Multi-Sensor Scheduling for State Estimation with Event-Based, Stochastic Triggers

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Abstract—In networked systems, state estimation is hampered by communication limits. Past approaches, which consider scheduling sensors through deterministic event-triggers, reduce communication and maintain estimation quality. However, these approaches destroy the Gaussian property of the state, making it computationally intractable to obtain an exact minimum mean squared error estimate. We propose a stochastic event-triggered sensor schedule for state estimation which preserves the Gaussianity of the system, extending previous results from the single-sensor to the multi-sensor case.

I. INTRODUCTION

Networked Control Systems (NCSs), spatially distributed systems where sensors, actuators, and controllers exchange information over a shared, bandlimited communication network, have become a topic of significant interest in both academia and industry. As noted by Astrom and Bernhardsson [9], the use of NCSs in practice provides for flexible architecture and reduces costs in installation and maintenance. Thus, NCSs have been used in several applications including public transportation, health care, and mobile sensor networks. Nonetheless, remote state estimation remains a significant challenge in NCSs [2]. Traditionally, state estimates are computed at an estimation center using information from sensors which sample and send measurements periodically. While it is reasonable to assume that remote state estimation centers are well equipped, in most cases, sensors have a limited power supply and are difficult to replace. Moreover, bandwidth constraints in a communication network may restrict the number of sensors which can communicate at any given time [3], [4], [5]. One way to address these issues is to simply reduce the communication rate. This solution however degrades estimation quality. In this paper, we propose a sensor scheduling scheme which allows us to achieve a desired tradeoff between communication rate and estimation performance. Specifically, we design a stochastic multi-sensor event-based schedule for the remote state estimation problem which extends the single sensor results from [6].

Before continuing, we briefly document recent attempts to address the problem of remote estimation via sensor scheduling. We first examine offline schemes where sensors are scheduled based on system parameters prior to use. Yang et. al. [7] determined that given fixed communication constraints, an optimal deterministic offline schedule should allocate sensor transmission times as uniformly as possible over a finite horizon time. Moreover, Shi et. al. [8] specifically considered the 2-sensor problem with bandwidth constraints and found that a periodic sensor schedule minimized average error covariance. In addition to offline designs, previous work has considered event-based designs, where sensor transmissions are scheduled in real time based on an occurrence related to a sensor measurement or current system parameters. Astrom and Bernhardsson [9] show that for certain systems, event based sampling offers better performance than periodic sampling. Additionally, Imer et. al. [10] consider a single sensor sequential estimation problem where the state is represented by an independent identically distributed (i.i.d) process. The authors assume communication is limited over a finite horizon and propose a stochastic solution. Furthermore, Xu et. al. [11] consider scheduling a single, smart sensor which computes and sends a local estimate of the state. The authors propose a stochastic event trigger, where the rate of transmission is a quadratic function of the difference between the state estimate computed at the sensor and the estimate computed at the remote estimator.

While not utilized in [6], [10], and [11], event-based approaches can allow the estimator to extract information about the state from the absence of a measurement, and thus improve its estimate. For instance, Ribeiro et. al. [12] require the transmission of a single bit per observation based on the sign of the innovation and derive an approximate minimum mean squared error (MMSE) estimator. Also, the authors in [13] design a threshold scheme on the normalized innovation vector to trigger communication to the remote estimator, and derive an approximate MMSE estimate. Deterministic schemes as discussed by [12], [13] destroy the Gaussian property of the innovation process in traditional Kalman filtering, thus rendering the closed-form derivation of the exact MMSE estimator computationally intractable. Symmetric triggers such as those proposed in [14] and [6] allow the remote estimator to compute an MMSE estimate. Here, the triggers are designed so that a priori and a posteriori estimates are identical if a measurement is dropped which implicitly requires that the sensor has access to the same information as the estimator. However, this is not feasible in the multi-sensor case without substantially increasing communication in the network.

Han et. al. in [6] incorporate a stochastic decision rule, which not only allows the remote estimator to use information contained in the absence of a measurement, but also maintains the Gaussian distribution of the current state. A key advantage of the proposed method over most deterministic triggers is that in addition to obtaining an exact MMSE estimator, by preserving Gaussianity, [6] maintains an exact distribution of the state $x_k$ and the estimation error $e_k$ for all time $k$. Thus, the proposed stochastic event-based trigger is useful in scenarios where real time error analysis is critical.

In this paper, we extend the same stochastic decision rule to the multi-sensor case where there exists a unique decision variable for each of $m$ sensors. The main contribution of this paper relative to [6], which considers a binary transmit or drop policy for a single trigger, is the derivation of a two-step estimation filter to account for multiple independent triggers, a modified optimization problem to design each trigger, and a realistic simulation example on data center energy management. For this scenario, we also obtain expressions for sensor communication rates and upper and lower bounds on the error covariance. A preliminary study for this paper was previously presented [15]. Here a three-step recursive filter is proposed which computes a state distribution conditioned on all previous information, newly received measurements, and the identity of sensors which do not transmit sequentially. In this article, we obtain an equivalent two-step recursive filter which combines the last two stages, allowing us to directly obtain an posteriori state distribution without any intermediary steps. We also extend [15] by accounting for vector sensor measurements with correlated sensor noise as well as through our optimization problem and simulation example.
The remainder of the paper is organized as follows. Section III formulates the multi-sensor state estimation problem and proposes a stochastic event-based sensor scheduling scheme. Section III introduces a recursive filtering algorithm to obtain the MMSE estimator of the state and its error covariance. Section IV derives results about communication rate and estimation performance. Section V proposes a semi-definite program to intelligently select trigger parameters. Section VI consists of a simulation. A conclusion at the end summarizes future work.

Notation: \( X' \) denotes the transpose of matrix \( X, S^n_k, S^n_{k+1} \) are the sets of \( n \times n \) positive semi-definite and positive definite matrices. When \( X \in S^n_k \), we simply write \( X \geq 0 \) (or \( X > 0 \) if \( X \in S^n_{k+1} \)). \( N(\mu, \Sigma) \) denotes a Gaussian distribution with mean \( \mu \) and covariance matrix \( \Sigma, \mathbb{E}[\cdot] \) denotes the expectation, \( \text{Pr}(\cdot) \) denotes the probability of a random event, \( \rho(\cdot) \) denotes the spectral radius of a matrix. \( \text{diag}(X_1, \ldots, X_s) \) is the block diagonal matrix with square submatrices \( X_1, \ldots, X_s, 1 \) and 0 denote vectors with entries 1 and 0 respectively and \( I_n \) is the identity matrix of size \( n \times n \). Finally, \( \{A\}_0 \) is the matrix obtained by deleting all 0 rows from the matrix \( A \).

