Weak and viscosity solutions for non-homogeneous fractional equations in Orlicz spaces

María L. de Borbón\textsuperscript{a}, Leandro M. Del Pezzo\textsuperscript{b}, Pablo Ochoa\textsuperscript{c}

\textsuperscript{a}Facultad de Ciencias Económicas, FCE, Universidad Nacional de Cuyo-CONICET, Parque Gra. San Martín SN (5500), Mendoza, Argentina.
\textsuperscript{b}Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Pabellon I, Ciudad Universitaria (C1428BCW), Buenos Aires, Argentina.
\textsuperscript{c}Facultad de Ingeniería, Universidad Nacional de Cuyo-CONICET, Parque Gra. San Martín SN (5500), Mendoza, Argentina.

Abstract

In this paper, we consider non-homogeneous fractional equations in Orlicz spaces, with a source depending on the spatial variable, the unknown function and its fractional gradient. The latter is adapted to the Orlicz framework. The main contribution of the article is to establish the equivalence between weak and viscosity solutions for such equations.

Keywords: 2000 MSC: 35D40,, 35D30,, 35R11,, 46E30 viscosity solutions, weak solutions, non-homogeneous problems, g-Laplace operator, Orlicz spaces.

1. Introduction

In this paper, we prove the equivalence between weak and viscosity solutions for the non-homogeneous fractional \( g \)-Laplace equation

\[
(-\Delta_{g})^s u = f(x, u, D^s_{g} u) \quad \text{in} \quad \Omega,
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded open domain. Given \( s \in (0,1) \) and a Young function \( G \) such that \( g := G' \), the fractional \( g \)-Laplace operator is defined by

\[
(-\Delta_{g})^s u(x) = \text{P.V.} \int_{\mathbb{R}^n} g\left( \frac{u(x) - u(y)}{|x-y|^s} \right) \frac{dy}{|x-y|^{n+s}}
\]

for any smooth function \( u \). Here P.V. is an abbreviation for “in the principal value sense”. In (1.1), \( D^s_{g} \) denotes the \( g \)-fractional gradient of \( u \in W^{s,G}(\Omega) \), that is defined as follows

\[
D^s_{g} u(x) := \int_{\mathbb{R}^n} G\left( \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dy}{|x-y|^{n}}, \quad x \in \Omega.
\]

Observe that \( D^s_{g} u(x) \) is finite a.e. since \( u \in W^{s,G}(\Omega) \). We refer the reader to Section 2 for for definitions and properties of the spaces that we use.
To state the equivalence of solutions, we will assume that $f: \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following growth conditions
\[ |f(x, r, \eta)| \leq \gamma(|r|) \tilde{G}^{-1}(|\eta|) + \phi(x), \tag{1.2} \]
where $\tilde{G}$ is the complementary function of $G$, $\gamma \geq 0$ is continuous, and $\phi \in L^\infty(\Omega)$.

If we take $g(t) = t$, then the fractional $g-$Laplace operator is the fractional Laplace operator, that is
\[ (-\Delta)^s u(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy. \]
This operator is the most basic elliptic linear integro-differential operator. Problems with non local diffusion that involve integro-differential operators have been intensively studied in the last years. These nonlocal operators appear when we model different physical situations as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics and relativistic quantum mechanics of stars (see \cite{3, 8} and the references therein). They also appear in mathematical finance \cite{2, 7}, elasticity problems \cite{24}, phase transition problems \cite{1} and crystal dislocation structures \cite{26}, among others.

On the other hand, the $g-$fractional gradient is the natural extension of the fractional gradient
\[ D^s u(x) := \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy, \quad x \in \Omega. \]
This $s-$gradient appears naturally when studying fractional harmonic maps to the sphere. See \cite{6} and the references therein.

Our first main result is that viscosity solutions of (1.1) are also weak solutions. We state the result for supersolutions. See Section 2 for details in the assumptions below.

**Theorem 1.1.** Assume that $G$ is a Young function satisfying (2.1) and so that its complementary $\tilde{G}$ verifies the $\Delta'$-condition. Also, suppose that $f = f(x, r, \eta)$ is non-increasing in $r$, uniformly continuous in $\Omega \times \mathbb{R} \times \mathbb{R}$, Lipschitz continuous in $\eta$, and it satisfies the growth (1.2). If $u \in L^\infty(\mathbb{R}^n)$ is lower semicontinuous in $\mathbb{R}^n$ and is a viscosity supersolution of (1.1), then $u$ is a weak supersolution of (1.1).

For the converse result we need to assume that a comparison principle holds for weak solutions of (1.1). Following \cite{4}, we define below the class of functions that satisfy the comparison principle property.

**Definition 1.1.** Let $u$ be a weak supersolution of (1.1) in $D \subset \Omega$. We say that the comparison principle property (CPP) holds in $D$ if for every weak subsolution $v$ of (1.1) in $D$ such that $u \geq v$ a.e. in $\mathbb{R}^n \setminus D$, we have $u \geq v$ a.e. in $D$.

Next we state the reverse result.

**Theorem 1.2.** Assume that $G$ is a Young function satisfying (2.1) and so that its complementary $\tilde{G}$ verifies the $\Delta'$-condition. Also, suppose that $f = f(x, r, \eta)$ is continuous in $\Omega \times \mathbb{R} \times \mathbb{R}$ and Lipschitz continuous in $\eta$. If $u \in W^{s, G}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a bounded weak supersolution of (1.1) and (CPP) holds, then $u$ is also a viscosity supersolution of (1.1).

The relation between different notions of solutions has been studied by several authors and for different operators in the last decades. For linear problems, the equivalence between distributional and viscosity solutions is given in \cite{13}. Later, for weak and viscosity solutions for the homogeneous $p-$Laplace equations, the equivalence was provided in \cite{10} (with a different proof in \cite{14}). Recently, for a source depending on all the lower-order terms, the relation between weak and viscosity solutions for the $p-$Laplace equation was given in \cite{21}, following some ideas from \cite{14}. Similar studies have been made for operator...
with non-standard growth \cite{17, 23}, and recently for non-local operators \cite{18} and \cite{4}. In this work, we propose to generalize the later results to the non-homogeneous fractional $g$–laplacian operator.

Applications of the equivalence between viscosity and weak solutions can be found in \cite{25} and \cite{15} to removability of sets and Rado type theorems. Also, the equivalence has been used in free-boundary problems \cite{12}, \cite{5}.

The paper is organized as follows. In Section 2 we give the definition of the Orlicz spaces, some technical results and the notions of solutions. In Section 3 we give previous results for viscosity and weak solutions that we shall use to state the equivalence of solutions. There we also provide some continuity properties necessary for the proof of Theorem 1.2. Afterwards, in Sections 4 and 5 we prove the main results of the paper. We end the paper with a brief appendix regarding some inequalities for Young functions.

2. Preliminaries

In this section, we gather some preliminary properties which will be useful in the forthcoming sections.

2.1. Notations

Throughout the paper, we use the notation

$$u_+(x) = \max\{u(x), 0\}, \quad u_-(x) = \max\{-u(x), 0\},$$

$$D_s u := \frac{u(x) - u(y)}{|x - y|^s} \quad \text{and} \quad d\mu := \frac{dx\, dy}{|x - y|^n}.$$ 

For all $a \in \mathbb{R}$ and $q > 0$, we set

$$a^q = |a|^{q-1}a.$$ 

2.2. Young functions

A function $G : [0, \infty) \to [0, \infty)$ is called a Young function if it has the following integral representation

$$G(t) = \int_0^t g(r) \, dr$$

where the right-continuous function $g : [0, \infty) \to [0, \infty)$ satisfies

1. $g(0) = 0$, $g(t) > 0$ for $t > 0$;

2. $g$ is non-decreasing in $[0, \infty)$;

3. $\lim_{t \to \infty} g(t) = \infty$.

Observe that we use the term Young function to denote N-functions as defined in \cite{19}.

From the above properties, it is easy to see that a Young function $G$ is continuous, nonnegative, strictly increasing, and convex in $[0, \infty)$. Also, without loss of generality we can assume that $G(1) = 1$.

In this article, we consider Young functions $G$ assuming that $g = G'$ is an absolutely continuous function such that

$$p^- - 1 \leq \frac{tg'(t)}{g(t)} \leq p^+ - 1, \quad \forall t > 0$$

with $1 < p^- < p^+ < \infty.$
Examples 1. The following functions are Young functions that satisfy (2.1):

(a) \( G(t) = t^p \) with \( p > 1 \).

(b) \( G(t) = t^p(|\log(t)| + 1) \) with \( p > \frac{3 + \sqrt{5}}{2} \).

Observe that, by integration by parts, (2.1) gives

\[
1 < p^{-} \leq \frac{tg(t)}{G(t)} \leq p^{+} < \infty \quad \forall t > 0.
\]

(2.2)

Therefore, see [19, Theorem 4.1], \( G \) satisfies the \( \Delta_2 \)-condition, that is, there exists a positive constant \( C \) such that

\[
G(2t) \leq CG(t)
\]

for any \( t \geq 0 \). Moreover, from (2.1) and (2.2) it follows that

\[
\min \left\{ a^{p^{-}-1}, a^{p^{+}-1} \right\} g(b) \leq g(ab) \leq \max \left\{ a^{p^{-}-1}, a^{p^{+}-1} \right\} g(b),
\]

(2.3)

and

\[
\min \left\{ a^{p-}, a^{p+} \right\} G(b) \leq G(ab) \leq \max \left\{ a^{p-}, a^{p+} \right\} G(b),
\]

(2.4)

for all \( a, b \geq 0 \).

On the other hand, we may extend \( g \) to the whole \( \mathbb{R} \) as following:

\[
g(t) = -g(-t) \quad \text{if } t < 0.
\]

Then, (2.1) also holds for negative \( t \), and it implies that there is a positive constant \( C \) such that

\[
C \min \{t^{p^{-}-1}, t^{p^{+}-1}\} \leq g(t) \leq C \max \{t^{p^{-}-1}, t^{p^{+}-1}\}, \quad t \in \mathbb{R}.
\]

(2.5)

By [19, Lemma 1.3], \( G \) is an absolutely continuous function. Then, applying [22, Lemma 2.2] to \( G \), and using (2.2) and Lemma [Appendix A.1] there is a positive constant \( C \) such that

\[
|G(b) - G(a)| \leq C|b - a|(g(|a|) + g(|b|)), \quad \forall a, b \in \mathbb{R}.
\]

(2.6)

We say that a Young function \( G \) satisfies the \( \Delta' \)-condition if there exists a positive constant \( C \geq 1 \) such that

\[
G(ab) \leq CG(a)G(b)
\]

for all \( a, b \geq 0 \). Observe that if \( G \) satisfies the \( \Delta' \)-condition, then it also satisfies the \( \Delta_2 \)-condition.

Examples of functions satisfying the \( \Delta' \)-condition are:

- \( G(t) = t^p, \ t \geq 0, \ p > 1 \);
- \( G(t) = t^p(\log(t)) + 1, \ t \geq 0, \ p > \frac{3 + \sqrt{5}}{2} \);
- \( G(t) = t^p \chi(0,1)(t) + t^q \chi(1,\infty)(t), \ t \geq 0, \ p, q > 1 \).
Necessary and sufficient conditions for the $\Delta'$-condition are given in [19, Chapter I, Sec. 5].

The complementary function of a Young function $G$ is defined on $[0, \infty)$ by

$$\tilde{G}(a) := \sup \{ at - G(t) : t > 0 \}.$$ 

The function $\tilde{G}$ plays the role that the conjugate function exponent has in the standard theory of Lebesgue and Sobolev spaces. Also, by [19, Chapter I, Theorem 4.3], the inequality $\tilde{G}$ satisfies the $\Delta_2$-condition. Moreover, by [19, Chapter I, Theorem 5.3], $\tilde{G}$ satisfies the $\Delta'$-condition if the function $h(t) := t g'(t)/g(t)$ does not decrease.

As a consequence of the inequality [9, Eq. (2.5)]

$$at \leq G(t) + \tilde{G}(a), \text{ for any } t, a \geq 0,$$

and the inequalities (from [21]),

$$G(at) \leq tG(a), \text{ for any } a \geq 0, 0 \leq t \leq 1,$$

$$G(at) \leq t^{p^+} G(a), \text{ for any } a \geq 0, t \geq 1,$$

we get the following Young’s inequality for $0 < \delta < 1$

$$at = (a\delta) \left( \frac{1}{\delta} t \right) \leq G(\delta a) + G \left( \frac{1}{\delta} t \right) \leq \delta \tilde{G}(a) + \left( \frac{1}{\delta} \right)^{p^+} G(t), \forall a, t > 0.$$

Finally, we quote the following useful lemma.

**Lemma 2.1.** [9, Lemma 2.9] Let $G$ be an Young function. If $G$ satisfies (2.1) then

$$\tilde{G}(g(t)) \leq (p^+ - 1) G(t),$$

where $g = G'$ and $\tilde{G}$ is the complementary function of $G$.

We will provide some more useful inequalities for Young functions in the Appendix.

### 2.3. Orlicz-fractional Sobolev spaces

Given a Young function $G$ with $g = G'$, $s \in (0, 1)$, and an open set $\Omega \subseteq \mathbb{R}^n$, we consider the spaces:

$$L^G(\Omega) := \{ u : \Omega \to \mathbb{R} : \Phi_{G,\Omega}(u) < \infty \},$$

$$W^{s,G}(\Omega) := \{ u \in L^G(\Omega) : \Phi_{s,G,\Omega}(u) < \infty \}, \text{ and}$$

$$L_g(\mathbb{R}^n) := \left\{ u \in L_{loc}^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} g \left( \frac{|u(x)|}{1 + |x|^s} \right) \frac{dx}{1 + |x|^{n+s}} < \infty \right\}.$$

Here, the modulars $\Phi_{G,\Omega}$ and $\Phi_{s,G,\Omega}$ are defined as

$$\Phi_{G,\Omega}(u) := \int_\Omega G(|u(x)|) \, dx, \quad \text{and} \quad \Phi_{s,G,\Omega}(u) := \int_\Omega \int_\Omega G(|D_s u|) \, d\mu.$$

The spaces $L^G(\Omega)$ and $W^{s,G}(\Omega)$ are endowed, respectively, with the following norms

$$\| u \|_{L^G(\Omega)} := \inf \left\{ \lambda > 0 : \Phi_{G,\Omega} \left( \frac{u}{\lambda} \right) \leq 1 \right\},$$

5
and
\[\|u\|_{W^{r,G}(\Omega)} := \|u\|_{L^G(\Omega)} + [u]_{W^{r,G}(\Omega)},\]
where
\[[u]_{W^{r,G}(\Omega)} := \inf \left\{ \lambda > 0 : \Phi_{s,G,\Omega} \left( \frac{u}{\lambda} \right) \leq 1 \right\}.\]

The space \(W^r_G(\Omega)\) will be the closure of \(C^\infty(\Omega)\) with respect to the norm \(\| \cdot \|_{W^{r,G}(\Omega)}\).

The following lemma relates modulars and norms in Orlicz spaces.

**Lemma 2.2.** Let \(G\) be a Young function satisfying (2.1), and let \(\xi^\pm : [0,\infty) \to \mathbb{R}\) be defined as
\[\xi^-(t) := \min \left\{ t^{p^-}, t^{p^+} \right\}, \quad \text{and} \quad \xi^+(t) := \max \left\{ t^{p^-}, t^{p^+} \right\}.\]

Then

1. \(\xi^-(\|u\|_{L^G(\Omega)}) \leq \Phi_G(u) \leq \xi^+(\|u\|_{L^G(\Omega)});\)
2. \(\xi^-([u]_{W^{r,G}(\Omega)}) \leq \Phi_{s,G}(u) \leq \xi^+([u]_{W^{r,G}(\Omega)}).\)

### 2.4. Notions of Solutions

Borrowing ideas from [18], we first introduce the definition of viscosity solution. For a given domain \(\Omega \subset \mathbb{R}^n\) and \(\beta > 2\), we let

\[C^2_\beta(\Omega) := \left\{ u \in C^2(\Omega) : \sup_{x \in \Omega} \left( \frac{\min\left\{ d_u(x), 1\right\}^{\beta-1}}{|\nabla u(x)|} + \frac{|D^2 u(x)|}{d_u(x)^{\beta-2}} \right) < \infty \right\}\]

where \(d_u(x) := \text{dist}(x, N_u)\) and \(N_u := \{x \in \Omega : \nabla u(x) = 0\}\) represents the set of critical points of the function \(u\).

**Definition 2.1.** We say that a function \(u : \mathbb{R}^n \to [-\infty, +\infty]\) is a viscosity supersolution (subsolution) of (1.1) in \(\Omega\) if:

(i) \(u < +\infty\) (\(u > -\infty\)) a.e. in \(\mathbb{R}^n\), \(u > -\infty\) (\(u < +\infty\)) a.e. in \(\Omega\),

(ii) \(u\) is lower (upper) semicontinuous in \(\Omega\),

(iii) if \(\psi \in C^2(B_r(x_0)) \cap L_g(\mathbb{R}^n)\) for some \(B_r(x_0) \subset \Omega\) such that \(\psi(x_0) = u(x_0)\) and \(\psi \leq u\) (\(u \leq \psi\)) in \(\mathbb{R}^n\) and one of the following holds:

(a) \(p^- > \frac{2}{\beta-2}\) or \(\nabla \psi(x_0) \neq 0\),

(b) \(1 < p^- \leq \frac{2}{\beta-2}\), \(\nabla \psi(x_0) = 0\) such that \(x_0\) is an isolated critical point of \(\psi\) in \(B_r(x_0)\), and \(\psi \in C^2_\beta(B_r(x_0))\) for some \(\beta > \frac{2p^-}{p^- - 1}\),

then
\[(-\Delta_g)^* \psi(x_0) \geq f(x_0, \psi(x_0), D^*_g \psi(x_0)),\]

(iv) \(u_- (u^+)\) belongs to \(L_g(\mathbb{R}^n)\).

