ON CELLULAR ALGEBRAS WITH JUCYS MURPHY ELEMENTS

FREDERICK M. GOODMAN AND JOHN GRABER

Abstract. We study analogues of Jucys–Murphy elements in cellular algebras arising from repeated Jones basic constructions. Examples include Brauer and BMW algebras and their cyclotomic analogues.

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1. Introduction

We recently developed a framework for proving cellularity of a tower of algebras $(A_n)_{n \geq 0}$ that is obtained from another tower of cellular algebras $(Q_n)_{n \geq 0}$ by repeated Jones basic constructions [16]. A key idea in this work is that of a tower of algebras with coherent cellular structures; coherence means that the cellular structures are well–behaved with respect to induction and restriction.

This paper continues our work on the themes of [16]. Here we refine the framework of [16] by taking into account the role played by Jucys–Murphy elements. We give conditions which allow lifting Jucys–Murphy elements from $Q_n$ to $A_n$.

At the same time, we give a new version of Andrew Mathas’s axiomatization [28] of cellular algebras with Jucys–Murphy elements, taking into account coherence of a sequence of such algebras. While Mathas posits the triangularity property of the action of the Jucys–Murphy elements on the cellular basis, we derive this property from simpler assumptions.

Examples of algebras covered by this theory are Jones–Temperley–Lieb algebras, Brauer algebras, BMW algebras, and their cyclotomic analogues. Our method yields an easy and uniform proof of the triangularity property of the
action of the Jucys–Murphy elements in these examples, recovering theorems of Enyang [11] and of Rui and Si [36] and [35].

2. Preliminaries

2.1. Algebras with involution, and their bimodules. Let $R$ be a commutative ring with identity. Recall that an involution $i$ on an $R$–algebra $A$ is an $R$–linear algebra anti–automorphism of $A$ with $i^2 = \text{id}_A$. If $A$ and $B$ are $R$–algebras and $\Delta$ is an $A$–$B$ bimodule, then we define a $B$–$A$ bimodule $i(\Delta)$ as follows. As an $R$–module, $i(\Delta)$ is a copy of $\Delta$ with elements marked with the symbol $i$. The $B$–$A$ bimodule structure is defined by $bi(x)a = i(axb)$. Then $i$ is a functor from the category of $A$–$B$ bimodules to the category of $B$–$A$ bimodules.

By the same token, we have a functor $i$ from the category of $B$–$A$ bimodules to the category of $A$–$B$ bimodules, and for an $A$–$B$ bimodule $\Delta$, we can identify $i \circ i(\Delta)$ with $\Delta$.

Suppose that $A$, $B$, and $C$ are $R$–algebras with involutions $i_A$, $i_B$, and $i_C$. Let $BPA$ and $AQ_C$ be bimodules. Then

$$i(P \otimes_A Q) \cong i(Q) \otimes_A i(P),$$

as $C$–$B$–bimodules. Note that if we identify $i(P \otimes_A Q)$ with $i(Q) \otimes_A i(P)$, then we have the formula $i(p \otimes q) = i(q) \otimes i(p)$.

In particular, let $M$ be a $B$–$A$–bimodule, and identify $i \circ i(M)$ with $M$, and $i(M \otimes_A i(M))$ with $i \circ i(M) \otimes_A i(M) = M \otimes_A i(M)$. Then we have the formula $i(x \otimes i(y)) = y \otimes i(x)$.

2.2. Cellularity. The definition of cellularity that we use is slightly weaker than the original definition of Graham and Lehrer in [20], see Remark 2.2.

**Definition 2.1.** Let $R$ be an integral domain and $A$ a unital $R$–algebra. A cell datum for $A$ consists of an algebra involution $i$ of $A$; a partially ordered set $(\Lambda, \geq)$ and for each $\lambda \in \Lambda$ a set $T(\lambda)$; and a subset $\mathcal{C} = \{c_{s,t}^{\lambda} : \lambda \in \Lambda$ and $s,t \in T(\lambda)\} \subseteq A$; with the following properties:

1. $\mathcal{C}$ is an $R$–basis of $A$.
2. For each $\lambda \in \Lambda$, let $\bar{\Lambda}$ be the span of the $c_{s,t}^{\mu}$ with $\mu \geq \lambda$. Given $\lambda \in \Lambda$, $s \in T(\lambda)$, and $a \in A$, there exist coefficients $r_{s}^{\lambda}(a) \in R$ such that for all $t \in T(\lambda)$:

$$ac_{s,t}^{\lambda} \equiv \sum_{\mu} r_{s}^{\lambda}(a)c_{s,t}^{\mu} \mod \bar{\Lambda}.$$

3. $i(c_{s,t}^{\lambda}) \equiv c_{i_{t},i_{s}}^{\lambda} \mod \bar{\Lambda}$ for all $\lambda \in \Lambda$ and $s,t \in T(\lambda)$.

$A$ is said to be a cellular algebra if it has a cell datum.

For brevity, we will write that $(\mathcal{C}, \Lambda)$ is a cellular basis of $A$.

**Remark 2.2.**

1. The original definition in [20] requires that $i(c_{s,t}^{\lambda}) = c_{i_{t},i_{s}}^{\lambda}$ for all $\lambda, s, t$. However, one can check that the basic consequences of the definition ([20], pages 7-13) remain valid with our weaker axiom.
In case 2 ∈ R is invertible, one can check that our definition is equivalent to the original.

We recall some basic structures related to cellularity, see [20]. Given λ ∈ Λ. Let $A^λ$ denote the span of the $c^\mu_{\alpha t}$ with $\mu \geq \lambda$. It follows that both $A^λ$ and $\tilde{A}^λ$ (defined above) are $t$-invariant two sided ideals of $A$. If $t \in \mathcal{T}(\lambda)$, define $C^\lambda_t$ to be the $R$-submodule of $A^\lambda/\tilde{A}^\lambda$ with basis $\{c^\lambda_s + \tilde{A}^\lambda : s \in \mathcal{T}(\lambda)\}$. Then $C^\lambda_t$ is a left $A$-module by Definition 2.1 (2). Furthermore, the action of $A$ on $C^\lambda_t$ is independent of $t$, i.e $C^\lambda_0 \cong C^\lambda_t$ for any $u, t \in \mathcal{T}(\lambda)$. The left cell module $\Delta^\lambda$ is defined as follows: as an $R$-module, $\Delta^\lambda$ is free with basis $\{c^\lambda_s : s \in \mathcal{T}(\lambda)\}$; for each $a \in A$, the action of $a$ on $\Delta^\lambda$ is defined by $ac^\lambda_s = \sum_ar^\lambda_s(a)c^\lambda_r$ where $r^\lambda_s(a)$ is as in Definition 2.1 (2). Then $\Delta^\lambda \cong C^\lambda_\emptyset$, for any $t \in \mathcal{T}(\lambda)$. For all $s, t \in \mathcal{T}(\lambda)$, we have a canonical $A$-$A$-bimodule isomorphism $\alpha : A^\lambda/\tilde{A}^\lambda \to \Delta^\lambda \otimes_R i(\Delta^\lambda)$ defined by $\alpha(c^\lambda_{\alpha t} + \tilde{A}^\lambda) = c^\lambda_{\alpha t} \otimes_R i(c^\lambda_t)$. Moreover, we have $i \circ \alpha = \alpha \circ i$, using the remarks at the end of Section 2.1 and point (2) of Definition 2.1.

In [16], we defined a coherent tower of cellular algebras as follows:

**Definition 2.3.** Let $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ be an increasing sequence of cellular algebras over an integral domain $R$. Let $\Lambda_n$ denote the partially ordered set in the cell datum for $H_n$. We say that $(H_n)_{n \geq 0}$ is a coherent tower of cellular algebras if the following conditions are satisfied:

1. The involutions are consistent; that is, the involution on $H_{n+1}$, restricted to $H_n$, agrees with the involution on $H_n$.
2. For each $n \geq 0$ and for each $\lambda \in \Lambda_n$, the induced module $\text{Ind}_{H_n}^{H_{n+1}}(\Delta^\lambda)$ has a filtration by cell modules of $H_{n+1}$. That is, there is a filtration

$$\text{Ind}_{H_n}^{H_{n+1}}(\Delta^\lambda) = M_t \supseteq M_{t-1} \supseteq \cdots \supseteq M_0 = (0)$$

such that for each $j \geq 1$, there is a $\mu_j \in \Lambda_{n+1}$ with $M_j/M_{j-1} \cong \Delta^{\mu_j}$.
3. For each $n \geq 0$ and for each $\mu \in \Lambda_{n+1}$, the restriction $\text{Res}_{H_n}^{H_{n+1}}(\Delta^\mu)$ has a filtration by cell modules of $H_n$. That is, there is a filtration

$$\text{Res}_{H_n}^{H_{n+1}}(\Delta^\mu) = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0)$$

such that for each $i \geq 1$, there is a $\lambda_i \in \Lambda_n$ with $N_i/N_{i-1} \cong \Delta^{\lambda_i}$.

The modification of the definition for a finite tower of cellular algebras is obvious. We call a filtration as in (2) and (3) a cell filtration. In the examples of interest to us, we will also have uniqueness of the multiplicities of the cell modules appearing as subquotients of the cell filtrations, and Frobenius reciprocity connecting the multiplicities in the two types of filtrations. We did not include uniqueness of multiplicities and Frobenius reciprocity as requirements in the definition, as they will follow from additional assumptions that we will impose later.

We introduce a stronger notion of coherence:
Definition 2.4. Say that a coherent tower of cellular algebras \((H_n)_{n \geq 0}\) is **strongly coherent** if in the cell filtrations (2) and (3) in Definition 2.3, we have

\[ \mu_t < \mu_{t-1} < \cdots < \mu_1 \]

in the partially ordered set \(\Lambda_{n+1}\), and

\[ \lambda_s < \lambda_{s-1} < \cdots < \lambda_1 \]

in the partially ordered set \(\Lambda_{n-1}\).

2.3. **Inclusions of split semisimple algebras and branching diagrams.** A finite dimensional split semisimple algebra over a field \(F\) is one which is isomorphic to a finite direct sum of full matrix algebras over \(F\).

Suppose \(A \subseteq B\) are finite dimensional split semisimple algebras over \(F\) (with the same identity element). Let \(A(i), i \in I\), be the minimal ideals of \(A\) and \(B(j), j \in J\), the minimal ideals of \(B\). We associate a \(J \times I\) inclusion matrix \(\Omega\) to the inclusion \(A \subseteq B\), as follows. Let \(W_j\) be a simple \(B(j)\)-module. Then \(W_j\) becomes an \(A\)-module via the inclusion, and \(\Omega(j, i)\) is defined to be the multiplicity of a simple \(A(i)\)-module in the decomposition of \(W_j\) as an \(A\)-module.

It is convenient to encode an inclusion matrix by a bipartite graph, called the **branching diagram**; the branching diagram has vertices labeled by \(I\) arranged on one horizontal line, vertices labeled by \(J\) arranged along a second (higher) horizontal line, and \(\Omega(j, i)\) edges connecting \(j \in J\) to \(i \in I\).

If \(A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots\) is a (finite or infinite) sequence of inclusions of finite dimensional split semisimple algebras over \(F\), then the branching diagram for the sequence is obtained by stacking the branching diagrams for each inclusion, with the upper vertices of the diagram for \(A_i \subseteq A_{i+1}\) being identified with the lower vertices of the diagram for \(A_{i+1} \subseteq A_{i+2}\).

For our purposes, it will suffice to restrict our attention to the case that \(A_0 \cong F\) and there are no multiple edges between vertices in the branching diagram (i.e. all entries of the inclusion matrices are 0 or 1). There is a unique vertex at level zero, which we denote by \(\emptyset\). For two vertices \(\lambda\) on level \(\ell\) of a branching diagram and \(\mu\) on level \(\ell + 1\), write \(\lambda \not\rightarrow \mu\) if \(\lambda\) and \(\mu\) are connected by an edge.

Let \(R\) be an integral domain with field of fractions \(F\). Let \(H\) be a cellular algebra over \(R\) and \(\Delta\) an \(H\)-module. Write \(H_n^F\) for \(H_n \otimes_R F\) and \(\Delta^F\) for \(\Delta \otimes_R F\).

Lemma 2.5 ([16], Lemma 2.20). Let \(R\) be an integral domain with field of fractions \(F\). Suppose that \((H_n)_{n \geq 0}\) is a coherent tower of cellular algebras over \(R\) and that \(H_n^F\) is split semisimple for all \(n\). Let \(\Lambda_n\) denote the partially ordered set in the cell datum for \(H_n\). Then

1. \(\{(\Delta^\lambda)^F : \lambda \in \Lambda_n\}\) is a complete family of simple \(H_n^F\)-modules.
2. Let \([\omega(\mu, \lambda)]_{\mu \in \Lambda_{n+1}, \lambda \in \Lambda_n}\) denote the inclusion matrix for \(H_n^F \subseteq H_{n+1}^F\).

Then for any \(\lambda \in \Lambda_n\) and \(\mu \in \Lambda_{n+1}\), and any cell filtration of \(\text{Res}_{H_{n+1}^F}(\Delta^\mu)\), the number of subquotients of the filtration isomorphic to \(\Delta^\lambda\) is \(\omega(\mu, \lambda)\).
Likewise, for any \( \lambda \in \Lambda_n \) and \( \mu \in \Lambda_{n+1} \), and any cell filtration of \( \text{Ind}^{H_{n+1}}_{H_n}(\Delta^\lambda) \), the number of subquotients of the filtration isomorphic to \( \Delta^\mu \) is \( \omega(\mu, \lambda) \).

