The energy partitioning of non-thermal particles in a plasma: the Coulomb logarithm revisited

Robert L Singleton Jr and Lowell S Brown
Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Received 28 May 2008, in final form 29 July 2008
Published 4 November 2008
Online at stacks.iop.org/PPCF/50/124016

Abstract
The charged particle stopping power in a highly ionized and weakly to moderately coupled plasma has been calculated exactly to leading and next-to-leading accuracy in the plasma density by Brown, Preston and Singleton (BPS). Since the calculational techniques of BPS might be unfamiliar to some, and since the same methodology can also be used for other energy transport phenomena, we will review the main ideas behind the calculation. BPS used their stopping power calculation to derive a Fokker–Planck equation, also accurate to leading and next-to-leading orders, and we will also review this. We use this Fokker–Planck equation to compute the electron–ion energy partitioning of a charged particle traversing a plasma. The motivation for this application is ignition for inertial confinement fusion—more energy delivered to the ions means a better chance of ignition, and conversely. It is therefore important to calculate the fractional energy loss to electrons and ions as accurately as possible. One method by which one calculates the electron–ion energy splitting of a charged particle traversing a plasma involves integrating the stopping power $dE/dx$. However, as the charged particle slows down and becomes thermalized into the background plasma, this method of calculating the electron–ion energy splitting breaks down. As a result, it suffers a systematic error that may be as large as $T/E_0$, where $T$ is the plasma temperature and $E_0$ is the initial energy of the charged particle. The formalism presented here is designed to account for the thermalization process and it provides results that are near-exact.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

When a fast charged particle with initial energy $E_0$ traverses a plasma, it loses energy at a rate $dE/dx$ per unit of distance, and it comes into thermal equilibrium after depositing its energy into the electrons and ions that make up the plasma. Using the formalism of Brown, Preston and Singleton (BPS) [1], which provides a means of regulating the kinetic equations at short and long distances in a consistent manner while treating quantum mechanical effects exactly,
we shall calculate this energy splitting in a uniform plasma to leading and next-to-leading order in the plasma coupling. Our motivation is the thermonuclear burn of deuterium and tritium in inertial confined fusion experiments where a fast $\alpha$ particle is born with an energy $E_0 = 3.54$ MeV. The more efficiently the ions are heated, the easier it will be to initiate the bootstrap heating process that triggers ignition and burn.

In this paper we shall always work with a plasma whose components have a single temperature $T$ (measured in energy units). We are currently generalizing our formalism to a plasma in which the electrons and ions are at different temperatures [2]. The energy partitioning is usually computed within the context of a fast charged particle traversing the plasma until it comes to a complete stop. The total stopping power $dE/dx$ is a function of energy $E$, and as the particle slows down, this can be inverted to obtain the distance travelled as a function of energy, $x = x(E)$. One may then express the stopping power as a function of $x$. The result of integrating along this trajectory gives the energy contribution to ions and electrons, which can be written schematically as

$$E_I = \int_0^{E_0} dE \frac{dE_I}{dx}$$

and

$$E_e = \int_0^{E_0} dE \frac{dE_e}{dx}.$$  

(1.1)

(1.2)

Here $dE_I/dx$ and $dE_e/dx$ are the stopping power contributions from the ions and electrons expressed as functions of $x$. We see that

$$\frac{dE}{dx} = \frac{dE_I}{dx} + \frac{dE_e}{dx},$$

(1.3)

and thus $E_I + E_e = E_0$. This is only an approximate description because the fast charged particle does not simply come to rest within the plasma, but rather it becomes thermalized at the temperature $T$. One should not extend the integrals in equations (1.1) and (1.2) down to zero, but rather a lower limit $E_{\text{min}}$ on the order of the thermal plasma energy, $E_{\text{min}} \sim T$. Consequently the systematic error in the above calculation of $E_I$ and $E_e$ is of order $T/E_0$, and as we shall see, the correct electron–ion energy partition relation reads

$$E_I + E_e + \frac{3}{2}T = E_0.$$  

(1.4)

Before examining the energy splitting, we need to discuss the calculational framework within which it appears. As we shall see, the correct expression for the energy splitting arises from a Fokker–Planck equation derived in BPS; however, before getting into the details, it will be useful to briefly review some salient features of the stopping power calculation.