II. PROBLEM SETUP

We define the following linear system:

\[
x_{k+1} = Ax_k + w_k, \quad y_k^{(i)} = C_k^{(i)} x_k + v_k^{(i)}, \quad i = 1, \ldots, m.
\]

Here \( x_k \in \mathbb{R}^n \) is the state vector, while \( y_k^{(i)} \in \mathbb{R}^s_i \) is the \( i \)-th of \( m \) vector sensor measurements. In addition, \( w_k \in \mathbb{R}^n \) and \( v_k^{(i)} \in \mathbb{R}^s_i \) are mutually uncorrelated Gaussian noises with covariances \( Q > 0 \) and \( R > 0 \), respectively and \( s_i = \sum_{j=1}^m s_j \). To simplify notation, we denote \( y_k \triangleq [y_k^{(1)} \ldots y_k^{(m)}]' \). The initial state \( x_0 \) is zero-mean Gaussian random variable with covariance matrix \( \Sigma_0 > 0 \) and is uncorrelated with \( w_0 \) and \( v_k^{(i)} \) for all \( k \geq 0 \). We assume that \((A, C)\) is detectable where we define \( C \triangleq [C_1^{(1)}, \ldots, C_m^{(m)}]' \).

To reduce the rate of sensor to estimator communication, we intelligently transmit a fraction of our sensor measurements. Note that we choose to transfer sensor measurements as opposed to local estimates. This reduces computation by a factor as well as possibly the size of packets for \( n > s_i \). We specify \( \gamma_k^{(i)} \in \{0, 1\} \) as the binary decision variable for sensor \( i \) at time \( k \). When \( \gamma_k^{(i)} = 1 \), a transmission occurs while when \( \gamma_k^{(i)} = 0 \), no measurement is sent. Collecting our decision variables over \( m \) sensors, we have \( \gamma_k = [\gamma_k^{(1)}, \ldots, \gamma_k^{(m)}]' \). Also, suppose at each time \( k \), \( I_{n-l_k} \) sensors drop their measurements and \( m - I_{n-l_k} \) sensors transmit their measurements. The sensors which transmit have indices \( p_1, \ldots, p_{m-l_k} \). Define the vector of received measurements \( y_k' \in \mathbb{R}^{m-l_k} \) at time \( k \) by \( y_k' = [y_k^{(p_1)} \ldots y_k^{(p_{m-l_k})}]' \).

To obtain a MMSE estimator given all previous and current measurements, we perform a two-step process. The first step is a time update where we obtain the MMSE estimator of \( x_k \) given the information set up to time \( k - 1 \). This is denoted by \( \hat{x}_{k-1} = \{y_0', \ldots, \gamma_{k-1} y_0', \ldots, y_{k-1}' \} \) where \( \gamma_{k-1} \triangleq 0 \). In the second step, we update our estimate of \( x_k \), using our previous information set, the received measurements at time \( k \), \( y_k' \), and the knowledge that certain sensors did not transmit a measurement at time \( k \), \( \gamma_k \). Thus, we update using \( I_k \).

Given the information set, we define the following estimation parameters:

\[
\hat{x}_{k-1} = \mathbb{E}[x_k | I_{k-1}], \quad P_k = \mathbb{E}((x_k - \hat{x}_{k-1})(x_k - \hat{x}_{k-1})'), \quad \hat{x}_{k} = \mathbb{E}[x_k | I_k], \quad \hat{P}_{k} = \mathbb{E}((x_k - \hat{x}_{k})(x_k - \hat{x}_{k}')), \quad \hat{x}_{k} = \mathbb{E}[x_k | I_k].
\]  

Here \( \hat{x}_{k-1} \) is an \( \text{a priori} \) MMSE estimate and \( \hat{x}_{k} \) is an \( \text{a posteriori} \) MMSE estimate. When all measurements are sent to the estimator, computation of \( \hat{x}_{k} \) and \( P_k \), the error covariance, reduces to the standard Kalman filter, where the Gaussian distribution of the state allows for a simple recursive filter. As done by [6], to maintain the Gaussian distribution of \( x_k \), we consider a stochastic trigger. A stochastic trigger takes a measurement \( y_k^{(i)} \) and computes a function \( \varphi^{(i)} : \mathbb{R}^s_i \rightarrow [0, 1] \) to determine the probability \( \gamma^{(i)} \) that sensor \( i \) does not transmit. While deterministic triggers assign probabilities equal to 1 or 0 for each measurement, the chosen trigger assigns probabilities in \([0, 1]\). To do this, at time \( k \), each sensor \( i \) generates an i.i.d. uniform random variable \( C_k^{(i)} \) over \([0, 1]\) and computes \( \gamma_k^{(i)} \):

\[
\gamma_k^{(i)} = \left\{ \begin{array}{ll}
0 & \text{if } C_k^{(i)} \leq \varphi^{(i)}(y_k^{(i)}), \\
1 & \text{if } C_k^{(i)} > \varphi^{(i)}(y_k^{(i)}),
\end{array} \right.
\]

\[
\varphi^{(i)}(\alpha) = \exp \left( -\frac{1}{2} \alpha' \Sigma_k^{(i)} \alpha \right).
\]

Here \( Y_k^{(i)} = \{y_k^{(i)} \} \) are trigger parameters and we define \( Y_k^{(i)} \) as \( Y_k^{(i)} = \mathbb{E}(|Y_k^{(i)}|) \) to simplify notation. Note that \( \varphi^{(i)}(0) = 0 \) has the shape of a scaled Gaussian distribution. In the next section, we will show this allows the state to remain Gaussian. For the chosen trigger we consider stable systems, i.e., \( \rho(A) < 1 \). If the system is unstable, any sensor \( i \) which measures an unstable state will have \( y_k^{(i)} \) grow unbounded. In this case, by [3] sensor \( i \) will always transmit.\[1\]
We prove Theorem 1 using induction on the distribution $f(x_k | I_{k-1}) \sim N(\hat{x}_k, P_k)$.  

**Case $n = 0$:** For $n = 0$, we have $I_{k-1} = 0$. Thus, $f(x_0 | I_{-1}) = f(x_0) \sim N(0, \Sigma_0)$ and the initial conditions holds.

