Geometry of Planar Quadratic Systems

Valery A. Gaiko

Department of Mathematics
Belarusian State University of Informatics and Radioelectronics
L. Beda Str. 6-4, Minsk 220040, Belarus

Abstract

In this paper, the global qualitative analysis of planar quadratic dynamical systems is established and a new geometric approach to solving Hilbert’s Sixteenth Problem in this special case of polynomial systems is suggested. Using geometric properties of four field rotation parameters of a new canonical system which is constructed in this paper, we present a proof of our earlier conjecture that the maximum number of limit cycles in a quadratic system is equal to four and the only possible their distribution is (3 : 1) [10]. Besides, applying the Wintner–Perko termination principle for multiple limit cycles to our canonical system, we prove in a different way that a quadratic system has at most three limit cycles around a singular point (focus) and give another proof of the same conjecture.

Keywords: Hilbert’s sixteenth problem; Wintner–Perko termination principle; planar quadratic dynamical system; field rotation parameter; bifurcation; limit cycle; separatrix cycle

1 Introduction

We consider planar dynamical systems

\[ \dot{x} = P_2(x, y), \quad \dot{y} = Q_2(x, y), \]

where \( P_2 \) and \( Q_2 \) are quadratic polynomials with real coefficients in the real variables \( x, y \). The main problem of qualitative theory of such systems is...
**Hilbert’s Sixteenth Problem** on the maximum number and relative position of their limit cycles, i.e., closed isolated trajectories of (1.1) \([10], [13]\). In this paper, we suggest a new geometric approach \([12]\) to studying limit cycle bifurcations of (1.1) and to solving the Problem in this special case of polynomial systems.

In particular, in Section 2, we construct two canonical systems with field rotation parameters, one of which contains four such parameters. In Section 3, using the canonical systems and geometric properties of the spirals filling the interior and exterior domains of limit cycles, we obtain the main result of this paper on the maximum number and relative position of limit cycles. In Section 4, we obtain the same result in a different way: applying the Wintner–Perko termination principle for multiple limit cycles. Finally, in Conclusion, we discuss some different approaches to Hilbert’s Sixteenth Problem.

### 2 Canonical systems

In \([2], [10]\), we constructed a canonical quadratic system with two field rotation parameters for studying limit cycle bifurcations:

\[
\dot{x} = P(x, y) + \alpha Q(x, y), \quad \dot{y} = Q(x, y) - \alpha P(x, y),
\]

where

\[
P(x, y) = -y + mxy + (n - \gamma)y^2, \quad Q(x, y) = x - x^2 + \gamma xy + cy^2.
\]

In \([2], [10]\), we show also by which linear transformations of the phase variables \(x, y\) arbitrary quadratic system (1.1) is reduced to form (2.1) and how the parameters of (2.1) are expressed via the parameters of (1.1). System (2.1) is especially convenient for the investigation of quadratic systems in the case two finite singularities when the parameters \(\alpha, \gamma\) rotate the field of (2.1) in the whole phase plane \(x, y\).

Later, we constructed a canonical system with three field rotation parameters, \(\alpha, \beta, \lambda\),

\[
\dot{x} = -(1 + x)y + \alpha Q(x, y), \quad \dot{y} = Q(x, y),
\]

where

\[
Q(x, y) = x + \lambda y + ax^2 + \beta(1 + x)y + cy^2,
\]

which, together with the system

\[
\dot{x} = -y + \nu y^2, \quad \dot{y} = Q(x, y), \quad \nu = 0; 1,
\]

can be used in an arbitrary case of finite singularities \([10]\).
Applying a similar approach, we can construct a canonical system with the maximum number of field rotation parameters, namely: with four such parameters. It is valid the following theorem.

**Theorem 2.1.** A quadratic system with limit cycles can be reduced to the canonical form

\[
\begin{align*}
\dot{x} &= -y (1 + x + \alpha y) \equiv P, \\
\dot{y} &= x + (\lambda + \beta + \gamma) y + a x^2 + (\alpha + \beta + \gamma) x y + c \gamma y^2 \equiv Q
\end{align*}
\] (2.4)

or

\[
\begin{align*}
\dot{x} &= -y (1 + \nu y), \quad \nu = 0; 1, \\
\dot{y} &= x + (\lambda + \beta + \gamma) y + a x^2 + (\beta + \gamma) x y + c \gamma y^2.
\end{align*}
\] (2.5)

**Proof.** In [10] is shown that an arbitrary quadratic system with limit cycles, by means of Erugun’s two-isocline method [6], can be reduced to the form

\[
\begin{align*}
\dot{x} &= -y + m x y + n y^2, \\
\dot{y} &= x + \lambda y + a x^2 + b x y + c y^2,
\end{align*}
\] (2.6)

where \( m = -1 \) or \( m = 0 \).

