Polar decomposition of a Dirac spinor

J G Sumner and P D Jarvis

School of Mathematics and Physics, University of Tasmania
GPO Box 252-21, Hobart Tas 7001, Australia

Abstract

Local decompositions of a Dirac spinor into ‘charged’ and ‘real’ pieces \( \psi(x) = M(x)\chi(x) \) are considered. \( \chi(x) \) is a Majorana spinor, and \( M(x) \) a suitable Dirac-algebra valued field. Specific examples of the decomposition in 2 + 1 dimensions are developed, along with kinematical implications, and constraints on the component fields within \( M(x) \) sufficient to encompass the correct degree of freedom count. Overall local reparametrisation and electromagnetic phase invariances are identified, and a dynamical framework of nonabelian gauge theories of noncompact groups is proposed. Connections with supersymmetric composite models are noted (including, for 2+1 dimensions, infrared effective theories of spin-charge separation in models of high-\( T_c \) superconductivity).
1 Introduction

The standard model of elementary particle physics has been highly successful in account-
ing for the structure and interactions of the currently known generations of fundamental
quarks and leptons and associated charge carrying gauge bosons. However, there is con-
tinued interest in alternative formulations, in the hope of reducing the arbitrariness and
hierarchical fixing of standard model parameters, seeking physics beyond the standard
model, or perhaps in improving the singular mathematical structure of quantum field
theory. Supersymmetric models, composite models and string-theory models have all
been investigated under these umbrellas.

A recurring theme in investigations of alternative formulations has been the nature of
spinorial quantities, given their double-valuedness and the fact that physical quantities
are always even functions of them. Frameworks purporting to eliminate the ‘phase’ of
the spinor wavefunction have included Kähler fermions [1], the Loukesto real Clifford
algebra [2] and the Hestenes geometric algebra constructions [3], and various attempts to
substitute the four complex components of a Dirac spinor in four dimensions (aside from
the unobservable overall phase), with seven bosonic, gauge invariant quantities (usually
defining a Lorentz frame, together with a scalar field) [4, 5, 6]. In the present note we take
these ideas in a related direction, in proposing a different rearrangement of the degrees
of freedom of a Dirac spinor wavefunction, which may entail insights into new dynamical
models of fermions. We study ‘polar’ decompositions of a Dirac spinor, into a ‘charged’
and a ‘neutral’ part, in the form of a local product $\psi(x) = M(x) \cdot \chi(x)$. Taking into
account charge conjugation properties, as well as Lorentz covariance, $\psi'(x) = M'(x) \cdot \chi'(x) = S(\Lambda)M(x)S(\Lambda)^{-1} \cdot S(\Lambda)\chi(x)$ leads to the Ansatz that $\chi(x)$ be a Majorana
spinor (in space-time dimensions $D$ and spinor dimensions $N$ where this is defined), and that $M(x)$
be an appropriate Dirac-algebra valued quantity (within the space $M_N(\mathbb{C})$ of $N \times N$
complex matrices) transforming in the adjoint representation.

In order to have a specific example to study we take the case of $D = 3$ (2 + 1-
dimensional Minkowski space), with spinor dimension $N = 2$. In the next section (§2) we
argue for a particular standard form of the parametrisation, and by studying aspects of
this are able to demonstrate the decomposition, the number and nature of the parameters
and electromagnetic interactions. Particular examples of the free particle solutions of
the Dirac equation are given in the appendix, §A, for both plane and circular waves.
In §3 below, we return to the general parametrisation from the point of view of local
gauge symmetries and suggest a dynamical picture of the polar decomposition. In the
concluding remarks (§4 below) the case of general space-time dimensions is discussed,
together with implications of the polar decomposition in the setting of supersymmetric
models, applications in condensed matter theory, as well as the context of nonabelian
gauge theory.

