Books in graphs

Béla Bollobás*†‡ and Vladimir Nikiforov†

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Abstract

A set of \(q\) triangles sharing a common edge is called a book of size \(q\). We write \(\beta(n, m)\) for the the maximal \(q\) such that every graph \(G(n, m)\) contains a book of size \(q\). In this note
1) we compute \(\beta(n, cn^2)\) for infinitely many values of \(c\) with \(1/4 < c < 1/3\),
2) we show that if \(m \geq (1/4 - \alpha)n^2\) with \(0 < \alpha < 17^{-3}\), and \(G\) has no book of size at least \(\left(\frac{1}{6} - 2\alpha^{1/3}\right)n\) then \(G\) contains an induced bipartite graph \(G_1\) of order at least \(\left(1 - \alpha^{1/3}\right)n\) and minimal degree
\[
\delta(G_1) \geq \left(\frac{1}{2} - 4\alpha^{1/3}\right)n,
\]
3) we apply the latter result to answer two questions of Erdős concerning the booksize of graphs \(G(n, n^2/4 - f(n)n)\) every edge of which is contained in a triangle, and \(0 < f(n) < n^{2/5-\varepsilon}\).

1 Introduction

Our notation and terminology are standard (see, e.g., [2]). Thus, \(G(n, m)\) is a graph of order \(n\) and size \(m\); for a graph \(G\) and a vertex \(u \in V(G)\) we write \(\Gamma(u)\) for the set of vertices adjacent to \(u\); \(d_G(u) = |\Gamma(u)|\) is the degree of \(u\); we write \(d(u)\) instead of \(d_G(u)\) if the graph \(G\) is implicit. However, somewhat unusually, we set \(\hat{\delta}(U) = |\bigcap_{x \in U} \Gamma(x)|\). Unless explicitly stated, all graphs are assumed to be defined on the vertex set \([n] = \{1, 2, \ldots, n\}\). Also, \(k_s(G)\) is the number of \(s\)-cliques of \(G\).

In 1962 Erdős [6] initiated the study of books in graphs. A book of size \(q\) consists of \(q\) triangles sharing a common edge. We write \(bk(G)\) for the size of the largest book in a graph \(G\) and call it the booksize of \(G\). Since 1962 books

*Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA
†Trinity College, Cambridge CB2 1TQ, UK
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have attracted considerable attention both in extremal graph theory (see, e.g., [10], [5], and [3]) and in Ramsey graph theory (see, e.g., [13], [9], and [11]).

Erdős, Faudree and Rousseau defined in [3] the function

\[ \beta(n, m) = \min \{ bk(G) \mid G = G(n, m) \} . \]

Our aim in this paper is to study the function \( \beta(n, m) \) and its variants. We shall prove a technical inequality about booksizes that we shall use to give bounds on \( \beta(n, m) \) and answer two questions of Erdős.

The paper is organized as follows: in section 2 we use a counting argument of Khadžiivanov and Nikiforov [10] to prove a bound on \( \beta(n, m) \) in terms of the degree sequence and other graph parameters. In particular, this result implies that \( \beta(n, \lfloor n^2/4 \rfloor + 1) > n/6 \), as conjectured by Erdős and proved by Edwards [3]. In addition, we determine \( \beta(n, cn^2) \) for infinitely many values of \( c \) with \( 1/4 < c < 1/3 \). In section 3 we prove that a graph \( G(n, (1/4 - \alpha)n^2) \) with \( 0 < \alpha < 17^{-3} \) either has a book of size about \( n/6 \) or has a large induced bipartite graph with minimal degree close to \( n/2 \). In the last section we make use of this structural property to answer two questions of Erdős concerning the booksize of graphs \( G(n, n^2/4 - f(n)n) \), every edge of which is contained in a triangle and \( 0 < f(n) \leq n^{2/5 - \varepsilon} \).

## 2 A lower bound on the booksize of a graph

In 1962 Erdős [6] conjectured that the booksize of a graph \( G \) of order \( n \) and size greater than \( \lfloor n^2/4 \rfloor \) is at least \( \lfloor n/6 \rfloor \), i.e., \( \beta(n, \lfloor n^2/4 \rfloor + 1) \geq n/6 \). This was proved by Edwards in an unpublished manuscript [3] and independently by Khadžiivanov and Nikiforov in [10].