Input the field rotation parameters into this system so that (2.4) corresponds to the case of \( m = -1 \) and (2.5) corresponds to the case of \( m = 0 \).

Compare (2.4) with (2.6) when \( m = -1 \). Firstly, we have changed several parameters: \( n \) by \(-\alpha\); \( b \) by \( \beta \); \( c \) by \( c \gamma \). Secondly, we have input additional terms into the expression for \( \dot{y} : (\beta + \gamma) y \) and \( (\alpha + \gamma) x y \). Similar transformations have been made in system (2.6) when \( m = 0 \); but in this case, we have denoted \( n \) by \( \nu \) assigning two principal values to this parameter: 0 and 1. It is obvious that all these transformations do not restrict generality of systems (2.4) and (2.5) in comparison with system (2.6), what proves the theorem.

System (2.4) will be a basic system for studying limit cycle bifurcations. It contains four field rotation parameters: \( \lambda, \alpha, \beta, \gamma \). The following lemma is valid for this system (a similar lemma is valid for system (2.5), with respect to the parameters \( \lambda, \beta, \gamma \)).

**Lemma 2.1.** Each of the parameters \( \lambda, \beta, \gamma, \) and \( \alpha \) rotates the vector field of (2.4) in the domains of existence of its limit cycles, under the fixed other parameters of this system, namely: when the parameter \( \lambda, \beta, \gamma, \) or \( \alpha \) increases (decreases), the field is rotated in positive (negative) direction, i.e., counterclockwise (clockwise), in the domains, respectively.
1 + x + α y < 0 (> 0);
(1 + x)(1 + x + α y) < 0 (> 0);
(1 + x + c y)(1 + x + α y) < 0 (> 0);
(λ + β + γ) y + (a - 1) x^2 + (β + γ) xy + c γ y^2 < 0 (> 0).

Proof. Using the definition of a field rotation parameter [3], [10] we can calculate the following determinants:

\[
\Delta_λ = PQ'_λ - QP'_λ = -y^2(1 + x + α y);
\]
\[
\Delta_β = PQ'_β - QP'_β = -y^2(1 + x)(1 + x + α y);
\]
\[
\Delta_γ = PQ'_γ - QP'_γ = -y^2(1 + x + c y)(1 + x + α y);
\]
\[
\Delta_α = PQ'_α - QP'_α = y^2((λ + β + γ) y + (a - 1) x^2 + (β + γ) xy + c γ y^2).
\]

Since, by definition, the vector field is rotated in positive direction (counter-clockwise) when the determinant is positive and in negative direction (clockwise) when the determinant is negative [3], [10] and since the obtained domains correspond to the domains of existence of limit cycles of (2.4), the lemma is proved.

3 The main result

By means of canonical systems (2.4) and (2.5), we will study global limit cycle bifurcations of (1.1). First of all, let us give a new proof of the following theorem.

**Theorem 3.1.** A quadratic system can have at least four limit cycles in the (3 : 1)-distribution.

Proof. To prove the theorem, consider the case of two finite anti-saddles and the only saddle at infinity when, for example, \(a = 1/2\) and \(c = -1\) in (2.4):

\[
\dot{x} = -y (1 + x + α y),
\]
\[
\dot{y} = x + (λ + β + γ) y + (1/2) x^2 + (α + β + γ) xy - γ y^2.
\]

Vanish all field rotation parameters in (3.1): \(α = β = γ = λ = 0\). Then we have got a system with two centers which is symmetric with respect to the \(x\)-axis.

