Sweeping processes with stochastic perturbations generated by a fractional Brownian motion

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Abstract
We study well-posedness of sweeping processes with stochastic perturbations generated by a fractional Brownian motion and convergence of associated numerical schemes. To this end, we first prove new existence, uniqueness and approximation results for deterministic sweeping processes with bounded $p$-variation and next we apply them to the stochastic case.

Key Words: sweeping process, differential inclusions, integral equations, fractional Brownian motion, $p$-variation, Skorokhod problem, stochastic differential equations with reflecting boundary condition.

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1 Introduction
In the present paper we study well-posedness of some variants of the so-called sweeping process introduced by Moreau in the early 70s with motivation in plasticity theory. In his original formulation the sweeping process coincides with a first order differential inclusion of the form

$$\begin{cases}
\frac{dx}{dt}(t) \in N(C_t; x(t)), \\
x(0) = x_0 \in C_0, \\
x(t) \in C_t,
\end{cases}$$

(1.1)

where $C_t$ is a given convex moving set and $N(C_t; x(t))$ is the inward normal cone to $C_t$ at point $x(t)$ (see [35, 36, 37]). Many attempts have been made to generalize Moreau’s results to larger class of moving sets or more general than (1.1) differential inclusions containing deterministic or stochastic perturbations. For instance, sweeping by prox-regular moving sets instead of convex sets was considered by Colombo and Goncharov [12], Benabdellah [4], Thibault [51], Colombo and Monteiro Marques [13]. The study of sweeping processes with perturbations was introduced by Castaing, Dúc, Ha and Valadier [9] and Castaing and Monteiro Marques [10]. The interest in the theory of sweeping processes comes from the fact that it has numerous practical applications in nonsmooth mechanics, analysis of hysteresis phenomena, mathematical economics and in the modeling of switched electrical circuits (see, e.g., the monographs by Acary, Bonnefon and Brogliato [1], Drábek, Krejčí and Takač [16], Monteiro Marques [34] and the references therein).

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In our paper we study sweeping processes with stochastic perturbations. This problem was considered earlier by Colombo [7, 8] and recently by Bernicot and Venel [5]. In the last paper the authors give conditions ensuring well-posedness of d-dimensional stochastic differential inclusions of the form

\[
\begin{align*}
&dX_t \in f(t, X_t)dt + g(t, X_t)dB_t + N(C_t; X_t), \\
&X_0 = x_0 \in C_0, \\
&X_t \in C_t,
\end{align*}
\]  

(1.2)

where \( C_t \) is a given prox-regular moving set and \( B = \{B_t\}_{t \in \mathbb{R}^+} \) is a standard Brownian motion. To do this, in proofs they combine the methods of deterministic sweeping process theory with the methods of stochastic differential equations (SDEs) with reflecting boundary conditions. The use of the methods of SDEs is possible, because one can observe that (1.2) is equivalent to the SDE with reflecting boundary condition of the form

\[
X_t = x_0 + \int_0^t f(s, X_s) \, ds + \int_0^t g(s, X_s) \, dB_s + K_t, \quad t \in \mathbb{R}^+,
\]  

(1.3)

where the integral with respect to \( B \) is the classical stochastic integral.

By a solution to (1.3) we mean a pair \((X, K)\) consisting of a process \( X = \{X_t\}_{t \in \mathbb{R}^+} \) such that \( X_t \in C_t \) and the process \( K = \{K_t\}_{t \in \mathbb{R}^+} \), called regulator term, such that \( dK_t \in N(C_t; X_t) \) in appropriately defined sense. Equation (1.3) was firstly investigated by Skorokhod [45] for \( C_t = [0, \infty), \ t \in \mathbb{R}^+ \). Extensions of Skorokhod’s results to larger class of domains was studied for instance by Tanaka [50], Lions and Sznitman [32], Saisho [43], Dupuis and Ishi [20], Slomiński [46] and Rozkosz [40]. Equations of the form (1.3) also have many applications, for instance in queueing systems, seismic reliability analysis and finance (see, e.g., [3, 21, 28, 44] and the references therein). Solutions of (1.3) are often called solutions of Skorokhod’s SDEs or of the Skorokhod problem.

In our paper a stochastic perturbation is generated not by a standard Brownian motion but by a fractional Brownian motion (fBm) \( B^H = \{B^H_t\}_{t \in \mathbb{R}^+} \) with Hurst index \( H > 1/2 \), i.e. by a continuous centered Gaussian process with covariance

\[
\text{EB}^H_{t_2}B^H_{t_1} = \frac{1}{2}(t_2^{2H} + t_1^{2H} - |t_2 - t_1|^{2H}), \quad t_1, t_2 \in \mathbb{R}^+.
\]

It is well known that \( B^H \) is not a semimartingale and therefore the classical stochastic integration theory for semimartingales cannot be applied. However, \( B^H \) has \( \lambda \)-Hölder continuous paths for all \( \lambda \in (0, H) \), which allows one to define the pathwise Riemann-Stieltjes integral with respect to fBm (see, e.g., [18, 19, 41]). The theory of SDEs without reflecting boundary condition driven by \( B^H \) with the pathwise Riemann-Stieltjes integral is at present quite well-developed. General results on existence and uniqueness of solutions one can find in Nualart and Răşcanu [39]. The viability property for such equations is considered in details in Ciotir and Răşcanu [11].

Let \( f : \mathbb{R}^d \to \mathbb{R}^d, \ g : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) be measurable functions and \( B^H \) be a d-dimensional fBm. Our main purpose is to study d-dimensional SDE with reflecting boundary condition of the form

\[
X_t = X_0 + \int_0^t f(s, X_s) \, ds + \int_0^t g(s, X_s) \, dB^H_s + K_t, \quad t \in \mathbb{R}^+,
\]  

(1.4)
where the integral with respect to $B^H$ is the pathwise Riemann-Stieltjes integral and $C_t = [L_t, U_t] = \times_{i=1}^d [L^i_t, U^i_t] \subset \mathbb{R}^d$ is a moving convex set (Here $L^i_t \leq U^i_t$, $t \in \mathbb{R}^+$, $i = 1, \ldots, d$).

We also study some generalizations of (1.4). Clearly, (1.4) is equivalent to sweeping process of the form (1.2) with stochastic perturbation generated by fBm. The integral form (1.4) is however more convenient because in general the process $K$ need not be of bounded variation and therefore the use of the differential $dK_t$ would require additional explanations.

In the recent paper by Ferrante and Rovira [24] the special case of (1.4) with $C_t = [0, \infty)^d$ was considered. Using quite natural in the context of SDEs driven by $B^H$ methods based on $\lambda$-Hölder norms they gave conditions ensuring the existence of solutions and their uniqueness for some small time interval. Some global uniqueness results for (1.4) with time homogenous coefficients $f, g$ and $C_t = \times_{i=1}^d [L^i_t, \infty)$ were proved in Falkowski and Śłomiński [23], where in contrast to [24] the $p$-variation norm is used. In the present paper we also use techniques using the $p$-variation norm. It is worth noting that we do not assume the so-called “interior ball condition”, which in our case means that there is $r > 0$ such that $U^i_t - L^i_t > r$, $t \in \mathbb{R}^+$. We even allow that $L^i_t = U^i_t$. Unfortunately, we are not able to extend our methods to more general moving convex sets or prox-regular moving sets (we think that it is not possible apart from the case of functions $g$ depending purely on time).

As a matter of fact, in the present paper we study more general than (1.4) equations in which the driving processes may have jumps, that is equations of the form

$$X_t = X_0 + \int_0^t f(s, X_s) \, dA_s + \int_0^t g(s, X_s) \, dZ_s + K_t, \quad t \in \mathbb{R}^+. \tag{1.5}$$

where $A$ is a one-dimensional càdlàg process with locally bounded variation and $Z$ is a $d$-dimensional càdlàg process with locally bounded $p$-variation for some $1 < p < 2$ (note that $B^H$ has locally bounded $p$-variation only for $p \in (1/H, \infty)$).

The paper is organized as follows.

In Section 2 we consider the deterministic extended Skorokhod problem $x = y + k$ associated with a càdlàg $d$-dimensional function $y$ (i.e. $y \in \mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)$) and time dependent barriers $l, u \in \mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that $l \leq u$, which means that $l_t \leq u_t$, $t \in \mathbb{R}^+$ and $l_0 \leq y_0 \leq u_0$. We show that for fixed $l, u$ the mapping $y \mapsto (x, k)$ is Lipschitz continuous in the $p$-variation norm. It is worth noting here that in [24] Remark 3.6 it is observed that $y \mapsto (x, k)$ is not Lipschitz continuous in the $\lambda$-Hölder norm and that for that reason in [24] the authors were not able to obtain global uniqueness.

In Section 3 we consider a deterministic version of (1.5). We give conditions ensuring the existence and uniqueness of solutions. In the proof we use an analogue of the Picard iteration method (we work in spaces equipped with the $p$-variation norm). Our assumptions on the coefficients $f, g$ are similar to those considered in [39]. Since our integrators are càdlàg with bounded $p$-variation and need not be $\lambda$-Hölder continuous, our theorem generalizes results from [39] even in the trivial case where $C_t = \mathbb{R}^d$, $t \in \mathbb{R}^+$.

Section 4 is devoted to the approximation of deterministic solutions considered in Section 3. We consider two methods of approximations. The first one is an easy to implement discrete-time method constructed in analogy with the classical Euler scheme (it is an analogue of the so-called “catching-up” algorithm introduced by Moreau to prove the existence of a solution to (1.1)). We prove the convergence of the scheme in the Skorokhod topology $J_1$ (in the case of continuous data we obtain uniform convergence on compact subsets of $\mathbb{R}^+$). The second method uses stability of solutions of deterministic versions of (1.5) with respect
to convergence of its coefficients. More precisely, we consider family of solutions with the coefficients \( f_\varepsilon, g_\varepsilon, \varepsilon > 0 \) instead of \( f, g \) and such that \( f_\varepsilon \xrightarrow{\mathcal{K}} f, g_\varepsilon \xrightarrow{\mathcal{K}} g \) as \( \varepsilon \to 0 \), which means that \( f_\varepsilon, g_\varepsilon \) tend to \( f, g \) uniformly on compact subsets of \( \mathbb{R}^d \). We show that under some mild additional assumptions on \( f_\varepsilon, g_\varepsilon \) the associated solutions converge in the \( p \)-variation norm to the solution of equation with coefficients \( f, g \).

In Section 5 we apply our deterministic results to show the existence and uniqueness of solutions of SDEs of the form \( \text{(1.5)} \). To illustrate how our results work in practice we consider fractional SDEs \( \text{(1.4)} \) and its simple generalizations. We give conditions ensuring the existence and uniqueness of their solutions and show how approximate them by a simply to implement numerical scheme.

Section 6 contains the proof of Theorem \ref{thm:main}.

