BLOW-UP PROFILE OF GROUND STATES FOR THE CRITICAL
BOSON STAR

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ABSTRACT. We analyze the asymptotic behavior of the ground state solution to the boson star equation when the interaction strength tends to a critical value.

1. Introduction

In this paper, we study the variational problem of a boson star

\[ I(a) := \inf \left\{ E_a(u) : u \in H^{1/2}(\mathbb{R}^3), \| u \|_{L^2}^2 = 1 \right\} \]

where

\[ E_a(u) := \| (-\Delta + m^2)^{1/4} u \|_{L^2}^2 - \frac{a}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|} \, dx \, dy. \]

Here \( m > 0 \) is the mass of particles and \( \sqrt{-\Delta + m^2} \) is the pseudo-differential operator defined as usual via Fourier transform, i.e.

\[ \| (-\Delta + m^2)^{1/4} u \|_{L^2}^2 = \int_{\mathbb{R}^3} \sqrt{\| \xi \|^2 + m^2} |\hat{u}(\xi)|^2 \, d\xi. \]

The coupling constant \( a > 0 \) describes the strength of the attractive interaction, which will play an essential role in our analysis.

The functional \( E_a \) effectively describes the energy per particle of a boson star. The rigorous derivation of this functional from many-body quantum theory can be found in [9], where one may interpret \( a = gN \) with \( g \) the gravitational constant and \( N \) the number of particles. It is a fundamental fact that the boson star collapses when \( N \) is larger than a critical number (the Chandrasekhar limit). In the effective model [13], the collapse simply boils down to the fact that \( I(a) = -\infty \) if \( a \) is larger than a critical value \( a^* \).

From the simple inequality

\[ \sqrt{-\Delta} \leq \sqrt{-\Delta + m^2} \leq \sqrt{-\Delta} + m \]

and a standard scaling argument, we can see that \( a^* \) is the optimal constant in the interpolation inequality

\[ \| (-\Delta)^{1/4} u \|_{L^2}^2 \| u \|_{L^2}^2 \geq \frac{a^*}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|} \, dx \, dy, \quad \forall u \in H^{1/2}(\mathbb{R}^3). \]

1We use the convention

\[ \hat{u}(\xi) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} u(x)e^{-i\xi \cdot x} \, dx. \]
It is well-known (see e.g. [1, Appendix A.2]) that the inequality $(1.4)$ has an optimizer $Q \in H^{1/2}(\mathbb{R}^3)$ which can be chosen to be positive, radially symmetric decreasing and satisfy
\[
\|(-\Delta)^{1/4} Q\|_{L^2} = \|Q\|_{L^2} = \frac{a^*}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|Q(x)|^2 |Q(y)|^2}{|x-y|} \, dx \, dy = 1. \tag{1.5}
\]
Moreover, $Q$ solves the nonlinear equation
\[
\sqrt{-\Delta} Q + Q - a^* \left(|\cdot|^{-1} * |Q|^2\right) Q = 0, \quad Q \in H^{1/2}(\mathbb{R}^3) \tag{1.6}
\]
and it satisfies the decay property (see [11])
\[
Q(x) \leq C(1 + |x|)^{-4}. \tag{1.7}
\]

The uniqueness (up to translation and dilation) of the optimizer for $(1.4)$, as well as the uniqueness (up to translation) of the positive solution to the equation $(1.6)$, is an open problem (see [9, 10] for related discussions). In the following, we denote by $G$ the set of all positive, radially symmetric decreasing functions satisfying $(1.5)$, $(1.6)$.

For the reader’s convenience, we recall the following well-known results.

**Theorem 1** (Existence and non-existence of minimizers). The variational problem $I(a)$ in $(1.2)$ has the following properties.

(i) If $a > a^*$, then $I(a) = -\infty$.

(ii) If $a = a^*$, then $I(a^*) = 0$, but it has no minimizer.

(iii) If $0 < a < a^*$, then $I(a) > 0$ and it has at least one minimizer. The minimizer can be chosen to be non-negative and radially symmetric decreasing.

As already explained, the assertion (i) is a direct consequence of the simple inequality $(1.3)$ and a standard scaling argument. The statement (ii) is a particular case of [2, Theorem 2.1 (ii)]. The existence result in (iii) was first proved in [10] by means of rearrangement inequalities (see also [2] for another proof based on the concentration-compactness method [10]).

