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String equation–2. Physical solution.

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References
1. Introduction

This paper is a continuation of [1].

String Equation is by definition the equation \([L, A] = 1\) for the coefficients of two linear ordinary differential operators \(L\) and \(A\).

This terminology appeared after the papers [2, 3, 4] in 1989-90. For the so-called "double-scaling limit of 1-matrix models" we always have \(L = -\partial_x^2 + u(x)\), \(A\) is some operator of the odd order \(2k + 1\). For \(k = 0\) we have trivial answer \(u = x\), \(A = \partial\).

First nontrivial case is \(k = 1\). We have here Gardner–Green–Kruscal–Miura–Lax operator

\[
A = -4\partial^3 + 6u\partial + 3u_x
\]

and the following differential equation (string equation) for \(u(x)\):

\[
u_{xxx} - 6uu_x = 1
\]

Last equation after one integration exactly coincides with the well–known Painlevé equation of the first type or simply \(P-1\) equation:

\[
u_{xx} - 3u^2 = x
\]

We are looking for real solutions on the \(x\)–line or at least on the halfline \(-x \to \infty\). Even more: the solutions which may be important for the Quantum String Theory should have asymptotics

\[
u(x) \sim +\sqrt{-x/3}, \ x \to -\infty
\]

For the mathematical foundation of the 'double–scaling limit' and of the selection of Physical Solution of P-1 equation from the finite matrix models see [5].

P-1 equation has two types of solutions in the class of formal series:

\[
u_\pm(x) = \pm\sqrt{-x/3}(1 + \sum_{i=1}^{\infty} a_i^\pm \tau^{-2i}), \ \tau = (-x)^{5/4}
\]

1 Russian version of this paper was published in the Journal 'Algebra and Analysis' (St–Petersburg, Russian Academy of Sciences, 1994), v.6, No 3, pp. 118-140, dedicated to the 60–ieth birthday of L.D.Faddeev
It was shown by Boutroux [6] that there exists an unique exact solutions of P-1, for which the formal series $u_-(x)$ is an asymptotic one.

For other solutions such that $u \sim -\sqrt{-x/3}$ even the second coefficient $a_{1i}$ has nothing in common with the real asymptotic behavior.

Situation is different for the physical solutions $u_+(x)$:

There exists a 1-parametric ‘separatrix’ family of solutions of the P-1 equation for which formal solution $u_+(x)$ is an asymptotic series.

More detailed information about these families can be found in the work [7].

In the papers [1, 8] a ‘Plank parameter’ has been introduced in the theory of P-1 equation. For studying its asymptotics it is good to write it in the form

$$[L, A] = \epsilon \cdot 1, \quad u_{xx} - 3u^2 = \epsilon \cdot x$$

Two asymptotic methods were developed in [1] for the equation above:

1. **Nonlinear semiclassics (or averaging or Bogolyubov–Whitham method)**—see Appendix in [1] written by Dubrovin and Novikov. This method has been improved essentially in [9]. However it gives some good information about some nonphysical solutions only. For example, Dubrovin and Novikov in [1], Appendix, studied by this method some solutions with infinite number of poles on the important real halfaxis $-x \to \infty$; Krichever investigated nonphysical solutions of the type $u \sim -\sqrt{-x/3}$ in [9].

We don’t see any way to apply this technic to the studying of the physical solutions for $k = 1$. The nonphysical solutions mentioned above have been studied also earlier in the work [10] and in the later papers of the St-Petersburg group (see the survey [12]; they were known earlier in the classical theory of P-1 equation).

2. **Linear semiclassics for the Lax pair.**

The idea of the ‘Isomonodromic’ method has been invented in the work [13]. It is based on the semiclassics. It has been applied to the different Painlevé types in [14]. P-1 equation was studied for example in [10, 8, 11, 12]. Some results concerning the behavior of physical solution along the special lines in complex $x$-space and monodromy (‘Stokes’) coefficients were obtained.

In the work [1] completely different semiclassical ideas were developed.
In particularly some strange beautiful identities connecting the theory of P-1 with families of elliptic curves were found. These ideas and their development may be found in the paragraphs 2–4 of the present work.

Some very specific features of the "Isomonodromic" method available for the Physical solution only will be developed in the present work (see the paragraphs 5 and 6). The comparison of our technic with St-Petersburg’s and Fokas group may be found in the Remark after the proof of the Corollary 1 in the paragraph 5.

By the conjecture of Novikov, these special 'Physical' solutions of the equations like $[L, A] = 1$, which are analytically exceptional, probably have much stronger 'hidden symmetry' than other solutions of the same equations and are much more 'exactly integrable' than others. However, this problem is not yet solved.

Our paper is written for the case $k = 1$ only. However there are no difficulties to develop the same staff for any $k$. For even values of $k = 2n$ the physical solutions can be described by the method 1 above, like in [1], but the case $k = 1$ is more complicated.
2. General Identities.

Let us construct a 'Zero-Curvature Representation' for the P-1 equation, starting from the zero-curvature representation of the KdV system found in the very first paper of Novikov on the periodic problem [15, 16] (it was more convenient for the studying of the so-called 'finite-gap solutions' than the ordinary Lax representation \( L = [L, A] \)). KdV is equivalent to the compatibility condition for the pair of the linear systems below:

\[
\begin{align*}
\Psi_t &= \Lambda \Psi, \quad \Psi_x = Q \Psi \\
\Lambda_x - Q_t &= [Q, \Lambda]
\end{align*}
\]

Here we have

\[
\Lambda = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix}
\]

and

\[a = -u_x, \ b = 2u + 4\lambda, \ c = -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx}\]

You may replace here \( u_{xx} \) by \( \epsilon \cdot x + 3u^2 \)

**Remark.** In general, for any \( k \) we have analogous representation with the same matrix \( Q \), \( b \) is a polinomial of the order \( k \) in \( \lambda \)

**Proposition 1** Replacing the derivative \( \partial_t \) by \( \epsilon \cdot \partial_\lambda \) we get a 'Zero-Curvature' representation for P-1 equation

\[
\begin{align*}
\epsilon \Psi_\lambda &= \Lambda \Psi, \quad \Psi_x = Q \Psi \\
\Lambda_x - \epsilon \cdot Q_\lambda &= -[\Lambda, Q]
\end{align*}
\]

For the proof, it is easy for example to check this by elementary calculation. In fact, before 1990 people studied P-1 system using some different representation (see [14, 11, 12]). This one directly follows from the Heisenberg type relation \([L, A] = 1\), as it was pointed out in [1]. It looks more natural.

Let \( R(\lambda) = -\det \Lambda = a^2 + bc = -16\lambda^3 - 4C\lambda - D \).

**Proposition 2** The following general identities are true:

\[
\begin{align*}
dR/dx &= -\epsilon \cdot b, \quad db/dx = -2a \\
dC/dx &= \epsilon, \quad dD/dx = 2\epsilon \cdot u \\
C &= u_{xx} - 3u^2, \quad D = -4u^3 - u_x^2 + 2uu_{xx} \\
u_x &= \sqrt{R(\lambda)}|_{\lambda = -u/2}
\end{align*}
\]
This statement also might be checked for $k = 1$ by the elementary calculation. Its analog follows easily from the 'Zero-Curvature Representation' for any $k$. It is obvious that all coefficients of the Riemann surface, who were constants for $\epsilon = 0$ will have $x$-derivatives proportional to $\epsilon$.

