Singular solutions to the Riemann problem for the pressureless Euler equations with discontinuous source term

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ABSTRACT
It is interesting and challenging to study conservation laws with discontinuous source terms and explore how the delta shock wave is influenced by the discontinuous source term. However, so far, few results have been obtained about it. In this paper, the Riemann problem for the pressureless Euler equations with a discontinuous source term is considered. The delta shock wave solution is obtained by combining the generalized Rankine-Hugoniot conditions, together with the method of characteristics for various kinds of different situations, and the impact of the discontinuous source term on the delta shock front are precisely illustrated. Moreover, during the process of constructing the Riemann solution, some interesting phenomena are also observed, such as the disappearance of the delta shock wave and the occurrence of the vacuum state, etc.

1. Introduction
It is well known that singular discontinuities (delta shock waves) may develop for hyperbolic conservation laws, which may result from the initial data or the degeneracy and coincidence of the characteristics. The propagation of such delta shock waves or singular discontinuities may be affected by source terms for nonlinear systems, such as the pressureless Euler equations with friction [1,2] or dissipation [3,4] or their combinations [5], the generalized pressureless Euler equations with various source term [6–8], and the (generalized) Chaplygin gas with friction term [9,10]. However, so far, few results have been obtained for conservation laws with discontinuous source terms. In this paper, we are concerned with the pressureless Euler equations with discontinuous source terms in the following form:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
u_t + \left(\frac{u^2}{2}\right)_x &= H(x - s(t))f(x, t, u) + H(s(t) - x)g(x, t, u),
\end{align*}
\]

(1)

where \( H \) is the standard Heaviside function, \( f(x, t, u) \) and \( g(x, t, u) \) are a given continuous functions with respect to \( x \) and \( t \), \( \rho > 0 \) and \( u \) denote the density and the velocity, respectively. From the second equation of (1), \( u \) is assumed to be a piecewise smooth function with a single jump discontinuity at the curve \( x = s(t) \) in the \( (x, t) \) plane.

There are also quite a number of physical phenomena that can be described by hyperbolic conservation laws with singular source terms, such as the shallow-water flow of gravity currents [11], atmospheric cold fronts [12] and radiative hydrodynamics [13]. Moreover, the regularization technique has been developed in [14] to deal with hyperbolic conservation laws with time-dependent
singular Dirac delta source terms. In the future, we will also consider the pressureless Euler equations with the above singular source term.

The main purpose of this paper is to consider the impact of the source terms \( f(x, t, u) \) and \( g(x, t, u) \) on the location of delta shock front. In order to give an explicit expression of the delta shock wave curve and display the effect of the source terms, we will take \( f(x, t, u) = 0 \), \( g(x, t, u) = 1 \) or \( -u \) and \( f(x, t, u) = 1 \), \( g(x, t, u) = -u \) as typical examples to study the system (1) with Riemann initial data

\[
(\rho, u)(x, 0) = \begin{cases} 
(\rho_-, u_-), & x < 0, \\
(\rho_+, u_+), & x > 0.
\end{cases}
\]

(2)

With the above discontinuous source terms, the delta shock wave solutions will display some interesting behaviors and the vacuum state may occur in some situations, which may provide some insights into more general source term situations.

Specifically, for \( f(x, t, u) = 0 \), \( g(x, t, u) = 1 \) or \( -u \), (1) is reduced to the following form:

\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
u_t + \left( \frac{u^2}{2} \right)_x = H(s(t) - x)g(x, t, u),
\end{cases}
\]

(3)

For smooth solutions, (3) is equivalent to the following system:

\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2)_x = H(s(t) - x)\rho g(x, t, u),
\end{cases}
\]

(4)

which is just the generalization of the pressureless Euler equations with friction or dissipation [1–4]. For other cases such as \( g(x, t, u) = 0 \) with \( u_- > u_+ \), it is well known that the Riemann problem owns a delta shock solution, which should satisfy the following generalized Rankine-Hugoniot condition (GRHC) [17,21–23]:

\[
\begin{align*}
\frac{ds(t)}{dt} &= \sigma(t) = u_8, \\
\frac{dw(t)}{dt} &= \sigma(t)[\rho] - [\rho u], \\
\sigma(t)[u] &= \left[ \frac{u^2}{2} \right],
\end{align*}
\]

(5)

whose Riemann solution can be expressed as

\[
(\rho, u)(x, t) = \begin{cases} 
(\rho_-, u_-), & x < s(t), \\
w(t)\delta(x - s(t)), u_8(t), & x = s(t), \\
(\rho_+, u_+), & x > s(t).
\end{cases}
\]

(6)

where \( x = s(t) = \frac{u_- + u_+}{2} t \), \( w(t) = \frac{1}{2}(\rho_- + \rho_+)(u_- - u_+) \) and \( \frac{ds(t)}{dt} = \sigma(t) = \frac{u_- + u_+}{2} \) respectively denote the location, weight and propagation speed of the delta shock wave, and \( u_8(t) \) indicates the assignment of \( u \) on the delta shock wave, and \([\rho] = \rho(x(t) + 0, t) - \rho(x(t) - 0, t)\) denotes the jump of the function \( \rho \) across the delta shock wave.

For nonhomogeneous situations, it is worthwhile to notice that the characteristic curves do not keep straight, and \( u \) along each of the characteristic curves is not a constant again on one side or both sides of the delta shock front. Furthermore, the source term will act on the delta shock front such that it will also bend and no longer be a straight line again. Thus, the characteristic curves and the
value of \( u \) along these characteristic curves should be solved by the method of characteristics, and then the generalized Rankine-Hugoniot conditions should be proposed to determine the location of the delta shock wave front. For the Riemann problem for hyperbolic conservation laws, one can refer to the standard textbooks such as Refs [24,25]. Recently, there are many works concentrating on how to use the method of characteristics to solve the Riemann problem for scalar conservation law with discontinuous coefficients or with source terms, such as Refs [26–28]. However, there are very few related works for conservation laws, such as Refs [29]. In this paper and in future work, we will focus on Riemann problem for conservation laws with discontinuous coefficients or with source terms.

It is clear to see that the system (1) depends on the existence of the delta shock wave. Thus, if the delta shock wave does not exist or the entropy condition cannot hold, then we shall cease to construct the delta shock wave solution. In fact, a delta shock wave is generated for the Riemann problem (1) and (2) provided that \( u_- > u_+ \). Some interesting phenomena can be observed in the structural behavior of the delta shock wave, such as the disappearance of the delta shock wave and the occurrence of the vacuum state, etc.

Obviously, the second equation of (1) is a special type of the quasilinear hyperbolic equations

\[
 u_t + F(x,t,u)_x = G(x,t,u).
\]

The weak solution to the Cauchy problem for the scalar situation for (7) was first studied in Refs [30,31]. For a general form of \( F(x,t,u) \) and \( G(x,t,u) \), the situations are more complicated and still far from a complete description of Riemann solutions. Therefore, some special choices of \( F(x,t,u) \) and \( G(x,t,u) \) are taken to study the possible structures and asymptotic profiles of solutions. The particular Riemann problem (7) and (2) has been extensively studied, such as Ref. [32] for \( F(x,t,u) = F(u) \) and \( G(x,t,u) = G(u) \), Refs [33–35] for \( F(x,t,u) = F(u) \) and \( G(x,t,u) = a'(x)u \), and Refs [36,37] for \( F(x,t,u) = F(u) \) and \( G(x,t,u) = a'(x) \), where \( a(x) \) is usually assumed to be discontinuous at \( x = 0 \) and its derivative is also a Dirac delta measure such that a standing wave discontinuity is developed on the line \( x = 0 \). For more cases about hyperbolic conservation laws with source terms, one can see [38,39] for \( F(x,t,u) = F(u) \) and \( G(x,t,u) = c(x- \alpha t)g(u) \) where \( \alpha \) is the speed of the source, [40] for \( F(x,t,u) = F(u) \) and \( G(x,t,u) = s(t)\delta(x) \) where the source models the inlet, etc. One can also refer to Refs [41,42] for related results about the Burgers equation with source terms.

