right)^{-\frac{\nu_2 + \tau}{d_2}} + K_4 \left( \frac{N}{q^{d_2}} \right)^{-\frac{\nu_1}{d_2}} \left( \frac{\alpha(N)}{t} \right)^{s}.
\]

Thus we finish the proof.

C Proof Related to Section 4

C.1 Section 4.1

Proof of Theorem 3

Proof of Theorem 3 is almost the same as the proof of Theorem 3 in Jenish (2012). Hence we only highlight the difference here. A basic decomposition of \( \hat{\theta}(x) - \theta(x) \) is as follow,

\[
\hat{\theta}(x) - \theta(x) = \left( \frac{\hat{m}(x) - m(x)}{\left( \nabla \hat{m}(x) - \nabla m(x) \right) h} \right) = S_N^{-1}(x) \left\{ T_N(x) - S_N(x) \left( \frac{m(x)}{\nabla m(x) h} \right) \right\},
\]

\[ := W_N(x). \]
where, for \( k = 1, \ldots, D + 1 \), by denoting \( v^k \) as the \( k \)-th element of vector \( v \) and location \( s_{iN} \) as \( iN \),

\[
W_N^k(x) = \frac{1}{Nh^D} \sum_{i=1}^{N} K \left( \frac{X_{iN} - x}{h} \right) U_i^k(x) \omega_{iN}
\]

\[
\omega_{iN} = Y_{iN} - m(x) - \nabla m(x)^T (X_{iN} - x).
\]

\( \nabla m(x) \) denotes the gradient of \( m(x) \). Hence, it suffice to finish the proof if we demonstrate the following three things,

1. \( \sup_{x \in \mathcal{X}} ||U^{-1}(x)|| = O_p(1) \),
2. \( \sup_{x \in \mathcal{X}} |E[W_1^1(x)]| = O_p(h^\beta) \),
3. \( \sup_{x \in \mathcal{X}} |W_N^1(x) - E[W_N^1(x)]| = \sqrt{\log N/Nh^D} \),

where \( ||A|| \) means Euclidean norm of matrix \( A \). Furthermore, according to Assumption 8, we have

\[
\sup_{x \in \mathcal{X}} |W_N^1(x) - E[W_N^1(x)]| \leq \sup_{x \in \mathcal{X}} ||m||_{\infty} \left| \frac{1}{Nh^D} \sum_{i=1}^{N} \left( K \left( \frac{X_{iN} - x}{h} \right) U_i^1(x) - E \left[ K \left( \frac{X_{iN} - x}{h} \right) U_i^1(x) \right] \right) \right| + \sup_{x \in \mathcal{X}} ||\nabla m|| \left| \frac{1}{Nh^D} \sum_{i=1}^{N} \left( (X_{iN} - x)K \left( \frac{X_{iN} - x}{h} \right) U_i^1(x) - E \left[ (X_{iN} - x)K \left( \frac{X_{iN} - x}{h} \right) U_i^1(x) \right] \right) \right| := \Delta_1 + \Delta_2 + \Delta_3
\]

**Step 1** (Union Upper Bound)

Define \( Z_{iN} = (X_{iN}, Y_{iN}) \),

\[
\mathcal{K}(\mathcal{X}) = \{ g_x(v) = K(v - x/h) : v \in \prod_{l=1}^{D} [x^{(l)}_l - 1, x^{(l)}_l + 1], y \in [-L_N, L_N], x \in \mathcal{X} \}.
\]

Since \( \mathcal{X} \) is compact in \( \mathbb{R}^D \), here we assume \( \exists M > 0 \), such that \( \mathcal{X} \subset [-M, M]^D \). In this step, we focus on deriving the convergence rate of

\[
\sup_{g \in \mathcal{K}(\mathcal{X})} \left| \frac{1}{Nh^D} \sum_{i=1}^{N} (g_x(X_{iN}) Y_{iN} - E[g_x(X_{iN}) Y_{iN}]) \right| := \sup_{g \in \mathcal{K}(\mathcal{X})} \left| \frac{1}{Nh^D} \sum_{i=1}^{N} h_x(Z_{iN}) \right|.
\]
Therefore, according to some basic union bound argument based on covering number, for any $t > 0$,

$$
P \left( \sup_{g \in \mathcal{K}(X)} \left| \frac{1}{N} \sum_{i=1}^{N} h_x(Z_iN) \right| > t \right) \leq \mathcal{N}(th^D/3, \mathcal{K}, \| \cdot \|_\infty) P \left( \left| \frac{1}{N} \sum_{i=1}^{N} h_x(Z_iN) \right| > \frac{th^D}{3} \right),$$

where $\mathcal{N}(th^D/3, \mathcal{K}, \| \cdot \|_\infty)$ denotes the $th^D/3$-covering number of set $\mathcal{K}(X)$ with respect to $\| \cdot \|_\infty$. Note that, based on Assumption 9 and some simple algebra, for any $z \neq z'$,

$$|h_x(z) - h_x(z')| \leq 2\sqrt{2M} \frac{L_N \vee \|K\|_\infty}{h} ||z - z'||_2,$$

which implies $Lip(h_x) = O(L_N/h)$. Then, according to Proposition 2, for any $x$ and $l$, $\{h_x(Z_iN)\}_{s_iN \in \Gamma}N$ is also a centered and real-valued $L^\ast$-NED random field on process $\epsilon$ and satisfies

$$||h_x(Z_iN) - E[h_x(Z_iN)|F_iN(r)]||_s \leq 4\sqrt{2M} \frac{L_N \vee \|K\|_\infty}{h} \psi_Z(r).$$

Furthermore, due to Assumption 8, for any $x$ and the given $p_0 > 2$,

$$||h_x||_{p_0} \leq ||K||_\infty L_N, \ Var(h_x(Z)) \leq h^D \sigma^2,$$

$$||h_x||_{p_0}^2 = O(h^{2D}), \ max_{i \neq j} Cov(h_x(Z_iN), h_x(Z_jN)) = O(h^{2D}).$$

Therefore, by using Theorem 2, for any $q \in \mathbb{N}$,

$$P \left( \frac{1}{N} \sum_{i=1}^{N} h_x(Z_iN) > th^D \right) \leq \exp \left( - \frac{Nt^2h^2D}{\nu(q) + \frac{NL_Nth^D}{q^d_2}} \right) + \sqrt{\frac{1}{th^D}} \left( \frac{N}{q^d_2} \right)^{-\frac{\nu_2}{\nu_2} + \gamma} q^d_2 + \left( \frac{N}{q^d_2} \right)^{-\frac{\nu_1}{\nu_1}} \left( \frac{L_N}{th^D+1} \right)^s$$

$$:= U_1 + U_2 + U_3. \quad (C.2)$$

**Step 2 (Covering Number)**

Define mapping $T(x) : x \to K(v - x/h)y$, $x \in \mathcal{X}$. Then, according to Assumption 9, it can be shown that, $T(x)$ is a Lipschitz mapping with Lipschitz modulo $2\sqrt{2M} \frac{L_N \vee \|K\|_\infty}{h}$. Thus, according to Proposition 13 of Hang et al., we have

$$\mathcal{N} \left( \frac{th^D}{3}, \mathcal{K}(X), || \cdot ||_\infty \right) \leq \frac{L_N}{th^D+1}. \quad (C.3)$$
Step 3 (Generic Uniform)

Let \( q = (N^{1 + \frac{1}{p_0}} t)^{\frac{1}{D}} \), \( t = \sqrt{\log N / Nh^D} \) and \( L_N = N^{\frac{1}{p_0}} \). Firstly, we calculate \( v(q) \). To calculate \( v(q) \), we only need to calculate \( B(N, q) h^{\frac{2D}{p_0}} \) and \( \frac{N}{q^{D/2}} h^{2D} \).

According to the (1)’s in Assumptions 10, 11 and 12, based on some simple calculation, we directly obtain

\[
B(N, q) h^{\frac{2D}{p_0}} = O(h^D), \quad \frac{N}{q^{D/2}} h^{2D} = O(h^D),
\]

which implies \( v(q) = O(h^D) \). Therefore,

\[
U_1 \lesssim \exp \left( - \frac{N t^2 h^{2D}}{h^D + N^{1 + \frac{1}{p_0}} t h^D} \right) \lesssim \exp(-N t^2 h^D) \lesssim \exp(-\log N) = \frac{1}{N}. \tag{C.4}
\]

Similarly, according to Assumption 10, we have

\[
U_2 \lesssim h^{\frac{\alpha^D}{2D}} (N \log N)^{\frac{1}{4}}, \quad U_3 \lesssim \left( h^{\frac{\alpha^D}{2D}} \frac{N}{\sqrt{\log N}} \right)^s. \tag{C.5, C.6}
\]

Combine (C.3) to (C.6) and Assumptions 10 and 11 and set \( t = \sqrt{\log N / Nh^D} \), we obtain

\[
\mathcal{N} \left( \frac{t h^D}{3}, \mathcal{K}(X), \| \cdot \|_{\infty} \right) (U_1 + U_3 + U_2) \lesssim \frac{1}{\sqrt{\log N}} + h^{\frac{\alpha^D}{2D} - \frac{\alpha^D}{2} + \frac{5D}{4} + 1} (N^5 \log N)^{\frac{1}{4}} \left( h^{\frac{\alpha^D}{2D}} \frac{N^2}{\log N} \right)^s = o(1).
\]

Furthermore, similar to (A.7) in Hansen (2008), under Assumption 8, we can also obtain

\[
E \left| \frac{1}{Nh^D} \sum_{i=1}^{N} K \left( \frac{X_{iN} - x}{h} \right) Y_{iN} 1\{|Y_{iN}| > N^{\frac{1}{p_0}}\} \right|^{p_0} = O \left( \frac{1}{N} \right).
\]
Hence, we manage to prove,
\[
\sup_{x \in \mathcal{X}} \left| \frac{1}{N h^D} \sum_{i=1}^{N} \left( K \left( \frac{X_{iN} - x}{h} \right) Y_{iN} - E \left[ K \left( \frac{X_{iN} - x}{h} \right) Y_{iN} \right] \right) \right| = O_p \left( \sqrt{\frac{\log N}{Nh^D}} \right).
\]
(C.7)

**Step 4**

Based on (C.7), Assumption 8 and compactness of \( \mathcal{X} \), it can be shown that
\[
\sup_{x \in \mathcal{X}} |W_{1N}(x) - E[W_{1N}(x)]| = O_p \left( \sqrt{\frac{\log N}{Nh^D}} \right).
\]

(2) can also be obtained by using standard argument of convolution kernel and Hölder continuity of \( m(x) \). Thus it is omitted here. To prove (1), it is equivalent to prove matrix \( U(x) \) is asymptotically invertable, which can be done by using Proposition 4. Since procedure of this is almost the same as the proof of Lemma 1 in Jenish, it is omitted as well. Then, we finish the proof.

**C.2 Section 4.2**

We firstly introduce the following to lemmas which are useful within our proof.

**Lemma 4** (Lemma 2.3.7 van der Vaart and Weller (1996)) Let \( Z_i(f) := f(X_i) - Ef(X_i) \), where \( f \in \mathcal{F} \), for some real valued function class \( \mathcal{F} \). For arbitrary process \( (Z_i(f))_{i=1}^{N} \) and arbitrary functions \( \mu_1, ..., \mu_N : \mathcal{F} \rightarrow \mathbb{R} \),
\[
\beta_N(\delta) P^* \left( \sup_{f \in \mathcal{F}} | \sum_{i=1}^{N} Z_i(f) | > \delta \right) \leq 2P^* \left( 4 \sup_{f \in \mathcal{F}} | \sum_{i=1}^{N} \epsilon_i(Z_i(f) - \mu_i) | > \delta \right),
\]
for every \( \delta > 0 \) and \( \beta_N(\delta) \leq \inf_{f \in \mathcal{F}} P( | \sum_{i=1}^{N} Z_i(f) | < \delta/2 ) \). Here \( P^* \) is outer measure and \( \epsilon_i \) denotes Rademacher random variable.

