The Topology of the Normalization of Complex Surface Germs

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Abstract

Let \((X, p)\) be a reduced complex surface germ and let \(L_X\) be its well defined link. If \((X, p)\) is normal at \(p\), D. Mumford [7] shows that \((X, p)\) is smooth if and only if \(L_X\) is simply connected. Moreover, if \(p\) is an isolated singular point, \(L_X\) is a three dimensional Waldhausen graph manifold. Then, the Plumbing Calculus of W. Neumann [8] shows that the homeomorphism class of \(L_X\) determines a unique plumbing in normal form and consequently, determines the topology of the good minimal resolution of \((X, p)\).

Here, we do not assume that \(X\) is normal at \(p\), and so, the singular locus \((\Sigma, p)\) of \((X, p)\) can be one dimensional. We describe the topology of the singular link \(L_X\) and we show that the homeomorphism class of \(L_X\) (Theorem 3.0.3) determines the homeomorphism class of the normalization and consequently the plumbing of the minimal good resolution of \((X, p)\). Moreover, in Proposition 4.0.1, we obtain the following generalization of the D. Mumford theorem [7]:

Let \(\nu : (X', p') \to (X, p)\) be the normalization of an irreducible germ of complex surface \((X, p)\). If the link \(L_X\) of \((X, p)\) is simply connected then \(\nu\) is a homeomorphism and \((X', p')\) is a smooth germ of surface.

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1 Introduction

Let \(I\) be a reduced ideal in \(\mathbb{C}\{z_1, \ldots, z_n\}\) such that the quotient algebra \(A_X = \mathbb{C}\{z_1, \ldots, z_n\}/I\) is two-dimensional. The zero locus, at the origin 0 of \(\mathbb{C}^n\), of a set of generators of \(I\) is an analytic surface germ embedded in \((\mathbb{C}^n, 0)\). Let \((X, 0)\) be its intersection with the compact ball \(B^2_\epsilon\) of radius a sufficiently small \(\epsilon\), centered at the origin in \(\mathbb{C}^n\), and let \(L_X\) be its intersection with the boundary \(S^2_{\epsilon n-1}\) of \(B^2_\epsilon\). Let \(\Sigma\) be the set of the singular points of \((X, 0)\).
As \( I \) is reduced, \( \Sigma \) is empty when \((X, 0)\) is smooth, \( \Sigma \) is equal to the origin when 0 is an isolated singular point and \( \Sigma \) is a curve when the germ has a non-isolated singular locus (in particular \((X, 0)\) can be a reducible germ).

If \( \Sigma \) is a curve, \( K_\Sigma = \Sigma \cap S_\epsilon^{2n-1} \) is the disjoint union of \( r \) one-dimensional circles (\( r \) being the number of irreducible components of \( \Sigma \)) embedded in \( L_X \). We say that \( K_\Sigma \) is the link of \( \Sigma \). By the conic structure theorem of J. Milnor [6], for a sufficiently small \( \epsilon \), \((X, \Sigma, 0)\) is homeomorphic to the cone on the pair \((L_X, K_\Sigma)\). When \( \Sigma = \{0\} \), \((X, 0)\) is homeomorphic to the cone on \( L_X \).

On the other hand, thanks to A. Durfee [3], the homeomorphism class of \((X, 0)\) depends only on the isomorphism class of the algebra \( A_X \) (i.e. is independent of a sufficiently small \( \epsilon \) and of the choice of the embedding in \((\mathbb{C}^n, 0)\)). The analytic type of \((X, 0)\) is given by the isomorphism class of \( A_X \), and its topological type is given by the homeomorphism class of \((X, 0)\).

**Definition 1.0.1** The link of \((X, 0)\) is the homeomorphism class of \( L_X \). The link of \((\Sigma, 0)\) is the homeomorphism class of the pair \((L_X, K_\Sigma)\).

Let \( \nu : (X', p') \to (X, p) \) be the normalization of a reduced germ of complex surface \((X, p)\).

**Remark 1.0.2**

1. If \((X, 0)\) is reducible, let \((\cup_{1 \leq i \leq r} X_i, 0)\) be its decomposition as a union of irreducible surface germs. Let \( \nu_i : (X'_i, p_i) \to (X_i, 0) \) be the normalization of the irreducible components of \((X, 0)\).
   The morphisms \( \nu_i \) induce the normalization morphism on the disjoint union \( \coprod_{1 \leq i \leq r} (X'_i, p_i) \).

2. If 0 is an isolated singular point of \((X, 0)\) then \( \nu \) is a homeomorphism.

**Definition 1.0.3** If \((\Sigma, 0)\) is a one-dimensional germ, let \( \sigma \) be an irreducible component of \( \Sigma \). Let \( \sigma'_j, 1 \leq j \leq n(\sigma) \), be the \( n(\sigma) \) irreducible components of \( \nu^{-1}(\sigma) \) and let \( d_j \) be the degree of \( \nu \) restricted to \( \sigma'_j \). The following number \( k(\sigma) \):

\[
k(\sigma) = d_1 + \ldots + d_j + \ldots + d_n(\sigma).
\]

is the total degree of \( \nu \) above \( \sigma \).

Let \( \Sigma_+ \) be the union of the irreducible components \( \sigma \) of \( \Sigma \) such that \( k(\sigma) > 1 \). In \( L_X \), let \( K_{\Sigma_+} \) be the link of \( \Sigma_+ \). We choose a compact regular neighbourhood \( N(K_{\Sigma_+}) \) of \( K_{\Sigma_+} \). Let \( E(K_{\Sigma_+}) \) be the closure of \( L_X \setminus N(K_{\Sigma_+}) \). By definition \( E(K_{\Sigma_+}) \) is the (compact) exterior of \( K_{\Sigma_+} \).

In Section 3 of [5], one can find a description of the topology of \( N(K_{\Sigma_+}) \) which implies the following lemma [3.0.1]. To be be self-contained, we begin Section 3 with a quick proof of it.

**Lemma (3.0.1)**

1. The restriction of \( \nu \) to \( \nu^{-1}(E(K_{\Sigma_+})) \) is an homeomorphism and \( (L_X \setminus K_{\Sigma_+}) \) is a topological manifold.
2. The link \( K_{\Sigma_+} \) is the set of the topologically singular points of \( L_X \).
3. The homeomorphism class of \( L_X \) determines the homeomorphism class of \( N(K_{\Sigma_+}) \) and \( E(K_{\Sigma_+}) \).
4. The number of connected components of \( E(K_{\Sigma_+}) \) is equal to the number of irreducible components of \((X, 0)\).

In Section 3, we prove the following theorem:

**Theorem (3.0.3)** Let \((X, 0)\) be a reduced surface germ. The homeomorphism class of \( L_X \) determines the homeomorphism class of the link \( L_X \), of the normalization of \((X, 0)\).
The proof of Theorem 3.0.3 gives an explicit construction to transform \( L_X \) in a Waldhausen graph manifold orientation preserving homeomorphic to \( L_{X'} \). Then, the plumbing calculus of W. Neumann [8] implies the following corollary.