Last equation demonstrates very nice interpretation of $P^{-1}$ as a special deformation of the elliptic Riemann surface $\Gamma(x)$:

$$y^2 = R(\lambda)$$

(Here $x$ plays a role of the parameter). We shall discuss this subject in the second paragraph.

Let us introduce a $\mathbb{Z}$, graded commutative and associative differential ring $A_\epsilon$ over $\mathbb{Z}[1/2]$ containing the symbols $u, \partial, \epsilon$ such that:

$$A_\epsilon = \sum_{i \geq 0} A_i, \ A_0 = \mathbb{Z}, \ A_1 = 0 \quad (13)$$

$$u \in A_2, \ u_x = \partial u \in A_3, \ u_{xx} = \partial^2 u \in A_4, \ldots, \epsilon \in A_5 \quad (14)$$

$$u_{xxx} = -6uu_x = \epsilon \quad (15)$$

**Theorem 1** Consider the graded rings $A_\epsilon$ and a trivial $\epsilon$-extension of the ring of elliptic functions holomorphic outside of the point $a = 0$ on the algebraic curve $\Gamma$ (the same as above)

$$\tilde{y}^2 = \tilde{R}(\mu) = 4\mu^3 - 2\mu - 3, \ \mu = -\lambda, \ \tilde{y} = y/2, \ g_2 = -C = -\epsilon \cdot x, \ g_3 = D/4 \quad (16)$$

and a substitution, which preserves the grading by definition:

$$u(x) = 2\wp(a)$$

(the grading of $\wp^{(n)}(a)$ we take as $2 + n$). The following formulas are true:

$$u = 2\wp(a), \ u_x = 2\wp'(a) = 2\partial_w \wp(w + a)|_{w=0}$$

$$u_{xx} = 2\wp''(a) = \partial_w^2 \wp(w + a)|_{w=0}$$

$$u_{xxx} = 2\wp'''(a) + \epsilon$$

$$u_{x...x}^{(4)} = 2\wp^{(4)}(a), \ u_{x...x}^{(5)} = 2\wp^{(5)}(a) + 6u\epsilon$$

$$u_{x...x}^{(6)} = 2\wp^{(6)}(a) + 24ux\epsilon$$

$$u_{x...x}^{(7)} = 2\wp^{(7)}(a) + (60u_{xx} + 36u^2)\epsilon$$

$$u_{x...x}^{(8)} = 2\wp^{(8)}(a) + (36 \cdot 30u_{xx} + 60u_x)\epsilon + 60\epsilon^2$$

$$\ldots$$

$$u_{x...x}^{(n)} = 2\wp^{(n)}(a) + \sum_{i \geq 1} P_{i,n}(u, u_x, u_{xx})\epsilon^i, \ P_i \in A_{n-5i+2} \quad (16)$$
The polynomials $P_{i,n}$ have constant integral coefficients. They depend only on the variables $u, u_x, u_{xx}$, which may be replaced by $\wp, \wp', \wp''$.

**Remark.** After the identification of $\epsilon$ with scalars we shall have a ring, graded modulo 5. Identifying the symbols $g_2, g_3$ with the scalars we shall destroy the grading, but the formulas above don’t contain these symbols.

The proof of this theorem is based on the equation (which is already established) $u_x = 2\sqrt{R(u/2)}$ and on the comparison of (4) with standard equations for the elliptic functions (see [17]):

\[
(\wp')^2 = 4(\wp)^3 - g_2\wp - g_3
\]

\[
g'_2 = g'_3 = \epsilon' = 0
\]

\[
\wp'' = 6(\wp)^2 - g_2/2
\]

The equality $u_{xx} = 2\wp''(a)$ follows from the relation $u_{xx}/2 = 6(u/2)^2 - g_2/2$, the same as for $\wp''$. Differentiating it and using the relation $u_{xxx} = 6uu_x + \epsilon$, we obtain the next relation for $u_{xxx}$ and so on. Each time when $u_{xx}$ appears, its derivative in $x$ will contain $\epsilon$. The total sum of terms in the final sum which don’t contain $\epsilon$ at all will exactly equal to the formula for $\wp^{(n)}$, because they are constructed exactly by the same rule. The theorem therefore is proven.
3. String equation and deformations of the Riemann Surfaces

Consider now the Zero-Curvature equation for P-1 (with $\epsilon$) and especially the first linear system in the variable $\lambda$. It contains a parameter $\epsilon$ as a Plank constant. It is natural therefore to develope a semiclassical approach to the studying of the $\Psi(\lambda)$, because the $\lambda$-dependence of the coefficients is already known (the corresponding matrix $\Lambda$ is polinomial and traceless, it does not depend on $\epsilon$).

The first principle of Semiclassical Approach is that it should be applied to systems which are diagonal neglecting the terms of the order $\epsilon$ in the righ-hand part.

Therefore we should make a corresponding transformation. Consider a matrix $U(\lambda)$, such that

$$U^{-1}\Lambda U = \begin{pmatrix} \sqrt{R} & 0 \\ 0 & -\sqrt{R} \end{pmatrix} = \Lambda'$$

$$R = -\det \Lambda = a^2 + bc$$

$$U = \begin{pmatrix} 1 & 1 \\ \chi_- & \chi_+ \end{pmatrix}$$

$$\chi_\pm = \frac{-a \mp \sqrt{R}}{b}$$

After the substitution

$$\Psi = U\tilde{\Psi}$$

we are coming to the new system through the Gauge Transformation

$$\epsilon \tilde{\Psi}_\lambda = \tilde{\Lambda}\tilde{\Psi} = (\Lambda' - \epsilon \cdot U^{-1}U_\lambda)\tilde{\Psi}$$

The last system is diagonal modulo $\epsilon$, but it is defined on the Riemann surface

$$\Gamma : y^2 = R(\lambda)$$

not on the complex plane.

After that we may write a formal semiclassical solution:

**Proposition 3** There exists a formal solution

$$\tilde{\Psi}_{sc} = (1 + \sum_{i \geq 1} A_i \epsilon^i)e^{\frac{1}{\epsilon}B_{-1} + B_0 + \sum_{i \geq 1} B_i \epsilon^i}$$

...
such that all matrices $A_i$ have diagonal elements equal to zero, all matrices $B_i$ are diagonal and may be found in the form:

$$
B_{-1,\lambda} = \sqrt{R} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

$$
B_{0,\lambda} = \frac{a}{b} \lambda \frac{b}{2\sqrt{R}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \left( \frac{R\lambda}{4R} - \frac{b\lambda}{2b} \right) \cdot 1
$$

(20)

All functions —matrix elements of $A_i$ are algebraic on $\Gamma$, all differential forms—matrix elements of $dB_i(\lambda)$ are abelian differentials on $\Gamma$.

**Remark 1.** There exist a formal solution of this form which satisfies to the both linear equation in the Zero-Curvature representation for the equation $\mathcal{L}$. We shall discuss this later.

**Remark 2.** It is easy to write algebraic formulas for all $A_i$ and $B_i,\lambda$, like in [1]. The last proposition in fact is extracted from this work.

The proof of this proposition may be obtained by the direct substitution in the equation. All formulas for matrix elements were written in [1].