**Lemma 5** Let \( \mathcal{G} \) be a class of functions \( g : \mathbb{R}^m \rightarrow [-A, A] \) with \( 2 \leq V_{\mathcal{G}}^+ < +\infty \). Let \( Q \) be any probability measure on \( \mathbb{R}^m \) and let \( 0 < \epsilon < A/2 \). Then for any \( p \geq 1 \), we have
\[
\mathcal{M}(\epsilon, \mathcal{G}, || \cdot ||_{L^p(Q)}) \leq 3 \left( \frac{2^{p+1} A^p}{\epsilon^p} \log \frac{3e(2A)^p}{\epsilon^p} \right)^{V_{\mathcal{G}}^+},
\]
where \( \mathcal{M}(\epsilon, \mathcal{G}, || \cdot ||_{L^p(Q)}) \) denotes the \( \epsilon \)-packing number of set \( \mathcal{G} \) with respect to \( || \cdot ||_{L^p(Q)} \).
Please note, as we mentioned previously, we have postponed the problem of measurability. Thus the outer probability in Lemma 4 is directly considered as usual probabilistic measure when we apply it. As for Lemma 5, this result is a direct consequence of Theorem 9.4 in Györfi et al. (2002). To prove this, we only need to note that, this is equal to prove the same argument works fine on the following set, \( \{ g : \mathbb{R}^m \to [0, 2A] \} := \mathcal{G}' \) on condition that \( VC_{\mathcal{G}'}^+ \geq 2 \). Hence we omit the proof here.

**Proof of Proposition 6**

**Step 1** (Symmetrization by a ghost sample)
Let \( X_N = (X_{siN})_{i=1}^N \) and introduce a shadow sample vector \( X'_N := (X'_{siN})_{i=1}^N \) such that \( X_N \) is an independent copy of \( X'_N \). Then, we introduce a Rademacher sequence, \( \{ \epsilon_i \}_{i=1}^N \), i.e. \( \epsilon_i \)'s are mutually independent and for each \( i \), \( P(\epsilon_i = 1) = P(\epsilon_i = -1) = \frac{1}{2} \). Let vector \( (\epsilon_i)_{i=1}^N \) is also independent from vector \( X_N \). Due to Corollary 1, we know, for \( \forall g \in \mathcal{G} \) and some constants \( C_0 \) and \( K_2 \) inherited from Theorem 1, for \( \forall t > 0 \)

\[
P \left( \left| N^{-1} \sum_{i=1}^N (g(X_{siN}) - E[g(X_{siN})]) \right| > t \right) < 4 \exp \left( -\frac{Nt^2}{K_2(\log N)^{\frac{d_2}{2}}} \left( (C_0A)^2 + \frac{8C_0At}{3} \right) \right),
\]

where the upper bound holds uniformly over function class \( \mathcal{G} \). Thus, according to Lemma 4, by letting \( \delta = Nt \),

\[
\beta_N(Nt) = \inf_{g \in \mathcal{G}} P \left( \left| N^{-1} \sum_{i=1}^N (g(X_{siN}) - E[g(X_{siN})]) \right| \leq \frac{t}{2} \right)
= \inf_{g \in \mathcal{G}} 1 - P \left( \left| N^{-1} \sum_{i=1}^N (g(X_{siN}) - E[g(X_{siN})]) \right| > \frac{t}{2} \right)
= 1 - \sup_{g \in \mathcal{G}} P \left( \left| N^{-1} \sum_{i=1}^N (g(X_{siN}) - E[g(X_{siN})]) \right| > \frac{t}{2} \right)
\geq 1 - 4 \exp \left( -\frac{Nt^2}{4K_2(\log N)^{\frac{d_2}{2}}} \left( (C_0A)^2 + \frac{8C_0At}{3} \right) \right).
\]

Then for any \( N \geq \min \{ N \in \mathbb{N} : \frac{N}{(\log N)^{d_2/2}} \geq 4K_2 \log 8C_0A((C_0A) + 8t/3)/t^2 \} \), we have \( \beta_N(Nt) \leq \frac{1}{2} \). Therefore, according to Lemma 4, by
letting $\mu_i = g(X'_{s_iN}) - E[g(X_{s_iN})]$,
\[
P \left( \sup_{g \in G} \left| N^{-1} \sum_{i=1}^{N} (g(X_{s_iN}) - E[g(X_{s_iN})]) \right| > t \right) \\
\leq 4P \left( \sup_{g \in G} \left| N^{-1} \sum_{i=1}^{N} \epsilon_i(g(X_{s_iN}) - g(X'_{s_iN})) \right| > t \right) \\
\leq 4P \left( \sup_{g \in G} \left| N^{-1} \sum_{i=1}^{N} \epsilon_i g(X_{s_iN}) - g(X'_{s_iN}) \right| > \frac{t}{2} \right) + 4P \left( \sup_{g \in G} \left| N^{-1} \sum_{i=1}^{N} \epsilon_i g(X'_{s_iN}) \right| > \frac{t}{2} \right) \\
\leq 8P \left( \sup_{g \in G} \left| N^{-1} \sum_{i=1}^{N} \epsilon_i g(X_{s_iN}) \right| > \frac{t}{2} \right).
\]

**Step 2 (Chaining)**

This step is similar to the proof of Theorem 19.1 in Györfi et al. (2002). But in order to maintain the completeness of our proof, we still demonstrate the detail here. Let $|| \cdot ||_2$ denotes the empirical $L^2$ norm on condition that $X_N = x_N$. Since due to Cauchy-Schwarz inequality, for any $g \in G$,
\[
\sup_{g \in G} \left| \frac{1}{N} \sum_{i=1}^{N} g(x_{s_iN}) \epsilon_i \right| \leq ||g||_2 \sqrt{\frac{1}{N} \sum_{i=1}^{N} \epsilon_i^2} \leq A.
\]

Hence we only need to consider the situation $t < 2A$. For any $s \in \mathbb{N}$, let $\{g_1^s, \ldots, g_N^s\} := G_s$ be a $|| \cdot ||_N$-cover of $G$ of radius $A/2^s$ on condition that $X_N = x_N$. Here $N_s = N(2^{-s}, G, || \cdot ||_N)$ is the $|| \cdot ||_N$-covering number of $G$ of radius $A/2^s$. Thus, given $X_N = x_N$, by using chaining argument, for $\forall g \in G$, we can rewrite it as
\[
g = g - g^0 = g - g^S + \sum_{s=1}^{S} (g_s - g_{s-1}),
\]
where for each $s$, $g^s \in G_s$ and obviously we can define $g^0 = 0$. For simplicity, denote $g(x_{s_iN})$ as $g_i$. Let $S = \min\{s \geq 1 : A/2^s \leq t/4\}$. Thus, due to the
chaining, we have
\[
\left| \frac{1}{N} \sum_{i=1}^{N} g_i \epsilon_i \right| = \left| \frac{1}{N} \sum_{i=1}^{N} (g_i - g_i^s) \epsilon_i + \sum_{s=1}^{S} \frac{1}{N} \sum_{i=1}^{N} (g_i^s - g_i^{s-1}) \epsilon_i \right|
\leq \left| \frac{1}{N} \sum_{i=1}^{N} (g_i - g_i^s) \epsilon_i \right| + \sum_{s=1}^{S} \left| \frac{1}{N} \sum_{i=1}^{N} (g_i^s - g_i^{s-1}) \epsilon_i \right|
\leq ||g - g^s|| \cdot \sqrt{\frac{1}{N} \sum_{i=1}^{N} \epsilon_i^2 + \sum_{s=1}^{S} \frac{1}{N} \sum_{i=1}^{N} (g_i^s - g_i^{s-1}) \epsilon_i}
\leq \frac{t}{4} + \sum_{s=1}^{S} \left| \frac{1}{N} \sum_{i=1}^{N} (g_i^s - g_i^{s-1}) \epsilon_i \right|.
\]

Therefore, for any \( \eta_s \geq 0, s = 1, \ldots, S \) and \( \sum_{s=1}^{S} \eta_s \leq 1 \), we have, by denoting \( P_{X_N}(\cdot) = P(\cdot | X_N = x_N) \),
\[
P_{X_N} \left( \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^{N} g_i \epsilon_i \right| > \frac{t}{2} \right) \leq P_{X_N} \left( \exists g \in \mathcal{G} : \left| \frac{1}{N} \sum_{i=1}^{N} (g_i^s - g_i^{s-1}) \epsilon_i \right| > \frac{t}{4} + \sum_{s=1}^{S} \eta_s \frac{t}{4} \right)
\leq \sum_{s=1}^{S} P_{X_N} \left( \exists g \in \mathcal{G} : \left| \frac{1}{N} \sum_{i=1}^{N} (g_i^s - g_i^{s-1}) \epsilon_i \right| > \eta_s \frac{t}{4} \right)
\leq \sum_{s=1}^{S} \eta_s N_{s-1} \max_{g \in \mathcal{G}} P_{X_N} \left( \left| \frac{1}{N} \sum_{i=1}^{N} (g_i^s - g_i^{s-1}) \epsilon_i \right| > \eta_s \frac{t}{4} \right).
\]

**Step 3 (Conditional Concentration)**

Note that, on condition of \( X_N = x_N \), \{ (g_i^s - g_i^{s-1}) \epsilon_i \}_{i=1}^{N} \) is an iid sequence. For each \( i \), the support is \([- (g_i^s - g_i^{s-1}), g_i^s - g_i^{s-1}]\). Thus, by using Hoeffding inequality under iid setting, we have
\[
P_{X_N} \left( \left| \frac{1}{N} \sum_{i=1}^{N} (g_i^s - g_i^{s-1}) \epsilon_i \right| > \eta_s \frac{t}{4} \right) \leq 2 \exp \left( - \frac{N \eta_s^2}{8 \| g^s - g^{s-1} \|^2_N} \right) \leq 2 \exp \left( - \frac{N \eta_s^2 t^2}{72 \frac{2}{N} \eta_s^2} \right),
\]
where the second inequality above ensures an upper bound independent of
the value of sample vector. Therefore,

\[
P \left( \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^{N} g(X_{s,N}) \epsilon_i \right| > \frac{t}{2} \right) = E \left[ P_{X_N} \left( \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^{N} g_i \epsilon_i \right| > \frac{t}{2} \right) \right] \\
\leq 2 \sum_{s=1}^{S} E \left[ N_s^2 \max_{g \in \mathcal{G}} P_{X_N} \left( \left| \frac{1}{N} \sum_{i=1}^{N} (g_s - g_{s-1}) \epsilon_i \right| > \eta_s \frac{t}{4} \right) \right] \\
\leq 2 \sum_{s=1}^{S} \exp \left( 2 \log E[N_s] - \frac{N(\eta_s t)^2}{72 A^2 s^2} \right).
\]

**Step 4 (Proper design of \( \eta_s \))**

In order to get rid of the covering number, we firstly let \( \eta_s \) satisfy

\[
2 \log E[N_s] \leq \frac{Nt^2 \eta_s^2}{144 A^2 s^2},
\]

which yields

\[
\eta_s \geq \frac{24 A^2 \sqrt{\log E[N(A/2^s, \mathcal{G}, \| \cdot \|_N)]}}{\sqrt{Nt}}.
\]

