Conformal Field Theory Approach to Quantum Impurity Problems

Ian Affleck

A brief review is given of a new method for studying the critical behavior of quantum impurity problems, based on conformal field theory techniques, which I developed with Andreas Ludwig. Some results on the overscreened Kondo problem are reviewed. It is shown that the simple open and periodic fixed points, which occur in quantum spin chain impurity models, are related to each other by fusion.

1. Introduction

Quantum impurity problems occur in various areas of physics. Some examples are: an impurity spin in a metal (the Kondo problem), the two-impurity Kondo problem, a defect or impurity spin in a spin chain, tunnelling through a barrier in a quantum wire proton-monopole scattering (the Callan-Rubakov effect). Some common features of these problems are:

1. gapless “bulk” excitations (ie. far from the impurity or in the absence of an impurity the excitations are gapless),
2. essentially harmonic behavior away from the impurity (Fermi or Luttinger liquids),
3. the impurity is localized and may carry quantum-mechanical degrees of freedom (eg. the two spin states of the impurity spin),
4. perturbation theory in the impurity-bulk coupling may be infrared divergent,
5. the problems are either initially defined in one dimension or can be mapped into one dimension (ie. some sort of partial wave expansion can be made and only a finite number of partial waves are important).

The first and last points imply that the systems are equivalent to $(1+1)$-dimensional conformal field theories with the impurity or defect at the origin of the one-dimensional position space. In the imaginary time path-integral formulation, we have a boundary or defect line at $(τ, 0)$. See Figure 1. John Cardy recently developed [1] a boundary conformal field theory to treat two-dimensional classical critical systems with boundaries. We have extended his technique to deal with the $(1+1)$ dimensional quantum case, allowing for the possibility of a dynamical defect [2-14].

In the next section we review a simple and venerable example to provide some motivation: the local Fermi liquid theory of the Kondo effect, developed by Nozières [15], following earlier ideas of Wilson and Anderson. In Section 3 we discuss our general approach to quantum impurity problems. In Section 4 we present some of our results on
the overscreened Kondo effect [2-6, 8-10], in order to demonstrate the power of the new method. In Section 5 we discuss impurities in the spin $s = 1/2$ Heisenberg antiferromagnetic chain [11, 12]. In particular we show that the simple open and periodic fixed points which occur in the problem are related by the “fusion” technique, which we used in our analysis of the Kondo problem.

2. A Simple Example: Local Fermi Liquid Theory of the Kondo Effect

The continuum form of the Hamiltonian density is:

$$H = \psi^\dagger \left(-\frac{\nabla^2}{2m}\right)\psi + J\delta^2(x)\psi^\dagger \vec{\sigma} \psi \cdot \vec{S}$$

(1)

Here $\vec{S}$ is an $s = 1/2$ spin operator and the fermion annihilation operator, $\psi_\alpha$ carries a spin index which is not written explicitly. Expanding in spherical harmonics, only the $s$-wave interacts with the impurity due to the $\delta$-function. Thus we obtain an effective one-dimensional model defined on the half line, $r \geq 0$ with the impurity at the origin. Alternatively, we may reflect the outgoing wave to the negative $r$-axis, obtaining a theory with a left-mover only on the entire real axis. (See Figure 2.) The corresponding Hamiltonian density, written in terms of a left-moving fermion operator, $\psi$ is:

$$H = \psi^\dagger \frac{d}{dx} \psi + \lambda \delta(x)\psi^\dagger \vec{\sigma} \psi \cdot \vec{S}$$

(2)

As discovered by Kondo [16], perturbation theory in the Kondo coupling, $\lambda$, is infrared divergent. In modern language, this corresponds to the $\beta$-function:

$$\frac{d\lambda}{d\ln L} = \lambda^2 + ...$$

(3)

It appears that $\lambda$ renormalizes to $\infty$ if it is initially positive. (See Figure 3.) What does an infinite effective coupling constant really mean? Nozières made this notion precise, [15], by considering a lattice version of the Kondo problem. Since the dimensionality of the lattice is unimportant we will consider the one-dimensional case:

$$H = t \sum_i \left(\psi_i^\dagger \psi_{i+1} + \text{h.c.}\right) + J\psi_0^\dagger \vec{\sigma} \psi_0 \cdot \vec{S}$$

(4)

This model is easy to study for $J >> t$. For $t = 0$ we must have precisely one electron at the origin, which forms a singlet with the impurity spin, $\vec{S}$. The electronic configuration on the other sites is arbitrary. For a relatively small non-zero $t$, this degeneracy is broken. The other electrons simply go into a Bloch wave (Slater determinant) state, but the single-particle wave-functions must all vanish at $x = 0$ in order to preserve the spin-singlet condition there. Thus is the even-parity sector, the zero-coupling wave function:

$$\phi(x) = \cos kx \quad (\lambda = 0)$$

(5)

gets modified to:

$$\phi(x) = |\sin kx| \quad (\lambda \to \infty).$$

(6)
This is a solution of the free-particle Schroedinger equation everywhere except at the origin where the vanishing boundary condition is imposed. On the other hand, the odd-parity wave-function:

\[ \phi(x) = \sin kx \]  

is the same for zero or infinite coupling. As Nozières observed [15], the strong coupling fixed point is the same as the weak coupling fixed point except that the impurity disappears (screened) and is replaced by a Boundary Condition,

\[ \psi(0) = 0. \]  

Equivalently, we have a \( \pi/2 \) phase shift in the even parity channel.

