Charging Quantum Batteries with a General Harmonic Driving Field

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A general harmonic driving field was considered for improving the charging efficiency of the quantum battery system. Charge saturation was used to describe the charging efficiency, where the charging mode is divided into saturated and unsaturated. The relationships between the time-dependent charge saturation and parameters of the general driving field were evaluated both analytically and numerically. The Floquet theorem was used to express time-dependent charge saturation with the quasienergy and Floquet states of the system. The analytical and numerical results were used to identify the best parameter values for optimizing the charging efficiency.

1. Introduction

In recent years, with the development of quantum thermodynamics and growing demand for device miniaturization, the quantum battery (QB), as a quantum energy storage device, has been receiving increasing attention. [1–17] The QB’s energy storage is known to be a charging process, where the state of the system is transferred from a low energy level to a high energy level. In general, the battery pack system comprises many battery units to supply adequate energy. The charging system is usually expressed as \( H(t) = H_0 + V(t) \), where \( H_0 \) is the Hamiltonian of QBs and \( V(t) \) is the charging field. [6,7] During the charging process, the charging field drives the QB from the initial state \( \rho_0 \) (i.e., low energy) to a higher energy state \( \rho(t) \). When the charging finishes at time \( t \), the stored energy of the QB is

\[
E(t) = \text{Tr} \left[ (\rho(t) - \rho(0)) H_0 \right] \tag{1}
\]

Previous studies have shown that choosing the proper type of charging field plays an important role in reducing the charging time. [19,10,12,13,18] For example, charging with a harmonic driving field is more efficient than charging with a steady one. [8,13] In addition, a charging process that contains entangling operations has a collective quantum advantage, where the charging speed increases with the number of battery units. [19,10,18] With a proper charging field \( V(t) \), the QBs can be fully charged, and the stored energy reaches the maximum \( E_{\text{max}} \) during the charging process. We introduce a concept as charge saturation and define it as \( \eta(t) = E(t)/E_{\text{max}} \). Just like the stored energy \( E(t) \), the charge saturation \( \eta(t) \) is also time-dependent during the charging process.

With some kinds of charging fields, the charge saturation can never reach 100%. However, unsaturated QBs can still provide enough energy when many charged battery units are available. Thus, we can set a threshold to define the minimum effective charge saturation. With different thresholds, the charging mode can be divided into saturated and unsaturated charging modes (see Figure 1). In the “saturated charging mode”, when the charging finishes, every battery unit is fully charged, and the charge saturation reaches 100%. In the “unsaturated charging mode”, only some of the battery units need to be charged to reach the preset effective charge saturation, which can be any positive number less than 100%.

The charging power is defined as the ratio between the charged energy and charging time \([6–9]\) and is always used to describe the charging efficiency. However, when the effective charge saturation is determined, the charging efficiency is directly reflected by the charging time. Thus, we can use the charging time to reach the effective charge saturation to describe the charging efficiency, and we do this in this paper.

The QB unit can be simplified as a two-level system because the quantum energy levels are discrete. It can be described by the charged state \( |1\rangle \) and the uncharged state \( |0\rangle \). [8,13] Thus, the Hamiltonian of QBs can be written as \( H_0 = \sum_i \frac{1}{2} \omega_i \sigma_i^z \) (setting \( \hbar = 1 \)), where \( \omega_i \) is the stored energy of a single charged QB unit. We assumed that the QB units are identical (i.e., \( \omega_0 = \omega_i \)), so \( H_0 = \frac{1}{2} \omega_0 \sum_i \sigma_i^z = \omega_0 J_z \). Then, we can choose the general harmonic charging field to be \( V(t) = \sum_{i=x,y,z} A_i \cos(\omega_i t + \phi_i) J_i \), where \( A_i, \omega_i, \phi_i \) are the driving strengths, driving frequencies and initial phases, respectively, in the three orthogonal directions. Thus, the Hamiltonian of the whole charging system is expressed as

\[
H(t) = \omega_0 J_z + \sum_{i=x,y,z} A_i \cos(\omega_i t + \phi_i) J_i \tag{2}
\]

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This is a quantum-driven system that can be described by the Dicke states \([s,m] = (s, s+1, \ldots, s)\) (i.e., the eigenstates of \(J^2\) and \(J_z\)). Analogous to the semi-classical Rabi model, it can be called the semi-classical Dicke model with a general driving field. We evaluated the driving field with a series of adjustable parameters and studied the relationships between the time-dependent charge saturation and parameters to identify the best parameter values to optimize the charging efficiency.

Obviously, the energy of our fully charged QB system is \(N\omega_0\). Thus, the time-dependent charge saturation is given by\(^{[7-10]}\)

\[
\eta(t) = \frac{E(t)}{N\omega_0} \tag{3}
\]

Furthermore, the strength of the input energy during charging should be considered because enhancing the ratio between the stored energy and input energy is crucial for improving the practicality of QBs. Therefore, we introduce \(A = |A|^2 = \sum_{A_x, A_y, A_z} A_x^2\) to describe the input energy and rewrite the driving strengths in the three directions as \(A = (A_x, A_y, A_z) = (A\cos\Theta\cos\Phi, A\cos\Theta\sin\Phi, A\sin\Theta)\), where \(A = |A|\) and \(\Theta, \Phi \in [0, 2\pi]\).