In the sequel we will use the following notation. \( \mathbb{R}^+ = [0, \infty) \), \( \mathbb{M}^d \) is the space of \( d \times d \) real matrices \( A \), with the matrix norm \( \| A \| = \sup \{|Au|; u \in \mathbb{R}^d, |u| = 1 \} \), where \( | \cdot | \) denotes the usual Euclidean norm in \( \mathbb{R}^d \), \( B(0, N) = \{ x \in \mathbb{R}^d; |x| \leq N \}, N \in \mathbb{R}^+ \), \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) is the space of càdlàg mappings \( x : \mathbb{R}^+ \to \mathbb{R}^d \), i.e. mappings which are right continuous and admit left-hands limits equipped with the Skorokhod topology \( J_1 \). For \( x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d), t > 0 \), we write \( x_{t-} = \lim_{s \uparrow t} x_s, \Delta x_i = x_t - x_{t-} \) and \( v_p(x)_{[a,b]} = \sup \sum_{i=1}^{n} \{|x_{t_i} - x_{t_{i-}}|^p < +\infty, \) where the supremum is taken over all subdivisions \( \pi = \{a = t_0 < \ldots < t_n = b\} \) of \([a,b] \), \( V_p(x)_{[a,b]} = V_p(x)_{[0,T]} + |x_0| \), where \( V_p(x)_{[a,b]} = (v_p(x)_{[a,b]})^{1/p} \), is the usual \( p \)-variation norm. Moreover, for simplicity of notation we write \( v_p(x)_{[0,T]} = v_p(x)_{[0,T]} \) and \( V_p(x)_{[0,T]} = V_p(x)_{[0,T]} \).

We write \( x \leq x', x, x' \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) if \( x_i \leq x_i', t \in \mathbb{R}^+, i = 1, \ldots, d \).

## 2 Main estimates

We begin with recalling the definition of the extended Skorokhod problem with time dependent reflecting barriers introduced in \cite{6}. Let \( y, l, u \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) be such that \( l \leq u \) and \( l_0 \leq y_0 \leq u_0 \). We say that a pair \( (x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \) is a solution of the extended Skorokhod problem associated with \( y \) and barriers \( l, u \) (and we write \( (x, k) = \text{ESP}(y, l, u) \)) if

\[(i) \ x_t = y_t + k_t \in [l_t, u_t], t \in \mathbb{R}^+, \]

\( (ii) \ k_0 = 0, k = (k^1, \ldots, k^d) \), where for every \( 0 \leq t \leq q \) and \( i = 1, \ldots, d \),

\[ k^i_t - k^i_0 \geq 0, \quad \text{if } x^i_s < u^i_s \text{ for all } s \in (t,q), \]

\[ k^i_q - k^i_t \leq 0, \quad \text{if } x^i_s > l^i_s \text{ for all } s \in (t,q), \]

and for every \( t \in \mathbb{R}^+ \), \( \Delta k^i_t \geq 0 \) if \( x^i_t < u^i_t \) and \( \Delta k^i_t \leq 0 \) if \( x^i_t > l^i_t \).

In \cite{6} Theorem 2.6 it is proved that for any \( y, l, u \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) such that \( l \leq u \) and \( l_0 \leq y_0 \leq u_0 \) there exists a unique solution \( (x, k) = \text{ESP}(y, l, u) \).

**Remark 2.1** \( (a) \) It is observed in \cite{45} that instead of (ii) the following system of conditions can be considered: for every \( 0 \leq t \leq q \) and \( i = 1, \ldots, d \) such that \( \inf_{s \in [t,q]} (u^i_s - l^i_s) > 0 \) the function \( k^i \) has bounded variation on \([t,q]\) and

\[ \int_{[t,q]} (x^i_s - l^i_s) \, dk^i_s \leq 0 \quad \text{and} \quad \int_{[t,q]} (x^i_s - u^i_s) \, dk^i_s \leq 0 \quad (2.1) \]
(Here \( \int_{t,q} w_s \, dv_s \) denotes the integral over the closed interval \([t,q]\), which is equal to \( w_t \Delta v_t + \int_t^q w_s \, dv_s \), where \( \int_t^q w_s \, dv_s \) denotes the usual integral over the half open interval \((t,q)\). Simple calculations show that the definitions from [6] and [48] are equivalent.

(b) By (2.1), if \( u_i^i > l_i^i \) then \( (x_i^i - u_i^i) \Delta k_i^i \leq 0 \) and \( (x_i^i - u_i^i) \Delta k_i^i \leq 0 \). Consequently, if \( \Delta k_i^i > 0 \) then \( x_i^i = l_i^i \) and if \( \Delta k_i^i < 0 \) then \( x_i^i = u_i^i \), \( i = 1, \ldots, d \). Therefore, for every \( t \in \mathbb{R}^+ \),

\[
x_t = \max(\min((x_t - \Delta y_t), u_t), l_t) \quad \text{and} \quad k_t = \max(\min(k_{t-}, u_t - y_t), l_t - y_t),
\]

which means that \( x_t \) is the projection of \( x_t - \Delta y_t \) on the interval \([u_t, l_t]\) and \( k_t \) is the projection of \( k_{t-} \) on the interval \([u_t - y_t, l_t - y_t]\).

(c) In the classical Skorokhod problem it is assumed that the function \( k \) has bounded variation on each bounded interval \([t,q]\), or, equivalently, \( k = k^{(+)} - k^{(-)} \), where \( k^{(+)}, k^{(-)} \) are nondecreasing right continuous functions with \( k_0 = k_0^{(+)} = k_0^{(-)} = 0 \) such that \( k^{(+)} \) increases only on \( \{t; x_i^t = l_i^t\} \) and \( k^{(-)} \) increases only on \( \{t; x_i^t = u_i^t\} \), \( i = 1, \ldots, d \). If \( (x,k) = ESP(y,l,u) \) and \( \inf_{t \leq T} (u_t - l_t) > \epsilon_T > 0 \), \( T \in \mathbb{R}^+ \) then \( k \) is a function of bounded variation and \( (x,k) \) is a solution of the classical Skorokhod problem (see, e.g., [6 Corollary 2.4]).

The Lipschitz continuity of the mapping \( (y,l,u) \mapsto (x,k) \) in the supremum norm is well known. Let \( (x,k) = ESP(y,l,u), (x',y') = ESP(k',l',u') \). By [48] Theorem 2.1,

\[
\sup_{t \leq T} |x_t - x'_t| \leq 2 \sup_{t \leq T} |y_t - y'_t| + \max(\{|l_t - l'_t|, |u_t - u'_t|\})
\]

and

\[
\sup_{t \leq T} |k_t - k'_t| \leq \sup_{t \leq T} |y_t - y'_t| + \max(\{|l_t - l'_t|, |u_t - u'_t|\}).
\]

From this one can deduce the following stability result for solutions of the extended Skorokhod problem in the topology \( J_1 \). Assume that \( (x^n, k^n) = ESP(y^n, l^n, u^n), n \in \mathbb{N}, (x,k) = ESP(y,l,u) \) and \( (y^n, l^n, u^n) \to (y,l,u) \) in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \). Then

\[
(x^n, k^n, y^n, l^n, u^n) \to (x,k,y,l,u) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{5d})
\]  
(2.2)

(see [6 Theorem 2.6] or [48 Theorem 2.8]). Below we show that in the case of fixed barriers \( l, u \) the Lipschitz continuity of the mapping \( y \mapsto (x,k) \) also holds in the \( p \)-variation norm. We first consider the case \( d = 1 \).

**Theorem 2.2** Assume that \( y^1, y^2, l, u \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}) \) are such that \( l_0 \leq y^1_0, y^2_0 \leq u_0 \) and \( l \leq u \). Let \( (x^j, k^j) = ESP(y^j, l, u), j = 1,2 \). Then for any \( T \in \mathbb{R}^+ \)

\[
\hat{V}_p(k^1 - k^2)_T \leq \hat{V}_p(y^1 - y^2)_T.
\]

Since our proof involves some technical one-dimensional arguments not associated with the rest of the paper, we defer the proof of Theorem 2.2 to Section 6.

**Remark 2.3**

(a) The case \( p = 1 \) was studied earlier in [48 Theorem 2.14] (see also [43]).

(b) In Ferrante and Rovira [24 Remark 3.6] it is observed that property stated in Theorem 2.2 does not hold in \( \lambda \)-Hölder norm.

(c) [48 Example 2.15] shows that it is not possible to omit the assumption that \( l = l' \) and \( u = u' \).
Corollary 2.4 Assume \( y, y', l, u \in D(\mathbb{R}^+, \mathbb{R}^d) \) are such that \( l_0 \leq y_0, y'_0 \leq u_0 \). Let \((x, k) = ESP(y, l, u) \) and \((x', k') = ESP(y', l, u) \). Then for any \( T \in \mathbb{R}^+ \),
\[
\hat{V}_p(x - x')_T \leq (d+1)\hat{V}_p(y - y')_T \quad \text{and} \quad \hat{V}_p(k - k')_T \leq d\hat{V}_p(y - y')_T.
\]
Proof. By Theorem [2.2]
\[
\hat{V}_p(k - k')_T \leq d^{(p-1)/p} \left( \sum_{i=1}^{d} v_p(k_i - k'_i)_T \right)^{1/p} \leq d^{(p-1)/p} \left( \sum_{i=1}^{d} \hat{V}_p(y_i - y'_i)_T \right)^{1/p}
\]
\[
\leq d \max_i \hat{V}_p(y_i - y'_i)_T \leq d\hat{V}_p(y - y')_T.
\]
Since \( \hat{V}_p(x - x')_T \leq \hat{V}_p(y - y')_T + \hat{V}_p(k - k')_T \), the proof is complete. \( \square \)

Corollary 2.5 Assume \( y, l, h, u \in D(\mathbb{R}^+, \mathbb{R}^d) \) are such that \( l_0 \leq y_0 \leq u_0, l \leq h \leq u \). Let 
\((x, k) = ESP(y, l, u) \). Then for any \( T \in \mathbb{R}^+ \),
\[
\hat{V}_p(x)_T \leq (d+1)\hat{V}_p(y)_T + d\hat{V}_p(h)_T \quad \text{and} \quad \hat{V}_p(k)_T \leq d\hat{V}_p(y)_T + d\hat{V}_p(h)_T,
\]
Proof. Note that \((h, 0) = ESP(h, l, u) \). By Corollary [2.3]
\[
\hat{V}_p(k)_T \leq d\hat{V}_p(y - h)_T \leq d\hat{V}_p(y)_T + d\hat{V}_p(h)_T,
\]
i.e. the second inequality of the corollary is satisfied. From the second inequality we immediately get the first one. \( \square \)

3 Deterministic equations with reflecting boundary condition

Let \( a \in D(\mathbb{R}^+, \mathbb{R}), z, l, u \in D(\mathbb{R}^+, \mathbb{R}^d) \) be such that \( V_1(a)_T, V_p(z)_T < \infty \) for \( T \in \mathbb{R}^+ \) and \( l \leq u \). We also assume that there is \( h \in D(\mathbb{R}^+, \mathbb{R}^d) \) such that \( l \leq h \leq u \) and \( V_p(h)_T < \infty \) for \( T \in \mathbb{R}^+ \). This additional assumption is indispensable to ensure that \((x, k) = ESP(y, l, u) \) have bounded \( p \)-variation for any bounded \( p \)-variation function \( y \) (it is automatically satisfied if \( \inf_{t \leq T}(u_t - l_t) \geq \varepsilon_T > 0, T \in \mathbb{R}^+ \), because in this case \( k \) is a function of bounded variation).

