SPECTRAL DATA ASYMPTOTICS FOR THE HIGHER-ORDER DIFFERENTIAL OPERATORS WITH DISTRIBUTION COEFFICIENTS

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Abstract
In this paper, the asymptotics of the spectral data (eigenvalues and weight numbers) are obtained for the higher-order differential operators with distribution coefficients and separated boundary conditions. Additionally, we consider the case when, for the two boundary value problems, some coefficients of the differential expressions and of the boundary conditions coincide. We estimate the difference of their spectral data in this case. Although the asymptotic behaviour of spectral data is well-studied for differential operators with regular (integrable) coefficients, to the best of the author’s knowledge, there were no results in this direction for the higher-order differential operators with distribution coefficients (generalized functions) in a general form. The technique of this paper relies on the recently obtained regularization and the Birkhoff-type solutions for differential operators with distribution coefficients. Our results have applications to the theory of inverse spectral problems as well as a separate significance.

Keywords
Higher-order differential operators · Distribution coefficients · Regularization · Eigenvalue asymptotics · Weight numbers

Mathematics Subject Classification
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Introduction

Consider the differential expression

$$\ell_{2m+\tau}(y) := \sum_{k=0}^{m-1} (-1)^{j_k} (\sigma_{2k} y (i_j))^2 + \sum_{k=0}^{m-2} (-1)^{2k+1+j} (\sigma_{2k+1} (i_j)^2 y (i_{j+1})),$$

where $$m \in \mathbb{N}, \tau = 0, 1, n = 2m + \tau; (i_{2k+j})_{2k+1}^{\tau-2}$$ are integers such that $$0 \leq i_{2k+j} \leq m - k - j, j = 0, 1; (i_{2k+j})_{2k+1}^{\tau-2}$$ are complex-valued functions satisfying

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\[ \sigma_v \in L_2(0, 1), \quad \nu = \frac{0, n - 2}{2}, \quad \sigma_{2k+j} \in L_2(0, 1) \quad \text{if } n = 2m, \ i_{2k+j} = m - k - j, \ j \in \{0, 1\}, \] (1.2)

and the derivatives \( \sigma^{(k)}_v \) are understood in the sense of distributions.

The paper aims to study spectral data asymptotics for the differential equation

\[ \ell_n(y) = \lambda y, \quad x \in (0, 1), \] (1.3)

subject to the separated boundary conditions. The results of this paper are applied in [1] to the inverse problem theory and also have a separate significance.

If the functions \( \sigma_v(x) \) are sufficiently smooth, then the differential expression Eq. 1.1 can be represented in the form

\[ y^{(n)} + \sum_{k=0}^{n-2} q_k(x)y^{(k)}, \] (1.4)

where \( (q_k)_{k=0}^{n-2} \) are some integrable functions. However, for differential operators with distribution coefficients, it is more convenient to consider the divergent form Eq. 1.1 following [2–4].

For regular differential operators Eq. 1.4, the standard approach to obtaining eigenvalue asymptotics is described in the classical monograph by Naimark [5]. In recent years, eigenvalue asymptotics of higher-order differential operators with non-smooth coefficients attract considerable attention (see, e.g. [6–9]).

For differential operators with distribution coefficients, the asymptotic behaviour of the eigenvalues has been studied much less. In [10, 11], asymptotic formulas have been obtained for the eigenvalues of the Sturm-Liouville operators with potentials of class \( W^{-1}_2(0, 1) \) (i.e. \( n = 2, i_0 = 1 \) in Eq. 1.1). In [12, 13], the eigenvalue asymptotics were studied for the even-order operator \( \frac{d^n}{dx^n} \) perturbed by distribution potential. For the higher-order differential operators with distribution coefficients of the general form Eq. 1.1, to the best of the author’s knowledge, the asymptotic behaviour of the eigenvalues has not been investigated before.

Our treatment of the differential expression Eq. 1.1 relies on the regularization methods of [2, 3, 14]. Mirzoev and Shkalikov have developed the regularization approach to the differential expression Eq. 1.1 with \( i_{2k+j} = m - k - j, \ j \in \{0, 1\} \) for an even order \( n = 2m \) in [2] and for an odd order \( n = 2m + 1 \) in [3]. Vladimirov [14] has obtained an alternative construction, which can be used for a wider class of differential operators than the results of [2, 3]. In particular, the approach of [14] has been applied to the differential expression of form Eq. 1.1 in [4]. It is worth mentioning that, in [2, 3, 14], the coefficients at \( y^{(n)} \) and \( y^{(n-1)} \) in the differential expression can be arbitrary functions of some classes. In this paper, we confine ourselves to the coefficients 1 and 0 at \( y^{(n)} \) and \( y^{(n-1)} \), respectively, because this case is natural for the inverse problem theory (see [1, 4, 15]).

Let us briefly describe the regularization of the differential expression Eq. 1.1. The matrix function \( F(x) = \{f_{kj}(x)\}_{k,j=1}^{n} \) associated with \( \ell_n(y) \) is constructed: \( F = F_{\sigma_0, \sigma_1, \ldots, \sigma_{n-2}}(\sigma_0, \sigma_1, \ldots, \sigma_{n-2}) \). The certain formulas for the associated matrix entries \( f_{kj}(x) \) are provided in the “Reduction to first-order systems” section. By using the quasi-derivatives

\[ y^{[0]} := y, \quad y^{[k]} := (y^{[k-1]})' - \sum_{j=1}^{k} f_{kj}y^{[j-1]}, \quad k = 1, n, \] (1.5)

Eq. 1.3 is reduced to the equivalent system

\[ y' = (F(x) + \Lambda)y, \quad x \in (0, 1), \] (1.6)

where \( y(x) = \text{col}(y^{[0]}(x), y^{[1]}(x), \ldots, y^{[n-1]}(x)) \), \( \Lambda \) is the \( (n \times n) \) matrix whose entry at the position \( (n, 1) \) equals \( \lambda \) and all the other entries equal 0.

Define the boundary conditions
where $r \in \{1, \ldots, n - 1\}$ is fixed, $p_s \in \{0, \ldots, n - 1\}$ for $s = 1, n$, $p_s \neq p_k$ for $1 \leq s < k \leq r$ and for $r + 1 \leq s < k \leq n$. It can be easily shown that the spectrum of the boundary value problem Eqs. 1.3, 1.7 is a countable set of eigenvalues (see [15]). In [15], the spectra of several boundary value problems of form Eqs. 1.3, 1.7 have been used as the spectral data of the inverse problem.

The first result of this paper is the following theorem, which describes the asymptotic behaviour of the eigenvalues.

**Theorem 1.1** The eigenvalues $\{\lambda_l\}_{l \geq 1}$ of the boundary value problem Eqs. 1.3, 1.7 satisfy the relation

$$
\lambda_l = \left( -1 \right)^{n-r} \left( \frac{\pi}{\sin \frac{\pi}{n}} (l + \chi + \varepsilon_j) \right)^n, \quad l \in \mathbb{N}, \quad \{\varepsilon_j\} \in l_2, \tag{1.8}
$$

where the constant $\chi$ depends only on $n, r, (p_y)^y_{y=0} \text{ and does not depend on } (\sigma_v)^{v-2}_{v=0} \text{ and } u_{s,j}, s = 1, n, j = 1, p_s$.

In the inverse problem theory [1], it is convenient to recover the coefficients $\sigma_{n-2}, \sigma_{n-3}, \ldots, \sigma_1, \sigma_0$ one-by-one. Therefore, the question arises:

*If $(\sigma_v)^{v-2}_{v=0}$ are known, then what can be said about the eigenvalue asymptotics?*

We give a rigorous answer to this question in Theorem 1.2. Denote by $L$ the boundary value problem Eqs. 1.3, 1.7 and by $\widetilde{L}$ the boundary value problem of the same form but with the coefficients $(\sigma_v)$ and $(u_{s,j})$ replaced by $(\tilde{\sigma}_v)$ and $(\tilde{u}_{s,j})$, respectively. The numbers $n, r, (i_v), \text{ and } (p_s)$ for the problems $L$ and $\widetilde{L}$ are the same. Throughout the paper, if a symbol $\gamma$ denotes an object related to the problem without tilde, then $\tilde{\gamma}$ denotes the similar object related to the problem with tilde, and $\gamma = \gamma - \tilde{\gamma}$. Consider the values $\rho_l$ and $\tilde{\rho}_l$ from the asymptotics Eq. 1.8 for the problems $L$ and $\widetilde{L}$, respectively.

**Theorem 1.2** Suppose that $\sigma_v(x) = \tilde{\sigma}_v(x)$ for a.e. $x \in (0, 1), v = v_0, n - 1$. Denote

$$
d := n - 1 - \max_{v=0, v_0=1} (v + i_v),
N_d := \{v = v_0, v_0 - 1 : d = n - 1 - (v + i_v)\}, \quad N_d^0 := \{v \in N_d : i_v = 0\},
$$

and assume that $u_{s,p_i-j} = \tilde{u}_{s,p_i-j}$ for $j = 0, d - 2, s = 1, n$. Then,

$$
\rho_l - \tilde{\rho}_l = l^d (\hat{\chi} + \hat{\delta}_l), \quad \delta_l = o(1), \quad l \to \infty,
$$

where the constant $\hat{\chi}$ depends on the numbers

$$
\int_0^1 \tilde{\sigma}_v(x) \, dx, \quad v \in N_d^0, \quad \text{and} \quad \tilde{u}_{s,p_i-d+1}, \quad s = 1, n. \tag{1.10}
$$

In particular, $\hat{\chi} = 0$ if all the numbers Eq. 1.10 equal zero. If $\tilde{\sigma}_v \in L_2(0, 1), v \in N_d$, then $\{\delta_l\} \subseteq l_2$.

