Existence and symmetries of solutions on Besov-Morrey spaces for a nonlinear parabolic-hyperbolic Volterra equation

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Abstract

This paper concerns with an interpolated nonlinear Parabolic-Hyperbolic PDE arising of an integro-differential equations. We showed global existence in Besov-Morrey spaces $\mathcal{N}_{p,\mu,\infty}^\sigma(\mathbb{R}^n)$ $(n \geq 1)$ and some qualitative aspects, like symmetries and positivity of solutions. It seems that our initial data $u_0$ (see 1.10) is larger than the previous works. Also, asymptotic behavior of solutions is proved in the framework of scaling invariant Besov-Morrey spaces and self-similarity of solutions is investigated.

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1. Introduction and Theorems

This paper concerns with a semilinear time-fractional partial differential equation, (FPDE for short) which describe diverse physical phenomena and mathematical models (see e.g. [13, 14, 27]). More precisely, in this paper we consider the semilinear integro-partial differential equation in $\mathbb{R}^n$, which reads as

\[
\begin{cases}
    u_t = \int_0^t r_\alpha(t-s)[P(D)u(s) + f(u(s))]ds, & (x \in \mathbb{R}^n \text{ and } t > 0) \\
    u(0, x) = u_0(x), & x \in \mathbb{R}^n
\end{cases}
\] (1.1)

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where \( u(t) = u(t, x) = (u_1(t, x), \ldots, u_n(t, x)) \) with \( n \geq 1 \), \( r_\alpha(t) = \nu t^{\alpha-1}/\Gamma(\alpha) \), \( \Gamma(\alpha) \) denotes the gamma function, \( P(D) = \Delta_x \) is the Laplacian operator on \( x \)-variable, \( \nu \) denotes the Newtonian viscosity and \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a function satisfying

\[
f(0) = 0 \text{ and } |f(a) - f(b)| \leq C|a - b| (|a|^\rho - 1 + |b|^\rho - 1), \tag{1.2}
\]

here \( \rho > 1 \) and \( C \) is a positive constant independent of \( a, b \in \mathbb{R} \). Typical examples of \( f(u) \) are given by \( \gamma|u|^\rho u \) and \( \gamma|u|^\rho \) for \( \gamma \in \{+, -\} \). This nonlinearities yield a scaling for (1.1) which is fundamental in our approach on Besov-Morrey spaces, modeled on Besov space, but with underlying norm is of Morrey type. This spaces have been introduced by H. Kozono and M. Yamazaki [15] for analysis of the Navier-Stokes equations and many authors have studied PDEs (see [5, 8, 23, 32, 38]) and Harmonic analysis ([31, 34]) in this framework, for survey see [29, 30] and references therein. As far as we know, the problem of existence of solutions to (1.1) in Besov-Morrey space is new as \( 1 < \alpha < 2 \). Formally the system (1.1) is equivalent to (FPDE),

\[
\partial_t^\alpha u = \nu P(D)u + f(u) \quad \text{in } (0, \infty) \times \mathbb{R}^n \tag{1.3}
\]

\[
u u_t(0) = 0 \quad \text{and } u(0) = u_0 \quad \text{in } \mathbb{R}^n, \tag{1.4}
\]

where \( \partial_t^\alpha u = D_{0|t}^{\alpha-1}u_t, \ u_t = \frac{\partial u}{\partial t} \) and \( D_{0|t}^{\alpha-1} \) stands for the Riemann-Liouville derivative of order \( \alpha - 1 \),

\[
D_{0|t}^{\alpha-1}u = \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{(t-s)^{\alpha-1}} ds, \quad (t > 0 \text{ and } 1 < \alpha < 2)
\]

for a suitable \( u(\cdot, x) \in L^1_{\text{loc}}(\mathbb{R}^n) \). Employing a Duhamel-type formula (see [10, Proposition 2.1]) in (1.3)-(1.4) (or (1.1)), formally we obtain the integral equation

\[
u u(t) = L_\alpha(t)u_0 + B_\alpha(u)(t), \tag{1.5}
\]

where

\[
B_\alpha(u)(t) = \int_0^t L_\alpha(t-s) \left( \int_0^s r_{\alpha-1}(s-\tau)f(u(\tau))d\tau \right) ds
\]

and \( \{L_\alpha(t)\}_{t \geq 0} \) stands for convolution operators (or diffusion-wave operator) given by

\[
\hat{L_\alpha(t)}\varphi(\xi) = E_\alpha(-t^\alpha|\xi|^2)\hat{\varphi}(\xi)
\]

(1.7)

for every Schwartz function \( \varphi \in \mathcal{S}(\mathbb{R}^n) \). Throughout this paper a mild solution for (1.3)-(1.4) (or (1.1)) is a function \( u(t, x) \) satisfying (1.5) for all \( x \in \mathbb{R}^n, \ t > 0 \) and \( u(t, x) \rightarrow u_0 \) in \( \mathcal{S}'(\mathbb{R}^n) \) as \( t \rightarrow 0^+ \), actually using the Proposition 3.2 and Sobolev embedding (2.13) we showed the weak convergence on homogeneous Besov space \( B^{2/(\rho-1)}_{\infty, \infty} \), see details on Lemma 3.4 and Lemma 3.5. Here \( E_\alpha(-t^\alpha|\xi|^2) \) stands for Mittag-Leffler function (see (2.17)) and \( \hat{\varphi} = \mathcal{F} \) stands for Fourier transform in Schwartz’s space \( \mathcal{S}(\mathbb{R}^n) \) which can be extended to tempered distributions’ spaces \( \mathcal{S}'(\mathbb{R}^n) \). The operator \( L_\alpha(t) \) does not
satisfy the semigroup property $L_\alpha(t+s) \neq L_\alpha(t)L_\alpha(s)$ (see e.g. [26]) unless if $\alpha = 1$. In this case, the operator $L_1(t) = S(t)$ is the heat semigroup, because $E_1(-t|\xi|^2) = e^{-t|\xi|^2}$. The kernel $k_\alpha$ of $L_\alpha(t)$ is the fundamental solution of ((1.3) with $f \equiv 0$) given by

$$k_\alpha(t, x) = \int_{\mathbb{R}^n} e^{ix\xi} E_\alpha(-t|x|^2) d\xi,$$  \hspace{1cm} (1.8)

which in one-dimensional case (see [6]) reads as

$$k_\alpha(t, x) = \frac{1}{\alpha} \int_{\mathbb{R}} \exp\{i\xi x - t|x|^2 e^{-i\frac{2\pi}{\alpha} \text{sgn}(\xi)}\} d\xi, \quad \left(\gamma = 2 - \frac{2}{\alpha}\right).$$

The FPDE (1.3)-(1.4) interpolate two groups of PDEs (see e.g. [9]), namely hyperbolic ($\alpha = 2$) and parabolic ($\alpha = 1$) PDEs, which have been widely investigated in the last years. Theses groups presents many differences in theory of the existence and asymptotic behavior of solutions in scaling invariant spaces. In the case $\alpha = 1$, the FPDE (1.3) reduces to the usual semilinear heat equation which is well documented in singular spaces, see e.g. [15]. Without making a complete list, we mention the weak-$L^p$ spaces, the Besov spaces $\dot{B}^s_{p,\infty}$, the Morrey spaces $\mathcal{M}_{p,\mu}$, the $L^p$-spaces, the Besov-Morrey spaces $\mathcal{N}^s_{p,\mu,\infty}$ and so on. However, there are few papers dealing with FPDEs on those spaces. In [10], the authors based in Mihlin-Hörmander’s theorem establish $L^p-L^r$ estimates to Mittag-Leffler’s family (1.7) and local well-posedness is obtained in a $L^r(\mathbb{R}^n)$-framework. Using the estimates of [10] and employing techniques of [3, 21], the authors of [19] showed the existence of self-similar global solution with initial data $u_0 \in \dot{B}^n_{p,\infty} \cap \mathcal{E}_{q(r, p_0), r}$ (for $\mathcal{E}_{q(r, p_0), r}$, see Remark 1.2-(ii)). In [1], the authors studied qualitative properties, like self-similarity, antisymmetry and positivity, of global solutions for small initial data in Morrey space $\mathcal{M}_{p,\lambda}, \lambda = n - \frac{2\rho}{\rho-1}$. Also their derive local-in-time version of its results. Now, let $P(D) f$ be Riesz’s potential $(-\Delta_\gamma)^{3/2} f = \mathcal{F}^{-1}|\xi|^3\mathcal{F} f$ and $f(u(t)) = h(x, t)|u(t)|^{\rho-2}u(t)$, it is worth mentioning the works [11, 16, 36] where the authors, motived by pioneering works of Fujita and Weissler [7, 33], established necessary or sufficient conditions for either blow up or global existence of weak nonnegative solutions. It is well know that solutions $u$ of the Navier-Stokes equations has not smoothness in $t$, but in [18] Lions showed the a priori estimate

