Positive solutions of a Kirchhoff–Schrödinger–Newton system with critical nonlocal term

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Abstract. This paper deals with the following Kirchhoff–Schrödinger–Newton system with critical growth

$$
\begin{aligned}
- M \left( \int_\Omega |\nabla u|^2 \, dx \right) \Delta u &= \phi |u|^{2^*-3} u + \lambda |u|^{p-2} u, \quad \text{in } \Omega, \\
- \Delta \phi &= |u|^{2^*-1}, \quad \text{in } \Omega, \\
u = \phi &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $M(t) = 1 + bt^{\theta-1}$ with $t > 0$, $1 < \theta < \frac{N+2}{N-2}$, $b > 0$, $1 < p < 2$, $\lambda > 0$ is a parameter, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. By using the variational method and the Brézis–Lieb lemma, the existence and multiplicity of positive solutions are established.

Keywords: Kirchhoff–Schrödinger–Newton, positive solutions, critical growth.

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1 Introduction and main result

Consider the following Kirchhoff–Schrödinger–Newton system involving critical growth

$$
\begin{aligned}
- M \left( \int_\Omega |\nabla u|^2 \, dx \right) \Delta u &= \phi |u|^{2^*-3} u + \lambda |u|^{p-2} u, \quad \text{in } \Omega, \\
- \Delta \phi &= |u|^{2^*-1}, \quad \text{in } \Omega, \\
u = \phi &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $M(t) = 1 + bt^{\theta-1}$ with $t > 0$, $1 < \theta < \frac{N+2}{N-2}$, $b > 0$, $1 < p < 2$, $\lambda > 0$ is a parameter, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

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This system is derived from the Schrödinger–Poisson system

\[
\begin{align*}
-\Delta u + V(x)u + \eta \varphi f(u) &= h(x,u), \quad \text{in } \mathbb{R}^3, \\
-\Delta \varphi &= 2F(u), \quad \text{in } \mathbb{R}^3.
\end{align*}
\] (1.2)

System as (1.2) has been studied extensively by many researchers because (1.2) has a strong physical meaning, which describes quantum particles interacting with the electromagnetic field generated by the motion. The Schrödinger–Poisson system (also called Schrödinger–Maxwell system) was first introduced by Benci and Fortunato in [6] as a physical model describing a charged wave interacting with its own electrostatic field in quantum mechanic. For more information on the physical aspects about (1.2), we refer the reader to [6, 7].

Many recent studies of (1.2) have focused on existence of multiple solutions, ground states, positive and non-radial solutions. When \( h(x,u) = |u|^{p-2}u \), Alves et al. in [4] considered the existence of ground state solutions for (1.2) with \( 4 < p < 6 \). In [10], Cerami and Vaira proved the existence of positive solutions of (1.2) when \( h(x,u) = a(x)|u|^{p-2}u \) with \( 4 < p < 6 \) and \( a(x) \) is a nonnegative function. The same result was established in [11, 18, 22, 23] for \( 2 < p < 6 \). In [20, 25, 26, 28], by using variational methods, the authors proved the existence of ground state solutions of (1.2) with subcritical and critical growths. In addition, the existence of solutions for Schrödinger–Poisson system involving critical nonlocal term has been paid much attention by many authors, we can see [2, 13, 16, 19, 24, 27] and so on.

In [5], Arora et al. considered a nonlocal Kirchhoff type equation with a critical Sobolev nonlinearity, using suitable variational techniques, the authors showed how to overcome the lack of compactness at critical levels. In [15], by using the variational method and the concentration compactness principle, Lei and Suo established the existence and multiplicity of nontrivial solutions. Luyen and Cuong [21] obtained the existence of multiple solutions for a given boundary value problem, using the minimax method and Rabinowitz’s perturbation method. In [29], Zhou, Guo and Zhang combined the variational method and the mountain pass theorem, to get the existence of weak solutions, this time on the Heisenberg group.

Specially, Azzollini, D’Avenia and Vaira [3] studied the following Schrödinger–Newton type system with critical growth

\[
\begin{align*}
-\Delta u &= \lambda u + |u|^{2^*-3}u\varphi, \quad \text{in } \Omega, \\
-\Delta \varphi &= |u|^{2^*-1}, \quad \text{in } \Omega, \\
u = \varphi &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a smooth bounded domain. By the variational method, they obtained the existence and nonexistence results of positive solutions when \( N = 3 \) and the existence of solutions in both the resonance and the non-resonance case for higher dimensions.

Lei and Gao [14] considered the Schrödinger–Newton system with sign-changing potential

\[
\begin{align*}
-\Delta u &= f_\lambda(x)|u|^{p-2}u + |u|^3u\varphi, \quad \text{in } \Omega, \\
-\Delta \varphi &= |u|^5, \quad \text{in } \Omega, \\
u = \varphi &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^3 \) is a smooth bounded domain, \( 1 < p < 2, f_\lambda = \lambda f^+ + f^-, \lambda > 0, f^\pm = \max\{\pm f, 0\} \). By using the variational method and analytic techniques, the authors proved the existence and multiplicity of positive solutions.
In [17], Li et al. proved the existence, nonexistence and multiplicity of positive radially symmetric solutions for the following Schrödinger–Poisson system

\[
\begin{cases}
-\Delta u + u + \lambda |u|^3 u = \mu |u|^{p-2} u, & \text{in } \mathbb{R}^3, \\
-\Delta \phi = |u|^5, & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( p \in (2, 6), \lambda \in \mathbb{R} \) and \( \mu \geq 0 \) are parameters.

