Spontaneously Generated Inhomogeneous Phases via Holography

James Alsup, Eleftherios Papantonopoulos, George Siopsis and Kubra Yeter

1 Computer Science, Engineering and Physics Department, The University of Michigan-Flint, Flint, MI 48502-1907, USA
2 Department of Physics, National Technical University of Athens, Zografou Campus GR 157 73, Athens, Greece
3 Department of Physics and Astronomy, The University of Tennessee, Knoxville, TN 37996 - 1200, USA

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Abstract

We discuss a holographic model consisting of a U(1) gauge field and a scalar field coupled to a charged AdS black hole under a spatially homogeneous chemical potential. By turning on an interaction term between the Einstein tensor and the scalar field, a one-dimensional lattice is generated by the spatially dependent profile of the scalar field. We calculate the critical temperature as a function of an effective lattice spacing (ultraviolet cutoff). By perturbing the holographic lattice below the critical temperature, we show that the dual gauge theory on the boundary develops spontaneously a spatially inhomogeneous phase.

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I. INTRODUCTION

There is a lot of activity recently in an attempt to study phenomena at strong coupling using their weakly coupled dual gravity description. The tool to carry out such a study is the gauge/gravity duality. This holographic principle [1] has many applications in string theory, where it is well founded, but it has also been applied to other physical systems like the systems encountered in condensed matter physics. One of the most extensively studied condensed matter system, using the gauge/gravity duality, is the holographic superconductor (for a review see [2]).

The gravity dual of a homogeneous superconductor consists of a system with a black hole and a charged scalar field, in which the black hole admits scalar hair at temperatures lower than a critical temperature [3], while there is no scalar hair at higher temperatures. According to the holographic principle this breaking of the abelian $U(1)$ symmetry corresponds in the boundary theory to a scalar operator which condenses at a critical temperature proportional to the charged density of the scalar potential, while fluctuations of the vector potential give the frequency dependent conductivity in the boundary theory [4]. Back reaction effects on the metric were also studied in [5]. In [6] an exact gravity dual of a gapless superconductor was discussed in which the charged scalar field responsible for the condensation is an exact solution of the field equations and below a critical temperature dresses up a vacuum black hole with scalar hair.

Apart from conventional homogeneous superconductors, extensions have been studied to unconventional superconductors which are characterized by higher critical temperatures, such as cuprates and iron pnictides. The interesting new features of these systems is that they exhibit competing orders that are related to the breaking of the lattice symmetries. This breaking introduces inhomogeneities and a study of the effect of inhomogeneity of the pairing interaction in a weakly coupled BCS system [7] as well as numerical studies of Hubbard models [8, 9] suggest that inhomogeneity might play a role in high-Tc superconductivity.

The recent discovery of transport anomalies in La$_{2-x}$Ba$_x$CuO$_4$ might be explained under the assumption that this cuprate is a superconductor with a unidirectional charge density wave, i.e., a “striped” superconductor [10]. Other studies using mean-field theory have also shown that unlike the homogeneous superconductor, the striped superconductor exhibits the existence of a Fermi surface in the ordered phase [11, 12] and its complex sensitivity to
quenched disorder \[10\]. Holographic striped superconductors were discussed in \[13\] where a modulated chemical potential was introduced, and it was shown that below a critical temperature superconducting stripes develop. Properties of the striped superconductors and back reaction effects were studied in \[14, 15\]. Striped phases were also found in electrically charged RN-AdS black branes that involve neutral pseudo-scalars \[16\].

Inhomogeneities also appear in condensed matter systems other than superconductors. These systems are characterized by additional ordered states which compete or coexist with superconductivity \[17, 18\]. The most important of them are charge and spin density waves (CDW and SDW, respectively) \[19\]. The development of these states corresponds to the spontaneous modulation of the electronic charge and spin density, below a critical temperature \(T_c\). Density waves are widely spread among different classes of materials. One may distinguish between them either orbitally \[20\], or through Zeeman driven \[21\], field-induced CDWs, confined \[22\], and even unconventional density waves \[23\].

To study the effect of inhomogeneity at strong coupling, the usual approach is to introduce a modulated chemical potential which according to the holographic principle is translated into a modulated boundary value for the electrostatic potential in the AdS black hole gravity background. Then from an Einstein-Maxwell scalar system solutions can be obtained, which below a critical temperature show that the system undergoes a phase transition and a condensate can develop with a non vanishing modulation. Depending on what symmetries are broken, the modulated condensate, according to the holographic principle, corresponds to ordered states like CDW or SDW in the boundary theory \[13, 24\].