Under increasing the parameter \(γ\) (\(0 < γ ≪ 1\)), the vector field of (3.1) is rotated in negative direction (clockwise) and the centers turn into foci: \((0, 0)\) becomes an unstable focus and \((-2, 0)\) becomes a stable one.
Fix $\gamma$ and take $\lambda$ satisfying the condition: $-1 \ll \lambda < -\gamma < 0$ ($-1 \ll \gamma + \lambda < 0$). Then, in the half-plane $x > -1$, the vector field of (3.1) is rotated in positive direction and the focus $(0, 0)$ changes the character of its stability generating an unstable limit cycle. In the half-plane $x < -1$, the field is rotated in negative direction again and the focus $(-2, 0)$ remains stable.

Fix the parameters $\gamma, \lambda$ and take $\alpha$ satisfying the condition: $\gamma + \lambda \ll \alpha < 0$. After rotation of the vector field of system (3.1) in positive direction, the straight line $x = 1$ is destroyed and two limit cycles are generated by the separatrix cycles formed by this line and two Poincaré hemi-circles: a stable limit cycle surrounding the focus $(0, 0)$ and an unstable one surrounding the focus $(-2, 0)$.

Finally, fix the parameters $\gamma, \lambda, \alpha$ and take $\beta$ satisfying the condition: $0 < -\gamma - \lambda < \beta \ll 1$ ($0 < \beta + \gamma + \lambda \ll -\alpha$). Then, after rotation of the vector field in negative direction in the whole phase plane, the focus $(0, 0)$ changes the character of its stability again generating a stable limit cycle, since the parameter $\alpha$ is non-rough and negative when $\beta = -\gamma - \lambda$. Thus, we have obtained at least three limit cycles surrounding the focus $(0, 0)$, under the co-existence of a limit cycle surrounding the focus $(-2, 0)$, what proves the theorem.

It is valid a much stronger theorem (the main result).

**Theorem 3.2.** A quadratic system has at most four limit cycles and only in the $(3:1)$-distribution.

**Proof.** Consider again the most interesting case of quadratic systems: with two finite anti-saddles and the only saddle at infinity when $a = 1/2$ and $c = -1$ in (2.4). All other cases of singular points can be considered in a similar way.

Vanish all field rotation parameters of system (3.1), $\alpha = \beta = \gamma = \lambda = 0$:

\[
\begin{align*}
\dot{x} &= -y (1 + x), \\
\dot{y} &= x + (1/2) x^2.
\end{align*}
\] (3.2)

We have got a system with two centers which is symmetric with respect to the $x$-axis. Let us input successively the field rotation parameters into (3.2).

Begin, for example, with the parameter $\gamma$ supposing that $\gamma > 0$:

\[
\begin{align*}
\dot{x} &= -y (1 + x), \\
\dot{y} &= x + \gamma y + (1/2) x^2 + \gamma xy - \gamma y^2.
\end{align*}
\] (3.3)

Under increasing $\gamma$, the vector field of (3.3) is rotated in negative direction (clockwise) and the centers turn into foci: $(0, 0)$ becomes an unstable focus and $(-2, 0)$ becomes a stable one.
Fix $\gamma$ and input a new parameter, for example, $\lambda < 0$ into (3.3):

$$\dot{x} = -y (1 + x),$$

$$\dot{y} = x + (\lambda + \gamma) y + (1/2) x^2 + \gamma x y - \gamma y^2. \tag{3.4}$$

Then, in the half-plane $x > -1$, the vector field of (3.4) is rotated in positive direction (counterclockwise) and the focus $(0, 0)$ changes the character of its stability (when $\lambda = -\gamma$) generating an unstable limit cycle. Under decreasing $\lambda$, this limit cycle will expand until it disappears in a Poincaré hemi-cycle with a saddle-node lying on the invariant straight line $x = -1$ [10]. In the half-plane $x < -1$, the field is rotated in negative direction again and the focus $(-2, 0)$ remains stable.

Denote the limit cycle by $\Gamma_1$, the domain inside the cycle by $D_1$, the domain outside the cycle by $D_2$ and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a “trajectory concentration” surrounding the focus $(0, 0)$. It is clear that under decreasing $\lambda$, a semi-stable limit cycle cannot appear in the domain $D_1$, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation in positive direction.