2.4. **Bases in strongly coherent towers.** Adopt the assumptions and notation of Lemma 2.5, but with \((H_n)_{n \geq 0}\) strongly coherent. Assume in addition that the branching diagram \( \mathfrak{B} \) for \((H_n^F)_{n \geq 0}\) has no multiple edges and that \( H_0^F = F \). A path on \( \mathfrak{B} \) from \( \lambda \in \Lambda_\ell \) to \( \mu \in \Lambda_m \) \((\ell < m)\) is a sequence \( (\lambda = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m-\ell)} = \mu) \) with \( \lambda^{(i)} \nless \lambda^{(i+1)} \) for all \( i \). A path \( s \) from \( \lambda \) to \( \mu \) and a path \( t \) from \( \mu \) to \( \nu \) can be concatenated in the obvious way; denote the concatenation \( s \circ t \). If \( t = (0 = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)} = \lambda) \) is a path from \( 0 \) to \( \lambda \in \Lambda_n \), and \( 0 \leq k < \ell \leq n \), write \( t_{[k,\ell]} \) for the path \((\lambda^{(k)}, \ldots, \lambda^{(\ell)})\). Write \( t' \) for \( t_{[0,n-1]} \).

For \( \lambda \in \Lambda_n \), the rank of the cell module \( \Delta^\lambda \) of \( H_n \) is the same as the dimension of the simple \( H_n^F \) module \((\Delta^\lambda)^F \), namely the number of paths on \( \mathfrak{B} \) from \( 0 \) to \( \lambda \). It follows that each \( H_n \) has a cell datum (perhaps different from the one initially given) with the same partially ordered set \( \Lambda_n \) but with \( \mathcal{T}(\lambda) \) equal to the set of paths on \( \mathfrak{B} \) from \( 0 \) to \( \lambda \).

Remark that the set \( \mathcal{T}(\lambda) \) of paths on \( \mathfrak{B} \) from \( 0 \) to \( \lambda \) can be partially ordered by (reverse) lexicographic order. Namely, if \( t_1 = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)}) \) and \( t_2 = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n)}) \) are two such paths with \( \lambda^{(i)} \nless \mu^{(i)} \in \Lambda_i \), then \( t_1 \prec t_2 \) if, for the last index \( j \) such that \( \lambda^{(j)} \neq \mu^{(j)} \), we have \( \lambda^{(j)} < \mu^{(j)} \) in \( \Lambda_j \). Similarly, we can order the paths going from level \( k \) to level \( n \) on \( \mathfrak{B} \) lexicographically.

**Example 2.6.** Fix an integral domain \( S \) and an invertible \( q \in S \). The Hecke algebra \( H_n(q) = H_{n,S}(q) \) is the associative, unital \( S \)-algebra with generators \( T_j \) for \( 1 \leq j \leq n - 1 \), satisfying the braid relations and the quadratic relation \((T_j - q)(T_j + 1) = 0\) for all \( j \). \( H_n(q) \) has an algebra involution \( x \mapsto x^* \) uniquely determined by \((T_j)^* = T_j \). \( H_n(q) \) has a cellular basis due to Murphy

\[
\{ m^\lambda_{\mathfrak{m},\mathfrak{s}} : \lambda \in Y_n; \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda) \}.
\]

where \( Y_n \) is the partially ordered set of all Young diagrams of size \( n \), with dominance order \( \succeq \), and \( \mathcal{T}(\lambda) \) is the set of all standard Young tableaux of shape \( \lambda \). By results of Murphy \([33]\), Dipper and James \([7, 8]\), and Jost \([24]\), the sequence of Hecke algebras \((H_{n,S}(q))_{n \geq 0}\) is strongly coherent.

The generic ground ring for the Hecke algebras is \( R = \mathbb{Z}[q, q^{-1}] \), where \( q \) is an indeterminate over \( \mathbb{Z} \); the Hecke algebra \( H_{n,S}(q) \) over any \( S \) is a specialization of \( H_{n,R}(q) \). If \( F = \mathbb{Q}(q) \) denotes the field of fractions of \( R \), then \( H_{n,F}(q) \) is split semisimple for all \( n \) and the branching diagram for the tower of Hecke algebras \((H_{n,F}(q))_{n \geq 0}\) is Young’s lattice \( \mathcal{Y} \). Standard Young tableaux of shape \( \lambda \) can be identified with paths on \( \mathcal{Y} \) from the empty diagram to \( \lambda \). The set \( \mathcal{T}(\lambda) \) of standard tableaux of shape \( \lambda \) can be partially ordered by dominance \( \succeq \) or by the (reverse) lexicographic order \( \succeq \) described above; if \( t_1 = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)}) \)
and \( t_2 = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n)}) \), then \( t_1 \preceq t_2 \) means that for the last \( j \) such that \( \lambda^{(j)} \neq \mu^{(j)} \), we have \( \lambda^{(j)} \triangleright \mu^{(j)} \). Note that \( t_1 \triangleright t_2 \) implies \( t_1 \preceq t_2 \).

Continuing the discussion preceding Example 2.6, we construct bases of the cell modules \( \Delta^\lambda \) (\( \lambda \in \cup \Lambda_n \)) indexed by the set of paths \( T(\lambda) \) by induction on \( n \), as follows. For \( \lambda \in \Lambda_0 \) or \( \lambda \in \Lambda_1 \), the cell module \( \Delta^\lambda \) is free of rank one, and we choose any basis. Suppose now that \( n > 0 \), and a basis \( \{ a_s^\mu : s \in T(\mu) \} \) for \( \Delta^\mu \) has been obtained for each \( \mu \in \Lambda_k \) for \( k \leq n - 1 \). Let \( \lambda \in \Lambda_n \), and consider the filtration

\[
\text{Res}^{H_n}_{H_{n-1}}(\Delta^\lambda) = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0),
\]

with \( N_j/N_{j-1} \cong \Delta^{\mu_j} \) and \( \mu_s < \mu_{s-1} < \cdots < \mu_1 \). For each \( j \), let \( \{ \bar{a}_{s}^{\mu_j} : s \in T(\mu_j) \} \) be any lifting to \( N_j \) of the basis \( \{ a_s^{\mu_j} : s \in T(\mu_j) \} \) of \( N_j/N_{j-1} \cong \Delta^{\mu_j} \). Then \( \bigcup_j \{ \bar{a}_{s}^{\mu_j} : s \in T(\mu_j) \} \) is a basis of \( \Delta^\lambda \). Note that \( t \mapsto t' \) is a bijection from \( T(\lambda) \) to \( \bigcup_j T(\mu_j) \). We define \( a_1^\lambda \) to be \( \bar{a}_{v}^{\mu_j} \) if \( t' \in T(\mu_j) \), so our basis is now denoted by \( \{ a_s^\lambda : s \in T(\lambda) \} \).

Under the hypotheses discussed above (in particular strong coherence of the tower \( (H_n)_{n \geq 0} \)), the bases \( \{ a_s^\lambda : s \in T(\lambda) \} \) of the cell modules \( \Delta^\lambda \) have the following property.

**Proposition 2.7.** Fix \( 0 \leq k < n \), \( \lambda \in \Lambda_n \), and \( t \in T(\lambda) \). Write \( \mu = t(k) \), \( t_1 = t|_{[0,k]} \), and \( t_2 = t|_{[k,n]} \). Let \( x \in H_k \), and let \( xa_1^\lambda = \sum_s r(x; s, t_1) a_s^\lambda \). Then

\[
xa_1^\lambda \equiv \sum_s r(x; s, t_1) a_s^\lambda \mod R-
\]

span of \( \{ a_0^\lambda : v_{[k,n]} \triangleright t_{[k,n]} \} \).

**Proof.** We prove this by induction on \( n - k \). Consider the case \( k = n - 1 \). Consider the filtration (2.1). If \( t' \in T(\mu_j) \), then by the construction of the basis \( \{ a_t^\lambda : t \in T(\lambda) \} \), we have

\[
xa_t^\lambda \equiv \sum_s r(x; s, t_1) a_s^\lambda \mod N_{j-1},
\]

while \( N_{j-1} \) equals the \( R \)-span of \( \{ a_0^\lambda : v_{[n-1,n]} \triangleright t_{[n-1,n]} \} \).

Now suppose that \( n - k > 1 \), and \( t' \in T(\mu_j) \). Then \( xa_t^\lambda = xa_t^{\mu_j} \). By a suitable induction hypothesis,

\[
xa_t^{\mu_j} \equiv \sum_s r(x; s, t_1) a_s^{\mu_j} \mod \text{span of } \{ a_0^{\mu_j} : v_{[k,n-1]} \triangleright t_{[k,n-1]} \}.
\]

But then

\[
xa_t^\lambda \equiv \sum_s r(x; s, t_1) a_s^\lambda \mod R-
\]

span of \( \{ a_0^\lambda : v_{[k,n-1]} \triangleright t_{[k,n-1]} \} \). This completes the proof.

\( \square \)
2.5. **Framework axioms and a theorem on cellularity.** We describe the framework axioms and main theorem of [16]. Let $R$ be an integral domain with field of fractions $F$. We consider two towers of $R$–algebras

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \quad \text{and} \quad Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots.$$ 

The framework axioms of [16] are the following:

(1) $(Q_n)_{n \geq 0}$ is a coherent tower of cellular algebras.

(2) There is an algebra involution $\iota$ on $\cup_n A_n$ such that $\iota(A_n) = A_n$.

(3) $A_0 = Q_0 = R$, and $A_1 = Q_1$ (as algebras with involution).

(4) For all $n$, $A_n^F := A_n \otimes_R F$ (and hence also $Q_n^F := Q_n \otimes_R F$) is split semisimple.

(5) For $n \geq 2$, $A_n$ contains an essential idempotent $e_{n-1}$ such that $i(e_{n-1}) = e_{n-1}$ and $A_n/(A_ne_{n-1}A_n) \cong Q_n$, as algebras with involution.

(6) For $n \geq 2$, $e_{n-1}$ commutes with $A_{n-2}$ and $e_{n-1}A_{n-1}^e_{n-1} = A_{n-2}^e_{n-1}$.

(7) For $n \geq 2$, $A_n e_{n-1} = A_n e_{n-1}^e$, and the map $x \mapsto xe_{n-1}$ is injective from $A_{n-1}$ to $A_{n-1} e_{n-1}$.

(8) For $n \geq 2$, $e_{n-1} \in A_{n+1} e_{n} A_{n+1}$.

Say that the pair of towers of algebras $(Q_k)_{k \geq 0}$ and $(A_k)_{k \geq 0}$ satisfy the *strong framework axioms*, if they satisfy the axioms with (1) replaced by

(1') $(Q_n)_{n \geq 0}$ is a strongly coherent tower of cellular algebras.

**Remark 2.8.**

(1) Let $\Lambda_n^{(0)}$ denote the partially ordered set in the cell datum for $Q_n$. It follows from axioms (1) and (4) and Lemma 2.5 that $\Lambda_n^{(0)}$ can be identified with the $n$–th row of vertices of the branching diagram for $(Q_n^F)_{n \geq 0}$.

(2) Applying the involution in axiom (7), we also have $e_{n-1} A_{n-1} = e_{n-1} A_n$, and the map $x \mapsto e_{n-1} x$ is injective from $A_{n-1}$ to $e_{n-1} A_{n-1}$. It follows from axioms (6) and (7) that $e_{n-1} A_n^e = A_{n-2}^e e_{n-1}$.

(3) From axiom (6), we have for every $x \in A_{n-1}$, there is a $y \in A_{n-2}$ such that $e_{n-1} x e_{n-1} = y e_{n-1}$; but by axiom (7), $y$ is uniquely determined, so we have a map $c_{n-1} : A_{n-1} \rightarrow A_{n-2}$ with $e_{n-1} x e_{n-1} = c_{n-1}(x) e_{n-1}$. It is easy to check that $c_{n-1}$ is an $A_{n-2} – A_{n-2}$–bimodule map, but it is not unital in general; if $e_{n-1}^2 = \delta e_{n-1}$, then $c_{n-1}(1) = \delta 1$. If $\delta$ is invertible in $R$, then $e_{n-1} = (1/\delta) c_{n-1}$ is a conditional expectation, i.e., a unital $A_{n-2} – A_{n-2}$–bimodule map.

In the following theorem, point (4) we use the notion of a branching diagram obtained by reflections from another branching diagram. We refer the reader to [16], Section 2.5 for this notion.

**Theorem 2.9 ([16], Theorem 3.2).** Let $R$ be an integral domain with field of fractions $F$. Let $(Q_k)_{k \geq 0}$ and $(A_k)_{k \geq 0}$ be two towers of $R$–algebras satisfying the framework axioms (resp. the strong framework axioms). Then
(1) \((A_k)_{k \geq 0}\) is a coherent tower of cellular algebras (resp. a strongly coherent tower of cellular algebras).

(2) For all \(k\), the partially ordered set in the cell datum for \(A_k\) can be realized as

\[
\Lambda_k = \prod_{i \leq k} \Lambda_i^{(0)} \times \{k\},
\]

with the following partial order: Let \(\lambda \in \Lambda_i^{(0)}\) and \(\mu \in \Lambda_j^{(0)}\), with \(i, j, \) and \(k\) all of the same parity. Then \((\lambda, k) > (\mu, k)\) if, and only if, \(i < j\), or \(i = j\) and \(\lambda > \mu\) in \(\Lambda_i^{(0)}\).

(3) Suppose \(k \geq 2\) and \((\lambda, k) \in \Lambda_i^{(0)} \times \{k\} \subseteq \Lambda_k\). Let \(\Delta^{(\lambda,k)}\) be the corresponding cell module. If \(i < k\), then \(A_k e_{k-1} A_k \Delta^{(\lambda,k)} = \Delta^{(\lambda,k)}\), while if \(i = k\) then \(A_k e_{k-1} A_k \Delta^{(\lambda,k)} = 0\).

(4) The branching diagram \(\mathcal{B}\) for \((A_k^F)_{k \geq 0}\) is that obtained by reflections from the branching diagram \(\mathcal{B}_0\) for \((Q_k^F)_{n \geq 0}\).

**Proof.** The theorem for coherent towers is proved in [16]. The modification for strongly coherent towers is straightforward.

\[\square\]

3. JM elements in coherent towers

**Example 3.1.** We recall the classical Jucys–Murphy elements in the Hecke algebra \(H_n(q)\), and some of their properties. The (multiplicative) Jucys–Murphy elements in \(H_n(q)\) are the elements \(\{L_1, \ldots, L_n\}\) defined by \(L_1 = 1\) and \(L_{j+1} = q^{-1} T_j L_j T_j\) for \(1 \leq j \leq n - 1\). The elements \(L_k\) are mutually commuting; in fact, \(L_k \in H_k(q) \subseteq H_n(q)\) for \(1 \leq k \leq n\), and for \(k \geq 2\), \(L_k\) commutes with \(H_{k-1}\).

