The Zero-Dispersion Limit for the Odd Flows in the Focusing Zakharov-Shabat Hierarchy

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Abstract

We present a numerical and theoretical study of the zero-dispersion limit of the focusing Zakharov-Shabat hierarchy, which includes NLS and mKdV flows as its second and third members. All the odd flows in the hierarchy are shown to preserve real-valued data. We establish the zero-dispersion limit of all the nontrivial conserved densities and associated fluxes for these odd flows for a large class of real-valued initial data which includes all “single hump” initial data. In particular, it is done for the “focusing” mKdV flow. The method is based on the Lax-Levermore KdV strategy, but here it is carried out in the context of a nonselfadjoint spectral problem. We find that after an algebraic transformation the limiting dynamics of the mKdV equation is identical to that of the KdV equation.
1. Introduction

1.1. Background. The asymptotic analysis of nonlinear wavepackets, extending the Fourier-theoretic treatment of the linear case, has developed into a prominent area of activity in both Analysis and Applied Mathematics. The first systematic approach to this problem is due to G. Whitham [W], who, in the 1960's, found various mechanisms (variational principles, averaged conservation laws) for formally deriving modulation equations which describe the evolution of parameters in a family of exact travelling wave solutions to a nonlinear equation. This is in analogy to the classical geometric optics approach for linear equations. There the asymptotic evolution of a linear wavepacket is described in terms of a deformation of parameters in the family of plane wave solutions corresponding to the dispersion relation of the equation.

It was fortuitous that at the same time that Whitham was formulating his theory, the subject of completely integrable partial differential equations (PDE’s), or soliton theory, became widely known. The latter concerns a class of physically significant PDE’s with large families of exact solutions, including multiphase waves, as opposed to just travelling wave subfamilies. This provided a rich and fertile testing ground in which to explore the techniques that Whitham had developed. Whitham studied a wide variety of these equations himself in the travelling wave setting: Korteweg-deVries (KdV), nonlinear Schrödinger (NLS), sine-Gordon (SG), etc. The extension of Whitham analysis to the class of multiphase solutions was accomplished by Flaschka, Forest and McLaughlin [FFM] initially for KdV. This was still in the setting of wavepacket analysis. The first work to place Whitham’s theory within the framework of an explicit singular limit problem was due to Lax and Levermore [LL2] who investigated the zero-dispersion limit of the Korteweg-deVries (KdV) equation. There one studies the limit as $\epsilon \to 0$ of the conserved densities for the scaled KdV equation

\[
\begin{align*}
\partial_t u^\epsilon - 6u^\epsilon \partial_x u^\epsilon + \epsilon^2 \partial_{xxx} u^\epsilon &= 0, \\
(u^\epsilon(x, 0) &= u_{in}(x). 
\end{align*}
\]

The limit is strong and given by the solution of the Hopf equation, which is hyperbolic,

\[
\begin{align*}
\partial_t u + 6u \partial_x u &= 0, \\
u(x, 0) &= u_{in}(x),
\end{align*}
\]

so long as its solution is classical. After the breaktime the limit is weak due to the development of regularizing small wavelength oscillations with an amplitude of order unity; thereafter its evolution is no longer governed by the Hopf equation (1.2a).

In their seminal paper, Gardner, Greene, Kruskal and Miura [GGKM] showed that the KdV equation is completely integrable using the inverse scattering transform.
associated with the selfadjoint Schrödinger operator

\begin{equation}
L_S = -\epsilon^2 \partial_{xx} + u^\epsilon.
\end{equation}

Lax and Levermore [LL] realized that this inverse transform would enable one to reduce the asymptotic analysis of nonlinear KdV wavepackets to classical short wave asymptotics. They analyzed the limiting behavior of the scattering and inverse scattering transform using a WKB analysis of (1.3) and a kind of steepest descent argument to obtain a characterization of the (weak) limits in terms of the solution of a variational problem. The solution of this variational problem was then constructed through the solution of a Riemann-Hilbert problem. Venakides [V] has analyzed the microstructure of the limiting solutions, bridging the gap to the local approach of modulation theory developed in [FFM]. These results are surveyed in [LLV].

Extensions of the Lax-Levermore strategy to other integrable PDE’s have been pursued, perhaps the most notable being a recent analysis [JLM2] of the defocusing NLS hierarchy. In this paper we are going to investigate the asymptotic behavior of a part of the hierarchy of commuting flows associated to the focusing nonlinear Schrödinger (NLS) equation (1.4). A significant difference between the asymptotic analysis of this last equation and that of the KdV or even the defocusing NLS hierarchy is that the inverse transform of the former is based on a non-selfadjoint scattering problem (1.5 +) whereas that of the latter is based on selfadjoint scattering problems. At a formal level this has the consequence that the Whitham modulation equations for focusing NLS can become elliptic and therefore ill-posed for general initial data.

This non-selfadjointness also presents an obstacle for knowing how to even begin a Lax-Levermore analysis. The work in this paper makes a first step towards confronting this obstacle. We do not try to tackle the focusing NLS equation proper; rather, we expand our view and look at the entire hierarchy of flows. By abandoning the even members of the flow, and the possibility of complex initial data, we see that the remaining members are a hierarchy in their own right, and that their limiting behavior is identified with that of the Korteweg-deVries hierarchy in the same limit. Numerical simulation suggests that this is a sharp result: after the initial shock, there are \(O(1)\) differences in the two flows which go weakly to zero as the dispersion is taken to zero.

1.2. The NLS and mKdV Equations. The one dimensional cubic nonlinear Schrödinger (NLS) focusing (+) and defocusing (−) equations

\begin{equation}
\begin{aligned}
i\epsilon \partial_t u^\epsilon + \frac{\epsilon^2}{2} \partial_{xx} u^\epsilon \pm |u^\epsilon|^2 u^\epsilon &= 0, \\
\end{aligned}
\end{equation}

with initial data in the amplitude-phase form

\begin{equation}
\begin{aligned}
u^\epsilon(x, 0) &= A(x) \exp \left( \frac{i}{\epsilon} S(x) \right),
\end{aligned}
\end{equation}
were first solved by Zakharov and Shabat [ZS] for some natural far field boundary conditions of the initial data \(A(x)\) (positive) and \(S(x)\) (real) by using the inverse scattering transform associated with a Dirac operator (ZS operator) [ZS1, ZS2]:

\[
\mathcal{L} = \begin{pmatrix}
-i\epsilon \partial_x & \pm i\bar{u}\epsilon \\
 i\bar{u}\epsilon & i\epsilon \partial_x
\end{pmatrix}.
\]

This ZS operator is selfadjoint if in the defocusing case, but nonselfadjoint if in the focusing case. The parameter \(\epsilon\) is introduced into the evolution and scattering problems with the zero-dispersion limit scaling; namely, replacing \(\partial_x\) and \(\partial_t\) by \(\epsilon \partial_x\) and \(\epsilon \partial_t\). The choice of phase scaling in the initial data is the natural balance for obtaining a non-trivial momentum density in the limit. This scaling, both in the data and the evolution equations, is completely analogous to the linear Schrödinger case of quantum mechanics, where the \(\epsilon\) is taken as a nondimensional version of Planck’s constant, \(\hbar\). In the linear case, there is no specific scattering operator to contend with.

For the nonlinear case, the solution of the initial-value problem centers around the eigenvalue problem

\[
\mathcal{L} f = \zeta f, \quad \text{where} \quad f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix},
\]

where now both \(t\) and \(\epsilon\) play roles as parameters, and are not explicitly differentiated. For convenience, we may sometimes represent the eigenvalue \(\zeta\) by its real part \(\xi\) and imaginary part \(\eta\).

The NLS equations are the second members, and first interesting members, of a sequence of flows associated with \(\mathcal{L}\) (1.5) [FNR]. The third members are the modified KdV equations (mKdV),

\[
\partial_t \epsilon u^\epsilon \pm \frac{3}{2} \epsilon u^\epsilon |u^\epsilon|^2 \partial_x u^\epsilon + \frac{\epsilon^2}{4} \partial_{xxx} u^\epsilon = 0,
\]

or, more generally, their complex forms

\[
\partial_t \epsilon u^\epsilon \pm \frac{3}{2} \epsilon u^\epsilon |u^\epsilon|^2 \partial_x u^\epsilon + \frac{\epsilon^2}{4} \partial_{xxx} u^\epsilon = 0,
\]

As members of the focusing (+) and defocusing (−) NLS hierarchies, they share the same respective ZS operator and the same respective conserved quantities.

There are in fact an infinite number of conserved densities for the NLS hierarchy. The first two of them are mass density \(\rho^\epsilon\) and momentum density \(\mu^\epsilon\) defined by

\[
\rho^\epsilon = |u^\epsilon|^2, \quad \mu^\epsilon = -i\frac{\epsilon}{2} (\bar{u}^\epsilon \partial_x u^\epsilon - u^\epsilon \partial_x \bar{u}^\epsilon).
\]
The corresponding conservation laws of the focusing (−) and defocusing (+) NLS equations are

\[
\begin{align*}
\frac{\partial_t\rho^\varepsilon}{\rho^\varepsilon} + \partial_x \left( \frac{\mu^\varepsilon}{\rho^\varepsilon} \pm \frac{1}{2} \rho^\varepsilon \right) &= 0, \\
\frac{\partial_t\mu^\varepsilon}{\mu^\varepsilon} + \partial_x \left( \frac{\mu^\varepsilon}{\rho^\varepsilon} \pm \frac{1}{2} \rho^\varepsilon \right) &= \frac{\varepsilon^2}{4} \partial_x \left[ \rho^\varepsilon \partial_{xx} \log(\rho^\varepsilon) \right] ;
\end{align*}
\]

while those of the focusing (+) and defocusing (−) mKdV equations are

\[
\begin{align*}
\frac{\partial_t\rho^\varepsilon}{\rho^\varepsilon} + \partial_x \left( \pm \frac{3}{4} \rho^\varepsilon - \frac{3}{4} \mu^\varepsilon^2 \rho^\varepsilon \right) &= \frac{\varepsilon^2}{4} \partial_x \left[ \frac{3}{4} (\partial_x \rho^\varepsilon)^2 - \partial_{xx} \rho^\varepsilon \right] , \\
\frac{\partial_t\mu^\varepsilon}{\mu^\varepsilon} + \partial_x \left( \pm \frac{3}{2} \rho^\varepsilon - \frac{3}{2} \mu^\varepsilon^3 \rho^\varepsilon \right) &= \frac{\varepsilon^2}{4} \partial_x \left[ \frac{3}{8} \mathcal{R}^\varepsilon - \partial_{xx} \mu^\varepsilon \right] ,
\end{align*}
\]

where

\[
\mathcal{R}^\varepsilon = \frac{\partial_x \rho^\varepsilon}{\rho^\varepsilon} \partial_x \mu - \frac{\partial_{xx} \rho^\varepsilon}{\rho^\varepsilon} \mu + \frac{(\partial_x \rho^\varepsilon)^2}{2 \rho^\varepsilon} \mu .
\]

The solutions of either the NLS equations or mKdV equations can then be determined by \( \rho^\varepsilon \) and \( \mu^\varepsilon \) up to a constant phase. The other conserved densities can be also represented in terms of \( \rho^\varepsilon \) and \( \mu^\varepsilon \); for example, the energy density \( \epsilon^\varepsilon \) for the focusing (−) and defocusing (+) NLS hierarchy, defined by

\[
\begin{align*}
\epsilon^\varepsilon &= \frac{\varepsilon^2}{2} |\partial_x u^\varepsilon|^2 \pm \frac{1}{2} |u^\varepsilon|^4 ,
\end{align*}
\]

can be determined if \( \rho^\varepsilon \) and \( \mu^\varepsilon \) are known because

\[
\epsilon^\varepsilon = \frac{1}{2} \frac{|\mu^\varepsilon|^2}{\rho^\varepsilon} + \frac{3 \varepsilon^2}{8} |\partial_x \rho^\varepsilon|^2 \rho^\varepsilon \pm \frac{1}{2} \rho^\varepsilon^2 .
\]

There are recursion formulas (see equation (7.5)) from which all of the NLS conserved densities can be explicitly generated by iteration.