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain $D_2$. Suppose it appears in this domain for some values of the parameters $\gamma^* > 0$ and $\lambda^* < 0$. Return to initial system (3.2) and change the order of inputting the field rotation parameters. Input first the parameter $\lambda < 0$:

$$\dot{x} = -y (1 + x),$$

$$\dot{y} = x + \lambda y + (1/2) x^2. \tag{3.5}$$

Fix it under $\lambda = \lambda^*$. In the half-plane $x > -1$, the vector field of (3.5) is rotated in negative direction and $(0, 0)$ becomes a stable focus. Inputting the parameter $\gamma > 0$ into (3.5), we have got again system (3.4), the vector field of which is rotated in positive direction in the half-plane $x > -1$. Under this rotation, an unstable limit cycle $\Gamma_1$ will appear from a Poincaré hemi-cycle with a saddle-node on the invariant straight line $x = -1$. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing the parameter $\gamma$ to the value $\gamma = \gamma^*$. It follows that there are no values of $\gamma = \gamma^*$ and $\lambda = \lambda^*$, for which a semi-stable limit cycle could appear in the domain $D_2$.

This contradiction proves the uniqueness of a limit cycle surrounding the focus $(0, 0)$ in system (3.4) for any values of the parameters $\gamma$ and $\lambda$ of different signs. Obviously, if these parameters have the same sign, system (3.4) has no limit cycles surrounding $(0, 0)$ at all, like there are no limit cycles surrounding the focus $(-2, 0)$ for the parameters $\gamma$ and $\lambda$ of different signs.
Let system (3.4) have the unique limit cycle $\Gamma_1$. Fix the parameters $\gamma > 0$, $\lambda < 0$ and input the third parameter, $\alpha < 0$, into this system:

\begin{align*}
\dot{x} &= -y (1 + x + \alpha y), \\
\dot{y} &= x + (\lambda + \gamma) y + (1/2) x^2 + (\alpha + \gamma) xy - \gamma y^2.
\end{align*}

(3.6)

The vector field of (3.6) is rotated in positive direction again, the invariant straight line $x = -1$ is immediately destroyed and two limit cycles appear from the corresponding Poincaré hemi-cycles containing this straight line: a stable cycle, denoted by $\Gamma_2$, surrounding the focus $(0, 0)$ and an unstable limit cycle, denoted by $\Gamma_3$, surrounding the focus $(-2, 0)$. Under further decreasing $\alpha$, the limit cycle $\Gamma_2$ will join with $\Gamma_1$ forming a semi-stable limit cycle, $\Gamma_{12}$, which will disappear in a “trajectory concentration” surrounding the origin $(0, 0)$. Can another semi-stable limit cycle appear around the origin in addition to $\Gamma_{12}$? It is clear that such a limit cycle cannot appear neither in the domain $D_1$ bounded by the origin and $\Gamma_1$ nor in the domain $D_3$ bounded on the inside by $\Gamma_2$ because of increasing the distance between the spiral coils filling these domains under decreasing $\alpha$.

To prove impossibility of the appearance of a semi-stable limit cycle in the domain $D_2$ bounded by the cycles $\Gamma_1$ and $\Gamma_2$ (before their joining), suppose the contrary, i.e., for some set of values of the parameters $\gamma^* > 0$, $\lambda^* < 0$, and $\alpha^* < 0$, such a semi-stable cycle exists. Return to system (3.2) again and input the parameters $\alpha < 0$ and $\lambda < 0$:

\begin{align*}
\dot{x} &= -y (1 + x + \alpha y), \\
\dot{y} &= x + \lambda y + (1/2) x^2 + \alpha xy.
\end{align*}

(3.7)

In the half-plane $x > -1$, both parameters act in a similar way: they rotate the vector field of (3.7) in positive direction turning the origin $(0, 0)$ into a stable focus. In the half-plane $x < -1$, they rotate the field in opposite directions generating an unstable limit cycle from the focus $(-2, 0)$.