With the help of the Lax–Milgram theorem, for every \( u \in H^1_0(\Omega) \), the second equation of system (1.1) has a unique solution \( \phi_u \in H^1_0(\Omega) \), we substitute \( \phi_u \) to the first equation of system (1.1), then system (1.1) transforms into the following equation

\[
\begin{cases}
-\Delta \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = \phi_u |u|^{2^*-3} u + \lambda |u|^{p-2} u, & \text{in } \Omega, \\
u = \phi = 0, & \text{on } \partial \Omega.
\end{cases}
\]

(1.3)

The variational functional associated with (1.3) is defined by

\[
I_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{b}{2\theta} \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\theta} - \frac{1}{2(2^*-1)} \int_{\Omega} \phi_u |u|^{2^*-1} \, dx - \frac{\lambda}{p} \int_{\Omega} |u|^p \, dx.
\]

We say that \( u \in H^1_0(\Omega) \) is a weak solution of (1.3), for all \( \psi \in H^1_0(\Omega) \), then \( u \) satisfies

\[
\left[ 1 + b \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\theta-1} \right] \int_{\Omega} \nabla u \nabla \psi \, dx = \int_{\Omega} \phi_u |u|^{2^*-3} u \psi \, dx + \lambda \int_{\Omega} |u|^{p-2} u \psi \, dx.
\]

Our technique based on the Ekeland variational principle and the mountain pass theorem. Since system (1.1) contains a nonlocal critical growth term, which leads to the cause of the lack of compactness of the embedding \( H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega) \) and the Palais–Smale condition for the corresponding energy functional could not be checked directly. Then we overcome the compactness by using the Brézis–Lieb lemma.

Now we state our main result.

**Theorem 1.1.** Assume that \( 1 < \theta < \frac{N+2}{N-2}, \frac{N}{N-2} < p < 2 \) and \( N > 4, b > 0 \) is small enough. Then there exists \( \Lambda_* > 0 \) such that for all \( \lambda \in (0, \Lambda_*), \) system (1.1) has at least two positive solutions.

Throughout this paper, we make use of the following notations:

- The space \( H^1_0(\Omega) \) is equipped with the norm \( \|u\|_{H^1_0(\Omega)}^2 = \int_{\Omega} |\nabla u|^2 \, dx \), the norm in \( L^p(\Omega) \) is denoted by \( \| \cdot \|_p \).

- Let \( D^{1,2}(\mathbb{R}^N) \) be the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm \( \|u\|_{D^{1,2}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \).

- \( C, C_1, C_2, \ldots \) denote various positive constants, which may vary from line to line.

- We denote by \( S_\rho \) (respectively, \( B_\rho \)) the sphere (respectively, the closed ball) of center zero and radius \( \rho \), i.e. \( S_\rho = \{ u \in H^1_0(\Omega) : \|u\| = \rho \}, \) \( B_\rho = \{ u \in H^1_0(\Omega) : \|u\| \leq \rho \} \).

- Let \( S \) be the best constant for Sobolev embedding \( H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega) \), namely

\[
S = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{(\int_{\Omega} |u|^2 \, dx)^{2/2^*}}.
\]
2 Proof of the theorem

Firstly, we have the following important lemma in [3].

**Lemma 2.1.** For all $u \in H^1_0(\Omega)$, there exists a unique solution $\phi_u \in H^1_0(\Omega)$ of

\[
\begin{cases}
-\Delta \phi = |u|^{2^*-1}, & \text{in } \Omega, \\
\phi = 0, & \text{on } \partial \Omega.
\end{cases}
\]

Moreover,

1. $\phi_u \geq 0$ for $x \in \Omega$ and for each $t > 0$, $\phi_{tu} = t^{2^*-1} \phi_u$.
2. \[\int_\Omega |\nabla \phi_u|^2 dx = \int_\Omega \phi_u |u|^{2^*-1} dx \leq S^{-2^*} \|u\|^{2(2^*-1)}.\]
3. If $u_n \rightharpoonup u$ in $H^1_0(\Omega)$, then

\[\int_\Omega \phi_{u_n} |u_n|^{2^*-1} dx - \int_\Omega \phi_{u_n} - u |u_n - u|^{2^*-1} dx = \int_\Omega \phi_u |u|^{2^*-1} dx + o_n(1).\]

**Lemma 2.2.** There exist constants $\delta, \rho, \Lambda_0 > 0$, for all $\lambda \in (0, \Lambda_0)$ such that the functional $I_\lambda$ satisfies the following conditions:

1. $I_\lambda|_{u \in S_\rho} \geq \delta > 0; \inf_{u \in S_\rho} I_\lambda(u) < 0$.
2. There exists $e \in H^1_0(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.

