Double series expression for the Stieltjes constants

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Abstract

We present expressions in terms of a double infinite series for the Stieltjes constants $\gamma_k(a)$. These constants appear in the regular part of the Laurent expansion for the Hurwitz zeta function. We show that the case $\gamma_k(1) = \gamma$ corresponds to a series representation for the Riemann zeta function given much earlier by Brun. As a byproduct, we obtain a parameterized double series representation of the Hurwitz zeta function.

Key words and phrases

Hurwitz zeta function, Stieltjes constants, series representation, Riemann zeta function, Laurent expansion, digamma function

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Introduction and statement of results

Herein we supplement our very recent work [5] showing how a method of Addison [2] for series representations of the Euler constant $\gamma$ may be generalized in many different ways. We focus on the Stieltjes constants $\gamma_k(a)$ and recall the defining Laurent expansion [6, 7, 8, 14, 16]

$$\zeta(s, a) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a)(s - 1)^n, \quad s \neq 1. \quad (1.1)$$

where $\zeta(s, a)$ is the Hurwitz zeta function. For $\text{Re } s > 1$ and $\text{Re } a > 0$ we have

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \quad (1.2)$$

and by analytic continuation $\zeta(s, a)$ extends to a meromorphic function throughout the whole complex plane. Notationally, we let $\zeta(s) = \zeta(s, 1)$ be the Riemann zeta function [9, 11, 13, 15] and $\psi = \Gamma'/\Gamma$ be the digamma function (e.g., [1, 3, 10]) with the Euler constant $\gamma = \gamma_0(1) = -\psi(1)$. In addition, $P_1(x) = B_1(x-\lfloor x \rfloor) = x-\lfloor x \rfloor - 1/2$ denotes the first periodized Bernoulli polynomial, with $\{x\} = x - \lfloor x \rfloor$ the fractional part of $x$.

The sequence $\{\gamma_k(a)\}_{k=0}^{\infty}$ exhibits complicated changes in sign with $k$. For instance, for both even and odd index, there are infinitely many positive and negative values. Furthermore, there is sign variation with the parameter $a$. These features, as well as the exponential growth in magnitude in $k$, are now well captured in an asymptotic expression ([12], Section 2). In fact, though initially derived for large values of $k$, this expression is useful for computational approximation even for small values of $k$. 

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In this paper, we give exact expressions for the Stieltjes constants as double infinite series, and one of the series has an exponentially fast rate of convergence. In a particular case for \( a = 1 \), we are able to recover a result that corresponds to an earlier series representation for the Riemann zeta function given by Brun [4]. Brun performed ‘horizontal’ and ‘vertical’ summations, and so our approach is very different.

**Proposition 1.** Let \( \text{Re} \ a > 0 \). Then we have

\[
\gamma_0(a) = -\psi(a) = -\ln a + \frac{1}{a} + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^j}{j + a2^n},
\]

and for \( \ell \geq 1 \)

\[
\gamma_{\ell}(a) = -\ln^{\ell+1} a \frac{1}{\ell + 1} + \frac{1}{a} \ln^{\ell} a + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^j}{j + a2^n} \ln^{\ell} \left( a + \frac{j}{2^n} \right).
\]

As a byproduct of our proof of Proposition 1, we obtain the following double series representation. We have

**Corollary 1.** For \( \text{Re} \ s > 1 \) and \( \text{Re} \ a > 0 \), we have

\[
\zeta(s, a) = a^{-s} + \frac{a^{1-s}}{s-1} + \sum_{n=1}^{\infty} 2^{n(s-1)} \sum_{j=1}^{\infty} \frac{(-1)^j}{(j + a2^n)^s}.
\]

Brun [4] realized the following representation of the Riemann zeta function for \( \text{Re} \ s > 1 \):

\[
\zeta(s) = \frac{1}{s-1} + 1 - \beta(s),
\]

where

\[
\beta(s) = \sum_{n,j=1}^{\infty} \frac{(-1)^j 2^{n(s-1)}}{(2^n + j)^s}.
\]