A viscosity solution of (1.1) is a function \(u\) which is a viscosity sub- and supersolution of (1.1).

**Remark 1.** Observe that when \(u \in L^\infty(\mathbb{R}^n)\), in Definition 2.1(iii), we may define \(\psi\) as \(u\) outside the ball \(B_r(x_0)\).
Remark 2. In Section 3 we shall show that \((-\Delta_g)^s\psi\) is well-defined for the class of test functions considered above.

We now give the definition of weak solutions.

Definition 2.2. A function \(u \in W^{s,G}(\Omega) \cap L_q(\mathbb{R}^n)\) is a weak supersolution (subsolution) of \((1.1)\) if
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(D_s u) D_s \psi \, d\mu \geq (\leq) \int_\Omega f(x,u,D_g u) \psi \, dx,
\]
for any non-negative \(\psi \in C_0^\infty(\Omega)\). We say that \(u\) is a weak solution if it is a weak sub- and supersolution.

Observe that by density, we may extend the above definition to test functions in \(W_0^{s,G}(\Omega)\).

2.5. Infimal convolutions

A standard smoothing operator in the theory of viscosity solutions is the infimal convolution.

Definition 2.3. Given \(\varepsilon > 0\), we define the infimal convolution of a function \(u: \mathbb{R}^n \to \mathbb{R}\) as
\[
u_{\varepsilon}(x) := \inf_{y \in \mathbb{R}^n} \left( u(y) + \frac{|x - y|^q}{q\varepsilon^{q-1}} \right)
\]
where \(q = 2\) if \(p^- > \frac{2}{2-s}\) and \(q > \frac{sp^-}{p^- - 1} \geq 2\) if \(1 < p^- \leq \frac{2}{2-s}\).

The infimal convolution is one of the main tools to prove that any viscosity solution is a weak solution. See, for instance [4, 14] and the references therein.

Lemma 2.3 (See [14]). Let \(u\) be a bounded and lower semicontinuous function in \(\mathbb{R}^n\). Then:

(i) There exists \(r(\varepsilon) > 0\) such that
\[
u_{\varepsilon}(x) = \inf_{y \in B_{r(\varepsilon)}(x)} \left( u(y) + \frac{|x - y|^q}{q\varepsilon^{q-1}} \right)
\]
where \(r(\varepsilon) \to 0\) as \(\varepsilon \to 0\).

(ii) The sequence \(\{\nu_{\varepsilon}\}_{\varepsilon > 0}\) is increasing as \(\varepsilon \to 0\) and \(\nu_{\varepsilon} \to u\) pointwise in \(\mathbb{R}^n\).

(iii) \(\nu_{\varepsilon}\) is locally Lipschitz and twice differentiable a.e. Actually, for almost every \(x, y \in \mathbb{R}^n\)
\[
u_{\varepsilon}(y) = \nu_{\varepsilon}(x) + \nabla \nu_{\varepsilon}(x) \cdot (x - y) + \frac{1}{2} D^2 \nu_{\varepsilon}(x)(x - y)^2 + o(|x - y|^2).
\]

(iv) \(\nu_{\varepsilon}\) is semiconcave, that is, there exists a constant \(C = C(q,\varepsilon, \text{osc}(u)) > 0\) such that the function \(x \mapsto \nu_{\varepsilon}(x) - C|x|^2\) is concave. In particular
\[
D^2 \nu_{\varepsilon}(x) \leq 2CI, \quad \text{a.e. } x \in \mathbb{R}^n.
\]

(v) The set \(Y_{\varepsilon}(x) := \left\{ y \in B_{r(\varepsilon)}(x) : \nu_{\varepsilon}(x) = u(y) + \frac{|x - y|^q}{q\varepsilon^{q-1}} \right\}\) is non empty and closed for every \(x \in \mathbb{R}^n\).

(vi) If \(\nabla \nu_{\varepsilon}(x) = 0\), then \(\nu_{\varepsilon}(x) = u(x)\).
3. Previous results

In this section we shall provide preliminary results to state the equivalence between weak and viscosity solutions of the equation (1.1). We divide the section into three parts: results related to viscosity solutions, results regarding the weak formulation of solutions and the last part is dedicated to certain continuity properties of $D^s_g$ and $(-\Delta_g)^s$. To prove the results, we borrow some calculations from [18] and [4]. However, due to some technical differences, we prove the results in detail.

3.1. Results on viscosity solutions

In this section, we prove that $(-\Delta_g)^s\psi$ is well-defined for the test functions introduced in Definition 2.1. Moreover, we will state the equations satisfied by the inf-convolution of a viscosity solution.

We start with some preliminary lemmas.

Lemma 3.1. Let $\rho > 0$ be such that $B_\rho(x) \subset D \subset \subset \Omega$ (with $D$ open), and $u \in C^2(D)$. If $p^- > \frac{2}{2-s}$ or $D \subset \subset \{d_u > 0\}$ then there is a positive constant $C_\rho$ independent of $x$ such that

$$\left| \text{P.V.} \int_{B_\rho(x)} g(D_su) \frac{dy}{|x-y|^{n+s}} \right| \leq C_\rho.$$ 

Moreover $C_\rho \to 0$ as $\rho \to 0^+$. \[\Box\]

Proof. Case 1: $\nabla u(x) = 0$ and $p^- > \frac{2}{2-s}$.

Since $u \in C^2(D)$, and $\nabla u(x) = 0$, for a given $\rho_0 > 0$ so that $B_{\rho_0}(x) \subset D$, there is a positive constant $C$ independent of $x$ and $y$ such that

$$|u(x) - u(y)| \leq C|x-y|^2$$

for all $y \in B_{\rho_0}(x)$. Then, for all $\rho \leq \rho_0$ so that $C\rho^{2-s} \leq 1$, we have

$$|D_s u| \leq C|x-y|^{2-s} \leq C\rho^{2-s} < 1.$$ 

Hence, by (2.1) that

$$|g(D_s u)| \leq g(D_s u) \leq (D_s u)^{p^--1} \leq C|x-y|^((2-s)(p^- - 1)).$$

Therefore, we have

$$\left| \text{P.V.} \int_{B_\rho(x)} g(D_s u) \frac{dy}{|x-y|^{n+s}} \right| \leq C \int_{B_\rho(x)} |x-y|^{(2-s)p^- - n-2} dy = C\rho^{(2-s)p^- - 2}.$$ 

Using that $p^- > \frac{2}{2-s}$, it follows that

$$C_\rho := C\rho^{(2-s)p^- - 2} \to 0,$$

as $\rho \to 0$.

Case 2: $\nabla u(x) \neq 0$.

Let $L(y) := u(x) + \nabla u(x) \cdot (y - x)$. Observe that, since $g$ is odd, we get

$$\text{P.V.} \int_{B_\rho(x) \setminus B_{\rho^e}(x)} g(D_s L) \frac{dy}{|x-y|^{n+s}} = 0.$$ 

8
for any \( \varepsilon < \rho \). Then, for any \( 0 < \varepsilon < \rho \), taking \( B_{\varepsilon, \rho} := B_{\varepsilon}(x) \setminus B_{\rho}(x) \) we have

\[
\left| \int_{B_{\varepsilon, \rho}} g(D_s u) \frac{dy}{|x - y|^{n+s}} \right| \leq \int_{B_{\varepsilon, \rho}} |g(D_s u) - g(D_s L)| \frac{dy}{|x - y|^{n+s}}.
\]

Thus, by Lemma \text{Appendix A.2}, there is a positive constant \( C \) independent of \( x, \varepsilon, \), and \( \rho \) such that

\[
\left| \int_{B_{\varepsilon, \rho}} g(D_s u) \frac{dy}{|x - y|^{n+s}} \right| \leq C \int_{B_{\varepsilon, \rho}} \max \left\{ \frac{H(u, L, x, y)^{p^{-2}}}{|x - y|^{s(p^{-2})}}, \frac{H(u, L, x, y)^{p^{-2}}}{|x - y|^{s(p^{-2})}} \right\} |u(y) - L(y)| \frac{dy}{|x - y|^{n+2s}}.
\]

where \( H(u, L, x, y) := |L(x) - L(y)| + |u(y) - L(y)| \). Then

\[
\left| \int_{B_{\varepsilon, \rho}} g(D_s u) \frac{dy}{|x - y|^{n+s}} \right| \leq C \left\{ \int_{B_{\varepsilon, \rho}} \frac{H(u, L, x, y)^{p^{-2}}}{|x - y|^{N+sp}} |u(y) - L(y)| dy + \int_{B_{\varepsilon, \rho}} \frac{H(u, L, x, y)^{p^{-2}}}{|x - y|^{N+sp}} |u(y) - L(y)| dy \right\}.
\]

The rest of the proof follows the same lines as that of \text{[18, Lemma 3.6]}.

\textbf{Lemma 3.2.} Let \( 1 < p^- \leq \frac{2}{2-s} \), \( D \subset \Omega \) be an open set, and \( u \in C^2_{D}(D) \) with \( \beta > \frac{sp^-}{p^- - 1} \). Then, for any \( \rho \in (0, 1) \) such \( B_{\rho}(x) \subset D \) and \( x \) is such that \( d_{u} < \rho \). Then there is a positive constant \( C_{\rho} \) independent of \( x \) such that

\[
\left| P.V. \int_{B_{\rho}(x)} g(D_s u) \frac{dy}{|x - y|^{n+s}} \right| \leq C_{\rho}.
\]

Moreover \( C_{\rho} \to 0 \) as \( \rho \to 0^+ \).

\textbf{Proof.} Case 1: \( \nabla u(x) = 0 \).

Since \( u \in C^2_{D}(D) \), and \( \nabla u(x) = 0 \), there is a positive constant \( C \) independent of \( x \) such that

\[
|u(x) - u(y)| \leq C|x - y|^{\beta}
\]

in \( B_{\rho}(x) \), for all \( \rho \) small enough. Then, as in the proof of Lemma \text{3.1}, using \text{25}, we have that

\[
|g(D_s u)| \leq g(D_s u) \leq (D_s u)^{p^- - 1} \leq C|x - y|^{(\beta-s)(p^- - 1)}.
\]

Therefore, using that \( \beta > \frac{sp^-}{p^- - 1} \), we have

\[
\left| P.V. \int_{B_{\rho}(x)} g(D_s u) \frac{dy}{|x - y|^{n+s}} \right| \leq C \int_{B_{\rho}(x)} |x - y|^{(\beta-s)p^- - \beta} dy = C \rho^{(\beta-s)p^- - \beta} =: C_{\rho} \to 0
\]

as \( \rho \to 0 \).

Case 2: \( \nabla u(x) \neq 0 \). Now, proceeding as Case 2 in the proof of Lemma \text{3.1}, we have that there is a constant \( C \) independent of \( x \) and \( \rho \) such that

\[
\left| \int_{B_{\varepsilon, \rho}} g(D_s u) \frac{dy}{|x - y|^{n+s}} \right| \leq C \left\{ \int_{B_{\varepsilon, \rho}} \frac{H(u, L, x, y)^{p^{-2}}}{|x - y|^{N+sp}} |u(y) - L(y)| dy + \int_{B_{\varepsilon, \rho}} \frac{H(u, L, x, y)^{p^{-2}}}{|x - y|^{N+sp}} |u(y) - L(y)| dy \right\}.
\]
where \( L(x) := u(x) + \nabla u(x) \cdot (y - x) \) and \( B_{\varepsilon, \rho} = B_\varepsilon(x) \setminus B_\rho(x) \) The rest of the proof follows the same line as that of [18, Lemma 3.7].

**Remark 3.** It is worth mentioning that, as far as we know, the notion of viscosity solution is new for the case \( 1 < p^- \leq \frac{n}{n-s} \). See, for instance, [10].

**Remark 4.** Lemmas [5.1] and [5.2] prove that the principal values are well-defined for functions \( \psi \) that are smooth enough. If additionally \( \psi \in L_g(\mathbb{R}^n) \), then for \( x \in \mathbb{R}^n \) and \( 0 < \rho < 1 \), we have

\[
\int_{\mathbb{R}^n \setminus B_\rho(x)} g \left( \frac{\psi(x) - \psi(y)}{|x - y|^s} \right) \frac{dy}{|x - y|^{n+s}} < \infty.
\]

Indeed, take \( R > 0 \) such that \( B_\rho(x) \subset B_R \). Then, using Lemma A.5 from [11] it holds that

\[
|x - y| \geq \frac{1 + R}{\rho} (1 + |y|), \quad y \in \mathbb{R}^n \setminus B_\rho(x).
\]

Therefore,

\[
\int_{\mathbb{R}^n \setminus B_\rho(x)} g \left( \frac{\psi(x) - \psi(y)}{|x - y|^s} \right) \frac{dy}{|x - y|^{n+s}} \\
\leq \left( \frac{\rho}{1 + R} \right)^{n+s} \int_{\mathbb{R}^n \setminus B_\rho(x)} g \left( \frac{\psi(x) + |\psi(y)|}{1 + |y|^s} \right) \frac{dy}{1 + |y|^{n+s}} \\
\leq C \left( \frac{\rho}{1 + R} \right)^{n+s} \int_{\mathbb{R}^n \setminus B_\rho(x)} g \left( \frac{\psi(x) + |\psi(y)|}{1 + |y|^s} \right) \frac{dy}{1 + |y|^{n+s}} < \infty,
\]

where in the last inequality we have used (2.3), Lemma [Appendix A.1] and the fact that the constant function \( \psi(x) \) and \( \psi \) are in \( L_g(\mathbb{R}^n) \). Therefore, we get that \( (-\Delta_g)^s \psi \) is well-defined for the test functions considered in Definition [2.1].

Next, we will prove that when \( u \) is a viscosity solution, then it is also the case for \( u_\varepsilon \) with a slightly different equation.

**Lemma 3.3.** Let \( u : \mathbb{R}^n \to \mathbb{R} \) be a bounded and lower semicontinuous function in \( \mathbb{R}^n \), and \( f = f(x,t,\eta) \) be continuous in \( \Omega \times \mathbb{R} \times \mathbb{R} \) and non increasing in \( t \). If \( u \) is a viscosity supersolution of

\[
(-\Delta_g)^s u = f(x,u,D_g^s u) \quad \text{in} \quad \Omega
\]

then its infimal convolution \( u_\varepsilon \) is a viscosity supersolution of

\[
(-\Delta_g)^s u_\varepsilon = f_\varepsilon(x,u_\varepsilon,D_g^s u_\varepsilon) \quad \text{in} \quad \Omega_{r(\varepsilon)},
\]

where \( \Omega_{r(\varepsilon)} := \{ x \in \Omega : \text{dist}(x,\partial\Omega) > r(\varepsilon) \} \) and

\[
f_\varepsilon(x,t,\eta) := \inf_{y \in B_{r(\varepsilon)}(x)} f(y,t,\eta).
\]

Moreover,

\[
(-\Delta_g)^s u_\varepsilon(x) \geq f_\varepsilon(x,u_\varepsilon(x),D_g^s u_\varepsilon(x)) \quad \text{a.e.} \quad x \in \Omega_{r(\varepsilon)}.
\]

**Proof.** Let \( z \in B_{r(\varepsilon)}(0) \) and define

\[
\phi_\varepsilon(x) := u(x + z) + \frac{|z|^q}{q \varepsilon^{q-1}}, \quad x \in \mathbb{R}^n.
\]
Then we claim that $\phi_z$ is a viscosity supersolution of (3.1). Indeed, since $u$ is bounded and lower semicontinuous, $\phi_z$ satisfies items (i), (ii) and (iv) from Definition 2.1.

