Control of mechanical systems on Lie groups and
ideal hydrodynamics

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Abstract

In contrast to the Euler-Poincaré reduction of geodesic flows of left- or right-invariant metrics on Lie groups to the corresponding Lie algebra (or its dual), one can consider the reduction of the geodesic flows to the group itself. The reduced vector field has a remarkable hydrodynamic interpretation: it is a velocity field for a stationary flow of an ideal fluid. Right- or left-invariant symmetry fields of the reduced field define vortex manifolds for such flows.

Consider now a mechanical system, whose configuration space is a Lie group and whose Lagrangian is invariant to left translations on that group, and assume that the mass geometry of the system may change under the action of internal control forces. Such system can also be reduced to the Lie group. With no controls, this mechanical system describes a geodesic flow of the left-invariant metric, given by the Lagrangian, and thus its reduced flow is a stationary ideal fluid flow on the Lie group. The standard control problem for such system is to find the conditions, under which the system can be brought from any initial position in the configuration space to another preassigned position by changing its mass geometry. We show that under these conditions, by changing the mass geometry, one can also bring one vortex manifold to any other preassigned vortex manifold.

Keywords: Ideal hydrodynamics, Lie groups, control.

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1 Introduction

For the Euler top, the Hamiltonian vector field on the cotangent bundle $T^*SO(3)$ can be uniquely projected onto the Lie algebra $\mathfrak{so}(3)$ – this is a classical reduction, known in the general case as the Euler-Poincaré reduction. In the 30’s, E.T.Whittaker suggested an "alternative" reduction procedure for the Euler top: by fixing values of the Noether integrals, the Hamiltonian vector field can be uniquely projected from $T^*SO(3)$ onto the group $SO(3)$ [28]. The Whittaker reduction is valid for any Hamiltonian system on a cotangent bundle $T^*G$ to a Lie group $G$, provided the Hamiltonian is invariant under the left (or right) shifts on the group $G$. An important example of such Hamiltonian systems is a geodesic flow of a left-(right-)invariant metric on a Lie group.

If we reduce a Hamiltonian system to the Lie group $G$, and then factorize the reduced vector field by the orbits of its symmetry fields, then, by the Marsden-Weinstein theorem, we get the same Hamiltonian system on a coadjoint orbit on the dual algebra $\mathfrak{g}^*$, as if we first reduced the system to the dual algebra $\mathfrak{g}^*$, and then to the coadjoint orbit (see also [1], Appendix 5). Thus, the Whittaker reduction can be regarded as a part of the Marsden-Weinstein reduction of Hamiltonian systems with symmetries [24].

In contrast to the Marsden-Weinstein reduction, it has not been paid much attention to the Whittaker reduction alone. However, it is itself worth studying. It turns out that a vector field, reduced to a Lie group $G$ has a remarkable hydrodynamic interpretation: it is a velocity field for a stationary flow of an ideal fluid, that flows on the group $G$ (viewed as a Riemannian manifold), and is incompressible with respect to some left-(or right-)invariant measure on $G$, see [9,16,17,18] for details. The reduction to a Lie group is also useful for a series of applications, which include stability theory, noncommutative integration of Hamiltonian systems, discretization, differential geometry of diffeomorphism groups and see, e.g., [18,19,8,9,11].
In this article, we first review the Whittaker reduction and its hydrodynamic essence, and provide an explicit expression for the reduction of a geodesic flow of a left- or right-invariant metric onto a Lie group. For any Lie group we find both the reduced vector field and its "symmetry fields", i.e., left- or right-invariant fields on the group that commute with our reduced vector field. These fields have also a hydrodynamical meaning: these are the vortex vector fields for our stationary flow (i.e., they annihilate the vorticity 2-form), cf. [18], [9]. The distribution of the vortex vector fields is always integrable, thus they define a manifold, that we call the vortex manifold. Typically, these manifolds are tori.

Next, we consider the following control problem. We study mechanical systems, whose configuration space is a Lie group and whose Lagrangian is invariant to left translations on that group, and we assume that the mass geometry of the system may change under the action of internal control forces. Such systems can also be reduced to the Lie group, and they also have an interesting hydrodynamic interpretation: the reduced vector field is the velocity of a stationary flow of an electron gas (with no controls, this mechanical system describes a geodesic flow of the left-invariant metric, given by the Lagrangian, and thus its reduced flow is a stationary ideal fluid flow). Notice that without relating to hydrodynamics, controlled systems on Lie groups were studied in many works, see, e.g., [7] and references therein.

The standard control problem for such systems is to find the conditions, under which the system can be brought from any initial position on the Lie group to another preassigned position by changing its mass geometry. We show that under these conditions, by changing the mass geometry, one can bring the whole vortex manifold to any other preassigned vortex manifold.

As an example, we consider the $n$-dimensional Euler top. We write down the reduced controlled system explicitly, find the vortex mani-
folds, which typically (when the momentum matrix has the maximal
rank) are tori, and show that, by changing the mass geometry, every
such vortex manifold can be transformed to any other vortex manifold.

In the Appendix we study the Whittaker reduction for nonholonomic
systems, and formulate and discuss the standard controllability condi-
tions.

2 Reduction of a geodesic flow to a Lie group

We start with some basic facts on coadjoint representations, inertia
operators on Lie algebras and the Euler equations (see, e.g., [3]). Let
G be an arbitrary Lie group, g be its Lie algebra, and g* be the corre-
sponding dual algebra. The group G may be infinite-dimensional, and
not necessarily a Banach manifold, but we assume that the exponential
map \( \exp : g \to G \) exists.

