The following question arises naturally: does a tubular neighborhood of \( C \) strongly dense in \( W \)?

We address in this paper the problem of density of smooth maps in the fractional Sobolev spaces \( W^{s,p}(Q^m; N^n) \) with values into manifolds. More precisely, let \( u : Q^m \rightarrow \mathbb{R}^\nu \) be a compact manifold of dimension \( \nu \) ison the cube \( Q^m \) with values into \( N^n \). The class of Sobolev maps \( \text{Sobolev spaces} \)

Proposition 1.1. If \( sp \geq m \), then the family of smooth maps \( C^\infty(Q^m; N^n) \) is strongly dense in \( W^{s,p}(Q^m; N^n) \).

Here is the sketch of the argument: given \( u \in W^{s,p}(Q^m; N^n) \), we consider the convolution \( \varphi_\epsilon * u \) with a smooth kernel \( \varphi_\epsilon \). If the range of \( \varphi_\epsilon * u \) lies in a small tubular neighborhood of \( N^n \), then we may project \( \varphi_\epsilon * u \) pointwisely into \( N^n \). We can always do this for \( \epsilon > 0 \) sufficiently small as long as \( sp \geq m \). Indeed, in this case \( W^{s,p}(Q^m; \mathbb{R}^\nu) \) imbeds into the space of functions of vanishing mean oscillation \( \text{VMO}(Q^m; \mathbb{R}^\nu) \), whence dist \( (\varphi_\epsilon * u, N^n) \) converges uniformly to \( 0 \) [9, Eq. (7)].

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**Abstract.** Brezis and Mironescu have announced several years ago that for a compact manifold \( N^n \subset \mathbb{R}^\nu \) and for real numbers \( 0 < s < 1 \) and \( 1 \leq p < \infty \), the class \( C^\infty(Q^m; N^n) \) of smooth maps on the cube with values into \( N^n \) is dense with respect to the strong topology in the Sobolev space \( W^{s,p}(Q^m; N^n) \) when the homotopy group \( \pi_{[sp]}(N^n) \) of order \([sp]\) is trivial. The proof of this beautiful result is long and rather involved. Under the additional assumption that \( N^n \) is \([sp]\) simply connected, we give a shorter proof of their result. Our proof for \( sp \geq 1 \) is based on the existence of a retraction of \( \mathbb{R}^\nu \) onto \( N^n \) except for a small subset in the complement of \( N^n \) and on the Gagliardo-Nirenberg interpolation inequality for maps in \( W^{1,q} \cap L^\infty \). In contrast, the case \( sp < 1 \) relies on the density of step functions on cubes in \( W^{s,p} \).

1. Introduction

We address in this paper the problem of density of smooth maps in the fractional Sobolev spaces \( W^{s,p} \) with values into manifolds. More precisely, let \( 0 < s < 1 \) and \( 1 \leq p < +\infty \), and let \( N^n \) be a compact manifold of dimension \( n \) imbedded in the Euclidean space \( \mathbb{R}^\nu \). The class of Sobolev maps \( W^{s,p}(Q^m; N^n) \) on the unit \( m \) dimensional cube \( Q^m \) with values into \( N^n \) is defined as the set of measurable maps \( u : Q^m \rightarrow \mathbb{R}^\nu \) such that

\[
\|u(x)\|_{N^n} \quad \text{for a.e. } x \in Q^m
\]

having finite Gagliardo seminorm [12],

\[
[u]_{W^{s,p}(Q^m)} = \left( \int_{Q^m} \int_{Q^m} \frac{|u(x) - u(y)|^p}{|x - y|^{m+sp}} \, dx \, dy \right)^{1/p}.
\]

The following question arises naturally: does \( W^{s,p}(Q^m; N^n) \) coincide with the closure of smooth maps \( C^\infty(Q^m; N^n) \) with respect to the distance given by

\[
d_{s,p}(u, v) = \|u - v\|_{L^p(Q^m)} + [u - v]_{W^{s,p}(Q^m)}?
\]

This is indeed the case when \( sp \geq m \):

**Proposition 1.1.** If \( sp \geq m \), then the family of smooth maps \( C^\infty(Q^m; N^n) \) is strongly dense in \( W^{s,p}(Q^m; N^n) \).

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The counterpart of Proposition 1.1 for $W^{1,p}(Q^m; N^n)$ and $p \geq m$ is due to Schoen and Uhlenbeck [25]. The role played by VMO functions in this problem has been first observed by Brezis and Nirenberg [9].

In the subtler case $sp < m$, the answer to the density problem only depends on the topology of the manifold $N^n$:

**Theorem 1.** If $sp < m$, then $C^\infty(Q^m; N^n)$ is strongly dense in $W^{s,p}(Q^m; N^n)$ if and only if $\pi_{[sp]}(N^n) \simeq \{0\}$.

We denote by $[sp]$ the integral part of $sp$ and for every $\ell \in \mathbb{N}$, $\pi_\ell(N^n)$ is the $\ell$th homotopy group of $N^n$. The topological assumption $\pi_{[sp]}(N^n) \simeq \{0\}$ means that every continuous map $f : S^{[sp]} \to N^n$ on the $[sp]$ dimensional sphere is homotopic to a constant map. The necessity of this condition has been known for some time [11, Theorem 3; 22, Theorem 4.4; 25, Section 4, Example].

Brezis and Mironescu have announced this beautiful result in a personal communication in April 2003 and a sketch of the proof can be found for instance in [21, pp. 205–206]. The analog of Theorem 1 for $W^{1,p}$ Sobolev maps had been obtained by Bethuel in his seminal paper [2] (see also [14]). Partial results for fractional Sobolev exponents $s$ were known when the manifold $N^n$ is a sphere with dimension $n \geq sp$ [11] and also in the setting of trace spaces with $s = 1 - \frac{1}{p}$ [3, 23].

The proof of Theorem 1 is long and quite involved. In this paper we prove the reverse implication of Theorem 1 in the case of $[sp]$ simply connected manifolds $N^n$. Under this assumption, we give a shorter argument which leads to the following:

**Theorem 2.** If $sp < m$ and if for every $\ell \in \{0, \ldots, [sp]\}$, 
$$\pi_\ell(N^n) \simeq \{0\},$$

then $C^\infty(Q^m; N^n)$ is strongly dense in $W^{s,p}(Q^m; N^n)$.

This condition has been used by Hajłasz [14] to give a simpler proof of Bethuel’s density result for $W^{1,p}$. In [6], we explain how Hajłasz’s strategy can be implemented for every Sobolev exponent $s \geq 1$ using some pointwise estimates involving the maximal function operator inspired from the work of Maz’ya and Shaposhnikova [10].

In order to treat the case $s < 1$, we introduce here another additional ingredient based on the density of maps which are smooth except for a small set. The case $sp \geq 1$ is covered by Proposition 2.1 below which relies on a projection argument due to Hardt and Lin [15] (Lemma 2.2 below) and on analytical estimates by Bourgain, Brezis and Mironescu [3]. The case $sp < 1$ is based on the density of step functions on cubes (Proposition 3.2 below) inspired by the works of Escobedo [11] and Bourgain, Brezis and Mironescu [3].

2. Strong density for $sp \geq 1$

The proof of Theorem 2 for $sp \geq 1$ is based on two main ingredients: (1) when the manifold $N^n$ is $[sp]$ simply connected, smooth maps are strongly dense in $W^{1,q}(Q^m; N^n)$ for every $1 \leq q < [sp] + 1$ and (2) locally Lipschitz continuous maps outside a set of dimension $m - [sp] - 1$ are dense in $W^{s,p}(Q^m; N^n)$.

