SEQUENTIAL PROPERTIES OF FUNCTION SPACES WITH THE COMPACT-OPEN TOPOLOGY

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Abstract. Let $M$ be the countably infinite metric fan. We show that $C_k(M, 2)$ is sequential and contains a closed copy of Arens space $S_2$. It follows that if $X$ is metrizable but not locally compact, then $C_k(X)$ contains a closed copy of $S_2$, and hence does not have the property AP.

We also show that, for any zero-dimensional Polish space $X$, $C_k(X, 2)$ is sequential if and only if $X$ is either locally compact or the derived set $X'$ is compact. In the case that $X$ is a non-locally compact Polish space whose derived set is compact, we show that all spaces $C_k(X, 2)$ are homeomorphic, having the topology determined by an increasing sequence of Cantor subspaces, the $n$th one nowhere dense in the $(n + 1)$st.

1. Introduction

Let $C_k(X)$ be the space of continuous real-valued functions on $X$ with the compact-open topology. $C_k(X)$ for metrizable $X$ is typically not a $k$-space, in particular not sequential. Indeed, by a theorem of R. Pol [8], for $X$ paracompact first countable (in particular, metrizable), $C_k(X)$ is a $k$-space if and only if $X$ is locally compact, in which case $X$ is a topological sum of locally compact $\sigma$-compact spaces and $C_k(X)$ is a product of completely metrizable spaces. A similar result holds for $C_k(X, [0, 1])$: it is a $k$-space if and only if $X$ is the topological sum of a discrete space and a locally compact $\sigma$-compact space, in which case $C_k(X)$ is the product of a compact space and a completely metrizable space. It follows that, for separable metric $X$, the following are equivalent:

1. $C_k(X)$ is a $k$-space;
2. $C_k(X)$ is first countable;
3. $C_k(X)$ is a complete separable metrizable space, i.e., a Polish space;
4. $X$ is a locally compact Polish space.

The same equivalences hold for $C_k(X, [0, 1])$. On the other hand, for Polish $X$, $C_k(X)$ always has the (strong) Pytkeev property [9].

A space $X$ has the property AP if whenever $x \in \overline{A} \setminus A$, there is some $B \subseteq A$ such that $x \in \overline{B} \subseteq A \cup \{x\}$. $X$ has the property WAP when a subset $A$ of $X$ is closed if and only if there is no $B \subseteq A$ such that $|\overline{B} \setminus A| = 1$. Thus, every Fréchet space is AP and every sequential space is WAP. It was asked in [5] whether $C_k(\omega^n)$ is WAP.

In this note, we first show that if $X$ is metrizable but not locally compact, then $C_k(X)$ contains a closed copy of Arens space $S_2$, and hence is not AP. In fact, such a closed copy of $S_2$ is contained in $C_k(M, 2)$, where $M$ is the countable metric fan. We then show that $C_k(M, 2)$ is sequential, in contrast to the full function space $C_k(M)$. Next we show that for a zero-dimensional Polish space $X$, if $C_k(X, 2)$ is
Theorem 7.1 observed that, for closed copies of copy of the space $M$

By Lemma 8.3 of [4], a first countable space $X$ is not metrizable (which is the case if and only if $X$ is not locally compact), then $C_k(X, 2)$ is sequential if and only if the derived set $X'$ is compact. We obtain a complete description of $C_k(X, 2)$ for a non-locally compact Polish $X$ such that $X'$ is compact: any such $C_k(X, 2)$ is homeomorphic to the space $(2^\omega)^\infty$, which is the space with the topology determined by an increasing sequence of Cantor sets, the $n$th one nowhere dense in the $(n + 1)$st.

2. When $C_k(X)$ Contains $S_2$

Arens’s space $S_2$ is the set

$\{(0, 0), (1/n, 0), (1/n, 1/nm): n, m \in \omega \setminus \{0\}\} \subseteq \mathbb{R}^2$

carrying the strongest topology inducing the original planar topology on the convergent sequences $C_0 = \{(0, 0), (\frac{1}{n}, 0): n > 0\}$ and $C_n = \{(\frac{1}{n}, 0), (\frac{1}{n}, \frac{1}{nm}): m > 0\}$, $n > 0$. The sequential fan is the quotient space $S_\omega = S_2/C_0$ obtained from the Arens space by identifying the points of the sequence $C_0$ [6]. $S_\omega$ is a non-metrizable Fréchet-Urysohn space, and $S_2$ is sequential and not Fréchet-Urysohn. In fact, any space which is sequential but not Fréchet-Urysohn contains $S_2$ as a subspace.