Any vector \( \dot{g} \in T_g G \) and any covector \( m \in T^*_g G \) can be translated to
the group unity by the left or the right shifts. As the result we obtain
the vectors \( \omega_c, \omega_s \in g \) and the momenta \( m_c, m_s \in g^* \):

\[
\omega_c = L_{g^{-1} \dot{g}}, \quad \omega_s = R_{g^{-1} \dot{g}}, \quad m_c = L^*_g m, \quad m_s = R^*_g m.
\]

The following relation plays the central role in the sequel:

\[
m_c = \text{Ad}^*_g m_s, \quad (2.1)
\]

\( \text{Ad}^*_g : g^* \to g^* \) being the group coadjoint operator. Let us fix the
"momentum in space" \( m_s \). Then relation (2.1) defines a coadjoint
orbit. The Casimir functions are the functions of the "momentum in
the body" \( m_c \), that are invariants of coadjoint orbits. For example,
for the Euler top, the Casimir function is the length of the kinetic
momentum.

Let \( A : g \to g^* \) be a positive definite symmetric operator (inertia
operator) defining a scalar product on the Lie algebra. This operator
defines a left- or right-invariant inertia operator $A_g$ (and thus a left- or right-invariant metric) on the group $G$. For example, in the left-invariant case, $A_g = L_g^{-1} A L_g^{-1}$. Let the metric be left-invariant. The geodesics of this metric are described by the Euler equations

$$\dot{m}_c = ad_{A^{-1} m_c}^\ast m_c,$$  \hspace{1cm} (2.2)$$

Here $ad^\ast_\xi : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is the coadjoint representation of $\xi \in \mathfrak{g}$. Given a solution of the Euler equations $\omega_c = A^{-1} m_c$, the trajectory on the group is determined by the relation

$$L_{g^{-1}} \dot{g} = \omega_c.$$  \hspace{1cm} (2.3)$$

The Euler equations follow from the fact that ”the momentum in space” $m_s$ is constant, whereas ”the momentum in the body” $m_c$ is obtained from $m_s$ by (2.1), see [3].

**Remark.** Strictly speaking, in the infinite dimensional case the operator $A$ is invertible only on a regular part of the dual algebra $\mathfrak{g}^*$. In our case this means, that some natural restriction on values of $m_s$ (or $m_c$) have to be imposed (see [3]).

The Euler equations can be considered as Hamilton’s equations on the dual algebra, where the Hamiltonian equals $H = \frac{1}{2}(A^{-1} m, m)$, $m \in \mathfrak{g}^*$, and the Poisson structure is defined by the following Poisson brackets. For two functions $F(m)$ and $G(m)$ on the dual algebra $\mathfrak{g}^*$,

$$\{F, G\} = (m, [dF(m), dG(m)]),$$

where $dF(m), dG(m) \in \mathfrak{g}$ are the differentials of functions $F$ and $G$, and $[\xi, \eta] = ad_{\xi}^\ast \eta$ is the commutator (adjoint action) on the Lie algebra $\mathfrak{g}$.

Let now

$$H = \frac{1}{2}(A^{-1} m_c, m_c) + (\lambda, m_c),$$

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where $\lambda \in \mathfrak{g}$ is a constant vector. Then Equation (2.2) becomes

\[
\dot{m}_c = ad_{A^{-1}m_c + \lambda}^* m_c, 
\]

and the velocity $\omega_c = A^{-1}m_c + \lambda$

In case of a right-invariant metric $m_c$ is constant, the Euler equations read $\dot{m}_s = -ad_{A^{-1}m_s + \lambda}^* m_s$, and the trajectory on the group is determined by the equation $R_{g^{-1}} \dot{g} = \omega_s$.

The result of the reduction onto the group is a vector field $v(g) \in TG$ such that the trajectory on the group is defined by the equation $\dot{g} = v(g)$. The field $v(g)$ will be referred to as reduced.

**Proposition 2.1. (The Whittaker reduction)** For the case of the left-invariant or the right-invariant metric, the vector field $v(g)$ has the form

\[
v(g) = L_g^*(A^{-1}Ad_g^*m_s + \lambda) \quad (2.5)
\]

and, respectively

\[
v(g) = R_g^*(A^{-1}Ad_{g^{-1}}^*m_c + \lambda). \quad (2.6)
\]

Here $m_s$, respectively $m_c$, is constant.

Notice that in Proposition 2.1 to find the reduced vector field we do not need the Hamiltonian equations on $T^*G$ and the explicit expression for the Noether integrals. We only need the Lie group structure and the inertial operator. This is important for generalizations to the infinite-dimensional case. Unlike for the Marsden-Weinstein reduction, we do not have to assume nondegeneracy conditions on the momenta $m_s$ or $m_c$.

**Proof.** We consider only the case of the left-invariant metric; for the right-invariant case the proof is similar. Relation (2.1) determines the function $m_c = m_c(m_s, g)$ on the group $G$ depending on $m_s$ as a
parameter. From the equality $\omega_c = A^{-1}m_c + \lambda$ and $L_{g^{-1}}\dot{g} = \omega_c$ follows that for any $g \in G$,

$$L_{g^{-1}}\dot{g} = A^{-1}m_c(m_s, g) + \lambda$$

which implies (2.5). \(\square\)

In Appendix A we consider the case, when the inertia operator is not left- or right-invariant, i.e, $A = A(g)$. Some nonholonomic systems have this form. It turns out that system of equations (2.3), (2.4) can still be reduced to the group $G$ (although now Equation (2.4) cannot be separated).

Even if $\lambda = 0$, the reduced vector fields (2.5), (2.6) are in general neither left- nor right-invariant. An important exception is when the inertia operator defines a Killing metric on the Lie algebra. However, the reduced covector fields are always right- or left-invariant.

**Proposition 2.2.** Let $\lambda = 0$. If the metric is left-invariant, then the reduced covector field $m(g) = A_g v(g)$ is right-invariant.