**Corollary 1.0.4** The homeomorphism class of \( L_X \) determines the topology of the plumbing of the good minimal resolution of the germ \((X,0)\).

The proof of [3.0.3] is based on a detailed description of a regular neighbourhood \( N(K_{\Sigma_+}) \) of the topologically singular locus \( K_{\Sigma_+} \) of \( L_X \) and on the topology of \( \nu \) restricted to \( \nu^{-1}(L_X \setminus K_{\Sigma_+}) \).

In Section 2, we describe the topology of a *d-curling* and the topology of a *singular pinched torus* which is defined as the mapping torus of an orientation preserving homeomorphism acting on a reducible germ of curves. Curls and singular pinched tori are already studied in [5]. But, to prove the new results 3.0.3, 3.0.2 and 4.0.1 of this paper, we need to insist on particular properties, given in 2.0.3 of these two topological objects. Section 2 contains also the presentation of the following example which is a typical example of d-curling.

**Example (2.0.4)** Let \( X = \{(x, y, z) \in \mathbb{C}^3 \text{ where } z^d - xy^d = 0 \text{ and } d > 1\} \). The normalization of \((X,0)\) is smooth i.e. the morphism \( \nu : (\mathbb{C}^2,0) \to (X,0) \) defined by \((u,v) \mapsto (u^d,v,uv)\) is a normalization morphism. The singular locus of \((X,0)\) is the line \( l_x = \{(x,0,0) \in \mathbb{C}^3, x \in \mathbb{C}\} \). Let \( T = \{(u,v) \in (S \times D) \subset \mathbb{C}^2\} \). We have \( n(l_x) = 1 \) because \( \nu^{-1}(l_x) \) is the line \( \{(u,0) \in \mathbb{C}^2, u \in \mathbb{C}\} \) and \( d_1 = d \). Moreover \( N(K_{\Sigma_+}) = \nu(T) \) is a tubular neighbourhood of \( K_{\Sigma_+} \) and \( \nu \) restricted to \( T \) is the quotient called d-curling. In this example \( L_X \) is not simply connected. In fact: \( H_1(L_X,\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z} \).

In Section 3, we show that each connected component of \( N(K_{\Sigma_+}) \) is a singular pinched torus (see also [5]). As stated in [4] and as proved in [5], it implies that \( \nu \) restricted to \( \nu^{-1}(L_X \setminus K_{\Sigma_+}) \) is the composition of two kind of quotients: curls and identifications. Here, we need the more detailed description of \( N(K_{\Sigma_+}) \) given Section 2, to prove the following Lemma 3.0.2 which shows how the topology of \( L_X \) determines the invariants \( n(\sigma) \) and \( d_j, 1 \leq j \leq n(\sigma) \), of the normalization morphism. More precisely:

**Lemma (3.0.2)** Let \( \sigma \) be an irreducible component of \( \Sigma_+ \), let \( K_{\sigma} \) be the link of \( \sigma \) in \( L_X \). We choose, in \( L_X \), a compact regular neighbourhood \( N(K_{\sigma}) \) of \( K_{\sigma} \). The link \( K_{\sigma} \) is a deformation retract of \( N(K_{\sigma}) \). If \( l_\sigma \) is the homotopy class of \( K_{\sigma} \) in \( \pi_1(N(K_{\sigma})) \), then \( \pi_1(N(K_{\sigma})) = \mathbb{Z}/l_\sigma \). We have:

1. The tubular neighbourhood \( \nu^{-1}(N(K_{\sigma})) \) of \( \nu^{-1}(K_{\sigma}) \) is the disjoint union of \( n(\sigma) \) solid tori \( T_j, 1 \leq j \leq n(\sigma) \), and the boundary of \( N(K_{\sigma}) \) is the disjoint union of \( n(\sigma) \) tori.
2. Let \( c \) be the permutation of \( k(\sigma) \) elements which is the composition of \( n(\sigma) \) disjoint cycles \( c_j \) of order \( d_j \). Then \( N(K_{\sigma}) \) is homeomorphic to a singular pinched torus \( T(k(\sigma)(D),c) \) which has \( n(\sigma) \) sheets \( T_j \) where \( T_j = \nu(T_j'), 1 \leq j \leq n(\sigma) \).
3. On each connected component \( \tau_j \) of the boundary of \( N(K_{\sigma}) \), the homeomorphism class of \( N(K_{\sigma}) \) determines a unique (up to isotopy) meridian curves \( m_j \). If \( l_j \) is a parallel on \( \tau_j \) the homotopy class of \( l_j \) in \( \pi_1(N(K_{\sigma})) \) is equal to \( d_j l_\sigma \).

In [7], D. Mumford proves that a normal surface germ which has a simply connected link is a smooth germ of surface. However, there exist surface germs with one dimensional singular locus and simply connected links. Obvious examples are obtained as follows:

Let \( f(x, y) \) be an irreducible element of \( \mathbb{C}\{x, y\} \) of multiplicity \( m > 1 \) at 0.
Let \( Z = \{(x, y, z) \in \mathbb{C}^3 \text{ such that } f(x, y) = 0\} \). The singular locus of \((Z,0)\) is the line \( l_z = \{(0,0,z), z \in \mathbb{C}\} \). The normalization \( \nu : (\mathbb{C}^2,0) \to (Z,0) \) can be given by a Puiseux expansion of \( f(x,y) \). So, the link \( L_Z \) is the sphere \( S^3 \). Lê’s conjecture states that this family is the only family of singular irreducible surface germs with one dimensional singular locus and simply connected links (see [4] and [11] for partial results).
In Section 4, we prove the following proposition which is a kind of generalization of Mumford’s theorem for non-normal surface germs:

**Proposition (4.0.1)** Let \((X, 0)\) be an irreducible surface germ. If the link \(L_X\) of \((X, 0)\) is simply connected then the normalization \(\nu : (X', p') \rightarrow (X, 0)\) is a homeomorphism and \((X', p')\) is smooth at \(p'\). In particular, the normalization is the good minimal resolution of \((X, 0)\).

**Remark 1.0.5** There exist reducible surface germs with simply connected link for which the normalization is not a homeomorphism. For example let \(Y\) be an irreducible surface germ. If the link \(L_Y\) of \((Y, 0)\) is simply connected then the normalization \(\nu : (Y', p') \rightarrow (Y, 0)\) is a homeomorphism and \((Y', p')\) is smooth at \(p'\). In particular, the normalization is the good minimal resolution of \((X, 0)\).

1.1 Conventions

The boundary of a topological manifold \(W\) will be denoted by \(b(W)\).

A **disc** (resp. an **open disc**) will always be an oriented topological manifold orientation preserving homeomorphic to \(D = \{z \in \mathbb{C}, |z| \leq 1\}\) (resp. to \(\hat{D} = \{z \in \mathbb{C}, |z| < 1\}\)). A **circle** will always be an oriented topological manifold orientation preserving homeomorphic to \(S = \{z \in \mathbb{C}, |z| = 1\}\).

