The Magic Number Problem for Subregular Language Families

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We investigate the magic number problem, that is, the question whether there exists a minimal \( n \)-state nondeterministic finite automaton (NFA) whose equivalent minimal deterministic finite automaton (DFA) has \( \alpha \) states, for all \( n \) and \( \alpha \) satisfying \( n \leq \alpha \leq 2^n \). A number \( \alpha \) not satisfying this condition is called a magic number (for \( n \)). It was shown in \cite{11} that no magic numbers exist for general regular languages, while in \cite{5} trivial and non-trivial magic numbers for unary regular languages were identified. We obtain similar results for automata accepting subregular languages like, for example, combinational languages, star-free, prefix-, suffix-, and infix-closed languages, and prefix-, suffix-, and infix-free languages, showing that there are only trivial magic numbers, when they exist. For finite languages we obtain some partial results showing that certain numbers are non-magic.

1 Introduction

Nondeterministic finite automata (NFAs) are probably best known for being equivalent to right-linear context-free grammars and, thus, for capturing the lowest level of the Chomsky-hierarchy, the family of regular languages. It is well known that NFAs can offer exponential saving in space compared with deterministic finite automata (DFAs), that is, given some \( n \)-state NFA one can always construct a language equivalent DFA with at most \( 2^n \) states \cite{23}. This so-called powerset construction turned out to be optimal, in general. That is, the bound on the number of states is tight in the sense that for an arbitrary \( n \) there is always some \( n \)-state NFA which cannot be simulated by any DFA with less than \( 2^n \) states \cite{17, 21, 22}. On the other hand, there are cases where nondeterminism does not help for the succinct representation of a language compared to DFAs. These two milestones from the early days of automata theory form part of an extensive list of equally striking problems of NFA related problems, and are a basis of descriptional complexity. Moreover, they initiated the study of the power of resources and features given to finite automata. For recent surveys on descriptional complexity issues of regular languages we refer to, for example, \cite{6, 7, 8}.

Nearly a decade ago a very fundamental question on the well known subset construction was raised in \cite{9}: Does there always exists a minimal \( n \)-state non-deterministic finite automaton (NFA) whose equivalent minimal deterministic finite automaton (DFA) has \( \alpha \) states, for all \( n \) and \( \alpha \) with \( n \leq \alpha \leq 2^n \)? A number \( \alpha \) not satisfying this condition is called a magic number for \( n \). The answer to this simple question turned out not to be so easy. For NFAs over a two-letter alphabet it was shown that \( \alpha = 2^n - 2^k \) or \( 2^n - 2^k - 1 \), for \( 0 \leq k \leq n/2 - 2 \) \cite{9}, and \( \alpha = 2^n - k \), for \( 5 \leq k \leq 2n - 2 \) and some coprimality condition for \( k \) \cite{10}, are non-magic. In \cite{12} it was proven that the integer \( \alpha \) is non-magic, if \( n \leq \alpha \leq 1 + n(n + 1)/2 \). This result was improved by showing that \( \alpha \) is non-magic for \( n \leq \alpha \leq 2 \sqrt{n} \) in \cite{13}. Further non-magic numbers for two-letter input alphabet were identified in \cite{4} and \cite{19}. It turned out that the problem becomes easier if one allows more input letters. In fact, for exponentially growing alphabets there are no magic numbers at all \cite{12}. This result was improved to less growing alphabets in \cite{4}, to constant alphabets of size four in \cite{11}, and very recently to three-letter
alphabet [15]. Magic numbers for unary NFAs were recently studied in [5] by revising the Chrobak normal-form for NFAs. In the same paper also a brief historical summary of the magic number problem can be found. Further results on the magic number problem (in particular in relation to the operation problem on regular languages) can be found, for example, in [13, 14].

To our knowledge the magic number problem was not systematically studied for subregular languages families (except for unary languages). Several of these subfamilies are well motivated by their representations as finite automata or regular expressions: finite languages (are accepted by acyclic finite automata), combinational languages (are accepted by automata modeling combinational circuits), star-free languages or regular non-counting languages (which can be described by regular-like expression using only union, concatenation, and complement), prefix-closed languages (are accepted by automata where all out-transitions of every accepting state go to a rejecting sink state), and infix-free languages (are accepted by non-returning and returning automata, that is, automata where the initial state does not have any in-transition), prefix-free languages (are accepted by non-exiting automata, that is, automata where all out-transitions of every accepting state go to a rejecting sink state), and infix-free languages (are accepted by non-returning and non-exiting automata, where these conditions are necessary, but not sufficient).

