LOCAL UNIVERSALITY OF DETERMINANTAL POINT PROCESSES ON RIEMANNIAN MANIFOLDS

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Abstract. We consider the Laplace-Beltrami operator $\Delta_g$ on a smooth, compact Riemannian manifold $(M, g)$ and the determinantal point process $\mathcal{X}_\lambda$ on $M$ associated with the spectral projection of $-\Delta_g$ onto the subspace corresponding to the eigenvalues up to $\lambda^2$. We show that the pull-back of $\mathcal{X}_\lambda$ by the exponential map $\exp_p : T^*_p M \to M$ under a suitable scaling converges weakly to the universal determinantal point process on $T^*_p M$ as $\lambda \to \infty$.

1. Introduction

Let $(M, g)$ be a smooth, compact, Riemannian manifold of dimension $m$ with no boundary. We fix an orthonormal basis $\{\phi_i\}_{i \geq 0}$ of eigenfunctions of the Laplace-Beltrami operator $\Delta_g$ acting on $L^2(M) := L^2(M, \text{vol}_g)$:

$$-\Delta_g \phi_i = \lambda_i^2 \phi_i, \quad \langle \phi_i, \phi_j \rangle_{L^2(M)} = \delta_{ij},$$

with $0 = \lambda_0^2 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots \nearrow \infty$. Here $\{\lambda_i\}_{i=0}^\infty$ are the eigenvalues of $\sqrt{-\Delta_g}$. We denote the eigenspace corresponding to an eigenvalue $\lambda_i$ by $W_{\lambda_i}$. The projection operator $E_\lambda$ on $L^2(M, \text{vol}_g)$ onto the closed subspace $W_{\leq \lambda} := \bigoplus_{\lambda_i \leq \lambda} W_{\lambda_i}$ admits the following integral kernel

$$E_\lambda(x, y) = \sum_{\lambda_i \leq \lambda} \phi_i(x) \overline{\phi_i(y)} \quad (x, y \in M).$$

The projection kernel $E_\lambda(x, y)$ is the reproducing kernel of $W_{\leq \lambda}$ and thus defines a determinantal point process (DPP) $\mathcal{X}_\lambda$ on $M$, which is a random simple point configuration on $M$ whose $n$-point correlation function with respect to $\text{vol}_g$ is given by

$$\rho_n(x_1, x_2, \ldots, x_n) = \det(E_\lambda(x_i, x_j))_{i,j=1}^n.$$

In particular, the 1-point correlation function, the density of points, is

$$\rho_1(x) = E_\lambda(x, x).$$

See Section 2 for the definition of DPP.

The number of points in $\mathcal{X}_\lambda$ on $M$ is equal to the eigenvalue counting function given by

$$N(\lambda) = \sum_{\lambda_i \leq \lambda} 1 = \text{rank } E_\lambda = \int_M E_\lambda(x, x) \text{vol}_g(dx).$$

Since $\{\lambda_i\}_{i=0}^\infty$ are the eigenvalues of $\sqrt{-\Delta_g}$, it is known as the classical Weyl law (cf. [10]) that

$$N(\lambda) \sim \frac{\lambda^m}{(2\pi)^m |B_1^m|} \text{vol}_g(M) \quad (\lambda \to \infty),$$

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where $|B_1^{(m)}|$ is the volume of a unit ball in $\mathbb{R}^m$, i.e., $|B_1^{(m)}| = \pi^{m/2}/\Gamma(m/2 + 1)$. This means that the points in $\mathcal{X}_\lambda$ on $M$ become dense as $\lambda \to \infty$.

**Example 1.** When $M = \mathbb{S}^1$, for every $\lambda > 0$, the DPP associated with $E_\lambda$ is the random eigenvalues of Circular Unitary Ensemble (CUE) of size $N(\lambda)$ (cf. [8]). More generally, when $M = \mathbb{S}^m$, the corresponding DPPs are called harmonic ensembles on $\mathbb{S}^m$ (cf. [12]). These point processes are homogeneous in the sense that they are invariant under the $O(m)$-action.

The quantum ergodicity theorem originated by Shnirel’man [18, 19] and also studied in [6, 21] states that if the geodesic flow on $M$ is ergodic then $N(\lambda)^{-1}E_\lambda(x, x)\operatorname{vol}_g(dx)$ converges weakly to $\operatorname{vol}_g(dx)$ as $\lambda \to \infty$, in other words, so does the normalized first correlation measure of the DPP $\mathcal{X}_\lambda$. This theorem describes the global behavior of random points of the DPP on $M$.

