There does not exist a strongly regular graph with parameters \((1911, 270, 105, 27)\)

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Abstract

In this paper we show that there does not exist a strongly regular graph with parameters \((1911, 270, 105, 27)\).

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1 Introduction

In this paper we show the following result:

Theorem 1.1. There does not exist a strongly regular graph with parameters \((1911, 270, 105, 27)\).

This was the largest open case of a set of feasible parameters of a strongly regular graph with smallest eigenvalue \(-3\). We conjecture:

Conjecture 1.2. Let \(G\) be a primitive strongly regular graph with parameters \((n, k, \lambda, \mu)\) and smallest eigenvalue \(-3\). Then either \(\mu \in \{6, 9\}\) or \(n \leq 276\).

On this moment, there are twelve cases of parameter sets of putative primitive strongly regular graphs with smallest eigenvalue \(-3\), \(n > 276\) and \(\mu \notin \{6, 9\}\) which are open. They are in Table 1 below (cf. [8]).

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To show the main result we find large cliques that intersect in many vertices. We are sure that the method we use in this paper can be generalized to the larger class of amply regular graphs (for a definition, see [6]).

| $(v, k, \lambda, \mu)$ | $\theta_0, [\theta_1]^{m(\theta_1)}, [\theta_2]^{m(\theta_2)}$ | $(v, k, \lambda, \mu)$ | $\theta_0, [\theta_1]^{m(\theta_1)}, [\theta_2]^{m(\theta_2)}$ |
|------------------------|------------------------------------------------|------------------------|------------------------------------------------|
| $(288, 105, 52, 30)$   | $105, [25]^{27}, [-3]^{260}$                     | $(476, 133, 60, 28)$   | $133, [35]^{34}, [-3]^{441}$                     |
| $(300, 117, 60, 36)$   | $117, [27]^{26}, [-3]^{273}$                     | $(540, 147, 66, 30)$   | $147, [39]^{35}, [-3]^{504}$                     |
| $(351, 140, 73, 44)$   | $140, [32]^{26}, [-3]^{324}$                     | $(550, 162, 75, 36)$   | $162, [42]^{33}, [-3]^{516}$                     |
| $(375, 102, 45, 21)$   | $102, [27]^{34}, [-3]^{340}$                     | $(575, 112, 45, 16)$   | $112, [32]^{46}, [-3]^{528}$                     |
| $(405, 132, 63, 33)$   | $132, [33]^{30}, [-3]^{374}$                     | $(703, 182, 81, 35)$   | $182, [49]^{37}, [-3]^{665}$                     |
| $(441, 88, 35, 13)$    | $88, [25]^{44}, [-3]^{396}$                     | $(1344, 221, 88, 26)$  | $221, [65]^{56}, [-3]^{1287}$                    |

Table 1: List of putative primitive strongly regular graphs with smallest eigenvalue $-3$ for $n > 276$, unknown whether they exist or not.

This paper is organized as follows: In the next section we give the preliminaries. In Section 3 we give restrictions on graphs with large cliques and smallest eigenvalue at least $-3$. In Section 4 we find large cliques in a putative strongly regular graph with parameters $(1911, 270, 105, 27)$ and apply the restrictions given in Section 3 to show the main result.

# 2 Preliminaries

## 2.1 Graphs

In this paper all the graphs are finite, undirected and simple. For definitions, we do not define, see [2]. The eigenvalues of a graph are the eigenvalues of its adjacency matrix $A(G)$ indexed by $V(G)$, such that $A_{xy} = 1$ if $xy \in E(G)$ and 0 otherwise. The smallest eigenvalue of a graph is denoted by $\lambda_{\text{min}}(G)$.

Let $G = (V(G), E(G))$ be a graph. The valency $k_x$ of a vertex $x$ of $G$ is the number of neighbours of $x$, i.e. the vertices $y \in V(G)$ such that $xy \in E(G)$. A graph $G$ is $k$-regular if $k_x = k$ for all vertices $x \in V(G)$. A graph $G$ is strongly regular with parameters $(n, k, \lambda, \mu)$ if $G$ has $n$ vertices, is $k$-regular and any two distinct vertices have exactly $\lambda$ (resp. $\mu$) common neighbours if they are adjacent (resp. non-adjacent). The strongly regular graph $G$ is called primitive if $G$ and its complement are both connected.

## 2.2 Interlacing

If $M$ (resp. $N$) is a real symmetric $m \times m$ (resp. $n \times n$) matrix with $\theta_1(M) \geq \theta_2(M) \geq \cdots \geq \theta_m(M)$ (resp. $\theta_1(N) \geq \theta_2(N) \geq \cdots \geq \theta_n(N)$) the eigenvalues of $M$ (resp. $N$) in nonincreasing
order. Assume $m \leq n$. Then we say that the eigenvalues of $M$ interlace the eigenvalues of $N$, if $	heta_{n-m+i}(N) \leq \theta_i(M) \leq \theta_i(N)$ for $i = 1, \ldots, m$.

The following result is a special case of interlacing.

**Lemma 2.1** (cf. [5, Theorem 9.1.1]). Let $B$ be a real symmetric $n \times n$ matrix and $C$ be a principal submatrix of $B$ of order $m$, where $m < n$. Then the eigenvalues of $C$ interlace the eigenvalues of $B$.