Combing Lemma 5 and Lemma 9.2 in Györfi et al. (2002), we can show that

\[
E[N(A/2^s, \mathcal{G}, \| \cdot \|_N)] \leq 3 \left( \frac{8e}{(1/2^s)^2} \log \frac{12e}{(1/2^s)^2} \right)^{V_{\overline{\rho}}^+}.
\]

Hence, we define

\[
\frac{24 A^2 \frac{\sqrt{3}}{2^s} \sqrt{\log 3 \left( \frac{8e}{(1/2^s)^2} \log \frac{12e}{(1/2^s)^2} \right)^{V_{\overline{\rho}}^+}}}{\sqrt{Nt}} := \bar{\eta}_s
\]

and let

\[
\eta_s = \max \left\{ \bar{\eta}_s, \sqrt{s} \frac{2^s}{4} \right\}.
\]

Thus, due to the facts

\[
\sum_{s=1}^{S} \frac{\sqrt{s}}{2^s 4} \leq \sum_{s=1}^{+\infty} \frac{\sqrt{s}}{2^s 4} \leq \frac{1}{2}
\]

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and

\[
\sum_{s=1}^{S} \eta_s \leq \sum_{s=1}^{S} \frac{24A \frac{1}{2} \sqrt{V_g^+ \log 3 \left( \frac{8e}{(1/2)^2} \log \frac{12e}{(1/2)^2} \right)}}{\sqrt{Nt}} \\
\leq 24A \sum_{s=1}^{S} \int_{\frac{1}{2}}^{\frac{1}{2}t} \sqrt{V_g^+ \log 3 \left( \frac{8e}{u^2} \log \frac{12e}{u^2} \right)} du \\
= 24A \int_{\frac{1}{2}}^{\frac{1}{2}t} \sqrt{V_g^+ \log 3 \left( \frac{8e}{u^2} \log \frac{12e}{u^2} \right)} du \\
\leq 24A \int_{\frac{1}{2}}^{\frac{1}{2}t} \sqrt{V_g^+ \log 3 \left( \frac{8e}{u^2} \log \frac{12e}{u^2} \right)} du.
\]

Note that for any \( t > 0 \), there exists some constant \( c > 0 \) such that,

\[
\int_{\frac{1}{2}}^{\frac{1}{2}t} \sqrt{V_g^+ \log 3 \left( \frac{8e}{u^2} \log \frac{12e}{u^2} \right)} du \leq c \int_{\frac{1}{2}}^{\frac{1}{2}t} \sqrt{\log \frac{1}{u^2}} du \\
= \sqrt{2c} \int_{\frac{1}{2}}^{\frac{1}{2}t} \sqrt{\log \frac{1}{u}} du \leq \sqrt{2c} \int_{\frac{1}{2}}^{\frac{1}{2}t} u^{-\frac{1}{2}} du \leq 2\sqrt{2c} < +\infty.
\]

Therefore, for any \( t > 0 \), \( \int_{\frac{1}{2}t}^{\frac{1}{2}t} \sqrt{V_g^+ \log 3 \left( \frac{8e}{u^2} \log \frac{12e}{u^2} \right)} du < +\infty \). This ensures the \( N_0 \) is always existed. Then, for any \( N \geq N_0 \), we have \( \sum_{s=1}^{S} \eta_s \leq 1 \), which coincides with the requirement of \( \eta_s \)'s.

**Step 5 (Plug in \( \eta_s \))**

Since \( \eta_s \geq \frac{\sqrt{s}}{\sqrt{2}} \sqrt{\bar{\eta}_s} \), we obtain

\[
P \left( \sup_{g \in G} \left| \frac{1}{N} \sum_{i=1}^{N} g(X_{st, sn}) \epsilon_i \right| > \frac{t}{2} \right) \leq 2 \sum_{s=1}^{S} \exp \left( -\frac{Nt^2 \eta_s^2}{144 \frac{A^2}{27}} \right) \\
\leq 2 \sum_{s=1}^{S} \exp \left( -\frac{sNt^2}{16 \cdot 144A^2} \right) \\
\leq 5 \exp \left( -\frac{Nt^2}{(48A)^2} \right),
\]

for any \( \sqrt{N}t \geq 36A \). Hence we finish the proof.
Proof of Proposition 7

Similar to the proof of Proposition 6, the major tool here is still chaining. Hence, we will skip some details and highlight the difference.

Step 1 (Chaining)

Let \( \{g_1^s, \ldots, g_N^s\} \) be the minimum \( \sigma/2^{2s} \)-net of \( \mathcal{G} \) with respect to infinite norm \( \| \cdot \|_\infty \). Then, for any \( g \in \mathcal{G} \), we can construct chaining

\[
g = g - g^S + \sum_{s=1}^{S} g^s - g^{s-1},
\]

where \( g^0 = 0 \) and \( S = \min \{ s \in \mathbb{N} : \frac{\sigma}{2^{2s}} \leq \frac{t}{4L} \} \). Then, we have

\[
|P_N(gY) - P(gY)| \leq |P_N((g - g^S)Y) - P((g - g^S)Y)| + \sum_{s=1}^{S} |P_N((g^s - g^{s-1})Y) - P((g^s - g^{s-1})Y)|
\]

\[
\leq 2L\|g - g^S\|_\infty + \sum_{s=1}^{S} |P_N((g^s - g^{s-1})Y) - P((g^s - g^{s-1})Y)|
\]

\[
\leq \frac{t}{2} + \sum_{s=1}^{S} |P_N((g^s - g^{s-1})Y) - P((g^s - g^{s-1})Y)|.
\]

Therefore, similar to Step 2 in the proof of Proposition 6, for any \( \eta_s, s = 1, \ldots, S \) such that \( \sum_{s=1}^{S} \eta_s \leq 1 \),

\[
P \left( \sup_{g \in \mathcal{G}} |P_N(gY) - P(gY)| \geq t \right) \leq \sum_{s=1}^{S} P \left( \exists \ g : |P_N((g^s - g^{s-1})Y) - P((g^s - g^{s-1})Y)| \geq \frac{\eta_st}{2} \right)
\]

\[
\leq \sum_{s=1}^{S} N_s N_{s-1} \max_{g \in \mathcal{G}} P \left( |P_N((g^s - g^{s-1})Y) - P((g^s - g^{s-1})Y)| \geq \frac{\eta_st}{2} \right)
\]

Step 2 (Application of Corollary 1)

Note that, for each \( s = 1, \ldots, S \), by letting \( Z_{iN}^s = (g^s(X_{siN}) - g^{s-1}(X_{siN}))Y_{siN} - E[(g^s(X_{siN}) - g^{s-1}(X_{siN}))Y_{siN}] \), due to the measurability of function \( g^s \), we know process \( \{Z_{iN}^s\}_{s \in \Gamma_N} \) is also a mean-zero geometric irregularly-spaced \( \alpha \)-mixing random field whose mixing coefficient is smaller than \( \{(X_{siN}, Y_{siN})\}_{s \in \Gamma_N} \)'s. In order to apply Corollary 1, we firstly need to investigate the upper bound of \( \sup_{iN} Var(Z_{iN}^s) \) and \( \sup_{iN} \|Z_{iN}^s\|_\infty \) for each \( s \).
For \( \text{Var}(Z_s^{i_N}) \), when \( s \geq 2 \), according to Step 1,

\[
\text{Var}(Z_s^{i_N}) = \| (g^s - g^{s-1})(X_{i_N}) Y_{i_N} \|^2_2 \leq L^2 \left( \| g^s - g^{s-1} \|_\infty \right)^2 \\
\leq L^2 \left( \| g^s - g \|_\infty + \| g^{s-1} - g \|_\infty \right)^2 \leq L^2 \left( \frac{\sigma}{2^s} + \frac{\sigma}{2^{s-1}} \right)^2 \\
\leq L^2 \left( \frac{\sigma}{2^s} + \frac{\sigma}{2^{s-1}} \right)^2 = \frac{9L^2 \sigma^2}{2^s}.
\]

When \( s = 1 \), since \( g^0 = 0 \),

\[
\text{Var}(Z_1^{i_N}) = \| g^s (X_{i_N}) Y_{i_N} \|^2_2 \leq L^2 \sigma^2 < \frac{9L^2 \sigma^2}{4}.
\]

Therefore, for any \( s \), we have \( \sup_{i_N} \text{Var}(Z_s^{i_N}) \leq \frac{9L^2 \sigma^2}{2^s} \). Based on the same type of argument, we can also show that, for any \( s \), \( \sup_{i_N} \| Z_s^{i_N} \|_\infty \leq \frac{10AL}{2^s} \).

Then, according to Corollary 1, we obtain

\[
P \left( \sup_{g \in G} \left| P_N(gY) - P(gY) \right| \geq t \right) \leq 4 \sum_{s=1}^{S} \exp \left( 2 \log N_s - \frac{Nt^2 \eta_s^2}{4K_2 (\log N)^{d_2} g (C_0^{2} 9L^2 \sigma^2 \gamma + \frac{40C_0 AL t}{3} \frac{2^s}{2^s})} \right).
\]

Together with the fact that \( C_0 \geq 1 \) and condition C1,

\[
P \left( \sup_{g \in G} \left| P_N(gY) - P(gY) \right| \geq t \right) \leq 4 \sum_{s=1}^{S} \exp \left( 2 \log N_s - \frac{Nt^2 \eta_s^2}{40C_0^2 K_2 (\log N)^{d_2} \frac{2^s}{2^s}} \right.
\]

\[= 4 \sum_{s=1}^{S} \exp(2 \log N_s - H(N)). \tag{C.8}\]

Finally, by following the same procedure as Step 4 and 5 in the proof of Proposition 6, we let

\[
\eta_s = \max \left\{ \sqrt{s}, \sqrt{\frac{\sigma}{2^s}} \right\},
\]

\[
\bar{\eta}_s = 4\sqrt{10K_2 \sigma C_0 L} \sqrt{\frac{\sigma}{2^s}} \sqrt{\log \mathcal{N}(\frac{\sigma}{2^s}, G, \| \cdot \|_\infty)} \sqrt{\frac{(\log N)^{d_2}}{Nt^2}} \gamma.
\]

Obviously, it’s obvious that, based on this definition of \( \eta_s \), we immediately get \( 2 \log N_s \leq H(N)/2 \). The only condition needs to be specified
is \( \sum_{s=1}^{S} \eta_s \leq 1 \). It has already been shown that \( \sum_{s=1}^{S} \frac{\sqrt{\eta_s}}{2\pi} \leq \frac{1}{2} \). We only need to calculate \( \sum_{s=1}^{S} \bar{\eta}_s \).

\[
\sum_{s=1}^{S} \bar{\eta}_s = 4 \sqrt{10K_2 \sigma C_0 L} \sqrt{\frac{\log N}{N t^2}} \sum_{s=1}^{S} \sqrt{\frac{\sigma}{2 \sigma_s^2}} \sqrt{\log \mathcal{N}(\sigma \frac{\sigma_s}{2 \sigma_s^2}, \mathcal{G}, \| \cdot \|_{\infty})}.
\]