Symmetric polynomials in the \(\{L_k\}\) are in the center of \(H_n(q)\). The Jucys–Murphy elements act on the Murphy bases of the cell module \(\Delta^\lambda\) as follows. Let \(\kappa(j, t) = c(j, t) - r(j, t)\), where \(c(j, t)\) is the column of \(j\) in the standard tableau \(t\) and \(r(j, t)\) is the row of \(j\) in \(t\). Then

\[
L_j m^\lambda_t = q^{\kappa(j,t)} m^\lambda_t + \sum_{s > t} r_s m^\lambda_s.
\]

For a cell \(x\) in the Young diagram \(\lambda\), let \(\kappa(x)\) denote its content, namely the column of \(x\) minus the row of \(x\). It follows from (3.1) that the product \(p = \prod_{j=1}^n L_j\) acts as a scalar \(\alpha_\lambda = q^{\sum_{x \in \lambda} \kappa(x)}\) on the cell module \(\Delta^\lambda\). Namely, if \(t_0\) is the most dominant standard tableau of shape \(\lambda\) then \(p m^\lambda_{t_0} = \alpha_\lambda m^\lambda_{t_0}\), by (3.1). But \(p\) is central and \(\Delta^\lambda\) is a cyclic module with generator \(m^\lambda_{t_0}\).

Abstracting from the Hecke algebra example, Mathas [28] defined a family of JM–elements in a cellular algebra as follows.

**Definition 3.2 ([28]).** Let \(A\) be a cellular algebra over \(R\); let \(\Lambda\) denote the partially ordered set in the cell datum for \(A\), and, for each \(\lambda \in \Lambda\), let \(\{a^\lambda_t : t \in \Lambda\}\)
\( T(\lambda) \) denote the basis of the cell module \( \Delta^\lambda \). Suppose that for each \( \lambda \in \Lambda \), the index set \( T(\lambda) \) is given a partial order \( \succeq \).

A finite family of elements \( \{ L_j : 1 \leq j \leq M \} \) in \( A \) is a \textit{JM–family in the sense of Mathas} if the elements \( L_j \) are mutually commuting and invariant under the involution of \( A \), and, for each \( \lambda \in \Lambda \), there is a set of scalars \( \{ \kappa(j, t) : 1 \leq j \leq n, t \in T(\lambda) \} \) such that for \( 1 \leq j \leq n \) and \( t \in T(\lambda) \),

\[
L_j a_1^\lambda = \kappa(j, t) a_1^\lambda + \sum_{s > t} r_s a_s^\lambda, 
\]

for some \( r_s \in R \), depending on \( j \) and \( t \). In addition, the family \( \{ L_j \} \) is said to be \textit{separating} if for each \( \lambda \in \Lambda_n \), \( t \mapsto (\kappa(j, t))_{1 \leq j \leq n} \) is injective on \( T(\lambda) \).

We are going to introduce a different abstraction of Jucys–Murphy elements that is appropriate for coherent towers of cellular algebras. We will see that our concept implies that of Mathas.

**Definition 3.3.** Let \( (A_n)_{n \geq 0} \) be a coherent tower of cellular algebras over \( R \). Let \( \Lambda_n \) denote the partially ordered set in the cell datum for \( A_n \).

A family of elements \( \{ L_n : n \geq 0 \} \) is a \textit{multiplicative JM–family} if for all \( n \geq 1 \),

1. \( L_n \in A_n, L_n \) is invariant under the involution of \( A_n \), and \( L_n \) commutes with \( A_{n-1} \). In particular, the elements \( L_j \) are mutually commuting.
2. For each \( n \geq 1 \) and each \( \lambda \in \Lambda_n \), there exists an invertible \( \alpha(\lambda) \in R \) such that the product \( L_1 \cdots L_n \) acts as the scalar \( \alpha(\lambda) \) on the cell module \( \Delta^\lambda \).

In particular, the elements \( L_j \) in Definition 3.3 must be invertible.

**Definition 3.4.** An \textit{additive JM–family} is defined similarly, except that (2) is replaced by

\( (\prime) \) For each \( n \geq 1 \) and each \( \lambda \in \Lambda_n \), there exists \( d(\lambda) \in R \) such that the sum \( L_1 + \cdots + L_n \) act as the scalar \( d(\lambda) \) on the cell module \( \Delta^\lambda \).

In the following, let \( R \) be an integral domain with field of fractions \( F \), and let \( (A_n)_{n \geq 0} \) be a strongly coherent tower of cellular algebras over \( R \). Suppose that \( A_n^F = A_n \otimes_R F \) is split semisimple for all \( n \), that \( A_0^F = F \) and that the branching diagram \( \mathcal{B} \) for the tower \( (A_n^F)_{n \geq 0} \) has no multiple edges. Let \( (\Lambda_n, >) \) denote the partially ordered set in the cell datum for \( A_n \). Without loss of generality, we can assume that for each \( \lambda \in \Lambda_n \), the index set \( T(\lambda) \) for the basis of the cell module \( \Delta^\lambda \) is the set of paths on \( \mathcal{B} \) from \( \emptyset \) to \( \lambda \), and we can give \( T(\lambda) \) the lexicographic order \( \succeq \) described in the remarks preceding Example 2.6. Let \( \{ a_1^\lambda : \lambda \in \Lambda_n, s, t \in T(\lambda) \} \) denote the cellular basis of \( A_n \), and let \( \{ a_s^\lambda \} \) be the corresponding basis of \( \Delta^\lambda \) for each \( \lambda \in \Lambda_n \). For \( t = (\emptyset = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)} = \lambda) \in T(\lambda) \), write \( t(j) = \lambda^{(j)} \).

**Proposition 3.5.** Let \( (A_n)_{n \geq 0} \) be a strongly coherent tower of cellular algebras over an integral domain \( R \). Adopt the hypotheses and notation of the previous paragraph. Suppose that \( \{ L_n : n \geq 0 \} \) is a multiplicative JM–family for the tower \( (A_n)_{n \geq 0} \).
(1) For \( n \geq 1 \) and \( \lambda \in \Lambda_n \), let \( \alpha(\lambda) \in R^\times \) be such that \( L_1 \cdots L_n \) acts by the scalar \( \alpha(\lambda) \) on the cell module \( \Delta^\lambda \). Then for all \( n \geq 1 \), \( \lambda \in \Lambda_n \), \( t \in T(\lambda) \), and \( 1 \leq j \leq n \), we have
\[
L_j a_\lambda^t = \kappa(j, t) a_\lambda^t + \sum_{s \geq t} r_s a_\lambda^s,
\]
(3.2)
for some elements \( r_s \in R \) (depending on \( j \) and \( t \)), with \( \kappa(j, t) = \frac{\alpha(t(j))}{\alpha(t(j - 1))} \).

(2) For each \( n \geq 1 \), \( L_1 \cdots L_n \) is in the center of \( A_n \).

Proof. We prove (1) by induction on \( n \). For \( n = 1 \), the statement follows from (2) of Definition 3.3. Assume \( n > 1 \) and adopt the appropriate induction hypothesis. For \( j < n, \lambda \in \Lambda_n \), and \( t \in T(\lambda) \), (3.2) holds by the induction hypothesis and Proposition 2.7, while
\[
L_n a_\lambda^t = (L_1 \cdots L_{n-1})^{-1}(L_1 \cdots L_n) a_\lambda^t
= \alpha(\lambda)(L_1 \cdots L_{n-1})^{-1} a_\lambda^t
= \alpha(\lambda)\alpha(t(n - 1))^{-1} a_\lambda^t + \sum_{s \geq t} r_s a_\lambda^s,
\]
using point (2) of Definition 3.3 and Proposition 2.7.

For all \( x \in A_n \), \( x(L_1 \cdots L_n) = (L_1 \cdots L_n)x \) on each cell module. But the direct sum of all cell modules is faithful. This proves (2). \( \square \)

The additive version of the proposition is the following. The proof is similar.

**Proposition 3.6.** Let \((A_n)_{n \geq 0}\) be a coherent tower of cellular algebras over an integral domain \( R \). Adopt the hypotheses and notation in the paragraph preceding Proposition 3.5. Suppose that \( \{L_n : n \geq 0\} \) is an additive JM-family for the tower \((A_n)_{n \geq 0}\).

(1) For \( n \geq 1 \) and \( \lambda \in \Lambda_n \), let \( d(\lambda) \in R \) be such that \( L_1 + \cdots + L_n \) acts by the scalar \( d(\lambda) \) on the cell module \( \Delta^\lambda \). Then for all \( n \geq 1 \), \( \lambda \in \Lambda_n \), \( t \in T(\lambda) \), and \( 1 \leq j \leq n \), we have
\[
L_j a_\lambda^t = \kappa(j, t) a_\lambda^t + \sum_{s \geq t} r_s a_\lambda^s,
\]
(3.3)
for some elements \( r_s \in R \) (depending on \( j \) and \( t \)), with \( \kappa(j, t) = \alpha(t(j)) - \alpha(t(j - 1)) \).

(2) For each \( n \geq 1 \), \( L_1 + \cdots + L_n \) is in the center of \( A_n \).

4. JM elements in algebras arising from the basic construction

**Theorem 4.1.** Consider two towers of \( R \)-algebras \((A_n)_{n \geq 0}\) and \((Q_n)_{n \geq 0}\) satisfying the strong framework axioms of Section 2.5. Suppose that \( \{L_j^{(0)} : j \geq 1\} \) is a multiplicative JM-family for the tower \((Q_n)_{n \geq 0}\), in the sense of Section 3, and that \( \{L_n : n \geq 1\} \) is a family of elements in \((A_n)_{n \geq 0}\) satisfying the following conditions:
Corollary 4.2. If \( \lambda \) is independent of \( j \), say \( \gamma_j = \gamma \) for all \( j \), then \( \beta((\lambda, n)) = \gamma^{(n-k)/2} \alpha(\lambda) \) when \( \lambda \in \Lambda_k^{(0)} \).

The additive version of the theorem is the following. The proof is similar.
Theorem 4.3. Consider two towers of $R$–algebras $(A_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ satisfying the strong framework axioms of Section 2.5. Suppose that $\{L_j^{(0)} : j \geq 1\}$ is an additive JM–family for the tower $(Q_n)_{n \geq 0}$, in the sense of Section 3, and that $\{L_n : n \geq 1\}$ is a family of elements in $(A_n)_{n \geq 0}$ satisfying the following conditions:

(1) $L_n \in A_n$, and $L_n$ commutes with $A_{n-1}$.

(2) $\pi_j(L_j) = L_j^{(0)}$, where $\pi_j : A_j \to Q_j$ is the quotient map.

(3) For each $j \geq 1$, there exists $\gamma_j \in R$ such that

$$(L_j + L_{j+1})e_j = e_j(L_j + L_{j+1}) = \gamma_j e_j.$$ 

Then $\{L_j : j \geq 1\}$ is an additive JM–family for the tower $(A_n)_{n \geq 0}$.

The additive analogue of the formula for $\beta$ developed in the proof of Theorem 4.1 is the following. For $n \geq 1$ and $\lambda \in \Lambda_n^{(0)}$, let $d(\lambda) \in R$ be such that $L_1^{(0)} + \cdots + L_n^{(0)}$ acts by the scalar $d(\lambda)$ on the cell module $\Delta^\lambda$ of $Q_n$. Then for $(\lambda, n) \in \Lambda_n$, with $\lambda \in \Lambda_k^{(0)}$, $L_1 + \cdots + L_n$ acts by the scalar

$$\beta((\lambda, n)) = \gamma_{n-1} + \cdots + \gamma_{k+1} + d(\lambda).$$ 

If $\gamma_j$ is independent of $j$, say $\gamma_j = \gamma$ for all $j$, then

$$\beta((\lambda, n)) = \frac{n - k}{2} \gamma + d(\lambda).$$

5. Examples

5.1. Preliminaries on tangle diagrams. Several of our examples involve tangle diagrams in the rectangle $\mathcal{R} = [0, 1] \times [0, 1]$. Fix points $a_i \in [0, 1], i \geq 1$, with $0 < a_1 < a_2 < \cdots$. Write $i = (a_i, 1)$ and $\overline{i} = (a_i, 0)$.

Recall that a knot diagram means a collection of piecewise smooth closed curves in the plane which may have intersections and self-intersections, but only simple transverse intersections. At each intersection or crossing, one of the two strands (curves) which intersect is indicated as crossing over the other.

An $(n, n)$–tangle diagram is a piece of a knot diagram in $\mathcal{R}$ consisting of exactly $n$ topological intervals and possibly some number of closed curves, such that:

(1) the endpoints of the intervals are the points $1, \ldots, n, \overline{1}, \ldots, \overline{n}$, and these are the only points of intersection of the family of curves with the boundary of the rectangle, and (2) each interval intersects the boundary of the rectangle transversally.

An $(n, n)$–Brauer diagram is a “tangle” diagram containing no closed curves, in which information about over and under crossings is ignored. Two Brauer diagrams are identified if the pairs of boundary points joined by curves is the same in the two diagrams. By convention, there is a unique $(0, 0)$–Brauer diagram, the empty diagram with no curves. For $n \geq 1$, the number of $(n, n)$–Brauer diagrams is $(2n - 1)!! = (2n - 1)(2n - 3) \cdots (3)(1)$. 

For any of these types of diagrams, we call $P = \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$ the set of vertices of the diagram, $P^+ = \{1, \ldots, n\}$ the set of top vertices, and $P^- = \{\overline{1}, \ldots, \overline{n}\}$ the set of bottom vertices. A curve or strand in the diagram is called a vertical or through strand if it connects a top vertex and a bottom vertex, and a horizontal strand if it connects two top vertices or two bottom vertices.

5.2. The BMW algebras. The BMW algebras were first introduced by Birman and Wenzl [5] and independently by Murakami [32] as abstract algebras defined by generators and relations. The version of the presentation given here follows [30] and [31].

