Quantifying Nonlocality Based on Local Hidden Variable Models

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We introduce a fresh scheme based on the local hidden variable models to quantify nonlocality for arbitrarily high-dimensional quantum systems. Our scheme explores the minimal amount of white noise that must be added to the system in order to make the system local and realistic. Moreover, the scheme has a clear geometric significance and is numerically computable due to powerful computational and theoretical methods for the class of convex optimization problems known as semidefinite programs.

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In a celebrated work in 1964, Bell showed surprisingly that different measurements on two separated parts of entangled quantum states would lead to stronger correlations, which are unexplainable by any local hidden variable (LHV) theory or shared randomness only [1]. This kind of stronger correlations is now well known as nonlocality.

In the last decade, nonlocality has attracted much interest in both theoretical and experimental works [2] not only because of its close relations to the foundations of quantum mechanics, but also because of the vital role it plays in many quantum computation and information processes [3, 4]. Nevertheless, we are still far away from fully understanding nonlocality. Many fundamental questions still remain open, one of which is how to quantify nonlocality. Up to now, there are various schemes to quantify entanglement [5, 6], while few works for nonlocality quantification exist [7].

A nature consideration is to use the violations of Bell inequalities as the measure of nonlocality, based on which some nonlocality distillation protocols are proposed recently in the framework of generalized nonsignaling theories [8]. However, these distillation protocols show exemplarily the advantages of this measure, they also drastically reveal its severe shortcomings: (i) For a specific system, there are many tight Bell inequalities that are not equivalent to each other. Then which inequality is the best one as the measure of nonlocality for this quantum system? (ii) Some quantum states do not violate a certain Bell inequality, but they do violate some other inequalities. What is worse, we even do not know whether these states violate Bell inequalities since we in principle need to find out all the Bell inequalities for the system to answer this question. (iii) Finding out all tight Bell inequalities for a quantum system is a NP hard problem [9], so it is not pragmatic to use Bell inequality as nonlocality measure. Due to these disadvantages and the rapid theoretical and experimental developments in this realm, it becomes more and more pressing to introduce a more reasonable measure for nonlocality.

In this Letter, based on the LHV models, we introduce a possible scheme with a clear geometric significance to quantify nonlocality for arbitrarily high-dimensional quantum systems. This scheme explores the minimal amount of white noise that must be added to the system so that the new system is local and realistic. In addition, by taking advantage of a brilliant work by Terhal et al. [10] and the established method for semidefinite programs, we show explicitly how to estimate efficiently this quantification by numerical method.

To start with, we should specify some notations and definitions. Let \( \mathcal{H}^{[d]} = \mathcal{H}^{[d_1]} \otimes \cdots \otimes \mathcal{H}^{[d_n]} \) denote a \( d \)-dimensional Hilbert space of \( n \) particle system. Here \( \mathcal{H}^{[d_k]} \), \( (k = 1, \cdots, n) \), denotes the \( d_k \)-dimensional Hilbert space of the \( k \)-th subsystem and \( d = d_1 \times \cdots \times d_k \). Denoting the set of all quantum states on \( \mathcal{H}^{[d]} \) that admit LHV models by \( \mathcal{L}_{lhv} \), then the scheme to quantify nonlocality is as follows: For a specific quantum state \( \rho \) on the Hilbert space \( \mathcal{H}^{[d]} \), let \( \rho_\lambda = \lambda \rho + (1 - \lambda) \rho_s \), \( (0 \leq \lambda \leq 1) \), then the definition of the nonlocality is:

\[
\mathcal{N}(\rho) = \min_{\rho_\lambda \in \mathcal{L}_{lhv}} \lambda.
\]

Here \( I_d \) is a \( d \)-by-\( d \) identity matrix. Obviously, this definition of nonlocality has the following properties:

(i) \( \mathcal{N}(\rho) \) is invariant under local unitary transformations, i.e., \( \mathcal{N}(\rho) = \mathcal{N}(U_1 \otimes \cdots \otimes U_n \rho U_1^\dagger \otimes \cdots \otimes U_n^\dagger) \).

(ii) \( \mathcal{N}(\rho) = 0 \) if the state \( \rho \) is separable. In fact, if \( \rho \) admits a LHV model, then it is obvious \( \mathcal{N}(\rho) = 0 \).

(iii) The \( \mathcal{N}(\rho) \) exists for any quantum state \( \rho \) since \( \rho_\lambda \) is separable as long as \( \lambda \) is large enough [11].

(iv) \( \mathcal{N}(\rho) < 1 \). Actually, for any state \( \rho \) on \( \mathcal{H}^{[d]} \), \( \rho_\lambda \) is separable for \( \lambda \geq 1 - 1/\sqrt{(d^2 - 1)(d - 1)} \) [12]. Thus we have a more accurate upper bound \( \mathcal{N}(\rho) \leq 1 - 1/\sqrt{(d^2 - 1)(d - 1)} \).

