Well-Posedness Results for the Continuum Spectrum Pulse Equation

Giuseppe Maria Coclite 1,* and Lorenzo di Ruvo 2

1 Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, via E. Orabona 4, 70125 Bari, Italy
2 Dipartimento di Matematica, Università di Bari, via E. Orabona 4, 70125 Bari, Italy; lorenzo.diruvo@uniba.it
* Correspondence: giuseppemaria.coclite@poliba.it

Received: 25 September 2019; Accepted: 21 October 2019; Published: 23 October 2019

Abstract: The continuum spectrum pulse equation is a third order nonlocal nonlinear evolutive equation related to the dynamics of the electrical field of linearly polarized continuum spectrum pulses in optical waveguides. In this paper, the well-posedness of the classical solutions to the Cauchy problem associated with this equation is proven.

Keywords: existence; uniqueness; stability; continuum spectrum pulse equation; Cauchy problem

MSC: 35G25; 35K55

1. Introduction

In this paper, we investigate the well-posedness of the classical solution of the following Cauchy problem:

\begin{align}
\partial_t u + 3gu^2\partial_x u - a\partial_x^3 u + q\partial_x(uv) &= bP, & t > 0, x \in \mathbb{R}, \\
\partial_x P &= u, & t > 0, x \in \mathbb{R}, \\
a\partial_x^2 v + \beta\partial_x v + \gamma v &= \kappa u^2, & t > 0, x \in \mathbb{R}, \\
P(t, -\infty) &= 0, & t > 0, \\
u(0, x) &= u_0(x), & x \in \mathbb{R},
\end{align}

where \( g, a, q, b, \alpha, \beta, \gamma, \kappa \in \mathbb{R} \).

On the initial datum, we assume that

\[ u_0 \in H^2(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x)dx = 0. \]  

Following [1–6], on the function

\[ P_0(x) = \int_{-\infty}^x u_0(y)dy, \]  

we assume that

\[ \int_{\mathbb{R}} P_0(x)dx = \int_{\mathbb{R}} \left( \int_{-\infty}^x u_0(y)dy \right)dx = 0, \]

\[ \|P_0\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left( \int_{-\infty}^x u_0(y)dy \right)^2 dx < \infty. \]
In addition, we assume that
\[
\frac{q^2 \kappa}{\gamma} \geq 0, \quad g \neq 0, \quad a \neq 0, \quad \alpha \neq 0.
\] (5)

Observe that, since \( \alpha \) cannot vanish, we can factorize it and deal with only three constants.

In the physical literature (1) is termed the continuum spectrum pulse equation (see [7–14]). It is used to describe the dynamics of the electrical field \( u \) of linearly polarized continuum spectrum pulses in optical waveguides, including fused-silica telecommunication-type or photonic-crystal fibers, as well as hollow capillaries filled with transparent gases or liquids.

The constants \( a, b, g, q, \alpha, \kappa, \beta, \gamma \), in (1), take into account the frequency dispersion of the effective linear refractive index and the nonlinear polarization response, the excitation efficiency of the vibrations, the frequency and the decay time (see [7,8,14]).

Moreover, (1) generalizes the following system:
\[
\begin{cases}
\partial_t u + q \partial_x (uv) = b P, & t > 0, \ x \in \mathbb{R}, \\
\partial_x P = u, & t > 0, \ x \in \mathbb{R}, \\
a \partial_x^2 v + \beta \partial_x v + \gamma v = \kappa u^2, & t > 0, \ x \in \mathbb{R}, \\
P(t, -\infty) = 0, & t > 0, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\] (6)

whose the well-posedness is studied in [15]. From a mathematical point of view, the presence of the term
\[3g u^2 \partial_x u - a \partial_x^3 u\]
in the first equation of (1) makes the analysis of such system more subtle than the one of (6).

Observe that, taking \( b = \alpha = \beta = 0 \), (1) becomes the modified Korteweg-de Vries equation (see [16–20])
\[
\partial_t u + \left( g + \frac{q \kappa}{\gamma} \right) \partial_x u^3 - a \partial_x^3 u = 0.
\] (7)

In [8,9,21–24], it is proven that (7) is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. In [6,18], the Cauchy problem for (7) is studied, while, in [16,19], the convergence of the solution of (7) to the unique entropy solution of the following scalar conservation law
\[
\partial_t u + \left( g + \frac{q \kappa}{\gamma} \right) \partial_x u^3 = 0
\] (8)
is proven.

On the other hand, taking \( \alpha = \beta = 0 \) in (1), we have the following equations
\[
\begin{cases}
\partial_t u + \left( g + \frac{q \kappa}{\gamma} \right) \partial_x u^3 = b P, & t > 0, \ x \in \mathbb{R}, \\
\partial_x P = u, & t > 0, \ x \in \mathbb{R},
\end{cases}
\] (9)

that were deduced by Kozlov and Sazonov [12] for the description of the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media and by Schäfer and Wayne [25] for the description of the propagation of ultra-short light pulses in silica optical fibers. Moreover, (9) is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons (see [22–24,26–28]), a particular Rabelo equation which describes pseudospherical surfaces (see [29–32]), and a model for the descriptions of the short pulse propagation in nonlinear metamaterials characterized by a weak Kerr-type nonlinearity in their dielectric response (see [33]).
Finally, (9) was deduced in [34] in the context of plasma physics and that similar equations describe the dynamics of radiating gases [35,36], in [37–40] in the context of ultrafast pulse propagation in a mode-locked laser cavity in the few-femtosecond pulse regime and in [41] in the context of Maxwell equations.

The Cauchy problem for (9) was studied in [42–44] in the context of energy spaces, [4,5,45,46] in the context of entropy solutions. The homogeneous initial boundary value problem was studied in [47–50]. Nonlocal formulations of (9) were analyzed in [15,51] and the convergence of a finite difference scheme proved in [52].

Observe that, taking \( \alpha = \beta = 0 \), (1) reads

\[
\begin{aligned}
\partial_t u + \left( g + \frac{qK}{\gamma} \right) \partial_x u^3 - a \partial_x^3 u &= bP, \quad t > 0, \ x \in \mathbb{R}, \\
\partial_x P &= u, \quad t > 0, \ x \in \mathbb{R}.
\end{aligned}
\]

(10)

It was derived by Costanzino, Manukian and Jones [53] in the context of the nonlinear Maxwell equations with high-frequency dispersion. Kozlov and Sazonov [12] show that (10) is a more general equation than (9) to describe the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media.

Mathematical properties of (10) are studied in many different contexts, including the local and global well-posedness in energy spaces [43,53] and stability of solitary waves [53,54], while, in [6], the well-posedness of the classical solutions is proven.

Observe that letting \( a \to 0 \) in (10), we obtain (9). Hence, following [19,55,56], in [5,57], the convergence of the solution of (10) to the unique entropy solution of (9).

The main result of this paper is the following theorem.

**Theorem 1.** Assume (2), (3), (4) and (5). Fix \( T > 0 \), there exists an unique solution \( (u, v, P) \) of (1) such that

\[
\begin{aligned}
u &\in L^\infty(0,T;H^2(\mathbb{R})), \\
v &\in L^\infty(0,T;H^4(\mathbb{R})), \\
P &\in L^\infty(0,T;H^3(\mathbb{R})).
\end{aligned}
\]

(11)

In particular, we have that

\[
\int_\mathbb{R} u(t,x) dx = 0, \quad t \geq 0.
\]

(12)

Moreover, if \( (u_1, v_1, P_1) \) and \( (u_2, v_2, P_2) \) are two solutions of (1), we have that

\[
\begin{aligned}
\| u_1(t,\cdot) - u_2(t,\cdot) \|_{L^2(\mathbb{R})} &\leq e^{C(T)t} \| u_{1,0} - u_{2,0} \|_{L^2(\mathbb{R})}, \\
\| v_1(t,\cdot) - v_2(t,\cdot) \|_{H^2(\mathbb{R})} &\leq e^{C(T)t} \| u_{1,0} - u_{2,0} \|_{L^2(\mathbb{R})}, \\
\| P_1(t,\cdot) - P_2(t,\cdot) \|_{H^1(\mathbb{R})} &\leq e^{C(T)t} \| P_{1,0} - P_{2,0} \|_{H^1(\mathbb{R})},
\end{aligned}
\]

(13)

where,

\[
\begin{aligned}
P_{1,0}(x) &= \int_{-\infty}^{x} u_{1,0}(y) dy, \quad P_{2,0}(x) = \int_{-\infty}^{x} u_{2,0}(y) dy,
\end{aligned}
\]

(14)

for some suitable \( C(T) > 0 \), and every \( 0 \leq t \leq T \).

The proof of Theorem 1 is based on the Aubin–Lions Lemma (see [58–60]).

The paper is organized as follows. In Section 2, we prove several a priori estimates on a vanishing viscosity approximation of (1). Those play a key role in the proof of our main result, that is given in...
Section 3. Appendix A is an appendix, where we prove the posedness of the classical solutions of (1), under the assumption
\[ u_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R}). \] (15)

2. Vanishing Viscosity Approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1).

Fix a small number \( 0 < \varepsilon < 1 \) and let \( u_\varepsilon = u_\varepsilon(t,x) \) be the unique classical solution of the following mixed problem [19,61,62]:
\[
\begin{aligned}
\partial_t u_\varepsilon + 3gu_\varepsilon^2 \partial_x u_\varepsilon - a \varepsilon \partial_x^2 u_\varepsilon + q \partial_x (v_\varepsilon u_\varepsilon) &= bP_\varepsilon - \varepsilon \partial_x^5 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\
\partial_x P_\varepsilon &= u_\varepsilon, & t > 0, x \in \mathbb{R}, \\
\alpha \partial_x^2 v_\varepsilon + \beta \partial_x v_\varepsilon + \gamma v_\varepsilon &= \kappa u_\varepsilon^2, & t > 0, x \in \mathbb{R}, \\
P_\varepsilon(t,-\infty) &= 0, & t > 0, \\
u_\varepsilon(0,x) &= u_{\varepsilon,0}(x), & x \in \mathbb{R},
\end{aligned}
\] (16)

where \( u_{\varepsilon,0} \) is a \( C^\infty \) approximation of \( u_0 \) such that
\[
\| u_{\varepsilon,0} \|_{H^2(\mathbb{R})} \leq \| u_0 \|_{H^2(\mathbb{R})}, \quad \int_\mathbb{R} u_{\varepsilon,0} dx = 0,
\]
\[
\| P_{\varepsilon,0} \|_{L^2(\mathbb{R})} \leq \| P_0 \|_{L^2(\mathbb{R})}, \quad \int_\mathbb{R} P_{\varepsilon,0} dx = 0.
\] (17)

Let us prove some a priori estimates on \( u_\varepsilon, P_\varepsilon \) and \( v_\varepsilon \). We denote with \( C_0 \) the constants which depend only on the initial data, and with \( C(T) \), the constants which depend also on \( T \).

**Lemma 1.** For each \( t \geq 0 \),
\[ P_\varepsilon(t,\infty) = 0. \] (18)

In particular, we have that
\[ \int_\mathbb{R} u_\varepsilon(t,x) dx = 0. \] (19)

**Proof.** We begin by proving (18). Thanks to the regularity of \( u_\varepsilon \) and the first equation of (16), we have that
\[
\lim_{x \to \infty} \left( \partial_t u_\varepsilon + 3gu_\varepsilon^2 \partial_x u_\varepsilon - a \varepsilon \partial_x^2 u_\varepsilon + q \partial_x (v_\varepsilon u_\varepsilon) - \varepsilon \partial_x^5 u_\varepsilon \right) = bP_\varepsilon(t,\infty) = 0,
\]
which gives (18).

Finally, we prove (19). Integrating the second equation of (16) on \((\infty,x)\), by (16), we have that
\[ P_\varepsilon(t,x) = \int_{-\infty}^{x} u_\varepsilon(t,y) dy. \] (20)

Equation (19) follows from (18) and (20). \( \square \)

Arguing as in ([15], Lemma 2.2), we have the following result.

**Lemma 2.** Assume (5). We have that
\[
\int_\mathbb{R} u_\varepsilon^2 \partial_x v_\varepsilon dx = \begin{cases} 
\frac{\beta}{\kappa} \| \partial_x v_\varepsilon(t,\cdot) \|_{L^2(\mathbb{R})}^2, & \text{if } \beta \neq 0, \\
0, & \text{if } \beta = 0.
\end{cases}
\] (21)
Lemma 3. Assume (5). If $\beta \neq 0$, then for each $t \geq 0$, there exists a constant $C_0 > 0$, independent on $\epsilon$, such that

$$
\|u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{q\beta}{K} \int_0^t \|\partial_s v_\epsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds + 2\epsilon \int_0^t \|\partial_s u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C_0. \tag{22}
$$

If $\beta = 0$, then for each $t \geq 0$,

$$
\|u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\epsilon \int_0^t \|\partial_s u_\epsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq \|u_0\|_{L^2(\mathbb{R})}^2. \tag{23}
$$

In particular, we have

$$
\|\partial_s v_\epsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}, \|\partial_s v_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_0,
$$

$$
\|v_\epsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}, \|v_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_0. \tag{24}
$$

Moreover, fixed $T > 0$, there exists a constant $C(T) > 0$, independent on $\epsilon$, such that

$$
\epsilon \int_0^t \|\partial_s u_\epsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T). \tag{25}
$$

Proof. Multiplying by $2u_\epsilon$ the first equation of (16), an integration on $\mathbb{R}$ gives

$$
\frac{d}{dt} \|u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 2 \int_{\mathbb{R}} u_\epsilon \partial_t u_\epsilon \, dx
$$

$$
= -6\epsilon \int_{\mathbb{R}} u_\epsilon^3 \partial_x u_\epsilon \, dx - 2q \int_{\mathbb{R}} u_\epsilon \partial_x (u_\epsilon v_\epsilon) \, dx + 2b \int_{\mathbb{R}} P_\epsilon u_\epsilon \, dx
$$

$$
+ 2a \int_{\mathbb{R}} u_\epsilon \partial_x^3 u_\epsilon \, dx - 2\epsilon \int_{\mathbb{R}} u_\epsilon \partial_x^4 u_\epsilon \, dx
$$

$$
= -2q \int_{\mathbb{R}} u_\epsilon \partial_x (u_\epsilon v_\epsilon) \, dx - 2a \int_{\mathbb{R}} \partial_x u_\epsilon \partial_x^2 u_\epsilon \, dx
$$

$$
+ 2b \int_{\mathbb{R}} P_\epsilon u_\epsilon \, dx + 2\epsilon \int_{\mathbb{R}} \partial_x u_\epsilon \partial_x^2 u_\epsilon \, dx
$$

$$
= -2q \int_{\mathbb{R}} u_\epsilon \partial_x (u_\epsilon v_\epsilon) \, dx - 2a \int_{\mathbb{R}} \partial_x u_\epsilon \partial_x^2 u_\epsilon \, dx
$$

$$
+ 2b \int_{\mathbb{R}} P_\epsilon u_\epsilon \, dx - 2\epsilon \|\partial_x^2 u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2. \tag{26}
$$

Therefore,

$$
\frac{d}{dt} \|u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\epsilon \|\partial_x^2 u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 2b \int_{\mathbb{R}} P_\epsilon u_\epsilon \, dx - 2q \int_{\mathbb{R}} u_\epsilon \partial_x (u_\epsilon v_\epsilon) \, dx.
$$

Arguing as in ([15], Lemma 2.2), we have (22), (23) and (24).

Finally, arguing as in (6), Lemma 2.3), we have (25). \qed

Lemma 4. Assume (5). Fix $T > 0$. There exists a constant $C_0 > 0$, independent on $\epsilon$, such that

$$
\|\partial_x^2 v_\epsilon\|_{L^\infty((0, T) \times \mathbb{R})} \leq C_0 \left( 1 + \|u_\epsilon\|_{L^\infty((0, T) \times \mathbb{R})}^2 \right). \tag{26}
$$

Proof. Let $0 \leq t \leq T$. Thanks to the third equation of (16), we have that

$$
\alpha \partial_x^2 v_\epsilon = \kappa u_\epsilon^2 - \beta \partial_x v_\epsilon - \gamma v_\epsilon. \tag{27}
$$
Therefore, by (24),
\[
|a| \|\partial_x^2 v\| = |\kappa u^2 - \beta \partial_x v - \gamma v| \leq |\kappa| u^2 + |\beta| \|\partial_x v\| + |\gamma| |v|
\]
\[
\leq |\kappa| \|u_\varepsilon\|_{L^2((0,T) \times \mathbb{R})} + |\beta| \|\partial_x v_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} + |\gamma| \|v_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}
\]
which gives (26). 

Arguing as in ([6], Lemma 2.2), we have the following result.