**Proof.**

$$m(g) = L_g^{-1}A L_{g^{-1}}v(g) = L_g^{-1}AA^{-1}L_g R_g^{-1}m_s = R_g^{-1}m_s.$$  

\(\square\)

Let $w(g) \in TG$ be a right-invariant vector field on the group $G$, which is defined by a vector $\xi \in g$: $w(g) = R_{g*}\xi$.

We fix a momentum $m_s$.

**Theorem 2.3.** For the momentum $m_s$ fixed, the vector field $w(g)$ on $G$ is a symmetry field of the reduced system $v(g)$ if and only if the vector $\xi$ satisfies the condition

$$ad^*_\xi m_s = 0.$$  \hspace{1cm} (2.7)
In the finite-dimensional case this means that the flows of the vector fields \( v(g) \), \( w(g) \) on the group commute. In the infinite-dimensional case one should be more accurate: the equation \( \dot{g} = v(g) \) is a partial integral-differential equation, rather than an ordinary differential equation, hence, strictly speaking, it is not clear if it has a solution. On the other hand, equation \( \dot{g} = R_{g*} \xi \) always has a solution, which is a one-parametric family of the left shifts on the group \( G: g \rightarrow (\exp \tau \xi)g \), see, for example, [27], as we have assumed that the exponential map exists. Notice also that, in view of Proposition 2.2 under the assumption of Theorem 2.3 the Lie derivative \( L_{w(g)}m(g) = 0 \).

\textbf{Proof of Theorem 2.3} The vector fields \( w(g) \) and \( L_{g*} \lambda \) commute, as right-invariant fields always commute with left-invariant fields. Thus, it is sufficient to show that

\[ v((\exp \tau \xi)g) = L_{(\exp \tau \xi)*}v(g) \]

if and only if the condition of the theorem is fulfilled. Indeed,

\[ v((\exp \tau \xi)g) = L_{(\exp \tau \xi)*}A^{-1}Ad_{(\exp \tau \xi)g}^*m_s \]
\[ = L_{\exp \tau \xi*}L_{g*}A^{-1}Ad_g^*(Ad_{\exp \tau \xi}^*m_s). \]

The last term equals \( L_{(\exp \tau \xi)*}v(g) \) for any \( g \in G \) if and only if

\[ Ad_{\exp \tau \xi}^*m_s = m_s \]

for all values of the parameter \( \tau \). Differentiating the last relation by \( \tau \) we arrive at the statement of the theorem. \( \Box \)

3 Stationary flows on Lie groups

We now formulate some results on the hydrodynamics character of the reduced vector fields from the previous section. Consider first the Euler equations for an ideal incompressible fluid, that flows on a Riemannian
manifold $M$:

$$\frac{\partial v}{\partial t} + \nabla_v v = -\nabla p, \quad \text{div} v = 0,$$

where $\nabla_v v$ is the covariant derivative of the fluid velocity vector $v$ by itself with respect to the Riemannian connection and $p$ is a pressure function.

Consider a geodesic vector field $u$ on the manifold $M$. Locally it always exists, but it may not be defined globally on $M$ – take a two-sphere as a simple example. Then $u$ is a stationary flow of the ideal fluid with a constant pressure. Indeed, as $u$ is a geodesic vector field, its derivative along itself is zero: $\nabla_u u = 0$.

**Remark.** The converse is of course not true: there are stationary flows that are not geodesics of the Riemannian metric.

The stationary flows with constant pressure form a background for hydrodynamics of Euler equations on Lie groups. Consider a Hamiltonian system on a finite-dimensional Lie group $G$, with a left-invariant Hamiltonian, which is quadratic in the momenta (in terms of Section 2, vector $\lambda = 0$). This Hamiltonian defines a left-invariant metric on the Lie group $G$. As we reduce this system to the group, the reduced vector field is globally defined on $G$, and is a geodesic vector field of the Riemannian metric, defined by the left-invariant Hamiltonian, and it defines a stationary flow of an ideal fluid on $G$.

Thus, the reduced vector field (2.5) (and (2.6)) is the velocity vector field for a stationary flow on the Lie group $G$ with left- (right-) invariant metric. An immediate corollary of Proposition 2.2 is

**Proposition 3.1.** There is an isomorphism between the stationary flows with constant pressure, defined by a left-invariant metric on a finite-dimensional Lie group $G$, and the space of right-invariant covector fields on this group.

**Remark.** Stationary flows with constant pressure play an important
role in studying the differential geometry of diffeomorphism groups, see \cite{5, 14, 25}: they define asymptotic directions on the subgroup of the volume-preserving diffeomorphisms of the group of all diffeomorphism. Proposition 3.1 is a generalization of \cite{26}, where it was shown that every left-invariant vector field on a compact Lie group equipped with a bi-invariant metric is asymptotic: if a Hamiltonian defines the bi-invariant metric on the Lie algebra, then the reduced vector field (2.5) is itself left-invariant. Moreover, its flow (which are right shifts on the Lie group $G$) are isometries of this metric (see, e.g., \cite{10}).

Recall now that the reduced covector field is right-invariant (Proposition 2.2). Thus, the condition $ad^{*}m_{s} = 0$ is equivalent to $L_{\eta}(m)(g) = 0$, where $m(g)$ is the right-invariant 1-form (being equal to $m_{s}$ at $g = id$), and $\eta(g) = R_{s\eta}^{*}\eta$ is the right-invariant symmetry field. By the homotopy formula,

$$0 = L_{\eta(g)}m(g) = i_{\eta(g)}dm(g) + d(\eta(g), m(g)) = i_{\eta(g)}dm(g),$$

as $(\eta(g), m(g)) = (\eta, m_{s}) = const$ for all $g$ (both vector and covector fields are right-invariant).

We now define a vortex vector field, as an annihilator of the vorticity 2-form. Then the condition $i_{\eta(g)}dm(g) = 0$ is exactly the definition of a vortex field. Thus, we have proved

**Proposition 3.2.** Any symmetry field to the reduced vector field is a vortex vector field.

Vortex vectors, i.e., vectors $\xi \in g$ that satisfy condition (2.7), are the isotropy vectors. We now review some classical results on the isotropy vectors and the Casimir functions, see, e.g., \cite{2} for details, and adapt them to our case.

