On the degrees of freedom of a black hole

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Abstract

By examining whether black holes fulfill the theorem of equipartition of energy we find that the notion of degrees of freedom, previously introduced for cosmic horizons, is meaningful in the case of Schwarzschild and Kerr black holes. However, for Reissner-Nördstrom and Kerr-Newman black holes this notion fails.

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I. INTRODUCTION

As is widely acknowledged nowadays, the complete gravitational collapse of matter typically produces a black hole [1, 2]. Very generally, a fraction of the energy of the pre-existing body is radiated away during the collapse [3] whereby the final entropy (the entropy carried in the outflow of energy plus the entropy of the resulting black hole) is necessarily larger than the entropy of the said body. Roughly speaking, the latter is expected to vary with the number of particles composing it and therefore with the number of unfrozen degrees of freedom.

In the case of a black hole the concept of “number of particles” is meaningless but the notion of “number of unfrozen degrees of freedom” may not. According to the equipartition theorem each unfrozen degree of freedom of a system at equilibrium at temperature $T$ contributes a fixed quantity, say $\xi k_B T$, to the energy of the system. Since black holes possess entropy and temperature it is natural to associate a certain number $N$ of (unfrozen) degrees of freedom to them. This number is usually taken as the area of the black hole’s event horizon over the Planck length to the square, $A/\ell_p^2$ —see e.g. [4]. However, presently, this remains an unsubstantiated conjecture. To the best of our knowledge, a proof from first principles based on quantum gravity is still lacking.

It is noteworthy that while the entropy of a gas is approximately proportional to the number of particles composing it [5] —therefore to the mass of the gas—, the entropy of the black hole resulting from the full gravitational collapse of the gas varies as the square of its mass. Loosely speaking, this sudden and huge increase may be regarded as a phase transition. This sharply contrasts with phase transitions in non-gravitational physics since in the latter, this entropy increase solely occurs when the system transits from a condensed state to a non-condensed one. It illustrates of how deeply affected the thermodynamic behavior of a system is when gravity dominates its evolution.

In the absence of a quantum gravity argument in favor of the said conjecture it seems worthwhile to study whether the area (in Planck’s units) of a classical black hole satisfies the equipartition of energy. If it does, then the notion of “degrees of freedom” of a black
hole should receive a strong support. In this paper we take the aforesaid conjecture (i.e., that the number of degrees of freedom of a black hole is given by $N = A/\ell_p^2$) as a working hypothesis and explore whether the equipartition theorem, $M = \xi N k_B T$, is satisfied. As it turns out, Schwarzschild black holes obey it with $\xi = 1/2$. Strictly speaking rotating black holes do not fulfil it though they satisfy a generalized version of the theorem. Charged black holes fail to comply with it. We use units such that $c = G = k_B = \hbar = 1$.

II. KERR AND SCHWARZSCHILD BLACK HOLES

We begin by considering whether rotating, uncharged black holes fulfill the equipartition of energy.

The Smarr’s formula in this case reads

$$M = \frac{\kappa}{4\pi} A + 2\Omega J,$$

(1)

where $J$ and $\Omega$ are the angular momentum of the black hole and the angular velocity of the event horizon, respectively, and

$$\kappa = \frac{1}{2M} \frac{\sqrt{1 - J^2/M^4}}{1 + \sqrt{1 - J^2/M^4}},$$

(2)

denotes the acceleration felt by a test particle on the event horizon. Related to it is the black hole temperature defined by $T = \kappa/2\pi$.

Using $J = M\Omega A/4\pi$ and identifying the number of unfrozen degrees of freedom with the horizon area leads to

$$M = \frac{\pi}{2\pi - \Omega^2 N} N T.$$

(3)

Notice that the condition $J^2 < M^4$ ensures that for regular black holes $N < 2\pi/\Omega^2$, hence the right hand side of last equation never becomes negative. For extreme Kerr black holes ($J^2 = M^4$) the first term on the right diverges and $N T$ vanishes. Also, the aforesaid side remains finite and equal to $M$, as it should.