**Definition 5.1.** Let $S$ be a commutative unital ring with invertible elements $\rho$ and $q$ and an element $\delta$ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$. The Birman–Wenzl–Murakami algebra $W_n(S; \rho, q, \delta)$ is the unital $S$–algebra with generators $g_i^{\pm 1}$ and $e_i$ ($1 \leq i \leq n - 1$) and relations:

1. (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$.
2. (Essential idempotent relation) $e_i^2 = \delta e_i$.
3. (Braid relations) $g_i g_{i+1} g_i = g_{i+1} g_i g_i + 1$ and $g_i g_j = g_j g_i$ if $|i - j| \geq 2$.
4. (Commutation relations) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.
5. (Tangle relations) $e_i e_{i \pm 1} e_j = e_j, g_i g_{i \pm 1} e_i = e_i g_{i \pm 1} e_i$, and $e_i g_{i \pm 1} g_i = e_i e_{i \pm 1}$.
6. (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.
7. (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$, and $e_i g_{i \pm 1} e_i = \rho e_i$.

The BMW algebra $W_n$ can also be realized as the algebra of $(n, n)$–tangle diagrams modulo regular isotopy and the following Kauffman skein relations:

1. Crossing relation: \[
\begin{array}{c|c|c}
\begin{array}{c} \bigtriangleup \bigtriangledown \end{array} & - & \begin{array}{c} \bigtriangledown \bigtriangleup \end{array} \\
\end{array} \rightleftharpoons (q^{-1} - q) \left( \begin{array}{c|c|c}
\begin{array}{c} \bigtriangleup \bigtriangledown \end{array} & - & \begin{array}{c} \bigtriangledown \bigtriangleup \end{array} \\
\end{array} \right).
\]
2. Untwisting relation: \[
\begin{array}{c|c|c}
\begin{array}{c} \bigcirc \bigcirc \end{array} & = & \begin{array}{c} \bigcirc \bigcirc \end{array} \\
\end{array} = \rho
\] and \[
\begin{array}{c|c|c}
\begin{array}{c} \bigcirc \bigcirc \end{array} & = & \begin{array}{c} \bigcirc \bigcirc \end{array} \\
\end{array} = \rho^{-1}
\].
3. Free loop relation: $T \cup \bigcirc = \delta T$, where $T \cup \bigcirc$ means the union of a tangle diagram $T$ and a closed loop having no crossings with $T$.

In the tangle picture, $e_j$ and $g_j$ are represented by the following $(n, n)$–tangle diagrams:

\[
e_j = \left\lfloor \begin{array}{c} j \bigtriangleup \bigtriangledown j + 1 \end{array} \right. \quad g_j = \left\lfloor \begin{array}{c} j \bigtriangledown \bigtriangleup j + 1 \end{array} \right.
\]

The realization of the BMW algebra as an algebra of tangles is from [31]. See [16], Section 5.4 for more details.

The quotient of the BMW algebra $W_n(S; \rho, q, \delta)$ by the ideal $J$ generated by $e_{n-1}$ is the Hecke algebra $H_n(S; q^2)$. If $\pi_n$ denotes the quotient map $\pi_n : W_n \rightarrow W_n/J$, take $T_i = \pi_n(q g_i)$ to obtain an isomorphism with the Hecke algebra as presented in Example 2.6.
The generic ground ring for the BMW algebras is
\[ R = \mathbb{Z}[\rho^\pm 1, q^\pm 1, \delta]/\langle \rho^{-1} - \rho = (q^{-1} - q)(\delta - 1) \rangle, \]

where \( \rho, q, \) and \( \delta \) are indeterminants over \( \mathbb{Z} \). \( R \) is an integral domain whose field of fractions is \( F \cong \mathbb{Q}(\rho, q) \) (with \( \delta = (\rho^{-1} - \rho)/(q^{-1} - q) + 1 \) in \( F \).) Write \( W_n \) for \( W_n(R; \rho, q, \delta) \) and \( H_n \) for \( H_n(R; q^2) \). It is shown in [16], Section 5.4, that the pair of towers \( (W_n)_{n \geq 0} \) and \( (H_n)_{n \geq 0} \) satisfy the framework axioms of Section 2.5. In fact, by Example 2.6, the tower of Hecke algebras is strongly coherent, so the pair satisfies the strong version of the framework axioms. Consequently, by Theorem 2.9, the sequence of BMW algebras is a strongly coherent tower of cellular algebras. The partially ordered set \( \Lambda_n \) in the cell datum of \( W_n \) is the set of pairs \( (\lambda, n) \), with \( \lambda \) a Young diagram of size \( k \leq n \) with \( n - k \) even. The set of paths \( \mathcal{T}(\lambda, n) \) can be identified with up–down tableaux of length \( n \) and shape \( \lambda \), see [11].

The following analogue of Jucys–Murphy elements for the BMW algebras were introduced by Leduc and Ram [26] and Enyang [11]. Define \( L_1 = 1 \) and \( L_{j+1} = g_j L_j g_j \) for \( j \geq 1 \). (Thus, for example, \( L_5 = g_4 g_3 g_2 g_1 g_1 g_3 g_4 \).) The involution on \( W_n \) is the unique algebra involution taking \( e_i \mapsto e_i \) and \( g_i \mapsto g_i \); it leaves each \( L_j \) invariant. One can check algebraically that \( L_n \) commutes with the generators of \( W_{n-1} \), but this is far easier to see using the geometric realization of \( W_n \). In fact, in the geometric picture, \( L_n \) is represented by the braid in which the \( n \)-th strand wraps once around the first through \( (n - 1) \)-st strands.

Let \( L_j(0) \) denote the classical JM elements in the Hecke algebras \( H_n \), as defined in Example 3.1. Then we have \( \pi_n(L_j) = L_j(0) \) for \( 1 \leq j \leq n \); this follows because \( \pi_n(L_1) = 1 \) and \( \pi_n(L_{j+1}) = q^{-2} T_j \pi_n(L_j) T_j \). (This is the correct recursion, because the Hecke algebra parameter \( q \) has been replaced by \( q^2 \).) One can check, using algebraic relations or by using tangle diagrams, that for all \( j \geq 1 \),
\[ L_j L_{j+1} e_j = e_j L_j L_{j+1} = \rho^{-2} e_j. \]

(The factor of \( \rho^{-2} \) comes from two applications of the untwisting relation (2) above.)

It now follows from Theorem 4.1 that \( \{ L_j : j \geq 0 \} \) is a multiplicative JM–family in \( (W_n)_{n \geq 0} \), with \( L_1 \ldots L_n \) acting by
\[ \beta((\lambda, n)) := \rho^{-(n-k)} \alpha(\lambda) \]
on the cell module \( \Delta^{(\lambda, n)} \), if \( \lambda \) is a Young diagram of size \( k \). By Proposition 3.5, the action of the elements \( L_j \) on the basis of \( \Delta^{(\lambda, n)} \) labelled by up–down tableaux is triangular:
\[
L_j a^\lambda_1 = \kappa(j, t) a^\lambda_1 + \sum_{s > t} r_s a^\lambda_s,
\]

where \( \kappa(j, t) \) and \( r_s \) are indeterminants over \( \mathbb{Z} \).
with \( \kappa(j, t) = \frac{\beta(t(j))}{\beta(t(j-1))} \), for some elements \( r_s \in R \), depending on \( j \) and \( t \). Moreover, if \( t(j) = (\nu, j) \) and \( t(j-1) = (\mu, j-1) \), then \( |\nu| = |\mu| \pm 1 \). If \( |\nu| = |\mu| + 1 \) and \( \nu \setminus \mu = x \), then

\[
\kappa(j, t) = \frac{\beta((\nu, j))}{\beta((\mu, j-1))} = \frac{\alpha(\nu)}{\alpha(\mu)} = q^{2\kappa(x)},
\]

where \( \kappa(x) \) is the content of \( x \), namely the column of \( x \) minus the row of \( x \). If \( |\nu| = |\mu| - 1 \) and \( \mu \setminus \nu = x \), then

\[
\kappa(j, t) = \frac{\beta((\nu, j))}{\beta((\mu, j-1))} = \rho^{-2} \frac{\alpha(\nu)}{\alpha(\mu)} = \rho^{-2} q^{-2\kappa(x)}.
\]

This recovers Theorem 7.8 of Enyang [11].

5.3. The Brauer algebras. The Brauer algebras were introduced by Brauer [6] as a device for studying the invariant theory of orthogonal and symplectic groups.

Let \( S \) be a commutative ring with identity, with a distinguished element \( \delta \). The Brauer algebra \( B_n(S, \delta) \) is the free \( S \)-module with basis the set of \( (n, n) \)-Brauer diagrams, with multiplication defined as follows. The product of two Brauer diagrams is defined to be a certain multiple of another Brauer diagram. Namely, given two Brauer diagrams \( a, b \), first “stack” \( b \) over \( a \); the result is a planar tangle that may contain some number of closed curves. Let \( r \) denote the number of closed curves, and let \( c \) be the Brauer diagram obtained by removing all the closed curves. Then \( ab = \delta^r c \).

**Definition 5.2.** For \( n \geq 1 \), the Brauer algebra \( B_n(S, \delta) \) over \( S \) with parameter \( \delta \) is the free \( S \)-module with basis the set of \( (n, n) \)-Brauer diagrams, with the bilinear product determined by the multiplication of Brauer diagrams. In particular, \( B_0(S, \delta) = S \).

Note that the Brauer diagrams with only vertical strands are in bijection with permutations of \( \{1, \ldots, n\} \), and that the multiplication of two such diagrams coincides with the multiplication of permutations. Thus the Brauer algebra contains the group algebra \( S\mathfrak{S}_n \) of the permutation group \( \mathfrak{S}_n \). The identity element of the Brauer algebra is the diagram corresponding to the trivial permutation. We will note below that \( S\mathfrak{S}_n \) is also a quotient of \( B_n(S, \delta) \).

The involution \( i \) on \( (n, n) \)-Brauer diagrams which reflects a diagram in the axis \( y = 1/2 \) extends linearly to an algebra involution of \( B_n(S, \delta) \).

Let \( e_j \) and \( s_j \) denote the \( (n, n) \)-Brauer diagrams:

\[
e_j = \begin{array}{|c|c|c|c|}
\hline
\&
\&
\&
\&
\hline
\end{array}
\quad s_j = \begin{array}{|c|c|c|c|}
\hline
\&
\&
\&
\&
\hline
\end{array}
\]

\[1\]The theorem is stated in [11] with dominance order rather than lexicographic order, but it appears that the proof only yields the statement with lexicographic order.
Note that \( e_j^2 = \delta e_j \), so \( e_j \) is an essential idempotent if \( \delta \neq 0 \), and nilpotent if \( \delta = 0 \). We have \( i(e_j) = e_j \) and \( i(s_j) = s_j \). It is easy to see that \( e_1, \ldots, e_{n-1} \) and \( s_1, \ldots, s_{n-1} \) generate \( B_n(S, \delta) \) as an algebra.

The products \( ab \) and \( ba \) of two Brauer diagrams have at most as many through strands as \( a \). Consequently, the span of diagrams with fewer than \( n \) through strands is an ideal \( J \) in \( B_n(S, \delta) \). The ideal \( J \) is generated by \( e_{n-1} \). We have \( B_n(S, \delta)/J \cong S\mathfrak{S}_n \), as algebras with involutions.

The generic ground ring for the Brauer algebras is \( R = \mathbb{Z}[\delta] \), where \( \delta \) is an indeterminant. Let \( F = \mathbb{Q}(\delta) \) denote the field of fractions of \( R \). Write \( B_n = B_n(R, \delta) \).

It is shown in [16], Section 5.2, that the pair of towers \( (B_n)_{n \geq 0} \) and \( (R\mathfrak{S}_n)_{n \geq 0} \) satisfy the framework axioms of Section 2.5. In fact, since the symmetric group algebra is a specialization of the Hecke algebra, the tower of symmetric group algebras is strongly coherent, so the pair satisfies the strong version of the framework axioms. Consequently, by Theorem 2.9, the sequence of Brauer algebras is a strongly coherent tower of cellular algebras. As for the BMW algebras, the partially ordered set \( \Lambda_n \) in the cell datum of \( B_n \) is the set of pairs \( (\lambda, n) \), with \( \lambda \) a Young diagram of size \( k \leq n \) with \( n - k \) even. The set of paths \( T((\lambda, n)) \) can be identified with up–down tableaux of length \( n \) and shape \( \lambda \).

We need to recall the Jucys–Murphy elements for the symmetric group algebras, which can be defined inductively by \( L_{1}^{(0)} = 0 \), \( L_{j+1}^{(0)} = s_j L_j s_j + s_j \). Thus, for example, \( L_5^{(0)} = (1, 5) + (2, 5) + (3, 5) + (4, 5) \). One has \( L_j^{(0)} \in R\mathfrak{S}_j \), and \( L_j^{(0)} \) commutes with \( R\mathfrak{S}_{j-1} \). \( L_1^{(0)} + \cdots + L_n^{(0)} \) is central in \( R\mathfrak{S}_n \) and acts as the scalar \( \alpha(\lambda) = \sum_{x \in \lambda} \kappa(x) \) on the cell module \( \Delta^\lambda \). Here, \( \lambda \) is a Young diagram of size \( n \) and for a cell \( x \) of \( \lambda \), \( \kappa(x) \) is the content of \( x \), namely the column co-ordinate minus the row co-ordinate of \( x \). In particular \( \{ L_j^{(0)} : j \geq 0 \} \) is an additive JM–family in the sense of Definition 3.4.