Theorem 1.2 helps to improve the asymptotics of Theorem 1.1 for odd $n$ and so leads to the following result.

**Corollary 1.3** For $n = 2m + 1$, the remainder $\varepsilon_l$ in Eq. 1.8 has the form

$$
\varepsilon_l = \frac{\chi_{\hat{\chi}}}{T} + \frac{\varepsilon_{l,1}}{T}, \quad \varepsilon_{l,1} = o(1), \quad l \to \infty, \tag{1.11}
$$
and the constant $\chi_1$ depends on \( \int_0^1 \sigma_{n-3}(x) \, dx \) and \( u_{s,p,s} \), $s = 1, n$. If \( \sigma_{n-3} \in L_2(0, 1) \) and \( \sigma_{n-3} \in L_2(0, 1), i_{n-3} = 1 \) or \( i_{n-3} = 0 \), then \( \{\epsilon_{1,1}\} \in l_2$. 

In order to prove Theorem 1.1, we use the approach of Naimark [5] and the Birkhoff-type solutions constructed by Savchuk and Shkalikov [16]. The proof of Theorem 1.2 relies on the special structure of the matrix function $F(x)$ associated with the differential expression $\epsilon_n(y)$. We develop the technique of [16] to study the difference of the corresponding Birkhoff-type solutions for the problem $\mathcal{L}$ and $\tilde{\mathcal{L}}$. Furthermore, we follow the proof strategy of Theorem 1.1 to analyze the difference of the eigenvalues. 

In addition, we obtain the analogs of Theorems 1.1 and 1.2 for the weight numbers defined in the “Asymptotics of weight numbers” section. The weight numbers together with the eigenvalues are used as spectral data for recovering higher-order differential operators with distribution coefficients in [1].

The paper is organized as follows. In the “Reduction to first-order systems” section, Eq. 1.3 is transformed to the first-order system Eq. 1.6 and then Eq. 1.6 is reduced to a more convenient form for studying solution asymptotics. The “Birkhoff-type solutions” section is devoted to the Birkhoff-type solutions of Eq. 1.3 with the certain asymptotic behaviour for large values of the spectral parameter. We formulate the necessary propositions from [16] and study the difference of the Birkhoff-type solutions. In the “Eigenvalue asymptotics” section, the proofs of Theorems 1.1 and 1.2 and Corollary 1.3 are provided. In the “Examples” section, the main results on the eigenvalue asymptotics are illustrated by several examples. The “Asymptotics of weight numbers” section contains the definition of the weight numbers and the analogs of Theorems 1.1 and 1.2 for the weight numbers supplied by the proofs. Throughout the paper, we use the following notations.

- The same symbol $C$ denotes various constants independent of $x$, $\rho$, etc.
- $I$ denotes the $(n \times n)$ unit matrix.
- $\delta_{jk}$ is the Kronecker delta.
- We use the following vector and matrix norms: 
  $$
  \|a\| = \max_j |a_j|, \quad a = [a_j]_{j=1}^n, \quad \|A\| = \max_{i,j} |a_{ij}|, \quad A = [a_{ij}]_{i,j=1}^n.
  $$
- $\text{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix with the entries $(d_j)_{j=1}^n$ on the main diagonal.
- For a matrix $A = [a_{kj}]_{k,j=1}^n$, we denote by $\text{diag}(A)$ the diagonal matrix $\text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.
- We use the same notation $L_{\mu}(0, 1), \mu \in [1, \infty)$, for the space of scalar functions, for the space of vector functions 
  $$
  Y = [y_j]_{j=1}^n, \quad y_j \in L_{\mu}(0, 1), \quad \|Y\|_{L_{\mu}} = \max_j \|y_j\|_{L_{\mu}},
  $$
  and for the space of matrix functions 
  $$
  A = [a_{kj}]_{k,j=1}^n, \quad a_{kj} \in L_{\mu}(0, 1), \quad \|A\|_{L_{\mu}} = \max_{k,j} \|a_{kj}\|_{L_{\mu}}.
  $$
- The notation $\{x_j\}$ is used for various sequences of $l_2$.
- $\lambda = \rho^n, \quad \dot{f}(\rho) = \frac{d}{d\rho} f(\rho)$.

### Reduction to first-order systems

In this section, Eq. 1.3 is reduced to the system Eq. 1.6 and then to the form Eq. 2.6, which is more convenient for studying the asymptotics of solutions. This section is based on the results of [2, 4, 14, 16].

The associated matrix $F(x)$ is defined by the coefficients $(\sigma_{s})_{s=0}^{m-2}$ of the differential expression Eq. 1.1 as follows.

**Definition 2.1** Define the matrix $Q(x) = [q_{x,j}(x)]_{j,0}^m$ by the following formulas:
\[Q(x) := \sum_{s=0}^{n-2} \sigma_s(x) x_s,i, \quad x_{s,i} = [x_{s,j}]_{s=1,j=0}^m,\]

\[x_{2k,s+k-j-s+k} = C_i^{s}, \quad s = 0, i,\]
\[x_{2k+1,s+k-j+1-s+k} = C^{s}_{+1} - 2C^{s-1}_{i+1}, \quad s = 0, i + 1,\]

and all the other entries \(x_{s,j}\) equal zero. Here and below, \(C_i^{s} := \frac{i!}{s!(i-s)!}\) are the binomial coefficients, \(C_i^{-1} := 0\). Then, by using the elements \(q_{s,j}\), define the elements of the matrix function \(F(x) = [f_{k,j}]_{k,j=1}^m\) as follows:

\[f_{m,j} := (-1)^{m+1} q_{j-1,m}, \quad j = 1, m,\]
\[f_{k,m+1} := (-1)^{k+1} q_{m,2m-k}, \quad k = m + 1, 2m,\]
\[f_{k,j} := (-1)^{k+1} q_{j-1,2m-k} + (-1)^{m+k} q_{j-1,m} q_{m,2m-k}, \quad k = m + 1, 2m, j = 1, m,\]

\[n = 2m + 1 : \quad f_{k,j} := (-1)^{k} q_{j-1,2m+1-k}, \quad k = m + 1, 2m + 1, j = 1, m + 1.\]

The other elements are defined as \(f_{k,j} = \delta_{k+1,j}\).

Definition 2.1, together with the condition Eq. 1.2, implies

\[f_{k,j} \in L_1(0, 1), \quad f_{k,k} \in L_2(0, 1), \quad 1 \leq j \leq k \leq n, \quad \text{trace}(F(x)) = 0. \quad (2.1)\]

Denote by \(D'\) the space of continuous linear functionals (generalized functions) on \(D = C_0^\infty(0, 1)\). Suppose that \(y \in W^m_{2,loc}(0, 1)\) if for some indices \(v \in \{0, \ldots, n - 2\}\) the condition Eq. 1.2 implies \(\sigma_v \in L_2(0, 1)\) and \(y \in W^m_{1,loc}(0, 1)\) otherwise. Then, \(\epsilon_n(y)\) is correctly defined in \(D'\) (see [4, Lemma 2.1]). If \(y \in D_F\),

\[D_F := \{y : y^{[k]} \in AC_{loc}(0, 1), k = 0, n - 1\},\]

then \(\epsilon_n(y)\) is a regular function and \(\epsilon_n(y) = y^{[n]}\) (see [4, Theorem 2.2]). Then, instead of Eq. 1.3, one can consider the system Eq. 1.6. Indeed, in view of Eq. 1.5, the first \((n - 1)\) rows of Eq. 1.6 coincide with the definition of the quasi-derivatives \(y^{[k]}, k = 1, n - 1\), and the last row is \(y^{[n]} = \lambda y\). Below, we say that \(y\) is a solution of Eq. 1.3 if \(y \in D_F\) and the vector function \(y(x) = \text{col}(y^{[0]}(x), y^{[1]}(x), \ldots, y^{[n-1]}(x))\) satisfies Eq. 1.6.

Let \(\lambda = \rho^n\). The change of variables \(y(x) = \text{diag}(1, \rho, \ldots, \rho^{n-1})u(x)\) transforms the system Eq. 1.6 into

\[u'(x) = F(x, \rho)u, \quad x \in (0, 1),\]

where

\[F(x, \rho) = \rho F_{-1} + F_0 = \rho F_{-1} + F_0(x) + \sum_{k=1}^{n-1} \rho^{-k} F_k(x),\]

\[F_{-1} = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix},\]

and the matrix functions \(F_k(x)\) are formed by the corresponding lower diagonals of \(F(x)\). The relations Eq. 2.1 imply \(F_0 \in L_2(0, 1), F_k \in L_4(0, 1), k = T, n - 1\).