The proof of the first assertion can be found in [6, 13]. Before giving the precise statement of the second assertion, we introduce for $j \in \{0, \ldots, m - 2\}$ the class $\mathcal{R}_j(Q^m; N^n)$ of maps $u : Q^m \to N^n$ such that

(i) there exists a finite union of $j$ dimensional submanifolds $T \subset \mathbb{R}^m$ such that $u$ is locally Lipschitz continuous in $Q^m \setminus T$, ...
(ii) for almost every \( x \in Q^n \setminus T \),
\[
|Du(x)| \leq \frac{C}{\text{dist}(x, T)},
\]
for some constant \( C > 0 \) depending on \( u \).

We observe that for every \( 1 \leq q < m - j \), \( \mathcal{R}_j(Q^m; N^n) \subset W^{1,q}(Q^m; N^n) \), whence by the Gagliardo-Nirenberg interpolation inequality [8; 20, Remark 1], for every \( 0 < s < 1 \),
\[
\mathcal{R}_j(Q^m; N^n) \subset W^{s,\frac{q}{s}}(Q^m; N^n).
\]
In particular, \( \mathcal{R}_{m-\lfloor sp \rfloor}(Q^m; N^n) \) is a subset of \( W^{s,p}(Q^m; N^n) \).

Assertion (2) above can be stated as follows:

**Proposition 2.1.** If \( 1 \leq sp < m \) and \( N^n \) is \( \lfloor sp \rfloor \) simply connected, then the class \( \mathcal{R}_{m-\lfloor sp \rfloor}(Q^m; N^n) \) is strongly dense in \( W^{s,p}(Q^m; N^n) \).

The proof of Theorem 1 by Brezis and Mironescu is based on the fact that \( \mathcal{R}_{m-\lfloor sp \rfloor}(Q^m; N^n) \) is strongly dense in \( W^{s,p}(Q^m; N^n) \) for every compact manifold \( N^n \). This is also known to be the case for every \( s \in \mathbb{N} \). A previous density result of this type for \( S^1 \) valued maps in \( W^{s,2} \) is due to Rivière [24] (see also [5]).

We temporarily assume Proposition 2.1 and complete the proof of Theorem 2.

**Proof of Theorem 2** when \( sp \geq 1 \). By Proposition 2.1 we only need to prove that any map \( u \in \mathcal{R}_{m-\lfloor sp \rfloor}(Q^m; N^n) \) can be approximated in the \( W^{s,p} \) norm by smooth maps.

Since \( u \in W^{1,q}(Q^m; N^n) \) for every \( 1 \leq q < \lfloor sp \rfloor + 1 \), by the topological assumption on the manifold \( N^n \) there exists a sequence of smooth maps converging to \( u \) in \( W^{1,q}(Q^m; N^n) \). When \( sp > 1 \), we may take \( q = sp \) and by the Gagliardo-Nirenberg interpolation inequality [8, Lemma D.1] the same sequence converges to \( u \) in \( W^{s,p}(Q^m; N^n) \). The Gagliardo-Nirenberg interpolation inequality fails for \( q = 1 \) in the sense that \( W^{1,1} \cap L^\infty \) is not continuously imbedded into \( W^{s,\frac{q}{s}} \). When \( sp = 1 \) we then take any fixed \( 1 < q < 2 \) and by the Gagliardo-Nirenberg interpolation inequality \( W^{s,p} \) is continuously imbedded in \( W^{1,q} \). This implies that the sequence converges to \( u \) in \( W^{s,p}(Q^m; N^n) \) as before.

We now turn ourselves to the proof of Proposition 2.1. The main geometric ingredient asserts the existence of a retraction from a cube \( Q^\nu_R \) onto \( N^n \) except for a small set [13, Lemma 6.1]:

**Lemma 2.2.** Let \( \ell \in \{0, \ldots, \nu - 2\} \). If \( N^n \) is \( \ell \) simply connected and contained in a cube \( Q^\nu_R \) for some \( R > 0 \), then there exists a closed subset \( X \subset Q^\nu_R \setminus N^n \) contained in a finite union of \( \nu - \ell - 2 \) dimensional planes and a locally Lipschitz retraction \( \kappa : Q^\nu_R \setminus X \to N^n \) such that for \( x \in Q^\nu_R \setminus X \),
\[
|D\kappa(x)| \leq \frac{C}{\text{dist}(x, X)},
\]
for some constant \( C > 0 \) depending on \( \nu \) and \( N^n \).

**Proof.** Let \( K \) be a triangulation of a polyhedral neighborhood \( K^\nu \) of \( N^n \) such that \( N^n \) is a Lipschitz deformation retract of \( K^\nu \). In particular, \( K^\nu \) and \( N^n \) are homotopically equivalent [16, p. 3] and there exists a Lipschitz retraction \( h : K^\nu \to N^n \). We extend \( K \) as a triangulation of \( Q^\nu_R \) that we denote by \( T \). Since for every \( j \in \{0, \ldots, \ell\} \),
\[
\pi_j(K^\nu) \simeq \pi_j(N^n) \simeq \{0\},
\]
there exists a Lipschitz retraction \( q : T^{\ell+1} \cup K^\nu \to K^\nu \). Denoting by \( \mathcal{L} \) a dual skeleton of \( T \) \cite[Chapter 6]{1}, let \( f : (T^\nu \setminus L^{\nu-\ell-2}) \cup K^\nu \to T^{\ell+1} \cup K^\nu \) be a locally Lipschitz retraction such that for every \( x \in (T^\nu \setminus L^{\nu-\ell-2}) \cup K^\nu \),

\[
|Df(x)| \leq C \frac{1}{\text{dist}(x, L^{\nu-\ell-2})}.
\]

The conclusion follows by taking

\[
X := L^{\nu-\ell-2} \setminus K^\nu \quad \text{and} \quad \kappa := h \circ g \circ f.
\]

The next lemma ensures that the approximation we construct in the proof of Proposition \ref{prop:approximation} belongs to a suitable class \( \mathcal{R}_j \).

**Lemma 2.3.** Let \( \Omega \subset \mathbb{R}^m \) be an open set, \( v \in C^\infty(\Omega; \mathbb{R}^\nu) \) and let \( \lambda \in \mathbb{N} \) be such that \( \lambda \leq \min \{m, \nu\} \). If \( Y \subset \mathbb{R}^\nu \) is a finite union of \( \nu - \lambda \) dimensional planes, then for almost every \( \xi \in \mathbb{R}^\nu \),

1. the set \( v^{-1}(Y + \xi) \) is a finite union of smooth submanifolds of \( \mathbb{R}^m \) of dimension \( m - \lambda \),
2. for every compact subset \( K \subset \Omega \) there exists a constant \( C > 0 \) such that for every \( x \in K \),

\[
\text{dist} \left( x, v^{-1}(Y + \xi) \right) \leq C \text{dist}(v(x), Y + \xi).
\]

**Proof.** We first assume that \( Y \) is a single \( \nu - \lambda \) dimensional plane and, without loss of generality,

\[
Y = \{0\}' \times \mathbb{R}^{\nu-\lambda}
\]

with \( 0' \in \mathbb{R}^\lambda \). Let \( P : \mathbb{R}^\lambda \times \mathbb{R}^{\nu-\lambda} \to \mathbb{R}^\lambda \) be the orthogonal projection on the \( \lambda \) first coordinates. For every \( \xi = (\xi', \xi'') \in \mathbb{R}^\lambda \times \mathbb{R}^{\nu-\lambda} \),

\[
v^{-1}(Y + \xi) = v^{-1}(Y + (\xi', 0'')) = v^{-1}(P^{-1}(\{\xi\})) = (P \circ v)^{-1}(\{\xi\}).
\]

By Sard’s lemma, almost every \( \xi' \in \mathbb{R}^\lambda \) is a regular value of the map \( P \circ v \). We deduce in this case that \( v^{-1}(Y + \xi) \) is an \( m - \lambda \) smooth submanifold of \( \Omega \).