1.3. The Zero-Dispersion Limit. We want to understand the behavior of solutions of the ZS hierarchy when \( \varepsilon \to 0 \). Formally, a natural guess is that the limits of \( \rho^\varepsilon \) and \( \mu^\varepsilon \), denoted by \( \rho \) and \( \mu \), would satisfy the following equations associated to the focusing (−) and defocusing (+) NLS (1.4a)

\[
\begin{align*}
\frac{\partial_t\rho}{\rho} + \partial_x \mu &= 0 , \\
\frac{\partial_t\mu}{\mu} + \partial_x \left( \frac{\mu^2}{\rho} \pm \frac{1}{2} \rho^2 \right) &= 0 ;
\end{align*}
\]
or the following equations for focusing (+) and defocusing (−) mKdV, (1.6),

\[
\begin{align*}
\partial_t \rho + \partial_x \left( \pm \frac{3}{4} \rho^2 - \frac{3}{4} \mu^2 \right) &= 0, \\
\partial_t \mu + \partial_x \left( \pm \frac{3}{2} \rho \mu - \frac{3}{4} \mu^3 \rho \right) &= 0,
\end{align*}
\]
(1.12)

with initial values which are the limits of the initial \( \rho^\epsilon \) and \( \mu^\epsilon \). We call them zero-dispersion limits because the formally small terms are dispersive. We call them semi-classical limits because these limits in general exist only in a weak sense.

Consider the equations for \( \rho \) and \( \mu \) given by (1.12) with the focusing sign. They are the formal limits of the first two conservation laws specialized to the focusing mKdV flow (1.9). In the case that \( \mu \equiv 0 \), the equation for \( \rho \) is nothing but the Hopf equation (1.2a) which governs the zero-dispersion limit of the KdV equation (1.1a) near the initial time, except for a trivial scaling: the time variable \( t \) in (1.12) is replaced by \( 4t \) in (1.2a).

With a brief calculation, one can find that the systems (1.11) and (1.12) are not always well-posed for initial value problems. They are both hyperbolic for the defocusing case, which is well-posed; but both elliptic for the focusing case, which is ill-posed for general initial data. This has been a principal obstacle to the further study of the dispersive limit of the focusing hierarchy. However, as we shall describe, there is a subset of the flows, namely the odd flows such as (1.12), which will propagate real data \( (S \equiv 0) \) preserving its reality. (This invariance is decidedly untrue for the general even flow; for instance, in the focusing NLS flow itself.) We will demonstrate how the analysis of the dispersive limit, on this invariant subspace, can be accomplished.

Motivated by the results presented here, Kamvissis, McLaughlin, and Miller [KMM] very recently undertook a careful analysis of the dispersive limit of focusing NLS for a very particular initial condition: \( A = \text{sech}(x) \) and \( S \equiv 0 \). For this data one has explicit formulae for the WKB scattering data. Using newly developed Riemann-Hilbert problem techniques, they are able to show that the asymptotic behavior of this particular evolution problem is correctly described by the Whitham equations, despite their ellipticity. Given this development, it now seems probable that this conclusion may hold more generally for analytic data.

1.4. Outline of the Paper. In Section 2, we display several numerical experiments to compare the different situations: well-posed and ill-posed system, complex valued and real valued system as well as the dynamics among the KdV, focusing mKdV and the defocusing mKdV initial value problems with the small parameter \( \epsilon \). In Section 3, we describe the direct scattering data of the focusing NLS equation given by (1.4) and its hierarchy. In Section 4, we describe, via WKB asymptotics, the class of “single
lobe” real initial data; this asymptotic description can be given in terms of reflectionless (multi-soliton) potentials. We analyze the formulae for \(N\)-soliton potentials and deduce asymptotic upper and lower bounds on these expressions as well convexity properties that enable us to establish a KdV-like limit. In Section 5 we present the rigorous derivation of this limit and its relation to a fundamental maximization problem. We then show how the densities and fluxes for the odd flows on real initial data can be directly expressed in terms of the maximizer of this problem. Section 6 then describes the limiting dynamics of these flows in terms of a Riemann-Hilbert boundary value problem for the maximizer. Finally, in Section 7, we introduce a “loop algebra” description of the phase space for the NLS system which emphasizes the Lax pair structure of this system. In this setting it becomes easy to understand the behavior of the odd flows when one restricts to real initial data. One can identify an invariant subspace for these odd flows, even at finite \(\epsilon\), which further collapses, formally as \(\epsilon \to 0\) onto a space isomorphic to the KdV dispersionless densities. This gives a different approach to the conclusions found in the previous sections. Finally, we make some concluding remarks in Section 8.

2. Numerical Experiments

Having introduced the problem, we now present a few numerical calculations that will serve to illustrate and motivate the forthcoming analysis. We will be considering the real initial value problem for focusing mKdV.

Figure (2.1) presents a sequence of three snapshots of the evolution of the focusing mKdV flow (1.6a-b), for the real, periodic initial data

\[
(2.1) \quad u(x, 0) = \cos^2(x),
\]

with \(\epsilon = 0.04\). In these figures one can see quite clearly the formation of a region of oscillations, of bounded amplitude, within the profile. A fourth panel in this figure shows a space-time contour plot of the evolution. For small times one sees that this evolution appears to move along straight line characteristics and that the oscillations begin to set in where characteristics focus and begin to form an envelope. Indeed this evolution very closely matches that of the inviscid Burgers equation (1.6a with \(\epsilon = 0\)) for the same initial data. From the method of characteristics one calculates the break time for this dispersionless equation to be \(t_B \approx 0.49\), which agrees well with where one sees the onset of oscillations. After this time, the solution develops a modulated periodic wave train (a so-called “Dispersive Shock”) which propagates away from the initial kink. Eventually the shock regions collide creating more complex modulational structures.

Figure 2.2 shows the evolution of the same initial data as \(\epsilon\) is varied between \(\epsilon = 0.08\), \(\epsilon = 0.04\) and \(\epsilon = 0.02\). Notice the structure of the shocks as \(\epsilon\) is varied.
It appears that the width, height, and speed of the modulational envelopes remain essentially the same as \( \epsilon \to 0 \), while the oscillations within a given envelope are roughly proportional to \( 1/\epsilon \). Because of this increasingly oscillatory structure, one does not expect to see pointwise convergence, as \( \epsilon \to 0 \), of the conserved densities (1.7, 1.10) mentioned in the previous section.

However, Figure (2.3) indicates that local averages of these densities might have a regular limit: the anti-derivatives of the mass density \( \rho^\epsilon \) and of the energy density \( \epsilon^\epsilon \) appear to be converging pointwise to a limit with smaller \( \epsilon \) producing higher frequency but smaller magnitude deviations. Note that \( \mu^\epsilon (x,t) = \epsilon \text{Im} (u^\ast u_x) = 0 \) for real initial data in the odd flows.

We conclude this section by mentioning how these simulations were done. Time was discretized using implicit midpoint or implicit-Runge-Kutta based on the fourth order Gauss points. Space was discretized pseudo-spectrally, with the nonlinear terms computed without de-aliasing, which was not necessary because we had adequately resolved the spectrum of the nonlinear terms. All these calculations were done in 16-digit precision, and the simulations preserved the first three conserved quantities to 7-digit accuracy.
Fig 2.1 Time-slices, $u(x, t)|_{t=0.5,1.0,2.0}$, and contour plot (lower right) $u(x, t) = \text{const.}$, $0 < x < 2\pi, 0 < t < 4$. $u(x, 0) = \cos^2 x$. The horizontal axis is $x$, and the vertical axis is either $u$ (slice), or $t$ (contour plot).
Fig 2.2. Contour plot, $u_\epsilon(x,t) = \text{const.}$, $0 < t < 1.5$ (left). Slice, $u_\epsilon(x,t)|_{t=1.5}$, (right). Here, $\epsilon = 0.08$ (top), 0.04 (center), and 0.02 (bottom), and $u_\epsilon(x,0) = \cos^2 x$. The horizontal axis is $x$, and the vertical axis is either $t$ (contour plot) or $u$ (slice). The graphs have the same scales for comparison.
Fig 2.3 Primitives of the first (left) and third (right) conserved densities. The cases for for $\epsilon = 0.08$ and $\epsilon = 0.02$, are overlaid, with the wider oscillations corresponding to large $\epsilon$. Here, the pseudo-inverse $\partial_x^{-1}$ is a periodic normalization of $\int^x$: each discrete Fourier mode of the density is multiplied by $\frac{1}{ik}$, except for mode 0, which is left unchanged. As $\epsilon$ decreases, both the wavelength and the magnitude of the deviating oscillations decrease. This suggests that the limit for the primitives will be classical.

3. The Focusing Zakharov-Shabat Hierarchy

3.1. The Scattering Transform. Zakharov and Shabat [ZS1] solved the Cauchy problem for the focusing NLS equation (1.4) with initial data $A(x)$ and $\partial_x S(x)$ that decay sufficiently rapidly as $|x| \to \infty$. They used the inverse scattering transform associated with the nonselfadjoint Dirac operator

$$\mathcal{L} = \begin{pmatrix} -i \epsilon \partial_x & i \bar{u} \\ iu & i \epsilon \partial_x \end{pmatrix}.$$  

Here we collect the relevant facts regarding their theory. Throughout this section $\epsilon > 0$ will be considered to be fixed, and hence, no explicit $\epsilon$ dependence will be indicated.

