GENERALIZED FRACTIONAL KINETIC EQUATIONS INVOLVING THE GENERALIZED MODIFIED $k$-BESSEL FUNCTION

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Abstract. Fractional kinetic equations are investigated in order to describe the various phenomena governed by anomalous reaction in dynamical systems with chaotic motion. Many authors have provided solutions of various families of fractional kinetic equations involving special functions. Here, in this paper, we aim at presenting solutions of certain general families of fractional kinetic equations associated with the generalized modified $k$-Bessel function of the first kind. It is also pointed out that the main results presented here are general enough to be able to be specialized to yield many known and (presumably) new solutions for fractional kinetic equations.

1. Introduction, Notations and Preliminaries

The Bessel function acting as a strong tool for investigating the solutions of various types of differential equations has attracted a large numbers of researchers such as mathematicians, physicists and engineers, due mainly to its importance in mathematical physics, nuclear physics, systems and control theory (see [19]). Various extensions and modifications of the Bessel function have been given and are involved in solutions of fractional differential equations. From a statistical point of view, the Bessel function is an unavoidable candidate to study various types of distribution theories. For more works of the Bessel function related to statistics, one may refer to [20]. The generalized Bessel function of the first kind $w_p(z)$ is defined for $z \in \mathbb{C} \setminus \{0\}$ and $b, c, p \in \mathbb{C}$ with $\Re(p) > -1$ by the following series representation (see [1]):

$$w_{p,b,c}(z) = w_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (z/2)^{2n+p}}{n! \Gamma(p + \frac{b+1}{2} + n)}.$$  \hfill (1.1)
where $\mathbb{C}$ is the set of complex numbers and $\Gamma(z)$ is the familiar Gamma function whose Euler’s integral is given by (see, e.g., [35, Section 1.1]):

$$
\Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} \, dt \quad (\Re(z) > 0). \tag{1.2}
$$

More details of $w_p(z)$ can be found in the recent works [2, 8, 24].

We consider three special cases of (1.1).

- The special case of (1.1) when $b = c = 1$ is easily seen to reduce to the Bessel function of the first kind of order $p$ which is defined by the series following representation (see, e.g., [1, 37]):

$$
J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+p}}{n! \Gamma(p+n+1)} \quad (z, p \in \mathbb{C}; \Re(p) > -1). \tag{1.3}
$$

- Setting $b = 1$ and $c = -1$ in (1.1) yields the following modified Bessel function of the first kind of order $p$ (see, e.g., [37, p. 77]):

$$
I_p(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+p}}{n! \Gamma(p+n+1)} \quad (z, p \in \mathbb{C}; \Re(p) > -1). \tag{1.4}
$$

- The special case of (1.1) when $b = 2$ and $c = 1$ gives the following spherical Bessel function of the first kind (see, e.g., [37, p. 77]):

$$
j_p(z) = \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+p}}{n! \Gamma(p+n+3/2)} \quad (z, p \in \mathbb{C}; \Re(p) > -3/2). \tag{1.5}
$$

In the above three cases, unless other restriction is given, it is assumed that the principal value of $\arg z$ is taken.

The generalized Bessel function $\varphi_{p,b,c}(z)$ is given by the following transformation (see [11]):

$$
\varphi_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b + 1}{2}\right) z^{1-p/2} w_p\left(\sqrt{z}\right)
= z + \sum_{n=1}^{\infty} \frac{(-c)^n z^{n+1}}{n! 4^n (\gamma)_n}, \tag{1.6}
$$

where $\gamma = p + (b + 1)/2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $(\lambda)_n$ is the familiar Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see, e.g., [35, p. 2 and p. 5]):

$$
(\lambda)_n := \begin{cases} 
1 & (n = 0) \\
\lambda(\lambda + 1) \ldots (\lambda + n - 1) & (n \in \mathbb{N}) 
\end{cases}
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{1.7}
$$
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$\mathbb{R}$ and $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{Z}_0$ being the sets of real and positive real numbers, positive and non-positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The $k$-Bessel function of the first kind is defined by the following series (see [28]):

$$J_{k,\mu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \mu + 1)} \frac{(-1)^n (z/2)^n}{(n!)^2}$$

