CONVERGENCE OF A DECOUPLED SPLITTING SCHEME FOR THE CAHN–HILLIARD–NAVIER–STOKES SYSTEM
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Abstract. This paper is devoted to the analysis of an energy-stable discontinuous Galerkin algorithm for solving the Cahn–Hilliard–Navier–Stokes equations within a decoupled splitting framework. We show that the proposed scheme is uniquely solvable and mass conservative. The energy dissipation and the $L^\infty$ stability of the order parameter are obtained under a CFL condition. Optimal a priori error estimates in the broken gradient norm and in the $L^2$ norm are derived. The stability proofs and error analysis are based on induction arguments and do not require any regularization of the potential function.

Key words. Cahn–Hilliard–Navier–Stokes, discontinuous Galerkin, stability, optimal error bounds

AMS subject classifications. 65M12, 65M15, 65M60

1. Introduction. The Cahn–Hilliard–Navier–Stokes (CHNS) system serves as a fundamental phase-field model extensively used in many fields of science and engineering. The simulation of the CHNS equations is a challenging computational task primarily because of: (i) the coupling of highly nonlinear equations; and (ii) the requirement of preserving certain physical principles, such as conservation of mass and dissipation of energy. A common approach to overcome these difficulties is to decouple the mass and momentum equations, and to further split the nonlinear convection from the incompressibility constraint [27]. The splitting scheme constructed from this strategy only requires the successive solution of several simpler equations at each time step. Thus, such an algorithm is both convenient for programming and efficient in large-scale simulations. A non-exhaustive list of several computational papers on the CHNS model include [9, 2, 6, 19, 32, 20].

The analysis of semi-discrete spatial formulations with continuous and discontinuous Galerkin (dG) methods for solving the CHNS equations has been extensively investigated. Without being exhaustive, we refer to the papers [10, 31, 8, 21] for the study of fully coupled schemes. For decoupled splitting algorithms based on projection methods, we mention a few papers [14, 4, 28, 29]. Han and Wang in [14] introduce a second order in time scheme and show unique solvability, but this work does not contain any theoretical proof of convergence. Cai and Shen in [4] formulate an energy-stable scheme and show convergence based on a compactness argument. In this work, in order to obtain energy dissipation, the authors introduced an additional stabilization term. Similar stabilizing strategies can be found in [28, 29]. Although this technique enforces a discrete energy law, it also introduces an extra consistency error. A major difficulty in proving optimal convergence error rates of a numerical scheme for the CHNS system arises from the nonlinear potential function. A widely used regularization technique is to truncate the potential and to extend it with a quadratic growth [4, 27, 17]. An important objective of our work is to obtain a rigorous
convergence analysis of the scheme without modifying and regularizing the potential function. To the best of our knowledge, the theoretical analysis of a decoupled splitting scheme in conjunction with interior penalty dG discretization without any regularization on the potential function is not available in literature.

The main contribution of this work is the stability and error analysis of a dG discretization of a splitting scheme for the CHNS model. We prove the energy stability, the $L^\infty$ stability of the order parameter, and we derive the optimal a priori error bounds in both the broken gradient norm and the $L^2$ norm. Our analysis is novel and general in sense that: (i) we successfully avoid using any artificial stabilizing terms (which introduce an extra consistency error) when discretizing the CHNS system, and (ii) no regularization (truncation and extension) assumptions on the potential function are needed for the analysis. The proofs are technical and rely on induction arguments. A priori error bounds are valid for convex domains because the convergence analysis utilizes dual problems. Our arguments can be extended to analyze splitting algorithms for other type of phase-field models.

The outline of this paper is as follows. In Section 2, the CHNS mathematical model is presented. In Section 3, we introduce the fully discrete numerical scheme. The unique solvability of the scheme is proved in Section 4. We show that our scheme is energy stable in Section 5, and we derive error estimates in Section 6. Numerical experiments validating our theoretical results are presented in Section 7. Concluding remarks follow.

2. Model problem. Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be an open bounded polyhedral domain and let $n$ denote the unit outward normal to the boundary $\partial \Omega$. In the context of incompressible immiscible two-phase flows, we introduce a scalar field order parameter as a phase indicator, which is defined as the difference between mass fractions. The unknown variables in the CHNS system are the order parameter $c$, the chemical potential $\mu$, the velocity $u$, and the pressure $p$, satisfying:

\begin{align*}
\partial_t c - \Delta \mu + \nabla \cdot (cu) &= 0 \quad \text{in } (0, T) \times \Omega, \\
\mu &= \Phi'(c) - \kappa \Delta c \quad \text{in } (0, T) \times \Omega, \\
\partial_t u + u \cdot \nabla u - \mu \Delta u &= -\nabla p - c \nabla \mu \quad \text{in } (0, T) \times \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } (0, T) \times \Omega. 
\end{align*}

The parameter $\kappa$ and shear viscosity $\mu_s$ are positive constants. The Ginzburg-Landau potential function $\Phi$ is defined by:

$$\Phi(c) = \frac{1}{4}(1-c^2)(1+c)^2.$$ 

This polynomial potential can be decomposed into the sum of a convex part $\Phi_+$ and a concave part $\Phi_-$. We have:

$$\Phi = \Phi_+ + \Phi_-, \quad \text{where } \Phi_+ = \frac{1}{4}(1+c^4) \text{ and } \Phi_- = -\frac{1}{2}c^2.$$ 

We supplement our model problem (2.1) with the following initial and boundary conditions:

\begin{align*}
(2.3a) \quad c &= c^0, \quad u = u^0 \quad \text{on } \{0\} \times \Omega, \\
(2.3b) \quad \nabla c \cdot n &= 0, \quad \nabla \mu \cdot n = 0, \quad u = 0 \quad \text{on } (0, T) \times \partial \Omega.
\end{align*}
Let \( \overline{c_0} \) denote the average of the initial order parameter. The model problem (2.1) satisfies the global mass conservation property:

\[
\frac{1}{|\Omega|} \int_{\Omega} c = \frac{1}{|\Omega|} \int_{\Omega} c^0 = \overline{c_0},
\]

as well as the energy dissipation property [30, 12]. Let \( F \) denote the total energy of the system.

\[
F(c, u) = \int_{\Omega} \frac{1}{2} |u|^2 + \int_{\Omega} \left( \Phi(c) + \frac{\kappa}{2} |\nabla c|^2 \right), \quad \frac{d}{dt} F(c, u) \leq 0.
\]

We end this section by briefly stating the functional setting used throughout the paper. For a given real number \( p \geq 1 \), on a domain \( \Omega \in \mathbb{R}^d \), where \( d = 2 \) or 3, the standard notation for the \( L^p(\Omega) \) spaces is employed. Let \( (\cdot, \cdot)_O \) denote the \( L^2 \) inner product and the \( L^2(\Omega) \) norm, when \( O \) is the whole computational domain.

\[
\|\cdot\|_2 = \left( \int_{\Omega} |\cdot|^2 \right)^{1/2}, \quad (\cdot, \cdot)_O = \left( \int_{\Omega} \frac{\kappa}{2} |\nabla c|^2 \right)^{1/2}.
\]

The usual Sobolev semi-norm \( |\cdot|_{W^{m,p}(\Omega)} \), norm \( \|\cdot\|_{W^{m,p}(\Omega)} \) are employed. We introduce the space \( H^m(\Omega) = W^{m,2}(\Omega) \) with the associated semi-norm \( |\cdot|_{H^m(\Omega)} = |\cdot|_{W^{m,2}(\Omega)} \) and norm \( \|\cdot\|_{H^m(\Omega)} = \|\cdot\|_{W^{m,2}(\Omega)} \).

For convenience, we use \( (\cdot, \cdot) \) and \( \|\cdot\| \) to denote the \( L^2 \) inner product and the \( L^2 \) norm.

\[
H^s(\Gamma_h) = \{ \omega \in L^2(\Omega) : \forall E \in \Gamma_h, \omega|_E \in H^s(E) \}.
\]

The average and jump operators of any scalar function \( \omega \in H^s(\Gamma_h) \) is defined for each interior face \( \varepsilon \in \Gamma_h \) by

\[
\{\omega\}_\varepsilon = \frac{1}{2} \omega|_{E_\varepsilon^-} + \frac{1}{2} \omega|_{E_\varepsilon^+}, \quad [\omega]_\varepsilon = \omega|_{E_\varepsilon^-} - \omega|_{E_\varepsilon^+}, \quad \varepsilon = \partial E_\varepsilon^- \cap \partial E_\varepsilon^+.
\]

If \( \varepsilon \) belongs to the boundary \( \partial \Omega \), the jump and average of \( \omega \) coincide with its trace on \( \varepsilon \). The related definitions of any vector quantity in \( H^s(\Gamma_h)^d \) are similar [26]. Fix an integer \( k \geq 1 \) and denote by \( P_k(E) \) the set of all polynomials of degree at most \( k \) on an element \( E \). Define the following discontinuous polynomial spaces for simplicial meshes:

\[
M_h^k = \{ \omega_h \in L^2(\Omega) : \forall E \in \Gamma_h, \omega_h|_E \in P_k(E) \},
\]

\[
M_{00}^k = \{ \omega_h \in M_h^k : (\omega_h, 1) = 0 \},
\]

\[
X_h^k = \{ \theta_h \in L^2(\Omega)^d : \forall E \in \Gamma_h, \theta_h|_E \in P_k(E)^d \}.
\]
For meshes with parallelograms or parallelepipeds, the space \( Q_k(E) \), namely the space of tensor product polynomials of degree at most \( k \) on an element \( E \), is used instead of \( P_k(E) \) in the above definitions. We now present the dG pressure projection algorithm for solving (2.1) with initial and boundary conditions (2.3). Uniformly partition \([0,T]\) into \( N_T \) intervals with length equal to \( \tau \) and for any \( 1 \leq n \leq N_T \) let \( t^n = n \tau \) and let \( \delta_\tau \) be the temporal backward finite difference operator \( \delta_\tau u^n = (u^n - u^{n-1})/\tau \). The scheme consists of four sequential steps.

Given \((c_h^{n-1}, u_h^{n-1}) \in M_h^k \times X_h^k\), compute \((c_h^n, u_h^n, v_h^n) \in M_h^k \times M_h^k \times X_h^k\), such that for all \( \chi_h \in M_h^k \) and for all \( q_h \in M_h^k \)

\[
\begin{align*}
(3.1) & \quad (\delta_\tau c_h^n, \chi_h) + a_{\text{diff}}(\mu_h^n, \chi_h) + a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \chi_h) = 0, \\
(3.2) & \quad (\Phi_+(c_h^n) + \Phi_-(c_h^{n-1}), q_h) + \kappa a_{\text{diff}}(c_h^n, q_h) - (\mu_h^n, q_h) = 0.
\end{align*}
\]

Second, given \((c_h^{n-1}, \mu_h^{n-1}, v_h^{n-1}, p_h^{n-1}) \in M_h^k \times M_h^k \times X_h^k \times M_h^{k-1}\), compute \( v_h^n \in X_h^k\), such that for all \( \theta_h \in X_h^k\)

\[
\frac{1}{\tau} (v_h^n - u_h^{n-1}, \theta_h) + a_C(u_h^{n-1}, u_h^{n-1}, v_h^n, \theta_h) + \mu_s a_D(v_h^n, \theta_h) = b_p(\theta_h, p_h^{n-1}) + b_I(c_h^{n-1}, \mu_h^{n-1}, \theta_h).
\]

Next, given \( v_h^n \in X_h^k\), compute \( \phi_h^n \in M_h^{k-1} \), such that for all \( \varphi_h \in M_h^{k-1} \)

\[
(3.4) \quad a_{\text{diff}}(\phi_h^n, \varphi_h) = -\frac{1}{\tau} b_p(v_h^n, \varphi_h).
\]

Finally, given \((v_h^n, p_h^n, \phi_h^n) \in X_h^k \times M_h^{k-1} \times M_h^{k-1}\), compute \((u_h^n, p_h^n) \in X_h^k \times M_h^{k-1}\), such that for all \( \chi_h \in M_h^{k-1} \) and for all \( \theta_h \in X_h^k\)

\[
\begin{align*}
(3.5) & \quad (p_h^n, \chi_h) = (p_h^{n-1}, \chi_h) + (\phi_h^n, \chi_h) - \sigma \chi \mu_s b_p(v_h^n, \chi_h), \\
(3.6) & \quad (u_h^n, \theta_h) = (v_h^n, \theta_h) + \tau b_p(\theta_h, \phi_h^n).
\end{align*}
\]

For the approximation of the initial values, let \( u_h^0 \) be the \( L^2 \) projection of \( u^0 \) and let \( c_h^0 \) be the elliptic projection of \( c^0 \), namely \( c_h^0 \) satisfies

\[
(3.7) \quad a_{\text{diff}}(c_h^0 - c^0, \chi_h) = 0, \quad \forall \chi_h \in M_h^k, \quad \text{with constraint} \quad (c_h^0 - c^0, 1) = 0.
\]

In addition, we set \( p_h^0 = \phi_h^0 = 0 \) and \( v_h^0 = u_h^0 \). The parameter \( \sigma \) is a (user-specified) positive number that can be chosen between 0 and \( 1/(4d) \).

The forms \( a_{\text{diff}} \) and \( a_D \) are the SIPG discretizations of the scalar and vector Laplace operator, \(-\Delta \omega \) and \(-\Delta \mathbf{v} \), respectively. Let \( \sigma \geq 1, \omega \geq 1 \) be given penalty parameters. We define

\[
\begin{align*}
(3.8) & \quad a_{\text{diff}}(\omega, \chi) = \sum_{e \in T_h} \int_E \mathbf{v} \cdot \nabla \chi - \sum_{e \in T_h} \int_e (\nabla \cdot \mathbf{n}_e)[\chi] \\
& \quad - \sum_{e \in T_h} \int_e \{\nabla \chi \cdot n_e\}[\omega] + \frac{\sigma}{h} \sum_{e \in T_h} \int_e [\omega][\chi], \quad \forall \omega, \chi \in H^2(T_h), \\
(3.9) & \quad a_D(\mathbf{v}, \theta) = \sum_{e \in T_h} \int_E \mathbf{v} \cdot \nabla \theta - \sum_{e \in T_h} \int_e \{\nabla \mathbf{v} \cdot n_e\} \cdot [\theta] \\
& \quad - \sum_{e \in T_h} \int_e (\nabla \cdot n_e)[\mathbf{v}] + \frac{\sigma}{h} \sum_{e \in T_h} \int_e [\mathbf{v}][\theta], \quad \forall \mathbf{v}, \theta \in H^2(T_h)^d.
\end{align*}
\]
The dG form \(a_C : H^2(T_h)^d \times H^2(T_h)^d \times H^2(T_h)^d \times H^2(T_h)^d \rightarrow \mathbb{R}\) of the convection term \(v \cdot \nabla v\) is

\[
\begin{align*}
(3.10) \quad a_C(w, v, z, \theta) &= \sum_{E \in T_h} \int_E (v \cdot \nabla z) \cdot \theta + \int_{\partial E} |v| \cdot n_E |(z^{\text{int}} - z^{\text{ext}}) \cdot \theta^{\text{int}}| \\
&+ \frac{1}{2} \sum_{E \in T_h} \int_E (v \cdot z) \cdot \theta - \frac{1}{2} \sum_{e \in E \cup \partial \Omega} \int_e [v \cdot n_e] (z \cdot \theta).
\end{align*}
\]

Here, the set \(\partial E^w\) is the inflow part of \(\partial E\), defined by \(\partial E^w = \{ x \in \partial E : \{ w(x) \} \cdot n_E < 0 \}\), and the superscript \(\text{int}\) (resp. \(\text{ext}\)) refers to the trace of the function on a face of \(E\) coming from the interior of \(E\) (resp. coming from the exterior of \(E\) on that face). In addition, if the face lies on the boundary of the domain, we take the exterior trace to be zero. The discretization of the linear advection term \(\nabla \cdot (c v)\) is done with the dG form \(a_{\text{adv}} : H^2(T_h) \times H^2(T_h)^d \times H^2(T_h)^d \rightarrow \mathbb{R}\):

\[
(3.11) \quad a_{\text{adv}}(c, v, \chi) = -\sum_{E \in T_h} \int_E c v \cdot \nabla \chi + \sum_{e \in E} \int \{c \} v \cdot n_e \chi.
\]

The dG form \(b_F : H^2(T_h) \times H^2(T_h)^d \times H^2(T_h)^d \rightarrow \mathbb{R}\) of the interface term \(-c \nabla \mu\) is equal to \(a_{\text{adv}}\) with switched arguments:

\[
(3.12) \quad b_F(c, \mu, \theta) = a_{\text{adv}}(c, \theta, \mu).
\]

Finally, for the discretization of the gradient and divergence terms, such as \(-\nabla p\), \(-\nabla \phi\), and \(\nabla \cdot v\), we introduce the dG bilinear form \(b_p : H^2(T_h)^d \times H^1(T_h) \rightarrow \mathbb{R}\):

\[
(3.13) \quad b_p(\theta, p) = \sum_{E \in T_h} \int_E p \nabla \cdot \theta - \sum_{e \in E} \int \{\theta, n_e\} [p].
\]

With Green’s theorem, an equivalent expression for \(b_p\) is:

\[
(3.14) \quad b_p(\theta, p) = -\sum_{E \in T_h} \int_E \nabla \cdot \theta \cdot p + \sum_{e \in E} \int \{\theta, n_e\} [p].
\]

The broken space \(H^1(T_h)\) and discrete space \(M_h^k\) are equipped with the semi-norm \(| \cdot |_{DG}\):

\[
|\omega|^2_{DG} = \sum_{E \in T_h} \|\nabla \omega\|^2_{L^2(E)} + \frac{\sigma}{h} \sum_{e \in T_h} \|\omega\|^2_{L^2(e)}, \quad \forall \omega \in H^1(T_h).
\]

Note, \(| \cdot |_{DG}\) is a norm on \(H^1(T_h) \cap L^2_0(\Omega)\). The vector space \(H^1(T_h)^d\) is equipped with the following norm:

\[
\|v\|^2_{DG} = \sum_{E \in T_h} \|\nabla v\|^2_{L^2(E)} + \frac{\sigma}{h} \sum_{e \in T_h \cup \partial \Omega} \|\nabla v\|^2_{L^2(e)}, \quad \forall v \in H^1(T_h)^d.
\]

We now recall several properties satisfied by the dG forms. The forms \(a_{\text{diff}}\) and \(a_D\) are coercive. There exist \(\sigma_0\) and \(\sigma_0\) such that for all \(\delta \geq \delta_0\) and \(\sigma \geq \sigma_0\), there exist \(K_{\text{a}} > 0\) and \(K_D > 0\) independent of \(h\) such that

\[
(3.15) \quad K_{\text{a}} |\omega_h|^2_{DG} \leq a_{\text{diff}}(\omega_h, \omega_h), \quad \forall \omega_h \in M_h^k,
\]

\[
(3.16) \quad K_D \|v_h\|^2_{DG} \leq a_D(v_h, v_h), \quad \forall v_h \in X_h^k.
\]
Without loss of generality, we can set \( K_a = K_D = 1/2 \). Indeed, this is true if \( \delta_0 = \sigma_0 = 2N_{\text{face}}C_{tr}^2 \) where \( C_{tr} \) is the trace constant in (3.33) (that depends on \( k \)) and \( N_{\text{face}} \) is the maximum number of faces of a mesh element. Let us denote

\[
M_k = 2N_{\text{face}}C_{tr}^2.
\]

Continuity also holds for these two forms. There exist constants \( C_a > 0 \) and \( C_D > 0 \) independent of mesh size \( h \), such that

\[
\begin{align*}
|a_{\text{diff}}(v_h, X_h)| &\leq C_a|v_h|_{\text{DG}}|X_h|_{\text{DG}}, \quad \forall v_h, X_h \in M_h^k, \\
|a_D(v_h, \theta_h)| &\leq C_D|v_h|_{\text{DG}}\|	heta_h\|_{\text{DG}}, \quad \forall v_h, \theta_h \in X_h^k.
\end{align*}
\]

The form \( a_{\text{adv}} \) satisfies the following bounds (see Lemma 3.2 in [21]). For all \( c_h, X_h \in M_h^k \), and for all \( v_h \in X_h^k \).

\[
\begin{align*}
|a_{\text{adv}}(c_h, v_h, X_h)| &\leq C_y\left(|c_h|_{\text{DG}} + \int_\Omega c_h\right)\|v_h\|_{\text{DG}}|X_h|_{\text{DG}}, \\
|a_{\text{adv}}(c_h, v_h, X_h)| &\leq C_y\left(|c_h|_{\text{DG}} + \int_\Omega c_h\right)\|v_h\|^{1/2}\|v_h\|_{\text{DG}}^{1/2}|X_h|_{\text{DG}}.
\end{align*}
\]

The form \( a_C \) satisfies the following positivity property:

\[
a_C(v_h, v_h, \theta_h, \theta_h) \geq 0, \quad \forall v_h, \theta_h \in X_h^k.
\]

We define lift operators \( R_h : X_h^k \rightarrow M_h^{k-1} \) and \( G_h : M_h^{k-1} \rightarrow X_h^k \) by

\[
(R_h(\theta_h), q_h) = \sum_{c \in \Gamma_h \cup \partial \Omega} \int_c (q_h)\{\theta_h\} \cdot n_c, \quad \forall \theta_h \in X_h^k, \quad \forall q_h \in M_h^{k-1},
\]

\[
(G_h(\chi_h), \theta_h) = \sum_{c \in \Gamma_h} \int_c \{\theta_h\}\{\chi_h\}, \quad \forall \chi_h \in M_h^{k-1}, \quad \forall \theta_h \in X_h^k.
\]

One can easily obtain the bounds [23, 7], recalling the definition (3.17)

\[
\begin{align*}
\|R_h(\theta_h)\| &\leq M_{k-1}\left(h^{-1} \sum_{c \in \Gamma_h \cup \partial \Omega} \|\theta_h\|_{L^2(c)}^2\right)^{1/2}, \quad \forall \theta_h \in X_h^k, \\
\|G_h(q_h)\| &\leq M_k\left(h^{-1} \sum_{c \in \Gamma_h} \|q_h\|_{L^2(c)}^2\right)^{1/2}, \quad \forall q_h \in M_h^{k-1}.
\end{align*}
\]

With the lift operators, we can rewrite \( b_p \) into two different ways:

\[
\begin{align*}
&b_p(\theta_h, p_h) = (\nabla_h \cdot \theta_h, p_h) - (R_h(\theta_h), p_h), \quad \forall \theta_h \in X_h^k, \quad \forall p_h \in M_h^{k-1}, \\
&b_p(\theta_h, p_h) = -(\nabla_h p_h, \theta_h) + (G_h(p_h), \theta_h), \quad \forall \theta_h \in X_h^k, \quad \forall p_h \in M_h^{k-1}.
\end{align*}
\]

In the above, \( \nabla_h \) and \( \nabla_h \cdot \) denote the broken divergence and gradient operators respectively.

**Proposition 3.1.** The \( dG \) pressure correction scheme satisfies the discrete global mass conservation property, i.e., for any \( 1 \leq n \leq N_T \), we have

\[
(c_h^n, 1) = (c_h^0, 1) = (c_0, 1) = (c(t^n), 1), \quad \forall 1 \leq n \leq N_T.
\]
Proof. The first equality is obtained by choosing $\chi_h = 1$ in (3.1) and by using $a_{\text{diff}}(\mu_h^n, 1) = 0$ and $a_{\text{adv}}(\mu_h^{n-1}, \mu_h^{n-1}) = 0$. The second equality is a constraint on the initial order parameter and the third equality is from (2.4).

