Mean first-passage time for random walks on the T-graph

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Abstract. For random walks on networks (graphs), it is a theoretical challenge to explicitly determine the mean first-passage time (MFPT) between two nodes averaged over all pairs. In this paper, we study the MFPT of random walks using the famous T-graph, linking this important quantity to the resistance distance in electronic networks. We obtain an exact formula for the MFPT that is confirmed by extensive numerical calculations. This interesting quantity is derived through the recurrence relations resulting from the self-similar structure of the T-graph. The obtained closed-form expression shows that the MFPT increases approximately as a power-law function of the number of nodes, with the exponent lying between 1 and 2. Our research may further a deeper understanding of random walks on the T-graph.
1. Introduction

As the paradigmatic discrete-time realization of Brownian motion and diffusive processes, the theory of which was formulated first by Einstein [1] and Smoluchowski [2], random walks have received considerable attention [3]–[5], uncovering a wide range of distinct applications. Thus far, random walks remains an active area of research [6]–[10]. However, due to the complexity and variety of real media, the theory of random walks on general graphs (networks) is not yet available. For this reason, studying random walks occurring on simple structures is a matter of exceptional importance. Fractal structures, in particular deterministic fractals, are valuable media in this content [7, 8], because their properties can be exactly studied.

Among various deterministic fractals [11]–[17], the T-fractal (T-graph) [18] is a typical candidate for an exactly solvable model, and many issues for random walks on this fractal have been studied [19]–[27]. A simple analytical approach was proposed in [18], showing that random walks on the T-graph can be mapped onto diffusion on a comb structure. The asymptotic behavior of the moments of the first-passage time (FPT), and survival probability for random walks on the T-graph, was computed in [19]. Particularly, random walks performed on the T-graph with a single trap were extensively studied by many groups [19, 24, 26, 27], revealing some scaling relations and dominating behavior. Despite the fact that these investigations uncovered many unusual and exotic features of the T-graph, providing useful insights into random walks on this fractal, they did not give a complete picture of random-walk dynamics on the T-fractal because only one special trap was considered. It was shown that in some real networks any node may be seen as a trap, and the location of traps strongly affects the behavior of random walks [28]. Irrespective of its obvious importance and ubiquity, this issue has not been addressed for the T-graph.

In this paper, using the connection between random walks and electrical networks [29], we investigate random walks on the T-graph. The random walk process addressed here may be considered as the trapping problem with the perfect trap uniformly located at all nodes. We derive analytically an exact formula for the mean first-passage time (MFPT) averaged over all pairs of nodes, which describes the efficiency of random walks on the T-graph. We show that the location of traps has no qualitative effect on scaling for the MFPT. We expect that our analytical method can be applied to other deterministic media, and that our results can lead to deeper insights into random walks on the T-graph.

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Figure 1. The iterative construction method for the T-graph. The next generation is obtained by performing the operation shown to the right of the arrow.

Figure 2. The T-graph for generation 4.

2. Brief introduction to the T-graph

The T-graph is built in an iterative way [18, 27]. We denote by $T_g$ ($g \geq 0$) the T-graph after $g$ iterations. Note that henceforth we will also call the number of iterations the generation of the T-graph. The construction of the T-graph starts from ($g = 0$) two nodes connected by an edge, which corresponds to $T_0$. For $g \geq 1$, $T_g$ is obtained from $T_{g-1}$ by performing the operation illustrated in figure 1. According to the construction algorithm, at each generation the number of edges in the system increases by a factor of 3. Thus, we can easily deduce that at generation $g$, the total number of edges in $T_g$ is $E_g = 3^g$. Since the T-graph is a tree, the total number of nodes in $T_g$ is $N_g = E_g + 1 = 3^g + 1$. Figure 2 shows schematically the structure of $T_4$.

The T-graph is a fractal with the fractal dimension and the random-walk dimension being $d_f = \frac{\ln 3}{\ln 2} \approx 1.585$ and $d_w = \frac{\ln 6}{\ln 2} = 1 + d_f$, respectively. The spectral dimension of the T-graph is $\tilde{d} = 2d_f/d_w = \frac{\ln 6}{\ln 9} \approx 1.226$. Note that for a general connected graph, the spectral dimension fundamentally influences the longtime behavior of a random walk on the graph. For example,
when a walker originates at a given node $i$ of the graph, the probability $P_{ii}(t)$ for returning back to $i$, at long time $t$, obeys the relation $P_{ii}(t) \sim t^{-\tilde{d}/2}$ [30]. In another instance, for an infinite graph with the spectral dimension $\tilde{d} \leq 2$, a walker starting from a given node will return to the node almost surely over the course of time. The random walk phenomenon on this graph is termed ‘persistence’. Because the spectral dimension $\tilde{d} < 2$ for the T-graph, a random walk on it is persistent.

