Q-stars in 2 + 1 dimensions

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Abstract

We study q-stars with one or two scalar fields, non-abelian, and fermion-scalar q-stars in 2 + 1 dimensions in an anti de Sitter or flat spacetime. We fully investigate their properties, such as mass, particle number, radius, numerically, and focus on the matter of their stability against decay to free particles and gravitational collapse. We also provide analytical solutions in the case of flat spacetime and other special cases.

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1 Introduction

The investigation of scalar field configurations coupled to gravity started with the work by Kaup, [1], and Ruffini and Bonazzola, [2]. They regarded a complex scalar field with no self-interactions. Quartic self interactions were taken into account in other works [3, 4]. Scalar fields with a local $U(1)$ symmetry were also investigated [5]. The results were generalized in a series of papers in other gravity theories [6, 7, 8, 9].

Soliton stars appeared in the literature as stable field configurations consisted of abelian or non-abelian scalar fields, containing sometimes fermions or gauge bosons coupled to gravity [10, 11, 12, 19, 20, 21, 22, 24]. These configurations are stable even in the absence of gravity. A certain class of soliton stars are very large field configurations with radius of the order of lightyears [10, 11, 12]. They consist of a very large interior with very low energy density of “kinetic” type resulting from the time variation of the scalar field. On the other hand the potential energy within the soliton is considered to be negligible. Also, the metric and matter fields vary very slowly in the soliton interior. A considerable amount of energy, comparable to that stored in the huge interior, is contained in the thin surface. Within the surface both the potential and the energy resulting from the radial variation of the matter field contribute to the total surface energy, but the contribution of the time variation of the field is very small, as it is supposed to rotate in its internal $U(1)$ space extremely slowly.

Another class of non-topological soliton stars, the q-stars, appeared as relativistic generalizations of a certain family of non-topological solitons, namely q-balls. Q-balls have a certain role in particle theory and especially in the baryogenesis through the flat MSSM directions [17]. They appear in Lagrangians [13, 14, 15, 16], with a global $U(1)$ or $SU(3)$, $SO(3)$ symmetry, or a local $U(1)$ [18], when the scalar field takes the special value which minimizes the $U/\phi^{2}$ quantity.

The properties of the q-stars have been studied thoroughly in [19, 20, 21, 22, 23, 25]. There are q-stars with one or two scalar fields, [19, 25], q-stars with non-abelian fields, [23], and q-stars with a fermion and a scalar field, [20, 21, 22]. These objects contain a large interior within which both metric and matter fields vary smoothly. The energy within the interior is the sum of the potential energy and of the “kinetic” energy resulting from the time variation of the matter field. In the case of q-balls and, consequently, the case q-stars the potential is always positive. We will name $U$, $W$ and $V$ the potential energy, the energy resulting from the time variation and the
energy resulting from the space variation respectively. There is a crucial relation that roughly describes the q-solitons and differentiates them from other classes of non-topological solitons, namely:

\[ \omega \sim \phi \sim m , \]  

(1.1)

where \( \omega \) is the frequency with which the soliton rotates in its internal \( U(1) \) space, or, with proper generalizations that we describe, in more complicated spaces, \( \phi \) is a typical value of the or one of the scalar field(s) within the soliton and \( m \) is the mass of the free particles. This relation means that \( W \) is of the same order of magnitude with \( U \), in contrast with other non-topological solitons for which the \( U \) is negligible in the interior and \( W \) negligible within the surface. The common feature between q-stars and other soliton stars is the smooth variation of the matter field in the interior and the rapid change, within the surface, from a \( \phi_0 \) value to zero. In non-topological soliton stars, the energy density within the surface is huge compared to the energy density in the interior, and so, although the surface is very thin, the energy, contained within it, is comparable to the energy stored in the interior. This does not hold true for the q-stars. They have an approximately constant energy density in the interior, the sum of \( W \) and \( U \). At the surface, these quantities retain the same order of magnitude and \( V \) in now added with the same order of magnitude as well. Because the energy density is everywhere of the same order of magnitude, either in the interior, or in the surface, and because the surface is of width of \( m^{-1} \), the total energy of the thin surface is negligible. It has been proved that q-stars are smaller than non-topological soliton stars but larger than boson stars with no soliton features in the absence of gravity.

Gravity in \( 2 + 1 \), [29, 30], dimensions coupled to scalar fields is studied for many reasons. One of them is the theoretical interest for a theory qualitatively different from the corresponding theory in \( 3 + 1 \) dimensions. The solutions in a \( 3- \)dimensional theory of gravity are much more simpler because there is no \( A/r^2 - 1 \) term in the Einstein tensor, where \( A \) is a metric field as we will see. This can lead even to analytical solutions for special field configurations composed of scalar fields (or fermion-scalar) coupled to \( 3- \)dimensional gravity [26, 27, 28].

The aim of the present article is to study the formation of q-stars in \( 2 + 1 \) dimensions in anti anti de Sitter or flat spacetime, as a limiting case. We want to investigate the matter of their stability with respect to gravitational collapse and to fission into free particles and to study the influence of the global spacetime curvature in the features of the q-star. We study four different kinds of q-stars, namely, with one and two scalar fields, non-abelian
q-stars and fermion-scalar q-stars. We thoroughly investigate their properties, such as soliton radius, mass, particle number, as functions of either the cosmological constant, which is an absolutely independent parameter, or as functions of their internal frequency, a quantity with crucial role in the theory of non-topological solitons. We also give analytical solutions for the important case that the cosmological constant tends to zero and also for other cases, which show a very close similarity in their behavior, when compared to solutions obtained with the usual numerical methods.

2 A q-star with one scalar field

We consider a static, spherically symmetric metric:

\[ ds^2 = -e^\nu dt^2 + e^\lambda d\rho^2 + \rho^2 d\alpha^2 \]

(2.1)

with \( g_{tt} = -e^\nu \). We regard a scalar field with the minimum-energy time dependence:

\[ \phi(\bar{\rho}, t) = \sigma(\rho)e^{-\nu t} \]

(2.2)

The action in natural units for a scalar field coupled to gravity in 2+1 dimensions is:

\[ S = \int d^3x \sqrt{-g} \left[ R - \frac{2\Lambda}{16\pi G} + g^{\mu\nu}(\partial_\mu \phi)^*(\partial_\nu \phi) - U \right], \]

(2.3)

where \( \Lambda \) stands for the cosmological constant, regarded here to be negative, or zero, as a limiting case. The energy-momentum tensor is:

\[ T_{\mu\nu} = (\partial_\mu \phi)^*(\partial_\nu \phi) + (\partial_\mu \phi)(\partial_\nu \phi)^* - g_{\mu\nu}[g^{\alpha\beta}(\partial_\alpha \phi)^*(\partial_\beta \phi)] - g_{\mu\nu}U. \]

(2.4)

The theory need not be fundamental and, thus, renormalizable but effective. The Euler-Lagrange equation for the matter field is:

\[ \frac{1}{\sqrt{|g|}} \partial_\mu(\sqrt{|g|}g^{\mu\nu} \partial_\nu) - \frac{dU}{d\sigma} \frac{1}{d\sigma^2} \phi = 0, \]

(2.5)

taking now the form:

\[ \sigma'' + \left[ \frac{1}{\rho} + (1/2)(\nu' - \lambda') \right] \sigma' + e^\lambda \omega^2 e^{-\nu} \sigma - e^\lambda \frac{dU}{d\sigma^2} \sigma = 0. \]

(2.6)

The Einstein equations are:

\[ G^\mu_\nu = R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R = 8\pi GT^\mu_\nu - \Lambda \delta^\mu_\nu. \]

(2.7)
With the above assumptions the Einstein equations take the form:

\[-e^{-\lambda} \frac{\lambda'}{2\rho} = 8\pi G (W - V - U) - \Lambda, \tag{2.8}\]

\[e^{-\lambda} \frac{\nu'}{2\rho} = 8\pi G (W + V - U) - \Lambda, \tag{2.9}\]

where

\[W \equiv e^{-\nu} \left( \frac{\partial \phi}{\partial t} \right)^* \left( \frac{\partial \phi}{\partial t} \right) = e^{-\nu} \omega^2 \sigma^2, \tag{2.10}\]

\[V \equiv e^{-\lambda} \left( \frac{\partial \phi}{\partial \rho} \right)^* \left( \frac{\partial \phi}{\partial \rho} \right) = e^{-\lambda} \sigma'^2.\]

