I. INTRODUCTION

Models that display spinless p-wave pairing are known to exist in both Abelian and non-Abelian topological phases. The systems are BdG (Bogoliubov de-Gennes) type topological insulator, and therefore support gapless chiral modes at the edges between Abelian and non-Abelian domains. When these edge modes have zero energy they are known to be Majorana fermions. In addition to this the bulk of a non-Abelian phase is capable of supporting Majorana zero modes. In particular we will see in what situations the exact zero energy Majorana edge modes exist. On a cylinder, and for the particular instances where the Abelian phase of the model is the full vacuum, we have been able to exactly solve for the systems edge energy eigensolutions and derive a recursive formula that exactly describes the edge mode structure. Penetration depth is also calculated and shown to be dependent on the momentum of the edge mode. These solutions also describe the overall character of the fully open non-Abelian domain and are excellent approximations at moderate distances from the corners.

FIG. 1. Vortices always appear at the end of a branch cut. Figures (a) and (b) are real vortex configurations. Note that the eigenvalues of the homologically non-trivial symmetries dictate which vortices are connected to each other. In the absence of any vortices the $x$ and $y$ anti-periodic homological conditions are encoded as lines $X(N_y, y) = -1 \forall y$ and $Y(N_x, x) = -1 \forall x$ respectively. With the conventions used in [7] the term $Y(N_x, 0) = -1$ dictates which vortices are connected by the branch cuts. Figures (c) and (d) are “simulated” vortex configurations obtained by varying couplings $J_x$ and $J_y$ and $\kappa$. Where the branch cuts are an afterthought to ensure that the modes are single-valued. We will see that, as expected, these branch cuts connect the topological defects (vortices) of the system. However, through out this story we will attempt to emphasize that it is the branch cuts that are the fundamental objects. For example, it is the branch cuts, and not the vortices, that dictate the fermionic behavior of the system. This perspective also holds on the boundaries between Abelian and...
non-abelian domains. For example we will see that it is the
number of branch cuts through those edges that dictate the
character of the modes found there. In the second half of
the paper we will analyse the zero energy bulk modes and the
zero energy and chiral edge modes found in the model. Our
analysis of edge modes is valid for both cylindrical and fully
open boundary conditions but is based on the consistency rela-
tions between homologically trivial excitations (vortices) and
the homologically non-trivial excitations on a torus. We first
introduce the cylindrical system and describe the general char-
acter of the modes found in this case. The general conclusion
is that exact zero modes only form on edges that are inter-
sected by an even number of branch cuts. In addition to this
we see for the hard boundary condition (i.e. where the Abelian
domain is exactly the vacuum), that there are exact solutions
for the BdG equations. We use these solutions to examine
the mode penetration depth as a function of the Hamiltonian
parameters and the mode momenta along the edge.

We finally extend the general analysis to fully open rectan-
gular boundary conditions we see that exact zero modes only
form in this case when there is an odd number of branch cuts
through the domain. The reason for the difference is an extra
phase factor that is contributed at the corners of the system. At
moderate distances from the corners however the exact solu-
tions for the cylindrical hard boundary system are an excellent
approximation for the open system eigenmodes. These results
are general agreement with Ref. 2, and Ref. 7.

II. FERMIONIC FORMULATION

It was shown in 21 that each vortex sector of the honeycomb
lattice model can be written as

\[ H = H_0 + \sum_q \sum_l P_{q}^{(l)} \]

where in terms of fermions we can write

\[
H_0 = J_x \sum_q X_q (c_q^\dagger - c_q)(c_{q+i}^\dagger + c_{q-i}) \\
+ J_y \sum_q Y_q (c_q^\dagger - c_q)(c_{q+\uparrow}^\dagger + c_{q-\uparrow}) \\
+ J_z \sum_q (2c_q^\dagger c_q - I),
\]

(1)

where we have used the shorthand \( q \rightarrow q + n_x, q \uparrow \rightarrow q + n_y \)
and \( q \nearrow \rightarrow q + n_y + n_x \). In the plane, \( Y_q = I \) for all \( q \) and
\( X_q \) is defined as

\[
X_{x,y} = \prod_{y=0}^{y-1} W_{x,y'}.
\]