We pursue the proof of the estimate in (ii) by assuming that \( \xi = 0 \) and \( Y \) is of the form \( \mathbb{R}^{\nu-\lambda} \) where every element of \( Y \) is a regular value of \( P \circ v \). Given \( a \in \Omega \) such that \( v(a) \in Y \), the linear transformation \( P \circ Dv(a) \) is surjective, whence there exist \( \delta > 0 \) with \( B_\delta^{\nu-\lambda}(a) \subset \Omega \) and a smooth diffeomorphism \( \psi : B_\delta^{\nu-\lambda}(a) \to \mathbb{R}^m \) such that for every \( x \in B_\delta^{\nu-\lambda}(a) \),

\[
P \circ v(x) = P \circ Dv(a)[\psi(x)].
\]

This is a consequence of the Inverse function theorem. Indeed, let \( \psi_1 \) be the orthogonal projection in \( \mathbb{R}^m \) onto \( \ker P \circ Dv(a) \) and let \( \psi_2 = (P \circ Dv(a)[\ker P \circ Dv(a)])^{-1} \circ P \circ v \). Then, \( D(\psi_1 + \psi_2)(a) = \text{id}_{\mathbb{R}^\nu} \), whence by the Inverse function theorem the function \( \psi = \psi_1 + \psi_2 \) is a smooth diffeomorphism in a neighborhood of \( a \) and satisfies \( P \circ v = P \circ Dv(a) \circ \psi \).

It follows from \ref{eq:inverse_function} that dist \((v(x), Y) = \text{dist}(Dv(a)(\psi(x)), Y)\).

Denoting by \( V = (Dv(a))^{-1}(Y) \),

we observe that for every \( y \in B_\delta^{\nu-\lambda}(a) \), \( v(y) \in Y \) if and only if \( \psi(y) \in V \). Since \( \psi \) is a diffeomorphism, there exist \( C_1 > 0 \) such that for \( x \in B_\delta^{\nu-\lambda}(a) \),

\[
\text{dist} \left( x, v^{-1}(Y) \cap B_\delta^{\nu-\lambda}(a) \right) \leq C_1 \text{dist}(\psi(x), V \cap \psi(B_\delta^{\nu-\lambda}(a))).
\]

By the counterpart of (ii) for linear transformations, there exists a constant \( C_2 > 0 \) such that for every \( z \in \mathbb{R}^m \),

\[
\text{dist}(z, V) \leq C_2 \text{dist}(Dv(a)[z], Y);
\]
To conclude the argument, take $0 < \delta \leq \delta$ such that for every $x \in B_\delta^m(a)$,
\[ \text{dist}(\psi(x), V) \leq C_2 \text{dist}(Dv(a)[\psi(x)], Y) = C_2 \text{dist}(v(x), Y). \]

We deduce from the above that for $x \in B_\delta^m(a)$,
\[ \text{dist}(x, v^{-1}(Y)) = \text{dist}(x, v^{-1}(Y) \cap B_\delta^m(a)) \]
and
\[ \text{dist}(\psi(x), V) = \text{dist}(\psi(x), V \cap \psi(B_\delta^m(a))). \]

Using a covering argument of $K \cap v^{-1}(Y)$, the conclusion follows when $Y$ is a single $\nu - \lambda$ dimensional plane.

We now assume that $Y$ is a finite union of $\nu - \lambda$ dimensional planes $Y_1, \ldots, Y_j$. The first assertion is true for almost every $\xi \in \mathbb{R}^\nu$. Concerning the second assertion, note that for every $x \in \Omega$ and for every $\xi \in \mathbb{R}^\nu$,
\[ \text{dist}(x, v^{-1}(Y + \xi)) \leq \min_{i \in \{1, \ldots, j\}} \text{dist}(x, v^{-1}(Y_i + \xi)) \]
and
\[ \text{dist}(v(x), Y + \xi) = \min_{i \in \{1, \ldots, j\}} \text{dist}(v(x), Y_i + \xi). \]

Let $\xi \in \mathbb{R}^\nu$. If the estimate holds for every $Y_i$ with some constant $C_i' > 0$, then for every $x \in K$,
\[ \text{dist}(x, v^{-1}(Y + \xi)) \leq \left( \max_{i \in \{1, \ldots, j\}} C_i' \right) \min_{i \in \{1, \ldots, j\}} \text{dist}(v(x), Y_i + \xi) \]
\[ = \left( \max_{i \in \{1, \ldots, j\}} C_i' \right) \text{dist}(v(x), Y + \xi). \]

This concludes the proof of the lemma. \hfill \Box

Given a domain $\Omega \subset \mathbb{R}^m$ and a measurable function $u : \Omega \to \mathbb{R}^\nu$, we now estimate the convolution function $\varphi \ast u$ and its derivative in terms of a fractional derivative of $u$. More precisely, given $0 < s < 1$ and $1 \leq p < +\infty$, define for $x \in \Omega$ [20],
\[ D^{s,p}u(x) = \left( \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{m+sp}} \, dy \right)^{1/p}. \]

We assume that $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a mollifier. In other words,
\[ \varphi \in C^\infty_c(B_1^m), \quad \varphi \geq 0 \text{ in } B_1^m \quad \text{and} \quad \int_{B_1^m} \varphi = 1. \]

For every $t > 0$, define $\varphi_t : \mathbb{R}^m \to \mathbb{R}$ for $h \in \mathbb{R}^m$ by
\[ \varphi_t(h) = \frac{1}{t^m} \varphi \left( \frac{h}{t} \right). \]

Using the notation above we have the following:

**Lemma 2.4.** If $u \in W^{s,p}(\Omega; \mathbb{R}^\nu)$, then for every $t > 0$ and for every $x \in \Omega$ such that $\text{dist}(x, \partial \Omega) > t$,
\begin{enumerate}[(i)]
\item $|\varphi_t \ast u(x) - u(x)| \leq C t^s D^{s,p}u(x)$,
\item $|D(\varphi_t \ast u)(x)| \leq C t^{-(1-s)} D^{s,p}u(x),$
\end{enumerate}
for some constants $C > 0$ depending on $\varphi$ and $C' > 0$ depending on $D\varphi$ and $p$. 
Proof. By Jensen’s inequality,
\[
|\varphi_t \ast u(x) - u(x)|^p \leq \int_{\mathbb{R}^m} \varphi_t(h)|u(x-h) - u(x)|^p \, dh
\]
\[
= \int_{\mathbb{R}^m} \varphi_t(h)|h|^{m+sp}|u(x-h) - u(x)|^p \, dh.
\]
Since \(\varphi_t\) is supported in \(B_t^n\), for every \(h \in \mathbb{R}^m\), \(\varphi_t(h)|h|^{m+sp} \leq C_1 t^p\). The first inequality follows.

Next, since \(\int_{\mathbb{R}^m} D\varphi_1 = 0\),
\[
|D(\varphi_1 \ast u)(x)| \leq \int_{\mathbb{R}^m} |D\varphi_1(h)||u(x-h) - u(x)| \, dh.
\]
Since
\[
\int_{\mathbb{R}^m} |D\varphi_1| \leq \frac{C_2}{t},
\]
by Jensen’s inequality,
\[
|D(\varphi_1 \ast u)(x)|^p \leq \frac{C_2^{p-1}}{t^{p-1}} \int_{\mathbb{R}^m} |D\varphi_1(h)||u(x-h) - u(x)|^p \, dh
\]
\[
= \frac{C_2^{p-1}}{t^{p-1}} \int_{\mathbb{R}^m} |D\varphi_1(h)||h|^{m+sp}|u(x-h) - u(x)|^p \, dh.
\]
Since for every \(h \in \mathbb{R}^m\), \(|D\varphi_1(h)||h|^{m+sp} \leq C_3 t^{sp-1}\), the second estimate follows.

