Abstract—We prove that the Bethe expression for the conditional input-output entropy of cycle LDPC codes on binary symmetric channels above the MAP threshold is exact in the large block length limit. The analysis relies on methods from statistical physics. The finite size corrections to the Bethe expression are expressed through a polymer expansion which is controlled thanks to expander and counting arguments.

I. INTRODUCTION

A few years ago Cherktov and Chernyak [1] devised a loop series which represents the partition function of a general vertex model as the product of the Bethe mean field expression and a residual partition function over a system of loops. In this representation all quantities are entirely expressible in terms of Belief Propagation (BP) marginals or messages. However it has not been clear so far if this representation leads to a controlled series expansions for the log-partition, in other words the free energy. If this is the case it should hopefully allow to control the difference between the true free energy and the Bethe free energy.

The loop expansion has a potential interest in coding theory since Low-Density-Parity-Check (LDPC) and Low-Density-Generator-Matrix (LDGM) codes on general binary-input memoryless symmetric (BMS) channels fit in the framework of (generalized) vertex models. In this context free energy is just another name for the conditional input-output Shannon entropy. For these models it is believed that the Bethe formula for the conditional entropy/free energy is exact. However there is no general proof, except for the cases of the binary erasure channel [2]. LDGM codes for high noise, and in special situations for LDPC codes at low noise [3].

We consider cycle LDPC codes for high noise (above the MAP threshold) on the binary symmetric channel (BSC). We show that, under the assumption that there exists a fixed point for the BP equations, the average conditional entropy/free energy is given by the Bethe expression. The novelty of the approach is to turn the loop expansion into a rigorous tool allowing to derive provably convergent polymer expansions [4]. Controlling the loop expansion is a non-trivial task because in most situations of interest the number of loops proliferates. For example, this is the case (for the system of fundamental cycles) in capacity approaching codes even under MAP decoding [5].

II. LOOP AND POLYMER REPRESENTATIONS

Let $\Gamma=(V,E)$ be a graph with vertices $a \in V$ of regular degree $d$ and edges $ab \in E$. The symbol $\partial a$ denotes the set of $d$ neighbors of $a$. In vertex models the degrees of freedom are spins $\sigma_{ab} \in \{-1,+1\}$ attached to each edge. At each function node $a \in V$ we attach a non-negative function $f_a(\sigma_{\partial a})$ depending only on neighboring variables $\sigma_{\partial a} \equiv (\sigma_{ab})_{b \in \partial a}$. We study probability distributions which can be factorized as

$$\mu_\Gamma(\vec{\sigma}) = \frac{1}{Z_{\Gamma}} \prod_{a \in V} f_a(\vec{\sigma}_{\partial a}), \quad Z_{\Gamma} = \sum_{\vec{\sigma}} \prod_{a \in V} f_a(\vec{\sigma}_{\partial a}),$$

and their associated free energy $f_\Gamma \equiv \frac{1}{n} \ln Z_{\Gamma}$.

For each edge $ab \in E$ we introduce two directed “messages” $\eta_{a \to b}$ and $\eta_{b \to a}$. For the moment these variables are arbitrary and are collectively denoted by $\vec{\eta}$. One has the identity

$$f_n = \frac{1}{n} \ln Z_{\text{Bethe}}(\vec{\eta}) + \frac{1}{n} \ln Z_{\text{corr}}(\vec{\eta}).$$

The first term is the Bethe free energy functional,

$$\ln Z_{\text{Bethe}}(\vec{\eta}) = \sum_{a \in V} \ln \left( \sum_{\vec{\sigma}_{\partial a}} f_a(\vec{\sigma}_{\partial a}) \prod_{b \in \partial a} e^{\eta_{a \to b} \sigma_{ab}} \right) - \sum_{ab \in E} \ln (2 \cosh(\eta_{a \to b} + \eta_{b \to a})).$$

The “partition function” in the second term can be expressed as a sum over all subgraphs (not necessarily connected) $g \subset \Gamma$

$$Z_{\text{corr}}(\vec{\eta}) = \sum_{g \subset \Gamma} K(g)$$

and $K(g) = \prod_{a \in g} K_a$ with

$$K_a = \sum_{\vec{\sigma}_{\partial a}} f_a(\vec{\sigma}_{\partial a}) \prod_{b \in \partial a \cap g} \sigma_{ab} e^{-\sigma_{ab} (\eta_{a \to b} + \eta_{b \to a})}.$$

