Radiation reaction in curved even-dimensional spacetime

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We develop a new method of computing radiation reaction for a point particle interacting with massless scalar and vector fields in curved space-time. It is based on the analysis of field Green’s functions with both points lying on the particle world-line and does not require integration of the field stresses outside the world line as was used in the DeWitt-Brehme approach, thus leading to a substantial simplification of the problem. We start with space-time of an arbitrary dimension and show that the Hadamard expansion of the massless scalar and vector Green’s functions contain only integer inverse powers of the Synge world function in even dimensions and only half-integer in the odd dimensions. The even-dimensional case then is treated in detail. We analyze divergencies, calculate higher-derivative counterterms, and find a recurrent formula for the local parts of the reaction force in neighboring dimensions. Higher-dimensional curved space counterterms are not simply the covariant generalizations of the flat ones, but contain additional curvature-dependent terms. We illustrate our formalism in four and six dimensions. In the first case we rederive the results of DeWitt-Brehme-Hobbs in a simpler way, in the second case we give a covariant generalization of the Kosyakov equation. The local part of the reaction force is found to contain a term proportional to the Riemann tensor which is absent in four dimensions.

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I. INTRODUCTION

Though already Abraham[1] gave a correct formula for radiation reaction force in classical electrodynamics

\[ m \dot{z}^\alpha = \frac{2}{3} e^2 \left( \dot{z}^\alpha \right)^2 + \frac{2}{3} e^2 \left( \dot{z}^\alpha - \dot{z}^\beta \dot{z}^\gamma \right) + \frac{2}{3} \left( R_{\beta \gamma} \dot{z} \dot{z} + R_{\beta \gamma} \dot{z} \dot{z} \right), \]

this problem, especially its generalization for curved space-time and gravitational radiation still attract attention. Physical status of the Eq. (1) also remains subject of discussion. In 1938 Dirac gave a consistent interpretation of the LD equation is obscured by the presence of the third derivative Schott term [12]

\[ f_{\text{Schott}} = \frac{2}{3} e^2 \ddot{z}^\alpha. \]

This term can be found as improvement of the radiation recoil term equal to the loss of the particle momentum due to radiation,

\[ f_{\text{emit}} = -\frac{2}{3} e^2 z^\beta \dot{z}^\alpha, \]

to ensure the orthogonality of the reaction force to the particle four-velocity. But, somewhat unexpectedly, the Schott term leads to disturbing modification of the particle momentum balance and creates the self-acceleration problem, for which reason the validity of the LD equation is questioned. A widespread opinion is that the resolution of these problems requires quantum theory.

Meanwhile, correct physical interpretation of the Schott term solves this controversy within the classical theory. As was emphasized long ago by Teitelboim [3], and more recently reconsidered (with all calculations explicit) in our paper [10], the Schott term must be treated as a part of the electromagnetic momentum, but not as a part of the particle momentum. The LD equation (and its generalization which we will discuss here) describes a composite object whose energy-momentum contains the mechanical part and the Coulomb field part bound to the charge, which varies once particle is accelerated. The energy-momentum balance can be verified globally for world-lines which asymptotically are purely geodesic (non-accelerated). The Coulomb coat varies with time when the charge experiences an acceleration, moreover, as a result of interaction with the charge, the coat energy can be transformed to that of radiation. This provides a natural explanation of the Born paradox.

The LD equation was covariantly generalized to curved space-time with an arbitrary metric in 1960 by DeWitt and Brehme [14] and completed with the Ricci term by Hobbs [15] (for a recent review see [16]). In this case, in addition to the local Abraham force [1], there is a tail integral over the past history of the particle:

\[ m \ddot{z}^\alpha = e F_{\text{ext} \beta} z^\beta + \frac{2}{3} e^2 \left( \dot{z}^\alpha - \dot{z}^\beta \dot{z}^\gamma \right) + \frac{2}{3} \left( R_{\beta \gamma} \dot{z} \dot{z} \right) + \frac{2}{3} \left( R_{\beta \gamma} \dot{z} \dot{z} \right) + c^2 \ddot{z} \int_{-\infty}^{\tau} f_{\beta \gamma} (\tau, \tau') \dot{z} \dot{z} d\tau', \]

where \( F_{\text{ext} \beta} \) is the external electromagnetic field and \( f_{\beta \gamma} (\tau, \tau') \) is some two-point tensor. It could be expected that the curved space LD equation which takes into account non-local effects, should violate the equivalence principle. But in the DeWitt-Brehme-Hobbs (DWBH) equation this violation is minimal: only the Ricci term is present in the local part of the LD force, not the Riemann tensor. In empty space the local part of the DWBH equation is just the co-
variantization of the flat-space LD equation. The tail term of course makes the reaction problem essentially non-local, but this term is due to the global properties of space-time.

Similar equation was also derived for the scalar radiation reaction \[ m(\tau) = \frac{q^2}{3}(z^\alpha - \ddot{z}^\alpha) + \frac{q^2}{6}(R^\alpha_\beta \dot{z}^\beta + R^\alpha_\beta \dot{z}^\beta \dot{z}^\alpha) - q^2 \int_{-\infty}^{\tau} G_{\text{ret}}(z(\tau), z(\tau'))d\tau', \] where \( G_{\text{ret}} \) is the retarded solution of the wave equation and \( m(\tau) = m_0 - q\phi(z) \) is the variable effective mass.

The derivation of these generalizations of the LD equation to curved space-time in the existing literature is based on lengthy calculations of the integral contribution of the bound and radiative energy-momentum fluxes within the world-tube surrounding the particle world line. Meanwhile, the resulting equation (including the tail term) depends only on the quantities localized on the world-line. This indicates that a simpler derivation should be possible which does not require considering the field outside the world line. Recently we have shown that similar derivation (restricted to four space-time dimensions) can be performed in curved space-time as well. Note that we discuss here only radiation of non-gravitational nature, for gravitational waves the situation is likely to be more complicated.

The purpose of the present paper is to generalize the approach of \[19\] to higher-dimensional space-times. Generalization to higher dimensions in the flat space-time was discussed recently in \[20–22\] in view of an interest to cosmological models with large extra dimensions. Here we consider the curved space of dimension \( D \geq 4 \). Note that the cases of odd and even dimensions are essentially different: in odd dimensions the tail terms appear already in the flat space due to violation of the Huygens’ principle. (This effect can be easily interpreted considering \( D \)-dimensional point particles as parallel strings in \( D + 1 \) dimensions \[21\].) We will present some general considerations valid in any dimensions and then specialize to an even-dimensional case. The mostly plus space-time signature is used.

## II. GREEN FUNCTIONS

As in \( D = 4 \) \[14\], the curved space Green’s functions for massless fields in arbitrary dimensions can be constructed starting with the Hadamard solution.

### A. Hadamard expansion

We start with the scalar Hadamard \[23, 24\] Green’s function \( G_H(x, x') \) which is a solution to the homogeneous scalar wave equation

\[ \Box_x G_H(x, x') = 0, \quad (6) \]

where \( \Box_x = g_{\mu\nu} \nabla^\mu \nabla^\nu \) is the curved space scalar D’Alembert operator. The procedure consists in expanding \( G_H(x, x') \) in terms of the Synge world function

\[ \sigma(x, x') = \frac{1}{2}(s_1 - s_0) \int_{s_0}^{s_1} g_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta d\tau, \quad (7) \]

where an integral is taken along the geodesic \( x = z(\tau) \), \( z(\tau_0) = x \), \( z(\tau_1) = x' \) connecting points \( x \) and \( x' \) and it is assumed that there is a unique such geodesic (which makes the construction essentially local). The sign is chosen such that \( \sigma \) be negative for time-like geodesics. The gradients of the Synge two-point function

\[ \sigma_{\mu\nu} = \partial \sigma(x, x')/\partial x^\mu, \quad \sigma_{\alpha\beta} = \partial \sigma(x, x')/\partial x'^\alpha \quad (8) \]

satisfy the Hamilton-Jacobi equation

\[ g^{\mu\nu} \sigma_{\mu\nu} = g^{\alpha\beta} \sigma_{\alpha\beta} = 2\sigma, \quad (9) \]

where (using DeWitt-Brehme conventions) we denote by the initial Greek letters \( \alpha, \beta, \ldots \) tensor indices transforming at \( x' \), and by the older Greek letters \( \mu, \nu, \ldots \) the indices transforming at \( x \). In the coincidence limit \( x \to x' \) (denoted by square brackets) one has

\[ [\sigma] = \lim_{x \to x'} \sigma(x, x') = 0, \quad [\sigma_{\alpha\beta}] = 0, \quad [\sigma_{\mu\nu}] = 0. \quad (10) \]

A mixed covariant derivative \( \nabla_\alpha \nabla_\beta \sigma(x, x') \), where according to the convention \( \nabla_\mu \) acts on \( x \) and \( \nabla_\alpha \) on \( x' \), defines the van Vleck determinant

\[ \Delta(x, x') = \frac{\det(\nabla_\alpha \nabla_\beta \sigma(x, x'))}{\sqrt{g(x)g(x')}}. \quad (11) \]

Higher covariant derivatives of the Synge function in the coincidence limit are expressed via the metric tensor and the Riemann tensor, this is used to find Taylor’s expansions of the two-point tensors \[14\].