As an easy consequence of Lemma 2.1, we have the following proposition.

**Proposition 2.2.** Let $G$ be a graph and $H$ a proper induced subgraph of $G$. Denote by $\theta_{\text{min}}(G)$ (resp. $\theta_{\text{min}}(H)$) the smallest eigenvalue of $G$ (resp. $H$). Then $\theta_{\text{min}}(G) \leq \theta_{\text{min}}(H)$.

Let $G = (V, E)$ be a graph and $\pi := \{V_1, \ldots, V_r\}$ be a partition of $V$. We say $\pi$ is an equitable partition with respect to $G$ if the number of neighbours in $V_j$ of a vertex $u$ in $V_i$ is a constant $q_{ij}$, independent of $u$, only dependent on $i$ and $j$. For an equitable partition $\pi$ with respect to $G$, the quotient matrix $Q$ of $G$ with respect to $\pi$ is defined as $Q = (q_{ij})_{1 \leq i, j \leq r}$.

**Lemma 2.3** (cf. [5, Theorem 9.3.3]). Let $G$ be a graph. If $\pi$ is an equitable partition of $G$ and $Q$ is the quotient matrix with respect to $\pi$ of $G$, then every eigenvalue of $Q$ is an eigenvalue of $G$.

### 2.3 Terwilliger graphs

A Terwilliger graph is a non-complete graph such that, for any two vertices $x$ and $y$ at distance 2, the subgraph induced by common neighbours of $x$ and $y$ forms a clique of order $c$ (for some fixed $c \geq 0$).

**Lemma 2.4** (cf. [1, Corollary 1.16.6 (ii)]). There is no strongly regular Terwilliger graph with parameters $(n, k, \lambda, \mu)$ for $k < 50(\mu - 1)$.

**Lemma 2.5.** If a strongly regular graph with parameters $(1911, 270, 105, 27)$ exist, then it contains an induced quadrangle.

**Proof.** Suppose there exist a strongly regular graph $G$ with parameters $(1911, 270, 105, 27)$ which does not contain an induced quadrangles. Then $G$ is a Terwilliger graph. As $\mu = 27$ then, by Lemma 2.4, the valency of $G$ is at least 1300, which is a contradiction as $k = 270$. This shows the lemma.

### 2.4 Join of graphs

Let $G_1$ and $G_2$ be two graphs such that $V(G_1) \cap V(G_2) = \emptyset$. The join of $G_1$ and $G_2$, denoted by $G_1 \nabla G_2$, has as vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{\{x_1, x_2\} \mid x_1 \in V(G_1), x_2 \in V(G_2)\}$. The following lemma is a consequence of Section 2.3.1 of [2].
Lemma 2.6. Let $G_i$ be a $k_i$-regular graph with $n_i$ vertices, for $i=1,2$, such that $V(G_1) \cap V(G_2) = \emptyset$. Then the smallest eigenvalue $\lambda_{\min}(G_1 \∇ G_2)$ of the join $G_1 \∇ G_2$ satisfies

$$\lambda_{\min} = \min \{ \lambda_{\min}(G_1), \lambda_{\min}(G_2), \lambda_{\min}(Q) \}$$

where

$$Q = \begin{pmatrix} k_1 & n_2 \\ n_1 & k_2 \end{pmatrix}.$$ 

The following lemma was inspired by Cao, Koolen, Munemasa, Yoshino [3].

Lemma 2.7. Let $G$ be a $k$-regular graph on $n$ vertices with smallest eigenvalue $\lambda_{\min}(G) \leq -1$. Consider $K_t \∇ G$ for some positive integer $t$. Then $\lambda_{\min}(K_t \∇ G) = \lambda_{\min}(G)$ if and only if $(\lambda_{\min}(G) - k)(\lambda_{\min}(G) + 1 - t) \geq nt$.

Proof. As $\lambda_{\min}(K_t) \geq -1$, by Lemma 2.6 we find $\lambda_{\min}(G) = \lambda_{\min}(K_t \∇ G)$ if and only if

$$\lambda_{\min} \left( \begin{array}{cc} t - 1 & n \\ t & k \end{array} \right) \geq \lambda_{\min}(G)$$

if and only if

$$\det \begin{pmatrix} t - 1 - \lambda_{\min}(G) & n \\ t & k - \lambda_{\min}(G) \end{pmatrix} \geq 0,$$

as $t \geq 1$.

This show the lemma. \qed

3 Large cliques

Let $G$ be a strongly regular graph with parameters $(n,k,\lambda,\mu)$ and smallest eigenvalue $-m$. Let $C$ be a clique of $G$ of order $c$. Then

$$c \leq 1 + \frac{k}{m}. \quad (1)$$

The inequality (1) is called the Delsarte bound. Moreover, if $c = 1 + \frac{k}{m}$, then $C$ is called a Delsarte clique.

Let $H(a,t)$ be the graph with $1+a+t$ vertices, consisting of a complete graph $K_{a+t}$ and a vertex adjacent to exactly $a$ vertices of $K_{a+t}$.

In [6], Greaves, Koolen and Park obtained the following lemma.