Note \( \sqrt{\frac{\sigma}{2 \sigma^2}} = \sqrt{\frac{\sigma^2}{2 \sigma (s-1)}} \) - \( \sqrt{\frac{\sigma}{2 \sigma^2}} \), then based on the definition of Stieltjes integral, we know

\[
\sum_{s=1}^{S} \bar{\eta}_s \leq 4 \sqrt{10K_2 \sigma C_0 L} \sqrt{\frac{\log N}{N t^2}} \int \sqrt{\frac{\sigma}{2 \sigma^2}} \sqrt{\log \mathcal{N}(u, \mathcal{G}, || \cdot ||_{\infty})} d \sqrt{u}.
\]

Even though the integral above is a Stieltjes integral, considering that for any interval \([a, b] \subseteq \mathbb{R}\), function \( \sqrt{u} \) is continuous and differentialble, Stieltjes integral is equal to Riemann integral. Thus, we have

\[
\int \sqrt{\frac{\sigma}{2 \sigma^2}} \sqrt{\log \mathcal{N}(u, \mathcal{G}, || \cdot ||_{\infty})} d \sqrt{u} = \int_{\sqrt{\frac{\sigma}{2 \sigma^2}}}^{\sqrt{\frac{\sigma}{2 \sigma^2}}} \sqrt{\log \mathcal{N}(u^2, \mathcal{G}, || \cdot ||_{\infty})} du
\]

\[
\leq \int_{\sqrt{\frac{\sigma}{2 \sigma^2}}}^{\sqrt{\frac{\sigma}{2 \sigma^2}}} \sqrt{\log \mathcal{N}(u^2, \mathcal{G}, || \cdot ||_{\infty})} d u,
\]

where the last inequality is due to the definition of \( S \). Thus, according to C2, we know \( \sum_{s=1}^{S} \bar{\eta}_s \leq \frac{1}{2} \). Therefore, we ensure the \( \eta_s \)'s are well defined. Finally, we obtain

\[
P \left( \sup_{g \in \mathcal{G}} |P_N(gY) - P(gY)| \geq t \right) \leq 4 \sum_{s=1}^{S} \exp \left( - \frac{sN t^2}{1440C_0^2 K_2 (\log N)^{\frac{d_2}{4}} L^2 \sigma^2} \right)
\]

\[
\leq \frac{4}{1 - \exp \left( - \frac{N t^2}{1440C_0^2 K_2 (\log N)^{\frac{d_2}{4}} L^2 \sigma^2} \right)} \exp \left( - \frac{N t^2}{1440C_0^2 K_2 (\log N)^{\frac{d_2}{4}} L^2 \sigma^2} \right)
\]

\[
\leq 8 \exp \left( - \frac{N t^2}{1440C_0^2 K_2 (\log N)^{\frac{d_2}{4}} L^2 \sigma^2} \right),
\]

where the last inequality is due to condition C3.
Proof of Proposition 8
Denote $X_{s_i,N} = X_i$. For any $l = 0, 1, \ldots, p$, let

$$g_l(z) = \left( \frac{z-x}{h} \right)^l K_h \left( \frac{z-x}{h} \right), \quad K_h \left( \frac{z-x}{h} \right) = \frac{1}{h^l} K \left( \frac{z-x}{h} \right),$$

$G_l = \{g_l(z) : [a,b] \to \mathbb{R}, K \left( \frac{z-x}{h} \right) \text{ satisfies Assumption 7, } x \in [a,b]\}.$

**Step 1**
Obviously, we have the following simple decomposition

$$\frac{1}{N} \sum_{i=1}^N g_l(X_i) F_N(X_i) = \frac{1}{N} \sum_{i=1}^N g_l(X_i) (F_N(X_i) - F(X_i)) + \frac{1}{N} \sum_{i=1}^N g_l(X_i) F(X_i),$$

and similarly,

$$E \left[ \left( \frac{X_i-x}{h} \right)^l K_h \left( \frac{X_i-x}{h} \right) F_N(X_i) \right] = E \left[ g_l(X_i) \triangle_N(X_i) \right] + E \left[ g_l(X_i) F(X_i) \right].$$

This implies,

$$\left| \frac{1}{N} \sum_{i=1}^N \left( g_l(X_i) F_N(X_i) - E \left[ g_l(X_i) F_N(X_i) \right] \right) \right| \leq \frac{1}{N} \sum_{i=1}^N \left| g_l(X_i) \triangle_N(X_i) - E \left[ g_l(X_i) \triangle_N(X_i) \right] \right|$$

$$+ \frac{1}{N} \sum_{i=1}^N \left| g_l(X_i) F(X_i) - E \left[ g_l(X_i) F(X_i) \right] \right| := S_1(x) + S_2(x).$$

Therefore,

$$\sup_{x \in [a,b]} \left| \frac{1}{N} \sum_{i=1}^N \left( g_l(X_i) F_N(X_i) - E \left[ g_l(X_i) F_N(X_i) \right] \right) \right| \leq \sup_{x \in [a,b]} S_1(x) + \sup_{x \in [a,b]} S_2(x).$$

**Step 2**($S_2(x)$)
In order to measure the capacity of set $G_l$, we firstly investigate the Lipschitz property of $g^l \in G_l$. Considering the support is $[a, b]$, due to Assumption 13, we have, for any $z \in [a, b]$,

$$|g^l_h(z - x) - g^l_h(z - x')| \leq K_h \left| \left( \frac{z - x}{h} \right)^l - \left( \frac{z - x'}{h} \right)^l \right| + \left| \frac{z - x'}{h} \right| K_h \left( \frac{z - x}{h} \right) - K_h \left( \frac{z - x'}{h} \right) \leq (K \vee l(b - a)^{l-1}) h^{-l+1} |x - x'| + (b - a)^l h^{-(l+1)} |x - x'|$$

$$\leq c h^{-(l+2)} |x - x'|,$$

for some $c > 0$.

Hence, define a mapping $T : ([a, b], | \cdot |) \to (G^l, || \cdot ||_\infty)$. The Lipschitz modulo of $T$ is $O(1/h^{l+2})$. Under Assumption 13, according to Proposition 13 of Hang et al. (2018), we have, there exists some $c'>0$ such that

$$N(u^2, G_l, || \cdot ||_\infty) \leq \frac{c'}{u^{2}h^{l+2}}.$$  

Also note that for any $g^l \in G_l$, $\text{Var}[g^l(X)] \leq O(1/h)$, $l = 0, 1, \ldots, p$. Then, to apply Proposition 7, we only need to ensure conditions C1-C3 hold in an asymptotic level. Let $t = \sqrt{\log N/Nh}$. Actually, under Assumption 14 and 16, C1 and C3 hold obviously. Hence, we only need to check C2.

By letting ”$A \preceq B$” denote that A is asymptotically smaller than B, we have

$$\sqrt{\sigma} \int_{\frac{1}{2}(\frac{t}{h})^{1/4}}^{1/4} \frac{\log N(u^2, K, || \cdot ||_\infty)}{du} \preceq \frac{1}{h} \int_{\frac{1}{2} (\log N_{Nh}^{1/8})}^{1/8} \sqrt{\log \frac{1}{u^{2}h^{l+2}}} du$$

$$= \left( \frac{1}{h} \right)^{\frac{1}{4}} \int_{\frac{1}{2} (\log N_{Nh}^{1/8})}^{1/8} \sqrt{2 \log \frac{1}{u} + (l + 2) \log \frac{1}{h}} du \preceq \frac{1}{h} \int_{\frac{1}{2} (\log N_{Nh}^{1/8})}^{1/8} \left( \sqrt{2 \log \frac{1}{u} + (l + 2) \log \frac{1}{h}} \right) du$$

$$\preceq \left( \frac{1}{h} \right)^{\frac{3}{8}} \log \frac{1}{h} + \left( \frac{1}{h} \right)^{\frac{1}{2}} \int_{\frac{1}{2} (\log N_{Nh}^{1/8})}^{1/8} u^{-\frac{3}{2}} du \preceq \left( \frac{1}{h} \right)^{\frac{3}{8}} \log \frac{1}{h}.$$  

On the other hand, by setting $t = \sqrt{\log N/Nh} \cdot \sqrt{\frac{Nt_{2}^{2}}{(\log N)^{\gamma}}}, \gamma = O(\sqrt{\log N}^{1-\delta_{2}/\gamma}/h)$. C2 is easily satisfied. According to Proposition 7, and $F(X_i) = Y_i$, for sufficiently large $N$,

$$\Pr \left( \sup_{x \in [a, b]} S_2(x) > \sqrt{\frac{\log N}{Nh}} \right) \leq 8 \exp \left( -\frac{(\log N)^{1-\delta_2}}{1440C_0^2K_2} \right),$$

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which implies \( \sup_{x \in [a,b]} S_2(x) = O_p \left( \sqrt{\frac{\log N}{Nh}} \right) \).

**Step 3** (*S_1(x)*)

Firstly, due to Corollary 2, an immediate consequence is

\[
\sup_{x \in [a,b]} |F_N(x) - F(x)| = \sup_{x \in [a,b]} \triangle_N(x) = O_{a.s.} \left( \frac{\log N}{N} \right).
\]

Then, according to some simple algebra, we almost surely have

\[
\sup_{x \in [a,b]} S_1(x) \leq \sup_{x \in [a,b]} \left( \left| \frac{1}{N} \sum_{i=1}^{N} g_h(X_i) \right| + \left| E[g_h(X)] \right| \right) \sqrt{\frac{\log N}{N}}.
\]

Please note the calculation of \( \sup_{x \in [a,b]} T_N^1(x) \) is exactly the same as Step 2 by considering \( Y_i = 1 \). Thus, according to Corollary 2, we obtain

\[
\sup_{x \in [a,b]} T_N^1(x) = O_p \left( \sqrt{\frac{\log N}{Nh}} \right).
\]

As for \( T_N^2(x) \), according to Assumption 15, we know the density function \( f \) at least has continuously derivative on support \([a,b]\) and is uniformly bounded. Moreover, due to Assumption 13, it can be shown that

\[
|E[g_h(X)]| \leq \int_{R} \left| \left( \frac{z-x}{h} \right)^l K_h \left( \frac{z-x}{h} \right) f(z) \right| dz \leq ||f||_\infty \int_{-1}^{1} |v^l K(v)| dv < +\infty
\]

Therefore, we obtain

\[
\sup_{x \in [a,b]} S_1(x) = O_p \left( \sqrt{\frac{\log N}{N}} \right) + o_p(1).
\]

Above all, we have shown that, for any \( l = 0, 1, \ldots, p \),

\[
\sup_{x \in [a,b]} \left| \frac{1}{N} \sum_{i=1}^{N} \left( g_h(X_i) - E[g_h(X)] \right) \right| = O_p \left( \sqrt{\frac{\log N}{Nh}} \right).
\]
Proof of Theorem 4
Firstly, we have the following decomposition existed,
\[
\hat{\theta}(x) - \theta(x) = \hat{\theta}(x) - U_N^{-1}(x)E(U_N(x)\theta(x)) + U_N^{-1}(x)E(U_N(x)\theta(x)) - \theta(x)
\]
\[
= U_N^{-1}(x)[V_N(x) - E(V_N(x))] + U_N^{-1}(x)E[V_N(x) - U_N(x)\theta(x)] + U_N^{-1}(x)[E(U_N(x)) - U_N(x)]\theta(x)
\]
\[
:= U_N^{-1}(x)(W_N^1(x) + W_N^2(x) + W_N^3(x))
\]

