Relationship between quandle shadow cocycle invariants and Vassiliev invariants of links for any finite quandles

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Abstract
As one of the problems in his list [24], T. Ohtsuki proposed to study relations between quandle cocycle invariants and quantum invariants. The aim of this paper is to answer one of those questions. We prove that all coefficients of the perturbative expansion of the quandle shadow cocycle invariant defined by any dihedral quandle are Vassiliev invariants for torus knots and double twist knots.

Keywords
Vassiliev invariants, Quandle (shadow) cocycle invariants.

1 Introduction
The present work is motivated by [24, §5.4 Quandle cocycle invariants], in particular the following problem:

Problem 1.1 ([24, Problem 5.7]). Find relations between quandle cocycle invariants and knots invariants known so far, such as quantum invariants.

M. Graña proved that the quandle cocycle invariants can be presented by knot invariants derived from certain ribbon categories [11]. A. Soloviev, develop a theory of non-unitary set-theoretical solutions to the Quantum Yang–Baxter equation [25]. S. Abe obtained Vassiliev invariants in the case where quandle (shadow) cocycles are calculated by using trivial quandles [1]. We knew only these three results, and Vassiliev invariants and quandle (shadow) cocycle invariants were thought to have little relation to each other.

In the rest of this section, “quandle shadow cocycle invariants” mean ones deduced by using dihedral quandles. We obtain the following main Theorems:

Theorem 1.2. Let \( p \) be an odd prime, \( K \) be an \((l,m)\)-torus knot, let \( \xi = \exp(2\pi\sqrt{-1}/p) \), let \( \phi \) be Mochizuki 3-cocycle, and let \( \Phi_\phi(K) \) be a quandle shadow cocycle invariant of \( K \). We have that

\[
\frac{\Phi_\phi(K)}{p^2}|_{t=\xi} = a_{p,0}(K) + a_{p,1}(K)(\xi - 1) + a_{p,2}(K)(\xi - 1)^2 + \cdots + a_{p,p-2}(K)(\xi - 1)^{p-2}
\in \mathbb{Z}[\xi] \cong \mathbb{Z}[t]/(1 + t + t^2 + \cdots + t^{p-1}).
\]

Here, for any \( d \in \mathbb{N} \), there exists an integer \( N \in \mathbb{N} \) such that

\[
\ell\lim_{p>N} a_{p,d}(K) = \lambda_d(K) \in \mathbb{Q}.
\]

Then, we obtain the following power series.

\[
\lambda_0(K) + \lambda_1(K)(t - 1) + \lambda_2(K)(t - 1)^2 + \cdots = \sum_{d=0}^{\infty} \lambda_d(K)(t - 1)^d \in \mathbb{Q}[[t - 1]].
\]
Thus, \( \lambda_d(K) \in \mathbb{Q} \) is a Vassiliev invariant of degree \( d \in \mathbb{Z}_{\geq 2} \) of \( K \), and we obtain the following equation:

\[
-6p\lambda_1(K) = \frac{d}{dt} \left( \Delta_K(t) \cdot t^{\frac{i+m-1}{2}} \right) |_{t=1}.
\]

Here \( \Delta_K(t) \) is an Alexander polynomial of \( K \).

**Theorem 1.3.** Let \( p \) be an odd prime, \( K \) be a double twist knot of Conway’s normal form \( C(2n, 2l) \), let \( \xi = \exp(2\pi\sqrt{-1}/p) \), let \( \phi \) be Mochizuki 3-cocycle, and let \( \Phi_{\phi}(K) \) be a quandle shadow cocycle invariant. We have that

\[
\Phi_{\phi}(K) |_{t=\xi} = a_{p,0}(K) + a_{p,1}(K)(\xi - 1) + a_{p,2}(K)(\xi - 1)^2 + \cdots + a_{p,p-2}(K)(\xi - 1)^{p-2} \in \mathbb{Z}[\xi] \cong \mathbb{Z}[t]/(1 + t + t^2 + \cdots + t^{p-1}).
\]

Here, for any \( d \in \mathbb{N} \), there exists \( N \in \mathbb{N} \) such that

\[
f_{p > N} \lim \ a_{p,d}(K) = \lambda_d(K) \in \mathbb{Q}.
\]

Then, we obtain the following power series.

\[
\lambda_0(K) + \lambda_1(K)(t - 1) + \lambda_2(K)(t - 1)^2 + \cdots = \sum_{d=0}^{\infty} \lambda_d(K)(t - 1)^d \in \mathbb{Q}[[t - 1]].
\]

Thus, \( \lambda_d(K) \in \mathbb{Q} \) is a Vassiliev invariant of degree \( d + 1 \in \mathbb{Z}_{\geq 2} \) of \( K \). For any torus knots and double twist knots, we can obtain Vassiliev invariants from quandle shadow cocycle invariants. This is expected to have applications to surface links and low-dimensional manifolds in the future.