Lemma 3.1. Let $G$ be a graph with smallest eigenvalue $\lambda = \lambda_{\min}(G)$. Assume that $G$ contains an induced $H(a,t)$. Then we have

$$(a - \lambda(\lambda + 1))(t - (\lambda + 1)^2) \leq \lambda(\lambda + 1)^2. \quad (2)$$
In this paper we need the following consequence of Lemma 3.1.

**Lemma 3.2.** Let $G$ be a graph with smallest eigenvalue at least $-3$. Let $C$ be a complete subgraph of $G$ with order $c$. Let $x$ be a vertex of $G$ not in $C$ with exactly $t$ neighbours in $C$. Then $t \leq t_{\text{min}}$ or $t \geq t_{\text{max}}$ where $t_{\text{min}}$ and $t_{\text{max}}$ are as in the Table 2.

| $c$ | $t_{\text{min}}$ | $t_{\max}$ | $c$ | $t_{\text{min}}$ | $t_{\max}$ |
|-----|-----------------|-------------|-----|-----------------|-------------|
| 29  | 8               | 23          | 31  | 7               | 26          |
| 30  | 8               | 24          | 32  | 7               | 27          |

Table 2: Values of $t_{\text{min}}$ and $t_{\text{max}}$

Using Lemma 3.1 Greaves et al. [6] derived a method restricting the order of maximal cliques in a strongly regular graph.

**Lemma 3.3** (cf. [6, Lemma 3.7]). Let $G$ be a strongly regular graph with parameters $(v, k, \lambda, \mu)$ having smallest eigenvalue $-m$. Let $C$ be a maximal clique of $G$ of order $c$. If $\mu > m(m-1)$ and $c > \frac{\mu^2}{\mu - m(m-1)} - m + 1$, then

$$(c+m-3)(k-c+1)-2(c-1)(\lambda-c+2)^2-(k-c+1)^2(c+m-1)(c-(m-1)(4m-1)) \geq 0. \hspace{1cm} (3)$$

We denote the polynomial on the left hand side of the inequality (3) by $M_G(c)$.

**Lemma 3.4.** If a strongly regular graph $G$ with parameters $(1911, 270, 105, 27)$ exists, then any clique in $G$ has order at most 32.

**Proof.** Let $G$ be a strongly regular graph with parameters $(1911, 270, 105, 27)$. Then, it has smallest eigenvalue $-3$. Let $C$ be a maximal clique in $G$ of order $c$. If $c > \frac{\mu^2}{27^2} - 3 + 1 = 32\frac{5}{7}$, then by Lemma 3.4, we have

$$M_G(c) = 672e^3 - 80784e^2 + 1468512e + 3277200 \geq 0$$

as $\mu = 27 > 6$. It is easily checked that $M_G(26) < 0$ and $M_G(97) < 0$. This means that $c \geq 98$. This gives a contradiction, as the Delsarte bound gives $c \leq 1 + \frac{k}{m} = 1 + \frac{270}{105} = 91$. So we obtain that any clique has order at most 32.

**Lemma 3.5.** Let $G$ be strongly regular graph with parameters $(1911, 270, 105, 27)$. Assume there are two cliques $C_1$ and $C_2$ such that $V(C_1) \neq V(C_2)$, each of order at least 29, intersecting in at least 22 vertices. Then one of the following holds:

1. There is at least one vertex $z$ in the symmetric difference $V(C_1) \Delta V(C_2)$ which is adjacent to all vertices in $V(C_1) \Delta V(C_2) \setminus \{z\}$,
(2) \( C_1 \) and \( C_2 \) intersect in exactly 27 vertices and both are maximal with order 29.

**Proof.** Let \( H \) be the induced subgraph on \( V(C_1) \cap V(C_2) \). If \( H \) is complete, then we are in Case (1). So we may assume \( H \) is not complete. This means that \( |V(C_1) \cap V(C_2)| =: t \in \{22, 23, \ldots, 27\} \) as \( \mu = 27 \).

Assume \( t = 22 \). Let \( C'_1 \) (resp. \( C'_2 \)) be a sub clique of \( C_1 \) (resp. \( C_2 \)) such that \( |V(C'_1) \cap V(C'_2)| \geq |V(C'_1) \cap V(C_2)| \) and \( |V(C'_1)| = |V(C'_2)| = 29 \). Let \( K \) be the induced subgraph on \( V(C'_1) \cup V(C'_2) \). By Proposition 2.1 we see that \( K \) has smallest eigenvalue at least \(-3\). Let \( \pi = \{V(C'_1) \cap V(C'_2), V(C'_1) \cap V(C_2)\} \) of \( V(C'_1) \cup V(C'_2) \) be a partition of \( K \) with quotient matrix

\[
Q = \begin{pmatrix}
21 & 14 \\
22 & \alpha + 6
\end{pmatrix}.
\]

By Lemma 2.3, we see that the smallest eigenvalue of \( Q \) is at least \(-3\). This implies that \( 24\alpha \geq 92 \), as \( \det(Q + 3I) \geq 0 \). So, \( \alpha \geq \frac{23}{6} \). This means that there are at least \( \lceil \frac{7 \times 23}{6} \rceil = 27 \) edges between \( V(C'_1) \setminus V(C'_2) \) and \( V(C'_2) \setminus V(C'_1) \). If Case (1) of the lemma does not happen, then all vertices \( V(C'_1) \setminus V(C'_2) \) have at most 5 neighbours in \( V(C'_2) \setminus V(C'_1) \), as \( \mu = 27 \).

