Average degrees of edge-chromatic critical graphs

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Abstract

Given a graph $G$, denote by $\Delta$, $\bar{d}$ and $\chi'$ the maximum degree, the average degree and the chromatic index of $G$, respectively. A simple graph $G$ is called edge-$\Delta$-critical if $\chi'(G) = \Delta + 1$ and $\chi'(H) \leq \Delta$ for every proper subgraph $H$ of $G$. Vizing in 1968 conjectured that if $G$ is edge-$\Delta$-critical, then $\bar{d} \geq \Delta - 1 + \frac{3}{\pi}$. We show that

$$\bar{d} \geq \begin{cases} 0.69241\Delta - 0.15658 & \text{if } \Delta \geq 66, \\ 0.69392\Delta - 0.20642 & \text{if } \Delta = 65, \text{ and} \\ 0.68706\Delta + 0.19815 & \text{if } 56 \leq \Delta \leq 64. \end{cases}$$

This result improves the best known bound $\frac{2}{3}(\Delta + 2)$ obtained by Woodall in 2007 for $\Delta \geq 56$. Additionally, Woodall constructed an infinite family of graphs showing his result cannot be improved by well-known Vizing’s Adjacency Lemma and other known edge-coloring techniques. To overcome the barrier, we follow the recently developed recoloring technique of Tashkinov trees to expand Vizing fans technique to a larger class of trees.

Keywords: edge-$k$-coloring; edge-critical graphs; Vizing’s Adjacency Lemma

1 Introduction

All graphs in this paper, unless otherwise stated, are simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $\Delta$ the maximum degree of $G$. An edge-$k$-coloring of a graph $G$ is a mapping $\varphi : E(G) \to \{1, 2, \cdots, k\}$ such that $\varphi(e) \neq \varphi(f)$ for any two adjacent edges $e$ and $f$. We call $\{1, 2, \cdots, k\}$ the color set of $\varphi$. Denote by

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\( \mathcal{C}^k(G) \) the set of all edge-\( k \)-colorings of \( G \). The chromatic index \( \chi'(G) \) is the least integer \( k \geq 0 \) such that \( \mathcal{C}^k(G) \neq \emptyset \). We call \( G \) class one if \( \chi'(G) = \Delta \). Otherwise, Vizing’ theorem [12] gives \( \chi'(G) = \Delta + 1 \) and \( G \) is said to be of class two. An edge \( e \) is called critical if \( \chi'(G - e) < \chi'(G) \), where \( G - e \) is the subgraph obtained from \( G \) by removing the edge \( e \). A graph \( G \) is called edge-\( \Delta \)-critical if \( \chi'(G) = \Delta + 1 \) and \( \chi'(H) \leq \Delta \) holds for any proper subgraph \( H \) of \( G \). Clearly, if \( G \) is edge-\( \Delta \)-critical, then \( G \) is connected and \( \chi'(G - e) = \Delta \) for any \( e \in E(G) \). Let \( \overline{d}(G) \) denote the average degree of a graph \( G \). Vizing [14] made the following conjecture in 1968, which is commonly referred as Vizing’s Average Degree Conjecture.

**Conjecture 1.** [Vizing [14]] If \( G \) is an edge-\( \Delta \)-critical graph of \( n \) vertices, then \( \overline{d}(G) \geq \Delta - 1 + \frac{3}{n} \).

The conjecture has been verified for \( \Delta \leq 6 \), see [3, 5, 6, 8]. For arbitrary \( \Delta \), there are a few results on the lower bound for \( \overline{d}(G) \). Let \( G \) be an edge-\( \Delta \)-critical graph. Fiorini [2] showed, for \( \Delta \geq 2 \),

\[
\overline{d}(G) \geq \begin{cases} 
\frac{1}{2}(\Delta + 1) & \text{if } \Delta \text{ is odd;} \\
\frac{1}{2}(\Delta + 2) & \text{if } \Delta \text{ is even.}
\end{cases}
\]

Haile [4] improved the bounds as follows.