To explore further the properties of spatial inhomogeneities in holographic superfluids, gravitational backgrounds which are not spatially homogeneous were introduced \[25–28\]. In \[29\] the breaking of the translational invariance is sourced by a scalar field with a non-trivial profile in the \(x\) direction. Then, perturbing the one-dimensional "lattice", the system of Einstein-Maxwell-scalar field equations was numerically solved at first order and the optical conductivity was calculated. Further properties of this construction were studied in \[30\].

In this work we study a system of a holographic superfluid in which a spatial inhomogeneous phase is spontaneously generated. The gravity sector consists of an AdS-Reissner-Nordström black hole, an electromagnetic field and a scalar field. In this background we introduce an interaction term between the Einstein tensor to the scalar field. The motivation in introducing this geometrical term \[31\] is that the Einstein tensor has naturally encoded
the information from the backreaction of the electric field to the metric and its presence is essential in generating spontaneously the inhomogeneous phase in the boundary theory.

We put this gravitational background on a one-dimensional “lattice” which is generated by an \( x \)-dependent profile of a scalar field. At the onset of the condensation of the scalar field we calculate the transition temperature. We find that as the momentum of the scalar field is increased, the transition temperature increases indefinitely, approaching an asymptotic value at infinite momentum. To remedy this behavior, we introduce in the interaction term an ultraviolet cutoff which effectively plays the role of the lattice spacing. Then we find a maximum transition temperature corresponding to a finite momentum, which is the critical temperature of our system, below which it undergoes a second order phase transition.

We perturb our lattice below the critical temperature. We expand our fields around the critical temperature, and solve analytically the coupled system of Einstein-Maxwell-scalar field equations at first order. We find that a spatially inhomogeneous phase is spontaneously generated in the boundary theory.

The paper is organized as follows. In section II we present the basic setup of the holographic model. In section III we calculate numerically the critical temperature of the system. In section IV we perturb the lattice below the critical temperature, and solve the Einstein-Maxwell-scalar field equations analytically at first order. Finally in section V we present our conclusions.

II. THE SETUP

In this section we introduce a holographic model whose main new feature is the spontaneous generation of spatial inhomogeneous phases in the boundary theory. This cures the main deficiency of an earlier proposal [24]. This is achieved by introducing a direct coupling of the Einstein tensor to the scalar field.

Consider a system consisting of a \( U(1) \) gauge field, \( A_\mu \), corresponding field strength \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and a real scalar field \( \Psi \) which is neutral under the \( U(1) \) group. They live in a spacetime of negative cosmological constant \( \Lambda = -6/L^2 \).

The action is

\[
S = \int d^4 \sqrt{-g} \left[ \frac{R + 6/L^2}{16\pi G} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \Psi \partial^\mu \Psi - \frac{m^2}{2} \Psi^2 \right].
\] (1)
For simplicity, we shall set $16\pi G = L = 1$.

Our main concern is to generate spatially inhomogeneous phases in the boundary theory. To this end, we introduce an interaction term of the form

$$S_{\text{int}} = \frac{\eta}{2} \int d^4x \sqrt{-g} \left[ (G^{\mu\nu} - 3g^{\mu\nu}) \partial_\mu \Psi \partial_\nu \{ J (-\alpha D_\sigma D_\sigma) \Psi \} \right],$$

(2)

coupling the scalar field $\Psi$ to the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}.$$  

(3)

This coupling is essential for the generation of spatial inhomogeneities [31].

We have also included an ultraviolet cutoff to suppress the high momentum (short wavelength) modes of the scalar field. The cutoff is implemented by the function $J(x)$ with the properties $J(0) = 1$, and $J \to 0$ as $x \to \infty$. The argument of this function contains a (small) parameter $\alpha$ which determines the size of the lattice spacing. We defined

$$D_\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\nu\rho} \partial_\sigma.$$  

(4)

As we discussed in the introduction, the presence of a lattice naturally introduces such a cutoff. In some sense, this is an effective way to introduce lattice effects in our model. A convenient choice of the function $J$ is

$$J(x) = e^{-x}.$$  

(5)