Now take $x_0 \in \Omega_{r(z)}$ and let $\varphi \in C^2(B_r(x_0)) \cap L^q(\mathbb{R}^n)$ such that $\varphi(x_0) = \phi_z(x_0)$, $\varphi \leq \phi_z$ in $\mathbb{R}^n$ with $B_r(x_0) \subset \Omega_{r(z)}$ and $\varphi$ satisfies (a) or (b) from Definition 2.1. We put $y_0 := z + x_0$. Then $y_0 \in B_{r(z)}(x_0) \subset \Omega$ since $z \in B_{r(z)}(0)$ and $x_0 \in \Omega_{r(z)}$. Now define

$$\bar{\varphi}(y) := \varphi(y - z) - \frac{|z|^q}{q^{\frac{q}{q-1}}}. $$

Observe that $\varphi(\xi) = \bar{\varphi}(\xi + z) + \frac{|z|^q}{q^{\frac{q}{q-1}}}$. Therefore

$$(-\Delta)_g^s \varphi(x_0) =\text{P.V.} \int_{\mathbb{R}^n} g \left( \frac{\varphi(x_0) - \varphi(\xi)}{|x_0 - \xi|^{n+s}} \right) \frac{d\xi}{|x_0 - \xi|^{n+s}} = \text{P.V.} \int_{\mathbb{R}^n} g \left( \frac{\bar{\varphi}(y_0) - \bar{\varphi}(\xi + z)}{|y_0 - \xi - z|^{n+s}} \right) \frac{d\xi}{|y_0 - \xi - z|^{n+s}}$$

(3.3)

$$= \text{P.V.} \int_{\mathbb{R}^n} g \left( \frac{\bar{\varphi}(y_0) - \bar{\varphi}(\xi)}{|y_0 - \xi|^{n+s}} \right) \frac{d\xi}{|y_0 - \xi|^{n+s}} = (-\Delta)_g^s \bar{\varphi}(y_0).$$

On the other hand,

$$\bar{\varphi}(y_0) = \varphi(x_0) - \frac{|z|^q}{q^{\frac{q}{q-1}}} = \phi_z(x_0) - \frac{|z|^q}{q^{\frac{q}{q-1}}} = u(y_0)$$

and

$$\bar{\varphi}(y) = \varphi(x) - \frac{|z|^q}{q^{\frac{q}{q-1}}} \leq \phi_z(x) - \frac{|z|^q}{q^{\frac{q}{q-1}}} = u(y).$$

Thus, by (3.3) and the fact that $u$ is a viscosity supersolution of (1.1) in $\Omega$ we get

$$(-\Delta)_g^s \varphi(x_0) = (-\Delta)_g^s \bar{\varphi}(y_0) \geq f(y_0, \varphi(y_0), D_g^s \bar{\varphi}(y_0)) = f(x_0 + z, \varphi(x_0) - \frac{|z|^q}{q^{\frac{q}{q-1}}}, D_g^s \bar{\varphi}(y_0))$$

\[ \geq f(x_0 + z, \varphi(x_0), D_g^s \bar{\varphi}(y_0)), \]

where in the last inequality we used that $f$ is non increasing in the second variable.

Now, reasoning as in (3.3) it holds that $D_g^s \bar{\varphi}(y_0) = D_g^s \varphi(x_0)$. Moreover, since $x_0 + z \in B_{r(z)}(x_0)$, the definition of $f_z$ yields

$$(-\Delta)_g^s \varphi(x_0) \geq f_z(x_0, \varphi(x_0), D_g^s \varphi(x_0)).$$

Hence, for every $z \in B_{r(z)}(0)$, $\phi_z$ is a viscosity supersolution of (3.1).

Now we check that $u_z$ satisfies Definition 2.1 for the equation (3.1). Again, since $u \in L^\infty(\mathbb{R}^n)$, $u_z$ fulfills conditions (i) and (iv). Moreover, from (iii) in Lemma 2.4, $u_z$ is locally Lipschitz, so assumption (ii) from Definition 2.1 is also satisfied.

We verify now condition (iii). Take $\psi \in C^2(B_{r(x_0)}) \cap L^q(\mathbb{R}^n)$ such that $B_r(x_0) \subset \Omega_{r(z)}$, $\psi(x_0) = u_z(x_0)$, $\psi \leq u_z$ in $\mathbb{R}^n$ and $\psi$ satisfies (a) or (b). By (i) and (v) in Lemma 2.4 we can write

$$u_z(x) = \inf_{y \in B_{r(z)}(x)} \left( u(y) + \frac{|x - y|^q}{q^{\frac{q}{q-1}}} \right) = \inf_{z \in B_{r(z)}(0)} \phi_z(x), \ x \in \mathbb{R}^n,$$

and there is $z \in B_{r(z)}(0)$ such that $u_z(x_0) = \phi_(x_0)$. Moreover, by definition $\psi \leq u_z \leq \phi_\bar{\psi}$ in $\mathbb{R}^n$. Thus we can employ $\psi$ as a test function for the problem satisfied by $\phi_\bar{\psi}$ and get

$$(-\Delta)_g^s \psi(x_0) \geq f_z(x_0, \psi(x_0), D_g^s \psi(x_0)).$$

Therefore $u_z$ is a viscosity supersolution of (3.1).
Now we prove \(3.2\). By (iii) from Lemma 2.3 we can fix \(x \in \Omega(r)\) such that \(u_r\) is twice differentiable at \(x\). We first assume that \(p > \frac{2}{n-2}\) or \(\nabla u_r(x) \neq 0\). Take \(r > 0\) such that \(B_r(x) \subset \Omega(r)\) and define

\[
\psi_\delta(y) := u_r(x) + \nabla u_r(x)(x - y) + \frac{1}{2}(D^2u_r(x) - \delta I)(x - y)^2,
\]

with \(\delta > 0\) and \(I\) the identity matrix. Notice that \(\psi_\delta(x) = u_r(x)\) and \(\psi_\delta \in C^2(B_r(x))\). Consider now the function

\[
\psi_r(y) := \begin{cases} 
\psi_\delta(y) & \text{if } y \in B_r(x), \\
u_r(y) & \text{if } y \in \mathbb{R}^n \setminus B_r(x).
\end{cases}
\]

Then, for \(\delta > 0\) big enough, \(\psi_r \leq u_r\). Also \(\psi_r \in L_g(\mathbb{R}^n)\) since \(\psi_\delta\) is bounded in \(B_r(x)\) and \(u_r \in L_g(\mathbb{R}^n)\). Finally observe that \(\nabla \psi_r(x) = \nabla u_r(x)\). Therefore we can use \(\psi_r\) as a test function for the problem solved by \(u_r\) and get

\[
(-\Delta_g)^s \psi_r(x) \geq f(x, \psi_r(x), D_g^s \psi_r(x)).
\]

Now observe that \(\psi_\delta \in C^2(\mathbb{R}^n)\), hence \(\psi_\delta \in C^2(B_1(x))\). Then, for \(y \in B_1(x)\),

\[
||\psi_\delta(x) - \psi_\delta(y)|| \leq \sup_{z \in B_1(x)} |\nabla \psi_\delta(z)||x - y| = C(\psi)|x - y|.
\]

Thus by \(2.7\) we have for \(0 < r < 1\)

\[
\int_{B_r(x)} G(|D_s \psi_\delta|) \frac{dy}{|x - y|^n} \leq \int_{B_r(x)} G(C|x - y|^{1-s}) \frac{dy}{|x - y|^n} \leq G(C) \int_{B_r(x)} |x - y|^{-n+1-s} \, dy = C(n, s, \psi)r^{1-s}.
\]

Therefore

\[
D^s_g \psi_r(x) = \int_{\mathbb{R}^n} G(|D_s \psi_r|) \frac{dy}{|x - y|^n} = \int_{B_r(x)} G(|D_s \psi_\delta|) \frac{dy}{|x - y|^n} + \int_{B_r(x)^c} G(|D_s u_r|) \frac{dy}{|x - y|^n}
\]

\[
= O(r^{1-s}) + \int_{\mathbb{R}^n \setminus B_r(x)} G(|D_s u_r|) \frac{dy}{|x - y|^n}.
\]

Then

\[
\lim_{r \to 0} D^s_g \psi_r(x) = D^s_g u_r(x). \tag{3.4}
\]

On the other hand, note that \(\nabla u_r(x) \neq 0\) implies that \(B_r(x) \subset \{d_\psi > 0\}\) for \(r\) small enough, since \(\psi_\delta \in C^2(\mathbb{R}^n)\) and \(\nabla \psi_\delta(x) = \nabla u_r(x)\). Hence, by Lemma 3.1

\[
(-\Delta_g)^s \psi_r(x) = \int_{\mathbb{R}^n} g(D_s \psi_r) \frac{dy}{|x - y|^{n+s}}
\]

\[
= \int_{\mathbb{R}^n \setminus B_r(x)} g(D_s u_r) \frac{dy}{|x - y|^{n+s}} + \int_{B_r(x)} g(D_s \psi_\delta) \frac{dy}{|x - y|^{n+s}}
\]

\[
\leq \int_{\mathbb{R}^n \setminus B_r(x)} g(D_s u_r) \frac{dy}{|x - y|^{n+s}} + o_r(1)
\]

where \(o_r(1) \to 0\) as \(r \to 0\). Then

\[
\int_{\mathbb{R}^n \setminus B_r(x)} g(D_s u_r) \frac{dy}{|x - y|^{n+s}} \geq (-\Delta_g)^s \psi_r(x) - o_r(1) \geq f(x, \psi_r(x), D_g^s \psi_r(x)) - o_r(1). \tag{3.5}
\]
Passing to the limit as \( r \to 0 \) in (3.5) and using (3.4) we get

\[
(-\Delta g)^s u_{\varepsilon}(x) = \operatorname{P.V.} \int_{\mathbb{R}^n} g(D_s u_{\varepsilon}) \frac{dy}{|x-y|^{n+s}} \geq f_{\varepsilon}(x, u_{\varepsilon}(x), D_s^* u_{\varepsilon}(x)).
\]

Now consider the case \( 1 < p^- \leq \frac{2}{2-s} \) and \( \nabla u_{\varepsilon}(x) = 0 \). By (2.7) we have for \( C \)

\[
u(x) = u_{\varepsilon}(x) \leq u(y) + \frac{|x-y|^q}{q^{q-1}}, \text{ for all } y \in \mathbb{R}^n, \quad q > \frac{sp^-}{p^- - 1} \geq 2.
\]

Define

\[
\zeta_r(y) := \begin{cases} 
u(x) - \frac{|x-y|^q}{q^{q-1}} & y \in B_r(x), \\ u_{\varepsilon}(y) & y \in \mathbb{R}^n \setminus B_r(x) \end{cases}
\]

Then \( \zeta_r \in C^2(B_r(x)) \cap L_g(\mathbb{R}^n) \) and clearly \( \zeta_r(x) = \nu(x) \) and \( \zeta_r \leq u \). Therefore we can use \( \zeta_r \) as a test function for the problem solved by \( u \) and get

\[
(-\Delta g)^s \zeta_r(x) \geq f(x, \zeta_r(x), D_s^* \zeta_r(x)) \geq f_{\varepsilon}(x, u_{\varepsilon}(x), D_s^* \zeta_r(x)). \tag{3.6}
\]

By Lemma 3.2 it holds that

\[
(-\Delta g)^s \zeta_r(x) \leq \int_{\mathbb{R}^n \setminus B_r(x)} g(D_s u_{\varepsilon}) \frac{dy}{|x-y|^{n+s}} + o_r(1).
\]

On the other hand, by (2.7) we have for \( 0 < r < 1 \)

\[
\int_{B_r(x)} G(|D_s \zeta_r|) \frac{dy}{|x-y|^n} = \int_{B_r(x)} G \left( \frac{|x-y|^q/q^{q-1}}{|x-y|^s} \right) \frac{dy}{|x-y|^n} \leq \int_{B_r(x)} G \left( \frac{1}{q^{q-1}} \right) |x-y|^{q-s-n} dy \leq C(q, \varepsilon, n, s) r^{q-s}
\]

and

\[
D_s^* \zeta_r(x) = \int_{\mathbb{R}^n \setminus B_r(x)} G(|D_s u_{\varepsilon}|) \frac{dy}{|x-y|^n} + O(r^{q-s}).
\]

Thus, passing to the limit in (3.6) we get (3.4). \( \square \)

### 3.2. Results on weak solutions

Our first result regarding weak solutions is a Caccioppoli type estimate.

**Proposition 3.1.** Let \( f \in C(\Omega \times \mathbb{R} \times \mathbb{R}) \) satisfy \( (1.2) \) and \( u \in L^\infty(\mathbb{R}^n) \) be a weak supersolution of \( (1.1) \). Then, there is a positive constant \( C = C(p, K, \varphi) \) such that

\[
\int_K \int_{\mathbb{R}^n} G(|D_s u|) G(\xi(x)) |\xi(x)| \, d\mu \leq C \left[ G(\operatorname{osc}(u)) \left( \int_K \int_{\mathbb{R}^n} G(|D_s \xi|) \, d\mu \right) + \operatorname{osc}(u) \right], \tag{3.7}
\]

for all \( \xi \in C^\infty_0(\Omega) \), \( \xi \in [0, 1] \), where \( K = \operatorname{supp}(\xi) \), and

\[
\gamma_{\infty, u} := \max \{ \gamma(t) : t \in [-\|u\|_{L^\infty(\mathbb{R}^n)}, \|u\|_{L^\infty(\mathbb{R}^n)}] \}.
\]

**Proof.** Let \( \xi \in C^\infty_0(\Omega) \), \( \xi \in [0, 1] \), and take \( K = \operatorname{supp}(\xi) \). Define

\[
\varphi(x) := \begin{cases} \sup_{\mathbb{R}^n} u - u(x) G(\xi(x)) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
Observe that
\[ \varphi(x) - \varphi(y) = -(u(x) - u(y))G(\xi(x)) + (G(\xi(x)) - G(\xi(y))) \left( \sup_{R^n} u - u(y) \right). \]
for any \( x, y \in \mathbb{R}^n \). Then, since \( u \) is a weak supersolution, we have
\[
\int_{\Omega} f(x, u, D^{s}_u) \varphi \, dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(D_s u) D_s \varphi \, d\mu
\]
\[
= -\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(D_s u) D_s u G(\xi(x)) \, d\mu + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(D_s u) D_s (G \circ \xi) \left( \sup_{R^n} u - u(y) \right) \, d\mu. \tag{3.8}
\]

Now, since \( g \) is odd, we have that \( g(t)t = g(|t|)|t| \). Hence, using the inequality \( |g(t)t| \leq p^+ \int g(|t|) \, dt \), we have
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(D_s u) D_s u G(\xi(x)) \, d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(|D_s u|) |D_s u| G(\xi(x)) \, d\mu
\]
\[
\geq p^- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G(|D_s u|) G(\xi(x)) \, d\mu.
\]
Thus, from (3.8) it follows that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G(|D_s u|) G(\xi(x)) \, d\mu \leq (I) - (II), \tag{3.9}
\]
where
\[
(I) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(D_s u) D_s (G \circ \xi) \left( \sup_{R^n} u - u(y) \right) \, d\mu, \quad \text{and} \quad (II) = \int_{\Omega} f(x, u, D^{s}_u) \varphi \, dx.
\]

We first treat the integral \((I)\). By (2.6) and (2.8), we have
\[
(I) \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(|D_s u|) |D_s \xi(g(\xi(x)) + g(\xi(y))) \, d\mu
\]
\[
\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(|D_s u|) |g(\xi(x))| |D_s \xi| \, d\mu
\]
\[
\leq C \left[ \delta \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{G}(g(|D_s u|)g(\xi(x))) \, d\mu + C_\delta G(\sup_{\mathbb{R}^n} u) \int_{\mathbb{R}^n} G(|D_s \xi|) \, d\mu \right]. \tag{3.10}
\]

By the \( \Delta' \)-condition for \( \tilde{G} \) and Lemma 2.1
\[
\int_{K} \int_{\mathbb{R}^n} \tilde{G}(g(|D_s u|)g(\xi(x))) \, d\mu \leq C \int_{K} \int_{\mathbb{R}^n} \tilde{G}(g(|D_s u|)) \tilde{G}(g(\xi(x))) \, d\mu
\]
\[
\leq C \int_{K} \int_{\mathbb{R}^n} G(|D_s u|) G(\xi(x)) \, d\mu. \tag{3.11}
\]

As a result, from (3.10) and (3.11), there holds
\[
(I) \leq C \left[ \delta \int_{K} \int_{\mathbb{R}^n} G(|D_s u|) G(\xi(x)) \, d\mu + C_\delta G(\sup_{\mathbb{R}^n} u) \int_{K} \int_{\mathbb{R}^n} G(|D_s \xi|) \, d\mu \right]. \tag{3.12}
\]

Now, we estimate the term \((II)\) in (3.9). By the assumption (1.2), we get
\[
(II) \leq \gamma_{\infty,u} \int_{K} \tilde{G}^{-1}(|D^s u|) \, osc(u) G(\xi(x)) \, dx + \|\phi\|_{L^\infty(\Omega)} |K| osc(u). \tag{3.13}
\]
Now, by (3.12) and (3.15), and choosing \( \delta \)

\[
\text{let } \delta > 0 \text{ small enough, we derive,}\]

\[
\text{Thus, combining (3.13)-(3.14), we get}\]

\[
\int_K \tilde{G}^{-1}(|D^s_y u|) \text{osc}(u)G(\xi(x)) \, dx \leq \int_K \tilde{G}^{-1}(|D^s_y u|) \text{osc}(u)G(\xi(x)) \, dx
\]

\[
\leq \delta \int_K \tilde{G} \left( \tilde{G}^{-1}(|D^s_y u|) \right) \, dx + C_G(\text{osc}(u))|K|
\]

\[
= \delta \int_K |D^s_y u| \tilde{G}(g(\xi(x))) \, dx + C_G(\text{osc}(u))|K|.
\]

Moreover, by Lemma 2.1, it follows that

\[
\int_K |D^s_y u| \tilde{G}(g(\xi(x))) \leq \int_K |D^s_y u| G(\xi(x)) \, dx = \int_K \int_{\mathbb{R}^n} G(|D^s_y u|) G(\xi(x)) \, d\mu \tag{3.14}
\]

Thus, combining (3.13) and (3.14), we get

\[
(II) \leq \gamma_{\infty,u} \left( \delta \int_K \int_{\mathbb{R}^n} G(|D^s_y u|) G(\xi(x)) \, d\mu + C_G(\text{osc}(u))|K| \right) + \|\phi\|_{L^\infty(\Omega)}|K| \text{osc}(u). \tag{3.15}
\]

From (3.13) and (3.14), and choosing \( \delta \) small enough, we derive (3.15).

