High-density limit of quasi-two-dimensional dipolar Bose gas

Volodymyr Pastukhov*
Ivan Franko National University of Lviv, Department for Theoretical Physics
12 Drahomanov St., Lviv, Ukraine

Abstract

We consider a simple model of the quasi-two-dimensional dipolar Bose gas confined in the one-dimensional square well potential. All dipoles are assumed to be oriented along the confining axis. By means of hydrodynamic approach it is shown that the general structure of the low-lying excitations can be analyzed exactly. We demonstrate that the problem significantly simplifies in the high-density limit for which the density profile in the confined direction as well as the leading-order contribution to the ground-state energy and spectrum of elementary excitations are calculated. The low-temperature result for the damping rate of the phonon mode is also presented.

1 Introduction

For more than decade the dipolar condensates of atoms with large magnetic moments can be realized experimentally [1, 2, 3, 4]. Such experimental progress stimulated extensive theoretical studies (see, for instance, reviews [5, 6, 7]). On the other hand, it is well-known that three-dimensional Bose gas with only dipole-dipole interaction is unstable, but presence of any trapping potential that strongly confines system in the direction of the external magnetic (electric) field stabilizes the system [8, 9, 10]. The reason underlying this property can be easily understood even on the mean-field level. In this approximation, which accurately describes only dilute systems, the stability condition is fully controlled by the sign of zero-momentum Fourier transform of the effective two-body interaction between particles. Although this potential functionally depends on the one-dimensional density profile in the confining direction, but at least weakly-interacting quasi-two-dimensional dipolar Bose systems are always stable. Another interesting feature of such objects is the existence of maxon-roton behavior of the excitation spectrum [11] even in the simplest Bogoliubov’s approximation. As the result the Beliaev damping of these quasi-particles disappear in

*e-mail: volodyapastukhov@gmail.com
the vicinity of the roton minimum \cite{12}. Moreover, the presence of roton-like behavior leads to modification of static \cite{13} and dynamic \cite{14} structure factors of the system, which can be measured using Bragg spectroscopy. Recently, roton instability as well as formation of stable droplets was observed experimentally \cite{15} in the dysprosium (\textsuperscript{164}Dy) dipolar Bose condensate. This stabilization effect is possible due to the existence of quantum fluctuations \cite{16,17} and cannot be understood on the mean-field level. The excitations with energy close to double roton gap are unstable to the spontaneous decay into two rotons \cite{18} leading to existence of the termination point in the spectrum at very low temperatures that was not observed in experiments with quasi-two-dimensional dipolar Bose gases yet. The roton excitations were also theoretically predicted in quasi-two-dimensional Bose condensate with quadrupolar as well as contact interaction between particles \cite{19}. The situation becomes more complicated but no less interesting in the case when dipole polarization forms non-zero angle with the confining direction. This regime of the so-called tilted dipolar Bose condensate includes the emergence of direction dependent superfluid properties \cite{20}, non-typical vortex-vortex interaction \cite{21}, striped-phase formation \cite{22}, anomalous atom-number fluctuations \cite{23} and anisotropic spectrum with maxon-roton behavior \cite{24,25}. Finite temperature properties of quasi-two-dimensional dipolar Bose systems were studied in Ref. \cite{26} within the Hartree-Fock-Bogoliubov method, and in \cite{27} using classical field approximation. Fully two-dimensional dipolar Bose gas with softened interaction in connection to the description of polarized excitons was considered in Ref. \cite{28} with the help of dielectric formalism. At finite temperatures the presence of the roton minimum also changes properties of the Landau damping both for phonon and maxon-roton parts of the excitation spectrum \cite{29}.

In the present paper, we consider quasi-two-dimensional dipolar Bose system confined in a square well potential at low temperatures. In particular, by means of the hydrodynamic approach we develop perturbative treatment of the problem in terms of inverse two-dimensional density of the system. Notwithstanding that the experimental realization of Bose systems in square well potential is complicated, our predictions may serve the starting point to describe the harmonically confined systems near center of the trap. Additionally we show that our findings for Bose systems obtained within hydrodynamic approach are consistent with the results of other formulations.