**Lemma 5.** For each \( t \geq 0 \), we have that
\[
\int_0^{-\infty} P_\varepsilon(t,x)dx = A_\varepsilon(t),
\] (28)
\[
\int_0^{\infty} P_\varepsilon(t,x)dx = A_\varepsilon(t),
\] (29)

where
\[
A_\varepsilon(t) = -\frac{1}{b} \partial_t P_\varepsilon(t,0) - \frac{\varphi}{b} u_\varepsilon^2(t,0) - \frac{a}{b} \partial_x^2 u_\varepsilon(t,0) - \frac{\varphi}{b} u_\varepsilon(t,0)v_\varepsilon(t,0) + \frac{\varepsilon}{b} \partial_x u_\varepsilon(t,0).
\]

In particular, we have
\[
\int_{\mathbb{R}} P_\varepsilon(t,x)dx = 0.
\] (30)

**Lemma 6.** Assume (5). Fix \( T > 0 \). There exists a constant \( C(T) > 0 \), independent on \( \varepsilon \), such that
\[
\|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} + 2\varepsilon e^{C_0 t} \int_0^t e^{-C_0 s} \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})} ds
\]
\[
\leq C(T) \left(1 + \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2\right).
\] (31)

for every \( 0 \leq t \leq T \). In particular, we have that
\[
\|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq \sqrt{C(T) \left(1 + \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2\right)}.
\] (32)

**Proof.** Let \( 0 \leq t \leq T \). We begin by observing that, by (28), we can consider the following function:
\[
F_\varepsilon(t,x) = \int_{-\infty}^{x} P_\varepsilon(t,y)dy.
\] (33)
Integrating the second equation of (16) on \((-\infty, x)\), we have
\[
P_\varepsilon(t,x) = \int_{-\infty}^{x} u_\varepsilon(t,y)dy.
\] (34)
Differentiating (34) with respect to \( t \), we get
\[
\partial_t P_\varepsilon(t,x) = \frac{d}{dt} \int_{-\infty}^{x} u_\varepsilon(t,y)dy = \int_{-\infty}^{x} \partial_t u_\varepsilon(t,x)dx.
\] (35)
Equation (33), (35) and an integration on \((-\infty, x)\) of the first equation of (16) give
\[
\partial_t P_\varepsilon = bF_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon - g u_\varepsilon^3 + a \partial_x^2 u_\varepsilon - \varphi v_\varepsilon u_\varepsilon.
\] (36)
Multiplying (36) by $-2P_t$, an integration on $\mathbb{R}$ gives
\[
\frac{d}{dt} \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 2b \int_{\mathbb{R}} F_t P_t dx - 2\varepsilon \int_{\mathbb{R}} P_t \partial_x^2 u_t dx - 2g \int_{\mathbb{R}} P_t u_t^3 dx + 2a \int_{\mathbb{R}} P_t \partial_x^2 u_t dx - 2a \int_{\mathbb{R}} P_t \partial_x u_t dx.
\] (37)

Observe that, by (18), (30), (33) and the second equation of (16),
\[
2b \int_{\mathbb{R}} F_t P_t dx = \frac{1}{2} \int_{\mathbb{R}} \partial_x F_t dx - bF_t^2(t, \infty) = b \left( \int_{\mathbb{R}} P_t(t, x) dx \right)^2 dx = 0,
\]
\[
-2\varepsilon \int_{\mathbb{R}} P_t \partial_x^2 u_t dx = -2\varepsilon \int_{\mathbb{R}} \partial_x P_t \partial_x^2 u_t dx = -2\varepsilon \int_{\mathbb{R}} u_t \partial_x^2 u_t dx = -2\varepsilon \|\partial_x u_t(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\]
\[
2a \int_{\mathbb{R}} P_t \partial_x^2 u_t dx = -2a \int_{\mathbb{R}} \partial_x P_t \partial_x u_t dx = -2a \int_{\mathbb{R}} u_t \partial_x u_t = 0,
\]
\[
-2g \int_{\mathbb{R}} P_t v_t u_t dx = -2g \int_{\mathbb{R}} P_t \partial_x P_t v_t dx = 2g \int_{\mathbb{R}} \partial_x v_t P_t^2 dx.
\]

Consequently, by (24) and (37),
\[
\frac{d}{dt} \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x u_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 2g \int_{\mathbb{R}} \partial_x v_t P_t^2 dx - 2g \int_{\mathbb{R}} P_t u_t^3 dx
\]
\[
\leq 2g \int_{\mathbb{R}} |\partial_x v_t| P_t^2 dx + 2g \int_{\mathbb{R}} |P_t| |u_t|^3 dx \leq 2|g| \int_{\mathbb{R}} |\partial_x v_t| P_t^2 dx + 2|g| \int_{\mathbb{R}} |P_t| |u_t|^3 dx \leq C_0 \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2|g| \int_{\mathbb{R}} |P_t| |u_t|^3 dx.
\] (38)

Due to Lemma 3 and the Young inequality,
\[
2|g| \int_{\mathbb{R}} |P_t| |u_t|^3 dx \leq \int_{\mathbb{R}} |2P_t u_t| |gu_t^2| dx \leq 2 \int_{\mathbb{R}} P_t^2 u_t^2 dx + \frac{g^2}{2} \int_{\mathbb{R}} u_t^4 dx \leq 2 \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|u_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{g^2}{2} \|u_t\|_{L^2((0,T)\times\mathbb{R})}^2 \|u_t(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_0 \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}^4 + C_0 \|u_t\|_{L^2((0,T)\times\mathbb{R})}^2 \leq \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}^4 + C_0 + C_0 \|u_t\|_{L^2((0,T)\times\mathbb{R})}^2.
\]

It follows from (38) that
\[
\frac{d}{dt} \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x u_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0 \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}^4 + C_0 \left( 1 + \|u_t\|_{L^2((0,T)\times\mathbb{R})}^2 \right).
\] (39)

Thanks to Lemma 3 and the Hölder inequality,
\[
\int_{\mathbb{R}} P_t^2(t, x) dx = \int_{-\infty}^\infty P_t u_t dy \leq 2 \int_{\mathbb{R}} |P_t| |u_t| dx \leq \|P_t(t, \cdot)\|_{L^2(\mathbb{R})} \|u_t(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_0 \|P_t(t, \cdot)\|_{L^2(\mathbb{R})}.
\]
Hence,
\[ \| P_\varepsilon(t, \cdot) \|_{L^\infty(\mathbb{R})}^4 \leq C_0 \| P_\varepsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2. \] (40)

It follows from (39) and (40) that
\[
\frac{d}{dt} \| P_\varepsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\varepsilon \| u_\varepsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
\leq C_0 \| P_\varepsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2 + C_0 \left( 1 + \| u_\varepsilon \|_{L^\infty((0, T) \times \mathbb{R})}^2 \right).
\]

The Gronwall Lemma and (17) give
\[
\| P_\varepsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{C_{\text{ext}}} \int_0^t e^{-C s} \| u_\varepsilon(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \\
\leq C_0 e^{C_{\text{ext}}} + C_0 \left( 1 + \| u_\varepsilon \|_{L^\infty((0, T) \times \mathbb{R})}^2 \right) e^{C_{\text{ext}}} \int_0^t e^{-C s} \, ds \\
\leq C(T) \left( 1 + \| u_\varepsilon \|_{L^\infty((0, T) \times \mathbb{R})}^2 \right),
\]
which gives (31).

Finally, (32) follows from (31) and (40). \(\square\)

Following ([6], Lemma 2.5), we prove the following result.

**Lemma 7.** Assume (5). Fix \( T > 0 \). There exists a constant \( C(T) > 0 \), independent on \( \varepsilon \), such that
\[ \| u_\varepsilon \|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T). \] (41)

In particular, we have
\[ \| \partial_x u_\varepsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{C_{\text{ext}}} \int_0^t e^{-C s} \| \partial_x^3 u_\varepsilon(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \leq C(T), \] (42)
for every \( 0 \leq t \leq T \). Moreover,
\[ \| \partial_x^2 v_\varepsilon \|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T), \]
\[ \| P_\varepsilon(t, \cdot) \|_{L^2(\mathbb{R})} \leq C(T), \]
\[ \| P_\varepsilon(t, \cdot) \|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T), \] (43)
for every \( 0 \leq t \leq T \).

**Proof.** Let \( 0 \leq t \leq T \). Multiplying the first equation of (1) by \(-2\partial_x^2 u_\varepsilon + \frac{2g}{a} u_\varepsilon^3\), we have that
\[
\left( -2\partial_x^2 u_\varepsilon + \frac{2g}{a} u_\varepsilon^3 \right) \partial_x u_\varepsilon + 3g \left( -2\partial_x^2 u_\varepsilon + \frac{2g}{a} u_\varepsilon^3 \right) u_\varepsilon^2 \partial_x^3 u_\varepsilon \\
- a \left( -2\partial_x^2 u_\varepsilon + \frac{2g}{a} u_\varepsilon^3 \right) \partial_x^3 u_\varepsilon + q \left( -2\partial_x^2 u_\varepsilon + \frac{2g}{a} u_\varepsilon^3 \right) \partial_x (v_\varepsilon u_\varepsilon) \\
= b \left( -2\partial_x^2 u_\varepsilon + \frac{2g}{a} u_\varepsilon^3 \right) P_\varepsilon - \varepsilon \left( -2\partial_x^2 u_\varepsilon + \frac{2g}{a} u_\varepsilon^3 \right) \partial_x^4 u_\varepsilon.
\] (44)

Observe that, by (18) and the second equation of (16),
\[ -2b \int_\mathbb{R} P_\varepsilon \partial_x^2 u_\varepsilon \, dx = 2b \int_\mathbb{R} \partial_x P_\varepsilon \partial_x u_\varepsilon \, dx = 2b \int_\mathbb{R} u_\varepsilon \partial_x u_\varepsilon \, dx = 0. \] (45)
Moreover,
\[
\int_{\mathbb{R}} \left( -2\partial_x^2 u_e + \frac{2\gamma}{a} u_e^3 \right) \partial_t u_e dx = \frac{d}{dt} \left( \| \partial_x u_e(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{\gamma}{2a} \int_{\mathbb{R}} u_e^4 dx \right),
\]
\[
3\gamma \int_{\mathbb{R}} \left( -2\partial_x^2 u_e + \frac{2\gamma}{a} u_e^3 \right) u_e^2 \partial_x u_e dx = -6\gamma \int_{\mathbb{R}} u_e^2 \partial_x u_e \partial_x^2 u_e dx,
\]
\[
-\epsilon \int_{\mathbb{R}} \left( -2\partial_x^2 u_e + \frac{2\gamma}{a} u_e^3 \right) \partial_x^3 u_e dx = -2\epsilon \left\| \partial_x^3 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]
\[
+ \frac{6\gamma \epsilon}{a} \int_{\mathbb{R}} u_e^2 \partial_x u_e \partial_x^3 u_e dx.
\]

Defined
\[
G(t) := \| \partial_x u_e(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{\gamma}{2a} \int_{\mathbb{R}} u_e^4 dx,
\]
it follows from (45), (46) and an integration on \( \mathbb{R} \) of (44) that
\[
\frac{dG(t)}{dt} + 2\epsilon \left\| \partial_x^3 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})} = \frac{2\gamma}{a} \int_{\mathbb{R}} p_t u_e^3 dx + \frac{6\gamma \epsilon}{a} \int_{\mathbb{R}} u_e^2 \partial_x u_e \partial_x^3 u_e dx
\]
\[
+ 2\epsilon \int_{\mathbb{R}} \partial_x (v_t u_e) \partial_x^3 u_e dx - 2qga \int_{\mathbb{R}} \partial_x (v_t u_e) u_e^3 dx.
\]

Observe that
\[
2\epsilon \int_{\mathbb{R}} \partial_x (v_t u_e) \partial_x^2 u_e dx = 2\epsilon \int_{\mathbb{R}} u_e \partial_x v_t \partial_x^2 u_e dx + 2\epsilon \int_{\mathbb{R}} v_t \partial_x u_e \partial_x^3 u_e dx
\]
\[
= -2\epsilon \int_{\mathbb{R}} \partial_x^2 v_t u_e \partial_x u_e dx - 3\epsilon \int_{\mathbb{R}} \partial_x v_t (\partial_x u_e)^2 dx,
\]
\[
-2qga \int_{\mathbb{R}} \partial_x (v_t u_e) u_e^3 dx = -2qga \int_{\mathbb{R}} \partial_x v_t u_e^4 dx - 2qga \int_{\mathbb{R}} v_t \partial_x u_e u_e^3 dx
\]
\[
- \frac{3qg}{2a} \int_{\mathbb{R}} \partial_x v_t u_e^4 dx.
\]

Consequently, by (48),
\[
\frac{dG(t)}{dt} + 2\epsilon \left\| \partial_x^3 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})} = \frac{2\gamma}{a} \int_{\mathbb{R}} p_t u_e^3 dx + \frac{6\gamma \epsilon}{a} \int_{\mathbb{R}} u_e^2 \partial_x u_e \partial_x^3 u_e dx
\]
\[
- 3\epsilon \int_{\mathbb{R}} \partial_x v_t (\partial_x u_e)^2 dx - 2\epsilon \int_{\mathbb{R}} \partial_x^2 v_t u_e \partial_x u_e dx
\]
\[
- \frac{3qg}{2a} \int_{\mathbb{R}} \partial_x v_t u_e^4 dx.
\]
Due to (26), (32), Lemma 3 and the Young inequality,

\[ \left\| \frac{2bg}{a} \int_{\mathbb{R}} p_t u_t^3 dx \right\| \leq 2 \int_{\mathbb{R}} p_t^2 u_t^2 dx + \frac{b^2 g^2}{2a^2} \int_{\mathbb{R}} u_t^4 dx \leq 2 \left\| p_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{b^2 g^2}{2a^2} \left\| u_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]

\[ \leq 2C_0 \left\| p_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 + C_0 \left\| u_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 + C_0 \left( 1 + \left\| u_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right) \]

3)\[ q \int_{\mathbb{R}} |\partial_x v_x| (|\partial_x u_x|)^2 dx \leq 3q \left\| \partial_x u_x(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \left\| \partial_x u_x(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]

\[ \leq C_0 \left\| \partial_x u_x(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]

2)\[ q \int_{\mathbb{R}} \left\| \partial^2_x v_x \partial_x u_x \right\|_{L^\infty(\mathbb{R})} \left\| \partial_x u_x \right\|_{L^\infty(\mathbb{R})} \left\| \partial_x u_x \right\|_{L^2(\mathbb{R})}^2 \]

\[ \leq q^2 \left\| \partial^2_x v_x \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| u_x(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x u_x(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]

\[ \leq C_0 \left\| \partial^2_x v_x \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 + \left\| \partial_x u_x(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]

\[ \leq C(T) \left( 1 + \left\| u_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right) \]

Consequently, by (49),

\[ \frac{dG(t)}{dt} + \frac{\epsilon}{a} \left\| \partial^2_x u_x(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 \left\| \partial_x u_x(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{9\epsilon^2}{a^2} \int_{\mathbb{R}} u_t^4 (\partial_x u_x)^2 dx + C(T) \left( 1 + \left\| u_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right) \]

Lemma 2.6 of [6] says that

\[ \int_{\mathbb{R}} u_t^4 (\partial_x u_x)^2 dx \leq 4 \left\| u_x(t, \cdot) \right\|_{L^2(\mathbb{R})}^4 \left\| \partial^2_x u_x(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
Hence, by Lemma 3, we have that
\[
\frac{9\gamma^2 \varepsilon}{a^2} \int u^4 \partial_x u \frac{dx}{2a} \leq \frac{36\gamma^2 \varepsilon}{a^2} \| u \|_{L^2([0,1])}^4 \| \partial_x^2 u \|_{L^2([R])}^2 \\
\leq C_0 \varepsilon \| \partial_x^2 u \|_{L^2([R])}^2 .
\]
Therefore, by (50),
\[
\frac{dG(t)}{dt} + \varepsilon \int \| \partial_x^2 u \|_{L^2([R])}^2 \leq C_0 \varepsilon \| \partial_x u \|_{L^2([R])}^2 + C_0 \varepsilon \| \partial_x^2 u \|_{L^2([R])}^2 \\
+ C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right).
\]
Observe that, by (47) and Lemma 3,
\[
C_0 \| \partial_x u \|_{L^2([R])} = C_0 G(t) - \frac{8C_0}{2a} \int u^4 dx \\
\leq C_0 G(t) + \frac{8C_0}{2a} \| u \|_{L^\infty((0,T) \times [0,1])} \| u \|_{L^2([R])}^2 \\
\leq C_0 G(t) + C_0 \| u \|_{L^\infty((0,T) \times [0,1])}^2 .
\]
It follows from (52) and (53) that
\[
\frac{dG(t)}{dt} + \varepsilon \int \| \partial_x^2 u \|_{L^2([R])}^2 \leq C_0 G(t) + C_0 \varepsilon \| \partial_x^2 u \|_{L^2([R])}^2 \\
+ C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right).
\]
The Gronwall Lemma, (17), (47) and Lemma 3 that
\[
\| \partial_x u \|_{L^2([R])}^2 + \frac{8}{2a} \int u^4 dx + \varepsilon C_0 t \int_0^t e^{-C_0 s} \| \partial_x^2 u \|_{L^2([R])}^2 ds \\
\leq C_0 e^{C_0 t} + C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right) e^{C_0 t} \int_0^t e^{-C_0 s} ds \\
+ C_0 \varepsilon e^{C_0 t} e^{-C_0 t} \| \partial_x^2 u \|_{L^2([R])}^2 ds \\
\leq C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right) + C(T) \varepsilon \int_0^t \| \partial_x^2 u \|_{L^2([R])}^2 ds \\
\leq C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right).
\]
Consequently, by Lemma 3,
\[
\| \partial_x u \|_{L^2([R])}^2 + 2 \varepsilon e^{C_0 t} \int_0^t e^{-C_0 s} \| \partial_x^2 u \|_{L^2([R])}^2 ds \\
\leq C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right) - \frac{8}{2a} \| u \|_{L^2([R])}^4 \\
\leq C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right) + \frac{8}{2a} \int u^4 dx \\
\leq C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right) + \frac{8}{2a} \| u \|_{L^2([0,1])}^4 \| u \|_{L^2([R])}^2 \\
\leq C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right) + C_0 \| u \|_{L^\infty((0,T) \times [0,1])}^2 \\
\leq C(T) \left( 1 + \| u \|_{L^\infty((0,T) \times [0,1])}^2 \right).
\]
We prove (41). Thanks to (54), Lemma 3 and the Hölder inequality,
\[
u_t^2(t, x) = 2 \int_{-\infty}^{x} u_t \partial_x u \, dx \leq \int_{\mathbb{R}} |u_t| \|\partial_x u_t\| \, dx
\]
\[
\leq \|u_t(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x u_t(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T) \sqrt{\left(1 + \|u_t\|_{L^\infty(0, T) \times \mathbb{R})}^2\right)}.
\]

Hence,
\[
\|u_t\|_{L^2(0, T) \times \mathbb{R})}^2 - C(T) \|u_t\|_{L^\infty(0, T) \times \mathbb{R})}^2 - C(T) \leq 0,
\]
which gives (41).