There is a Noether current:

\[j^\mu = \sqrt{-g} g^{\mu\nu} \left( \phi^* \partial_\nu \phi - \phi \partial_\nu \phi^* \right). \tag{2.11}\]

The current is conserved according to the equation:

\[j^\mu_{;\mu} = 0. \tag{2.12}\]

The total charge is defined as:

\[Q = \int d^2 x j^0, \tag{2.13}\]

now taking the form

\[Q = 4\pi \int \rho d\rho \omega \sigma^2 e^{-\nu/2} e^{\lambda/2}. \tag{2.14}\]

This charge represents the total particle number if the charge of each particle is unity. In the present case the energy of the free particles with the same charge is \(mQ\), where \(m\) is the mass of the free particles. We rescale all the parameters of the Lagrangian with respect to the mass, and set \(m\) equal to unity. So, if the total charge is less than the total mass of the field configuration, the star is stable with respect to decay into free particles. We define for convenience:

\[B = e^{-\nu}, \quad A = e^{-\lambda}. \tag{2.15}\]
and rescale:

\[
\begin{align*}
\tilde{\rho} &= \rho m , \quad \tilde{\omega} = \omega / m , \quad \tilde{\sigma} = \sigma / m^{1/2} , \\
\tilde{U} &= U / m^3 , \quad \tilde{W} = W / m^3 , \quad \tilde{V} = V / m^3 ,
\end{align*}
\]  

(2.16)

where \( m \) in generally is a mass scale, here the mass of the scalar field. We also make the rescalings

\[
\tilde{\Lambda} 8\pi Gm^3 \equiv \Lambda , \quad \tilde{r} = \sqrt{8\pi Gm\tilde{\rho}} .
\]  

(2.17)

Inside the soliton both metric and matter fields vary very slowly with respect to the radius. This leads to a considerable simplification to both Einstein and Lagrange equations. We see that \( V \propto \phi(0)^2 / P^2 \propto m^3 \sigma(0)^2 / \tilde{P}^2 \propto Gm^4 \tilde{\sigma}^2 / \tilde{R}^2 \) which is very small compared to the other energy quantities. \( P \) is the (unrescaled) soliton radius and \( \tilde{P} \) and \( \tilde{R} \) correspond to the rescaled radii, according to the rescalings under 2.16 and 2.17 respectively.

So, ignoring the \( O(Gm^4) \) quantities and dropping the tildes the Einstein equations take the form:

\[
\begin{align*}
- \frac{dA}{dr} \frac{1}{2r} &= \omega^2 \sigma^2 B + U + \Lambda , \\
- \frac{A dB}{B dr} \frac{1}{2r} &= \omega^2 \sigma^2 B - U - \Lambda ,
\end{align*}
\]  

and the Euler-Lagrange equation is:

\[
\omega^2 \sigma B - \frac{1}{2} \frac{dU}{d\sigma} = 0 .
\]  

(2.20)

A rescaled potential admitting q-ball type solutions in the absence of gravity is:

\[
U = |\phi|^2 - |\phi|^4 + \frac{1}{3} |\phi|^6 = \sigma^2 \left( 1 - \sigma^2 + \frac{1}{3} \sigma^4 \right) .
\]  

(2.21)

The solution to the eq. 2.20 is:

\[
\sigma^2 = 1 + \omega \sqrt{B}
\]  

(2.22)

and \( U \) is:

\[
U = \frac{1}{3} (1 + \omega^3 B^{3/2}) .
\]  

(2.23)
Within the thin surface the metric and matter fields change rapidly. We can write the Lagrange equation in the following form (unrescaled quantities):

\[
\left( \frac{A}{\rho} - \frac{1}{2} \frac{A}{B} \frac{dB}{d\rho} \right) \frac{d\sigma}{d\rho} = \frac{1}{2} \frac{\delta(U - W - V)}{\delta \sigma}. \tag{2.24}
\]

There is a certain form for the \( g_{11}(r) \) metric component (unrescaled quantities) [31], [32]:

\[-g_{11}^{-1}(\rho) \equiv A(\rho) = 1 - \frac{2GE_\rho}{\rho^{D-3}} - \frac{2\Lambda \rho^2}{(D-2)(D-1)}, \tag{2.25}\]

where \( E_\rho \) is the total energy density contained within a sphere of radius \( \rho \) and \( D \) is the spacetime dimensionality, here equal to 3. The above relation can be verified numerically and will be discussed more extensively when dealing with q-stars with two scalar fields, which offer a richer variety of analytical solutions. Using the relation holding within the surface approximately namely the: \( A \sim 1/B \) and rescaling in the usual way, we find that:

\[
\frac{1}{2} \frac{\delta(U - W - V)}{\delta \sigma} \sim \sqrt{8\pi G m} \left[ -\frac{1}{\pi} \int_0^\tilde{r} d^2 \tilde{r} \tilde{\varepsilon} - \frac{1}{\pi} \tilde{\varepsilon} \tilde{r} \frac{d}{d\tilde{r}} \int_0^\tilde{r} d^2 \tilde{r} \tilde{\varepsilon} - 2\tilde{\Lambda} \tilde{r}^2 \right] m^4, \tag{2.26}\]

where we regarded that within the surface the field decreases from a \( \sigma_0 \) value to zero and that \( \sigma_0 \sim m^{1/2} \). Also, the surface width is \( \sim m^{-1} \). So, dropping the \( O(\sqrt{Gm}) \) quantities, we find:

\[
\frac{\delta(W - U - V)}{\delta \sigma} = 0. \tag{2.27}\]

Because at the outer edge of the surface all energy quantities are zero, we find a first integral to the above equation:

\[
V + U - W = 0. \tag{2.28}\]

Eqs. 2.26[2.28] hold true only within the surface. At the inner edge of the surface \( \sigma' \) is zero in order to match the interior with the surface solution. So, at the inner edge of the surface the equality \( U = W \), together with eqs. 2.22[2.23] gives the eigenvalue equation for \( \omega \):

\[
\omega = \frac{B_{\text{sur}}^{-1/2}}{2} = \frac{A_{\text{sur}}^{1/2}}{2}. \tag{2.29}\]

which has the right limiting value when gravity is absent, i.e.: \( A_{\text{sur}} \to 1 \). So, \( \omega \), apart from a certain parameter referring to the soliton properties, is
also a measure for the gravity strength, equivalent to the metrics. The same discussion, eqs. 2.24 and 2.29, holds true in any case of q-star.

Solving the Einstein equations and in order to find the soliton parameters, we use the $G^0_0$ component of the Einstein tensor for the calculation of the energy or, equivalently, the form

$$E = \pi (1 - A(r) - \Lambda r^2) , \quad r \to \infty ,$$

(rescaled quantities) arising from eq. 2.25 and eq. 2.14 for the calculation of the charge. The thin surface has a negligible energy and charge contribution. The independent variables are two: The cosmological constant, a global feature of the spacetime, and the soliton frequency.

Outside the soliton the energy quantities are all zero. Einstein equations can be solved analytically:

$$A(r) = A_{\text{sur}} - \Lambda r^2 + \Lambda R^2 = 1 - \frac{E}{\pi} - \Lambda r^2 , \quad (2.30)$$

$$B(r) = \frac{A_{\text{sur}} - \Lambda r^2 + \Lambda R^2}{A^2_{\text{sur}}} = \left(1 - \frac{E}{\pi} - \Lambda r^2 \right)^{-1} , \quad (2.31)$$

with $R$ the star radius and $E$ the total mass. The same relations hold true for every case of q-star we discuss.