Our new result concerns the asymptotic behavior of the minimizers as $a \searrow a^*$.

**Theorem 2** (Blow-up profile of minimizers). Let $u_a$ be a non-negative minimizer of $I(a)$ for $0 \leq a < a^*$. Then for every sequence $\{a_k\} \with a_k \searrow a^*$ as $k \to \infty$, there exist a subsequence of $\{a_k\}$ (still denoted by $\{a_k\}$ for simplicity) and a sequence $\{x_k\} \subset \mathbb{R}^3$, such that
\[
\lim_{k \to \infty} (a^* - a_k)^{3/4} u_{a_k}(x^* - a_k)^{1/2} + x_k) = \lambda^{3/2} Q(\lambda x)
\]
strongly in $H^{1/2}(\mathbb{R}^3)$, where $Q \in G$ and $\lambda > 0$ satisfies
\[
\lambda = m \sqrt{\frac{a^*}{2}} \inf_{W \in G} \|(-\Delta)^{-1/4} W\|_{L^2} = m \sqrt{\frac{a^*}{2}} \|(-\Delta)^{-1/4} Q\|_{L^2}.
\]

Our proof is based on a detailed analysis of the Euler-Lagrange equation associated to variational problem $I(a)$. As a by-product of our proof, we also obtain the asymptotic behavior of the energy
\[
\lim_{a \searrow a^*} \frac{I(a)}{(a^* - a)^{2}} = m \lambda \sqrt{\frac{2}{a^*}}. \tag{1.8}
\]

Our work is inspired by the recent studies in [3, 4, 11] on the concentration of the Bose-Einstein condensate described by the 2D focusing Gross-Pitaevskii equation.
However, unlike the Gross-Pitaevskii equation, our boson star equation (1.6) is nonlocal and the uniqueness of its solution is unknown (it is an open problem). This complicates our analysis in several ways.

2. Behavior of Minimizers

In this section, we prove the blow-up profile of minimizers of $I(a)$ in Theorem 1. Let $a_k \nearrow a^*$ as $k \to \infty$ and let $u_k := u_{a_k}$ be a non-negative minimizer for $I(a_k)$. We have the following Lemmas.

**Lemma 3.** There exist positive constants $M_1 < M_2$ independent of $a_k$ such that

$$M_1(a^* - a_k)^{\frac{1}{2}} \leq I(a_k) \leq M_2(a^* - a_k)^{\frac{1}{2}}. \tag{2.1}$$

**Proof.** We start with proof of the upper bound in (2.1). Let $Q \in G$, by the operator inequality

$$\frac{m^2}{2\sqrt{-\Delta + m^2}} \leq \sqrt{-\Delta + m^2} - \sqrt{-\Delta} = \frac{m^2}{\sqrt{-\Delta + m^2} + \sqrt{-\Delta}} \leq \frac{m^2}{2\sqrt{-\Delta}}, \tag{2.2}$$

we deduce that

$$I(a_k) \leq \mathcal{E}_a(\tau^{3/2} Q(\lambda x)) \leq \tau \left(1 - \frac{a_k}{a^*}\right) + \frac{m^2}{2\tau} \|(-\Delta)^{-1/4} Q\|_{L^2}^2,$$

where $\|(-\Delta)^{-1/4} Q\|_{L^2} < \infty$ because of decay rate of $Q$ in (1.7). By taking $\tau = (a^* - a_k)^{-\frac{1}{2}}$ we arrive at the desired upper bound.

Next we prove the lower bound in (2.1). From (1.4) and the upper bound of $I(a_k)$ in (2.1) we see that

$$M_2(a^* - a_k)^{\frac{1}{2}} \geq \left(1 - \frac{a_k}{a^*}\right) \|(-\Delta + m^2)^{1/4} u_k\|_{L^2}^2,$$

which implies that

$$\|(-\Delta + m^2)^{1/4} u_k\|_{L^2}^2 \leq M_2 a^*(a^* - a_k)^{-\frac{1}{2}}.$$

Again from (1.4), (2.2) and Hölder’s inequality we have

$$I(a_k) = \mathcal{E}_{a_k}(u_k) \geq \frac{2m^2}{2} \|(-\Delta + m^2)^{-1/4} u_k\|_{L^2}^2 \geq \frac{m^2}{2}\|u_k\|_{L^2}^2 \geq M_1(a^* - a_k)^{\frac{1}{2}}.$$

