Saturation and recurrence of quantum complexity in random quantum circuits

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Quantum complexity is a measure of the minimal number of elementary operations required to approximately prepare a given state or unitary channel. Recently, this concept has found applications beyond quantum computing—in studying the dynamics of quantum many-body systems and the long-time properties of AdS black holes. In this context Brown and Susskind [1] conjectured that the complexity of a chaotic quantum system grows linearly in time up to times exponential in the system size, saturating at a maximal value, and remaining maximally complex until undergoing recurrences at doubly-exponential times. In this work we prove the saturation and recurrence of the complexity of quantum states and unitaries in a model of chaotic time-evolution based on random quantum circuits, in which a local random unitary transformation is applied to the system at every time step. Importantly, our findings hold for quite general random circuit models, irrespective of the gate set and geometry of qubit interactions. Our results advance an understanding of the long-time behaviour of chaotic quantum systems and could shed light on the physics of black hole interiors. From a technical perspective our results are based on establishing new quantitative connections between the Haar measure and high-degree approximate designs, as well as the fact that random quantum circuits of sufficiently high depth converge to approximate designs.

I. INTRODUCTION

Over the past few years, the notion of complexity has emerged from the confines of theoretical computer science and into the lexicon of modern theoretical physics. Intuitively speaking, the circuit complexity of a quantum state or unitary allows one to distinguish between feasible and intractable computational tasks for quantum computers [2–4]. From classifying topological phases of matter [5] to grappling with the dynamics of black hole interiors [1, 6–9] and other interacting quantum systems [10, 11], the quantum complexity of states and unitaries has become a central concept in quantum information and quantum many-body physics.

The circuit complexity of a state or unitary transformation is defined as the minimal number of elementary operations required to approximately prepare the state from an unentangled initial state, or to implement the unitary to within a set tolerance. Specifically, the elementary operations we consider are a set of universal gates, where each gate only acts on a few qubits. Despite being a seemingly innocuous concept, the quantum complexity of a specific state or unitary is extremely difficult to compute [4, 9]. Naively, it requires one to list all possible circuits in order to correctly identify the shortest one.

The complexity of unitary time-evolutions appears connected to some deep ideas in theoretical physics. The broad interest in understanding the behaviour of complexity of time-evolving states and unitaries is motivated by the AdS/CFT correspondence, a duality between a theory of quantum gravity in asymptotically Anti-de Sitter space (AdS) in $D$ dimensions and a Conformal Field Theory (CFT) defined on its $(D - 1)$-dimensional boundary. Intriguing consequences seem to emerge if this conjectured duality is applied to a specific state on two copies on the CFT [12, 13]. The holographic picture of this state is a two-sided (or maximally extended) AdS-Schwarzschild black hole. In this geometry, the region behind the horizon of the AdS black hole, a wormhole connecting the two asymptotic regions, appears to grow linearly in time for an exponentially long time, up to a time exponential in the black hole entropy $t \sim e^{S_{BH}}$. The claim is that the quantum complexity of the dual CFT state is the quantity whose exponentially long linear growth in

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time describes the growth of the black hole interior \[6, 14\]. More specifically, there have been a number of conjectures for the quantity in the bulk which precisely computes the complexity of the time-evolved thermofield double state of the boundary CFTs. One proposal is that the complexity of the time-evolved state equals spatial volume of region behind the horizon on a maximal time slice of the geometry \[15\]. A second proposal conjectures that the action computed on the Wheeler-DeWitt patch, a region of the bulk geometry anchored on the two boundaries, which extends behind the black hole horizon, is equal to the complexity \[7\].

More generally in quantum many-body physics, it is believed that long-time complexity growth is a universal aspect of chaotic systems. Specifically, for unitary evolutions \(e^{-iHt}\) generated by a chaotic Hamiltonian \(H\), one expects the circuit complexity of the unitary time-evolution operator, as well as that of a state reached by evolving from an initial unentangled state, to grow linearly in time for an exponentially long time. This behaviour of quantum complexity for a time-independent chaotic evolution has been conjectured by Brown and Susskind \[1\]. The quantum complexity of an \(n\) qubit state is expected to (i) grow linearly in time for exponential time \(t \sim e^{\Theta(n)}\), (ii) saturate at its maximal possible value, and (iii) undergo a recurrence back to trivial complexity at times doubly-exponential in \(n\), \(t \sim e^{e^{\Theta(n)}}\); see Figure 1. The intuition for the initial growth is that we should not be able to approximate a chaotic many-body evolution by a substantially shorter circuit. But this reasoning needs to break down for deep circuits as the complexity saturates and collisions start to dominate. The evidence for this is the observation that at exponential depths the number of circuits one can generate from a finite set of universal quantum gates is comparable to the volume of the unitary group \[1\] (or rather, under the suitable normalization of the Haar measure, the number of pairwise distinct \(\varepsilon\)-balls in the unitary group). Lastly, recurrence of complexity after double exponential time can be seen as expected consequence of the Poincaré recurrence of states and unitaries in isolated quantum systems \[16–18\].

Proving the Brown-Susskind conjecture for a concrete time-evolution generated by a specific Hamiltonian is unfortunately out of reach using currently available techniques. In the light of this it is interesting to consider its generalization to other models of chaotic quantum evaluations such as local random quantum circuits \[19\]. Random quantum circuits are known to rapidly scramble quantum information \[20, 21\] and efficiently approximate moments of the Haar measure \[19, 22–26\]. Their ability to generate pseudo-randomness is the theoretical basis for recent experimental demonstrations of quantum advantage \[27\]. In the context of strongly-interacting quantum dynamics, random quantum circuits have been utilized to investigate quantum chaos and the spread of entanglement \[28–30\].

The goal of this work is to provide rigorous results on the long-time behaviour of complexity for random quantum circuits (RQCs). Specifically, we consider local RQCs constructed from the random application of local 2-site unitaries to a system of \(n\) qudits. In this model the depth (number of gates) of the quantum circuit plays the role of time in the chaotic evolution discussed above. The main results of our work are rigorous

FIG. 1. The conjectured \[1\] time evolution of the circuit complexity in an \(n\) qubit system. The complexity exhibits a (linear) growth for an exponentially-long time until it saturates at its maximal value in time \(t_{\text{sat}} = \exp(\Theta(n))\). Afterwards the complexity remains maximal until it undergoes a recurrence at doubly exponential times \(t_{\text{rec}} = \exp(\exp(\Theta(n)))\).
proves of the saturation and recurrence of quantum complexity in random unitaries and states generated by this model. Importantly, our findings do not depend on the details of the RQC model and show that the timescales of saturation $t_{\text{sat}}$ and recurrence $t_{\text{rec}}$ are, respectively, exponential and doubly-exponential in number of qudits $n$ (size of the system), as expected, with an essentially optimal dependence on the accuracy parameter $\varepsilon$. Furthermore, we prove that in RQCs the complexity recurrences last an exponential amount of time. This supports the conjectured connection [31] between the evolution of complexity in chaotic systems and the Penrose diagram (tracking the evolution of an AdS wormhole) and points towards even deeper connections between complexity recurrences and regimes in which quasi-classical descriptions of spacetime can be used.

Previous work has made some progress in understanding the evolution of complexity in random quantum circuits. Specifically in Ref. [32], by proving bounds on the complexity of structured ensembles called unitary designs, and using the known convergence of random quantum circuits to approximate designs, they were able to lower bound the complexity of random circuits. Notably, they used a strong definition of circuit complexity, which, for quantum states, captures the difficulty of implementing a measurement to distinguish a given quantum state from the maximally mixed state.\(^1\) More recently, Refs. [34, 35] used algebraic-geometric techniques to prove a long-time linear growth of the complexity of random circuits for qubit systems, but only for the exact complexity, i.e. demanding that the target unitary be implemented exactly by quantum circuits capable of implementing arbitrary two qubit gates. Note that this notion of complexity is very stringent and of limited physical relevance as in an arbitrarily small neighborhood of the identity there exist unitary transformations of maximal complexity.\(^2\) The notion of approximate complexity adopted by us does not exhibit this problem and assigns trivial (zero) complexity to the $\varepsilon$ ball around the identity.

There are other approaches to proving lower bounds on the complexity of unitary transformations. First, the seminal work of Nielsen and others [36, 37] connected complexity to the geodesic distance in a suitably-defined sub-Riemannian geometry. This approach however cannot be used to directly compute (or estimate) complexity due to the immense difficulty of the optimization problem of finding the shortest geodesic. Note, however, that in Refs. [38, 39] Nielsen’s geometric approach was used to compute complexity in free fermionic and bosonic theories. Furthermore, Ref. [40] used this perspective to lower bound (exact) complexity in terms of the entangling power of quantum gates. Additionally, complexity-theoretic assumptions (such as $\text{PSPACE} \neq \text{BQP}$) can be used to show that the complexity of specific Hamiltonian evolutions must be super-polynomial at exponentially long times, see e.g. Refs. [4, 8, 41, 42].

In contrast to these works, our proof techniques crucially depend on a novel property called approximate equidistribution, which we introduce for probability measures on the space of unitaries and the space of pure quantum states. The property captures an ensemble’s ability to approximate the Haar measure on small discretized scales and mimic its behavior on small regions. We prove that ensembles of unitaries exhibiting this property are characterized by maximal (exponential) complexity. Furthermore, we show that realizations of random circuits resulting from iterations of equidistributed ensembles typically give recurrences of complexity after doubly-exponential time. Importantly, we use approximate complexity, a notion with clear physical significance. In order to make practical use of these results, we prove that high-degree approximate designs exhibit approximate equidistribution by adapting techniques from [26, 43], which were used to connect approximate unitary designs and $\varepsilon$-nets in the set of unitaries. Together with exponential convergence bounds of the deep random quantum circuits to approximate $k$-expanders [19], this allows us to establish (exponential) upper bounds on equidistribution times for local random quantum circuits. A careful application of the approximate equidistribution property then allows us to prove that typical recurrences of complexity have exponential duration.

We emphasize that our results hold for very general classes of local random quantum circuits, which indicates the universality of quantum complexity saturation and recurrences. Many of the results in the literature regarding, for instance, approximate designs, decoupling, and scrambling in random quantum circuits are tailored to specific geometries (often 1D or a complete graph) with Haar-random 2-site unitary gates. Our results apply to local random quantum circuits with any arrangement of qubits defined on a graph $\mathcal{G}$ with a Hamiltonian path and with gates chosen randomly from any universal gate set $\mathcal{G}$ (without

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\(^1\) We note that this definition of complexity necessarily assumes dichotomic measurements. If this condition is relaxed it can be shown [33] that computational basis measurements preceded by local random circuits of depth $O(n)$ suffice to efficiently distinguish any pure state from maximally mixed state.

\(^2\) Indeed, general dimension counting arguments show that almost all (with respect to the Haar measure) unitary circuits have maximal exact complexity (scaling as $\varepsilon^{O(n)}$). As a result, in an arbitrarily small neighborhood of the computationally-trivial identity channel there exist unitary transformations of that maximal exact complexity.
requiring inverses, as in [19]). Furthermore, we believe that due to the universality of the approximate equidistribution property, our techniques could likely be adapted to other models of chaotic evolution, such as those generated by stochastic local Hamiltonians [44].

**Organization of work** After fixing notation and presenting some definitions, in Section II we give an overview of our results regarding late-time complexity, including the novel notion of approximate equidistribution, which we define. We also present a list of possible directions for future work. In Section III we introduce the technical tools that we require in our proofs, including volumes in the space of unitary channels and states as well as approximate unitary designs. In Section IV we relate approximate equidistribution and approximate designs, from which it follows that deep random circuits constitute approximately equidistributed distributions. In Section V we prove that typical unitaries and states picked at random from equidistributed distributions have (essentially) maximal complexity. In Section VI, we show that if random quantum circuits equidistribute at some time scale \( \tau \), then their complexity undergoes a recurrence at doubly exponential times. Building on previous sections, we use this to prove that complexity of random quantum circuits experiences a recurrence to trivial complexity at (and not before) doubly-exponential times. In Section VII we prove that with high probability, the duration of recurrences dips is exponential in the number of qubits. Finally, in Section VIII we prove a linear complexity growth for exponentially deep circuits. In Appendix A we generalize results of [26] and connect approximate unitary designs (expanders) with approximate equipartition property for states and unitaries. In Appendix B and Appendix C we present proofs of some additional technical results used in the main body of the work.

**Setting**

We consider systems of \( n \) qudits, with local dimension \( q \). The dimension of the corresponding Hilbert space \( \mathcal{H} = (\mathbb{C}^q)^{\otimes n} \) is then \( d = q^n \). Every unitary operator \( U \) on \( \mathcal{H} \) defines a unitary channel \( U \) via conjugation \( U(\rho) = U\rho U^\dagger \). Note that all unitary operators \( \exp(i\varphi)U \), which differ from \( U \) by a global phase, define the same unitary channel \( U \). We denote the set of quantum channels on \( \mathcal{H} \) by \( \mathcal{U}(d) \). Similarly, let \( \mathcal{S}(d) \) denote the set of pure quantum states in \( \mathcal{H} \), i.e. rank-one projectors of the form \( \psi = |\psi\rangle\langle\psi| \). We choose to work with the following definitions of distance between unitary channels and quantum states

\[
D(U, V) = \min_{\varphi \in [0, 2\pi)} \|U - \exp(i\varphi)V\|_{\infty} \quad \text{and} \quad D(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1 ,
\]

where \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_1 \) are the operator norm and 1-norm, respectively. These distances describe optimal statistical distinguishability of unitary channels and states. For quantum states this follows from the operational definition of trace distance while for unitaries it follows from the relation between the operator norm and the diamond norm

\[
D(U, V) \leq \|U - V\|_\diamond \leq 2D(U, V) ,
\]

for a proof of this see, for instance, Ref. [26].

We consider circuits formed from a universal discrete gate set \( \mathcal{G} \) of size \( |\mathcal{G}| = \text{poly}(n) \). Assume that each gate in \( \mathcal{G} \) is \( \ell \)-local, i.e. acts non-trivially on at most \( \ell \) qudits, where \( \ell = \text{O}(1) \). Moreover, we will make the assumption that the gates in \( \mathcal{G} \) are universal for any subset of qudits of size \( \ell \). Let \( \mathcal{G}^r \) \((r = 0, 1, 2, \ldots)\) be the set of all size \( r \) circuits built from \( \mathcal{G} \). We can now define the complexity of a unitary and state.

**Definition 1** (Unitary complexity). For \( \varepsilon \in [0, 1] \), we say that a unitary \( U \) has \( \varepsilon \)-unitary complexity equal to \( r \) if and only if \( r = \min \{ l : \exists V \in \mathcal{G}^l \text{ s.t. } D(U, V) \leq \varepsilon \} \), which we denote as \( C_u(U) = r \).

**Definition 2** (State complexity). For \( \varepsilon \in [0, 1] \), we say that a pure state \( \psi \) has \( \varepsilon \)-state complexity equal to \( r \) if and only if \( r = \min \{ l : \exists V \in \mathcal{G}^l \text{ s.t. } D(\psi, V(\psi_0)) \leq \varepsilon \} \), which we denote as \( C_s(\psi) = r \).

In the definition of state complexity we assume that \( \psi_0 \) is chosen to be an unentangled product state, for instance \( \psi_0 = |0\rangle\langle 0|^{\otimes n} \). Let \( \mathcal{G}^r_e = \{U : \exists V \in \mathcal{G}^l, l \leq r \text{ s.t. } D(U, V) \leq \varepsilon \} \) denote the set of unitary channels with complexity at most \( r \), i.e. unitaries that can be approximated to an accuracy \( \varepsilon \) by circuits of size at most \( r \). Likewise let \( \mathcal{S}^r_\varepsilon = \{\psi : \exists V \in \mathcal{G}^l, l \leq r \text{ s.t. } D(\psi, V(\psi_0)) \leq \varepsilon \} \) be the set of states that can be \( \varepsilon \)-approximated by the states generated by circuits of size at most \( r \) acting on an initial state \( \psi_0 \).
In this work we are interested in understanding the complexity of states and unitaries generated by random quantum circuits. Specifically, let $\nu$ be a probability measure on $\mathcal{U}(d)$ that corresponds to one layer of a random quantum circuit, e.g. a single time step. Then by $\nu_t$ and $(\nu_t)_S$ we denote measures describing the distributions of unitary transformations and quantum states realized by depth $t$ random circuit:

$$U_t = V^{(t)} V^{(t-1)} \ldots V^{(1)}, \quad \psi_t = V^{(t)} V^{(t-1)} \ldots V^{(1)}(\psi_0),$$

where unitaries $V^{(i)}$ are i.i.d. and distributed according to measure $\nu$. Later we shall consider two versions of $\nu$: one corresponding to local circuits in one dimension, which will be denoted by $\nu^{(n)}$ and a more general one, corresponding to circuits on connected graphs, denoted by $\nu^{(G)}$. We shall also denote $(\nu_t)_S$ by $\nu_{t,S}$, hence also $(\nu_t^{(n)})_S$ by $\nu_{t,S}^{(n)}$, and $(\nu_t^{(G)})_S$ by $\nu_{t,S}^{(G)}$.

II. OVERVIEW OF RESULTS

FIG. 2. Diagram depicting our results for the circuit complexity $C_\varepsilon(U_t)$ of depth-$t$ local random quantum circuits acting on $n$ qudits. For typical realizations of random quantum circuits, complexity grows until an exponential-time, at which point it saturates to its maximal value at time $t_{sat} \approx \exp(\Theta(n)) \log(1/\varepsilon)$. Thereafter, the complexity remains maximal until it undergoes a recurrence at doubly exponential times $t_{rec} \approx (1/\varepsilon)^{\exp(\Theta(n))}$. The recurrence process typically takes exponential amount of time.

The summary of our contributions to late-time behaviour of complexity in random quantum circuits is presented in Figure 1. In what follows we give precise formulation of each of the results indicated in the figure. Our findings crucially depend on a new (to the best of our knowledge) property of probability measures on sets of unitary channels and pure quantum states, which we call the approximate equidistribution property.

Definition 3 (Approximate equidistribution on unitary channels and pure states). Let $\alpha, \beta$ be real numbers such that $0 < \alpha \leq 1 \leq \beta$. We say that a probability measure $\nu$ on $\mathcal{U}(d)$ is $(\alpha, \beta)$-equidistributed on $\mathcal{U}(d)$ at the scale $\varepsilon$ if, for every ball $B(V, R) \subset \mathcal{U}(d)$ of radius $R \geq \varepsilon$, we have

$$\text{Vol}(\alpha \cdot R) \leq \nu(B(V, R)) \leq \text{Vol}(\beta \cdot R).$$

Similarly, a probability measure $\nu_S$ on pure states $S(d)$ is $(\alpha, \beta)$-equidistributed on $S(d)$ at the scale $\varepsilon$ if, for every ball $B(\psi, R) \subset S(d)$ of radius $R \geq \varepsilon$, we have

$$\text{Vol}_S(\alpha \cdot R) \leq \nu_S(B(\psi, R)) \leq \text{Vol}_S(\beta \cdot R).$$

Informally, a measure $\nu$ satisfies this approximate equipartition property on a given scale if the value of the measure it assigns the to every ball of size $\varepsilon$ is comparable to the Haar volume of the of balls of similar size, e.g. see Figure 3.
We prove a number of properties suggesting that the approximate equipartition property is central for establishing the saturation and recurrence of complexity for states and unitary channels. Recall we assumed that the complexity of states and unitaries is defined with respect to a fixed local universal \( \ell \)-local gate set \( \mathcal{G} \) of cardinality \( |\mathcal{G}| = \text{poly}(n) \).

**Result 1** (Typical unitary channels and states generated by ensembles exhibiting approximate equidistribution have almost maximal complexity). Let \( \nu \) and \( \nu_\mathcal{S} \) be probability measures on \( \mathcal{U}(d) \) and \( \mathcal{S}(d) \) which \((\alpha, \beta)\)-equidistribute on their respective spaces. Then, with high probability, unitary channels \( \mathbf{U} \) and states \( \psi \) selected at random from \( \nu \) and \( \nu_\mathcal{S} \) satisfy

\[
C_\varepsilon(\mathbf{U}) \gtrsim \frac{d^2}{\log |\mathcal{G}|} \log \left( \frac{1}{\beta \varepsilon} \right) \quad \text{and} \quad C_\varepsilon(\psi) \gtrsim \frac{d}{\log |\mathcal{G}|} \log \left( \frac{1}{\beta \varepsilon} \right),
\]

where \( c_0 = 87 \) is an absolute constant (see Eq. (17)). Both lower bounds appearing in Eq. (6) saturate (up to terms polynomial in \( n \)) the known upper bounds on complexities of unitary states and unitaries in \( n \) qudit systems (see Lemma 8).

**Result 2** (Approximately equidistributed ensembles of unitary channels and states have doubly-exponential distinct high-complexity elements). Let \( \nu \) and \( \nu_\mathcal{S} \) be probability measures on \( \mathcal{U}(d) \) and \( \mathcal{S}(d) \) which \((\alpha, \beta)\)-equidistribute on their respective spaces. Let \( N_\mathcal{U}^\varepsilon(\nu) \) be the maximal number of pairwise \( \varepsilon \)-distinct unitary channels \( \mathbf{U} \) from the support of \( \nu \) on unitaries with complexity greater than \( r \). Likewise, let \( N_\mathcal{S}^\varepsilon(\nu_\mathcal{S}) \) be the maximal number of pairwise \( \varepsilon \)-distinct states \( \psi \) from the support of \( \nu_\mathcal{S} \) on states with complexity greater than \( r \). Then for

\[
\begin{align*}
\nu_{\text{unitary}} &\approx \frac{d^2 \log(A_\beta \varepsilon)}{\log |\mathcal{G}|}, & \nu_{\text{states}} &\approx \frac{d \log (\frac{1}{4 \beta \varepsilon})}{\log |\mathcal{G}|}.
\end{align*}
\]

we have

\[
N_\mathcal{U}^\varepsilon(\nu) \approx \left( \frac{1}{2 \beta c_0} \right)^{d^2}, \quad N_\mathcal{S}^\varepsilon(\nu_\mathcal{S}) \approx \left( \frac{1}{2 \beta c_0} \right)^{2d},
\]

where \( c_0 = 87 \) is an absolute constant (see Eq. (17)) and \( A_\beta > 0 \) is an explicit constant depending on \( \beta \). As \( d = q^n \), we get that both \( N_\mathcal{U}^\varepsilon(\nu) \) and \( N_\mathcal{S}^\varepsilon(\nu_\mathcal{S}) \) exhibit a doubly exponential scaling with the number of qudits (for fixed accuracy parameter \( \varepsilon \)).

Our first result, proven in Lemma 9, is that the complexity of typical states and unitary channels chosen from the Haar measure are close to maximal with overwhelming probability. The second result, proven in Lemma 10, ensures that the number of distinct high-complexity unitaries is comparable to the packing number of the set all unitary channels \( \mathcal{U}(d) \) (on the appropriate logarithmic scale). In this sense the Haar measure is an ensemble consisting of unitaries and states with essentially maximal complexity. The above two results show that probability measures which approximately equidistribute, describe ensembles of states or unitary channels with essentially maximal complexity. Furthermore, the number of distinct high complexity unitaries and states in the support of such measures scales doubly-exponentially with the number of qudits \( n \). Crucially the dependence on \( \varepsilon \) in Eq. (6) is optimal.
Similar results can be derived using techniques based on moments and approximate k-designs \cite{32}, but they yield weaker results not capturing the correct scaling in \( \varepsilon \). In the current work we utilize these techniques based on moments (and Markov’s inequality) to: (i) show linear growth of complexity for exponentially long times (see Result 7) and (ii) to find explicit lower bounds on the packing number of \( C^r_{\varepsilon} \) (i.e. circuits of depth \( r \) generated by universal gate set \( \mathcal{G} \)), which is needed in the technical formulation of Result 3 (see Lemma 13).