Fix these parameters under $\alpha = \alpha^*$, $\lambda = \lambda^*$ and input the parameter $\gamma > 0$ into (3.7) getting again system (3.6). Since, on our assumption, this system has two limit cycles for $\gamma < \gamma^*$, there exists some value of the parameter, $\gamma_{12}$ ($0 < \gamma_{12} < \gamma^*$), for which a semi-stable limit cycle, $\Gamma_{12}$, appears in system (3.6) and then splits into an unstable cycle, $\Gamma_1$, and a stable cycle, $\Gamma_2$, under further increasing $\gamma$. The formed domain $D_2$ bounded by the limit cycles $\Gamma_1$, $\Gamma_2$ and filled by the spirals will enlarge, since, on the properties of a field rotation parameter, the interior unstable limit cycle $\Gamma_1$ will contract and the exterior stable limit cycle $\Gamma_2$ will expand under increasing $\gamma$. The distance between the spirals of the domain $D_2$ will naturally increase, what will prohibit from the appearance of a semi-stable limit cycle in this domain for $\gamma > \gamma_{12}$. Thus, there are no such values of the parameters, $\gamma^* > 0, \lambda^* < 0, \alpha^* < 0$, for which system (3.6) would have an additional semi-stable limit cycle.
Obviously, there are no other values of the parameters $\gamma$, $\lambda$, $\alpha$, for which system (3.6) would have more than two limit cycles surrounding the origin $(0,0)$ and simultaneously more than one limit cycle surrounding the point $(-2,0)$ (on the same reasons). It follows that system (3.6) can have at most three limit cycles and only in the $(2:1)$-distribution.

Suppose that system (3.6) has two limit cycle, $\Gamma_1$ and $\Gamma_2$, around the origin $(0,0)$ and the only limit cycle, $\Gamma_3$, around the point $(-2,0)$. Fix the parameters $\gamma > 0$, $\lambda < 0$, $\alpha < 0$ and input the fourth parameter, $\beta > 0$, into (3.6) getting system (3.1). Under increasing $\beta$, the vector field of (3.1) is rotated in negative direction, the focus $(0,0)$ changes the character of its stability (when $\beta = -\gamma - \lambda$) and a stable limit cycle, $\Gamma_0$, appears from the origin. Suppose it happens before the cycle $\Gamma_1$ disappears in $(0,0)$ (this is possible by Theorem 3.1). Under further increasing $\beta$, the cycle $\Gamma_0$ will join with $\Gamma_1$ forming a semi-stable limit cycle, $\Gamma_{01}$, which will disappear in a “trajectory concentration” surrounding the origin $(0,0)$; the other cycles, $\Gamma_2$ and $\Gamma_3$, will expand tending to Poincaré hemi-cycles with the straight line $x = -1$.

Let system (3.1) have four limit cycles: $\Gamma_0$, $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$. Can an additional semi-stable limit cycle appear around the origin under increasing the parameter $\beta$? It is clear that such a limit cycle cannot appear neither in the domain $D_0$ bounded by the origin and $\Gamma_0$ nor in the domain $D_2$ bounded by $\Gamma_1$ and $\Gamma_2$ because of increasing the distance between the spiral coils filling these domains under increasing $\beta$. Consider two other domains: $D_1$ bounded by the cycles $\Gamma_0$, $\Gamma_1$ and $D_3$ bounded on the inside by the cycle $\Gamma_2$. As before, we will prove impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters, $\gamma^* > 0$, $\lambda^* < 0$, $\alpha^* < 0$, and $\beta^* > 0$, such a semi-stable cycle exists. Return to system (3.2) again and input first the parameters $\beta > 0$, $\gamma > 0$ and then the parameter $\alpha < 0$:

\[
\begin{align*}
\dot{x} &= -y (1 + x + \alpha y), \\
\dot{y} &= x + (\beta + \gamma) y + (1/2) x^2 + (\alpha + \beta + \gamma) xy - \gamma y^2.
\end{align*}
\]

(3.8)

Fix the parameters $\beta$, $\gamma$ under the values $\beta^*$, $\gamma^*$, respectively. Under decreasing the parameter $\alpha$, two limit cycles immediately appear from Poincaré hemi-cycles with the straight line $x = -1$: a stable cycle, $\Gamma_2$, around $(0,0)$ and an unstable one, $\Gamma_3$, around $(-2,0)$. Fix $\alpha$ under the value $\alpha^*$ and input the parameter $\lambda < 0$ into (3.8) getting system (3.1).