There is a very interesting analogy between semiclassics for this system and the so-called 'Baker–Akhiezer’ function:

**Proposition 4** The semiclassical solution above is essentially scalar function on the Riemann surface $\tilde{\Gamma}$ which is a covering over the surface $\Gamma$ with some branching point. It means exactly that the permutation of the matrix indices 1 and 2 is equivalent to the permutation $\kappa$ of of sheets of the surface $\Gamma$. In particularly, we have

$$
\kappa^* d_{b_{11}}(\lambda, +) = d_{b_{22}}(\lambda, -)
$$

$$
\kappa^* a_{12}(\lambda, +) = a_{21}(\lambda, -)
$$

(21)

for all matrices $A_i$, $d_\lambda B_i$ in $\tilde{\Psi}_{sc}$

The proof immediately follows from the fact that this permutation acts in the same way on all our equations and substitutions including the original system in the variable $\lambda$ and matrix $U$.

Consider now the scalar function $\Phi$ which determines $\tilde{\Psi}_{sc}$ neglecting the terms of the order one in $\epsilon$

$$
\tilde{\Psi}_{sc}^1 = \exp\{1/\epsilon B_{-1} + B_0\} = R^{-1/4} b^{1/2} \begin{pmatrix} \Phi & 0 \\ 0 & \Phi^* \end{pmatrix}
$$
\[ d_\lambda \ln \Phi = \Omega = \left( \frac{1}{\epsilon} \sqrt{R} + \frac{a/b}{2\sqrt{R}} \right) d\lambda \]  
\hspace{1cm} \text{(22)}

From the definition of \( \Omega \), \( a \), \( b \), \( R \) above we obtain immediately the following result:

**Proposition 5** The following 'Baker–Akhiezer' type asymptotic is true for the differential form \( \Omega \) and function \( \Phi \)

\[
\begin{align*}
\Omega & \sim \left( \frac{8}{\epsilon z^6} + \frac{C}{\epsilon z^2} - \frac{D}{4\epsilon} + O(z^2) \right) dz \\
\Phi & \sim \exp \left( -\frac{8}{5\epsilon} k^5 - kx \right) \left( 1 - \frac{D}{4\epsilon} z + \frac{D^2}{32\epsilon} z^2 + O(z^3) \right)
\end{align*}
\]  
\hspace{1cm} \text{(23)}

Consider now the analytical properties of the functions \( \Phi \), \( \Phi^2 \) and \( \tilde{\Phi} = \sqrt{b\Phi/\sqrt{-4}} \)

**Theorem 2** Let the equation (3) be satisfied. In this case the following properties are valid: function \( \Phi \) is a well defined locally one-valued function on the double covering \( \Gamma^* \) over the surface \( \Gamma \) with the branching points \( R(\lambda) = 0 \) and \((-u/2, \pm)\) determined by the function \( R^{-1/4}b^{1/2} \); functions \( \Phi^2 \) and \( \tilde{\Phi} \) are well-defined locally one-valued meromorphic functions on \( \Gamma \) and globally one-valued on some its covering with a free abelian monodromy group. The following formula is true for the function \( \tilde{\Phi} \):

\[
\begin{align*}
\ln \tilde{\Phi}(w) & = \frac{8}{5\epsilon w^5} - \frac{4g_2}{5\epsilon w} - \ln w + \frac{8g_3 w}{7\epsilon} - \frac{uw^2}{4} + O(w^3), \ w \to 0 \\
\tilde{\Phi}(w) & = \frac{\sigma(w+a)}{\sigma(w)\sigma(a)} \exp \left\{ -\frac{4}{5\epsilon} \varphi'(w) - \frac{4g_2}{\epsilon} \zeta(w) + \frac{6g_3 w}{5\epsilon} - \zeta(a)w \right\}
\end{align*}
\]  
\hspace{1cm} \text{(24)}

The equality

\[
d_w(\ln \tilde{\Phi}) = \Omega + d_w(\ln b^{1/2}), \ \varphi(w) = \mu \\
-g_2 = \epsilon x, \ g_3x = \epsilon u/2
\]  
\hspace{1cm} \text{(25)}

is equivalent to the \( P-1 \) equation and gives nonlinear elliptic representation of \( P-1 \), depending on the parameter \( w \).
Proof. The asymptotics of these functions near the infinite point follows from the proposition above. In particularly, the residue of the logarithmic derivatives of $\Phi^2$ in the variable $\lambda$ is equal to zero. The corresponding logarithmic residue for the function $\tilde{\Phi}$ is equal to $-1$ by definition. Consider now the singularities of these forms in the finite points of $\Gamma$. By definition we have

$$d_\lambda (\ln \Phi^2) = 2\Omega$$
$$d_\lambda (\ln \tilde{\Phi}) = \Omega + \frac{b_\lambda}{2b}$$

(26)

Using the P–1 equation in the form

$$u_x = 2\sqrt{R}|_{\lambda=u/2}$$

and the exact values of $a$ and $b$

$$a = -u_x, \quad b = 2u + 4\lambda$$

we are coming to the following conclusion: these differential forms may have poles in the points $(\mu, \pm) = (u/2, \pm)$ only, except infinity.

It is obvious that $\Omega$ has the residues in both these points equal to $\pm 1/2$.

The second form $\Omega + d_\lambda \ln b^{1/2}$ has a pole in one of these two points only, because the second term exactly cancels the pole on one sheet. It has a nontrivial residue at infinity (equal to $-1$) coming from this second term.

All the residues of the both forms (who are the logarithmic derivatives of the functions in the theorem ) are equal to $\pm 1$. Therefore the exponent from the integral for the both of these forms will be locally one–valued function on the Riemann surface $\Gamma$.

Both of them will be obviously one–valued globally on some abelian covering over $\Gamma$. The theorem is proved.

All these identities are valid for all solutions of the P-1 equation.
For the \textbf{Physical Solution} Novikov formulated some very special
\textbf{Conjecture}: There exists a function $F$ from the variables $g_2, g_3$ such
that

$$u = 2\wp(F||g_2, g_3)$$

Here we have

$$g_2 = -\epsilon x, \ g_3x = \epsilon u/2$$
as above. For the real $x \to -\infty$ we should have

$$F(g_2, g_3) = (\omega)/2 + \delta$$

It is more convenient to write everything on the reduced algebraic curve $\tilde{\Gamma}$ which has a good limit for $x \to -\infty$:

$$g_2 = t^4 \tilde{g}_2 = 12, \ g_3 = t^6 \tilde{g}_3, \ F = t^{-1} \tilde{F} = \tilde{\omega}/2 + \tilde{\delta}$$

For the function $u$ we have:

$$u = 2t^2 \wp(\tilde{\omega} + O((-x)^{-5/4}))||12, \ \tilde{g}_3, \ t^2 = (-\epsilon x)/12$$

The limiting algebraic curve for $-x \to \infty$ is rational

$$y^2 = 4(s + 2)(s - 1)^2$$

. For $\tilde{\Gamma}$ we have

$$y^2 = 4(s + 2)(s - 1)^2 + O((-x)^{-5/4})$$

for $\epsilon = 1$.

For $\tilde{g}_3$ we have a strictly negative correction which is coming from the first

term $a_1^+$ in the asymptotic serie for the Physical Solution (see Introduction)

$$\tilde{g}_3 = -8 + O((-x)^{-5/4}), \ \epsilon = 1$$

Therefore we have one real period $2\tilde{\omega}$ of the lattice corresponding to
the physical solution for $x \to -\infty$ (it is generated by two complex adjoint
periods).