In Section 2, we review the quandle (shadow) cocycle invariants. A quandle is a set with a binary operation satisfying certain axioms, which is analogous to a group with conjugation. The cohomology groups of quandles were introduced by Carter, Jelsovsky, Kamada, Langford and Saito [6] as an analogy of group cohomology. The \( n \)-th cohomology group of quandle \( X \) with coefficient group \( A \) shall be denoted by \( H^n_{Q}(X; A) \). Furthermore, the quandle shadow cocycle invariants of classical links were defined using 3-cocycles of the cohomology groups of quandles [6, 7].

In Section 3, we review the Vassiliev invariants. The notion of Vassiliev invariant was introduce by V.A. Vassiliev [27]. We know that after a suitable change of variables each coefficient in the Taylor expansion of the Jones polynomial is a Vassiliev invariant [4].

In Section 4, we drive perturbative expansion from quandle shadow cocycle invariants. It is the first time that we have ever derived perturbative expansion from quandle shadow cocycle invariants.

In Section 5, we prove that Vassiliev invariants can be deduced quandle shadow cocycle invariants by using Fermat-limits.

## 2 Quandle and Quandle shadow cocycle invariants

Firstly, we review the definitions of quandles and their cohomology groups of S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito [6].

A quandle is a set with a binary operation satisfying certain axioms, which is analogous to a group with the conjugation. It is a set with a self-distributive binary operation whose definition was motivated from knot theory.

A quandle is a set \( X \) with a binary operation \( * : X \times X \to X \) such that the following three conditions are satisfied:
(i) For any \( a \in X \), \( a \ast a = a \).

(ii) For any \( a, b \in X \), there exists a unique \( c \in X \) such that \( a = c \ast b \).

(iii) For any \( a, b, c \in X \), \( (a \ast b) \ast c = (a \ast c) \ast (b \ast c) \).

Cohomology groups of quandles were introduced by S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito [6], as an analogy of group cohomology; we denote by \( H^n_Q(X; A) \) the \( n \)-th cohomology group of a quandle \( X \) with a coefficient group \( A \).

We review the quandle cochain complex [6]. Let \((X, \ast)\) be a finite quandle and \( A \) be an abelian group. Consider an abelian group,

\[
C^n_Q(X; A) := \{ f : X^n \to A \mid f(x_1, \ldots, x_n) = 0 \text{ when } x_i = x_{i+1} \text{ for some } i \}.
\]

For \( n \geq 1 \), we define the coboundary map of the set above, \( \delta_n : C^n_Q(X; A) \to C^{n+1}_Q(X; A) \), as follows:

\[
\delta_n(f)(x_1, \ldots, x_{n+1}) := \sum_{i=2}^{n+1} (-1)^i \left( f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{n+1}) - f(x_1 \ast x_i, \ldots, x_{i-1} \ast x_i, x_{i+1}, \ldots, x_{n+1}) \right).
\]

We easily see that \( \delta_{n+1} \circ \delta_n = 0 \). The cohomology of this complex \( (C^n_Q(X; A), \delta_n) \) is denoted by \( H^n_Q(X; A) \) and called the \textit{quandle cohomology} of \( X \) with coefficient group \( A \).

Further, S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito defined invariants of links using the quandle cocycle. Especially, by using 3-cocycles of the cohomology groups of quandles, \textit{quandle shadow cocycle invariants} of links can be defined [6, 7]. To define the quandle shadow cocycle invariants, we first need to define the \textit{X-shadow coloring}.

