Sharp multiple testing boundary for sparse sequences

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Abstract: This work investigates multiple testing from the point of view of minimax separation rates in the sparse sequence model, when the testing risk is measured as the sum FDR+FNR (False Discovery Rate plus False Negative Rate). First using the popular beta-min separation condition, with all nonzero signals separated from 0 by at least some amount, we determine the sharp minimax testing risk asymptotically and thereby explicitly describe the transition from “achievable multiple testing with vanishing risk” to “impossible multiple testing”. Adaptive multiple testing procedures achieving the corresponding optimal boundary are provided: the Benjamini–Hochberg procedure with properly tuned parameter, and an empirical Bayes ℓ-value (‘local FDR’) procedure. We prove that the FDR and FNR have non-symmetric contributions to the testing risk for most procedures, the FNR part being dominant at the boundary. The optimal multiple testing boundary is then investigated for classes of arbitrary sparse signals. A number of extensions, including results for classification losses, are also discussed.

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1. Introduction

1.1. Background

Multiple testing is a prominent topic of contemporary statistics, with a wide spectrum of applications including for example molecular biology, neuro-imaging and astrophysics. In this framework, many individual tests have to be performed simultaneously while controlling global error rates that take into account the multiplicity of the tests. A primary aim is to build procedures that guarantee control of a form of type I error, the most popular being the False Discovery Rate (FDR, see (3) below). For instance, the celebrated Benjamini–Hochberg (BH) procedure controls the FDR under independence [8]. Given an FDR controlling procedure, one may then ask whether it has a controlled type II error (or equivalently good power), measured for instance by the False Negative Rate (FNR, see (4) below).

In this context, a natural question is that of optimality: what is the best sum of type I and type II errors that is achievable by any multiple testing procedure? From a testing perspective and when a single test is considered, this can be answered via minimax separation rates between the two considered hypotheses, which has been investigated for a variety of nonparametric (see, e.g., [30], and [25] for an overview and further references) and high-dimensional models and loss functions, see, e.g., [6], [31], [37]. The analogous question for multiple testing has received attention only very recently: the case of a familywise error risk is studied in [23]; in the case of FDR and FNR risks for deterministic sparse signals, aspects of this problem have being investigated in [4] and subsequently also in [17], [39], [38], [7]. More precise connections to these works are made below, see Section 1.7.

1.2. Sparse sequence model

Consider observing data $X = (X_1, \ldots, X_n)$, where

$$X_i = \theta_i + \varepsilon_i, \quad i = 1, \ldots, n,$$

with $\varepsilon_1, \ldots, \varepsilon_n$ i.i.d. $\mathcal{N}(0,1)$ variables. (While the Gaussian case is our default noise distribution, we will later also consider Subbotin-type distributions, see (15).) The vector $\theta = (\theta_1, \ldots, \theta_n)$ is assumed to be sparse, that is, to belong to the set

$$\ell_0[s_n] = \{ \theta \in \mathbb{R}^n, \#\{i : \theta_i \neq 0\} \leq s_n \}$$

consisting of vectors that have at most $s_n$ nonzero coordinates, where $0 \leq s_n \leq n$. We write $P_{\theta}$ for the law of $X$ with parameter $\theta$ in (1) and $E_{\theta}$ for the corresponding expectation.

The multiple testing problem consists of testing simultaneously, for each $1 \leq i \leq n$, the null hypothesis that there is no signal against the alternative hypothesis:

$$H_{0,i} : \theta_i = 0 \quad \text{vs.} \quad H_{1,i} : \theta_i \neq 0.$$

1.3. Multiple testing risks

A multiple testing procedure is formally defined as a measurable function of the data $\varphi : x \in \mathbb{R}^n \mapsto (\varphi_i(x))_{1 \leq i \leq n} \in \{0,1\}^n$, where, by convention, $\varphi_i(X) = 1$ corresponds to rejecting the null $H_{0,i}$. As such, the procedure will depend on $n$, and with some slight
abuse of terminology, when dealing with asymptotics in terms of $n$, a sequence of such procedures is sometimes simply referred to as a ‘procedure’ for short.

For any $\theta \in \mathbb{R}^n$ and any procedure $\varphi$, the false discovery rate (FDR) and the false discovery proportion (FDP) of $\varphi$ at the parameter $\theta$ are respectively defined as

$$FDR(\theta, \varphi) = E_\theta [\text{FDP}(\theta, \varphi)], \quad \text{FDP}(\theta, \varphi) = \frac{\sum_{i=1}^n 1\{\theta_i = 0\} \varphi_i(X)}{1 \lor \sum_{i=1}^n \varphi_i(X)}.$$  \hfill (3)

The false negative rate (FNR) at $\theta$ is here defined as (see, e.g., [4])

$$\text{FNR}(\theta, \varphi) = E_\theta \left[ \sum_{i=1}^n 1\{\theta_i \neq 0\} (1 - \varphi_i(X)) \right] \left(1 \lor \sum_{i=1}^n 1\{\theta_i \neq 0\} \right).$$ \hfill (4)

The (multiple testing) combined risk at $\theta \in \mathbb{R}^n$ of a procedure $\varphi$ is the sum

$$\mathcal{R}(\theta, \varphi) = \text{FDR}(\theta, \varphi) + \text{FNR}(\theta, \varphi).$$

Given the popularity of FDR and FNR, this can be considered as a canonical notion of testing risk in the multiple testing context: it has indeed been considered for example in [4] and the papers mentioned above. While the above is the main notion of risk used in this paper, other choices, including the classification risk, are discussed in Section 5.

1.4. Separation of hypotheses and minimax testing risk

To investigate “separation” rates for multiple testing, one needs to define a notion of separation of alternatives $H_{1,i}$ from nulls $H_{0,i}$. Indeed, one cannot hope to have a vanishing combined risk when the nonzero $\theta_i$’s are arbitrarily close to 0. In the present high-dimensional setting, perhaps the most popular way to separate hypotheses is via a ‘beta-min’ condition (see, e.g., [11], Section 7.4) meaning that all nonzero signals are above a certain “large” threshold value. For instance, this condition is used for the (related but different) task of consistent model selection for estimators such as the LASSO; see Section 1.7 for more details on this, and on other losses.

A natural optimality benchmark is the minimax multiple testing risk defined as

$$\mathcal{R}(\Theta) = \inf_{\varphi} \sup_{\theta \in \Theta} \mathcal{R}(\theta, \varphi),$$ \hfill (5)

where the infimum is over all multiple testing procedures and the parameter set $\Theta$ is some appropriate subset of $\ell_0[s_n]$. Typically, interesting parameter sets $\Theta$ are given to be “as large as possible”, while keeping $\mathcal{R}(\Theta)$ in (5) at least strictly smaller than 1. Noting that the risk of the trivial procedure $\varphi_i = 0$ for all $i$ is equal to 1, the latter corresponds to parameter configurations for which ‘non-trivial multiple testing’ is achievable.

To fix ideas, let us consider the collection $\Theta(M)$ of vectors $\theta \in \ell_0[s_n]$ with nonzero coordinates taking only one possible value $M = M(n, s_n)$. (In the main body of the paper we consider more general parameter sets.) It follows from results in [4] (Theorem 2 therein) that if

$$M > a \sqrt{n \log(n/s_n)} \quad \hfill (6)$$

for some constant $a > 1$, then the BH procedure with appropriately vanishing parameter has a vanishing $\mathcal{R}$-risk. In addition, it is proved that no thresholding-type procedure can have a non-trivial $\mathcal{R}$-risk uniformly over $\Theta(M)$ if $a < 1$ (Theorem 1 therein). These results suggest that, at least for thresholding procedures, the boundary of possible multiple testing is “close to” the threshold $\sqrt{2 \log(n/s_n)}$, which we refer to as oracle threshold in the sequel.
1.5. Questions of interest

The previous discussion raises the following natural questions:

- how does the minimax risk $R(\Theta)$ behave for separated alternatives (e.g., for $\Theta = \Theta(M)$ with $M$ as in (6)) when the infimum in (5) if taken over all possible procedures, not only thresholding ones?

- since multiple testing is “easy” when $M$ is large and impossible when $M$ is too small, what is the precise (asymptotic) boundary of signal strength $M$ for which $R(\Theta)$ goes from 0 to 1? In other words, can one describe the transition from 0 to 1 in $R$ when $M$ decreases? This requires investigating the sharp minimax risk $R(\Theta)$.

- are there procedures that achieve the minimax risk, at least asymptotically, without knowledge of the sparsity parameter $s_n$?

- suppose that rather than the $s_n$ nonzero coordinates all equalling one signal value $M$, they can be divided into two different values $M_1$ and $M_2$ (e.g., as in Figure 1, middle column). Is this an easier or more difficult multiple testing problem than the former?

These and more general questions are addressed in the sequel.

As a specific aspect of testing problems in general, and of multiple testing in particular, it is common in practice to allow for a tolerance level, especially for the type I error, here typically for the FDR. Hence, for the combined risk under study here, it is not only the case where $R(\Theta) = o(1)$ that is of interest, but also the one where $R(\Theta)$ is of the order of a (possibly small) constant $c \in (0,1)$. In addition, since the FDR and FNR account for errors that have different interpretations, it is also of interest to study the contribution of each error rate in the combined risk $R(\Theta)$. The results below will show that the contributions of FDR and FNR are not symmetric. Finally, we present our results asymptotically for simplicity – in particular this eases the presentation of results in the new setting of multiple signal strengths considered in Section 3 – but the proofs also give non-asymptotic bounds.

1.6. Popular procedures: BH and empirical Bayes $\ell$-values

Here, we describe two procedures that will be considered in the sequel.

First, probably the most widely used multiple testing procedure is the so-called BH procedure, introduced in [8]. For some level $\alpha$, it is defined as $\varphi^{BH}_\alpha = \{ |X_i| \geq \hat{t} \}_{1 \leq i \leq n}$ where the threshold $\hat{t}$ is defined as a specific intersection point between the empirical upper-tail
distribution function of the $X_i$’s and a quantile curve of the noise distributions (see Appendix D for details). To achieve good performance with respect to the combined risk, we will make use of the BH procedure where $\alpha$ is chosen to be slowly decreasing with $n$, as in [4, 10, 36]. Herein we make the typical choice $\alpha = \alpha_n \asymp 1/\sqrt{\log n}$.

The second procedure uses Bayesian $\ell$-values (often called local FDR values) with an empirical Bayes calibration. For a particular spike-and-slab prior $\Pi_w$ on $\mathbb{R}^n$ (see Appendix C for details), we consider the empirical Bayes $\ell$-value procedure defined by thresholding posterior probabilities of null hypotheses at some specified level $t \in (0, 1)$, i.e.,

$$\varphi_t = (1\{\Pi_w(\theta_i = 0 \mid X) < t\})_{1 \leq i \leq n},$$

where $\hat{w}$ is the marginal maximum likelihood estimator for $w$, as in [34, 33]. The choice of $t$ is not critical for obtaining a small combined risk, e.g., $t = 0.3$ or $t = 1/2$ are possible choices. The widely used empirical Bayes spike-and-slab posterior distribution has been investigated in terms of estimation properties in [33, 16], confidence sets in [18], and the resulting $\ell$-value procedure was recently shown to control the FDR in [17]; see also [5] for an overview on the analysis of Bayesian high-dimensional posteriors and [1] for a related $\ell$-value multiple testing algorithm.

### 1.7. Related literature, other modelling assumptions and risks

In the sparse sequence model, the interesting recent works [4, 39] provide bounds for the $\mathfrak{R}$-risk. In [4], the separation condition (6) is considered and the risk is shown to asymptotically converge to 0 for some procedures when $a > 1$, while it converges to 1 for any thresholding based procedure when $a < 1$. In [39], non-asymptotic lower bounds and upper bounds are further derived in the regime where $a = a_n > 1$ (possibly approaching 1). This allows the authors to show a limiting convergence rate, which is proved to be achieved by the BH procedure with a suitably decreasing level. This analysis is further broadened and extended to more general models in the recent preprint [38]. In these works, the case where the risk converges to an arbitrary constant is not studied, hence the problem of identifying the sharp transition of the minimax risk from 0 to 1 was left open.

Note that unlike the minimax risk, which involves an infimum over all multiple testing procedures, the optimality results stated in [4, 39] are restricted to thresholding-type procedures. Further, bounds are obtained only under a polynomial sparsity assumption.

In the multiple testing literature, a related way to measure optimality consists of finding a solution that minimizes the FNR while controlling a FDR-type error rate at level $\alpha$, see, e.g., [41, 29, 20] in case of weighted procedures. This task is often done under the so-called ‘two-group mixture model’, introduced in [21], which assumes that each null hypothesis is true with some probability, and relies on specific $\ell$-value (or local FDR) thresholding procedures, see [44, 45, 14, 15] and the recent work [28]. One way the present work differs from these references is in seeking not to minimise the FNR under a constraint, but rather to minimise the combined risk $\mathfrak{R}$. A more important difference is that we do not posit a mixture distribution for the true parameter $\theta$, but rather assume it is deterministic and arbitrary (up to the sparsity constraint). Despite these differences, we will see that that an $\ell$-value thresholding based procedure (namely, $\varphi_t$ as mentioned above) still achieves minimax performance.

Regarding related testing problems, the important work by Donoho and Jin [19] studies the detection problem for a single null and multiple alternatives, see also the subsequent works [26, 27, 3, 35]. In addition, the sparse sequence model has been much studied in terms of estimation for quadratic or $\ell^p$-losses, here we only mention [2, 43] for their connections to
multiple testing, where the authors use estimators related to the BH procedure.

Let us also mention that another FDR+FNR risk has been considered in [24], in a model with a single alternative distribution and using the knowledge of the number of true nulls. However, the FNR definition there is different from here: the denominator equals the number of accepted nulls, rather than the number of alternatives as here. In the case of sparse signal, this two FNR notions scale very differently, and using the FNR notion considered herein enables us to exhibit a sharp phase transition phenomenon. The recent work [7] derives some robust results for model selection based procedures, including for various notions of sums FDR+FNR, but only in a range where the multiple testing risk tends to zero at a certain rate.

Finally, a different but somewhat related loss function is the Hamming or classification loss. The corresponding classification risk is considered in [10, 36] in a two-group mixture model, with lower bounds restricted to thresholding classifiers. There, it is proved that the BH procedure with a suitably vanishing level achieves the (Bayes) oracle performance. However, these results study the Bayes risk (i.e., the minimum average risk) in this mixture model and do not provide a complete minimax analysis. Further extensions are derived in [32], e.g., by handling more general dependent models. Coming back to the Gaussian sequence model (1) (with non-random \( \theta \)'s), minimax Hamming estimators are derived in [12] (see also the earlier work [13]), where the authors study the boundary for exact recovery (the true support is recovered with high probability) and almost sure recovery (an overwhelming proportion of the true support is recovered with high probability). Connections between the \( \Phi \)-risk and almost sure recovery are further investigated in Section 5.

1.8. Outline and notation

The paper is organized as follows. Section 2 presents some main results of the paper: the asymptotic minimax risk is computed and adaptive procedures achieving this risk are given, with the FNR shown to be the dominating term in the risk for most procedures. These results are further generalized in Section 3 for arbitrary sparse signals and Subbotin noise, with illustrations provided in Section 4. Some other risks are investigated in Section 5 and related open issues are discussed in Section 6. While one prominent result is proved in Section 7, other proofs are postponed to the appendices.

Notation. The density of a standard normal variable is denoted by \( \phi \) and we write \( \Phi \) for the cumulative distribution function \( \Phi(x) = \int_{-\infty}^{x} \phi(t)dt \) and \( \overline{\Phi} = 1 - \Phi \) for the tail probabilities. We use \( \lim_{n \to \infty}, \overline{\lim}_{n \to \infty} \) to denote the liminf or the limsup, respectively. We use \( x_n \prec y_n, x_n \succ y_n \) or \( x_n = O(y_n) \) to signify that there exists \( C > 0 \) such that \( x_n \leq C y_n \) for all \( n \) large; we write \( x_n \asymp y_n \) if \( x_n \preceq y_n \) and \( y_n \preceq x_n \); we write \( x_n \sim y_n \) if \( x_n / y_n \to 1 \); and we write \( x_n \ll y_n, y_n \gg x_n \) or \( x_n = o(y_n) \) if \( x_n / y_n \to 0 \). We use this notation correspondingly for functions \( \phi(x), \Phi(x) \), with the limits taken either as \( x \to \infty \) or as \( x \to 0 \) depending on context. A sequence of random variables is said to be a \( o_P(1) \) if it converges to 0 in probability under the data generating law \( P_\theta \). For reals \( a, b \), we set \( a \vee b = \max(a, b) \) and \( a \wedge b = \min(a, b) \). Finally, for a finite set \( A \), we denote by \( |A| \) or \#\( A \) its cardinality.

2. Sharp boundaries: beta-min condition and Gaussian noise

To state our main results, we consider the sparse asymptotic setting where

\[
\frac{n}{s} \to \infty, \frac{s}{n} \to \infty \text{ and } \frac{n}{s} \to \infty. \quad (7)
\]
To evaluate the minimax risk, we define a class of configurations for $\theta$ that measures how the alternatives are separated from the null hypothesis: for a given $a \in \mathbb{R}$, set

$$\Theta = \Theta(a, s_n) = \left\{ \theta \in \ell_0[s_n] : |\theta_i| \geq a \text{ for } i \in S_\theta, \ |S_\theta| = s_n \right\},$$

(8)

where $S_\theta = \{i : \theta_i \neq 0\}$ denotes the support of $\theta$ (recall $\ell_0[s_n]$ is defined by (2)). This choice corresponds to a so-called beta-min type condition, meaning that all intensities of nonzero coefficients are required to be above a certain value $a > 0$. For example, each of the signals depicted in Figure 1 belongs to some class $\Theta(a, s_n)$; in the first panel, one may take $a$ to be the shared value of the nonzero $\theta_i$’s, whereas in the third $a$ can at most be taken to be the smallest nonzero value. More general classes, leading to more refined results (particularly in the latter case), are considered in Section 3.

2.1. Minimax multiple testing risk

Previous works in the literature [4, 39] suggest that the phase transition for the combined risk over $\Theta(a, s_n)$ arises when $a$ is close to the oracle threshold $\sqrt{2 \log(n / s_n)}$. We identify here a sharp formulation for this boundary, by considering for $b \in \mathbb{R}$,

$$\Theta_b = \Theta(ab, s_n), \quad ab = \sqrt{2 \log(n / s_n)} + b,$$

(9)

with $\Theta(a, s_n)$ as in (8). The following result holds.

**Theorem 1.** Consider $\Theta_b = \Theta(ab, s_n)$ as in (9). For any fixed $b \in \mathbb{R}$, under the sparse asymptotics (7) the minimax $\mathcal{R}$-risk over $\Theta_b$ verifies

$$\inf_{\varphi} \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi) = \Phi(b) + o(1).$$

The result also holds in the limiting cases $b = b_n \to +\infty$ and $b = b_n \to -\infty$.

Since $\Phi(b)$ increases from 0 to 1 when $b$ decreases from $+\infty$ to $-\infty$, Theorem 1 exhibits an asymptotic phase transition and shows that the considered boundary (9) is sharp. It follows from the proof of the result that the oracle thresholding rule

$$\varphi^*_i = 1\{|X_i| \geq \sqrt{2 \log(n / s_n)}\}, \ 1 \leq i \leq n,$$

(10)

is asymptotically minimax, independent of the value of $b$.

2.2. Applications: non-trivial testing and conservative testing

We provide below two consequences of Theorem 1, both considering $\Theta_b = \Theta(ab, s_n)$ as in (9) and the sparse asymptotics (7).

**Corollary 1.** For any fixed $b \in \mathbb{R}$, asymptotic non-trivial testing is possible over $\Theta_b$, in that there exists a procedure $\varphi$ such that

$$\lim_{n \to \infty} \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi) < 1.$$

In contrast, for any sequence $b = b_n \to -\infty$, we have

$$\lim_{n \to \infty} \inf_{\varphi} \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi) = 1,$$

that is, asymptotic non-trivial testing is impossible.
We thus identify a new sharp boundary for asymptotic non-trivial testing: \( a_b = \sqrt{2 \log(n/s_n)}+ b \) with \( b \to -\infty \) corresponds to the regime where it is impossible to build a multiple testing procedure doing (asymptotically) better than the trivial ones \( \varphi_i = 0 \) for all \( i \) or \( \varphi_i = 1 \) for all \( i \). On the contrary, provided that \( b \) is a finite constant (which may even be negative), some non-trivial control of the risk is possible.

**Corollary 2.** If \( b = b_n \to +\infty \), asymptotic conservative testing is possible over \( \Theta_b \), in that there exists a procedure \( \varphi \) such that, as \( n \to \infty \),

\[
\sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi) \to 0.
\]

In contrast, for any fixed \( b \in \mathbb{R} \), we have

\[
\lim_{n \to \infty} \inf_{\varphi} \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi) > 0,
\]

that is, asymptotic conservative testing is impossible over \( \Theta_b \) for any procedure.

We thus find the following sharp boundary for asymptotic conservative testing: it is possible when \( a_b = \sqrt{2 \log(n/s_n)}+ b \) with \( b = b_n \to +\infty \), but becomes impossible when \( b \) is finite.

It is interesting to note that a similar boundary with \( b \to +\infty \) was identified in [12] for the so-called *almost-full recovery* problem when the risk is the Hamming loss and the goal is to correctly classify up to a \( o(n) \) proportion of nonzero signals. In this view, Corollary 2 can be seen as an analogue of Theorem 4.3 in [12] for the multiple testing risk. A more detailed connection to the classification problem is made in Section 5.

### 2.3. Adaptation

The asymptotically minimax procedure (10) requires the knowledge of the sparsity parameter \( s_n \). A natural question is whether there exists an adaptive multiple testing procedure that achieves the minimax risk without using the knowledge of \( s_n \). Such procedures do exist. Some will require an additional *polynomial sparsity* assumption: for some unknown \( c < 1 \),

\[
s_n \lesssim n^c.
\]

**Theorem 2.** Consider \( \Theta_b = \Theta(a_b, s_n) \) as in (9) where \( b \) is either fixed in \( \mathbb{R} \) or is a sequence \( b = b_n \) tending to \( +\infty \) (in which case one replaces \( \Phi(b) \) by 0 in the next display). Then there exists a multiple testing procedure \( \varphi \), not depending on \( s_n \) or \( b \), such that in the sparse asymptotics (7)

\[
\lim_{n \to \infty} \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi) = \Phi(b).
\]

In particular this holds for the empirical Bayes \( \ell \)-value procedure (48) taken at any fixed threshold \( t \in (0, 1) \). In addition, under (11), this also holds for the BH procedure (69) taken at a vanishing level \( \alpha = \alpha_n = o(1) \) that satisfies \( -\log(\alpha_n) = o(\sqrt{\log n}) \).

Theorem 2 establishes that two popular (and simple) procedures are asymptotically sharp minimax adaptive. In particular, they automatically achieve the non-trivial and conservative boundaries described in Corollaries 1 and 2. One advantage of the empirical Bayes \( \ell \)-value procedure over the BH procedure is that it does not need any further parameter tuning in order to be valid. Nevertheless, an advantage of the BH procedure is that its FDR is always equal to \( (1 - |S_\theta|/n)\alpha \) (see [9]) so that one has more explicit information about the first term of the combined risk.
Remark 1 (The case $b = b_n \to -\infty$). When $b_n \to -\infty$, the lower bound of Theorem 1 shows that non-trivial testing is impossible. One could in principle have $R(\theta, \varphi) = 2 > 1$ so that the upper bounds in Theorem 2 are not automatic. However, the BH procedure does achieve the limit $\Phi(-\infty)$ in this case, as a consequence of the explicit control of its FDR noted above. Under the polynomial sparsity condition (11) the same is true of the $\ell$-value procedure (with $t \leq 3/4$) as an immediate consequence of Theorem 1 in [17].

Remark 2 (Polynomial sparsity assumption for the BH procedure). The condition $- \log(\alpha_n) = o(\sqrt{\log n})$ for the BH procedure is typically achieved by choosing $\alpha_n = 1/\sqrt{\log n}$. A rate of convergence of the risk to $\Phi(b)$ can be obtained for such choice, see Remark 9. In addition, the polynomial sparsity assumption can be relaxed slightly. Indeed, a weaker sufficient condition for the asymptotic minimaxity of BH is that $- \log(\alpha_n) = o(\sqrt{\log \log(n)})$, see Appendix D.2. This means, for instance, that the sparse regimes $s_n \asymp n/(\log(n))^d$, $d > 0$, can also be permitted under the stronger constraint $- \log(\alpha_n) = o(\sqrt{\log \log(n)})$, satisfied for example by $\alpha_n = 1/\log \log n$.

2.4. Sparsity preserving procedures and dominating FNR at the boundary

Theorem 1 determines the minimax optimal combined risk $R$, i.e. the sum FDR + FNR, but not the balance between the two terms struck by optimal procedures. In this section, we investigate this tradeoff in more details. For instance, the oracle procedure (10) achieves an optimal multiple testing risk by satisfying

$$
\sup_{\theta \in \Theta_b} \text{FDR}(\theta, \varphi^*) = o(1), \quad \sup_{\theta \in \Theta_b} \text{FNR}(\theta, \varphi^*) = \Phi(b) + o(1),
$$

that is, the FNR "spends" all the allowed "budget" from the overall minimax risk. In this section, we show that this phenomenon holds true for most "reasonable" procedures. Clearly, some restriction is required on the class of procedures as the trivial test $\varphi \equiv 1$ that always rejects the null achieves the optimal asymptotic risk (of 1) over $\Theta_b$ for $b = b_n \to -\infty$ and has FNR zero. We see below that for any sparsity preserving procedure, that is, one not overshooting the true sparsity index $s_n$ by more than some large multiplicative factor, the FNR alone cannot go below $\Phi(b)$ asymptotically.

