Supersymmetric non conservative systems

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We give the generalization of a recent variational formulation for nonconservative classical mechanics, for fermionic and supersymmetric systems. Both cases require slightly modified boundary conditions. The supersymmetric version is given in the superfield formalism. The corresponding Noether theorem is formulated. As expected, like the energy, the supersymmetric charges are not conserved. Examples are discussed.

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I. INTRODUCTION

The study of a general mechanical system includes influences of external factors, whose origin and detailed description may be partially or fully unknown. The evolution of such systems is frequently irreversible and non invariant under time reversal. There are fundamental questions related to this issue, like time direction, the second principle of thermodynamics, etc. A general formulation for such systems has been given in terms of equations of motion. For instance, for a nonconservative subsystem of a conservative system, we can obtain a set of equations of motion by substitution of the solution of the equations of the rest of the system into the ones of the subsystem. This reasoning is extended to write “effective” equations of motion of nonconservative systems.

Hamilton’s principle gives a way to obtain the equations of motion of conservative systems by the variation of the action, with the variables fixed at the initial and final times. The physical trajectory goes along a curve through these initial and final points and minimizes the action. Thus this trajectory is determined by conditions in the past and in the future. For conservative, time reversible, systems, this situation is not a problem. However if one wishes to describe nonconservative systems, not invariant under time reversal, such a conditions could lead to violations of causality.

Classical mechanics deals with the evolution of systems described by real quantities like the position, angles, and so on. However, if we see it as a limit of quantum mechanics, it is natural to ask about fermionic degrees of freedom, which have properties properly described by anticommutative quantities. This has lead to the formulation of fermionic classical mechanics [1], whose quantization leads to the quantum mechanics of fermionic systems [2]. Supersymmetry transforms bosons into fermions and its basic entities are supermultiplets, which contain bosons and fermions, see e.g. [3] and [4]. The profound difference between bosonic and fermionic systems has been one of the principal reasons which made supersymmetry so attractive. On the other side, supersymmetry imposes strong restrictions, like the cancellation of divergences in field theory, one of the main reasons of its success, or the equality of the masses of particles in the same supermultiplet, one of the principal objections against it as a symmetry of nature.

Supersymmetry in mechanics is an extension of time translations, by means of transformations with anticommutative parameters. Thus supersymmetry is intimately related to energy conservation and a question which arises naturally is what happens with non-closed, nonconservative systems, with broken time translational symmetry. On the other side, if a supersymmetric system is exposed to external forces, these forces could be due to supersymmetric unknown factors, and could have a supersymmetric structure.

In two papers recently publicated by Galley [5, 6], a systematic proposal has been made for a lagrangian formulation for nonconservative systems. This proposal is similar to the closed time path formalism, and amounts to a modification of Hamilton’s variational principle, introducing a generalized nonconservative “potential”. The closed time path formalism has been originally proposed by Schwinger [7] for field theory, for further development see [8]. It has been developed recently for classical and quantum mechanics [9].

We follow Galley’s formulation to address the study of supersymmetric nonconservative systems. We consider first fermionic systems, whose boundary conditions are determined by only one parameter [2]. Thus their nonconservative generalization requires a slight modification of the boundary conditions. The generalization for supersymmetric systems is done in the superspace formalism. The boundary conditions must be also modified, and are given in terms

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of superfields. Considering that supersymmetry in classical mechanics is an extension of time translational symmetry, and that the last is broken by the nonconservative interaction, the same must happen for supersymmetry, as is verified by means of the Noether theorem. We maintained the supersymmetric structure for the nonconservative generalized potential, i.e. it is written in terms of superfields. In Section II we give a short review of Galley’s formulation. In section III we consider the generalization for fermionic systems. In Section IV we perform the supersymmetric generalization and formulate the corresponding Noether theorem. We consider along the work the examples of two coupled oscillators and a damped oscillator. In the last Section we draw some conclusions.

II. LAGRANGIAN FORMULATION

Hamilton’s principle in mechanics establishes that the trajectory of a system in an arbitrary time interval \([t_i, t_f]\), on which act conservative forces, is an extremum of the action. The variation is done on curves which go through two fixed points, at the initial and final times. Instead of it, we could consider curves that at the initial time go through a fixed point, with a fixed velocity. In this case the point at the final time should be fixed also, although it cannot be arbitrary, as is determined by the initial position and velocity. It can be seen that in this way we obtain the Euler-Lagrange equations as well \[10\]. As far as the kinetic and potential terms do not depend explicitly on time, the E-L equations are symmetric under time reversal, although the integration constants of the solutions can be chosen asymmetric. However, if we are interested on an effective action, where a part of the degrees of freedom have been integrated, by substituting their solutions into the action, the resulting action will be as well time reversal invariant. For instance \[5\], for a system of two coupled oscillators, after integrating one of them, the equation of motion of the integrated, by substituting their solutions into the action, the resulting action will be as well time reversal invariant. It can be seen that in this way we obtain the effective lagrangian has a reversible interaction, with a time symmetric Green’s function, meaning that, in general, in this way the equations of motion of an irreversible, nonconservative, system cannot be obtained.

The lagrangian formulation of Galley supposes that there is a conservative lagrangian \(L(q, \dot{q})\), where \(q\) are \(n\)-dimensional vectors. Nonconservativity is attained by a doubling of the degrees of freedom \(q \rightarrow (q_1, q_2)\) and the variational principle is modified in such a way that \(q_2(t)\) effectively runs back in time, and appears as a continuation of \(q_1(t)\). Thus the boundary conditions are imposed only at the initial time. After the variation, at the level of the equations of motion, the doubling is reverted, by the so called physical limit \(q_1 \rightarrow q_2 = q\). This is similar to the Closed Path-Time approach \[7, 8\]. Further, Hamilton’s principle is applied, with the condition that the values of both variables coincide at \(t_f\) and have fixed values at \(t_i\). In fact, both variables could be arranged as one, beginning and finishing at \(t_i\), after a closed time path \(t_i \rightarrow t_f \rightarrow t_i\ \[9\]. Galley avoids the closed time and instead of it modifies the setting of the action

\[
S = \int_{t_i}^{t_f} L(q_1, \dot{q}_1)dt + \int_{t_f}^{t_i} L(q_2, \dot{q}_2)dt = \int_{t_i}^{t_f} L(q_1, \dot{q}_1)dt - \int_{t_i}^{t_f} L(q_2, \dot{q}_2)dt. \tag{1}
\]

This setting allows to add to the action a nonconservative “generalized potential”, \(K(q_1, \dot{q}_1, q_2, \dot{q}_2)\), which depends on both variables and is antisymmetric under \(1 \leftrightarrow 2\). The nonconservative lagrangian is given by

\[
\Lambda(q_1, \dot{q}_1, q_2, \dot{q}_2) = L(q_1, \dot{q}_1) - L(q_2, \dot{q}_2) + K(q_1, \dot{q}_1, q_2, \dot{q}_2). \tag{2}
\]