Let \( D \) be a diagram of the oriented link \( L \), \( x \) be a crossing of the diagram \( D \), and \( A(D) \) be the set of arcs of \( D \). An \textit{X-coloring} \[7, \text{Definition}4.1\] of \( D \) is a map \( C : A(D) \to X \) satisfying the condition \( C(\gamma_1) \ast C(\gamma_2) = C(\gamma_3) \) in Figure 1 at each crossing \( x \) of \( D \). If \( X \) is a finite quandle, then the number of \( X \)-colorings of \( D \) is an invariant of link \( L \) \[14\]. Let \( R(D) \) be the set of regions of the underlying immersed curves of \( A(D) \). The map \( C : A(D) \sqcup R(D) \to X \) is an \textit{X-shadow coloring} of \( D \) \[\text{Definition}4.3 \[7\]\], as shown in Figure 1 for \( C|_{A(D)} : A(D) \to X \) and in Figure 2 for \( C|_{R(D)} : R(D) \to X \).

![Figure 1: The coloring at each crossing x.](image)

![Figure 2: Boxes indicate the colorings of regions](image)
Let a finite quandle $X$ and a map $\phi : X^3 \to A$ be given. A (Boltzmann) weight $W_\phi(x; C)$, at crossing $x$ is defined as follows:

$$W_\phi\left(\frac{x}{z} \bigg\downarrow \begin{array}{c} z \times y \end{array} \right) = \phi(z, x, y) \in A,$$

$$W_\phi\left(\frac{z}{x} \bigg\downarrow \begin{array}{c} x \times y \end{array} \right) = \phi(z, x, y)^{-1} \in A.$$

The following state sum is:

$$\sum_C \prod_x W_\phi(x; C) \in \mathbb{Z}[A].$$

If $\phi : X^3 \to A$ is a quandle 3-cocycle of $X$, then the above state sum is an invariant of link $L$. We denote this invariant by

$$\Phi_\phi(L).$$

$\Phi_\phi(L)$ is called the quandle shadow cocycle invariant.

S. Asami and S. Satoh calculated the quandle shadow cocycle invariant of torus knots [3]. M. Iwakiri calculated the quandle shadow cocycle invariant of double twist knots [12]. See the results in Section 5.

3 Vassiliev invariants

Secondly, we explain the definition of Vassiliev invariants. Let $\mathcal{K}$ be a vector space over $\mathbb{C}$ freely spanned by the isotopy classes of oriented knots in $S^3$. A singular knot is an immersion of $S^1$ into $S^3$, whose singularities are transversal double points. We regard a singular knot as a linear sum in $\mathcal{K}$ obtained by the relation shown in the following Figure 3.

![Figure 3: Singular point.](image)

Let $B$ be an abelian group, $\mathcal{K}_d$ denote the vector subspace of $\mathcal{K}$ spanned by singular knots with $d$ double points. A homomorphism map $v : \mathcal{K} \to B$ is called a Vassiliev invariant of degree $d$ if $v|_{\mathcal{K}_{d+1}} = 0$ [9, 10, 27].

The construction of the general quantum invariants is explained below.

**Theorem 3.1** ([2]). *The braid group on n strands can be presented as*

$$B_n := \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2, \quad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ for } |i - j| = 1 \rangle,$$

*where the braid $\sigma_i$ performs a positive crossing of the strands $i$ and $i+1$. Similarly, the braid $\sigma_i^{-1}$ performs a negative crossing of the strands $i$ and $i+1.*
Definition 3.2 ([16]). Let $V$ be a vector space over $\mathbb{C}$. We obtain a representation, $\psi_n : B_n \to \text{End}(V \otimes n)$, defined as

$$
\psi_n(\sigma_i) = (id_V) \otimes (i-1) \otimes R \otimes (id_V) \otimes (n-i-1).
$$

(1)

Such a map $\psi_n$ given in (1) always satisfies $\psi_n(\sigma_i \sigma_j) = \psi_n(\sigma_j \sigma_i)$ ($|i - j| \geq 2$). To obtain $\psi_n(\sigma_i \sigma_{i+1} \sigma_i) = \psi_n(\sigma_{i+1} \sigma_i \sigma_{i+1})$, the matrix $R$ is required to satisfy the following equation:

$$
(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R).
$$

We call this equation the Yang-Baxter equation, and its solution is called an $R$-matrix. For an $R$-matrix we obtain a representation $\psi_n$ of the braid group $B_n$ given by (1).