For \( r \geq 3 \) and \( 0 \leq j < r \), we write \( K^{(j)}_r \) for the graph consisting of a complete graph \( K_{r-1} \) and an additional vertex joined to precisely \( r - j - 1 \) vertices of the \( K_{r-1} \). We denote by \( k^{(j)}_r(G) \) the number of induced subgraphs of \( G \) that are isomorphic to \( K^{(j)}_r \), e.g., \( k^{(3)}_4(G) \) is the number of induced subgraphs of \( G \) that are isomorphic to a triangle with an isolated vertex.

**Theorem 1** Let \( G = G(n, m) \) be a graph with degree sequence \( d(1), \ldots, d(n) \). Then,

\[
6k_3(G) - \sum_{i=1}^{n} d^2(i) + nm \leq bk(G) \geq nk_3(G) + 8k_4(G) + 2k^{(3)}_4(G).
\]

**Proof** In the proof we use some arguments from [10]. Set \( \beta = bk(G) \). Clearly \( G \) contains exactly \( (n - 3) k_3(G) \) pairs \((v, T)\) where \( v \in V(G) \) and \( T \) is a triangle in \( G \). Also, a \( K_4 \) subgraph of \( G \) contains exactly 4 such pairs; a \( K^{(j)}_4 \) subgraph contains two such pairs for \( j = 1 \), and one such pair for \( j = 2 \) and 3. Therefore,

\[
(n - 3) k_3(G) = 4k_4(G) + 2k^{(1)}_4(G) + k^{(2)}_4(G) + k^{(3)}_4(G).
\]  

\[ (1) \]
We have
\[ \sum_{(i,j) \in E(G)} \left( \frac{\hat{d}(ij)}{2} \right) = 6k_4(G) + k_4^{(1)}(G), \]
yielding
\[ \sum_{(i,j) \in E(G)} \left( \hat{d}^2(ij) - \hat{d}(ij) \right) = 12k_4(G) + 2k_4^{(1)}(G). \]
Since
\[ \sum_{(i,j) \in E(G)} \hat{d}(ij) = 3k_4(G), \] we see that
\[ \sum_{(i,j) \in E(G)} \hat{d}^2(ij) = 12k_4(G) + 2k_4^{(1)}(G) + 3k_4(G). \]
Subtracting (1) from the last equality and rearranging the terms, we obtain
\[ nk_3(G) = \sum_{(i,j) \in E(G)} \hat{d}^2(ij) - 8k_4(G) + k_4^{(2)}(G) + k_4^{(3)}(G). \] (3)
Next we shall eliminate the term \( k_4^{(2)}(G) \) from (3). For every \( i \in V(G) \) set \( \Gamma'(i) = V(G) \setminus \Gamma(i) \). The sum \( \sum_{ij \in E(G)} \hat{d}(ij) |\Gamma'(i) \cap \Gamma'(j)| \) counts each \( K_4^{(2)} \) once and each \( K_4^{(3)} \) three times, so
\[ \sum_{(i,j) \in E(G)} \hat{d}(ij) |\Gamma'(i) \cap \Gamma'(j)| = k_4^{(2)}(G) + 3k_4^{(3)}(G). \] (4)
Subtracting (1) from (3), we see that
\[ nk_3(G) = \sum_{(i,j) \in E(G)} \hat{d}^2(ij) + \sum_{(i,j) \in E(G)} \hat{d}(ij) |\Gamma'(i) \cap \Gamma'(j)| - 8k_4(G) - 2k_4^{(3)}(G) \]
\[ = \sum_{(i,j) \in E(G)} \hat{d}(ij) \left( \hat{d}(ij) + |\Gamma'(i) \cap \Gamma'(j)| \right) - 8k_4(G) - 2k_4^{(3)}(G). \] (5)
Noting that \( \hat{d}(ij) \leq \beta \) for every edge \((i,j)\) and recalling (4), inequality (5) implies that
\[ nk_3(G) \leq \beta \sum_{(i,j) \in E(G)} \left( \hat{d}(ij) + |\Gamma'(i) \cap \Gamma'(j)| \right) - 8k_4(G) - 2k_4^{(3)}(G) \]
\[ = \beta \left( 3k_4(G) + \sum_{(i,j) \in E(G)} |\Gamma'(i) \cap \Gamma'(j)| \right) - 8k_4(G) - 2k_4^{(3)}(G) \] (6)
Since
\[ |\Gamma'(i) \cap \Gamma'(j)| = n - d(i) - d(j) + \hat{d}(ij). \]
we find that
\[
\sum_{(i,j) \in E(G)} |\Gamma'(i) \cap \Gamma'(j)| = \sum_{(i,j) \in E(G)} \left( n - d(i) - d(j) + \hat{d}(ij) \right) = 3k_3(G) + nm - \sum_{i=1}^{n} d^2(i).
\]