To help describe what follows, we define the central node in figure 2 as the innermost node, and we describe those nodes farthest from the central node as the outermost nodes. The T-graph can also be constructed using another method (see figure 3). Given the generation $g$, $T_{g+1}$ may be obtained by joining three copies of $T_g$, denoted as $T_g^{(1)}$, $T_g^{(2)}$ and $T_g^{(3)}$, respectively. In other words, to obtain $T_{g+1}$ one can merge together the discrete outermost nodes of the three replicas of $T_g$. The three outermost nodes merge into a single new node, which is then the innermost node of $T_{g+1}$.

3. Formulation of the problem

In this section, we investigate a minimal model for the discrete-time random walks of a particle on the T-graph $T_g$. At each time step, the walker can move from its current location to any of its nearest neighbors with equal probability. A key quantity characterizing such a random walk is the FPT, in terms of which many other quantities can be expressed. We are interested in the MFPT between two distinct nodes, averaged over all pairs.

It is well known that the random walks addressed here can be described using Markov chains [31], where the fundamental matrices can be used to express the FPT between any pair of nodes. However, the fundamental-matrix method for calculating the MFPT in $T_g$ requires calculating the inversion of $N_g$ matrices of size $(N_g - 1) \times (N_g - 1)$, making it prohibitively difficult to calculate the quantity concerned for all but small networks.
To reduce the high computational demands of the fundamental matrix method, one can apply the method of the pseudoinverse of the Laplacian matrix [32] for the graph $T_g$ that random walks are performed on, which allows us to compute the FPT between two arbitrary nodes directly from the network topology, and this only requires inversion of a single $N_g \times N_g$ matrix. The elements $l_{ij}^g$ of the Laplacian matrix $L_g$ for graph $T_g$ are defined as follows: $l_{ij}^g = -1$, if the pair of two different nodes $i$ and $j$ is connected by a link, otherwise $l_{ij}^g = 0$; while $l_{ii}^g = k_i$ (i.e. degree of node $i$). Then, the pseudoinverse of the Laplacian matrix $L_g$ is [33]

$$L_g^\dagger = \left( L_g - \frac{e_g e_g^\top}{N_g} \right)^{-1} + \frac{e_g e_g^\top}{N_g},$$

(1)

where $e_g$ is the $N_g$-dimensional ‘one’ vector, i.e. $e_g = (1, 1, \ldots, 1)^\top$.

Let $F_{ij}(g)$ denote the FPT for a walker in the $T$-graph $T_g$, starting from node $i$, to arrive at node $j$ for the first time. Since the graph $T_g$ is connected, the pseudoinverse matrix $L_g^\dagger$ is well defined and the entries $l_{ij}^g$ can be applied to express the FPT $F_{ij}(g)$ as follows [34]:

$$F_{ij}(g) = \sum_{m=1}^{N_g} \left( l_{im}^{g^\top} - l_{ij}^{g^\top} - l_{jm}^{g^\top} + l_{jj}^{g^\top} \right) l_{mm}^{g},$$

(2)

where $l_{mm}^{g}$ is the $m$ entry of the diagonal of the Laplacian matrix $L_g$, as defined above. Then the sum $F_{\text{tot}}(g)$ for the FPT between the two nodes over all node pairs in graph $T_g$ reads

$$F_{\text{tot}}(g) = \sum_{i\neq j}^{N_g} \sum_{j=1}^{N_g} F_{ij}(g),$$

(3)

and the MFPT, $\langle F \rangle_g$, is

$$\langle F \rangle_g = \frac{F_{\text{tot}}(g)}{N_g(N_g - 1)^{-1}} = \frac{1}{N_g(N_g - 1)} \sum_{i\neq j}^{N_g} \sum_{j=1}^{N_g} F_{ij}(g).$$

(4)