The variation is given now with the boundary conditions that at the initial time \(q_1(t_i) = q_2(t_i)\) and \(\delta q_1(t_i) = \delta q_2(t_i) = 0\), and at the final time both variables coincide \(q_1(t_f) = q_2(t_f)\) and have nonvanishing variations \(\delta q_1(t_f) = \delta q_2(t_f)\). Consistently \(\dot{q}_1(t_i) = \dot{q}_2(t_i)\). Thus

\[
\delta S = \int_{t_i}^{t_f} \delta \Lambda(q_1, \dot{q}_1, q_2, \dot{q}_2)dt = \left[\delta q_1 \left(\frac{\partial L}{\partial \dot{q}_1} + \frac{\partial K}{\partial q_1}\right) + \delta q_2 \left(-\frac{\partial L}{\partial \dot{q}_2} + \frac{\partial K}{\partial q_2}\right)\right]_{t=t_f} + \int_{t_i}^{t_f} \left\{\delta q_1 \left[\frac{\partial}{\partial \dot{q}_1} (L + K) - \frac{d}{dt} \frac{\partial}{\partial q_1} (L + K)\right] - \delta q_2 \left[\frac{\partial}{\partial \dot{q}_2} (L - K) - \frac{d}{dt} \frac{\partial}{\partial q_2} (L - K)\right]\right\} dt, \tag{3}
\]

where summation indices are understood. The boundary terms vanish after taking into account the boundary conditions and the antisymmetry of \(K\), from which in particular follow

\[
\frac{\partial K}{\partial q_1} + \frac{\partial K}{\partial q_2} = 0 \quad \text{and} \quad \frac{\partial K}{\partial \dot{q}_1} + \frac{\partial K}{\partial \dot{q}_2} = 0. \tag{4}
\]

Thus, the equations of motion are

\[
\frac{\partial}{\partial \dot{q}_1} (L + K) - \frac{d}{dt} \frac{\partial}{\partial q_1} (L + K) = 0, \tag{5}
\]

\[
\frac{\partial}{\partial \dot{q}_2} (L - K) - \frac{d}{dt} \frac{\partial}{\partial q_2} (L - K) = 0. \tag{6}
\]
Taking into account (11) and making the physical limit \( q_1 = q_2 = q \), we see that (5) and (6) coincide, giving the corresponding nonconservative equations of motion

\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = F_K \equiv \left. \left( \frac{\partial}{\partial q_2} - \frac{d}{dt} \frac{\partial}{\partial \dot{q}_2} \right) K(q_1, \dot{q}_1, q_2, \dot{q}_2) \right|_{q_1=q_2=q}.
\]

(7)

A. Damped oscillator

As an example we give the damped oscillator [5]. The conservative lagrangian is the one of the oscillator \( L(q, \dot{q}) = m/2(q^2 - \omega^2 \dot{q}^2) \) and \( K(q_1, \dot{q}_1, q_2, \dot{q}_2) = c/2(q_1 \dot{q}_2 - q_2 \dot{q}_1) \). Hence

\[
\Lambda(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{m}{2}(q_1^2 - \omega^2 \dot{q}_1^2) - \frac{m}{2}(q_2^2 - \omega^2 \dot{q}_2^2) + \frac{c}{2}(q_1 \dot{q}_2 - q_2 \dot{q}_1).
\]

(8)

Hence the equation of motion, in the physical limit is

\[
m\ddot{q} - c\dot{q} + m\omega^2 = 0.
\]

(9)

III. FERMIONIC SYSTEMS

Fermionic systems in classical mechanics are formulated by means of anticommuting variables. We consider variables \( \psi \) and \( \bar{\psi} \), where \( \psi \) is the complex conjugated. These variables can have an \( n \)-dimensional index and satisfy \( \psi_i \psi_j = -\psi_j \psi_i, \bar{\psi}_i \bar{\psi}_j = \bar{\psi}_j \bar{\psi}_i \) and \( \psi_i \bar{\psi}_j = -\bar{\psi}_j \psi_i \), otherwise they commute with bosonic quantities. Thus they are nilpotent, \( \psi_i \bar{\psi}_i = 0 \). Complex conjugation reverses the order like hermitian conjugation, thus giving a sign, e.g. \( (\psi_i \psi_j)^\dagger = -\bar{\psi}_j \bar{\psi}_i \).

In the following we will not indicate the \( n \)-dimensional indices. We will adopt the convention that fermionic derivatives act on the left, as

\[
\frac{\partial}{\partial \psi_i} \psi_j = \delta_{ij} - \psi_j \frac{\partial}{\partial \psi_i}.
\]

(10)

The kinetic term of fermionic systems is first order, which means that the trajectory is determined by fixing only one parameter. For this reason, in [2] a boundary term has been added to fermionic actions. Consider for example a system described by a fermionic variable \( \psi(t) \) with lagrangian \( L(\psi, \dot{\psi}) = i/2(\dot{\psi} \psi + \psi \dot{\psi}) + \psi \dot{\psi} \). We use an economical notation, where the dependence of the Lagrangian on the first time derivatives is not indicated. According to [2], the action is

\[
S = \int_{t_1}^{t_2} \left[ \frac{i}{2}(\dot{\psi} \psi + \psi \dot{\psi}) - \omega \psi \dot{\psi} \right] dt - \frac{i}{2} \left[ \psi(t_1) \psi(t_2) + \bar{\psi}(t_1) \bar{\psi}(t_2) \right],
\]

(11)

whose variation with the boundary conditions \( \delta \dot{\psi}(t_1) + \delta \psi(t_2) = 0 \) and \( \delta \bar{\psi}(t_1) + \delta \bar{\psi}(t_2) = 0 \) gives the expected equations of motion \( i\dot{\psi} + \omega \psi = 0 \) and \( i\bar{\psi} - \omega \bar{\psi} = 0 \). However this procedure depends on the action and would be incompatible with supersymmetry.

II.}

Following the observations at the beginning of the previous section, instead of it we fix, as usual, the values of the fermionic coordinates at the initial and final times, but with the consistency restriction that one of both values depends on the other. As long as, for Hamilton’s principle, we only need that the variations of the coordinates at the initial and final times vanish, there is no problem if we do not bother about how is this dependence, which would turn out after solving the equations of motion. However, these considerations require that the nonconservative formulation for fermions must be somewhat modified. After the doubling of the variables \( \psi \rightarrow (\psi_1, \psi_2) \) and \( \bar{\psi} \rightarrow (\bar{\psi}_1, \bar{\psi}_2) \), the variation requires that \( \psi_1(t_f) = \psi_2(t_f), \bar{\psi}_1(t_f) = \bar{\psi}_2(t_f) \). This means that for consistency with the equations of motion, also the initial time values should coincide \( \psi_1(t_i) = \psi_2(t_i), \bar{\psi}_1(t_i) = \bar{\psi}_2(t_i) \). Therefore the action is

\[
S = \int_{t_1}^{t_2} \left[ L(\psi_1, \bar{\psi}_1) - L(\psi_2, \bar{\psi}_2) + K(\psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2) \right] dt,
\]