It is well known that the stationary points of (3) satisfy the BP fixed point equations,

$$\eta_{a \to c} = \frac{\sum_{\vec{\sigma}_{\partial c}} \sigma_{ac} f_a(\vec{\sigma}_{\partial a}) \prod_{b \in \partial c \setminus \partial a} e^{\eta_{c \to b} \sigma_{ab}}}{\sum_{\vec{\sigma}_{\partial c}} f_a(\vec{\sigma}_{\partial a}) \prod_{b \in \partial c \setminus \partial a} e^{\eta_{c \to b} \sigma_{ab}}}.$$

Remarkably, for any solution of (5), $K(g) = 0$ if $g$ contains a degree one node. Thus if $\vec{\eta}$ is a fixed point of the BP equations then $Z_{\text{corr}}(\vec{\eta})$ is given by the sum in (4) over $g \subset \Gamma$ with no degree one nodes. Such graphs are called loops (see figure 1).

One can recognize that $Z_{\text{corr}}$ can be interpreted as the partition function of a system of polymers. A loop $g \subset \Gamma$ can be decomposed into its connected parts in a unique way as...
illustrated on figure \[\text{Fig. 1. Left: an example of a loop graph } g \text{ with no dangling edge. Right: decomposition of } g \text{ into its connected parts } \gamma_i.\]

each polymer \( \gamma \) we define a weight (also called activity), \( K(\gamma) = \prod_{a \in \gamma} K_a \). Let \( g = \bigcup_{i = 1}^M \gamma_i \). Since the \( \gamma_i \) are disjoint, \( \prod_{a \in g} K_a = \prod_{i = 1}^M K(\gamma_i) \). Thus equation (4) can be cast in the form

\[
Z_{\text{corr}}(\vec{\eta}) = \sum_{M \geq 0} \frac{1}{M!} \prod_{i = 1}^M K(\gamma_i) \prod_{i < j} (1 - I(\gamma_i \cap \gamma_j = \emptyset)),
\]

In this sum each \( \gamma_i \) runs over all connected subgraphs with no dangling edges of the underlying graph \( \Gamma \). The sum over the number of polymers \( M \) has a finite number of terms because the polymers cannot intersect.

In the next paragraphs \( \Gamma \) is a random \( d \)-regular graph. We denote by \( \mathbb{P} \) and \( \mathbb{E} \) the relevant probability and expectation over this ensemble.

III. POLYMER EXPANSION

We wish to compute the correction to the Bethe free energy in (2), namely \( f_{\text{corr}}(\vec{\eta}) \equiv \frac{1}{n} \ln Z_{\text{corr}}(\vec{\eta}) \) when \( \vec{\eta} \) is a BP fixed point. Using (6) the logarithm can be expanded as a power series in \( K(\gamma_i) \)'s. This yields the polymer (or Mayer) expansion

\[
f_{\text{corr}}(\vec{\eta}) = \frac{1}{n} \sum_{M = 1}^\infty \frac{1}{M!} \prod_{i = 1}^M K(\gamma_i) \times \sum_{G \subseteq \overline{G}_M \ (i,j) \in G} \prod_{G \subseteq \overline{G}_M \ (i,j) \in G} \{-1, I(\gamma_i \cap \gamma_j \neq \emptyset)\}.
\]

The third sum is over the set \( \overline{G}_M \) of all connected graphs with \( M \) vertices labeled by \( \gamma_1, \ldots, \gamma_M \), and at most one edge between each pair of vertices. The product of indicator functions is over edges \((i,j) \in G\). It constrains the set of polymers \( \gamma_1, \ldots, \gamma_M \) to intersect according to the structure of \( G \). In this expansion one sums over an infinite number of terms so it is important to address the question of convergence. A criterion which ensures the convergence of the expansion uniformly in system size \( n \) (and thus ensures convergence in the infinite size limit) is

\[
\sum_{t=0}^{+\infty} \frac{1}{t} \sup_{\gamma} \sum_{\gamma} |\gamma|^t |K(\gamma)| < 1
\]

(8)