\[
\overline{d}(G) \geq \begin{cases} 
\frac{3}{5}(\Delta + 2) & \Delta = 9, 11, 13; \\
\frac{\Delta + 6}{15 + \sqrt{2\Delta}} - \frac{12}{\Delta + 4} & \Delta \geq 10, \Delta \text{ is even;} \\
\frac{\Delta + 7}{2} - \frac{16}{\Delta + 9} & \Delta = 15; \\
\frac{\Delta + 7}{2} & \Delta \geq 17, \Delta \text{ is odd.}
\end{cases}
\]

Sanders and Zhao [9] showed \( \overline{d}(G) \geq \frac{1}{2}(\Delta + \sqrt{2\Delta} - 1) \) for \( \Delta \geq 2 \). Woodall [16] improved the bound to \( \overline{d}(G) \geq \frac{t(\Delta + t - 1)}{2t - 1} \), where \( t = \lceil \sqrt{\Delta/2} \rceil \). Improving Vizing’s Adjacency Lemma, Woodall [15] improved the coefficient of \( \Delta \) from \( \frac{1}{2} \) to \( \frac{2}{3} \) as follows.

\[
\overline{d}(G) \geq \begin{cases} 
\frac{2}{3}(\Delta + 1) & \text{if } \Delta \geq 2; \\
\frac{2}{3}\Delta + 1 & \text{if } \Delta \geq 8; \\
\frac{2}{3}(\Delta + 2) & \text{if } \Delta \geq 15.
\end{cases}
\]

In the same paper, Woodall provided the following example demonstrating that the above result cannot be improved by the use of his new adjacency Lemmas (see Lemma 2 and Lemma 3) and Vizing’s Adjacency Lemma alone.
Let $G$ be a graph comprising $k$ vertices of degree 4, all of whose neighbors have degree $\Delta$, and $2k$ vertices of degree $\Delta$, each of which is adjacent to two vertices of degree 4 and $\Delta - 2$ vertices of degree $\Delta$. Graph $G$ can be chosen to be triangle-free, and indeed to have arbitrarily large girth. Then $G$ may not be edge-\(\Delta\)-critical, but it satisfies the conclusions of all the existing lemmas at that time including two mentioned above, and it has average degree $\frac{2}{3}(\Delta + 2)$. So, using these known results, it is impossible to prove that the example is not edge-\(\Delta\)-critical. On the other hand, we note that using our new result, Claim 3.4 in Section 3, it is readily seen that if $\Delta \geq 6$ then the above example is not edge-\(\Delta\)-critical. By proving a few stronger properties of edge-\(\Delta\)-critical graphs, we improve Woodall’s result as below for $\Delta \geq 56$.

**Theorem 1.** If $G$ is an edge-\(\Delta\)-critical graph, then

$$
d(G) \geq \begin{cases} 0.69241\Delta - 0.15658 & \text{if } \Delta \geq 66, \\ 0.69392\Delta - 0.20642 & \text{if } \Delta = 65, \text{ and} \\ 0.68706\Delta + 0.19815 & \text{if } 56 \leq \Delta \leq 64. \end{cases}
$$

We will prove a few technic lemmas in Section 2 and give the proof of Theorem 1 in Section 3. We will use the following terminology and notation. Let $G$ be a graph. Denote by $N(x)$ the neighborhood of $x$ for any $x \in V(G)$, and $d(x)$ the degree of $x$, i.e., $d(x) = |N(x)|$. For any nonnegative integer $m$, we call a vertex $x$ an $m$-vertex if $d(x) = m$, a ($< m$)-vertex if $d(x) < m$, and ($> m$)-vertex if $d(x) > m$. Correspondingly, we call a neighbor $y$ of $x$ an $m$-neighbor if $d(y) = m$, etc. Let $k$ be a positive integer such that $C_k(G - e) \neq \emptyset$, and let $\varphi \in C_k(G - e)$ and $v \in V(G)$. Let $\varphi(v) = \{\varphi(e) : e \text{ is incident with } v\}$ and $\bar{\varphi}(v) = \{1, \ldots, k\} \setminus \varphi(v)$. We call $\varphi(v)$ the set of colors seen by $v$ and $\bar{\varphi}(v)$ the set of colors missing at $v$. A set $X \subseteq V(G)$ is called **elementary** with respect to $\varphi$ if $\bar{\varphi}(u) \cap \bar{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. For any color $\alpha$, let $E_\alpha$ denote the set of edges assigned color $\alpha$. Clearly, $E_\alpha$ is matching of $G$. For any two colors $\alpha$ and $\beta$, the components of induced by edges in $E_\alpha \cup E_\beta$, named $(\alpha, \beta)$-chains, are even cycles and paths with alternating color $\alpha$ and $\beta$. For a vertex $v$ of $G$, we denote by $P_v(\alpha, \beta, \varphi)$ the unique $(\alpha, \beta)$-chain that contains the vertex $v$. Let $\varphi/P_v(\alpha, \beta, \varphi)$ denote the edge-$k$-coloring obtain from $\varphi$ by switching colors $\alpha$ and $\beta$ on the edges on $P_v(\alpha, \beta, \varphi)$.