\textbf{Some good characterization of the Physical solution may come}
\textbf{from the studying of the $x$–dependence of the muptiplicators of the function $\tilde{\Phi}$ along the global cycles on the Riemann surface $\tilde{\Gamma}$.}
4.Semiclassical approximation and foliations of the Riemann surfaces.

Consider a linear system

$$\epsilon \Psi_t = V(t, \epsilon) \Psi$$

with small parameter $\epsilon$ and the right-hand part of the form

$$V = V_0(t) + V_1(t) \epsilon + \ldots$$

such that $V_0(t)$ is a polynomial or a rational function in the variable $t$.

As it was pointed out above (see chapter 3), for the construction of the formal semiclassical solution we should at first reduce our system to the form which is diagonal in the first approximation. After the substitution

$$\Psi = U(t) \tilde{\Psi}, \quad U^{-1}V_0U = \tilde{V}_0 = diag$$

we are coming to the admissible form

$$\epsilon \tilde{\Psi}_t = \tilde{V} \tilde{\Psi} = (U^{-1}VU - \epsilon U^{-1}U_t) \tilde{\Psi}$$

The formal semiclassical solution was written in the form

$$\tilde{\Psi}_{sc} = (1 + \sum_{i \geq 1} \epsilon^i A^i) \exp\{B_{-1}^{-1} \epsilon + B_0 + \sum \epsilon^i B_i\}$$

Here all matrices $A_i$ have diagonal elements equal to zero, all matrices $B_i$ are diagonal. For simplicity we may think that all matrices here are of the order 2, and the trace of $V$ is equal to zero.

The first diagonal term is

$$\tilde{V}_0 = diag(\sqrt{R}, -\sqrt{R}), \quad R = - \det V_0(t)$$

Therefore this formula presents us some structure on the Riemann surface $\Gamma$ as above, the form

$$\sqrt{R}dt = \Omega_0$$

is meromorphic on the surface $y^2 = R(t)$.

Let us to ask:
In which cases the formal semiclassical expression above or at least its first two exponential terms

\[
\psi_{sc+} = \exp \left\{ \int_{t_0}^{t} \frac{(\sqrt{R(t')} + \epsilon b_{11}^{(1)}) dt'}{\epsilon} \right\} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(\epsilon) \right\} \tag{27}
\]

\[
\psi_{sc-} = \exp \left\{ \int_{t_0}^{t} \frac{(-\sqrt{R(t')} + \epsilon b_{22}^{(1)}) dt'}{\epsilon} \right\} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\epsilon) \right\} \tag{28}
\]

gives the right asymptotic in $\epsilon$ for some exact solution $\psi_{+}(t, \epsilon)$ along all the path of integration?

The following theorem gives an answer for this question.

**Theorem 3** Consider a hamiltonian foliation of the surface $\Gamma$ given by the zeroes of the harmonic form which is a real part of the abelian differential 1–form on the Riemann surface $\Gamma$

\[
\omega = 0, \sqrt{R}dt = \omega + i\omega' \tag{29}
\]

and two different orientations in it: The orientation + is such that

\[
\int_{t_0}^{t} \omega
\]

increases along this path in its positive direction; the opposite orientation is –. The formulas above describe the right asymptotic for some exact solutions increasing along the positive path of integration $\gamma$ and increasing for $\epsilon \to 0$ if this path is transversal to foliation in all its points. It should be oriented in the direction of the increasing of the function $f_{t_0}^{t} \omega$ for the $\psi_{sc+}$ and in the opposite direction for $\psi_{sc-}$. Two different paths with the same endpoints describe the same solution along the union of these paths if they are homotopic in the class of paths everywhere transversal to foliation with fixed endpoints.

Remark 1. By definition the decreasing asymptotics are the same one but directed in the opposite way.

Remark 2. If there is no small parameter $\epsilon$ in the linear system above, we may use a variable $t^{-1}$ as a small parameter in many cases and apply this statement for the large $t \to \infty$. 

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There is no need to give a proof for this statement. In fact, it is known many years (in different terminology) for special cases in the physical and mathematical literature (see [R]), but things never had been put in this terminology, connecting this problem with the theory of algebraic Riemann surfaces and the topology of these beautiful and nontrivial foliations.

I believe, this form gives a beautiful motivation for the studying very nontrivial topology of special Hamiltonian foliations on the Riemann surfaces given by the real parts of holomorphic and meromorphic forms.

Even for the simplest generalization of the straight line flow on the torus as a foliation generated by the real part of holomorphic 1-form on hyperelliptic Riemann surface of genus 2 nothing has been done.
Example 1. Let $R(t)$ be a polynomial of the odd degree $n = 2m + 1$.

We have a following foliation near the infinite point (see Fig 1). There exist exactly $2(2m + 3)$ sectors near the point $v = \infty$, $v^2 = t-1$ on the Riemann surface near infinity, which exactly cover the $2m + 3$ sectors on the $t$–plane, separated by 'separatrices':

$$t^{-1} = \exp\{\rho + i\phi\}, \quad 2k\pi/j \leq \phi \leq 2(k + 1)\pi/j, \quad k = 0, 1, 2, \ldots, 2m + 2$$
In each sector we have a picture like in Fig 1. Each trajectory starts in the infinite point \( t^{-1} = 0 \) asymptotically tangent to the separatrix and turns to another separatrix on the boundary of this sector. Finally it returns to the infinite point asymptotically tangent to the last separatrix. Let us introduce the orientation (as above) in this foliation.

We may observe that the neighboring sectors have the opposite orientations: if some transversal path is directed to the \( \infty \) in some sector and it is positive, so in the neighboring sectors the positive transversal paths will be directed out of the \( \infty \).

Starting from some point \( t_0 \) (‘very closed to infinity’) in the sector (i) we may come by the transversal paths \( \gamma'_{i+1} \) or by \( \gamma''_{i-1} \) for the proper orientation \( \pm \) (which admits the positive transversal path out of \( \infty \) in this sector) to the neighboring sectors \((i-1)\) and \((i+1)\) and to follow along these paths in the direction of the infinite point. Therefore the solution which increases along these paths (and decreases in the inverse direction to the infinite point in the sector \((i)\)) has a natural continuation in the neighboring sectors. We shall denote it by \( \psi_i \)

**Proposition 6** There is a ‘Stokes relation’ between the solutions:

\[
\psi_{i+1} - \psi_{i-1} = \alpha_i \psi_i
\]

Here \( \alpha_i = \text{const} \) such that

\[
\alpha_i = \alpha_{i+2^{m+3}}
\]

The proof of this relation follows from the fact that there is exactly one-dimensional family of solutions in each sector which have a decreasing behavior for \( t \to \infty \). Both solutions \( \psi_{i\pm1} \) have a continuation in the sector \((i)\) as the increasing ones, described by the same asymptotics. Their difference is proportional to the decreasing solution in this sector with some constant coefficient because the system has an order two. Therefore the first part is proved.