**Lemma 4.** There exist positive constants $K_1 < K_2$ independent of $a_k$ such that

$$K_1(a^* - a_k)^{-\frac{1}{2}} \leq \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u_k(x)|^2 |u_k(y)|^2}{|x - y|} dxdy \leq K_2(a^* - a_k)^{-\frac{1}{2}}. \tag{2.3}$$

**Proof.** From (1.4), we have

$$I(a_k) = \mathcal{E}_{a_k}(u_k) \geq \frac{a^* - a_k}{2} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u_k(x)|^2 |u_k(y)|^2}{|x - y|} dxdy.$$

Hence, the upper bound in (2.3) follows immediately from upper bound of $I(a_k)$ in (2.1).
We choose a constant $k$ and $\tilde{k}$ from the above two inequalities we obtain that
\begin{equation}
\int_{\mathbb{R}^3} \frac{|u_k(x)|^2 |u_k(y)|^2}{|x-y|} \, dx \, dy \geq \int_{\mathbb{R}^3} \frac{|u_b(x)|^2 |u_b(y)|^2}{|x-y|} \, dx \, dy. \tag{2.5}
\end{equation}

From (2.4) and Lemma 3, we deduce that there exist two positive constants $M_1 < M_2$ such that for any $0 < b < a_k < a^*$,
\begin{align*}
\int_{\mathbb{R}^3} \frac{|u_k(x)|^2 |u_k(y)|^2}{|x-y|} \, dx \, dy &\geq 2 \frac{I(b) - I(a_k)}{a_k - b} \\
&\geq 2 \frac{M_1 (a^* - b)^{\frac{1}{2}} - M_2 (a^* - a_k)^{\frac{1}{2}}}{a_k - b}.
\end{align*}

Choosing $b = a_k - \gamma (a^* - a_k)$ with $\gamma > 0$, we arrive at
\begin{equation}
\int_{\mathbb{R}^3} \frac{|u_k(x)|^2 |u_k(y)|^2}{|x-y|} \, dx \, dy \geq (a^* - a_k)^{\frac{\gamma}{2}} - M_2 2^{-1} \frac{M_1 (1 + \gamma)^{\frac{1}{2}}}{\gamma}.
\end{equation}

The last fraction is positive for $\gamma$ large enough. For $a_k$ closes to $a^*$, then there exists a positive constant $k_1$ such that
\begin{equation}
\int_{\mathbb{R}^3} \frac{|u_k(x)|^2 |u_k(y)|^2}{|x-y|} \, dx \, dy \geq k_1 (a^* - a_k)^{-\frac{1}{2}}.
\end{equation}

For smaller $a_k$, we use (2.4) to obtain that for any $0 \leq a_k \leq a^*$,
\begin{equation}
\int_{\mathbb{R}^3} \frac{|u_k(x)|^2 |u_k(y)|^2}{|x-y|} \, dx \, dy \geq \int_{\mathbb{R}^3} \frac{|u_0(x)|^2 |u_0(y)|^2}{|x-y|} \, dx \, dy.
\end{equation}

We choose a constant $k_2 > 0$ such that
\begin{equation}
\int_{\mathbb{R}^3} \frac{|u_0(x)|^2 |u_0(y)|^2}{|x-y|} \, dx \, dy \geq k_2 (a^* - a_k)^{-\frac{1}{2}}.
\end{equation}

Therefore, we arrive at the desired lower bound by choosing $K_1 = \min \{k_1, k_2\}$.

Now let $\epsilon_k := (a^* - a_k)^{\frac{1}{2}} > 0$, we see that $\epsilon_k \to 0$ as $k \to \infty$. We define $\tilde{w}_k(x) := \epsilon_k^{3/2} u_k(\epsilon_k x)$ be $L^2$-normalized of $u_k$. It follows from Lemma 3 that
\begin{equation}
K_1 \leq \int_{\mathbb{R}^3} \frac{1}{|x-y|} |\tilde{w}_k(x)\tilde{w}_k(y)|^2 \, dx \, dy \leq K_2. \tag{2.6}
\end{equation}

