Higher Divergence Functions in Heisenberg Groups

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16th October 2014

Abstract. We prove a Filling Theorem for the Heisenberg Group $H^{2n+1}$: For a given $k$-cycle $a$ we construct a $(k+1)$-chain $b$ (the filling) with boundary $\partial b = a$ and controlled volume. For this filling $b$ we prove a bound on the distance of points in $b$ to its boundary $a$. Using this we compute the higher divergence functions in and below the critical dimension $n$ for the Heisenberg Group $H^{2n+1}$. Further we generalise these results to the Jet-Groups $J^m(\mathbb{R}^n)$.

1 Introduction and main results

For studying the large scale geometry of a group or another metric space one searches quasi isometry invariants. One example are the filling functions, which describe the difficulty to find a chain with a given boundary. The most basic filling function in groups is the Dehn function, which measures the area needed to fill a loop by a disk and are well understood. There is a natural generalisation to higher-order Dehn functions, which describe the difficulty of filling $k$-cycles by $(k+1)$-chains. They are much harder to compute, but give a finer distinction. For example see [11],[12] where the higher Dehn functions of the Heisenberg Groups $H^{2n+1}$ are computed. These prove, at least in higher dimensions, to be different from those of $\mathbb{R}^n$, while the usual Dehn functions are of equal type.

An other way to study the geometry at infinity are the higher divergence functions, which, roughly speaking, measure the difficulty to fill an outside an $r$-ball lying $k$-cycle with a $(k+1)$-chain outside a $\rho r$-ball, $0 < \rho \leq 1$. A formal definition will be given in section 1.1. They were first time introduced by Brady and Farb [3] for Hadamard manifolds. Later results of Leuzinger [7] and Hindawi [6] proves, that higher divergence functions detect the euclidean rank of a symmetric space and Wenger [9] generalised this to CAT(0)-spaces. Abrams, Brady, Dani, Duchin and Young [1] established bounds for the higher divergence functions for right angled Artin groups. In the same paper they compute the higher divergence functions of $\mathbb{R}^n$, which show, that filling a cycle with an $r$-ball-avoidant chain is roughly as difficult as with a non-avoidant chain. We will adapt the idea of them to compute the higher divergence functions of the Heisenberg Groups (or more generally of the Jet-Groups) in and below the critical dimension.
1.1 Higher Divergence Functions

We give the basic definitions of the higher Divergence-Functions, an extensive introduction can be found in [1].

Let $X$ be a metric space with basepoint $o \in X$.

**Definition.** Let $r > 0$. We call a lipschitz chain $a$ in $X$ $r$-avoidant, if $\text{image}(a) \cap B_r(o) = \emptyset$.

**Definition.** For $0 < \rho \leq 1$ we call $X (\rho, k)$-acyclic at infinity, if every $r$-avoidant $m$-cycle has a $\rho r$-avoidant filling, $0 \leq m \leq k$.

We define the divergence dimension $\text{divdim}(X)$ of $X$ as the largest integer $n$, such that $X$ is $(\rho, n)$-acyclic at infinity for some $\rho$.

In the following let $k$ be always less or equal to $\text{divdim}(X)$ and we define the mass of a lipschitz chain as the total volume of its summands.

**Definition.** For $0 < \rho \leq 1$ and $\alpha > 0$ we define

$$\text{div}^k_{\rho, \alpha}(r) := \text{div}^k_{\rho}(\alpha r^k, r) := \sup_{a} \text{div}^k_{\rho}(a, r) := \sup_{a} \inf_{b} \text{mass}(b)$$

where the infimum is taken over all $\rho r$-avoidant $(k + 1)$-chains $b$ with $\partial b = a$ and the supremum is taken over all $r$-avoidant $k$-cycles $a$ with $\text{mass}(a) \leq \alpha r^k$.

Then we define for $k \in \mathbb{N}$ the $k$th-Divergence of $X$ as the 2-parameter family

$$\text{Div}^k(X) := \{\text{div}^k_{\rho, \alpha}\}_{\rho, \alpha}$$

To make $\text{Div}^k(X)$ a quasi-isometry invariant, we have to define a equivalence relation:

**Definition.**

a) A positive 2-parameter $k$-family is a 2-parameter family $F = \{f_{s,t}\}$, indexed over $0 < s \leq 1, t > 0$, of functions $f_{s,t} : \mathbb{R}^+ \to \mathbb{R}^+$ together with an fixed integer $k$.

b) Let $k \in \mathbb{N}$ and let $F = \{f_{s,t}\}$ and $H = \{h_{s,t}\}$ be two positive 2-parameter $k$-families, indexed over $0 < s \leq 1, t > 0$.

Then we write $F \preceq H$, if there exists thresholds $0 < s_0 \leq 1$ and $t_0 > 0$, as well as constants $L, M \geq 1$, such that for all $s \leq s_0$ and all $t \geq t_0$ there is a constant $A \geq 1$ with:

$$f_{s,t}(x) \leq Ah_{Ls,Mt}(Ax + A) + O(x^k)$$

If both $F \preceq H$ and $H \preceq F$, so we write $F \sim H$. This defines an equivalence relation.

**Remark.** In the definition of the equivalence relation in [1, 2.2] is a little error (making equivalence classes much smaller than wanted: e.g. $\{r^{k+1}\}_{\rho, \alpha} \not\sim \{\alpha r^{k+1}\}_{\rho, \alpha}$, which implies $\text{Div}^k(\mathbb{R}^n) \not\sim r^{k+1}$, in contrast to [1, Prop.2.5] ). We repaired it by choosing the constant $A$ individually for any fixed pair of parameters.
Remark. We consider $\text{Div}^k$ as positive 2-parameter $k$-family, indexed over $\rho$ and $\alpha$.