Another useful two-point tensor is the bivector of parallel transport, which can be expressed through the vielbein associated with metric: \( g_{\mu\nu}(x) = e^m_\mu(x) e^n_\nu(x) \eta_{mn} \) as

\[ \tilde{g}_{\mu}^{\alpha}(x, x') = e^m_\mu(x) e^n_\alpha(x') \delta^s_m. \quad (12) \]

This operator transforms indices of the type \( \alpha \) into indices of the type \( \mu \), in particular,

\[ \tilde{g}_{\mu}^{\alpha}(x, x') \sigma_{\alpha}(x, x') = -\sigma_{\mu}(x, x'), \quad (13) \]

where it was taken into account that the two gradients have opposite directions.

The world function has a dimension of the length squared. In flat space-time

\[ \sigma(x, x') = \frac{1}{2}(x - x')^2. \quad (14) \]

A two-point bitensor (in particular, a biscalar) can be expanded in powers of Synge’s world function and its gradients. It is worth noting that this type of expansion is different from the Taylor’s covariant expansion
of a two-point bitensor. In particular, vanishing of \( \sigma(x, x') \) does not necessarily mean \( x = x' \).

Let us write an appropriate expansion for the Hadamard’s two-point scalar function \( G_H(x, x') \) which is singular in the coincidence limit. For \( D = 4 \) the Hadamard expansion contains two terms singular in \( \sigma \), namely, \( \sigma^{-1} \) and \( \ln \sigma \), the remaining part being regular in \( \sigma \). In higher dimensions one has to add other singular terms, and by dimensionality it is easy to guess that each dimension introduces an additional factor \( \sigma^{-1/2} \). Thus, the Hadamard expansion in \( D = 2d \) dimensions \((d \geq 3/2\) is integer or half-integer\) generically must read

\[
G_H(x, x') = \frac{1}{(2\pi)^d} \left[ \sum_{n=1}^{D} g_n \sigma^{1-n/2} + v \ln \sigma \right],
\]

where \( g_n = g_n(x, x') \), \( v = v(x, x') \) are coefficient two-point scalar functions. The logarithmic term, as will be clear shortly, is present only in even dimensions. Let us show, that in odd dimensions we actually have only odd powers of \( \sigma^{-1/2} \), and in even dimensions — only even powers, that is, an expansion in terms of inverse integer powers of \( \sigma \).

Substituting \((15)\) into \((10)\), in the leading singular order we will have:

\[
2\sigma^{\mu} \partial_{\mu} g_D + g_D(\Box \sigma - D) = 0.
\]

Using the relation

\[
\Box \sigma = D - \frac{\sigma^{\mu} \partial_{\mu} \Delta}{\Delta}
\]

we arrive at the equation

\[
\sigma^{\mu}(2 \partial_{\mu} g_D - g_D \partial_{\mu} \ln \Delta) = 0,
\]

which is equivalent to

\[
2 \partial_{\mu} g_D - g_D \partial_{\mu} \ln \Delta = \left( g^{\mu \nu} - \frac{\sigma^{\mu} \sigma^{\nu}}{2\sigma} \right) h_{\nu},
\]

where \( h_{\nu}(x) \) is an arbitrary regular vector field. The left hand side of this equation is analytic in the coincidence limit, while the right hand side is not, unless \( h_{\nu}(x) = 0 \), so with the normalization condition \([g_D = 1]\) we obtain

\[
g_D = \Delta^{1/2}.
\]

In the next singular order we obtain:

\[
2 \partial_{\mu} g_{D-1} \sigma^{\mu} + g_{D-1} \Box \sigma - (D-1) g_{D-1} = 0.
\]

For \( D = 3 \), this equation holds not for the \( g_{D-1} \), but for \( v \). The Eq. \((21)\) has a solution

\[
g_{D-1} = C \frac{\Delta^{1/2}}{\sigma^{1/4}},
\]

which does not satisfy the required analyticity for \( g_n \) unless \( C = 0 \), and hence

\[
g_{D-1} = 0.
\]

For \( D = 3 \) this means the absence of the logarithmic term. Similarly, considering the equation for \( g_{(D-1-2k)} \), \( k \in \mathbb{N} \) we find

\[
g_{D-1-2k} = 0.
\]

This means that for an even-dimensional space-time the Hadamard Green’s function contains only integer negative powers of \( \sigma \) plus logarithm and a regular part, while in the odd-dimensional case — only half-integer powers of \( \sigma \) plus a regular part.

**B. Dimensional recurrent relation**

For the sequence of Green’s functions in the flat space-time, the one in \( D+2 \) dimension is proportional to the derivative of the Green’s function in the twice preceding dimension \( D \). In particular, in even dimensions the symmetric Green’s function is the derivative of the order \( d - 2 \) of the delta-function:

\[
G^D \sim \delta^{d-2}(-\sigma), \quad \sigma = (x - x')^2/2
\]

and thus, \( G^{D+2} \sim dG^D/d\sigma \). Applying a convenient regularization of the delta function,

\[
\delta((x - x')^2) = \lim_{\varepsilon \to 0} \delta(|(x - x')^2| - \varepsilon^2),
\]

we obtain

\[
G^{D+2} \sim dG^D/d\varepsilon^2.
\]

This relation has a consequence that the Laurent expansion of the Lorentz-Dirac force in terms of \( \varepsilon \) in the even-dimensional Minkowski space has only odd negative powers, and no even terms.

Now consider the Hadamard function in the curved space. Taking into account the Eq. \((24)\), we change notation for \( g_n \) in \((15)\) in even dimensions as follows:

\[
G_H = \frac{1}{(2\pi)^d} \left[ \sum_{k=0}^{d-2} \frac{u_k}{\sigma^{d-1-k}} + v \ln \sigma + w \right],
\]

where \( u_0 = \Delta^{1/2} \) and we also denote \( v = u_{d-1}, \ w = u_d \). Applying the curved space operator \( \Box \) with respect to \( x^\mu \), we obtain from \( \Box G_H = 0 \) the following equation:

\[
\frac{d-2}{\sigma^{d-3}} \left( \frac{\Box u_0}{(d-2)} - 2(\partial_{\mu} u_1 - u_1 \partial_{\mu} \ln u_0) \sigma^{\mu} - 2u_1 \right) + \frac{d-3}{\sigma^{d-2}} \left( \frac{\Box u_1}{(d-3)} - 2(\partial_{\mu} u_2 + u_2 \partial_{\mu} \ln u_0) \sigma^{\mu} - 4u_2 \right) + \ldots + \frac{1}{\sigma^2} (\Box u_{d-2} - 2(\partial_{\mu} u_{d-2} - 2 \partial_{\mu} \ln u_0) \sigma^{\mu}) + \frac{1}{\sigma} (\Box u_{d-1} + 2(\partial_{\mu} u_{d-1} - 2u_{d-1} \partial_\mu \ln u_0) \sigma^{\mu} + (D-2)u_{d-1} + \sigma \Box u_d) + \ln \sigma \Box u_{d-1} = 0.
\]

Each term in the right hand side must vanish independently, which gives the system of recurrent differential equations for \( u_i(x, x') \). Integrating them along the geodesic connecting two points \( x, x' \), one can uniquely
extract \( u_1(x, x') \) through \( u_0(x, x') \). Furthermore, \( u_2 \) is determined through \( u_1 \), etc.