Therefore, we recover his result as the special case at \( a = 1 \).
Corollary 2. We have

\[ \gamma = \frac{1}{2} \sum_{n=0}^{\infty} \left[ \psi \left( \frac{2^{n+1} + 1}{2} \right) - \psi(2^n) \right] \]

\[ = \sum_{n=0}^{\infty} [\psi(2^{n+1}) - \psi(2^n) - \ln 2]. \quad (1.7) \]

Corollary 3. We have for \( \Re a > 0 \)

\[ \gamma_0(a) = - \ln a + \sum_{n=0}^{\infty} [\psi(a2^{n+1}) - \psi(a2^n) - \ln 2]. \quad (1.8) \]

This result agrees with (2.24) of [5].

Proposition 2. We have for \( \ell \geq 0, \Re a > 0, \) and integers \( k \geq 2 \)

\[ \gamma_{\ell}(a) = \frac{1}{2} \ln^\ell a - \frac{\ln^\ell a}{\ell + 1} - \sum_{n=0}^{\infty} \frac{1}{k^n} \sum_{j=0}^{\infty} \left\{ \frac{1}{2} \left( \frac{1}{k} - 1 \right) \left[ \frac{\ln^\ell (bj + a)}{(bj + a)} + \frac{\ln^\ell (b(j + 1) + a)}{(b(j + 1) + a)} \right] \right. \]

\[ + \frac{1}{k} \sum_{m=1}^{k-1} \ln^\ell \left[ b(j + m/k + a) \right] \left\}_{b=k-n} \]

\[ = - \frac{\ln^\ell a}{\ell + 1} + \frac{1}{k} \sum_{n=0}^{\infty} \sum_{m=1}^{k-1} \ln^\ell \left[ \frac{bm/k + a}{m/k + ak^n} \right] \left\}_{b=k-n} \]

\[ - \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \left\{ \left( \frac{1}{k} - 1 \right) \frac{\ln^\ell (bj + a)}{j + ak^n} + \frac{1}{k} \sum_{m=1}^{k-1} \ln^\ell \left[ \frac{b(j + m/k + a)}{j + m/k + ak^n} \right] \right\} \] \quad (1.9a)

\[ \left. \left\}_{b=k-n} \right. \]

\[ - \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \left\{ \left( \frac{1}{k} - 1 \right) \frac{\ln^\ell (bj + a)}{j + ak^n} + \frac{1}{k} \sum_{m=1}^{k-1} \ln^\ell \left[ \frac{b(j + m/k + a)}{j + m/k + ak^n} \right] \right\} \] \quad (1.9b)

Our results have application to Dirichlet \( L \) functions, as these may be written as a linear combination of Hurwitz zeta functions. For instance, for \( \chi \) a principal (nonprincipal) character modulo \( m \) and \( \Re s > 1 \) (\( \Re s > 0 \)) we have

\[ L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s} = \frac{1}{m^s} \sum_{k=1}^{m} \chi(k) \zeta \left( s, \frac{k}{m} \right). \quad (1.10) \]
Proof of Propositions

**Proposition 1.** We apply Lemma 1. We have for $\text{Re } s > 1$,

$$
\zeta(s,a) - \frac{a^{1-s}}{s-1} = a^{-s} \frac{1}{2} + \frac{1}{4} \sum_{n=0}^{\infty} 2^{-n} \sum_{j=0}^{\infty} \left[ \frac{1}{(2^{-n}j + a)^s} + \frac{1}{(2^{-n}j + 2^{-n} + a)^s} - \frac{2}{(2^{-n}(j + 1/2) + a)^s} \right]
$$

$$
= a^{-s} + \frac{1}{2} \sum_{n=0}^{\infty} 2^{n(s-1)} \sum_{j=0}^{\infty} \left[ \frac{1}{(j + 1 + a2^n)^s} - \frac{1}{(j + 1/2 + a2^n)^s} \right]. \tag{2.1}
$$