**Proof.** (i) Using the Hölder inequality and the Sobolev inequality, we get

\[
\int_\Omega |u|^p dx \leq \left( \int_\Omega |u|^{2^*} dx \right)^{\frac{p}{2^*}} \left( \int_\Omega \frac{2^*}{p} dx \right)^{\frac{2}{2^*}} \leq |\Omega|^{\frac{2^*}{2^*} - \frac{p}{2^*}} \|u\|^p. \tag{2.1}
\]

Therefore, it follows from (2.1) and the Sobolev inequality that

\[
I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{b}{2^*} \left( \int_\Omega |\nabla u|^2 dx \right)^{\frac{2^*}{2}} - \frac{1}{2(2^* - 1)} \int_\Omega \phi_u |u|^{2^*-1} dx - \frac{\lambda}{p} \int_\Omega |u|^p dx \\
\geq \frac{1}{2} \|u\|^2 - \frac{1}{2(2^* - 1)} S^{-2^*} \|u\|^{2(2^*-1)} - \frac{\lambda}{p} |\Omega|^{\frac{2^*}{2^*} - \frac{p}{2^*}} \|u\|^p \\
= \|u\| \left( \frac{1}{2} \|u\|^{2^*-p} - \frac{1}{2(2^* - 1)} S^{-2^*} \|u\|^{2(2^*-1)-p} - \frac{\lambda}{p} |\Omega|^{\frac{2^*}{2^*} - \frac{p}{2^*}} \right).
\]

Let $H(t) = \frac{1}{2} t^{2^*-p} - \frac{1}{2(2^* - 1)} S^{-2^*} t^{2(2^*-1)-p}$ for $t > 0$, thus, there exists a constant

\[
\rho = \left[ \frac{(2^* - 1)(2 - p) S^{2^*/p}}{(2(2^* - 1) - p)} \right]^{\frac{1}{2^* - p}} > 0
\]

such that $\max_{t > 0} h(t) = h(\rho) > 0$. Setting $\Lambda_0 = \frac{pS^{2^*/p}}{|\Omega|^{2^*/p}} h(\rho)$, there exists a constant $\delta > 0$ such that $I_\lambda|_{u \in S_\rho} \geq \delta$ for each $\lambda \in (0, \Lambda_0)$. Moreover, for every $u \in H^1_0(\Omega) \setminus \{0\}$, we get

\[
\lim_{t \to 0^+} \frac{I_\lambda(tu)}{t^p} = -\frac{\lambda}{p} \int_\Omega |u|^p dx < 0.
\]
So we obtain \( I_\lambda(tu) < 0 \) for all \( u \neq 0 \) and \( tu \) small enough. Hence, for \( \|u\| \) small enough, we have
\[
m \triangleq \inf_{u \in B_\rho} I_\lambda(u) < 0.
\]

(ii) Set \( u \in H^1_0(\Omega) \), for all \( t > 0 \), we get
\[
I_\lambda(tu) = \frac{t^2}{2} \|u\|^2 + \frac{bt^{2\theta}}{2\theta} \|u\|^{2\theta} - \frac{t^{2(2^*-1)}}{2(2^*-1)} \int_\Omega \phi_u |u|^{2^*-1} dx - \frac{\lambda t^p}{p} \int_\Omega |u|^p dx \to -\infty
\]
as \( t \to \infty \), which implies that \( I_\lambda(tu) \to -\infty \) as \( t \to 0 \) large enough. Consequently, we can find \( e \in H^1_0(\Omega) \) with \( \|e\| > \rho \) such that \( I_\lambda(e) < 0 \). The proof is complete. \( \square \)

**Definition 2.3.** A sequence \( \{u_n\} \subset H^1_0(\Omega) \) is called \((PS)_c\) sequence of \( I_\lambda \) if \( I_\lambda(u_n) \to c \) and \( I'_\lambda(u_n) \to 0 \) as \( n \to \infty \). We say that \( I_\lambda \) satisfies \((PS)_c\) condition if every \((PS)_c\) sequence of \( I_\lambda \) has a convergent subsequence in \( H^1_0(\Omega) \).

**Lemma 2.4.** Assume that \( 1 < \theta < \frac{N+2}{N-2} \) and \( 1 < p < 2 \), the functional \( I_\lambda \) satisfies the \((PS)_c\) condition for each \( c < c_* = \frac{2}{N+2} S^\frac{2}{N} - D\lambda^\frac{2}{p-2} \), where
\[
D = \frac{(2^*-1)\|\lambda\|^\frac{2}{2^*}}{2(2^*-1)(2^*-2)^\frac{2}{p-2}} (S^\frac{2}{p} |\Omega|^\frac{2}{p} S^\frac{2}{p-2})^\frac{2}{p-2}.
\]