For finite (or even small) Kondo coupling this picture still holds but only at low energies and long distances. The boundary condition is a fixed point. i.e., we have an effective boundary condition which holds in the asymptotic regime. Our main new result is that All Quantum Impurity Problems work this way!

### 3. General Approach

Far from the impurity we expect the low energy physics to be described by a scale-invariant, time-independent boundary condition. This should be true outside a boundary layer whose width, \( a \), is set by the longest microscopic or crossover length scale in the problem. Further from the impurity than this it is reasonable to expect scale invariance. However the critical behavior may still be affected, in a universal way, by the presence of the boundary. Consider, for example, a two-point Green’s function. As illustrated in Figure 4 there are two special limits. In the limit where both points are far from the impurity relative to their distance from each other we expect to recover the bulk critical behavior, unaffected by the boundary. However, in the opposite limit where the two points are far apart compared to their distance from the boundary (which is still large compared to the width, \( a \), of the boundary layer), the critical behavior can be quite different. i.e. boundary critical phenomena can occur. In general, to each bulk universality class corresponds several boundary universality classes. We expect essentially arbitrary boundary interactions should renormalize to one of these. A simple example is provided by the classical two dimensional Ising model, at temperature, \( T = T_c \), defined on the half-plane, \( x > 0 \). There are only three universality classes of boundary conditions, spin up, spin down and free [1]. The correlation exponent, \( \eta = 1/4 \) far from the boundary, but \( \eta = 1 \) in the boundary limit, for free boundary conditions. Furthermore, the universal crossover function, depending only on the ratios of distances from the boundary to distance between the points has been calculated exactly, by Cardy [1]. If we apply a weak magnetic field near the boundary, the system should cross over to the spin up universality class at large distances.

For analogous reasons to the bulk case, we expect not only translational and scale invariance but the infinite-dimensional conformal symmetry to hold at the critical point. We introduce a complex co-ordinate:

\[ z \equiv \tau + ix \]  

### 3
A general conformal transformation is of the form: \( z \rightarrow w(z) \) where \( w(z) \) is an analytic function. This has the Taylor expansion:

\[
w(z) = \sum_{n=1}^{\infty} a_n z^n
\]  

where the \( a_n \)'s are arbitrary complex numbers. With a boundary at \( x = 0 \) we can, at most, expect the subgroup of the conformal group which leave the boundary invariant to be a symmetry of the problem. i.e. we require \( w(\tau)^* = w(\tau) \). This implies that the expansion coefficients, \( a_n \) are all real. Since the \( a_n \)'s correspond to generators of the conformal group, we see that one half the conformal group remains in the presence of the conformally invariant boundary condition. From the real-time, Hamiltonian viewpoint, the boundary relates left and right movers. Consequently, the left and right energy density, \( T \) and \( \bar{T} \) are no longer independently conserved. There is now only one Virasoro algebra, not two as in bulk conformal field theory.

More explicitly, let us assume that the problem is defined on the positive half-plane, \( x > 0 \) only. (We can always represent the problem this way by reflecting the other half-plane, if necessary.) We will then assume the condition:

\[
T(0, t) = \bar{T}(0, t)
\]  

This results from assuming that the momentum density vanishes at the boundary; it is essentially a unitarity condition. Since \( T \) is a function of \( t + x \) only and \( \bar{T} \) is a function of \( t - x \) only, Eq. (11) implies that we may regard \( \bar{T} \) as the analytic continuation of \( T \) to the negative \( x \)-axis; i.e.

\[
T(t, x) \equiv \bar{T}(t, -x) \quad \text{for } x < 0
\]  

Thus we obtain a problem defined on the entire real axis with left-movers only. This identification of left with right leads to a modification of Green’s functions near the boundary. An arbitrary operator, \( O \) consists of left and right factors, depending on \( t + x \) and \( t - x \) only:

\[
O(x, t) = O_L(t + x)\bar{O}_R(t - x)
\]  

Using the left-right identification we obtain (at \( t = 0 \)):

\[
O(x) \rightarrow O_L(x)\bar{O}_L(-x)
\]  

i.e. the local operator \( O \) becomes effectively bilocal in the presence of a boundary. This is similar to the method of image charges in electrostatics. An immediate consequence of this is that operators pick up non-vanishing one-point functions near the boundary (even if they vanish in the bulk). i.e.

\[
< O(x) \rightarrow < O_L(x)\bar{O}_L(-x) > = \frac{C}{(2ix)^n}
\]  

i.e. a one-point function becomes a two-point function. Similarly two-point functions becomes four-point functions. In general we can characterize the effects of the boundary by the operator product expansion:
The set of operators, $O_j$ and exponents, $\eta_j$ simply correspond to the left-moving Hilbert Space of bulk operators; they do not depend on the particular boundary conditions. The operator product expansion coefficients, $C_j$ on the other hand, do depend on the boundary condition.