Following the standard ideas described in the book of Naimark [5], we consider the partition of the \(\rho\)-plane into the sectors.
Fix a sector $\Gamma_\kappa$. Denote by $\{\omega_k\}_{k=1}^n$ the roots of the equation $\omega^n = 1$ numbered so that
\[
\text{Re}(\omega_1) < \text{Re}(\omega_2) < \cdots < \text{Re}(\omega_n), \quad \rho \in \Gamma_\kappa.
\] (2.4)

We also define the extended sector (see Fig. 1):
\[
\Gamma_{\kappa,h} := \left\{ \rho \in \mathbb{C} : \rho + h \exp\left(\frac{i\pi(k-1/2)}{n}\right) \in \Gamma_\kappa \right\}, \quad h > 0.
\] (2.5)

Put $B := \text{diag}\{\omega_1, \omega_2, \ldots, \omega_n\}, \Omega := [\omega_k^{j-1}]_{j,k=1}^n$. Obviously, $\Omega^{-1}F_{-1}\Omega = B$. By changing the variables $w(x) := \Omega^{-1}u(x)$, we reduce the system Eq. 2.2 to the form
\[
w' = \rho Bw + A(x, \rho)w, \quad x \in (0, 1),
\] (2.6)
where
\[
A(x, \rho) = A_0(x) + \sum_{k=1}^{n-1} \rho^{-k}A_k(x),
\] (2.7)
\[
A_k(x) = \Omega^{-1}F_k(x)\Omega, \quad k = 0, n - 1.
\]

Thus, instead of Eq. 1.3, one can consider the system Eq. 2.6. It follows from Eqs. 2.1 and 2.7 that $A_0 \in L_2(0, 1)$, $A_k \in L_1(0, 1), k = 1, n - 1$, $\text{diag}(A_0(x)) \equiv 0$. 

Fig. 1 Sectors
Birkhoff-type solutions

In this section, we study the Birkhoff-type solutions with certain asymptotic behaviour as $|\rho| \to \infty$ for the system Eq. 2.6 and for Eq. 1.3. The Birkhoff-type fundamental systems of solutions (FSS) for the first-order systems which generalize Eq. 2.6 have been constructed in [16, 17]. In this paper, we use the approach of Savchuk and Shkalikov [16]. First, we provide the necessary notations and results of [16] specified for the system Eq. 2.6. Secondly, we consider the problems $L$ and $\tilde{L}$ satisfying the conditions of Theorem 1.2 and investigate the difference of the corresponding Birkhoff-type solutions. The main results for the latter case are formulated in Theorem 3.4 and Corollary 3.6.

Suppose that $A(x, \rho)$ is an arbitrary matrix function of form Eq. 2.7, where $A_k \in L_1(0, 1), k = 0, n - 1$, $\text{diag}(A_0(x)) \equiv 0$, $\Gamma_{x, h}$ is a fixed sector of form Eq. 2.5, $(\rho_j)_{j=1}^n$ are the roots of the equation $\omega^h = 1$ numbered in the order Eq. 2.4.

Denote the elements of the matrices $A_k(x)$ and $A(x, \rho)$ by $a_{k, l}$ and $v_{jl}(x, \rho) = a_{0, l}(x) + r_{jl}(x, \rho)$, $j, l = 1, n$, respectively. Put

$$v_{jl}(s, x, \rho) := (\pm)_{jk}(s)_{lk} \int a_{0, l}(t) \exp(\rho[(\omega_l - \omega_k)(t - s) + (\omega_j - \omega_k)(x - t)]) dt, \quad (3.1)$$

$$o_{jk}(s, x, \rho) := (\pm)_{jk}(s)_{lk} \int r_{jl}(t, \rho) \exp(\rho[(\omega_l - \omega_k)(t - s) + (\omega_j - \omega_k)(x - t)]) dt, \quad (3.2)$$

where $(\pm)_{jk} = \begin{cases} 1, & j < k, \\ -1, & j \geq k, \end{cases}$ the integration is taken over the intervals

$$\begin{align*}
(x, s), & \quad \text{if } j, l < k, \\
(\max \{x, s\}, 1), & \quad \text{if } j < k \leq l, \\
(0, \min \{x, s\}), & \quad \text{if } l < k \leq j, \\
(s, x), & \quad \text{if } k \leq j, l,
\end{align*}$$

and the integrals are assumed to be zero if the upper limit is less than the lower one. In view of Eq. 2.4, the exponents in Eqs. 3.1 and 3.2 are bounded:

$$|\exp(\rho[(\omega_l - \omega_k)(t - s) + (\omega_j - \omega_k)(x - t)])| \leq C, \quad \rho \in \Gamma_{x, h}.$$

Introduce the region $\mathcal{G} := \{ \rho \in \Gamma_{x, h} : |\rho| \geq \rho^* \}$ for some $\rho^* > 0$. In view of Eq. 2.7, we have

$$\|r_{jl}(\cdot, \rho)\|_{L_1} \leq C|\rho|^{-1}, \quad \rho \in \mathcal{G},$$

and so

$$\max_{j, k, l, s} |o(s, x, \rho)| \leq C|\rho|^{-1}. \quad (3.3)$$

Denote

$$\mathcal{Y}(\rho) := \max_{j, k, l, x} |v_{jk}(s, x, \rho)|. \quad (3.4)$$

**Proposition 3.1** ([16]) For any fixed sector $\Gamma_{x, h}$ and some $\rho^* > 0$, the system Eq. 2.6 has a fundamental solution matrix $w(x, \rho)$ of the form

$$w(x, \rho) = (I + \mathcal{E}(x, \rho)) \exp(\rho B x), \quad (3.5)$$

where $\mathcal{E}(x, \rho)$ is continuous for $x \in [0, 1], \rho \in \mathcal{G}$, analytic in $\rho$ for each fixed $x \in [0, 1], \rho \in \mathcal{G}$, and

$$\max_x \|\mathcal{E}(x, \rho)\| \leq C(\mathcal{Y}(\rho) + |\rho|^{-1}), \quad \rho \in \mathcal{G}. \quad (3.6)$$

**Proposition 3.2** ([16]) $\mathcal{Y}(\rho) \to 0$ as $|\rho| \to \infty, \rho \in \Gamma_{x, h}$. 


We call a sequence \( \{s_k\}_{k=1}^{\infty} \) non-condensing if
\[
\beta := \sup(N(t + 1) - N(t)) < \infty, \quad N(t) := \#\{k \in \mathbb{N} : |\rho_k| \leq t\}.
\]

**Proposition 3.3** ([18]) Suppose that \( A_0 \in L_{\mu} (0, 1), \mu \in (1, 2], \) and \( \{s_k\}_{k=1}^{\infty} \) is a non-condensing sequence in \( G. \) Then, the sequence \( \{\Upsilon(s_k)\}_{k=1}^{\infty} \) belongs to \( L_{\mu'}, \mu' = 1, \) and
\[
\|\{\Upsilon(s_k)\}\|_{L_{\mu'}} \leq C\|A_0\|_{L_{\mu}},
\]
where the constant \( C \) depends only on \( h, \rho^*, \) and \( \beta. \)

Along with the system Eq. 2.6, consider the system
\[
\dot{\vec{w}}' = \rho B\vec{w} + \vec{A}(x, \rho)\vec{w}, \quad x \in (0, 1),
\]
\[
\vec{A}(x, \rho) = \vec{A}_0(x) + \sum_{k=1}^{n-1} \rho^{-k}\vec{A}_k(x).
\]
(3.7)

Suppose that \( \vec{A}_k \in L_{l}(0, 1), k = 0, n - 1, \) and \( \text{diag}(\vec{A}_k(x)) \equiv 0. \)
Consider the difference \( \dot{w}(x, \rho) = w(x, \rho) - \vec{w}(x, \rho) \) of the fundamental solutions defined by Proposition 3.1 for the systems Eqs. 2.6 and 3.7.

**Theorem 3.4** Suppose that
\[
A_k(x) = \vec{A}_k(x) \text{ a.e. on } (0, 1), \quad k = 0, d - 1,
\]
(3.8)
for a fixed \( d \in \{1, \ldots, n - 1\}. \) Then,
\[
\dot{w}(x, \rho) = \hat{E}(x, \rho) \exp(\rho Bx),
\]
\[
\max_x \left\| \rho^d\hat{E}(x, \rho) - \int_0^x \text{diag}(\hat{A}_d(t)) \, dt \right\| \leq C(\psi(\rho) + \psi_d(\rho) + |\rho|^{-1}), \quad \rho \in \overline{G},
\]
(3.9)
where
\[
\psi(\rho) := \max_{j,k,d,t,x} \{ |\nu_{j,k}(x, x, \rho)|, |\nu_d(x, x, \rho)| \},
\]
(3.10)
\[
\psi_d(\rho) := \max_{j,k,d,t,x} |\alpha_{j,k}(x, \rho)|, \quad \alpha_{j,k}(x, \rho) := \int_{b_{j,k}}^x \hat{A}_{d,j,k}(t) \exp(\rho(\omega_j - \omega_k)(x - t)) \, dt,
\]
(3.11)
\[
b_{j,k} := \begin{cases} 0, & j \geq k, \\ 1, & j < k. \end{cases}
\]
(3.12)
Clearly, Propositions 3.2 and 3.3 are valid for \( \psi(\rho) \) defined by Eq. 3.10. Proposition 3.2 can be similarly proved for \( \psi_d(\rho). \) Proposition 3.3 is valid for \( \psi_d(\rho) \) if \( A_0 \) is replaced with \( A_d. \)

**Proof of Theorem 3.4** In this proof, we apply the technique of [16]. By changing the variables \( w(x, \rho) = \omega(x, \rho) \exp(\rho Bx), \)
\[
z(x, \rho) = [z_{j,k}(x, \rho)]_{j,k=1}^{n},
\]
we reduce the system Eq. 2.6 to the form
\[
z' = \rho(Bz - zB) + A(x, \rho)z, \quad x \in (0, 1).
\]
(3.13)
By integrating the latter system with the initial conditions
we obtain the integral equations

$$z_{jk}(x, \rho) - \delta_{jk} = \sum_{l=1}^{n} \int_{b_{jk}}^{x} v_{jl}(t, \rho) \exp(\rho(\omega_j - \omega_k)(x - t)) z_{lk}(t, \rho) dt, \quad j, k = 1, n,$$

(3.13)

where \(b_{jk}\) are defined by Eq. 3.12. The matrix function \(w(x, \rho) = z(x, \rho) \exp(\rho Bx)\) that is constructed by the solution \(z(x, \rho)\) of the system Eq. 3.13 is the fundamental matrix of Proposition 3.1.

