DETERMINATION OF THE FRICKE FAMILIES

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Abstract. For a positive integer $N$ divisible by 4, let $\mathcal{O}^1_N(\mathbb{Q})$ be the ring of weakly holomorphic modular functions for the congruence subgroup $\Gamma^1(N)$ with rational Fourier coefficients. We present explicit generators of the ring $\mathcal{O}^1_N(\mathbb{Q})$ over $\mathbb{Q}$ in terms of both Fricke functions and Siegel functions, from which we are able to classify all Fricke families of such level $N$.

1. Introduction

The group $\text{SL}_2(\mathbb{R})$ acts on the complex upper half-plane $\mathbb{H} = \{ \tau \in \mathbb{C} | \text{Im}(\tau) > 0 \}$ by fractional linear transformations, that is,

$$
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} (\tau) = \frac{a\tau + b}{c\tau + d}.
$$

For a positive integer $N$, let $\mathcal{F}_N$ be the field of meromorphic modular functions for the principal congruence subgroup $\Gamma(N) = \{ \gamma \in \text{SL}_2(\mathbb{Z}) | \gamma \equiv I_2 \pmod{N} \}$ of $\text{SL}_2(\mathbb{Z})$ whose Fourier coefficients belong to the $N$th cyclotomic field $\mathbb{Q}(\zeta_N)$, where $\zeta_N = e^{2\pi i/N}$. It is well known that $\mathcal{F}_1$ is generated over $\mathbb{Q}$ by the elliptic modular function $j(\tau)$, and $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ with

$$
\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \cong \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}
$$

(see §2). For $N \geq 2$, let

$$
\mathcal{V}_N = \{ \mathbf{v} \in \mathbb{Q}^2 | \mathbf{v} \text{ has primitive denominator } N \}.
$$

We call a family $\{h_\mathbf{v}(\tau)\}_{\mathbf{v} \in \mathcal{V}_N}$ of functions in $\mathcal{F}_N$ a Fricke family of level $N$, if it satisfies the following three conditions:

(F1) Each $h_\mathbf{v}(\tau)$ is weakly holomorphic (that is, holomorphic on $\mathbb{H}$).

(F2) $h_\mathbf{v}(\tau)$ depends only on $\pm \mathbf{v} \pmod{\mathbb{Z}^2}$.

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(F3) \( h_\nu(\tau) = h_\nu(\tau)^t \) for all \( \alpha \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \), where \( \alpha^t \) means the transpose of \( \alpha \).

There are two important kinds of Fricke families \( \{f_\nu(\tau)\}_\nu \) and \( \{g_\nu(\tau)^{12N}\}_\nu \), one consisting of Fricke functions and the other consisting of \( 12N \)th powers of Siegel functions (see §3). They are building blocks of fields of modular functions and groups of modular units ([7, Chapter 2] and [8, Chapter 6]). Since their special values at imaginary quadratic arguments generate class fields over the corresponding imaginary quadratic fields (see [3], [4] and [8, Chapter 10]), it would be meaningful by themselves and also worth of investigating the structure of Fricke families as a ring.

As far as we understand, there is no known result on constructing all such families. In this paper, we shall first classify all Fricke families of level \( N \), when \( N \equiv 0 \pmod{4} \) (Theorems 4.3, 6.2 and Corollary 6.4). Furthermore, if we constrain the condition (F1) to

\[(F1') \, \text{every} \, h_\nu(\tau) \, \text{is holomorphic on} \, \mathbb{H} \, \text{except for the set} \, \{\gamma(\zeta_3), \gamma(\zeta_4) | \gamma \in \text{SL}_2(\mathbb{Z})\},
\]

then we can also determine all weak families \( \{h_\nu(\tau)\}_{\nu \in \mathcal{V}_N} \) of functions in \( \mathcal{F}_N \) satisfying (F1'), (F2) and (F3) for arbitrary level \( N \geq 2 \) (Theorem 7.4 and Remark 7.5).

### 2. Galois actions on functions

In this section, we shall briefly describe the actions of the group \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \) on the field \( \mathcal{F}_N \).

For a positive integer \( N \), the group \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \) has a unique decomposition

\[ G_N \cdot \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \quad \text{with} \quad G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \mid d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}. \]

This group acts on the field \( \mathcal{F}_N \) as follows ([9, §6.1–6.2]): Let

\[ h(\tau) = \sum_{n \geq \infty} c_n q^n/N \in \mathcal{F}_N \quad (c_n \in \mathbb{Q}(\zeta_N), \, q = e^{2\pi i \tau}). \]

(A1) The matrix \( \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in G_N \) acts on \( h(\tau) \) as

\[ h(\tau) \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} = \sum_{n \geq \infty} c_n^{\sigma_d} q^n/N, \]

where \( \sigma_d \) is the automorphism of \( \mathbb{Q}(\zeta_N) \) given by \( \zeta_N^{d} = \zeta_N \).

(A2) The matrix \( \gamma \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \) acts on \( h(\tau) \) as

\[ h(\tau) \gamma = (h \circ \tilde{\gamma})(\tau), \]

where \( \tilde{\gamma} \) is any preimage of the reduction \( \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \) considered as a fractional linear transformation.
Lemma 2.1. Let \( \{h_\nu(\tau)\}_{\nu \in \mathcal{V}_N} \) be a Fricke family of level \( N \geq 2 \). Then \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \) acts on \( \{h_\nu(\tau)\}_\nu \) transitively.