We will now give the analytic solution holding true when $\Lambda \to 0$. In this case the metric $B$ is constant everywhere and the other parameters of the soliton are given in the following equations:

$$B(r) = 1/A_{\text{sur}} = \frac{1}{4\omega^2} , \quad (2.32)$$

$$A(r) = 1 - \frac{3}{4} r^2 , \quad (2.33)$$

$$R = \sqrt{\frac{4}{3}(1 - A_{\text{sur}})} = \sqrt{\frac{4}{3}(1 - 4\omega^2)} , \quad (2.34)$$

$$E = \pi (1 - A_{\text{sur}}) = \pi (1 - 4\omega^2) , \quad (2.35)$$

$$Q = 4\pi (1 - \sqrt{A_{\text{sur}}}) = 4\pi (1 - 2\omega) . \quad (2.36)$$

In the above equations $R$ is the soliton radius determined when $A(r) = 1/B(r)$ and $A_{\text{sur}}$ is the value of the $A(r)$ at the surface of the star. There
Figure 1: Energy as a function of charge for several values of the cosmological constant for a q-star with one scalar field. The numbers within the figures denote the absolute value of the cosmological constant. We start from a large value of the metric at the surface, $A_{\text{sur}}$, near unity, corresponding to the weak gravity limit. The values of the energy and charge are small. As we decrease the metric the values of energy and charge increase. If $\Lambda \neq 0$ and when reaching a certain value of $A_{\text{sur}}$, then energy and charge decrease. The initial increase in the energy and charge values is not visible in the $\Lambda = -2.5$ case, because it takes place nearly to the maximum energy and for a short range of the surface gravity parameter.
Figure 2: The matter field value at the center of the soliton as a function of the cosmological constant for a q-star with one scalar field for four different values of $A_{\text{sur}}$. The frequency can be obtained using relation 2.29. For zero cosmological constant the field is everywhere constant within the soliton, equal to $(1 + \omega/A_{\text{sur}}^{1/2})^{1/2} = 1.5^{1/2}$. As the cosmological constant increases in absolute values the field takes larger values as a reflection of the increase in the gravity strength. Also, the field has larger values when $\omega$ decreases, because small $\omega$ indicates stronger gravity.

Figure 3: The soliton radius as a function of the cosmological constant for a q-star with one scalar field for four different values of $A_{\text{sur}}$. 
Figure 4: The total soliton mass as a function of the cosmological constant for a q-star with one scalar field for four different values of $A_{\text{sur}}$.

Figure 5: The particle number as a function of the cosmological constant for a q-star with one scalar field for four different values of $A_{\text{sur}}$. We find that radius, energy and charge increase for small absolute values of the cosmological constant but for larger values they decrease.
are some interesting results obtained when investigating the properties of the analytical solution. The total soliton energy is:

\[ E = \frac{3\pi}{4} R^2. \quad (2.37) \]

This agrees with the intuitive result that the total energy is analogous to the "volume" of the field configuration. Also, \( E/Q < 1 = \) the rescaled mass of the free particles for every \( 0 < A_{\text{sur}} \leq 1 \) for the case of zero cosmological constant. This means that there is no possibility for the soliton fission to free particles, because this procedure is energetically forbidden. From the \( \sigma^2 = 1 + \omega B^{1/2} \) relation and eq. 2.32 we find that in a flat spacetime, the scalar field has an everywhere constant absolute value, here equal to 1.5. So, when \( A_{\text{sur}} \to 0 \), i.e.: when a horizon is going to be formed, there is no anomaly at the center of the star.

In our figures, every dashed line corresponds to an analytical solution. The total energy is in \( 8\pi G \) units, the particle number in \( 8\pi G \tilde{m} \) units, with \( \tilde{m} = 1 \) and the soliton radius in \( (8\pi G \tilde{m}^3)^{1/2} \) units. In figures 2-5 starting with an initial zero value of the cosmological constant, we increase its absolute value and terminate our calculations when the soliton energy equates to the energy of free particles with the same charge. In figure 1 we start from a large value of the metric field \( A_{\text{sur}} \), near unity, or, equivalently, a large value of the characteristic soliton frequency, near 1/2 (which according to eq. 2.29 is the maximum limiting value of the frequency when gravity is absent, i.e., when \( A_{\text{sur}} \to 0 \)) and gradually decrease the frequency.

We can observe, as a general comment of our numerical results depicted in figures 1-5, that the effect of the increase, in absolute values, of the cosmological constant causes a consequent increase in the energy, charge and radius of the q-star. This is a counter-balancing effect: Negative cosmological constant means that different particles within the soliton, corresponding to the \( \phi \) field, tend to move along deviating geodesics. So, the increase in mass and the other soliton parameters is necessary in order the object to avoid this kind of "deviation", or, in other words, to generate "positive" gravity as opposed to the "negative" gravity, generated by the negative cosmological constant. But when cosmological constant becomes considerably large (i.e. when \( \Lambda \) is larger than \( 8\pi G \tilde{U} \) or \( 8\pi G \tilde{W} \) or, equivalently, when \( \tilde{\Lambda} \) is larger than \( \tilde{U} \) or \( \tilde{W} \)) then no soliton energy contribution can prevent this "deviation", apart from a shrinking in the soliton magnitude and a consequent decrease in the soliton mass and particle number.

Figure 1 gives more interesting results. We start from an \( \omega \) near the critical value 1/2 holding true in the absence of gravity. The field configuration
is a small one, with small energy and charge. When decreasing $\omega$ (or $A_{\text{sur}}$) gravity becomes more important and the soliton becomes larger and more massive. There is an interesting point, common in similar diagrams of other kinds of q-stars. Decreasing frequency both energy and charge increase, but only until the frequency reaches a certain point, below which the two quantities decrease. So for a certain charge there are usually two values of the energy, the lower one corresponding to high frequency (lower branch of the $E = E(Q)$ diagram) and the higher one corresponding to low frequency (upper branch of the $E = E(Q)$ diagram). So, a soliton star with a certain charge and a high energy amount can emit the energy excess, falling down to the lower branch, changing of course its frequency. We will study this case in more details when treating fermion-scalar q-stars, for which there is an analytical solution for a special $\Lambda \neq 0$ case and the obtained relations are simpler than in the case of q-stars with two scalar fields. This change in the energy and charge variation with respect to the frequency happens only when the cosmological constant differs from zero. The above analytical solution given for the zero cosmological constant case predicts that both energy and charge are monotone decreasing functions of the frequency. When $A_{\text{sur}} \to 0$ one expects that a black hole is formed. But the calculations depicted in figure 1 interrupted for an $A_{\text{sur}} > 0$ value, because below that certain value the star decays into free particles as the energetically favorable choice. This means that before the formation of an horizon, the q-star decays. So, when $\Lambda \neq 0$ gravitational collapse is impossible, at least in the usual way of decreasing frequency holding true for q-stars in $3+1$ dimensions, [19]-[23]. There may be a possibility of forming q-type black holes, if one takes into account the spatial variation of the scalar field, but this demands a considerably different way of solving the equations of motion to the one known so far. The same discussion holds true for any sort of q-star.

3  A q-star with two scalar fields

A first simple generalization to the above described soliton is a field configuration with two scalar fields, one N-carrying $\phi$, and, consequently, complex, and the other, $\sigma$, used to constrain the N-carrying field within a certain region, generating an appropriate potential, taken to be real for simplicity. The Lagrangian in the case under discussion is:

$$L = g^{\mu\nu}(\partial_\mu \phi)^*(\partial_\nu \phi) + \frac{1}{2}g^{\mu\nu}(\partial_\mu \sigma)(\partial_\nu \sigma) - U ,$$  (3.1)
and the metric is supposed to have the same form as in the q-star with one scalar field, i.e. to be static and spherically symmetric. The potential has the general form:

$$U = a\phi^2 \sigma^2 + b\phi^4 + c(a^2 - d)^2 .$$  \hspace{1cm} (3.2)

We parametrize \( d \) as:

$$d = \mu^{-3/2}a^{-1/2}b^{1/2} ,$$  \hspace{1cm} (3.3)

where the parameter \( \mu \) has mass dimensions, is nothing more than a way of re-expressing \( d \), is of the same order of magnitude as the potential parameters and will simplify the rescalings of the energy quantities. We now rescale the matter fields

$$\tilde{\phi} = (2b)^{1/4} \mu^{-1/3} \phi , \quad \tilde{\sigma} = a^{1/2}(2b)^{-1/4} \mu^{3/4} \sigma .$$  \hspace{1cm} (3.4)

We also define:

$$\lambda = a^{-1}bc ,$$  \hspace{1cm} (3.5)

and the potential takes the simple form:

$$U = \tilde{U}/\mu^3 , \quad \tilde{U} = \tilde{\phi}^2 \tilde{\sigma}^2 + \frac{1}{2} \tilde{\phi}^4 + \frac{\lambda}{2}(\tilde{\sigma}^2 - 1)^2 .$$  \hspace{1cm} (3.6)

The rescalings in the other quantities (spacetime and frequency) are the same as in eq. 2.16.