2. The BPS formalism

Calculating Coulomb energy exchange processes in a hot plasma is notoriously difficult because of the subtleties of the Coulomb interaction, which produce logarithmic divergences at both long and short distance scales. This problem was first spelled out and solved to leading order by Landau and then Spitzer in the context of electron–ion temperature equilibration, and later by Corman et al for the charged particle stopping power [3–5]. Since the divergences are only logarithmic, one introduces ad hoc short and long-distance cutoffs $b_{\text{min}}$ and $b_{\text{max}}$, and the rate
of energy loss of some process (such as temperature equilibration or stopping power) can be cast in the form

\[
\frac{dE}{dt} = K \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{db}{b} = K \ln \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right). \tag{2.1}
\]

The prefactor \(K\) is easy to compute exactly. The logarithmic term, conventionally called the Coulomb logarithm, can only be approximated within the above scheme. The long-distance scale \(b_{\text{max}}\) is set by the relevant Debye screening length, while the short distance scale \(b_{\text{min}}\) is determined by either the Landau length or the thermal de Broglie wavelength (or some interpolation between them). As such, this method suffers a systematic uncertainty in the argument of the Coulomb logarithm\(^1\). In the language of perturbation theory, equation (2.1) is accurate to leading order in the plasma coupling constant. This accuracy was extended to subleading order by BPS [1] which performed a first principles controlled calculation, including the exact terms under the logarithm and a rigorous treatment of the quantum to classical transition.

Reference [6] discusses at length the manner in which one can expand thermodynamic quantities as a perturbation series in integer powers (plus additional logs) of a dimensionless plasma coupling parameter \(g\). This approach was applied in BPS [1] to the stopping power and temperature equilibration problems. This coupling is generally defined by

\[
g = \frac{e^2 \kappa}{T}, \tag{2.2}
\]

where \(\kappa\) is a typical Debye wave number. This parameter is just the ratio of the potential energy of two electrons a Debye distance apart to the thermal kinetic energy of the plasma, and it is related to the usual plasma parameter by \(g \propto \Gamma^{3/2}\). For this discussion, it is not critical whether one chooses the electron or the total Debye wave number, as we are primarily interested in constructing a small dimensionless quantity that will indicate how a systematic perturbative calculation should be organized. One may consult the Introduction of [1] for more details concerning this coupling constant. The main point is that quantities expand in integer powers of \(g\), except for possible \(\ln g\) terms, and the rate of energy exchange for the stopping power takes the form

\[
\frac{dE}{dt} = -A g^2 \ln g + B g^2 + \mathcal{O}(g^3) = A g^2 \ln \Lambda_{\text{coul}} + \mathcal{O}(g^3). \tag{2.3}
\]

We have indicated the leading order (LO) and the next-to-leading order (NLO) terms in the \(g\)-expansion, and in a weakly coupled plasma these two contributions provide a good approximation to the exact result since the cubic order term will be quite small. Here \(\Lambda_{\text{coul}} = -\ln(Cg)\), with \(C\) defined by \(B = -A \ln C\), and therefore knowing the next-to-leading order coefficient \(B\) is equivalent to knowing the exact coefficient \(C\) under the logarithm. To get a feel for the numbers, at the centre of the sun \(g = 0.04\), and the error term in equation (2.3) is consequently small.

The problem with directly calculating \(A\) and \(B\) is that the kinetic equations diverge and must be regularized in the appropriate manner. Furthermore, to find the coefficient under the logarithm, this regularization procedure must preserve the delicate balance between the long and short distance physics. Indeed, the BPS calculation [1] includes both short distance physics and dynamic collective long distance physics, joined together exactly and unambiguously.

\(^1\) The constant under the logarithm sometimes varies by an order of magnitude from paper to paper within the literature, depending upon the choices of \(b_{\text{min}}\) and \(b_{\text{max}}\).
The rigorous starting point is the BBGKY hierarchy (or its quantum generalization), which is finite and well defined and does not suffer from the aforementioned divergences. One must of course truncate this vast number of equations to something manageable, such as the Boltzmann or Lenard–Balescu equations, and it is this truncation process that renders the various three-dimensional integrals divergent. However, as shown in [7], in $\nu$ spatial dimensions these divergences become simple poles of the form $1/(\nu - 3)$. In spatial dimensions $\nu > 3$ the BBGKY hierarchy reduces to the Boltzmann equation (BE) to leading order in $g$ (the BE is finite and does not have the usual long-distance divergence for $\nu > 3$). Calculating the rate of energy loss using the $\nu$-dimensional BE gives a result of the form

$$\frac{dE}{dt} > H(\nu) g^2 \frac{3}{\nu - 3} + \mathcal{O}(\nu - 3): \text{LO in } g \text{ when } \nu > 3. \quad (2.4)$$