By the origin of this system on \( \Gamma \) from the original \( t \)-plane we have the same (or proportional) solutions over the sectors which cover the same domain in the \( t \)-plane up to permutation of the matrix elements. From this the proof of the last relation between the Stokes coefficients follows immediately.
Example 2. Consider now the asymptotics for the Physical and Non-physical solutions for $x \to -\infty$ as above:

$$u_\pm \sim \pm \sqrt{-\frac{e x}{3}}$$

After the substitution

$$x = -\tau^{4/5}, \quad u = \frac{\tau^{2/5}}{\sqrt{3}} \tilde{u}(\tau)$$

$$\lambda = -\frac{\tau^{2/5}}{2\sqrt{3}} \mu$$

$$\tilde{\psi}_1 = \psi_1, \quad \psi_2 = -\frac{5}{4} \tau^{1/5} \tilde{\psi}_2 \quad (31)$$

we are coming to the system

$$\frac{1}{\tau} \tilde{\Psi}_\mu = \tilde{\Lambda}_\pm \tilde{\Psi}$$

such that

$$\tilde{\Lambda}_\pm = -\frac{1}{12} \left( \begin{array}{ccc} 0 & -5(\pm 1 - \mu) \\ \frac{8}{5\sqrt{3}}(3 - \mu^2 - \mu(\pm 1) - 1) & 0 \end{array} \right) + O\left(\frac{1}{\tau}\right) \quad (32)$$

Here the quantity $\tau^{-1}$ plays a role of small parameter. Riemann surface has a genus, equal to 0. The corresponding foliations you may see on the Fig 2 (for the Physical Solution) and on the Fig 3 (for the Nonphysical one).
We have exactly two finite critical points of these foliations in the following points (only one of them is nondegenerate which is located in the points $\mu = \pm 1$):

\begin{align*}
\mu &= 1, \mu = -2 \\
\mu &= -1, \mu = 2
\end{align*}

for the Physical solution (there is no separatrices joining two finite critical points here. All trajectories including separatrices are coming to infinity).

for the Nonphysical Solution (there exist a separatrix line joining two finite critical points along the real line. All other trajectories including separatrices are coming to infinity.)
5. Special semiclassics for the Lax pairs in the case of Physical Solution

We shall start our studying of the Physical solution from the substitution (rescaling) described in the formulas (31) at the end of the paragraph 4 using the same notations

\[ \tilde{u}, \mu, \tau, \tilde{\Psi} = (\tilde{\psi}_1, \tilde{\psi}_2), \tilde{\Lambda}, \tilde{Q} \]

Let us write more explicit formulas

\[ \tilde{\Lambda} = -\frac{1}{12} \begin{pmatrix} \frac{2\tilde{u}(\tau) + 5\tau \tilde{u}'(\tau)}{8(3 - \mu^2 - \mu \tilde{u}(\tau) - \tilde{u}^2)} & \frac{5(\mu - \tilde{u}(\tau))}{2\tau} \\ \frac{5(\mu - \tilde{u}(\tau))}{2\tau} & \frac{-2\tilde{u}(\tau) - 5\tau \tilde{u}'(\tau)}{2\tau} \end{pmatrix} = \]

\[ = -\frac{1}{12} \begin{pmatrix} 0 & \frac{5(\mu - 1)}{\sqrt{3}} \\ \frac{5(\mu - 1)}{\sqrt{3}} & 0 \end{pmatrix} + O\left(\frac{1}{\tau}\right) \]

\[ \tilde{Q} = \begin{pmatrix} 0 & 1 \\ \frac{16\tilde{u} + 8\mu}{25\sqrt{3}} & -\frac{1}{5\tau} \end{pmatrix} + \frac{2}{5} \mu \tilde{\Lambda} \]

\[ \frac{1}{\tau} \mu = \tilde{\Lambda} \tilde{\Psi}, ; \Psi_\tau = \tilde{Q} \tilde{\Psi} \quad (33) \]

This is a very interesting special Lax Pair (or Zero–Curvature Representation) such that its second equation is written in terms of the parameter \( \tau \) or \( \tau^{-1} \) which plays a role of the small parameter ('Plank constant') for the first equation of this pair written in the variable \( \mu \).

 Especially interesting point here which is valid for the Physical Solution only is the following: there exist an asymptotic serie (3) for this solution \( \tilde{u}_+ \) in the variable \( \tau \). Therefore we shall try to look for the Semiclassical Solution \( \tilde{\Psi}_{sc}(\mu, \tau) \) for the \( \mu \)-equation of this pair which is a formal serie in the variable \( \tau^{-1} \) and satisfies to the \( \tau \)-equation also.

More precisely, after the substitution below we are coming as before to the equations which are diagonal modulo \( \frac{1}{\tau} \):

\[ \tilde{\Psi} = U \Phi, \quad U = \begin{pmatrix} \frac{1}{\sqrt{\alpha(\mu + 2)}} & \frac{1}{\sqrt{\alpha(\mu + 2)}} \\ -\sqrt{\alpha(\mu + 2)} & \sqrt{\alpha(\mu + 2)} \end{pmatrix} \]

\[ \alpha = \frac{8}{25\sqrt{3}}, \quad \frac{1}{\tau} \Phi_\mu = \tilde{\Lambda} \Phi, \quad \Phi_\tau = \tilde{Q} \Phi \]
\[
\bar{\Lambda} = \left( \begin{array}{cc} r_+(\mu) & 0 \\ 0 & r_-(\mu) \end{array} \right) + O\left( \frac{1}{\tau} \right)
\]
\[
\bar{Q} = \left( \begin{array}{cc} q_+(\mu) & 0 \\ 0 & q_-(\mu) \end{array} \right) + O\left( \frac{1}{\tau} \right)
\] (34)

It is useful to observe that
\[
q_+(\mu) = \int_{-2}^{\mu} r_+(\mu')d\mu' = \int_{-3}^{\mu} r_+(\mu')d\mu'
\]
Here we have
\[
q_\pm(\mu) = \frac{\pm\sqrt{\alpha}}{6}(\mu - 3)(\mu + 2)^{3/2}, \quad r_\pm(\mu) = \pm\frac{5}{12}(\mu - 1)\sqrt{\alpha(\mu + 2)}
\] (35)

Let us formulate the following main theorems of this section:

**Theorem 4** The last 'rescaled' Lax pair (34) admit a unique formal quasi-classical matrix-solution of the form
\[
\Phi_{sc} = \tau^{-\frac{1}{4}}(\mu + 2)^{-1/4}(1 + \sum_{i\geq 1} \frac{A_i}{\tau^i}) \exp\{\tau B_{-1} + \sum_{i\geq 1} B_i \tau^{-i}\}
\] (36)
such that all \(A_i\) are off-diagonal, all \(B_i\) are diagonal, \(A_i\) and \(B_i\) are algebraic functions on the Riemann surface \(\Gamma^0\) of genus zero
\[
\Gamma^0 = \{y^2 = \mu + 2\}
\] (37)
For the most important term we have
\[
B_{-1} = \left( \begin{array}{cc} q_+(\mu) & 0 \\ 0 & q_-(\mu) \end{array} \right), \quad q_- = -q_+, \quad B_0 = 0
\] (38)

**Theorem 5** Consider the 'Exact Stokes Sectors' bounded by the lines
\[
\text{Re}[q_\pm(\mu)] = 0
\] (39)
on the Riemann surface \(\Gamma^0\) (see Fig 4). For each Exact Stokes Sector there exist a unique up to a constant factor exact vector-solution for the rescaled Lax pair (34) which is decaying as \(\tau \to \infty\). Its asymptotics may be described by the first exponential term of the semiclassical solution (modulo terms of the order \(O\left( \frac{1}{\tau} \right) \) inside and outside of the exponent) in the previous theorem.
We may see on the Fig 4 that there are two Exact Stokes Curves (39) on the Riemannian surface $\Gamma^0$ which are closed and project on the closed curves in the extended $\mu$-plane plus infinity. They are passing through the points ($\mu = 3, \pm$) and $\infty$. Asymptotically near infinity these two closed curves give us four different asymptotic Stokes lines with the numbers 1 and 2 (which bound a sector number 2) and with the numbers 6 and 7 (which bound a sector number 7). A Stokes Sector number $k$ is bounded by the lines number $(k - 1, k)$, $k = 1, 2, \ldots, 10$.