$$\int_{0}^{T} \|D_0^\gamma u(t)\|_{L^2(\mathbb{R}^n)} dt \leq \text{const.}(J + J^\frac{3}{2}), \quad J = \|u_0\|_{L^2(\mathbb{R}^n)}$$  \hspace{1cm} (1.9)

where $0 \leq \gamma < 1/4$ and $u$ is a weak solution in $L^2((0, T); L^2(\mathbb{R}^n))$ associated to data $u_0 \in L^2(\mathbb{R}^n) = C_0^\infty ||L^2$, for $n \leq 4$. In [28, Theorem 5.3], Shinbrot gave a step ahead showing (1.9) for all dimensions $n$ and $0 \leq \gamma < 1/2$. This show us that solutions of the Navier-Stokes have more smoothness in $t$ than at first appears. It seems that our initial data $u_0$ (see Theorem 1.1) is larger than the previous works and can be taken as
strongly singular function (see Remark 1.2-(iii)). Indeed, if \( \mu = n - \frac{2}{p-1} \) and \( \lambda = n - \frac{2p}{p-1} \), we obtain the continuous inclusions

\[
L^q \subset \text{weak-}L^q \subset \mathcal{M}_{p,\lambda} \subset \mathcal{N}^\sigma_{p,\mu,\infty} \quad \text{and} \quad \dot{B}^k_{r,\infty} \subset \mathcal{N}^\sigma_{p,\mu,\infty}
\]

provided that \( \frac{n}{q} = \frac{n-\lambda}{p} = -\sigma + \frac{n-\mu}{p} = -k + \frac{n}{r} \), where \( \sigma = \frac{n-\mu}{p} - \frac{2}{p-1} \), \( k = \frac{n}{r} - \frac{2}{p-1} \) and \( 1 \leq q \leq r \leq p < \frac{n(p-1)}{2} \) (all spaces in (1.10) are invariant by scaling (1.12), i.e., have the same scaling). 

One of the aims of this paper is to establish the existence of solutions for (1.3)-(1.4) in the framework of Besov-Morrey spaces. In order to show our theorems, we present key estimates in the Sobolev-Morrey and Besov-Morrey spaces for diffusion-wave operator \( L_\alpha(t) \), see Lemma 3.1, which has an interest of its own. The technical point on those estimates is the lack of semigroup’s property of \( L_\alpha(t) \), for \( 1 < \alpha < 2 \), that is, we can not write \( L_\alpha(t)f = L_\alpha(t/2)L_\alpha(t/2)f \) to get \( \| L_\alpha(t)f \|_{\mathcal{M}^\beta_{p,\mu}} \leq C(t/2)^{\frac{\eta}{2} - \frac{n-\mu}{q}} \| L_\alpha(t/2)f \|_{\mathcal{M}^\beta_{p,\mu}} \leq CA t^{\frac{\eta}{2} - \frac{n-\mu}{q}} \| f \|_{\mathcal{M}^\beta_{p,\mu}} \) which by an interpolation yield the estimates (3.2)-(3.3), see [5, Lemma 2.2]. Furthermore, will be investigated qualitative aspects of solutions like positivity, symmetries and self-similarity, under certain conditions on datum \( u_0 \). For that matter, we performed a scaling analysis in order to choose the correct indexes of spaces such that their norms are invariant by scaling (1.11). Indeed, it is well known that if \( u \) solves (1.3) with \( f(u) = \gamma |u|^{p-1}u \) then, for each \( \lambda > 0 \), the rescaled function \( u_\lambda(t, x) = \Lambda^{2p-2} u(\lambda^{\frac{2}{p}} t, \lambda x) \) is also a solution. This led us to define a scaling map for (1.3) as

\[
u(t, x) \mapsto u_\lambda(t, x).
\]

Making \( t \to 0^+ \) in (1.11) this map induces the following scaling for initial data \( u_0(x) \),

\[
u_0(x) \mapsto u_{\lambda_0}(x) = \lambda^{\frac{2}{p}} u_0(\lambda x).
\]

Solutions invariant by (1.11), namely \( u(t, x) = u_\lambda(t, x) \), are called forward self-similar solutions.

Let \( BC((0, \infty), X) \) be the class of the bounded functions from \( (0, \infty) \) into a Banach space \( X \). We define our ambient spaces based on Besov-Morrey type spaces (see (2.8)) as follows

\[
X_q^p = \{ u \in BC((0, \infty); \mathcal{N}^\sigma_{p,\mu,\infty}) : t^{\eta} u \in BC((0, \infty); \mathcal{M}^\mu_{q,\mu}) \},
\]

which endowed with norm

\[
\| u \|_{X_q^p} := \sup_{t>0} \| u(t, \cdot) \|_{\mathcal{N}^\sigma_{p,\mu,\infty}} + \sup_{t>0} t^{\eta} \| u(t, \cdot) \|_{\mathcal{M}^\mu_{q,\mu}}
\]

is a Banach space for \( (1 < p \leq q < \infty) \). Here \( \eta \in \mathbb{R} \) and \( \sigma < 0 \),

\[
\eta = \alpha \left( \frac{2}{p-1} - \frac{n-\mu}{q} \right) \quad \text{and} \quad \sigma = \frac{n-\mu}{p} - \frac{2}{p-1}
\]

having been chosen such that the norm (1.14) is invariant by scaling map (1.11).
1.1. Main results

Our main results are stated as follows.

**Theorem 1.1** (Well-posedness). Let \( n \geq 1 \) and \( 1 \leq \alpha < 2 \) and let \( 1 < \{\rho, p\} \leq q < \infty\), \( 0 \leq \mu < n \) be such that

\[
\frac{2}{\rho - 1} - \frac{2}{\alpha \rho} < \frac{n - \mu}{q} < \frac{2}{\alpha (\rho - 1)} \quad \text{and} \quad \frac{n - \mu}{p} < \frac{2}{\rho - 1}.
\] (1.16)

**(i)** (Existence and uniqueness) Let \( \varepsilon > 0 \) and \( \delta = \delta(\varepsilon) \) be such that \( \|u_0\|_{N^\sigma_{\rho,\mu,\infty}} \leq \delta \). Then the problem (1.1) has a mild solution \( u \in X^p_\rho \) which is unique, if \( u \) lies in closed small balls \( D_r \subset X^p_\rho \), \( r < 2\varepsilon \). Also, \( u(t) \rightharpoonup u_0 \) in the weak-\( * \) topology of \( B^2_{\rho,(\rho-1)} \) as \( t \to 0^+ \).

**(ii)** (Continuous dependence on data) Let \( D_\delta \subset N^\sigma_{\rho,\mu,\infty} \) be closed ball and let \( u \in X^p_\rho \) be a mild solution associated to data \( u_0 \in D_\delta \). Then data-solution map \( u_0 \in D_\delta \mapsto u \in X^p_\rho \) is Lipschitz continuous.

**Remark 1.2.**

**(i)** Let \( l > 0 \) be such that \( \{p, \rho\} \leq q \leq l \) and \( (n - \mu)/q = n/l \). By (1.16) it follows that \( \alpha n (\rho - 1) < 2l < \alpha n (\rho - 1) \rho \) as \( 1 \leq \alpha < 2 \). For every \( a \in N^\sigma_{\rho,\mu,\infty} \) satisfying the assumptions of the Theorem 1.1, there exists a unique solution \( u(t, x) \) of (1.3) in \( L^\infty((0, \infty); N^\sigma_{\rho,\mu,\infty}) \) such that \( \|u(t, \cdot)\|_{M^\sigma_{\rho,\mu}} \leq C t^{-\alpha/\rho - 1 + \alpha(n - \mu)/2q} \). In particular, we recover the [15, Theorem 1].

**(ii)** Under the assumptions of the Theorem 1.1, for \( \mu = 0 \) and \( q \leq r \leq p \), we recover the Theorem 1.1 of [19]. Indeed, in view of \( N^\sigma_{r,0,\infty} = B^\infty_{r,\infty} \) and proceeding as in Lemma 3.4 with \( (p, q) = (r, q) \) one has

\[
\|u_0\|_{e_{q(r,p_0),r}} := \sup_{t > 0} t^{q(q(p_0, r))} \|L_\alpha(t)u_0\|_r \\
= \sup_{t > 0} t^{\frac{q}{p_0} - \frac{q}{r}} \|L_\alpha(t)u_0\|_r \leq C \|u_0\|_{B^\infty_{r,\infty}}^{\frac{n}{2} - \frac{2}{p_0}} 
\]

where \( \frac{1}{q(p_0, r)} = \frac{n}{2} \left( \frac{1}{p_0} - \frac{1}{r} \right) \) and \( p_0 = \frac{n(\rho - 1)}{2} \). The assumption \( \|u_0\|_{B^\infty_{r,\infty}}^{\frac{n}{2} - \frac{2}{p_0}} \leq \delta \) on Theorem 1.1(i) yield [19, Theorem 1.1].