## 2 Lemmas

Let $q$ be a positive number, $G$ be an edge-$\Delta$-critical graph and $x \in V(G)$. For each $y \in N(x)$, let $\sigma_q(x, y) = |\{z \in N(y) \setminus \{x\} : d(z) \geq q\}|$, the number of neighbors of
y (except x) with degree at least q. Vizing studied the case \( q = \Delta \) and obtained the following result.

**Lemma 1.** [Vizing’s Adjacency Lemma [13]] Let \( G \) be an edge-\( \Delta \)-critical graph. Then \( \sigma_\Delta(x, y) \geq \Delta - d(x) + 1 \) holds for every \( xy \in E(G) \).

Woodall [15] studied \( \sigma_q(x, y) \) for the case \( q = 2\Delta - d(x) - d(y) + 2 \) and obtained the following two results. For convention, we let \( \sigma(x, y) = \sigma_q(x, y) \) when \( q = 2\Delta - d(x) - d(y) + 2 \).

**Lemma 2.** [Woodall [15]] Let \( xy \) be an edge in an edge-\( \Delta \)-critical graph \( G \). Then there are at least \( \Delta - \sigma(x, y) \geq \Delta - d(y) + 1 \) vertices \( z \in N(x) \setminus \{y\} \) such that \( \sigma(x, z) \geq 2\Delta - d(x) - \sigma(x, y) \).

Let \( x \) be a vertex in a graph \( G \) and \( y \in N(x) \). Vizing’s Adjacency Lemma shows that \( \sigma_\Delta(x, y) \geq \Delta - d(x) + 1 \). So, \( \sigma(x, y) \geq \Delta - d(x) + 1 \). Woodall studied their difference through the following two parameters.

\[
p_{\text{min}}(x) := \min_{y \in N(x)} \sigma(x, y) - \Delta + d(x) - 1 \quad \text{and} \quad p(x) := \min\{ p_{\text{min}}(x), \left\lceil \frac{d(x)}{2} \right\rceil - 1 \}.
\]

Clearly, \( p(x) < d(x)/2 - 1 \). As a corollary, the following lemma shows that there are about \( d(x)/2 \) neighbors \( y \) of \( x \) such that \( \sigma(x, y) \geq \Delta/2 \). In general, for any positive number \( q \) with \( q \leq \Delta \), we define the following two parameters.

\[
p_{\text{min}}(x, q) := \min_{y \in N(x)} \sigma_q(x, y) - \Delta + d(x) - 1 \quad \text{and} \quad p(x, q) := \min\{ p_{\text{min}}(x, q), \left\lceil \frac{d(x)}{2} \right\rceil - 3 \}.
\]

**Lemma 3.** [Woodall [15]] Every vertex \( x \) in an edge-\( \Delta \)-critical graph has at least \( d(x) - p(x) - 1 \) neighbors \( y \) for which \( \sigma(x, y) \geq \Delta - p(x) - 1 \).

When \( d(x) \leq \Delta - 4 \), we generalize the above results by allowing \( q \) taking various values and obtain the following two results, which serve as key ingredients in our proof of Theorem 1.

