Tree-Partitions with Bounded Degree Trees*

Marc Distel‡ David R. Wood‡

Abstract

A tree-partition of a graph $G$ is a partition of $V(G)$ such that identifying the vertices in each part gives a tree. It is known that every graph with treewidth $k$ and maximum degree $\Delta$ has a tree-partition with parts of size $O(k\Delta)$. We prove the same result with the extra property that the underlying tree has maximum degree $O(\Delta)$.

1 Introduction

For a graph $G$ and a tree $T$, a $T$-partition of $G$ is a partition $(V_x: x \in V(T))$ of $V(G)$ indexed by the nodes of $T$, such that for every edge $vw$ of $G$, if $v \in V_x$ and $w \in V_y$, then $x = y$ or $xy \in E(T)$. The width of a $T$-partition is $\max\{|V_x|: x \in V(T)\}$. The tree-partition-width of a graph $G$ is the minimum width of a tree-partition of $G$.

Tree-partitions were independently introduced by Seese [31] and Halin [24], and have since been widely investigated [6–8, 15, 16, 21, 32, 33]. Applications of tree-partitions include graph drawing [12, 14, 19, 20, 34], graphs of linear growth [11], nonrepetitive graph colouring [2], clustered graph colouring [1, 27], monadic second-order logic [26], network emulations [3, 4, 9, 22], statistical learning theory [35], and the edge-Erdős-Pósa property [13, 23, 28]. Tree-partitions are also related to graph product structure theory since a graph $G$ has a $T$-partition of width at most $k$ if and only if $G$ is isomorphic to a subgraph of $T \boxtimes K_k$ for some tree $T$; see [10, 17, 18] for example.

‡School of Mathematics, Monash University, Melbourne, Australia ({marc.distel,david.wood}@monash.edu). Research of D.W. supported by the Australian Research Council. Research of M.D. supported by an Australian Government Research Training Program Scholarship.

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†Tree-partition-width has also been called strong treewidth [7, 31].
Bounded tree-partition-width implies bounded treewidth\(^2\), as noted by Seese [31]. In particular, for every graph \(G\),
\[
tw(G) \leq 2 \text{tpw}(G) - 1.
\]
Of course, \(tw(T) = \text{tpw}(T) = 1\) for every tree \(T\). But in general, \(\text{tpw}(G)\) can be much larger than \(\text{tw}(G)\). For example, fan graphs on \(n\) vertices have treewidth 2 and tree-partition-width \(\Omega(\sqrt{n})\). On the other hand, the referee of [15] showed that if the maximum degree and treewidth are both bounded, then so is the tree-partition-width, which is one of the most useful results about tree-partitions. A graph \(G\) is trivial if \(E(G) = \emptyset\). Let \(\Delta(G)\) be the maximum degree of \(G\).

Theorem 1 ([15]). For any non-trivial graph \(G\),
\[
\text{tpw}(G) \leq 24(tw(G) + 1)\Delta(G).
\]

Theorem 1 is stated in [15] with “\(tw(G)\)” instead of “\(tw(G) + 1\)”, but a close inspection of the proof shows that “\(tw(G) + 1\)” is needed. Wood [33] showed that Theorem 1 is best possible up to the multiplicative constant, and also improved the constant 24 to \(9 + 6\sqrt{2} \approx 17.48\).

This paper considers the maximum degree of \(T\) in a \(T\)-partition. Consider a tree-partition \((B_x : x \in V(T))\) of a graph \(G\) with width \(k\). For each node \(x \in V(T)\), there are at most \(\sum_{v \in B_x} \deg(v)\) edges between \(B_x\) and \(G - B_x\). Thus we may assume that \(\deg_T(x) \leq |B_x|\Delta(G) \leq k\Delta(G)\), otherwise delete an ‘unused’ edge of \(T\) and add an edge to \(T\) between leaf vertices of the resulting component subtrees. It follows that if \(\text{tpw}(G) \leq k\) then \(G\) has a \(T\)-partition of width at most \(k\) for some tree \(T\) with maximum degree at most \(\max\{k\Delta(G), 2\}\). By Theorem 1, every graph \(G\) has a \(T\)-partition of width at most \(24(tw(G) + 1)\Delta(G)\) for some tree \(T\) with maximum degree at most \(24(tw(G) + 1)\Delta(G)^2\). This fact has been used in several applications of Theorem 1 (see [12, 20] for example). The following theorem improves this upper bound on \(\Delta(T)\). Indeed, \(\Delta(T)\) is independent of \(\text{tw}(G)\).

Theorem 2. Every non-trivial graph \(G\) has a \(T\)-partition of width at most
\[
18(tw(G) + 1)\Delta(G)
\]
for some tree \(T\) with \(\Delta(T) \leq 6\Delta(G)\).