Recently, Fang et al. [27] investigated the Riemann problem for the Burgers equation with a discontinuous source term as follows:

\[
 u_t + \left( \frac{u^2}{2} \right)_x = g(x,t),
\]

where \( g(x,t) = g^- H(-x) + g^+ H(x) \) and \( g^-, g^+ \) are two constants. The Riemann problem (8) and (2) have been constructed completely in Ref. [27], where some interesting phenomena were observed there. Moreover, they have also studied the behavior of shock waves for \( 2 \times 2 \) balance laws with discontinuous source terms [29]. In this paper, similar to Ref. [43], we assume that the discontinuity of the source term always accompanies with the location of the delta shock front, so we only need to concern the delta shock wave solution to the Riemann problem (1) and (2). Moreover, we find that this assumption in our paper simplifies our study greatly that enables us to explicitly construct the delta shock wave solution to (1) and (2), and furthermore clearly display the effect of the source term on the delta shock wave.

The paper is organized as follows. In Section 2, the generalized Rankine-Hugoniot condition for (1) is derived in order to be selfcontained. In Section 3, the Riemann problem (1) and (2) is considered and the delta shock wave solution is constructed when \( u_- > u_+ \), \( f(x,t,u) = 0 \) and \( g(x,t,u) = 1 \) or \( -u \). In Section 4, the delta shock wave solution to the Riemann problem (1) and (2) is constructed when \( u_- > u_+ \), \( f(x,t,u) = 1 \) and \( g(x,t,u) = -u \). Finally, discussions are carried out and conclusions are drawn in Section 5.
2. The generalized Rankine-Hugoniot jump conditions

For nonlinear hyperbolic conservation laws with a source term that is discontinuous at the location of the shock front, Montgomery and Moodie [11] proposed a generalized Rankine-Hugoniot jump condition to determine the shock front. For the case when the source term is continuous, Shen [1] has given the generalized Rankine-Hugoniot jump condition for the delta shock wave front. In this paper, for the convenience of study, we only restrict ourselves to a particular form (1).

**Proposition 2.1:** Assume that

\[
(\rho, u)(x, t) = \begin{cases} 
(\rho_-, u_-), & x < s(t), \\
(w(t) \delta(x - s(t)), u_\delta(t)), & x = s(t) \\
(\rho_+, u_+), & x > s(t),
\end{cases}
\]

is a delta shock wave solution of (1) which has a single jump discontinuity at the position of the delta shock front \(x = s(t)\). Then, at \(x = s(t)\), the jump conditions (5) still hold and the following \(\delta\)-entropy condition is satisfied:

\[u(s(t) + 0) < \frac{ds}{dt} < u(s(t) - 0).\]  

**Proof:** We will only prove the third identity of (5). For the proof of the others, one can refer to Ref. [17]. Let us assume that \(\Gamma\) is the delta shock wave curve. Let \(\Omega\) be any region containing \(\Gamma\). Then, we use \(\Omega_- : \{(x, t) \mid x < s(t)\}\) to denote the left-hand side region of \(\Gamma\), and \(\Omega_+ : \{(x, t) \mid x > s(t)\}\) to denote the right-hand side region of \(\Gamma\). It is clear to see from (1) that

\[u_t + \left( \frac{u^2}{2} \right)_x = g(x, t, u), \quad \text{in } \Omega_-,
\]

\[u_t + \left( \frac{u^2}{2} \right)_x = f(x, t, u), \quad \text{in } \Omega_+.
\]

Let \(\varphi(x, t) \in C_\infty(R_+^2)\) be a test function, such that the following equality

\[
\left\langle u_t + \left( \frac{u^2}{2} \right)_x, \varphi \right\rangle = \left\langle H(x - s(t))f(x, t, u) + H(s(t) - x)g(x, t, u), \varphi \right\rangle
\]

holds in the sense of distributions. It is clear to see that the left-hand side of (13) can be calculated by

\[
\left\langle u_t + \left( \frac{u^2}{2} \right)_x, \varphi \right\rangle = -\langle u, \varphi_t \rangle - \left\langle \frac{u^2}{2}, \varphi_x \right\rangle
\]

\[
= -\int \int_{\Omega_-} (u\varphi_t + \frac{u^2}{2} \varphi_x) \, dx \, dt - \int \int_{\Omega_+} (u\varphi_t + \frac{u^2}{2} \varphi_x) \, dx \, dt
\]

\[
= -\int \int_{\Omega_-} (u\varphi_t + \frac{u^2}{2} \varphi_x) \, dx \, dt - \int \int_{\Omega_+} g\varphi \, dx \, dt
\]

\[
\quad - \int \int_{\Omega_+} (u\varphi_t + \frac{u^2}{2} \varphi_x) \, dx \, dt + \int \int_{\Omega_-} f\varphi \, dx \, dt
\]

\[
= \int_{\delta\Omega} \frac{u^2}{2} \varphi \, dt - u\psi \, dx + \int \int_{\Omega_-} g\varphi \, dx \, dt
\]
\[ \int_{\Omega_+} \frac{u^2}{2} \phi \, dt - u \psi \, dx + \int \int_{\Omega_+} f \phi \, dx \, dt \]
\[ = \int_0^{\infty} \left( \sigma[u] - \left[ \frac{u^2}{2} \right] \right) \phi \, dt + \int \int_{\Omega_-} g \phi \, dx \, dt + \int \int_{\Omega_+} f \phi \, dx \, dt, \quad (14) \]

in which (11) and (12) have been used.

On the other hand, the right-hand side of (13) can be calculated by

\[ \langle H(x - s(t)) f(x, t, u) + H(s(t) - x) g(x, t, u), \varphi \rangle = \int \int_{\Omega_-} g \phi \, dx \, dt + \int \int_{\Omega_+} f \phi \, dx \, dt. \quad (15) \]

Since \( \varphi \) is arbitrary, then the third identity of (5) can be obtained by combining (14) with (15) together. The proof is completed. \( \square \)

3. The situation for \( f(x, t, u) = 0 \) or \( g(x, t, u) = 0 \)

Without loss of generality, we assume that \( f(x, t, u) = 0 \) and \( g(x, t, u) \neq 0 \). Specially, for simplicity and practice meaning in [1,3,4], we take \( g(x, t, u) = 1 \) or \(-u\) in the following two subsections.

3.1. The situation for \( g(x, t, u) = 1 \)

In this section, we consider the Riemann problem (1) and (2) for the situation \( f(x, t, u) = 0 \) and \( g(x, t, u) = 1 \). When \( u_- > u_+ \), it is clear to see that the Riemann problem (1) and (2) has a delta shock solution with a single jump discontinuity across the path \( x = s(t) \), which crosses the \( x-\)axis at \( s(0) = 0 \) and satisfies the generalized Rankine-Hongniot condition (5). So the Riemann solutions to (1) and (2) can be found by considering the initial value problem on either side of the delta shock with the method of characteristics and then implementing the generalized Rankine-Hongniot condition (5) to determine the position of the delta shock front.