**Lemma 4.** Let \( xy \) be an edge in an edge-\( \Delta \)-critical graph \( G \) and \( q \) be a positive number. If \( \Delta/2 < q \leq \Delta - d(x)/2 - 2 \), then \( x \) has at least \( \Delta - \sigma_q(x, y) - 2 \) vertices \( z \in N(x) \setminus \{y\} \) such that \( \sigma_q(x, z) \geq 2\Delta - d(x) - \sigma_q(x, y) - 4 \).
Due to its length, the proof of Lemma 4 will be placed at the end of this section. The following is a consequence of it.

**Lemma 5.** Let \( G \) be an edge-\( \Delta \)-critical graph, \( x \in V(G) \) and \( q \) be a positive number. If \( \Delta/2 < q \leq \Delta - d(x)/2 - 2 \), then \( x \) has at least \( d(x) - p(x, q) - 3 \) neighbors \( y \) for which \( \sigma_q(x, y) \geq \Delta - p(x, q) - 5 \).

**Proof.** Let \( y \in N(x) \) such that \( p_{\min}(x, q) = \sigma_q(x, y) - \Delta + d(x) - 1 \).

If \( p(x, q) = p_{\min}(x, q) \), by Lemma 4, \( x \) has at least \( \Delta - \sigma_q(x, y) - 2 = \Delta - (\Delta - d(x) + p_{\min}(x, q) + 1) - 2 = d(x) - p_{\min}(x, q) - 3 \) vertices \( z \in N(x) \setminus \{y\} \) such that \( \sigma_q(x, z) \geq 2\Delta - d(x) - 4 - \sigma_q(x, y) = \Delta - p_{\min}(x, q) - 5 \).

If \( p(x, q) = \left\lfloor \frac{d(x)}{2} \right\rfloor - 3 < p_{\min}(x, q) \), then for every \( y \in N(x) \), \( \sigma_q(x, y) > \Delta - d(x) + 1 + \left\lfloor \frac{d(x)}{2} \right\rfloor - 3 \geq \Delta - \left\lfloor \frac{d(x)}{2} \right\rfloor - 3 = \Delta - p(x, q) - 6 \). So \( \sigma_q(x, y) \geq \Delta - p(x, q) - 5 \). \( \square \)

Let \( G \) be a graph (in this paragraph, \( G \) may be a multigraph), \( e_1 = y_0y_1 \in E(G) \) and \( \varphi \in C^k(G - e_1) \). A Tashkinov tree \( T \) is a sequence \( T = (y_0, e_1, y_1, \ldots, e_p, y_p) \) with \( p \geq 1 \) consisting of edges \( e_1, e_2, \ldots, e_p \) and vertices \( y_0, y_1, \ldots, y_p \) such that the following two conditions hold:

- The edges \( e_1, e_2, \ldots, e_p \) are distinct and \( e_i = y_ry_i \) for each \( 1 \leq i \leq p \), where \( r < i \);
- For every edge \( e_i \) with \( 2 \leq i \leq p \), there is a vertex \( y_h \) with \( 0 \leq h < i \) such that \( \varphi(e_i) \in \varphi(y_h) \).

Clearly, a Tashkinov tree is indeed a tree of \( G \). Tashkinov [11] proved that if \( G \) is edge-\( k \)-critical with \( k \geq \Delta + 1 \), then \( V(T) \) is elementary. In the above definition, if \( e_i = y_0y_i \) for every \( i \) (i.e., \( T \) is a star with \( y_0 \) as the center), then \( T \) is a Vizing fan. The classic result of Vizing [10] show that for every Vizing fan \( T \) the set \( V(T) \) is elementary if \( G \) is edge-\( k \)-critical for every \( k \geq \Delta \), which includes edge-\( \Delta \)-critical graphs. In the definition of Tashkinov tree, if \( e_i = y_{i-1}y_i \) for every \( i \) (i.e. \( T \) is a path with end-vertices \( y_0 \) and \( y_p \)), then \( T \) is a Kierstead path, which was introduced by Kierstead [7]. Kierstead proved that for every Kierstead path \( P \) the set \( V(P) \) is elementary if \( G \) is an edge-\( k \)-critical with \( k \geq \Delta + 1 \). For simple graphs, following Kierstead’s proof, Zhang [17] noticed that for a Kierstead path \( P \) the set \( V(P) \) is elementary if \( G \) is edge-\( \Delta \)-critical and \( d(y_i) < \Delta \) for every \( i \) with \( 2 \leq i \leq p \). Clearly, every Kierstead path \( P \) with three vertices is a Vizing fan, so \( V(P) \) is elementary if \( G \) is edge-\( \Delta \)-critical.
Lemma 6. [Kostochka and Stiebitz [10]] Let $G$ be a graph with maximum degree $\Delta$ and $\chi'(G) = \Delta + 1$. Let $e_1 \in E(G)$ be a critical edge and $\varphi \in C^\Delta(G - e_1)$. If $K = (y_0, e_1, y_1, e_2, y_2, e_3, y_3)$ is a Kierstead path with respect to $e_1$ and $\varphi$, then the following statements hold:

1. $\bar{\varphi}(y_0) \cap \bar{\varphi}(y_1) = \emptyset$;

2. if $d(y_2) < \Delta$, then $V(K)$ is elementary with respect to $\varphi$;

3. if $d(y_1) < \Delta$, then $V(K)$ is elementary with respect to $\varphi$;

4. if $\Gamma = \bar{\varphi}(y_0) \cup \bar{\varphi}(y_1)$, then $|\bar{\varphi}(y_3) \cap \Gamma| \leq 1$.

In the definition of Tashkinov tree $T = (y_0, e_1, y_1, e_2, y_2, \ldots, y_p)$, we call $T$ a broom if $e_2 = y_1y_2$ and for each $i \geq 3$, $e_i = y_2y_i$, i.e., $y_2$ is one of the end-vertices of $e_i$ for each $i \geq 3$. Moreover, we call $T$ a simple broom if $\varphi(e_i) \in \bar{\varphi}(y_0) \cup \bar{\varphi}(y_1)$ for each $i \geq 3$, i.e., $(y_0, e_1, y_1, e_2, y_2, e_i, y_i)$ is a Kierstead path.

Lemma 7. [Chen, Chen, Zhao [1]] Let $G$ be an edge-$\Delta$-critical graph, $e_1 = y_0y_1 \in E(G)$ and $\varphi \in C^\Delta(G - e_1)$ and $B = \{y_0, e_1, y_1, e_2, y_2, \ldots, e_p, y_p\}$ be a simple broom. If $|\bar{\varphi}(y_0) \cup \bar{\varphi}(y_1)| \geq 4$ and $\min\{d(y_1), d(y_2)\} < \Delta$, then $V(B)$ is elementary with respect to $\varphi$.

Lemma 8. Let $G$ be an edge-$\Delta$-critical graph, $xy \in E(G)$, and $\varphi \in C^\Delta(G - xy)$. Let $q$ be a positive number with $d(x) < q \leq \Delta - 1$ and $Z = \{z \in N(x) \setminus \{y\} : d(z) > q, \varphi(xz) \in \bar{\varphi}(y)\}$. Then for every $z \in Z$ the following three inequalities hold.

\[
|Z| \geq \Delta - d(y) + 1 - \left\lfloor \frac{d(x) + d(y) - \Delta - 2}{\Delta - q} \right\rfloor \tag{1}
\]

\[
\sum_{z \in Z} (d(z) - q) \geq (\Delta - d(y) + 1)(\Delta - q) - d(x) - d(y) + \Delta + 2 \tag{2}
\]

\[
\sigma_{q}(x, z) \geq 2\Delta - d(x) - d(y) + 1 - \left\lfloor \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \right\rfloor \tag{3}
\]

Proof. Since $xy$ is a critical edge of $G$, $\bar{\varphi}(x) \cap \bar{\varphi}(y) = \emptyset$. Let $Z_y := \{z \in N(x) \setminus \{y\} : \varphi(xz) \in \bar{\varphi}(y)\}$. Clearly, $Z \subseteq Z_y$ and $|Z_y| = \Delta - d(y) + 1$. Since $\{x, y\} \cup Z_y$ forms a Vizing fan with center $x$, it is elementary, so $|\bar{\varphi}(x)| + |\bar{\varphi}(y)| + \sum_{z \in Z_y} |\bar{\varphi}(z)| \leq \Delta$ holds. Since $|\bar{\varphi}(x)| = \Delta - d(x) + 1$ and $|\bar{\varphi}(y)| = \Delta - d(y) + 1$, we have