(12)

with \( K(\psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2) \) antisymmetric under \( 1 \leftrightarrow 2 \). The boundary conditions are \( \delta \bar{\psi}_a(t_i) = \delta \bar{\psi}_a(t_i) = 0 \ (a = 1, 2), \psi_1(t_f) = \psi_2(t_f), \bar{\psi}_1(t_f) = \bar{\psi}_2(t_f), \delta \psi_1(t_f) = \delta \psi_2(t_f) \) and \( \delta \bar{\psi}_1(t_f) = \delta \bar{\psi}_2(t_f) \). Following the same steps as in the preceding section and considering the relations (11)

\[
\frac{\partial K}{\partial \psi_1} + \frac{\partial K}{\partial \bar{\psi}_2} = 0 \quad \text{and} \quad \frac{\partial K}{\partial \bar{\psi}_1} + \frac{\partial K}{\partial \psi_2} = 0,
\]

(13)
we get the equations of motion
\[
\frac{\partial}{\partial \psi_1}(L + K) - \frac{d}{dt} \frac{\partial}{\partial \dot{\psi}_1}(L + K) = 0, \\
\frac{\partial}{\partial \psi_2}(L - K) - \frac{d}{dt} \frac{\partial}{\partial \dot{\psi}_2}(L - K) = 0, 
\]
and their complex conjugated \( \bar{\psi} \rightarrow \bar{\dot{\psi}} \). After the physical limit \( \psi_1 = \psi_2 = \psi \) and \( \bar{\psi}_1 = \bar{\dot{\psi}}_2 = \bar{\dot{\psi}} \) both equations coincide, and the nonconservative equations of motion are
\[
\frac{\partial L}{\partial \dot{\psi}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\psi}} = F_K = \left( \frac{\partial}{\partial \dot{\psi}_2} - \frac{d}{dt} \frac{\partial}{\partial \dot{\psi}_2} \right) K(\psi_1, \bar{\psi}_1, \dot{\psi}_1, \psi_2, \bar{\dot{\psi}}_2, \dot{\psi}_2) \bigg|_{\psi_1 = \psi_2 = \psi}, \]
\[
\frac{\partial L}{\partial \dot{\bar{\psi}}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\bar{\psi}}} = \bar{F}_K = \left( \frac{\partial}{\partial \dot{\bar{\psi}}_2} - \frac{d}{dt} \frac{\partial}{\partial \dot{\bar{\psi}}_2} \right) K(\psi_1, \bar{\psi}_1, \dot{\psi}_1, \psi_2, \bar{\dot{\psi}}_2, \dot{\psi}_2) \bigg|_{\psi_1 = \psi_2 = \psi}. 
\]

A. Coupled fermionic oscillators

As an example, let us consider two coupled fermionic oscillators with Lagrangian
\[
L(\psi, \bar{\psi}, \Psi, \bar{\Psi}) = -i \frac{m}{2} \left( \dot{\bar{\psi}} \psi + \bar{\psi} \psi \right) + m \omega \psi \bar{\psi} - \lambda \left( \psi \bar{\Psi} + \bar{\psi} \Psi \right) - i \frac{M}{2} \left( \bar{\Psi} \bar{\Psi} + \Psi \Psi \right) + M \Omega \Psi \bar{\Psi}. 
\]

Similar to the bosonic case (see e.g. [3]), the effective action obtained by means of Hamilton’s principle by the integration of \( \Psi \) and \( \bar{\Psi} \) is given by
\[
S_{\text{eff}} = \int_{t_i}^{t_f} \left[ -i \frac{m}{2} \left( \dot{\bar{\psi}} \psi + \bar{\psi} \psi \right) + m \omega \psi \bar{\psi} - \lambda \left( \psi \bar{\Psi} + \bar{\psi} \Psi \right) - i \frac{M}{2} \left( \bar{\Psi} \bar{\Psi} + \Psi \Psi \right) \right] dt + \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \left[ \Gamma_{\text{ret}}(t - t') + \Gamma_{\text{adv}}(t - t') \right] \bar{\psi}(t') \psi(t),
\]
where \( \Psi^{(h)} = e^{i t \bar{\Psi}} \Psi_0 \) is the solution of the homogeneous equation, \( \Gamma_{\text{ret}}(t - t') = -i \theta(t - t')e^{i \Omega(t - t')} \) and \( \Gamma_{\text{adv}}(t - t') = i \theta(t - t')e^{i \Omega(t - t')} \) are the retarded and advanced fermionic Green functions. This effective action is time reversal symmetric. In order to obtain an irreversible effective action, the previous fermionic formulation can be used, with a Lagrangian \( \Lambda = L(\psi_1, \bar{\psi}_1, \Psi_1, \bar{\Psi}_1) - L(\psi_2, \bar{\psi}_2, \Psi_2, \bar{\Psi}_2) \). If we use the notation \( \psi_{\pm} = 1/\sqrt{2}(\psi_1 \pm \psi_2) \), this Lagrangian can be written as
\[
\Lambda = -i \frac{m}{2} \left( \psi_+ \dot{\bar{\psi}}_- + \psi_- \dot{\bar{\psi}}_+ \right) + m \omega \psi_+ \bar{\psi}_- - \lambda \left( \psi_+ \bar{\Psi}_- + \psi_- \bar{\Psi}_+ \right) - i \frac{M}{2} \left( \psi_+ \bar{\Psi}_- + \psi_- \bar{\Psi}_+ \right) + M \Omega \Psi_+ \bar{\Psi}_- + c.c. 
\]
The boundary conditions with these variables are \( \delta \psi_+(t_1) = \delta \psi_-(t_2) = \delta \Psi_+(t_1) = \delta \Psi_- (t_2) = 0 \), \( \delta \bar{\psi}_+(t_2) = \delta \bar{\psi}_-(t_1) = 0 \), \( \psi_-(t_2) = \bar{\psi}_+(t_2) = 0 \) and their complex conjugated. The variation of [20] in order to integrate the variables \( \Psi \) and \( \bar{\Psi} \) leads to the following solutions
\[
\Psi_\pm(t) = \Psi^{(h)}_\pm(t) + \frac{\lambda}{M} \int_{t_i}^{t_f} dt' \Gamma_{\text{ret}}(t - t') \psi_\pm(t'), \\
\bar{\Psi}_\pm(t) = \bar{\Psi}^{(h)}_\pm(t) + \frac{\lambda}{M} \int_{t_i}^{t_f} dt' \Gamma_{\text{adv}}(t - t') \bar{\psi}_\pm(t'), 
\]
where \( \Psi^{(h)}_\pm(t) = \bar{\Psi}^{(h)}_\pm(t) = 0 \) due to the boundary conditions. Therefore the effective action is
\[
S_{\text{eff}} = \int_{t_i}^{t_f} dt \left[ -i \frac{m}{2} \left( \psi_+ \dot{\bar{\psi}}_- + \psi_- \dot{\bar{\psi}}_+ \right) + m \omega \psi_+ \bar{\psi}_- - \lambda \psi_- \bar{\Psi}^{(h)}_\pm \right] - \frac{\lambda^2}{M} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \left[ \psi_-(t) \Gamma_{\text{ret}}(t - t') \bar{\psi}_+(t') - \bar{\psi}_-(t) \Gamma_{\text{adv}}(t - t') \psi_+(t') \right] + c.c., 
\]
whose equations of motion in the physical limit, $\psi_+ = \sqrt{2}\psi$, $\psi_- = 0$, are

$$i\dot{\psi} + \omega \psi - \frac{\lambda}{M} \psi^{(h)} - \frac{\lambda^2}{M^2} \int_{t_i}^{t_f} dt' \bar{\Gamma}_{ret}(t - t') \psi(t') = 0,$$

$$i\dot{\bar{\psi}} - \omega \bar{\psi} + \frac{\lambda}{M} \bar{\psi}^{(h)} + \frac{\lambda^2}{M^2} \int_{t_i}^{t_f} dt' \bar{\Gamma}_{ret}(t - t') \bar{\psi}(t') = 0.$$  \hfill (24)