From the action (1), together with the interaction term (2), we obtain the Einstein equations

$$G_{\mu\nu} - 3g_{\mu\nu} = \frac{1}{2} T_{\mu\nu},$$

(6)

where $T_{\mu\nu}$ is the stress-energy tensor,

$$T_{\mu\nu} = T_{\mu\nu}^{(EM)} + T_{\mu\nu}^{(\Psi)} + \Theta_{\mu\nu},$$

(7)

containing a gauge, scalar, and interaction term contributions. Assuming that the effect of the cutoff term is negligible beyond determining the lattice spacing (to be confirmed by the
results), we obtain the explicit expressions

\[
T^{(EM)}_{\mu\nu} = F^\mu_\rho F^{\rho}_\nu - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma},
\]

\[
T^{(\Psi)}_{\mu\nu} = \nabla_\mu \Psi \nabla_\nu \Psi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \Psi \nabla^\alpha \Psi - m^2 g_{\mu\nu} \Psi^2,
\]

\[
\Theta_{\mu\nu} = - \frac{1}{2} (R + 12) \nabla_\mu \Psi \nabla_\nu \Psi + R^\alpha_\mu \nabla^\alpha \Psi \nabla_\nu \Psi + \Psi - \frac{1}{2} G^\mu_{\alpha\nu} \nabla_\alpha \Psi \nabla^\alpha \Psi
\]

\[
+ R_{\mu\alpha\nu\beta} \nabla^\alpha \Psi \nabla^\beta \Psi + \nabla_\mu \nabla^\alpha \Psi \nabla_\nu \nabla^\alpha \Psi - \nabla_\mu \nabla_\nu \Psi \square \Psi
\]

\[
+ g_{\mu\nu} \left[ - \frac{1}{2} \nabla^\alpha \nabla^\beta \Psi \nabla_\alpha \nabla_\beta \Psi + \frac{1}{2} (\square \Psi)^2 - (R^{\alpha\beta} - 3 g^{\alpha\beta}) \nabla_\alpha \Psi \nabla_\beta \Psi \right]. \quad (8)
\]

Varying the Lagrangian with respect to $A_\mu$ we find the Maxwell equations (with $\alpha = 0$),

\[
\nabla_\rho F^{\rho\mu} = 0. \quad (9)
\]

Finally, the equation of motion for the scalar field is

\[
\square \Psi - m^2 \Psi = \eta \left( G^{\mu\nu} - 3 g^{\mu\nu} \right) \nabla_\mu \nabla_\nu \mathcal{J} (-\alpha D_\sigma D^\sigma) \Psi, \quad (10)
\]

where we used the fact that the Einstein tensor is conserved ($\nabla_\mu G^{\mu\nu} = 0$). It is important to include the cutoff in the scalar equation, because its solution will determine the lattice spacing.

Our aim is to study the Einstein-Maxwell-scalar system of equations first at the critical temperature and then below the critical temperature perturbatively.

### III. THE CRITICAL TEMPERATURE

At the critical temperature, we have $\Psi = 0$. The Einstein-Maxwell system has a static solution with metric of the form

\[
ds^2 = \frac{1}{z^2} \left[ -h(z) dt^2 + \frac{dz^2}{h(z)} + dx^2 + dy^2 \right]. \quad (11)
\]

The system possesses a scaling symmetry. The arbitrary scale is often taken to be the radius of the horizon. It is convenient to fix the scale by using a radial coordinate $z$ so the horizon is at $z = 1$. Since the scale has been fixed, we should only be reporting on scale-invariant quantities.

The Maxwell equations admit the solution

\[
A_t = \mu (1 - z), \quad (12)
\]
so that $U(1)$ gauge field has an electric field in the $z$-direction equal to the chemical potential, $E_z = \mu$.

The Einstein equations are then solved by

$$h(z) = 1 - \left(1 + \frac{\mu^2}{4}\right) z^3 + \frac{\mu^2}{4} z^4 .$$

The temperature is given by

$$\frac{T_c}{\mu} = -\frac{h'(1)}{4\pi \mu} = \frac{3}{4\pi \mu} \left(1 - \frac{\mu^2}{12}\right) ,$$

where we divided by $\mu$ to create a scale-invariant quantity.