Since, on our assumption, system (3.1) has three limit cycles around the origin $(0,0)$ for $\lambda > \lambda^*$, there exists some value of the parameter, $\lambda_{01}$ ($\lambda^* < \lambda_{01} < 0$), for which a semi-stable limit cycle, $\Gamma_{01}$, appears in this system and then splits into a stable cycle, $\Gamma_0$, and an unstable cycle, $\Gamma_1$, under further decreasing $\lambda$. The formed domain $D_1$ bounded by the limit cycles $\Gamma_0$, $\Gamma_1$ and also the domain
$D_3$ bounded on the inside by the limit cycle $\Gamma_2$ will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there, i.e., at most three limit cycles can exist around the origin $(0, 0)$. On the same reasons, a semi-stable limit cannot appear around the point $(-2, 0)$ under decreasing the parameter $\lambda$, i.e., at most one limit cycle can exist around this point simultaneously with three limit cycles surrounding $(0, 0)$.

All other combinations of the parameters $\lambda, \alpha, \beta, \gamma$ are considered in a similar way. It follows that system (3.1) has at most four limit cycles and only in the $(3 : 1)$-distribution. Applying the same approach to canonical system (2.5), we can complete the proof of the theorem.

4 The Wintner–Perko termination principle

For the global analysis of limit cycle bifurcations in [10], we used the Wintner–Perko termination principle which was stated for relatively prime, planar, analytic systems and which connected the main bifurcations of limit cycles [14], [16]. Let us formulate this principle for the polynomial system

$$\dot{x} = f(x, \mu),$$  \hspace{1cm} (4.1µ)

where $x \in \mathbb{R}^2; \mu \in \mathbb{R}^n; f \in \mathbb{R}^2 \ (f$ is a polynomial vector function).

**Theorem 4.1 (Wintner–Perko termination principle).** Any one-parameter family of multiplicity-$m$ limit cycles of relatively prime polynomial system (4.1µ) can be extended in a unique way to a maximal one-parameter family of multiplicity-$m$ limit cycles of (4.1µ) which is either open or cyclic.

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of (4.1µ), which is typically a fine focus of multiplicity $m$, or on a (compound) separatrix cycle of (4.1µ), which is also typically of multiplicity $m$.

The proof of the Wintner–Perko termination principle for general polynomial system (4.1µ) with a vector parameter $\mu \in \mathbb{R}^n$ parallels the proof of the planar termination principle for the system

$$\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda)$$  \hspace{1cm} (4.1λ)

with a single parameter $\lambda \in \mathbb{R}$ (see [10], [14]), since there is no loss of generality in assuming that system (4.1µ) is parameterized by a single parameter $\lambda$; i.e., we can assume that there exists an analytic mapping $\mu(\lambda)$ of $\mathbb{R}$ into $\mathbb{R}^n$ such
that \((4.1_{\mu})\) can be written as \((4.1_{\mu(\lambda)})\) or even \((4.1_{\lambda})\) and then we can repeat everything, what had been done for system \((4.1_{\lambda})\) in [14]. In particular, if \(\lambda\) is a field rotation parameter of \((4.1_{\lambda})\), it is valid the following Perko’s theorem on monotonic families of limit cycles.

**Theorem 4.2.** If \(L_0\) is a nonsingular multiple limit cycle of \((4.1_0)\), then \(L_0\) belongs to a one-parameter family of limit cycles of \((4.1_{\lambda})\); furthermore:

1) if the multiplicity of \(L_0\) is odd, then the family either expands or contracts monotonically as \(\lambda\) increases through \(\lambda_0\);

2) if the multiplicity of \(L_0\) is even, then \(L_0\) befurcates into a stable and an unstable limit cycle as \(\lambda\) varies from \(\lambda_0\) in one sense and \(L_0\) disappears as \(\lambda\) varies from \(\lambda_0\) in the opposite sense; i.e., there is a fold bifurcation at \(\lambda_0\).

Using Theorems 4.1 and 4.2 in [7]–[11], we proved a theorem on three limit cycles around a singular point for canonical systems \((2.2)\) and \((2.3)\). Let us prove the same theorem using systems \((2.4)\) and \((2.5)\).

**Theorem 4.3.** There exists no quadratic system having a swallow-tail bifurcation surface of multiplicity-four limit cycles in its parameter space. In other words, a quadratic system cannot have neither a multiplicity-four limit cycle nor four limit cycles around a singular point (focus), and the maximum multiplicity or the maximum number of limit cycles surrounding a focus is equal to three.

**Proof.** The proof of this theorem is carried out by contradiction. Consider canonical systems \((2.4)\) and \((2.5)\), where system \((2.5)\) represents two limit cases of \((2.4)\).