\(^2\)A tree-decomposition of a graph \(G\) is a collection \((B_x \subseteq V(G) : x \in V(T))\) of subsets of \(V(G)\) (called bags) indexed by the nodes of a tree \(T\), such that: (a) for every edge \(uv \in E(G)\), some bag \(B_x\) contains both \(u\) and \(v\); and (b) for every vertex \(v \in V(G)\), the set \(\{x \in V(T) : v \in B_x\}\) induces a non-empty subtree of \(T\). The width of a tree-decomposition is the size of the largest bag, minus 1. The treewidth of a graph \(G\), denoted by \(\text{tw}(G)\), is the minimum width of a tree-decomposition of \(G\). Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [5, 25, 29] for surveys.
Theorem 2 enables a \( \text{tw}(G)\Delta(G)^2 \) term to be replaced by a \( \Delta(G) \) term in various results [12, 20].

As mentioned above, Wood [33] improved the constant 24 to \( 9 + 6\sqrt{2} \) in Theorem 1. By tweaking the constants in the proof of Theorem 2, we match this constant with a small increase in the bound on \( \Delta(T) \); see Appendix A. We choose to present the proof with integer coefficients for ease of understanding.

Our final result shows that the linear upper bound on \( \Delta(T) \) in Theorem 2 is best possible even for trees.

**Proposition 3.** For any integer \( \Delta \geq 3 \) there exist \( \alpha > 0 \) such that there are infinitely many trees \( X \) with maximum degree \( \Delta \) such that for every tree \( T \) with maximum degree less than \( \Delta \), every \( T \)-partition of \( X \) has width at least \( |V(X)|^\alpha \). Moreover, if \( \Delta = 3 \) then \( \alpha \) can be taken to be arbitrarily close to 1.

## 2 Proofs

The proof of Theorem 2 is identical to the proof of Theorem 1, except that we pay attention to \( \Delta(T) \). Let \( \mathbb{N} = \{1, 2, \ldots\} \).

**Lemma 4.** Fix \( k, d \in \mathbb{N} \). Let \( G \) be a graph with \( \text{tw}(G) \leq k - 1 \) and \( \Delta(G) \leq d \). Then \( G \) has a tree-partition \( \langle B_x : x \in V(T) \rangle \) of width at most \( 18kd \) such that \( \Delta(T) \leq 6d \). Moreover, for any set \( S \subseteq V(G) \) with \( 4k \leq |S| \leq 12kd \), there exists a tree-partition \( \langle B_x : x \in V(T) \rangle \) of \( G \) with width at most \( 18kd \), such that \( \Delta(T) \leq 6d \) and there exists \( z \in V(T) \) such that:

- \( S \subseteq B_z \),
- \( |B_z| \leq \frac{3}{2}|S| - 2k \),
- \( \deg_T(z) \leq \frac{|S|}{2k} - 1 \).

**Proof.** We proceed by induction on \( |V(G)| \).

**Case 1.** \( |V(G)| < 4k \): Then \( S \) is not specified. Let \( T \) be the 1-vertex tree with \( V(T) = \{x\} \), and let \( B_x := V(G) \). Then \( \langle B_x : x \in V(T) \rangle \) is the desired tree-partition, since \( |B_x| = |V(G)| < 4k \leq 18kd \) and \( \Delta(T) = 0 \leq 6d \).

Now assume that \( |V(G)| \geq 4k \). If \( S \) is not specified, then let \( S \) be any set of \( 4k \) vertices in \( G \).

**Case 2.** \( |V(G - S)| \leq 18kd \): Let \( T \) be the 2-vertex tree with \( V(T) = \{y, z\} \) and \( E(T) = \{yz\} \). Note that \( \Delta(T) = 1 \leq 6d \) and \( \deg_T(z) = 1 \leq \frac{|S|}{2k} - 1 \). Let \( B_z := S \) and \( B_y := V(G - S) \). Thus \( |B_z| = |S| \leq \frac{3}{2}|S| - 2k \leq 18kd \) and
We have shown that \(|B_y| \leq |V(G - S)| \leq 18kd\). Hence \((B_x : x \in V(T))\) is the desired tree-partition of \(G\). Now assume that \(|V(G - S)| \geq 18kd\).