\[
\sum_{z \in Z_y} |\bar{\varphi}(z)| \leq \Delta - |\bar{\varphi}(x)| - |\bar{\varphi}(y)| \leq d(x) + d(y) - \Delta - 2. \tag{4}
\]
Since $d(z) \leq q$ for all $z \in Z_y - Z$, $\sum_{z \in Z_y} |\bar{\varphi}(z)| \geq (|Z_y| - |Z|)(\Delta - q)$. Solving for $|Z|$, we get $|Z| \geq |Z_y| - \left[\frac{d(x) + d(y) - \Delta - 2}{\Delta - q}\right]$. Since $|Z_y| = \Delta - d(y) + 1$, inequality (1) holds.

Plugging $|\bar{\varphi}(z)| = \Delta - d(z)$ for each $z \in Z_y$ in inequality (4), we get
\[
\sum_{z \in Z_y} d(z) \geq |Z_y|\Delta - (d(x) + d(y) - \Delta - 2).
\]

Since $d(z) \leq q$ for every $z \in Z_y - Z$, we have
\[
\sum_{z \in Z} (d(z) - q) \geq \sum_{z \in Z_y} (d(z) - q) \geq |Z_y|\Delta - (d(x) + d(y) - \Delta - 2) - |Z_y|q.
\]

Plugging $|Z_y| = \Delta - d(y) + 1$, we get (2).

For each $z \in Z$, let $U^*_z = \{u \in N(z) \backslash \{x\} : \varphi(zu) \in \varphi(x) \cup \varphi(y) \backslash \{\varphi(xz)\}\}$ and $U_z = \{u \in U^*_z : d(u) > q\}$. Clearly, $|U^*_z| = 2\Delta - d(x) - d(y) + 1$ and $\{y, x, z\} \cup U^*_z$ forms a simple broom. Since $d(x) < q \leq \Delta - 1$, we have $d(x) \leq \Delta - 2$. Thus $|\bar{\varphi}(x) \cup \varphi(y)| \geq 4$ and $\min\{d(x), d(z)\} = d(x) < \Delta$. By Lemma 7, $\{y, x, z\} \cup U^*_z$ is elementary with respect to $\varphi$. So
\[
\sum_{u \in U^*_z} |\bar{\varphi}(u)| + |\varphi(x)| + |\varphi(y)| + |\bar{\varphi}(z)| \leq \Delta,
\]
which in turn gives $\sum_{u \in U^*_z} |\bar{\varphi}(u)| \leq d(x) + d(y) + d(z) - 2\Delta - 2$. Since $d(u) \leq q$ for every $u \in U^*_z - U_z$, $\sum_{u \in U^*_z} |\bar{\varphi}(u)| \geq (|U^*_z| - |U_z|)(\Delta - q)$. So,
\[
(|U^*_z| - |U_z|)(\Delta - q) \leq d(x) + d(y) + d(z) - 2\Delta - 2.
\]
Solving the above inequality with $|U^*_z| = 2\Delta - d(x) - d(y) + 1$, we get
\[
|U_z| \geq 2\Delta - d(x) - d(y) + 1 - \left[\frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q}\right].
\]
Since $\sigma_q(x, z) \geq |U_z|$, the inequality (3) holds. \hfill \Box

2.1 Proof of Lemma 4

Lemma 4. Let $xy$ be an edge in an edge-$\Delta$-critical graph $G$ and $q$ be a positive number. If $\Delta/2 < q \leq \Delta - d(x)/2 - 2$, then $x$ has at least $\Delta - \sigma_q(x, y) - 2$ vertices $z \in N(x) \backslash \{y\}$ such that $\sigma_q(x, z) \geq 2\Delta - d(x) - \sigma_q(x, y) - 4$. 