Proof. Note by (F3) that \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \) acts on the family \( \{h_\nu(\tau)\}_\nu \). Let \( \nu = \begin{bmatrix} a/N \\ b/N \end{bmatrix} \in \mathcal{V}_N \) so that \( \gcd(a, b) \) is relatively prime to \( N \). If we take any \( \alpha = \begin{bmatrix} a \ b \\ c \ d \end{bmatrix} \in M_2(\mathbb{Z}) \) such that \( \det(\alpha) \) is relatively prime to \( N \), then we see by (F3) that

\[
\begin{align*}
\left[ \begin{array}{cc} 1/N \\ 0 \end{array} \right] (\tau) & = h_{\nu \alpha}
\left[ \begin{array}{cc} 1/N \\ 0 \end{array} \right] (\tau) = h_{\nu N}(\tau) = h_\nu(\tau).
\end{align*}
\]

This implies that \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \) acts on \( \{h_\nu(\tau)\}_\nu \) transitively. \( \square \)

Remark 2.2. Roughly speaking, this family \( \{h_\nu(\tau)\}_\nu \) is completely determined by its component \( h_{\nu \nu}(\tau) \).

3. Fricke and Siegel functions

For a lattice \( \Lambda \) in \( \mathbb{C} \), we let

\[
g_2(\Lambda) = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^2}, \quad g_3(\Lambda) = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^4} \quad \text{and} \quad \Delta(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2.
\]

The elliptic modular function \( j(\tau) \) is defined by

\[
j(\tau) = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)} = 1728 \left( 1 + \frac{27}{\Delta(\tau)} \right) \quad (\tau \in \mathbb{H}),
\]

where \( g_2(\tau) = g_2([\tau, 1]), \quad g_3(\tau) = g_3([\tau, 1]) \) and \( \Delta(\tau) = \Delta([\tau, 1]) \). This generates the ring of weakly holomorphic functions in \( \mathcal{F}_1 \) over \( \mathbb{Q} \) ([8, Theorem 2 in Chapter 3]).

The Weierstrass \( \wp \)-function relative to \( \Lambda \) is given by

\[
\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right) \quad (z \in \mathbb{C}).
\]

For each \( \nu = [v_1 v_2] \in \mathbb{Q}^2 \setminus \mathbb{Z}^2 \), we define the Fricke function \( f_\nu(\tau) \) by

\[
f_\nu(\tau) = -2^73^5 \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)} \wp_\nu(\tau) \quad (\tau \in \mathbb{H}),
\]

where \( \wp_\nu(\tau) = \wp(v_1\tau + v_2; [\tau, 1]) \).

By the Weierstrass \( \sigma \)-function relative to \( \Lambda \), we mean the infinite product

\[
\sigma(z; \Lambda) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left( 1 - \frac{z}{\lambda} \right) e^{z/\lambda + (1/2)(z/\lambda)^2} \quad (z \in \mathbb{C}).
\]

Taking logarithmic derivative, we achieve the Weierstrass \( \zeta \)-function as

\[
\zeta(z; \Lambda) = \frac{\sigma'(z; \Lambda)}{\sigma(z; \Lambda)} = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right) \quad (z \in \mathbb{C}).
\]
Proposition 3.1. If \( \zeta(z; \Lambda) = -\varphi(z; \Lambda) \) is periodic with respect to \( \Lambda \), for every \( \lambda \in \Lambda \) there is a constant \( \eta(\lambda; \Lambda) \) which satisfies
\[
\zeta(z + \lambda; \Lambda) - \zeta(z; \Lambda) = \eta(\lambda; \Lambda) \quad (z \in \mathbb{C}).
\]
For any \( v = \left[ \frac{v_1}{v_2} \right] \in \mathbb{Q}^2 \setminus \mathbb{Z}^2 \), we then define the Siegel function \( g_v(\tau) \) by
\[
g_v(\tau) = e^{-\pi i v_2 (\tau + v_1)/2} \eta(v_1, v_2, \tau; \mathbb{Z}) (\tau \in \mathbb{H}),
\]
where
\[
\eta(\tau) = \sqrt{2\pi} \zeta_8 q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (\tau \in \mathbb{H})
\]
is the Dedekind \( \eta \)-function which is a 24th root of \( \Delta(\tau) \) ([8, Theorem 5 in Chapter 18]). By using the \( q \)-product expansion of the Weierstrass \( \sigma \)-function, we get the expression
\[
g_v(\tau) = \frac{-e^{\pi i v_2 (\tau - 1)} q^{(1/2) B_2(v_1)} (1 - q^{v_1} e^{2\pi i v_2})}{\prod_{n=1}^{\infty} (1 - q^{n+v_1} e^{2\pi i v_2})(1 - q^{n-v_1} e^{-2\pi i v_2})},
\]
where \( B_2(x) = x^2 - x + 1/6 \) is the second Bernoulli polynomial ([8, Chapter 19, §2]). Observe that \( g_v(\tau) \) has neither zeros nor poles on \( \mathbb{H} \).

**Proposition 3.1.** If \( N \geq 2 \), then \( \{ f_v(\tau) \}_{v \in \mathbb{V}_N} \) and \( \{ g_v(\tau)^{12N} \}_{v \in \mathbb{V}_N} \) are Fricke families of level \( N \).

**Proof.** See [8, Chapter 6, §2–3] and [7, Proposition 1.3 in Chapter 2]. \( \square \)

**Remark 3.2.** We call a function \( h(\tau) \) in \( \mathcal{F}_N \) a modular unit of level \( N \geq 1 \), if both \( h(\tau) \) and \( h(\tau)^{-1} \) are integral over \( \mathbb{Q}[j(\tau)] \). As is well known, \( h(\tau) \) is a modular unit if and only if it has neither zeros nor poles on \( \mathbb{H} \) ([7, p. 36] or [2, Proposition 2.3]). Thus \( g_v(\tau)^{12N} \) is a modular unit of level \( N \) for every \( v \in \mathbb{V}_N \) with \( N \geq 2 \). Moreover, \( g_v(\tau) \) is a modular unit of level \( 12N^2 \) ([7, Theorems 5.2 and 5.3 in Chapter 3]).