The following analogues of Jucys-Murphy elements for the Brauer algebras were introduced by Nazarov [34]. Let \( L_1 = 0 \) and \( L_{j+1} = s_j L_j s_j + s_j - e_j \). Observe that \( \pi_n(L_j) = L_j^{(0)} \) for \( 1 \leq j \leq n \), where \( \pi_n : B_n \to R\mathfrak{S}_n \) is the quotient map. Evidently, \( L_n \in B_n \). By [34], Proposition 2.3, \( L_n \) commutes with \( B_{n-1} \), and for all \( j \geq 1 \),

\[
(L_j + L_{j+1})e_j = e_j(L_j + L_{j+1}) = (1 - \delta)e_j.
\]

It now follows from Theorem 4.3 that \( \{ L_j : j \geq 0 \} \) is an additive JM–family in \( (B_n)_{n \geq 0} \), with \( L_1 + \cdots + L_n \) acting by

\[
\beta((\lambda, n)) := \frac{n-k}{2}(1 - \delta) + \alpha(\lambda)
\]

on the cell module \( \Delta^{(\lambda,n)} \), if \( \lambda \) is a Young diagram of size \( k \).
By Proposition 3.6, the action of the elements $L_j$ on the basis of $\Delta^{(\lambda,n)}$ labelled by up–down tableaux is triangular:

\[ L_j a_{\lambda} = \kappa(j, t) a_{\lambda} + \sum_{s \geq t} r_s a_{\lambda}, \]

with $\kappa(j, t) = \beta(t(j)) - \beta(t(j - 1))$, for some elements $r_s \in R$, depending on $j$ and $t$. Moreover, if $t(j) = (\nu, j)$ and $t(j - 1) = (\mu, j - 1)$, then $|\nu| = |\mu| \pm 1$. If $|\nu| = |\mu| + 1$ and $\nu \setminus \mu = x$, then

\[ \kappa(j, t) = \beta((\nu, j)) - \beta((\mu, j - 1)) = \alpha(\nu) - \alpha(\mu) = \kappa(x). \]

If $|\nu| = |\mu| - 1$ and $\mu \setminus \nu = x$, then

\[ \kappa(j, t) = \beta((\nu, j)) - \beta((\mu, j - 1)) = (1 - \delta) + \alpha(\nu) - \alpha(\mu) = (1 - \delta) - \kappa(x). \]

This recovers Theorem 10.7 of Enyang [11].

5.4. Cyclotomic BMW algebras. The cyclotomic Birman–Wenzl–Murakami algebras are BMW analogues of cyclotomic Hecke algebras [2, 1]. The cyclotomic BMW algebras were defined by Hāring–Oldenburg in [22] and have recently been studied by three groups of mathematicians: Goodman and Hauschild–Mosley [17, 18, 19, 13, 14], Rui, Xu, and Si [37, 36], and Wilcox and Yu [39, 40, 41, 42].

5.4.1. Definition of cyclotomic BMW algebras.

**Definition 5.3.** Fix an integer $r \geq 1$. A ground ring $S$ is a commutative unital ring with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$), and $u_1, \ldots, u_r$, with $\rho$, $q$, and $u_1, \ldots, u_r$ invertible, and with $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$.

**Definition 5.4.** Let $S$ be a ground ring with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$), and $u_1, \ldots, u_r$. The cyclotomic BMW algebra $W_{n,S,r}(u_1, \ldots, u_r)$ is the unital $S$–algebra with generators $g_{1}^{\pm 1}$, $g_{i}^{\pm 1}$ and $e_{i}$ ($1 \leq i \leq n - 1$) and relations:

1. (Inverses) $g_{i}^{-1} = g_{i+1}^{-1}$ and $y_{1}^{-1} = y_{1}^{-1} y_{1} = 1$.
2. (Idempotent relation) $e_{i}^{2} = \delta_{0} e_{i}$.
3. (Affine braid relations)
   - (a) $g_{i} g_{i+1} g_{i} = g_{i+1} g_{i} g_{i+1}$ and $g_{i} g_{j} = g_{j} g_{i}$ if $|i - j| \geq 2$.
   - (b) $y_{1} y_{1} y_{1} g_{1} = g_{1} y_{1} y_{1} y_{1}$ and $y_{1} g_{j} = g_{j} y_{1}$ if $j \geq 2$.
4. (Commutation relations)
   - (a) $g_{i} e_{j} = e_{j} g_{i}$ and $g_{i} e_{j} = e_{j} g_{i}$ if $|i - j| \geq 2$.
   - (b) $y_{1} e_{j} = e_{j} y_{1}$ if $j \geq 2$.
5. (Affine tangle relations)
   - (a) $e_{i} e_{i+1} e_{i} = e_{i}$.
   - (b) $g_{i} g_{i+1} e_{i} = e_{i} g_{i+1}$ and $e_{i} g_{i+1} g_{i} = e_{i} e_{i+1}$.
   - (c) For $j \geq 1$, $e_{1} y_{1}^{j} e_{1} = \delta_{j} e_{1}$.
6. (Kauffman skein relation) $g_{i} - g_{i}^{-1} = (q - q^{-1})(1 - e_{i})$.
7. (Untwisting relations) $g_{i} e_{i} = e_{i} g_{i} = \rho^{-1} e_{i}$ and $e_{i} g_{i+1} e_{i} = \rho e_{i}$.

---

2The same caution about lexicographic order versus dominance order applies here, as in the BMW case.
(8) (Unwrapping relation) \( e_1 y_1 g_1 y_1 = \rho e_1 = y_1 g_1 y_1 e_1 \).

(9) (Cyclotomic relation) \( (y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0 \).

Thus, a cyclotomic BMW algebra is the quotient of the affine BMW algebra \([17]\), by the cyclotomic relation \( (y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0 \).

The cyclotomic BMW algebra has a unique algebra involution \( i \) fixing each of the generators.

5.4.2. Geometric realization. It is shown in \([19]\) and in \([41]\) that the cyclotomic BMW algebra has a geometric realization as the “cyclotomic Kauffman tangle (KT) algebra,” assuming admissibility conditions on the ground ring (see below). The cyclotomic KT algebra is described in terms of “affine tangle diagrams,” which are just ordinary tangle diagrams with a distinguished vertical strand connecting \( 1 \) and \( \mathbf{T} \), as in the following figure.

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{affine-tangle-diagram.png}
\end{array}
\]

The cyclotomic KT algebra is the algebra of affine tangle diagrams, modulo regular isotopy, Kauffman skein relations, and a cyclotomic skein relation, which is a “local” version of the cyclotomic relation of Definition 5.4 (9). See \([18]\) for the precise definition.

In the geometric realization, the generators \( g_i, e_i, \) and \( x_1 = \rho^{-1} y_1 \) are represented by the following affine tangle diagrams:

\[
\begin{align*}
x_1 & = \includegraphics[width=0.2\textwidth]{affine-tangle-diagram-x1.png} \\
g_i & = \includegraphics[width=0.2\textwidth]{affine-tangle-diagram-g.png} \\
e_i & = \includegraphics[width=0.2\textwidth]{affine-tangle-diagram-e.png}
\end{align*}
\]

In the geometric picture, the algebra involution \( i \) is given on the level of affine tangle diagrams by the map that flips an affine tangle diagram over the horizontal line \( y = 1/2 \).

5.4.3. Admissibility. The cyclotomic BMW algebras can be defined over arbitrary ground rings. However, it is necessary to impose conditions on the parameters in order to get a satisfactory theory. Two apparently different “admissibility” conditions have been proposed, one by Wilcox and Yu \([39]\), and another by Rui and Xu \([37]\). It has been shown in \([14]\) that the two conditions are equivalent in the case of greatest interest, when \( S \) is an integral domain with \( q - q^{-1} \neq 0 \). We consider only this case from now on.
Definition 5.5. Let $S$ be an integral ground ring with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$) and $u_1, \ldots, u_r$, with $q - q^{-1} \neq 0$. One says that $S$ is admissible (or that the parameters are admissible) if $\{e_1, y_1e_1, \ldots, y_1^{r-1}e_1\} \subseteq W_{2,S,r}$ is linearly independent over $S$.

It is shown in [39] that admissibility is equivalent to finitely many (explicit) polynomial relations on the parameters. Moreover, these relations give $\rho$ and $(q - q^{-1})\delta_j$ as Laurent polynomials in the remaining parameters $q, u_1, \ldots, u_r$; see [39] and [19] for details.

5.4.4. Generic ground ring. There is a universal admissible integral ground ring $R$ for cyclotomic BMW algebras, which is a little more complicated to describe than the generic ground rings for the other algebras we have encountered. We refer to [19], Theorem 3.19 for details. Suffice it to say that the field of fractions $F$ of $R$ is $\mathbb{Q}(q, u_1, \ldots, u_r)$, where $q, u_1, \ldots, u_r$ are algebraically independent indeterminants over $\mathbb{Q}$; the remaining parameters are given by certain Laurent polynomials in $q, u_1, \ldots, u_r$, and $(q - q^{-1})^{-1}$, and $R$ is the subring of $F$ generated by all the parameters. Any other admissible integral ground ring $S$ is a module over $R$, and $W_{n,S,r} \cong W_{n,R,r} \otimes_R S$. We will write $W_n$ for $W_{n,R,r}$.

5.4.5. Cyclotomic BMW algebras and cyclotomic Hecke algebras. We recall the definition of the affine and cyclotomic Hecke algebras, see [1].

Definition 5.6. Let $S$ be a commutative unital ring with an invertible element $q$. The affine Hecke algebra $\widehat{H}_{n,S}(q)$ over $S$ is the $S$–algebra with generators $T_0, T_1, \ldots, T_{n-1}$, with relations:

1. The generators $T_i$ are invertible, satisfy the braid relations, and the Hecke relations $(T_i - q)(T_i + q) = 0$.
2. The generator $T_0$ is invertible, $T_0T_1T_0T_1 = T_1T_0T_1T_0$ and $T_0$ commutes with $T_j$ for $j \geq 2$.

Let $u_1, \ldots, u_r$ be additional elements in $S$. The cyclotomic Hecke algebra $H_{n,S,r}(q; u_1, \ldots, u_r)$ is the quotient of the affine Hecke algebra $\widehat{H}_{n,S}(q)$ by the polynomial relation $(T_0 - u_1) \cdots (T_0 - u_r) = 0$.

We remark that since the generator $T_0$ can be rescaled by an arbitrary invertible element of $S$, only the ratios of the parameters $u_i$ have invariant significance in the definition of the cyclotomic Hecke algebra. The cyclotomic Hecke algebra has a unique algebra involution $i$ leaving each generator invariant. By [2], the cyclotomic Hecke algebras $H_{n,S,r}$ are free $S$–modules of rank $r^n n!$ and $H_{n,S,r}$ imbeds in $H_{n+1,S,r}$.

The cyclotomic Hecke algebras were shown to be cellular algebras in [20]. In [9], a cellular basis was given generalizing the Murphy basis of the ordinary Hecke algebra. The partially ordered set $\Lambda_n^{(0)}$ in the cell datum for $H_{n,S,r} = H_{n,S,r}(q; u_1, \ldots, u_r)$ is the set of $r$–tuples of Young diagrams with total size $n$, ordered by dominance. For each $\lambda \in \Lambda_n^{(0)}$, the index set $T(\lambda)$ in the cell datum
is the set of standard tableaux of shape $\lambda$; this has the usual meaning: fillings with the numbers $1, \ldots, n$, so that the numbers increase in each row and column (separately in each component Young diagram). The cyclotomic Hecke algebras are generically split semisimple; in the semisimple case, the branching diagram has vertices at level $n$ labelled by all $r$–tuples of Young diagrams of total size $n$, and $\lambda \succ \mu$ if $\mu$ is obtained from $\lambda$ by adding one box in one component of $\lambda$. Standard tableaux of shape $\lambda$ can be identified with paths on the generic branching diagram from $\emptyset$ (the $r$–tuple of empty Young diagrams) to $\lambda$.

By results of Ariki and Mathas ([3], Proposition 1.9) and Mathas [29], the sequence of cyclotomic Hecke algebras $(H_{n,S,r})_{n \geq 0}$ is a strongly coherent tower of cellular algebras.

Let $J$ be the ideal in $W_n = W_{n,R,r}$ generated by $e_{n-1}$. It is not hard to show that the quotient $W_n/J$ is isomorphic to $H_{n,R,r}(q^2; u_1, \ldots, u_r)$. If $\pi_n$ denotes the quotient map $\pi_n : W_n \to W_n/J$, take $T_j = \pi_n(q g_j)$ for $j \geq 1$, and $T_0 = \pi_n(y_1)$ to obtain an isomorphism with the cyclotomic Hecke algebra as presented above. We will write $H_n$ for $H_{n,R,r}(q^2; u_1, \ldots, u_r)$.

It is shown in [16], Section 5.5, that the pair of towers of algebras $(W_n)_{n \geq 0}$ and $(H_n)_{n \geq 0}$ satisfies the framework axioms of Section 2.5. Since the sequence of Hecke algebras is strongly coherent, the pairs satisfies the strong version of the framework axioms. Therefore, it follows from Theorem 2.9 that the sequence of cyclotomic BMW algebras is a strongly coherent tower of cellular algebras.

The partially ordered set $\Lambda_n$ in the cell datum of $W_n$ is the set of pairs $(\lambda, n)$, with $\lambda$ an $r$–tuple of Young diagrams of total size $k \leq n$ with $n - k$ even. The set of paths $T((\lambda, n))$ is the set of standard tableaux of length $n$ and shape $\lambda$, that is sequences of $r$–tuples of Young diagrams in which each successive $r$–tuple is obtained from the previous one by either adding or removing one box from one component Young diagram.

5.4.6. JM elements for cyclotomic BMW and Hecke algebras. In the cyclotomic Hecke algebra $H_{n,S,r} = H_{n,S,r}(q; u_1, \ldots, u_r)$, define $L_1^{(0)} = T_0$ and $L_j^{(0)} = q^{-1} T_j L_j^{(0)} T_j$ for $j \geq 1$. Then $L_n^{(0)} \in H_{n,S,r}$, $L_n^{(0)}$ is invariant under the involution on $H_{n,S,r}$, and $L_n^{(0)}$ commutes with $H_{n-1,S,r}$. The product $L_1^{(0)} \cdots L_n^{(0)}$ is central in $H_{n,S,r}$.