(1.8)

$(k \in \mathbb{R}; \gamma, \lambda, \mu \in \mathbb{C}; \min\{\Re(\lambda), \Re(\mu)\} > 0)$,

where $(\gamma)_{n,k}$ is the $k$-Pochhammer symbol defined as follows (see [12]):

$$(\gamma)_{n,k} = \gamma(\gamma + k)(\gamma + 2k) \cdots (\gamma + (n-1)k) \quad (\gamma \in \mathbb{C}; n \in \mathbb{N})$$

(1.9)

and $\Gamma_k(z)$ is the $k$-Gamma function defined by (see [12]):

$$\Gamma_k(z) = \int_0^{\infty} e^{-\frac{t}{k}} t^{z-1} dt \quad (k \in \mathbb{R}^+; \Re(z) > 0).$$

(1.10)

It is easy to see that the $\Gamma_k(z)$ in (1.10) with $k = 1$ reduces to the classical Gamma function $\Gamma(z)$. Also the $\Gamma_k$ satisfies the following relations (see [4]):

$$\Gamma_k(z + k) = z\Gamma_k(z);$$

(1.11)

$$\Gamma_k(z) = k^{\frac{z}{k}} \Gamma\left(\frac{z}{k}\right);$$

(1.12)

$$(\gamma)_{n,k} = \frac{\Gamma_k(\gamma + nk)}{\Gamma_k(\gamma)}.$$  

(1.13)

Nisar and Saiful [25] introduced and defined the following new generalization of $k$-Bessel function which is called generalized modified $k$-Bessel function of the first kind:

$$J_{b,k,\mu}^{c,\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k}}{\Gamma_k(\lambda n + \mu + \frac{b+1}{2})} \frac{(z/2)^{\mu+2n}}{(n!)^2}$$

(1.14)

$(k \in \mathbb{R}^+; \gamma, \lambda, \mu, b, c \in \mathbb{C}; \min\{\Re(\lambda), \Re(\mu)\} > 0)$.

Fractional calculus has found many demonstrated applications in extensive fields of engineering and science such as electromagnetics, fluid mechanics, electrochemistry, biological population models, optics, signal processing and control theory. It has been used to model physical and engineering processes that are found to be best described by fractional differential equations. The fractional derivative models are used for modeling of those systems that require accurate modeling of damping. Recent studies showed that the solutions of fractional order differential equations can model real-life situations better, particularly, in reaction-diffusion type problems. Due to its potential applicability to a
wide variety of problems, fractional calculus has been developed for applications in a wide range of mathematics, physics and engineering (see, e.g., [3, 7, 18, 27, 29]).

During the last several decades, fractional kinetic equations of different forms have been widely used in describing and solving several important problems of physics and astrophysics. Many researchers have investigated and derived the solutions of the fractional kinetic equations associated with various types of special functions (see, e.g., [5, 6, 9, 10, 14, 16, 17, 26, 30, 31, 32, 33]). Motivated by a large number of the above-cited investigations on the fractional kinetic equation, in this sequel, we propose to investigate solution of a certain generalized fractional kinetic equation associated with the generalized modified $k$-Bessel function of the first kind. It is also pointed out that the main results presented here can include, as their special cases, solutions of many fractional kinetic equations which are (presumably) new and known.

Consider an arbitrary reaction characterized by a time-dependent quantity $N = N(t)$. It is possible to calculate the rate of change $\frac{dN}{dt}$ to be a balance between the destruction rate $\vartheta$ and the production rate $p$ of $N$, that is, $dN/dt = -\vartheta + p$. In general, through feedback or other interaction mechanism, destruction and production depend on the quantity $N$ itself, that is,

$\vartheta = \vartheta(N) \quad \text{and} \quad p = p(N)$.

