On the geometry of the compactification of the universal Picard variety

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Abstract

Here we focus on the geometry of $P_{d,g}$, the compactification of the universal Picard variety constructed by L. Caporaso. In particular, we show that the moduli space of spin curves constructed by M. Cornalba naturally injects into $P_{d,g}$ and we give generators and relations of the rational divisor class group of $P_{d,g}$, extending previous work by A. Kouvidakis.

1 Introduction

The universal Picard variety $P_{d,g}$ is the coarse moduli space for line bundles of degree $d$ on smooth algebraic curves of genus $g$. Even though one is mainly interested in the behaviour of line bundles on smooth curves, nevertheless it is often useful to control their degenerations on singular curves. Perhaps the most celebrated example of proof by degeneration is provided by the Brill-Noether-Petri theorem (see [2] and the references therein). Another very recent achievement of degeneration techniques is the proof given by L. Caporaso and E. Sernesi (see [5] and [6]) that a general curve of genus $g \geq 3$ can be recovered from its odd theta-characteristics. In particular, in order to control degenerations of curves with prescribed theta-characteristics, a key role is played in [6] by the moduli space of spin curves $S_g$ constructed by M. Cornalba in [7]. This perspective suggests the deep mathematical interest (both in itself and as a tool) of a geometrically meaningful compactification of the moduli spaces parameterizing pairs of curves and line bundles. Let $\overline{P}_{d,g}$ denote the compactification of $P_{d,g}$ constructed by L. Caporaso in [3] via geometric invariant theory. The boundary points of $\overline{P}_{d,g}$ correspond to certain line bundles on Deligne-Mumford semistable curves, while all previously known compactifications of the generalized Jacobian of an integral nodal curve used torsion free sheaves of rank one. From this point of view, a strict analogy emerges between $\overline{P}_{d,g}$ and $S_g$: even though the techniques used
in the two constructions are completely different, in both cases the resulting compactification is given in terms of line bundles on the same kind of singular curves. We will see that this analogy has a precise explanation: namely, in section 3 we introduce a subscheme of $\overline{P}_{d,g}$ which compactifies the locus in $P_{d,g}$ corresponding to curves with theta characteristics and we investigate how it is related to $\overline{S}_g$. Indeed, after having established the existence of a natural morphism from $\overline{S}_g$ to $\overline{P}_{d,g}$ (see Theorem 1), we prove that it is an injection (see Theorem 2) and we give a precise combinatorial description of its image (see Theorem 3). In the future we hope to address similar questions also for moduli spaces of higher spin curves, which were introduced by T. J. Jarvis in a rather different style (see [11], [12], and [13]). In section 4 instead, we obtain a complete description of the rational divisor class group of $\overline{P}_{d,g}$ (see Theorem 5): its rational Picard group is determined whenever $\overline{P}_{d,g}$ turns out to be a geometric quotient (see Corollary 1). The strategy of proof is straightforward: first of all, we deduce from the basic properties of $\overline{P}_{d,g}$ a rough description of its boundary (see Proposition 4); next, we recall a theorem proved by A. Kouvidakis in [14] on the Picard group of $P_{d,g}$ (see Theorem 4). Hence the result on generation follows and in order to exclude nontrivial relations we simply lift to $\overline{P}_{d,g}$ the families of curves constructed by E. Arbarello and M. Cornalba in [11].

We work over the field $\mathbb{C}$ of complex numbers.

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2 Notation and preliminaries

Let $X$ be a Deligne-Mumford semistable curve and let $E$ be a complete, irreducible subcurve of $X$. One says that $E$ is exceptional if it is smooth, rational, and meets the other components in exactly two points. Moreover, one says that $X$ is quasistable if any two distinct exceptional components of $C$ are disjoint. In the sequel, $\tilde{X}$ will denote the subcurve $X \setminus \bigcup E_i$ obtained from $X$ by removing all exceptional components.

A spin curve of genus $g$ (see [7], [8]) is the datum of a quasistable genus $g$ curve $X$ with an invertible sheaf $\zeta_X$ of degree $g-1$ on $X$ and a homomorphism of invertible sheaves

$$\alpha_X : \zeta_X^{\otimes 2} \rightarrow \omega_X$$

such that
(i) $\zeta_X$ has degree 1 on every exceptional component of $X$;
(ii) $\alpha_X$ is not zero at a general point of every non-exceptional component of $X$.

From the definition it follows that $\alpha_X$ vanishes identically on all exceptional components of $X$ and induces an isomorphism

$$\tilde{\alpha}_X : \zeta_X^\otimes 2 |_{\tilde{X}} \rightarrow \omega_{\tilde{X}}.$$  

In particular, when $X$ is smooth, $\zeta_X$ is just a theta-characteristic on $X$.

By definition (see §2), two spin curves $(X, \zeta_X, \alpha_X)$ and $(X', \zeta_{X'}, \alpha_{X'})$ are isomorphic if there are isomorphisms $\sigma : X \rightarrow X'$ and $\tau : \sigma^*(\zeta_X) \rightarrow \zeta_X$ such that $\tau$ is compatible with the natural isomorphism between $\sigma^*(\omega_{X'})$ and $\omega_X$.

However, we point out the following fact.