## 2 Formulation

The considered model is described by the following Euclidian action

\[
S = \int dx \psi^*(x) \left( \frac{\partial}{\partial \tau} + \frac{\hbar^2 \nabla^2}{2m} + \mu \right) \psi(x) - \frac{1}{2} \int dx \int dx' \Phi(x-x')\psi^*(x)\psi^*(x')\psi(x')\psi(x),
\]  

(1)
where \( x = (\tau, \mathbf{r}) \), \( \mu \) is chemical potential and the following notation is used
\[
\int d\mathbf{x} = \int_0^\beta d\tau \int_A d\mathbf{r} \int_0^a dz .
\]
Here \( \beta = 1/T \) and \( Aa \) are inverse temperature and volume of the system, respectively. Complex field \( \psi(x) \) is \( \beta \)-periodic function of imaginary time \( \tau \) and the confining potential imposes boundary conditions
\[
\psi(x)|_{z=0} = \psi(x)|_{z=a} = 0 .
\]
In transverse direction we apply periodic boundary conditions with large area \( A \). This situation is approximately realized in pancake traps with small frequencies in the plane perpendicular to the external field direction. The second term of the action describes two-body interaction between particles. We also introduce function \( \Phi(x - x') = \delta(\tau - \tau')\Phi(\mathbf{r} - \mathbf{r}') \), where dipole-dipole potential
\[
\Phi(\mathbf{r}) = \frac{g}{4\pi} \left\{ \frac{1}{r^3} - \frac{3z^2}{r^5} \right\} .
\]
Of course, in experimentally relevant cases the dipolar Bose gas usually contains a short-range contact interaction as well, but the presence of such a term in (2) would not change our further consideration.

Bearing in mind low-temperature description we can use one-mode approximation and for convenience pass to density-phase variables [30, 31, 32]
\[
\psi(x) = \chi_0(z) \sqrt{\rho(x)} e^{i\varphi(x)} ,
\]
where \( x = (\tau, \mathbf{r}_\perp) \) and \( \chi_0(z) \) is the variational function with boundaries \( \chi_0(0) = \chi_0(a) = 0 \) and normalization condition
\[
\int_0^a dz |\chi_0(z)|^2 = 1 .
\]
The multi-mode generalization of this approach required for finite-temperature description of quasi-two-dimensional dipolar Bose systems is straightforward, but necessarily leads to considerable technical complications during calculations [26]. After making use of Fourier transformation
\[
\rho(x) = \rho + \frac{1}{\sqrt{BA}} \sum_K e^{i(\omega_k \tau + \mathbf{k} \cdot \mathbf{r}_\perp)} \rho_K ,
\]
\[
\varphi(x) = \frac{1}{\sqrt{BA}} \sum_K e^{i(\omega_k \tau + \mathbf{k} \cdot \mathbf{r}_\perp)} \varphi_K ,
\]
where summations are carried out over \( K = (\omega_k, \mathbf{k}) \) (\( \omega_k \) is bosonic Matsubara frequency, \( \mathbf{k} \neq 0 \)), we can write down the effective action for quasi-two-dimensional system
\[
S_{\text{eff}} = S_0 + S_G + S_{\text{int}} ,
\]
where constant mean-field part is
\[
S_0 = \beta A \rho \left\{ \mu - \frac{\hbar^2}{2m} \int_0^a dz \left( \frac{d\chi_0}{dz} \right)^2 - \frac{1}{2} \rho \nu(0) \right\} .
\]
The second term of Eq. (6) is action of non-interacting two-dimensional quasi-
particles

\[ S_G = -\frac{1}{2} \sum_K \left\{ \omega_k \varphi_K \rho_{-K} - \omega_k \varphi_{-K} \rho_K + \frac{\hbar^2 k^2}{m} \rho \varphi_K \varphi_{-K} \right. \]
\[ + \left[ \frac{\hbar^2 k^2}{4m \rho} + \nu(k) \right] \rho_K \rho_{-K} \left\}, \tag{8} \]

and the last one takes into account collisions between them

\[ S_{\text{int}} = \frac{1}{2\sqrt{\beta A}} \sum_{K,Q} \frac{\hbar^2}{m} k q \rho_{-K-Q} \varphi_K \varphi_Q \]
\[ + \frac{1}{3!\sqrt{\beta A}} \sum_{K+Q+P=0} \frac{\hbar^2}{8m \rho^2} (k^2 + q^2 + p^2) \rho_K \rho_Q \rho_P \]
\[ - \frac{1}{8\beta A} \sum_{K,Q} \frac{\hbar^2}{2m \rho^3} (k^2 + q^2) \rho_K \rho_{-K} \rho_Q \rho_{-Q}. \tag{9} \]

