On the complexity of deciding whether the distinguishing chromatic number of a graph is at most two

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Abstract

In an article [3] published recently in this journal, it was shown that when \( k \geq 3 \), the problem of deciding whether the distinguishing chromatic number of a graph is at most \( k \) is NP-hard. We consider the problem when \( k = 2 \). In regards to the issue of solvability in polynomial time, we show that the problem is at least as hard as graph automorphism but no harder than graph isomorphism.

1 Introduction

We consider simple undirected graphs. A nontrivial automorphism of a graph is an automorphism that is not the identity mapping. We use the abbreviation NTA for nontrivial automorphism. A graph that has no nontrivial automorphism is said to be asymmetric. A vertex \( k \)-coloring of graph \( G = (V, E) \) is a mapping \( V \to \{1, 2, \ldots, k\} \). A vertex \( k \)-coloring of graph \( G \) is proper if no two adjacent vertices of \( G \) receive the same color. \( \chi(G) \)

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is the chromatic number of graph $G$, namely, the smallest positive integer $k$ such that $G$ admits a proper vertex $k$-coloring. We use $\leq_m$ and $\equiv_m$ to denote polynomial-time many-one reducibility and equivalence, respectively, and $\leq_T$ for polynomial-time Turing reducibility.

A vertex $k$-coloring of graph $G$ is distinguishing if the only automorphism of $G$ that preserves the coloring is the identity automorphism. The distinguishing number of graph $G$, denoted $D(G)$, is the smallest positive integer $k$ such that $G$ admits a $k$-coloring (not necessarily proper) that is distinguishing. Similarly, the distinguishing chromatic number of graph $G$, denoted $\chi_D(G)$, is the smallest positive integer $k$ such that $G$ admits a proper $k$-coloring that is distinguishing. The concept of distinguishing number of a graph was introduced by Albertson and Collins in [1]. Later, Collins and Trenk [4] introduced the notion of distinguishing chromatic numbers of graphs.

The computational complexities of the problems of computing $D(G)$ and $\chi_D(G)$ have been investigated in the recent past. It was shown in [7] that given a graph $G$ and integer $k$, deciding whether $D(G) \leq k$ belongs to AM, the set of languages for which there exist Arthur and Merlin games. In a more recent paper [2] it has been shown that given a planar graph $G$ and an integer $k$, whether $D(G) \leq k$ can be decided in polynomial time. Cheng [3] has shown that given an interval graph $G$ and an integer $k$, whether $\chi_D(G) \leq k$ can be decided in polynomial time. In contrast to this, Cheng [3] also established that given an arbitrary graph $G$ and an integer $k$, where $k \geq 3$, deciding whether $\chi_D(G) \leq k$ is NP-hard. Further, the problem remains NP-hard when $k = 3$ and the input graph is planar with maximum degree at most five [3]. In regards to the problem of deciding whether $\chi_D(G) \leq 2$, given a graph $G$, Cheng remarked in [3] that “it will be interesting to consider what the corresponding results are” for deciding whether $\chi_D(G) \leq 2$.

We show that given a connected graph $G$, deciding whether $\chi_D(G) \leq 2$ is polynomial-time Turing equivalent to the problem of deciding whether a given graph $H$ has a NTA. Thus, given an arbitrary graph $G$, deciding whether $\chi_D(G) \leq 2$ is at least as hard as deciding whether a graph $H$ has any NTA. We then show that given an arbitrary graph $G$, the problem of deciding whether $\chi_D(G) \leq 2$ is no harder than deciding whether given graphs $G_1$ and $G_2$ are isomorphic to each other.

Next, we introduce the definitions of some needed problems. Then, we present our main results. Finally, we conclude with some discussion.
2 Graph automorphism and graph isomorphism

Consider the following decision problems each of which is known to be in NP, but neither of which is known to be in P or NP-complete. Graph isomorphism has long been considered a candidate to be in NP but neither in P nor NP-complete (such problems are known to exist if \( P \neq NP \) [6]).

**Graph Automorphism (GA)**
Instance: Graph \( G \).
Question: Does \( G \) have a nontrivial automorphism?

**Graph Isomorphism (GI)**
Instance: Graphs \( G_1 \) and \( G_2 \).
Question: Is \( G_1 \cong G_2 \)?

It is known that \( GA \leq_m GI \) [5]; however, as stated in [5], “GI does not seem to be reducible to GA”. Thus, it is possible that GA is easier to compute than GI.