(v) \( \mathcal{N}(\rho) > 0 \) if \( \rho \) is an entangled pure state. This is obvious since any pure entangled state violates certain Bell inequality [13].

Interestingly, this definition bears a geometric significance as showed in the Fig. 1. It is obvious that the
nonlocality $N(\rho)$ indicates the distance between $\rho$ and $\rho^*$ if we normalize the distance between nonlocal state $\rho$ and the white noise state $\rho_w$. Moreover, from the experimental point of view, $N(\rho)$ also implies the minimal amount of white noise that must be added to the system in order to hide the nonlocal character of the state $\rho$. Its physical meaning is that it provides a lower bound of how much noise is needed. In other words, if the noise added to the system exceeds the bound, then the nonlocal correlations of this system will disappear.

Now, we have the scheme to quantify nonlocality. However, from the definition of $N(\rho)$, it is very difficult to compute $N(\rho)$ analytically since the analytical construction of LHV models for some entangled states is itself extremely difficult [14]. Fortunately, Terhal et al. have introduced a simple and efficient algorithmic approach for LHV-model construction for quantum states [10]. Their approach is based on the construction of a symmetric quasiextension of the quantum state. They showed explicitly that a symmetric quasiextension might lead to a LHV model for the specific state, depending only on the number of local measurement settings for each observer. In the following, we shall show how to estimate numerically $N(\rho)$ by taking advantage of this method. We will focus on the two-qubit and two-qutrit systems. The generalization to multipartite higher-dimensional systems is straightforward.

For convenience and completeness, we first recapitulate the main results of Terhal et al. [10]. Consider the following Bell-type scenario: Two distant parties, Alice and Bob, has a set of local measurements. Denote the number of measurements for Alice as $M_a$ and let each measurement has $O_a$ outcomes. Similarly, we denote the number of measurements for Bob as $M_b$ and let each measurement has $O_b$ outcomes. Moreover, let’s define

$$S(\rho) = \frac{1}{M!} \sum_\Lambda \Lambda_\rho \Lambda^\dagger,$$

where $\Lambda$: $\mathcal{H}^\otimes M \rightarrow \mathcal{H}^\otimes M$ is a permutation of spaces $\mathcal{H}$ in $\mathcal{H}^\otimes M$. We say that $\rho$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ has a $(M_a, M_b)$-symmetric quasiextension when there exists a multipartite entanglement witness $W_\rho$ on $\mathcal{H}_A^\otimes M_a \otimes \mathcal{H}_B^\otimes M_b$ such that $\text{Tr}_{\mathcal{H}_A^\otimes M_a-1, \mathcal{H}_B^\otimes M_b-1} W_\rho = \rho$ and $W_\rho = \mathcal{S}_A \otimes \mathcal{S}_B (W_\rho)$. Terhal et al. showed that if $\rho$ has a $(M_a, M_b)$-symmetric quasiextension, then $\rho$ definitely admits a LHV model when Alice and Bob have $M_a$ and $M_b$ arbitrary measurements [10].

Now, the task of constructing LHV models is transformed to a new task of finding symmetric quasiextension for a specific state. According to Ref. [10], the new task can be stated as a particular case of the class of convex optimization known as semidefinite programs (SDP) [10], which correspond to the optimization of a linear function subject to a linear matrix inequality. Vandenberghe and Boyd [15] write the typical SDP as:

$$\begin{align*}
\text{minimize} & \quad c^T x, \\
\text{subject to} & \quad F(x) \geq 0,
\end{align*}$$

where $c$ is a given vector of length $\nu$, and $F(x) = F_0 + \sum_i x_i F_i$; $F_i$ ($i = 1, \cdots, \nu$) are some fixed $\mu$-by-$\mu$ Hermitian matrices. The vector $x$, also of length $\nu$, is the variable over which the minimization is performed. The SDP is said to be strictly feasible if there exists a vector $x$ such that $F(x) > 0$ is satisfied. The dual problem corresponding to a SDP, also a SDP, reads:

$$\begin{align*}
\text{maximize} & \quad -\text{Tr}[F_i Z], \\
\text{subject to} & \quad \text{Tr}[F_i Z] = c_i, \quad Z \geq 0,
\end{align*}$$

