Similarity solutions for a class of Fractional Reaction-Diffusion equation

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This work studies exact solvability of a class of fractional reaction-diffusion equation with the Riemann-Liouville fractional derivatives on the half-line in terms of the similarity solutions. We derived the conditions for the equation to possess scaling symmetry even with the fractional derivatives. Relations among the scaling exponents are determined, and the appropriate similarity variable introduced. With the similarity variable we reduced the differential partial differential equation to a fractional ordinary differential equation. Exactly solvable systems are then identified by matching the resulted ordinary differential equation with the known exactly solvable fractional ones. Several examples involving the three-parameter Mittag-Leffler function (Kilbas-Saigo function) are presented. The models discussed here turn out to correspond to superdiffusive systems.

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I. INTRODUCTION

The well-known Brownian motion is characterized by a mean-squared displacement relation $\langle x^2 \rangle \sim t$ between displacement $x$ and time $t$. However, there are situations in which $\langle x^2 \rangle \sim t^\gamma (\gamma \neq 1)$ is not linear in time. Such are called the anomalous diffusions – superdiffusive for $\gamma > 1$, subdiffusive for $\gamma < 1$. Anomalous diffusions have been found in many physical situations. For examples, charge carrier transport in amorphous semiconductors, nuclear magnetic resonance diffusometry in percolative, and porous systems, Rouse or reptation dynamics in polymeric systems, transport on fractal geometries, and many others (see [1] for a nice review).

The Brownian motion is Markovian in nature. This means each new step in the motion depends only on the present state, and is independent of the the previous states. Anomalous diffusion arises, on the contrary, from some memory effect of previous states, or as a result of some fractal structure of the background space, or due to some non-linear interaction inherent in the system, etc. Accordingly, the most commonly employed approaches to model anomalous diffusion are by fractional differential equations for diffusion equations [2–13], Fokker-Planck equations [14–22], and reaction-diffusion equations [23–29]; by various nonlinear extensions of the Fokker-Planck equations [30–34]; or both [35].

Among various ways to model anomalous diffusion, the approach based on differential equations with fractional differential operators [36–39] has attracted much attention in recent years. This is mainly because these equations, despite the presence of fractional derivatives, can still be linear (except the nonlinear type), and as such many familiar techniques in ordinary calculus can still be employed. Other than statistical physics, fractional differential equations have also found applications recently in many other areas, such as quantum mechanics [40–42], fluid mechanics [43–48], and nonlinear science [49–55].

Several models of anomalous diffusions, mostly with different definitions of time-fractional derivatives, have been proposed and studied in the past. In this work we would like to consider exact solvability of the following class of spatial fractional Reaction-Diffusion equations (FRDE) (with the Fokker-Planck equation as its special case),

$$\frac{\partial}{\partial t} P(x, t) = s D_x^\beta (D(x, t)P(x, t)) + R(P, x, t),$$

(1)

where $P(x, t)$ is the particle distribution function, $D(P, x, t)$ is the diffusion coefficient, and $R(P, x, t)$ the reaction term which accounts for the local reaction or external force. This equation describes the anomalous change of concentration/population of a substance/species distributed in space under the influence of two processes: local reaction which modify the concentration/population, and diffusion which causes the substances/species to spread in space. We use the term “particle” to denote generally the number of basic member of a substance or a species. Our interest in spatial fractional derivative is motivated by the fact that diffusion equations with spatial fractional derivatives allow one to account for certain superdiffusive processes, called the Lévy flights, which have divergent mean squared displacement $\langle x^2 \rangle \sim t \gamma$. Here we also include possible effect of the external reaction .