We finish this section by recalling Poincaré’s, inverse and trace inequalities. For $p < +\infty$ when $d = 2$ and for $p \leq 6$ when $d = 3$, we have [13, 7]:

\begin{align}
\|a_h - \frac{1}{|\Omega|} \int_\Omega a_h \|_{L^p(\Omega)} &\leq C_P |a_h|_{DG}, \quad \forall a_h \in M^k_h, \\
\|\varphi\|_{L^p(\Omega)} &\leq C_P \left( |\varphi|_{DG}^2 + \frac{1}{|\Omega|} \int_\Omega |\varphi|^2 \right)^{1/2}, \quad \forall \varphi \in H^1(\Omega), \\
\|v_h\|_{L^p(\Omega)} &\leq C_P \|v_h\|_{DG}, \quad \forall v_h \in X^k_h.
\end{align}

Here, $C_P > 0$ denote a constant independent of the mesh size $h$. The following are two well-known inverse inequalities, for $p, q \in [1, \infty]$ and $p \geq q$,:

\begin{align}
\|v_h\|_{L^q(\Omega)} &\leq C_{\text{inv}} h^{d/p-d/q} \|v_h\|_{L^p(\Omega)}, \quad \forall v_h \in X^k_h, \\
\|v_h\|_{DG} &\leq C_{\text{inv}} h^{-1} \|v_h\|, \quad \forall v_h \in X^k_h,
\end{align}

where $C_{\text{inv}}$ is a constant independent of $h$. We also use the following trace estimates:

\begin{align}
\|v_h\|_{L^r(\partial E)} &\leq C_{\text{tr}} h^{-1/r} \|v_h\|_{L^1(\Omega)}, \quad \forall v_h \in X^k_h, \ r \geq 1, \ e \in \partial E, \\
\|v\|_{L^2(\partial E)} &\leq \tilde{C}_{\text{tr}} h^{-1/2} (\|v_h\|_{L^1(\Omega)} + h \|v\|_{L^2(\partial E)}), \quad \forall v \in H^1(\Omega), \ e \in \partial E.
\end{align}

We remark that the above inverse and trace estimates also hold for scalar valued functions. For brevity, we may refer to the above bounds for both scalar and vector valued functions. For any function $\xi$ we denote by $\overline{\xi}$ the average of $\xi$:

$$
\overline{\xi} = \frac{1}{|\Omega|} \int_\Omega \xi.
$$

We end this section by stating a discrete Sobolev inequality.

**Lemma 3.2.** Define $\Theta_{h,d}$ as follows:

\begin{equation}
\Theta_{h,d} = \begin{cases} 
(1 + |\ln(h)|)^{1/2}, & d = 2, \\
h^{-1/2}, & d = 3.
\end{cases}
\end{equation}

There exists a constant $\tilde{C}_P$ independent of $h$ such that

\begin{equation}
\|a_h - \overline{a_h}\|_{L^\infty(\Omega)} \leq \tilde{C}_P \Theta_{h,d} |a_h|_{DG}, \quad \forall a_h \in M^k_h.
\end{equation}

Proof. For $d = 2$, bound (3.36) is proved in [3] (see (19) in Theorem 5). For $d = 3$, inequality (3.36) follows from (3.31) (with $p = \infty$ and $q = 6$) and (3.28).

\[\|a_h - \overline{a_h}\|_{L^\infty(\Omega)} \leq C_{\text{inv}} h^{-1/2} \|a_h - \overline{a_h}\|_{L^2(\Omega)} \leq C_{\text{inv}} C_P h^{-1/2} |a_h|_{DG}.\]

Throughout the paper, $C$ denotes a generic constant that takes different values at different places and that is independent of mesh size $h$ and time step size $\tau$. 
4. Existence and uniqueness. We show that the discrete problem (3.1)-(3.6) is well posed in three steps. First, we show equivalence of (3.1)-(3.2) to a problem posed in $M^k_{h0}$ in Lemma 4.1. Similar to the arguments found in [14, 21], we then use the Browder–Minty theorem to show existence and uniqueness of a solution to (4.1)-(4.2) in Lemma 4.2. The existence of solutions to (3.3)-(3.6) follow by showing uniqueness in Lemma 4.3.

**Lemma 4.1.** The unique solvability of (3.1)-(3.2) is equivalent to the unique solvability of the following problem: given $(c_h^{n-1}, u_h^{n-1}) \in M^k_h \times X_h$, compute $(y_h^n, w_h^n) \in M^k_{h0} \times M^k_{h0}$, such that for all $\bar{\chi}_h \in M^k_h$ and for all $\bar{\varphi}_h \in M^k_{h0}$

\begin{align}
(\delta_t y_h^n, \bar{\chi}_h) + a_{\text{diff}}(w_h^n, \bar{\chi}_h) + a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \bar{\chi}_h) &= 0, \\
(\Phi_+(y_h^n + \bar{c}_0) + \Phi_-(c_h^{n-1}), \bar{\varphi}_h) + \kappa a_{\text{diff}}(y_h^n, \bar{\varphi}_h) - (w_h^n, \bar{\varphi}_h) &= 0,
\end{align}

where $y_h^n = c_h^{n-1} - \bar{c}_0$.

**Proof.** Assume that the system (4.1)-(4.2) has a unique solution $(y_h^n, w_h^n)$. Let $f(y_h^n) = \Phi_+(y_h^n + \bar{c}_0) + \Phi_-(c_h^{n-1})$. Define $c_h^n = y_h^n + \bar{c}_0$ and $\mu_h^n = w_h^n + f(y_h^n)$. To see that $(c_h^n, \mu_h^n)$ solves (3.1)-(3.2), take any $\chi_h \in M^k_h$ and let $\bar{\chi}_h = \chi_h - \bar{\chi}_h \in M^k_{h0}$ in (4.1). Then, since $a_{\text{diff}}(z_h, q) = a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, q) = 0$ for any constant $q$ and any $z_h \in M^k_h$, we have by (4.1)

\begin{align}
(\delta_t y_h^n, \chi_h - \bar{\chi}_h) + a_{\text{diff}}(\mu_h^n, \chi_h) + a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \chi_h) &= 0, \quad \forall \chi_h \in M^k_h.
\end{align}

By Proposition 3.1 and the definitions of $c_h^n$ and $y_h^n$, we have

\begin{align}
(\delta_t y_h^n, \chi_h - \bar{\chi}_h) = (\delta_t c_h^n, \chi_h - \bar{\chi}_h) = (\delta_t c_h^{n-1}, \chi_h).
\end{align}

Hence, (3.1) is satisfied. Similarly, let $\bar{\phi}_h = \phi_h - \overline{\phi}_h$ for any $\phi_h \in M^k_h$. Since $w_h^n = \mu_h^n - f(y_h^n) \in M^k_{h0}$, we have

\begin{align}
(f(y_h^n), \bar{\phi}_h - \overline{\phi}_h) - (\mu_h^n - f(y_h^n), \bar{\phi}_h - \overline{\phi}_h) = (f(y_h^n), \phi_h) - (\mu_h^n, \phi_h).
\end{align}

This implies that (3.2) is satisfied. To see that this solution is unique, assume that there exists a different pair $(c_h^{1,n}, \mu_h^{1,n})$ that satisfies (3.1)-(3.2). Define $(y_h^{1,n}, w_h^{1,n}) = (c_h^{1,n} - \overline{c}_0, \mu_h^{1,n} - \overline{\mu}_h)$. This pair also solves (4.1)-(4.2) which is a contradiction. Hence, the solution to (3.1)-(3.2) is unique.

Conversely, assume that (3.1)-(3.2) has a unique solution $(c_h^n, \mu_h^n)$. Then, it is easy to see that $(y_h^n, w_h^n) = (c_h^n - \overline{c}_0, \mu_h^n - \overline{\mu}_h)$ solves (4.1)-(4.2). To show uniqueness, assume there is a different pair $(y_h^{1,n}, w_h^{1,n})$ which solves (4.1)-(4.2). Then, the pair $(c_h^{1,n}, \mu_h^{1,n}) = (y_h^{1,n} + \overline{c}_0, w_h^{1,n} + f(y_h^{1,n}))$ also solves (3.1)-(3.2). This provides a contradiction. Hence, the solution to (3.1)-(3.2) is unique.

**Lemma 4.2.** The system (4.1)-(4.2) is uniquely solvable for any fixed time step size $\tau$ and mesh size $h$.

**Proof.** For any $w_h \in M^k_{h0}$, let $y_h = y_h(w_h) \in M^k_{h0}$ be the unique function such that

\begin{align}
(\Phi_+(y_h + \overline{c}_0) + \Phi_-(c_h^{n-1}), \bar{\phi}) + \kappa a_{\text{diff}}(y_h, \bar{\phi}) - (w_h, \bar{\phi}) &= 0, \quad \forall \bar{\phi} \in M^k_{h0}.
\end{align}
Existence and uniqueness of such a $y_h(w_h)$ is proved in Lemma 3.14 of [21]. If we choose $\phi_h = y_h$ in Eq. (4.6) and use the coercivity property (3.15), we obtain

$$\langle \Phi_\ast(y_h + \xi_h), y_h \rangle + K_\alpha \| y_h \|_{DG}^2 \leq \langle w_h, y_h \rangle.$$  

By convexity of $\Phi_\ast$ and the fact that $\langle y_h, 1 \rangle = 0$, we have

$$\langle \Phi_\ast(y_h + \xi_h), y_h \rangle = \langle \Phi_\ast(\xi_h), y_h \rangle + \langle \Phi_\ast''(\xi_h), y_h^2 \rangle = \langle \Phi_\ast''(\xi_h), y_h^2 \rangle \geq 0.$$  

Therefore, (3.28) and Cauchy–Schwarz’s inequality yield

$$\| y_h(w_h) \|_{DG} \leq \frac{C^2}{K_\alpha} |w_h|_{DG} + \frac{C^3}{K_\alpha} \| \Phi_\ast'(c_h^{n-1}) \|.$$  

Let $M_{h0}^{k*}$ denote the dual space of $M_{h0}^{k}$ and let $\mathcal{F} : M_{h0}^{k} \to M_{h0}^{k*}$ be a mapping defined as follows: for all $\hat{\chi}_h$ in $M_{h0}^{k}$,

$$\langle \mathcal{F}(w_h), \hat{\chi}_h \rangle = \langle y_h(w_h) - y_h^{n-1}, \hat{\chi}_h \rangle + \tau a_{\text{diff}}(w_h, \hat{\chi}_h) + \tau a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \hat{\chi}_h).$$  

First, let us check the boundedness of $\mathcal{F}$. For any $\hat{\chi}_h \in M_{h0}^{k}$ with $|\hat{\chi}_h|_{DG} = 1$, by Cauchy–Schwarz’s inequality, (3.30) with $p = 2$, (3.18), (3.20), we have

$$|\langle \mathcal{F}(w_h), \hat{\chi}_h \rangle| \leq C_a \tau |w_h|_{DG} + \frac{C^2}{K_\alpha} (|y_h|_{DG} + |y_h^{n-1}|_{DG})$$

$$+ C_v \tau (|c_h^{n-1}|_{DG} + |\Omega|_{DG}) \| u_h^{n-1} \|_{DG}.$$  

Using (4.10) and (4.9), we have

$$\| \mathcal{F}(w_h) \|_{M_{h0}^{k}} \leq \left( C_a \tau + \frac{C^4}{K_\alpha} \right) |w_h|_{DG} + \frac{C^3}{K_\alpha} \| \Phi_\ast'(c_h^{n-1}) \|$$

$$+ \frac{C^2}{K_\alpha} |y_h^{n-1}|_{DG} + C_v \tau (|c_h^{n-1}|_{DG} + |\Omega|_{DG}) \| u_h^{n-1} \|_{DG}.$$  

Thanks to (4.11), we have shown that the operator $\mathcal{F}$ maps bounded sets in $M_{h0}^{k}$ to bounded sets in $M_{h0}^{k*}$, i.e., we have proved boundedness of the operator. Second, we show the coercivity of $\mathcal{F}$. With (3.15), (3.30) and (3.20), we have

$$\langle \mathcal{F}(w_h), w_h \rangle \geq \langle y_h, w_h \rangle + \tau |w_h|_{DG}^2 - C_v |y_h^{n-1}| |w_h|_{DG}$$

$$- C_v \tau (|c_h^{n-1}|_{DG} + |\Omega|_{DG}) \| u_h^{n-1} \|_{DG} |w_h|_{DG}.$$  

With (4.7), (4.8), (3.30) and Young’s inequality, we have

$$\langle y_h, w_h \rangle \geq K_\alpha \| y_h \|_{DG}^2 + \langle \Phi_\ast'(c_h^{n-1}), y_h \rangle \geq - \frac{C^2}{4K_\alpha} \| \Phi_\ast'(c_h^{n-1}) \|^2.$$  

Therefore we obtain

$$\langle \mathcal{F}(w_h), w_h \rangle \geq \tau |w_h|_{DG}^2 - C_v |y_h^{n-1}| |w_h|_{DG} - \frac{C^2}{4K_\alpha} \| \Phi_\ast'(c_h^{n-1}) \|^2$$

$$- C_v \tau (|c_h^{n-1}|_{DG} + |\Omega|_{DG}) \| u_h^{n-1} \|_{DG} |w_h|_{DG}.$$  

This implies coercivity of $\mathcal{F}$:

$$\lim_{|w_h|_{DG} \to +\infty} \frac{\langle \mathcal{F}(w_h), w_h \rangle}{|w_h|_{DG}} = +\infty.$$
Third, let us check the monotonicity of \( \mathcal{F} \). For any \( w_h \) and \( s_h \) in \( M_{h0}^k \), we have:

\[
(4.13) \quad \langle \mathcal{F}(w_h) - \mathcal{F}(s_h), w_h - s_h \rangle = (y_h(w_h) - y_h(s_h), w_h - s_h) + \tau a_{\text{diff}}(w_h - s_h, w_h - s_h).
\]

Due to the coercivity of \( a_{\text{diff}} \), the second term in the right-hand side is nonnegative, which means we only need to check the sign of the first term. From (4.6), we have, for any \( \phi_h \in M_{h0}^k \):

\[
(w_h - s_h, \phi_h) = (\Phi_+'(y_h(w_h) + \alpha_0) - \Phi_+'(y_h(s_h) + \alpha_0), \phi_h) + \kappa a_{\text{diff}}(y_h(w_h) - y_h(s_h), \phi_h).
\]

Choosing \( \phi_h = y_h(w_h) - y_h(s_h) \in M_{h0}^k \) and using (3.15) and the convexity of \( \Phi_+ \), we have

\[
(4.14) \quad (w_h - s_h, y_h(w_h) - y_h(s_h)) \geq K_a \kappa |y_h(w_h) - y_h(s_h)|_{DG}^2 \geq 0.
\]

Substituting (4.14) into (4.13), considering \( \cdot \|_{DG} \) is a norm in \( M_{h0}^k \), the following inequality is strict whenever \( w_h \neq s_h \),

\[
\langle \mathcal{F}(w_h) - \mathcal{F}(s_h), w_h - s_h \rangle \geq K_a \tau |w_h - s_h|^2_{DG} > 0.
\]

Thus we have established the strict monotonicity of \( \mathcal{F} \). Finally, let us check the continuity of \( \mathcal{F} \). For any \( \tilde{x}_h \in M_{h0}^k \) with \( |\tilde{x}_h|_{DG} = 1 \), by Cauchy–Schwarz’s inequality, (3.30), (3.18), we have

\[
(4.15) \quad |\langle \mathcal{F}(w_h) - \mathcal{F}(s_h), \tilde{x}_h \rangle| \leq C_a \tau |w_h - s_h|_{DG} + C_T^2 |y_h(w_h) - y_h(s_h)|_{DG}.
\]

To bound the second term in the right-hand side, we revert to (4.14) and use (3.30) to obtain:

\[
(4.16) \quad |y_h(w_h) - y_h(s_h)|_{DG} \leq \frac{C_T^2}{K_a \kappa} |w_h - s_h|_{DG}.
\]

Combining (4.15) and (4.16), we have

\[
\|\mathcal{F}(w_h) - \mathcal{F}(s_h)\|_{M_{h0}^k} = \sup_{\tilde{x}_h \in M_{h0}^k} \|\langle \mathcal{F}(w_h) - \mathcal{F}(s_h), \tilde{x}_h \rangle\| \leq \left( C_a \tau + \frac{C_T^2}{K_a \kappa} \right) |w_h - s_h|_{DG},
\]

which means \( \|\mathcal{F}(w_h) - \mathcal{F}(s_h)\|_{M_{h0}^k} \) tends to zero whenever \( |w_h - s_h|_{DG} \) tends to zero, i.e., we proved the continuity of the operator \( \mathcal{F} \). We can then apply the Browder–Minty theorem to conclude that there exists a unique solution \( w_h^n \) such that \( \langle \mathcal{F}(w_h^n), \tilde{x}_h \rangle = 0 \) for all \( \tilde{x}_h \in M_{h0}^k \). This implies that \( (y_h(w_h^n), y_h^n) \) is the unique solution of system (4.1)–(4.2).

**Lemma 4.3.** Given \( (c_h^{n-1}, \mu_h^n, \nu_h^{n-1}, \sigma_h^{n-1}, p_h^{n-1}, p_h^n) \in M_{h0}^k \times M_{h0}^k \times X_h^k \times X_h^k \times M_{h0}^{k-1} \), there exists a unique solution \( (\nu_h^n, u_h^n, p_h^n) \in X_h^k \times X_h^k \times M_{h0}^{k-1} \) to problem (3.3)-(3.6).

**Proof.** Let us first show that \( p_h^n \) belongs to \( M_{h0}^{k-1} \) by induction on \( n \). The statement trivially holds for \( n = 0 \). Assume \( p_h^{n-1} \in M_{h0}^{k-1} \). Then, take \( \chi_h = 1 \) in (3.5). Since \( \phi_h^n \) has zero average, we have

\[
\int_{\Omega} p_h^n = \int_{\Omega} p_h^{n-1} - \sigma_b \mu_h(b_p(\nu_h^n, 1) = 0.
\]
The last equality is obtained by applying the induction hypothesis and the fact that
\( b_\varphi(v, 1) = 0 \) for any \( v \). Second, let us show the existence of the intermediate velocity \( v^n_h \). Since this is a linear problem in finite dimensions, it suffices to show uniqueness of the solution. Suppose there exist two solutions \( v^n_h \) and \( \tilde{v}^n_h \) and let \( w^n_h = v^n_h - \tilde{v}^n_h \) denote the difference. Choosing \( \theta_h = w^n_h \) in (3.3) and using (3.22) and (3.16) yield
\[
\|w^n_h\|^2 + K_D \tau \mu_s \|w^n_h\|_{DG}^2 \leq 0.
\]
This implies that \( w^n_h = 0 \), which yields uniqueness and existence of \( v^n_h \). The existence of \( \phi^n_h \in M^{k-1}_{h0} \) follows by similar arguments, and the fact that \( |\cdot|_{DG} \) is a norm for the space \( M^{k-1}_{h0} \). Existence and uniqueness of \( p^n_h \) and \( u^n_h \) is trivial.

Lemma 4.1, Lemma 4.2 and Lemma 4.3 imply existence and uniqueness of a solution to problem (3.1)-(3.6).

5. Stability. We will prove that our scheme satisfies a discrete energy dissipation property in two steps. First we obtain the energy dissipation property under an assumption on the boundedness of the order parameter. Second, we use an induction argument to show that this assumption holds true. Define the discrete total energy at time \( t^n \) as follows.

\[
F_h(c^n_h, u^n_h) = \frac{1}{2} (u^n_h, u^n_h) + (\Phi(c^n_h), 1) + \frac{\kappa}{2} a_{diff}(c^n_h, c^n_h).
\]

We can bound the initial discrete energy by assuming enough regularity on the initial conditions and by using stability of the \( L^2 \) projection and elliptic projection.

\[
F_h(c^0_h, u^0_h) \leq \frac{1}{2} \|u^0\|^2 + C \|c^0\|^4_{H^1(\Omega)} + C \|c^0\|^2_{H^1(\Omega)} + C \leq C.
\]

We introduce auxiliary variables that play an important role in the analysis, similar variables were introduced in [25, 23]. Define \( S^0_h = 0 \) and \( \zeta^0_h = p^0_h \). For any \( 1 \leq n \leq N_T \), we define \( S^n_h \in M^{k-1}_{h0} \) and \( \zeta^n_h \in M^{k-1}_{h0} \) as follows:

\[
S^n_h = \sigma_h \mu_s \sum_{i=1}^{n} \left( \nabla_h \cdot v^i_h - R_h([v^i_h]) \right), \quad \zeta^n_h = p^n_h + S^n_h.
\]

We now show a discrete energy dissipation inequality.

**Theorem 5.1.** Assume that \( \tau \) and \( h \) satisfy the following CFL condition.

\[
\tau \leq \min \left( \frac{K_a}{8 C^2_p \max \left( \frac{2}{K_{\kappa}}, 2 \right) F_h(c^0_h, u^0_h) \Theta_{h,d}}, \frac{K_a}{8 s_0^2} \right),
\]

where \( \Theta_{h,d} \) is given (3.35). Fix \( n \geq 1 \) and assume that \( c^{n-1}_h \) satisfies the bound:

\[
|c^{n-1}_h|_{DG}^p \leq \max \left( \frac{2}{K_{\kappa}}, 2 \right) F_h(c^0_h, u^0_h).
\]
If $\sigma > M_{k-1}^2/d, \tilde{\sigma} > \tilde{M}_k^2$, and $\sigma_\chi < K_D/(2d)$, the discrete dissipation inequality holds:

\begin{equation}
F_h(c_h^n, u_h^n) + \frac{\tau}{2\sigma\mu_h} \|S_h^n\|^2 + \frac{\tau^2}{2} a_{\text{diff}}(\zeta_h^n, c_h^n) \\
- F_h(c_h^{n-1}, u_h^{n-1}) - \frac{\tau}{2\sigma\mu_h} \|S_h^{n-1}\|^2 - \frac{\tau^2}{2} a_{\text{diff}}(c_h^{n-1}, c_h^{n-1}) \\
\leq - \frac{K_a \tau}{2} |\mu_h^n|^2_{DG} - \frac{K_D \mu_a \tau}{2} \|v_h^n\|^2_{DG} - \frac{1}{4} \|v_h^n - u_h^{n-1}\|^2.
\end{equation}

**Proof.** Fix $n \geq 1$ and take $\chi_h = \mu_h^n$ in (3.1), $q_h = \delta \tau c_h^n$ in (3.2), $\theta_h = v_h^n$ in (3.3)

\begin{align*}
& (\delta \tau c_h^n, \mu_h^n) + a_{\text{diff}}(\mu_h^n, \mu_h^n) + a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \mu_h^n) = 0, \\
& (\Phi_+ c_h^n, \Phi_- (c_h^{n-1}), \delta \tau c_h^n) + \kappa a_{\text{diff}}(c_h^n, \delta \tau c_h^n) - (\mu_h^n, \delta \tau c_h^n) = 0, \\
& \frac{1}{\tau} (v_h^n - u_h^{n-1}, v_h^n) + a(c_h^{n-1}, u_h^{n-1}, v_h^n, v_h^n) + \mu_a a_D(v_h^n, v_h^n) \\
& \quad = b\Phi(v_h^n, p_h^{n-1}) + b f(c_h^{n-1}, \mu_h^n, v_h^n).
\end{align*}

Adding the equations above, and using (3.22), (3.15) and (3.16), we have

\begin{equation}
\frac{1}{\tau} (v_h^n - u_h^{n-1}, v_h^n) + (\Phi_+ c_h^n, \Phi_- (c_h^{n-1}), \delta \tau c_h^n) + \kappa a_{\text{diff}}(c_h^n, \delta \tau c_h^n) \\
+ K_a |\mu_h^n|^2_{DG} + K_D \mu_a \|v_h^n\|^2_{DG} = b\Phi(v_h^n, p_h^{n-1}) - a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \mu_h^n) + b f(c_h^{n-1}, \mu_h^n, v_h^n).
\end{equation}

With Taylor’s expansions and the fact that $\Phi_+$ is convex and $\Phi_-$ is concave, we have for some $\xi_h$ and $\eta_h$ between $c_h^{n-1}$ and $c_h^n$

\begin{equation}
(\Phi_+ c_h^n, \Phi_- (c_h^{n-1}), \delta \tau c_h^n) (\delta \tau, \Phi(c_h^n), 1) + \frac{1}{2\tau} (\Phi_+''(\xi_h), (c_h^{n-1} - c_h^n)^2)
\end{equation}

$$= -\frac{1}{2\tau} (\Phi_-''(\eta_h), (c_h^n - c_h^{n-1})^2) \geq (\delta \tau, \Phi(c_h^n), 1).$$