### B. Damped fermionic oscillator

Let us now consider a damped fermionic oscillator. For the lagrangian we consider the fermionic oscillator

$$L(\psi, \bar{\psi}) = -i \frac{m}{2} (\dot{\bar{\psi}} \psi + \psi \bar{\dot{\psi}}) + m \omega \bar{\psi} \psi$$  \hfill (25)

and

$$K(\psi_1, \bar{\psi}_1, \psi_2, \bar{\psi}_2) = \frac{\tilde{c}}{2} (\psi_1 \bar{\dot{\psi}}_2 - \bar{\psi}_2 \dot{\psi}_1) + \text{c.c.}.$$  \hfill (26)

Hence $\Lambda = L(\psi_1, \bar{\psi}_1) - L(\psi_2, \bar{\psi}_2) + K(\psi_1, \bar{\psi}_1, \psi_2, \bar{\psi}_2)$, which can be rewritten as

$$\Lambda = -i \frac{m}{2} (\psi \psi_- + \bar{\psi} \bar{\psi}_+ + m \omega \psi \bar{\psi} - \frac{\tilde{c}}{2} (\psi_+ \bar{\psi}_- - \psi_- \bar{\psi}_+) + \text{c.c.}$$

$$= -i \left[ (m + i\tilde{c})\psi_- \bar{\psi}_+ + (m - i\tilde{c})\psi_+ \bar{\psi}_- + m \omega \bar{\psi} \psi + \text{c.c.} \right].$$  \hfill (27)

The equations of motion in the physical limit are

$$i\dot{\psi} + \bar{\omega} \left( 1 + \frac{i\tilde{c}}{m} \right) \psi = 0,$$

$$i\bar{\psi} - \omega \left( 1 - \frac{i\tilde{c}}{m} \right) \bar{\psi} = 0,$$  \hfill (28, 29)

where $\bar{\omega} = \omega/(1 + \tilde{c}^2/m^2)$, with solutions

$$\psi(t) = e^{-i\bar{\omega}(1 - i\frac{\tilde{c}}{m})t} \psi_0 \quad \text{and} \quad \bar{\psi}(t) = e^{i\omega(1 + i\frac{\tilde{c}}{m})t} \bar{\psi}_0.$$  \hfill (30)

The energy change rate can be computed as in [6], by means of the Noether theorem, which for a conservative system of fermionic variables works in the same way as for bosonic variables.

### IV. SUPERSYMMETRIC FORMULATION

Supersymmetric mechanics can be realized by an extension of time to a Grassmann space, or superspace, $t \to z \equiv (t, \theta, \bar{\theta})$, where $\theta$ and $\bar{\theta}$ are anticommuting variables. There are derivatives defined in these spaces by the rules $\{\partial_\theta, \theta\} = 1$, $\{\partial_{\bar{\theta}}, \bar{\theta}\} = 1$, $\{\partial_{\bar{\theta}}, \theta\} = 0$ and $\{\partial_\theta, \bar{\theta}\} = 0$, and integration $\int d\theta = 0$, $\int d\bar{\theta} = 1$, $\int d\theta d\bar{\theta} = 1$. The dynamical variables are extended to superfields $q(t) \to \phi(t, \theta, \bar{\theta})$, which are real and are given by a finite expansion in the fermionic variables as

$$\phi(t, \theta, \bar{\theta}) = q(t) + \theta \psi(t) + \bar{\theta} \bar{\psi}(t) + \theta \bar{\theta} b(t),$$  \hfill (31)

where the variables $q(t), \psi(t), \bar{\psi}(t)$ and $b(t)$ are called the components of the superfield. Note that $\theta \bar{\theta}$ is real. The transformations of supersymmetry are generated by the fermionic charges $Q = \frac{d}{dt} - i\theta \frac{d}{dt}$ and $\bar{Q} = -\frac{d}{dt} + i\theta \frac{d}{dt}$, which satisfy $\{Q, \bar{Q}\} = \bar{Q} + Q$, $\bar{Q} + Q = 2i \frac{d}{dt}$ Thus a supersymmetric transformation is written as $\delta \phi(t, \theta, \bar{\theta}) = (\xi Q - \bar{\xi} \bar{Q}) \phi(t, \theta, \bar{\theta}) = \xi \psi - \bar{\xi} \bar{\psi} + \theta (b + i\hat{q}) - \bar{\theta} (b - i\hat{q}) + i\theta \bar{\theta} (\xi \bar{\psi} + \bar{\xi} \psi)$, from which, by comparison of the components, we get the infinitesimal transformations

$$\delta_\xi q = \bar{\xi} \psi - \bar{\xi} \bar{\psi}, \quad \delta_\xi \bar{\psi} = \xi (b + i\hat{q}), \quad \delta_\xi \psi = \xi (b - i\hat{q}) \quad \text{and} \quad \delta_\xi b = i(\xi \bar{\psi} + \bar{\xi} \psi).$$  \hfill (32)
or in finite form

\[ \phi'(t, \theta, \bar{\theta}) = \phi(t - i\xi - i\bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}). \]  

(33)

Thus supersymmetry transformations can be seen as a certain type of superspace translation. There are covariant derivatives \( D = \frac{d}{dt} + i\theta \frac{d}{dt} \) and \( \bar{D} = -\frac{d}{dt} + i\bar{\theta} \frac{d}{dt} \), which satisfy \( \{D, \bar{D}\} = -2i\frac{d}{dt}, \) \( \{Q, \bar{D}\} = \{Q, D\} = \{\bar{Q}, \bar{D}\} = 0 \). Thus, quantities obtained from superfields by the action of covariant derivatives will transform as superfields. The integral of the covariant derivatives of a superfield \( D\phi(t, \theta, \bar{\theta}) = \psi + \bar{\theta}(i\bar{q} - b) - i\theta\bar{\theta}\bar{\psi} \) and \( \bar{D}\phi(t, \theta, \bar{\theta}) = \bar{\psi} - \theta(i\bar{q} + b) + i\theta\bar{\theta}\bar{\psi} \) give

\[
\int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} D\phi(t, \theta, \bar{\theta}) = -i\psi|_{t_i}^{t_f}, \\
\int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \bar{D}\phi(t, \theta, \bar{\theta}) = i\bar{\psi}|_{t_i}^{t_f},
\]

hence such terms can be added to the Lagrangian without changing the equations of motion. Moreover, integration by parts can be done \( \int dt d\theta d\bar{\theta} \phi_1 D\phi_2 = -\int dt d\theta d\bar{\theta} D\phi_1 \phi_2 + \) boundary terms.