**Theorem 3.1:** For the situation \( f(x, t, u) = 0 \) and \( g(x, t, u) = 1 \), if \( u_- > u_+ \), then the Riemann solution to (1) and (2) is a delta shock wave solution which can be expressed as

\[ (\rho, u)(x, t) = \begin{cases} 
(\rho_-, u_- + t), & x < s(t), \\
(w(t)\delta(x - s(t)), u_\delta(t)), & x = s(t) \\
(\rho_+, u_+), & x > s(t), 
\end{cases} \quad (16) \]

with

\[ s(t) = \frac{1}{4} t^2 + \frac{1}{2} (u_- + u_+) t, \quad (17) \]

\[ w(t) = \frac{1}{2} (\rho_+ + \rho_-) \left\{ (u_- - u_+) t + \frac{1}{2} t^2 \right\}, \quad (18) \]

\[ u_\delta(t) = \frac{1}{2} (u_- + u_+ + t). \quad (19) \]

Furthermore, under the influence of discontinuous source term, for all cases, the delta shock wave exists for all the time by taking account of the entropy condition.
Proof: Since \( f(x, t, u) = 0 \) and \( g(x, t, u) = 1 \), (1) is simplified into
\[
\begin{cases}
  \rho_t + (\rho u)_x = 0, \\
  u_t + \left( \frac{u^2}{2} \right)_x = H(s(t) - x).
\end{cases}
\]

If \( x > s(t) \), then we have \( H(s(t) - x) = 0 \), and the initial value problem (1) and (2) becomes
\[
\begin{cases}
  \rho_t + (\rho u)_x = 0, \\
  u_t + uu_x = 0,
\end{cases}
\]
with initial condition
\[
(\rho, u)(x_0, 0) = (\rho_+, u_+), \quad \text{where} \ x_0 > s(0) = 0.
\]
It is obvious to see that the initial value problem (20) and (21) has a trivial solution as
\[
(\rho, u)(x, t) = (\rho_+, u_+), \quad \text{for all} \ x > s(t), t \geq 0.
\]
with \( x = u_+ t + x_0 \), for \( t > 0, x_0 > 0 \). For details, one can refer to Refs [17,21–23].

On the other hand, if \( x < s(t) \), then we have \( H(s(t) - x) = 1 \), and the initial value problem (1) and (2) becomes
\[
\begin{cases}
  \rho_t + (\rho u)_x = 0, \\
  u_t + uu_x = 1,
\end{cases}
\]
with initial condition
\[
(\rho, u)(x_0, 0) = (\rho_-, u_-), \quad \text{where} \ x_0 < s(0) = 0.
\]
By the method of characteristics, it is obvious to see that the initial value problem (23) and (24) has a trivial solution as
\[
(\rho, u)(x, t) = (\rho_-, u_- + t), \quad \text{for all} \ x < s(t), t \geq 0,
\]
with \( x = \frac{1}{2} t^2 + u_- t + x_0 \), for \( t > 0, x_0 < 0 \). For details, one can refer to Ref. [1].

To connect the solutions of (22) and (25) as a delta shock wave solution to the Riemann problem (1) and (2), the generalized Rankine-Hongniot condition (5) should be imposed at the position of the delta shock front \( x = s(t) \) as follows:
\[
\frac{ds}{dt}(u_+ - (u_- + t)) = \frac{1}{2}(u_+^2 - (u_- + t)^2),
\]
\[
\text{namely,}
\]
\[
u_\delta(t) = \mu(t) = \frac{ds}{dt} = \frac{1}{2}(u_+ + u_- + t).
\]
With \( s(0) = 0 \) in mind, we can obtain the expression of the delta shock front (17).

Moreover, the weight of the delta shock can be got from the second equality of (5) with
\[
\frac{dw(t)}{dt} = \frac{1}{2}(u_+ + u_- + t)(\rho_+ - \rho_-) - \{(\rho_+ u_+ - \rho_- (u_- + t))
\]
\[
= \frac{1}{2}(\rho_+ + \rho_-)((u_- - u_+) + t),
\]
from which we obtain (18).
The delta shock wave solution to (1) and (2) when \( f(x, t, u) = 0 \) and \( g(x, t, u) = 1 \). (a) \( u_- > u_+ > 0 \). (b) \( u_- > 0 \geq u_+ \).

Since the \( \delta \)-entropy condition
\[
\frac{ds}{dt} = \frac{1}{2}(u_+ + u_- + t) < u_- + t, \quad \text{for } t \geq 0,
\]
is always satisfied, so the delta shock wave always exists. The proof is completed.

Case 1 If \( u_- + u_+ \geq 0 \), taking account of \( u_- > u_+ \), it is easy to obtain \( u_- > 0 \). Moreover, from (27), we can obtain \( \frac{ds}{dt} > 0 \) and \( \frac{d^2s}{dt^2} = \frac{1}{2} > 0 \) for any \( t > 0 \). In other words, \( s(t) \) is always convex and increases along with \( t \) such that the delta shock wave should always move forward. Let us draw Figure 1(a,b) for the situations \( u_- > u_+ > 0 \) and \( u_- > 0 \geq u_+ \), respectively.

Case 2 If \( u_- + u_+ < 0 \), taking account of \( u_- > u_+ \), it is easy to obtain \( u_+ < 0 \). So, from (29), the delta shock wave has a negative speed in the beginning. Since \( \frac{d^2s}{dt^2} = \frac{1}{2} > 0 \) for any \( t > 0 \) from (27), \( s(t) \) is always convex, which implies that the speed of the delta shock wave speeds up. It is clear that there exists a time \( t_1 = -(u_- + u_+) \), such that \( \frac{ds}{dt} \big|_{t=t_1} = 0 \). In other words, the delta shock wave moves backward for \( 0 < t < t_1 \) and moves forward for \( t > t_1 \). It follows from (17) that there also exists a time \( t_2 = -2(u_- + u_+) \) such that \( s(t_2) = 0 \), which means that the delta shock wave front...
intersects with the $t$-axis at the time $t_2$. Let us draw Figure 1(c,d) for the situations $u_+ > 0 > u_-$ and $0 > u_- > u_+$, respectively, where $t = t_1^2$ is the symmetry axis of the characteristic curves on the left hand side of the delta shock wave.

**Remark 3.1:** Similarly, we can consider the situation $g(x, t, u) = -1$.

### 3.2. The situation for $g(x, t, u) = -u$

In this section, we consider the Riemann problem (1) and (2) for the situation $f(x, t, u) = 0$ and $g(x, t, u) = -u$. Similar to Section 3.1, we have the following theorem to depict the delta shock wave solution to the Riemann problem (1) and (2) when $u_- > u_+$.

**Theorem 3.2:** For the situation $f(x, t, u) = 0$ and $g(x, t, u) = -u$, if $u_- > u_+$, then the Riemann solution to (1) and (2) is a delta shock solution which can be expressed as

$$(\rho, u)(x, t) = \begin{cases} 
(\rho_-, u_-e^{-t}), & x < s(t), \\
(w(t)\delta(x - s(t)), u_5(t)), & x = s(t) \\
(\rho_+, u_+), & x > s(t),
\end{cases}$$

(30)

with

$$s(t) = \frac{1}{2}[u_- (1 - e^{-t}) + u_+ t],$$

(31)

$$w(t) = \frac{1}{2}(\rho_- + \rho_+) \{u_- (1 - e^{-t}) - u_+ t\},$$

(32)

$$u_5(t) = \frac{1}{2}(u_- e^{-t} + u_+).$$

(33)

Furthermore, under the influence of discontinuous source term, for $u_- > u_+ > 0$, the delta shock wave will disappear after some time and the vacuum occurs; while for other cases, the delta shock wave exists all the time by taking account of the entropy condition.