Definition 1. We say that a multiple-testing procedure $\varphi = \varphi(X) \in \{0, 1\}^n$ (or strictly, a sequence of such procedures, indexed by $n$), is sparsity-preserving over $\Theta = (\Theta_n)_n$, with $\Theta_n \subset \ell_0[s_n]$ up to a multiplicative factor $A = (A_n)_n$ if, as $n \to \infty$,

$$
\sup_{\theta \in \Theta_n} P_\theta \left[ \sum_{i=1}^n \varphi_i(X) > A_n s_n \right] = o(1).
$$

We denote by $S_A(\Theta) = S_A((s_n)_n, \Theta)$ the set of all such procedure sequences.

The sparsity preserving property entails a total number of rejections not exceeding $A_n s_n$ with high probability, and can be interpreted as a weak notion of type I error rate control. Many procedures (or also estimators) encountered in the literature on sparse classes verify this condition for $A_n$ either a large enough constant or going to infinity slowly, even when taking the supremum in (13) over the whole sparse class $\ell_0[s_n]$. Several examples are provided in Appendix E, including the BH procedure (with fixed level $\alpha < 1$ or with $\alpha = \alpha_n \to 0$), the $\ell$-value procedure, the oracle thresholding procedure used to prove Theorem 1, and more generally procedures controlling the FDP in a specific sense.
Theorem 3. With the notation of Theorem 1, consider a fixed real $b$, a sequence $B = (B_n)_n$ with $B_n \in [2, e^{(\log(n/s_n)^{1/4})}]$, and $S_B(\Theta_b)$ as in Definition 1. Then, we have in the sparse asymptotics (7),
\begin{equation}
\inf_{\varphi \in S_B(\Theta_b)} \sup_{\theta \in \Theta_b} \text{FNR}(\theta, \varphi) = \Phi(b) + o(1).
\end{equation}

The result continues to hold if $b = b_n \to +\infty$ or $b = b_n \to -\infty$.

Since Theorem 2 shows that the (sparsity preserving) ℓ-value and BH procedures have an FNR suitably upper-bounded, these procedures both achieve the bound (14), under polynomial sparsity (11) for the BH procedure (note that the bound (14) trivially holds for any procedure in the case $b = b_n \to -\infty$).

Theorem 3 sharpens the findings of Theorem 1 by stating that the combined risk $\mathcal{R}$ can in fact be replaced by the FNR (i.e. the type II error) only, if one is willing to restrict slightly the class of multiple testing procedures to $\varphi$’s that do not often reject more than $B_n s_n$ hypotheses, where $B_n$ is permitted to tend to infinity at some rate specified above. Some intuition behind this result is given in Section 4 (in particular, see Figure 2).

An interesting consequence of Theorem 3 is that no sparsity preserving procedure can “trade” some loss in FDR for some improvement of the FNR while still staying close to the optimal risk: if its FDR is equal to $\alpha \in (0, 1)$ asymptotically (such as the standard BH procedure for a fixed level $\alpha$), it must miss the sharp combined risk $\mathcal{R}$ over $\Theta_b$ by at least an additive factor $\alpha$.

Corollary 3. In the setting of Theorem 3, for some $\alpha \in (0, 1)$ and $b \in \mathbb{R}$, let $\varphi_\alpha$ be any sparsity preserving procedure $\varphi_\alpha \in S_B(\Theta_b)$ such that $\lim_{n} \inf_{\theta \in \Theta_b} \text{FDR}(\theta, \varphi_\alpha) \geq \alpha > 0$. Then $\varphi_\alpha$ must miss the asymptotically minimax combined risk by at least $\alpha$, that is,
\begin{equation}
\lim_{n} \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi_\alpha) \geq \alpha + \Phi(b).
\end{equation}

This is in particular the case of the BH procedure with fixed level $\alpha$, as defined in (69).

Corollary 3 follows from Theorem 3, the fact that the BH procedure with fixed parameter is sparsity preserving (Appendix E) and the explicit expression for the FDR of the BH procedure, see (71).

Finally, for completeness, let us mention that for a procedure which is not sparsity preserving, trading FDR for FNR is formally possible. A (somewhat degenerate) example is given in Appendix E, see Example 3.

3. Sharp boundaries: arbitrary signal strengths and beyond Gaussian noise

This section presents more refined results. The generalization is two-fold: first, we consider more general Subbotin noise distributions, which involve techniques broadly similar to the Gaussian case in our proofs. Second, we study more precise classes of sparse signals, which enable us to derive more refined risk results. In particular, one would like to understand how relaxing the beta-min condition and allowing for nearly arbitrary signals affects the testing bound. We start by introducing the extended frameworks in Sections 3.1 and 3.2, and present the corresponding results in Section 3.3.
3.1. Subbotin noise distribution

For a normalising constant $L_\zeta$, the centred ‘Subbotin’ (generalised Gaussian) noise has density function

$$f(x) = f_\zeta(x) = L_\zeta^{-1}e^{-|x|^{\zeta}/\zeta}, \quad \zeta > 1.$$  \hfill (15)

The case $\zeta = 2$ corresponds to standard Gaussian noise, while the excluded case $\zeta = 1$ would correspond to Laplace noise. Write $F = F_\zeta$ for the associated cumulative distribution function, and $\bar{F} = \bar{F}_\zeta$ for the tail probabilities $\bar{F}(x) = 1 - F(x)$.

In this section, we work under this more general noise distribution. We still denote by $P_\theta$ the joint law of $X_i$’s given by $X_i = \theta_i + \varepsilon_i$, now with $\varepsilon_i$ drawn i.i.d. from $F$. The risk $R$ is defined accordingly taking the expectation under $P_\theta$.

3.2. Arbitrary sparse signals

The beta-min condition as in (8) is most natural when all nonzero signals have close to the same (absolute) value. However, when this is not the case (as in the second and third graphs of Figure 1), the multiple testing lower bound of Theorem 1, which is sharp for constant nonzero signals at the boundary of $\Theta(a, s_n)$, can be improved by considering smaller classes that take into account the specific nonzero signal values.

Let $b = (b_j)_{1 \leq j \leq s_n}$ be a sequence of reals satisfying

$$b_j > -(\zeta \log(n/s_n))^{1/\zeta}, \quad 1 \leq j \leq n,$$

and define

$$\Theta_b = \left\{ \theta \in \ell_0[s_n] : \exists i_1, \ldots, i_{s_n} \text{ all distinct,} \right\}

|\theta_{i_j}| \geq (\zeta \log(n/s_n))^{1/\zeta} + b_j, \quad 1 \leq j \leq s_n \right\}.$$  \hfill (17)

In words, $\Theta_b$ consists of sparse signals such that the $s_n$ nonzero coordinates are above some arbitrary (nonzero) given values. Note each $b_j$ is permitted to vary with $n$: for instance, one may set $b_1 = -(\zeta \log(n/s_n))^{1/\zeta} + 1$. Henceforth, we use the terminology ‘signal strength’ to refer either to the $b_j$’s or to the $|\theta_j|$’s themselves.

Further define, for $b$ as above,

$$\Lambda_n(b) = \frac{1}{s_n} \sum_{j=1}^{s_n} \bar{F}(b_j).$$  \hfill (18)

Let us assume that there exists $\Lambda_\infty \in [0, 1]$ such that

$$\lim_{n \to \infty} \Lambda_n(b) = \Lambda_\infty.$$  \hfill (19)

This condition is for ease of presentation and can be removed up to considering subsequences, see Remark 3.

Example 1 (Single signal strength). For $b = (b, \ldots, b)$, we have $\Theta_b = \Theta_b$ and we recover the beta-min parameter set. Note condition (19) is satisfied: for $b \in \mathbb{R}$, $\Lambda_n(b) = \bar{F}(b) \to \Lambda_\infty = \bar{F}(b) \in (0, 1)$, while when $b = b_n \to \infty$ (resp. $-\infty$), $\Lambda_n(b) \to \Lambda_\infty = 0$ (resp. $\Lambda_n(b) \to \Lambda_\infty = 1$).
Example 2 (Two signal strengths). Let \( x, y \in \mathbb{R}^2 \) correspond to two signal strength values and let \( M = \max(x, y), m = \min(x, y) \). Define \( b \in \mathbb{R}^{s_n} \) by \( b_j = M, 1 \leq j \leq \lfloor s_n \beta \rfloor \) and \( b_j = m, \lfloor s_n \beta \rfloor < j \leq s_n \), for a given proportion \( \beta \in (0, 1) \) of the stronger signal. Then we have, as \( n \to \infty \),

\[
\Lambda_n(b) = \frac{1}{s_n} \sum_{j=1}^{s_n} F(b_j) = \frac{\lfloor s_n \beta \rfloor}{s_n} F(M) + \frac{s_n - \lfloor s_n \beta \rfloor}{s_n} F(m)
\]

\[
\rightarrow \beta F(M) + (1 - \beta) F(m) = \Lambda_{\infty}.
\]

Level sets of \( \Lambda_{\infty} = \Lambda_{\infty}(x, y) \) for this \( b \) are displayed in Figure 3 below.

Remark 3. Note that (19) is only assumed for clarity of the results: the limit \( \Lambda_{\infty} \) can be replaced by the limsup, which always exists, in which case the convergences in all our results should be formulated with a limsup, and feasibility of testing at a given level is understood to require asymptotic control of the risk along all subsequences.

3.3. Main results in this setting

Below, Theorems 4, 5 and 6 extend Theorems 1, 2 and 3, respectively, to the more general framework of this section.

Theorem 4. Consider the sequence model with a Subbotin noise defined by (15) for some \( \zeta > 1 \) and for a parameter set \( \Theta_b \) defined by (17). For any \( b \in \mathbb{R}^{s_n} \) such that (16) and (19) hold, the minimax \( \mathcal{R} \)-risk over \( \Theta_b \) verifies, in the sparse asymptotics (7),

\[
\inf_{\varphi} \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi) = \Lambda_{\infty} + o(1).
\]

Theorem 4 shows that the asymptotic minimax risk is equal to \( \Lambda_{\infty} \). Since the latter depends on the limiting behavior of \( b \) via (19), this result allows us to quantify precisely how the different signal strengths are linked to the combined risk.

Example 2 (continued). Note that \( \Lambda_{\infty}(x, y) \) is no larger – and potentially much smaller – than the risk bound \( F(\min(x, y)) \) we could attain using Theorem 1 in the setting of Example 2. Specialising to the case \( \beta = 1/2 \) for simplicity, we have \( \Lambda_{\infty} = \Lambda_{\infty}(x, y) = (F(x) + F(y))/2 \). This allows one to determine which signal strengths are (asymptotically) the easiest from the \( \mathcal{R} \)-risk point of view in that case. For instance, the signal strengths \( (x, -x) \) and \( (0, 0) \) are equally risky \( (\Lambda_{\infty}(0, 0) = \Lambda_{\infty}(x, -x) \) for \( x > 0 \) \) while \( (1, 3) \) is more risky than \( (2, 2) \) \( (\Lambda_{\infty}(x, y) > \Lambda_{\infty}((x + y)/2, (x + y)/2) \) when \( x, y > 0 \) with \( x \neq y \), by convexity of \( F \) on \( (0, +\infty) \).

The procedure exhibited to prove the upper bound in Theorem 4, like that of Theorem 1, is an ‘oracle’ procedure requiring knowledge of the sparsity \( s_n \). The following result shows that the risk bound can be attained adaptively.

Theorem 5. Consider the setting of Theorem 4. Assume \( \Lambda_{\infty} < 1 \). Then there exists a multiple testing procedure \( \varphi \), not depending on \( s_n \) or \( b \), such that

\[
\sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi) = \Lambda_{\infty} + o(1)
\]

in the two following cases:
(i) Gaussian case, that is, $\zeta = 2$, for $\varphi$ being the empirical Bayes $\ell$-value procedure (48) with fixed threshold $t \in (0, 1)$;

(ii) Under polynomial sparsity (11), with $\varphi$ being the BH procedure (69) taken at a vanishing level $\alpha = \alpha_n = o(1)$ satisfying $-\log(\alpha_n) = o((\log n)^{1-1/\zeta})$.

In Theorem 5, Case (i) thus generalizes the $\ell$-value statement of Theorem 2 to the case of multiple signals. Case (ii) generalizes the BH statement of Theorem 2 both to the case of multiple signals and Subbotin noise. In the latter, we see that the value of $\zeta$ modifies the condition on $\alpha_n$: higher values of $\zeta$ make the condition less stringent.

The remarks made just after Theorem 2 also apply here: for the BH procedure, it can handle the case $\Lambda_\infty = 1$ while the polynomial sparsity assumption can be somewhat relaxed, while the $\ell$-value procedure (with $t \leq 3/4$) also handles $\Lambda_\infty = 1$ by adding the polynomial sparsity assumption. In addition to these remarks, we conjecture that the $\ell$-value procedure also achieves the bound with Subbotin noise, but proving it would require substantial extra technical work, so we refrain from presenting such a result here.

The following result shows that, still in the general framework of this section, the FNR is the dominating term at the boundary when focusing on sparsity preserving procedures. Both the BH procedure and empirical Bayes $\ell$-value procedure are sparsity preserving hence achieve the bound adaptively under the conditions of Theorem 5.

**Theorem 6.** Consider the setting of Theorem 4 and fix $\kappa \in (0, 1 - 1/\zeta)$. For any sequence $B = (B_n)_{n}$ with $B_n \in [2, e^{\log^*(n/s_n)}]$, and for $S_B(\Theta_b)$ the set of sparsity preserving procedures as in Definition 1,

$$\inf_{\varphi \in S_B(\Theta_b)} \sup_{\theta \in \Theta_b} \text{FNR}(\theta, \varphi) = \Lambda_\infty + o(1).$$

Let us finally mention that this result entails the sub-optimality of any sparsity preserving procedure with FDR asymptotically above some $\alpha > 0$. That is, Corollary 3 also extends to the more general framework of this section.

### 3.4. Application to pointwise control of the multiple testing risk

Since Theorem 5 is fully adaptive to $s_n$ and to arbitrary lower bounds $b$ on the signal strengths, we in fact obtain pointwise control of the $\mathcal{R}$-risk of the BH or empirical Bayes procedures.

**Corollary 4.** Let $\theta \in \ell_0(s_n)$ be arbitrary up to its support, denoted by $S_\theta$, having size $s_n$. Then, writing $a_n^* = (\zeta \log(n/s_n))^{1/\zeta}$, we have for any $\varphi$ as in Theorem 5 (and assuming the corresponding conditions delineated therein)

$$\lim_{n} \mathcal{R}(\theta, \varphi) \leq \lim_{n} s_n^{-1} \sum_{j \in S_\theta} F(|\theta_j| - a_n^*),$$

whenever the right side is less than 1 (or without this condition under (11)).

**Remark 4.** Corollary 4 may be contrasted with the weaker upper bound – which is sharp over the ‘beta-min’ classes $\Theta_b$ but not in general – that can be obtained from Theorem 2,

$$\lim_{n} \mathcal{R}(\theta, \varphi) \leq \lim_{n} F(\min_{j \in S_\theta} |\theta_j| - a_n^*).$$
It can be deduced from the proof of Theorem 4 that the new bound (20) is sharp, in the sense that
\[
\liminf_{n} \inf_{\tilde{\varphi}} \sup_{\sigma} \mathfrak{R}(\theta_{\sigma}, \tilde{\varphi}) = \lim_{n} s_{n}^{-1} \sum_{j \in S_{\theta}} F(|\theta_{j}| - a_{n}^*),
\]
where the infimum is over multiple testing procedures \(\tilde{\varphi}\), the supremum is over permutations \(\sigma\), and where \(\theta_{\sigma}\) denotes the vector \((\theta_{\sigma(j)} : j \leq n)\). Using that the Bayesian and BH procedures are invariant under permutations \((\varphi_{\sigma(i)}(X_{\sigma}) = \varphi_{i}(X))\) we deduce that (20) holds with equality. In words, up to the presence of limsup\(^s\) which reflect the fact that testing may be easier along some subsequences than others, our result quantify the exact pointwise risk of the procedures in question.

**Remark 5 (Testability).** Corollary 4 and Remark 4 discuss pointwise risk convergence for specific procedures. A rephrasing of these results characterises testability for a given collection of signal strengths. Suppose we know \(\theta \in \ell_{0}[s_{n}]\) has \(s_{n}\) non–zero coordinates with arbitrary absolute values \(|\theta_{j}|, j \in S_{\theta}\), whose values may be known or unknown but whose positions are unknown. Then non-trivial multiple testing is achievable for this problem – in that there exists an (adaptive to the unknown value of \(s_{n}\)) multiple testing procedure with \(\mathfrak{R}\) risk asymptotically less than 1 – if and only if, for \(a_{n}^* = (\zeta \log(n/s_{n}))^{1/\zeta}\),
\[
\lim_{n} \frac{1}{s_{n}} \sum_{j \in S_{\theta}} F(|\theta_{j}| - a_{n}^*) < 1.
\]
More generally, there exists a multiple testing procedure of \(\mathfrak{R}\)–risk asymptotically at most \(\alpha \in [0, 1]\) for this problem if and only if the limsup in the last display is at most \(\alpha\). Hence, given different arbitrary shapes of signals (again as in Figure 1), to compare the difficulty of the corresponding multiple testing problems, it suffices to compare the corresponding limsup\(^s\) in the last display: the value of the limsup characterises the (asymptotic) difficulty of the multiple testing problem for any arbitrary \(\theta\) with \(s_{n}\) nonzero coordinates.

4. Illustrations

In this section, we present numerical illustrations for the results stated above, as well as provide some intuition behind these.

4.1. Illustrating FDR/FNR tradeoff

Theorems 3 and 6 establish that the FNR is asymptotically the driving force for optimal procedures: any (sparsity preserving) procedure has an FDR vanishing when \(n\) goes to infinity and only the FNR matters in the minimax risk.

This phenomenon is illustrated in Figure 2 for thresholding based procedures \(\varphi_{t} = 1_{|X_{i}| \geq t}, t \in \mathbb{R}\). For ease of computations, instead of the FDR, we consider here a closely related quantity, the marginal FDR, denoted by \(m\text{FDR}\). Hence, we consider the marginal risk

\[
\begin{align*}
m\mathfrak{R}(\varphi_{t}) &= m\text{FDR}(\varphi_{t}) + \text{FNR}(\varphi_{t}); \\
\text{FNR}(\varphi_{t}) &= F(t - a_{n}^*) + F(t + a_{n}^*); \\
m\text{FDR}(\varphi_{t}) &= \frac{2(n - s_{n})F(t)}{2(n - s_{n})F(t) + s_{n}(1 - \text{FNR}(\varphi_{t}))}.
\end{align*}
\]
for which the parameter $\theta$ is taken at the border of $\Theta_b = \Theta(\theta_b, s_n)$ (for $b = 0$) as follows: $\theta_i = a_n^* = \{\zeta \log n / s_n\}^{1/\zeta}$ if $1 \leq i \leq s_n$ and $\theta_i = 0$ otherwise. (It can be shown that our results also hold with the marginal risk $\mathfrak{m}\mathcal{R}$ instead of the original risk $\mathcal{R}$.)

We can make the following general comments: first, as the threshold $t$ increases (along the $x$-axis on Figure 2), fewer rejections are made, so that $\text{FNR}(\varphi_t)$ increases with $t$ and $\text{mFDR}(\varphi_t)$ decreases, hence there is a tradeoff for the finite sample (marginal) risk $\mathfrak{m}\mathcal{R}(\varphi_t)$: the minimum is displayed with the light dashed horizontal line. This finite sample risk can be compared to its asymptotic counterpart (thin solid horizontal line). We note that the asymptotic regime is not reached for the current choice of $n, s_n$ if $\zeta = 1.5$, but is approached as $\zeta$ increases, with almost a perfect matching when $\zeta = 5$. This is well expected from the convergence rate found to be $1/(\log n / s_n)^{1-1/\zeta}$ in some cases (see Remark 9), so that the case $\zeta = 5$ well approximates the asymptotic picture. Second, when the asymptotics are (close to being) reached, we see that the dominating part in the risk at the point where the risk is minimum is the FNR, as Theorems 3 and 6 establish. The pictures above give an interpretation of these results: since the transition from 0 to 1 is much more abrupt for
(m)FDR, we should make the (m)FDR close to 0 to make a good tradeoff; the mFDR is close to a step function, so achieving mFDR < 1−ε requires essentially the same threshold as achieving mFDR < ε, hence to make the optimal tradeoff will require mFDR close to zero.

4.2. Illustrating minimax risks for two signal strengths

The asymptotic minimax risk $\Lambda_\infty$ found in Section 3 is a function of the multiple signal strengths that we propose to illustrate in this section. For simplicity, we focus on the case of two signal strengths here which corresponds to considering the functional

$$\Lambda_\infty : (x, y) \in \mathbb{R}^2 \mapsto \Lambda_\infty(x, y) = \beta F(\max(x, y)) + (1 - \beta) F(\min(x, y)), \quad (24)$$

see Example 2. Level sets of this function are displayed in Figure 3 for $\beta \in \{1/4, 1/2\}$ and $\zeta \in \{2, 4\}$. As we can see, while the behaviour on the diagonal $x = y$ matches that of the single signal strength case, the asymptotic minimax risk has various behaviours off this diagonal, when there are two different signal strengths. In general, the shape of the level sets depends on the proportion $\beta$ and on $\zeta$. First, $\beta$ strongly affects the level sets: $\Lambda_\infty$ is larger for $\beta = 1/4$ than for $\beta = 1/2$ outside the diagonal $x = y$. This is to be
expected because when \( \beta = 1/4 \), only a proportion \( 1/4 \) of the nonzero means are equal to the larger value \( \{ \zeta \log n/s_n \}^{1/\zeta} + \max(x, y) \) while the other nonzero values are equal to \( \{ \zeta \log n/s_n \}^{1/\zeta} + \min(x, y) \). This is clearly a less favorable situation compared to the case where the proportions are balanced (\( \beta = 1/2 \)). Second, \( \zeta \) also affects the level sets (although less severely): increasing \( \zeta \) makes the level sets flatter in the center of the \((x, y)\)-picture. This difference comes from the fact that the tails of the \( \zeta \)-Subbotin distribution grow lighter as \( \zeta \) increases.

Figure 4. Same as Figure 3 (top-left, Gaussian case with \( \beta = 1/2 \)) with finite sample risk \((n = 10^6, s_n = 20) \) at some particular configurations (displayed by black dots). The risk is computed via 100 Monte-Carlo simulations. Two (asymptotically) minimax procedures are implemented: the \( \ell \)-value procedure \((t = 0.3, \text{risk displayed below each dot}) \) and the BH procedure \((\alpha = 0.1, \text{risk displayed above each dot}) \). Left: dots are located on the \( \Lambda_{\infty} \)-level sets of values in \( \{0.7, 0.5, 0.2\} \) of the boundary function. Right: dots are located on lines such that the average of \( x \) and \( y \) is kept constant (not \( \Lambda_{\infty} \)-level sets).

Figure 4 further illustrates how the (asymptotic) level sets of \( \Lambda_{\infty} \) are approached for large finite samples: it displays some finite-sample risks \( R(\theta, \varphi) \) \((n = 10^6, s_n = 20) \) of the minimax procedures described in Section 1.6: \( \ell \)-value for \( t = 0.3 \) and BH procedure at level \( \alpha = 0.1 \), for some parameter \( \theta \) corresponding to configurations \((x, y)\) taken from the Gaussian and \( \beta = 1/2 \) plot (top-left panel of Figure 3, see again Example 2). These pointwise risks are estimated via 100 Monte-Carlo simulations. As expected from the slow convergence already discussed above, the finite sample risk has not yet converged to the minimax risk. However, we can see that the global monotonicity of the risk is maintained. More precisely, the left panel of Figure 4 displays the finite sample risks at points taken along asymptotic level sets. We can see that the values reported for the finite sample risks are near-constant along asymptotic level sets (up to the Monte-Carlo errors), suggesting that the finite sample level sets are close to their asymptotic counterparts. One exception for which the convergence appears to be slower is for configuration \((x, y) = (-3, 3)\): non-asymptotically this configuration seems to be easier than for example \((x, y) = (1, -1)\) (although asymptotically equivalent). One possible explanation is that an extreme value of \( y \) makes it easily detectable for finite \( n \). On the other hand, the right panel of Figure 4 displays the finite sample risks for points taken along lines which are not asymptotic level sets. The results are markedly different from the left panel: the values vary much more along these lines. Hence, these simple lines, which are not asymptotic level sets, are also far from being finite-sample levels sets. This supports that the conclusion of Remark 5 is
also valid non-asymptotically (for \( n \) large): the functional \( \Lambda_\infty \) can be used for comparing testing difficulties of given collections of signal strengths (such as those given in Figure 1).

5. Extensions to other risks: FDR-controlling procedures and classification

Results in this section are for simplicity stated in the original setting of Section 2, but versions also exist in the extended setting of Section 3.