**Case 3.** \(4k \leq |S| \leq 12k\): Let \(S' := \bigcup\{N_G(v) \setminus S : v \in S\}\). Thus \(|S'| \leq d|S| \leq 12kd\). If \(|S'| < 4k\) then add \(4k - |S'|\) vertices from \(V(G - S - S')\) to \(S'\), so that \(|S'| = 4k\). This is well-defined since \(|V(G - S)| \geq 18kd \geq 4k\), implying \(|V(G - S - S')| \geq 4k - |S'|\). By induction, there exists a tree-partition \((B_x : x \in V(T'))\) of \(G - S\) with width at most \(18kd\), such that \(\Delta(T') \leq 6d\) and there exists \(z' \in V(T')\) such that:

- \(S' \subseteq B_{z'}\),
- \(|B_{z'}| \leq \frac{3}{2} |S'| - 2k \leq 18kd - 2k\),
- \(\deg_{T'}(z') \leq \frac{|S|}{2k} - 1 \leq 6d - 1\).

Let \(T\) be the tree obtained from \(T'\) by adding one new node \(z\) adjacent to \(z'\). Let \(B_z := S\). So \((B_x : x \in V(T))\) is a tree-partition of \(G\) with width at most \(\max\{18kd, |S|\}\) \(\leq 18kd\). By construction, \(\deg_T(z) = 1 \leq \frac{|S|}{2k} - 1\) and \(\deg_{T'}(z') = \deg_T(z') + 1 \leq (6d - 1) + 1 = 6d\). Every other vertex in \(T\) has the same degree as in \(T'\). Hence \(\Delta(T) \leq 6d\), as desired. Finally, \(S = B_z\) and \(|B_z| = |S| \leq \frac{3}{2}|S| - 2k\).

**Case 4.** \(12k \leq |S| \leq 12kd\): By the separator lemma of Robertson and Seymour [30, (2.6)], there are induced subgraphs \(G_1\) and \(G_2\) of \(G\) with \(G_1 \cup G_2 = G\) and \(|V(G_1 \cap G_2)| \leq k\), where \(|S \cap V(G_i)| \leq \frac{2}{3}|S|\) for each \(i \in \{1, 2\}\). Let \(S_i := (S \cap V(G_i)) \cup V(G_1 \cap G_2)\) for each \(i \in \{1, 2\}\).

We now bound \(|S_i|\). For a lower bound, since \(|S \cap V(G_1)| \leq \frac{2}{3}|S|\), we have \(|S_2| \geq |S \setminus V(G_1)| \geq \frac{1}{3}|S| \geq 4k\). By symmetry, \(|S_1| \geq 4k\). For an upper bound, \(|S_i| \leq \frac{2}{3}|S| + k \leq 8kd + k \leq 12kd\). Also note that \(|S_1| + |S_2| \leq |S| + 2k\).

We have shown that \(4k \leq |S_i| \leq 12kd\) for each \(i \in \{1, 2\}\). Thus we may apply induction to \(G_i\) with \(S_i\) the specified set. Hence there exists a tree-partition \((B_{z_i}^i : x \in V(T_i))\) of \(G_i\) with width at most \(18kd\), such that \(\Delta(T_i) \leq 6d\) and there exists \(z_i \in V(T_i)\) such that:

- \(S_i \subseteq B_{z_i}\),
- \(|B_{z_i}| \leq \frac{3}{2} |S_i| - 2k\),
- \(\deg_{T_i}(z_i) \leq \frac{|S_i|}{2k} - 1\).

Let \(T\) be the tree obtained from the disjoint union of \(T_1\) and \(T_2\) by merging \(z_1\) and \(z_2\) into a vertex \(z\). Let \(B_z := B_{z_1}^1 \cup B_{z_2}^2\). Let \(B_x := B_x^i\) for each \(x \in V(T_i) \setminus \{z_i\}\). Since \(G = G_1 \cup G_2\) and \(V(G_1 \cap G_2) \subseteq B_{z_1}^1 \cap B_{z_2}^2 \subseteq B_z\), we have that \((B_x : x \in V(T))\) is a tree-partition of \(G\). By construction, \(S \subseteq B_z\) and since \(V(G_1 \cap G_2) \subseteq B_{z_i}^i\) for each \(i\),

\(|B_z| \leq |B_{z_1}^1| + |B_{z_2}^2| - |V(G_1 \cap G_2)|\)
\[ \leq \left( \frac{3}{2}|S_1| - 2k \right) + \left( \frac{3}{2}|S_2| - 2k \right) - |V(G_1 \cap G_2)| \]
\[ = \frac{3}{2}(|S_1| + |S_2|) - 4k - |V(G_1 \cap G_2)| \]
\[ \leq \frac{3}{2}(|S| + 2|V(G_1 \cap G_2)|) - 4k - |V(G_1 \cap G_2)| \]
\[ \leq \frac{3}{2}|S| + 2|V(G_1 \cap G_2)| - 4k \]
\[ \leq \frac{3}{2}|S| - 2k \]
\[ \leq 18kd. \]

Every other part has the same size as in the tree-partition of \( G_1 \) or \( G_2 \). So this tree-partition of \( G \) has width at most \( 18kd \). Note that

\[ \deg_T(z) = \deg_{T_1}(z_1) + \deg_{T_2}(z_2) \leq \frac{|S_1|}{2k} - 1 + \frac{|S_2|}{2k} - 1 \]
\[ = \frac{|S_1| + |S_2|}{2k} - 2 \]
\[ \leq \frac{|S| + 2k}{2k} - 2 \]
\[ = \frac{|S|}{2k} - 1 \]
\[ < 6d. \]

Every other node of \( T \) has the same degree as in \( T_1 \) or \( T_2 \). Thus \( \Delta(T) \leq 6d \). This completes the proof.