**Proof.** Let \( \{u_n\} \subset H^1_0(\Omega) \) be a \((PS)\) sequence for \( I_\lambda \) at the level \( c \), that is
\[
I_\lambda(u_n) \to c \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty.
\]
Combining with (2.1) and (2.2), we have
\[
c + o(\|u_n\|) \geq I_\lambda(u_n) - \frac{1}{2(2^*-1)} \langle I'_\lambda(u_n), u_n \rangle
\]
\[
= \left( \frac{1}{2} - \frac{1}{2(2^*-1)} \right) \|u_n\|^2 + b \left( \frac{1}{2\theta} - \frac{1}{2(2^*-1)} \right) \|u_n\|^{2\theta}
\]
\[
- \lambda \left( \frac{1}{p} - \frac{1}{2(2^*-1)} \right) |\Omega|^\frac{2^*-2}{p} S^\frac{2}{p-2} \|u_n\|^{p}
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{2(2^*-1)} \right) \|u_n\|^2 - \frac{1}{2(2^*-1)} \|u_n\|^{p} S^\frac{2}{p-2} \|u_n\|^{p}.
\]
Therefore \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \) for all \( 1 < p < 2 \). Thus, we may assume up to a subsequence, still denoted by \( \{u_n\} \), that there exists \( u \in H^1_0(\Omega) \) such that
\[
\begin{align*}
u_n &\rightharpoonup u, \quad \text{weakly in} \ H^1_0(\Omega), \\
u_n &\to u, \quad \text{strongly in} \ L^q(\Omega) \ (1 \leq q < 2^*), \\
u_n(x) &\to u(x), \quad \text{a.e. in} \ \Omega,
\end{align*}
\]
as \( n \to \infty \). By (2.1) and the Young inequality, one has
\[
\lambda \int_\Omega |u|^p dx \leq \lambda S^\frac{2}{p} |\Omega|^\frac{2}{p} \|u\|^{p} \leq \eta \|u\|^2 + C(\eta) \lambda^\frac{2}{p},
\]
where
\[
C(\eta) = \eta^{-\frac{2}{p}} (S^\frac{2}{p} |\Omega|^\frac{2}{p})^\frac{2}{p-2},
\]
and it follows from (2.2) and (2.4) that
\[
I_\lambda(u) = I_\lambda(u) - \frac{1}{2(2^*-1)} \langle I'_\lambda(u), u \rangle
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{2(2^*-1)} \right) \|u\|^2 - \frac{1}{2(2^*-1)} \lambda \int_\Omega |u|^p dx
\]
\[
\geq \left( \frac{2^*-2}{2(2^*-1)} - \frac{2(2^*-1)-p}{2(2^*-1)p} \right) \|u\|^2 - \frac{2(2^*-1)-p}{2(2^*-1)p} C(\eta) \lambda^\frac{2}{p}.
\]
Letting $\eta = \frac{p(2^*-2)}{2(2^*-1)-p}$ and $D = \frac{|2(2^*-1)-p|^{2\gamma}}{2(2^*-1)(2^*-2)\gamma} (S^{-\frac{2\gamma}{p}})_{\frac{2^*}{p} \gamma}^2$, we have $I_\lambda(u) \geq -D \lambda^{\frac{2\gamma}{p}}$.

Next, we prove that $u_n \to u$ strongly in $H^1_0(\Omega)$. Set $w_n = u_n - u$ and $\lim_{n \to \infty} \|w_n\| = l$, by using the Brézis–Lieb lemma [9], we have

$$
\|w_n\|^2 + \|u\|^2 + o(1) - \int \phi_{w_n} |w_n|^{2^*-1} dx - \int \phi_u |u|^{2^*-1} dx - l \int |u|^p dx = o(1),
$$

(2.5)

and

$$
\|u\|^2 + b\|u\|^{2\theta} - \int \phi_u |u|^{2^*-1} dx - \lambda \int |u|^p dx = 0.
$$

(2.6)

It follows from (2.5) and (2.6) that

$$
\|w_n\|^2 + b \left[ (\|w_n\|^2 + \|u\|^2 + o(1))^{\theta} - \|u\|^{2\theta} \right] - \int \phi_{w_n} |w_n|^{2^*-1} dx = o(1).
$$

(2.7)

Since $\|w_n\| \to l$, we have

$$
(\|w_n\|^2 + \|u\|^2 + o(1))^{\theta} - \|u\|^{2\theta} \to (l^2 + \|u\|^2 + o(1))^{\theta} - \|u\|^{2\theta} = l_1 \geq 0, \quad \text{as} \quad n \to \infty.
$$

If follows from (2.7) that

$$
\int \phi_{w_n} |w_n|^{2^*-1} dx \to l^2 + bl_1.
$$

Applying the Sobolev inequality, we get

$$
\|w_n\|^{2(2^*-1)} \geq S^{2^*} \int \phi_{w_n} |w_n|^{2^*-1} dx + o(1).
$$

(2.8)

Thus, by (2.8), we can deduce that

$$
l^{2(2^*-1)} \geq S^{2^*} (l^2 + bl_1) \geq S^{2^*} l^2 \quad \text{as} \quad n \to \infty,
$$

which implies that $l \geq S^{\frac{N}{2}}$ as $n \to \infty$. Since $I(u_n) = c + o(1)$, we obtain

$$
\frac{1}{2} \|w_n\|^2 + \frac{b}{2^{\theta}} (\|w_n\|^2 + \|u\|^2 + o(1))^{\theta} - \|u\|^{2\theta} - \frac{1}{2(2^*-1)} \int \phi_{w_n} |w_n|^{2^*-1} dx = c - I_\lambda(u) + o(1).
$$

Hence, there holds

$$
c = \left( \frac{1}{2} - \frac{1}{2(2^*-1)} \right) l^2 + \left( \frac{1}{2^{\theta}} - \frac{1}{2(2^*-1)} \right) bl_1 + I_\lambda(u)
\geq \frac{2}{N + 2} S^{\frac{N}{2}} - D \lambda^{\frac{2\gamma}{p}} \geq c_\star,
$$

as $n \to \infty$. This is a contradiction. Hence, we can conclude that $u_n \to u$ in $H^1_0(\Omega)$. The proof is complete.
Choose the extremal function
\[ U_\epsilon(x) = \frac{[N(N-2)e^2]^{\frac{N-2}{2}}}{(\epsilon^2 + |x|^2)^{\frac{N-2}{2}}}, \quad x \in \mathbb{R}^N, \ \epsilon > 0. \]