The terms \( P_{q}^{(l)} \) are explicit \( T \)-symmetry breaking terms, the
fermionic form of which was also derived in 21. For simplic-
ity in this work we will retain only terms \( P_{q}^{(1)}, P_{q}^{(2)}, P_{q}^{(3)} \) and
\( P_{q}^{(4)} \). These terms are sufficient to generate the required non-
abelian phase and lead to more symmetrical solutions. Explicitly
these terms are

\[
P_{q}^{(1)} = -i\kappa X_q (c_q^\dagger - c_q)(c_{q+i}^\dagger - c_{q-i}) \\
P_{q}^{(2)} = -i\kappa X_q (c_q^\dagger + c_q)(c_{q+i}^\dagger + c_{q-i}) \\
P_{q}^{(3)} = +i\kappa Y_q (c_q^\dagger - c_q)(c_{q+\uparrow}^\dagger - c_{q-\uparrow}) \\
P_{q}^{(4)} = +i\kappa Y_q (c_q^\dagger + c_q)(c_{q+\uparrow}^\dagger + c_{q-\uparrow})
\]

(3)

The Jordan-Wigner convention used to define the fermions
is directly responsible for how vorticity is encoded in the
fermionic system. For the string convention chosen in 21 the
vorticity is encoded in the fermionic Hamiltonian through the
condition (2). On a torus there are additional homologically
non-trivial degrees of freedom which also need to be deter-
mined consistently with the condition (2). These homolog-
ically non-trivial are encoded the \( X_q \) and \( Y_q \) values at the
boundary of the system. Recently we have extended this
Jordan-Wigner method to deal with the Yao-Kivelson 3-12 lat-
tice variant of the model.

The consistency relations provided in Ref. 21 have an
interesting pictorial representation which leads us naturally to
the concept of branch cuts and a less restrictive understand-
ing of vorticity. For any vortex arrangement we see that there
are lines of \( X_q = -1 \) and \( Y_q = -1 \) which together connect
vortices in pairs. In Figure 1 we have provided a number of
examples.

On an open plane we no longer have these homologically
non-trivial symmetries but neither do we have the condition
that vortices are created in pairs: \( \prod_q W_q = 1 \). In this case
valid vortex sectors can be encoded using the following guide-
lines.

- The vortex free sector (\( W_q = 1 \forall q \)) is encoded as \( X_q = 1 \forall q \).
- A single isolated vortex at position \( q \) is encoded with
  \( X_q = 1 \) everywhere except for a single line of \( X_{x,y} = -1 \)
  starting at \( y + n_y \) and extending to infinity.
- When two vortices occur at different \( x \)-positions there
  are two unique strands of \( X_q = -1 \) connecting them
  both to infinity.
- If two vortices occur at different \( y \)-positions but with
  the same \( x \) a line of \( X_q = -1 \) connects them together.

One can ‘simulate’ the change of vortex sectors by altering
the coupling constants (the \( J_x \) and \( J_y \)) on unique link. Thus
by changing the sign of \( J_x \) at \( q \) one effectively changes the
gauge encoding \( X_q \). Strictly speaking this does not change the
vortex sector of the Hamiltonian however. With our fermion-
ization convention, and on a plane, there is no vortex sector
which would correspond to the change \( J_y \rightarrow -J_y \) at \( q \).

From now on we will take \( J_x = J_y = J \), dropping the sub-
script and take the viewpoint used in 24 where, by changing
the coupling strengths, we can simulate changing the vortex
configurations. In what follows however, and only for conve-
nience, we will generally continue to regard the \( J \) and \( \kappa \) terms
as constant across the lattice and allow vorticity to be encoded in the $X$ and $Y$ terms. With this perspective it is easier to appreciate that truly meaningful objects in this story are not the vortices themselves but the connected strings of $-1$'s defined on the $X_q$ and $Y_q$ matrices. Indeed as we have already shown these strings take on the role of branch cuts in our fermionic Hamiltonian and will see later that it is their ends that give rise to localized zero modes. From this perspective we can say that zero modes are only associated with vortices because a branch cut always happens to end there.