where the Hermitian matrix $Z$ is the variable over which the maximization is performed. Similarly, if there exists a matrix $Z > 0$ satisfying the trace constraints, the dual SDP is also said to be strictly feasible. A very important relation between the primal and dual optimizations was shown by Vandenberghe and Boyd [15]: If both the primal and dual forms of a SDP are strictly feasible, their optima are equal and achieved by some feasible pair $(x_{\text{opt}}, Z_{\text{opt}})$. The constructions of symmetric extensions correspond to the dual form of a SDP [10] with $F_i = \mathcal{S}_A \otimes \mathcal{S}_B (\lambda_i \otimes I)$, $c_i = r_i = \text{Tr}[\lambda_i \rho]$ and $F_0 = I_{d_A} \otimes I_{d_B} \otimes I$. Here $\{\lambda_i\}$ is the basis for the space of Hermitian matrices that operate on $\mathcal{H}_A \otimes \mathcal{H}_B$. This basis is orthogonal in the trace inner product $\text{Tr}[\lambda_i \lambda_j] = \delta_{ij}$ and $\lambda_0 = I_{d_A} \otimes I_{d_B}/\sqrt{d_A d_B}$. The corresponding $\nu = (d_A d_B)^2 - 1$ and $\mu = d_A M_a \cdot d_B M_b$.

Based on the discussion above, in order to calculate $N(\rho)$, we only need to find the minimal $\lambda$ so that $\rho_\lambda$ admits LHV model. Nevertheless, the above numerical construction of LHV model depends on the number of local
measurement settings for each observer. Consequently, it might be more convenient to introduce a transitional definition of nonlocality also based on the number of local measurement settings for each observer:

$$N(M_a, M_b)|\rho\rangle = \min_{\rho \in S(M_a, M_b)} \lambda,$$

where $S(M_a, M_b)$ is the set of all the states that admit LHV models for $(M_a, M_b)$-settings. We term $N(M_a, M_b)|\rho\rangle$ the $(M_a, M_b)$-nonlocality of state $\rho$. Mathematically, one has

$$N(\rho) = \lim_{M_a \to \infty, M_b \to \infty} N(M_a, M_b)|\rho\rangle.$$

From the relationship stated in Eq. (4), we can always use $N(M_a, M_b)|\rho\rangle$ to estimate $N(\rho)$ as long as $M_a$ and $M_b$ are big enough. Actually, numerical results show that for many states, $N(M_a, M_b)|\rho\rangle$ vary very slightly with different $(M_a, M_b)$. So, we only need to compute $N(M_a, M_b)|\rho\rangle$ with small $M_a$ and $M_b$. Consequently, the scheme to quantify nonlocality introduced above can be carried out numerically due to the efficient method to construct LHV models with $(M_a, M_b)$-settings.

To illustrate explicitly how the method works, we give some examples here. We have implemented the mentioned corresponding semidefinite program using SeDuMi [16]. The first example is about a set of two-qubit states: $|\psi\rangle = \sin \theta |00\rangle + \cos \theta |11\rangle$. The nonlocality $N(|\psi\rangle)$ with different $\theta$ is showed in Fig. 2 (a). Note that here we use $N^{[2,2]}(|\psi\rangle)$ to estimate $N(|\psi\rangle)$. Also drawn are the entanglement of formation (EOF) [6] and Bell-CHSH [17] inequality violation of this set of states. From the figure, the maximal nonlocality occurs at $\theta = \pi/4$, i.e., the maximally entangled state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and the maximal nonlocality $N(|\Psi\rangle)$ equals to 1/3. For two-qubit pure state, the nonlocality increases monotonically as the EOF goes from 0 to 1. Interestingly, based on the fact that $N(|\Psi\rangle) = 1/3$, we suspect that there might exist some two-qubit Bell inequalities with an improved visibility 2/3, which is stronger than the CHSH inequality. However, we have not found such an inequality and the question proposed by Gisin is still open [18].

Another example involves the so called maximally entangled mixed states (MEMS), which have the maximum amount of entanglement for a given linear entropy [19]:

$$\rho_{\text{mems}} = \begin{pmatrix} f(\gamma) & 0 & 0 & \gamma/2 \\ 0 & 1 - 2f(\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma/2 & 0 & 0 & f(\gamma) \end{pmatrix},$$

where $f(\gamma) = \begin{cases} \gamma/2, & \text{for } \gamma \geq 2/3 \\ 1/3, & \text{for } \gamma < 1/3 \end{cases}$. The nonlocality $N(\rho_{\text{mems}})$, entanglement of formation $E(\rho_{\text{mems}})$ and Bell-CHSH violation $B(\rho_{\text{mems}})$ for this set of states with different $\gamma$ are plotted in Fig. 2 (b). From the figure, the difference between our scheme and the one using Bell inequality violation to quantify nonlocality is obvious: according to our quantification, the nonlocality of $\rho_{\text{mems}}$ arises when $\gamma > 0.35$, while the Bell inequality approach fails to quantify the nonlocality for $0.35 \leq \gamma \leq \frac{\pi}{2}$ since the state does not violate the CHSH inequality for this region. Moreover, it is worthwhile to note that $E(\rho_{\text{mems}}) > 0$ and $N(\rho_{\text{mems}}) = 0$ for the region $0 < \gamma < 0.35$, indicating that for this region, the state $\rho_{\text{mems}}$ is entangled, but it admits LHV model. This provides us a fresh evidence from another aspect that entanglement and nonlocality are different.