For each fixed \(k\), the system Eq. 3.13 implies

$$z_k = z_k^0 + V_k z_k,$$

(3.14)

where \(z_k = z_k(x, \rho)\) is the \(k\)-th column of \(z(x, \rho)\), \(z_k^0\) is the \(k\)-th column of \(I\), and \(V_k = V_k(\rho)\) is the integral operator given by the right-hand side of Eq. 3.13. The similar relation can be obtained for the system Eq. 3.7:

$$\tilde{z}_k = \tilde{z}_k^0 + \tilde{V}_k \tilde{z}_k,$$

(3.15)

Subtracting Eq. 3.15 from Eq. 3.14, we get

$$\hat{z}_k = \hat{V}_k \hat{z}_k + \hat{V}_k \hat{z}_k,$$

where \(\hat{z}_k = z_k - \tilde{z}_k\), \(\hat{V}_k = V_k - \tilde{V}_k\). Formal calculations show that

$$\hat{z}_k = \hat{V}_k \hat{z}_k + \sum_{\nu=1}^{\infty} \hat{V}_k^{\nu} \hat{V}_k \hat{z}_k$$

(3.16)

$$= \hat{V}_k z_k^0 + \hat{V}_k V_{k} z_k^0 + \hat{V}_k V_{k}^2 z_k + \sum_{\nu=1}^{\infty} \hat{V}_k^{\nu} (\hat{V}_k V_{k} z_k + \hat{V}_k V_{k} z_k).$$

It has been proved in [16] that

$$\|z_k\| \leq C,$$

(3.17)

$$\|V_k z_k^0\|_{L_\infty} \leq C(\gamma(\rho) + |\rho|^{-1}),$$

(3.18)

$$\|V_k^2\|_{L_\infty} \leq C(\gamma(\rho) + |\rho|^{-1}),$$

(3.19)

for \(\rho \in \Gamma_{\kappa,d}, |\rho| \geq \rho^*\). We suppose that \(\rho\) belongs to this region everywhere below in this proof. Here and below, the notation \(\|\|_{L_\infty} \) is used for the operator norm in the vector space \(L_\mu(0, 1)\).

By virtue of Eq. 3.8, we have

$$\hat{A}(x, \rho) = \sum_{k=0}^{\infty} \rho^{-k} \hat{A}_k(x).$$

Hence

$$\|\hat{v}(\cdot, \rho)\|_{L_1} \leq C|\rho|^{-1}.$$

(3.20)

Using this estimate together with Eqs. 3.1 and 3.13, we obtain

$$\|\hat{V}_k\|_{L_\infty} \leq C|\rho|^{-d}. (3.21)$$

The estimates Eqs. 3.18 and 3.19 together imply
\[ \| \hat{V}_k V_k z_k^0 \|_{L_w} \leq C |\rho|^{-d} (Y(\rho) + |\rho|^{-1}). \]  
(3.22)

By using Eqs. 3.17, 3.19, and 3.21, we get
\[ \| \hat{V}_k V_k^2 z_k^0 \|_{L_w}, \| \hat{V}_k^2 V_k z_k^0 \|_{L_w} \leq C |\rho|^{-d} (Y(\rho) + |\rho|^{-1}). \]  
(3.23)

It remains to estimate the term \( \hat{V}_k V_k z_k \). For this purpose, we will show that
\[ \| \hat{V}_k V_k \|_{L_w} \leq C |\rho|^{-d} (Y(\rho) + |\rho|^{-1}). \]  
(3.24)

Let \( f \) be an arbitrary vector function of \( L_w(0, 1) \) and \( g = \hat{V}_k V_k f \). In the element-wise form
\[ g_j(x, \rho) = \sum_{l,m=1}^n \int_{b_k}^x \hat{V}_{jl}(t, \rho) \exp(\rho(\omega_j - \omega_k)(x - t)) \]
\[ \times \int_{b_k}^t \hat{V}_{lm}(s, \rho) \exp(\rho(\omega_l - \omega_k)(t - s)) f_m(s) ds \, dt. \]

By changing the integration order and taking Eqs. 3.1, 3.2 into account, we derive
\[ g_j(x, \rho) = \sum_{l,m=1}^n \int_0^1 \left( \sum_{l,m=1}^n \hat{V}_{jl}(s, \rho)(\hat{V}_{jk}(s, x, \rho) + \tilde{a}_{jk}(s, x, \rho)) \right) f_m(s) ds. \]

By using Eq. 3.3 for \( \tilde{a}_{jk} \), Eqs. 3.10, and 3.20, we obtain the estimate
\[ \max_{x,t} |g_j(x, \rho)| \leq C |\rho|^{-d} (Y(\rho) + |\rho|^{-1}) \max_{x,t} |f_k(s)|, \]
which yields Eq. 3.24.

In view of Eq. 3.19 and Proposition 3.2, one can choose \( \rho^* \) so that
\[ \| \hat{V}_k^2(\rho) \|_{L_w} \leq \frac{1}{2}, \quad |\rho| \geq \rho^*. \]  
(3.25)

Combining Eqs. 3.16, 3.21–3.25, we obtain
\[ \| \hat{z}_k - \hat{V}_k z_k^0 \|_{L_w} \leq C |\rho|^{-d} (Y(\rho) + |\rho|^{-1}). \]  
(3.26)

Now, consider the vector function \( \varepsilon_k = \hat{V}_k z_k^0 \) with the elements
\[ \varepsilon_{jk}(x, \rho) = \int_{b_k}^x \tilde{v}_{jk}(t, \rho) \exp(\rho(\omega_j - \omega_k)(x - t)) dt \]

Since
\[ \max_{j,k} \| \rho^d \tilde{v}_{jk}(., \rho) - \tilde{a}_{d,\tilde{k}}(., \rho) \|_{L_1} \leq C |\rho|^{-1}, \]

we have
\[ \max_{j,k} |\rho^d \varepsilon_{jk}(x, \rho) - \tilde{a}_{d,j}(x)| \leq C |\rho|^{-1}, \quad \max_{j,k} \| \varepsilon_{jk}(x, \rho) \| \leq C |\rho|^{-d} (Y, \rho) + |\rho|^{-1}). \]

Combining the latter estimates with Eq. 3.26, we obtain Eq. 3.9 for \( \tilde{\varepsilon}(x, \rho) = \tilde{z}(x, \rho). \) \hfill \Box

Now, we apply the obtained results to Eq. 1.3. Returning from the system Eq. 2.6 back to Eq. 2.2 and then to Eq. 1.6 (which is equivalent to Eq. 1.3), we arrive at Proposition 3.5, which is an immediate corollary of Proposition 3.1. For the Mirzoev-Shkalikov case \( n = 2m, \ i_{2k+j} = m - k - j, \ j = 0, 1 \), Proposition 3.5 has been obtained in [16].
Proposition 3.5 For any fixed sector $\Gamma_{1,\rho}$ and some $\rho^* > 0$, Eq. 1.3 has a FSS $\{y_k(x, \rho)\}_{k=1}^n$ whose quasi-derivatives $y_k^{[\rho]}(x, \rho)$, $k = 1, n$, $j = 0, n - 1$, are continuous for $x \in [0, 1]$, $\rho \in \bar{G}$, analytic in $\rho \in G$, for each fixed $x \in [0, 1]$, and satisfy the relation

$$y_k^{[\rho]}(x, \rho) = (\rho \omega_k y) \exp(\rho \omega_k x)(1 + \zeta_{jk}(x, \rho)),$$  \hspace{1cm} (3.27)

where

$$\max_{j,k,x} |\zeta_{jk}(x, \rho)| \leq C(Y(\rho) + |\rho|^{-1}), \quad \rho \in \bar{G},$$  \hspace{1cm} (3.28)

and $Y(\rho)$ is defined by Eq. 3.4.

Consider the differential expressions $\sigma_n(y)$ and $\tilde{\sigma}_n(y)$ of form Eq. 1.1 with the coefficients $(\sigma_n)$ and $(\tilde{\sigma}_n)$, respectively, and $i_v = \tilde{i}_v$, $v = 0, n - 2$. Suppose that

$$\sigma_n(y) = \tilde{\sigma}_n(y) \text{ a.e. on } (0, 1), \quad v = v_0, n - 2,$$  \hspace{1cm} (3.29)

for a fixed $v_0 \in \{1, \ldots, n - 2\}$. Let us study the influence of this condition on the matrices $F(x)$ and $\tilde{F}(x)$.