Now we want to compare \( u_1^D(x, x') \) in different dimensions \( D \). First of all we observe that \( u_0^D = \Delta^{1/2} = 1 + 1/2 \left( R_{\alpha\beta}\sigma^\alpha\sigma^\beta + \ldots \right) \) the tensor indices \( \alpha, \beta \) will run different range of values. Keeping this subtlety in mind, we can write

\[
\begin{align*}
\frac{d}{d^\prime} u_0^D &= \frac{d}{d^\prime} u_0^D, \\
\frac{d}{d^\prime} u_1^D &= \frac{d}{d^\prime} u_1^D, \\
\frac{d}{d^\prime} u_2^D &= \frac{d}{d^\prime} u_2^D
\end{align*}
\]

for any \( D, D' \). Now, equating the first line in (29) to zero, we see that the product \( u_1(d - 2) \) is subject to the same master equation in any dimension, so for \( D, D' \geq 5 \) one has

\[
\frac{d}{d^\prime} u_0^D = \frac{d}{d^\prime} u_0^D. 
\]

The second line in (29) gives similarly for \( D, D' \geq 7 \):

\[
\frac{d}{d^\prime} u_2^D = \frac{d}{d^\prime} u_2^D. 
\]

If \( D' = D + 2 \), this relation can be rewritten as

\[
\frac{d}{d^\prime} u_2^{D+2} = \frac{d}{d^\prime} u_2^D. 
\]

For \( u_2^{D+2} \) we obtain

\[
\frac{d}{d^\prime} u_2^{D+2} = \frac{d}{d^\prime} u_2^D. 
\]

Comparing the equation for the \( \sigma^{-1} \) term in \( D \) dimensions and the equations for the \( \sigma^{-2} \) term in \( D' = D + 2 \) dimensions

\[
\frac{d}{d^\prime} u_2^{D+2} = \frac{d}{d^\prime} u_2^D. 
\]

Denote the part of the Hadamard function (28) without the logarithmic term as “direct” (it will correspond to the non-scattered propagation of waves)

\[
G_H = \frac{1}{2\pi^d} \sum_{k=0}^{d-2} \frac{d}{d-1-k}. 
\]

Differentiating \( G_H \) with respect to \( \sigma \) we find that the functions \( u_k, v, w \) are not differentiated over \( \sigma \) we find

\[
(2\pi)^d \frac{d}{d\sigma} G_H = \sum_{k=0}^{d-2} \frac{d}{d-1-k} (d - 1 - k) + v = \sum_{k=0}^{d-2} \frac{d}{d-1-k} (d - 1 - k). 
\]

Taking into account the relation (36), one obtains

\[
\frac{-d}{d^\prime} (2\pi)^d \frac{d}{d\sigma} G^D_H = \sum_{k=0}^{d-2} \frac{d}{d-1-k} (d - 1 - k) + v = \sum_{k=0}^{d^\prime-2} \frac{d}{d^\prime-1-k} (2\pi)^d G^D_H. 
\]

C. Retarded Green’s functions

To define the retarded and advanced Green’s function one has to specify boundary conditions. For a general space-time this is not suggestive, so usually one uses a quasilocal definition proposed by DeWitt and Brehme [14] in the case of four dimensions. We follow the same approach here for arbitrary \( D \).

The prescription consists in constructing the Feynman’s propagator \( G_F \) by means of a shift of \( \sigma \) in the Hadamard expansion onto the upper complex plane \( \sigma \to \sigma + i0 \) and then separating real and imaginary parts

\[
G_F = G(1) - 2iG_{self}. 
\]

This prescription works both for odd and even \( D \), here we will concentrate on \( D \) even. In this case the relations

\[
(\sigma + i0)^{-n} = \mathcal{P} \left( \frac{1}{\sigma^n} - \frac{(-1)^{n-1}i\pi}{(n-1)!} \right) \delta^{(n-1)}(\sigma) 
\]

\[
\ln(\sigma + i0) = \ln|\sigma| + i\pi \theta(-\sigma) 
\]

are useful and we find for the symmetric (self) Green’s function:

\[
G_{self} = -\frac{1}{2} \text{Im} G_F = \frac{1}{4\pi^{d-1}} \times \sum_{m=0}^{d-2} \left( -1 \right)^m u_{d-2-m} \delta(m)(\sigma) - v\theta(-\sigma). 
\]

To construct the retarded and advanced Green’s functions one applies the formal global definition of the Heaviside function \( \theta[\Sigma(x), x'] \) which is equal to one if \( x' \) is in the past of any space-like hypersurface \( \Sigma(x) \) containing \( x \), and zero otherwise. Then the symmetric Green’s function is split into the sum of the retarded and advanced ones like in the flat space:

\[
G_{ret}(x, x') = 2\theta[\Sigma(x), x']G_{self}(x, x'), 
\]

\[
G_{adv}(x, z) = 2\theta[x', \Sigma(x)]G_{self}(x, x'). 
\]

The retarded and advanced Green’s functions satisfy the inhomogeneous wave equation:

\[
\square G(x, x') = -\frac{\delta^D(x - x')}{g^{1/4}(x)g^{1/4}(x')(d - 2)!}, 
\]

where a symmetrization of the delta-density is performed. Following the above prescription one finds the retarded Green’s function in terms of the same \( u_k, v \) (note that the regular term \( w \) does not enter):

\[
G_{ret} = \frac{1}{2\pi^{d-1}} \Theta(x', \Sigma(x)) \times \sum_{m=0}^{d-2} \left( -1 \right)^m u_{d-2-m} \delta(m)(\sigma) - v\theta(-\sigma). 
\]

Here the sum contains terms proportional to derivatives of the delta function constitute the direct part of
the retarded function with the support on the “light cone” $\sigma = 0$:

$$G_{\text{dir}} = \frac{1}{2(2\pi)^{d-1}} \Theta(\sigma) \sum_{m=0}^{d-2} \frac{(-1)^m u_{d-2-m} \delta^{(m)}(\sigma)}{m!}.$$ (48)

The $v$-term is localized inside the light cone and represents a tail resulting from the scattering of waves on the curvature.

Using the recurrent relation found for the Hadamard function in the previous section, we obtain for the retarded Green’s function a similar relation

$$\frac{\partial G_D^\text{ret}}{\partial \sigma} = -2(1-d) G_{\text{dir}}^{D+2}.$$ (49)

This relation can be rewritten using the regularized delta-functions in terms of differentiation with respect to the point-splitting distance $\epsilon = \sqrt{2}/2$,

$$\delta(\sigma) = \lim_{\epsilon \to +0} \delta(|\sigma| - \epsilon).$$ (50)

Obviously we will have

$$G_{\text{dir}}^{D+2} = \frac{1}{2\pi(1-d)} \frac{\partial G_D^\text{ret}}{\partial \epsilon}.$$ (51)

D. Vector field

Now consider the massless vector field $A_\mu$ in $D$ dimensions governed by the usual Maxwell action quadratic in $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The homogeneous Maxwell equation

$$F_{\mu\nu}^{\text{ret}} = 0$$ (52)

in terms of $A_\mu$ in the gauge $A_{\mu}^{\text{ret}};\nu = 0$ reads

$$\Box A_\mu - R^{\mu}_{\nu\lambda} A_\nu = 0,$$ (53)

where $R^{\mu}_{\nu\lambda}$ is the Ricci tensor (we use the definition as $R^{\mu\nu} = g_{\rho\sigma} R^{\rho\sigma\mu\nu}$). Similarly to the scalar case, we define the Hadamard function as a solution to the equation

$$\Box G_{\mu\alpha}(x, x') - R^{\mu}_{\nu\lambda}(x) G_{\nu\alpha}(x, x') = 0.$$ (54)

We look for a solution in the form ($D$ even)

$$G^{\mu\alpha}_H = \frac{1}{(2\pi)^d} \left( \frac{u^{\mu\alpha}_0}{\sigma^2} + \frac{u^{\mu\alpha}_1}{\sigma^3} + \frac{u^{\mu\alpha}_2}{\sigma^4} + \frac{u^{\mu\alpha}_3}{\sigma^5} + \frac{u^{\mu\alpha}_4}{\sigma^6} + \frac{u^{\mu\alpha}_5}{\sigma^7} + \cdots \right),$$ (55)

where all the quantities $u^{\mu\alpha}_k(x, x')$, $v^{\mu\alpha}(x, x')$ are two-point tensors regular in the coincidence limit. Now the lowest order term is

$$u^{\mu\alpha}_0 = \Delta^{1/2} g^{\mu\alpha}.$$ (56)

By the same reasoning as in the scalar case we find relations between $u^{\mu\alpha}_k$ in different dimensions and establish the differentiation rule essentially equivalent to (51).