How do we find all possible conformally invariant boundary conditions and calculate Green’s functions? Consider the system in a box of length $l$ with arbitrary conformally invariant boundary conditions $A$ and $B$ at $x = 0$ and $x = l$, at inverse temperature $\beta$. The path integral is defined on a cylinder of circumference $\beta$ and length $l$. (See Figure 5.) Letting $H_{AB}^l$ denote the Hamiltonian for the finite system with boundary conditions $A$ and $B$; the path integral corresponds to the partition function:

$$Z_{AB} = \text{tr} e^{-\beta H_{AB}^l}$$  \hspace{1cm} (17)

Alternatively, we may regard $l$ as the time interval and $\beta$ as the space interval, ie. make a modular transformation. The system is now periodic in space; we write the corresponding Hamiltonian as $H_{P}^\beta$, $P$ denoting periodic. Now the system propagates for a time interval $l$ so the imaginary time evolution operator, $e^{-lH_{P}^\beta}$ occurs. However, we do not write a trace in this case since the system is not periodic in time. Instead, the system evolves between some initial and final states, $|A>$ and $|B>$; ie.:

$$Z_{AB} = <A|e^{-lH_{P}^\beta}|B>$$  \hspace{1cm} (18)

$|A>$ and $|B>$ are called boundary states. To each boundary condition corresponds a boundary state. It turns out to be easier to find all possible boundary states than boundary conditions. Equating the two expressions for $Z_{AB}$ gives an equation (true for all $l/\beta$) which determines all conformally invariant boundary states (and hence boundary conditions). The boundary states determine the Greens functions. (Matrix elements of operators between the boundary state and the vacuum give the needed operator product expansion coefficients.) To solve for the critical behavior of an arbitrary quantum impurity problem we “just” have to find (guess?) the corresponding boundary state. Frequently the number of possibilities consistent with the symmetries of a given bulk critical system is very small. We have found boundary states for the various problems listed in the introduction.

4. Multi-Channel Kondo Effect

We generalize the Kondo Hamiltonian of Eq. (2) to include $k$ channels of electrons interacting with a spin-$s$ impurity:

$$\mathcal{H} = \sum_{j=1}^{\infty} \left[ \psi_{j}^{\dagger}(i \frac{d}{dx}) \psi_{j} + \lambda \delta(x) \psi_{j}^{\dagger} \vec{\sigma} \cdot \vec{S} \right]$$  \hspace{1cm} (19)

Note that we have made the (in general unrealistic) assumption that each channel has identical Fermi velocity (set to 1), identical Kondo coupling and identical potential scattering term (assumed to be 0). Thus the model has an $SU(k)$ symmetry corresponding
to interchanging the channels. When the number of channels exceeds twice the impurity spin, $k > 2s$, the impurity is overscreened and the Kondo coupling does not renormalize to $\infty$. Consider, for example the simplest overscreened case, $k = 2$, $s = 1/2$. See Figure 6. If we assume that the on-site Kondo coupling goes to infinity, then one electron from each channel would sit on the impurity site with spins anti-parallel to that of the impurity. This overscreened complex has spin $1/2$. Now consider the electrons on the next site. More correctly we must consider an even parity combination of the two neighboring sites. In the three-dimensional setting we are considering spherical shells around the impurity, as drawn in Figure 6. These electrons feel a weak antiferromagnetic Kondo coupling with the spin complex at the origin. This coupling also renormalizes to large values. If we assume it also goes to infinity then we obtain yet another, larger, spin complex with $s = 1/2$. This process would keep on going forever. This basically tells us that the assumption that the Kondo coupling flows to $\infty$ is not correct.

We have found the boundary states for all values of $k$ and $s$. From these we can calculate the single-particle Green’s function [6, 9]:

$$< \psi_L^\dagger(z_1) \psi_R(z_2) > \rightarrow \frac{S_{(1)}}{z_1 - z_2}$$

(20)

$S_{(1)}$ measures the correlation of incoming and outgoing electron. From it we can determine the self-energy for a dilute random array of impurities, and hence the zero-temperature resistivity. This is given by:

$$\rho = \rho_{un} \frac{1 - S_{(1)}}{2}$$

(21)

Here $\rho_{un}$ is the resitivity in the unitary limit. i.e., it is the maximum possible resistivity that could occur for potential scattering, corresponding to a $\pi/2$ phase shift. $S_{(1)}$ has the interpretation of the 1 particle $\rightarrow$ 1 particle S-matrix at zero energy. For potential scattering, we would have, $S_{(1)} = e^{2i\delta}$. This is also true for the one-channel Kondo effect, with $\delta = \pi/2$. In fact, with a particle-hole symmetric Hamiltonian as in Eq. (19), $S_{(1)}$ must be real. However, in the overscreened case, we find $|S_{(1)}| < 1!!$ Unitarity then implies that there must be inelastic scattering even at zero energy, violating the basic assumption of Landau’s Fermi liquid theory. Thus we call this a non Fermi liquid fixed point. Our explicit calculation from the boundary state gives:

$$S_{(1)} = \frac{\cos[\pi (2s + 1)/(2 + k)]}{\cos[\pi/(2 + k)]}$$

(22)

We can also calculate the electron pair-operator Green’s function [6, 10], for example. For the case, $k = 2$, $s = 1/2$, we define the spin-singlet, flavor-singlet pair operator:

$$O(z) \equiv \psi_{L,\alpha i}(z) \psi_{R,\beta j}^\dagger(\bar{z}) \epsilon^{\alpha\beta} \epsilon^{ij}$$