Another important property of equidistributed ensembles of unitaries is that they naturally yield doubly-exponential recurrence times for complexity. This is proven in Theorem 12 in Section VI and summarized below.

**Result 3** (From approximate equidistribution to saturation in exponential time and recurrence in doubly-exponential time). Let \( \nu \) be a probability measure on \( \mathcal{U}(d) \) describing a single layer of a random quantum circuit, i.e. a single time step. Let \( \tau \) be a circuit depth such that the distribution of depth \( \tau \) random quantum circuits \( U_\tau \), generated by \( \nu_\tau \), \((\alpha, \beta)\)-distribute on scale \( \varepsilon \). Then, with high probability, sequences of random unitary circuits \((U_t)_{t \in \mathbb{N}}\):

(i) Achieve complexity \( C_\varepsilon(U_t) \gtrsim d^2 \log(1/(\beta c_\varepsilon \varepsilon)) \) for \( t = \tau \) (this follows from Result 1).

(ii) Have a recurrence time \( t_{\text{rec}} \) —the first time after reaching maximal complexity as in (i) that the complexity exhibits a recurrence, i.e. \( C_\varepsilon(U_{t_{\text{rec}}}) = O(1) \)—which satisfies

\[
\left( \frac{1}{\beta c_\varepsilon \varepsilon} \right)^{d^2} \lesssim t_{\text{rec}} \lesssim \tau \left( \frac{1}{\alpha c_\varepsilon \varepsilon} \right)^{d^2},
\]

where \( c_\varepsilon = 87 \) and \( c_\varepsilon = \frac{1}{\pi^2} \) are absolute constants.

Noting that \( d = q^n \), we observe that the complexity of random circuits undergoes recurrence at a circuit depth \( e^{\alpha \varepsilon \varepsilon} \) in the multiples \( \tau \). An analogous result holds for sequences of states \((\psi_t)_{t \in \mathbb{N}}\) generated by \( \nu \), the only difference being the maximal complexity and dimension of the space proportional to \( d \), not \( d^2 \).

In order to make practical use of the above results we lower bound equidistribution times \( \tau \) and \( \tau_S \) for local random quantum circuits by establishing a connection between high degree approximate k-designs and approximate equidistribution for unitary channels and pure quantum states.

**Result 4** (High-degree approximate k-expanders satisfy approximate equidistribution). Let \( \nu \) be a \( \delta \)-approximate unitary k-expander with

\[
\delta \lesssim \left( \frac{\varepsilon^2 \Gamma^2}{d} \right)^{d^2}, \quad k \approx \frac{d^{6/2}}{\varepsilon \Gamma},
\]

then the distribution \( \nu \) is \((1 - \Gamma, 1 + \Gamma)\)-equidistributed in \( \mathcal{U}(d) \) on a scale \( \varepsilon \). Furthermore, if

\[
\delta \lesssim (\varepsilon \Gamma)^{6d}, \quad k \approx \frac{d}{(\varepsilon \Gamma)^4},
\]

then the measure \( \nu_S \) induced by \( \nu \) on \( S(d) \) is \((1 - \Gamma, 1 + \Gamma)\)-equidistributed in \( S(d) \) on a scale \( \varepsilon \).

The proof of the above result is based on adapting techniques from Refs. \cite{26, 43}, which were used to establish when unitary ensembles forming approximate expanders yield \( \varepsilon \)-nets in the space of unitaries. Furthermore, these techniques are generalized in a manner that allows us to characterize when approximate unitary designs will induce a measure on the set of pure quantum states which would exhibit approximate equidistribution. The technical formulation of the above result is given in Theorem 5.

Our quantitative statements about the recurrence of complexity for more general local random quantum circuits follow from combining Result 3 and Result 4 with the exponential convergence bounds of local RQCs to approximate k-designs, which are valid for arbitrary degree \( k \) \cite{19} (see Proposition 3 for details).

**Result 5** (Saturation and recurrence of complexity for random quantum circuits). Let \( \nu^{(n)} \) be the probability distribution describing a single layer of local quantum circuits on the 1D architecture, acting on an \( n \) qudit
system $\mathcal{H} = (\mathbb{C}^n)^\otimes n$ of dimension $d = q^n$. Then, the measure $\nu_i^{(n)}$ becomes $(1 - \Gamma, 1 + \Gamma)$-equidistributed on scale $\varepsilon$ when the circuit depth is

$$\tau \approx d^4 \log \frac{1}{\varepsilon \Gamma}.$$  

(12)

Moreover, the induced measure $\nu_{i, S}^{(n)}$ on the space of pure states $S(d)$ becomes $(1 - \Gamma, 1 + \Gamma)$-equidistributed on scale $\varepsilon$ in this set after a circuit depth

$$\tau_S \approx d^3 \log \frac{1}{\varepsilon \Gamma}.$$  

(13)

By inserting $d = q^n$ and using Result 3, we find that the complexity $C_ε$ evaluated on sequences of random unitary channels $(U_t)_{t \in \mathbb{N}}$ and states $(\psi_t)_{t \in \mathbb{N}}$ generated by $\nu^{(n)}$ with high probability

(i) Saturates at an essentially maximal value, equal to $C_{\text{unitary}} \approx d^2 \log(1/\varepsilon)$ and $C_{\text{state}} \approx d \log(1/\varepsilon)$ for unitaries and states respectively, at time $t_{\text{sat}} \lesssim e^{\Theta(n)} \log(1/\varepsilon)$.

(ii) The first time after saturation that complexity reaches $C_ε = O(1)$ is $t_{\text{rec}} \approx e^{\Theta(n)}$.

A technical statement of formulas Eq. (12) and Eq. (13) is contained in Eq. (49) of Lemma 7. There we also give results for the saturation and recurrence times of complexity for more general $G$-local RQCs, where the qudits are coupled according a connected graph $\mathcal{G}$ and random two-qudit gates are chosen from a general universal gate set $\mathcal{G}$. The dependence of $\tau$ and $\tau_S$ on the total dimension $d$ is the same but, if one does not assume algebraic entries in $\mathcal{G}$, the leading dependence on $\varepsilon$ becomes $\sim \log(1/\varepsilon)^3$.

Interestingly, careful usage of approximate equidistribution allows us to gain insight into timescales on which recurrences of complexity occur. The following result shows that their typical duration scales exponentially with the system size, in accordance to what was recently conjectured in Ref. [31].

**Result 6** (Typical duration of recurrences of complexity). Let $\nu$ be a probability measure on $U(d)$ describing a single layer of a random quantum circuit on an $n$ qudit system of dimension $d = q^n$. Let $t$ be a natural number such that $t > 2T$, where $T$ denotes the time at which the distribution of $U_T$ is $(1 - \Gamma, 1 + \Gamma)$-equidistributed on a scale $\varepsilon$ with $\Gamma \leq e^{-\Theta(d^2)}$. Assume that $C_ε(U_t) = 0$ at the time $t$. Then, conditioned on this event, with high probability (over the distribution of trajectories $(U_t)_{t \in \mathbb{N}}$) we have

$$C_ε(U_{t-T}) \approx d^2 \log(1/\varepsilon) \quad \text{and} \quad C_ε(U_{t+T}) \approx d^2 \log(1/\varepsilon).$$  

(14)

From Result 5 we know that local random quantum circuits on the 1D architecture (i.e. $v = \nu^{(n)}$) have $T = e^{\Theta(n)} \log(1/\varepsilon)$. Since complexity $C_ε$ cannot change faster than linearly with $t$, we get that with high probability recurrences of complexity have a duration exponential in the system size. An analogous result holds for sequences of states $(\psi_t)_{t \in \mathbb{N}}$ generated by $\nu$.

We provide a technical formulation and formal proof of the above claim in Section VII. We note that similar results hold also for the general model of $G$-local quantum circuits discussed above.

Lastly, we combine some previous results to show that an exponentially-small but moment-independent spectral gap for random quantum circuits implies a linear complexity growth for deep circuits.

**Result 7** (Linear growth at exponential depths). For local random quantum circuits of depth $t$, $\nu_i^{(n)}$, when the circuit depth is $t \gtrsim d^2 \log(1/\delta)$, then $\nu_i^{(n)}$ forms a $\delta$-approximate $k$-expander for all moments, which implies a linear complexity growth up until the saturation time $t \approx d^3$. Specifically, with very high probability, a depth $t$ random quantum circuit has complexity lower bounded as $C_ε(U_t) \gtrsim (t/d^2)$. Similarly, the states generated by RQCs of the same depth have complexity $C_ε(\psi) \gtrsim (t/d^2)$, such that a linear growth of state complexity begins at $t \approx d^3$ and proceeds until saturation. We emphasize that in contrast to the results involving approximate equidistribution, these complexity lower bounds have a trivial $\varepsilon$ dependence.
Our work leaves open a number of interesting directions for future work:

First, we have focused on time-evolutions generated by random quantum circuits, leveraging the fact that they form approximate unitary designs to prove statements about the late-time behavior of complexity. Specifically, we use convergence to designs of exponential degree, \( k = e^{\Omega(n)} \), to establish approximate equidistribution. It is also known that other time-dependent Hamiltonian evolutions, such as the disordered spin systems with time-dependent random couplings in Refs. [44, 45], converge to approximate designs of subexponential degree. If one could extend these results to show that exponentially-long stochastic Hamiltonian evolution generates high-degree designs, rigorous statements about the saturation and recurrence of complexity would directly follow from our results on approximate equidistribution.

Second, the results presented in this work, along with those in Ref. [32], provide a consistent framework for understanding the growth and saturation of quantum complexity. The techniques directly apply to time-dependent unitary evolutions that form approximate designs. Nevertheless, in physics we are often interested in the energy-conserving evolution of time-independent Hamiltonians, which will not (in general) form unitary designs [46]. Proving statements about the complexity growth of the ensemble \( \{ e^{-iHt}, H \in \mathcal{E}_H \} \), for an ensemble of Hamiltonians, demands a different proof technique.

An exponentially-small lower bound on the spectral gap for random circuits, \( \Delta \geq e^{-\Omega(n)} \) and independent of \( k \), gave that random quantum circuits form designs of exponential degree. Combined with the conditions for distributions to equidistribute over the space of unitaries, this gives the circuit depth at which complexity saturates and recurs. Can one substantially improve the \( n \)-dependence of the spectral gap bound for random circuits? An even more challenging question: can one prove an unconditional constant lower bound on the spectral gap? This would prove the optimal design depth and, via the results in Ref. [32] and in this work, would directly imply the conjectured linear complexity growth for random circuits.

Recently, Ref. [47] studied another long-time property of random circuit evolution—the fluctuations of subsystem entropies of states evolved by random circuits, which become increasingly suppressed for an exponentially long time. The results here should give the timescale at which the subsystem entropy recurs and the evolved-state becomes unentangled. More generally, it would be intriguing to rigorously explore the connection between quantum complexity and other long-time properties of many-body systems.

Furthermore, in our analysis we rely heavily on the approximate equidistribution property to estimate the measure on balls \( \nu(B(V, \epsilon)) \), \( \nu_S(B(\psi, \epsilon)) \) in the set of unitary channels and in the space of pure quantum states. The use of traditional techniques to estimate these quantities—e.g. based on (approximate) designs, moment bounds, or Levy’s lemma—does not give the correct scaling with \( \epsilon \). This suggests that in the regime of high-degree designs it should be possible to develop stronger concentration inequalities, interpolating between the known design bounds [48] and the properties of the Haar measure.
Lastly, there are a number of interesting questions regarding the scale of complexity fluctuations at various regimes of quantum complexity growth. It would be interesting to apply our methods to get precise bounds on the probability and size of a typical fluctuation in the complexity of a given random circuit instance away from the ensemble ‘average.’ More speculatively, for holographic complexity there are various expectations of the behavior of circuit complexity coming from explicit gravity calculations in the bulk. It would be interesting to see if our methods were able to identify any particular features in the long-time behavior of the behavior of circuit complexity coming from explicit gravity calculations in the bulk. It would be interesting to apply our methods to get precise bounds in particular subtleties that arise at exponential time scales.

### III. TECHNICAL TOOLS

**Haar measure, volumes in the space of unitary channels and pure states**

The manifold of unitary channels $\mathcal{U}(d)$ inherits the unique invariant probability measure from the Haar measure on the unitary group $\mathcal{U}(d)$ according to the following prescription: for $A \subset \mathcal{U}(d)$ we set $\mu_P(A) = \mu(A')$, where $\mu_P, \mu$ are Haar measures on $\mathcal{U}(d)$ and $\mathcal{U}(d)$ respectively, and $A'$ denotes collection of all unitary operators defining quantum channels from $A$. In what follows we will not differentiate between unitary channels and unitary operators, as well as Haar measures defined on these sets, unless it leads to an ambiguity. In particular, we let $\text{Vol}(A)$ denote the Haar measure of a subset $A$ of unitary channels $\mathcal{U}(d)$ or unitary group $\mathcal{U}(d)$, depending on the context. Finally, the Haar measure $\mu$ on $\mathcal{U}(d)$ induces the unique invariant normalized measure $\mu_S$ on the set of pure quantum states by $S(d)$. This measure can be defined by $\mu_S(B) = \mu(\varphi^{-1}(B))$, where $B \subset S(d)$ and $\varphi : \mathcal{U}(d) \rightarrow S(d)$ is defined by $\varphi(U) = U(\phi_0)$, for arbitrarily chosen pure state $\phi_0 \in S(d)$. In analogous way, for a fixed choice of a referential state $\phi_0 \in S(d)$, any measure $\nu$ on $\mathcal{U}(d)$ defines a measure on the set of pure states via

$$\nu_S(B) = \nu(\varphi^{-1}(B)).$$  \hspace{1cm} (15)

We will be frequently working with balls of radius $\varepsilon$ in the space of unitary channels $\mathcal{U}(d)$, induced by the diamond norm

$$B(U, \varepsilon) = \{V : D(V, U) \leq \varepsilon\}.$$  \hspace{1cm} (16)

We denote the Haar measure of such a ball by $\text{Vol}(B(U, \varepsilon)) = \text{Vol}(\varepsilon)$. From the results in [49, 50], we have

$$(c_o \varepsilon)^{d^2 - 1} \leq \text{Vol}(\varepsilon) \leq (c^o \varepsilon)^{d^2 - 1},$$  \hspace{1cm} (17)

where $c_o$ and $c^o$ are absolute constants. In Ref. [26] one finds that $c_o = \frac{1}{\pi^2}$ and $c^o = 87$. Similarly for pure states, we have balls of radius $\varepsilon$ in the space of states $B(\psi, \varepsilon) = \{\phi : D(\phi, \psi) \leq \varepsilon\}$. We denote the volume of this set according to the uniform measure $\mu_S$ in $S$ induced from the Haar measure on $\mathcal{U}(d)$ by $\text{Vol}_S(B(\psi, \varepsilon)) = \text{Vol}_S(\varepsilon)$. In this case it is possible to find an exact expression for the volume

$$\text{Vol}_S(\varepsilon) = \varepsilon^{2(d-1)}.$$  \hspace{1cm} (18)

The above equation arises from the distribution of the overlaps of pure states $x = \text{tr}(\psi \phi)$. The explicit expression given by Ref. [51] for this distribution is

$$p(x) = (d - 1)(1 - x)^{d-2}.$$  \hspace{1cm} (19)

**Packing and covering numbers**

Throughout this work we will make use of the *packing* and *covering* numbers; see, for instance, Ref. [52]. They are defined as follows: Let $Y$ be a subset $Y \subset X$ of a metric space $X$, equipped with distance measure $D$. We will consider $X = \mathcal{U}(d)$ or $X = S(d)$ with distances defined in the previous subsection. The packing...
number $N_{\text{pack}}(\mathcal{Y}, \varepsilon)$ quantifies the maximal number of points in $\mathcal{Y}$ that are pairwise distant. The covering number counts the minimal number of balls of size $\varepsilon$ centered at points in $\mathcal{Y}$ needed to cover the whole set. For an illustration of these concepts see Figure 5. Formally, we have

\begin{align}
N_{\text{pack}}(\mathcal{Y}, \varepsilon) &:= \max \left\{ |S| : \forall x, y \in S \ D(x, y) \geq \varepsilon, \ S \subset \mathcal{Y} \right\} \quad (20) \\
N_{\text{cov}}(\mathcal{Y}, \varepsilon) &:= \min \left\{ |S| : \mathcal{Y} \subset \bigcup_{x \in S} B(x, \varepsilon), \ S \subset \mathcal{Y} \right\}. \quad (21)
\end{align}

Note that dependence of above numbers on the metric $D$ is suppressed.

\textbf{FIG. 5.} Illustrations of the packing and covering numbers. On the left: a diagram of a maximal packing of balls of radius $\varepsilon/2$ in a subset $\mathcal{Y}$ of a metric space $X$. On the right: a minimal covering of the same subset, built from balls of radius $\varepsilon$.

Covering and packing numbers are related as follows:

\begin{equation}
N_{\text{cov}}(\mathcal{Y}, 2\varepsilon) \leq N_{\text{pack}}(\mathcal{Y}, \varepsilon) \leq N_{\text{cov}}(\mathcal{Y}, \varepsilon). \quad (22)
\end{equation}

Moreover, we have

\begin{equation}
N_{\text{pack}}\left( \bigcup_{i} \mathcal{Y}_i, \varepsilon \right) \leq \sum_i N_{\text{pack}}(\mathcal{Y}_i, \varepsilon). \quad (23)
\end{equation}

In the cases where $\mathcal{X} = \mathcal{Y} = \mathcal{U}(d)$ and $\mathcal{X} = \mathcal{Y} = \mathcal{S}(d)$, both the metric and the measure are \textit{unitary invariant} and therefore

\begin{equation}
N_{\text{pack}}(\mathcal{U}(d), \varepsilon) \leq \text{Vol}(\varepsilon)^{-1}, \quad N_{\text{pack}}(\mathcal{S}(d), \varepsilon) \leq \text{Vol}_{\mathcal{S}}(\varepsilon)^{-1}. \quad (24)
\end{equation}

Similarly, we have

\begin{equation}
N_{\text{cov}}(\mathcal{U}(d), \varepsilon) \geq \text{Vol}(\varepsilon)^{-1}, \quad N_{\text{cov}}(\mathcal{S}(d), \varepsilon) \geq \text{Vol}_{\mathcal{S}}(\varepsilon)^{-1}. \quad (25)
\end{equation}

Analogous bounds can be derived if we want to control the covering and packing numbers of a measure $\nu$ that is not invariant, but where we have control over the measure it assigns to balls of radius of size of order $\varepsilon$ (like in the definition of approximate equidistribution). In the following lemma we collect bounds for the covering and packing numbers for unitary channels and states that result from Eqs. (17) and (18) and bounds given in Eq. (17).

\textbf{Lemma 1} (Bounds on covering and packing numbers for unitary channels and states). \textit{We have the following bounds on packing and covering numbers of $\mathcal{U}(d)$ and $\mathcal{S}(d)$ equipped with distances defined in Eq. (1):}

\begin{equation}
\left( \frac{1}{\varepsilon \alpha^d} \right)^{d^2 - 1} \leq N_{\text{cov}}(\mathcal{U}(d), \varepsilon) \leq \left( \frac{2}{\varepsilon \alpha^d} \right)^{d^2 - 1}, \quad (26)
\end{equation}

\begin{equation}
\left( \frac{1}{\varepsilon \alpha^d} \right)^{d^2 - 1} \leq N_{\text{pack}}(\mathcal{U}(d), \varepsilon) \leq \left( \frac{2}{\varepsilon \alpha^d} \right)^{d^2 - 1}, \quad (27)
\end{equation}

\begin{equation}
\left( \frac{1}{\varepsilon \alpha^d} \right)^{d^2 - 1} \leq N_{\text{cov}}(\mathcal{S}(d), \varepsilon) \leq \left( \frac{2}{\varepsilon \alpha^d} \right)^{d^2 - 1}, \quad (28)
\end{equation}

\begin{equation}
\left( \frac{1}{\varepsilon \alpha^d} \right)^{d^2 - 1} \leq N_{\text{pack}}(\mathcal{S}(d), \varepsilon) \leq \left( \frac{2}{\varepsilon \alpha^d} \right)^{d^2 - 1}. \quad (29)
\end{equation}
\[
\left( \frac{1}{2 \varepsilon c^2} \right)^{d^2 - 1} \leq N_{\text{pack}}(U(d), \varepsilon) \leq \left( \frac{1}{\varepsilon c^2} \right)^{d^2 - 1},
\]
(27)

\[
\left( \frac{1}{\varepsilon} \right)^{2d - 2} \leq N_{\text{cov}}(S(d), \varepsilon) \leq \left( \frac{2}{\varepsilon} \right)^{2d - 2},
\]
(28)

\[
\left( \frac{1}{2 \varepsilon} \right)^{2d - 2} \leq N_{\text{pack}}(S(d), \varepsilon) \leq \left( \frac{1}{\varepsilon} \right)^{2d - 2}.
\]
(29)

**Approximate unitary design**

The pseudo-random properties of probability distributions over the unitary group are elegantly captured by the notions of (approximate) unitary \( k \)-designs. Here we provide basic concepts and terminology regarding unitary designs. For a more comprehensive treatment of this subject see, for instance, Ref. [53].

To any measure \( \nu \) on \( U(d) \) we associate the \( k \)-fold channel, defined as

\[
\Phi^{(k)}_{\nu}(O) := \int_{U(d)} d\nu(U) U \otimes^k (O),
\]
(30)

where \( O \) is an operator acting on \( (C^d)^{\otimes k} \). A measure \( \nu \) is said to form an exact \( k \)-design if \( \Phi^{(k)}_{\nu}(O) = \Phi^{(k)}_{\mu}(O) \), where \( \mu \) denotes the Haar measure on \( U(d) \). Alternatively, for any measure \( \nu \) we can define the \( k \)-th moment operator as

\[
M_{\nu,k} := \int_{U(d)} d\nu(U) U \otimes^k \bar{U} \otimes^k.
\]
(31)

The property of \( \nu \) being an exact \( k \)-design can be equivalently expressed by demanding \( M_{\nu,k} = M_{\mu,k} \) (this is because a moment operator \( M_{k,\nu} \) is a vectorization of a \( k \)-fold channel \( \Phi^{(k)}_{\nu} \)). Approximate \( k \)-designs are measures for which the aforementioned equalities are satisfied approximately. It is common to define approximate designs such that the channels are close in diamond norm \( \| \Phi^{(k)}_{\nu} - \Phi^{(k)}_{\mu} \|_\diamond \), or even stronger definitions involving small relative error. In this work it will be sufficient to work with a weaker definition of approximate design in terms of the operator norm.

**Definition 4** (Approximate unitary designs/expanders). Let \( \delta > 0 \) be a positive number. A probability distribution \( \nu \) on \( U(d) \) is called a \( \delta \)-approximate unitary \( k \)-design (or approximate \( k \)-expander) if

\[
\| M_{\nu,k} - M_{\mu,k} \|_\infty \leq \delta.
\]
(32)

The quantity \( g(\nu, k) := \| M_{\nu,k} - M_{\mu,k} \|_\infty \) is called the expander norm of \( \nu \).

Given our definition of design in terms of the expander norm, we will use the terms approximate \( k \)-design and approximate \( k \)-expander interchangeably.