We use a simple ansatz for the matter fields, namely we regard that within the soliton holds:

$$\tilde{\phi}(\tilde{\rho},t) = \tilde{\varphi}(\tilde{\rho})e^{-i\tilde{\omega}t} , \quad \tilde{\sigma} \approx 0 .$$  \hspace{1cm} (3.7)

Redefining: \( \tilde{r} = \tilde{\rho}8\pi G\mu \), we have:

$$\tilde{V} = A \left[ \left( \frac{d\tilde{\varphi}}{d\tilde{\rho}} \right)^2 + \left( \frac{d\tilde{\sigma}}{d\tilde{\rho}} \right)^2 \right] \approx \left( \frac{d\varphi}{d\rho} \right)^2 \propto \left( \frac{d\tilde{\varphi}}{d\tilde{r}} \right)^2 8\pi G\mu \ll \tilde{W} , \tilde{U} .$$  \hspace{1cm} (3.8)

with

$$\tilde{W} = B\tilde{\omega}^2\tilde{\varphi}^2 .$$  \hspace{1cm} (3.9)

The Euler-Lagrange equation for the complex field is:

$$A \left[ \frac{d^2\tilde{\varphi}}{d\tilde{r}^2} + \left( \frac{1}{\tilde{r}} + \frac{1}{2} \frac{d(A - B)}{d\tilde{r}} \right) \frac{d\tilde{\varphi}}{d\tilde{r}} \right] 8\pi G\mu = \frac{1}{2} \frac{\delta(\tilde{U} - \tilde{W})}{\delta\tilde{\varphi}} .$$  \hspace{1cm} (3.10)
Regarding the metric and matter fields as smoothly varying within the star, we find from the Lagrange equation for the $\varphi$ field:

$$\tilde{\varphi}^2 = \tilde{\omega}^2 B ,$$  \hspace{1cm} (3.11)

$$\tilde{U} = \frac{1}{2}(\tilde{\omega}^4 B^2 + \lambda)$$

$$\tilde{W} = \tilde{\omega}^4 B^2 .$$  \hspace{1cm} (3.12)

Within the surface the metric and matter fields vary rapidly. Repeating the discussion (eqs 2.24-2.28) of the previous section we find the eigenvalue equation for the frequency:

$$\tilde{\omega} = \left( \frac{\lambda}{B^2_{\text{sur}}} \right)^{1/4} .$$  \hspace{1cm} (3.13)

We will choose $\lambda = 1/9$ for simplicity. Dropping from now on the tildes, the Einstein equations take the form:

$$-\frac{1}{2r} \frac{dA}{dr} = \frac{3}{2} \omega^4 B^2 + \frac{1}{2} \lambda + \Lambda ,$$  \hspace{1cm} (3.14)

$$-\frac{1}{2r} \frac{dB}{dA} = \frac{1}{2} \omega^4 B^2 - \frac{1}{2} \lambda - \Lambda .$$  \hspace{1cm} (3.15)

The soliton mass is given by the $G^0_0$ component of the Einstein tensor and the particle number is:

$$Q = 4\pi \int dr r \omega \varphi^2 \sqrt{\frac{B}{A}} = 4\pi \int dr r \omega^3 B^{3/2} A^{-1/2} .$$  \hspace{1cm} (3.16)

Outside the soliton we take $\varphi$ to be everywhere zero and $\sigma$ equal to unity, so that both the energy density and Noether current are zero. These values are also solutions to the Lagrange equations of the matter fields.

We will now find some analytical solutions. The case of zero cosmological constant is one of them. In this case holds:

$$B(r) = 1/A_{\text{sur}} , \quad A(r) = 1 - 2\lambda r^2 .$$  \hspace{1cm} (3.17)

The soliton radius $R$ can be found equal to:

$$R = \left( \frac{1 - A_{\text{sur}}}{2\lambda} \right)^{1/2} = \left( \frac{1 - \omega^2/\lambda^{1/2}}{2\lambda} \right)^{1/2} .$$  \hspace{1cm} (3.18)
Figure 6: The total mass of the field configuration as a function of the particle number for several values of the cosmological constant for a q-star with two scalar fields. Dashed lines depict analytical solutions.

Figure 7: The absolute value of the $N$–carrying field at the center of the field configuration as a function of the cosmological constant for a q-star with two scalar fields for four values of $A_{\text{sur}}$, or equivalently, the frequency. For flat spacetime the field $\phi$ has a constant value equal to $\lambda^{1/4}$ (here $1/9^{1/4}$) in the soliton interior.
Figure 8: The radius of a q-star with two scalar fields as a function of the cosmological constant for four values of $A_{\text{sur}}$.

Figure 9: The total mass as a function of the cosmological constant for a q-star with two scalar fields for four values of $A_{\text{sur}}$. 
Figure 10: The particle number as a function of the cosmological constant for a q-star with two scalar fields for four values of the $A_{\text{sur}}$.

and the energy and charge

$$E = (1 - A_{\text{sur}})\pi = (1 - \omega^2/\lambda^{1/2})\pi,$$  \hspace{1cm} (3.19)

$$Q = 2\sqrt{3}\pi(1 - A_{\text{sur}}^{1/2}) = 2\sqrt{3}\pi(1 - \omega/\lambda^{1/4}).$$ \hspace{1cm} (3.20)

We can see that the equation $E = Q$ has no solution, so, the soliton energy can not be equal to the energy of free particles and consequently the star can not decay into free particles. We can also find the relation for the energy $E_r$ stored within a surface of radius $r$:

$$E_r = 2\lambda\pi r^2,$$ \hspace{1cm} (3.21)

which verifies relation 2.25 when combined with eq. 3.17.

We will now examine a different case admitting analytical solutions. The case:

$$\frac{1}{2}\lambda + \Lambda = 0.$$ \hspace{1cm} (3.22)

If $R$ is the soliton radius (i.e. the solution to the equation: $A(r) = 1/B(r)$) the soliton parameters take the form:

$$R = 3\sqrt{2}(A_{\text{sur}}^{2/3} - A_{\text{sur}})^{1/2} = 3\sqrt{2}(-3\omega^2 + 3^{2/3}\omega^{4/3})^{1/2},$$ \hspace{1cm} (3.23)

$$A(r) = \frac{(18A_{\text{sur}} - r^2 + R^2)^3}{5832A_{\text{sur}}^2} = \frac{(18A_{\text{sur}}^{2/3} - r^2)^3}{5832A_{\text{sur}}^2} = \frac{(-r^2 + 18 \cdot 3^{2/3}\omega^{4/3})^3}{52488\omega^4},$$ \hspace{1cm} (3.24)
\[ B(r) = \frac{18A_{\text{sur}} - r^2 + R^2}{18A_{\text{sur}}^2} = \frac{18A_{\text{sur}}^{2/3} - r^2}{18A_{\text{sur}}^2} = \frac{-r^2 + 18 \cdot 3^{2/3} \omega^{4/3}}{162 \omega^4} , \] (3.25)

\[ E = (1 + A_{\text{sur}}^{2/3} - 2A_{\text{sur}}) \pi = (1 - 6 \omega^2 + 3^{2/3} \omega^{4/3}) \pi , \] (3.26)

\[ Q = 4\sqrt{3} \pi (A_{\text{sur}}^{1/6} - A_{\text{sur}}^{1/2}) = 4\sqrt{3} \pi (3^{1/6} \omega^{1/3} - 3^{1/2} \omega) . \] (3.27)

The equation \( E = Q \) admits analytical solution numerically equal to \( A_{\text{sur}} = 1.03484 \cdot 10^{-5} \). Then, \( E \approx Q \approx \pi \) and below this value of the surface metric the energy of the free particles with the same charge is less than the soliton energy, making in this way stars with \( A_{\text{sur}} \) below that critical value unstable.