The ‘greater-than’ superscript is to remind us that the calculation has been performed in dimensions $\nu > 3$. In a similar manner, to leading order in $g$ the BBGKY hierarchy reduces to the Lenard–Balescu equations (LBE) for $\nu < 3$ (the LBE is finite and does not suffer from short distance divergences when $\nu < 3$). A calculation of the energy rate with the LBE gives the form

$$\frac{dE}{dt} < G(\nu) g^{\nu - 3} + \mathcal{O}(\nu - 3): \text{LO in } g \text{ when } \nu < 3. \quad (2.5)$$

Note that both rates are of order $g^2$ in three dimensions, and they both suffer from a divergent simple pole. The coefficients $H(\nu)$ and $G(\nu)$ can be expanded in powers of $\epsilon = \nu - 3$, with

$$H(\nu) = -A + \epsilon H_1 + \mathcal{O}(\epsilon^2) \quad \text{and} \quad G(\nu) = -A + \epsilon G_1 + \mathcal{O}(\epsilon^2). \quad (2.6)$$

The leading terms must be equal, $H(3) = G(3) = -A$. This arises from the calculation itself and is not imposed by hand, and it makes the short- and long-distance poles cancel, thereby giving a finite result.

Since the rates $dE^>/dt$ and $dE^-/dt$ were calculated in mutually exclusive dimensional regimes, one might think that they cannot be compared. However (and this is perhaps the most crucial step in the method, and certainly the most subtle), we can analytically continue the quantity $dE^>/dx$ to dimensional values $\nu > 3$, after which we can directly compare the rates (2.4) and (2.5) in a common dimension $\nu$, and the limit $\nu \to 3$ may then be taken. Upon writing the $g$-dependence of equation (2.5) as $g^{2(\nu - 3)}$, when $\nu > 3$ we see that the rate (2.5) is indeed of higher order in $g$ than equation (2.4) since $\epsilon = \nu - 3 > 0$:

$$\frac{dE^-}{dt} = -G(\nu) g^{2(\nu - 3)} \frac{3}{\nu - 3} + \mathcal{O}(\nu - 3): \text{NLO in } g \text{ when } \nu > 3. \quad (2.7)$$

The individual pole-terms in equations (2.4) and (2.7) will cancel giving a finite result when the leading and next-to-leading order terms are added. Summing terms (2.4) and (2.7), using the relation $g^4 = \exp(\epsilon \ln g) = 1 + \epsilon \ln g + \mathcal{O}(\epsilon^2)$, and taking the $\epsilon \to 0$ limit gives

$$\frac{dE}{dt} = -Ag^2 \ln g + Bg^2 + \mathcal{O}(g^3), \quad (2.8)$$

with $B = H_1 - G_1$. This is in agreement with equation (2.3). In this way, BPS has calculated the charged particle stopping power accurate to leading order and next-to-leading order in $g$. In the context of inertial confinement fusion and ignition, these arguments are presented in more detail in [8], while in the context of temperature equilibration one may consult [9].
BPS also derived a Fokker–Planck equation accurate to leading and next-to-leading order in the plasma coupling [1]. Denoting the phase space density for the dilute collection of charged particles by 

\[ f(r, p, t) = \sum_b \frac{\partial}{\partial r} C_b^{\ell \ell}(p) \left[ \beta v^\ell + \frac{\partial}{\partial p^\ell} \right] f(r, p, t), \tag{2.9} \]

where the sum runs over the plasma components \( b \), \( \beta = 1/T \), the vector \( v = p/m \) is the velocity of a particle with momentum \( p \) and the summation convention is used for repeated vector indices. The symmetric tensor \( C_b^{\ell \ell} \) has longitudinal and transverse components,

\[ C_b^{\ell \ell}(p) = A_b(E) \frac{\hat{v}^\ell \hat{v}^\ell}{\beta v} + B_b(E) \frac{1}{2} (\delta^{\ell \ell} - \hat{v}^\ell \hat{v}^\ell), \tag{2.10} \]

where \( v = |v| \) and \( \hat{v} = v/v \) and \( E = \frac{1}{2} m v^2 \). We denote the sum of the ion components of the \( A \)-coefficients by \( A_I = \sum_i A_i \) and the electron component by \( A_e \), with \( A = A_I + A_e \). Expressions for the \( A_b \) can be found in BPS [1]. With our conventions, the number and kinetic energy densities of the charged particles is given by

\[ n(r, t) = \int \frac{d^3 p}{(2\pi \hbar)^3} f(r, p, t) \tag{2.11} \]

and

\[ E(r, t) = \int \frac{d^3 p}{(2\pi \hbar)^3} \frac{p^2}{2m} f(r, p, t). \tag{2.12} \]