If \(u \in W^{s,p}(\Omega; \mathbb{R}^\nu)\) and \(\kappa : \mathbb{R}^\nu \to \mathbb{R}^\nu\) is Lipschitz continuous, then \(\kappa \circ u \in W^{s,p}(\Omega; \mathbb{R}^\nu)\) and
\[
[\kappa \circ u]_{W^{s,p}(\Omega)} \leq [\kappa]_{\text{Lip}(\mathbb{R}^\nu)} [u]_{W^{s,p}(\Omega)},
\]
where \([\kappa]_{\text{Lip}(\mathbb{R}^\nu)}\) denotes the best Lipschitz constant of \(\kappa\). The next lemma gives the continuity of the composition operator \(u \mapsto \kappa \circ u\) in \(W^{s,p}\):

**Lemma 2.5.** Let \(\Omega \subset \mathbb{R}^m\) be a bounded open set and \(u \in W^{s,p}(\Omega; \mathbb{R}^\nu)\). For every \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \(\kappa : \mathbb{R}^\nu \to \mathbb{R}^\nu\) is Lipschitz continuous, \(v \in W^{s,p}(\Omega)\) and \(\|u - v\|_{W^{s,p}(\Omega, \mathbb{R}^\nu)} \leq \delta\), then
\[
[\kappa \circ u - \kappa \circ v]_{W^{s,p}(\Omega)} \leq [\kappa]_{\text{Lip}(\mathbb{R}^\nu)} \varepsilon.
\]

By a result of Marcus and Mizel [18, Theorem 1] in the scalar case \(\nu = 1\), the map \(u \in W^{1,p}(\Omega; \mathbb{R}) \mapsto \kappa \circ u \in W^{1,p}(\Omega; \mathbb{R})\) is continuous. Lemma 2.5 has been proved by Bourgain, Brezis and Mironescu [5, Claim (5.43)]. For the convenience of the reader we present their proof, organized differently.

**Proof of Lemma 2.5.** For \(u, v \in W^{s,p}(\Omega; \mathbb{R}^\nu)\) and \(\kappa : \mathbb{R}^\nu \to \mathbb{R}^\nu\), define for \(x, y \in \Omega\),
\[
I(x, y) = \frac{|\kappa(u(x)) - \kappa(v(x)) - \kappa(u(y)) + \kappa(v(y))|^p}{|x - y|^{m+sp}},
\]
so that
\[
[\kappa \circ u - \kappa \circ v]_{W^{s,p}(\Omega)} = \int_{\Omega} \int_{\Omega} I(x, y) \, dx \, dy.
\]
Observe that
\[ I(x, y) \leq 2^{p-1} \frac{|\kappa(u(x)) - \kappa(v(x))|^{p} + |\kappa(u(y)) - \kappa(v(y))|^{p}}{|x - y|^{m+sp}} \leq 2^{p-1}|\kappa|_{\text{Lip}(\mathbb{R}^p)}^{p} \frac{|u(x) - v(x)|^{p} + |u(y) - v(y)|^{p}}{|x - y|^{m+sp}} \]
and that
\[ I(x, y) \leq 2^{p-1} \frac{|\kappa(u(x)) - \kappa(u(y))|^{p} + |\kappa(v(x)) - \kappa(v(y))|^{p}}{|x - y|^{m+sp}} \leq 2^{p-1}|\kappa|_{\text{Lip}(\mathbb{R}^p)}^{p} \frac{|u(x) - u(y)|^{p} + |v(x) - v(y)|^{p}}{|x - y|^{m+sp}} \leq C_{1}|\kappa|_{\text{Lip}(\mathbb{R}^p)}^{p} \left( \frac{|u(x) - u(y)|^{p}}{|x - y|^{m+sp}} + \frac{|u(x) - v(x) - u(y) + v(y)|^{p}}{|x - y|^{m+sp}} \right). \]

Given \( \epsilon > 0 \), let
\[ A_{\epsilon} = \{ (x, y) \in \Omega \times \Omega : |u(x) - v(x)|^{p} + |u(y) - v(y)|^{p} \geq \epsilon |x - y|^{m+sp} \}. \]
Using the first upper bound of \( I(x, y) \) on the set \( \Omega \times \Omega \) \( \setminus A_{\epsilon} \) and the second one on the set \( A_{\epsilon} \), we get
\[ [\kappa \circ u - \kappa \circ v]_{W^{p, p}(\Omega)}^{p} \leq |\kappa|_{\text{Lip}(\mathbb{R}^p)}^{p} \left( 2^{p-1}\epsilon |\Omega|^{2} + C_{1} \int_{A_{\epsilon}} \frac{|u(x) - u(y)|^{p}}{|x - y|^{m+sp}} \, dx \, dy + C_{1}[u - v]_{W^{p, p}(\Omega)}^{p} \right). \]

Since \( u \in W^{p, p}(\Omega) \) and \( |A_{\epsilon}| \to 0 \) as \( \epsilon \to 0 \) in \( W^{p, p}(\Omega) \), the conclusion follows from the Dominated convergence theorem. \( \square \)

Despite of the estimate (2.3), when \( \kappa \) is not affine there is no inequality of the form
\[ [\kappa \circ u - \kappa \circ v]_{W^{p, p}(\Omega)}^{p} \leq C|\kappa|_{\text{Lip}(\mathbb{R}^p)}[u - v]_{W^{p, p}(\Omega)}. \]

In fact, the map \( u \mapsto \kappa \circ u \) is not even uniformly continuous in \( W^{p, p} \). We explain the argument when the domain is the unit cube \( Q^{n} \). For this purpose, let \( \varphi \in C^{\infty}_{c}(Q^{n}; \mathbb{R}^n) \) and denote by \( \bar{\varphi} \) the periodic extension of \( \varphi \) to \( \mathbb{R}^n \). Define for \( j \in \mathbb{N}_{+}, \)
\[ v_{j}(x) = \bar{\varphi}(jx) \]
and, for some fixed \( \xi \in \mathbb{R}^n, \)
\[ u_{j}(x) = \bar{\varphi}(jx) + \xi. \]
We observe that
\[ \|u_{j} - v_{j}\|_{W^{p, p}(Q^{m})} = \|u_{j} - v_{j}\|_{L^{p}(Q^{m})} = 2^{m}||\xi|| \]
whereas
\[ (2.4) \quad [\kappa \circ u_{j} - \kappa \circ v_{j}]_{W^{p, p}(Q^{m})}^{p} \geq j^{sp} \int_{Q^{m}} \int_{Q^{m}} \frac{|\kappa(\varphi(x) + \xi) - \kappa(\varphi(x)) - \kappa(\varphi(y) + \xi) + \kappa(\varphi(y))|^{p}}{|x - y|^{m+sp}} \, dx \, dy. \]

When \( \kappa \) is not affine, there exist \( \xi, \tau, \sigma \in \mathbb{R}^n \) such that
\[ \kappa(\tau + \xi) - \kappa(\tau) \neq \kappa(\sigma + \xi) - \kappa(\sigma). \]
Taking \( \varphi \in C_c^\infty(Q^m; \mathbb{R}^p) \) for which both sets \( \varphi^{-1}(\{\sigma\}) \) and \( \varphi^{-1}(\{\tau\}) \) have positive measure, we have
\[
\int_{Q^m} \int_{Q^m} \frac{|\kappa(\varphi(x) + \xi) - \kappa(\varphi(x)) - \kappa(\varphi(y) + \xi) + \kappa(\varphi(y))|}{|x - y|^{m + sp}} \, dx \, dy > 0.
\]
As we let \( j \) tend to infinity in (2.4), we conclude that \( u \mapsto \kappa \circ u \) is not uniformly continuous in \( W^{s,p} \).

**Proof of Proposition 2.1** Let \( u \in W^{s,p}(Q^m; N^n) \). The restrictions to \( Q^m \) of the maps \( u_\gamma \in W^{s,p}(Q^m_{1+2\gamma}; N^n) \) defined for \( x \in Q^m_{1+2\gamma} \) by \( u_\gamma(x) = u(x/(1 + 2\gamma)) \) converge strongly to \( u \) in \( W^{s,p}(Q^m; N^n) \) as \( \gamma \) tends to 0. We can thus assume from the beginning that \( u \in W^{s,p}(Q^m_{1+2\gamma}; N^n) \) for some \( \gamma > 0 \).