In fact action \( S_G + S_{\text{int}} \) describes two-dimensional Bose system and the Fourier transform of the effective two-particle interaction \( \nu(k) \) contains all information about confining potential

\[ \nu(k) = \int_0^a dz \int_0^a dz' |\chi_0(z)|^2 |\nu_k(z - z')| \chi_0(z')|^2, \tag{10} \]

where for sake of simplicity we have introduced two-dimensional Fourier transform of the dipole-dipole interaction

\[ \nu_k(z) = \int_A d\mathbf{r} e^{i\mathbf{r} \cdot \mathbf{k}} \Phi(r) = g \left\{ \frac{2}{3} \delta(z) - \frac{1}{2} k e^{-k|z|} \right\}. \tag{11} \]

The last two terms of Eq. (9) are usually omitted in the long-wavelength limit, but their contribution is significant at finite temperatures [33]. Thermodynamic relation for the grand canonical potential \(-\partial \Omega/\partial \mu = N\) together with normalization condition on \(\chi_0(z)\) fix the quantity \(\rho\) to be equilibrium area density \(N/A\). Actually this fact allows to provide further consideration of the translation-invariant systems in canonical ensemble. But in our case when the normalization condition has to be taken into account the last step is to minimize the grand canonical potential of the system with respect to function \(\chi_0(z)\). These calculations lead to Gross-Pitaevskii-type equation

\[ \left\{ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + \int_0^a dz' |\chi_0(z')|^2 \phi(z' - z) \right\} \chi_0(z) = \mu \chi_0(z), \tag{12} \]

where we denote the effective one-dimensional potential in the confined direction

\[ \phi(z) = \rho \nu_0(z) + \frac{1}{A} \sum_{k \neq 0} \nu_k(z) [S_k - 1]. \tag{13} \]
Here $S_k = \frac{1}{\beta} \sum_{\omega_k} \langle \rho_K \rho_{-K} \rangle / \rho$ is the static structure factor of two-dimensional bosons with density $\rho$ and interacting via potential $\nu(k)$. The function $\langle \rho_K \rho_{-K} \rangle$ is related to the dynamic structure factor of the system and its poles determine the spectrum of collective modes. Taking into account only first two terms of the action (6) we correctly reproduce the Bogoliubov theory with undamped spectrum $E_q = \sqrt{\left(\frac{\hbar^2 q^2}{2m}\right)^2 + \frac{\hbar^2 q^2}{m} \rho \nu(q)}$. The further consideration is cumbersome, but hydrodynamic approach allows us to build perturbation theory free of infrared divergences [34] which greatly simplifies the calculations of the spectral properties of Bose systems. On the other hand, it is not hard to show that general structure of the low-lying excitations and consequently the long-range behavior of the one-particle density matrix can be found exactly. This also gives information about leading-order asymptote of the effective one-dimensional potential $\phi(z)$ at large particle separations.

2.1 Low-energy excitations

For the first step we introduce the matrix correlation function

$$F(K) = \left( \begin{array}{cc} \langle \varphi_K \varphi_{-K} \rangle & \langle \rho_K \varphi_{-K} \rangle \\ \langle \varphi_K \rho_{-K} \rangle & \langle \rho_K \rho_{-K} \rangle \end{array} \right),$$

that satisfies Dyson equation

$$F^{-1}(K) = F_0^{-1}(K) - \Pi(K),$$

with zero-order approximation determined by the Gaussian part of the action [34]

$$F_0(K) = \left( \begin{array}{cc} \frac{\hbar^2 k^2}{m} \rho & \omega_k \\ -\omega_k & \frac{\hbar^2 k^2}{4mp} + \nu(k) \end{array} \right)^{-1}.$$  

The elements $\Pi_{\varphi\varphi}(K), \Pi_{\rho\rho}(K)$ of matrix self-energy are even functions of frequency and $\Pi_{\rho\varphi}(K)$ is odd function with additional constrain $\Pi_{\rho\varphi}(-K) = \Pi_{\varphi\rho}(K)$. Moreover, taking into account structure of the anharmonic terms of the action [34] and absence of infrared divergences in perturbation theory it is readily seen that $\Pi_{\rho\rho}(K) \propto \omega_k k^2$ and consequently has no effect on the real part of phonon mode. The leading-order contribution to the diagonal matrix elements of correlation function [34] can also be found in the long-length limit. To derive the low-energy limit of $\Pi_{\rho\rho}(K)$ we use the fact that differentiation of every bare vertex function with respect to density while keeping $\nu(k)$ fixed gives the vertex with one more zero-momentum ($K = 0$) $\rho$ line. Of course, this conclusion holds for exact vertices. The latter observation immediately leads to the identity

$$\nu(0) - \Pi_{\rho\rho}(0) = \left( \frac{\partial \mu}{\partial \rho} \right)_T,$$  