It is a positive solution of the following problem
\[ -\Delta U_\epsilon = U_\epsilon^{\frac{2^* - 1}{2}} \quad \text{in} \ \mathbb{R}^N, \]
and satisfies
\[ \int_{\mathbb{R}^N} |\nabla U_\epsilon|^2 dx = \int_{\mathbb{R}^N} U_\epsilon^{2^*} dx = S_{2^*}. \]

Pick a cut-off function \( \varphi \in C_0^\infty(\Omega) \) such that \( \varphi(x) = 1 \) on \( B(0, \frac{r}{2}) \), \( \varphi(x) = 0 \) on \( \mathbb{R}^N - B(0,r) \) and \( 0 \leq \varphi(x) \leq 1 \) on \( \mathbb{R}^N \). Set \( u_\epsilon(x) = \varphi(x)U_\epsilon(x) \), from [8], we have
\[ \begin{aligned}
\int_\Omega |\nabla u_\epsilon|^2 dx &= S_{2^*} + O(\epsilon^{N-2}), \\
\int_\Omega |u_\epsilon|^{2^*} dx &= S_{2^*} + O(\epsilon^N).
\end{aligned} \tag{2.9} \]

To estimate the value \( c \) observe that, multiplying the second equation of system (1.1) by \( |u| \) and integrating, we get
\[ \int_\Omega |u|^{2^*} dx = \int_\Omega \nabla \varphi_u \nabla |u| dx \leq \frac{1}{2} \|\varphi_u\|^2 + \frac{1}{2} \|u\|^2. \tag{2.10} \]

Then, we define a new functional \( H_\lambda : H_0^1(\Omega) \to \mathbb{R} \) by
\[ H_\lambda(u) \triangleq \frac{2^*}{2(2^* - 1)} \|u\|^2 + \frac{b}{2\theta} \|u\|^{2\theta} - \frac{1}{2^* - 1} \int_\Omega |u|^{2^*} dx - \frac{\lambda}{p} \int_\Omega |u|^p dx \]
\[ = \frac{2^*}{2^* - 1} \left[ \int_\Omega \frac{1}{2} |u|^{2^*} dx + \frac{(2^* - 1)b}{2\theta 2^*} \|u\|^{2\theta} - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx - \frac{\lambda}{2^* - 1} \int_\Omega |u|^p dx \right] \]
\[ \triangleq \frac{2^*}{2^* - 1} J_\lambda(u), \]
where
\[ J_\lambda(u) = \frac{1}{2} \|u\|^2 + \frac{(2^* - 1)b}{2\theta 2^*} \|u\|^{2\theta} - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx - \frac{\lambda}{2^* - 1} \int_\Omega |u|^p dx. \]

By (2.10), which implies that
\[ I_\lambda(u) \leq H_\lambda(u) = \frac{2^*}{2^* - 1} J_\lambda(u), \tag{2.11} \]
for every \( u \in H_0^1(\Omega) \), and \( c \leq \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \sup_{t \geq 0} I_\lambda(tu) \). If we consider the following problem
\[ \begin{cases}
- \left[ 1 + \frac{(2^* - 1)b}{2^*} \left( \int_\Omega |\nabla u_\epsilon|^2 dx \right)^{\theta - 1} \right] \Delta u = |u|^{2^* - 2} u + \lambda \frac{2^* - 1}{2^*} |u|^{p-2} u, & \text{in} \ \Omega, \\
u = 0, & \text{on} \ \partial \Omega.
\end{cases} \tag{2.12} \]

Then we find that the weak solution of problem (2.12) correspond to the critical points of the functional \( J_\lambda \). Next, we compute \( \sup_{t \geq 0} J_\lambda(tu_\epsilon) = J_\lambda(tu_\epsilon) \).
Lemma 2.5. Assume that $1 < \theta < \frac{N+2}{N-2}$, $\frac{N}{N-2} < p < 2$ and $N > 4$, then there exist $\Lambda_3, b_0 > 0$ such that for all $\lambda \in (0, \Lambda_3)$ and $b \in (0, b_0)$, it holds

\[
\sup_{t \geq 0} J_\lambda(tu_\lambda) < \frac{1}{N} S_0^N - \frac{N+2}{2N} D\lambda \frac{2}{2p}.
\]

In particular,

\[
\sup_{t \geq 0} I_\lambda(tu_\varepsilon) < \frac{2}{N+2} S_0^N - D\lambda \frac{2}{2p}.
\]

Proof. For convenience, we consider the functional $J_b^* : H_0^1(\Omega) \to \mathbb{R}$ as follows

\[
J_b^*(u) = \frac{1}{2} \|u\|^2 + \frac{(2^* - 1)b}{2\theta 2^*} \|u\|^{2\theta} - \frac{1}{2\theta} \int_\Omega |u|^{2\theta} dx.
\]

Define

\[
h_b(t) = J_b^*(tu_\varepsilon) = \frac{t^2}{2} \|u_\varepsilon\|^2 + \frac{(2^* - 1)b t^{2\theta}}{2\theta 2^*} \|u_\varepsilon\|^{2\theta} - \frac{t^{2\theta}}{2\theta} \int_\Omega |u_\varepsilon|^{2\theta} dx, \quad \text{for all } t \geq 0.
\]