Putting this into (6) we see that
\[
nk_3(G) + 8k_4(G) + 2k_4^{(3)}(G) \leq 6\beta k_3(G) + \beta \left( -\sum_{i=1}^{n} d^2(i) + nm \right),
\]
as claimed. \qed

The following corollary is due to Edwards [3].

**Corollary 2** For every graph \( G = G(n, m) \) with \( m > n^2/4 \)
\[
bk(G) \geq \frac{2m}{n} - \frac{n}{3}.
\]

**Proof** With \( \beta = bk(G) \), Theorem 1 implies that
\[
\left( 6k_3(G) - \sum_{i=1}^{n} d^2(i) + nm \right) \beta \geq nk_3(G) + 8k_4(G) + 2k_4^{(3)}(G) \geq nk_3(G),
\]
and so
\[
(6\beta - n) k_3(G) \geq \beta \left( \sum_{i=1}^{n} d^2(i) - nm \right).
\]

Since \( \sum_{i=1}^{n} d(i) = 2m \), we have
\[
\sum_{i=1}^{n} d^2(i) \geq \frac{4m^2}{n} > nm;
\]
in particular,
\[
\sum_{i=1}^{n} d^2(i) - nm > 0.
\]

Hence, (8) implies that \( 6\beta > n \). Furthermore, as \( 3k_3(G) \leq \beta m \), we see from (8) and (9) that
\[
\frac{1}{3} (6\beta - n) \beta m \geq \beta \left( \frac{4m^2}{n} - nm \right),
\]
implying (7). \qed

As a consequence of Corollary 2 we easily obtain the following bound.
Corollary 3 For every graph $G \left( n, \left\lceil \frac{n^2}{4} \right\rceil + 1 \right)$ we have $bk(G) > n/6$. □

The graph $H_{s,t}$ below, constructed by Erdős, Faudree and Rousseau in [4], shows that the bound in Corollary 2 is essentially best possible.

Example 4 Let $t \geq 1$, $s > 3$ be fixed integers. Partition the vertex set $V = [n]$ with $n = 3st$ into $3s$ sets $V_{ij}$ ($i \in [3], j \in [r]$) of cardinality $t$. Join two vertices $v \in V_{ij}$ and $u \in V_{kl}$ iff $i \neq k$ and $j \neq l$.

By straightforward counting we see that

$$e(H_{s,t}) = 3s(s-1)t^2 = 3s(s-1)\left(\frac{n}{3s}\right)^2 = \frac{s-1}{3s}n^2,$$

and

$$bk(H_{s,t}) = (s-2)t = \left(\frac{s-2}{3s}\right)n.$$

On the other hand, from Corollary 2 we have

$$bk(H_{s,t}) \geq \frac{2e(H_{s,t})}{n} - \frac{n}{3} = \frac{2(s-1)n}{3s} - \frac{n}{3} = \left(\frac{s-2}{3s}\right)n,$$

thus, the bound in Corollary 2 is tight for $n, m$ with $3s|n$, $s > 8$, and $m = (s-1)n^2/3s$.

A different extremal graph ([3], [10]) is defined as follows.

Example 5 Select 6 disjoint sets $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$ with $|A_{11}| = |A_{12}| = |A_{13}| = k-1$ and $|A_{21}| = |A_{22}| = |A_{23}| = k+1$. Set $V(G)$ to be the union of all these sets. For every $1 \leq j < k \leq 3$ join every vertex of $A_{1j}$ to every vertex of $A_{2j}$ and for $j = 1, 2, 3$ join every vertex of $A_{1j}$ to every vertex of $A_{2j}$.