The countably infinite metric fan is the space $M = (\omega \times \omega) \cup \{\infty\}$, where points of $\omega \times \omega$ are isolated, and the basic neighborhoods of $\infty$ are $U(n) = \{\infty\} \cup ((\omega \setminus n) \times \omega)$, $n \in \omega$. $M$ is not locally compact at its non-isolated point $\infty$.

Lemma 2.1. $C_k(M, 2)$ contains a closed copy of $S_2$.

Proof. For each $n > 1$ and each $k$, let

$U(n, k) = \{(0) \times n\} \cup ((n \setminus \{0\}) \times k) \cup U(n),$

and let $f_{n,k}$ be the member of $C_k(M)$ which is 0 on $U(n, k)$ and 1 otherwise (i.e., the characteristic function of $M \setminus U(n, k)$).

Let $f_n \in C_k(M)$ be the function which is 1 on $\{0\} \times (\omega \setminus n)$ and 0 otherwise, and let $c_0$ be the constant 0 function. For each $n$, $\lim_k f_{n,k} = f_n$, and $\lim_n f_n = c_0$. Thus, $c_0$ is a limit point of the set $A = \{f_{n,k}: n > 1, k \in \omega\}$. Let $S = \{f_n: n > 1\}$, and $X = \{c_0\} \cup S \cup A$.

We claim that $X$ is homeomorphic to the Arens space $S_2$. It suffices to show that for each sequence $(k_n)_{n>1}$, $c_0$ is not in the closure of the set $\{f_{n,k}: k < k_n, n > 1\}$. Given $(k_n)_{n>1}$, set $K = \{(n - 1, k_n): n > 1\} \cup \{\infty\}$. Then $K$ is a sequence convergent to $\infty$, and for each $f_{n,k} \in \{f_{n,k}: k < k_n, n > 1\}$ there exists $x \in K$, namely, $x = (n - 1, k_n)$, such that $f(x) = 1$. Therefore $\{f_{n,k}: k < k_n, n > 1\}$ does not intersect the neighborhood $\{f \in C_k(M, 2): f \upharpoonright K \equiv 0\}$ of $c_0$, and hence does not contain $c_0$ in its closure.

By [1] Corollary 2.6, if every point $z$ in a topological space $Z$ is regular $G_\delta$ (i.e., $\{z\}$ is equal to $\bigcap_n U_n$ for some open neighborhoods $U_n$ of $z$), and $Z$ contains a copy of $S_2$, then $Z$ contains a closed copy of $S_2$. Since every point of $C_k(M, 2)$ is regular $G_\delta$, $C_k(M, 2)$ contains a closed copy of $S_2$. In fact, the space $X$ constructed above is closed, even in $C_p(M, 2)$. □

Theorem 2.2. If $X$ is metrizable and not locally compact, then $C_k(X)$ contains closed copies of $S_2$ and $S_\omega$.

Proof. By Lemma 8.3 of [4], a first countable space $X$ contains a closed topological copy of the space $M$ if and only if $X$ is not locally compact. E.A. Michael [7] Theorem 7.1 observed that, for $Y$ a closed subspace of a metrizable space $X$, the
linear extender \( e : C(Y) \to C(X) \) given by the Dugundji extension theorem is a homeomorphic embedding when both \( C(Y) \) and \( C(X) \) are given the compact-open topology (or the topology of uniform convergence, or pointwise convergence). Thus we have that for each metrizable space \( X \) which is not locally compact, \( C_k(M, 2) \) is closely embedded in \( C_k(X) \), and hence \( C_k(X) \) contains a closed copy of \( S_2 \). Finally, \( C_k(X) \) also contains a closed copy of \( S_\omega \) because for any topological group \( G, G \) contains a (closed) copy of \( S_\omega \) if and only if it contains a closed copy of \( S_\omega \). \( \square \)

**Remark.** C.J.R. Borges [2] showed that the Dugundji extension theorem holds for the class of stratifiable spaces, and hence Theorem 2.2 holds more generally for first countable stratifiable spaces.