We can derive a relation between the stopping power and the \( A \)-coefficients in the following manner. For a single particle at \( r_p \) moving with velocity \( v_p \), the distribution function takes the form

\[ f(r, p, t) = \frac{1}{(2\pi \hbar)^3} \delta(r - r_p) \delta(p - p_p), \]

and the Fokker–Planck equation gives the particle’s rate of energy loss as

\[ v_p \frac{dE}{dx} = \frac{dE}{dt} = \sum_b \left[ \beta v_p^\ell - \frac{\partial}{\partial p_p^\ell} \right] v_p^\ell C_b^{\ell \ell}(p_p). \tag{2.13} \]

Upon substituting the decomposition (2.10) for the scattering tensor and dropping the projectile subscript \( p \), the contribution from species \( b \) appears as

\[ \frac{dE_b}{dx} = \left[ 1 - \frac{1}{\beta m v} \frac{\partial}{\partial v} \right] A_b(E). \tag{2.14} \]

As \( v \) gets large (\( \beta m v^2 \gg 1 \)) note that \( dE_b/dx \rightarrow A_b \).

3. Formulation of the problem

With this background in hand, we turn now to the energy splitting problem. Rather than tracking an individual charged particle slowing down in the plasma, it is much simpler to examine a homogeneous and isotropic source of charged particles of a single energy \( E_0 \). The distribution function will therefore depend only upon the energy and time, \( f = f(E, t) \). Furthermore, the homogeneity and isotropy conditions will greatly simplify the form of the Fokker–Planck equation: only the transverse coefficients \( A_b \) will enter the diffusion kernel on the right-hand side of equation (2.9), while the convective term on the left-hand side will of course vanish. Thus, we now consider the inhomogeneous Fokker–Planck equation with a source, which, after some algebra reads

\[ \left\{ \frac{\partial}{\partial t} - \frac{2}{mv} \frac{\partial}{\partial E} A(E)E \left[ 1 + T \frac{\partial}{\partial E} \right] \right\} f(E, t) = \delta(E - E_0) s(t). \tag{3.1} \]
Figure 1. The waterfall analogy. The small green rocks represent the plasma electrons while the larger red rocks are the plasma ions, with the flowing ‘water’ in blue representing the evolution of the produced charged particles ($\alpha$ particles for DT fusion). As ‘water’ falls down the electron–ion slope at a constant rate determined by $\tilde{f}(E)$, energy is deposited into electrons and ions. At the bottom of the fall is a lake into which the excess ‘water’ drains, representing the final thermalized particles, with height $n(t)$ and Maxwell–Boltzmann distribution $\propto e^{-\beta E}$.

We find [2] that the relevant solution is of the form

$$f(E, t) = n(t) Ne^{-\beta E} + \tilde{f}(E).$$

(3.2)

We proceed now to motivate the structure of the solution (3.2) and the nature of the time-independent function $\tilde{f}(E)$. The particles will eventually thermalize to a Maxwell–Boltzmann distribution, and the first term of equation (3.2) merely represents the buildup and subsequent thermalization of the particles produced by the source. The normalization factor $N$ is chosen so that $n(t)$ is the number density of the produced particles once they have thermalized into the Maxwell–Boltzmann distribution $\propto e^{-\beta E}$. The time-independent piece $\tilde{f}(E)$ describes the steady state of non-thermal particles losing energy to the plasma, i.e. particles cascading down ‘energy bins’ from the initial energy $E_0$ to the final thermal energy. The situation described here can be pictured as the flow of water over a rocky waterfall that slows the motion of the water as it descends. The initial rate of flow of the river corresponds to the rate of produced particles; the height of the waterfall to the initial energy $E_0$. The energy dissipated in the fall corresponds to the energy lost to the ions and electrons and is determined by $\tilde{f}(E)$. The final flow into a horizontal lake corresponds to the build up of the particles into their final thermal equilibrium state. This is illustrated in figure 1.

We now focus on the relevant case in which the source $s(t)$ slowly turns on and attains a constant value. In this case, particles are produced at a constant rate per unit volume $\dot{n}_0$, and from equation (2.12) the rate of change in energy density becomes

$$\dot{E}(t) = E_0 \dot{n}_0 - \int \frac{d^3p}{(2\pi\hbar)^3} [A_1 + A_2] \left( v + \frac{1}{\beta m} \frac{\partial}{\partial v} \right) \tilde{f}(E).$$

(3.3)

The fraction of energy lost to the ions and electrons is now identified as [2]

$$\frac{E_1}{E_0} = \frac{1}{\tilde{n}_0 E_0} \int \frac{d^3p}{(2\pi\hbar)^3} A_1 \left( v + \frac{1}{\beta m} \frac{\partial}{\partial v} \right) \tilde{f}(E)$$