The hierarchy of these and some further subregular families is well known. We study all families mentioned with respect to the magic number problem, and show—except for finite languages, where only some partial results will be presented—that there are only trivial magic numbers, whenever they exist.

2 Definitions

Let $\Sigma^*$ denote the set of all words over the finite alphabet $\Sigma$. For $n \geq 0$ we write $\Sigma^n$ for the set of all words of length $n$. The empty word is denoted by $\lambda$ and $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$. A language $L$ over $\Sigma$ is a subset of $\Sigma^*$. For the length of a word $w$ we write $|w|$. Set inclusion is denoted by $\subseteq$ and strict set inclusion by $\subset$. We write $2^\Sigma$ for the power set and $|\Sigma|$ for the cardinality of a set $\Sigma$.

A nondeterministic finite automaton (NFA) is a quintuple $A = (Q, \Sigma, \delta, q_0, F)$, where $Q$ is the finite set of states, $\Sigma$ is the finite set of input symbols, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of accepting states, and $\delta : Q \times \Sigma \to 2^Q$ is the transition function. As usual the transition function is extended to $\delta : Q \times \Sigma^* \to 2^Q$ reflecting sequences of inputs: $\delta(q, \lambda) = \{q\}$ and $\delta(q, aw) = \bigcup_{q' \in \delta(q, a)} \delta(q', w)$, for $q \in Q$, $a \in \Sigma$, and $w \in \Sigma^*$. A word $w \in \Sigma^*$ is accepted by $A$ if $\delta(q_0, w) \cap F \neq \emptyset$. The language accepted by $A$ is $L(A) = \{w \in \Sigma^* \mid w \text{ is accepted by } A\}$.

A finite automaton is deterministic (DFA) if and only if $|\delta(q, a)| = 1$, for all $q \in Q$ and $a \in \Sigma$. In this case we simply write $\delta(q, a) = p$ for $\delta(q, a) = \{p\}$ assuming that the transition function is a mapping $\delta : Q \times \Sigma \to Q$. So, any DFA is complete, that is, the transition function is total, whereas for NFAs it is possible that $\delta$ maps to the empty set. Note that a sink state is counted for DFAs, since they are always complete, whereas it is not counted for NFAs, since their transition function may map to the empty set. In the sequel we refer to the DFA obtained from an NFA $A = (Q, \Sigma, \delta, q_0, F)$ by the powerset construction as $A' = (2^Q, \Sigma, \delta', \{q_0\}, F')$, where $\delta'(P, a) = \bigcup_{p \in P} \delta(p, a)$, for $P \subseteq Q$ and $a \in \Sigma$, and $F' = \{P \subseteq Q \mid P \cap F \neq \emptyset\}$.

As already mentioned in the introduction, in [11] it was shown that for all integers $n$ and $\alpha$ such that $n \leq \alpha \leq 2^n$, there exists an $n$-state nondeterministic finite automaton $A_{n, \alpha}$ whose equivalent minimal deterministic finite automaton has exactly $\alpha$ states. Since some of our constructions rely on this proof
and for the sake of completeness and readability we briefly recall the sketch of the construction. In the following we call the NFA $A_{n,\alpha}$ the Jirásek-Jirásková-Szabari automaton, or for short the JJS-automaton.

**Theorem 1 ([11])** For all integers $n$ and $\alpha$ such that $n \leq \alpha \leq 2^n$, there exists an $n$-state nondeterministic finite automaton $A_{n,\alpha}$ whose equivalent minimal deterministic finite automaton has exactly $\alpha$ states.