Other six Exact Stokes Lines project in the graph on the $\mu$-plane which contains 3 curves starting from the point $\mu = -2$ in the directions of the angles $\pi$, $\pm \frac{\pi}{3}$. One of them is exactly a negative part of real line below the point $\mu = -2$. The total preimage of this graph on $\Gamma^0$ gives the Exact Stokes Curves which asymptotically near infinity exactly coincide with the Stokes lines number 3 and 8, 4 and 9, 5 and 10 (see numeration on the Fig 4).
From the theorem 5 and this picture we have immediately the following conclusions:

**Corollary 1** The Stokes coefficients defined in the Proposition 6 are constant (they don’t depend on μ, τ); the Stokes coefficients number 2 and 7 are equal to zero

\[ \alpha_2 = \alpha_7 = 0 \]

For other coefficients we have

\[ \alpha_k = \alpha_l, \; k - l = 0 (mod 5) \]

\[ \alpha_j = i, \; j = 4, 5, 9, 10 \]

\[ \alpha_1 + \alpha_3 = \alpha_6 + \alpha_8 = i \]

The proof of this corollary is the following. The ordinary Stokes Sectors with the numbers 1 and 3 in fact are connected with each other because they belong to the same domain (39). Therefore we have \( \phi_1 = \phi_3 \) for the appropriate solutions which have a \( \tau \)-decay in the sectors 1 and 3 correspondingly and \( \alpha_2 = 0 \) by definition. We have also \( \alpha_2 = \alpha_7 \)—see proposition 6.

Let us eliminate now the closed exact Stokes lines from the Riemann surface \( \Gamma_0 \). Consider the following functions \( F_k \):

\[ \phi_1 = \phi_3 = F_1, \; \phi_4 = F_2, \; \phi_5 = F_3, \; \phi_6 = \phi_8 = F_4, \; \phi_9 = F_5, \; \phi_{10} = F_6 \]

For the last solutions \( F_k \) we have practically the same Stokes coefficients but numeration is modulo 6.

\[ F_{k+1} - F_{k-1} = \beta_k F_k \]

\[ \beta_k = \beta_l, \; F_k \sim F_l, \; k - l = 0 (mod 3) \]

\[ \beta_k = i = \sqrt{-1}, \; k = 1, 2, 3, 4, 5, 6 \]

They coincide with the Stokes coefficients for the equation describing the Airy function where the infinite point meets exactly six Stokes sectors. This is true just because it follows from the algebraic symmetry:

\[ \beta_{k+3} = \beta_k, \; F_{k+3} \sim F_k \]
and from the property that semiclassical solution has an $\infty$ as a double branching point (see a theorem 2). It means that we have to put sign $-1$ after the passing the full angle $2\pi$ around $\infty$.

For the proof we are writing a sequence of formulas:

$$F_2 = F_0 + \beta_1 F_1, \ \phi_3 = F_1 + \beta_2 F_2, \ldots$$

and using this algebraic symmetry. The full angle we shall pass after the six steps. Expressing the functions $F_k$ through the previous ones and finally through the $F_1$, $F_2$ we shall get the algebraic equations for the coefficients $\beta_k$. Our formulas for them follows automatically.

By definition we have:

$$\alpha_k = \beta_{k-2} = \beta_{k+1}, \ k = 4, 5$$

We have also

$$\beta_1 = \alpha_1 + \alpha_3$$

because

$$F_2 - F_6 = \phi_4 - \phi_{10} = (\phi_4 - \phi_2) + (\phi_2 - \phi_{10}) = (\alpha_1 + \alpha_3)\phi_1$$

The corollary is proved.

**Remark.** In the work [10] (see also [11, 12]) this result about the Stokes coefficients for this special solution has been found. These authors used different (more complicated) Lax-type representation for the P–1 equation extracted from [14], but the most important difference from our work is the following:

the authors of [11, 12] and others in St–Petersburg group did not worked with the common solution for the both equations for the Lax pair. They worked with one linear system (which is in the variables $\lambda$ or $\mu$ after rescaling) and used the following fact: Stokes coefficients don’t depend on $\tau$ or $x$ iff $u$ satisfies to the P–1 equation. It means in fact that these authors worked with picture like in Fig 2 and had a need to investigate the neighborhood of the critical point $\mu = 1$ of our foliation and to prove that it has in fact a trivial monodromy.

In our approach this difficulty does not exist because this point is nonsingular at all for the special semiclassics for the (rescaled) Lax pair in the case of the Physical Solution.
Let us consider now the behavior of the semiclassical solution in the interval of real line between the points \( \mu = -2 \) and \( \mu = 3 \). It is the most important interval because its small shift in the variable \( \mu \) in the negative imaginary direction connects two Exact Stokes Sectors: the sector number 2 bounded by the separatrices number 1 and 2 near infinity (or by 6 and 7 on the second sheet) and the sector number 0 bounded by the separatrices number 9 and 10 near infinity (or by 4 and 5 on the second sheet)—see Fig 4.

There is only one essentially nontrivial Stokes coefficient \( \alpha_1 \) which determines the difference between two Physical Solutions in the one-parametric family of them. It is exactly equal to the difference between the solutions \( \phi_0 \) and \( \phi_2 \) divided by the solution \( \phi_1 \). These special solutions \( \phi_i \) have decay in the corresponding exact Stokes sectors in the variable \( \tau \) and decay in the variable \( \mu \) if \( \mu \to \infty \) in this sector.

The points \( \mu = -2 \) and \( \mu = 3 \) are singular for the \( \tau \)-dynamics in the pair of the Systems (33).

We are starting from the original nondiagonal form (33) because the transformation (34) is singular in the point \( \mu = -2 \). Modulo \( \tau^{-2} \) we have:

\[
\tilde{\Psi}_\tau \tilde{\Psi}^{-1} = -\frac{1}{6} (\mu - 3)(\mu + 2) \begin{pmatrix} 0 & 1 \\ \alpha(\mu + 2) & 0 \end{pmatrix} + \\
+ \begin{pmatrix} -\frac{\mu}{36\tau} & 0 \\ 0 & \frac{\mu^6}{36\tau} \end{pmatrix} + O\left(\frac{1}{\tau^2}\right) \tag{40}
\]

After the substitutions

\[
\mu = 3, \ \mu = -2
\]

we are coming to the following proposition

**Proposition 7** The exact solution \( \tilde{\Psi} \) in the variable \( \tau \) has a behavior like some power of \( \tau \) for \( \tau \to \infty \) in the points \( \mu = -2 \) and \( \mu = 3 \). Consider the special solution

\[
\tilde{\psi}_1 = U(\mu)\phi_1
\]

for the system (33) which has an exponential decay for \( \tau \to \infty \) in the Exact Stokes Sector containing the open interval \((-2, 3)\) in the variable \( \mu \) by the theorem 5. Here \( U \) is given by the formula (34). This solution is increasing for \( \mu \to -2 \) and \( \mu \to 3 \) for \( \tau \) large enough.
Proof of this statement is obvious.
The following result might be immediately extracted from this:

**Corollary 2** Consider the Linear Operator

\[ \hat{A} = \frac{1}{\tau} \partial_\mu - \Lambda(\mu, \tau) \]

The spectral problem

\[ \hat{A}\psi = \lambda\psi \]

with the 'Plank parameter' \( h = \frac{1}{\tau} \) has a point \( \lambda = 0 \) as a discreet semiclassical eigenvalue for the Dirichlet Problem on the interval \((-2, 3)\) in the variable \( \mu \). It means that the boundary conditions are satisfied modulo exponentially small nonaccuracy in the endpoints \( \mu = -2, 3 \).