Approximate \( k \)-expanders yield approximate designs defined in terms of stronger norms when the error is exponentially small [19]. Nevertheless, approximate \( k \)-expanders, despite being a weaker notion, are often more convenient to work with. This is because \( g(\nu, k) \) can be connected to the spectral gap of \( \nu \), \( \Delta(\nu, k) = 1 - g(\nu, k) \), defined as the second highest eigenvalue of the moment operator \( M_{\nu,k} \). This allows one to estimate \( g(\nu, k) \) for local random circuits via the application of tools from many-body physics, which, after iterating the property \( g(\nu^t, k) = g(\nu, k)^t \) gives practical estimates on the circuit depth needed to ensure a strong design-like properties.
Designs from local quantum circuits

It is known that random quantum circuits, circuits on $n$ qudits formed from the random application of 2-local gates, converge to unitary $k$-designs in a depth which scales polynomially in $n$ and $k$ \cite{19, 22–25, 54}. For simplicity, we will first consider the random quantum circuit architecture defined on a 1D chain of $n$ qudits with local dimension $q$. We will then introduce a more general circuit architecture.

**Definition 5** (Local random quantum circuits). Let $\nu^{(n)}$ denote the probability distribution on $\mathcal{U}(d)$ for a single step of the random circuit, defined by choosing a site $i$ uniformly at random from $[1, \ldots, n-1]$ and applying a Haar random 2-site unitary $U_{i,i+1}$, drawn from $\mathcal{U}(q^2)$, to an adjacent pair of qudits. We are assuming open boundary conditions. Depth $t$ random quantum circuits are then given by the probability distribution $\nu^{(n)}_t := (\nu^{(n)})^{*t}$. The measure $\nu^{(n)}_t$ induces the measure on states $\nu^{(n)}_{t,S}$ as defined previously in Section I.

We will need two statements regarding the convergence of 1D RQCs to designs for low (sub-exponential) moments as well as high moments.

**Theorem 2** (RQCs form unitary designs \cite{19}). For $\delta > 0$ and $4k \leq q^{2/5}$, local random quantum circuits on $n$ qudits form $\delta$-approximate unitary $k$-designs when the circuit depth is

$$ t \geq c_1 n \left( \log(4k) \right)^2 k^{0.5} \log(1/\delta), $$

where the constant is taken to be $c_1 = 4 \times 10^7$.

We note that there is evidence that the polynomial dependence on $k$ can likely be substantially improved \cite{24, 25}. Specifically, it is known that if one takes the local dimension to be large (e.g. $q \geq 6k^2$) the spectral gap is constant and the design depth scales linearly in $k$. Quite recently, the exponent in Theorem 2 was improved from $k^{9.5}$ to $k^{4+o(1)}$ via an improved RQC spectral gap \cite{55}.

As the above theorem only holds for moments up to the square root of the dimension, we can use an exponentially small but $k$-independent spectral gap, proved in Refs. \cite{19, 55}, to establish that exponentially deep random circuits form high degree designs.

**Proposition 3** (Deep RQCs form high-degree designs). For $\delta > 0$ and for any $k$, local random quantum circuits on $n$ qudits with local dimension $q$ form $\delta$-approximate unitary $k$-designs if the circuit depth is

$$ t \geq c_2 n^3 d^2 \log(1/\delta), $$

where $d = q^n$ and the constant here is taken to be $c_2 = 10^5$.

The proofs of these statements follow from a number of Lemmas in Ref. \cite{19}, which we present for completeness in Appendix B.

We might also consider random circuits with parallelized application of local gates, those defined on higher dimensional lattices, or non-local circuits defined on a complete graph. In all cases it is also known that such circuits constitute unitary designs in a depth which scales polynomially in $n$ and $k$ for polynomial moments \cite{19, 23–25}.

Moreover, a deep result by Bourgain and Gamburd \cite{56} on the independence of the spectral gap for universal gate sets (including inverses and with gates consisting of algebraic entries) ensures that Theorem 2 holds for random circuits consisting of gates drawn randomly from such discrete gate sets, with the same polynomial dependence on $n$ and $k$. The choice of universal gate set $\mathcal{G}$ only changes the constant factor. Recent work \cite{26, 43} extended this to any universal gate set of 2-qudit unitaries, without requiring inverses or algebraic entries, at the expense of polynomial factors.

The 1D RQCs with local Haar-random gates introduced above are very concrete in that they allow for an explicit calculation of the scaling and constants involved. To demonstrate the applicability of our results, we also introduce a very general random circuit architecture, with notably few restrictions on the local gate set and graph on which the qudits live. We consider $n$ qudits on a graph $\mathfrak{g}$ which contains a Hamiltonian path, i.e. there is no restriction on the geometric locality of the qudit interaction graph but the graph is connected and there exists a path that visits each qudit exactly once.
Definition 6 (G-local random quantum circuits). Consider $n$ qudits defined on a graph $\mathcal{g}$ which contains a Hamiltonian path. Let $\nu_G$ be the distribution for a single step of the random circuit, choosing a pair $(i,j)$ uniformly at random from the set of edges of $\mathcal{g}$, and applying a random 2-site gate $g_{i,j}$, drawn uniformly at random from $\mathcal{G}$, to the pair of qudits $(i,j)$. Here $\mathcal{G}$ is a set of universal 2-qudit gates, which does not necessarily contain inverses. The distribution defining a single step of $G$-local random quantum circuits will be denoted by $\nu^{(G)}$, where $G$ is understood to denote the pair $(\mathcal{G}, \mathcal{g})$, i.e. the qudit interaction graph $\mathcal{g}$ and the set of 2-local gates $\mathcal{G}$. Depth $t$ $G$-local random quantum circuits are defined by the probability distribution $\nu^{(G)}_t := (\nu^{(G)})^\ast t$.

For these random quantum circuits on more general graphs with arbitrary universal gate sets, we can also establish convergence to unitary designs. This follows from the fact that the spectral gap on a connected graph can be lower bounded by the 1D gap, as well as the aforementioned results on more general gate sets.

Proposition 4 (G-local RQCs form approximate unitary designs). Let $\delta > 0$ and $4k \leq d^{2/5}$ (where $d = q^n$). Then $G$-local random quantum circuits on $n$ qudits with local dimension $q$ form $\delta$-approximate unitary $k$-designs when the circuit depth is

$$t \geq c(\mathcal{G}) n^3 \log^4(k) k^{0.5} \log(1/\delta),$$

where $c(\mathcal{G})$ is a constant that depends on the gate set $\mathcal{G}$. For arbitrarily high moments, i.e. with no restriction on $k$, $G$-local random quantum circuits on $n$ qudits form $\delta$-approximate unitary $k$-designs when the circuit depth is

$$t \geq c(\mathcal{G}) n^7 \log^2(k) d^2 \log(1/\delta).$$

We present proofs of the above statements in Appendix B.

IV. APPROXIMATE EQUIDISTRIBUTION PROPERTY FOR ENSEMBLES OF UNITARY CHANNELS AND STATES

As advertised in Section II, our technical results depend on the approximate equidistribution property of measures on $\mathcal{U}(d)$ and $\mathcal{S}(d)$ (see Definition 3). The following technical theorem establishes a connection between approximate equidistribution and approximate $k$-designs (expanders), a familiar concept in quantum information and quantum computing. It was stated informally in Result 4.

Theorem 5 (Equidistribution of states and unitaries from approximate designs). Consider $\varepsilon$, $\alpha$, and $\beta$ satisfying $0 < \varepsilon \leq 1/4$ and $1/2 \leq \alpha \leq 1 \leq \beta \leq 3/2$, and let $\Gamma = \min\{1 - \alpha, \beta - 1\}$. Let $\nu$ be a $\delta$-approximate unitary $k$-expander in $\mathcal{U}(d)$ for

$$k = 30 \frac{d^{5/2}}{\varepsilon \Gamma} \eta(\varepsilon \Gamma, d) \quad \text{and} \quad \delta \leq \left( \frac{c_0 \Gamma^2 \varepsilon^2}{36d \sqrt{\log \frac{6}{c_0 \varepsilon}} \eta(\varepsilon \Gamma, d) \log \frac{6}{c_0 \varepsilon}} \right)^{d^2 - 1},$$

where

$$\eta(\varepsilon, d) = \sqrt{\frac{\log \frac{6}{c_0 \varepsilon}}{\log \frac{6}{c_0 \varepsilon}} \log \left( \frac{6d}{\varepsilon \log \frac{6}{c_0 \varepsilon}} \right)},$$

i.e. for

$$k \approx \frac{d^{5/2}}{\varepsilon \Gamma}, \quad \delta \approx \left( \frac{e^{2 \Gamma^2}}{d} \right)^{d^2}.$$ 

Then $\nu$ is $(\alpha, \beta)$-equidistributed on $\mathcal{U}(d)$ on a scale $\varepsilon$. 


Moreover, let \( \nu \) be a \( \delta \)-approximate unitary \( k \)-expander for

\[
k = 1296 \frac{d}{(\varepsilon \Gamma)^4} \log \left( \frac{3}{\varepsilon \Gamma} \right) \quad \text{and} \quad \delta = \left( \frac{\varepsilon \Gamma}{9 \log \left( \frac{3}{\varepsilon \Gamma} \right)} \right)^{6d}, \tag{40}
\]

i.e. for

\[
k \approx \frac{d}{(\varepsilon \Gamma)^4}, \quad \delta \approx (\varepsilon \Gamma)^{6d}, \tag{41}
\]

Then the induced measure \( \nu_{S} \) on \( S(d) \) is \((\alpha, \beta)\)-equidistributed on \( S(d) \) on a scale \( \varepsilon \).

The above theorem is simply a combination of Proposition 25 and Proposition 31, which are stated and proven in Appendix A. The techniques used there are inspired by the ones used in Ref. [26]. The essential ingredient of this approach is the construction of the suitable polynomial approximation of Dirac delta on appropriate manifold of interest.

**Remark.** While we prove equidistribution for state designs that are induced by unitary designs, one could consider a more direct definition of state designs. Namely, one can say that measure \( \xi \) is state \( k \)-design if

\[
\int_{S(d)} d\xi(\psi) |\psi\rangle|\psi\rangle^{\otimes k} = \int_{S(d)} d\mu_{S}(\psi) |\psi\rangle|\psi\rangle^{\otimes k}, \tag{42}
\]

where \( \mu_{S} \) is uniform measure on states (induced by the Haar measure). One can prove equidistribution for such designs (both exact and approximate) as well. However, we are primarily interested in distributions over states that come from quantum circuits, and therefore we do not consider such directly defined state designs.

Recall that we are interested in the complexity properties of sequences of random circuits and states:

\[
U_{t}, \psi_{t}, \; t = 0, 1, 2, \ldots \tag{43}
\]

with initial conditions \( U_{0} = I \) and \( \psi_{0} = |0\rangle|0\rangle^{\otimes n} \) (a computationally trivial state). The distributions of \( U_{t} \) and \( \psi_{t} \) are controlled by the \( t \)-fold convolution of \( \nu^{*t} \):

\[
\nu_{t} = \nu^{*t}, \; \nu_{t,S} = \nu^{*t}(\psi_{0}). \tag{44}
\]

**Definition 7** (Equidistribution time). Let \( \nu \) be a probability distribution on \( U(d) \). We define unitary and state equidistribution times as follows

\[
\tau(\nu, \varepsilon) = \min\{t : \nu_{t} \; (\alpha, \beta)\text{-equidistributes in } U(d) \text{ on a scale } \varepsilon\}, \tag{45}
\]

\[
\tau_{S}(\nu, \varepsilon) = \min\{t : \nu_{t,S} \; (\alpha, \beta)\text{-equidistributes in } S(d) \text{ on a scale } \varepsilon\}. \tag{46}
\]

**Fact 6.** If a probability measure \( \nu \) on \( U(d) \) is \((\alpha, \beta)\)-equidistributed on \( U(d) \) on the scale \( \varepsilon \) then so does measure the \( \nu \ast \vartheta \) for any measure \( \vartheta \) on \( U(d) \). Similarly, if \( \nu_{S} \) is \((\alpha, \beta)\)-equidistributed in \( S(d) \), then so is \((\nu \ast \vartheta)_{S}\) for any measure \( \vartheta \) on \( U(d) \).

**Proof.** Assume first that \( \vartheta = \delta_{U} \), i.e. a Dirac delta concentrated at \( U \in U(d) \). For arbitrary \( V \in U(d) \) we have

\[
\nu \ast \delta_{U} (B(V, \varepsilon)) = \nu (U^{-1} [B(V, \varepsilon)]) = \nu (B(U^{-1} V, \varepsilon)), \tag{47}
\]

where the first equality follows from definition of convolution while in the second equality we used \( U^{-1} [B(U, \varepsilon)] = B(V^{-1} U, \varepsilon) \), which follows from the unitary invariance of the distance measure \( D(\cdot, \cdot) \). By invoking the approximate equipartition property (as in Eq. (4)) of \( \nu \) to a ball \( B(V^{-1} U, \varepsilon) \), we get that the measure \( \nu \ast \delta_{U} \) also satisfies this property. For general measures \( \vartheta \) the claim follows because an arbitrary
measure \( \vartheta \) can be written as a (possibly infinite) convex combination of Dirac deltas: 
\[
\vartheta = \sum \alpha p_\alpha \delta_{U_\alpha}.
\]
We then have
\[
\nu \ast \vartheta (B(V, \varepsilon)) = \sum \alpha p_\alpha \nu (B(U_{\alpha}^{-1}V, \varepsilon)),
\]
and approximate equidistribution follows by employing approximate equidistribution of \( \nu \) for every summand and using \( \sum \alpha p_\alpha = 1 \). The proof for pure states is completely analogous.

The above fact shows that Definition 7 is well-justified since equidistribution for time (circuit depth) \( t \) implies equidistribution for time \( t + 1 \) (for both states and unitaries). We are now ready to formulate the lemma which constitutes a technical version of Result 5.

**Lemma 7** (Equidistribution time for random quantum circuits). Let \( \alpha, \beta \) be fixed parameters entering the definition of approximate equidistribution, \( \Gamma = \max \{ 1 - \alpha, \beta - 1 \} \), and let \( \varepsilon \in (0, \frac{1}{4}) \). Let \( \nu^{(n)} \) be a single layer of a local random quantum circuit (see Definition 5). Then we have
\[
\tau_{\nu^{(n)}, \varepsilon} \lesssim c_{\delta} n^{6} d^{4} \log \frac{1}{\varepsilon \Gamma}, \quad \tau_{S} (\nu^{(n)}, \varepsilon) \lesssim c_{\delta} n^{6} d^{4} \log \frac{1}{\varepsilon \Gamma}.
\]
Moreover, let \( \nu^{(G)} \) be a single layer of \( G \)-local random quantum circuits (see Definition 6). Then we have
\[
\tau_{\nu^{(G)}, \varepsilon} \lesssim c(G) n^{10} d^{4} \log^{3} \frac{1}{\varepsilon \Gamma}, \quad \tau_{S} (\nu^{(G)}, \varepsilon) \lesssim c(G) n^{9} d^{3} \log^{3} \frac{1}{\varepsilon \Gamma}.
\]

**Proof.** To prove the above estimates we combine Theorem 5 (saying that \( \delta \)-approximate \( k \)-expanders equidistribute for high enough \( k \) and low enough \( \delta \)) and the results in Proposition 3 and Proposition 4 relating high-degree designs with very deep random quantum circuits. We simply insert the values \( \delta \) and \( k \) from Theorem 5 (Eq. (39) for unitaries and Eq. (41) for states) into the expressions bounding lengths of circuits from Proposition 3 and Proposition 4. Specifically, we use Eq. (34) to get Eq. (49), and Eq. (36) to obtain Eq. (50).

**Remark.** Qualitatively Lemma 7 states that for local random quantum circuits acting on an \( n \)-qudit system, the timescale needed for equidistribution on scale \( \varepsilon \) with parameters \( \alpha, \beta \) satisfying \( \Gamma = \max \{ 1 - \alpha, \beta - 1 \} \) is
\[
\tau_{RQC} = \exp (\Theta(n)) \log \left( \frac{1}{\varepsilon \Gamma} \right) \lesssim c_{\delta} n^{6} d^{4} \log \frac{1}{\varepsilon \Gamma},
\]
for \( \nu = \nu^{(n)} \) and
\[
\tau_{G} = \exp (\Theta(n)) \log \left( \frac{1}{\varepsilon \Gamma} \right)^{3} \lesssim c(G) n^{9} d^{3} \log^{3} \frac{1}{\varepsilon \Gamma},
\]
for \( \nu = \nu^{(G)} \), i.e. general \( G \)-local circuits for which global (i.e. moment independent) gap is not necessarily established.

## V. MAXIMAL COMPLEXITY AND APPROXIMATE EQUIDISTRIBUTION

In this section we prove that ensembles of quantum states and unitaries which exhibit the approximate equipartition property describe high-complexity unitary states and channels. Specifically, we prove that

- Typical states and unitaries from these ensembles have essentially maximal complexity (Lemma 9).
- The ensembles themselves contain doubly-exponentially many (in the number of qudits \( n \)) high-complexity unitaries and states (Lemma 10).
A. Typical values of complexity for approximately equidistributing ensembles

We begin by stating the following upper bounds on the complexity of arbitrary states and unitaries with respect to a general \( \ell \)-local gate set \( \mathcal{G} \).

Lemma 8. Let \( \mathcal{G} \) be an arbitrary \( \ell \)-local universal gate set on a Hilbert space \( \mathcal{H} = (\mathbb{C}^q)^\otimes n \) of dimension \( d = q^n \). Then for an arbitrary unitary \( U \in \mathcal{U}(d) \) and a state \( \psi \in \mathcal{S}(d) \) we have

\[
C_\varepsilon(U) \leq A d^2 \log \left( \frac{d^2}{\varepsilon^\gamma} \right), \quad C_\varepsilon(\psi) \leq A (2d - 2) \log \left( \frac{d}{\varepsilon^\gamma} \right),
\]

(53)

where \( A \) is a constant depending on \( \mathcal{G} \) and \( \gamma < 3 \) is the constant appearing in the Solovay-Kitaev theorem [2].

Proof. The proof of this result can be found in [57] but we reproduce it here for completeness. The idea is to use an exact procedure for compilation of unitary gates given in Ref. [58] and then apply Solovay-Kitaev theorem. We explicitly consider the case of unitaries as the argument for states is virtually identical. Ref. [58] gives an algorithm that allows one to write every unitary \( U \in \mathcal{U}(d) \) as a product \( U = \prod_{i=1}^{N_{\text{gates}}} U_i \), where the number of gates \( N_{\text{gates}} = O(d^2) \) and the gates \( U_i \) act non-trivially on at most two qubits. Every such two qudit gate can approximated to an accuracy of \( \tilde{\varepsilon} \) by a sequence of gates from \( \mathcal{G} \) using the Solovay-Kitaev theorem (which also holds for gate sets that are not symmetric, i.e. do not necessitate the inclusion of inverses). Specifically, it suffices to use \( L_{\text{SK}}(\tilde{\varepsilon}) = O(\log(1/\tilde{\varepsilon})^γ) \) gates from \( \mathcal{G} \) to exactly realize a gate \( \tilde{U}_i \) that approximates \( U_i \) to accuracy \( \tilde{\varepsilon} \) in distance \( D \). Using the standard telescopic argument, we obtain that the gate \( \tilde{U} = \prod_{i=1}^{N_{\text{gates}}} \tilde{U}_i \) approximates the target unitary \( U \) to accuracy \( \delta = N_{\text{gates}} \tilde{\varepsilon} \). By setting \( \tilde{\varepsilon} = \varepsilon / d^2 \) we see that it is possible to approximate the target gate to an accuracy \( \varepsilon \) using a total of \( N_{\text{gates}} L_{\text{SK}}(d^2/\varepsilon) \) gates. Inserting bounds for these quantities establishes first inequality in Eq. (53). The argument for pure quantum states is analogous. The only difference is that in this case \( N_{\text{gates}} = O(d) \) two qudit gates suffice to obtain any pure quantum state from \( |0\rangle^\otimes n \) [58].

Remark. We note that in light of the results in Ref. [43] (see also [26]) the gate set \( \mathcal{G} \) does not have to be symmetric, i.e. it is not required to contain inverses. Moreover, for the so-called magic gate sets \( \mathcal{G} \) (for which the corresponding transition operator has a gap) one can chose \( \gamma = 1 \) [59].

Lemma 9. Let \( \mathcal{G} \) be an arbitrary \( \ell \)-local universal gate set acting on a Hilbert space \( \mathcal{H} = (\mathbb{C}^q)^\otimes n \) of dimension \( d = q^n \). Let \( U \) be a random unitary channel chosen according to a measure \( \nu \), which \((\alpha, \beta)\)-equidistributes on scale \( \varepsilon \). Then

\[
\Pr_{U \sim \nu} \left( C_\varepsilon(U) \geq \frac{(d^2 - 1) \log \frac{1}{\varepsilon^\alpha} - \log \frac{1}{\Delta}}{\log |\mathcal{G}|} \right) \geq 1 - \Delta.
\]

(54)

Moreover, let \( \psi \equiv \psi \) a random pure state chosen according to \( \nu_S \) which \((\alpha, \beta)\)-equidistributes on scale \( \varepsilon \). We then have

\[
\Pr_{\psi \sim \nu_S} \left( C_\varepsilon(\psi) \geq \frac{(2d - 2) \log \frac{1}{\varepsilon^\alpha} - \log \frac{1}{\Delta}}{\log |\mathcal{G}|} \right) \geq 1 - \Delta.
\]

(55)

Note that up to a factor polynomial in \( n \) and the factor of \( 1 / \log |\mathcal{G}| \), the lower bounds matches upper bound from Lemma 8. The latter does not play a significant role as the gate sets used in quantum computing have \( |\mathcal{G}| = \text{poly}(n) \). Thus, Lemma 9 shows that most unitaries and states coming from measures that exhibit approximate equidistribution are, in essence, maximally complex. Finally, let us remark that setting \( \beta = 1 \) in the above lemma allows us to derive statements for the Haar measure \( \mu \) on \( \mathcal{U}(d) \) as well as the induced uniform measure \( \mu_S \) on the set of pure states \( \mathcal{S}(d) \).

Proof of Lemma 9. We first tackle the case of unitary complexity. We shall bound the volume of all unitaries that have complexity at most \( r \): \( C_\varepsilon^r = \{ U : \exists V \in \mathcal{G}^r, \, l \leq r \, \text{ s.t. D}(U, V) \leq \varepsilon \}. \) By definition, the set of
unitaries with $\varepsilon$-complexity smaller than $r$ is a union of $\varepsilon$-balls centered on circuits from $\mathcal{G}^l$ for $l \leq r$,

$$C^r_\varepsilon = \bigcup_{l \leq r} \bigcup_{\mathbf{v} \in \mathcal{G}^l} B(\mathbf{v}, \varepsilon).$$  \hfill (56)

Therefore, we have

$$\nu(C^r_\varepsilon) \leq \sum_{l \leq r} \nu \left( \bigcup_{\mathbf{v} \in \mathcal{G}^l} B(\mathbf{v}, \varepsilon) \right) \leq \sum_{l \leq r} |\mathcal{G}^l| \nu(B(\mathbf{I}, \varepsilon)) \leq |\mathcal{G}|^{r+1} (\beta c \varepsilon)^{d^2-1},$$  \hfill (57)

where we use the fact that for $a \geq 2$ we have $1 + a + \ldots + a^r \leq a^{r+1}$, and the last inequality comes from the equidistribution property and Eq. (17). Thus, if

$$|\mathcal{G}|^{r+1} (\beta c \varepsilon)^{d^2-1} \leq \Delta,$$  \hfill (58)

then with probability at least $1 - \Delta$, a unitary $\mathbf{U} \in \mathcal{U}(d)$ has complexity larger than $r$. We obtain the lemma by solving the above inequality for $r$. The case of state complexity is proven in a completely analogous manner along with the use of Eq. (18). \hfill \Box

**B. Number of high complexity unitaries and states**

Let us draw our attention to another feature that justifies the claim that probability distributions which approximately equidistribute are, in a sense, maximally complex. The following lemma gives a lower bound on the packing number of high complexity unitaries and states in such ensembles. Interestingly, unless $r$ is very large this number is comparable to the packing number of unitary channels $\mathcal{U}(d)$ and quantum states $\mathcal{S}(d)$, respectively.