The proof of the following proposition can be found in [1, Prop. 2.2].

Proposition 1.1. Let $k \in \mathbb{N}$ and $X,Y$ sufficiently connected (i.e. $\text{divdim}(X), \text{divdim}(X) \geq k$) metric spaces with basepoints. Then:

$$X \text{ quasi-isometric to } Y \implies \text{Div}^k(X) \sim \text{Div}^k(Y)$$

1.2 Heisenberg Groups

We will state the definition of the higher Heisenberg Groups and some basic facts of their sub-riemannian structure.

A detailed introduction to Heisenberg Groups and sub-riemannian geometry give [4] and [5].

Definition. The Heisenberg Group $H^{2n+1}$ of dimension $2n + 1$ can be defined as

$$H^{2n+1} := \mathbb{C}^n \times \mathbb{R}$$

with the group law

$$(z,x)(w,y) := (z + w, x + y - \frac{1}{2} \sum_{i=1}^{n} \text{Im}(z_i w_i))$$

This is a Lie group with (real) Lie algebra $\mathfrak{h}^{2n+1} = V_1 \oplus V_2$ where $V_1 = \mathbb{C}^n, V_2 = \mathbb{R}$ and with the bracket

$$[(Z,X),(W,Y)] = (0, \sum_{i=1}^{n} \text{Im}(Z_i \overline{W_i}))$$

Remark. $H^{2n+1}$ (resp. $\mathfrak{h}^{2n+1}$) is 2-step nilpotent and $[V_1,V_1] = V_2$, $[V_1,V_2] = [V_2,V_2] = \{0\}$.

Definition.

a) Let $M$ be a manifold and $f : M \rightarrow H^{2n+1}$ a lipschitz map and $dL_g$ the differential of the left-translation with $g \in H^{2n+1}$.

Then $f$ is horizontal, if all its tangent vectors lie in the subbundle

$$H := \bigcup_{g \in H^{2n+1}} dL_g V_1$$

of the tangentbundle of $H^{2n+1}$.

b) Let $X$ be a simplicial complex and $f : X \rightarrow H^{2n+1}$ a lipschitz map. Then $f$ is $k$-horizontal, if $f$ is horizontal on the interior of all $m$-simplices, $m \leq k$, of $X$.

Remark. To be horizontal is a left-invariant property, i.e. is $f$ a horizontal map, so is $L_g \circ f$, $g \in H^{2n+1}$ (see [4,2.2.1]).

Definition. Let $\pi : H^{2n+1} \rightarrow \mathbb{C}^n \cong \mathbb{R}^{2n}$ be the canonical projection.

The length of an horizontal path $\gamma$ is defined as $L(\gamma) := L_{\text{eucl}}(\pi(\gamma))$

Now we can define a left-invariant metric $d$ on $H^{2n+1}$ by

$$d(P,Q) := \inf \{ L(\gamma) \mid \gamma \text{ horizontal path from } P \text{ to } Q \}$$

This metric is called the Carnot-Carathéodory-Metric.
Proposition 1.2. Let \( s_t : H^{2n+1} \rightarrow H^{2n+1} ; (z,x) \mapsto (tz,t^2x) \), \( t > 0 \), the family of dilations. Then each \( s_t \) is an automorphism and \( d(s_t(P),s_t(Q)) = t \cdot d(P,Q) \) \( \forall P,Q \in H^{2n+1} \)

In the following we will call this maps scaling automorphisms.

Remark. To be horizontal is invariant under scaling automorphisms, i.e. is \( f \) a horizontal map, so is \( s_t \circ f \), \( t > 0 \) (see [4,2.2.1]).

1.3 Statement of the main results

We will now state the two main theorems. Theorem 1 is the analogue for the Heisenberg Groups of the filling theorem proved in [10] in the CAT(0) case. It arises from the filling theorem proved in [11,Thm.1] for Jet-Groups. The improvement is the bound for the distance of points in the filling to its boundary stated in the 3. property.

Theorem 1. Let \( H^{2n+1} \) be the Heisenberg Group of dimension \( 2n + 1 \) and \( k < n \). Then there is a constant \( C > 0 \), such that for every \( k \)-cycle \( a \) with \( \text{mass}(a) = l \) there exists a \((k+1)\)-chain \( b \) with

1. \( \partial b = a \)
2. \( \text{mass}(b) \leq C \cdot l^{\frac{k+1}{k+2}} \)
3. \( \text{image}(b) \subset U_{\varepsilon}(\text{image}(a)) \) with \( \varepsilon = C \cdot \sqrt[l]{l} \)

In the case \( k = n \) we get (while 1. and 3. stay unchanged)

2'. \( \text{mass}(b) \leq C \cdot l^{\frac{n+2}{n}} \)

We will use Theorem 1 to compute the equivalence classes of the higher divergence functions for the Heisenberg Group \( H^{2n+1} \) in and below the critical dimension \( n \).