For later use, we need the following lemmas.

**Lemma 3.3.** Let \( u, v \in \mathbb{Q}^2 \setminus \mathbb{Z}^2 \).

(i) We have the assertion that \( f_u(\tau) = f_v(\tau) \) if and only if \( u \equiv \pm v \pmod{\mathbb{Z}^2} \).

(ii) If \( u \not\equiv \pm v \pmod{\mathbb{Z}^2} \), then we get the relation
\[
(f_u(\tau) - f_v(\tau))^6 = 2^{12} 3^6 j(\tau)^2 j(\tau - 1728)^3 g_u + v(\tau)^6 g_u - v(\tau)^6 g_u(\tau)^{12} g_v(\tau)^{12}.
\]

**Proof.** (i) See [1, Lemma 10.4] and definition (3).

(ii) See [8, Theorem 2 in Chapter 18] and definitions (2), (3) and (4). \( \square \)
Remark 3.4. For \( N \geq 2 \), let \( u, v, u', v' \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2 \) such that \( u \not\equiv \pm v \pmod{\mathbb{Z}^2} \) and \( u' \not\equiv \pm v' \pmod{\mathbb{Z}^2} \). Then, the function
\[
\frac{f_u(\tau) - f_v(\tau)}{f_{u'}(\tau) - f_{v'}(\tau)} = \frac{\wp_u(\tau) - \wp_v(\tau)}{\wp_{u'}(\tau) - \wp_{v'}(\tau)}
\]
in \( \mathcal{F}_N \) has neither zeros nor poles on \( \mathbb{H} \) by Lemma 3.3(ii). Thus it is a modular unit of level \( N \) by Remark 3.2, called a Weierstrass unit of level \( N \).

Lemma 3.5. Let \( v \in \mathbb{Q}^2 \setminus \mathbb{Z}^2 \).

(i) We have \( g_{-v}(\tau) = -g_v(\tau) \).

(ii) If \( s = \begin{bmatrix} s_1 & s_2 \\ 0 & 1 \end{bmatrix} \in \mathbb{Z}^2 \), then we get \( g_v + s(\tau) = (-1)^{s_1 s_2} (e^{e^{-\pi i (s_1 v_2 - s_2 v_1)}}) g_v(\tau) \).

(iii) For each \( \gamma \in \text{SL}_2(\mathbb{Z}) \), we obtain \( (g_v \circ \gamma)(\tau) = \zeta g_{\gamma v}(\tau) \) for some 12th root of unity \( \zeta \) depending only on \( \gamma \).

Proof. See [6, Proposition 2.4]. \( \square \)

4. Rings of weakly holomorphic functions

For an integer \( N \geq 2 \), we denote by \( \text{Fr}_N \) the set of all Fricke families of level \( N \). Then, \( \text{Fr}_N \) becomes a ring under the operations
\[
\{h_v(\tau)\}_v + \{k_v(\tau)\}_v = \{(h_v + k_v)(\tau)\}_v,
\]
\[
\{h_v(\tau)\}_v \cdot \{k_v(\tau)\}_v = \{(h_v k_v)(\tau)\}_v.
\]

For a positive integer \( N \), let \( \mathcal{F}_N^1(\mathbb{Q}) \) be the field of meromorphic modular functions for the congruence subgroup
\[
\Gamma^1(N) = \left\{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \pmod{N} \right\}
\]
with rational Fourier coefficients. Further, we let \( \mathcal{O}_N^1(\mathbb{Q}) \) its subring consisting of weakly holomorphic functions.

Lemma 4.1. Let \( \{h_v(\tau)\}_v \in \text{Fr}_N \) with \( N \geq 2 \). Then, \( h_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}}(\tau) \) belongs to \( \mathcal{O}_N^1(\mathbb{Q}) \).

Proof. For any \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^1(N) \), we see that
\[
(h_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}} \circ \gamma)(\tau) = h_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}}(\tau) \gamma \quad \text{by (A2)}
\]
\[
= h_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}}(\tau) \quad \text{by (F3)}
\]
\[
= h_{\begin{bmatrix} a/N \\ b/N \end{bmatrix}}(\tau)
\]
\[
= h_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}}(\tau) \quad \text{by the fact } a \equiv 1, \; b \equiv 0 \pmod{N} \text{ and (F2)}.
\]

Thus \( h_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}}(\tau) \) is modular for \( \Gamma^1(N) \).
Now, let $\beta = [\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}] \in G_N$. We get by (F3) and (F2) that

$$h_{\begin{pmatrix} 1/N \\ 0 \\ 0 \end{pmatrix}}(\tau)^\beta = h_{\beta h_{\begin{pmatrix} 1/N \\ 0 \\ 0 \end{pmatrix}}}(\tau) = h_{\begin{pmatrix} 1/N \\ 0 \\ 0 \end{pmatrix}}(\tau),$$

which shows that $h_{\begin{pmatrix} 1/N \\ 0 \\ 0 \end{pmatrix}}(\tau)$ has rational Fourier coefficients by (A1).