Finally, (42) follows from (41) and (54), while (26), (31), (32) and (41) give (43). □

Arguing as in (15), Lemmas 2.8 and 2.9, we have the following result.

**Lemma 8.** Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\epsilon$, such that
\[
\left\|\partial_x^2 v_t(t, \cdot)\right\|_{L^2(\mathbb{R})} \leq C(T),
\]
for every $0 \leq t \leq T$.

**Lemma 9.** Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\epsilon$, such that
\[
\|\partial_x u_t\|_{L^\infty(0, T) \times \mathbb{R})} \leq C(T),
\]
In particular, we have that
\[
\left\|\partial_x^2 u_t(t, \cdot)\right\|_{L^2(\mathbb{R})}^2 + 2 e^{C^2 t} \int_0^t e^{-C^2 s} \left\|\partial_x^4 u_t(s, \cdot)\right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T),
\]
for every $0 \leq t \leq T$.

**Proof.** Let $0 \leq t \leq T$. Consider two real constants $D$, $E$ which will be specified later. Multiplying the first equation of (16) by
\[
2a^2 \partial_x^4 u_t + Dagu_t(\partial_x u_t)^2 + Eagu_t^2 \partial_x^2 u_t,
\]
we have that
\[
\left(2a^2 \partial_x^4 u_t + Dagu_t(\partial_x u_t)^2 + Eagu_t^2 \partial_x^2 u_t\right) \partial_t u_t
\]
\[
+ 3g \left(2a^2 \partial_x^4 u_t + Dagu_t(\partial_x u_t)^2 + Eagu_t^2 \partial_x^2 u_t\right) u_t \partial_x u_t
\]
\[
- a \left(2a^2 \partial_x^4 u_t + Dagu_t(\partial_x u_t)^2 + Eagu_t^2 \partial_x^2 u_t\right) \partial_x^2 u_t
\]
\[
+ q \left(2a^2 \partial_x^4 u_t + Dagu_t(\partial_x u_t)^2 + Eagu_t^2 \partial_x^2 u_t\right) u_t \partial_x v_t
\]
\[
+ \epsilon \left(2a^2 \partial_x^4 u_t + Dagu_t(\partial_x u_t)^2 + Eagu_t^2 \partial_x^2 u_t\right) \partial_x^2 u_t
\]
\[
= b \left(2a^2 \partial_x^4 u_t + Dagu_t(\partial_x u_t)^2 + Eagu_t^2 \partial_x^2 u_t\right) P_t
\]
\[
- \epsilon \left(2a^2 \partial_x^4 u_t + Dagu_t(\partial_x u_t)^2 + Eagu_t^2 \partial_x^2 u_t\right) \partial_x^2 u_t.
\]
Observe that
\[
\int_{\mathbb{R}} \left( 2a^2 \partial_1^2 u_\varepsilon + \operatorname{Dag}_u \partial_v (\partial_v u_\varepsilon) + E ag u^2 \partial_2^2 u_\varepsilon \right) \partial_v u_\varepsilon \, dx \\
= a^2 \frac{d}{dt} \left\| \partial_1^2 u_\varepsilon (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \operatorname{Dag} \int_{\mathbb{R}} u_\varepsilon (\partial_v u_\varepsilon)^2 \partial_v u_\varepsilon \, dx + E ag \int_{\mathbb{R}} u_\varepsilon^2 \partial_2^2 u_\varepsilon \partial_v u_\varepsilon \, dx,
\]

\[
3g \int_{\mathbb{R}} \left( 2a^2 \partial_1^3 u_\varepsilon + \operatorname{Dag}_u \partial_v (\partial_v u_\varepsilon) + E ag u^2 \partial_2^3 u_\varepsilon \right) u_\varepsilon \partial_v u_\varepsilon \, dx \\
= -12a^2g \int_{\mathbb{R}} u_\varepsilon (\partial_v u_\varepsilon)^2 \partial_v u_\varepsilon \, dx + 6a^2g \int_{\mathbb{R}} u_\varepsilon^2 \partial_2^2 u_\varepsilon \partial_v u_\varepsilon \, dx \\
+ (3D - 6E) ag^2 \int_{\mathbb{R}} u_\varepsilon^3 (\partial_v u_\varepsilon)^3 \, dx,
\]

\[
q \int_{\mathbb{R}} \left( 2a^2 \partial_1^3 u_\varepsilon + \operatorname{Dag}_u (\partial_v u_\varepsilon)^2 + E ag u^2 \partial_2^3 u_\varepsilon \right) u_\varepsilon \partial_v u_\varepsilon \, dx \\
= -2a^2q \int_{\mathbb{R}} \partial_v u_\varepsilon \partial_v \partial_v u_\varepsilon \partial_v u_\varepsilon \, dx - 2a^2q \int_{\mathbb{R}} u_\varepsilon^2 \partial_2^2 u_\varepsilon \partial_v u_\varepsilon \, dx \\
+ (D - 3E) agq \int_{\mathbb{R}} u_\varepsilon^2 (\partial_v u_\varepsilon)^2 \partial_v u_\varepsilon \, dx - agqE \int_{\mathbb{R}} u_\varepsilon^2 \partial_v u_\varepsilon \partial_2^2 u_\varepsilon \, dx \\
= 2a^2q \int_{\mathbb{R}} \partial_v u_\varepsilon \partial_v (\partial_v u_\varepsilon)^2 \partial_v u_\varepsilon \, dx + 4a^2q \int_{\mathbb{R}} \partial_v u_\varepsilon \partial_2^2 u_\varepsilon \partial_v u_\varepsilon \, dx \\
+ 2a^2q \int_{\mathbb{R}} u_\varepsilon^2 \partial_2^2 u_\varepsilon \partial_v u_\varepsilon \, dx + ag (D - 3E) \int_{\mathbb{R}} u_\varepsilon^3 (\partial_v u_\varepsilon)^2 \partial_v u_\varepsilon \, dx,
\]

\[
q \int_{\mathbb{R}} \left( 2a^2 \partial_1^3 u_\varepsilon + \operatorname{Dag}_u (\partial_v u_\varepsilon)^2 + E ag u^2 \partial_2^3 u_\varepsilon \right) v_\varepsilon \partial_v u_\varepsilon \, dx \\
= -2a^2q \int_{\mathbb{R}} \partial_v u_\varepsilon \partial_v \partial_v u_\varepsilon \partial_v u_\varepsilon \, dx - 2a^2q \int_{\mathbb{R}} v_\varepsilon \partial_2^2 u_\varepsilon \partial_v u_\varepsilon \, dx \\
+ (D - E) agq \int_{\mathbb{R}} u_\varepsilon v_\varepsilon (\partial_v u_\varepsilon)^3 \partial_v u_\varepsilon \, dx + \frac{E ag}{2} \int_{\mathbb{R}} u_\varepsilon^2 (\partial_v u_\varepsilon)^3 \partial_v^2 u_\varepsilon \, dx \\
= 2a^2q \int_{\mathbb{R}} \partial_2^2 v_\varepsilon \partial_v u_\varepsilon \partial_v u_\varepsilon \, dx + 3a^2q \int_{\mathbb{R}} \partial_v v_\varepsilon (\partial_v u_\varepsilon)^2 \partial_v u_\varepsilon \, dx \\
+ (D - E) agq \int_{\mathbb{R}} u_\varepsilon v_\varepsilon (\partial_v u_\varepsilon)^3 \partial_v u_\varepsilon \, dx + \frac{E ag}{2} \int_{\mathbb{R}} u_\varepsilon^2 (\partial_v u_\varepsilon)^3 \partial_v^2 u_\varepsilon \, dx,
\]

\[
- \epsilon \int_{\mathbb{R}} \left( 2a^2 \partial_1^3 u_\varepsilon + \operatorname{Dag}_u (\partial_v u_\varepsilon)^2 + E ag u^2 \partial_2^3 u_\varepsilon \right) \partial_1^4 u_\varepsilon \, dx \\
= -2a^2 \epsilon \left\| \partial_1^4 u_\varepsilon (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \operatorname{Dag} \int_{\mathbb{R}} u_\varepsilon (\partial_v u_\varepsilon)^2 \partial_1^4 u_\varepsilon \, dx \\
+ E ag \int_{\mathbb{R}} u_\varepsilon^2 \partial_2^2 u_\varepsilon \partial_1^4 u_\varepsilon \, dx.
\]
Consequently, an integration on $\mathbb{R}$ of (58) gives
\[
\frac{d}{dt} \left\| \nabla^2 u(t) \right\|^2_{L^2(\mathbb{R})} + \text{Dag} \int_{\mathbb{R}} u_t(\partial_x u)^2 \partial_t u dx + \text{Eag} \int_{\mathbb{R}} u_t^2 \partial_4^2 u_t \partial_4 u dx + 2a^2 \left\| \partial_t^4 u(t) \right\|^2_{L^2(\mathbb{R})} = -a^2 (30 + 2D + E) \int_{\mathbb{R}} u_t \partial_t u_e (\partial_t^2 u_e)^2 dx - (3D - 6E) \text{ag} \int_{\mathbb{R}} u_t^3 (\partial_x u_e)^3 dx - 5a^2 \int_{\mathbb{R}} \partial_x v_e (\partial_t^2 u_e)^2 dx - 6a^2 \int_{\mathbb{R}} \partial_x^2 v_e \partial_x u_e \partial_5^2 u_e dx - 2a^2 \int_{\mathbb{R}} u_t \partial_t^3 u_e dx - aq (D - 3E) \int_{\mathbb{R}} u_t^2 (\partial_x u_e)^2 \partial_x v_e dx - (D - E) \text{ag} \int_{\mathbb{R}} u_t v_e (\partial_x u_e)^3 dx + \text{Dag} \int_{\mathbb{R}} P_e \partial_t^4 u_e dx - \text{Eag} \int_{\mathbb{R}} P_e u^2 (\partial_x u_e)^2 dx + \text{Eag} \int_{\mathbb{R}} P_e u_t^2 \partial_4^2 u_t dx - \text{Dag} \int_{\mathbb{R}} u_t (\partial_x u_e)^2 \partial_4^4 u_e dx + \text{Eag} \int_{\mathbb{R}} u_t^2 \partial_4^2 u_t \partial_4^4 u_e dx.
\tag{59}
\]

Thanks to the second equation of (16) and (18), we have that
\[
2a^2 b \int_{\mathbb{R}} P_e \partial_t^4 u_e dx = -2a^2 b \int_{\mathbb{R}} \partial_x P_e \partial_t^3 u_e = -2a^2 b \int_{\mathbb{R}} u_t \partial_t^3 u_e dx = 0,
\]
\[
\text{Eag} \int_{\mathbb{R}} P_e u^2 \partial_4^2 u_e dx = -\text{Eag} \int_{\mathbb{R}} P_e u_t^2 \partial_4^3 u_e dx - 2\text{Eag} \int_{\mathbb{R}} P_e u_t (\partial_x u_e)^2 dx = -2\text{Eag} \int_{\mathbb{R}} P_e u_t (\partial_x u_e)^2 dx - \text{Eag} \int_{\mathbb{R}} u_t \partial_4^2 u_e dx = -2\text{Eag} \int_{\mathbb{R}} P_e u_t (\partial_x u_e)^2 dx.
\]

Therefore, by (59),
\[
\frac{d}{dt} \left\| \nabla^2 u(t) \right\|^2_{L^2(\mathbb{R})} + \text{Dag} \int_{\mathbb{R}} u_t(\partial_x u)^2 \partial_t u dx + \text{Eag} \int_{\mathbb{R}} u_t^2 \partial_4^2 u_t \partial_4 u dx + 2a^2 \left\| \partial_t^4 u(t) \right\|^2_{L^2(\mathbb{R})} = -a^2 (30 + 2D + E) \int_{\mathbb{R}} u_t \partial_t u_e (\partial_t^2 u_e)^2 dx - (3D - 6E) \text{ag} \int_{\mathbb{R}} u_t^3 (\partial_x u_e)^3 dx - 5a^2 \int_{\mathbb{R}} \partial_x v_e (\partial_t^2 u_e)^2 dx - 6a^2 \int_{\mathbb{R}} \partial_x^2 v_e \partial_x u_e \partial_5^2 u_e dx - 2a^2 \int_{\mathbb{R}} u_t \partial_t^3 u_e dx - aq (D - 3E) \int_{\mathbb{R}} u_t^2 (\partial_x u_e)^2 \partial_x v_e dx - (D - E) \text{ag} \int_{\mathbb{R}} u_t v_e (\partial_x u_e)^3 dx + \text{Dag} \int_{\mathbb{R}} P_e \partial_t^4 u_e dx - \text{Eag} \int_{\mathbb{R}} P_e u^2 (\partial_x u_e)^2 dx + \text{Eag} \int_{\mathbb{R}} P_e u_t^2 \partial_4^2 u_t dx - \text{Dag} \int_{\mathbb{R}} u_t (\partial_x u_e)^2 \partial_4^4 u_e dx + \text{Eag} \int_{\mathbb{R}} u_t^2 \partial_4^2 u_t \partial_4^4 u_e dx.
\tag{60}
\]
Observe that

\[
\begin{align*}
\mathcal{D} & \int_a \partial_t (u_t^2) (\partial_x u_t) \, dx + E \mathcal{A} \int_a u_t^2 \partial_x^2 u_t \partial_t u_t \, dx \\
= & \frac{D \mathcal{A}}{2} \int_a \partial_t (u_t^2) (\partial_x u_t) \, dx - E \mathcal{A} \int_a \partial_x u_t \partial_x (u_t^2) \partial_t u_t \, dx \\
= & \frac{D \mathcal{A}}{2} \int_a \partial_t (u_t^2) (\partial_x u_t) \, dx - 2E \mathcal{A} \int_a u_t (\partial_x u_t)^2 \partial_t u_t \, dx - E \mathcal{A} \int_a u_t^2 \partial_x u_t \partial_x^2 u_t \, dx \\
\frac{d}{dt} & \left( D \mathcal{A} \int_a \partial_t (u_t^2) (\partial_x u_t) \, dx \right)
\end{align*}
\]