As several definitions of fractional derivatives are available, we shall adopt the more commonly discussed Riemann-
Here we have used the identity
\[ c \] where the scaling exponents \( \tau \) and \( \delta \) are some real parameters. Suppose under this transformation, the distribution function, the diffusion coefficient and the reaction term scale as
\[ \epsilon \] where the scale factor \( \epsilon \) and the two scaling exponents \( a \) and \( b \) are real parameters. Suppose under this transformation, the distribution function, the diffusion coefficient and the reaction term scale as
\[ P(x, t) = \epsilon^c \hat{P}(\tilde{x}, \tilde{t}), \]
\[ D(P, x, t) = \epsilon^d \hat{D}(\hat{P}, \tilde{x}, \tilde{t}), \]
\[ R(P, x, t) = \epsilon^e \hat{R}(\hat{P}, \tilde{x}, \tilde{t}), \]
where the scaling exponents \( c, d \) and \( e \) are some real parameters. Then
\[ \frac{\partial}{\partial t} P(x, t) = \epsilon^{-b} \frac{\partial \hat{P}}{\partial \tilde{t}}, \]
\[ s D_x^\beta ((D(x, t)P(x, t))) = \epsilon^{c+d} s D_x^{\beta} (\hat{D}(\tilde{x}, \tilde{t})\hat{P}(\tilde{x}, \tilde{t})) \]
\[ = \epsilon^{-\beta a+c+d} s D_x^{\beta} (\hat{D}(\tilde{x}, \tilde{t})\hat{P}(\tilde{x}, \tilde{t})), \]
\[ s = \epsilon^{-a} s. \]
Here we have used the identity
\[ s D_x^\beta f(\tilde{x} = bx) = b^\beta \frac{\partial}{\partial b} D_x^\beta f(\tilde{x}), \]
which can be easily checked from Eq.\[ 3. \]
Written in the transformed variables, Eq.\[ 1 \] becomes
\[ \epsilon^{-b} \frac{\partial \hat{P}}{\partial \tilde{t}} = \epsilon^{-\beta a+c+d} s D_x^{\beta} (\hat{D}(\tilde{P}) + \epsilon^e \hat{R}). \]
For simplicity and clarity of presentation, here and below we shall often omit the independent variables in a function. Eq. (7) has the same functional form as Eq. (1), if and only if the scaling indices satisfy $b = \beta a - d = c - e$, and $s = \epsilon^{-s} - s = s$. The latter condition implies that $s$ can only be $s = 0$ (for the half real line) or $s = -\infty$ (for the whole real line). In this work, we shall only consider $s = 0$ as it is in this case that several exact solutions are available [38, 39].

III. SIMILARITY VARIABLE AND SCALING FORMS

When Eq. (1) possess scaling symmetry, it can be solved by the similarity method. Similarity method allows one to transform the partial differential equation to an ordinary differential equation through some new independent variable (called similarity variables), which are certain combination of the old independent variables such that they are scaling invariant, i.e., no appearance of parameter $\epsilon$, as a scaling transformation is performed.

The similarity variable $z$ is defined by

$$z \equiv \frac{x}{t^\alpha}, \quad \text{where } \alpha = \frac{a}{b}, \quad b \neq 0.$$ (8)

Eq. (5) is realized if we assume the following scaling forms of the distribution function, the diffusion and the reaction terms in terms of $z$:

$$P(x, t) = t^\mu y(z), \quad D(P, x, t) = t^\nu \rho(z), \quad R(P, x, t) = t^\lambda \sigma(z).$$ (9)

From Eq. (5) together with the scaling conditions $b = \beta a - d = c - e$, one has

$$\mu = \frac{c}{b}, \quad \nu = \frac{d}{b} = \beta \alpha - 1, \quad \lambda = \frac{e}{b} = \mu - 1.$$ (10)

Thus $\alpha$ and $\mu$ are the only two independent scaling exponents of the RDE.

In terms of these scaling forms, Eq. (11) reduces to an ordinary fractional differential equation

$$sD_z^\beta (py) + \alpha z \frac{dy}{dz} - \mu y + \sigma(z) = 0.$$ (11)

Note that when $\mu = -\alpha$, which we will encounter below, Eq. (11) reduces to

$$sD_z^\beta (py) + \frac{d}{dz} (\alpha z y) + \sigma(z) = 0.$$ (12)

The results discussed thus far can be employed in two ways.

First, given a specific fractional reaction-diffusion system of interest in applied sciences or engineering, one can use the procedures described here to check if this model possesses scaling symmetry, and if it does, reduce it to an fractional ODE [11] by introducing the appropriate similarity variable, and then solve the resulted ODE by some appropriate methods, eg., analytic, asymptotic, approximate, and numerical.

Second, by matching the Eq. (11) with an ODE with appropriate domain, initial conditions and boundary condition, one can determine the corresponding D, and R functions, and then construct a fractional reaction-diffusion system that can be solved as long as the ODE can be solved by any method. Here we will concentrate on exactly solvable ODEs.

Before we proceed to discussing some exactly solvable systems, let us first consider the conditions imposed by the continuity in the change of the particle number of the system, i.e., the equation of continuity.