We consider equations with reflecting time-dependent barriers of the form
\[
x_t = x_0 + \int_0^t f(s, x_{s-}) \, da_s + \int_0^t g(s, x_{s-}) \, dz_s + k_t, \quad t \in \mathbb{R}^+,
\]
(3.1)
where \( f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d \), \( g : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{M}^d \) are given functions, the integral with respect to \( z \) is a Riemann-Stieltjes integral and \( l_0 \leq x_0 \leq u_0 \). We recall that if \( w \in D(\mathbb{R}^+, \mathbb{M}^d) \), \( z \in D(\mathbb{R}^+, \mathbb{R}^d) \) are such that \( V_q(w)_T < +\infty, V_p(z)_T < +\infty, T \in \mathbb{R}^+ \), where \( 1/p + 1/q > 1, p, q \geq 1 \), then the Riemann-Stieltjes integral \( \int_0^t w_{s-} \, dz_s \) is well defined (see, e.g., [17]). Moreover, it is well known that for any \( a < b \),
\[
V_p\left( \int_a^b w_{s-} \, dz_s \right)_{[a,b]} \leq C_{p,q} V_q(w)_{[a,b]} V_p(z)_{[a,b]},
\]
(3.2)
where \( C_{p,q} = \zeta(p^{-1} + q^{-1}) \) and \( \zeta \) denotes the Riemann zeta function, i.e. \( \zeta(x) = \sum_{n=1}^\infty 1/n^x \).
Definition 3.1 We say that a pair \((x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})\) is a solution of \((3.1)\) if \(V_p(x)_T < \infty\) for \(T \in \mathbb{R}^+\) and \((x, k) = ESP(y, l, u)\), where

\[
y_t = x_0 + \int_0^t f(s, x_{s-}) \, da_s + \int_0^t g(s, x_{s-}) \, dz_s, \quad t \in \mathbb{R}^+.
\]

We will need the following conditions on \(f, g\).

(F) (a) There exists \(L > 0\) such that

\[
|f(t, x)| \leq L(1 + |x|), \quad x \in \mathbb{R}^d, t \in \mathbb{R}^+.
\]

(b) For every \(N \in \mathbb{R}^+\) there exists \(L_N > 0\) such that

\[
|f(t, x) - f(t, y)| \leq L_N|x - y|, \quad x, y \in B(0, N), t \in \mathbb{R}^+.
\]

(G) (a) There exist \(\beta \in (1 - 1/p, 1]\) and \(C^\beta > 0\) such that

\[
|g(t, x) - g(s, y)| \leq C^\beta(|t - s|^{\beta} + |x - y|), \quad x, y \in \mathbb{R}^d, t, s \in \mathbb{R}^+
\]

(b) \(g\) is differentiable in \(x\) and for every \(N \in \mathbb{R}^+\) there exist \(\alpha_N \in (p - 1, 1]\) and \(C_N > 0\) such that

\[
|\nabla_x g(t, x) - \nabla_x g(s, y)| \leq C_N(|t - s|^{\beta} + |x - y|^{\alpha_N}), \quad x, y \in B(0, N), t, s \in \mathbb{R}^+, \quad N > 0
\]

where \(\nabla_x g(t, x) = (\nabla_x g^{i,j}(t, x))_{i,j=1,...,d}\) and

\[
|\nabla_x g(t, x)|^2 = \sum_{k=1}^{d} \sum_{i,j=1}^{d} |\partial_{x_k}^{i,j}(t, x)|^2.
\]

Similar sets of conditions were considered in papers on equations without reflecting boundary condition driven by functions (processes) with bounded \(p\)-variation (see [17, 29, 30, 33, 39, 41]).

Remark 3.2 \(\text{Note that under (G)(a) for every } T \in \mathbb{R}^+, \text{ there exists } C^{\beta, T} > 0\) such that for every \(t \in [0, T]\) and \(x \in \mathbb{R}^d\),

\[
|g(t, x)| \leq C^{\beta, T}(1 + |x|), \quad (3.3)
\]

and for \(q = p \lor (1/\beta)\) and every \(w \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)\),

\[
\tilde{V}_p(g(\cdot, w))_t \leq C^{\beta, T}t^\beta + C^{\beta}pV_p(w)_t + |g(0, x_0)| \leq C^{\beta, T}(1 + \tilde{V}_p(x)_t). \quad (3.4)
\]

Moreover, \(C^{\beta, T} = C^{\beta}(T^\beta + 1) + |g(0, 0)|\).

We will approximate solutions of \((3.1)\) by using an analogue of the Picard iteration method. Set \((x^0, k^0) = ESP(x_0, l, u)\) and for any \(n \in \mathbb{N}\) set

\[
\begin{aligned}
&y^n = x_0 + \int_0^t f(s, x_{s-}) \, da_s + \int_0^t g(s, x_{s-}) \, dz_s, \\
&(x^n, k^n) = ESP(y^n, l, u),
\end{aligned}
\]

where the integral with respect to \(z\) is the Riemann-Stieltjes integral. Note that \((3.5)\) is well defined if (F)(a) and (G)(a) are satisfied. Indeed, by Corollary 2.5

\[
\tilde{V}_p(x^0)_T \leq (d + 1)|x_0| + d\tilde{V}_p(h)_T, \quad T \in \mathbb{R}^+ \quad (3.6)
\]
and for any $n \in \mathbb{N}$,
\[
\tilde{V}_p(x^n)_T \leq (d + 1)\tilde{V}_p(y^n)_T + d\tilde{V}_p(h)_T
\leq (d + 1) \left[ |x_0| + \int_0^T f(s, x_{s-}^{n-1}) \, ds_T + \int_0^T g(s, x_{s-}^{n-1}) \, dz_T \right] + d\tilde{V}_p(h)_T.
\]
Moreover,
\[
V_p(\int_0^T f(s, x_{s-}^{n-1}) \, ds_T) \leq \sup_{s \leq T} |f(s, x_{s-}^{n-1})|V_1(a)_T \leq L(1 + \tilde{V}_p(x^{n-1}))V_1(a)_T
\]
and by (3.2) and (3.3) for $q = p \lor (1/\beta)$ we have
\[
V_p(\int_0^T g(s, x_{s-}^{n-1}) \, dz_T) \leq C_{p,q} \tilde{V}_q(g(\cdot, x^{n-1}))_T V_p(z)_T \leq C_{p,q} C^{\beta,T}(1 + \tilde{V}_p(x^{n-1}))V_p(z)_T.
\]
Hence, in particular, $\tilde{V}_p(x^n)_T < \infty$ for $n \in \mathbb{N}$, $T \in \mathbb{R}^+$. In fact, under (F)(a) and (G)(a) we have
\[
\sup_n \tilde{V}_p(x^n)_T < \infty, \quad T \in \mathbb{R}^+.
\] (3.7)
To check this, fix $T \in \mathbb{R}^+$ and set $C_0 = (d + 1)|x_0| + d\tilde{V}_p(h)_T$, $C_1 = (d + 1)\max(L, C_{p,q} C^{\beta,T})$. Observe that by the above estimates for any $t \leq T$ we have $\tilde{V}_p(x^0)_t \leq C_0$ and
\[
\tilde{V}_p(x^n)_t \leq C_0 + C_1(1 + \tilde{V}_p(x^{n-1}))_t (V_1(a)_t + V_p(z)_t), \quad n \in \mathbb{N}.
\]
If we set $t_1 = \inf \{ t; C_1(V_1(a)_t + V_p(z)_t) > 1/2 \} \wedge T$ then
\[
\tilde{V}_p(x^n)_{t_1} \leq C_0 + \frac{1}{2} \tilde{V}_p(x^{n-1})_{t_1}, \quad n \in \mathbb{N},
\]
which implies that $\sup_n \tilde{V}_p(x^n)_{t_1} \leq 2(C_0 + 1/2)$. Since
\[
|\Delta x^n_{t_1}| \leq |f(t_1, x_{t_1-}^{n-1})\Delta a_{t_1}| + |g(t_1, x_{t_1-}^{n-1})\Delta z_{t_1}| + \max(|\Delta t_1|, |\Delta u_{t_1}|),
\]
it is clear that
\[
\sup_n \tilde{V}_p(x^n)_{t_1} < \infty.
\] (3.8)
Set $t_k = \inf \{ t > t_{k-1}; C_1(V_1(a)_{t_{k-1}} + V_p(z)_{t_{k-1}}) > 1/2 \} \wedge T$, $k \geq 2$. Modifying slightly the proof of (3.8) on can show that $\sup_n \tilde{V}_p(x^n)_{[t_{k-1},t_k]} < \infty$. What is left is to show that $m = \inf \{ k; t_k = T \}$ is finite. To see this, without loss of generality assume that $C_1 \geq 1$. Observe that $1/2 < C_1(V_1(a)_{t_{k-1}} + V_p(z)_{t_{k-1}}) \leq 2 \max(V_1(a)_{t_{k-1}}, V_p(z)_{t_{k-1}})$ for each $k$, which implies that $(1/4)^p < V_1(a)_{t_{k-1}} + v_p(z)_{t_{k-1}}$, $k \in \mathbb{N}$. Consequently,
\[
m(\frac{1}{4})^p < \sum_{k=1}^m (V_1(a)_{t_{k-1}} + v_p(z)_{t_{k-1}}) \leq (V_1(a)_T + v_p(z)_T) < \infty,
\] (3.9)
which completes the proof of (3.7).
Theorem 3.3 Assume (F), (G) and that there exists \( h \in D(\mathbb{R}^+, \mathbb{R}^d) \) such that \( l \leq h \leq u \) and \( V_p(h)_T < \infty, T \in \mathbb{R}^+ \). Let \( \{x^n, k^n\} \) denote the sequence of Picard’s iterations defined by (3.5). Then for every \( T \in \mathbb{R}^+ \),

\[
V_p(x^n - x)_T \to 0 \quad \text{and} \quad V_p(k^n - k)_T \to 0,
\]

where \((x, k)\) is a unique solution of (3.7).

Proof. Step 1. Convergence of Picard’s iteration. Fix \( T \in \mathbb{R}^+ \). Since \( x_0^n = x_0^{n-1} = x_0 \), applying Corollary 2.4 we get

\[
V_p(x^n - x^{n-1}) = V_p(x^n - x^{n-1}) \leq (d + 1)V_p(\int_0^T f(s, x^{n-1}_s) - f(s, x^{n-2}_s) \, ds) + \int_0^T (s, x^{n-1}_s) - g(s, x^{n-2}_s) \, dz_s) \leq (d + 1)V_p(\int_0^T f(s, x^{n-1}_s) - f(s, x^{n-2}_s) \, ds) + (d + 1)V_p(\int_0^T g(s, x^{n-1}_s) - g(s, x^{n-2}_s) \, dz_s)
\]

for \( t \in [0, T] \). By (3.7), \( \sup_{t \leq T} |x^n_t| \leq N \) for \( n \in \mathbb{N} \), where \( N = \sup_n V_p(x^n)_T \). Therefore

\[
V_p(\int_0^T f(s, x^{n-1}_s) - f(s, x^{n-2}_s) \, ds) \leq L_N V_1(a)_T \sup_{s \leq t} |x^{n-1}_s - x^{n-2}_s| \leq L_N V_1(a)_T V_p(x^{n-1} - x^{n-2})_T
\]

and by (3.2),

\[
V_p(\int_0^T g(s, x^{n-1}_s) - g(s, x^{n-2}_s) \, dz_s) \leq C_p \beta V_p(g(\cdot, x^{n-1} - g(\cdot, x^{n-2}))_T V_p(z)_T,
\]

where \( r = (p/\alpha_N) \vee (1/\beta) \). To estimate the right hand-side of the last inequality we will use the following lemma.

