Tsallis entropy on fractal sets

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ABSTRACT

In this article, we review fractal calculus (\(F^{\alpha}\)-calculus) and define generalized Tsallis entropy on the fractal sets which is called fractal Tsallis entropy. We define \(q\)-fractal calculus to obtain \(q\)-Gaussian or generalized stable and Lévy distribution on fractal sets. The conditions for the non-linear coupling of the statistical states are given. The relationship of fractal dimension which is defined in \(F^{\alpha}\)-calculus with \(q\)-parameter of the Tsallis entropy of the Hadron system is proposed.

ARTICLE HISTORY

Received 30 September 2020
Revised 13 March 2021
Accepted 16 April 2021

KEYWORDS

Fractal Tsallis entropy; fractal Lévy distribution; \(q\)-fractal calculus; fractal \(q\)-Gaussian; thermo-fractal

MSC2010S

28A80; 94A17; 60E07; 82C41

1. Introduction

Fractal geometry is a new branch of mathematics that includes shapes and objects with self-similarity and fractal dimensions \([1,2]\). Analysis on fractals was formulated by some researchers using different approaches \([3–18]\). Fractal calculus is formulated in seminal papers by generalizing ordinary calculus to include the fractal sets and curves which is simple and algorithmic \([19–24]\).

Entropy has different definitions in the branches of science. For example, it shows disorderly and uncertainty in a system and it increases in the direction of time \([25–27]\). In equilibrium statistical mechanics, thermodynamic coordinates of a system can be obtained with the help of Boltzmann–Gibbs entropy \([28]\). The Gaussian probability distribution maximizes Boltzmann–Gibbs entropy \([29]\). The Central Limit Theorem (CLT) has an important role in probability theory and found application in science, i.e. the zig-zag movement of the small particles suspended in a liquid or a gas is called Brownian Motion that modelled by the sum of independent and identically distributed (i.i.d) random walks which leads to Gaussian probability distribution of the number of particles vs. distance \([30,31]\).

Tsallis entropy of the discrete and continuous random variables is defined (see in Refs \([29,32]\)), which is called non-extensive entropy. Tsallis entropy involves the index \(q\) which a real number that used to explain long-range interacting many-body classical Hamiltonians \([29]\). The \(q\)-Gaussian probability distribution maximizes Tsallis entropy and was defined and applied to model nonlinear systems and complex systems \([33]\). Using \(q\)-Gaussian probability distribution the \(q\)-Central Limit Theorem has been established to include dependent variables also. Nonextensive statistical mechanics deals with strongly correlated random variables \([32,34–41]\).

The \(q\)-parameter in the Tsallis entropy has no physical meaning so far, but does not diminish its importance because it can be used to obtain the properties of systems for which the Boltzmann–Gibbs statistic is not valid \([42]\). Researchers have tried to show that nonextensive statistical mechanics on the basis of Tsallis entropy is a good framework for studying systems with fractal structures \([43,44]\). The relationship between the \(q\)-parameter of Tsallis entropy and the fractal dimensions of the Cantor sets was expressed and Cantor sets were classified based on that relationship \([45]\). Recently, random variables on Cantor sets and their corresponding probability distribution and Shannon entropy have been defined which shows the effect fractal dimension on Shannon entropy \([46]\). Stochastic processes and the fractal mean-square calculus were defined and the stochastic differential equation has been solved \([47]\).

In some processes in nature, complexity increases with decreasing temperature. For example when raindrops freeze and turn to snow flake which have fractal structure, so we see that the entropy decreasing by having fractal structure \([48–50]\).

According to the mentioned research above, our main goal is to define fractal Tsallis entropy which can include both long-range interacting and fractal structures. The relationship between the fractal dimension and the \(q\)-parameter indicates that the possible distribution for stochastic processes can be changed by changing the correlation of random variables or fractal
dimension. This relationship may present both geometrical meaning and physical meaning for $q$-parameter.

In this paper, we review fractal calculus and suggest a generalized Tsallis entropy. We relate $q$-parameter of the Tsallis entropy to $\gamma$-dimension ($\omega$) which is defined in fractal calculus. The generalized $q$-Gaussian Lévy distribution conditions for the nonlinear coupling of the $q$-infractal calculus. The generalized vision of $J$ where $W$ is defined by $\gamma_\alpha(b)$.