This dependence is complicated, since the destruction or the production at a time $t$ depends not only on $N(t)$, but also on the past history $N(\eta)$ ($\eta < t$) of the variable $N$. This may be formally represented by the following equation (see [17]):

$$\frac{dN}{dt} = -\vartheta \left(N_i\right) + p \left(N_i\right), \quad (1.15)$$

where $N_i$ denotes the function defined by

$$N_i(t^*) = N(t - t^*) \quad (t^* > 0).$$

Haubold and Mathai [17] studied a special case of the equation (1.15) in the following form:

$$\frac{dN_i}{dt} = -c_i N_i(t) \quad (1.16)$$

with the initial condition that $N_i(t = 0) = N_0$ is the number density of species $i$ at time $t = 0$ and the constant $c_i > 0$. This is known as a standard kinetic equation. The solution of the equation (1.16) is easily seen to be given by

$$N_i(t) = N_0 e^{-c_i t}. \quad (1.17)$$

Integration gives an alternative form of the equation (1.16) as follows:

$$N(t) - N_0 = c \cdot 0D_t^{-1}N(t), \quad (1.18)$$
where $0D_t^{-1}$ is the standard integral operator and $c$ is a constant.

The fractional generalization of the equation (1.18) is given as in the following form (see [17]):

\[ N(t) - N_0 = c' 0D_t^{-\nu} N(t), \]

where $0D_t^{-\nu}$ is the familiar Riemann-Liouville fractional integral operator (see, e.g., [18] and [21]) defined by

\[ 0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u) \, du \quad (\Re(\nu) > 0). \]

Suppose that $f(t)$ is a real- or complex-valued function of the (time) variable $t > 0$ and $s$ is a real or complex parameter. The Laplace transform of the function $f(t)$ is defined by

\[ F(p) = \mathcal{L} \{ f(t) : p \} = \int_0^\infty e^{-pt} f(t) \, dt = \lim_{\tau \to \infty} \int_0^\tau e^{-pt} f(t) \, dt, \]

whenever the limit exits (as a finite number).

Since Mittag-Leffler introduced the so-called Mittag-Leffler function $E_\alpha(z)$ (see [23]):

\[ E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (z, \alpha \in \mathbb{C}; |z| < 0, \Re(\alpha) \geq 0), \]

a large number of its extensions and generalizations have been presented. The following rather simpler extension is recalled (see [38]):

\[ E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (z, \alpha, \beta \in \mathbb{C}; |z| < 0, \min\{\Re(\alpha), \Re(\beta)\} > 0). \]

2. SOLUTION OF GENERALIZED FRACTIONAL KINETIC EQUATIONS

We investigate the solution of the generalized fractional kinetic equations involving the generalized modified $k$-Bessel function of the first kind (1.14).

**Theorem 1.** Let $e, t, k, \nu \in \mathbb{R}^+$. Also let $b, c, \gamma, \lambda, \mu \in \mathbb{C}$ with $\Re(\mu) > -1$. Then the solution of the following generalized fractional kinetic equation:

\[ N(t) - N_0 J_{b,k,\mu}^{c,\gamma,\lambda}(t) = -e^{\nu} 0D_t^{-\nu} N(t) \]

is given by

\[ N(t) = N_0 \sum_{n=0}^{\infty} \frac{e^n(\gamma)_{n,k} \Gamma(\mu + 2n + 1)}{(n!)^2 \Gamma_k(\lambda n + \mu + b + 1)} \left( \frac{t}{2} \right)^{\mu + 2n} E_{\nu,2n+\mu+1}(-e^{\nu} t^\nu), \]
where $E_{\nu,2n+\mu+1}(\cdot)$ is the generalized Mittag-Leffler function in (1.23).