Moreover, since $h_{\begin{pmatrix} 1/N \\ 0 \\ 0 \end{pmatrix}}(\tau)$ is weakly holomorphic by (F1), it belongs to $\mathcal{O}_N^1(\mathbb{Q})$. \hfill \Box

Hence we obtain by Lemma 4.1 a ring homomorphism

$$\phi_N : \text{Fr}_N \rightarrow \mathcal{O}_N^1(\mathbb{Q})$$

(6)

$$\{h_{\nu}(\tau)\}_\nu \mapsto h_{\begin{pmatrix} 1/N \\ 0 \end{pmatrix}}(\tau).$$

**Lemma 4.2.** For $N \geq 2$, let $a$ and $b$ be a pair of integers such that $\gcd(a, b)$ is relatively prime to $N$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ be matrices in $M_2(\mathbb{Z})$ such that $\det(\gamma) \equiv \det(\gamma') \equiv 1 \pmod{N}$. Then, there is a matrix $\delta \in \Gamma^1(N)$ satisfying $\delta \gamma \equiv \gamma' \pmod{N}$.

**Proof.** Take $\delta = \begin{pmatrix} c' & 1 \\ -ad' & c \end{pmatrix} \in \Gamma^1(N)$. One can then show that

$$\delta \gamma \equiv \begin{pmatrix} a & b \\ c' \det(\gamma) + c(-\det(\gamma') + 1) & d' \det(\gamma) + d(-\det(\gamma') + 1) \end{pmatrix} \equiv \gamma' \pmod{N}$$

due to the fact $\det(\gamma) \equiv \det(\gamma') \equiv 1 \pmod{N}$. \hfill \Box

**Theorem 4.3.** If $N \geq 2$, then two rings $\text{Fr}_N$ and $\mathcal{O}_N^1(\mathbb{Q})$ are isomorphic via the map $\phi_N$ stated in (6).

**Proof.** Let $\{h_{\nu}(\tau)\}_\nu \in k(\phi)$, and so $\phi_N(\{h_{\nu}(\tau)\}_\nu) = h_{\begin{pmatrix} 1/N \\ 0 \end{pmatrix}}(\tau) = 0$. Then we attain by Lemma 2.1 that $h_{\nu}(\tau) = 0$ for all $\nu \in V_N$. This shows that $\phi_N$ is one-to-one.

Now, let $h(\tau) \in \mathcal{O}_N^1(\mathbb{Q})$. For each $\nu = \begin{pmatrix} a/N \\ b/N \end{pmatrix} \in V_N$, we take any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ such that $\det(\gamma) \equiv 1 \pmod{N}$, and set $h_{\nu}(\tau) = h(\tau)^\gamma$. We first claim that $h_{\nu}(\tau)$ is well-defined, independent of the choice of $\gamma$. Indeed, if $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ is another matrix in $M_2(\mathbb{Z})$ such that $\det(\gamma') \equiv 1 \pmod{N}$, then we see that

$$h(\tau)^\gamma = h(\tau)^{\delta \gamma} \quad \text{for some } \delta \in \Gamma^1(N) \text{ by Lemma 4.2 and (1)}$$

$$= h(\tau)^\gamma \quad \text{because } h(\tau) \text{ is modular for } \Gamma^1(N).$$

Since $h(\tau)$ is weakly holomorphic, so is $h_{\nu}(\tau) = h(\tau)^\gamma$ by (A2). Furthermore, $h_{\nu}(\tau)$ depends only on $\pm \nu \pmod{\mathbb{Z}^2}$ by (1). Let $\alpha = [x \ y] \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$. We then derive by considering $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as an element of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ that

$$h_{\nu}(\tau)^\alpha = \left( h(\tau) \begin{pmatrix} x & y \\ c & d \end{pmatrix} \right)^\alpha.$$
DETERMINATION OF THE FRICKE FAMILIES

\[ h(\tau) \left[ \begin{array}{cc} ax+by & az+dw \\ cx+dy & cy+dw \end{array} \right] = h(\tau) \left[ \begin{array}{cc} 1 & 0 \\ 0 & \text{det}(\alpha) \end{array} \right] \left[ \begin{array}{cc} ax+by & az+dw \\ \text{det}(\alpha)^{-1}(cx+dy) & \text{det}(\alpha)^{-1}(cy+dw) \end{array} \right] \]

since \( h(\tau) \) has rational Fourier coefficients

\[ h(\tau) \left[ \begin{array}{cc} a/N & b/N \\ x/N & y/N \end{array} \right] = h(\tau) \frac{1}{\sqrt{N}} \begin{bmatrix} a/b \end{bmatrix} \]

Thus the family \( \{ h_\nu(\tau) \} \) satisfies (F3). Lastly, since \( \phi_N(\{ h_\nu(\tau) \} \nu) = h[1/N] \tau) \), \( \phi_N \) is surjective.

Therefore, we conclude that \( \text{Fr}_N \) and \( \mathcal{O}_N^1(Q) \) are isomorphic via \( \phi_N \).

5. Conjugate subgroups of \( \text{SL}_2(\mathbb{R}) \)

For a positive integer \( N \), let

\( \Gamma_1(N) = \{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{N} \} \) and \( \omega_N = \begin{bmatrix} 1/\sqrt{N} & 0 \\ 0 & \sqrt{N} \end{bmatrix} \).