Equation (3) suggests that every degree of freedom contributes $\xi T$, where

$$\xi = \frac{\pi}{2\pi - \Omega^2 N},$$

(4)
to the black hole mass. Thus, strictly speaking, the equipartition theorem does not hold for Kerr black holes, since $\xi$ depends on $N$. Nevertheless, they satisfy a direct generalization of the theorem since, given $N$, each degree of freedom contributes to $M$ by the same amount.

Consider two black holes of the same area but with different angular velocities (and therefore, different masses and temperatures). The black hole with the larger $\Omega$ will have the larger $\xi$ and, because of (2) and the relationship $J = M\Omega A/4\pi$, the lower temperature. This is most reasonable from the point of view of the equipartition theorem.

On the other hand, the fact that $\xi$ remains finite for non-extreme Kerr black holes is fully consistent with the third law of black hole thermodynamics (i.e., that $\kappa$ cannot be made vanish by a continuous process of absorption of matter that satisfies the weak energy condition [7]). We may conclude that the third law and the non-divergence of $\xi$ mutually imply each other.

Further, this is in keeping: (i) with the well known fact that in the absorption of a particle the variation of the black hole parameters is constrained by the relationship $\delta J < \delta M/\Omega$, where $\delta J$ and $\delta M$ coincide with the angular momentum and energy, respectively, of the particle measured by an observer at infinity -see, e.g. [8]. And (ii), with Page result that in Hawking radiance the angular momentum of the black hole is emitted faster than its energy [9].

Clearly, one can write the quantity $\xi$ in terms of $J$ and $M$ but the simple formula (3) for the mass gets lost. Instead, one obtains a cubic equation for $M$, not an expression of the equipartition theorem since the mass does not longer appear proportional to the temperature when $\Omega$ is replaced by $J$. To see this from a different angle, let us assume that (thanks to astrophysical measurements) the area, angular velocity and temperature of a Kerr black hole are experimentally known, but neither its angular momentum nor its mass (though they can be derived). Then, while $M$ can be obtained directly by (3) it cannot by the Smarr’s formula, Eq. (1).

From (1) it is immediately seen that for Schwarzschild black holes ($J = 0$) the dimensionless
quantity $\xi$ reduces to a constant ($1/2$ in this case). Therefore, we can say that non-rotating, uncharged black holes satisfy the equipartition theorem. Intriguing enough, this $\xi$ value coincides with the corresponding one to systems whose Hamiltonian is a quadratic function of the linear momentum of its particles —see e.g. [10]; something far removed from fully gravitationally collapsed objects.

III. KERR-NEWMAN BLACK HOLES

In the case of rotating charged black holes Smarr’s formula generalizes to

$$M = \frac{\kappa}{4\pi} A + 2\Omega J + \Phi Q,$$

where

$$\Phi = \frac{1}{M} \left[ \frac{Q}{2} + \frac{2\pi Q^3}{A} \right].$$

stands for the electrostatic potential on the black hole event horizon generated by the charge $Q$.

In view of this it is not possible to express the black hole mass as $M = \xi N T$ being $\xi$ a function of $J$ and $Q$ but not of $M$. This implies that the equipartition theorem does not hold for charged black holes and, therefore, they must possess additional degrees of freedom whose contribution to the black hole mass do not obey the simple $\xi T$ rule. Hence a baffling situation arises. Considers a Kerr black hole. There, as we have seen, a generalized version of the equipartition theorem is satisfied. However, it suffices the fall of a single electron on the black hole for the said theorem to break down right away.