Proof. Consider any solution \( \tilde{\psi} \) for the rescaled Lax Pair (33) in the variables \( \mu, \tau \) different from the decreasing solution \( \tilde{\psi}_1 \) above. This solution has an increasing asymptotics for \( \tau \to +\infty \) because it has a semiclassical behavior written in the theorem 4 which obviously describes its asymptotics (at least its first main exponentially increasing term does). Therefore the modulus of this solution has a maximum somewhere near the point

\[ \max_{-2 \leq \mu \leq 3} q_+(\mu) = 1 \]

Its growth will be less and less for \( \mu \to -2 \) and \( \mu \to 3 \). After the proper normalization of \( \psi \) (making the maximum of modulus equal to 1 or the norm in \( L^2[-2, 3] \) equal to 1) we shall see that the normalized solution of the equation \( \hat{A}\psi = 0 \) has some decay for \( \mu \to -2, 3 \) and therefore may be treated as a semiclassical eigenfunction corresponding to the Dirichlet Problem on this interval. The corollary is proved.

**Conjecture.** The property of the Operator \( \hat{A} \) described in the previous Corollary determines all formal serie (3) in the Physical Solution.
6. Proof of the main theorems of the p.5

We shall give in this paragraph the complete proof for the theorems 4 and 5.

Proof of the theorem 4.
Let us write the system (34) for the matrix $\Phi$ in the following form:

\[
\Phi_{\mu} = \tau (V_0 + V_1^d \tau^{-1} + V_1^a \tau^{-1} + V_{rest}) \Phi \\
\Phi_{\tau} = (W_0 + W_1^d \tau^{-1} + W_1^a \tau^{-1} + W_{rest}) \Phi \\
W_{rest} = \sum_{i \geq 2} W_i \tau^{-i}, \quad W_i = W_i(\mu)
\]

(41)

Here 'd' means diagonal and 'a' means offdiagonal parts of the matrices, $W_0$ and $V_0$ are already diagonal.

Our strategy now is to solve this system in the variable $\tau$ instead of $\mu$ as a formal serie. It is easy to check the following formulas for the solution (36):

\[
B_{-1} = W_0, \quad A_1 B_{-1} = W_1^a + W_0 A_1, \quad B_0 = 0,
\]

\[
A_2 B_{-1} = W_0 A_2 + A_1 + B_1 + W_1^a A_1 + W_2, \quad \ldots
\]

\[
A_n B_{-1} - B_{-1} A_n = (n - 1) B_{n-1} + P_n(W_1^a, W_2, \ldots, W_n, A_1, \ldots, A_{n-1}, B_1, \ldots, B_{n-2})
\]

(42)

Here $P_n$ are some noncommutative polynomials.

From this formulas we deduce that the solution in the form of the desired formal serie in the variable $\tau$ exists and is unique. In particularly we have:

\[
A_1 = Ad[W_0]^{-1}(W_1^a), \quad Ad[X](Y) = [X,Y]
\]

\[
B_n = -n^{-1} P_{n+1}^d, \quad A_n = Ad[W_0]^{-1}(P_n^a)
\]

(43)

Let us now substitute this formal serie in the first equation of the system (34) in the variable $\mu$. Consider the expression

\[
X(\mu, \tau) = (\partial_\mu - \tau \bar{A}) \Phi_{sc}
\]

Obviously we have:

\[
\tau^{\frac{1}{10}} \mu^{1/4} X = (c_1 \tau^{-1} + c_2 \tau^{-2} + \ldots) \exp\{B_{-1} \tau + B_1 \tau^{-1} + \ldots\}
\]
Let us remark now that $X$ also satisfies to the equation in $\tau$ of the system (34), as $\Phi_{sc}$, because the operators $\partial_\mu - \tau\bar{A}$ and $\partial_\tau - \bar{Q}$ commute with each other.

Adding $X$ to the solution $\Phi_{sc}$ above we shall get a new solution

$$\Phi_{sc} + X = \tau^{-1/4} \mu^{-1/4} (1 + \frac{A_1 + c_1}{\tau} + \ldots) \exp\{B_{-1}\tau + B_1\tau^{-1} + \ldots\}$$

which is easily to rewrite in the admissible form

$$\Phi_{sc} + X = \tau^{-1/4} \mu^{-1/4} (1 + \frac{A_1 + c_1}{\tau} + \ldots) \exp\{B_{-1}\tau + (B_1 + c_1^d)\tau^{-1} + \ldots\}$$

like the solution $\Phi_{sc}$. From the uniqueness of the admissible semiclassical solutions in this form we deduce that

$$X = 0$$

Therefore the theorem 4 is proved.
Proof of the theorem 5.
Consider now any exact Stokes Sector in which we have
\[ \text{Re}[q_+(\mu)] \geq 0 \]
Let us introduce for any point \((\mu, \pm)\) in this sector a special semiclassical solution for the equation (42) in the variable \(\tau\):
\[
\phi_{\text{dec}} = \tau^{-\frac{1}{10}} \mu^{-1/4} \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] + \sum_{i \geq 1} \phi_i^{-}\tau^{-i} + \ldots \] \(e^{q_-(\mu)\tau} \quad (44)\)
Let
\[
\tilde{\phi} = \tau^{-\frac{1}{10}} \phi
\]
and
\[
L_0\tilde{\phi} = \left[ \frac{W_1^a}{\tau} + W_{\text{rest}} \right] \tilde{\phi}
\]
\[ L_0 = \partial_\tau - W_0 \quad (45) \]

The corresponding Green function of the Operator \(L_0\) may be written in the form
\[
G_{\text{dec}} = \begin{cases} 
\left( \begin{array}{cc} -1 & 0 \\ 0 & 0 \end{array} \right) e^{(\tau-y)q_+(\mu)}, \tau < y \\
\left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) e^{(\tau-y)q_-(\mu)}, \tau > y 
\end{cases} \quad (46) \\
L_0G_{\text{dec}} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \delta(x-y) \quad (47) 
\]

**Lemma 1** For any point \(\mu\) inside of the Exact Stokes Sector \(\text{Re}(q_+(\mu)) > 0\) and \(\tau_0\) sufficiently large there exists a unique exact solution \(\phi_{\text{dec}}(\tau, \mu)\) for the equations (34), (41) in the variable \(\tau\) such that it has an asymptotics (44) above for \(\tau \to +\infty\).