Lemma 3.4 If \( g : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R} \) satisfies (G) then for any \( x, y \in D(\mathbb{R}^+, \mathbb{R}^d), T, N \in \mathbb{R}^+ \) such that \( V_p(x)_T < \infty, V_p(y)_T < \infty, \sup_{t \leq T} |x_t|, \sup_{t \leq T} |y_t| \leq N \) and \( r = (p/\alpha_N) \vee (1/\beta) \) we have

\[
V_r(g(\cdot, x) - g(\cdot, y))_T \leq C^\beta V_r(x - y)_T + C_N \sup_{t \leq T} |x_t - y_t| \left( T^\beta + V_p(x)_T \alpha_N + V_p(y)_T \alpha_N \right).
\]

Proof of the Lemma. For every \( t, s \in [0, T] \),

\[
|g(t, x_t) - g(t, y_t) - g(s, x_s) + g(s, y_s)| = \left| \int_0^1 \nabla g(t, \theta x_t + (1 - \theta)y_t)(x_t - y_t) \cdot \nabla g(s, \theta x_s + (1 - \theta)y_s)(x_s - y_s) \, d\theta \right|
\]

Hence

\[
|g(t, x_t) - g(t, y_t) - g(s, x_s) + g(s, y_s)| \leq \int_0^1 |\nabla g(t, \theta x_t + (1 - \theta)y_t)|(x_t - y_t) \cdot |x_s - y_s| \, d\theta \]

\[
+ \int_0^1 |\nabla g(t, \theta x_t + (1 - \theta)y_t) - \nabla g(s, \theta x_s + (1 - \theta)y_s)|(x_s - y_s) \, d\theta \]

\[
\leq C^\beta |x_t - y_t - x_s + y_s| + C_N |x_s - y_s|(1 - |s - t|^\beta + |x_t - x_s|^\alpha_N + |y_t - y_s|^\alpha_N).
\]

9
Applying this estimate to each pair \( t = t_i, s = t_{i-1} \) from an arbitrary partition \( 0 = t_0 < t_1 < \cdots < t_n = T \) of \([0, T]\) and using Minkowski’s inequality, we obtain the desired result. \( \square \)

By Lemma 3.4, for \( i, j = 1, \ldots, d \) we have

\[
V_r(g_{i,j}(\cdot, x^{n-1}) - g_{i,j}(\cdot, x^{n-2}))_t \leq C^3 V_r(x^{n-1} - x^{n-2})_t
\]
\[
+ C_N \sup_{s \leq t} |x^{n-1}_s - x^{n-2}_s| \left( t^\beta + V_p(x^{n-1})^{\alpha_N} + V_p(x^{n-2})^{\alpha_N} \right),
\]

which implies that

\[
V_r(g(\cdot, x^{n-1}) - g(\cdot, x^{n-2}))_t \leq \sum_{i,j=1}^{d} V_r(g_{i,j}(\cdot, x^{n-1}) - g_{i,j}(\cdot, x^{n-2}))_t
\]
\[
\leq (C^3)^2 V_r(x^{n-1} - x^{n-2})_t
\]
\[
+ (C_N)^2 \sup_{s \leq t} |x^{n-1}_s - x^{n-2}_s| \left( t^\beta + V_p(x^{n-1})^{\alpha_N} + V_p(x^{n-2})^{\alpha_N} \right).
\]

From the above estimates, (3.7) and fact that

\[
\sup_{s \leq t} |x^{n-1}_s - x^{n-2}_s| \leq V_p(x^{n-1} - x^{n-2})_t, \quad t \in \mathbb{R}^+
\]

we conclude that there exists \( D > 0 \) depending only on \( C_{p,r}, C^\beta, C_N, \alpha_N, \beta, L_N, T \) and \( d \) such that for every \( n \in \mathbb{N} \),

\[
\bar{V}_p(x^n - x^{n-1})_t \leq D(V_1(a)_t + V_p(z)_t)\bar{V}_p(x^{n-1} - x^{n-2})_t.
\]

(3.10)

Set \( t_1 = \inf\{t > 0; (D(V_1(a)_t + V_p(z)_t)) \geq \frac{1}{2}\} \wedge T \) and observe that by induction,

\[
\bar{V}_p(x^n - x^{n-1})_{t_1} \leq 2^{-(n-1)} \bar{V}_p(x^1 - x^0)_{t_1}, \quad n \in \mathbb{N}.
\]

Thus \( \{x^n\} \) is a Cauchy sequence in the space of càdlàg functions on \([0, t_1]\) with the \( p \)-variation norm. Therefore there is a càdlàg function \( x \) such that \( \bar{V}_p(x^n - x)_{t_1} \longrightarrow 0 \). This implies that \( V_p(\int_0^1 (f(s, x^n_s) - f(s, x_{s-})) \, ds)_{t_1} \longrightarrow 0 \) and \( V_p(\int_0^1 (g(s, x^n_s) - g(s, x_{s-})) \, dz)_{t_1} \longrightarrow 0 \), and hence that there exists a càdlàg function \( k \) such that \( \bar{V}_p(k^n - k)_{t_1} \longrightarrow 0 \) and \( (x, k) \) is a solution of (3.1) on the interval \([0, t_1]\). If we set

\[
x_{t_1} = \max(\min(x_{t_1-} + f(t_1, x_{t_1-}) \Delta a_{t_1} + g(t_1, x_{t_1-}) \Delta z_{t_1}, u_{t_1}), l_{t_1})
\]

and \( k_{t_1} = k_{t_1-} + \Delta x_{t_1} - (f(t_1, x_{t_1-}) \Delta a_{t_1} + g(t_1, x_{t_1-}) \Delta z_{t_1}) \) then by Remark 2.1(b), \((x, k)\) is a solution of (3.1) on the closed interval \([0, t_1]\). Moreover,

\[
x_{t_1} = \max(\min(x_{t_1-} + f(t_1, x_{t_1-}) \Delta a_{t_1} + g(t_1, x_{t_1-}) \Delta z_{t_1}, u_{t_1}), l_{t_1})
\]
\[
\longrightarrow \max(\min(x_{t_1-} + f(t_1, x_{t_1-}) \Delta a_{t_1} + g(t_1, x_{t_1-}) \Delta z_{t_1}, u_{t_1}), l_{t_1}) = x_{t_1},
\]

which implies that \( \bar{V}_p(x^n - x)_{t_1} \longrightarrow 0 \) and \( \bar{V}_p(k^n - k)_{t_1} \longrightarrow 0 \). It is easy to see that we can apply the arguments used above to the interval \([t_1, t_2]\) with \( t_2 = \inf\{t > 0; (D(V_1(a)_{t_1} + V_p(z)_{t_1})) \geq \frac{1}{2}\} \wedge T \), and then to intervals \([t_2, t_3], [t_3, t_4], \ldots \). Since \( V_1(a)_T < \infty \) and \( V_p(z)_T < \infty \), in finitely many steps we are able to construct the solution \((x, k)\) of (3.1) on the whole interval \([0, T]\) and to show that \( \bar{V}_p(x^n - x)_T \longrightarrow 0 \) and \( \bar{V}_p(k^n - k)_T \longrightarrow 0 \).
Step 2. Uniqueness of solutions of (3.1). Assume that there exists two solutions \((x^1, k^1)\) and \((x^2, k^2)\). Let \(t_1\) be defined as in Step 1. Using arguments from Step 1 we show that \(V_p(x^1 - x^2)_{t_1} \leq \frac{1}{2} \bar{D}_p(x^1 - x^2)_{t_1} \), which implies that \(x^1 = x^2\) on \([0, t_1] \). Since by Remark 2.1(b) we know that

\[
x^j_{t_1} = \max(\min(x^j_{t_1-} + f(t_1, x^j_{t_1-})\Delta a_{t_1} + g(t_1, x^j_{t_1-})\Delta z_{t_1}, u_{t_1}), l_{t_1}), \quad j = 1, 2,
\]

it is clear that \(x^1 = x^2\) on the closed interval \([0, t_1]\). Applying the above argument to intervals \([t_1, t_2], [t_2, t_3], \ldots\) we show in finitely many steps that \(x^1 = x^2\) on \([0, T]\) for every \(T \in \mathbb{R}^+\). □

4 Discrete-time approximation and stability of solutions

We assume that \(l, h, u \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)\) are such that \(l \leq h \leq u, V_p(h)_T < \infty\) for \(T \in \mathbb{R}^+\). Let \(x_0 \in [l_0, u_0]\) and \(a \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}), z \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)\) be such that \(V_1(a)_T < \infty\) and \(V_p(z)_T < \infty\), \(T \in \mathbb{R}^+\).

Set \(x^n_0 = x_0, k^n_0 = 0\) and

\[
\begin{align*}
\Delta y^n_{(k+1)/n} & = f(k/n, x^n_{(k+1)/n})(a(k+1)/n - a_k/n) + g(k/n, x^n_{(k+1)/n})(z(k+1)/n - z_k/n), \\
x^n_{(k+1)/n} & = \max(\min(x^n_{k/n} + \Delta y^n_{(k+1)/n}, u_{(k+1)/n}), l_{(k+1)/n}), \\
k^n_{(k+1)/n} & = k^n + (x^n_{(k+1)/n} - x^n_{k/n}) - \Delta y^n_{(k+1)/n}
\end{align*}
\]

and \(x^n_t = x^n_{k/n}, k^n_t = k^n_{(k+1)/n}, t \in [k/n, (k + 1)/n], k \in \mathbb{N} \cup \{0\}\). Since

\[
x^n_{(k+1)/n} = \Pi_{C(k+1)/n}(x^n_{k/n} + \Delta y^n_{(k+1)/n}), \quad k \in \mathbb{N} \cup \{0\},
\]

where \(\Pi_{C(k+1)/n}\) denotes the projection on the set \(C_{(k+1)/n} = [l_{(k+1)/n}, u_{(k+1)/n}]\), (4.1) is the well known Euler scheme for (3.1) (see, e.g., [47]). It is also an analogue of the so-called “catching-up” algorithm introduced by Moreau to prove the existence of a solution of (1.1) (see, e.g., [2]).

Theorem 4.1 Let \(\{(x^n, k^n)\}\) be a sequence of approximations defined by (4.1). If \(f, g\) satisfy (F), (G) and moreover \(f\) is continuous then

\[
(x^n, k^n, l^n, u^n) \longrightarrow (x, k, l, u) \quad \text{in} \quad \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),
\]

where \(l^n_t = l_{k/n}, u^n_t = u_{k/n}, t \in [k/n, (k + 1)/n], k \in \mathbb{N} \cup \{0\}\) and \((x, k)\) is a unique solution of (3.1).