To illustrate the use of the polymer expansion in a simple case, consider a vertex model at high temperature defined by

\[
f_a(\sigma_{\partial a}) = \frac{1}{2} (1 + \tanh J_a \sum_{\bar{b} \in \partial a} \sigma_{\bar{b}}) e^{h_{\bar{a}} \sigma_{\bar{a}}}
\]

where \( J_a \) and \( h_{\bar{a}} \) are in \( \mathbb{R} \) with \( \sup_{a \in V} J_a = J < 1 \) and \( \sup_{a \in E} h_a < h < +\infty \). For \( J \) small enough the BP (5) equations have a unique fixed point solution [6]. We call \( \vec{\eta}^*_n \) this fixed point. The subscript \( n \) indicates (with some abuse of notation) that this fixed point depends on the finite instance, that is, the graph \( \Gamma \), and \( J_n, h_n \). For the activities of the polymers, computed at the fixed point, we have the bounds \( |K(\gamma)| \leq (2J)^{|\gamma|} \). Moreover the number of polymers \( \gamma \ni a \) is (for each \( a \)) at most \( c_\bar{a} |\gamma| \) with \( c_\bar{a} > 0 \) a numerical constant depending only on \( d \). Using also that the smallest polymer must have \( |\gamma| \geq 3 \), it is then easily shown that the left hand side of (8) is \( O(J^3) < 1 \). By standard methods one can then estimate the sum over \( M \) in (7) term by term, which yields

\[
|f_{\text{corr}}(\vec{\eta}_n^c)| \leq |1 + O(J^3)| \frac{1}{n} \sum_{a \in V} \sum_{\gamma \ni a} (2J)^{|\gamma|} |\gamma|^{-1}
\]

(9)

Proposition 3.1: For \( J < J_0(h) \) small enough, we have \( \lim_{n \to +\infty} \mathbb{E} |f_{\text{corr}}(\vec{\eta}_n^c)| = 0 \).

Proof idea: From (9) \( \mathbb{E} \|f_{\text{corr}}(\vec{\eta}_n^c)\| \leq (1 + O(J^3)) \mathbb{E} |f_{\text{corr}}(\vec{\eta}_n^c)| \). Here \( o \) is any specified node in the graph. In order to conclude it suffices to use the fact that on a random \( d \)-regular graph, with probability \( 1 - c_\bar{a} |1| \), polymers have a size \( |\gamma| \geq a_d \ln n \) \( (a_d \geq 0 \) a positive numerical constant).

IV. CYCLE LDPC CODES OVER THE BSC

Random \( d \)-regular graphs are equivalent to the LDPC(2, 2) ensemble of cycle codes. Code bits \( \sigma_{ab} = 0, 1 \) are attached to the edges \( \partial b \in E \). In the spin language bits are \( \sigma_{ab} = \pm 1 \) and the parity check constraints are \( \prod_{b \in \partial a} \sigma_{ab} = 1 \). For definiteness we assume transmission over the BSC(p) \( p \in [0, \frac{1}{2}] \).

Without loss of generality one can assume that the transmitted word is \( (1, \ldots, 1) \) so that MAP decoding is based on the posterior distribution (10) with

\[
f_a(\sigma_{\partial a}) = \frac{1}{2} (1 + \sum_{\bar{b} \in \partial a} \sigma_{\bar{b}}) \prod_{b \in \partial a} e^{h_{\bar{a}} \sigma_{\bar{a}}}
\]

(10)

Here \( h_{\bar{a}} \) is the half-log-likelihood for the bit \( \sigma_{ab} = \pm 1 \), based on the channel output. The Shannon conditional input-output entropy and free energy are essentially equivalent, and related by the simple formula,

\[
\frac{1}{n} H(X|Y) = \mathbb{E}_{\vec{\eta}}[f_a(\vec{h})] - \frac{1 - 2p}{2} \ln \frac{1 - p}{p}
\]

(11)
where $E_p$ is the average over channel outputs (or the log-likelihood vector).

We interested in the high noise regime where $p$ is close to 1/2. Therefore we seek solutions of the BP equations such that $\sup_{a,b,c} |h_{a,b}| \leq h$ where $h > 0$ is a fixed small number. We assume that for $h$ small enough there exists a fixed point of the BP equations for each finite instance. Denote we denote it $\hat{n}_n$ as before. Note that assuming its unicity is not needed.

**Proposition 4.1:** Assuming the existence of a fixed point $\hat{n}_n$ of the BP equations for $h$ small enough, we have $\lim_{n \to +\infty} \E[\frac{1}{n} \ln Z_{corr}(\hat{n}_n)] = 0$.