The quantity of MFPT $\langle F \rangle_g$ is very important, because it measures the efficiency of the random walks on $T_g$: the smaller the value $\langle F \rangle_g$, the higher the efficiency, and vice versa. Equations (2) and (4) show that the problem of calculating $\langle F \rangle_g$ is reduced to determining the elements of the pseudoinverse matrix $L_g^\dagger$, whose complexity involves inverting an $N_g \times N_g$ matrix, and can be easily obtained by utilizing a standard software package, Mathematica 5.0. However, since $N_g$ increases exponentially with $g$, for large $g$ it becomes difficult to obtain $\langle F \rangle_g$ through direct calculation using the pseudoinverse matrix because of the demands on time and computer memory. Therefore, one can compute the MFPT directly only for the first generations (see figure 4). Fortunately, the construction method for the T-graph and the connection [35, 36] between effective resistance and FPT allow us to calculate the MFPT analytically to obtain an explicit formula. Details will be given below.

4. Rigorous solution to the MFPT

In the following text, we use the connection between electrical networks and random walks to derive the closed-form expression for MFPT $\langle F \rangle_g$, avoiding the computation process for inverting a matrix.

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Figure 4. MFPT $\langle F \rangle_g$ as a function of the iteration $g$ on a semilogarithmic scale. The red squares are the numerical results obtained by direct calculation from equations (2) and (4), while the full blue squares represent the exact values from equation (23), both of which agree with each other.

4.1. Relation for commute time and resistance distance between two nodes

Given a graph, the underlying electrical network [29] can be obtained by replacing each edge of the original graph with a unit resistor. The effective resistance between any node pair $i$ and $j$ is defined as the voltage when a unit current enters one node and leaves the other. It has been proved that the effective resistance is a distance measure, and so the quantity is also called resistance distance [37]. Thus, we can view $T_g$ as a resistor network by considering all edges of $T_g$ to be unit resistors. Previous work [35, 36] has shown that many problems on a resistor network are closely related to the classic random walks on the original graph. For example, the computation of effective resistance between two nodes in a resistor network can be expressed by the FPT between the two nodes on the corresponding network: for an arbitrary connected graph, the effective resistance $R_{ij}$ between a pair of nodes $i$ and $j$ is equal to $(F_{ij} + F_{ji})/(2E)$, where $E$ is the total number of edges in the graph, and $F_{ij}$ is the expected time for a walker starting at node $i$ to first reach $j$. In fact, $F_{ij} + F_{ji}$ is the average time that a walker originating at node $i$ will take to hit node $j$ and return to $i$, and it is often called the commute time [38] between $i$ and $j$, denoted as $C_{ij}$, i.e. $C_{ij} = F_{ij} + F_{ji}$.

According to the close relation between commute time and effective resistance, we deduce that for $T_g$ the effective resistance $R_{ij}(g)$ between a pair of nodes $i$ and $j$ is $R_{ij}(g) = C_{ij}(g)/(2E_g)$, where $C_{ij}(g) = F_{ij}(g) + F_{ji}(g)$ is the commute time between $i$ and $j$. Thus $C_{ij}(g) = C_{ji}(g) = 2E_g R_{ij}(g)$. Then, equation (3) can be rewritten as

$$F_{tot}(g) = \frac{1}{2} \sum_{i\neq j=1}^{N_g} C_{ij}(g) = E_g \sum_{i\neq j=1}^{N_g} R_{ij}(g),$$

(5)

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and equation (4) can be recast as
\[
(F)_g = \frac{F_{\text{tot}}(g)}{N_g(N_g - 1)} = \frac{1}{N_g} \sum_{i \neq j} R_{ij}(g).
\] (6)

Equation (6) shows that if we know how to determine the effective resistances, then we have a method to find the MFPT.

For a general graph with the order \(N\), the complexity of resistance computation is that of inverting an \(N \times N\) matrix, identical to that of calculating the MFPT. However, for a treelike graph, the effective resistance between two nodes is exactly the usual shortest-path distance (also called geodesic distance) between the corresponding nodes. Since the T-graph has a treelike structure and is also self-similar, we can use these properties to find its geodesic distance, obtaining a rigorous expression.