We now prove the lower bound. For \( \Delta, d \in \mathbb{N} \) with \( \Delta \geq 2 \), let \( X_{\Delta,d} \) be the tree rooted at a vertex \( r \) such that every leaf is at distance \( d \) from \( r \) and every non-leaf vertex has degree \( \Delta \). Observe that \( X_{\Delta,d} \) has the maximum number of vertices in a tree with maximum degree \( \Delta \) and radius \( d \), where

\[ |V(X_{\Delta,d})| = 1 + \Delta \sum_{i=0}^{d-1} (\Delta - 1)^i. \]

Note that \( |V(X_{2,d})| = 2d + 1 \), and if \( \Delta \geq 3 \) then

\[ (\Delta - 1)^d \leq |V(X_{\Delta,d})| = 1 + \frac{\Delta}{\Delta-2}((\Delta - 1)^d - 1) \leq 3(\Delta - 1)^d. \]

**Proposition 3.** For any integer \( \Delta \geq 3 \) there exist \( \alpha > 0 \) such that there are infinitely many trees \( X \) with maximum degree \( \Delta \) such that for every tree \( T \) with maximum degree less than \( \Delta \), every \( T \)-partition of \( X \) has width at least \( |V(X)|^\alpha \). Moreover, if \( \Delta = 3 \) then \( \alpha \) can be taken to be arbitrarily close to 1.

**Proof.** First suppose that \( \Delta \geq 4 \). Let \( d_0 \in \mathbb{N} \) be sufficiently large so that \( \left( \frac{\Delta - 1}{\Delta - 2} \right)^{d_0} > 3 \). Let \( \alpha := 1 - \log_{\Delta-1}(3^{1/d_0}(\Delta - 2)) \), which is positive by the choice of \( d_0 \). Let \( d \in \mathbb{N} \) with \( d \geq d_0 \). It follows that \( (\Delta - 1)^{(1-\alpha)d} \geq 3(\Delta - 2)^d \). Consider any tree-partition \( (B_u : u \in V(T)) \) of \( X_{\Delta,d} \), where \( T \) is any tree with maximum degree at most \( \Delta - 1 \). Let \( z \) be the vertex of \( T \) such that the root \( r \in B_z \). Since adjacent
vertices in $X_{\Delta,d}$ belong to adjacent parts or the same part in $T$, every vertex in $T$ is at distance at most $d$ from $z$. Thus $T$ has radius at most $d$, and

$$|V(T)| \leq |V(X_{\Delta-1,d})| \leq 3(\Delta - 2)^d \leq (\Delta - 1)^{(1-\alpha)d} \leq |V(X_{\Delta,d})|^{1-\alpha}.$$  

By the pigeon-hole principle, there is a vertex $u \in V(T)$ such that $|B_u| \geq \frac{|V(X_{\Delta,d})|}{|V(T)|} \geq |V(X_{\Delta,d})|^\alpha$.

Now assume that $\Delta = 3$. Let $\alpha \in (0,1)$, let $d_0 \in \mathbb{N}$ be sufficiently large so that $2d_0 + 1 \leq 2^{(1-\alpha)d_0}$, and let $d \geq d_0$. So $2d + 1 \leq 2^{(1-\alpha)d}$. Consider any tree-partition $(B_u : u \in V(T))$ of $X_{3,d}$, where $T$ is any tree with maximum degree at most 2. By the argument above, $T$ has radius at most $d$, implying

$$|V(T)| \leq |V(X_{2,d})| = 2d + 1 \leq 2^{(1-\alpha)d} \leq |V(X_{3,d})|^{1-\alpha}.$$  

Again, there is a vertex $u \in V(T)$ such that $|B_u| \geq \frac{|V(X_{3,d})|}{|V(T)|} \geq |V(X_{3,d})|^\alpha$.□