We claim that there exist a sequence $\{y_k\} \subset \mathbb{R}^3$ and positive constants $R_0$ such that
\begin{equation}
\liminf_{k \to \infty} \int_{B(y_k, R_0)} |\tilde{w}_k(x)|^2 \, dx > 0. \tag{2.7}
\end{equation}
On the contrary, we assume that for any $R$ there exists a subsequence of $\{a_k\}$, still denoted by $\{a_k\}$, such that
\[
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B(y, R)} |\tilde{w}_k(x)|^2 \, dx = 0. \tag{2.8}
\]
It follows from [7, Lemma 9] that $\lim_{k \to \infty} \|\tilde{w}_k\|_{L^q} = 0$ for any $2 \leq q < 3$. By the Hardy-Littlewood-Sobolev inequality (see, e.g., [8, Theorem 4.3]) we have
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\tilde{w}_k(x)|^2 |\tilde{w}_k(y)|^2}{|x - y|} \, dx \, dy \to 0
\]
as $k \to \infty$, which contradicts to (2.6). Thus (2.8) does not occur, and hence (2.7) holds true.

Let $w_k$ be non-negative $L^2$-normalize of $u_k$, defined by
\[
w_k(x) = \tilde{w}_k(x + y_k) = \epsilon_k^{3/2} u_k(\epsilon_k x + \epsilon_k y_k).
\]
It follows from (1.6) and Lemma 4 that there exist constant $C > 0$ such that
\[
\|(-\Delta)^{1/4} w_k\|^2_{L^2} = \epsilon_k \|(-\Delta)^{1/4} u_k\|^2_{L^2} \leq C.
\]
Thus $w_k$ is bounded in $H^{1/2}(\mathbb{R}^3)$, and hence we may assume that $w_k \rightharpoonup w$ weakly in $H^{1/2}(\mathbb{R}^3)$ and $w_k \to w$ pointwise almost everywhere. Extracting a subsequence if necessary, we have $w_k \to w$ strongly in $L^q_{\text{loc}}(\mathbb{R}^3)$ for $2 \leq q < 3$, thanks to a Rellich-type theorem (see, e.g., [3, Theorem 8.6]).

Since $u_k$ is a non-negative minimizer of $I(a_k)$, it solves the Euler-Lagrange equation
\[
\sqrt{-\Delta + m^2} u_k(x) = \mu_k u_k(x) + a_k(|-1 \ast |u_k|^2)(x)u_k(x), \tag{2.9}
\]
where
\[
\mu_k = I(a_k) - \frac{a_k}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u_k(x)|^2 |u_k(y)|^2}{|x - y|} \, dx \, dy. \tag{2.10}
\]
We see that $w_k$ is a non-negative solution to
\[
\sqrt{-\Delta + m^2} \epsilon_k w_k(x) = \epsilon_k \mu_k w_k(x) + a_k(|-1 \ast |w_k|^2)(x)w_k(x). \tag{2.11}
\]
We deduce from Lemma 3 and 4 that $\epsilon_k \mu_k$ is bounded uniformly and strictly negative as $k \to \infty$. By passing to a subsequence if necessary, we can thus assume that $\epsilon_k \mu_k$ converges to some number $-\lambda < 0$ as $k \to \infty$.

Passing (2.11) to weak limit, we have $w$ be a non-negative solution to
\[
\sqrt{-\Delta} w(x) = -\lambda w(x) + a^*(|^{-1 \ast |w|^2})(x)w(x).
\]
By a simple scaling we see that
\[
Q(x) = \lambda^{-3/2} w(\lambda^{-1} x)
\]
is a non-negative solution of (1.0). It is well-known that the kernel of the resolvent $((-\Delta)^{1/2} + 1)^{-1}$ is strictly positive on $\mathbb{R}^3$ (see [3 eq. (A.11)]), hence we obtain from (1.6) that either $Q \equiv 0$ or $Q > 0$ on $\mathbb{R}^3$. We infer from (2.7) and the convergence of $w_k$ in $L^2_{\text{loc}}(\mathbb{R}^3)$ that
\[
\int_{B(0, R_0)} |w(x)|^2 \, dx = \lim_{k \to \infty} \int_{B(0, R_0)} |w_k(x)|^2 \, dx = \lim_{k \to \infty} \int_{B(y_k R_0)} |\tilde{w}_k(x)|^2 \, dx > 0.
\]
This implies that $w \neq 0$, and hence $Q \neq 0$. 