In the construction for some fixed integer $n$ the cases $\alpha = n$ and $\alpha = 2^n$ are treated separately by appropriate witness languages. For the remaining cases it is first shown that every $\alpha$ satisfying $n < \alpha < 2^n$ can be written as a specific sum of powers of two. In particular, for all integers $n$ and $\alpha$ such that $n < \alpha < 2^n$, there exist integers $k$ and $m$ with $1 \leq k < n - 1$ and $1 \leq m < 2^k$, such that

$$\alpha = n - (k + 1) + 2^k + m$$

and

$$m = (2^k - 1) + (2^{k_2} - 1) + \cdots + (2^{k_\ell - 1} - 1) + \left\{ \begin{array}{ll} 2^{k_\ell} - 1 & \text{if } 1 \leq \ell \leq \ell - 1 \\ 2 \cdot 2^{k_\ell} - 1 & \text{if } \ell = \ell \text{ and } m \text{ is of the first form} \\ 2 \cdot 2^{k_\ell} - 1 & \text{if } \ell = \ell + 1 \text{ and } m \text{ is of the second form} \\ 0 & \text{otherwise.} \end{array} \right.$$ 

where $1 \leq \ell \leq k - 1$ and $k \geq k_1 > k_2 > \cdots > k_\ell \geq 1$. Then NFAs are constructed such that the powerset construction yields DFAs whose number of states is exactly one of these powers of two, which finally have to be combined appropriately to lead to a single $n$-state NFA $A_{n,\alpha}$ whose equivalent minimal DFA has exactly $\alpha$ states. Automaton $A_{n,\alpha}$ is depicted in Figure 1 where the following $d$-transitions are not shown:

![Diagram](image_url)

Figure 1: Jirásek-Jirásková-Szabari’s (JJS) nondeterministic finite automaton $A_{n,\alpha}$ with $n$ states ($d$-transitions are not shown) accepting a language for which the equivalent minimal DFA needs exactly $\alpha = n - (k + 1) + m$ states.

### 3 Results

We systematically investigate the magic number problem for the aforementioned subregular language families. For the remaining theorems of this paper, when speaking of an $n$-state NFA we always mean a
minimal NFA. Given a subregular language family, if \( f(n) \) is the number of states that is sufficient and necessary in the worst case for a DFA to accept the language of an \( n \)-state NFA belonging to the family, then a number \( \alpha \) with \( f(n) < \alpha \leq 2^n \) is called a trivial magic number. Similarly, if \( g(n) \) is the number of states that is necessary for any DFA simulating an arbitrary \( n \)-state NFA, then all numbers \( \alpha \) with \( \alpha < g(n) \) is also called a trivial magic number. For example, for infix-free languages \( g(n) \) is shown to be \( n + 1 \) in Theorem 5 while \( f(n) \) is known to be \( 2^{n-2} + 2 \) \([1]\). Due to space constraints most proofs are omitted.

An observation from \([1]\) shows that the magic number problem for elementary and combinational languages is trivial.

### 3.1 Star-Free Languages and Power Separating Languages

A language \( L \subseteq \Sigma^* \) is star-free (or regular non-counting) if and only if it can be obtained from the elementary languages \( \{a\} \), for \( a \in \Sigma \), by applying the Boolean operations union, complementation, and concatenation finitely often. These languages are exhaustively studied, for example, in \([20]\). Since regular languages are closed under Boolean operations and concatenation, every star-free language is regular. On the other hand, not every regular language is star free.

Here we use an alternative characterization of star-free languages by so called permutation-free automata \([20]\): A regular language \( L \subseteq \Sigma^* \) is star-free if and only if the minimal DFA accepting \( L \) is permutation-free, that is, there is no word \( w \in \Sigma^* \) that induces a non-trivial permutation of any subset of the set of states. Here a trivial permutation is simply the identity permutation. Observe that a word \( uw \) induces a non-trivial permutation \( \{q_1, q_2, \ldots, q_n\} \subseteq Q \) in a DFA with state set \( Q \) and transition function \( \delta \) if and only if \( wu \) induces a non-trivial permutation \( \{\delta(q_1, u), \delta(q_2, u), \ldots, \delta(q_n, u)\} \) in the same automaton. Further, if one finds a non-trivial permutation consisting of multiple disjoint cycles, it suffices to consider a single cycle. Before we show that no magic numbers exist for star-free languages we prove a useful lemma on permutations in (minimal) DFAs obtained by the powerset construction.

**Lemma 2** Let \( A \) be a nondeterministic finite automaton with state set \( Q \) over alphabet \( \Sigma \), and assume that \( A' \) is the equivalent minimal deterministic finite automaton, which is non-permutation-free. If the word \( w \) in \( \Sigma^* \) induces a non-trivial permutation on the state set \( \{P_0, P_1, \ldots, P_{n-1}\} \subseteq 2^Q \) of \( A' \), that is, \( \delta'(P_i, w) = P_{i+1}, \) for \( 0 \leq i < n-1 \), and \( \delta'(P_{n-1}, w) = P_0 \), then there are no two states \( P_i \) and \( P_j \) with \( i \neq j \) such that \( P_i \subseteq P_j \).