**Lemma 1.** Let $(X, \zeta_X, \alpha_X)$ and $(X', \zeta_{X'}, \alpha_{X'})$ be two spin curves and assume that there are isomorphisms $\sigma : X \rightarrow X'$ and $\tau : \sigma^*(\zeta_X') \rightarrow \zeta_X$. Then $(X, \zeta_X, \alpha_X)$ and $(X', \zeta_{X'}, \alpha_{X'})$ are isomorphic as spin curves.

**Proof.** Let $\beta_X : \zeta_X^\otimes 2 \rightarrow \omega_X$ be defined by the following commutative diagram:

$$\begin{align*}
\zeta_X^\otimes 2 & \xrightarrow{\beta_X} \omega_X \\
\uparrow \tau^\otimes 2 & \uparrow \cong \\
(\sigma^*\zeta_X')^\otimes 2 & \xrightarrow{\sigma^*\alpha_{X'}} \sigma^*\omega_{X'}.
\end{align*}$$

Then $(X, \zeta_X, \beta_X)$ is a spin curve, which is isomorphic to $(X', \zeta_{X'}, \alpha_{X'})$ by definition and to $(X, \zeta_X, \alpha_X)$ by [7], Lemma (2.1), so the thesis follows.

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A family of spin curves is a flat family of quasistable curves $f : \mathcal{X} \rightarrow S$ with an invertible sheaf $\zeta_f$ on $\mathcal{X}$ and a homomorphism

$$\alpha_f : \zeta_f^\otimes 2 \rightarrow \omega_f$$

such that the restriction of these data to any fiber of $f$ gives rise to a spin curve.

Two families of spin curves $f : \mathcal{X} \rightarrow S$ and $f' : \mathcal{X}' \rightarrow S$ are isomorphic if there are isomorphisms $\sigma : \mathcal{X} \rightarrow \mathcal{X}'$ and $\tau : \sigma^*(\zeta_f) \rightarrow \zeta_f$ such that $f = f' \circ \sigma$ and $\tau$ is compatible with the natural isomorphism between $\sigma^*(\omega_{f'})$ and $\omega_f$ (see [8] p. 212).

Let $\mathfrak{S}_g$ be the contravariant functor from schemes to sets, which to every scheme $S$ associates the set $\mathfrak{S}_g(S)$ of isomorphism classes of families of spin curves of genus $g$. 

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Let \( S_g \) be the set of isomorphism classes of spin curves of genus \( g \) and \( S_g \) be the subset consisting of classes of smooth curves. One can define a natural structure of analytic variety on \( \overline{S}_g \) (see [7], §5) and from [7], Proposition (4.6), it follows that \( \overline{S}_g \) is a coarse moduli variety for \( S_g \).

Let now \( g \geq 3 \). For every integer \( d \), there is a universal Picard variety 
\[ \psi_d : \mathcal{P}_{d,g} \rightarrow \mathcal{M}_{g}^{0} \]
whose fiber \( J_d(X) \) over a point \( X \) of \( \mathcal{M}_{g}^{0} \) parametrizes line bundles on \( X \) of degree \( d \) modulo isomorphism.

Assume \( d \geq 20(g - 1) \), but notice that this is not a real restriction because of the natural isomorphism \( \mathcal{P}_{d,g} \cong \mathcal{P}_{d+n(2g-2),g} \). Then \( \mathcal{P}_{d,g} \) has a natural compactification 
\[ \phi_d : \overline{\mathcal{P}}_{d,g} \rightarrow \overline{\mathcal{M}}_g \]
such that \( \phi_d^{-1}(\mathcal{M}_g^{0}) = \mathcal{P}_{d,g} \). Namely, let \( \text{Hilb}^{dx-g+1}_{d-\overline{g}} \) be the Hilbert scheme parametrizing closed subschemes of \( \mathbb{P}^{d-\overline{g}} \) having Hilbert polynomial \( dx-g+1 \) and set \( H_d := \{ h \in \text{Hilb}^{dx-g+1}_{d-\overline{g}} : h \text{ is } G\text{-semistable and the corresponding curve is connected } \} \). Then \( \overline{\mathcal{P}}_{d,g} \) was constructed in [3] as a GIT quotient 
\[ \pi_d : H_d \rightarrow H_d/G = \overline{\mathcal{P}}_{d,g}, \]
where \( G = SL(d - \overline{g} + 1) \).

Moreover, one can define (see [3], §8.1) the contravariant functor \( \overline{\mathcal{P}}_{d,g} \) from schemes to sets, which to every scheme \( S \) associates the set \( \overline{\mathcal{P}}_{d,g}(S) \) of equivalence classes of polarized families of quasistable curves of genus \( g \)
\[ f : (\mathcal{X}, \mathcal{L}) \rightarrow S \]
such that \( \mathcal{L} \) is a relatively very ample line bundle of degree \( d \) whose multidegree satisfies the following Basic Inequality on each fiber.

**Definition 1.** Let \( X = \bigcup_{i=1}^{n} X_i \) be a projective, nodal, connected curve of arithmetic genus \( g \), where the \( X_i \)'s are the irreducible components of \( X \). We say that the multidegree \( (d_1, \ldots, d_n) \) satisfies the Basic Inequality if for every complete subcurve \( Y \) of \( X \) of arithmetic genus \( g_Y \) we have 
\[ m_Y \leq d_Y \leq m_Y + k_Y \]
where
\[ d_Y = \sum_{X_i \subseteq Y} d_i \]
\[ k_Y = |Y \cap X \setminus Y| \]
\[ m_Y = \frac{d}{g-1} \left( g_Y - 1 + \frac{k_Y}{2} \right) - \frac{k_Y}{2} \]

(see [3] p. 611 and p. 614).