2 Polar decomposition in 2+1 dimensions

For completeness, a summary of notational conventions relating to the Dirac algebra in
2 + 1 space-time dimensions is given in the appendix. The most important observation
is that the set of $2 \times 2$ complex matrices (to which the ‘charged’ part $M(x)$ of the Dirac wavefunction belongs) is spanned by the four matrices $\mathbb{1}, \gamma^\mu, \mu = 0, 1, 2$. Thus we begin by assuming a decomposition of the form of a product of a bosonic field and a (Majorana) fermionic field,

$$\psi(x) = M(x)\chi(x) = [a(x) + b_\mu(x)\gamma^\mu]\chi(x),$$

where $a$ and $b_\mu$ are real. For now, this can be interpreted as a general parametrisation, in that it is the sum of a Majorana spinor $a\chi$ and an anti-Majorana spinor $b_\mu\gamma^\mu\chi$ (the general case is discussed in §4 below). Clearly the parts transform under the Lorentz group variously as a scalar, a vector and a spinor field. This decomposition has too many parameters if it is to represent the four real degrees of freedom of a Dirac spinor. The scalar $a$ can always be scaled out, but $b_\mu$ is a three vector, so there must be a condition which will reduce the number of free parameters to 2. Fortunately, requiring form invariance under local $U(1)$ gauge transformations leads to such a condition. These transformations are given in terms of $\psi$ and the electromagnetic vector potential by

$$A_\mu \rightarrow A_\mu' = A_\mu - \partial_\mu \alpha(x),$$
$$\psi \rightarrow \psi' = e^{i\alpha(x)}\psi.$$  

The situation is complicated by the fact that the components of the polar decomposition get mixed up in order to absorb the phase. The appropriate procedure is to split the decomposition into Majorana and anti-Majorana parts, apply the gauge transformation and then collect respective Majorana and anti-Majorana parts back together. This leads to

$$\psi' = e^{i\alpha}\psi = (a \cos q\alpha + ib_\mu\gamma^\mu \sin q\alpha)\chi + (b_\mu\gamma^\mu \cos q\alpha + ia \sin q\alpha)\chi.$$  

The above implies the transformations

$$a\chi \rightarrow (a\chi)' = (a \cos q\alpha + ib_\mu\gamma^\mu \sin q\alpha)\chi,$$
$$b_\mu\gamma^\mu \chi \rightarrow (b_\mu\gamma^\mu \chi)' \equiv \frac{b_\mu'\gamma^\mu}{a'} (a \cos q\alpha + ib_\mu\gamma^\mu \sin q\alpha)\chi = (b_\mu'\gamma^\mu \cos q\alpha + ia \sin q\alpha)\chi.$$  

A necessary condition for a solution is

$$\frac{b_\mu'\gamma^\mu}{a'} = \frac{(b_\mu\gamma^\mu \cos q\alpha + ia \sin q\alpha)(a \cos q\alpha - ib_\mu\gamma^\mu \sin q\alpha)}{a^2 \cos^2 q\alpha + b^2 \sin^2 q\alpha}$$
$$= \frac{ab_\mu\gamma^\mu + i \sin q\alpha \cos q\alpha(a^2 - b^2)}{a^2 \cos^2 q\alpha + b^2 \sin^2 q\alpha}.$$  

Thus under gauge transformations, the vector part of the assumed decomposition has covariance problems, unless $a^2 = b_\mu b^\mu$. Take this as a constraint, so that the polar decomposition can be written (after choosing without loss of generality $a > 0$ for nonzero $\psi(x)$),

$$\psi(x) = M(x)\chi(x), \quad M(x) = \frac{1}{2} [1 + n_\mu(x)\gamma^\mu], \quad n_\mu n^\mu = 1$$
where $a$ has been scaled out. In this form the polar decomposition takes on the form of a true Dirac spinor, with both Majorana and anti-Majorana parts.

The decomposition now has a particularly elegant behaviour under gauge transformations:

$$\chi \rightarrow \chi' = \exp \left( i q n_\beta \gamma^\beta \alpha \right) \chi,$$

$$n_\mu \rightarrow n'_\mu = n_\mu.$$  \hfill (3)

So not only has form invariance under a gauge transformation been achieved, but a new restriction on the decomposition has been found so that the correct number of free parameters is present. Notice also that the gauge transformation is in fact a local Lorentz transformation on the Majorana spinor (see (A.15)).

Let us consider the implications of (2) for charge conjugation, $\psi \rightarrow \psi_c$. We have

$$\psi \rightarrow \psi_c = M_c \chi = \frac{1}{2} (1 - n_\mu \gamma^\mu) \chi.$$  \hfill (4)

Thus, the bosonic part of the wavefunction $n_\mu$ goes to $-n_\mu$ under charge conjugation\footnote{The apparent gauge invariance of $n_\mu$ in this parametrisation is the result of additional reparametrisation invariance (see below).}. This is consistent with the observation that the bosonic part of the wavefunction carries the interaction with the electromagnetic field. The constraint $n_\mu n^\mu = 1$ entails

$$(n_0)^2 = 1 + \mathbf{n} \cdot \mathbf{n},$$

so that the parameter space of $n_0$ is the two disconnected regions $n_0 > 1$ and $n_0 < 1$, corresponding to a two-sheeted hyperboloid. Thus it is actually the sign of $n_0$ alone that characterises the charge conjugation symmetry of the wavefunction.