**Proposition 3.3.** The distribution of the isotropy vectors in integrable.

The Proposition says that if vectors $\xi_{1}, \xi_{2} \in g$ satisfy condition (2.7), then the vector $[\xi_{1}, \xi_{2}] = ad_{\xi_{1}}\xi_{2}$ also satisfies this condition, which is a
simple consequence of the Jacobi identity. The integrable distribution of the isotropy vectors defines a manifold (at least locally), that we, following [18], call a vortex manifold.

The isotropy vectors $\xi \in \mathfrak{g}$ form a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, called an isotropy algebra for the coadjoint orbit $m = \text{Ad}_g^* m_s$. If the differentials of the Casimir functions form a basis of the isotropy algebra $\mathfrak{h}$, then $\mathfrak{h}$ is Abelian. In general, an isotropy algebra is not necessarily Abelian. A very simple example is $G = SO(3)$: if the “momentum in space” $m_s = 0$, then $\mathfrak{h} = \mathfrak{so}(3)$. However, in the finite-dimensional case isotropy algebras are Abelian on an open and dense set in $\mathfrak{g}^*$ (the Duflo theorem). Thus, the corresponding vortex manifolds (that pass through the group unity) are commutative subgroups of the Lie group $G$. Notice that in the infinite-dimensional case, vortex fields can still define a certain commutative subgroup, which can also be referred to as ”vortex manifold”.

Vortex manifolds have always the dimension of the same parity as the Lie group dimension. This is a simple corollary of the fact that coadjoint orbits are always even-dimensional (also the degenerate ones), see, e.g., [2].

One can show that if the Hamiltonian has also terms, linear in the momenta (in the other words, if $\lambda \neq 0$), then the reduced field has the following hydrodynamic sense: it is the velocity of the stationary flow for the electron gas, which satisfies an ”infinite conductivity equation”, again, with a constant pressure, see [3].

4 Control on Lie groups and vortex manifolds

Consider a Lagrangian system on a tangent bundle $T G$ to a Lie algebra $G$, with the Lagrangian, which is left-invariant under the action of the Lie group $G$. 
In order to introduce the controls in our system, we consider Lagrangians on $T \mathbb{G}$ of the following form:

$$L(\omega, u) = \frac{1}{2} \left( A(\omega + \sum_{i=1}^{k} u_i \lambda_i), \omega + \sum_{i=1}^{k} u_i \lambda_i \right),$$

where $\omega \in \mathfrak{g}$ is the system velocity, $\lambda_i \in \mathfrak{g}$ are constant vectors, $u_i(t) \in \mathbb{R}$ are controls, and $A : \mathfrak{g} \to \mathfrak{g}^*$ is the inertia operator. Notice that the dimension $k$ of the control vector $u(t)$ may be lower than the dimension of the Lie algebra. We assume that there is a positive constant $\epsilon$, such that $\|u(t)\| \leq \epsilon$, i.e., our controls are always bounded. Physically, these controls mean that we can change the system mass geometry by internal forces.

The Euler equations (2.4) are:

$$\dot{m} = ad^*_{\omega} m,$$

where the momentum $m = A(\omega + \sum_{i=1}^{k} u_i \lambda_i) \in \mathfrak{g}^*$. The system, reduced to the group $\mathbb{G}$, is (cf. (2.5):

$$\dot{g} = L_{g^*} \left( A^{-1} Ad^*_{g} m_s - \sum_{i=1}^{k} u_i \lambda_i \right) = v_\lambda(g). \quad (4.1)$$

From Theorem 2.3 and Proposition 3.3 follows the following result.

Suppose that System (4.1) is controllable (we formulate corresponding conditions in the Appendix B), and we assume that the controls $u(t)$ are piecewise constant functions. We fix the ”momentum in space” $m_s$: with $m_s$ fixed, so are the vortex manifolds.

**Theorem 4.1.** By applying controls $u(t)$, one can transform any vortex manifold $H_1$ to any other prescribed vortex manifold $H_2$, such that the following diagram is commutative:

$$
\begin{array}{ccc}
H_1 & \xrightarrow{g^*_\lambda} & H_2 \\
g^*_w \downarrow & & \downarrow g^*_w \\
H_1 & \xrightarrow{g^*_\lambda} & H_2,
\end{array}
$$

(4.2)
where by we denote vortex vector fields for the given momentum $m_s$, $g_{w}^s$ being its phase flow, and $g_{v_{\lambda}}^s$ is the phase flow of System (4.1).

This theorem is a reflection of a well-known fact that vortex lines are frozen into the flow of an ideal fluid.

**Proof.** By Theorem 2.3 the vector fields $v_{\lambda}(g)$ and the vortex fields $w(g)$ commute (the vortex fields are right-invariant, while the vectors $L_{g_{s}^n} \lambda_i$ are left-invariant, and we have also assumed that $u(t)$ is piecewise constant). Pick up the controls (i.e., functions $u(t)$), that send a point $h_1 \in H_1$ to a point $h_2 \in H_2$. Then the same controls send a point $g_{w}^s h_1$ to $g_{w}^s h_2$, due to commutativity, which proves the theorem. \( \square \)

A simple corollary is that all vortex manifolds, that correspond to the same value of the momentum $m_s$, are homotopic to each other. Another observation is that an electron gas, flowing on a Lie group, can be controlled by changing an external electro-magnetic field.

As an example, we consider the control problem for an $n$-dimensional rigid body with a fixed point in $\mathbb{R}^n$ ($n$-dimensional top). We follow the reduction procedure, suggested in [8].