We need to consider two cases. There exist a vertex \( x \in V(C'_1) \setminus V(C'_2) \) such that \( x \) has 5 neighbours in \( V(C'_2) \setminus V(C'_1) \) or there is no vertex \( x \in V(C'_1) \setminus V(C'_2) \) with 5 neighbours in \( V(C'_2) \setminus V(C'_1) \).

In the first case, let \( y_1, y_2, \ldots, y_5 \) be the 5 neighbours of \( x \) in \( V(C'_2) \setminus V(C'_1) \). Then \( y_1, y_2, \ldots, y_5 \) have each at most 5 neighbours in \( V(C'_1) \setminus V(C'_2) \) and \( V(C'_2) \setminus V(C'_1) \), such that \( x \in \{V(C'_1) \setminus V(C'_2)\} \setminus \{x\} \) and \( u \in \{V(C'_2) \setminus V(C'_1)\} \setminus \{y_1, y_2, \ldots, y_5\} \). Then \( u \) and \( x \) are at distance two and they have at least 28 common neighbours, a contradiction. Now assume that all vertices of \( V(C'_1) \setminus V(C'_2) \) (resp. \( V(C'_2) \setminus V(C'_1) \)) have at most 4 neighbours in \( V(C'_2) \setminus V(C'_1) \) (resp. \( V(C'_1) \setminus V(C'_2) \)). As there are at least 27 edges between \( V(C'_1) \setminus V(C'_2) \) and \( V(C'_2) \setminus V(C'_1) \), then there exist \( x \in V(C'_1) \setminus V(C'_2) \) and \( y \in V(C'_2) \setminus V(C'_1) \) such that \( x \) and \( y \) are at distance two and they have at least 8 + 22 = 30 common neighbours, a contradiction. This shows that, if \( t = 22 \), then we are in Case (1) of the lemma.

In similar fashion, it can be shown that, if \( t \in \{23, 24, \ldots, 26\} \), then we are in Case (1) of the lemma.

Now assume \( t = 27 \). If \( |V(C_1)| \geq 30 \) and \( |V(C_2)| \geq 29 \), then the quotient matrix \( Q \) of \( \pi = \{V(C_1) \cap V(C_2), V(C_1) \setminus V(C_2), V(C_2) \setminus V(C_1)\} \) satisfies

\[
\begin{pmatrix}
26 & t_1 & t_2 \\
27 & t_1 - 1 & 0 \\
27 & 0 & t_2 - 1
\end{pmatrix},
\]

where \( t_1 + 27 = |V(C_1)| \) and \( t_2 + 27 = |V(C_2)| \), or we are in Case (1) of the lemma.
As the smallest eigenvalue of \( Q \) is at least \(-3\) we obtain that
\[
29(t_1 + 2)(t_2 + 2) - 27(t_1(t_2 + 2) + t_2(t_1 + 2)) \geq 0
\]
This means
\[
-25t_1t_2 + 4(t_1 + t_2) + 116 \geq 0,
\]
and hence
\[
25(t_1 - \frac{4}{25})(t_2 - \frac{4}{25}) < 117
\]
But, as \( t_1 \geq 3 \) and \( t_2 \geq 2 \) we have \( 25(t_1 - \frac{4}{25})(t_2 - \frac{4}{25}) > 130 \), a contradiction. This shows the lemma.

\[\square\]

4 On the local graph of \( G \)

For a graph \( G \) and \( x \in V(G) \) let \( \Delta(x) \) be the induced subgraph on \( \{y \in V(G) \mid x \sim y\} \). The graph \( \Delta(x) \) is called the local graph of \( G \) with respect to \( x \).

**Lemma 4.1** (cf. [4, 7]). Let \( G \) be a primitive strongly regular graph with parameters \((v, k, \lambda, \mu)\). Let \( x \) be a vertex of \( G \) and \( \Delta(x) \) be the local graph of \( G \) with respect to \( x \). Let \( \bar{C} = \{y_1, y_2, \ldots, y_{\bar{c}}\} \) be an independent set of \( \Delta(x) \) of order \( \bar{c} \). Then
\[
\binom{C - 1}{2}(\mu - 1) \geq \bar{c}(\lambda + 1) - k
\]
holds.

For distinct non-adjacent vertices \( w_1, w_2 \in \Delta(x) \) define \( C(w_1, w_2) := \{z \sim x \mid z \sim w_1, z \sim w_2\} \) and \( c(w_1, w_2) := \) number of elements of \( C(w_1, w_2) \).

**Lemma 4.2.** Assume a strongly regular graph \( G \) with parameters \((1911, 270, 105, 27)\) exists such that \( G \) has an induced quadrangle, say \( x \sim u \sim y \sim v \sim x \). Then there is no independent set \( S \) of order 5 inside \( \Delta(x) \) such that \( u, v \in S \).

