\textbf{C*-algebras coming from some buildings.}

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\textbf{Abstract}

We construct compact polyhedra with triangular faces whose links are generalized 3-gons. They are interesting compact spaces covered by Euclidean buildings of type $A_2$. Those spaces give us two-dimensional subshifts, which can be used to construct some $C^*$-algebras.

\section{Introduction}

Given a graph $G$ we assign to each edge the length 1. The diameter of the graph is its diameter as a length metric space, its injectivity radius is half of the length of the smallest circuit.

The following definition is equivalent to the usual one.

\textbf{Definition 1.1.} For a natural number $m$ we call a connected graph $G$ a generalized $m$-gon, if its diameter and injectivity radius are both equal to $m$.

A graph is \textit{bipartite} if its set of vertices can be partitioned into two disjoint subsets $P$ and $L$ such that no two vertices in the same subset lie on a common edge. Such a graph can be interpreted as a planar geometry, i.e. a set of points $P$ and a set of lines $L$ and an incidence relation $R \subset P \times L$. On the other hand each planar geometry can be considered as a bipartite graph.

Under this correspondence projective planes are the same as generalized 3-gons.

Let $G$ be a planar geometry. For a line $y \in L$ we denote by $I(y)$ the set of all points $x \in P$ incident to $y$. If no confusion can arise we shall
write \( x \in y \) instead of \( x \in I(y) \) and \( y_1 \cap y_2 \) instead of \( I(y_1) \cap I(y_2) \). A subset \( S \) of \( P \) is called collinear if it is contained in some set \( I(y) \), i.e. if all points of \( S \) are incident to a line.

Given a planar geometry \( G \) we shall denote by \( G' \) its dual geometry arising by calling lines resp. points of \( G \) points resp. lines of \( G' \). The graphs corresponding to \( G \) and \( G' \) are isomorphic.

We will call a polyhedron a two-dimensional complex which is obtained from several oriented \( p \)-gons by identification of corresponding sides. Consider a point of the polyhedron and take a sphere of a small radius at this point. The intersection of the sphere with the polyhedron is a graph, which is called the link at this point.

**Definition.** Let \( \tilde{A}_2 \) be a tessellation of the Euclidean plane by regular triangles. A Euclidean building of type \( \tilde{A}_2 \) is a polygonal complex \( X \), which can be expressed as the union of subcomplexes called apartments such that:

1. Every apartment is isomorphic to \( \tilde{A}_2 \).
2. For any two polygons of \( X \), there is an apartment containing both of them.
3. For any two apartments \( A_1, A_2 \in X \) containing the same polygon, there exists an isomorphism \( A_1 \to A_2 \) fixing \( A_1 \cap A_2 \).

If we consider a polyhedron with triangular faces and incidence graphs of finite projective planes as links, the universal covering of the polyhedron is an Euclidean building \( \Delta \), see [1], [2].

So, to construct Euclidean buildings with compact quotients, it is sufficient to construct finite polyhedra with appropriate links.

We construct a family of compact polyhedra with 3-gonal faces whose links are generalized 3-gons. Then we will use those polyhedra to construct systems of 2-dimensional words, which give examples of new \( C^* \)-algebras.

We recall the definition of the polygonal presentation, given in [3].

**Definition.** Suppose we have \( n \) disjoint connected bipartite graphs \( G_1, G_2, \ldots, G_n \). Let \( P_i \) and \( L_i \) be the sets of black and white vertices respectively in \( G_i \), \( i = 1, \ldots, n \); let \( P = \cup P_i, L = \cup L_i, P_i \cap P_j = \emptyset \) \( L_i \cap L_j = \emptyset \) for \( i \neq j \) and let \( \lambda \) be a bijection \( \lambda : P \to L \).

A set \( K \) of \( k \)-tuples \( (x_1, x_2, \ldots, x_k), \ x_i \in P \), will be called a polygonal presentation over \( P \) compatible with \( \lambda \) if

\[(1) \quad (x_1, x_2, x_3, \ldots, x_k) \in K \text{ implies that } (x_2, x_3, \ldots, x_k, x_1) \in K;\]
(2) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in K$ for some $x_3, \ldots, x_k$ if and only if $x_2$ and $\lambda(x_1)$ are incident in some $G_i$;

(3) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in K$ for at most one $x_3 \in P$.

If there exists such $K$, we will call $\lambda$ a *basic bijection*.

Polygonal presentations for $n = 1$, $k = 3$ were listed in [3] with the incidence graph of the finite projective plane of order two or three as the graph $G_1$.

One can associate a polyhedron $X$ on $n$ vertices with each polygonal presentation $K$ as follows: for every cyclic $k$-tuple $(x_1, x_2, x_3, \ldots, x_k)$ from the definition we take an oriented $k$-gon on the boundary of which the word $x_1 x_2 x_3 \ldots x_k$ is written. To obtain the polyhedron we identify the sides with the same label of our polygons, respecting orientation. We will say that the polyhedron $X$ *corresponds* to the polygonal presentation $K$.