In this paper, we focus on the local statistics of points in the DPP by taking a scaling as in (1.4) below so that we define a DPP $\Xi_{\lambda, p}$ on the cotangent space $T_p^*M$ by taking the pull-back of the DPP $\mathcal{X}_\lambda$ on $M$ by the exponential map.

We denote the Riemannian metric on $T_p^*M$ by $\langle \cdot, \cdot \rangle_{g_p}$, and the corresponding norm by $|\cdot|_{g_p}$. Here $|\xi|_{g_p}$ is the same as the principal symbol of $\sqrt{-\Delta_g}$ locally given by

$$|\xi|_{g_p} = \left( \sum_{i,j=1}^m g^{ij}(p) \xi_i \xi_j \right)^{1/2},$$

and $(g^{ij}(p))_{i,j=1}^m$ is the inverse matrix $g_p^{-1}$ of $g_p = (g_{ij}(p))_{i,j=1}^m$. The so-called pointwise Weyl law can be expressed as follows: as $\lambda \to \infty$,

$$E_\lambda(x, x) = \frac{1}{(2\pi)^m} \int_{|\xi|_{g_p} < \lambda} \frac{d\xi}{\sqrt{|\det g_x|}} + R_\lambda(x)$$

$$= \frac{|B_1^{(m)}|}{(2\pi)^m} \lambda^m + R_\lambda(x)$$

with uniform bound $\sup_{x \in M} |R_\lambda(x)| \leq C\lambda^{m-1}$ [10], which leads to the classical Weyl law (1.2). Since $M$ is compact, the injectivity radius $\operatorname{inj}_p^*(M)$ is positive, i.e., the exponential map $\exp_p : T_p^*M \to M$ is injective on the subset $\{ \xi \in T_p^*M : |\xi|_{g_p} < \operatorname{inj}_p^*(M) \}$ for any $p \in M$. We fix a point $p \in M$ and positive $\epsilon < \operatorname{inj}_p^*(M)$. Let $B_\epsilon$ be the open ball of radius $\epsilon$ in $T_p^*M$ centered at the origin and denote the image $\exp_p(B_\epsilon)$ by $\mathcal{B}_{p, \epsilon}$. For $\lambda > 0$, we define a point process $\Xi_{\lambda, p, \epsilon}$ on the cotangent space $T_p^*M$ by

$$\Xi_{\lambda, p, \epsilon} := \sum_{x \in \mathcal{X}_\lambda \cap \mathcal{B}_{p, \epsilon}} \delta_{x \exp_p^{-1}(x)},$$

which defines the pull-back of $\mathcal{X}_\lambda$ restricted on $\mathcal{B}_{p, \epsilon}$ by the exponential map and is scaled by $\lambda$. Here, we identified $\mathcal{X}_\lambda$ with a subset in $M$ (see Section 2.1). It turns out again to be a DPP on $T_p^*M$ (see Lemma 3).

Our main assertion in this paper is the following.

**Theorem 1.** As $\lambda \to \infty$, the point process $\Xi_{\lambda, p, \epsilon}$ converges weakly to the DPP $\Xi_p$ on $T_p^*M$ associated with the kernel

$$K_{g_p}^{(m)}(u, v) = \frac{1}{(2\pi)^m |u - v|_{g_p}^{m/2}} J_{m/2}(|u - v|_{g_p})$$
and the reference measure $\text{vol}_{G(p)}$, where $J_\alpha(x)$ is the Bessel function of the first kind defined by

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k+\alpha}$$

and $\text{vol}_{G(p)}$ is the Riemannian measure on $T^*_pM$ with respect to the constant Riemannian metric $G^{(p)} = (G^{(p)}_u)_{u \in T^*_pM}$ such that $G^{(p)}_u = (\exp_p^* g)_0$ for every $u \in T^*_pM$.

We remark that the limiting DPP $\Xi_p$ does not depend on $\epsilon > 0$.