It is easy to check that the resulting graph has $n = 6k$ vertices, $9k^2 + 3 > n^2/4$ edges and its booksize is precisely $k + 1 = n/6 + 1$.

3 A stability theorem for graphs without large books

In this section we give a structural property of graphs having substantial size and whose booksizes is small.

In [11] Andrásfai, Erdős and Sós proved that if $G$ is a $K_{r+1}$-free graph of order $n$ with minimal degree

$$\delta(G) > \left(1 - \frac{3}{3r-1}\right)n$$

then $G$ is $r$-chromatic. We shall use this theorem to obtain a structural result related to the stability theorems of Simonovits (see, e. g., [12]).
Theorem 6 For every \( \alpha \) with \( 0 < \alpha < 10^{-5} \) and every graph \( G = G(n, m) \) with
\[
m \geq \left( \frac{1}{4} - \alpha \right) n^2
\] (10)
either
\[
bk(G) > \left( \frac{1}{6} - 2\alpha^{1/3} \right) n
\] (11)
or \( G \) contains an induced bipartite graph \( G_1 \) of order at least \( (1 - \alpha^{1/3}) n \) and with minimal degree
\[
d(G_1) \geq \left( \frac{1}{2} - 4\alpha^{1/3} \right) n.
\] (12)

Proof If \( m > n^2/4 \) then Corollary 3 implies that \( bk(G) > n/6 \), which is stronger than (11), so we may assume that \( m \leq n^2/4 \). Furthermore, if \( \sum_{i=1}^{n} d^2(i) > nm \) then Theorem 1 implies that
\[
(6bk(G) - n) k_3(G) > 0,
\]
and so again \( bk(G) > n/6 \). Therefore, we may assume
\[
\sum_{i=1}^{n} d^2(i) \leq nm.
\]

Clearly, from (10),
\[
\frac{4m^2}{n} \geq m(n - 4\alpha n) = nm - 4\alpha nm,
\]
and so,
\[
\sum_{i=1}^{n} \left( d(i) - \frac{2m}{n} \right)^2 = \sum_{i=1}^{n} d^2(i) - \frac{4m^2}{n} \leq 4\alpha nm \leq \alpha n^3.
\] (13)

Set \( \varepsilon = \alpha^{1/3} \), \( M = \{ u \in V(G) : d(u) < \frac{2m}{n} - \varepsilon n \} \) and \( G_1 = G[V \setminus M] \). We claim that \( G_1 \) has the required properties. First we show that its minimal degree satisfies (12). From (13),
\[
|M| \varepsilon^2 n^2 \leq \sum_{v \in M} \left( d(v) - \frac{2m}{n} \right)^2 < \sum_{i=1}^{n} \left( d(i) - \frac{2m}{n} \right)^2 \leq \alpha n^3.
\]

Hence, \( |M| < (\alpha/\varepsilon^2) n = \alpha^{1/3} n \), i.e., \( v(G_1) > (1 - \alpha^{1/3}) n \). Also, for \( v \in V \setminus M \), we have
\[
\begin{align*}
d_{G_1}(v) &\geq d(v) - |M| > \left( \frac{2m}{n} - \varepsilon n \right) - |M| = \frac{n}{2} - 2\alpha n - \alpha^{1/3} n - |M| \\
&> \left( \frac{1}{2} - 2\alpha n - 2\alpha^{1/3} \right) n \geq \left( \frac{1}{2} - 4\alpha^{1/3} \right) n.
\end{align*}
\] (14)
All that remains to prove is that \( G_1 \) is bipartite. Suppose first that \( G_1 \) contains a triangle with vertices \( u, v, w \), say. Since

\[
n \geq d(u) + d(v) + d(w) - \tilde{d}(uv) - \tilde{d}(uw) - \tilde{d}(vw)
\]

we find that

\[
\tilde{d}(uv) + \tilde{d}(uw) + \tilde{d}(vw) \geq d(u) + d(v) + d(w) - n
\geq 3 \left( \frac{1}{2} - \alpha - \sqrt[3]{\alpha} \right) n - n.
\]