Then, we see from the observation

\[ \omega_N \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \omega_N^{-1} = \left[ \begin{array}{cc} a & b/N \\ c & d \end{array} \right] \text{ for all } \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in \text{SL}_2(\mathbb{R}) \]

that \( \Gamma^1(N) \) and \( \Gamma_1(N) \) are conjugate in \( \text{SL}_2(\mathbb{R}) \), namely,

\[ \omega_N \Gamma^1(N) \omega_N^{-1} = \Gamma_1(N). \]

Let \( \mathcal{F}_{1,N}(Q) \) be the field of meromorphic modular functions for \( \Gamma_1(N) \) with rational Fourier coefficients. One can readily check that the relation (7) gives rise to an isomorphism

\[ h(\tau) = \sum_{n \gg -\infty} c_n q^n \mapsto (h \circ \omega_N)(\tau) = h(\tau/N) = \sum_{n \gg -\infty} c_n q^{n/N} \]

with inverse map \( f(\tau) \mapsto (f \circ \omega_N^{-1})(\tau) = f(N\tau) \). Furthermore, let \( \mathcal{O}_{1,N}(Q) \) be the subring of \( \mathcal{F}_{1,N}(Q) \) consisting of weakly holomorphic functions. Since the map in (8) preserves weakly holomorphicity, it induces an isomorphism

\[ \mathcal{O}_{1,N}(Q) \approx \mathcal{O}_N^1(Q). \]
Let $X_1(4)$ be the modular curve corresponding to the congruence subgroup $\Gamma_1(4)$. It is well known that $X_1(4)$ has genus 0 with three inequivalent cusps 0, 1/2 and $\infty$ ([5, p. 131]). Moreover, the function

$$g_{1,4}(\tau) = \left(\frac{g_{1,2/4}(\tau)}{g_{1,4/4}(\tau)}\right)^8 = q^{-1}(1+q)\prod_{n=1}^{\infty} \left(\frac{1-q^{4n+2}}{1-q^{4n+1}}\right)^4 \left(\frac{1-q^{4n+2}}{1-q^{4n-1}}\right)^4$$

generates the function field $\mathbb{C}(X_1(4))$ of $X_1(4)$ over $\mathbb{C}$, having values 16, 0 and $\infty$ at the cusps 0, 1/2 and $\infty$, respectively ([5, Theorem 3(ii)] and [6, Tables 2 and 3]). Since $g_{1,4}(\tau)$ has rational Fourier coefficients, we deduce by [5, Lemma 4.1]

$$F_{1,4}(\mathbb{Q}) = \mathbb{Q}(g_{1,4}(\tau)).$$

Lemma 5.1. Let $c \in \mathbb{C}$. Then, $(g_{1,4}(\tau) - c)$ has neither zeros nor poles on $\mathbb{H}$ if and only if $c \in \{0, 16\}$.

Proof. See [2, (4)]. □

Theorem 5.2. We get the following structures.

(i) $O_{1,4}(\mathbb{Q}) = \mathbb{Q}[g_{1,4}(\tau), g_{1,4}(\tau)^{-1}, (g_{1,4}(\tau) - 16)^{-1}]$.

(ii) $O_{1,4}^1(\mathbb{Q}) = \mathbb{Q}[g_{1,4}^1(\tau), g_{1,4}^1(\tau)^{-1}, (g_{1,4}^1(\tau) - 16)^{-1}]$, where $g_{1,4}^1(\tau) = g_{1,4}(\tau/4) = \frac{1}{g_{1,4}(\tau)}(\tau)^8 g_{1,4/2}(\tau)^8$.

Proof. (i) Since $g_{1,4}(\tau)$ and $(g_{1,4}(\tau) - 16)$ are modular units in $F_{1,4}(\mathbb{Q})$ by Lemma 5.1 and (10), we obtain the inclusion $O_{1,4}(\mathbb{Q}) \supseteq \mathbb{Q}[g_{1,4}(\tau), g_{1,4}(\tau)^{-1}, (g_{1,4}(\tau) - 16)^{-1}]$.

Conversely, let $h(\tau) \in O_{1,4}(\mathbb{Q})$. By (10), we can express $h(\tau)$ as $h(\tau) = A(g_{1,4}(\tau))B(g_{1,4}(\tau))$ for some polynomials $A(x), B(x) \in \mathbb{Q}[x]$ which are relatively prime. Suppose that $B(x)$ has a zero $c \in \mathbb{Q}$ not equal to 0 or 16. We see by Lemma 5.1 that $g_{1,4}(\tau_0) = c = 0$ for some $\tau_0 \in \mathbb{H}$, from which we have $B(g_{1,4}(\tau_0)) = 0$. But, since $A(x)$ is not divisible by $(x-c)$ in $\mathbb{Q}[x]$, we achieve $A(g_{1,4}(\tau_0)) \neq 0$. This contradicts that $h(\tau)$ is weakly holomorphic. Thus $B(x)$ has no zeros other than 0 and 16, which implies the converse inclusion $O_{1,4}(\mathbb{Q}) \subseteq \mathbb{Q}[g_{1,4}(\tau), g_{1,4}(\tau)^{-1}, (g_{1,4}(\tau) - 16)^{-1}]$.

(ii) It follows immediately from (i) and the isomorphism given in (9). □

6. Generators for $N \equiv 0 \pmod{4}$

Now, we are ready to present explicit generators of the ring $O_N^1(\mathbb{Q})$ over $\mathbb{Q}$, when $N \equiv 0 \pmod{4}$. This amounts to classifying all Fricke families of such level $N$ by Theorem 4.3.

Proposition 6.1. If $N \geq 2$, then we obtain $F_N^1(\mathbb{Q}) = F_1(f_{1/N}(\tau))$. 