**Proof.** Assume that \( \Delta(x) \) contains an independent set \( S = \{u_1, u_2, \ldots, u_5\} \) of order 5. Then by Lemma 4.1 we have \( 260 = \binom{5}{2}26 = 5 \times 106 - 270 = 260 \). So we have equality in (4). This means that \( c(u_i, u_j) = 26 \) in \( \Delta(x) \) for all \( 1 \leq i < j \leq 5 \). As \( C(u, v) \leq 25 \) in \( \Delta(x) \), we obtain that \( u \) and \( v \) are not both elements in an independent set \( S \) of order 5 in \( \Delta(x) \). This shows the lemma \(\square\)

**Lemma 4.3.** Assume a strongly regular graph \( G \) with parameters \((1911, 270, 105, 27)\) exists such that \( G \) has an induced quadrangle, say \( x \sim u \sim y \sim v \sim x \). Let \( U = \{u, v, w_1, w_2\} \) be an independent set of \( \Delta(x) \). Let \( A_i = \{a_{2i-1}, a_{2i}\} \) for \( i = 1, 2, \ldots, 6 \) such that \( A_i \in \binom{U}{2} \), \( A_i \neq A_j \) if \( 1 \leq i < j \leq 6 \) and \( A_1 = \{u, v\} \). Then \( c(u, v) \in \{24, 25\} \) and \( \sum_{\{u_1, u_2\} \in \binom{U}{2}} c(u_1, u_2) \in \{154, 155\} \).

Then one of the following hold:
(1) \(c(u, v) \in \{24, 25\} \text{ and } \sum_{\{u_1, u_2\} \in \binom{U}{2}} c(u_1, u_2) = 154. \) Then, without loss of generality, 
\((c(a_1, a_2), c(a_3, a_4), \ldots, c(a_{11}, a_{12})) \in \{(24, 26, 26, \ldots, 26), (25, 25, 26, \ldots, 26)\}.

Moreover, any vertex \(w\) of \(\Delta(x)\) has at most 2 neighbours in \(U\).

(2) \(c(u, v) = 25 \text{ and } \sum_{\{u_1, u_2\} \in \binom{U}{2}} c(u_1, u_2) = 155. \) Then 
\((c(a_1, a_2), c(a_3, a_4), \ldots, c(a_{11}, a_{12})) = (25, 26, 26, \ldots, 26).

In this case there is a unique vertex \(z\) of \(\Delta(x)\) with exactly 3 neighbours in \(U\) and any other vertex \(w\) of \(\Delta(x)\) has at most 2 neighbours in \(U\).

**Proof.** As \(k = 270\) and \(\lambda = 105\), we have \(4(\lambda + 1) - k = 154\). This means that the number of vertices \(w \in \Delta(x)\) such that \(w\) is adjacent to at least two vertices in \(U\) is 154, by Lemma 4.1. This implies
\[
\sum_{\{u_1, u_2\} \in \binom{U}{2}} c(u_1, u_2) \geq 154.
\]
As \(c(u, v) \leq 25\) and \(c(u_1, u_2) \leq 26\) for all \(\{u_1, u_2\} \in \binom{U}{2}\) we find that
\[
154 \leq \sum_{\{u_1, u_2\} \in \binom{U}{2}} c(u_1, u_2) \leq 155.
\]
We also find that \(c(u, v) \geq 154 - 5 \times 26 = 24\).

If \(c(u, v) = 24\), then \(c(a_{2i-1}, a_{2i}) = 26\) for \(i = 2, 3, \ldots, 6\) and any vertex \(w\) in \(\Delta(x)\) has at most two neighbours in \(U\).

If \(c(u, v) = 25\) and \(\sum_{\{u_1, u_2\} \in \binom{U}{2}} c(u_1, u_2) = 154\), then there exists at most one \(i\) in \(\{2, 3, \ldots, 6\}\) such that \(c(a_{2i-1}, a_{2i}) = 25\) and for the other \(i\)'s in \(\{2, 3, \ldots, 6\}\) we have \(c(a_{2i-1}, a_{2i}) = 26\).

If \(c(u, v) = 25\) and \(\sum_{\{u_1, u_2\} \in \binom{U}{2}} c(u_1, u_2) = 155\), then \(c(a_{2i-1}, a_{2i}) = 26\) for all \(i = 2, 3, \ldots, 6\) and there exists a unique vertex \(z\) of \(\Delta(x)\) such that \(z\) has exactly 3 neighbours in \(U\). Any other vertex \(w\) of \(\Delta(x)\) has at most 2 neighbours in \(U\). This shows the lemma. 

As a consequence of Lemma 4.3 we have the following.

**Lemma 4.4.** Assume a strongly regular graph \(G\) exists with parameters \((1911, 270, 105, 27)\) such that \(G\) has an induced quadrangle, say \(x \sim u \sim y \sim v \sim x\). Let \(W := \{w \sim x \mid w \neq u, w \neq v, w \neq u, w \neq v\}\) and let \(Z := \{z \in W \mid \text{there exists a vertex } z' \in W \setminus \{z\} \text{ such that } z' \not\sim z\}\). Let \(\Gamma_W\) be the subgraph induced on \(W\). For \(w \in W\), let \(k_w\) be the valency of \(w\) in \(\Gamma_W\). For \(z \in Z\), let \(K_z\) (resp. \(K_z\)) be the subgraph induced on \(\{w \in W \mid w \neq z, w \not\sim z\}\) (resp. \(\{w \in W \mid w \neq z, w \not\sim z\} \cup \{x\}\)).