**Proof:** Since $f(x, t, u) = 0$ and $g(x, t, u) = -u$, (1) is simplified into

$$\begin{cases} 
\rho_t + (\rho u)_x = 0, \\
u_t + \left(\frac{u^2}{2}\right)_x = -H(s(t) - x)u.
\end{cases}$$

If $x > s(t)$, then we have $H(s(t) - x) = 0$. Similar to Section 3.1, we have

$$(\rho, u)(x, t) = (\rho_+, u_+), \quad \text{for all } x > s(t), t \geq 0.$$ 

(34)

with $x = u_+ t + x_0$, for $t > 0$, $x_0 > 0$.

On the other hand, if $x < s(t)$, then we have $H(s(t) - x) = 1$ and the initial value problem (1) and (2) becomes

$$\begin{cases} 
\rho_t + (\rho u)_x = 0, \\
u_t + uu_x = -u,
\end{cases}$$

(35)

with initial condition

$$\begin{cases} 
(\rho, u)(x_0, 0) = (\rho_-, u_-), \\
\text{where } x_0 < s(0) = 0.
\end{cases}$$

(36)

By the method of characteristics, it is obvious to see that the problem (35) and (36) has a trivial solution as

$$\begin{cases} 
(\rho, u)(x, t) = (\rho_-, u_- e^{-t}), \\
\text{for all } x < s(t), t \geq 0,
\end{cases}$$

(37)

with $x = u_- (1 - e^{-t}) + x_0$, for $t > 0$, $x_0 < 0$. For details, one can refer to Refs [3,4].
It is clear to see that for some situation, after some time $t$ the delta shock wave will disappear for the reason that the $\delta$-entropy condition cannot be satisfied. However, for a sufficiently small time $t$, the delta shock wave may exist and should satisfy the generalized Rankine-Hongnuiot condition (5), so we have

$$\frac{ds}{dt}(u_+ - u_- e^{-t}) = \frac{1}{2}(u_+^2 - (u_- e^{-t})^2),$$

(38)

namely,

$$u_\delta(t) = \sigma(t) = \frac{ds}{dt} = \frac{1}{2}(u_+ + u_- e^{-t}).$$

(39)

So (33) is obtained. With $s(0) = 0$ in mind, we can obtain the expression of the delta shock wave front (31).

Moreover, the weight of the delta shock wave can be obtained from the second equality of (5) with

$$\frac{dw(t)}{dt} = \frac{1}{2}(u_+ + u_- e^{-t})(\rho_+ - \rho_-) - (\rho_+ u_+ - \rho_- u_- e^{-t})$$

$$= \frac{1}{2}(\rho_+ + \rho_-)(u_- e^{-t} - u_+),$$

(40)

from which we obtain (32) with $w(0) = 0$.

It can be derived easily from (39) that

$$\frac{d^2 s}{dt^2} = -\frac{1}{2}u_- e^{-t}.$$

(41)

If the delta shock wave exists, the following $\delta$-entropy condition

$$u_+ < \frac{ds}{dt} < u_- e^{-t},$$

(42)

should be satisfied.

It follows from (39) that

$$\frac{ds}{dt} \bigg|_{t=0} = \frac{1}{2}(u_- + u_+),$$

(43)

so the $\delta$-entropy condition (42) is satisfied in the beginning for $u_- > u_+$ when $g(x, t, u) = -u$.

Since

$$\frac{ds}{dt} - u_+ = u_- e^{-t} - \frac{ds}{dt} = \frac{1}{2}(u_- e^{-t} - u_+),$$

(44)

for simplicity, we introduce the notation

$$p(t) = u_- e^{-t} - u_+.$$ 

(45)

Differentiating (45) with respect to $t$ yields

$$p'(t) = -u_- e^{-t}, \quad p''(t) = u_- e^{-t}.$$ 

(46)

It follows from (45)

$$p(0) = u_- - u_+ > 0.$$ 

(47)

In order to check the inequality (42), our discussions should be divided into the following four cases according to the values of $u_-$ and $u_+$. 
Figure 2. Figures for $p(t)$. (a) $u_- > 0 \geq u_+$. (b) $u_- > u_+ > 0$. (c) $u_+ < u_- < 0$ and (d) $u_+ < u_- = 0$.

(1) If $u_- > 0 \geq u_+$, then $p'(t) < 0$ and $p''(t) > 0$ for $t \geq 0$, so $p(t)$ is convex and strictly decreasing. Moreover, $\lim_{t \to +\infty} p(t) = -u_+$, which implies $y = p(t)$ has the line $y = -u_+$ as its asymptotic line in the $(t,y)$-plane. So $p(t) > -u_+ \geq 0$ for $t \geq 0$ (see Figure 2(a)), which means that the $\delta$-entropy condition (42) always holds for $t \geq 0$ when $u_- > 0 \geq u_+$, and the delta shock wave always exists.

(2) If $u_- > u_+ > 0$, it is easy to see that $p'(t) < 0$ and $p''(t) > 0$ for $t \geq 0$. Since $p(0) > 0$ and $\lim_{t \to +\infty} p(t) = -u_+ < 0$, there exist a unique $t_3$ such that $p(t_3) = 0$, i.e. $t_3 = \ln \frac{u_-}{u_+}$. Moreover, it is easy to obtain that $p(t) > 0$ for $0 \leq t < t_3$ and $p(t) < 0$ for $t > t_3$ (see Figure 2(b)). Thus the $\delta$-entropy condition (42) holds for $0 \leq t < t_3$ when $u_- > 0 \geq u_+$, and the delta shock wave disappears at the time $t = t_3$.

(3) If $u_- < 0$, then $p'(t) > 0$ and $p''(t) < 0$ for $t \geq 0$ (see Figure 2(c)), so $p(t)$ is concave and strictly increasing. So $p(t) > p(0) > 0$ for $t \geq 0$, which means that the $\delta$-entropy condition (42) always holds for $t \geq 0$ when $u_- < 0$, and the delta shock wave always exists.

(4) If $u_- = 0 > u_+$, then $p(t) = -u_+ > 0$ for $t \geq 0$ (see Figure 2(d)), which means that the $\delta$-entropy condition (42) always holds for $t \geq 0$ when $u_- = 0 > u_+$ and the delta shock wave always exists.

In conclusion, the delta shock wave always exists except for the case $u_- > u_+ > 0$, where the delta shock disappears after some time and the vacuum occurs. The proof is completed.

\[\square\]

For the nonhomogeneous situation $g(x, t, u) = -u$, the discussion about the path of the delta shock wave for the Riemann problem (1) and (2) can be carried out as before and should be divided into the following two cases.
Case 1 If \( u_- + u_+ > 0 \), taking account of \( u_+ > u_- \), it is easy to obtain \( u_- > 0 \). So \( \frac{d^2 s}{dt^2} < 0 \) for \( t \geq 0 \), which means that the delta shock wave curve is always concave, and the speed of the delta shock wave slows down. In the following, there are three subcases needed to be considered.

(i) If \( u_- > u_+ > 0 \), from the result obtained in (2), the delta shock wave disappears at the time \( t_3 \) and \( \frac{ds}{dt}|_{t=t_3} = u_+ = u_- e^{-t_3} > 0 \). Taking account of \( \frac{d^2 s}{dt^2} < 0 \) for \( t \geq 0 \), we have \( \frac{ds}{dt} > \frac{ds}{dt}|_{t=t_3} > 0 \) for \( 0 < t < t_3 \). So the delta shock wave curve is concave and \( s(t) \) increases monotonically along with \( t \) until it reaches the time \( t_3 \). Moreover, \( \frac{ds}{dt}|_{t=t_3} = u_+ = u_- e^{-t_3} \), so the delta shock wave curve is tangent with the characteristic curves at the time \( t_3 \) on both sides of it and then disappears. After the time \( t_3 \), the vacuum appears. We can draw Figure 3(a) to depict this situation.