With (5.7) and the symmetry of $a_{\text{diff}}$, we obtain

\begin{align*}
& \frac{1}{2\tau} \|v_h^n\|^2 - \frac{1}{2\tau} \|u_h^{n-1}\|^2 + \frac{1}{2\tau} \|v_h^n - u_h^{n-1}\|^2 + (\delta \tau, \Phi(c_h^n), 1) \\
& + \frac{K_a}{2\tau} a_{\text{diff}}(c_h^n, c_h^n) - \frac{K_a}{2\tau} a_{\text{diff}}(c_h^{n-1}, c_h^{n-1}) + \frac{K_a}{2\tau} a_{\text{diff}}(c_h^n - c_h^{n-1}, c_h^n - c_h^{n-1}) \\
& + K_a |\mu_h^n|^2_{DG} + K_D \mu_a \|v_h^n\|^2_{DG} \leq b\Phi(v_h^n, p_h^{n-1}) - a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \mu_h^n) + b f(c_h^{n-1}, \mu_h^n, v_h^n).
\end{align*}

Next, we choose $\theta_h = u_h^n$ in (3.6)

\begin{equation}
\frac{1}{2} \|u_h^n\|^2 - \frac{1}{2} \|v_h^n\|^2 + \frac{1}{2} \|u_h^n - v_h^n\|^2 = \tau b\Phi(u_h^n, \phi_h^n).
\end{equation}

Let $\theta_h = u_h^n - v_h^n$ in (3.6), then

\begin{equation}
\|u_h^n - v_h^n\|^2 = \tau b\Phi(u_h^n - v_h^n, \phi_h^n).
\end{equation}

Therefore (5.9) becomes

\begin{equation}
\frac{1}{2} \|u_h^n\|^2 - \frac{1}{2} \|v_h^n\|^2 - \frac{\tau}{2} b\Phi(v_h^n, \phi_h^n) = \frac{\tau}{2} b\Phi(u_h^n, \phi_h^n).
\end{equation}
We can also show (see Lemma 5.1 in [23])

\[ (5.12) \quad b_p(u^n_h, q_h) = -\tau \sum e \in e \Delta K e \int \Phi^n_h \left( q_h \right) + \tau \left( G_h(\phi^n_h), G_h(q_h) \right), \quad \forall q_h \in M^{k-1}_h. \]

Using (3.4) (with \( q_h = \phi^n_h \)) and choosing \( q_h = \phi^n_h \) in (5.12), we rewrite (5.11) as

\[ (5.13) \quad \frac{1}{2} \| u^n_h \|^2 - \frac{1}{2} \| v^n_h \|^2 + \frac{\tau}{2} a_{\text{diff}}(\phi^n_h, \phi^n_h) + \frac{\tau^2}{2} \sum e \in e \Delta K e \| \phi^n_h \|_L_2^2(c) = \frac{\tau^2}{2} \| G_h(\phi^n_h) \|^2. \]

With (3.24), we have

\[ (5.14) \quad \frac{1}{2\tau} \| u^n_h \|^2 - \frac{1}{2\tau} \| v^n_h \|^2 + \frac{\tau}{2} a_{\text{diff}}(\phi^n_h, \phi^n_h) + \tau \frac{\tilde{\sigma}}{2\tau} \sum e \in e \Delta K e \| \phi^n_h \|_L_2^2(c) \leq 0. \]

Adding (5.8) and (5.14) and choosing \( \tilde{\sigma} > \tilde{M}_h^2 \) yields

\[ (5.15) \quad \frac{1}{2\tau} \| u^n_h \|^2 - \frac{1}{2\tau} \| u^{n-1}_h \|^2 + \frac{1}{2\tau} \| v^n_h - u^{n-1}_h \|^2 + \left( \delta_1, \Phi(c^n_h), 1 \right) \]
\[ + \frac{k}{2} a_{\text{diff}}(c^n_h, c^n_h) - \frac{k}{2} a_{\text{diff}}(c^{n-1}_h, c^{n-1}_h) + \frac{k}{2} a_{\text{diff}}(c^n_h - c^{n-1}_h, c^n_h - c^{n-1}_h) \]
\[ + \frac{k}{4} a_{\text{diff}}(\phi^n_h, \phi^n_h) \]
\[ \leq b_p(v^n_h, p^{n-1}_h) - a_{\text{adv}}(c^{n-1}_h, u^{n-1}_h, \mu^{n-1}_h) + b_f(c^{n-1}_h, \mu^{n-1}_h, v^{n}_h). \]

To proceed with the term \( b_p(v^n_h, p^{n-1}_h) \), we rewrite it using the variables \( S^n_h \) and \( \zeta^n_h \).

\[ (5.16) \quad b_p(v^n_h, p^{n-1}_h) = b_p(v^n_h, \zeta^{n-1}_h) - b_p(v^n_h, S^{n-1}_h). \]

Using (5.2), (3.5) and (3.25), we note that

\[ (5.17) \quad \zeta^n_h - \zeta^{n-1}_h = p^n_h - p^{n-1}_h + \sigma \chi_h \mu_s(\nabla_h v^n_h - R_h[v^n_h]) = \phi^n_h. \]

Therefore, by (3.4), we have

\[ (5.18) \quad b_p(v^n_h, \zeta^{n-1}_h) = -\tau a_{\text{diff}}(\phi^n_h, \zeta^{n-1}_h) = -\tau a_{\text{diff}}(\zeta^n_h - \zeta^{n-1}_h, \zeta^{n-1}_h) 
- \frac{\tau}{2} a_{\text{diff}}(\zeta^n_h, \zeta^n_h) + \frac{\tau}{2} a_{\text{diff}}(\zeta^{n-1}_h, \zeta^{n-1}_h) + \frac{\tau}{2} a_{\text{diff}}(\phi^n_h, \phi^n_h). \]

For the second term of the right-hand side of (5.16), by (5.2) and (3.25), we have

\[ (5.19) \quad b_p(v^n_h, S^{n-1}_h) = (\nabla_h v^n_h - R_h[v^n_h], S^{n-1}_h) = \frac{1}{\sigma \chi_h \mu_s} (S^n_h - S^{n-1}_h, S^{n-1}_h) = \frac{1}{2\sigma \chi_h \mu_s} (\| S^n_h \|^2 - \| S^{n-1}_h \|^2 - \| S^n_h - S^{n-1}_h \|^2). \]

Substitute (5.18) and (5.19) into (5.16) to obtain

\[ (5.20) \quad b_p(v^n_h, p^{n-1}_h) = -\frac{\tau}{2} a_{\text{diff}}(\zeta^n_h, \zeta^n_h) + \frac{\tau}{2} a_{\text{diff}}(\zeta^{n-1}_h, \zeta^{n-1}_h) + \frac{\tau}{2} a_{\text{diff}}(\phi^n_h, \phi^n_h) 
- \frac{1}{2\sigma \chi_h \mu_s} (\| S^n_h \|^2 - \| S^{n-1}_h \|^2 - \| S^n_h - S^{n-1}_h \|^2). \]
In addition, if we choose parameters $\sigma_\chi$ and $\sigma$ such that $\sigma_\chi < K_\partial/(2d)$ and $\sigma > M_{k-1}^2/d$, we have
\begin{equation}
\frac{1}{2\sigma_\chi \mu_s} \|S_h^n - S_h^{n-1}\|^2 \leq \frac{K_\partial \mu_s}{2} \|\psi_h^n\|_{DG}^2. \tag{5.21}
\end{equation}
Thus, with (3.12), (5.20), and (5.21), the (5.15) becomes
\begin{equation}
\frac{1}{2\tau} \|\psi_h^n\|^2 - \frac{1}{2\tau} \|\psi_h^{n-1}\|^2 + \|\psi_h^n - \psi_h^{n-1}\|^2 + (\delta_t \Phi(c_h^n), 1) + \frac{K}{2\tau} a_{diff}(c_h^n, c_h^{n-1}) + \frac{\kappa}{2\tau} a_{diff}(c_h^n - c_h^{n-1}, c_h^n - c_h^{n-1}) + K_\partial |\mu_h^n|_{DG}^2 - \frac{K_\partial \mu_s}{2} \|\psi_h^n\|_{DG}^2 + \frac{\tau}{2} a_{diff}(c_h^n, c_h^n - c_h^{n-1}) - \frac{\tau}{2} a_{diff}(c_h^n, c_h^n - c_h^{n-1}) + \frac{1}{2\sigma_\chi \mu_s} \|S_h^{n-1}\|^2 - \frac{1}{2\sigma_\chi \mu_s} \|S_h^{n-1}\|^2 \leq a_{adv}(c_h^{n-1}, \psi_h^n, -\psi_h^{n-1}, \mu_h^n). \tag{5.22}
\end{equation}
Using the definition of $a_{adv}$, Holder’s inequality, (3.33), and the fact that $\tilde{\sigma} > \tilde{M}_k^2$ and $\tilde{M}_k = \sqrt{2} C_n N_{\text{face}}^{1/2}$, we have
\begin{equation}
a_{adv}(c_h^{n-1}, \psi_h^n, -\psi_h^{n-1}, \mu_h^n) \leq \|c_h^{n-1}\|_{L^\infty(\Omega)} \|\psi_h^n - \psi_h^{n-1}\|_{L^2(\Omega)} \|\mu_h^n\|_{DG}. \tag{5.23}
\end{equation}
Using (3.36), Young’s inequality, (3.27), and (5.4), we obtain
\begin{equation}
a_{adv}(c_h^{n-1}, \psi_h^n, -\psi_h^{n-1}, \mu_h^n) \leq \frac{1}{4\tau} \|\psi_h^n - \psi_h^{n-1}\|^2 + \tau \|c_h^{n-1}\|_{L^2(\Omega)}^2 \|\mu_h^n\|_{DG}^2 \leq \frac{1}{4\tau} \|\psi_h^n - \psi_h^{n-1}\|^2 + \tau \left(2\tilde{C}_\partial \Theta_{h,d}^2 |c_h^{n-1}|_{DG}^2 + 2\tau_0\right) \|\mu_h^n\|_{DG}^2 \leq \frac{1}{4\tau} \|\psi_h^n - \psi_h^{n-1}\|^2 + \tau \left(2\tilde{C}_\partial \Theta_{h,d}^2 max\left(\frac{2}{K_\partial \kappa}, 2\right) F_h(c_h^0, u_h^0) + 2\tau_0\right) \|\mu_h^n\|_{DG}^2. \tag{5.24}
\end{equation}
Substitute (5.24) into the right-hand side of (5.22). The condition (5.3) implies
\begin{equation}
\tau \left(2\tilde{C}_\partial \Theta_{h,d}^2 max\left(\frac{2}{K_\partial \kappa}, 2\right) F_h(c_h^0, u_h^0) + 2\tau_0\right) \leq \frac{K_\partial \mu_s}{2}. \tag{5.25}
\end{equation}
Then, multiply by $\tau$. The result follows.

Remark 5.2. In 3D, as $h$ tends to 0, the CFL constraint (5.3) simply reads $\tau \leq Ch$.

In 2D, the CFL constraint is milder: $\tau \leq C(1 + |\ln h|)^{-1}$.

Theorem 5.3. Assume that $\tau$ and $h$ satisfy the CFL condition (5.3). If $\sigma > M_{k-1}^2/d$, $\bar{\sigma} > M_k^2$ and $\sigma_\chi < K_\partial/(2d)$, for any $1 \leq l \leq N_T$, we have
\begin{equation}
\frac{1}{2} \|u_h^0\|^2 + (\Phi(c_h^0), 1) + \frac{K_\partial \mu_s}{2} \|S_h^l\|_{DG}^2 + \frac{\tau}{2\sigma_\chi \mu_s} \|S_h^l\|^2 + \frac{\tau_0^2}{2} a_{diff}(c_h^l, c_h^l) \leq \frac{K_\partial \mu_s}{2} \|\psi_h^n\|_{DG}^2 + \frac{K_\partial \mu_s}{2} \|\psi_h^n\|_{DG}^2 + \frac{\tau_0}{2} a_{diff}(c_h^0, c_h^0) + \frac{K_\partial \mu_s}{2} \|\psi_h^n\|_{DG}^2. \tag{5.26}
\end{equation}

Proof. To prove (5.26) we use an induction argument on $l$. The positivity of the chemical energy density and (3.15) imply
\begin{equation}
F_h(c_h^0, u_h^0) = \frac{1}{2} (u_h^0, u_h^0) + (\Phi(c_h^0), 1) + \frac{\kappa}{2} a_{diff}(c_h^0, c_h^0) \geq \frac{K_\partial \mu_s}{2} |c_h^0|^2. \tag{5.27}
\end{equation}
Therefore, assumption (5.4) holds for $n = 1$. We apply Theorem 5.1 to obtain (5.5) for $n = 1$, which implies (5.26) for $\ell = 1$ since $S_0^0 = C_0^0 = 0$.

Fix $j \geq 1$ and assume that (5.26) holds for all $1 \leq \ell \leq j$. This means that (5.4) is valid for all $1 \leq n \leq j + 1$. With Theorem 5.1, we have that (5.5) is valid for any $1 \leq n \leq j + 1$. Summing (5.5) over $n$ yields (5.26) for $\ell = j + 1$.

Remark 5.4. Using a triangle inequality, Poincare's inequality (3.28), and mass conservation (3.27), stability bound (5.26) implies that for any $p \leq 6$

\begin{equation}
\|c^\ell_h\|_{L^p(\Omega)} \leq C, \quad 1 \leq \ell \leq N_T.
\end{equation}

6. Error analysis. In the remainder of the paper, we assume that $\Omega$ is convex. The goal of this section is to show the following convergence result.

**Theorem 6.1.** Assume that $\sigma \geq \tilde{M}^2_{k-1}/d$, $\tilde{\sigma} \geq 4\tilde{M}^2_k$, and $\sigma_\gamma \leq K_D/(2d)$. Fix $0 < \delta < 1$. There exist constants $\gamma, C_{\text{err}}, h_0, \tau_0$ independent of $h$ and $\tau$, such that if $h \leq h_0, \tau \leq \tau_0$ and

\begin{equation}
\tau \leq \gamma h^{1+\delta},
\end{equation}

then the following error estimate holds. For $1 \leq m \leq N_T$,

\begin{equation}
K_a \sum_{n=1}^m \mu^n_h - \mu^n u^{n}_{\text{DG}} + K_D \mu \tau \sum_{n=1}^m \|v^n_h - u^n\|_{\text{DG}}^2 + \tau \sum_{n=1}^m \|\phi^n_h\|_{\text{DG}}^2 
+ \kappa K_a \|c^n_h - c^m\|_{\text{DG}}^2 + \|u^n_h - u^m\| \leq C_{\text{err}}(\tau + h^{2k}).
\end{equation}

In addition, there exists a constant $\tilde{C}_{\text{err}}$ independent of $h, \tau$, such that the following improved estimate holds. For $1 \leq m \leq N_T$,

\begin{equation}
\|c^n_h - c^m\|^2 + \tau \sum_{n=1}^m \|\mu^n_h - \mu^n\|^2 
+ \mu \tau \sum_{n=1}^m (\|v^n_h - u^n\|^2 + \|u^n_h - u^n\|^2) \leq \tilde{C}_{\text{err}}(\tau^2 + \tau h^2 + h^{2k+2}).
\end{equation}

The above estimates hold under the following regularity assumptions: $\nabla c^0 \cdot n = 0$ on $\partial \Omega$ and

\begin{align*}
c, \mu &\in L^\infty(0, T; H^{k+1}(\Omega)), \quad \partial_t c \in L^2(0, T; H^{k+1}(\Omega)), \quad \partial_t \mu \in L^2(0, T; L^2(\Omega)), \\
u &\in L^\infty(0, T; H^{k+1}(\Omega)^d), \quad \partial_t u \in L^2(0, T; H^{k+1}(\Omega)^d), \quad p \in L^\infty(0, T; H^k(\Omega)).
\end{align*}

Remark 6.2. Hereinafter, we use an induction argument to prove (6.2). In each induction iteration, the constants $\gamma, C_{\text{err}}, h_0, \tau_0$ are unchanged. Therefore, the algorithm (3.1)-(3.6) is suited in simulations with any prescribed end time $T$.

Remark 6.3. The bound (6.3) is optimal for $k = 1$ since $\tau h^2 \leq (\tau^2 + h^4)/2$. For $k \geq 2$, the reverse CFL condition "$h^2 \leq \tau^{\frac{1}{k}}"$ is required for optimality.

**Proof Outline:** Since the proof of this result requires several intermediate Lemmas, we provide a brief outline here. The proof of (6.2) is in Section 6.4 and it is a consequence of the following bound, valid for $1 \leq m \leq N_T$:

\begin{equation}
\tau^2 \sum_{n=0}^{m-1} \|\phi^n_h\|_{\text{DG}}^2 + \tau \sum_{n=0}^{m-1} \|v^n_h - \Pi_h u^n\|_{\text{DG}}^2 \leq \tau \frac{1}{1+\tau} + h^{2k+2}.
\end{equation}
where $\Pi_h$ is a suitable interpolant, see (6.5). We will show (6.4) by induction on $m$. For the starting value $m = 1$, the bound (6.4) is easy to obtain since $\phi_h^0 = 0$ and $v_h^0 = u_h^0$ which is the $L^2$ projection of $u^0$.

Next, we fix $m \geq 1$ and assume that the induction hypothesis (6.4) holds true. To show (6.2), our induction argument contains the following steps:

(i) The induction hypothesis (6.4) implies an $L^\infty(\Omega)$ bound for the discrete order parameter $c_h^m$ (see Lemma 6.7).

(ii) We then obtain a bound on the dG norm of $c_h^m - P_h c^m$, where $P_h$ is a suitable projection (see Section 6.1). This bound is proved in Lemma 6.9 and uses Lemma 6.7.

(iii) A bound on the dG norm of $\mu_h^m - P_h \mu^m$ easily follows (see Lemma 6.10).

(iv) We then show that (6.2) holds true (see Lemma 6.11 and Section 6.4 for more details).

(v) We complete the induction proof by then showing the induction hypothesis for $m + 1$ (see Lemma 6.12).

Finally, to show (6.3), we use several duality arguments and (6.2). The main proof of (6.3) is provided in subsection 6.5.1.

6.1. Approximation operators. As a prelude to showing the above steps, we introduce the several approximations of the exact solution that are employed in the error analysis. Let $P_h : H^2(\mathcal{T}_h) \to M_h^k$ be the elliptic projection operator. For $\phi \in H^2(\mathcal{T}_h)$, define $P_h \phi$ as the solution of the following problem.

(6.5) \[ a_{\text{diff}}(P_h \phi - \phi, \chi_h) = 0, \quad \forall \chi_h \in M_h^k, \quad \int_\Omega (P_h \phi - \phi) = 0. \]

We have the following error bounds which can be derived from the dG error analysis for elliptic problems on convex domains [26].

(6.6) \[ \|\phi - P_h \phi\| + h\|\phi - P_h \phi\|_{\text{DG}} \leq Ch^{k+1}\|\phi\|_{H^{k+1}(\mathcal{T}_h)}, \quad \forall \phi \in H^{k+1}(\mathcal{T}_h). \]

For functions in $H^1(\mathcal{T}_h)^d$, we will make use of the operator $\Pi_h : H^1(\mathcal{T}_h)^d \to X_h^k$. For $u(t) \in H^1(\mathcal{T}_h)^d$, this operator satisfies the following:

(6.7) \[ b_{PF}(\Pi_h u(t) - u(t), q_h) = 0, \quad \forall q_h \in M_h^{k-1}. \]

The proof for the existence of this operator and the following approximation bounds can be found in [5].

**Lemma 6.4.** For $E \in \mathcal{T}_h$, $1 \leq p \leq \infty$, $1 \leq s \leq k + 1$, $0 \leq n \leq N_T$, and $u(t) \in (W^{s,p}(E) \cap H^1(D))^d$,

(6.8) \[ \|\Pi_h u(t) - u(t)\|_{L^p(E)} + \|E\|_{L^p(E)} \|\nabla (\Pi_h u(t) - u(t))\|_{L^p(E)} \leq C h^p \|u(t)\|_{W^{s,p}(E)}. \]

where $\Delta E$ is a macro element that contains $E$.

For $0 \leq t \leq T$, if $u(t) \in (W^{s,p}(\Omega) \cap H^1_0(\Omega))^d$ for $1 \leq s \leq k + 1$, then the above bound yields the global estimates:

(6.9) \[ \|\Pi_h u(t) - u(t)\|_{L^p(\Omega)} \leq C h^p \|u(t)\|_{W^{s,p}(\Omega)}, \]

(6.10) \[ \|\Pi_h u(t) - u(t)\|_{\text{DG}} \leq C h^{s-1} \|u(t)\|_{H^s(\Omega)}. \]
Define the $L^2$ projection $\pi_h : L^2(\Omega) \rightarrow M_h^k$ as follows: For $0 \leq n \leq N_T$, a given function $p(t) \in L^2(\Omega)$, and any $E \in T_h$,

\begin{equation}
(6.11) \quad \int_E (\pi_h p(t) - p(t)) q_h = 0, \quad \forall q_h \in P_{k-1}(E).
\end{equation}

The following error bound for the $L^2$ projection holds. For $t \in [0, T]$ and $p(t) \in H^s(\Omega)$,

\begin{equation}
(6.12) \quad \|\pi_h p(t) - p(t)\| + h\|\nabla p(\pi_h p(t) - p(t))\| \leq Ch^\min(k,s)\|p(t)\|_{H^s(\Omega)}.
\end{equation}

We will also make use of a linear operator $\mathcal{J} : M_{h_0}^k \rightarrow M_{h_0}^k$. Given $\chi_h \in M_{h_0}^k$, define $\mathcal{J}(\chi_h) \in M_{h_0}^k$ as the solution of the following elliptic problem:

\begin{equation}
(6.13) \quad a_{\text{diff}}(\mathcal{J}(\chi_h), \varphi_h) = (\chi_h, \varphi_h), \quad \forall \varphi_h \in M_{h_0}^k.
\end{equation}

The following property for this operator is shown in [21].

\textbf{Lemma 6.5.} For a function $\lambda \in H^1(T_h)$, there exists a constant $C_1$ independent of $h$ such that

\begin{equation}
(6.14) \quad |(\lambda, \chi_h)| \leq C_1 |\mathcal{J}(\chi_h)|_{DG} |\lambda|_{DG}, \quad \forall \chi_h \in M_{h_0}^k.
\end{equation}

We introduce a discrete Laplacian operator $\Delta_h : M_h^k \rightarrow M_h^k$ via the following variational problem: Given $z_h \in M_h^k$, find $\Delta_h z_h \in M_h^k$ such that

\begin{equation}
(6.15) \quad (\Delta_h z_h, \chi_h) = -a_{\text{diff}}(z_h, \chi_h), \quad \forall \chi_h \in M_h^k.
\end{equation}

We now show discrete broken versions to the following Agmon’s and Gagliardo–Nirenberg inequalities in $d = 2$ and $d = 3$. For $z \in H^2(\Omega)$,

\begin{align}
(6.16) \quad & \|z\|_{L^\infty(\Omega)} \leq C\|z\|^{|1-d/4|}\|z\|_{H^2(\Omega)}^{1/4}, \\
(6.17) \quad & \|\nabla z\|_{L^2(\Omega)} \leq C\|z\|^{|1/2-d/12|}\|z\|_{H^2(\Omega)}^{1/2+d/12}.
\end{align}

For a proof for the above inequalities, we refer to Theorem 5.8 in [1] for (6.16) and to Lecture 2 in [24] for (6.17). The proof of the following Lemma follows the arguments in [16] presented for $d = 2$. For completeness, we provide a proof here for $d \in \{2, 3\}$.