3 Results

It can be observed based on the definitions that \( \chi(G) \leq \chi_D(G) \) and that \( D(G) \leq \chi_D(G) \). If \( G \) is asymmetric, then \( \chi_D(G) = \chi(G) \). Clearly, \( D(G) = 1 \) if and only if \( G \) is asymmetric. Therefore, given graph \( G \), deciding whether \( D(G) = 1 \) is polynomial-time equivalent to GA. In contrast, given graph \( G \), deciding whether \( \chi_D(G) = 1 \) is trivial; \( G = K_1 \) is the only graph with \( \chi_D(G) = 1 \). When \( \chi_D(G) = 2 \), \( G \) is necessarily bipartite. In the remainder of the paper, we use 2-coloring to refer to a proper 2-coloring.

Our focus is on the following problem:

**Distinguishing 2-Colorability (D2C)**
Instance: Graph \( G \).
Question: Is \( \chi_D(G) \leq 2 \)?

We first consider the problem D2C restricted to connected graphs. Note that a connected bipartite graph \( G \) has a unique (up to renaming the colors) 2-coloring and, therefore, either every 2-coloring of \( G \) is distinguishing or none of them is. Consequently, D2C for connected graphs is polynomial-time many-one equivalent to the problem: Given a graph \( G \) and a 2-coloring
of \( G \), is \( c \) a distinguishing coloring? Thus, since a given coloring \( c \) of graph \( G \) is not distinguishing if and only if there is a NTA of \( G \) that preserves \( c \), the complement of \( \text{D2C} \) restricted to connected graphs seems closely related to \( \text{GA} \). The next theorem shows that those two problems are in fact polynomial-time many-one equivalent.

In the remainder of the paper, we refer to the *complement* of \( \text{D2C} \) restricted to connected graphs as \( \text{CC} \):

**Complement of \( \text{D2C} \) on connected graphs (\( \text{CC} \))**

**Instance:** Connected graph \( G \).

**Question:** Is \( \chi_D(G) \not\leq 2 \)?

**Theorem 1** Problems \( \text{CC} \) and \( \text{GA} \) are polynomial-time many-one equivalent.

**Proof.** First, we show that \( \text{GA} \leq_m \text{CC} \).

Since a graph has a NTA if and only if its complement has a NTA, and the complement of a disconnected graph is connected, we may assume that the given instance \( G = (V, E) \) of \( \text{GA} \) is connected.

If \( G = K_1 \), \( G \) is a no instance of \( \text{GA} \) and we can easily construct a no instance \( G' \) of \( \text{CC} \). Otherwise, let \( G' = (V' = V \cup E, E') \) be the graph obtained from \( G \) by subdividing each edge of \( G \) once. We note that for an edge \( xy \) of \( G \), we use \( xy \) to refer to the edge of \( G \) as well as the vertex of \( G' \) that subdivides the edge \( xy \) of \( G \). Clearly, \( G' \) is connected and bipartite, and every vertex in \( E \) has degree 2. In order to complete the reduction from \( \text{GA} \) to \( \text{CC} \), we prove that \( G \) has a NTA if and only if \( \chi_D(G') \not\leq 2 \).

If all the vertices of \( V' \) have degree 2 then \( G \) is a chordless cycle of length \( \geq 3 \) and therefore has a NTA. In this case, for every 2-coloring of \( G' \), the vertices in \( V \) are mapped to one color, the vertices in \( E \) are mapped to the other color, and also there is a NTA of \( G' \) that preserves the coloring. Therefore, \( G \) has a NTA and \( \chi_D(G') \not\leq 2 \).

In the remaining case, one color class of \( G' \) consists entirely of degree 2 vertices (vertices in \( E \)) and the other color class (vertices in \( V \)) contains a vertex of degree \( \neq 2 \). Thus, every NTA of \( G' \) must map \( V \) to \( V \) and \( E \) to \( E \).