Let $\mathfrak{so}(n)$ be the Lie algebra of $SO(n)$, $R \in SO(n)$ be the rotation matrix of the top, $\Omega_c = R^{-1} \dot{R} \in \mathfrak{so}(n)$ be its angular velocity in the moving axes, and $M_c \in \mathfrak{so}^*(n)$ be its angular momentum with respect to the fixed point of the top, which is also represented in the moving axes.

The angular momentum in space $M_s = Ad^*_{R^{-1}} M_c \equiv RM_c R^{-1}$ is a constant matrix, and the Euler equations have the following matrix form generalizing the classical Euler equations of the rigid body dynamics

$$\dot{M}_c + [\Omega_c, M_c] = 0.$$  \hspace{1cm} (4.3)

We assume, that the inertia operator of $A : \mathfrak{so}(n) \rightarrow \mathfrak{so}^*(n)$ is defined by the relation $\Omega_c = A^{-1} M = UM + MU$, where $U$ is any constant
nondegenerate operator. Thus the system \[ (4.3) \] is a closed system of \( n(n - 1)/2 \) equations, which was first written in an explicit form by F. Frahm (1874) [12]. As was shown in [22] (for \( n = 4 \), in [12]), with the above choice of the inertia tensor, the system \( (4.3) \) is a completely integrable Hamiltonian system on the coadjoint orbits of the group \( SO(n) \) in \( \mathfrak{so}^*(n) \).

Now we fix the angular momentum \( M_s \) (and, therefore, the coadjoint orbit) and assume that rank \( M_s = k \leq n \) (\( k \) is even). Then, according to the Darboux theorem (see, e.g., [2]), there exist \( k \) mutually orthogonal and fixed in space vectors \( x^{(l)}, y^{(l)} \), \( l = 1, \ldots, k/2 \) such, that \( |x^{(l)}|^2 = |y^{(l)}|^2 = h_l \), \( h_l = \text{const} \), and the momentum can be represented in the form

\[
M_s = \sum_{l=1}^{k/2} x^{(l)} \wedge y^{(l)}, \quad \text{that is} \quad M_s = \mathcal{X}^T \mathcal{Y} - \mathcal{Y}^T \mathcal{X}, \quad (4.4)
\]

where \( \mathcal{X}^T = (x^{(1)} \cdots x^{(k/2)}) \), \( \mathcal{Y}^T = (y^{(1)} \cdots y^{(k/2)}) \), \( x^{(l)} \wedge y^{(l)} = x^{(l)} \otimes y^{(l)} - y^{(l)} \otimes x^{(l)} \), and \( (\cdot)^T \) denotes transposition. Under these conditions on \( x^{(l)}, y^{(l)} \) the set of \( k \times n \) matrices \( Z = (x^{(1)} y^{(1)} \cdots x^{(k/2)} y^{(k/2)})^T \) forms the Stiefel variety \( \mathcal{V}(k, n) \) (see, for example, [10]).

The momentum in the body \( M_c \) has the same expression as \( (4.4) \), but here the components of matrices \( \mathcal{X}, \mathcal{Y} \) are taken in a frame attached to the body, see [22].

Since the above vectors are fixed in space, in the moving frame they satisfy the Poisson equations, which are equivalent to matrix equations

\[
\dot{\mathcal{X}} = \mathcal{X} \Omega_c, \quad \dot{\mathcal{Y}} = \mathcal{Y} \Omega_c. \quad (4.5)
\]

Now we set \( \Omega_c = U M_c + M_c U \) and substitute this expression into \( (4.5) \). Then taking into account \( (4.4) \), we obtain the following dynamical system on \( \mathcal{V}(k, n) \)

\[
\begin{align*}
\dot{\mathcal{X}} &= \mathcal{X}[U(\mathcal{X}^T \mathcal{Y} - \mathcal{Y}^T \mathcal{X}) + \mathcal{X}^T \mathcal{Y} U] , \\
\dot{\mathcal{Y}} &= \mathcal{Y}[U(\mathcal{X}^T \mathcal{Y} - \mathcal{Y}^T \mathcal{X}) - \mathcal{Y}^T \mathcal{X} U]. \quad (4.6)
\end{align*}
\]
Notice that in the case of maximal rank \( k \) (\( k = n \) or \( k = n - 1 \)), the Stiefel variety is isomorphic to the group \( SO(n) \), and the components of vectors \( x^{(1)} \, y^{(1)} \ldots x^{(k/2)} \, y^{(k/2)} \) form redundant coordinates on it. Thus the system (4.6) describes required reduced flow (2.5) on \( SO(n) \).

The representation (4.4) is not unique: rotations in 2-planes spanned by the vectors \( x^{(l)}, y^{(l)} \) in \( \mathbb{R}^n \) (and only they), leave the angular momentum \( M \) invariant (in the case of the maximal rank). As a result, the system (4.6) on \( SO(n) \) has \( k/2 \) vortex vector fields \( w_1(g), \ldots, w_{k/2}(g) \), which are generated by the right shifts of vectors \( \xi^l \in so(n) \), such that \( ad_{\xi^l} M_s \equiv [\xi^l, M] = 0 \), cf. Section 3. In the redundant coordinates the fields take the form

\[
\dot{x}^{(l)} = (x^{(l)}, x^{(l)}) y^{(l)}, \quad \dot{y}^{(l)} = -(y^{(l)}, y^{(l)}) x^{(l)}, \quad l = 1, \ldots, k/2.
\]

One can easily see that in the case of maximal rank of the momentum matrix, the corresponding vortex manifolds are \( k/2 \)-dimensional tori. This is a general fact: if a Lie group is compact, then the vortex manifolds are compact manifolds, and, by the Duflo theorem, for a dense set of the momenta \( m_s \), the vortex manifolds are tori (in our case, this dense set is determined by the condition that the momentum rank is maximal). The torus, that passes through the group unity, is called the maximal torus for the Lie group; maximal tori play an important role in classification of compact Lie groups.