(23)

We obtain the Green’s function:

$$< O(z_1) O^\dagger(z_2) > = \frac{4\eta^{-1/2}}{|z_1 - \bar{z}_2|^2} \left[ \frac{1}{1 - \eta} + 3 \right]$$

(24)

Here
\[ \eta \equiv \frac{4r_1r_2}{|\tau_1 - \tau_2|^2 + (r_1 + r_2)^2}, \quad z_j \equiv \tau_j + ir_j, \quad (r_j > 0) \]  
(25)

This universal cross-over function gives the non-interacting result in the bulk limit:

\[ <O(z_1)O^\dagger(z_2)> \rightarrow \frac{4}{|z_1 - z_2|^2} \quad (r_1, r_2 >> |z_1 - z_2|) \]  
(26)

and gives non-trivial singular behavior in the boundary limit:

\[ <O(z_1)O^\dagger(z_2)> \rightarrow \frac{8}{|\tau_1 - \tau_2|\sqrt{r_1r_2}} \quad (a << r_1, r_2 << |\tau_1 - \tau_2|) \]  
(27)

(Here \( a \) is the width of the boundary layer, or short-distance cut off.) Note that the \( 1/\tau \) fall off is more singular than the \( 1/\tau^2 \) behavior in the non-interacting case. Fourier-transforming with respect to time for fixed \( r_i \) we obtain a logarithmically divergent local pair susceptibility, \( \chi(\omega) \propto \ln \omega \). This has been used as the basis of models of superconductivity [17].

So far, we have calculated the leading temperature dependence of Green’s functions, using only properties of the fixed point. The flow to the fixed point is also important and determines various temperature dependent corrections [2, 4, 6, 9]. These are obtained by considering the leading irrelevant operator, \( O \), at the stable fixed point. i.e., we write the effective Hamiltonian density:

\[ \mathcal{H} = \mathcal{H}_{FP} + g\delta(x)O \]  
(28)

Here the fixed point Hamiltonian, \( \mathcal{H}_{FP} \) is defined with the corresponding boundary condition. \( g \) is the (non-universal) leading irrelevant coupling constant. For \( k = 2, s = 1/2, \) \( O \) has scaling dimension \( 3/2 \). It then follows from dimensional analysis that \( g \) has dimensions of (energy)\(^{-1/2}\). (Note that the \( \delta \)-function contributes 1 to the dimension.) The corresponding energy scale is, by definition, the Kondo temperature:

\[ g \equiv \frac{1}{\sqrt{T_K}} \]  
(29)

We perform low order perturbation theory in \( g \). This is infrared finite, since \( g \) is irrelevant. It gives a series in \((T/T_K)^{1/2}\), by dimensional analysis. We can express the leading temperature dependence of several quantities in terms of one unknown parameter, ratio, exactly for small \( T \). For \( k = 2, s = 1/2 \), we find the resistivity correction [6, 9]:

\[ \rho(T) = \frac{1}{2}\rho_{un}[1 + 4g\sqrt{\pi T}] \]  
(30)

the impurity specific heat [4]:

\[ C_{imp}(T) = 9\pi^3Tg^2\ln(T/T_K) \]  
(31)

and the impurity susceptibility [4]:

\[ \chi_{imp}(T) = 18\pi\mu_B^2g^2\ln(T_K/T) \]  
(32)

Notice that \( g = 1/\sqrt{T_K} \), drops out of the Wilson ratio, \( \chi_{imp}/C_{imp} \) and also the ratio of the square of the finite temperature resistivity correction to the specific heat. The analogous
ratio for the Fermi liquid Kondo fixed point was first calculated by Nozières [15]. In that case the approach to the fixed point is quite different. In particular, the resistivity then takes the form:

\[ \rho_{FL}(T) = \rho_{un}[1 - g^2 T^2] \] (33)

Differences occur both in the dimension of the leading irrelevant operator (2 instead of 3/2) and the lowest order in perturbation theory at which various quantities become non-zero. Note the much more singular T-dependence in this non-Fermi liquid case.

5. Impurities in Spin-1/2 Heisenberg Chains: A Fusion Rule Approach

Recently, Sebastian Eggert and I applied the boundary conformal field theory technique to a localized impurity in an s=1/2 antiferromagnetic chain [11]. The impurities we considered were modified exchange couplings on a single link or a pair of links or the coupling of an extra spin (possibly with s > 1/2) to the chain. The boundary fixed points that occurred were very simple, corresponding to a periodic (ie. unperturbed) chain or an open chain. Unlike in our analysis of the multi-channel or two-impurity Kondo problems [3, 4], we did not use the fusion rules to find the fixed point to which the system renormalized upon adding the impurity. In this section, I wish to show that, in fact, it is possible to pass between these fixed points using fusion, at least in the case of an SU(2) invariant system, ie. the Heisenberg model. We do not know how to do this in the case of the \( xxz \) model; it appears likely that fusion is not a general procedure that works in all cases [13, 14]. We hope that this approach to the Heisenberg model may shed light on this general question. It also brings out analogies between the spin chain impurity problem and both two-channel [3, 4] and two-impurity [7] Kondo problems. We begin by briefly reviewing the fusion rules approach to finding boundary states and then turn to the spin chain example.