First, we show that if \( f : V \mapsto V \) is a NTA of \( G \), then for every 2-coloring of \( G' \) there exists a NTA \( f' : V' \mapsto V' \) that preserves the coloring of \( G' \) (and hence \( \chi_D(G') \not\leq 2 \)). Note that as \( G' \) is connected, it is enough to consider a particular 2-coloring of \( G' \).
Suppose \( f : V \rightarrow V \) is a NTA of \( G \). Define \( f' : V' \rightarrow V' \) where

\[
f'(x) = \begin{cases} 
  f(x) & \text{if } x \in V \\
  f(u)f(v) & \text{if } x = uv \in E
\end{cases}
\]

We now show that \( f' \) is an NTA that preserves every 2-coloring of \( G' \). Since \( f \) is an automorphism, \( f' \) is a bijection. To see that \( f' \) is an automorphism, observe that:

\[
uw \in E' \iff u \in V \text{ and } v \in E \text{ and } v = uw \text{ for some } w \in V \text{ (or vice versa)}
\]

\[
\iff f'(u) = f(u) \in V \text{ and } f'(v) = f(u)f(w)
\]

\[
\iff f'(u)f'(v) \in E' \text{ since } f'(v) \text{ corresponds to an element of } E \text{ that is incident with } f'(u) \text{ in } G.
\]

Since \( G' \) is connected, it has a unique 2-coloring, and that 2-coloring is preserved by \( f' \) since \( f' \) maps \( V \) to \( V \) and \( E \) to \( E \). Finally, as \( f \) is a NTA of \( G \), \( f' \) is a NTA of \( G' \).

Next, we show that if \( \chi_D(G') \not\leq 2 \), then \( G \) has a NTA. Suppose \( \chi_D(G') \not\leq 2 \). Let \( c \) be the unique 2-coloring of \( G' \) and let \( f' : V' \rightarrow V' \) be a NTA of \( G' \) that preserves \( c \). Define \( f : V \rightarrow V \) such that \( f(x) = f'(x) \) for all \( x \in V \). Since \( f' \) preserves \( c \), it maps \( V \) to \( V \) and \( E \) to \( E \). As \( f' \) is a NTA of \( G' \), the \( V \) to \( V \) mapping of \( f \) is a NTA of \( G \).

This completes the proof of \( GA \leq_m CC \).

We now prove the reduction in the other direction, that is, \( CC \leq_m GA \). Let \( G = (X, Y, E) \) be a connected bipartite graph that is not \( K_1 \) or \( K_2 \). \( K_1 \) and \( K_2 \) are no instances of \( CC \) and any connected non-bipartite graph \( G \) is an YES instance of \( CC \). In these cases, we can construct \( G' \) accordingly. Note that \( G \) has a unique 2-coloring with color classes \( X \) and \( Y \). Define

\[
G' = \begin{cases} 
  G = (X, Y, E) & \text{if } |X| \neq |Y| \\
  (X', Y', E') & \text{otherwise}
\end{cases}
\]

where \( a, b, c \notin X \cup Y \) and

\[
X' = X \cup \{b\} \quad Y' = Y \cup \{a, c\} \quad E' = E \cup \{ax \mid x \in X\} \cup \{ab, bc\}
\]

We prove that \( \chi_D(G) \not\leq 2 \) if and only if \( G' \) has a NTA.
Suppose $\chi_D(G) \leq 2$. Then there exists a NTA $f$ of $G$ that preserves the unique 2-coloring of $G$. In the case that $G' = G$, $f$ is also a NTA of $G'$. In the case that $G' \neq G$ define the mapping $f' : X' \cup Y' \mapsto X' \cup Y'$ where

$$f'(x) = \begin{cases} f(x) & \text{if } x \in X \cup Y \\ x & \text{if } x \in \{a, b, c\}. \end{cases}$$

It is easily seen that $f'$ is a NTA of $G'$.

Now suppose $f$ is a NTA of $G' = (X', Y', E')$. Since $|X'| \neq |Y'|$ and $G'$ is connected, $f$ preserves the unique 2-coloring of $G'$. Further, $f(a) = a$, $f(b) = b$, and $f(c) = c$ by the vertex degrees, the connectedness of $G$, and the fact that $G \not\cong K_2$. Thus, $f$ maps $X$ to $X$ and $Y$ to $Y$ and therefore $f$ restricted to $G$ is a NTA of $G$ that preserves the unique 2-coloring of $G$. Therefore, $\chi_D(G) \leq 2$ and the proof of the theorem is complete. □

The following proposition allows us to analyze the complexity of problem D2C for graphs that are not necessarily connected. We again use the fact that a connected bipartite graph has a unique (up to renaming the colors) 2-coloring.