**Proof of Lemma 1.** We write the equation for finding the desired solution in the form of Integral Equation:
\[
\tilde{\phi} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) e^{q_-(\mu)} + T_{\text{dec}}\tilde{\phi}
\]
\[
(T_{\text{dec}}f)(\tau) = \int_{\tau_0}^{\infty} G_{\text{dec}}(\mu, \tau, y) \left[ \frac{W_1^a}{y} + W_{\text{rest}}(\mu, y) \right] f(y)dy \quad (48)
\]
28
Here $f$ is a vector–function

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Let us introduce a norm

$$||f||_c = \delta + c\kappa$$

depending on the parameter $c > 0$. Here we have by definition:

$$\delta = \sup_{\tau_0 \leq \tau} |f_2(\tau)e^{\tau q_+ (\mu)}| < \infty$$

$$\kappa = \sup_{\tau_0 \leq \tau} |\tau f_1(\tau)e^{\tau q_+ (\mu)}| < \infty$$

Therefore we have

$$|f_1(\tau)| \leq \frac{c_1}{c\tau}|e^{-\tau q_+ (\mu)}|$$

$$|f_2(\tau)| \leq c_1|e^{-\tau q_+ (\mu)}|$$

if $||f||_c = c_1$.

Inside of the Exact Stokes Sector we may estimate the coefficients of the equations (34), (42). For $\tau_0$ large enough we have:

$$\left| \frac{W^a_1}{\tau} + W_{\text{rest}} \right| \leq \left| \frac{d_1}{\tau^2} \frac{d_3}{\tau} \right|$$

where $d_i$ are some negative constants (depending on $\mu$ in principle).

This follows in the obvious elementary way from the formulas for these matrices.

Applying the operator $T_{dec}$ to the vector $f$ with norm $||f||_c = c_1$ we are coming to the following inequalities:

$$|g_1| \leq c_1 \int_{\tau}^{\infty} |e^{q_+(\mu) (\tau - 2y)}| \, dy \left( \frac{d_1}{c\tau^3} + \frac{d_2}{\tau} \right) =$$

$$= c_1 \left| e^{-\tau q_+ (\mu)} \right| \frac{1}{2\Re q_+(\mu)} \frac{1}{\tau} \left( d_2 + \frac{d_1}{c\tau^2} \right)$$

$$|g_2| \leq c_1 \int_{\tau_0}^{\tau} \left( \frac{d_3}{c} + d_4 \right) \frac{dy}{y^2} \left| e^{\tau q_- (\mu)} \right| \leq$$

$$\leq \frac{c_1}{\tau_0} \left| e^{\tau q_- (\mu)} \right| \left( \frac{d_3}{c} + d_4 \right)$$

(49)
For the norm of the image \( g = T_{dec} f \) we obtain from (49) inequality:

\[
\left\| \frac{g_1}{g_2} \right\|_c \leq \left( \frac{c}{2 \text{Re} q_0(\mu)} \left( d_2 + \frac{d_1}{c \tau_0^2} \right) + \frac{1}{\tau_0} \left( \frac{d_3}{c} + d_4 \right) \right) \left\| \frac{f_1}{f_2} \right\|_c
\]  

(50)

Choosing a constant \( c \) small enough and \( \tau_0 \) big enough we see that the norm of Operator \( T_{dec} \) is small. Therefore the Operator \( T_{dec} - 1 \) is invertible and the Integral equation (48) above is uniquely solvable.

**Lemma 1 is proved.**

We know now that in any point \( \mu \) inside of the Exact Stokes Sector there exists a special one-dimensional subspace of the solutions proportional to \( \phi_{dec} \) in all 2-dimensional linear space of solutions of the equation (41) in the variable \( \tau \) constructed in the lemma 1 above.

**Lemma 2** Consider any solution \( \phi(\mu, \tau) \) for the system (34), (41) in the variable \( \mu \) inside of some Exact Stokes Sector such that it coincides with the special solution \( \phi_{dec} \) in one point \( \mu_0 \). In this case the solution \( \phi \) is proportional to \( \phi_{dec} \) everywhere inside of the Exact Stokes Sector.

**Proof of Lemma 2.** Let us compare two points \( \mu_0, \mu_1 \) inside of this sector which are connected by the 'negative' path \( \mu(t), 0 \leq t \leq 1 \), inside this sector with the property

\[
\text{Re} \left[ r_+(\mu(t)) \frac{d\mu}{dt} \right] \leq 0
\]

Starting from the solution \( \phi_{dec}(\mu_0, \tau) \) we find the solution \( \phi(\mu, \tau) \) for all the points on this curve and for all \( \tau \geq \tau_0 \) solving the equation (34),(41) along the variable \( \mu \) with fixed \( \tau \) large enough. It is possible because the equations in \( \mu \) and \( \tau \) commute with each other.

We have an obvious estimate for the equation along the curve:

\[
||z(t)|| \leq ||z(0)|| ||A(t)||
\]

where

\[
z(t) = \phi(\mu(t), \tau)
\]

and \( A(t) \) is the operator of evolution generated by the linear system (34), (41) along the curve \( \mu(t) \)

\[
\frac{dz}{dt} = \left( \tau \Lambda \frac{d\mu(t)}{dt} \right) z(t)
\]
after the diagonalization of the first nontrivial term.

There exists such constant $c$ that the following inequality is true for the operator of evolution along the 'negative' path:

$$||A(t)|| \leq \exp\left\{ \int_0^t \left( \tau \left| Re\left[ r_+ (\mu) \frac{d\mu}{dt} \right] \right| + c \right) \right\}$$

because our matrix in the left side of the $\mu$-equation along the curve is diagonal modulo terms of the order $0$. We know that the initial value is exactly the solution $\phi_{dec}(\mu_0, \tau)$.

Starting from the point $\mu_0, \tau$ and from $\phi_{dec} = z(0)$ as an initial value we apply the inequality above for the norm $||z(t)||$. We are coming to the following estimate

$$|\phi(\mu, \tau)| \leq c_2 \exp\left\{ -Re[q_+(\mu_0)] \tau + \left| Re\left[ \int_{\mu_0}^\mu r_+ (\mu') d\mu' \right] \right| \right\} \leq c_2 \exp(-\tau Re[q_+(\mu)])$$

for the solution under investigation and for any $\tau$ large enough.

Combining this with the previous inequality we are coming to the following conclusion: this solution exactly coincides with the solution $\phi_{dec}(\mu, \tau)$ in the variable $\tau$ because it has a right asymptotics for $\tau \to \infty$ in the point $\mu$. (Let us to point out that any other solution $\phi(\mu, \tau)$ in fact has an exponential growth in $\tau$ for $\tau \to \infty$ inside of this Exact Stokes Sector.)

Consider now an arbitrary point $\mu_1$ in the same sector. We know already that for all such points $\mu_1$ which may be connected with $\mu_0$ by the 'negative' path $\mu(t)$ our lemma is true. This lemma therefore is true for 'positive' paths also changing the role of $\mu_1$ and $\mu_0$ and using the uniqueness of the solution with decay in $\tau$ for any given $\mu$. We may connect any 2 points in the sector by the finite product of 'positive' and 'negative' paths. This gives the proof of lemma for our sector.

Exact Stokes Sectors with $Re[q_-(\mu)] > 0$ may be considered in the same way up to inessential changes.

**Lemma 2 is proved.**

Proof of the theorem 5 follows from the lemmas 4 and 5.
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Fig 1.
The pathes $\gamma_1$ and $\gamma_2$ are transversal to the foliation. Here $\mu_0=-2$ or $\mu_0=3$. The Exact Stokes Lines coincides with the integral trajectories of the foliations passing through the points $\mu=-2$ and $\mu=3$. 

Fig 2.
Fig 3.
Fig 4.