5.1. Combined testing risk with given FDR control

From the multiple testing perspective, practitioners are often willing to accept a fixed small rate \( \alpha > 0 \) of false discoveries. Hence, an interesting alternative risk to \( R \) is, with
\[
z_+ = z \vee 0 = \max(z, 0),
\]

\[
R_\alpha(\theta, \varphi) = (\text{FDR}(\theta, \varphi) - \alpha)_+ + \text{FNR}(\theta, \varphi).
\]

Proposition 1 shows that allowing for some false discoveries by targeting \( R_\alpha \to 0 \) rather than \( R = R_0 \to 0 \) does not allow for weaker boundary conditions.

Proposition 1. For \( \alpha \in [0, 1) \) and a fixed real \( b \), let us consider the set of sparsity preserving procedures \( S_B(\Theta) \) as in Definition 1. Then, under the conditions of Theorem 3,
\[
\inf_{\varphi \in S_B(\Theta)} \sup_{\theta \in \Theta} R_\alpha(\theta, \varphi) = \Phi(b) + o(1).
\]

Proposition 1 is an immediate consequence of Theorems 1 and 3 and of the bounds \( \text{FNR} \leq R_\alpha \leq R_\alpha \). From the multiple testing literature perspective, this result is relatively counterintuitive as allowing a looser FDR control is generally considered as beneficial for the power of the procedure. This is of course true, but as was shown in Figure 2 and the discussion thereof, for large \( n \) the FDR curve is much steeper than the FNR curve, so that the two cannot be traded efficiently (at least for the noise considered here).

5.2. Classification: sharp adaptive minimaxity in probability

The classification (Hamming) loss in terms of classes \( \{0\} \) and \( \mathbb{R} \setminus \{0\} \) is defined by
\[
L_C(\theta, \varphi) = \sum_{i=1}^n \left( 1 \{ \theta_i = 0 \} 1 \{ \varphi_i = 0 \} + 1 \{ \theta_i \neq 0 \} 1 \{ \varphi_i \neq 0 \} \right).
\] (25)

Theorem 7. If \( a_b = \sqrt{2 \log n/s_n} + b \) for a fixed real \( b \), then for \( \Theta_b = \Theta(a_b; s_n) \) and any \( \eta > 0 \),
\[
\inf_{\varphi} \sup_{\theta \in \Theta_b} P_\theta \left[ L_C(\theta, \varphi)/s_n \geq \Phi(b) - \eta \right] = 1 + o(1).
\]

If \( b = b_n \to -\infty \), the last display holds with \( \Phi(b) \) replaced by 1.

There exist procedures \( \varphi \) achieving this bound, in that for any fixed real \( b \) and any \( \eta > 0 \)
\[
\sup_{\theta \in \Theta_b} P_\theta \left[ L_C(\theta, \varphi)/s_n \geq \Phi(b) + \eta \right] = o(1),
\]
and the same holds with \( \Phi(b) \) replaced by 0 when \( b = b_n \to +\infty \) or by 1 when \( b = b_n \to -\infty \).

In particular this is true for \( \varphi = \varphi^k_\ell \) the empirical Bayes \( \ell \)-value procedure (48) or, under the polynomial sparsity assumption (11), the BH procedure \( \varphi_{\alpha_n}^{BH} \) (69) for \( \alpha = \alpha_n = o(1) \) with \( -\log \alpha = o((\log n)^{1/2}) \).
Note in this result and Theorem 8 below, in contrast to Theorem 2, we are able to include the case $b = b_n \to -\infty$ for the $\ell$-value procedure without extra assumptions. More generally, one could also derive results for weighted classification losses such as $L_\rho$ defined in (32), and similarly for the results in expectation below.

5.3. Classification: sharp adaptive minimaxity in expectation

In [12], the authors define a notion of almost full recovery with respect to the Hamming loss, which turns out to be related (though not identical) to the notion of conservative testing as considered in Corollary 2. A testing procedure $\varphi \in \{0,1\}^n$ is said to achieve almost full recovery with respect to the Hamming loss over a subset $\Theta_{s_n} \subset \ell_0[s_n]$ if, as $n \to \infty$,

$$\sup_{\theta \in \Theta_{s_n}} \frac{1}{s_n} E_{\theta} L_C(\theta, \varphi) = o(1).$$

Here to allow for a direct comparison with [12], we work with the class considered therein, defined by

$$\Theta'_b = \Theta'_b(s_n) = \bigcup_{0 \leq s \leq s_n} \Theta(a_b, s). \quad (26)$$

Note that all our lower bound results imply the same bounds for this larger class $\Theta'_b \supset \Theta_b = \Theta(a_b, s_n)$, and the following result gives a corresponding upper bound on this class.

**Theorem 8.** If $a_b = \sqrt{2 \log(n/s_n)} + b$ for an arbitrary real $b$ or a sequence $b = b_n \to \pm \infty$, then for $\Theta'_b = \Theta'_b(s_n)$ as in (26), the sharp asymptotic minimax risk for classification is given by

$$\inf_{\varphi} \sup_{\theta \in \Theta'_b} E_{\theta} L_C(\theta, \varphi)/s_n = \Phi(b) + o(1).$$

For $\varphi = \varphi^\ell$ the empirical Bayes $\ell$-value procedure (48) or $\varphi = \varphi^{BH}$ the BH procedure (69) at a level $\alpha = \alpha_n = o(1)$ with $-\log \alpha = o((\log n)^{1/2})$ (additionally assuming the polynomial sparsity (11)), the bound is achieved: for any real $b$, or for $b = b_n \to \pm \infty$,

$$\sup_{\theta \in \Theta'_b} E_{\theta} L_C(\theta, \varphi)/s_n = \Phi(b) + o(1).$$

These results complement some of the results of Sections 4–5 in [12] in the following sense: the focus in that work is on almost full recovery (as well as on the even more stringent ‘exact recovery’ setting) which requires a higher signal strength than the classes $\Theta'_b$ for fixed $b$, needing that $b = b_n \to +\infty$, as already noted in [12]. Theorem 5.2 therein states that there exists an adaptive procedure that achieves almost full recovery if $b_n \gtrsim \log \log n$. Our Theorem 8 shows that the $\ell$-value procedure achieves it under the weakest possible condition $b_n \to +\infty$, not requiring a particular rate. Further, Theorem 8 investigates the setting of a finite $b$. Over the class $\Theta'_b$, Theorem 4.2(ii) in [12] provides an in-expectation lower bound that is asymptotically similar to that from Theorem 8, but only for the case of $b \geq 0$ (which essentially amounts to $W > 0$ in the notation from [12]); Theorem 8 provides the sharp asymptotic constant for any real $b$, and asserts that it can be achieved by an adaptive (i.e. not dependent on $s_n$ or $b$) procedure. A finite sample lower bound can also be derived from (43).
6. Discussion

Overview of the results. This work derived new results for multiple testing from a minimax point of view, in particular the sharp minimax constant for the sum risk $R = \text{FDR} + \text{FNR}$. Allowing for a variety of possible noise distributions including standard Gaussian noise, we considered first the beta-min conditions on signals, and next allowed for (essentially) arbitrary sparse signals. This enables one to compare qualitatively difficulties of multiple testing problems, such as the ones depicted on Figure 1. For such general signals, a prominent finding is that the overall testing difficulty can be expressed in terms of an average-type limit, $\Lambda_\infty$ in (19), where the strength of a signal is formulated by comparison to the ‘oracle threshold’ (e.g., $\sqrt{2 \log(n/s_n)}$ for Gaussian noise).

For this problem, it follows from this work that the “boundary” for the testing problem corresponds to slightly weaker signals (i.e., class $\Theta_b$ with fixed $b = F^{-1}(t)$ for a target risk $t$ in the Gaussian case) compared to classification with almost sure support recovery, for which one needs to be well above the oracle threshold ($b = b_n \to +\infty$).

The reason is that one allows, as is common in (multiple) testing, for a certain percentage of error in the overall testing error. An important message regarding this tolerance level is that asymptotically, the two types of errors, FDR and FNR, are not symmetric in terms of their contribution to this level: for any optimal procedure, as long as it is sparsity preserving, the FNR is dominating, at least for the noise distributions considered here.

The techniques of proof make it possible to deal with different related risks. Indeed, the $R$-risk is closely related to in-probability bounds for some weighted classification (Hamming) risk. Although the focus of the paper is on multiple testing risks, this observation enabled us to derive also new bounds and boundaries for the classification risk, for which the key messages from the two preceding paragraphs remain true.

Adaptive procedures: comparing BH and $\ell$-values. We have shown that two popular procedures reach the bounds in an adaptive manner: the $\ell$-value procedure and a properly tuned BH procedure. Both have optimal behaviour under relatively similar conditions, but each has its own merits in certain settings. One important message is that, even if the overall risk $R$ is allowed to be at least $\alpha$, taking the standard BH procedure with fixed $\alpha$ parameter does not lead to an optimal risk: taking $\alpha = \alpha_n$ to go to zero is really needed in order to achieve optimality (see Corollary 3).

The $\ell$-value procedure does not need a tuning parameter going to zero: it can be applied, for example, for $t = 1/2$: one can interpret this as the fact that it is already on the correct “scale” for the $R$-risk (this is perhaps unsurprising, as this procedure relates to the Bayes classifier for the Hamming risk when the prior is correct), while BH is scaled for the FDR, i.e. it essentially prescribes the FDR value. Since the FDR has to be negligible for certain parameters for optimal procedures with respect to $R$ (for sparsity preserving ones, as directly follows from combining Theorems 4 and 6), this explains why one needs to take $\alpha = \alpha_n \to 0$ to achieve optimality for BH.

On the other hand, since $\ell$-values are in principle not immediately designed for a (frequentist) FDR-control, proving that the $\ell$-value procedure controls the FDR can be nontrivial; in [17] it was shown to be the case for arbitrary sparse signals, but this requires significant technical work (a result that is invoked here only to handle the somewhat degenerate case $\Lambda_\infty = 1$).

From a more practical point of view, choosing between these two competitors depends
on the pursued aim: a user interested solely in the combined risk may use the \( \ell \)-value procedure because it is more intrinsic and is asymptotically optimal for a fixed positive value of the parameter \( t \) (in the case of the BH-\( \alpha \) procedure the level \( \alpha \) parameter needs to be tuned appropriately), while if the FDR value should be also controlled or known, they could use the BH procedure with a level \( \alpha \) satisfying the convergence requirements.

**Future directions.** This work paves the way for several future investigations. First, we expect many ideas relevant here for the Gaussian sequence model to transport to multiple testing for more complex models, such as high dimensional linear regression. Second, since the combined risk involves two parts with markedly different behaviors at the boundary, it could be valuable to find another notion of risk making a better balance between these two terms. Deriving such a risk notion would have interesting consequences for building semi-supervised machine learning algorithm achieving an appropriate FDR/FNR tradeoff. Finally, the multiple signal framework introduced here is worth investigating for estimation-type risks, where the constant \( \Lambda_\infty \) is expected to still play a key role.

### 7. Proof of Theorems 1 and 4

In this section, we give the proof of Theorems 1 and 4; other proofs are postponed to the appendices. In fact, we shall restrict our attention to proving Theorem 4, because Theorem 1 is a direct consequence of it. Indeed, once Theorem 4 is proved, for fixed \( b \in \mathbb{R} \) (resp. \( b = b_n \to +\infty \)), for \( n \) large enough that \( b > -2 \log(n/s_n) \) we may apply Theorem 4 with \( \zeta = 2\), \( b = (b_1, \ldots, b_k) \), \( \Lambda_n(b) = \overline{\Phi}(b) \), \( \Lambda_\infty = \overline{\Phi}(b) \) (resp. 0) to obtain these cases of Theorem 1. For \( b = b_n \to -\infty \) we replace \( b_n \) by a sequence \( b_n' \to -\infty \) satisfying \( b_n' > \max(-2 \log(n/s_n)^{1/2}, b_n) \) before applying Theorem 4, obtaining in this way \( \inf_x \sup_{\Theta \in \Theta_{b_n}} \mathfrak{R}(\theta, \varphi) = 1 + o(1) \). Since \( \Theta_{b_n} \subset \Theta_{b_n} \) we deduce \( \inf_x \sup_{\Theta \in \Theta_{b_n}} \mathfrak{R}(\theta, \varphi) \geq 1 + o(1) \); since the upper bound in this case is obtained using the trivial test \( \varphi \equiv 0 \) we deduce Theorem 1 in all cases.

#### 7.1. Upper bound

We first focus on bounding the minimax risk \( \mathfrak{R} \) from above. If \( \Lambda_\infty = 1 \) there is nothing to prove for the upper bound (one may use the trivial test \( \varphi \equiv 0 \)) so we assume \( \Lambda_\infty < 1 \). Define the ‘oracle threshold’

\[
a_n^* = (\zeta \log(n/s_n))^{1/\zeta},
\]

and the ‘oracle thresholding procedure’

\[
\varphi^*_k = 1_{|X_i| \geq a_n^*}.
\]

If \( V^* = \sum_{i \notin S_{b_n}} \varphi^* \) is the number of false discoveries of \( \varphi^* \), we have \( V^* \sim \text{Bin}(n-s_n, 2\overline{F}(a_n^*)) \).

One notes, using Lemma 23 and the fact that \( a_n^* \to \infty \), \( f(a_n^*) = L_{\zeta}^{-1}s_n/n \), that

\[
\mathcal{E}_\theta V^* = 2(n-s_n)\overline{F}(a_n^*) \asymp \frac{s_n}{(\log(n/s_n))^{1-1/\zeta}}.
\]

Bernstein’s inequality (Lemma 5) gives, for any \( M > 0 \), and \( c > 0 \) a small constant,

\[
P_\theta\left[|V^* - \mathcal{E}_\theta V^*| \geq M \sqrt{\mathcal{E}_\theta V^*}\right] \leq 2 \exp\left\{-c[M^2 \wedge (M\sqrt{\mathcal{E}_\theta V^*})]\right\}.
\]

Taking \( M \) such that \( M\sqrt{\mathcal{E}_\theta V^*} = \sqrt{s_n} \) (i.e. \( M^2 \asymp (\log(n/s_n))^{1-1/\zeta} \to \infty \)) gives \( V^* \leq \mathcal{E}_\theta V^* + \sqrt{s_n} \) with high probability.
On the other hand, the variable $\varphi^*_i = \varphi^*_i(X_i)$ is $\text{Ber}(p^*_i)$ with

$$p^*_i = P_\theta[|\theta_i + \varepsilon_i| \geq a^*_i] = F(a^*_i - \theta_i) + F(a^*_i + \theta_i).$$

From the definition (17) of $\Theta_b$, we deduce that we may write $S_\theta = \{i_1, \ldots, i_{s_n}\}$ with $p^*_{i_j} \geq F(-b_{i_j})$. The number of true discoveries $S^* = \sum_{i \in S_\theta} \varphi^*_i$ is controlled by again using Bernstein’s inequality: since by symmetry $F(-x) = 1 - F(x)$, we see by (19) that

$$E_\theta S^* = \sum_{i \in S_\theta} p^*_i \geq \sum_{j=1}^{s_n} F(-b_j) = s_n (1 - \Lambda_\infty + o(1)), \quad (29)$$

and we note that for any $t > 0$ we have

$$P_\theta[|S^* - E_\theta S^*| \geq t] \leq 2 \exp\left\{-t^2/(2s_n(1 - \Lambda_\infty + o(1)) + 2t/3)\right\}.$$ 

Taking $t = s_n^{3/4}$ leads to the desired concentration, yielding the bound $2 \exp(-cs_n^{1/2})$ for the above display.

We deduce that, if $A$ denotes an event of probability tending to 1 on which both $V^*$ and $S^*$ concentrate as above,

$$\text{FDR}(\theta, \varphi^*) \leq \frac{E_\theta[V^*] + \sqrt{s_n}}{E_\theta[S^*] - s_n^{3/4}} + P_\theta[A] \leq C \left\{(\log(n/s_n))^{-(1/\zeta)} + s_n^{-1/2}\right\}/(1 - \Lambda_\infty + o(1)) + P_\theta[A],$$

uniformly over $\theta \in \Theta_b$. Also, by definition $\text{FNR}(\theta, \varphi) = 1 - E_\theta[S^*]/s_n \leq \Lambda_\infty + o(1)$, using (29). Combining the previous estimates leads to

$$\lim_{n \to \infty} \sup_{\theta \in \Theta_b} \text{FNR}(\theta, \varphi^*) \leq \Lambda_\infty, \quad \sup_{\theta \in \Theta_b} \text{FDR}(\theta, \varphi^*) \to 0. \quad (30)$$

We deduce that

$$\lim_{n \to \infty} \inf_{\varphi} \sup_{\theta \in \Theta_b} \mathfrak{R}(\theta, \varphi) \leq \lim_{n \to \infty} \sup_{\theta \in \Theta_b} \mathfrak{R}(\theta, \varphi^*) \leq \Lambda_\infty,$$

and the upper bound follows.

### 7.2. Lower bound

Here we prove, in the setting of Theorem 4,

$$\lim_{n \to \infty} \inf_{\varphi} \sup_{\theta \in \Theta_b} \mathfrak{R}(\theta, \varphi) \geq \Lambda_\infty. \quad (31)$$

While this result is asymptotic, a non-asymptotic counterpart can be derived by combining Theorem 9 and Lemma 1 below.

Let us first introduce additional notation. Define, for $\rho > 0$, the weighted, non-symmetric, loss function

$$L_\rho(\theta, \varphi) = \sum_{i=1}^{n} \{\mathbf{1}\{\theta_i = 0\}\mathbf{1}\{\varphi_i \neq 0\} + \rho \mathbf{1}\{\theta_i \neq 0\}\mathbf{1}\{\varphi_i = 0\}\}.$$ 

(32)
Standard classification loss $L_C$ corresponds to $\rho = 1$. Recall that we work with data $X_i = \theta_i + \varepsilon_i$, $1 \leq i \leq n$, with $\varepsilon_i$ drawn i.i.d. from the Subbotin noise (15). Recalling the definition (9) of $a_b$ in the Gaussian case, we here generalise to the Subbotin case, defining
\[
a_b = \{\zeta \log(n/s_n)\}^{1/\zeta} + b, \quad b \in \mathbb{R}.
\]
(33)

Recall for $b = (b_j)_{1 \leq j \leq s_n}$ a sequence of reals such that $a_{b_j} > 0$ the definition (17) of $\Theta_b$,
\[\Theta_b = \left\{ \theta \in \ell_0[s_n] : \exists i_1, \ldots, i_{s_n} \text{ all distinct, } |\theta_{i_j}| \geq a_{b_j}, 1 \leq j \leq s_n \right\}.\]

For $\rho \geq 1$ an integer let
\[p_n(b, \rho) = P[\#\{2 \leq i \leq n/s_n : \varepsilon_i > a_b + \varepsilon_1\} \geq \rho].\]
(34)

and
\[\mathcal{F}_n(b, \rho) = \frac{1}{s_n} \sum_{j=1}^{s_n} p_n(b_j, \rho).\]
(35)

Equation (31) follows from applying a number of auxiliary results that are given below, after the core of the proof.

**Proof of (31).** Fix $\varepsilon, \delta > 0$ and choose $\rho_n = \rho_n(\delta)$ tending to infinity according to Lemma 2 (page 28). Then Lemma 1 applied with $\rho = \rho_n$ and $\lambda = \min(1-(1+\varepsilon)s_n^{-1/2}, \mathcal{F}_n(b, \rho_n) - \varepsilon)$, together with Theorem 9, yields
\[\liminf_{n \to \infty} \inf_{\varphi, \theta \in \Theta_b} \mathcal{R}(\theta, \varphi) \geq \lambda \wedge \frac{\rho_{\lambda}}{1 + \rho_{\lambda}} (1 - \exp(-\sqrt{s_n}\varepsilon^2/3)).\]

Taking the liminf as $n$ tends to infinity yields, in view of Lemma 2 and the fact that $\Lambda = \mathcal{F}_n(b, \rho_n) - \varepsilon$ for $n$ large enough,
\[\lim_{n \to \infty} \inf_{\varphi, \theta \in \Theta_b} \mathcal{R}(\theta, \varphi) \geq \lim_{n \to \infty} \left( \lambda \wedge \frac{\rho_{\lambda}}{1 + \rho_{\lambda}} \right) = \lim_{n \to \infty} (\mathcal{F}_n(b, \rho_n) - \varepsilon) \geq \Lambda_{\infty} - C_f \delta - \varepsilon.
\]
The left side does not depend on $\delta$ or $\varepsilon$, so taking the limit as these tend to zero yields the result. \(\square\)

**Lemma 1.** For any procedure $\varphi$, any $\theta \in \ell_0[s]$ with $s \geq 1$, and any $\lambda \in (0, 1)$ and $\rho > 0$,
\[\mathcal{R}(\theta, \varphi) \geq \left( \lambda \wedge \frac{\rho_{\lambda}}{1 + \rho_{\lambda}} \right) P_{\theta}(L_{\rho}(\theta, \varphi) \geq \lambda \rho s).\]

**Proof.** Let us write $D_n(X) = \sum_{i=1}^{n} \varphi_i(X)$ for the total number of rejections, and denote by $s_\theta$ the number of nonzero coefficients of $\theta$. Define
\[Q(X) = Q(X, \varphi) = \sum_{i=1}^{n} \left\{ 1\{\theta_i = 0\} \frac{\varphi_i(X)}{1 \lor D_n(X)} + 1\{\theta_i \neq 0\} \frac{1 - \varphi_i(X)}{s} \right\},\]
so that $\mathcal{R}(\theta, \varphi) \geq E_{\theta} Q(X)$, using $s_\theta \leq s$. Let $A_n = \{D_n(X) \leq (1 + \delta)s\}$, for $\delta > 0$. On the one hand, since $1 \lor D_n(X) \leq (1 + \delta)s$ on $A_n$,
\[Q(X)1\{A_n\} \geq 1\{A_n\} \sum_{i=1}^{n} \left\{ 1\{\theta_i = 0\} \frac{\varphi_i(X)}{(1 + \delta)s} + \frac{\rho}{\rho} 1\{\theta_i \neq 0\} \frac{1 - \varphi_i(X)}{s} \right\} \]
\[\geq 1\{A_n\} \left( \frac{1}{1 + \delta} \wedge \frac{1}{\rho} \right) \frac{1}{s} L_{\rho}(\theta, \varphi).\]
On the other hand, if \( \mathcal{A}_n^c \) denotes the complement of \( \mathcal{A}_n \),

\[
Q(X)1\{\mathcal{A}_n^c\} \geq 1\{\mathcal{A}_n^c\} \sum_{i=1}^n 1\{\theta_i = 0\} \frac{\varphi_i(X)}{D_n(X)} \\
\geq 1\{\mathcal{A}_n^c\} \frac{1}{D_n(X)} \sum_{i=1}^n \varphi_i(X) - \sum_{i=1}^n 1\{\theta_i \neq 0\} \varphi_i(X) \\
\geq 1\{\mathcal{A}_n^c\} \frac{D_n(X) - s_\theta}{D_n(X)} \geq 1\{\mathcal{A}_n^c\} \frac{D_n(X) - s_\theta}{D_n(X)} \geq \frac{1}{1 + \delta} 1\{\mathcal{A}_n^c\}.
\]

Combining the previous bounds and setting \( C_n = \{L_\rho(\theta, \varphi) \geq \lambda \rho s\} \), we obtain

\[
Q(X) \geq \left( \frac{1}{1 + \delta} \wedge \frac{1}{\rho} \right) \frac{1}{s} L_\rho(\theta, \varphi) 1\{\mathcal{A}_n\} + \frac{\delta}{1 + \delta} 1\{\mathcal{A}_n^c\} \\
\geq \left( \frac{1}{1 + \delta} \wedge \frac{1}{\rho} \right) \rho \lambda 1\{\mathcal{A}_n\} 1\{C_n\} + \frac{\delta}{1 + \delta} 1\{\mathcal{A}_n^c\} 1\{C_n\} \\
\geq \left[ \left( \frac{\lambda \rho}{1 + \delta} \wedge \lambda \right) \wedge \frac{\delta}{1 + \delta} \right] 1\{L_\rho(\theta, \varphi) \geq \lambda \rho s\},
\]

where the second line uses that \( 1 \geq 1\{C_n\} \) and uses the definition of \( C_n \) to bound \( L_\rho(\theta, \varphi) \) from below, and the third line that \( 1\{\mathcal{A}_n\} + 1\{\mathcal{A}_n^c\} = 1 \). Setting \( \delta = \lambda \rho \) and taking the expectation under \( P_\theta \) on both sides of the inequality leads to the result. \( \square \)

**Theorem 9** (Lower bound for weighted classification losses). For any integer \( \rho \geq 1 \), any \( 0 \leq \varepsilon \leq 1 \), any \( s_n > 4 \), and any \( b \in \mathbb{R}^{s_n} \) satisfying (16),

\[
\inf_{\varphi} \sup_{\theta \in \Theta_b} P_\theta \left( L_\rho(\theta, T) \geq \min \left( 1 - \frac{\varepsilon}{\sqrt{s_n}}, \mathcal{F}_n(b, \rho) - \varepsilon \right) \rho s_n \right) \geq 1 - \exp(-\sqrt{s_n} \varepsilon^2 / 3).
\]

where the infimum is over all possible multiple testing procedures \( \varphi = \varphi(X) \).