Two families over \( S, (\mathcal{X}, \mathcal{L}) \) and \( (\mathcal{X}', \mathcal{L}') \) are equivalent if there exists an \( S \)-isomorphism

\[ \sigma : \mathcal{X} \to \mathcal{X}' \]

and a line bundle \( M \) on \( S \) such that

\[ \sigma^* \mathcal{L}' \cong \mathcal{L} \otimes f^* M. \]

By [3], Proposition 8.1, there is a morphism of functors:

\[ \mathcal{P}_{d,g} \to \text{Hom}(., \mathcal{P}_{d,g}) \] (1)

and \( \mathcal{P}_{d,g} \) coarsely represents \( \mathcal{P}_{d,g} \) if and only if

\[ (d - g + 1, 2g - 2) = 1. \] (2)

The relationship between \( S_g \) and \( \mathcal{P}_{d,g} \) can be expressed as follows.

**Theorem 1.** For every integer \( t \geq 10 \) there is a natural morphism:

\[ f_t : S_g \to \mathcal{P}_{(2t+1)(g-1),g}. \]

**Proof.** First of all, notice that in this case (2) does not hold, so the points of \( \mathcal{P}_{d,g} \) are not in one-to-one correspondence with equivalence classes of very ample line bundles of degree \( d \) on quasistable curves, satisfying the Basic Inequality (see [3], p. 654). However, we claim that the thesis can be deduced from the existence of a morphism of functors:

\[ F_t : S_g \to \mathcal{P}_{(2t+1)(g-1),g}. \] (3)

Indeed, since \( S_g \) coarsely represents \( S_g \), any morphism of functors \( S_g \to \text{Hom}(., S) \) induces a morphism of schemes \( S_g \to S \), so the claim follows from (1). Now, a morphism of functors as (3) is the datum for any scheme \( S \) of a set-theoretical map

\[ F_t(S) : S_g(S) \to \mathcal{P}_{(2t+1)(g-1),g}(S), \]

satisfying obvious compatibility conditions. Let us define

\[ F_t(S)([f : \mathcal{X} \to S, \zeta_f, \alpha_f]) := [f : (\mathcal{X}, \zeta_f \otimes \omega_f^{\otimes t}) \to S]. \]

In order to prove that \( F_t(S) \) is well-defined, the only non-trivial matter is to check that the multidegree of \( \zeta_f \otimes \omega_f^{\otimes t} \) satisfies the Basic Inequality on each fiber, so the thesis follows from the next Lemma.
Lemma 2. If $Y$ is a complete subcurve of $X$ and $d_Y$ is the degree of $\zeta_X \otimes \omega_X^{\otimes t}|_Y$, then $m_Y \leq d_Y \leq m_Y + k_Y$ in the notation of the Basic Inequality. Moreover, if we set

$$k_Y := |\tilde{Y} \cap X \setminus Y|,$$

we have $d_Y = m_Y$ if and only if $k_Y = 0$ and all exceptional components in $Y$ do not intersect $X \setminus Y$, and $d_Y = m_Y + k_Y$ if and only if $k_Y = 0$ and all exceptional components in $X \setminus Y$ do not intersect $Y$.

Proof. Let $Y_1, \ldots, Y_\nu$ be the irreducible components of $Y$, of arithmetic genus $g_1, \ldots, g_\nu$ respectively. We may assume that the first $\tilde{\nu}$ ones are non-exceptional and the last $(\nu - \tilde{\nu})$ ones are exceptional, so that $\tilde{Y} = Y_1 \cup \ldots \cup Y_\nu$. Next, let $\{p_1, \ldots, p_\delta\}$ be the points of intersection between two distinct irreducible components of $Y$. Again, we may assume that the first $\tilde{\delta}$ ones involve two non-exceptional components and the last $(\delta - \tilde{\delta})$ ones are between a non-exceptional and an exceptional component. We have

$$g_Y = \sum_{i=1}^{\nu} g_i + \delta - \nu + 1 = \sum_{i=1}^{\tilde{\nu}} g_i + \delta - \nu + 1$$

and since $\zeta_X^{(\otimes 2)}|_{\tilde{Y}} \cong \omega_X|_{\tilde{Y}}$ we may compute

$$\deg \zeta_X|_{\tilde{Y}} = \frac{1}{2} \deg \omega_X|_{\tilde{Y}} = \frac{1}{2} \left( \sum_{i=1}^{\tilde{\nu}} (2g_i - 2) + 2\tilde{\delta} + \tilde{k}_Y \right).$$

Hence we deduce

$$d_Y = \deg (\zeta_X \otimes \omega_X^{\otimes t})|_Y = \deg \zeta_X|_Y + t \deg \omega_X|_Y =$$

$$= \deg \zeta_X|_Y + \deg \zeta_X|_{\tilde{Y} \setminus Y} + t \deg \omega_X|_Y =$$

$$= \frac{1}{2} \left( \sum_{i=1}^{\tilde{\nu}} (2g_i - 2) + 2\tilde{\delta} + \tilde{k}_Y \right) + (\nu - \tilde{\nu}) + t(2g_Y - 2 + k_Y) =$$

$$= g_Y - 1 - (\delta - \tilde{\delta}) + 2(\nu - \tilde{\nu}) + \frac{k_Y}{2} + 2t \left( g_Y - 1 + \frac{k_Y}{2} \right).$$