Additionally, at the critical temperature the scalar field satisfies the wave equation,

$$\partial_z \left( \frac{h \hat{f} \pm}{z^2} \partial_z \Psi \right) + \frac{\hat{f} \pm}{z^2} \nabla_2^2 \Psi - \frac{m^2}{z^4} \Psi = 0 ,$$

where

$$\hat{f}_\pm = 1 \pm \eta \frac{\mu^2}{4} z^4 e^{-\alpha \mu^2 z^6 \nabla_2} , \quad \nabla_2^2 = \partial_x^2 + \partial_y^2 .$$

The wave equation can be solved by separating variables,

$$\Psi(z, x) = \Phi(z) Y(x, y) ,$$

where $Y$ is an eigenfunction of the two-dimensional Laplacian,

$$\nabla_2^2 Y = -\tau Y .$$
We will keep the translation invariance in the $y$-direction, and concentrate on the one-dimensional “lattice” defined by

$$ Y = \cos kx , \tag{19} $$

with $\tau = k^2$, and leave the two-dimensional lattices for future study.

The radial function $\Phi(z)$ satisfies the wave equation

$$ \Phi'' + \left[ \frac{h'}{h} + \frac{f_+}{f_+} - \frac{2}{z} \right] \Phi' - \frac{\tau f_-}{hl_+} \Phi - \frac{m^2}{z^2 f_+} \Phi - 3\alpha\mu^2 \tau z^5 \left[ \left( \frac{h'}{h} + \frac{3}{z} \right) \left( 1 - \frac{1}{f_+} \right) + \frac{f'_+}{f_+} \right] \Phi = 0 \tag{20} $$

where

$$ f_\pm = 1 \pm \frac{\mu^2}{4} e^{-\alpha\mu^2 \tau z^6} . \tag{21} $$

The asymptotic behavior (as $z \to 0$) is $\Phi \sim z^\Delta$, where $\Delta(\Delta - 3) = m^2$. It is convenient to write

$$ \Phi(z) = \frac{\langle O_\Delta \rangle}{\sqrt{2}} z^\Delta F(z) , \quad F(0) = 1 . \tag{22} $$

For general $\Delta$, we obtain

$$ F'' + \left[ \frac{2(\Delta - 1)}{z} + \frac{f'_+}{f_+} + \frac{h'}{h} \right] F' + \left[ -\frac{\tau}{f_+} + \frac{\Delta f'_+}{f_+} + \frac{(\Delta - 3)\Delta}{z^2} \left( 1 - \frac{1}{f_+} \right) + \frac{\Delta h'}{zh} \right] F \tag{23} $$

The maximum transition temperature of the system can be calculated by solving (23) numerically, and using the expression (14) for the temperature. This maximum transition temperature is the critical temperature $T_c$ of the system. Due to the presence of the $\eta$-dependent interaction term, the critical temperature $T_c$ of the system also depends on the coupling constant $\eta$.

In Fig. 1 we show the transition temperature $T$ as a function of the wavenumber $k$. Both quantities are divided by the chemical potential $\mu$ to render them dimensionless. The critical temperature of the system is the maximum transition temperature. In the absence of an ultraviolet cutoff ($\alpha = 0$), this maximum transition temperature corresponds to $k \to \infty$ as can be seen in Fig. 2 where we show the dependence of the critical temperature $T_c$, and wavenumber $k$ on the cutoff parameter $\alpha$. The value of the wavenumber decreases with increasing cutoff parameter $\alpha$ as shown on the right panel of Fig. 2.

Summarizing, at the critical temperature the electric field backreacts on the system, the Einstein-Maxwell field equations admit solutions with a spatially dependent scalar field while
the electric field attains a constant value equal to the chemical potential. In the next section we will perturb the system around the critical temperature, and show that below the critical temperature the system develops a spatially inhomogeneous phase in the boundary theory.

IV. BELOW THE CRITICAL TEMPERATURE

In this section we will study the system below the critical temperature. We will write the field equations resulting from the considered action and then perturb the system near the critical temperature with spatially dependent perturbations. We will study the behavior of the system analytically leaving a full numerical study for the future. To simplify the discussion, we shall assume that the effects of the cutoff are negligible, i.e., \( \mathcal{J} \approx 1 \) near the critical temperature \( (T \approx T_c) \). It is straightforward, albeit tedious, to include the effects of the cutoff below the critical temperature.