(ii) If \( u_- > 0 > u_+ \), from the result obtained in (1), the delta shock wave always exists for \( t > 0 \). It is easy to obtain \( \lim_{t \to +\infty} \frac{ds}{dt} = \frac{1}{2} u_+ < 0 \). Taking account of \( \frac{ds}{dt}|_{t=0} > 0 \) and \( \frac{d^2 s}{dt^2} < 0 \) for \( t \geq 0 \), there exists a unique \( t_4 \) such that \( \frac{ds}{dt}|_{t=t_4} = 0 \), i.e. \( t_4 = \ln(-\frac{u_-}{u_+}) \). Moreover, \( \frac{ds}{dt} > 0 \) for \( 0 < t < t_4 \) and \( \frac{ds}{dt} < 0 \) for \( t > t_4 \). So the delta shock wave curve is always concave, moves forward for \( 0 < t < t_4 \), changes its direction at the time \( t = t_4 \) and moves backward for \( t > t_4 \). Moreover, \( \lim_{t \to +\infty} s(t) = -\infty \), so the delta shock wave curve intersects with the \( t \)-axis at the time \( \hat{t} \) such that \( s(\hat{t}) = 0 \). We can draw Figure 3(b) to depict this situation.

(iii) If \( u_- > 0 = u_+ \), from the result obtained in (1), the delta shock wave always exists for \( t > 0 \). Moreover, \( \frac{ds}{dt} = \frac{1}{2} u_- e^{-t} > 0 \) and \( \frac{d^2 s}{dt^2} = -\frac{1}{2} u_- e^{-t} < 0 \) for \( t \geq 0 \). So the delta shock wave curve is always concave and moves forward for \( t > 0 \). Furthermore, the delta shock wave front never intersects with the \( t \)-axis since \( s(t) = \frac{1}{2} u_-(1 - e^{-t}) > 0 \) for \( t > 0 \). We can draw Figure 3(c) to depict this situation.

Case 2 If \( u_- + u_+ \leq 0 \), taking account of \( u_- > u_+ \), it is easy to obtain \( u_+ < 0 \). In the following, there are three subcases needed to be considered.

(i) If \( u_- > 0 > u_+ \), from the result obtained in (1), the delta shock wave always exists for \( t > 0 \). It is easy to obtain \( \lim_{t \to +\infty} \frac{ds}{dt} = \frac{1}{2} u_+ < 0 \). Taking account of \( \frac{ds}{dt}|_{t=0} \leq 0 \) and \( \frac{d^2 s}{dt^2} < 0 \) for \( t \geq 0 \), we have \( \frac{ds}{dt} < 0 \) for \( t > 0 \). So the delta shock wave curve is always concave and moves backward for \( t > 0 \). Moreover, the delta shock wave front never intersects with the \( t \)-axis. We can draw Figure 4(d) to depict this situation.

(ii) If \( u_+ < u_- < 0 \), from the result obtained in (3), the delta shock wave always exists for \( t > 0 \). It is easy to obtain \( \lim_{t \to +\infty} \frac{ds}{dt} = \frac{1}{2} u_+ < 0 \). Taking account of \( \frac{ds}{dt}|_{t=0} \leq 0 \) and \( \frac{d^2 s}{dt^2} > 0 \) for \( t \geq 0 \), we have \( \frac{ds}{dt} < \frac{1}{2} u_+ < 0 \) for \( t > 0 \). So the delta shock wave curve is always convex and moves backward for \( t > 0 \). Moreover, the delta shock wave front never intersects with the \( t \)-axis since \( s(t) = \frac{1}{2}(t_- (1 - e^{-t}) + u_+ t) < 0 \) for \( t > 0 \). We can draw Figure 5(e) to depict this situation.

(iii) If \( u_- = 0 > u_+ \), from the result obtained in (4), the delta shock wave always exists for \( t > 0 \). The delta shock wave curve is \( s(t) = \frac{1}{2} u_+ t \) for \( t > 0 \). We can draw Figure 6(f) to depict this situation.

Remark 3.2: Similarly, we can consider the situation \( g(x, t, u) = u \).

4. The situation for \( f(x, t, u) \neq 0 \) and \( g(x, t, u) \neq 0 \)

In this section, we consider the case that \( f(x, t, u) \neq 0 \), \( g(x, t, u) \neq 0 \) and \( f(x, t, u) \neq g(x, t, u) \). For simplicity, we take \( f(x, t, u) = 1 \) and \( g(x, t, u) = -u \) to display how the delta shock wave front develop under the effect of the source term. Similar to Section 3.1, we have the following theorem to depict the delta shock wave solution to the Riemann problem (1) and (2) when \( u_- > u_+ \).
Figure 3. The delta shock wave solution to (1) and (2) when \( f(x, t, u) = 0 \) and \( g(x, t, u) = -u \). (a) \( u_- + u_+ > 0 \) for \( u_- > u_+ > 0 \). (b) \( u_- + u_+ > 0 \) for \( u_- > 0 > u_+ \). (c) \( u_- + u_+ > 0 \) for \( u_- > 0 = u_+ \). (d) \( u_- + u_+ \leq 0 \) for \( u_- > 0 > u_+ \). (e) \( u_- + u_+ \leq 0 \) for \( u_+ < u_- < 0 \) and (f) \( u_- + u_+ \leq 0 \) for \( u_+ = 0 > u_- \).

Theorem 4.1: For the situation \( f(x, t, u) = 1 \) and \( g(x, t, u) = -u \), if \( u_- > u_+ \), then the Riemann solution to (1) and (2) is a delta shock solution which can be expressed as

\[
(\rho, u)(x, t) = \begin{cases} 
(\rho_-, u_- e^{-t}), & x < s(t), \\
(w(t) \delta(x - s(t)), u_\delta(t)), & x = s(t), \\
(\rho_+, u_+ t), & x > s(t),
\end{cases}
\]

with

\[
s(t) = \frac{1}{2} \left\{ u_- (1 - e^{-t}) + u_+ t + \frac{1}{2} t^2 \right\},
\]

\[
w(t) = \frac{1}{2} (\rho_- + \rho_+) \left\{ u_- (1 - e^{-t}) - u_+ t - \frac{1}{2} t^2 \right\},
\]
\begin{align}
    u_\delta(t) &= \frac{1}{2} (u_- e^{-t} + u_+ + t). \quad (51)
\end{align}

Furthermore, under the influence of discontinuous source term, the delta shock wave will disappear after some time and the vacuum occurs for all cases by taking account of the entropy condition.

**Proof:** Since \( f(x, t, u) = 1 \) and \( g(x, t, u) = -u \), (1) is simplified into

\[
\begin{cases}
    \rho_t + (\rho u)_x = 0, \\
    u_t + \left( \frac{u^2}{2} \right)_x = H(x - s(t)) - H(s(t) - x)u.
\end{cases}
\]

For \( x < s(t) \), similar to Section 3.2, we have

\[
(\rho, u)(x, t) = (\rho_-, u_- e^{-t}), \quad \text{for all } x < s(t), \ t \geq 0,
\]

with \( x = u_- (1 - e^{-t}) + x_0 \), for \( t \geq 0, x_0 < 0 \).