**Proof:** Assume to the contrary that \( P_0 \subseteq P_i \) (possibly after a cyclic shift), for some \( 0 < i \leq n-1 \). Then one can show by induction that \( \delta'(P_0, v) \subseteq \delta'(P_i, v) \), for every word \( v \in \Sigma^* \). In particular, this also holds true for the word \( w \) that induces the non-trivial permutation on the state set \( \{P_0, P_1, \ldots, P_{n-1}\} \). But then \( P_{ki \mod n} = \delta'(P_0, w^{ki}) \subseteq \delta'(P_{i}, w^{ki}) = P_{(k+1)i \mod n} \), for \( k \geq 0 \), and one finds the chain of inclusions \( P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_{(k+1)i \mod n} \subseteq P_0 \), which implies \( P_0 = P_i \), a contradiction. \( \square \)

Now we are prepared for the main theorem, which utilizes Lemma 2.

**Theorem 3** For all integers \( n \) and \( \alpha \) such that \( n \leq \alpha \leq 2^n \), there exists an \( n \)-state nondeterministic finite automaton accepting a star-free language whose equivalent minimal deterministic finite automaton has exactly \( \alpha \) states.

The previous theorem generalizes to all language families that are a superset of the family of star-free languages such as, for example, the family of power separating languages introduced in \([25]\).
3.2 Stars and Comet Languages

A language $L \subseteq \Sigma^*$ is a star language if and only if $L = H^*$, for some regular language $H \subseteq \Sigma^*$, and $L \subseteq \Sigma^*$ is a comet language if and only if it can be represented as concatenation $G^*H$ of a regular star language $G^* \subseteq \Sigma^*$ and a regular language $H \subseteq \Sigma^*$, such that $G \neq \{\lambda\}$ and $G \neq \emptyset$. Star languages and comet languages were introduced in [2] and [3]. Next, a language $L \subseteq \Sigma^*$ is a two-sided comet language if and only if $L = EG^*H$, for a regular star language $G^* \subseteq \Sigma^*$ and regular languages $E, H \subseteq \Sigma^*$, such that $G \neq \{\lambda\}$ and $G \neq \emptyset$. So, (two-sided) comet languages are always infinite. Clearly, every star language not equal to $\{\lambda\}$ is also a comet language and every comet is a two-sided comet language, but the converse is not true in general.

**Theorem 4** For all integers $n$ and $\alpha$ such that $n \leq \alpha \leq 2^n$, there exists an $n$-state nondeterministic finite automaton accepting a star language whose equivalent minimal deterministic finite automaton has exactly $\alpha$ states. The statement remains valid for (two-sided) comet languages.

3.3 Subword Specific Languages

In this section we consider languages for which for every word in the language either all or none of its prefixes, suffixes or infixes belong to the same language. Again, there are only trivial magic numbers. We start with subword-free languages.

A language $L \subseteq \Sigma^*$ is prefix-free if and only if $y \in L$ implies $yz \notin L$, for all $z \in \Sigma^+$, infix-free if and only if $y \in L$ implies $xyz \notin L$, for all $x \in \Sigma^+$, and suffix-free if and only if $y \in L$ implies $xy \notin L$, for all $x \in \Sigma^+$.

**Theorem 5** Let $A$ be a minimal $n$-state NFA accepting a non-empty prefix-, suffix- or infix-free language. Then any equivalent minimal DFA accepting language $L(A)$ needs at least $n+1$ states.

In the following we show that no non-trivial magic numbers exist for subword-free languages. The upper bound for the deterministic blow-up in prefix- and suffix-free languages is $2^{n-1}+1$ and for infix-free languages it is $2^{n-2}+2$, so all numbers above are trivially magic.

**Theorem 6** For all integers $n$ and $\alpha$ such that $n < \alpha \leq 2^{n-1} + 1$, there exists an $n$-state nondeterministic finite automaton accepting a prefix-free language whose equivalent minimal deterministic finite automaton has exactly $\alpha$ states. The statement remains true for NFAs accepting suffix-free languages.

For infix-free regular languages the situation is slightly different compared to above.