**(iii)** Let \( \rho > 1 + 2/n \) and \( \lambda = n - 2p/(\rho - 1) \) for \( p > 1 \). It follows that

\[
M_{p, \lambda} \subset M_{1, n - \frac{2}{\rho}} \subset N_{1, \mu, \infty}^0 \subset N_{p, \mu, \infty}^\sigma, \quad \mu = n - \frac{2}{\rho - 1}
\]

in view of (2.11) and (2.13). Our initial data can be taken strictly larger than those in [1], see [15, Example 0.10].
Suppose that $n = 1$ and let $P(D) = D_0^β$ be the Riesz-Feller operator which is given by $D_0^β \varphi(ξ) = ψ_0^β(ξ)\hat{\varphi}(ξ)$, where $ψ_0^β(ξ) = -|ξ|^β e^{i(\text{sgn } ξ)\frac{πθ}{2}}$ with $0 < β ≤ 2$ and $|θ| ≤ \min\{β, 2 - β\}$, $ξ ∈ R$. Hence (see e.g. [20]), the diffusion-wave operator $L_{α}(t)$ reads as

$$\hat{L}_{β,α}^θ(t) \varphi(ξ) = E_{α}[-t^α|ξ|^β e^{i(\text{sgn } ξ)\frac{πθ}{2}}] \hat{\varphi}(ξ)$$

which have kernel

$$k_{β,α}^θ(t, x) = \begin{cases} \int_{R} \exp\{iξx - t|ξ|^β e^{-i\frac{θπ}{2} \text{sgn } ξ}\} dξ, & α = 1 \\ \int_{R} e^{iξx} E_{α}[-t^αψ_0^β(ξ)] dξ, & 1 < α < 2. \end{cases}$$

If $(1 ≤ α < 2)$ and $(β = 2)$, the Theorem 1.1 gives us an insight on how to proceed on the study of SFPDEs (stochastic fractional partial differential equations)

$$\frac{∂^αu}{∂t}(t, x) = D_{0}^β u(t, x) + g(t, x, u(t, x)) + \sum_{k=1}^{n} \frac{∂^k h_k}{∂x^k} + f(t, x, u(t, x)) \frac{∂^2W(t, x)}{∂t∂x}$$

with datum $u_0$ in spaces more singular than $L^p(R)$ spaces. Here, the functions $f, g, h_k$ satisfy Lipschitz and certain growth conditions (see e.g. [2] and [24]).

Let $O(n)$ be the orthogonal matrix group in $R^n$ and let $G$ be a subset of $O(n)$. A function $h$ is said even (or symmetric) and odd (or antisymmetric) under the action of the subset $G$ if $h(x) = h(Mx)$ and $h(x) = -h(Mx)$, respectively, for every $M ∈ G$.

**Theorem 1.3.** Under the hypotheses of Theorem 1.1.

(i) **(Self-similarity)** Let $f(u) = γ|u|^{p-1}u$ and let $u_0$ be a homogeneous function of degree $-\frac{2}{p-1}$, then the mild solution given in Theorem 1.1 is self-similar.

(ii) **(Symmetry and antisymmetry)** The solution $u(x, t)$ is antisymmetric (resp. symmetric) for $t > 0$, when $u_0$ is antisymmetric (resp. symmetric) under $G$.

(iii) **(Positivity)** If $u_0 ≠ 0$ and $u_0(x) ≥ 0$ (resp. $u_0(x) ≤ 0$) then $u$ is positive (resp. negative).

**Remark 1.4.** If $G = O(n)$ we have radial symmetry. Indeed, it follows from Theorem 1.3(ii) that if $u_0$ radially symmetric then $u(x, t)$ is radially symmetric for all $t > 0$.

Also we have provided an asymptotic behavior in our ambient space $X_{θ}^0$ of the solutions of Theorem 1.1 as $t → ∞$. 

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Theorem 1.5. Under the hypotheses of Theorem 1.1, let \( u \) and \( v \) be two global mild solutions for (1.1) given by Theorem 1.1, with respective data \( u_0 \) and \( v_0 \). We have that
\[
\lim_{t \to +\infty} \|u(\cdot, t) - v(\cdot, t)\|_{\mathcal{N}^{p,\mu}_{\infty}} = \lim_{t \to +\infty} t^\eta \|u(\cdot, t) - v(\cdot, t)\|_{\mathcal{M}^{p,\mu}} = 0
\]
if and only if
\[
\lim_{t \to +\infty} \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{N}^{p,\mu}_{\infty}} + t^\eta \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{M}^{p,\mu}} = 0.
\]

The manuscript is organized as follows. In the section 2, basic properties of Sobolev-Morrey, Besov-Morrey spaces and Mittag-Leffler functions are reviewed. In the section 3, the main estimates of the paper are obtained. In the section 4, the proofs of our theorems are performed.

2. Preliminaries

In this section we collect some well known properties about Sobolev-Morrey and Besov-Morrey spaces. Also, we recall properties of the Mittag-Leffler functions.

2.1. Besov-Morrey space

The basic properties of Morrey and Besov-Morrey spaces is reviewed in the present subsection for the reader convenience, more details can be found in [12, 15, 17, 25, 35].

Let \( D_r(x_0) \) be the open ball in \( \mathbb{R}^n \) centered at \( x_0 \) and with radius \( r > 0 \). For two parameters \( 1 \leq p < \infty \) and \( 0 \leq \mu < n \), we define the Morrey spaces \( \mathcal{M}^{p,\mu} = \mathcal{M}^{p,\mu}(\mathbb{R}^n) \) as the set of functions \( f \in L^p(D_r(x_0)) \) such that
\[
\|f\|_{p,\mu} := \sup_{x_0 \in \mathbb{R}^n} \sup_{r > 0} r^{-\mu/p} \|f\|_{L^p(D_r(x_0))} < \infty
\]
which is a Banach space. For \( s \in \mathbb{R} \) and \( 1 \leq p < \infty \), the homogeneous Sobolev-Morrey space \( \mathcal{M}^{s,p,\mu} = (-\Delta)^{-s/2} \mathcal{M}^{p,\mu} \) is a Banach space with norm
\[
\|f\|_{\mathcal{M}^{s,p,\mu}} = \left\|(-\Delta)^{s/2} f\right\|_{p,\mu}.
\]
Taking \( p = 1 \), \( \|f\|_{L^1(D_r(x_0))} \) stands for the total variation of \( f \) on \( D_r(x_0) \) and \( \mathcal{M}_{1,\mu} \) is a space of signed measures. In particular, when \( \mu = 0 \), \( \mathcal{M}_{1,0} = \mathcal{M} \) is the space of finite measures. For \( p > 1 \), \( \mathcal{M}^{p,0} = L^p \) and \( \mathcal{M}^{s,0} = L^s \) is the homogeneous Sobolev space. With the natural adaptation in (2.1) for \( p = \infty \), the space \( L^\infty \) corresponds to \( \mathcal{M}_{\infty,\mu} \). Morrey spaces present the following scaling
\[
\|f(\lambda \cdot)\|_{p,\mu} = \lambda^{-\frac{n\mu}{p}} \|f\|_{p,\mu}
\]
and
\[
\|f(\lambda \cdot)\|_{\mathcal{M}^{s,p,\mu}} = \lambda^{-\frac{n\mu}{p}} \|f\|_{\mathcal{M}^{s,p,\mu}}.
\]
where the exponent $s - \frac{n-\mu}{p}$ is called scaling index. We have that

$$(-\Delta)^{1/2} \mathcal{M}_{p,\mu}^s = \mathcal{M}_{p,\mu}^{s-1}. \quad \text{(2.5)}$$

Morrey spaces contain Lebesgue and weak-$L^p$ with the same scaling indexes. Precisely, we have the continuous proper inclusions

$$L^p(\mathbb{R}^n) \subset \text{weak-}L^p(\mathbb{R}^n) \subset \mathcal{M}_{r,\mu}(\mathbb{R}^n) \quad \text{(2.6)}$$

where $r < p$ and $\mu = n(1 - r/p)$ (see e.g. [22]). Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz space and the tempered distributions, respectively. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be nonnegative radial function such that

$$\text{supp}(\varphi) \subset \{\xi \in \mathbb{R}^n; \frac{1}{2} < |\xi| < 2\} \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \varphi_j(\xi) = 1, \text{ for all } \xi \neq 0 \quad \text{(2.7)}$$

where $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Let $\phi(x) = F^{-1}(\varphi)(x)$ and $\phi_j(x) = F^{-1}(\varphi_j)(x) = 2^{jn}\phi(2^jx)$ where $F^{-1}$ stands for inverse Fourier transform. For $1 \leq q < \infty$, $0 \leq \mu < n$ and $s \in \mathbb{R}$, the homogeneous Besov-Morrey space $\mathcal{N}_{q,\mu,r}^s(\mathbb{R}^n)$ (or $\mathcal{N}_{q,\mu}^s$ for short) is defined to be the set of $u \in \mathcal{S}'(\mathbb{R}^n)$, modulo the space of polynomials $\mathcal{P}$, such that $F^{-1}\varphi_j(x)Fu \in \mathcal{M}_{q,\mu}$ for all $j \in \mathbb{Z}$ and

$$\|u\|_{\mathcal{N}_{q,\mu,r}^s} = \left\{ \begin{array}{ll} \left(\sum_{j \in \mathbb{Z}} (2^{js}\|\phi_j \ast u\|_{q,\mu})^r \right)^{1/r} < \infty, & 1 \leq r < \infty \\ \sup_{j \in \mathbb{Z}} 2^{js}\|\phi_j \ast u\|_{q,\mu} < \infty, & r = \infty. \end{array} \right. \quad \text{(2.8)}$$