III. RADIATION REACTION

A. Lorentz-Dirac force

Consider a point particle of mass $m$ interacting with massless scalar and vector fields ($q, e$ being the corresponding couplings). The particle world-line will be denoted as $x^\mu = z^\mu(s)$ where $s$ is the interval. The action consists of the field part

$$S_F = -\frac{1}{\Omega_{D-2}} \int \left( (\nabla \phi)^2 + \frac{1}{4} F^2 \right) \sqrt{-g} d^D x,$$ (57)

and the particle term

$$S_p = -m_0 \int (1 + q \phi) \sqrt{-\gamma} d\tau - e \int A_\mu \dot{z}^\mu d\tau,$$ (58)

where $\phi$ and $A^\mu$ are taken at the point $x = z(\tau)$. The field equations read

$$\Box \phi = \Omega_{D-2} \rho,$$ (59)

$$F^{\mu\nu} = \Omega_{D-2} J^\mu,$$ (60)

while the equation of motion of a particle is

$$m_0 (1 + q \phi) \ddot{z}^\mu = -e F^{\mu\nu} \dot{z}_\nu - m_0 q \Pi^{\mu\nu} \dot{z}_\nu,$$ (61)

where the covariant differentiation along the world line is denoted by dot, $\phi = \partial_\tau \phi$, and $\Pi^{\mu\nu} = g^{\mu\nu} + \dot{z}^\mu \dot{z}^\nu$ is a projector on the subspace orthogonal to the world line. The natural parameter $\tau$ is assumed, so that $\dot{z}^2 = -1$.

To get the radiation reaction force we insert at the right hand side of the Eq. (51) the retarded solution of the wave equations [54,60]

$$\phi(x) = \int G_{\text{ret}}(x, x') \rho(x') \sqrt{-g} g_{D-2} d\tau,$$ (62)

$$A^\mu(x) = \int G_{\text{ret}}^{\mu\alpha}(x, x') j_\alpha(x') \sqrt{-g} g_{D-2} d\tau,$$ (63)

with the sources

$$\rho(x') = m_0 q \int \frac{\delta(x' - z(\tau))}{\sqrt{-g}} d\tau,$$ (64)

$$j_\mu(x') = e \int \frac{\delta(x' - z(\tau)) \partial_\alpha \dot{z}^\alpha(\tau)}{\sqrt{-g}} d\tau.$$ (65)

The resulting Lorentz-Dirac self-force will be by definition the right hand side of the equation of motion in absence of external fields

$$m_0 \ddot{z}^\mu = f_L^{\mu\nu},$$ (66)

where the mass parameter at the left hand side is constant (contrary to another frequently used in presence of the scalar field definition of the effective mass in [5]). The Lorentz-Dirac force consists of a scalar part

$$f_{\text{sc}}^{\mu} = -m_0 q \Pi^{\mu\nu}(\phi_\nu + \phi \dot{z}_\nu),$$ (67)

and a vector part

$$f_{\text{em}}^{\mu} = -e F_{\nu\mu} \dot{z}^\nu.$$ (68)
According to the above decomposition of the retarded Green’s function into the direct part and the tail part, we will have a similar decomposition for the Lorentz-Dirac force. Then for the direct part we obtain the following differentiation rule relating the values of the force in twice neighboring dimensions:

\[ f_{D+2}^{\mu \text{dir}} = -\frac{1}{D-1} \frac{\partial f_{D}^{\mu \text{dir}}(s, \mathcal{E})}{\partial \mathcal{E}}. \]  

The limit \( \mathcal{E} \to +0 \) has to be taken after the differentiation. The direct force is due to the light cone of the theta function in the tail term \( \psi \theta(-\sigma) \). In the scalar case this contribution vanishes, but in the electromagnetic case an extra local term arises:

\[ f_{\text{loc}}^{\mu} = e^2 \left( \{\psi_{\sigma\alpha}\} \hat{z}_\alpha - \{\psi_{\sigma\alpha}\} \hat{z}_\mu \right) \varepsilon^\nu \hat{z}_\nu, \]

where the coincidence limit \( \{\psi_{\sigma\alpha}\} \) depends on the dimension. In the cases for \( D = 4 \) it is given by \( \varepsilon \). The remaining contribution from the tail term will have the form of an integral along the past half of the particle world line.

### B. Divergences

The direct part of the Lorentz-Dirac force contains divergences. To separate the divergent terms one can use the decomposition of the retarded potential suggested in the case of four dimensions by Detweiler and Whiting. In higher even dimensions we can follow essentially the same procedure. We define the “singular” part \( G_S \) of the retarded Green’s function as the sum of the symmetric part (“self”) and the tail function \( \psi \) as follows

\[ G_S(x, x') = G_{\text{self}}(x, x') + \frac{1}{4(2\pi)^{d-1}} v(x, x') = \]

\[ = G_{\text{self dir}}(x, x') + \frac{1}{4(2\pi)^{d-1}} v(x, x') \theta(\sigma). \]  

Here the direct part of the self function means its part without the tail \( v \)-term. Taking into account \( \nabla v = 0 \), it is clear that \( G_S \) satisfies the same inhomogeneous equation as \( G_{\text{self}} \). The \( v \)-term in the second line of the Eq. (72) is localized outside the light cone. Therefore the corresponding field (for instance, scalar), at an arbitrary point \( x \) will be given by

\[ \phi_S(x) = \phi_{\text{self dir}} + \frac{m_0 g \Omega}{4(2\pi)^{d-1}} \int_{\tau_{\text{ret}}}^\tau v(x, z(\tau)) d\tau, \]

where the retarded and advanced proper time values \( \tau_{\text{ret}}(x), \tau_{\text{adv}}(x) \) are the intersection points of the past and future light cones centered at \( x \) with the world line.

The remaining part of the Green’s function

\[ G_R(x, x') = G_{\text{ret}}(x, x') - G_S(x, x') \]

satisfies a free wave equation and is regular.

Let us investigate divergent terms in the Lorentz-Dirac force and show that they can be eliminated adding counterterms into the action (in flat space this was demonstrated for \( D = 6 \) by Kosyakov and generalized to arbitrary \( D \) in [20]). Inserting (72) into the Lorentz-Dirac force defined on the world-line \( x = z(\tau) \), we observe that the integral contribution from the tail term vanishes and so the divergent part is entirely given by \( \phi_{\text{self dir}} \). For the delta-functions we use the point-splitting regularization [50]. All divergent terms will then arise as negative powers of \( \varepsilon \).

To prove the existence of the counterterms we consider the interaction term in the action substituting the field as the \( S \)-part of the retarded solution to the wave equation. In the scalar case we will have:

\[ S_S = \frac{1}{2} \int G_S(x, x') \rho(x) \rho(x') \sqrt{g(z) g(z')} \, dx \, dx', \]

where a factor one half is introduced to avoid double counting when self-interaction is considered. Substituting the currents we get the integral over \( d\tau, d\nu \):

\[ S_S = \frac{1}{2} m_0^2 \varepsilon^2 \Omega \int G_S(z(\tau), z(\tau')) d\tau d\nu'. \]

Since the Green’s function is localized on the light cone (by virtue of (72)), we can expand the integrand in terms of the difference \( t = \tau - \nu' \) around the point \( z(\tau) \):

\[ S_S \sim \int d\tau \int \sum_{k,l} B_{kl}(\tau) \delta(k)(t^2 - \varepsilon^2)^l \, dt. \]

Here the coefficients \( B_{kl}(z) \) depend on the curvature, while the delta-functions are flat: \( \delta(k)(t^2 - \varepsilon^2) \) (the derivatives with respect to the argument are understood). By virtue of parity, the integrals with odd \( l \) vanish, so only the odd inverse powers of \( \varepsilon \) will be present in the expansion. Moreover, if we know the divergent terms in some dimension \( D \), we can obtain all divergent terms in \( D + 2 \) except for 1/\( \varepsilon \) term. The linearly divergent term corresponds to \( l = 2k \). The integral is equal to

\[ \int_0^\infty \delta(k)(t^2 - \varepsilon^2)^l \, dt = \frac{1}{2k+1} \frac{1}{\varepsilon}. \]