**Case assume for $n = k$:** We assume that $f(x_k | I_{k-1}) \sim N(\hat{x}_k, P_k)$.

**Case prove for $n = k + 1$:** We first verify the measurement update step.

**Measurement Update Step:** Consider the joint conditional pdf of $x_k$ and $\Lambda_k y_k$ given $I_k$

$$f(x_k, \Lambda_k y_k | I_k) = f(x_k, \Lambda_k y_k | y_k', \Gamma_k \gamma_k = 1, \Lambda_k \gamma_k = 0, I_{k-1}) = f(x_k, \Lambda_k y_k | y_k', \Gamma_k \gamma_k = 0, I_{k-1}).$$

Thus, the joint pdf of our state and unknown measurements $\Gamma_k y_k$ implies that the decision variables $\Gamma_k \gamma_k = 1$.

The second equality follows since the knowledge of the values of sent measurements $\Gamma_k y_k$ implies that the decision variables $\Gamma_k \gamma_k = 1$.

The last equality is derived from Bayes rule.

By our induction assumption, $f(x_k, \Lambda_k y_k, \Gamma_k y_k) \sim \text{Gaussian distributed given } I_{k-1}$. As a result, the conditional distribution $f(x_k, \Lambda_k y_k | y_k', I_{k-1})$ is also Gaussian. We first observe $f(x_k, \Lambda_k y_k, \Gamma_k y_k | I_{k-1})$ has mean $[\hat{x}_k', (\Lambda_k C \hat{x}_k'), (\Gamma_k C \hat{x}_k')]'$ and covariance

$$
\begin{bmatrix}
    P_k^{-1} & P_k^{-1} C^T \Lambda_k' & P_k^{-1} C^T \Gamma_k' \\
    \Lambda_k C P_k^{-1} & \Lambda_k (C P_k C' + R) \Lambda_k' & \Lambda_k (C P_k C' + R) \Gamma_k' \\
    \Gamma_k C P_k^{-1} & \Gamma_k (C P_k C' + R) \Lambda_k' & \Gamma_k (C P_k C' + R) \Gamma_k'
\end{bmatrix}.
$$

Given a joint Gaussian distribution $f(x_k, \Lambda_k y_k, \Gamma_k y_k | I_{k-1})$, it is easy to compute $f(x_k, \Lambda_k y_k, \Gamma_k y_k | I_{k-1})$ which is also Gaussian [16]. The conditional means are

$$
\mu_x = \hat{x}_k + P_k^{-1} (\Gamma_k C)' (\Gamma_k (C P_k C' + R) \Gamma_k')^{-1} \Gamma_k (y_k - C \hat{x}_k),
$$

$$
\mu_y = \Lambda_k C \hat{x}_k
$$

$$
+ \Lambda_k (C P_k C' + R) \Gamma_k (\Gamma_k (C P_k C' + R) \Gamma_k')^{-1} \Gamma_k (y_k - C \hat{x}_k).
$$

Furthermore, the covariance of $x_k$ and $\Lambda_k y_k$ given $\Gamma_k y_k$ and $I_{k-1}$ is

$$
\Sigma_{xx} = P_k^{-1} - P_k^{-1} (\Gamma_k C)' (\Gamma_k (C P_k C' + R) \Gamma_k')^{-1} (\Gamma_k C) P_k^{-1},
$$

$$
\Sigma_{yy} = \Lambda_k (C P_k C' + R) \Lambda_k
$$

$$
- \Lambda_k (C P_k C' + R) \Gamma_k (\Gamma_k (C P_k C' + R) \Gamma_k')^{-1} \Gamma_k (C P_k C' + R) \Lambda_k',
$$

$$
\Sigma_{xy} = P_k^{-1} (\Lambda_k C)' - P_k^{-1} (\Gamma_k C)' (\Gamma_k (C P_k C' + R) \Gamma_k')^{-1} \Gamma_k (C P_k C' + R) \Lambda_k'.
$$

Now that we have obtained $f(x_k, \Lambda_k y_k | \Gamma_k y_k, I_{k-1})$, we also observe that

$$
\begin{align*}
\Pr(\Lambda_k \gamma_k = 0 | x_k, y_k, I_{k-1}) &= \Pr(\Lambda_k \gamma_k = 0 | \Lambda_k y_k) \\
&= \exp \left(-\frac{1}{2} y_k' \Lambda_k' \Lambda_k y_k + \frac{1}{2} \lambda_k \lambda_k y_k y_k' \right).
\end{align*}
$$

Using [8], we can thus obtain the joint probability density function for the state $x_k$ and the dropped measurements $\Lambda_k y_k$. That is we have

$$
f(x_k, \Lambda_k y_k | I_k) = \beta_k^{-1} \exp(-\frac{1}{2} \theta_k),$$

where $\beta_k \in \mathbb{R}$ and $\theta_k \in \mathbb{R}$ are defined respectively as

$$
\beta_k \triangleq \Pr(\Lambda_k \gamma_k = 0 | \Gamma_k y_k, I_{k-1}) \sqrt{\det(\Phi_k)} (2\pi)^n + \Sigma_{k}^\frac{1}{2} s_j),
$$

$$
\theta_k \triangleq 
\begin{bmatrix}
    x_k - \hat{x}_k \\
    \Lambda_k y_k - \hat{y}_k
\end{bmatrix}' 
\begin{bmatrix}
    \Sigma_{xx} & \Sigma_{xy} \\
    \Sigma_{xy} & \Sigma_{yy}
\end{bmatrix}^{-1} 
\begin{bmatrix}
    x_k - \hat{x}_k \\
    \Lambda_k y_k - \hat{y}_k
\end{bmatrix}
+ (\Lambda_k y_k)\Lambda_y \Lambda_y' (\Lambda_k y_k).
$$

We now introduce the following Lemma with proof found in the appendix.