On the other hand,

$$m_Y = (2t + 1) \left( g_Y - 1 + \frac{k_Y}{2} \right) - \frac{k_Y}{2} = 2t \left( g_Y - 1 + \frac{k_Y}{2} \right) + g_Y - 1,$$

so

$$d_Y = m_Y - (\delta - \tilde{\delta}) + 2(\nu - \tilde{\nu}) + \frac{k_Y}{2}.$$
and the Basic Inequality is satisfied if and only if
\[ 0 \leq 2(\nu - \bar{\nu}) - (\delta - \bar{\delta}) + \frac{k_Y}{2} \leq k_Y. \]

Now, since every exceptional component meets the other components in exactly two points, there are obvious inequalities
\[ \delta - \bar{\delta} \leq 2(\nu - \bar{\nu}) \]
and
\[ (\delta - \bar{\delta}) + (k_Y - \bar{k}_Y) \geq 2(\nu - \bar{\nu}), \]

hence the thesis follows.

\[ \square \]

**Remark 1.** If \( t_1 \) and \( t_2 \) are integers \( \geq 10 \), then Lemma 8.1 of [3] yields an isomorphism \( \tau : \overline{P}^{(2t_1+1)(g-1),g} \rightarrow \overline{P}^{(2t_2+1)(g-1),g} \). We point out that by the definitions of \( \tau \) (see [3], proof of Lemma 8.1) and of \( f_t \) (see proof of Theorem 7), there is a commutative diagram:

\[
\begin{array}{ccc}
S_g & \xrightarrow{f_{t_1}} & \overline{P}^{(2t_1+1)(g-1),g} \\
\| & \downarrow \tau & \\
S_g & \xrightarrow{f_{t_2}} & \overline{P}^{(2t_2+1)(g-1),g}.
\end{array}
\]

\[ \square \]

### 3 Spin curves in \( \overline{P}_{d,g} \)

For every integer \( t \geq 10 \) we define

\[ K^{(2t+1)(g-1)} := \{ h \in \text{Hilb}^{(2t+1)(g-1),g-1} : \text{there is a spin curve} \]
\[ (X, \zeta_X, \alpha_X) \text{ and an embedding } h_t : X \rightarrow \mathbb{P}^{(2t+1)(g-1)-g} \]
\[ \text{induced by } \zeta_X \otimes \omega_X \otimes^{\otimes t} \text{ such that } h = h_t(X) \}. \]

By applying [3], Proposition 6.1, from the first part of Lemma 2 we deduce

\[ K^{(2t+1)(g-1)} \subset H^{(2t+1)(g-1)} \]

(the definition of \( H_d \) was recalled above in section 2). Moreover, the second part of the same Lemma provides a great amount of information on Hilbert points corresponding to spin curves.
Proposition 1. Let \((X, \zeta_X, \alpha_X)\) be a spin curve. Then \(h_t(X)\) is GIT-stable if and only if \(X\) is connected.

Proof. We are going to apply the stability criterion of [3], Lemma 6.1, which says that \(h_t(X)\) is GIT-stable if and only if the only subcurves \(Y\) of \(X\) such that \(d_Y = m_Y + k_Y\) are union of exceptional components.

If \(X\) is connected, then for every subcurve \(Y\) of \(X\) which is not union of exceptional components we have \(k_Y > 0\), so from Lemma 2 it follows that \(d_Y < m_Y + k_Y\) and \(h_t(X)\) turns out to be GIT-stable.

If instead \(X\) is not connected, pick any connected component \(Z\) of \(X\) and take \(Y\) to be the union of \(Z\) with all exceptional components of \(X\) intersecting \(Z\). It follows that \(k_Y = 0\) and all exceptional components in \(X \setminus Y\) do not intersect \(Y\), so Lemma 2 yields \(d_Y = m_Y + k_Y\) and \(h_t(X)\) is not GIT-stable. \(\square\)

Proposition 2. If \((X, \zeta_X, \alpha_X)\) is a spin curve, then the orbit of \(h_t(X)\) is closed in the semistable locus.

Proof. Just recall the first part of [3], Lemma 6.1, which says that the orbit of \(h_t(X)\) is closed in the semistable locus if and only if \(k_Y = 0\) for every subcurve \(Y\) of \(X\) such that \(d_Y = m_Y\), so the thesis is a direct consequence of Lemma 2.

\(\square\)

The sublocus of \(\overline{\mathcal{P}}_{(2t+1)(g-1), g}\) obtained by projection from \(K_{(2t+1)(g-1)}\) is indeed the GIT analogue of \(\overline{S}_g\) we are looking for. Namely, if we set

\[\Sigma_t := \pi_{(2t+1)(g-1)}(K_{(2t+1)(g-1)})\]

then the following holds.