We consider the following correlation kernel on $\mathbb{R}^m$,

$$K^{(m)}(u,v) := \frac{1}{(2\pi |u-v|)^m/2} J_{m/2}(|u-v|)$$

$$= \frac{1}{(2\pi)^m} \int_{|\xi|<1} e^{\sqrt{-1}(u-v,\xi)} d\xi,$$

where $(\cdot,\cdot)$ (resp. $|\cdot|$) is the standard inner product (resp. norm) on $\mathbb{R}^m$. The DPP on $\mathbb{R}^m$ associated with $K^{(m)}(u,v)$ is invariant under the action of the Euclidean motion group. When $n = 1$, $K^{(1)}(u,v)$ coincides with the sinc kernel

$$K^{(1)}(u,v) = \frac{\sin(u-v)}{\pi(u-v)},$$

which is the reproducing kernel of the classical Paley-Wiener space (see also Example 2 for $K^{(m)}$ given in Section 2.2). It is well known that the point process of eigenvalues of CUE (also GUE) under suitable scaling converges to the DPP associated with the sinc kernel. This DPP is also one of the most important examples of the class of DPPs associated with de Branges spaces discussed in [3]. In [12], we proved a special case of Theorem 1 when $M = \mathbb{S}^m$ by using spherical harmonics. Theorem 1 can be regarded as a generalization of these results to compact Riemannian manifolds. For the proof of Theorem 1 the pointwise Weyl law (1.3) plays a central role.

Theorem 1 shows the local universality of DPPs on Riemannian manifolds. This type of universality has been discussed as the asymptotic local structure of Szegő kernels, which is used to analyze random spherical harmonics and random section of holomorphic line bundles over a compact Kähler manifold. The former corresponds to the Euclidean class (real case) while the latter does the Heisenberg class (complex case) (cf. [2, 22, 23]). The terms “Euclidean” and “Heisenberg” are related to representations of the Euclidean and Heisenberg motion groups. The result in this paper falls in the Euclidean class in this terminology.

Theorem 1 can also be generalized to the case where the spectral projections of Laplace-Beltrami operators are replaced by those of general elliptic operators.

2. Determinantal point processes

For the necessary background for determinantal point processes, see e.g. [14, 15, 16, 17, 20, 11, 13].

2.1. Definition. Let $S$ be a locally compact Hausdorff space with countable base. A configuration $\Xi$ on $S$ is a non-negative integer-valued Radon measure and it can be expressed as $\Xi = \sum_i \delta_{x_i}$ ($x_i \in S$). We denote by $\text{Conf}(S)$ the totality of configurations on $S$, which we call a configuration space over $S$. An element $\Xi$ of $\text{Conf}(S)$ is sometimes regarded as an at most countable subset in $S$ without accumulation, possibly with multiple points.
Thus, $\Xi(A)$ is equal to the number of points in $A \in \mathcal{B}(S)$ with counted multiplicity, where $\mathcal{B}(S)$ is the totality of all bounded (i.e., relatively compact) sets in $S$. The configuration space $\text{Conf}(S)$ equipped with vague topology turns out to be a Polish space, i.e., a complete, separable metrizable space. We equip the configuration space $\text{Conf}(S)$ with the Borel structure with respect to this topology, which coincides with the Borel structure generated by the mapping $\text{Conf}(S) \ni \Xi \mapsto \Xi(A) \in S$ for all bounded $A \in \mathcal{B}(S)$. A point process on $S$ is a $\text{Conf}(S)$-valued random variable $\Xi = \Xi_\omega$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $\Xi(\{x\}) \leq 1$ for every $x \in S$ a.s., then $\Xi$ is called a simple point process. In this case, by identifying $\Xi$ with its support, we use the notation $x \in \Xi$ meaning that $\Xi(\{x\}) = 1$.

We fix a Radon measure $\nu$ on $S$ as a reference measure. A symmetric measure $\nu_n$ on $S^n$ is called the $n$-th correlation measure if it satisfies

$$
\mathbb{E} \left[ \prod_{i=1}^{p} \frac{\Xi(A_i)!}{\Xi(A_i) - k_i)!} \right] = \nu_n(A_1^{k_1} \times \cdots \times A_p^{k_p})
$$

for any disjoint bounded sets $A_1, \ldots, A_p \in \mathcal{B}(S)$ and any $k_1, \ldots, k_p \in \mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$ with $\sum_{i=1}^{p} k_i = n$. If $\nu_n$ is absolutely continuous with respect to the product measure $\nu^{\otimes n}$, the Radon-Nikodym derivative $\rho_n := d\nu_n/d\nu^{\otimes n}$ is called the $n$-point correlation function with respect to the reference measure $\nu$:

$$
\rho_n(dx_1 \ldots dx_n) = \rho^n(x_1, \ldots, x_n)\nu^{\otimes n}(dx_1 \ldots dx_n).
$$