IV. EQUATION OF CONTINUITY

The total number $N$ of the system is related to the density function $P(x, t)$ by

$$N = \int_{\mathcal{D}} P(x, t) \, dx = t^{\alpha+\mu} \int_{\mathcal{D}} y(z) \, dz,$$ (13)

where $\mathcal{D}$ is the domain of the independent variable. For simplicity, we use the same notation $\mathcal{D}$ for both the variable $x$, and the corresponding similarity variable $z$. 
Eq. (13) distinguishes two different situations: \( \alpha + \mu \neq 0 \) and \( \alpha + \mu = 0 \): \( N \) is conserved if and only if \( \mu = -\alpha \).

The time rate of change of \( N \) is

\[
\frac{dN}{dt} = (\alpha + \mu)t^{\alpha+\mu-1}\left(\int_D y(z) \, dz\right).
\]

But from Eq. (14) one has

\[
\frac{dN}{dt} = \int_D \frac{\partial P}{\partial t} \, dx
\]

\[
= \int_D sD_z^\beta((D(x,t)P(x,t)) + \int_D R(P,x,t) \, dx.
\]

In terms of the similarity variable \( z \),

\[
(\mu + \alpha)t^{\alpha+\mu-1}\int_D y \, dz
\]

\[
= t^{\alpha+\mu-1}\int_D sD_z^\beta (py) \, dz + t^{\alpha+\lambda}\int_D \sigma(z) \, dz.
\]

In view of \( \lambda = \mu - 1 \), one has

\[
(\mu + \alpha)\int_D y \, dz = \int_D sD_z^\beta (py) \, dz + \int_D \sigma(z) \, dz.
\]

With the identity

\[
sD_z^\beta (py) = \frac{d}{dz} sD_z^{\beta-1}(py),
\]

which follows from the definition (13), we have

\[
(\alpha + \mu)\int_D y(z) \, dz = \int_D \sigma(z) \, dz + \Delta[\int sD_z^{\beta-1}(py)]_{\partial D}.
\]

Here \( \partial D \) denotes the boundaries of the domain \( D \), and \( \Delta[\cdot]_{\partial D} \) the difference of the terms in the bracket at the boundaries. But one can rewrite Eq. (11) as

\[
\frac{d}{dz} \left[ sD_z^{\beta-1}(py) + \alpha zy \right] - (\mu + \alpha)y - \sigma(z) = 0.
\]

By integrating Eq. (20) and comparing the resulted equation with (19), we see that the equation of continuity is equivalent to

\[
\Delta[zy]_{\partial D} = 0.
\]

This is the same as that obtained for the ordinary convection-diffusion-reaction equation, which includes the Fokker-Planck equation and the reaction-diffusion equation as sub-cases [61].

It should be noted that in cases where the number of particle \( N \) cannot be defined, i.e., if the integral in Eq. (13) is divergent, then Eq. (19) or (21) are not satisfied. Two such examples are presented in [60].

In what follows, we shall present some exactly-solvable examples for \( \mu = -\alpha \) and \( \mu \neq -\alpha \).

V. CASES WITH \( \mu = -\alpha \) AND \( \sigma = 0 \)

The case with \( \mu = -\alpha \) and \( \sigma = 0 \) corresponds to fractional diffusion. In this case Eq. (20) gives

\[
sD_z^{\beta-1}(py) + \alpha zy = \text{constant}.
\]

In this work we shall only consider cases with the constant equals zero, i.e.,

\[
sD_z^{\beta-1}(py) + \alpha zy = 0.
\]
As mentioned in Sect. III, we will only be interested in obtaining exactly solvable reaction-diffusion system by matching Eq. (11) with an exactly solvable ODE. Unfortunately, there are not many known exact fractional ODEs in the literature. So we shall present examples based on the exact fractional ODEs available from [35].

Consider a special class of problems on the half-line \( x \geq 0 \), so \( z \geq 0 \) and \( s = \bar{s} = 0 \), with \( \rho(z) = z^{-q} \) (a real constant). Let \( Y(z) = \rho y(z) \), or \( y(z) = z^q Y(z) \). Eq. (23) becomes

\[
\frac{\partial}{\partial z} z^{\beta-1} Y(z) + \alpha z^{\alpha+1} Y(z) = 0. \tag{24}
\]

The diffusion coefficient is \( D(t) = t^{-\nu} z^{-q} = t^{1-\beta\alpha-q} \). The order \( \beta \) of the fractional differential operator, the scaling exponent \( \alpha \), and \( q \) are the three independent parameters in this system.