Note that the right side in the statement of Theorem 9 only depends on \( s_n, \varepsilon \), so that one may optimise the left side with respect to \( \rho \).

Some ideas of the proof are inspired from [43], who derived an asymptotically sharp bound for the minimax in-probability risk in terms of the quadratic loss and Gaussian noise. Weighted classification loss is, like quadratic loss, a sum over coordinates, so one can split the global loss into blocks and define a least favourable prior in a similar way as for the former. There are two important differences: firstly, our working with weighted classification losses leads to the study of different Bayes estimators, and secondly, we shall study the Bayes risk globally instead of reducing the problem to the study of Bayes risks over blocks. The latter turns out to be necessary, as unlike for the quadratic loss, there is no “concentration” for the loss over a given block for (weighted or unweighted) classification losses. A further benefit of the argument below is that the derived lower bound is non-asymptotic.

**Proof.** Since \( L_\rho \geq 0 \), there is nothing to prove if \( \mathcal{F}_n(b, \rho) \leq \varepsilon \), so assume that \( \mathcal{F}_n(b, \rho) > \varepsilon \). Let \( q = \lfloor n/s_n \rfloor \) and let \( n' = q s_n \). Let \( \mathcal{P}_c \) be the set of all prior distributions on the set \( \Theta_b \). For any \( \lambda \in (0, 1) \) we have

\[
\inf_{\varphi} \sup_{\theta \in \Theta_b} P_\theta \left( L_\rho(\theta, \varphi) \geq \lambda \rho s_n \right) \geq \sup_{\pi \in \mathcal{P}_c} \inf_{\varphi} P_{\pi} \left( L_\rho(\theta, \varphi) \geq \lambda \rho s_n \right), \tag{36}
\]
where $P_\pi$ denotes the distribution of $(\theta, X)$ in the Bayesian setting $\theta \sim \pi$ and $X \mid \theta \sim P_\theta$.

Let us define a specific prior $\pi$ as a product prior over $s_n$ blocks of consecutive coordinates $Q_1 = \{1, 2, \ldots, q\}, Q_2 = \{q + 1, \ldots, 2q\}, \ldots, Q_{s_n} = \{(s_n - 1)q + 1, \ldots, n\}$. We write $Q_\infty$ for the (possibly empty) set $\{n'+1, \ldots, n\}$. Let $\beta[m, b]$ denote the vector of $\mathbb{R}^q$, with coordinates defined, for $1 \leq i, m \leq q$ and $a_b$ as in (33), by

$$
(\beta[m, b])_i = \begin{cases} a_b & \text{if } i = m; \\
0 & \text{if } i \neq m.
\end{cases}
$$

In words, $\beta[m, b] \in \mathbb{R}^q$ is the 1-sparse vector with its only nonzero coordinate, at position $m$, equalling $a_b$. Over each block $Q_j$, $1 \leq j \leq s_n$, one takes the following prior: first draw an integer $I_j$ from the uniform distribution $U(Q_j)$ over the block $Q_j$ and next set $\beta_{I_j} = a_{b_j}$ as above. That is, $\pi_j$ generates $\beta^j \in \mathbb{R}^q$, to be identified with $(\beta_i)_{i \in Q_j}$, according to

$$I_j \sim U(Q_j), \quad \beta^j \mid I_j \sim \delta_{\beta[I_j \; \text{mod} \; q, b]}.
$$

with ‘mod $q$’ meaning modulo the integer $q$. If $Q_\infty$ is non-empty, set $\beta^\infty \sim \delta_0$. By definition, $\pi$ belongs to $\mathcal{P}_L$. Using the fact that the classification loss is a sum of losses over all coordinates, one may rewrite $L_\rho(\theta, \varphi) = \sum_{j=1}^{s_n} L_j + L_\infty$, with $L_j$ the contribution of block $Q_j$. For $1 \leq j \leq s_n$, we note that for $\theta$ sampled from $\pi$, $L_j = L_j(\theta, \varphi) = \rho \mathbf{1}\{\varphi_{I_j} = 0\} + \sum_{i \in Q_j, i \neq I_j} \mathbf{1}\{\varphi_i \neq 0\}$.

Then for any integer $\rho$ and $\lambda \in (0, 1)$, noting that $L_\infty \geq 0$ we see that

$$P_\pi [L_\rho(\theta, \varphi) \geq \lambda \rho s_n] \geq P_\pi \left[ \sum_{j=1}^{s_n} L_j \geq \lambda \rho s_n \right] = 1 - P_\pi \left[ \sum_{j=1}^{s_n} L_j < \lambda \rho s_n \right].
$$

For a multiple testing procedure $\varphi$, let $A^\varphi = \{i : \varphi_i(X) \neq 0\}$ denote its support and $A_j^\varphi = A^\varphi \cap Q_j$ its support within the $j$th block. Let us consider the event $\mathcal{E}$ defined as

$$\mathcal{E} = \mathcal{E}(\varphi) = \mathcal{E}(\varphi, \rho, \lambda) = \{L_1 + \cdots + L_{s_n} < \lambda \rho s_n\}.
$$

To complete the proof, it is enough to bound $\sup_{\varphi} P_\pi[\mathcal{E}]$ from above for a suitable $\lambda$. Let us further define

$$N_{\varphi}(X, \theta) = \sum_{j=1}^{s_n} \mathbf{1}\{I_j \in A_j^\varphi, |A_j^\varphi| \leq \rho\}.
$$

One next notes that the following inequality holds:

$$L_1 + \cdots + L_{s_n} \geq (s_n - N_{\varphi}(X, \theta))\rho.
$$

To check this, it suffices to verify that for any index $j$ for which the indicator equals 0 in the sum defining $N_{\varphi}(X, \theta)$, the corresponding loss $L_j$ is at least $\rho$. The latter is true because if $I_j \notin A_j^\varphi$ we have $\rho \mathbf{1}\{\varphi_{I_j} = 0\} = \rho$, and if $|A_j^\varphi| \geq \rho + 1$ then $\mathbf{1}\{\varphi_i \neq 0\}$ equals 1 at least $\rho$ times for $i \in Q_j, i \neq I_j$. This leads to the desired inequality, which itself further implies that, for any integer $\rho \geq 1$,

$$\mathcal{E} \subset \{N_{\varphi}(X, \theta) \geq s_n - \lambda s_n\}.$$
We wish to bound \( P_\pi[\mathcal{E}] = E_\pi(P_\pi[\mathcal{E} | X]) \) from above uniformly in \( \varphi \), where \( E_\pi \) denotes the (Bayesian) expectation under \( P_\pi \), and it suffices to bound from above

\[
\sup_{\varphi} E_\pi P_\pi[\mathcal{E} | X] \leq \sup_{\varphi} E_\pi P_\pi[N_{\varphi}(X, \theta) \geq (1 - \lambda)s_n | X].
\]

Conditional on \( X \), the distribution of \( N_{\varphi}(X, \theta) \) is Poisson-binomial, that is, the law of a sum of \( s_n \) independent Bernoulli variables \( Z_j \) with parameters \( p_j^\varphi(X) \) given by

\[
p_j^\varphi(X) = 1\{|A_j^\varphi| \leq \rho\} P_\pi[I_j \in A_j^\varphi | X].
\]

Let us now investigate the posterior distribution \( P_\pi[\cdot | X] \), restricted to coordinates \( \theta_i, \ i \in \{1, \ldots, n'\} = \cup_{j=1}^{s_n} Q_j \). By definition the prior distribution is a product over the blocks \( Q_j \). The model, that is the law of \( X | \theta \), is also of product form, so by Bayes’ formula the posterior \( \pi[\cdot | X] \) is a product \( \otimes_{j=1}^{s_n} \pi_j[\cdot | X^j] \), where \( X^j \) denotes the observations \( (X_i : i \in Q_j) \) over the block \( Q_j \). Writing \( w_j^\varphi(X) = P_\pi(I_j = i | X^j) \) for the posterior probability that \( I_j = i \), we have

\[
\pi_j[\cdot | X^j] = \sum_{i \in Q_j} w_j^\varphi(X) \delta_{\beta_i \mod q, a_{b_j}},
\]

\[
w_j^\varphi(X) = \frac{f(X_i - a_{b_j})/f(X_i)}{\sum_{k \in Q_j} f(X_k - a_{b_j})/f(X_k)} = \frac{h_\zeta(X_i, a_{b_j})}{\sum_{k \in Q_j} h_\zeta(X_k, a_{b_j})},
\]

\[
h_\zeta(x, a) := \exp(\zeta^{-1}(\|x\| \zeta - \|x - a\| \zeta)).
\]

By definition, \( P_\pi[I_j \in A_j^\varphi | X] = \sum_{i \in A_j^\varphi} h_\zeta(X_i, a_{b_j})/\sum_{i \in Q_j} h_\zeta(X_i, a_{b_j}) \). Among (frequentist) estimators \( \varphi(X) \) such that \( |A_j^\varphi| = \#\{i : \varphi_i(X) \neq 0\} \leq \rho \), this expression is maximal for any estimator whose support \( A_j^\varphi \) over the block \( Q_j \) is equal to the set \( A_j^\varphi \) consisting of indices \( i_1', \ldots, i_s' \) corresponding to the \( \rho \) largest observations among \( (X_i : i \in Q_j) \); to see this, note that the real map \( x \to \|x\| \zeta - \|x - \tau\| \zeta \) is increasing for \( \tau \geq 0 \).

Recalling \( p_j^\varphi(X) = 1\{|A_j^\varphi| \leq \rho\} P_\pi[I_j \in A_j^\varphi | X] \), for any procedure \( \varphi \) and all \( j \), we have

\[
p_j^\varphi(X) \leq P_\pi[I_j \in A_j^\varphi | X] = \pi_j[I_j \in A_j^\varphi | X^j] =: p_j^\varphi(X^j).
\]

Write \( \mathbf{p}^\varphi \) for the vector of \( p_j^\varphi \)'s, and similarly for \( \mathbf{p}^\varphi \). Lemma 3 implies that, given \( X \), \( \text{PBin}[\mathbf{p}^\varphi] \) stochastically dominates \( \text{PBin}[\mathbf{p}^\varphi] \) (see the definition of the Poisson-Binomial distribution just before Lemma 3), so that

\[
P_\pi[N_{\varphi}(X, \theta) \geq (1 - \lambda)s_n | X] = P_\pi[\text{PBin}[\mathbf{p}^\varphi] \geq (1 - \lambda)s_n | X] \leq P_\pi[\text{PBin}[\mathbf{p}^\varphi] \geq (1 - \lambda)s_n | X],
\]

for any procedure \( \varphi \). Let us note that \( E_\pi p_j^\varphi(X^j) = P_\pi[I_j \in A_j^\varphi] \) and

\[
P_\pi[I_j \in A_j^\varphi] = P_\pi[X_{I_j} = a_{b_j} + \varepsilon_{I_j} \text{ belongs to the } \rho \text{ largest among coordinates of } X^j]
\]

\[
= P(\varepsilon_{I_j} \leq 1 \text{ belongs to the } \rho \text{ largest among } a_{b_j} + \varepsilon_{I_j}, \varepsilon_{I_2}, \ldots, \varepsilon_{I_q}) = 1 - p_n(b_j, \rho),
\]

using that \( I_j \) has a uniform distribution over \( Q_j \) and the fact that \( \varepsilon_i \) are iid and independent of \( I_j \). This implies that the parameters \( p_j^\varphi(X^j) \) of the Poisson-binomial \( \text{PBin}[\mathbf{p}^\varphi] \) have expectations given by the vector \( \mathbf{p} = (1 - p_n(b_j, \rho))_j \) in the Bayesian model under \( E_\pi \).

Taking the expectation under \( E_\pi \) in the last but one display and using Lemma 4 leads to

\[
\sup_{\varphi} P_\pi[N_{\varphi}(X, \theta) \geq (1 - \lambda)s_n] \leq P[\text{PBin}[\mathbf{p}] \geq (1 - \lambda)s_n].
\]
If $1 - \mathcal{F}_n(b, \rho) \geq s_n^{-1/2}$, define $\lambda$ by

$$1 - \lambda = (1 + \varepsilon)(1 - \mathcal{F}_n(b, \rho)),$$

which, recalling that we have excluded already the case $\mathcal{F}_n(b, \rho) \leq \varepsilon$, indeed has a solution $\lambda = \lambda_\varepsilon(b, \rho, \rho) \in (0, 1)$. Noting that PBin$(p)$ has mean $s_n(1 - \mathcal{F}(b, \rho))$, we apply Bernstein’s inequality for deviations of the Poisson Binomial distribution from its mean (Lemma 5) to obtain

$$P[\text{PBin}[p] \geq (1 - \lambda)s_n] \leq \exp(-s_n(1 - \mathcal{F}_n(b, \rho))\varepsilon^2/3) \leq \exp(-s_n^1/\varepsilon^2/3).$$

If instead $1 - \mathcal{F}_n(b, \rho) < s_n^{-1/2}$, we set

$$1 - \lambda = (1 + \varepsilon)s_n^{-1/2}$$

and Bernstein’s inequality (Lemma 5), with the upper bound $s_n^{1/2}$ for the variance and expectation of PBin$[p]$, yields

$$P[\text{PBin}[p] \geq (1 - \lambda)s_n] \leq \exp\left(-\frac{\varepsilon^2 s_n^{1/2}/2}{1 + \varepsilon/3}\right) \leq \exp(-\varepsilon^2 s_n^{1/2}/3).$$

Combining, and returning to (36), we deduce that

$$\inf_{\theta \in \Theta_b} L_\rho(\theta, T) \geq (1 - \lambda)\rho s_n \geq 1 - \exp(-\sqrt{n}\varepsilon^2/3),$$

for $\lambda = \max\left\{\frac{1+\varepsilon}{\sqrt{n}}, (1+\varepsilon)(1 - \mathcal{F}_n(b, \rho))\right\}$. Noting that $1 - (1 + \varepsilon)(1 - \mathcal{F}_n(b, \rho)) \geq \mathcal{F}_n(b, \rho) - \varepsilon$ concludes the proof. \hfill \Box

Given $\delta \geq 0$, and $q \geq 2$, let us set

$$\omega = \omega_{\delta, q} = \exp\left(-(|q| - 1)\mathcal{F}(\zeta \log q)^{1/\zeta} - \delta\right).$$

(39)

Lemma 2 gives the asymptotic behaviour of $\mathcal{F}_n(b, \rho_n(\delta))$ defined in (35) and appearing in Theorem 9.

**Lemma 2.** Fix $\delta > 0$ and let $\rho_n = \rho_n(\delta) \to \infty$ be a sequence of integers such that $\rho_n \leq \log(1/\omega_{\delta, n/s_n})/2$. Then for any $b$ such that (16) and (19) are satisfied we have

$$\Lambda_\infty - C_f\delta \leq \lim_{n \to \infty} \mathcal{F}_n(b, \rho_n(\delta)) \leq \lim_{n \to \infty} \mathcal{F}_n(b, \rho_n(\delta)) \leq \Lambda_\infty,$$

where $C_f = C_f(\zeta)$ denotes the Lipschitz constant of the noise c.d.f. $F$. In particular this holds when $\rho_n$ is the largest integer smaller than $\log(1/\omega_{\delta, n/s_n})/2$, or when it is the largest integer smaller than the square root of this quantity.

Also,

$$\lim_{n \to \infty} \mathcal{F}_n(b, 1) = \Lambda_\infty.$$

**Proof.** First note that the validity of the particular choices of $\rho_n$ are a consequence of Lemma 6 (which shows that $1/\omega_{\delta, n/s_n} \to \infty$). Then for the first assertion, Lemma 8 tells us that

$$p_n(b, \rho_n) \leq \mathcal{F}(b) + e^{-\rho_n}.$$
Inserting into the definition (35) of $F_n(b, \rho)$ and taking the limsup yields the claimed upper bound. Similarly, Lemma 7 yields

$$F_n(b, \rho) \geq \frac{1}{s_n} \sum_{i=1}^{s_n} F(b_i + \delta) - \omega_{\delta, n/s_n}^{1/10}$$

and the lower bound follows by noting $F(b_i + \delta) \geq F(b_i) - C_f \delta$ for any $1 \leq i \leq s_n$ (as $F$ is $C_f$-Lipschitz over $\mathbb{R}$) and taking the liminf.

For the second assertion, the same argument applies to give the limiting lower bound

$$\lim_{n} F_n(b, 1) \geq \Lambda_\infty - C_f \delta.$$ 

For the upper bound, applying the second inequality of Lemma 8 and using again that $F$ is $C_f$-Lipschitz gives the limiting upper bound

$$\lim_{n} F_n(b, 1) \leq \Lambda_\infty + C_f \delta.$$ 

One now takes $\delta \to 0$ to conclude.

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Appendix A: Remaining proofs

A.1. Proof of Theorem 2

Theorem 2 is an immediate consequence of Theorem 5 (by taking \( \zeta = 2, b = (b, b, \ldots, b) \)), since for \( n \) large \( b > −(2 \log (n/s_n))^1/2 \) so that this choice is permitted, the proof of which can be found in Appendix C (\( \ell \)-value case) and in Appendix D (BH case).

A.2. Proof of Theorems 3 and 6

Up to minor technical details in the case \( b = b_n \to −\infty \) (as given in the proof of Theorem 1), Theorem 3 follows from Theorem 6 by taking \( \zeta = 2, \kappa = 1/4 \) and \( b = (b, b, \ldots, b) \), hence we restrict our attention to proving the latter result. We begin by proving that the limsup of the quantity in question is upper bounded by \( \Lambda_{\infty} \). It suffices to check that the oracle procedure \( \varphi^* \) considered in the proof of Theorem 4 belongs to the class \( S_B \), since its FNR is controlled at level \( \Lambda_{\infty} \) by (30). [Note, for the case \( \Lambda_{\infty} = 1 \), that the trivial procedure \( \varphi = 0 \) is sparsity preserving.] That \( \varphi^* \in S_B \) follows from Lemma 21 in Appendix E since \( \text{FDR}(\varphi^*) \to 0 \) (recall (30)). [One could also argue directly, using Bernstein’s inequality (Lemma 5) as in the proof of Theorem 4 that the number of false discoveries is \( o_p(s_n) \) (recall (28) and the concentration argument thereafter), hence with probability tending to 1 the total number of rejections of \( \varphi^* \) is at most \( s_n + o(s_n) \leq 2s_n \leq B_n s_n \).

To show that the liminf of the quantity in question is lower bounded by \( \Lambda_{\infty} \), the idea is to derive a bound for a weighted risk. Suppose one can show, for a suitable sequence \( M_n \to \infty \), that

\[
\lim_{n \to \infty} \inf_{\varphi \in S_B} \sup_{\theta \in \Theta_b} \left[ M_n^{-1} \text{FDR}(\theta, \varphi) + \text{FNR}(\theta, \varphi) \right] \geq \Lambda_{\infty}. \tag{40}
\]

Then by bounding the FDR from above by 1 one gets

\[
\lim_{n \to \infty} \left[ M_n^{-1} + \inf_{\varphi \in S_B} \sup_{\theta \in \Theta_b} \text{FNR}(\theta, \varphi) \right] \geq \Lambda_{\infty},
\]

which gives the desired lower bound since \( M_n \to \infty \). To prove (40), we adapt the argument of Lemma 1. Fix \( \varphi \in S_B \), denote by \( D_n(X) = \sum_{i=1}^n \varphi_i(X) \) its total number of rejections, and define

\[
q(X) = q(X, \varphi) = \sum_{i=1}^n \left( 1\{\theta_i = 0\} M_n^{-1} \frac{\varphi_i(X)}{1 \lor D_n(X)} + 1\{\theta_i \neq 0\} \frac{1 - \varphi_i(X)}{s_n} \right),
\]

so that the quantity inside the brackets in (40) equals \( E_{\theta}q(X) \). Further set \( B_n = \{ D_n(X) \leq B_n s_n \} \), and recall that by assumption \( \sup_{\theta \in \Theta_b} P_{\theta}(B^c_n) \to 0 \). On \( B_n \) we have \( 1 \lor D_n(X) \leq B_n s_n \), so that for any \( \rho > 0 \)

\[
q(X) \geq 1\{B_n\} \sum_{i=1}^n \left( 1\{\theta_i = 0\} \frac{\varphi_i(X)}{B_n M_n s_n} + \frac{\rho M_n}{\rho M_n} 1\{\theta_i \neq 0\} \frac{1 - \varphi_i(X)}{s_n} \right)
\]

\[
\geq 1\{B_n\} \left( \frac{1}{B_n} \lor \frac{1}{\rho} \right) \frac{1}{M_n s_n} L_{\rho M_n}(\theta, \varphi),
\]
where \( L_r \) (for \( r > 0 \)) is the weighted classification loss as in (32). For \( \rho \geq B_n \) to be chosen set \( M_n = \rho \). Then for any \( \lambda \in (0,1) \),

\[
q(X) \geq 1 \{ B_n \} \frac{1}{\rho M_n s_n} L_{\rho M_n}(\theta, \varphi) 1 \{ L_{\rho M_n}(\theta, \varphi) \geq \lambda \rho M_n s_n \} \\
\geq \lambda 1 \{ B_n \} 1 \{ L_{\rho M_n}(\theta, \varphi) \geq \lambda \rho M_n s_n \} \geq \lambda 1 \{ L_{\rho M_n}(\theta, \varphi) \geq \lambda \rho M_n s_n \} - 1 \{ B_n \}.
\]

Now observing that the last term in the above display is \( o_P(1) \) uniformly over \( \Theta_B \) for \( \varphi \in S_B \) by definition of the class, and then taking expectations, one obtains, for any \( \varphi \in S_B \) and \( \theta \in \Theta_B \),

\[
M_n^{-1} \text{FDR}(\theta, \varphi) + \text{FNR}(\theta, \varphi) \geq \lambda P_\theta[L_{\rho M_n}(\theta, \varphi) \geq \lambda \rho M_n s_n] - o(1).
\]

Setting \( r_n = \rho M_n = M_n^2 \), and noting that the infimum over \( \varphi \in S_B \) is larger that the unrestricted infimum,

\[
\inf_{\varphi \in S_B} \sup_{\theta \in \Theta_B} M_n^{-1} \text{FDR}(\theta, \varphi) + \text{FNR}(\theta, \varphi) \geq \inf_{\varphi} \sup_{\theta \in \Theta_B} \lambda P_\theta[L_{r_n}(\theta, \varphi) \geq \lambda r_n s_n] - o(1).
\]

Let \( r_n = r_n(\delta) \) to be the largest square number smaller than \( \log(1/\omega_{\delta,n}/s_n)/2 \), where \( \omega_{\delta,q} \) is defined as in (39) for some \( \delta > 0 \). The proof of Lemma 6 shows that \( r_n \leq C \exp(\delta [a_n^* - \delta(\zeta^{-1})(a_n^* - \delta)^{-\zeta^{-1}}] \) for some constant \( C \), where \( a_n^* = (\zeta \log(n/s_n))^{1/\zeta} \). In particular \( r_n \geq \exp(2(\log n/s_n)^{\zeta}) \), for \( n/s_n \) large enough (recall that \( \kappa < 1 - 1/\zeta \)); we deduce that \( \rho = \sqrt{r_n} \geq B_n \) hence this choice is valid.

From here, arguing exactly as in the proof of (31) yields the result. Indeed, define \( \mathcal{F}(b, \rho) \in [0,1] \) as in (35) and take \( \lambda = \min(1 - (1 + \varepsilon)n^{-1/2}, \mathcal{F}(b, \rho) - \varepsilon) \); note for \( n \) large enough we always have \( \lambda = \mathcal{F}(b, \rho) - \varepsilon \). Then Theorem 9 tells us that

\[
\inf_{\varphi} \sup_{\theta \in \Theta_B} P_\theta[L_{\rho}(\theta, \varphi) \geq \lambda \rho s_n] \geq 1 - \exp(-s_n^{1/2} \varepsilon^2/3),
\]

while Lemma 2 tells us that

\[
\lim_{n \to \infty} \mathcal{F}(b, \rho_n(\delta)) \geq \Lambda_\infty - CF \delta.
\]

Combining what we have so far we deduce

\[
\inf_{\varphi \in S_B} \sup_{\theta \in \Theta_B} M_n^{-1} \text{FDR}(\theta, \varphi) + \text{FNR}(\theta, \varphi) \geq \lim_{n \to \infty} [\lambda(1 - \exp(-s_n^{1/2} \varepsilon^2/3)) - o(1)] = \lim_{n \to \infty} \mathcal{F}(b, \rho) - \varepsilon \geq \Lambda_\infty - CF \delta - \varepsilon.
\]

Then (40) follows by letting \( \varepsilon, \delta \to 0 \), concluding the proof.

**A.3. Proof of Theorem 7 (lower bound)**

The proof of the upper bound for the \( \ell \)-value procedure and for the BH procedure can be found in Appendix C.3 and Appendix D.3 respectively. We here prove a more general version of the lower bound, allowing for multiple signals and Subbotin noise, as in Section 3.