Let $\mathcal{J}_1(S, \nu)$ be the ideal of trace class operators $K: L_2(S, \nu) \to L_2(S, \nu)$; we denote the $\mathcal{J}_1$-norm of the operator $K$ by $\|K\|_{\mathcal{J}_1}$. Let $\mathcal{J}_{1,\text{loc}}(S, \nu)$ be the space of operators $K: L_2(S, \nu) \to L_2(S, \nu)$ such that $1_A K 1_A \in \mathcal{J}_1(S, \nu)$ for any bounded Borel subset $A \subset S$, where $1_A$ is the indicator function of a set $A$. Such an operator $K$ is called a locally trace class operator. We endow the space $\mathcal{J}_{1,\text{loc}}(S, \nu)$ with a countable family of semi-norms $\|1_A K 1_A\|_{\mathcal{J}_1}$ where $A$ runs through an exhausting family $A_n$ of bounded sets, i.e., $A_n$ is increasing and $\bigcup_{n=1}^{\infty} A_n = S$. A locally trace class operator $K$ admits a kernel (cf. [2] [13]), for which, slightly abusing notation, we use the same symbol $K$.

A point process is called a determinantal point process associated with $K$ and $\nu$ if there exists an operator $K \in \mathcal{J}_{1,\text{loc}}(S, \nu)$ such that for any bounded measurable function $h$, for which $h - 1$ is supported in a bounded set $A$, we have

$$
(2.1) \quad \mathbb{E} \Psi_h = \det \left( 1 + (h - 1)K 1_A \right),
$$

where $\Psi_h(\Xi) = \prod_{x \in \Xi} h(x)$ for $\Xi \in \text{Conf}(S)$. The Fredholm determinant in (2.1) is well-defined since $K \in \mathcal{J}_{1,\text{loc}}(S, \nu)$. For example, if $K$ is a positive contraction operator $K \in \mathcal{J}_{1,\text{loc}}(S, \nu)$, then there exists a DPP associated with $K$ and $\nu$. The equation (2.1) determines the law of the DPP uniquely (15) [16] [20]. For the DPP associated with $K$, the $n$-th correlation function with respect to $\nu$ is given by

$$
\rho_n(x_1, \ldots, x_n) = \det(K(x_i, x_j))_{i,j=1}^{n}.\]

$K(x, y)$ is often called the correlation kernel and $\nu$ the reference measure. When $S = \mathbb{R}^m$, if $\nu$ is the Lebesgue measure and $K(x, y) = k(x - y)$ for some $k$, then the law of the DPP associated with $K$ and $\nu$ is invariant under the action of the Euclidean motion group.

Weak convergence for DPPs is characterized by the convergence of operators (cf. Proposition 3.10 in [16]) as follows.
Lemma 2. Let $\Xi_n$ (resp. $\Xi$) be a DPP on $S$ associated with $K_n$ (resp. $K$) and $\nu$. Suppose $K_n$ converges $K$ in $\mathcal{F}_{1,loc}(S,\nu)$ as $n \to \infty$. Then $\Xi_n$ converges weakly to $\Xi$ as $n \to \infty$. In particular, if the kernel $K_n(x,y)$ converges to $K(x,y)$ uniformly on any compact set in $S \times S$, then the convergence above takes place.

2.2. DPPs associated with reproducing kernel Hilbert spaces. Let $S$ be a non-empty subset and $\mathcal{F}(S)$ be a linear space of functions on $S$, i.e., $\mathcal{F}(S) := \{f : S \to \mathbb{C}\}$. A subspace $H$ of $\mathcal{F}(S)$ is called a reproducing kernel Hilbert space (RKHS) if $H$ is endowed with an inner product $\langle \cdot , \cdot \rangle_H$ which makes $H$ a Hilbert space and the evaluation functional $E_s : H \to \mathbb{C}$ defined by $E_s(f) := f(s)$ is bounded for every $s \in S$. By the Riesz representation theorem, for each $s \in S$, there exists a unique element $k_s \in H$ such that $E_s(f) = \langle f, k_s \rangle_H = f(s)$. We define a kernel $K : S \times S \to \mathbb{C}$ by

$$K(s, t) := k_t(s) = \langle k_t, k_s \rangle_H,$$

which is called the reproducing kernel for $H$ (see [1] for more details about RKHS). The integral operator $K$ with kernel $K(s, t)$ defines an orthogonal projection onto $H$. Therefore, the DPP is associated with reproducing kernel $K(s, t)$, or equivalently, RKHS $H$.