Consequently, by (60),

\[
\left\| \partial_t^2 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{D \mathcal{A}}{2} \left( \frac{D}{2} - E \right) \int_a \partial_t (u_t^2) (\partial_x u_t) \, dx
\]

\[
- \frac{E \mathcal{A}}{2} \left( \int_a u_t^2 \partial_t (\partial_x u_t)^2 \, dx + 2\alpha^2 \left\| \partial_t^2 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right)
\]

\[
= -a^2 (30 + 2D + E) \int_a u_t \partial_x u_t (\partial_x u_t)^2 \, dx - (3D - 6E) \mathcal{A}^2 \int_a u_t^3 (\partial_x u_t)^3 \, dx
\]

\[
- 5a^2 \mathcal{A} \int_a \partial_x \partial_t u_t (\partial_x u_t)^2 \, dx - 6a^2 \mathcal{A} \int_a \partial_t \partial_x \partial_t u_t \partial_x^2 u_t \, dx
\]

\[
- 2a^2 \mathcal{A} \int_a u_t \partial_x^3 \partial_t u_t \, dx - \mathcal{A} (D - 3E) \int_a u_t^2 (\partial_x u_t)^2 \partial_x^2 v_t \, dx
\]

\[
+ (D - 2E) \mathcal{A} b \int_a P_t u_t (\partial_x u_t)^2 \, dx + \mathcal{D} \mathcal{A} \int_a u_t (\partial_x u_t)^2 \partial_t^4 u_t \, dx
\]

\[
+ E \mathcal{A} g \int_a u_t^2 \partial_x^2 u_t \partial_t^4 u_t \, dx.
\]

We search \( D, E \) such that

\[
\frac{D}{2} - E = - \frac{E}{2}, \quad 30 + 2D + E = 0,
\]

that is

\[
D = E, \quad 30 + 2D + E = 0.
\]

Since \( D = E - 10 \) is the unique solution of (62), it follows from (61) that

\[
\left\| \partial_t^2 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 5a^2 \int_a \partial_t (u_t^2) (\partial_x u_t) \, dx + 5a^2 \int_a u_t^2 \partial_t (\partial_x u_t)^2 \, dx
\]

\[
+ 2\alpha^2 \left\| \partial_t^2 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
= -30a^2 \int_a u_t^3 (\partial_x u_t)^3 \, dx - 5a^2 \alpha \int_a \partial_x \partial_t u_t (\partial_x u_t)^2 \, dx - 2\alpha^2 \int_a u_t \partial_x^3 \partial_t u_t \, dx
\]

\[
- 2a^2 \alpha \int_a u_t \partial_x^3 \partial_t u_t \, dx - 20a^2 \int_a u_t^2 (\partial_x u_t)^2 \partial_t \partial_x \partial_t u_t \, dx
\]

\[
+ 10a^2 b \int_a P_t u_t (\partial_x u_t)^2 \, dx - 10a^2 \mathcal{A} \int_a u_t (\partial_x u_t)^2 \partial_t^4 u_t \, dx
\]

\[
- 10a^2 g \int_a u_t^2 \partial_x^2 u_t \partial_t^4 u_t \, dx,
\]
that is
\[
\frac{d}{dt} \left( a^2 \left\| \partial^4_x u_t(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 5aq \int_{\mathbb{R}} \partial_x^2 u^2(x) dx \right) + 2a^2 \varepsilon \left\| \partial^4_x u_t(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
= -30aq^2 \int_{\mathbb{R}} u^3 \partial_x u^3 dx - 5a^2 q \int_{\mathbb{R}} \partial_x v^2 \partial^2_x u^2 dx \\
- 6a^2 q \int_{\mathbb{R}} \partial_x^2 v \partial_x u \partial^2_x u dx - 2a^2 q \int_{\mathbb{R}} u \partial_x^2 v \partial^2_x u dx \\
- 20aq \int_{\mathbb{R}} u^2 \partial_x u \partial^2_x u dx - 5aq \int_{\mathbb{R}} u^2 \partial_x u \partial^2_x u dx \\
+ 10aq \int_{\mathbb{R}} P_t u \partial_x u^2 dx - 10aq \int_{\mathbb{R}} u \partial_x u \partial^2_x u dx \\
- 10aq \int_{\mathbb{R}} u^2 \partial^2_x u \partial^4_x u dx.
\]

Due to (41), (42), (43), (55), Lemma 3 and the Young inequality,
\[
|30aq^2| \int_{\mathbb{R}} |u_t|^3 |\partial_x u_t|^3 dx \leq |30aq^2| \left\| u_t \right\|^3_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x u_t|^3 dx \\
\leq C(T) \int_{\mathbb{R}} |\partial_x u_t|^3 dx \\
\leq C(T) \left\| \partial_x u_t(t, \cdot) \right\|^3_{L^2(\mathbb{R})} + C(T) \int_{\mathbb{R}} (\partial_x u_t)^4 dx \\
\leq C(T) + C(T) \left\| \partial_x u_t(t, \cdot) \right\|^3_{L^2(\mathbb{R})} \left\| \partial_x u_t(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq C(T) \left( 1 + \left\| \partial_x u_t \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \right),
\]
\[
|5a^2 q| \int_{\mathbb{R}} |\partial_x v| |(\partial^2_x u_t)^2| dx \leq |5a^2 q| \left\| \partial_x v(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \left\| \partial^2_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq C_0 \left\| \partial^2_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})},
\]
\[
|6a^2 q| \int_{\mathbb{R}} |\partial_x^2 v \partial_x u_t| \left\| \partial^2_x u_t \right\|^2 dx \leq 3a^4 q^2 \int_{\mathbb{R}} (\partial^2_x v)^2 |(\partial_x u_t)|^2 dx + 3 \left\| \partial^2_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq 3a^4 q^2 \left( \left\| \partial^2_x v \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x u_t(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 3 \left\| \partial^2_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq C(T) + 3 \left\| \partial^2_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})},
\]
\[
|2a^2 q| \int_{\mathbb{R}} |u_t \partial^2_x v| \left\| \partial^2_x u_t \right\|^2 dx \leq a^4 q^2 \int_{\mathbb{R}} u^2 (\partial^2_x v)^2 dx + \left\| \partial^2_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq a^4 q^2 \left\| u \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial^2_x v(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \left\| \partial^2_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq C(T) + \left\| \partial^2_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})},
\]
\[
|20aq| \int_{\mathbb{R}} u^2 \partial_x u^2 |(\partial_x v)| dx \leq |20aq| \left\| \partial_x v(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \left\| u \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq C(T) \left\| \partial_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \leq C(T),
\]
\[ 5agq \int_{\mathbb{R}} u_t^2 |\partial_x u_e|^3 |\partial^2_x v_e| \, dx \leq |5agq| \left\| u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| \partial^2_x v_e \right\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x u_e|^3 \, dx \\
\leq C(T) \int_{\mathbb{R}} |\partial_x u_e|^3 \, dx \\
\leq C(T) \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \int_{\mathbb{R}} (\partial_x u_e)^4 \, dx \\
\leq C(T) + C(T) \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right), \\
|10agb| \int_{\mathbb{R}} |P_t u_e| (|\partial_x u_e|^2)^2 \, dx \leq |10agb| \left\| P_t \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| u_e \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T), \\
|10agc| \int_{\mathbb{R}} |u_e (\partial_x u_e)^2| |\partial^4_x u_e| \, dx = \varepsilon \int_{\mathbb{R}} |10gu_e (\partial_x u_e)^2| |\partial^4_x u_e| \, dx \\
\leq 50g^2 \varepsilon \int_{\mathbb{R}} u_t^2 (\partial_x u_e)^4 \, dx + \frac{a^2 \varepsilon}{2} \left\| \partial^4_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq 50g^2 \varepsilon \left\| u_e \right\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} (\partial_x u_e)^4 \, dx + \frac{a^2 \varepsilon}{2} \left\| \partial^4_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) \varepsilon \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
+ \frac{a^2 \varepsilon}{2} \left\| \partial^4_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
|10agd| \int_{\mathbb{R}} |u_t^2 (\partial_x^2 u_e)| |\partial^4_x u_e| \, dx = \varepsilon \int_{\mathbb{R}} |10gu_t^2 (\partial_x^2 u_e)| |\partial^4_x u_e| \, dx \\
\leq 50g^2 \varepsilon \int_{\mathbb{R}} u_t^4 (\partial_x^2 u_e)^2 \, dx + \frac{a^2 \varepsilon}{2} \left\| \partial^4_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq 50g^2 \varepsilon \left\| u_e \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{a^2 \varepsilon}{2} \left\| \partial^4_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) \varepsilon \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{a^2 \varepsilon}{2} \left\| \partial^4_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \\
\]

Therefore, defining

\[ G_1(t) = a^2 \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 5ag \int_{\mathbb{R}} u_t^2 (\partial_x u_e)^2 \, dx, \tag{64} \]

by (63) and (64), we have

\[ \frac{dG_1(t)}{dt} + \varepsilon \left\| \partial^4_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 \left\| \partial^2_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right) \\
+ C(T) \varepsilon \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
+ C(T) \varepsilon \left\| \partial^4_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{65} \]
Observe that by (41), (42) and (64),

\[ C_0 \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = \frac{C_0 a^2}{d^2} \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]

\[ = \frac{C_0}{a} G_1(t) - \frac{5C_0 g}{a} \int_{\mathbb{R}} u_e^2(\partial_x u_e)^2 dx \]

\[ \leq C_0 G_1(t) + \frac{5C_0 g}{a} \int_{\mathbb{R}} u_e^2(\partial_x u_e)^2 dx \]

\[ \leq C_0 G_1(t) + \frac{5C_0 g}{a} \left\| u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^{2} \]

\[ \leq C_0 G_1(t) + C(T). \quad (66) \]

It follows from (65) and (66) that

\[
\frac{dG_1(t)}{dt} + \varepsilon \left\| \partial_x^4 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 G_1(t) + C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \right) \]

\[ + C(T) \varepsilon \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^{2} \]

\[ + C(T) \varepsilon \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^{2}. \]

The Gronwall Lemma, (17) and Lemma 3 give

\[
\left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^{2} + 5ag \int_{\mathbb{R}} u_e^2(\partial_x u_e)^2 dx + 2\varepsilon C_0 t \int_{0}^{t} e^{-C_0 s} \left\| \partial_x^4 u_e(s, \cdot) \right\|_{L^2(\mathbb{R})}^{2} ds \]

\[ \leq C_0 e^{C_0 t} + C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \right) \]

\[ e^{C_0 t} \int_{0}^{t} e^{-C_0 s} ds \]

\[ + C(T) \varepsilon \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^{2} \]

\[ + C(T) \varepsilon \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^{2} \]

\[ \leq C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \right) \]

\[ + C(T) \varepsilon \int_{0}^{t} \left\| \partial_x^2 u_e(s, \cdot) \right\|_{L^2(\mathbb{R})}^{2} ds \]

\[ \leq C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \right). \]

Therefore, thanks to (41) and (42),

\[
\left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^{2} + 2\varepsilon C_0 t \int_{0}^{t} e^{-C_0 s} \left\| \partial_x^4 u_e(s, \cdot) \right\|_{L^2(\mathbb{R})}^{2} ds \]

\[ = C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \right) - 5ag \int_{\mathbb{R}} u_e^2(\partial_x u_e)^2 dx \]

\[ \leq C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \right) + \left| 5ag \right| \int_{\mathbb{R}} u_e^2(\partial_x u_e)^2 dx \]

\[ \leq C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \right) + \left| 5ag \right| \left\| u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^{2} \]

\[ \leq C(T) \left( 1 + \left\| \partial_x u_e \right\|_{L^\infty((0,T) \times \mathbb{R})}^{2} \right). \quad (67) \]
We prove (56). Due to (42), (67) and the Hölder inequality,
\[
(\partial_x u_\varepsilon(t,x))^2 = 2 \int_{-\infty}^{\infty} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx \leq 2 \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_x^2 u_\varepsilon| dx \\
\leq \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(T) \sqrt{1 + \|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2}.
\]
Therefore,
\[
\|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 - C(T) \|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2 - C(T) \leq 0,
\]
which gives (56).

Finally, (57) follows from (56) and (67). 

**Lemma 10.** Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\varepsilon$, such that

\[
\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \leq C(T),
\]
for every $0 \leq t \leq T$. In particular, we have that

\[
\left\|\partial_x^4 v_\varepsilon\right\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T).
\]

**Proof.** Let $0 \leq t \leq T$. Differentiating the third equation of (16) twice with respect to $x$, we have

\[
\alpha \partial_x^4 v_\varepsilon = 2\kappa (\partial_x u_\varepsilon)^2 + 2\kappa u_\varepsilon \partial_x^2 u_\varepsilon - \beta \partial_x^4 v_\varepsilon - \gamma \partial_x^2 v_\varepsilon.
\]

Since

\[
u_\varepsilon(t, \pm \infty) = \partial_x u_\varepsilon(t, \pm \infty) = \partial_x^2 u_\varepsilon(t, \pm \infty) = 0,
\]
it follows from (24) and (55) that

\[
\partial_x^4 v_\varepsilon(t, \pm \infty) = 0.
\]

Multiplying (70) by $2 \alpha \partial_x^4 v_\varepsilon$, an integration on $\mathbb{R}$ gives

\[
2\alpha^2 \left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 = 2\kappa \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^4 v_\varepsilon dx + 2\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^4 v_\varepsilon dx - 2\beta \alpha \int_{\mathbb{R}} \partial_x^2 v_\varepsilon \partial_x^4 v_\varepsilon dx - 2\gamma \alpha \int_{\mathbb{R}} \partial_x^2 v_\varepsilon \partial_x^4 v_\varepsilon dx - 2\gamma \alpha \int_{\mathbb{R}} \partial_x^2 v_\varepsilon \partial_x^4 v_\varepsilon dx.
\]

Observe that, thanks to (24), (55) and (72),

\[
-2\beta \alpha \int_{\mathbb{R}} \partial_x^3 v_\varepsilon \partial_x^4 v_\varepsilon dx = 0,
\]

\[
-2\gamma \alpha \int_{\mathbb{R}} \partial_x^2 v_\varepsilon \partial_x^4 v_\varepsilon dx = 2\gamma \alpha \left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\]

Therefore, by (55) and (73),

\[
2\alpha^2 \left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \leq 2|\kappa| \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 |\partial_x^4 v_\varepsilon| dx + 2|\kappa| \int_{\mathbb{R}} |u_\varepsilon \partial_x^2 u_\varepsilon| |\partial_x^4 v_\varepsilon| dx \\
+ 2|\gamma| \left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \leq 2|\kappa| \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 |\partial_x^4 v_\varepsilon| dx + 2|\kappa| \int_{\mathbb{R}} |u_\varepsilon \partial_x^2 u_\varepsilon| |\partial_x^4 v_\varepsilon| dx + C(T).
\]
Due to (41), (42), (56), (57) and the Young inequality,

\[
2|\alpha| \int_{\mathbb{R}} (\partial_x u_e)^2 |\partial_x^4 v_e| dx \leq \kappa^2 \int_{\mathbb{R}} (\partial_x u_e)^4 dx + \alpha^2 \left\| \partial_x^4 v_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq \kappa^2 \left\| \partial_x u_e \right\|_{L^\infty((0, \infty) \times \mathbb{R})}^2 \left\| \partial_x u_e \right\|_{L^2(\mathbb{R})}^2 + \alpha^2 \left\| \partial_x^4 v_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) + \alpha^2 \left\| \partial_x^4 v_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]

which gives (68).

Finally, we prove (69). Due to (55), (68) and the Hölder inequality,

\[
(\partial_x^4 v_e(t, x))^2 = 2 \int_{-\infty}^{x} \partial_x^3 v_e \partial_x^4 v_e dx \leq 2 \int_{\mathbb{R}} |\partial_x^4 v_e| |\partial_x^4 v_e| dx \\
\leq \left\| \partial_x^4 v_e(t, \cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^4 v_e(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C(T).
\]

Hence,

\[
\left\| \partial_x^4 v_e \right\|_{L^\infty((0, \infty) \times \mathbb{R})}^2 \leq C(T),
\]

which gives (69). \qed

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1.

We begin by proving the following lemma.