**Theorem 7** For all integers $n$ and $\alpha$ such that $n < \alpha \leq 2^{n-2} + 2$, there exists an $n$-state nondeterministic finite automaton accepting an infix-free language whose equivalent minimal deterministic finite automaton has exactly $\alpha$ states.

Next, we consider prefix-, infix-, and suffix-closed languages. A language $L \in \Sigma^*$ is prefix-closed if and only if $xy \in L$ implies $x \in L$, for $x \in \Sigma^*$, infix-closed if and only if $xyz \in L$ implies $y \in L$, for $x, z \in \Sigma^*$, and suffix-closed if and only if $yz \in L$ implies $z \in L$, for $z \in \Sigma^*$. We use the following results from [16].

**Theorem 8** (1) A nonempty regular language is prefix-closed if and only if it is accepted by some nondeterministic finite automaton with all states accepting. (2) A nonempty regular language is infix-closed if and only if it is accepted by some nondeterministic finite automaton with multiple initial states with all states both initial and accepting.
Prefix-closed languages reach the upper bound of $2^n$ states, and for infix-closed languages it is $2^{n-1} + 1$. Up to these bounds the only magic number for both language families is $n$ (except for $n = 1$). The upper bound for suffix-closed languages is $2^{n-1} + 1$, and up to this, no magic numbers exist.

**Theorem 9** For all integers $n$ and $\alpha$ such that $n < \alpha \leq 2^{n-1} + 1$, there exists an $n$-state nondeterministic finite automaton accepting an infix-closed language whose equivalent minimal deterministic finite automaton has exactly $\alpha$ states. The case $\alpha = n$ can only be reached for $n = 1$.

**Proof**: For the second statement, note that each DFA accepting a language $L \neq \Sigma^*$ needs a non-accepting state, which the minimal NFA cannot have, due to Theorem 8. So, $\Sigma^*$ is the only infix-closed language, for which the size of the minimal DFA equals the size of an equivalent minimal NFA. Both have a single state. The case $\alpha = 2^{n-1} + 1$ is discussed in [1]. For the remaining, assume $n < \alpha \leq 2^{n-1}$. In this case, the JJS-automaton $A_{n,\alpha} = (Q, \Sigma, \delta, n-1, \{k\})$ has a non-empty initial tail of states, that is, the initial state is equal to state $n-1$. From $A_{n,\alpha}$ we construct an automaton $A_1 = (Q, \Sigma \cup \{\#, \}$, $\delta_1, Q, Q)$ with all states initial and accepting and transition function $\delta_1(k, \#) = \{k\}$, $\delta_1(q, \#) = \{q-1\}$ if $k+2 \leq q \leq n-1$, $\delta_1(k+1, \#) = \{1\}$ and $\delta_1(q, a) = \delta(q, a)$ for $0 \leq q \leq k$ and $a \in \Sigma$. This NFA with multiple initial states can be converted into an equivalent NFA $A_2$ with initial state $n-1$ and the transition function $\delta_2(n-1, a) = \bigcup_{q \in Q} \delta_1(q, a)$ and $\delta_2(q, a) = \delta_1(q, a)$ for all $a \in \Sigma \cup \{\#, \}$ and $q \in Q \setminus \{n-1\}$.

With $S_1 = \{(S^i, S^{n-(k+1)-i}c^k) \mid 0 \leq i \leq n-(k+1)\}$, $S_2 = \{(S^{n-(k+1)}c, c^{k-1}) \mid 1 \leq i \leq k-1\}$ and $S_3 = \{(S^{n-(k+1)}d, c^k)\}$, one can easily check, that $S = S_1 \cup S_2 \cup S_3$ is a fooling set for $L(A_2)$: Different pairs from $S_1$ result in a word beginning with more than $n-(k+1)$ $\#$-symbols, pairs from $S_2$ result in too many c-symbols, $c^k$ from $S_3$ cannot be combined with any other word and mixing pairs from $S_1$ and $S_2$ either results in a word containing the infix $Sc^k$ or, if $(S^{n-(k+1)}, c^{k-1})$ is chosen from $S_1$, in $S^{n-(k+1)}c^k$, which ends with too many c-symbols.