In particular $\mathcal{N}_{q,0}^s = \tilde{B}_{q,r}^s$ denotes the homogeneous Besov spaces and for $1 \leq q \leq p < \infty$ such that $\mu = n(1 - q/p)$ we obtain $\mathcal{N}_{q,\mu,r}^s \equiv \mathcal{N}_{p,q,r}^s$, because $\mathcal{M}_{q,\mu} = \mathcal{M}_{p,\mu}^0$ (see [35]). The space $\mathcal{N}_{q,\mu,r}^s$ is a Banach space and have the real interpolation properties

$$\mathcal{N}_{q,\mu,r}^s = (\mathcal{M}_{q,\mu}^{s_1}, \mathcal{M}_{q,\mu}^{s_2})_{\theta,r} \quad \text{(2.9)}$$

and

$$\mathcal{N}_{q,\mu,r}^s = (\mathcal{N}_{q,\mu,r_1}^{s_1}, \mathcal{N}_{q,\mu,r_2}^{s_2})_{\theta,r}, \quad \text{(2.10)}$$

for all $s_1 \neq s_2$, $0 < \theta < 1$ and $s = (1 - \theta)s_1 + \theta s_2$. Here $(X,Y)_{\theta,r}$ stands for the real interpolation space between $X$ and $Y$ constructed via the $K_{\theta,q}$-method. Recall that $(\cdot, \cdot)_{\theta,r}$ is an exact interpolation functor of exponent $\theta$ on the category of normed spaces.

In the next lemmas, we collect basic facts about Morrey spaces and Besov-Morrey spaces (see e.g. [12, 15, 35]).

**Lemma 2.1.** Suppose that $s_1, s_2 \in \mathbb{R}$, $1 \leq p_1, p_2, p_3 < \infty$ and $0 \leq \mu_i < n$, $i = 1, 2, 3$.

(i) **(Inclusion)** If $\frac{n-\mu_1}{p_1} = \frac{n-\mu_2}{p_2}$ and $p_2 \leq p_1$,

$$\mathcal{M}_{p_1,\mu_1} \subset \mathcal{M}_{p_2,\mu_2} \quad \text{and} \quad \mathcal{N}_{p_1,\mu_1}^{0} \subset \mathcal{N}_{p_2,\mu_2}^{0} \subset \mathcal{N}_{p_1,\mu_1}^{0} \subset \mathcal{N}_{p_1,\mu_1}^{0}. \quad \text{(2.11)}$$
(ii) (Sobolev-type embedding) Let $j = 1, 2$ and $p_j, s_j$ be $p_2 \leq p_1$, $s_1 \leq s_2$ such that $s_2 - \frac{n-\mu_2}{p_2} = s_1 - \frac{n-\mu_1}{p_1}$, we obtain
\[ M^{s_2}_{p_2, \mu} \subset M^{s_1}_{p_1, \mu}, (\mu = \mu_1 = \mu_2) \tag{2.12} \]
and for every $1 \leq r_2 \leq r_1 \leq \infty$, we have
\[ N^{s_2}_{p_2, \mu_2, r_2} \subset N^{s_1}_{p_1, \mu_1, r_1} \quad \text{and} \quad N^{s_2}_{p_2, \mu_2, r_2} \subset B^{s_2 - \frac{n-\mu_2}{p_2}}_{\infty, r_2}. \tag{2.13} \]

(iii) (Hölder inequality) Let $\frac{1}{p_3} = \frac{1}{p_2} + \frac{1}{p_1}$ and $\frac{\mu_3}{p_3} = \frac{\mu_2}{p_2} + \frac{\mu_1}{p_1}$. If $f_j \in M_{p_j, \mu_j}$ with $j = 1, 2$, then $f_1 f_2 \in M_{p_3, \mu_3}$ and
\[ \|f_1 f_2\|_{p_3, \mu_3} \leq \|f_1\|_{p_1, \mu_1} \|f_2\|_{p_2, \mu_2}. \tag{2.14} \]

We finish this subsection recalling an estimate for certain multiplier operators on $M^s_{q, \mu}$ (see e.g. [17]).

**Lemma 2.2.** Let $m, s \in \mathbb{R}$ and $0 \leq \mu < n$ and $P(\xi) \in C^{[n/2]+1}(\mathbb{R}^n \setminus \{0\})$. Assume that there is $A > 0$ such that
\[ \left| \frac{\partial^k P}{\partial \xi^k}(\xi) \right| \leq A |\xi|^{m-|k|}, \tag{2.15} \]
for all $k \in (\mathbb{N} \cup \{0\})^n$ with $|k| \leq [n/2] + 1$ and for all $\xi \neq 0$. Then the multiplier operator $P(D)f = F^{-1}P(\xi)Ff$ on $S'/\mathcal{P}$ satisfies boundedness estimates
\[ \|P(D)f\|_{M^{s-m}_{q, \mu}} \leq CA \|f\|_{M^s_{q, \mu}}, \quad (1 < q < \infty) \tag{2.16} \]
where $C > 0$ is a constant independent of $f$, and the set $S'/\mathcal{P}$ is the one of equivalence classes in $S'$ modulo polynomials with $n$ variables.

### 2.2. Mittag-Leffler function

In this part we collect some basic properties for Mittag-Leffler functions $E_\alpha(-t^\alpha|\xi|^2)$ as well as the fundamental solution $k_\alpha$ (see (1.8)), more details can be obtained in [1, 6, 10] and references there in.

Recall that Mittag-Leffler’s function $E_\alpha(-t^\alpha|\xi|^2)$ can be defined via complex integral as
\[ E_\alpha(-t^\alpha|\xi|^2) = \frac{1}{2\pi i} \int_{\zeta} e^{\zeta t^\alpha|\xi|^2} \zeta^{\alpha-1} d\zeta, \quad (\alpha > 0) \tag{2.17} \]
where $\zeta$ is any Hankel’s path in complex plane $\mathbb{C}$. The integrand in (2.17) has simple poles given by
\[ a_\alpha(\xi) = |\xi|^\frac{\alpha}{\alpha} e^{i\alpha}, \quad b_\alpha(\xi) = |\xi|^\frac{\alpha}{\alpha} e^{-i\alpha}, \quad \text{for} \ \xi \in \mathbb{R}^n. \]
Lemma 2.3. Let \(1 < \alpha < 2\) and \(k_\alpha\) be as in (1.8). We have that

\[
L^1(\mathbb{R}^n) \ni E_\alpha(-|x|^2) = \frac{1}{\alpha} (\exp(a_\alpha) + \exp(b_\alpha)) + l_\alpha(x) \ (n \geq 1)
\]

where

\[
l_\alpha(x) = \begin{cases} 
\frac{\sin(\alpha \pi)}{\pi} \int_0^\infty \frac{|s|^{\alpha - 1} e^{-s}}{s^{\alpha + 2|\alpha| |\cos(\alpha \pi)| + |\xi|^2}} \, ds & \text{if } \xi \neq 0 \\
1 - \frac{2}{\alpha} & \text{if } \xi = 0.
\end{cases}
\]

Moreover,

\[
\frac{\partial^k k_\alpha}{\partial x_i^k}(t, x) = t^{-\frac{n}{2}(k+n)} \frac{\partial^k}{\partial x_i^k} k_\alpha(1, t^{-\frac{n}{2}} x), \ (t > 0)
\]

\(k_\alpha(t, x) \geq 0, P_\alpha(1, |x|) = \alpha k_\alpha(1, x)\) is a probability measure.

Lemma 2.4. \(1 \leq \alpha < 2\) and \(0 \leq \delta < 2\). There exists \(A > 0\) such that

\[
\left| \frac{\partial^k}{\partial \xi^k} \left[ |\xi|^\delta E_\alpha(-|\xi|^2) \right] \right| \leq A |\xi|^{-|k|}, \ \xi \neq 0,
\]

for all \(k \in (\mathbb{N} \cup \{0\})^n\) with \(|k| \leq \lfloor n/2 \rfloor + 1\).