\textbf{Lemma 6.6.} There exists a constant $C$ independent of $h$ such that

\begin{align}
(6.18) \quad & \|z_h - \frac{1}{|\Omega|} \int_\Omega z_h\|_{L^\infty(\Omega)} \leq C\|z_h\|^{1-d/4}\|\Delta_h z_h\|^{d/4}, \quad \forall z_h \in M_h^k, \\
(6.19) \quad & \|\nabla z_h\|_{L^2(\Omega)} \leq C\|z_h\|^{1/2-d/12}\|\Delta_h z_h\|^{1/2+d/12}, \quad \forall z_h \in M_h^k.
\end{align}

\textbf{Proof.} We first define a Green’s operator $\mathcal{G} : M_{h_0}^k \rightarrow H^1(\Omega) \cap L_0^2(\Omega)$ as the solution to $-\Delta(\mathcal{G}(\varphi_h)) = \varphi_h$ in $\Omega$ with homogeneous Neumann boundary conditions. By elliptic regularity, we have

\begin{equation}
(6.20) \quad \|\mathcal{G}(\varphi_h)\|_{H^2(\Omega)} \leq C\|\varphi_h\|, \quad \varphi_h \in M_{h_0}^k.
\end{equation}

Since $a_{\text{diff}}$ is symmetric and $\Omega$ is convex, we have [26]

\begin{equation}
(6.21) \quad \|\mathcal{J}(\varphi_h) - \mathcal{G}(\varphi_h)\| \leq Ch^2\|\mathcal{G}(\varphi_h)\|_{H^2(\Omega)}, \quad \varphi_h \in M_{h_0}^k.
\end{equation}
To simplify notation, for \( z_h \in M_h^k \), let \( \xi_h = \Delta_h z_h \). From the definitions, it can be readily deduced that (see (2.18) in [16])

\[
\mathcal{F}(\xi_h) = z_h - \frac{1}{|\Omega|} \int_{\Omega} z_h.
\]

Let \( \mathcal{I}_h \) denote the Scott–Zhang interpolation operator. Using the approximation properties of the Scott–Zhang operator, (3.31) (with \( p = +\infty \) and \( q = 2 \), (6.16) and (6.21),

\[
\| \mathcal{F}(\xi_h) \|_{L^\infty(\Omega)} \leq \| \mathcal{F}(\xi_h) - \mathcal{I}_h(\mathcal{G}(\xi_h)) \|_{L^\infty(\Omega)} + \| \mathcal{I}_h(\mathcal{G}(\xi_h)) - \mathcal{G}(\xi_h) \|_{L^\infty(\Omega)} + \| \mathcal{G}(\xi_h) \|_{L^\infty(\Omega)}
\]

(6.23)

\[
\leq C h^{2+d/2} \| \mathcal{G}(\xi_h) \|_{H^2(\Omega)} + C \| \mathcal{G}(\xi_h) \|_{H^2(\Omega)}^{1-d/4} \| \mathcal{G}(\xi_h) \|_{H^2(\Omega)}^{d/4}.
\]

Note that with triangle inequality, (6.21) and (6.22),

(6.24)

\[
\| \mathcal{G}(\xi_h) \| \leq 2 \| z_h \| + Ch^2 \| \mathcal{G}(\xi_h) \|_{H^2(\Omega)}.
\]

Using the above bound, (6.20), and the definition of \( \xi_h \) in (6.23) yields

(6.25)

\[
\| \mathcal{F}(\xi_h) \|_{L^\infty(\Omega)} \leq C \| z_h \|^{1-d/4} \| \Delta_h z_h \|^{d/4} + Ch^{2+d/2} \| \Delta_h z_h \|
\]

\[
= C \| z_h \|^{1-d/4} \| \Delta_h z_h \|^{d/4} + C(h^2 \| \Delta_h z_h \|^{1-d/4} \| \Delta_h z_h \|^{d/4}.
\]

Taking \( \chi_h = \Delta_h z_h \) in (6.15) and using (3.18) and (3.32) yields

(6.26)

\[
\| \Delta_h z_h \|^2 = a_{\text{diff}}(z_h, \Delta_h z_h) \leq C \| z_h \|_{DG} \| \Delta_h z_h \|_{DG} \leq Ch^{-2} \| z_h \| \| \Delta_h z_h \|.
\]

Using (6.26) and (6.22) in (6.25) proves (6.18). To show (6.19), we proceed in a similar way. With Holder’s inequality, (3.31), (3.32), approximation properties, and (6.21):

\[
\| \nabla \mathcal{F}(\xi_h) \|_{L^3(\Omega)} \leq \| \nabla \mathcal{G}(\xi_h) \|_{L^3(\Omega)} + \| \nabla \mathcal{G}(\xi_h) - \mathcal{I}_h(\mathcal{G}(\xi_h)) \|_{L^3(\Omega)} + \| \mathcal{I}_h(\mathcal{G}(\xi_h)) - \mathcal{F}(\xi_h) \|_{L^3(\Omega)}
\]

\[
\leq C \| \mathcal{G}(\xi_h) \|_1^{12-d/12} \| \mathcal{G}(\xi_h) \|_{H^2(\Omega)}^{1-d/12} + Ch^{1-d/6} \| \mathcal{G}(\xi_h) \|_{H^2(\Omega)}
\]

\[
+ Ch^{-d/6} \| \mathcal{I}_h(\mathcal{G}(\xi_h)) - \mathcal{F}(\xi_h) \|
\]

\[
\leq C \| \mathcal{G}(\xi_h) \|_1^{12-d/12} \| \mathcal{G}(\xi_h) \|_{H^2(\Omega)}^{1-d/12} + Ch^{1-d/6} \| \mathcal{G}(\xi_h) \|_{H^2(\Omega)}.
\]

Using (6.24), (6.20), and (6.26) yields

(6.27)

\[
\| \nabla \mathcal{F}(\xi_h) \|_{L^3(\Omega)} \leq C \| z_h \|^{1-d/12} \| \Delta_h z_h \|^{12+d/12}
\]

\[
+ C(h^2 \| \Delta_h z_h \|^{1/2-d/12} \| \Delta_h z_h \|^{1/2+d/12} \| \Delta_h z_h \|^{1/2-d/12} \| \Delta_h z_h \|^{1/2+d/12} \| \Delta_h z_h \|^{1/2+d/12} \leq C \| z_h \|^{1/2-d/12} \| \Delta_h z_h \|^{1/2+d/12}.
\]

The result is concluded by recalling (6.22).

We end this section by stating the consistency properties of our scheme. For readability, for any function \( g \in L^1(0, T; H^2(\Omega)^d) \) we denote \( g^n = g(t^n) \) and use a similar notation for scalar functions. The weak solution of model problem (2.1) satisfies the following. For any \( 1 \leq n \leq N_T \), for any \( \chi_h \in M_h^k, \phi_h \in M_h^k \), and \( \theta_h \in X_h^k \),

(6.28)

\[
((\partial t)^n, \chi_h) + a_{\text{diff}}(\mu^n, \chi_h) + a_{\text{adv}}(c^n, u^n, \chi_h) = 0,
\]

(6.29)

\[
(\Phi, (c^n) + \Phi_c, (c^n), \phi_h) + a_{\text{diff}}(c^n, \phi_h) - (\mu^n, \phi_h) = 0,
\]

(6.30)

\[
((\partial t)^n, \theta_h) + a_C(u^n, u^n, u^n, \theta_h) + \mu_s a_D(u^n, \theta_h)
\]

\[
= b_p(\theta_h, p^n) + b_T(c^n, \mu^n, \theta_h).
\]
6.2. The $L^\infty$ bound (Step (i)).

**Lemma 6.7.** Fix $m$, with $1 \leq m \leq N_T$, and assume that (6.4) holds. In addition, assume that $\tau$ satisfies (6.1). Then, there exists a constant $C$ independent of $h$, $\tau$, and $m$, but depending linearly on $T$, such that

$$
\max_{1 \leq n \leq m} \left( \|\mu^n_h\|^2 + \|c^n_h\|^2 + \|\nabla c^n_h\|^2_{L^\infty(\Omega)} \right) + \kappa \tau \sum_{n=1}^{m} \|\delta c^n_h\|^2 \leq C.
$$

**Proof.** Let $1 \leq n \leq m$. From (3.27), (6.18), (6.19) and Young’s inequality (of the form $ab \leq (1/p)a^p + (1/q)b^q$ for $p > 1, q > 1$ and $1/p + 1/q = 1$), we have

$$
\|c^n_h - \overline{c}_n\|_{L^\infty(\Omega)} \leq C\|c^n_h\|^{1-d/4}\|\Delta_h c^n_h\|^{d/4} \leq C(\|c^n_h\| + \|\Delta_h c^n_h\|),
$$

$$
\|\nabla c^n_h\|_{L^1(\Omega)} \leq C\|c^n_h\|^{1/2-d/12}\|\Delta_h c^n_h\|^{1/2+d/12} \leq C(\|c^n_h\| + \|\Delta_h c^n_h\|).
$$

With (5.28), the above bounds yield

$$
\|c^n_h\|^2_{L^\infty(\Omega)} + \|\nabla c^n_h\|^2_{L^1(\Omega)} \leq C(1 + \|\Delta_h c^n_h\|^2).
$$

Choosing $\varphi_h = \Delta_h c^n_h$ in (3.2), and using the definition of the discrete Laplacian operator (6.15), Cauchy–Schwarz’s and Young’s inequalities yield

$$
\kappa \|\Delta_h c^n_h\|^2 = -\kappa a_{\text{diff}}(c^n_h, \Delta_h c^n_h) = (\Phi'_+(c^n_h) + \Phi'_-(c^n_{h-1}), \Delta_h c^n_h) - (\mu^n_h, \Delta_h c^n_h)
\leq \frac{\kappa}{2} \|\Delta_h c^n_h\|^2 + C\|\Phi'_+(c^n_{h}) + \Phi'_-(c^n_{h-1})\|^2 + C\|\mu^n_h\|^2.
$$

Since $\Phi'_+(c) = c^3$ and $\Phi'_-(c) = -c$, by (5.28), we obtain

$$
\|\Phi'_+(c^n_h) + \Phi'_-(c^n_{h-1})\|^2 \leq 2(\|c^n_h\|^2_{L^\infty(\Omega)} + \|c^n_{h-1}\|^2) \leq C.
$$

Thus, from (6.32), (6.33), and (6.34), it follows that for all $1 \leq n \leq m$,

$$
\|c^n_h\|^2_{L^\infty(\Omega)} + \|\nabla c^n_h\|^2_{L^1(\Omega)} \leq C(1 + \|\mu^n_h\|^2).
$$

Therefore to obtain (6.31), it suffices to bound $\max_{1 \leq n \leq m} \|\mu^n_h\|^2 + \tau \sum_{n=1}^{m} \|\delta c^n_h\|^2$, which is done below via Gronwall’s lemma. We first set by convention $c^0_{h-1} = 0$. Then, we introduce $\mu^0_h \in M^k_h$ via the following variational problem

$$
(\mu^0_h, \varphi_h) = (\Phi'_+(c^0_h) + \Phi'_-(c^0_{h-1}), \varphi_h) + \kappa a_{\text{diff}}(c^0_h, \varphi_h), \quad \forall \varphi_h \in M^k_h.
$$

Note, $\mu^0_h$ is well-defined by Riesz representation theorem. Choose $\varphi_h = \mu^0_h$ above, use (3.7) and a bound similar to (6.34), to obtain

$$
\|\mu^0_h\|^2 \leq C\|\mu^0_h\| \left( \|c^0_h\|_{L^\infty(\Omega)} + \|c^0_h\|^2 \right)^{1/2} + \kappa \|a_{\text{diff}}(c^0_h, \mu^0_h)\|.
$$

Under the assumption $c^0 \in H^2(\Omega)$ and $\nabla c^0 \cdot n = 0$ on $\partial \Omega$, we have

$$
a_{\text{diff}}(c^0, \mu^0_h) = -\langle \Delta c^0, \mu^0_h \rangle \leq \|c^0\|_{H^2(\Omega)} \|\mu^0_h\|.
$$

Then by (5.28), we obtain

$$
\|\mu^0_h\| \leq C.
$$
Now, we subtract (3.2) at time \( t^n \) from itself at time \( t^{n-1} \) (for \( n = 1 \) we use (6.36)) and choose \( \varphi_h = \mu_h^n / \tau \). We have
\[
(\delta_t \mu_h^n, \mu_h^n) = \frac{1}{\tau} (\Phi_\tau'(c_h^n) - \Phi_\tau'(c_h^{n-1}), \mu_h^n) + \frac{1}{\tau} (\Phi_\tau'(c_h^{n-1}) - \Phi_\tau'(c_h^{n-2}), \mu_h^n) + \kappa a_{\text{diff}}(\delta_t \epsilon_h^n, \mu_h^n).
\]
Choosing \( \chi_h = \kappa \delta_t c_h^n \) in (3.1), we have
\[
\kappa (\delta_t c_h^n, \delta_t c_h^n) = -\kappa a_{\text{diff}}(\mu_h^n, \delta_t c_h^n) - \kappa a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \delta_t c_h^n).
\]
Adding the two equations above yields
\[
(6.38) \quad \kappa \|\delta_t c_h^n\|^2 + \frac{1}{2\tau} \|\mu_h^n\|^2 - \frac{1}{2\tau} \|\mu_h^{n-1}\|^2 \leq \frac{1}{\tau} (\Phi_\tau'(c_h^n) - \Phi_\tau'(c_h^{n-1}), \mu_h^n) + \frac{1}{\tau} (\Phi_\tau'(c_h^{n-1}) - \Phi_\tau'(c_h^{n-2}), \mu_h^n) - \kappa a_{\text{diff}}(\mu_h^n, \delta_t c_h^n).
\]
We separately bound the terms on the right-hand side of (6.38). Using (5.28), Holder’s and Poincaré’s inequalities (3.28), we obtain
\[
(\frac{1}{\tau} (\Phi_\tau'(c_h^n) - \Phi_\tau'(c_h^{n-1}), \mu_h^n) \leq \|\delta_t c_h^n\| (\|c_h^n\|^2 + c_h^{n-1}) + (c_h^{n-1})^2 \|\mu_h^n\| L^2(\Omega) \|\mu_h^n\| L^6(\Omega)

\leq \frac{K}{6} \|\delta_t c_h^n\|^2 + C (\|c_h^n\| L^2(\Omega) + \|c_h^{n-1}\| L^4(\Omega)) \|\mu_h^n\|^2

\leq \frac{K}{6} \|\delta_t c_h^n\|^2 + C \|\mu_h^n\|^2.
\]
Since \( \Phi_\tau'(c) = -c \), with Cauchy–Schwarz’s and Young’s inequalities, we obtain
\[
(6.40) \quad \frac{1}{\tau} (\Phi_\tau'(c_h^{n-1}) - \Phi_\tau'(c_h^{n-2}), \mu_h^n) \leq \frac{K}{6} \|\delta_t c_h^{n-1}\|^2 + C \|\mu_h^n\|^2.
\]
Handling the last term on the right-hand side of (6.38) is technical. Here, we provide an outline of the proof and suppress details for brevity. Considering the definition of \( a_{\text{adv}} \), integrating by parts and and rearranging terms, yield
\[
a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \delta_t c_h^n) = \sum_{E \in T_h} \int_E \nabla c_h^{n-1} : \nabla u_h^{n-1} \delta_t c_h^n + \sum_{E \in T_h} \int_E c_h^{n-1} \nabla \cdot u_h^{n-1} \delta_t c_h^n

- \sum_{e \in T_h(\partial \Omega)} \int_e (c_h^{n-1} \delta_t c_h^n) \{n_e \cdot u_h^{n-1} - n_e \cdot c_h^{n-1}) - \sum_{e \in T_h} \int_E \{c_h^{n-1} \delta_t c_h^n \{n_e \cdot n_e \} \delta_t c_h^n = \sum_{i=1}^{4} T_i.
\]
For the terms \( T_1 \) to \( T_3 \), we apply Holder’s, Poincaré’s, trace and triangle inequalities. We also use (3.6), (3.26), and inverse estimate (3.32) to deduce that
\[
\|u_h^{n-1}\|_{DG} \leq C_{\text{inv}} h^{-1} \|u_h^{n-1} - v_h^{n-1}\| + \|v_h^{n-1}\|_{DG} \leq C \tau h^{-1} |\phi_h^{n-1}|_{DG} + \|v_h^{n-1}\|_{DG}.
\]
Thus, with the above bound, we obtain
\[
\sum_{i=1}^{3} |T_i| \leq C (\|\nabla c_h^{n-1}\| L^2(\Omega) + \|c_h^{n-1}\| L^2(\Omega)) (\tau h^{-1} |\phi_h^{n-1}|_{DG} + \|v_h^{n-1}\|_{DG}) \|\delta_t c_h^n\|

\leq \frac{1}{12} \|\delta_t c_h^n\|^2 + C (\|\nabla c_h^{n-1}\| L^2(\Omega) + \|c_h^{n-1}\| L^2(\Omega)) (\tau h^{-1} |\phi_h^{n-1}|_{DG} + \|v_h^{n-1}\|_{DG}).
\]
To handle $T_4$, we add and subtract the approximation $\Pi_h u^{n-1}$

\[
T_4 = - \sum_{e \in \mathcal{T}_h} \int_{e} [(c_h^{n-1})^2 (u_h^{n-1} - \Pi_h u^{n-1}) \cdot n_e] \{\delta_t c_h^n\} - \sum_{e \in \mathcal{T}_h} \int_{e} [(c_h^{n-1})^2 (\Pi_h u^{n-1}) \cdot n_e] \{\delta_t c_h^n\}
\]

\[
= T_4^1 + T_4^2.
\]

For $T_4^1$, we split it by inserting $v_h^{n-1}$. We use trace inequality (3.33), inverse estimate (3.31), Poincaré’s and Young’s inequalities. With (3.6), (5.26), and (3.26), we have

\[
|T_4^1| \leq C h^{-1} \|c_h^{n-1}\|_{L^\infty(\Omega)} \|u_h^{n-1} - v_h^{n-1}\| \|\delta_t c_h^n\| + C \|c_h^{n-1}\|_{DG} \|v_h^{n-1} - \Pi_h u^{n-1}\|_{L^2(\Omega)} \|\delta_t c_h^n\|_{L^2(\Omega)}
\]

\[
\leq \frac{1}{24} \|\delta_t c_h^n\|^2 + C \tau^2 h^{-2} \|c_h^{n-1}\|_{L^2(\Omega)} \|\phi_h^{n-1}\|_{DG}^2 + C h^{-d/3} \|v_h^{n-1} - \Pi_h u^{n-1}\|_{DG}^2.
\]

For $T_4^2$, we simply have

\[
|T_4^2| \leq C \|\Pi_h u^{n-1}\|_{L^\infty(\Omega)} \|c_h^{n-1}\|_{DG} \|\delta_t c_h^n\| \leq \frac{1}{24} \|\delta_t c_h^n\|^2 + C.
\]

We combine the bounds on $T_1$ to $T_4$ with (6.35). We obtain

\[
\|u^n - \Pi_h u^{n-1}\|_{DG}^2 \leq C T + C \kappa h^{-d/3} \sum_{n=0}^{\ell-1} \|v_h^n - \Pi_h u^n\|_{DG}^2 + C \kappa T h^{-d/3} \sum_{n=0}^{\ell-1} \|v_h^n - \Pi_h u^n\|_{DG}^2 + C \kappa T h^{-d/3} \sum_{n=0}^{\ell-1} \|v_h^n - \Pi_h u^n\|_{DG}^2.
\]

We substitute bounds (6.39), (6.40), and (6.41) into (6.38), multiply by $2\tau$, sum from $n = 1$ to $\ell$, with $\ell \leq m$ (recall that $\delta_t c_h^n = 0$ by convention):

\[
\|
\mu_h^n\|^2 + \kappa \tau \|
\delta_t c_h^n\|^2 + \frac{\kappa^2}{3} \|
\delta_t c_h^n\|^2 \leq C T + C \kappa T h^{-d/3} \sum_{n=0}^{\ell-1} \|v_h^n - \Pi_h u^n\|_{DG}^2
\]

\[
+ C \kappa T \sum_{n=0}^{\ell} (1 + \|\mu_h^n\|^2)(\|v_h^n\|_{DG}^2 + \tau^2 h^{-2} \|\phi_h^n\|_{DG}^2) + C \tau \sum_{n=1}^{\ell} (\|\mu_h^n\|^2 + |\mu_h^n|_{DG}^2) + C \|\mu_h^n\|^2.
\]

Take $\phi_h = \mu_h^n$ in (3.2), use (3.18), Cauchy–Schwarz’s and Young’s inequalities, we have

\[
\|
\mu_h^n\|^2 \leq \|c_h^n\|^3 - c_h^{n-1} \|\mu_h^n\| + C a \kappa \|c_h^n\|_{DG} |\mu_h^n|_{DG}
\]

\[
\leq \frac{1}{2} \|
\mu_h^n\|^2 + \|c_h^n\|_{L^2(\Omega)}^6 + \|c_h^{n-1}\|^2 + \frac{C a}{2} \|c_h^n\|_{DG}^2 + \frac{C a}{2} |\mu_h^n|_{DG}^2.
\]

Multiply by $\tau$ the above inequality and sum from $n = 1$ to $\ell$. By stability bounds (5.26), (5.28), and (6.37), we have

\[
\tau \sum_{n=1}^{\ell} \|\mu_h^n\|^2 \leq C + C T.
\]

With the induction hypothesis (6.4) and (6.1), we have

\[
\sum_{n=0}^{\ell-1} (\tau^2 h^{-2} |\phi_h^n|_{DG}^2 + \tau h^{-d/3} \|v_h^n - \Pi_h u^n\|_{DG}^2) \leq (\tau h^{-2} + \tau h^{-d/3})(\tau h^{-2} + h^{2+\alpha})
\]

\[
\leq \gamma h^{-\alpha} \gamma h + h^{1/2} \leq C.
\]
With the above bounds and (5.26), we obtain

\begin{equation}
\|\mu_h^l\|^2 + \kappa T \sum_{n=1}^{l} \|\delta \tau c_n^m\|^2 \leq CT + C\kappa T \sum_{n=0}^{l-1} (\|v_h^n\|_{DG}^2 + \tau^2 h^{-2} \|\phi_h^n\|_{DG}^2)\|\mu_h^n\|^2.
\end{equation}

We now apply the Gronwall’s inequality [15] to obtain

\[\|\mu_h^l\|^2 + \kappa T \sum_{n=1}^{l} \|\delta \tau c_n^m\|^2 \leq C(1 + T),\]

which concludes the proof. Finally, it is important to point out that, in the above proof, every generic constant $C$ is independent of the induction iteration index $m$. In other words, at each induction iteration the constant $C$ in (6.31) is unchanged. \[\square\]

### 6.3. Intermediate results (Steps (ii) and (iii))

To simplify the writeup, we define the following projection and discretization error functions.

\[\eta^n_c = \mathcal{P}_h c^n - c^n, \quad \eta^n_p = \mathcal{P}_h \mu^n - \mu^n, \quad \eta^n = \pi_h p^n - p^n,\]

\[\xi^n_c = c^n - \mathcal{P}_h c^n, \quad \xi^n = \mu^n - \mathcal{P}_h \mu^n, \quad \xi^n = p^n - \pi_h p^n,\]

\[\eta^n_u = \Pi_h u^n - u^n, \quad \xi^n_u = v^n_h - \Pi_h u^n, \quad \xi^n_u = u^n_h - \Pi_h u^n.\]

From the consistency equations (6.28)-(6.30), the fully discrete scheme (3.1)-(3.6), and the definition of $\mathcal{P}_h$ (6.5), we obtain the following error equations for all $1 \leq n \leq N_T$.