Below the critical temperature the scalar field backreacts to the metric. Consider the following ansatz

\[
ds^2 = \frac{1}{z^2} \left[ -h(z,x)e^{-\alpha(z,x)}dt^2 + \frac{dz^2}{h(z,x)} + e^{\beta(z,x)}dx^2 + e^{-\beta(z,x)}dy^2 \right].
\]

To solve the equations of motion (6), (9), and (10) below the critical temperature \( T_c \), we expand in the order parameter

\[
\xi = \frac{\langle O_\Delta \rangle}{\sqrt{2}}.
\]
and write

\[ h(z, x) = h_0(z) + \xi^2 h_1(z, x) + \mathcal{O}(\xi^4), \]
\[ \alpha(z, x) = \xi^2 \alpha_1(z, x) + \mathcal{O}(\xi^4), \]
\[ \beta(z, x) = \xi^2 \beta_1(z, x) + \mathcal{O}(\xi^4), \]
\[ \Psi(z, x) = \xi \Psi_0(z, x) + \xi^3 \Psi_1(z, x) + \mathcal{O}(\xi^5), \]
\[ A_t(z, x) = A_{t0}(z) + \xi^2 A_{t1}(z, x) + \mathcal{O}(\xi^4), \]

(26)

where \( h_0, \xi \Psi_0, \) and \( A_{t0} \) are defined at the critical temperature \( T_c \) (\( A_{t0} \) and \( h_0 \) are given by (12) and (13), respectively; \( \xi \Psi_0 \) is a solution of eq. (15)). We also have for the chemical potential

\[ \mu \equiv A_t(0, x) = \mu_0 + \xi^2 \mu_1 + \mathcal{O}(\xi^2), \quad \mu_0 = A_{t0}(0), \quad \mu_1 = A_{t1}(0, x). \]

(27)

It should be noted that we are working with an ensemble of fixed chemical potential, which seems to contradict eq. (27) in which the chemical potential appears to receive corrections below the critical temperature. However, the reported chemical potential is measured in units in which the radius of the horizon is 1, so a change in \( \mu \) in these units is due to a change in our scale as we lower the temperature.

At each given order of the order parameter \( \xi \), only a finite number of modes of the various fields are generated. At \( \mathcal{O}(\xi^2) \), we have only 0 and 2\( k \) Fourier modes,

\[ h_1(z, x) = z^3 \left( h_{10}(z) + h_0^2(z) h_{11}(z) \cos 2kx \right), \]
\[ \alpha_1(z, x) = \alpha_{10}(z) + z^3 h_0(z) \alpha_{11}(z) \cos 2kx, \]
\[ \beta_1(z, x) = \beta_{10}(z) + z^3 \beta_{11}(z) \cos 2kx, \]
\[ A_{t1}(z, x) = A_{t10}(z) + z h_0(z) A_{t11}(z) \cos 2kx. \]

(28)

where we included explicit factors of \( z \) and \( h_0(z) \) for convenience. From (27), we obtain the boundary condition

\[ A_{t10}(0) = \mu_1. \]

(29)

From the Maxwell equation (9) and the boundary condition (29), we find

\[ A_{t10}(z) = C(1 - z) + \frac{\mu_0}{2} \int_z^1 dw \alpha_{10}(w), \quad C = \mu_1 - \frac{\mu_0}{2} \int_0^1 dz \alpha_{10}(z). \]

(30)
Thus the integration constant $C$ is expressed in terms of the chemical potential parameters $\mu_0$ and $\mu_1$. While $\mu_0$ is determined at the critical temperature, $\mu_1$ still needs to be determined. Subsequently, we will determine $C$ using the scalar equation and then use that value in eq. (30) to find $\mu_1$.

After some algebra, from the zero mode Einstein equations we find that the mode function $h_{10}(z)$ is given by

$$h_{10}(z) = -\frac{\mu_0 C}{2}(1 - z) - \frac{1}{16} \int_z^1 \frac{dw}{w^2} \tilde{h}_{10}(w),$$

where

$$\tilde{h}_{10}(z) = \left[ \frac{m^2}{z^2} + k^2 \right] \Phi_0'(z) + h_0(z)[\Phi_0'(z)]^2$$

$$+ \eta \left[ 2k^2 h_0(z)\Phi_0'(z) - \frac{h_0(z)(-8 + 8k^2 z^2 - 10(4 + \mu_0^2)z^3 + 13\mu_0^2 z^4)}{4} [\Phi_0'(z)]^2 ight.$$ 

$$- 4z h_0'(z)\Phi_0'(z)\Phi_0''(z) - \frac{k^2 z(16 + 8\mu_0^2 z^4 - 7(4 + \mu_0^2)z^3)\Phi_0(z)\Phi_0'(z)}{4}$$

$$- 2k^2 z^2 h_0(z)\Phi_0(z)\Phi_0''(z) \right].$$

(32)