Theorem 2. The morphism \(f_t\) is an injection:

\[f_t : \overline{S}_g \hookrightarrow \Sigma_t.\]

Proof. It is easy to check that \(f_t(\overline{S}_g) = \Sigma_t\). Indeed, if \([([X, \zeta_X, \alpha_X]) \in \overline{S}_g\), then any choice of a base for \(H^0(X, \zeta_X \otimes \omega_X^{\otimes t})\) induces an embedding \(h_t : X \to \mathbb{P}^{(2t+1)(g-1)-g}\) and \(f_t(([X, \zeta_X, \alpha_X]]) = \pi_{(2t+1)(g-1)}(h_t(X)) \in \Sigma_t\); conversely, if \(\pi_{(2t+1)(g-1)}(h) \in \Sigma_t\), then there is a spin curve \((X, \zeta_X, \alpha_X)\) and an embedding \(h : X \to \mathbb{P}^{(2t+1)(g-1)-g}\) such that \(h = h_t(X)\) and \(f_t(([X', \zeta_{X'}, \alpha_{X'}]]) = \pi_{(2t+1)(g-1)}(h)\).

Next we claim that \(f_t\) is injective. Indeed, let \((X, \zeta_X, \alpha_X)\) and \((X', \zeta_{X'}, \alpha_{X'})\) be two spin curves and assume that \(f_t(([X, \zeta_X, \alpha_X]]) = f_t(([X', \zeta_{X'}, \alpha_{X'}]))\). Choose bases for \(H^0(X, \zeta_X \otimes \omega_X^{\otimes t})\) and \(H^0(X', \zeta_{X'} \otimes \omega_{X'}^{\otimes t})\) and embed \(X\) and \(X'\)
in \( \mathbb{P}^{(2t+1)(g-1)-g} \). If \( h_t(X) \) and \( h_t(X') \) are the corresponding Hilbert points, then \( \pi_{(2t+1)(g-1)}(h(X)) = \pi_{(2t+1)(g-1)}(h(X')) \) and the Fundamental Theorem of GIT implies that \( \mathcal{O}_G(h_t(X)) \) and \( \mathcal{O}_G(h_t(X')) \) intersect in the semistable locus. It follows from Proposition \( \mathbb{P}^2 \) that \( \mathcal{O}_G(h_t(X)) \cap \mathcal{O}_G(h_t(X')) \neq \emptyset \), so \( \mathcal{O}_G(h_t(X)) = \mathcal{O}_G(h_t(X')) \) and there are isomorphisms \( \sigma : X \to X' \) and \( \tau : \sigma^* (\zeta_{X'}) \to \zeta_X \). Now the claim follows from Lemma \( \mathbb{P}^1 \) and the proof is over.

Next we are going to derive an explicit combinatorial description of \( \Sigma_t \). We omit the proof of the following easy Lemma, referring to the proof of Lemma \( \mathbb{P}^3 \) for a similar computation.

**Lemma 3.** Let \((X, \zeta_X, \alpha_X)\) be a spin curve. Fix a decomposition

\[
X = \bigcup_{i=1}^n \tilde{X}_i \cup \bigcup_{j=1}^m E_j
\]

where the \( \tilde{X}_i \)'s are the irreducible components of \( \tilde{X} \) and the \( E_j \)'s are the exceptional components of \( X \). Set \( k_i := |\tilde{X}_i \cap X \setminus \tilde{X}_i| \) and \( \tilde{k}_i := |\tilde{X}_i \cap X \setminus \tilde{X}_i| \). Then the multidegree of \( \zeta_X \otimes \omega_X^{\otimes t} \) on \( X \) is \( d = (d_1, \ldots, d_n, d_{n+1}, \ldots, d_{n+m}) \) with

\[
d_i = (2t+1)(p_a(\tilde{X}_i) - 1) + tk_i + \frac{1}{2} \tilde{k}_i \quad 1 \leq i \leq n
\]

\[
d_{n+j} = 1 \quad 1 \leq j \leq m.
\]

As in \( \mathbb{P}^4 \), \( \S 5.1 \), we set

\[
M_C^d := \{ h \in H_d, h = \text{hilb}(C, L) : \deg L = d \}
\]

and

\[
V_C^d := \overline{M_C^d} \cap H_d.
\]

By \( \mathbb{P}^5 \), Corollary 5.1 (but see also on p. 627), if \([C] \in \overline{M}_g\) then the \( V_C^d \)'s are exactly the irreducible components of the fiber over \([C]\) of the natural morphism

\[
\psi_d : H_d \longrightarrow \overline{M}_g.
\]

**Proposition 3.** Let \( d = (d_1, \ldots, d_n) \) be a multidegree and let \( C = \bigcup_{i=1}^n C_i \) be a stable curve, where the \( C_i \)'s are the irreducible components of \( C \). Set \( g_i := p_a(C_i) \) and \( k_{ij} := |C_i \cap C_j| \) if \( i \neq j \), \( k_{ij} := 0 \) if \( i = j \).
Then there exists a spin curve \((X, \zeta_X, \alpha_X)\) such that \(h_t(X) \in V^d_C\) if and only if for every \(1 \leq i, j \leq n\) there are integers \(s_{ij}\) and \(\sigma_{ij}\) with

\[
0 \leq s_{ij} \leq k_{ij} \quad s_{ij} = s_{ji} \quad \sum_{j=1}^{n} (k_{ij} - s_{ij}) \equiv 0 \mod 2
\]

such that

\[
d_i = (2t + 1)(g_i - 1) + t \sum_{j=1}^{n} k_{ij} + \frac{1}{2} \sum_{j=1}^{n} (k_{ij} - s_{ij}) + \sum_{j=1}^{n} \sigma_{ij}.
\]