Proof. Fix \(T \in \mathbb{R}^+\) and set \(b_T = V_1(a)_T + V_p(z)_T + \bar{V}_p(h)_T + \sup_{t \leq T} \max(\mid\Delta l_t\mid, \mid\Delta u_t\mid)\). First we show that if \((x, k)\) satisfies (3.1) then

\[
\bar{V}_p(x)_T \leq D \quad \text{where} \quad D \text{ depends only on} \quad d, x_0, L, C^\beta, \beta \quad \text{and} \quad b_T.
\]

By Corollary 2.5 for any \(t \leq T\),

\[
\bar{V}_p(x)_t \leq (d + 1)\bar{V}_p(y)_t + d\bar{V}_p(h)_t \\
\leq (d + 1)\left[ x_0 + V_p\left(\int_0^t f(s, x_{s-})\,ds\right)_t + V_p\left(\int_0^t g(s, x_{s-})\,dz_s\right)_t \right] + d\bar{V}_p(h)_t.
\]
We have \( V_p\left(\int_0^1 f(s, x_{s-}) \, ds\right)_t \leq V_1(a)_t \sup_{s \leq t} |f(s, x_{s-})| \leq LV_1(a)_t (1 + V_p(x)_t) \) and, by (3.2) and (3.4),

\[
V_p\left(\int_0^1 g(s, x_{s-}) \, ds\right)_t \leq C_{p,p\nu(1/\beta)} V_{p\nu(1/\beta)}(g(\cdot, x)) V_p(z)_t \\
\leq C_{p,p\nu(1/\beta)} C^{\beta,T} (1 + V_p(x)_t) V_p(z)_t.
\]

Hence there is \( C_0 > 0 \) depending only on \( d, x_0, V_p(h)_T \) and \( C_1 > 0 \) depending on \( d, L, \beta, C^{\beta,T} \) such that

\[
\bar{V}_p(x)_t \leq C_0 + C_1 (1 + V_p(x)_t)(V_1(a)_t + V_p(z)_t).
\]

Set \( t_1 = \inf\{t; C_1(V_1(a)_t + V_p(z)_t) > \frac{1}{2}\} \wedge T \). By the above,

\[
\bar{V}_p(x)_{[0,t_1]} \leq C_0 + \frac{1}{2} + \frac{1}{2} \bar{V}_p(x)_{[0,t_1]},
\]

which implies that \( \bar{V}_p(x)_{[0,t_1]} \leq 2(C_0 + 1/2) \). Since by (3.3),

\[
|\Delta x_{t_1}| \leq f(t_1, x_{t_1}) |\Delta t_1| + |g(t_1, x_{t_1})| \Delta z_{t_1}| + \max(|\Delta l_{t_1}|, |\Delta u_{t_1}|) \\
\leq LT(1 + |x_{t_1-}|) |\Delta t_1| + C_{\beta,T}(1 + |x_{t_1-}|) |\Delta z_{t_1}| + \max(|\Delta l_{t_1}|, |\Delta u_{t_1}|),
\]

it is clear that \( \bar{V}_p(x)_{[0,t_1]} \leq D \) where \( D \) depends only on \( d, x_0, L, C^{\beta,T}, \beta \) and \( b_T \). If we set \( t_k = \inf\{t > t_{k-1}; C_1(V_1(a)|_{[t_{k-1}, t]} + V_p(z)|_{[t_{k-1}, t]} > \frac{1}{2}\} \wedge T, k = 2, 3, \ldots \), then for by the same arguments \( V_p(x)_{[t_{k-1}, t_k]} \leq D \) where \( D \) is depending only on \( d, x_{t_{k-1}}, L, C^{\beta,T}, \beta \) and \( b_T \). Since \( m = \inf\{k; t_k = T\} \) is bounded (similarly to (3.3)) one can check that \( m \leq 4^p(V_1(1/a) + v_p(z)(T)) \), this completes the proof of (4.3).

Now set \( a^n_t = a_{k/n}, z^n_t = z_{k/n}, h^n_t = h_{k/n}, \rho^n_t = k/n, t \in [k/n, (k+1)/n], k \in \mathbb{N} \cup \{0\} \). It is an elementary check that

\[
\{ (x^n, k^n) = ESP(y^n, l, u), \quad \text{where } y^n = x_0 + \int_0^t f(\rho^n_{s-}, x^n_{s-}) \, da^n_s + \int_0^t g(\rho^n_{s-}, x^n_{s-}) \, dz^n_s, \quad t \in \mathbb{R}^+ \},
\]

Clearly, for any \( n \in \mathbb{N} \),

\[
V_1(a^n)_T \leq V_1(a)_T, \quad V_p(z^n)_T \leq V_p(z)_T \quad \text{and} \quad V_p(h^n)_T \leq V_p(h)_T.
\]

Combining (4.3) with (4.4) we get

\[
\sup_n V_p(x^n)_T < \infty, \quad T \in \mathbb{R}^+.
\]

Let \( x_t^{(n)} = x_{k/n}, h_t^{(n)} = k_{k/n}, t \in [k/n, (k+1)/n], k \in \mathbb{N} \cup \{0\} \), denote the discretization of the solution \( (x, k) \). By using [22, Chapter 3, Proposition 6.5] and [25, Chapter VI, Proposition 2.2] one can check that \( (x^n, a^n, z^n, l^n, u^n, \rho^n) \rightarrow (x, a, z, l, u, I) \) in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d+2}) \), where \( I_s = s, s \in \mathbb{R}^+ \). From this and an easy extension of [27, Proposition 2.9] to functions with bounded \( p \)-variation it follows that

\[
\left( \bar{y}^n = x_0 + \int_0^1 f(\rho^n_{s-}, x^n_{s-}) \, da^n_s + \int_0^1 g(\rho^n_{s-}, x^n_{s-}) \, dz^n_s, l^n, u^n \right) \\
\rightarrow \left( y = x_0 + \int_0^1 f(s, x_{s-}) \, ds + \int_0^1 g(s, x_{s-}) \, dz_s, l, u \right) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).
\]
By the above and (4.2),
\[(\bar{x}^n, \bar{k}_n, \bar{y}^n, l^n, u^n) \to (x, k, y, l, u) \text{ in } D(\mathbb{R}^+, \mathbb{R}^{5d}),\] (4.6)
where \((\bar{x}^n, \bar{k}_n) = ESP(\bar{y}^n, l^n, u^n), \ n \in \mathbb{N}.\) Moreover, analysis similar to that in the proof of (4.3) shows that
\[
\sup_n \bar{V}_p(x^n)_T < \infty, \quad T \in \mathbb{R}^+.
\] (4.7)

By (4.6) and Chapter VI, Proposition 2.2, \((\bar{x}^n, x^{(n)}) \to (x, x) \text{ in } D(\mathbb{R}^+, \mathbb{R}^{2d}),\) which implies that
\[
\sup_{t \leq T} |\bar{x}^n_t - x^{(n)}_t| \to 0, \quad T \in \mathbb{R}^+.
\]
Combining the above convergence with (4.7) and the fact that \(\bar{V}_p(x^{(n)})_T \leq \bar{V}_p(x)_T < \infty, \) for every \(\epsilon > 0\) we obtain
\[
\bar{V}_{p+\epsilon}(\bar{x}^n - x^{(n)})_T \leq \text{Osc}(\bar{x}^n - x^{(n)})^{1-p/(p+\epsilon)}\bar{V}_p(\bar{x}^n - x^{(n)})^{p/(p+\epsilon)} \to 0, \quad T \in \mathbb{R}^+,
\] (4.8)
where \(\text{Osc}(x)_T = \sup_{s, t \leq T} |x_t - x_s|\). Fix \(\epsilon > 0.\) By Corollary 2.4, for any \(n \in \mathbb{N}\) and \(t \leq T,
\[
\bar{V}_{p+\epsilon}(\bar{x}^n - \bar{x}^n)_t
\leq (d+1)\bar{V}_{p+\epsilon}\left(\int_0^t f(\rho^n_{s-}, x^n_{s-}) - f(\rho^n_{s-}, x^{(n)}_{s-}) \, da^n + \int_0^t g(\rho^n_{s-}, x^n_{s-}) - g(\rho^n_{s-}, x^{(n)}_{s-}) \, dz^n_{s-t}\right)
\leq (d+1)\bar{V}_{p+\epsilon}\left(\int_0^t f(\rho^n_{s-}, x^n_{s-}) - f(\rho^n_{s-}, x^{(n)}_{s-}) \, da^n + \int_0^t g(\rho^n_{s-}, \bar{x}^n_{s-}) - g(\rho^n_{s-}, x^{(n)}_{s-}) \, dz^n_{s-t}\right)
\leq (d+1)\bar{V}_{p+\epsilon}\left(\int_0^t f(\rho^n_{s-}, x^n_{s-}) - f(\rho^n_{s-}, \bar{x}^n_{s-}) \, da^n + \int_0^t g(\rho^n_{s-}, \bar{x}^n_{s-}) - g(\rho^n_{s-}, x^{(n)}_{s-}) \, dz^n_{s-t}\right)
\]
\[
= I^{n, 1}_t + I^{n, 2}_t.
\]
From (4.7), the estimates from the proof of Theorem 3.3 and the fact that \(\bar{V}_{p+\epsilon}(v)_T \leq \bar{V}_p(v)_T\) one can deduce that there is \(D > 0\) such that for any \(n \in \mathbb{N},\)
\[
I^{n, 1}_T \leq D(V_1(a^n)_T + V_p(z^n)_T) \bar{V}_{p+\epsilon}(x^n - x^{(n)})_T.
\]
This together with (4.3) and (4.8) shows that \(\lim_{n \to \infty} I^{n, 1}_T = 0.\) The same arguments and (4.5) show that there is \(D > 0\) such that for every \(t \leq T,\)
\[
I^{n, 2}_t \leq D(V_1(a^n)_t + V_p(z^n)_t) \bar{V}_{p+\epsilon}(x^n - \bar{x}^n)_t.
\]
If we set \(t_1 = \inf\{t; D(V_1(a)_t + V_p(z)_t) > 1/2\} \land T\) then by (4.4),
\[
\bar{V}_{p+\epsilon}(x^n - \bar{x}^n)_{t_1-} \leq I^{n, 1}_T + \frac{1}{2} \bar{V}_{p+\epsilon}(x^n - \bar{x}^n)_{t_1-},
\]
which implies that \(\bar{V}_{p+\epsilon}(x^n - \bar{x}^n)_{t_1-} \to 0.\) This and (4.8) imply that \(\bar{V}_{p+\epsilon}(x^n - x^{(n)})_{t_1-} \to 0.\) Note that
\[
x^{(n)}_{t_1} = \max(\min(x^{(n)}_{t_1} + f(t_1, x^{(n)}_{t_1}) \Delta a^n_{t_1} + g(t_1, x^{(n)}_{t_1}) \Delta z^n_{t_1}, u^n_{t_1}), l^n_{t_1}), \quad n \in \mathbb{N}.
If \( t_1 \) is a nonrational number then \( \Delta a^n_{t_1} = \Delta x^n_{t_1} = \Delta l^n_{t_1} = \Delta u^n_{t_1} = \Delta x^n_{t_1} = 0 \). Hence \( x^n_{t_1} = x^n_{t_1} \) and \( x^n_{t_1} = x^n_{t_1} \) which implies that 

\[
\bar{V}_{p+\epsilon}(x^n - x^{(n)})_{t_1} \to 0. \tag{4.9}
\]

In case \( t_1 \) rational, set \( I = \{ n; \text{there is } k \text{ such that } t_1 = k/n \} \) and observe that if \( n \in I \) then \( \Delta a^n_{t_1} = \Delta a_{t_1}, \Delta x^n_{t_1} = \Delta x_{t_1}, \Delta l^n_{t_1} = \Delta l_{t_1}, \Delta u^n_{t_1} = \Delta u_{t_1} \) and \( \Delta x^n_{t_1} = \Delta x_{t_1} \). Consequently, if \( n \in I \) then \( x^n_{t_1} = x^n_{t_1} \) and \( x^n_{t_1} = x^n_{t_1} \), which completes the proof of (4.9) in case of rational \( t_1 \). By the same arguments in finitely many steps we show that 

\[
\bar{V}_{p+\epsilon}(x^n - x^{(n)})_{T} \to 0.
\]

Therefore \( \sup_{t \leq T} |x^n_t - x^{(n)}_t| \to 0 \), from which we deduce that \( \sup_{t \leq T} |k^n_t - k^{(n)}_t| \to 0 \), which together with (4.6) completes the proof of (4.2). \[ \square \]

**Corollary 4.2** Under the assumptions of Theorem 4.1, for any \( T \in \mathbb{R}^+ \), 

\[
\max_{k/n \leq T} |x^n_{k/n} - x_{k/n}| \to 0 \quad \text{and} \quad \max_{k/n \leq T} |k^n_{k/n} - k_{k/n}| \to 0, \tag{4.10}
\]

where \((x, k)\) is a unique solution of (3.7).