Let \( \kappa : \mathbb{R}^\nu \setminus X \to N^n \) be the locally Lipschitz retraction of Lemma 2.2 with \( \ell = |sp| - 1 \); we may assume that \( \nu \geq \ell + 2 \). For every \( \xi \in \mathbb{R}^\nu \), consider the map \( \kappa_\xi : \mathbb{R}^\nu \setminus (X + \xi) \to N^n \) defined by
\[
\kappa_\xi(x) = \kappa(x - \xi).
\]
Given a mollifier \( \varphi \) (see p. 5 above), the map \( \kappa_\xi \circ (\varphi_1 \ast u) \) is locally Lipschitz continuous in \( Q^m_{1+\gamma} \setminus (\varphi_1 \ast u)^{-1}(X + \xi) \). Moreover, by the chain rule and by the pointwise estimate satisfied by \( D\kappa \),
\[
(2.5) \quad |D[\kappa_\xi \circ (\varphi_1 \ast u)]| \leq C_1 \frac{|D(\varphi_1 \ast u)|}{\text{dist}(\varphi_1 \ast u, X + \xi)}.
\]
The set \( X \) is contained in a finite union of \( \nu - |sp| - 1 \) dimensional planes \( Y \) in \( \mathbb{R}^\nu \). Applying Lemma 2.3 to \( v = \varphi_1 \ast u \in C^\infty(Q^m_{1+\gamma}; \mathbb{R}^\nu) \), we obtain that for every \( 0 < t \leq \gamma \) and for almost every \( \xi \in \mathbb{R}^\nu \), the set \( (\varphi_1 \ast u)^{-1}(X + \xi) \) is contained in a finite union of \( m - |sp| - 1 \) dimensional submanifolds,
\[
T = (\varphi_1 \ast u)^{-1}(Y + \xi).
\]
By (2.5) and the inclusion \( X \subset Y \),
\[
|D[\kappa_\xi \circ (\varphi_1 \ast u)]| \leq C_2 \frac{1}{\text{dist}(\varphi_1 \ast u, X + \xi)} \leq C_2 \frac{1}{\text{dist}(\varphi_1 \ast u, Y + \xi)}.
\]
By the second part of Lemma 2.3 we conclude that for \( x \in \overline{Q^m} \setminus (\varphi_1 \ast u)^{-1}(Y + \xi) \),
\[
|D[\kappa_\xi \circ (\varphi_1 \ast u)](x)| \leq C_3 \frac{1}{\text{dist}(x, (\varphi_1 \ast u)^{-1}(Y + \xi))} = \frac{C_3}{\text{dist}(x, T)}.
\]
In particular, for every \( 0 < t \leq \gamma \) and for almost every \( \xi \in \mathbb{R}^\nu \), the map \( \kappa_\xi \circ (\varphi_1 \ast u) \) belongs to \( \mathcal{R}_{m - |sp| - 1}(Q^m; N^n) \).

We proceed using an idea from [5] for \( W^{1/2,2} \) maps with values into the circle \( \mathbb{S}^1 \). Let
\[
\alpha = \frac{1}{4} \text{dist}(X, N^n),
\]
let \( \theta : \mathbb{R}^\nu \to \mathbb{R} \) be a Lipschitz continuous function such that
(a) for \( \text{dist}(x, X) \leq 2\alpha \), \( \theta(x) = 1 \),
(b) for \( \text{dist}(x, X) \geq 3\alpha \), \( \theta(x) = 0 \),
and let
\[
\kappa_\xi = (1 - \theta) \kappa_\xi \quad \text{and} \quad \bar{\kappa}_\xi = \theta \kappa_\xi.
\]
Since $\kappa_\xi = \bar{\kappa}_\xi$ on $u(Q_{1+2\gamma}^n) \subset N^n$, we have by the triangle inequality,

\begin{equation}
(2.6) \quad \|\kappa_\xi \circ (\varphi_t \ast u) - u\|_{W^{s,p}(Q^m)} \\
\leq \|\bar{\kappa}_\xi \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)} + \|\kappa_\xi \circ \bar{\kappa}_\xi - u\|_{W^{s,p}(Q^m)} + \|\kappa_\xi \circ u - u\|_{W^{s,p}(Q^m)}.
\end{equation}

Since $\kappa$ is Lipschitz continuous on a neighborhood of $N^n$ and $\kappa_\xi \circ u = \kappa(u - \xi)$, we have by continuity of the composition operator in $W^{s,p}$ (Lemma 2.5),

\begin{equation}
(2.7) \quad \lim_{\xi \to 0} \|\kappa_\xi \circ u - u\|_{W^{s,p}(Q^m)} = 0.
\end{equation}

By Lemma 2.5 as the maps $\bar{\kappa}_\xi$ are uniformly Lipschitz continuous and $\varphi_t \ast u$ converges to $u$ in $W^{s,p}(Q^m),$

\begin{equation}
(2.8) \quad \lim_{t \to 0} \|\bar{\kappa}_\xi \circ (\varphi_t \ast u) - \bar{\kappa}_\xi \circ u\|_{W^{s,p}(Q^m)} = 0,
\end{equation}

uniformly with respect to $\xi$.

It remains to estimate the first term in the right hand side of (2.6). This is done in the following:

**Claim.** For every $0 < t \leq \gamma$,

\[
\int_{B^m_{\gamma}} \|\bar{\kappa}_\xi \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)}^p \, d\xi \leq C \int_{\{\|\varphi_t \ast u - u\| \geq \alpha\}} (D^{s,p}u)^p
\]

We assume temporarily the claim, and complete the proof of Proposition 2.1. Since $D^{s,p}u \in L^p(Q^m)$ and $\varphi_t \ast u$ converges to $u$ in measure as $t$ tends to zero, by the claim we have

\[
\lim_{t \to 0} \int_{B^m_{\gamma}} \|\bar{\kappa}_\xi \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)}^p \, d\xi = 0.
\]

By the Chebyshev inequality,

\[
\lim_{t \to 0} \left\{ \xi \in B^m_{\alpha} : \|\bar{\kappa}_\xi \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)}^p \geq \left( \int_{B^m_{\gamma}} \|\bar{\kappa}_\xi \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)}^p \, d\xi \right)^{\frac{1}{p}} \right\} = 0.
\]

Thus, for every $0 < t \leq \gamma$, there exists $\xi_t \in B^m_{\alpha}$ such that $\lim_{t \to 0} \xi_t = 0$ and

\[
\lim_{t \to 0} \|\bar{\kappa}_\xi_t \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)} = 0.
\]

We conclude from (2.6), (2.7) and (2.8) that

\[
\lim_{t \to 0} \|\kappa_\xi_t \circ (\varphi_t \ast u) - u\|_{W^{s,p}(Q^m)} = 0.
\]

This gives the conclusion of Proposition 2.1.