Theorem 2. Let \( H^{2n+1} \) be the Heisenberg Group of dimension \( 2n + 1 \). Then holds:

\[
\text{Div}_k(H^{2n+1}) \sim r^{k+1} \quad \text{for } k < n
\]

and

\[
\text{Div}_n(H^{2n+1}) \sim r^{n+2} \quad \text{for } k = n
\]

Remark. We see, that for the Heisenberg Group we get a similar result as in the case of \( \mathbb{R}^n \): It is, in the low dimensions (\( k \leq n \)), roughly as difficult to fill a cycle with an avoidant chain as with a non-avoidant chain.
**Corollary.** Let $\Gamma$ be a group acting properly discontinuously, freely and cocompactly on $H^{2n+1}$ via isometries. Then holds:

\[
\text{Div}_k(\Gamma) \sim r^{k+1} \quad \text{for } k < n \\
\text{Div}_n(\Gamma) \sim r^{n+2} \quad \text{for } k = n
\]

**Proof.** By the lemma of Švarc-Milnor, $\Gamma$ is quasi-isometric to $H^{2n+1}$. The claim follows with Proposition 1.1. \qed

## 2 Proof of Theorem 1

### 2.1 Federer-Fleming

A main tool for the proof of Theorem 1 is the Theorem of Federer Fleming about simplicial approximations of lipschitz chains. A proof, near the version stated below, can be found in [5, Thm. 10.3.3].

**Theorem 2.1** (Deformation-Theorem of Federer-Fleming). Let $X$ be a simplicial complex. Then there is a constant $c_X > 0$, such that for every lipschitz $k$-chain $a$ in $X$ there is a simplicial $k$-chain $P(a)$, a lipschitz $(k+1)$-chain $Q(a)$ and a lipschitz $k$-chain $R(a)$ with:

1. $\text{mass}(P(a)) \leq c \cdot \text{mass}(a)$
2. $\text{mass}(Q(a)) \leq c \cdot \text{mass}(a)$
3. $\text{mass}(R(a)) \leq c \cdot \text{mass}(\partial a)$
4. $\partial Q(a) = a - P(a) - R(a)$
5. $\partial R(a) = \partial a - \partial P(a)$

Furthermore, the chains $P(a)$ and $Q(a)$ are included in the smallest subcomplex of $X$, that contains $a$, and $R(a)$ is included in the smallest subcomplex of $X$, that contains $\partial a$.

### 2.2 Young’s Construction

In [11] Young gives a construction for fillings of cycles in Carnot groups with specific preconditions. This fillings fulfil property 2. (resp. 2’.) in Theorem 1. Young further shows, that the Heisenberg Groups satisfy the preconditions for this construction (see [11, Lem. 4.13]).

In these section we give the main steps of the construction and state some properties, that we will use to prove Theorem 1. The proofs can be found in [11,Section 3].

Below $G$ is a Carnot group (or just the Heisenberg Group $H^{2n+1}$) and $(\tau, f : \tau \rightarrow G)$ a triangulation of $G$, i.e. $\tau$ is a simplicial complex and $f$ a bilipschitz homeomorhism. Further let $\phi : \tau \rightarrow G$ (one assume the existence for the moment, for the Heisenberg Groups it is proved in [11,Lem.4.13]) a $k$-horizontal map in bounded distance of $f$, i.e. there is a constant $c$ with $d(f(x), \phi(x)) < c$ for all $x \in \tau$.

Now let $a$ be a lipschitz $k$-cycle in $G$. 

STEP 1

At first one approximates the $k$-cycle $a$ and constructs an interpolating $(k+1)$-chain. Define

$$P_{\phi(\tau)}(a) := \phi(P_{\tau}(f^{-1}(a))) \quad (P_{\tau} \text{ as in Theorem 2.1})$$

Then $P_{\phi(\tau)}(a)$ is a horizontal $k$-cycle.

Because $\phi$ is in bounded distance to $f$ there is a lipschitz homotopy $h : G \times [0,1] \to G$ between $id_G$ and $\phi \circ f^{-1}$.

Now one defines

$$Q_{\phi(\tau)}(a) := h(a \times [0,1]) + \phi(Q_{\tau}(f^{-1}(a)))$$

Lemma 2.2 ([11, Lem.3.1]). Let $c_\tau$ be the constant from the Deformation-Theorem 2.1 and let $c_Q = \text{lip}(h)^{k+1} + \text{lip}(f^{-1})^k \text{lip}(\phi)^{k+1} c_\tau$. Then holds:

- $\partial Q_{\phi(\tau)}(a) = P_{\phi(\tau)}(a) - a$
- $\text{mass}(Q_{\phi(\tau)}(a)) \leq c_Q \cdot \text{mass}(a)$

Remark. In the above Lemma, $c_Q$ only depends on the triangulation and $\phi$.

STEP 2

In this step one defines coarser approximations of $a$.

Let $(s_t)_{t \in \mathbb{R}}$ be the family of scaling automorphisms of $G$.

Define for $i \in \mathbb{N}_0$

$$P_i(a) := s_i(P_{\phi(\tau)}(s_{2-i}(a)))$$

Lemma 2.3 ([11, Lem.3.2]). There is a constant $c_P$, only depending on the triangulation and $\phi$, such that:

$$\text{mass}(P_i(a)) \leq c_P \cdot \text{mass}(a) \quad \forall i$$

STEP 3

Now one constructs a chain, interpolating between the first two approximations.

Let $(\tau, f)$ and $(\tau, s_2 \circ f)$ triangulations of $G$. And let $(\eta, g : \eta \to G \times [0,1])$ be a triangulation of $G \times [0,1]$, such that $g^{-1}(G \times j) \equiv \tau$, $j \in \{0,1\}$ via the isomorphisms $\iota_j$ with $g \circ \iota_0 = f$ and $g \circ \iota_1 = s_2 \circ f$.