Notice that the mode function $h_{10}$ contributes at $O(z^3)$ near the boundary, because $h_{10} \sim \text{const.}$ at the boundary, and we removed a factor of $z^3$ in the definition (28). We fixed one of the integration constants by setting $h_{10}(1) = 0$, so that the horizon remains at $z = 1$. $C$ is the remaining integration constant to be determined.

The mode function $\alpha_{10}(z)$ is given by

$$\alpha_{10} = \int_0^z dw \left[ \Phi_0'(w) \frac{h_0'(w)}{h_0(w)} h_{10}(w) + w \hat{\alpha}_{10}(w) \right],$$

(33)

where

$$\hat{\alpha}_{10}(z) = [\Phi_0'(z)]^2 + \eta \left[ - k^2 \Phi_0^2(z) - 2 \left( 1 + \frac{k^2}{2} z^2 - 5(1 + \mu_0^2)z^3 + \frac{5\mu_0^2}{4} z^4 \right) [\Phi_0'(z)]^2 ight.$$ 

$$- 2z h_0(z)\Phi_0'(z)\Phi_0''(z) - 4k^2 z\Phi_0(z)\Phi_0'(z) - k^2 z^2 \Phi_0(z)\Phi_0''(z) \right].$$

(34)

Notice that the mode function $\alpha_{10}$ contributes to $\alpha_1$ at an order higher than $O(z^3)$ near the boundary for $\Delta > \frac{3}{2}$.

Finally, the mode function $\beta_{10}(z)$ is given by

$$\beta_{10}(z) = \frac{k^2}{16} \int_0^z \frac{dw}{h_0(w)} \int_w^1 dw' \hat{\beta}_{10}(w'),$$

(35)
where
\[
\hat{\beta}_{10}(z) = \frac{2}{z^2} \Phi_0^2(z) + \eta \left[ \left( \frac{2}{z^2} - 2 \mu_0^2 z^2 + (4 + \mu_0^2) z^3 \right) \Phi_0^2(z) - 2 h_0(z) \Phi_0^2(z) \right]^2 \\
+ \frac{-8 - 6 \mu_0^2 z^4 + 5(4 + \mu_0^2) z^3}{2z} \Phi_0(z) \Phi_0(z) - 2 h_0(z) \Phi_0(z) \Phi_0''(z) \right].
\]

Notice that the mode function \( \beta_{10} \) also contributes at \( O(z^3) \) near the boundary, because \( \beta_{10} \sim z^3 \) at the boundary.