However, a charging system with a general harmonic charging field is difficult to solve analytically. In a previous study, our group considered an analytically solvable situation by using \(\Theta = 0\), \(\Phi = 0\), \(\phi = 0\)^\(\text{[11]}\). In this study, we identified more analytical solutions by charging in one or two directions. We then compared the charging efficiencies of these situations in the saturated and unsaturated charging modes.

According to the Floquet theorem, the charging system can be described by the Floquet Hamiltonian because it is periodic\(^{[19-22]}\). With the quasienergy and Floquet states of the periodic system (i.e., the eigenvalues and eigenstates of the Floquet Hamiltonian), the time-dependent charge saturation of the system can be expressed analytically. The rest of this paper is structured as follows. We review some analytically solvable QB charging systems, including the results of the system given in the previous work\(^{[11]}\). We compare the charging systems in terms of the charging efficiency. Furthermore, we use the framework of the Floquet theorem to give analytical expressions for the time-dependent charge saturation with the quasienergy and Floquet states of the system.

We then present an analysis on the charging process based on numerical results. We use the analytical and numerical results to identify the best parameter values for optimizing the charging efficiency. Finally, we present our conclusions and further discussion to end the paper.

2. Charging Efficiencies of QB Systems with Charging in One or Two Directions

As the uncharged state, the initial state of the system is \(|\psi(0)\rangle = \frac{1}{\sqrt{N}} \otimes^n \phi\rangle\), which can be rewritten as \(|\psi(0)\rangle = \frac{N}{2^n} \otimes^{n-1} \phi\rangle\) to represent the Dicke states. The initial density matrix can be obtained as \(\rho(0) = |\psi(0)\rangle \langle \psi(0)|\).

2.1. Analytical Results with Charging in the Parallel Direction

First, we investigated charging in the parallel direction. By setting \(\Theta = \frac{\pi}{2}\), the Hamiltonian of our charging system turns into \(H(t) = \alpha_0 J_z + A\cos(\alpha t + \phi)J_x\), which is merely with driving in the parallel direction. With the Schrödinger equation, the final state of the QB system is easy to obtain

\[
|\psi(t)\rangle = \exp\left\{i\frac{2k\pi}{2\sin \phi} \left[\frac{\alpha_0\alpha t}{A} + \sin(\alpha t + \phi)\right]\right\} \left|\frac{N}{2}, \frac{N}{2}\right\rangle \tag{4}
\]

where \(k \in \mathbb{Z}\) is an arbitrary integer. This indicates that the QB system cannot be charged with a harmonic driving field in the parallel direction because it causes only a phase evolution in this state.

2.2. Analytical Results with Charging in One Vertical Direction

We set \(\Theta = 0\) and \(\alpha_{x,y} = \alpha, \phi_{x,y} = \phi\) to obtain the charging system with a harmonic driving field in one vertical direction. Consequently, the Hamiltonian of the charging system is written as \(H(t) = \alpha_0 J_z + A\cos(\alpha t + \phi)J_x\), where \(J_x = \cos \Phi J_x + \sin \Phi J_y\) represents the action in an arbitrary vertical direction. By introducing the unitary transformation \(U_j = e^{\phi J_j}\) (many of the unitary transformations in the \(su(2)\) system \(U = e^{i\psi J_z}\) are used in this section, so we derived the results for a general situation and give the details in Appendix A), the Hamiltonian can be simplified as

\[
\tilde{H}(t) = \alpha_0 J_z + A\cos(\alpha t + \phi)J_x \tag{5}
\]

where \(|\tilde{\psi}(t)\rangle = e^{\phi J_j}|\psi(t)\rangle\). It is easy to demonstrate that the charge saturation remains unchanged after the transformation: \(\tilde{\eta}(t) = \langle \tilde{\psi}(t)| e^{-\phi J_j} |\psi(t)\rangle + \frac{1}{2} = \langle \psi(t)| \frac{1}{2} J_x |\psi(t)\rangle + \frac{1}{2} = \eta(t)\). Thus, we can use the Hamiltonian in Equation (5) (i.e., \(\tilde{H}\) system) to describe the charging process with an arbitrary harmonic driving field in one vertical direction.