Further let $\psi : \eta \to G$ be a $(k+1)$-horizontal map with $\psi \circ \iota_0 = \phi$ and $\psi \circ \iota_1 = s_2 \circ \phi$.

Define

$$R_{\psi(\eta)}(a) := \psi\left(P_{\eta}(Q_{\tau}(f^{-1}(a)) + g^{-1}(\alpha \times [0,1]) - Q_{\tau}((s_2 \circ f)^{-1}(a)))\right)$$

Lemma 2.4 ([11, Lem.3.3]). $R_{\psi(\eta)}(a)$ is horizontal and there is a constant $c_R$, only depending on $\psi$ and $(\eta, g)$, such that:

- $\partial R_{\psi(\eta)}(a) = P_{(s_2 \circ \phi)(\tau)}(a) - P_{\phi(\tau)}(a)$
- $\text{mass}(R_{\psi(\eta)}(a)) \leq c_R \cdot \text{mass}(a)$

Remark. The existence of the triangulation $(\eta, g)$ and the horizontal map $\psi$ is not clear for an arbitrary Carnot group. But for the Heisenberg Group of dimension $2n+1$ it is proved for $k < n$ in [11, Lem. 4.13].
**Step 4**

In this step one interpolates between two consecutive coarser approximations. Define for \( i \in \mathbb{N}_0 \)
\[
R_i(a) := s_{2^i}(R_{\psi(\eta)}(s_{2^{-i}}(a)))
\]

**Lemma 2.5** ([11, Lem.3.4]). With the above definition holds:

- \( \partial R_i(a) = P_{i+1}(a) - P_i(a) \)
- \( \text{mass}(R_i(a)) \leq 2^i c_R \cdot \text{mass}(a) \)

**Step 5**

To bound the mass of the constructed filling, one needs a 'coarsest' approximation, i.e. an index, such that all rougher approximations are 0.

**Lemma 2.6.** Let \( c_\tau \) be the constant in the Deformation-Theorem 2.1 and \( c' := c_\tau \text{lip}(f^{-1})^k(\text{vol}(\Delta_k))^{-1} \). Further let \( i_0 \) be the integer with
\[
2^{(i_0-1)k} \leq c' \cdot \text{mass}(a) < 2^{i_0k}.
\]
Then:
\[
P_i(a) = 0 \quad \text{for all } i \geq i_0
\]

**Step 6**

In this last step, one collects the constructions to define a filling of the cycle \( a \) with controlled mass. Define
\[
b := -(Q_{\phi(\tau)}(a) + \sum_{i=0}^{i_0-1} R_i(a))
\]

**Proposition 2.7** ([11, Thm.3]). For the \((k + 1)\)-chain \( b \) holds:

- \( \partial b = a - P_{i_0} = a \)
- \( \text{mass}(b) \leq (c_Q + 2c_R) \text{mass}(a)^{\frac{k+1}{k}} \)

If one omits the horizontal-condition (for \( \phi \) and \( \psi \)) one gets instead of Lemma 2.5 and Proposition 2.7 (see [11,Section 5]):

**Lemma 2.8.** Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be a function, such that for any \((k + 1)\)-simplex \( \Delta \) of \( \eta \) we have \( \text{mass}(s_t(\psi(\Delta))) \leq f(t) \). Then

- \( \partial R_i(a) = P_{i+1}(a) - P_i(a) \)
- \( \text{mass}(R_i(a)) \leq 2^{-ik} f(2^i c_R) \cdot \text{mass}(a) \)

**Proposition 2.9.** If \( f(t) = c_f \cdot t^{k+2} \) with some constant \( c_f > 0 \), then for the \((k + 1)\)-chain \( b \) holds:

- \( \partial b = a - P_{i_0} = a \)
- \( \text{mass}(b) \leq (c_Q + 2c_R) c_f \cdot \text{mass}(a)^{\frac{k+2}{k}} \)

**Remark.** For the Heisenberg Group \( H^{2n+1} \) there is for \( k = n \) a suitable map \( \psi \) and an constant \( \hat{c} \) with \( \text{mass}(s_t(\psi(\Delta))) \leq \hat{c} t^{k+2} \) for all \((k + 1)\)-simplices \( \Delta \) in \( \eta \). This is proved in [11, previous to Thm. 8].
2.3 Proof of the neighbourhood-condition

To check that the filling constructed above is contained in the desired neighbourhood of \( a \), we compute the maximal distance of points in the summands of \( b \) to \( a \).
Without loss of generality, we assume \( \operatorname{diam}(\Delta) = 1 \) for all \( m \)-simplices \( \Delta \in \eta , m \geq 1 \).

**Lemma 2.10.** Let \( c \) be the distance between \( f \) and \( \phi \) and define \( \delta := \psi + \phi + c \).
Then \( Q_{\phi(f)}(a) \) lies in the \( \delta \)-neighbourhood of \( a \).

**Proof.** At first remember the definition of \( Q_{\phi(f)}(a) := \psi + \phi + c \).
Because \( h \) is a lipschitz homotopy, it moves points not farer away than \( \psi + \phi + c \).
\( Q_{\phi} \) interpolates between \( f^{-1}(a) \) and \( P_{\phi}(-1)(a) \) by Federer-Fleming, and it is contained in the smallest subcomplex of \( \tau \) that contains \( f^{-1}(a) \).
We map it with \( f \) to \( G \) and get that \( f(Q_{\phi}(f^{-1}(a))) \) is contained in the \( \psi \)-neighbourhood of \( a \).
The maps \( f \) and \( \phi \) are in bounded distance with \( d(f(x), \phi(x)) < c \ \forall x \in \tau \).
It follows that \( Q_{\phi}(f^{-1}(a)) \) is contained in the \( \psi \)-neighbourhood of \( a \).