Step 1 \((U_N(x))\)
Since the calculation of expectation has nothing to do with dependence, we here directly borrow Lemma 1 in the supplementary of Cattaneo et al. (2020) which yields,
\[
E[U_N(x)] = \begin{cases}
    f(x)\int_{-c}^{1} r_p(u)r_p(u)^TK(u)du + O_p(1/\sqrt{Nh}), & \text{if } x \text{ is interior}, \\
    f(x)\int_{-c}^{1} r_p(u)r_p(u)^TK(u)du + O_p(1/\sqrt{Nh}), & \text{if } x \text{ is lower boundary, } c \in (0, 1), \\
    f(x)\int_{-c}^{1} r_p(u)r_p(u)^TK(u)du + O_p(1/\sqrt{Nh}), & \text{if } x \text{ is upper boundary, } c \in (0, 1),
\end{cases}
\]
where \(r_p(u) = (1, u, u^2, \ldots, u^p)\). By using \(U_N(x)_{ij}\) to indicate the \((i,j)\)-th element of matrix \(U_N(x)\), its variance is
\[
Var(U_N(x)_{ij}) = \frac{1}{N^2} Var \left( \sum_{i=1}^{N} g_{h}^{i+j-2}(X_i) \right)
\]
\[
\leq \frac{1}{N^2} \sum_{i=1}^{N} Var(g_{h}^{i+j-2}(X_i)) + \frac{1}{N^2} \sum_{i \neq j} Cov(g_{h}^{i+j-2}(X_i), g_{h}^{i+j-2}(X_j)) := V_1 + V_2.
\]
According to Assumptions 13 and 15,
\[
V_1 \leq \frac{1}{Nh} \|f\|_{\infty} \int_{-1}^{1} u^{2i+2j-4}K^2(u)du \leq \frac{1}{Nh} \|f\|_{\infty} \int_{-1}^{1} K^2(u)du = O\left(\frac{1}{Nh}\right).
\]
Based on Lemma 3 and Assumption 14, we have, for some \(\delta > 0\),
\[
V_2 = \frac{1}{N^2} \sum_{i \neq j} Cov(g_{h}^{i+j-2}(X_i), g_{h}^{i+j-2}(X_j))
\]
\[
= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{m=1}^{+\infty} \sum_{s_{jN},\rho(s_{iN},s_{jN})\in[m,m+1]} Cov(g_{h}^{i+j-2}(X_i), g_{h}^{i+j-2}(X_j))
\]
\[
\leq \frac{1}{N^2} \sum_{i=1}^{N} \sum_{m=1}^{+\infty} m^{d-1} \exp\left(-\frac{\delta bm^\gamma}{2 + \delta}\right) \|g^{i+j-2}\|_{2+\delta}^2 \leq \frac{M}{Nh},
\]
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for some $M > 0$, which yields, $V_1 + V_2 = O\left(\frac{1}{Nh}\right)$. Then, $U_N(x)$ is asymptotically positive definite. Hence, the uniform convergence rate of the k-th element of vector $\hat{\theta}(x) - \theta(x)$ is determined by $\sup_{x \in [a,b]} |W_N^k(x)_k|$, $i = 1, 2, 3$, where $W_N^k(x)_k$ denotes the k-th entry of vector $W_N^k(x)$.

**Step 2**($W_N^k(x)_k$)

Due to Assumption 15, CDF $F(x)$ is $p + 1$ times continuously differentiable at $x \in [a, b]$. Thus, for any $l = 0, 1, \ldots, p$, we have $||F^l(x)||_\infty < +\infty$. Then, for any $k = 1, 2, \ldots, p + 1$,

$$|W_N^3(x)_k| = \sum_{l=0}^p \frac{1}{N} \sum_{i=1}^N \left[ \left( \frac{X_i - x}{h} \right)^{l+k-1} K_h \left( \frac{X_i - x}{h} \right) - E \left[ \left( \frac{X_i - x}{h} \right)^{l+k-1} K_h \left( \frac{X_i - x}{h} \right) \right] \right] F^{(l)}(x)h^l \leq \sum_{l=0}^p \frac{1}{N} \sum_{i=1}^N \left[ \left( \frac{X_i - x}{h} \right)^{l+k-1} K_h \left( \frac{X_i - x}{h} \right) - E \left[ \left( \frac{X_i - x}{h} \right)^{l+k-1} K_h \left( \frac{X_i - x}{h} \right) \right] \right] ||F^{(l)}||_\infty h^l$$

which implies,

$$\sup_{x \in [a, b]} |W_N^3(x)_k| \leq \sum_{l=0}^p \max_{0 \leq l \leq p} ||F^{(l)}||_\infty \sup_{g \in G_{l+k}} \frac{1}{N} \sum_{i=1}^N \left[ g_h^{l+k-1}(X_i) - E[g_h^{l+k-1}(X_i)] \right] \leq p \max_{0 \leq l \leq p} ||F^{(l)}||_\infty \sqrt{\log N \frac{N}{Nh}},$$

where the last inequality is due to “$\sup_{x \in [a, b]} T_N^1(x)$” in Step 3 of the proof of Proposition 8.

**Step 3**($W_N^1(x)$ and $W_N^2(x)$)

For $W_N^2(x)$, Lemma 2 in Cattaneo et al. (2020) has already shown that $\sup_{x \in [a,b]} |W_N^2(x)_k| = O(h^{p+1})$, for any $k = 1, 2, \ldots, p + 1$.

For $W_N^1(x)$, according to Proposition 8, we know, $\sup_{x \in [a,b]} |W_N^2(x)_k| = O(\sqrt{\log N/Nh})$, for any $k = 1, 2, \ldots, p + 1$. 

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Finally, based on all the results above, we prove, for any $v = 0, 1, \ldots, p$

$$\sup_{x \in [a, b]} |\hat{F}^v(x) - F^v(x)| = h^{-v} \sup_{x \in [a, b]} |\hat{\theta}_{v+1}(x) - \theta_{v+1}(x)| = O\left(\sqrt{\frac{\log N}{Nh^{2v+1}}} + h^{p+1-v}\right).$$

**D Proof Related to Section 5**

**Proof of Proposition 9**

Generally speaking, even though the proof of Proposition 9 is quite similar to the proof of Proposition 7, things are still quite different since we have an additional polynomial term in the inequality. Hence we here will only sketch the similar parts and concentrate on the different parts. Without loss of generality, we only consider the case when $C_{\beta_0} = 1$.

**Step 1 (Complexity of $G$)**

Define $g(\cdot, \theta) = \{g(x, \theta) : x \in \mathcal{X}\}$, for $\forall \theta \in \Theta$. Then, according to Assumption 20, mapping $T : \theta \rightarrow g(\cdot, \theta)$ is a Lipschitz continuous mapping from space $(\Theta, ||\cdot||_E)$ to space $(G, ||\cdot||_\infty)$ and its Lipschitz constant is $C_{Lip}$.

Based on Lemma 19 in Hang et al. (2018), we have, for any $u > 0$,

$$\mathcal{N}(u, G, ||\cdot||_\infty) = \mathcal{N}(u, T(\Theta), ||\cdot||_\infty) \leq \mathcal{N}(u C_{Lip}, \Theta, ||\cdot||_E).$$

Furthermore, please note when $B$ is a Banach space with norm $||\cdot||$, for $\forall r > 0$ and $A \subset B$ there holds

$$\mathcal{N}(u, rA, ||\cdot||) = \mathcal{N}(\frac{u}{r}, A, ||\cdot||).$$

Together with $\Theta \subset [-L, L]^m$, we instantly have

$$\mathcal{N}(\frac{u}{C_{Lip}}, \Theta, ||\cdot||_E) \leq \mathcal{N}(\frac{u}{C_{Lip}}, B_{L'}, ||\cdot||_E) = \mathcal{N}(\frac{u}{C_{Lip}L'}, B_1, ||\cdot||_E),$$

where $L' = \sqrt{mL}/2$ and $B_r$ denotes a $m$-dimensional Euclidean ball centered as origin with radius $r$. By applying Lemma 2.7 in Sen (2018), we know there exits a constant $C_1 > 0$ such that

$$\mathcal{N}(\frac{u}{C_{Lip}L'}, B_1, ||\cdot||_E) \leq C_1 \left(\frac{C_{Lip}}{u}\right)^m.$$

Above all, we manage to show

$$\mathcal{N}(u, G, ||\cdot||_\infty) \leq C_1 \left(\frac{C_{Lip}}{u}\right)^m. \quad (23)$$
Step 2 (Chaining)
In this step, we follow exactly the way we construct "chaining" in Step 1 of the proof of Proposition 7. That is to say, for ∀ \( t > 0 \), we have
\[
P \left( \sup_{g \in \Theta} |P_N(gY) - P(gY)| \geq t \right)
\leq \sum_{s=1}^{S} N_2^2 \max_{g^s, g^{s-1}} P \left( |P_N((g^s - g^{s-1})Y) - P((g^s - g^{s-1})Y)| \geq \frac{\eta_s t}{2} \right). \tag{24}
\]
Here \( S, N_2, g^s \) and \( \eta_s \) are all inherited from the proof of Proposition 7.

Step 3 (Application of Theorem 1) Firstly, we define
\[
h^s(x, y; \theta) = (g^s(x, \theta) - g^{s-1}(x, \theta))y - P((g^s - g^{s-1})Y)
:= (g^s - g^{s-1})(x, \theta)y - P_s,
\]
\[
W^{s}_{iN} := h^s(X_{iN}, Y_{iN}; \theta).
\]
Due to stationarity, \( P_s \) is a constant only associated with \( s \). Furthermore, similar to the Step 2 in the proof of Proposition 7, we can obtain that, for any \( s = 1, 2, \ldots, S \),
\[
Var(W^s_{iN}) \leq \frac{9L^2 \sigma^2}{22s} \quad \text{and} \quad ||W^s_{iN}|| \leq \frac{10LA}{22s}.
\]
Now, to apply Theorem 1, we only need to specify the NED condition of process \( \{W^s_{iN}\}_{s, iN \in \Gamma_N}, \forall s = 1, 2, \ldots, S \). According to Assumption 20, by using simple triangle inequality, we obtain that, for any given \( \theta \) and \( (x, y) \neq (x', y') \),
\[
|h^s(x, y; \theta) - h^s(x', y'; \theta)|
\leq |(g^s - g^{s-1})(x, \theta)y - (g^s - g^{s-1})(x', \theta)y|
\leq |(g^s - g^{s-1})(x, \theta) - (g^s - g^{s-1})(x', \theta)||y| + |(g^s - g^{s-1})(x', \theta)||y - y'|
\leq 2LD_{Lip} + 4AL \leq 4LN^\alpha,
\]
where the last inequality is due to condition C5. Thus, \( h^{x, y; \theta} \) is a Lipschitz continuous function on \( \mathcal{X} \times \mathcal{Y} \) with constant \( 4LN^\alpha \). Then, according to Proposition 2, we know, for any fixed \( \theta \in \Theta \), \( \{W^s_{iN}\}_{s, iN \in \Gamma_N} \) is a centered and
real valued geometric \( L^p \)-NED\((p \geq 1)\) random field on \( \epsilon \). Furthermore, for \( p \geq 1 \),
\[
||W_{\epsilon N}^s - E[W_{\epsilon N}^s|\mathcal{F}_{\epsilon N}(r)]||_p \leq 4LN^{\alpha}\psi_Z(r).
\]
Therefore, by applying Theorem 1, for any \( s \),
\[
P\left( |P_{\epsilon N}((g^s - g^{s-1})Y) - P((g^s - g^{s-1})Y)| \geq \frac{\eta_s t}{2} \right)
\leq 2K_1 \left( \frac{1}{N^{\alpha+2\beta+\tau+1}} \right)^p + 4 \exp \left( -\frac{Nt^2\eta_s^2}{4K_2(\log N)^{\frac{d_2}{2}} (C_0^22^2\sigma^2 + 40C_0ALt)} \right)
\]
\[
\leq 2K_1 \left( \frac{1}{N^{\alpha+2\beta+\tau+1}} \right)^p + 4 \exp \left( -\frac{Nt^2\eta_s^2}{40C_0^2K_2(\log N)^{\frac{d_2}{2}} L^2\sigma^2} \right)
\]
Together with (23), we have
\[
P\left( \sup_{g \in \mathcal{G}} |P_{\epsilon N}(gY) - P(gY)| \geq t \right)
\leq 2K_1 \left( \frac{1}{N^{\alpha+2\beta+\tau+1}} \right)^p \left( \sum_{s=1}^S \eta_s^2 \right) + 4 \sum_{s=1}^S \exp \left( 2\log N - \frac{Nt^2\eta_s^2}{40C_0^2K_2(\log N)^{\frac{d_2}{2}} L^2\sigma^2} \right)
\]
\[= M_1 + M_2.\]
Obviously, based on condition C6 and the technique used in the proof of Proposition 7, we immediately get that
\[
M_2 \leq 8 \exp \left( -\frac{Nt^2}{1440(C_0L)^2K_2(\log N)^{\frac{d_2}{2}} \sigma^2} \right).
\]
Therefore, from now on, we only concentrate on bounding \( M_1 \).