For \( x > s(t) \), similar to Section 3.1, we have

\[
(\rho, u)(x, t) = (\rho_+, u_+ + t), \quad \text{for all } x > s(t), \ t \geq 0,
\]

with \( x = \frac{1}{2} t^2 + u_+ t + x_0 \), for \( t \geq 0, x_0 > 0 \).

It is clear to see that after some time \( t \) the delta shock wave will disappear for the reason that the \( \delta \)-entropy condition cannot be satisfied. However, for a sufficiently small time \( t \), the delta shock wave
Figure 5. The delta shock wave solution to (1) and (2) when \( f(x,t,u) = 1 \) and \( g(x,t,u) = -u \) for \( u_- + u_+ > 0 \).

(a) \( u_+ < 0 < u_- \leq 1 \). (b) \( 0 \leq u_+ < u_- \leq 1 \). (c) \( u_- > 1 \) with \( 1 + u_+ + \ln u_- = 0 \). (d) \( u_- > 1 \) with \( 1 + u_+ + \ln u_- < 0 \). (e) \( u_- > 1 \) with \( 1 + u_+ + \ln u_- > 0 \), \( t_5 \leq t_6 \) and \( u_+ \geq 0 \) and (f) \( u_- > 1 \) with \( 1 + u_+ + \ln u_- > 0 \), \( t_5 \leq t_6 \) and \( u_+ < 0 \).

Figure 6. The delta shock wave solution to (1) and (2) when \( f(x,t,u) = 1 \) and \( g(x,t,u) = -u \) for \( u_- + u_+ > 0 \) and \( u_- \leq 1 \).

(a) \( u_- + u_+ < 0 \) for \( u_+ < u_- \leq 0 \) and (b) \( u_- + u_+ < 0 \) for \( u_+ < 0 < u_- \leq 1 \).
may exist and should satisfy the generalized Rankine-Hongniot condition (5), so we have
\[
\frac{ds}{dt} (u_+ + t - u_- e^{-t}) = \frac{1}{2} (u_+ + t^2 - (u_- e^{-t})^2),
\]  
(54)
namely,
\[
\begin{aligned}
u_\delta(t) &= \sigma(t) = \frac{ds}{dt} = \frac{1}{2} (u_- e^{-t} + u_+ + t).
\end{aligned}
\]  
(55)
So (51) is obtained. With \(s(0) = 0\) in mind, we can obtain the expression of the delta shock wave front (49) in finite time.

Moreover, the weight of the delta shock can be obtained from the second equality of (5) with
\[
\frac{dw}{dt} (t) = \frac{1}{2} (u_- e^{-t} + u_+ + t) (\rho_+ - \rho_-) - \{\rho_+ (u_+ + t) - \rho_- u_- e^{-t}\}
= \frac{1}{2} (\rho_+ + \rho_-) \{u_- e^{-t} - (u_+ + t)\},
\]  
(56)
from which we obtain (50) with \(w(0) = 0\).

It can be derived easily from (55) that
\[
\frac{d^2 s}{dt^2} = \frac{1}{2} (1 - u_- e^{-t}).
\]  
(57)
If the delta shock wave exists, the following \(\delta\)-entropy condition
\[
u_+ + t < \frac{ds}{dt} < u_- e^{-t},
\]  
(58)
should be satisfied.

It follows from (55) that
\[
\frac{ds}{dt} \bigg|_{t=0} = \frac{1}{2} (u_- + u_+),
\]  
(59)
so the \(\delta\)-entropy condition (58) is satisfied in the beginning for \(u_- > u_+\) when \(f(x, t, u) = 1\) and \(g(x, t, u) = -u\).

Since
\[
\frac{ds}{dt} - (u_+ + t) = u_- e^{-t} - \frac{ds}{dt} = \frac{1}{2} (u_- e^{-t} - (u_+ + t)),
\]  
(60)
for simplicity, we introduce the notation
\[
q(t) = u_- e^{-t} - (u_+ + t).
\]  
(61)
Differentiating (61) with respect to \(t\) yields
\[
q'(t) = -u_- e^{-t} - 1, \quad q''(t) = u_- e^{-t}.
\]  
(62)
It follows from (61) that
\[
q(0) = u_- - u_+ > 0.
\]  
(63)
Since \(\lim_{t \to +\infty} q(t) = -\infty\), there exists \(t_5 > 0\) such that \(q(t_5) = 0\), i.e. \(u_- e^{-t_5} - (u_+ + t_5) = 0\). Next, we will prove that \(t_5\) is unique and \(q(t) > 0\) for \(0 < t < t_5\). Our discussions should be divided into the following two cases according to the values of \(u_-\) and \(u_+\).
(1) If \( u_- \geq 0 \), then \( q'(t) < 0 \) and \( q''(t) \geq 0 \), which means that \( q(t) \) is strictly decreasing and convex for \( t \geq 0 \). Obviously, \( t_5 \) is unique and \( q(t) > q(t_5) = 0 \) for \( 0 < t < t_5 \) (see Figure 4(a)).

(2) If \( u_- < 0 \), since \( q'(0) = -u_- - 1 \), then it should be divided into the following two subcases.

(2a) If \(-1 \leq u_- < 0\), then \( q'(0) \leq 0 \). Since \( q''(t) < 0 \) for \( t \geq 0 \), so \( q'(t) < q'(0) \leq 0 \), which means that \( q(t) \) is strictly decreasing for \( t \geq 0 \). Obviously, \( t_5 \) is unique and \( q(t) > q(t_5) = 0 \) for \( 0 < t < t_5 \) (see Figure 4(b)).

(2b) If \( u_- < -1 \), then \( q'(0) > 0 \). Since \( q''(t) < 0 \) for \( t \geq 0 \) and \( \lim_{t \to +\infty} q'(t) = -1 < 0 \), there exists \( \bar{t} > 0 \) such that \( q'(\bar{t}) = 0 \). Moreover, \( q'(t) > q'(0) > 0 \) for \( 0 < t < \bar{t} \), and \( q'(t) < q'(\bar{t}) = 0 \) for \( t > \bar{t} \). Since \( q(0) > 0 \), we have \( q(t) > 0 \) for \( 0 < t \leq \bar{t} \). Since \( q(t_5) = 0 \), we conclude that \( t_5 > \bar{t} \). Combined with \( q'(t) < 0 \) for \( t > \bar{t} \), which means that \( q(t) \) is strictly decreasing for \( t > \bar{t} \), we can claim that \( t_5 \) is unique and \( q(t) > q(t_5) = 0 \) for \( \bar{t} < t < t_5 \). In a word, \( q(t) > 0 \) for \( 0 < t < t_5 \) (see Figure 4(c)).

From the above discussion, we can conclude that there exists a unique \( t_5 > 0 \) such that \( q(t_5) = 0 \) and \( q(t) > 0 \) for \( 0 < t < t_5 \), which means that the \( \delta \)-entropy condition (58) holds for \( 0 \leq t < t_5 \). Moreover, \( \frac{ds}{dt}|_{t=t_5} = u_- e^{-t_5} = u_+ + t_5 \), So the delta shock wave is tangent with the characteristic curves at the time \( t_5 \) on both sides of it and then disappears. The proof is completed.

For the nonhomogeneous situation \( f(x, t, u) = 1 \) and \( g(x, t, u) = -u \), from Theorem 4.1, the delta shock wave disappears at the time \( t_5 \). Next, the discussion about the path of the delta shock wave for the Riemann problem (1) and (2) can be carried out like as before. Since \( \frac{ds}{dt}|_{t=0} = \frac{1}{2} (u_- + u_+) \), then it should be divided into the following three cases. Hereafter, for \( u_+ < 0 \), we denote \( t_{5_2} = -u_+ \dot{} \) the symmetry axes of the characteristics on the right-hand side of the delta shock wave.

