A NEW TIME CONTOUR FOR
THERMAL FIELD THEORIES

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ABSTRACT

A simpler new time contour for path-ordered approaches to real-time thermal field
theories is presented. In doing so the use of a so called ‘asymptotic’ condition as seen
in existing derivations is seen to be incorrectly applied but luckily unnecessary.

In this talk, I shall be concerned with path ordered approaches to equilib-
rium thermal field theory. These include both PORTF (path-ordered real-time
formalisms) of the various types and the ITF (imaginary-time formalism or
Matsubara method). The path-ordered approach to a real-time formalism is to
be distinguished from the Thermo Field Dynamics type approaches to real-time
thermal field theories. The latter includes the approach due to Umezawa and
collaborators as well as the axiomatic field theory version involving $C^*$-algebra
methods, since the two approaches are identical. Thermo Field Dynamics and
PORTF are related but they have important differences.

The starting point for path-ordered approaches to thermal field theory is the
thermal generating functional, $Z[j]$.\(^\dagger\)

$$Z[j] := \text{Tr} \{ e^{-\beta H} \exp \{ T_C \int_\tau d\tau \int d^3 \vec{x} \, j(\tau, \vec{x}) \phi(\tau, \vec{x}) \} \}$$

The sources $j$ are coupled to the fields, here generically denoted by $\phi$. In principle
there is a source for every field but for simplicity this will be represented by a
single $j\phi$ term. These sources are unphysical and are set to zero at the end of the
calculation. The $T_c$ indicates that the fields are path ordered with respect to the
relative order of their time arguments along a directed path, $C$, in the complex time
plane. Then, by using one’s favourite method, such as the path-integral or
operator methods, one can obtain Feynman rules, the effective action or whatever
else is required.

In order for path-ordered methods to work, the path $C$ starts at some arbitrary
time, say $\tau_{in}$, and then must end at a time $\tau_{out} = \tau_{in} - i\beta$. It has been suggested on
formal grounds that the path must also always have a decreasing imaginary part

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but this limitation is not seen in the Feynman rules. Here we stay also within this limitation. Any \( C \) satisfying these conditions may be choosen. Physical results are therefore independent of \( C \). One of the great advantages of the path-ordered approach to thermal field theory is that the different FTFT formalisms simply correspond to different choices for \( C \).

The Green functions which are generated from \( Z[j] \) are path ordered expectation values of fields where the fields are ordered according to the position of their time arguments on \( C \). Thus the physics is encoded in different ways for different choices of \( C \) and thus for different FTFT formalisms. All the thermodynamic information can be obtained from calculating the partition function \( Z = Z[j = 0] \).

The ITF approach uses the curve \( C = C_I \) of figure 1. ITF is good for static quantities which include all the bulk macroscopic information contained in the partition function. It is harder to extract dynamical information from ITF where real-times are required and thus difficult to extend the formalism to non-equilibrium.

\[ \begin{align*}
\Im m(\tau) & \quad C_I \\
0 & \quad C_1 \\
-\tau/2 & \quad C_2 \\
C_3 & \quad C_4 \\
\Re e(\tau) & \quad -\tau(1 - \alpha)\beta \\
& \quad -\tau\beta
\end{align*} \]

Figure 1: The curves for imaginary-time and old real-time approaches.

\(^1\)For instance the general curve I described elsewhere can slope upwards yet the Feynman rules are independent of this factor.
For these reasons a different curve $C$ was sought which would lead to a real-time formalism, similar to that obtained in the Thermo Field Dynamics approaches.

For a real-time formalism, one part of $C$ must run along the whole of the real time axis. The question is then how one completes $C$ which must finish $-i\beta$ below its starting point.

The traditional curve for PORTF is $C = C_{\text{old}} = C_1 \oplus C_2 \oplus C_3 \oplus C_4$ shown in figure [1]. The limit $T \to \infty$ is taken and the figure suggests that $0 \leq \alpha \leq 1$. Again no physical result depends on $\alpha$ suggesting any value can be taken provided we make use of the periodic or anti-periodic boundary conditions when redrawing the curve.

The sections running parallel with the real time axis, $C_1, C_2$, are not too bad to deal with as they can be parameterised by a real time. Thus a formalism close to familiar Minkowskii field theory is obtained from these sections. It is the vertical portions that are unfamiliar and which cause problems. In Thermo Field Dynamics approaches there are only two real fields in their formalisms, and so there is nothing which corresponds to the vertical sections $C_3, C_4$ of $C_{\text{old}}$. This suggests that only the $C_1$ and $C_2$ sections should be kept in PORTF. This is achieved in the literature through a process called factorisation, namely

$$Z[j] \rightarrow Z_{12}[j].Z_{34}[j]$$

where in $Z_{ab}$ all the fields and sources are limited to lie on $C_{ab}$,

$$C_{ab} = C_a \oplus C_b.$$ (3)

We can follow the usual derivation of this result in the path integral approach to PORTF. In this case

$$Z[j] = \exp\{i \int_C d\tau \, V[-i \frac{\partial}{\partial j}]\} \cdot Z_0$$

$$= \exp\{i \int_{C_{12}} d\tau \, V[-i \frac{\partial}{\partial j}]\}\exp\{i \int_{C_{34}} d\tau \, V[-i \frac{\partial}{\partial j}]\} \cdot Z_0,$$ (4)

$$Z_0[j] = \exp\{-\frac{i}{2} \int_c d\tau d\tau' \, j(\tau)\Delta_c(\tau - \tau')j(\tau').$$ (5)

Here the interaction terms in the Lagrangian are represented by the functional $V$ and it is easy to see that the interaction part factorises in Eq. (3).