The solution strategy centers on the spectral problem

$$\mathcal{L} f = \zeta f \quad \text{where} \quad f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}.$$
Given \( u(x,0) \), the asymptotics of the eigenfunctions \( f(\zeta, x, 0) \) as \( |x| \to \infty \), referred to as the scattering data, can be calculated in principle. The evolution of the scattering data is governed by simple formulas and \( u(x,t) \) is then obtained from the knowledge of the large \( |x| \) asymptotics of \( f(\zeta, x, t) \) using inverse scattering theory.

More specifically, if \( u(x,0) \) has the form (1.4b) where \( A(x) \) and \( \partial_x S(x) \) decay sufficiently rapidly as \( x \to \pm \infty \) (say, faster than any power of \( x \)), then the \( L^2 \) spectrum of \( L \) consists of the whole real axis, comprising the continuous spectrum, along with a finite set (possibly empty) of simple eigenvalues in the complex plane. It is easily seen that if \( f \) and \( \zeta \) satisfy (3.1) then so do \( \tilde{f} \) and \( \tilde{\zeta} \) where

\[
(3.2) \quad \tilde{f} = \left( \begin{array}{c} \tilde{f}^{(2)} \\ \tilde{f}^{(1)} \end{array} \right).
\]

We will therefore only consider \( \zeta_1, \cdots, \zeta_N \), the \( N \) eigenvalues of \( L \) located in the upper half of the complex plane.

To describe the asymptotic behavior of the \( L^2 \) scattering data we first introduce the Jost fundamental solution \( F(\zeta, x) \). For each \( \zeta \in \mathbb{C} \), this matrix solution is uniquely defined by the following two conditions:

\[
\begin{align*}
  i) & \quad F(\zeta, x) \exp(-i\sigma_3 \zeta x/\epsilon) \to I, & x \to +\infty \\
  ii) & \quad \sup_{x \in \mathbb{R}} \|F(\zeta, x)\| < \infty,
\end{align*}
\]

where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). This Jost solution is meromorphic in the open upper half-plane, as well as in the open lower half-plane, and has limiting values,

\[
F_{\pm}(\lambda, x) = \lim_{\epsilon \to 0} F(\lambda \pm i\epsilon, x),
\]

from above and below respectively, on the real axis \( \lambda \in \mathbb{R} \), which are continuous in \( \lambda \). These two limiting values are related by a “jump matrix” as follows:

\[
F_+(\lambda) = F_-(\lambda) \left( 1 + \frac{|R(\lambda)|^2}{R(\lambda)} \begin{pmatrix} R(\lambda) \\ \overline{R(\lambda)} \end{pmatrix} \right),
\]

where the function \( R(\lambda) \) appearing in the jump matrix is called the reflection coefficient. It is complex-valued and depends only on \( \lambda \).

The first and second columns of \( F(\zeta, x) \), respectively denoted by \( f_1 \) and \( f_2 \), have the following asymptotics in \( \zeta \):

\[
\begin{align*}
  f_1(\zeta, x) &= \exp \left( \frac{i\zeta x}{\epsilon} \right) \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(\zeta^{-1}) \right] & \text{for } \zeta \to \infty, \\
  f_2(\zeta, x) &= \exp \left( -\frac{i\zeta x}{\epsilon} \right) \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\zeta^{-1}) \right] & \text{for } \zeta \to \infty,
\end{align*}
\]
The eigenvalues $\zeta_j$ in the upper half-plane are determined by the condition that these two solutions should be linearly dependent. (These are also precisely the locations where the Jost function in the upper half plane has a pole. Away from these points the Jost function is analytic.)

At an eigenvalue, the proportionality between these two solutions

$$f_1(\zeta_j, x) = \exp\left(-\frac{\chi_j}{\epsilon}\right) f_2(\zeta_j, x)$$

is specified in terms of $\chi_j$, which is referred to as the norming exponent.

The inverse theory prescribes that the fundamental scattering data consists of the reflection coefficient $R(\lambda)$, the $N$ eigenvalues $\zeta_j$ and the $N$ norming exponents $\chi_j$.

As $u(x, t)$ evolves according to the focusing NLS equation (1.4), the eigenvalues $\{\zeta_j\}$ remain unchanged while the time dependence of the other scattering data is

$$\chi_j(t) = \chi_j + i\zeta_j^2 t, \quad R(\lambda, t) = R(\lambda) \exp\left(\frac{2i\lambda^2 t}{\epsilon}\right).$$

Given $R(\lambda)$, $\zeta_j$, and $\chi_j$ computed from the initial data $u(x, 0)$, the solution $u(x, t)$, as well as all of the conserved densities defined in the next subsection, of the NLS equation (1.4), are then determined by inverse scattering from $R(\lambda, t)$, $\zeta_j$, and $\chi_j(t)$ given by (3.3) [ZS1, KMM].

3.2. The Hierarchy. The complete integrability of this cubic Schrödinger equation implies the existence of an infinite family of independent, conserved quantities [ZS1],

$$H_m = \int_{-\infty}^{\infty} \rho_m \, dx \quad \text{for } m = 0, 1, \ldots,$$

which are in involution with respect to the Poisson bracket,

$$0 = \{H_m, H_n\} = \frac{1}{i\epsilon} \int_{-\infty}^{\infty} \left( \frac{\delta H_m}{\delta u} \frac{\delta H_n}{\delta \bar{u}} - \frac{\delta H_m}{\delta \bar{u}} \frac{\delta H_n}{\delta u} \right) \, dx.$$

The first three densities $\rho_m$ are essentially those given by (1.7) and (1.10):

$$\rho_0 = -\rho = -|u|^2;$$

$$\rho_1 = \mu = -i\frac{\epsilon}{2} \left( \bar{u} \partial_x u - u \partial_x \bar{u} \right);$$

$$\rho_2 = \frac{3}{2} \epsilon \bar{\epsilon} = \frac{3}{4} \left( \epsilon^2 |\partial_x u|^2 - |u|^4 \right).$$

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A recursion formula that generates all the conserved densities is given by (7.5).

All of the $H_m$ are Hamiltonians that generate flows which commute with the focusing cubic NLS flow (1.4a), the so-called focusing Zakharov-Shabat (ZS) hierarchy. Letting $t_m$ denote the time variable associated with the $m^{th}$ flow, its evolution is then given by

$$i\epsilon \partial_{t_m} u = \frac{\delta H_m}{\delta \bar{u}} \quad \text{for } m = 0, 1, \ldots .$$

The $t_0$ flow is just a phase rotation, the $t_1$ flow is just a spatial translation, the $t_2$ flow is given by the focusing NLS equation (1.4a) with $t = t_2$, while the $t_3$ flow is that of the complex mKdV equation

$$\partial_{t_3} u + \frac{3}{2} |u|^2 \partial_x u + \epsilon^2 \frac{1}{4} \partial_{xxx} u = 0 .$$

Every $H_n$ is conserved by each flow; their densities satisfy the local conservation laws

$$(3.5) \begin{align*}
\partial_{t_0} \rho_{n-1} &= 0 , \\
\partial_{t_n} \rho_{n-1} + \partial_x \mu_{m,n} &= 0 , \quad \text{for } m, n = 1, 2, \ldots .
\end{align*}$$

Here $\mu_{m,n}$ is the flux for the $(n - 1)^{st}$ conserved quantity under the $m^{th}$ flow.

Because all these flows commute, they may be solved simultaneously for $u(x, t)$, having the form of the initial condition (1.4b), where $t = (t_0, t_1, \ldots )$ such that all but finitely many $t_m$ are zero. Associated with each $t$ is a polynomial $p(\cdot, t)$ defined by

$$(3.6) p(\zeta, t) = \sum_{m=0}^{\infty} t_m \zeta^m .$$

The simultaneous evolution of the scattering data is then given by

$$(3.7) \begin{align*}
\chi_j(t) &= \chi_j + ip(\zeta_j, t) , \\
R(\lambda, t) &= R(\lambda) \exp \left( \frac{2ip(\lambda, t)}{\epsilon} \right) ,
\end{align*}$$

and $u(x, t)$ is determined by inverse scattering.

In Section 7 we show that the odd flows in this hierarchy preserve real symmetry. More specifically, we show that $u(x, t)$ is real whenever $u(x, 0)$ is real and $t_m = 0$ for every even $m$. We show moreover that when $u$ is real that $\rho_n = 0$ for every odd $n$.

The scope of the zero-dispersion limit for the NLS equation can then be enlarged to consider the solution $u^\epsilon(x, t)$ of the whole hierarchy that satisfies the initial condition

$$u^\epsilon(x, 0) = A(x) \exp \left( \frac{iS(x)}{\epsilon} \right) ,$$

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where $A(x)$ is positive, $S(x)$ is real, both are independent of $\epsilon$, and $A(x)$ and $\partial_x S(x)$ are Schwartz class. The goal is then to determine the limiting behavior of all the conserved densities $\rho_n^\epsilon$ and fluxes $\mu_{m,n}^\epsilon$ associated with the entire focusing ZS hierarchy of flows as $\epsilon$ tends to zero. In this paper we achieve this goal only for the case when $u$ is real and we restrict to the odd flows of the hierarchy.

4. Analysis of the Scattering Transform

4.1. Asymptotic Analysis of the Initial Scattering Data. For initial data

$$u^\epsilon(x,0) = A(x) \exp\left(\frac{i}{\epsilon} S(x)\right),$$

of the class described at the end of the previous section, one can asymptotically determine the initial scattering data by a WKB analysis.

The $L^2$ eigenvectors as well as the reflection coefficient, $R(\lambda)$, for the Zakharov-Shabat eigenvalue problem (3.1) were defined in Section 3. The WKB ansatz is that the eigenvectors have a leading order asymptotic (in $\epsilon$) behavior of the form

$$f^{\epsilon (1)} = r^{(1)} \exp\left(\frac{i(\theta - \frac{1}{2} S)}{\epsilon}\right) + \cdots,$$

$$f^{\epsilon (2)} = r^{(2)} \exp\left(\frac{i(\theta + \frac{1}{2} S)}{\epsilon}\right) + \cdots,$$

where $r^{(1)}$, $r^{(2)}$, and $\theta$ do not depend on $\epsilon$. (There is a similar ansatz for the Jost functions.) When this ansatz is substituted into $\mathcal{L} f^\epsilon = \zeta f^\epsilon$, the leading order gives

$$\begin{pmatrix}
\zeta + (\partial_x \theta - \frac{1}{2} \partial_x S) & iA \\
iA & \zeta - (\partial_x \theta + \frac{1}{2} \partial_x S)
\end{pmatrix}
\begin{pmatrix}
r^{(1)} \\
r^{(2)}
\end{pmatrix} = 0.$$

This implies that

$$0 = \det\begin{pmatrix}
\zeta + (\partial_x \theta - \frac{1}{2} \partial_x S) & iA \\
iA & \zeta - (\partial_x \theta + \frac{1}{2} \partial_x S)
\end{pmatrix}
= (\zeta - \frac{1}{2} \partial_x S)^2 - (\partial_x \theta)^2 + A^2.$$

Suppose now that the initial value of $u_\epsilon$ is real. That means $u_\epsilon(x,0) = A(x)$ and

$$(\partial_x \theta)^2 = (A^2 + \zeta^2).$$

If the WKB asymptotics are valid then, in order for $f^\epsilon$ to be an eigenvector, $\partial_x \theta(x)$ should be pure imaginary for sufficiently large values of $|x|$. It follows from the preceding
equation that $\zeta$ should then be pure imaginary. Thus, we would expect the point spectrum to converge, as $\epsilon \to 0$, onto the imaginary axis.