### 4.2. Exact expression for the MFPT

After reducing the determination of the MFPT to finding the average geodesic distance, the next step is to explicitly determine the latter quantity. To this end, we represent the shortest-path distance of \(T_g\) as a matrix \(\Omega\), where the entry \(d_{ij}(g)\) is the shortest distance from node \(i\) to node \(j\), which is the minimum length for the path connecting the two nodes. The maximum value \(D_g\) of \(d_{ij}(g)\) is called the diameter of \(T_g\). Then, the average geodesic distance of \(T_g\) is defined as the mean of shortest-path distances over all pairs of nodes:

\[
\langle L \rangle_g = \frac{S_g}{N_g(N_g - 1)/2}.
\] (7)

where
\[
S_g = \sum_{i \in T_g, j \in T_g, i \neq j} d_{ij}(g)
\] (8)
denotes the sum of the geodesic distances between two nodes over all pairs. Note that in equation (8) for the two nodes \(i\) and \(j\) (\(i \neq j\)) we only count \(d_{ij}(g)\) or \(d_{ji}(g)\), not both.

We continue by showing the procedure for determination of the total shortest-path distance and present the recurrence formula, which allows us to obtain \(S_{g+1}\) of the \(g+1\) generation from \(S_g\) of the \(g\) generation. According to the second construction method, we can easily see that the T-graph is self-similar. This obvious self-similar structure allows us to calculate \(S_g\) analytically. It is not difficult to establish that the total shortest-path distance \(S_{g+1}\) satisfies the recursion relation

\[
S_{g+1} = 3 S_g + \Omega_g,
\] (9)

where \(\Omega_g\) is the sum over all the shortest-path distances whose end-points are not in the same \(T_g\) branch. The solution of equation (9) is

\[
S_g = 3^g S_0 + \sum_{m=0}^{g-1} [3^{g-m-1} \Omega_m].
\] (10)

Thus, all that is left to obtain \(S_g\) is to compute \(\Omega_m\).

The paths that contribute to \(\Omega_g\) must all go through the innermost node (e.g., \(X\) in figure 3) of \(T_{g+1}\). To find \(\Omega_g\), we denote \(\Omega_g^{x,\beta}\) as the sum of all the shortest paths with end-points in \(T_g^{(x)}\)
and $\mathbb{T}^{(\beta)}$, respectively. Note that $\Omega_{g}^{\alpha,\beta}$ rules out the paths where the end-point is the node shared by $\mathbb{T}^{(\alpha)}$ and $\mathbb{T}^{(\beta)}$. By symmetry, $\Omega_{g}^{1,2} = \Omega_{g}^{1,3} = \Omega_{g}^{2,3}$. Then, the total sum $\Omega_{g}$ is given by

$$
\Omega_{g} = \Omega_{g}^{1,2} + \Omega_{g}^{1,3} + \Omega_{g}^{2,3} = 3 \Omega_{g}^{1,2}.
$$

(11)

In order to calculate the path length $\Omega_{g}^{1,2}$, we give the following notation. Let $s_{g}$ be the geodesic distance of an outermost node of $\mathbb{T}_{g}$ [40], which is defined as the sum of geodesic distances between the outermost node and all nodes of $\mathbb{T}_{g}$, including the outermost node itself. Prior to determining $s_{g}$, we compute the diameter $D_{g}$ of $\mathbb{T}_{g}$. In fact, the diameter $D_{g}$ is equal to the path length between an arbitrary pair of the outermost nodes previously belonging to two different $D_{g-1}$ branches. Obviously, the following recursive relation holds:

$$
D_{g+1} = 2 \cdot D_{g}.
$$

(12)

Considering the initial condition $D_{0} = 1$, equation (12) is solved inductively to obtain

$$
D_{g} = 2^{g}.
$$

(13)

We now calculate the quantity $s_{g+1}$. Let $K$ denote an outermost node of $\mathbb{T}_{g+1}$, which is in the branch $\mathbb{T}_{g}^{(1)}$ (see figure 3). By definition, $s_{g+1}$ can be given by the sum

$$
s_{g+1} = \sum_{j \in \mathbb{T}_{g+1}} d_{Kj}(g + 1)
= \sum_{j \in \mathbb{T}_{g}^{(1)}} d_{Kj}(g + 1) + \sum_{j \in \mathbb{T}_{g}^{(2)}, j \neq X} d_{Kj}(g + 1) + \sum_{j \in \mathbb{T}_{g}^{(3)}, j \neq X} d_{Kj}(g + 1)
= s_{g} + 2 \sum_{j \in \mathbb{T}_{g}^{(2)}, j \neq X} d_{Kj}(g + 1),
$$

(14)

where we have made use of $\sum_{j \in \mathbb{T}_{g}^{(2)}, j \neq X} d_{Kj}(g + 1) = \sum_{j \in \mathbb{T}_{g}^{(3)}, j \neq X} d_{Kj}(g + 1)$, which is obvious from the symmetry. We denote the second term in equation (14) by $2h_{g}$. Then, $s_{g+1} = s_{g} + 2h_{g}$.