**Proof.** To get a quasistable curve \(X\) starting from \(C = \bigcup_{i=1}^{n} C_i\), for every \(1 \leq i, j \leq n\) choose \(r_i\) nodes of \(C_i\) and \(s_{ij}\) contact points between \(C_i\) and \(C_j\) and blow them up. In the notation of Lemma 3 notice that \(p_a(X_i) = g_i - r_i, k_i = \sum_{j=1}^{n} k_{ij} + 2r_i\) and \(\tilde{k}_i = \sum_{j=1}^{n} (k_{ij} - s_{ij})\). As pointed out in [7], § 3, in order for a spin curve having \(X\) as underlying curve to exist, a necessary and sufficient condition is that \(\tilde{k}_i \equiv 0 \mod 2\) for every \(1 \leq i \leq n\). Moreover, by [3], Proposition 5.1, \(h_t(X) \in V^d_C\) if and only if there is a partition \(X = \bigcup_{i=1}^{n} X_i\) such that \(X_i\) is a complete connected subcurve of \(X\) whose stable model is \(C_i\) and \(d_i = \deg_X(\zeta_X \otimes \omega_X^{\otimes t})\). So the thesis follows from Lemma 3.

Let \(V_C := \bigcup_d V^d_C\) and let \(\overline{P}_{d,C} := \phi^{-1}_d(C)\). By [3], proof of Corollary 5.1, we have

\(\overline{P}_{d,C} = V_C / G\)

and for every irreducible component \(I\) of \(\overline{P}_{d,C}\) there is a unique multidegree \(d\) such that \(V^d_C\) dominates \(I\) via the quotient map

\(V_C \twoheadrightarrow \overline{P}_{d,C}\).

**Theorem 3.** Let \(C = \bigcup_{i=1}^{n} C_i\) be a stable curve, where the \(C_i\)'s are the irreducible components of \(C\). Set \(g_i := p_a(C_i)\) and \(k_{ij} := |C_i \cap C_j|\) if \(i \neq j\), \(k_{ij} := 0\) if \(i = j\). Let \(I\) be an irreducible component of \(\overline{P}_{d,C}\) and let \(d = (d_1, \ldots, d_n)\) be the multidegree such that \(I\) is dominated by \(V^d_C\).

Then there exists a spin curve \((X, \zeta_X, \alpha_X)\) such that \(f_t([X, \zeta_X, \alpha_X]) \in I\) if and only if for every \(1 \leq i, j \leq n\) there are integers \(s_{ij}\) and \(\sigma_{ij}\) with

\[
0 \leq s_{ij} \leq k_{ij} \quad s_{ij} = s_{ji} \quad \sum_{j=1}^{n} (k_{ij} - s_{ij}) \equiv 0 \mod 2
\]

such that

\[
d_i = (2t + 1)(g_i - 1) + t \sum_{j=1}^{n} k_{ij} + \frac{1}{2} \sum_{j=1}^{n} (k_{ij} - s_{ij}) + \sum_{j=1}^{n} \sigma_{ij}.
\]
such that

\[ d_i = (2t + 1)(g_i - 1) + t \sum_{j=1}^{n} k_{ij} + \frac{1}{2} \sum_{j=1}^{n} (k_{ij} - s_{ij}) + \sum_{j=1}^{n} \sigma_{ij}. \]

**Proof.** By the Fundamental Theorem of GIT,

\[ f_t([(X, \zeta_X, \alpha_X)]) = \pi_{(2t+1)(g-1)}(h_t(X)) \in I \]

if and only if there is \( h \in V^d_C \) such that \( \overline{O_G(h_t(X))} \) and \( \overline{O_G(h)} \) intersect in the semistable locus.

Since \( O_G(h_t(X)) \) is closed in the semistable locus by Proposition 2, we have

\[ O_G(h_t(X)) \cap \overline{O_G(h)} \neq \emptyset \]

and since \( \overline{O_G(h)} \) is a union of orbits we may rephrase the above condition as

\[ h_t(X) \in \overline{O_G(h)}. \]

On the other hand, we have \( V^d_C = \bigcup_{h \in V^d_C} O_G(h) \) since \( V^d_C \) is \( G \)-invariant and \( V^d_C = \bigcup_{h \in V^d_C} \overline{O_G(h)} \) since \( V^d_C \) is closed.

Summing up, we see that \( f_t([(X, \zeta_X, \alpha_X)]) \in I \) if and only if \( h_t(X) \in V^d_C \).

Now the thesis follows from Proposition 3.

**Example.** Let \( C \) be a *split curve* of genus \( g \), i.e. the union of two nonsingular rational curves meeting transversally at \( g + 1 \) points. Such curves are particularly interesting, for a number of reasons (see [4] and [6]). According to Theorem 3 \( \Sigma_t \) meets an irreducible component \( I \) of \( \mathcal{T}_{d,C} \) if and only if \( I \) corresponds to a bidegree \((d_1, d_2)\) with

\[
\begin{align*}
d_1 &= \left( t + \frac{1}{2} \right)(g + 1) - (2t + 1) - \frac{1}{2} s + \sigma \\
d_2 &= \left( t + \frac{1}{2} \right)(g + 1) - (2t + 1) + \frac{1}{2} s - \sigma
\end{align*}
\]

where \( s \) and \( \sigma \) are nonnegative integers satisfying

\[ \sigma \leq s \leq g + 1 \quad \text{and} \quad s \equiv g + 1 \mod 2. \]