Theorem 3.3 (chap.I [26] and chap.X [17]). Let $L$ be an oriented link and $b \in B_n$ be a braid such that the closure is isotopic to $L$. If an $R$-matrix $R$ and a linear map $h \in \text{End}(V)$ satisfy the following relations:

- trace$_2 ((id_V \otimes h) \cdot R^\pm) = id_V$,
- $R : (h \otimes h) = (h \otimes h) \cdot R$,
- $h = \left( \begin{array}{cc} t^{-1/2} & 0 \\ 0 & t^{1/2} \end{array} \right)$,

then trace$(h^{\otimes n} \cdot \psi_n(b))$ is invariants of $L$.

Jones polynomial, which is a quantum invariant, can be derived from the above invariant.

If

$$
R = \left( \begin{array}{cccc} t^{1/2} & 0 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & t & t^{1/2} - t^{3/2} & 0 \\ 0 & 0 & 0 & t^{1/2} \end{array} \right)
$$

and

$$
h = \left( \begin{array}{cc} t^{-1/2} & 0 \\ 0 & t^{1/2} \end{array} \right),
$$

then trace$(h^{\otimes n} \cdot \psi_n(b))$ is a Jones polynomial. Moreover, Vassiliev invariants are derived from Jones polynomial by the following theorem.

Theorem 3.4 ([4]). For any link $L$ and any $k \in \mathbb{C}$, let $J_L(t)$ be the Jones polynomial of $L$. Then, the coefficient of $h^d$ in $J_L(t)|_{t = e^{kh}}$ is a Vassiliev invariant of degree $d$.

Proof. Let $D$ be a diagram of $L$. We associate matrix $R$ and its inverse, $R^{-1}$, to the positive and negative crossings of $D$ respectively. We substitute $t = e^{kh}$ in $J_L(t)$. Since

$$
R = \left( \begin{array}{cccc} t^{1/2} & 0 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & t & t^{1/2} - t^{3/2} & 0 \\ 0 & 0 & 0 & t^{1/2} \end{array} \right)
$$

and

$$
R^{-1} = \left( \begin{array}{cccc} t^{-1/2} & 0 & 0 & 0 \\ 0 & t^{-1/2} - t^{-3/2} & t^{-1} & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & 0 & t^{-1/2} \end{array} \right),
$$

$R = R^{-1}$ when $h = 0$. Therefore, $R - R^{-1}$ is a matrix whose entries are divisible by $h$. This difference is associated to the double point of a singular link that occurs in the definition of Vassiliev invariants. Theorem 3.3 shows that $J_L(t)$ is composed by the product of $R$-matrix $R$. Therefore, if $L$ is a singular link with exactly $d + 1$ singular points, then $J_L(t)|_{t = e^{kh}}$ is divisible by $h^{d+1}$. Hence, the coefficient of $h^d$ is equal to 0 for such singular links. \qed
4 Perturbative expansion of quandle shadow cocycle invariants

Thirdly, quandle shadow cocycle invariants have perturbative expansion. T. Le, J. Murakami and T. Ohtsuki showed that the quantum invariants of 3-manifolds have a perturbative expansion [18]. We review the Fermat-limit necessary to show that quandle shadow cocycle invariants $\Phi_\phi(L)$ have perturbative expansion.

**Definition 4.1.** Let $p$ be an odd prime. If $a \in \mathbb{Z}$, we denote by $\chi_p(a)$ its image in numbers in $\mathbb{Z}/p\mathbb{Z}$. If $b$ is not divisible by $p$ then $\chi_p(b)$ is invertible in $\mathbb{Z}/p\mathbb{Z}$ and we denotes its inverse by $\chi_p(1/b)$.

$$Q_p := \{ \frac{a}{b} \in \mathbb{Q} \mid \text{gcd}(a, b) = 1, \text{gcd}(p, b) = 1 \},$$

we define the homomorphism $\chi_p : Q_p \to \mathbb{Z}/p\mathbb{Z}$ by

$$\chi_p(\frac{a}{b}) = \chi_p(a)\chi_p(\frac{1}{b}).$$

Let $\{u_p\}_{p\text{ prime}}$ be a sequence of numbers in $\mathbb{Z}/p\mathbb{Z}$. Then we say that $\{u_p\}_{p\text{ prime}}$ admits a Fermat-limit if there exists a rational number $u \in Q_p$ and $N \in \mathbb{N}$ such that $\chi_p(u) \equiv u_p$ for all $p > N$. This is represented by

$$\lim_{p \to \infty} u_p = u.$$

We prove that all coefficient are unique when $\Phi_\phi(L)_{t = \xi}$ has a perturbative expansion.