To solve the system, we need to extend the counter-rotating hybridized rotating wave approximation (CHRWA) from a two-level system to a \(su(2)\) system\(^{[23-26]}\). As a first step, the unitary transformation \(U_j(t) = \exp[2i\tilde{\gamma} \sin(\alpha t + \phi) J_j]\) is introduced with \(\tilde{\gamma}\) as a
regulating variable. After the unitary evolution, the Hamiltonian becomes

$$\hat{H}_1 = \hbar \omega_0 \left\{ \cos \left( \frac{\Lambda}{\omega} \xi \sin (\omega t + \phi) \right) \hat{J}_z + \sin \left( \frac{\Lambda}{\omega} \xi \sin (\omega t + \phi) \right) \hat{J}_y \right\} + A(1 - \xi) \cos (\omega t + \phi) \hat{J}_x$$

Note that \(\hat{H}_1\) can be expanded as an infinite series with

$$\cos(\xi \sin \phi) = J_0(\xi) + \frac{1}{1 \cdot 2} J_2(\xi) \cos(2\xi \phi)$$

and

$$\sin(\xi \sin \phi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} J_{2n+1}(\xi) \sin((2n+1)\phi),$$

where \(J_n(\xi)\) is the Bessel function. If the higher-order harmonic terms are neglected, the Hamiltonian can be approximated as \(t^{24-26}\)

$$\hat{H}_1 = \hbar \omega_0 J_0 \left( \frac{\Lambda}{\omega} \xi \right) \hat{J}_z + A(1 - \xi) \cos (\omega t + \phi) \hat{J}_x$$

$$+ 2 \hbar \omega_0 J_1 \left( \frac{\Lambda}{\omega} \xi \right) \sin (\omega t + \phi) \hat{J}_y$$

$$= \hbar \omega_0 J_0 \left( \frac{\Lambda}{\omega} \xi \right) \hat{J}_z + \hat{A} \left[ \cos (\omega t + \phi) \hat{J}_x + \sin (\omega t + \phi) \hat{J}_y \right]$$

(7)

where \(\hat{A} = A(1 - \xi) = 2 \hbar \omega_0 J_1 \left( \frac{\Lambda}{\omega} \xi \right)\) is regulated by \(\xi\). Furthermore, the unitary evolution \(\hat{U}_z(t) = \exp[i(\omega t + \phi)\hat{J}_z]\) makes the Hamiltonian time-independent

$$\hat{H}_2 = \hat{A} \hat{J}_z + \hat{A} \hat{J}_x$$

(8)

where \(\hat{A} = \hbar \omega_0 J_0 \left( \frac{\Lambda}{\omega} \xi \right) - \hbar \omega\) is the effective detuning. \(t^{24-26}\)

Finally, when \(\hat{U}_2(t) = \exp(-i\hat{H}_2 t)\) is the time evolution operator of the \(\hat{H}_2\) system, the time-dependent state \(|\psi(t)\rangle\) can be expressed as \(|\psi(t)\rangle = \hat{U}_2(t)|\psi(0)\rangle\). With \(|\psi(0)\rangle = \hat{U}_1(0)|\psi(0)\rangle\), the initial state \(|\psi(0)\rangle\) is obtained as \(|\psi(0)\rangle = \hat{U}_2(0)|\psi(0)\rangle\) by setting \(t = 0\). Thus, the state of the \(\hat{H}\) system can be approximated as

$$|\psi(t)\rangle = \hat{U}_2(t)|\psi(0)\rangle = \hat{U}_2(t)|\psi(0)\rangle$$

(9)

Especially, when the initial phase \(\phi = 0\), the state is expressed as \(|\psi(t)\rangle = \hat{U}_1(t)|\psi(0)\rangle = P_z \frac{N}{2} |z - N\rangle\), which is a system that has been studied.\(^{113}\) After the state is substituted into Equation (1) and Equation (3), the time-dependent charge saturation of the \(\hat{H}\) system can be written as

$$\hat{\eta}(t) = \frac{1}{2} \left[ 1 - \cos \left( \frac{\omega t}{\Omega_k} \right) \hat{A} \cos (\omega t) \sin \phi \right.$$  

$$+ \frac{\cos \left( \frac{\omega t}{\Omega_{k'}} \right)}{\Omega_{k'}} \left( \hat{A} \hat{J}_x \sin (\omega t) \sin \phi - \hat{A}^2 \cos \phi \right) \left(10\right)$$

$$\hat{\eta} = \frac{\omega}{\Omega} \sin(\omega t), \Omega_k = \sqrt{\frac{\omega^2}{2} + \frac{\Lambda^2}{2}}, \text{and } \xi = \text{the root of the regulating formula } A(1 - \xi) = 2 \hbar \omega_0 J_1 \left( \frac{\Lambda}{\omega} \xi \right).$$

With Equation (10), we demonstrate that the QBs are fully charged (i.e., \(\hat{\eta} = 1\)) when \(\Omega_k = \hat{A}, \omega t = k \pi, \text{and } \Omega_{k'} = (2k' + 1) \pi\), where \(k, k' \in \mathbb{N}\). Hence, the charging should stop at \(t = (2k' + 1) \pi / \hat{A}\), and the minimum charging time \(t_{\text{min}}\) is obviously when \(k' = 1\). By introducing \(z = A^2 \xi\), the parameter conditions to obtain the maximum charging efficiency can be rewritten as \(A = \omega_0 J_0(z) (z + 1)\) and \(\omega = \omega_0 J_0(z)\), where \(z\) is the root of the transcendental equation \(J_0(z) = 2k_0 J_1(z)\). We solve the equation with the graphical method as given in Figure 2a, when \(k = 1\) and \(k = 2\), the series of roots of the equation are marked by the intersection points between the graph of the function \(y = 2k_0 J_1(z) - J_0(z)\) and \(H\text{-axis} = \omega_0 J_0(z)\). By comparing the roots, we find that the first positive root with \(k = 1\) is optimal because it lead to a larger value on the dotted red line than the others, which is represented by \(\gamma_{\text{max}}\). Then, the shortest charging time is given by \(t_{\text{min}} = \frac{\pi}{\omega_0 J_0(z)}\) when compared to the situation at \(k > 1\).