Thus,

\[
\delta(G_1) \geq \left( \frac{1}{6} - \alpha n - \alpha^{1/3} \right) n \geq \left( \frac{1}{6} - 2\alpha^{1/3} \right) n,
\]

and so \( \text{(12)} \) holds. Finally, assume that \( G_1 \) is triangle-free. Since \( \alpha < 10^{-5} \),

\[
\delta(G_1) \geq \left( \frac{1}{2} - 4\alpha^{1/3} \right) n > \frac{2}{5} v(G_1).
\]

Hence, the case \( r = 2 \) of the theorem of Andrásfai, Erdős and Sós mentioned above implies that \( G_1 \) is indeed bipartite, completing the proof of Theorem 6. \( \square \)

It is easily seen that if we are a little more careful in our proof of \( \delta(G_1) > v(G_1) \) then the condition on \( \alpha \) can be relaxed to \( 0 < \alpha < 17^{-3} \).

## 4 Two problems of Erdős

Erdős and Rothschild suggested the study of the booksize of graphs in which every edge is contained in a triangle. In \[7\] and \[8\] Erdős himself gave some results on such graphs. Suppose \( f(n) \) is a fixed positive function of \( n \), and let \( TG(n, f) \) be the set of all graphs \( G = G(n, m) \) such that every edge of \( G \) is contained in a triangle and \( m > \max \{n^2/4 - f(n)n, 0\} \). Set

\[
\gamma(n, f) = \min \{bk(G) \mid G \in TG(n, f)\}.
\]

In \[7\], p. 91, Erdős proved that for every \( c > 0 \) there exists some \( c_1 > 0 \) such that

\[
\gamma(n, c) \geq c_1 n
\]

for \( n \) sufficiently large. Hence, setting

\[
\lim_{n \to \infty} \frac{\gamma(n, c)}{n} = \sigma(c),
\]

we see that for every \( c > 0 \), \( \sigma(c) > 0 \). Erdős asked how large \( \sigma(c) \) is. Our next theorem gives an answer that is asymptotically tight when \( c \) tends to 0.
Theorem 7  For every function $f(n)$ with $0 < f(n) < n/4$,

$$\gamma(n, f) > \frac{n}{12f(n) + 6}.$$

Proof  From Theorem 7 we have for $\beta = bk(G)$

$$\left(6k_3(G) - \sum_{i=1}^{n}d^2(i) + nm\right)\beta \geq nk_3(G),$$

and hence,

$$(6\beta - n)k_3(G) \geq \beta \left(\sum_{i=1}^{n}d^2(i) - nm\right).$$

From $\sum_{i=1}^{n}d(i) = 2m$ we have $\sum_{i=1}^{n}d^2(i) \geq 4m^2/n$ and thus,

$$(6\beta - n)k_3(G) \geq \beta \left(\frac{4m^2}{n} - nm\right) > -4f(n)\beta m.$$  

Clearly $3k_3 \geq m$; hence, assuming $6\beta \leq n$,

$$12f(n)\beta m > (n - 6\beta)k_3(G) \geq (n - 6\beta)m,$$

and the desired result follows.  \(\square\)

Applying Theorem 7 with $f(n) = c$, we obtain

$$\sigma(c) \geq \frac{1}{12c + 6}. \quad (15)$$

On the other hand, a slight modification of the graphs described in Example 4 gives a graph $G = G(n, n^2/4 - O(1))$, such that every edge of $G$ is contained in a triangle and

$$bk(G) \leq \frac{n}{6},$$

and this, together with (15), implies

$$\lim_{c \to 0} \sigma(c) = \frac{1}{6}.$$  

However, for large $c$ Theorem 7 is not precise enough. Prior to obtaining a lower bound on $\gamma(n, f)$ that is valid in a more general case of a function $f$, we recall the graph that Erdős outlined in [5].

Example 8  Suppose $f(n)$ with $0 < f(n) < n/4$ tends to infinity with $n$; set $l_n = f(n)^{1/2}$. Define a graph $G$ as follows: let $V(G) = [n] = A \cup B \cup C$, with $|A| = l_n^2$, $|B| = |C| = (n - l_n^2)/2$. Join every vertex of $B$ to every vertex of $C$. Divide $B$ and $C$ into $l_n$ roughly equal disjoint sets $B_i$ and $C_i$. Join every vertex $x_{ij} \in A$ to every vertex of $B_i$ and $C_i$.  