**Lemma 2.11.** Set \( \lambda := \psi + \phi + (\psi + \phi + 1) \).
Then \( R_{\psi, \phi}(a) \) lies in the \( \lambda \)-neighbourhood of \( a \).

**Proof.** \( Q_{\phi} \) is an interpolating chain by Federer-Fleming, so it is contained in the smallest subcomplex of \( \tau \) that contains \( g^{-1}(a) \).
We map it with \( f \) to \( G \) and, because \( g \) is in bounded distance to \( f \), get that \( f(Q_{\phi}(g^{-1}(a))) \) is contained in the \( \psi \)-neighbourhood of \( a \).

**Lemma 2.12.** Set \( \lambda_i := 2^i \lambda \). Then \( \psi_{\phi}(s_{2^i}(a)) \) lies in the \( \lambda \)-neighbourhood of \( a \).

**Proposition 2.13.** Let \( b = -(Q_{\phi(f)}(a) + \sum_{i=0}^{n-1} R_{\phi}(s_{2^i}(a))) \) be the constructed filling of \( a \) with \( \operatorname{mass}(b) \leq \sqrt[2n]{\sqrt[2n]{\sqrt[n]{\lambda} + 2c}} \cdot \sqrt[n]{\lambda} \), let \( c' \) be as in Lemma 2.6, \( \delta \) as in Lemma 2.10 and \( \lambda \) as in Lemma 2.11 and define \( \mu := ((\delta + \lambda)^{b_{c'}} + b_{c'}) \cdot \sqrt[n]{\lambda} \).
Then \( b \) lies in the \( \mu \)-neighbourhood of \( a \).
Proof. We connect the above results: As each $R_i(a)$, $0 \leq i < i_0$ is contained in the $2^i\lambda$-neighbourhood ($\subset 2^{i-1}\lambda$-neighbourhood) of a it follows that $\sum_{i=0}^{i_0-1} R_i(a)$ is contained in the $(2^{i_0-1}\lambda)$-neighbourhood of a. $Q_{\delta(\tau)}(a)$ is contained in the $\delta$-neighbourhood of a by Lemma 2.10. And by definition of $i_0$ we have $1 \leq 2^{i_0-1} \leq \sqrt[7]{c} \cdot \max(\mathcal{a})$.

So $b \subset U_{\max\{\delta,\lambda,\sqrt[7]{\max(\mathcal{a})}\}} \subset U_{\delta + \lambda, \sqrt[7]{\max(\mathcal{a})}} \subset U_{((\delta + \lambda) \cdot \sqrt[7]{\max(\mathcal{a})}) \cdot \sqrt[7]{\max(\mathcal{a})}}$.

\[ \Box \]

Remark. In section 2.3 we nowhere used horizontality and so the neighbourhood condition is proved for all $k \leq n$.

With the constant defined as $C := \max\{c_Q + 2c_R, (c_Q + 2c_R)\cdot (\delta + \lambda) \cdot \sqrt[7]{\mathcal{a}}\}$, this proves Theorem 1: By Proposition 2.7 (resp. 2.9) we get a filling $b$ with $\max(b) \leq (c_Q + 2c_R)\cdot \max(a)^{\frac{k+1}{k}} \leq C \cdot \max(a)^{\frac{k+1}{k}}$ (resp. $\max(b) \leq (c_Q + 2c_R)\cdot \max(a)^{\frac{k+1}{k}} \leq C \cdot \max(a)^{\frac{k+1}{k}}$) and by Proposition 2.13 this filling is contained in $U_{((\delta + \lambda) \cdot \sqrt[7]{\max(\mathcal{a})}) \cdot \sqrt[7]{\max(\mathcal{a})}}$.

3 PROOF OF THEOREM 2

3.1 Horizontal Triangulations

Definition. A triangulation $(\tau, f)$ of $H^{2n+1}$ is horizontal, if $f : \tau \rightarrow H^{2n+1}$ is $n$-horizontal.

Lemma 3.1. There is a horizontal triangulation $(\tau, f)$ of $H^{2n+1}$ and a $\rho > 0$, such that

1. $\text{diam}(f(\Delta)) < \frac{1}{2} \quad \forall \Delta \in \tau$

2. $d(f(\tau^{(k)}), 0) > \rho \quad \forall k \leq n$, where $\tau^{(k)}$ denotes the $k$-skeleton of $\tau$.

Proof. Let $(\tau_0, f_0)$ be an arbitrary horizontal triangulation (Existence see [5, 3.4.B], or [11, Lem. 4.13]) and $\delta \in \mathbb{R}$ with $\delta \geq \text{diam}(f_0(\Delta)) \quad \forall \Delta \in \tau_0$.

Now let $x \in \text{Int}(f_0(\Delta_{2n+1}))$ for an arbitrary $(2n+1)$-simplex of $\tau_0$ and choose $\rho_0$ with $0 < \rho_0 < d(x, f(\Delta_{2n+1}))$.

Then $(\tau_1, f_1) := (\tau_0, L_{x-1} \circ f_0)$ is a horizontal triangulation with $\delta > \text{diam}(f_1(\Delta)) \quad \forall \Delta \in \tau_1$ and $d(f(\tau_1^{(k)}), 0) > \rho_0 \quad \forall k \leq n$.