**Lemma 10** (The number of distinct high-complexity unitaries and states is large in equidistributed ensembles). Let $\nu$ be a measure on $\mathcal{U}(d)$ that $(\alpha, \beta)$-equidistributes on a scale $\varepsilon$. Let $C^r_\varepsilon(\nu) = \{ \mathbf{U} \in \text{supp}(\nu) : C^r_\varepsilon(\mathbf{U}) > r \}$ be the set of unitaries from the support of measure $\nu$ with complexity larger than $r$. Then, provided

$$r < (d^2 - 1) \frac{\log(A/\varepsilon)}{\log(|\mathcal{G}|)} - 2, \text{ where } A = \frac{c_o}{4 \beta (c\varepsilon)^2},$$  \hfill (59)

the packing number of $C^r_\varepsilon(\nu)$, i.e. the number of high-complexity unitaries from the support of $\nu$, is lower bounded by

$$N_{\text{pack}}(C^r_\varepsilon(\nu), \varepsilon) \geq \frac{1}{2} N_{\text{pack}}(\text{supp}(\nu), \varepsilon),$$  \hfill (60)

where $N_{\text{pack}}(\text{supp}(\nu), \varepsilon) \geq \text{Vol}(2\varepsilon)^{-1} \geq (\frac{1}{2\varepsilon})^{d^2-1}$.

Moreover, let $\nu_S$ be a measure on $\mathcal{S}(d)$ that $(\alpha, \beta)$-equidistributes on scale $\varepsilon$. Furthermore, let $S^r_\varepsilon(\nu_S) = \{ \psi \in \text{supp}(\nu_S) : C^r_\varepsilon(\psi) > r \}$ be the set of states from $\text{supp}(\nu_S)$ with complexity larger than $r$. Then, provided

$$r < (2d^2 - 2) \frac{\log(1/(4\beta\varepsilon))}{\log(|\mathcal{G}|)} - 2,$$  \hfill (61)

the packing number of $S^r_\varepsilon(\nu_S)$, i.e. the number of high-complexity states from the support of $\nu_S$, is lower bounded by

$$N_{\text{pack}}(S^r_\varepsilon(\nu_S), \varepsilon) \geq \frac{1}{2} N_{\text{pack}}(\text{supp}(\nu_S), \varepsilon),$$  \hfill (62)

where $N_{\text{pack}}(\text{supp}(\nu_S), \varepsilon) \geq \text{Vol}_S(2\beta\varepsilon)^{-1} \geq (\frac{1}{2\varepsilon\beta})^{2d^2-2}$. 
Proof. We describe in detail the case of unitary channels. The arguments for quantum states are analogous. We have \( \text{supp}(\nu) = C^{>\nu}_\varepsilon(\nu) \cup C^{<\nu}_\varepsilon(\nu) \), where \( C^{>\nu}_\varepsilon(\nu) = \varepsilon C^{>\nu}_\varepsilon(\nu) \cap \text{supp}(\nu) \) are unitaries from the support of \( \nu \) with complexity at most \( r \). Using the inequality in Eq. (23) we have

\[
N_{\text{pack}}(\text{supp}(\nu), \varepsilon) \leq N_{\text{pack}}(C^{>\nu}_\varepsilon(\nu), \varepsilon) + N_{\text{pack}}(C^{<\nu}_\varepsilon(\nu), \varepsilon).
\]  

(63)

Then using the characterization of \( C^{>\nu}_\varepsilon \) given in Eq. (57) and again employing the bound Eq. (23), we find

\[
N_{\text{pack}}(C^{>\nu}_\varepsilon(\nu), \varepsilon) \leq \sum_{l \leq r} \sum_{U \in G^l} N_{\text{pack}}(B(U, \varepsilon) \cap \text{supp}(\nu), \varepsilon) \leq |G|^{r+1} N_{\text{pack}}(B(U, \varepsilon), \varepsilon),
\]

(64)

where in the last inequality we used the monotonicity of the packing number subject to inclusion of sets. Furthermore, since any disjoint balls of radius \( \varepsilon \) that have centers in \( B(U, \varepsilon) \) are contained in \( B(U, 2\varepsilon) \), a simple volume comparison gives that \( N_{\text{pack}}(B(U, \varepsilon), \varepsilon) \text{Vol}(\varepsilon) \leq \text{Vol}(2\varepsilon) \), which in turn yields

\[
N_{\text{pack}}(B(U, \varepsilon), \varepsilon) \leq \frac{\text{Vol}(2\varepsilon)}{\text{Vol}(\varepsilon)} \leq \left( \frac{2r^d}{c_o} \right)^{d-1},
\]

(65)

where in the second inequality we used Eq. (17). Altogether, we obtain

\[
N_{\text{pack}}(C^{>\nu}_\varepsilon(\nu), \varepsilon) \geq N_{\text{pack}}(\text{supp}(\nu), \varepsilon) - |G|^{r+1} \left( \frac{2r^d}{c_o} \right)^{d-1}.
\]

(66)

Now, choosing \( r \) in such a way that \( |G|^{r+1} \left( \frac{2r^d}{c_o} \right)^{d-1} \leq \frac{1}{2} N_{\text{pack}}(\text{supp}(\nu), \varepsilon) \) subsequently ensures that \( N_{\text{pack}}(C^{>\nu}_\varepsilon(\nu), \varepsilon) \geq \frac{1}{2} N_{\text{pack}}(\text{supp}, \varepsilon) \). To estimate \( r \) we use the following bound lower on \( N_{\text{pack}}(\text{supp}(\nu), \varepsilon) \):

\[
N_{\text{pack}}(\text{supp}(\nu), \varepsilon) \geq \frac{1}{\text{Vol}(\beta \cdot 2\varepsilon)}.
\]

(67)

The above inequality follows from the relation \( N_{\text{pack}}(\text{supp}(\nu), \varepsilon) \geq N_{\text{cov}}(\text{supp}(\nu), 2\varepsilon) \), which in turn follows from the fact that the number of balls needed to fully cover the space cannot increase if we increase their radii, as well as the bound

\[ 1 \leq N_{\text{cov}}(\text{supp}(\nu), 2\varepsilon) \text{Vol}(\beta \cdot 2\varepsilon), \]

(68)

which follows from the definition of the covering number and the equidistribution property. Finally, from Eq. (67) and Eq. (27), we find that

\[ |G|^{r+1} \left( \frac{2r^d}{c_o} \right)^{d-1} \leq \frac{1}{2} \left( \frac{1}{\text{Vol}(\beta \cdot 2\varepsilon)} \right)^{d-1}. \]

(69)

It can be easily seen that Eq. (69) is implied by Eq. (59). This concludes the proof of the first part of the lemma.

The proof for pure states proceeds identically. Just like for unitaries we separate the support \( \text{supp}(\nu_S) \) into states of high complexity, \( S^{>\nu}_\varepsilon(\nu_S) \), and low complexity, the complement of \( S^{>\nu}_\varepsilon(\nu_S) \) denoted by \( (S^{>\nu}_\varepsilon(\nu_S))^c \). For these states we apply Eq. (23) and proceed as before by (i) upper bounding of \( N_{\text{pack}}((S^{>\nu}_\varepsilon(\nu_S))^c, \varepsilon) \) and (ii) lower bounding \( N_{\text{pack}}(\text{supp}(\nu_S), \varepsilon) \). Upon implementing (i) we use the estimate

\[
N_{\text{pack}}(B(\psi, \varepsilon), \varepsilon) \leq \frac{\text{Vol}_S(2\varepsilon)}{\text{Vol}_S(\varepsilon)} = 2^{2d-2},
\]

(70)

which is proven analogously to Eq. (65).
VI. RECURRENCE OF COMPLEXITY FROM PROBABILITY DISTRIBUTIONS EXHIBITING APPROXIMATE EQUIDISTRIBUTION

Here we show that the notion of equidistribution allows us to derive strong quantitative statements concerning the recurrence of circuit complexity in random quantum circuits. Before moving on to formal results and proofs, we first present informal reasoning that justifies, on an intuitive level, why approximate equidistribution can be relevant for studying the recurrence of complexity.

The main idea is that the space of quantum states or unitaries gets divided into objects of low and high complexity (for unitaries we denoted them as $C^r_\varepsilon$ and $C^{>r}_\varepsilon$, respectively). These sets have well-defined volumes, computed according to the Haar measure on $U(d)$ and the uniform measure on $S(d)$. Consider now the ensemble of unitaries $U_t$ induced by a measure $\nu_t = \nu^t$ resulting from $t$ steps of a random walk $\nu$ (specified by the random quantum circuit model) and initialized at identity $I$. Assume that at time $\tau$ the measure $\nu_\tau$ approximately equidistributes on the scale $\varepsilon$. Then the probability of a unitary $U_\tau$ having complexity smaller than $r$ roughly scales like the Haar measure of low-complexity unitaries, i.e. $\text{Vol}(C^r_\varepsilon)\cdot\frac{1}{\varepsilon}$. Importantly, the approximate equidistribution property does not depend on the initial state of a random walk and therefore the probability that $U_{2\tau}$ has complexity smaller than $r$ again exhibits the same behaviour, regardless of the state of the walker at time $\tau$. The same holds for the subsequent multiples of $\tau$. It follows that, when observed every $\tau$ times, the walker effectively behaves as if at every interrogation time $t = j \cdot \tau$ the unitaries $U_t$ are independent random variables distributed according to the Haar measure on $U(d)$, as illustrated in Figure 6. Therefore, the expected time of return to a low complexity region goes as $\text{Vol}(C^r_\varepsilon)^{-1}$, which scales doubly-exponentially in the number of qudits $n$. Importantly, our findings do not depend on the details of the measure $\nu$ on $U(d)$, which is used to specify the walk.

We start with the following crucial lemma that gives insight into the typical long-time behaviour of the complexity of the random sequences of unitaries $(U_t)_{t \in \mathbb{N}}$ (resp. states $(\psi_t)_{t \in \mathbb{N}}$) generated by a random circuit architecture $\nu$. The lemma will allow us to prove the main result of this section contained in Theorem 12.
Lemma 11 (Bounds on recurrence events for random quantum circuits). Consider a system of $n$ qudits described by a Hilbert space $\mathcal{H} = (\mathbb{C}^q)^{\otimes n}$ of dimension $d = q^n$. Let $\nu$ be a measure (random quantum circuit architecture) on $\mathcal{U}(d)$ that defines walks $U, \psi$ in their respective spaces, as in Eq. (43). Let $\tau$ and $\tau_S$ be the corresponding equidistribution times of $\nu_U$ and $\nu_S$, on the scale of $\varepsilon$ (with parameters $\alpha, \beta$, as in Definition 3). We then have the following inequalities

$$\Pr \left( \text{There exists } k = 1, \ldots, K \text{ s.t. } C_{\varepsilon}(U_{\tau+k}) \leq r \right) \leq K |\mathcal{G}|^{r+1} \text{Vol}(\beta \varepsilon), \quad (71)$$

Moreover, for $K \geq \tau$ (resp. $K \geq \tau_S$ for states) we also have bounds in the opposite direction

$$\Pr \left( \text{There exists } k = 1, \ldots, K \text{ s.t. } C_{\varepsilon}(\psi_{\tau+k}) \leq r \right) \geq 1 - \left(1 - \mathcal{N}_{\text{pack}}(\mathcal{G}', \varepsilon) \text{Vol}(\alpha \varepsilon) \right)^{\frac{K}{\tau}}, \quad (72)$$

Proof. We start with the proof for unitary channels. Consider first the inequality Eq. (71). We apply the union bound: \begin{equation}
\Pr \left( \text{There exists } k = 1, \ldots, K \text{ s.t. } C_{\varepsilon}(U_{\tau+k}) \leq r \right) \leq \sum_{k=1}^{K} \Pr(C_{\varepsilon}(U_{\tau+k}) \leq r) = \sum_{k=1}^{K} \nu^{(r+k)}(C_{\varepsilon}^r). \quad (73)
\end{equation}

Now, by definition we have $C_{\varepsilon}^r = \bigcup_{l \leq r} \bigcup_{\mathcal{G}' \in \mathcal{G}} B(V, \varepsilon)$ and as a result

$$\nu^{(r+k)}(C_{\varepsilon}^r) \leq \sum_{l \leq r} \sum_{\mathcal{G}' \in \mathcal{G}} \nu^{(r+k)}(B(V, \varepsilon)) \leq \sum_{l \leq r} |\mathcal{G}'|^{l} \text{Vol}(\beta \varepsilon). \quad (74)$$

In the first inequality we again used the union bound, while in the second we applied Fact 6, which implies that since $\nu^{(r+k)}$ equidistributes, so does $\nu^{(r+k)}$. Consider now the inequality in Eq. (73). To prove it, we will upper bound the probability of the opposite event

$$\Pr \left( \text{For all } k = 1, \ldots, K \text{ } C_{\varepsilon}(U^{(r+k)}) > r \right). \quad (75)$$

Denoting $\mathcal{A} = \{ U : C_{\varepsilon}(U) > r \}$ and using the Markov property we obtain

$$\Pr \left( \text{For all } k = 1, \ldots, K \text{ } C_{\varepsilon}(U_{\tau+k}) > r \right) = \Pr (\text{For all } k = 1, \ldots, K \text{ } U_{\tau+k} \in \mathcal{A})$$

$$\leq \Pr (U_{2\tau} \in \mathcal{A}, U_{3\tau} \in \mathcal{A}, \ldots, U_{\tau+K/\tau} \in \mathcal{A})$$

$$= \Pr (U_{2\tau} \in \mathcal{A}) \Pr (U_{3\tau} \in \mathcal{A}|U_{2\tau} \in \mathcal{A}) \ldots \Pr (U_{\tau+K/\tau} \in \mathcal{A}|U_{K/\tau} \in \mathcal{A}), \quad (76)$$

where for clarity, we have assumed that $K/\tau$ is an integer. We now have

$$\Pr (U_{(l+1)\tau} \in \mathcal{A}|U_{l\tau} \in \mathcal{A}) = \Pr (VU \in \mathcal{A}|U \in \mathcal{A}), \quad (77)$$

where $V$ is distributed according to $\nu^{(r)}$ and $U$ according to $\nu^{(r)}$. To avoid complicated notation, we shall now assume that $\nu^{(r)}$ is a discrete measure (the proof for the case of a continuous measure is analogous) and denote it by $\{p_{\alpha}, U_{\alpha}\}$. We then write

$$\Pr (VU \in \mathcal{A}|U \in \mathcal{A}) = \sum_{U_{\alpha} \in \mathcal{A}} p_{\alpha} \nu^{(r)}_{U_{\alpha}}(\mathcal{A}), \quad (78)$$

Proof (continued).
where \( \nu^*_\alpha = \nu^* \ast \delta_U \), with \( \delta_U \) being a Dirac delta measure concentrated on \( U \). Since, by definition, \( \mathcal{A} = \mathcal{U}(d) \setminus C_r^c \) and the set \( C_r^c \) contains an \( \varepsilon \)-ball centered around some \( U_0 \), we have
\[
\nu^*_\alpha(A) = 1 - \nu^*_\alpha(C_r^c) = 1 - \nu^*_\alpha(B(U_0, \varepsilon)) \leq 1 - \text{Vol}(\alpha \varepsilon),
\]
where we used the equidistribution property of \( \nu^* \). From Eqs. (79) and (80) we get that for integer \( l \geq 1 \)
\[
\Pr(U_{(l+1)r} \in \mathcal{A} | U_{lr} \in \mathcal{A}) \leq 1 - \text{Vol}(\alpha \varepsilon). \tag{82}
\]
Since \( \nu^{2\pi} \) equidistributes, we have the same estimate for \( \Pr(U_{2\pi r} \in \mathcal{A}) \), and we obtain
\[
\Pr(\text{For all } k = 1, \ldots, K \ C_\varepsilon(U_{\tau+k}) > r) \leq (1 - \text{Vol}(\alpha \varepsilon))^{K/\tau}, \tag{83}
\]
which gives the sought-after lower bound
\[
\Pr(\text{There exists } k = 1, \ldots, K \text{ s.t. } C_\varepsilon(U_{\tau+k}) \leq r) \geq 1 - (1 - \text{Vol}(\alpha \varepsilon))^{K/r}. \tag{84}
\]
If \( K \) is not a multiple of \( \tau \), the formula remains valid if we replace \( K/\tau \) with \( |K/\tau| \). But we can obtain a better bound on \( \nu^*_\alpha(C_r^c) \) by considering the set \( X \subset C_r^c \) of disjoint \( \varepsilon \) balls centered in \( \mathcal{G}^r \), i.e., words of length \( r \) formed from unitaries in \( \mathcal{G} \). The maximal number of such balls is the packing number \( N_{\text{pack}}(\mathcal{G}^r, \varepsilon) \). Then we have that
\[
\nu^*_\alpha(C_r^c) \geq \sum_{W \in X} \nu^*_\alpha(B(W, \varepsilon/2)) \geq N_{\text{pack}}(\mathcal{G}^r, \varepsilon) \text{Vol}(\alpha \varepsilon), \tag{85}
\]
which gives us that
\[
\Pr(\text{There exists } k = 1, \ldots, K \text{ s.t. } C_\varepsilon(U_{\tau+k}) \leq r) \geq 1 - (1 - N_{\text{pack}}(\mathcal{G}^r, \varepsilon) \text{Vol}(\alpha \varepsilon/2))^{K/r}. \tag{86}
\]
The proof for the case of pure states follows very similar reasoning, with the only notable difference being that it requires repeated usage of the equidistribution property of \( (\nu^* \ast 1)^{\alpha} \) and the measures related to it. \( \square \)

We are now ready to present the main result of this section, which is a technical formulation of items (i) and (ii) in Result 5. To establish timescales claimed there it suffices to substitute the timescales \( \tau, \tau_S \) in the bounds on the equidistribution time given in Lemma 7.

**Theorem 12** *(Recurrence behavior for random walks)*. Consider a system of \( n \) qudits described by a Hilbert space \( \mathcal{H} = (\mathbb{C}^q)^\otimes n \) of dimension \( d = q^n \). Let \( \nu \) be a measure (random quantum circuit architecture) on \( \mathcal{U}(d) \) that defines walks \( U_t, \psi_1 \) in the space of unitary channels (as in Eq. (43)). Let \( \tau, \tau_S \) be the corresponding \((\alpha, \beta)\)-equidistribution times of these walks on the scale of \( \varepsilon \). Let \( r_1 > r_2 \) and
\[
T_1 = \frac{\Delta_1}{|\mathcal{G}|^{r_1+1}} \left( \frac{1}{\beta \varepsilon} \right)^{d-1}, \quad T_2 = \tau \log(1/\Delta_2) \left( \frac{2\alpha(r_2)}{d^2(1 - \varepsilon^2)} \right)^{a(r_2)} \left( \frac{1}{\alpha \varepsilon} \right)^{2d-1}, \tag{87}
\]
\[
T_{1,S} = \frac{\Delta_1}{|\mathcal{G}|^{r_1+1}} \left( \frac{1}{\beta \varepsilon} \right)^{2d-2}, \quad T_{2,S} = \tau_S \log(1/\Delta_2) \left( \frac{\alpha(r_2)}{d(1 - \varepsilon^2)} \right)^{a(r_2)} \left( \frac{1}{\alpha \varepsilon} \right)^{2d-2}, \tag{88}
\]
where \( \alpha(r) = \lfloor (r/n^2 \log(\mathcal{G}))^{1/11} \rfloor \).

Then with probability greater than \( 1 - \Delta - \Delta_1 - \Delta_2 \) we have

(a) Saturation of complexity at time \( \tau (\tau_S) \)
\[
C_\varepsilon(U_\tau) \geq \frac{(d^2 - 1) \log \frac{1}{\beta \varepsilon} - \log \frac{1}{\Delta_1}}{\log |\mathcal{G}|}, \quad C_\varepsilon(\psi_1) \geq \frac{(2d - 2) \log \frac{1}{\beta \varepsilon} - \log \frac{1}{\Delta_1}}{\log |\mathcal{G}|}, \tag{89}
\]
(b) Large complexity for all times \( t \) satisfying \( \tau + 1 \leq t \leq T_1 \) \((\tau_S + 1 \leq t \leq T_{S,1})\)
\[
C_\varepsilon(U_t) > r_1, \quad C_\varepsilon(\psi_t) > r_1.
\]  

(90)

c) Small complexity for some time \( t \) satisfying \( T_1 < t \leq T_2 \) \((T_{1,S} < t \leq T_{2,S})\)
\[
C_\varepsilon(U_t) \leq r_2, \quad C_\varepsilon(\psi_t) \leq r_2.
\]  

(91)

Remark. For \( r_2 = 0 \) (recurrence to trivial complexity), the expressions for the above time-scales simplify significantly:
\[
T_1 = \frac{\Delta_1}{|G|^{r_1+1}} \left( \frac{1}{\beta \epsilon \epsilon} \right)^{d^2-1}, \quad T_2 = \tau \log (1/\Delta_2) \left( \frac{1}{\alpha \epsilon \epsilon} \right)^{d^2-1},
\]
\[
T_{1,S} = \frac{\Delta_1}{|G|^{r_1+1}} \left( \frac{1}{\beta \epsilon} \right)^{2d-2}, \quad T_{2,S} = \tau_S \log (1/\Delta_2) \left( \frac{1}{\epsilon \epsilon} \right)^{2d-2}.
\]

(92)

(93)

It is clear that both lower bounds \((T_1, T_{1,S})\) as well as upper bounds \((T_2, T_{2,S})\) for the recurrence time exhibit the same exponential dependence on the inverse of the accuracy parameter \( \epsilon \) (ignoring the dependence on the equidistribution times \( \tau, \tau_S \)).

Proof. We present the proof just for unitaries as the reasoning for states is proceeds analogously. Let us define the following three events in the space of realizations of the random walk \((U_t)_{t \in \mathbb{N}}\):
\[
I = \left\{ C_\varepsilon(U_\tau) \geq \frac{(d^2 - 1) \log \frac{1}{\epsilon \epsilon} - \log \frac{1}{\Delta}}{\log |G|} \right\},
\]
\[
II = \left\{ C_\varepsilon(U_t) \geq r_1 \text{ for all } t \text{ s.t. } \tau + 1 \leq t \leq T_1 \right\},
\]
\[
III = \left\{ C_\varepsilon(U_t) < r_2 \text{ for some } t \text{ s.t. } T_1 \leq t \leq T_2 \right\}.
\]

(94)

As for event \( I \), by the definition of the equidistribution time \( \tau \), Lemma 9 ensures that \( \Pr(I) \geq 1 - \Delta \).

To control the probability of event \( II \), we use Lemma 11. In conjunction with lower and upper bounds for the volume of balls given in Eq. (17), Eq. (71) implies that, for \( T_1 \) as given by first inequality of Eq. (87), we have \( \Pr(II) \geq 1 - \Delta_1 \).

Next, to deal with event \( III \) we use Lemma 11 and the lower bounds on the packing number \( \mathcal{N}_{\text{pack}}(G^r, \epsilon) \) from Lemma 13 (the same lemma contains the lower bound of the packing number \( \mathcal{N}_{\text{pack}}(S^r, \epsilon) \), relevant for states). Namely, the inequalities Eq. (73) and Eq. (95) along with Eq. (17) imply that, for \( T_2 \) as given by the second inequality in Eq. (87), \( \Pr(III) \geq 1 - \Delta_2 \). Finally, twice iterating \( \Pr(A \cap B) \geq \Pr(A) + \Pr(B) - 1 \), we find \( \Pr(I \cap II \cap III) \geq 1 - \Delta - \Delta_1 - \Delta_2 \).