Lemma 2 tells us that, for \( b \) such that (16) and (19) hold,

\[
\lim_{n \to \infty} \mathcal{F}(b, 1) \to \Lambda_\infty.
\]
For the case \( b \in \mathbb{R} \), set \( b = (b, \ldots, b) \); then for \( \varepsilon > 0 \) and \( n \) large enough that \( b > -(\zeta \log(n/s_n))^{1/\zeta} \) and
\[
\min \{1 - (1 + \varepsilon)s_n^{-1/2}, \Phi(0,1) - \varepsilon\} \Lambda \geq \Lambda_\infty - 2\varepsilon,
\]
we apply Theorem 9 with \( \rho = 1 \), for which \( L_\rho(\theta, \varphi) = L_C(\theta, \varphi) \), to obtain
\[
\inf \sup_{\theta, \varphi} P_\theta \{ L_C(\theta, T) \geq (\Lambda_\infty - 2\varepsilon)s_n \} \geq 1 - \exp \left\{ -\frac{s_n^{1/2}\varepsilon^2}{3} \right\},
\]
yielding the claimed lower bound in this case by setting \( b = (b, \ldots, b) \) and \( \eta = 2\varepsilon \), and noting that \( \Lambda_\infty = \Phi(b) \). In the case \( b = b_n \to -\infty \), we replace \( b \) by some \( b' = b_n' \geq \max(b_n, -b/\log(n/s_n))^{1/\xi} \) still tending to \( -\infty \) and apply the same argument, noting that \( \Lambda_\infty = 1 \) and that taking the supremum over \( \Theta_b \supset \Theta_{b'} \) can only increase probability.

**A.4. Proof of Theorem 8**

For the lower bound one may simply appeal to Theorem 7 (for \( \eta = \eta_n \) tending slowly to zero) and Markov’s inequality; because \( \Theta_b \subset \Theta_{b'} \) this gives a sharper lower bound than required. An alternative proof involving a new inequality is provided in Remark 6.

For the upper bound, given \( \theta \in \Theta_b'(s_n) \) write \( S_\theta = \{ i : \theta_i \neq 0 \} \) for the support of \( \theta \) and write \( s_\theta = |S_\theta| \). We can decompose the classification loss \( L_C(\theta, \varphi) \) as the sum \( V + (s_\theta - S) \), where \( V = V(\varphi, \theta) = \sum_{i \in S_\theta} \varphi_i \) and \( S = S(\varphi, \theta) = \sum_{i \in S_\theta} \varphi_i \). It suffices to show that
\[
\sup_{\theta \in \Theta_b'(s_n)} E_\theta |s_\theta - S| \leq s_n \Phi(b) + o(s_n),
\]
\[
\sup_{\theta \in \Theta_b'(s_n)} E_\theta V = o(s_n).
\]

We begin with (41). This holds trivially when \( b = b_n \to -\infty \) since \( 0 \leq s_\theta - S \leq s_\theta \leq s_n \) for \( \theta \in \Theta_b'(s_n) \), so we restrict to the cases \( b \in \mathbb{R} \) fixed or \( b = b_n \to -\infty \). It suffices to show, for a sequence \( s_n' \) satisfying \( 1 \ll s_n' \ll s_n \), that
\[
\max_{s_n' \leq s \leq s_n} \sup_{\theta \in \Theta_b'(s_n)} E_\theta (s - S) \leq s_n \Phi(b) + o(s_n),
\]
since for \( \theta \in \Theta_b(s) \) with \( 0 \leq s \leq s_n' \), we have \( 0 \leq s_\theta - S \leq s_\theta = s \leq s_n' = o(s_n) \). Let \( s_n'' \) denote a sequence such that
\[
\max_{s_n' \leq s \leq s_n} \sup_{\theta \in \Theta_b'(s_n)} E_\theta (s - S) = \sup_{\theta \in \Theta_b'(s_n'')} E_\theta (s_n'' - S).
\]

We now apply Theorem 2 with \( s_n'' \) in place of \( s_n \) (note the asymptotics (7) hold). We have
\[
(s_n'')^{-1} E_\theta (s_n'' - S) \leq \Pi(\theta, \varphi) \leq \Phi(b) + o(1)
\]
and we deduce that \( E_\theta (s_n'' - S) \leq s_n'' (\Phi(b) + o(1)) \leq s_n (\Phi(b) + o(1)) \) as required. For (42), the proof differs depending on whether \( \varphi \) is the \( \ell \)-value procedure or the BH procedure. For the former one notes that the proof of (68) in Appendix C.3 holds for \( \theta \in \Theta_b'(s_n) \) not just \( \theta \in \Theta_b(s_n) \). Taking the supremum yields the claim. For the latter one uses Lemma 20 in Appendix D.4 and the fact that \( \alpha = \alpha_n = o(1) \). An alternative proof of this upper bound in the \( \ell \)-value case is provided in Remark 8.
Remark 6. Let us also give another proof of the lower bound, proceeding via the following non-asymptotic result, interesting in its own right: For any \( b \in \mathbb{R}^n \) and \( \overline{F}_n(b, 1) \) as in (35) with \( \rho = 1 \),

\[
\inf_{\varphi} \sup_{\theta \in \Theta_b} E_\theta L_C(\theta, \varphi) \geq \overline{F}_n(b, 1) s_n. \tag{43}
\]

One follows the proof of Theorem 9, from which the notation is borrowed. Recalling the inequality (38), and applying it for \( \rho = 1 \) (which gives the classification risk), one gets \( L_C(\theta, \varphi) \geq s_n - N_\varphi(X, \theta) \) for any possible procedure \( \varphi \) and \( \theta \) sampled from the same prior \( \pi \) as in the proof of Theorem 9. To bound \( E_\pi L_C(\theta, \varphi) \), it is thus enough to focus on \( E_\pi N_\varphi(X, \theta) \). Noting \( E_\pi p_j^*(X^j) = P_\pi[I_j \in A_j^\pi] \),

\[
E_\pi N_\varphi(X, \theta) = E_\pi[P_{\text{Bin}}(p^*) \mid X] \leq E_\pi[P_{\text{Bin}}(p^*) \mid X] = E_\pi[\sum_{j=1}^{s_n} p_j^*(X^j) \mid X] = \sum_{j=1}^{s_n} P_\pi[I_j \in A_j^\pi] = \sum_{j=1}^{s_n} p_\pi(b_j, 1) = s_n - \overline{F}_n(b, 1) s_n,
\]

from which (43) follows. It now follows from the last identity of Lemma 2, taking \( b \) to have all coordinates equal to a fixed real \( b \) and noting \( \Lambda_\infty = \overline{\Phi}(b) \), that

\[
\lim_{n \to \infty} \inf_{\varphi} \sup_{\theta \in \Theta_b} E_\theta L_C(\theta, \varphi)/s_n \geq \overline{\Phi}(b),
\]

which is again an even more precise result than needed, since \( \Theta_b \subset \Theta^\rho_b \). The case \( b = b_n \to +\infty \) works similarly (with \( \Lambda_\infty = 0 \)), while in the case \( b = b_n \to -\infty \) the same argument works after the usual technical adjustments as in proving Theorem 1, replacing \( b_n \) by \( b'_n \to -\infty \) such that \( b'_n > -(2 \log(n/s_n))^{1/2} \).

Appendix B: Auxiliary lemmas

Let \( \text{PBin}[a] \) denote the Poisson-binomial distribution of parameter \( a = (a_1, \ldots, a_S) \), for \( S \geq 1 \): it is the distribution of \( \sum_{i=1}^S Z_i \), where \( Z_i \) are independent \( \text{Be}(a_i) \) random variables.

Lemma 3. Let \( a = (a_1, \ldots, a_S) \), \( b = (b_1, \ldots, b_S) \) be vectors in \( [0, 1]^S \) for some \( S \geq 1 \). Suppose \( a_i \leq b_i \) for all \( 1 \leq i \leq S \). Then, for any integer \( k \),

\[
P[\text{PBin}[a] \geq k] \leq P[\text{PBin}[b] \geq k].
\]

That is, \( \text{PBin}[b] \) stochastically dominates \( \text{PBin}[a] \).

Proof. By transitivity it is enough to prove the result in the case \( a \) and \( b \) differ only by one coordinate, say (by symmetry) the first one: that is, let \( a = (a_1, \ldots, a_S) \) and \( b = (b_1, a_2, \ldots, a_S) \). If \( X_1 \sim \text{PBin}[a] \) and \( X_2 \sim \text{PBin}[b] \), one can write \( X_1 = \varepsilon + T_1 \) and \( X_2 = \xi + T_2 \) (in distribution), where \( T_1, T_2 \) are equal in law, and \( \varepsilon \sim \text{Be}(a_1), \xi \sim \text{Be}(b_1) \), independently of \( T_1, T_2 \) respectively. For any integer \( k \),

\[
P[X_1 \geq k] = P[T_1 + 1 \geq k \mid \varepsilon = 1]a_1 + P[T_1 \geq k \mid \varepsilon = 0](1 - a_1) \\
\leq P[T_1 + 1 \geq k \mid \varepsilon = 1]b_1 + P[T_1 \geq k \mid \varepsilon = 0](1 - b_1) = P[X_2 \geq k],
\]

where the second line uses that \( P[T_1 + 1 \geq k \mid \varepsilon = 1] - P[T_1 \geq k \mid \varepsilon = 0] = P[T_1 \geq k - 1] - P[T_1 \geq k] \geq 0 \) and \( a_1 \leq b_1 \), which concludes the proof. \( \square \)
Lemma 4. Let $Y = (Y_1, \ldots, Y_S)$ be a vector of independent random variables taking values in $[0, 1]$, and let $p = (p_j) \in [0, 1]^S$ be a deterministic vector such that $E[Y_j] = p_j$ for all $1 \leq j \leq S$. Conditionally on $Y$, let $\xi_j \sim \text{Be}(Y_j)$, $j \leq S$ be independent Bernoulli random variables. Then

$$\sum_{i=1}^S \xi_j \sim \text{PBin}[p].$$

Proof. Observe that the variables $\xi_j$, defined to be independent given $Y$, are also independent unconditionally, because the $Y_j$’s are independent. It now suffices to note that $\xi_j$ has Bernoulli distribution, with parameter $E[\xi_j] = E[E[\xi_j | Y_j]] = E[Y_j] = p_j$. □

Lemma 5 (Bernstein’s inequality). Let $W_i, 1 \leq i \leq n$ centered independent variables with $|W_i| \leq M$ and $\sum_{i=1}^n \text{Var}(W_i) \leq V$, then for any $A > 0$,

$$P\left[\sum_{i=1}^n W_i > A\right] \leq \exp\left\{-\frac{1}{2}A^2/(V + MA/3)\right\}.$$

In particular, let $P \sim \text{PBin}[p]$, with $p \in [0, 1]^S$, $S \geq 1$, and set $\mu := \sum_{j=1}^S p_j$. Then for any $0 \leq \delta \leq 1$,

$$P[P \geq \mu(1 + \delta)] \leq e^{-\mu \delta^2/3}.$$

Proof. The first part of the Lemma is the standard Bernstein inequality. The second part follows, for $\xi_j$ independent Bernoulli variables with parameter $p_j$, by setting $W_j = \xi_j - E[\xi_j], A = \mu \delta, V = \sum_{j=1}^S p_j + \mu, M = 1$ and noting, since $0 \leq \delta \leq 1$,

$$\exp\left(-\frac{\mu^2 \delta^2}{2\mu + 2\mu \delta/3}\right) \leq \exp(-3\mu \delta^2/8) \leq \exp(-\mu \delta^2/3).$$

Recall the definitions (33), (34) and (39), in the Subbotin case, of $a_b, p_n$ and $\omega$ respectively,

$$a_b = (\zeta \log(n/s_n))^{1/\zeta} + b,$$

$$p_n(b, \rho) = P[\#\{2 \leq i \leq n/s_n : \varepsilon_i > a_b + \varepsilon_1 \geq \rho\}],$$

$$\omega = \omega_{s,q} = \exp\left(-(\lfloor q \rfloor - 1)F((\zeta \log q)^{1/\zeta} - \delta)\right).$$

Lemma 6. For $\delta > 0$, we have $\log(1/\omega_{s_n/s_n}) \rightarrow \infty$.

Proof. Write $a_n^* = (\zeta \log(n/s_n))^{1/\zeta}$ and note, since $n/s_n \rightarrow \infty$, that to see as required that

$$\log(1/\omega_{s_n/s_n}) = (\lfloor n/s_n \rfloor - 1)F((\zeta \log(n/s_n))^{1/\zeta} - \delta) \rightarrow \infty,$$

it suffices to show that $(n/s_n)F(a_n^* - \delta) \rightarrow \infty$. By Lemma 23,

$$F(x) \asymp \frac{f(x)}{x^{\zeta-1}}, \quad \text{as} \quad x \to \infty,$$

and we deduce, noting that $f(a_n^*) = L_{\zeta}^{-1}s_n/n$ from the definition (15) of $f$, we have

$$\frac{n}{s_n}F(a_n^* - \delta) \asymp \frac{f(a_n^* - \delta)}{(a_n^* - \delta)^{\zeta-1}f(a_n^*)}.$$
Lemma 24 tells us that for $\delta \geq 0$ we have
\[
    f(a_n^* - \delta)/f(a_n^*) \geq \exp(\delta|a_n^* - \delta|^2 \text{sign}(a_n^* - \delta)).
\]
Thus
\[
    \frac{n}{s_n} F(a_n^* - \delta) \gtrsim (a_n^* - \delta)^{-2\zeta} \exp(\delta|a_n^* - \delta|^2) \to \infty.
\]

**Lemma 7.** For any $\delta \geq 0$, any integer $\rho \leq \log(1/\omega_{\delta,n/s_n})/2$ and any $b \in \mathbb{R}$ we have
\[
    p_n(b, \rho) \geq F(b + \delta) - \omega_{\delta,n/s_n}^{1/10}.
\]

**Proof.** For $y_n = a_b - b - \delta$, we decompose
\[
    1\{\#\{2 \leq i \leq n/s_n : \varepsilon_i > a_b + \varepsilon_1\} \leq \rho - 1\}
    \leq 1\{\#\{2 \leq i \leq n/s_n : \varepsilon_i > y_n\} \leq \rho - 1\} + 1\{a_b + \varepsilon_1 > y_n\}.
\]
We have $P(a_b + \varepsilon_1 > y_n) = F(y_n - a_b) = F(-b - \delta)$, so that it is enough to upper bound
\[
    p := P[\#\{2 \leq i \leq n/s_n : \varepsilon_i > y_n\} \leq \rho - 1].
\]
Noting that $B_n = \#\{2 \leq i \leq n/s_n : \varepsilon_i > y_n\}$ is binomially distributed with parameters $[n/s_n] - 1$ and $F(y_n)$, Bernstein’s inequality (Lemma 5) tells us that for $m_n := E[B_n] = ([n/s_n] - 1)F(y_n)$ we have
\[
    P[B_n \leq m_n/2] \leq \exp\left(-\frac{m_n^2/8}{m_n + m_n/6}\right) \leq \exp(-m_n/10).
\]
This bound is valid for the probability $p$ above as soon as $\rho \leq m_n/2$ with $m_n = \log(1/\omega_{\delta,n/s_n})$. □

**Lemma 8.**
1. There exists $C = C(\zeta)$ such that for any $\rho_n \geq 1$, $n/s_n \geq C$ and $b \in \mathbb{R}$,
\[
    p_n(b, \rho_n) \leq F(b) + e^{-\rho_n}.
\]
2. If $\rho = 1$, for any $\delta > 0$ we have for $n/s_n \geq C = C(\delta, \zeta)$, and any $b \in \mathbb{R}$
\[
    p_n(b, 1) \leq F(b - \delta) + e^{-\delta(\log(n/s_n))^{1-\zeta}}.
\]

**Proof.** For some $\delta \geq 0$ let $y_n = a_b - b + \delta$. For any integer $\rho \geq 1$, we may decompose
\[
    1\{\#\{2 \leq i \leq n/s_n : \varepsilon_i > a_b + \varepsilon_1\} \leq \rho - 1\} \geq 1\{\#\{2 \leq i \leq n/s_n : \varepsilon_i > y_n\} \leq \rho - 1\} 1\{\varepsilon_1 > \delta - b\};
\]
so that
\[
    1 - p_n(b, \rho) = P[\#\{2 \leq i \leq n/s_n : \varepsilon_i > a_b + \varepsilon_1\} \leq \rho - 1]
    \geq P[\#\{2 \leq i \leq n/s_n : \varepsilon_i > y_n\} \leq \rho - 1, \varepsilon_1 > \delta - b]
    \geq 1 - P[\#\{2 \leq i \leq n/s_n : \varepsilon_i > y_n\} \leq \rho - 1]P[\varepsilon_1 > \delta - b],
\]
using the independence of the $\varepsilon_i$s. Note that $P[\varepsilon_1 > \delta - b] = F(\delta - b) = 1 - F(b - \delta)$ and that $\#\{2 \leq i \leq n/s_n : \varepsilon_i > y_n\}$ is binomially $\text{Bin}([n/s_n] - 1, F(y_n))$ distributed.
By Lemma 23, for $x \geq 1$ we have
\[
\bar{F}(x) \leq \frac{f(x)}{x^{\zeta - 1}},
\]
so that, writing $a_n^* = (\zeta \log(n/s_n))^{1/\zeta} = y_n - \delta$ and appealing also to Lemma 24 and the fact from the definition (15) that $f(a_n^*) = L_{\zeta}^{-1}(s_n/n)$, we have as $n \to \infty$
\[
\frac{n}{s_n} \bar{F}(y_n) \leq \frac{f(a_n^* + \delta)}{(a_n^* + \delta)^{\zeta - 1} f(a_n^*)} \leq \exp(-\delta |a_n^*|^{\zeta - 1} (a_n^* + \delta)^{-(\zeta - 1)}).
\]

For the first claim we take $\delta = 0$ and note the upper bound above equals $(a_n^*)^{-(\zeta - 1)}$ which tends to zero as $n/s_n$ tends to infinity. There exists $C = C(\zeta) \geq 0$ such that for $n/s_n \geq C$ we have $\frac{n}{s_n} \bar{F}(y_n) \leq 1/6 \leq \rho_n/6$, hence Bernstein’s inequality (Lemma 5) implies
\[
P[\text{Bin}([n/s_n] - 1, \bar{F}(y_n)) \geq \rho_n] \leq e^{-\rho_n}.
\]

Putting the previous bounds together leads to the first assertion of the Lemma, as
\[
p_n(b, \rho_n) \leq 1 - (1 - \bar{F}(b))(1 - e^{-\rho_n}) \leq \bar{F}(b) + e^{-\rho_n}.
\]

When $\rho = 1$, we instead directly bound from below the probability
\[
P[\text{Bin}([n/s_n] - 1, \bar{F}(y_n)) = 0] = (1 - \bar{F}(y_n))^{[n/s_n] - 1} \geq 1 - 2([n/s_n] - 1) \bar{F}(y_n),
\]
where one uses $(1 - u)^n \geq 1 - 2au$, valid for any $u \in [0, 1]$ and any $a \geq 1$. Feeding in our bound for $\bar{F}(y_n)$, we see that for $n/s_n$ larger than some constant $C$ we have
\[
P[\text{Bin}([n/s_n] - 1, \bar{F}(y_n)) = 0] \geq 1 - \exp(-\delta (\zeta \log(n/s_n))^{1-\zeta^{-1}}).
\]

One deduces the second assertion of the Lemma, as
\[
p_n(b, 1) \leq 1 - (1 - \bar{F}(b - \delta))(1 - \exp(-\delta (\zeta \log(n/s_n))^{1-\zeta^{-1}})) \leq \bar{F}(b - \delta) + \exp(-\delta (\zeta \log(n/s_n))^{1-\zeta^{-1}}).
\]

**Appendix C: An empirical Bayes multiple testing procedure**

In this section we define concretely a Bayesian ‘ℓ-values procedure’ and show that it achieves the minimax risk adaptively, proving Theorem 5(i) – and consequently also (a part of) Theorem 2 – as well as Theorem 7. Results in this section can be seen as generalisations of Lemmas 5, 7 and 9 in [1] and corresponding earlier results in [17].

**C.1. Definitions**

For $w \in (0, 1)$, let $\Pi_w = \Pi_{w, \gamma}$ denote a spike and slab prior for $\theta$, where, for $\mathcal{G}$ a distribution with density $\gamma$,
\[
\Pi_w = ((1 - w)\delta_0 + w \mathcal{G})^{\otimes n}.
\]

That is, the coordinates of a draw $\theta'$ from $\Pi_w$ are independent, and are either exactly equal to 0, with probability $(1 - w)$, or are drawn from the ‘slab’ density $\gamma$. When the Bayesian model holds, the data $X$ follows a mixture distribution, with each coordinate $X_i$ independently having density $(1 - w)\phi + wg$, where $g$ denotes the convolution $\phi \ast \gamma$. [In
keeping with the rest of this paper, and in contrast to many papers on empirical Bayesian procedures including [1, 17], we will reserve \( \theta \) for the “true” parameter under which we analyse the performance of procedures \( \varphi \), and so we use the notation \( \theta' \) to denote a draw from the prior/posterior.] We consider a ‘quasi-Cauchy’ alternative as in [34], where \( \gamma \) is defined implicitly such that

\[
g(x) = (2\pi)^{-1/2}x^{-2}(1 - e^{-x^2/2}), \quad x \in \mathbb{R}.
\]  

(45)

The posterior distribution \( \Pi_w(\cdot \mid X) \) can be explicitly derived hence, taking an empirical Bayes approach, one may estimate \( \hat{w} \) by maximising the log-likelihood

\[
\hat{w} = \arg\max_{w \in [1/n, 1]} L(w),
\]

(46)

\[
L(w) = \sum_{i=1}^{n} \log \phi(X_i) + \sum_{i=1}^{n} \log(1 + w\beta(X_i)), \quad \beta(x) := \frac{g(\phi(x))}{\phi(x)} - 1.
\]

(47)

Note that a maximiser can be seen to exist under the current assumptions by taking derivatives (see Lemma 12).

For an arbitrary level \( t \in (0, 1) \), we consider the multiple testing procedure given by thresholding the posterior probabilities of coming from the null (also known as ‘\( \ell \)-values’),

\[
\varphi^\hat{\ell}(X) = (1\{\ell_{i,w}(X) < t\})_{i \leq n},
\]

(48)

\[
\ell_{i,w}(X) = \Pi_w(\theta' = 0 \mid X) = \frac{(1 - w)\phi(X_i)}{(1 - w)\phi(X_i) + wg(X_i)}.
\]

(49)

Let us gather various other definitions as in [17] and [33]. Useful properties of these quantities are given in Appendix C.4.

\[
\beta(x, w) = \frac{\beta(x)}{1 + w\beta(x)}, \quad x \in \mathbb{R}, \ w \in (0, 1)
\]

(50)

\[
\tilde{m}(w) = -E_{\theta_1=0}[\beta(X_1, w)], \quad w \in (0, 1)
\]

(51)

\[
m_1(\tau, w) = E_{\theta_1=\tau}[\beta(X_1, w)], \quad \tau \in \mathbb{R}, \ w \in (0, 1)
\]

(52)

\[
m_2(\tau, w) = E_{\theta_1=\tau}(\beta(X_1, w)^2), \quad \tau \in \mathbb{R}, \ w \in (0, 1)
\]

(53)

\[
\xi(u) = (\phi/g)^{-1}, \quad u \in (0, (\phi/g)(0))
\]

(54)

\[
\zeta(u) = \beta^{-1}(1/u), \quad u \in (0, 1)
\]

(55)

C.2. Proof of Theorem 5(i)

The lower bound is given by Theorem 4, and it remains to prove that

\[
sup_{\theta \in \Theta_b} \Re(\theta, \varphi^\hat{\ell}) \leq \Lambda_\infty + o(1).
\]

Step 1 (concentration of \( \hat{w} \)) We construct \( w_- \leq w_+ \) satisfying \( w_- \asymp w_+ \asymp (s_n/n)(\log(n/s_n))^{1/2} \) such that for \( \theta \in \Theta_b \) with \( \Lambda_\infty < 1 \),

\[
P_{\theta}(\hat{w} \notin (w_-, w_+)) = o(1).
\]
For a constant $\nu \in (0, 1/2)$ we let $w_-, w_+$ be the (almost surely unique) solutions to

\begin{align}
\sum_{i \in S_\theta} m_1(\theta_i, w_-) &= (1 + \nu)(n - s_n)\tilde{m}(w_-), \\
\sum_{i \in S_\theta} m_1(\theta_i, w_+) &= (1 - \nu)(n - s_n)\tilde{m}(w_+),
\end{align}

whose existence (for $n$ large) is yielded by Lemma 9. By Lemma 10, for a constant $c > 0$ we have

\[ P_\theta(\tilde{w} \in (w_-, w_+)) \geq 1 - e^{-cs_n} = 1 - o(1). \]

**Step 2 (FNR control)** Note that $\ell_{i, w}$ monotonically decreases as $w$ increases (see Lemma 12), so that on the event $\tilde{w} \in (w_-, w_+)$ we have $\ell_{i, w_+} \leq \ell_{i, \tilde{w}} \leq \ell_{i, w_-}$.