We scale this triangulation with $\frac{1}{2}$, then $(\tau, f) := (\tau_1, s_{\frac{1}{2}} \circ f_1)$ is a horizontal triangulation with $\text{diam}(f(\Delta)) < \frac{1}{2} \quad \forall \Delta \in \tau$ and $d(f(\tau^{(k)}), 0) > \frac{\rho_0}{2} \quad \forall k \leq n$.

The claim follows with $\rho = \frac{\rho_0}{2}$.

\[ \Box \]

If we start the construction in 2.2 with a horizontal triangulation, we can choose $\phi = f$. In the following we assume this situation.

We are interested how the constant $c_Q$, defined in Lemma 2.2, changes, if we replace a horizontal triangulation $(\tau, f)$ by the horizontal triangulation $(\tau, s_t \circ f)$.

For this purpose we denote by $c_{Q,f}$ the constant for the triangulation $(\tau, f)$ and by $c_{Q,s \circ f}$ the constant for the triangulation $(\tau, s_t \circ f)$.

Lemma 3.2. Let $(\tau, f)$ and $(\tau, s_t \circ f)$ triangulations of $H^{2n+1}$ as above.

Then holds:

$c_{Q,s \circ f} = t \cdot c_{Q,f}$
Proof. First we recall the definition of $c_{Q,f}$:

$$c_{Q,f} = \text{lip}(h)^{k+1} + \text{lip}(f^{-1})^k \text{lip}(\phi)^{k+1} c_{\tau}$$

Because we have $\phi = f$, the homotopy $h$ drops out and $c_{Q,f} = \text{lip}(f^{-1})^k \text{lip}(\phi)^{k+1} c_{\tau}$.

For the other lipschitz constants we use again the fact $d(s_t(x), s_t(y)) = t \cdot d(x, y)$ and get:

$$\text{lip}(s_t \circ f) = t \cdot \text{lip}(f)$$

$$\text{lip}((s_t \circ f)^{-1}) = \text{lip}(f^{-1} \circ s_{t^{-1}}) = t^{-1} \cdot \text{lip}(f^{-1})$$

This implies:

$$c_{Q,s \circ f} = \text{lip}((s_t \circ f)^{-1})^k \text{lip}(s_t \circ f)^{k+1} c_{\tau}$$

$$= t^{-k} \text{lip}(f)^{-1)^k t^{k+1} \text{lip}(f)^{k+1} c_{\tau}$$

$$= t \cdot \text{lip}(f)^{-1)^k \text{lip}(f)^{k+1} c_{\tau}$$

$$= t \cdot c_{Q,f}$$

\[ \square \]

Remark. In the proof above it is important, that $c_{\tau}$ only depends on $\tau$ and is not influenced by changing $f$.

3.2 The proof in the case $k < n$

The proof of Theorem 2, which is inspired by the computation of the higher Divergence functions of $\mathbb{R}^n$ seen in [1, 2, 3], consists of two parts: One for the upper bound and one for the lower bound.

**Proposition 3.3.** Let $H^{2n+1}$ be the $(2n+1)$-dimensional Heisenberg Group and $k < n$. Then:

$$r^{k+1} \leq \text{Div}^k(H^{2n+1})$$

**Proof.** Let $r, l > 0$ and $a$ be the 'hard to fill' $k$-cycle of mass($a$) = $l$ constructed by Burillo in [2, Proof of Thm. 2.1]. Then there is a constant $\tilde{c}$, independent from $a$, such that every $(k+1)$-chain $b$ with $\partial b = a$ has mass($b$) $\geq \tilde{c} \cdot l^{k+1}$. By left-translation we can transport $a$ out of $B_r(0)$. The left-invariance of the metric $d$ now implies the claim. \[ \square \]

Before we start with the proof of the upper bound, we state a helpful lemma:

**Lemma 3.4.** Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function and let $0 < \rho_0 \leq 1$ and $\alpha_0 \geq 0$, such that $\text{div}^{k}_{\rho_0, \alpha}(r) \leq h(r)$ for all $\alpha \geq \alpha_0$.

Then holds:

$$\text{Div}^k(X) \leq h(r)$$

**Proof.** It's clear by definition, that $\text{div}^{k}_{\rho, \alpha}(r) \leq \text{div}^{k}_{\rho_0, \alpha}(r)$ for all $\rho \leq \rho_0$.

The constants $L, M, A$ can all be taken as 1. \[ \square \]

**Proposition 3.5.** Let $H^{2n+1}$ be the $(2n+1)$-dimensional Heisenberg Group and $k < n$.

Then:

$$\text{Div}^k(H^{2n+1}) \leq r^{k+1}$$
Proof. Let \( r > 0 \) and \( a \) be an \( r \)-avoidant \( k \)-cycle with mass\((a) = l \). Theorem 1 ensures the existence of a \((k + 1)\)-chain \( b \) such that \( \partial b = a \), mass\((b) \leq C \cdot \text{mass}(a)^{\frac{k+1}{k+1}} \) and image\((b) \subset U_\varepsilon(\text{image}(a)) \) with \( \varepsilon = C \cdot \frac{1}{\sqrt{k+1}} \).

Case 1: Let \( l \leq \left(\frac{1}{2C}\right)^{k+1} \). Then \( b \) is \( \frac{1}{2}r \)-avoidant.

Case 2: Let \( l > \left(\frac{1}{2C}\right)^{k+1} \) and w.l.o.g. \( r \geq 1 \).