Here we present a lower bound on the packing number that we have used in the proof of Theorem 12

Lemma 13 (Lower bounds for packing number of low complexity states and unitaries). The packing number of the set \( G^r \) (circuits of depth \( r \) generated by gates in \( G \)) satisfies
\[
\mathcal{N}_{\text{pack}}(G^r, \epsilon) \geq \left( \frac{d^2(1 - 4\epsilon^2)}{\alpha(r)} \right)^\alpha(r),
\]

(95)

where \( \alpha(r) := [(r/n^2c(G))]^{1/11} \) and \( c(G) \) is a constant depending on the gate set \( G \).

Similarly, the packing number of the set \( S^r \) (states generated by depth \( r \) circuits constructed from \( G \) and applied to computationally trivial state \( \psi_0 \)) satisfies
\[
\mathcal{N}_{\text{pack}}(S^r, \epsilon) \geq \left( \frac{d(1 - 4\epsilon^2)}{\alpha(r)} \right)^\alpha(r).
\]

(96)
Remark. For small depth $r$, Eqs. (95) and (96) appear singular. However, in this regime one can easily lower bound $N_{\text{pack}}(G^r, \varepsilon)$ and $N_{\text{pack}}(S^r, \varepsilon)$ by 1, by considering a single $\varepsilon$-ball centered around $I$ (resp. $\psi_0$).

Proof of Lemma 13. For the gate set $G$, the set $G^r$ contains the support of the measure given by random quantum circuits generated from gate set $G$, denoted as $\text{supp}(\nu_{\text{RQC}})$. To proceed with the proof, we first analyze unitary expanders, and then use the fact that random quantum circuits are approximate $k$-expanders. From Lemma 33 we get that for $\delta$-approximate $k$-expanders and arbitrary $V \in U(d)$ we have $\nu(B(V, \varepsilon)) \leq \frac{k! + 1}{d^{2k}(1 - \varepsilon^2)^k}$. For local random quantum circuits, we now employ Theorem 2 to conclude that circuits of size $T \geq c(G)n^2k^{11}$ form $\delta$-approximate $k$-expanders with $\delta = 1/d^{2k}$. For random quantum circuits $\nu_{\text{RQC}}$ of size $r$, it follows that

$$
\nu_{\text{RQC}}(B(V, \varepsilon)) \leq \frac{k! + 1}{d^{2k}(1 - \varepsilon^2)^k} \leq \left( \frac{(r/n^2c(G))^{1/11}}{d^2(1 - \varepsilon^2)} \right)^{(r/n^2c(G))^{1/11}} =: f(\varepsilon, r). \tag{97}
$$

We now construct a subset of elements $\text{supp}(\nu_{\text{RQC}}) = G^r$ that are pairwise far apart by more than a distance $\varepsilon$. First, take an arbitrary element $V_1$ of $\text{supp}(\nu_{\text{RQC}})$. Since $\nu_{\text{RQC}}(B(V_1, 2\varepsilon)) \leq f(2\varepsilon, r) < 1$ is strictly less than one, a condition which can easily be satisfied provided $r$ is not too large, there must be some $V_2$ from $\text{supp}(\nu_{\text{RQC}})$ which does not belong to the ball, i.e. $D(V_1, V_2) > 2\varepsilon$. By a union bound we then have

$$
\nu_{\text{RQC}}(B(V_1, 2\varepsilon) \cup B(V_2, 2\varepsilon)) \leq 2f(2\varepsilon, r) < 1 \tag{98}
$$

There must then exist a $V_3$ that does not belong to both balls, $D(V_3, V_1) > 2\varepsilon$ and $D(V_3, V_2) > 2\varepsilon$. Continuing this reasoning, we obtain $N$ unitaries $\{V_i\}_{i=1, \ldots, N}$ such that the distance $D(V_i, V_j) > 2\varepsilon$ for all pairs $(i, j)$, provided that $Nf(2\varepsilon, r) < 1$. Thus the support of $\nu_{\text{RQC}}$ contains at least $1/f(2\varepsilon, r) - 1$ unitaries that are pairwise far apart by $2\varepsilon$. If we consider balls of radius $\varepsilon$ with centers placed at those unitaries, the balls will be pairwise disjoint and so the packing number of $G^r$ is bounded as follows

$$
N_{\text{pack}}(G^r, \varepsilon) \geq \frac{1}{f(2\varepsilon, r)}, \tag{99}
$$

which gives the sought after estimate.

The proof of the lemma for the set of states is proceeds identically to the unitary case. We construct, using a probabilistic argument, a set of pairwise disjoint $\varepsilon$ balls centered in states generated by depth $r$ quantum circuits acting on $\psi_0$. The only difference is a different bound on the measure of an $\varepsilon$-ball according to probability distribution $\nu_S$ induced on $S(d)$ from $\delta$-approximate unitary $k$-designs on $U(d)$: $\nu_S(B(\phi, \varepsilon)) \leq \frac{k! + d^k\delta}{d^{2k}(1 - \varepsilon^2)^k}$, which we prove in Lemma 34.

\[ \square \]

\section{VII. Recurrences of Complexity Have Exponential Size}

In this section we quantify timescales on which recurrences of complexity take place for local random quantum circuits. We approach this problem by showing that under certain conditions

$$
\Pr\left(C_{\varepsilon}(U_{i-T}) > r, C_{\varepsilon}(U_{i+T}) > r \mid C_{\varepsilon}(U_i) = 0\right) \approx 1. \tag{100}
$$

The quantity appearing above is the conditional probability in the space of realizations of a random walk $\{U_i\}_{i \in \mathbb{N}}$ (see Definition 5). We want to show that this probability is close to 1 for a large enough time $t$, and for $r$ comparable to the maximal complexity $C_{\text{max}}$ and times $T$ comparable to the equidistribution time $\tau$. Establishing this together with the “temporal stability” of complexity, which is proved in Lemma 14 below, shows that recurrences of complexity in random quantum circuits indeed take place on exponential timescales, as conjectured in recent work by Susskind [31].

Lemma 14 (Stability of complexity under action of local gates). Consider a Hilbert space of $n$ qudits with local dimension $q$, $\mathcal{H} = (\mathbb{C}^q)^{\otimes n}$ of dimension $d = q^n$. Let $G$ be a universal gate set used to define complexity,
as in Definition 1 and Definition 2. Let \( U \in \mathcal{U}(d) \) and \( \psi \in \mathcal{S}(d) \). Furthermore, let \( W_1, W_2, \ldots, W_m \) be a sequence of \( l \)-local gates acting in \( \mathcal{H} \) and let \( \varepsilon > 0 \). Then we have

\[
C_{2\varepsilon}(W_m W_{m-1} \ldots W_1 U) \leq C_{\varepsilon}(U) + Am \log((m/\varepsilon)^{\gamma}) \, , \quad C_{2\varepsilon}(W_m W_{m-1} \ldots W_1(\psi)) \leq C_{\varepsilon}(\psi) + Am \log((m/\varepsilon)^{\gamma}) ,
\]

where \( A > 0 \) depends on \( \mathcal{G} \) and \( \gamma < 3 \) is an absolute constant (\( \gamma = 1 \) provided \( \mathcal{G} \) is symmetric).

**Proof.** We present the proof for unitaries as the reasoning for states is analogous. By definition of the complexity \( C_{\varepsilon} \), a unitary channel \( U \) can be approximated to accuracy \( \varepsilon \) by sequence gates from \( \mathcal{G} \) of length \( r = C_{\varepsilon}(U) \). Furthermore, each of the gates \( W_i \) (\( i = 1, \ldots, m \)) can be approximated to accuracy \( \varepsilon' = \varepsilon/m \) via sequences of gates from \( \mathcal{G} \) of length \( r_i = A \log(1/\varepsilon')^{\gamma} \) (as in the proof of Lemma 8). Putting together this gives an \( \varepsilon + m \cdot \varepsilon' = 2\varepsilon \) approximation of \( W_m W_{m-1} \ldots W_1 U \) via a sequence of length \( r + r_1 + \ldots + r_m = C_{\varepsilon}(U) + Am \log((m/\varepsilon)^{\gamma}) \).

**Remark.** In the case when unitaries \( W_1, W_2, \ldots, W_m \) in the above lemma can be exactly obtained from gates in \( \mathcal{G} \), then the bounds in Eq. (101) can be strengthened to

\[
C_{\varepsilon}(W_m W_{m-1} \ldots W_1 U) \leq C_{\varepsilon}(U) + Am \, , \quad C_{\varepsilon}(W_m W_{m-1} \ldots W_1(\psi)) \leq C_{\varepsilon}(\psi) + Am .
\]

Note that in the above inequalities complexities are computed on the same scale \( \varepsilon \) and no \( \varepsilon \)-dependence is present in the upper bounds.

The following proposition gives a sufficient condition on the time \( T \) such that the complexity of the ‘past’ unitary \( U_{t-T} \) is essentially maximal, conditioned on \( U_t \) having the trivial (i.e. zero) complexity.

**Proposition 15.** Let \( \mathcal{G} \) be a fixed universal gate set used in the definition of unitary and state complexity on a qudit system \( \mathcal{H} = (\mathbb{C}^q)^{\otimes n} \) of dimension \( d = q^n \). Let \( \nu \) be a probability distribution on \( \mathcal{U}(d) \) generating random walks \( U_t \) and \( \psi_t \) on the space of unitary channels and quantum states. Let \( \varepsilon \in (0, 1/4) \) and \( \Delta \in (0, 1) \). Assume that \( T \) is a time such that the measure \( \nu^{\ast T} \) is \((\alpha, \beta)\)-equidistributed on a scale \( \varepsilon \) for \( \Gamma = \max\{1 - \alpha, \beta - 1\} \leq \frac{2}{9}(c_{\alpha}/3)^{d^2-1} \). Then for \( t > 2T \) and

\[
r = \frac{(d^2 - 1) \log \frac{1}{\varepsilon^{ps}} - \log \frac{2}{\Delta}}{\log |\mathcal{G}|} ,
\]

we have

\[
\Pr\left( C_{\varepsilon}(U_{t-T}) > r \mid C_{\varepsilon}(U_t) = 0 \right) \geq 1 - \delta .
\]

Similarly for pure quantum states, we let \( T_S \) be a time such that the measure \( (\nu^{\ast T_S})_S \) is \((\alpha, \beta)\)-equidistributed on a scale \( \varepsilon \) for \( \Gamma_S = \max\{1 - \alpha, \beta - 1\} \leq \frac{2}{225} \), where \( \Delta \in (0, 1) \). Then for \( t > 2T_S \) and

\[
r_S = \frac{(2d - 2) \log \frac{1}{\varepsilon^{ps}} - \log \frac{2}{\Delta}}{\log |\mathcal{G}|} ,
\]

we have

\[
\Pr\left( C_{\varepsilon}(\psi_{t-T_S}) > r_S \mid C_{\varepsilon}(\psi_t) = 0 \right) \geq 1 - \Delta .
\]

**Proof.** In order to simplify the notation let us denote \( \mathcal{A} = \{ U \in \mathcal{U}(d) \mid C_{\varepsilon}(U) > r \} \), \( \mathcal{B} = \{ U \in \mathcal{U}(d) \mid C_{\varepsilon}(U) = 0 \} \). We then write

\[
\Pr\left( C_{\varepsilon}(U_{t-T}) > r \mid C_{\varepsilon}(U_t) = 0 \right) = \Pr(U_{t-T} \in \mathcal{A} \mid U_t \in \mathcal{B}) = \frac{\Pr(U_{t-T} \in \mathcal{A} \; U_t \in \mathcal{B})}{\Pr(U_t \in \mathcal{B})} ,
\]

where in the second equality we used Bayes theorem. Let us assume for simplicity that \( \nu \) is a discrete measure (the proof in the general case is analogous). It then follows that the distribution of \( U_{t-T} \) is also
discrete. For the measure $\nu_{t-T} = \{p_i, V_i\}$, we use the definition of $U_t$ to obtain

$$\Pr(U_{t-T} \in A, \ U_t \in B) = \sum_{i: V_i \in A} p_i \Pr(U_t \in B | U_{t-T} = V_i).$$

(108)

The distribution of $U_t$ conditioned on $U_{t-T} = V_i$ is simply $\nu_T * \delta_{V_i}$ and therefore

$$\Pr(U_{t-T} \in A, \ U_t \in B) = \sum_{i: V_i \in A} p_i \nu_T * \delta_{V_i}(B).$$

(109)

Using $\mathcal{B} = B(I, \varepsilon)$ and the fact that for every $i$ the measure $\nu_T * \delta_{V_i}$ approximately equidistributes on scale $\varepsilon$ (this follows from the assumption about $\nu_T$ and Fact 6), we obtain

$$\Pr(U_{t-T} \in A, \ U_t \in B) \geq \Pr(U_{t-T} \in \mathcal{A}) \text{Vol}(\alpha \varepsilon).$$

(110)

Since $\nu_t$ is also $(\alpha, \beta)$-equidistributed we can upper bound the denominator of Eq. (107) by $\text{Vol}(\beta \varepsilon)$, which gives

$$\Pr(C_{\varepsilon}(U_{t-T}) > r \mid C_{\varepsilon}(U_t) = 0) \geq \Pr(C_{\varepsilon}(U_{t-T}) > r) \frac{\text{Vol}(\alpha \varepsilon)}{\text{Vol}(\beta \varepsilon)}. $$

(111)

Furthermore, we have

$$\frac{\text{Vol}(\beta \varepsilon)}{\text{Vol}(\alpha \varepsilon)} = 1 + \frac{\text{Vol}(A(\alpha \varepsilon, \beta \varepsilon))}{\text{Vol}(\alpha \varepsilon)},$$

(112)

where $A(\alpha \varepsilon, \beta \varepsilon) = \{U \in U(d) | \alpha \varepsilon \leq D(U, I) \leq \beta \varepsilon\}$ is an annulus in the space of unitaries. Using Proposition 36 for $\kappa = \alpha \varepsilon$ and $\lambda = \beta / \alpha$ we find

$$\text{Vol}(A(\alpha \varepsilon, \beta \varepsilon)) \leq (2 \beta \varepsilon)^{d^2-1}(\lambda - 1).$$

(113)

Combining this bound with $\text{Vol}(\alpha \varepsilon) \geq (c_o \beta \varepsilon)^{d^2-1}$ we obtain

$$\frac{\text{Vol}(\beta \varepsilon)}{\text{Vol}(\alpha \varepsilon)} \leq 1 + \left(\frac{2 \beta}{\alpha c_o}\right)^{d^2-1}\left(\lambda - 1\right).$$

(114)

It is straightforward to see that $\Gamma \leq \frac{\Delta}{6} (c_o/3)^{d^2-1}$ implies

$$\lambda = \frac{\beta}{\alpha} \leq \frac{1 + \Gamma}{1 - \Gamma} \leq (1 + \Gamma)^2 \leq \frac{3}{2}, \quad 1 - \Gamma \geq \frac{2}{3}. $$

(115)

$$\left(\lambda - 1\right) \left(\frac{2 \beta}{c_o \alpha}\right)^{d^2-1} \leq \left(\frac{3}{c_o}\right)^{d^2-1} \frac{2 \Gamma}{1 - \Gamma} \leq 3 \Gamma \left(\frac{3}{c_o}\right)^{d^2-1} \leq \frac{\Delta}{2}.$$ 

(116)

On the other hand, from Lemma 9 and the fact that $\nu_{t-T}$ equidistributes it follows that for $r$ as in Eq. (103) we have $\Pr(C_{\varepsilon}(U_{t-T}) > r) \geq 1 - \Delta / 2$. Combining these estimates with Eq. (111) gives the desired result:

$$\Pr(C_{\varepsilon}(U_{t-T}) > r \mid C_{\varepsilon}(U_t) = 0) \geq \frac{1 - \Delta / 2}{1 + \Delta / 2} \geq 1 - \Delta. $$

(117)

The considerations for quantum states are similar. By repeating the same steps we obtain

$$\Pr(C_{\varepsilon}(\psi_{t-T}) > r \varepsilon \mid C_{\varepsilon}(\psi_t) = 0) \geq \Pr(C_{\varepsilon}(\psi_{t-T}) > r \varepsilon) \frac{\text{Vol}(\alpha \varepsilon)}{\text{Vol}(\beta \varepsilon)}. $$

(118)
In this case volumes of balls in the space of pure states are explicitly known: $\text{Vol}_S(\varepsilon) = \varepsilon^{2d-2}$. We can then bound the ratio of volumes as

$$\frac{\text{Vol}_S(\beta\varepsilon)}{\text{Vol}_S(\alpha\varepsilon)} \leq \left(\frac{1 + \Gamma_S}{1 - \Gamma_S}\right)^{2(d-1)} \leq (1 + 2\Gamma_S)^{4(d-1)} \leq \exp(8d\Gamma_S),$$

(119)

where in the second inequality we used $\Gamma_S \leq 1$, which is implied by the assumptions. Using the inequality $\log(1 + x) \geq \log(2)x$, valid for $x \in [0,1]$, it is easy to show that requiring

$$8d\Gamma_S \leq \frac{\log(2)\Delta}{2},$$

(120)

which is equivalent to $\Gamma_S \leq \frac{\Delta}{8d}$, implies that $\exp(8d\Gamma_S) \leq 1 + \Delta/2$. Similarly as for the case of unitaries, using Lemma 9 and the fact that $\nu_{t-T,S}$ equidistributes it follows that for $r_S$ as in Eq. (105) we have $\Pr(C_\varepsilon(\psi_{t-T}) > r_S) \geq 1 - \Delta/2$. Proceeding as before we get

$$\Pr\left(C_\varepsilon(\psi_{t-T}) > r_S \mid C_\varepsilon(\psi_1) = 0\right) \geq \frac{1 - \Delta/2}{1 + \Delta/2} \geq 1 - \Delta.$$

(121)

\[ \square \]

**Remark.** We note that the assumptions on $\Gamma$ in the above proposition are much more stringent for unitary channels, compared to pure states. We believe that this is solely an artifact of our proof and that the assumptions in the former case can be greatly weakened. In order to achieve this one would likely need much stronger control over the ratio of volumes $\text{Vol}(\beta\varepsilon)/\text{Vol}(\alpha\varepsilon)$ in the space of unitary channels.

We are ready to formulate and prove the main result of this section, which constitutes technical formulation of Result 6. To establish the timescales claimed there for local random quantum circuits, it suffices to substitute as timescales $T, T_S$ the bounds on equidistribution time given in Lemma 7 (for suitable parameters $\Gamma$ required by the Theorem).

**Theorem 16** (Timescales of complexity recurrences). Let $\mathcal{G}$ be a fixed universal gate set used in the definition of unitary and state complexity on $n$ qudit system $\mathcal{H} = (\mathbb{C}^q)^\otimes n$ of dimension $d = q^n$. Let $\nu = \{\nu_i, \mathcal{V}_i\}$ be a probability distribution on $\mathcal{U}(d)$ generating random walks $\mathcal{U}_t$ and $\psi_t$ on the space of unitary channels and quantum states. Let $\varepsilon \in (0,1/4)$ and $\Delta \in (0,1)$ and assume that $T$ is a time such that the measure $\nu^{*T}$ is $(\alpha,\beta)$-equidistributed on a scale $\varepsilon$ for $\Gamma = \max\{1 - \alpha, \beta - 1\} \leq \frac{\Delta}{6}(c_\alpha/3)^{d^2-1}$. Then for $t > 2T$ and

$$r = \frac{(d^2 - 1) \log \frac{1}{\varepsilon \alpha \beta} - \log \frac{2}{\Delta}}{\log |\mathcal{G}|},$$

(122)

we then have

$$\Pr\left(C_\varepsilon(\mathcal{U}_{t-T}) > r, C_\varepsilon(\mathcal{U}_{t+T}) > r \mid C_\varepsilon(\mathcal{U}_t) = 0\right) \geq 1 - \frac{3}{2}\Delta.$$

(123)

Furthermore, assuming that each of the gates $\{\mathcal{V}_i\}, \{\mathcal{V}_i^{-1}\}$ are implemented exactly using at most $k$ gates from $\mathcal{G}$, then conditioned on $C_\varepsilon(\mathcal{U}_t) = 0$ we have

$$\max\{C_\varepsilon(\mathcal{U}_{t-m}), C_\varepsilon(\mathcal{U}_{t+m})\} \leq A \cdot k \cdot m,$$

(124)

where the constant $A$ depends only on $\mathcal{G}$. Taken together, Eq. (123) and Eq. (124) show that for typical realizations of a random walk $\mathcal{U}_t$ complexity recurrences occur for $m_*$ satisfying

$$\exp(\Theta(n)) \log(1/\varepsilon) \leq m_* \leq T.$$

(125)

Analogous results holds for recurrences of complexity in pure quantum states $\psi_t$. The only difference is that in this case the role of timescale $T$ is replaced by $T_S$ required for measure $\nu_{t,S}$ to distribute on scale $\varepsilon$.
with parameters satisfying \( \Gamma_S = \max\{1 - \alpha, \beta - 1\} \leq \frac{A}{247} \) (see Proposition 15). Furthermore, the value of \( r \) is replaced by \( r_S \) from Eq. (105).

Proof. Using a union bound we obtain

\[
\Pr\left(C_\varepsilon(U_{t-T}) > r, C_\varepsilon(U_{t+T}) > r \mid C_\varepsilon(U_t) = 0\right) \\
\geq \Pr\left(C_\varepsilon(U_{t-T}) > r \mid C_\varepsilon(U_t) = 0\right) + \Pr\left(C_\varepsilon(U_{t+T}) > r \mid C_\varepsilon(U_t) = 0\right) - 1. \tag{126}
\]

For \( r \) as in Eq. (122) we then obtain Eq. (123) by lower bounding both conditional probabilities appearing on the right hand side of above inequality using Proposition 15 and Eq. (57), as well as

\[
\text{Lemma 9}. \quad \text{On the other hand, the inequality Eq. (124) follows directly from the assumptions and considerations following the proof Lemma 14, particularly Eq. (102).} \]

\[
\square
\]

\section{VIII. Linear Growth of Complexity at Late Times}

Thus far we have primarily focused on the saturation and recurrence of complexity in random quantum circuits, but there has been significant interest in proving the growth of quantum complexity, specifically establishing a long-time strictly linear growth of complexity. Such a long-time linear growth has been conjectured by Brown and Susskind for random quantum circuits and chaotic quantum many-body systems more generally. Ref. [32] proved a sublinear algebraic growth and Ref. [34] proved linear growth until exponential times for the restrictive notion of exact complexity. But it is worth emphasizing that RQC spectral gaps are sufficient to prove a linear growth which begins at exponential times, from \( t \approx d^2 \) until \( t \approx d^4 \). This linear growth follows from the exponentially small gaps proved in Refs. [19, 55].

\begin{proposition}[Linear complexity growth] For depth \( t \) local random quantum circuits on \( n \) qudits, \( \nu_\|^{(n)} \), if the circuit depth obeys \( t \leq 2c_2n^6d^4 \) then with probability greater than \( 1 - \Delta \) a given random circuit instance has circuit complexity lower bounded as

\[
C_\varepsilon(U_t) \geq t \frac{\log(2(1 - \varepsilon^2))}{d^2c_2n^6\log(|G|)} - \frac{\log(1/\Delta)}{\log(|G|)}, \tag{127}
\]

where the constant \( c_2 = 10^5 \). This implies a linear complexity growth from circuit depth \( t = \tilde{O}(d^2) \) until \( t = \tilde{O}(d^4) \).