**Proposition 1** Let $G$ be a graph. $\chi_D(G) \leq 2$ if and only if

- $G$ is bipartite and
- for every component $C$ of $G$:
  - $\chi_D(C) \leq 2$,
  - $C$ is isomorphic to at most one other component of $G$, and
  - if $C$ is isomorphic to some other component of $G$ then $C$ is asymmetric.

**Proof.** The proposition clearly holds when $G = K_1$. Therefore, we now assume $G$ has at least two vertices.

⇒ Suppose that $\chi_D(G) \leq 2$. Then, $G$ is bipartite by an earlier observation. By the definition of $\chi_D$, there is a 2-coloring $c$ of $G$ that is distinguishing.

Let $C$ be a component of $G$. It is clear that $c$ restricted to $C$ is a distinguishing coloring or else we contradict the choice of $c$.

Suppose that $C_1 = (X_1, Y_1, E_1)$, $C_2 = (X_2, Y_2, E_2)$, and $C_3 = (X_3, Y_3, E_3)$ are three distinct isomorphic components of $G$ and that there are isomorphisms mapping $X_1$ to $X_2$ and $X_2$ to $X_3$. No matter how the vertices of $C_1$,
$C_2$, and $C_3$ are 2-colored, two of $X_1$, $X_2$, $X_3$ will be in the same color class and therefore for every 2-coloring of $G$, there is an NTA that preserves the coloring (specifically, an automorphism that maps the two $X_i$'s that are in the same color class to one another), contradicting that $\chi_D(G) \leq 2$. Therefore, each component can be isomorphic to at most one other component.

Suppose that $C = (X_C, Y_C, E_C)$ is isomorphic to another component $C' = (X_{C'}, Y_{C'}, E_{C'})$ and that some isomorphism $f$ maps $X_C$ to $X_{C'}$. If $C$ is not asymmetric, then it has a NTA $g$, and every NTA of $C$ maps $X_C$ to $Y_C$ and vice versa, or else we contradict that $c$ is distinguishing. But, now there are two isomorphisms from $C$ to $C'$, namely, $f$ and $g \circ f$, one of which preserves $c$, a contradiction.

$\Leftarrow$ Let bipartite graph $G$, with components $C_1, C_2, \ldots, C_k$, satisfy the conditions. Suppose $\chi_D(G) \not\leq 2$. Then, for every 2-coloring of $G$, there is a NTA that preserves the coloring. Let $c$ be a 2-coloring of $G$ in which isomorphic pairs of components are colored such that if there is an isomorphism mapping $X_1$ to $X_2$ then $X_1$ and $X_2$ have opposite colors in $c$. Now, every NTA swaps colors within single components and/or swaps colors in pairs of components but, in any case, $c$ is not preserved, a contradiction. ■

**Corollary 1** $D2C \leq_T GI$

**Proof.**

By Proposition 1, an algorithm for $D2C$ can be constructed from algorithms for $CC$, $GI$, and $GA$. Since $CC \leq_m GA$ (Theorem 1) and $GA \leq_m GI$, the result follows. ■

### 4 Discussion

Combining Theorem 1, Corollary 1, and the observation that $CC \leq_T D2C$, we have $GA \equiv_m CC \leq_T D2C \leq_T GI$. That is, $D2C$ is at least as hard as $GA$ and no harder than $GI$, in terms of Turing reductions.

Our results imply that $CC \in NP$ and $D2C \in co-NP$. In addition, a direct consequence of Corollary 1 is that for a graph $G$ belonging to a class $\mathcal{C}$ such that the isomorphism problem can be solved in polynomial time for $\mathcal{C}$, deciding whether $\chi_D(G) \leq 2$ can be done in polynomial time.

A question that arises from Theorem 1 and Corollary 1 is: is problem $D2C$ polynomial-time equivalent to $GA$ or to $GI$, or does its complexity lie in between those of problems $GA$ and $GI$?
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