References

[1] Noga Alon, Guoli Ding, Bogdan Oporowski, and Dirk Vertigan. Partitioning into graphs with only small components. J. Combin. Theory Ser. B, 87(2):231–243, 2003.
[2] János Barát and David R. Wood. Notes on nonrepetitive graph colouring. Electron. J. Combin., 15:R99, 2008.
[3] Hans L. Bodlaender. The complexity of finding uniform emulations on fixed graphs. Inform. Process. Lett., 29(3):137–141, 1988.
[4] Hans L. Bodlaender. The complexity of finding uniform emulations on paths and ring networks. Inform. and Comput., 86(1):87–106, 1990.
[5] Hans L. Bodlaender. A partial $k$–arboretum of graphs with bounded treewidth. Theoret. Comput. Sci., 209(1-2):1–45, 1998.
[6] Hans L. Bodlaender. A note on domino treewidth. Discrete Math. Theor. Comput. Sci., 3(4):141–150, 1999.
[7] Hans L. Bodlaender and Joost Engelfriet. Domino treewidth. J. Algorithms, 24(1):94–123, 1997.
[8] Hans L. Bodlaender, Carla Groenland, and Hugo Jacob. On the parameterized complexity of computing tree-partitions. 2022, arXiv:2206.11832.
[9] Hans L. Bodlaender and Jan van Leeuwen. Simulation of large networks on smaller networks. Inform. and Control, 71(3):143–180, 1986.
[10] Rutger Campbell, Katie Clinch, Marc Distel, J. Pascal Gollin, Kevin Hendrey, Robert Hickingbotham, Tony Huynh, Freddie Illingworth, Youri Tamitegama, Jane Tan, and David R. Wood. Product structure of graph classes with bounded treewidth. 2022, arXiv:2206.02395.
[11] Rutger Campbell, Marc Distel, J. Pascal Gollin, Daniel J. Harvey, Kevin Hendrey, Robert Hickingbotham, Bojan Mohar, and David R. Wood. Graphs of linear growth have bounded treewidth. 2022, arXiv:tba.
[12] Paz Carmi, Vida Dujmović, Pat Morin, and David R. Wood. Distinct distances in graph drawings. Electron. J. Combin., 15:R107, 2008.
[13] Dimitris Chatzidimitriou, Jean-Florent Raymond, Ignasi Sau, and Dimitrios M. Thilikos. An $O(\log OPT)$-approximation for covering and packing minor models of $\theta_r$. *Algorithmica*, 80(4):1330–1356, 2018.

[14] Emilio Di Giacomo, Giuseppe Liotta, and Henk Meijer. Computing straight-line 3D grid drawings of graphs in linear volume. *Comput. Geom. Theory Appl.*, 32(1):26–58, 2005.

[15] Guoli Ding and Bogdan Oporowski. Some results on tree decomposition of graphs. *J. Graph Theory*, 20(4):481–499, 1995.

[16] Guoli Ding and Bogdan Oporowski. On tree-partitions of graphs. *Discrete Math.*, 149(1–3):45–58, 1996.

[17] Vida Dujmović, Louis Esperet, Pat Morin, Bartosz Walczak, and David R. Wood. Clustered 3-colouring graphs of bounded degree. *Combin. Probab. Comput.*, 31(1):123–135, 2022.

[18] Vida Dujmović, Gwenael Joret, Piotr Micek, Pat Morin, and David R. Wood. Bad news for product structure of bounded-degree graphs. 2022, arXiv:19a.

[19] Vida Dujmović, Pat Morin, and David R. Wood. Layout of graphs with bounded tree-width. *SIAM J. Comput.*, 34(3):553–579, 2005.

[20] Vida Dujmović, Matthew Suderman, and David R. Wood. Graph drawings with few slopes. *Comput. Geom. Theory Appl.*, 38:181–193, 2007.

[21] Anders Edenbrandt. Quotient tree partitioning of undirected graphs. *BIT*, 26(2):148–155, 1986.

[22] John P. Fishburn and Raphael A. Finkel. Quotient networks. *IEEE Trans. Comput.*, C-31(4):288–295, 1982.

[23] Archontia C. Giannopouloou, O-Joung Kwon, Jean-Florent Raymond, and Dimitrios M. Thilikos. Packing and covering immersion models of planar subcubic graphs. In Pinar Heggernes, ed., *Proc. 42nd Int’l Workshop on Graph-Theoretic Concepts in Computer Science* (WG 2016), vol. 9941 of Lecture Notes in Comput. Sci., pp. 74–84, 2016.

[24] Rudolf Halin. Tree-partitions of infinite graphs. *Discrete Math.*, 97:203–217, 1991.

[25] Daniel J. Harvey and David R. Wood. Parameters tied to treewidth. *J. Graph Theory*, 84(4):364–385, 2017.

[26] Dietrich Kuske and Markus Lohrey. Logical aspects of Cayley-graphs: the group case. *Ann. Pure Appl. Logic*, 131(1–3):263–286, 2005.