**Lemma 4.2.** Fermat-limit is unique.

**Proof.** If there is another rational number $u' \in Q_p$ such that $\chi_p(u) = \chi_p(u')$. Let $u = a/b$ and $u' = c/d$. Since $u - u' = (ad - bc)/bd$ is divisible by all $p > N$, $ad - bc \equiv 0 \ (p)$ for all $p > N$. Hence,

$$u = \frac{a}{b} \equiv \frac{c}{d} = u'.$$

Let $\varepsilon := |u - u'|$. There exist primes $p > \max\{N, \varepsilon\}$ such that $\varepsilon \equiv 0 \ (p)$ and $p > \varepsilon > 0$. This is a contradiction. Hence, $u = u'$.

Let $p$ be an odd prime and let $\xi = \exp(2\pi\sqrt{-1}/p)$. We define the map $\varphi : \mathbb{Z}[t] \to \mathbb{Z}[\xi]$ by $\varphi(f(t)) := f(\xi)$ i.e. just replacing $t$ by $\xi$. Since $\varphi$ is a surjective homomorphism and Ker $\varphi = (1 + t + t^2 + \cdots + t^{p-1})$, we obtain the ring $\mathbb{Z}[t] / (T(t))$ is isomorphic to $\mathbb{Z}[\xi]$ where

$$T(t) = 1 + t + t^2 + \cdots + t^{p-1}.$$

Since $\Phi_\phi(L)$ is an invariant of $L$, $\Phi_\phi(L)_{t = \xi}$ is also an invariant of $L$.

**Lemma 4.3** ([23], Lemma 9.7.). Let $p$ be an odd prime and let $\xi = \exp(2\pi\sqrt{-1}/p)$. We consider $\Phi_\phi(L)_{t = \xi} \in \mathbb{Z}[\xi]$. We put

$$\Phi_\phi(L)_{t = \xi} = a_{p,0}(L) + a_{p,1}(L)(\xi - 1) + a_{p,2}(L)(\xi - 1)^2 + \cdots + a_{p,p-2}(L)(\xi - 1)^{p-2},$$

for some integers $a_{p,n}(L)$'s. Then $(a_{p,n}(L) \mod p) \in \mathbb{Z}/p\mathbb{Z}$ ($0 \leq n \leq p - 2$) is uniquely determined by $\Phi_\phi(L)_{t = \xi}$. 

6
Proof. The ring $\mathbb{Z}[\xi]$ is isomorphic to $\mathbb{Z}[t]/(T(t))$ where
\[
T(t) = 1 + t + t^2 + \cdots + t^{p-1} = \binom{p}{1} + \binom{p}{2}(t-1) + \cdots + \binom{p}{p}(t-1)^{p-1}.
\]
Since $p$ is an odd prime, $\binom{p}{k}$ is divisible by $p$ for $1 \leq k \leq p-1$. Hence $a_{p,k}$ is uniquely determined modulo $p$ by $\Phi_\varphi(L)|_{t=\xi} \in \mathbb{Z}[\xi]$.

5 Main Theorems

We prove the main Theorems. For any torus knots and double twist knots, Vassiliev invariants derive from quandle shadow cosycle invariants by using dihedral quandles.

A dihedral quandle is defined to be $\mathbb{Z}/p\mathbb{Z}$ with a binary operation given by $x \ast y = 2y-x$, where $\mathbb{Z}/p\mathbb{Z}$ means a cyclic group of order $p$.

Moreover, T. Mochizuki obtained the following equation \[20, 21\]:
\[
(x-y)\frac{1}{p}(y^p-2z^p+(2z-y)^p) \in H^3_Q((\mathbb{Z}/p\mathbb{Z}, \ast); \mathbb{F}_q).
\]
We call this cocycle the Mochizuki 3-cocycle.