(3.4)

and

$$\frac{E_\text{e}}{E_0} = \frac{1}{\tilde{n}_0 E_0} \int \frac{d^3p}{(2\pi\hbar)^3} A_2 \left( v + \frac{1}{\beta m} \frac{\partial}{\partial v} \right) \tilde{f}(E).$$

(3.5)
In the steady state, the energy density build up of final particles is their thermal energy per particle times the increase in number density, $\dot{\varepsilon} = \frac{3}{2} T \dot{n}_0$. And so, in view of equations (3.4) and (3.5), the energy balance expression (3.3) now appears as

$$\frac{3}{2} T \dot{n}_0 = [E_0 - E_I - E_e] \dot{n}_0,$$

which gives expression (1.4): the original energy of a produced particle is lost to the ions and electrons with the remainder being the thermal energy of a free particle.

We can simplify equations (3.4) and (3.5) by calculating the action of the operator in square brackets on $\bar{f}(E)$, and one finds [2]

$$\frac{E_I}{E_0} = \int_0^{E_0} \frac{dE}{E_0 A(E)} \left[ \text{erf}(\sqrt{\beta E}) - \sqrt{\frac{4\beta E}{\pi}} e^{-\beta E} \right]$$

(3.7)

and

$$\frac{E_e}{E_0} = \int_0^{E_0} \frac{dE}{E_0 A(E)} \left[ \text{erf}(\sqrt{\beta E}) - \sqrt{\frac{4\beta E}{\pi}} e^{-\beta E} \right],$$

(3.8)

where $\text{erf}(x)$ is the error function. Since $dE_0/dx \to A_b$ for large energies, and the error function approaches unity, at high energies $E$ we see that equations (3.7) and (3.8) approach the same form as the more intuitive but less accurate results (1.1) and (1.2). The primary differences occur for $E \sim T$.

Let us compare these near-exact results with the less precise but well known result of Fraley et al (FR) [10]. Starting with a model of the stopping power in an equimolar DT plasma, these authors show that the simple rule

$$\frac{E_I}{E_0} = \frac{1}{1 + T_c/T_e}$$

(3.9)
Figure 3. The percentage change of the ion fractional energy loss between Fraley et al and BPS for a 3.54 MeV α particle in an equimolar DT plasma for three densities $n_e = 10^{24}$ cm$^{-3}$, $10^{25}$ cm$^{-3}$, $10^{26}$ cm$^{-3}$, which correspond to $T_c = 31$ keV, 30 keV, 20 keV, respectively.

provides a good fit to their calculations. The crossover temperature $T_c$, where the electron and ion fraction is 50–50%, can be determined from their figure 1(b). Fraley et al find $T_c = 32$ keV at the density $\rho = 0.213$ g cm$^{-3}$, or a corresponding electron number density $n_e = 5 \times 10^{22}$ cm$^{-3}$. At the number densities $n_e = 10^{23}$ cm$^{-3}$, $10^{25}$ cm$^{-3}$, $10^{26}$ cm$^{-3}$, we roughly estimate that $T_c = 31$ keV, 30 keV, 28 keV, respectively. Figure 2 shows the fraction energy loss to ions for BPS and FR for a DT plasma with electron number density $n_e = 10^{26}$ cm$^{-3}$. In figure 3 we compare the percentage difference between BPS and FR over a wide range of densities. We see that FR somewhat overestimates the energy deposited to ions for temperatures above 100 keV over this wide range of densities.

References

[1] Brown L S, Preston D L and Singleton R L Jr 2005 Phys. Rep. 410 237–333 (Preprint physics/0501084)
[2] Brown L S, Preston D L and Singleton R L Jr LA-UR-07-8096 in progress
[3] Landau L D 1937 Phys. Z. Sowjetunion 10 154
[4] Landau L D 1937 Sov. Phys.—JETP 7 203
[5] Spitzer L 1965 The Physics of Fully Ionized Gasses (New York: Interscience)
[6] Corman E G, Loewe W E, Cooper G E and Winslow A M 1975 Nucl. Fusion 15 377
[7] Brown L S 2000 Phys. Rev. D 62 045026 (Preprint physics/9911055)
[8] Brown L S 2000 Phys. Plasmas 15 056302 (Preprint 0712.1023)
[9] Brown L and Singleton R 2007 Phys. Rev. E 76 066404 (Preprint 0707.2370)
[10] Fraley G S, Linnebur E J, Mason R J and Morse R L. 1974 Phys. Fluids 17 474