We have (see Lemma 18)

\[ \xi(u) \leq (2 \log(1/u) + 2 \log \log(1/u) + 6 \log 2)^{1/2}. \]

The right side is decreasing in $u$ hence, using that $w_- \geq s_n/n$ for $n$ large enough, we have for any $t \in (0, 1)$, a constant $c = c(t)$ and a sequence $\eta_n = \eta_n(t) \to 0$

\[ \xi(t w_-/2) \leq \sqrt{2 \log(n/s_n) + 2 \log \log(n/s_n)} + c = \sqrt{2 \log(n/s_n)} + \eta_n. \]

We deduce, using that $\xi(u) = (\phi/g)^{-1}(u)$ is decreasing (Lemma 12), that if $|X_i| \geq \sqrt{2 \log(n/s_n)} + \eta_n$ then

\[ \ell_{i, w_-} = \left(1 + \frac{w_-}{1 + w_-} \frac{g(|X_i|)}{\phi} \right)^{-1} \leq \left(1 - \frac{w_-}{2\xi^{-1}(|X_i|)} \right)^{-1} \leq t, \]

hence, using that $\Phi$ is Lipschitz, there exists a sequence $\delta_n \to 0$ not depending on $\theta$ such that

\[ P_\theta(\ell_{i, w_-} \geq t) \leq P(-\varepsilon_i \geq \theta_i - \sqrt{2 \log(n/s_n)} - \eta_n) \leq \Phi(\theta_i - \sqrt{2 \log(n/s_n)}) + \delta_n. \]

We deduce, writing $S = \#\{i \in S_\theta : \ell_{i, \tilde{w}} < t\}$ and appealing to Lemma 3, that we may upper bound $s_n - S$ on the event $\tilde{w} \geq w_-$ by a variable $N$ following a Poisson binomial distribution with parameter vector $p = (\Phi(b_j) + \delta_n)_{j \leq s_n}$. (Note that $a_n = \sqrt{2 \log(n/s_n)} + b$ here because we are in the Gaussian case $\zeta = 2$.) For some sequence $\delta_n \to 0$ we see that the false negative rate is then

\[ \text{FNR}(\theta, \varphi, \tilde{\ell}) = s_n^{-1}E_\theta[s_n - S] \leq P_\theta(\tilde{w} < w_-) + s_n^{-1}E[N] \leq \Lambda_\infty + \delta_n'. \]

A concentration argument further yields that on an event of probability tending to 1,

\[ S \geq s_n(1 - \Lambda_\infty)/2, \]

which will be used in the next step. Indeed observe, using the assumption $1 - \Lambda_\infty > 0$, that for $n$ large

\[ EN = s_n(\Lambda_n(b) + \delta_n) \leq s_n\Lambda_\infty + (1/4)s_n(1 - \Lambda_\infty). \]

Using this bound and Bernstein’s inequality (Lemma 5) yields

\[ P(N > s_n\Lambda_\infty + \frac{1}{4}s_n(1 - \Lambda_\infty)) \leq P(N - EN > \frac{1}{4}s_n(1 - \Lambda_\infty)) \to 0, \]

hence

\[ P_\theta(S < s_n(1 - \Lambda_\infty)/2) = P_\theta(s_n - S > s_n\Lambda_\infty + s_n(1 - \Lambda_\infty)/2) \leq P(N > s_n\Lambda_\infty + \frac{1}{2}s_n(1 - \Lambda_\infty)) + P_\theta(\tilde{w} < w_-) \to 0. \]
Step 3 (FDR control) Let $V$ denote the number of false positives,

$$V = \#\{i \notin S_\theta : \ell_{i,w} < t\}.$$  

By (59), let $A$ be an event of probability tending to 1 on which $S \geq s_n(1 - \Lambda_\infty)/2$ and $\tilde{w} < w_+$. Define

$$V' = \#\{i \notin S_\theta : \ell_{i,w_+} < t\}.$$  

Using monotonicity and applying Jensen’s inequality to the convex function $x \mapsto x/(a+x)$ we obtain

$$\text{FDR}(\theta, \varphi^\ell) = \mathbb{E}_\theta \left[ \frac{V}{(V + S) \wedge 1} \right] \leq \mathbb{E}_\theta \left[ \frac{V}{V + S} 1_A \right] + \mathbb{P}(A^c)$$

$$\leq \mathbb{E}_\theta \left[ \frac{V'}{V' + s_n(1 - \Lambda_\infty)/2} \right] + o(1) \leq \frac{\mathbb{E}_\theta V'}{\mathbb{E}_\theta V' + s_n(1 - \Lambda_\infty)/2} + o(1). \tag{60}$$

Next, Lemma 13 yields

$$\mathbb{E}_\theta V' = (n - s_n)P_\theta(\ell_{i,w_+} \leq t) \leq 2(n - s_n)r(w_+, t)\xi(r(w_+, t))^{-3}, \tag{61}$$

where $r(w_+, t) = w_+ t(1 - w_+)^{-1}(1 - t)^{-1} \sim w_+$. We note that $\xi(u) \sim (2 \log(1/u))^{1/2}$ as $u \to 0$ by Lemma 18, so that using $w_+ \asymp (s_n/n)(\log n/s_n)^{1/2}$, we have $
\mathbb{E}_\theta V' \lesssim s_n \log(n/s_n)^{-1} = o(s_n)$. Since we are assuming $\Lambda_\infty < 1$, this yields the bound

$$\text{FDR}(\theta, \varphi^\ell) \leq \frac{o(s_n)}{o(s_n) + s_n(1 - \Lambda_\infty)/2} + o(1) = o(1).$$

Combined with the bound on the false negative rate, this concludes the proof.

Lemma 9. Under the conditions of Theorem 5(i), there exist solutions $w_- \leq w_+$ to (56) and (57) respectively. Moreover, these solutions are almost surely unique and satisfy

$$w_+ \asymp s_n(n - s_n)^{-1}\tilde{m}(w_+)^{-1} \asymp s_n(n - s_n)^{-1}(\log n/s_n)^{1/2} \asymp (s_n/n)(\log n/s_n)^{1/2}.$$  

Proof. We follow the proof of [1, Lemma 5], adapting to allow for the weaker and mixed signals considered here, and taking advantage of not targeting a rate of convergence to simplify some aspects. We claim that, for some constants $c, C > 0$,

$$\sum_{i \in S_\theta} m_1(\theta_i, c(n/s_n)(\log n/s_n)^{1/2}) > (1 + \nu)(n - s_n)\tilde{m}(c(n/s_n)(\log n/s_n)^{1/2}) \tag{62}$$

$$\sum_{i \in S_\theta} m_1(\theta_i, C(n/s_n)(\log n/s_n)^{1/2}) < (1 - \nu)(n - s_n)\tilde{m}(C(n/s_n)(\log n/s_n)^{1/2}), \tag{63}$$

at least for $n$ large enough. It will follow by the intermediate value theorem that there exist unique $w_-, w_+$ solving (56) and (57) respectively, and satisfying $(s_n/n)(\log n/s_n)^{1/2} \lesssim w_- \leq w_+ \lesssim (s_n/n)(\log n/s_n)^{1/2}$, since $\tilde{m}$ is continuous, increasing and non-negative and $m_1(\tau, \cdot)$ is continuous and decreasing for each fixed $\tau$ (see Lemma 15).

To prove the claim, note that also by Lemma 15 we have, for some $c_0, C_0 > 0$, any $\mu \in \mathbb{R}$ and asymptotically as $w \to 0$,

$$c_0(\log(1/w))^{-1/2} \leq \tilde{m}(w) \leq C_0(\log(1/w))^{-1/2},$$

$$m_1(\mu, w) \leq 1/w.$$
It follows that
\[
\sum_{i \in S_0} m_1(\theta_i, C(s_n/n)(\log n/s_n))^{1/2} \leq C^{-1} n(\log(n/s_n))^{-1/2},
\]

\[
(1 - \nu)(n - s_n) \tilde{m}(C(s_n/n)(\log n/s_n))^{1/2} \geq (n - s_n)(\log(n/s_n))^{-1/2},
\]
where the suppressed constant can be chosen independently of \(C > 0\) and \(\nu \in (0, 1/2)\), for \(n\) larger than some \(N = N(C)\). The inequality (63) follows, for \(C\) large enough.

For the lower bound on (62) we observe that Lemma 15 further yields, for some constants \(\omega_0 \in (0, 1)\), \(M_0 > 0\) and all \(n \leq \omega_0\), \(\mu \geq M_0\),

\[
m_1(\mu, w) \geq c_1 \frac{\Phi(\zeta(w) - \mu)}{w} T_\mu(w),
\]
where \(T_\mu(w)\) is a function bounded below by 1. Recall that \(\zeta\) is a decreasing function satisfying \(\zeta(w) \leq (2 \log(1/w) + 2 \log(1/w) + C)^{1/2}\) for a constant \(C > 0\) (see Lemma 17). In particular, note by a Taylor expansion that \(\zeta(c(s_n/n)(\log n/s_n))^{1/2} \leq \sqrt{2 \log(n/s_n) + o(1)}\) and hence, using also that \(\Phi\) is Lipschitz, for some \(C_1 > 0\) and some \(o(1)\) not depending on \(b\) or \(c\) we have for \(n\) large and \(a_{b_j} \geq M_0\)

\[
m_1(s_n/n)(\log(n/s_n))^{1/2} \geq C_1^{-1}(n/s_n)(\log(n/s_n))^{-1/2} (\Phi(-b_j) - o(1)).
\]
Observe also, recalling the definition (50), that \(\beta(x, w) \geq -|\beta(0)|(1 - |\beta(0)|)\) is lower bounded by a constant, hence the same is true of \(m_1(\tau, w)\) for all \(\tau\) and \(w\) (including \(\tau < M_0\)). Let \(\sigma\) denote a bijection taking \(i \in S_0\) to \(\sigma(i) = j \in \{1, \ldots, s_n\}\) such that \(|\theta_i| \geq b_j\). Noting that \(\Phi(-b_j) = o(1)\) uniformly in \(j\) such that \(a_{b_j} \leq M_0\), we deduce that for some constants \(C_2, C_3 > 0\) not depending on \(c\)

\[
\sum_{i \in S_0} m_1(\theta_i, c_s n(\log(n/s_n))^{1/2}) \leq \sum_{i \in S_0, a_{\sigma(i)} < M_0} m_1(\theta_i, c_s n(\log(n/s_n))^{1/2}) + \sum_{i \in S_0, a_{\sigma(i)} \geq M_0} m_1(\theta_i, c_s n(\log(n/s_n))^{1/2})
\]
\[
\geq -C_2 s_n + \frac{c_2 n}{c_s n(\log(n/s_n))^{1/2}} \sum_{j \leq s_n, a_{b_j} \geq M_0} (\Phi(-b_j) - o(1))
\]
\[
\geq -C_2 s_n + \frac{c_2 n}{c_s n(\log(n/s_n))^{1/2}} (\sum_{j \leq s_n} (\Phi(-b_j) - o(1)))
\]
\[
= -C_2 s_n + \frac{c_2 n}{c (\log n/s_n)^{1/2}} (1 - \Lambda_\infty - o(1)).
\]
Note that \(s_n = o(n)\) implies \(s_n = o(n/(\log(n/s_n)))^{1/2}\). Since \(\Lambda_\infty < 1\) we deduce that the left side of (62) is lower bounded by a constant not depending on \(c\) multiplied by \(c^{-1} n(\log n/s_n)^{-1/2}\), while the right side, using the earlier upper bound on \(\tilde{m}(w)\), is upper bounded by a constant not depending on \(c\) multiplied by \(n \log(n/s_n)^{-1/2}\). Taking \(c\) small enough yields the claim.

\begin{lemma}
Under the assumptions of Lemma 9, define \(\hat{w}, w_-\) and \(w_+\) as in (46),(56) and (57) respectively. Then there exists \(c > 0\) such that\n
\[\sup_{\theta \in \Theta} P_{\theta}(\hat{w} \not\in (w_-, w_+)) \leq e^{-c s_n}.\]
\end{lemma}
Proof. We adapt the proof of [17, Lemma S-4] or [1, Lemma 7] to the current setting with no polynomial sparsity and multiple signal levels $b_j$.

Let us prove, for a constant $c > 0$ depending only on an upper bound for $\nu < 1$, that

$$P_\theta(\hat{w} < w_-) \leq e^{-cn}.$$ 

The proof that $P_\theta(\hat{w} > w_+) \leq e^{-cn}$ is similar (see also the similar proof below of Lemma 11), yielding the claim up to a factor of 2 which can be removed by initially considering a $c' > c$.

Let $S = L'$ be the score function, that is, the derivative of the likelihood $L$ defined in (47). Since $\hat{w}$ maximises $L(w)$, necessarily $S(\hat{w}) \leq 0$ or $\hat{w} = 1$. If $\hat{w} < w_-$ then only the former may hold, so that applying the strictly monotonic function $S$ (Lemma 12) we obtain $\{\hat{w} < w_-\} = \{S(w_-) < S(\hat{w})\} \subseteq \{S(w_-) < 0\}$. Hence,

$$P_\theta(\hat{w} < w_-) \leq P_\theta(S(w_-) < 0) = P_\theta(S(w_-) - E_\theta S(w_-) < -E_\theta S(w_-))$$

$$= P_\theta\left(\sum_{i=1}^n W_i < -E\right),$$

where we have introduced the notation $W_i = \beta(X_i, w_-) - m_1(\theta_i, w_-)$ and $E = E_\theta S(w_-) = \sum_{i=1}^n m_1(\theta_i, w_-)$. For $n$ large, $|W_i| \leq M := 2/w_-$ a.s. (see Lemma 14), so that we may scale the variables $W_i$ to apply the Bernstein inequality (Lemma 5) and obtain

$$P_\theta(\hat{w} < w_-) \leq e^{-0.5E^2/(V_2 + ME/3)},$$

where $V_2 = \sum_{i=1}^n \text{Var}(W_i) \leq \sum_{i=1}^n m_2(\theta_i, w)$, for $m_2(\theta_i, w) = E_\theta(\beta(X_i, w)^2)$. In view of the definition (56) of $w_-$, we have

$$E = \sum_{i \in S_0} m_1(\theta_i, w_-) - (n - s_n)\tilde{m}(w_-) = \nu(n - s_n)\tilde{m}(w_-).$$

We also note, using the bounds on $m_2$ in Lemma 16 that for some constants $C, M_0 > 0$ and $n$ larger than some universal threshold,

$$V_2 \leq \sum_{i \leq n: |\theta_i| > M_0} m_2(\theta_i, w_-) + \sum_{i \leq n: |\theta_i| \leq M_0} m_2(\theta_i, w_-)$$

$$\leq \frac{C}{w_-} \sum_{i \in S_0: |\theta_i| > M_0} m_1(\theta_i, w_-) + C \sum_{i \leq n: |\theta_i| \leq M_0} \frac{\Phi(\zeta(w_-) - |\theta_i|)}{w_-^2}$$

$$\leq \frac{C}{w_-} \sum_{i \in S_0: |\theta_i| \leq M_0} m_1(\theta_i, w_-) - \frac{C}{w_-} \sum_{i \in S_0: |\theta_i| \leq M_0} m_1(\theta_i, w_-) + C s_n \frac{\Phi(\zeta(w_-) - M_0)}{w_-^2} + C n \frac{\Phi(\zeta(w_-))}{w_-^2},$$

with $\zeta$ defined as in (55). For the first term we use the definition (56) of $w_-$, and for the second we use that $m_1(\theta_i, w_-)$ is bounded below by a (negative) constant. By a standard normal tail bound (included in Lemma 23), the bounds on $w_-$, the fact that $\sqrt{\log(1/w_-)} \leq \zeta(w_-) \lesssim \sqrt{\log(n/s_n)}$ as $n \to 0$ (Lemmas 9 and 17) and that $\tilde{m}(w_-) \asymp \zeta(w_-)^{-1}$ (Lemma 15), we deduce for some constant $M_1$ that

$$\Phi(\zeta(w_-) - M_0) \asymp \frac{\Phi(\zeta(w_-) - M_0)}{\zeta(w_-)} \lesssim w_- \tilde{m}(w_-) e^{M_1\sqrt{\log(n/s_n)}}.$$
so that, bounding $M_1 \sqrt{\log(n/s_n)}$ by $\log(n/s_n)$, for $n$ large the third term is upper bounded by a constant multiple of

$$s_n w\tilde{m}(w_\cdot)e^{M_1 \sqrt{\log(n/s_n)}} \leq n w\tilde{m}(w_\cdot).$$

For the fourth term, by the same normal tail bound and the definition of $\zeta$ we have $\tilde{\Phi}(\zeta(w_\cdot)) \asymp \phi(\zeta(w_\cdot))/\zeta(w_\cdot) \asymp w_\cdot g(\zeta(w_\cdot))/\zeta(w_\cdot)$, which is of order $w_\cdot(\zeta(w_\cdot))^{-3} \asymp w_\cdot/\zeta(w_\cdot)^3$, hence of order $w_\cdot\tilde{m}(w_\cdot)/\zeta(w_\cdot)^2$ because $\tilde{m}(w_\cdot) \asymp \zeta(w_\cdot)^{-1}$. We deduce, recalling that $s_n = o(n)$ implies automatically that $s_n \ll n(\log(n/s_n))^{-1/2} \asymp n\tilde{m}(w_\cdot)$,

$$V_2 \lesssim n w\tilde{m}(w_\cdot) + w\tilde{m}(w_\cdot)/\zeta(w_\cdot)^2 \lesssim n w\tilde{m}(w_\cdot),$$

so that

$$\frac{V_2 + M E/3}{E^2} \lesssim \frac{n w\tilde{m}(w_\cdot)}{(\nu(n - s_n)\tilde{m}(w_\cdot))^2} + \frac{1}{\nu w_\cdot(n - s_n)\tilde{m}(w_\cdot)} \lesssim \frac{1}{\nu^2 n w\tilde{m}(w_\cdot)}.$$

This implies that $P_{\theta}(\tilde{w} < w_\cdot) \leq e^{-c\nu^2 n w\tilde{m}(w_\cdot)}$ for some constant $c > 0$. Now, by Lemma 9, we have $n w\tilde{m}(w_\cdot) \asymp s_n$, yielding the desired bound on the probability.

We adapt Lemmas 9 and 10 slightly to apply even when $\Lambda_\infty = 1$ or $|S_\theta| < s_n$, in order to accommodate the settings of Theorems 7 and 8.

**Lemma 11.** Consider the setting of Theorem 8. For constants $C, D > 0$ let $\omega_1 = C(\sigma_n/n)(\log(n/\sigma_n))^{1/2}$, where $\sigma_n = \max(s_n, D \log n)$. Then if $C$ and $D$ are suitably large, for all $n$ large enough we have

$$\sup_{\theta \in \Theta_\theta'} P_{\theta}(\tilde{w} > \omega_1) \leq n^{-1}. \quad (64)$$

**Proof.** We begin with a corresponding upper bound to (63). Using the bounds, found in Lemma 15, that for some $c_0, \omega_0 > 0$, any $\mu \in \mathbb{R}$ and any $w < \omega_0$

$$\tilde{m}(w) \geq c_0 (\log(1/w))^{-1/2}, \quad m_1(\mu, w) \leq 1/w,$$

we note that $\tilde{m}(\omega_1) \asymp (\log(1/\omega_1))^{-1/2} \asymp (\log(n/\sigma_n))^{-1/2}$ because $n/\sigma_n \to \infty$, and we further deduce that for for $n$ large enough, for any $\theta \in \Theta_\theta'$ with support $S_\theta$ of size $s_\theta \leq s_n$ we have

$$\sum_{i \in S_\theta} m_1(\theta_i, \omega_1) < (1 - \nu)(n - s_\theta)\tilde{m}(\omega_1), \quad (65)$$

provided the constant $C$ in the definition of $\omega_1$ is large enough.

We now argue as in proving Lemma 10. Let $S = L'$, for $L$ as in (47), denote the score function. Necessarily $S(\tilde{w}) \geq 0$ or $\tilde{w} = 0$, and we deduce that $\{\tilde{w} > \omega_1\} \subset \{S(\omega_1) > 0\}$, hence

$$P_{\theta}(\tilde{w} > \omega_1) \leq P_{\theta}(S(\omega_1) - E_0 S(\omega_1) > -E_0 S(\omega_1)) = P_{\theta} \left( \sum_{i=1}^{n} W_i > -E \right),$$

where $W_i = \beta(X_i, \omega_1) - m_1(\theta_i, \omega_1)$ and $E = E_0 S(\omega_1) = \sum_{i=1}^{n} m_1(\theta_i, \omega_1)$. For $n$ large $|W_i| \leq M := 2/\omega_1$ a.s. (see Lemma 14), so that we may scale the variables $W_i$ to apply the Bernstein inequality (Lemma 5) and obtain

$$P_{\theta}(\tilde{w} > \omega_1) \leq e^{-0.5E^2/(V_2 + ME/3)},$$

as required.

where $V_2 = \sum_{i=1}^n \text{Var}(W_i) \leq \sum_{i=1}^n m_2(\theta_i, \omega_1)$, for $m_2(\theta_i, w) = E_\theta(\beta(X_i, w)^2)$. In view of (65),

$$-E = (n-s_\theta)\bar{m}(\omega_1) - \sum_{i \in S_\theta} m_1(\theta_i, \omega_1) > \nu(n-s_\theta)\bar{m}(\omega_1).$$

Arguing as in the proof of Lemma 10, one obtains

$$V_2 \lesssim \omega_1^{-1} \sum_{i \in S_\theta} m_1(\theta_i, \omega_1) + \omega_1^{-1}s_n + n\omega_1^{-1}\bar{m}(\omega_1)/\zeta(\omega_1)^2 \lesssim n\omega_1^{-1}\bar{m}(\omega_1),$$

with the last bound following from (65). Inserting the bounds for $E, V_2$ and $M$ into the obtained bound on $P_\theta(\hat{w} > \omega_1)$ yields for a constant $c = c(\nu)$

$$P_\theta(\hat{w} > \omega_1) \leq e^{-cn\omega_1 \bar{m}(\omega_1)}.$$

Recalling that $\bar{m}(\omega_1) \asymp (\log(n/\sigma_n))^{-1/2}$ we see for $c'$ not depending on $D$ that $P_\theta(\hat{w} > \omega_1) \leq e^{-c'\sigma_n}$, and (64) follows upon choosing $D = D(c')$ large enough. \hfill \Box

**C.3. Proof of Theorem 7 (ℓ-value upper bound)**

Suppose $\theta \in \Theta_b$ and let us write the classification loss $L_{C}(\theta, \varphi^\ell)$ as the sum $V + (s_n - S)$, where as in the proof of Theorem 5 we write $V = V(\hat{w}) = \sum_{i \notin S_{\hat{\theta}}} \varphi^\ell_i$, $S = S(\hat{w}) = \sum_{i \in S_{\hat{\theta}}} \varphi^\ell_i$, with $S_{\hat{\theta}} = \{i : \theta_i \neq 0\}$ denoting the support of $\theta$. It suffices to show that uniformly over $\theta \in \Theta_b$, for some positive sequences $\delta_n \to 0$ and $\nu_n \to 0$,

$$P_\theta[s_n - S \geq (\Phi(b) + \delta_n)s_n] = o(1), \tag{66}$$

$$P_\theta[V > \nu_n s_n] = o(1). \tag{67}$$

To prove (66), for $b = (b, b, \ldots)$, we may embed the setting of Theorem 7 into that of Theorem 5(i). In the case of a fixed $b \in \mathbb{R}$, recall that we argued in proving Theorem 5 (see before (58)) that on the event $\hat{w} \geq w_-$, we have $s_n - S \leq N$, for $N$ a Poisson binomial $N$ with parameter vector $p = (\Phi(b) + \delta_n)_{\ell \leq s_n}$ and for a sequence $\delta_n \to 0$. Then (66) follows by recalling that $P(\hat{w} < w_-) \to 0$ from Lemma 10, and by applying Bernstein’s inequality (Lemma 5). In the case $b = b_n \to +\infty$ the same argument applies upon replacing $\Phi(b)$ by $\Lambda_\infty = \lim \Phi(b) = 0$. In the case $b = b_n \to -\infty$, since $S \geq 0$ we automatically have

$$P_\theta[s_n - S \geq (1 + \delta_n)s_n] = 0,$$

so that (66) holds in all cases (with suitable substitutions).

To prove (67), one could similarly use the bound (61). Here, to allow for the case $b = b_n \to -\infty$, we adapt this bound by appealing to Lemma 11, which yields the existence of constants $C, D$ such that for large $n$

$$P_\theta(\hat{w} > \omega_1) \leq n^{-1}, \quad \omega_1 = C\frac{\sigma_n}{n}(\log(n/\sigma_n))^{1/2}, \quad \sigma_n = \max(s_n, D \log n).$$

Then, writing

$$V' = \#\{i \notin S_{\hat{\theta}} : \ell_i, \omega_1 < t\},$$

we apply Lemmas 13 and 18 as in proving Theorem 5 to deduce that

$$E_\theta V' \lesssim n\omega_1 \log(1/\omega_1)^{-3/2} \lesssim \sigma_n/(\log(n/\sigma_n)).$$
where we have used that \( n/\sigma_n \to \infty \) to see that \( \log(1/\omega_1) \asymp \log(n/\sigma_n) \). When \( \sigma_n = D\log(n) \) we note that \( \sigma_n/\log(n/\sigma_n) \) is upper bounded by a constant. For some \( C' > 0 \) we therefore have

\[
E_{\theta} V \leq nP_{\theta}(\hat{\omega} > \omega_1) + E_{\theta} V' \leq 1 + E_{\theta} V' \leq C' \max(1, \frac{s_n}{\log(n/s_n)}) = o(s_n),
\]

(68)

where we have used that \( s_n \to \infty \) and \( n/s_n \to \infty \). Applying Markov’s inequality we deduce \( P_{\theta}[V > \nu_n s_n] \leq \nu_n^{-1}s_n^{-1}E_{\theta} V \) tends to zero if \( \nu_n \) tends to zero sufficiently slowly. This proves (67), including the cases \( b = b_n \to \pm \infty \), and thus concludes the proof.