We can also find \( E_r \):

\[ E_r = \frac{\pi r^2 (792A_{\text{sur}}^{4/3} + 324A_{\text{sur}}^2 - 54A_{\text{sur}}^{2/3} r^2 + r^4)}{5832A_{\text{sur}}^2} . \] (3.28)

It is a matter of simple algebra to verify equation 2.25.

As one can see from figure 6, the analytical solutions behave in the same way as the solutions obtained in numerical methods. Summarizing our results, we see that decay into free particles is forbidden for the flat spacetime case, energetically favorable for an anti de Sitter spacetime, but only below a certain value of the frequency. Solitons can not be extremely large. For any value of the independent parameters (\( \omega \) and \( \Lambda \)) there is a certain region in the \( E-Q \) phase space, fully depicted in our figures.

4 Non-abelian q-stars

The Lagrangian in this case is:

\[ \mathcal{L} = \frac{1}{2} \text{Tr}(\partial_{\mu}\phi)(\partial^{\nu}\phi) - \text{Tr}U(\phi) . \] (4.1)

The Enstein equations are:

\[ -\frac{1}{2\rho} \frac{dA}{d\rho} = 8\pi G \left[ U + \text{Tr} \frac{1}{2} B \left( \frac{\partial \phi}{\partial t} \right)^2 + \text{Tr} \frac{1}{2} A \left( \frac{\partial \phi}{\partial \rho} \right)^2 \right] + \Lambda , \] (4.2)

\[ -\frac{1}{2\rho B} \frac{dB}{d\rho} = 8\pi G \left[ \text{Tr} \frac{1}{2} B \left( \frac{\partial \phi}{\partial t} \right)^2 + \text{Tr} \frac{1}{2} A \left( \frac{\partial \phi}{\partial \rho} \right)^2 - U \right] - \Lambda , \] (4.3)
with a general renormalizable potential

\[ U = \frac{\mu^2}{2} \phi^2 + \frac{g}{3!} \phi^3 + \frac{\lambda}{4!} \phi^4 , \quad (4.4) \]

with \( \phi \) in the \( SO(3) \) 5 representation. We make the following rescalings:

\[ g = \mu \tilde{g} , \quad \phi = (\mu/\tilde{g}) \tilde{\phi} , \quad \lambda = \tilde{g}^2 \tilde{\lambda} , \quad \rho = \tilde{\rho} \mu^{-1} , \quad \omega = \mu \tilde{\omega} . \quad (4.5) \]

All energy quantities are rescaled by \( \mu^4/\tilde{g}^2 \). The potential takes the simple form:

\[ U = \frac{\mu^4}{\tilde{g}^2} \tilde{U} , \quad \tilde{U} = \frac{\tilde{\phi}^2}{2} + \frac{\tilde{\phi}^3}{3!} + \frac{\tilde{\lambda}}{4!} \tilde{\phi}^4 , \quad (4.6) \]

where \( \tilde{\lambda} = \lambda/\tilde{g}^2 \) Redefining

\[ \tilde{r} = \tilde{\rho} \sqrt{8 \pi G \mu^2 \tilde{g}^2} , \]

\[ 8 \pi G \frac{\mu^4}{\tilde{g}^2} \tilde{\Lambda} = \Lambda , \]

the independent Einstein equations take the simple form:

\[ -\frac{1}{2\tilde{r}} \frac{dA}{d\tilde{r}} = U + W + V + \tilde{\Lambda} , \quad (4.7) \]

\[ -\frac{1}{2\tilde{r}} A \frac{dB}{d\tilde{r}} = W + V - U - \tilde{\Lambda} , \quad (4.8) \]

where:

\[ U = \text{Tr} \tilde{U} , \]

\[ W = \text{Tr} \frac{1}{2} B \left( \frac{\partial \tilde{\phi}}{\partial t} \right)^2 = \text{Tr} B [\tilde{\Omega}, \tilde{\phi}]^2 , \quad (4.9) \]

\[ V = \text{Tr} \frac{1}{2} A \left( \frac{\partial \tilde{\phi}}{\partial \tilde{\rho}} \right)^2 = \text{Tr} \frac{1}{2} A \left( \frac{\partial \tilde{\phi}}{\partial \tilde{r}} \right)^2 8 \pi G \mu^2 \frac{1}{\tilde{g}^2} . \]

It is obvious that for a large soliton with smoothly varying fields within the interior the \( V \) terms are of \( O(G) \) order and, thus, can be neglected. We define \( \Omega \) through:

\[ \frac{\partial \phi}{\partial t} = i[\Omega, \phi] . \quad (4.10) \]
We also define:
\[
\Omega \equiv \mu \tilde{\Omega} \equiv -i\tilde{\omega}_\mu \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right).
\] (4.11)

With the above definitions, the Euler-Lagrange equation takes the form:
\[
- B[\tilde{\Omega}, [\tilde{\Omega}, \tilde{\phi}]] + \frac{\partial U}{\partial \tilde{\phi}} - \frac{1}{3} \text{Tr} \left( \frac{\partial U}{\partial \tilde{\phi}} \right) = 8\pi G \mu^2 \frac{2}{g^2} \left[ \frac{\partial^2 \tilde{\phi}}{\partial \tilde{r}^2} + \frac{\partial \tilde{\phi}}{\partial \tilde{r}} \left( \frac{1}{\tilde{r}} + \frac{1}{2A} \frac{dA}{d\tilde{r}} - \frac{1}{2A} \frac{dA}{d\tilde{r}} \right) \right],
\] (4.12)

where \(1\) is the unity matrix. We can always diagonalize \(\phi = e^{iR} \phi_{\text{diag}} e^{-iR}\). The rigid rotation condition means that \(R(\rho, t) = \Omega t + C\) where we can eliminate the constant \(C\) through a global \(SO(3)\) rotation and write:
\[
\phi = -\frac{1}{2} \tilde{\phi}_2 \cdot \text{diag}(1+y, -2, 1-y) \Leftrightarrow \tilde{\phi} = -\frac{1}{2} \tilde{\phi}_2 \cdot \text{diag}(1+y, -2, 1-y).
\] (4.13)

The charge is defined by the equation:
\[
Q = \int d^2 \rho \sqrt{-g} j^0 = -i \int d^2 \rho \sqrt{-g} B[\phi, \tilde{\phi}].
\] (4.14)

The equation of motion for the matter field within the soliton dropping the \(O(8\pi G \mu^2 / g^2)\) takes the form:
\[
\tilde{\omega}^2 B(\tilde{\phi}_2) = 2 \tilde{\omega}^2 B(\tilde{\phi}_1 - \tilde{\phi}_3) \text{diag}(1, 0, -1).
\] (4.15)

At the inner edge of the surface we obtain the usual relation (repeating eqs 2.24, 2.25):
\[
(U - W)_{\text{sur}} = 0.
\] (4.16)

The usual relation \(\delta(W = U) / \delta \phi = 0\) is no longer valid. We can see that that neglecting the \(O(8\pi G \mu^2 / g^2)\) in the Euler-Lagrange equation, multiplying by \(\tilde{\phi}\) and tracing yields \(2W = \text{Tr}(\tilde{\phi} \cdot \partial U / \partial \tilde{\phi})\). Applying equation 4.16 we obtain for the fields at the inner edge of the surface, with \(R\) the soliton radius:
\[
\tilde{\phi}_2(R) = \frac{12}{\lambda} \left( \frac{y^2 - 1}{(y^2 + 3)^2} \right)_{\text{sur}},
\] (4.17)
\[
\tilde{\omega}^2 B(R) = \left[ \frac{1}{4} \left( 1 + \frac{3}{y^2} \right) - \frac{3}{4\lambda} \frac{1}{y^2} \frac{(y^2 - 1)^2}{(y^2 + 3)^2} \right]_{\text{sur}}.
\] (4.18)
Taking the \((2,2)\) component of the Euler-Lagrange, eq. 4.15 we take:

\[
y^2(\tilde{r}) = \frac{3 \tilde{\lambda} \tilde{\phi}_2^2 + 2 \tilde{\phi}_2 + 8}{2 - \tilde{\lambda} \tilde{\phi}_2^2}.
\]  (4.19)