In summary, the QBs are fully charged at the time

$$t_{\text{min}} = \frac{\pi}{\omega J_0(z)}$$

(11)

when the charging strength and charging frequency of the \(\hat{H}\) system are respectively set as

$$\begin{cases} A = \omega_0 J_0(z)(z + 1), \\ \omega = \omega_0 J_0(z) \end{cases}$$

(12)

where \(z \approx 0.90\) is the first positive root of the transcendental equation \(J_0(z) = 2k_1 J_1(z)\).

### 2.3. Analytical Results with Charging in Two Vertical Directions

Similarly, we can set \(\Theta = 0, \omega_0 J_0 = \omega, \text{and } \phi_0 = \phi, \phi_0 = \phi - \frac{\Lambda}{\omega} \xi\) to obtain a charging system with the harmonic driving field in two vertical directions

$$H(t) = \omega_0 J_2(z) + \frac{\omega}{\sqrt{2}} \frac{A^2}{2} \sin (\omega t + \phi) \hat{J}_x$$

(13)

We designate this as the \(\hat{H}\) system.
In the same manner, we can find a unitary transformation $U(t) = \hat{U}(t) = \exp[\im \omega t + \phi]J_z$ to make the Hamiltonian time-independent

$$H_1 = (\omega_0 - \omega)J_z + \frac{\sqrt{2}}{2} A \hat{s}_z$$

(14)

Thus, the wave function becomes $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. With $|\psi(t)\rangle = e^{-i\hat{H}_1t}|\psi(0)\rangle$, we can obtain the wave function of the $H$ system

$$|\psi(t)\rangle = U'_1(t)e^{-i\hat{H}_1t}U_1(0)|\frac{N}{2}, -\frac{N}{2}\rangle$$

(15)

Furthermore, the analytical expression of the charge saturation in the $H$ system is given by

$$\eta(t) = \frac{A^2}{4\Omega^2}[1 - \cos(\Omega t)]$$

(16)

where $\Omega = \sqrt{(\omega_0 - \omega)^2 + \frac{1}{4}A^2}$. This implies that the charge saturation is independent of the initial phase. When $\omega = \omega_0$, the $H$ system is fully charged ($\eta = 1$) at

$$t_{\text{min}} = \frac{2\pi}{A}$$

(17)

Thus, the time taken to charge the QBs fully in the $H$ system is inversely proportional to the charging strength $A$.

We compared the results with the charging efficiency of the $HF$ system. In the saturated charging mode, the charging strength and charging frequency are set to $A \approx 1.53\omega_0$ and $\omega \approx 0.81\omega_0$, respectively, for the $HF$ system. The charging strength for the $H$ system is the same when the charging frequency is set to $\omega = \omega_0$. Thus, the QBs are fully charged with the same charging strength at $t_{\text{min}} \approx 3.88\omega_0^{-1}$ in the $H$ system and at $t_{\text{min}} \approx 2.90\omega_0^{-1}$ in the $HF$ system. Further analytical and numerical results for the time-dependent charge saturation are shown in Figure 2b. The results indicate that the $H$ system is better in the saturated charging mode.

For further exploration of the relations between the charge saturation and parameters in the unsaturated charging mode, we set $A = 1.53\omega_0$ and give the numerical results for the charge saturation $\eta$ with a varying initial phase $\phi$ and driving frequency $\omega$ in Figure 3a, b, and c, respectively.

The results in Figure 3 imply that the $H$ system can be more efficient with proper adjustment of the charging frequency $\omega$ and initial phase $\phi$ in the unsaturated charging mode. However, when an arbitrary effective charge saturation is set, the threshold for the $H$ system is always reached more quickly than the threshold for the $HF$ system.

Thus, the $H$ system is advantageous for charging in both the saturated and unsaturated charging modes compared with the $HF$ system.

3. Expression of the Charge Saturation with the Floquet Theorem

After discussing the charging in one or two directions, we now return to the general harmonic driving field. For the charging frequencies $\omega_{\text{opt}}$, if the ratios between arbitrary two of them are rational—or irrational but analyzed with finite precision—we can find an $\omega$ to set $\omega = n\omega_0$, $i = x, y, z$, where $n_{x,y,z}$ are positive integers. Thus, the Hamiltonian of the charging system is periodic with $T = \frac{2\pi}{\omega}$, which implies that it can be expanded with the Fourier series to $H(t) = \sum_{n=-\infty}^{\infty} H_n e^{\im nt}$, where

$$H_n = \omega_0 \delta_{n,0} + \frac{1}{2} \sum_{i=x,y,z} A_i (e^{\im \phi} \delta_{n,0} + e^{-\im \phi} \delta_{n,-n})J_i$$

(18)

Then, the Floquet Hamiltonian in the frequency space $H_F$ can be built with $H_n$, and the element in row $n'$ and column $n$ of the Floquet Hamiltonian is written as $[19,20,27]

$$(H_F)_n^{n'} = H_{n-n'} + \delta_{n',n} \omega_n E$$

(19)

where $E$ is the identical matrix. In this manner, the quasenergy and Floquet states of the periodic system (i.e., the eigenvalues $\epsilon_\alpha$ and corresponding eigenstates $|\Phi_\alpha\rangle$ of $H_F$) can be obtained.