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It is easily seen that \( e(G) = n^2/4 - f(n) n \), every edge of \( G \) is contained in a triangle and \( bk(G) = o(n) \).

In order to obtain a precise estimate of \( bk(G) \) we shall describe more accurately the graph \( G \). Suppose \( f(n) \) is a function of \( n \) with \( 4 < f(n) < n/4 \.

Set \( k = \left(\frac{2f(n)}{n}\right)^{1/2} \), so that \( k^2 < 2f(n) < (k+1)^2 \). Let \( n = 2kt + k^2 + s \), where \( 0 \leq s < 2k \). Set \( V(G) = [n] \) and partition \( [n] \) into \( 2k + 2 \) sets \( A, B_1, ..., B_k, C_1, ..., C_k, S \) such that

\[
|A| = k^2, \ |B_1| = ... = |B_k| = |C_1| = ... = |C_k| = t, \ |S| = s.
\]

Join every vertex of \( \cup_{i=1}^{k} B_i \) to every vertex of \( \cup_{i=1}^{k} C_i \); label the members of \( A \) by \( a_{ij} \) \((i, j \in [k])\), and, for every \( i, j \in [k] \), join \( a_{ij} \) to all vertices of \( B_i \cup C_j \). By straightforward calculations we obtain

\[
e(G) = \frac{(n-s-k^2)^2}{4} + k^2 \frac{(n-s-k^2)}{2} \geq \frac{(n-2k-k^2)^2}{4} + k \left(n - 2k - k^2\right)
\]

\[
\geq \frac{n^2}{4} - \frac{k^2n}{2} + \frac{k^4 - 4k^2}{4} > \frac{n^2}{4} - f(n)n,
\]

and

\[
bk(G) \leq \frac{n-s-k^2}{2k} < \frac{n}{2k} \leq \frac{n}{2\sqrt{2f(n)}}.
\]

Since, obviously, \( G \in TG(n,f) \), we immediately obtain the bound

\[
\gamma(n,f) < \frac{n}{2\sqrt{2f(n)}}. \tag{16}
\]

Our next aim is to show that, for a wide class of functions \( f \), (16) is essentially tight.

**Theorem 9** Let \( 0 < c < 2/5 \) and \( 0 < \varepsilon < 1 \) be constants, and \( 0 < f(n) < n^c \). Then, if \( n \) is sufficiently large,

\[
\gamma(n,f) > (1-\varepsilon) \frac{n}{2\sqrt{2f(n)}}.
\]

**Proof** Let us start with a brief sketch of our proof. Suppose the graph \( G \) is a counterexample to our assertion. Then, from Theorem 6 \( G \) has an induced bipartite graph \( G_1 \) of order at least \( n - \alpha^{1/3}n \) and large minimal degree. We show that each part of \( G_1 \) has cardinality close to \( n/2 \) and then consider an edge from \( G_1 \); by assumption it is contained in a triangle whose third vertex \( w \) is not in \( G_1 \). We bound the degree of \( w \) from above and then bound the number of all such vertices from below. Dropping a carefully selected number of such vertices we obtain a graph of order \( n_1 \) and size greater than \( n_1^2/4 \), such that \( n_1 \) is close to \( n \). Then, by Corollary 3 this graph contains a book of size \( n_1/6 \), completing the proof.

Now let us give the complete proof. Set \( \beta = bk(G) \) and \( \alpha = f(n)/n \). Assume the assertion does not hold, i.e., there is some \( \varepsilon > 0 \) such that for every
$F$ and every $N$ there is an $n > N$ with $f(n) > F$ and a graph $G = G(n, m)$ satisfying the conditions of the theorem and with

$$\beta \leq (1 - \varepsilon) \frac{1}{2} \sqrt{\frac{n}{2\alpha}}. \quad (17)$$

Then, as $\beta < n/8$, Theorem 6 implies that $G$ has an induced bipartite graph $G_1$ of order at least $n - \alpha^{1/3}n$ and

$$\delta (G_1) > \left( \frac{1}{2} - 4\alpha^{1/3} \right) n = \frac{n}{2} - 4\alpha^{1/3}n. \quad (18)$$