It remains to establish the claim:

**Proof of the claim.** Let $1 < q < p < r$ be such that

\begin{equation}
(2.9) \quad \frac{1}{p} = \frac{1-s}{r} + \frac{s}{q}.
\end{equation}

By the Gagliardo-Nirenberg interpolation inequality,

\begin{equation}
(2.10) \quad \|\bar{\kappa}_\xi \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)} \leq C_q \|\bar{\kappa}_\xi \circ (\varphi_t \ast u)\|^{1-s}_{L^q(Q^m)} \|\bar{\kappa}_\xi \circ (\varphi_t \ast u)\|^{s}_{L^q(Q^m)}.
\end{equation}

As $N^m$ is compact, we observe that the functions $\kappa_\xi \circ (\varphi_t \ast u)$ are uniformly bounded and supported on the set $\{ \text{dist} (\varphi_t \ast u, X) \leq 3\alpha \}$. Moreover,

\[
\{ \text{dist} (\varphi_t \ast u, X) \} \subset \{ |\varphi_t \ast u - u| \geq \alpha \}.
\]
Thus,
\begin{equation}
(2.11) \quad \|\kappa \circ (\varphi_t \ast u)\|_{L^r(Q^m)} \leq C_4 \left[\|\varphi_t \ast u - u\| \geq \alpha\right]^{\frac{1}{r}}.
\end{equation}
Next, by the Leibniz rule and by (2.5),
\begin{align*}
|D(\kappa \circ (\varphi_t \ast u))| &\leq \left( |D\theta(\varphi_t \ast u)| |\kappa(\varphi_t \ast u)| + |\theta(\varphi_t \ast u)| |D\kappa(\varphi_t \ast u)| \right) |D(\varphi_t \ast u)| \\
&\leq C_5 \left( 1 + \frac{1}{\text{dist}(\varphi_t \ast u, X + \xi)} \right) |D(\varphi_t \ast u)|.
\end{align*}
Since the functions $D(\kappa \circ (\varphi_t \ast u))$ are also supported in the set $\{|\varphi_t \ast u - u| \geq \alpha\}$, we get
\begin{align*}
\|\kappa \circ (\varphi_t \ast u)\|_{W^{1,q}(Q^m)} &\leq C_6 \int_{\{|\varphi_t \ast u - u| \geq \alpha\}} \left[ 1 + \left( 1 + \frac{1}{\text{dist}(\varphi_t \ast u, X + \xi)} \right) |D(\varphi_t \ast u)|^q \right] \, d\xi.
\end{align*}
For $q \geq sp$, we have by Hölder’s inequality and by Fubini’s theorem,
\begin{align*}
&\int_{B^\nu} \|\kappa \circ (\varphi_t \ast u)\|^p_{W^{1,q}(Q^m)} \, d\xi \\
&\leq |B^\nu|^{1 - \frac{sp}{q}} \left( \int_{B^\nu} \|\kappa \circ (\varphi_t \ast u)\|^q_{W^{1,q}(Q^m)} \, d\xi \right)^{\frac{sp}{q}} \\
&\leq C_7 \left( \int_{\{|\varphi_t \ast u - u| \geq \alpha\}} \int_{B^\nu} \left[ 1 + \left( 1 + \frac{1}{\text{dist}(\varphi_t \ast u(x), X + \xi)} \right) |D(\varphi_t \ast u)(x)| \right] \, d\xi \, dx \right)^{\frac{sp}{q}}.
\end{align*}
We have
\begin{align*}
\int_{B^\nu} \frac{1}{\text{dist}(\varphi_t \ast u(x), X + \xi)} \, d\xi &= \int_{B^\nu} \frac{1}{\text{dist}(\varphi_t \ast u(x) - X, \xi)} \, d\xi \\
&= \int_{B^\nu + \varphi_t \ast u(x)} \frac{1}{\text{dist}(X, \xi)} \, d\xi \\
&\leq \int_{B^\nu_R} \frac{1}{\text{dist}(X, \xi)} \, d\xi,
\end{align*}
where $R > 0$ is such that for every $x \in Q^m$, $B^\nu + \varphi_t \ast u(x) \subset B^\nu_R$. Since $X$ is a closed subset of a finite union of $\nu - [sp] - 1$ dimensional planes, assuming in addition that
\[ q \geq [sp] + 1, \]
then the last integral is finite. Thus,
\begin{align*}
&\int_{B^\nu} \|\kappa \circ (\varphi_t \ast u)\|^p_{W^{1,q}(Q^m)} \, d\xi \leq C_8 \left( \int_{\{|\varphi_t \ast u - u| \geq \alpha\}} \left[ 1 + |D(\varphi_t \ast u)|^q \right] \right)^{\frac{sp}{q}}.
\end{align*}
Inserting this estimate and (2.11) into (2.10), we deduce that

\[
\int_{B_3} \|E_{t} \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)}^p \, d\xi \\
\leq C_9 \left\{ \{\varphi_t \ast u - u \geq \alpha\} \right\}^{(1-s)p} \left( \int_{\{\varphi_t \ast u - u \geq \alpha\}} \left[ 1 + |D(\varphi_t \ast u)|^p \right] \right)^{\frac{sp}{r}}. 
\]

Since \( q < p \), by Hölder’s inequality and by the identity (2.9) satisfied by the exponents \( r, p \) and \( q \),

\[
\int_{B_3} \|E_{t} \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)}^p \, d\xi \\
\leq C_{10} \left\{ \{\varphi_t \ast u - u \geq \alpha\} \right\}^{1-s} \left( \int_{\{\varphi_t \ast u - u \geq \alpha\}} \left[ 1 + |D(\varphi_t \ast u)|^p \right] \right)^s. 
\]

By the Chebyshev inequality and by Lemma 2.4,

\[
\left\{ \{\varphi_t \ast u - u \geq \alpha\} \right\} \leq \frac{1}{\alpha^p} \int_{\{\varphi_t \ast u - u \geq \alpha\}} |\varphi_t \ast u - u|^p \\
\leq C_{11} t^{sp} \int_{\{\varphi_t \ast u - u \geq \alpha\}} (D^{s,p}u)^p. 
\]

By Lemma 2.4, we also have

\[
\int_{\{\varphi_t \ast u - u \geq \alpha\}} |D(\varphi_t \ast u)|^p \leq \frac{C_{12}}{t(1-s)p} \int_{\{\varphi_t \ast u - u \geq \alpha\}} (D^{s,p}u)^p. 
\]

We conclude that

\[
\int_{B_3} \|E_{t} \circ (\varphi_t \ast u)\|_{W^{s,p}(Q^m)}^p \, d\xi \leq C_{13} (t^{sp} + 1) \int_{\{\varphi_t \ast u - u \geq \alpha\}} (D^{s,p}u)^p. 
\]

This proves the claim. \( \square \)

3. **Strong density for \( sp < 1 \)**

The proof of Theorem 2 when \( sp < 1 \) relies on the density of step functions in \( W^{s,p} \) based on a Haar projection \( H \). This analytical step is developed in Propositions 3.1 and 3.2 below. Then, a standard tool from Differential topology (Proposition 3.3) allows us to reduce the problem to an approximation of a map with values in a convex set and this can be carried out by convolution.

Given a function \( v \in L^1(G; \mathbb{R}^d) \), we consider the Haar projection \( E_j(v) : G \to \mathbb{R}^d \) defined almost everywhere on \( G \). More precisely, denoting by \( K_{2^{-j}} \), the standard cubication of \( G \) in \( 2^m \) cubes of radius \( 2^{-j} \), for every \( \sigma \in K_{2^{-j}} \), the function \( E_j(v) \) is constant in \( \sigma \) and for \( x \in \text{int} \, \sigma \),

\[
E_j(v)(x) = \frac{1}{|\sigma|} \int_{\sigma} v. 
\]

In particular, \( E_j(v) \) is a step function.

**Proposition 3.1.** Let \( v \in L^p(G; \mathbb{R}^d) \). Then, for every \( j \in \mathbb{N}_+ \),

\[
\|E_j(v)\|_{L^p(G)} \leq \|v\|_{L^p(G)} \\
\text{and the sequence} \ (E_j(v))_{j \in \mathbb{N}_+} \text{ converges strongly to} \ v \text{ in} \ L^p(G; \mathbb{R}^d). 
\]
Proof. The estimate follows from Hölder’s inequality. To prove the convergence of the sequence \( (E_j(v))_{j \in \mathbb{N}} \), we write
\[
\|E_j(v) - v\|_{L^p(Q^m)} = \sum_{\sigma \in \mathcal{K}^{m-j}_{2^j}} \int |v(x) - \frac{1}{|\sigma|} \int v dx| dx \\
\leq \sum_{\sigma \in \mathcal{K}^{m-j}_{2^j}} \frac{1}{|\sigma|} \int |v(x) - v(y)|^p dx dy.
\]
Approximating \( v \) in \( L^p(Q^m; \mathbb{R}^n) \) by a continuous function, we deduce that the right-hand side converges to 0 as \( j \) tends to infinity. This gives the conclusion. \( \square \)

The counterpart of the previous proposition still holds in the case of fractional Sobolev spaces \( W^{s,p} \) for \( sp < 1 \) and is due to Bourgain, Brezis and Mironescu [1] Corollary A.1:

**Proposition 3.2.** Let \( v \in W^{s,p}(Q^m; \mathbb{R}^n) \). If \( sp < 1 \), then for every \( j \in \mathbb{N} \),
\[ [E_j(v)]_{W^{s,p}(Q^m)} \leq C[v]_{W^{s,p}(Q^m)} \]
for some constant \( C > 0 \) depending on \( s, p \) and \( m \). In addition, the sequence \( (E_j(v))_{j \in \mathbb{N}} \) converges strongly to \( v \) in \( W^{s,p}(Q^m; \mathbb{R}^n) \).