The third example concerns the two-qutrit system. Let us first briefly introduce the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [20] and the von Neumann entropy. For the two-qutrit system, the CGLMP inequality reduces to:

$$I_3 = [P(A_1 \dagger B_1) + P(B_1 \dagger A_2 + 1) + P(A_2 \dagger B_2)$$
$$+ P(B_2 \dagger A_1)] - [P(A_1 \dagger B_1 - 1) + P(B_1 \dagger A_2)$$
$$+ P(A_2 \dagger B_2 - 1) + P(B_2 \dagger A_1 - 1)] \leq 2.$$
entropy and the Bell-CGLMP inequality violation with different $\xi$ and $\beta$ in Fig. 3. Note that the nonlocality $\mathcal{N}(\ket{\psi})$ is estimated by $\mathcal{N}^{[2,2]}(\ket{\phi})$. For the convenience of drawing, the Bell-CGLMP inequality violation and the von Neumann entropy of the reduced density operators are divided by $4\sqrt{2}$ and $2\ln 3$, respectively.

In other words, $\mathcal{B}(\ket{\phi}) = \frac{\mathcal{L}^2(\ket{\phi})}{4\sqrt{2}}$ and $\mathcal{V}(\ket{\phi}) = \frac{\mathcal{V}(\ket{\phi})}{2\ln 3}$. From the figure, $\mathcal{N}(\ket{\phi})$, $\mathcal{B}(\ket{\phi})$ and $\mathcal{V}(\ket{\phi})$ behave quite differently as $\xi$ and $\beta$ vary. In this case, the maximal nonlocality $\mathcal{N}(\ket{\phi})$ does not occur at the maximal entangled state. Actually, for the maximal two-qutrit entangled state $\ket{\Phi} = \frac{1}{\sqrt{2}} (\ket{00} + \ket{11} + \ket{22})$, the nonlocality is $\mathcal{N}(\ket{\Phi}) = \frac{1}{5}$, which is less than $1/2$, the nonlocality of the state $\ket{\phi}_{\xi=\pi/2, \beta=\pi/4} = \frac{1}{\sqrt{2}} (\ket{00} + \ket{11})$.

It is worthwhile to point out that for all the three examples, the nonlocality $\mathcal{N}(\rho)$ is estimated by $\mathcal{N}^{[2,2]}(\rho)$. Of course, one can use $\mathcal{N}^{[M_a,M_b]}(\rho) (M_a, M_b > 2)$ as the approximation of $\mathcal{N}(\rho)$ and the bigger $M_a$ and $M_b$, the better the approximation. However, as $M_a$ and $M_b$ become bigger, it might need more time and EMS memory to run the SeDuMi program on the computer. Actually, our numerical results show that for the states under discussion, the variations of $\mathcal{N}^{[M_a,M_b]}(\rho)$ with different $M_a$ and $M_b$ are very small. Thus, we believe $\mathcal{N}^{[2,2]}(\rho)$ is sufficient for a rough estimation of $\mathcal{N}(\rho)$.

In summary, we introduced a reasonable scheme with a clear geometric significance to quantify nonlocality based on the LHV models. The scheme is numerically computable due to the systematic approach of constructing LHV models for quantum states and the powerful numerical methods for SDP problems. To illustrate how the method works, we have provided some examples concerning two-qubit and two-qutrit systems. Our approach does not need any Bell inequality, thus it naturally circumvent the previously mentioned disadvantages of using Bell inequality violation as nonlocality measure. This scheme sheds a new light on the quantitative understanding of quantum nonlocality. Interestingly, we have found that it may be very useful in quantum phase transitions [21]. It would also be interesting and significant to explore its possible applications in quantum information and computation science, which we shall investigate subsequently.

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FIG. 3: (Color online) Numerical results of nonlocality, Bell-CGLMP inequality violation and von Neumann entropy for the two-qutrit state $\ket{\phi} = \sin \xi \sin \beta \ket{000} + \sin \xi \cos \beta \ket{111} + \cos \xi \ket{222}$. The nonlocality $\mathcal{N}(\ket{\phi})$ is estimated by $\mathcal{N}^{[2,2]}(\ket{\phi})$. The Bell-CGLMP inequality violation and the von Neumann entropy are normalized by dividing $4\sqrt{2}$ and $2\ln 3$, respectively. In this figure, we have only plotted the curves with $\xi = \pi/6$, $\xi = \pi/3$ and $\xi = \pi/2$. 

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