Similarly, consider the states generated by depth \( t \) local random quantum circuits \( \nu_\|^{(n)} \), if the circuit depth is \( t \leq c_2n^6d^3 \) then with probability greater than \( 1 - \Delta \) a state has complexity lower bounded as

\[
C_\varepsilon(\psi_t) \geq t \frac{\log(2(1 - \varepsilon^2))}{d^2c_2n^6\log(|G|)} - \frac{\log(1/\Delta)}{\log(|G|)} \tag{128},
\]

which implies a linear state complexity growth from \( t = \tilde{O}(d^2) \) until \( t = \tilde{O}(d^3) \).

Note that in the above proposition, as we are interested in regimes where \( t \geq d^2 \) for unitaries and \( t \geq d \) for states, we may take \( \Delta \) to be exponentially small in the total dimension \( d \).

Similar complexity lower bounds hold for \( G \)-local random quantum circuits, albeit with a slightly sublinear scaling in \( t \), where the quantum complexity is instead growing as \( C_\varepsilon(U) \gtrsim t/\log(t) \) due to the \( \log(k) \) factors in the design depth for \( G \)-local random circuits.

Proof. We start by computing complexity lower bounds for elements of high degree designs. Let \( \nu \) be a \( \delta \)-approximate unitary \( k \)-expander. The probability that a unitary \( U \) drawn from \( \nu \) has complexity at most \( r \) can be union bounded, similar to the proof of Lemma 9 and Eq. (57), as

\[
\Pr\left(C_\varepsilon(U) \leq r\right) = \nu\left(\bigcup_{\ell \leq r} \bigcup_{V \in G^\ell} B(V, \varepsilon)\right) \leq \sum_{\ell \leq r} \sum_{V \in G^\ell} \nu(B(V, \varepsilon)). \tag{129}
\]
Using the upper bound on the volumes of balls according to $\nu$ in Lemma 33, we have that for unitaries drawn from a $\delta$-approximate unitary $k$-expander

$$\Pr(C_\epsilon(U) \leq r) \leq |G|^{r+1} \frac{k! + d^k \delta}{d^k (1 - \epsilon^2)^k}. \quad (130)$$

First, we take $\delta = 1/d^k$ and then for any $k \geq 6$ we have that $k! + 1 \leq (k/2)^k$, thus

$$\frac{k! + 1}{d^k (1 - \epsilon^2)^k} \leq \left(\frac{k}{2d^2 (1 - \epsilon^2)}\right)^k. \quad (131)$$

Restricting to $k \leq d^2$, we find

$$\Pr(C_\epsilon(U) \leq r) \leq |G|^{r+1} \frac{1}{(2(1 - \epsilon^2))^k}. \quad (132)$$

Computing the value of $r$ for which $\Pr(C_\epsilon(U) \leq r) \leq \Delta$ and taking the negation, we find that the circuit complexity of a unitary $k$-design element is

$$C_\epsilon(U) \geq \frac{1}{\log(|G|)} (k \log(2(1 - \epsilon^2)) - \log(1/\Delta)) \quad (133)$$

with probability greater than $1 - \Delta$. Furthermore, Proposition 3 gives that local RQCs of circuit depth $t = 2c_2d^2n^6k$ form $\delta$-approximate $k$-expanders with $\delta = 1/d^k$, which establishes the first statement of the proposition with the restriction that $6 \leq k \leq d^2$. Note the irrelevance of the lower restriction on $k$ as we wish to apply the proposition to exponentially deep circuits $t = \Omega(d^2)$, but the upper bound restricts to circuits of depth $t \leq \tilde{O}(d^4)$.

The proof for states proceeds similarly. Let $\nu_S$ be a $\delta$-approximate state $k$-expander. Again using a union bound and the upper bound on state design volumes in Lemma 34, the probability that a state $\psi$ drawn from $\nu$ has complexity at most $r$ can be upper bounded as

$$\Pr(C_\epsilon(\psi) \leq r) \leq |G|^{r+1} \frac{k! + d^k \delta}{d^k (1 - \epsilon^2)^k} \leq |G|^{r+1} \frac{1}{(2(1 - \epsilon^2))^k}, \quad (134)$$

where we take $\delta = 1/d^k$ and restrict to $6 \leq k \leq d$. Proceeding, we get that the circuit complexity of a state drawn from an approximate state $k$-design is lower bounded as

$$C_\epsilon(\psi) \geq \frac{1}{\log(|G|)} (k \log(2(1 - \epsilon^2)) - \log(1/\Delta)). \quad (135)$$

Proposition 3 establishes that local random circuits of depth $t = c_2d^2n^6k$ form $\delta$-approximate state $k$-expanders with $\delta = 1/d^k$, which yields the second statement of the proposition for circuits of depth $t \leq \tilde{O}(d^3)$.

We conclude by noting that bounding the volumes of $\epsilon$-balls, as prescribed by the distribution $\nu$, using moments of exponential degree allows one to see some aspects of complexity saturation. Specifically, Result 1 and Result 4 establish that unitaries drawn from high degree designs have nearly maximal complexity, which similarly follows from Lemma 33 for $k = O(d^2)$ and Lemma 34 for $k = O(d)$, albeit with trivial $\epsilon$ dependence. The volume bounds increase negligibly as we take $\epsilon \to 0$, in stark contrast to the behavior which follows from approximate equidistribution.

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**Appendix A: Equidistribution from unitary designs**

In this section we prove equidistribution using $\delta$-approximate $k$-expanders for unitaries and states. For unitaries, to improve readability, we first prove equidistribution for exact $k$-expanders and subsequently extend to approximate expanders, as in Proposition 21. For states, we prove the property directly for approximate expanders in Proposition 31. The two propositions are then collected into Theorem 5, which is a technical formulation of our Result 4.

1. **Equidistribution on $U(d)$ from exact designs**

   We quote the following theorem from Ref. [26].

   **Theorem 18** (Efficient polynomial approximation of the Dirac delta on unitary channels [26]). Let $\varepsilon \in (0, 2/3]$ and $\sigma \leq \frac{\varepsilon}{6\sqrt{d}}$. There exists a function $F^\sigma_k : U(d) \to \mathbb{R}$ with the following properties.

   1. Normalization: $\int_{U(d)} d\mu(U)|F^\sigma_k(U)| = 1$.
   2. Vanishing integral of modulus outside of the ball $B(I, \varepsilon)$: for $k \geq \frac{5d^2}{\sigma^2} \left( \frac{1}{2} \right)$ we have
      \[
      \int_{B(I, \varepsilon)^c} d\mu(U)|F^\sigma_k(U)| \leq 9 \exp \left( -\frac{\varepsilon^2}{4\sigma^2} \right) \left( \frac{\pi}{2} \right)^{d(d-1)}. \tag{A2}
      \]
   3. Low degree polynomial: $F^\sigma_k(U)$ can be represented as a balanced polynomial in $U$ and $\bar{U}$ of degree $k$.
   4. Bounded $L^2$-norm: for $k \geq d/\sigma$ we have
      \[
      \|F^\sigma_k\|_2 = \sqrt{\int_{U(d)} d\mu(U)|F^\sigma_k(U)|^2} \leq 8 \times 2^{d^2} \sigma^{-d(d-\frac{1}{2})}. \tag{A3}
      \]
   5. $L^1$-norm close to 1: for $k$ satisfying Eq. (A1), we have
      \[
      1 \leq \|F^\sigma_k\|_1 \leq 1 + 6 \exp \left( -\frac{\varepsilon^2}{4\sigma^2} \right) \left( \frac{\pi}{2} \right)^{d(d-1)}. \tag{A4}
      \]

   In order to simplify the notation let us denote
   \[
   g^\sigma_k(\varepsilon) = \int_{B(I, \varepsilon)^c} d\mu(U)|F^\sigma_k(U)|, \quad h^\sigma_k = \|F^\sigma_k\|_1 - 1. \tag{A5}
   \]
Under the assumptions of the theorem we then have
\[ g_k^\sigma(\varepsilon) \leq 9 \exp \left( -\frac{\varepsilon^2}{4\sigma^2} \right) \left( \frac{\pi}{2} \right)^{d(d-1)}, \quad h_k^\sigma \leq 6 \exp \left( -\frac{\varepsilon^2}{4\sigma^2} \right) \left( \frac{\pi}{2} \right)^{d(d-1)}. \] (A6)

Further note that \( h \) does not depend on \( \varepsilon \) explicitly, but under the assumptions of the theorem \( k \) depends on \( \varepsilon \). Hence the bound on \( h_k^\sigma \) depends on \( \varepsilon \) as well. The following lemma will be crucial for establishing the connection between equidistribution and unitary designs.

Lemma 19. Let \( \nu = \{ \nu_\alpha, U_\alpha \} \) be exact \( k \)-design. Then for any \( U \in U(d) \), \( \sigma > 0 \), and for any \( \kappa_-, \kappa_+ \) and \( \varepsilon \) satisfying \( 0 < \kappa_- < \varepsilon < \kappa_+ \) we have
\[ \operatorname{Vol}(\kappa_-) - g_k^\sigma(\varepsilon - \kappa_-) - h_k^\sigma \leq \nu(B(U, \varepsilon)) \leq \operatorname{Vol}(\kappa_+) + g_k^\sigma(\kappa_+ - \varepsilon) + h_k^\sigma. \] (A7)

Proof. We first prove the result for \( U = I \) and then will argue that for general \( U \) the proof is analogous. For unitary channels \( V \) and \( W \), let us denote \( F_k^\sigma(V^{-1}W) = F_k^\sigma(V)F_k^\sigma(W) \), so that \( F_k^\sigma(V) \) is centered around \( V \).

To prove a lower bound, we consider two balls around \( I \), with radii \( \kappa_- \) and \( \varepsilon \) (recall that \( \varepsilon > \kappa_- \)). Using the exact \( k \)-design property, we have that for every \( U \in U(d) \), \( \sum_\alpha \nu_\alpha F_{k,U_\alpha}(U) = 1 \), and therefore
\[ \operatorname{Vol}(\kappa_-) = \int_{B(I, \kappa_-)} d\mu(U) \sum_\alpha \nu_\alpha F_{k,U_\alpha}(U) = \sum_\alpha \nu_\alpha \int_{B(I, \kappa_-)} d\mu(U) F_{k,U_\alpha}(U) + \sum_\alpha \nu_\alpha \int_{B(I, \kappa_-)} d\mu(U) F_{k,U_\alpha}(U). \] (A8)

For \( U_\alpha \) satisfying \( D(U_\alpha, I) > \varepsilon \), we estimate
\[ \sum_{\alpha: D(U_\alpha, I) > \varepsilon} \nu_\alpha \leq 1, \] (A9)

and
\[ \int_{B(I, \kappa_-)} d\mu(U) F_{k,U_\alpha}(U) \leq \int_{B(I, \kappa_-)} d\mu(U) |F_{k,U_\alpha}(U)| \leq \int_{B(I, |\kappa_-|)} d\mu(U) |F_k^\sigma| = g_k^\sigma(\varepsilon - \kappa_-). \] (A10)

Furthermore, for \( U_\alpha \) with \( D(U_\alpha, I) \leq \varepsilon \) we have
\[ \sum_{\alpha: D(U_\alpha, I) \leq \varepsilon} \nu_\alpha = \nu(B(I, \varepsilon)). \] (A11)

We then estimate
\[ \int_{B(I, \kappa_-)} d\mu(U) F_{k,U_\alpha}(U) \leq \int_{B(I, \kappa_-)} d\mu(U) |F_{k,U_\alpha}(U)| \leq \int_{U(d)} d\mu(U) |F_{k,U_\alpha}(U)| = \|F_k^\sigma\|_1, \] (A12)

where in the last equality we used the identity \( \|F_k^\sigma\|_1 = \|F_k^\sigma\|_1 \). Inserting the above estimates into Eq. (A8), we obtain
\[ \operatorname{Vol}(\kappa_-) \leq \|F_k^\sigma\|_1 \nu(B(I, \varepsilon)) + g_k^\sigma(\varepsilon - \kappa_-) \leq \nu(B(I, \varepsilon)) + g_k^\sigma(\varepsilon - \kappa_-) + h_k^\sigma. \] (A13)

where in the last inequality we have used Eq. (A4) and definition of \( h_k^\sigma \). This gives the required lower bound.

The upper bound is proven analogously. We consider balls with radii \( \varepsilon \) and \( \kappa_+ \) (recall that this time
\[ \varepsilon \leq \kappa >. \] We have

\[
1 - \text{Vol}(\kappa) = \int_{B(I, \kappa^>)^c} d\mu(U) \sum_{\alpha} \nu_\alpha F_{k, U_\alpha}^\sigma(U) \quad (A14)
\]

\[
= \sum_{\alpha: D(U_\alpha, I) > \varepsilon} \nu_\alpha \int_{B(I, \kappa^>)^c} d\mu(U) F_{k, U_\alpha}^\sigma(U) + \sum_{\alpha: D(U_\alpha, I) \leq \varepsilon} \nu_\alpha \int_{B(I, \kappa^>)^c} d\mu(U) F_{k, U_\alpha}^\sigma(U).
\]

As before, consider the first term. We have

\[
\sum_{\alpha: D(U_\alpha, I) > \varepsilon} \nu_\alpha = 1 - \nu(B(I, \varepsilon)) \quad (A15)
\]

and for \( U_\alpha \) with \( D(U_\alpha, I) > \varepsilon \) we estimate

\[
\int_{B(I, \kappa^>)^c} d\mu(U) F_{k, U_\alpha}^\sigma(U) \leq \| F_{k}^\sigma \|_1. \quad (A16)
\]

For the second term use

\[
\sum_{\alpha: D(U_\alpha, I) \leq \varepsilon} \nu_\alpha \leq 1, \quad (A17)
\]

and for \( U_\alpha \) with \( D(U_\alpha, I) \leq \varepsilon \) we estimate

\[
\int_{B(I, \kappa^>)^c} d\mu(U) F_{k, U_\alpha}^\sigma(U) \leq \int_{B(I, \kappa^>)^c} d\mu(U)|F_{k, U_\alpha}^\sigma(U)| \leq \int_{B(I, \kappa^>)^c} d\mu(U)|F_{k}^\sigma(U)| = g_k^\sigma(\kappa > - \varepsilon). \quad (A18)
\]

Putting the estimates together we obtain

\[
1 - \text{Vol}(\kappa) \leq (1 - \nu(B(I, \varepsilon))) \| F_{k}^\sigma \|_1 + g_k^\sigma(\kappa > - \varepsilon). \quad (A19)
\]

Subsequently, from the definition of \( h_k^\sigma \) and its positivity, we arrive at

\[
\nu(B(I, \varepsilon)) \leq \text{Vol}(\kappa) + g_k^\sigma(\kappa > - \varepsilon) + h_k^\sigma, \quad (A20)
\]

which gives the required upper bound. To finish the proof, we have to argue that the same holds for \( \nu(B(U, \varepsilon)) \) for all \( U \). One obtains this by taking the original function not just \( F_{k}^\sigma \) but \( F_{k, U}^\sigma \), so that it is centered around \( U \), and considering balls around \( U \).

Now we proceed to proving equidistribution. To this end we first need to establish the following technical lemma.

**Lemma 20.** For any \( \varepsilon \in (0, 2/3) \), \( d \geq 16 \), and

\[
k \geq k^* (\varepsilon) = 5 \frac{d^2}{\sigma} \sqrt{\frac{1}{8} \frac{\varepsilon^2}{d^2 \sigma^2} + \log \left( \frac{1}{\sigma} \right)} \quad (A21)
\]

with

\[
\sigma \leq \sigma^* (d, \varepsilon) = \frac{\varepsilon}{2d} \frac{1}{\log(2/(c_\varepsilon \varepsilon))^{1/2}}, \quad (A22)
\]

it holds that \( \sigma \leq \frac{\varepsilon}{6\sqrt{d}} \) as well as that

\[
g_k^\sigma (\varepsilon) \leq \frac{1}{2} \text{Vol}(\varepsilon), \quad h_k^\sigma \leq \frac{1}{3} \text{Vol}(\varepsilon). \quad (A23)
\]
Remark. The restriction \( d \geq 16 \) is needed to make sure that \( \sigma \leq \varepsilon / (6 \sqrt{d}) \), which in turn is needed later when we discuss approximate designs. However, in the original paper the bound \( \varepsilon / (6 \sqrt{d}) \) is a strict joint lower bound on \( 4/(\pi \sqrt{d}) \) and \( 1/4 \).

Proof. First, one directly verifies that \( \sigma \) of Eq. (A22) with \( \varepsilon \leq 1 \) satisfies \( \sigma \leq \varepsilon / (6 \sqrt{d}) \). Let us now prove the first bound of Eq. (A23) using Eq. (A6). Due to Eq. (17), we need to show that for \( \sigma \) satisfying Eq. (A22) we have

\[
9 \exp \left( -\frac{\varepsilon^2}{4\sigma^2} \right) \left( \frac{\pi}{2} \right)^{d(d-1)} \leq \frac{1}{2} \left( \frac{c_0}{\varepsilon} \right)^{d^2-1} .
\]  

(A24)

To prove the above estimate, we drop \(-1\) from exponent on the right hand side, insert \( \sigma^* \), and get that the following estimate is to be proven:

\[
18 \frac{1}{2^{d^2}} \left( \frac{\pi}{2} \right)^{d^2-d} \leq 1 ,
\]  

(A25)

which indeed holds for \( d \geq 3 \). To prove the second estimate of Eq. (A23) it is enough to see that, according to bound on \( h \) in Eq. (A6), changing 9 into 6 in Eq. (A24) we get \( 1/3 \) on right hand side.

Note that we have a fixed constant \( c_0 = \frac{1}{9\pi} \), but the above lemma and following proposition would hold for any such constant in the volume lower bound so long as it is less than one.

Proposition 21. Let \( \nu \) be exact \( k \)-design. Then for \( 0 < \varepsilon < 1/4 \), \( 1/2 \leq \alpha < 1 < \beta \leq 3/2 \), \( d \geq 16 \) and

\[
k \geq 30 \frac{d^{5/2}}{\varepsilon \Gamma} \eta(\varepsilon \Gamma, d)
\]  

(A26)

with

\[
\eta(\varepsilon, d) = \sqrt{\log \frac{6}{c_0 \varepsilon} \sqrt{\frac{1}{2} \log \frac{6}{c_0 \varepsilon} + \log \left( \frac{6d}{\varepsilon} \log \frac{6}{c_0 \varepsilon} \right) }}, \quad \Gamma = \min \{1 - \alpha, \beta - 1\} ,
\]  

(A27)

we have

\[
\text{Vol}(\alpha \varepsilon) \leq \nu(B(\mathbf{U}, \varepsilon)) \leq \text{Vol}(\beta \varepsilon) .
\]  

(A28)

Proof. Upper bound. Lemma 19 tells us that

\[
\nu(B(\mathbf{U}, \varepsilon)) \leq \text{Vol}(\kappa_\gamma) + g_\gamma^\kappa(\kappa_\gamma - \varepsilon) + h_\gamma^\kappa .
\]  

(A29)

From Lemma 20 it follows that for any \( k \geq k_\gamma(\kappa_\gamma - \varepsilon) \) with \( k_\gamma \) given by Eq. (A21), and \( \sigma = \sigma_\gamma(\kappa_\gamma - \varepsilon) \), with \( \sigma_\gamma \) given by Eq. (A22), we have that

\[
g_\gamma^\kappa(\kappa_\gamma - \varepsilon) + h_\gamma^\kappa \leq \text{Vol}(\kappa_\gamma - \varepsilon) ,
\]  

(A30)

where we have dropped the factor of 5/6. From this we obtain

\[
\nu(B(\mathbf{U}, \varepsilon)) \leq \text{Vol}(\kappa_\gamma) + \text{Vol}(\kappa_\gamma - \varepsilon) .
\]  

(A31)

Combining this statement with Fact 22, we find that

\[
\nu(B(\mathbf{U}, \varepsilon)) \leq \text{Vol}(\kappa_\gamma + 2(\kappa_\gamma - \varepsilon)) .
\]  

(A32)

Choosing \( \kappa_\gamma = \varepsilon(\beta + 2)/3 \) so that \( \kappa_\gamma > \varepsilon \) as required, implies

\[
\text{Vol}(\kappa_\gamma) + g_\gamma^\kappa(\kappa_\gamma - \varepsilon) + h_\gamma^\kappa \leq \text{Vol}(\beta \varepsilon) .
\]  

(A33)
Together with Eq. (A29) this gives the desired bound for \( k \geq k^* ((\beta - 1)\varepsilon/3) \), with \( \sigma = \sigma^*(((\beta - 1)\varepsilon/3)) \), and where we used that for \( \kappa > \) as above we have \( \kappa - \varepsilon = (\beta - 1)\varepsilon/3 \).

**Lower bound.** Again from Lemma 19, we know that

\[
\text{Vol} (\kappa_\varepsilon) - g_k^\alpha (\varepsilon - \kappa_\varepsilon) - h_k^\alpha \leq \nu (B(U, \varepsilon)). \tag{A34}
\]

As before, from Lemma 20 we get

\[
g_k^\alpha (\varepsilon - \kappa_\varepsilon) + h_k^\alpha \leq \text{Vol} (\varepsilon - \kappa_\varepsilon), \tag{A35}
\]

but this time for \( k \geq k_\varepsilon (\varepsilon - \kappa_\varepsilon) \) with \( \sigma = \sigma_\varepsilon (\varepsilon - \kappa_\varepsilon) \), which gives

\[
\text{Vol} (\kappa_\varepsilon) - \text{Vol} (\varepsilon - \kappa_\varepsilon) \leq \nu (B(U, \varepsilon)). \tag{A36}
\]

We want the left hand side to be bounded from below by \( \text{Vol}(\alpha \varepsilon) \), which would complete the proof. Equivalently, we want

\[
\text{Vol} (\kappa_\varepsilon) \geq \text{Vol} (\varepsilon - \kappa_\varepsilon) + \text{Vol}(\alpha \varepsilon). \tag{A37}
\]

To show this, we set \( \kappa_\varepsilon = \varepsilon (\alpha + 2)/3 \), so that \( \kappa_\varepsilon = \alpha \varepsilon + 2 (\varepsilon - \kappa_\varepsilon) \) and \( \varepsilon - \kappa_\varepsilon = (1 - \alpha)\varepsilon/3 \). For such a \( \kappa_\varepsilon \) we can then use Fact 22 to get the required bound Eq. (A37), which works for \( k \geq k^* ((1 - \alpha)\varepsilon/3) \) with \( \sigma = \sigma^* ((1 - \alpha)\varepsilon/3) \).

Since \( k^* \), upon inserting \( \sigma = \sigma^* (\varepsilon) \), is a decreasing function of \( \varepsilon \), we get that both the upper and lower bounds hold when \( k \geq k^* (\varepsilon \Gamma/3) \) with \( \sigma = \sigma^* (\varepsilon \Gamma/3) \), where

\[
\Gamma = \min \{ 1 - \alpha, \beta - 1 \}. \tag{A38}
\]

The formula Eq. (A26) is then obtained by inserting \( \sigma^* (\varepsilon \Gamma/3) \) into \( k^* (\varepsilon \Gamma/3) \) and neglecting one of square roots.

Note that using Fact 22 requires \( \kappa_\varepsilon \leq 1/2 \), which, since \( \kappa_\varepsilon < \varepsilon \), only holds when \( \varepsilon \leq 1/2 \) as the diameter of \( \mathcal{U} \) with our distance \( D \) is greater than 1. It also requires that \( \kappa_\varepsilon + 2 (\kappa_\varepsilon - \varepsilon) \leq 1/2 \), i.e. \( \varepsilon \beta \leq 1/2 \). Since \( \beta \leq 3/2 \), we need to take \( \varepsilon \leq 1/3 \). We have also used Lemma 20, which requires \( \varepsilon - \kappa_\varepsilon \leq 2/3 \) and \( \kappa_\varepsilon - \varepsilon \leq 2/3 \). Since \( \beta \leq 3/2 \) and \( \alpha \geq 1/2 \), we thus require \( \varepsilon \leq 1/4 \).