In four dimensions this term is unique. Applying our recurrence chain we obtain the inverse cubic divergence in six dimensions and calculate again the linearly divergent term. Thus in \( D = 2d \) dimensions we will get \( d - 1 \) divergent terms from which \( d - 2 \) can be obtained by the differentiation of the previous-dimensional divergence, and the linearly divergent will be new. This linearly divergent self-action term in the action will have generically the form

\[ S_{S_S}^{(-1)} \sim \frac{1}{\varepsilon} \int \sum_{k,n} \frac{(-1)^k(2k - 1)!!}{2^{k+1}} B_{k,2k}(\tau) d\tau. \]

Here the coefficient functions are obtained taking the coincidence limits of the two-point tensors involved in
the expansion of the Hadamard solution. They actually depend on the derivatives of the world line embedding function \( z(\tau) \) as well as the curvature terms taken on the world line:

\[
B_{k,2k}(\tau) = B_{k,2k}(\dot{z}, \ddot{z}, ..., R(z(\tau)), R_{\mu\nu}(z(\tau)), ...).
\]

Finally we have to rewrite the expression in the curved reparametrization-invariant form. This amounts to the replacement:

\[
\dot{z}(\tau) \rightarrow Dz \equiv \frac{1}{\sqrt{-\dddot{z}^2}} \frac{dz}{d\tau},
\]

\[
\ddot{z}(\tau) \rightarrow D^2 z, \quad \ldots ,
\]

\[
d\tau \rightarrow \sqrt{-\dddot{z}^2} d\tau,
\]

where \( D \) is the covariant derivative along the world line.

The vector case is technically the same, now one has to expand in powers \( t \) in the integrand of

\[
S^S = \frac{1}{2} \int G^S_{\mu\alpha}(x,x') j^\mu(x) j^\alpha(x') \sqrt{g(z\cdot g(z')} \ dx \ dx'.
\]

From this analysis it follows that in any dimension the highest divergent term can be absorbed by the renormalization of the mass as in the generating four dimensional case. To absorb the remaining \( d-2 \) divergences one has to add to the initial action the sum of \( d-2 \) counterterms depending on higher derivatives of the particle velocity.

### C. Finite reaction terms

To obtain local part of the finite radiation reaction terms one can used the recurrent procedure from \( D \) to \( D + 2 \) dimensions. So if one performs expansions in terms of \( t = \tau - \tau' \) in the integrals up to sufficiently high order, one can find the generating expression to get finite reaction terms in higher dimensions. In addition one obtains the integral tail term which is essentially the same (in terms of function \( v \)) in all dimensions.

### IV. FOUR DIMENSIONS

Here we apply our formalism to rederive the radiation reaction force in four dimensions. This approach is much simpler than the non-local DeWitt-Brehme type calculations (integration of the stress-tensor flux over the world-tube surrounding the particle) and gives a more transparent understanding of the renormalization involved. Brief account of our approach in four dimensions was presented in [19]. Calculation in four dimensions may also serve as a starting point for higher-dimensional problems using the recurrent relations found above.

#### A. Scalar force

In four dimensions the scalar retarded Green’s function contains a single direct term localized on the light cone and a tail term:

\[
G_{\text{ret}} = \frac{1}{4\pi} \theta[\Sigma(x), x'] \left[ \Delta^{1/2} \delta(\sigma) - v \theta(-\sigma) \right].
\]

The retarded solution for the scalar field reads

\[
\phi_{\text{ret}}(x) = m_0 q \int_{-\infty}^{\tau_{\text{ret}}(x)} \left[ -\Delta^{1/2} \delta(\sigma) + v \theta(-\sigma) \right] d\tau'.
\]

Differentiating this expression we obtain \( \phi_\nu \) on the world line:

\[
\phi_\nu(z(\tau)) = m_0 q \int_{-\infty}^{\tau} \left[ -\Delta^{1/2} \delta'(\sigma) \sigma_\nu - \Delta^{1/2} \delta(\sigma) - v \delta(\sigma) \sigma_\nu + v_\nu \right] d\tau',
\]

where integration is performed along the past history of the particle. All the two-point functions are taken on the world-line at the points \( x = z(\tau) \) (“observation” point) and \( z' = z(\tau') \) (“emission” point).

To compute local contributions from the terms proportional to delta-functions and its derivative, it is enough to expand the integrand in terms of the difference \( s = \Delta \tau = \tau - \tau' \) around the point \( z(\tau) \). The Taylor (covariant) expansion of the fundamental biscalar \( \sigma(z(\tau), z(\tau')) \) is given by [27]:

\[
\sigma(z(\tau), z(\tau')) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k \sigma(\tau, \tau)(\tau - \tau')^k,
\]

where \( D \) is a covariant derivative along the world-line:

\[
D^2 \sigma \equiv \sigma_\alpha \partial_\alpha, \quad D^2 \sigma \equiv \sigma_\alpha z^\alpha + \sigma_\alpha z^\alpha,
\]

etc. Such an expansion exists since the difference \( s = \tau - \tau' \) is a two-point scalar itself: this is the integral from the scalar function \( (\delta(\sigma)) \) over the world-line from \( z(\tau) \) to \( z(\tau') \). Taking the limits and using \( \dddot{z}(\tau) = -1 \), we find:

\[
\sigma(s) = -\frac{s}{2} - \frac{\dddot{z}^2}{2} + O(s^3).
\]

To obtain an expansion of the derivative of \( \sigma \) over \( z^\mu(\tau) \) one can expand \( \sigma^\mu(\tau, \tau - s) \) in powers of \( s \). This quantity transforms as a vector at \( z(\tau) \) and a scalar at \( z(\tau') \), this

\[
\sigma^\mu(s) = s \left( \dddot{z}^\mu - \dddot{z}^\mu \frac{s^2}{2} + \dddot{z}^\mu \frac{s^2}{6} \right) + O(s^4),
\]

where the index \( \mu \) corresponds to the point \( z(\tau) \): \( \sigma^\mu = \partial \sigma(z, z')/\partial z_\mu \). Recall that the initial Greek indices correspond to \( z(\tau') \). The expansion of \( \delta(-\sigma) \) will read:

\[
\delta(-\sigma) = \delta(s^2/2) + s^4 \frac{\dddot{z}^2}{2} \delta'(s^2/2) + \ldots
\]

where the derivative of the delta-function is taken with respect to the full argument. Since the most singular term is \( \Delta^{1/2} \delta'(\sigma) \sigma_\mu \), the maximal order giving the non-zero result after the integration is \( s^3 \). (Note, that in order to use our dimensional recurrent relations to obtain a reaction force in higher dimensions we should...
perform an expansion up to higher orders in $s$.) Thus, with the required accuracy, $\delta(\sigma) = 2\delta(s^2)$, and all the integrals for the delta-derivatives are the same as in the flat space-time. This allows us to use the same regularization of the delta-functions with double roots $\Delta^{1/2}/2$ by the point-splitting. Expanding the biscalar $\Delta^{1/2}$ and its gradient at $z$ we have:

$$\Delta^{1/2} = 1 + \frac{s^2}{12} R_{\sigma\tau} \zeta^\sigma \zeta^\tau, \quad \partial_\nu \Delta^{1/2} = \frac{s}{6} R_{\nu\tau} \zeta^\tau. \quad (85)$$

Combining all the contributions we obtain finally for the field strength:

$$\phi_\nu = m_0 q \left( \frac{1}{2\varepsilon} \zeta_\nu - \frac{1}{3} \zeta_\nu \frac{1}{6} R_{\nu\tau} \zeta^\tau + \frac{1}{12} R_\nu \zeta^\nu - \frac{1}{12} R_{\gamma\delta} \zeta^\gamma \zeta^\delta \zeta^\nu + \int_\nu v_\nu d\tau' \right), \quad (86)$$

where the first term is divergent. Note that the terms $-1/6 R_{\mu\nu} \zeta^\mu \zeta^\nu + 1/12 R \zeta^\nu$ are annihilated by the projector $\Pi^{\mu\nu}$. The retarded field $\phi$ itself is also singular on the world-line:

$$\phi(z) = -\frac{m_0 q}{\varepsilon} - \int_{-\infty}^\nu v d\tau'. \quad (87)$$