Suppose that system \((2.4)\) with four field rotation parameters, \(\lambda, \alpha, \beta,\) and \(\gamma\), has four limit cycles around the origin (system \((2.5)\) is considered in a similar way). Then we get into some domain of the field rotation parameters being restricted by definite conditions on two other parameters, \(a\) and \(c\), corresponding to one of the six cases of finite singularities which we considered in [10]. Without loss of generality, we can fix both of these parameters. Thus, there is a domain bounded by three fold bifurcation surfaces forming a swallow-tail bifurcation surface of multiplicity-four limit cycles in the space of the field rotation parameters \(\lambda, \alpha, \beta,\) and \(\gamma\).

The corresponding maximal one-parameter family of multiplicity-four limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity five (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-five limit cycles through this
point and parameterizing the corresponding maximal one-parameter family of multiplicity-five limit cycles by a field-rotation parameter, according to Theorem 4.2, we will obtain a monotonic curve which, by the Wintner–Perko termination principle (Theorem 4.1), terminates either at the origin or on some separatrix cycle surrounding the origin. Since we know absolutely precisely at least the cyclicity of the singular point (Bautin’s result [11]) which is equal to three, we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate.

If the maximal one-parameter family of multiplicity-four limit cycles is not cyclic, on the same principle (Theorem 4.2), this again contradicts to Bautin’s result not admitting the multiplicity of limit cycles higher than three. This contradiction completes the proof.

As was shown in [10], to complete the solution of Hilbert’s Sixteenth Problem for quadratic systems (1.1), it is sufficient to prove impossibility of the (2 : 2)-distribution of limit cycles only in the case of two finite foci and a saddle at infinity. In [2] (see also [10]), using canonical system (2.1) with two field rotation parameters, $\alpha$ and $\gamma$, in the case of two foci and a saddle at infinity, we constructed a quadratic system with at least four limit cycles in the (3 : 1)-distribution. If to let this system have only three limit cycles in the (2 : 1)-distribution, i.e., two cycles around the focus $(0,0)$ and the only one around the focus $(1,0)$, it is easy to show impossibility of obtaining the second limit cycle around $(1,0)$ by means of the parameters $\alpha$ and $\gamma$. Logically, we can suppose only that a semi-stable cycle appears around the focus $(1,0)$ under the variation of a field rotation parameter, for example, $\alpha$. Then, applying the Wintner–Perko termination principle (Theorem 4.1), we can show that the maximal one-parameter family of multiplicity-three limit cycles parameterized by another field rotation parameter, $\gamma$, cannot terminate in the focus $(1,0)$, since it will be a rough focus for any $\alpha \neq 0$ (see [2, 10]). The same proof could be given for canonical system (2.4). Thus, we have given one more proof of Theorem 3.2 on at most four limit cycles in the only (3 : 1)-distribution for quadratic systems (1.1).

5 Conclusion

In [10], applying the methods of catastrophe theory and the Wintner–Perko termination principle for multiple limit cycles, we have developed the global bifurcation theory of planar polynomial dynamical systems and, basing on this theory, we have suggested a program on the complete solution of Hilbert’s Sixteenth Problem for the case of quadratic systems. In principal, the program
has been realized in [10] (see the previous section). In this paper, we have presented a new (geometric) approach to its realization.

Our program is an alternative to the program which is put forward in [4], [5] and which is often called as “Roussarie’s program” by the name of its ideological inspirer [15]. Roussarie’s program is reduced to the classification of separatrix cycles, determining their cyclicity and finding an upper bound of the number of limit cycles for quadratic systems. Unfortunately, there are some serious problems in the realization of this program: for example, it is not clear how to determine the cyclicity of non-monodromic separatrix cycles when there is no return map in the neighborhood of these cycles and there is no general approach to the study of the cyclicity of separatrix cycles in the case of center when the return map is identical zero. Besides, even in the case of realization of the program, as its authors note themselves [4], the obtained upper bound of the number of limit cycles obviously can not be optimal, since the used pure analytic methods cannot ensure neither the global control of limit cycle bifurcations around a singular point nor, especially, the simultaneous control of the bifurcations around different singular points.