Using the above equation combined with the \(2W = \text{Tr}(\dot{\phi} \cdot \partial U/\partial \tilde{\phi})\) relation, we find for the \(\tilde{\phi}_2\) field:

\[
\tilde{\phi}_2(\tilde{r}) = \frac{1 - 4\tilde{\lambda} \tilde{\omega}^2 B(\tilde{r})}{2\tilde{\lambda}} + \frac{1}{2\tilde{\lambda}} \left( [1 - 4\tilde{\lambda} \tilde{\omega}^2 B(\tilde{r})]^2 + 8\tilde{\lambda} [4\tilde{\omega}^2 B(\tilde{r}) - 1] \right)^{1/2}.
\]  (4.20)

Using now eqs 4.17, 4.19 we find:

\[
[(\tilde{\lambda} - 1)(y^2 + 3)^3 + 16(y^2 + 3)^2 - 72(y^2 + 3) + 96]_{\text{sur}} = 0.
\]  (4.21)

\(\tilde{\lambda} < 1\) holds in order 4.19 to have real solutions. The solution to the above equation is:

\[
y^2(R) = -3 + \frac{16}{3(1 - \tilde{\lambda})} + \frac{2}{3} \frac{2^{1/3}}{-1 + \lambda} \left[ -13 + 162\tilde{\lambda} - 81\tilde{\lambda}^2 + 9(-1 + \tilde{\lambda})^2 \left( \frac{-1 + 81\tilde{\lambda}^2}{(-1 + \tilde{\lambda})^2} \right)^{1/3} + \frac{2}{3} \frac{2^{1/3}}{1 - \lambda} \left[ 13 + 162\tilde{\lambda} + 81\tilde{\lambda}^2 + 9(-1 + \tilde{\lambda})^2 \left( \frac{-1 + 81\tilde{\lambda}^2}{(-1 + \tilde{\lambda})^2} \right)^{1/3} \right] \right].
\]  (4.22)

The idea for the non-abelian solitons is the following: We give a certain value to the free parameter \(\tilde{\lambda}\), in our calculations \(\tilde{\lambda} = 2/3\), and using eq. 4.22 we find \(y^2(R)\). Substituting in eq. 4.18 we find the eigenvalue for the frequency, in our case equal to 0.4956\(A_{\text{sur}}^{1/2}\). We then substitute the values of the \(\tilde{\phi}_2\) and \(y_2\) matter fields (eqs 4.19, 4.20) in Einstein equations and solve them numerically.

From the figures 11-15 we see that for larger values of the cosmological constant the matter field \((\tilde{\phi}_2)\) is larger, but the soliton radius, charge and energy decrease with the increase of the cosmological constant in absolute values as in other kinds of q-stars.

We now investigate the analytical solution to the Einstein equations in the case of zero cosmological constant. We find:

\[
A = 1 - 2.82529\tilde{r}^2,
\]  (4.23)
Figure 11: Energy as a function of the particle number for four values of the cosmological constant, for a non-abelian q-star.

Figure 12: The matter field value (\(\tilde{\phi}_2(0)\)) at the center of the field configuration as a function of the cosmological constant for a non-abelian q-star for four values of the \(A_{\text{sur}}\).
Figure 13: The soliton radius as a function of the cosmological constant for a non-abelian q-star for four values of the $A_{\text{sur}}$.

Figure 14: The total mass as a function of the cosmological constant for a non-abelian q-star for four values of the $A_{\text{sur}}$. 
Figure 15: The particle number as a function of the cosmological constant for a non-abelian q-star for four values of the $A_{\text{sur}}$.

\begin{align*}
R &= \left[\frac{(1 - A_{\text{sur}})}{2.8259}\right]^{1/2}, \\
E &= \pi (1 - A_{\text{sur}}), \\
Q &= 16.7482 \left(1 - A_{\text{sur}}^{1/2}\right).
\end{align*}

5 Fermion-Scalar q-stars

Fermion-scalar q-stars are realistic field configurations of a scalar and a fermionic field. These field configurations are supposed, [21], to describe certain stellar objects known as neutron stars. The fermionic field carries the charge needed to stabilize the soliton and the real scalar field generates an everywhere positive potential. In the star interior the scalar field is approximately zero but outside the soliton takes a certain value $\sigma_0$, vanishing the potential. The potential energy density outside the soliton vanishes in this way (eq. 5.9) and the fermion mass acquires its vacuum value $m$.

We regard the fermion-scalar q-star as a zero temperature fermionic sea with local Fermi energy and momentum $\varepsilon_F^2 + k_F^2 + m^2(\sigma)$. With spherical symmetry, the local fermion density is:

$$
\langle \psi^\dagger \psi \rangle = \frac{1}{4\pi^2} \int n_k d^2k = \frac{k_F^2}{4\pi},
$$

(5.1)
and the scalar density:

$$ \langle \bar{\psi} \psi \rangle = \frac{1}{4\pi^2} \int n_k d^2k \frac{m}{2(k^2 + m^2)^{1/2}} = \frac{m(-m + \varepsilon_F)}{2\pi}. \quad (5.2) $$

We also find using the definitions for the energy and pressure density:

$$ P_\psi \equiv \frac{1}{2} \int n_k d^2k \frac{k^2}{2(k^2 + m^2)^{1/2}} = \frac{1}{3} (\varepsilon_F \langle \psi^\dagger \psi \rangle - m \langle \bar{\psi} \psi \rangle), \quad (5.3) $$

$$ E_\psi \equiv \frac{1}{4\pi^2} \int n_k d^2k(k^2 + m^2)^{1/2} = 2P_\psi + m \langle \bar{\psi} \psi \rangle. \quad (5.4) $$

The Lagrangian density is:

$$ L = \bar{\psi}(i\partial / - m(\sigma))\psi - \frac{1}{2}(\partial_\mu \sigma)^2 - U(\sigma), \quad (5.5) $$

with

$$ m(\sigma) = g\sigma. \quad (5.6) $$

We choose $g = 2$ for our calculations. The Euler-Lagrange equation for the scalar field is:

$$ \frac{d^2 \sigma}{d\rho^2} + \frac{1}{\rho} \frac{d\sigma}{d\rho} = -\frac{\partial}{\partial \sigma}(P_\psi - U). \quad (5.7) $$

An appropriate form for the potential is:

$$ U = \frac{1}{4} \lambda (\sigma^2 - \sigma_0^2)^2. \quad (5.8) $$

The proper generalization in order to include gravity in our discussion is to replace the Fermi energy with a global chemical potential, according to the equation 5.14, [12]. In this way the Fermi energy takes the proper generally covariant form, being the zero component of the energy-momentum 4–vector. The generalization of the Lagrangian with the inclusion of gravity is straightforward:

$$ \mathcal{L}/\sqrt{-g} = \frac{i}{2}(\bar{\psi} \gamma^\mu \psi;\mu - \bar{\psi} \gamma^\mu \psi) - m(\sigma)\bar{\psi} \psi + \frac{1}{2} \sigma_\mu \sigma^\mu - U(\sigma). \quad (5.9) $$

The proper rescalings for the case under discussion are made with respect to a new mass scale $\mu$ equal to $\sigma_0^2$, simplifying in this way the potential.
and extracting a $\mu^3$ factor from the Lagrangian. So the quantities in the Lagrangian are rescaled in the following way:

\[
\tilde{\sigma} = \frac{\sigma}{\sigma_0}, \quad \tilde{\psi} = \frac{\psi}{\sigma_0}, \quad \tilde{\lambda} = \frac{\lambda}{\sigma_0}, \quad \tilde{m} = \frac{m}{\sigma_0}, \quad \tilde{\rho} = \rho \sigma_0^2, \quad \tilde{L} = \frac{L}{\sigma_0^3}.
\]  