Example 2. (1) For a given $a > 0$,

$$\text{PW}_a := \{f \in C(\mathbb{R}) : \text{supp } \hat{f} \subset [-a, a]\}$$

is called a Paley-Wiener space or the space of band-limited functions. Here $\hat{f}$ is the Fourier transform of $f$ defined as

$$\hat{f}(\xi) := \int_{\mathbb{R}^m} f(x)e^{-\sqrt{-1}(x,\xi)}dx.$$

The corresponding reproducing kernel $K_a$ is given by

$$K_a(x, y) = \frac{\sin a(x-y)}{\pi(x-y)},$$

and the corresponding DPP is the limiting DPP obtained from CUE (also GUE) eigenvalues.

(2) A generalized Paley-Wiener space is similarly defined as follows: for a bounded Borel set $\Omega \subset \mathbb{R}^m$,

$$\text{PW}_\Omega := \{f \in C(\mathbb{R}^m) : \text{supp } \hat{f} \subset \overline{\Omega}\}.$$

When $\Omega = B_1^{(m)} \subset \mathbb{R}^m$, the corresponding reproducing kernel is $K^{(m)}(x, y)$ which appeared in Theorem [1].

(3) Let $(M, g)$ be a compact, smooth, Riemannian manifold and $\Delta_g$ be the Laplace-Beltrami operator on $L^2(M, \text{vol}_g)$. We denote the resolution of the identity for $\Delta_g$ by $\{E(A) : A \in \mathcal{B}(\mathbb{R})\}$. Then the integral operator $E_\lambda$ with kernel $E_\lambda(x, y)$ given in (12) coincides with the projection $E([0, \lambda^2])$ and $W_{\leq \lambda}$ turns out to be a RKHS admitting the reproducing kernel $E_\lambda(x, y)$.

3. PROOF OF THE MAIN THEOREM

We define $\phi_\lambda : T_p^* M \to M$ by $\phi_\lambda(u) = \exp_p(u/\lambda)$ for $u \in T_p^* M$. For $u, v \in T_p^* M$, we write $U_\lambda = \phi_\lambda(u)$ and $V_\lambda = \phi_\lambda(v)$. We consider the kernel

$$K_{\lambda, p, c}(u, v) = \frac{1}{\lambda^m} E_\lambda(U_\lambda, V_\lambda) 1_{B_{p, c}}(U_\lambda) 1_{B_{p, c}}(V_\lambda).$$

(3.1)
We have the following.

**Lemma 3.** The scaled point process \( \Xi_{\lambda,p,e} \) defined by (1.4) is the DPP on \( T^*pM \) associated with the kernel \( K_{\lambda,p,e}(u,v) \) of (3.1) and \( \lambda^m \phi^*_\lambda \operatorname{vol}_g \).

**Proof.** We note that \( X_{\phi}|_{B_{p,\epsilon}} \) is the DPP associated with the kernel \( E_{\lambda}(x,y)1_{B_{p,\epsilon}}(x)1_{B_{p,\epsilon}}(y) \) and the reference measure \( \operatorname{vol}_g \). Then the pull-back \( \phi^*_\lambda X_{\phi}|_{B_{p,\epsilon}} \) is the DPP associated with the kernel \( E_{\lambda}(\phi_{\lambda}(u),\phi_{\lambda}(v))1_{B_{p,\epsilon}}(\phi_{\lambda}(u))1_{B_{p,\epsilon}}(\phi_{\lambda}(v)) \) and \( \phi^*_\lambda \operatorname{vol}_g \) since \( \phi_{\lambda}|_{B_{\epsilon}} : B_{\epsilon} \rightarrow B_{p,\epsilon} \) is a diffeomorphism. The law of this DPP is the same as that of the DPP associated with the kernel (3.1) and \( \lambda^m \phi^*_\lambda \operatorname{vol}_g \) through the measure change by the factor \( \lambda^m \) (cf. [13, Section 2.3]). \( \square \)