Remark 7. A straightforward corollary of the above proof shows that the \( \ell \)-value procedure is sparsity preserving. It suffices to show that \( P(V + S > 2s_n) = o(1) \), or that \( P(V > s_n) = o(1) \), which follows from the proof, taking \( \nu_n = 1 \).

Remark 8. Let us provide an alternative proof of the \( \ell \)-value upper bound part of Theorem 8, similar in spirit to the just obtained in-probability bounds. Write \( S_{\theta} = \{i:\theta_i \neq 0\} \), \( s_{\theta} = |S_{\theta}| \), \( V = V(\hat{\omega}) = \sum_{i \in S_{\theta}} \phi_i^\ell \) and \( S = S(\hat{\omega}) = \sum_{i \in S_{\theta}} \phi_i^\ell \), and note as in the previous proof of Theorem 8 (in Appendix A.4) that it suffices to show

\[
\sup_{\theta \in \Theta_b'(s_n)} E_{\theta} V = o(s_n), \quad \sup_{\theta \in \Theta_b'(s_n)} E_{\theta}[s_{\theta} - S] \leq s_n \overline{\Phi}(b) + o(s_n).
\]

The first of these follows from taking a supremum in (68), whose proof we note applies for \( \theta \in \Theta_b'(s_n) \) not just \( \theta \in \Theta_b(s_n) \).

The second is trivial in the case \( b = b_n \to -\infty \) since \( 0 \leq s_{\theta} - S \leq s_{\theta} \leq s_n \) for \( \theta \in \Theta_b'(s_n) \). In the cases \( b \in \mathbb{R} \) fixed and \( b = b_n \to +\infty \), note firstly an examination of the proofs of Lemmas 9 and 10 reveals that for some constants \( c, c' > 0 \), for any \( \theta \in \Theta_b(s) \) with \( s \leq s_n \) we have

\[
P_{\theta}(\hat{\omega} < w_-) \leq e^{-cs}, \quad w_- := c'(s/n)(\log(n/s))^{1/2}.
\]

In this setting the bound (58) reads that, for some sequence \( \delta_n \to 0 \) which can be chosen independently of \( s \),

\[
E_{\theta}[s - S] \leq sP_{\theta}(\hat{\omega} < w_-) + s(\overline{\Phi}(b) + \delta_n) \leq sc^{-cs} + s(\overline{\Phi}(b) + \delta_n) \leq C(c) + s_n \overline{\Phi}(b) + o(s_n).
\]

The bound at stake then follows by taking the maximum over \( 0 \leq s \leq s_n \) in the last display.

C.4. Background material for the \( \ell \)-value procedure

We gather results from [17] which are used in the proofs for this appendix. Some of these results were originally formulated with dependence on \( g \) and a related parameter \( \kappa \in [1,2] \); as in [1], we simplify such expressions here by substituting the explicit form (45) for \( g \), which has \( \kappa = 2 \), and using the bounds \( \sup_x |g(x)| \leq 1/\sqrt{2\pi} \) and, for \( |x| \geq 2 \),

\[
x^{-2}/(2\sqrt{2\pi}) \leq g(x) \leq x^{-2}/\sqrt{2\pi}.
\]

Lemma 12. The following functions are strictly decreasing (with probability 1 in the case
of random functions).

\[
\begin{align*}
  w &\mapsto S(w) = L'(w), \\
  w &\mapsto \ell_{t,w}(X), \\
  w &\mapsto -\tilde{m}(w), \\
  w &\mapsto m_1(\tau,w), \quad \tau \in \mathbb{R} \text{ fixed}, \\
  u &\mapsto \xi(u) = (\phi/g)^{-1}(u), \\
  w &\mapsto \zeta(u).
\end{align*}
\]

These monotonicity results can be found in [17], and see [1, Lemma 4] for most proofs collected in one place.

**Lemma 13** (Proposition 3 in [17]). \( P_\theta(\ell_{t,w} \leq t) \leq 2r(w,t)\xi(r(w,t))^{-3} \), where \( r(w,t) = wt(1-w)^{-1}(1-t)^{-1} \) and \( \xi \) is as in (54).

**Lemma 14** (Lemma S-20 in [17]). Define \( \beta(x,w) \) as in (50). Then there exists \( c_1 > 0 \) such that for any \( x \in \mathbb{R} \) and \( w \in (0,1] \), \( |\beta(x,w)| \leq (\min(w,c_1))^{-1} \).

**Lemma 15** (Lemmas S-21, S-23 and S-27 in [17], and using Lemma 17 below). Define \( \tilde{m} \) and \( m_1 \) as in (51), (52). Then \( \tilde{m} \) is continuous, non-negative and increasing. For fixed \( \tau \) the function \( w \mapsto m_1(\tau,w) \) is continuous and decreasing. There exist \( \omega_0, c, c' > 0 \) such that for all \( \tau \in \mathbb{R} \) and all \( w < \omega_0 \)

\[
c(\log(1/w))^{-1/2} \leq \tilde{m}(w) \leq c'(\log(1/w))^{-1/2},
\]

\[
m_1(\tau,w) \leq 1/w.
\]

There exist constants \( M_0, C_1 > 0 \) and \( \omega_0 \in (0,1) \) such that for any \( w \leq \omega_0 \), and any \( \mu \geq M_0 \), with \( T_\mu(w) = 1 + |\mu|^{-1}|\zeta(w) - |\mu||, \)

\[
m_1(\mu,w) \geq C_1 \left( \frac{\Phi(\zeta(w) - |\mu|)}{w} \right) T_\mu(w).
\]

**Lemma 16** (Lemma S-26 and Corollary S-28 in [17]). Define \( m_2 \) as in (53). There exist constants \( C > 0 \) and \( \omega_0 \in (0,1) \) such that for any \( w \leq \omega_0 \) and any \( \mu \in \mathbb{R}, \)

\[
m_2(\mu,w) \leq C \left( \frac{\Phi(\zeta(w) - |\mu|)}{w^2} \right).
\]

There exist \( M_0, C' > 0 \) and \( \omega_0 \in (0,1) \) such that for any \( w \leq \omega_0 \) and any \( \mu \geq M_0 \)

\[
m_2(\mu,w) \leq C' \frac{m_1(\mu,w)}{w}.
\]

**Lemma 17** (Lemma S-14 in [17]). Consider \( \zeta(w) \) as in (55). Then \( \zeta(w) \sim (2\log(1/w))^{1/2} \) as \( w \to 0 \). More precisely, for constants \( c, C \in \mathbb{R} \) and for \( w \) small enough,

\[
(2\log(1/w) + 2\log\log(1/w) + c)^{1/2} \leq \zeta(w) \leq (2\log(1/w) + 2\log\log(1/w) + C)^{1/2}.
\]

**Lemma 18** (Lemma S-12 in [17]). Consider \( \xi(u) \) as in (54). Then \( \xi(u) \sim (2\log(1/u))^{1/2} \), and more precisely, for \( u \) small enough,

\[
\begin{align*}
  \xi(u) &\geq \left( 2\log(1/u) + 2\log\log(1/u) + 2\log 2 \right)^{1/2} \\
  \xi(u) &\leq \left( 2\log(1/u) + 2\log\log(1/u) + 6\log 2 \right)^{1/2}.
\end{align*}
\]
The above bounds come from Lemma S-12 in [17]

\[ \xi(u) \geq \left( -2 \log u - 2 \log \left( \sqrt{-2 \log(Cu)} \right) - \log(2\pi) \right)^{1/2} ; \]

\[ \xi(u) \leq \left( -2 \log u - 2 \log \left( \sqrt{-4 \log u} \right) - \log(2\pi) \right)^{1/2} , \]

where \( C = \sqrt{2\pi} \sup_x |g(x)| \), by using \( x^{-2}/(2\sqrt{2\pi}) \leq g(x) \leq x^{-2}/(\sqrt{2\pi}) \) for \( |x| \geq 2 \). For instance, for the lower bound we use

\[ -2 \log \left( \sqrt{-2 \log(Cu)} \right) - \log(2\pi) \geq 2 \log(\sqrt{2\pi}) + 2 \log(-2 \log(Cu)) - \log(2\pi) \]

\[ \geq 2 \log(2) + 2 \log(\log(1/u)) + \log(1/C) \]

\[ \geq 2 \log(2) + 2 \log(\log(1/u)) \]

where the last inequality holds because \( C \leq 1 \).

**Appendix D: Materials for BH procedure**

**D.1. Definition**

Recall that the BH procedure of level \( \alpha \) is defined as follows (see, e.g., [40]):

\[ \varphi_{BH}^{\alpha} = \left( 1 \{|X_i| \geq \hat{t} \} \right)_{1 \leq i \leq n} ; \quad \hat{t} = \min\{t \in \mathbb{R} \cup \{-\infty\} : \hat{G}_n(t) \geq 2F(t)/\alpha \}; \]

\[ \hat{G}_n(t) = \frac{1}{n-1} \sum_{i=1}^{n} 1\{|X_i| \geq t\}. \]

**D.2. Proof of Theorem 5(ii)**

Let us prove, under the condition \( \alpha = \alpha_n = o(1) \) with \( -\log \alpha = o((\log(n/s_n))^{1-1/\zeta}) \), that

\[ \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi_{BH}^{\alpha}) = \sup_{\theta \in \Theta_b} \mathcal{FNR}(\theta, \varphi_{BH}^{\alpha}) + o(1) = \Lambda_{\infty} + o(1). \]  

(70)

The proof of Theorem 5(ii) follows because under polynomial sparsity, the condition \( -\log \alpha = o((\log(n/s_n))^{1-1/\zeta}) \) is satisfied if \( -\log \alpha = o((\log n)^{1-1/\zeta}) \).

First, by [8, 9] we have for all \( \theta \in \Theta_b \),

\[ \text{FDR}(\theta, \varphi_{BH}^{\alpha}) = \frac{\alpha(n - |S_{\theta}|)}{n} \leq \alpha, \]

(71)

where \( S_{\theta} \) denotes the support of \( \theta \). This entails, since \( \alpha \) tends to 0,

\[ \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi_{BH}^{\alpha}) = \sup_{\theta \in \Theta_b} \mathcal{FNR}(\theta, \varphi_{BH}^{\alpha}) + o(1). \]

By Theorem 4, we have

\[ \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi_{BH}^{\alpha}) \geq \inf_{\varphi} \sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi) = \Lambda_{\infty} + o(1). \]

It remains to prove

\[ \sup_{\theta \in \Theta_b} \mathcal{FNR}(\theta, \varphi_{BH}^{\alpha}) \leq \Lambda_{\infty} + o(1). \]

(72)
To prove this bound we rely on the following inequalities: for all $t \geq 0$,

$$P_\theta(\hat{t} > t) \leq P_\theta(\hat{G}_n(t) < 2\bar{F}(t)/\alpha) = P_\theta(\hat{G}_n(t) - G_n(t) < 2\bar{F}(t)/\alpha - G_n(t)),$$

where $G_n(t) = E_\theta \hat{G}_n(t) = (1 - s_n/n)2\bar{F}(t) + n^{-1} \sum_{i \in S_\theta} (\bar{F}(t + \theta_i) + \bar{F}(t - \theta_i))$.

Let $a_n^* = (\zeta \log(n/s_n))^{1/\zeta}$. Note that $G_n(t) \geq (1 - s_n/n)2\bar{F}(t) + (s_n/n)s_n^{-1} \sum_{j=1}^{s_n} \bar{F}(t - a_n^* - b_j)$ and observe that $t \mapsto s_n^{-1} \sum_{j=1}^{s_n} \bar{F}(t - a_n^* - b_j)/\bar{F}(t)$ is a continuous increasing bijection from $[0, \infty]$ to $[1, \infty]$ by Lemma 19. Hence, denoting by $t_n^* > 0$ the only point $t > 0$ (which exists because $3/\alpha - 2 > 1$) such that $(1 - s_n/n)2\bar{F}(t) + (s_n/n)s_n^{-1} \sum_{j=1}^{s_n} \bar{F}(t - a_n^* - b_j) = 3\bar{F}(t)/\alpha$, we obtain

$$P_\theta(\hat{t} > t_n^*) \leq P_\theta(\hat{G}_n(t_n^*) - G_n(t_n^*) < -\bar{F}(t_n^*)/\alpha) \leq \exp \left( -\frac{n^{2} F^2(t_n^*)/\alpha^2}{nG_n(t_n^*) + (n/3)\bar{F}(t_n^*)/\alpha} \right) \leq \exp \left( -\frac{n^{2} F^2(t_n^*)/\alpha^2}{s_n/n + (2\alpha + 1/3)\bar{F}(t_n^*)/\alpha} \right),$$

by using Bernstein’s inequality (Lemma 5) and the fact that $G_n(t) \leq 2\bar{F}(t) + s_n/n$ for all $t$. Now, by Lemma 19 (using the conditions on $\alpha = \alpha_n$), we have that $t_n^* \to \infty$ with $t_n^* - a_n^* = o(1)$. Hence, with $C_f = C_f(\zeta)$ denoting the Lipschitz constant of the noise c.d.f. $F$,

$$s_n^{-1} \sum_{j=1}^{s_n} |\bar{F}(t_n^* - a_n^* - b_j) - \bar{F}(b_j)| \leq C_f |t_n^* - a_n^*|.$$

This entails

$$s_n^{-1} \sum_{j=1}^{s_n} \bar{F}(t_n^* - a_n^* - b_j) \to 1 - \Lambda_\infty,$$

which is positive by assumption. Since by definition of $t_n^*$ and $\alpha = o(1)$ we have $\bar{F}(t_n^*)/\left(s_n^{-1} \sum_{j=1}^{s_n} \bar{F}(t_n^* - a_n^* - \alpha s_n/n)\right)$, this gives $\bar{F}(t_n^*) \asymp \alpha s_n/n$ and

$$P_\theta(\hat{t} > t_n^*) \leq e^{-c_n},$$

for some universal constant $c > 0$. It follows that

$$\text{FNR}(\theta, \varphi_\alpha^{BH}) \leq s_n^{-1} E_\theta \left( \sum_{i \in S_\theta} 1\{|X_i| < t_n^*\} \right) + e^{-c_n}.$$

Now, since for all $i \in S_\theta$, $P_\theta(|X_i| < t_n^*) = 1 - P_\theta(|X_i| \geq t_n^*) \leq 1 - \bar{F}(t_n^* - |\theta_i|)$,

$$s_n^{-1} E_\theta \left( \sum_{i \in S_\theta} 1\{|X_i| < t_n^*\} \right) \leq 1 - s_n^{-1} \sum_{j=1}^{s_n} \bar{F}(t_n^* - a_n^* - b_j) \to \Lambda_\infty,$$

by (73). We have shown (72), and this finishes the proof of Theorem 5(ii).
Remark 9. The following convergence rate can be obtained when $\Lambda_\infty < 1$: for a vector $b$ that has coordinates with a fixed value in $\mathbb{R}$, and taking $\alpha \propto 1/(\log(n/s_n))^{1-1/\zeta}$, we have

$$\sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi^{BH}_\alpha) - \Lambda_n(b) \lesssim 1/(\log(n/s_n))^{1-1/\zeta} + e^{-c s_n}.$$ 

Indeed, from the computations of the above proof, we get

$$\sup_{\theta \in \Theta_b} \mathcal{R}(\theta, \varphi^{BH}_\alpha) \leq \alpha + 1 - s_n^{-1} \sum_{j=1}^{s_n} F(t^*_n - a^*_n - b_j) + e^{-c s_n}$$

$$\leq \alpha + 1 - s_n^{-1} \sum_{j=1}^{s_n} F((t^*_n - a^*_n)_+ - b_j) + e^{-c s_n}$$

$$\leq \Lambda_n(b) + \alpha + O((t^*_n - a^*_n)_+) + e^{-c s_n},$$

and the claim follows from Lemma 19 point (ii).

D.3. Proof of Theorem 7 (BH upper bound)

Consider the BH procedure $\varphi^{BH}_\alpha$ (69), for $\alpha = \alpha_n = o(1)$ with $-\log \alpha = o((\log(n/s_n))^{1/2})$. Recalling the definition (25) of the classification loss $L_C$, let us prove that for any fixed real $b$ and any $\eta > 0$,

$$\sup_{\theta \in \Theta_n} P_\theta \left[ L_C(\theta, \varphi^{BH}_\alpha) \geq (\bar{\Phi}(b) + \eta)s_n \right] = o(1),$$

and that the same holds with $\bar{\Phi}(b)$ replaced by 0 when $b = b_n \to +\infty$ or by 1 when $b = b_n \to -\infty$. This entails the BH part of Theorem 7 under polynomial sparsity (11), because $-\log \alpha = o((\log n)^{1/2})$ implies $-\log \alpha = o((\log(n/s_n))^{1/2})$ in that case.

Fix $\theta \in \ell_0[s_n]$ and denote by $V = \sum_{i=1}^n 1\{\theta_i = 0\} 1\{\varphi^{BH}_i \neq 0\}$ the number of false discoveries of $\varphi^{BH}$ and by $W = \sum_{i=1}^n 1\{\theta_i \neq 0\} 1\{\varphi^{BH}_i = 0\}$ the number of false non-discoveries of $\varphi^{BH}$. Assume first $b \in \mathbb{R}$. We have

$$P_\theta \left[ L_C(\theta, \varphi^{BH}_\alpha) \geq (\bar{\Phi}(b) + \eta)s_n \right] \leq P_\theta \left[ V \geq \eta s_n/2 \right] + P_\theta \left[ W \geq s_n \bar{\Phi}(b) + \eta s_n/2 \right].$$

We have by Markov’s inequality and Lemma 20,

$$P_\theta \left[ V \geq \eta s_n/2 \right] \leq (\eta/2)^{-1} E_\theta V / s_n \leq (\eta/2)^{-1} \left( \frac{\alpha}{1-\alpha} + \frac{\alpha}{s_n(1-\alpha)^2} \right),$$

hence $\sup_{\theta \in \Theta_n} P_\theta \left[ V \geq \eta s_n/2 \right] = o(1)$. On the other hand, we have by using (74) (note that we use $\Lambda_\infty = \bar{\Phi}(b) < 1$ here)

$$P_\theta \left[ W \geq s_n \bar{\Phi}(b) + \eta s_n/2 \right] = P_\theta \left[ \sum_{i \in S_\theta} 1\{|X_i| < t^*_n\} \geq s_n \bar{\Phi}(b) + \eta s_n/2 \right]$$

$$\leq P_\theta \left[ \sum_{i \in S_\theta} 1\{|X_i| < t^*_n\} \geq s_n \bar{\Phi}(b) + \eta s_n/2 \right] + e^{-c s_n},$$

where $t^*_n > 0$ is defined as in Lemma 19, that is, as the only point $t > 0$ such that (78) holds. By (76), the random variable $\sum_{i \in S_\theta} 1\{|X_i| < t^*_n\}$ is stochastically upper-bounded
by a Binomial distribution with parameter \( s_n \) and \( q_n := 1 - \Phi(t_n^* - a_n^* - b) = \Phi(b) + o(1) \) by using Lemma 19. Applying Bernstein’s inequality (Lemma 5), we obtain

\[
P_\theta \left[ \sum_{i \in S_n} 1\{|X_i| < t_n^*\} \geq s_n \Phi(b) + \eta s_n/2 \right]
\leq P_\theta \left[ B(s_n, q_n) \geq s_n \Phi(b) + \eta s_n/2 \right]
= P_\theta \left[ B(s_n, q_n) - s_n q_n \geq s_n (\Phi(b) - q_n) + \eta s_n/2 \right]
\leq \exp \left( -0.5 s_n \frac{(\Phi(b) - q_n + \eta/2)^2}{q_n (\Phi(b) - q_n + \eta/2)/3} \right) \leq e^{-c's_n},
\]

for \( n \) large enough and some constant \( c' > 0 \) only depending on \( \eta \) (using the bound \( q_n \leq 1 \)). This gives (77) for \( b \in \mathbb{R} \).

When \( b = b_n \to +\infty \) (so \( \Phi(b) \to 0 \)) and \( \eta \in (0,1) \), the same techniques as above lead to

\[
P_\theta \left[ L_C(\theta, \varphi_\alpha^{BH}) \geq \eta s_n \right] \leq \eta/2 \left( \frac{\alpha}{1 - \alpha} + \frac{\alpha}{s_n (1 - \alpha)^2} \right) + e^{-c's_n} + P_\theta \left[ B(s_n, q_n) \geq \eta s_n/2 \right],
\]

and this again shows (77) (with \( \Phi(b) \) replaced by \( 0 \)) because

\[
P_\theta \left[ B(s_n, q_n) \geq \eta s_n/2 \right] \leq 2q_n/\eta = o(1)
\]

by Markov’s inequality.

Finally, when \( b = b_n \to -\infty \) (so \( \Phi(b) \to 1 \)), we directly use \( W \leq s_n \), so that

\[
P_\theta \left[ L_C(\theta, \varphi_\alpha^{BH}) \geq (1 + \eta)s_n \right] \leq P_\theta \left[ V \geq \eta s_n \right] \leq \eta \left( \frac{\alpha}{1 - \alpha} + \frac{\alpha}{s_n (1 - \alpha)^2} \right),
\]

by using again Markov’s inequality and Lemma 20. This shows (77) (with \( \Phi(b) \) replaced by \( 1 \)) in that case.

**D.4. Useful lemmas**

**Lemma 19.** In the setting of Theorem 5 with \( a_n^* = (\zeta \log(n/s_n))^{1/\zeta} \) (recall \( b \in \mathbb{R}^{s_n} \) is such that \( b_j > -a_n^*, \ 1 \leq j \leq s_n \)), the map

\[
\Psi : t \mapsto s_n^{-1} \sum_{j=1}^{s_n} F(t - a_n^* - b_j)/F(t)
\]

is continuous increasing from \( \mathbb{R} \) to \([1, \infty)\) (\( n \) being kept fixed).

Let \( t_n^* > 0 \) be the unique point \( t > 0 \) such that

\[
(1 - s_n/n)2F(t) + (s_n/n)s_n^{-1} \sum_{j=1}^{s_n} F(t - a_n^* - b_j) = 3F(t)/\alpha.
\]

Then the following holds when \( \Lambda_\infty < 1 \):

(i) Choosing \( \alpha \to 0 \) with \( -\log \alpha = o((\log(n/s_n))^{1-1/\zeta}) \), we have \( t_n^* - a_n^* \to 0 \).
(ii) Choosing \( \alpha \asymp 1/(\log(n/s_n))^{1-1/\zeta} \), we moreover have

\[
(t^*_n - a^*_n)^+ \lesssim 1/(\log(n/s_n))^{1-1/\zeta}.
\]

Point (ii) above allows us to obtain the convergence rate mentioned in Remark 9.

The proof of Lemma 19 takes inspiration from Section S-2.3 in [36], but requires new arguments, because here the model is not a mixture model, and different signal strengths are allowed. Throughout the proof, we denote \( t^*_n \) by \( t_n \) for short. First observe that \(- \log \mathcal{F}\) is increasing on \( \mathbb{R} \); its derivative is \( f/\mathcal{F} > 0 \). It is also strictly convex: its second derivative is \((f\mathcal{F} + f^2)/\mathcal{F}^2\) which, noting that \( f'(x) = -|x|^{\zeta-1}f(x)\) \( \sign(x) \), is given by

\[
(- \log \mathcal{F})'' = \frac{f(x)|x|^{-1}}{\mathcal{F}^2} \left( f(x)/|x|^{-1} - \sign(x) \mathcal{F}(x) \right),
\]

which is positive (using Lemma 23 for the case \( x > 0 \)), so that \( f/\mathcal{F} \) is strictly increasing and \(- \log \mathcal{F} \) is strictly convex.

Let us write \( \Psi_j(t) = \mathcal{F}(t - a^*_n - b_j)/\mathcal{F}(t) \) for \( 1 \leq j \leq s_n \). Since the function \( f/\mathcal{F} \) is increasing on \( \mathbb{R} \), for all \( t \in \mathbb{R} \), we have

\[
(\log \Psi_j)'(t) = \frac{f(t)}{\mathcal{F}(t)} - \frac{f(t - a^*_n - b_j)}{\mathcal{F}(t - a^*_n - b_j)} > 0
\]

hence each \( \Psi_j \) is increasing and thus so is \( \Psi \). Also, by Lemma 23, when \( t \) is large (\( n \) is fixed here), we have

\[
(\log \Psi_j)'(t) \geq \zeta^{-1} - (t - a^*_n - b_j)\zeta^{-1}(1 + (\zeta - 1)t^{-\zeta})
\]

\[
= \zeta^{-1}(1 - (1 - (a^*_n + b_j)/t)\zeta^{-1}(1 + (\zeta - 1)t^{-\zeta})) \geq \zeta^{-1}(\zeta - 1)(a^*_n + b_j)/2,
\]

so that \( \Psi_j(t) \) has limit \( \infty \) when \( t \to \infty \). This proves the first claim of the lemma.

Now, let us denote

\[
\tau_n = \frac{3/\alpha - 2(1 - s_n/n)}{s_n/n}; \quad \tau^*_n = \Psi(a^*_n)
\]

so that the threshold under study is \( t_n = \Psi^{-1}(\tau_n) \) while the ‘oracle’ threshold is \( a^*_n = \Psi^{-1}(\tau^*_n) \). Let us prove point (i), that is, \( t_n - a^*_n = o(1) \).