In the corresponding powerset automaton $A'_2$, by reading prefixes of $S^{n-(k+1)}$, one reaches $n-(k+1)$ states $\{n-1\}, \{n-2, \ldots, k+1, 1\}, \ldots, \{k+1\}$. After reading $S^{n-(k+1)}$, $A'_2$ is in state $\{1\}$ and from there, according to the JJS-construction, $2^k + m$ states from $2^{\{0,1,\ldots,k\}}$ are reachable. So we have exactly $\alpha$ states. To see that no further states can be reached, note that the transition function differs from the one of the JJS-automaton only in states $k+1, \ldots, n-1$ and state $k$. The #-transition in state $k$ gives no new reachable states and reading $S$ always leads to either a state $\{n-i, \ldots, k+1, 1\}$, for some $1 \leq i \leq n-k+1$, or to state $\{1\}$ or the empty set. So, the only interesting transitions are those of the initial state $\{n-1\}$ on the input symbols $a, b, c$ and $d$. Reading $a$ or $b$ leads to $\{0, \ldots, k\}$, reading $c$ to $\{1, \ldots, k\}$ and on input $d$, $A'_2$ enters the state $\delta(q, d)$ for the largest $q \in Q$ for which this transition is defined. All these states were already counted.

To prove that any two distinct states $M, N \subseteq Q \setminus \{n-1\}$ are pairwise inequivalent, without loss of generality, pick an element $q \in M \setminus N$. If $q \leq k$, the word $c^{k+q}#$ distinguishes $M$ and $N$. Otherwise, if $q \geq k+1$, one can drive it to state 1 by reading $S$-symbols, and then $c^{k-1}$ distinguishes the two states. Finally, state $\{n-1\}$ is inequivalent with any state $N \subseteq Q \setminus \{n-1\}$ by the input word $S^{n-(k+1)}c^{k-1}$.

The family of infix-closed languages is a subset of the family of suffix-closed languages, so the previous theorem generalizes to the latter language family, except for $n$ which is not magic for $n \geq 1$ anymore:

**Corollary 10** For all integers $n$ and $\alpha$ such that $n \leq \alpha \leq 2^{n-1} + 1$, there exists an $n$-state nondeterministic finite automaton accepting a suffix-closed language whose equivalent minimal deterministic finite automaton has exactly $\alpha$ states.
Since the upper bound for the deterministic blow-up of prefix-closed languages is greater than that of infix-closed languages, we need to treat them separately here.

**Theorem 11** For all integers $n$ and $\alpha$ such that $n < \alpha \leq 2^n$, there exists an $n$-state nondeterministic finite automaton accepting a prefix-closed language whose equivalent minimal deterministic finite automaton has exactly $\alpha$ states. The case $\alpha = n$ can only be reached for $n = 1$.

### 3.4 Finite Languages

For finite languages the magic number problem turns out to be more challenging which seems to coincide with the fact, that the upper bounds for the deterministic blow-up of finite languages differ much from these of infinite language families. In [24] it was shown that for each $n$-state NFA over an alphabet of size $k$, there is an equivalent DFA with at most $O(k^n/\log(k+1))$ states. This matches an earlier result of $O(2^{n/2})$ for finite languages over binary alphabets [13].

In this section we give some partial results for finite languages over a binary alphabet, that is, we show that a roughly quadratic interval beginning at $n+1$ contains only non-magic numbers and that numbers of some exponential form $2^{(n-1)/2} + 2^j$ are non-magic, too. Note that for finite languages, $n$ is a trivial magic number, since any DFA needs a non-accepting sink state which is not necessary for an NFA.

**Theorem 12** For all integers $n$ and $\alpha$ such that $n + 1 \leq \alpha \leq (\frac{n}{2})^2 + \frac{n}{2} + 1$ if $n$ is even, and $n + 1 \leq \alpha \leq (\frac{n-1}{2})^2 + n + 1$ if $n$ is odd, there exists an $n$-state nondeterministic finite automaton accepting a finite language over a binary alphabet whose equivalent minimal deterministic finite automaton has exactly $\alpha$ states.

**Proof**: The case $\alpha = n + 1$ can be seen with the witness language $\{a, b\}^n$. So, assume $n + 1 < \alpha$. Then there exist integers $k$ and $m$ such that

$$k = \max\{x \geq 0 \mid \alpha > 1 + \sum_{i=0}^{x} n - 2i\} \quad \text{and} \quad m = \alpha - 1 - \sum_{i=0}^{k} n - 2i.$$

Let $A = (\{1, \ldots, n\}, \{a, b\}, \delta, 1, \{n\})$ be an NFA with $\delta(q, a) = \{q + 1, 2q + 1, 2q + 2, \ldots, n\}$, $1 \leq q \leq k$, $\delta(k + 1, a) = \{(k + 1) + 1, n - (m - 1), n - (m - 1) + 1, \ldots, n\}$, $\delta(q, a) = \{q + 1\}$, for $k + 1 < q < n$, and $\delta(q, b) = \{q + 1\}$, for $q < n$.