Lemma 11. Fix T > 0. Then,

the sequence \( \{u_e\}_{\varepsilon>0} \) is compact in \( L^2_{loc}((0, \infty) \times \mathbb{R}) \). \hspace{1cm} (75)

Consequently, there exists a subsequence \( \{u_{e_k}\}_{k \in \mathbb{N}} \) of \( \{u_e\}_{\varepsilon>0} \) and \( u \in L^2_{loc}((0, \infty) \times \mathbb{R}) \) such that, for each compact subset \( K \) of \((0, \infty) \times \mathbb{R}) \),

\[
u_{e_k} \rightarrow v \text{ in } H^1((0, T) \times \mathbb{R}), \hspace{1cm} (77)
\]

\[
P_{e_k} \rightarrow P \text{ in } L^2((0, T) \times \mathbb{R}), \hspace{1cm} (78)
\]

Moreover, \((u, v, P)\) is a solution of (1) satisfying (11) and (12).
**Proof.** We begin by proving (75). To prove (75), we rely on the Aubin–Lions Lemma (see [58–60]). We recall that
\[
H^1_{\text{loc}}(\mathbb{R}) \hookrightarrow \hookrightarrow L^2_{\text{loc}}(\mathbb{R}) \hookrightarrow H^{-1}_{\text{loc}}(\mathbb{R}),
\]
where the first inclusion is compact and the second is continuous. Owing to the Aubin–Lions Lemma [60], to prove (75), it suffices to show that
\[
\{ u_\epsilon \}_{\epsilon > 0} \text{ is uniformly bounded in } L^2(0, T; H^1_{\text{loc}}(\mathbb{R})), \quad (79)
\]
\[
\{ \partial_t u_\epsilon \}_{\epsilon > 0} \text{ is uniformly bounded in } L^2(0, T; H^{-1}_{\text{loc}}(\mathbb{R})). \quad (80)
\]
We prove (79). Thanks to (42), (57) and Lemma 3,
\[
\| u_\epsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2 = \| u_\epsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \| \partial_x u_\epsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \| \partial_x^2 u_\epsilon(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C(T).
\]
Therefore,
\[
\{ u_\epsilon \}_{\epsilon > 0} \text{ is uniformly bounded in } L^\infty(0, T; H^2(\mathbb{R})),
\]
which gives (79).
We prove (80). By the first equation of (16),
\[
\partial_t u_\epsilon = \partial_x \left( -g u_\epsilon^3 + a \partial_x^2 u_\epsilon - q v_\epsilon u_\epsilon - \epsilon \partial_x^3 u_\epsilon \right) + b P_\epsilon.
\]
We have that
\[
\| u_\epsilon \|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T). \quad (82)
\]
Indeed, thanks to (41) and Lemma 3,
\[
g^2 \int_0^T \int_{\mathbb{R}} u_\epsilon^2 \, dt \, dx \leq g^2 \| u_\epsilon \|_{L^\infty((0, T) \times \mathbb{R})}^4 \int_0^T \int_{\mathbb{R}} u_\epsilon^2 \, dt \, dx \leq C(T) \int_0^T \int_{\mathbb{R}} u_\epsilon^2 \, dt \, dx \leq C(T).
\]
We prove that
\[
q^2 \| v_\epsilon u_\epsilon \|_{L^2((0, T) \times \mathbb{R})}^2 \leq C(T). \quad (83)
\]
Due to Lemma 3,
\[
q^2 \int_0^T \int_{\mathbb{R}} v_\epsilon^2 u_\epsilon^2 \, dt \, dx \leq q^2 \| v_\epsilon \|_{L^\infty((0, T) \times \mathbb{R})}^2 \int_0^T \int_{\mathbb{R}} u_\epsilon^2 \, dt \, dx \leq C(T) \int_0^T \int_{\mathbb{R}} u_\epsilon^2 \, dt \, dx \leq C(T).
\]
Observe that, since \(0 < \epsilon < 1\), thanks to (42) and (57),
\[
\epsilon \| \partial_x^2 u_\epsilon \|_{L^2((0, T) \times \mathbb{R}), \beta^2 \| \partial_x^2 u_\epsilon \|_{L^2((0, T) \times \mathbb{R})}^2 \leq C(T). \quad (84)
\]
Therefore, by (82), (83) and (84),
\[
\left\{ \partial_x \left( -g u_\epsilon^3 + a \partial_x^2 u_\epsilon - q v_\epsilon u_\epsilon - \epsilon \partial_x^3 u_\epsilon \right) \right\}_{\epsilon > 0} \text{ is bounded in } H^1((0, T) \times \mathbb{R}). \quad (85)
\]
Moreover, by (43), we have that
\[
b^2 \| P_\epsilon \|_{L^2((0, T) \times \mathbb{R})}^2 \leq C(T). \quad (86)
\]
Equation (80) follows from (85) and (86).

Thanks to the Aubin–Lions Lemma, (75) and (76) hold.

Observe that, (77) follows from Lemma 3, while, by (43), we have (78). Consequently, \((u, v, P)\) solves (1).

Observe again that, thanks to Lemmas 3, 7, 8, 9, (10) and the second equation of (16), we obtain (11).

Finally, we prove (12). Thanks to Lemmas 3 and 7, we have

\[
\nu_t \rightarrow u \text{ in } H^1((0, T) \times \mathbb{R}). \quad (87)
\]

Therefore, (12) follows from (19) and (87). \[\square\]

We are ready for the proof of Theorem 1.

**Proof of Theorem 1.** Lemma 11 gives the existence of a solution of (1) satisfying (11) and (12). Let \((u_1, P_1)\) and \((u_2, P_2)\) be two solutions of (1) satisfying (11) and (12), namely

\[
\begin{align*}
\partial_t u_1 + 3gu_1^2 \partial_x u_1 - a\partial_x^3 u_1 + g \partial_x (u_1 v_1) &= bP_1, & t > 0, x \in \mathbb{R}, \\
\partial_t P_1 &= u_1, & t > 0, x \in \mathbb{R}, \\
\alpha \partial_x^2 v_1 + \beta \partial_x v_1 + \gamma v_1 &= \kappa u_1^2, & t > 0, x \in \mathbb{R}, \\
P_1(t, -\infty) &= 0, & t > 0, \\
u_1(0, x) &= u_{1,0}(x), & x \in \mathbb{R}, \\
\partial_t u_2 + 3gu_2^2 \partial_x u_2 - a\partial_x^3 u_2 + g \partial_x (u_2 v_2) &= bP_2, & t > 0, x \in \mathbb{R}, \\
\partial_t P_2 &= u_2, & t > 0, x \in \mathbb{R}, \\
\alpha \partial_x^2 v_2 + \beta \partial_x v_2 + \gamma v_2 &= \kappa u_2^2, & t > 0, x \in \mathbb{R}, \\
P_2(t, -\infty) &= 0, & t > 0, \\
u_2(0, x) &= u_{2,0}(x), & x \in \mathbb{R}.
\end{align*}
\]

Then, the triad \((\omega, V, \Omega)\) defined by

\[
\begin{align*}
\omega(t, x) &= u_1(t, x) - u_2(t, x), & V(t, x) &= v_1(t, x) - v_2(t, x), \\
\Omega(t, x) &= \int_{-\infty}^{x} \omega(t, y) dy = \int_{-\infty}^{x} u_1(t, y) dy - \int_{-\infty}^{x} u_2(t, y) dy, \\
\Omega(0, x) &= \int_{-\infty}^{x} \omega(0, y) dy = \int_{-\infty}^{x} u_1(0, y) dy - \int_{-\infty}^{x} u_2(0, y) dy.
\end{align*}
\]

is solution of the following Cauchy problem:

\[
\begin{align*}
\partial_t \omega + 3g \left( u_1^2 \partial_x u_1 - u_2^2 \partial_x u_2 \right) - a\partial_x^3 \omega + g \partial_x (u_1 v_1 - u_2 v_2) &= b\Omega, & t > 0, x \in \mathbb{R}, \\
\partial_x \Omega &= \omega, & t > 0, x \in \mathbb{R}, \\
\alpha \partial_x^2 V + \beta \partial_x V + \gamma V &= \kappa (u_1^2 - u_2^2), & t > 0, x \in \mathbb{R}, \\
\Omega(t, -\infty) &= 0, & t > 0, \\
\omega(0, x) &= u_{1,0} - u_{2,0}(x), & x \in \mathbb{R}.
\end{align*}
\]
Arguing as in ([15], Theorem 1.1), we have that

\[
\|V(t, \cdot)\|_{L^2(R)}^2 \leq C(T) \|\omega(t, \cdot)\|_{L^2(R)}^2, \\
\|V(t, \cdot)\|_{L^\infty(R)}^2 \leq C(T) \|\omega(t, \cdot)\|_{L^2(R)}^2, \\
\|\partial_x V(t, \cdot)\|_{L^\infty(R)}^2 \leq C(T) \|\omega(t, \cdot)\|_{L^2(R)}^2.
\]

Moreover, by (12) and (88),

\[
\Omega(t, \infty) = \int_R \omega(t, x) dx = \int_R u_1(t, y) dy - \int_R u_2(t, x) dx = 0.
\]

Observe that, by (88)

\[
3g \left( u_1^2 \partial_x u_1 - u_2^2 \partial_x u_2 \right) = 3g \left( u_1^2 \partial_x u_1 - u_2^2 \partial_x u_1 + u_2^2 \partial_x u_1 - u_2^2 \partial_x u_2 \right) \\
= 3g \left( \partial_x u_1 \left( u_1^2 - u_2^2 \right) + u_2^2 \partial_x \omega \right) \\
= 3g \left( \partial_x u_1 \left( u_1 + u_2 \right) \omega + u_2^2 \partial_x \omega \right).
\]

Moreover, arguing as in ([15], Theorem 1.1),

\[
q \partial_x (u_1 v_1 - u_2 v_2) = \partial_x (u_1 v_1 - u_2 v_1 + u_2 v_1 - u_2 v_2) = \partial_x (v_1 \omega + q \partial_x (u_2 V)).
\]

Therefore, thanks to (94) and (95), the first equation of (89) is equivalent to the following one:

\[
\partial_t \omega = b \Omega - 3g \partial_x u_1 \left( u_1 + u_2 \right) \omega - 3g u_2^2 \partial_x \omega + a \partial_t \omega - q \partial_x (v_1 \omega) - q \partial_x (u_2 V).
\]

Multiplying (96) by 2\omega, an integration on R gives

\[
\frac{d}{dt} \|\omega(t, \cdot)\|_{L^2(R)}^2 = 2b \int_R \Omega \partial_x \omega dx - 6g \int_R \partial_x u_1 \left( u_1 + u_2 \right) \omega^2 dx - 2a \int_R \omega \partial_t^2 \omega dx \\
- 6g \int_R u_2^2 \omega \partial_x \omega dx - 2q \int_R \partial_x (v_1 \omega) \omega dx - 2q \int_R \partial_x (u_2 V) \omega dx.
\]

Observe that, by (88) and (93),

\[
2b \int_R \Omega \partial_x \omega dx = 2b \int_R \Omega \partial_x \Omega dx = b \Omega^2 (t, \infty) = 0, \\
- 6g \int_R u_2^2 \omega \partial_x \omega dx = 6g \int_R u_2 \partial_x u_2 \omega^2 dx, \\
- 2a \int_R \omega \partial_t \omega dx = 2a \int_R \partial_x \omega \partial_t \omega = 0, \\
- 2q \int_R \partial_x (v_1 \omega) \omega dx = 2q \int_R v_1 \omega \partial_x \omega dx = -q \int_R \partial_x v_1 \omega^2 dx, \\
- 2q \int_R \partial_x (u_2 V) \omega dx = - 2q \int_R \partial_x u_2 V \omega dx - 2q \int_R u_2 \partial_x V \omega dx.
\]

It follows from (97) and (98) that

\[
\frac{d}{dt} \|\omega(t, \cdot)\|_{L^2(R)}^2 = - 6g \int_R \partial_x u_1 \left( u_1 + u_2 \right) \omega^2 dx + 6g \int_R u_2 \partial_x u_2 \omega^2 dx \\
- q \int_R \partial_x v_1 \omega^2 dx - 2q \int_R \partial_x u_2 V \omega dx - 2q \int_R u_2 \partial_x V \omega dx.
\]
Since (11) holds, we have that

\[
\|u_1\|_{L^\infty((0,T) \times \mathbb{R})}, \|H_2\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T),
\]

\[
\|\partial_x u_1\|_{L^\infty((0,T) \times \mathbb{R})}, \|\partial_x u_2\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T),
\]

\[
\|\partial_x v_1\|_{L^\infty((0,T) \times \mathbb{R})}, \|u_2(t, \cdot)\|_{L^2(\mathbb{R})}, \|\partial_x u_2(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T),
\]

(100)

for every $0 \leq t \leq T$. Consequently, by (91), (100) and the Hölder inequality,

\[
|6g| \int_\mathbb{R} |\partial_x u_1||u_1 + u_2|\omega^2 dx \leq |6g| \|\partial_x u_1\|_{L^\infty((0,T) \times \mathbb{R})} \int_\mathbb{R} |u_1 + u_2|\omega^2 dx
\]

\[
\leq C(T) \left( \|u_1\|_{L^\infty((0,T) \times \mathbb{R})} + \|u_2\|_{L^\infty((0,T) \times \mathbb{R})} \right) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2
\]

\[
\leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\]

\[
|6g| \int_\mathbb{R} |u_2| |\partial_x u_2|\omega^2 dx \leq |6g| \|u_2\|_{L^\infty((0,T) \times \mathbb{R})} \int_\mathbb{R} |\partial_x u_2|\omega^2 dx
\]

\[
\leq C(T) \|\partial_x u_2\|_{L^\infty((0,T) \times \mathbb{R})} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2
\]

\[
\leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\]

\[
|q| \int_\mathbb{R} |\partial_x v_1|\omega^2 dx \leq |q| \|\partial_x v_1\|_{L^\infty((0,T) \times \mathbb{R})} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2
\]

\[
\leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\]

\[
|2q| \int_\mathbb{R} |\partial_x u_2| |V|\omega dx \leq |2q| \|V(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_\mathbb{R} |\partial_x u_2|\omega dx
\]

\[
\leq C(T) \|\partial_x u_2(t, \cdot)\|_{L^2(\mathbb{R})} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2
\]

\[
\leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\]

\[
|2q| \int_\mathbb{R} |u_2| |\partial_x V|\omega dx \leq |2q| \|\partial_x V(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_\mathbb{R} |u_2|\omega dx
\]

\[
\leq C(T) \|u_2(t, \cdot)\|_{L^2(\mathbb{R})} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2
\]

\[
\leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]

It follows from (99) that

\[
\frac{d}{dt} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]

(101)

The Gronwall Lemma and (89) give

\[
\|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq e^{C(T)t} \|\omega(0, x)\|_{L^2(\mathbb{R})}^2.
\]

(102)

Since (11) holds, by (88), arguing as in Lemma 5, $\Omega(t, \cdot)$ is integrable at $\pm \infty$. Moreover, thanks to (93) and Lemma 5, we have that

\[
\int_\mathbb{R} \Omega(t, x) dx = 0.
\]

(103)

Consider the following function:

\[
\Omega_1(t, x) = \int_x^\infty \Omega(t, y) dy,
\]

(104)

since, by the second equation of (89),

\[
\partial_t \Omega = \frac{d}{dt} \int_{-\infty}^x \omega(t, y) dy = \int_{-\infty}^x \partial_t \omega(t, y) dy,
\]

(105)
integrating the first equation of (89) on \((-\infty, x)\), by (104) and (105), we have that
\[
\partial_t \Omega = b\Omega_1 - g \left( u_1^3 - u_2^3 \right) + \alpha \partial_x^2 \omega - q(u_1 v_1 - u_2 v_2). \tag{106}
\]
Observe that, by (88),
\[
u_1^3 - u_2^3 = \left( u_1^2 + u_2^2 + u_1 u_2 \right) \omega,
\]
u_1 v_1 - u_2 v_2 = \nu_1 \omega + u_2 V.
Consequently, by (106),
\[
\partial_t \Omega = b\Omega_1 - g \left( u_1^2 + u_2^2 + u_1 u_2 \right) \omega + \alpha \partial_x^2 \omega - q u_1 \omega - q u_2 V. \tag{107}
\]
It follows from (88), (93), (103) and (104) that
\[
\begin{align*}
2b \int_{\mathbb{R}} \Omega_1 \Omega dx = & 2b \int_{\mathbb{R}} \Omega_1 \partial_x \Omega_1 dx = b \Omega^2_1(t, \infty) = b \left( \int_{\mathbb{R}} \Omega(t, x) dx \right)^2 = 0, \\
2a \int_{\mathbb{R}} \partial_x^2 \omega \omega dx = & -2a \int_{\mathbb{R}} \partial_x \Omega \partial_x \omega dx = -2a \int_{\mathbb{R}} \omega \partial_x \omega dx = 0.
\end{align*}
\]
Therefore, multiplying (107) by 2\(\Omega\), an integration on \(\mathbb{R}\) gives
\[
\frac{d}{dt} \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 = -2g \int_{\mathbb{R}} \left( u_1^2 + u_2^2 + u_1 u_2 \right) \omega \Omega dx \\
- 2q \int_{\mathbb{R}} v_1 \omega \Omega dx - 2q \int_{\mathbb{R}} u_2 V \Omega dx. \tag{108}
\]
Due to (91), (100) and the Young inequality,
\[
\begin{align*}
|2g| \int_{\mathbb{R}} \left| u_1^2 + u_2^2 + u_1 u_2 \right| |\omega| \|\Omega\| dx \\
\leq g^2 \int_{\mathbb{R}} \left( u_1^2 + u_2^2 + u_1 u_2 \right)^2 \omega^2 dx + \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\end{align*}
\]
\[
|\omega| \int_{\mathbb{R}} \left| v_1 \omega \right| \|\Omega\| dx \\
\leq q^2 \int_{\mathbb{R}} v_1^2 \omega^2 dx + \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
\leq q^2 \|v_1\|_{L^\infty((0, T) \times \mathbb{R})} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\]
\[
|\omega| \int_{\mathbb{R}} \left| u_2 V \right| \|\Omega\| dx \\
\leq q^2 \int_{\mathbb{R}} V^2 u_2^2 dx + \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
\leq q^2 \|V(t, \cdot)\|_{L^\infty(\mathbb{R})} \|u_2(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) \|V(t, \cdot)\|_{L^\infty(\mathbb{R})} \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]
Therefore, by (108),
\[
\frac{d}{dt} \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 3 \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2. \tag{109}
\]
Adding (101) and (109), by (88) and the second equation of (89), we have that
\[
\frac{d}{dt} \| \Omega(t, \cdot) \|^2_{H^1(\mathbb{R})} \leq C(T) \| \omega(t, \cdot) \|^2_{L^2(\mathbb{R})} + 3 \| \Omega(t, \cdot) \|^2_{L^2(\mathbb{R})} \leq C(T) \| \Omega(t, \cdot) \|^2_{H^1(\mathbb{R})}
\]
and
\[
\| \Omega(t, \cdot) \|^2_{H^1(\mathbb{R})} \leq e^{C(T)t} \| \Omega(0, \cdot) \|^2_{H^1(\mathbb{R})}. \tag{110}
\]
Therefore, (13) follows (14), (88), (89), (90), (102) and (110). □

4. Conclusions

In this paper we studied the Cauchy problem for the Spectrum Pulse equation. It is a third order nonlocal nonlinear evolutive equation related to the dynamics of the electrical field of linearly polarized continuum spectrum pulses in optical waveguides. Our existence analysis is based on on passing to the limit in a fourth order perturbation of the equation. If the initial datum belongs to $H^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and has zero mean we use the Aubin–Lions Lemma while if it belongs to $H^3(\mathbb{R}) \cap L^1(\mathbb{R})$ and has zero mean we use the Sobolev Immersion Theorem. Finally, we directly prove a stability estimate that implies the uniqueness of the solution.