**Fact 22.** Let \( \text{Vol}(\varepsilon) \) be volume of an \( \varepsilon \)-ball in the set of unitary channels \( \mathcal{U}(d) \) equipped with distance \( D(U, V) \), as in Eq. (1). Let \( \varepsilon_1, \varepsilon_2 > 0 \) be such that \( \varepsilon_1 + \varepsilon_2 \leq 2 = \text{diam}(\mathcal{U}(d)) \). Then we have

\[
\text{Vol}(\varepsilon_1 + 2\varepsilon_2) \geq \text{Vol}(\varepsilon_1) + \text{Vol}(\varepsilon_2). \tag{A39}
\]

**Proof.** Let \( U_1 \in \mathcal{U}(d) \) be a fixed unitary channel. By assumption there exist \( U_2 \) such that \( D(U_1, U_2) = \varepsilon_1 + \varepsilon_2 \). Using the triangle inequality for distance \( D \) it is straightforward show that balls \( B(U_1, \varepsilon_1), B(U_2, \varepsilon_2) \) intersect at most in boundary. Indeed, let \( D(V, U_1) \leq \varepsilon_2 \). Then it follows

\[
D(V, U_2) \geq D(U_1, U_2) - D(V, U_1) \geq \varepsilon_1 + \varepsilon_2 - \varepsilon_1 = \varepsilon_2. \tag{A40}
\]

Therefore \( \text{Vol}(B(U_1, \varepsilon_1) \cap B(U_2, \varepsilon)) = 0 \). Furthermore, by definition \( B(U_1, \varepsilon_1) \cup B(U_2, \varepsilon) \subset B(U_1, \varepsilon_1 + 2\varepsilon_2) \). Combining these two facts and we get

\[
\text{Vol}(B(U_1, \varepsilon_1)) + \text{Vol}(B(U_2, \varepsilon_2)) = \text{Vol}(B(U_1, \varepsilon_1) \cup B(U_2, \varepsilon)) \leq \text{Vol}(B(U_1, \varepsilon_1 + 2\varepsilon_2)) \tag{A41}
\]

We conclude the proof by using the unitary invariance of the volume (Haar measure) on \( \mathcal{U}(d) \).

**Remark.** An analogous argument to the one above can be applied to any unitary invariant metric on \( \mathcal{U}(d) \) and to manifold of pure states \( S(d) \).
2. Equidistribution on $U(d)$ from approximate designs

**Lemma 23.** Let $\nu = \{\nu_\alpha, U_\alpha\}$ be a $\delta$-approximate unitary $k$-expander. Then for any $U \in U(d)$ and for any $\kappa_\prec, \kappa_\succ$, and $\varepsilon$ satisfying $0 < \kappa_\prec < \varepsilon < \kappa_\succ$, we have

$$\text{Vol}(\kappa_\prec) - g_k^\varepsilon(\varepsilon - \kappa_\prec) - h_k^\varepsilon - \delta\|F_k^{\varepsilon}\|_2 \leq \nu(B(U, \varepsilon)) \leq \text{Vol}(\kappa_\succ) + g_k^\varepsilon(\kappa_\succ - \varepsilon) + h_k^\varepsilon + \delta\|F_k^{\varepsilon}\|_2. \quad (A42)$$

**Proof.** From Lemma 3 of Ref. [26] we have that for $\kappa \in [0, 2]$

$$\left| \text{Vol}(\kappa) - \int_{B(I, \kappa)} d\mu(U) \sum_\alpha \nu_\alpha F_{k, B, U, \alpha}(U) \right| \leq \delta\|F_k^{\varepsilon}\|_2 \sqrt{\text{Vol}(\kappa)}. \quad (A43)$$

Since $\text{Vol}(\kappa) \leq 1$ we choose to drop $\sqrt{\text{Vol}(\kappa)}$ obtaining

$$\text{Vol}(\kappa) - \delta\|F_k^{\varepsilon}\|_2 \leq \int_{B(I, \kappa)} d\mu(U) \sum_\alpha \nu_\alpha F_{k, B, U, \alpha}(U) \leq \text{Vol}(\kappa) + \delta\|F_k^{\varepsilon}\|_2. \quad (A44)$$

We now repeat the proof of Lemma 19 replacing the first equality in Eq. (A8) with the inequality

$$\text{Vol}(\kappa_\prec) - \delta\|F_k^{\varepsilon}\|_2 \leq \int_{B(I, \kappa_\prec)} d\mu(U) \sum_\alpha \nu_\alpha F_{k, B, U, \alpha}(U) \quad (A45)$$

and replacing the first equality in Eq. (A14) with

$$1 - (\text{Vol}(\kappa_\succ) + \delta\|F_k^{\varepsilon}\|_2) \leq \int_{B(I, \kappa_\succ)} d\mu(U) \sum_\alpha \nu_\alpha F_{k, B, U, \alpha}(U). \quad (A46)$$

\[\square\]

Before formulating the main result of this section, let us present an estimate that one can obtain from Eq. (A3).

**Lemma 24.** For $0 < \sigma \leq 1/d$, $d \geq 7$ and $k \geq d/\sigma$, the following implication holds. If

$$\delta \leq \left( c_0 \frac{\varepsilon \sigma}{2} \right)^{d^2 - 1}, \quad (A47)$$

then

$$\delta\|F_k^{\varepsilon}\|_2 \leq \frac{1}{6} \text{Vol}(\varepsilon). \quad (A48)$$

**Proof.** Due to bound in Eq. (A3), we need to ensure that

$$\delta \leq \frac{1}{6(c_0 \varepsilon)^{d^2 - 1} / 8 \times 2^{d^2} \sigma^{-d(d-\frac{1}{2})}}. \quad (A49)$$

One finds, that for $\sigma \leq 1/d$ and $d \geq 7$ we have

$$6 \times 8 \times 2^{d^2} \sigma^{-d(d-\frac{1}{2})} \leq \left( \frac{2}{\sigma} \right)^{d^2 - 1}. \quad (A50)$$

This implies that if

$$\delta \leq \left( c_0 \frac{\varepsilon \sigma}{2} \right)^{d^2 - 1}, \quad (A51)$$
then Eq. (A49) is satisfied, which ends the proof.

We now formulate the main proposition.

**Proposition 25.** Let $0 < \varepsilon < 1/4$, $1/2 \leq \alpha < 1 < \beta \leq 3/2$, $c_0$ be constant from Eq. (17), and let $\nu = \{\nu_\alpha, U_\alpha\}$ be a $\delta$-approximate $k$-expander with

$$
\delta \leq \left( \frac{c_0 \Gamma^2 \varepsilon^2}{d \sqrt{\log \frac{d}{c_0 \varepsilon \Gamma}}} \right)^{d^2 - 1}, \quad \Gamma = \min\{1 - \alpha, \beta - 1\},
$$

(A52)

and

$$
k \geq \frac{30 \varepsilon^{5/2}}{\varepsilon \Gamma} \eta(\varepsilon \Gamma, d),
$$

(A53)

with

$$
\eta(\varepsilon, d) = \sqrt{\log \frac{6}{c_0 \varepsilon}} \left[ \frac{1}{2} \log \frac{6}{c_0 \varepsilon} + \log \left( \frac{6d}{\varepsilon} \log \frac{6}{c_0 \varepsilon} \right) \right].
$$

(A54)

It follows that

$$
\text{Vol}(\alpha \varepsilon) \leq \nu(B(U, \varepsilon)) \leq \text{Vol}(\beta \varepsilon).
$$

(A55)

**Proof. Lower bound.** From left hand side of Eq. (A42) of Lemma 23 we have

$$
\text{Vol}(\kappa_<) - g_k^\varepsilon(\varepsilon - \kappa_<) - h_k^\varepsilon - \delta \| F_k^\varepsilon \|_2 \leq \nu(B(U, \varepsilon)).
$$

(A56)

We proceed as in the proof of Proposition 21. The only difference is that for $k \geq d/\sigma$, we bound $\delta \| F_k^\varepsilon \|_2$ by $1/6 \text{Vol}(\varepsilon - \kappa_<)$, which is possible if $\delta$ satisfies Eq. (A47) with $\varepsilon - \kappa_<$ in place of $\varepsilon$ and with $\sigma = \sigma^*(\varepsilon - \kappa_<)$.

Here, as in the proof of Proposition 21, $\sigma^*$ is given by Eq. (A22) and $\varepsilon - \kappa_< = (1 - \alpha)\varepsilon/3$). Note that the required lower bound for $k$ given by $d/\sigma$ is smaller than one needed to bound $g_k^\varepsilon(\varepsilon - \kappa_<) + h_k^\varepsilon$, we can thus keep the bound for $k$ the same as in Proposition 21 given by $k^*(\varepsilon - \kappa_<)$ with $k^*$ of Eq. (A21). The expression $\varepsilon - \kappa<$ we replace by $\varepsilon \Gamma/3$. For such conditions for $k$, $\sigma$, and $\delta$ we find

$$
\nu(B(U, \varepsilon)) \geq \text{Vol}(\kappa_<) - \frac{5}{6} \text{Vol}(\varepsilon - \kappa_<) - \frac{1}{6} \text{Vol}(\varepsilon - \kappa_<) - \text{Vol}(\kappa_<) - \text{Vol}(\varepsilon - \kappa_<).
$$

(A57)

We have thus obtained the same estimate as Eq. (A36) in the proof of Proposition 21. Therefore, by the subsequent reasoning from that proposition leads to the required lower bound.

**Upper bound.** Similarly as in the proof for lower bound, we have

$$
\nu(B(U, \varepsilon)) \leq \text{Vol}(\kappa_> + g_k^\varepsilon(\kappa_> - \varepsilon) + h_k^\varepsilon + \delta \| F_k^\varepsilon \|_2,
$$

(A58)

so that

$$
\nu(B(U, \varepsilon)) \leq \text{Vol}(\kappa_> + \frac{5}{6} \text{Vol}(\kappa_> - \varepsilon) + \frac{1}{6} \text{Vol}(\kappa_> - \varepsilon) \leq \text{Vol}(\kappa_> + \text{Vol}(\kappa_> - \varepsilon).
$$

(A59)

We have thus arrived at the estimate in Eq. (A31), and we can continue exactly as in Proposition 21, which concludes the proof.
3. Equidistribution on $S(d)$ induced by approximate unitary designs

We consider the set of states $S(d)$ equipped with measure $\mu_S$ (induced by the Haar measure on unitaries). The trace distance can be rewritten in terms of the overlap as

$$D(\psi, \phi) = \sqrt{1 - \text{tr}(\psi\phi)}.$$  \hfill (A60)

The ball $B(\phi, \varepsilon) = \{ \phi : D(\phi, \psi) \leq \varepsilon \}$ can be equivalently described as follows

$$B(\phi, \varepsilon) = \{ \psi : \text{tr}(\psi\phi) \geq \sqrt{1 - \varepsilon^2} \}.$$  \hfill (A61)

We introduce the following polynomial approximation of Dirac delta centered around $\phi$

$$F_{k, \phi}(\psi) = \frac{1}{I_k} \text{tr}(\psi\phi)^k,$$  \hfill (A62)

where

$$I_k = \int d\mu_S(\psi) \text{tr}(\psi\phi)^k = \binom{d + k - 1}{k}^{-1}.$$  \hfill (A63)

If the index $\phi$ is omitted, this means that we consider $\phi = \psi_0$, where $\psi_0$ is arbitrarily but fixed chosen reference state. Let us further denote

$$J_k(\varepsilon) = \int_{B^c(\psi_0, \varepsilon)} d\mu_S(\psi) F_k(\psi).$$  \hfill (A64)

We have the following bound on $J_k(\varepsilon)$:

**Lemma 26.** For $0 < \varepsilon \leq 1$ and $k \geq \frac{4d}{\varepsilon^2}$ we have $J_k(\varepsilon) \leq \exp\left(- k \frac{\varepsilon^4}{8} \right)$.

**Proof.** Directly from the definition of $J_k(\varepsilon)$ and Eq. (A61) we obtain

$$J_k(\varepsilon) = \int_{B^c(\psi_0, \varepsilon)} d\mu_S(\psi) F_k(\psi) = \left( \frac{d + k - 1}{k} \right) \int_{\psi : \text{tr}(\psi\psi_0) \leq \sqrt{1 - \varepsilon^2}} d\mu_S(\psi) \text{tr}(\psi\psi_0)^k.$$  \hfill (A65)

Passing to the variable $x = \text{tr}(\psi\psi_0)$, whose distribution is given by Eq. (19) we get

$$J_k(\varepsilon) = \left( \frac{d + k - 1}{k} \right) \int_0^{\sqrt{1 - \varepsilon^2}} dx x^k (d - 1)(1 - x)^{d-1}.$$  \hfill (A66)

The distribution appearing in above equation is known as Beta distribution (supported on interval $[0, 1]$) with two parameters

$$p_{\text{Beta}}(x) = N(a, b) x^{a-1}(1 - x)^{b-1},$$  \hfill (A67)

where $N$ is normalization constant. In our case $a = k + 1$, $b = d - 1$, and so $N(a, b)$ is equal to $I_k^{-1}$ in Eq. (A63). The average of this distribution is given by $E X = \frac{a}{a + b} = \frac{k + 1}{d + k}$. In Ref. [60] (see also [61]) the following tail inequality is provided for the Beta distributed random variable $X$

$$\max \{ \Pr(X \geq EX + \Delta), \Pr(X \leq EX - \Delta) \} \leq \exp\left(- 2(a + b + 1)\Delta^2 \right),$$  \hfill (A68)

which gives in our case as $\Pr(X \leq \frac{k + 1}{d + k} - \Delta) \leq \exp(-2(d + k + 1)\Delta^2)$. Since $J_k(\varepsilon) = \Pr(X \leq \sqrt{1-\varepsilon^2})$, we choose

$$\Delta = \left( 1 - \frac{k + 1}{d - 1} \right) - \sqrt{1 - \varepsilon^2} \geq \frac{k + 1}{d + k} - \varepsilon^2,$$  \hfill (A69)
where we applied $\sqrt{1-\epsilon^2} \leq 1 - \epsilon^2/2$, so that we obtain

$$J_k(\epsilon) \leq \exp \left\{ -2(d+k+1) \left( \frac{\epsilon^2}{2} - \frac{d-1}{d+k} \right)^2 \right\}. \tag{A70}$$

We now restrict to $k$ satisfying $k \geq 4d/\epsilon^2$. This implies that $\frac{d+1}{d+2} \leq \frac{\epsilon^2}{4}$ and hence

$$J_k(\epsilon) \leq \exp \left\{ -2(d+k+1) \left( \frac{\epsilon^2}{2} - \frac{d-1}{d+k} \right)^2 \right\} \leq \exp \left\{ -2(d+k+1) \frac{\epsilon^4}{16} \right\} \leq e^{-k \frac{\epsilon^4}{4}}. \tag{A71}$$

Moreover, we have the following estimate for second norm of $F_k$.

**Lemma 27.** For $k \geq d$ we have that

$$\|F_k\|_2 \leq \left( \frac{6k}{d} \right)^d. \tag{A72}$$

**Proof.** We start by noting that the 2-norm of $F_k$ can be written as

$$\|F_k\|_2 = \sqrt{\frac{I_{2k}}{I_k}} \leq \frac{1}{I_k} = \left( \frac{d+k-1}{k} \right). \tag{A73}$$

Proceeding, we write

$$\binom{d+k-1}{k} = \binom{d+k-1}{d-1} \leq \binom{d+k}{d} = \frac{(d+k)(d+k-1)\ldots(k+1)}{d!}. \tag{A74}$$

In the regime where $k$ is larger than $d$, we have

$$\binom{d+k}{d} = \frac{(k+d)(k+d-1)\ldots(k+1)}{d!} \leq \frac{(2k)^d}{d!} \leq \left( \frac{2ek}{d} \right)^d, \tag{A75}$$

which gives the desired estimate for $1/I_k$ and hence also for $\|F_k\|_2$. \hfill \qed

To prove equidistribution we need a relation between $\|F_k\|_2$ and $J_k(\epsilon)$ and the volumes of balls. This is contained in the following lemma.

**Lemma 28.** For $0 < \epsilon < \frac{1}{4}$ and $k$ satisfying

$$k \geq k_{**}(\epsilon) = 16 \frac{d}{\epsilon^4} \log(1/\epsilon), \tag{A76}$$

we have

$$J_k(\epsilon) \leq \frac{1}{2} \text{Vol}_S(\epsilon). \tag{A77}$$

Moreover, for $k$ as above and $\delta$ satisfying

$$\delta \leq \left( \frac{\epsilon}{3 \log(1/\epsilon)} \right)^{6d}, \tag{A78}$$

we have

$$\delta \|F_k\|_2 \leq \frac{1}{2} \text{Vol}_S(\epsilon). \tag{A79}$$
Proof. Bound for $J_k$. Given the expression for the volume $\text{Vol}_S(\varepsilon) = \varepsilon^{2d-2}$ and the bound on $J_k$ in Lemma 26, we need to choose $k$ such that

$$e^{-k\frac{4}{6k}} \leq \frac{1}{2} \left(\frac{\varepsilon}{2}\right)^{2d-2}.$$  

(A80)

Taking logarithm of both sides, solving for $k$ and noting that $\varepsilon^2 \leq 1/2$ if $\varepsilon \leq 1/4$, we get that for $k \geq k_{**}(\varepsilon)$ the inequality Eq. (A80) is satisfied. Also note, that $k \geq k_{**}$ implies that $k \geq 4d/\varepsilon^2$ as needed in order to use Lemma 26.

Bound for $\|F_k\|_2$. We use the bound in Lemma 27, with $k$ as in assumptions, and want $\delta$ to satisfy

$$\delta \left(\frac{d}{6k}\right)^d \leq \frac{1}{2} \varepsilon^{2d-2},$$  

(A81)

or after inserting $k_{**}$

$$\delta \leq \frac{1}{2} \varepsilon^{6d-2} \left(\frac{1}{96 \log(1/\varepsilon)}\right)^d.$$  

(A82)

Since we can bound the right hand side as follows

$$\frac{1}{2} \varepsilon^{6d-2} \left(\frac{1}{96 \log(1/\varepsilon)}\right)^d \geq \frac{1}{2} \varepsilon^{6d} \left(\frac{1}{3 \log(1/\varepsilon)}\right)^{6d} \geq \left(\frac{\varepsilon}{3 \log(1/\varepsilon)}\right)^{6d},$$

(A83)

where the last estimate holds for $\varepsilon \leq 1/4$, we obtain the required result.

Finally, we need the following lemma (proven in Ref. [43] and also in Ref. [26]):

Lemma 29. Let $\nu = \{\nu_\alpha, U_\alpha\}$ be $\delta$-approximate unitary $k$-expander. Let $\psi_0 = U_\alpha(\psi_0)$. Then

$$\left| \text{Vol}_S(\varepsilon) - \int_{B(\psi_0,\varepsilon)} \sum_\alpha \nu_\alpha F_{k,\psi_\alpha}(\psi) d\mu_\varepsilon \right| \leq \delta \|F_k\|_2.$$  

(A84)

Proof. For arbitrary measure $\xi$, and function $f$ on states, let us denote

$$(T^S_\xi f)(\psi) = \int d\xi(U)f(U^{-1}(\psi)).$$

(A85)

We then notice that

$$(T^S_{\mu_S} F_k)(\psi) = 1, \quad (T^S_{\mu_S} F_k)(\psi) = \sum_\alpha \nu_\alpha F_{k,\psi_\alpha}(\psi).$$

(A86)

Thus we have

$$\left| \text{Vol}_S(\varepsilon) - \int_{B(\psi_0,\varepsilon)} \sum_\alpha \nu_\alpha F_{k,\psi_\alpha}(\psi) d\mu_\varepsilon \right| = \left| \int_{B(\psi_0,\varepsilon)} (T^S_{\mu_S} F_k - T^S_{\nu_S} F_k) d\mu_\varepsilon \right| = \left| \langle (T^S_{\mu_S} - T^S_{\nu_S}) F_k, I_{B(\psi_0,\varepsilon)} \rangle \right|.$$  

(A87)

Here $I_{B(\psi_0,\varepsilon)}$ is indicator function of the ball $B(\psi_0,\varepsilon)$, and scalar product is with respect to square integrable functions on $S(d)$ with measure $\mu_S$. We next apply the Cauchy-Schwartz inequality

$$\left| \langle (T^S_{\mu_S} - T^S_{\nu_S}) F_k, I_{B(\psi_0,\varepsilon)} \rangle \right| \leq \| (T^S_{\mu_S} - T^S_{\nu_S}) F_k \|_2 \| I_{B(\psi_0,\varepsilon)} \|_2 \leq \| (T^S_{\mu_S} - T^S_{\nu_S}) \|_2 \| F_k \|_2 \sqrt{\text{Vol}_S(\varepsilon)},$$

(A88)

where by $|k|$ we denote the restriction to the set of the polynomials of degree $k$. Next we notice that our
polynomials on \(\mathcal{S}(d)\) embed isometrically into polynomials on \(\mathcal{U}(d)\) so that
\[
\|T^{S}_\mu|k - T^{S}_\nu|k\| \leq \|T_\mu|k - T_\nu|k\|, \tag{A89}
\]
where \(T^{S}_\xi\) is defined analogously as \(T^{S}_\xi\), but on unitary channels, and with Haar measure \(\mu\) in place of \(\mu_S\).

In terms of notation from Ref. [26], \(\|T^{S}_\mu|k - T^{S}_\nu|k\| = \|M_{\mu,k} - M_{\nu,k}\|\) which in turn is equal to \(\delta\) since \(\nu\) is a \(\delta\)-approximate \(k\)-design. Returning to Eq. (A87), we see that this ends the proof. \(\square\)

We can now turn to the topic of equidistribution for states. In analogy to the unitary channels case, we obtain the following lemma.

**Lemma 30.** Let \(\nu = \{\nu_\alpha, U_\alpha\}\) be a unitary \(\delta\)-approximate \(k\)-expander. Consider the measure \(\nu_S = \{\nu_\alpha, U_\alpha(\psi_0)\}\) on the set of states \(\mathcal{S}(d)\). Then for any \(\psi \in \mathcal{S}(d)\), and for any \(\kappa_-, \kappa_+\) and \(\varepsilon\) satisfying \(0 < \kappa_- < \varepsilon < \kappa_+\), we have
\[
\text{Vol}_{\mathcal{S}}(\kappa_-) - J_k(\varepsilon - \kappa_-) - \delta\|F_k\|_2 \leq \nu_S(B(\psi, \varepsilon)) \leq \text{Vol}(\kappa_+) + J_k(\kappa_+ - \varepsilon) + \delta\|F_k\|_2 \tag{A90}
\]

As we see the only difference (in comparison with unitary designs) is that there is no \(h\) function. This is because our function is now positive, and the \(h\) function vanishes. Thus the proof of this lemma is the same as the proof of Lemma 23 with the function \(h\) set to zero.