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2 The topology of \(d\)-curlings

In this section we study in details the topological properties of \(d\)-curlings because they are the key to make the proofs of the two original statements of this paper self-contained. In particular, we need well defined, up to isotopy, meridian curves on the boundary of curlings since the proof of Theorem 3.0.3 is based on Dehn fillings associated to these meridian curves. In this section, we suppose that \(d > 1\) to avoid the trivial case \(d = 1\).

**Definition 2.0.1** 1. A **\(d\)-curling** \(C_d\) is a topological space homeomorphic to the following quotient of a solid torus \(S \times D\):

\[
C_d = S \times D / (u, 0) \sim (u', 0) \Leftrightarrow u^d = (u')^d.
\]

Let \(q : (S \times D) \rightarrow C_d\) be the associated quotient morphism. By definition, \(l_0 = q(S \times \{0\})\) is the **core of \(C_d\)**. By definition \(C_d\) is given with the following orientations: The oriented circle \(S\) and the oriented disc \(D\) induce an orientation on the circles \(l_0, q(S \times \{z\}), z \in D\), and on the topological discs \(q([u] \times D), u \in S\).
2. A d-pinched disc, $d(D)$, is orientation preserving homeomorphic to the quotient of the disjoint union of $d$ oriented and ordered discs $D_i, 1 \leq i \leq d$ with origin $0_i$, by the relation $0_i \sim 0_j$ for all $i, 1 \leq i \leq d$, and $j, 1 \leq j \leq d$. So, all the origins $0_i, 1 \leq i \leq d$ are identified in a unique point 0. By definition 0 is the origin of $d(D)$. The class $D_i$ of each disc $D_i$ in $d(D)$ is an irreducible component of $d(D)$.

The kernel of the homomorphism $i_1 : \pi_1(S \times S) \to \pi_1(S \times D)$ induced by the inclusion $(S \times S) \subset (S \times D)$ is infinite cyclic generated by the class of the closed simple curve

$$m = (\{u\} \times S) = b(\{u\} \times D), u \in S.$$ 

Moreover, $m$ is oriented as the boundary of the oriented disc $(\{u\} \times D)$. This defines a unique generator $m^+$ of the kernel of $i_1$. So, any closed simple curve on the boundary of $(S \times D)$ which generates the kernel of $i_1$, can be oriented to be isotopic to $m$. By definition it is a meridian curve of $(S \times D)$. The $d$-curling $C_d$ is defined by the quotient $q : (S \times D) \to C_d$. But, $q$ restricted to $(S \times S)$ is the identity. Moreover $q(\{u\} \times D)$ is a topological disc in $C_d$.

So, the kernel of the homomorphism $\tilde{i}_1 : \pi_1(b(C_d)) \to \pi_1(C_d)$ induced by the inclusion $b(C_d) \subset (C_d)$ is infinite cyclic generated by the class $m^+$ of the oriented simple closed curve $q(m)$. As for the solid torus, any simple closed curve in $b(C_d)$ which generates the kernel of $\tilde{i}_1$ can be oriented to be isotopic to $q(m)$.

**Definition 2.0.2** An oriented simple closed curve $m$, on the boundary of a $d$-curling $C_d$, is a meridian curve of $C_d$ if the class of $m$, in the kernel of $\tilde{i}_1$, is equal to $m^+$.

Let $m$ be a meridian curve of $C_d$. An oriented simple closed curve $l$ on the boundary of $C_d$ is a parallel curve of $C_d$ if $m \cap l = +1$.

We gather together the topological properties of a $d$-curling $C_d$ in the following remark.

**Remark 2.0.3**

1. A consequence of the definition 2.0.3 is: A meridian curve on the torus $b(C_d)$ is unique up to isotopy and depends only on the homomorphism class of $C_d$.

A simple closed curve $\gamma$ on a torus $\tau$ is essential if $\gamma$ does not bound a disc in $\tau$. But $b(C_d)$ is a torus and, by definition, a meridian curve $m$ of $C_d$ is a generator of $\pi_1(b(C_d))$. So, $m$ is essential on $b(C_d)$.

Let $(u, z) \in S \times S$, we have seen that $m = q(\{u\} \times D)$ is a meridian curve of $C_d$. If $z \in S = b(D)$, let us consider $l_z = q(S \times \{z\})$. So, we have $m \cap l_z = +1$ and $l_z$ is a parallel curve of $C_d$. But, contrary to meridians, parallels are not unique up to isotopy.

2. A $d$-curling $C_d$ can be retracted by deformation onto its core $l_0$. Let $l^+$ be the class of $l_0$ in $\pi_1(C_d)$. If $z \in (D \setminus \{0\})$, the homotopy class of $q(S \times \{z\})$ is equal to $d.l^+$. Moreover, whatever the parallel curve we choose, its class in $\pi_1(C_d)$ is always equal to $d.l^+$.

3. A germ of complex curve $(\Gamma, p)$ with $d$ irreducible components is a $d$-pinched disc and $p$ is its origin.

4. Let $q : (S \times D) \to C_d$ be a $d$-curling. Let $\pi_d : (S \times D) \to (S \times D)$ be the covering of degree $d$ defined, for all $(u, z) \in (S \times D)$, by $\pi_d(u, z) = (u^d, z)$. But, $\pi_d$ induces a unique topological morphism $\tilde{\pi}_d : C_d \to (S \times D)$ such that $\pi_d = \tilde{\pi}_d \circ q$. By construction, for all $t \in S$, $D_t = \pi^{-1}_d(t \times D)$ is a $d$-pinched disc. If $u^t = t$, the origin of $D_t$ is $q(u, 0)$.

5. The circles $(S \times \{z\})$, $z \in D$, equip the solid torus $T = (S \times D)$ with a trivial fibration in oriented circles. If we choose $t \in S$, the first return map along these circles induces the identity on the disc $(\{t\} \times D)$. Using $\pi^{-1}_d$, we can lift these fibration by circles on $C_d$. Let $h$ be the automorphism of $D_t$ defined by the first return map along these circles. So, $h$ is an orientation preserving homeomorphism of $D_t$ which induces a cyclic permutation of the $d$ irreducible components of $D_t$. Obviously $h$ keeps the origin of $D_t$ fixed. So, the $d$-curling $C_d$ is the mapping torus of an orientation preserving homeomorphism which induces a cyclic permutations of the $d$ irreducible components of $D_t$.
6. As \( d > 1 \), a homeomorphism of a \( d \)-pinched disc keeps always the origin fixed. There is, up to isotopy, a unique orientation preserving homeomorphism which induces a cyclic permutation of the \( d \) discs irreducible components of a \( d \)-pinched disc. So, the mapping torus of an orientation preserving homeomorphism which induces a cyclic permutation of the \( d \) irreducible components of a \( d \)-pinched disc, is always orientation preserving homeomorphic to a \( d \)-curling.

The following example illustrates Definitions 2.0.1 and Remarks 2.0.3.