**Proof.** We begin by recalling the Laplace transform of the Riemann-Liouville fractional integral operator (see, e.g., [15, 36]):

$$
\mathcal{L}\{\alpha D_t^{-\nu} f(t) ; p\} = p^{-\nu} F(p),
$$

(2.3)

where, just as in the definition (1.21),

$$
F(p) = \mathcal{L}\{f(t) ; p\}.
$$

Taking the Laplace transform of both sides of (2.1) and using (1.14) and (2.3), we obtain

$$
\mathcal{N}(p) = \mathcal{N}_0 \int_0^\infty e^{-pt} \sum_{n=0}^\infty \frac{c_n (\gamma)_{n,k} (t/2)^{\mu+2n}}{(n!)^2} \Gamma_k (\lambda n + \mu + \frac{b+1}{2}) \Gamma(\mu + 2n + 1) dt - e^\nu p^{-\nu} \mathcal{N}(p),
$$

(2.4)

where

$$
\mathcal{N}(p) = \mathcal{L}\{\mathcal{N}(t) ; p\}.
$$

Integrating the integral in (2.4) term by term, which is guaranteed under the given restrictions, and using (1.2), we get: For $\Re(p) > 0$,

$$
(1 + (e/p)^\nu) \mathcal{N}(p) = \mathcal{N}_0 \sum_{n=0}^\infty \frac{c_n (\gamma)_{n,k} (t/2)^{\mu+2n}}{(n!)^2} \Gamma_k (\lambda n + \mu + \frac{b+1}{2}) \Gamma(\mu + 2n + 1) \int_0^\infty e^{-pt} t^{\mu+2n} dt
$$

$$
= \mathcal{N}_0 \sum_{n=0}^\infty \frac{c_n (\gamma)_{n,k}}{(n!)^2} \Gamma_k (\lambda n + \mu + \frac{b+1}{2}) \Gamma(\mu + 2n + 1) \frac{2^{-\mu-2n}}{p^{\mu+2n+1}} \sum_{r=0}^\infty (-1)^r (e/p)^{\nu r}.
$$

Expanding $(1 + (e/p)^\nu)^{-1}$ as a geometric series, we have: For $e < |p|$,

$$
\mathcal{N}(p) = \mathcal{N}_0 \sum_{n=0}^\infty \frac{c_n (\gamma)_{n,k}}{(n!)^2} \Gamma_k (\lambda n + \mu + \frac{b+1}{2}) \Gamma(\mu + 2n + 1) \frac{2^{-\mu-2n}}{p^{\mu+2n+1}} \sum_{r=0}^\infty (-1)^r (e/p)^{\nu r}.
$$

Taking the inverse Laplace transform and using the following known formula:

$$
\mathcal{L}^{-1}\{p^{-\alpha}\} = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \quad (\Re(\alpha) > 0),
$$

(2.5)

we obtain

$$
\mathcal{N}(t) = \mathcal{L}^{-1}\{\mathcal{N}(p)\}
$$

$$
= \mathcal{N}_0 \sum_{n=0}^\infty \frac{c_n (\gamma)_{n,k}}{(n!)^2} \Gamma(\mu + 2n + 1) \Gamma_k (\lambda n + \mu + \frac{b+1}{2}) \left(\frac{t}{2}\right)^{\mu+2n} \sum_{r=0}^\infty (-1)^r (et)^{\nu r} \Gamma(\nu r + \mu + 2n + 1),
$$

which, upon using (1.22), yields the desired result (2.2). \qed
Since the generalized modified $k$-Bessel function of the first kind (1.14) includes many known functions as its special cases (see Section 1), so does the result in Theorem 1. Here, we give just one example. Setting $k = \gamma = \lambda = 1$ and replacing $c$ by $-c$ in the result in Theorem 1 with the notations in Section 1, we obtain a known result asserted by the following corollary (see [13, Eq. (18)]).