The next lemma treats the convergence of the sources \( f_\varepsilon \).

**Lemma 3.4.** Let \( u \in W^{s,G}(\Omega) \cap L^\infty(\Omega) \). Suppose that \( f = f(x,t,\eta) \) is uniformly continuous in \( \Omega \times \mathbb{R} \times \mathbb{R} \), Lipschitz continuous in \( \eta \), and satisfies (1.2). Let \( \psi \in C^1_0(\Omega) \), \( \psi \geq 0 \) with \( K = \text{supp}(\psi) \subset \Omega \). If

\[
\lim_{\varepsilon \to 0} \int_K \int_{\mathbb{R}^n} \left( \frac{|u_\varepsilon(y) - u_\varepsilon(x) - (u(x) - u(y))|}{|x - y|^s} \right) \, d\mu = 0, \tag{3.16}
\]

then

\[
\lim_{\varepsilon \to 0} \int_K f_\varepsilon(x, u_\varepsilon, D^s_y u_\varepsilon) \psi \, dx = \int_K f(x, u, D^s_y u) \psi \, dx.
\]

**Proof.** Let \( \varepsilon > 0 \), \( \psi \) and \( K \) as in the statement. By the uniformly continuity of \( f \), for every \( \rho > 0 \), there exists \( \delta > 0 \) such that

\[
|f(x, u_\varepsilon, D^s_y u_\varepsilon) - f(y, u_\varepsilon, D^s_y u_\varepsilon)| \leq \rho, \quad y \in B_\delta(x).
\]

Hence,

\[
\int_K |f(x, u_\varepsilon, D^s_y u_\varepsilon) - f(y, u_\varepsilon, D^s_y u_\varepsilon)| \psi \, dx \leq \rho \|\psi\|_{L^\infty(K)}|K|.
\]

Since \( \|u_\varepsilon\|_{L^\infty} \leq C \) for all \( \varepsilon \), it follows that

\[
\max_{[-\|u\|_{L^\infty},\|u\|_{L^\infty}]} |\gamma(t)| \leq \max_{[-\|u\|_{L^\infty},\|u\|_{L^\infty}]} |\gamma(t)|,
\]

and then we have

\[
|f(x, u_\varepsilon, D^s_y u)| \leq C \tilde{G}^{-1}(|D^s_y u|) + \varphi(x) \in L^\infty(K) \subset L^1(K)
\]

for a constant \( C \) independent of \( \varepsilon \). Then, by Lebesgue Convergence Theorem,

\[
\lim_{\varepsilon \to 0} \int_K f(x, u_\varepsilon, D^s_y u) \psi \, dx = \int_K f(x, u, D^s_y u) \psi \, dx. \tag{3.17}
\]
Moreover, the Lipschitz assumption of \( f \) in \( \eta \) gives

\[
\int_K |f(x, u_\varepsilon, D_\varepsilon^s u_\varepsilon) - f(x, u_\varepsilon, D_\varepsilon^s u)| \psi \, dx \leq C \int_K |D_\varepsilon^s u_\varepsilon - D_\varepsilon^s u| \, dx \\
\leq C \int_K \int_{\mathbb{R}^n} G \left( \frac{|u_\varepsilon(x) - u_\varepsilon(y)|}{|x - y|^s} \right) - G \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \, d\mu \\
\leq C \int_K \int_{\mathbb{R}^n} |h_\varepsilon(x, y)| g \left( \frac{|u(x) - u(y)| + h_\varepsilon(x, y)}{|x - y|^s} \right) \, d\mu, \tag{3.18}
\]

where, in the last inequality, we have used

\[
|G(a + b) - G(b)| \leq |b|g(|a| + |b|), \quad a, b \in \mathbb{R},
\]

and

\[
h_\varepsilon(x, y) = u_\varepsilon(x) - u_\varepsilon(y) - (u(x) - u(y)), \quad x \in K, \ y \in \mathbb{R}^n.
\]

Using Young's inequality with \( \delta > 0 \) and Lemma 2.1, we get

\[
\int_K \int_{\mathbb{R}^n} |h_\varepsilon(x, y)| g \left( \frac{|u(x) - u(y)| + h_\varepsilon(x, y)}{|x - y|^s} \right) \, d\mu + \delta \int_K \int_{\mathbb{R}^n} G \left( \frac{|u(x) - u(y)| + h_\varepsilon(x, y)}{|x - y|^s} \right) \, d\mu. \tag{3.19}
\]

Observe that by the inequality \( G(s + t) \leq C(G(s) + G(t)) \) and the assumption (3.16), the integrals

\[
\int_K \int_{\mathbb{R}^n} G \left( \frac{|u(x) - u(y)| + h_\varepsilon(x, y)}{|x - y|^s} \right) \, d\mu
\]

remains uniformly bounded. Hence, taking \( \limsup \) as \( \varepsilon \to 0 \) in (3.19), using (3.16), and then \( \delta \to 0 \), we get from (3.18) that

\[
\int_K |f(x, u_\varepsilon, D_\varepsilon^s u_\varepsilon) - f(x, u_\varepsilon, D_\varepsilon^s u)| \psi \, dx \to 0 \quad \text{as} \ \varepsilon \to 0. \tag{3.20}
\]

Then, combining (3.17) and (3.20), we have

\[
\lim_{\varepsilon \to 0} \int_K |f_\varepsilon(x, u_\varepsilon, D_\varepsilon^s u_\varepsilon) - f(x, u_\varepsilon, D_\varepsilon^s u)| \psi \, dx \\
\leq \lim_{\varepsilon \to 0} \int_K |f_\varepsilon(x, u_\varepsilon, D_\varepsilon^s u_\varepsilon) - f(x, u_\varepsilon, D_\varepsilon^s u)| \psi \, dx + \int_K |f(x, u_\varepsilon, D_\varepsilon^s u) - f(x, u_\varepsilon, D_\varepsilon^s u)| \psi \, dx = 0.
\]

This concludes the proof. \( \square \)

In the next two results we will study the relation between the weak and the pointwise formulation of solutions. We distinguish two cases: \( p^- > \frac{2}{n} \) and \( 1 < p^- \leq \frac{2}{n} \).

**Lemma 3.5.** Assume \( p^- > \frac{2}{n} \) and let \( u \in L_g(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). Then, for all \( \psi \in C_0^\infty(\Omega_{r(\varepsilon)}) \), \( \psi \geq 0 \), we have

\[
\int_K \int_{\mathbb{R}^n} g(D_s u_\varepsilon) \, D_s \psi \, d\mu \geq \int_K ((-\Delta_g)^{s} u_\varepsilon) \, \psi \, dx
\]

where \( u_\varepsilon \) is the infimal convolution of \( u \), \( K \) is the support of \( \psi \), and

\[
Q_K := (K \times K) \cup ([\mathbb{R}^n \setminus K] \times K) \cup [K \times (\mathbb{R}^n \setminus K)].
\]
Proof. Let $u_{\varepsilon, \delta}$ be smooth and semiconcave functions converging to $u_\varepsilon$ given by

$$u_{\varepsilon, \delta} := (u_\varepsilon \ast \eta_\delta) \chi_{\Omega_{r(\varepsilon)}} + u_\varepsilon \chi_{\mathbb{R}^n \setminus \Omega_{r(\varepsilon)}},$$

where $\eta_\delta$ is the standard mollifier with support in $B_1$. Observe that $u_{\varepsilon, \delta} \in C^2(\Omega_{r(\varepsilon)}) \cap L^2_2(\mathbb{R}^n)$. Moreover, for any $\psi \in C_0^\infty(\Omega_{r(\varepsilon)})$, with $K = \text{supp} \psi$, we have recalling that $g$ is odd that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(D_s u_{\varepsilon, \delta}) \frac{\psi(x) - \psi(y)}{|x-y|^{s+n}} \, dx \, dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(D_s u_{\varepsilon, \delta}) \frac{\psi(y)}{|x-y|^{n+s}} \, dx \, dy = \int_K (\Delta g)^s u_{\varepsilon, \delta}) \psi \, dx.$$  

We shall take the limit as $\delta \to 0$. First, we show that

$$\lim_{\delta \to 0} \int_{Q_K} g(D_s u_{\varepsilon, \delta}) \frac{\psi(x) - \psi(y)}{|x-y|^{s+n}} \, dx \, dy = \int_{Q_K} g(D_s u_\varepsilon) \frac{\psi(x) - \psi(y)}{|x-y|^{s+n}} \, dx \, dy. \tag{3.21}$$

Let

$$F_\delta(x, y) := g(D_s u_{\varepsilon, \delta}) \frac{\psi(x) - \psi(y)}{|x-y|^{s+n}}.$$  

By symmetry,

$$\int_{Q_K} F_\delta(x, y) \, dx \, dy = \left(2 \int_{K \times (\mathbb{R}^n \setminus K)} + \int_{K \times K} \right) F_\delta(x, y) \, dx \, dy.$$  

Hence, in order to get (3.21), it is enough to prove that

$$|F_\delta(x, y)| \leq F(x, y)$$

for any $\delta > 0$ and some $F \in L^1(K \times \mathbb{R}^n)$. Let us consider

$$r := \text{dist} \left( K, \partial \Omega_{r(\varepsilon)} \right) \quad \text{and} \quad K_{r/2} := \left\{ y \in \Omega_{r(\varepsilon)} : \text{dist} \ (y, K) \leq \frac{r}{2} \right\}.$$  

By Young’s inequality with $\delta = 1$, and Lemma 2.1, we have for $(x, y) \in K \times K_{r/2}$

$$|F_\delta(x, y)| \leq g(D_s u_{\varepsilon, \delta}) \frac{\psi(x) - \psi(y)}{|x-y|^{s+n}} \leq \frac{C}{|x-y|^n} \left( \hat{G} \left( g(|D_s u_{\varepsilon, \delta}|) \right) + G \left( |D_s \psi| \right) \right)$$

$$\leq \frac{C}{|x-y|^n} \left( G \left( |D_s u_{\varepsilon, \delta}| \right) + G \left( |D_s \psi| \right) \right)$$

$$\leq C \left( |x-y|^{(1-s)p^+ - n} + |x-y|^{(1-s)p^- - n} \right) \in L^1(K \times K_{r/2})$$

with a constant $C$ independent of $\delta$. Next, we consider the case $(x, y) \in K \times (\mathbb{R}^n \setminus K_{r/2})$. Since $\text{dist} \ (K, \mathbb{R}^n \setminus K_{r/2}) > 0$ and $K$ is bounded, by [11] Lemma A.5 there is $C > 0$ such that

$$|x-y| \geq C(1+|y|).$$

Hence, for any $\delta > 0$,

$$|F_\delta(x, y)| \leq \frac{C}{1+|y|^{n+s}} g \left( \frac{1+|u_\varepsilon(y)|}{1+|y|^s} \right) \leq \frac{C}{1+|y|^{n+s}} \left[ g \left( \frac{1}{1+|y|^s} \right) + g \left( \frac{|u_\varepsilon(y)|}{1+|y|^s} \right) \right]. \tag{3.22}$$

and thus, since $u_\varepsilon \in L_g(\mathbb{R}^n)$, we get that the right-hand side of (3.22) belongs to $L^1(K \times (\mathbb{R}^n \setminus K_{r/2}))$. As a result, by dominated convergence Theorem, (3.21) holds.
The next step is to prove that
\[
\liminf_{\delta \to 0} \int_{K} (-\Delta g)^* u_{\delta, e} \psi \, dx \geq \int_{K} \liminf_{\delta \to 0} (-\Delta g)^* u_{\delta, e} \psi \, dx,
\] (3.23)
by appealing to Fatou Lemma. Hence, it is enough to find \( C \) so that
\[
(-\Delta g)^* u_{\delta, e}(x) \geq -C,
\]
for all \( x \in K \) and all \( \delta \). So let \( r > 0 \) so that \( r < \text{dist} \left( K, \partial \Omega_{r(z)} \right) \). Then \( B_{r}(x) \subset \Omega_{r(z)} \) for all \( x \in K \), and for \( y \in \mathbb{R}^n \setminus B_{r}(x) \), there holds \( |x - y| > r \). Now, again by \([1],[8], \text{Lemma A. 5}\]
\[
\left| \int_{\mathbb{R}^n \setminus B_{r}(x)} g(D_{s} u_{\delta, e}) \frac{dy}{|x - y|^{n+s}} \right| \leq C \int_{\mathbb{R}^n \setminus B_{r}(x)} g \left( \frac{1 + |u_{e}(y)|}{1 + |y|^{s}} \right) \frac{dy}{|x - y|^{n+s}},
\]
and the latter integral is uniformly bounded in \( x \) since \( u_{e} \in L_{q}(\mathbb{R}^n) \). Let us now estimate the integral over the ball \( B_{r}(x) \). By Lemma 2.4 there is \( C > 0 \) such that
\[
D^2 u_{e}(x) \leq CI, \quad a.e. \ x \in \Omega_{r(z)},
\] (3.24)
Hence, by symmetry and the fact that \( g \) is odd, we have
\[
I_{B_{r}(x)} := P.V. \int_{B_{r}(x)} g(D_{s} u_{\delta, e}) \frac{dx \, dy}{|x - y|^{n+s}}
= \int_{B_{r}(x)} \left[ g(D_{s} u_{\delta, e}) - g \left( -\nabla u_{e, \delta}(y - x) \right) \right] \frac{dx \, dy}{|x - y|^{n+s}}.
\]
Since for all \( a, b \in \mathbb{R} \),
\[
g(b) - g(a) = (a - b) \int_{0}^{1} g'(ta + (1 - t)b) \, dt,
\]
putting \( a = D_{s} u_{\delta, e} \), and \( b = -\nabla u_{e, \delta}(y - x) \), we have
\[
I_{B_{r}(x)} = \int_{B_{r}(x)} \frac{u_{e, \delta}(y) - u_{e, \delta}(y) + \nabla u_{e, \delta}(y - x)}{|x - y|^{n+2s}} \left( \int_{0}^{1} g'(ta + (1 - t)b) \, dt \right) dy
= \int_{B_{r}(x)} \frac{-D^2 u_{e, \delta}(z)(x - y)^2}{|x - y|^{n+2s}} \left( \int_{0}^{1} g'(ta + (1 - t)b) \, dt \right) dy
\geq \int_{B_{r}(x)} \frac{-D^2 u_{e, \delta}(z)(x - y)^2}{|x - y|^{n+2s}} \left( \int_{0}^{1} g'(ta + (1 - t)b) \, dt \right) dy,
\] (3.25)
where \( z \in B_{r}(x) \) and \( B_{r}(x)^{+} := \{ y \in B_{r}(x) : D^2 u_{e, \delta}(y) \geq 0 \} \). Next, assume that \( p^{-} \geq 2 \). Then, by (2.1) and (2.5), we get
\[
\left| \int_{0}^{1} g'(ta + (1 - t)b) \, dt \right| \leq C \int_{0}^{1} \left| \frac{g(ta + (1 - t)b)}{ta + (1 - t)b} \right| \, dt
\leq C \left( \int_{0}^{1} \left| ta + (1 - t)b \right|^{p^{-} - 2} \, dt + \int_{0}^{1} \left| ta + (1 - t)b \right|^{p^{-} - 2} \, dt \right).
\]
By Lemma 2.4 in \([4]\) and since \( \| \nabla u_{e, \delta} \|_{L^{\infty}(K)} \leq C \) independently of \( \delta \), it follows that
\[
\int_{0}^{1} \left| ta + (1 - t)b \right|^{p^{-} - 2} \, dt + \int_{0}^{1} \left| ta + (1 - t)b \right|^{p^{-} - 2} \, dt
\leq C \left( |a|^{p^{-} - 2} + |b|^{p^{-} - 2} + |a|^{p^{-} - 2} + |b|^{p^{-} - 2} \right)
\leq C \left( |x - y|^{1-s(p^{-} - 2)} + |x - y|^{1-s(p^{-} - 2)} \right) \quad (y \in B_{r}(x)).
\] (3.26)
Plugging (3.26) into (3.25), we get

$$I_{B_r(x)} \geq -C \int_{B_r(x)} \left( |x-y|(1-s)(p^- - 2)^2 + 2 - n^{-2s} + |x-y|(1-s)(p^- - 2)^2 + 2 - n^{-2s} \right) dy \geq -C,$$

uniformly in $\delta$.

Next, assume that $\frac{2}{p^-} < p^- < 2$. Then, we have again from Lemma 2.4 in [4] that

$$0 \leq \int_0^1 |g'(at + (1 - t)b)| \, dt \leq C \left( \int_0^1 |at + (1 - t)b|^{p^- - 2} \, dt + \int_0^1 |at + (1 - t)b|^{p^- + p^- - 2} \, dt \right) \leq C \left( \int_0^1 |at + (1 - t)b|^{p^- - 2} \, dt + \int_0^1 \left( |a|^{p^- - 2} + |b|^{p^- - 2} \right) |at + (1 - t)b|^{p^- - 2} \, dt \right) \leq C \left( |a|^{p^- - 2} + |b|^{p^- - 2} + 1 \right) |a - b|^{p^- - 2}.$$

Thus, by (3.24), and the assumption $p^- > \frac{2}{2-s}$, it follows

$$I_{B_r(x)} \geq -C \int_{B_r(x)} |x-y|(2 - s)(p^- - 2)^2 + 2 - n^{-2s} dy \geq -C,$$

with $C$ independent of $\delta$.