Furthermore, the final state of our QB system is given by the Floquet theorem (see Appendix B for details)[19-21]

$$|\Psi(t)\rangle = \sum_{n=1}^{N+1} \sum_{\alpha=1}^{\infty} e^{\im \epsilon_\alpha t} |\Phi_\alpha\rangle \langle \Phi_\alpha|^\frac{N}{2}, -\frac{N}{2}\rangle$$

(20)

where $\epsilon_{\alpha}, \alpha = 1, 2, \ldots, N + 1$ are the $N + 1$ eigenvalues of the Floquet Hamiltonian $H_F$ in one period of the frequency space (e.g., $\epsilon_{\alpha} \in \left[ -\frac{\omega_0}{2} + k\omega_0, \frac{\omega_0}{2} + k\omega_0 \right], k \in \mathbb{Z}$) and $|\Phi_\alpha\rangle$ is the element in the corresponding eigenstate $|\Phi_\alpha\rangle = (\cdots, |\Phi_\alpha^n\rangle, |\Phi_\alpha^{n+1}\rangle, \cdots)^T$. 

Figure 3. a) Relationship between the charge saturation and initial phase $\phi$ in the $HF$ system $\eta_1(\phi, t)$ when $A = 1.53\omega_0$ and $\omega = 0.81\omega_0$ (the charging time $t$ takes $\omega_0^{-1}$ as the unit for all figures in this paper). b) Relationship between the charge saturation and initial phase $\phi$ in the $H$ system $\eta_2(\phi, t)$ when $A = 1.53\omega_0$ and $\omega = \omega_0$. c) Relationship between the charge saturation and charging frequency $\omega$ in the $HF$ system $\eta(\omega, t)$ when $A = 1.53\omega_0$ and $\phi = 0$. d) Relationship between the charge saturation and charging frequency $\omega$ in the $H$ system $\eta(\omega, t)$ when $A = 1.53\omega_0$ and $\phi = 0$. 

In the same manner, we can find a unitary transformation $U(t) = \hat{U}(t) = \exp[\im \omega t + \phi]J_z$ to make the Hamiltonian time-independent

$$H_1 = (\omega_0 - \omega)J_z + \frac{\sqrt{2}}{2} A \hat{s}_z$$

(14)

Thus, the wave function becomes $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. With $|\psi(t)\rangle = e^{-i\hat{H}_1t}|\psi(0)\rangle$, we can obtain the wave function of the $H$ system

$$|\psi(t)\rangle = U'_1(t)e^{-i\hat{H}_1t}U_1(0)|\frac{N}{2}, -\frac{N}{2}\rangle$$

(15)

Furthermore, the analytical expression of the charge saturation in the $H$ system is given by

$$\eta(t) = \frac{A^2}{4\Omega^2}[1 - \cos(\Omega t)]$$

(16)

where $\Omega = \sqrt{(\omega_0 - \omega)^2 + \frac{1}{4}A^2}$. This implies that the charge saturation is independent of the initial phase. When $\omega = \omega_0$, the $H$ system is fully charged ($\eta = 1$) at

$$t_{\text{min}} = \frac{2\pi}{A}$$

(17)

Thus, the time taken to charge the QBs fully in the $H$ system is inversely proportional to the charging strength $A$.

We compared the results with the charging efficiency of the $HF$ system. In the saturated charging mode, the charging strength and charging frequency are set to $A \approx 1.53\omega_0$ and $\omega \approx 0.81\omega_0$, respectively, for the $HF$ system. The charging strength for the $H$ system is the same when the charging frequency is set to $\omega = \omega_0$. Thus, the QBs are fully charged with the same charging strength at $t_{\text{min}} \approx 3.88\omega_0^{-1}$ in the $H$ system and at $t_{\text{min}} \approx 2.90\omega_0^{-1}$ in the $HF$ system. Further analytical and numerical results for the time-dependent charge saturation are shown in Figure 2b. The results indicate that the $H$ system is better in the saturated charging mode.

For further exploration of the relations between the charge saturation and parameters in the unsaturated charging mode, we set $A = 1.53\omega_0$ and give the numerical results for the charge saturation $\eta$ with a varying initial phase $\phi$ and driving frequency $\omega$ in Figure 3a, b, and c, d, respectively.

The results in Figure 3 imply that the $H$ system can be more efficient with proper adjustment of the charging frequency $\omega$ and initial phase $\phi$ in the unsaturated charging mode. However, when an arbitrary effective charge saturation is set, the threshold for the $H$ system is always reached more quickly than the threshold for the $HF$ system.