(5.10)

We also make two new redefinitions:

\[
\tilde{r} = 8\pi G \sigma_0^2, \quad \tilde{\Lambda} = \frac{\Lambda}{8\pi G \sigma_0^6}.
\]  

(5.11)

So, dropping the tildes we can regard that inside the soliton the matter field derivative with respect to the radius is of order of $O(G\mu^2)$ so the Euler-Lagrange equation for the scalar field can be written:

\[
\frac{\partial}{\partial \sigma}(P_\psi - U) = 0.
\]  

(5.12)

The energy-momentum tensor is:

\[
T^{\mu\nu} = \frac{1}{2}(\bar{\psi} \gamma^{(\mu} \psi^{\nu)}) - (\bar{\psi}^{(\mu} \gamma^{\nu)} \psi) + \sigma^{(\mu} \sigma^{\nu)} - g^{\mu\nu}[1/2\sigma_\alpha \sigma^\alpha - U(\sigma)].
\]  

(5.13)

We define a global chemical potential $\omega_\psi$ through the equation:

\[
\omega^2_\psi = \varepsilon^2_F B^{-1}(r).
\]  

(5.14)

This quantity retains a role similar to the frequency of the other kinds of soliton stars. The particle number is:

\[
Q_\psi = \int \sqrt{-g} dx^1 dx^2 j^0 = 2\pi \int rdr \sqrt{\frac{1}{A}} \langle \psi^\dagger \psi \rangle.
\]  

(5.15)

The Einstein equations take the form:

\[
-\frac{1}{2r} \frac{dA}{dr} = E_\psi + U + \Lambda,
\]  

(5.16)

\[
-\frac{1}{2r} \frac{dB}{dr} = P_\psi - U - \Lambda.
\]  

(5.17)

Within the surface we repeat the discussion of eqs. 2.24-2.28 and find from eq. 5.12

\[
U = P_\psi \Rightarrow \frac{\lambda}{4} = \frac{1}{12\pi} \omega^3_\psi B_{\text{sur}}^{3/2}.
\]  

(5.18)
Figure 16: Energy as a function of charge for four values of the $A_{\text{sur}}$ or, equivalently, the chemical potential for a fermionic q-star. We start from a small value of the cosmological constant. Both energy and charge have small values. When increasing the cosmological constant, in absolute values, both energy and charge increase in order the soliton to generate “positive” gravity as a counter effect to the “negative” gravity from the spacetime structure. But when $\Lambda$ takes very large absolute values, a large soliton can not be stable and mass, charge and radius decrease as one can see from figures [16][19].

This is the eigenvalue equation for the chemical potential. The Einstein equations take the soluble form:

$$\frac{1}{2r} \frac{dA}{dr} = \frac{1}{6\pi} \omega_\psi B^{3/2} + \frac{\lambda}{4} + \Lambda ,$$

(5.19)

$$\frac{1}{2r} \frac{dB}{dr} = \frac{1}{12\pi} \omega_\psi B^{3/2} - \frac{\lambda}{4} - \Lambda ,$$

(5.20)

The total charge is:

$$Q = \frac{1}{2} \int drr\omega_\psi B A^{-1/2} .$$

(5.21)

For our numerical calculations we use $\lambda = 1/2$.

Outside the soliton we regard that $\sigma$ is everywhere unity and $\psi$ zero, so both energy density and current are zero.

Let us now take $\Lambda \to 0$. In that case the Einstein equations admit the following analytical solution, also verified numerically:

$$B(r) = 1/A_{\text{sur}} = \left(\frac{3\pi}{2}\right)^{2/3} \omega_\psi^{-2} , \quad A(r) = 1 - \frac{3}{8}r^2 .$$

(5.22)
Figure 17: The soliton radius as a function of the chemical potential for a fermionic q-star for five values of the cosmological constant. When $\omega_\psi$ is large the gravity strength according to the eq. 5.18 is small.

Figure 18: The total soliton mass as a function of the chemical potential for five values of the cosmological constant for a fermionic q-star.
Figure 19: The particle number as a function of the chemical potential for five values of the cosmological constant for a fermionic q-star.

Figure 20: The total mass of the field configuration as a function of the charge for a fermionic q-star with constant cosmological constant when varying the chemical potential. We start from large values of this quantity (i.e: with $A_{\text{sur}}$ close to unity, according to eq. 5.18) and, consequently, weak gravity. Then both charge and energy have small values. Decreasing $\omega_\psi$ both energy and charge increase up to a certain value of the chemical potential, below which they both decrease.
We can easily find that the parameters of the soliton are:

\[
R = [(1 - A_{\text{sur}})8/3]^{1/2} = \left[1 - \left(\frac{2}{3\pi}\right)^{2/3} \omega_{\psi}^{2}\right]^{8/3} \frac{1}{2}, \quad (5.23)
\]

\[
E = (1 - A_{\text{sur}})\pi = \left[1 - \left(\frac{2}{3\pi}\right)^{2/3} \omega_{\psi}^{2}\right] \pi, \quad (5.24)
\]

\[
Q = \left(\frac{3\pi}{2}\right)^{2/3} \frac{4}{3}(1 - A_{\text{sur}}^{1/2}) = \left(\frac{3\pi}{2}\right)^{2/3} \frac{4}{3} \left[1 - \left(\frac{2}{3\pi}\right)^{1/3} \omega_{\psi}\right], \quad (5.25)
\]

where \(R\) is the soliton radius, \(E\) the total mass and \(Q\) the soliton charge.

We will now examine a different case admitting analytical solutions, namely:

\[
\Lambda = -\frac{\lambda}{4} \quad (5.26)
\]

In that case Einstein equations take the form

\[
-\frac{1}{2r} \frac{dA}{dr} = \frac{2}{12\pi} \omega_{\psi}^{3}B^{3/2} \quad (5.27)
\]

\[
-\frac{1}{2r} \frac{dB}{dr} = \frac{1}{12\pi} \omega_{\psi}^{3}B^{3/2} \quad (5.28)
\]

We can find the soliton radius:

\[
R = 4(A_{\text{sur}}^{3/4} - A_{\text{sur}})^{1/2} = 4\sqrt{-\left(\frac{2}{3\pi}\right)^{2/3} \omega_{\psi}^{2} + \left(\frac{2}{3\pi}\right)^{1/2} \omega_{\psi}^{3/2}}. \quad (5.29)
\]

The metrics take the form:

\[
A(r) = \frac{(16A_{\text{sur}} - r^2 + R^2)^4}{65536A_{\text{sur}}^3}
\]

\[
= \frac{(16A_{\text{sur}}^{3/4} - r^2)^4}{65536A_{\text{sur}}^3} = \frac{\pi^2(3r^2 - 16\sqrt{6/\pi}\omega_{\psi}^{3/2})^4}{2359296\omega_{\psi}^6}, \quad (5.30)
\]

\[
B(r) = \frac{1}{256} \frac{(16A_{\text{sur}} - r^2 + R^2)^2}{A_{\text{sur}}^3}
\]

\[
= \frac{1}{256} \frac{(16A_{\text{sur}}^{3/4} - r^2)^2}{A_{\text{sur}}^3} = \frac{(3\pi r^2 - 16\sqrt{6\pi}\omega_{\psi}^{3/2})^2}{1024\omega_{\psi}^6}. \quad (5.31)
\]
We can also find the energy and charge:

\[
E = (1 + 2A_{\text{sur}}^{3/4} - 3A_{\text{sur}})\pi = \pi - 2^{2/3}(3\pi)^{1/3}\omega_{\psi}^{2} + 2\sqrt{\frac{2\pi}{3}}\omega_{\psi}^{3/2}, \quad (5.32)
\]

\[
Q = 2^{4/3}(3\pi)^{2/3}(A_{\text{sur}}^{1/4} - A_{\text{sur}}^{1/2}) = 2\sqrt{6\pi\omega_{\psi}^{1/2}} - 2^{5/3}(3\pi)^{1/3}\omega_{\psi}. \quad (5.33)
\]

For both the analytically soluble cases the chemical potential \(\omega_{\psi}\) varies from the maximum value obtained by the eq. 5.18 (equal to \(3\pi/2\) for these stars) when the minimum value is obtained equating the soliton energy with the energy of the free particles with the same charge. When \(B = A_{\text{sur}}^{-1} = \text{const.}\) this equation has no solution and no decay to free particles is possible. In the latter case the minimum value is \(\omega_{\psi} = 0.0481\).