Collecting all the orders [86][87], we obtain the following expression for the scalar part of the self-force [87]:

$$f_{\text{s.c.}}^{\mu} = m_0^2 q^2 \left[ \Pi^{\mu\nu} \left( \frac{1}{3} \zeta_\nu + \frac{1}{6} R_{\nu\tau} \zeta^\tau - \int_\nu v_\nu d\tau' \right) \right.$$  

$$\left. + \zeta^\mu \left( \frac{1}{2\varepsilon} - \int_{-\infty}^\nu v d\tau' \right) \right]. \quad (88)$$

**B. Vector force**

Now consider the vector contribution. The Hadamard function in four dimensions is

$$G_{H,\mu\nu} = \frac{1}{(2\pi)^2} \left( \frac{u_{\nu\alpha}}{\sigma} + v_{\alpha\nu} \ln \sigma + w_{\alpha\nu} \right), \quad (89)$$

where $u_{\mu\nu}(x, x'), \ v_{\mu\nu}(x, x')$ and $w_{\mu\nu}(x, x')$ are bivectors. One finds $u_{\mu\alpha} = \tilde{\eta}_{\mu\alpha} \Delta^{1/2}$. We look for power expansions

$$v_{\mu\alpha} = \sum_{n=0}^{\infty} v_{\mu\alpha}^{(n)} \sigma^n, \quad w_{\mu\alpha} = \sum_{n=0}^{\infty} w_{\mu\alpha}^{(n)} \sigma^n, \quad (90)$$

we find the following equation for $v_{\mu\alpha}^{(0)}$:

$$2v_{\mu\alpha}^{(0)} + (2v_{\mu\alpha}^{(0)} + v_{\mu\alpha}^{(0)} \Delta^{-1} \Delta_{\mu}) \sigma^{\nu\mu} = -\Box u_{\mu\alpha} + R_{\mu\nu} \nu_{\nu\alpha}. \quad (91)$$

Substituting the covariant derivative of the two-point tensors, we obtain the following coincidence limit for $v_{\mu\alpha}$:

$$[v_{\mu\alpha}^{(0)}] = [v_{\mu\alpha}] = \frac{1}{2} \bar{g}_\mu^\alpha \left( R_{\alpha\beta} - \frac{1}{6} g_{\alpha\beta} R \right). \quad (92)$$

The symmetric Green’s function is given by

$$G_{\text{self}}^{\mu\nu} = \frac{1}{8\pi} \left[ u_{\mu\alpha} \delta(\sigma) - v_{\mu\alpha} \theta(-\sigma) \right], \quad (93)$$

while the retarded one is

$$G_{\text{ret}}(x, x') = 2\theta(\Sigma) G_{\text{self}}(x, x'). \quad (94)$$

The retarded vector-potential is

$$A_{\mu}^{\text{ret}}(x) = -4\pi \epsilon \int c_{\mu\alpha}(x, z(\tau')) \zeta^\alpha d\tau' \left. \right|_{s_{\text{ret}}(x)}^{s_{\text{ret}}(z)} = e \int_{-\infty}^\nu \left[ -u_{\mu\alpha} \delta(\sigma) + v_{\mu\alpha} \theta(-\sigma) \right] \zeta^\alpha d\tau'. \quad (95)$$

The field strength on the world-line $x = z(s)$ will be

$$f_{\mu\alpha}^{\text{ret}}(z(s)) = e \int_{-\infty}^s \left[ u_{\mu\alpha;\nu} \delta(\sigma) + u_{\mu\alpha} \sigma_{\nu} \delta(\sigma) + v_{\nu\alpha,\mu} + v_{\nu\alpha} \sigma_{\nu} \delta(\sigma) - \{\mu \leftrightarrow \nu\} \right] \zeta^\alpha d\tau', \quad (96)$$

so the Lorentz-Dirac force will read

$$f_{\text{em}}^{\mu}(s) = -e F^{\mu\nu}(z(s)) \zeta^\nu(s). \quad (97)$$

Substituting all the expansions

$$\tilde{g}_{\mu\alpha}(\tau, s) \zeta^\alpha(\tau - s) = -1 - s^2 \frac{\tilde{g}^2}{2} + O(s^3)$$

$$u_{\mu\nu \alpha}(\tau) \zeta^\alpha(\tau) = -s \frac{R_{\mu\nu \alpha} \zeta^\alpha}{2} + O(s^2)$$

$$u_{\mu\nu \alpha \nu}(\tau) \zeta^\alpha(\tau - s) = \frac{8}{6} R_{\mu\nu \alpha \nu} \zeta^\alpha + O(s^2)$$

$$v_{\nu\alpha \sigma}(\tau) \zeta^\alpha(\tau - s) = -s \tilde{g}^\mu + \frac{2}{2} \zeta^\mu + s^2 \left[ -1 - \frac{1}{12} R_{\mu\nu \alpha \nu \sigma} \zeta^\alpha \zeta^\mu - \frac{1}{12} \zeta^\alpha \zeta^\mu \right] + O(s^4)$$

$$u_{\nu\alpha \sigma \nu \zeta^\alpha}(\tau - s) = -s \tilde{g}^\mu + \frac{2}{2} \zeta^\mu + s^2 \left[ -1 - \frac{1}{12} R_{\mu\nu \alpha \nu \sigma} \zeta^\alpha \zeta^\mu - \frac{1}{12} \zeta^\alpha \zeta^\mu \right] + O(s^4)$$

$$v_{\nu\alpha \sigma \nu \zeta^\alpha}(\tau - s) = -s \tilde{g}^\mu + \frac{2}{2} \zeta^\mu + s^2 \left[ -1 - \frac{1}{12} R_{\mu\nu \alpha \nu \sigma} \zeta^\alpha \zeta^\mu - \frac{1}{12} \zeta^\alpha \zeta^\mu \right] + O(s^4)$$

after integration we find

$$f_{\mu\nu}^{\text{ret}} = e^2 \left[ -\frac{1}{2\varepsilon} \zeta^\mu + \Pi^{\mu\nu} \left( \frac{2}{3} \zeta^\nu + \frac{1}{3} R_{\nu\alpha \sigma} \zeta^\alpha \right) + \zeta^\nu(\tau) \int_{-\infty}^\nu v_{\nu \alpha \nu - v_{\nu \alpha \nu} \mu} \zeta^\alpha d\tau' \right]. \quad (99)$$

**C. Renormalization and total force**

Combining the divergent parts of the expressions [90] and [97] into the unique mass-renormalization term, we get

$$m - m_0 = \frac{1}{2\varepsilon}(e^2 - m_0^2q^2). \quad (100)$$
Clearly, this is just the flat-space result. Note that under the condition:

\[ m_0|q| = |e|, \]  
(101)

the model is free from singularities and does not require mass-renormalization. The final form of curved space Lorentz-Dirac equation in this case will read:

\[
m(\tau)\mathbf{z}^\mu = e^2 \left[ \Pi^{\mu\nu} \left( \frac{\partial}{\partial \tau} + \frac{1}{2} R_{\nu\alpha\gamma} \mathbf{z}^\alpha - \int_{-\infty}^{\tau} v_{\nu} d\tau' \right) + \frac{\partial}{\partial \tau} \left( \frac{v^\mu_{\alpha\nu} - v^\nu_{\alpha\mu}}{2} \right) - \int_{-\infty}^{\tau} \left( \frac{v^\mu_{\alpha\nu} - v^\nu_{\alpha\mu}}{2} \right) d\tau' \right],
\]

(102)

where

\[ m(\tau) = m + \int_{-\infty}^{\tau} v d\tau' \]

is the \( \tau \)-dependent “mass”.

This result coincides with the sum of the DeWitt-Brehme-Hobbs and the corresponding scalar equations \[17\] which was previously obtained within the DeWitt-Brehme non-local approach.

## V. SIX DIMENSIONS

The next even-dimensional case is \( D = 6 \). The vector reaction force in the flat space-time was previously given by Kosyakov \[20\]. Here we give the curved space treatment and also add the scalar field.