Thus, the $H$ system is advantageous for charging in both the saturated and unsaturated charging modes compared with the $HF$ system.
Thus, with the final state, the charge saturation is written as

\[ \eta(t) = \frac{1}{N} \sum_{n=-N}^{N} \sum_{r=0}^{\infty} e^{-i\frac{\omega_0 t}{\hbar}} \langle \Phi_{n,r}^\prime \rangle \langle \Phi_{n,r}^\prime | J_z | \Phi_0^\prime \rangle + \frac{1}{2} \]  

(21)

where \( \omega_{n,r} = \omega_0 + (n - r + n')\omega \) is the difference between the quasienergies of different periods and \( \Phi_{n,r}^\prime = \langle \Phi_{n,r}^\prime | \rangle \) is the last element of \( | \Phi_{n,r}^\prime \rangle \). Figure 4 gives some examples of the results obtained with the Floquet theorem, which strongly agree with the exact numerical results.

4. Further Analysis with Some Numerical Results

After analytically investigating the charging system, we further explored the relationship between the parameters and time-dependent charge saturation numerically.

4.1. Numerical Results When Varying The Charging Strength and Its Distribution in The Vertical Direction

We first set \( \Theta = 0, \omega_\alpha = \omega_\beta = \omega, \phi_\alpha = 0, \phi_\beta = -\frac{\pi}{2} \) to investigate the charging efficiency when the charging strength and its distribution were varied in the vertical direction. This resulted in a Hamiltonian of

\[ H(t) = \omega_0 J_z + A \left[ \cos \Phi \cos (\omega_\chi t + \phi_\chi) J_z + \sin \Phi \sin (\omega_\chi t) J_z \right] \]  

(22)

where \( A \) and \( \Phi \) can be adjusted.

This model clearly contains both the \( H \) system (by setting \( \Phi = \frac{\pi}{2} \)) and \( A \) system (by setting \( \Phi = 0 \)) when the influence of the initial phase \( \phi_\chi \) is neglected. The numerical results for the charge saturation \( \eta(\Phi,t) \) are given in Figure 5, and the range of \( \Phi \) moves from \([0, \pi]\) to \([-\frac{\pi}{2}, \frac{\pi}{2}]\), which indicates the symmetry of the results.

In the saturated charging mode, the charging time was proved to be inversely proportional to the charging strength \( A \) in the \( H \) system, as shown in Equation (17) and Figure 5a–c. These results also indicate that the charging time was shortened when the charging strength was increased in the unsaturated charging mode. Furthermore, the shortest time for fully charging was clearly when \( \Phi = \pi/4 \), which means that the \( H \) system is the most effective charging system when the distribution of the charging strength is varied.

When \( \Phi = -\frac{\pi}{2} + k\pi \), the Hamiltonian turned into \( H = \omega_0 J_z + \frac{\sqrt{2}}{2} A [\cos(\omega_\chi t) J_z - \sin(\omega_\chi t) J_z] \), which can be obtained by flipping the driving in the \( y \) direction from the \( H \) system. The numerical results showed that the QBSs were hardly charged in this situation, which is useless for our QB charging system. However, it provides a solution for shielding the quantum system from the influence of the harmonic driving field in the vertical direction by introducing another harmonic driving field.

4.2. Numerical Results When A Charging Field is Added in The Parallel Direction to The H System

We took the \( H \) system as the base and added a general charging field in the parallel direction: \( A \sin \Theta \cos (\omega_\chi t + \phi_\chi) J_z \). The Hamiltonian of the charging system became

\[ H(t) = \omega_0 J_z + A \left[ \cos \Phi \cos (\omega_\chi t + \phi_\chi) J_z \right. \]  

\[ \left. + \frac{\sqrt{2}}{2} A \cos \Theta \left[ \cos (\omega_\chi t) J_z + \sin (\omega_\chi t) J_z \right] \right] \]  

(23)

First, we set \( \omega_\alpha = \omega \) and \( \phi = 0 \) to obtain the numerical results for \( \eta(\Phi,t) \) shown in Figure 6a, which describes the relationship between the charge saturation and distribution of the charging power. The results indicated that the charge saturation decreased when the harmonic power from the vertical direction was distributed in the parallel direction.

For further analysis, we set \( \Theta = \arccos(0.8) \) to obtain systems with \( A = 1 \) and \( A_{xy} = A_y = 0.8 \). This made them easy to compare with the results in Figure 5a and b. We numerically calculated the saturation with varying \( \phi_\chi \) and \( \omega_\chi \), as shown in Figure 6a and b, respectively. The results indicated that the charging power can
be enhanced by setting $\phi_i$ and $\omega_i$ to proper values. However, the comparison of the results with Figure 5a and b indicated that the $H$ system is still more effective, which means that arbitrary driving in the parallel direction has a negative effect on our charging system.