**Proof.** By (4.2) and [25] Chapter VI, Proposition 2.2,

\[
(x^n, x^{(n)}, k^n, k^{(n)}) \to (x, x, k, k) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d).
\]

This implies that \( x^n - x^{(n)} \to 0 \) in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) and \( k^n - k^{(n)} \to 0 \) in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \), which is equivalent to (4.10). \[ \square \]

**Theorem 4.3** Assume (F), (G) and that there exists \( h \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) such that \( l \leq h \leq u \) and \( V_p(h)_T < \infty \) for \( T \in \mathbb{R}^+ \). For \( \epsilon > 0 \) let \( l_0 \leq x_0 \leq u_0 \) and let \( f, g \) be functions satisfying (F), (G) with constants \( L, \beta, C^\beta \) not depending on \( \epsilon \). If \((x^\epsilon, k^\epsilon)\) denotes a solution of (3.7) with \( x_0, f, g \) replaced by \( x_0^\epsilon, f^\epsilon, g^\epsilon \) and \( x_0 \to x_0, f \to f^\epsilon, g \to g^\epsilon \) then for every \( T \in \mathbb{R}^+ \),

\[
\bar{V}_p(x^\epsilon - x)_T \to 0 \quad \text{and} \quad \bar{V}_p(k^\epsilon - k)_T \to 0,
\]

where \((x, k)\) is a unique solution of (3.7).

**Proof.** First observe that by (4.3),

\[
\sup_{\epsilon > 0} \bar{V}_p(x^\epsilon)_T < \infty, \quad T \in \mathbb{R}^+. \tag{4.11}
\]

Fix \( T \in \mathbb{R}^+ \). By Corollary 2.4 for every \( t \in [0, T] \),

\[
\bar{V}_p(x^\epsilon - x)_t \leq (d + 1) \left( |x^n_0 - x_0| + \bar{V}_p \left( \int_0^t f^\epsilon(s, x^\epsilon_{s-}) - f(s, x_{s-}) \, ds \right)_t \right.
\]

\[
+ \bar{V}_p \left( \int_0^t g^\epsilon(s, x^\epsilon_{s-}) - g(s, x_{s-}) \, dz_s \right)_t \right)
\]

\[
\leq (d + 1) \left( |x^n_0 - x_0| + \bar{V}_p \left( \int_0^t (f^\epsilon - f)(s, x^\epsilon_{s-}) \, ds \right)_t \right. + \bar{V}_p \left( \int_0^t (g^\epsilon - g)(s, x^\epsilon_{s-}) \, dz_s \right)_t
\]

\[
+ \bar{V}_p \left( \int_0^t (f(s, x^\epsilon_{s-}) - f(s, x_{s-}) \, ds \right)_t + \bar{V}_p \left( \int_0^t (g(s, x^\epsilon_{s-}) - g(s, x_{s-}) \, dz_s \right)_t
\]

\[
= (d + 1) \left( |x^n_0 - x_0| + I_1^1 + I_2^1 + I_3^1 + I_4^1 \right).
\]
Let \( p < 2 \), there exists \( \gamma \in (1 - 1/p, \beta \wedge (1/p)) \). Therefore by (3.2), (3.4) and (4.11) there is \( C > 0 \) depending only on \( \gamma, C_{p,1}/\gamma, L, \beta, C^\beta, C^{\beta,T} \) and \( V_p(z)_T \) such that
\[
I^{e,2}_T = V_p(\int_0^1 (g - g)(s, x^e_s) \, ds)_T \leq C_{p,1}/\gamma \bar{V}/\gamma ((g - g)(\cdot, x^e))_T V_p(z)_T \leq C_{p,1}/\gamma \text{Osc}((g - g)(\cdot, x^e))(1/\gamma - p\gamma(1/\beta)) \gamma V_p(1/\beta)(g - g)(\cdot, x^e))_T \gamma + |(g - g)(0, x^e)|_T V_p(z)_T \leq C \sup_{v \in B(0,N)} \| (g - g)(t, v) - g(t, v)\|_p - (p\gamma(1/\beta)) \gamma + |g - g(0, v)|_p.
\]
Consequently,
\[
I^{e,1}_T + I^{e,2}_T \xrightarrow[\epsilon \to 0]{} 0. \tag{4.12}
\]
Using once again (4.11) and estimates from the proof of Theorem 3.3 we check that there is \( D > 0 \) such that for every \( t \leq T \),
\[
(d + 1)(I^{e,3}_T + I^{e,4}_T) \leq D(V_1(A)_t + V_p(z)_t) \bar{V}_p(x^e - x)_t.
\]
Similarly to the proof of Theorem 3.3 we set \( t_1 = \inf\{t; D(V_1(A)_t + V_p(z)_t) > 1/2\} \wedge T \). Observe that
\[
\bar{V}_p(x^e - x)_{t_1-} \leq (d + 1)(|x^e_0 - x_0| + I^{e,1}_T + I^{e,2}_T) + \frac{1}{2} \bar{V}_p(x^e - x)_{t_1-}.
\]
From this and (4.12) we deduce that \( \bar{V}_p(x^e - x)_{t_1-} \to 0 \). Using arguments from the proof of Theorem 3.3 we show that this implies that \( \bar{V}_p(x^e - x)_{t_1} \xrightarrow[\epsilon \to 0]{} 0 \). Applying this argument to (finitely many) intervals \([t_i, t_{i+1}]\) we prove the theorem. \( \Box \)

5. Applications to stochastic processes

In this section we apply our deterministic results to SDEs with reflecting boundary condition. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) be a filtered probability space and let \( A \) be an \((\mathcal{F}_t)\) adapted process with trajectories in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}) \). Z, L, U, H be \((\mathcal{F}_t)\) adapted processes with trajectories in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) such that \( L \leq H \leq U \) and \( P(V^1(A)_T < \infty) = 1, P(V^p(Z)_T < \infty) = 1, P(V^p(H)_T < \infty) = 1 \) for every \( T \in \mathbb{R}^+ \). Note that \( Z \) need not be a semimartingale. However, it is a Dirichlet process and a \( p \)-semimartingale in the sense considered in [15] and [29, 30].

**Definition 5.1** Let \( L \leq U \) and \( X_0 \) be an \( \mathcal{F}_0 \) measurable random vector such that \( L \leq X_0 \leq U_0 \). We say that a pair \((X, K)\) of \((\mathcal{F}_t)\) adapted processes with trajectories in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) such that \( P(V_p(X)_T < \infty) = 1 \) for \( T \in \mathbb{R}^+ \) is a strong solution of (1.1) if \((X, K) = ESP(Y, L, U)\), where
\[
Y_t = X_0 + \int_0^t f(s, X_{s-}) \, ds + \int_0^t g(s, X_{s-}) \, dZ_s, \quad t \in \mathbb{R}^+.
\]
Theorem 5.2 Assume (F), (G). Let $L, U, H$ be $(\mathcal{F}_t)$ adapted processes with trajectories in $D(\mathbb{R}^+, \mathbb{R}^4)$ such that $L \leq H \leq U$ and $P(V_p(H)_T < \infty) = 1$. If $X_0$ is an $\mathcal{F}_0$ measurable random vector such that $L_0 \leq X_0 \leq U_0$ then \[(1.1)\] has a unique strong solution $(X, K)$. Moreover, if we define \{$(X^n, K^n)$\} to be a sequence of Picard’s iterations for \[(1.1)\], i.e. $(X^n, K^n) = ESP(X_0, L, U)$ for each $n \in \mathbb{N}$, $Y^n = X_0 + \int_0^T f(s, X^n_s) \, dA_s + \int_0^T g(s, X^n_s) \, dZ_s$ and $(X^n, K^n) = ESP(Y^n, L, U)$ then for every $T \in \mathbb{R}^+$,

$$
\hat{V}_p(X^n - X)_T \to 0, \quad P\text{-a.s. and } \hat{V}_p(K^n - K)_T \to 0, \quad P\text{-a.s.}
$$

Proof. From Theorem 5.3 we deduce that for every $\omega \in \Omega$ there exists a unique solution $(X(\omega), K(\omega)) = ESP(Y(\omega), L(\omega), U(\omega))$ and for every $T \in \mathbb{R}^+$,

$$
\hat{V}_p(X^n(\omega) - X(\omega))_T \to 0, \quad P\text{-a.s. and } \hat{V}_p(K^n(\omega) - K(\omega))_T \to 0, \quad P\text{-a.s.}
$$

Since for each $n \in \mathbb{N}$ the pair $(X^n, K^n)$ is $(\mathcal{F}_t)$ adapted, the pair of limit processes $(X, K)$ is $(\mathcal{F}_t)$ adapted as well, which completes the proof. \hfill $\square$

Let $B^H$ be a fractional Brownian motion (fBm) with Hurst index $H > 1/2$ and let $\sigma : \mathbb{R}^+ \to \mathbb{R}$ be a measurable function such that

$$
\|\sigma\|_{L_{1/2,H}} := \left( \int_0^T |\sigma_s|^{1/H} \, ds \right)^H < \infty, \quad T \in \mathbb{R}^+.
$$

One can observe that the process $Z^H = \int_0^T \sigma_s \, dB^H_s$ is a centered Gaussian process with continuous trajectories. Moreover, by [38 Theorem 1.1], for every $r > 0$,

$$
E|Z^H_{t_2} - Z^H_{t_1}|^r \leq C(r, H) \left( \int_{t_1}^{t_2} |\sigma_s|^{1/H} \, ds \right)^{rH}
$$

for all $0 \leq t_1 \leq t_2$. Hence for any subdivision $\pi = \{0 = t_0 < \ldots < t_n = T\}$ of $[0, T]$ we have

$$
\sum_{i=1}^n (E|Z^H_{t_i} - Z^H_{t_{i-1}}|)^{1/H} \leq (C(1,H))^{1/H} \sum_{i=1}^n \left( \int_{t_{i-1}}^{t_i} |\sigma_s|^{1/H} \, ds \right) = (C(1,H))^{1/H} \|\sigma\|_{L_{1/2,H}}.
$$

Therefore from [26 Theorem 3.2] it follows that if $p > 1/H$ then $P(V_p(Z^H)_T < \infty) = 1$ for $T \in \mathbb{R}^+$ (note also that $Z^H$ is a Dirichlet process from the class $D^{1/H}$ studied in [14]. To approximate $Z^H$ one can use the methods developed in [49].