5
where \((\partial \mu / \partial \rho)_T\) is inverse susceptibility of two-dimensional system with density \(\rho\) and fixed potential \(\nu(k)\). Making use of gauge transformation \(\varphi(x_\perp) \to \varphi(x_\perp) + m \nu_{\perp}/\hbar\) and mentioning that the derivative \(\frac{\hbar^2 k^2}{m} \varphi^\prime(K)\) line with zero frequency and vanishingly small \(k\) to every vertex, one can show that in low-energy limit

\[
\frac{\hbar^2 k^2}{m} \rho - \Pi \varphi(K \to 0) = \frac{\hbar^2 k^2}{m} \rho_s,
\]

where \(\rho_s\) coincides with two-dimensional superfluid density of the system. Equations (17), (18) connect the low-energy spectrum of collective modes to the macroscopic quantities of the system and will be tested below within perturbation theory. The above analysis shows that equal-time one-particle Green function of the system reveals non-diagonal long-range behavior \(\langle \psi(x) \psi^\dagger(x') \rangle |_{\tau = \tau'} \propto \chi_0(z) \chi_0^\dagger(z') |r_\perp - r'_\perp|^{-\alpha}\) with exponent \(\alpha = mT/2\hbar^2 \rho_s\) [35, 36] which is typical for Berezinskii-Kosterlitz-Thouless phase of two-dimensional superfluid.

3 High-density solution

The main problem to be solved in our approach is finding the solution of Eq. (12) and calculating the Fourier transform of the two-dimensional potential (10) that determines the properties of the system. The situation is complicated by the fact that \(\phi(z)\) functionally depends on \(\chi_0(z)\). Of course, in the weak-coupling limit when dimensionless parameter \(\gamma = \rho g a / \hbar^2 m a^2\) is small the solution is trivial \(\chi_0(z) = \sqrt{2} \sin(\pi z/a)\) with \(\mu = \frac{\hbar^2 \pi}{2m a^2}\), but physically interesting is the inverse limit \(\gamma \gg 1\) where excitation spectrum exhibits maxon-roton behavior. The analysis simplifies greatly for the high-density systems \(\rho a^2 \gg 1\) not too close to roton instability. More precisely the condition of validity of our approximation can be formulated introducing dimensionless roton gap \(\delta = \Delta / \frac{\hbar^2}{2ma^2}\) and temperature \(t = T / \frac{\hbar^2}{2ma^2}\). Then the last term of the one-dimensional potential (13) can be omitted in the limits \(\rho a^2 \ln \delta^{-1} \ll 1\) and \(\frac{\rho a^2 \delta}{\rho a^2} \ll 1\) [37] at zero and finite temperatures, respectively. These conditions are quite general, i.e., independent on the specific perturbation scheme and do not necessary require the interaction parameter \(\gamma\) to be small. Actually, for this high-density limit the mean-field approximation becomes exact one determining thermodynamic properties of the system, and function \(\chi_0(z)\) is given by the solution of the one-dimensional Gross-Pitaevskii equation

\[
-\frac{\hbar^2}{2m} \frac{d^2}{dz^2} \chi_0(z) + \frac{2}{3} \rho g |\chi_0(z)|^2 \chi_0(z) = \mu \chi_0(z),
\]

with zero boundary conditions. For such simple geometry of confining potential the solution reads

\[
\chi_0(z) = \frac{C(\kappa)}{\sqrt{a}} \sin(2K(\kappa)z/a, \kappa),
\]
\[ \kappa = \sqrt{\frac{\gamma}{6}} C(\kappa)/K(\kappa), \quad \mu = \frac{2\hbar^2}{ma^2} K^2(\kappa)(1 + \kappa^2), \]

where \( \text{sn}(z, \kappa) \) is the Jacobi elliptic sine, \( C(\kappa) \) is the normalization constant and \( K(\kappa) \) is the elliptic integral of the first kind. Having found one-dimensional mean-field density profile and chemical potential, we can calculate the ground-state energy of the quasi-two-dimensional dipolar Bose system

\[ \frac{E_0}{N} = \frac{1}{\rho_0} \int_0^\rho \mu d\rho, \]

or in our approximation

\[ \frac{E_0}{N} = \mu - \frac{\rho g}{3} \int_0^a \rho_0(z)dz |\chi_0(z)|^4. \]  