There are also chiral superfields which are complex and satisfy the covariant condition \( \bar{D}\phi = 0 \), and which can be written as \( \phi(t, \theta) = A(t) + \theta \psi(t) \).

The superfield formalism allows to write supersymmetric actions as superspace integrals of superfield Lagrangians, taking advantage that the supersymmetry transformation of the superspace integral of a superfield is \( \int d\theta d\bar{\theta} \delta \phi(t, \theta, \bar{\theta}) = \int d\theta d\bar{\theta} (\xi Q - \bar{\xi} \bar{Q}) \phi(t, \theta, \bar{\theta}) \), and that, similar to (34) and (35),

\[
\int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} Q(t, \theta, \bar{\theta}) = i\psi|_{t_i}^{t_f}, \\
\int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \bar{Q}(t, \theta, \bar{\theta}) = -i\bar{\psi}|_{t_i}^{t_f}.
\]

Therefore, a lagrangian will be a function of superfields and their covariant derivatives of first order, and the corresponding action will be of the general form

\[ S = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} L(\phi, D\phi, \bar{D}\phi), \]

(38)

whose supersymmetry transformation is

\[
\delta S = \int d\theta d\bar{\theta} \delta \xi L(\phi, D\phi, \bar{D}\phi) = \int d\theta d\bar{\theta} (\xi \bar{Q} - \bar{\xi} Q) L = i \left( \xi L_\theta - i\bar{\xi} L_{\bar{\theta}} \right)|_{t_i}^{t_f},
\]

(39)

where \( L_\theta \) and \( L_{\bar{\theta}} \) are the \( \theta \) and \( \bar{\theta} \) components of the expansion \( \bar{L} \) of the superfield \( L \). Action \( \bar{L} \) can be varied under a variation of the superfields \( \delta \phi \)

\[
\delta S = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left( \delta \phi \frac{\partial L}{\partial \phi} + \delta D\phi \frac{\partial L}{\partial D\phi} + \delta \bar{D}\phi \frac{\partial L}{\partial \bar{D}\phi} \right) \\
= \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \delta \phi \left( \frac{\partial L}{\partial \phi} - D \frac{\partial L}{\partial D\phi} - \bar{D} \frac{\partial L}{\partial \bar{D}\phi} \right) + \text{B.T.},
\]

(40)

where the boundary terms are

\[
\text{B.T.} = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left[ D \left( \delta \phi \frac{\partial L}{\partial D\phi} + \delta \bar{D} \frac{\partial L}{\partial \bar{D}\phi} \right) \right] \\
= \left\{ \delta \phi \left[ \left( \frac{\partial L}{\partial D\phi} \right)_{\phi} + \left( \frac{\partial L}{\partial \bar{D}\phi} \right)_{\bar{\phi}} \right] + \delta \psi \left( \frac{\partial L}{\partial D\phi} \right)_{\bar{\phi}} + \delta \bar{\psi} \left( \frac{\partial L}{\partial \bar{D}\phi} \right)_{\bar{\phi}} \right\}|_{t_i}^{t_f}.
\]

(41)

Therefore, it is enough if \( \delta q(t_i, f) = 0 \) and \( \delta \bar{q}(t_i, f) = \delta \bar{\psi}(t_i, f) = 0 \), which require by consistency with supersymmetry, that \( \delta b(t_i, f) = 0 \), hence \( \delta \phi(t_i, \theta, \bar{\theta}) = \delta \phi(t_f, \theta, \bar{\theta}) = 0 \). Therefore the equations of motion are

\[
\frac{\partial L}{\partial \phi} - D \frac{\partial L}{\partial D\phi} - \bar{D} \frac{\partial L}{\partial \bar{D}\phi} = 0.
\]

(42)

From these equations, the equations of the components are obtained from the \( \theta \)-expansion. These equations can be also obtained directly from \( \bar{L} \) written in components, i.e. after integrating the fermionic superspace variables.
A. Nonconservative formulation

As mentioned in the introduction, we are interested on the on the preceding system under the influence of nonconservative forces. We assume that these forces have a supersymmetric structure. This can be done by means of the superfield formulation, following the lines of the bosonic formalism. As a first step, the variables are duplicated \( \phi(t, \theta, \bar{\theta}) \rightarrow (\phi_1(t, \theta, \bar{\theta}), \phi_2(t, \theta, \bar{\theta})) \). Thus the noncommutative action is

\[
S = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left[ L(\phi_1, D\phi_1, \bar{D}\phi_1) - L(\phi_2, D\phi_2, \bar{D}\phi_2) + K(\phi_1, D\phi_1, \bar{D}\phi_1, \phi_2, D\phi_2, \bar{D}\phi_2) \right],
\]

where \( K(\phi_1, D\phi_1, \bar{D}\phi_1, \phi_2, D\phi_2, \bar{D}\phi_2) \) is antisymmetric under the exchange \( 1 \leftrightarrow 2 \). Its variation is

\[
\delta S = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left\{ \delta \phi_1 \left[ \frac{\partial(L + K)}{\partial \phi_1} - D \frac{\partial(L + K)}{\partial D\phi_1} - \bar{D} \frac{\partial(L + K)}{\partial \bar{D}\phi_1} \right] \\
- \delta \phi_2 \left[ \frac{\partial(L - K)}{\partial \phi_2} - D \frac{\partial(L - K)}{\partial D\phi_2} - \bar{D} \frac{\partial(L - K)}{\partial \bar{D}\phi_2} \right] \right\} + \text{B.T.}
\]

The boundary terms are

\[
\text{B.T.} = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left\{ D \left[ \delta \phi_1 \frac{\partial(L + K)}{\partial D\phi_1} + \bar{D} \frac{\partial(L + K)}{\partial \bar{D}\phi_1} \right] \\
- D \left[ \delta \phi_2 \frac{\partial(L - K)}{\partial D\phi_2} - \bar{D} \frac{\partial(L - K)}{\partial \bar{D}\phi_2} \right] \right\},
\]

which, taking into account \( 44 \) and \( 45 \), vanish with the boundary conditions \( \delta \phi_1(t_i, \theta, \bar{\theta}) = \delta \phi_2(t_i, \theta, \bar{\theta}) = 0 \), \( \phi_1(t_f, \theta, \bar{\theta}) = \phi_2(t_f, \theta, \bar{\theta}) \), \( D\phi_1(t_f, \theta, \bar{\theta}) = D\phi_2(t_f, \theta, \bar{\theta}) \) and \( \bar{D}\phi_1(t_f, \theta, \bar{\theta}) = \bar{D}\phi_2(t_f, \theta, \bar{\theta}) \). Further, the consistency equality conditions of the fermionic variables at the initial time, shown in the previous section, lead to the conditions \( \phi_1(t_i, \theta, \bar{\theta}) = \phi_2(t_i, \theta, \bar{\theta}) \). This can be seen considering that, for a supersymmetric invariant theory, if one of the components of two superfields coincide, then both superfields must coincide, as can be seen from \( 32 \).