**Theorem 5.1.** Theorem 1.2 is true. i.e. Let $p$ be an odd prime, $K$ be an $(l,m)$-torus knot, and let $\xi = \exp(2\pi\sqrt{-1}/p)$. If $l \in \mathbb{Z}$ is divisible by $p$, and if $m \in \mathbb{Z}$ is even, then S. Asami and S. Satoh \[3\] calculated quandle shadow cocycle invariants $\Phi_\varphi(K)$ as follows:
\[
\Phi_\varphi(K) = p \sum_{0 \leq i \leq p-1} t^{-\frac{i\pi}{p}} \in \mathbb{Z}[t]/(t^p-1) \cong \mathbb{Z}[\xi];
\]
otherwise, the invariant $\Phi_\varphi(K)$ is trivial. We have that
\[
\frac{\Phi_\varphi(K)}{p^2}|_{t=\xi} = a_{p,0}(K) + a_{p,1}(K)(\xi-1) + a_{p,2}(K)(\xi-1)^2 + \cdots + a_{p,p-2}(K)(\xi-1)^{p-2}
\]
\[
\in \mathbb{Z}[\xi] \cong \mathbb{Z}[t]/(1+t+t^2+\cdots+t^{p-1}).
\]
Here, for any $d \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that
\[
\lim_{p>N} a_{p,d}(K) = \lambda_d(K) \in \mathbb{Q}.
\]
Then, we obtain the following power series.
\[
\lambda_0(K) + \lambda_1(K)(t-1) + \lambda_2(K)(t-1)^2 + \cdots = \sum_{d=0}^{\infty} \lambda_d(K)(t-1)^d \in \mathbb{Q}[[t-1]].
\]
Thus, $\lambda_d(K) \in \mathbb{Q}$ is a Vassiliev invariant of degree $d \in \mathbb{Z}_{\geq 2}$ of $K$, and we obtain the following equation:
\[
-6p\lambda_1(K) = \frac{d}{dt}(\Delta_K(t) \cdot t^{\frac{p-1}{2}})|_{t=1}.
\]
Here $\Delta_K(t)$ is an Alexander polynomial of $K$. 


Proof. Firstly, we show that $\lambda_d(K)$ is a Vassiliev invariants of degree $d$ of $K$. We know that
\[ J_K(t) = t^{(l(l+1)/2)} \frac{1 - t^{l+1} - t^{m+1} + t^{l+m}}{1 - t^2} \in \mathbb{Z}[t, t^{-1}]. \]
We set
\[ J_K(e^{kh}) := \sum_{d=0}^{\infty} b_d(l, m)k^dh^{d} \in \mathbb{Q}[h]. \]
Because of Lemma 3.4, $b_d(l, m)k^d$ is a Vassiliev invariant of degree $d$. According to [3], $l$ is divisible by $p$ and $m$ is even, then
\[ \frac{\Phi_\varphi(K)}{p^2} = \frac{1}{p} \sum_{0 \leq i < p-1} t^{-\frac{ln}{4p}i^2} \in \mathbb{Z}[t]/(t^p - 1). \]
For any $d \in \mathbb{N}$ there exists an integer $N \in \mathbb{N}$ such that
\[ f\lim_{p \to \infty (p > N)} \frac{\Phi_\varphi(K)}{p^2} |_{t = \varepsilon} = f\lim_{p \to \infty (p > N)} \frac{1}{p} \sum_{i=0}^{p-1} t^\frac{ln}{4p}i^2 \in \mathbb{Z}[\varepsilon]/(\varepsilon^p - 1). \]
Here, $\ell = -\frac{ln}{2p}i^2 \in \mathbb{Z}/p\mathbb{Z}$ and
\[ iP_d = \begin{cases} 0 & (d = 0), \\ \ell(\ell - 1) \cdots (\ell - d + 1) & (\text{otherwise}). \end{cases} \]
We define $\deg_j(f)$ as the degree of the variable $j$ of the multivariable polynomial $f$. Because of the definition of Vassiliev invariants, for any $d \in \mathbb{Z}_{\geq 2}$,
\[ \deg_j(b_d(l, m)) = \deg_i(\lambda_d(K)) \quad \text{and} \quad \deg_m(b_d(l, m)) = \deg_m(\lambda_d(K)) \]
must hold, and then it is true.
Since $b_d(l, m)k^d$ and $\lambda_d(K)$ are multivariate polynomials with rational number coefficients of $k$, $l$, $m$ and $d$, when the value of $l$, $m$ and $d$ are fixed, there exists $k \in \mathbb{C}$ satisfying
\[ k = \sqrt[4]{\frac{\lambda_d(K)}{b_d(l, m)}}, \]
\[ \sqrt{-1} = e^{\frac{\pi\sqrt{-1}}{d}}. \]
Secondly, because of Lemma 4.2 and Lemma 4.3, the value of $\lambda_d(L) \in \mathbb{Q}$ is unique.
Thirdly, we prove that $-6p\lambda_1(K) = \frac{d}{dt} \left( \Delta_K(t) \cdot \frac{t^{l+n-1}}{p} \right) |_{t=1}$ in Example 5.3. \qed