**Step 4**\((M_1)\)

It suffice to only consider the case \( p = 1 \). Note, for function \( m(x) = \sqrt{x} \),
we have \( m'(x) = \frac{1-2\ln x}{2x+1} \sqrt{x} \), which is negative for all \( x \geq 1 \). Then, for any \( s = 1, 2, \ldots, S \),
\[
\eta_s \geq \frac{\sqrt{s}}{2s \cdot 4} \geq \frac{\sqrt{S}}{2S \cdot 4} \Rightarrow \eta_s^2 \geq \frac{S}{2S \cdot 16}.
\]
Note that based on the definition of $S$, we have
\[
\frac{\sigma}{2^{2S}} \leq \frac{t}{4L} \Rightarrow S \geq \frac{1}{2} \log_2 \frac{4L\sigma}{t},
\]
\[
\frac{\sigma}{2^{2(S-1)}} > \frac{t}{4L} \Rightarrow \frac{1}{2^{2S}} > \frac{t}{16L\sigma},
\]
which yields
\[
S < \frac{1}{2} \log_2 \frac{16L\sigma}{t} \quad \text{and} \quad \eta_s^2 \geq \frac{S}{2^{2S} \cdot 16} \geq \frac{t \log_2 \frac{4L\sigma}{t}}{1024L\sigma}.
\]

Thus, for any $s$,
\[
\eta_s^3 = (\eta_s^2)^{\frac{3}{2}} \geq \frac{S}{2^{2S} \cdot 16} \geq \frac{(t \log_2 \frac{4L\sigma}{t})^{\frac{3}{2}}}{2^{14}(L\sigma)^{\frac{3}{2}}},
\]
\[
\frac{2}{N^{\alpha+2\beta+\tau+1} \eta_s t} \leq \frac{2^{15} \sigma^3 \eta_s^2}{N^{\alpha+\beta+\tau+1} R_1},
\]
\[
R_1 = N^\beta t^{\frac{2}{3}} (\log_2 \frac{4L\sigma}{t})^{\frac{2}{3}}.
\]

Then, together with condition C5, we have, for any $\beta > \beta_0$,
\[
M_1 \leq \frac{2^{15} \sigma^3 \eta_s^2}{N^{\alpha+\beta+\tau+1}} \sum_{s=1}^S (\eta_s N_s)^2.
\]

Since $eta_s = \max\{\frac{\sqrt{s}}{2^{2.4} \eta_s}\}$, we only need to calculate them separately. Based on condition C5, on one hand, we have
\[
\sum_{s=1}^S \left( \frac{\sqrt{s}}{2^{2.4}} \right)^2 N_s^2 = \frac{1}{16} \sum_{s=1}^S \frac{N_s^2}{2^{2s}} \leq \frac{1}{16\sigma} \sum_{s=1}^S \frac{\sigma}{2^{2s}} N_s^2 = \frac{1}{16\sigma} \sum_{s=1}^S \frac{\sigma}{2^{2s}} N_s^2 (\frac{\sigma}{2^{2s}}, \mathcal{G}, \| \cdot \|_{\infty})
\]
\[
\leq \frac{S}{16\sigma} \int_{\sigma/2^{2s}}^{\sigma/4} \mathcal{N}^2(u, \mathcal{G}, \| \cdot \|_{\infty}) du \leq \frac{C^2 \log_2 \frac{16L\sigma}{t}}{32\sigma} \int_{t/16L}^{\sigma/4} \left( \frac{C_{\text{Lip}}}{u} \right)^{2m} du \leq N^{\beta_0} < N^\beta.
\]

Then again,
\[
\sum_{s=1}^S (\eta_s N_s)^2 \leq 160 C_1^3 C_0^2 K_2 \sigma \left( \frac{\log N}{N t^2} \right)^{\frac{d}{2}} \int_{t/16L}^{\sigma/4} \left( \frac{C_{\text{Lip}}}{u} \right)^{3m} du \leq N^{\beta_0} < N^\beta.
\]

Above all, we can show
\[
M_1 \leq \frac{1}{N^{\alpha+\beta+\tau+1}}.
\]
Proof of Theorem 5 Denote
\[ G_h := G_h(z' - z/h) = h^{-(D+1)}G(z' - z/h), \]
\[ \mathcal{G} = \{G_h(z' - z/h) : z, z' \in \mathcal{Z}\}. \]

Thus, we can regard function \( G_h \) as a function defined on \( \mathcal{Z}^2, \mathcal{Z} = \mathcal{X} \times \mathcal{Y} \). Since \( G(u) \) is Lipshitz continuous on \( \mathbb{R}^{D+1} \) with constant of order \( O(h^{-(D+2)}) \), which indicates that there exists an \( \alpha > 0 \) such that \( h^{-(D+2)} \lesssim N^\alpha \). Thus Assumption 20 is satisfied. Based on (23), we know
\[ \mathcal{N}(u, \mathcal{G}, \| \cdot \|_\infty) \lesssim \left( \frac{1}{h^{D+1}} \right)^m. \]

Furthermore, to prove Theorem 5, we only need to apply Proposition 9. Hence we need to investigate, when \( t = \sqrt{\log N Nh^{D+1}} \), whether conditions C1, C3 and C4-C6 can be satisfied asymptotically. C5 is trivial. Please note
\[ \left\| G_h \left( \frac{Z_N - z}{h} \right) \right\|_2 \lesssim h^{-\frac{D+1}{2}}, \quad \|G_h\|_\infty \lesssim h^{-(D+1)}.\]

Thus it’s obvious that conditions C1 is satisfied asymptotically. As for condition C3, note that, since \( \gamma > d_2 \), we have
\[ \sqrt{\frac{N t^2}{(\log N)^{d_2/2}}} = \sqrt{\frac{(\log N)^{\frac{\gamma - d_2}{\gamma}}}{h^{D+1}}} \gtrsim 24 \sqrt{10 K_2 C_0 L h^{-\frac{D+1}{2}}}, \]

which ensures C3 holds asymptotically. As for C4, it suffice to show than the growth condition of \( \max\{A, B, C\} \) is \( O(N^{\beta_0}) \), for some \( \beta_0 > 0 \). Based on some simple calculation, we can show that
\[ A \lesssim h^{-\frac{D+1}{2}}, \]
\[ B \lesssim h^{(D+1) - 2m(D+1)} \left( \frac{Nh^{D+1}}{\log N} \right)^{\frac{2m+1}{2}}, \]
\[ C \lesssim \frac{N^{3m-1}}{(\log N)^{1-d_2/2} + \frac{3m-1}{2} - \frac{3m(D+3)}{2}}. \]

Recall that, for kernel density estimator, a basic requirement is that \( Nh^{D+1} = o(1) \). Thus there must be some positive \( \beta_0 > 0 \) such that \( \max A, B, C \lesssim N^{\beta_0} \). For condition C6, this actually has already been checked within Step 2 of
the proof of Proposition 8. Thus we omit it here. Then, based on all the argument above, by using Proposition 9, we can finish the proof.

**Proof of Theorem 6**

According to Assumption 21, we immediately have, for any \( \xi > 0, \exists a > 0, \)

\[
P\left( \sup_{x \in \mathcal{X}} |\hat{y}_x - y_x| > \xi \right) \leq P\left( \sup_{x \in \mathcal{X}} |f(\hat{y}_x|x) - f(y_x|x)| > a \right).
\]

Note that

\[
\sup_{x \in \mathcal{X}} |f(\hat{y}_x|x) - f(y_x|x)| \leq \sup_{x \in \mathcal{X}} |f(\hat{y}_x|x) - \hat{f}(y_x|x)| + \sup_{x \in \mathcal{X}} |\hat{f}(\hat{y}_x|x) - f(y_x|x)|
\]

\[
\leq \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} |\hat{f}(y|x) - f(y|x)| + \sup_{x \in \mathcal{X}} |\hat{f}(\hat{y}_x|x) - f(y_x|x)|.
\]

Recall the definition of \( \hat{y}_x \) and \( y_x \), we have \( \hat{f}(\hat{y}_x|x) = \sup_{y \in \mathcal{Y}} \hat{f}(y|x), f(y_x|x) = \sup_{y \in \mathcal{Y}} f(y|x) \). Then,

\[
\sup_{x \in \mathcal{X}} |\hat{f}(\hat{y}_x|x) - f(y_x|x) |
\]

\[
\leq \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} |\hat{f}(y|x) - f(y|x)| + \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} |\hat{f}(\hat{y}_x|x) - f(y_x|x)|
\]

\[
\leq \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} |\hat{f}(y|x) - f(y|x)| + \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} |\hat{f}(\hat{y}_x|x) - f(y_x|x)|
\]

\[
\leq 2 \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} |\hat{f}(y|x) - f(y|x)|.
\]

Therefore,

\[
P\left( \sup_{x \in \mathcal{X}} |\hat{y}_x - y_x| > \xi \right) \leq P\left( \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} |\hat{f}(y|x) - f(y|x)| > \frac{a}{2} \right),
\]

which shows that, to prove the uniform convergence rate of uni-modal regression, it suffice to investigate the \( L^\infty \) loss of conditional density estimator.
Furthermore, according to the construction of \( \hat{f}(y|x) \), we have

\[
|\hat{f}(y|x) - f(y|x)| = \left| \frac{\hat{f}(x, y) - f(y|x)}{f(x)} \right| = \left| \frac{\hat{f}(x, y) - f(y|x)}{f(x)} \right| + \left| \frac{\hat{f}(x) - f(y|x)}{f(x)} \right| + \left| \frac{\hat{f}(x) - f(x)}{f(x)} \right|
\]

\[
\leq \frac{|f(x)\hat{f}(x, y) - f(x, y)|}{f^2(x)} + \frac{|(\hat{f}(y|x) - f(y|x))(\hat{f}(x) - f(x))|}{f(x)}
\]

\[
:= A_N(x, y) + A_2(x, y) + A_3N(x, y).
\]

Obviously, \( \sup_{x,y} |\hat{f}(y|x) - f(y|x)| = O(\sup_{x,y} A_1(x, y)) \). Together with assumption \( \inf f(x) > 0 \) and Theorem 5, we finish the proof.