**Example 2.0.4** Let \( X = \{(x, y, z) \in \mathbb{C}^3 \mid z^d - xy^d = 0\} \). The normalization of \((X, 0)\) is smooth i.e. \(\nu : (\mathbb{C}^2, 0) \to (X, 0)\) is given by \((u, v) \mapsto (u^d, v, uv)\). Here \(B = \nu(D \times D)\) is a good semi-analytic neighbourhood of \((X, 0)\) in the sense of A. Durfee [M]. So, the link of \((X, 0)\) can be defined as \(L_X = X \cap \nu((S \times D) \cup (D \times S))\). Let \( T = \{(u, v) \in (S \times D) \subseteq \mathbb{C}^2\} \). Let \(\pi_x : \nu(T) \to S\) be the projection \((x, y, z) \mapsto x\) restricted to \(\nu(T)\). Here the singular locus of \((X, 0)\) is the line \(l_x = \{(x, 0, 0) \in \mathbb{C}^3, x \in \mathbb{C}\}\) and \(N(K_{l_x}) = L_X \cap (\pi_x^{-1}(S)) = \nu(T)\) is a tubular neighbourhood of \(K_{l_x}\).

Let \( q : T \to C_d \) be the quotient morphism which defined a \(d\)-curling (see 2.0.7). There exists a well defined homeomorphism \( f : C_d \to N(K_{l_x}) \) which satisfies \( f(q(u, v)) = (u^d, v, uv)\). So, \(N(K_{l_x})\) is a \(d\)-curling and \(K_{l_x}\) is its core. Moreover, \( f \) restricted to the core \( l_0 \) of \(C_d\) is a homeomorphism onto \(K_{l_x}\).

Let us take \( s = e^{2\pi i/d} \). The intersection \( D_1 = N(K_{l_x}) \cap \{x = 1\} = \{(1, y, z) \in \mathbb{C}^3 \mid z^d - y^d = 0\} \) is a plane curve germ at \((1, 0, 0)\) with \( d \) irreducible components given by \(\nu(s^k \times D)\), \( 1 \leq k \leq d \).

On the torus \(\tau = b(N(K_{l_x})) = \nu(S \times S)\), \( m = \nu(\{1\} \times S) \) is a meridian curve of \(N(K_{l_x})\) and \( l_1 = \nu(S \times \{1\}) \) is a parallel. Moreover, \(N(K_{l_x})\) is saturated by the foliation in oriented circles \( l_v = \nu(S \times \{v\}) \) which cuts \( D_1 \) transversally at the \( d \) points \( \{(1, v, s^k v), 1 \leq k \leq d\} \), when \( v \neq 0 \) and at \((1, 0, 0)\) when \( v = 0 \). So, \(N(K_{l_x})\) is the mapping torus of the homeomorphism defined on the \(d\)-pinched disc \( D_1 \) by the first return map along the circles \( l_v \).

To compute \( H_1(L_X, \mathbb{Z}) \), we use the Mayer-Vietoris sequence associated to the decomposition of \(L_X\) as the union \( N(K_{l_x}) \cup \nu(D \times S) \). The homology classes \(\bar{m}\) and \(\bar{l}_1\) of the curves \( m \) and \( l_1 \) form a basis of \( H_1(b(N(K_{l_x})), \mathbb{Z}) \). But, \(\bar{m}\) is a generator of \( H_1(\nu(D \times S), \mathbb{Z}) \) and is equal to 0 in \( H_1(N(K_{l_x}), \mathbb{Z}) \). As the class \( \bar{l}_0 \) of \( K_{l_x} \) is a generator of \( H_1(N(K_{l_x}), \mathbb{Z}) \) and as \( l_1 = d \bar{l}_0 \), the Mayer-Vietoris sequence has the following shape:

\[
\cdots \to H_1(b(N(K_{l_x})), \mathbb{Z}) \xrightarrow{\Delta_0} \mathbb{Z} \bar{m} \oplus \mathbb{Z} \bar{l}_0 \xrightarrow{i_1} H_1(L_X, \mathbb{Z}) \to 0
\]

where \( \Delta_1(\bar{l}_1) = (0, d \bar{l}_0) \) and \( \Delta_1(\bar{m}) = (\bar{m}, 0) \). So, \( H_1(L_X, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}/d \mathbb{Z} \).

We generalize the notion of \(d\)-curlings as follows. Let \( k(D) \) be the \( k \)-pinched disc, where \( k > 1 \), quotient by identification of their centre of \( k \) oriented and ordered discs \( D_i, 1 \leq i \leq k \). Let \( c = c_1 \circ c_2 \circ \ldots \circ c_n \) be a permutation of the indices \(\{1, \ldots , k\}\) given as the composition of \( n \) disjoint cycles \( c_j, 1 \leq j \leq n \), where \( c_j \) is the cycle \( (1(j), \ldots , i(j), \ldots , d(j)) \) of order \( d_j, 1 \leq d_j \). Let \( \tilde{h}_c \) be an orientation preserving homeomorphism of the disjoint union of the \( k \) discs \( D_i, 1 \leq i \leq k \), such that \( \bar{h}_c(D_i) = D_{c(i)}, \bar{h}_c(0_i) = 0_{c(i)} \) and \( \tilde{h}_c^{d_j} \) restricted to \( \prod_{1 \leq i \leq d_j} D_{c(i)} \) is the identity. Let \( h_c \) be the orientation preserving homeomorphism of \( k(D) \) induces by \( \tilde{h}_c \). By construction we have \( h_c(0) = 0 \). Up to isotopy the homeomorphisms \( \bar{h}_c \) and \( h_c \) depend only on the permutation \( c \). So, the homeomorphism class of the mapping torus of \( h_c \) acting on \( k(D) \) depends only on \( c \). So, we can state the following definition.
**Definition 2.0.5** A singular pinched solid torus associated to the permutation $c$ is a topological space orientation preserving homeomorphic to the mapping torus $T(k(D), c)$ of $h_c$ acting on $k(D)$:

$$T(k(D), c) = [0, 1] \times k(D)/(1, x) \sim (0, h(x))$$

The core of $T(k(D), c)$ is the oriented circle $l_0 = [0, 1] \times \hat{0}/(1, \hat{0}) \sim (0, \hat{0})$. A homeomorphism between two singular pinched solid tori is orientation preserving if it preserves the orientation of $k(D) \setminus \{0\}$ and the orientation of the core $l_0$.

**Definition 2.0.6** Let $T(k(D), c)$ be the singular pinched torus associated to the permutation $c$. A sheet of $T(k(D), c)$ is the closure of a connected component of $(T(k(D), c) \setminus l_0)$.

**Remark 2.0.7** By construction $(T(k(D), c) \setminus l_0)$ has $n$ connected components, each of them is homeomorphic to a torus minus its core $(S \times D) \setminus (S \times \{0\})$. The closures, in $T(k(D), c)$, of these connected components, are the mapping tori of the homeomorphisms which permute cyclically the $d_j$ irreducible components of the pinched discs $d_j(D)$. So, Point 6 of Remark 2.0.3 implies that a sheet $T(d_j(D), c_j), 1 \leq j \leq n$, of $T(k(D), c)$, is a $d_j$-curling.