For all $\chi_h \in M_h^k$,

\begin{equation}
(\delta \tau \xi^n_c, \chi_h) + a_{\text{diff}}(\xi^n_c, \chi_h)
= (\delta \tau c^n, (\partial_t c) - \delta \tau \eta^n_c, \chi_h) - a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \chi_h) + a_{\text{adv}}(c^n, u^n, \chi_h).
\end{equation}

For all $\varphi_h \in M_h^k$,

\begin{equation}
\kappa a_{\text{diff}}(\xi^n, \varphi_h) - (\xi^n, \varphi_h)
= (\eta^n_u, \varphi_h) + (\Phi_\mu(c^n) - \Phi_\mu(c_h^n), \varphi_h) + (\Phi_\mu(c^n) - \Phi_\mu(c_h^{n-1}), \varphi_h).
\end{equation}

For all $\theta_h \in X^k_h$,

\begin{equation}
\frac{1}{\tau} (\xi^n_u - \xi_{u}^{n-1}, \theta_h) + a_C(u_h^{n-1}, u_h^{n-1}, \xi^n_u, \theta_h) + \mu_a a_D(\xi^n_u, \theta_h) = -\mu_a a_D(\eta^n_u, \theta_h)
+ b_p(\theta_h, p_h^{n-1} - p^n) + a_C(u^n, u^n, \theta_h) - a_C(u_h^{n-1}, u_h^{n-1}, \Pi_h u^n, \theta_h)
+ (\delta \tau u^n - \delta \tau u^n - \delta \tau \eta^n_u, \theta_h) + b_f(c_h^{n-1}, \mu_h^n, \theta_h) - b_f(c^n, \mu^n, \theta_h).
\end{equation}

The next lemma gives a bound on the last two terms in (6.46).

**Lemma 6.8.** There exists a constant $C$ independent of $h$, $\tau$ such that for any $\epsilon > 0$ and $\chi_h \in M_h^k$, the bound holds for all $n \geq 1$:

\begin{equation}
|a_{\text{adv}}(c^n, u^n, \chi_h) - a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \chi_h)| \leq 5\epsilon |\chi_h|_{DG}^2 + \frac{C}{\epsilon} h^{2k+2}
+ \frac{C}{\epsilon} \tau \int_{t_{n-1}}^{t_n} \left( \|\partial_t c\|_{H^1(\Omega)}^2 + \|\partial_t u\|_{H^1(\Omega)}^2 \right) + \frac{C}{\epsilon} \left( \|\xi^n_c\|^2 + \|\xi^n_u\|^2 \right).
\end{equation}
The proof is in Appendix A. We are now ready to complete Step (ii).

**Lemma 6.9.** Fix $m$, with $1 \leq m \leq N_T$. Assume that (6.4) and (6.1) hold. Then, for any $n$ with $1 \leq n \leq m$, we have for a constant $C$ independent of $h$ and $\tau$

\[
\frac{K_a}{2} \tau |\mathcal{F}(\delta_t \xi^n_c)|^2_{DG} + (a_{diff}(\xi^n_c, \xi^n_c) - a_{diff}(\xi^{n-1}_c, \xi^{n-1}_c)) + \kappa K_a |\xi^n_c - \xi^{n-1}_c|^2_{DG}
\]

\[
\leq C \tau \int_{\tau^{n-1}}^{\tau^n} \left( \tau \|\partial_t c\|^2 + \tau^{-1} h^2 |\partial_t c|^2_{H^2(\Omega)} + \tau \|\partial_t u\|^2_{H^1(\Omega)} \right) + C \tau h^2 + C \tau \left( |\xi^n_c|^2_{DG} + |\xi^{n-1}_c|^2_{DG} + \|\xi^{n-1}_c\|^2 \right).
\]

**Proof.** Note that with the definition of the operator $\mathcal{F}$ in (6.13), for any $\chi_h \in M^{h_0}_{DG}$ and $\phi_h \in M^{h_0}_{DG}$, we have:

\[
(\delta_t \xi^n_c, \mathcal{F}(\chi_h)) = a_{diff}(\mathcal{F}(\chi_h), \phi_h - \phi_h) = (\chi_h, \phi_h - \phi_h).
\]

Thus, we obtain

\[
(\delta_t \xi^n_c, \mathcal{F}(\delta_t \xi^n_c)) = a_{diff}(\mathcal{F}(\delta_t \xi^n_c), \delta_t \xi^n_c), \quad a_{diff}(\mathcal{F}(\delta_t \xi^n_c), \xi^n_c) = (\xi^n_c, \delta_t \xi^n_c).
\]

Let $\chi_h = \mathcal{F}(\delta_t \xi^n_c)$ in (6.46), and use the coercivity properties of $a_{diff}$ (3.15). With the above equalities, we obtain

\[
K_a|\mathcal{F}(\delta_t \xi^n_c)|^2_{DG} + (\xi^n_c, \delta_t \xi^n_c) \leq (-\delta_t c^n + (\partial_t c)^n - \delta_t \eta^n_c, \mathcal{F}(\delta_t \xi^n_c))
\]

\[
- a_{adv}(c^n_h, u^{n-1}_h, \mathcal{F}(\delta_t \xi^n_c)) + a_{adv}(c^n, u^n, \mathcal{F}(\delta_t \xi^n_c)) = T_1 + T_2 + T_3.
\]

Let $\varphi_h = \delta_t \xi^n_c$ in (6.47). With the symmetry and coercivity of $a_{diff}$, we have:

\[
\frac{K_a}{2\tau} (a_{diff}(\xi^n_c, \xi^n_c) - a_{diff}(\xi^{n-1}_c, \xi^{n-1}_c)) + K_a|\xi^n_c - \xi^{n-1}_c|^2_{DG} - (\xi^n_c, \delta_t \xi^n_c)
\]

\[
\leq (\eta^n_c, \delta_t \xi^n_c) + (\Phi_\ast'(c^n) - \Phi_\ast'(c^n_h), \delta_t \xi^n_c) + (\Phi_\ast'(c^n) - \Phi_\ast'(c^{n-1}_h), \delta_t \xi^n_c) = T_4 + T_5 + T_6.
\]

Adding the above two inequalities yields:

\[
K_a|\mathcal{F}(\delta_t \xi^n_c)|^2_{DG} + \frac{K_a}{2\tau} (a_{diff}(\xi^n_c, \xi^n_c) - a_{diff}(\xi^{n-1}_c, \xi^{n-1}_c))
\]

\[
+ \frac{\kappa K_a}{2\tau} |\xi^n_c - \xi^{n-1}_c|^2_{DG} \leq \sum_{i=1}^6 T_i.
\]

With Cauchy–Schwarz’s, Poincaré’s and Young’s inequalities, Taylor expansions, and (6.6), we obtain

\[
|T_1| \leq \frac{C}{K_a} (\|\delta_t c^n - (\partial_t c)^n\|^2 + \|\delta_t \eta^n_c\|^2) + \frac{K_a}{18} |\mathcal{F}(\delta_t \xi^n_c)|^2_{DG}
\]

\[
\leq \frac{C}{K_a} \int_{\tau^{n-1}}^{\tau^n} \left( \tau \|\partial_t c\|^2 + \tau^{-1} h^2 |\partial_t c|^2_{H^2(\Omega)} + \tau \|\partial_t u\|^2_{H^1(\Omega)} \right) + \frac{K_a}{18} |\mathcal{F}(\delta_t \xi^n_c)|^2_{DG}.
\]

The terms $T_2 + T_3$ are bounded via Lemma 6.8 with $\chi_h = \mathcal{F}(\delta_t \xi^n_c)$ and $\epsilon = K_a/18$. We also use Lemma 6.7 and obtain

\[
\frac{5K_a}{18} |\mathcal{F}(\delta_t \xi^n_c)|^2_{DG} + \frac{C}{K_a} \tau \int_{\tau^{n-1}}^{\tau^n} \left( \|\partial_t c\|^2_{H^1(\Omega)} + \|\partial_t u\|^2_{H^1(\Omega)} \right)
\]

\[
+ \frac{C}{K_a} h^2 + \frac{C}{K_a} (|\xi^n_c|^2_{DG} + \|\xi^{n-1}_c\|^2).
\]
The term $T_4$ is bounded with Lemma 6.5, Young’s inequality, and (6.6).

\begin{equation}
(6.56)
|T_4| \leq \frac{C}{K_\alpha} h^{2k} \|\mu\|_{L^\infty((0,T;H^{k+1}(\Omega))}^2 + \frac{K_\alpha}{18} |\mathcal{F}(\delta \epsilon_\tau^n)|_{DG}^2.
\end{equation}

For $T_5$ and $T_6$, we insert $\Phi_-'(c^{n-1})$ and apply Lemma 6.5 to obtain:

\begin{equation}
(6.57)
|T_5| + |T_6| \leq C |\Phi_+'(c^n) - \Phi_+'(c_h^n)|_{DG} |\mathcal{F}(\delta \epsilon_\tau^n)|_{DG}
+ C (|\Phi_-'(c^n) - \Phi_-'(c_h^{n-1})|_{DG} + |\Phi_-'(c^{n-1}) - \Phi_-'(c_h^{n-1})|_{DG}) |\mathcal{F}(\delta \epsilon_\tau^n)|_{DG}.
\end{equation}

For the first term and for the Ginzburg–Landau potential, we have Using the $L^\infty$ bounds on $c^n$ and $c_h^n$ and Hölder’s inequality, we can write

\begin{equation}
(|c^n|^3 - (c_h^n)^3)|_{DG} \leq C |\nabla_h(c^n - c_h^n)| + C \|c^n - c_h^n\|_{L^2(\Omega)} (\|\nabla c^n\|_{L^2(\Omega)} + \|\nabla h c_h^n\|_{L^2(\Omega)}).
\end{equation}

Lemma 6.7 and Poincaré’s inequality applied to $c^n - c_h^n$ with (3.27) yield

\begin{equation}
(|c^n|^3 - (c_h^n)^3)|_{DG} \leq C |c_h^n - c^n|_{DG}.
\end{equation}

We substitute the above bound in (6.57), and we apply a Taylor’s expansion, (6.6) and Young’s inequality. We obtain

\begin{equation}
|T_5| + |T_6| \leq C |c_h^n - c^n|_{DG} + |c^n - c^{n-1}|_{DG} + |c_h^{n-1} - c_h^n|_{DG} |\mathcal{F}(\delta \epsilon_\tau^n)|_{DG}
\leq \frac{C}{K_\alpha} (|\epsilon_\tau^n|^2_{DG} + |\epsilon_{\tau}^{n-1}|_{DG}^2 + h^{2k} |\epsilon_\tau^n|^2_{L^\infty((0,T;H^{k+1}(\Omega))})
+ \frac{C}{K_\alpha} \tau \int_{t_{n-1}}^{t_n} \|\partial_t c\|_{H^1(\Omega)}^2 + \frac{K_\alpha}{18} |\mathcal{F}(\delta \epsilon_\tau^n)|_{DG}^2.
\end{equation}

Using the bounds on the terms $T_i$’s in (6.52) and multiplying by $2\tau$ yield the result.\[]

Next, we derive a bound for $\epsilon_{\mu}^{n}$ in the energy norm.

**Lemma 6.10.** Fix $m$, with $1 \leq m \leq N_T$. Assume that (6.4) and (6.1) hold. Then, for any $n$ with $1 \leq n \leq m$, we have for a constant $C$ independent of $h$ and $\tau$

\begin{equation}
(6.58)
\frac{K_\alpha}{2} |\epsilon_{\mu}^{n}|_{DG}^2 \leq \frac{4C_\alpha^2}{K_\alpha} |\mathcal{F}(\delta \epsilon_\tau^n)|_{DG}^2 + C \tau \int_{t_{n-1}}^{t_n} \left( \|\partial_t c\|_{H^1(\Omega)} + \|\partial_t u\|_{H^1(\Omega)} + \|\partial_t c\|_{H^1(\Omega)} \right)
+ C h^{2k} \left( 1 + \tau^{-1} \int_{t_{n-1}}^{t_n} \|\partial_t c\|_{H^{k+1}(\Omega)}^2 \right) + C \left( |\epsilon_{\tau}^{n-1}|_{DG}^2 + \|\epsilon_{\tau}^{n-1}\|_{DG}^2. \right)
\end{equation}

**Proof.** Let $\chi_h = \epsilon_{\mu}^{n}$ in (6.46). With the coercivity property of $a_{\text{diff}}$, we obtain:

\begin{equation}
(6.59)
K_\alpha |\epsilon_{\mu}^{n}|_{DG} \leq -\langle \delta \epsilon_\tau^n, \epsilon_{\mu}^{n} \rangle + (\delta \epsilon_\tau^n, \epsilon_{\mu}^{n}) - \delta \epsilon_\tau^n \cdot \epsilon_{\mu}^{n} - \delta \epsilon_\tau^n \cdot \epsilon_{\mu}^{n}
\end{equation}

\begin{equation}
- a_{\text{diff}}(c_h^{n-1}, u_h^{n-1}, \epsilon_\tau^{n-1}) + a_{\text{adv}}(c^n, u^n, \epsilon_\tau^n).
\end{equation}

The first term is bounded by Lemma 6.5 and Young’s inequality. We have:

\begin{equation}
|\langle \delta \epsilon_\tau^n, \epsilon_{\mu}^{n} \rangle| \leq \frac{K_\alpha}{16} |\epsilon_{\mu}^{n}|_{DG}^2 + \frac{4C_\alpha^2}{K_\alpha} |\mathcal{F}(\delta \epsilon_\tau^n)|_{DG}^2.
\end{equation}
By taking $\chi_h = 1$ in (6.46), we observe that the average of $\delta_c \epsilon_c^n - (\partial_c \epsilon)^n + \delta_r \eta^p$ is zero. Hence, we bound the second term with Cauchy–Schwarz’s and Poincaré’s inequalities.

\begin{equation}
(6.60) \quad |(\delta_c \epsilon_c^n - (\partial_c \epsilon)^n + \delta_r \eta^p, \xi^n)| = |(\delta_c \epsilon_c^n - (\partial_c \epsilon)^n + \delta_r \eta^p, \xi^n - \overline{\xi^n})| \\
\leq \frac{K_\alpha}{16} \|\epsilon^n_m\|_{\Gamma h}^2 + \frac{C}{K_\alpha} \int_{t_{n-1}}^{t_n} (\tau \|\partial_c \epsilon\|^2 + \tau^{-1} h^{2k} |\partial_c \epsilon|^2)_{\Omega h_{n+1}}(\Omega).
\end{equation}

We use the above two bounds in (6.59) and Lemma 6.8 with $\chi_h = \xi^n_m$ and $\epsilon = K_\alpha/16$ to bound the last two terms. We then conclude by using Lemma 6.7.

We now show an estimate involving the errors $\xi_u$ and $\xi_v$. To this end, we denote

\begin{equation}
A^n_1 = \sum_{\Gamma h} \tilde{\sigma} \|\phi_h^n\|_{\Gamma h_{\sigma}}^2 - \sum_{\Gamma h} \tilde{\sigma} \|\phi_h^{n-1}\|_{\Gamma h_{\sigma}}^2, \quad A^n_2 = \|G_h([\phi_h^n])\|^2 - \|G_h([\phi_h^{n-1}])\|^2.
\end{equation}

In the following lemma, the notation $\delta_{n,1}$ is the Kronecker symbol: $\delta_{1,1} = 1$ and $\delta_{n,1} = 0$ for $n > 1$.

**Lemma 6.11.** Assume that $\sigma \geq \tilde{M}^2 d_{\sigma}^{-1} / \Gamma_h$, $\tilde{\sigma} \geq 4 \tilde{M}^2$, and $\sigma = K_D/(2d)$. There exists a constant $C$ independent of $h$ and $\tau$ such that for all $n \geq 1$ and any $\epsilon > 0$, we have

\begin{equation}
(6.61) \quad \frac{1}{2\tau} (\|\xi_u^n\|^2 - \|\xi_u^{n-1}\|^2 + \|\xi_v^n - \xi_v^{n-1}\|^2) + \frac{K_D \mu_s}{4} \|\xi_u^n\|_{\Gamma h}^2 + \frac{\tau}{16} |\phi_h^n|_{\Gamma h}^2 \\
+ \frac{1}{2\sigma \mu_s} (\|S_h^n\|^2 - \|S_h^{n-1}\|^2) + \frac{\tau}{2} (\|\partial_t u\|^2 + \|\partial_t v\|^2 + \|\partial_{\Gamma} \epsilon\|^2 + \tau^{-1} h^{2k} |\partial_t u|^2_{\Omega h_{n+1}}(\Omega) + \delta_{n,1} |b_\phi(\xi_u^n, \phi_h^n)|.
\end{equation}

**Proof.** Taking $\theta_h = \xi_v^n$ in (6.48) and using (3.22) and (3.16) yield:

\begin{equation}
(6.62) \quad \frac{1}{2\tau} (\|\xi_u^n\|^2 - \|\xi_u^{n-1}\|^2 + \|\xi_v^n - \xi_v^{n-1}\|^2) + \mu_s K_D \|\xi_v^n\|_{\Gamma h}^2 \\
\leq -\mu_s a_{DH}(\eta_u^n, \xi_v^n) + b_\phi(\xi_v^n, p^{n-1} - p^n) + a_C(u^n, u^n, u^n, \xi_v^n) - a_C(u_h^{n-1}, u_h^{n-1}, \Pi_h u^n, \xi_v^n) \\
+ ((\partial_t u)^n - \delta_r \eta^u_h, \xi_v^n) + b_\phi(c_h^{n-1}, \mu_h^n, \xi_v^n) - b_\phi(\epsilon_h^n, \mu_h^n, \xi_v^n) = \sum_{i=1}^N W_i.
\end{equation}

Following similar arguments to the proof of Theorem 1 in [23], we can prove (for completeness, we provide some details in the Appendix B)

\begin{equation}
(6.63) \quad \frac{1}{2\tau} (\|\xi_u^n\|^2 - \|\xi_u^{n-1}\|^2 + \|\xi_v^n - \xi_v^{n-1}\|^2) + \frac{\tau}{4} |\phi_h^n|_{\Gamma h}^2 + \frac{\tau}{2} |\phi_h^n|_{\Gamma h}^2 \\
+ \frac{\tau}{2} (A^n_1 - A^n_2) + \frac{\tau}{4} \sum_{\Gamma h} \tilde{\sigma} \|\phi_h^n - \phi_h^{n-1}\|_{\Gamma h_{\sigma}}^2 \leq \frac{1}{2\tau} \|\xi_v^n - \xi_v^{n-1}\|^2 + \delta_{n,1} |b_\phi(\xi_u^n, \phi_h^n)|.
\end{equation}
We add (6.63) with (6.62) to obtain

\[
\begin{align*}
(6.64) \quad & \quad \frac{1}{2\tau} \left( \left\| \xi_u^n \right\|^2 - \left\| \xi_u^{n-1} \right\|^2 + \left\| \xi_u^n - \xi_u^{n-1} \right\|^2 \right) + \mu_s K_D \left\| \xi_v^n \right\|^2_{DG} + \frac{\tau}{2} (A_1^n - A_2^n) \\
& + \frac{\tau}{4} |\phi_h^n|_{DG}^2 + \frac{\tau}{4} \sum_{c \in \mathcal{E}_h} \left( \left\| \phi_h^n - \phi_h^{n-1} \right\|_{L^2(c)}^2 + \frac{\tau}{2} a_{\text{diff}}(\phi_h^n, \phi_h^{n-1}) \right) \leq \delta_{n,1} |b_P(\xi_u^n, \phi_h^n)| + \sum_{i=1}^7 W_i.
\end{align*}
\]

The bounds for \( W_i \) for \( i = 1, \ldots, 5 \) are handled in a similar way as in [23]. We recall the main ideas for completeness. Using the continuity of \( a_D \), we have

\[
(6.65) \quad |W_1| \leq C \mu_s h^{2k} \left\| u^n \right\|^2_{H^{1+1}(\Omega)} + \frac{K_D \mu_s}{32} \left\| \xi_v^n \right\|^2_{DG}.
\]

To handle \( W_2 \), we split it as follows.

\[
(6.66) \quad W_2 = b_P(\xi_v^n, p_h^n - p^n) = b_P(\xi_v^n, p_h^n - p^n) - b_P(\xi_v^n, \pi_h p^n) + b_P(\xi_v^n, \pi_h p^n - p^n).
\]

For the first term, recall that \( b_P(\Pi_h u^n, p_h^n - p^n) = b_P(u^n, p_h^n - p^n) = 0 \). From (5.20),

\[
b_P(\xi_v^n, p_h^n - p^n) = b_P(\xi_v^n, p_h^n - p^n) = -\frac{\tau}{2} a_{\text{diff}}(\xi_v^n, \xi_v^n) + \frac{\tau}{2} a_{\text{diff}}(\nabla_h \xi_v^n, \nabla_h \xi_v^n) + \frac{\tau}{2} a_{\text{diff}}(\phi_h^n, \phi_h^n)
\]

\[
\leq -\frac{1}{2\sigma h \mu_s} \left( \left\| S_h^n \right\|^2 - \left\| S_h^{n-1} \right\|^2 - \left\| S_h^n - S_h^{n-1} \right\|^2 \right).
\]

Using the fact \( \nabla_h \cdot (\Pi_h u^n - R_h([\Pi_h u^n])) = 0 \), taking \( \sigma h \leq K_D/(2d) \) and \( \sigma \geq M_k/(2d) \), by (3.23), we have

\[
\frac{1}{2\sigma h \mu_s} \left\| S_h^n - S_h^{n-1} \right\|^2 = \frac{\sigma h \mu_s}{2} \left\| \nabla_h \cdot (\xi_v^n + R_h([\xi_v^n])) \right\|^2 \leq \frac{K_D \mu_s}{2} \left\| \xi_v^n \right\|^2_{DG}.
\]

Since \( \pi_h p^n \in M_0^{k-1} \), we use (3.4), (3.18), and stability of the \( L^2 \) projection.

\[
\left| -b_P(\xi_v^n, \pi_h p^n) \right| = \tau |a_{\text{diff}}(\phi_h^n, \pi_h p^n)| \leq C a \tau |\phi_h^n|_{DG} |\pi_h p^n|_{DG} \leq \frac{\tau}{8} |\phi_h^n|_{DG}^2 + C \tau |p^n|_{H^{1}(\Omega)}^2.
\]

Since \( \nabla \cdot \xi_v^n \in M^{k-1} \), by the definition of \( \pi_h p^n \) the first term in \( b_P(\xi_v^n, \pi_h p^n - p^n) \) is zero. Hence, by using trace estimate (3.34) and (6.12), we obtain

\[
\left| b(\pi_h p^n - p^n, \xi_v^n) \right| \leq C \left( |\pi_h p^n - p^n| + h \left\| \nabla_h (\pi_h p^n - p^n) \right\| \right) \left\| \xi_v^n \right\|_{DG} \leq \frac{C}{\mu_s} h^{2k} |p^n|_{H^{1+1}(\Omega)}^2 + \frac{K_D \mu_s}{32} \left\| \xi_v^n \right\|^2_{DG}.
\]

With the above bounds and expressions, (6.64) becomes:

\[
(6.67) \quad \frac{1}{2\tau} \left( \left\| \xi_u^n \right\|^2 - \left\| \xi_u^{n-1} \right\|^2 + \left\| \xi_u^n - \xi_u^{n-1} \right\|^2 \right) + \frac{K_D \mu_s}{2} \left\| \xi_v^n \right\|^2_{DG} + \frac{\tau}{2} (A_1^n - A_2^n) \\
+ \frac{1}{2\sigma_h \mu_s} \left( \left\| S_h^n \right\|^2 - \left\| S_h^{n-1} \right\|^2 \right) + \frac{\tau}{2} (A_1^n - A_2^n) + \frac{\tau}{4} \sum_{c \in \mathcal{E}_h} \left( \phi_h^n - \phi_h^{n-1} \right)_{L^2(c)}^2 \\
+ \frac{\tau}{2} \left( a_{\text{diff}}(\xi_v^n, \xi_v^n) - a_{\text{diff}}(\xi_v^{n-1}, \xi_v^{n-1}) \right) \leq \delta_{n,1} |b_P(\xi_u^n, \phi_h^n)| + \sum_{i=1}^7 W_i.
\]
The terms $W_3 + W_4$ are bounded by Lemma 6.6 in the paper [23]. We have:

$$\|W_3 + W_4\| \leq C \tau \int_{t_{n-1}}^{t_n} \|\partial_t u\|^2 + CH^{2k} + C\|\xi^{n-1}_m\|^2 + \frac{K_D\mu_s}{16} \|\xi^n_m\|^2_{DG}. \quad (6.68)$$