It is clear that $\lim_{t \to 0} h_b(t) = 0$ and $\lim_{t \to \infty} h_b(t) = -\infty$. Therefore there exists $t_{b,\varepsilon} > 0$ such that $h(t_{b,\varepsilon}) = \max_{t \geq 0} h_b(t)$, that is

\[
0 = h_0'(t_{0,\varepsilon}) = t_{0,\varepsilon} \left( \|u_{\varepsilon}\|^2 - t_{0,\varepsilon}^2 - \int_\Omega |u_{\varepsilon}|^{2\theta} dx \right),
\]

one has

\[
t_{0,\varepsilon} = \left( \int_\Omega |u_{\varepsilon}|^{2\theta} dx \right)^{-1/2\theta}.
\]

Hence, we deduce from (2.9) that

\[
\sup_{t \geq 0} J_b^*(tu_\varepsilon) = h_b(t_{b,\varepsilon} u_\varepsilon) \leq h_0(t_{b,\varepsilon} u_\varepsilon) \leq h_0(t_{0,\varepsilon} u_\varepsilon)
\]

\[
= \frac{1}{N} S_0^N - \frac{N+2}{2N} D\lambda \frac{2}{2p} + \frac{N+2}{2N} D\lambda \frac{2}{2p} + O(\varepsilon^{N-2}).
\]

(2.13)

By using the definitions of $J$ and $u_\varepsilon$, we have

\[
J_\lambda(tu_\varepsilon) \leq \frac{t^2}{2} \|u_\varepsilon\|^2 + \frac{b(2^* - 1)t^{2\theta}}{2\theta 2^*} \|u_\varepsilon\|^{2\theta},
\]

for all $t \geq 0$ and $\lambda > 0$. It follows from (2.9) that there exist $T \in (0, 1)$, $\Lambda_1$, $b_0 > 0$ and $\varepsilon_1 > 0$ such that

\[
\sup_{0 \leq t \leq T} J_\lambda(tu_\varepsilon) < \frac{1}{N} S_0^N - \frac{N+2}{2N} D\lambda \frac{2}{2p},
\]

for every $0 < \lambda < \Lambda_1$, $0 < b < b_0$ and $0 < \varepsilon < \varepsilon_1$. According to the definition of $u_\varepsilon$, there

\[\text{...}\]
exists $C_1 > 0$, such that we have
\[
\int_{\Omega} |u_t|^p \, dx \geq C \int_{B_{r/2}(0)} \frac{\varepsilon^{p(N-2)/2} (\varepsilon^2 + |x|^2)^{N-2}}{(\varepsilon^2 + |x|^2)^{N-1}} \, dx
\]

\[
= C \varepsilon^{p(N-2)/2} \int_0^{r/2} \frac{\varepsilon^{N-1}}{(\varepsilon^2 + t^2)^{N-2}} \, dt
\]

\[
= C \varepsilon^{N-(p(N-2))/2} \int_0^{r/2 \sqrt{2}} \frac{y^{N-1}}{\left(1 + y^2\right)^{N-2}} \, dy
\]

\[
\geq C \varepsilon^{N-(p(N-2))/2} \int_0^1 \frac{y^{N-1}}{\left(1 + y^2\right)^{N-2}} \, dy
\]

\[
\geq C_1 \varepsilon^{N-(p(N-2))/2}.
\]

Thus, it follows from (2.13) and (2.14) that
\[
\sup_{t \geq T} \int \left( f_b(tu_t) - \lambda \frac{2^* - 1}{2^*} t^p \right) |u_t|^p \, dx \leq \frac{1}{N} S_{\frac{2}{2-p}}^N - \frac{N+2}{2N} D \lambda^{\frac{2}{2-p}}
\]

\[
+ \frac{N+2}{2N} D \lambda^{\frac{2}{2-p}} + C_2 \varepsilon^{N-2} - C_1 \lambda \varepsilon^{N-(p(N-2))/2}
\]

\[
< \frac{1}{N} S_{\frac{2}{2-p}}^N - \frac{N+2}{2N} D \lambda^{\frac{2}{2-p}},
\]

where the constant $C_2 > 0$. Here we have used the fact that $\frac{N}{N-2} < p < 2$ and $\frac{(N-2)(2-2p)+2N}{(N-2)(2-p)} < \frac{2}{2-p}$, let $\varepsilon = \lambda^{\frac{2}{(N-2)(2-p)}}$, $0 < \lambda < \Lambda_2 = \min \left\{ 1, \left( \frac{C_1}{C_2} \right)^{\frac{(N-2)(2-p)}{2p(N-2)}} \right\}$, then
\[
\frac{N+2}{2N} D \lambda^{\frac{2}{2-p}} + C_2 \varepsilon^{N-2} - C_1 \lambda \varepsilon^{N-(p(N-2))/2} \leq C_3 \lambda^{\frac{2}{2-p}} - C_1 \lambda \varepsilon^{N-(p(N-2))/2}
\]

\[
= C_3 \lambda^{\frac{2}{2-p}} - C_1 \lambda \varepsilon^{N-(p(N-2))/2} < 0,
\]

where $C_3 > 0$. Therefore, we have
\[
\sup_{t \geq 0} I_{\lambda}(tu_t) < \frac{1}{N} S_{\frac{2}{2-p}}^N - \frac{N+2}{2N} D \lambda^{\frac{2}{2-p}},
\]

for all $0 < \lambda < \Lambda_3 = \min \{ \Lambda_1, \Lambda_2, \varepsilon_1 \}$ and $0 < b < b_0$. The proof is complete.}

**Theorem 2.6.** Assume that $0 < \lambda < \Lambda_0$ ($\Lambda_0$ is as in Lemma 2.2). Then system (1.1) has a positive solution $u_\lambda$ satisfying $I_\lambda(u_\lambda) < 0$.