BLOW-UP PROFILE OF GROUND STATES FOR THE CRITICAL BOSON STAR
We prove that \( Q \in G \). First, since \( Q \) is a positive solution of (1.6), it follows from [1, Theorem 1.1] that \( Q \) is positive, radially symmetric decreasing up to translation. It remains to prove that \( Q \) satisfies (1.5). Indeed, by the same reason that \( Q \) is a solution of (1.6), we have

\[
\| (-\Delta)^{1/4} Q \|_{L^2}^2 + \| Q \|_{L^2}^2 = a^* \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|Q(x)|^2 |Q(y)|^2}{|x-y|} \, dx \, dy.
\]  

(2.12)

Thus, by (1.4) and the Cauchy-Schwarz inequality we have

\[
1 \leq \frac{2}{a^*} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|Q(x)|^2 |Q(y)|^2}{|x-y|} \, dx \, dy \leq \frac{a^*}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|Q(x)|^2 |Q(y)|^2}{|x-y|} \, dx \, dy.
\]  

(2.13)

Let \( V \) be in \( G \), we have

\[
1 = \| V \|_{L^2}^2 = \| (-\Delta)^{1/4} V \|_{L^2}^2 = \frac{a^*}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|V(x)|^2 |V(y)|^2}{|x-y|} \, dx \, dy.
\]

We define \( v(x) = \left( \frac{\lambda}{\epsilon_k} \right)^{3/2} V \left( \frac{\lambda}{\epsilon_k} x \right) \). Then we have \( \| v \|_{L^2}^2 = \| V \|_{L^2}^2 = 1 \) and

\[
\epsilon_k \mathcal{E}_{a_k} (v) = \lambda \| (-\Delta + m^2 \lambda^{-2})^{1/4} V \|_{L^2}^2 - \frac{\lambda a_k}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|V(x)|^2 |V(y)|^2}{|x-y|} \, dx \, dy
\]

\[
= \lambda \left( 1 - \frac{a_k}{a^*} \right) + m^2 \lambda^{-1} \| (-\Delta + m^2 \lambda^{-2})^{1/2} + (-\Delta)^{1/4} V \|_{L^2}^2
\]

\[
\leq \lambda \left( 1 - \frac{a_k}{a^*} \right) + m^2 \lambda^{-1} \| (-\Delta)^{-1/4} V \|_{L^2}^2.
\]

Since \( w_k \) satisfies (2.11) we have

\[
\epsilon_k \mathcal{E}_{a_k} (u_k) = \| (-\Delta + m^2 \lambda^{-2})^{1/4} w_k \|_{L^2}^2 - \frac{a_k}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w_k(x)|^2 |w_k(y)|^2}{|x-y|} \, dx \, dy
\]

\[
= \epsilon_k \mu_k \int_{\mathbb{R}^3} |w_k(x)|^2 \, dx + \frac{a_k}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w_k(x)|^2 |w_k(y)|^2}{|x-y|} \, dx \, dy.
\]

By assumption that \( u_k \) is a minimizer of \( I(a_k) \), we have \( \mathcal{E}_{a_k} (u_k) \leq \mathcal{E}_{a_k} (v) \) and hence

\[
\liminf_{k \to \infty} \epsilon_k \mathcal{E}_{a_k} (u_k) \leq \liminf_{k \to \infty} \epsilon_k \mathcal{E}_{a_k} (v).
\]

From the above estimates and by Fatou’s Lemma we deduce that

\[
\frac{a^*}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|Q(x)|^2 |Q(y)|^2}{|x-y|} \, dx \, dy = \frac{a^*}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w(x)|^2 |w(y)|^2}{|x-y|} \, dx \, dy
\]

\[
\leq \liminf_{k \to \infty} \frac{a_k}{2\lambda} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w_k(x)|^2 |w_k(y)|^2}{|x-y|} \, dx \, dy \leq 1.
\]

This inequality together with (2.12) and (2.13), imply that

\[
1 = \frac{a^*}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|Q(x)|^2 |Q(y)|^2}{|x-y|} \, dx \, dy = \| (-\Delta)^{1/4} Q \|_{L^2}^2 = \| Q \|_{L^2}^2.
\]

Thus we have proved that \( Q \in G \).