Proof. We first recall that $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ with

$$\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \cong \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \cong G_N \cdot \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}.$$ 

Observe by (A1) and (A2) that $F_N$ is a Galois extension of $\mathcal{F}_N^1(\mathbb{Q})$ with

$$\text{Gal}(\mathcal{F}_N/F_N^1(\mathbb{Q})) \cong G_N \cdot \{ \gamma \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \mid \gamma \equiv \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{N} \},$$

Let $F = \mathcal{F}_1(f_{[1/0]}^{1/N})(\tau))$. Since $\{f_\nu(\tau)\}_{\nu \in \text{Fr}_N}$ by Proposition 3.1, we have the inclusion $F \subseteq \mathcal{F}_N^1(\mathbb{Q})$ by Lemma 4.1. Suppose that $\alpha = \beta\gamma$ with $\beta \in G_N$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ leaves $f_{[1/0]}^{1/N}(\tau)$ fixed. Then we derive that

$$f_{[1/0]}^{1/N}(\tau) = f_{[1/0]}^1(\tau)^\alpha 
= (f_{[1/0]}^1(\tau)^\beta)^\gamma 
= f_{[1/0]}^1(\tau)^\gamma \text{ because } f_{[1/0]}^{1/N}(\tau) \text{ has rational Fourier coefficients} 
= f_{\gamma^\nu}[1/0]^{1/N}(\tau) \text{ by (F2) and (F3) for } \{f_\nu(\tau)\}_\nu 
= f_{[a/N][b/N]}^{a/N}(\tau).$$

Thus we get $b \equiv 0 \pmod{N}$ and $a \equiv d \equiv 1 \pmod{N}$ by Lemma 3.3(i) and the fact $\gamma \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$. This yields $F \supseteq \mathcal{F}_N^1(\mathbb{Q})$ by Galois theory. Therefore, we conclude $F = \mathcal{F}_1(f_{[1/0]}^{1/N}(\tau)) = \mathcal{F}_N^1(\mathbb{Q})$. \hfill $\Box$

When $N \geq 8$ and $N \equiv 0 \pmod{4}$, we consider a function

$$f_N^1(\tau) = \frac{f_{[1/0]}^{1/N}(\tau) - f_{[1/0]}^{1/2}(\tau)}{f_{[1/0]}^{1/4}(\tau) - f_{[1/0]}^{1/2}(\tau)} \quad (\tau \in \mathbb{H}).$$

It is a modular unit belonging to $\mathcal{O}_N^1(\mathbb{Q})$ by Proposition 3.1, Remark 3.4 and Lemma 4.1.

**Theorem 6.2.** If $N \geq 8$ and $N \equiv 0 \pmod{4}$, then we attain

$$\mathcal{O}_N^1(\mathbb{Q}) = \mathcal{O}_1^1(\mathbb{Q})[f_N^1(\tau)] = \mathbb{Q}[g_1^1(\tau), g_2^1(\tau)^{-1}, (g_1^1(\tau) - 16)^{-1}, f_N^1(\tau)].$$

Proof. It is obvious that $\mathcal{O}_N^1(\mathbb{Q}) \supseteq \mathcal{O}_1^1(\mathbb{Q})[f_N^1(\tau)]$.

As for the converse inclusion, let $h(\tau) \in \mathcal{O}_N^1(\mathbb{Q})$. Note by Proposition 6.1 and Lemma 4.1 that

$$\mathcal{F}_N^1(\mathbb{Q}) = \mathcal{F}_1(f_{[1/0]}^{1/N}(\tau)) = \mathcal{F}_N^1(\mathbb{Q})(f_N^1(\tau)).$$

So, we can express $h = h(\tau)$ as

\begin{equation}
(11) \quad h = c_0 + c_1 f + \cdots + c_{d-1} f^{d-1},
\end{equation}
where \( f = f_N^1(\tau), \) \( d = \left[ F_N^1(\mathbb{Q}) : F_4^1(\mathbb{Q}) \right] \) and \( c_0, c_1, \ldots, c_{d-1} \in F_4^1(\mathbb{Q}) \). Multiplying both sides of (11) by \( 1, f, \ldots, f^{d-1} \), respectively, we have a linear system (with unknowns \( c_0, c_1, \ldots, c_{d-1} \))

\[
\begin{bmatrix}
1 & f & f^2 & \cdots & f^{d-1} \\
f & 0 & f & \cdots & f^{d-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f^{d-1} & f^{d-2} & f^{d-3} & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{d-1} \\
\end{bmatrix}
= 
\begin{bmatrix}
h \\
fh \\
\vdots \\
f^{d-1} \cdot h \\
\end{bmatrix}
\]

By taking the trace \( \text{Tr} = \text{Tr}_{F_N^1(\mathbb{Q})/F_4^1(\mathbb{Q})} \) on both sides, we obtain

\[
T \begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{d-1} \\
\end{bmatrix}
= 
\begin{bmatrix}
\text{Tr}(h) \\
\text{Tr}(fh) \\
\vdots \\
\text{Tr}(f^{d-1}h) \\
\end{bmatrix}
\text{ with } T = 
\begin{bmatrix}
\text{Tr}(1) & \text{Tr}(f) & \cdots & \text{Tr}(f^{d-1}) \\
\text{Tr}(f) & \text{Tr}(f^2) & \cdots & \text{Tr}(f^d) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Tr}(f^{d-1}) & \text{Tr}(f^d) & \cdots & \text{Tr}(f^{2d-2}) \\
\end{bmatrix}
\]

Since every \( \text{Tr}(\ast) \), appeared in the above expression, lies in \( O_4^1(\mathbb{Q}) \), we get (12) \( c_0, c_1, \ldots, c_{d-1} \in \text{det}(T)^{-1}O_4^1(\mathbb{Q}) \).