1.1. Local fractal calculus

Suppose $b_1 < b_2$ are real numbers and $J = [b_1, b_2]$ a closed interval. Let $W$ denote a subdivision of $J$ that is, a finite ordered sequence of points $b_1 = x_0 < x_1 < \cdots < x_n = b_2$. We refer reader to see the details in [19–23].

For the every Cantor set the Hausdorff dimension is defined by

$$\dim_W(C^\omega) = \frac{\log 2}{\log 2 - \log(1-\omega)},$$

(1)

$H(C^\omega)$ indicates the Hausdorff measure [24].

The flag function is defined in [19–23] by

$$F(C^\omega, J) = \begin{cases} 1 & \text{if } C^\omega \cap J \neq \emptyset \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $J = [b_1, b_2]$. Then, $\rho_\alpha[C^\omega, W]$ is defined in [19–23] by

$$\rho_\alpha[C^\omega, W] = \sum_{i=1}^n \Gamma(\alpha+1)(t_i - t_{i-1})^\alpha F(C^\omega, [t_{i-1}, t_i]),$$

where $W_{[b_1, b_2]} = \{b_1 = t_0, t_1, t_2, \ldots, t_n = b_2\}$ is a subdivision of $J$ and $\Gamma(*)$ indicate the usual Gamma function.

The mass function $\gamma_\alpha(C^\omega, b_1, b_2)$ is defined by [19–23]

$$\gamma_\alpha(C^\omega, b_1, b_2) = \lim_{\delta \to 0} \left( \inf_{W_{[b_1, b_2]} \subseteq \delta} \rho_\alpha[C^\omega, W] \right) = \lim_{\delta \to 0} K_\delta^\alpha,$$

(3)

here $|W| := \max_{1 \leq i < n}(t_i - t_{i-1}) \leq \delta$, and

$$K_\delta^\alpha = \left( \inf_{W_{[b_1, b_2]} \subseteq \delta} \rho_\alpha[C^\omega, W] \right).$$

The staircase function $S_{C^\omega}^\alpha(t)$ is given [19–23] by

$$S_{C^\omega}^\alpha(t) = \begin{cases} \gamma_\alpha(C^\omega, b_0, t) & \text{if } t \geq b_0 \\ -\gamma_\alpha(C^\omega, b_0, t) & \text{otherwise,} \end{cases} \tag{5}$$

where $b_0$ is the fixed number (see Figure 1b).

The $\gamma$-dimension of a Cantor set is defined by

$$\dim_{\gamma}(C^\omega \cap [b_1, b_2]) = \inf \{\alpha : \gamma_\alpha(C^\omega, b_1, b_2) = 0\} = \sup \{\alpha : \gamma_\alpha(C^\omega, b_1, b_2) = \infty\}. \tag{6}$$

Figure 1(c) gives the $\gamma$-dimension for $\omega = 1/2$.

The fractal limit ($C^\omega$-limit) of a function $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$\forall \epsilon > 0, \exists \delta > 0 \ z \in C^\omega \quad \text{and} \quad |z - t| < \delta \Rightarrow |g(z) - l| < \epsilon. \tag{7}$$

If $l$ exists, then we have

$$l = C^\omega \lim_{z \to t} g(z). \tag{8}$$

The $C^\omega$-continuity of a function $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$g(t) = C^\omega \lim_{z \to t} g(z). \tag{9}$$

Remark 1.1: Note that the definition of fractal limit doesn’t include values of the function at $z$ if $z \notin C^\omega$. Also, $C^\omega$-limit is not defined at points $z \notin C^\omega$.

Remark 1.2: Note that for the function on $C^\omega$ at some points has one side fractal limit. The fractal derivative of $f(t)$ at $t$ is defined by Parvate and Gangal [19,20], Golmankhaneh et al. [21], Golmankhaneh and Baleanu [22], and Golmankhaneh [23]:

$$D_{C^\omega}^\alpha f(t) = \begin{cases} \frac{C^\omega \lim_{y \to t} f(y) - f(t)}{\gamma_\alpha(C^\omega, y) - \gamma_\alpha(C^\omega, t)}, & \text{if } t \in C^\omega, \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

if the limit exists [19–23].