(21)

The results of numerical evaluation for the dimensionless ground-state energy per particle \( E_0/N = \frac{\hbar^2}{2ma^2} e_0 \) and chemical potential \( \mu = \frac{\hbar^2}{2ma^2} \lambda \) of the system are presented in Fig. 1. It is worth noting that beyond mean-field calculation of the thermodynamic properties of a quasi-two-dimensional dipolar Bose system is tedious. First of all, one has to find the next-to-leading order correction to the density profile in confined direction. And secondly, these calculations require knowledge of the quasi-two-dimensional dipole-dipole scattering properties \cite{38, 39, 40}.

### 3.1 Spectrum and damping rate

We are in position now to calculate the function \( \nu(k) \) and study the spectral properties of the system. For reasons of convenience we introduce function \( f(\xi) = \nu(k)/\kappa \), where \( \xi = ka \) is the dimensionless wave-vector. In two limiting
Figure 2: Bogoliubov spectrum in units of $\frac{\hbar^2}{2ma^2}$ for $\gamma = 35$. Solid line presents exact calculation with solution (20). Dotted and dashed lines are the results for weak- and strong-coupling limits, respectively.

cases Fourier transform of the effective two-dimensional interaction and as a result the Bogoliubov spectrum can be calculated analytically. The first one is the above-mentioned weak-coupling limit and the second one is known as the Thomas–Fermi approximation which is applicable for $\gamma \gg 1$ where the function $\chi_0(z)$ tends to constant $1/\sqrt{a}$ on the interval $[0, a]$. The explicit expressions for function $f(\xi)$ for these two cases are the following

$$f_0(\xi) = \frac{1 - e^{-\xi}}{\xi} - \frac{1}{2} \frac{\xi}{\xi^2 + (2\pi)^2} \left[ \xi + 3(1 - e^{-\xi}) - \frac{\xi^2 - (2\pi)^2}{\xi^2 + (2\pi)^2} (1 - e^{-\xi}) \right],$$

$$f_{TF}(\xi) = -\frac{1}{3} + \frac{1 - e^{-\xi}}{\xi}.$$

The results of numerical calculations for the spectrum $E_k = \frac{\hbar^2}{2ma^2} \varepsilon(\xi)$, $\varepsilon(\xi) = \xi \sqrt{\xi^2 + 4\gamma f(\xi)}$ of Bogoliubov excitations are presented in Fig. 2. In the long-wavelength limit we have $E_k = \frac{\hbar^2}{\pi a^2} \xi (1 - \xi/4f(0) + \ldots)$. Here we introduce the sound velocity $c = \sqrt{\rho u(0)/m} = \sqrt{\frac{\rho g}{ma}} u(\gamma)$ where $u(\gamma)$ is dimensionless monotonically decreasing function (see Fig. 3). For the maximal and minimal values we obtained analytically $u(0) = 1$ and $u(\infty) = \sqrt{2/3} \simeq 0.816$, respectively. Note that the second term of the low-energy spectrum is quadratic. It is general feature of the quasi-two-dimensional dipolar Bose systems that does not depend on the nature of confining potential. Our numerical calculations show that at $\gamma_c \simeq 39.86$ the roton instability occurs. In fact, this value of coupling parameter determines the limits of applicability of our approximation.
Finally, we have to make sure that damping of the spectrum is small, i.e., elementary excitations are well-defined. The calculation of the damping rate requires the knowledge of the self-energy parts for which the explicit formulae on the one-loop level are given in the Appendix. Although in our approximation the real parts of self-energies do not affect on the properties of the system, but to verify the identities (17), (18) we calculate the leading-order contribution to the function $\Pi_{\phi\phi}(K)$ in the long-length limit ($K \to 0$)

$$\Pi_{\phi\phi}(K) = \frac{\hbar^2 k^2}{m} \frac{1}{2A} \sum_{q \neq 0} \frac{\hbar^2 q^2}{m} \left[ -\frac{\partial}{\partial E_q} n(\beta E_q) \right],$$