According to Definition 2.1, the coefficient $\sigma_n$ influences on the lower diagonal of the matrix $F(x)$ with the index $d_v = n - 1 - (v + i_v)$ and, in some cases, on the diagonals with greater indices. We mean that the diagonal containing the entry $f_{k,j}$, $k \geq j$, has the index $(k - j)$. That is, the main diagonal has index 0, the next lower diagonal, index 1, etc. (see Fig. 2). Consequently, the condition Eq. 3.29 implies that the corresponding diagonals of the matrices $F(x)$ and $\tilde{F}(x)$ with indices $0, 1, \ldots, (d - 1)$ coincide, where

$$d := n - 1 - \max_{v=0,v_0-1} (v + i_v).$$  \hspace{1cm} (3.30)

The $d$-th diagonal of $\tilde{F}(x) = F(x) - \tilde{F}(x)$ contains linear combinations of $\hat{\sigma}_v(x)$ with indices $v \in N_d$,

$$N_d = \{v = 0, v_0 - 1 : n - 1 - (v + i_v) = d\}.$$

Transforming the systems of form Eq. 1.6 to the form Eq. 2.6, we conclude that Eq. 3.8 holds and $\hat{\lambda}_d$ depends on $\tilde{\sigma}_v$ with $v \in N_d$. More precisely,

$$\hat{\lambda}_d(x) = \Omega^{-1} \hat{F}_d(x)\Omega.$$

Hence

$$\hat{a}_{d,i}(x) = \frac{1}{n} \sum_{k=\max d} \hat{f}_{k,i}(x)\omega_i^{-k}, \quad i = 1, n.$$

By using Definition 2.1, we obtain the relation

Fig. 2 Indices of diagonals
Consider the boundary value problem
\[ \text{Eq. 1.3 with the boundary conditions Eq. 1.7.} \]
For \( k = 1, n \), denote by \( C(x, \lambda) \) the solution of Eq. 1.3 under the initial conditions \( C_k^{(j-1)}(0, \lambda) = \delta_{jk}, j = 1, n \). The solutions \( \{ C_k(x, \lambda) \}_{k=1}^n \) form a FSS of Eq. 1.3. Therefore, the eigenvalues of the problem \( \mathcal{L} \) coincide with the zeros of the characteristic function
\[ \Delta(\lambda) = \det(\mathcal{L}(C_k))_{k,j=1}^n. \]

**Proof of Theorem 1.1** Step 1. Expansion in the Birkhoff FSS. Fix a sector \( 1_{x,h}^\lambda \) with the property Eq. 2.4. Then, for \( \rho \in 1_{x,h}^\lambda, |\rho| \geq \rho^* \), Eq. 1.3 with \( \lambda = \rho^* \) has a FSS \( \{ \gamma_j(x, \rho) \}_{j=1}^n \) from Proposition 3.5.

Consider the matrix functions \( C(x, \lambda) := [C_k^{(j-1)}(x, \lambda)]_{j,k=1}^n \) and \( Y(x, \rho) := [\gamma_j(x, \rho)]_{j,k=1}^n \). Obviously,
\[ C(x, \lambda) = Y(x, \rho) \mathcal{A}(\rho), \quad \mathcal{A}(\rho) := \det[\mathcal{L}(\gamma_j)]_{j,k=1}^n. \]

By virtue of Propositions 3.1 and 3.5,
\[ Y(x, \rho) = \text{diag}\{1, \rho, \ldots, \rho^{n-1}\} \Omega(I + \mathcal{E}(x, \rho)) \exp(\rho Bx), \quad (4.3) \]

where \( \mathcal{E}(x, \rho) \) satisfies Eq. 3.6. In particular, Proposition 3.2 implies \( \mathcal{E}(0, \rho) \to 0 \) as \( |\rho| \to \infty, \rho \in \Gamma_{x,h} \). Consequently, using Eqs. 4.1, 4.3, the initial condition \( C(0, \lambda) = I \), we derive

\[ \det A(\rho) = (\det \Omega)^{-1} \rho^{-n(n-1)/2}(1 + o(1)), \quad |\rho| \to \infty, \quad \rho \in \Gamma_{x,h}. \quad (4.4) \]

Hence, for sufficiently large \( |\rho| \), we have \( \det A(\rho) \neq 0 \).

Consider values of \( \rho \in \Gamma_{x,h} \) with sufficiently large \( |\rho| \). In view of Eq. 4.2, a number \( \lambda = \rho^n \) is a zero of the characteristic function \( \Delta(\lambda) \) if and only if \( \rho \) is a zero of \( D(\rho) \).

**Step 2. Asymptotics of \( D(\rho) \).** Introduce the notation \( [1] = 1 + \epsilon(\rho) \), where \( \epsilon(\rho) \) can denote various functions satisfying

\[ |\epsilon(\rho)| \leq C(Y(\rho) + |\rho|^{-1}), \quad \rho \in \mathcal{G}. \quad (4.5) \]

Substituting Eq. 3.27 into Eq. 1.7 and taking Eq. 3.31 into account, we obtain

\[ U_s(y_k) = \begin{cases} (\rho \omega_k)^{y_1}[1], & s \leq r, \\ (\rho \omega_k)^{y_2} \exp(\rho \omega_k)[1], & s > r. \end{cases} \]

Thus

\[ D(\rho) = \rho^p \begin{array}{cccc} \omega_1^{y_1}[1] & \omega_2^{y_1}[1] & \cdots & \omega_n^{y_1}[1] \\ \omega_1^{y_2} \exp(\rho \omega_1)[1] & \omega_2^{y_2} \exp(\rho \omega_2)[1] & \cdots & \omega_n^{y_2} \exp(\rho \omega_n)[1] \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{y_n} \exp(\rho \omega_1)[1] & \omega_2^{y_n} \exp(\rho \omega_2)[1] & \cdots & \omega_n^{y_n} \exp(\rho \omega_n)[1] \end{array}, \quad p := \sum_{s=1}^{n} p_s. \quad (4.6) \]

For definiteness, consider the case when \( n - r \) is even and \( \kappa = 1 \). The other cases can be treated similarly. It is worth noting that one has to study two neighbouring sectors \( \Gamma_{x} \) and \( \Gamma_{x+1} \) so that their images cover the whole \( \lambda \)-plane. By analyzing the asymptotics of \( D(\rho) \) as \( |\rho| \to \infty \), one can show that all its zeros in \( \Gamma_{1,h} \) for sufficiently large \( |\rho| \) lie in the strip

\[ S_R := \{ \rho : \Re \rho > 0, |\Im \rho| < R \} \subset \Gamma_{1,h} \quad (4.7) \]

if \( h \) and \( R \) are chosen to be sufficiently large (see Fig. 3).

For \( \rho \in S_R \), we have the asymptotics

\[ D(\rho) = \rho^p \exp(\rho \omega)D_1(\rho), \quad (4.8) \]

\[ D_1(\rho) = D^p_1(\rho) + \epsilon(\rho), \quad D^0_1(\rho) := (c_1 - c_2 \exp(\rho \omega_0 - \omega_{n+1})), \quad \omega := \sum_{k=r+1}^{n} \omega_k, \quad c_1 := \det[a_{\omega_0}^{y_0}]_{k=1} \cdot \det[a_{\omega_0}^{y_{n+1}}]_{k=r+1} \neq 0, \quad c_2 := \det[a_{\omega_0}^{y_{n+1}}]_{k=1} \cdot \det[a_{\omega_0}^{y_0}]_{k=r+1} \neq 0. \quad (4.9) \]

**Step 3. Asymptotics of zeros for large \( |\rho| \).** Clearly, the zeros of \( D^0_1(\rho) \) have the form

\[ \rho^0_1 = \frac{\pi}{\sin \frac{\pi}{n}} (l + \chi), \quad \chi := -\frac{1}{2\pi i} \log(c_1/c_2), \quad l \in \mathbb{Z}. \]

By using Eqs. 4.8, 4.9 and the standard method based on Rouche’s Theorem, we conclude that the zeros of \( D_1(\rho) \) and of \( D(\rho) \) for sufficiently large \( |\rho| \) are simple and have the asymptotics

\[ \rho_l = \frac{\pi}{\sin \frac{\pi}{n}} (l + \chi + \epsilon_l), \quad \epsilon_l = o(1), \quad l \to +\infty. \quad (4.10) \]
Substituting Eq. 4.10 into the relation \( D_l(\rho_i) = 0 \) and using Eq. 4.9, we derive \( \epsilon_l = \epsilon(\rho_i) \), where \( \epsilon(\rho) \) satisfies Eq. 4.5. Clearly, \( \{ \rho_i \} \) is a non-condensing sequence in \( \mathcal{G} \). Recall that \( A_0 \in L_2(0, 1) \). It follows from Eq. 4.5 and Proposition 3.3 with \( \mu = 2 \) that \( \{ \epsilon_l \} \in l_2 \). In this way, we consider the two neighbouring sectors \( \Gamma_1 \) and \( \Gamma_2 \). Returning to the \( \lambda \)-plane, we conclude that, in our case, the eigenvalues have the form \( \lambda_l = \rho_l^n, \ l \geq l_0 \), where \( \rho_l \) satisfy Eq. 4.10. It remains to prove that \( l_0 \) depends only on \( n, r \), and \( (\rho_l)^n \).