Remark 1.3: Note that main difference between the Hausdorff measure and the staircase function is that the Hausdorff measure involves sums over a countable covers of $C^\omega$, while the staircase function includes sums over finite subdivision. Since possible finite subdivisions are smaller than all countable covers of fractal sets so the staircase function is algorithmic and it is easier to calculate (see Refs. [3,4,51]).

The fractal Jackson derivative is defined by

$$d_{C^\omega}^\alpha f(t) = \begin{cases} \frac{f(qy) - f(t)}{\gamma_\alpha^{\omega}(qy) - \gamma_\alpha^{\omega}(t)}, & \text{if } t \in C^\omega, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

where $q \in \mathbb{R}$. In Figure 1, we have shown the definitions are given above for especial case $\omega = 1/2$.

The fractal integral of $f(t)$ on $J = [b_1, b_2]$ is defined in [19–23] and approximately given by

$$\int_{b_1}^{b_2} f(t) d_{C^\omega}^\alpha t \approx \sum_{i=1}^{n} f(t_i)(S_{C^\omega}^{\alpha}(t_i) - S_{C^\omega}^{\alpha}(t_{i-1})). \tag{12}$$

For more details, we refer the reader to [19–23]. The Characteristic function of the Cantor set is defined by Parvate and Gangal [19,20], Golmankhaneh
Figure 1. Graphs corresponding to the Cantor set with $\omega = 1/2$: (a) the Cantor set ($\omega = 1/2$) by iteration, (b) the integral staircase function for the Cantor set with $\omega = 1/2$, (c) $\gamma$-dimension of the Cantor set with $\omega = 1/2$ and (d) characteristic function of the Cantor set with $\omega = 1/2$.

The coupled probability distribution is defined by Gell–Mann and Tsallis [29] and Nelson and Umarov [35]:

$$f_q(t) = \frac{[f(t)]^{1-q}}{\int_{-\infty}^{+\infty} [f(t)]^{1-q} dt},$$

(18)

The q-mean and q-variance of the using the coupled probability distribution are defined by

$$E_q(X) = \int_{-\infty}^{+\infty} t f_q(t) dt,$$

(19)

$$V_q(X) = \int_{-\infty}^{+\infty} (t - E_q(X))^2 f_q(t) dt,$$

(20)

where $E_q(X)$ is called $q$-mean of random variable and $V_q(X)$ is named $q$-variance of random variable.

**Example 1.1:** Fractal survival function on the Cantor-like set is defined by

$$p(x) = \begin{cases} \frac{1}{\Gamma(1+\alpha)}, & x \in C^\omega < 1; \\ \frac{1}{x^{-\alpha}}, & x \in C^\omega \geq 1, \end{cases}$$

(21)
where $\alpha$ is the fractal dimension. The fractal survival function is the probability density function of patient and devices will be survive on fractal time set.

In Figure 2, we have plotted Equation (21).

### 2. Fractal Tsallis entropy

In this section, we suggest the fractal Tsallis statistics which may useful for studying the nonlinear systems and with strong correlation and fractal structure but not ergodic nor close systems [35,37]. We connect the $q$-parameter in Tsallis statistics to $\gamma$-dimension in fractal calculus which might be considered a model for the system with fractal structure.

Consider the fractal differential equation in the following form:

$$D_{C^\alpha} q y(x) = y^q, \quad x \in C^\alpha, \quad q \in \mathbb{N}. \quad (22)$$

The solution of Equation (22) by using the conjugation between fractal calculus and ordinary calculus [20] is given by

$$y(x) = [1 + (1 - q)S^q_{C^\alpha}(x)]^{1/q}, \quad (23)$$

$$\approx [1 + (1 - q)x^\alpha]^{1/q}. \quad (24)$$

In Figure 3, we have sketched Equation (24) for the cases of $\alpha = 0.6, q = 0.8$ (blow), and $q = 0.5$ (up) which shows the effect of $q$-parameter and $\alpha$-dimension.