We now show how to apply Theorem 5.2 and our approximation results of Section 4 to fractional SDEs with reflecting boundary condition. Let $B^H = (B^{H,1}, \ldots, B^{H,d})$, where $B^{H,1}, \ldots, B^{H,d}$ are independent fractional Brownian motions, and let $Z^H = (Z^{H,1}, \ldots, Z^{H,d})$, where $Z^{H,i} = \int_0^T \sigma^i_s \, dB^{H,i}_s$ with $\sigma^i : \mathbb{R}^+ \to \mathbb{R}$ such that $\|\sigma^i\|_{L_{1/2,H}} < \infty$ for $T > 0$, $i = 1, \ldots, d$.

Let $a : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with locally bounded variation. We consider fractional SDEs of the form

$$
X_t = X_0 + \int_0^t f(s, X_s) \, da_s + \int_0^t g(s, X_s) \, dZ^H_s + K_t, \quad t \in \mathbb{R}^+.
$$

Clearly, \[(5.1)\] generalizes classical fractional SDEs driven by $B^H$ and is a particular case of \[(1.5)\].
For any \( n \in \mathbb{N} \) we set
\[
X^n_t = X^n_{k/n}, \quad K^n_t = K^n_{k/n}, \quad t \in [k/n, (k+1)/n), \quad k \in \mathbb{N} \cup \{0\},
\]
where \( X^n_0 = X_0, \ K^n_0 = 0 \) and
\[
\begin{aligned}
\Delta Y^n_{(k+1)/n} &= f(k/n, X^n_{k/n})(a(k+1)/n - a/k/n) + g(k/n, X^n_{k/n})(Z^H_{(k+1)/n} - Z^H_{k/n}), \\
X^n_{(k+1)/n} &= \max \left( \min(X^n_{k/n} + \Delta Y^n_{(k+1)/n}, U_{(k+1)/n}), L_{(k+1)/n} \right), \\
K^n_{(k+1)/n} &= K^n_{k/n} + (X^n_{(k+1)/n} - X^n_{k/n}) - \Delta Y^n_{(k+1)/n}.
\end{aligned}
\]

**Corollary 5.3** Assume (F), (G). Let \( L, U, H \) be \((\mathcal{F}_t)\) adapted processes with continuous trajectories such that \( L \leq H \leq U \) and \( P(V_p(H)_T < \infty) = 1 \) for \( T \in \mathbb{R}^+ \). If \( X_0 \) is an \( \mathcal{F}_0 \)-measurable random vector such that \( L_0 \leq X_0 \leq U_0 \) then \((5.1)\) has a unique strong solution \((X, K)\). Moreover, if \( \{(X^n, K^n)\} \) is a sequence of approximation defined by \((5.2)\) and \((5.3)\) then for every \( T \in \mathbb{R}^+ \),
\[
\sup_{t \leq T} |X^n_t - X_t| \to 0, \quad P\text{-a.s.}, \quad \sup_{t \leq T} |K^n_t - K_t| \to 0, \quad P\text{-a.s.}
\]

**Proof.** It suffices to apply Theorem 5.2 and Theorem 4.1. The uniform convergence follows from the fact that if \( a \) is a continuous function and \( Z^H, L, U \) have continuous trajectories then also the solution \((X, K)\) has continuous trajectories. \( \square \)

Note that in the case where \( L, U \) may have jumps Theorem 4.1 implies weaker then \((5.4)\) convergence. Namely, we then have
\[
(X^n, K^n) \to (X, K), \quad P\text{-a.s. in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).
\]

### 6 Proof of Theorem 2.2

We follow the proof of \cite[Theorem 2.1]{23}.

**Step 1.** We assume additionally that \( y^1, y^2 \) and the barriers \( l, u \) are step functions of the form
\[
y^j_i = Y^j_i, \ l_t = L_i, \ u_t = U_i, \quad t \in [t_{i-1}, t_i), \quad i = 1, \ldots, n - 1
\]
and \( y^j_t = Y^j_t, \ l_t = L_n, \ u_t = U_n, \ t \in [t_{n-1}, t_n = T], \ j = 1, 2 \), for some partition \( 0 = t_0 < t_1 < \cdots < t_n = T \) of the interval \([0, T]\). By Remark 2.1(b), \( K^j_t = K^j_i, \ t \in [t_{i-1}, t_i), \ i = 1, \ldots, n - 1, \ k^j_i = K^j_n, \ t \in [t_{n-1}, t_n = T], \ j = 1, 2 \), where \( K^1_0 = K^2_0 = 0 \) and \( K^1_i = \max(\min(K^j_{i-1}, U_i - Y^j_i), L_i - Y^j_i), \ i = 1, \ldots, n, \ j = 1, 2 \). Clearly
\[
L_i - Y^j_i \leq K^j_i \leq U_i - Y^j_i, \quad i = 1, \ldots, n, \ j = 1, 2.
\]

Without loss of generality we may and will assume that \( v_p(k^1_s - k^2_s)_T > 0 \). Hence there exists \( i \) such that \( K^1_i \neq K^1_{i-1} \) or \( K^2_i \neq K^2_{i-1} \). Later on, without loss of generality we will assume that for any \( i = 1, \ldots, n - 1, \)
\[
K^1_i \neq K^1_{i-1} \text{ or } K^2_i \neq K^2_{i-1}
\]
(If \((6.2)\) does not hold then we set \( v_0 = 0, \)
\[
v_k = \inf \{i > v_{k-1}; K^1_i \neq K^1_{i-1} \text{ or } K^2_i \neq K^2_{i-1} \} \land n, \quad k = 1, \ldots, n,
\]
\[ \tilde{\eta} = \inf\{ k; v_k = n \}, \; \tilde{\gamma}_t^j = Y_{t,k}^j, \; \tilde{t}_k = L_{v_k}, \; \tilde{u}_t = U_{v_k}, \; t \in [t_{v_{k-1}}, t_{v_k}) \text{ for } k = 1, \ldots, \tilde{n} - 1, \]

\[ \tilde{\gamma}_t^j = Y_{\tilde{n}}, \; \tilde{t}_k = L_{\tilde{n}}, \; \tilde{u}_t = U_{\tilde{n}}, \text{ for } t \in [t_{v_{k-1}}, t_{v_k}) = T, \; j = 1, 2. \] Then (6.2) holds true for the functions \( \tilde{\gamma}_j, (\tilde{x}^1, \tilde{x}^2) = \text{ESP}(\tilde{\gamma}_j, \tilde{t}_k, \tilde{u}_t), \; j = 1, 2, \) and moreover, \( \tilde{\nu}_p(k^1 - k^2)_T = \tilde{\nu}_p(k^1 - \tilde{k}^2)_T \)

and \( \tilde{\nu}_p(y^1 - \tilde{y}^2)_T \leq \tilde{\nu}_p(y^1 - y^2)_T. \) Consequently,

\[ \tilde{\nu}_p(k^1 - \tilde{k}^2)_T \leq \tilde{\nu}_p(y^1 - \tilde{y}^2)_T \implies \tilde{\nu}_p(k^1 - k^2)_T \leq \tilde{\nu}_p(y^1 - y^2)_T. \]

It is clear that there exist numbers \( 0 = i_0 < i_1 < \ldots < i_m = n \) such that

\[ \nu_p(k^1 - k^2)_T = \sum_{k=1}^{m} \left| (K^1_{i_k} - K^1_{i_{k-1}}) - (K^2_{i_k} - K^2_{i_{k-1}}) \right|^p \]  

and

\[ (K^1_{i_k} - K^1_{i_{k-1}}) - (K^2_{i_k} - K^2_{i_{k-1}}) \neq 0, \; k = 1, \ldots, m. \]  

Hence, if \( m \geq 2 \) then for \( k = 2, \ldots, m \) we have

\[ ((K^1_{i_{k-1}} - K^1_{i_{k-2}}) - (K^2_{i_{k-1}} - K^2_{i_{k-2}})) \left( (K^1_{i_k} - K^1_{i_{k-1}}) - (K^2_{i_k} - K^2_{i_{k-1}}) \right) < 0. \]  

Indeed, if (6.5) is not satisfied then by (6.4),

\[ \left| (K^1_{i_{k-1}} - K^1_{i_{k-2}}) - (K^2_{i_{k-1}} - K^2_{i_{k-2}}) \right|^p + \left| (K^1_{i_k} - K^1_{i_{k-1}}) - (K^2_{i_k} - K^2_{i_{k-1}}) \right|^p \]

\[ < \left| (K^1_{i_{k-1}} - K^1_{i_{k-2}}) - (K^2_{i_{k-1}} - K^2_{i_{k-2}}) + (K^1_{i_k} - K^1_{i_{k-1}}) - (K^2_{i_k} - K^2_{i_{k-1}}) \right|^p \]

\[ = \left| (K^1_{i_k} - K^1_{i_{k-1}}) - (K^2_{i_k} - K^2_{i_{k-2}}) \right|^p, \]

which contradicts (6.3).

We will show that there exists \( 0 = i_0 \leq i_1 \) (resp. \( 0 = i_0 \leq i_1 \)) such that if \( K^2_{i_1} - K^2_{i_1} \leq 0 \) (resp. \( K^1_{i_1} - K^1_{i_1} \geq 0 \)) then

\[ Y^2_{r^1_{i_1}} - Y^1_{r^1_{i_1}} \leq K^1_{i_1} - K^2_{i_1} \] (resp. \( K^1_{i_1} - K^2_{i_1} \leq Y^2_{r^2_{i_1}} - Y^1_{r^1_{i_1}} \))

and for \( k = 2, \ldots, m \) there exist \( i_{k-1} \leq r^1_k, r^2_k \leq i_k \) such that

\[ Y^2_{r^1_k} - Y^1_{r^1_k} \leq K^1_{i_k} - K^2_{i_k} \leq Y^2_{r^2_k} - Y^1_{r^2_k}. \]

Fix \( k = 1, \ldots, m. \) Set \( r^j_k = \max\{ i \leq i_k : K^j_{i_k} = U_i - Y^j_i \text{ or } K^j_{i_k} = L_i - Y^j_i \}, \; j = 1, 2, \) with the convention that \( \max \emptyset = 0. \) By (6.2), \( r^1_k = i_k \) or \( r^2_k = i_k. \) Without loss of generality we may and will assume that \( r^1_k = i_k. \) Then we have three cases:

(a) \( K^1_{i_k} - K^2_{i_k} = (L_{i_k} - Y^1_{i_k}) - (L_{r^1_{i_k}} - Y^2_{r^1_{i_k}}) \) (or \( K^1_{i_k} - K^2_{i_k} = (U_{i_k} - Y^1_{i_k}) - (U_{r^2_{i_k}} - Y^2_{r^2_{i_k}}) \)),

(b) \( K^1_{i_k} - K^2_{i_k} = L_{i_k} - Y^1_{i_k} \) and \( r^2_k = 0 \) (or \( K^1_{i_k} - K^2_{i_k} = U_{i_k} - Y^1_{i_k} \) and \( r^2_k = 0 \)),

(c) \( K^1_{i_k} - K^2_{i_k} = (L_{i_k} - Y^1_{i_k}) - (U_{r^2_{i_k}} - Y^2_{r^2_{i_k}}) \) (or \( K^1_{i_k} - K^2_{i_k} = (U_{i_k} - Y^1_{i_k}) - (L_{r^2_{i_k}} - Y^2_{r^2_{i_k}}) \)).