Case 1 If \( u_- + u_+ > 0 \), taking account of \( u_- > u_+ \), it is easy to obtain \( u_- > 0 \). So \( \frac{ds}{dt}|_{t=t_5} = u_- e^{-t_5} = u_+ + t_5 > 0 \), from which we have \( t_5 - t_{5_2} = u_- e^{-t_5} > 0 \), i.e. \( t_5 > t_{5_2} \). From (57), we have \( \frac{d^2s}{dt^2}|_{t=0} = \frac{1}{2} u_- e^{-t_5} > 0 \) for \( t \geq 0 \) and \( \frac{d^2s}{dt^2}|_{t=0} = \frac{1}{2} (1 - u_-) \). In the following, there are two subcases needed to be considered.

(i) If \( 0 < u_- \leq 1 \), then \( \frac{d^2s}{dt^2}|_{t=0} = \frac{1}{2} (1 - u_-) \geq 0 \). Since \( \frac{d^2s}{dt^2} = \frac{1}{2} u_- e^{-t} > 0 \), \( \frac{d^2s}{dt^2} > 0 \) for \( t > 0 \). Taking account of \( \frac{d^2s}{dt^2} |_{t=0} = \frac{1}{2} (u_- + u_+) > 0 \), we have \( \frac{d^2s}{dt^2} > 0 \) for \( t > 0 \). Since the delta shock wave disappears at the time \( t_5 \), the delta shock wave is convex and increases along with \( t \) such that the delta shock wave moves forward for \( 0 < t < t_5 \) and then disappears. We draw Figure 5(a,b) to depict this situation according to \( u_+ < 0 \) and \( u_- \geq 0 \), where \( t = t_{5_2} = -u_+ \) is the symmetry axis of the characteristic curves on the right-hand side of the delta shock waves and \( t_{5_2} < t_5 \).

(ii) If \( u_- > 1 \), then \( \frac{d^2s}{dt^2}|_{t=0} = \frac{1}{2} (1 - u_-) < 0 \). Since \( \frac{d^2s}{dt^2} = \frac{1}{2} u_- e^{-t} > 0 \), and \( \lim_{t \to +\infty} \frac{d^2s}{dt^2} = \frac{1}{2} > 0 \), there exists a unique \( t_6 \) such that \( \frac{d^2s}{dt^2}|_{t=t_6} = 0 \), i.e. \( t_6 = \ln u_- \). Moreover, \( \frac{d^2s}{dt^2} < 0 \) for \( 0 \leq t < t_6 \) and \( \frac{d^2s}{dt^2} > 0 \) for \( t > t_6 \). At the time \( t_6 \), \( \frac{d^2s}{dt^2}|_{t=t_6} = \frac{1}{2} (u_- e^{-t_6} + u_+ + t_6) = \frac{1}{2} (1 + u_+ + \ln u_-) \). Then, it should be divided into the following three subcases.

(a) If \( 1 + u_+ + \ln u_- = 0 \), which is equivalent to \( u_- e^{-t_6} + u_+ + t_6 = 0 \). Furthermore, it is easy to obtain \( u_+ < 0 \). Since \( q(t_5) = u_- e^{-t_5} - (u_+ + t_5) = 0 \), we have \( t_5 - t_6 = u_- (e^{-t_5} + e^{-t_6}) > 0 \), i.e. \( t_5 > t_6 \), which means that the delta shock wave disappears after the time \( t_6 \). Moreover, \( \frac{ds}{dt} > 0 \) for \( 0 < t < t_6 \) and \( t_6 < t < t_5 \), \( \frac{d^2s}{dt^2} < 0 \) for \( 0 < t < t_6 \) and \( \frac{d^2s}{dt^2} > 0 \) for \( t_6 < t < t_5 \), which means that the delta shock wave is concave for \( 0 \leq t < t_6 \), convex for \( t_6 < t < t_5 \), always increases along with \( t \) such that the delta shock wave should always moves forward for \( 0 \leq t < t_5 \) and then disappears. We draw Figure 5(c) to depict this situation, where \( t_6 < t_{5_2} < t_5 \).

(b) If \( 1 + u_+ + \ln u_- < 0 \), which is equivalent to \( u_- e^{-t_6} + u_+ + t_6 < 0 \). Furthermore, it is easy to obtain \( u_+ < 0 \). Since \( q(t_5) = u_- e^{-t_5} - (u_+ + t_5) = 0 \), we have \( t_5 - t_6 > u_- (e^{-t_5} + e^{-t_6}) > 0 \), i.e. \( t_5 > t_6 \), which means that the delta shock wave disappears after the time \( t_6 \).
Moreover, $\frac{d^2 s}{dt^2}|_{t=0} > 0$, $\frac{d^2 s}{dt^2}|_{t=t_6} < 0$, $\frac{d^2 s}{dt^2}|_{t=t_5} = 0$, combining with $\frac{d^2 s}{dt^2} < 0$ for $0 < t < t_6$ and $\frac{d^2 s}{dt^2} > 0$ for $t_6 < t < t_5$, there exists a unique $t_7$ with $t_7 < t_5$ and a unique $t_8$ with $t_6 < t_8 < t_5$ such that $\frac{d^2 s}{dt^2}|_{t=t_7} = \frac{d^2 s}{dt^2}|_{t=t_8} = 0$. So $\frac{d s}{dt} > 0$ for $0 < t < t_7$ and $t_8 < t < t_5$, and $\frac{d s}{dt} < 0$ for $t_7 < t < t_8$. In a word, the delta shock wave is concave for $0 \leq t < t_6$, convex for $t_6 < t < t_5$. Furthermore, the delta shock wave increases along with $t$ such that the delta shock wave moves forward for $0 \leq t < t_7$ and $t_8 < t < t_5$, decreases along with $t$ such that the delta shock wave moves backward for $t_7 < t < t_8$ and then disappears. We draw Figure 5(d) to depict this situation, where $t_8 < t_2^* < t_5$.

(c) If $1 + u_+ + \ln u_- > 0$, which is equivalent to $u_- e^{-t_6} + u_+ + t_6 > 0$ and $u_+ > -(1 + \ln u_-)$. If $t_5 > t_6$, the delta shock wave in this section is similar to Case 1(ii)(a), see Figure 5(c). If $t_5 \leq t_6$, $\frac{d s}{d t} > 0$ and $\frac{d^2 s}{dt^2} < 0$ for $0 < t < t_5$. In other words, the delta shock wave is concave, increases along with $t$ such that the delta shock wave moves forward for $0 \leq t < t_5$, and then disappears. We draw Figure 5(e,f) to depict this situation according to $u_+ \geq 0$ and $-(1 + \ln u_-) < u_+ < 0$, where $t_2^* < t_5$.

Case 2 If $u_- + u_+ < 0$, taking account of $u_- > u_+$, it is easy to obtain $u_+ < 0$. According to (57),

$$\frac{d^2 s}{dt^2}|_{t=0} = \frac{1}{2}(1 - u_-),$$

there are two subcases needed to be considered as follows.

(i) If $u_- \leq 1$, then $\frac{d^2 s}{dt^2}|_{t=0} = \frac{1}{2}(1 - u_-) > 0$. we claim that $\frac{d^2 s}{dt^2} > 0$ for $t > 0$. In fact, the discussions should be divided into the following two subcases according to $u_- \leq 0$ and $0 < u_- \leq 1$.