According to Theorem 4.13 in Section 4.2.6 of [35] (summarized in Appendix C), the general solution of (23) is a linear combination of the following solutions

\[
Y_j(z) = z^{\beta-1-j} E_{\beta-1,1+\frac{\alpha+1}{\nu+1}} \left( -\alpha z^{\beta+q} \right), \quad j = 1, 2, \ldots, n, \quad \beta > 1,
\]

with the boundary condition (21)

\[
\Delta \left[ z^{q+1} Y(z) \right]_{\nu D} = 0. \tag{26}
\]

Here \( E_{\alpha,m} \) is the three-parameter Mittag-Leffler function (or the Kilbas-Saigo function) (Appendix A). The general solution of Eq. (11) is \( P(x, t) = t^{-\alpha} \sum_j c_j y_j(z) \), or \( y(z) = z^q Y_j(z) \). For normalized \( P(x, t) \) one must have \( \beta + q - n - 1 \geq 0 \), or \( q \geq n + 1 - \beta \).

Fig. 1 depicts graphs of \( y_j(z) \) for \( 1 < \beta \leq 2 \) (with \( n = 1, j = 1 \) and \( q = n + 1 - \beta = 2 - \beta \)), \( n = 2, j = 1, 2 \) and \( q = n + 1 - \beta = 3 - \beta \). The numerical results indicate that for \( \beta > 2 \), \( y_j(z) \) could be negative for some values of \( z \). As \( P(x, t) \geq 0 \), being a distribution function, the fractional diffusion equation discussed here only makes sense for \( \beta \leq 2 \). So \( n = j = 1 \), and

\[
P(x, t) = N \frac{x^{\beta+q-2}}{\nu^{\beta+q-1}} \times E_{\beta-1,1+\frac{\alpha+1}{\nu+1}} \left( -\alpha z^{\beta+q} \right), \tag{27}
\]

where \( N \) is the normalization constant.

Normalized \( P(x, t) \) requires \( \beta + q - 2 \geq 0 \), or \( q \geq 2 - \beta \). There are two possible situations: 1) if \( q = 2 - \beta \), then \( P(x, t) > 0 \) at \( x = 0 \); 2) if \( q > 2 - \beta \), then \( P(x, t) = 0 \) at \( x = 0 \).

The diffusion coefficient in the standard diffusion equation for the Brownian motion is a constant. Without loss of generality we set \( D(x, t) = 1 \). This implies \( \nu = 1 \) and \( \beta = \alpha - 1 = 0 \). So \( \alpha \) and \( \beta \) are tied by \( \alpha = 1/\beta \). The fractional diffusion equation reduces to the standard diffusion equation for \( \beta = 2 \) (\( n = 1 \)) and \( q = 0 \). In this case Eq. (27) becomes, according to Eqs. (B1), (B4) and (B6),

\[
P(x, t) \propto \frac{1}{\sqrt{t}} E_{1,2+\frac{\alpha+1}{\nu+1}} \left( -\alpha z^2 \right) \propto \frac{1}{\sqrt{t}} E_{1,1} \left( -\alpha z^2 \right) \propto \frac{1}{\sqrt{t}} e^{-\alpha z^1}, \tag{28}
\]

which is the well-known solution of the diffusion equation.

We will present some examples with \( 0 < \beta \leq 2 \) and \( \alpha = 1/\beta \).

In Figs. 2 and 3 we show the evolution of the distribution function \( P(x, t) \) with \( q = 2 - \beta \) and \( q > 2 - \beta \), respectively. The former case can be considered as describing the deformed Brownian motion (on the half-line) by \( \beta \). It is seen that at large times \( t \), \( P(x, t) \) for \( \beta < 2 \) is slightly larger than that for \( \beta = 2 \) (the standard Brownian). So the fractional diffusion equation discussed here can be considered as a model for superdiffusive motion.

We note here that a type of superdiffusive motion, so-called the \( \text{Lévy} \) flight, has been modeled by a diffusion equation with the Riesz-Weyl type fractional space-derivative [4,5], which is different from that employed here.