The proof of Bourgain, Brezis and Mironescu is based on a characterization of the fractional Sobolev spaces \( W^{s,p} \) for \( sp < 1 \) due to Bourdaud [10] in terms of the Haar basis. We present an alternative argument relying directly on the Gagliardo seminorm. The main ingredient is the following:

**Claim.** If \( sp < 1 \), then for every \( \sigma, \rho \in \mathcal{K}^m_{2^j} \),
\[
\int \int_{\sigma \cap \rho} \frac{1}{|x - y|^{m+sp}} dx dy \leq C' \frac{|\sigma||\rho|}{\delta(\sigma, \rho)^{m+sp}},
\]
where
\[ \delta(\sigma, \rho) = \sup \{|x - y| : x \in \sigma \text{ and } y \in \rho| \]
and the constant \( C' > 0 \) depends on \( m \) and \( sp \).

**Proof of the claim.** For every \( (x, y) \in \sigma \times \rho \),
\[ |x - y| \geq \delta(\sigma, \rho) - \text{diam } \sigma - \text{diam } \rho = \delta(\sigma, \rho) - 2^{-j+2} \sqrt{m}.
\]
If \( \delta(\sigma, \rho) \geq 2^{-j+3} \sqrt{m} \), then \( \frac{1}{2} \delta(\sigma, \rho) \leq |x - y| \leq \delta(\sigma, \rho) \) and the result follows in this case. Since the indicator function of the unit cube \( \chi_{Q^m} \) belongs to \( W^{s,p}(\mathbb{R}^m) \) for \( sp < 1 \), a scaling argument leads to the following estimate
\[
\frac{1}{|\sigma||\rho|} \int \int_{\sigma \cap \rho} \frac{1}{|x - y|^{m+sp}} dx dy \leq C_1 2^{(m+sp)}.
\]
In turn, this implies the claim when \( \delta(\sigma, \rho) < 2^{-j+3} \sqrt{m} \). \( \square \)

**Proof of Proposition 3.2.** Let \( \sigma, \rho \in \mathcal{K}^m_{2^j} \). For \( x \in \sigma \) and \( y \in \rho \),
\[ |E_j(v)(x) - E_j(v)(y)| \leq \frac{1}{|\sigma||\rho|} \int \int_{\sigma \cap \rho} |v(x) - v(y)| d\tilde{x} d\tilde{y}.
\]
Thus, by Jensen’s inequality,
\[ |E_j(v)(x) - E_j(v)(y)|^p \leq \frac{1}{|\sigma||\rho|} \int \int_{\sigma \cap \rho} |v(x) - v(y)|^p d\tilde{x} d\tilde{y}.
\]
We deduce that
\[
\int \int_{D_\lambda} \frac{|E_j(v)(x) - E_j(v)(y)|^p}{|x - y|^{m+sp}} \, dx \, dy \leq C' \sum_{\sigma, \rho \in K_{2^{-j}}} \int \int_{\sigma \times \rho} \frac{|v(x) - v(y)|^p}{|x - y|^{m+sp}} \, dx \, dy
\]
(3.1)
\[
\leq C' \int \int_{\sigma \times \rho} \frac{|v(x) - v(y)|^p}{|x - y|^{m+sp}} \, dx \, dy.
\]

The desired estimate follows from (3.1) by summation over dyadic cubes in \(K_{2^{-j}}\).

To prove the convergence in \(W^{s,p}\) we write for every \(\lambda > 0\),
\[
|E_j(v) - v|^p_{W^{s,p}(Q^m)}
\]
\[
\leq 2^{p-1} \int \int_{D_\lambda} \frac{|E_j(v)(x) - E_j(v)(y)|^p + |v(x) - v(y)|^p}{|x - y|^{m+sp}} \, dx \, dy
\]
\[
+ \frac{2^p|Q^m|}{\lambda^{m+sp}} \int |E_j(v) - v|^p_{Q^m},
\]
where
\[
D_\lambda = \{(x, y) \in Q^m \times Q^m : |x - y| \leq \lambda\}.
\]

By estimate (3.1),
\[
\int \int_{D_\lambda} \frac{|E_j(v)(x) - E_j(v)(y)|^p}{|x - y|^{m+sp}} \, dx \, dy \leq C_1 \sum_{\sigma, \rho \in K_{2^{-j}}} \int \int_{\sigma \times \rho} \frac{|v(x) - v(y)|^p}{|x - y|^{m+sp}} \, dx \, dy
\]
\[
\leq C_1 \int \int_{D_\lambda + Q_{2^{-j+1}}^m} \frac{|v(x) - v(y)|^p}{|x - y|^{m+sp}} \, dx \, dy.
\]

Hence,
\[
|E_j(v) - v|^p_{W^{s,p}(Q^m)}
\]
\[
\leq C_2 \int \int_{D_\lambda + Q_{2^{-j+1}}^m} \frac{|v(x) - v(y)|^p}{|x - y|^{m+sp}} \, dx \, dy
\]
\[
+ \frac{2^p|Q^m|}{\lambda^{m+sp}} \int |E_j(v) - v|^p_{Q^m}.
\]

By Proposition 3.3, the last integral tends to zero as \(j\) tends to infinity. Thus,
\[
\limsup_{j \to \infty} |E_j(v) - v|^p_{W^{s,p}(Q^m)} \leq C_2 \int \int_{D_\lambda} \frac{|v(x) - v(y)|^p}{|x - y|^{m+sp}} \, dx \, dy.
\]

The conclusion follows by choosing \(\lambda > 0\) small enough.

In the proof of Theorem 2 we need the following property from Differential topology:

**Proposition 3.3.** Let \(N^n\) be a connected manifold. Then, for every finite subset \(A\) in \(N^n\), there exists an open neighborhood of \(A\) in \(N^n\) which is diffeomorphic to the Euclidean ball \(B^n\).

**Proof.** Let \(U \subset N^n\) be an open set which is diffeomorphic to the Euclidean ball \(B^n\). There exists a diffeomorphism \(f : N^n \to N^n\) mapping \(A\) into \(U\) [17, Lemma 5.2.6]; in dimension \(n \geq 2\) this follows from the multi-transitivity in the group of diffeomorphism of \(N^n\) [1, Lemma 2.1.10]. The set \(f^{-1}(U)\) is thus diffeomorphic to \(B^n\) and contains \(A\). □
Proof of Theorem\(\PageIndex{2}\) when \(sp < 1\). Let \(u \in W^{s,p}(Q^m; N^n)\) and let \(\iota > 0\) be such that the nearest point projection \(\Pi\) into \(N^n\) is smooth on \(N^n + \mathbb{B}'\).