**Lemma 1.** The scalar $\theta_k \in \mathbb{R}$ is given by

$$
\theta_k = \left[ \begin{array}{c}
    x_k - \hat{x}_k \\
    \Lambda_k y_k - \hat{y}_k
\end{array} \right]' \Theta_k^{-1} \left[ \begin{array}{c}
    x_k - \hat{x}_k \\
    \Lambda_k y_k - \hat{y}_k
\end{array} \right] + c_k,
$$

where $\hat{x}_k \in \mathbb{R}^n$, $\hat{y}_k \in \mathbb{R}^n$, $c_k \in \mathbb{R}$ and $\Theta_k \in \mathbb{S}_+^{n+1}$ are given by

$$
\begin{align*}
\hat{x}_k &= \hat{x}_k + P_k^{-1} C (P_k C')^{-1} P_k^{-1} (y_k - C \hat{x}_k), \\
\hat{y}_k &= \hat{y}_k + P_k^{-1} \Sigma_{yy} P_k^{-1} (y_k - C \hat{x}_k),
\end{align*}
$$

$$
\Theta_k = \left[ \begin{array}{cc}
    \Theta_{x x, k} & \Theta_{x y, k} \\
    \Theta_{x y, k}' & \Theta_{y y, k}
\end{array} \right].
$$

Thus, the joint pdf of our state and unknown measurements are given as follows

$$
f(x_k, \Lambda_k y_k | I_k) = \frac{1}{\beta_k} \exp \left(-\frac{1}{2} \theta_k \right)
$$

$$
\times \exp \left(-\frac{1}{2} \begin{bmatrix}
    x_k - \hat{x}_k \\
    \Lambda_k y_k - \hat{y}_k
\end{bmatrix}' \Theta_k^{-1} \begin{bmatrix}
    x_k - \hat{x}_k \\
    \Lambda_k y_k - \hat{y}_k
\end{bmatrix} \right).
$$

Since $f(x_k, \Lambda_k y_k | I_k)$ is a pdf, its integral normalizes to one which implies that $f(x_k, \Lambda_k y_k | I_k)$ are jointly Gaussian. Moreover, this implies that $x_k$ is conditionally Gaussian given $I_k$ with mean $\hat{x}_k$ and covariance $P_k$. Therefore, [5] and [6] hold for the measurement update step.

**Time Update Step:** We have proved $f(x_k | I_k) \sim N(\hat{x}_k, P_k)$. By the conditional independence of $x_k$ and $w_k$, we can verify the time update step

$$
f(x_{k+1} | I_k) = f(A x_k + w_k | I_k) \sim N(A \hat{x}_k, A P_k A' + Q).
$$

Thus, [3] holds. By induction, $f(x_k | I_{k-1}) \sim N(\hat{x}_k, P_k^{-1})$. Moreover, from this result, and the proof of the measurement update step, $f(x_k | I_k) \sim N(\hat{x}_k, P_k)$, which concludes the proof.

**Remark 1.** The estimation filter can be formulated as a Kalman filter with time-varying sensor noise $R + (I - \Psi_k) Y^{-1}$ and innovation $\Psi_k y_k - C \hat{x}_k$. The similarity between the stochastic schedule and Kalman filtering allows for computational simplicity and easy implementation.

**Remark 2.** With an imperfect channel, the estimator will have to differentiate between intended packet drops by the sensor due to the stochastic trigger and unintended drops due to the channel. If packet drops are IID Bernoulli, the state will be distributed according to a Gaussian mixture model corresponding to each possible trajectory of $\gamma_k$. The resulting distribution however is intractable as $k \to \infty$. 

IV. PERFORMANCE ANALYSIS

In proposing an event-based trigger, our goal is to address the trade-off between estimation performance and power consumed through communication by sensor nodes. The communication rate \( \lambda(i) \in [0, 1] \) for sensor \( i \) can be defined as

\[
\lambda(i) \triangleq \lim_{T \to \infty} \sup_{T} \frac{1}{T+1} \sum_{k=0}^{T} \mathbb{E}[v_{1k}(i)].
\]  

(25)

Knowledge of the communication rate \( \lambda(i) \) of each sensor will allow designers to determine the required system bandwidth and to estimate the lifetime of each sensor. To obtain an expression for the communication rate \( \lambda(i) \) for each sensor, we first define \( \Sigma \in \mathbb{R}^{n \times n} \), \( \Pi(i) \) by

\[
\Sigma \triangleq \lim_{k \to \infty} \text{Cov}(x_k) = A \Sigma A' + Q,
\]

\[
\Pi(i) \triangleq \lim_{k \to \infty} \text{Cov}(y_k(i)) = C(i) \Sigma C(i)' + R(i),
\]

where \( R(i) \triangleq \mathbb{E}[v_{1k}(i) v_{1k}'] \). With these results, we now can arrive at an expression for the communicate rate of each sensor with proof in [6].

Theorem 2. Consider a stable linear system (1) with a stochastic event-based sensor schedule given by (3). The communication rate \( \lambda(i) \) for each sensor \( i = 1, \cdots, m \) is given by

\[
\lambda(i) = 1 - \frac{1}{\sqrt{\det(I + \Pi(i)Y(i)^{[i]}(i))}}.
\]  

(26)

We next verify that the properties established for the expected communication rate over several runs, apply to a single sample path, the proof of which is found in [6].

Theorem 3. The following equality almost surely holds.

\[
\lim_{N \to \infty} \frac{1}{T+1} \sum_{k=0}^{T} \frac{\gamma_k(i)}{a_s} = \lambda(i).
\]  

(27)

Furthermore, for any finite integer \( l \geq 0 \), define the event of \( l \) sequential packed drops over all \( m \) sensors \( E_{k,l} \) and the event of \( l \) sequential packet arrivals over all \( m \) sensors \( E'_{k,l} \) as follows

\[
E_{k,l} \triangleq \{ \gamma_k = 0, \cdots, \gamma_{k+l-1} = 0 \},
\]

\[
E'_{k,l} \triangleq \{ \gamma_k = 1, \cdots, \gamma_{k+l-1} = 1 \}.
\]

Then almost surely \( E_{k,l} \) and \( E'_{k,l} \) happen infinitely often.

We next examine the estimation performance by analyzing the statistical properties of \( P_{k}^{-1} \).

Theorem 4. Consider a stable system (1) with scheduler given by (3). Let \( g_{w}(X) \triangleq AXA' + Q - AXC(i)CXC'(i)' + W^{-1} \).

1) There exists an \( M \in \mathbb{R}^{n \times n} \), such that for all \( k \), \( P_{k}^{-1} \) is uniformly bounded above by \( M \).

2) For any \( \epsilon > 0 \), there exists an \( N \) such that for all \( k \geq N \), the following inequalities hold

\[
\lambda - \epsilon I \leq P_{k}^{-1} \leq \lambda + \epsilon I,
\]  

(28)

where \( \lambda \) and \( \lambda \) are the unique solutions \( X = g_{R}(X) \) and \( X = g_{R+Y^{-1}}(X) \) respectively.