The transitions on $b$ ensure the minimality of $A$ and the inequivalence of states in the corresponding powerset automaton $A'$. To count all reachable states of $A'$, we partition the set $\{a, b\}^*$ as follows:

$$\{a, b\}^* = \bigcup_{i=0}^{k} \{b^i a\} \{a, b\}^* \cup \{b^{k+1}\} \{a, b\}^* \cup \{b\}^*.$$

With words from $\{b\}^*$, the singletons $\{1\}, \ldots, \{n\}$ and $\emptyset$ are reachable—which gives $n + 1$ states. Next, let $w = b'^i aw'$ and $w' \in \{a, b\}^i$ for some integers $0 \leq i \leq k - 1$ and $j \geq k$. Then

$$\delta'((1), w) = \delta'(\{i + 1\}, aw')$$

$$= \delta'(\{i + 2, 2(i + 1) + 1, 2(i + 1) + 2, \ldots, n\}, w')$$

$$= \{i + j + 2, 2(i + 1) + j + 1, 2(i + 1) + j + 2, \ldots, n\}. \quad (1)$$

Since we already counted the singleton sets and the empty set, we have to count sets of the form $\{1\}$ having at least two elements. We conclude that the set $\{i + j + 2, 2(i + 1) + j + 1, 2(i + 1) + j + 2, \ldots, n\}$...
Theorem 13 For all integers \(n\) and \(\alpha\) such that \(\alpha = 3 \cdot 2^{(n/2)-1} + \beta\) if \(n\) is even and \(\alpha = 2^{(n+1)/2} + \beta\) if \(n\) is odd, with \(\beta = 2^i - 1\) for some integer \(1 \leq i \leq \lceil \frac{n-1}{2} \rceil\), there exists an \(n\)-state nondeterministic finite automaton accepting a finite language over a binary alphabet whose equivalent minimal deterministic finite automaton has exactly \(\alpha\) states.

Proof: Let \(n, \alpha\) and \(\beta\) be as required and \(x = n + 1 - \log(\beta + 1)\). We construct a minimal automaton \(B_{n,\beta}\) adapting \(A_{n-1} = (\{1, \ldots, n-1\}, \{a, b\}, \delta_1, 1, \{n-1\})\) from above by taking a new initial state 0 and setting the transition function \(\delta\) to \(\delta(0,b) = \{1\}\), \(\delta(0,a) = \{1,x\}\), and \(\delta(q,c) = \delta_1(q,c)\), for \(1 \leq q \leq n-1\) and letter \(c \in \{a, b\}\).

Let \(A'_{n-1}\) and \(B'_{n,\beta}\) be the powerset automata of \(A_{n-1}\) and \(B_{n,\beta}\). Then, by reading words \(bw'\) for \(w' \in \{a, b\}^*\), all states of \(A'_{n-1}\) are reachable in \(B'_{n,\beta}\). Together with the initial state \(\{0\}\), these
Figure 2: The NFA $A_n$ from [18] and its powerset automaton that builds a binary tree. In the DFA on the right the transitions of states $\{3\}$ and $\{3, 4\}$ are the same as for $\{3, 5\}$ and $\{3, 4, 5\}$, respectively.

are $2^{(n-1)/2+1}$ states if $n$ is odd, and $3 \cdot 2^{n/2-1}$ states if $n$ is even. For considering words of the form $w = aw'$, for $w' \in \{a, b\}^*$, let $k = \lceil \frac{n-1}{2} \rceil$. Then $k + 1 \leq x \leq n$ and we reach the states

$$\delta'(\{0\}, aw) = \delta'(\{1, x\}, w) = \delta'(\{1\}, w) \cup \delta'(\{x\}, w).$$

These states differ from the ones in $A'_{n-1}$ as long as $\delta'(\{x\}, w) \neq \emptyset$, and this holds if and only if $|w| \leq n - x$. There are $2^{n-x+1} - 1 = \beta$ such words, so there are $\beta$ additional states.

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