In fact there is a natural class of initial data, the so-called “single lobe” potentials for which the eigenvalues are always pure imaginary independent of the value of $\epsilon$. By a single lobe potential we shall mean a real-valued potential for which $A(x)$ is nonnegative, smooth and integrable on $\mathbb{R}$ and moreover $A(x)$ should be non-decreasing on the left of $x = 0$ and non-increasing on the right. It has recently been demonstrated [KS] that for such data the eigenvalues are always pure imaginary. In fact these authors show that this result holds under a somewhat weakened definition of single lobe, but the above will suffice for our purposes. For such data, $A(x)$ will have two turning points, $x^\pm(\eta)$ (Fig. 4.1). We will assume that $A(x)$ has maximum height 1. In this setting the WKB ansatz implies that the asymptotic scattering data should satisfy:

1. The reflection coefficient $R(\lambda) \equiv 0$.

2. The eigenvalues are independent of time. They are purely imaginary with $\zeta_j = i\eta_j$.

The density of states, for the positive imaginary eigenvalues, is

$$
\phi(\eta) = \frac{1}{\pi} \frac{d}{d\eta} \text{Re} \int \sqrt{A^2(y) - \eta^2} \, dy = \frac{1}{\pi} \frac{d}{d\eta} \int_{x^-(\eta)}^{x^+(\eta)} \sqrt{A^2(y) - \eta^2} \, dy ;
$$

the eigenvalues are then defined by

$$(j + \frac{1}{2})\epsilon = -\int_{\eta_j}^{\eta_{\text{max}}} \phi(\eta) \, d\eta$$

for $j = 1, 2, \ldots, N^\epsilon$ and $\eta_1 < \eta_2 < \cdots < \eta_{N^\epsilon}$, where

3. $N^\epsilon$ is the total number of eigenvalues in the upper half plane and is determined by

$$N^\epsilon = \left[ \frac{1}{2\pi \epsilon} \text{Re} \int A(y) \, dy \right].$$

4. The initial norming exponents can be approximated by

$$\chi_j(0) = \chi_j = \eta_j x^\pm(\eta_j) + \int_{x^+(\eta_j)}^{\infty} \left( \eta_j - \sqrt{\eta_j^2 - A^2(x)} \right) \, dx.$$
**Fig 4.1** The initial data $A(x)$. Note its critical value $\eta_{max}$ and the indicated defining relations for $x^{\pm} = x^{\pm}(\eta)$.

Thus, if one just considers the odd flows, i.e. $t = (-t_{1}, t_{3}, \ldots (-1)^{k}t_{2k-1}, \ldots)$ then all the scattering data are real. The formula for $\chi_{j}$ given by (3.7) can be rewritten as

\[ \chi_{j}(t) = \chi_{j} + \sum_{k=1}^{\infty} \eta_{j}^{2k-1}t_{2k-1}. \]

We observe that this asymptotic data is the same as that for the Schrödinger operator (1.3) with $A^{2}(x)$ playing the role of $u_{in}(x)$ [LLV]. Henceforth, we consider the Cauchy problem for focusing NLS (1.4a) with initial data corresponding to the above asymptotic scattering data.

**4.2. Reflectionless Approximation.** Motivated by the previous section, we choose to replace the exact real-valued initial data $A$ by the real-valued reflectionless potential $A^{\epsilon}$ corresponding to the WKB scattering data given above. This allows us to use the reconstruction formulas for reflectionless potentials [FNR] that give the associated conserved densities and fluxes by

\[ \rho_{n-1}^{\epsilon}(x, t) = \epsilon^{2}\partial_{xt} \log(\tau^{\epsilon}(x, t)), \]
\[ \mu_{m,n}^{\epsilon}(x, t) = -\epsilon^{2}\partial_{t_{m}t_{n}} \log(\tau^{\epsilon}(x, t)), \]

where the so-called tau-function $\tau^{\epsilon}(x, t)$ is given by the $N^{\epsilon} \times N^{\epsilon}$ determinant

\[ \tau^{\epsilon}(x, t) = \det \left( I + \epsilon^{2}G^{\epsilon}(x, t)\bar{G}^{\epsilon}(x, t) \right). \]
Here $I$ is the identity matrix while the $N^\varepsilon \times N^\varepsilon$ matrix $G^\varepsilon$ has the form

$$(4.5) \quad G^\varepsilon(x, t) = \left( \frac{i}{\zeta_j - \zeta_k} \exp \left( \frac{a(i\zeta_j, x, t) + a(i\zeta_k, x, t)}{\varepsilon} \right) \right)_{j,k=1}^{N^\varepsilon},$$

where

$$(4.6) \quad a(\eta, x) \equiv -\eta x + \chi_j(t).$$

All the eigenvalues $\{\zeta_j\}_1^N$ lie on the positive imaginary axis and are bounded as follows:

$$\zeta_j = i\eta_j^\varepsilon, \quad 0 < \eta_1^\varepsilon < \eta_2^\varepsilon < \cdots < \eta_N^\varepsilon < 1,$$

then $E$ is real for all the odd flows $(t = t_1, t_3, \cdots)$ and

$$\frac{i}{\zeta_j - \zeta_k} = \frac{1}{\eta_j^\varepsilon + \eta_k^\varepsilon}.$$

From this, we see that $G^\varepsilon$ is real, positive definite and symmetric.

In this setting we define

$$B(x, t) = G^\varepsilon(x, t) = G^\varepsilon(x, t).$$

### 4.3. Convexity of the Tau-Function

We will make use of the following property of the tau-function.

**Lemma 4.1.** For each $\varepsilon > 0$ the function

$$(x, t) \mapsto \tau(x, t)$$

is convex.

**Proof.** Set $W = \varepsilon^2 \log \det(I + \varepsilon^2 B^2)$. It is sufficient to show that $W$ is convex in $x$ for general real parameters $\eta_j$ and $\chi_j$. This is true because convexity is a property along line segments, and examining $W$ along a general line, parameterized by $\bar{x}$, of the form $P(\bar{x}) = (x(\bar{x}), t(\bar{x})) = P_0(1 - \bar{x}) + P_1 \bar{x}$ results in the function

$$W(\bar{x}) = \varepsilon^2 \log \det(I + \varepsilon^2 B(\bar{x})^2).$$

The convexity of $W(x, t)$ follows from the convexity of $W(\bar{x})$. For the remainder of the argument, we drop the tildes and denote differentiation in $\bar{x}$ with a prime ($'$).
Using the definition of $B$, we find $B' = -\frac{1}{2}(\Lambda B + BA)$, where $\Lambda$ is the diagonal matrix of the $\eta_j$. From this and the general facts that $\det(a)' = \det(a)(\text{tr}(a^{-1}a')$ and $\text{tr}(ab) = \text{tr}(ba)$, we find by direct calculation that

\begin{equation}
W' = -4\epsilon \text{ tr}\left((I + \epsilon^2 B^2)^{-1} \epsilon^2 B^2 \Lambda\right).
\end{equation}

It is more convenient to write $W'$ as

$$W' = -4\epsilon \text{ tr}\left((I + (\epsilon^2 B^2)^{-1})^{-1} \Lambda\right)$$

before differentiating to obtain

\begin{equation}
W'' = 4 \text{ tr } [(A)(\Lambda B^2 A\Lambda)] + 8 \text{ tr } [(AB)(\Lambda A\Lambda)] + 4 \text{ tr } [(AB^2)(\Lambda A\Lambda)],
\end{equation}

where

$$A = (I + (\epsilon B)^{-2})^{-1} (\epsilon B)^{-2}.$$ 

Both $A$ and $B$ are symmetric positive definite (SPD), and $A$ commutes with $B$, so products of $A$ with powers of $B$ are also SPD. Thus the terms of $W''$ in (4.8) are traces of the products of two SPD matrices, each grouped in parentheses in the equation.

Traces of products of SPD matrices are positive, and so the sum is positive. Thus $W'' \geq 0$, and so $W(\tilde{x})$ is convex, and, so, $W(x, t)$ is convex, which proves Lemma 4.1.

We remark that (4.7) for the full system yields

$$\epsilon^2 \partial_x \log \tau^\epsilon(+\infty, t) = 0, \quad \epsilon^2 \partial_x \log \tau^\epsilon(-\infty, t) = -4\epsilon \text{ tr}(\Lambda).$$

One can similarly find $\epsilon^2 \partial_{\xi_m} \log \tau^\epsilon(\pm\infty, t)$ for any positive, odd number $m$.

5. Establishing the Limit

5.1. The Limit of the Tau-Functions. In this subsection we establish the limit, as $\epsilon$ tends to zero, of $\epsilon^2 \log(\tau^\epsilon(x, t))$ in the topology of uniform convergence over compact subsets of $\mathbb{R} \times T_o$, characterizing the limit in terms the solution of a maximization problem. Here, $T_o$ denotes the subspace of vectors of the form $t = (-t_1, t_3, \ldots (-1)^k t_{2k-1}, \ldots)$ which are real and have finite support.
We adopt the following notation. Given any $N^\epsilon \times N^\epsilon$ matrix $A$ indexed by the set $J^\epsilon = \{1, \ldots, N^\epsilon\}$, and given any two nonempty subsets $S \subset J^\epsilon$ and $T \subset J^\epsilon$, let $A_{ST}$ denote the $|S| \times |T|$ submatrix of $A$ obtained by retaining only those entries whose row index belongs to $S$ and column index belongs to $T$. We abbreviate $A_{SS}$ as $A_S$. Finally, for every $S \subset J^\epsilon$, we let $S'$ denote the compliment of $S$ in $J^\epsilon$.

The key new result is the following.