We start from the representation

$$
\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} - s \int_0^{\infty} \frac{P_1(x)}{(x + a)^{s+1}} dx, \quad \text{Re } s > 0, \tag{2.2}
$$

and employ the functions $f(x) = -P_1(x)$ and $g_2(x) = f(x) - f(2x)/2$, with $\sum_{n=0}^{\infty} g_2(2^n x)/2^n = f(x)$. We find

$$
\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + s \sum_{n=0}^{\infty} 2^{-n} \int_0^{\infty} \frac{g_2(2^n x)}{(x + a)^{s+1}} dx
$$

$$
= \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + s \sum_{n=0}^{\infty} 4^{-n} \sum_{j=0}^{\infty} \int_j^{j+1} \frac{g_2(y)dy}{(2^{-n}y + a)^{s+1}}
$$

$$
= \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + s \sum_{n=0}^{\infty} \frac{2^{-n}}{4} \sum_{j=0}^{\infty} \left( \int_j^{j+1/2} - \int_{j+1/2}^{j+1} \right) \frac{dy}{(2^{-n}y + a)^{s+1}}. \tag{2.3}
$$

Carrying out the integrations then gives the first line of (2.1). Elementary manipulations then yield

$$
\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \frac{1}{4} \sum_{n=0}^{\infty} 2^{n(s-1)} \sum_{j=0}^{\infty} \left[ \frac{1}{(j + 1 + a2^n)^s} + \frac{1}{(j + a2^n)^s} - \frac{2}{(j + 1/2 + a2^n)^s} \right]
$$

$$
= \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \frac{1}{4} \sum_{n=0}^{\infty} 2^{n(s-1)} \left\{ \frac{1}{a^s 2^{ns}} + 2 \sum_{j=0}^{\infty} \left[ \frac{1}{(j + 1 + a2^n)^s} - \frac{1}{(j + 1/2 + a2^n)^s} \right] \right\}. \tag{2.4}
$$
giving the rest of the Lemma. We remark that (2.3) is generalized in (2.14) for values of \( k \geq 2 \).

We now use standard expansions about \( s = 1 \), including

\[
(j + \beta + a2^n)^{s-1+1} = (j + \beta + a2^n) \exp[(s - 1) \ln(j + \beta + a2^n)],
\]

(2.5)

to write

\[
\zeta(s,a) - \frac{1}{s-1} = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell!} \ln^{\ell} a(s-1)^{\ell-1} + \frac{1}{a} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} \ln^{\ell} a(s-1)^{\ell}
\]

\[
+ \frac{1}{2} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(s-1)^{\ell}}{\ell!} \left\{ \left[ \frac{n \ln 2 - \ln(j + 1 + a2^n)}{j + 1 + a2^n} \right]^\ell - \left[ \frac{n \ln 2 - \ln(j + 1/2 + a2^n)}{j + 1/2 + a2^n} \right]^\ell \right\}.
\]

Comparing to the expansion (1.1) we have

\[
\gamma(t) = -\frac{\ln^{t+1} a}{t+1} + \frac{1}{a} \ln^t a
\]

\[
+ \frac{(-1)^{t}}{2} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \left\{ \left[ \frac{n \ln 2 - \ln(j + 1 + a2^n)}{j + 1 + a2^n} \right]^t - \left[ \frac{n \ln 2 - \ln(j + 1/2 + a2^n)}{j + 1/2 + a2^n} \right]^t \right\}
\]

\[
= -\frac{\ln^{t+1} a}{t+1} + \frac{1}{a} \ln^t a + \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \left[ \ln^t \left( \frac{a + j}{2j + 2 + a2^n} \right) - \ln^t \left( \frac{a + j+1}{2j + 1 + a2^{n+1}} \right) \right]
\]

\[
= -\frac{\ln^{t+1} a}{t+1} + \frac{1}{a} \ln^t a + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \left[ \ln^t \left( \frac{a + j}{2j + a2^n} \right) - \ln^t \left( \frac{a + j-1/2}{2j - 1 + a2^n} \right) \right].
\]

(2.6)

(2.7)

This completes the Proposition.