One can furthermore show that if the rank of the momentum is not maximal, the vortex manifolds would be products of a torus and a certain \( SO(m) \) Lie group.

We now introduce the controls in System (4.6) by the above scheme. Using Equation (4.1) and the fact that any left-invariant vector field on the Lie group \( SO(n) \) in our redundant coordinates can be written as

\[
\dot{\mathbf{x}} = \mathbf{X}\Lambda, \quad \dot{\mathbf{y}} = \mathbf{Y}\Lambda, \quad \Lambda \in so(n),
\]
we get at once the following controlled system on the group:

\[ \dot{X} = \mathcal{X}[U(X^T Y - Y^T X) + X^T Y U] - \mathcal{X} \left( \sum_i u_i \Lambda_i \right), \]
\[ \dot{Y} = \mathcal{Y}[U(X^T Y - Y^T X) - Y^T X U] - \mathcal{Y} \left( \sum_i u_i \Lambda_i \right). \] (4.7)

This system describes an \( n \)-dimensional rigid body with "symmetric flywheels", which is a direct generalization of the Liouville problem of the rotation of a variable body [21].

**Proposition 4.2.** On can choose two vectors \( \Lambda_1 \) and \( \Lambda_2 \), such that for any choice of the inertia operator \( U \), one can transform any vortex manifold to any other vortex manifold for any momentum in space \( M_s \), using the corresponding two control functions \( u_1(t) \) and \( u_2(t) \).

**Proof.** First, we notice that System (4.6) preserves volume in the phase space of the redundant variables \( X, Y \) (this can be checked by the direct computation, but the general result of the existence of an invariant measure for a reduced system (2.5) or (2.6) with \( \lambda = 0 \) follows from [18]). As the Lie algebra \( \mathfrak{so}(n) \) is semi-simple, controllability of System (4.7) follows from Corollary B.1, Appendix B. Proposition 4.2 follows now from Theorem 4.1. \( \square \)

## 5 Conclusion and acknowledgements

In this article, we considered the reduction of geodesic flows of left- or right-invariant metrics on Lie groups to the group. The reduced vector field has a remarkable hydrodynamic interpretation: it is a velocity field for a stationary flow of an ideal fluid, the the right- or left-invariant symmetry fields of the reduced field being vortex vector fields, i.e., they annihilate the vorticity 2-form. The distribution of the vortex fields is always integrable, thus it defines a manifold (at least locally), that we call a vortex manifold. Typically, the vortex manifolds are tori.
We studied the following control problem. Consider a mechanical system, whose configuration space is a Lie group and whose Lagrangian is invariant to left translations on that group, and assume that the mass geometry of the system may change under the action of internal control forces. Such system can also be reduced to the Lie group; with no controls, it describes a geodesic flow of the left-invariant metric, given by the Lagrangian, and thus its reduced flow is a stationary flow of an ideal fluid.

The control problem for such system is to find the conditions, under which the system can be brought from any initial position in the configuration space to another preassigned position by changing its mass geometry. We showed that under these conditions, by changing the mass geometry, one can also bring one vortex manifold to any other preassigned vortex manifold. As an example, we considered the $n$-dimensional Euler top. We wrote down the reduced controlled system explicitly, showed that the vortex manifolds are tori, and proved that, by changing the mass geometry, every such torus can be transformed to any other torus.

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Appendix

A Reduction to the Lie group for nonholonomic systems

Consider the following equations, that we will refer to as the generalized Euler equations (the left-invariant case):

\[
\dot{m}_c = \text{ad}_{A(g)}^* m_c, \quad (A.1)
\]

\[
L_{g^{-1}} \dot{g} = A(g) m_c. \quad (A.2)
\]
Here $A(g) : g^* \rightarrow g$ is positive definite symmetric operator.

**Example.** Consider the Chaplygin problem of a rigid ball rolling on a horizontal plane. The equations of motion are:

\[
\begin{align*}
\dot{M} &= M \times \omega, \quad \dot{\gamma} = \gamma \times \omega, \\
M &= I \omega + D \gamma \times (\omega \times \gamma),
\end{align*}
\]

where $M$ is the ball momentum with respect to the moving axes, fixed in the ball (momentum in the body), $\omega$ is the angular velocity, $\gamma$ is the unit vertical vector, also written with respect to the moving axes, matrix $I$ is the inertia tensor and $D$ is a constant. One can see that these equations are of the form (A.1).

**Theorem A.1. (The Euler theorem)** The momentum in space $m_s$ is constant for the generalized Euler equations (A.1-A.2).

**Proof.** Differentiate relation (2.1) by time and apply (A.2), cf. [3]. □

Proposition 2.1 relied only on relation (2.1), which turns out to be true also for this case. Thus, reduction to the group is possible, and the reduced vector field is

\[
\begin{align*}
v(g) &= L_{g^*}A(g)Ad^*_g m_s \quad \text{(A.3)} \\
v(g) &= R_{g^*}A(g)Ad^*_{g^{-1}} m_c \quad \text{(A.4)}
\end{align*}
\]

correspondingly in the left- or right-invariant case. Here $m_s$, respectively $m_c$, is constant.

It would be interesting to find hydrodynamic description of the reduced field.
\section*{B \ Controllability conditions}

As system (4.1) has the standard form
\[ \dot{x} = f(x) + \sum_{i} u_{i} g_{i}(x), \quad |u(t)| \leq \epsilon \]  
(B.1)
of a classical control system, one can apply general theorems to it.

\textbf{Theorem B.1.} Let the Lie group $G$ be compact. Then system (4.1) is controllable for all $\epsilon > 0$, if the minimal Lie subalgebra of vector fields on $G$, which contains both vector fields $L_{g} \lambda_{i}$ and $v(g)$, spans the tangent space $T_{g}G$ at any $g \in G$.