### E Proof Related to Section 6

Generally speaking, the proof of Theorem 7 is essentially same as the proof of Theorem 3.1 in Rigollet and Vert (2009). Major differences are some technical details, since we have additional remainder term "(N^\delta)^p" (see Definition 6). Without loss of generality, we here assume \( B = 1 \).

Before we prove Theorem 7, we firstly introduce the following useful lemma.

**Lemma 6** Under the conditions of Theorem 7, for any given \( l_N \), the plug-in level set estimator \( \hat{\Gamma}_f(\lambda) := \{z \in Z : \hat{f}(z) \geq \lambda + l_N\} \) satisfies that, for some \( C' > 0 \),

\[
\sup_{f \in \mathcal{P}} E[d_H(\hat{\Gamma}, \Gamma)] \leq C'(\varphi_N \vee l_N)^{1+\rho}.
\]  

**(25)**

**Proof of Lemma 6**

Note that \( \hat{\Gamma} \Delta \Gamma = (\hat{\Gamma} \cap \Gamma^c) \cup (\hat{\Gamma}^c \cap \Gamma) \). Then, due to the definition of \( d_H \), we have

\[
d_H(\hat{\Gamma}, \Gamma) = \int_{\Gamma \cap \Gamma^c} |f(z) - \lambda|d\mu + \int_{\Gamma^c \cap \Gamma} |f(z) - \lambda|d\mu := I_1 + I_2.
\]

Define \( \alpha_N = C_\alpha(\varphi_N \vee l_N) \) and \( \beta_N = C_\beta(\varphi_N \vee \varphi_N) \sqrt{\log N} \), where \( C_\alpha = 2 \max(c_\varphi, c_\phi, 1) \) and \( C_\beta \geq C_\alpha \max(c_\epsilon, 2(1+\rho)/c_2, 1) \). Let \( N_0 \) be the smallest
integer satisfying $\alpha_N < \beta_N < \eta \land \epsilon_0 \land \triangle$. Without loss of generality, we always assume $N \geq N_0$ in the remainder of the proof.

Since $\hat{\Gamma}_c \cap \Gamma = \{ \hat{f} < \lambda + l_N, f > \lambda \}$, by defining

$$A_1 = \{ \hat{f} < \lambda + l_N, \lambda < f \leq \alpha_N \},$$

$$A_2 = \{ \hat{f} < \lambda + l_N, \lambda + \alpha_N < f \leq \lambda + \beta_N \},$$

$$A_3 = \{ \hat{f} < \lambda + l_N, f > \lambda + \beta_N \},$$

we immediately have $\hat{\Gamma}_c \cap \Gamma \subset A_1 \cup A_2 \cup A_3$. This yields

$$I_1 \leq \sum_{i=1}^{3} \int_{A_i} |f(z) - \lambda| d\mu := I_{11} + I_{12} + I_{13}.$$

For $I_{11}$, since $A_1 \subset \{0 < |f - \lambda| \leq \alpha_N\}$, we have

$$I_{11} \leq \int_{\{0 < |f - \lambda| \leq \alpha_N\}} |f - \lambda| d\mu \leq \alpha_N \mu \{ |f - \lambda| \leq \alpha_N\}.$$

Since we have assumed that $N \geq N_0$ which indicates $\alpha_N < \eta \land \epsilon_0$, together with the condition that $f$ is $\rho$-exponent at level $\lambda$, we have

$$I_{11} \leq c_0 \alpha_N^{1+\rho}. \tag{26}$$

For $I_{12}$, we firstly partition $A_2$ as follow,

$$A_2 = \bigcup_{j=1}^{J_N} B_j \cap A_2,$$

where $B_j = \{ \hat{f} < \lambda + l_N, \lambda + 2^j-1\alpha_N < f \leq \lambda + 2^j\alpha_N \} \cap D(\eta \land \epsilon_0)$ and $J_N = \lfloor \log(\beta_N/\alpha_N) \rfloor + 2$. Thus we can rewrite

$$I_{12} = \sum_{j=1}^{J_N} E \int_{B_j \cap A_2} |f - \lambda| d\mu := \sum_{j=1}^{J_N} I_{12}^j.$$

For each $j = 1, \ldots, J_N$, since $l_N \leq \alpha_N/2$, we have

$$B_j \subset \{|\hat{f} - f| > 2^{j-2}\alpha_N\} \cap \{|f - \lambda| < 2^j\alpha_N\} \cap D(\eta \land \epsilon_0).$$
Then based on Fubini’s Theorem, we obtain, for each $j$,

$$I_{12}^j = \int \int_{B_j \cap A_2} |f(z) - \lambda| d\mu dP$$

$$\leq \int_{\{|f-\lambda|<2^j\alpha_N\} \cap D(\eta \wedge \epsilon_0)} |f(z) - \lambda| \int 1_{\{|f-\lambda|>2^j-2\alpha_N\}}(\omega) d\mu dP$$

$$\leq 2^j \alpha_N \int_{D(\eta \wedge \epsilon_0) \cap \{|f-\lambda|<2^j\alpha_N\}} P(|\hat{f}(z) - f(z)| > 2^j-2\alpha_N) d\mu.$$  

Since for any $j$, $c_\varphi \varphi_N \leq 2^j-2\alpha_N \leq \beta_N < \Delta$ and $z \in D(\eta)$, we obtain

$$P(|\hat{f}(z) - f(z)| > 2^j-2\alpha_N) \leq c_3 \exp \left(-c_4 \left(\frac{2^j-2\alpha_N}{\varphi_N}\right)^2\right) + \left(\frac{1}{N^\theta 2^j-2\alpha_N}\right)^\rho, \forall \theta > 0,$$

which yields

$$I_{12}^j \leq c_0 c_3 (2^j \alpha_N)^{1+\rho} \left(\exp \left(-c_4 \left(\frac{2^j-2\alpha_N}{\varphi_N}\right)^2\right) + \left(\frac{1}{N^\theta 2^j-2\alpha_N}\right)^\rho\right).$$  \hspace{1cm} (27)$$

According to (27), we instantly have

$$I_{12} \leq c_0 c_3 \sum_{j=1}^{+\infty} (2^j \alpha_N)^{1+\rho} \left(\exp \left(-c_4 \left(\frac{2^j-2\alpha_N}{\varphi_N}\right)^2\right) + \left(\frac{1}{N^\theta 2^j-2\alpha_N}\right)^\rho\right)$$

$$\leq c_0 c_3 \alpha_N^{1+\rho} \sum_{j=1}^{+\infty} 2^j(1+\rho)e^{-c_42^j-4} + c_0 c_3 \frac{1+\rho-\rho}{4^p N^\theta p} \sum_{j=1}^{+\infty} \left(\frac{1}{2^p-1-\rho}\right)^j.$$  \hspace{1cm} (28)$$

Due to Assumption 22, we know there exits some positive constants $C'_1$ and $C'_2$ such that,

$$\left[\frac{(25)}{25}\right] \leq c_0 c_3 C'_1 \alpha_N^{1+\rho} + c_0 c_3 C'_2 \frac{1}{N^\theta p \alpha_N^{p-1-\rho}}.$$  \hspace{1cm} (29)$$

Recall $\alpha_N = O(\varphi_N \vee l_N)$. Thus $\alpha_N \geq CN^{-\nu}$. Hence, since $\theta$ is allowed to be arbitrary large, there exits sufficiently large $\theta_1 > 0$, such that, for any $\theta > \theta_1$, we have

$$I_{12} \leq c_0 c_3 (C'_1 \vee C'_2) \alpha_N^{1+\rho}.  \hspace{1cm} (30)$$

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For $I_{13}$, since $A_3 \subset \{ f > \lambda \}$, due to Markov inequality with respect to measure $\mu$, we have $\mu(A_3) \leq \frac{1}{\lambda}$. Furthermore, since $\beta_N/2 > l_N$,

$$A_3 \subset \{ f > \lambda + \beta_N \} \cap \{ |\hat{f} - f| \geq \beta_N/2 \}.$$ 

By using Funibi's Theorem, we have

$$I_{13} \leq \int \int_{\{ f > \lambda + \beta_N \} \cap \{ |\hat{f} - f| \geq \beta_N/2 \}} |f - \lambda| d\mu dP$$

$$= \int_{\{ f > \lambda + \beta_N \}} |f(z) - \lambda| P(|\hat{f}(z) - f(z)| > \beta_N/2) d\mu$$

$$\leq \int_{\{ f > \lambda + \beta_N \}} (f(z) - \lambda) \left( c_1 \exp \left( -c_2 \left( \frac{\beta_N}{2\phi_N} \right)^2 \right) + \left( \frac{1}{N^\theta \beta_N} \right)^p \right) d\mu$$

$$\leq c_1 \exp \left( -c_2 \left( \frac{\beta_N}{2\phi_N} \right)^2 \right) + \left( \frac{1}{N^\theta \beta_N} \right)^p .$$

Since $p > 1$, $\beta_N = C_\beta (\varphi_N \vee \phi_N) \sqrt{\log N}$ and $C_\beta = 2 \max(c_\varphi, c_\phi, 1) C_\alpha := 2 C'_\beta$ and $C'_\beta > C^2_\beta > 2 \nu(1 + \rho)/c_2$,

$$I_{13} \leq c_1 CN^{-\nu(1 + \rho)} + \frac{1}{N^{\theta - \nu} \sqrt{\log N}} \leq c_1 \alpha_1^{1 + \rho} + \frac{1}{N^{\theta - \nu} \sqrt{\log N}} .$$

Obviously there exists sufficiently large $\theta_2 > 0$, such that, for any $\theta > \theta_2$, we have $(N^{\theta - \nu} \sqrt{\log N})^{-1} \leq c_1 CN^{-\nu(1 + \rho)} \leq c_1 \alpha_1^{1 + \rho}$. Above all, we obtain

$$I_{13} \leq 2c_1 \alpha_1^{1 + \rho} . \quad (31)$$

By combining (25), (30) and (31), we finish the proof. As for the proof of $I_2$, in the same manner it can be shown that, for any $N \geq N_0$, $I_2 \lesssim \varphi_1^{1 + \rho}$. The only difference with the part of the proof detailed above is that in the step that corresponds to proving the upper bound of $I_{13}$, we use the assumption that $\mu(\hat{f} > \lambda) \leq M$ almost surely with respect to $P$, instead of Markov inequality.

**Proof of Theorem 7**

Obviously, (19) is a direct consequence of Lemma 6 applied with $l_N \lesssim \varphi_N$. Hence, we concentrate on proving (20). In order to apply (16, we need to show the existence of such a constant $L_\mu$ satisfying $\mu(\hat{\Gamma} \Delta \Gamma)$. This is indeed true, since we have

$$\mu(\hat{\Gamma} \Delta \Gamma) \leq \mu(\hat{f} \geq \lambda) + \mu(f \geq \lambda) \leq M + \frac{1}{\lambda} .$$

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Therefore, we can choose $L_\mu = M + \frac{1}{\lambda}$. Thus, according to (16), we instantly know, there exist some positive constant $C$ such that

$$E[d_\triangle(\hat{\Gamma}, \Gamma)] \leq E[\mu(\hat{\Gamma} \triangle \Gamma \cap \{ f = \lambda \})] + CE[\|d_H(\hat{\Gamma}, \Gamma)\|^{\rho}]$$

$$\lesssim E[\mu(\hat{\Gamma} \triangle \Gamma \cap \{ f = \lambda \})] + C(\varphi_N \sqrt{\log N})^\rho,$$  \hspace{1cm} (32)

where the second term of the right side of the second inequality above is due to Jensen inequality and the Lemma 6. Additionally, please note that $\hat{\Gamma} \triangle \Gamma \cap \{ f = \lambda \} \subset \{ |\hat{f} - f| > l_N \} \cap \{ f = \lambda \}$.