**Theorem 5.1.** Let

\[
q^\epsilon(x, t) = \max \left\{ \epsilon^2 \log \left( \det \left( G_S^\epsilon(x, t) \right) \right) : S \subset J^\epsilon \right\},
\]

where the maximum ranges over all index sets, with the understanding that $\det(G_S^\epsilon) = 1$ when $S$ is the empty set. Then

\[
\lim_{\epsilon \to 0} \left| \epsilon^2 \log \left( \tau^\epsilon(x, t) \right) - q^\epsilon(x, t) \right| = 0
\]

uniformly over $\mathbf{R} \times \mathbf{T}_o$.

**Proof.** Below we will derive the crude bounds

\[
\epsilon^2 N^\epsilon \exp \left( \frac{q^\epsilon(x, t)}{\epsilon} \right) \leq \tau^\epsilon(x, t) \leq 2^{\epsilon N^\epsilon} \exp \left( \frac{q^\epsilon(x, t)}{\epsilon} \right).
\]

Given these, by taking their logarithm, multiplying the result by $\epsilon^2$, and subtracting $q^\epsilon(x, t)$, one arrives at

\[
\epsilon^2 2N^\epsilon \log(\epsilon) \leq \epsilon^2 \log(\tau^\epsilon(x, t)) - q^\epsilon(x, t) \leq \epsilon^2 2N^\epsilon \log(2).
\]

Because $N^\epsilon = O(1/\epsilon)$ by the WKB ansatz, the limit (5.2) follows. Hence, the theorem follows once the bounds (5.3) are established.

To derive the upper bound on $\tau^\epsilon$ in (5.3), first use the fact that $G^\epsilon$ is Hermitian positive to obtain

\[
\tau^\epsilon = \det \left( I + \epsilon^2 G^\epsilon \right) \leq \det \left( I + 2\epsilon G^\epsilon + \epsilon^2 G^\epsilon \right) = \left[ \det \left( I + \epsilon G^\epsilon \right) \right]^2.
\]

The last determinant in (5.5) is just the characteristic polynomial of $\epsilon G^\epsilon$ evaluated at $-1$ and can be expanded in terms of the determinants of its minor matrices $\epsilon G_S^\epsilon$ as

\[
\det(\epsilon G^\epsilon) = \sum_{S \subset J^\epsilon} \epsilon^{|S|} \det(G_S^\epsilon).
\]
Because every principal minor of $G^\epsilon$ has the same form as $G^\epsilon$, each $G^\epsilon_S$ is Hermitian positive, and hence, has a positive determinant. The sum in (5.6) therefore contains $2^{N^*}$ positive terms, each of which is bounded above (for $\epsilon \leq 1$) by

$$(5.7) \quad \epsilon^{|S|} \det(G^\epsilon_S) \leq \exp\left(\frac{q^\epsilon(x,t)}{2\epsilon}\right),$$

The upper bound on $\tau^\epsilon$ in (5.3) therefore follows by combining (5.5), (5.6), and (5.7).

To derive the lower bound on $\tau^\epsilon$ in (5.3), first use the fact that

$$(5.8) \quad \tau^\epsilon = \det(I + \epsilon^2 B) = \sum_{S \subset J^\epsilon} \det(\epsilon^2 B_S).$$

Because $G^\epsilon$ is Hermitian positive, so is $B = G^\epsilon^2$. Hence, each term of the sum in (5.8) is positive. We then use the fact that

$$B^\epsilon_S = B^\epsilon_{SS} = (G^\epsilon_{SS} G^\epsilon_{SS} + G^\epsilon_{SS} G^\epsilon_{SS}'),$$

and the fact that $G^\epsilon_{SS} G^\epsilon_{SS}$ and $G^\epsilon_{SS} G^\epsilon_{SS}'$ are Hermitian nonnegative matrices to bound each term below by

$$(5.9) \quad \det(\epsilon^2 B_S) \geq \det(\epsilon^2 G^\epsilon_{SS} G^\epsilon_{SS})$$

$$= \epsilon^{|S|} \left[ \det(G^\epsilon_S) \right]^2$$

$$\geq \epsilon^{2N^*} \left[ \det(G^\epsilon_S) \right]^2.$$ 

Because each term of the sum in (5.8) is positive, the sum may be bounded below by the largest lower bound on any of its terms. By (5.1) that is achieved for the $S$ where

$$\left[ \det(G^\epsilon_S) \right]^2 = \exp\left(\frac{q^\epsilon(x,t)}{\epsilon}\right),$$

whereby the desired lower bound on $\tau^\epsilon$ in (5.3) is obtained, and the proof is complete.

**5.3. Properties of the Maximization Problem.**

Before we can characterize the semiclassical limit of the conserved densities and fluxes, more basic facts concerning the maximization problem in (5.1) must be established, including the uniqueness of the density at which the maximum is attained. By introducing the scalar product

$$(5.10) \quad (\alpha | \beta) \equiv \frac{1}{\pi} \int_0^1 \alpha(\eta) \beta(\eta) d\eta,$$
and the integral operator

\begin{equation}
L\psi(\eta) = \frac{1}{2\pi} \int_{0}^{1} \log\left(\frac{\eta - \nu}{\eta + \nu}\right)^2 \psi(\nu) d\nu.
\end{equation}

The underlying functional being maximized in (5.1) can be recast more abstractly as the quadratic form

\begin{equation}
Q(\psi; x, t) = \frac{1}{2} (\psi | L\psi) + (a(x, t) | \psi).
\end{equation}

where \(a\) is again given by (4.6) but here we suppress the \(\eta\) dependence. The properties of these objects that we will need are presented in the following three lemmas.

The maximization problem (5.1) in this form is then posed over the set \(\mathcal{A}\) of admissible \(L^1\) densities,

\[
\mathcal{A} = \{ \psi \in L^1([0, 1]) : 0 \leq \psi(\eta) \leq \phi(\eta) \},
\]

which naturally inherits a topology from the weak-* topology of measures. However, the densities in \(\mathcal{A}\), being all bounded above by \(\rho(\eta)\), are equi-integrable and, hence, comprise a relatively compact set in the weak topology of \(L^1([0,1])\). This means that the weak-* topology of measures and the weak topology of \(L^1\) coincide on \(\mathcal{A}\). We will always consider \(\mathcal{A}\) equipped with this topology. The key fact we will need is stated in the first lemma, the proof of which is omitted.

**Lemma 5.2 (\(\mathcal{A}\)-compactness).** The set \(\mathcal{A}\) is compact.

The operator \(L\) defined by (5.11) and the functional \(Q\) defined by (5.12) are well behaved over the set \(\mathcal{A}\). Indeed, given that \(\psi\) is in \(\mathcal{A}\), it can be shown [LL2] that \(L\psi\) is in the class of continuous, odd, functions of \(\eta\), which we denote as \(C_{odd}(\mathbb{R})\). Moreover, it is clear from (4.6) that for every \((x, t) \in \mathbb{R} \times \mathcal{T}_o\) the function \(\eta \mapsto a(\eta, x, t)\) is also in \(C_{odd}(\mathbb{R})\). Consequently, \(Q\) defined by (5.12) takes values in \(\mathbb{R}\). Moreover, \(L\) and \(Q\) possess continuity properties that are stated in the next lemma.

**Lemma 5.3 (\(\mathcal{A}\)-continuity).**

(a) The operator \(L : \mathcal{A} \to C_{odd}(\mathbb{R})\) is continuous.

(b) The functional \(Q : \mathcal{A} \times \mathbb{R} \times \mathcal{T}_o \to \mathbb{R}\) is continuous.

**Proof.** The hardest part is the proof of (a) which resembles that of Theorem 3.4 in [LL2]. The proof of (b) follows from (a) and the continuity of \(a\). //

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Remark. By combining Lemma 5.2 with Lemma 5.3, it is seen that the image of $\mathcal{A}$ under $L$ is a compact subset of $C_{\text{odd}}(\mathbb{R})$ and that for every $(x, t) \in \mathbb{R} \times \mathcal{T}_0$ the map $\psi \mapsto Q(\psi; x, t)$ has a compact range and, hence, attains a maximum over $\mathcal{A}$. The existence of the maximum for (5.1), which has already been established through the limiting procedure of Theorem 5.1, is thereby re-established intrinsically.

The uniqueness of the maximum in (5.1) follows from the third lemma.

Lemma 5.4 (strict concavity).

(a) The quadratic form $(\psi | L \psi)$ is negative definite over $\mathcal{A} \ominus \mathcal{A}$.

(b) For every $(x, t) \in \mathbb{R} \times \mathcal{T}_0$, the map $\psi \mapsto Q(\psi; x, t)$ is strictly concave over $\mathcal{A}$.

Proof. The hardest part is again the proof of (a) which here resembles that of Theorem 3.7 in [LL2]. The proof of (b) follows directly from (a).

With the above pieces in place, the main result of this subsection can now be presented.

Theorem 5.5 (uniqueness and regularity). For each $(x, t) \in \mathbb{R} \times \mathbb{R}^\infty$ there exists a unique $\psi^*(x, t)$ in the admissible class $\mathcal{A}$ such that

\begin{equation}
q(x, t) = Q(\psi^*(x, t); x, t).
\end{equation}

Moreover, this maximizing density satisfies the following.

(a) The map $(x, t) \mapsto \psi^*(x, t)$ is continuous from $\mathbb{R} \times \mathcal{T}_0$ into $\mathcal{A}$ equipped with the weak $L^1$ topology.

(b) The map $(x, t) \mapsto L \psi^*(x, t)$ is continuous from $\mathbb{R} \times \mathcal{T}_0$ into $C_{\text{odd}}(\mathbb{S})$ equipped with the uniform topology.

(c) The map $(x, t) \mapsto q(x, t)$ is differentiable from $\mathbb{R} \times \mathcal{T}_0$ into $\mathbb{R}$ with its continuous partial derivatives given by

\begin{align}
\partial_x q(x, t) &= -(\eta | \psi^*(x, t)) , \\
\partial_{t_{2^n-1}} q(x, t) &= (\eta^{2^n-1} | \psi^*(x, t)).
\end{align}

Proof. Lemma 5.4 states that for every $(x, t)$ the functional $Q(\cdot; x, t)$ is strictly concave over $\mathcal{A}$, thereby insuring the uniqueness of the point in $\mathcal{A}$ at which the maximum in (5.1) is attained. Denote this point as $\psi^*(x, t)$. 23
To prove the continuity asserted in (a), let \((x, t)\) be any point and \(\{(x_k, t_k)\}\) any sequence converging to that point in \(\mathbb{R} \times T_o\). Because, by Lemma 5.2, \(\mathcal{A}\) is compact, the sequence \(\{(\psi^*(x_k, t_k))\}\) has cluster points, all of which lie in \(\mathcal{A}\). Let \(\psi_*\) denote one such cluster point. Upon passing to a subsequence if necessary, part (b) of Lemma 5.3 implies

\[
(5.15) \quad \lim_{k \to \infty} q(x_k, t_k) = \lim_{k \to \infty} Q(\psi^*(x_k, t_k); x_k, t_k) = Q(\psi_*; x, t).
\]

On the other hand, the continuity of \(q\) implies

\[
(5.16) \quad \lim_{k \to \infty} q(x_k, t_k) = q(x, t).
\]

Comparing (5.15) and (5.16) shows that \(Q(\psi_*; x, t) = q(x, t)\), whereby we conclude that \(\psi_* = \psi^*(x, t)\). Hence, the original sequence \(\{(\psi^*(x_k, t_k))\}\), having \(\psi^*(x, t)\) as the only cluster point, must converge to \(\psi^*(x, t)\). The continuity asserted in (a) follows immediately, while that of (b) does so after invoking the part (a) of Lemma 5.3.