If we let \( f_1, f_2, \ldots, f_d \) be all the Galois conjugates of \( f \) over \( F_4^1(\mathbb{Q}) \), then we derive that

\[
\text{det}(T) = \sum_{k=1}^{d} f_k^0 \sum_{k=1}^{d} f_k^1 \cdots \sum_{k=1}^{d} f_k^{d-1} \\
\sum_{k=1}^{d} f_k^0 \sum_{k=1}^{d} f_k^1 \cdots \sum_{k=1}^{d} f_k^{d-1} \\
\vdots \\
\sum_{k=1}^{d} f_k^{d-1} \sum_{k=1}^{d} f_k^0 \cdots \sum_{k=1}^{d} f_k^{d-1} \\
\]

\[
= \prod_{1 \leq m < n \leq d} (f_m - f_n)^2 \quad \text{by the Vandermonde determinant formula.}
\]

On the other hand, since \( f_{[1/2]}^1(\tau) \) and \( f_{[1/4]}^1(\tau) \) belong to \( F_4^1(\mathbb{Q}) \) by Lemma 4.1, each \( (f_m - f_n) \) is of the form

\[
\frac{f_{[a/N]}^1(\tau) - f_{[1/2]}^1(\tau)}{f_{[a/N]}^1(\tau) - f_{[1/4]}^1(\tau)} \quad \frac{f_{[c/N]}^1(\tau) - f_{[1/2]}^1(\tau)}{f_{[c/N]}^1(\tau) - f_{[1/4]}^1(\tau)}
\]

for some \( \left[ a/N \right], \left[ c/N \right] \in V_4^1 \) such that \( \left[ a/N \right] \not\equiv \left[ c/N \right] (\mod \mathbb{Z}^2) \) by Lemma 3.3(i). Thus \( \text{det}(T) \) is a modular unit in \( O_4^1(\mathbb{Q}) \) by Remark 3.4, from which it follows by (11) and (12) that \( h(\tau) \in O_4^1(\mathbb{Q})[f_N^1(\tau)] \). Therefore we establish the inclusion \( O_N^1(\mathbb{Q}) \subseteq O_4^1(\mathbb{Q})[f_N^1(\tau)] \), as desired. \( \square \)
Question 6.3. Whenever $N \not\equiv 0 \pmod{4}$, determine whether the ring $\mathcal{O}_{1}^{N}(\mathbb{Q})$ is also generated by both Fricke and Siegel functions, or not.

Corollary 6.4. Let $N \geq 8$ and $N \equiv 0 \pmod{4}$. For each $v = \begin{bmatrix} a/N \\ b/N \end{bmatrix} \in \mathcal{V}_{N}$, let

$$r_{v}(\tau) = \left( \frac{g(N/2)v(\tau)}{g(N/4)v(\tau)} \right)^{8} \quad \text{and} \quad s_{v}(\tau) = \frac{f_{v}(\tau) - f_{1/(N/2)v}(\tau)}{f_{1/(N/4)v}(\tau) - f_{1/(N/2)v}(\tau)}.$$

Then, a family $\{h_{v}(\tau)\}_{v \in \mathcal{V}_{N}}$ of functions in $\mathcal{F}_{N}$ is a Fricke family of level $N$ if and only if there is a polynomial $P(x, y, z, w) \in \mathbb{Q}[x, y, z, w]$ for which

$$h_{v}(\tau) = P(r_{v}(\tau), s_{v}(\tau))^{-1}, (r_{v}(\tau) - 16)^{-1}, s_{v}(\tau) \quad \text{for all } v \in \mathcal{V}_{N}.$$

Proof. For each $v = \begin{bmatrix} a/N \\ b/N \end{bmatrix} \in \mathcal{V}_{N}$, we take any $\gamma = \begin{bmatrix} a & b \\ d & c \end{bmatrix} \in M_{2}(\mathbb{Z})$ and $\tilde{\gamma} \in SL_{2}(\mathbb{Z})$ such that $det(\gamma) \equiv 1 \pmod{N}$ and $\tilde{\gamma} \equiv \gamma \pmod{N}$. Note that $t^{4}u \equiv \pm t^{4}u \pmod{2}$ for all $u \in (1/N)\mathbb{Z}^{2}$. We then see by (A2) and Lemma 3.5 that

$$g_{1}^{v}(\tau) = (g_{1}^{v} \circ \tilde{\gamma})(\tau) = \left( \frac{g_{\gamma}[1/2][1/4](\tau)}{g_{\gamma}[1/0][1/4](\tau)} \right)^{8} = \left( \frac{g_{\gamma}[1/2][1/4](\tau)}{g_{\gamma}[1/0][1/4](\tau)} \right)^{8} = \frac{g[1/2](\tau)}{g[1/4](\tau)} = r_{v}(\tau).$$

Furthermore, we get by Proposition 4.1 that

$$f_{N}^{v}(\tau) = \frac{f_{\gamma}[1/0](\tau) - f_{\gamma}[1/2](\tau)}{f_{\gamma}[1/4](\tau) - f_{\gamma}[1/2](\tau)} = \frac{f_{[a/N][1/2]}(\tau) - f_{[a/N][1/2]}(\tau)}{f_{[a/4][1/2]}(\tau) - f_{[a/4][1/2]}(\tau)} = s_{v}(\tau).$$

Now, the corollary follows from Theorems 4.3 (with its proof) and 6.2. \hfill \Box

7. Weak Fricke families

Let $\mathbb{H}' = \mathbb{H} \setminus \{ \gamma(\zeta_{3}), \gamma(\zeta_{4}) \mid \gamma \in SL_{2}(\mathbb{Z}) \}$. For a positive integer $N$, we let $\mathcal{O}_{1}^{N}(\mathbb{Q})$ be the ring of functions in $\mathcal{F}_{N}(\mathbb{Q})$ which are holomorphic on $\mathbb{H}'$.