In addition to the vortex zero-modes we will also see in what follows that it is the branch cuts that are directly responsible for the appearance of the single extended zero mode that occurs at the interface between abelian and non-abelian phases when an odd number of (ordinary localized) zero-modes are in the non-abelian bulk. The parameter $J$ dictates which phase we are in. For $J < J_z/2$ we are in the abelian phase and for $J > J_z/2$ we are in the non-Abelian phase if $\kappa \neq 0$. In what follows we will specify the $J$ and $\kappa$ values in the Abelian domains as $J_A$ and $\kappa_A$ respectively.

### III. BULK MAJORANA FERMION ZERO MODES

In this section we will briefly discuss the bulk Majorana modes found at the end of the branch-cuts. We will not however discuss the detailed structure of the bulk modes other than to present some numerical calculations. In later sections however we will demonstrate how the structure can be seen as a limiting case of edge modes found between domains of Abelian and non-Abelian topological phase.

We begin by presenting the Bogoliubov-De Gennes formalism. The full position space Hamiltonian can be written in the form

$$ H = \frac{1}{2} \sum_{q,q'} \left[ c^\dagger_{q} c_{q'} \right] \left[ \begin{array}{cc} \xi_{qq'} & \Delta_{qq'} \xi_{qq'}^* \\ \Delta_{qq'} & -\xi_{qq'}^* \end{array} \right] \left[ \begin{array}{c} c_{q'} \\ c^\dagger_{q} \end{array} \right]. $$

(7)

This system can be diagonalized by solving the Bogoliubov-De Gennes eigenvalue problem

$$ \left[ \begin{array}{cc} \xi^* & \Delta \\ \Delta^* & -\xi \end{array} \right] \left[ \begin{array}{c} U \ V^* \\ V \ U^* \end{array} \right] = \left[ \begin{array}{c} E \ 0 \\ 0 \ -E \end{array} \right] \left[ \begin{array}{c} U \ V^* \\ V \ U^* \end{array} \right], $$

(8)

where the non-zero entries of the diagonal matrix $E_{nm} = E_n \delta_{nm}$ are the quasi-particle excitation energies. The Bogoliubov-Valatin quasi-particle excitations are

$$ \left[ a_1^\dagger, ..., a_M^\dagger, \ a_1, ..., a_M \right], $$

(9)

$$ \left[ c_1^\dagger, ..., c_M^\dagger, \ c_1, ..., c_M \right], $$

(10)

which after inversion and substitution into (7) give

$$ H = \sum_{n=1}^{M} E_{n} (a_n^\dagger a_n - \frac{1}{2}). $$

(11)

In spinless $p$-waves it is guaranteed by an index theorem that in the the case of the $2N$ well separated vortices we have $2N$ zero energy ($E = 0$) fermionic modes of which $N$ must be identified as $a^\dagger$'s and $N$ as $a^\dagger$'s. It is rather remarkable that one can always choose a superposition of the $2N$ $a^\dagger$ and $a$ zero-modes such that the resulting modes are fully localized around the vortex excitations.

$$ \gamma_j = \sum_{n=1}^{N} \alpha_{jn} a_n^\dagger + \alpha_{j,n+N} a_n $$

$$ = \left[ a_1^\dagger, ..., a_M^\dagger, \ c_1, ..., c_M \right] \left[ \begin{array}{c} u_{q,j} \\ v_{q,j} \end{array} \right]. $$

(12)

It is interesting to note that this localization condition also enforces the condition that $u_{q,j} = e^{-i \Omega n} v_{q,j}^*$. However, if one wishes to call this a Majorana mode $\gamma_j = \gamma_j^*$ it is necessary to multiply the states $(u, v)^T$ by the overall phase $e^{-i \Omega n/2}$ such that $u_{q,j} = v_{q,j}^*$. It was pointed out by Stone and Chung\cite{Stone2006} that the Majorana condition therefore fixes the global phase of the states $\gamma_j$ up to an overall sign $\pm 1$. Understanding how and when this overall sign changes is crucial to understanding how non-abelian statistics arise in this degenerate subspace. In Figure 3 we show the $|u_j|$ and $|v_j|$ position space structure for some different values of $J$ and $\kappa$.