As mentioned in Sec. 3, the quantum impurity problems that we consider can always be formulated so as to obey the boundary condition \( T(t, 0) = \tilde{T}(t, 0) \). In many cases they also obey an analogous condition on some conserved chiral currents:

\[ J^a(t, 0) = \tilde{J}^a(t, 0) \] (34)

In the spin chain problem \( a \) will run over the three generators of \( SU(2) \). As before, we may regard \( \tilde{J}^a \) as the analytic continuation of \( J^a \) to \( x < 0 \). If we consider the cylinder geometry of Sec. 3, with such boundary conditions at each end, then it follows that

\[ J^a(t, -l) = J^a(t, l) \] (35)

and the same for \( T \). Thus all eigenstates of \( H^A_{AB} \) will be eigenstates of \( T \) defined periodically on an interval of length \( 2l \). It follows that \( Z_{AB} \) can be expanded in Kac-Moody characters of \( T \), corresponding to the algebra of the currents, \( J^a \). These can be written as:

\[ \chi_i(q) \equiv \sum_{x; t} q^x \] (36)
where the $x$ label the scaling dimensions in the $i^{th}$ conformal tower, and, $q \equiv e^{-\pi \beta/l}$. i.e., in general:

$$Z_{AB} = \sum_i n_{iAB}^i \chi_i(q).$$  \hspace{1cm} (37)

Only the integer multiplicities, $n_{iAB}^i$, depend on the particular conformally invariant boundary conditions $A, B$.

Eqs. (11) and (35) correspond to conditions on the boundary states, $A >$, after the modular transformation:

$$[T(x) - \bar{T}(x)]|A> = [J^a(x) + \bar{J}^a(x)]|A> = 0$$  \hspace{1cm} (38)

There is one solution of these equations, called an Ishibashi state [18], for each Kac-Moody conformal tower:

$$|i> = \sum_N |i, N >_L \otimes \Omega |i, N >_R.$$  \hspace{1cm} (39)

Here $N$ is summed over all members of the conformal tower; the anti-unitary operator $\Omega$ takes $\bar{J}^a \rightarrow -\bar{J}^a$. All boundary states, $|A>$, can be expanded in Ishibashi states:

$$|A> = \sum_i |i > < i, 0|A>$$  \hspace{1cm} (40)

where $|i, 0 > \equiv |i, 0 >_L \otimes \Omega |i, 0 >_R$. Thus, the apparently formidable problem of classifying all boundary states and spectra is reduced to finding a finite number of multiplicites, $n_{iAB}^i$, or a finite number of matrix elements $< i, 0|A >$, one for each conformal tower. By equating the two expressions for $Z_{AB}$, Eq. (17) and (18) we obtain an important set of equations which can be used to determined these parameters. Using the modular transformation:

$$\chi_i(q) = S^j_i \chi_j(\tilde{q}),$$  \hspace{1cm} (41)

where

$$\tilde{q} \equiv e^{-4\pi l/\beta},$$  \hspace{1cm} (42)

and assuming linear independence of the characters, we obtain:

$$\sum_j S^i_j n_{jAB}^j = < A|i, 0 > < i, 0|B >,$$  \hspace{1cm} (43)

Cardy’s equation [1].

Cardy discovered an important property of this equation which allows new boundary states to be constructed from old ones. Given any consistent boundary state, $|A>$, and any conformal tower, $i$, we construct a new boundary state, $|A, i>$, as follows. With some fixed boundary state $|B>$ at the other end of the cylinder, the new spectrum is given by:

$$n_{A,i,B}^j = N_{ik}^j n_{iAB}^k.$$  \hspace{1cm} (44)

Here the non-negative integers, $N_{ik}^j$, are the fusion rule coefficients; they give the number of occurrences of the primary field $\phi_j$ in the operator product expansion of $\phi_i$ with $\phi_k$. 

The new boundary state is given by:

\[ < A, i | j, 0 > = \frac{S^i_j}{S^0_0} < A | j, 0 >. \] (45)

The consistency of Eqs. (44) and (45) follows from the Verlinde formula relating the fusion rule coefficients to the modular S-matrix:

\[ S^j_i N^k_{i\ell} = \frac{S^j_l S^j_i}{S^j_0}. \] (46)

This strategy for constructing new boundary states was used in our solution for the critical behavior of the multi-channel and two-impurity Kondo problems. We began with a trivial boundary state corresponding to a simple vanishing boundary condition on the free fermions. We then constructed non-trivial boundary states by fusion with particular operators, \( \phi_i \), motivated by physical considerations.