The following lemma was proved in [6]:

**Lemma 1.1.** A polyhedron $X$ which corresponds to a polygonal presentation $K$ has graphs $G_1, G_2, \ldots, G_n$ as the links.

**Remark.** Consider a polygonal presentation $K$. Let $s_i$ be the number of vertices of the graph $G_i$ and $t_i$ be the number of edges of $G_i$, $i = 1, \ldots, n$. If the polyhedron $X$ corresponds to the polygonal presentation $K$, then $X$ has $n$ vertices (the number of vertices of $X$ is equal to the number of graphs), $k \sum_{i=1}^{n} s_i$ edges and $\sum_{i=1}^{n} t_i$ faces, all faces are polygons with $k$ sides.

**2 Construction of the polyhedra.**

Let $G$ be a finite projective plane and let $P$ resp. $L$ denote the set of its points resp. lines.

Assume that a bijection $T : P \to L$ is given and satisfies the following properties

1. For each $x \in P$ the point $x$ and the line $T(x)$ are not incident.
2. For each pair $x_1, x_2$ of different points in $P$ the points $x_1, x_2$ and $T(x_1) \cap T(x_2)$ are not collinear.

Projective planes with such bijections exist, see for example [7].
Lemma 2.1. Let $T : P \to L$ be as above, $y \in L$ a line. Then the map $T^* : I(y) \to I(y)$ given by $T^*(x) = T(x) \cap I(y)$ is a bijection.

Proof. By the first property of $T$ the map $T^*$ is well defined, by the second property it must be injective. Since $I(y)$ is finite, the statement follows. $\Box$

Let $G, P, L, T : P \to L$ be as above. Let $P = \{x_1, \ldots, x_p\}$ be a labeling of points in $P$ and set $y_i = T(x_i)$. Consider the following set $O \subset P \times P \times P$, consisting of all triples $(x_i, x_j, x_k)$ satisfying $x_i \in y_k$, $x_j \in y_i$ and $x_j \in y_k$.

Remark 2.1. The conditions on $(x_i, x_j, x_k) \in K$ are not cyclic. We require $x_j \in y_k$ and not $x_k \in y_j$ !! For this reason in the polygonal presentations defined below dual graphs of $G$ appear.

The following lemma is crucial for the later construction:

Lemma 2.2. A pair $(x_i, x_k)$ resp. $(x_i, x_j)$ resp. $(x_j, x_k)$ is a part of at most one triple $(x_i, x_j, x_k) \in K$ and such a triple exists iff $x_i \in y_k$ resp. $x_j \in y_i$ resp. $x_j \in y_k$ holds.

Proof. The conditions stated at the end are certainly necessary.

1) Let $x_i \in y_k$ be given. Then $y_i$ and $y_k$ are different and the point $x_j = y_i \cap y_k$ is uniquely defined.

2) Let $x_j \in y_i$ be given. Then $x_j$ and $x_i$ are different, so there is exactly one line $y_k$ containing $x_j$ and $x_i$.

3) Let $x_j \in y_k$ be given. Then $(x_i, x_j, x_k) \in K$ iff for the map $T^* : I(y_k) \to I(y_k)$ of Lemma 2.1 the equality $T^*(x_i) = x_j$ holds. By Lemma 2.1 the point $x_i$ is uniquely defined. $\Box$

Now we are ready for the polygonal presentations. Let the notations be as above, $G_1$ and $G_2$ two projective planes with isomorphism $J^1 : G \to G_1$ and $G_3$ a projective plane with an isomorphism $J^3 : G' \to G_3$ of the dual projective plane $G'$ of $G$. For $t = 1, 2$ we set $x_i^t = J^t(x_i)$, $y_i^t = J^t(y_i)$ and for $t = 3$ we set $x_i^3 = J^3(y_i)$ and $y_i^3 = J^3(x_i)$.

Let $P_t$ resp. $L_t$ be the set of lines of $G_t$. For $P = \bigcup P_t$ and $L = \bigcup L_t$ we consider the bijection $\lambda : P \to L$ given by $\lambda(x_i^t) = y_i^{t+1}$ ($t + 1$ is taken modulo 3).

Now consider the subset $T$ of $P \times P \times P$ consisting of all triples $(x_i^1, x_j^2, x_k^3)$ with $(x_i, x_j, x_k) \in K$ and all cyclic permutation of such triples.

The statement of Lemma 2.2 can be now reformulated as:
PROPOSITION 2.3. The subset $T$ of $P \times P \times P$ defines a polygonal presentation compatible with $\lambda$.

The polyhedron $X$ which corresponds to $T$ by the construction of Lemma 1.1 has triangular faces and exactly three vertices with two links naturally isomorphic to $G$ and one link naturally isomorphic to the dual $G'$ of $G$. By [1], [2] the universal covering of $X$ is a Euclidean building of type $\tilde{A}_2$.