[27] Chun-Hung Liu and Sang-il Oum. Partitioning $H$-minor free graphs into three subgraphs with no large components. *J. Combin. Theory Ser. B*, 128:114–133, 2018.

[28] Jean-Florent Raymond and Dimitrios M. Thilikos. Recent techniques and results on the Erdős–Pósa property. *Discrete Appl. Math.*, 231:25–43, 2017.

[29] Bruce A. Reed. Algorithmic aspects of tree width. In *Recent advances in algorithms and combinatorics*, vol. 11, pp. 85–107. Springer, 2003.

[30] Neil Robertson and Paul Seymour. Graph minors. II. Algorithmic aspects of tree-width. *J. Algorithms*, 7(3):309–322, 1986.

[31] Detlef Seese. Tree-partite graphs and the complexity of algorithms. In Lothar Budach, ed., *Proc. Int’l Conf. on Fundamentals of Computation Theory*, vol. 199 of Lecture Notes in Comput. Sci., pp. 412–421. Springer, 1985.

[32] David R. Wood. Vertex partitions of chordal graphs. *J. Graph Theory*, 53(2):167–172, 2006.

[33] David R. Wood. On tree-partition-width. *European J. Combin.*, 30(5):1245–1253, 2009.
A Tweaking the Constants

Consider the following generalisation of Lemma 4.

**Lemma 5.** Fix \( k, d \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \) with \( \alpha > 2 \). Let \( G \) be a graph with \( \text{tw}(G) \leq k - 1 \) and \( \Delta(G) \leq d \). Then \( G \) has a tree-partition \( (B_x : x \in V(T)) \) of width at most

\[
\frac{3\alpha(a-1)}{a-2}kd - \frac{\alpha}{a-2}k
\]

such that \( \Delta(T) \leq \frac{3\alpha}{a-2}d + \frac{\alpha-4}{a-2} \). Moreover, for any set \( S \subseteq V(G) \) with \( \alpha k \leq |S| \leq 3\alpha kd \), there exists a tree-partition \( (B_x : x \in V(T)) \) of \( G \) with width at most \( \frac{3\alpha(a-1)}{a-2}kd - \frac{\alpha}{a-2}k \), such that \( \Delta(T) \leq \frac{3\alpha}{a-2}d + \frac{\alpha-4}{a-2} \) and there exists \( z \in V(T) \) such that:

- \( S \subseteq B_z \),
- \( |B_z| \leq \frac{a-1}{a-2}|S| - \frac{\alpha}{a-2}k \),
- \( \deg_T(z) \leq \frac{1}{(a-2)k}|S| - \frac{2}{a-2} \).

**Proof.** We proceed by induction on \( |V(G)| \).

**Case 1.** \( |V(G)| < \alpha k \): Then \( S \) is not specified. Let \( T \) be the 1-vertex tree with \( V(T) = \{x\} \), and let \( B_x := V(G) \). Then \( (B_x : x \in V(T)) \) is the desired tree-partition, since \( |B_x| = |V(G)| < \alpha k \leq \frac{3\alpha(a-1)}{a-2}kd - \frac{\alpha}{a-2}k \) and \( \Delta(T) = 0 \leq \frac{3\alpha}{a-2}d + \frac{\alpha-4}{a-2} \).

Now assume that \( |V(G)| \geq \alpha k \). If \( S \) is not specified, then let \( S \) be any set of \( \lceil \alpha k \rceil \) vertices in \( G \) (implying \( |S| \leq 3\alpha kd \)).

**Case 2.** \( |V(G - S)| \leq \frac{3\alpha(a-1)}{a-2}kd - \frac{\alpha}{a-2}k \): Let \( T \) be the 2-vertex tree with \( V(T) = \{y, z\} \) and \( E(T) = \{yz\} \). Note that \( \Delta(T) = 1 \leq \frac{3\alpha}{a-2}d + \frac{\alpha-4}{a-2} \) and \( \deg_T(z) = 1 \leq \frac{1}{(a-2)k}|S| - \frac{2}{a-2} \). Let \( B_x := S \) and \( B_y := V(G - S) \). Thus \( |B_x| = |S| \leq \frac{a-1}{a-2}|S| - \frac{\alpha}{a-2}k \leq \frac{3\alpha(a-1)}{a-2}kd - \frac{\alpha}{a-2}k \) and \( |B_y| \leq |V(G - S)| \leq \frac{3\alpha(a-1)}{a-2}kd \). Hence \( (B_x : x \in V(T)) \) is the desired tree-partition of \( G \). Now assume that \( |V(G - S)| \geq \frac{3\alpha(a-1)}{a-2}kd - \frac{\alpha}{a-2}k \).