3. **Sequentiality of \( C_k(X, 2) \)**

A topological space \( X \) carries the inductive topology with respect to a closed cover \( C \) of \( X \), if for each \( F \subseteq X \), \( F \) is closed whenever \( F \cap C \) is closed in \( X \) for each \( C \in C \). A topological space is a \( k \)-space (respectively, sequential space) if it carries the inductive topology with respect to its cover by compact (respectively, compact metrizable) subspaces. \( X \) is sequential if and only if for every non-closed \( A \subseteq X \), there exists a sequence in \( A \) converging to a point in \( X \setminus A \).

Since the metric fan \( M \) is not locally compact, \( C_k(M) \) and \( C_k(M, [0, 1]) \) are not \( k \)-spaces [3]. However, we have the following.

**Theorem 3.1.** \( C_k(M, 2) \) is sequential.

**Proof.** Suppose not. Then there is \( A \subseteq C_k(M, 2) \) which is not closed and yet contains all limit points of convergent sequences of its elements. As \( M \) is zero-dimensional, \( C_k(M, 2) \) is homogeneous. Thus, without loss of generality, we may assume that \( c_0 \in \overline{A} \setminus A \), where \( c_0 \) is the constant 0 function. We may additionally assume that \( f(\infty) = 0 \) for all \( f \in A \). Let \( A_n = \{ f \in A : f(U(n)) = \{0\} \} \).

Note that the sets \( A_n \) are increasing with \( n \), and their union is \( A \).

**Claim 3.2.** There exists a sequence \( (k_n)_{n \in \omega} \) such that for each \( n \) with \( f \in A_{n+1} \), \( 1 \in f(\bigcup_{i \leq n} \{i\} \times k_i) \).

**Proof.** By induction. Assume that for all \( i < n \), there are \( k_i \) such that \( f \in A_{i+1} \) implies \( 1 \in f(\bigcup_{i \leq j} \{j\} \times k_j) \), but that for each \( k \), there is \( f_k \in A_{n+1} \) such that \( f_k((\bigcup_{i < n} \{i\} \times k_i) \cup (\{n\} \times k)) = \{0\} \). Let \( f'_k = f_k \upharpoonright (n+1) \times \omega \). As \( 2^{(n+1)} \times \omega \) is homeomorphic to the Cantor space, there is a subsequence \( \{f'_{k'_i}\} \) of \( \{f'_k\} \), converging to an element \( f' \in 2^{(n+1)} \times \omega \). As \( f_k \in A_{n+1} \), \( f_k(U(n+1)) = \{0\} \). Define \( g \in C_k(M) \) by \( g(U(n+1)) = \{0\} \) and \( g \upharpoonright (n+1) \times \omega = f' \). Then in \( C_k(M) \), \( g = \lim f_{k_i} \), and therefore \( g \in A \). As \( f'_k(\{n\} \times k) = \{0\} \), \( g(\{n\} \times \omega) = \{0\} \). As \( g(U(n+1)) = \{0\} \), \( g(U(n)) = \{0\} \), and thus \( g \in A_n \). But \( g(\bigcup_{i \leq n-1} \{i\} \times k_i) = \{0\} \) (indeed, this holds for all \( f_k \)'s), contradicting the induction hypothesis. \( \square \)

Let

\[
K = (\bigcup_{i \in \omega} \{i\} \times k_i) \cup \{\infty\}.
\]

Let \( V \) be the set of all functions which map \( K \) into the interval \((-1/2, 1/2)\). Then \( V \) is a neighborhood of \( c_0 \) which misses \( A \), a contradiction. \( \square \)
We proceed to characterize the zero-dimensional Polish spaces $X$ such that $C_k(X, 2)$ is sequential.