As $k > 1$, a homeomorphism of $T(k(D), c)$ preserves the core $l_0$ which is the set of the topologically singular points of $T(k(D), c)$. So the homeomorphism classes of the sheets $T(d_j(D), c_j), 1 \leq j \leq n$, of $T(k(D), c)$, depend only on the permutation $c$. The identifications point by point of the $n$ circles which are the $n$ cores $l_j$ of $T(d_j(D), c_j), 1 \leq j \leq n$, determine a quotient morphism

$$\delta : \left( \coprod_{1 \leq j \leq n} T(d_j(D), c_j) \right) \to T(k(D), c).$$

Conclusion: Up to homeomorphism $T(k(D), c)$ can be obtained by the composition of the $d_j$-curling morphisms $q_j : (S \times D) \to C_{d_j}, 1 \leq j \leq n$, defined on the disjoint union of $n$ solid tori, followed by the identification $\delta$ of their cores.

### 3 The topology of the normalization

Let $(X, 0)$ be a reduced surface germ, let $(\Sigma, 0)$ be its singular locus and let $\nu : (X', p') \to (X, p)$ be its normalization. As in Definition 1.0.3 if $\sigma$ is an irreducible component of $\Sigma$, let $\sigma_j, 1 \leq j \leq n(\sigma)$, be the $n(\sigma)$ irreducible components of $\nu^{-1}(\sigma)$, and let $d_j$ be the degree of $\nu$ restricted to $\sigma_j$. Moreover let $k(\sigma) = d_1 + \ldots + d_j + \ldots + d_n(\sigma)$ be the total degree of $\nu$ above $\sigma$.

Let $\Sigma_+$ be the union of the irreducible components $\sigma$ of $\Sigma$ such that $k(\sigma) > 1$. In $L_X$, let $K_{\Sigma_+}$ be the link of $\Sigma_+$. We choose a compact regular neighbourhood $N(K_{\Sigma_+})$ of $K_{\Sigma_+}$. Let $E(K_{\Sigma_+})$ be the closure of $L_X \setminus N(K_{\Sigma_+})$. By definition $E(K_{\Sigma_+})$ is the (compact) exterior of $K_{\Sigma_+}$.

**Lemma 3.0.1** 1. The restriction of $\nu$ to $\nu^{-1}(E(K_{\Sigma_+}))$ is an homeomorphism and $(L_X \setminus K_{\Sigma_+})$ is a topological manifold.

2. The link $K_{\Sigma_+}$ is the set of the topologically singular points of $L_X$.

3. The homeomorphism class of $L_X$ determines the homeomorphism class of $N(K_{\Sigma_+})$ and $E(K_{\Sigma_+})$.

4. The number of connected components of $E(K_{\Sigma_+})$ is equal to the number of irreducible components of $(X, 0)$.

**Proof:**
If $(X, 0)$ has an isolated singular point at the origin, the normalization is bijective and as the links $L_X$ and $L_{X'}$ are compact, $\nu$ restricted to $L_{X'}$ is a homeomorphism.
If $\Sigma$ is one dimensional, let $K_{\Sigma}$ be the link of $\Sigma$. In $L_X$ we choose a compact regular neighbourhood $N(K_{\Sigma})$ of $K_{\Sigma}$. Let $E(K_{\Sigma})$ be the closure of $L_X \setminus N(K_{\Sigma})$. By definition $E(K_{\Sigma})$ is the (compact) exterior of $K_{\Sigma}$. As $\nu$ restricted to $(X \setminus \Sigma)$ is bijective, $\nu$ restricted to the compact $\nu^{-1}(E(K_{\Sigma}))$ is a homeomorphism.

Let $\sigma$ be an irreducible component of $\Sigma$ and let $N(K_{\sigma})$ be the connected component of $N(K_{\Sigma})$ which contains the link $K_{\sigma}$. When $k(\sigma) = 1$, $\nu$ restricted to $\nu^{-1}(N(K_{\sigma}))$ is a bijection. So, the restriction of $\nu$ to $\nu^{-1}(E(K_{\Sigma_{\sigma}}))$ is an homeomorphism. Moreover $\nu$ restricted to $\nu^{-1}(X \setminus \Sigma)$ is an analytic isomorphism. So, $(L_X \setminus K_{\Sigma_{\sigma}})$ is a topological manifold. This ends the proof of Statement 1.

If $k(\sigma) > 1$, $\nu$ restricted to $\nu^{-1}(K_{\sigma})$ is not injective. Let $p$ be a point of $K_{\sigma}$. The number of the irreducible components $\sigma_j'$ of $\nu^{-1}(\sigma)$ is denoted $n(\sigma)$. So, $\nu^{-1}(\sigma) = \cup_{1 \leq j \leq n(\sigma)} \sigma_j'$. Let $d_j$ be the degree of $\nu$ restricted to $\sigma_j'$. The intersection $\nu^{-1}(p) \cap \sigma_j'$ has $d_j$ points $\{p_{i(j)}, 1 \leq i \leq d_j\}$. As $(X', p')$ is normal, $(X' \setminus p')$ is smooth and $(\sigma_j' \setminus p')$ is a smooth curve at any point $z_j \in (\sigma_j' \setminus p')$. In $(X' \setminus p')$, we can choose at the points $p_{i(j)}$, a smooth germ of curve $(\gamma_{i(j)}, p_{i(j)})$ which cuts $\sigma_j'$ transversally at $p_{i(j)}$ and such that $D_{i(j)}' = \nu^{-1}(N(K_{\sigma})) \cap \gamma_{i(j)}$ is a disc centered at $p_{i(j)}$. Let $D_{i(j)}$ be $\nu(D_{i(j)}')$. By construction $p$ is the common center of the topological discs $D_{i(j)}$. So, $\cup_{1 \leq j \leq n(\sigma)} (\cup_{1 \leq i \leq d_j} D_{i(j)})$ is a $k(\sigma)$-pinched disc centered at $p$. As $k(\sigma) > 1$, $L_X$ is not a topological manifold at $p$. This ends the proof of Statement 2.

Statements 1 and 2 imply that $(K_{\Sigma_{\sigma}})$ is the set of the topologically singular points of $L_X$. It implies 3.

The number $r$ of irreducible components of $(X, 0)$ is equal to the number of connected components of $L_X$. But, $L_X$ and $\nu^{-1}(E(K_{\Sigma_{\sigma}}))$ have the same number of connected components since $(L_X \setminus \nu^{-1}(E(K_{\Sigma_{\sigma}})))$ is a regular neighbourhood of the differential link $\nu^{-1}(K_{\Sigma_{\sigma}})$. Statement 1 implies that $r$ is also the number of connected components of $E(K_{\Sigma_{\sigma}})$. This proves 4.

End of proof.