Then the following hold:

1. The graph \(K_z\) and \(K_z\) are complete and \(K_z\) has at least 28 vertices,
Proof. (1): If \( K_z \) is not complete, then there would be an independent set \( U \) of order 5 containing \( u \) and \( v \), a contradiction with Lemma 4.1. As \( K_z \) is a subgraph of \( \Delta(x) \), it is clear that \( \tilde{K}_z \) is complete as well. We will show later that \( K_z \) has at least 28 vertices.

(2): We have \( |W| = k - 2(\lambda + 1) + c(u,v) \). As \( k = 270, \lambda = 105 \) and \( c(u,v) \in \{24,25\} \), we find \( |W| \in \{82,83\} \). This shows (2).

(3): Let \( w \in W \). If \( w \not\in Z \), then \( k_w = |W| - 1 \). So, let \( w \in Z \) and \( w' \in Z \setminus \{z\} \) such that \( w \not\sim w' \). We have \( c(u,v) \in \{24,25\} \).

If \( c(u,v) = 24 \), then \( |W| = 82 \), and \( c(a_1,a_2) = 26 \) for all \( \{a_1,a_2\} \in \binom{\{u,v,w,w'\}}{2} \) and \( \{a_1,a_2\} \neq \{u,v\} \), by Lemma 4.3(1). In particular, we have \( c(w,w') = 26 \) and \( k_w = k_{w'} = 105 - 2 \times 26 = 53 \). In this case, \( K_w \) has 82 - 53 - 1 = 28 vertices.

If \( c(u,v) = 25 \), then \( |W| = 83 \). We have \( c(u,w),c(v,w) \in \{25,26\} \). If one of \( c(u,w) \) or \( c(v,w) \) is equal to 25, then \( k_w = 105 - c(u,w) - c(v,w) = 105 - 25 - 26 = 54 \), by Lemma 4.3(1). If \( c(u,w) = c(v,w) = 26 \), then \( 54 = 105 - c(u,w) - c(v,w) + 1 \geq 105 - c(u,w) - c(v,w) = 53 \), by Lemma 4.4(2). This implies \( k_w \) has at least 83 - 54 - 1 = 28 vertices. This shows the lemma. \( \square \)

Lemma 4.5. Assume a strongly regular graph \( G \) with parameters \( (1911,270,105,27) \) exists containing an induced quadrangle, say \( x \sim u \sim y \sim v \sim x \). Let \( W := \{w \sim x \mid w \neq u, w \neq v, w \not\sim u, w \not\sim v\} \). Then \( |W| \neq 83 \).

Proof. Let \( \Gamma_W \) be the subgraph of \( G \) induced on \( W \). In Lemma 4.4(3), we have seen that the valency \( k_w \) in \( \Gamma_W \) of a vertex \( w \in W \) satisfies \( k_w \in \{82,54,53\} \).

Let \( Y := \{w \in W \mid k_w \in \{54,82\}\} \cup \{x\} \).

Claim 4.6. The induced subgraph of \( \Gamma_W \) on \( Y \) is complete.

Proof of Claim 4.6. Clearly \( x \) is adjacent to all the other vertices in \( Y \). Let \( w \in W \) such that \( k_w \in \{82,54\} \). If \( k_w = 82 \), then \( w \) is adjacent to all other vertices of \( \Gamma_W \). So, we only need to show that, if \( w \) and \( w' \) are distinct vertices in \( W \) such that \( k_w = k_{w'} = 54 \), then \( w \sim w' \).

Let \( w \) be such that \( k_w = 54 \). Let \( w' \in W \setminus \{w\} \) be such that \( w \not\sim w' \). There are, without loss of generality, two cases namely, \( c(u,w) = 25 \) and \( c(v,w) = 26 \) or, \( c(u,w) = c(v,w) = 26 \). In the first case \( c(u,w') = 26 = c(w',v) \) and \( k_{w'} = 105 - c(u,w') - (v,w') = 53 \), by Lemma 4.3(1). In the second case there exists a vertex \( z \sim x \) such that \( u \sim z \sim v \) and \( z \sim w \). This means that \( c(u,w') = 26 = c(v,w') \) and \( k_{w'} = 105 - c(u,w') - c(v,w') = 53 \), by Lemma 4.3(2). \( \square \)
By Lemma 3.4, a clique of $G$ has order at most 32, so $|V| \leq 32$. This means that $|\{w \in W \mid k_w = 53\}| \geq 83 - 31 = 52$. So there are two distinct non-adjacent vertices $z$ and $z'$ in $W$ with $k_z = k_{z'} = 53$. We have $c(z, z') \in \{25, 26\}$. There is no vertex $\tilde{z}$ that is adjacent to $u, v$ and one of $z$ and $z'$. This means that, if $c(z, z') = 26$, then there exists a vertex $\tilde{z}$ adjacent to $z, z'$ and one of $u$ and $v$. This means $|C(z, z') \cap W| = 25$ holds whether $c(z, z') = 25$ or $c(z, z') = 26$. This means that the cliques $K_z$ and $K_{z'}$ both have exactly 29 vertices, where $K_z$ is defined as in Lemma 4.4.