An expression for $\chi$ is calculated easily as

$$\chi = \frac{1}{2} (\psi + \psi_c);$$  \hfill (5)

the Majorana part of the wavefunction is simply the superposition of the original spinor with its charge conjugate. Consider the scalar and vector covariants

$$\overline{\psi} \psi = \frac{1}{2} \overline{\chi} \chi, \quad \overline{\psi} \gamma^\mu \psi = \frac{1}{2} n^\mu \overline{\chi} \chi.$$  

Hence the scalar density of the Dirac field is carried exclusively by the Majorana-spinor component of the field, while the Noether current of the Dirac field is represented by $n_\mu$ scaled by the scalar density $\overline{\chi} \chi$. A remarkably simple form for $n_\mu$ follows as

$$n^\mu = \frac{\overline{\psi} \gamma^\mu \psi}{\overline{\psi} \psi} = \frac{\tilde{j}^\mu}{\overline{\psi} \psi},$$  \hfill (6)

so that $n^\mu$ can be interpreted as the 3-velocity of the Dirac field.

Finally, notice that both $M$ and its charge conjugate $M_c = \frac{1}{2} (1 - n_\mu \gamma^\mu)$ are singular,

$$\det M = \det M_c = \frac{1}{4} (1 - n_\mu n^\mu) = 0.$$  \hfill (7)
This seems to imply that if we take $\psi = M\chi$ then $\chi$ is not unique. However, (7) is equivalent to
\[
M^2 = M, \quad M^2_c = M_c, \quad MM_c = 0,
\]
showing that $M$ and $M_c$ are orthogonal projectors. The null space of $M$ is then any spinor written in the form $M_c\varphi$. For the decomposition this means that
\[
\psi = M\chi = M(\chi + M_c\varphi).
\]
However it is clear that $M_c\varphi$ cannot be Majorana so that $\chi$ is in fact unique. Notice also that $\psi$ is orthogonal to its charge conjugate:
\[
\frac{1}{2} \bar{\psi}_1 \psi_2 = - \bar{\psi}_c = \bar{\chi}MM_c\chi = \chi_2\chi_1 \det M = 0,
\]
due to the projective property of $M$; similarly the complex vector $\bar{\psi}\gamma^\mu\psi_c$ vanishes, for real $n^\mu$.

3 Local gauge symmetries

In the previous section the discussion focussed on choosing a minimally constrained set of parameters for the polar-type decomposition in the 2 + 1-dimensional case. In this section we reconsider the general Ansatz $\psi = M\cdot\chi$ from the point of view of local gauge symmetries which carry redundant parameters, and which may also play a dynamical role. Namely, the product decomposition admits any local reparametrisations which respect the form of $\chi$ as a Majorana spinor. Given $\chi$ as in (A.14), the requirement that $L(x)\chi$ be of the same form necessitates
\[
L(x) = \begin{pmatrix} P & Q \\ Q^* & P^* \end{pmatrix}, \quad PP^* - QQ^* \neq 0, \quad P, Q \in \mathbb{C}.
\]
– identical in this case to the pseudo-unitary transformations acting on bispinors $\psi$ as in (A.15), preserving the (Dirac) norm $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi = u^*u - v^*v$ up to proportionality. These are nothing but local Lorentz transformations, in fact, elements of the covering group $SU(1, 1)$ composed with local field rescalings belonging to the multiplicative group of dilatations, $\mathbb{R} - \{0\} \simeq GL(1, \mathbb{R})$. Thus with our assumption that $\chi$ is a Majorana spinor, we have the following semidirect product of global Lorentz transformations $A \in SO(2, 1)$ (in the spinor representation $S(A)$), local Weyl group reparametrisations $L \in SU(1, 1) \times GL(1, \mathbb{R})$ and local electromagnetic phase transformations:
\[
M \to S(A)MS(A)^{-1}, \quad \chi \to S(A)\chi,
\]
\[
M \to ML(x)^{-1}, \quad \chi \to L(x)\chi,
\]
\[
M \to e^{i\theta}M, \quad \chi \to \chi.
\]
From this point of view the minimal parameters for $M$ of the previous section can be seen as an appropriate gauge choice with respect to the local Weyl group. Generically,
the independent degrees of freedom of \( M \) are associated with the orbit structure of \( \mathbb{M}_2(\mathbb{C}) \) under the right action of \( L \), and the minimal parameters for \( M \) correspond to a choice of section of \( \mathbb{M}_2(\mathbb{C}) \) under the induced fibration. For example, the apparent non-invariance of \( \chi \) under local phase transformations of \( \psi \) – contradictory if \( \chi \) is a neutral field – is the result of a combined local phase transformation and local Weyl (actually \( SU(1,1) \)) transformation in order to remain within the minimal gauge class:

\[
e^{i\theta} M \chi = M e^{i\theta n^\mu \gamma_\mu} \chi = M(e^{i\theta n^\mu \gamma_\mu} \chi)
\]

where the projection property of \( M = \frac{1}{2}(1 + n^\mu \gamma_\mu) \) with \( n \cdot n = 1 \) has been used, to compensate the local phase transformation with a local Lorentz transform with parameter \( q\theta(x)n^\mu \), so that effectively \( M' = M \) and \( \chi' = e^{i\theta n^\mu \gamma_\mu} \chi \) as discussed above.

The orbits of \( M_2(\mathbb{C}) \) are most easily studied via the unitary change of basis

\[
\chi \rightarrow \tilde{\chi} = \Theta \chi, \quad M \rightarrow \tilde{M} = \Theta M \Theta^{-1},
\]

where \( \Theta = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \), so \( \tilde{\chi} = \begin{pmatrix} r \\ s \end{pmatrix} \). (9)

with \( u = r + is \) in terms of the original components of \( \chi \). In this real basis\(^3\) the Lorentz group clearly acts as \( 2 \times 2 \) nonsingular real matrices (the group \( SL(2,\mathbb{R}) \)), with the local Weyl group being \( GL(2,\mathbb{R}) \). Under \( M \rightarrow M' = M \cdot L(x) \), \( \det M' = \det M \cdot \det L \), so that orbits can be classified\(^4\) according to whether \( \det M = 0 \) or \( \det M \neq 0 \).

Generically let

\[
\tilde{M} = \begin{pmatrix} a + i\alpha & b + i\beta \\ c + i\gamma & d + i\delta \end{pmatrix}
\]

(corresponding to an expansion \( M = A + B^\mu \gamma_\mu \) in the Dirac algebra, with complex coefficients). Then \( \det M = \det \tilde{M} = (\Delta - \Delta') + i(\Gamma + \Gamma') \) with \( \Delta = ad - bc \), \( \Delta' = \alpha \delta - \beta \gamma \), \( \Gamma = a \delta - b \gamma \), \( \Gamma' = \alpha d - \beta c \). Consider the case \( \Delta \neq 0 \), \( \det M \neq 0 \). Let \( \tilde{M} = e^{i\phi} \tilde{N} \) where \( \det \tilde{N} \) is real, and set

\[
L(x) = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\]

Transforming back to the standard basis, we find for some \( b^\mu \)

\[
M' = e^{i\phi}(1 + b^\mu \gamma_\mu), \quad \det M' = e^{2i\phi}(1 - b \cdot b) \neq 0.
\]

On the other hand in the case \( \Delta \neq 0 \), \( \det M = 0 \) the same choice of \( L(x) \) yields

\[
M' = (1 + n^\mu \gamma_\mu), \quad \det M' = 1 - n \cdot n = 0.
\]

Clearly the latter case\(^4\) coincides with the minimal parametrisation of the previous section; the two real constraints \( \det M = 0 \) entail \( 8 - 2 = 6 \) parameters, or effectively \( 6 - 4 = 2 \) degrees of freedom resident in \( M \), after allowing for the gauge transformations

\(^2\)The Dirac matrices are pure imaginary in this basis.

\(^3\)The determinants of the real and imaginary parts of \( M \) also scale with \( \det L(x) \).
by elements of \( GL(2, \mathbb{R}) \) to fix a further four real components. If \( \det M \) is real but nonzero, there are three field components corresponding to the parametrisation \( M = (a + b^\mu \gamma^\mu) \) for real \( a, b^\mu \) constrained to some hyperboloid, say \( a^2 - b \cdot b = 1 \). Finally in case (10) above there are four fields \( \phi, b^\mu \).