For an $r$–tuple of Young diagrams $\lambda$ of total size $n$ and a cell $x \in \lambda$, the multiplicative content of the cell is

$$\kappa(x) = u_j q^{b-a}$$

if $x$ is in row $a$ and column $b$ of the $j$–th component of $\lambda$. For a standard tableau $t$ of shape $\lambda$, and $1 \leq j \leq n$, let $\kappa(j, t) = \kappa(x)$, where $x$ is the cell occupied by $j$ in $t$. Let $\{a_t^\lambda\}$ be the Murphy type basis of the cell module $\Delta^\lambda$ indexed by standard
tableaux of shape $\lambda$. Then $L^{(0)}_j$ acts by

\begin{equation}
L^{(0)}_j a^\lambda_t = \kappa(j,t) a^\lambda_t + \sum_{s \geq t} r_s a^\lambda_s,
\end{equation}

where the sum is over standard tableaux $s$ greater than $t$ in dominance order (hence in lexicographic order). These results are from [23], Section 3. It follows that the product $L^{(0)}_1 \cdots L^{(0)}_n$ acts as the scalar $\alpha(\lambda) = \prod_{x \in \lambda} \kappa(x)$ on the cell module $\Delta^\lambda$. Thus $\{L^{(0)}_n : n \geq 0\}$ is a multiplicative JM–family in the strongly coherent tower of cellular algebras $(H_{n,S,r})_{n \geq 0}$.

Define elements $L_j$ in the cyclotomic BMW algebras $W_n = W_{n,R,r}(q; u_1, \ldots, u_r)$ over the generic integral admissible ground ring $R$ by $L_1 = y_1$, $L_{j+1} = g_j L_j g_j$ for $j \geq 1$. These are the same as the elements $y_j$ in [19]. We have $L_n \in W_n$ and $L_n$ commutes with $W_{n-1}$. One can verify that $L_j L_{j+1} e_j = e_j L_j L_{j+1} = e_j$. The computations can be done at the level of the affine BMW algebra, using the algebraic relations or using affine tangle diagrams.

We have $\pi_n(L_1) = T_0 = L^{(0)}_1$, and $\pi_n(L_{j+1}) = q^{-2} T_j \pi_n(L_j) T_j$. Hence $\pi_n(L_j)$ satisfy the recursion for $L^{(0)}_j$ in $H_n = H_{n,R,r}(q^2; u_1, \ldots, u_r)$.

It now follows from Theorem 4.1 that $\{L_j : j \geq 0\}$ is a multiplicative JM–family in $(W_n)_{n \geq 0}$, with the product $L_1 \cdots L_n$ acting by

\[ \beta((\lambda, n)) := \alpha(\lambda) \]

on the cell module $\Delta^{(\lambda,n)}$, if $\lambda$ is an $r$–tuple of Young diagrams of total size $k$. By Proposition 3.5, the action of the elements $L_j$ on the basis of $\Delta^{(\lambda,n)}$ labelled by up–down tableaux is triangular:

\begin{equation}
L_j a^\lambda_t = \kappa(j,t) a^\lambda_t + \sum_{s \geq t} r_s a^\lambda_s,
\end{equation}

with $\kappa(j,t) = \frac{\beta(t(j))}{\beta(t(j-1))}$, for some elements $r_s \in R$, depending on $j$ and $t$.

Moreover, if $t(j) = (\nu, j)$ and $t(j-1) = (\mu, j-1)$, then $|\nu| = |\mu| \pm 1$. If $|\nu| = |\mu| + 1$ and $\nu \setminus \mu = x$, where $x$ is the cell in row $a$ and column $b$ of the $\ell$–th component of $\nu$, then

\[ \kappa(j,t) = \frac{\alpha(\nu)}{\alpha(\mu)} = \kappa(x) = u_b q^{2(b-a)}. \]

If $|\nu| = |\mu| - 1$ and $\mu \setminus \nu = x$, then

\[ \kappa(j,t) = \frac{\alpha(\nu)}{\alpha(\mu)} = \kappa(x)^{-1} = u_b^{-1} q^{-2(b-a)}. \]

This recovers Theorem 3.17 of Rui and Si [36].
5.5. **Degenerate cyclotomic BMW algebras (cyclotomic Nazarov Wenzl algebras).** Degenerate affine BMW algebras were introduced by Nazarov [34] under the name *affine Wenzl algebras.* The cyclotomic quotients of these algebras were introduced by Ariki, Mathas, and Rui in [4] and studied further by Rui and Si in [35], under the name *cyclotomic Nazarov–Wenzl algebras.* We propose to refer to these algebras as degenerate affine (resp. degenerate cyclotomic) BMW algebras instead, to bring the terminology in line with that used for degenerate affine and cyclotomic Hecke algebras.

5.5.1. **Definition of the degenerate cyclotomic BMW algebras.** Fix a positive integer $n$ and a commutative ring $S$ with multiplicative identity. Let $\Omega = \{\omega_a : a \geq 0\}$ be a sequence of elements of $S$.

**Definition 5.7** (Nazarov [34]; Ariki, Mathas, Rui [4]). The *degenerate affine BMW algebra* $W^{\text{aff}}_{n,S} = W^{\text{aff}}_{n,S}(\Omega)$ is the unital associative $R$–algebra with generators 

\[
\{s_i, e_i, x_j : 1 \leq i < n \text{ and } 1 \leq j \leq n\}
\]

and relations:

1. (Involutions) $s_i^2 = 1$, for $1 \leq i < n$.
2. (Affine braid relations)
   - $s_is_j = s_js_i$ if $|i - j| > 1$,
   - $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, for $1 \leq i < n - 1$,
   - $s_ix_j = x_js_i$ if $j \neq i, i + 1$.
3. (Idempotent relations) $e_i^2 = \omega_0e_i$, for $1 \leq i < n$.
4. (Commutation relations)
   - $s_ie_j = e_js_i$, if $|i - j| > 1$,
   - $e_ie_j = e_je_i$, if $|i - j| > 1$,
   - $e_ix_j = x je_i$, if $j \neq i, i + 1$,
   - $x_ix_j = x_jx_i$, for $1 \leq i, j \leq n$.
5. (Skein relations) $s_ix_i - x_{i+1}s_i = e_i - 1$, and $x_is_i - s_ix_{i+1} = e_i - 1$, for $1 \leq i < n$.
6. (Unwrapping relations) $e_1x_i^ae_1 = \omega_a e_1$, for $a > 0$.
7. (Tangle relations)
   - $e_is_i = e_i = s_ie_i$, for $1 \leq i \leq n - 1$,
   - $s_is_{i+1}e_i = s_{i+1}e_i$, and $e_is_{i+1}s_i = e_is_{i+1}$, for $1 \leq i \leq n - 2$,
   - $e_is_{i+1}e_i = e_{i+1}s_i$, and $s_{i+1}e_i = e_is_{i+1}$, for $1 \leq i \leq n - 2$.
8. (Untwisting relations) $e_{i+1}e_ie_i = e_{i+1}$, and $e_ie_{i+1}e_i = e_i$, for $1 \leq i \leq n - 2$.
9. (Anti–symmetry relations) $e_i(x_i + x_{i+1}) = 0$, and $(x_i + x_{i+1})e_i = 0$, for $1 \leq i < n$.

**Definition 5.8** (Ariki, Mathas, Rui [4]). Fix an integer $r \geq 1$ and elements $u_1, \ldots, u_r$ in $S$. The *degenerate cyclotomic BMW algebra* $W^{\text{cyc}}_{n,S,r} = W^{\text{cyc}}_{n,S,r}(u_1, \ldots, u_r)$ is the quotient of the degenerate affine BMW algebra $W^{\text{aff}}_{n,S}(\Omega)$ by the relation

\[
(x_1 - u_1) \cdots (x_1 - u_r) = 0.
\]
Due to the symmetry of the relations, \( W_{n,S}^{\text{aff}} \) has a unique \( S \)-linear algebra involution \( i \) fixing each of the generators. The involution passes to cyclotomic quotients.

5.5.2. Admissibility. As for the cyclotomic BMW algebras, it is necessary to impose an admissibility condition on the parameters in order to get a satisfactory theory. Ariki, Mathas and Rui [4] proposed a condition called \( u \)-admissibility. It was recently shown [12] that their condition is equivalent to an analogue of the admissibility condition of Wilcox and Yu [39] for the cyclotomic BMW algebras, and to \( W_{r,2e1} \) being free of rank \( r \). This assumes that 2 is invertible in the ground ring. In an admissible ground ring, the parameters \( \omega_a \) are given by specific polynomial functions of \( u_1, \ldots, u_r \). There is a generic admissible ground ring \( R = \mathbb{Z}[1/2, u_1, \ldots, u_r] \), where the \( u_j \) are algebraically independent indeterminants. The field of fractions \( F \) of \( R \) is \( \mathbb{Q}(u_1, \ldots, u_r) \). Henceforth, we assume that we work over an admissible integral domain containing 1/2.

5.5.3. Some basic properties of degenerate cyclotomic BMW algebras. We establish some elementary properties of degenerate cyclotomic BMW algebras. Several of the properties can be shown for degenerate affine BMW algebras instead. Let \( S \) be any appropriate ground ring for the degenerate affine or cyclotomic BMW algebras, and write \( W_n^{\text{aff}} \) for \( W_{n,S}^{\text{aff}} \) and \( W_n \) for \( W_{n,S,r} \).

**Lemma 5.9** (see [4], Lemma 2.3). In the affine BMW algebra \( W_n^{\text{aff}} \), for \( 1 \leq i < n \) and \( a \geq 1 \), one has

\[
 s_i x_i^a = x_i^{a+1}s_i + \sum_{b=1}^{a} x_{i+1}^{b-1} (e_i - 1)x_i^{a-b}.
\]

**Lemma 5.10.** For \( n \geq 1 \), \( W_n^{\text{aff}} \) is contained in the span of \( W_{n-1}^{\text{aff}} \) and of elements of the form \( ax_n \chi_n b \), where \( a, b \in W_{n-1}^{\text{aff}} \) and \( \chi_n \in \{ e_{n-1}, s_{n-1}, x_n^\alpha : \alpha \geq 1 \} \).

**Proof.** We do this by induction on \( n \). The base case \( n = 1 \) is clear since \( W_{1,S,r} \) is generated by \( x_1 \). Suppose now that \( n > 1 \) and make the appropriate induction hypothesis. We have to show that a word in the generators having at least two occurrences of \( e_{n-1}, s_{n-1} \), or a power of \( x_n \) can be rewritten as a linear combination of words with fewer occurrences.

Consider a subword \( \chi_n y \chi'_n \) with \( \chi_n, \chi'_n \in \{ e_{n-1}, s_{n-1}, x_n^\alpha : \alpha \geq 1 \} \) and \( y \in W_{n-1}^{\text{aff}} \). If one of \( \chi_n, \chi'_n \) is a power of \( x_n \), then it commutes with \( y \), say without loss of generality \( \chi_n = x_n^a \). Then \( \chi_n y \chi'_n = y x_n^a \chi'_n \). Now consider the cases \( \chi_n = e_{n-1}, \chi'_n = s_{n-1} \), and \( \chi_n = x_n^\beta \). We have \( y x_n^a e_{n-1} = y(-1)^a x_n^a e_{n-1} \) and \( y x_n^a s_{n-1} = y x_n^a + y \). Finally, \( y x_n^a s_{n-1} = y x_n^a + y \). Finally, \( y x_n^a s_{n-1} = y x_n^a + y \). Finally, \( y x_n^a s_{n-1} = y x_n^a + y \).

Suppose both of \( \chi_n, \chi'_n \) are in \( \{ e_{n-1}, s_{n-1} \} \). If \( y \in W_{n-2,S}^{\text{aff}} \), then \( \chi_n y \chi'_n = y \chi_n \chi'_n \). But the product of any two of \( e_{n-1}, s_{n-1} \) is either 1 or a multiple of \( e_{n-1} \). If \( y \not\in W_{n-2,S}^{\text{aff}} \), then we can assume, using the induction hypothesis, that \( y = y' \chi y'' \), where \( y', y'' \in W_{n-2,S}^{\text{aff}} \), and \( \chi \) is one of \( e_{n-2}, s_{n-2} \), or a power of \( x_{n-1} \). Since
Definition 5.14. Degenerate cyclotomic Hecke algebras.

\( \chi_n, \chi'_n \) commute with \( y', y'' \), we are reduced to considering \( \chi_n \chi'_n \). Moreover, if \( \chi \) is not a power of \( x_{n-1} \), then essentially we are dealing with a computation in the Brauer algebra, which was done in \([38]\), Proposition 2.1. If one of \( \chi_n, \chi'_n \) is \( s_{n-1} \), then the computation can be done using Lemma 5.9. Thus the only interesting case is \( e_{n-1} x_{n-1}^{\alpha} e_{n-1} \). But by Lemma 4.15 in \([4]\), \( e_{n-1} x_{n-1}^{\alpha} e_{n-1} = \omega e_{n-1} \), where \( \omega \) is in the center of \( W_{n-2}^{\text{aff}} \).

Lemma 5.11.

(1) For \( n \geq 3 \), \( e_{n-1} W_{n-1}^{\text{aff}} e_{n-1} = W_{n-2}^{\text{aff}} e_{n-1} \).

(2) \( e_1 W_1^{\text{aff}} e_1 = (\omega_j : j \geq 0) e_1 \), where \( (\omega_j : j \geq 0) \) denotes the ideal in \( S \) generated by all \( \omega_j \).

(3) For \( n \geq 2 \), \( e_{n-1} \) commutes with \( W_{n-2}^{\text{aff}} \).

Proof. First we have to show that if \( y \in W_{n-1}^{\text{aff}}_n \), then \( e_{n-1} y e_{n-1} \in W_{n-2}^{\text{aff}} e_{n-1} \).

Using Lemma 5.10, we can suppose that either \( y \in W_{n-2}^{\text{aff}}_n \) or \( y = y' \chi_n y'' \), with \( y', y'' \in W_{n-2}^{\text{aff}}_n \), and \( \chi_{n-1} \in \{ e_{n-2}, s_{n-2}, x_{n-1}^{\alpha} : \alpha \geq 1 \} \). For \( \chi_{n-1} \) a power of \( x_{n-1} \), apply Lemma 4.15 from \([4]\). In all other cases, the result follows from the defining relations of \( W_{n}^{\text{aff}} \). Thus we have \( e_{n-1} W_{n-1}^{\text{aff}}_n e_{n-1} \subseteq W_{n-2}^{\text{aff}} e_{n-1} \).