**Corollary 1.** Let $e, t, \nu \in \mathbb{R}^+$. Also let $b, c, \mu \in \mathbb{C}$ with $\Re(\mu) > -1$. Then the solution of the following generalized fractional kinetic equation involving the generalized Bessel function of the first kind (1.1):

$$N(t) - N_0 w_{b,c} (t) = -e^{\nu} \, _0D_t^{-\nu} N(t)$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(\mu + 2n + 1)}{n! \Gamma(n + \mu + 1)} \left( \frac{t}{2} \right)^{\mu+2n} E_{\nu,2n+\mu+1} (-e^{\nu} t^\nu),$$

where $E_{\nu,2n+\mu+1} (\cdot)$ is the generalized Mittag-Leffler function in (1.23).

We also provide two more general results than that in Theorem 1, which are given in Theorems 2 and 3. They can be proved in parallel with the proof of Theorem 1. So the details of their proofs are omitted.

**Theorem 2.** Let $e, t, k, \nu \in \mathbb{R}^+$. Also let $b, c, \gamma, \lambda, \mu \in \mathbb{C}$ with $\Re(\mu) > -1$. Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0 J_{b,k,\mu}^{\nu,\gamma,\lambda} (e^{\nu} t^\nu) = -e^{\nu} \, _0D_t^{-\nu} N(t)$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k} \Gamma(\mu + 2n + 1)}{(n!)^2 \Gamma_k (\lambda n + \mu + \frac{n+1}{2})} \left( \frac{e^{\nu} t^\nu}{2} \right)^{\mu+2n} E_{\nu,2n+\mu+1} (-e^{\nu} t^\nu),$$

where $E_{\nu,2n+\mu+1} (\cdot)$ is the generalized Mittag-Leffler function in (1.23).

**Theorem 3.** Let $a, e, t, k, \nu \in \mathbb{R}^+$. Also let $b, c, \gamma, \lambda, \mu \in \mathbb{C}$ with $\Re(\mu) > -1$. Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0 J_{b,k,\mu}^{\nu,\gamma,\lambda} (e^{\nu} t^\nu) = -a^{\nu} \, _0D_t^{-\nu} N(t)$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k} \Gamma(\mu + 2n + 1)}{(n!)^2 \Gamma_k (\lambda n + \mu + \frac{n+1}{2})} \left( \frac{e^{\nu} t^\nu}{2} \right)^{\mu+2n} E_{\nu,2n+\mu+1} (-a^{\nu} t^\nu),$$

where $E_{\nu,2n+\mu+1} (\cdot)$ is the generalized Mittag-Leffler function in (1.23).
3. Concluding Remarks

The case $a = e$ in Theorem 3 reduces to the result in Theorem 2. The main results given in Section 2 are general enough to be specialized to yield many new and known solutions of the corresponding generalized fractional kinetic equations, as in Corollary 1.

We conclude this paper by illustrating such a special case of Theorem 3 as in the following corollary.

**Corollary 2.** Let $a, e, t, \nu \in \mathbb{R}^+$. Also let $\mu \in \mathbb{C}$ with $\Re(\mu) > -3/2$. Then the solution of the following generalized fractional kinetic equation involving the spherical Bessel function of the first kind in (1.5):

$$N(t) - N_0 j_\mu(e^{\nu}t^\nu) = -a^{\nu} \frac{D^{-\nu}t}{} N(t)$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\mu + 2n + 1)}{\Gamma(n + \mu + 3/2)} \left( \frac{e^{\nu}t^\nu}{2} \right)^{n+2n} E_{\nu,2n+\mu+1}(-a^{\nu}t^\nu),$$

where $E_{\nu,2n+\mu+1}(-\cdot)$ is the generalized Mittag-Leffler function in (1.23).

*Proof.* Setting $b = 2$, $c = -1$, and $k = \lambda = \gamma = 1$ in Theorem 3 and considering (1.5) is easily seen to yield the desired result.

\[ \square \]

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