3) For any \( \epsilon > 0 \), almost surely for infinitely many \( k \)'s, we have \( P_{k}^{-1} \geq \lambda - \epsilon I \) and almost surely for infinitely many \( k \)'s, we have \( P_{k}^{-1} \leq \lambda + \epsilon I \).

The first statement shows that regardless of the choice of \( Y(i) \) (communication rate), the error covariance is bounded. The second statement obtains upper and lower bounds while the third statement shows that during a sample path, \( P_{k}^{-1} \) will approach these bounds infinitely many times, a consequence of Theorem [3] where we expect long strings of transmissions and drops.

V. OPTIMIZATION OF TRIGGER PARAMETERS

Before we continue, we introduce the following Corollary with proof found in the appendix.

Corollary 1. Define \( \mathcal{T} \triangleq \mathcal{X}C(i)'C(i)X + R + Y^{-1}C(i)X \).

1) For any \( \epsilon > 0 \), \( \exists \) an \( N \) such that for all \( k \geq N \), \( P_{k} \leq \mathcal{T} + \epsilon I \).

2) For any \( \epsilon > 0 \), almost surely for infinitely many \( k \)'s, we have \( P_{k} \geq \mathcal{T} - \epsilon I \).

Thus, it is a worthy goal to design \( Y(i) \) to limit \( \mathcal{T} \). We address the estimation and communication tradeoff by minimizing the system communication rate subject to this bound.

\[
\text{Problem 1: } Y_{\ast}^{(i)} = \arg \min_{Y^{(i)} \geq 0, \ 1 = 1, \cdots, m} \sum_{i=1}^{m} \lambda_{i}, \ 	ext{subject to } \mathcal{T} \leq \Delta.
\]  

(29)

Here, the matrix \( \Delta \) serves as an upper bound on our worse case error covariance, thus providing a robust bound on our estimation quality. Unfortunately, this formulation deals with a nonconvex minimization problem which cannot easily be solved. However, we observe the following result.

Lemma 2. Define \( f(x) \triangleq 1 - (1+x)^{-\frac{1}{2}} \) and \( g(x) = 1 - \exp(x)^{-\frac{1}{2}} \). Given \( \lambda^{(i)} \) from (25), \( \Pi(i) > 0 \) and \( Y(i) > 0 \), the following inequality holds

\[
f \left( \frac{m}{m} \sum_{i=1}^{m} \lambda_{i} \right) \leq \lambda^{(i)} \leq \frac{m}{m} \sum_{i=1}^{m} \lambda_{i}.
\]  

(30)

where \( \lambda^{(i)} \) is the global minimum of Problem 1.

Proof. Let \( u = \sum_{i=1}^{m} u_{i} \) and \( u_{i} = \text{tr} \left( \Pi(i) Y_{\ast}^{(i)} \right) \). We observe that

\[
f(u) \leq \sum_{i=1}^{m} f(u_{i}) \leq \lambda^{(i)} \leq \sum_{i=1}^{m} g(u_{i}) \leq mg \left( \frac{u}{m} \right).
\]  

(31)

The first equality holds for \( u = 0 \). The inequality holds since partial derivatives of \( \sum_{i=1}^{m} f(u_{i}) \) with respect to \( u_{i} \) are greater than or equal to those of \( f(u) \). The second and third inequalities are proved in [6]. Applying Jensen’s inequality to \( g \) which is concave, we get the last inequality.

Since the optimum value of our objective function can be bounded by two increasing functions of \( \sum_{i=1}^{m} \text{tr} \left( \Pi(i) Y^{(i)} \right) \), we propose the following convex relaxation to Problem 1.

\[
\text{Problem 2: } Y_{\ast}^{(i)} = \arg \min_{Y^{(i)} \geq 0, \ 1 = 1, \cdots, m} \sum_{i=1}^{m} \text{tr} \left( \Pi(i) Y^{(i)} \right), \ 	ext{subject to } \mathcal{T} \leq \Delta.
\]  

(32)

There exist challenges with the constraint since \( \mathcal{T} \) is only defined through \( \mathcal{X} \) which itself is defined through an implicit function \( g_{w} \). The following theorem allows us to obtain an equivalent set of constraints and thus formulate the problem as a semi-definite program.

2. \( Y_{\ast}^{(i)} \geq 0 \) is chosen to ensure the problem is feasible for solvers. To ensure \( Y \in \mathbb{S}_{++}^{n} \), consider \( Y_{\ast}^{(i)} \geq \epsilon I \) where \( \epsilon > 0 \).
Theorem 5. The optimal $Y^{(i)}$ satisfying Problem 2 can be found by solving the following problem.

\[
\text{Solve: } Y^{(i)} = \arg \min_{Y^{(i)} \geq 0, \ i = 1, \ldots, m} \sum_{i=1}^{m} \text{tr} \left( \Pi^{(i)} Y^{(i)} \right).
\]

\[
\begin{bmatrix}
Q^{-1} - S + C'R^{-1}C & Q^{-1}A \\
A'Q^{-1} & A'Q^{-1}A + S
\end{bmatrix}
\begin{bmatrix}
R^{-1}C \\
0
\end{bmatrix}
\geq 0,
\]

\[
y^{(i)} \geq 0, \quad S \geq \Delta^{-1}.
\]

The proof is found in the appendix.

VI. Numerical Analysis

To assess performance, we consider a thermal model for data centers, introduced in [17]. The size of data centers has been growing both in number and capacity, resulting in rising energy costs. To conserve energy, [17] considers the following thermal model for energy control.

\[
\begin{bmatrix}
T_{o}^{\text{out}} \\
T_{o}^{\text{in}}
\end{bmatrix}
= \begin{bmatrix}
k_{s} \Psi_{ss} - 1 & k_{s} \Psi_{sc} & k_{s} \Psi_{so} \\
k_{c} \Psi_{cs} & k_{c} \Psi_{cc} - 1 & k_{c} \Psi_{co} \\
k_{o} \Psi_{os} & k_{o} \Psi_{oc} & k_{o} \Psi_{oo} - 1
\end{bmatrix}
\begin{bmatrix}
T_{o}^{\text{out}} \\
T_{o}^{\text{in}}
\end{bmatrix}
+ Bu,
\]

\[
\begin{bmatrix}
T_{a}^{\text{out}} \\
T_{a}^{\text{in}}
\end{bmatrix}
= \begin{bmatrix}
\Psi_{aa} & \Psi_{ac} & \Psi_{ao} \\
\Psi_{ca} & \Psi_{cc} - 1 & \Psi_{co} \\
\Psi_{oa} & \Psi_{oc} & \Psi_{oo}
\end{bmatrix}
\begin{bmatrix}
T_{a}^{\text{out}} \\
T_{a}^{\text{in}}
\end{bmatrix}
+ Du.
\]