For the opposite inclusion, let \( x \in W_{n-2}^{\text{aff}}_n \). Then \( x e_{n-1} = e_{n-1} x e_{n-2} e_{n-1} \in e_{n-1} W_{n-1}^{\text{aff}}_n e_{n-1} \). Points (2) and (3) are obvious.

Lemma 5.12. For \( n \geq 2 \), \( W_n e_{n-1} = W_n e_{n-1} \).

Proof. The proof is similar to the proof of Lemma 5.3 in \([16]\). Using Lemma 5.10, if \( x \in W_n \) and \( x \notin W_{n-1} \), then we can assume that \( x = y' \chi_n y'' \), with \( y', y'' \in W_{n-1} \), and \( \chi_{n} \in \{ e_{n-1}, s_{n-1}, x_{n-1}^{\alpha} : \alpha \geq 1 \} \). Likewise, we can assume that either \( y'' \in W_{n-2} \) or \( y'' = z' \chi_{n-1} z'' \) with \( z', z'' \in W_{n-2} \) and \( \chi_{n-1} \in \{ e_{n-2}, s_{n-2}, x_{n-1}^{\beta} : \beta \geq 1 \} \). The problem reduces to showing that \( \chi_{n-1} e_{n-1} \) and \( \chi_{n-1} e_{n-1} \) lie in \( W_{n-1} e_{n-1} \) for the various choices of \( \chi_{n}, \chi_{n-1} \). Most of the cases follow directly from the defining relations, while \( s_{n-1} x_{n-1}^{\beta} e_{n-1} \) must be reduced using Lemma 5.9, and \( e_{n-1} x_{n-1}^{\beta} e_{n-1} \) requires the use of Lemma 4.15 in \([4]\).

Lemma 5.13. Let \( R \) be the universal admissible ring. For \( n \geq 1 \), the map \( x \mapsto x e_{n} \) is injective from \( W_{n,R} \) to \( W_{n,R} e_{n} \).

Proof. Note that \( e_{n+1}(x e_{n}) e_{n+1} = x e_{n+1} \), so it suffices to show that \( x \mapsto x e_{n+1} \) is injective. It follows from Proposition 2.15 and Theorem A in \([4]\) that \( W_{n,R} \) has a basis of "\( r \)-regular monomials". The map \( x \mapsto x e_{n+1} \) takes the basis elements of \( W_{n,R} \) to distinct basis elements of \( W_{n+2,R} \), so is injective.

5.5.4. Degenerate cyclotomic Hecke algebras.

Definition 5.14. Let \( S \) be a commutative ring with identity. The degenerate affine Hecke algebra \( \bar{H}_{n,S} \) is the unital associative \( S \)-algebra with generators

\[ \{ s_i, x_j : \ 1 \leq i < n \text{ and } 1 \leq j \leq n \} \]
and relations:

1. (Involutions) \( s_i^2 = 1 \), for \( 1 \leq i < n \).
2. (Affine braid relations)
   
   a. \( s_i s_j = s_j s_i \) if \( |i - j| > 1 \),
   
   b. \( s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \), for \( 1 \leq i < n - 1 \),
3. (Commutation relations) \( x_i x_j = x_j x_i \), for \( 1 \leq i, j \leq n \) and \( s_i x_j = x_j s_i \) if \( j \neq i, i + 1 \).
4. (Skein relations) \( s_i x_i - x_{i+1} s_i = -1 \), and \( x_i s_i - s_i x_{i+1} = -1 \), for \( 1 \leq i < n \).

Let \( u_1, \ldots, u_r \) be elements of \( S \). The degenerate cyclotomic Hecke algebra \( H_{n,S,r}(u_1, \ldots, u_r) \) is the quotient of \( \tilde{H}_n \) by the relation

\[
(x_1 - u_1)(x_2 - u_2) \cdots (x_1 - u_r) = 0.
\]

The degenerate cyclotomic Hecke algebra is a free \( S \)-module of rank \( r^n n! \), and \( H_{n,S,r}(u_1, \ldots, u_r) \) is the algebra with involution \( i \) fixing the generators; the involutions on the tower of degenerate cyclotomic Hecke algebras are consistent.

It is observed in [4], Section 6, that the Murphy type cellular basis of the cyclotomic Hecke algebra from [9] can be easily adapted to the degenerate cyclotomic Hecke algebras. Recall that the partially ordered set \( \Lambda_n^{(0)} \) in the cell datum for \( H_{n,S,r} = H_{n,S,r}(q; u_1, \ldots, u_r) \) is the set of \( r \)-tuples of Young diagrams with total size \( n \), ordered by dominance. For each \( \lambda \in \Lambda_n^{(0)} \), the index set \( T(\lambda) \) in the cell datum is the set of standard tableaux of shape \( \lambda \). The proof of strong coherence of the sequence of cyclotomic Hecke algebras in [3], Proposition 1.9, and [29] also applies to the degenerate cyclotomic Hecke algebras.

Let \( J \) be the ideal in the degenerate cyclotomic BMW algebra \( W_{n,S,r}(u_1, \ldots, u_r) \) generated by \( e_{n-1} \). It is straightforward to show that \( W_{n,S,r}(u_1, \ldots, u_r)/J \cong H_{n,S,r}(u_1, \ldots, u_r) \), as algebras with involution.

5.5.5. Verification of the framework axioms for the degenerate cyclotomic BMW algebras. Let \( R \) be the generic admissible integral ground ring, \( R = \mathbb{Z}[1/2, u_1, \ldots, u_r] \). In this section, we write \( W_n \) for \( W_{n,R,r}(u_1, \ldots, u_r) \) and \( H_n \) for \( H_{n,R,r}(u_1, \ldots, u_r) \). The field of fractions of \( R \) is \( F = \mathbb{Q}(u_1, \ldots, u_r) \).

**Proposition 5.15.** The two sequences of algebras \( (W_n)_{n \geq 0} \) and \( (H_n)_{n \geq 0} \) satisfy the strong framework axioms of Section 2.5.

**Proof.** As observed above, \( (H_n)_{n \geq 0} \) is a strongly coherent tower of cellular algebras, so the strong version of axiom (1) holds. Axioms (2) and (3) are evident. \( W_n^F \) is semisimple by [4], Theorem 5.3. Thus axiom (4) holds.

We observed above that \( W_n / W_n e_{n-1} W_n \cong H_n \), as algebras with involutions. Thus axiom (5) holds. Axiom (6) follows from Lemma 5.11 and axiom (7) from Lemmas 5.12 and 5.13. Finally, axiom (8) holds because of the relation \( e_{n-1} e_n e_{n-1} = e_{n-1} \). □
We have the generic integral admissible ground ring. It is noted in [35], Theorem 4.15. The proof of both results here is shorter.

5.5.6. JM elements for degenerate cyclotomic BMW and Hecke algebras. The analogue of Jucys–Murphy elements for the degenerate cyclotomic Hecke algebras $H_{n,S,r} = H_{n,S,r}(u_1, \ldots, u_r)$ are just the generators $x_k$. In order to eventually distinguish between JM elements in the degenerate cyclotomic Hecke algebras and the degenerate cyclotomic BMW algebras, let us introduce the slightly superfluous notation $L_j^{(0)} = x_j$. It follows from the defining relations that $L_1^{(0)} + \cdots + L_n^{(0)}$ is central in $H_{n,S,r}$.

For an $r$–tuple of Young diagrams $\lambda$ of total size $n$ and a cell $x \in \lambda$, the additive content of the cell is

$$\kappa(x) = u_j + b - a$$

if $x$ is in row $a$ and column $b$ of the $j$–th component of $\lambda$. For a standard tableau $t$ of shape $\lambda$, and $1 \leq j \leq n$, let $\kappa(j,t) = \kappa(x)$, where $x$ is the cell occupied by $j$ in $t$. Let $\{a_t^\lambda\}$ be the Murphy type basis of the cell module $\Delta^\lambda$ indexed by standard tableaux of shape $\lambda$. Then $L_j^{(0)}$ acts by

$$L_j^{(0)} a_t^\lambda = \kappa(j,t) a_t^\lambda + \sum_{s > t} r_s a_s^\lambda,$$

where the sum is over standard tableaux $s$ greater than $t$ in dominance order (hence in lexicographic order). It is noted in [4], Lemma 6.6, that this follows by the argument in [23], Section 3. It follows that the sum $L_1^{(0)} + \cdots + L_n^{(0)}$ acts as the scalar $\alpha(\lambda) = \sum_{x \in \lambda} \kappa(x)$ on the cell module $\Delta^\lambda$. Thus $\{L_n^{(0)} : n \geq 0\}$ is an additive JM–family in the strongly coherent tower of cellular algebras $(H_{n,S,r})_{n \geq 0}$.

In the degenerate cyclotomic BMW algebras $W_n = W_{n,R,r}(u_1, \ldots, u_r)$ over the generic integral admissible ground ring $R$, we define $L_j = x_j$ for $1 \leq j \leq n$. We have $L_n \in W_n$ and $L_n$ commutes with $W_{n-1}$. We have $(L_j + L_{j+1})e_j = e_j(L_j + L_{j+1}) = 0$ by the defining relations. It is clear that $\pi_n(L_j) = L_j^{(0)}$, where $\pi_n : W_n \twoheadrightarrow H_n = H_{n,R,r}(u_1, \ldots, u_r)$ is the quotient map.

It now follows from Theorem 4.3 that $\{L_j : j \geq 0\}$ is an additive JM–family in $(W_n)_{n \geq 0}$, with the sum $L_1 + \cdots + L_n$ acting by

$$\beta((\lambda,n)) := \alpha(\lambda)$$

on the cell module $\Delta^{(\lambda,n)}$, if $\lambda$ is an $r$–tuple of Young diagrams of total size $k$. By Proposition 3.6, the action of the elements $L_j$ on the basis of $\Delta^{(\lambda,n)}$ labelled
by up–down tableaux is triangular:

\[(5.7) \quad L_j a_1^{(\lambda, n)} = \kappa(j, t) a_1^{(\lambda, n)} + \sum_{s > t} r_s a_s^{(\lambda, n)},\]

with \(\kappa(j, t) = \beta(t(j)) - \beta(t(j - 1))\), for some elements \(r_s \in R\), depending on \(j\) and \(t\). Moreover, if \(t(j) = (\nu, j)\) and \(t(j - 1) = (\mu, j - 1)\), then \(|\nu| = |\mu| + 1\). If \(|\nu| = |\mu| + 1\) and \(\nu \setminus \mu = x\), where \(x\) is the cell in row \(a\) and column \(b\) of the \(\ell\)-th component of \(\nu\), then

\[
\kappa(j, t) = \alpha(\nu) - \alpha(\mu) = \kappa(x) = u_\ell + (b - a).
\]

If \(|\nu| = |\mu| - 1\) and \(\mu \setminus \nu = x\), then

\[
\kappa(j, t) = \alpha(\nu) - \alpha(\mu) = -\kappa(x)^{-1} = -u_\ell - (b - a).
\]

This recovers Theorem 5.12 of Rui and Si [35].

5.6. The Jones–Temperley–Lieb algebras. Let \(S\) be a commutative ring with identity, with distinguished element \(\delta\). The Jones–Temperley–Lieb algebra \(A_n(S, \delta)\) is the unital \(S\)-algebra with generators \(e_1, \ldots, e_{n-1}\) satisfying the relation:

\[
(1) \quad e_j^2 = \delta e_j,
(2) \quad e_j e_{j+1} e_j = e_j,
(3) \quad e_j e_k = e_k e_j, \text{ if } |j - k| \geq 2,
\]

whenever all indices involved are in the range from 1 to \(n - 1\).

The Jones–Temperley–Lieb algebra can also be realized as the subalgebra of the Brauer algebra, with parameter \(\delta\), spanned by Brauer diagrams without crossings. If \(J_n\) denotes the ideal in \(A_n(S, \delta)\) generated by \(e_{n-1}\) (or, equivalently, by any \(e_j\)), then \(A_n(S, \delta)/J_n \cong S\).

The generic ground ring for the Jones–Temperley–Lieb algebras is \(R_0 = \mathbb{Z}[\delta]\), where \(\delta\) is an indeterminant over \(\mathbb{Z}\). It is shown in [16], Section 5.3, that the pair of towers of algebras \((A_n(R_0, \delta))_{n \geq 0}\) and \((R_0)_{n \geq 0}\) satisfies the framework axioms of Section 2.5. It follows from Theorem 2.9 that the sequence of Jones–Temperley–Lieb algebras is a strongly coherent tower of cellular algebras. Moreover, the partially ordered set in the cell datum for \(A_n\) is naturally realized as

\[(5.8) \quad \{(k, n) : k \leq n \text{ and } n - k \text{ even}\}, \text{ with}
(k, n) \leq (k', n) \iff k \geq k'.\]

**Proposition 5.17.** Fix \(S\) and \(\delta\) and write \(A_n\) for \(A_n(S, \delta)\). For \(n \geq 0\) and \(k \leq n\), \(A_n^{(k, n)}\) is the ideal in \(A_n\) generated by \(e_{k+1} e_{k+3} \cdots e_{n-1}\).

**Proof.** For \(k = n\), we interpret \(e_{k+1} e_{k+3} \cdots e_{n-1}\) as 1, so the statement is trivial. In particular, the statement is true for \(n = 0, 1\). Let \(n \geq 2\) and suppose the statement is true for \(A_{n'}\) with \(n' < n\). By the proof of Theorem 3.2 in [16],
in particular Proposition 4.7, for \( k < n \) we have \( A_n^{(k,n)} = A_ne_{n-1}A_{n-2}^{(k,n-2)}A_n \).