Now turn to the differentiability (c). It suffices to establish (5.14b). Let \(t\) and \(t'\) differ only in the coordinate \(t_{2n-1}\). For every \(\eta\) in \(\mathcal{A}\), a direct calculation then yields

\[
(5.17) \quad Q(\psi; x, t') - Q(\psi; x, t) = (\eta^{2n-1} | \eta) (t'_{2n-1} - t_{2n-1}).
\]

However, because \(\psi^*\) maximizes \(Q\), one has the general two-sided inequality

\[
(5.18) \quad Q(\psi^*(x, t); x, t') - Q(\psi^*(x, t); x, t) \leq q(x, t') - q(x, t) \\
\leq Q(\psi^*(x, t'); x, t') - Q(\psi^*(x, t'; x, t),
\]

which, when combined with (5.17), gives

\[
(5.19) \quad (\eta^{2n-1} | \psi^*(x, t)) (t'_{2n-1} - t_{2n-1}) \leq q(x, t') - q(x, t) \\
\leq (\eta^{2n-1} | \psi^*(x, t')) (t'_{2n-1} - t_{2n-1}).
\]

The result now follow by the continuity of \(\eta^*\). //

It will prove useful that the solution \(\psi^*(x, t)\) of the maximization problem can be characterized in terms of variational conditions.

**Theorem 5.6 (variational conditions).** If \(\eta \in \mathcal{A}\) then \(\eta = \eta^*(x, t)\) if and only if \(\eta\) satisfies the variational conditions

\[
(5.20) \quad \psi = \begin{cases} 
0 & \text{where } a(\eta, x, t) + L\psi < 0, \\
\phi(\eta) & \text{where } a(\eta, x, t) + L\psi > 0.
\end{cases}
\]
Proof. The proof follows that of Theorem 3.12 in [LL2]. It uses the continuity of $a + L\psi$ in $\eta$ to assert that the conditionals in (5.20) define open sets. //

5.4. The Limit of the Densities and Fluxes. By combining the differentiability result of Theorem 5.5 with the following elementary, but nontrivial, lemma will yield a strengthening in the sense of the convergence for the tau-functions in (5.2).

**Lemma 5.7 (converging derivatives).** Let $\{h_n\}$ be a sequence of differentiable convex functions over $\mathbb{R}^\infty$ such that $h_n(x) \rightarrow h(x)$ uniformly over compact subsets of $x \in \mathbb{R}^\infty$ and $h$ is differentiable. Let $\partial_s = \dot{x} \cdot \partial_x$ denote the directional derivative of $x$ in a given direction $\dot{x} \in \mathbb{R}^\infty$. Then

$$(5.21) \quad \partial_s h_n(x) \rightarrow \partial_s h(x),$$

uniformly over compact subsets of $x \in \mathbb{R}^\infty$.

In particular, we can now give the main result of this section.

**Theorem 5.8 (limit of densities and fluxes).** The limit of the tau-function $\tau^\epsilon$ is given by

$$(5.22) \quad \lim_{\epsilon \to 0} \epsilon^2 \partial_{t_{2n-1}} \log \tau^\epsilon(x, t) = \partial_{t_{2n-1}} q(x, t) = (\eta^{2n-1} \mid \psi^*(x, t)), \nonumber$$

uniformly over compact subsets of $(x, t) \in \mathbb{R} \times T_0$. The densities $\rho^\epsilon$ and fluxes $\mu^\epsilon_{m,n}$ have the distributional limits

$$(5.23a) \quad \mathcal{D}'(dx) - \lim_{\epsilon \to 0} \rho^\epsilon_{2n-2} = \partial_x (\eta^{2n-1} \mid \psi^*),$$

$$(5.23b) \quad \mathcal{D}'(dt_{2m-1}) - \lim_{\epsilon \to 0} \mu^\epsilon_{2m-1,2n-1} = -\partial_{t_{2m-1}} (\eta^{2n-1} \mid \psi^*).$$

Proof. Assertion (5.22) follows directly from (5.2) of Theorem 5.1 (tau-function limit) and Theorem 5.14 (uniqueness and regularity) upon applying Lemma 5.7 (converging derivatives). Assertion (5.23) then follows by the usual integration by parts argument. //

**Remark.** The significance of (5.23) is that one can now pass to the limit in the local conservation laws

$$(5.24) \quad \partial_{t_{2m-1}} \rho^\epsilon_{2n-2} + \partial_x \mu^\epsilon_{2m-1,2n-1} = 0, \quad \text{for } m, n = 1, 2, \ldots.$$
By the results of the previous section we have seen that the limit related to $\tau$ can be converted to a variational problem. This can in turn be converted to a Riemann-Hilbert boundary value problem which can be solved with the use of the Hilbert transform. We will list some related results but omit most proofs. The interested reader can find more details in [LL] and [LLV].

The variational conditions (5.20) for the maximization problem can be expressed

$$
\begin{align*}
L \psi(\eta) + a(\eta, x, t) &< 0, & \text{when } \psi(\eta, x, t) = 0, \\
L \psi(\eta) + a(\eta, x, t) &= 0, & \text{when } 0 < \psi(\eta, x, t) < \phi(\eta), \\
L \psi(\eta) + a(\eta, x, t) &> 0, & \text{when } \psi(\eta, x, t) = \phi(\eta).
\end{align*}
$$

Let $I$ be the interior of the set of $(\eta, x, t)$ in which equality holds and $\bar{I}$ be its closure. Directly differentiating (6.1) while using formula (4.6) for the function $a$ leads to

$$
\begin{align*}
L \psi_x(\eta) &= \eta, & L \psi_t(\eta) &= -\partial_t \chi(\eta; t), & \text{when } (\eta, x, t) \in I, \\
\psi_x(\eta) &= 0, & \psi_t(\eta) &= 0, & \text{when } (\eta, x, t) \notin \bar{I},
\end{align*}
$$

where $\chi(\eta; t)$ is defined as $\chi_j(t)$ if $\eta = \eta_j$.

These differentiated variational conditions do not have any explicit dependence on $(x, t)$, but rather their dynamics, as well as all their memory of the initial data, is contained in the set $I$. Consequently, these conditions have more general validity than the variational conditions (6.1). Indeed, the latter vary (although in a sense insignificantly) when the initial data are considered in different classes (such as periodic or tending to different limits as $x \to \pm\infty$), while the differentiated conditions remain unchanged.

Making the ansatz that at fixed $(x, t)$ the set $I(x, t)$, defined as

$$
I(x, t) \equiv \left\{ \eta \in [0, 1) : (\eta, x, t) \in I \right\},
$$

consists of a finite union of disjoint open intervals, one can uniquely determine the functions $\psi_x(\eta)$ and $\psi_t(\eta)$ in terms of the endpoints of these intervals. More precisely, $I(x, t)$ is assumed to take the form

$$
I(x, t) = [0, \beta_1) \cup (\beta_2, \beta_3) \cup \cdots \cup (\beta_{2g}, \beta_{2g+1})
$$

for some nonnegative integer $g$, where the $\beta_i$ and $g$ depend on $(x, t)$ and

$$
0 < \beta_1 < \beta_2 < \cdots < \beta_{2g+1} < 1.
$$

The operator $L$ given by (5.11) plays an important role for seeking the unique solution of the maximum problem.
Extend $\psi(\eta)$ to be an odd function over the real axis $\mathbb{R}$ that vanishes when $|\eta| \geq 1$, then the $\eta$-derivative of $L$ is the Hilbert transform:

$$\frac{d}{d\eta} L\psi(\eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\psi(\mu)}{\eta - \mu} d\mu \equiv H\psi(\eta).$$

This observation allows the differentiated variational conditions (6.2) to be transformed into Riemann-Hilbert problems through which they can be solved explicitly for $\psi_x$ and $\psi_t$ in terms of hyperelliptic functions involving a radical function $R$.

This $R$ has the following form

$$R(\eta, x, t) = \left( \prod_{i=1}^{2g+1} \left( \beta_i^2 - \eta^2 \right) \right)^{1/2},$$

where the $\{\beta_i\}$ are the boundaries of the set $I(x, t)$ as given by (6.3).

Note that $I(x, t)$ is the set over which $R(\eta)$ is real valued and $g$ is just the genus of the associated Riemann surface.

The variational problem (5.12) is uniquely solvable, say, by $\psi^*$. It follows from (5.14) and (5.23), for $n=1$, that all the limits of nontrivial conserved densities along odd flows can be represented by the derivatives of $\psi^*$ in the sense of distributional convergence:

$$\rho(x, t) = -\langle \eta | \psi^*_x \rangle,$$
$$\epsilon(x, t) = \frac{2}{3} \langle \eta | \psi^*_t \rangle,$$
$$\rho_{2m-2}(x, t) = \langle \eta | \psi^*_{t_{2m-1}} \rangle,$$

for $m = 1, 2, \cdots$.

The integrals above are calculated on $I(x, t)$, the particular domain of $\eta$ as given by (6.3). These $\{\beta_i\}$, as the boundary points of $I(x, t)$, are then governed by a genuine nonlinear hyperbolic system under any odd flow. Different flow makes different time evolution of the domain $I(x, t)$ through its boundary $\{\beta_i\}$, and enforces different time evolution of the semiclassical limit.

The dependence of the $\beta_i$ on $(x, t)$ is then derived from the compatibility constraint $\partial_t \psi_x = \partial_x \psi_t$. This reduces to the first order system of hyperbolic equations in the Riemann invariant (diagonal) form

$$\partial_{t_m} \beta_i + S_{m1}(\beta_1, \cdots, \beta_{2g+1}) \partial_x \beta_i = 0,$$

for $i = 1, \cdots, 2g + 1$ and any positive odd number $m$. 