Here the state $x$ is a collection of output temperatures of devices while the measured values $y$ are the input temperatures of devices which require multiple sensors. The subscripts represent different nodes under consideration, where ‘s’ corresponds to servers, ‘c’ corresponds to air conditioners, and ‘o’ corresponds to other devices. The inputs include a reference temperature for the air conditioners, power consumed, and temperature of heat sources. $\Psi$ gives weight to how the temperature output of each node affects the temperature into each node and $k$ is a set of thermal constants. Addressing the trade-off between estimation and communication in this example will reduce energy expenditures and data storage necessary for thermal control.

To obtain a model consistent with [1], we linearize the system around its stable equilibrium, and assume the inputs remain at or near their equilibrium values for all time, a valid assumption during the night or backup periods. Furthermore, we sample the system at a rate of $1/100$ Hz. We consider a system with 16 servers, 3 air conditioners, and 1 other device. The matrices $Q$ and $R$ are generated as a product of a random matrix with entries uniform from 0 to 1 multiplied by its transpose. The matrices are scaled so that the average magnitude of error in $w_{k}$ is 0.1 Kelvin and in $v_{k}$ is 0.5 Kelvin. In Fig 1, we plot the mean squared error in the state estimate as a function of the average communication rate, where each data point is obtained over a run of 10,000 trials. We consider 3 main designs. We first consider a random design where for each sensor at each time step, the probability of transmission is $\lambda_{avg}$. We also consider a stochastic design where each sensor communicates at the same rate, and an optimized design from Problem 2. Also shown are upper and lower bounds for the un-optimized approach. In Fig 2, we plot the percent improvement of the stochastic designs relative to the random design in terms of the mean squared error plotted in Fig 1. An un-optimized design provides as much as 15% improvement, while the optimized design offers as much as 30% improvement.

VII. Conclusion

In this paper we considered a stochastic event trigger for the sensor scheduling problem in multi-sensor networked systems. The stochastic trigger has inherent advantages over offline triggers which can not improve estimates using information contained by the absence of a measurement. Moreover, it maintains the Gaussian properties of the state, an advantage over previous event triggered approaches. We thus could derive a recursive filter to obtain the MMSE estimator and error covariance. Additionally, we obtained an expression for sensor communication rate as well as asymptotic bounds for our error covariance. Finally, we introduced an optimization problem that will allow designers to reduce the overall communication rate in the system subject to some upper bound on the worst case error covariance. Future work consists of considering the stochastic trigger in a system with control inputs and incorporating inter-sensor cooperation.

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VIII. APPENDIX

Proof. (Lemma 1)

We begin by observing the following identities associated with $\Lambda_k$ and $\Gamma_k$.

1) $\Gamma_k \Gamma_k^T = I_{n \times n} - \mu_k \mu_k^T$
2) $\Lambda_k \Lambda_k^T = I_{n \times n} - \mu_k \mu_k^T$
3) $\Gamma_k \Gamma_k = I_m - \Lambda_k \Lambda_k = \Psi_k$
4) $\Lambda_k \Gamma_k = 0$.

We note that sum of following two quadratic forms can be expressed as

$$
(x - \mu_1)' \Sigma_1 (x - \mu_1) + x' \Sigma_2 x
= x' (\Sigma_1 + \Sigma_2) x - 2x' (\Sigma_1 \mu_1 + \mu_1 ' \Sigma_1 \mu_1),
= x' (\Sigma_1 + \Sigma_2)^{-1} \Sigma_1 (x - \Sigma_1 + \Sigma_2)^{-1} \Sigma_1 x
+ \mu_1 ' (\Sigma_1 - \Sigma_1 (\Sigma_1 + \Sigma_2)^{-1} \Sigma_1 \mu_1)1.
$$

Setting $Y \triangleq \begin{bmatrix} 0 & 0 & \Lambda_k Y \Lambda_k' \end{bmatrix}$, and noting that

$$
y_k' \Lambda_k' \Lambda_k y_k = \begin{bmatrix} x_k' \Lambda_k' y_k & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \Lambda_k Y \Lambda_k' \\ \Lambda_k' y_k & \Lambda_k Y \Lambda_k' \end{bmatrix},
$$

we can directly apply (35) to (17) to obtain

$$
\theta_k = \begin{bmatrix} x_k - \bar{x}_k \Lambda_k y_k - \bar{y}_k \end{bmatrix}^T \Theta_k^{-1} \begin{bmatrix} x_k - \bar{x}_k \Lambda_k y_k - \bar{y}_k \end{bmatrix} + c_k,
$$

where we have

$$
\Theta_k = \begin{bmatrix} \Phi_k + \hat{Y} \end{bmatrix}^{-1}, \begin{bmatrix} \bar{x}_k \bar{y}_k \end{bmatrix} = \begin{bmatrix} \mu_k \mu_y \end{bmatrix},
$$

$$
c_k = \begin{bmatrix} \mu_x \mu_y \end{bmatrix}^T \begin{bmatrix} \Phi_k - \Phi_k^{-1} (\Phi_k + \hat{Y})^{-1} \Phi_k \end{bmatrix} \begin{bmatrix} \mu_x \mu_y \end{bmatrix}.
$$

We now attempt to verify (21). From Lemma 1 of [13], we can directly obtain

$$
\Theta_k = \begin{bmatrix} \Sigma_{xx} & 0 \\
0 & 0 \end{bmatrix} + \begin{bmatrix} -\Sigma_{xy} (\Sigma_{yy} + \Lambda_k Y \Lambda_k' \Sigma_{yy} - 1)^{-1} \Sigma_{xy} \\
\Sigma_{xx} (I + \Lambda_k Y \Lambda_k' \Sigma_{yy} - 1)^{-1} \end{bmatrix},
$$