It is well known that \(E_\alpha(-|x|^2)\) coincides with \(\sum_{k=0}^{\infty} \frac{(-|\xi|^2)^k}{\Gamma(\alpha k + 1)}\). Indeed, recalling that the Gamma function \(\Gamma(x)\) can be defined as \(\frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_\zeta e^z \zeta^{-x} \, dz, \ x \in \mathbb{C}\), dominated convergence theorem yields

\[
E_\alpha(-|x|^2) = \frac{1}{2\pi i} \int_\zeta e^z \zeta^{-1} \left[ \lim_{n \to \infty} \frac{1 - (z^{-\alpha} |\xi|^2)^{n+1}}{1 + z^\alpha |\xi|^2} \right] \, dz
\]

\[
= \frac{1}{2\pi i} \int_\zeta e^z \zeta^{-1} \left[ \lim_{n \to \infty} \sum_{k=0}^{n} (z^{-\alpha} |\xi|^2)^k \right] \, dz
\]

\[
= \lim_{n \to \infty} \sum_{k=0}^{n} (-|\xi|^2)^k \frac{1}{2\pi i} \int_\zeta e^z z^{-(\alpha k + 1)} \, dz = \sum_{k=0}^{\infty} \frac{(-|\xi|^2)^k}{\Gamma(\alpha k + 1)},
\]

for all \(\alpha \geq 1\). In view of (2.21) we have \(E_\alpha(-|\xi|^2) \in \mathbb{R}\).

3. Key estimates

The goal of this section is to derive estimates for Mittag-Leffler convolution operators \(\{L_\alpha(t)\}_{t \geq 0}\) on Sobolev-Morrey spaces and Besov-Morrey spaces. Here and below the letter \(C\) will denote constants which can be change line to line.

Lemma 3.1. Let \(s, \beta \in \mathbb{R}, 1 \leq \alpha < 2, 1 < p \leq q < \infty, 0 \leq \mu < n,\) and \((\beta - s) + \frac{n - \mu}{p} - \frac{n - \mu}{q} < 2\) where \(\beta \geq s\).
Lemma 4.5

Proof. Let $\delta = (\beta - s) + l$, where $l = \frac{n - \mu}{p} - \frac{n - \mu}{q}$. Recalling that $\hat{f}_\lambda(\xi) = \lambda^{-n} \hat{f}(\xi/\lambda)$ for $f_\lambda(x) := f(\lambda x)$, we let $(-\Delta)^{\frac{s}{2} L_\alpha(t)$ be the Fourier multiplier defined as follows

$$\langle (-\Delta)^{\frac{s}{2} L_\alpha(t)\rangle} = \hat{h}_\alpha(\xi, t) \hat{f}(\xi)$$

where the symbol of $P(D)$ is $\hat{h}_\alpha(\xi, 1) = |\xi|^{\vartheta} E_\alpha(-|\xi|^2)$. Noticing that $0 \leq \vartheta < 2$, follow from Lemma 2.4 that $P(\xi)$ shall to satisfy (2.15) with $m = 0$. Using (2.4) and (2.16) we obtain

$$\|P(D)(f_{\xi/2})\|_{L^{-\alpha/2}}\|M_{p,\mu}^s\| = t^{-\frac{\vartheta}{2} \left( s - \frac{n - \mu}{p} \right)}\|P(D)(f_{\xi/2})\|_{M_{p,\mu}^s}$$

$$\leq C A t^{-\frac{\vartheta}{2} \left( s - \frac{n - \mu}{p} \right)}\|f_{\xi/2}\|_{M_{p,\mu}^s}$$

$$= C A \|f\|_{M_{p,\mu}^s}^s.$$
where (3.7) is obtained of (3.6) via inequality (3.5). In order to obtain (3.2) we recall of the real interpolation \( \mathcal{N}_{q,\mu}^\beta = (\mathcal{M}_{q,\mu}^{\beta_1}, \mathcal{M}_{q,\mu}^{\beta_2})_{\theta, r}, \mathcal{N}_{p,\mu}^s = (\mathcal{M}_{p,\mu}^{\beta_1}, \mathcal{M}_{p,\mu}^{\beta_2})_{\theta, r} \) where \( \beta = (1 - \theta)\beta_1 + \theta \beta_2, \beta_1 \neq \beta_2 \) and \( s = (1 - \theta)s_1 + \theta s_2, s_1 \neq s_2 \). Then, we have

\[
\|L_\alpha(t)f\|_{\mathcal{N}_{q,\mu}^{\beta}}, \leq m_0^{1-\theta} m_1^\theta \|f\|_{\mathcal{N}_{p,\mu}^s}, \quad 0 < \theta < 1,
\]

(3.8)

where \( m_i = \|L_\alpha(t)f\|_{\mathcal{M}_{q,\mu}^{\beta_i}} \). In view of (3.1), we obtain

\[
m_i \leq C A t^{-\frac{1}{q}(\beta_i - s_i) - \frac{2}{2} \left(\frac{n-p}{p} - \frac{n-q}{q}\right)}
\]

inserting this into (3.8) yield (3.2). Now using (3.2) we obtain

\[
\|L_\alpha(t)f\|_{\mathcal{N}_{q,\mu,\infty}^{2\beta-s}} \leq \frac{1}{t} C t^{-\frac{1}{q}\left(\frac{n-p}{p} - \frac{n-q}{q}\right)} \|f\|_{\mathcal{N}_{p,\mu,\infty}^s}
\]

and

\[
\|L_\alpha(t)f\|_{\mathcal{N}_{q,\mu,\infty}^s} \leq \frac{1}{t} C t^{-\frac{1}{q}\left(\frac{n-p}{p} - \frac{n-q}{q}\right)} \|f\|_{\mathcal{N}_{p,\mu,\infty}^s}.
\]

In view of (2.10) and \((2\beta - s)(1 - 1/2) + s(1/2) = \beta\) we have \( \mathcal{N}_{q,\mu,1}^\beta = (\mathcal{N}_{q,\mu,\infty}^{2\beta-s}, \mathcal{N}_{q,\mu,\infty}^s)_{1/2,1} \) which yield (3.3).

\[\blacksquare\]

**Proposition 3.2.** Let \( \xi \in \mathbb{R}^n \). If \( 1 < \alpha < 2 \), we have \( |E_\alpha(-|\xi|^2)| \leq 1 \) and \( E_\alpha(-|t^{\alpha/2}\xi|^2) \rightarrow 1 \) as \( t \to 0^+ \).

**Proof.** It is enough to make a proof for \( 1 < \alpha < 2 \), because the Lemma holds for \( E_1(-|t|\xi|^2) = e^{-|t|\xi|^2} \). To this end, let \( t = |\xi|^2 t^{1/\alpha} \) for \( \xi \neq 0 \) and using Lemma 2.3 and \( |\exp(a_\xi(\xi))| = |\exp(b_\alpha(\xi))| = \exp(|\xi|^{2/\alpha} \cos(\pi/\alpha)) \leq 1 \) we obtain

\[
|E_\alpha(-|\xi|^2)| \leq \frac{1}{\alpha} + |l_\alpha(\xi)|
\]

\[
\leq \frac{1}{\alpha} + \sin(\alpha \pi) \int_0^\infty \frac{e^{-s}}{s^{2} + 2s \cos(\alpha \pi) + 1} ds
\]

\[
\leq \frac{1}{\alpha} + \sin(\alpha \pi) \int_0^\infty \frac{1}{s^{2} + 2s \cos(\alpha \pi) + 1} ds
\]

\[
= \frac{1}{\alpha} + (1 - \frac{2}{\alpha}) = 1.
\]

Let \( \Phi(t) = \frac{e^{\frac{\alpha}{2} - 1}}{t^{\alpha + 1}|\xi|^{2\alpha}} \). Note that \( |\Phi(t)| \in L^1(0, \infty) \) and \( \Phi(t) \rightarrow e^{z^\alpha/2} \) as \( t \to 0^+ \), using dominated converge theorem and residue theorem we have

\[
E_\alpha(-t^{\alpha}|\xi|^2) \rightarrow \frac{1}{2\pi i} \int_0 e^{z^\alpha/2} dz = \frac{1}{2\pi i} \left\{ 2\pi i \mathrm{res} \left( e^{z^\alpha/2}; z = 0 \right) \right\} = 1,
\]

(3.9)

because \( \mathrm{res} \left( e^{z^\alpha/2}; z = 0 \right) := \mathrm{residue} \left( e^{z^\alpha/2}; z = 0 \right) = 1 \).

\[\blacksquare\]
3.1. Linear estimates

We start by recalling an elementary fixed point lemma whose proof can be found in [4].

**Lemma 3.3.** Let \((X, \| \cdot \|)\) be a Banach space and \(1 < \rho < \infty\). Suppose that \(B : X \to X\) satisfies \(B(0) = 0\) and

\[
\|B(x) - B(z)\| \leq K\|x - z\|\|x\|^{\rho - 1} + \|z\|^{\rho - 1}.
\]

Let \(R > 0\) be the unique positive root of \(2^\rho K R^{\rho - 1} - 1 = 0\). Given \(0 < \varepsilon < R\) and \(y \in X\) such that \(\|y\| \leq \varepsilon\), there exists a solution \(x \in X\) for the equation \(x = y + B(x)\) which is the unique one in the closed ball \(D_{2\varepsilon} = \{z \in X; \|z\| \leq 2\varepsilon\}\). Moreover, if \(\|y\| \leq \varepsilon\) and \(\bar{x} \in D_{2\varepsilon}\) satisfies the equation \(\bar{x} = \bar{y} + B(\bar{x})\) then

\[
\|x - \bar{x}\| \leq \frac{1}{1 - 2^\rho K \varepsilon^{\rho - 1}}\|y - \bar{y}\|.
\]  (3.10)

The integral equation (1.5) has the form \(u = y + B(u, u)\) on the space \(X = X^p_0\) where \(y = L_\alpha(t)u_0\) and \(B(u, u)\) is given by (3.14). We invoke the Lemma 3.3 in our proofs, hence the estimates for linear and nonlinear part of (1.5) will be necessary.