One may try to solve this puzzle as follows. When the charge is small ($Q^2 \ll M^2$), the Smarr’s formula can be approximated by

$$M \simeq \frac{\kappa}{4\pi} A + \frac{\Omega^2 A}{2\pi} M + \frac{Q^2}{2M},$$

where we have used the relationship $J = M\Omega A/4\pi$. Solving for $M$, discarding the minus sign before the square root and expanding the latter in terms of $Q^2$, we arrive to $M \simeq \xi N T$ with

$$\xi = \frac{\pi}{2\pi - \Omega^2 N} \left[ 1 + \frac{2\pi - \Omega^2 N}{4\pi \left( \frac{\kappa N}{4\pi} \right)^2} Q^2 \right] + \mathcal{O}(Q^4).$$
Nevertheless, this does not solve the problem at all because the black hole temperature enters (via $\kappa$) the expression for $\xi$. Therefore, the generalized equipartition theorem fails also when the electric charge is small; i.e., it does not cease to hold smoothly but abruptly and the puzzle remains.

Altogether, the fulfillment of the equipartition theorem by Schwarzschild and a generalized version of it for Kerr black holes strongly suggests that the notion of degrees of freedom makes sense for these black holes and that the area of their horizon gives (in Planck units) the number of their unfrozen degrees freedom. For charged black holes, however, the theorem breaks down and the black hole area does no longer counts the aforesaid number.

IV. DISCUSSION

In the absence of a proof from first principles of the conjecture that the number of degrees of freedom of a black hole is given by the area (in Planck units) of its event horizon, it seems reasonable to explore whether the said number is consistent with the equipartition theorem of statistical physics. A positive answer would lend support to that conjecture. However, the overall result is inconclusive. While Schwarzschild black holes fulfill the theorem and Kerr black holes satisfy a generalized version of it, charged black holes do not.

As we have seen in the second section, there is a strong connection between $\xi$ and $\kappa$. The natural requirement that the former should remain finite implies that the latter cannot vanish (and vice versa). So, in a way, the third law of black hole thermodynamics sets an upper limit, for a given angular momentum, $J$, on $\xi$. To the best of our knowledge, this interesting feature was never noticed.

One cannot avoid wonder why neither Reissner-Nordström nor Kerr-Newman black holes comply with the equipartition theorem. Why the electric charge behaves so dissimilarly to the angular momentum in this respect? It may be related to the fact that in Schwarzschild spacetimes there is one killing vector, $t^a = \partial x^a / \partial t$, directly connected to existence of the black hole mass. In the case of Kerr spacetimes there is one further Killing vector entirely related to the rotation of the event horizon, i.e., $\phi^a = \partial x^a / \partial \phi$. By contrast, no Killing
vector is related to the presence of the electric charge, neither in Reissner-Nordström nor Kerr-Newman spacetimes. Nevertheless, for the time being the relationship, if it exists, between the mentioned Killing fields and the equipartition theorem remains a mystery. Hopefully, quantum gravity will provide someday a convincing answer to it.

One may argue that, actually, the equipartition theorem is confined to equilibrium systems and, strictly speaking, this is not the case of isolated black holes since their mass steadily diminish via Hawking emission. We do not think this is a serious hurdle. The mass of a classical black hole is much larger than the Planck mass. Hence the rate of mass loss is negligible even when compared to the Hubble constant (bear in mind that $-\dot{M} \sim M^{-2}$). On the other hand, a Kerr black hole can be brought to a stable thermodynamic equilibrium by enclosing it in a box filled with radiation at the temperature of the black hole and rotating at the angular speed of the latter, provided that the radiation energy in the box does not exceed one fourth of the black hole mass [11].

Thus far our interest was restricted to classical black holes. When quantization is incorporated the equipartition theorem still holds, provided that $M \gg 1$, but the number of degrees of freedom differs though slightly. Recalling that the area spectrum of quantum Schwarzschild black holes is [12]

$$A_n = 4(\ln 3) n \quad (n = 1, 2, ...),$$

and that $N$ has to be an integer, it follows $N = (\ln 3) A_n$. Thereby,

$$M = \frac{\ln 3}{2} N T$$

valid for $n \gg 1$. Thus, each degree of freedom contributes $(\ln 3)T/2$ to the mass of a large Schwarzschild quantum black hole.
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