We now consider the example of the \( s = 1/2 \) Heisenberg spin chain. Our technique can be applied to this problem because, in the absence of a boundary, the system corresponds to a harmonic Luttinger liquid, i.e., a free spin boson, or correspondingly a \( k = 1, SU(2) \) Wess-Zumino-Witten (WZW) non-linear \( \sigma \)-model. (See, for example, [19].) The spin operators, in the bulk system are represented in terms of the \( SU(2) \) matrix field, \( g \), of the WZW model and the current operators as:

\[ \vec{S}_i = (-1)^i \text{constant} \times \text{tr}(g) \vec{\sigma} + (\vec{J}_L + \vec{J}_R). \] (47)

The field \( g \) may be regarded as a product of left and right-moving factors:

\[ g^\alpha_\beta = g^L_\alpha g^R_\beta. \] (48)

Impurity interactions generally involve the field \( g \) at the origin. A crucial complication of this Luttinger liquid system, compared to free fermions, is that the impurity interaction involves both left and right-moving fields and cannot be reduced to a single type of field. This is unlike the Kondo problem. If we consider, for example, the Kondo problem in a one-dimensional system of left and right-moving electrons, the Kondo interaction only involves the sum of left and right-moving fermion fields, \([\psi_L(0) + \psi_R(0)]\). We may define even and odd parity channels,

\[ \psi_{e,o}(x) \equiv \frac{1}{\sqrt{2}} [\psi_L(x) \pm \psi_R(-x)], \] (49)

and only \( \psi_e(0) \) appears in the Kondo interaction. No analogous reduction of the number of degrees of freedom can be made in the Heisenberg model case so we are faced with what is fundamentally a two-channel problem. It is actually convenient to regard the right-movers as being a second channel of left movers. i.e., we define:

\[ g_{L\alpha 1}(x) \equiv g_{L\alpha}(x), \quad g_{L\alpha 2}(x) \equiv g_{R\alpha}(-x) \] (50)

We now have two left-moving \( k = 1 \) WZW models, a theory with central charge \( c = 2 \). This model is known to be equivalent to a single \( k = 2 \) WZW model \( (c = 3/2) \) together with an Ising model \( (c = 1/2) \). This representation is useful because the total spin current, \( \vec{J}_1 + \vec{J}_2 \equiv \vec{J} \), is conserved whereas the original left and right-moving currents are
not separately conserved in general, with a boundary present. Thus only a finite number of characters and Ishibashi states can occur, corresponding to direct products of $SU(2)_k$ and Ising conformal towers. Each theory has three conformal towers. Those of the WZW model are labeled by the spin of the highest weight state, $j = 0, 1/2, 1$. Those of the Ising model correspond to the identity operator, the order parameter, and the energy operator.

It is convenient to also label these by a second index $j_l = 0, 1/2, 1$ respectively because the modular $S$-matrices are the same with this identification:

$$S = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}$$

Let us now consider a boundary. It turns out that there are apparently only two relevant boundary conditions corresponding to a periodic system (i.e. no boundary at all) or else an open system, i.e. a break in the spin chain at one point. As discussed previously, we can always formulate the problem in terms of left-movers only on the entire real axis or left and right-movers on the positive axis. In this problem, in the first formulation, we have two channels of left-movers and in the second formulation we have two channels each of left and right-movers. The latter formulation is convenient for studying the cylinder geometry with two boundary conditions $A$ and $B$ discussed in Sec. 4. We now wish to determine the partition function $Z_{AB}$ for $A$ and $B$ either the periodic or open boundary condition, labelled $P$ and $O$ respectively. Note that $Z_{PP}$ is the partition function for a periodic spin chain of length $2l$; $Z_{PO}$ corresponds to a chain of length $2l$ with open boundary conditions and $Z_{OO}$ corresponds to two decoupled chains with open boundary conditions, each of length $l$. Although we ignore corrections of $O(1/l)$ to the energies, we must carefully distinguish the case of even or odd $l$. These spectra were worked out in [11] by mapping the boundary conditions on the spins into boundary conditions on the abelian boson which is equivalent to the $k = 1$ WZW model. Periodic boundary conditions on the spins map into periodic boundary conditions on the boson and open map into vanishing boundary conditions. From this we determined the partition functions, written in terms of $k = 1$ WZW characters, which are labelled by highest weight states, $j = 0$ and $j = 1/2$.

We label the corresponding characters, $\chi_j^{(1)}$; the superscript (1) labels the Kac-Moody central charge $k = 1$. The partition functions depend on whether the chains have even or odd length. For a periodic chain of even length $2l$ the partition function is given by $[\chi_0^{(1)}(q)]^2 + [\chi_1^{(1)}(q)]^2$. For odd length $2l + 1$ it is given by $2\chi_0^{(1)}(q)\chi_1^{(1)}(q)$. For an open chain of even length, $l$ it is given by $\chi_0^{(1)}(q)$ and for odd length, $l + 1$ it is given by $\chi_1^{(1)}(q)$.

We can reformulate these spectra in terms of the $SU(2)_2 \times$ Ising representation. We now give the spectra for the three possible pairs of boundary conditions ($P$ or $O$ at each end) and for even or odd length of the line segments (denoted by a superscript e or o.). These are given by:

$$Z_{PP}^e(q) = [\chi_0^{(2)}(q) + \chi_1^{(2)}(q)] \cdot [\chi_0^{I}(q) + \chi_1^{I}(q)]$$
$$Z_{PO}^e(q) = 2\chi_1^{I/2}(q) \cdot \chi_1^{I/2}(q)$$
$$Z_{PO}^o(q) = [\chi_0^{(2)}(q) + \chi_1^{(2)}(q)] \cdot \chi_1^{I}(q)$$
$$Z_{OO}^e(q) = \chi_0^{(2)}(q) \cdot [\chi_0^{I}(q) + \chi_1^{I}(q)]$$
$$Z_{OO}^o(q) = \chi_0^{(2)}(q) \cdot [\chi_0^{I}(q) + \chi_1^{I}(q)]$$
\[ Z_{oo}^e(q) = \chi_{1/2}^{(2)}(q) \cdot \chi_{1/2}^I(q) \]
\[ Z_{oo}^{1o}(q) = \chi_0^{(2)}(q) \cdot \chi_1^I(q) + \chi_1^{(2)}(q) \cdot \chi_0^I(q) \]  

(51)

It is straightforward to check all these identities explicitly since the \( SU(2)_2 \) and Ising characters all have simple expressions in terms of \( \theta \)-functions and the \( SU(2)_1 \) characters are given by simple free boson formulas.