Thereupon, it makes sense to say some words on Roussarie’s review MR2023976 (2005d:37102) on [10]. The only concrete remark in this “awkward” review is the following: “I just mention the hazardous claim made in Theorem 4.12, page 137, that there exists no quadratic system having a swallow-tail bifurcation surface of multiplicity-four limit cycles. Looking at the proof, it seems that the author unfortunately confuses two different notions: paths of limit cycles, as defined in Definition 4.7, page 112, and lines of multiple limit cycles, as defined by Perko (and recalled in Definition 4.13, page 127). In fact, there is nothing forbidding that a path begin at a parameter value with a multiplicity-four limit cycle and end at a focus point”. So, Roussarie’s remark is related to a swallow-tail bifurcation surface of multiplicity-four limit cycles. However, Definition 4.13, page 127, is a definition of a cusp bifurcation surface of multiplicity-three limit cycles. This is an evident lack of correspondence! Maybe, the reviewer means Definition 4.14, pages 128-129? Then it seems that he did not pay attention for our remark on page 132, following just after Theorem 4.10, which could perhaps settle his doubts. Moreover, there is a reference to the corresponding work by Perko in this remark (see also [14]). Or the reviewer has complaints against Perko’s work, too? Besides, his “claim” that “there is nothing forbidding that a path begin at a parameter value with a multiplicity-four limit cycle and end at a focus point” says that he unfortunately does not see (or does not want to see) Bautin’s result [1] (Theorem 2.1, page 45) on the cyclicity of a singular point of the focus or center type, which is an obstacle on such a path. Or, maybe, Bautin’s result is also “questionable”? 

12
References

[1] N. N. Bautin, E. A. Leontovich, *Methods and Examples of the Qualitative Analysis of Dynamical Systems in a Plane*, Nauka, Moscow, 1990 (Russian).

[2] L. A. Cherkas, V. A. Gaiko, Bifurcations of limit cycles of a quadratic system with two critical points and two field-rotation parameters, *Diff. Equat.* **23** (1987), 1062–1069.

[3] G. F. D. Duff, Limit-cycles and rotated vector fields, *Ann. Math.* **67** (1953), 15–31.

[4] F. Dumortier, R. Roussarie, C. Rousseau, Hilbert’s 16th problem for quadratic vector fields, *J. Diff. Equat.* **110** (1994), 86–133.

[5] F. Dumortier, R. Roussarie, C. Rousseau, Elementary graphics of cyclicity 1 and 2, *Nonlinearity* **7** (1994), 1001–1041.

[6] N. P. Erugin, Some questions of motion stability and qualitative theory of differential equations on the whole, *Prikl. Mat. Mekh.* **14** (1950), 459–512 (Russian).

[7] V. A. Gaiko, Qualitative theory of two-dimensional polynomial dynamical systems: problems, approaches, conjectures, *Nonlin. Anal.* **30** (1997), 1385–1394.

[8] V. A. Gaiko, Hilbert’s sixteenth problem and global bifurcations of limit cycles, *Nonlin. Anal.* **47** (2001), 4455–4466.

[9] V. A. Gaiko, Global bifurcation families of multiple limit cycles in polynomial dynamical systems, *Nonlin. Phenom. Compl. Syst.* **6** (2003), 734–745.

[10] V. A. Gaiko, *Global Bifurcation Theory and Hilbert’s Sixteenth Problem*, Kluwer, Boston, 2003.

[11] V. A. Gaiko, Wintner–Perko termination principle, parameters rotating a field, and limit-cycle problem, *J. Math. Sci.* **126** (2005), 1259–1266.

[12] V. A. Gaiko, F. Botelho, Dynamical systems: bifurcations and applications, *Probl. Nonlin. Anal. Eng. Syst.* **11** (2005), 100–107.

[13] Yu. Ilyashenko, Centennial history of Hilbert’s 16th problem, *Bul. Amer. Math. Soc.* **39** (2002), 301–354.

[14] L. Perko, *Differential Equations and Dynamical Systems*, Springer, New York, 2002.

[15] R. Roussarie, *Bifurcations of Planar Vector Fields and Hilbert’s Sixteenth Problem*, Birkhäuser, Basel, 1998.

[16] A. Wintner, Beweis des E. Stromgrenschen dynamischen Abschlusprinzips der periodischen Bahngruppen im restriingierten Dreikörperproblem, *Math. Zeit.* **34** (1931), 321–349.