(22)

where $n(x) = (e^x - 1)^{-1}$ is the Bose distribution function, and it is easy to recognize the well-known Landau formula for the normal density of the super-fluid in two dimensions. In the same manner one can show that the first-order calculations for $\Pi_{\rho\rho}(K)$ lead to the result

$$\Pi_{\rho\rho}(0) = \frac{1}{A} \sum_{q \neq 0} \left[ \frac{\hbar^2 q^2 \nu(q)}{2mE_q} \right]^2 \left\{ \frac{1}{2E_q} + \frac{1}{E_q} n(\beta E_q) - \frac{\partial}{\partial E_q} n(\beta E_q) \right\},$$

(23)

which up to a sing coincides with the correction to the inverse compressibility of the two-dimensional system. For the damping rate of low-energy quasi-particles we have

$$\frac{\Gamma_k}{E_k} = \frac{\rho}{2mc^2} \Im \Pi_{\rho\rho}(E_k, k) - \frac{1}{\hbar c k} \Re \Pi_{\phi\rho}(E_k, k) + \frac{m}{2\hbar^2 k^2 \rho} \Im \Pi_{\phi\phi}(E_k, k),$$

(24)
where $\Im \Pi_{\rho\rho}(\omega, k)$, $\Im \Pi_{\varphi\varphi}(\omega, k)$ are imaginary parts and $\Re \Pi_{\rho\rho}(\omega, k)$ is real part of appropriate self-energies after analytical continuation $i\omega_k \rightarrow \omega + i0$. Due to the sign of the second term in the low-wavelength expansion of the Bogoliubov spectrum the Beliaev damping of phonon mode is strongly exhausted [41]. At finite temperatures the damping rate is fully controlled by the so-called Landau mechanism of quasi-particle decay. So, in Appendix we present the only terms of matrix self-energy responsible for the Landau damping of low-lying excitations [see Eqs. (29)-(31)]. After substitution in the equation (24) we obtain $\Gamma_k/E_k = i(t)/\rho a^2$, where dimensionless damping rate reads

$$
\left. \begin{array}{c}
i(t) = \gamma \frac{2\pi tf(0)}{2\pi tf(0)} \Theta\left(\frac{\varepsilon'(\xi)}{\sqrt{\varepsilon'^2(\xi)/4\gamma f(0) - 1}} - 1\right) \left[\frac{f(\xi)\xi}{\varepsilon(\xi)} + 2f(0)\varepsilon(\xi)\right] \right. \\
\times n(\varepsilon(\xi)/t)[1 + n(\varepsilon(\xi)/t)],
\end{array} \right. 
$$

(25)

here $\Theta(x)$ is the Heaviside step function and we use notation for the derivative of dimensionless spectrum $\varepsilon'(\xi)$ with respect to $\xi$. This result is similar to that obtained with the help of other approach [29]. If we set $f(\xi) = 1$ it reproduces the damping rate of the two-dimensional Bose gas with contact interaction obtained with the help of the Hartree-Fock-Bogoliubov approximation in Ref. [42] and using modified hydrodynamic approach in [33]. Our findings prove the consistency of Popov’s hydrodynamic description of Bose systems at finite temperatures. As it is seen from formula (25) the only contribution to the damping of low-energy excitations comes from the quasi-particles with the velocity $\partial E_q/\partial(q)$ greater than sound velocity of the system [29]. This peculiarity of the damping rate

Figure 4: Temperature dependence of the damping rate of the phonon mode calculated with exact (solid line), Thomas-Fermi (dashed line) and weak-coupling (dotted line) density profiles at $\gamma = 35$. 
is confirmed by numerical calculations and illustrated in Fig. 4. Similarly to Bose systems with contact interaction, at high temperatures the curves clearly demonstrate linear behavior on the reduced temperature.

4 Conclusions

In conclusion, we have studied properties of quasi-two-dimensional dipolar Bose system confined in square well potential. At low temperatures such a simple geometry of the external potential allows to construct systematic perturbation theory in terms of the inverse two-dimensional (area) density of the system. In the leading order the mean-field equation that determines one-dimensional density profile is solved exactly and the effective potential of inter-particle interaction in two dimensions is calculated. For the calculations of spectral properties of the system we used the hydrodynamic approach. This formulation is naturally-suited for two-dimensional Bose systems and allows to analyze the low-energy spectrum of collective modes exactly.