**Case 3.** \( \alpha k \leq |S| \leq 3\alpha k \): Let \( S' := \bigcup_{v \in S} N_G(v) \setminus S \). Thus \( |S'| \leq d|S| \leq 3\alpha kd \). If \( |S'| < \alpha k \) then add \( \alpha k - |S'| \) vertices from \( V(G - S - S') \) to \( S' \), so that \( |S'| = \alpha k \). This is well-defined since \( |V(G - S)| \geq \frac{3\alpha(a-1)}{a-2}kd - \frac{\alpha}{a-2}k \geq \alpha k \), implying \( |V(G -
We have shown that $|S - S'| \geq \alpha k - |S'|$. By induction, there exists a tree-partition $(B_x : x \in V(T'))$ of $G - S$ with width at most $\frac{3\alpha(\alpha - 1)}{2}kd - \frac{\alpha}{2}k$, such that $\Delta(T') \leq \frac{3\alpha}{2}d + \frac{\alpha - 4}{2}$ and there exists $z' \in V(T')$ such that:

- $S' \subseteq B_{z'}$,
- $|B_{z'}| \leq \frac{\alpha}{2} - \frac{\alpha - 2}{2}k - \frac{3\alpha(\alpha - 1)}{2}kd - \frac{\alpha}{2}k$,
- $\deg_T(z') \leq \frac{1}{(\alpha - 2)k}|S'| - \frac{2}{\alpha - 2} \leq \frac{3\alpha}{2}d - \frac{2}{\alpha - 2}$.

Let $T$ be the tree obtained from $T'$ by adding one new node $z$ adjacent to $z'$. Let $B_z := S$. So $(B_x : x \in V(T))$ is a tree-partition of $G$ with width at most

$$\max\left\{\frac{3\alpha(\alpha - 1)}{2}kd - \frac{\alpha}{2}k, |S|\right\} \leq \max\left\{\frac{3\alpha(\alpha - 1)}{2}kd - \frac{\alpha}{2}k, 3\alpha k\right\} = \frac{3\alpha(\alpha - 1)}{2}kd - \frac{\alpha}{2}k.$$

By construction,

$$\deg_T(z) = 1 \leq \frac{1}{(\alpha - 2)k}|S| - \frac{2}{\alpha - 2} \quad \text{ and } \quad \deg_T(z') = \deg_T(z') + 1 \leq \frac{3\alpha}{2}d - \frac{2}{\alpha - 2} + 1 = \frac{3\alpha}{2}d + \frac{\alpha - 4}{2}.$$

Every other vertex in $T$ has the same degree as in $T'$. Hence $\Delta(T') \leq \frac{3\alpha}{2}d + \frac{\alpha - 4}{2}$, as desired. Finally, $S = B_z$ and $|B_z| = |S| \leq \frac{\alpha - 1}{2} |S| - \frac{\alpha}{2}k$.

Case 4. $3\alpha k \leq |S| \leq 3\alpha kd$: By the separator lemma of Robertson and Seymour [30, (2.6)], there are induced subgraphs $G_1$ and $G_2$ of $G$ with $G_1 \cup G_2 = G$ and $|V(G_1 \cap G_2)| \leq k$, where $|S \cap V(G_i)| \leq \frac{2}{3}|S|$ for each $i \in \{1, 2\}$. Let $S_i := (S \cap V(G_i)) \cup V(G_i \cap G_2)$ for each $i \in \{1, 2\}$.

We now bound $|S_i|$. For a lower bound, since $|S \cap V(G_1)| \leq \frac{2}{3}|S|$, we have $|S_2| \geq |S \setminus V(G_1)| \geq \frac{1}{3}|S| \geq \frac{1}{3}3\alpha k \geq \alpha k$. By symmetry, $|S_1| \geq \alpha k$. For an upper bound, $|S_i| \leq \frac{2}{3}|S| + k \leq 2\alpha kd + k \leq 3\alpha kd$. Also note that $|S_1| + |S_2| \leq |S| + 2|V(G_1 \cap G_2)| \leq |S| + 2k$.

We have shown that $\alpha k \leq |S_i| \leq 3\alpha kd$ for each $i \in \{1, 2\}$. Thus we may apply induction to $G_i$ with $S_i$ the specified set. Hence there exists a tree-partition $(B^i_x : x \in V(T_i))$ of $G_i$ with width at most $\frac{3\alpha(\alpha - 1)}{2}kd - \frac{\alpha}{2}k$, such that $\Delta(T_i) \leq \frac{3\alpha}{2}d + \frac{\alpha - 4}{2}$ and there exists $z_i \in V(T_i)$ such that:

- $S_i \subseteq B_{z^i}$,
- $|B_{z^i}| \leq \frac{\alpha - 1}{2} |S_i| - \frac{\alpha}{2}k$,
- $\deg_{T_i}(z^i) \leq \frac{1}{(\alpha - 2)k}|S_i| - \frac{2}{\alpha - 2}$.