Author Contributions: Writing—original draft, G.M.C. and L.d.R.

Funding: This research received no external funding.

Acknowledgments: The authors are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. $u_0 \in H^3(\mathbb{R}) \cap L^1(\mathbb{R})$

In this appendix, we consider the Cauchy problem (1), where, on the initial datum, we assume
\[
u_0(x) \in H^3(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_\mathbb{R} u_0(x) dx = 0, \tag{A1}
\]
while on the function $P(x)$, defined in (3), we assume (4). Moreover, we assume (5). The main result of this appendix is the following theorem.

**Theorem A1.** Assume (3), (4), (5) and (A1). Fix $T > 0$, there exists an unique solution $(u, v, P)$ of (1) such that
\[
\begin{align*}
u &\in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^3(\mathbb{R})), \\
\nu &\in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^5(\mathbb{R})) \cap W^{1, \infty}((0, T) \times \mathbb{R}), \\
\partial_t^2 v &\in L^\infty((0, T) \times \mathbb{R}) \cap L^\infty(0, T; L^2(\mathbb{R})), \\
P &\in L^\infty(0, T; H^4(\mathbb{R})).
\end{align*} \tag{A2}
\]
Moreover, (12) and (13) hold.

To prove Theorem A1, we consider the approximation (16), where $u_{\varepsilon, 0}$ is a $C^\infty$ approximation of $u_0$ such that
\[
\begin{align*}
\| u_{\varepsilon, 0} \|_{H^1(\mathbb{R})} &\leq \| u_0 \|_{H^1(\mathbb{R})}, \quad \int_\mathbb{R} u_{\varepsilon, 0} dx = 0, \\
\| P_{\varepsilon, 0} \|_{L^2(\mathbb{R})} &\leq \| P_0 \|_{L^2(\mathbb{R})}, \quad \int_\mathbb{R} P_{\varepsilon, 0} dx = 0, \\
\varepsilon \| \partial_t^2 u_{\varepsilon}(t, \cdot) \|^2_{L^2(\mathbb{R})} &\leq C_0,
\end{align*} \tag{A3}
\]
where $C_0$ is a positive constant, independent on $\epsilon$.

Let us prove some a priori estimates on $u_\epsilon$, $v_\epsilon$ and $P_\epsilon$.

Since $H^2(\mathbb{R}) \subset H^3(\mathbb{R})$, then Lemmas 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 are still valid.

We prove the following result.

**Lemma A1.** Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\epsilon$, such that

$$\left\| \partial_t^2 u_\epsilon \right\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T). \quad (A4)$$

In particular, we have that

$$\left\| \partial_t^2 u_\epsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\epsilon C(T)t \int_0^t e^{C(T)s} \left\| \partial_t^2 u_\epsilon(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \quad (A5)$$

for every $0 \leq t \leq T$.

**Proof.** Let $0 \leq t \leq T$. Multiplying the first equation of (16) by $-2\partial_t^6 u_\epsilon$, we have that

$$-2\partial_t^6 u_\epsilon \partial_t u_\epsilon = -2bP_\epsilon \partial_t^6 u_\epsilon + 2\partial_t^6 u_\epsilon \partial_t^4 u_\epsilon + 6\epsilon u_\epsilon^2 \partial_\epsilon^2 u_\epsilon \partial_t^6 u_\epsilon$$

$$-2\partial_t^3 u_\epsilon \partial_t^6 u_\epsilon + 2\epsilon u_\epsilon \partial_t^2 u_\epsilon \partial_t^6 u_\epsilon + 2\epsilon u_\epsilon \partial_t^2 u_\epsilon \partial_t^6 u_\epsilon.$$  \hspace{1cm} (A6)

Observe that by (18) and the second equation of (16),

$$-2b \int_{\mathbb{R}} P_\epsilon \partial_t^6 u_\epsilon dx = 2b \int_{\mathbb{R}} \partial_\epsilon P_\epsilon \partial_t^6 u_\epsilon dx = 2b \int_{\mathbb{R}} \partial_t^6 u_\epsilon dx$$

$$= -2b \int_{\mathbb{R}} \partial_t u_\epsilon \partial_t^5 u_\epsilon dx = 2b \int_{\mathbb{R}} \partial_t^5 u_\epsilon \partial_t^4 u_\epsilon dx = 0. \quad (A7)$$

Moreover,

$$-2 \int_{\mathbb{R}} \partial_t^6 u_\epsilon \partial_t u_\epsilon = \frac{d}{dt} \left\| \partial_t^2 u_\epsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,$$

$$2\epsilon \int_{\mathbb{R}} \partial_t^5 u_\epsilon \partial_t^4 u_\epsilon dx = -2\epsilon \left\| \partial_t^2 u_\epsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,$$

$$-2\epsilon \int_{\mathbb{R}} \partial_t^3 u_\epsilon \partial_t^6 u_\epsilon dx = 2\epsilon \int_{\mathbb{R}} \partial_t^5 u_\epsilon \partial_t^4 u_\epsilon dx = 0. \quad (A8)$$

It follows from (A7), (A8) and an integration of (A6) on $\mathbb{R}$ that

$$\frac{d}{dt} \left\| \partial_t^2 u_\epsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\epsilon \left\| \partial_t^2 u_\epsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2$$

$$= 6\epsilon \int_{\mathbb{R}} u_\epsilon^2 \partial_\epsilon u_\epsilon \partial_t^6 u_\epsilon dx + 2\epsilon \int_{\mathbb{R}} u_\epsilon \partial_\epsilon v_\epsilon \partial_t^6 u_\epsilon dx + 2\epsilon \int_{\mathbb{R}} v_\epsilon \partial_\epsilon u_\epsilon \partial_t^6 u_\epsilon dx. \quad (A9)$$
Observe that

\[
6g \int_R \partial_x \partial_x u \partial_x^5 u \, dx = -12g \int_R u_x (\partial_x u_x)^2 \partial_x^5 u \, dx - 6g \int_R u_x^2 \partial_x^2 u \partial_x^3 u \, dx \\
= 12g \int_R (\partial_x u_x)^3 \partial_x^5 u \, dx + 36g \int_R u_x \partial_x u_x \partial_x^2 u \partial_x^4 u \, dx \\
+ 6g \int_R u_x^2 \partial_x^2 u \partial_x^4 u \, dx \\
= -72g \int_R (\partial_x u_x)^2 \partial_x^2 u \partial_x^3 u \, dx - 36g \int_R (\partial_x^2 u_x)^2 \partial_x^3 u \, dx \\
- 42g \int_R u_x \partial_x u_x (\partial_x^3 u_x)^2 \, dx,
\]

\[
2q \int_R u_x \partial_x v_x \partial_x^5 u \, dx = -2q \int_R \partial_x u_x \partial_x v_x \partial_x^5 u \, dx - 2q \int_R u_x^2 \partial_x v_x \partial_x^5 u \, dx \\
= 2q \int_R \partial_x^2 u_x \partial_x v_x \partial_x^4 u \, dx + 4q \int_R \partial_x u_x \partial_x^2 v_x \partial_x^3 u \, dx \\
+ 2q \int_R u_x \partial_x^2 v_x \partial_x^2 u \, dx \\
= -2q \int_R \partial_x v_x (\partial_x^3 u_x)^2 \, dx - 6q \int_R \partial_x^2 u_x \partial_x^2 v_x \partial_x^2 u \, dx \\
- 6q \int_R \partial_x u_x \partial_x^3 v_x \partial_x^2 u \, dx - 2q \int_R u_x \partial_x v_x \partial_x^3 u \, dx \\
= -2q \int_R \partial_x v_x (\partial_x^3 u_x)^2 \, dx + 3q \int_R \partial_x^2 v_x (\partial_x^2 u_x)^2 \, dx \\
- 6q \int_R \partial_x u_x \partial_x^3 v_x \partial_x^2 u \, dx - 2q \int_R u_x \partial_x v_x \partial_x^3 u \, dx,
\]

\[
2q \int_R v_x \partial_x u_x \partial_x^5 u \, dx = -2q \int_R \partial_x v_x \partial_x u_x \partial_x^5 u \, dx - 2q \int_R v_x^2 \partial_x^2 u_x \partial_x^3 u \, dx \\
= 2q \int_R \partial_x^2 v_x \partial_x^2 u_x \partial_x^4 u \, dx + 4q \int_R \partial_x v_x \partial_x^2 u_x \partial_x^3 u \, dx \\
+ 2q \int_R v_x \partial_x^2 u_x \partial_x^2 u \, dx \\
= -2q \int_R \partial_x v_x \partial_x^2 u_x \partial_x^4 u \, dx - 6q \int_R \partial_x^2 v_x \partial_x^2 u_x \partial_x^3 u \, dx \\
- 5q \int_R \partial_x v_x (\partial_x^3 u_x)^2 \, dx \\
= -2q \int_R \partial_x^2 v_x \partial_x^2 u_x \partial_x^3 u \, dx + 3q \int_R \partial_x^3 v_x (\partial_x^2 u_x)^2 \, dx \\
- 5q \int_R \partial_x v_x (\partial_x^3 u_x)^2 \, dx.
\]

Consequently, by (A9),

\[
\frac{d}{dt} \left\| \partial_x^3 u(x, \cdot) \right\|_{L^2(R)}^2 + 2\varepsilon \left\| \partial_x^5 u(x, \cdot) \right\|_{L^2(R)}^2 \\
= -72g \int_R (\partial_x u_x)^2 \partial_x^2 u \partial_x^2 u \, dx - 36g \int_R (\partial_x^2 u_x)^2 \partial_x^3 u \, dx \\
- 42g \int_R u_x \partial_x u_x (\partial_x^3 u_x)^2 \, dx - 7q \int_R \partial_x v_x (\partial_x^3 u_x)^2 \, dx \\
+ 6q \int_R \partial_x^3 v_x (\partial_x^2 u_x)^2 \, dx - 5q \int_R \partial_x v_x \partial_x^2 u_x (\partial_x^3 u_x)^2 \, dx.
\]
Due to (24), (41), (42), (56), (57), (68), (69) and the Young inequality,

\[
|72g| \int_{\mathbb{R}} |
\partial_x u| \partial_x^2 u \partial_x^3 u | dx \\
\leq 36g^2 \left( \int_{\mathbb{R}} (\partial_x u)^4 (\partial_x^2 u)^2 dx + 36 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
\leq 36g^2 \left( \int_{\mathbb{R}} (\partial_x u)^4 (\partial_x^2 u)^2 dx + 36 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
\leq C(T) + 36 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 ,
\]

\[
|36g| \int_{\mathbb{R}} |u_x (\partial_x^2 u)^2 | \partial_x^3 u | dx \\
\leq 18g^2 \left( \int_{\mathbb{R}} u_x^2 (\partial_x^2 u)^4 dx + 18 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
\leq 18g^2 \left( \int_{\mathbb{R}} u_x^2 (\partial_x^2 u)^4 dx + 18 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
\leq C(T) \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 ,
\]

\[
|7q| \int_{\mathbb{R}} |\partial_x v| (\partial_x^2 u)^2 dx \leq |7q| \left\| \partial_x v(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 ,
\]

\[
|6q| \int_{\mathbb{R}} |\partial_x^2 v| (\partial_x^2 u)^2 dx \\
\leq |6q| \left\| \partial_x^2 v \right\|_{L^\infty((0, T) \times \mathbb{R})} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T),
\]

\[
|8q| \int_{\mathbb{R}} |\partial_x^2 v \partial_x u| \partial_x^3 u | dx \\
\leq 4q^2 \left( \int_{\mathbb{R}} (\partial_x^2 v)^2 (\partial_x u)^2 dx + 4 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
\leq 4q^2 \left( \int_{\mathbb{R}} (\partial_x^2 v)^2 \left\| \partial_x u \right\|_{L^\infty((0, T) \times \mathbb{R})}^2 + 4 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
\leq C(T) + 4 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 ,
\]

\[
|2q| \int_{\mathbb{R}} |u_x \partial_x^4 v| \partial_x^3 u | dx \\
\leq q^2 \left( \int_{\mathbb{R}} u_x^2 (\partial_x^4 v)^2 dx + \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
\leq q^2 \left( \int_{\mathbb{R}} u_x^2 \left\| \partial_x^4 v(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 ,
\]

It follows from (A10) that

\[
\frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2x \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left( 1 + \left\| \partial_x^2 u \right\|_{L^\infty((0, T) \times \mathbb{R})}^2 \right).
\]
The Gronwall Lemma and (A3) give
\[
\left\| \partial_x^3 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\epsilon e^{C(T)t} \int_0^t e^{-C(T)s} \left\| \partial_x^5 u_e(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \\
\leq C_0 e^{C(T)t} + C(T) \left( 1 + \left\| \partial_x^2 u_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right) e^{C(T)t} \int_0^t e^{-C(T)s} \, ds \quad \text{(A11)}
\]

We prove (A4). Thanks to (57), (A11) and the H"older inequality,

\[
(\partial_x^2 u_t(t, x))^2 = 2 \int_{-\infty}^x \partial_x^2 u_t \partial_x^3 u_t dx \\
\leq \left\| \partial_x^2 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^3 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) \sqrt{\left( 1 + \left\| \partial_x^2 u_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right)}.
\]

Hence,
\[
\left\| \partial_x^2 u_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^4 - C(T) \left\| \partial_x^2 u_t \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 - C(T) \leq 0,
\]
which gives (A4).