We define $q$-fractal exponent as follows:

$$e^q_q(x) = \begin{cases} (1 + qS^q_{C^\alpha}(x))^{1/q}, & 1 + qS^q_{C^\alpha}(x) \geq 0; \\ 0, & 1 + qS^q_{C^\alpha}(x) < 0. \end{cases} \quad (25)$$

Here, if $q = 0$ we have $e^q_0(x) = e^x$. Then we can rewrite Equation (24) as follows:

$$y(x) = e^q_{1-q}(x). \quad (26)$$

The inverse function of $y$ is denoted by $y^{-1}(x) = \ln_q^\alpha(x)$ and defined by

$$\ln_q^\alpha(x) = \frac{S^q_{C^\alpha}(x) - 1}{q}, \quad \ln_q^\alpha(0) = \ln^\alpha(0), \quad (27)$$

which is called $q$-fractal logarithm function. This function has the following property:

$$\ln_q^\alpha (S^q_{C^\alpha,A}(x) S^q_{C^\alpha,B}(x)) = \ln_q^\alpha (S^q_{C^\alpha,A}(x)) + \ln_q^\alpha (S^q_{C^\alpha,B}(x)) + (1 - q)[\ln^\alpha_q(S^q_{C^\alpha,A}(x)) + \ln^\alpha_q(S^q_{C^\alpha,B}(x))] \quad (28)$$

which is called fractal pseudo-additivity.

Fractal derivative of $q$-fractal exponent function is

$$D^\alpha_{C^\alpha} e^q_q(\eta x) = D^\alpha_{C^\alpha} (1 + \eta^\alpha qS^q_{C^\alpha}(x))^{1/q}$$

$$= \eta^\alpha (1 + \eta^\alpha qS^q_{C^\alpha}(x))^{1/q - 1}$$

$$= \eta^\alpha (1 + \frac{q}{1 - q}) \eta^\alpha (1 - q)S^q_{C^\alpha}(x))^{1/q - 1}$$

$$= \eta^\alpha e^q_q(\eta x), \quad q \neq 1. \quad (29)$$

The extending of Equation (29) to $n$ th fractal derivative we have

$$(D^\alpha_{C^\alpha} e^q_q(\eta x) = \left[ \eta^\alpha n \prod_{i=1}^{n} (1 - (1 - 1)q) \right]$$

$$\times e^q_q \frac{1}{n!} ((1 - q)(n)\eta x), \quad q \neq 1. \quad (30)$$

The fractal integral of the $q$-exponential is defined by

$$\int e_q(x) \, d_{C^\alpha} x$$

$$= \frac{1}{\eta(1 + q)} \left(1 + \frac{q}{1 + q} \eta(1 + q)S^q_{C^\alpha}(x) \right)^{1/q - q} \quad (31)$$

$$= \frac{1}{\eta(1 + q)} e_{\frac{q}{1 - q}} \left(1 + q\eta x\right) + c_1, \quad q \neq -1. \quad (32)$$
Then $n$ th-fractal integral is given by
\[
\int \cdots \left( \int e_q(x) \right) \cdots d_{C_C}^q x
= \left( \frac{1}{n^q} \prod_{i=1}^{n} \frac{1}{1 + nq} \right) e_{\frac{q}{1 + nq}}((1 + nq)x) \tag{33}
\]
\[
= + \sum_{i=1}^{n} c_i S_{C_C}^q(x) x^{-i - 1}, \quad q \neq - \frac{1}{n}. \tag{34}
\]

The fractal coupled probability density function is defined by
\[
p_{C_C}^q(x) = \frac{[p(x)]^{1 - q}}{\int_{-\infty}^{\infty} [p(x)]^{1 - q} d_{C_C}^q x}, \quad x \in C^\downarrow. \tag{35}
\]

The fractal Tsallis Entropy is defined on fractal sets as follows:
\[
S_{C_C}^q(x) = -1 + \int \frac{p(x)}{q} d_{C_C}^q x, \quad x \in C^\downarrow, \tag{36}
\]
where we have (see in reference [45])
\[
q = \frac{1}{\alpha - 1}. \tag{37}
\]

**Remark 2.1:** Note that the fractal Tsallis Entropy proposes a branch of thermodynamics that might called thermo-fractal/thermodynamics with fractal structure.