In view of (2), (11) the proposition implies that the average conditional entropy is given by the average of the Bethe expression computed at the fixed point (for $|p - \frac{1}{2}| < 1$).

In order to prove the proposition we will use the identity

$$\ln Z_{corr}(\tilde{\eta}) = \ln Z_p(\tilde{\eta}) + \ln \left\{ 1 + \sum_{\gamma \in \Gamma} K(\gamma) \frac{Z_p(\tilde{\eta} | \gamma)}{Z_p(\tilde{\eta})} \right\}$$

(12)

where

$$Z_p(\tilde{\eta} | \gamma) = \sum_{M \geq 0} \frac{1}{M!} \prod_{\gamma_1, \ldots, \gamma_{M}} K(\gamma_i | \gamma = 0) \prod_{i<j} \{ 1 + \{ \gamma_i | \gamma_j \} \} \leq \sum_{M \geq 0} \frac{1}{M!} \prod_{\gamma_1, \ldots, \gamma_{M}} K(\gamma_i | \gamma = 0) \prod_{i<j} \{ 1 + \{ \gamma_i | \gamma_j \} \}$$

(13)

and $Z_p(\tilde{\eta}) \equiv Z_p(\tilde{\eta} | \emptyset)$. This identity is derived by splitting the sum over $\{ \gamma_1, \ldots, \gamma_{M} \}$ in (6), into a sum where all polymers are small (all $|\gamma_i| < n/2$), and a sum where there exists at least one large polymer (all $|\gamma_i| \geq n/2$); and by noting that when there exists a large polymer it has to be unique.

We will need three lemmas.

**Lemma 4.2:** For $h$ small enough we have $\lim_{n \to +\infty} \E[\frac{1}{n} \ln Z_p(\hat{n}_n)] = 0$.

**Sketch of Proof:** It is possible to estimate the activities computed at the fixed point,

$$|K(\gamma)| \leq (1 - \alpha_d h^2) \alpha_d^{d-1} \omega(\gamma) \leq \frac{1}{2} \alpha_d^{d-1} \omega(\gamma)$$

(14)

Here $0 < \alpha_d < 1$, and $\alpha_1 > 0$, $i = 2, \ldots, d - 1$ are fixed numerical constants (that we can take close to 1). The $\omega(\gamma)$ denotes the number of nodes of degree $i$ in the polymer $\gamma$. Estimate (13) is essentially optimal for $h$ small, as can be checked by Taylor expanding $K(\gamma)$ in powers of $h_{a,b}$. Hard constraints manifest themselves in the factor $(1 - \alpha_d h^2)^{\omega(\gamma)}$ which is not small enough to compensate the entropic term $e^{\alpha_d |\gamma|}$ in the convergence criterion. However for polymers of size $|\gamma| < \frac{n}{2}$ we can use argument on pages 10 to 11 to circumvent this problem. Let $e(\gamma)$ the set of edges in $E$ connecting $g$ to $\Gamma \setminus g$. We say that $\Gamma$ is an expander if for all $g \subset \Gamma$ with $|g| < \frac{n}{2}$ we have $e(\gamma) \geq \kappa |g|$. For all $d \geq 3$, we have

$$\Pr[\lfloor \frac{n}{2} \rfloor \geq \alpha_d h^2 \frac{n}{2} \omega(\gamma) + 2d] \leq \frac{1}{2} \alpha_d^{d-1} \omega(\gamma)$$

(15)

This assumption can be relaxed by softening the hard constraint in (9) and using existence results [9]. Indeed all our estimates are uniform in the softening parameter. We omit this discussion here due to lack of space.

Now note that for polymers $c(\gamma) = \sum_{i=2}^{d-1} (d - i) n_i(\gamma) \leq d \sum_{i=2}^{d-1} n_i(\gamma)$. Therefore we deduce thanks to (15) that with high probability $\sum_{i=2}^{d-1} n_i(\gamma) \geq 0.18 |\gamma|$ and $K(\gamma) \leq (2h)^{|\gamma|/2} |\gamma|$. This is sufficient to control the convergence criterion, and achieve the proof of this lemma by methods similarly to the high temperature case.

**Lemma 4.3:** Fix $\epsilon > 0$. Then

$$\Pr[\forall \gamma \subset \Gamma : e^{-\epsilon n_g} \leq \frac{Z_p(\tilde{\eta} | \gamma)}{Z_p(\tilde{\eta})} \leq e^{\epsilon n_g} \geq 1 - \frac{1}{\epsilon} \alpha_n(1)$$

The proof uses rather trivial bounds on the partition functions. We omit the details.