Finally, (A5) follows from (A4) and (A11). \(\square\)

**Lemma A2.** Assume (5). Fix \(T > 0\). There exists a constant \(C(T) > 0\), independent on \(\epsilon\), such that
\[
\left\| \partial_x^5 v_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T), \quad \text{(A12)}
\]
for every \(0 \leq t \leq T\). In particular, we have that
\[
\left\| \partial_x^4 v_t \right\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T). \quad \text{(A13)}
\]

**Proof.** Let \(0 \leq t \leq T\). Differentiating (70) with respect to \(x\), we have
\[
a \partial_x^5 v_e = 6\alpha \partial_x u_t \partial_x^4 u_e + 2\epsilon u_t \partial_x^3 u_e - \beta \partial_x^4 v_e - \gamma \partial_x^3 v_e. \quad \text{(A14)}
\]

Since \(\partial_x^3 u_e(t, \pm \infty) = 0\), by (55), (71) and (72), we have that
\[
\partial_x^5 v_e(t, \pm \infty) = 0. \quad \text{(A15)}
\]

Observe that
\[
-2\beta \alpha \int_{\mathbb{R}} \partial_x^4 v_t \partial_x^5 v_e dx = 0,
\]
\[
-2\alpha \gamma \int_{\mathbb{R}} \partial_x^3 v_t \partial_x^5 v_e dx = 2\alpha \gamma \left\| \partial_x^4 v_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

Consequently, multiplying (A14) by \(2a\partial_x^5 v_e\), an integration on \(\mathbb{R}\) gives
\[
2a^2 \left\| \partial_x^5 v_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = 12\alpha \kappa \int_{\mathbb{R}} \partial_x u_t \partial_x^2 u_t \partial_x^5 v_e dx + 4\alpha \kappa \int_{\mathbb{R}} u_t \partial_x^3 u_t \partial_x^5 v_e dx \\
+ 2\alpha \gamma \left\| \partial_x^4 v_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \quad \text{(A16)}
\]
Due to (41), (56), (57), (A4) and the Young inequality,

\[
12|\alpha| \int_{\mathbb{R}} |\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| |\partial_x^5 v_\varepsilon| \, dx \\
= \int_{\mathbb{R}} |12\kappa \partial_x u_\varepsilon \partial_x^2 u_\varepsilon| |\alpha \partial_x^5 v_\varepsilon| \, dx \\
\leq 72 \kappa^2 \int_{\mathbb{R}} \langle \partial_x u_\varepsilon \rangle^2 \langle \partial_x^2 u_\varepsilon \rangle^2 \, dx + \frac{\alpha^2}{2} \left\| \partial_x^5 v_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq 72 \kappa^2 \left\| \partial_x u_\varepsilon \right\|_{L^\infty((0, T) \times \mathbb{R})} \left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\alpha^2}{2} \left\| \partial_x^5 v_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq C(T) + \frac{\alpha^2}{2} \left\| \partial_x^5 v_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})},
\]

Finally, we prove (A13). Thanks to (68), (A12) and the H"older inequality,

\[
(\partial_x^4 v_\varepsilon(t, x))^2 = 2 \int_{-\infty}^x \partial_x^4 v_\varepsilon \partial_x^5 v_\varepsilon \, dx \leq 2 \int_{\mathbb{R}} |\partial_x^4 v_\varepsilon| |\partial_x^5 v_\varepsilon| \, dx \\
\leq \left\| \partial_x^4 v_\varepsilon \right\|_{L^2(\mathbb{R})} \left\| \partial_x^5 v_\varepsilon \right\|_{L^2(\mathbb{R})} \leq C(T),
\]

which gives (A13). □

Lemma A3. Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\varepsilon$, such that,

\[
\varepsilon \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \varepsilon^2 \int_0^t \left\| \partial_x^6 u_\varepsilon(s, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds \leq C(T),
\]

(A17)

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying the first equation of (16) by $2\varepsilon \partial_x^3 u_\varepsilon$, we have

\[
2\varepsilon \partial_x^3 u_\varepsilon \partial_t u_\varepsilon = 2\varepsilon e \partial_x^2 u_\varepsilon - 2\varepsilon^2 \partial_x^4 u_\varepsilon \partial_x^6 u_\varepsilon - 6\varepsilon u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^6 u_\varepsilon \\
+ 2\varepsilon \partial_x^3 u_\varepsilon \partial_x^6 u_\varepsilon - 2\varepsilon u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^6 u_\varepsilon - 2\varepsilon u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^6 u_\varepsilon.
\]

(A18)
Observe that by the second equation of (16) and (18),

\[
2be \int_{\mathbb{R}} P_{\varepsilon} \partial_x^6 u_t \, dx = -2be \int_{\mathbb{R}} \partial_x P_{\varepsilon} \partial_x^6 u_t \, dx = 2be \int_{\mathbb{R}} \partial_x u_t \partial_x^6 u_t \, dx = -2be \int_{\mathbb{R}} \partial_x^2 u_t \partial_x^5 u_t \, dx = 2be \int_{\mathbb{R}} \partial_x^3 u_t \partial_x^4 u_t \, dx = 0. \tag{A19}
\]

Moreover,

\[
2e \int_{\mathbb{R}} \partial_x^6 u_t \, dx = \epsilon \frac{d}{dt} \left\| \partial_x^4 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]

\[-2e^2 \int_{\mathbb{R}} \partial_x^4 u_t \partial_x^5 u_t \, dx = -2e^2 \left\| \partial_x^6 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \tag{A20}
\]

\[2ae \int_{\mathbb{R}} \partial_x^3 u_t \partial_x^8 u_t \, dx = 0.
\]

Therefore, (A19), (A20) and an integration of (A18) on \( \mathbb{R} \) give

\[
\epsilon \frac{d}{dt} \left\| \partial_x^4 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2e^2 \left\| \partial_x^6 u_t(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -6e \int_{\mathbb{R}} u_t^2 \partial_x u_t \partial_x^6 u_t \, dx - 2ae \int_{\mathbb{R}} \partial_x v_t \partial_x^8 u_t \, dx, \tag{A21}
\]

Observe that

\[
-6e \int_{\mathbb{R}} u_t^2 \partial_x u_t \partial_x^6 u_t \, dx = 12e \int_{\mathbb{R}} u_t (\partial_x u_t)^2 \partial_x^6 u_t \, dx + 6e \int_{\mathbb{R}} u_t^2 \partial_x^2 u_t \partial_x^5 u_t \, dx
\]

\[= -12e \int_{\mathbb{R}} (\partial_x u_t)^3 \partial_x^6 u_t \, dx - 36e \int_{\mathbb{R}} u_t \partial_x u_t \partial_x^2 u_t \partial_x^5 u_t \, dx
\]

\[= 6e \int_{\mathbb{R}} u_t^2 \partial_x^3 u_t \partial_x^6 u_t \, dx,
\]

\[-2ae \int_{\mathbb{R}} \partial_x v_t \partial_x^8 u_t \, dx = 2ae \int_{\mathbb{R}} \partial_x u_t \partial_x v_t \partial_x^7 u_t \, dx + 2ae \int_{\mathbb{R}} u_t \partial_x^2 v_t \partial_x^7 u_t \, dx
\]

\[= -2ae \int_{\mathbb{R}} \partial_x^2 u_t \partial_x v_t \partial_x^6 u_t \, dx - 4ae \int_{\mathbb{R}} \partial_x u_t \partial_x^3 v_t \partial_x^6 u_t \, dx
\]

\[= 2ae \int_{\mathbb{R}} u_t \partial_x^2 v_t \partial_x^6 u_t \, dx, \tag{A22}
\]

\[-2ae \int_{\mathbb{R}} v_t \partial_x^3 u_t \partial_x^6 u_t \, dx = 2ae \int_{\mathbb{R}} \partial_x v_t \partial_x^2 u_t \partial_x^6 u_t \, dx + 2ae \int_{\mathbb{R}} v_t \partial_x^2 u_t \partial_x^7 u_t \, dx
\]

\[= -2ae \int_{\mathbb{R}} \partial_x v_t \partial_x^3 u_t \partial_x^6 u_t \, dx - 4ae \int_{\mathbb{R}} \partial_x v_t \partial_x^4 u_t \partial_x^6 u_t \, dx
\]

\[= 2ae \int_{\mathbb{R}} v_t \partial_x^3 u_t \partial_x^6 u_t \, dx.
\]
Consequently, by (A21),

\[
\frac{d}{dt} \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\epsilon^2 \left\| \partial_x^6 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
= -12\epsilon \int_{\mathbb{R}} (\partial_x u_e)^3 \partial_x^6 u_e dx - 36\epsilon \int_{\mathbb{R}} u_e \partial_x u_e \partial_x^2 u_e \partial_x^6 u_e dx
- 6\epsilon \int_{\mathbb{R}} u_e^2 \partial_x^3 u_e \partial_x^5 u_e dx - 6\epsilon \int_{\mathbb{R}} \partial_x v_e \partial_x^2 u_e \partial_x^6 u_e dx
- 6\epsilon \int_{\mathbb{R}} \partial_x^2 v_e \partial_x u_e \partial_x^5 u_e dx - 2\epsilon \int_{\mathbb{R}} u_e \partial_x^3 v_e \partial_x^5 u_e dx
- 2\epsilon \int_{\mathbb{R}} v_e \partial_x^3 u_e \partial_x^5 u_e dx.
\] (A23)

Due to (24), (41), (42), (43), (56), (57), (A5) and the Young inequality,

\[
12|\epsilon| \int_{\mathbb{R}} |\partial_x u_e|^3 |\partial_x^3 u_e| dx = 12 \int_{\mathbb{R}} \left| \frac{g(\partial_x u_e)^3}{\sqrt{D_1}} \right| \left| \epsilon \sqrt{D_1} \partial_x^3 u_e \right| dx
\leq 6\epsilon^2 \int_{\mathbb{R}} (\partial_x u_e)^3 dx + 6D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\leq 6\epsilon^2 \left\| \partial_x u_e \right\|_{L^6((0, T) \times \mathbb{R})}^4 \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 6D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\leq C(T) \frac{D_1}{D_1} + 6D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]

\[
36|\epsilon| \int_{\mathbb{R}} |u_e \partial_x u_e | |\partial_x^3 u_e| dx = 36 \int_{\mathbb{R}} \frac{g} {\sqrt{D_1}} \left| \epsilon \sqrt{D_1} \partial_x^3 u_e \right| dx
\leq 18\epsilon^2 \int_{\mathbb{R}} u_e^2 (\partial_x u_e)^2 |\partial_x^3 u_e|^2 dx + 18D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\leq 18\epsilon^2 \left\| u_e \right\|_{L^6((0, T) \times \mathbb{R})}^4 \left\| \partial_x u_e \right\|_{L^\infty((0, T) \times \mathbb{R})} \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 18D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\leq C(T) \frac{D_1}{D_1} + 18D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]

\[
6|\epsilon| \int_{\mathbb{R}} |u_e^2 \partial_x^3 u_e| dx = 6 \int_{\mathbb{R}} \frac{g} {\sqrt{D_1}} \left| \epsilon \sqrt{D_1} \partial_x^3 u_e \right| dx
\leq 3\epsilon^2 \int_{\mathbb{R}} u_e^4 |\partial_x^3 u_e|^2 dx + 3D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\leq 3\epsilon^2 \left\| u_e \right\|_{L^6((0, T) \times \mathbb{R})}^4 \left\| \partial_x u_e \right\|_{L^\infty((0, T) \times \mathbb{R})} \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 3D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\leq C(T) \frac{D_1}{D_1} + 3D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]

\[
6|\epsilon| \int_{\mathbb{R}} |\partial_x v_e \partial_x^2 u_e| dx = 6 \int_{\mathbb{R}} \left| \epsilon \sqrt{D_1} \partial_x^3 u_e \right| dx
\leq 3\epsilon^2 \int_{\mathbb{R}} (\partial_x v_e)^2 (\partial_x^2 u_e)^2 dx + 3D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\leq 3\epsilon^2 \left\| \partial_x v_e \right\|_{L^\infty((0, T) \times \mathbb{R})} \left\| \partial_x^2 u_e \right\|_{L^2(\mathbb{R})} \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 3D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\leq C(T) \frac{D_1}{D_1} + 3D_1 \epsilon^2 \left\| \partial_x^3 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]
\[ 6q \varepsilon \int_{\mathbb{R}} \left| \frac{\partial^2 v \partial x u_e}{\sqrt{D_1}} \right| \sqrt{D_1 \varepsilon \partial^2 u_e} \, dx = 6 \int_{\mathbb{R}} \left| \frac{q \partial^2 v \partial x u_e}{\sqrt{D_1}} \right| \sqrt{D_1 \varepsilon \partial^2 u_e} \, dx \]

\[ \leq \frac{3q^2}{D_1} \int_{\mathbb{R}} (\partial^2 u_e)^2 \, dx + 3D_1 \varepsilon^2 \left\| \partial^2 u_e \right\|_{L^2(\mathbb{R})}^2 \]

\[ \leq \frac{3q^2}{D_1} \left\| \partial^2 u_e \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial x u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 3D_1 \varepsilon^2 \left\| \partial^2 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]

\[ \leq \frac{C(T)}{D_1} + 3D_1 \varepsilon^2 \left\| \partial^2 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})}, \]

where \( D_1 \) is a positive constant, which will be specified later. Consequently, by \((A23)\),

\[ \varepsilon \frac{d}{dt} \left\| \partial^2 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + (2 - 35D_1) \varepsilon^2 \left\| \partial^2 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \leq \frac{C(T)}{D_1}. \]

Taking \( D_1 = \frac{1}{35} \), we have that

\[ \varepsilon \frac{d}{dt} \left\| \partial^2 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \varepsilon^2 \left\| \partial^2 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \leq C(T). \]

\[ \varepsilon \left\| \partial^2 u_e(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \varepsilon^2 \int_0^t \left\| \partial^2 u_e(s, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds \leq C_0 + C(T) t \leq C(T), \]

that is \((A17)\). \( \square \)

**Lemma A4.** Assume \((5)\). Fix \( T > 0 \). There exists a constant \( C(T) > 0 \), independent on \( \varepsilon \), such that,

\[ \left\| \partial_t u_e(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C(T), \quad (A24) \]

for every \( 0 \leq t \leq T \).
Proof. Let $0 \leq t \leq T$. Multiplying the first equation of (16) by $2\partial_t u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} = 2b \int_{\mathbb{R}} P_\varepsilon \partial_t u_\varepsilon dx - 2\varepsilon \int_{\mathbb{R}} \partial_t^4 u_\varepsilon \partial_t u_\varepsilon dx - 6\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_t u_\varepsilon dx$$
$$+ 2a \int_{\mathbb{R}} \partial_t^2 u_\varepsilon \partial_t u_\varepsilon dx - 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x v_\varepsilon \partial_t u_\varepsilon dx$$
$$- 2\varepsilon \int_{\mathbb{R}} v_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon dx. \tag{A25}$$

Since $0 < \varepsilon < 1$, thanks to (24), (41), (42), (43), (A5), (A17) and the Young inequality,

$$|2b| \int_{\mathbb{R}} |P_\varepsilon| |\partial_t u_\varepsilon| dx = 2 \int_{\mathbb{R}} \frac{hP_\varepsilon}{\sqrt{D_2}} \left| \sqrt{D_2} \partial_t u_\varepsilon \right| dx$$
$$\leq \frac{h^2}{D_2} \left\| P_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq C(T) + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})},$$

$$2\varepsilon \int_{\mathbb{R}} |\partial_t^4 u_\varepsilon| |\partial_t u_\varepsilon| dx = 2 \int_{\mathbb{R}} \frac{\varepsilon^2 h}{\sqrt{D_2}} \left| \sqrt{D_2} \partial_t u_\varepsilon \right| dx$$
$$\leq \frac{\varepsilon^2}{D_2} \left\| \partial_t^4 u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{\varepsilon}{D_2} \left\| \partial_t^4 u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{C(T)}{D_2} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})},$$

$$6\varepsilon \int_{\mathbb{R}} |u_\varepsilon^2 \partial_x u_\varepsilon| |\partial_t u_\varepsilon| dx = 6 \int_{\mathbb{R}} \left| \frac{g u_\varepsilon^2 \partial_x u_\varepsilon}{\sqrt{D_2}} \right| \left| \sqrt{D_2} \partial_t u_\varepsilon \right| dx$$
$$\leq \frac{3\varepsilon^2}{D_2} \int_{\mathbb{R}} u_\varepsilon^4 (\partial_x u_\varepsilon)^2 dx + 3D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{3\varepsilon^2}{D_2} \left\| u_\varepsilon \right\|^4_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 3D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{C(T)}{D_2} + 3D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})},$$

$$2a \int_{\mathbb{R}} |\partial_t^2 u_\varepsilon| |\partial_t u_\varepsilon| dx = 2 \int_{\mathbb{R}} \frac{a h^2}{\sqrt{D_2}} \left| \sqrt{D_2} \partial_t u_\varepsilon \right| dx$$
$$\leq \frac{a^2}{D_2} \left\| \partial_t^2 u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{a}{D_2} \left\| \partial_t^2 u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{C(T)}{D_2} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})},$$

$$2\varepsilon \int_{\mathbb{R}} |u_\varepsilon \partial_x v_\varepsilon| |\partial_t u_\varepsilon| dx = 2 \int_{\mathbb{R}} \frac{\varepsilon^2 h}{\sqrt{D_2}} \left| \sqrt{D_2} \partial_t u_\varepsilon \right| dx$$
$$\leq \frac{\varepsilon^2}{D_2} \int_{\mathbb{R}} u_\varepsilon^2 (\partial_x v_\varepsilon)^2 dx + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{\varepsilon^2}{D_2} \left\| u_\varepsilon \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x v_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{C(T)}{D_2} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})},$$

$$2\varepsilon \int_{\mathbb{R}} |v_\varepsilon \partial_x u_\varepsilon| |\partial_t u_\varepsilon| dx = 2 \int_{\mathbb{R}} \frac{\varepsilon^2 h}{\sqrt{D_2}} \left| \sqrt{D_2} \partial_t u_\varepsilon \right| dx$$
$$\leq \frac{\varepsilon^2}{D_2} \int_{\mathbb{R}} v_\varepsilon^2 (\partial_x u_\varepsilon)^2 dx + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{\varepsilon^2}{D_2} \left\| v_\varepsilon \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{C(T)}{D_2} + D_2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|^2_{L^2(\mathbb{R})},$$
\begin{align*}
|2q| \int_{\mathbb{R}} |v_\epsilon \partial_x u_\epsilon| |\partial_t u_\epsilon| \, dx &= 2 \int_{\mathbb{R}} \left| \frac{q v_\epsilon \partial_x u_\epsilon}{\sqrt{D_2}} \right| \left| \sqrt{D_2} \partial_t u_\epsilon \right| \, dx \\
&\leq \frac{q^2}{D_2} \int_{\mathbb{R}} v_\epsilon^2 (\partial_x u_\epsilon)^2 \, dx + D_2 \| \partial_t u_\epsilon (t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
&\leq \frac{q^2}{D_2} \| v_\epsilon (t, \cdot) \|_{L^\infty(\mathbb{R})}^2 \| \partial_x u_\epsilon (t, \cdot) \|_{L^2(\mathbb{R})}^2 + D_2 \| \partial_t u_\epsilon (t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
&\leq \frac{C(T)}{D_2} + D_2 \| \partial_t u_\epsilon (t, \cdot) \|_{L^2(\mathbb{R})}^2,
\end{align*}

where \( D_2 \) is a positive constant, which will be specified later. Therefore, by (A25),

\[ 2 \left( 1 - \frac{1}{4} D_2 \right) \| \partial_t u_\epsilon (t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)}{D_2}, \]

Choosing \( D_2 = \frac{1}{8} \), we have that

\[ \| \partial_t u_\epsilon (t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C(T), \]

which gives (A24). \( \square \)

Arguing as in ([15], Lemma 2.12), we have the following result.