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In this way, the initial value of $\psi^*$ can be represented in the form

$$
\psi^*(x, 0) = \begin{cases} 
0 & \text{for } x \geq x^+ (\eta), \\
\int_x^{x^+ (\eta)} \frac{\eta}{(A^2(y) - \eta^2)^{1/2}} \, dy & \text{for } x^+ (\eta) \geq x \geq x^- (\eta), \\
\phi(\eta) & \text{for } x \leq x^- (\eta),
\end{cases}
$$

where $x^- (\eta) \leq x^+ (\eta)$ are implicitly defined by

$$
A(x^+) = A(x^-) = \eta.
$$

Consider the semiclassical limit along the $t_3$ flow, i.e. the mKdV flow (1.9). Note that its formal limit equation (1.12) has no classical solution beyond a finite time, say $t = t_b$, because of its hyperbolic nature. If $\rho(x, t)$ is the solution of equation (1.12) (focusing) with the initial value $\rho(x, 0) = A^2(x)$, then $t_b$ is given by

$$
t_b = \left[ \max_x (3AA_x) \right]^{-1}.
$$

For $t < t_b$, i.e. $g = 0$, we have that $\beta_1^2(x, t) = \rho(x, t)$, an exact solution of (1.2), and

$$
L^2 - \lim_{\epsilon \to 0} \rho(t, x, t) = \beta_1^2(x, t)
$$

uniformly in $t$.

For $t \geq t_b$, the limits of all conserved densities and their fluxes exist in the sense of $L^2$ weak convergence. They are determined by the $2g + 1$ functions $\{\beta_i\}$ which are governed by a strict genuinely nonlinear hyperbolic system (6.4) and will break down again after a finite time.

7. An Invariant Subspace for the Hierarchy

Loop algebras provide a natural “phase space” in which to study the focusing Zakharov-Shabat (ZS) hierarchy. We will see that, when $u_{in}$ is restricted to be real, the correspondingly restricted phase space is invariant under the action of the odd flows of the hierarchy. Reduction to this invariant subspace provides a framework in which to geometrically understand the dispersionless limit of these odd flows onto the corresponding limit for KdV.

7.1. Loop Algebra Structure. We begin by summarizing the loop algebra structure for these soliton equations. Further details may be found in [FNR]. To facilitate our presentation, we introduce the following notation for the Pauli basis of $su(2)$:

$$
\mathcal{H} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
$$
With respect to this basis the ZS spectral problem can be expressed as

\begin{equation}
\mathcal{L} - \zeta = -\epsilon \mathcal{H} \partial_x + \alpha \mathcal{E} + \beta \mathcal{F} - \zeta,
\end{equation}

where \( u^\epsilon(x) = \alpha(x) + i\beta(x) \). Upon multiplying this expression by \( \mathcal{H} \), we get an equivalent operator,

\begin{equation}
\epsilon \partial_x - \alpha \mathcal{F} + \beta \mathcal{E} - \zeta \mathcal{H}.
\end{equation}

Observe that the nondifferential part of this operator is a real linear combination of the Pauli matrices, and so is a family of elements of \( su(2) \). This observation motivates the definition of the loop algebra for the hierarchy, which is defined as the vector space of all formal Laurent series, \( Q \), with coefficients in \( su(2) \):

\begin{equation}
Q = \sum_{n=-\infty}^{\infty} \zeta^n (h_n \mathcal{H} + a_n \mathcal{F} + b_n \mathcal{E}).
\end{equation}

The “algebra” structure is a Lie algebra structure induced by the usual Lie bracket, \([\cdot, \cdot]\) for \( su(2) \): if \( Q = \sum_{j=-\infty}^{\infty} \zeta^j Q_{-j} \), and \( R = \sum_{k=-\infty}^{\infty} \zeta^k R_{-k} \), then

\begin{equation}
[Q, R] = \sum_{m=-\infty}^{\infty} \sum_{j+k=m} \zeta^m [Q_{-j}, R_{-k}].
\end{equation}

This loop algebra is the natural setting in which to develop the Lax pair formulation of the NLS flows. For example, in the case of the original focusing NLS flow if we let

\begin{equation}
\epsilon \partial_x + Q^{(1)}
\end{equation}

denote the operator (7.1), then \( Q^{(1)} \) is an element of the loop algebra (7.3) which is linear in \( \zeta \). There is another operator,

\begin{equation}
\epsilon \partial_t + Q^{(2)},
\end{equation}

where

\begin{equation}
Q^{(2)} = \zeta^2 Q_0 + \zeta Q_1 + Q_2
\end{equation}

\begin{equation}
= \zeta^2 \mathcal{H} + \zeta \begin{pmatrix} 0 & -u \\ \bar{u} & 0 \end{pmatrix} - \begin{pmatrix} i/2 |u|^2 & i\epsilon/2 \partial_x u \\ i\epsilon/2 \partial_x \bar{u} & -i/2 |u|^2 \end{pmatrix}
\end{equation}

is a quadratic polynomial in the loop algebra (7.3) such that the compatibility condition,

\begin{equation}
[\partial_x - Q^{(1)}, \partial_t - Q^{(2)}] = Q^{(1)}_t - Q^{(2)}_x + [Q^{(1)}, Q^{(2)}] = 0,
\end{equation}

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known as a Lax equation, is equivalent to the requirement that \( u_\epsilon \) should solve (1.4a).

### 7.2. The Hierarchy of Densities and Flows

In Section 3 we took an approach for constructing the hierarchy of flows which is based on a Hamiltonian formalism applied to a generating function for the conserved densities. However, this recursive construction can be encoded [FNR] in the loop algebra representation as follows. The hierarchy of conserved densities, \( \rho_m = h_{m+2} \) is recursively defined by

\[
(7.5a) \quad h_n = i \int (ue_n - \bar{u}\bar{e}_n) \, dx ,
\]

where

\[
(7.5b) \quad e_n = i/2(\epsilon \partial_x e_{n-1} - 2i\bar{u}h_{n-1}) ,
\]

with \( h_0 = -1 \) and \( e_0 = 0 \). Setting \( e_n = a_n + ib_n \) then determines the coefficients, \( a_n, b_n, h_n \) of a loop element (7.3).

We now introduce the matrix

\[
(7.6) \quad Q_n = \begin{pmatrix} ih_n & e_n \\ -\bar{e}_n & -ih_n \end{pmatrix} ,
\]

where \( e_n \) and \( h_n \) are the quantities we have just defined. One sees immediately from the definition of these quantities that \( Q_n \) is an element of \( su(2) \); i.e., it is skew-Hermitian. Using these recursive definitions it is also a straightforward exercise, which we leave to the reader, to show that the \( Q_n \) satisfy a hierarchy of coupled ODE’s:

\[
(7.7) \quad \partial_x Q_n - [Q_1, Q_n] = [Q_0, Q_{n+1}] .
\]

This hierarchy can be encoded in a single “generating” ODE

\[
(7.8) \quad \partial_x Q - [\zeta Q_0 + Q_1, Q] = 0 ,
\]

where \( Q = Q_0 + \zeta^{-1}Q_1 + \zeta^{-2}Q_2 + \cdots \) is a Laurent element of our loop algebra, as is again straightforward to check. The coefficient of \( \zeta^{-n} \) on the left-hand side of (7.8) is equivalent to the \( n \)th ODE of (7.7). Note that the linear polynomial in the bracket of (7.8), \( \zeta Q_0 + Q_1 = Q^{(1)} \) is the nondifferential part of \( \mathcal{H}(L - \zeta) \) which is equivalent to the original Zakharov-Shabat eigenvalue problem. Moreover, if \( \Psi \) be a fundamental solution of the ZS problem, then [FNR] \( Q = \Psi \mathcal{H} \Psi^{-1} \) solves (7.8). Thus one sees that the loop algebra elements have a realization in terms of “squared eigenfunctions” of the ZS eigenvalue problem and, therefore, the densities \( h_n \) and \( e_n \) can, in principle, be read off from the asymptotic expansion in \( \zeta \) of the Jost function for the ZS problem.
7.3. Invariance of Real Symmetry for the Odd Flows. Before investigating how the loop algebra structure reflects a restriction to real initial data, we need to introduce one other ingredient: a natural inner product on the underlying vector space of loop elements. For any two loop algebra elements, \( Q = \sum_{j=-\infty}^{\infty} \zeta^j Q_j \) and \( R = \sum_{k=-\infty}^{\infty} \zeta^k R_k \), this symmetric form is defined by

\[
(Q, R) = -\frac{1}{2} \text{tr} \left( \sum_{j+k=0} Q_{-j} R_{-k} \right).
\]

Henceforth it will be assumed that \( u(x) \) is real-valued; i.e. that \( u(x) = A(x) \) and \( S(x) \equiv 0 \). Let’s use the recursion formulas (7.5) to calculate the first few densities under the reality constraint:

\[
e_0 = 0 \quad h_0 = -1 \\
e_1 = uh_0 = -u \quad h_1 = 0 \\
e_2 = i/2 \partial_x (-u) \quad h_2 = i \int u(-i/2 \partial_x u) - u(i/2 \partial_x u) \, dx = \frac{1}{2} u^2.
\]

By using this as a base step, we can prove

**Proposition 7.1.** If \( u \in \mathbb{R} \), then \( e_{ev} \in i\mathbb{R}, e_{od} \in \mathbb{R} \) and \( h_{od} \equiv 0 \). Here the subscripts \( ev \) or \( od \) respectively stand for any subscript that is even or odd.

**Proof.** The proposition is proved by induction with the base step given by (7.10). Now assume that the proposition has been established for \( e_{2n-2} \) and \( h_{2n-2} \) which are thus, respectively, pure imaginary and real. Because, by (7.5b) \( e_{2n-1} = i/2 \partial_x e_{2n-2} + uh_{2n-2} \) it follows that \( e_{2n-1} \) is real. From this it follows, using (7.5a), that \( h_{2n-1} \) vanishes identically. Using (7.5b) again, we have \( e_{2n} = i/2 \partial_x e_{2n-1} \in i\mathbb{R} \) which completes the induction.

Now let \( Q^{(ev)} \) and \( Q^{(od)} \), denote any linear combination of loop coefficients of the form \( Q_{2j} \) and \( Q_{2j-1} \) respectively. (This includes, as a particular instance, loop algebra elements which are even in \( \zeta \) and odd in \( \zeta \) respectively.) Then based on the preceding proposition one has the following corollary.

**Corollary 7.2.** If \( u \in \mathbb{R} \), then \( Q^{(ev)} \in \mathbb{R} \cdot \mathcal{H} \bigoplus \mathbb{R} \cdot \mathcal{E} \) and thus in particular has pure imaginary components. \( Q^{(od)} \in \mathbb{R} \cdot \mathcal{F} \) and therefore has real components. Moreover, because \( \mathcal{H}, \mathcal{E}, \mathcal{F} \) are an orthonormal basis of \( su(2) \) with respect to the inner product introduced earlier, one always has \( Q^{(ev)} \perp Q^{(od)} \) with respect to this inner product.
**Proof.** By the proposition, $h_{2j-1}$ is identically zero while $e_{2j-1}$ is real. Using this in (7.6) one immediately observes that $Q_{2j-1}$ is in the span of $\mathcal{F}$. Because the $h$-coefficients are always real, $h_{2j}\mathcal{H}$ is pure imaginary. Because, by the proposition, $e_{2j}$ is pure imaginary, it follows that $Q_{2j}$ is in the span of $\mathcal{H}$ and $\mathcal{E}$.