The remaining first-order modes \( \alpha_{11}, \beta_{11}, h_{11}, A_{111} \) are determined by a system of coupled linear ordinary differential equations,
\[
\begin{align*}
\alpha_{11}' + \frac{z h_0' + 3 h_0 + 2 k^2 z^2}{z h_0} \alpha_{11} - &\frac{4 z h_0' + 12 (h_0 - 1) + \mu^2 z^4 - 8 k^2 z^2}{2 z h_0} A_{11}(z) = 0, \\
\beta_{11}' + \frac{1}{2} z \mu A_{111}' + &\frac{3}{2} \frac{2 k^2 z^2 + h_0}{z h_0} \beta_{11} - \frac{\mu (5 h_0 + 3 z h_0')}{2 h_0} A_{111} + \frac{1}{4} \left( -8 k^2 z^3 + 3 \mu^2 z^3 + 2 h_0' \right) \alpha_{11} \\
+ \frac{(12 - \mu^2 z^4 + 20 k^2 z^2 - 14 h_0 - 12 z h_0)}{h_0} \beta_{11} - &\frac{z^2 \Delta^4}{h_0} A_2(z) = 0, \\
A_{111}'' + 2 \left[ \frac{1}{z} + \frac{h_0'}{h_0} \right] A_{111}' - &\frac{2 h_0' + z (-4 k^2 + h_0'^2)}{z h_0} A_{111} - \frac{\mu}{2} z^2 \alpha_{11}' - \frac{\mu}{2} \frac{z^2}{z} \left[ \frac{3}{z} + \frac{h_0'}{h_0} \right] \alpha_{11} = 0.
\end{align*}
\]
where
\[
\begin{align*}
A_1(z) &= \frac{\Delta}{4} z (-1 - 3 \eta + (4 + 3 \Delta) \eta h_0) F F' - \frac{z^2}{8} \left( 1 + \eta \left( k^2 z^2 + 3 - (5 + 4 \Delta) h_0 \right) \right) F'^2 \\
+ \frac{1}{8} \left( -\eta (1 + \Delta) k^2 z^2 + \Delta^2 (2 \Delta + 3) \eta h_0 - \Delta^2 (1 + 3 \eta) \right) F^2 \\
+ \frac{\eta}{8} z^2 \left( k^2 z^2 + 2 \Delta h_0 \right) F F'' + \frac{\eta}{4} z^3 h_0 F' F''', \\
A_2(z) &= -\frac{\eta}{2} z^2 h_0 \left( 2 z h_0 F' + (k^2 z^2 + 2 \Delta h_0) F \right) F'^2 \\
+ \frac{1}{16} z \left[ 2 \eta (24 \Delta^2 + \Delta - 3) h_0^2 + 3 \eta k^2 z^3 h_0' \\
+ 2 (1 + 3 \eta) (1 - 5 \Delta) - 6 \eta k^2 z^2 + \eta (1 + 10 \Delta) z h_0' \right] F' \Phi \\
+ \frac{1}{16} \left( \eta (1 - 16 \Delta^2 + 7 \Delta + 6) h_0^2 + 3 \Delta (2 \Delta + 3) - 3 k^2 z^2 (1 + 3 \eta + \Delta \eta z h_0' \Delta) \right) F^2 \\
+ \frac{1}{16} \left( 5 \eta k^2 (1 + 4 \Delta) z^2 + (1 + 3 \eta) \Delta (5 \Delta - 2) - 2 \eta \Delta (1 + 5 \Delta) z h_0' \right) h_0 F^2 \\
+ \frac{1}{16} \left( 5 + \eta \left( 15 + 8 k^2 z^2 - (9 + 32 \Delta) h_0 - 10 z h_0' \right) \right) h_0 F'^2, \\
A_3(z) &= \frac{z}{8} \left( 1 + \eta (3 - (3 + 4 \Delta) h_0 + (1 + \Delta) z h_0') \right) F F' \\
+ \frac{\eta}{16} z^2 (-4 h_0 + z h_0') F'^2 - \frac{\Delta}{16} (2 + \eta (6 - 2 (3 + 2 \Delta) h_0 + (2 + \Delta) z h_0')) F^2.
\end{align*}
\]
FIG. 3: Profiles of the Fourier modes $\alpha_{11}$ (upper left panel), $h_{11}$ (upper right panel), $\beta_{11}$ (lower left panel), and $A_{11}$ (lower right panel) for $\eta = 180$, $\Delta = 1.7$.

The system of equations (37) can easily be seen to possess a unique solution by requiring finiteness of all functions in the entire domain $z \in [0, 1]$. Notice that the unknown parameter $C$ is absent, which is due to the fact that at first-order the 10 modes decouple from the 11 modes (see eq. (28) for the definition of the Fourier modes). However, explicit solutions can only be obtained numerically; see Fig. 3 for an example. A complete numerical analysis will be presented elsewhere.

To complete the determination of the first order modes, we need to find the parameter $C$ (or, equivalently, the chemical potential parameter $\mu_1$). To this end, we turn to the scalar wave equation. At zeroth order, the chemical potential parameter $\mu_0$ was obtained as an eigenvalue of the scalar wave equation. $\mu_1$ is its first-order correction and is determined by the first order equation of the scalar wave equation.

Considering (10) below the critical temperature, we note that the scalar field at first order has two Fourier modes,

$$\Psi_1(z, x) = \Phi_{10}(z) \cos kx + \Phi_{11}(z) \cos 3kx. \quad (41)$$

The first (10) mode satisfies the equation

$$\left( \frac{h f_{+}}{z^2} \Phi_{10}'' \right) - k^2 \frac{f}{z^2} \Phi_{10} - \frac{m^2}{z^4} \Phi_{10} = B_2 \Phi_0'' + B_1 \Phi_0' + B_0 \Phi_0, \quad (42)$$
where \( f_\pm = 1 - \eta \mu_0^2 \), and the coefficients \( B_i \) \((i = 0, 1, 2)\) are given by lengthy expressions which will not be included here.