**Lemma 3.4.** Under the assumptions of Theorem 1.1, there exists \(L > 0\) such that

\[
\|L_\alpha(t)u_0\|_{X^p_0} \leq L\|u_0\|_{\mathcal{N}^\sigma_{p, \mu, \infty}},
\]  (3.11)

for all \(u_0 \in \mathcal{N}^\sigma_{p, \mu, \infty}\). Let \(s = 2/(\rho - 1)\), if \(u_0 \in \dot{B}^{s}_{\infty, \infty}\) we obtain \(L_\alpha(t)u_0 \rightharpoonup u_0\) in the weak-* topology of \(\dot{B}^{s}_{\infty, \infty}\) as \(t \to 0^+\).

**Proof.** Notice that by (1.15) we obtain \(\eta + \frac{\alpha}{2} \sigma = \frac{\rho}{2} \left(\frac{n - \mu}{p} - \frac{n - \mu}{q}\right)\). Using (1.16) one has \(\frac{n - \mu}{p} - \frac{n - \mu}{q} - \sigma = \frac{2}{\rho - 1} - \frac{n - \mu}{q} < \frac{2}{\rho \alpha} < 2\) and \(\sigma < 0\) which by (3.2) and afterwards by (2.11) and (3.3), respectively, give us

\[
\sup_{t > 0} \|L_\alpha(t)u_0\|_{\mathcal{N}^\sigma_{p, \mu, \infty}} + \sup_{t > 0} \|L_\alpha(t)u_0\|_{\mathcal{N}^\sigma_{q, \mu}} \leq C\|u_0\|_{\mathcal{N}^\sigma_{p, \mu, \infty}} + \sup_{t > 0} \|L_\alpha(t)u_0\|_{\mathcal{N}^\sigma_{q, \mu, 1}}
\]

\[
\leq C\|u_0\|_{\mathcal{N}^\sigma_{p, \mu, \infty}} + C\sup_{t > 0} \|u_0\|_{\mathcal{N}^\sigma_{q, \mu, 1}},
\]

\[
\leq L\|u_0\|_{\mathcal{N}^\sigma_{p, \mu, \infty}}.
\]

this yield (3.11). It remains to check the weak-* convergence. To this end, let \(v \in \dot{B}^{-s}_{1, 1}\) the predual space of \(\dot{B}^{s}_{\infty, \infty}\). Using Proposition 3.2 we have

\[
\|L_\alpha(t)v - v\|_{\dot{B}^{-s}_{1, 1}} = \sum_{j = -\infty}^{\infty} \{2^{-j} \|F^{-1}\varphi_j [E_\alpha(-t^\alpha|\xi|^2) - 1]Fv\|_{L^1(\mathbb{R}^n)}\} \to 0
\]

as \(t \to 0^+\). Thanks to (2.19) and (2.21) one has \(E_\alpha(-t^\alpha|\xi|^2) = t^{-\frac{n}{\alpha}}E_\alpha(-t^{-\frac{n}{\alpha}}|\xi|^2) \in \mathbb{R}\) for all \(t > 0\) and \(\xi \in \mathbb{R}^n\), it follows that

\[
|\langle L_\alpha(t)u_0 - u_0, v \rangle| = |\langle u_0, L_\alpha(t)v - v \rangle| \leq \|u_0\|_{\dot{B}^{s}_{\infty, \infty}} \|L_\alpha(t)v - v\|_{\dot{B}^{-s}_{1, 1}} \to 0,
\]  (3.12)

as \(t \to 0^+\).
3.2. Nonlinear estimates

To treat the nonlinear term, we recall
\[ B_{\alpha}(u)(t) = \int_0^t L_{\alpha}(t-s) \int_0^s r_{\alpha-1}(s-\tau)f(u(\tau))d\tau ds. \] (3.13)

**Lemma 3.5** (Nonlinear estimate). Under the assumptions of Theorem 1.1. There is a positive constant $K$ such that
\[ \|B_{\alpha}(u) - B_{\alpha}(v)\|_{X_q^p} \leq K\|u - v\|_{X_q^p}(\|u\|_{X_q^p}^{p-1} + \|v\|_{X_q^p}^{p-1}), \] (3.14)
for $K = K_1 + K_2$. Moreover, we have $B_{\alpha}(u)(t) \to 0$ in the weak-* topology of $\dot{B}_{\infty,\infty}^{2/(p-1)}$ as $t \to 0^+$.

**Proof.** The proof is divided in three steps.

**First step.** Let $\tilde{s} \in \mathbb{R}$ be such that $\sigma - \frac{n-\mu}{p} = \tilde{s} - \frac{n-\mu}{q/\rho}$. In view of (1.16) and $\alpha \geq 1$ we have $\frac{n-\mu}{q/\rho} \geq \frac{2}{p-1} > \frac{n-\mu}{p}$, it follows that $\sigma < 0 \leq \tilde{s}$ and $p \geq q/\rho$. Applying (2.12) and (3.2) afterwards (2.11) and (2.14), respectively, we have
\[ \|B_{\alpha}(u)(t)-B_{\alpha}(v)(t)\|_{\Lambda_{q/p,\mu,\infty}^p} \leq \|B_{\alpha}(u)(t) - B_{\alpha}(v)(t)\|_{\Lambda_{q/p,\mu,\infty}^p} \]
\[ \leq C \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau)\|f(u(\tau)) - f(v(\tau))\|_{\Lambda_{q/p,\mu,\infty}^p} d\tau ds \]
\[ \leq C \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau)\|f(u(\tau)) - f(v(\tau))\|_{\Lambda_{q/p,\mu,\infty}^p} d\tau ds \]
\[ \leq C \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau)\|u - v\|_{\Lambda_{q/p,\mu}^p}(\|u\|_{\Lambda_{q/p,\mu}^p}^{p-1} + \|v\|_{\Lambda_{q/p,\mu}^p}^{p-1}) d\tau ds \] (3.15)
\[ := \psi_1(t) \sup_{t>0}\|u(t) - v(t)\|_{\Lambda_{q/p,\mu}^p} \sup_{t>0}\|u(t)\|_{\Lambda_{q/p,\mu}^p}^{p-1} + \|v(t)\|_{\Lambda_{q/p,\mu}^p}^{p-1} \] (3.16)
where $r_{\alpha}(s) = s^{\alpha-1}/\Gamma(\alpha), f(u(\tau)) = |u(\tau)|^{p-1}u(\tau), \gamma_1 = -\frac{q}{2}\tilde{s} = -\frac{q}{2}\sigma - \frac{q}{2}(\frac{n-\mu}{q/\rho} - \frac{n-\mu}{p})$ and
\[ \psi_1(t) = C \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau)\tau^{-\eta p}d\tau ds \]
\[ = C \beta(1-\eta p, \alpha - 1)\beta(\alpha - \eta, \gamma_1 + 1), \] (3.17)
for $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ if $x, y > 0$. Indeed, by change of variables $\tau = zs$ and $s = tw$, respectively, we get
\[ \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau)\tau^{-\eta p}d\tau ds = \int_0^t (t-s)^{\gamma_1} s^{\alpha-1-\eta p} \left( \int_0^1 (1-z)^{\alpha-2\omega-\eta p}dz \right) ds \]
\[ = \beta(1-\eta p, \alpha - 1)\omega^{\alpha-1-\eta p} \int_0^1 (1-\omega)^{\gamma_1} \omega^{\alpha-1-\eta p}d\omega, \]
\[ = \beta(1-\eta p, \alpha - 1)\beta(\alpha - \eta, \gamma_1 + 1), \] (3.18)
because by \( \gamma_1 = -\frac{\alpha}{2}\sigma - \frac{\alpha}{2}(\frac{n-\mu}{q/\rho} - \frac{n-\mu}{\rho}) \) and (1.15) we have

\[
\alpha + \gamma_1 - \eta \rho = \alpha + \frac{\alpha}{2}\left(\frac{2}{\rho - 1} - \frac{n-\mu}{q}\right) - \frac{\alpha}{2}\left(\frac{2}{\rho - 1} - \frac{n-\mu}{q}\right)\rho
= \alpha + \frac{\alpha}{\rho - 1} - \alpha \rho - 1 - \eta = -\eta.
\]

Inserting (3.17) into (3.16) yields

\[
\|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{X_{p,\infty}^{\sigma}} \leq K_1 \sup_{t > 0} t^\eta \|u(t) - v(t)\|_{M_{q,\mu}} \sup_{t > 0} t^{\gamma(\rho - 1)}(\|u(t)\|_{M_{q,\mu}}^{\rho-1} + \|v(t)\|_{M_{q,\mu}}^{\rho-1}).
\]  