We remark that since \( (d\phi_{\lambda})_u = \lambda^{-1}(d\phi_1)_{u/\lambda} \), the pull-back of the Riemannian metric \( g \) on \( M \) is expressed as

\[
\lambda^2(\phi^*_\lambda g)_u = (\phi^*_1 g)_{u/\lambda}
\]

for \( u \in T^*_uM \). Therefore, \( \lambda^m \phi^*_\lambda \operatorname{vol}_g \) is equal to the Riemannian measure with respect to \( (\phi^*_1 g)_{/\lambda} \). For the proof of Theorem 1 we appeal to the pointwise Weyl law (1.3), which gives an off-diagonal asymptotics for the projection kernel \( E_{\lambda}(x,y) \) as \( \lambda \rightarrow \infty \) as follows: if \( x \) is close enough to \( y \), i.e., \( x \in B_{y,\epsilon} \) with \( \epsilon < \operatorname{inj}^*(M) \), then

\[
E_{\lambda}(x,y) = \lambda^m \int_{|\xi|_{g_y} < 1} e^{-\Delta_{g_y}} \psi(x,y,\xi) \frac{d\xi}{\sqrt{\det g_y}} + R_{\lambda}(x,y),
\]

where \( \psi(x,y,\xi) \) is a phase function which is adapted, in Hörmander’s terminology [10], to the principal symbol \( |\xi|_{g_y} \sqrt{-\Delta_y} \), vanishing on the diagonal \( x = y \). This type of asymptotics for the spectral function was initiated by Hörmander [10] as an application of the theory of pseudo-differential operators and recovers the classical Weyl law (1.2). The choice of a phase function is not unique, and one can take

\[
\psi(x,y,\xi) = \langle \exp_y^{-1}(x),\xi \rangle_{g_y}
\]

in a coordinate-independent way [24, 4]. Indeed, the integral on the right-hand side of (3.2) with (3.3) is taken over the cotangent fiber \( T^*_yM \) and it is coordinate-independent since the measure \( d\xi/\sqrt{\det g_y} \) is the quotient of the canonical symplectic form \( d\xi \wedge dy \) on \( T^*M \) by the Riemannian volume form \( \sqrt{\det g_y} dy \) on \( M \). There are many papers estimating the remainder term \( R_{\lambda}(x,y) \). From [4, Theorem 2], the remainder term is uniformly estimated as follows.

**Theorem 4 ([10, 4, 5]).** We assume (3.3). Then, for any fixed \( r > 0 \), as \( \lambda \rightarrow \infty \),

\[
\sup_{d_g(x,y) < r/\lambda} |R_{\lambda}(x,y)| = O(\lambda^{m-1}),
\]

where \( d_g(x,y) \) is the Riemannian distance.

Before giving a proof of the main theorem, we see a generalization of the following formula (cf. [13])

\[
\frac{1}{(2\pi)^{m/2}} \int_{|\omega| < 1} e^{-\frac{1}{2}(\eta,\omega)} d\omega = F_{m/2}(|\eta|),
\]

where \( F_\alpha(t) = J_\alpha(t)/t^\alpha \) for \( \alpha > 0 \).
Lemma 5. Let \( m = \dim M \). For \( \eta \in T^*_pM \),
\[
\frac{1}{(2\pi)^{m/2}} \int_{|\xi|_{g_p} < 1} e^{\sqrt{-1} \langle \eta, \xi \rangle_{g_p}} \frac{d\xi}{\sqrt{\det g_p}} = F_{m/2}(|\eta|_{g_p}).
\]

Proof. We note that
\[
\langle \eta, \xi \rangle_{g_p} = (g_p^{-1/2} \eta, g_p^{-1/2} \xi),
\]
where \( g_p^{-1/2} \) is the positive definite square root of the inverse matrix \( g_p^{-1} \). In particular, \( |\eta|_{g_p} = |g_p^{-1/2} \eta| \). From (3.4), by change of variables \( \omega = g_p^{-1/2} \xi \), we have
\[
F_{m/2}(|\eta|_{g_p}) = \frac{1}{(2\pi)^{m/2}} \int_{|\omega| < 1} e^{\sqrt{-1} \langle g_p^{-1/2} \eta, \omega \rangle} d\omega
\]
\[
= \frac{1}{(2\pi)^{m/2}} \int_{|\xi|_{g_p} < 1} e^{\sqrt{-1} \langle \eta, \xi \rangle_{g_p}} \frac{d\xi}{\sqrt{\det g_p}}.
\]
We obtain the assertion. \( \square \)

Remark 1. We have a similar formula
\[
\frac{1}{(2\pi)^{m/2}} \int_{|\xi|_{g_p} = 1} e^{\sqrt{-1} \langle \eta, \xi \rangle_{g_p}} \frac{d\xi}{\sqrt{\det g_p}} = F_{(m-2)/2}(|\eta|_{g_p}).
\]

We need one more fact for the local behavior of the Riemannian distance function.