Thus the equations of motion are

\[
\left( \frac{\partial}{\partial \phi_1} - D \frac{\partial}{\partial D\phi_1} - \bar{D} \frac{\partial}{\partial \bar{D}\phi_1} \right) [L(\phi_1) + K(\phi_1, \phi_2)] = 0,
\]

\[
\left( \frac{\partial}{\partial \phi_2} - D \frac{\partial}{\partial D\phi_2} - \bar{D} \frac{\partial}{\partial \bar{D}\phi_2} \right) [L(\phi_2) - K(\phi_1, \phi_2)] = 0.
\]

In the physical limit, \( \phi_1(t, \theta, \bar{\theta}) = \phi_2(t, \theta, \bar{\theta}) = \phi(t, \theta, \bar{\theta}) \), and both equations coincide. Thus, if we define the nonconservative forces

\[
F_K(\phi, D\phi, \bar{D}\phi) = \left. \left( D \frac{\partial}{\partial D\phi_1} + \bar{D} \frac{\partial}{\partial \bar{D}\phi_1} - \frac{\partial}{\partial \phi_1} \right) K(\phi_1, \phi_2) \right|_{\phi_1 = \phi_2 = \phi},
\]

then, the nonconservative equations of motion are

\[
\left( \frac{\partial}{\partial \phi} - D \frac{\partial}{\partial D\phi} - \bar{D} \frac{\partial}{\partial \bar{D}\phi} \right) L(\phi, D\phi, \bar{D}\phi) = F_K(\phi, D\phi, \bar{D}\phi).
\]

B. Noether theorem

Before we turn to examples, let us work out, following \( 6 \), Noether’s theorem for a general supersymmetry transformation, which includes time translations. Thus the transformations are

\[
t \rightarrow t' = t + \delta t - i\xi \bar{\theta} - i\xi \theta,
\]

\[
\phi'(t', \theta', \bar{\theta}') = \phi(t, \theta, \bar{\theta}) + \delta \phi(t, \theta, \bar{\theta}) + (\xi Q - \xi \bar{Q}) \phi(t, \theta, \bar{\theta}).
\]
Thus the transformation of the action is
\[
\delta \xi S = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left\{ \delta t \left[ D \left( \frac{\partial L}{\partial D\phi} \right) + \bar{D} \left( \frac{\partial L}{\partial D\phi} \right) - \frac{dL}{dt} + \frac{\partial L}{\partial t} \right] + \xi \left[ (Q\phi) \frac{\partial L}{\partial \phi} + (QD\phi) \frac{\partial L}{\partial D\phi} + (Q\bar{D}\phi) \frac{\partial L}{\partial \bar{D}\phi} \right] - \bar{\xi} \left[ \bar{Q}\phi \frac{\partial L}{\partial \phi} + \bar{Q}D\phi \frac{\partial L}{\partial D\phi} + \bar{Q}\bar{D}\phi \frac{\partial L}{\partial \bar{D}\phi} \right] \right\},
\]
which, after some arrangements and considering that the supersymmetry transformations anticommute with the covariant derivatives, turns to
\[
\delta \xi S = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left\{ \delta t \left[ D \left( \frac{\partial L}{\partial D\phi} \right) + \bar{D} \left( \frac{\partial L}{\partial D\phi} \right) - \frac{dL}{dt} + \frac{\partial L}{\partial t} + \frac{\partial L}{\partial \phi} - D \frac{\partial L}{\partial D\phi} - \bar{D} \frac{\partial L}{\partial \bar{D}\phi} \right] \right\}. \tag{53}
\]
Further, taking into account (59) and (61), (65), we get for supersymmetry transformations
\[
\delta \xi S = - \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left( \xi DL - \bar{\xi} \bar{D}L \right). \tag{54}
\]
Therefore, the conservation laws of the conservative theory of lagrangian \( L \) are
\[
\int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left[ D \left( \frac{\partial L}{\partial D\phi} \right) + \bar{D} \left( \frac{\partial L}{\partial D\phi} \right) \right] = 0, \tag{55}
\]
\[
\int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left[ D \left( Q\phi \frac{\partial L}{\partial D\phi} - L \right) + \bar{D} \left( Q\phi \frac{\partial L}{\partial D\phi} \right) \right] = 0, \tag{56}
\]
\[
\int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left[ D \left( Q\phi \frac{\partial L}{\partial D\phi} \right) + \bar{D} \left( Q\phi \frac{\partial L}{\partial D\phi} - L \right) \right] = 0. \tag{57}
\]
From which we obtain the expressions for the energy and the supersymmetric charges
\[
E = -i \left( \frac{\partial L}{\partial \phi} \right)_{\theta} + i \left( \frac{\partial L}{\partial \bar{\phi}} \right)_{\bar{\theta}}, \tag{58}
\]
\[
J = - \left( Q\phi \frac{\partial L}{\partial D\phi} - L \right)_{\theta} + \left( Q\phi \frac{\partial L}{\partial D\phi} \right)_{\bar{\theta}}, \tag{59}
\]
\[
\bar{J} = - \left( Q\phi \frac{\partial L}{\partial D\phi} \right)_{\theta} + \left( Q\phi \frac{\partial L}{\partial D\phi} - L \right)_{\bar{\theta}}. \tag{60}
\]
which satisfy \( \dot{E} = 0, \dot{J} = 0 \) and \( \dot{\bar{J}} = 0 \). \( L_{\theta\bar{\theta}} \) is the \( \theta\bar{\theta} \) component of the superfield \( L \). Actually, these conservation laws can be given in a supersymmetric covariant way by the vanishing of the superfields inside the square brackets in (58), (59) and (60).

Thus, for the nonconservative system, from (49) and (53) we get
\[
\frac{dE}{dt} = \left( -\frac{\partial L}{\partial t} + \phi F_K \right)_{\theta\bar{\theta}}, \quad \frac{dJ}{dt} = [(Q\phi)F_K]_{\theta\bar{\theta}} \quad \text{and} \quad \frac{d\bar{J}}{dt} = [(Q\bar{\phi})F_K]_{\theta\bar{\theta}}. \tag{61}
\]
If the conservative theory has additional, internal symmetries, say \( \phi'(t, \theta, \bar{\theta}) = \phi(t, \theta, \bar{\theta}) + \xi^a T_a \phi(t, \theta, \bar{\theta}) \), which satisfy \( [T_a, Q] = [T_a, \bar{Q}] = 0 \), then
\[
\delta \xi S = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \xi^a \left[ -D \left( T_a \phi \frac{\partial L}{\partial D\phi} \right) - \bar{D} \left( T_a \phi \frac{\partial L}{\partial D\phi} \right) + T_a \phi \left( \frac{\partial L}{\partial \phi} - D \frac{\partial L}{\partial D\phi} - \bar{D} \frac{\partial L}{\partial \bar{D}\phi} \right) \right]. \tag{62}
\]
Therefore, the corresponding conserved current of the conservative theory is

\[ J_\alpha = D \left( T_\alpha \phi \frac{\partial L}{\partial D\phi} \right) + \bar{D} \left( T_\alpha \phi \frac{\partial L}{\partial D\phi} \right), \]  

whence conservation law for the nonconservative theory as

\[ \frac{dJ_\alpha}{dt} = \left[ (T_\alpha \phi) F_\phi \right]_{\theta \bar{\theta}}. \]  

Unlike the case of time translations and supersymmetry transformations, a nontrivial nonconservative generalized potential \( K \) can be invariant under the internal symmetry transformations of the conservative system, and \([64]\) can be zero.