3 Subshift coming from a polygonal presentation.

Let $T$ be a polygonal presentation with $n = 3$, $k = 3$, where all there graphs $G_1$, $G_2$ and $G_3$ are incidence graphs of finite projective planes of order $q$. The polyhedron, which corresponds to $T$, has triangular faces and three vertices. We will consider polyhedra such that all three vertices of each triangle have different graphs as links. In this case we can give a Euclidean metric to every face. In this metric all sides of the triangles are geodesics of the same length. The universal covering of the polyhedron is an Euclidean building $\Delta$, see [1], [2]. Each element of $T$ may be identified with an oriented basepointed triangle in $\Delta$. We now construct a 2-dimensional shift system associated with $T$.

The transition matrices $M_1, M_2$ in the way, defined as in [5]: if $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3) \in T$ say that $M_1(\alpha, \beta) = 1$ if and only if there exists $\psi = (x_2, y_1, z_3)$ and $M_1(\alpha, \beta) = 0$ otherwise. In a similar way, $M_2(\alpha, \gamma) = 1$ for $\alpha = (x_1, x_2, x_3), \gamma = (z_1, z_2, z_3)$ if and only if there exists $\psi = (x_2, y_1, z_3)$ and $M_2(\alpha, \gamma) = 0$ otherwise. The matrices $M_1, M_2$ of order $3(q+1)(q^2+q+1) \times 3(q+1)(q^2+q+1)$ are nonzero $\{0,1\}$ matrices. We will use $T$ as an alphabet and $M_1, M_2$ as transition matrices to build up 2-dimensional words as in [4]. Let $[m,n]$ denote $\{m, m+1, \ldots, n\}$, where $m \leq n$ are integers. If $m, n \in \mathbb{Z}^2$, say that $m \leq n$ if $m_j \leq n_j$ for $j=1,2$, and when $m \leq n$ let $[m,n] = [m_1,n_1] \times [m_2,n_2]$. In $\mathbb{Z}^2$, let $0$ denote the zero vector and let $e_j$ denote the $j$-th standard unit basis vector. If $m \in \mathbb{Z}^2_+ = \{m \in \mathbb{Z}^2; m \geq 0\}$, let $W_m = \{w : [0,m] \to T; M_j(w(l+e_j), w(l)) = 1 \text{ where } l, l+e_j \in [0,m]\}$ and call the elements of $W_m$ words.

In order to apply the theory from [4] we need the matrices $M_1, M_2$ to satisfy the following conditions:
(H0) Each $M_i$ is a nonzero $\{0,1\}$-matrix.
(H1a) $M_1 M_2 = M_2 M_1$.
(H1b) $M_1 M_2$ is a $\{0,1\}$-matrix.
(H2) The directed graph with vertices $\alpha \in T$ and directed edges $(\alpha, \beta)$ whenever $M_i(\alpha, \beta) = 1$ for some $i$ is irreducible.
(H3) For any nonzero $p \in \mathbb{Z}^2$, there exists a word $w \in W$ which is not $p$–periodic, i.e., there exists $l$ so that $w(l)$ and $w(l+p)$ are both defined but not equal.

In [3] some $C^*$-algebra is defined by the system of words $W_m$, where $m \in \mathbb{Z}_2^2$. It is proved there, that if the matrices $M_1, M_2$ satisfy the conditions (H0),(H1a,b),(H2),(H3), then this algebra is simple, purely infinite and nuclear.

Now we prove the conditions (H0),(H1a,b),(H2),(H3) for our two-dimensional shift. By definition of matrices $M_1, M_2$ they are nonzero $\{0,1\}$ matrices, so (H0) holds. If we have $\alpha, \beta, \psi$, such that $M_1(\alpha, \beta) = 1$, $M_2(\beta, \psi) = 1$, then $\gamma$ such that $M_2(\alpha, \gamma) = 1$, $M_1(\gamma, \psi) = 1$, is uniquely defined because of properties of finite projective planes. Conditions (H1a,b) follow. To prove (H2) we need to color sides of triangles in three different colors. This is possible since there are three vertices in the polyhedron with different graphs as links. So, all triangles from $T$ have one of three possible colorings. We need to show, that for any $\alpha, \beta \in T$ we can choose $r > 0$ such that $M_j^r(\alpha, \beta) > 0$, where $j = 1, 2$. Geometrically it means that any $\alpha, \beta \in T$ can be realized so that $\beta$ lies in some sector with base $\alpha$ (for more details see [3]).

Without loss of generality we can assume, that $j = 1$. We will say, that $\beta \in T$ is reachable from $\alpha \in T$ in $r$ steps, if there is $r > 0$ such that $M_j^r(\alpha, \beta) > 0$. It is easy to see, that every triangle is reachable from some triangle of other color in one or two steps. So, to prove (H2) we need to show, that any triangle is reachable from another one of the same color. Now we can use the proof of the Theorem 1.3 from [3], since at each step of this proof it is only used, that the link at each vertex of the building is an incidence graph of a finite projective plane, which is true in our case too. The proof of (H3) is identical to the proof of (H3) in the case of the subshift considered in [3].

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