From Theorem [B.1] follows, that the minimal number of controls is mainly defined by the Lie algebra structure. This minimal number of controls should not necessarily be equal to the number of the degrees of freedom of the system \textit{unless the Lie algebra is commutative}. If the dimension of the Lie algebra $g$ is greater than 1, and we are interested in controllability for \textit{all} inertia operators and all momenta $m_{s}$, then the minimal number of controls should necessarily be greater than 1.

\textbf{Example.} For controllability of a reduced Euler top on $SO(3)$ it is enough to have only one control, provided all the principal axes of the inertia ellipsoid are different, and the momentum is the space $m_{s} \neq 0$ is not directed along any of the principal axes – see [23]. Obviously, if the ellipsoid of inertia is a sphere, then the minimal number of controls is 2.

\textbf{Proof.} We only have to check that the vector field
\[ v(g) = L_{g} \cdot A^{-1} \text{Ad}_{g}^{*} m_{s} \]
is Poisson-stable (i.e., almost all trajectories come back to the vicinity of the initial conditions infinitely many times). Indeed, if the minimal Lie subalgebra, which contains vectors $L_{g} \lambda_{i}$ and $v(g)$, spans the tangent space $T_{g}G$ at any $g \in G$, then for each point $g_{0} \in G$, the set of
points \( g(t, g_0, u(t)) \), accessible by the controls \( u(t) \) for \( 0 < t < T \), form an open set (the point \( g_0 \) itself may belong to the boundary). Under the condition of the Poisson stability, these sets can be joined together to get the necessary trajectory, see, e.g., [20] for details. The Poisson stability follows from the existence of a smooth invariant measure of the reduced system \( \dot{g} = v(g) \), as we have assumed that the group \( G \) is compact. But this is exactly the case: if the Lie group \( G \) is compact, the reduced system always preserves a bi-invariant Haar measure on \( G \), see [18]. □

**Corollary B.1.** Under conditions of Theorem [B.1], let the Lie algebra \( g \) be real and semi-simple. Then two controls is sufficient for controllability for all values of the inertia operator and of the momentum \( m_s \).

**Proof.** It is well known that a real semisimple Lie algebra is generated by 2 elements, see, e.g., [6]. □

**Remark.** If the momentum \( m_s = 0 \), then under conditions of Theorem [B.1] system (4.1) is controllable even if the Lie group is noncompact – this is the classical Rashevsky-Chow theorem, see, e.g., [13].

The condition of Theorem [B.1] is usually referred to as the Lie algebra rank condition (see, e.g., [23]). In real systems, it may be difficult to check it directly, as, in principle, the number of commutators one has to take is not bounded from above. We suggest using a “transversality” condition, which is in the next section.

**C Transversality conditions**

The Lie algebra rank condition may be difficult to check, as the number of commutators one should take to check it is not bounded from below. We introduce a “transversality” condition, which seems to be easier to verify in applications. This condition will also provide stronger con-
trollability results (in the non-analytic case): we give a (rather trivial) example, when the Lie algebra rank condition is not fulfilled, while the system is controllable.

Suppose that $N$ is a domain or a submanifold of $M$. We call a vector field $v(x)$ transversal to $N$, if every phase trajectory of the field $v(x)$, that starts in $N$ at $t = 0$, escapes from $N$ both for some $t < 0$ and some $t > 0$. Notice that it is enough to claim only one inequality (i.e., for example, say that the trajectory leaves $N$ for some $t > 0$), if $v(x)$ preserves measure on $M$ and both $M$ and $N$ are compact: if at least one inequality is fulfilled, there may not be any stationary points of the field $v$ in $N$.

By a finite system of commutators we will understand a system of the vector fields, which consists of the original fields $f(x), g_i(x)$ and a finite number of vector fields, obtained by taking some fixed number of commutators of $f, g_i, [f, g_i]$, etc. If the finite system of commutators has rank $n$ at least in one point of $M$, then it has rank $n$ almost everywhere on $M$.

**Theorem C.1.** Suppose that all functions are analytic, and there exists a finite system of commutators, which has rank $n$ on $M/N$, $N \in M$. Suppose that the field $f(x)$ is transversal to $N$. Then the Lie algebra rank condition is fulfilled.

**Proof.** Obviously the dimension of $N$ is less than $n$ – due to the analyticity condition. At any point $x_0 \in N$, take local coordinates, such that $f(x) = (1, 0, \ldots, 0)$, and $N$ is given by condition $x_1 = 0$.

We give a proof in a simple situation, when the rank of a finite system of commutators falls by 1 on $N$. At $x = x_0$, let $h_1, \ldots, h_{n-1}$ be independent vector fields, obtained as linear combinations of $g_i, [f, g_i]$, etc., such that on $N$, $h_j^2 = 0$ for all $j$. Let the rank of the vector fields $f, h_2, \ldots, h_{n-1}$ be $n - 1$ at $x = x_0$. Then the rank of the vector fields $f, [f, [f, \ldots [f, h_1] \ldots]], h_2, \ldots, h_{n-1}$ is $n$ at $x = x_0$: otherwise, all the
derivatives $\partial^k h_1^2 / \partial x_1^k = 0$, while $h_1^2 \neq 0$ for $x_1 \neq 0$. □

**Example.** Consider the vector fields

$$f(x) = (1, 0), \ g(x) = (0, 1 - \cos(x_1)); \quad x = (x_1 \mod 2\pi, x_2).$$

These fields are independent only for $x_1 \neq 0$. The vector field $f$ is transversal to the line $x_1 = 0$, and one can easily check that the vector fields $f$ and $[f, [f, g]]$ are independent in the neighbourhood of $x_1 = 0$.

In the non-analytic case, Theorem C.1 is not true (we give a simple example below). However, a system may still be controllable, even if the rank condition is not satisfied. One can for example imagine a situation, when a system cannot be controlled on some domain of the phase space.