If we show \( \text{div}_{\rho,\alpha}^k(1) \leq h(1) \) we get \( \text{div}_{\rho,\alpha}^k(r) \leq h(r + 1) \) for all \( r \leq 1 \), because \( \text{div}_{\rho,\alpha}^k \) and \( h \) are increasing.

Now let \((\tau, f)\) be any triangulation of \( H^{2n+1} \) with \( \text{diam}(f(\Delta)) < \frac{1}{2} \) \( \forall \Delta \in \tau \) and \( d(f(\tau(k)), 0) > \rho \ \forall k \leq n \), as in Lemma 3.1 and define \( \gamma := s_{\tau-1}(a) \).

Then \( \gamma \) is a 1-avoidant \( k \)-cycle with mass\((\gamma) \leq r^{-k}l \). Further let \( \beta \) be the filling of \( \gamma \) given by Theorem 1.

We approximate \( \beta \) using Theorem 2.1 and get \( b' := f(P_\tau(f^{-1}(\beta))) \). By definition of \((\tau, f)\) it follows that \( b' \) is horizontal and \( \rho \)-avoidant. Theorem 2.1 yields

\[
\text{mass}(b') \leq c_r C (r^{-k}l)^{\frac{k+1}{k+1}} = c_r C r^{-k+1}l^{\frac{k+1}{k+1}}
\]

\[
\partial b' = f(P_\tau(f^{-1}(\gamma))) = f(P_\tau(f^{-1}(s_{\tau-1}(a))))
\]

We now push out \( b' \) with \( s_\tau \) and get \( b := s_\tau(b') \), which is \( \rho r \)-avoidant with:

\[
\text{mass}(b) = r^{-k+1} \text{mass}(b') \leq c_r C l^{\frac{k+1}{k+1}}
\]

\[
\partial b = s_\tau(\partial b') = s_\tau\left(f(P_\tau(f^{-1}(s_{\tau-1}(a))))\right) = P_{(s_\tau \circ f)(\tau)}(a)
\]

So Young’s construction gives us an interpolating \((k + 1)\)-chain \( Q := Q_{(s_\tau \circ f)(\tau)}(a) \) with

\[
\partial Q = P_{(s_\tau \circ f)(\tau)}(a) - a
\]

\[
\text{mass}(Q) \leq c_{Q,s_\tau \circ f} \text{mass}(a) = r c_{Q,f} l \leq 2C \sqrt[2k+1]{c_{Q,f}} l^\frac{k+1}{k+1}
\]

Where the second inequality follows by the lower bound of \( l \).

\( Q \) is contained in the smallest subcomplex of \((s_\tau \circ f)(\tau)\), which contains \( a \).

By definition of \((\tau, f)\), in particular \( \text{diam}(f(\Delta)) \leq \frac{1}{2} \), and the fact \( d(s_t(x), s_t(y)) = t \cdot d(x, y) \) we get, that \( \text{diam}(s_\tau \circ f(\Delta)) \leq \frac{1}{2}r \) and so \( Q \) is \( \frac{1}{2}r \)-avoidant.

Altogether \( b + Q \) is a \( \rho r \)-avoidant filling of \( a \) with

\[
\text{mass}(b + Q) = \text{mass}(b) + \text{mass}(Q) \leq (c_r C + 2C c_{Q,f}) l^{\frac{k+1}{k+1}}
\]

The claim follows with the definitions of \( \text{div}_{\rho,\alpha}^k \) and Lemma 3.4.

\[
3.3 \text{ The proof in the case } k = n
\]

**Proposition 3.6.** Let \( H^{2n+1} \) be the \((2n + 1)\)-dimensional Heisenberg Group.

Then:

\[
r^{n+2} \preceq \text{Div}^n(H^{2n+1})
\]

**Proof.** Let \( r, l > 0 \) and \( a \) be the ‘hard to fill’ \( n \)-cycle of mass\((a) = l \) constructed by Burillo in [2, Proof of Thm. 2.1]. Then there is a constant \( \tilde{c} \), independent from \( a \), such that every \((n + 1)\)-chain \( b \) with \( \partial b = a \) has mass\((b) \geq \tilde{c} \cdot l^{\frac{n+2}{n+2}} \). By left-translation we can transport \( a \) out of \( B_r(0) \). The left-invariance of the metric \( d \) now implies the claim.
Proposition 3.7. Let $H^{2n+1}$ be the $(2n + 1)$-dimensional Heisenberg Group. Then:

$$\text{Div}^n(H^{2n+1}) \leq r^{n+2}$$

Proof. Let $r > 0$ and $a$ be an $r$-avoidant $n$-cycle with mass($a$) = $l$. Theorem 1 ensures the existence of a $(n + 1)$-chain $b$ such that $\partial b = a$, mass($b$) $\leq C \cdot \text{mass}(a)^\frac{n+2}{2}$ and image($b$) $\subset U_r(\text{image}(a))$ with $\varepsilon = C \cdot \sqrt{l}$.

Case 1: Let $l \leq \left(\frac{1}{2C}\right)^n r^n$. Then $b$ is $\frac{1}{2}r$-avoidant.