**Proposition 31.** Let \(0 < \varepsilon \leq 1\), and let \(\nu = \{\nu_\alpha, U_\alpha\}\) be a unitary \(\delta\)-approximate \(k\)-expander with
\[
\delta \leq \left(\frac{\varepsilon \Gamma}{9 \log(3/(\varepsilon \Gamma))}\right)^{6d} \tag{A91}
\]
and
\[
k \geq 1296 \frac{d}{(\varepsilon \Gamma)^4} \log(3/(\varepsilon \Gamma)) \tag{A92}
\]
with \(\Gamma = \min\{\beta - 1, 1 - \alpha\}\). Consider the measure \(\nu_S = \{\nu_\alpha, U_\alpha(\psi_0)\}\) on the set of states \(\mathcal{S}(d)\). We then have for any \(\psi\)
\[
\text{Vol}_{\mathcal{S}}(\alpha \varepsilon) \leq \nu_S(B(\psi, \varepsilon)) \leq \text{Vol}_{\mathcal{S}}(\beta \varepsilon). \tag{A93}
\]

**Proof.** We know from Lemma 28 that for \(k\) satisfying Eq. (A92) and \(\delta\) satisfying Eq. (A91)
\[
J_k(\varepsilon - \kappa_-) \leq \frac{1}{2} \text{Vol}_{\mathcal{S}}(\varepsilon - \kappa_-), \quad J_k(\kappa_+ - \varepsilon) \leq \frac{1}{2} \text{Vol}_{\mathcal{S}}(\kappa_+ - \varepsilon), \tag{A94}
\]
\[
\delta\|F_k\|_2 \leq \frac{1}{2} \text{Vol}_{\mathcal{S}}(\kappa_+ - \varepsilon), \quad \delta\|F_k\|_2 \leq \frac{1}{2} \text{Vol}_{\mathcal{S}}(\varepsilon - \kappa_-).
\]
We apply these inequalities to Lemma 30 and get
\[
\text{Vol}_{\mathcal{S}}(\kappa_-) - \text{Vol}_{\mathcal{S}}(\varepsilon - \kappa_-) \leq \nu_S(B(\psi, \varepsilon)) \leq \text{Vol}_{\mathcal{S}}(\kappa_+) + \text{Vol}_{\mathcal{S}}(\kappa_+ - \varepsilon). \tag{A95}
\]

which are analogues of estimates Eqs. (A31) and (A36) from the proof of Proposition 21 concerning unitaries. Then using Fact 22 (or actually its immediate modification for states), and knowing that diameter of \(\mathcal{S}(d)\) is equal to 1 we obtain the desired estimate in the same way as in the proof of Proposition 21. \(\square\)

**Appendix B: Proofs for random quantum circuits and unitary designs**

Here we present some delayed proofs for theorems and propositions in Section III regarding the convergence of various random quantum circuit models to approximate unitary designs.
The spectral gap of the moment operators has a convenient reinterpretation as the spectral gap of a local Hamiltonian [54, 62, 63]. Moreover, this Hamiltonian is frustration-free, powerful techniques from many-body physics may be employed to lower bound the spectral gap, as in [19, 25, 54]. We start by reviewing the Hamiltonian and its relation to the gap of the random circuit moment operators.

For $n$ qudits on a graph $g$, we can define the corresponding local Hamiltonian acting on $n$ subsystems each of dimension $q^{2k}$

$$H_{n,k}^{(g)} = \sum_{(i,j) \in g} h_{i,j} \quad \text{with} \quad h_{i,j} := (I - P_H)_{i,j},$$

where the local term $h_{i,j}$ acts nontrivially only on sites $i$ and $j$, and $P_H$ is the $k$-fold Haar moment operator the 2-sites: $P_H = \int_{\mathcal{U}(q^2)} d\mu(U) U \otimes \cdots \otimes U \otimes \bar{U}. \otimes \bar{U}$. In the Hamiltonian $H_{n,k}^{(g)}$, we sum over all edges of the graph $g$. This Hamiltonian is translation-invariant, has ground state energy zero, and is frustration-free. Moreover, the dimension of the ground space is $\dim \ker(H_{n,k}^{(g)}) = k!$. Most importantly, the spectral gap of the Hamiltonian is directly related to the spectral gap of a single time step of the random quantum circuit with Haar-random local gates. In one dimension we have $g(\nu_n, k) = 1 - \Delta(H_{n,k}^{(1D)})/n$ as in Ref. [19], which for a general interaction graph $g$ becomes $g(\nu_{U(q^2)}, g, k) = 1 - \Delta(H_{n,k}^{(g)})/|E|$, where $|E|$ is the cardinality of the edge set of the graph $g$.

Using the amplification property $g(\nu^t, k) = g(\nu, k)^t$, it follows that 1D random quantum circuits $\nu_{U(q^2), 1D}$ with Haar random local gates form $\delta$-approximate designs when

$$t \geq \frac{n}{\Delta(H_{n,k}^{(1D)})} \log(1/\delta).$$

We now continue to proving the two propositions.

Proof of Proposition 3. The proof this follows from an exponentially small spectral gap, which in turn follows from a number of Lemmas in Ref. [19] and an improved bound in Ref. [55].

Using a method for bounding the mixing time of Markov chains, a technique called path-coupling and specifically a version for random walks on the unitary group, Lemmas 19 and 20 in Ref. [19] prove a $k$-independent lower bound on the spectral gap of $H_{n,k}^{(1D)}$ of

$$\Delta(H_{n,k}^{(1D)}) \geq \frac{1}{n} \frac{1}{e(q^2 + 1)^n}.$$  

(B3)

The spectral gap is exponentially small, but holds for all moments $k$. For local random quantum circuits on $n$ qudits, the design depth is related to the spectral gap as in Eq. (B2). Noting that $q \geq 2$, this gap gives that RQCs generate expanders for all $k$ when the circuit depth is

$$t \geq n^2 q^{3.78} \log(1/\delta).$$

More recently, Ref. [55] gave an improved lower bound for an exponentially-small but $k$-independent gap, proving that the spectral gap of $H_{n,k}^{(1D)}$

$$\Delta(H_{n,k}^{(1D)}) \geq \frac{1}{c_2 n^2} \frac{1}{d^2},$$

(B4)

where $c_2 = 10^5$. Again, we find that the claim follows from the relation $g(\nu_n, k) = 1 - \Delta(H_{n,k}^{(1D)})/n$, given that exponentially deep circuits generate expanders for all $k$. □

Proof of Proposition 4. Theorem 2 and Proposition 3 show that 1D local random quantum circuits form $\delta$-approximate unitary $k$-designs using lower bounds on the spectral gap of $\Delta(H_{n,k}^{(1D)})$, a 1D frustration-free Hamiltonian. Importantly, the Hamiltonian here is defined with open boundary conditions. We now show that the 1D spectral gap bounds that of the Hamiltonian on any graph $g$ with a Hamiltonian path.

First, we note that the Hamiltonians $H_{n,k}^{(1D)}$ and $H_{n,k}^{(g)}$ have the same ground space, spanned by permutations. As the graph $g$ contains a Hamiltonian path, up to relabeling of the vertices, $g$ contains the 1D chain.
The $g$-local Hamiltonian can then be obtained by adding projectors to the 1D Hamiltonian. The operator inequality $H_{n,k}^{(g)} \geq H_{n,k}^{(1D)}$ then implies that $\Delta(H_{n,k}^{(g)}) \geq \Delta(H_{n,k}^{(1D)})$.

The independence of the spectral gap for universal gate sets containing inverses and comprised of algebraic entries [56] extends the result at the expense of a constant $c(\mathcal{G})$, which depends on the gate set. Following the results of Refs. [26, 43], we can drop the restrictions on the universal gate set at the additional expense of polynomial factors (Result 5 in [26]). Specifically, we have for the spectral gaps of the distributions for a single step of the walks

$$
(1 - g(\nu(\mathcal{G}, g), k)) \geq \frac{1}{c(\mathcal{G}) n \log^2(k)} (1 - g(\nu(U(\mathcal{G}, g), k))).
$$

The conclusion is that $G$-local random quantum circuits on $n$ qubits form $\delta$-approximate unitary $k$-designs, in operator norm, when the circuit depth is

$$
t \geq c(\mathcal{G}) \log^2(k) \frac{n^3 \log(1/\delta)}{\Delta(H_{n,k}^{(1D)})},
$$

where we use that $|E| \leq n^2$. Using the bounds on the 1D spectral gap from Ref. [19] as well as the exponentially small gap in Eq. (B4), the claims in the proposition then follow.

**Appendix C: Technical results concerning distances and volumes in the set of unitary channels**

**Lemma 32.** If two unitary channels $U$ and $V$ have $D(U, V) \leq \varepsilon$, then Hilbert-Schmidt inner product of the corresponding unitary matrices satisfies $|\text{tr}(UV)|^2 \geq d^2(1 - \varepsilon^2)$.

**Proof.** The proof of this claim follows from a similar statement in Ref. [32, Lemma 3]. We proceed by lower bounding the distance in terms of the diamond distance as

$$
D(U, V) \geq \frac{1}{2} ||U - V||_o \geq \frac{1}{2} ||(U \otimes I - V \otimes I)(|\Omega\rangle\langle\Omega|)||_1 = \sqrt{1 - \frac{1}{d^2} |\text{tr}(UV^\dagger)|^2},
$$

where $|\Omega\rangle$ is again the maximally entangled state on two copies of the system. It thus follows that $|\text{tr}(UV^\dagger)|^2 \geq d^2(1 - \varepsilon^2)$ is a necessary condition for $D(U, V) \leq \varepsilon$. \hfill $\square$

**Lemma 33** (Volumes of balls in $\mathcal{U}(d)$ according to approximate unitary expanders). Let $\nu$ be $\delta$-approximate unitary $k$-expander. For any unitary channel $V \in \mathcal{U}(d)$ and any $\varepsilon > 0$, we then have

$$
\nu(B(V, \varepsilon)) \leq \frac{k! + d^2 k \delta}{d^{2k}(1 - \varepsilon^2)^k}.
$$

**Proof.** We first note that if $D(U, V) \leq \varepsilon$, then it follows from Lemma 32 that

$$
|\text{tr}(UV^\dagger)|^2 \geq d^2(1 - \varepsilon^2).
$$

Therefore

$$

\nu(B(V, \varepsilon)) \leq \nu \left( \{ U : |\text{tr}(UV^\dagger)|^2 \geq d^2(1 - \varepsilon^2) \} \right) = \nu \left( \{ U : |\text{tr}(UV^\dagger)|^{2k} \geq d^{2k}(1 - \varepsilon^2)^k \} \right)
$$

Applying Markov’s inequality, we find that

$$
\nu(B(V, \varepsilon)) \leq \frac{\mathbb{E}_{U \sim \nu} [|\text{tr}(UV^\dagger)|^{2k}]}{d^{2k}(1 - \varepsilon^2)^k}.
$$
We now want to bound this quantity for approximate unitary designs. First we note that we may rewrite the quantity in the expectation as

\[
\left| \text{tr}(UV^\dagger) \right|^2 = d^2 \text{tr}(|\Omega\rangle\langle\Omega|UV^{-1} \otimes I(|\Omega\rangle\langle\Omega|))
\]

\[
= d^2 \langle \Omega | \left( U \otimes I \right) \left( \left( V^{-1} \otimes I \right) |\Omega\rangle\langle\Omega| \right) |\Omega\rangle,
\]

where \( |\Omega\rangle = \frac{1}{\sqrt{d}} \sum_j |j\rangle \otimes |j\rangle \) is the maximally entangled state on two copies of the system and \( \{|j\rangle \}_{j=1}^d \) is the computational basis of \( \mathbb{C}^d \).

Let \( \nu \) be an \( \delta \)-approximate unitary \( k \)-expander, it follows from Eq. (7) that

\[
\mathbb{E}_{U \sim \nu} \left[ |\text{tr}(UV^\dagger)|^2 \right] = \mathbb{E}_{U \sim \nu} \left[ |\text{tr}(UV^\dagger)|^2 \right] - \mathbb{E}_{U \sim \mu} \left[ |\text{tr}(UV^\dagger)|^2 \right] + \mathbb{E}_{U \sim \mu} \left[ |\text{tr}(UV^\dagger)|^2 \right]
\]

\[
= d^2 \text{tr} \left( |\Omega\rangle\langle\Omega| \otimes 2k \left( \Phi_v(k) - \Phi_\mu(k) \right) \left( V^{-1}|\Omega\rangle\langle\Omega| \right) \right) + \mathbb{E}_{U \sim \mu} \left[ |\text{tr}(UV^\dagger)|^2 \right]
\]

\[
\leq d^2 k \left( \| \Phi_v(k) - \Phi_\mu(k) \|_2 - k! \right) + \mathbb{E}_{U \sim \mu} \left[ |\text{tr}(UV^\dagger)|^2 \right]
\]

\[
\leq d^2 k \| \Phi_v(k) - \Phi_\mu(k) \|_2 - 2k + k!
\]

\[
\leq d^2 k \| \Phi_v(k) - \Phi_\mu(k) \|_2 - 2k + k!
\]

where we use the fact that for any \( k \) the Haar expectation of moments of traces is \( \mathbb{E}_{U \sim \mu} \left[ |\text{tr}(U)|^2 \right] \leq k! \), with equality if \( k \leq d \). Furthermore, in the third line we use H"older’s inequality, in the fourth line we employ the definition of the superoperator \( 2 \)-norm, and in the last line we use the equality between \( k \)-fold channels and moment operators \( \| \Phi_v(k) - \Phi_\mu(k) \|_2 - 2k \) and then Definition 4. Subsequently, it follows that

\[
\nu(B(V, \varepsilon)) \leq \frac{k! + d^2 k \delta}{d^2 k (1 - \varepsilon^2)^k}.
\]

**Lemma 34** (Volumes of balls in \( S(d) \) according to measures induced by approximate designs). Let \( \nu \) be \( \delta \)-approximate unitary \( k \)-expander in \( U(d) \) and let \( \nu_S \) be the induced measure on the set of pure quantum states \( S(d) \) (see Section III). For \( \varepsilon > 0 \) and for a pure state \( \phi \in S(d) \) we have

\[
\nu_S(B(\phi, \varepsilon)) \leq \frac{k! + d^2 \delta}{d^2 (1 - \varepsilon^2)^k}.
\]

**Proof.** First, we note that for two pure states \( \psi \) and \( \phi \), we have \( \text{D}(\psi, \phi) = \sqrt{1 - \text{tr}(\psi \phi)} \), and thus for a distribution \( \nu_S \) on the set of pure state

\[
\nu_S(B(\phi, \varepsilon)) \leq \nu_S \left( \{ \psi : \text{tr}(\psi \phi) \geq 1 - \varepsilon^2 \} \right) = \nu_S \left( \{ \psi : |\text{tr}(\psi \phi)|^k \geq (1 - \varepsilon^2)^k \} \right) \leq \frac{\mathbb{E}_{\psi \sim \nu_S} \left[ |\text{tr}(\psi \phi)|^k \right]}{(1 - \varepsilon^2)^k},
\]

where in the last step we used Markov’s inequality. If the distribution on states \( \nu_S \) is a \( \delta \)-approximate state design, then it follows that

\[
\mathbb{E}_{\psi \sim \nu_S} \left[ |\text{tr}(\psi \phi)|^k \right] = \mathbb{E}_{\psi \sim \nu_S} \left[ |\text{tr}(\psi \phi)|^k \right] - \mathbb{E}_{\psi \sim \mu_S} \left[ |\text{tr}(\psi \phi)|^k \right] + \mathbb{E}_{\psi \sim \mu_S} \left[ |\text{tr}(\psi \phi)|^k \right]
\]

\[
= \text{tr} \left[ \phi \otimes k \left( \Phi_v(k) - \Phi_\mu(k) \right) \left( \phi \otimes k \right) \right] + \left( \frac{d + k - 1}{k} \right)^{-1} \text{tr} \left( \phi \otimes k \Pi_v \right),
\]

\[
\leq \| \Phi_v(k) - \Phi_\mu(k) \|_2 - 2k + \left( \frac{d + k - 1}{k} \right)^{-1},
\]

where we used the fact that the \( k \)-fold average of Haar random states is proportional to the projector onto the symmetric subspace \( \Pi_v \), and in the last line used the same reasoning as in the proof of Lemma 33. We
then use \( \| \Phi^{(k)}_{\nu} - \Phi^{(k)}_{\mu} \|_{2\to2} = \| M^{(k)}_{\nu} - M^{(k)}_{\mu} \|_{\infty} \leq \delta \) to upper bound the expression in Eq. (C15). To conclude, we use the fact that \( \binom{d+k-1}{k} \geq \frac{d^k}{k!} \).

It is interesting to further comment on the forms of Lemma 33 and Lemma 34. Let us consider the respective Haar volumes, for which the lemmas apply with \( \delta = 0 \) and for all \( k \). For large enough \( k \), Lemma 33 fails to improve as the numerator grows. However for \( k = O(d^2) \), we recover the correct \( d \) dependence of \( \text{Vol}(\varepsilon) \), albeit with a much weaker \( \varepsilon \) dependence which does not diminish as we take \( \varepsilon \to 0 \). On the other hand, Lemma 34 is quite different; before taking the final bound in the proof we had for states \( \nu_\mathcal{S}(B(\phi, \varepsilon)) \leq (\frac{d+k-1}{k})^{-1} (1 - \varepsilon^2)^{-k} \), where the inverse binomial coefficient continues to decrease for larger \( k \). From the expression we recover the correct \( d \) dependence of \( \text{Vol}_\mathcal{S}(\varepsilon) = \varepsilon^{2(d-1)} \), but the bound for states appears to know much more about the dependence on the radius of balls in the space of states.

We conclude this section by proving Proposition 36, an important technical proposition that allows one to pinpoint the timescales at which a complexity recurrence takes place for the case of quantum channels. We start by stating a useful relation between the distance \( D(\mathbf{U}, \mathbf{I}) \) and the length of the shortest arc containing the eigenvalues of a unitary matrix \( \mathbf{U} \in \mathcal{U}(d) \) that gives rise to a channel \( \mathbf{U} \in \mathcal{U}(d) \).

**Lemma 35.** Let \( \text{Eig}(\mathbf{U}) = \{\exp(i\alpha_1), \exp(i\alpha_2), \ldots, \exp(i\alpha_d)\} \) be eigenvalues of \( \mathbf{U} \in \mathcal{U}(d) \) corresponding to unitary channel \( \mathbf{U} \in \mathcal{U}(d) \). Furthermore, let \( \text{Arc}(X) \) denote the length of the shortest arc containing a subset \( X \subset S^1 \). Assuming that \( D(\mathbf{U}, \mathbf{I}) \leq \sqrt{2} \), we then have

\[
D(\mathbf{U}, \mathbf{I}) = \kappa \iff \text{Arc}(\text{Eig}(\mathbf{U})) = 4 \arcsin \left( \frac{\kappa}{2} \right). \tag{C19}
\]

The proof of the above follows from elementary geometric considerations given in Proposition 6 and illustrated in Figure 3 of [26].

**Proposition 36** (Upper bound for the volume of an annulus in \( \mathcal{U}(d) \)). Let \( \lambda \geq 1 \) and \( \kappa > 0 \), such that \( \lambda \kappa \leq \sqrt{2} \), and let \( \mathcal{A} \) be an annulus in \( \mathcal{U}(d) \)

\[
A(\kappa, \lambda \kappa) = \{ \mathbf{U} \in \mathcal{U}(d) \mid \kappa \leq D(\mathbf{U}, \mathbf{I}) \leq \lambda \kappa \}. \tag{C20}
\]

Then we have

\[
\text{Vol}(A(\kappa, \lambda \kappa)) \leq (2\lambda \kappa)^{d^2-1}(\lambda - 1). \tag{C21}
\]

**Proof.** We first recall that for subset \( \mathcal{A} \subset \mathcal{U}(d) \) we have \( \text{Vol}(\mathcal{A}) = \mu_{\mathcal{U}(d)}(\mathcal{A}') \), where \( \mu_{\mathcal{U}(d)} \) is the Haar measure on \( \mathcal{U}(d) \) and \( \mathcal{A}' = \{ \mathbf{V} \in \mathcal{U}(d) \mid \mathbf{V} \in \mathcal{A} \} \). Consequently by setting \( f(x) := 4 \arcsin \left( \frac{x}{2} \right) \) and using Lemma 35 we get

\[
\text{Vol}(A(\kappa, \lambda \kappa)) = \mu_{\mathcal{U}(d)}(\{ \mathbf{V} \in \mathcal{U}(d) \mid f(\mathbf{V}) \leq \text{Arc}(\text{Eig}(\mathbf{V})) \leq f(\lambda \kappa) \}) \tag{C22}
\]

Since \( \text{Arc}(\text{Eig}(\mathbf{V})) \) is a class function on \( \mathcal{U}(d) \), i.e. \( \text{Arc}(\text{Eig}(\mathbf{WVW}^\dagger)) = \text{Arc}(\text{Eig}(\mathbf{V})) \) for all \( \mathbf{V}, \mathbf{W} \in \mathcal{U}(d) \), we can use Weyl integration formula [64] to rewrite the above expression in terms of integral over phases

\[
\text{Vol}(A(\kappa, \lambda \kappa)) = N_d \int_{\alpha \in [0, 2\pi]^d} \prod_{1 \leq i < j \leq d} |\exp(i\alpha_i) - \exp(i\alpha_j)|^2 d\alpha, \tag{C23}
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_d), \ d\alpha = d\alpha_1 \ldots d\alpha_d, \) and \( N_d = \frac{1}{\pi(2\pi)^d} \). Using \( |\exp(i\alpha_i) - \exp(i\alpha_j)| \leq 2\lambda \kappa \) (see Figure 3 of [26]) we obtain

\[
\text{Vol}(A(\kappa, \lambda \kappa)) \leq \frac{(2\lambda \kappa)^{d(d-1)}}{d!} \Pr_\alpha[f(\kappa) \leq \text{Arc}(\alpha) \leq f(\lambda \kappa)], \tag{C24}
\]

where \( \Pr_\alpha \) denotes probability with respect to phases \( \alpha = (\alpha_1, \ldots, \alpha_d) \) uniformly and independently distributed on \( S^1 \). The distribution of the random variable \( \text{Arc}(\alpha) \), the length of the shortest arc containing randomly and independently distributed points on a unit circle, has been studied previously in [65, Section
3.6], where it was shown that for \( r \leq \pi \)

\[
\Pr_{\alpha} [\text{Arc}(\alpha) \leq r] = d \left( \frac{r}{2\pi} \right)^{d-1}.
\]  \hspace{1cm} (C25)

Since \( f(x) \leq x \) for \( x \in [0, \sqrt{2}] \) we can apply the above formula to Eq. (C24) and obtain

\[
\text{Vol}(A(\kappa, \lambda)) \leq \frac{(2\lambda\kappa)^{d(d-1)}}{(2\pi)^{d-1}(d-1)!} (f(\lambda\kappa)^{d(d-1)} - f(\kappa)^{d(d-1)}) \hspace{1cm} (C26)
\]

To simplify the above formula we use that for \( x > y \geq 0 \) we have the following inequalities

\[
x^{d-1} - y^{d-1} \leq (d-1)x^{d-2}(x-y), \quad f(x) - f(y) \leq 2\sqrt{2}(x-y),
\]  \hspace{1cm} (C27)

where the second bound follows from integral representation of arcsin: \( \arcsin(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}} \). Application of Eq. (C27) in Eq. (C26) finally gives

\[
\text{Vol}(A(\kappa, \lambda)) \leq \frac{(2\lambda\kappa)^{d(d-1)}f(\lambda\kappa)^{d(d-2)}}{(2\pi)^{d-1}(d-2)!} (f(\lambda\kappa) - f(\kappa)) \leq \frac{(2\lambda\kappa)^{d(d-1)}(\sqrt{2})^{d-1}}{(2\pi)^{d-1}(d-2)!}(\lambda - 1) \leq (2\lambda\kappa)^{d(d-1)}(\lambda - 1),
\]  \hspace{1cm} (C28)

where in the last inequality we used \( \lambda \geq 1 \).

\[\]
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