Defining $Z_k \triangleq CP_k C' + R$ and $W_k \triangleq \Gamma_k' (\Gamma_k Z_k \Gamma_k')^{-1} \Gamma_k$ and substituting equations (13), (14), (15), we obtain

$$
\Sigma_{xx} - \Sigma_{xy} (\Lambda_k Y \Lambda_k')^{-1} \Sigma_{xy} = P_k - P_k C' (V_k + U_k) C P_k^-,
$$

where

$$
V_k \triangleq W_k + (\Lambda_k' - W_k Z_k \Lambda_k') \times (\Lambda_k (Z_k - Z_k W_k \Lambda_k')^{-1} (\Lambda_k - \Lambda_k Z_k W_k)),
$$

$$
U_k \triangleq \Lambda_k' - W_k Z_k \Lambda_k' (\Lambda_k (Z_k - Z_k W_k Z_k \Lambda_k')^{-1} (\Lambda_k - \Lambda_k Z_k W_k))^{-1} \times (\Lambda_k Y \Lambda_k' + (\Lambda_k Z_k - Z_k W_k \Lambda_k')^{-1} (\Lambda_k - \Lambda_k Z_k W_k)),
$$

(42)

Since $\Lambda_k Z_k (\Lambda_k' - W_k Z_k \Lambda_k') (\Lambda_k (Z_k - Z_k W_k Z_k \Lambda_k')^{-1} = I$ and $\Lambda_k$ has a unique right inverse equal to $\Lambda_k'$, we have

$$
(\Lambda_k' - W_k Z_k \Lambda_k') (\Lambda_k (Z_k - Z_k W_k Z_k \Lambda_k')^{-1} = Z_k^{-1} \Lambda_k'.
$$

(43)
Furthermore, we observe that
\[
(\Lambda_k Z_k^{-1} A_k'(\Lambda_k(Z_k - Z_k W_k)Z_k) A_k') = \\
= \Lambda_k Z_k^{-1} A_k' A_k Z_k A_k' - \Lambda_k Z_k^{-1} \Lambda_k A_k Z_k W_k Z_k A_k' = \\
= \Lambda_k Z_k^{-1} (I - \Gamma_k^T \Gamma_k) Z_k A_k' = \\
= -\Lambda_k Z_k^{-1} (I - \Gamma_k^T \Gamma_k) Z_k \Gamma_k^T (\Gamma_k^T (I - \Lambda_k Z_k W_k)Z_k A_k') \\
= -\Lambda_k A_k' - \Lambda_k Z_k^{-1} \Gamma_k^T \Gamma_k Z_k A_k' + \Lambda_k Z_k^{-1} \Gamma_k^T \Gamma_k Z_k A_k' \\
= -\Lambda_k A_k' + Z_k^{-1} \Gamma_k^T \Gamma_k Z_k A_k' \\
= I. 
\]
(44)
Thus, from (43), we obtain
\[
V_k = W_k + Z_k^{-1} A_k'(\Lambda_k - \Lambda_k A_k W_k), \\
= Z_k^{-1} A_k' A_k + (I - Z_k^{-1} A_k' A_k) W_k, \\
= Z_k^{-1} A_k' A_k + (I - Z_k^{-1} (I - \Gamma_k^T \Gamma_k) Z_k) W_k, \\
= Z_k^{-1} A_k' A_k + Z_k^{-1} \Gamma_k^T \Gamma_k Z_k (\Gamma_k^T (\Gamma_k^T Z_k) Z_k A_k')^{-1} -\Gamma_k, \\
= Z_k^{-1} A_k' A_k + Z_k^{-1} \Gamma_k^T \Gamma_k, \\
= Z_k^{-1}. 
\]
Moreover, from (43) and (44), we have
\[
U_k = -Z_k^{-1} A_k'(\Lambda_k Y A_k' + \Lambda_k Z_k^{-1} A_k')^{-1} A_k Z_k^{-1}. 
\]
From the matrix inversion lemma
\[
U_k + V_k = (CP_k) C' + R + (I - \Psi_k Y)^{-1}. 
\]
As such, (27) is verified. Next from (27)
\[
\begin{bmatrix}
\bar{x}_k \\
\bar{y}_k
\end{bmatrix} = \left( I + \Phi_k \bar{Y} \right)^{-1} \begin{bmatrix}
\mu_x \\
\mu_y
\end{bmatrix}, \\
= \left( I \begin{bmatrix}
\Sigma_{xx} & \Sigma_{xy} \\
\Sigma_{yx} & \Sigma_{yy}
\end{bmatrix} \begin{bmatrix}
0 \\
0
\end{bmatrix} \Lambda_k Y A_k' \right)^{-1} \begin{bmatrix}
\mu_x \\
\mu_y
\end{bmatrix}, \\
= I \begin{bmatrix}
\Sigma_{xy} (\Lambda_k Y A_k' + \Lambda_k Y^{-1} A_k')^{-1} \\
(I + \Sigma_{yy} A_k Y A_k')^{-1}
\end{bmatrix} \begin{bmatrix}
\mu_x \\
\mu_y
\end{bmatrix}. 
\]
(48)
Therefore, it can be seen that
\[
\bar{x}_k = \bar{x}_k + P_k C' S_k, 
\]
where
\[
S_k = W_k \Sigma_k (y_k - C \bar{x}_k) \\
- (\Lambda_k' - W_k Z_k A_k') (\Lambda_k (Z_k - Z_k W_k) Z_k) A_k' + \Lambda_k Y^{-1} A_k' \\
\times ((\Lambda_k - \Lambda_k Z_k W_k) C \bar{x}_k + \Lambda_k Z_k W_k y_k).
\]
Noting that \( W_k = W_k \Gamma_k \Gamma_k^T \) and \( \Lambda_k \Gamma_k^T = 0 \), we have
\[
S_k = (W_k - (\Lambda_k' - W_k Z_k A_k') (\Lambda_k (Z_k - Z_k W_k) Z_k)) A_k' \\
+ \Lambda_k Y^{-1} A_k' -1 (\Lambda_k - \Lambda_k Z_k W_k)) (\Gamma_k^T \Gamma_k y_k - C \bar{x}_k). 
\]
Utilizing the matrix inversion lemma
\[
S_k = (U_k + V_k)(\Psi_k y_k - C \bar{x}_k). 
\]
Thus, (19) and (20) associated with \( \bar{x}_k \) and \( \bar{y}_k \) immediately follow. It now remains to verify the expression for \( c_k \) in (20). We first observe by the matrix inversion lemma that
\[
(\Phi_k^{-1} + \bar{Y})^{-1} = \left( \Phi_k^{-1} + \begin{bmatrix}
0 \\
I
\end{bmatrix} \Lambda_k Y A_k' \begin{bmatrix}
0 \\
I
\end{bmatrix} \right)^{-1}, \\
= \Phi_k - \Phi_k \begin{bmatrix}
0 \\
I
\end{bmatrix} \begin{bmatrix}
0 \\
I
\end{bmatrix} \Phi_k \begin{bmatrix}
0 \\
I
\end{bmatrix} \Lambda_k Y^{-1} A_k' \begin{bmatrix}
0 \\
I
\end{bmatrix} \Phi_k, 
\]
Applying (38),
\[
c_k = \left[ \begin{bmatrix}
\mu_x \\
\mu_y
\end{bmatrix} \right] \begin{bmatrix}
0 \\
0 \Lambda_k Y^{-1} A_k' \end{bmatrix}^{-1} \begin{bmatrix}
\mu_x \\
\mu_y
\end{bmatrix}, \\
= \mu_y \Lambda_k Y^{-1} A_k' \end{bmatrix}^{-1} \begin{bmatrix}
\mu_x \\
\mu_y
\end{bmatrix}. 
\]
Proof. (Corollary 1) To begin we define function \( h(X, \Psi) : S_{n}^+ \times (0, 1)^m \to S_{n}^+ \) as
\[
h(X, \Psi) \triangleq X - XC'C + R + (I - \Psi Y)^{-1} C X. 
\]
Using the matrix inversion lemma
\[
h(X, \Psi) = \left( X^{-1} + C' (R + (I - \Psi Y)^{-1} C) C' \right)^{-1}. 
\]
This implies \( h \) is monotonically increasing in \( X \), and maximized for \( \Psi = 0 \). From Theorem 1, we observe that
\[
P_k = h(P_k, \gamma_k^{(1)} \gamma_k^{(2)} \cdots \gamma_k^{(n)} Y^{-1}). 
\]
(51)
By Theorem 4.2, we have that \( P_k \leq \bar{X} + \bar{e}I \) for \( k \geq \tilde{N}(\bar{e}) \). By the monotonicity of \( h \), we obtain
\[
P_k \leq h(\bar{X} + \bar{e}I, 0) \leq \bar{X} - \bar{X} C'C (\bar{X} C'C + R + Y^{-1})^{-1} C \bar{X} + \bar{e} I \\
= \bar{P} + \bar{e} I, 
\]
for \( k \geq \tilde{N}(\bar{e}) = N(\bar{e}) \). We must now show that \( P_k \) approaches this upper bound infinitely many times. To do this, define function \( \bar{h} : S_{n}^+ \to S_{n}^+ \) as
\[
\bar{h}(X) \triangleq (AXA' + Q - (AXA' + Q) C' (CXA' + Q) C + R + Y^{-1})^{-1} C(XA' + Q) = h(AXA' + Q, 0). 
\]
(53)
Note that \( h \) is monotonically increasing in \( X \) since \( h \) is monotonically increasing in its first argument and \( AXA' + Q \) is monotonically increasing in \( X \). Utilizing Proposition 1 of [6], we know there exists an \( \ell > 0 \) such that
\[
\bar{h}(0) \geq \bar{X} - \bar{X} C'C (\bar{X} C'C + R + Y^{-1})^{-1} C \bar{X} - \ell I = \bar{P} - \ell I. 
\]
If event \( \bar{E}_{k,t} \) occurs, then we know that
\[
P_{k+1} = \bar{h}(P_k) \geq \bar{h}(0) \geq \bar{P} - \ell I. 
\]
(54)
By Theorem 3, the event \( \bar{E}_{k,t} \) almost surely occurs infinitely often and thus the result holds.