Let $T$ be the tree obtained from the disjoint union of $T_1$ and $T_2$ by merging $z_1$ and $z_2$ into a vertex $z$. Let $B_z := B^1_{z_1} \cup B^2_{z_2}$. Let $B_z := B^i_z$ for each $x \in V(T_i) \setminus \{z_i\}$. Since $G = G_1 \cup G_2$ and $V(G_1 \cap G_2) = B^1_{z_1} \cap B^2_{z_2} \subseteq B_z$, we have that $(B_x : x \in V(T))$ is a tree-partition of $G$. By construction, $S \subseteq B_z$ and since $V(G_1 \cap G_2) \subseteq B^i_{z^i}$ for each $i$,

$$|B_z| \leq |B^1_{z_1}| + |B^2_{z_2}| - |V(G_1 \cap G_2)|$$
\[
\begin{aligned}
&\leq \left(\frac{\alpha-1}{\alpha^2} |S_1| - \frac{\alpha}{\alpha^2} k \right) + \left(\frac{\alpha-1}{\alpha^2} |S_2| - \frac{\alpha}{\alpha^2} k \right) - |V(G_1 \cap G_2)| \\
&= \frac{\alpha-1}{\alpha^2} (|S_1| + |S_2|) - \frac{2\alpha}{\alpha^2} k - |V(G_1 \cap G_2)| \\
&\leq \frac{\alpha-1}{\alpha^2} (|S| + 2|V(G_1 \cap G_2)|) - \frac{2\alpha}{\alpha^2} k - |V(G_1 \cap G_2)| \\
&= \frac{\alpha-1}{\alpha^2} |S| - \frac{2\alpha}{\alpha^2} k + \frac{\alpha}{\alpha^2} |V(G_1 \cap G_2)| \\
&\leq \frac{\alpha-1}{\alpha^2} |S| - \frac{2\alpha}{\alpha^2} k + \frac{\alpha}{\alpha^2} k \\
&= \frac{\alpha-1}{\alpha^2} |S| - \frac{\alpha}{\alpha^2} k \\
&\leq \frac{3\alpha(\alpha-1)}{\alpha-2} kd - \frac{\alpha}{\alpha^2} k.
\end{aligned}
\]

Every other part has the same size as in the tree-partition of \(G_1\) or \(G_2\). So this tree-partition of \(G\) has width at most \(\frac{3\alpha(\alpha-1)}{\alpha-2} kd - \frac{\alpha}{\alpha^2} k\). Note that

\[\deg_T(z) = \deg_{T_1}(z_1) + \deg_{T_2}(z_2)\]
\[\leq \frac{1}{(\alpha-2)k} |S_1| - \frac{2}{\alpha^2} + \frac{1}{(\alpha-2)k} |S_2| - \frac{2}{\alpha^2} \]
\[\leq \frac{1}{(\alpha-2)k} (|S_1| + |S_2|) - \frac{4}{\alpha^2} \]
\[\leq \frac{1}{(\alpha-2)k} (|S| + 2k) - \frac{4}{\alpha^2} \]
\[\leq \frac{1}{(\alpha-2)k} |S| - \frac{2}{\alpha^2} \]
\[\leq \frac{3\alpha}{\alpha^2} d - \frac{2}{\alpha^2} \]
\[\leq \frac{3\alpha}{\alpha^2} d + \frac{\alpha-4}{\alpha-2}.
\]

Every other node of \(T\) has the same degree as in \(T_1\) or \(T_2\). Thus \(\Delta(T) \leq \frac{3\alpha}{\alpha^2} d + \frac{\alpha-4}{\alpha-2} \).

This completes the proof. \(\square\)

Lemma 5 with \(\alpha = 4\) implies the following slight strengthening of Theorem 2.

**Theorem 6.** Every non-trivial graph \(G\) has a \(T\)-partition of width at most

\[2(tw(G) + 1)(9\Delta(G) - 1),\]

for some tree \(T\) with \(\Delta(T) \leq 6\Delta(G)\).

Lemma 5 with \(\alpha = 2 + \sqrt{2}\) (chosen to minimise \(\frac{3\alpha(\alpha-1)}{\alpha^2-2}\)) implies the next result.

**Theorem 7.** Every non-trivial graph \(G\) has a \(T\)-partition of width at

\[(1 + \sqrt{2})(tw(G) + 1)(3 + \sqrt{2})\Delta(G) - 1),\]

for some tree \(T\) with \(\Delta(T) \leq (3 + 3\sqrt{2})\Delta(G) - 3(\sqrt{2} - 1)\).