**Proof.** Applying Lemma 2.2, we have
\[
m \triangleq \inf_{u \in B_{\rho}(0)} I_\lambda(u) < 0.
\]

By the Ekeland variational principle [12], there exists a minimizing sequence $\{u_n\} \subset B_{\rho}(0)$ such that
\[
I_\lambda(u_n) \leq \inf_{u \in B_{\rho}(0)} I_\lambda(u) + \frac{1}{n}, \quad I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n} \|v - u_n\|, \quad v \in B_{\rho}(0).
\]
Thus, we obtain that $I_\lambda(u_n) \to m$ and $I'_\lambda(u_n) \to 0$. By Lemma 2.4, we have $u_n \to u_\lambda$ in $H^1_0(\Omega)$ with $I_\lambda(u_n) \to m < 0$, which implies that $u_\lambda \not\equiv 0$. Note that $I_\lambda(u_n) = I_\lambda(|u_n|)$, we have $u_\lambda \geq 0$. Then, by using the strong maximum principle, we obtain that $u_\lambda$ is a positive solution of system (1.1) such that $I_\lambda(u_\lambda) < 0$. □

Theorem 2.7. Assume that $0 < \lambda < \Lambda_\ast(\Lambda_\ast = \min\{\Lambda_0, \Lambda_3\})$. Then the system (1.1) has a positive solution $u_\ast \in H^1_0(\Omega)$ with $I_\lambda(u_\ast) > 0$.

Proof. According to the mountain pass theorem [1] and Lemma 2.2, there exists a sequence $\{u_n\} \subset H^1_0(\Omega)$ such that

$$I_\lambda(u_n) \to c > 0 \text{ and } I'_\lambda(u_n) \to 0 \text{ as } n \to \infty,$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \left\{ \gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = e \right\}.$$  

From Lemma 2.4, we know that $\{u_n\} \subset H^1_0(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_\ast$ in $H^1_0(\Omega)$ as $n \to \infty$,

$$I_\lambda(u_\ast) = \lim_{n \to \infty} I_\lambda(u_n) = c > 0,$$

which implies that $u_\ast \not\equiv 0$. It is similar to Theorem 2.6 that $u_\ast > 0$, we obtain that $u_\ast$ is a positive solution of system (1.1) such that $I_\lambda(u_\ast) > 0$. Combining the above facts with Theorem 2.6 the proof of Theorem 1.1 is complete. □

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References

[1] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14(1973), No. 4, 349–381. https://doi.org/10.1016/0022-1236(73)90051-7; MR0370183; Zbl 0273.49063

[2] A. Azzollini, P. D’Avenia, On a system involving a critically growing nonlinearity, J. Math. Anal. Appl. 387(2012), No. 1, 433–438. https://doi.org/10.1016/j.jmaa.2011.09.012; MR2845762; Zbl 1229.35060

[3] A. Azzollini, P. D’Avenia, G. Vaira, Generalized Schrödinger–Newton system in dimension $N \geq 3$: Critical case, J. Math. Anal. Appl. 449(2017), No. 1, 531–552. https://doi.org/10.1016/j.jmaa.2016.12.008; MR3595217; Zbl 1373.35120

[4] C. O. Alves, M. A. S. Souto, S. H. M. Soares, Schrödinger–Poisson equations without Ambrosetti–Rabinowitz condition, J. Math. Anal. Appl. 377(2011), No. 2, 584–592. https://doi.org/10.1016/j.jmaa.2010.11.031; MR2769159; Zbl 1211.35249
Positive solutions of a Schrödinger–Newton system for Kirchhoff type

[5] R. Arora, A. Fiscella, T. Mukherjee, P. Winkert, On critical double phase Kirchhoff problems with singular nonlinearity, *Rend. Circ. Mat. Palermo* (2), published online, 2022. https://doi.org/10.1007/s12215-022-00762-7

[6] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger–Maxwell equations, *Topol. Methods Nonlinear Anal.* 11(1998), No. 2, 283–293. https://doi.org/10.12775/tmna.1998.019; MR1659454; Zbl 0926.35125

[7] V. Benci, D. Fortunato, Solitons in Schrödinger–Maxwell equations, *J. Fixed Point Theory Appl.* 15(2014), No. 1, 101–132. https://doi.org/10.1007/s11784-014-0184-1; MR3282784; Zbl 1307.35292

[8] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, *Comm. Pure. Appl. Math.* 36(1983), No. 4, 437–477. https://doi.org/10.1002/cpa.3160360405; MR0709644; Zbl 0541.35029

[9] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* 88(1983), No. 3, 486–490. https://doi.org/10.1007/978-3-642-55925-9_42; MR0699419; Zbl 0526.46037

[10] G. Cerami, G. Vaira, Positive solution for some non-autonomous Schrödinger–Poisson systems, *J. Differential Equations* 248(2010), No. 3, 521–543. https://doi.org/10.1016/j.jde.2009.06.017; MR2557904