The quantity $h_{g}$ is evaluated as follows:

$$
h_{g} = \sum_{j \in \mathbb{T}_{g}^{(2)}, j \neq X} [d_{KX}(g + 1) + d_{Xj}(g + 1)]
= s_{g} + (N_{g} - 1) D_{g},
$$

(15)

where $d_{KX}(g + 1) = D_{g}$ was used. Hence, equation (14) becomes

$$
s_{g+1} = 3s_{g} + 2(N_{g} - 1) D_{g}.
$$

(16)

Using $N_{g} = 3^{g} + 1$, $D_{g} = 2^{g}$ and $s_{0} = 1$, equation (16) is resolved by induction

$$
s_{g} = 3^{g-1}(2^{g+1} + 1).
$$

(17)

With the above results, we can determine $\Omega_{g}^{1,2}$, which can then be expressed in terms of the previously explicitly determined quantities. By definition, $\Omega_{g}^{1,2}$ is given by the sum

$$
\Omega_{g}^{1,2} = \sum_{i \in \mathbb{T}_{g}^{(1)}, i \neq X} \sum_{j \in \mathbb{T}_{g}^{(2)}, j \neq X} d_{ij}(g + 1)
$$

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expression for the total distance

$$\langle L \rangle = \sum_{i \in T_g^{(1)}, i \neq \chi} d_{i\chi}(g+1) + \sum_{j \in T_g^{(2)}, j \neq \chi} d_{\chi j}(g+1)$$

$$= (N_g - 1) \sum_{i \in T_g^{(1)}, i \neq \chi} d_{i\chi}(g+1) + (N_g - 1) \sum_{j \in T_g^{(2)}, j \neq \chi} d_{\chi j}(g+1)$$

$$= 2 (N_g - 1) s_g$$

$$= 2 \times 3^{2g-1}(2^{g+1} + 1), \quad (18)$$

where we have used the equivalence relation \( \sum_{i \in T_g^{(1)}, i \neq \chi} d_{i\chi}(g+1) = \sum_{j \in T_g^{(2)}, j \neq \chi} d_{\chi j}(g+1) \).

Inserting equation (18) into (11), we have

$$\Omega_g = 2 \times 3^{2g}(2^{g+1} + 1). \quad (19)$$

Substituting equation (19) into (10) and using the initial value \( S_0 = 1 \), we can obtain the exact expression for the total distance

$$S_g = \frac{1}{5} \times 3^{g-1}(4 \times 6^g + 5 \times 3^g + 6). \quad (20)$$

Then the analytic expression for mean geodesic distance can be obtained as

$$\langle L \rangle_g = \frac{8 \times 6^g + 10 \times 3^g + 12}{15 \times (3^g + 1)}. \quad (21)$$

Since for \( T_g \), the shortest-path distance is equivalent to the resistance distance, i.e. \( d_{ij}(g) = R_{ij}(g) \), according to the above obtained results shown in equations (20) and (21), we can easily establish that the totals for FPT between all \( N_g(N_g - 1) \) pairs of nodes and the MFPT are

$$F_{tot}(g) = 2E_g S_g = \frac{2}{15} \times 9^g(4 \times 6^g + 5 \times 3^g + 6) \quad (22)$$

and

$$\langle F \rangle_g = \frac{2S_g}{N_g} = E_g \langle L \rangle_g = \frac{3g}{15(3^g + 1)}(8 \times 6^g + 10 \times 3^g + 12), \quad (23)$$

respectively.

We have checked our analytical formula against numerical values (see figure 4). For the range of \( 1 \leq g \leq 8 \), the values obtained from equation (23) are in complete agreement with the results derived through the pseudoinverse matrix method discussed in section 3. This agreement serves as an independent test of our theoretical formula.