We note that \( \| w \|_{L^2} = \| Q \|_{L^2} = 1 \). From the norm preservation, we conclude that \( w_k \to w \) strongly in \( L^2(\mathbb{R}^3) \). In fact, \( w_k \to w \) strongly in \( L^q(\mathbb{R}^3) \) for \( 2 \leq q < 3 \).
because of $H^{1/2}(\mathbb{R}^3)$ boundedness. By the Hardy-Littlewood-Sobolev inequality we have
\[
\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w_k(x)|^2 |w_k(y)|^2}{|x-y|} \, dx \, dy \rightarrow \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w(x)|^2 |w(y)|^2}{|x-y|} \, dx \, dy
\]
as $k \to \infty$. We deduce from this convergence and the inequality
\[
\epsilon_k \mathcal{E}_{a_k} (\epsilon_k^{-3/2} w_k (\epsilon_k^{-1} x - y_k)) = \epsilon_k \mathcal{E}_{a_k} (u_k) \leq \epsilon_k \mathcal{E}_{a_k} (\epsilon_k^{-3/2} w (\epsilon_k^{-1} x))
\]
that
\[
\limsup_{k \to \infty} \|(-\Delta)^{1/4} w_k\|^2_{L^2} \leq \limsup_{k \to \infty} \|(-\Delta + m^2 \epsilon_k^2)^{1/4} w_k\|^2_{L^2}
\]
\[
\leq \limsup_{k \to \infty} \|(-\Delta + m^2 \epsilon_k^2)^{1/4} w\|^2_{L^2} = \|(-\Delta)^{1/4} w\|^2_{L^2}.
\]
On the other hand, since $w_k \to w$ weakly in $H^{1/2}(\mathbb{R}^3)$, by Fatou’s Lemma we have
\[
\liminf_{k \to \infty} \|(-\Delta)^{1/4} w_k\|^2_{L^2} \geq \|(-\Delta)^{1/4} w\|^2_{L^2}.
\]
Therefore we have proved that
\[
\lim_{k \to \infty} \|(-\Delta)^{1/4} w_k\|^2_{L^2} = \|(-\Delta)^{1/4} w\|^2_{L^2},
\]
which implies that $w_k \to w$ strongly in $H^{1/2}(\mathbb{R}^3)$.

We have thus shown that there exist a subsequence of $\{a_k\}$ and a sequence $\{x_k\} \subset \mathbb{R}^3$ where $x_k = \epsilon_k y_k = (a^* - a_k)^{1/2} y_k$, such that
\[
(a^* - a_k)^{3/4} u_{a_k} (x(a^* - a_k)^{1/2} + x_k) = w_k(x) \to w(x) = \lambda^{3/2} Q (\lambda x)
\]
strongly in $H^{1/2}(\mathbb{R}^3)$, where $Q \in G$ and $\lambda > 0$.

To complete the proof of Theorem 2 we compute the exact value of $\lambda$. Since $u_k(x) = \epsilon_k^{-3/2} w_k (\epsilon_k^{-1} x - y_k)$ is a minimizer of $I(a_k)$ we have
\[
I(a_k) = \frac{1}{\epsilon_k} \left( \|(-\Delta + m^2 \epsilon_k^2)^{1/4} w_k\|^2_{L^2} - \frac{\alpha^*}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w_k(x)|^2 |w_k(y)|^2}{|x-y|} \, dx \, dy \right)
+ \frac{\alpha^* - a_k}{2 \epsilon_k} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w_k(x)|^2 |w_k(y)|^2}{|x-y|} \, dx \, dy
\]
\[
\geq m^2 \epsilon_k \|((-\Delta + m^2 \epsilon_k^2)^{1/2} + (-\Delta)^{1/2})^{-1/2} w_k\|^2_{L^2}
\]
\[
+ \frac{\epsilon_k}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w_k(x)|^2 |w_k(y)|^2}{|x-y|} \, dx \, dy,
\]
where we have used (1.4) for the term in the brackets.