Applying the induction hypothesis,

\[
A_n^{(k,n)} = A_ne_{n-1}A_{n-2}^{(k,n-2)}A_n \\
= A_ne_{n-1}A_{n-2}(e_{k+1}e_{k+3} \cdots e_{n-3})A_{n-2}A_n \\
= A_n(e_{k+1}e_{k+3} \cdots e_{n-3}e_{n-1})A_n.
\]

Let \( R_0 \) be as above, and let \( q^{1/2} \) be a solution to \( q^{1/2} + q^{-1/2} = \delta \) in an extension of \( R_0 \). Define \( R = \mathbb{Z}[q^{\pm 1/2}] \) and let \( F = \mathbb{Q}(q^{\pm 1/2}) \). Let \( H_n \) denote the Hecke algebra \( H_{n,R}(q) \). Then \( \varphi : T_j \mapsto q^{1/2}e_j - 1 \) defines a homomorphism from \( H_{n,R}(q) \) to \( A_n(R, \delta) \), respecting the algebra involutions. The kernel of \( \varphi \) is the ideal in \( H_n \) generated by

\[
(5.9) \quad \xi = T_1T_2T_1 + T_1T_2 + T_2T_1 + T_1 + T_2 + 1,
\]

see [15], Corollary 2.11.2.

Recall from Example 2.6 that the Hecke algebra \( H_n \) has a cell datum whose partially ordered set is the the set \( Y_n \) of Young diagrams of size \( n \) with dominance order. The set \( \Gamma_n \) of Young diagrams with at least three columns is an order ideal in \( Y_n \); let \( I_n = H_n(\Gamma_n) \) denote the corresponding \( i \)-invariant two sided ideal of \( H_n \).

The proof of the following lemma is straightforward.

**Lemma 5.18.** Let \( A \) be a cellular algebra. Let \( \Lambda \) denote the partially ordered set in the cell datum for \( A \), let \( \Gamma \) be an order ideal in \( \Lambda \), and let \( A(\Gamma) \) be the corresponding ideal of \( A \). Then \( A/A(\Gamma) \) is a cellular algebra, with cellular basis \( \{ e_{s,t} + A(\Gamma) : \lambda \in \Lambda \setminus \Gamma; \ s, t \in T(\lambda) \} \).

Applying the lemma to the Hecke algebra, we have that \( H_n/I_n \) is a cellular algebra, with cellular basis \( \{ m^\lambda_{s,t} + H_n(\Gamma_n) : \lambda \in Y_n \setminus \Gamma_n; \ s, t \in T(\lambda) \} \). The set \( Y_n \setminus \Gamma_n \) is the set of Young diagrams of size \( n \) with no more than 2 columns. It is totally ordered by dominance. Write \( \lambda(k,n) = (2^{(n-k)/2},1^k) \), i.e. the Young diagram with \((n-k)/2\) rows with two boxes and \( k \) rows with one box. Then

\[
(5.10) \quad \lambda(k,n) \leq \lambda(k',n) \iff k \geq k';
\]

compare (5.8).

**Lemma 5.19.** \( H_n/I_n \cong A_n(R, \delta) \).

**Proof.** For \( n = 1, 2 \), \( \Gamma_n = \emptyset \) and \( I_n = (0) \). On the other hand, \( H_n \cong A_n(R, \delta) \cong R \). For \( n \geq 3 \), let \( \mu = (3,1^{n-3}) \). In the notation of [27], chapter 3, \( \xi = m_\mu = m^\mu_{\mu,\mu} \in I_n \), where \( \xi \) is the element in Equation (5.9). Hence the ideal \( \langle \xi \rangle \) generated by \( \xi \) in \( H_n \) is contained in \( I_n \). Therefore, we have a surjective homomorphism of involutive algebras \( A_n \cong H_n/\langle \xi \rangle \twoheadrightarrow H_n/I_n \). Both algebras are free of
rank $\sum_{\lambda} (f_{\lambda})^2 = \frac{1}{n+1} \binom{2n}{n}$, where the sum is over Young diagrams of size $n$ and no more than two columns, and $f_{\lambda}$ is the number of standard Young tableaux of shape $\lambda$. Hence, the homomorphism is an isomorphism. □

We identify $H_n/I_n$ with $A_n$. By slight abuse of notation, we write $T_j$ for the image of $T_j$ in $A_n$, namely $T_j = q^{1/2}e_j - 1$. Thus $T_j + 1 = q^{1/2}e_j$. We now have potentially two cellular structures on $A_n$, one inherited from the Hecke algebra and one obtained by the construction of [16], Section 5.3.

By the description of the cellular structure on the Hecke algebra in [27], chapter 3, we have that $A_{\lambda}^{(k,n)}$ is the span of $A_n m_{\lambda(j,n)} A_n$ with $j \leq k$, where

$$m_{\lambda(j,n)} = (1 + T_1)(1 + T_3) \cdots (1 + T_{n-j-1}) = q^{(n-j)/2}e_1e_3 \cdots e_{n-j-1}.$$ 

Thus, in fact,

$$A_{\lambda}^{(k,n)} = A_n (e_1 \cdots e_{n-k-1}) A_n = A_n (e_{k+1} \cdots e_{n-1}) A_n = A_n^{(k,n)}.$$

Moreover, the cell modules from the two cellular structures are explicitly isomorphic:

$$\Delta_{\lambda}^{(k,n)} = A_n (e_1 \cdots e_{n-k-1}) + \tilde{A}_n^{\lambda} \cong A_n (e_{k+1} \cdots e_{n-1}) + \tilde{A}_n^{(k,n)} = \Delta^{(k,n)}.$$

We can now import the JM elements from the Hecke algebras (see Example 3.1) to the Jones–Temperley–Lieb algebras. Set $L_1 = 1$ and $L_{j+1} = q^{-1}T_j L_j T_j$ for $j \geq 1$. Since the cell modules for the Jones–Temperley–Lieb algebra $A_n$ are in fact cell modules for the Hecke algebra $H_n$, the triangularity property (3.1) follows, and the product $\prod_{j=1}^n L_j$ acts as the scalar

$$\alpha(\lambda(k,n)) = \sum_{x \in \lambda(k,n)} \kappa(x)$$

on the cell module $\Delta^{\lambda(k,n)} = \Delta^{(k,n)}$. One can check that

$$\frac{\alpha(\lambda(k,n))}{\alpha(\lambda(k,n-2))} = q^{-n+3},$$

independent of $k$, for $n \geq 2$. It follows from this that $L_n L_{n+1} e_n = e_n L_n L_{n+1} = q^{-n+2} e_n$ for $n \geq 1$.

Remark 5.20. The same or similar analogues of Jucys–Murphy elements for the Jones–Temperley–Lieb algebras have been considered in [21] and [10]. Those in [10] are defined over the generic ring $R_0 = \mathbb{Z}[\delta]$, but it is not clear that they have, or can be modified to have, the multiplicative property (resp. additive property) of Definition 3.3 or 3.4.
References

1. Susumu Ariki, *Representations of quantum algebras and combinatorics of Young tableaux*, University Lecture Series, vol. 26, American Mathematical Society, Providence, RI, 2002, Translated from the 2000 Japanese edition and revised by the author. MR MR1911030 (2004b:17022)

2. Susumu Ariki and Kazuhiko Koike, *A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations*, Adv. Math. 106 (1994), no. 2, 216–243. MR MR1279219 (95h:20006)

3. Susumu Ariki and Andrew Mathas, *The number of simple modules of the Hecke algebras of type $G(r,1,n)$*, Math. Z. 233 (2000), no. 3, 601–623. MR MR1750939 (2001e:20007)

4. Joan S. Birman and Hans Wenzl, *Braids, link polynomials and a new algebra*, Trans. Amer. Math. Soc. 313 (1989), no. 1, 249–273. MR 90g:57004

5. Richard Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. (2) 38 (1937), no. 4, 857–872. MR MR1503378

6. Richard Dipper and Gordon James, *Representations of Hecke algebras of general linear groups*, Proc. London Math. Soc. (3) 52 (1986), no. 1, 20–52. MR MR812444 (88b:20065)

7. Richard Dipper and Gordon James, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. (3) 54 (1987), no. 1, 57–82. MR MR872250 (88m:20084)

8. Richard Dipper, Gordon James, and Andrew Mathas, *Cyclotomic $q$-Schur algebras*, Math. Z. 229 (1998), no. 3, 385–416. MR MR1658581 (2000a:20033)

9. John Enyang, *Representations of Temperley–Lieb algebras*, preprint (2007), arXiv:0710.3218.

10. John Enyang, *Specht modules and semisimplicity criteria for Brauer and Birman-Murakami-Wenzl algebras*, J. Algebraic Combin. 26 (2007), no. 3, 291–341. MR MR2348099

11. Frederick M. Goodman, *Admissibility conditions for degenerate cyclotomic BMW algebras*, preprint (2009), arXiv:0905.4253.

12. Frederick M. Goodman, *Cellularity of cyclotomic Birman-Wenzl-Murakami algebras*, Journal of Algebra 321 (2009), no. 11, 3299 – 3320, Special Issue in Honor of Gus Lehrer.

13. Frederick M. Goodman, *Comparison of admissibility conditions for cyclotomic Birman–Wenzl–Murakami algebras*, preprint (2009), arXiv:0905.4258.

14. Frederick M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications, vol. 14, Springer-Verlag, New York, 1989. MR MR999799 (91c:46082)

15. Frederick M. Goodman and John Graber, *Cellularity and the Jones basic construction*, preprint (2009), arXiv:0906.1496.

16. Frederick M. Goodman and John Graber, *Cellularity and the Jones basic construction II*, preprint (2009), arXiv:0906.1496.

17. Frederick M. Goodman and Holly Hauschild, *Affine Birman–Wenzl–Murakami algebras and tangles in the solid torus*, Fund. Math. 190 (2006), 77–137. MR MR2232856

18. Frederick M. Goodman and Holly Hauschild Mosley, *Cyclotomic Birman-Wenzl-Murakami algebras I: Freeness and realization as tangle algebras*, J. Knot Theory Ramifications (to appear), arXiv:math/0612064.

19. Frederick M. Goodman and Holly Hauschild Mosley, *Cyclotomic Birman-Wenzl-Murakami algebras II: Admissibility relations and freeness*, Algebras and Representation Theory (to appear), arXiv:math/0612064.

20. J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. 123 (1996), no. 1, 1–34. MR MR1376244 (97h:20016)

21. Tom Halverson, Manuela Mazzocco, and Arun Ram, *Commuting families in Temperley-Lieb algebras*, preprint (2007), arXiv:0710.0596.

22. Reinhard Häring-Oldenburg, *Cyclotomic Birman-Murakami-Wenzl algebras*, J. Pure Appl. Algebra 161 (2001), no. 1-2, 113–144. MR MR1834081 (2002c:20055)
23. Gordon James and Andrew Mathas, *The Jantzen sum formula for cyclotomic $q$-Schur algebras*, Trans. Amer. Math. Soc. **352** (2000), no. 11, 5381–5404. MR MR1665333 (2001b:16017)

24. Thomas Jost, *Morita equivalence for blocks of Hecke algebras of symmetric groups*, J. Algebra **194** (1997), no. 1, 201–223. MR MR1614877 (98h:20014)

25. Alexander Kleshchev, *Linear and projective representations of symmetric groups*, Cambridge Tracts in Mathematics, vol. 163, Cambridge University Press, Cambridge, 2005. MR MR2165457 (2007b:20006)

26. Robert Leduc and Arun Ram, *A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: the Brauer, Birman-Wenzl, and type A Iwahori-Hecke algebras*, Adv. Math. **125** (1997), no. 1, 1–94. MR MR1427801 (98c:20015)

27. Andrew Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, University Lecture Series, vol. 15, American Mathematical Society, Providence, RI, 1999. MR MR1711316 (2001g:20006)

28. , *Seminormal forms and Gram determinants for cellular algebras*, J. Reine Angew. Math. **619** (2008), 141–173, With an appendix by Marcos Soriano. MR MR2414949 (2009e:16059)

29. , *A Specht filtration of an induced Specht module*, preprint (2009), arXiv:0903.4493.

30. Hugh Morton and Paweł Traczyk, *Knots and algebras*, Contribuciones Matematicas en homenaje al profesor D. Antonio Plans Sanz de Brendon (E. Martín-Peindador and A. Rodez Usan, eds.), University of Zaragoza, Zaragoza, 1990, pp. 201–220.

31. Hugh Morton and Antony Wassermann, *A basis for the Birman-Wenzl algebra*, Unpublished manuscript (1989, revised 2000), 1–29.

32. Jun Murakami, *The Kauffman polynomial of links and representation theory*, Osaka J. Math. **24** (1987), no. 4, 745–758. MR MR927059 (89c:57007)

33. G. E. Murphy, *The representations of Hecke algebras of type $A_n$*, J. Algebra **173** (1995), no. 1, 97–121. MR MR1327362 (96b:20013)

34. Maxim Nazarov, *Young’s orthogonal form for Brauer’s centralizer algebra*, J. Algebra **182** (1996), no. 3, 664–693. MR MR1398116 (97m:20057)

35. Hebing Rui and Mei Si, *On the structure of cyclotomic Nazarov-Wenzl algebras*, J. Pure Appl. Algebra **212** (2008), no. 10, 2209–2235. MR MR2418167

36. Hebing Rui and Mei Si, *The representation theory of cyclotomic BMW algebras II*, preprint (2008), arXiv:0807.4149.

37. Hebing Rui and Jie Xu, *The representations of cyclotomic BMW algebras*, J. Pure Appl. Algebra (to appear), arXiv:0801.0465.

38. Hans Wenzl, *On the structure of Brauer’s centralizer algebra*, Ann. of Math. (2) **128** (1988), no. 1, 173–193. MR MR951511 (89h:20059)

39. Stewart Wilcox and Shona Yu, *The cyclotomic BMW algebra associated with the two string type B braid group*, preprint (2006, revised 2009), arXiv:math/0611518.

40. , *On the cellularity of the cyclotomic Birman-Murakami-Wenzl algebras*, preprint (2009).

41. , *On the freeness of the cyclotomic BMW algebras: admissibility and an isomorphism with the cyclotomic Kauffman tangle algebras*, preprint (2009).

42. Shona Yu, *The cyclotomic Birman-Murakami-Wenzl algebras*, Ph.D. Thesis, University of Sydney (2007).

**Department of Mathematics, University of Iowa, Iowa City, Iowa**

**E-mail address:** goodman@math.uiowa.edu

**Department of Mathematics, University of Iowa, Iowa City, Iowa**

**E-mail address:** jgraber@math.uiowa.edu