Our approximation leads to the Bogoliubov excitations with maxon-roton behavior of the spectrum which we calculate numerically using exact solution of the Gross-Pitaevskii equation for the ground-state density profile. The same calculations were also performed with approximate solutions of Eq. (19) for weak-coupling and Thomas-Fermi limits, respectively. Although our findings clearly demonstrate qualitative agreement in the behavior of the spectrum and damping rate in these three cases, but quantitative differences between curves (see Figs. 2,4) are noticeable.

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Appendix

In this section we present the results of calculation for self-energies to the first order of perturbation theory

$$\Pi_{\varphi \varphi}(K) = \frac{1}{\beta A} \sum_{Q} \frac{\hbar^4}{m^2} \left\{ (kq)^2 \langle \varphi Q \varphi_{-Q} \rangle \langle \rho_{K+Q} \rho_{-K-Q} \rangle \right. $$

$$\left. -kq(kq+k^2) \langle \varphi Q \rho_{-Q} \rangle \langle \varphi_{K+Q} \rho_{-K-Q} \rangle \right\} ,$$

(26)
\[ \Pi_{\varphi\rho}(K) = \frac{1}{2\beta A} \sum_{Q} \frac{\hbar^4}{m^2} \left\{ (q^2 + qk)^2 \langle \varphi_{Q\varphi} - \varphi_{-Q} \rangle \langle \varphi_{K+Q\varphi} - \varphi_{-K-Q} \rangle \right. \\
+ \frac{1}{16\rho^4} (k^2 + q^2 + kq)^2 \langle \rho_{Q\rho} - \rho_{-Q} \rangle \langle \rho_{K+Q\rho} - \rho_{-K-Q} \rangle \\
+ \frac{1}{2\rho^2} (k^2 + q^2 + kq)(q^2 + qk) \langle \varphi_{Q\rho} - \varphi_{-Q} \rangle \langle \varphi_{K+Q\rho} - \varphi_{-K-Q} \rangle \left. \right\} - \frac{1}{4\beta A} \sum_{Q} \frac{\hbar^2}{m^2} (k^2 + q^2) \langle \rho_{Q\rho} - \rho_{-Q} \rangle, \tag{27} \]

\[ \Pi_{\varphi\varphi}(K) = \frac{1}{\beta A} \sum_{Q} \frac{\hbar^4 kq}{m^2} \left\{ (q^2 + kq)^2 \langle \varphi_{Q\varphi} - \varphi_{-Q} \rangle \langle \varphi_{K+Q\varphi} - \varphi_{-K-Q} \rangle \right. \\
+ \frac{1}{4\rho^2} (k^2 + q^2 + kq) \langle \varphi_{Q\rho} - \varphi_{-Q} \rangle \langle \varphi_{K+Q\rho} - \varphi_{-K-Q} \rangle \left. \right\}. \tag{28} \]

where appropriate correlation functions should be taken neglecting self-energy corrections, i.e., given by Eq. (16). After performing the Matsubara frequency summations and passing to the upper complex half-plane \( i\omega_k \rightarrow \omega + i0 \) we obtain for self-energies

\[ \Im \Pi_{\varphi\varphi}(\omega, k) = -\omega \frac{\pi}{A} \sum_{q \neq 0} \frac{\hbar^2 kq}{m^2} \left[ \frac{\partial}{\partial E_q} n(\beta E_q) \right] \delta(E_{q+k} - E_q - \omega), \tag{29} \]

\[ \Im \Pi_{\varphi\rho}(\omega, k) = -\omega \frac{\pi}{A} \sum_{q \neq 0} \left[ \frac{\hbar^2 q^2 \nu(q)}{2m E_q} \right] \left[ \frac{\partial}{\partial E_q} n(\beta E_q) \right] \delta(E_{q+k} - E_q - \omega), \tag{30} \]

\[ \Re \Pi_{\varphi\rho}(\omega, k) = \omega \frac{\pi}{A} \sum_{q \neq 0} \frac{\hbar^2 q^2 \nu(q) \hbar^2 kq}{m^2} \left[ \frac{\partial}{\partial E_q} n(\beta E_q) \right] \delta(E_{q+k} - E_q - \omega). \tag{31} \]

The above formulae are valid only in the long-wavelength limit.

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