For \( \beta = 1 \), Eq. (27) is not valid. However, in this case Eq. (22) can be solved easily. Now we have

\[
\frac{\partial}{\partial t} P(x, t) = \frac{\partial}{\partial x} \left( D(x, t) P(x, t) \right). \tag{29}
\]

which is the reduced fractional ODE for the equation

\[
\frac{\partial}{\partial t} P(x, t) = \frac{\partial}{\partial x} \left( D(x, t) P(x, t) \right). \tag{30}
\]
As $\kappa D_z^\beta(\rho y) = \rho y$, the solution is

$$y(z) = \frac{C}{\rho(z) + \alpha z}.$$  \hfill (31)

For the choice $\alpha = 1/\beta = 1$, $C = 2/\pi$, and $\rho(z) = 1 - z + z^2$, we have the Cauchy distribution (on the half-line)

$$y(z) = \frac{2}{\pi (1 + z^2)}.$$ \hfill (32)

The probability density $P(x,t) = t^{-1}y(z)$ turns out to correspond to the distribution function of the Lévy flight described by the fractional diffusion equation based on the Riesz-Weyl fractional space-derivative with constant diffusion coefficient [4, 5].

We note here that for some choices of $\rho(z)$, the solution $y(z)$ in (31) may describe systems in which the number of particle $N$ is not defined, and hence the boundary condition (21) is not valid. For example, if we take $\rho(z) = 1$.

VI. CASES WITH $\mu = -\alpha$ AND $\sigma \neq 0$

Eq. (12) is not easy to solve exactly in general when $\sigma(z) \neq 0$. But we can find some exactly solvable systems for $\sigma(z)$ a gradient of some function $\tau(z)$, i.e., $\sigma(z) = -d\tau(z)/dz$. Under this situation, Eq. (12) becomes

$$\frac{d}{dz} [s D_z^{\beta-1}(\rho y) + \alpha z y - \tau] = 0,$$ \hfill (33)

or

$$s D_z^{\beta-1}(\rho y) + \alpha z y - \tau = 0,$$ \hfill (34)

where we have absorbed the constant of integration into the function $\tau(z)$.

- **Fokker-Planck type**

  If $\tau(z)$ is proportional to $y(z)$ and $z$, i.e., $\tau(z) = -\gamma y(z)$ for some constant $\gamma$, the diffusion equation is

  $$s D_z^{\beta-1}(\rho y) + (\alpha + \gamma) z y = 0,$$ \hfill (35)

  which is just Eq. (23) with $\alpha$ replaced by $\alpha + \gamma$. Hence solutions can be found as described in Sect. V with appropriate change of parameters.

- **Non-Fokker-Planck type**

  Let us consider solution of (34) with $s = \bar{s} = 0$, $y(z) = z^q Y(z)$, $\rho(z) = z^{-q}$ and $\tau(z) = \sum_{r=1}^l f_r z^{\mu_r}$ ($f_r, \mu_r$ are real constants),

  $$\kappa D_z^{\beta-1}Y(z) + \alpha z^{q+1} Y(z) = \sum_{r=1}^l f_r z^{\mu_r}.$$ \hfill (36)

  According to Theorem 4.13 of [38] (Appendix C), for $n - 1 < \beta - 1 < n, q + 1 > -1(\beta - 1), \mu_r > -1(r = 1, \ldots, l)$ the general solution of (36) is $Y(z) = Y_h(z) + Y_p(z)$, $Y_h(z) = \sum_{j=1}^n c_j Y_j(z)$ [Y_j(z) given in (25)] is the solution of the homogeneous equation (23), and $Y_p(z)$ is a particular solution of (36) ($y_0(z)$ in Appendix C),

  $$Y_p(z) = \sum_{r=1}^l \frac{\Gamma(\mu_r + 1)}{\Gamma(\mu_r + \beta)} \Gamma(\mu_r + \beta) z^{-\beta+1+\mu_r} \times E_{\beta-1+\frac{q+1}{\beta-1},1+\frac{\alpha+1}{\beta-1}-\frac{\mu_r}{\beta-1}}(-\alpha z^{\beta+q}).$$ \hfill (37)
Then the $P(x,t)$ is

$$P(x,t) = N t^{-\alpha} z^q Y(z),$$

(38)

where $N$ is the normalization constant.

In Fig. 4 we present $P(x,t)$ for three different sets of $1 < \beta \leq 2$ with $l = 1$ and $\mu_r = -0.5$. Again it is seen that the for the choice of parameters, the smaller the value of $\beta$, the higher the value of $P(x,t)$ at large $x$ - the reaction-diffusion system is superdiffusive.