Let \(b \in N^n\). For every \(j \in \mathbb{N}_+\), we define \(u_j : Q^m \to \mathbb{R}^\nu\) for \(x \in Q^m\) by

\[
u_j(x) = \begin{cases} E_j(u)(x) \quad \text{if } \text{dist}(E_j(u)(x), N^n) < \iota, \\ b \quad \text{otherwise.} \end{cases}
\]

Then, \((u_j)_{j \in \mathbb{N}_+}\) is a sequence of step functions with values into \(N^n + B'\). By the triangle inequality,

\[
\|u_j - u\|_{W^{s,p}(Q^m)} \leq \|E_j(u) - u_j\|_{W^{s,p}(Q^m)} + \|E_j(u) - u\|_{W^{s,p}(Q^m)}.
\]

We need to estimate the first term in the right hand side of this inequality. Since the range of \(E_j(u)\) is contained in a fixed bounded set — for instance the convex hull of \(N^n\) —, for every \(j \in \mathbb{N}_+\),

\[
\|E_j(u) - u_j\|_{L^p(Q^m)} = \|E_j(u) - b\|_{L^p(\{\text{dist}(E_j(u), N^n) \geq \iota\})} 
\leq C_1 \|x : \text{dist}(E_j(u)(x), N^n) \geq \iota\|^\frac{1}{p}
\]

Since \(|E_j(u)(x) - u(x)| \geq \iota\) on \(\{x : \text{dist}(E_j(u)(x), N^n) \geq \iota\}\), we get

\[
\|E_j(u) - u_j\|_{L^p(Q^m)} \leq C_1 \|x : |E_j(u)(x) - u(x)| \geq \iota\|^\frac{1}{p}.
\]

Thus, by the Chebyshev inequality,

\[
\|E_j(u) - u_j\|_{L^p(Q^m)} \leq C_1 \|x : |E_j(u)(x) - u(x)| \geq \iota\|^\frac{1}{p}.
\]

We need a similar estimate for the Gagliardo seminorm \(W^{s,p}\):

**Claim.** There exist \(C > 0\) depending on \(s, p\) and \(m\) such that for every \(j \in \mathbb{N}_+\),

\[
[E_j(u) - u_j]_{W^{s,p}(Q^m)} \leq C([E_j(u) - u]_{W^{s,p}(Q^m)} + [u]_{W^{s,p}(A_j)}),
\]

where \(A_j = \{x \in Q^m : \text{dist}(E_j(u)(x), N^n) \geq \iota\}\).

**Proof of the claim.** First note that

\[
[E_j(u) - u_j]_{W^{s,p}(Q^m)}^p = 2 \sum_{\sigma \in A} \sum_{\rho \in K^m_{m-j} \setminus A} \int_{\rho} \int_{\sigma} \frac{|E_j(u)(x) - b|^p}{|x - y|^{mp + sp}} \, dx \, dy 
+ \sum_{\sigma \in A} \sum_{\rho \in A} \int_{\rho} \int_{\sigma} \frac{|E_j(u)(x) - E_j(u)(y)|^p}{|x - y|^{mp + sp}} \, dx \, dy,
\]

where

\[
A = \{\sigma \in K^m_{m-j} : \text{dist}(E_j(u)(x), N^n) \geq \iota \text{ for } x \in \sigma\}.
\]

By \(\ref{3.1}\), we have

\[
\sum_{\sigma \in A} \sum_{\rho \in A} \int_{\rho} \int_{\sigma} \frac{|E_j(u)(x) - E_j(u)(y)|^p}{|x - y|^{mp + sp}} \, dx \, dy \leq C_1 [u]_{W^{s,p}(A_j)}^p.
\]

We now estimate the term

\[
I = \sum_{\sigma \in A} \sum_{\rho \in K^m_{m-j} \setminus A} \int_{\rho} \int_{\sigma} \frac{|E_j(u)(x) - b|^p}{|x - y|^{mp + sp}} \, dx \, dy.
\]

Since the image of \(u\) is contained in \(N^n\) and \(N^n\) is bounded, there exists a constant \(C_2 > 0\) such that for every \(j \in \mathbb{N}_+\),

\[
|E_j(u) - b| \leq C_2.
\]
Applying Proposition 3.2, we deduce that
\[ I \leq C_2 \sum_{\sigma \in \mathcal{A}} \sum_{\rho \in K_m^{sp} \setminus \mathcal{A}} \int_{\mathcal{A}} \frac{1}{|x - y|^{m+sp}} \, dx \, dy \]
\[ \leq C_3 \sum_{\sigma \in \mathcal{A}} \sum_{\rho \in K_m^{sp} \setminus \mathcal{A}} \frac{|\sigma|}{\delta(\sigma, \rho)^{m+sp}}. \]

For every \( \sigma \in \mathcal{A} \),
\[ \int_{\mathcal{A}} |E_j(u) - u|^p \geq t^p |\sigma|. \]

Thus,
\[ I \leq C_3 \frac{t^p}{|\rho|} \sum_{\sigma \in \mathcal{A}} \sum_{\rho \in K_m^{sp} \setminus \mathcal{A}} \frac{|\rho|}{\delta(\sigma, \rho)^{m+sp}} \int_{\mathcal{A}} |E_j(u) - u|^p. \]

Since \( E_j(u) = \frac{1}{|\rho|} \int_{\rho} u \in \rho \), for \( x \in \sigma \) we have by the triangle inequality,
\[ |E_j(u)(x) - u(x)| \leq \frac{1}{|\rho|} \int_{\rho} |E_j(u)(x) - u(x) - E_j(u)(y) + u(y)| \, dy. \]

Thus, by Jensen’s inequality,
\[ |E_j(u)(x) - u(x)|^p \leq \frac{1}{|\rho|} \int_{\rho} |E_j(u)(x) - u(x) - E_j(u)(y) + u(y)|^p \, dy. \]

We deduce that
\[ I \leq C_3 \frac{t^p}{|\rho|} \sum_{\sigma \in \mathcal{A}} \sum_{\rho \in K_m^{sp} \setminus \mathcal{A}} \int_{\mathcal{A}} \int_{\rho} \frac{|E_j(u)(x) - u(x) - E_j(u)(y) + u(y)|^p}{|x - y|^{m+sp}} \, dy \, dx \]
and the claim follows. \( \square \)

By the triangle inequality (3.2), by estimate (3.3) and by the previous claim, we have for every \( j \in \mathbb{N}_+ \),
\[ \|u_j - u\|_{W^{s,p}(Q^n)} \leq C_4 \|E_j(u) - u\|_{W^{s,p}(Q^n)} + C[u]_{W^{s,p}(A_j)}. \]

Since \( (E_j(u))_{j \in \mathbb{N}_+} \) converges to \( u \) in measure and \( u(x) \in N^n \) for a.e. \( x \in Q^n \), the sequence \( (\{A_j\})_{j \in \mathbb{N}_+} \) converges to zero. Since \( u \in W^{s,p}(Q^n) \), by the Dominated convergence theorem we get
\[ \lim_{j \to +\infty} [u]_{W^{s,p}(A_j)} = 0. \]

Applying Proposition 3.2 we deduce that \( (u_j)_{j \in \mathbb{N}_+} \) converges strongly to \( u \) in \( W^{s,p}(Q^n; \mathbb{R}^n) \). Since \( u_j(Q^n) \subset N^n + B'_1 \), the sequence \( (\Pi \circ u_j)_{j \in \mathbb{N}_+} \) converges strongly to \( u \) in \( W^{s,p}(Q^n; N^n) \).

To conclude the proof of Theorem 2 we may then assume that \( u \) is a step function. In this case, \( u(Q^n) \) is a finite set of points in \( N^n \). By Proposition 3.3 there exists an open neighborhood \( \mathcal{U} \) of \( u(Q^n) \) in \( N^n \) and a smooth diffeomorphism \( \Phi : \mathcal{U} \to \mathcal{B}' \). Since the set \( \mathcal{B}' \) is convex, there exists a sequence of smooth maps \( (u_i)_{i \in \mathbb{N}} \subset C^\infty(Q^n; \mathcal{B}') \) which converges strongly to \( \Phi \circ u \subset W^{s,p}(Q^n; \mathcal{B}') \). Hence, the sequence \( (\Phi^{-1} \circ u_i)_{i \in \mathbb{N}} \) converges strongly to \( u \) in \( W^{s,p}(Q^n; N^n) \). This completes the proof of Theorem 2 for \( sp < 1 \). \( \square \)

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