In any case, we have

$$(-\Delta_g)^s u_{\varepsilon, \delta} \geq -C$$

in $K$.

Hence, applying Fatou’s Lemma, we obtain (3.27).

Finally, we will show that

$$\int_K \lim_{\delta \to 0} (-\Delta_g)^s u_{\varepsilon, \delta} \psi \, dx \geq \int_K (-\Delta_g)^s u_{\varepsilon} \psi \, dx. \tag{3.27}$$

Write

$$(-\Delta_g)^s u_{\varepsilon, \delta}(x) = \int_{\mathbb{R}^n \setminus B_r(x)} g(D_s u_{\varepsilon, \delta})(y) \frac{dy}{|x-y|^{n+s}} + \int_{B_r(x)} g(D_s u_{\varepsilon, \delta}) - g \left( \frac{\nabla u_{\varepsilon, \delta}(y-x)}{|x-y|^{s}} \right) \frac{dy}{|x-y|^{n+s}}.$$

Then, we first have in $\mathbb{R}^n \setminus B_r(x)$ that

$$g(D_s u_{\varepsilon, \delta}) \frac{1}{|x-y|^{n+s}} \geq -C g \left( \frac{1 + |u_{\varepsilon}(y)|}{1 + |y|^{s}} \right) \frac{1}{1 + |y|^{n+s}}.$$

Now, in the ball $B_r(x)$,

$$\left[ g(D_s u_{\varepsilon, \delta}) - g \left( \frac{\nabla u_{\varepsilon, \delta}(y-x)}{|x-y|^{s}} \right) \right] \frac{1}{|x-y|^{n+s}} \geq -C \mathcal{F}(x,y),$$

where

$$\mathcal{F}(x,y) := \begin{cases} |x-y|(1-s)(p^- - 2)^2 + 2 - n^{-2s} + |x-y|(1-s)(p^- - 2)^2 + 2 - n^{-2s} & \text{if } p^- \geq 2, \\ |x-y|(2-s)(p^- - 2)^2 + 2 - n^{-2s} & \text{if } p^- < 2. \end{cases}$$
Since 
\[ g \left( \frac{1 + |u_z(y)|}{1 + |y|^s} \right) \frac{1}{1 + |y|^{n+s}} \in L^1(\mathbb{R}^n \setminus B_r(x)) \quad \text{and} \quad \mathcal{F}(x, \cdot) \in L^1(B_r(x)), \]
by Fatou’s Lemma, we have that \((3.27)\) holds. This ends the proof of the lemma. \(\blacksquare\)

Now, we state the counterpart of Lemma 3.5 for the range \(1 < p^- \leq \frac{2}{2-s}\).

**Lemma 3.6.** Assume \(1 < p^- \leq \frac{2}{2-s}\) and let \(u \in L_p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\). Then, for all \(\psi \in C_0^\infty(\Omega_{r(c)}), \psi \geq 0\), we have
\[ \int \int_{Q_K} g(D_s u_z) D_s \psi \, d\mu \geq \int_K ((-\Delta)^s u_z) \, \psi \, dx \]
where \(u_z\) is the infimal convolution of \(u\), \(K\) is the support of \(\psi\), and
\[ Q_K = (K \times K) \cup [([\mathbb{R}^n \setminus K] \times K) \cup [K \cup ([\mathbb{R}^n \setminus K])]. \]

**Proof.** For \(\rho > 0\) and non negative \(\psi \in C_0^\infty(\Omega)\), we have
\[ \int \int_{Q_K} g \left( \frac{u_z(x) - u_z(y)}{|x - y| + \rho}^s \right) D_s \psi \, d\mu \geq \int_K \int_{\mathbb{R}^n} g \left( \frac{u_z(x) - u_z(y)}{|x - y| + \rho}^s \right) \frac{\psi(x)}{|x - y|^{n+s}} \, dx \, dy, \quad (3.28) \]
since
\[ Q_K = (K \times K) \cup [([\mathbb{R}^n \setminus K] \times K) \cup [K \cup ([\mathbb{R}^n \setminus K])]. \]

Now, for \(x, y \in K\), it holds since \(u_z\) is locally Lipschitz, that
\[ g \left( \frac{u_z(x) - u_z(y)}{|x - y| + \rho}^s \right) \frac{\psi(x) - \psi(y)}{|x - y|^{n+s}} \leq C \left( \frac{|u_z(x) - u_z(y)|^{p^- - 1}}{|x - y|^{n+s} p^-} + \frac{|u_z(x) - u_z(y)|^{p^- - 1}}{|x - y|^{n+s} p^-} \right) \]
\[ \leq C \left( |x - y|^{1-s} p^- - n + |x - y|^{(1-s) p^- - n} \right). \quad (3.29) \]
On the other hand, if \(x \in K\) and \(y \in \mathbb{R}^n \setminus K\), recalling that supp \(\psi \subset \subset K\), we have
\[ \left| g \left( \frac{u_z(x) - u_z(y)}{|x - y| + \rho}^s \right) \frac{\psi(x) - \psi(y)}{|x - y|^{n+s}} \right| \leq C g \left( \frac{1 + |u_z(y)|}{1 + |y|^s} \right) \frac{1}{1 + |y|^s}. \quad (3.30) \]
Hence, from \((3.29)\) and \((3.30)\), we may apply Lebesgue theorem to let \(\rho \to 0\) in \((3.28)\) and get
\[ \lim_{\rho \to 0} \int \int_{Q_K} g \left( \frac{u_z(x) - u_z(y)}{|x - y| + \rho}^s \right) D_s \psi \, d\mu = \int \int_{Q_K} g \left( D_s u_z \right) D_s \psi \, d\mu. \]
To treat the right-hand side of \((3.28)\), we introduce the function
\[ \mathcal{F}_\rho(x) := \int_{\mathbb{R}^n} g \left( \frac{u_z(x) - u_z(y)}{|x - y| + \rho}^s \right) \frac{1}{|x - y|^{n+s}} \, dy, \quad x \in K. \]
We will prove that there is \(G \in L^1(K)\) such that
\[ \mathcal{F}_\rho(x) \psi(x) \geq -G(x), \quad (3.31) \]
in \(K\). Hence, we will apply Fatou’s Lemma to \(\mathcal{F}_\rho(x) \psi(x) + G(x)\) to get
\[ \liminf_{\rho \to 0} \int_K \mathcal{F}_\rho(x) \psi(x) \, dx \geq \int_K \liminf_{\rho \to 0} \mathcal{F}_\rho(x) \psi(x) \, dx. \]
In order to prove (3.31), let \( x \in K \) and choose \( \eta > 0 \) such that \( B_\eta(x) \subset \Omega_{r(\varepsilon)} \). By Lemma A.5 in [11], for any \( y \in \mathbb{R}^n \setminus B_\eta(x) \)

\[
|x - y| \geq \left( \frac{\eta}{1 + \eta + R} \right) (1 + |y|)
\]

where \( R > 0 \) satisfies that the ball \( B_R(0) \) contains \( K \). Hence, there is a constant \( C > 0 \) independent of \( x \) and \( \rho \) such that

\[
\left| \int_{\mathbb{R}^n \setminus B_\eta(x)} g \left( \frac{u_\varepsilon(x) - u_\varepsilon(y)}{|x - y| + \rho} \right) \frac{dy}{|x - y|^{n+s}} \right| \leq C \int_{\mathbb{R}^n \setminus B_\eta(x)} g \left( \frac{1 + |u_\varepsilon(y)|}{1 + |y|^s} \right) \frac{dy}{1 + |y|^{s+n}}.
\]

(3.32)

Now, we study the integrals in the ball \( B_\eta(x) \). By Lemma 2.3 for each \( x \in \Omega_{r(\varepsilon)} \), there is \( \hat{x} \in B_{r(\varepsilon)}(x) \)

such that

\[
u(x) = \varphi_\varepsilon(x),
\]

where

\[
\varphi_\varepsilon(y) := u(z) + |y - z|^q + \frac{|y - z|^q}{q^{q-1}}, \quad y, z \in \mathbb{R}^n.
\]

with \( q > \frac{sp}{(p - 1)} \). Then

\[
u(x) - u_\varepsilon(y) = \varphi_\varepsilon(x) - \inf_{z \in \mathbb{R}^n} \varphi_\varepsilon(y) \geq \varphi_\varepsilon(x) - \varphi_\varepsilon(y).
\]

Since \( g \) is non-decreasing,

\[
g \left( \frac{\nu(x) - u_\varepsilon(y)}{|x - y| + \rho} \right) \geq g \left( \frac{\varphi_\varepsilon(x) - \varphi_\varepsilon(y)}{|x - y| + \rho} \right).
\]

Hence

\[
\int_{B_\eta(x)} g \left( \frac{\nu(x) - u_\varepsilon(y)}{|x - y| + \rho} \right) \frac{dy}{|x - y|^{s+n}} \geq \int_{B_\eta(x)} g \left( \frac{\varphi_\varepsilon(x) - \varphi_\varepsilon(y)}{|x - y| + \rho} \right) \frac{dy}{|x - y|^{s+n}}.
\]

(3.33)

Observe that by (3.32) and by (3.33), the inequality (3.31) is stated if we additionally prove that

\[
\left| \int_{B_\eta(x)} g \left( \frac{\varphi_\varepsilon(x) - \varphi_\varepsilon(y)}{|x - y| + \rho} \right) \frac{dy}{|x - y|^{s+n}} \right| \leq C.
\]

(3.34)

Let \( \eta_0 \leq \eta \) small. First, suppose that \( x \notin B_{\eta_0}(\hat{x}) \). By Lemma 2.3

\[
\|\varphi_\varepsilon\|_{L^\infty(\Omega)} \leq C, \quad \|\nabla \varphi_\varepsilon(y)\| = \frac{|\hat{x} - y|^{q-1}}{\varepsilon^{q-1}} \leq C,
\]

and

\[
-CI \leq -\frac{q - 1}{\varepsilon^{q-1}} |\hat{x} - y|^{q-2} I \leq D^2 \varphi_\varepsilon(y) \leq \frac{q - 1}{\varepsilon^{q-1}} |\hat{x} - y|^{q-2} I \leq C(\varepsilon)I.
\]

Observe that \( |\nabla \varphi_\varepsilon(x)| \neq 0 \) since \( x \neq \hat{x} \). Now, since \( p^- < 2 \), we have reasoning as in the proof of [4].
Lemma 3.5] and letting \( L(y) = \varphi_\pm(x) + \nabla \varphi_\pm(x)(y - x) \), that
\[
\left| \int_{B_\eta(x)} g \left( \frac{\varphi_\pm(x) - \varphi_\pm(y)}{|x - y| + \rho} \right) \frac{dy}{|x - y|^{s+n}} \right| \\
\leq C \int_{B_\eta(x)} g \left( \frac{-\nabla \varphi_\pm(x)(y - x) - (y - x)^T D^2 \varphi_\pm(z)(y - x)}{|x - y| + \rho} \right) \frac{dy}{|x - y|^{s+n}} \quad \text{(for } z \in B_\eta(x)) \\
\leq C \int_{B_\eta(x)} g \left( \frac{-\nabla \varphi_\pm(x)(y - x) - (y - x)^T D^2 \varphi_\pm(z)(y - x)}{|x - y| + \rho} \right) - g \left( \frac{L(x) - L(y)}{|x - y| + \rho} \right) \frac{dy}{|x - y|^{s+n}} \\
\leq C \left[ \int_{B_\eta(x)} \frac{|\nabla \varphi_\pm(x)(y - x)| + |D^2 \varphi_\pm(z)||x - y|^2}{|x - y|^{n+sp^+}} dy \\
+ \int_{B_\eta(x)} \frac{|\nabla \varphi_\pm(x)(y - x)| + |D^2 \varphi_\pm(z)||x - y|^2}{|x - y|^{n+sp^-}} dy \right],
\tag{3.35}
\end{equation}
\end{align}
\end{equation}
\end{align}
\end{equation}

where in the last inequality we have used Lemma 3.7. Next, observe that since \( p^+ \geq p^- \) we have
\[
H(x, y, p^+):= \frac{|\nabla \varphi_\pm(x)(y - x)| + |D^2 \varphi_\pm(z)||x - y|^2}{|x - y|^{n+sp^+}} \\
= \frac{|\nabla \varphi_\pm(x)(y - x)| + |D^2 \varphi_\pm(z)||x - y|^2}{|x - y|^{n+(p^+-p^-)}} H(x, y, p^-) \\
\leq C(\varepsilon) \left( |x - y|^{(1-s)(p^+-p^-)} + |x - y|^{2s(p^+-p^-)} \right) H(x, y, p^-) \\
\leq C(\varepsilon) H(x, y, p^-)
\]
for any \( y \in B_\eta(x) \). Then
\[
\left| \int_{B_\eta(x)} g \left( \frac{\varphi_\pm(x) - \varphi_\pm(y)}{|x - y| + \rho} \right) \frac{dy}{|x - y|^{s+n}} \right| \leq C(\varepsilon) \int_{B_\eta(x)} H(x, y, p^-) dy
\]
for \( z \in B_\eta(x) \). Thus, taking \( \tau_\infty(z) := \sup_{B_\eta(x)} |D^2 \varphi_\pm| \), we have
\[
\int_{B_\eta(x)} H(x, y, p^-) dy \leq \tau_\infty(z) \int_{B_\eta(x)} \frac{|\nabla \varphi_\pm(x)||x - y| + \tau_\infty |y - x|^2}{|x - y|^{n+sp^-}} |x - y|^2 dy, \\
\leq C \tau_\infty(z) \int_{0}^{\tau} \left( 1 + \frac{r}{|\nabla \varphi_\pm(x)|} \right)^{p^- - 2} |\nabla \varphi_\pm(x)|^{p^- - 1} \frac{dr}{r} \leq C(\varepsilon).
\tag{3.36}
\end{equation}
\end{align}
\end{equation}
\end{equation}

Assume now that \( x \in B_{\eta}(\hat{x}) \). Since \( p^- < 2 \), we proceed following the proof of [18, Lemma 3.7]. Notice first that
\[
\sup_{y \in B_{r}(x)} |D^2 \varphi_\pm(y)| \leq \sup_{y \in B_{r}(x)} C|y - \hat{x}|^{q-2} \leq C(r + |x - \hat{x}|)^{q-2},
\]
for any \( x \in B_{\eta}(\hat{x}) \).
for a constant $C$ depending on $q$ and $\varepsilon$. Working as in (3.35), we have

$$
\int_{B_n(x)} \frac{(|\nabla \varphi_\beta(x)(y-x)| + |D^2 \varphi_\beta(z)||x-y|^2)^{p-2}}{|x-y|^{n+sp}} |D^2 \varphi_\beta(z)||x-y|^2 \, dy
$$

$$
\leq \int_0^{\eta} \left(1 + \frac{(r + |x - \hat{x}|)^{q-2}}{|\nabla \varphi_\beta(x)|}\right)^{p-2} (r + |x - \hat{x}|)^{q-2}|\nabla \varphi_\beta(x)|^{p-2} r^{p-(1-s)-1} dr
$$

$$
\leq C \left( \int_0^{\eta} \left(1 + \frac{(r + |x - \hat{x}|)^{q-2}}{|\nabla \varphi_\beta(x)|}\right)^{p-2} (r + |x - \hat{x}|)^{q-2}|\nabla \varphi_\beta(x)|^{p-2} r^{p-(1-s)-1} dr + \int_0^{\eta} \frac{r^{n-1}}{|\nabla \varphi_\beta(x)|} r^{p-2} |\nabla \varphi_\beta(x)|^{p-2} r^{p-(1-s)-1} dr \right)
$$

$$
\leq C \left( \eta^{(p-1)-sp} + \eta^{(p-1)-sp} \right),
$$

taking $q > \frac{sp}{p-1}$. We point out that $C$ depends on $\varepsilon$, but is independent of $x$, $\hat{x}$ and $\rho$.

We apply the arguments in to get (3.34). Hence, combining (3.32) and (3.34), we finally get (3.31).

Now, we will apply Fatou’s Lemma again to get

$$
\lim_{\rho \to 0} \mathcal{F}_\rho(x) \geq \int_{\mathbb{R}^n} g(D_n u_\varepsilon) \frac{1}{|x-y|^{n+s}} \, dy.
$$

Observe that by symmetry,

$$
\mathcal{F}_\rho(x) = \int_{\mathbb{R}^n} \left[ g \left( \frac{u_\varepsilon(x) - u_\varepsilon(y)}{|x-y| + \rho} \right) - g \left( \frac{-\nabla \varphi_\beta(x)(y-x) \chi_{B_n}(y)}{|x-y| + \rho} \right) \right] \frac{1}{|x-y|^{n+s}} \, dy.
$$

We will prove that there is a $\mathcal{F} \in L^1(\mathbb{R}^n)$ such that

$$
I(y) := \left[ g \left( \frac{u_\varepsilon(x) - u_\varepsilon(y)}{|x-y| + \rho} \right) - g \left( \frac{-\nabla \varphi_\beta(x)(y-x) \chi_{B_n}(y)}{|x-y| + \rho} \right) \right] \frac{1}{|x-y|^{n+s}} \geq -\mathcal{F}(y)
$$

in $\mathbb{R}^n$. First, if $y \in \mathbb{R}^n \setminus B_\eta(x)$, then

$$
|I(y)| \leq C g \left( \frac{1 + |u_\varepsilon(y)|}{1 + |y|} \right) \frac{1}{1 + |y|^{n+s}}.
$$

Hence, the function

$$
\mathcal{F}(y) := C g \left( \frac{1 + |u_\varepsilon(y)|}{1 + |y|} \right) \frac{1}{1 + |y|^{n+s}}, \quad y \in \mathbb{R}^n \setminus B_\eta(x),
$$

is in $L^1(\mathbb{R}^n \setminus B_\eta(x))$. Now, for $y \in B_\eta(x)$ and by Lemma [Appendix A.2] we get reasoning as in (3.35), and (3.36) that

$$
I(y) \geq -C \left( \frac{|\nabla \varphi_\beta(x)(y-x) + \tau_\infty|x-y|^2|^{p-2}\tau_\infty|x-y|^2}{|x-y|^{n+sp}} + \frac{|\nabla \varphi_\beta(x)(y-x) + \tau_\infty|x-y|^2|^{p-2}\tau_\infty|x-y|^2}{|x-y|^{n+sp}} \right) \in L^1(B_\eta(x)).
$$

23
Therefore, defining the function $\mathcal{F}$ as the right-hand side in (3.39) over $B_\eta(x)$, and recalling (3.38), we prove $\mathcal{F} \in L^1(\mathbb{R}^n)$ and (3.37). This ends the proof of the lemma.