Case 2: Let $l > \left(\frac{1}{2C}\right)^n r^n$ and w.l.o.g. $r \geq 1$. (See above)
Now let $(\tau, f)$ be any triangulation of $H^{2n+1}$ with diam($f(\Delta)) < \frac{1}{2^n} \forall \Delta \in \tau$ and $d(f(\tau^k), 0) > \rho \quad \forall k \leq n$, as in Lemma 3.1 and define $\gamma := s_{r-1}(a)$.
Then $\gamma$ is an 1-avoidant $n$-cycle with mass($\gamma$) $\leq r^{-n}l$. Further let $\beta$ be the filling of $\gamma$ given by Theorem 1.
We approximate $\beta$ using Theorem 2.1 and get $b' := f(P_r(f^{-1}(\beta)))$ and $R := f(R_r(f^{-1}(\beta))).$
By definition of $(\tau, f)$ follows that $b'$ and $R$ are $\rho$-avoidant. Theorem 2.1 yields

$$\text{mass}(b') \leq c_r C(r^{-n}l) \frac{n+1}{n} = c_r C r^{-n+1} l \frac{n+1}{n}$$
$$\partial b' = f(P_r(f^{-1}(\gamma))) = f(P_r(f^{-1}(s_{r-1}(a))))$$
$$\text{mass}(R) \leq c_r \text{mass}(\partial \beta) \leq c_r r^{-n} l$$

We now push out $b'$ and $R$ with $s_r$ and get $b := s_r(b') + s_r(R)$, which is $\rho r$-avoidant with:

$$\text{mass}(b) \leq r^{n+2}(\text{mass}(b') + \text{mass}(R)) \leq c_r(C l^{n+2} + r^2 l)$$
$$\leq c_r(C l^{n+2} + 4C^2 \sqrt{l}) = (c_r C + 4c_r C^2) l^{n+2}$$

Where the last inequality follows by the lower bound of $l$.

$$\partial b = s_r(\partial b' + \partial R) = s_r \left( f(P_r(f^{-1}(a))) + (s_{r-1}(a) - f(P_r(f^{-1}(a)))) \right) = a$$

Altogether $b$ is a $\rho r$-avoidant filling of $a$ with

$$\text{mass}(b) \leq (c_r C + 4c_r C^2) l^{n+2} \frac{n+1}{n}$$

The claim follows with the definitions of $\text{div}^n_{\rho, a}$, $\leq$ and Lemma 3.4. 

4 GENERALISATION

In [11] it is shown, that the Jet-Groups $J^m(\mathbb{R}^n)$ (for definition see [11, 4.2], [8]) fulfil the preconditions for the construction in 2.2. in the dimensions $k < n$. For the critical dimension $k = n$ we use again lemma 2.8 with $f(t) = t^{k+m+1}$ (see [11, Thm.8]).
As we nowhere in the proof of Theorem 1 used anything else than the construction of Young we get an similar version for the Jet-Groups:

Theorem 3. Let $J^m(\mathbb{R}^n)$ be the Jet-Group and $k < n$.
Then there is a constant $C > 0$, such that for every $k$-cycle $a$ with mass($a$) = $l$ there exists a $(k + 1)$-chain $b$ with
1. \( \partial b = a \)

2. \( \text{mass}(b) \leq C \cdot t^{\frac{k+1}{m}} \)

3. \( \text{image}(b) \subset U_\varepsilon(\text{image}(a)) \) with \( \varepsilon = C \cdot \frac{1}{\sqrt{t}} \)

In the case \( k = n \) we get

\[ 2'. \quad \text{mass}(b) \leq C \cdot t^{\frac{n+m+1}{n}} \]

If we define horizontal triangulations for the Jet-Groups in the same manner as in 3.1 for the Heisenberg Groups, we can translate lemma 3.1 and lemma 3.2 to the Jet-Groups by using the existence of a horizontal triangulation of the Jet-Group (see \cite[Lem. 4.13]{11}).

As we now have all the tools, used in the proof of the upper bound of the divergence function of the Heisenberg Groups, translated to the Jet-Groups we get:

**Proposition 4.1.**

\[
\begin{align*}
\text{Div}^k(J^m(\mathbb{R}^n)) & \preceq r^{k+1} \quad \text{for } k < n \\
\text{Div}^n(J^m(\mathbb{R}^n)) & \preceq r^{n+m+1} \quad \text{for } k = n
\end{align*}
\]

For the lower bound we use again a 'hard-to-fill' cycle (see preliminaries for Theorem 10 in \cite{11} and correct the scaling exponent of \( w_{k+1} \) to \( k + m + 1 \) (because \( z \) is in the \((m+1)\)-th summand of the grading)) and translate it out of the \( r \)-ball. So we finally get:

**Theorem 4.** Let \( J^m(\mathbb{R}^n) \) be the Jet-Group.

Then holds:

\[
\begin{align*}
\text{Div}_k(H^{2n+1}) & \sim r^{k+1} \quad \text{for } k < n \\
\text{Div}_n(H^{2n+1}) & \sim r^{n+m+1} \quad \text{for } k = n
\end{align*}
\]

**Corollary.** Let \( \Gamma \) be a group acting properly discontinuously, freely and cocompactly on \( J^m(\mathbb{R}^n) \) via isometries. Then holds:

\[
\begin{align*}
\text{Div}_k(\Gamma) & \sim r^{k+1} \quad \text{for } k < n \\
\text{Div}_n(\Gamma) & \sim r^{n+2} \quad \text{for } k = n
\end{align*}
\]

**Proof.** By the lemma of Švarc-Milnor, \( \Gamma \) is quasi-isometric to \( J^m(\mathbb{R}^n) \). The claim follows with Proposition 1.1.

**Acknowledgment**

I want to thank Enrico Leuzinger for introducing me to the topic and let me participate in his experience.
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