**Lemma 4.4:** Fix $\delta > 0$. There exists a numerical constant $C > 0$ such that for $h$ small enough

$$\Pr[\left| \sum_{g \subset \Gamma \cap K} \frac{Z_p(\tilde{\eta})}{Z_p(\tilde{\eta})} |K(g)| \cap | \hat{\gamma}| \geq \delta \right| \leq \frac{C}{\delta} e^{-n \alpha \hat{\gamma}/2}$$

(16)

This inequality is a fortiori valid for $g$’s replaced by $\gamma$’s in the sum.

**Sketch of Proof:** We denote by $K_n$ the complete graph with $n$ vertices. By Markov’s inequality,

$$\Pr[\left| \sum_{g \subset \Gamma \cap K} \frac{Z_p(\tilde{\eta})}{Z_p(\tilde{\eta})} |K(g)| \cap | \hat{\gamma}| \geq \delta \right| \leq \frac{1}{\delta} \sum_{g \subset \Gamma \cap K} \E[|K(g)| \cap | \hat{\gamma}|]$$

(17)

Consider graphs $g$ with $n_i(\gamma), i = 2, \ldots, d$ fixed. Mackay [9] provides a bound for the probability of finding a particular subgraph into a regular graph $\Gamma$. Namely for

$$\frac{\sum_{i=2}^{d} i \sum_{g \subset \Gamma \cap K} \frac{n_i(\gamma)}{2} \left( 2^d - 1 \right) \alpha_i h^{d-1} i n_i(\gamma)}{2^d - 1} \geq 1$$

(18)

The number of subgraphs of $K_n$ with given $n_i(\gamma)$ is

$$\frac{n!}{(n - \sum_{i=2}^{d} n_i(\gamma))! \prod_{i=2}^{d} n_i(\gamma)!} \alpha_i h^{d-1} i n_i(\gamma)$$

(19)

Replacing (18) in (17), setting $x_i = \frac{n_i(\gamma)}{n}$, and performing an asymptotic calculation for $n$ large, we have (here $\overline{x} \equiv (x_2, \ldots, x_d)$ and $\Delta \equiv \{ \overline{x} \} \overline{\frac{1}{2} \leq x_i \leq 1 \}$$

$$\Pr[\left| \sum_{g \subset \Gamma \cap K} \frac{Z_p(\tilde{\eta})}{Z_p(\tilde{\eta})} |K(g)| \cap | \hat{\gamma}| \geq \delta \right| \leq \frac{1}{\delta} \int_{\Delta} d\overline{x} \exp(n \{ \alpha \} \exp(\{ \alpha \})) \exp(n \{ f_n(\overline{x}) \} + d \overline{x} \ln(1 - \alpha_d h^2) + \sum_{i=2}^{d-1} x_i \ln(\alpha_i h^{d-1} i \overline{x}^{d-1}))$$

(20)
The large $n$ behavior of the integral asymptotic is controlled by $f_{n}(\vec{x})$, and $\eta_{n}(\vec{x})$ gives sub-dominant contributions that do not concern us here. We have

$$f_{n}(\vec{x}) = \frac{1}{2}(\sum_{i=2}^{d} x_{i}) \ln(\frac{1}{2}(\sum_{i=2}^{d} x_{i})) - \frac{d}{2} x_{1} \ln(\frac{x_{1}}{\eta})$$

$$(1 - \sum_{i=2}^{d} x_{i}) \ln(1 - \sum_{i=2}^{d} x_{i}) - (\frac{r}{2} - \frac{2r^{2}}{n}) \ln(\frac{r}{2} - \frac{2r^{2}}{n})$$

$$+ \frac{r}{2} - \frac{2r^{2}}{n} - \frac{1}{2} \sum_{i=2}^{d} x_{i} - \frac{2r^{2}}{n} \ln(\frac{r}{2} - \frac{2r^{2}}{n})$$

(21)

For $h$ small enough, in the domain $\Delta$, the exponent in (20) is strictly negative and attains its maximum at the corner point $x_{2} = \cdots = x_{d-1} = 0$, $x_{d} = 1$. At this point it is equal to $\ln(1 - \alpha_{d}^{2} h^{2})$ which allows to conclude (16).

We are now in a position to prove proposition 4.1.