Lemma 6. For \( u, v \in T^*_pM \), let \( c_1 \) and \( c_2 \) be \( C^1 \) curves in \( M \) such that \( c_1(0) = c_2(0) = p \), \( c_1'(0) = u \) and \( c_2'(0) = v \). Then,
\[
\lim_{t \to 0+} \frac{d_g(c_1(t), c_2(t))}{t} = |u - v|_{g_p}.
\]

Proof. See Corollary 3.1 in [7] for instance. \( \square \)

Now we are in a position to give a proof of the main theorem.

Proof of Theorem 4 It suffices to show that the DPP associate with \( K_{\lambda, p, \epsilon}(u, v) \) and \( \lambda^m \phi^* \vol_g \) converges as \( \lambda \to \infty \). Suppose \( d_g(x, y) \) is small enough. First we note that there exists \( \zeta \in T^*_yM \) such that \( |\zeta|_{g_y} = 1 \) and \( \exp_y^{-1}(x) = d_g(x, y) \zeta \). By using Lemma 5 we see that
\[
\frac{1}{(2\pi)^{m/2}} \int_{|\xi|_{g_y} < 1} e^{\sqrt{-1} \langle \exp_y^{-1}(x), \xi \rangle_{g_y}} \frac{d\xi}{\sqrt{\det g_y}} = F_{m/2}(d_g(x, y)).
\]
From Lemma 3 with (3.1), (3.2) with (3.3), Theorem 4 and Lemma 5 as \( \lambda \to \infty \), we have
\[
K_{\lambda, p, \epsilon}(u, v) = \frac{1}{(2\pi)^m} \int_{|\xi|_{g^\lambda} < 1} e^{\sqrt{-1} \lambda \langle \exp_{g^\lambda}^{-1}(U_{\lambda}), \xi \rangle_{g^\lambda}} \frac{d\xi}{\sqrt{\det g^\lambda}} 1_{B_{p, \epsilon}}(U_{\lambda}) 1_{B_{p, \epsilon}}(V_{\lambda}) + O(\lambda^{-1})
\]
\[
= \frac{1}{(2\pi)^{m/2}} F_{m/2}(\lambda d_g(U_{\lambda}, V_{\lambda})) 1_{B_{p, \epsilon}}(U_{\lambda}) 1_{B_{p, \epsilon}}(V_{\lambda}) + O(\lambda^{-1}).
\]
We note that \( \lim_{t \to 0} F_{\alpha}(t) = 2^{-\alpha} \Gamma(\alpha + 1)^{-1} \) and so \( F_{\alpha}(t) \) is a bounded continuous function on \( \mathbb{R} \). Since \( \lambda d_g(U_{\lambda}, V_{\lambda}) \to |u - v|_{g_p} \) by Lemma 3 and \( 1_{B_{p, \epsilon}}(U_{\lambda}) 1_{B_{p, \epsilon}}(V_{\lambda}) \) is equal to 1 for any sufficiently large \( \lambda \), we have
\[
K_{\lambda, p, \epsilon}(u, v) \to \frac{1}{(2\pi)^{m/2}} F_{m/2}(|u - v|_{g_p})
\]
uniformly on compacts in $T^*_p M$. From the remark after Lemma 3, the reference measure is the Riemannian measure with respect to $(\phi^*_p g)/\lambda$ and the Radon-Nikodym derivative relative to the Riemannian measure with respect to $(\phi^*_p g)_0$ is uniformly close to 1 on any compact set as $\lambda \to \infty$. Therefore, it follows from Lemma 2 that the scaled point process $\Xi_{\lambda, p, \epsilon}$ converges weakly to the DPP associated with the kernel $K^{(m)}_{\phi^*_p}(u, v)$ given by (1.5) and the reference measure given by the Riemannian measure with respect to the constant metric $(\phi^*_p g)_0$. The proof is completed. □

4. Concluding remarks

We have seen the local universality of DPPs on Riemannian manifolds. From this discussion, we came to several other questions.

(1) What is the universality when we consider the Heisenberg case in Zelditch’s terminology? We only discussed the Euclidean case in this article. One can expect that the Bergman kernel is involved as in [2, 22, 23, 12].

(2) We dealt with Laplace-Beltrami operators corresponding to the principal symbol $|\xi|_{g_p}$. What is the local universality result when we consider more general DPPs associated with the spectral projections of elliptic differential operators possibly with potentials?