**Step 4. Estimate of \( |\Delta(\lambda)| \) from below.** Consider the region

\[ \mathcal{G}_\delta := \{ \rho \in \mathcal{G} : |\rho - \rho_l| \geq \delta, \ l \geq l_0 \}, \ \delta > 0. \]

Using Eqs. 4.6–4.10, we obtain the estimate

\[ |D(\rho)| \geq C_\delta |\rho|^n \exp(\text{Re}(\rho\omega)), \ \rho \in \mathcal{G}_\delta. \]  

(4.11)

where \( C_\delta \) is a constant depending on \( \delta \) and \( \mathcal{G} \). By using Eqs. 4.4 and 4.11, we get

\[ |\Delta(\rho^n)| \geq C_\delta |\rho|^{p-n(n-1)/2} \exp(\text{Re}(\rho\omega)), \ \rho \in \mathcal{G}_\delta. \]  

(4.12)

**Step 5. Difference \( (\Delta(\lambda) - \Delta^0(\lambda)) \).** Consider the problem \( L^0 \) of the same form as \( L \) with the zero coefficients \( \sigma^0 = 0, v = 0, n - 2, a^0_j = 0, j = 1, r_s, s = 1, n. \) Let \( \Delta^0(\lambda) \) be the characteristic function of \( L^0 \). By using the formulas Eqs. 4.2 and 4.6, we obtain

\[ \Delta(\rho^n) - \Delta^0(\rho^n) = o(\rho^{p-n(n-1)/2} \exp(\rho\omega)), \ \rho \in \Gamma_{k,h}, \ |\rho| \to \infty. \]  

(4.13)

Combining Eqs. 4.12 and 4.13, we get

\[ |\Delta(\lambda) - \Delta^0(\lambda)| < |\Delta(\lambda)|, \]  

(4.14)

for \( \lambda = \rho^n, \rho \in \mathcal{G}_\delta \), and sufficiently large \( |\rho| \). Clearly, the inequality Eq. 4.14 can be obtained for the two neighbouring sectors \( \Gamma_{k,h} \) and \( \Gamma_{k+1,h} \), whose images cover the whole \( \lambda \)-plane. Consequently, Eq. 4.14 holds on some contour \( \{ \lambda : |\lambda| = R \} \) with a sufficiently large \( R \). Since the functions \( \Delta(\lambda) \) and \( \Delta^0(\lambda) \) are entire in \( \lambda \), then, by virtue of Rouche’s Theorem, these two functions have the same number of zeros in the circle \( \{ \lambda : |\lambda| < R \} \). Thus, the numeration of the zeros \( \{ \lambda_i \} \) and \( \{ \lambda_i^0 \} \) of \( \Delta(\lambda) \) and \( \Delta^0(\lambda) \), respectively, starts from the same index \( l_0 \). The shift of numeration leads to Eq. 1.8.

Furthermore, we need the following technical lemma.
Lemma 4.1 Let $D_1(\rho)$ be a function of form Eq. 4.9, where $\varepsilon(\rho)$ is an analytic function in $\mathcal{G}$ satisfying Eq. 4.5, and $\{\rho_l\}_{l \geq 0} \subset \mathcal{G}$ be an arbitrary sequence of form Eq. 4.10 with $\{\varepsilon_l\} \in l_2$. Then,

$$\dot{D}_1(\rho_l) = c + \chi_l, \quad c \in \mathbb{C}, \quad \{\chi_l\} \in l_2,$$

where $\dot{D}_1(\rho) = \frac{d}{d\rho}D_1(\rho)$.

Proof. It follows from Eq. 4.9 that

$$\dot{D}_1(\rho_l) = \dot{D}_1^0(\rho_l) + \varepsilon(\rho_l).$$

Obviously, $\dot{D}_1(\rho_l) = c + \chi_l$. The Cauchy formula yields

$$\dot{\varepsilon}(\rho_l) = \frac{1}{2\pi i} \int_{|\rho-\rho_l| = \delta} \frac{\varepsilon(\rho)}{(\rho - \rho_l)^2} d\rho,$$

where $\delta > 0$ is so small that $\{\rho : |\rho - \rho_l| = \delta\} \subset \mathcal{G}$. Hence,

$$|\dot{\varepsilon}(\rho_l)| \leq \delta^{-1} \max_{|\rho - \rho_l|=\delta} |\varepsilon(\rho)|.$$

Denote by $\{\rho_l^\ast\}$ the points such that

$$|\rho_l^\ast - \rho_l| = \delta, \quad |\varepsilon(\rho_l^\ast)| = \max_{|\rho - \rho_l|=\delta} |\varepsilon(\rho)|.$$

Clearly, $\{\rho_l^\ast\}_{l \geq 0}$ is a non-condensing sequence in $\mathcal{G}$. Therefore, it follows from Eq. 4.5 and Proposition 3.3 that $\{\varepsilon(\rho_l^\ast)\} \in l_2$. Consequently, $\{\dot{\varepsilon}(\rho_l)\} \in l_2$. This completes the proof. \qed

Proof. Proof of Theorem 1.2 Suppose that the problems $L$ and $\hat{L}$ satisfy the conditions of Theorem 1.2, that is, $\sigma_\varepsilon(s) = \hat{\sigma}_\varepsilon(s)$ a.e. on $(0, 1)$, $\nu = \nu_0, n - \frac{2}{d}$, $u_{s, p, -j} = u_{s, p, -j}^\ast$, $j = 0, d - 2$, $d := n - 1 - \max_{s=0} \nu_{s, n-1}$. By virtue of Theorem 1.1, the eigenvalues have the form $\lambda_l = (-1)^{n-r} \rho_l^i$ and $\hat{\lambda}_l = (-1)^{n-r} \hat{\rho}_l^i$, $l \geq 1$, where $\rho_l$ and $\hat{\rho}_l$ have the asymptotics Eq. 4.10 with $\chi = \hat{\chi}$. For definiteness, consider the case of even $(n - r)$ and $k = 1$. According to the proof of Theorem 1.1, the numbers $\rho_l$ and $\hat{\rho}_l$ for sufficiently large $l$ are the zeros of the functions $D_1(\rho)$ and $\hat{D}_1(\rho)$, respectively, defined by Eq. 4.9. In order to estimate $\hat{\rho}_l$, we analyze the difference $\hat{D}_1(\rho)$.

Using the conditions of Theorem 1.2, Corollary 3.6, and Eq. 1.7, we obtain

$$U_s(\tilde{y}_k) - \hat{U}_s(\hat{y}_k) = \begin{cases} (\rho \omega_k)^{n-r} (\varepsilon_{i, k} + \varepsilon(\rho)), & s \leq r, \\ (\rho \omega_k)^{n-r} \exp(\rho \omega_k) \rho^{-d} (\varepsilon_{i, k} + \varepsilon(\rho)), & s > r. \end{cases}$$

(4.15)

Here and below in this proof, we denote by $\hat{\varepsilon}(\rho)$ various functions satisfying

$$|\hat{\varepsilon}(\rho)| \leq C(\varepsilon(\rho) + Y_\varepsilon(\rho) + |\rho|^{-1}),$$

and by $\hat{c}$ with and without indices constants depending on the values

$$\int_0^1 \hat{\theta}_k(t) dt, \quad j = 0, n - 1, \quad k = 1, n, \quad \text{and} \quad \hat{u}_{s, p, r-d+1}, \quad s = 1, n,$$

(4.16)

where $\theta_j(t)$ are the functions from Eq. 3.31.

Repeating the arguments of Step 2 in the proof of Theorem 1.1, we obtain

$$\hat{D}_1(\rho) = \rho^{-d} (\varepsilon_1 - \varepsilon_2 \exp(\rho (\omega_r - \omega_{r+1})) + \varepsilon(\rho)), \quad \rho \in S_R,$$

(4.17)

for sufficiently large $|\rho|$.

It follows from $D_1(\rho_l) = 0$ and $\hat{D}_1(\hat{\rho}_l) = 0$ that
The complex Taylor formula implies

\[ D_1(\rho_l) - D_1(\tilde{\rho}_l) + \hat{D}_1(\tilde{\rho}_l) = 0. \]  

(4.18)

The proof is complete.

Proof Corollary 1.3 Let us apply Theorem 1.2 to an arbitrary problem \( L \) with \( n = 2m + 1, \ L' = \mathcal{L}'^0 \) and \( \nu_0 = n - 1 \). The inequality \( \ell_{2k+1} \leq m - k - j \) implies that the minimal value of \( d \) equals 1. In other words, the main diagonal in the corresponding matrix \( F(x) \) always equals zero in the odd case. We have \( i_{n-2} = 0 \) and \( i_{n-3} \in \{0, 1\} \). For \( i_{n-3} = 0 \), we have \( \mathcal{N}_d = \mathcal{N}_d^0 = \{(n-2)\} \), and for \( i_{n-3} = 1 \), \( \mathcal{N}_d = \{(n-3), (n-2)\} \). \( \mathcal{N}_d^0 = \{(n-2)\} \). Thus, Theorem 1.2 immediately yields the claim.

Examples

This section illustrates the application of Theorems 1.1 and 1.2 to various classes of differential operators with distribution coefficients.