**Lemma 3.0.2** Let $\sigma$ be an irreducible component of $\Sigma_{\sigma}$, let $K_{\sigma}$ be the link of $\sigma$ in $L_X$. We choose, in $L_X$, a compact regular neighbourhood $N(K_{\sigma})$ of $K_{\sigma}$. The link $K_{\sigma}$ is a deformation retract of $N(K_{\sigma})$. If $l_{\sigma}$ is the homotopy class of $K_{\sigma}$ in $\pi_1(N(K_{\sigma}))$, then $\pi_1(N(K_{\sigma}) = \mathbb{Z}, l_{\sigma}$. We have:

1. The tubular neighbourhood $\nu^{-1}(N(K_{\sigma}))$ of $\nu^{-1}(K_{\sigma})$ is the disjoint union of $n(\sigma)$ solid tori $T_j', 1 \leq j \leq n(\sigma)$, and the boundary of $N(K_{\sigma})$ is the disjoint union of $n(\sigma)$ tori.

2. Let $c$ be the permutation of $k(\sigma)$ elements which is the composition of $n(\sigma)$ disjoint cycles $c_j$ of order $d_j$. Then $N(K_{\sigma})$ is homeomorphic to a singular pinched torus $T(k(\sigma)(D), c)$ which has $n(\sigma)$ sheets $T_j$ where $T_j = \nu(T_j')$, $1 \leq j \leq n(\sigma)$.

3. On each connected component $\tau_j$ of the boundary of $N(K_{\sigma})$, the homeomorphism class of $N(K_{\sigma})$ determines a unique (up to isotopy) meridian curves $m_j$. If $l_j$ is a parallel on $\tau_j$ the homotopy class of $l_j$ in $\pi_1(N(K_{\sigma}))$ is equal to $d_j, l_{\sigma}$.

**Proof of Lemma 3.0.2.**
The link $L_X$ of the normalization $\nu : (X', p') \to (X, p)$ of $(X, 0)$ is a three dimensional Waldhausen graph manifold. Let $\sigma$ is an irreducible component of $\Sigma_{\sigma}$. In $L_X$, $\nu^{-1}(K_{\sigma})$ is a differentiable one dimensional link with $n(\sigma)$ irreducible components $K_{\sigma_j'}, 1 \leq j \leq n(\sigma)$. But, $\nu^{-1}(N(K_{\sigma}))$ is a regular compact neighbourhood of the link $(\prod_{1 \leq j \leq n(\sigma)} K_{\sigma_j'})$. So, th $\nu^{-1}(N(K_{\sigma}))$ is the disjoint union of $n(\sigma)$ solid tori that we denote by $T_j, 1 \leq j \leq n(\sigma)$. Moreover let $\tau_j'$ be the boundary of $T_j'$. As $\nu$ restricted to the boundary of $\nu^{-1}(N(K_{\sigma}))$ is a homeomorphism, the boundary of $N(K_{\sigma})$ is the disjoint union of the $n(\sigma)$ tori $\tau_j = \nu(\tau_j')$. Statement 1 is proved.
As in the proof of Lemma 3.0.1, we consider a point $p \in K_\sigma$ and $d_j$ meridian discs $D'_{(j)}$ of $T'_j$ centered at the $d_j$ points of $\nu^{-1}(p) \cap \sigma_j'$. We equip $T'_j$ with a trivial fibration in oriented circles which has $K_\sigma'$ as central fiber. The first return homeomorphism $h'_j : (\bigcup_i D'_{(j)}) \to (\bigcup_i D'_{(j)})$, along the chosen circles permutes cyclically the $d_j$ discs $D'_{(j)}$ and $(h'_j)^{d_j}$ is the identity. So, $h_j$ provides a $d_j$-cycle $c_j$. 

The direct image by $\nu$, of the fibration of $T'_j$ in circles, equip $T_j = \nu(T'_j)$ with a foliation in oriented circles which has $K_\sigma$ as singular leave. But $\nu(\bigcup_i D'_{(j)})$ is a $d_j$-pinched disc with origin $p \in K_\sigma$ and irreducible components $D_{(j)} = \nu(D'_{(j)})$. Let $h_j$ be the homeomorphism defined on the $d_j$-pinched disc $(\bigcup_i D_{(j)})$ by the first return along the given circles. By construction, $h_j(p) = p$ and $h_j$ permutes cyclically the $d_j$ irreducible components of the $d_j$-pinched disc $(\bigcup_i D_{(j)})$. So, $T_j$ is the mapping torus of $h_j$ acting on the $d_j$-pinched disc $(\bigcup_i D_{(j)})$. By Point 6 of Remark 2.0.3, $T_j$ is a $d_j$-curling.

But $h_j(p) = p$ for all $j$, $1 \leq j \leq n(\sigma)$. So, if we consider $h(z) = h_j(z)$ for all $z \in (\bigcup_i D_{(j)})$, $h$ is a well defined homeomorphism $$h : (\bigcup_{1 \leq j \leq n(\sigma)}(\bigcup_i D_{(j)})) \to (\bigcup_{1 \leq j \leq n(\sigma)}(\bigcup_i D_{(j)})).$$

By construction, $h$ induces on the $k(\sigma)$-pinched disc $(\bigcup_{1 \leq j \leq n(\sigma)}(\bigcup_i D_{(j)}))$ a permutation $c$, of its irreducible components, which is the composition of the disjoint cycles $c_j$. Then, $N(K_\sigma)$ is homeomorphic to the singular pinched torus $T(k(\sigma)(D), c)$. But $T_j$, $1 \leq j \leq n(\sigma)$, is the closure of a connected components of $(N(K_\sigma) \setminus K_\sigma)$. By definition the family of the $d_j$-curlings $\{C_{d_j} = T_j, 1 \leq j \leq n(\sigma)\}$, is the family of the sheets of $N(K_\sigma)$.

A pinched disc can be retracted by deformation onto its center. Such a retraction by deformation can be extended in a retraction by deformation of $T(k(\sigma)(D), c)$ onto its core. By 2, $N(K_\sigma)$ is homeomorphic to a singular pinched torus $T(k(\sigma)(D), c)$ and $K_\sigma$ is its core. Then, we can retract $N(K_\sigma)$ by deformation onto its core $K_\sigma$. So, $\pi_1(N(K_\sigma)) = \mathbb{Z} \cdot l_\sigma$ where $l_\sigma$ is the homotopy class of $K_\sigma$ in $\pi_1(N(K_\sigma))$. Moreover each connected component $\tau_j$ of the boundary of $N(K_\sigma)$ is the boundary of the $d_j$-curling $T_j$ which is a sheet of $N(K_\sigma)$. On $\tau_j$ we choose a meridian curve $m_j$ and a parallel curve $l_j$ as defined in 2.0.4. As $T_j$ is a $d_j$-curling, $m_j$ is unique up to isotopy by 1 of Remark 2.0.3. By 2 of Remark 2.0.3 the class of $l_j$ in $\pi_1(N(K_\sigma)$ is equal to $d_j \cdot l_\sigma$.

End of proof of Lemma 3.0.2.

Now we are ready to prove the main theorem of this paper.