Proof of proposition 4.1. In view of (12), we must show that for $h$ small enough,

$$\frac{1}{n} \mathbb{E} \left| \ln \left( 1 + \sum_{\gamma \in \Gamma} |K(\gamma)| \frac{Z_{p}(\vec{\eta})}{Z_{p}(\eta)} \right) \right| = o_{n}(1).$$

(22)

Call $I_{\zeta}$ the event

$$\sum_{\gamma \in \Gamma} |K(\gamma)| \frac{Z_{p}(\vec{\eta})}{Z_{p}(\eta)} < \zeta$$

where $\zeta$ is a positive constant that will be adjusted later on. We split the expectation in two terms $A + B$ by conditioning over $I_{\zeta}$ and its complement $I_{\zeta}^{c}$, and estimate each contribution. For the first contribution, using $\left| \ln(1 + x) \right| \leq |\ln(1 - x)|$ for $|x| < 1$, $A \leq \frac{1}{n} \mathbb{E} \left| \ln(1 - |\zeta|) \right| \leq \frac{1}{n} \mathbb{E} \left| \ln(1 - \zeta) \right|$. For the second contribution we have to estimate $\mathbb{P}[I_{\zeta}^{c}]$. The events

$$\{ \forall \gamma \in \Gamma : e^{-2n\varepsilon} \leq |Z_{p}(\vec{\eta})/Z_{p}(\eta)| \leq e^{2n\varepsilon} \} \text{ and } \{ \sum_{\gamma \in \Gamma} |K(\gamma)| \leq \delta \}$$

imply $I_{\delta_{3n\varepsilon}}$. Therefore $I_{3n\varepsilon}$ implies the union of the complementary events, so that applying the union bound together with lemmas 4.3 and 4.4

$$\mathbb{P}[I_{3n\varepsilon}] \leq \frac{C}{\delta} e^{-n\alpha_{d}^{2} h^{2}} + \frac{1}{\epsilon} o_{n}(1).$$

Now suppose for a moment that there exist a positive constant independent of $n$ such that

$$\frac{1}{n} \mathbb{E} \left| \ln \left( 1 + \sum_{\gamma \in \Gamma} K(\gamma) \frac{Z_{p}(\vec{\eta})}{Z_{p}(\eta)} \right) \right| \leq C'$$

(23)

Then $B \leq C'' \mathbb{E}[I_{\zeta}^{c}]$. Setting $\zeta = \delta e^{2n\varepsilon}$, the above arguments imply

$$\mathbb{E} \left| \frac{1}{n} \mathbb{E} \left| \ln \left( 1 + \sum_{\gamma \in \Gamma} K(\gamma) \frac{Z_{p}(\vec{\eta})}{Z_{p}(\eta)} \right) \right| \right| = A + B \leq \frac{1}{n} \mathbb{E} \left| \ln(1 - \delta e^{2n\varepsilon}) + C \frac{e^{-n\alpha_{d}^{2} h^{2}}}{\delta} + \frac{1}{\epsilon} o_{n}(1) \right|.$$

We are free to choose $\delta = e^{-n\alpha_{d}^{2} h^{2}}$ and $c = \alpha_{d}^{2} h^{2}$ (lemmas 4.3, 4.4 hold) and this choice $A + B = o_{n}(1)$, which proves (22).

It remains to justify (23). From the convergence of the polymer expansion we deduce that $\frac{1}{n} \mathbb{E} \ln Z_{p}(\eta_{n})$ is bounded uniformly in $n$. From (1), (10) we easily show that $\frac{1}{n} \mathbb{E} \ln Z_{p}(\eta_{n}) \leq \ln 2 + \frac{4h}{\alpha}$ the high noise regime the BP messages are bounded so that from (3) we deduce that $\frac{1}{n} \mathbb{E} \ln Z_{\text{Bethe}}(\eta_{n})$ is bounded by a constant independent of $n$. Finally the triangle inequality implies that $\frac{1}{n} \mathbb{E} \left| \ln Z_{\Gamma} - \ln Z_{\text{Bethe}}(\eta_{n}) - \ln Z_{p}(\eta_{n}) \right|$ is bounded uniformly in $n$. This is precisely the statement (23).

V. CONCLUSION

The approach is quite general and can hopefully be generalized to standard irregular LDPC codes with bounded degrees and binary-input memoryless output-symmetric channels with bounded log-likelihood variables. This will be the subject of future work.

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