Example 5.1 Suppose that \( n = 3, i_0 = 1, i_1 = 0, \sigma_x \in L_2(0, 1), \nu = 0, 1 \). Then, the differential expression Eq. 1.1 takes the form

\[ \mathcal{E}_3(y) = y^{(3)} - (\sigma_1(x)y')' - \sigma_1(x)y' - \sigma_1'(x)y, \quad x \in (0, 1), \]
and the associated matrix equals

\[
F(x) = \begin{bmatrix}
0 & 1 & 0 \\
(\sigma_0 + \sigma_1) & 0 & 1 \\
0 & -\sigma_0 - \sigma_1 & 0
\end{bmatrix}.
\]

Clearly,

\[
F_0(x) \equiv 0, \quad F_1(x) = \begin{bmatrix}
0 & 0 & 0 \\
(\sigma_0 + \sigma_1) & 0 & 0 \\
0 & -\sigma_0 - \sigma_1 & 0
\end{bmatrix}.
\]

(5.1)

Consider the sector \( \Gamma_1 = \{ \rho : 0 < \arg \rho < \pi / 3 \} \). Then, \( \omega_1 = \exp(-2\pi i/3) \), \( \omega_2 = \exp(2\pi i/3) \), \( \omega_3 = 1 \). It follows from Eq. 5.1 that \( A_0(x) \equiv 0 \) and

\[
\text{diag}(A_1(x)) = \text{diag}(\Omega^{-1}F_1(x)\Omega) = \frac{2}{3} \sigma_1(x) \begin{bmatrix}
\omega_1^{-1} & 0 & 0 \\
0 & \omega_2^{-1} & 0 \\
0 & 0 & \omega_3^{-1}
\end{bmatrix}.
\]

Applying Theorem 3.4 to the systems Eqs. 2.6 and 3.7 with \( \tilde{A}(x, \rho) \equiv 0 \) and \( d = 1 \), we obtain the following asymptotics for the fundamental solution matrix:

\[
w(x, \rho) = \left( I + \frac{2}{3j} \int_0^x \sigma_1(t) \, dt B^{-1} + \frac{\gamma(x, \rho)}{\rho} \right) \exp(\rho B x),
\]

where

\[
\max_x ||\gamma(x, \rho)|| \leq C(\gamma(\rho) + \gamma_1(\rho) + |\rho|^{-1}), \quad \rho \in \overline{\gamma}.
\]

(5.2)

Passing to the FSS \( \{y_k(x, \rho)\}_k \) of the equation \( \ell_2^r(y) = \rho^r y \), we obtain the asymptotics

\[
y_k^{[r]}(x, \rho) = (\rho \omega_k)^j \exp(\rho \omega_k x) \left( 1 + \frac{2}{3j} \int_0^x \sigma_1(t) \, dt + \frac{\gamma_k(x, \rho)}{\rho} \right), \quad k = 1, n, j = 0, n - 1,
\]

(5.3)

where scalar functions \( \gamma_k(x, \rho) \) satisfy the same estimate Eq. 5.2 as the matrix function \( \gamma(x, \rho) \).

Consider the differential equation \( \ell_2^r(y) = \lambda y \) with the following boundary conditions \( (r = 1) \):

\[
y(0) = 0, \quad y(1) = 0, \quad y^{[1]}(1) = 0.
\]

By virtue of Theorem 1.1 and Corollary 1.3, the eigenvalues of this problem have the asymptotics

\[
\lambda_l = \left( \frac{2\pi}{\sqrt{3}} \left( l + \chi + \frac{\delta_l}{l} \right) \right)^{n}, \quad \delta_l = o(1), \quad l \to \infty,
\]

(5.4)

where \( \chi \) depends on \( \int_0^1 \sigma_1(x) \, dx \) and, if \( \sigma_\nu \in L_2(0, 1), \nu = 0, 1 \), then \( \{\delta_l\} \in l_2 \).

The function \( D(\rho) \) defined by Eq. 4.2 has the form

\[
D(\rho) = \begin{bmatrix}
y_1(0, \rho) & y_2(0, \rho) & y_3(0, \rho) \\
y_1(1, \rho) & y_2(1, \rho) & y_3(1, \rho) \\
y_1^{[1]}(1, \rho) & y_2^{[1]}(1, \rho) & y_3^{[1]}(1, \rho)
\end{bmatrix}.
\]

(5.5)

Substituting the asymptotics Eqs. 5.3 into 5.5 and finding the asymptotics of the zeros \( \{\rho_l\} \) of \( D(\rho) \) in the strip \( S_0 \), we obtain the values
\[ \chi = \frac{1}{6}, \quad \chi_1 = \frac{1}{\pi^2} \int_{0}^{1} \sigma_1(x) \, dx \]

of the constants in Eq. 5.4.

In the next examples, for the sake of simplicity, we assume that \( u_{ij} = \tilde{u}_{ij}, s = 1, n, j = 1, p_j. \)

**Example 5.2** Suppose that \( n = 2m, i_+, = 0, \sigma_v \in L_1(0, 1), \nu = 0, n - 2. \) Due to Definition 2.1, the entries of the associated matrix \( F(x) = [f_{k,j}(x)]_{k,j=1}^{n} \) are given by the relations

\[ f_{n-k,k+1} = (-1)^{k+1} \sigma_{2k}, \quad k = 0, m - 1, \]
\[ f_{n-k-1,k+2} = (-1)^{k} \sigma_{2k+1}, \quad k = 0, m - 2, \]

and all the other entries are defined as \( f_{k,j} = \delta_{k+1,j}. \) For instance,

\[ \varepsilon_6(y) = y^{(6)} + (\sigma_3 y')' + \cdots + (\sigma_3 y')' - (\sigma_2 y')' - (\sigma_1 y') + \sigma_0 y, \]

and the associated matrix is

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\sigma_3 & -\sigma_4 & 0 & 1 & 0 \\
\sigma_1 & \sigma_2 & -\sigma_3 & 0 & 0 & 1 \\
-\sigma_0 & \sigma_1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Clearly, for each \( d = 1, n - 1, \) the \( d \)-th diagonal contains only \( \sigma_{n-d+1}. \) Since the main diagonal is zero, the remainder term \( \varepsilon \) of the asymptotics Eq. 1.8 has the form Eq. 1.11, similarly to the case of odd \( n. \)

Consider problems \( \mathcal{L} \) and \( \mathcal{L} \) such that \( \sigma_v(x) = \tilde{\sigma}_v(x) \) a.e. on \( (0, 1) \) for \( \nu = v_0, n - 2, v_0 \in \{1, \ldots, n - 1\}. \) Then, in Theorem 1.2, \( d = n - v_0, N_0 = N_0^0 = \{v_0 - 1\}. \) Hence,

\[ \hat{\rho}_1 = l^{-(n-v_0)}(\hat{c} + \delta_1), \quad \delta_1 = o(1), \]

and the constant \( \hat{c} \) linearly depends on \( \int_{0}^{1} \hat{\sigma}_{v_0-1}(x) \, dx. \) In addition, if \( \hat{\sigma}_{v_0-1} \in L_2(0, 1), \) then \( \{\delta_1\} \in l_2. \)

**Example 5.3** Suppose that \( n = 2m, i_+ = 1, \sigma_v \in L_2(0, 1), \nu = 0, n - 2, v_0 \in \{1, \ldots, n - 1\}. \) For instance,

\[ \varepsilon_4(y) = y^{(4)} + (\sigma_2 y')' + (\sigma_1 y')' - (\sigma_0 y')' \]

and the associated matrix equals

\[
F(x) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\sigma_1 & -\sigma_2 & 1 & 0 \\
(\sigma_0 - \sigma_1 \sigma_2) & -\sigma_2 & \sigma_2 & 1 \\
-\sigma_1 & (-\sigma_0 - \sigma_1 \sigma_2) & \sigma_1 & 0
\end{bmatrix}
\]

Suppose that, for the problems \( \mathcal{L} \) and \( \tilde{\mathcal{L}}, \) we have \( \sigma_v(x) = \tilde{\sigma}_v(x) \) a.e. on \( (0, 1) \) for \( \nu = v_0, n - 2. \) Then, \( d = n - v_0 - 1, N_0 = \{v_0 - 1\} \). Therefore, Theorem 1.2 implies \( \hat{\rho}_1 = l^{(n-v_0-1)} \chi_i, \{x_i\} \in l_2. \)
Example 5.4 Consider the case of Mirzoev and Shkalikov [2]: \( n = 2m, i_{2k+1} = m - k, j \in \{0, 1\}, \sigma_v \in L_2(0, 1), v = 0, n - 2 \). The structure of the associated matrix \( F(x) \) is provided in [2]. Suppose that for the problems \( L \) and \( \tilde{L} \), we have \( \sigma_v(x) = \bar{\sigma}_v(x) \) a.e. on \( (0, 1) \) for \( v = 2v_1, n - 2 \). Then, \( d = m - v_1, N_{\delta} = \{(2v_1 - 2), (2v_1 - 1)\}, N_{\delta}^0 = \emptyset \). Hence, Theorem 1.2 implies \( \beta_l = \ell^{-m_{\delta}(\nu)} \nu \{ \nu \} \in l_2 \).

The cases similar to Examples 5.2–5.4 can be considered for odd \( n \).

**Asymptotics of weight numbers**

In this section, we define the weight numbers \( \{ \beta_l \} \) and obtain for them results analogous to Theorems 1.1 and 1.2 for the eigenvalues.