**Theorem 5.2.** Theorem 1.3 is true. i.e. Let $p$ be an odd prime, $K$ be a double twist knot of Conway’s normal form $C(2n, 2l)$, and let $\xi = \exp(2\pi\sqrt{-1}/p)$. If $4ln + 1 \in \mathbb{Z}$ is divisible by $p$, then M. Iwakiri [12] calculated quandle shadow cocycle invariants $\Phi_\varphi(K)$ as follows:
\[ \Phi_\varphi(K) = p^2 \sum_{0 \leq i \leq p-1} t^{-\frac{(4ln+1)s^2}{p}} \in \mathbb{Z}[t]/(t^p - 1) \cong \mathbb{Z}[F_2], \]
where $2ls \equiv 1 \pmod{4ln + 1}$ and $s < 4ln + 1$; otherwise, the invariant $\Phi_\varphi(K)$ is trivial. We have that
\[ \frac{\Phi_\varphi(K)}{p^3} |_{t = \varepsilon} = a_{p, 0}(K) + a_{p, 1}(K)(\xi - 1) + a_{p, 2}(K)(\xi - 1)^2 + \cdots + a_{p, p-2}(K)(\xi - 1)^{p-2} \]
∈ \mathbb{Z}[\xi] \cong \mathbb{Z}[t]/(1 + t + t^2 + \cdots + t^{p-1}).

Here, for any \( d \in \mathbb{N} \), there exists \( N \in \mathbb{N} \) such that

\[
\lim_{p > N} a_{p,d}(K) = \lambda_d(K) \in \mathbb{Q}.
\]

Then, we obtain the following power series.

\[
\lambda_0(K) + \lambda_1(K)(t-1) + \lambda_2(K)(t-1)^2 + \cdots = \sum_{d=0}^{\infty} \lambda_d(K)(t-1)^d \in \mathbb{Q}[|t-1|].
\]

Thus, \( \lambda_d(K) \in \mathbb{Q} \) is a Vassiliev invariant of degree \( d+1 \in \mathbb{Z}_{\geq 2} \) of \( K \).

**Proof.** We know that

\[
J_{K}(t) = 1 - \frac{(t^2-1)(t^2n-1)(t^2 + t + 1)}{(1+t)^2} \in \mathbb{Z}[t^\pm].
\]

We can prove that \( \lambda_d(K) \in \mathbb{Q} \) is a Vassiliev invariant of degree \( d+1 \in \mathbb{Z}_{\geq 2} \) of \( K \) in the same way as Theorem 1.2. We set

\[
J_K(e^{\hbar}) := \sum_{d=0}^{\infty} b_{d,l,n} k^d \hbar^d \in \mathbb{Q}[\hbar].
\]

Because of Lemma 3.4, \( b_{d,l,n} k^d \) is a Vassiliev invariant of degree \( d \). According to [12], \( 4ln + 1 \) is divisible by \( p \), then

\[
\Phi_{\phi}(K) \equiv 1 \pmod{p} = \frac{\sum_{i=0}^{p-1} \ell (\ell - i)^d}{p} \in \mathbb{Z}/(t^p - 1).
\]

For any \( d \in \mathbb{N} \) there exists an integer \( N \in \mathbb{N} \) such that

\[
\lim_{p > N} \frac{\Phi_{\phi}(K)}{p^3} \big|_{t=\xi} = \lim_{p > N} \sum_{i=0}^{p-1} \ell P_d (\xi - 1)^d = \sum_{d=0}^{\infty} \lambda_d(K)(t-1)^d \in \mathbb{Q}[|t-1|].
\]

Here, \( \ell = -\frac{(4ln + 1)s}{p} i^2 \in \mathbb{Z}/p \mathbb{Z} \) and

\[
\ell P_d = \begin{cases} 0 & (d = 0), \\ \ell (\ell - 1) \cdots (\ell - d + 1) & \text{(otherwise)}. \end{cases}
\]

Because of the definition of Vassiliev invariants, for any \( d+1 \in \mathbb{Z}_{\geq 2} \),

\[
\deg_k (b_{d+1}(l,n)) = \deg_k (\lambda_d(K)) \quad \text{and} \quad \deg_n (b_{d+1}(l,n)) = \deg_n (\lambda_d(K))
\]

must hold, and then it is true.