By using the zeroth order wave equation (43), we obtain
\[
\int_0^1 dz \Phi_0 [B_2 \Phi_0'' + B_1 \Phi_0' + B_0 \Phi_0] = 0 .
\]
which is a linear equation in \( C \) (or \( \mu_1 \)), and can easily be solved for \( C \). Once \( C \) is found, the remaining unknown parameter \( \mu_1 \) is calculated using eq. (30).

The temperature of our system below the critical temperature \( T_c \) can be calculated using
\[
\frac{T}{\mu} = -\frac{h'(1)e^{-\alpha(1)}}{4\pi \mu} .
\]
We obtain
\[
\frac{T}{T_c} = 1 - \xi^2 \left( \alpha_{10}(1) + \frac{\mu_1}{\mu_0} \right) - \frac{\xi^2}{3 - \frac{\mu_0^2}{4}} h'_{10}(1) ,
\]
where \( \xi \) is given by (25).

Eq. (45) can be inverted to find the energy gap as a function of temperature near the critical temperature,
\[
\langle O_\Delta \rangle^{1/\Delta} \approx \gamma \left( 1 - \frac{T}{T_c} \right)^{\frac{1}{2\Delta}} , \quad \gamma = \frac{4\pi}{3 - \frac{\mu_0^2}{4}} \left( \frac{\alpha_{10}(1)}{2} + \frac{\mu_1}{2\mu_0} + \frac{h'_{10}(1)}{2(3 - \frac{\mu_0^2}{4})} \right)^{-\frac{1}{2\Delta}} .
\]
Thus, as the temperature of the system is lowered below the critical temperature \( T_c \) the condensate is spontaneously generated. The dependence of the condensate on the temperature is of the same form as in conventional holographic superconductors.

Finally, the charge density of the system is determined by using
\[
\frac{\rho}{\mu^2} = -\frac{\partial_x A_t(0, x)}{[A_t(0, x)]^2} = \frac{\rho_0 + \xi^2 \rho_1(x)}{\mu_0^2} ,
\]
where \( \rho_0 = \mu_0 \) is the charge density at or above the critical temperature, and
\[
\rho_1(x) = -2\mu_1 - A'_{1t10}(0) - A_{1t11}(0) \cos 2kx = -2\mu_1 + C - A_{1t11}(0) \cos 2kx ,
\]
where we used (33). This is an important result showing the generation of a spatially inhomogeneous charge density below the critical temperature in the presence of a \textit{spatially homogeneous} (constant) chemical potential. This is the case provided \( A_{1t11}(0) \neq 0 \), which is guaranteed analytically from the system of equations (37) for the 11 modes. Indeed, from the
last equation in (37), we obtain \( A_{11}'(0) = 0 \). Moreover, there is a boundary condition at the horizon \( z = 1 \) where we demand finiteness of \( A_{11}(1) < \infty \). If additionally \( A_{11}(0) = 0 \), then the second-order differential equation is overdetermined and has no solution. Thus, a general solution has \( A_{11}(0) \neq 0 \). This is confirmed by numerical analysis (see Fig. 3).

V. CONCLUSIONS

We have discussed a holographic model in which the gravity sector consists of a U(1) gauge field and a scalar field coupled to an AdS charged black hole under a constant chemical potential. We introduced an interaction term between the Einstein tensor and the scalar field. A gravitational lattice was generated spontaneously by a spatially dependent profile of the scalar field. We calculated the transition temperature and by introducing an ultraviolet cutoff which was effectively proportional to the lattice spacing, we calculated the critical temperature below which the system undergoes a second order phase transition and a scalar condensate forms.

We perturbed the system below the critical temperature and solved the full coupled system of Einstein-Maxwell-scalar field equations at first order. We found that a spatial inhomogeneous phase is generated at the boundary. In particular, we showed both analytically and numerically that a spatially inhomogeneous charge density is spontaneously generated in the system while it is held at constant potential.

In our work we only considered a uni-directional lattice. It would be interesting to extent our discussion to a more general two-dimensional lattice and determine which configuration is energetically favorable. Work in this direction is in progress.

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