There are equations similar to (7.8) which give the evolution of the density generating function, $Q$, under the higher flows. These have the form [FNR]:

\begin{equation}
Q_{tn} = [Q_{2n-1}, Q].
\end{equation}

The loop $Q$ has a natural decomposition $Q = Q^{(ev)} + Q^{(od)}$ into, respectively, the even and odd terms of the series. For $\zeta$ real, these are also, respectively, the imaginary and real components of $Q$. Moreover, these two parts are orthogonal with respect to the inner product, $(\cdot, \cdot)$ on the loop algebra. This decomposition can be applied to (7.11) (for instance by equating imaginary and real parts on both sides of the equation). We apply this in the case of the odd flows; i.e., when $n = 2m - 1$ is odd. This yields two equations:

\begin{align}
Q_{(ev)}^{(2m-1)} &= [Q^{(ev)}, Q_{2m-1}], \\
Q_{(od)}^{(2m-1)} &= [Q^{(od)}, Q_{2m-1}] = 0.
\end{align}

The right-hand side of the last equation is identically zero because $[\mathcal{F}, \mathcal{F}] = 0$.

Thus the subspace of even densities is preserved under the odd flows. In particular

\begin{equation}
Q_2 = \begin{pmatrix}
\frac{i}{2}u^2 & \frac{i}{2}\epsilon\partial_x u \\
\frac{i}{2}\epsilon\partial_x u & -\frac{i}{2}u^2
\end{pmatrix},
\end{equation}

which carries the densities that comprise the Miura transformation (2.7), $v = u^2 + i\epsilon\partial_x u = 2(h_2 + e_2)$, preserves its form under the odd flows. In Section 2 we only considered the evolution of these densities under the $t_3$-flow, the mKdV flow.

Notice that for any of the elements in $Q^{(ev)}$, the coefficients of terms proportional to $\mathcal{E}$ are formally $O(\epsilon)$. Hence, in the dispersionless limit the odd NLS flows acting on real initial data collapse onto the Abelian subalgebra spanned by $\mathcal{H}$ which is comprised of the Hamiltonian densities introduced in Section 3. This extends to the higher odd flows our formal observations about the dispersionless limit of the mKdV flow which motivated the rigorous analysis of the zero-dispersion limit carried out in Sections 5 and 6.

8. Concluding Remarks
There is an alternative approach to showing that the limits of the focusing mKdV equation (1.6) are very close to those of the KdV equation (1.1).

One is the Miura transform [Mr]. The simple relation

\[-u = M(v) \equiv v^2 \pm \epsilon v_x\]

provides a link between the semiclassical limits of the KdV equation and of the defocusing mKdV equation:

\[(u_t + 6uu_x + \epsilon^2 u_{xxx}) + (2v \pm \epsilon \partial_x)(v_t - 6v^2v_x + \epsilon^2 v_{xxx}) = 0.\]

This shows that the limit of \(v^2\), the mass density, (1.7), of the defocusing mKdV equation (1.6), converges to \(\lim_{\epsilon \to 0} u\), the zero dispersion limit of the solution to the KdV equation (1.1), in the sense of distributions. This relation of semiclassical limits, between the defocusing NLS hierarchy and the KdV hierarchy, works at least in the case when their initial values depend only on the continuous spectrum of the associated ZS Dirac operator and the KdV Schrödinger operator \(\mathcal{L}_S\) respectively [JLM2].

A similar transform,

\[(8.1)\]

\[u = -M(iv) = v^2 \mp i\epsilon v_x\]

gives a relation of solutions between a real valued focusing mKdV equation and a complex valued KdV equation:

\[u_t + 6uu_x + \epsilon^2 u_{xxx} = (2v \mp i\epsilon \partial_x)(v_t + 6v^2v_x + \epsilon^2 v_{xxx}).\]

It suggests that the limit of \(v^2\), the mass density of the focusing mKdV equation ,

\[v_t + 6v^2v_x + \epsilon^2 v_{xxx} = 0,\]

should converge to the zero dispersion limit of the KdV equation (1.1) as \(\epsilon \to 0\) in the distributional sense as we have in fact shown, for a reasonable class of initial data, in this paper.

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References.

[ELZ] N.M. Ercolani, C.D. Levermore, and T. Zhang, The Behavior of the Weyl Function in the Zero Dispersion KdV Limit, Commun. Math. Phys. 183 (1997), 119–143.

[FFM] H. Flaschka, M.G. Forest, and D.W. McLaughlin, Multiphase Averaging and the Inverse Spectral Solutions of the Korteweg-de Vries Equation, Comm. Pure Appl. Math. 33 (1980), 739–784.

[FNR] H. Flaschka, A.C. Newell, and T. Ratiu, Kac-Moody Lie Algebras and Soliton Equations II: Lax Equations Associated with $A_1^{(1)}$, Physica D 9 (1983), 300–323.

[FL] M.G. Forest and J.-E. Lee, Geometry and Modulation Theory for the Periodic Nonlinear Schrödinger Equation, in Oscillation Theory, Computation, and Methods of Compensated Compactness, C. Defermos, J.L. Erickson, D. Kinderleher, and M. Slemrod, eds., IMA Volumes on Mathematics and Its Applications 3, Springer-Verlag, New York, 1986, 35–69.

[GGKM] C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura, Method for Solving the Korteweg-deVries Equation, Phys. Rev. Lett. 19 (1967), 1095–1097.

[JLM1] S. Jin, C.D. Levermore, and D.W. McLaughlin, The Behavior of Solutions of the NLS Equation in the Semiclassical Limit, in Singular Limits of Dispersive Waves, N.M. Ercolani, I.R. Gabitov, C.D. Levermore, and D. Serre eds., NATO ASI Series B 320, Plenum, New York, 1994, 235–255.

[JLM2] S. Jin, C.D. Levermore, and D.W. McLaughlin, The Semiclassical Limit for the Defocusing Nonlinear Schrödinger Hierarchy, Commun. Pure & Appl. Math. 52 (1999), 613–654.

[KMM] S. Kamvissis, K. T.-R. McLaughlin, and P.D. Miller, Analysis of an Unstable Semiclassical Soliton Ensemble for the Focusing Nonlinear Schrödinger Equation, Analys of Mathematics Studies, Princeton University Press (to appear).

[KS] M. Klaus and J.K. Shaw, Purely Imaginary Eigenvalues of Zakharov-Shabat Systems, Phys. Rev. E 65 (2002).

[LL1] P.D. Lax and C.D. Levermore, The Zero Dispersion Limit of the Korteweg-deVries Equation, Proc. Nat. Acad. Sci. USA 76 (1979), 3602–3606.

[LL2] P.D. Lax and C.D. Levermore, The Small Dispersion Limit of the Korteweg-deVries Equation I, II, III, Commun. Pure & Appl. Math. 36 (1983), 253–290, 571–593, 809–829.

[LLV] P.D. Lax, C.D. Levermore, and S. Venakides, The Generation and Propagation of
Oscillations in Dispersive IVPs and Their Limiting Behavior, in Important Developments in Soliton Theory, T. Fokas and V.E. Zakharov eds., Springer-Verlag, Berlin, 1993, 205–241.

[Lev] C.D. Levermore, The KdV Zero-Dispersion Limit and Densities of Dirichlet Spectra, in Recent Advances in Partial Differential Equations and Applications (a conference celebrating the 70th birthdays of Peter D. Lax and Louis Nirenberg, Venice, Italy, 10–14 June 1996), Proceedings of Symposia in Applied Mathematics 54, R. Spigler and S. Venakides eds., American Mathematical Society, Providence, 1998, 187–210.

[Mr] R.M. Miura, Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation, J. Mathematical Phys. 9 (1968), 1202–1204.

[V1] S. Venakides, The Generation of Modulated Wavetrains in the Solution of the Korteweg-de Vries Equation, Commun. Pure & Appl. Math. 38 (1985), 883–909.

[V2] S. Venakides, The Zero Dispersion Limit of the Korteweg-de Vries Equation with Periodic Initial Data, AMS Trans. 301 (1987), 189–225.

[V3] S. Venakides, Higher Order Lax-Levermore Theory, Commun. Pure & Appl. Math. 43 (1990), 335–362.

[W] G.B. Whitham, Nonlinear Dispersive Waves, Proc. Royal Soc. London Ser. A 283 (1965), 238–261.

[ZS1] V.E. Zakharov and A.B. Shabat, Exact Theory of Two-dimensional Self-focusing and One-dimensional Self-modulation of Waves in Nonlinear Media, Sov. Phys. JETP 34 (1972), 62–69.

[ZS2] V.E. Zakharov and A.B. Shabat, Interaction Between Solitons in a Stable Medium, Sov. Phys. JETP 37 (1973), 823–828.
Fig 2.1 Time-slices, $u(x, t)|_{t=0.5,1.0,2.0}$, and contour plot (lower right) $u(x, t) = \text{const.}$, $0 < x < 2\pi, 0 < t < 4$. $u(x, 0) = \cos^2 x$. The horizontal axis is $x$, and the vertical axis is either $u$ (slice), or $t$ (contour plot).
Fig 4.1 The initial data $A(x)$. Note its critical value $\eta_{\text{max}}$ and the indicated defining relations for $x^{\pm} = x^{\pm}(\eta)$. 
Fig 2.2. Contour plot, $u_\varepsilon(x,t) = \text{const.}$, $0 < t < 1.5$ (left). Slice, $u_\varepsilon(x,t)|_{t=1.5}$, (right). Here, $\varepsilon = 0.08$ (top), 0.04 (center), and 0.02 (bottom), and $u_\varepsilon(x,0) = \cos^2 x$. The horizontal axis is $x$, and the vertical axis is either $t$ (contour plot) or $u$ (slice). The graphs have the same scales for comparison.
Fig 2.3 Primitives of the first (left) and third (right) conserved densities. The cases for for $\epsilon = 0.08$ and $\epsilon = 0.02$, are overlaid, with the wider oscillations corresponding to large $\epsilon$. Here, the pseudo-inverse $\partial_z^{-1}$ is a periodic normalization of $\int^x$: each discrete Fourier mode of the density is multiplied by $\frac{1}{ik}$, except for mode 0, which is left unchanged. As $\epsilon$ decreases, both the wavelength and the magnitude of the deviating oscillations decrease. This suggests that the limit for the primitives will be classical.