The surfeit of allowed parameters of \( M \) in the last two cases would be expected to be reconciled in a dynamical picture. For example, the role of the four-dimensional inner product \( a^2 - b \cdot b \) in the above discussion perhaps points to a higher conformal symmetry formulation of these bosonic concomitants of the Dirac spinor field (a projective null cone has codimension two, effecting the desired reduction from four to two degrees of freedom). An obvious first step is to promote the Weyl reparametrisation freedom of the polar decomposition to a local gauge principle, by introducing gauge fields \( \omega^{ab}_{\mu} \) in the internal Lorentz algebra, together with scale gauge fields \( \varphi^\mu \), and covariant derivatives

\[
\nabla_\mu \chi = \partial_\mu \chi + \frac{1}{2} \omega^{ab}_{\mu} \sigma_{ab} \chi + \varphi^\mu \chi
\]

\[
\nabla_\mu M = \partial_\mu M - \frac{1}{2} \omega^{ab}_{\mu} M \sigma_{ab} - \varphi^\mu M.
\]

Furthermore, we anticipate a total covariant derivative including electromagnetic potential

\[
D^\mu = \nabla^\mu + iq A^\mu
\]

so that the ordinary covariant derivative of the Dirac spinor \( \psi = M \cdot \chi \) becomes

\[
(i \gamma^\mu D^\mu + m) M \cdot \chi = [(i \gamma^\mu D^\mu + m) M] \cdot \chi + M \cdot (i \gamma^\mu \nabla^\mu \chi) + [i \gamma^\mu, M] \partial^\mu \chi.
\]

Regarding \( M \) and \( \chi \) as the fundamental fields, it is natural to interpret the Dirac equation for \( \psi \) as the result of separate ‘free’ equations

\[
(i \gamma^\mu D^\mu + m) M = 0, \quad i \gamma^\mu \nabla^\mu \chi = 0 \tag{12}
\]

plus coupling or ‘interaction’ terms, which taken together would re-constitute the third, commutator, piece of the above expansion (which is not locally Weyl covariant as it stands).

4 Conclusions

In this note we have followed through, for the particular case of \( 2+1 \) dimensions, a proposal to re-interpret the degrees of freedom of Dirac spinors in terms of ‘polar’ type decompositions, involving a neutral, Majorana part, and a charged companion. Here we have simply considered the kinematical implications of the decompositions. For the present case, the two degrees of freedom of the Majorana spinor are accompanied by a (possibly complex) scalar and vector field, which encapsulate the dynamics of a further two degrees of freedom. Under additional conditions, these fields may be constrained, for example to the surface of a hyperboloid or null cone in higher dimensions. As an aside notice that (2) implies that there is no sense, under this scheme, in which a generic Dirac spinor can be approximated as some kind ‘charge perturbation’ of a Majorana spinor. This is because the Majorana and anti-Majorana parts of the wavefunction are of the
same order of magnitude under all circumstances. Therefore, from the point of view of classical equations of motion, solutions of the Dirac equation under minimal coupling with the electromagnetic field cannot be treated as perturbations to Majorana solutions of the free equation.

In $3+1$ spacetime dimensions the polar decomposition $\psi = M \cdot \chi$ of a Dirac spinor involves, apart from a Majorana spinor (4 real degrees of freedom), the orbit structure of $M_4(\mathbb{C})$ under the local Weyl group $GL(4,\mathbb{R})$, which can be expected to fix at most $32 - 16 = 16$ real parameters. The dynamics of the remaining component fields and local gauge fields must therefore reduce the additional number of degrees of freedom resident in $M$ from 16 to 4.

Further work to develop a concrete dynamical picture is required. The connection fields implementing local Weyl reparametrisation invariance parametrise a non-compact gauge algebra. Standard Yang-Mills actions are inconsistent (because the Killing form occurring for terms quadratic in the field strength in the action is non-positive definite for hermitean fields), and appeal must therefore be made to other formulations\[7\]. Alternatively, the analogy between the connection forms $\omega^{ab\mu}$ and $\varphi_{\mu}$ and the vierbein formulation of relativity (where local Lorentz invariance occurs in the representation of the metric tensor) might be exploited, so that Einstein-type actions for these fields may be possible.