Let $V(G_1) = B \cup C$ be a bipartition of $G_1$ and set $A = V(G) \setminus V(G_1)$. From (18),

$$|B| \geq \left( \frac{1}{2} - 4\alpha^{1/3} \right) n, \quad |C| \geq \left( \frac{1}{2} - 4\alpha^{1/3} \right) n, \quad (19)$$

$$e(G_1) = e(B, C) \geq \frac{1}{2} \left( 1 - \alpha^{1/3} \right) n \left( \frac{1}{2} - 4\alpha^{1/3} \right) n = \frac{n^2}{4} \left( 1 - \alpha^{1/3} \right) \left( 1 - 8\alpha^{1/3} \right) > \frac{n^2}{4} \left( 1 - 9\alpha^{1/3} \right).$$

Consider the set $T$ of triangles containing an edge of $G_1$. Since every edge of $G_1$ is contained in a triangle and $G_1$ is bipartite, we see that

$$|T| \geq e(G_1) > \frac{n^2}{4} \left( 1 - 9\alpha^{1/3} \right). \quad (20)$$

Let $D \subset A$ be the set of vertices of $A$ that are contained in some triangle of $T$. We claim that for every $w \in D$, and $n$ sufficiently large,

$$d(w) < \sqrt{\frac{n}{2\alpha}}. \quad (21)$$

Indeed, by definition, every vertex $w \in D$ is joined to some $u \in B$ and some $v \in C$. Then,

$$\beta \geq |\Gamma (u) \cap \Gamma (w)| \geq |\Gamma (u) \cap \Gamma (w) \cap C| \geq d_C (w) + d_C (u) - |C| \geq d_C (w) + \delta (G_1) - |C|,$n

and, similarly,

$$\beta \geq |\Gamma (v) \cap \Gamma (w)| \geq |\Gamma (v) \cap \Gamma (w) \cap B| \geq d_B (w) + \delta (G_1) - |B|.$$

Hence, summing the last two inequalities and taking into account (18),

$$2\beta \geq d_B (w) + d_C (w) + 2\delta (G_1) - n + |A| \geq d_B (w) + d_C (w) + |A| - 8\alpha^{1/3}n \geq d(w) - 8\alpha^{1/3}n.$$
To complete the proof of (21), observe that from (17), we have

\[2\beta \leq (1 - \varepsilon) \sqrt{\frac{n}{2\alpha}}.\]

For every \(w \in D\), let \(t(w)\) be the number of triangles of \(T\) containing \(w\). Clearly, we have

\[t(w) = \frac{1}{2} \sum_{u \in \Gamma(w)} |\Gamma(u) \cap \Gamma(w)| \leq \frac{1}{2} d(w) \beta \leq \frac{1}{4} d(w) (1 - \varepsilon) \sqrt{\frac{n}{2\alpha}}.\]

This, together with (21), gives

\[t(w) < (1 - \varepsilon) \frac{n}{8\alpha}.\] (22)

Summing (22) for all \(w \in D\), in view of (20), we obtain

\[\frac{n^2}{4} \left(1 - 9\alpha^{1/3}\right) < |T| = \sum_{w \in D} t(w) < |D| \frac{n(1 - \varepsilon)}{8\alpha}.\]

Hence,

\[|D| > 2\alpha \frac{(1 - 9\alpha^{1/3}) n}{(1 - \varepsilon)}.\]

Observe that, as \(\alpha = f(n)/n < n^{c-1}\) and \(c < 2/5\), we have \(\lim_{n \to \infty} \alpha^{1/3} = 0\). Then, for \(n\) sufficiently large, we see that

\[|D| > 2(1 + \varepsilon) \alpha n.\]

Select a set \(D_0 \subset D\) with

\[(2 + \varepsilon) \alpha n < |D_0| < (2 + 2\varepsilon) \alpha n.\] (23)