Note that the result provided in equation (23) is consistent with the recent proposal by Condamin et al [41] for a very general scaling form for the FPT, \( \langle F \rangle \), of a random walk as a function of the distance, \( L \), between the origin and the trap location. Condamin et al proved that, for complex invariant networks with \( d_w > d_t \), the leading behavior of the FPT behaves as \( \langle F \rangle \sim NL^{d_w-d_t} \). For the T-graph \( d_w - d_t = 1 \), according to the conclusion of Condamin et al, one has \( \langle F \rangle_g \sim N_g \langle L \rangle_g \) as found in equation (23).

Next, we will express the MFPT \( \langle F \rangle_g \) as a function of the network order \( N_g \) in order to obtain the scaling between these two quantities. From the relation \( N_g = 3^g + 1 \) and \( g = \log_3(N_g - 1) \). From this, we can rewrite equation (23) as

$$\langle F \rangle_g = \frac{2(N_g - 1)N_g}{15} \left( 4(N_g - 1)^{1+\log_3} + 5(N_g - 1) + 6 \right), \quad (24)$$
which proves the explicit dependence of $\langle F \rangle_g$ on network order $N_g$. In the infinite network order, namely $N_g \to \infty$,

$$
\langle F \rangle_g \approx \frac{8}{15} (N_g)^{1+\log_2 3} = \frac{8}{15} (N_g)^{2/\bar{d}}.
$$

(25)

Thus, for the large T-graph, the MFPT grows as a power-law function of the network order with the exponent larger than 1 and less than 2, implying that the MFPT increases superlinearly with the number of network nodes.

Equation (25) encodes the speed of the random walks on $T_g$, which may be quantified by the coverage $C_g(t)$, standing for the mean number of different nodes visited by a walker at time $t$, averaged for distinct walks initially starting from different sources [42, 43]. As $t \to \infty$, the leading asymptotic behavior for $C_g(t)$ is

$$
C_g(t) \sim t^{3/2},
$$

(26)

which may be established from the following heuristic argument [44, 45]. Since, irrespective of the starting location, the random walker will visit all nodes of $T_g$ with probability one in the longtime limit, $N_g$ plays the role of $C_g(t)$, and $\langle F \rangle_g$ is akin to $t$ (the step number of the walker). Then, inverting the relation in equation (26) leads to asymptotic scaling $\langle F \rangle_g \sim (N_g)^{2/\bar{d}}$, as found above and given by equation (25), which also provides the exact coefficient of proportionality.

Note that in [27] Agliari studied the trapping problem with an immobile trap located at the innermost (central) node, and showed that in the asymptotic limit, the average trapping time (ATT) to first reach the trap, averaged over all nodes except the absorbing node itself, exhibited a similar behavior (the same exponent) to that of equation (25), but with a different prefactor. Also note that, due to this symmetry, the ATT in this case corresponds to the mean time for random walks with a fixed trap situated at an outmost node in the T-graph of generation $g-1$ [27]. That is to say, random walks in these two extreme cases have an almost identical ATT. On the other hand, the random walks discussed here may be considered as trapping issues with the trap uniformly distributed throughout all nodes on the T-graph. Thus, we can conclude that the trap’s location has no qualitative impact on the leading behavior of the ATT for random walks on the T-graph, which is in sharp contrast to trapping problem on some scale-free networks [28], where the ATT depends on trap position: it is much smaller in cases where traps are located at a large-degree node than where traps have been uniformly placed.

5. Conclusions

In this paper, we have studied standard discrete-time random walks on the T-graph. By making use of the link between electric networks and random walks, we have studied analytically the MFPT averaged over all pairs of nodes, based on the recursive relations derived from the self-similar structure of the T-graph. We have determined a rigorous solution for the MFPT, which in the large limit graph order increases algebraically with the number of nodes. We showed that trap location has little influence on the scaling of the MFPT for random walks on the T-graph. We expect that by providing a paradigm for computing the MFPT, our analytical technique could guide and advance related studies of random walks using other deterministic media, especially some scale-free fractals [46, 47] that have received much recent attention [48, 49]. We also expect that our work, especially the exact solution, could prompt study of the diffusion process.
on random fractals, other stochastic graphs, and even complex networks [50], by giving a guide to and a test of approximate methods for random media.

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