**Lemma A5.** Assume (5). Let \( T > 0 \). There exists a constant \( C(T) > 0 \), independent on \( \epsilon \), such that

\[
\| \partial_{x\epsilon}^2 v_\epsilon (t, \cdot) \|_{L^\infty(\mathbb{R})}^2, \| \partial_{x\epsilon}^2 v_\epsilon (t, \cdot) \|_{L^2(\mathbb{R})} \leq C(T),
\]

\[
\| \partial_t v_\epsilon (t, \cdot) \|_{L^\infty(\mathbb{R})} \| \partial_t v_\epsilon (t, \cdot) \|_{L^2(\mathbb{R})} \leq C(T),
\]

for every \( 0 \leq t \leq T \).

Using the Sobolev Immersion Theorem, we begin by proving the following result.

**Lemma A6.** Fix \( T > 0 \). There exist a subsequence \( \{ (u_{\epsilon,k}, v_{\epsilon,k}, P_{\epsilon,k}) \}_{k \in \mathbb{N}} \) of \( \{ (u_\epsilon, v_\epsilon, P_\epsilon) \}_{\epsilon > 0} \) and an a limit triplet \( (u, v, P) \) which satisfies (11) such that

\[
u_{\epsilon,k} \to v \text{ a.e. and in } L^p_{loc}((0, T) \times \mathbb{R}), 1 \leq p < \infty,
\]

\[
u_{\epsilon,k} \to u \text{ in } H^1((0, T) \times \mathbb{R}),
\]

\[
u_{\epsilon,k} \to v \text{ a.e. and in } L^p_{loc}((0, T) \times \mathbb{R}), 1 \leq p < \infty,
\]

\[
u_{\epsilon,k} \to v \text{ in } H^1((0, T) \times \mathbb{R}),
\]

\[
u_{\epsilon,k} \to P \text{ in } L^2((0, T) \times \mathbb{R}).
\]

Moreover, \( (u, v, P) \) is solution of (1) satisfying (12).

**Proof.** Let \( 0 \leq t \leq T \). We begin by observing that, thanks to Lemmas 3, 7, 9, A1 and A4,

\[ \{ u_{\epsilon} \}_{\epsilon > 0} \text{ is uniformly bounded in } H^1((0, T) \times \mathbb{R}). \] (A28)

Lemmas 3 and A5 say that

\[ \{ v_{\epsilon} \}_{\epsilon > 0} \text{ is uniformly bounded in } H^1((0, T) \times \mathbb{R}). \] (A29)
Instead, by Lemma 7, we have that
\[ \{ P_0 \} \in L^2((0, T) \times \mathbb{R}) \] (A30)

Equation (A28), (A29) and (A30) give (A27).

Observe that, thanks to Lemmas 3, 7, 9, A1 and the second equation of (16), we have that
\[ P \in L^\infty((0, T); H^4(\mathbb{R})). \]

Lemmas 3, 7, 9, A1 say that
\[ u \in L^\infty((0, T); H^3(\mathbb{R})). \]

Instead, thanks to Lemmas 3, 8, 10, A2 and (A26), we get
\[ v \in L^\infty((0, T); H^5(\mathbb{R})) \cap W^{1,\infty}((0, T) \times \mathbb{R}). \]

Moreover, Lemmas A5 says also that
\[ \partial_{t^2} v \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \]
for every \( 0 \leq t \leq T \). Therefore, (11) holds and \( (u, v, P) \) is solution of (1). Finally, (12) follows from (19) and (A27). \( \square \)

Now, we prove Theorem A1.

**Proof of Theorem A1.** Lemma A6 gives the existence of a solution of (1) such that (12) and (A27) hold. Arguing as in Theorem 1, we have (13). \( \square \)

**References**

1. Coclite, G.M.; di Ruvo, L. Convergence of the Ostrovsky Equation to the Ostrovsky-Hunter One. *J. Differ. Equ.* 2014, 256, 3245–3277. [CrossRef]
2. Coclite, G.M.; di Ruvo, L. Oleinik type estimate for the Ostrovsky-Hunter equation. *J. Math. Anal. Appl.* 2015, 423, 162–190. [CrossRef]
3. Coclite, G.M.; di Ruvo, L. Well posedness of bounded solutions of the non-homogeneous initial boundary value problem for the Ostrovsky-Hunter equation. *J. Hyperbolic Differ. Equ.* 2015, 12, 221–248. [CrossRef]
4. Coclite, G.M.; di Ruvo, L. Well-posedness results for the short pulse equation. *Z. Angew. Math. Phys.* 2015, 66, 1529–1557. [CrossRef]
5. Coclite, G.M.; di Ruvo, L. Well-posedness and dispersive/diffusive limit of a generalized Ostrovsky-Hunter equation. *Milan J. Math.* 2018, 86, 31–51. [CrossRef]
6. Coclite, G.M.; di Ruvo, L. Classical solutions for an Ostrovsky type equation. Submitted.
7. Bespalov, V.G.; Kozlov, S.A.; Shpolyanskiy, Y.A. Method for analyzing the propagation dynamics of femtosecond pulses with a continuum spectrum in transparent optical media. *J. Opt. Technol.* 2000, 67, 5–11. [CrossRef]
8. Bespalov, V.G.; Kozlov, S.A.; Shpolyanskiy, Y.A.; Walmsley, I.A. Simplified field wave equations for the nonlinear propagation of extremely short light pulses. *Phys. Rev. A* 2002, 66, 013811. [CrossRef]
9. Bespalov, V.G.; Kozlov, S.A.; Sutyagin, A.N.; Shpolyansky, Y.A. Spectral super-broadening of high-power femtosecond laser pulses and their time compression down to one period of the light field. *J. Opt. Technol.* 1998, 65, 823–825.
10. Gagarskii, S.V.; Prikhod’ko, K.V. Measuring the parameters of femtosecond pulses in a wide spectral range on the basis of the multiphoton-absorption effect in a natural diamond crystal. *J. Opt. Technol.* 2008, 75, 139–143. [CrossRef]
11. Konev, L.S.; Shpolyanskiì, Y.A. Calculating the field and spectrum of the reverse wave induced when a femtosecond pulse with a superwide spectrum propagates in an optical waveguide. *J. Opt. Technol.* 2014, 81, 6–11. [CrossRef]
12. Kozlov, S.A.; Sazonov, S.V. Nonlinear propagation of optical pulses of a few oscillations duration in dielectric media. *J. Exp. Theor. Phys.* 1997, 84, 221–228. [CrossRef]
13. Melnik, M.V.; Tsyupkin, A.N.; Kozlov, S.A. Temporal coherence of optical supercontinuum. *Rom. J. Phys.* 2018, 63, 203.

14. Shpolyanskiy, Y.A.; Belov, D.I.; Bakhtin, M.A.; Kozlov, S.A. Analytic study of continuum spectrum pulse dynamics in optical waveguides. *Appl. Phys. B* 2003, 77, 349–355. [CrossRef]

15. Coclite, G.M.; di Ruvo, L. A non-local elliptic-hyperbolic system related to the short pulse equation. *Nonlinear Anal.* 2020, 190, 111606. [CrossRef]

16. Coclite, G.M.; di Ruvo, L. Convergence of the solutions on the generalized Korteweg-de Vries equation. *Math. Model. Anal.* 2016, 21, 239–259. [CrossRef]

17. Colliander, J.; Keel, M.; Staffilani, G.; Takaoka, H.; Tao, T. Sharp global well posedness for KDV and modified KDV on R and T. *J. Am. Math. Soc.* 2003, 16, 705–749. [CrossRef]

18. Kenig, C.E.; Ponce, G.; Vega, L. Wellposedness and scattering results for the generalized Korteweg-de Vries Equation via the contraction principle. *Commun. Pure Appl. Math.* 1993, 46, 527–620. [CrossRef]

19. Schonbek, M.E. Convergence of solutions to nonlinear dispersive equations. *Commun. Part. Differ. Equ.* 1982, 7, 959–1000.

20. Tao, T. Nonlinear dispersive equations. In CBMS Regional Conference Series in Mathematics; Local and Global Analysis; Conference Board of the Mathematical Sciences: Washington, DC, USA, 2006; Volume 106.

21. Belashenkov, N.R.; Drozdov, A.A.; Kozlov, S.A.; Shpolyanskiy, Y.A.; Tsyupkin, A.N. Phase modulation of femtosecond light pulses whose spectra are superbroadened in dielectrics with normal group dispersion. *J. Opt. Technol.* 2008, 75, 611–614. [CrossRef]

22. Leblond, H.; Mihalache, D. Few-optical-cycle solitons: Modified Korteweg-de Vries sine-Gordon equation versus other non-slowly-varying-envelope-approximation models. *Phys. Rev. A* 2009, 79, 063835. [CrossRef]

23. Leblond, H.; Sanchez, F. Models for optical solitons in the two-cycle regime. *Phys. Rev. A* 2003, 67, 013804. [CrossRef]

24. Schäfer, T.; Wayne, C.E. Propagation of ultra-short optical pulses in cubic nonlinear media. *Physica D* 2004, 196, 90–105. [CrossRef]

25. Amiranashvili, S.; Vladimirov, A.G.; Bandelow, U. A model equation for ultrashort optical pulses. *Eur. Phys. J. D* 2010, 58, 219. [CrossRef]

26. Amiranashvili, S.; Vladimirov, A.G.; Bandelow, U. Solitary-wave solutions for few-cycle optical pulses. *Phys. Rev. A* 2008, 77, 063821. [CrossRef]

27. Coclite, G.M.; di Ruvo, L. Discontinuous solutions for the generalized short pulse equation. *Evol. Equ. Control Theory* 2019, 8, 737–753. [CrossRef]

28. Beals, R.; Rabelo, M.; Tenenblat, K. Bäcklund transformations and inverse scattering solutions for some pseudospherical surface equations. *Stud. Appl. Math.* 1989, 81, 125–248. [CrossRef]

29. Sakovich, A.; Sakovich, S. On the transformations of the Rabelo equations. *SIGMA* 3 2007, 8. [CrossRef]

30. Lattanzio, C.; Marcati, P. Global well-posedness and relaxation limits of a model for radiating gas. *J. Differ. Equ.* 2013, 190, 439–465. [CrossRef]

31. Serre, D. $L^1$-stability of constants in a model for radiating gases. *Commun. Math. Sci.* 2003, 1, 197–205. [CrossRef]

32. Coclite, G.M.; di Ruvo, L. Discontinuous solutions for the short-pulse master mode-locking equation. *AIMS Math.* 2019, 4, 437–462. [CrossRef]

33. Farnum, E.D.; Kutz, J.N. Master mode-locking theory for few-femtosecond pulses. *J. Opt. Soc. Am. B* 2010, 35, 3033–3035. [CrossRef]
39. Farnum, E.D.; Kutz, J.N. Dynamics of a low-dimensional model for short pulse mode locking. *Photonics* 2015, 2, 865–882. [CrossRef]

40. Farnum, E.D.; Kutz, J.N. Short-pulse perturbation theory. *J. Opt. Soc. Am. B* 2013, 30, 2191–2198. [CrossRef]

41. Pelinovsky, D.; Schneider, G. Rigorous justification of the short-pulse equation. *Nonlinear Differ. Equ. Appl.* 2013, 20, 1277–1294. [CrossRef]

42. Davidson, M. Continuity properties of the solution map for the generalized reduced Ostrovsky equation. *J. Differ. Equ.* 2012, 252, 3797–3815. [CrossRef]

43. Pelinovsky, D.; Sakovich, A. Global well-posedness of the short-pulse and sine-Gordon equations in energy space. *Commun. Part. Differ. Equ.* 2010, 35, 613–629. [CrossRef]

44. Davidson, M. Continuity properties of the solution map for the generalized reduced Ostrovsky equation. *J. Differ. Equ.* 2012, 252, 3797–3815. [CrossRef]

45. Pelinovsky, D.; Sakovich, A. Global well-posedness of the short-pulse and sine-Gordon equations in energy space. *Commun. Part. Differ. Equ.* 2010, 35, 613–629. [CrossRef]

46. Di Ruvo, L. Discontinuous Solutions for the Ostrovsky–Hunter Equation and Two Phase Flows. Ph.D. Thesis, University of Bari, Bari, Italy, 2013. Available online: www.dm.uniba.it/home/dottorato/dottorato/tesi/ (accessed on 1 June 2013).

47. Coclite, G.M.; di Ruvo, L. Wellposedness of bounded solutions of the non-homogeneous initial boundary for the short pulse equation. *Boll. Unione Mater. Ital.* 2015, 8, 31–44. [CrossRef]

48. Coclite, G.M.; di Ruvo, L. A note on the non-homogeneous initial boundary problem for an Ostrovsky-Hunter type equation. *Discr. Contin. Dyn. Syst. Ser. S* To Appear.

49. Coclite, G.M.; di Ruvo, L.; Karlsen, K.H. The initial-boundary-value problem for an Ostrovsky-Hunter type equation. In *Hyperbolic Conservation Laws and Related Analysis with Applications*; Springer Proceedings in Mathematics & Statistics; Springer: Heidelberg, Germany, 2014; Volume 49, pp. 143–159.

50. Coclite, G.M.; di Ruvo, L. Wellposedness of bounded solutions of the non-homogeneous initial boundary for the short pulse equation. *Boll. Unione Mater. Ital.* 2015, 8, 31–44. [CrossRef]

51. Coclite, G.M.; di Ruvo, L. Dispersive and Diffusive limits for Ostrovsky-Hunter type equations. *Nonlinear Differ. Equ. Appl.* 2015, 22, 1733–1763. [CrossRef]

52. Coclite, G.M.; Garavello, M. A Time Dependent Optimal Harvesting Problem with Measure Valued Solutions. *SIAM J. Control Optim.* 2017, 55, 913–935. [CrossRef]

53. Coclite, G.M.; Garavello, M.; Spinolo, L.V. Optimal strategies for a time-dependent harvesting problem. *Discr. Contin. Dyn. Syst. Ser. S* 2016, 11, 865–900. [CrossRef]

54. Simon, J. Compact sets in the space $L_p(0, T; B)$. *Ann. Mater. Pura Appl.* 1987, 4, 65–94.

55. Coclite, G.M.; Garavello, M.; Spinolo, L.V. Optimal strategies for a time-dependent harvesting problem. *Discr. Contin. Dyn. Syst. Ser. S* 2016, 11, 865–900. [CrossRef]

56. Simon, J. Compact sets in the space $L_p(0, T; B)$. *Ann. Mater. Pura Appl.* 1987, 4, 65–94.

57. Coclite, G.M.; Garavello, M.; Spinolo, L.V. Optimal strategies for a time-dependent harvesting problem. *Discr. Contin. Dyn. Syst. Ser. S* 2016, 11, 865–900. [CrossRef]

58. Coclite, G.M.; Holden, H.; Karlsen, K.H. Wellposedness for a parabolic-elliptic system. *Discr. Contin. Dyn. Syst.* 2005, 13, 659–682.

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).