VII. CASES WITH $\mu \neq -\alpha$

From Eq. (14), $dN/dt \neq 0$ if $\mu \neq -\alpha$, i.e., $N$ need not be conserved. So in this case $P(x,t)$ is not interpreted as a probability density function, and so $\beta$ need not be restricted to be $\beta \leq 2$. The reaction-diffusion equation (11) is in general difficult to solve. Here we present a way to obtain a special class of exactly solvable system for (11).

Suppose

$$\sigma(z) \equiv (\mu + \alpha)y - \frac{d}{dz} \tau(z),$$

(39)

for some function $\tau(z)$, then Eq. (20) is integrated to be (with the integration constant absorbed into $\tau(z)$)

$$sD_z^{\beta-1}(py) + \alpha zy - \tau = 0,$$

(40)

which is just Eq. (34).

Hence any choice of $\tau(z)$ that renders (34) exactly solvable defines an exactly solvable fractional reaction-diffusion system (11) with $\sigma(z)$ defined by (39). Particularly, the examples discussed in Sect. VI can be employed to define the corresponding systems in this case.

VIII. SUMMARY

In this work we have studied exact solvability of a class of FRDE with the Riemann-Liouville fractional derivatives on the half-line in terms of the similarity solutions. We have derived the conditions for the FRDE to possess scaling symmetry even with the fractional derivatives. Relations among the scaling exponents are determined, and the appropriate similarity variable introduced. With the similarity variable we reduced the partial differential equation to a fractional ordinary differential equation. Exactly solvable systems are then identified by matching the resulted ODE with known exactly solvable fractional ODEs. Several examples were presented, which involve the three-parameter Mittag-Leffler function (Kilbas-Saigo function). The FRDE’s discussed here turn out to correspond to superdiffusive diffusion.

All the cases discussed in this work can be easily extended to FRDE’s defined on the whole real line by simply changing $x$ to $|x|$, i.e.,

$$\frac{\partial}{\partial t} P(x,t) = sD_{|x|}^\beta (D(x,t)P(x,t)) + R(P,|x|,t).$$

(41)

The function $P(x,t)$ in the negative half-line is just mirror image of those on the positive half-line (with the height reduced by a factor of two for normalization).

We note here that the procedure presented here can be adapted to fractional reaction-diffusion equations defined with other types of fractional derivative.

Also, it would be interesting to extend the present analysis to fractional versions of differential equations in other areas, such as those in fluid mechanics, nonlinear science as mentioned in the Introduction, and mathematical biology [63].

Appendix A: Classical and generalized Mittag-Leffler Functions [38 39]

- Mittag-Leffler function
\[
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad \text{Re} \, \alpha > 0.
\]

\textbf{Two-parameter Mittag-Leffler function}

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad \text{Re} \, \alpha > 0,
\]

\textbf{Three-parameter Mittag-Leffler function (Prabhakar function) \cite{62}}

\[
E_{\gamma,\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k! \Gamma(\alpha k + \beta)}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \quad \text{Re} \, \alpha, \gamma > 0,
\]

where \((\gamma)_k = \gamma(\gamma + 1) \cdots (\gamma + k - 1) = \Gamma(\gamma + k)/\Gamma(\gamma)\) is the Pochhammer symbol.

\textbf{Three-parameter Mittag-Leffler function (Kilbas-Saigo function) \cite{13}}

\[
E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \quad z, l \in \mathbb{C}, \quad \alpha, m \in \mathbb{R}, \quad \alpha, m > 0,
\]

where

\[
c_0 = 1, \quad c_k = \prod_{i=1}^{k-1} \frac{\Gamma(a[im + l] + 1)}{\Gamma(a[im + l + 1] + 1)} (k = 1, 2, \ldots),
\]

such that

\[
\alpha(jm + l) \neq -1, -2, \ldots (j = 0, 1, 2, \ldots).
\]

\textbf{Appendix B: Some identities \cite{39}}

\textbf{•} \(\alpha > 0\)

\[
E_{\alpha,1}(z) = E_\alpha(z).
\]

\textbf{•} \(n \in \mathbb{N}, \quad \beta > 0\)

\[
E_{n,1,l}(z) = \Gamma(nl + 1) E_{n,nl+1}(z).
\]