Moreover, from the proof of Proposition 3 it follows that \( s \) is the number of exceptional components of a quasi-stable curve \( X \) underlying a spin curve \((X, \zeta_X, \alpha_X)\) such that \( f_t([(X, \zeta_X, \alpha_X)]) \in \Sigma_t \cap I \).
4 Divisors on $\overline{P}_{d,g}$

In order to understand the boundary of $\overline{P}_{d,g}$, we recall the decomposition of the boundary of $\overline{M}_g$ into its irreducible components:

$$\partial \overline{M}_g = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_{\lfloor g/2 \rfloor}$$

and we define

$$D_i := \phi_d^{-1}(\Delta_i)$$

for $i = 0, \ldots, \lfloor g/2 \rfloor$. Notice that, since $\phi_d$ is surjective, each $D_i$ turns out to be a divisor on $\overline{P}_{d,g}$. Moreover, if $X \in \overline{M}_g$, we set as usual $\overline{P}_{d,X} := \phi_d^{-1}(X)$.

**Lemma 4.** For every $i$, if $X$ is a general element in $\Delta_i$ then $\overline{P}_{d,X}$ is irreducible.

**Proof.** If $i \geq 1$, a general element $X$ of $\Delta_i$ is the union of two smooth curves $X_1$ and $X_2$ meeting at one node, so $X$ is of compact type and $\overline{P}_{d,X}$ is irreducible (see [3], footnote on p. 594). If instead $i = 0$, a general element $X \in \Delta_0$ is an irreducible curve and also $\overline{P}_{d,X}$ turns out to be irreducible (see [3] 7.1).

As a consequence, we obtain a complete description of the boundary of $\overline{P}_{d,g}$.

**Proposition 4.** For every $i$, $D_i$ is irreducible.

**Proof.** By [3], Corollary 5.1 (2), for every $X \in \overline{M}_g$ all irreducible components of $\phi_d^{-1}(X)$ have dimension $g$. Let $I$ be an irreducible component of $D_i$; by applying the theorem on the dimensions of the fibers (see for instance [10], II, Ex. 3.22 (b) p. 95) to the map

$$\phi_d|_I : I \rightarrow \phi_d(I) \subseteq \Delta_i$$

we obtain

$$\dim I - \dim \phi_d(I) \leq \dim(\phi_d^{-1}(X) \cap I) \leq g.$$  

Hence

$$3g - 4 = \dim \Delta_i \geq \dim \phi_d(I) \geq 4g - 4 - \dim(\phi_d^{-1}(X) \cap I) \geq 3g - 4$$

and all the above inequalities turn out to be equalities. In particular, we have

$$\dim(\phi_d^{-1}(X) \cap I) = g$$
and $\phi_d^{-1}(X) \cap I$ is a closed subscheme of $\phi_d^{-1}(X)$ of maximal dimension. By Lemma 4 there is a dense open subset $U_i \subset \Delta_i$ such that $\phi_d^{-1}(X)$ is irreducible for every $X \in U_i$. It follows that $\phi_d^{-1}(X) \subset I$ for every $X \in U_i$ and $\phi_d^{-1}(U_i)$ is a dense open subset of $I$. Hence $I = \overline{\phi_d^{-1}(U_i)}$ is uniquely determined.

We recall that the class of any line bundle $L$ on $P_{d,g}$ restricted to a fiber $J^d(X)$ is a multiple $m\theta$ of the class $\theta$ of the $\Theta$ divisor (see [14] p. 840); it seems therefore natural to define the class of $L$ to be the integer $m$.

The Picard group of $P_{d,g}$ is completely described by the following result, due to A. Kouvidakis (see [14], Theorem 4 p. 849).

**Theorem 4.** If $g \geq 3$ then the Picard group of the universal Picard variety $\psi_d : P_{d,g} \to M_g^0$ is freely generated over $\mathbb{Q}$ by the line bundles $L_{d,g}$ and $\psi_d^*(\lambda)$, where $L_{d,g}$ is any line bundle on $P_{d,g}$ with class

$$k_{d,g} = \frac{2g - 2}{\gcd(2g - 2, g + d - 1)}$$

and $\lambda$ is the Hodge bundle on $M_g^0$.

The analogous result for $\overline{P}_{d,g}$ is the following:

**Theorem 5.** Assume $g \geq 3$ and $d \geq 20(g - 1)$. Then the divisor class group of the universal Picard variety $\phi_d : \overline{P}_{d,g} \to \overline{M}_g$ is freely generated over $\mathbb{Q}$ by the classes $L_{d,g}$, $\phi_d^*(\lambda)$ and $D_i$ ($i = 0, \ldots, [g/2]$), where $D_i$ denotes the linear equivalence class of $D_i$.

**Proof.** Since $P_{d,g}$ is smooth and irreducible, there is a natural identification $\text{Pic}(P_{d,g}) = A_{4g-4}(P_{d,g})$, so using the exact sequence

$$A_{4g-4}(\overline{P}_{d,g} \setminus P_{d,g}) \to A_{4g-4}(\overline{P}_{d,g}) \to A_{4g-4}(P_{d,g}) \to 0$$

we may deduce from Theorem 4 that $A_{4g-4}(\overline{P}_{d,g})$ is generated by $L_{d,g}$, $\phi_d^*(\lambda)$ and $A_{4g-4}(\overline{P}_{d,g} \setminus P_{d,g})$. Now, we have

$$\overline{P}_{d,g} \setminus P_{d,g} = D_0 \cup \ldots \cup D_{[g/2]} \cup \phi_d^{-1}(\mathcal{M}_g \setminus \mathcal{M}_g^0).$$