### IV. UNPAIRED MAJORANA MODES AND EDGE STATES

The non-Abelian phase of Kitaev models are Topological insulators of the BdG class, see for example\cite{Kitaev2001}. Roughly speaking this means that we have a bulk energy spectrum which is ‘insulating’ (does not cross the Fermi energy at $E = 0$) and an edge spectrum which is ‘conducting’ (does cross the Fermi energy at $E = 0$). For a careful choice of edge conditions it is possible to analytically treat the ‘conducting’ edge modes.

In order to determine the structure of the modes let us consider an element of an arbitrary eigenstate $a_n^\dagger$ of the BdG Hamiltonian. Each value $u_{x,y}$ is connected to $u_{x+y,\pm 1}$ and $u_{x,y,\pm 1}$ through the non-zero elements of the $\xi$ matrix and to $v_{x+y,\pm 1}$ and $v_{x,y,\pm 1}$ through the non-zero elements of the $\Delta$ matrix. It is quite difficult to say anything generic about the position space structure

$$ (2J_z - E)u_{x,y} + $$

$$ J(u_{x+1,y} + u_{x-1,y} + u_{x,y+1} + u_{x,y-1}) + $$

$$ (J - 2i\kappa)v_{x+1,y} + (J - 2i\kappa)v_{x-1,y} + $$

$$ (J + 2i\kappa)v_{x,y+1} + (J - 2i\kappa)v_{x,y-1} = 0. $$

For edge states on a cylinder we make the reasonable assumption that, in the direction of edge, our modes are plane waves (momentum eigenstates). For example along the lower
edge of a cylindrical non-Abelian domain we have BdG excitations of the form

\[ a_n^\dagger = \mathcal{N} \sum_q e^{\pm ik_x x} (u(y-y_0)c_{q}^\dagger + v(y-y_0)c_{q}) \]  

This state corresponds to a superposition of left (right) moving particles and right (left) moving holes. On a cylinder the allowed values of \( k_x \) are \( 2n\pi/N_y \) when there is an even number of branch cuts through the edge and \( 2(n+1/2)\pi/N_y \) when the number is odd. The basic reasoning is this. A branch cut is accommodated in (15) by a change in signs of the elements \( u \) and \( v \). To keep the energy low then the phase of the mode \( a_n^\dagger \) should abruptly change sign to counteract the sudden sign change in the fermionic Hamiltonian.

On a cylinder this has interesting consequences. Let us start from the toroidal case and open up the \( y \)-boundary above and below the \( y = 0 \) line. We now have two edges which are some distance apart. Translation invariance remains in the \( x \)-direction but is broken in the \( y \)-direction. Recall now that the anti-periodic \( x \)-boundary condition is encoded as a single line \( X_q = -1 \). Thus in the periodic vortex free sector we therefore have chiral edge states with \( k_x = \pm 2n\pi/N_y \). This includes two edge zero-modes, one on each edge. In the anti-periodic vortex free case we have no zero modes. This is because we have a single branch cut intersecting both edges and thus \( k_x = \pm 2(n+1/2)\pi/N_y \).

If a single vortex exists inside the cylinder there must be a branch cut connecting it to either infinity or some other vortex outside the cylinder. If we were originally in the periodic system then the introduction of a branch cut through one wall would destroy periodicity on this edge and we could not have Majorana zero modes. The other edge however would remain unaffected. In the opposite sense if we were originally in the anti-periodic sector then the introduction of a vortex would restore periodicity to one of the edges and thus allow values of \( k_x = \pm 2n\pi/N_y \) to propagate along this wall.

We can extend this reasoning to deal with fully open boundaries (non-Abelian domains within Abelian domains and vice versa). However it is useful to first solve the system exactly on a hard interface \( J_A = 0 \) where the Abelian side of the edge is the full vacuum. In this scenario numerical calculation shows that all low-energy modes satisfy \( u_q = e^{i\theta}v_q \). Thus for modes along the lower edge at \( y = y_0 \) we have

\[ a_n^\dagger = \mathcal{N} \sum_q f(y-y_0)e^{\pm ik_x x}(e^{-i\theta/2}c_q^\dagger + e^{i\theta/2}c_q) \]  