A topological space $Y$ has the **strong Pytkeev property** [9] (respectively, **countable cs*-character**) if for each $y \in Y$, there is a countable family $\mathcal{N}$ of subsets of $Y$, such that for each neighborhood $U$ of $y$ and each $A \subseteq Y$ with $y \in \overline{A} \setminus A$ (respectively, each sequence $A$ in $Y \setminus \{y\}$ converging to $y$), there is $N \in \mathcal{N}$ such that $N \subseteq U$ and $N \cap A$ is infinite.

For every Polish space $X$ the space $C_k(X)$ has the strong Pytkeev property [9, Corollary 8]. Thus, any subspace of $C_k(X)$ has the strong Pytkeev property, and therefore has countable cs*-character.

An $mk_\omega$-space is a topological space which carries the inductive topology with respect to a countable cover of compact metrizable subspaces. A topological group $G$ is an $mk_\omega$-group if $G$ is an $mk_\omega$-space.

**Theorem 3.3** ([3]). *Let $G$ be a sequential non-metrizable topological group with countable cs*-character. Then $G$ contains an open $mk_\omega$-subgroup $H$ and thus is homeomorphic to the product $H \times D$ for some discrete space $D$.***

**Corollary 3.4.** *Let $G$ be a sequential separable topological group with countable cs*-character. If $G$ is not metrizable, then $G$ is $\sigma$-compact.*

**Lemma 3.5.** *Let $X$ be a zero-dimentional first countable space. Then $C_k(X, 2)$ is metrizable if and only if $X$ is locally compact and $\sigma$-compact.*

**Proof.** ($\Rightarrow$) Assume that $C_k(X, 2)$ is a topological group, its metrizability is equivalent to its first countability at $c_0$, the constant zero function.

($\Leftarrow$) Assume that $C_k(X, 2)$ is metrizable and fix a countable base $\{W_n : n \in \omega\}$ at $c_0$. Without loss of generality, $W_n = \{f \in C_k(X, 2) : f \upharpoonright K_n \equiv 0\}$ for some compact $K_n \subseteq X$, and $K_n \subseteq K_{n+1}$ for all $n$. It suffices to prove that for every $x \in X$ there are a neighborhood $U$ of $x$ and $n \in \omega$, such that $U \subseteq K_n$. If not, we can find $x \in X$ and a sequence $(x_n)_{n \in \omega}$ of elements of $X$ such that $x_n \in U_n \setminus K_n$, where $\{U_n : n \in \omega\}$ is a decreasing base at $x$. Set $K = \{x\} \cup \{x_n : n \in \omega\}$ and $W = \{f \in C_k(X, 2) : f \upharpoonright K \equiv 0\}$. Since $K_n \cap K$ is finite for every $n \in \omega$, there exists a function $f \in C_k(X, 2)$ such that $f \upharpoonright K_n \equiv 0$ but $f \upharpoonright K \not\equiv 0$, and hence $W_n \not\subseteq W$ for all $n \in \omega$. This contradicts our assumption that $\{W_n\}$ is a local base at $c_0$. $\square$

For a topological space $X$, $X'$ is the set of all non-isolated points of $X$.

**Theorem 3.6.** *Let $X$ be a zero-dimentional Polish space which is not locally compact. Then $C_k(X, 2)$ is sequential if and only if the derived set $X'$ is compact.*

**Proof.** Assume that $X'$ is compact and consider the subgroup $H = \{f \in C_k(X, 2) : f \upharpoonright X' \equiv 0\}$. $H$ is an open subgroup of $C_k(X, 2)$, and thus it suffices to prove that $H$ is sequential. Since $X$ is not locally compact, there is a clopen base $\{U_n : n \in \omega\}$ of $X'$ such that $U_0 = X$, $U_{n+1} \subseteq U_n$, and $U_n \setminus U_{n+1}$ is infinite for all $n \in \omega$. Let $f : X \to M$ be a map such that $f(X') = \{\infty\}$ and $f \upharpoonright (U_n \setminus U_{n+1})$ is an injective map onto $\{n\} \times \omega$. Then the map

$$f^* : \{g \in C_k(M, 2) : g(\infty) = 0\} \to H$$

assigning to $g$ the composition $g \circ f$ is easily seen to be a homeomorphism, and hence $H$ is sequential.
Now assume that $X'$ is not compact. Then there exists a countable closed discrete subspace $T \subseteq X'$, and hence there exists a discrete family $\{U_t : t \in T\}$ of clopen subsets of $X$ such that $t \in U_t$ for all $t \in T$. $C_k(X, 2)$ contains a closed copy of the product $\Pi_{t \in T} C_k(U_t, 2)$.