By Fatou’s Lemma, we have
\[
\liminf_{k \to \infty} \|((-\Delta + m^2 \epsilon_k^2)^{1/2} + (-\Delta)^{1/2})^{-1/2} w_k\|^2_{L^2}
\]
\[
\geq \frac{1}{2} \|(-\Delta)^{-1/4} w\|^2_{L^2} = \frac{1}{2 \lambda} \|(-\Delta)^{-1/4} Q\|^2_{L^2},
\]
(2.15)
We note that, for any $Q \in G$, by Höder’s inequality we have
\[
\|(-\Delta)^{-1/4} Q\|_{L^2} \geq \frac{\|Q\|^2_{L^2}}{\|(-\Delta)^{1/4} Q\|_{L^2}} = 1,
\]
which implies that $\inf_{Q \in G} \|(-\Delta)^{-1/4} Q\|_{L^2}$ is well defined.
By the Hardy-Littlewood-Sobolev inequality, we have
\[
\lim_{k \to \infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w_k(x)|^2 |w_k(y)|^2}{|x-y|^2} \, dx \, dy = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w(x)|^2 |w(y)|^2}{|x-y|^2} \, dx \, dy = \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|Q(x)|^2 |Q(y)|^2}{|x-y|^2} \, dx \, dy = \frac{2\lambda}{a^*}.
\]
(2.16)
We deduce from (2.14), (2.15) and (2.16) that
\[
\lim \inf_{k \to \infty} \frac{I(a_k)}{\epsilon_k} \geq \frac{m^2}{2\lambda} \|(\Delta)^{-1/4} Q\|_{L^2}^2 + \frac{\lambda}{a^*}.
\]
Thus
\[
\lim \inf_{k \to \infty} \frac{I(a_k)}{\epsilon_k} \geq \inf_{\lambda > 0} \left( \frac{m^2}{2\lambda} \|(\Delta)^{-1/4} Q\|_{L^2}^2 + \frac{\lambda}{a^*} \right) = m \sqrt{\frac{2}{a^*}} \|(\Delta)^{-1/4} Q\|_{L^2}.
\]
(2.17)
To see the matching upper bound in (2.17), one simply takes
\[
u_k(x) = \left( \frac{\tilde{\lambda}}{\epsilon_k} \right)^{3/2} W \left( \frac{\tilde{\lambda}}{\epsilon_k} x \right)
\]
where \(\tilde{\lambda} > 0\) and \(W \in G\), as trial state for \(E_{a_k}\). We use (1.5) and operator inequality (2.2) to obtain
\[
I(a_k) \leq \frac{\tilde{\lambda}}{\epsilon_k} \left( \|(-\Delta + m^2 \epsilon_k \tilde{\lambda}^{-2})^{1/4} W\|_{L^2}^2 - \frac{a^*}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|W(x)|^2 |W(y)|^2}{|x-y|} \, dx \, dy \right) + \frac{\tilde{\lambda} a^* - a_k}{2\epsilon_k} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|W(x)|^2 |W(y)|^2}{|x-y|} \, dx \, dy
\]
\[
\leq \frac{m^2 \epsilon_k}{2\lambda} \|(\Delta)^{-1/4} W\|_{L^2}^2 + \frac{\epsilon_k \tilde{\lambda}}{a^*}.
\]
This implies that
\[
\lim \sup_{k \to \infty} \frac{I(a_k)}{\epsilon_k} \leq \frac{m^2}{2\lambda} \|(\Delta)^{-1/4} W\|_{L^2}^2 + \frac{\tilde{\lambda}}{a^*}.
\]
Thus, taking the infimum over \(W \in G\) and \(\tilde{\lambda} > 0\) we see that
\[
\lim \sup_{k \to \infty} \frac{I(a_k)}{\epsilon_k} \leq m \sqrt{\frac{2}{a^*}} \inf_{W \in G} \|(\Delta)^{-1/4} W\|_{L^2} \leq m \sqrt{\frac{2}{a^*}} \|(\Delta)^{-1/4} Q\|_{L^2}.
\]
(2.18)
From (2.17) and (2.18) we conclude that
\[
\lim_{k \to \infty} \frac{I(a_k)}{\epsilon_k} = m \sqrt{\frac{2}{a^*}} \|(\Delta)^{-1/4} Q\|_{L^2}, \quad \inf_{W \in G} \|(\Delta)^{-1/4} W\|_{L^2} = \|(\Delta)^{-1/4} Q\|_{L^2}
\]
and
\[
\lambda = m \sqrt{\frac{a^*}{2}} \|(\Delta)^{-1/4} Q\|_{L^2} = m \sqrt{\frac{a^*}{2}} \inf_{W \in G} \|(\Delta)^{-1/4} W\|_{L^2}.
\]
We note that the limit of \(I(a_k)/\epsilon_k\) is independent of the subsequence \(\{a_k\}\). Therefore, we have the convergence of the whole family in (1.8)
\[
\lim_{a \searrow a^*} \frac{I(a)}{(a^* - a)^{1/2}} = m\lambda \sqrt{\frac{2}{a^*}}.
\]
The proof is complete.

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