### C. Interacting oscillators

Let us consider two interacting oscillators. As a first step we work out the supersymmetric conservative action

\[ S = \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \left[ \frac{m}{2} (\dot{\phi} + i \omega \phi^2) - \lambda \phi \Phi + \frac{M}{2} (\dot{\Phi} D\Phi + \Omega \Phi^2) \right], \]  

where \( \phi(t, \theta, \bar{\theta}) = q(t) + \theta \psi(t) - \bar{\theta} \bar{\psi}(t) + \theta \bar{\theta} b(t) \) and \( \Phi(t, \theta, \bar{\theta}) = Q(t) + \theta \Psi(t) - \bar{\theta} \bar{\Psi}(t) + \bar{\theta} \theta B(t) \), not confuse the variable \( Q(t) \) with the supersymmetric charge \( Q \).

In components this action can be written as \( S_b = S_b + S_f \), where

\[ S_b = \int_{t_i}^{t_f} \left[ \frac{m}{2} \dot{\theta}^2 + m \omega q \theta - \lambda (qB + Qb) + \frac{M}{2} \dot{Q}^2 + M \Omega QB + \frac{M}{2} B^2 \right], \]

\[ S_f = \int_{t_i}^{t_f} \left[ -i \frac{m}{2} (\dot{\psi} \bar{\psi} + \bar{\psi} \rho) + m \omega \psi \bar{\psi} - \lambda (\psi \bar{\Psi} + \bar{\psi} \Psi) - i \frac{M}{2} (\dot{\Psi} \bar{\Psi} + \bar{\Psi} \dot{\Psi}) + M \Omega \Psi \bar{\Psi} \right]. \]

The equations of motion of \( Q, \Psi, \bar{\Psi} \) and \( B \) are

\[ \ddot{Q} + \left( Q^2 + \frac{\lambda^2}{M^2} \right) Q - \frac{\lambda}{M} (\omega + \Omega) q = 0, \]  

\[ i \dot{\Psi} + \Omega \Psi - \frac{\lambda}{M} \psi = 0, \]  

\[ -i \dot{\bar{\Psi}} + \Omega \bar{\Psi} - \frac{\lambda}{M} \bar{\psi} = 0, \]  

\[ B + \Omega Q - \frac{\lambda}{M} q = 0. \]

From them follow effective actions invariant under time reversal, as in previous section. Instead of doing that, we show the computations in the superfield formalism. The equation of motion of the superfield \( \Phi \) is

\[ \frac{1}{2} \left[ \dot{D}, D \right] \Phi - \Omega \Phi + \frac{\lambda}{M} \phi = 0. \]  

Its solution can be written as

\[ \Phi(z) = \Phi(z) - \frac{\lambda}{M} \int dz' \Delta(z, z') \phi(z), \]  

where \( z \equiv (t, \theta, \bar{\theta}) \) and \( \int dz \equiv \int_{t_i}^{t_f} dt \int d\theta d\bar{\theta} \) and

\[ \left( \frac{1}{2} \left[ \dot{D}, D \right] - \Omega \right) \Delta(z, z') = \delta^{(3)}(z - z'). \]  

The Dirac function for anticommutative variables is defined as \( \delta(\theta - \theta') = \theta - \theta' \) and satisfies \( \int d\theta f(\theta) \delta(\theta - \theta') = f(\theta') \). Its generalization is obvious, hence \( \delta^{(3)}(z - z') = \delta(t - t') \delta(\theta - \theta') \delta(\bar{\theta} - \bar{\theta}') = \delta(t - t')(\theta - \theta')(\bar{\theta} - \bar{\theta}'). \) The rather lengthy expansion of \( \Delta(z, z') \) in its anticommutative variables, can be written in a simpler form as
\[ \Delta(z, z') = \Delta_0(t, t', \theta', \theta) + \theta \Delta_c(t, t', \theta', \theta') - \theta \Delta_c(t, t', \theta', \theta') + \theta \theta \Delta_{\phi^2}(t, t', \theta', \theta') \], which substituted into (74) and solved, component by component, leads to
\[ \Delta_{\text{ret}}(z, z') = (1 - \Omega \theta \theta) \left( 1 - \Omega \theta \theta \right) G_{\text{ret}}(t - t') + \theta \theta \Gamma_{\text{ret}}(t - t') + \theta \theta \Gamma_{\text{ret}}(t - t') - \theta \theta \theta \theta \delta(t - t'), \]
where \( G_{\text{ret}}(t - t') = \pi / \Omega \sin \Omega(t - t') \) and \( \Gamma_{\text{ret}}(t - t') \) as given in the previous section. Substituting (75) into (73) leads to the correct solutions of the equations (83-71). Note that (75) does not depend on the difference \( \theta - \theta' \).

Therefore the effective superfield action of (65) is
\[ S = \int dz \left\{ \frac{m}{2} \left[ \tilde{D}\phi(z) D\phi(z) + \omega \phi^2(z) \right] - \frac{\lambda}{2} \phi(z) \Phi^{(h)}(z) + \frac{\lambda^2}{2M} \int dz' \phi(z) \Delta_{\text{ret}}(z, z') \phi(z') \right\}, \]
which is time reversal symmetric, hence conservative.

The nonconservative formulation can be obtained following [E], in the superfield formalism. Thus the superfields are doubled and the action \( \Lambda (\Phi_1, \Phi_2, D\Phi_1, D\Phi_2) = L(\Phi_1, D\Phi_1) - L(\Phi_2, D\Phi_2) \) is defined. This action can be written with the superfields \( \phi_{\pm} = 1/\sqrt{2}(\phi_1 \pm \phi_2) \) and \( \Phi_{\pm} = 1/\sqrt{2}(\Phi_1 \pm \Phi_2) \), giving
\[ S = \int dz \left[ \frac{m}{2} \left( \tilde{D}\phi_+ D\phi_- + \tilde{D}\phi_- D\phi_+ \right) + mw \phi_+ \phi_- - \lambda \phi_+ \Phi_+^{(h)} + \frac{M}{2} \left( \tilde{D}\phi_+ D\phi_- + \tilde{D}\phi_- D\phi_+ \right) + M \Omega \phi_+ \Phi_- \right]. \]
We are interested only on the equations of \( \Phi_{\pm} \), which are the same as (72), \[ \frac{1}{2} [\tilde{D}, D] \Phi_{\pm} - \Omega \Phi_{\pm} + \frac{\lambda}{M} \phi_\pm = 0, \] with solution (73). In this case the effective action can be chosen to be causal
\[ S = \int dz \left[ \frac{m}{2} \left( \tilde{D}\phi_+ D\phi_- + \tilde{D}\phi_- D\phi_+ \right) + mw \phi_+ \phi_- - \lambda \phi_+ \Phi_+^{(h)} + \frac{\lambda^2}{M} \int dz' \phi_- (z') \Delta_{\text{ret}}(z, z') \phi_+(z') \right], \]
which in components, after elimination of the auxiliary fields \( b(t) \) and \( B(t) \) is
\[ S_{b, \text{eff}} = \int_{t_i}^{t_f} dt \left[ m \dot{q}_b q_- - m \left( \omega^2 + \frac{\lambda^2}{m} \right) q_+ q_- + \lambda (\omega + \Omega) q_- Q_+^{(h)} + \frac{\lambda^2}{M} (\omega + \Omega)^2 \int_{t_i}^{t_f} dt' \dot{q}_- (t') G_{\text{ret}}(t - t') q_+(t') \right] \]
\[ S_{f, \text{eff}} = \int_{t_i}^{t_f} dt \left[ -i \frac{m}{2} \left( \psi_+ \psi_- + \bar{\psi}_+ \bar{\psi}_- \right) + mw \psi_+ \psi_- - \lambda \psi_+ \bar{\psi}_+^{(h)} \right] \]
\[ - \frac{\lambda^2}{M} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \left[ \psi_- (t) \Gamma_{\text{ret}}(t - t') \psi_+(t') - \bar{\psi}_- (t) \bar{\psi}_+(t') \right] + c.c., \]
where \( G_{\text{ret}}(t - t') = 1/\sqrt{\Omega^2 + \frac{\lambda^2}{M} \sin \Omega^2 + \frac{\lambda^2}{M} (t - t') \text{and} \Gamma_{\text{ret}}(t - t') = -i \theta(t - t') e^{i \Omega(t - t')}. \]
The generalization for the interaction of more than one oscillator \( Q \) can be done straightforwardly following [E].