3.3. Certain continuity properties

The following lemmas will be useful for the proof of Theorem 1.2.

Lemma 3.7. Let $r > 0$, $x_0 \in \mathbb{R}^n$ and $F \in L^r(\mathbb{R}^n)$ Lipschitz in $B_r(x_0)$. For each $\varepsilon > 0$, $0 < \rho < r$ and $\eta \in C^2_0(B_r(x_0))$ with $0 \leq \eta \leq 1$, there exists $\tilde{\theta} = \tilde{\theta}(\varepsilon, G, \eta)$ such that $F_{\theta} := F + \theta \eta$ satisfies

$$\sup_{B_\rho(x_0)} |D^n F - D^n F_{\theta}| < \varepsilon \quad \text{for all } 0 \leq \theta < \tilde{\theta}. \quad (3.40)$$

Proof. Let $\varepsilon > 0$, $0 < \rho < r$ and $\eta \in C^2_0(B_r(x_0))$ such that $0 \leq \eta \leq 1$. Take $0 < \theta < 1$, $x \in B_\rho(x_0)$ and $0 < \delta < r - \rho$ small enough to choose later. We have

$$|D^n F(x) - D^n F_{\theta}(x)| = \left| \int_{\mathbb{R}^n} \left[ G(|D_y F|) - G(|D_y F_{\theta}|) \right] \frac{dy}{|x-y|^n} \right|$$

$$\leq \int_{B_\delta(x)} |G(|D_y F|) - G(|D_y F_{\theta}|)| \frac{dy}{|x-y|^n} + \int_{\mathbb{R}^n \setminus B_\delta(x)} |G(|D_y F|) - G(|D_y F_{\theta}|)| \frac{dy}{|x-y|^n}$$

$$\leq \int_{B_\delta(x)} |G(|D_y F|)| \frac{dy}{|x-y|^n} + \int_{B_\delta(x)} |G(|D_y F_{\theta}|)| \frac{dy}{|x-y|^n}$$

$$+ \int_{\mathbb{R}^n \setminus B_\delta(x)} |G(|D_y F|) - G(|D_y F_{\theta}|)| \frac{dy}{|x-y|^n}$$

$$= I_1 + I_2 + I_3.$$

Observe that, for $x \in B_\rho(x_0)$, $y \in B_\delta(x)$ and $\delta < r - \rho$ it is true that $y \in B_{\rho+\delta}(x_0) \subset B_r(x_0)$. Then, since $F$ is Lipschitz in $B_r(x_0)$

$$|F(x) - F(y)| \leq K_F |x-y|.$$

Taking $\delta < 1$ and using (2.4) we have for $I_1$

$$\int_{B_\delta(x)} G(|D_y F|) \frac{dy}{|x-y|^n} \leq \int_{B_\delta(x)} G(K_F |x-y|^{1-s}) \frac{dy}{|x-y|^n}$$

$$\leq CG(K_F) \int_{B_\delta(x)} \max\{|x-y|^{(1-s)p^-}, |x-y|^{(1-s)p^+}\} \frac{dy}{|x-y|^n}$$

$$\leq CG(K_F) \left[ \int_{B_\delta(x)} |x-y|^{(1-s)p^-} \, dy + \int_{B_\delta(x)} |x-y|^{(1-s)p^+} \, dy \right]$$

$$= C(G, K_F, n, s) \left[ \delta^{(1-s)p^-} + \delta^{(1-s)p^+} \right]. \quad (3.41)$$

On the other hand, since $\eta \in C^2_0(B_r(x_0))$, we have

$$|F(x) + \theta \eta(x) - F(y) - \theta \eta(y)| \leq K_F |x-y| + \theta \sup_{z \in B_{\rho+\delta}(x_0)} |\nabla \eta(z)||x-y|.$$
Then, by (A.4), (2.4), and recalling that \( \theta < 1 \) we get for \( I_2 \)

\[
\int_{B_2(x)} G \left( \left| D_x F_\theta \right| \right) \frac{dy}{|x - y|^n} \\
\leq \int_{B_2(x)} G \left( K_F \left| x - y \right| + \theta \sup_{z \in B_{r+\delta}(x_0)} \left| \nabla \eta(z) \right| \left| x - y \right| \right) \frac{dy}{|x - y|^n} \\
\leq C \left[ \int_{B_2(x)} G \left( K_F \left| x - y \right|^{1-\delta} \right) \frac{dy}{|x - y|^n} + \int_{B_2(x)} G \left( \theta \sup_{z \in B_{r+\delta}(x_0)} \left| \nabla \eta(z) \right| \left| x - y \right|^{1-\delta} \right) \frac{dy}{|x - y|^n} \right] \\
\leq C \left( G, K_F, \sup_{B_{r+\delta}(x_0)} \left| \nabla \eta \right|, n, s \right) \left[ \delta^{(1-s)p^-} + \delta^{(1-s)p^+} \right].
\] (3.42)

Finally, for \( I_3 \) we use the inequalities

\[
\left| G(a) - G(b) \right| \leq C |a - b| g(a + |a - b|), \quad a, b \geq 0,
\]
and \( |a| - |b| \leq |a - b| \), and recalling that \( 0 \leq \eta \leq 1 \) we get

\[
\int_{\mathbb{R}^n \setminus B_2(x)} \left| G \left( \left| D_x F \right| \right) - G \left( \left| D_x F_\theta \right| \right) \right| \frac{dy}{|x - y|^n} \\
\leq C \int_{\mathbb{R}^n \setminus B_2(x)} \theta \eta(x) - \eta(y) \frac{g \left( |F(x) - F(y)| + \theta \eta(x) - \eta(y) \right)}{|x - y|^s} \frac{dy}{|x - y|^n} \\
\leq C 2\theta \int_{\mathbb{R}^n \setminus B_2(x)} \frac{1}{|x - y|^s} g \left( \frac{2\theta + |F(x)| + |F(y)|}{|x - y|^s} \right) \frac{dy}{|x - y|^n}.
\]

Now, by (A.2) we have

\[
\int_{\mathbb{R}^n \setminus B_2(x)} \left| G \left( \left| D_x F \right| \right) - G \left( \left| D_x F_\theta \right| \right) \right| \frac{dy}{|x - y|^n} \\
\leq C \theta \left[ \int_{\mathbb{R}^n \setminus B_2(x)} g \left( \frac{2\theta}{|x - y|^s} \right) \frac{dy}{|x - y|^n} + \int_{\mathbb{R}^n \setminus B_2(x)} g \left( \frac{|F(x)|}{|x - y|^s} \right) \frac{dy}{|x - y|^n} \right] \\
+ \int_{\mathbb{R}^n \setminus B_2(x)} g \left( \frac{|F(y)|}{|x - y|^s} \right) \frac{dy}{|x - y|^n} \\
= C \theta (I_3^1 + I_3^2 + I_3^3).
\]

For \( I_3^1 \) we use (2.5)

\[
\int_{\mathbb{R}^n \setminus B_2(x)} g \left( \frac{2\theta}{|x - y|^s} \right) \frac{dy}{|x - y|^{n+s}} \\
\leq C \int_{\mathbb{R}^n \setminus B_2(x)} \max \left\{ \frac{(2\theta)^{p^- - 1}}{|x - y|^{n+sp^-}}, \frac{(2\theta)^{p^+ - 1}}{|x - y|^{n+sp^+}} \right\} \frac{dy}{|x - y|^n} \\
\leq C \left[ (2\theta)^{p^- - 1} \int_{\mathbb{R}^n \setminus B_2(x)} \frac{dy}{|x - y|^{n+sp^-}} + (2\theta)^{p^+ - 1} \int_{\mathbb{R}^n \setminus B_2(x)} \frac{dy}{|x - y|^{n+sp^+}} \right] \leq C \left[ (2\theta)^{p^- - 1} \delta^{-sp^-} + (2\theta)^{p^+ - 1} \delta^{-sp^+} \right] \leq C \delta^{-sp^-} + \delta^{-sp^+}.
\] (3.43)
Reasoning in the same way and taking $A = \|F\|_{L^\infty(B_\rho(x_0))}$, we have for $I_3^2$

$$
\int_{\mathbb{R}^n \setminus B_\delta(x)} g \left( \frac{|F(x)|}{|x-y|^s} \right) \frac{dy}{|x-y|^{n+s}} \leq C \max \left\{ A^{p^- - 1}, A^{p^+ - 1} \right\} [\delta^{-sp^-} + \delta^{-sp^+}] \quad (3.44)
$$

Finally, take $R > 0$ such that $B_\tau(x_0) \subset B_R$. Then, since $\delta < 1$ and $F \in L_\rho(\mathbb{R}^n)$, by Remark 3 for $B_\delta(x) \subset B_R$, we get for $I_3$

$$
\int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{dy}{|x-y|^s} \leq C \left( \frac{\delta}{1+R} \right)^{n+sp^-} \int_{\mathbb{R}^n \setminus B_\delta(x)} g \left( \frac{|F(y)|}{1+|y|^s} \right) \frac{dy}{1+|y|^{n+s}} \leq C.
$$

Observe that, because of the choice of $R$, the constant $C$ does not depend on $x$. Moreover, since $\delta < 1$, $\delta^{-sp^-} + \delta^{-sp^+} > 1$. Thus we can finally get for $I_3^3$

$$
\int_{\mathbb{R}^n \setminus B_\delta(x)} g \left( \frac{|F(y)|}{|x-y|^s} \right) \frac{dy}{|x-y|^{n+s}} \leq C[\delta^{-sp^-} + \delta^{-sp^+}]. \quad (3.45)
$$

Hence, by (3.43), (3.44) and (3.45) we have

$$
\int_{\mathbb{R}^n \setminus B_\delta(x)} |G(|D_\delta F|) - G(|D_\delta F_0|)| \frac{dy}{|x-y|^n} \leq C\theta[\delta^{-sp^-} + \delta^{-sp^+}] \quad (3.46)
$$

and joining (3.41), (3.42), and (3.46) we finally get

$$
|D_\delta^s F(x) - D_\delta^s F_\theta(x)| \leq C(\delta^{(1-s)p^-} + \delta^{(1-s)p^+} + \theta \delta^{-sp^-} + \theta \delta^{-sp^+}) \\
= C(\delta^{-sp^-} (\delta^{p^-} + \theta) + \delta^{-sp^+} (\delta^{p^+} + \theta)).
$$

Taking

$$
0 < \delta < \min \left\{ r - \rho, 1, \left( \frac{\varepsilon}{2C} \right)^{1/(1-s)p^-}, \left( \frac{\varepsilon}{2C} \right)^{1/(1-s)p^+} \right\} \quad \text{and}
$$

$$
0 \leq \theta < \min \left\{ \frac{\varepsilon \delta^{sp^+}}{2C} - \delta^{p^+}, \frac{\varepsilon \delta^{sp^-}}{2C} - \delta^{p^-}, 1 \right\},
$$

we obtain (3.40). \hfill \Box

**Lemma 3.8.** Let $B_\tau(x_0) \subset \Omega$ and $\psi \in C^2(B_\tau(x_0)) \cap L^\infty(\mathbb{R}^n)$ satisfying Definition 2.7 (iii) (a) or (b) with $\beta > \frac{sp^-}{p^- - 1}$. Then for all $\varepsilon > 0$ and $\rho' > 0$ there are $\theta' > 0$, $\rho \in (0, \rho')$ and $\eta \in C^0_\rho(B_\rho/2(x_0))$ with $0 \leq \eta \leq 1$ and $\eta(x_0) = 1$ such that $\psi_\theta = \psi + \theta \eta$ satisfies

$$
\sup_{B_\rho(x_0)} |(-\Delta_\rho)^s \psi - (-\Delta_\rho)^s \psi_\theta| < \varepsilon \quad (3.47)
$$

for $0 \leq \theta < \theta'$.

**Proof.** Take $\varepsilon > 0$, $\rho' > 0$ and first assume that $\nabla \psi(x_0) \neq 0$. Then there is $\rho \in (0, \rho')$ such that $|\nabla \psi| > \tau$ in $B_{2\rho}(x_0)$ for some $\tau > 0$. Now let $\eta \in C^0_\rho(B_{\rho/2}(x_0))$ with $0 \leq \eta \leq 1$ and $\eta(x_0) = 1$. Then there is $\theta'' > 0$ such that $|\nabla \psi_\theta| > \tau/2$ in $B_{2\rho}(x_0)$ when $0 \leq \theta < \theta''$. Now observe that, for $x \in B_\rho(x_0)$, $B_{\rho/2}(x_0) \subset B_{3\rho/2}(x_0) \subset \subset B_{2\rho}(x_0) \subset \{ |\psi_\theta | > 0 \}$. Then we may apply Lemma 5.1 to $\psi_\theta \in C^2(D)$ for $D = B_{3\rho/2}(x_0)$. Therefore we can take $\delta > 0$ small enough such that, for every $x \in B_\rho(x_0)$ and $0 \leq \theta < \theta''$

$$
P.V. \int_{B_\rho(x)} g \left( D_\rho \psi_\theta \right) \frac{dy}{|x-y|^{n+s}} < \frac{\varepsilon}{4}. \quad (3.48)
$$
If \( p^- > \frac{2}{n+2} \), we can get (3.48) using Lemma 3.1, whatever the value of \( \nabla \psi(x_0) \) is. Now consider the case \( 1 < p^- \leq \frac{2}{n+2} \), \( |\nabla \psi(x_0)| = 0 \) with \( x_0 \) an isolated critical point and \( \psi \in C^2_B(B_r(x_0)) \). Then, we can take \( \rho > 0 \) small enough such that \( |\nabla \psi| \neq 0 \) in \( B_{3\rho}(x_0) \setminus \{x_0\} \). Let \( \eta \in C^2_\partial(B_{\rho/2}(x_0)) \) such that \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) in \( B_{\rho/4}(x_0) \) and \( |D^2\eta| \leq M|\nabla\eta|^{\alpha - 2} \) for some \( M > 0 \). Then \( \nabla \psi = 0 \) in \( B_{2\rho}(x_0) \setminus \{x_0\} \) for \( \theta \) small enough and, therefore, \( d_\psi = d_{\psi_0} \) in \( B_\rho(x_0) \) for all such \( \theta \). Also, since \( \eta \in C^2_\partial(B_{\rho/2}(x_0)) \) and \( |\nabla \psi| \neq 0 \) in \( B_{3\rho}(x_0) \setminus \{x_0\} \), we may take \( \theta \) small enough such that \( \theta|\nabla \eta| \leq \frac{1}{2}|\nabla \psi| \) in \( B_\rho(x_0) \).

\[
\frac{1}{2}|\nabla \psi| \leq |\nabla \psi| - \theta|\nabla \eta| \leq |\nabla \psi| + \theta|\nabla \eta| \leq 2|\nabla \psi|, \quad \text{in} \ B_\rho(x_0).
\]

Moreover, since \( d_\eta \leq d_\psi = d_{\psi_0} \) in \( B_\rho(x_0) \) and \( \psi \in C^2_\partial(B_\rho(x_0)) \) it holds

\[
|D^2\psi_\theta| \leq |D^2\psi| + \theta|D^2\eta| \leq \|\psi\|C^2_\partial(B_\rho(x_0))d_\psi^{\beta - 2} + \theta MD_\eta^{\alpha - 2} \leq cd_\psi^{\beta - 2}.
\]

Therefore, since \( d_\psi = d_{\psi_0} \) in \( B_\rho(x_0) \) and \( \psi \in C^2_\partial(B_\rho(x_0)) \) we get, by (3.49) and (3.50), that \( \psi_\theta \in C^2_\partial(B_\rho(x_0)) \). Then we may apply Lemma 3.2 to find some \( \delta \in (0,\rho) \) such that (3.48) also holds in this case.