The term $W_5$ is bounded with Cauchy–Schwarz’s, Young’s and Poincaré’s inequalities, a Taylor expansion, and Lemma 6.4. We obtain:

$$\|W_5\| \leq C(\|\partial_t u\|^n - \delta_1 u^n\| + \|\delta_t \eta^n_m\|)\|\xi^n_0\|_{DG}$$

$$\leq \frac{C}{\mu_s} \int_{t^\prime_n}^{t_n} (\|\partial_t u\|^n + \tau^{-1} h^{2k}|\partial_t u|^2_{H^1(\Omega)}) + \frac{K_D\mu_s}{32} \|\xi^n_m\|^2_{DG}. \quad (6.69)$$

It remains to handle $W_6 + W_7$. Using (3.12) and following a similar approach to [21], we write:

$$W_6 + W_7 = a_{adv}(c^n_{h}, \xi^n_m, \xi^n_0) + a_{adv}(c^n_{h}, \xi^n_m, \eta^n_m)$$

$$- a_{adv}(c^n - c^{n-1}, \xi^n_0, \mu^n) + a_{adv}(c^n_{c}, \xi^n_0, \mu^n) + a_{adv}(\eta^n_{c}, \xi^n_0, \mu^n) = \sum_{i=1}^{5} B_i. \quad (6.70)$$

For the term $B_1$, use (3.21), Theorem 5.3, triangle inequality, and Young’s inequality, for any $\epsilon > 0$, we have

$$\|B_1\| \leq C\|\xi^n_m\|^{1/2}\|\xi^n_0\|^{1/2}\|\xi^n_m\|_{DG} \leq \frac{C}{\epsilon} (\|\xi^n_m - \xi^n_0\| + \|\xi^n_0\|)\|\xi^n_m\|_{DG} + \epsilon \|\xi^n_m\|^2_{DG}. \quad (6.71)$$

With (3.6), (3.26) and (3.24), we have

$$\|\xi^n_m - \xi^n_0\| \leq \|\tau u_{h\phi^n_0} - \tau G_h([\phi^n_0])\| \leq (1 + \tilde{M}_k)\tau|\phi^n_0|_{DG}. \quad (6.72)$$

Then, by Young’s inequality, we obtain

$$\|B_1\| \leq \frac{C}{\epsilon^2\mu_s} \left(\|\xi^n_0\|^2 + \tau^2 (1 + \tilde{M}_k)^2|\phi^n_0|_{DG}^2\right) + \frac{K_D\mu_s}{32} \|\xi^n_m\|^2_{DG} + \epsilon \|\xi^n_m\|^2_{DG}. \quad (6.73)$$

For the term $B_2$, apply Holder’s, trace and Poincaré’s inequalities, (5.28), (6.6), and Young’s inequality.

$$\|B_2\| \leq C\|c^n_{h}\|_{L^1(\Omega)}\|\xi^n_0\|_{L^3(\Omega)}|\eta^n_m|_{DG} \leq \frac{C}{\mu_s} h^{2k}\|\mu\|^2_{L^\infty(0,T;H^{k+1}(\Omega))} + \frac{K_D\mu_s}{32} \|\xi^n_m\|^2_{DG}. \quad (6.74)$$

The terms $B_3, B_4,$ and $B_5$ simplify since the jumps of $\mu^n$ are zero. With Holder’s and Poincaré’s inequalities, we have

$$\|B_3 + |B_4| + |B_5| \leq (\|c^n - c^{n-1}\| + \|\xi^{n-1}_c\| + \|\eta^{n-1}_c\|)\|\xi^n_0\|_{L^3(\Omega)}|\mu^n|_{W^{1,\infty}(\Omega)}$$

$$\leq \frac{K_D\mu_s}{32} \|\xi^n_0\|^2_{DG} + \frac{C}{\mu_s} \left(\|c^{n-1}_c\|^2_{DG} + h^{2k} + \tau \int_{t_{n-1}}^{t_n} \|\partial_t u\|^2\right). \quad (6.75)$$

Substituting bounds (6.68)-(6.73) into (6.75) yield the result. \hfill \Box

With the above intermediate results, we are now ready to provide the proof of (6.2) in the following section.
6.4. Proof of (6.2) in Theorem 6.1.

Proof. As mentioned in the outline, the proof is based on an induction argument. We suppose that (6.4) holds and we show that (6.2) holds at time step $m$. We multiply (6.61) by $\tau$, choose $c = K^2_h/(32C^2_D)$, denote $\alpha = 1/(K_D\tau^2\mu_s)$, and substitute bound (6.58) into (6.61). After adding the resulting bound to (6.50), we obtain

$$K_h^2 \tau \sum_{n=0}^m |(\partial_t \xi_c^n)|^2_{1DG} + \kappa K_h \sum_{n=1}^m |\xi^n_c - \xi^{n-1}_c|^2_{1DG} + \frac{1}{2} \sum_{n=1}^m \|\xi^n_u - \xi^{n-1}_u\|^2_{1DG}$$

$$+ \frac{\tau^2}{2} (a_{\text{diff}}(\xi^n_h, \xi^n_h) - a_{\text{diff}}(\xi^{n-1}_h, \xi^{n-1}_h) + A^n_1 - A^n_2) + \frac{1}{2} \|\xi^n_u - \xi^{n-1}_u\|^2$$

$$+ \frac{\tau^2}{2} \phi^n_h|_{1DG} \leq C(\tau^2 + \tau h^{2k}) + C\alpha \tau \|\xi^n_u\|^2 + 2\tau |\phi^n_h|_{1DG}^2$$

$$+ C\tau (|\xi^n_c|_{1DG}^2 + |\xi^n_u|_{1DG}^2 + |\xi^{n-1}_c|_{1DG}^2 + \delta_{n,1} \tau |b_\mathcal{P}(\xi^n_u, \phi^n_h)| + \Lambda^n)$$

where

$$\Lambda^n = C\tau^2 \int_{t^n}^{t^{n+1}} \left( \|\partial_t c\|_{H^1(\Omega)}^2 + \|\partial_t c\|_{H^1(\Omega)}^2 + \|\partial_t u\|_{H^1(\Omega)}^2 + \|\partial_t u\|_{H^1(\Omega)}^2 \right)$$

$$+ C\tau^2 h^{2k} \int_{t^n}^{t^{n+1}} \left( \|\partial_t c\|_{H^1(\Omega)}^2 + \|\partial_t u\|^2_{H^1(\Omega)} \right).$$

We sum the above inequality from $n = 1$ to $n = m$. Recalling that $S^0_h = 0$ and $\xi^0_h = 0$, we obtain

$$K_h^2 \tau \sum_{n=1}^m |(\partial_t \xi^n_c)|^2_{1DG} + \kappa K_h \sum_{n=1}^m |\xi^n_c - \xi^{n-1}_c|^2_{1DG} + \frac{1}{2} \sum_{n=1}^m \|\xi^n_u - \xi^{n-1}_u\|^2_{1DG}$$

$$+ (\kappa K_h - C\tau)|\xi^n_c|_{1DG}^2 + \left( \frac{1}{2} - C\alpha \tau \right) \|\xi^n_u\|^2_{1DG} + \left( \frac{1}{16} - C\alpha \tau \right) \tau^2 \sum_{n=1}^m \|\phi^n_h\|^2_{1DG}$$

$$+ \frac{K_D\mu_s}{4} \tau \sum_{n=1}^m \|\xi^n_u\|^2_{1DG} + \frac{\tau^2}{2} \sum_{n=1}^m (A^n_1 - A^n_2) \leq C(\tau + h^{2k})$$

$$+ C\tau \sum_{n=0}^{m-1} |\xi^n_c|_{1DG}^2 + C\tau \sum_{n=0}^{m-1} |\xi^n_u|_{1DG}^2 + \tau |b_\mathcal{P}(\xi^n_u, \phi^n_h)| + \kappa a_{\text{diff}}(\xi^n_c, \xi^n_c) + \frac{1}{2} \|\xi^n_u\|^2_{1DG}.$$
we have

\begin{equation}
K_{\alpha} \tau \sum_{n=1}^{m} |F(\delta_{\tau} \xi_{c}^{n})|_{DG}^{2} + \kappa K_{\alpha} \sum_{n=1}^{m} |\xi_{c}^{n} - \xi_{c}^{n-1}|_{DG}^{2} + \frac{1}{2} \sum_{n=1}^{m} ||\xi_{u}^{n} - \xi_{u}^{n-1}||^{2} \\
+ \frac{\tau^{2}}{32} \sum_{n=1}^{m} |\phi_{h}^{n}|_{DG}^{2} + \kappa K_{\alpha} \sum_{n=1}^{m} |\xi_{c}^{n}|_{DG}^{2} + \frac{1}{4} \sum_{n=1}^{m} ||\xi_{u}^{n}||^{2} + \frac{K_{\delta} \mu_{s}}{4} \tau \sum_{n=1}^{m} ||\xi_{u}^{n}||_{DG}^{2}
\leq CT (\tau + h^{2k}) + C\tau \sum_{n=0}^{m-1} ||\xi_{c}^{n}||_{DG}^{2} + C\tau \sum_{n=0}^{m-1} ||\xi_{u}^{n}||^{2}.
\end{equation}

Use Gronwall’s inequality, we obtain

\begin{equation}
K_{\alpha} \tau \sum_{n=1}^{m} |F(\delta_{\tau} \xi_{c}^{n})|_{DG}^{2} + \kappa K_{\alpha} \sum_{n=1}^{m} |\xi_{c}^{n} - \xi_{c}^{n-1}|_{DG}^{2} + \frac{1}{2} \sum_{n=1}^{m} ||\xi_{u}^{n} - \xi_{u}^{n-1}||^{2} + \tau^{2} \sum_{n=1}^{m} |\phi_{h}^{n}|_{DG}^{2}
+ \kappa K_{\alpha} |\xi_{c}^{n}|_{DG}^{2} + ||\xi_{u}^{n}||^{2} + K_{\delta} \mu_{s} \tau \sum_{n=1}^{m} ||\xi_{u}^{n}||_{DG}^{2} \leq CT e^{CT} (\tau + h^{2k}).
\end{equation}

Using triangle inequalities and approximation results, the bound above will yield the desired error estimate (6.2) except for the bound on the first term $$\sum_{n=1}^{m} |\mu_{h}^{n} - \mu_{s}^{n}|_{DG}^{2}$$.

To obtain a bound on the error of the chemical potential, we multiply (6.58) by $$\tau$$, sum the inequality from $$n = 1$$ to $$n = m$$, and use (6.78). We note that every generic constant $$C$$ in the proof is independent of the induction iteration index $$m$$.

Finally, we complete the induction proof by verifying the induction hypothesis for $$m + 1$$.

**Lemma 6.12.** There exists a constant $$C_{err}$$ independent of $$h$$ and $$\tau$$, such that under the conditions $$\tau \leq \gamma \leq C_{err}^{-(1+\delta)/\delta}$$ and $$h \leq C_{err}^{-(1+\delta)/\delta}$$, the bound (6.4) holds.

**Proof.** From the previous proof, it is straightforward to see that,

\begin{equation}
\tau^{2} \sum_{n=0}^{(m+1)-1} |\phi_{h}^{n}|_{DG}^{2} + \tau \sum_{n=0}^{(m+1)-1} ||\xi_{u}^{n}||_{DG}^{2} \leq C_{err} (\tau + h^{2k}) \leq (C_{err} \tau^{1+\delta})^{1/\delta} + (C_{err} h^{1+\delta})^{1/\delta}.
\end{equation}

Therefore, the induction hypothesis holds for sufficiently small time step length and mesh size, namely $$C_{err} \tau^{1+\delta} \leq 1$$ and $$C_{err} h^{1+\delta} \leq 1$$.

**6.5. Improved Estimate.** In this section, we use duality arguments to obtain estimate (6.3) in Theorem 6.1. We start by deriving bounds for $$||\xi_{u}^{n}||$$ and $$||\xi_{c}^{n} - \xi_{c}^{n-1}||$$. Note that for any $$\chi_{h} \in M_{k}$$ and $$\phi_{h} \in M_{k,0}$$, we have

\begin{equation}
a_{\text{diff}} (F(\chi_{h} - \chi_{h}), \phi_{h}) = (\chi_{h} - \chi_{h}, \phi_{h}) = (\chi_{h}, \phi_{h}).
\end{equation}

**Lemma 6.13.** There exists a constant $$C$$ independent of $$h$$ and $$\tau$$ such that for all $$m \geq 1$$, we have

\begin{equation}
\kappa ||\xi_{c}^{n}||^{2} + \tau \sum_{n=1}^{m} ||\delta_{\nabla} F(\xi_{c}^{n})||^{2} \leq C (\tau^{2} + h^{2k+2}) + C \tau \sum_{n=0}^{m} ||\xi_{u}^{n}||^{2} + C \tau \sum_{n=0}^{m-1} ||\xi_{u}^{n}||^{2}.
\end{equation}
Proof. For readability, let us denote $\hat{c}_h^n = \mathcal{J}(\xi^n_t)$. We note that the linear operator $\mathcal{J}$ is commutative with operator $\delta_t$. Recalling that $\delta_t\xi^n_t \in M^k_{h_0}$ and $\delta_t\hat{c}_h^n \in M^k_{h_0}$, we have

\begin{align}
(6.81) \quad (\delta_t\xi^n_t, \mathcal{J}(\delta_t\hat{c}_h^n)) &= a_{\text{diff}}(\delta_t\xi^n_t, \mathcal{J}(\delta_t\hat{c}_h^n)) = \|\delta_t\xi^n_t\|^2, \\
(6.82) \quad a_{\text{diff}}(\xi^n_t - \xi^n_{\mu}, \mathcal{J}(\delta_t\hat{c}_h^n)) &= a_{\text{diff}}(\xi^n_t - \xi^n_{\mu}, \mathcal{J}(\delta_t\hat{c}_h^n)) = (\delta_t\xi^n_t, \xi^n_t - \xi^n_{\mu}) = (\delta_t\xi^n_t, \xi^n_t), \\
(6.83) \quad a_{\text{diff}}(\xi^n_t, \delta_t\hat{c}_h^n) &= (\xi^n_t, \delta_t\hat{c}_h^n).
\end{align}

Choose $\chi_h = \mathcal{J}(\delta_t\hat{c}_h^n)$ in (6.46) and $\varphi_h = \delta_t\hat{c}_h^n$ in (6.47), and add the resulting equations. With the identities above, we obtain

\begin{align}
(6.84) \quad \|\delta_t\xi^n_t\|^2 + \frac{K}{2\tau} (\|\xi^n_t\|^2 - \|\xi^{n-1}_t\|^2 + \|\xi^n_t - \xi^{n-1}_t\|^2) = \\
(-\delta_t c^n - (\partial_t c^n - \delta_t\eta^n_t, \mathcal{J}(\delta_t\hat{c}_h^n)) - a_{\text{adv}}(c^n, u^{n-1}_h, \mathcal{J}(\delta_t\hat{c}_h^n)) + a_{\text{adv}}(c^n, u^n, \mathcal{J}(\delta_t\hat{c}_h^n)) \\
+ (\eta^n_t, \delta_t\hat{c}_h^n) + (\Phi_\text{c}(c^n) - \Phi_\text{c}(c^n), \delta_t\hat{c}_h^n) + (\Phi_\text{c}(c^n) - \Phi_\text{c}(c^{n-1}), \delta_t\hat{c}_h^n) = \sum_{i=1}^6 W_i.
\end{align}

With (3.30), (3.15) and (6.13), it follows that

\begin{align}
(6.85) \quad |\mathcal{J}(\delta_t\hat{c}_h^n)|_{\text{DG}} &\leq \frac{C_p}{K_\alpha} \|\delta_t\hat{c}_h^n\|.
\end{align}

Hence, the terms $W_1$ and $W_4$ are bounded with Cauchy–Schwarz’s inequality, Taylor’s expansions, Poincaré’s inequality, and the optimal bounds on $\|\eta^n_t\|$ and on $\|\delta_t\eta^n_t\|$. We have

\begin{align}
|W_1| + |W_4| &\leq C \|\delta_t c^n - (\partial_t c^n - \delta_t\eta^n_t, \mathcal{J}(\delta_t\hat{c}_h^n))\|_{\text{DG}} + C \|\eta^n_t\| \|\delta_t\hat{c}_h^n\| \\
&\leq \frac{1}{8} \|\delta_t\hat{c}_h^n\|^2 + C \int_{n-1}^n \left( \tau \|\partial_t c\|^2 + \tau^{-1} h^{2k+2} \|\partial_t c\|^2_{H^{k+1}(\Omega)} \right) + Ch^{2k+2} \|\mu\|^2_{L^\infty(0,T;H^{k+1}(\Omega))}.
\end{align}

To handle $W_2 + W_3$, we use Lemma 6.8, with $\epsilon = K^2_\alpha/(40C_p^2)$, and (6.85).

\begin{align}
|W_2 + W_3| \leq \frac{1}{8} \|\delta_t\hat{c}_h^n\|^2 + C \int_{n-1}^n \left( \|\partial_t c\|^2_{H^{1}(\Omega)} + \|\partial_t u\|^2_{H^{1}(\Omega)} \right) \\
+ Ch^{2k+2} + C (\|\xi^n_{t-1}\|^2 + \|\xi^{n-1}_{t-1}\|^2).
\end{align}

We bound $W_5$ and $W_6$ by applying Cauchy–Schwarz’s inequality and Lemma 6.7. We have

\begin{align}
|W_5| + |W_6| &\leq C (\|c^n - c^{n-1}\|^2 + \|c^n - c^{n-1}\|^2_{L^\infty(\Omega)} + \|c^n - c^{n-1}\|) \|\delta_t\hat{c}_h^n\| \\
&\leq \frac{1}{8} \|\delta_t\hat{c}_h^n\|^2 + C \left( \|\xi^n_{t-1}\|^2 + \|\xi^{n-1}_{t-1}\|^2 + h^{2k+2} \|c\|^2_{L^\infty(0,T;H^{k+1}(\Omega))} + \tau \int_{n-1}^n \|\partial_t c\|^2 \right).
\end{align}

We substitute the above bounds into (6.84), multiply by $2\tau$, sum from $n = 1$ to $n = m$, and conclude the proof by noticing $\|\xi^n_t\| = 0$.

We now show an $L^2$ bound on $\hat{c}_h^n - \xi^n_{\mu}$. 

Lemma 6.14. There exists a constant $C$ independent of $h$ and $\tau$ such that for all $m \geq 1$, we have
\begin{equation}
\tau \sum_{n=1}^{m} \| \xi^n_m - \xi^n_{m+1} \|^2 \leq C(\tau^2 + h^{2k+2}) + C\tau \sum_{n=0}^{m} \| \xi^n_c \|^2 + C\tau \sum_{n=0}^{m-1} \| \xi^n_{\omega} \|^2.
\end{equation}

**Proof.** Observe that (with (6.79) for the second equation):
\begin{equation}
\begin{align*}
a_{\text{diff}}(\xi^n_{\mu}, \hat{p}^n_h) &= a_{\text{diff}}(\xi^n_{\mu} - \xi^n_{\mu}^\tau, \hat{p}^n_h) = \| \xi^n_{\mu} - \xi^n_{\mu}^\tau \|^2, \\
(\xi^n_c, J(\delta_t \xi^n_c)) &= a_{\text{diff}}(J(\xi^n_{\mu} - \xi^n_{\mu}^\tau), J(\delta_t \xi^n_c)) = (\delta_t \xi^n_c, J(\xi^n_{\mu} - \xi^n_{\mu}^\tau)).
\end{align*}
\end{equation}

Choose $\chi_h = J(\xi^n_{\mu} - \xi^n_{\mu}^\tau)$ in (6.46) and $\varphi_h = J(\delta_t \xi^n_c) = \delta_t \hat{c}^n_h$ in (6.47). Adding the resulting equations, using the identities above and (6.83), we obtain
\begin{equation}
\begin{align*}
&\| \xi^n_{\mu} - \xi^n_{\mu}^\tau \|^2 + \frac{K}{2\tau} (\| \xi^n_c \|^2 - \| \xi^n_{c-1} \|^2 + \| \xi^n_{\omega} - \xi^n_{\omega-1} \|^2) = \\
&-\delta_t c^n + (\partial_t c^n - \delta_t \eta^n_c, J(\xi^n_{\mu} - \xi^n_{\mu}^\tau)) - a_{\text{adv}}(c^n, \hat{u}^n_h, J(\xi^n_{\mu} - \xi^n_{\mu}^\tau)) + a_{\text{adv}}(\hat{c}^n_h, \hat{u}^n, J(\xi^n_{\mu} - \xi^n_{\mu}^\tau)) \\
&+ (\eta^n_{\mu}, \delta_t \hat{c}^n_h) + (\Phi \cdot (c^n) - \Phi \cdot (\hat{c}^n_h), \delta_t \hat{c}^n_h) + (\Phi \cdot (c^n) - \Phi \cdot (\hat{c}^n_{\omega-1}), \delta_t \hat{c}^n_{\omega-1}) = \sum_{i=1}^{6} Z_i.
\end{align*}
\end{equation}

Comparing with (6.84), we remark that $Z_i = W_i$ for $i \in \{4, 5, 6\}$ and we will use the same bounds in the proof of Lemma 6.13. For the other terms, $Z_1, Z_2, Z_3$, we use similar bounds as for the terms $W_1, W_2, W_3$ where $\delta_t \hat{c}^n_h$ is replaced by $\xi^n_{\mu} - \xi^n_{\mu}^\tau$. Thus, we obtain
\begin{equation}
\begin{align*}
\frac{1}{4} \| \xi^n_{\mu} - \xi^n_{\mu}^\tau \|^2 &+ \frac{K}{2\tau} (\| \xi^n_c \|^2 - \| \xi^n_{c-1} \|^2 + \| \xi^n_{\omega} - \xi^n_{\omega-1} \|^2) \\
&\leq C h^{2k+2} + C (\| \delta_t \hat{c}^n_h \|^2 + \| \xi^n_u \|^2 + \| \xi^n_{c-1} \|^2 + \| \xi^n_{\omega-1} \|^2) + C \int_{t^{n-1}}^{t^n} \tau^{-1} h^{2k+2} |\partial_t c|^2_{H^{k+1}(\Omega)}.
\end{align*}
\end{equation}

Multiplying the above bound by $4\tau$, summing from $n = 1$ to $n = m$, and applying the results of Lemma 6.13, we conclude the proof.