The fractal $q$-mean is defined by
\[
E_{C_C}^q[X] = \int S_{C_C}^q(x) p_{C_C}^q(x) d_{C_C}^q x. \tag{38}
\]

The fractal $q$-variance is defined by
\[
V_{C_C}^q(x) = \int \left( S_{C_C}^q(x) - E_{C_C}^q[X] \right)^2 d_{C_C}^q x. \tag{39}
\]

The fractal maximum $q$-entropy with fractal finite $q$-variance is given by
\[
G_{C_C}^q(x) = \frac{B_{C_C}^q}{N_{C_C}^q} e_q \left[ B_{C_C}^q S_{C_C}^q(x) - E_{C_C}^q[X] \right] - \frac{V_{C_C}^q(x)}{2}, \quad S_{C_C}^q(x) < x^2 \tag{40}
\]
\[
\approx \frac{B_{C_C}^q}{N_{C_C}^q} e_q \left[ B_{C_C}^q (x^2 - E_{C_C}^q[X]) \right] - \frac{V_{C_C}^q(x)}{2} \tag{41}
\]

where
\[
B_{C_C}^q = [(2 + q) V_{C_C}^q(X)]^{-1} \tag{42}
\]
and
\[
N_{C_C}^q = \begin{cases}
\frac{\Gamma^q(1 + \frac{1}{q})}{\Gamma^q(1 + \frac{2 + 3q}{2q})}, & q > 0; \\
\sqrt{\frac{\pi}{q}}, & q = 0; \\
\left( \frac{\Gamma^q(1 + \frac{1}{q})}{\Gamma^q(1 + \frac{1}{q})} \right)^{q^2}, & -2 < q < 0,
\end{cases} \tag{43}
\]
where $\Gamma^q(\cdot)$ [22] is the gamma function with fractal support and $G_{C_C}^q(x)$ is called the fractal-$q$-Gaussian or generalized $\gamma$-stable Lévy distribution on fractal sets [35].

Equation (43) indicates nonlinear fractal coupling of statistical states. For $q > 0$ nonlinear fractal coupling between the states strengthens is decayed, namely:
\[
|S_{C_C}^q(x) - E_{C_C}^q[X]| > \frac{1}{q B_{C_C}^q}, \tag{44}
\]
\[
|x^2 - E_{C_C}^q[X]| > \frac{1}{q B_{C_C}^q} \Rightarrow G_{C_C}^q(x) \approx 0. \tag{45}
\]

In the case of $q = 0$ we get fractal Gaussian distribution. For the $-2 < q < 0$ the decoupled states leads to a fractal heavy-tail distribution [35].

3. Hadron system with fractal structure

Hadron is a subatomic composite particle made of two or more quarks held together by the strong force. Now, we want to connect $\alpha$-dimension of the fractal calculus to the Tsallis entropy of Hadron system (see reference [37]). Let us define $R$ as follows [37]:
\[
R = \frac{<E>}{N' < U>} = \frac{(q - 1)N'/N'}{3 - 2q + (q - 1)N'}, \quad N' = 2, 3, \tag{46}
\]
where $<*>$ indicate the average of $*$, $U$ is the total energy of system, $E$ is the internal energy of $N'$ number of subsystems, $N$ is the effective number of Hadron subsystems [37], and $<E>$ is the average energy of the Hadron system. In view of the results in [37], we conclude
\[
\alpha = \text{dim}_{\mathbb{R}}(C^\downarrow) = 1 + \frac{\log N'}{\log R}, \tag{47}
\]
where $\alpha = \text{dim}_{\mathbb{R}}(C^\downarrow)$ which holds for thin Cantor sets [19].

**Remark 3.1:** Note that by choosing $\alpha = 1$ we can obtain the standard Tsallis entropy and statistics[33,35,37].

4. Conclusion

In this work, we have generalized the Tsallis entropy on the fractal sets which includes both the fractal structure and strong correlations of systems. The relationship between the $q$-parameter in the Tsallis entropy and the fractal dimension ($\alpha$) of the random variable indicates that a change in dimension or a change in correlation can develop the central limit theorem. The $q$-Gaussian, Lévy distribution and their conditions for the nonlinear coupling of the fractal $q$-statistics have been obtained. We suggest a relation between $\alpha$-dimension and $q$-parameter in the Tsallis entropy for the Hadron system. Further applications will be the main goal of a future study.
Acknowledgements

This paper is dedicated to the memory of Professor Mohamad Ali Asadi-Golmankhaneh a brilliant mathematician, working in Algebraic & Geometric Topology. He was my supervisor and my uncle and passed away on 26 August 2021.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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