Lemma 7.1. $j(\tau)$ gives rise to a bijection $j(\tau) : SL_{2}(\mathbb{Z})\backslash \mathbb{H} \rightarrow \mathbb{C}$ such that $j(\zeta_{3}) = 0$ and $j(\zeta_{4}) = 1728$.

Proof. See [8, Theorem 4 in Chapter 3]. \hfill \Box

Theorem 7.2. We have $\mathcal{O}_{1}^{N}(\mathbb{Q}) = \mathbb{Q}[j(\tau), j(\tau)^{-1}, (j(\tau) - 1728)^{-1}]$. 

Proof. By Lemma 7.1, we get the inclusion $O_N^I(Q) \supseteq Q[j(\tau), j(\tau)^{-1}, (j(\tau) - 1728)^{-1}]$.

Now, let $h(\tau) \in O_N^I(Q)$. Since $F_N^I(Q) = F_1 = Q(j(\tau))$, we may write $h(\tau) = A(j(\tau))/B(j(\tau))$ for some polynomials $A(x), B(x) \in \mathbb{Q}[x]$ which are relatively prime. Suppose that $B(x)$ has a zero $c \in \mathbb{Q}$ not equal to 0 or 1728. Since $j(\tau_N) = c$ for some $\tau_N \in \mathbb{H}$ by Lemma 7.1, we attain $B(j(\tau_N)) = 0$. But, since $A(x)$ is not divisible by $(x - c)$, we see that $A(j(\tau_N)) \neq 0$, which contradicts that $h(\tau)$ is holomorphic on $\mathbb{H}$. Thus we conclude that 0 and 1728 are the only possible zeros of $B(x)$, which proves the converse inclusion $O_N^I(Q) \subseteq Q[j(\tau), j(\tau)^{-1}, (j(\tau) - 1728)^{-1}]$. $\square$

**Lemma 7.3.** Modular units of level 1 are exactly nonzero rational numbers.

**Proof.** See [6, Lemma 2.1]. One can also justify by using Lemma 7.1. $\square$

**Theorem 7.4.** If $N \geq 2$, then we obtain

$$O_N^I(Q) = O_N^I(Q)[f_{[1/N]}(\tau)] = Q[j(\tau), j(\tau)^{-1}, (j(\tau) - 1728)^{-1}, f_{[1/N]}(\tau)].$$

**Proof.** Since $f_{[1/N]}(\tau)$ is weakly holomorphic, we get the inclusion $O_N^I(Q) \supseteq O_N^I(Q)[f_{[1/N]}(\tau)]$.

For the converse inclusion, let $h = h(\tau) \in O_N^I(Q)$. Since $F_N^I(Q)$ is generated by $f = f_{[1/N]}(\tau)$ over $F_1 = F_1^I(Q)$ by Proposition 6.1, we can write

$$h = c_0 + c_1f + \cdots + c_{d-1}f^{d-1},$$

where $d = [F_N^I(Q) : F_1^I(Q)]$ and $c_0, c_1, \ldots, c_{d-1} \in F_1^I(Q)$. If $f_1, f_2, \ldots, f_d$ are all the Galois conjugates of $f$ over $F_1^I(Q)$ and $D = \prod_{1 \leq m, n \leq d} (f_m - f_n)^2$, then one can show that

$$c_0, c_1, \ldots, c_{d-1} \in D^{-1}O_N^I(Q)$$

as in the proof of Theorem 6.2. By Lemma 3.3, we see that each $(f_m - f_n)^6$ is of the form

$$(f_m - f_n)^6 = 2^{12}3^6j(\tau)^2(j(\tau) - 1728)^3g(u+v(\tau))^6g(u-v(\tau))^6$$

for some $u, v \in V_N$ such that $u \equiv \pm v \pmod{\mathbb{Z}^2}$. It then follows from Lemma 7.3 that

$$D = c_0j(\tau)^{d(d-1)/3}(j(\tau) - 1728)^{d(d-1)/2}$$

for some nonzero $c \in \mathbb{C}$.

Now that $D \in F_N^I(Q) = Q(j(\tau))$, we must have $d(d - 1)/3 \in \mathbb{Z}$ and $c \in \mathbb{Q}$. Hence we achieve by Theorem 7.2, (13) and (14) that $h(\tau) \in O_N^I(Q)[f_{[1/N]}(\tau)]$. Therefore, the inclusion $O_N^I(Q) \subseteq O_N^I(Q)[f_{[1/N]}(\tau)]$ also holds. $\square$
Remark 7.5. For $N \geq 2$, let $F_{N}^{'}$ be the set of weak Fricke families of level $N$, namely, the families $\{h_{v}(\tau)\}_{v \in \mathcal{V}_{N}}$ of functions in $\mathcal{F}_{N}$ satisfying (F1$'$), (F2) and (F3). It is also a ring under the operations stated in (5). In a similar way to the proof of Theorem 4.3, one can claim that $F_{N}^{'}$ is isomorphic to $O_{N}^{'}(\mathbb{Q})$. Therefore, we deduce by Theorem 7.4 that a family $\{h_{v}(\tau)\}_{v \in \mathcal{V}_{N}}$ of functions in $\mathcal{F}_{N}$ is a weak Fricke family of level $N$ if and only if there is a polynomial $P(x, y, z, w) \in \mathbb{Q}[x, y, z, w]$ so that $$h_{v}(\tau) = P(j(\tau), j(\tau)^{-1}, (j(\tau) - 1728)^{-1}, f_{v}(\tau)) \quad \text{for all} \quad v \in \mathcal{V}_{N}.$$ 

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