(3.19)

**Second step.** Let \( \beta = s = 0 \). By estimate (3.1) and Hölder inequality (2.14) we obtain

\[
\|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{M_{q,\mu}} \leq C \int_0^t (t - s)\gamma_2 \theta(s) ds
\]

(3.20)

where \( \gamma_2 = -\frac{\alpha}{2}(\frac{n-\mu}{q/\rho} - \frac{n-\mu}{\rho}) \) and \( \theta(s) \) is given by

\[
\theta(s) = \int_0^s r_{\sigma - 1}(s - \tau)\|u(\tau) - v(\tau)\|_{M_{q,\mu}}(||u(\tau)||_{M_{q,\mu}}^{\rho-1} + ||v(\tau)||_{M_{q,\mu}}^{\rho-1}) d\tau.
\]

Mimicking the **First step** we get

\[
\|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{M_{q,\mu}} \leq \psi_2(t) \sup_{t > 0} t^\eta \|u(t) - v(t)\|_{M_{q,\mu}} \sup_{t > 0} t^{\gamma(\rho - 1)}(\|u(t)\|_{M_{q,\mu}}^{\rho-1} + \|v(t)\|_{M_{q,\mu}}^{\rho-1})
\]

(3.21)

where \( \psi_2(t) \) can be estimated as

\[
\psi_2(t) \leq C \beta(1 - \eta \rho, \alpha - 1) \beta(\alpha - \eta \rho, \gamma_2 + 1)t^{\alpha + \gamma_2 - \eta \rho} = K_2 t^{-\eta},
\]  

(3.22)

because in view of (1.15) we have

\[
\alpha + \gamma_2 - \eta \rho = \alpha - \frac{\alpha n - \mu}{2} - \frac{\alpha 2 \rho}{2} \rho - 1 = \alpha + \frac{\alpha \rho}{\rho - 1} - \frac{\alpha \rho}{\rho - 1} - \eta = -\eta.
\]

Inserting (3.22) into (3.21) it follows that

\[
t^\eta \|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{M_{q,\mu}} \leq K_2 \sup_{t > 0} t^\eta \|u(t) - v(t)\|_{M_{q,\mu}} \sup_{t > 0} t^{\gamma(\rho - 1)}(\|u(t)\|_{M_{q,\mu}}^{\rho-1} + \|v(t)\|_{M_{q,\mu}}^{\rho-1}).
\]  

(3.23)

The convergence of the beta functions appearing in (3.18) and (3.22) is obtained by restrictions (1.15) and \( \alpha \geq 1 \), because this yields in \( \gamma_1, \gamma_2 > -1 \) and \( \eta \rho < 1 \leq \alpha \). It follows that \( (\frac{n-\mu}{q/\rho} - \frac{n-\mu}{\rho}) < \frac{2}{\alpha} \leq 2 \) which we have used in **Second step.** Recalling (1.14) and using (3.19) and (3.23) we obtain (3.14) with \( K = K_1 + K_2 \).

**Third step.** As \( S(\mathbb{R}^n) \) is dense in \( \tilde{B}^{2/(\rho - 1)}_{1,1} \), (see [37, p. 48]) the weak-* convergence can be obtained by estimate

\[
\langle B_\alpha(u)(t), v \rangle = \langle B_\alpha(u)(t), v - \varphi \rangle + \langle B_\alpha(u)(t), \varphi \rangle
\leq \|B_\alpha(u)(t)\|_{\tilde{B}^{2/(\rho - 1)}_{1,1}} \|v - \varphi\|_{\tilde{B}^{2/(\rho - 1)}_{1,1}} + \|B_\alpha(u)(t), \varphi \|
\leq C \|u\|_{X_q^\rho} + C \|u\|_{M_{q,\mu}} \|\varphi\|_{\tilde{B}^{2/(\rho - 1)}_{1,1}} t^\alpha \leq C \|u\|_{X_q^\rho} \text{ as } t \to 0^+,
\]  

(3.24)
because for $v \in \dot{B}_{1,1}^{\alpha-(n-\mu)/p} = \dot{B}_{1,1}^{2/(p-1)}$ one has $\|v - \varphi\|_{\dot{B}_{1,1}^{2/(p-1)}} \leq \varepsilon$ for all $\varepsilon > 0$. Moreover, by embedding $N_{t,\mu,\infty}^s \subset \dot{B}_{\infty,\infty}^{s-(n-\mu)/l}$ (see (2.13)) and $\frac{n-\mu}{p} < \frac{2}{p-1} < \frac{n-\mu}{q/\rho}$ one has

$$|\langle B_\alpha(u)(t), \varphi \rangle| = \left| \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \left( f(u(\tau)), L_\alpha(t-s)\varphi \right) d\tau ds \right| \leq C \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \|u(\tau)\|^{p-1} u(\tau) \|L_\alpha(t-s)\varphi\|_{\dot{B}_{1,1}^{2/(p-1)}} d\tau ds \leq C \int_0^t \int_0^s (s-\tau) \alpha-2(t-s) \frac{\alpha}{2} (\frac{n-\mu}{q/\rho} - \frac{2}{p-1}) \|u(\tau)\|^{p-1} u(\tau) \|\varphi\|_{\dot{B}_{1,1}^{2/(p-1)}} d\tau ds \leq C \int_0^t \int_0^s (s-\tau) \alpha-2(t-s) \frac{\alpha}{2} (\frac{n-\mu}{q/\rho} - \frac{2}{p-1}) \|u(\tau)\|^{p-1} u(\tau) \|\varphi\|_{\dot{B}_{1,1}^{2/(p-1)}} d\tau ds \leq C t^\alpha \|u\|_{\dot{X}_q^p} \|\varphi\|_{\dot{B}_{1,1}^{2/(p-1)}} \rightarrow 0 \text{ as } t \to 0^+ \quad (3.25)$$

and

$$\|B_\alpha(u)(t)\|_{\dot{B}_{\infty,\infty}^{s-(n-\mu)/p}} \leq C \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \|L_\alpha(t-s)\|^{p-1} u(\tau) \|\varphi\|_{\dot{B}_{1,1}^{2/(p-1)}} d\tau ds \leq \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \|L_\alpha(t-s)\|^{p-1} u(\tau) \|\varphi\|_{N_{q/\rho,\infty}^0} d\tau ds \leq \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \|L_\alpha(t-s)\|^{p-1} u(\tau) \|\varphi\|_{N_{q/\rho,\infty}^0} d\tau ds \leq C \int_0^t \int_0^s r_{\alpha-1}(s-\tau) (t-s)^\gamma_1 \|u(\tau)\|^{p-1} u(\tau) \|\varphi\|_{N_{q/\rho,\infty}^0} d\tau ds \leq C \int_0^t \int_0^s r_{\alpha-1}(s-\tau) (t-s)^\gamma_1 \tau^{-\eta_2} d\tau ds \|u\|_{\dot{X}_q^p} \leq C \|u\|_{\dot{X}_q^p}, \quad (3.26)$$

and we finish our proof.

4. Proof of theorems

4.1. Proof of Theorem 1.1

Let $0 < \varepsilon < R = (1/2^p K)^{p-1}$, where $K > 0$ and $L > 0$ are the constants obtained in Lemma 3.4 and Lemma 3.5, respectively. Let $\delta = \varepsilon / L$, the Lemma 3.3 with $X = X_q^p$ and $y = L_\alpha(t)u_0$ yields the existence of an unique global mild solution $u \in X_q$ such that $\|u\|_{X_q} \leq \varepsilon$. Moreover, the Lemmas 3.4 and 3.5 yield $u(t) \rightharpoonup u_0$ in the weak-* topology of $\dot{B}_{\infty,\infty}^{2/(p-1)}$ as
$t \to 0^+$. The dependence of the initial data can be obtained from Lemma 3.4 and Lemma 3.3. Indeed, let $\tilde{y} = L_\alpha(t)\tilde{u}_0$ where $\tilde{u}_0 \in \mathcal{N}_{\rho}^{\sigma}$, then

$$
\|u(t) - \tilde{u}(t)\|_{X_q^p} \leq \frac{1}{1 - 2^p K^{\rho-1}} \|L_\alpha(t)(u_0 - \tilde{u}_0)\|_{X_q^p} \leq \frac{1}{1 - 2^p K^{\rho-1}} \|u_0 - \tilde{u}_0\|_{\mathcal{N}_{\rho}^{\sigma}}.
$$

4.2. Proof of Theorem 1.3

The parts (i), (ii) and (iii) follows from analogous argument found in [1, Theorem 3.3]. For reader convenience, we indicate the main steps of the proof.