Now we proceed with the proof of (3.47). Take \( x \in B_\rho(x_0) \). Then, by (3.48) and Lemma Appendix A.2, taking \( T = |D_s\psi| + |D_s\psi - D_s\psi_\theta| \) we have

\[
|(-\Delta_g)^s\psi(x) - (-\Delta_g)^s\psi_\theta(x)| = \left| \text{P.V.} \int_{\mathbb{R}^n} \frac{g(D_s\psi) - g(D_s\psi_\theta)}{|x-y|^{n+sp}} dy \right|
\]

\[
\leq \left| \text{P.V.} \int_{B_\delta(x)} \frac{g(D_s\psi) - g(D_s\psi_\theta)}{|x-y|^{n+sp}} dy \right| + \left| \text{P.V.} \int_{B_\delta(x) \setminus B_{\delta}(x)} \frac{g(D_s\psi) - g(D_s\psi_\theta)}{|x-y|^{n+sp}} dy \right|
\]

\[
\leq \frac{e}{2} + C \int_{B_\delta(x) \setminus B_{\delta}(x)} \frac{|D_s\psi - D_s\psi_\theta|}{|x-y|^{n+sp}} \max\{T^{p^- - 2}, T^{p^- - 2}\} dy
\]

Now we use the monotonicity of \( (a + b)^{p^- - 2}b \) for \( a, b \geq 0 \) and \( p > 1 \), the fact that \( |D_s\psi - D_s\psi_\theta| \leq \frac{2\rho}{|x-y|^p} \), and (5.31) to get

\[
\frac{1}{2}|\nabla \psi| \leq |\nabla \psi| - \theta|\nabla \eta| \leq |\nabla \psi| + \theta|\nabla \eta| \leq 2|\nabla \psi|, \quad \text{in} \ B_\rho(x_0).
\]

Next we estimate I assuming first that \( 1 < p^- < 2 \). Then,

\[
C\theta \int_{\mathbb{R}^n \setminus B_{\delta}(x)} \frac{|\psi(x) - \psi(y)| + 2\theta|^{p^- - 2}}{|x-y|^{n+sp^+}} dy \leq C\theta^{p^- - 1}\delta^{-sp^+} < \frac{\varepsilon}{4}.
\]
for all \( \theta \) small enough. Now suppose \( p^+ \geq 2 \). Observe that \( \theta^{p^+-1} < \theta \) for \( \theta < 1 \). Therefore, since \( \psi \in L^\infty(\mathbb{R}^n) \) we can estimate (I) for \( \theta \) small enough as

\[
C \theta \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{(|\psi(x) - \psi(y)| + 2\theta)^{p^+-2}}{|x - y|^{n+sp^+}} \, dy \leq C \theta \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{(2||\psi||_{L^\infty(\mathbb{R}^n)} + 2\theta)^{p^+-2}}{|x - y|^{n+sp^+}} \, dy
\]

\[
\leq C \theta (||\psi||_{L^\infty(\mathbb{R}^n)}^{p^+-2} + \theta^{p^+-2}) \delta^{-sp^+} \leq C \theta \delta^{-sp^+} < \varepsilon \frac{4}{\delta}.
\]

Reasoning in the same way, we can estimate II for all \( \theta \) small enough and finally use (3.52) to get (3.54).

**Lemma 3.9.** Let \( B_\delta(x_0) \subset \Omega \) and \( \psi \in C^2(B_r(x_0)) \cap L_g(\mathbb{R}^n) \). We also assume \( \psi \in C^2_\beta(B_r(x_0)) \) for some \( \beta > \frac{sp^-}{p^+-1} \) if \( 1 < p^- \leq \frac{2}{n} \) and \( \nabla \psi(x_0) = 0 \) with \( x_0 \) an isolated point in \( B_r(x_0) \). Then \( (-\Delta_g)^s \psi \) is continuous in \( B_\delta(x_0) \).

**Proof.** Take \( x \in B_\delta(x_0) \) and \( \varepsilon > 0 \). First suppose \( p^- > \frac{2}{n} \) and choose \( \delta > 0 \) such that \( B_\delta(x) \subset B_r(x_0) \). Then, for \( y \in B_\delta(x) \), there is \( \delta' > 0 \) such that \( B_{\delta'}(y) \subset B_\delta(x) \subset \subset \Omega \). Hence, by Lemma 3.1, there is \( \rho > 0 \) such that

\[
\int_{B_\delta(y)} g(D_s \psi) \frac{dz}{|z - y|^{n+s}} < \varepsilon \frac{4}{\delta}
\]

whenever \( |x - y| < \delta \).

Now assume \( p^- \leq \frac{2}{n} \). If \( \nabla \psi(x_0) \neq 0 \) we take \( r > 0 \) such that \( \nabla \psi(z) \neq 0 \) for all \( z \in B_r(x_0) \) and then \( \nabla \psi(x) \neq 0 \). Hence, by continuity there is \( \delta > 0 \) such that \( \nabla \psi(y) \neq 0 \) for all \( y \in B_\delta(x) \subset B_r(x_0) \). Therefore, we can take \( \delta' > 0 \) such that \( B_{\delta'}(y) \subset B_\delta(x) \subset \subset \{ d_\psi > 0 \} \) and we may apply again Lemma 3.1 to find \( \rho > 0 \) such that (3.53) holds for all \( y \in B_\delta(x) \).

If on the contrary we have \( p^- \leq \frac{2}{n} \) and \( \nabla \psi(x_0) = 0 \) we choose \( r > 0 \) such that \( \nabla \psi(z) \neq 0 \) for all \( z \in B_r(x_0) \setminus \{ x_0 \} \). Then, if \( x \neq x_0 \), \( \nabla \psi(x) \neq 0 \) and we proceed as we did before. Now, if \( x = x_0 \), \( |x_0 - y| < \delta \) implies \( d_\psi(y) < \delta \) and we also have \( \psi \in C^2_\beta(B_r(x_0)) \). Take \( 0 < \delta' < 1 \) such that (3.53) holds for the first two cases and also impose \( \delta < r/2 \). Then \( B_\delta(y) \subset B_{2\delta}(x_0) \subset B_\delta(x_0) \) for all \( y \in B_\delta(x_0) \) and \( d_\psi(y) < \delta \). Hence we may use Lemma 3.2 to find \( \rho > 0 \) such that (3.53) also holds in this case for all \( y \in B_\delta(x) \).

Now we may suppose that \( |x - y| < \rho/3 \). Thus, using (3.2) we have

\[
g(D_s \psi) \frac{\lambda_{\mathbb{R}^n \setminus B_\delta(y)}(z)}{|y - z|^{n+s}} \leq c \left[ g \left( \frac{|\psi(y)|}{|y - z|^s} + g \left( \frac{|\psi(z)|}{|y - z|^s} \right) \right) \right] \frac{\lambda_{\mathbb{R}^n \setminus B_\delta(y)}(z)}{|y - z|^{n+s}} \leq c \left[ g \left( \frac{\|\psi\|_{L^\infty(B_{2\delta}(x))}}{\rho^s} + g \left( \frac{|\psi(z)|}{|y - z|^s} \right) \right) \right] \frac{\lambda_{\mathbb{R}^n \setminus B_\delta(y)}(z)}{|y - z|^{n+s}}.
\]

If we take \( \rho < 1 \), then the right hand side of (3.54) is integrable in \( \mathbb{R}^n \) by Remark 4, since \( \psi \in L_g(\mathbb{R}^n) \), \( g \left( \frac{|\psi(z)|}{|y - z|^s} \right) \frac{1}{|y - z|^{n+s}} \) belongs to \( L^1(\mathbb{R}^n \setminus B_\rho(y)) \). Hence, using dominated convergence theorem and the continuity of \( g \left( \frac{|\psi(z)|}{|y - z|^s} \right) \frac{1}{|y - z|^{n+s}} \) in \( \mathbb{R}^n \setminus \{ z \} \) we obtain

\[
\int_{\mathbb{R}^n \setminus B_\rho(y)} g \left( \frac{|\psi(y) - \psi(z)|}{|y - z|^s} \right) \frac{dz}{|y - z|^{n+s}} \rightarrow \int_{\mathbb{R}^n \setminus B_\rho(x)} g \left( \frac{|\psi(x) - \psi(z)|}{|x - z|^s} \right) \frac{dz}{|x - z|^{n+s}}
\]
as $y \to x$. Then there is $\delta > 0$ such that
\[
\left| \int_{\mathbb{R}^n \setminus B_\rho(y)} g \left( \frac{\psi(y) - \psi(z)}{|y - z|^s} \right) \frac{dz}{|y - z|^{n+s}} - \int_{\mathbb{R}^n \setminus B_\rho(x)} g \left( \frac{\psi(x) - \psi(z)}{|x - z|^s} \right) \frac{dz}{|x - z|^{n+s}} \right| < \frac{\varepsilon}{2}
\] (3.55)
whenever $|x - y| < \delta$.

Finally by (3.53) and (3.55) we get
\[
|(-\Delta_g)^s \psi(x) - (-\Delta_g)^s \psi(y)| < \varepsilon
\]
if $|x - y|$ is small enough. \hfill $\square$

4. Proof of Theorem 1.1

In this section, we prove that viscosity solutions to (1.1) are also weak solutions. We will follow in the proof some calculations from [18] and [4] adapted to the Orlicz framework.

Proof of Theorem 1.1. By Lemma 3.3, $u_\varepsilon$ is a viscosity supersolution of
\[
(-\Delta_g)^s w = f_\varepsilon(x, w, D^s_g w) \quad \text{in } \Omega(\varepsilon),
\]
and hence it satisfies
\[
(-\Delta_g)^s u_\varepsilon \geq f_\varepsilon(x, u_\varepsilon, D^s_g u_\varepsilon) \quad \text{a.e. in } \Omega(\varepsilon).
\]
By Lemmas 3.5 and 3.6,
\[
\int \int_{Q_{\Omega(\varepsilon)}} g \left( D_s u_\varepsilon \right) \frac{\psi(x) - \psi(y)}{|x - y|^{n+s}} dx dy \geq \int \int_{\Omega(\varepsilon)} f_\varepsilon(x, u_\varepsilon, D^s_g u_\varepsilon) \psi dx.
\] (4.1)
Hence, $u_\varepsilon$ is a weak supersolution in $\Omega(\varepsilon)$. Let now $\varphi \in C^\infty_0(\Omega)$, non-negative with $K := \text{supp } \varphi$. Then, $K \subset \Omega(\varepsilon)$ for all $\varepsilon$ small enough. The goal is to take the limit as $\varepsilon \to 0$ in (4.1).

Let $\xi \in C^\infty_0(\Omega(\varepsilon))$, $0 \leq \xi \leq 1$ such that $\xi = 1$ in $K' \subset K''$, where $K'$ contains $K$ and $K''$ is a compact set containing the support of $\xi$. By Proposition 3.1,
\[
\int_{K'} \int_{\mathbb{R}^n} G(\|D_s u_\varepsilon\|) \mu = \int_{K'} \int_{\mathbb{R}^n} G(\|D_s u_\varepsilon\|) G(\xi(x)) d\mu \\
\quad \leq C \left[ G(\text{osc } u_\varepsilon) \left( \int_{K''} \int_{\mathbb{R}^n} G(\|D_s \xi\|) d\mu + \gamma_{\infty, \varepsilon} + \text{osc}(u_\varepsilon) \right) \right],
\]
with
\[
\gamma_{\infty, \varepsilon} := \max_{\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}, \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}} \gamma(t).
\]

Since $u_\varepsilon$ is increasing as $\varepsilon \to 0^+$ and $u \in L^\infty(\mathbb{R}^n)$, we have
\[
\text{osc}(u_\varepsilon) \leq \sup_{\mathbb{R}^n} u - \inf_{\mathbb{R}^n} u_{\varepsilon_0},
\]
for all $\varepsilon \leq \varepsilon_0$, and also $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leq \|u\|_{L^\infty(\mathbb{R}^n)}$. Thus,
\[
\int_{K'} \int_{\mathbb{R}^n} G(\|D_s u_\varepsilon\|) d\mu \leq C \quad \text{for } \varepsilon \leq \varepsilon_0.
\] (4.2)

Hence, up to a subsequence, it holds
\[
u_\varepsilon \to u \quad \text{in } L^G(K')
\] (4.3)
and
\[ D_s u_\varepsilon \rightarrow D_s u \quad L^G_\mu(K' \times \mathbb{R}^n). \] (4.4)

For further reference observe that (4.2) and Lemma 2.1 imply
\[ \int_{K'} \int_{\mathbb{R}^n} \tilde{G}(g(D_s u_\varepsilon)) \, d\mu \leq C \int_{K'} \int_{\mathbb{R}^n} G(D_s u_\varepsilon) \, d\mu \leq C \]
for all \( \varepsilon \). Thus,
\[ g(D_s u_\varepsilon) \rightarrow g(D_s u) \quad \text{in} \quad L^\tilde{G}_\mu(K' \times \mathbb{R}^n). \] (4.5)

Consider now
\[ \psi(x) := (u(x) - u_\varepsilon(x))\theta(x), \]
where \( \theta \in C_0^\infty(\Omega), \supp \theta \subset K', \theta \in [0,1] \) and \( \theta = 1 \) in \( K \subset K' \). Observe that
\[ \psi(x) - \psi(y) = \theta(x)(u(x) - u_\varepsilon(x) - (u(y) - u_\varepsilon(y))) + (\theta(x) - \theta(y))(u(y) - u_\varepsilon(y)). \]

Then, by (4.1)
\[
\int_{K'} \int_{\mathbb{R}^n} f(x, u_\varepsilon, D^2 u_\varepsilon) \psi \, dx \leq \int_{Q_{K'}} g(D_s u_\varepsilon) \, D_s \psi \, d\mu
\]
\[
= \int_{Q_{K'}} g(D_s u_\varepsilon) \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \, d\mu
\]
\[
+ \int_{K'} \int_{\mathbb{R}^n} g(D_s u) \frac{\theta(x)(u(x) - u(y) - (u_\varepsilon(x) - u_\varepsilon(y)))}{|x - y|^s} \, d\mu
\]
\[
- \int_{K'} \int_{\mathbb{R}^n} [g(D_s u_\varepsilon) - g(D_s u)] \frac{\theta(x)(u(x) - u(y) - (u_\varepsilon(x) - u_\varepsilon(y)))}{|x - y|^s} \, d\mu. \] (4.6)

Now, by Hölder’s inequality
\[
\int_{Q_{K'}} g(D_s u_\varepsilon) \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \, d\mu
\]
\[
\leq C \|g(D_s u_\varepsilon)\|_{L^G_{\mu(Q_{K'})}} \left\| \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \right\|_{L^G_{\mu(Q_{K'})}} \|D_s \psi\|_{L^G_{\mu(Q_{K'})}} \leq C \left\| \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \right\|_{L^G_{\mu(Q_{K'})}}. \] (4.7)

Also, suppose without loss of generality that
\[
\min \left\{ \left\| \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \right\|_{L^G_{\mu(Q_{K'})}}, \left\| \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \right\|_{L^G_{\mu(Q_{K'})}} \right\}
\]
\[
= \left\| \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \right\|_{L^G_{\mu(Q_{K'})}}. \]