Together with \( U_j(y), s = \bar{1, n} \), consider the linear form

\[
U_0(y) = y^{p_0}l(0) + \sum_{j=1}^{p_0} U_{0,j} y^{l_{j-1}}(0), \quad p_0 \neq p_s, \ s = \bar{1, r}.
\]

Denote by \( L^* \) the boundary value problem for Eq. 1.3 with the boundary conditions \( U_j(y) = 0, s = \bar{0, n \setminus r} \). The eigenvalues of \( L^* \) coincide with the zeros of the characteristic functions \( \Delta^*(\lambda) := \det[U_j(C_k)]_{y=0, n \setminus r, k=1}^{1, n} \).

Define the weight numbers \( \{ \beta_l \} \) as follows:

\[
\beta_l := \text{Res}_{\lambda_{j,t}} \frac{\Delta^*(\lambda)}{\Delta(\lambda)}.
\]

By Theorem 1.2, for sufficiently large \( l \), the eigenvalues \( \{ \lambda_l \} \) of the problem \( L \) are simple. Therefore,

\[
\beta_l = \frac{\Delta^*(\lambda_l)}{d\Delta(\lambda_l)} \tag{6.1}
\]

for such values of \( l \). It is worth considering the weight numbers only for sufficiently large indices \( l \).

Example 6.1 Let \( n = 2, r = 1, p_1 = p_2 = 0, p_0 = 1, u_{0,1} = 0 \). Then, \( L \) and \( L^* \) are the boundary value problems for the Sturm-Liouville equation

\[
y'' - q(x)y = \lambda y, \quad x \in (0, 1), \tag{6.2}
\]

with the boundary conditions \( y(0) = y(1) = 0 \) and \( y^{(1)}(0) = y^{(1)}(1) = 0 \), respectively. Hence,

\[
\beta_l = \frac{C_1(1, \lambda_l)}{d\frac{d}{dx} C_2(1, \lambda_l)}.
\]

where \( C_1(x, \lambda) \) are the solutions of Eq. 6.2 under the initial conditions \( C_k^{(j-1)}(0, \lambda) = \delta_{jk}, j, k = 1, 2 \). One can easily show that \( C_2(x, \lambda_l) \) are the eigenfunctions of \( L \) and

\[
\beta_l = -\alpha_l^{-1}, \quad \alpha_l := \int_0^1 C_2^2(x, \lambda_l) \ dx. \tag{6.3}
\]

For a real-valued potential \( q \in L_2(0, 1) \), the numbers \( \{ \lambda_l, \alpha_l \} \) are the classical spectral data of the inverse Sturm-Liouville problem (see, e.g. [19, 20]). For the case of complex-valued \( q \in L_2(0, 1) \), the so-called generalized spectral data have been introduced in [21]. In the Dirichlet-Dirichlet case, the generalized weight numbers coincide with \( \alpha_l \) defined by Eq. 6.3 for sufficiently large \( l \) (see [22]).
Theorem 6.2 For sufficiently large $l$, the following relation holds:

$$\beta_l = l^{-1+p_{0}-p_r}(\beta_0^{l} + x_l), \quad \{x_l\} \in l_2,$$

where the constant $\beta_0^{l}$ depends only on $n$, $r$, and $(p_j)_{j=0}^{n}$.

**Proof** For definiteness, consider the case of even $(n-r)$. Recall that the eigenvalues of $L$ have the form $\lambda_l = \rho_l^{n}$, where $\{\rho_l\}$ for sufficiently large $l$ belong to $S_R$ and fulfill Eq. 4.10.

Similarly to the proof of Theorem 1.1, we obtain the formulas for $\rho \in S_R$:

$$\Delta^*(\lambda) = D'(\rho) \det \mathcal{A}(\rho), \quad D'(\rho) = \det [U_s(C_k)]_{s=0, n, k=1, R},$$

$$D'(\rho) = \rho^{l_p-2} \exp(\rho \omega) D_1'(\rho),$$

$$D_1'(\rho) = D_1^0(\rho) + \epsilon(\rho), \quad D_1^0(\rho) = c_1^* - c_2^* \exp(\rho(\omega_{l_r} - \omega_{l_r+1})),$$

where $\epsilon(\rho)$ is a function satisfying Eq. 4.5, not necessarily equal to $\epsilon(\rho)$ in Eq. 4.9, and $c_1^*, c_2^*$ are some constants different from $c_1, c_2$. Substituting Eqs. 4.2, 4.8, 6.5, 6.7 into 6.1, we derive

$$\beta_l = \frac{n \rho_l^{n-1} D'(\rho_l)}{d D(\rho_l)} = n \rho_l^{n-1+p_{0}-p_r} \frac{d D_1'(\rho_l)}{d \rho_1},$$

Using Eqs. 4.9, 4.10, Lemma 4.1 for $D_1'(\rho_l)$ and taking into account that $\{\epsilon(\rho_l)\} \in l_2$, we obtain

$$D_1'(\rho_l) = s_1 + x_{l,1}, \quad \frac{d}{d \rho_1} D_1'(\rho_l) = s_2 + x_{l,2},$$

where $s_1, s_2 \neq 0$ are constants and $\{x_{l,1}\}, \{x_{l,2}\} \in l_2$. Hence, we arrive at Eq. 6.4. \qed

**Remark 6.3** Theorems 1.1 and 6.2 are valid for the eigenvalues and the weight numbers, respectively, of the boundary value problems for the system $y'' = (F(x) + \Lambda)y$ with the appropriate boundary conditions generated by the linear forms $U_s(y), s = \bar{0}, \bar{n}$, and with an arbitrary matrix function $F(x) = [f_{j,k}]_{k,j=1}^{n}$ (not necessarily related to the differential expression $\mathcal{E}_n(y)$) satisfying the conditions:

$$f_{j,k} = \delta_{k+1,j}, \quad k < j, \quad f_{k,k} \in L_2(0, 1), \quad f_{k,j} \in L_2(0, 1), \quad k > j, \quad \text{trace}(F(x)) \equiv 0.$$

Furthermore, we formulate and prove the theorem for the weight numbers $\{\beta_l\}$ similar to Theorem 1.2 for the eigenvalues.

**Theorem 6.4** Suppose that $\sigma_s(x) = \tilde{\sigma}_s(x)$ for a.e. $x \in (0, 1), v = \bar{v}_0, n - 2$, and $u_{s,p_{r},j} = \bar{u}_{s,p_{r},j}, j = 0, d-2, s = 0, \bar{n}$, where $d$ is defined by Eq. 1.9. Then,

$$\beta_l - \tilde{\beta}_l = l^{-1+p_{0}-p_r}c, \quad \tilde{\beta}_l = o(1), \quad l \to \infty,$$

where $c$ and $\tilde{\beta}_l$ have the properties similar to the ones in Theorem 1.2, where $s = 1, \bar{n}$ in Eq. 1.10 is replaced with $s = \bar{0}, \bar{n}$.

The notations $c$ and $\tilde{\beta}_l$ in Theorems 1.2 and 6.3 are used for different values. However, we use the same notations in the both cases to emphasize the similar remainder properties.

**Proof of Theorem 6.4.** Assume that the conditions of the theorem hold for $L$ and $\tilde{L}$. Similarly to the previous proofs, we consider the case of even $(n-r)$. The odd case is analogous. Let us use the formula Eq. 6.8 for $\beta_l$ and $\tilde{\beta}_l$. For shortness, put $q = n - 1 + p_{0} - p_r$. Then,
\[
\frac{\beta_l - \tilde{\beta}_l}{n} = D'_{\lambda_l}(\rho_l) - D'_{\lambda_l}(\tilde{\rho}_l) = D'_{\lambda_l}(\rho_l) - D'_{\lambda_l}(\tilde{\rho}_l) + D'_{\lambda_l}(\tilde{\rho}_l).
\]

(6.9)

for sufficiently large \( l \).

Consider the first term in Eq. 6.9. Obviously,

\[
D'_{\lambda_l}(\rho_l) - D'_{\lambda_l}(\tilde{\rho}_l) = D'_{\lambda_l}(\rho_l) - D'_{\lambda_l}(\tilde{\rho}_l) + D'_{\lambda_l}(\tilde{\rho}_l).
\]

For \( D'_{\lambda_l}(\tilde{\rho}_l) \), the asymptotics similar to Eq. 4.21 holds. Using the Taylor formula, we obtain

\[
D'_{\lambda_l}(\tilde{\rho}_l) = D'_{\lambda_l}(\rho_l)(\tilde{\rho}_l - \rho_l) + R(\rho_l, \tilde{\rho}_l),
\]

\[
|R(\rho_l, \tilde{\rho}_l)| \leq C|\rho_l - \tilde{\rho}_l|^2.
\]

Thus, similarly to the proof of Theorem 1.2, we obtain

\[
D'_{\lambda_l}(\rho_l) - D'_{\lambda_l}(\tilde{\rho}_l) = l^{-d}(\hat{c} + \delta_l), \quad \delta_l = o(1), \quad l \to \infty.
\]

Using Eq. 4.10 and Lemma 4.1, we arrive at the asymptotics \( b^l t^{-d}(\hat{c} + \delta_l) \) for the first term of Eq. 6.9. For the second and the third terms, the same asymptotics can be obtained analogously. This completes the proof of Theorem 6.3.

\[ \square \]

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Declarations

Conflict of interest The author declares no competing interests.

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