We will now examine the \(\Lambda + \lambda/4 = 0\) analytical solution in more details because in the fermion-scalar soliton stars we face the simpler relations when compared with those resulting from the investigation of the solitons with two scalar fields. With the help of figure 20 we see that the \(E = E(Q)\) function has two branches, a lower corresponding to large values of \(A_{\text{sur}}\), or, equivalently, \(\omega_{\psi}\), and the upper one corresponding to small values of the above quantities. The boundary separating the two areas is determined by the solution of the equations

\[
\frac{dE}{d\omega_{\psi}} = 0 , \quad \frac{dQ}{d\omega_{\psi}} = 0 .
\]

The solution to both of the above equations is only one:

\[
\omega_{\psi}^{\text{cr}} = \frac{1}{4} \left( \frac{3\pi}{2} \right)^{1/3}, \quad A_{\text{sur}}^{\text{cr}} = \frac{1}{16}. \quad (5.34)
\]

For \(\omega_{\psi} > \omega_{\psi}^{\text{cr}}\) energy and charge are monotone decreasing functions of the chemical potential, but they are both monotone increasing for \(\omega_{\psi} < \omega_{\psi}^{\text{cr}}\). The charge shows a more rapid decrease.

Solving eq. 5.33 in terms of \(A_{\text{sur}}\) for simplicity we find two solutions, called \(A_{\text{sur}}^{(1)}\) and \(A_{\text{sur}}^{(2)}\) corresponding to the upper and lower branches respectively of the \(E = E(Q)\) function. These solutions are:

\[
A_{\text{sur}}^{(1)} = \frac{1}{72\pi^{3}} \left( 2^{1/3}3^{2/3}Q^{2}\pi^{5/3} - 12 \cdot 2^{2/3}3^{1/3}Q^{3}\pi^{7/3} + 36\pi^{3} - 6\sqrt{2\pi^{2}} \times \right.
\]

\[
\left. \sqrt{-2Q^{3} - 12 \cdot 2^{2/3}3^{1/3}Q^{4/3} + 18\pi^{2} + 5 \cdot 2^{1/3}Q^{2}(3\pi)^{2/3}} \right) \quad (5.35)
\]
\[ A_{\text{sur}}^{(2)} = \frac{1}{72\pi^3} \left( 2^{1/3}3^{2/3}Q^2\pi^{5/3} + 12 \cdot 2^{2/3}3^{1/3}Q\pi^{7/3} + 36\pi^3 - 6\sqrt{2}\pi^2 \times \right. \]
\[ \sqrt{-2Q^3 - 12 \cdot 2^{2/3}3^{1/3}Q\pi^{4/3} + 18\pi^2 + 5 \cdot 2^{1/3}Q^2(3\pi)^{2/3}} \right) \] (5.36)

The metric of the eq. 5.35 is the small one (less than 1/16) when the the metric of eq. 5.36 is the large one. Substituting in eq. 5.32 we find the energies corresponding to the two branches:

\[ E^{(1)} = -\frac{\pi}{2} + Q \left( \frac{2\pi}{2} \right)^{1/3} - \frac{Q^2}{4 \cdot 2^{3/2}(3\pi)^{1/3}} + \]
\[ \frac{1}{4} \sqrt{-4Q^3 - 24 \cdot 2^{2/3}3^{1/3}Q\pi^{4/3} + 36\pi^2 + 10 \cdot 2^{1/3}Q^2(3\pi)^{2/3} + \}
\[ \frac{1}{6 \cdot 2^{1/4}3^{1/2}\pi^{5/4}} \left( 2^{1/3}3^{2/3}Q^2\pi^{5/3} - 12 \cdot 2^{2/3}3^{1/3}Q\pi^{7/3} + 6\pi^2 \times \right. \]
\[ \left. (6\pi - \sqrt{-4Q^3 - 24 \cdot 2^{2/3}3^{1/3}Q\pi^{4/3} + 36\pi^2 + 10 \cdot 2^{1/3}Q^2(3\pi)^{2/3}} \right)^{3/4}, \] (5.37)

the large value of the energy corresponding to the upper branch and:

\[ E^{(2)} = -\frac{\pi}{2} + Q \left( \frac{2\pi}{2} \right)^{1/3} - \frac{Q^2}{4 \cdot 2^{3/2}(3\pi)^{1/3}} + \]
\[ \frac{1}{4} \sqrt{-4Q^3 - 24 \cdot 2^{2/3}3^{1/3}Q\pi^{4/3} + 36\pi^2 + 10 \cdot 2^{1/3}Q^2(3\pi)^{2/3} + \}
\[ \frac{1}{6 \cdot 2^{1/4}3^{1/2}\pi^{5/4}} \left( 2^{1/3}3^{2/3}Q^2\pi^{5/3} - 12 \cdot 2^{2/3}3^{1/3}Q\pi^{7/3} + 6\pi^2 \times \right. \]
\[ \left. (6\pi + \sqrt{-4Q^3 - 24 \cdot 2^{2/3}3^{1/3}Q\pi^{4/3} + 36\pi^2 + 10 \cdot 2^{1/3}Q^2(3\pi)^{2/3}} \right)^{3/4}, \] (5.38)

the small one.

6 Conclusions

Our discussion can be summarized as follows:

1. There are stable, gravitating field configurations, in 2 + 1 dimensions, consisted of a N–carrying field, fermionic, or scalar (abelian or non-abelian). An always positive potential, generated either by the the N–carrying field, or by an additional real scalar field, is necessary in order to stabilize the soliton. The potential as a function of the field \( \phi \) increases as \( \phi^3 \) for small
values of the field, then slower than $\phi^2$ providing in this way a local minimum for some value of the field differing from zero and for large values of the field as $\phi^4$, or $\phi^6$ usually, in order to ensure the positivity of the potential for large values of the field. The soliton solution lies near this minimum and the total energy of the star is smaller than $m^2 \phi^2$, i.e. the potential energy of the free particles in any field configuration under consideration. There is always a certain frequency, minimizing the total energy of the configuration, obtained by the equality of the $W$—energy to the $U$—energy. This equality holds true in the absence of gravity and is generalized in a generally covariant way.

2. When $\Lambda \neq 0$ and $A_{\text{sur}} \to 0$ the particle number decreases rapidly but the total energy decreases slowly, as one can see from figures 1, 6, 11 and 20. So, the $A_{\text{sur}} = 0$ case (formation of horizon) is impossible, because below a certain value of $A_{\text{sur}}$ fission into free particles is energetically favorable. This result is supported by both numerical and analytical solutions. Decay into free particles is energetically forbidden for zero cosmological constant, as can be proved by the analytical solutions for every special kind of q-star. For cosmological constant differing from zero, the energy and the charge of the soliton increase when the gravity becomes stronger (i.e.: when $A_{\text{sur}}$ decreases) but below a certain value of the frequency energy decreases slowly and charge more rapidly. The result is that for a certain charge there are usually two different values of the energy. So the soliton with the larger energy can emit the energy excess and goes to the lower-energy state. Due to the above behavior of the $E = E(Q)$ function, we found analytically and numerically that below a certain value of the frequency the soliton decay to free particles is energetically favorable, so when the cosmological constant differs from zero there is a certain range of the soliton parameters (the independent parameter, frequency, and the derivative ones, radius, energy and charge) which verify the soliton stability. This range of the phase space parameters is fully depicted in our figures.

3. When the cosmological constant is zero or small with respect to the energy densities (namely $U$ and $W$), the soliton parameters, radius, mass and particle number, increase when the frequency decreases, i.e. when the gravity becomes stronger. When the cosmological constant becomes more important than the energy densities, the soliton becomes smaller and the mass and particle number show a corresponding decrease.

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