The free part is expressed in terms of the free propagator $\Delta_C(\tau - \tau')$. For many systems of interest the propagator, $\Delta_C$, tends to zero as the real part of the time difference tends to infinity

$$\lim_{\text{Re} \{\tau - \tau'\} \to \infty} \Delta_c(\tau - \tau') = 0$$ (6)

This is essential in the usual derivations of PORTF as the $C_3$ and $C_4$ sections are at infinity whereas most of the $C_1$ and $C_2$ are not. For that perennial example, the relativistic scalar field, this condition is satisfied provided the solution of the
Klein-Gordon equation is suitably regularised\cite{3,5,10,11}. The Feynman $\epsilon$ regularisation achieves this. Strictly, $\epsilon$ must be left finite till the end of the calculation.

However, $Z_0$ in Eq. (3) includes non-zero contributions from regions where one integral is near an end of $C_1$ or $C_2$ and the other integral is running along $C_3$ or $C_4$. In these situations Eq. (3) can not be used but such contributions to $Z_0$ must be zero if factorisation is to be true. The usual solution, termed an “asymptotic condition”, is to say that the sources tend to zero for times lying at the ends of $C_1$ and $C_2$. Unfortunately this turns out to be unacceptable. The whole point of a generating function and of the separation in Eq. (4) is that the sources $j$ must not be fixed. In particular infinitesimal variations are needed for the derivatives in the interaction terms in Eq. (6). It makes the expression Eq. (6) meaningless if $j$ is set to zero in Eq. (5) in some regions.

One might try to get round this by switching off all interactions at the ends of $C_1$ and $C_2$, a proper asymptotic condition c.f.\cite{12}. However this means we have a time dependent Hamiltonian which invalidates the fundamental assumption of an equilibrium situation. It also means that the partition function, which is also a normalisation factor for the connected Green functions, is being disturbed. Finally, since the asymptotic condition is not used in ITF, it seems strange that PORTF would need this extra boundary condition.

In fact, however factorisation of Eq. (2) is enforced, it leads to a series of inconsistencies\cite{11}. The inescapable conclusion is that the generating function of PORTF does not factorise. However this does not alter the fact that if we do drop the $C_3$ and $C_4$ parts of the old PORTF curve, one obtains the same Feynman rules as one finds in the Thermo Field Dynamics approaches are obtained\cite{2,5}.

There are two ways out of this dilemma. One is to keep working with the whole of the old PORTF curve of figure 1 and to just use Eq. (3) to get the usual answers. For instance if this is done problems encountered with the normalisation of Green functions and with the partition function\cite{11} are avoided. However it is a bit cumbersome.

The alternative way out is to use a different curve for the PORTF. Such a curve must run along the whole real axis to get all real times accessible within the formalism. There must be only two sections, each parameterised by a real time parameter that runs between $\pm\infty$, by analogy with Thermo Field Dynamics methods and the existing successful Feynman rules for real-time formalisms. Likewise there should be no dropping of any sections. One curve, $C_{\text{new}} = C_1 \oplus C_{n2}$ which satisfies these criteria is where the curve is run straight back to the end point as shown in figure 2. This is a special case of the curves presented elsewhere\cite{1}. While we could of course parameterise $C_{n2}$ in terms of a real parameter, the gradient would surely hit you somewhere. In this case one must remember that the ends of the curve are going to be taken to infinity, $T \to \infty$, so that the gradient is going to become zero. The Feynman rules can be derived in time coordinates, with $T$ kept finite if required. The usual real-time Feynman rules in four-momentum space are obtained on taking $T \to \infty$ and then doing the Fourier transform\cite{1}.

It has been suggested that the vertical sections $C_3, C_4$ of the traditional PORTF
Figure 2: A new curve for real-time formalisms.

The gradient of $C_{n2}$ of the new curve can be ignored in calculating thermal Green functions provided Eq. (6) holds. It cannot, however, be ignored when calculating vacuum diagrams e.g. in a diagrammatic expansion of the partition function or Free energy. In this case one can use the trick in which a vacuum diagram is treated as a time integral multiplied by a tadpole type Green function diagram.

It turns out that the $C_3, C_4$ vertical sections of the old PORTF curve can be ignored in exactly the same circumstances that the gradient of the new curve can be neglected. Likewise when the gradient must be included, the effect of the vertical pieces is important. Thus the same problems are hiding in both methods. The new curve does however allow a much simpler and cleaner derivation of the Feynman
rules. It also emphasises that there is no need for any sort of “asymptotic condition”. This is true whatever sort of curve is used in path-ordered methods despite what is found in the literature on PORTF using the old curve. In particular parts of the standard PORTF derivations are clearly wrong yet the answers the resulting formalism gives is correct.

Finally when the vertical sections are needed, it not easy to see how to include their effects in the old PORTF whereas it is straightforward to keep the gradient terms in the new approach when they are needed. This is most important when long time correlations are not zero, i.e. Eq. (3) no longer holds for some of the fields in the problem. This occurs in certain models such as the Anderson model, and in certain physical situations e.g. near critical points. The new approach to PORTF using the curve of figure 2 is therefore likely to be of practical benefit.

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