[11] P. D’Avenia, A. Pomponio, G. Vaira, Infinitely many positive solutions for a Schrödinger–Poisson system, *Appl. Math. Lett.* 24(2011), No. 5, 661–664. https://doi.org/10.1016/j.aml.2010.12.002; MR2765114; Zbl 1211.35104

[12] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* 47(1974), No. 2, 324–353. https://doi.org/10.1016/0022-247x(74)90025-0; MR0346619; Zbl 0286.49015

[13] C. Y. Lei, G. S. Liu, H. M. Suo, Positive solutions for a Schrödinger–Poisson system with singularity and critical exponent, *J. Math. Anal. Appl.* 483(2020), No. 2, 123647. https://doi.org/10.1016/j.jmaa.2019.123647; MR4037579; Zbl 1433.35073

[14] C. Y. Lei, G. S. Liu, Multiple positive solutions for a Schrödinger–Newton system with sign-changing potential, *Comput. Math. Appl.* 77(2019), No. 3, 631–640. https://doi.org/10.1016/j.camwa.2018.10.001; MR3913619; Zbl 1442.35114

[15] J. Lei, H. Suo, Multiple solutions of Kirchhoff type equations involving Neumann conditions and critical growth, *AIMS Math.* 6(2021), No. 4, 3821–3837. https://doi.org/10.3934/math.2021227; MR4209614; Zbl 1442.35114

[16] F. Y. Li, Y. H. Li, J. P. Shi, Existence of positive solutions to Schrödinger–Poisson type systems with critical exponent, *Commun. Contemp. Math.* 16(2014), No. 6, 1450036. https://doi.org/10.1142/s0219199714500369; MR3277956; Zbl 1309.35025

[17] Y. H. Li, F. Y. Li, J. P. Shi, Existence and multiplicity of positive solutions to Schrödinger–Poisson type systems with critical nonlocal term, *Calc. Var. Partial Differential Equations* 56(2017), No. 5, 1–17. https:// doi.org/10.1007/s00526-017-1229-2; MR3690005; Zbl 1454.35116
[18] G. B. Li, S. J. Peng, S. S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger–Poisson system, *Commun. Contemp. Math.* 12(2010), No. 6, 1069–1092. https://doi.org/10.1142/s0219199710004068; MR2748286; Zbl 1206.35082

[19] H. D. Liu, Positive solutions of an asymptotically periodic Schrödinger–Poisson system with critical exponent, *Nonlinear Anal. Real World Appl.* 32(2016), No. 1, 198–212. https://doi.org/10.1016/j.nonrwa.2016.04.007; MR3514921; Zbl 1350.35086

[20] Z. Liu, S. Guo, On ground state solutions for the Schrödinger–Poisson equations with critical growth, *J. Math. Anal. Appl.* 412(2014), No. 1, 435–448. https://doi.org/10.1016/j.jmaa.2013.10.066; MR3145812; Zbl 1312.35059

[21] D. T. Luyen, P. V. Cuong, Multiple solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations, *Rend. Circ. Mat. Palermo (2)* 71(2022), No. 1, 495–513. https://doi.org/10.1007/s12215-021-00594-x; MR4398002; Zbl 1486.35197

[22] D. Ruiz, The Schrödinger–Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.* 237(2006), No. 1, 655–674. https://doi.org/10.1016/j.jfa.2006.04.005; MR2230354; Zbl 1136.35037

[23] J. Seok, On nonlinear Schrödinger–Poisson equations with general potentials, *J. Math. Anal. Appl.* 401 (2013), No. 2, 672–681. https://doi.org/10.1016/j.jmaa.2012.12.054; MR3018016; Zbl 1307.35103

[24] L. Shao, Non-trivial solutions for Schrödinger–Poisson systems involving critical non-local term and potential vanishing at infinity, *Open Math.* 17(2019), No. 1, 1156–1167. https://doi.org/10.1515/math-2019-0091; MR4023113; Zbl 1430.35069

[25] J. Zhang, Ground state and multiple solutions for Schrödinger–Poisson equations with critical nonlinearity, *J. Math. Anal. Appl.* 440(2016), No. 2, 466–482. https://doi.org/10.1016/j.jmaa.2016.03.062; MR3484979; Zbl 1432.35059

[26] J. Zhang, On ground state and nodal solutions of Schrödinger–Poisson systems with critical growth, *J. Math. Anal. Appl.* 428(2015), No. 1, 387–404. https://doi.org/10.1016/j.jmaa.2015.03.032; MR3326993; Zbl 1325.35024

[27] J. F. Zhang, C. Y. Lei, L. T. Guo, Positive solutions for a nonlocal Schrödinger–Newton system involving critical nonlinearity, *Comput. Math. Anal.* 76(2018), No. 8, 1966–1974. https://doi.org/10.1016/j.camwa.2018.07.042

[28] L. G. Zhao, F. K. Zhao, Positive solutions for Schrödinger–Poisson equations with a critical exponent, *Nonlinear Anal.* 70(2009), No. 6, 2150–2164. https://doi.org/10.1016/j.na.2008.02.116; MR2498302; Zbl 1156.35374

[29] J. Zhou, L. Guo, B. Zhang, Kirchhoff-type problems involving the fractional $p$-Laplacian on the Heisenberg group, *Rend. Circ. Mat. Palermo (2)*, published online: 4 June 2022, 25 pp. https://doi.org/10.1007/s12215-022-00763-6