We now wish to demonstrate that we can pass between the different spectra by fusion. The fusion rules are the same for \( SU(2)_2 \) and for the Ising model using the \( j_I \) representation. They are:

\[
\begin{align*}
0 \times j & \rightarrow j \\
1/2 \times 1/2 & \rightarrow 0 + 1 \\
1 \times 1/2 & \rightarrow 1/2
\end{align*}
\]

(52)

Let us consider starting with two open sections, both of even length, corresponding to \( Z_{oo}^e \). Suppose we now couple one extra \( s = 1/2 \) variable infinitesimally with equal strength to the two open chains at \( x = 0 \). We expect this weak link to “heal”; ie. to renormalize to a periodic boundary condition. We should then obtain the partition function, \( Z_{PO}^e \). (The total length is now odd since we have added one extra spin.) It is natural to suspect that this process should correspond to fusion with the \( j = 1/2 \) primary field in the \( SU(2)_2 \) sector. It can be verified explicitly from Eqs. (51) and (52) that this is the case. If we now couple the remaining open ends to another \( s = 1/2 \) variable then we obtain the \( Z_{PP}^e \) spectrum. This again can be obtained by fusion with the \( j = 1/2 \) primary field. Alternatively, we may start with two open sections both of odd length or one odd and one even. Adding extra \( s = 1/2 \)'s, coupled with equal strength at both ends, producing crossover to \( Z_{PO}^e \) and then \( Z_{PP}^e \) corresponds to fusion with \( j = 1/2 \) in all cases. This process is formally identical to a spin-only version of the two-channel, \( s = 1/2 \) Kondo effect [4]. In that problem the non-trivial fixed point is also obtained by fusion with \( j = 1/2 \) in the \( SU(2)_2 \) sector. The only difference lies in the presence of charge excitations in the Kondo problem; but these play a passive role anyway.

Alternatively, we may again start with two open even length sections and now couple them together with two intervening spins. This is analogous to the two-impurity Kondo problem; the coupling of the two extra spins to the ends of the open chains is analogous to the Kondo coupling and the self-coupling of the two extra spins is analogous to the RKKY coupling. (The two Kondo couplings are assumed equal.) The stable fixed points in this problem correspond to open chains, with the two extra spins either coupling together to form a singlet or else with one fastening onto the end of each chain. In between these two stable fixed points we expect an unstable one, which can be obtained by adjusting the ratio of the inter-impurity coupling to the Kondo coupling. This fixed point corresponds to the periodic boundary condition, with spectrum \( Z_{PO}^e \). It can be checked from Eq. (51) and (52) that this is obtained from fusion with the Ising primary, \( j_I = 1/2 \). Critically coupling the two remaining free ends to two more impurity spins produces \( Z_{PP}^e \), again from fusion with \( j_I = 1/2 \). Again the same process occurs if we start with two open chains of odd length or one even and one odd. This fusion process, with the \( j_I = 1/2 \) field is again identical to the one that determines the non-trivial unstable critical point in the two-impurity Kondo problem [7].
Rather remarkably, both two channel one impurity and one channel two impurity processes occur in this simple model, in a simplified form with no charge degrees of freedom present. In both cases the “non-trivial” fixed point is simply the periodic one, the “healing” process discovered in [11] corresponds to the flow to the non-trivial fixed point in both types of Kondo problem, a stable flow for one impurity but unstable for two impurities.

We may also write down boundary states corresponding to periodic and open boundary conditions, related to each other by fusion. Subtleties arise because of the distinction between even and odd length chains and we find it necessary to introduce (at least) two types of periodic boundary states and three types of open ones, in order to obtain all seven partition functions. Ishibashi states can be labelled either in terms of the two $SU(2)_1$ algebras or in terms of $SU(2)_2 \times \text{Ising}$. In the latter formulation ($|jj_j \rangle$ basis), the boundary states are:

$$
|\text{per } 1 \rangle = |0, 0 > |0, 1 > + |1, 0 > |1, 1 >
$$

$$
|\text{per } 2 \rangle = |0, 0 > + |0, 1 > |1, 0 > - |1, 1 >
$$

$$
|\text{open } 1 \rangle = \frac{1}{\sqrt{2}} [ |0, 0 > + |0, 1 > + |1, 0 > + |1, 1 > + |1/2, 1/2 \rangle_1 + |1/2, 1/2 \rangle_2]
$$

$$
|\text{open } 2 \rangle = \frac{1}{\sqrt{2}} [ |0, 0 > + |0, 1 > + |1, 0 > + |1, 1 > - |1/2, 1/2 \rangle_1 - |1/2, 1/2 \rangle_2]
$$

$$
|\text{open } 3 \rangle = \frac{1}{\sqrt{2}} [ |0, 0 > - |0, 1 > - |1, 0 > + |1, 1 > + |1/2, 1/2 \rangle_1 - |1/2, 1/2 \rangle_2] \quad (53)
$$