**6.5.1. Proof of the improved estimate (6.3).** We now proceed to obtain bounds on $\| \xi^n_u \|$ and $\| \xi^n_c \|$ via constructing a dual Stokes problem and its dG discretization and we follow the argument in [22]. To this end, define the error function
\begin{equation}
\chi_u(t) = u^n_h - u(t), \quad \forall t^{n-1} < t \leq t^n, \quad \forall n \geq 1, \quad \chi_u(0) = u^n_0 - u^0.
\end{equation}

Further, for $t \geq 0$, define $(U(t), P(t)) \in H^1(\Omega)^d \times L^2(\Omega)$ the weak solution of the dual Stokes problem:
\begin{equation}
-\Delta U(t) + \nabla P(t) = \chi_u(t) \quad \text{in } \Omega,
\end{equation}
\begin{equation}
\nabla \cdot U(t) = 0 \quad \text{in } \Omega,
\end{equation}
\begin{equation}
U(t) = 0 \quad \text{on } \partial \Omega,
\end{equation}
and $(U_h(t), P_h(t)) \in X^k_h \times M^{k-1}_{h0}$ its dG solution:
\begin{equation}
a_{\ast}(U_h(t), \theta_h) - b_P(\theta_h, P_h(t)) = (\chi_u(t), \theta_h) \quad \forall \theta_h \in X^k_h,
\end{equation}
\begin{equation}
b_P(U_h(t), q_h) = 0 \quad \forall q_h \in M^{k-1}_{h0}.
\end{equation}
Existence and uniqueness of \((\mathbf{U}_h(t), P_h(t))\) for all \(t > 0\) is a consequence of the coercivity of \(a_{\varepsilon}^\ast\) and the inf-sup condition for the pair of spaces \(\left(\mathbf{X}^1_h, M_{1,0}^{k-1}\right)\) [26]. We take \(\theta_h = \mathbf{U}_h^n = \mathbf{U}_h(t^n)\) in (6.48). We begin with handling the last two terms. Namely, we write

\[
\begin{align*}
&b T (c_h^{n-1}, \mu_h^n, \mathbf{U}_h^n) - b T (c^n, \mu^n, \mathbf{U}_h^n) = a_{\text{adv}}(c_h^{n-1}, \mathbf{U}_h^n, \xi_h^n) + a_{\text{adv}}(c_h^{n-1}, \mathbf{U}_h^n, \eta_h^n) \\
&\quad - a_{\text{adv}}(c^n, c_h^{n-1}, \mathbf{U}_h^n, \mu^n) + a_{\text{adv}}(\xi_h^{n-1}, \mathbf{U}_h^n, \mu^n) + a_{\text{adv}}(\eta_h^{n-1}, \mathbf{U}_h^n, \mu^n) = \sum_{i=1}^5 K_i.
\end{align*}
\]

We rewrite the term \(K_1\) by applying Green’s theorem.

\[
K_1 = \sum_{e \in \mathcal{T}_h} \int_e \left( \nabla c_h^{n-1} \cdot \mathbf{U}_h^n + c_h^{n-1} \mathbf{V} \cdot \mathbf{U}_h^n \right) \left( \frac{\xi_h^n}{\xi_h^\mu} - \frac{\xi_h^n}{\xi_h^{\mu_0}} \right)
\]

\[
- \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \left( c_h^{n-1} (\xi_h^n - \xi_h^{\mu_0}) \right) \left( \mathbf{U}_h^n \cdot \mathbf{n}_e \right) - \sum_{e \in \Gamma_h} \int_e \left( c_h^{n-1} (\xi_h^n - \xi_h^{\mu_0}) \right) \left( \mathbf{U}_h^n \cdot \mathbf{n}_e \right)
\]

With Holder’s and Poincaré’s inequalities and trace estimates, we have

\[
|K_1| \leq C \left( \|\nabla c_h^{n-1}\|_{L^3(\Omega)} + \|c_h^{n-1}\|_{L^6(\Omega)} \right) \|\mathbf{U}_h^n\|_{DG} \|\xi_h^n - \xi_h^{\mu_0}\| + C|c_h^{n-1} - c^{n-1}|_{DG} \|\mathbf{U}_h^n\|_{L^6(\Omega)} \|\xi_h^n - \xi_h^{\mu_0}\|.
\]

Note that with (6.2) in Theorem 6.1, inverse estimate (3.31), Poincaré’s inequality (3.30), and (6.1), we obtain

\[
|c_h^{n-1} - c^{n-1}|_{DG} \|\mathbf{U}_h^n\|_{L^6(\Omega)} \leq \left( \frac{C_{\varepsilon}}{\delta K_\alpha} \right)^{1/2} \left( \tau + h^{2k} \right) \|\mathbf{U}_h^n\|_{DG} \leq C \|\mathbf{U}_h^n\|_{DG}.
\]

We remark that for \(h \leq h_0 \leq \min(1, C_{\varepsilon}^{-1/2}, C_{\varepsilon}^{-1/\delta})\), the constant \(C\) in (6.94) is independent of \(C_{\varepsilon}\).

The term \(K_2\) is handled similarly. That is, we use the same integration by parts formula. Here, we use the trace estimates for functions in \(H^2(\mathcal{T}_h)\) and approximation properties. We have

\[
|K_2| \leq C h^{k+1} |\mu^n|_{H^{k+1}(\Omega)} \|\mathbf{U}_h^n\|_{DG}.
\]

From Lemma 6.7 and Young’s inequality, we have for \(\varepsilon > 0\),

\[
|K_1| + |K_2| \leq \varepsilon \mu_\ast \|\xi_h^n - \xi_h^{\mu_0}\|^2 + C \left( \frac{T + 1}{\varepsilon \mu_\ast} + 1 \right) \|\mathbf{U}_h^n\|^2_{DG} + C \tau^2 \|\mu^n\|^2_{L^6(0,T;H^{k+1}(\Omega))}.
\]

For \(K_3, K_4,\) and \(K_5\), the jumps of \(\mu^n\) evaluate to zero. Thus, with Holder’s inequality, we have

\[
\begin{align*}
|K_3| + |K_4| + |K_5| &\leq C \left( \|c^n - c^{n-1}\| + \|\xi_h^{n-1}\| + \|\xi_h^n\| + \|\eta_h^n\| \right) \|\mathbf{U}_h^n\|_{DG} \|\mu^n\|_{L^{6}(\Omega)} \\
&\leq \varepsilon \mu_\ast \|\xi_h^{n-1}\|^2 + C \left( \frac{1}{\varepsilon \mu_\ast} + 1 \right) \|\mathbf{U}_h^n\|^2_{DG} + C \tau^2 \|\mu^n\|^2_{L^6(0,T;H^{k+1}(\Omega))} + C \tau \int_{t^{n-1}}^{t^n} \|\partial_t c\|^2.
\end{align*}
\]
We refer to the proof of Theorem 1 in [22] to handle all the remaining terms in (6.48) with $\theta_h = U_h^n$ since the same arguments can be used here. For completeness, most details are given in Appendix C. We have

\begin{align}
&K_D \|U_h^n\|_{DG}^2 + \mu_s \tau \sum_{n=1}^m \|\xi_c^n\|^2 \leq C(\tau^2 + h^{2k+2} + \tau h^2)
\end{align}

\begin{align}
&+ C\tau \left( \frac{1}{\mu_s} + 1 \right) \sum_{n=1}^m \|U_h^n\|_{DG}^2 + \tau \sum_{n=1}^m (b_f(c_{h}^{n-1}, \mu_h^n, U_h^n) - b_f(c^n, \mu^n, U_h^n)).
\end{align}

With the bounds on $K_i$’s, we obtain

\begin{align}
&K_D \|U_h^n\|_{DG}^2 + \mu_s \tau \sum_{n=1}^m \|\xi_c^n\|^2 \leq C(\tau^2 + h^{2k+2} + \tau h^2)
\end{align}

\begin{align}
&+ C\left( \frac{T + 1}{\tau} + 1 \right) \tau \sum_{n=1}^m \|U_h^n\|_{DG}^2 + \epsilon \mu_s \tau \sum_{n=1}^m \|\xi_c^{n-1}\|^2 + \epsilon \mu_s \tau \sum_{n=1}^m \|\xi_c^n - \xi_c^{n-1}\|^2.
\end{align}

We next use Lemma 6.14 to bound the last bound above. We then multiply the bound in Lemma 6.13 by $\epsilon \mu_s$ and add to the resulting inequality.

\begin{align}
&K_D \|U_h^n\|_{DG}^2 + \epsilon \mu_s \kappa \|\xi_c^n\|^2 + \mu_s(1 - \hat{C} \epsilon) \tau \sum_{n=1}^m \|\xi_c^n\|^2 \leq C((1 + \epsilon)(\tau^2 + h^{2k+2}) + \tau h^2)
\end{align}

\begin{align}
&+ C\left( \frac{T + 1}{\mu_s} + 1 \right) \tau \sum_{n=1}^m \|U_h^n\|_{DG}^2 + C\epsilon \mu_s \tau \sum_{n=1}^m \|\xi_c^n\|^2 + C\tau \epsilon \mu_s \|\xi_c^0\|^2.
\end{align}

We choose $\epsilon = 1/(2\hat{C})$ and note that $\|\xi_c^0\|^2 \leq Ch^{2k+2}$.

\begin{align}
&K_D \|U_h^n\|_{DG}^2 + \frac{\mu_s \kappa}{2\hat{C}} \|\xi_c^n\|^2 + \frac{\mu_s}{2} \tau \sum_{n=1}^m \|\xi_c^n\|^2 \leq C(\tau^2 + h^{2k+2} + \tau h^2)
\end{align}

\begin{align}
&+ C\left( \frac{T + 1}{\mu_s} + 1 \right) \tau \sum_{n=1}^m \|U_h^n\|_{DG}^2 + C\mu_s \tau \sum_{n=1}^m \|\xi_c^n\|^2.
\end{align}

Therefore, choosing $\tau$ small enough, and applying Gronwall’s inequality, we obtain

\begin{align}
&\mu_s \tau \sum_{n=1}^m \|\xi_c^n\|^2 \leq C(\tau^2 + h^{2k+2} + \tau h^2).
\end{align}

To obtain a bound on $\|\xi_c^0\|$, we use (6.72), (6.96) and (6.2):

\begin{align}
&\mu_s \tau \sum_{n=1}^m \|\xi_c^n\|^2 \leq \mu_s \tau \sum_{n=1}^m \|\xi_c^n\|^2 + \mu_s \tau^3 \sum_{n=1}^m \|\phi_{h,DG}^n\| \leq C(\tau^2 + h^{2k+2} + \tau h^2).
\end{align}

Use (6.96) in Lemma 6.13 and apply Gronwall’s inequality to have

\begin{align}
&\kappa \|\xi_c^n\|^2 \leq C(\tau^2 + h^{2k+2} + \tau h^2).
\end{align}

This bound with Lemma 6.14 and (6.96) yields

\begin{align}
&\mu_s \tau \sum_{n=1}^m \|\xi_c^n - \xi_c^{n-1}\|^2 \leq C(\tau^2 + h^{2k+2} + \tau h^2).
\end{align}
To obtain a bound on $\|\bar{e}_h^n\|$, it suffices to derive a bound on $\bar{e}_h^n$. Let $\varphi_h = 1$ in (6.47). Since the averages of $\eta_{h\mu}$ and of $(c^n - c_h^{n-1})$ are zero, we obtain:

$$\int_{\Omega} \bar{e}_h^n \leq \| (c^n)^2 + c^n c_h^n + (c_h^n)^2 \| \| c^n - c_h^n \| \leq C (\tau^2 + h^{2k+2} + \tau h^2)^{1/2}.$$  

Finally, we conclude the proof of (6.3) by using the bounds above, triangle inequality and approximation error bounds.

7. Numerical Experiments. In this section, our numerical method is first verified via convergence rate tests. Next, the spinodal decomposition simulation shows the proposed algorithm enjoys mass conservation and energy dissipation properties. For all the numerical results, we choose $\sigma_x = 1/12$.

7.1. Convergence study. We utilize the manufactured solution method for convergence study on the unit cube $\Omega = (0, 1)^3$. The simulation end time is $T = 1$. For convenience, we select parameters $\mu_u = 1$ and $\kappa = 1$. The prescribed solution is defined as follows [20].

$$c(t, x, y, z) = \exp(-t) \sin(2\pi x) \sin(2\pi y) \sin(2\pi z),$$

$$v_1(t, x, y, z) = -\exp(-t + x) \sin(y + z) - \exp(-t + z) \cos(x + y),$$

$$v_2(t, x, y, z) = -\exp(-t + y) \sin(x + z) - \exp(-t + x) \cos(y + z),$$

$$v_3(t, x, y, z) = -\exp(-t + z) \sin(x + y) - \exp(-t + y) \cos(x + z),$$

$$p(t, x, y, z) = -\exp(-2t) \left( \exp(x + z) \sin(y + z) \cos(x + y) + \exp(x + y) \sin(x + z) \cos(y + z) + \exp(y + z) \sin(x + y) \cos(x + z) + \frac{1}{2} \exp(2x) + \frac{1}{2} \exp(2y) + \frac{1}{2} \exp(2z) - \bar{p}^0 \right),$$

where, $\bar{p}^0 = 7.63958172715414$, which guarantees zero average pressure over $\Omega$ (up to machine precision). Here in above, the order parameter field is taken from [11]. The chemical potential is an intermediate variable, which value is derived by the order parameter. The velocity and pressure fields are borrowed from the Beltrami flow [18], which enjoys the property that the nonlinear convection is balanced by the pressure gradient and the velocity is parallel to vorticity. In addition, the initial conditions and Dirichlet boundary condition for velocity are imposed by the manufactured solutions.

For order parameter and chemical potential, we apply Neumann boundary condition.

We obtain spatial rates of convergence by computing the solutions on a sequence of uniformly refined meshes (see the second column of Table 1 for $h$). We fix $\tau = 1/2^{10}$ for $k = 1$ ($P1$–$P1$–$P1$–$P0$ scheme); we fix $\tau = 1/2^{13}$ for $k = 2$ ($P2$–$P2$–$P2$–$P1$ scheme); and we fix $\tau = 1/2^{15}$ for $k = 3$ ($P3$–$P3$–$P3$–$P2$ scheme) to guarantee the spatial error dominates. We use SIPG and add subscript to distinguish the penalty parameter $\tilde{\sigma}$ in form $a_{diff}$, namely in (3.1)-(3.2) by $\tilde{\sigma}_{CH}$ and in (3.4) by $\tilde{\sigma}_{ellip}$, respectively. Recall the penalty parameter in form $a$ is $\sigma$. For $P1$–$P1$–$P1$–$P0$ scheme, we set $\tilde{\sigma}_{CH} = 2$, $\tilde{\sigma}_{ellip} = 1$, $\sigma = 8$ on $\Gamma_h$ and $\sigma = 16$ on $\partial\Omega$. For $P2$–$P2$–$P2$–$P1$ scheme, we set $\tilde{\sigma}_{CH} = 4$, $\tilde{\sigma}_{ellip} = 2$, $\sigma = 64$ on $\Gamma_h$ and $\sigma = 128$ on $\partial\Omega$. For $P3$–$P3$–$P3$–$P2$ scheme, we set $\tilde{\sigma}_{CH} = 8$, $\tilde{\sigma}_{ellip} = 8$, $\sigma = 128$ on $\Gamma_h$ and $\sigma = 256$ on $\partial\Omega$. If $err_h$ is the error on a mesh with resolution $h$, then the rate is defined by $\ln(err_h/err_{h/2})/\ln(2)$. We show the errors and rates in Table 1. The convergence rates are optimal.
The polynomial degree is \( k \). The elements of edge length equal to \( 10^{-2} \) and the total energy is dissipated. Throughout this process, the mass of the system is conserved and the total energy is dissipated.

The computational domain, \( \Omega = (0, 1)^3 \), is partitioned uniformly into cubic elements of edge length equal to \( 10^{-2} \). We select the time step size \( \tau = 10^{-3} \). The initial order parameter field is generated by sampling numbers from a discrete uniform distribution, namely, \( \tilde{c}_i^0 \sim U\{-1, +1\} \). The initial velocity field is taken to be zero. The polynomial degree is \( k = 1 \). We choose the viscosity \( \mu_s = 1 \) and parameter \( \kappa = 10^{-4} \). Following the same notation in Section 7.1, the penalty parameters are \( \tilde{\sigma}_{\text{CH}} = 2 \), \( \tilde{\sigma}_{\text{slip}} = 1 \), and \( \sigma = 8 \). We compute \( 2^{15} \) time steps in total, which is equivalent to the end time \( T = 32.768 \). Figure 1 displays 3D snapshots of the order parameter field. Figure 2 shows that the mass conservation and energy dissipation properties are satisfied.

8. Conclusion. The main contribution of this paper is the analysis of a fully decoupled splitting dG algorithm for solving the CHNS equations. Existence and uniqueness of the discrete solution are shown. We prove mass conservation, energy stability, and \( L^\infty \) stability of the discrete order parameter. Optimal a priori error estimates in both the broken \( H^1 \) norm and the \( L^2 \) norm are derived, under a CFL condition. The analysis relies on induction and requires multiple steps and intermediate results. The analysis is novel and the results are stronger because they do not assume any regularization of the potential function. Numerical experiments verify the theoretical results. Extending the numerical analysis to a higher order in time stepping method will be the object of future work.

| \( k \) | \( h \) | \( \|c_h^N - c(T)\| \) rate | \( \|u_h^N - u(T)\| \) rate | \( \|p_h^N - p(T)\| \) rate |
|---|---|---|---|---|
| 1 \( 1/2^2 \) | 6.363 E-2 | — | 2.306 E-2 | — | 1.449 E-1 | — |
| \( 1/2^3 \) | 2.599 E-2 | 1.292 | 3.342 E-3 | 2.787 | 3.417 E-1 | -1.238 |
| \( 1/2^4 \) | 8.242 E-3 | 1.657 | 7.918 E-4 | 2.078 | 1.381 E-1 | 1.307 |
| \( 1/2^5 \) | 2.241 E-3 | 1.879 | 1.988 E-4 | 1.994 | 4.082 E-2 | 1.758 |
| \( 1/2^6 \) | 5.767 E-4 | 1.958 | 5.071 E-5 | 1.971 | 1.136 E-2 | 1.845 |
| 2 \( 1/2^1 \) | 4.939 E-2 | — | 6.827 E-3 | — | 1.810 E-1 | — |
| \( 1/2^2 \) | 2.413 E-2 | 1.033 | 1.014 E-3 | 2.751 | 3.061 E-1 | -0.758 |
| \( 1/2^3 \) | 3.112 E-3 | 2.955 | 1.327 E-4 | 2.934 | 4.210 E-2 | 2.862 |
| \( 1/2^4 \) | 3.285 E-4 | 3.244 | 1.512 E-5 | 3.134 | 3.377 E-3 | 3.640 |
| \( 1/2^5 \) | 3.741 E-5 | 3.135 | 1.638 E-6 | 3.207 | 2.972 E-4 | 3.506 |
| 3 \( 1/2^0 \) | 1.553 E-1 | — | 5.130 E-3 | — | 9.792 E-1 | — |
| \( 1/2^1 \) | 6.856 E-2 | 1.180 | 1.035 E-3 | 2.309 | 9.538 E-1 | 0.038 |
| \( 1/2^2 \) | 5.764 E-3 | 3.572 | 1.761 E-4 | 2.555 | 6.414 E-2 | 3.895 |
| \( 1/2^3 \) | 3.279 E-4 | 4.136 | 6.186 E-6 | 4.832 | 2.365 E-3 | 4.762 |
| \( 1/2^4 \) | 1.989 E-5 | 4.043 | 1.829 E-7 | 5.080 | 7.923 E-5 | 4.900 |

Table 1

Errors and spatial convergence rates of order parameter, velocity, and pressure.

7.2. Spinodal decomposition. Spinodal decomposition serves as a widely used benchmark test for the validation of numerical algorithms on solving the CHNS equations. As a mechanism of phase separation, an initially thermodynamically unstable homogeneous mixture decomposes into two separated phases, which are more thermodynamically favorable. Throughout this process, the mass of the system is conserved and the total energy is dissipated.

The computational domain, \( \Omega = (0, 1)^3 \), is partitioned uniformly into cubic elements of edge length equal to \( 10^{-2} \). We select the time step size \( \tau = 10^{-3} \). The initial order parameter field is generated by sampling numbers from a discrete uniform distribution, namely, \( c^0_i \sim U\{-1, +1\} \). The initial velocity field is taken to be zero. The polynomial degree is \( k = 1 \). We choose the viscosity \( \mu_s = 1 \) and parameter \( \kappa = 10^{-4} \). Following the same notation in Section 7.1, the penalty parameters are \( \tilde{\sigma}_{\text{CH}} = 2 \), \( \tilde{\sigma}_{\text{slip}} = 1 \), and \( \sigma = 8 \). We compute \( 2^{15} \) time steps in total, which is equivalent to the end time \( T = 32.768 \). Figure 1 displays 3D snapshots of the order parameter field. Figure 2 shows that the mass conservation and energy dissipation properties are satisfied.
Fig. 1. Selected snapshots for order parameter in spinodal decomposition at time step $2^n$, where $n = 1, 3, \cdots, 15$. The phase A (order parameter $c_h^n = +1$) is displayed in red and the phase B (order parameter $c_h^n = -1$) is in transparent. The green surface corresponds to the center of diffusive interface (order parameter $c_h^n = 0$).

Fig. 2. The total mass of the mixture system (left) and log-log plot of the total energy (right).

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Appendix A. Proof of Lemma 6.8.

Proof. Let $\mathcal{A} = a_{\text{adv}}(c^n, u^n, \chi_h) - a_{\text{adv}}(c_h^{n-1}, u_h^{n-1}, \chi_h)$ and write

$$
\mathcal{A} = a_{\text{adv}}(c^n - c^{n-1}, u^n, \chi_h) + a_{\text{adv}}(c^{n-1}, u^n - u^{n-1}, \chi_h) - a_{\text{adv}}(\eta_0^{n-1}, u^{n-1}, \chi_h) - a_{\text{adv}}(\xi_0^{n-1}, u^{n-1}, \chi_h) - a_{\text{adv}}(c_h^{n-1}, \xi_0^{n-1}, \chi_h) - a_{\text{adv}}(c_h^{n-1}, \eta_0^{n-1}, \chi_h) = \sum_{i=1}^{6} \mathcal{A}_i.
$$
We begin with bounding $\mathcal{A}_1$. With Holder’s inequality, trace estimate (3.34), Young’s inequality and a Taylor expansion at $t^{n-1}$, we obtain

\begin{equation}
|\mathcal{A}_1| \leq C\|u^n\|_{L^\infty(\Omega)}|\chi_h|_{DG}(\|u^n - u^{n-1}\| + h\|\nabla(u^n - u^{n-1})\|)
\leq \frac{C}{\epsilon}\|u\|^2_{L^\infty(0,T;H^2(\Omega))}\int_{t^{n-1}}^{t^n} \|\partial_t c\|^2_{H^1(\Omega)} + \epsilon|\chi_h|^2_{DG}.
\end{equation}

For $\mathcal{A}_2$, similar arguments yield

\begin{equation}
|\mathcal{A}_2| \leq C\|c^{n-1}\|_{L^\infty(\Omega)}|\chi_h|_{DG}(\|\nabla u^{n-1}\| + h\|\nabla(u^n - u^{n-1})\|)
\leq \frac{C}{\epsilon}h^{2k+2}\|u\|^2_{L^\infty(0,T;H^2(\Omega))}\|c\|^2_{L^\infty(0,T;H^2(\Omega))} + \epsilon|\chi_h|^2_{DG}.
\end{equation}

For $\mathcal{A}_3$, we utilize the convexity of the computational domain. Namely, using (6.6), we have

\begin{equation}
|\mathcal{A}_3| \leq C\|c^{n-1}\|_{L^\infty(\Omega)}|\chi_h|_{DG}(\|\nabla u^{n-1}\| + h\|\nabla u^{n-1}\|)
\leq \frac{C}{\epsilon}h^{2k+2}\|u\|^2_{L^\infty(0,T;H^2(\Omega))}\|\nabla u^{n-1}\|^2_{L^\infty(0,T;H^2(\Omega))} + \epsilon|\chi_h|^2_{DG}.
\end{equation}

For $\mathcal{A}_4$, with Holder’s inequality, trace estimate (3.33), and Young’s inequality, we obtain the following bound.

\begin{equation}
|\mathcal{A}_4| \leq C\|c^{n-1}\|\|u^{n-1}\|_{L^\infty(\Omega)}|\chi_h|_{DG} \leq \frac{C}{\epsilon}\|c^{n-1}\|^2\|u\|^2_{L^\infty(0,T;H^2(\Omega))} + \epsilon|\chi_h|^2_{DG}.
\end{equation}

For $\mathcal{A}_5$ and $\mathcal{A}_6$, we use trace inequalities (3.34) and (3.33) respectively. We obtain

\begin{equation}
|\mathcal{A}_5| + |\mathcal{A}_6| \leq C\|c^{n-1}\|_{L^\infty(\Omega)}(\|\nabla u^{n-1}\| + h\|\nabla u^{n-1}\| + \|\xi^{n-1}\|)|\chi_h|_{DG}
\leq \frac{C}{\epsilon}\|c^{n-1}\|^2_{L^\infty(\Omega)}(h^{2k+2}\|u\|^2_{L^\infty(0,T;H^{k+1}(\Omega))} + \|\xi^{n-1}\|^2)^2 + \epsilon|\chi_h|^2_{DG}.
\end{equation}

Combining the bounds on $\mathcal{A}_i$ for $i = 1, \ldots, 6$ yields the result.