### A. Scalar force

Using the general formalism of Sec. \[11\] we can present the Hadamard Green’s function of the scalar field in six dimensions as follows

\[
G_H = \frac{1}{(2\pi)^3} \left( \frac{\Delta^{1/2}}{\sigma^2} + u + v \ln \sigma + w \right).
\]

(103)

The corresponding retarded Green’s function will be

\[
G_{\text{ret}} = -\frac{\delta(\Sigma)}{8\pi^2} \left( \frac{\Delta^{1/2}}{\sigma^2} - u\delta(\sigma) + v\theta(-\sigma) \right).
\]

(104)

so that the retarded solution is

\[
\phi_{\text{ret}}(x) = \frac{1}{\tau_{\text{ret}}(x)} \int_{-\infty}^{\tau_{\text{ret}}} \left( \frac{\Delta^{1/2}}{\sigma^2} - u\delta(\sigma) + v\theta(-\sigma) \right) d\tau'.
\]

(105)

where \( \sigma = \sigma(x, z, \tau) \).

We have to expand the two-point scalars \( \Delta^{1/2}(x, z), u(x, z), v(x, z) \) around \( x \) and then set \( x = z(\tau), z = z(\tau') \). To get all the local terms explicitly it is sufficient to expand \( u(x, z) \) up to the second order in \( \sigma^\mu \) keeping only the leading term (coincidence limit) for \( v(x, z) \).

The defining equations are:

\[
\Box \Delta^{1/2} - 2u_\mu \sigma^\mu - u \left( 2 - \frac{\Delta_\mu \sigma^\mu}{\Delta} \right) = 0,
\]

(106)

\[
u + \frac{2v_\mu \sigma^\mu + v \left( 4 - \frac{\Delta_\mu \sigma^\mu}{\Delta} \right) + \sigma \Box w = 0,
\]

(107)

\[
u = 0.
\]

(108)

Note that the representation of the Hadamard function in inverse powers of \( \sigma \) is non-unique, but this amounts only to a redefinition of the regular part \( w \) which does not contribute to the retarded solution

\[
\phi_{\text{ret}}(x) = -\frac{2\pi^2}{3} m_q \int G_{\text{ret}}(x, x')(x') dx',
\]

(109)

where

\[
G_{\text{ret},\mu} = \frac{1}{8\pi^2} \left[ \Delta^{1/2} \delta^{\mu}(\sigma)\sigma_\nu + \Delta^{1/2} \delta'(\sigma) - u\delta(\sigma)\sigma_\nu - w\theta(\sigma)\sigma_\nu \right] + \frac{1}{\delta(\Sigma)} \left( \frac{\Delta^{1/2}}{\sigma^2} - u\delta(\sigma) + v\theta(-\sigma) \right).
\]

(110)

The subsequent calculations consist in expanding two-point functions into powers of \( \sigma \), which are now needed up to higher orders than in the four-dimensional case since the Green’s function has higher degree of singularity. In particular, we get

\[
\Delta^{1/2} = 1 + \frac{1}{12} R_{\alpha\beta} \sigma^\alpha \sigma^\beta - \frac{1}{24} R_{\alpha\beta\gamma\delta} \sigma^\alpha \sigma^\beta \sigma^\gamma \sigma^\delta + \left( \frac{1}{288} - \frac{1}{360} - \frac{1}{60} \right) R_{\alpha\beta\gamma\delta} R_{\mu\nu\rho\tau} \sigma^\alpha \sigma^\beta \sigma^\gamma \sigma^\delta.
\]

(111)

Also, an expansion of the argument of delta-functions will be

\[
\sigma = \frac{1}{2} \sigma^\mu \sigma_\mu = \frac{1}{2} \left( -\tau^2 - \frac{1}{12} \bar{z}^2 \tau^4 + \frac{1}{12} (\bar{z} \cdot \bar{z}) \tau^5 \right),
\]

\[
\delta(\sigma) = \delta(-\tau^2/2) + \delta'(-\tau^2/2) \left( -\frac{1}{24} \tau^4 + ... \right).
\]

(112)

Omitting further calculations, we present the final result for the scalar Lorentz-Dirac-DeWitt-Brehme force:

\[
m_0 \bar{z}^\mu = m^2 q^2 \left( f_{\text{flat}} + f_{\text{curv, div}} + f_{\text{curv, fin}} + f_{\text{tail}} \right),
\]

where the “flat-space” part is given by

\[
f_{\text{flat}} = \frac{1}{6\varepsilon^3} \bar{z}^\mu + \frac{1}{\varepsilon} \left( -\frac{1}{8} \bar{z}^2 \bar{z}_\mu + \Pi^{\mu\nu} \frac{1}{24} \bar{z}^2 \bar{z}_\nu \right) \frac{1}{\varepsilon^3} \left( \bar{z}^2 \bar{z}_\nu + \frac{1}{45} \bar{z}^2 \bar{z}_\nu \right).
\]

(113)

It has the same form as in the Minkowski space, but the dots denote covariant derivatives along the worldline. It includes two divergent terms proportional to \( \varepsilon^{-3} \) and \( \varepsilon^{-1} \), and it is the only part of the total force which survives in the flat space limit.
The divergent part induced by the curvature contains only $\varepsilon^{-1}$ term and is given by

$$f_{\text{curv div}}^\mu = -\frac{1}{72\varepsilon} \Pi^{\mu\nu} \left( 2R_{\nu\alpha} \dot{z}^\alpha + 2R_{\nu\alpha\beta} \dot{z}^\alpha \dot{z}^\beta - R_{\alpha\beta\nu\mu} \dot{z}^\alpha \dot{z}^\beta - R_{\alpha\beta\nu\mu} \dot{z}^\alpha \dot{z}^\beta + R_{\nu\mu} + R_{\nu\nu} \right).$$

(114)

The leading divergence $\varepsilon^{-3}$ is absorbed by the renormalization of mass,

$$m = m_0 - \frac{m_0^2 \varepsilon^2}{6\varepsilon^3}. \quad (115)$$

it is not affected by the curvature. To eliminate the second divergence $\varepsilon^{-1}$ one has to introduce the following counterterm into the action:

$$S^\infty_c = -\frac{\kappa_0}{72} \int \left( \frac{3}{2} (\tilde{\Omega} \mathbf{z})^2 + (\hat{R} - R^{\alpha\beta} Dz^\alpha Dz^\beta) \right) \sqrt{-g} d\tau. \quad (116)$$

The bare coupling constant $\kappa$ entering the counterterm is renormalized to

$$\kappa = \kappa_0 - \frac{m_0 \mu^2}{\varepsilon}. \quad (117)$$

Non-vanishing $\kappa$ leads to rigid particle dynamics investigated in a number of papers (see [11] for the references).

The local part of the finite force induced by the curvature depends both on the Ricci and the Riemann tensor, it does not vanish therefore in the vacuum region of the space-time:

$$f^{\mu}_{\text{curv fin}} = \Pi^{\mu\nu} \left[ \left( -\frac{1}{45} R_{\nu\alpha\beta\gamma} \dot{R} + \frac{1}{36} R_{\nu\lambda\rho} R_{\mu\gamma\nu} - \frac{1}{72} R_{\nu\alpha\beta\gamma} R_{\mu\nu\rho\gamma} - \frac{1}{20} R_{\nu\rho\beta\nu\gamma} - \frac{1}{20} R_{\nu\alpha\beta\nu\gamma} + \frac{1}{30} R_{\nu\alpha\beta\nu\gamma} \right) \dot{z}^\alpha \dot{z}^\beta \dot{z}^\gamma + \left( \frac{1}{72} R_{\nu\alpha\beta\gamma} R_{\mu\nu\rho\gamma} + \frac{1}{72} R_{\nu\rho\beta\nu\gamma} + \frac{1}{72} R_{\nu\alpha\beta\nu\gamma} \right) \dot{z}^\alpha \dot{z}^\beta \dot{z}^\gamma \right]. \quad (118)$$

In the case of the geodesic motion the last four lines in this expression vanish. The non-vanishing part is quadratic in curvature.