\textbf{•} \(m > 0, l \in \mathbb{R}, jm + l \neq -1, -2, \ldots (j = 0, 1, 2, \ldots)\)

\[
E_{1,m,l}(z) = \Gamma\left(\frac{l + 1}{m}\right) E_{1,\frac{l+1}{m}}\left(\frac{z}{m}\right).
\]

\textbf{•} \(l \neq -1, -2, \ldots\)

\[
E_{1,1,l}(z) = \Gamma(l + 1) E_{1,l+1}(z).
\]

\textbf{•} \(l = 0, 1, 2, \ldots\)

\[
E_{1,1,l}(z) = l! E_{1,l+1}(z).
\]

Particularly,

\[
E_{1,1,0}(z) = E_1(z) = e^z.
\]
Here we summarize Theorem 4.13 in Section 4.2.6 of [38].

Consider the Cauchy problem for the Inhomogeneous differential equation of fractional order $\alpha > 0$ with a quasi-polynomial free term

$$a^+D_x^\alpha y(x) = \lambda(x-a)^\beta y(x) + \sum_{r=1}^{l} f_r(x-a)^{\mu_r},$$

\hspace{1cm} \text{ \text{(C1)}}

$$a^+D_x^{\alpha-k} y(a^+) = b_k, \quad k = 1, 2, \ldots, n \equiv -[-\alpha],\quad \text{\text{(C2)}}$$

where $\lambda, \beta, f_r, \mu_r (r = 1, 2, \ldots, l), b_k (k = 1, \ldots, n) \in \mathbb{R}$ are given real constants.

Let $n - 1 < \alpha < n$, $\beta > -\{\alpha\}$, and $\mu_r > -1(r = 1, \ldots, l)$. Then the Cauchy problem (B.1) and (B.2) has a unique solution in the space of functions locally integrable on $(a, b)$

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(1 + \alpha - j)} (x-a)^{\alpha-j} \times E_{\alpha,1+\beta/\alpha,1+(\beta-j)/\alpha} \left(\lambda(x-a)^{\alpha+\beta}\right) + y_0(x),$$

\hspace{1cm} \text{\text{(C3)}}

where

$$y_0(x) = \sum_{r=1}^{l} \frac{\Gamma(\mu_r + 1)}{\Gamma(\mu_r + \alpha + 1)} \frac{f_r(x-a)^{\alpha+\mu_r}}{\Gamma(\mu_r + \alpha + 1)} \times E_{\alpha,1+\beta/\alpha,1+(\beta+\mu_r)/\alpha} \left(\lambda(x-a)^{\alpha+\beta}\right).$$

\hspace{1cm} \text{\text{(C4)}}

\[\]
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FIG. 1: Plots of $y_j(z)$ with $\alpha = 1/\beta, q = n + 1 - \beta$ for different $\beta$ (solid line for $\beta = 2, n = j = 1$) : (a) $\beta = 1.9$ (dashed), 1.8 (dot-dashed), and 1.7 (dotted); (b) $\beta = 2.2$; (c) $\beta = 2.4$, (d) $\beta = 2.6$. For (b)-(d), $n = 2, j = 1$ (dashed), and 2 (dotted). Number of terms used in the series expansion for the Kilbas-Saigo function is 400.
FIG. 2: Plots of $P(x, t)$ at different times for deformed Brownian-type diffusion, Eq. (27) with $q = 2 - \beta$, $\alpha = 1/\beta$, and $\beta = 2$ (solid), $1.8$ (dashed), and $1.7$ (dotted). Number of terms used in the series expansion for the Kilbas-Saigo function is 400.
FIG. 3: Plots of $P(x, t)$ at different times for deformed non-Brownian type diffusion Eq. (27) with $\alpha = 1, q = 1 \neq 2 - \beta$, and $\beta = 2$ (solid), 1.9 (dashed), and 1.7 (dotted). Number of terms used in the series expansion for the Kilbas-Saigo function is 400.
FIG. 4: Plots of $P(x, t)$ at different times for deformed non-Fokker-Planck type reaction-diffusion Eq. (38) with $\mu_r = -0.5, \alpha = 1/\beta, q = 2 - \beta, c_1 = c_2 = 1$, and $\beta = 2$ (solid), 1.9 (dashed), and 1.8 (dotted). Number of terms used in the series expansion for the Kilbas-Saigo function is 400.