### D. Damped supersymmetric oscillator

As a last example we consider the damped oscillator, and compute its conservation laws. The conservative lagrangian is the one of the oscillator \( L(\phi, D\phi, \tilde{D}\phi) = \frac{\ell}{2} (\tilde{D}\phi D\phi + \omega \phi^2) \) and \( K(\phi_1, D\phi_1, \tilde{D}\phi_1, \phi_2, D\phi_2, \tilde{D}\phi_2) = i \ell / 2 (\tilde{D}\phi_1 D\phi_2 - D\phi_2 D\phi_1) \). The action of the nonconservative lagrangian \( \Lambda = L(\phi_1) - L(\phi_2) + K(\phi_1, \phi_2) \) is in components
\[ S = \int_{t_i}^{t_f} dt \left[ \frac{m}{2} \left( \dot{q}_1^2 - \dot{q}_2^2 \right) - i \frac{m}{2} \left( \psi_1 \dot{\psi}_1 + \bar{\psi}_1 \dot{\bar{\psi}}_1 - \bar{\psi}_2 \dot{\psi}_2 - \psi_2 \dot{\bar{\psi}}_2 \right) + mw (q_1 b_1 + \psi_1 \bar{\psi}_1 - q_2 b_2 - \psi_2 \bar{\psi}_2) \]
\[ + \frac{m}{2} (\dot{b}_1^2 - \dot{b}_2^2) - \bar{c} (b_1 \dot{q}_2 - b_2 \dot{q}_1) + \frac{\bar{c}}{2} \left( \dot{\psi}_2 \dot{\bar{\psi}}_1 - \dot{\bar{\psi}}_2 \dot{\psi}_1 + \psi_1 \dot{\bar{\psi}}_2 - \psi_2 \dot{\bar{\psi}}_1 \right) \].
As usual, the auxiliary fields can be eliminated by their equations of motion \( b_1 + \omega q_1 - c/m \dot{q}_2 = 0 \) and \( b_2 + \omega q_2 - c/m \dot{q}_1 = 0 \), with the resulting action
\[ S = \int_{t_i}^{t_f} dt \left[ \frac{m}{2} \left( 1 + \frac{\dot{q}_1^2}{m^2} \right) (q_1^2 - \dot{q}_2^2) - \frac{m}{2} (q_1^2 - \dot{q}_2^2) - \frac{m}{2} \left( \psi_1 \dot{\psi}_1 + \bar{\psi}_1 \dot{\bar{\psi}}_1 - \bar{\psi}_2 \dot{\psi}_2 - \psi_2 \dot{\bar{\psi}}_2 \right) + mw (\psi_1 \dot{\psi}_1 - \bar{\psi}_2 \dot{\bar{\psi}}_2) + \omega \bar{c} (q_1 \dot{q}_2 - q_2 \dot{q}_1) + \frac{\bar{c}}{2} \left( \dot{\psi}_2 \dot{\bar{\psi}}_1 - \dot{\bar{\psi}}_2 \dot{\psi}_1 + \psi_1 \dot{\bar{\psi}}_2 - \psi_2 \dot{\bar{\psi}}_1 \right) \].
The equations of motion in the physical limit are (28), (29) and
\[ \ddot{q} - 2\tilde{\omega}\frac{\dot{q}}{m} + \tilde{\omega}^2 \left( 1 + \frac{\tilde{\omega}^2}{m^2} \right) q = 0. \] (83)

The energy and the supersymmetric charges of the conservative system are
\[ E = \frac{m}{2} \left( \dot{q}^2 + \omega^2 q^2 \right) - \omega \psi \bar{\psi}, \] (84)
\[ J = (i\dot{q} + \omega q) \psi, \] (85)
\[ \bar{J} = - (i\dot{q} - \omega q) \bar{\psi}. \] (86)

Their time variations are
\[ \frac{dE}{dt} = 2\tilde{c} \left( -\omega \dot{q}^2 + \dot{\psi} \bar{\psi} \right) = 2\tilde{c} \left[ - \left( 1 + \frac{\tilde{\omega}^2}{m^2} \right) \dot{q}^2 + \omega \psi \bar{\psi} \right], \] (87)
\[ \frac{dJ}{dt} = \dot{\psi} \left[ 2i \dot{q} \bar{\psi} + \omega \left( q \dot{\psi} - \dot{q} \bar{\psi} \right) \right] = \frac{1}{\tilde{\omega}} \left[ \dot{q} - 3i \tilde{\omega} \left( 1 - i \frac{\tilde{\omega}}{m} \right) \dot{q} \right] \psi, \] (88)
and the complex conjugate of the second equation. For the r.h.s. of both equations, the equations of motion (28) and (83) have been used.

V. CONCLUSIONS

We have generalized the variational formalism of Galley for nonconservative systems for fermionic and supersymmetric systems, in the last case in superspace language. Consistency with the first order equations of motion of fermionic variables, requires that the boundary conditions are slightly modified in this and in the supersymmetric case, with no further consequences. Otherwise the generalization is straightforward. Similar to the case of time translational symmetry, for supersymmetric theories we maintain the supersymmetric structure for the nonconservative generalized potential, which is written in terms of superfields. The Noether theorem is evaluated and as expected, the supersymmetric charges are not conserved, unlike the case of internal symmetries, which could be conserved. In the case of coupled oscillators, the effective lagrangian can be written as well in superfield language and can be worked out on-shell or off-shell, i.e. keeping or eliminating the auxiliary fields. It would be interesting to generalize this approach for local supersymmetry, and explore consequences in supersymmetric quantum mechanics, as well as for field theory, in particular at non-zero temperatures.

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