The non-local term is given by

$$f^{\mu}_{\text{tail}} = -\frac{1}{3} \left( \Pi^{\mu\nu} \int_{-\infty}^{\tau} v_{\nu} d\tau' + \dot{z}^\mu \int_{-\infty}^{\tau} v d\tau' \right). \quad (119)$$

B. **Vector field**

In six dimensions the Hadamard function is

$$G^{\alpha\alpha}_{\text{flat}} = \frac{1}{(2\pi)^3} \left[ \frac{\Delta^{1/2} \rho^{\alpha\alpha}}{\sigma^2} + \frac{u^{\alpha\alpha}}{\sigma} + v^{\alpha\alpha} \ln \sigma + w^{\alpha\alpha} \right]. \quad (120)$$

while the retarded Green’s function reads:

$$G_{\text{ret}} = -\frac{1}{8\pi^2} \theta(x', \Sigma(x)) \times$$

$$\times \left[ \Delta^{1/2} \tilde{g}^{\alpha\alpha} \delta'(\sigma) - u^{\alpha\alpha} \delta(\sigma) + v^{\alpha\alpha} \theta(-\sigma) \right]. \quad (121)$$

The retarded potential will be

$$A_{\text{ret}}^\nu(x) = \frac{\epsilon}{3} \int_{-\infty}^{\tau_{\text{ret}}(x)} \left[ \Delta^{1/2} \tilde{g}^{\alpha\beta} \delta'(\sigma) - u^{\alpha\beta} \delta(\sigma) + v^{\alpha\beta} \theta(-\sigma) \right] \dot{z}^\alpha d\tau'. \quad (122)$$

Omitting the details of the calculation we present the final result in the form

$$m_0 \ddot{z}^\mu = e^2 \left( f^{\mu}_{\text{flat}} + f^{\mu}_{\text{curv div}} + f^{\mu}_{\text{curv fin}} + f^{\mu}_{\text{tail}} \right). \quad (123)$$

where again the “flat” part is the covariantization of the flat-space expression

$$f^{\mu}_{\text{flat}} = -\frac{1}{6\varepsilon} \ddot{z}_\nu + \frac{1}{16} \Pi^{\mu\nu} \left( \frac{3}{16} \ddot{z}^2 \ddot{z}_\nu + \frac{1}{8} \varepsilon^{(4)} \ddot{z}_\nu \right) + \frac{2}{3} (\ddot{z})^2 \ddot{z}_\nu - \Pi^{\mu\nu} \left( \frac{4}{45} \ddot{z}_\nu + \frac{2}{9} \varepsilon \ddot{z}^2 \dot{z}_\nu \right). \quad (124)$$
The divergent term is

\[ f_{\text{curv div}} = -\frac{5}{72\varepsilon} \Pi^{\mu \nu} \left( 2 R_{\nu \alpha} \dot{z}^{\alpha} + 2 R_{\nu \alpha; \beta} \dot{z}^{\alpha} \dot{z}^{\beta} - R_{\alpha \beta; \nu} \dot{z}^{\alpha} \dot{z}^{\beta} + R_{\alpha \beta; \nu} \dot{z}^{\alpha} \dot{z}^{\beta} + \frac{1}{5} (R_{\gamma \nu} + R_{\nu \gamma}) \right). \]

The leading divergence is absorbed by the mass renormalization, while the non-leading one is eliminated introducing the following counterterm:

\[ S_{\text{c}}^{\text{em}} = \frac{k^0}{72} \int \left( \frac{9}{2} (D^2 z)^2 - (R + 5 R^\alpha \beta D z^\alpha D z^\beta) \right) \sqrt{-g} \, dt. \] (126)

The finite part of the self-force \( f_{\text{curv fin}}^\mu \) can be cast into the sum of the following parts: a) the quadratic in curvature terms:

\[ f_{\text{quad}}^\mu = \Pi^{\mu \nu} \left( \frac{4}{36} R_{\nu \sigma} R_{\tau \rho} \dot{z}^{\sigma} \dot{z}^{\tau} \dot{z}^{\rho} - \frac{1}{27} \frac{7}{864} R R_{\nu \sigma} \dot{z}^{\sigma} + \frac{1}{1080} R_{\nu \sigma \tau \rho} R_{\alpha \beta \tau \sigma} \dot{z}^{\alpha} + \frac{29}{4320} R_{\nu \sigma} R_{\alpha \beta} \dot{z}^{\sigma} \dot{z}^{\gamma} + \frac{1}{540} R_{\nu \sigma \tau \rho} R_{\alpha \beta} \dot{z}^{\sigma} \dot{z}^{\gamma} \right); \] (127)

b) the Ricci-terms:

\[ f_{\text{Ric}}^\mu = \Pi^{\mu \nu} \left( \frac{4}{27} R_{\nu \sigma} R_{\tau \rho} \dot{z}^{\sigma} \dot{z}^{\tau} \dot{z}^{\rho} - \frac{1}{27} \frac{2}{z^2} R_{\nu \sigma} \dot{z}^{\sigma} + \frac{2}{27} R_{\tau \rho} \dot{z}^{\tau} \dot{z}^{\rho} \dot{z}^{\nu} + \frac{2}{9} R_{\nu \sigma \tau \rho} \dot{z}^{\sigma} \dot{z}^{\tau} \dot{z}^{\rho} + \frac{1}{60} \Box R_{\nu \sigma} \dot{z}^{\sigma} + \frac{1}{18} R_{\nu \sigma \tau \rho} \dot{z}^{\sigma} \dot{z}^{\tau} \dot{z}^{\rho} - \frac{1}{18} R_{\sigma \tau \rho} \dot{z}^{\sigma} \dot{z}^{\tau} \dot{z}^{\rho} + \frac{1}{12} R_{\sigma \tau \rho} \dot{z}^{\sigma} \dot{z}^{\tau} \dot{z}^{\rho} - \frac{1}{36} R_{\nu \sigma \tau \rho} \dot{z}^{\sigma} \dot{z}^{\tau} \dot{z}^{\rho} \right); \] (128)

c) the Ricci-scalar terms:

\[ f_{\text{Rec}}^\mu = \Pi^{\mu \nu} \left( -\frac{1}{54} R_{\nu \sigma} \dot{z}^{\sigma} \dot{z}^{\nu} - \frac{1}{36} R_{\nu \sigma \tau \rho} \dot{z}^{\sigma} \dot{z}^{\tau} \dot{z}^{\rho} + \frac{1}{72} R_{\nu \sigma \tau \rho} \dot{z}^{\sigma} \dot{z}^{\tau} \dot{z}^{\rho} \right); \]
d) the Riemann term

\[ f_{\text{Riem}}^\mu = - \frac{1}{9} R_{\sigma \nu \tau \rho} \dot{z}^{\sigma} \dot{z}^{\nu} \dot{z}^{\tau} \dot{z}^{\rho}. \] (129)

The appearance of the Riemann tensor in the finite local force in a new feature in six dimensions, which is absent in the scalar case. This term is non-zero in the vacuum space-time, but it still vanishes for geodesic motion.

Finally, we observe that the tail term has the same form as in four dimensions.

VI. CONCLUSIONS

In this paper we presented a purely local calculation of the radiation reaction for a point particle in curved space of an even dimension \( D \geq 4 \). The possibility of such a calculation means that scattering of radiation on curvature in the vicinity of a charge is irrelevant for the reaction force. Clearly the scattering can substantially modify global properties of radiation. In particular, emitted waves can be reflected back and act on a particle once again, but within our approach this should be rather interpreted as an action of the external field, than a modification of the reaction force. Note, however, that such a description may not be natural if the problem is treated globally in terms of the field modes defined on the full manifold. In particular, for space-times with reflection of massless fields from infinity (like anti-de Sitter) the action of modes (subject to reflection conditions) on a particle will include the effect of scattering, thus leading to a global definition of the reaction force. In terms of the above analysis this means that there are several geodesics connecting two points of space-time. Our approach is valid only under the assumption that this is not the case. It excludes, therefore, many physically interesting situations, which require a special global analysis. For an example of the global definition of the reaction force see [28] where the radiation reaction problem was considered for particles moving in the Kerr space-time, (for the case of a cosmic string see [29, 30]).

Another point worth to be discussed is that of elimination of divergences. Classical renormalizability of the equations for point particles interacting with massless scalar and vector fields in flat space time is well-known for \( D = 4 \). In higher dimensional Minkowski space it also holds modulo introduction of the higher derivative counterterms into the action \[ \Pi^D = 6 \text{ case.} \] One observes that the renormalization of the higher-dimensional reaction force contains not only higher derivatives of the particle velocity, but also higher derivatives of the Ricci tensor, as well as the Riemann tensor, a feature which is not present in four dimensions.
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