Proof. (Theorem 5) We first assert that the following two statements are equivalent.

1. \( \bar{P} \leq \Delta \).
2. There exists \( 0 < U \leq \Delta \) such that \( U \leq \bar{h}(U) \).

The first statement implies the second by taking \( U = \bar{P} \). Noting the monotonicity of \( \bar{h} \) and the convergence of \( \bar{h} \) to the fixed point \( \bar{P} \), the second statement implies the first by repeatedly applying \( \bar{h} \). Take \( S = U^{-1} \). Then by the matrix inversion lemma, the following statements are equivalent.

1. \( \bar{P} \leq \Delta \),
2) There exists $S \geq \Delta^{-1}$ such that $0 \leq (AS^{-1}A' + Q)^{-1} + C'(R + Y^{-1})^{-1}C - S$.

By the matrix inversion lemma,

$$(AS^{-1}A' + Q)^{-1} = Q^{-1} - Q^{-1}A(S + A'Q^{-1}A)^{-1}A'Q^{-1}.$$  

Thus, using Schur’s condition for positive definiteness we have that the following statements are equivalent.

1) $\bar{P} \leq \Delta$,
2) There exists $S \geq \Delta^{-1}$ such that

$$
\begin{bmatrix}
Q^{-1} + C'(R + Y^{-1})^{-1}C - S & Q^{-1}A \\
A'Q^{-1} & A'Q^{-1}A + S
\end{bmatrix} \geq 0, \\
A'Q^{-1}A + S > 0.
$$

(55)

Thus, by Schur’s condition for positive definiteness we have that the following statements are equivalent

1) $\bar{P} \leq \Delta$,
2) There exists $S \geq \Delta^{-1}$ such that

$$
\begin{bmatrix}
Q^{-1} + C'R^{-1}C - S & Q^{-1}A \\
A'Q^{-1} & A'Q^{-1}A + S
\end{bmatrix} - \\
\begin{bmatrix}
C'R^{-1} \\
0
\end{bmatrix}(R^{-1} + Y)^{-1} \begin{bmatrix}
R^{-1}C \\
0
\end{bmatrix} \geq 0.
$$

Thus, by Schur’s condition for positive definiteness we have that the following statements are equivalent

1) $\bar{P} \leq \Delta$,
2) There exists $S \geq \Delta^{-1}$ such that

$$
\begin{bmatrix}
Q^{-1} + C'R^{-1}C - S & Q^{-1}A & C'R^{-1} \\
A'Q^{-1} & A'Q^{-1}A + S & 0 \\
R^{-1}C & 0 & R^{-1} + Y
\end{bmatrix} \geq 0, \\
R^{-1} + Y > 0.
$$

Note that the latter statement is given for free since $R^{-1} > 0$. The theorem follows immediately. \qed