Since \( b_{d+1}(l,n) k^{d+1} \) and \( \lambda_d(K) \) are multivariate polynomials with rational number coefficients of \( k, l, n \) and \( d \), when the value of \( l, m \) and \( d \) are fixed, there exists \( k \in \mathbb{C} \) satisfying

\[
k = \frac{\lambda_d(K)}{b_{d+1}(l,n)},
\]

\[
d+1 \sqrt{-1} = e^{2\pi i/d+1}.
\]

Secondly, because of Lemma 4.2 and Lemma 4.3, the value of \( \lambda_d(K) \in \mathbb{Q} \) is unique. \( \square \)
Let $p$ be an odd prime number. We know that $H_3^2((\mathbb{F}_q, \omega); \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ [22]. Hence, we need Theorem 1.2 to deduce Vassiliev invariants from quandle shadow cocycle invariants by using the quandle cocycle of [20, 21].

**Example 5.3.** Let $K$ be $(l, n)$-torus knots.

$$J_K(e^{kh}) = 1 - \frac{1}{8}(l^2 - 1)(m^2 - 1)k^2h^2 - \frac{1}{24}ln(t^2 - 1)(m^2 - 1)k^3h^3 + O(h^4).$$

In addition, we consider the following quandle shadow cocycle invariant:

$$\Phi_{\phi}(K) = \frac{1}{p^2} \sum_{0 \leq i \leq p-1} t^{-\frac{lm}{2p^2}} \in \mathbb{Z}[t]/(t^p - 1).$$

(i) In the case of $d = 1$; Since $\Delta_K(t) = \frac{(1 - t)(1 - t^m)}{(1 - t^l)(1 - t^m)}$,

$$d \frac{d}{dt} \left( \frac{(1 - t)(1 - t^m)}{(1 - t^l)(1 - t^m)} \right)_{t=1} = \frac{lm}{2},$$

$$\frac{a_{p,1}(K)}{p} = -\left( \frac{lm}{2p^2} \right) \sum_{0 \leq i \leq p-1} i^2 = -\left( \frac{lm}{2p} \right) \frac{1}{6}(p-1)(2p-1).$$

Hence,

$$\lambda_1(K) = \lim_{p \to \infty} -\left( \frac{lm}{2p} \right) \frac{1}{6}(p-1)(2p-1) = -\frac{lm}{12p}.$$ 

Therefore,

$$-6p\lambda_1(K) = \frac{lm}{2}.$$ 

(ii) In the case of $d = 2$; $k = \sqrt{\frac{lm(lm - 10p)}{15(l^2 - 1)(m^2 - 1)p^2}},$

$$\frac{a_{p,2}(K)}{p} = -\left( \frac{lm}{2p^2} \right) \sum_{0 \leq i \leq p-1} \left( -\left( \frac{lm}{2p} \right) i^4 - i^2 \right)$$

$$= -\left( \frac{lm}{2p} \right) \left( -\left( \frac{lm}{2p} \right) \frac{1}{30}(p-1)(2p-1)(3p^2 + 3p - 1) - \frac{1}{6}(p-1)(2p-1) \right).$$

Hence,

$$\lim_{p \to \infty} \left( -\left( \frac{lm}{2p} \right) \left( -\left( \frac{lm}{2p} \right) \frac{1}{30}(p-1)(2p-1)(3p^2 + 3p - 1) - \frac{1}{6}(p-1)(2p-1) \right) \right)$$

$$= -\frac{l^2m^2}{120p^2} + \frac{lm}{12p} = b_2(l, m)k^2.$$ 

Because of Theorem 1.2, $-l^2m^2/120p^2 + lm/12p$ is one of the Vassiliev invariants of degree 2.

We obtain the 2-twist-spun knots [3] of the $(l, m)$-torus knots of Vassiliev invariants through Example 5.3. Thus, we consider the following problem of surface 2-knots:

**Problem 5.4.** (1) Do singular curves corresponding to singular points exist?

(2) Can one define quantum invariants for 2-knots?
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