**Theorem 3.0.3** Let $(X, 0)$ be a reduced surface germ. The homeomorphism class of $L_X$ determines the homeomorphism class of the link $L_{X'}$ of the normalization of $(X, 0)$.

**Proof:**

If $L_X$ is a topological manifold, Statements 1 and 2 of Lemma 3.0.1 state that $K_{\Sigma_+}$ is empty. So, $E(K_{\Sigma_+}) = L_X$ and $\nu$ is a homeomorphism. When $L_X$ is a topological manifold the theorem is trivial.

If $L_X$ is not a topological manifold, let $\mathcal{K}$ be the set of its singular points. By Statement 2 of Lemma 3.0.1 we know that $\mathcal{K} = K_{\Sigma_+}$ is a disjoint union of circles. Let $N(\mathcal{K})$ be a regular compact neighbourhood of $\mathcal{K}$. By definition, the exterior $E(\mathcal{K})$ of $\mathcal{K}$ is the closure of $(L_X \setminus N(\mathcal{K}))$.

Let $K$ be a connected component of $\mathcal{K}$ and let $N(K)$ be the connected component of $N(\mathcal{K})$ which contains $K$. There exist an irreducible component $\sigma$ of $\Sigma_+$ such that $K = K_\sigma$. By Lemma 3.0.2 $N(K)$ is homeomorphic to a singular pinched torus $T(k(\sigma)(D), c)$ which has $n(\sigma)$ sheets homeomorphic to $d_j$-curlings $C_{d_j}$, $1 \leq j \leq n(\sigma)$.

As explained Section 2, the integers $d_j$ and $n(\sigma)$, the permutation $c$ and the homeomorphism class of the family of the $d_j$-curlings $C_{d_j}$, sheets of $N(K)$, depend only on the homeomorphic class of $N(K)$. In particular the boundary $b(N(K))$ has $n(\sigma)$ tori $\tau_j$, $1 \leq j \leq n(\sigma)$, as connected components.
As defined in \[2.0.2\] and shown in Statement 1 of Remark \[2.0.3\] on each torus \(\tau_j\) we have a well defined meridian curve \(m_j\) associated to the corresponding sheet of \(N(K)\). It is the key point of this proof. The existence of well defined meridian curves of \(N(K)\) on each torus of the boundary of the exterior \(E(K)\) of \(K\) allows us to perform Dehn fillings. As justify below, to take \(E(K)\) and to close it by performing Dehn fillings associated to the given meridian curves of \(N(K)\), produce a closed manifold homeomorphic to \(L_X\). Let us be more precise.

**The Dehn filling construction:**

Let \(T\) be a solid torus given with a meridian disc \(D\) and let \(m_T\) be the boundary of \(D\). By definition \(m_T\) is a meridian curve on the boundary of \(T\). Let \(U(D)\) be a compact regular neighbourhood of \(D\) in \(T\) and let \(B\) be the closure of \(T \setminus U(D)\). By construction \(B\) is a 3-dimensional ball. In the boundary \(b(T)\) of \(T\), the closure of the complement of the annulus \(U(m_T) = U(D) \cap b(T)\) is also an annulus \(E(m_T) \subset b(B)\).

On the other hand, we suppose that a torus \(\tau\) is a boundary component of an oriented compact three-dimensional manifold \(M\). Let \(\gamma\) be an oriented essential simple closed curve on \(\tau\). So, \(\tau\) is the union of two annuli, \(U(\gamma)\), a compact regular neighbourhood of \(\gamma\), and the closure \(E(\gamma)\) of \(\tau \setminus U(\gamma)\). There is a unique way to glue \(T\) to \(M\) by an orientation reversing homeomorphism between the boundary of \(T\) and \(\tau\) which send \(m_T\) to \(\gamma\). Indeed, the gluing of \(U(m_T)\) onto \(U(\gamma)\) determines a union \(M'\) between \(U(D)\) and \(M\). This gluing extends to the gluing of \(E(m_T)\) onto \(E(\gamma)\) which determines a union between \(B\) and \(M'\).

So, the result of such a gluing is unique up to orientation preserving homeomorphism and it is called the Dehn filling of \(M\) associated to \(\gamma\).

One can find a presentation of the Dehn filling construction in S. Boyer [2].

The topology of the link \(L_X\) determines the exterior \(E(K)\) of the singular locus \(K\) of \(L_X\) and also the well defined meridian curves of \(N(K)\) on each connected component of the boundary of \(E(K)\). Let \(\sigma_i, 1 \leq i \leq r\) be the \(r\) irreducible components of \(\Sigma_+\). So, the boundary of \(E(K)\) has \(n = \sum_{1 \leq i \leq r} n(\sigma_i)\) connected components. Let \(T\) be a solid torus and let \(m_T\) be a meridian curve of \(T\). Let \(\tau\) be one connected component of the boundary of \(E(K)\) given with its already chosen curve \(m\) which is a meridian curve of \(N(K)\). By Remark \[2.0.3\], \(m\) is an essential simple closed curve on \(\tau\). We glue \(T\) to \(E(K)\) with the help of an orientation reversing homeomorphism

\[ f : b(T) \to \tau \]

defined on the boundary \(b(T)\) of \(T\) such that \(f(m_T) = m\).

We perform such a Dehn filling associated to \(m\) on each of the \(n\) connected components of the boundary of \(E(K)\). So, we obtain a closed 3-dimensional Waldhausen graph manifold \(L\).

But, \(\nu\) restricted to \(E' = \nu^{-1}(E(K))\) is a homeomorphism. Moreover, \(\nu^{-1}(N(K))\) is a tubular neighbourhood of the differential link \(\nu^{-1}(K)\) which has \(n = \sum_{1 \leq i \leq r} n(\sigma_i)\) connected components. So, \(\nu^{-1}(N(K))\) is a disjoint union of \(n\) solid tori. As in the proof of Lemma \[3.0.2\], let \(T_j'\) be one of these solid tori. Then, \(T_j = \nu(T_j')\) is a sheet of \(N(K_r)\) where \(\sigma\) is an irreducible component of \(\Sigma_+\). But \(T_j\) is a \(d_j\)-curling and \(\nu\) restricted to \(T_j'\) is a quotient morphism associated to this \(d_j\)-curling. Let \(m_j\) be the chosen meridian on \(T_j\). Definition \[2.0.1\] implies that \(\nu^{-1}(m_j)\) is a meridian curve of \(T_j'\). By the unicity of the Dehn filling construction, there exists an orientation preserving homeomorphism between \(L_{X'}\) and \(L\).

Conclusion: The construction of \(L\) only depends on the topology of \(L_X\) and \(L\) is homeomorphic to the link \(L_{X'}\) of the normalization \((X', p')\) of \((X, 0)\). So, Theorem \[3.0.3\] is proved.

*End of proof.*
4 Surface germs with simply connected links

This section is devoted to the proof of the following proposition.

**Proposition 4.0.1** Let \((X, 0)\) be an irreducible surface germ. If the link \(L_X\) of \((X, 0)\) is simply connected then the normalization \(\nu : (X', p') \to (X, p)\) is a homeomorphism and \((X', p')\) is smooth at \(p'\). In particular, the normalization is the good minimal resolution of \((X, 0)\).