(a) If $u_- \leq 0$, it is obvious that $\frac{d^2 s}{dt^2} = \frac{1}{2}(1 - u_- e^{-t}) > 0$ for $t > 0$. Since at the time $t_5$ when the delta shock wave disappears $\frac{d s}{dt}|_{t=t_5} = u_- e^{-t_5} \leq 0$, combined with $\frac{d s}{dt}|_{t=0} = \frac{1}{2}(u_- + u_+) < 0$, we have $\frac{d s}{dt} < 0$ for $0 < t < t_5$. So the delta shock wave is convex for $0 \leq t < t_5$, decreases along with $t$ such that the delta shock wave moves backward for $0 \leq t < t_5$ and then disappears. We draw Figure 6(a) to depict this situation, where $t_5 \leq t_2^*$. If $0 < u_- < 1$, since $\frac{d^2 s}{dt^2} = \frac{1}{2}u_- e^{-t} > 0$, combining with $\frac{d^2 s}{dt^2}|_{t=0} \geq 0$, we have $\frac{d^2 s}{dt^2} > 0$ for $t > 0$. Taking account of $\frac{d s}{dt}|_{t=0} > 0$ and at the time $t_5$ when the delta shock wave disappears $\frac{d s}{dt}|_{t=t_5} = u_- e^{-t_5} > 0$, there exists a unique $t_9 < t_5$ such that $\frac{d s}{dt}|_{t=t_9} = 0$. Moreover, $\frac{d s}{dt} < 0$ for $0 \leq t < t_9$ and $\frac{d s}{dt} > 0$ for $t_9 < t < t_5$. So the delta shock wave is convex for $0 \leq t < t_5$, decreases along with $t$ such that the delta shock wave moves backward for $0 \leq t < t_9$, increases along with $t$ such that the delta shock wave moves forward for $t_9 < t < t_5$, and then disappears. We draw Figure 6(b) to depict this situation, where $t_9 < t_2^* < t_5$.

(ii) If $u_- > 1$, then $\frac{d^2 s}{dt^2}|_{t=0} = \frac{1}{2}(1 - u_-) < 0$. Taking account of $\lim_{t \to +\infty} \frac{d^2 s}{dt^2} = \frac{1}{2} > 0$ and $\frac{d^2 s}{dt^2} = \frac{1}{2}u_- e^{-t} < 0$, we can conclude that there exists a unique $t_6$ such that $\frac{d^2 s}{dt^2}|_{t=t_6} = 0$, with $\frac{d^2 s}{dt^2} < 0$ for $0 < t < t_6$ and $\frac{d^2 s}{dt^2} > 0$ for $t > t_6$. Combined with $\frac{d s}{dt}|_{t=0} = \frac{1}{2}(u_- + u_+) < 0$, we have $\frac{d s}{dt}|_{t=t_6} = \frac{1}{2}(u_- e^{-t_6} + u_+ + t_6) < \frac{d s}{dt}|_{t=0} < 0$. Similar to Case 1(ii)(b), we can prove $t_6 < t_5$, which means that the delta shock wave disappears after the time $t_6$. Taking account of $\frac{d s}{dt}|_{t=t_5} = u_- e^{-t_5} > 0$ and $\frac{d^2 s}{dt^2} > 0$ for $t_6 < t < t_5$, there exists a unique $t_9$ such that $\frac{d s}{dt}|_{t=t_9} = 0$ and $t_6 < t_9 < t_5$. Moreover, $\frac{d s}{dt} < 0$ for $0 \leq t < t_9$ and $\frac{d s}{dt} > 0$ for $t_9 < t < t_5$. So the delta shock wave is concave for $0 < t < t_6$, convex for $t_6 < t < t_5$, decreases along with $t$ such that the delta shock wave moves backward for $0 \leq t < t_9$, increases along with $t$ such that the delta shock wave moves forward for $t_9 < t < t_5$, and then disappears. We draw Figure 7 to depict this situation, where $t_9 < t_2^* < t_5$. 
Case 3 If \( u_- + u_+ = 0 \), taking account of \( u_- > u_+ \), it is easy to obtain \( u_- > 0 > u_+ \). According to (57), \( \frac{d^2 u}{dt^2} |_{t=0} = \frac{1}{2} (1 - u_-) \), there are two subcases needed to be considered as follows.

(i) If \( 0 < u_- \leq 1 \), then \( \frac{d^2 u}{dt^2} |_{t=0} = \frac{1}{2} (1 - u_-) \geq 0 \). Since \( \frac{d^2 u}{dt^2} = \frac{1}{2} u_- e^{-t} > 0 \), we have \( \frac{d^2 u}{dt^2} > 0 \) for \( t > 0 \). Taking account of \( \frac{ds}{dt} |_{t=0} = 0 \), we have \( \frac{ds}{dt} > 0 \) for \( t > 0 \). But the delta shock wave disappears at \( t_5 \), so we have \( \frac{ds}{dt} > 0 \) and \( \frac{d^2 u}{dt^2} > 0 \) for \( 0 < t < t_5 \). So the delta shock wave is convex for \( 0 \leq t < t_5 \), increases along with \( t \) such that the delta shock wave moves forward for \( 0 < t < t_5 \) and then disappears. For the delta shock wave curve, one can refer to Figure 6(b) for \( t \geq t_9 \).

(ii) If \( u_- > 1 \), then \( \frac{d^2 u}{dt^2} |_{t=0} = \frac{1}{2} (1 - u_-) < 0 \). Next, since the discussion is similar to case 2 (ii), one can also refer to Figure 7 for the delta shock wave curve. So we omit it.

Remark 4.1: Similarly, we can consider other situations with source terms combined with \( f(x, t, u) = \pm 1 \) and \( g(x, t, u) = \pm u \). Different from the situation for \( f(x, t, u) \neq g(x, t, u) \), it is easy to prove that the delta shock wave always exists and never disappears in some situations if \( f(x, t, u) = g(x, t, u) = \pm 1 \) or \( \pm u \), for which one can refer to Refs [1,4].

5. Discusions and conclusions

In this paper, we have constructed the delta shock wave solution to the Riemann problem (1) and (2) in all cases, provided the delta shock wave exists initially for the discontinuous source term \( H(x - s(t))f(x, t, u) + H(s(t) - x)g(x, t, u) \) with special choices of \( f(x, t, u) \) and \( g(x, t, u) \). To obtain the delta shock solution and the influence of the discontinuous source term on the delta shock front, we have combined the generalized Rankine-Hugoniot conditions with the generalized characteristic method for various situations. It turns out that the discontinuity of the source term has clear influences and produces some new and interesting phenomena, such as the disappearance of the delta shock wave and the occurrence of the vacuum state, etc.

It is noticed that the special form of the discontinuous source term \( H(x - s(t))f(x, t, u) + H(s(t) - x)g(x, t, u) \) is just for convenience of demonstration and calculation in this paper. Moreover, the special choices of functions of \( f(x, t, u) \) and \( g(x, t, u) \) are propitious since they are only linearly dependent on \( u \). It is expected to adopt the more general discontinuous source term \( H(x)f(x, t, u) + H(-x)g(x, t, u) \) with \( f(x, t, u) \) and \( g(x, t, u) \) highly nonlinear in \( u \) or also dependent on \( x \) and \( t \) in our later work. We will focus on the effects of the discontinuous source term on the delta shock front. We hope that the method developed here would also be useful to explore new results about the pressureless Euler equations, the Chaplygin gas equations or the shallow water equations with more rigorous singular source term. Furthermore, we believe that this paper will provide some valuable insights.
into our future challenging study about the Cauchy problem for the pressureless Euler equations, the Chaplygin gas equations or the shallow water equations with more complex source terms.

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Disclosure statement
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