By (6.1) in all the cases

\[ K^1_{i_k} - K^2_{i_k} = L_{i_k} - Y^1_{i_k} - K^2_{i_k} \leq L_{i_k} - Y^1_{i_k} - K^2_{i_k} + K^2_{i_k} - (L_{i_k} - Y^2_{i_k}) = Y^2_{i_k} - Y^1_{i_k}, \]

18
which implies that we can put \( r_k^\wedge = i_k \). In order to find \( r_k^\wedge \) we consider the cases (a), (b), (c) separately.

In case (a), if \( r_k^2 = i_k \) then

\[
K_{ik_1}^1 - K_{ik_2}^2 = L_{ik_1} - Y_{ik_1}^1 - (L_{ik_2} - Y_{ik_2}^2) = Y_{ik_1}^2 - Y_{ik_2}^1
\]

and we put \( r_k^\wedge = i_k \). If \( r_k^2 < i_k \) then we set \( r^* = \max\{i < i_k : K_i^1 = U_{i} - Y_{i}^1\} \lor r_k^2 \). Observe that \( K_{i_k}^2 = K_{i_k+1}^2 = \ldots = K_{i_k}^2 \). Since for \( r^* < v \leq i_k \), \( K_v = \max(K_{v-1}, L_v - Y_v^1) \), it follows by (6.2) that

\[
K_{r^*}^1 - K_{r^*+1}^2 < K_{r^*+1}^2 < \ldots < K_{i_k}^1 - K_{i_k}^2.
\]

From this it also follows that \( Y_{i_k}^2 - Y_{i_k}^1 < K_{i_k}^1 - K_{i_k}^2 \). Indeed, if \( r^* > r_k^2 \) (resp. \( r^* = r_k^2 \)) then \( K_{i_k}^1 = U_{r^*} - Y_{r^*}^1 \) (resp. \( K_{i_k}^2 = L_{r^*} - Y_{r^*}^2 \)) and by (6.1),

\[
K_{i_k}^1 - K_{i_k}^2 > K_{r^*}^1 - K_{r^*+1}^2 > \ldots > K_{i_k}^1 - K_{r^*}^1 > K_{i_k}^1 - K_{r^*}^1 = i_k - i_{k-1}.
\]

What is left is to put \( r_k^\wedge = r^* \) and show that \( i_{k-1} \leq r^* \). This is obvious if \( k = 1 \), so assume that \( k \geq 2 \) and \( i_{k-1} > r^* \). By (6.3), \( (K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2) > 0 \). From this (6.5) it follows that

\[
(K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2) < 0,
\]

which together with (6.8) implies that \( i_{k-2} \leq r^* \).

Using once again (6.8) we see that \( (K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2) < 0 \). Consequently,

\[
0 > (K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2) = (K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2).
\]

and

\[
|(K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2)|^p < |(K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2)|^p.
\]

Similarly,

\[
0 < (K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2) = (K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2).
\]

which implies that

\[
|(K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2)|^p < |(K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2)|^p.
\]

Combining (6.9) with (6.10) we obtain

\[
|(K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2)|^p + |(K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2)|^p
< |(K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2)|^p + |(K_{i_k}^1 - K_{i_k}^1) - (K_{i_k}^2 - K_{i_k}^2)|^p,
\]

which contradicts (6.3) and completes the proof of the fact that \( i_{k-1} \leq r^* \). Consequently, in case (a) we put \( r_k^\wedge = r^* \).
In case (b) (resp. (c)) we set \( r^* = \max\{i < i_k : K^1_i = U_i - Y^1_i\} \) (resp. \( r^* = \max\{i < i_k : K^2_i = L_i - Y^2_i\) or \( K^1_i = U_i - Y^1_i\}) For \( r^* < v \leq i_k \) we have \( K^1_v = \max(K^1_{v-1}, L_v - Y^1_v) \) and \( K^2_v = \min(K^2_{v-1}, U_v - Y^2_v) \). As in case (a) we conclude from this and (6.2) that that
\[
K^1_{r^*} - K^2_{r^*} < K^1_{r^*+1} - K^2_{r^*+1} < \cdots < K^1_{i_{k-1}} - K^2_{i_{k-1}}.
\]
By the argument used in case (a) we also show that \( i_{k-1} \leq r^* \). Moreover, if \( r^* > 0 \) and \( K^2_{r^*} = L_{r^*} - Y^2_{r^*} \) (resp. \( K^1_{r^*} = U_{r^*} - Y^1_{r^*} \)) then by (6.1),
\[
K^1_{i_{k-1}} - K^2_{i_{k-1}} > (L_{r^*} - Y^2_{r^*}) - (U_{r^*} - Y^1_{r^*}) \geq (L_{r^*} - Y^2_{r^*}) - (L_{r^*} - Y^1_{r^*}) = Y^2_{r^*} - Y^1_{r^*}
\]
(resp. \( K^1_{i_{k-1}} - K^2_{i_{k-1}} > (U_{r^*} - Y^1_{r^*}) - K^2_{i_{k-1}} \geq (U_{r^*} - Y^1_{r^*}) - (U_{r^*} - Y^2_{r^*}) = Y^1_{r^*} - Y^2_{r^*} \)). Therefore we put \( \tilde{r}_k = r^* \). Since \( k = 1 \) if \( r^* = 0 \), the proof of (6.6) and (6.7) is complete.
Now observe that by (6.7), if \( K^1_{i_k} - K^2_{i_k} > K^1_{i_{k-1}} - K^2_{i_{k-1}} \) for some \( k = 2, \ldots, m \) then
\[
0 < (K^1_{i_k} - K^2_{i_k}) - (K^1_{i_{k-1}} - K^2_{i_{k-1}}) \leq (Y^2_{r_k} - Y^2_{r_k}) - (Y^1_{r_{k-1}} - Y^1_{r_{k-1}}),
\]
which implies that
\[
|(K^1_{i_k} - K^2_{i_k}) - (K^1_{i_{k-1}} - K^2_{i_{k-1}})|^p \leq |(Y^1_{r_k} - Y^2_{r_k}) - (Y^1_{r_{k-1}} - Y^2_{r_{k-1}})|^p. \tag{6.11}
\]
Similarly, if \( K^1_{i_k} - K^2_{i_k} < K^1_{i_{k-1}} - K^2_{i_{k-1}} \) then
\[
|(K^1_{i_k} - K^2_{i_k}) - (K^1_{i_{k-1}} - K^2_{i_{k-1}})|^p \leq |(Y^1_{r_k} - Y^2_{r_k}) - (Y^1_{r_{k-1}} - Y^2_{r_{k-1}})|^p. \tag{6.12}
\]
In case \( k = 1 \), if \( K^1_{i_1} - K^2_{i_1} > 0 \) then
\[
|(K^1_{i_1} - K^2_{i_1}) - (K^1_{i_0} - K^2_{i_0})|^p = |K^1_{i_1} - K^2_{i_1}|^p \leq |Y^1_{r_1} - Y^2_{r_1}|^p \tag{6.13}
\]
and if \( K^1_{i_1} - K^2_{i_1} < 0 \) then
\[
|(K^1_{i_1} - K^2_{i_1}) - (K^1_{i_0} - K^2_{i_0})|^p \leq |Y^1_{r_1} - Y^2_{r_1}|^p. \tag{6.14}
\]
Putting together (6.11) – (6.14) we conclude that
\[
\sum_{k=1}^m |(K^1_{i_k} - K^2_{i_k}) - (K^1_{i_{k-1}} - K^2_{i_{k-1}})|^p \leq |Y^1_{r_1} - Y^2_{r_1}|^p + \sum_{k=2}^m |(Y^1_{r_k} - Y^2_{r_k}) - (Y^1_{r_{k-1}} - Y^2_{r_{k-1}})|^p,
\]
where \( \tilde{r}_k = r^*_k \) or \( \tilde{r}_k = r^\lor_k \) and \( i_{k-1} \leq \tilde{r}_k \leq i_k, k = 1, \ldots, m \). Hence
\[
\tilde{V}_p(k^1 - k^2)_T = V_p(k^1 - k^2)_T
\]
\[
= \left( \sum_{k=1}^m |(K^1_{i_k} - K^2_{i_k}) - (K^1_{i_{k-1}} - K^2_{i_{k-1}})|^p \right)^{1/p}
\]
\[
\leq \left( |Y^1_{r_1} - Y^2_{r_1}|^p + \sum_{k=2}^m |(Y^1_{r_k} - Y^2_{r_k}) - (Y^1_{r_{k-1}} - Y^2_{r_{k-1}})|^p \right)^{1/p}
\]
\[
\leq |y^1_0 - y^2_0| + \left( \sum_{k=1}^m |(y^1_{r_k} - y^2_{r_k}) - (y^1_{r_{k-1}} - y^2_{r_{k-1}})|^p \right)^{1/p}
\]
\[
\leq |y^1_0 - y^2_0| + \left( \sum_{k=1}^m |(y^1_{r_k} - y^2_{r_k}) - (y^1_{r_{k-1}} - y^2_{r_{k-1}})|^p \right)^{1/p}
\]
for some partition $0 = t_{\tilde{r}_0} < t_{\tilde{r}_1} < \cdots < t_{\tilde{r}_n} \leq T$, which proves the theorem in the case of step functions $y^1, y^2$ and step barriers $l, u$.

\textit{Step 2.} The general case.

Let $\{y^{1,n}\}, \{y^{2,n}\}, \{l^n\}$ and $\{u^n\}$ be sequences of discretizations of $y^1$, $y^2$, $l$ and $u$, respectively, i.e. $y^{1,n}_{t_i} = y^{1}_{k/n}$, $y^{2,n}_{t_i} = y^{2}_{k/n}$, $l^n_t = l_{k/n}$, $u^n_t = u_{k/n}$ $t \in [k/n, (k + 1)/n)$, $k \in \mathbb{N} \cup \{0\}$. By [21, Chapter VI, Proposition 2.2], $(y^{1,n}, y^{2,n}, l^n, u^n) \rightarrow (y^1, y^2, l, u)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^4)$.

Let $(x^{j,n}, k^{j,n}) = \text{ESP}(y^{j,n}, l^n, u^n)$, $n \in \mathbb{N}$, $j = 1, 2$. By (2.2), $(k^{1,n}, k^{2,n}, y^{1,n}, y^{2,n}) \rightarrow (k^1, k^2, y^1, y^2)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^4)$, which implies that

$$k^{1,n} - k^{2,n} \rightarrow k^1 - k^2 \quad \text{in} \ \mathbb{D}(\mathbb{R}^+, \mathbb{R}). \quad (6.15)$$

By Step 1, for $n \in \mathbb{N}$ and $T \in \mathbb{R}^+$ we have $\bar{V}_p(k^{1,n} - k^{2,n}T) \leq \bar{V}_p(y^{1,n} - y^{2,n}T)$. Clearly, $\bar{V}_p(y^{1,n} - y^{2,n}T) \leq \bar{V}_p(y^1 - y^2)T$, $n \in \mathbb{N}$, $T \in \mathbb{R}^+$. From this and (6.15) it follows that for every $T \in \mathbb{R}^+$ such that $\Delta k^1_T = \Delta k^2_T = \Delta y^1_T = \Delta y^2_T = 0$,

$$\bar{V}_p(k^1 - k^2) \leq \lim_{n \to \infty} \sup_n \bar{V}_p(y^{1,n} - y^{2,n})_T \leq \bar{V}_p(y^1 - y^2)_T.$$ 

To obtain the desired result for arbitrary $T \in \mathbb{R}^+$ we use right continuity of $\bar{V}_p(k^1 - k^2)$ and $\bar{V}_p(y^1 - y^2)$.

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