**Proof:**
By Lemma 3.0.1 (or Proposition 3.12 in [4]), if \(L_X\) is a topological manifold the normalization \(\nu : (X', p') \to (X, p)\) is a homeomorphism. Then the link \(L_X\) is also simply connected and by Mumford’s theorem [7] \((X', p')\) is smooth at \(p'\).

Now, we suppose that \(L_X\) is not a topological manifold. Then, the following two statements I and II prove that \(L_X\) is not simply connected.

As before, \(\Sigma_+\) is the union of the irreducible components \(\sigma\) of the singular locus of \((X, 0)\), which have a total degree, \(k(\sigma) = d_1 + \ldots + d_1 + \ldots + d_{n(\sigma)}\), strictly greater than one. By Lemma 3.0.1 if \(L_X\) is not a topological manifold, \(\Sigma_+\) has at least one irreducible component \(\sigma\).

**Statement I** If there exists an irreducible component \(\sigma\) of \(\Sigma_+\) with \(n(\sigma) > 1\), then the rank \(r\) of \(H_1(L_X, \mathbb{Z})\) is greater than or equal to \((n(\sigma) - 1)\), in particular \(H_1(L_X, \mathbb{Z})\) has infinite order.

**Proof of Statement I.** Let \(\Sigma_+ = (\sigma_1 \cup \ldots \cup \sigma_{n(\sigma)})\), be the decomposition of \(\Sigma_+\) as the union of its irreducible components. As \((X, 0)\) is irreducible, \(E(K_{\Sigma_+})\) is connected by Lemma 3.0.1. Then, \(E(K_\sigma) = E(K_{\Sigma_+}) \cup \ldots \cup N(K_\sigma)\) and \(L_X\) are also connected. But \(N(K_\sigma)\) which is a singular pinched torus (Lemma 3.0.2) is connected with \(n(\sigma) > 1\) boundary components. We consider the Mayer-Vietoris exact sequence associated to the decomposition of \(L_X\) as the union \(E(K_\sigma) \cup N(K_\sigma)\).

\[\ldots \to H_1(L_X, \mathbb{Z}) \xrightarrow{d_1} H_0(E(K_{\Sigma_+}) \cap N(K_\sigma), \mathbb{Z}) \xrightarrow{\Delta_0} H_0(E(K_{\Sigma_+}), \mathbb{Z}) \oplus H_0(N(K_\sigma), \mathbb{Z}) \xrightarrow{i_0} H_0(L_X, \mathbb{Z}) \to 0\]

But \(E(K_{\Sigma_+}) \cap N(K_\sigma)\) is the disjoint union of \(n(\sigma)\) disjoint tori. So, the rank of \(H_0(E(K_\sigma) \cap N(K_\sigma), \mathbb{Z})\) is equal to \(n(\sigma)\). Since \(\sigma\) is irreducible \(H_0(N(K_\sigma), \mathbb{Z})\) has rank one. Since \((X, 0)\) is irreducible, \(H_0(E(K_{\Sigma_+}), \mathbb{Z})\) and \(H_0(L_X, \mathbb{Z})\) have rank one. So, the rank of \(\text{Ker}(\Delta_0) = d_1(H_1(L_X, \mathbb{Z}))\) is equal to \((n(\sigma) - 1)\). This ends the proof of Statement I.

**Statement II** If there exists an irreducible component \(\sigma\) of \(\Sigma_+\) with \(n(\sigma) = 1\) and \(k(\sigma) = d > 1\) the order of \(H_1(L_X, \mathbb{Z})\) is at least \(d\).

**Proof of Statement II.** By Lemma 3.0.2 if \(n(\sigma) = 1\) and \(d > 1\), \(N(K_\sigma)\) is a \(d\)-curling and the boundary of \(N(K_\sigma)\) is a torus \(\tau\). By Remark 2.0.3 \(\tau\) is given with a meridian curve \(m\) and a parallel curve \(l\). Let \(\tilde{m}\), \(\tilde{l}\), and \(l_\sigma\) be the classes of \(m\), \(l\) and \(K_\sigma\) in \(H_1(N(K_\sigma), \mathbb{Z})\). Moreover, we have \(H_1(N(K_\sigma), \mathbb{Z}) = \mathbb{Z}, l_\sigma\) and \(\tilde{l} = d_{\delta_1}\). We consider the Mayer-Vietoris exact sequence associated to the decomposition of \(L_X\) as the union \(E(K_\sigma) \cup N(K_\sigma)\).

\[\ldots \to H_2(L_X, \mathbb{Z}) \xrightarrow{d_2} H_1(E(K_{\Sigma_+}) \cap N(K_\sigma), \mathbb{Z}) \xrightarrow{\Delta_1} H_1(E(K_{\Sigma_+}), \mathbb{Z}) \oplus H_1(N(K_\sigma), \mathbb{Z}) \xrightarrow{i_1} H_1(L_X, \mathbb{Z}) \to \ldots\]

As \(E(K_{\Sigma_+}) \cap N(K_\sigma) = \tau\), the image of \(\Delta_1\) is generated by \(\Delta_1(\tilde{m}) = (x, 0)\) and \(\Delta_1(\tilde{l}) = (y, d_{\delta_1})\) where \(x\) and \(y\) are in \(H_1(E(K_{\Sigma_+}), \mathbb{Z})\). So, the image of \(\Delta_1\) is included in \(H_1(E(K_{\Sigma_+}), \mathbb{Z}) \oplus \mathbb{Z}, d_{\delta_1}\). It implies that the order of the cokernel of \(\Delta_1\) is at least \(d\). This ends the proof of Statement II.

The two statements above imply Proposition 4.0.1.

**End of proof.**
References

[1] J. Fernandez de Bobadilla: *A reformulation of Le’s conjecture*, Indag. Math., N.S., 17 (2006), p. 345-352.

[2] S. Boyer: *Dehn Surgery on knots*, Knot Theory and its applications. 9 (1998), p. 657-670.

[3] A. Durfee: *Neighborhoods of algebraic sets*, Trans. Amer. Math. Soc. 276 (1983), p. 517–530.

[4] I. Luengo and A. Pichon: *Lê’s conjecture for cyclic covers*, Séminaires et congrès 10 (2005), p. 163-190. Publications de la SMF, Ed. J.-P. Brasselet and T. Suwa.

[5] F. Michel: *The Topology of Surface Singularities*, ArXiv Mathematics 1910.14600v1, 31 (Oct 2019.) To appear as Chapter 2 in *Handbook of Singularities*, Springer, Ed. J.L. Cisneros-Molina, D.T. Lê and J. Seade.

[6] J. Milnor: *Singular Points of Complex Hypersurfaces*, Annals of Mathematical Studies 61 (1968), Princeton Univ. Press.

[7] D. Mumford: *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Etudes Sci. Publ. Math. 9 (1961), p. 5–22.

[8] W. Neumann: *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. 268 (1981), p. 299–344.