Two different Ishibashi states $|1/2, 1/2 \rangle_i, i = 1, 2$ occur in this formulation, corresponding to the $SU(2)_1 \otimes SU(2)_1$ Ishibashi states $|1/2, 0 \rangle$ and $|0, 1/2 \rangle$. Using the ratio of $S$-matrix elements:

$$
\frac{S\beta_{1/2}}{S\beta_0} = (\sqrt{2}, 0, -\sqrt{2}), \quad (54)
$$

we see from Eq. (45) that each of the $|\text{open } i \rangle$ states goes into one of the $|\text{per } j \rangle$ states under fusion with either the $j = 1/2$ or $jj_j = 1/2$ field. The 15 matrix elements of the form $\langle A | e^{-iH\beta} | B \rangle$ obtained from these five states are all equal to one of the seven different partition functions given in Eq. (51). For instance, $Z_{\text{per } i, \text{ per } j} = Z_{PP}^e$ for $i = j$, and $= Z_{PP}^o$ for $i \neq j$.

To conclude this section, the simple problem of an impurity in a Heisenberg spin chain can be understood using the fusion rules. It represents a spin-only version of both the two channel and two impurity Kondo problems.

I would like to thank Andreas Ludwig for his collaboration in much of this work and Eugene Wong for checking the identities in Eq. (51). I would also like to thank my other collaborators in this field: Sebastian Eggert, Jacob Sagi and Erik Sørensen. This research was supported in part by NSERC of Canada.

**Figure Captions**

1. Imaginary time formulation of quantum impurity problems.
2. Two possible formulation of quantum impurity problems with left and right-movers on the positive real axis or with left movers only on the entire axis.
3. Renormalization group flow of the Kondo coupling.
4. Boundary and bulk limit of correlation functions.
5. The cylinder geometry: $A$ and $B$ represent conformally invariant boundary conditions.
6. The two-channel, $s = 1/2$ Kondo effect. Successive electron orbitals each overscreen the impurity always leaving behind an effective $s = 1/2$.

References

* To appear in the proceedings of the Taniguchi Symposium held at Mie Prefecture, Japan, October, 1993.

[1] J.L. Cardy, Infinite Lie Algebras and Conformal Invariance in Condensed Matter and Particle Physics (ed. K. Dietz and V. Rittenberg, World Scientific, Singapore, 1987), 81; Nucl. Phys. B324, 581 (1989); D. Lewellen and J.L. Cardy, Phys. Lett. B259, 274 (1991).

[2] I. Affleck, Nucl. Phys. B336, 517 (1990).

[3] I. Affleck and A.W.W. Ludwig, Nucl. Phys. B352, 849 (1991).

[4] I. Affleck and A.W.W. Ludwig, Nucl. Phys. B360, 641 (1991).

[5] I. Affleck and A.W.W. Ludwig, Phys. Rev. Lett. 67, 161 (1991).

[6] A.W.W. Ludwig and I. Affleck, Phys. Rev. Lett., 67, 3160 (1991).

[7] I. Affleck and A.W.W. Ludwig, Phys. Rev. Lett. 68, 1046 (1992).

[8] I. Affleck, A.W.W. Ludwig, H.-B. Pang and D.L. Cox, Phys. Rev. B45, 7918 (1992).

[9] I. Affleck and A.W.W. Ludwig, Phys. Rev. B48, 7297 (1993).

[10] A.W.W. Ludwig and I. Affleck, in preparation.

[11] S. Eggert and I. Affleck, Phys. Rev. B46, 10866 (1992).

[12] E. Sørensen, S. Eggert and I. Affleck, preprint, UBCTP-93-07 (cond-mat@babbage.sissa.it, no. 9306003) to appear in J. Phys. A.

[13] I. Affleck and J. Sagi, preprint, UBCTP-93-18 (hep-th@babbage.sissa.it, no. 9311056).

[14] E. Wong and I. Affleck, preprint, UBCTP-93-20 (cond-mat@babbage.sissa.it, no. 931140).

[15] P. Nozières, J. Low Temp. Phys. 17, 31 (1974); P. Nozières, Proceedings of the 14th Int.'l Conf. on Low Temp. Phys. (ed. M. Krusius and M. Vuorio, North Holland, Amsterdam, 1974) V.5, p. 339.

[16] J. Kondo, Prog. Theor. Phys. 32, 37 (1964).

[17] D.L. Cox, Phys. Rev. Lett. 67 2883 (1987); V.J. Emery and S. Kivelson, Phys. Rev. B46, 10812 (1992).

[18] N. Ishibashi, Mod. Phys. Lett. A4, 251 (1989); T. Onogi and N. Ishibashi, Nucl. Phys. B318, 239 (1989).
[19] I. Affleck, *Fields, Strings and Critical Phenomena* (ed. E. Brézin and J. Zinn-Justin, North-Holland, Amsterdam), 563 (1990).