As, from (21), for every vertex \(w \in D_0\) and \(n\) sufficiently large, we have

\[d(w) < \sqrt{\frac{n}{2\alpha}},\]

then the graph \(G[V \setminus D_0]\) has at least

\[e(G) - |D_0| \sqrt{\frac{n}{2\alpha}}\]

edges. We shall prove that if \(n\) is large enough then

\[\frac{n^2}{4} - n\alpha^2 - |D_0| \sqrt{\frac{n}{2\alpha}} > \frac{(n - |D_0|)^2}{4}.\] (24)
Assume that (24) does not hold. Then, from (23),

\[
\frac{n^2}{4} - \alpha n^2 - (2(1 + \varepsilon)\alpha n) \sqrt{\frac{n}{2\alpha}} \leq \frac{n^2}{4} - \alpha n^2 - |D_0| \sqrt{\frac{n}{2\alpha}} \leq \frac{(n - |D_0|)^2}{4}
\]

and thus, after some simple algebra,

\[
\varepsilon \leq (2(1 + \varepsilon)) \frac{1}{\sqrt{2an}} + \frac{(2 + \varepsilon)^2 \alpha^2}{4} < \frac{4}{\sqrt{2f(n)}} + 4n^{2\varepsilon - 2},
\]

which is a contradiction if \( n \) is large enough. Thus, (24) holds. Then, if \( n \) is sufficiently large, Corollary 8 implies that

\[
\beta^k (G[V \setminus D_0]) > \frac{n - |D_0|}{6} \geq \sqrt{\frac{n}{2\alpha}}
\]

This contradiction completes our proof. \( \square \)

In [7], p. 235, Erdős asked how large \( \gamma (n, n^c) \) is for \( 0 < c < 1 \). Putting \( f(n) = n^{c-1} \) for \( 1 < c < 7/5 \) and applying Theorem 9 together with (16), we obtain the following.

**Corollary 10** If \( 0 < c < 1 \) and \( n \) is sufficiently large,

\[
\gamma (n, n^c) < \frac{1}{2\sqrt{2}} n^{1-c/2}.
\]

Also, if \( 0 < c < 2/5 \), \( \varepsilon > 0 \) and \( n \) is sufficiently large,

\[
\gamma (n, n^c) > \frac{1 - \varepsilon}{2\sqrt{2}} n^{1-c/2}.
\]

**References**

[1] B. Andrásfai, P. Erdős and V. T. Sós, On the connection between chromatic number, maximal clique and minimal degree of a graph, *Discrete Math.* 8 (1974), 205–218.

[2] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998), xiv+394 pp.

[3] C. S. Edwards, A lower bound for the largest number of triangles with a common edge, unpublished manuscript, 1977.

[4] P. Erdős, R. Faudree and E. Győri, On the book size of graphs with large minimum degree, *Studia Sci. Math. Hungar.* 30 (1995), 25–46.
[5] P. Erdős, R. Faudree and C. Rousseau, Extremal problems and generalized
degrees, *Graph Theory and Applications (Hakone, 1990)*, *Discrete Math.* 127 (1994), 139–152.

[6] P. Erdős, On a theorem of Rademacher-Turán, *Illinois J. Math.* 6 (1962),
122–127.

[7] P. Erdős, Some of my favourite problems in various branches of combi-
natorics, *Combinatorics 92 (Catania, 1992)*, *Matematiche (Catania)* 47 (1992), 231–240.

[8] P. Erdős, Problems and results in combinatorial analysis and graph theory,
*Proceedings of the First Japan Conference on Graph Theory and Applica-
tions (Hakone, 1986)*, *Discrete Math.* 72 (1988), 81–92.

[9] R. J. Faudree, C. C. Rousseau and J. Sheehan, More from the good book,
*Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph
Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1978)*,
pp. 289–299, *Congress. Numer.*, XXI, Utilitas Math., Winnipeg, Man.,
1978.

[10] N. Khadžiivanov and V. Nikiforov, Solution of a problem of P. Erdős about
the maximum number of triangles with a common edge in a graph (Rus-
sian), *C. R. Acad. Bulgare Sci.* 32 (1979), 1315–1318.

[11] V. Nikiforov and C. C. Rousseau, A note on Ramsey numbers for books,
submitted.

[12] M. Simonovits, A method for solving extremal problems in graph theory,

stability problems, in: *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pp.
279–319, Academic Press, New York, 1968.

[13] C. C. Rousseau and J. Sheehan, On Ramsey numbers for books, *J. Graph
Theory* 2 (1978), 77–87.