**Theorem C.2.** Let $M$ be compact, and let a finite system of commutators span the tangent space $T_x M$ at any $x \in M/N, N \in M$. Suppose that the vector field $f(x)$ is transversal to $N$. Then system (B.1) is controllable for all $\epsilon > 0$.

**Proof.** Under the conditions of the theorem, we can reach every point $x \in M/N$. For any point $x_0 \in N$, take a phase trajectory that starts at $x_1 \in M/N$ and passes through $x_0$, which exists due to the transversality condition. □

**Example.** Consider the following system:

$$\dot{x}_1 = 1, \quad \dot{x}_2 = u(t)g(x_1),$$

where we take $x_1 \mod 2\pi$. The function $u(t)$ is a control, which satisfies the condition $|u(t)| < \epsilon$, and

$$g(x_1) = 0 \quad \text{for} \quad 0 < x_1 < \pi, \quad g(x_1) = \sin x_1 \quad \text{for} \quad \pi < x_1 < 2\pi.$$

The vector field $(1, 0)$ is transversal to the domain $0 \leq x \leq \pi$, and is obviously Poisson-stable. This system is controllable for all $\epsilon > 0$. 

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References

[1] Arnold V.I. Mathematical methods in classical mechanics. Springer-Verlag.
[2] Arnold V.I., Givental A.B. Symplectic Geometry. Dynamical Systems 4, Springer-Verlag.
[3] Arnold V.I., Khesin B.A. Topological Methods in Hydrodynamics. Springer-Verlag. 1998.
[4] Arnold V.I., Kozlov V.V., Neishtadt A.I. Mathematical aspects of classical and celestial mechanics. Springer-Verlag, 1988. 291 p.
[5] Bao D., Ratiu T. On the geometrical origin and the solutions of a degenerate Monge–Ampère equation. Proc. Symp. Pure Math. AMS, Providence 54 (1993), 55–68.
[6] N. Bourbaki, Groupes et algèbres de Lie, Hermann, 1975, Chapitres 7 et 8.
[7] Cardetti, F., Mittenhuber, D. Local controllability for linear systems on Lie groups. J. Dyn. Contr. Sys., 11, 3, July 2005, 353-373.
[8] Deryabin M.V., Fedorov Yu.N. On Reductions on Groups of Geodesic Flows with (Left-) Right-Invariant Metrics and Their Fields of Symmetry. Doklady Mathematics. Interperiodica Translation, 68, 1, (2003) 75–78.
[9] Deryabin M.V. Ideal hydrodynamics on Lie groups. Physica D, 221 (2006), 84–91.
[10] Dubrovin B.A., Novikov S.P., Fomenko A.T. Modern geometry. Vol. 2. Springer.
[11] Fedorov Yu. N. Integrable flows and Bäcklund transformations on extended Stiefel varieties with application to the Euler top on the Lie group SO(3), J. Nonlinear Math. Phys. 12 (suppl. 2) (2005) 7794.
[12] Frahm F. Über gewisse Differentialgleichungen. *Math. Ann.* **8** (1874), 35–44

[13] Hermann, R. (1968). Accessibility problems for path systems (2nd ed.). In *Differential geometry and the calculus of variations* (pp. 241-257). Brookline, MA: Math Sci Press.

[14] Khesin B., Misiolek G. Asymptotic Directions, Monge-Ampère Equations and the Geometry of Diffeomorphism Groups. *Journal of Math. Fluid Mech.* **7** (2005) 365-375.

[15] Khesin B.A., Chekanov Yu.Y. Invariants of the Euler equations for ideal or barotropic hydrodynamics and superconductivity in d dimensions. *Phys. D.* **40** (1989), 119–131.

[16] Kozlov V.V. Hydrodynamics of Hamiltonian systems. *Vestn. Moskov. Univ., Ser. I. Mat. Mekh.*, No. 6 (1983) 10–22 (Russian)

[17] Kozlov V.V. The vortex theory of the top. *Vestn. Moskov. Univ., Ser. I. Mat. Mekh.* No. 4 (1990) 56–62 (Russian)

[18] Kozlov V.V. Dynamical systems X. General vortex theory. Springer-Verlag, 2003.

[19] Kozlov V.V. Dynamics of variable systems and Lie groups. *J. Appl. Math. Mech.* **68** (2004), 803–808

[20] Lian, K. Y., Wang, L. S., and Fu, L. C. Controllability of spacecraft systems in a central gravitational field. *IEEE Transactions on Automatic Control*, **39** (12), (1994), 2426-2441.

[21] Liouville, J., Deveoppements sur un chapitre de la ”Mechanique” de Poisson. *J. Math. Pures et Appl.*, (1858), **3**, 1-25.

[22] Manakov S.V. Note on the integration of Euler’s equations of the dynamics of an n-dimensional rigid body. *Funkts. Anal. Prilozh.* **10** (1976), 93–94. English transl. in: *Funct. Anal. Appl.* **10** (1976), 328–329.

[23] Manikondaa V., Krishnaprasad P.S. Controllability of a class of underactuated mechanical systems with symmetry. *Automatica* **38** (2002), 1837-1850.
[24] Marsden J.E., Weinstein A. Reduction of symplectic manifolds with symmetry. *Rep. Math. Phys.* **5** (1974), 120–121.

[25] Misiolek G. Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms, *Indiana Univ. Math. J.* **42** (1993), 215-235.

[26] B. Palmer. The Bao-Ratiu equations on surfaces. *Proc. Roy. Soc. London Ser. A* **449** (1995), no. 1937, 623-627.

[27] Warner F.W. Foundations of differentiable manifolds and Lie groups. Springer-Verlag, 1983.

[28] Whittaker E.T. A treatise on analytical dynamics. 4-d ed. , Cambridge Univ. Press, Cambridge 1960