We can rearrange (2.1) in the form

\[
\zeta(s,a) - \frac{a^{1-s}}{s-1} = a^{-s} + \frac{1}{2} \sum_{n=1}^{\infty} 2^{(n-1)(s-1)} 2^s \sum_{j=1}^{\infty} \left[ \frac{1}{(2j + a^n)^s} - \frac{1}{(2j - 1 + a^n)^s} \right].
\]

(2.8)

From this follows Corollary 1.
Remark. It is easily verified that from Corollary 1 we may obtain the property
\( \partial_a \zeta(s, a) = -s \zeta(s + 1, a) \). Likewise it follows that \( \zeta(s, 1/2) = (2^s - 1) \zeta(s) \), and we provide a demonstration of this property from (1.5). We have
\[
\zeta \left( s, \frac{1}{2} \right) = 2^s + \frac{2^{s-1}}{s - 1} + \sum_{n=1}^{\infty} 2^{n(s-1)} \sum_{j=1}^{\infty} \frac{(-1)^j}{(j + 2^n)^s}.
\]
We transform the latter double sum to
\[
\sum_{n=0}^{\infty} 2^{(n+1)(s-1)} \sum_{j=1}^{\infty} \frac{(-1)^j}{(j + 2^n)^s} = 2^{s-1} \left[ \sum_{j=1}^{\infty} \frac{(-1)^j}{(j + 1)^s} + \sum_{n=1}^{\infty} 2^{n(s-1)} \sum_{j=1}^{\infty} \frac{(-1)^j}{(j + 2^n)^s} \right]
\]
\[
= 2^{s-1} \left[ \zeta(s) - 1 - 2^{1-s} \zeta(s) + \sum_{n=1}^{\infty} 2^{n(s-1)} \sum_{j=1}^{\infty} \frac{(-1)^j}{(j + 2^n)^s} \right].
\]
Then from (2.9) and (1.5) with \( a = 1 \),
\[
\zeta \left( s, \frac{1}{2} \right) = 2^s + \frac{2^{s-1}}{s - 1} + 2^{s-1} \zeta(s) - 2^{s-1} - \zeta(s) + 2^{s-1} \sum_{n=1}^{\infty} 2^{n(s-1)} \sum_{j=1}^{\infty} \frac{(-1)^j}{(j + 2^n)^s}
\]
\[
= 2^s + \frac{2^{s-1} - 2^{s-1}}{s - 1} + 2^s \zeta(s) - 2^s - \zeta(s)
\]
\[
= (2^s - 1) \zeta(s).
\]
Above, we used the alternating form of the zeta function so that
\[
\sum_{j=1}^{\infty} \frac{(-1)^j}{(j + 1)^s} = - \left( \sum_{j=1}^{\infty} \frac{(-1)^j}{j^s} + 1 \right) = (1 - 2^{1-s}) \zeta(s) - 1.
\]

Corollary 2. The first expression for \( \gamma \) follows by performing one of the sums in the Brun result (1.6),
\[
\gamma = 1 - \beta(1) = 1 - \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2^n + j}.
\]
In order to obtain the second expression for $\gamma$, we apply the duplication formula of the digamma function, $2\psi(2x) = 2 \ln 2 + \psi(x) + \psi(x + 1/2)$.

Remark. The second expression for $\gamma$ in Corollary 2 is precisely (2.24) of [5] with $a = 1$.

**Corollary 3.** We have from (1.3)

$$
\gamma_0(a) = -\psi(a) = -\ln a + \frac{1}{a} + \frac{1}{2} \sum_{n=0}^{\infty} \left[ \frac{1}{j + 1 + a2^n} - \frac{1}{j + 1/2 + a2^n} \right]. \tag{2.14}
$$

Summing over $j$ then gives

$$
\gamma_0(a) = -\ln a + \frac{1}{a} + \frac{1}{2} \sum_{n=0}^{\infty} \left[ \psi \left( a2^n + \frac{1}{2} \right) - \psi \left( a2^n + 1 \right) \right]. \tag{2.15}
$$

We then use both the functional equation $\psi(x + 1) = \psi(x) + 1/x$ and the duplication formula of the digamma function to obtain

$$
\gamma_0(a) = -\ln a + \frac{1}{a} + \sum_{n=0}^{\infty} \left[ \psi \left( a2^n + 1 \right) - \psi \left( a2^n \right) - \ln 2 - \frac{1}{a2^{n+1}} \right]. \tag{2.16}
$$

The Corollary follows.