Assume that there are two distinct vertices in $C(z, z')$ that are adjacent to all vertices in $K_z$. Then they have at least 28 common neighbours and hence are adjacent. This implies there are at most 2 vertices in $C(z, z')$ such that they are adjacent to all vertices in $\tilde{K}_z$ and at most 2 vertices in $C(z, z')$ such that they are adjacent to all vertices in $\tilde{K}_{z'}$, where $K_z$ and $\tilde{K}_z$ are as defined in Lemma 4.4. So there exists a vertex $w \in C(z, z')$ that is not adjacent to all vertices of $\tilde{K}_z$ and not adjacent to all vertices of $\tilde{K}_{z'}$. We find $k_w \in \{53, 54\}$. Either $w$ is adjacent to all other vertices in $C(z, z')$ or there exist $w' \in C(z, z') \setminus \{w\}$ such that $w \not\sim w'$.

**Claim 4.7.** $|G(w) \cap G(x) \cap G(z) \cap G(z')| \geq 22$.

**Proof of Claim 4.7.** We may assume that there exists a vertex $w' \in C(z, z') \setminus \{w\}$ such that $w \not\sim w'$, as otherwise we are done. We have $|G(x) \cap G(z) \cap G(z')| = c(z, z') = 25$, $|G(z) \cap G(z')| = 27$ and $|G(w) \cap G(z) \cap G(z')| \geq 24$. The last statement follows from Lemma 4.3, as $w \sim z \sim w' \sim z' \sim w$ is an induced quadrangle of $G$. Now $|G(w) \cap G(x) \cap G(z) \cap G(z')| \geq 25 + 24 - 27 = 22$. This shows Claim 4.7.

Claim 4.7 implies that $w$ has at least 21 neighbours in $W \cap C(z, z')$. So, without loss of generality, the vertex $w$ has at least $\frac{53 \times 25}{2} = 14$ neighbours in $K_z$ and hence at least 15 vertices in $\tilde{K}_z$. By Table 2, we find that $w$ has at least 24 neighbours in $\tilde{K}_z$ and hence at least 23 neighbours in $K_z$. This means that $w$ has at most $54 - 21 - 23 = 10$ neighbours in $K_{z'}$, by Claim 4.7. By Table 2, we find that $w$ has at most 8 neighbours in $\tilde{K}_{z'}$.

Now we consider $K_w$. Then $K_w$ has at least 28 vertices, and $\tilde{K}_w$ and $\tilde{K}_{z'}$ intersect in at least $30 - 8 = 22$ vertices.

Now consider a maximal clique $C_1$ of $G$ containing $\tilde{K}_w$ and a maximal clique $C_2$ of $G$ containing $\tilde{K}_{z'}$. We have $w \sim z \in V(K_{z'}).$ This means that $C_2$ does not contain $C_1$. Also $K_w \cap K_z \neq \emptyset$, so $C_1$ does not contain $C_2$. This means that $C_1 \neq C_2$. As $|V(C_1)| \geq 29$ and $|V(C_2)| \geq 30$, by Lemma 3.5, we find that $C_1$ or $C_2$ is not maximal, a contradiction. This shows the lemma.

**Lemma 4.8.** Assume a strongly regular graph $G$ with parameters $(1911, 270, 105, 27)$ exists containing an induced quadrangle, say $x \sim u \sim y \sim v \sim x$. Let $W := \{w \sim x \mid w \neq u, w \neq v, w \neq u, w \neq v\}$. Then $|W| \neq 82$. 

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Proof. Let $\Gamma_W$ be the subgraph of $G$ induced on $W$. In Lemma 4.4(3), we have seen that the valency $k_w$ in $\Gamma_W$ of a vertex $w \in W$ satisfies $k_w \in \{82, 53\}$. There are two distinct vertices $z, z' \in W$ such that $z \not\sim z'$ and hence $k_z = k_{z'} = 53$, as $\Gamma_W$ is not complete.

If there exists a vertex $w' \in W$ such that $k_{w'} = 81$, then consider the induced subgraphs $\hat{K}_z$ (resp. $\hat{K}_{z'}$) on $\{w \in W \mid w \neq z, w \not\sim z\} \cup \{w'\} \cup \{x\}$ (resp. $\{w \in W \mid w \neq z', w \not\sim z'\} \cup \{w'\} \cup \{x\}$). As $u$ and $v$ does not lie in an independent set of order 5 inside $\Delta(x)$, we see that $\hat{K}_z$ and $\hat{K}_{z'}$ are complete. Now the proof follows the proof of Lemma 4.5 in this case, as $|V(\hat{K}_z)| = 30 = |V(K)_{z'}|$. We leave the details for the reader.

So we may assume that all vertices of $\Gamma_W$ have valency 53 and any two distinct non-adjacent vertices in $\Gamma_W$ have exactly 26 common neighbours. Note that, by Lemma 3.4, any clique $C$ in $\Gamma_W$ has at most 31 vertices, as the induced subgraph of $G$ on $\{x\} \cup V(C)$ is complete.