**Claim 3.7.** Let $Z$ be a non-discrete metrizable separable zero-dimensional space. Then $C_k(Z, 2)$ is not compact.

**Proof.** If $Z$ is locally compact, then it contains a clopen infinite compact subset $C$. Then $C_k(C, 2)$ is a closed subset of $C_k(Z, 2)$ homeomorphic to $\omega$, and hence $C_k(Z, 2)$ is not compact.

If $Z$ is not locally compact, then $Z$ contains a closed copy $Y$ of $M$. By Lemma 2.1 $C_k(Y, 2)$ contains a closed copy of $S_2$, and is thus not compact. As restriction to $Y$ is a continuous map from $C_k(Z, 2)$ onto $C_k(Y, 2)$, $C_k(Z, 2)$ is not compact. □

**Claim 3.8.** If none of the spaces $X_i$, $i \in \omega$, is compact, then the product $\Pi_{i \in \omega} X_i$ is not $\sigma$-compact.

**Proof.** A simple diagonalization argument. □

Since $T \subseteq X'$, $U_t$ is not discrete for all $t \in T$. By Claims 3.8 and 3.7 the product $\Pi_{t \in T} C_k(U_t, 2)$ is not $\sigma$-compact. Thus, $C_k(X, 2)$ is not $\sigma$-compact.

As $X$ is Polish, $C_k(X)$ has the strong Pytkeev property [9], and thus has countable cs*-character. Consequently, so does its subspace $C_k(X, 2)$. As $C_k(X, 2)$ is separable and $X$ is not locally compact, $C_k(X, 2)$ is not first countable, and hence it is not metrizable. Apply Corollary 3.9. □

**Corollary 3.9.** $C_k(\omega \times M, 2)$ is not sequential.

Let $(0) \in 2^\omega$ be the constant zero sequence. Following [1], let $(2^\omega)^\omega$ be the space $\bigcup_{n \in \omega} (2^\omega)^n$, where $(2^\omega)^n$ is identified with the subspace $(2^\omega)^n \times \{(0)\}$ of $(2^\omega)^{n+1}$, with the inductive topology with respect to the cover $\{(2^\omega)^n : n \in \omega\}$.

**Theorem 3.10** (Banakh [1]). Every non-metrizable uncountable zero-dimensional $mk_\sigma$-group is homeomorphic to $(2^\omega)^\omega$.

**Corollary 3.11.** For zero-dimensional Polish spaces $X$, the following are equivalent:

1. $C_k(X, 2)$ is sequential but not metrizable;
2. $C_k(X, 2)$ is homeomorphic to $(2^\omega)^\omega$;
3. $X$ is not locally compact but $X'$ is compact.

**Proof.** (1) $\rightarrow$ (3). Since $C_k(X, 2)$ is not metrizable, $X$ is not locally compact (Lemma 3.8). By Theorem 3.6 $X'$ is compact.

(3) $\rightarrow$ (1). Since $X$ is not locally compact, $C_k(X, 2)$ is not metrizable (Lemma 3.8). By Theorem 3.6 the compactness of $X'$ implies that $C_k(X, 2)$ is sequential.

(1) $\rightarrow$ (2). By [9, Corollary 8], $C_k(X, 2)$ has countable cs*-character. Applying Theorem 3.3 we have that $C_k(X, 2)$ contains an open $mk_\sigma$-subgroup. Since $C_k(X, 2)$ is separable, it is an $mk_\sigma$-group. Apply Theorem 3.10.

$C_k(\omega \times M, 2)$ is homeomorphic to $C_k(M, 2)^\omega$, and hence to $((2^\omega)^\omega)^\omega$. Thus, a negative answer to the following question would imply that $C_k(P)$ is not WAP for “most” Polish spaces, including $\omega$ and some $\sigma$-compact ones.

**Question 3.12.** Does the space $((2^\omega)^\omega)^\omega$ have the WAP property? What about $S_2^\omega$?
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