**Appendix B. Proof of bound (6.63).** From Lemma 6.3 in [23], for all $q_h \in M_h^{k-1}$ and $n \geq 1$, the following equalities holds:

\begin{equation}
b_p(\xi^{n}_\nu, q_h) = b_p(\xi^{n}_\nu, q_h) + \tau a_{\text{diff}}(\phi^n H^\nu, q_h)
- \tau \sum_{e \in \Gamma_h} \bar{\sigma} \int_{e} [\phi^n H^\nu \|q_h\| + \tau(G_h([\phi^n H^\nu]), G_h([q_h])),
\end{equation}

\begin{equation}
b_p(\xi^{n}_\nu, q_h) = -\tau \sum_{e \in \Gamma_h} \bar{\sigma} \int_{e} [\phi^n H^\nu \|q_h\| + \tau(G_h([\phi^n H^\nu]), G_h([q_h])),
\end{equation}

In addition, for $n \geq 1$ we have [23]

\begin{equation}
\|\xi^{n}_\nu - \xi^{n-1}_\nu\|^2 = \|\xi^{n}_\nu - \xi^{n-1}_\nu\|^2 + \tau^2(\|\nabla\phi^n H^\nu\|^2 + \|G_h([\phi^n H^\nu])\|^2) + \tau^2(A^n - A^{n-1})
+ \tau^2\left(\sum_{e \in \Gamma_h} \bar{\sigma} \int_{e} [\phi^n H^\nu - \phi^{n-1}_H\|q_h\| - \tau(G_h([\phi^n H^\nu]), G_h([q_h])),
\end{equation}

\begin{equation}
- 2\tau^2(\nabla\phi^n H^\nu, G_h([\phi^n H^\nu])) + 2\delta_{n,1} \tau b_p(\xi^{n}_\nu, \phi^n H^\nu).
\end{equation}
Using (3.24) and assuming $\tilde{\sigma} > 2\tilde{M}_k^2$, we have

$$
\|G_h(\phi^n_h - \phi^{n-1}_h)\|^2 \leq \frac{1}{4} \sum_{e \in \mathcal{H}} \tilde{M}_k^2 \|G_h(\phi^n_h - \phi^{n-1}_h)\|_{L^2(e)}^2 \leq \frac{1}{2} \sum_{e \in \mathcal{H}} \tilde{\sigma} \|\phi^n_h - \phi^{n-1}_h\|_{L^2(e)}^2.
$$

Similarly, with Cauchy–Schwarz’s inequality, Young’s inequality, (3.24), and the assumption that $\tilde{\sigma} \geq 4\tilde{M}_k^2$, we obtain

\[
|\langle \nabla \phi^n_h, G_h(\phi^n_h) \rangle| \leq \frac{1}{4} \|\nabla \phi^n_h\|^2 + \|G_h(\phi^n_h)\|^2 \leq \frac{1}{4} \|\nabla \phi^n_h\|^2 + \frac{1}{4} \sum_{e \in \mathcal{H}} \tilde{\sigma} \|\phi^n_h\|_{L^2(e)}^2.
\]

Using the above bounds in (B.3), we obtain:

\[
(B.4) \quad \frac{1}{2} \|\xi^n_u - \xi^{n-1}_u\|^2 + \frac{\tau^2}{4} \|\nabla \phi^n_h\|^2 + \frac{\tau^2}{2} \|G_h(\phi^n_h)\|^2 + \frac{\tau^2}{2} (A_1 - A_2) + \frac{\tau^2}{4} \sum_{e \in \mathcal{H}} \tilde{\sigma} \|\phi^n_h - \phi^{n-1}_h\|_{L^2(e)}^2 \leq \frac{1}{2} \|\xi^n_u - \xi^{n-1}_u\|^2 + \frac{\tau^2}{2} \sum_{e \in \mathcal{H}} \tilde{\sigma} \|\phi^n_h\|_{L^2(e)}^2 + \delta_{n,1}\|b_P(\xi^n_u, \phi^n_h)\|.
\]

Inserting $\Pi_h u^n$ in (6.95) yields:

\[
(B.5) \quad (\xi^n_u, \theta_h) = (\xi^n_v, \theta_h) + \tau b_P(\theta_h, \phi^n_h), \quad \forall \theta_h \in X^k_h.
\]

Let $\theta_h = \xi^n_u$ in (B.5) and use (B.2), we obtain

\[
(B.6) \quad \frac{1}{2} \|\xi^n_u - \xi^n_v\|^2 + \frac{1}{2} \|\xi^n_u - \xi^n_u\|^2 + \frac{\tau^2}{2} \sum_{e \in \mathcal{H}} \tilde{\sigma} \|\phi^n_h\|_{L^2(e)}^2 = \tau^2 \|\nabla \phi^n_h\|^2 + \frac{\tau^2}{2} \sum_{e \in \mathcal{H}} \tilde{\sigma} \|\phi^n_h\|_{L^2(e)}^2.
\]

We let $\theta_h = \xi^n_u - \xi^n_v$ in (B.5) and use (B.1). We have

\[
\|\xi^n_u - \xi^n_v\|^2 = \tau^2 b(\xi^n_u - \xi^n_v, \phi^n_h)
\]

\[
\quad = \tau^2 a_{\text{diff}}(\phi^n_h, \phi^n_h) - \tau^2 \sum_{e \in \mathcal{H}} \tilde{\sigma} \|\phi^n_h\|_{L^2(e)}^2 + \tau^2 \|G_h(\phi^n_h)\|^2.
\]

Hence, (B.6) reads

\[
(B.7) \quad \frac{1}{2} \|\xi^n_u - \xi^n_v\|^2 \leq \tau^2 a_{\text{diff}}(\phi^n_h, \phi^n_h) + \tau^2 \sum_{e \in \mathcal{H}} \tilde{\sigma} \|\phi^n_h\|_{L^2(e)}^2 = \tau^2 \|G_h(\phi^n_h)\|^2.
\]

Taking the sum of (B.4) with (B.7) and multiplying by $1/\tau$ yield the result.

**Appendix C. Proof of bound (6.95).** We define the norm $\|\cdot\| = \|\cdot\|_{L^\infty(\Omega)} + \|\cdot\|_{W^{1,2}(\Omega)}$. To start the proof, we first note that:

\[
(C.1) \quad \|\mathbf{U}(t) - \mathbf{U}_h(t)\| + h\|\mathbf{U}(t) - \mathbf{U}_h(t)\|_{DG} + h\|P(t) - P_h(t)\| \leq C h^2 \|\chi_u(t)\|,
\]

\[
(C.2) \quad \|\mathbf{U}(t)\| + \|\mathbf{U}_h(t)\|_{L^\infty(\Omega)} \leq C \|\chi_u(t)\|.
\]
The above estimates result from the error analysis of the dG formulation for the dual problem (6.89)-(6.91). For more details, we refer to Lemma 5 in [22]. Choosing $\mathbf{\theta}_h = \mathbf{U}_h^n$ in (6.48) and multiplying by $\tau$ yields

\[(C.3) \quad (\mathbf{v}_h^n - \mathbf{u}^n - \mathbf{\chi}_u^{n-1}, \mathbf{U}_h^n) + \tau \tilde{R}_C(\mathbf{U}_h^n) + \tau \mu_a a_D(\mathbf{n}_h^n - \mathbf{u}^n, \mathbf{U}_h^n) = \tau b_f(\mathbf{U}_h^n, p_h^n - p^n) + \tau (\partial_t \mathbf{u}^n - (\mathbf{u}^n - \mathbf{u}^{n-1}), \mathbf{U}_h^n) + \tau b_f(c_h^{n-1}, \mu_h^{n}, \mathbf{U}_h^n) - \tau b_f(c^n, \mu^n, \mathbf{U}_h^n),\]

where

\[\tilde{R}_C(\mathbf{U}_h^n) = a_C(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{\nu}_h^n) - a_C(\mathbf{u}_h^n; \mathbf{u}_h^n, \mathbf{\nu}_h^n).\]

Let us begin with the first term on the right-hand side of (C.3). We use the fact that $\mathbf{U}_h^n$ satisfies (6.93) and the definition of $\pi_h p^n$ to obtain the following.

\[b_f(\mathbf{U}_h^n, p_h^n - p^n) = -b_f(\mathbf{U}_h^n, p^n - \pi_h p^n) = \sum_{\epsilon \in \Gamma_h \cup \partial \Omega} \int_\epsilon \{p^n - \pi_h p^n\}(\mathbf{U}_h^n) : \mathbf{n}_\epsilon.\]

Let $\Delta_e$ denote the union of the two elements sharing a face $\epsilon$. By a trace inequality, approximation property (6.12), (C.1), and the fact that $[\mathbf{U}^n] = 0$ a.e. on any face $\epsilon \in \Gamma_h \cup \partial \Omega$ since $\mathbf{U}^n \in H^2_h(\Omega)^d$, we obtain for any $\epsilon > 0$.

\[(C.4) \quad |b_f(\mathbf{U}_h^n, p_h^{n-1} - p^n)| \leq C h^{k+1}|p^n|_{H^{k+1}(\Omega)}||\mathbf{x}_h^n|| \leq \epsilon \mu_a ||\mathbf{x}_h^n||^2 + C h^{2k+2} \left(1 + \frac{1}{\epsilon \mu_a}\right).\]

In the above, we used that

\[(C.5) \quad ||\mathbf{x}_h^n|| \leq C(h^{k+1}|\mathbf{u}^n|_{H^{k+1}(\Omega)} + ||\mathbf{x}_h^n||),\]

which is obtained by applying the triangle inequality and approximation property (6.9). This bound will be used repeatedly in this proof. For the second term on the right-hand side of (C.3), we simply have:

\[(C.6) \quad |(\tau(\partial_t \mathbf{u})^n - (\mathbf{u}^n - \mathbf{u}^{n-1}), \mathbf{U}_h^n)| \leq C \tau^2 \int_{t_{n-1}}^{t_n} ||\partial_t \mathbf{u}||^2 + \tau \|\mathbf{U}_h^n\|_{DG}^2.\]

We now consider the terms on the left-hand side of (C.3). With (3.6) and (6.93), we have

\[(v_h^n - \mathbf{u}^n - \mathbf{\chi}_u^{n-1}, \mathbf{U}_h^n) = (\mathbf{\chi}_u^n - \mathbf{\chi}_u^{n-1}, \mathbf{U}_h^n) - \tau b_f(\mathbf{U}_h^n, \phi_h^n) = (\mathbf{\chi}_u^n - \mathbf{\chi}_u^{n-1}, \mathbf{U}_h^n),\]

Note that from (6.92), (6.93), the above equality, and the symmetry of $a_D(\cdot, \cdot)$, we have

\[(C.7) \quad (v_h^n - \mathbf{u}^n - \mathbf{\chi}_u^{n-1}, \mathbf{U}_h^n) = a_D(\mathbf{U}_h^n - \mathbf{U}_h^{n-1}, \mathbf{U}_h^n)\]

\[= \frac{1}{2} (a_D(\mathbf{U}_h^n, \mathbf{U}_h^n) - a_D(\mathbf{U}_h^{n-1}, \mathbf{U}_h^{n-1}) + a_D(\mathbf{U}_h^n - \mathbf{U}_h^{n-1}, \mathbf{U}_h^n - \mathbf{U}_h^{n-1})).\]

In addition, we write

\[(C.8) \quad a_D(v_h^n - \mathbf{u}^n, \mathbf{U}_h^n) = a_D(\mathbf{x}_h^n - \mathbf{x}_h^{n-1}, \mathbf{U}_h^n) + a_D(\mathbf{x}_h^n, \mathbf{U}_h^n) + a_D(\mathbf{\xi}_h^n, \mathbf{U}_h^n).\]

To handle the last term in above equality, let $Q_h \mathbf{u}^n$ be the elliptic projection of $\mathbf{u}^n$ onto the space $X_h$. Since the domain is convex, this projection satisfies [26]:

\[(C.9) \quad \forall \theta_h \in X_h, \quad a_D(\mathbf{u}^n - Q_h \mathbf{u}^n, \theta_h) = 0 \quad \text{and} \quad ||\mathbf{u}^n - Q_h \mathbf{u}^n|| \leq C h^{k+1}|\mathbf{u}^n|_{H^{k+1}(\Omega)}.\]
Let \( \theta_h = \Pi_h u^n - Q_h u^n \) in (6.92). We obtain
\[
a_D(\eta^n_h, U^n_h) = a_D(\Pi_h u^n - Q_h u^n, U^n_h) = \langle \chi^n, \Pi_h u^n - Q_h u^n \rangle + b_F(\Pi_h u^n - Q_h u^n, P^n_h).
\]

We have
\[
|b_F(\Pi_h u^n - Q_h u^n, P^n_h)| \leq C \|\Pi_h u^n - Q_h u^n\| \|P^n_h\|_{DG}.
\]

Further, with approximation properties and (C.5), we obtain
\[
(C.10) \quad \|P^n_h\|_{DG} \leq \|P^n - \pi_h P^n\|_{DG} + \|\pi_h P^n\|_{DG} \leq C h^{-1} \|P^n - \pi_h P^n\| + C \|P^n\|_{H^1(\Omega)}
\]
\[
\leq C(\|\Sigma^n_{\Sigma_u}\| + h^{k+1} |u^n|_{H^{k+1}(\Omega)}).
\]

Hence, Cauchy–Schwarz’s inequality, the above bounds, the regularity assumption that \( u \in L^\infty(0,T;H^{k+1}(\Omega)^d) \), and Young’s inequality yield for any \( \epsilon > 0 \):
\[
(C.11) \quad |a_D(\Pi_h u^n - u^n, \u^n_h)| \leq C \|\Pi_h u^n - Q_h u^n\|\|\Sigma^n_{\Sigma_u}\| + h^{k+1} |u^n|_{H^{k+1}(\Omega)}.
\]
\[
\leq C h^{k+1} |u^n|_{H^{k+1}(\Omega)}(\|\Sigma^n_{\Sigma_u}\| + h^{k+1} |u^n|_{H^{k+1}(\Omega)}) \leq \epsilon \|\Sigma^n_{\Sigma_u}\|^2 + C \left( \frac{1}{\epsilon} + 1 \right) h^{2k+2}.
\]

Consider now the second term in (C.8). Letting \( \theta_h = \Sigma^n_{\Sigma_u} \) in (6.92), we obtain
\[
(C.12) \quad a_D(\Sigma^n_{\Sigma_u}, U^n_h) = \|\Sigma^n_{\Sigma_u}\|^2 + (\chi^n - \Sigma^n_{\Sigma_u}, \Sigma^n_{\Sigma_u}) + b_F(\Sigma^n_{\Sigma_u}, P^n_h).
\]

With (C.4), (C.6), (C.7), (C.8), (C.11), and (C.12), the equality (C.3) becomes
\[
(C.13) \quad \frac{1}{2} \left( a_D(U^n_h, U^n_h) - a_D(U^n_h - U^{n-1}_h, U^n_h - U^{n-1}_h) \right) + \tau \mu_u \|\Sigma^n_{\Sigma_u}\|^2
\]
\[
\leq C \left( \mu_u \left( 1 + \frac{1}{\epsilon} \right) + \frac{1}{\epsilon \mu_u} \right) \tau h^{2k+2} + C \tau^2 \int_{t_{n-1}}^{t_n} \|\partial_t u\|^2 + C \tau \|\Sigma^n_{\Sigma_u}\|^2
\]
\[
- \tau \tilde{R}_{C}(\Sigma^n_{\Sigma_u}) + b_F(c^n, \mu^n, \Sigma^n_{\Sigma_u}) - b_F(c^n, \mu^n, \Sigma^n_{\Sigma_u}) + 2\epsilon \tau \mu_u \|\Sigma^n_{\Sigma_u}\|^2
\]
\[
- \tau \mu_a a_D(\Sigma^n_{\Sigma_u}, \Sigma^n_{\Sigma_u}) - \mu_u (\Sigma^n_{\Sigma_u}, \Sigma^n_{\Sigma_u}) - \mu_u b_F(\Sigma^n_{\Sigma_u}, P^n_h).
\]

We now handle the last three terms in the above bound. Let \( \theta_h = \Sigma^n_{\Sigma_u} - \Sigma^n_{\Sigma_u} \) in (6.92).

\[
(C.14) \quad a_D(\Sigma^n_{\Sigma_u} - \Sigma^n_{\Sigma_u}, U^n_h) = (\chi^n - \Sigma^n_{\Sigma_u}, \Sigma^n_{\Sigma_u}) + b_F(\Sigma^n_{\Sigma_u} - \Sigma^n_{\Sigma_u}, P^n_h).
\]

Recall that by (3.6), (B.2), (3.24), and the assumption that \( \tilde{\alpha} \geq \tilde{M}^2 \), we have
\[
(C.15) \quad (\Sigma^n_{\Sigma_u} - \Sigma^n_{\Sigma_u}) = -\tau b_F(\Sigma^n_{\Sigma_u}, \phi^n) = \tau^2 \sum_{e \in E_h} \frac{\tilde{\alpha}}{h_e} \|\phi^n\|_{L^2(e)}^2 - \tau^2 \|G_h([\phi^n])\|^2 \geq 0.
\]

Using Cauchy–Schwarz’s inequality, (6.9) and (6.72), we have the following bound,
\[
|\phi^n - \Sigma^n_{\Sigma_u}| + b_F(\Sigma^n_{\Sigma_u} - \Sigma^n_{\Sigma_u}, P^n_h) \leq C \|\phi^n - \Sigma^n_{\Sigma_u}\|\|h^{k+1} |u^n|_{H^{k+1}(\Omega)} + \|P^n_h\|_{DG})
\]
\[
(C.16) \quad \leq C \tau \|\phi^n\|_{DG}(h^{k+1} |u^n|_{H^{k+1}(\Omega)} + \|P^n_h\|_{DG}).
\]

Therefore with (C.10), the regularity assumption that \( u \in L^\infty(0,T;H^{k+1}(\Omega)^d) \), and Young’s inequality, the bound (C.16) becomes
\[
(C.17) \quad |\phi^n - \Sigma^n_{\Sigma_u}| + b_F(\Sigma^n_{\Sigma_u} - \Sigma^n_{\Sigma_u}, P^n_h) \leq C \tau \|\phi^n\|_{DG}(h^{k+1} |u^n|_{H^{k+1}(\Omega)} + \|\Sigma^n_{\Sigma_u}\|)
\]
\[
\leq \epsilon \|\Sigma^n_{\Sigma_u}\|^2 + C \left( \frac{1}{\epsilon} + 1 \right) \tau^2 \phi^n_{DG}^2 + Ch^{2k+2}.
\]
With (6.9), we have

\[(C.18) \quad |(u_{n}, \xi_n^u)| \leq \frac{C}{\varepsilon} h^{2k+2} |u_n^m|_{H^{k+1}}^2.\]

To handle the last term in (C.13), we use (B.2), (3.24), and (C.10).

\[(C.19) \quad |b_P(\xi_n^u, P_n^m)| = \left| - \tau \sum_{e \in F_h} \frac{\tilde{h}}{\tilde{h}_e} \int_{e} [\phi_n(U) + \tau(G_h([\phi_n^u]), G_h([P_n^m]))] \right| \leq C \tau |\phi_n^u|_{DG} |P_n^m|_{DG} \leq \varepsilon |\xi_n^u|^2 + C \left( 1 + \frac{1}{\varepsilon} \right) \tau^2 |\phi_n^u|_{DG}^2 + Ch^{2k+2}.\]

With the above bounds combined, (C.13) becomes

\[(C.20) \quad \frac{1}{2} (a_D(U_h^n, U_h^n) - a_D(U_h^{n-1}, U_h^{n-1}) + a_D(U_h^n - U_h^{n-1}, U_h^n - U_h^{n-1})) + \tau \mu_s |\xi_n^u|^2 \leq C \left( \mu_s \left( 1 + \frac{1}{\varepsilon} \right) + 1 + \frac{1}{\varepsilon \mu_s} \right) \tau h^{2k+2} + C \tau^2 \int_{n-1}^n \|\partial_n u\|^2 + C \tau |U_h^n|_{DG}^2 - \tau \tilde{R}_C(U_h^n) + \tau b_f(c_h^{n-1}, \mu_h^{n-1}, U_h^{n-1}) - \tau b_f(c_h^n, \mu_h^n, U_h^n) + 5\varepsilon \tau \mu_s |\xi_n^u|^2 + C \mu_s \left( 1 + \frac{1}{\varepsilon} \right) \tau^3 |\phi_n^u|_{DG}^2.\]

The bound for the nonlinear term \(\tilde{R}_C(U_h^n)\) is technical and can be found in [22]. Namely, we have

\[(C.21) \quad |\tilde{R}_C(U_h^n)| \leq 7\varepsilon \mu_s |\xi_n^u|^2 + 2\varepsilon |\xi_n^{n-1} - \xi_n^u|^2 + C \left( \frac{h^2}{\varepsilon \mu_s} + 1 \right) (\tau^2 |\phi_n^u|_{DG}^2 + h^{2k+2}) + C \left( \frac{1}{\varepsilon \mu_s} + 1 \right) (h^2 (|\xi_n^u|^2 + |\xi_n^{n-1}|^2) + (|\xi_n^{n-1}|^2 + h^2) |\xi_n^u|^2_{DG}) + C \tau \int_{n-1}^n \|\partial_n u\|^2 + C \left( \frac{1}{\varepsilon \mu_s} + 1 \right) |U_h^n|_{DG}^2.\]

We use (C.21) in (C.20), use the coercivity property (3.16), and choose \(\varepsilon = 1/24\). We sum the resulting equation, from \(n = 1\) to \(n = m\), use the regularity assumptions, and obtain the following.

\[(C.22) \quad \frac{1}{2} a_D(U_h^m, U_h^m) - \frac{1}{2} a_D(U_h^0, U_h^0) + \frac{K_D}{4} \sum_{n=1}^{m} \|U_h^n - U_h^{n-1}\|_{DG}^2 + \frac{\mu_s \tau}{2} \sum_{n=1}^{m} |\xi_n^u|^2 \leq C \left( 1 + \frac{1}{\mu_s} + \mu_s \right) h^{2k+2} + C \tau^2 + C \tau^3 \left( \frac{h^2}{\mu_s} + 1 + \mu_s \right) \sum_{n=1}^{m} |\phi_n^u|_{DG}^2 + \frac{\tau}{12} \sum_{n=1}^{m} |\xi_n^u - \xi_n^{n-1}|^2 + C \tau \left( \frac{1}{\mu_s} + 1 \right) \sum_{n=1}^{m} |U_h^n|_{DG}^2 + C \left( \frac{1}{\mu_s} + 1 \right) \tau \sum_{n=1}^{m} \left( h^2 (|\xi_n^u|^2 + |\xi_n^{n-1}|^2) + (|\xi_n^{n-1}|^2 + h^2) |\xi_n^u|^2_{DG} \right) + \tau \sum_{n=1}^{m} \left( b_f(c_h^{n-1}, \mu_h^{n-1}, U_h^{n-1}) - b_f(c_h^n, \mu_h^n, U_h^n) \right).\]
Note that by (6.92)-(6.93), we easily see that $\mathbf{u}_h^0 = \mathbf{0}$. With (6.2) and coercivity of $a_D$ (3.16), we have

$$
K_D \| \mathbf{u}_h^m \|^2_{DG} + \mu_s \tau \sum_{n=1}^m \| \xi^n_s \|^2 \leq C(C_{err})(\tau^2 + h^{2k+2} + \tau h^2)
$$

$$
+ C \tau \left( \frac{1}{\mu_s} + 1 \right) \sum_{n=1}^m \| \mathbf{u}_h^n \|^2_{DG} + \tau \sum_{n=1}^m \left( b_f(c_h^{n-1}, \mu_h^n, \mathbf{u}_h^n) - b_f(c^n, \mu^n, \mathbf{u}_h^n) \right).
$$

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