Then, by Lemma 2.2,
\[
\left\| \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \right\|_{L^G_{\mu(Q_{K'})}} \leq \left[ \Phi_{\mu,G} \left( \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \right) \right]^{1/p^-} \]
\[
= \left[ \int_{Q_{K'}} G \left( \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x - y|^s} \right) \, d\mu \right]^{1/p^-}. \] (4.8)
Next, we will show that the last integral converges to zero. Observe that (2.4) and the uniform global handedness of \( u_\varepsilon \) imply that
\[
G \left( \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x-y|^s} \right) \leq C(2\|u\|_{L^\infty(\mathbb{R}^n)} + 1)^p G(|D_s\theta|).
\]
Since
\[
G(|D_s\theta|) \leq C \max \left\{ |D_s\theta|^p, |D_s\theta|^p \right\} \leq C \left( |D_s\theta|^p + |D_s\theta|^p \right),
\]
we obtain by the smoothness of \( \theta \) that
\[
G \left( \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x-y|^s} \right) \in L^1_\mu(Q_{K'}).
\]
By dominated convergence theorem, the last integral in (4.8) goes to zero and hence recalling (4.7), it holds
\[
\int \int_{Q_{K'}} g(D_s u_\varepsilon) \frac{(\theta(x) - \theta(y))(u(y) - u_\varepsilon(y))}{|x-y|^s} \, d\mu \to 0 \quad \text{as} \quad \varepsilon \to 0^+.
\] (4.9)
Now, we treat the following integral in (4.6):
\[
\int_{K'} \int_{\mathbb{R}^n} g(D_s u) \frac{\theta(x)(u(x) - u(y) - (u_\varepsilon(x) - u_\varepsilon(y)))}{|x-y|^s} \, d\mu.
\]
Observe that by Lemma 2.1
\[
g(D_s u) \theta \in L^G_\mu(K' \times \mathbb{R}^n).
\]
Hence, by (4.4) it holds
\[
\int_{K'} \int_{\mathbb{R}^n} g(D_s u) \frac{\theta(x)(u(x) - u(y) - (u_\varepsilon(x) - u_\varepsilon(y)))}{|x-y|^s} \, d\mu \to 0 \quad \text{as} \quad \varepsilon \to 0^+.
\] (4.10)
Finally, we consider the integral
\[
\int_{K'} \int_{\mathbb{R}^n} \left[ g(D_s u_\varepsilon) - g(D_s u) \right] \frac{\theta(x)(u(x) - u(y) - (u_\varepsilon(x) - u_\varepsilon(y)))}{|x-y|^s} \, d\mu.
\]
By the convexity of \( G \), we have
\[
G(|D_s u|) \leq G \left( \frac{D_s u + D_s u_\varepsilon}{2} \right) + g(|D_s u|) \frac{D_s u}{|D_s u|} \left( \frac{D_s u - D_s u_\varepsilon}{2} \right)
\]
and
\[
G(|D_s u_\varepsilon|) \leq G \left( \frac{D_s u + D_s u_\varepsilon}{2} \right) + g(|D_s u_\varepsilon|) \frac{D_s u_\varepsilon}{|D_s u_\varepsilon|} \left( \frac{D_s u_\varepsilon - D_s u}{2} \right).
\]
Adding the above expressions gives
\[
\frac{1}{2} \left[ g(D_s) - g(D_s u_\varepsilon) \right] (D_s u - D_s u_\varepsilon) \geq G(|D_s u|) + G(|D_s u_\varepsilon|) - G \left( \frac{D_s u + D_s u_\varepsilon}{2} \right)
\]
\[
\geq G \left( \frac{D_s u - D_s u_\varepsilon}{2} \right),
\]
(4.11)
where the last inequality follows again by convexity of \( G \) (see [2], Lemma 2.9). Hence, by (4.6), (4.9), (4.10), and (4.11), we obtain
\[
0 \leq \lim_{\varepsilon \to 0} \int_{K'} \int_{\mathbb{R}^n} \left[ g(D_s u_\varepsilon) - g(D_s u) \right] \frac{\theta(x)(u(x) - u(y) - (u_\varepsilon(x) - u_\varepsilon(y)))}{|x-y|^s} \, d\mu
\]
\[
\leq \limsup_{\varepsilon \to 0} \left( - \int_{K'} f_\varepsilon(x, u_\varepsilon, D_s^* u_\varepsilon) \right).
\]
From the assumption \((1.2)\), we have
\[
- \int_{K'} f_\varepsilon(x, u_\varepsilon, D_y^\varepsilon u_\varepsilon) \, dx \leq \gamma_\varepsilon \int_{K'} \tilde{G}^{-1}\left(|D_y^\varepsilon u_\varepsilon|\right)(u - u_\varepsilon) \theta \, dx + \|\phi\|_{L^\infty(K')} \int_{K'} (u - u_\varepsilon) \theta \, dx.
\]
Now, by Hölder’s inequality,
\[
\int_{K'} \tilde{G}^{-1}\left(|D_y^\varepsilon u_\varepsilon|\right)(u - u_\varepsilon) \theta \, dx \leq \|\tilde{G}^{-1}(|D_y^\varepsilon u_\varepsilon|)\|_{L^{\infty}(K')} \|u - u_\varepsilon\|_{L^\infty(K')}.
\]
By \((4.3)\), which in particular holds for \(B = G\),
\[
\|u - u_\varepsilon\|_{L^\infty(K')} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
On the other hand,
\[
\|\tilde{G}^{-1}(D_y^\varepsilon u_\varepsilon)\|_{L^\infty(K')} \leq \Phi_G(\tilde{G}^{-1}(D_y^\varepsilon u_\varepsilon))^{1/p^+} + \Phi_G(\tilde{G}^{-1}(D_y^\varepsilon u_\varepsilon))^{1/p^-}
\]
\[
= \left(\int_{K'} \int_{\mathbb{R}^n} G(D_s u_\varepsilon) \, d\mu\right)^{1/p^+} + \left(\int_{K'} \int_{\mathbb{R}^n} G(D_s u_\varepsilon) \, d\mu\right)^{1/p^-},
\]
and the last terms are uniformly bounded by \((4.3)\). Therefore
\[
\limsup_{\varepsilon \to 0} \left( - \int_{K'} f_\varepsilon(x, u_\varepsilon, D_y^\varepsilon u_\varepsilon) \, dx \right) = 0
\]
and thus
\[
\lim_{\varepsilon \to 0} \int_{K'} \int_{\mathbb{R}^n} [g(D_s u_\varepsilon) - g(D_s u)] \frac{\theta(x)(u(x) - u(y) - (u_\varepsilon(x) - u_\varepsilon(y)))}{|x - y|^s} \, d\mu = 0. \quad (4.12)
\]
Now, from \((4.11)\), we get
\[
\frac{1}{2} [g(D_s) - g(D_s u_\varepsilon)] (D_s u - D_s u_\varepsilon) \geq G \left( \frac{|D_s u - D_s u_\varepsilon|}{2} \right) \geq \left( \frac{1}{2} \right)^{-p^-} G(|D_s u - D_s u_\varepsilon|),
\]
where the last inequality follows from \((2.4)\). In this way, \((4.12)\) and Lemma \((3.4)\) imply that we may pass to the limit as \(\varepsilon \to 0\) in \((4.11)\) to get
\[
\int_{K'} \int_{Q_K} g(D_s u_\varepsilon) D_s \psi \, d\mu \geq \int_{K} f(x, u, D_y^\varepsilon u) \psi \, dx,
\]
where in the left-hand side we use \((1.9)\). This ends the proof of the theorem. \(\square\)

5. Proof of Theorem \((1.2)\)

Lastly, we show that a weak solution is also a viscosity solution.

**Proof of Theorem \((1.2)\)** We proceed by contradiction. Assume that \(u \in L^\infty(\mathbb{R}^n)\) is a continuous weak supersolution, but it is not a viscosity supersolution. Hence, according to Remark \((1)\), there is a point \(x_0 \in \Omega\) and a test function \(\psi \in C^2(B_r(x_0)) \cap L^1(\mathbb{R}^n)\), that we may take equal to \(u\) outside \(B_r(x_0)\) and thus in \(L^\infty(\mathbb{R}^n)\) since \(u\) is bounded, satisfying (iii) from Definition \((2.1)\) and
\[
(-\Delta_g)^s \psi(x_0) < f(x_0, \psi(x_0), D_y^\varepsilon \psi(x_0)).
\]
By Lemma \ref{lem:continuous}, the mapping
\[ x \rightarrow f(x, \psi(x), D^*_g \psi(x)) \]
is continuous in \( B_r(x_0) \). Also, by Lemma \ref{lem:comparison} there exist \( \delta \) and \( 0 < r_1 < r \) such that
\[
(-\Delta_g)^* \psi(x) \leq f(x, u, D^*_g \psi(x)) - \delta, \quad x \in B_{r_1}(x_0).
\] (5.1)

Since \( \psi \in C^2(B_r(x_0)) \cap L^\infty(\mathbb{R}^n) \), by Lemma \ref{lem:comparison} for every \( \varepsilon > 0 \) and \( \rho > 0 \), there exist \( 0 < \theta_1 = \theta(\varepsilon, \rho) \), \( 0 < r_2 < \rho \) and \( \eta \in C^\infty_0(B_{r_2/2}(x_0)), \eta \in [0, 1] \), with \( \eta(x_0) = 1 \) such that
\[
\sup_{B_{r_2}(x_0)} |(-\Delta_g)^* \psi - (-\Delta_g)^*(\psi + \theta \eta)| < \varepsilon
\]
for every \( 0 < \theta < \theta_1 \). Taking \( \varepsilon = \delta/2 \) and \( \rho = r_1 \) and using the Lipschitz assumption on \( f \), we get by Lemma \ref{lem:continuous} and (5.1) that
\[
(-\Delta_g)^*(\psi + \theta \eta)(x) \leq (-\Delta_g)^* \psi(x) + \frac{\delta}{2} \leq f(x, u(x), D^*_g \psi(x)) - \frac{\delta}{2}
\]
(5.2)

for \( x \in B_{r_2/2} \) and \( 0 < \theta < \min \left\{ \theta_1(\delta/2, r_1), \tilde{\theta}(\delta/(2K)), r_2 \right\} \), where \( \tilde{\theta} \) is given by Lemma \ref{lem:continuous} and \( K \) is the Lipschitz constant of \( f \). Let now
\[
\bar{f}(x, v) := f(x, u(x), v).
\]

Then, (5.2) also holds weakly in \( B_{r_2/2}(x_0) \) and hence \( \psi + \theta \eta \) is a weak subsolution of the equation
\[
(-\Delta_g)^* v(x) = \bar{f}(x, D^*_g v(x)), \quad x \in B_{r_2/2}(x_0).
\]

Since \( u \) is a weak supersolution of the same equation and \( \psi + \theta \eta \leq u \) in \( \mathbb{R}^n \setminus B_{r_2/2}(x_0) \), by the (CPP), we get \( \psi + \theta \eta \leq u \) in \( \mathbb{R}^n \). In particular,
\[
u(x_0) \geq \psi(x_0) + \theta \eta(x_0) = \psi(x_0) + \theta > \psi(x_0)
\]
which is a contradiction with the fact that \( u(x_0) = \psi(x_0) \).

\[ \square \]

\subsection{A comparison principle}

In this section, we provide a comparison principle for non-homogeneous \( g \)-Laplace equations. It is worth to point out that the general comparison principle needed in the proof of Theorem \ref{thm:comparison} remains an open problem.

\begin{theorem}
Let \( f = f(x, r) \) be non-increasing in \( r \). Assume that \( u \) and \( v \) are weak sub and supersolutions, respectively, of
\[
(-\Delta_g)^* w = f(x, w) \quad \text{in} \ \Omega,
\]
with \( u \leq v \) in \( \mathbb{R}^n \setminus \Omega \). Then \( v \geq u \) in \( \mathbb{R}^n \).
\end{theorem}

\begin{proof}
Using \( \psi = (u - v)^+ \) as a test function for \( u \) and \( v \), and subtracting give
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (g(D_s v) - g(D_s u)) D_s \psi \, d\mu \geq \int_{\Omega} (f(x, v) - f(x, u)) \psi(x) \, dx.
\] (5.3)

Observe that the latter integral is indeed over the set \{ \( u \geq v \) \} and hence, since \( f(x, r) \) is non-increasing in \( r \), the integral is greater or equal than 0. The left-hand side of (5.3) may be treated as in \cite[Lemma C.4]{11} to get that \( (u - v)^+ = 0 \). Hence, comparison follows.
\[ \square \]
Appendix A. Some inequalities for Young functions

Lemma Appendix A.1. Suppose that

\[ 1 < p^- \leq \frac{tg(t)}{G(t)} \leq p^+ < \infty. \] (A.1)

with \( 1 < p^- < p^+ < \infty \). Then, there is a positive constant \( C \) such that

\[ g(s + t) \leq C(g(s) + g(t)) \] (A.2)

for all \( s, t \geq 0 \).

Proof. By Theorem 4.1 from [19], (A.1) implies that \( G \) satisfies the \( \Delta_2 \)-condition, that is, there exists a constant \( C > 2 \) such that

\[ G(2t) \leq CG(t), \quad t > 0. \] (A.3)

The convexity of \( G \) and (A.3) give

\[ G(s + t) \leq \frac{C}{2}(G(s) + G(t)), \quad s, t > 0. \] (A.4)

Then, by (A.1) and (A.4) we get, for \( s, t > 0 \)

\[ g(s + t) \leq \frac{p^+}{s + t}G(s + t) \leq \frac{p^+ C}{2(s + t)}(G(s) + G(t)) \]

\[ \leq \frac{p^+ C}{2p^-(s + t)}(sg(s) + tg(t)) = \frac{p^+ C}{2p^-}(\frac{s}{s + t}g(s) + \frac{t}{s + t}g(t)) \leq \frac{p^+ C}{2p^-}(g(s) + g(t)). \]

Finally, recalling that \( g(0) = 0 \) we obtain (A.2) for all \( s, t \geq 0 \).

Lemma Appendix A.2. Let \( p^- > 1 \). Then there is a positive constant \( C \) such that

\[ |g(a + b) - g(b)| \leq C \max\left\{ (|b| + |a|)^{p^- - 2}, (|b| + |a|)^{p^+ - 2}\right\} |a| \]

for any \( a, b \in \mathbb{R} \).

Proof. Observe that

\[ |g(a + b) - g(b)| = \left| a \int_0^1 g'(b + ta)dt \right| \leq |a| \int_0^1 |g'(b + ta)|dt \leq |a| \int_0^1 g'(|b + ta|)dt. \]

Then, by (2.1) and (2.5), there is a positive constant \( C_1 = C_1(p^-, p^+) \) such that

\[ |g(a + b) - g(b)| \leq C_1 |a| \max\left\{ \int_0^1 |a + tb|^{p^- - 2}dt, \int_0^1 |a + tb|^{p^+ - 2}dt, \right\} \]

Then, by [18, Lemma 3.2], there is a positive constant \( C_2 = C_2(p^-, p^+) \) such that

\[ |g(a + b) - g(b)| \leq C_2 \max\left\{ (|b| + |a|)^{p^- - 2}, (|b| + |a|)^{p^+ - 2}\right\} |a|. \]
Acknowledgements

M. L. de Borbón and P. Ochoa have been supported by CONICET and Grant B080, UNCuyo, Argentina.

References

[1] G. Alberti and G. Bellettini, A nonlocal anisotropic model for phase transitions. I. The optimal profile problem. Math. Ann. 310 3 (1998), 527–560.

[2] D. Applebaum, Lévy processes and stochastic calculus. Second edition. Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, 2009. xxx+460 pp. ISBN: 978-0-521-73865-1.

[3] J.P. Bouchaud, and A. Georges, Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications. Phys. Rep. 195 4-5 (1990), 127–293.

[4] B. Barrios and M. Medina, Equivalence of weak and viscosity solutions in fractional non-homogeneous problems. Mathematische Annalen (2020). https://doi.org/10.1007/s00208-020-02119-w.

[5] L. Braga, R. Leitao and J. Oliveira, Free boundary theory for singular/degenerate nonlinear equations with right hand side: a non-variational approach. Calculus of Variations (2020), 59-86.

[6] L. Caffarelli and G. Dávila, Interior regularity for fractional systems. Ann. Inst. Henri Poincaré, Anal. Non Linéaire 36 1 (2019), 165-180.

[7] R. Cont, and P. Tankov, Financial modelling with jump processes. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004. xvi+535 pp. ISBN: 1-5848-8413-4

[8] P. Constantin, Euler equations, Navier-Stokes equations and turbulence. Mathematical foundation of turbulent viscous flows, 1–43, Lecture Notes in Math., 1871, Springer, Berlin, 2006.

[9] J. Fernández Bonder and A. Salort, Fractional order Orlicz-Sobolev spaces. Journal of Functional Analysis 277 (2019), 333-367.

[10] J. Fernández Bonder, M. Pérez–Llanos, and A. Salort. A Hölder infinity Laplacian obtained as limit of Orlicz fractional Laplacians. Rev Mat Complut (2021). https://doi.org/10.1007/s13163-021-00390-2

[11] J. Fernández Bonder, A. Salort and H. Vivas. Interior and boundary regularity for the fractional $g$-Laplacian. Preprint.

[12] F. Ferrari and C. Lederman, Regularity of flat free boundaries for a $p(x)$-Laplacian problem with right hand side. Nonlinear Analysis 212 (2021), 1-25.

[13] H. Ishii. On the equivalence of two notions of solutions, viscosity solutions and distribution solutions. Funkcislaj Ekvacioj 38 (1995), 101-120.

[14] V. Julin and P. Juutinen., A new proof for the equivalence of weak and viscosity solutions for the $p$–Laplace equation. Communications in PDE 37 5 (2012), 934–946.

[15] V. Julín and P. Lindqvist., A theorem of Radò type for solutions of a quasi-linear equation. Mathematical Research Letters 11 (2004), 31–34.
[16] P. Juutinen, P. Lindqvist and J. Manfredi. *On the equivalence of viscosity solutions and weak solutions for a quasilinear equation*. SIAM J. Math. Anal. 33 3 (2001), 699–717.

[17] P. Juutinen, T. Lukkari and M. Parviainen. *Equivalence of viscosity solutions and weak solutions for the p(x)-Laplacian*. Ann. Ins. H. Poincaré Anal. Non Linéaire 27 6 (2010), 1471–1487.

[18] J. Korvenpää, T. Kuusi and E. Lindgren, *Equivalence of solutions to fractional p−Laplace type equations*. J. Math. Pures Appl. 132 (2019), 1–26.

[19] M. A. Krasnoselskii and Ja. B Rutickii, [Convex functions and Orlicz spaces](https://www.worldcat.org/title/convex-functions-and-orlicz-spaces/oclc/790751758). P. Noordhoff Ltd. Groningen, 1961, translated from the first Russian edition by Leo F. Boron, MR 0126722.

[20] J. Lamperti, On the isometries of certain function-spaces. Pacific J. Math, 8 3 (1958), 459-466.

[21] M. Medina and P. Ochoa. *On viscosity and weak solutions for non-homogeneous p-Laplace equations*. Adv. Nonlinear Anal. 8 (2019), 468-481.

[22] S. Molina, A. Salort and H. Vivas, *Maximum principles, Liouville theorem and symmetry results for the fractional g−Laplacian*. Nonlinear Analysis 212 (2021), 112465.

[23] A. Salort, *Lower bounds for Orlicz eigenvalues*. Preprint [https://arxiv.org/pdf/2104.07562](https://arxiv.org/pdf/2104.07562)

[24] A. Signorini, *Questioni di elasticita. Statica non lineare; Vincoli unilaterali, statica semilinearizzata; Complementi*. (Italian) Confer. Sem. Mat. Univ. Bari 48-49 (1959), 42 pp.

[25] J. Siltakoski. *Equivalence of viscosity solutions and weak solutions for the normalized p(x)-Laplacian*. Calculus of Variations and PDEs 57 95 (2018), 1-20.

[26] J. F. Toland, *The Peterls-Nabarro and Benjamin-Ono equations*. J. Funct. Anal. 145 1 (1997), 136–150.