Then, for any $\theta > 0$ and $p > 1 + \rho$, by using Fubini's Theorem we have

$$E[\mu(\hat{\Gamma} \triangle \Gamma \cap \{ f = \lambda \})] \leq c_1 \exp \left(-c_2 \left(\frac{l_N}{2 \varphi_N}\right)^2 \right) + \left(\frac{1}{N^\theta l_N}\right)^p.$$  

Recall that $l_N = O(\varphi_N \sqrt{\log N})$ and $l_N \geq C N^{-\nu}$. We thus obtain, there exists sufficiently large $\theta_3$, such that for any $\theta > \theta_1 \lor \theta_2 \lor \theta_3$,

$$E[\mu(\hat{\Gamma} \triangle \Gamma \cap \{ f = \lambda \})] \lesssim \varphi_N^\rho.$$  \hspace{1cm} (33)

Based on (32) and (33), the proof is finished.

**Proof of Theorem 8**

We prove Theorem 8 by following Theorem 7.

Denote $\bar{K}(u) = \prod_{k=1}^D K(u^k)$, $u = (u^1, \ldots, u^D)$, where $K$ satisfies Assumption 24 with $\alpha = \beta^*$. Thus, it's easy to specify that $\bar{K}$ also satisfies (1) and (2) in Assumption 24.

**Step 1** (Specify $\exists M > 0$ such that $\mu(\hat{f} \geq \lambda) \leq M$ a.s. w.r.t $P$)

In this step, instead of $d\mu$, we use $\mu(dz)$. Note that since $\mu$ is the Lebesgue measure and $K \in L_\mu^1(\mathbb{R}^D)$, we instantly have

$$\infty > \int_{\mathbb{R}^D} |\bar{K}(z)| \mu(dz) \geq \int_{\{f \geq \lambda\}} |\bar{K}(z)| \mu(dz).$$

Recall that $\hat{f} = N^{-1} \sum_{i=1}^N \bar{K}_h((Z_{s_i} - z)/h)$. For arbitrary realization of sample, i.e $Z_{s_i} = z_i$, we have, for any $i$,

$$\int_{\{f \geq \lambda\}} |\bar{K}(z)| \mu(dz) = \int_{\{f \geq \lambda\}} |\bar{K}((z_i - z)/h)| \mu(d(z_i - z)/h)$$

$$= \int_{\{f \geq \lambda\}} |\bar{K}_h((z_i - z)/h)| \mu(dz).$$
This indicates, for any realization of sample $(z_i)_{i=1}^N$ 
\[
\int_{\{f \geq \lambda\}} |\bar{K}(z)|\mu(dz) = \frac{1}{N} \sum_{i=1}^N \int_{\{f \geq \lambda\}} |\bar{K}_h((z_i - z)/h)|\mu(dz).
\]
Then based on simple triangle inequality and arbitrary of this realization, we manage to obtain
\[
\infty > \int_{\mathbb{R}^d} |\bar{K}(z)|\mu(dz) \geq \int_{\{f \geq \lambda\}} |\hat{f}(z)|\mu(dz) \geq \lambda \mu(\{\hat{f} \geq \lambda\}).
\]
Hence, we can set $M = \int |\bar{K}|/\lambda$.

**Step 2** ($\hat{f}$ satisfies Definition 5)

Note that, for any given $z_0 \in Z$,
\[
|\hat{f}(z_0) - f(z_0)| = \left| \frac{1}{N} \sum_{i=1}^N \bar{K}_h \left( \frac{Z_{siN} - z_0}{h} \right) - f(z_0) \right| = \left| \frac{1}{N} \sum_{i=1}^N W_{iN}(z_0) \right|
\leq \frac{1}{N} \sum_{i=1}^N (W_{iN}(z_0) - E[W_{iN}(z_0)]) + \max_i |E[W_{iN}(z_0)]|
\leq \frac{1}{N} \sum_{i=1}^N \tilde{W}_{iN}(z_0) + \max_i |E[W_{iN}(z_0)]|,
\]
where $W_{iN}(z_0) = \bar{K}_h \left( \frac{Z_{siN} - z_0}{h} \right) - f(z_0)$.

Furthermore, due to the proof of Lemma 4.1 in Rigollet and Vert (2009), under Assumptions 23 and 24, it’s easy to obtain that, for any $i$,
\[
|E[W_{iN}(z_0)]| \leq Lc_5 h^{\beta}, \text{ when } z_0 \in D(\eta),
\]
\[
|E[W_{iN}(z_0)]| \leq Lc_5 h^{\beta'}, \text{ when } z_0 \in D(\eta)^c,
\]
where $c_5 = \int_{\mathbb{R}^d} ||t||^{\beta} |\bar{K}(t)|dt$. Then, provided that $z_0 \in D(\eta)$, for any $\delta > 0$ such that $Lc_5 h^{\beta \wedge \beta'} \leq \delta/2$, we have
\[
P \left( |\hat{f}(z_0) - f(z_0)| \geq \delta \right) \leq P \left( \left| \frac{1}{N} \sum_{i=1}^N \tilde{W}_{iN}(z_0) \right| \geq \delta - Lc_5 h^{\beta} \right)
\leq P \left( \left| \frac{1}{N} \sum_{i=1}^N \tilde{W}_{iN}(z_0) \right| \geq \frac{\delta}{2} \right). \quad (34)
\]
Similarly, when \( z_0 \in D(\eta)^c \), under condition \( Lc_5 h^{\beta' \land \beta'} \leq \delta/2 \), we can obtain (34) as well. Since the following argument is nearly the same no matter \( z_0 \in D(\eta) \) or not, we concentrate solely on the case that \( z_0 \in D(\eta) \).

In order to apply Theorem 1, we need to ensure that \( (\tilde{W}_{iN}(z_0))_{s_i \in \Gamma_N} \) is a centered real-valued and geometric \( L^p \)-NED random field on process \( \epsilon \).

Due to the definition of \( \tilde{W}_{iN}(z_0) \), it is certainly centered. Note that \( \tilde{W}_{iN}(z_0) \) can be regarded as a function of random vector \( Z_{s_iN} \), for each \( i \). According to stationarity of \( (Z_{s_iN})_{s_i \in \Gamma_N} \), we know \( E[\tilde{W}_{iN}(z_0)] = C \), for some constant \( C \). Now considering function

\[
\tilde{W}(z, z_0) = \bar{K}_h \left( \frac{z - z_0}{h} \right) - f(z_0) - C, z \in \mathbb{R}^D,
\]

we have, for any \( z_1 \neq z_2 \) and some positive constant \( E \),

\[
|\tilde{W}(z_1, z_0) - \tilde{W}(z_2, z_0)| = \left| \bar{K}_h \left( \frac{z_1 - z_0}{h} \right) - \bar{K}_h \left( \frac{z_2 - z_0}{h} \right) \right|
\leq h^{-D}||K||_{D, \infty}^D \sum_{k=1}^{D} \left| K \left( \frac{z_1 - z_0}{h} \right) - K \left( \frac{z_2 - z_0}{h} \right) \right|
\leq h^{-(D+1)||K||_{D, \infty} E} ||z_1 - z_2||_2.
\]

Thus, \( \text{Lip}(\tilde{W}(z, z_0)) = ||K||_{D, \infty} M Eh^{-(D+1)} := m_1 h^{-(D+1)} \). Then according to Proposition 2 and Assumption 20, we know \( \{\tilde{W}_{iN}(z_0)\} \) is a centered \( L^p \)-NED random field on \( \epsilon \) such that

\[
\left\| \tilde{W}_{iN}(z_0) - E[\tilde{W}_{iN}(z_0)|\mathcal{F}_{iN}(r)] \right\|_p \leq 2m_1 h^{-(D+1)} e^{-br^\gamma} \leq 2N^{\kappa_0} e^{-br^\gamma}, \quad (35)
\]

for some \( \kappa_0 > 0 \). Meanwhile, due to some simple calculation, we can easily obtain that, for any \( i \),

\[
||\tilde{W}_{iN}(z_0)||_\infty \leq (||K||_{D, \infty}/h)^D + L^* + Lc_5 h^{\beta} \leq c_7 h^{-D}, \quad c_7 = ||K||_{D, \infty} + L^* + Lc_5;
\]

\[
\text{Var}(\tilde{W}_{iN}(z_0)) \leq h^{-D} \int \bar{K}(u)^2 f(z_0 + hu)du \leq L^* ||\bar{K}||_2^2 h^{-D} := c_8 h^{-D}.
\]

(37)
Based on (31)-(33), by applying Theorem 1, provided that \( \delta \leq \Delta \) and

\[
\Delta = \frac{3C_0c_8}{4c_7} = \frac{3C_0L^*||\hat{K}||^2}{4(||K||_\infty + L^* + L \int_{\mathbb{R}^D} ||t||^\beta |\hat{K}(t)|dt)},
\]

we have

\[
P\left(\frac{1}{N} \sum_{i=1}^{N} \tilde{W}_{iN}(z_0) \geq \delta\right)
\leq 2 \exp\left(-\frac{N\delta^2}{4K_2(\log N)^{\frac{d_1}{2}}} \left(\frac{C_0^2c_8h^{-D} + 4C_0c_7\delta}{3h^D}\right)\right) + 2K_1 \left(\frac{2}{N^{\kappa_0 + 2\vartheta + \tau + 1}\delta}\right)^p
\]

\[
= 2 \exp\left(-\frac{N\delta^2h^D}{8K_2(\log N)^{\frac{d_1}{2}}} C_0^2c_8\right) + 2K_1 \left(\frac{2}{N^{\kappa_0 + 2\vartheta + \tau + 1}\delta}\right)^p
\]

\[
= 2 \exp\left(-c_6 \left(\frac{\delta}{\sqrt{(\log N)^{d_2/\gamma}/Nh^D}}\right)^2\right) + \left(\frac{B}{N^{R(\vartheta)\delta}}\right)^p,
\]

\[
c_6 = 8K_2C_0^2c_8, \quad R(\vartheta) = \kappa_0 + 2\vartheta + \tau + 1, \quad B = 2^{1+\frac{1}{p}}K_1^{\frac{1}{p}}.
\]

Here \( \vartheta \) is used to replace the \( \beta \) used in Theorem 1 so that we won’t confuse it with the smoothness of density function. \( K_1 \) and \( K_2 \) are defined the same as Theorem 1 by setting \( \kappa = \kappa_0 \). Note that \( R(\vartheta) \) here can be arbitrary large. Therefore, we prove that, when \( z_0 \in D(\eta) \), \( \hat{f} \) satisfies Definition 6. Similarly, when \( z_0 \in D(\eta) \), the arguments are the same.

**Step 3**

Set \( \varphi_N = \sqrt{\log N/Nh^D} \). To prove (21) and (22), let "\( h = N^{-\frac{1}{2\vartheta + D}} \), \( l_N = N^{-\frac{1}{2\vartheta + D}} \)" and "\( h = (\log N/N)^{\frac{1}{2\vartheta + D}}, \ l_N = N^{-\frac{1}{2\vartheta + D} \sqrt{\log N}} \)" respectively. Hence, it’s easy to specify that the \( \nu \) defined in Theorem 7 existed. Then, by following Theorem 7, we can instantly obtain the results.