4.3. Numerical Results with Perturbations to The Frequency and Phase for The $H$ System

The previous results showed that the $H$ system is the optimal charging system. We perturbed the $H$ system by changing the driving frequency and initial phase in two vertical directions to determine if the charging efficiency could be improved. The Hamiltonian was set as follows with different perturbations

\[ H_i(\delta \omega, t) = \omega_i J_z + \frac{\sqrt{2}}{2} A \cos[(\omega + \delta \omega)t] J_x + \sin[(\omega)t] J_y \]  

\[ H_i(\delta \phi, t) = \omega_i J_z + \frac{\sqrt{2}}{2} A \cos[(\omega + \delta \phi)t] J_x + \sin[(\omega)t] J_y \]  

\[ H_i(\delta \phi, t) = \omega_i J_z + \frac{\sqrt{2}}{2} A \cos[(\omega + \delta \phi)t] J_x + \sin[(\omega - \delta \phi)t] J_y \]  

\[ H_i(\delta \phi, t) = \omega_i J_z + \frac{\sqrt{2}}{2} A \cos[(\omega + \delta \phi)t] J_x + \sin[(\omega - \delta \phi)t] J_y \]  

\[ H_i(\delta \phi, t) = \omega_i J_z + \frac{\sqrt{2}}{2} A \cos[(\omega + \delta \phi)t] J_x + \sin[(\omega - \delta \phi)t] J_y \]  

Figure 7 shows the numerical results for $\eta(\delta \omega, t)$ and $\eta(\delta \phi, t)$, i = 1, 2, 3, which represent the time-dependent charge saturation with perturbation to the frequency or phase in one or two directions, respectively.

The results indicated that the $H$ system is optimal for the saturated charging mode. However, for the unsaturated charging mode with a threshold set to indicate the effective charge saturation, the charging efficiency can be enhanced by adjusting the parameters. For instance, when the effective charge saturation was set to $\eta = 0.4$, the charging time could be shortened by adding a suitable negative perturbation to the frequency in Equation (26), as shown in Figure 7c. This can also be achieved by adding a suitable positive perturbation to the phase in Equation (28) or negative perturbation to the phase in Equation (29), as shown in Figure 7e and f, respectively.

5. Conclusions

We built a QB system with $N$ two-level atoms and charged it with a controlled general harmonic driving field. Analytical and numerical analyses of our QB charging system showed that the QBs can be fully charged when the driving field is set with the proper parameters. With the same charging strength $A$, the QBs are fully charged in the shortest amount of time when the other parameters of the driving field are set to $\Theta = 0$, $\Phi = \frac{\pi}{2}$, and $\Delta \phi = \phi_x - \phi_y = \frac{\pi}{2}$. Thus, this is the optimal charging system for the saturated charging mode.

The results in Section 4.3 indicate that the charging efficiency in the unsaturated charging mode can be further enhanced by introducing proper perturbations to the frequency or phase of the charging system, as shown in Figure 7.

We believe that QB is an important quantum device that will step from theory into practice in the near future. Our QB charging model can be realized physically; for instance, we can take...
a solid-state platform to build our battery and charge it with Raman laser beams as the driving field. The bidirectional and bi-phase charging mode represented by the $H$ system and unsaturated charging mode in this paper will provide reasonable reference for many schemes aimed at improving the charging efficiency of this kind of QB.

Appendix A: Unitary Transformation in The $\text{Su}(2)$ System

The arbitrary unitary transformation operator in the $\text{su}(2)$ system can be constructed as $U = e^{	heta}$ with $A = a \cdot J$, where $J = (J_x, J_y, J_z)$ is the generator of the $\text{su}(2)$ algebra. With the commutation relation for $\text{su}(2)$ algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$, any operators in the space can be transformed as follows

$$e^{A} Be^{-A} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (ax)^n b \cdot J \tag{A1}$$

where $B = b \cdot J$ and $a = |a|$, $b = |b|$. Then if $a \perp b$, the result can be simplified as

$$e^{A} Be^{-A} = \cos a(b \cdot J) - \frac{\sin a}{a}(1 - \cos a)(ax)^2 b \cdot J \tag{A2}$$

Appendix B: Floquet Theorem in Quantum System

For a quantum system of $N$ dimensions, the Schrödinger equation $i\hbar \frac{\partial}{\partial t}\Psi(t) = H(t)\Psi(t)$ has $N$ linearly independent solutions $|\Psi_n(t)\rangle = \sum_{n=1}^{N} c_n|\Psi_n(t)\rangle$, where $|c_n\rangle$ is determined by the initial state $|\Psi(0)\rangle$. When the Hamiltonian of the system is periodic and given by $H(t + T) = H(t)$, the Floquet theorem indicates that we can find a real number $\omega$ and periodic function $|\Phi_n(t)\rangle$ with the same period as $T = \frac{2\pi}{\omega}$ to rewrite $|\Psi_n(t)\rangle$ as$^{19,20,27}$

$$|\Psi_n(t)\rangle = e^{-i\omega t}|\Phi_n(t)\rangle \tag{B1}$$

By substituting this expression into the Schrödinger equation, we can obtain $|\Phi_n(t)\rangle$

$$[H(t) - i\hbar] |\Phi_n(t)\rangle = \varepsilon_n |\Phi_n(t)\rangle \tag{B2}$$

We can expand $H(t)$ and $|\Phi_n(t)\rangle$ with the frequency $\alpha$ into a Fourier series

$$H(t) = \sum_{n=-\infty}^{\infty} H_n e^{i\alpha t} \tag{B3}$$

$$|\Phi_n(t)\rangle = \sum_{n=-\infty}^{\infty} |\Phi_n^\alpha\rangle e^{i\alpha t} \tag{B4}$$