(3) In this paper we have considered a point process $\Xi_{\lambda, p, \epsilon}$ on $T^*_p M$ at each ‘point’ $p \in M$. We expect that the collection $\Xi_{\lambda, \epsilon} = \{\Xi_{\lambda, p, \epsilon}\}_{p \in M}$ will be regarded as a ‘random field’ on the cotangent bundle $T^* M = \{T^*_p M\}_{p \in M}$. Theorem 1 determines the limit $\Xi_{\lambda, p, \epsilon} \to \Xi_p$ in $\lambda \to \infty$. How can we describe the limiting random field $\Xi_{\lambda, \epsilon} \to \Xi$ in $\lambda \to \infty$?

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References

[1] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68 (1950) 337–404.
[2] P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds, Invent. Math. 142 (2000) 351–395.
[3] A.I. Bufetov and T. Shirai, Quasi-symmetries and rigidity for determinantal point processes associated with de Branges spaces, Proceedings of the Japan Academy, Ser. A Math. Sci. 93 (2017), no.1, 1–5.
[4] Y. Canzani and B. Hanin, Scaling limit for the kernel of the spectral projector and remainder estimates in the pointwise Weyl law, Analysis & PDE 8 (2015), 1707–1731.
[5] Y. Canzani and B. Hanin, $C^\infty$ scaling asymptotics for the spectral projector of the Laplacian, J. Geom. Anal. 28 (2018), 111–122.
[6] Y. Colin de Verdiere, Ergodicité et fonctions propres du laplacien, Comm. Math. Phys., 102 (1985), 497–502.
[7] J.X. da Cruz Neto, O.P. Ferreira and L.R. Lucambio Pérez, Monotone point-to-set vector fields, Balkan J. Geom. Appl. 5 (2000), 69–79.
[8] P. J. Forrester, Log-gases and Random Matrices, London Math. Soc. Monographs, Princeton University Press, Princeton, 2010.
[9] H.-O. Georgii and H. J. Yoo, Conditional intensity and Gibbsianness of determinantal point processes, J. Statist. Phys. 118 (2005), 55–84.
[10] L. Hörmander, The spectral function of an elliptic operator, Acta Math. 121 (1968), 193–218.
[11] J.B. Hough, M. Krishnapur, Y. Peres, B. Virág, Determinantal processes and independence, Probab. Surv. 3 (2006), 206–229.
[12] M. Katori and T. Shirai, Scaling limit for determinantal point processes on spheres, RIMS Kôkyûroku Bessatsu B79 (2020), 123–138.
[13] M. Katori and T. Shirai, Partial isometry, duality, and determinantal point processes, Random Matrices: Theory and Applications (2022), 2250025 (70 pages).
[14] O. Macchi, The coincidence approach to stochastic point processes, Advances in Appl. Probability, 7 (1975), 83–122.
[15] T. Shirai, Y. Takahashi, Fermion process and Fredholm determinant, Proceedings of the Second ISAAC Congress, vol. I, 15–23, Kluwer 2000.
[16] T. Shirai, Y. Takahashi, Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes, J. Funct. Anal. 205 (2003), no. 2, 414–463.
[17] T. Shirai, Y. Takahashi, Random point fields associated with certain Fredholm determinants. II. Fermion shifts and their ergodic and Gibbs properties, Ann. Probab. 31 (2003), no. 3, 1533–1564.
[18] A. I. Shnirel’man, Ergodic properties of eigenfunctions, Uspekhi Math. Nauk 29/6 (1974), 181–182.
[19] A. I. Shnirelman, On the asymptotic properties of eigenfunctions in the regions of chaotic motion, Addendum to “KAM Theory and Semiclassical Approximations to Eigenfunctions” by V. F. Lazutkin, Ergebnisse der Mathematik 24, 313–337, Springer-Verlag, Berlin, 1993.
[20] A. Soshnikov, Determinantal random point fields. (Russian) Uspekhi Mat. Nauk 55 (2000), no. 5(335), 107–160; translation in Russian Math. Surveys 55 (2000), no. 5, 923–975.
[21] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, Duke Math. J., 55 (1987), 9190–941.
[22] S. Zelditch, From random polynomials to symplectic geometry, XIIIth International Congress on Mathematical Physics (London, 2000), 367–376, Int. Press, Boston, MA, 2001.
[23] S. Zelditch, Local and global analysis of eigenfunctions on Riemannian manifolds, in Handbook of Geometric Analysis, No. 1, L. Ji, P. Li, R. Schoen and L. Simon (eds.), Advanced Lectures in Mathematics (ALM), 7 (Intl. Pr. of Boston Inc., 2008) pp. 545–658.
[24] S. Zelditch, Real and complex zeros of Riemannian random waves. Contemp. Math. 484 (2009), 321–342.

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