Composite models of quarks and leptons commonly involve either binding of multi-particle sub-quark or sub-lepton states by a hypercolour gauge confinement mechanism\[8\], or two-body bound states of scalars and fundamental $Ur$-fermions. Our proposal is evidently consistent with the latter scenario, but motivated by a deconstruction of the existing degrees of freedom of Dirac fields, rather than invoking mere repetition of standard degrees of freedom at the preonic level. Thus the role of the Majorana particles is to carry spin degrees of freedom, while charge attributes are conferred by the quantum numbers of the bosonic binding partners. Whether ‘confinement’ exists for this type of model is unknown.

As noted above, our gauge bosons belong to a non-compact gauge symmetry, and standard Yang-Mills theory cannot directly be applied. What the degree of freedom count does guarantee is a balance between fermionic and bosonic parts for each Dirac field, the hallmark of possible supersymmetric formulations. In any case, the existence of fermion generations is attributed, in standard preon model terms, to the bound state spectrum. For example, if (12) are regarded as ‘free’ equations of motion, different fermion generations may be associated with zero modes of a covariant Dirac-type operator, which then bind with the charged scalar. In turn, the equation of motion of the latter acquires curvature terms (in the noncompact gauge symmetry) which affect its mass, and hence ultimately the mass of known quarks and leptons.

There is a striking analogy between the foregoing considerations (for relativistic theories) and the known situation in the condensed matter context, again in $2+1$ dimensions, in which spin-charge separation occurs into ‘spinon’ and ‘holon’ degrees of freedom. Relativistic formulations are appropriate at particular nodes of the Fermi surface, and indeed ‘supersymmetric composite models’ have been advocated to describe an effective infrared field theory of the phenomena of high-$T_c$ superconductivity, \[9\]. Again, the implication for the relativistic context would be that there is indeed a phase of matter wherein fun-
fundamental fermions fractionate into their constituents, with the local gauge fields taking on the dynamical role played by the lattice. In 2+1 dimensions the ingredients identified in this work (Majorana spinor, and bosonic accompaniments accounting for 2 degrees of freedom) are necessarily the same as for matter superfields (for \( N = 1 \) supersymmetry). The analogy with supersymmetry is even closer in terms of the specific fields \( a(x), b_\mu(x) \) parameterizing the matrix \( M(x) \), if, as the result of algebraic equations of motion and for appropriate gauge fixing conditions, constraints such as \( b_\mu = \partial_\mu a \) (for complex \( a, b_\mu \)) arise. Further development of our model would involve consideration of possible supersymmetric formulations [10], and dynamical aspects.

In the case of the ‘polar’ decomposition \( \psi = M \cdot \chi \) of Dirac spinors in a nonabelian gauge theory, it is natural to adopt a generalised Ansatz in which quantum numbers are shared between \( M \) and \( \chi \) (only internal symmetries possessing suitable real representations can be implemented on multiplets of Majorana spinors \( \chi \)). An experimental indication of the viability of such nonabelian polar decompositions might be the confirmation of Majorana mass terms in the lepton sector, suggesting the possibility of admixtures with the \( Ur \)-field \( \chi \). Further consideration of such generalisations is deferred to future work.

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Appendix

A The Dirac algebra in 2 + 1 dimensions.

The fundamental relation defining the gamma matrices in 2+1 dimensions is

\[
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},
\]

(A.13)

where the metric is

\[
\eta = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

The anti-commutation relation (A.13) can be satisfied by a set of 2 × 2 matrices generated from the Pauli matrices

\[
\sigma^1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \sigma^2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad \sigma^3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

Any generic spinor in 2+1 dimensions has two components

\[
\psi = \begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}.
\]
These independent components can be used to construct independent positive and negative energy solutions. For the purposes of explicit calculation it is useful to take a particular representation of the gamma matrices. Using the Pauli matrices a standard representation can be chosen as

$$
\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2.
$$

In odd dimensions there is no $\gamma^5$ matrix. The charge conjugation matrix $C$ can be chosen to be,

$$
C = -\gamma^2 = -i\sigma^2, \\
C^{-1} = -C = C^\dagger = C^\top.
$$

A Majorana spinor then takes the form

$$
\chi = \begin{pmatrix} v \\ v^* \end{pmatrix}. \tag{A.14}
$$

Taking solutions of the Dirac equation as anti-commuting field operators it follows that for Majorana fields

$$
\bar{\chi}\gamma^\mu\chi = \bar{\chi}\sigma^{\mu\nu}\chi = 0,
$$

indicating that the Majorana field is truly non-interacting with respect to the usual couplings.