**Proposition 2.** We simply outline the proof. We start again from the integral representation (2.2). Now we make use of the functions for $k \geq 2$ and $f(x) = -P_1(x)$ [5],

$$
g_k(x) = f(x) - \frac{1}{k} f(kx), \tag{2.17}
$$

with $\sum_{n=0}^{\infty} \frac{g_k(k^n x)}{k^n} = f(x)$. Then

$$
\zeta(s, a) - \frac{a^{-s}}{2} - \frac{a^{1-s}}{s-1}
= s \sum_{n=0}^{\infty} \frac{1}{k^n} \int_{0}^{\infty} \frac{g_k(k^n x)}{(x+a)^{s+1}} \, dx
$$

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\[
= s \sum_{n=0}^{\infty} \frac{1}{k^{2n}} \int_0^\infty \frac{g_k(y)dy}{(k^{-n}y + a)^{s+1}}
= s \sum_{n=0}^{\infty} \frac{1}{k^{2n}} \int_0^{j+1} \frac{g_k(y)dy}{(k^{-n}y + a)^{s+1}}
= s \sum_{n=0}^{\infty} \frac{1}{k^{2n}} \sum_{j=0}^\infty \left[ \frac{1}{2} \left( 1 - \frac{1}{k} \right) \int_j^{j+1/k} + \frac{1}{2} \left( 1 - \frac{3}{k} \right) \int_j^{j+2/k} + \frac{1}{2} \left( 1 - \frac{5}{k} \right) \int_j^{j+3/k} \right. 
+ \ldots + \left. \frac{1}{2} \left( 1 - \frac{1}{k} \right) \int_{j+(k-1)/k} \right] \frac{dy}{(k^{-n}y + a)^{s+1}}.
\]

(2.18)

In the last step we have used the values of \( g_k(x) \) on subintervals \( \left[ \frac{j-1}{k}, \frac{j}{k} \right) \) for \( j = 1, 2, \ldots, k, \)

\[
g_k(x) = \frac{1}{2} \left( 1 - \frac{1}{k} \right) - \frac{(j-1)}{k}, \quad x \in \left[ \frac{j-1}{k}, \frac{j}{k} \right).
\]

(2.19)

In particular, the difference of these values on consecutive subintervals is simply \( 1/k \).

We then perform the integrations and collect the terms to find for \( \text{Re} \ s > 1 \)

\[
\zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1}
\]

\[
= -\sum_{n=0}^{\infty} \frac{1}{k^n} \sum_{j=0}^{\infty} \left\{ \frac{1}{2} \left( \frac{1}{k} - 1 \right) \left[ \frac{1}{(bj + a)^s} + \frac{1}{(b(j+1) + a)^s} \right] + \frac{1}{k} \sum_{m=1}^{k-1} \frac{1}{b(j + m/k) + a} \right\}
\]

(2.20)

Expanding about \( s = 1 \) and using the definition (1.1) gives the Proposition.

Summary

Among other results, we have found computationally useful series representations of the Stieltjes constants. Although the series representations are doubly infinite, one of the series converges exponentially quickly with parameter \( k \geq 2 \). We have obtained in (2.16) a parameterized representation of the Hurwitz zeta function. In
the very special case of \( k = 2 \) and \( a = 1 \), we recover the much earlier result of Brun [4] of a double series expression for the Riemann zeta function. Since \( \gamma_0(a) = -\psi(a) \), where \( \psi \) is the digamma function, even at the lowest order, we effectively have found series representations of the harmonic numbers, generalized harmonic numbers, and other mathematical constants. We recall that the generalized harmonic numbers

\[
H_n^{(r)} \equiv \sum_{k=1}^{n} \frac{1}{k^r}
\]

may be readily found from the polygamma functions \( \psi^{(r-1)} \). Very special limit cases of our results include parameterized series representations for the expressions \( \zeta(0, a) = 1/2 - a \) and \( \zeta'(0, a) = \ln \Gamma(a) - (1/2) \ln(2\pi) \). Further, our results have application to series representation and expansions of Dirichlet \( L \) functions.
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