Consider the join $K_4 \nabla \Gamma_W$. Then $\lambda_{\min}(K_4 \nabla \Gamma_W) \geq -3$ by Lemma 2.7, as $\lambda_{\min}(\Gamma_W) \geq \lambda_{\min}(G) = -3$.

Let $w$ be a vertex of $\Gamma_W$. Let $K_w$ be the subgraph on $\{z \in W \mid z \neq w, z \not\sim w\}$. As before $K_w$ is complete and has 28 vertices. Then, in $K_4 \nabla \Gamma_W$, we consider the clique $K_4 \nabla K_w$. By Table 2, we find that any vertex $z$ of $K_4 \nabla \Gamma_W$ outside $K_4 \nabla K_w$ has at most 7 neighbours or at least 27 neighbours in $K_4 \nabla K_w$. This means that $z$ has at most 3 neighbours or at least 23 neighbours in $K_w$.

Let $w, w'$ be two distinct non-adjacent vertices in $\Gamma_W$. There are at most 3 vertices in $C(w, w')$ that are adjacent to all vertices in $K_w$, and similarly there are at most 3 vertices in $C(w, w')$ that are adjacent to all vertices in $k_{w'}$, as $|V(K_w)| = 28 = |V(K_{w'})|$. So there exist a vertex $z \in C(w, w')$ that is not adjacent to some vertex $p$ (resp. $p'$) in $K_w$ (resp. $K_{w'}$). As $z$ has at least 22 neighbours in $C(w, w')$, as in Claim 4.7 of Lemma 4.5, without loss of generality, we see that $z$ has at most $\lceil \frac{53 - 22}{2} \rceil = 15 < 23$ neighbours in $K_w$. So $z$ has at most 3 neighbours in $K_w$. So the two cliques $K_z$ and $K_{w'}$ intersect in at least 25 vertices.

Now consider the induced subgraphs $\hat{K}_z$ and $\hat{K}_w$ on $\{x\} \cup V(K_z)$ and $\{x\} \cup V(K_w)$ respectively. Then $\hat{K}_z$ and $\hat{K}_w$ both have 29 vertices and intersect in at least 26 vertices. As $z' \in V(K_z) \setminus V(K_w)$ and $p' \in V(K_w) \setminus V(K_z)$ we see that $\hat{K}_z$ and $\hat{K}_w$ must intersect in precisely 27 vertices and both are maximal, by Lemma 3.5. We have $z' \in V(K_z) \setminus V(K_w)$ and $p' \in V(K_w) \setminus V(K_z)$ and, $K_z$ and $K_{w'}$ intersect in exactly 26 vertices.

Now consider $K_{z'}$. The clique $K_{z'}$ and $K_w$ intersects in exactly two vertices under which $p'$. Consider any vertex $q$ of the clique $K_{p'}$. Then

(1) $q$ has 26 neighbours in $K_{z'}$,

(2) $q$ has 26 neighbours in $K_w$, or
(3) $q$ has at most 25 neighbours in $K_{z'}$ and at most 25 neighbours in $K_w$.

There are at most 5 vertices in case (1), and at most 5 vertices in case (2). This means that there exists a vertex $q$ in the clique $K_{p'}$ which has at most 25 neighbours in $K_{z'}$ and at most 25 neighbours in $K_w$. Without loss of generality, we may assume that $q$ has at most $\frac{26}{2} = 13$ neighbours in $K_{z'}$. Then $q$ has at most 3 neighbours in $K_{z'}$ and, $K_q$ and $K_{z'}$ intersect in at least 25 vertices. This implies the induced subgraphs $\tilde{K}_q$ and $\tilde{K}_{z'}$, on $\{x\} \cup V(K_q)$ and $\{x\} \cup V(K_{z'})$ respectively, have both 29 vertices and they intersect in 27 vertices, by Lemma 3.5. But this means that $K_q$ and $K_{z'}$ intersect in exactly 26 vertices and, $K_q$ and $K_{z'}$ intersect in exactly 4 vertices.

We obtain that every vertex in $V(K_q)\setminus V(K_w)$ has at least 23 neighbours in $V(K_w)$. So this means that there are at least $24 \times 21$ edges between $V(K_q)\setminus V(K_w)$ and $V(K_w)\setminus V(K_q)$.

This implies that the number of edges between $V(K_{z'}) \cup V(K_w)$ and $V(K_{p'})$ is at most $2 \times 26 \times 26 - 2 \times 24 \times 21 = 344$. On the other hand every vertex in $K_{p'}$ has valency 53, so the number of edges between $V(K_{z'}) \cup V(K_w)$ and $V(K_{p'})$ is exactly $26 \times 28 = 728 > 344$, a contradiction. This shows the lemma.

Now we give the proof of the main theorem:

**Proof of Theorem 1.1.** By Lemma 2.5, we see that $G$ has an induced quadrangle, say $x \sim u \sim y \sim v \sim x$. Let $W = \{w \sim x \mid w \neq u, w \neq v, w \not\sim u, w \not\sim v\}$. Then, by Lemma 4.4, we have $|W| \in \{82, 83\}$. By Lemmas 4.5 and 4.8, we see that this is not possible. This finishes the proof.

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