**Lower bounding** \( t_n \) Observe that \( \tau_n \geq \frac{n}{s_n \alpha} \to \infty \) so that \( t_n = \Psi^{-1}(\tau_n) \) tends to \( +\infty \). Since \( s_n^{-1} \sum_{j=1}^{s_n} \mathcal{F}(t_n - a^*_n - b_j)/\mathcal{F}(t_n) = \tau_n \), we have \( \tau_n \mathcal{F}(t_n) \leq 1 \) and thus \( t_n \geq \mathcal{F}^{-1}(\tau^*_n) \geq \mathcal{F}^{-1}(s_n \alpha/n) \), which by Lemma 23 implies

\[
t_n \geq \left( \zeta \log \left( \frac{n}{s_n \alpha} \right) - \zeta \log(\zeta L \zeta) - (\zeta - 1) \log \left( \zeta \log \left( \frac{s_n \alpha}{n} \right) - \zeta \log \log L \zeta \right) \right)^{1/\zeta}.
\]

By Taylor expanding this gives that for \( n \) large \( a^*_n - t_n \leq c \left( \log \log(n/(s_n \alpha)) \right)^{(\zeta - 1)/\zeta} \leq c \left( \log \log(n/s_n) \right)^{(\zeta - 1)/\zeta}
\]

for some constants \( c, c' > 0 \) depending only of \( \zeta \) (note that we have used \( - \log \alpha = o(\log(n/s_n)) \)).
Upper bounding $t_n$ Obtaining the upper bound on $t_n$ is slightly more involved and relies on the following inequality: for all $\eta \geq 0$,

$$\log \left( \frac{\Psi(a_n^* + \eta)}{\Psi(a_n^*)} \right) \geq \log \left( \frac{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j + \eta)}{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)} \right) + \eta (a_n^*)^{\zeta - 1}. \quad (79)$$

The proof of $(79)$ is as follows: since $-\log F$ is increasing convex, we have

$$\log \left( \frac{\Psi(a_n^* + \eta)}{\Psi(a_n^*)} \right) = \log \left( \frac{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j + \eta)}{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)} \right) - \log \left( \frac{F(a_n^* + \eta)}{F(a_n^*)} \right) \geq \log \left( \frac{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j + \eta)}{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)} \right) + \eta f(a_n^*) \frac{f(a_n^*)}{F(a_n^*)},$$

and $(79)$ comes from $f(a_n^*) \geq (a_n^*)^{\zeta - 1}$ by Lemma 23.

Now we propose to show the following claim: for any constant $B > 0$, we have

$$(t_n - a_n^*)_+ \leq \frac{(\log(\tau_n/\tau_n^*))_+ - \log \left( \frac{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j + B)}{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)} \right)}{(a_n^*)^{\zeta - 1}} \quad (80)$$

provided that

$$\log(\tau_n/\tau_n^*) \leq \log \left( \frac{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j + B)}{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)} \right) + B(a_n^*)^{\zeta - 1}. \quad (81)$$

For this, assume $t_n \geq a_n^*$ (or equivalently $\tau_n \geq \tau_n^*$), otherwise the result is trivial. We apply $(79)$ firstly with $\eta = B$ to obtain

$$\log \left( \frac{\Psi(a_n^* + B)}{\Psi(a_n^*)} \right) \geq \log \left( \frac{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j + B)}{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)} \right) + B(a_n^*)^{\zeta - 1} \geq \log(\tau_n/\tau_n^*) = \log \left( \frac{\Psi(t_n)}{\Psi(a_n^*)} \right),$$

which gives $t_n \leq a_n^* + B$. Hence applying $(79)$ a second time, now with $\eta = t_n - a_n^* \in [0, B]$, we obtain

$$\log \left( \frac{\tau_n}{\tau_n^*} \right) = \log \left( \frac{\Psi(t_n)}{\Psi(a_n^*)} \right) \geq \log \left( \frac{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j + B)}{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)} \right) + (t_n - a_n^*)(a_n^*)^{\zeta - 1},$$

which proves the claim.

Let us now compute an upper bound for $\log(\tau_n/\tau_n^*)$. By Lemma 23 and by definition of $\Psi$, we have

$$\tau_n^* = \Psi(a_n^*) = s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)/F(a_n^*) \geq s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)(a_n^*)^{\zeta - 1}/f(a_n^*)$$

$$= L \frac{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)(a_n^*)^{\zeta - 1} n}{s_n}.$$
Hence, we get, since \( \tau_n \leq 3n/(s_n\alpha) \),
\[
\log(\tau_n/\tau_n^s) \leq \log \left( \frac{3/\alpha}{L_\xi s_n^{-1} \sum_{j=1}^{s_n} F(-b_j) (a_n^*)^{\zeta-1}} \right)
\]
\[= C - \log \left( s_n^{-1} \sum_{j=1}^{s_n} F(-b_j) \right) + \log \left( \frac{1}{\alpha(\log(n/s_n))^{1-1/\zeta}} \right),
\]
for some constant \( C > 0 \) only depending on \( \zeta \). Now observe that, with \( C_f = C_f(\zeta) \) denoting the Lipschitz constant of the noise c.d.f. \( F \),
\[
\log \left( \frac{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j + B)}{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)} \right) \geq \log \left( 1 - \frac{C_f B}{s_n^{-1} \sum_{j=1}^{s_n} F(-b_j)} \right) \geq -\log 2,
\]
for \( n \) large by choosing for \( B \) any positive constant smaller than \( (1 - \Lambda_\infty)/(2C_f) \) (recall \( \lim_n s_n^{-1} \sum_{j=1}^{s_n} F(-b_j) = 1 - \Lambda_\infty > 0 \) by assumption). In particular, the condition (81) is satisfied when \( -\log \alpha = o((\log(n/s_n))^{1-1/\zeta}) \) (and for this choice of \( B \)). Hence, (80) holds.

Now, combining the latter with the bounds (82) and (83), we obtain that \( (t_n - a_n^*)_+ = o(1) \), which establishes point (i) (recall \( (a_n^* - t_n)_+ = o(1) \) from the lower bound part). In addition, when \( \alpha \approx 1/(\log(n/s_n))^{1-1/\zeta} \), we obtain by combining again (80), (82) and (83), that \( (t_n^* - a_n^*)_+ \leq 1/(\log(n/s_n))^{1-1/\zeta} \), which establishes point (ii).

**Lemma 20.** For all \( \theta \in \mathbb{R}^n \), considering the BH procedure \( \varphi_{\alpha}^{BH} \) (69) at some level \( \alpha \in (0, 1) \), then \( V = \sum_{i=1}^{n} 1\{\theta_i=0\} 1\{\varphi_{\alpha}^{BH} \neq 0\} \) satisfies
\[
E_\theta V \leq \frac{\alpha}{1-\alpha} \sum_{i=1}^{n} 1\{\theta_i \neq 0\} + \frac{\alpha}{(1-\alpha)^2}.
\]

**Proof.** To prove (84), we follow the proof of Proposition 7.2 in [36] (or alternatively, that of Lemma 7.1 in [10]). Recall that we have
\[
E_\theta V = E_\theta \sum_{i=1}^{n} 1\{\theta_i = 0\} 1\{|X_i| \geq \hat{t}\}
\]
with \( \hat{t} = \max\{t \in \mathbb{R} \cup \{\infty\} : \hat{G}_n(t) \geq 2\bar{F}(t)/\alpha \} \) and \( \hat{G}_n(t) = n^{-1} \sum_{i=1}^{n} 1\{|X_i| \geq t\} \).

Now consider the BH procedure applied to the \( X_i \)’s where we have plugged \( X_i = +\infty \) on \( S_\theta = \{i : \theta_i \neq 0\} \), that is, with threshold \( \hat{t}^0 = \max\{t \in \mathbb{R} \cup \{\infty\} : |S_\theta|/n + n^{-1} \sum_{i \notin S_\theta} 1\{|X_i| \geq t\} \geq 2\bar{F}(t)/\alpha \} \). Clearly, we have \( \hat{t}^0 \leq \hat{t} \), so that
\[
E_\theta V \leq E_\theta \sum_{i=1}^{n} 1\{\theta_i = 0\} 1\{|X_i| \geq \hat{t}^0\}.
\]

Denote \( n_0 = |S_\theta| \) for short. From classical multiple testing theory (see, e.g., Lemma 7.1 in [42]), the latter is the expected number of rejections of the step-up procedure with critical values \( (\alpha(k+|S_\theta|)/n)_{1 \leq k \leq n_0} \) and restricted to the \( p \)-value set \( \{2\bar{F}(|X_i|), i \notin S_\theta\} \). Now from Lemma 4.2 in [22] (applied with “\( n = n_0 \)”, “\( \beta = \alpha \)” and “\( \tau = \alpha/n \)”), the latter equals
\[
\alpha \frac{n_0}{n} n_0^{-1} \sum_{i=0}^{n_0-1} \binom{n_0-1}{i} (|S_\theta| + i + 1)i! (\alpha/n)^i \\
\leq \alpha \sum_{i \geq 0} (|S_\theta| + i + 1)\alpha^i = \frac{\alpha}{1-\alpha} |S_\theta| + \alpha/(1-\alpha)^2.
\]
\[\blacksquare\]
Appendix E: Sparsity preserving procedures

Here we show that a large class of procedures are sparsity preserving in the sense of Definition 1. The following lemma will be useful.

**Lemma 21.** For any $\Theta \subset \ell_0[sn]$ and any multiple testing procedure $\varphi$, we have for any $u > 1$ and $\theta \in \Theta$,

$$P_\theta \left[ \sum_{i=1}^n \varphi_i(X) > us_n \right] \leq P_\theta \left[ \text{FDP}(\theta, \varphi) > 1 - u^{-1}, \sum_{i=1}^n \varphi_i(X) \geq s_n \right] \leq \frac{u}{u-1} \text{FDR}(\theta, \varphi).$$

(85)

In particular, we have

(i) for any sequence $A = (A_n)_n$ for which $A_n \geq 2$, any sequence of procedures $\varphi$ with vanishing FDR in the sense of

$$\sup_{\theta \in \Theta} \text{FDR}(\theta, \varphi) = o(1)$$

is sparsity preserving up to the multiplicative factor $A = (A_n)_n$ over $\Theta$, that is, $\varphi \in S_A(\Theta)$ with Definition 1.

(ii) for any sequence $A = (A_n)_n$ for which $A_n \rightarrow \infty$, any sequence of procedures $\varphi$ with an FDP bounded uniformly in probability away from 1 when at least $s_n$ rejections are made, in the sense that for some fixed $t < 1$,

$$\sup_{\theta \in \Theta} P_\theta \left( \text{FDP}(\theta, \varphi) > t, \sum_{i=1}^n \varphi_i(X) \geq s_n \right) = o(1)$$

is sparsity preserving up to the multiplicative factor $A = (A_n)_n$ over $\Theta$, that is, $\varphi \in S_A(\Theta)$ with Definition 1.

An easy consequence of Lemma 21(i) is that the BH procedure taken at a level $\alpha_n \rightarrow 0$ satisfies the sparsity preserving condition on $\Theta = \ell_0[sn]$ for any sequence $A = (A_n)_n$ with $A_n \geq 2$ for all $n$ (recall that the FDR of this procedure is at most $\alpha_n$, see (71)). By (30) the same is true of the oracle procedure defined after (27). Using Theorem 1 of [17], the $\ell$-value procedure also has a vanishing FDR (under polynomial sparsity), so that Lemma 21(i) also ensures that the $\ell$-value procedure is sparsity preserving on $\Theta = \ell_0[sn]$. Recall that Remark 7 in Appendix 8 noted sparsity-preservingness of the $\ell$-value procedure on $\Theta = \ell_0[sn]$ without requiring polynomial sparsity. Finally Corollary 5 to follow uses Lemma 21(ii) to prove that the BH procedure at a fixed level $\alpha < 1$ is sparsity preserving.

**Proof.** Letting $V = \sum_{i=1}^n 1\{\theta_i = 0\} \varphi_i(X)$, we have for all $u > 1$,

$$\left\{ \sum_{i=1}^n \varphi_i(X) > us_n \right\} \subset \left\{ V > (u-1)s_n \right\} \cap \left\{ \sum_{i=1}^n \varphi_i(X) \geq s_n \right\}$$

$$\subset \left\{ \frac{V}{V + s_n} > \frac{(u-1)s_n}{(u-1)s_n + s_n} \right\} \cap \left\{ \sum_{i=1}^n \varphi_i(X) \geq s_n \right\}$$

$$\subset \left\{ \text{FDP}(\theta, \varphi) > 1 - u^{-1} \right\} \cap \left\{ \sum_{i=1}^n \varphi_i(X) \geq s_n \right\},$$
because the function $x \in [0, \infty) \mapsto \frac{x}{s + x}$ is non-decreasing. We conclude the proof of (85) by taking probabilities and using Markov’s inequality. Parts (i) and (ii) are immediate consequences.

**Corollary 5.** The BH procedure taken at a fixed level $\alpha < 1$ is sparsity preserving up to any multiplicative factor $A = (A_n)_{n}$ with $A_n \to \infty$ over any parameter set $\Theta \subset \ell_0[s_n]$.

**Proof.** From (69), letting $\hat{k} = \sum_{i=1}^{n} \varphi_i^{BH}$, we have $2\mathcal{F}(\hat{t}) = \alpha \hat{k}/n$. Thus, for all $\epsilon \in (0, 1-\alpha)$,

$$P_\theta \left( \text{FDP}(\theta, \varphi_i^{BH}) > \alpha + \epsilon, \sum_{i=1}^{n} \varphi_i^{BH} \geq s_n \right)$$

$$= P_\theta \left( \sum_{i=1}^{n} \mathbf{1}\{\theta_i = 0\} \mathbf{1}\{2\mathcal{F}(|X_i|) \leq \alpha \hat{k}/n\} > (\alpha + \epsilon) \hat{k}, \hat{k} \geq s_n \right)$$

$$\leq \sum_{k \geq s_n} P_\theta \left( \sum_{i=1}^{n} \mathbf{1}\{\theta_i = 0\} \mathbf{1}\{2\mathcal{F}(|X_i|) \leq \alpha k/n\} > (\alpha + \epsilon) k \right)$$

$$\leq \sum_{k \geq s_n} P_\theta \left( \sum_{i=1}^{n} \mathbf{1}\{U_i \leq \alpha k/n\} > (\alpha + \epsilon) k \right) \leq \sum_{k \geq s_n} P_\theta \left( \sum_{i=1}^{n} \mathbf{1}\{U_i \leq \alpha k/n\} > (\alpha + \epsilon) k \right),$$

for $U_i$, $1 \leq i \leq n$, i.i.d. variables uniformly distributed on $(0, 1)$. Now, we have by Bernstein’s inequality (Lemma 5), for any $k \geq s_n$,

$$P_\theta \left( \sum_{i=1}^{n} \mathbf{1}\{U_i \leq \alpha k/n\} > (\alpha + \epsilon) k \right) \leq P_\theta \left( \sum_{i=1}^{n} \mathbf{1}\{U_i \leq \alpha k/n\} - \alpha k > \epsilon k \right)$$

$$\leq \exp \left( -0.5 \frac{(\epsilon k)^2}{\alpha k + \epsilon k/3} \right) \leq \exp(-c\epsilon^2k),$$

for some universal constant $c > 0$ (which can be chosen independent of $\alpha$ by bounding $\alpha$ by 1). It follows that

$$P_\theta \left( \text{FDP}(\theta, \varphi_i^{BH}) > \alpha + \epsilon, \sum_{i=1}^{n} \varphi_i^{BH} \geq s_n \right) \leq \sum_{k \geq s_n} \exp(-c\epsilon^2k) \leq (1 - e^{-c\epsilon^2})^{-1} e^{-c\epsilon^2s_n} = o(1).$$

The result follows from Lemma 21(ii). 

Let us now give an example of a (randomized) non-sparsity-preserving procedure for which some FDR/FNR tradeoff is possible.

**Example 3.** Given a real $b$ and for $\varphi^*$ a procedure satisfying (12), define a (randomized) procedure $\tilde{\varphi}$ as follows: given a Bernoulli variable $Z$ with success probability $\pi \in [0, 1]$, let

$$\tilde{\varphi} = \tilde{\varphi}(X, Z) = (1 - Z)\varphi^*(X) + Z.$$

With probability $\pi$, the test $\tilde{\varphi}$ equals the trivial test 1, so rejects all null hypotheses. Then

$$\sup_{\theta \in \Theta_n} \text{FDR}(\theta, \tilde{\varphi}) = \pi \frac{n - s_n}{n} + (1 - \pi) \sup_{\theta \in \Theta_n} \text{FDR}(\theta, \varphi^*) = \pi (1 + o(1)).$$
while the FNR is controlled at level
\[
\sup_{\theta \in \Theta_h} \text{FNR}(\theta, \tilde{\varphi}) = \pi \cdot 0 + (1 - \pi) \sup_{\theta \in \Theta_h} \text{FNR}(\theta, \varphi^*) = (1 - \pi) \widetilde{T}(b)(1 + o(1)).
\]
In particular, for \( \pi > 0 \), the procedure \( \tilde{\varphi} \) trades a gain of \( \pi \widetilde{T}(b) \) in terms of the FNR with a loss of \( \pi \) in terms of the FDR. The procedure \( \varphi^* \) is not sparsity preserving, as it makes \( n \gg s_n \) rejections with probability \( \pi > 0 \).

**Appendix F: Link between conservative testing and almost full recovery**

We state a lemma to illustrate the link between conservative testing and almost full recovery.

**Lemma 22.** If a test \( \varphi \) achieves almost full recovery with respect to the Hamming loss over a subset \( \Theta_{s_n} \subset \ell_0[s_n]\{\ell_0[s_n - 1] \), that is, \( \sup_{\theta \in \Theta_{s_n}} E_{\theta} L_C(\theta, \varphi)/s_n = o(1) \), then it also allows for conservative testing over \( \Theta_{s_n} \), that is, \( \sup_{\theta \in \Theta_{s_n}} \Omega(\theta, \varphi) \rightarrow 0 \).

Conversely, suppose \( \varphi \) allows for conservative testing over \( \Theta_{s_n} \), and that the number of false discoveries \( V(\varphi) = \sum_{i: \theta_i = 0} \varphi_i \) of \( \varphi \) concentrates around its mean in that,
\[
\sup_{\theta \in \Theta_{s_n}} P_{\theta}[|V(\varphi) - E_{\theta} V(\varphi)| > E_{\theta} V(\varphi)/2] = o(1).
\]
(86)

Then \( \varphi \) achieves almost full recovery with respect to the Hamming loss over \( \Theta_{s_n} \).

**Proof of Lemma 22.** Recall that the FDP of \( \varphi \) can be written \( V/\{\max(V + S, 1)\} \), where \( V = V(\varphi) \) is the number of false discoveries of \( \varphi \) and \( S = S(\varphi) \) the number of true discoveries.

First suppose \( \varphi \) achieves almost full recovery with respect to the Hamming loss. This can be written \( E[V] + E[s_n - S] = o(s_n) \) uniformly over \( \Theta \). Let \( A = \{V + S > s_n/2\} \), then using Markov’s inequality,
\[
P_{\theta}[A^c] \leq P_{\theta}[s_n - S \geq s_n/2] \leq \frac{2}{s_n} E_{\theta}[s_n - S]
\]
so that \( P_{\theta}[A^c] = o(1) \). On the other hand,
\[
\text{FDR}(\theta, \varphi) = E_{\theta} \left[ \frac{V}{(V + S) \lor 1} \right] \leq \frac{2}{s_n} E_{\theta}[V 1_A] + P_{\theta}[A^c],
\]
which implies \( \text{FDR}(\theta, \varphi) = o(1) \) over \( \Theta \), while \( \text{FNR}(\theta, \varphi) = o(1) \) follows from \( E_{\theta}[s_n - S] = o(s_n) \).

Conversely, suppose \( \varphi \) allows for conservative testing over \( \Theta \). Then \( \text{FNR}(\theta, \varphi) = o(1) \) implies \( E_{\theta}[s_n - S] = o(s_n) \). Set \( B = \{\ |V(\varphi) - E_{\theta} V(\varphi)| \leq E_{\theta} V(\varphi)/2 \} \). By assumption \( P_{\theta}[B^c] = o(1) \) over \( \Theta \). Then
\[
\text{FDR}(\theta, \varphi) = E_{\theta} \left[ \frac{V}{(V + S) \lor 1} \right] \geq \frac{E_{\theta} V/2}{3 E_{\theta} V/2 + s_n P_{\theta}[B]} = \psi(E_{\theta} V/s_n)(1 + o(1)),
\]
where \( \psi \) is the bijective continuous map \( x \rightarrow x/(3x + 2) \) from \( (0, \infty) \) to \( (0, 1/3) \). By assumption \( \text{FDR}(\theta, \varphi) = o(1) \). As \( \psi^{-1} \) is continuous at 0 with limit 0, one gets \( EV_{\theta}/s_n = o(1) \) which implies combined with the control of \( E[s_n - S] \) that \( \varphi \) achieves almost full recovery with respect to the Hamming loss. \( \Box \)
Appendix G: Properties of Subbotin distributions

Lemma 23. Let $f(\cdot)$ be defined by (15). Then $F(x) = \int_x^{+\infty} f(u)du$ has the following properties:

- for any $x > 0$, we have
  
  $$F(x) < f(x)/x^\zeta - 1;$$  
  (87)

- for any $t \in (0,1/2)$ s.t. $F^{-1}(t) \geq 1$, we have
  
  $$F^{-1}(t) \leq (\zeta \log(1/t) - \zeta \log L_\zeta)^{1/\zeta};$$  
  (88)

The following holds:

- for any $x > 0$,
  
  $$F(x) \geq \frac{f(x)}{x^\zeta} \left[1 + (\zeta - 1)x^{-\zeta}\right]^{-1};$$  
  (89)

  $$F(x) \geq \frac{f(x)}{x^\zeta_1} \zeta_1^{-1} \text{ if } x \geq 1;$$  
  (90)

- for any $t \in (0,1/2)$ s.t. $F^{-1}(t) \geq 1$, we have
  
  $$F^{-1}(t) \geq \left(0 \lor \left\{ \zeta \log(1/t) - \zeta \log(L_\zeta) - (\zeta - 1) \log(\zeta \log(1/t) - \zeta \log L_\zeta) \right\}\right)^{1/\zeta}. $$  
  (91)

Proof. Let $\psi(u) = u^\zeta/\zeta + \log L_\zeta$, so that, restricting to $u > 0$,

$$f(u) = e^{-\psi(u)}, \quad \psi(u) = u^\zeta - 1, \quad \psi^{-1}(v) = (\zeta v - \zeta \log L_\zeta)^{1/\zeta}, \quad \psi''(u) = (\zeta - 1)u^\zeta - 2,$$

and thus for instance $\psi' \circ \psi^{-1}(v) = (\zeta v - \zeta \log L_\zeta)^{1-1/\zeta}, \psi''(u)/(\psi'(u))^2 = (\zeta - 1)u^{-\zeta}$.

Inequality (87) holds because $F(x) = \int_x^{+\infty} e^{-\psi(u)}du < (\psi'(x))^{-1} \int_x^{+\infty} \psi'(u)e^{-\psi(u)}du = f(x)/\psi'(x)$. Expression (88) follows from (87) applied with $x = F^{-1}(t) \geq 1$. Indeed, the latter entails $L_\zeta^{-1}\exp(-F^{-1}(t)\zeta/\zeta_t > tx^{-1}) \geq t$, from which (88) follows by applying the function $\log(\cdot)$ to both sides of the inequality. To prove (89), write for any $x > 0$,

$$\frac{\psi''(x)}{\psi'(x)^2} F(x) \geq \int_x^{+\infty} \frac{\psi''(u)}{\psi'(u)^2} e^{-\psi(u)}du = \left[-\frac{e^{-\psi(u)}}{\psi'(u)}\right]^{+\infty}_x F(x) = \frac{f(x)}{\psi'(x)} - F(x),$$

by using an integration by parts. Expressions (89) and (90) follow. Finally, let us prove (91). From (90) used with $x = F^{-1}(t)$, we get $\zeta t(F^{-1}(t)\zeta_1^{-1} \geq e^{-\psi(F^{-1}(t))}$ and thus $-\log(\zeta t) - (\zeta - 1) \log(F^{-1}(t)) \leq \psi(F^{-1}(t))$. Hence, by (89), we obtain

$$\log L_\zeta \lor (-\log(\zeta t) - (1 - 1/\zeta) \log(\log(1/t) - \zeta \log L_\zeta)) \leq \psi(F^{-1}(t))$$

and thus (91). \qed

Lemma 24. For any $a \geq 0$ and any $x \in \mathbb{R}$,

$$f(x-a)/f(x) \geq \exp(a|x-a|^{\zeta-1} \text{sign}(x-a)).$$

Proof. Since $\zeta > 1$, observe that $-\log f(x) = \zeta^{-1}|x|^\zeta + c$ is differentiable with increasing derivative $|x|^\zeta \text{sign}(x)$. It follows, using $a \geq 0$, that $-\log f(x) \geq -\log f(x-a) + a|x-a|^{\zeta-1} \text{sign}(x-a)$, and hence the claim. \qed