Moreover, by applying the theorem on the dimensions of the fibers to the map

$$\phi_d |_{\phi_d^{-1}(\mathcal{M}_g \setminus \mathcal{M}_g^0)} : \phi_d^{-1}(\mathcal{M}_g \setminus \mathcal{M}_g^0) \to \mathcal{M}_g \setminus \mathcal{M}_g^0$$

we obtain that $\text{codim}(\phi_d^{-1}(\mathcal{M}_g \setminus \mathcal{M}_g^0), \overline{P}_{d,g}) \geq \text{codim}(\mathcal{M}_g \setminus \mathcal{M}_g^0, \overline{M}_g) = g - 2$. Hence, if $g \geq 4$, Proposition 5 implies that $A_{4g-4}(\overline{P}_{d,g} \setminus P_{d,g})$ is generated by the $D_i$'s. If, instead, $g = 3$, we recall that the hyperelliptic locus $H$ is the
unique divisor in $\mathcal{M}_3$ contained in $\mathcal{M}_3 \setminus \mathcal{M}_3^0$ (see [2], Ex. 2.27, 3)). Since $[H] = 18\lambda$ in $\text{Pic}(\mathcal{M}_3 \otimes \mathbb{Q})$ (see [2] p. 164), we have $[\phi^{-1}_d(H)] = 18\phi_d^*(\lambda)$ and the result on generation is completely proved. As for relations, let

$$a\mathcal{L}_{d,g} + b\phi_d^*(\lambda) + \sum c_iD_i = 0$$

be a relation in $A_{4g-4}(\mathcal{T}_{d,g})$. If $J^d(X)$ is the fiber over a curve $X$ in $\mathcal{M}_g^0$, then restricting (1) to $J^d(X)$ yields $ak_{d,g}\theta = 0$ and we get $a = 0$. So we may rephrase (1) as $\phi_d^*(b\lambda + \sum c_i\delta_i) = 0$. The thesis is now a direct consequence of the following Lemma.

**Lemma 5.** Assume $g \geq 3$ and $d \geq 20(g-1)$. Then

$$\phi_d^*: A_{3g-4}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q} \longrightarrow A_{4g-4}(\mathcal{T}_{d,g}) \otimes \mathbb{Q}$$

is injective.

**Proof.** It is well-known that $A_{3g-4}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$ is generated by the Hodge class $\lambda$ and the boundary classes $\delta_i$ (see for instance [1], Proposition 2). A natural way to check that these classes are independent is to construct families of stable curves $h: X \rightarrow S$, with $S$ a smooth complete curve, such that the vectors $(\deg h(\lambda), \deg h(\delta_0), \ldots)$ are linearly independent. Such a construction is carried out in detail in [1] and it is applied in [7] to show the injectivity of $\chi^*: \text{Pic}(\overline{\mathcal{M}}_g) \rightarrow \text{Pic}(\overline{S}_g)$. We are going to mimic the same idea in our case. Namely, in order to prove that $\phi_d^*(\lambda)$ and the $\phi_d^*(\delta_i)$’s are independent in $A_{4g-4}(\mathcal{T}_{d,g}) \otimes \mathbb{Q}$, we will lift to $\mathcal{T}_{d,g}$ the families $h: X \rightarrow S$ constructed in [1]. The key observation is that each of them is equipped with many sections passing through the smooth locus of the general curve $C$ of the family. Indeed, for every irreducible component $C_i$ of $C$ we easily find a section $\sigma_i$ of $h$ which cuts on $C$ a smooth point $P_i \in C_i$. Next, we decompose the integer $d$ as a sum of $d_i$’s in such a way that the multidegree determined by the $d_i$’s satisfies the Basic Inequality. Finally, we endow $C$ with the line bundle $\mathcal{L} := \otimes_i \mathcal{O}_{C_i}(d_ip_i)$. Since $\mathcal{T}_{d,g}$ is proper over $\overline{\mathcal{M}}_g$, this construction uniquely determines a lifting of $h: X \rightarrow S$, so the proof is over.

**Corollary 1.** Assume $g \geq 3$, $d \geq 20(g-1)$ and $(d-g+1, 2g-2) = 1$. Then $\text{Pic}(\mathcal{T}_{d,g}) \otimes \mathbb{Q}$ is freely generated by $\mathcal{L}_{d,g}$, $\phi_d^*(\lambda)$ and $D_i$ ($i = 0, \ldots, \lfloor g/2 \rfloor$).

**Proof.** By [3], Lemma 2.2 (1), $\mathcal{T}_{d,g}$ is the quotient of a nonsingular scheme, so in particular it is normal and there is an injection:

$$\text{Pic}(\mathcal{T}_{d,g}) \hookrightarrow A_{4g-4}(\mathcal{T}_{d,g}),$$

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Moreover, since \((d - g + 1, 2g - 2) = 1\), the singularities of \(\overline{P}_{d,g}\) are all of finite quotient type (see [3], Proposition on p. 594). It follows that every Weil divisor is \(\mathbb{Q}\)-Cartier, so we get a surjective morphism:

\[ \text{Pic} (\overline{P}_{d,g}) \otimes \mathbb{Q} \to A_{4g-4} (\overline{P}_{d,g}) \otimes \mathbb{Q}. \]

Hence Theorem 5 yields the thesis.

\[ \square \]

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