**Part (ii):** Let $M \in \mathcal{G}$ and $u_0$ be antisymmetric, then $\Phi(x, t) = L_\alpha(t)u_0$ and $B_\alpha(u)$ is antisymmetric. Indeed, in view of the orthogonality of $M$ and $u_0(Mx) = -u_0(x)$ we have

$$
-\tilde{u}_0(\xi) = [u_0(M\cdot)]^\wedge(\xi) = \tilde{u}_0(M^{-1}\xi),
$$

it follows that

$$
[\Phi(Mx, t)]^\wedge(\xi) = E_\alpha(-t^\alpha |M^{-1}\xi|^2)\tilde{a}(M^{-1}\xi)
$$

$$
= -E_\alpha(-t^\alpha |\xi|^2)\tilde{a}(\xi)
$$

$$
= -\Phi(x, t)(\xi),
$$

this shows us that $L_\alpha(t)u_0$ is antisymmetric for each fixed $t > 0$. Similarly, we can show that $B_\alpha(u)$ is antisymmetric whether $u$ is also. So, employing an induction argument, one can prove that each element $u_k$ of the Picard sequence

$$
u_1(x, t) = \Phi(x, t)$$

$$
u_k(x, t) = \Phi(x, t) + B_\alpha(u_{k-1})(x, t), \quad k = 2, 3, \cdots
$$

is antisymmetric. It follows that $u(x, t)$ is antisymmetric, for all $t > 0$. The symmetric property is analogous.

**Part (i):** Let $\Phi(x, t) = L_\alpha(t)u_0$. In view of $u_0$ be homogeneous of degree $-\frac{2}{\rho-1}$ we set $u_0(\lambda x) = \lambda^{-2/(\rho-1)}u_0(x)$. It follows that

$$
[\Phi(\lambda\cdot, t)]^\wedge(\xi) = E_\alpha(-t^\alpha |\xi/\lambda|^2)\tilde{u}_0(\xi/\lambda)
$$

$$
= \lambda^{-\frac{2}{\rho-1}} \lambda^{-n} E_\alpha(-t^\alpha |\xi/\lambda|^2)\tilde{u}_0(\xi)
$$

$$
= \lambda^{-\frac{2}{\rho-1}} E_\alpha(-t^\alpha |\xi|^2)\tilde{u}_0(\xi)
$$

$$
= \lambda^{-\frac{2}{\rho-1}} \Phi(\cdot, t)(\xi),
$$

that is, $\Phi(\lambda x, t) = \lambda^{-\frac{2}{\rho-1}} \Phi(x, t)$. Now proceeding like **Part (ii)** we have that

$$
u(x, t) \equiv u_\lambda(x, t), \quad \text{for every } \lambda > 0,$$

in other words, $u$ is forward self-similar solution.
4.3. Proof of Theorem 1.5

We only show that (1.18) implies (1.17), because the converse statement follows analogously. Subtracting the integral equations verified by \( u \) and \( v \), and then taking the norms \( t^\eta \| u \|_{\mathcal{M}_{q,\mu}} \) and \( \| v \|_{\mathcal{N}_{p,\mu,\infty}} \) we obtain

\[
 t^\eta \| u(\cdot, t) - v(\cdot, t) \|_{\mathcal{M}_{q,\mu}} \leq t^\eta \| L_\alpha(t)(u_0 - v_0) \|_{\mathcal{M}_{q,\mu}} + t^\eta \| B_\alpha(u) - B_\alpha(v) \|_{\mathcal{M}_{q,\mu}} \\
 := t^\eta \| L_\alpha(t)(u_0 - v_0) \|_{\mathcal{M}_{q,\mu}} + J_1(t) \tag{4.4}
\]

and

\[
 \| u(\cdot, t) - v(\cdot, t) \|_{\mathcal{N}_{p,\mu,\infty}} \leq \| L_\alpha(t)(u_0 - v_0) \|_{\mathcal{N}_{p,\mu,\infty}} + \| B_\alpha(u) - B_\alpha(v) \|_{\mathcal{N}_{p,\mu,\infty}} \\
 \leq \| L_\alpha(t)(u_0 - v_0) \|_{\mathcal{N}_{p,\mu,\infty}} + J_2(t). \tag{4.5}
\]

Recalling that \( \| u \|_{X_q} \leq 2\varepsilon \) and \( \| v \|_{X_q} \leq 2\varepsilon \), and the inequality (3.20), we can estimate the term \( J_1(t) \) as

\[
 J_1(t) \leq C t^\eta \int_0^t (t - s)^{\gamma_2} \int_s^t r_{\alpha - 1}(s - \tau) \| u(\tau) - v(\tau) \|_{\mathcal{M}_{q,\mu}} \left( \| u(\tau) \|_{\mathcal{M}_{q,\mu}}^{\rho - 1} + \| v(\tau) \|_{\mathcal{M}_{q,\mu}}^{\rho - 1} \right) d\tau ds \\
 \leq t^\eta 2(2\varepsilon)^{\rho - 1} C \int_0^t (t - s)^{\gamma_2} \int_s^t r_{\alpha - 1}(s - \tau) \tau^{-\eta\rho} \Sigma_1(\tau) d\tau ds, \tag{4.6}
\]

where \( \Sigma_1(\tau) = t^\eta \| u(\tau) - v(\tau) \|_{\mathcal{M}_{q,\mu}} \) and \( \gamma_2 = \eta\rho - \alpha - \eta \). Similarly, in view of (3.15), the integral \( J_2(t) \) can be estimated as

\[
 J_2(t) \leq (2\varepsilon)^{\rho - 1} C \int_0^t (t - s)^{\gamma_1} \int_s^t r_{\alpha - 1}(s - \tau) \tau^{-\eta\rho} \Sigma_2(\tau) d\tau ds, \tag{4.7}
\]

where \( \Sigma_2(\tau) = \| u(\tau) - v(\tau) \|_{\mathcal{N}_{p,\mu,\infty}} \) and \( \gamma_1 = \eta\rho - \alpha \).

Now setting \( \Sigma(\tau) = \Sigma_1(\tau) + \Sigma_2(\tau) \) and making the changes \( \tau = sz \) and \( s = t\sigma \) in (4.6) and (4.7), we get

\[
 J_1(t) + J_2(t) \leq (2\varepsilon)^{\rho - 1} C \int_0^1 (1 - \sigma)^{\gamma_2} \sigma^{\alpha - 1 - \eta\rho} \int_0^1 r_{\alpha - 1}(1 - z) z^{-\eta\rho} \Sigma(t\sigma z) dz d\sigma + \\
 + (2\varepsilon)^{\rho - 1} C \int_0^1 (1 - \sigma)^{\gamma_1} \sigma^{\alpha - 1 - \eta\rho} \int_0^1 r_{\alpha - 1}(1 - z) z^{-\eta\rho} \Sigma(t\sigma z) dz d\sigma. \tag{4.8}
\]

Notice that \( \limsup_{t \to +\infty} \Sigma(t) < \infty \) because \( u, v \in X_q \). We claim that

\[
 \Pi := \limsup_{t \to +\infty} \Sigma(t) = 0, \tag{4.9}
\]
which is equivalent to (1.17). To see this, we take $\limsup_{t \to +\infty}$ in (4.8) to get

$$
\limsup_{t \to +\infty} [J_1(t) + J_2(t)] \leq (2^p \varepsilon^{p-1}) C \int_0^1 (1-\sigma)^{\gamma_2} \sigma^{\alpha-1-\eta p} \sigma \limsup_{t \to +\infty} \int_0^1 r_{\alpha-1}(1-z) z^{-\eta p} \Sigma(t \sigma z) dz
$$

$$
\quad + (2^p \varepsilon^{p-1}) C \int_0^1 (1-\sigma)^{\gamma_1} \sigma^{\alpha-1-\eta p} \sigma \limsup_{t \to +\infty} \int_0^1 r_{\alpha-1}(1-z) z^{-\eta p} \Sigma(t \sigma z) dz
$$

$$
\leq (2^p \varepsilon^{p-1}) C \left( \int_0^1 (1-\sigma)^{\gamma_2} \sigma^{\alpha-1-\beta p} \sigma \int_0^1 r_{\alpha-1}(1-z) z^{-\eta p} dz \right) \Pi
$$

$$
\quad + (2^p \varepsilon^{p-1}) C \left( \int_0^1 (1-\sigma)^{\gamma_1} \sigma^{\alpha-1-\eta p} \sigma \int_0^1 r_{\alpha-1}(1-z) z^{-\eta p} dz \right) \Pi
$$

$$
= (K_1 + K_2) (2^p \varepsilon^{p-1}) \Pi. \quad (4.10)
$$

Thanks to the inequalities (4.4), (4.5), (4.10) and the hypothesis (1.18), we obtain

$$
\Pi \leq \limsup_{t \to +\infty} (t^\eta \| L_{\alpha}(t)(u_0 - v_0) \|_{M_{\alpha, \mu}} + \| L_{\alpha}(t)(u_0 - v_0) \|_{\mathcal{N}_{\phi, \mu, \infty}}) + \limsup_{t \to +\infty} [J_1(t) + J_2(t)]
$$

$$
\leq 0 + (K_1 + K_2) (2^p \varepsilon^{p-1}) \Pi = (2^p \varepsilon^{p-1} K) \Pi
$$

(4.11)

which leads us to $\Pi = 0$, because $2^p \varepsilon^{p-1} K < 1$.

\[\square\]

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