Then we substitute them into Equation (B2) to obtain

$$\sum_{n=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} H_n |\Phi_n^\alpha\rangle e^{i\alpha t} + \sum_{n=-\infty}^{\infty} n\omega |\Phi_n^\alpha\rangle e^{i\alpha t} = \varepsilon_n |\Phi_n^\alpha\rangle e^{i\alpha t} \tag{B5}$$

With $\frac{1}{T} \int_0^T dt e^{-i\omega t} e^{i\alpha t} = \delta_{\omega\alpha}$, this time-dependent equation for $|\Phi_n^\alpha\rangle$ can then be transformed into a time-independent one

$$\sum_{n=-\infty}^{\infty} (H_n - \delta_{\omega\alpha}) |\Phi_n^\alpha\rangle = \varepsilon_n |\Phi_n^\alpha\rangle \tag{B6}$$

for any $k \in Z$. By introducing the Floquet Hamiltonian $H_{\omega}$ defined by Equation (19), the matrix form of Equation (B6) can be written as

$$\begin{pmatrix}
\vdots & \vdots & \vdots \\
(H_{\omega}^k)_{n_1} & (H_{\omega}^{k+1})_{n_1} & \vdots \\
(H_{\omega}^k)_{n_2} & (H_{\omega}^{k+1})_{n_2} & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
(H_{\omega}^k)_{N} & (H_{\omega}^{k+1})_{N} & \vdots
\end{pmatrix}
\begin{pmatrix}
|\Phi_0^n\rangle \\
|\Phi_1^n\rangle \\
\vdots \\
|\Phi_{N-1}^n\rangle
\end{pmatrix}
= \varepsilon_n
\begin{pmatrix}
|\Phi_0^n\rangle \\
|\Phi_1^n\rangle \\
\vdots \\
|\Phi_{N-1}^n\rangle
\end{pmatrix} \tag{B7}$$

where $k_i$ and $n_i$ are indices that are introduced to describe the matrix. Equation (B7) indicates that $\varepsilon_n$ and $|\Phi_n\rangle = |\Phi_n^{n+1}\rangle$ are eigenvalues and eigenstates, respectively, of $H_{\omega}$.

Equation (B7) clearly has infinite solutions, but we only need $N$ of them. The solution expansion is based on the periodic relationships of $\varepsilon_n$ and $|\Phi_n(t)\rangle$ as $\begin{pmatrix}
\varepsilon_n + \omega n \\
|\Phi_n(t)\rangle
\end{pmatrix}
= e^{i\omega t} |\Phi_n(t)\rangle$. Thus, we can obtain the $N$ solutions of $\varepsilon_n$ and $|\Phi_n\rangle$ with $\alpha = 1, \ldots, N$ by setting $\varepsilon_n$ in one period of the frequency space, such as $\varepsilon_n \in [-\frac{\omega}{2}, \frac{\omega}{2})$.

With the results of $\varepsilon_n$ and $|\Phi_n\rangle$, the wave function can be calculated as $|\Psi(t)\rangle = \sum_{n=-\infty}^{\infty} c_n |\Phi_n(t)\rangle e^{i\alpha t}$. By setting $t = 0$, we get

$$|\Psi(0)\rangle = \sum_{n=-\infty}^{\infty} c_n |\Phi_n^0\rangle = (\sum_n |\Phi_0^n\rangle, \sum_n |\Phi_1^n\rangle, \ldots, \sum_n |\Phi_{N-1}^n\rangle)\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_N
\end{pmatrix}$$
In this manner, the series of coefficients \( \{c_n\} \) can be expressed as

\[
\begin{pmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_N \\
\end{pmatrix} = \left( \sum_n |\Phi_n^c\rangle \langle \Phi_n^c|, \sum_n |\Phi_n^c\rangle \langle \Phi_n^c|, \ldots, \sum_n |\Phi_n^c\rangle \langle \Phi_n^c| \right)^{-1} |\Psi(0)\rangle 
\]

(B8)

By substituting the coefficients into the expression of the wave function, we can get the final state as \( |\Psi(t)\rangle = U(t)|\Psi(0)\rangle \), where

\[
U(t) = \sum_{n=1}^{N} \sum_{n'=0}^{\infty} |\Phi_n^a\rangle \langle \Phi_n^a| e^{i(n-n')\omega t} e^{-\epsilon t} 
\]

(B9)

With the initial state of our system as \( |\Psi(0)\rangle = |\frac{N}{2}, -\frac{N}{2}\rangle \), we can obtain the charge saturation as Equation (21).

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Conflict of Interest

The authors declare no conflict of interest.

Keywords

charge saturation, driven systems, Floquet theorem, harmonic driving, quantum batteries, su(2), unsaturated charging mode