The algebra is completed by the relations

$$
\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - ie^{\mu\nu\alpha}\gamma^\alpha, \\
e^{012} = 1, \\
\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu} = \eta^{\mu\alpha}\gamma^{\nu} - \eta^{\mu\nu}\gamma^{\alpha} + \eta^{\alpha\nu}\gamma^{\mu} - i\epsilon^{\mu\alpha\nu}.
$$

The generators of Lorentz transformations are then given by

$$
\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] = \epsilon^{\mu\nu\lambda}\gamma^\lambda,
$$

and the spinor representation of the Lorentz group can be expressed as

$$
S(\Lambda) = e^{i\omega_{\mu\nu}\sigma^{\mu\nu}} = e^{i\theta_{\mu}\gamma^\mu},
$$

$$
\theta^\lambda = \epsilon^{\mu\nu\lambda}\sigma_{\mu\nu}. \tag{A.15}
$$

As an example of polar decompositions, consider the free particle solutions

$$
\psi = N \left( \frac{1}{ip_+ E+m} \right) e^{-ip_+x^\mu}, \quad \psi_c = N \left( \frac{-ip_+}{E+m} \right) e^{ip_+x^\mu}, \tag{A.16}
$$

for $E = +\sqrt{p^2 + m^2}$, $N$ is a normalization factor and $p_+ = p^1 + ip^2 = p^r$. Analogously free solutions in polar coordinates, $(x, y) \to (r, \theta)$, can be defined in terms of Bessel functions as

$$
\psi = N \left( \frac{J_\alpha(pr)}{E+m J_{\alpha+1}(pr)} \right) e^{i(\alpha\theta - Et)}, \quad \psi_c = N \left( \frac{-pe^{-i\theta}}{E+m J_{\alpha+1}(pr)} \right) e^{-i(\alpha\theta - Et)}. \tag{A.17}
$$
where \( p = \sqrt{E^2 - m^2} \) and \( J_\alpha(pr) \) are the Bessel functions of the first kind. From the plane wave solutions (A.16) and using \( \chi \) as in (A.14) the polar decomposition takes on the form

\[
v = \frac{1}{2} N \left[ (E + m - ip_-) \cos p_\mu x^\mu - i(E + m + ip_-) \sin p_\mu x^\mu \right] (A.18)
\]

\[
n^0 = \frac{E}{m}, \quad n^1 = \frac{p^1}{m}, \quad n^2 = \frac{p^2}{m}. (A.19)
\]

Whereas for the charge conjugate solution \( \psi_c \)

\[
n^0_c = -\frac{E}{m}, \quad n^1_c = -\frac{p^1}{m}, \quad n^2_c = -\frac{p^2}{m}. (A.20)
\]

Thus the property \( n_\mu \rightarrow -n_\mu \) under charge conjugation is in keeping with what happens under charge conjugation to the plane wave solution \( \psi \rightarrow \psi_c \).

For the solutions (A.17) we have

\[
v = \frac{1}{2} N \left[ e^{i(\alpha \theta - Et)} J_\alpha(pr) - \frac{p e^{-i\theta}}{E+m} J_{\alpha+1}(pr) e^{-i(\alpha \theta - Et)} \right],
\]

\[
j^0 = N^2 \left[ J_\alpha^2(pr) + \frac{p^2}{(E+m)^2} J_{\alpha+1}^2(pr) \right] = j^0_c,
\]

\[
j^1 = \frac{2N^2 p}{E+m} J_\alpha J_{\alpha+1} \sin \theta = j^1_c,
\]

\[
j^2 = -\frac{2N^2 p}{E+m} J_\alpha J_{\alpha+1} \cos \theta = j^2_c,
\]

\[
j^r = \frac{2N^2 p}{E+m} J_\alpha J_{\alpha+1} = j^r_c,
\]

\[
j^\theta = (\theta - \frac{\pi}{2}) = j^\theta_c,
\]

\[
\bar{\psi} \psi = N^2 \left[ J_\alpha^2(pr) - \frac{p^2}{(E+m)^2} J_{\alpha+1}^2(pr) \right] = -\bar{\psi}_c \psi_c. (A.21)
\]

So again we have \( n^\mu \rightarrow -n^\mu \) under charge conjugation.

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