Note that under the conditions \( k_x = 0 \) and \( Im(f) = 0 \) this ansatz is already a Majorana fermion. If one now substitutes this expression into (15) we observe that

\[ E(J, \kappa, k_x) = \frac{8J\kappa}{\sqrt{J^2 + 4\kappa^2}} \sin k_x, \]  

and that, along the bottom edge, \( \theta = \tan^{-1}(2\kappa/J) \). Furthermore one sees that the function \( f \) follows from the recursive relation

\[ f(y_{n+2}) = \frac{1}{\sqrt{J^2 + 4\kappa^2} - J}[d_1 f(y_{n+1}) + d_2 f(y_n)] \]  

where

\[ d_1 = 2Jz + 2J \cos(k_x) - i2J^2 - 4\kappa^2 \sqrt{J^2 + 4\kappa^2} \sin(k_x) \]

\[ d_2 = \sqrt{J^2 + 4\kappa^2} + J \]

Interestingly the structure of the mode depends on the parameter \( J_z \) but the associated energy does not. However this feature is present for the \( (J_A = 0) \) hard boundary condition only. Indeed numerical calculation shows that even the \( \sin(k_x) \) dependence is not exact once the hard boundary condition is relaxed \( (J_A \neq 0) \).

The mode penetration depth can be calculated easily from the recursive relationship (18), see for example FIG. 2. The most salient point is that this depth depends on \( k_x \) and therefore on \( E \). Loosely speaking we can say that the further the energy is from \( E = 0 \) the further it extends into the bulk. An upper limit for the momenta \( k_x \) of the edge modes can be calculated from the condition that \( |d_1 + d_2| < 1 \). Note that this condition also says that we must be inside the non-Abelian domain \( |J| < |J_z|/2 \) for the solution to be normalized.
FIG. 3. (Color online) (a) The function $|f(y_n)|$ for different $k_x$ with $J_z = 1$, $J = -0.7$ and $\kappa = -0.4$. (b) A log plot of the same function $f(y_n)$ again with $J_z = 1$, $J = -0.7$ and $\kappa = -0.4$. (c) The penetration depth $\delta$ as a function of $k_x$ for different values of $J$ and fixed $J_z$ and $\kappa$. Penetration depth goes to infinity approximately when $|d_1 + d_2| > 1$

FIG. 4. A schematic of how $\theta$ in the Majorana edge zero mode varies around an isolated domain of non-Abelian phase. In this model $\theta = \tan^{-1}(2\kappa/J)$. Inside the bulk, and at the corners, we indicate the phase that must be picked up as we move around that corner in the direction indicated by the arrows.

V. FULLY OPEN BOUNDARY CONDITIONS

If we surround a non-Abelian domain with an Abelian domain we have no zero energy states if there are no vortices inside the non-Abelian domain. If we place an odd number of vortices inside the non-Abelian domain then we do have one zero energy edge mode even though an odd number of branch cuts intersect the domain wall.

The key to understanding all this is that phases are also picked up when the wall direction is changed and that these phases all add up to $\pi$, canceling the branch cut phase. A schematic of the phases picked up for the zero mode in a rectangular shaped system is shown in Figure 4. This picture can be arrived at by analyzing each of the edges separately and assuming the appropriate plane wave momentum.

VI. CONCLUSION

We have analysed edge mode structure of the Kitaev Honeycomb model using a Jordan-Wigner fermionization procedure. We see that the branch cuts are naturally defined for us with the single particle Hamiltonian $\xi$ and the order pa-
In these figures we see a clear dependence on penetration depth on the mode momenta. We have also outlined how to apply the cylindrical structure on edge between a vacuum and non-Abelian domain. Although our general conclusions are in agreement with other methodologies we feel there is an inherent simplicity to the above arguments that make them an important part of the overall story.

For the specific model we have chosen we have been able to derive a simple recursive relation that exactly dictates the structure on edge between a vacuum and non-Abelian domain. A number of key features are present. Firstly the solutions are only normalized in the non-Abelian domain. Secondly we see a clear dependence on penetration depth on the mode momenta. We have also outlined how to apply the cylindrical solutions for the hard boundary to a fully open system.

In future work we will attempt to analyse the edge mode momentum dependency further and to extend these results to softer boundaries. We will also attempt to identify enough properties to exactly formulate the mode structure at the corners.

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