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Proof. Let graph $G$, edge $xy \in E(G)$ and $q$ be defined as in Lemma 4. A neighbor $z \in N(x) \setminus \{y\}$ is called feasible if there exits a coloring $\varphi \in \mathcal{C}^\Delta(G - xy)$ such that $\varphi(xz) \in \varphi(y)$, and such a coloring $\varphi$ is called $z$-feasible. Denote by $C_z$ the set of all $z$-feasible colorings. For each $\varphi \in C_z$, let

$$Z(\varphi) = \{v \in N(z) \setminus \{x\} : \varphi(vz) \in \varphi(x) \cup \varphi(y)\},$$

$$C_z(\varphi) = \{\varphi(vz) : v \in Z(\varphi) \text{ and } d(v) < q\},$$

$$Y(\varphi) = \{v \in N(y) \setminus \{x\} : \varphi(vy) \in \varphi(x) \cup \varphi(z)\},$$

$$C_y(\varphi) = \{\varphi(vy) : v \in Y(\varphi) \text{ and } d(v) < q\}.$$  

Note that $Z(\varphi)$ and $Y(\varphi)$ are vertex sets while $C_z(\varphi)$ and $C_y(\varphi)$ are color sets. For each color $k \in \varphi(z)$, let $z_k \in N(z)$ such that $\varphi(z_kz) = k$. Similarly, we define $y_k$ for each $k \in \varphi(y)$. Let $T(\varphi) = \{k \in \varphi(x) \cap \varphi(y) \cap \varphi(z) : d(y_k) < q \text{ and } d(z_k) < q\}.$

Since $G$ is edge-$\Delta$-critical, $\{x, y, z\}$ is elementary with respect to $\varphi$. So $\varphi(x), \varphi(y), \varphi(z)$ and $\varphi(x) \cap \varphi(y) \cap \varphi(z)$ are mutually exclusive and

$$\varphi(x) \cup \varphi(y) \cup \varphi(z) \cup (\varphi(x) \cap \varphi(y) \cap \varphi(z)) = \{1, 2, \ldots, \Delta\}.$$  

Recall that $\sigma_q(x, y)$ and $\sigma_q(x, z)$ are number of vertices with degree $\geq q$ in $N(y) \setminus \{x\}$ and $N(z) \setminus \{x\}$, respectively. So, the following inequalities hold.

$$\sigma_q(x, y) + \sigma_q(x, z) \geq |Y(\varphi)| - |C_y(\varphi)| + |Z(\varphi)| - |C_z(\varphi)| + |\varphi(x) \cap \varphi(y) \cap \varphi(z)| - |T(\varphi)|$$

$$= |\varphi(x) \cup \varphi(y)| + |\varphi(x) \cup \varphi(y)| - 1 + |\varphi(x) \cap \varphi(y) \cap \varphi(z)| - |C_y(\varphi)| - |C_z(\varphi)| - |T(\varphi)|$$

$$= \Delta + |\varphi(x)| - |C_y(\varphi)| - |C_z(\varphi)| - |T(\varphi)| - 1$$

$$= 2\Delta - d(x) + 1 - |C_y(\varphi)| - |C_z(\varphi)| - |T(\varphi)| - 1.$$ 

So, Lemma 4 follows the three statements below.

I. For any $\varphi \in C_z$, $|C_z(\varphi)| \leq 1$ and $|C_y(\varphi)| \leq 1$;

II. there exists a $\varphi \in C_z$ such that $|T(\varphi)| \leq 2$; and

III. there are $\Delta - \sigma_q(x, y) - 2$ feasible vertices $z \in N(x) \setminus \{y\}$.

For every $z$-feasible coloring $\varphi \in \mathcal{C}^\Delta(G - xy)$, let $\varphi^d \in \mathcal{C}^\Delta(G - xz)$ obtained from $\varphi$ by assigning $\varphi^d(xy) = \varphi(xz)$ and keeping all colors on other edges unchange. Clearly, $\varphi^d$ is a $y$-feasible coloring and $Z(\varphi^d) = Z(\varphi)$, $Y(\varphi^d) = Y(\varphi)$, $C_z(\varphi^d) = C_z(\varphi)$ and
$C_y(\varphi^d) = C_y(\varphi)$. We call $\varphi^d$ the dual coloring of $\varphi$. Considering dual colorings, we see that some properties for vertex $z$ also hold for vertex $y$.

The condition $q \leq \Delta - d(x)/2 - 2$ implies $2(\Delta - q) + (\Delta - d(x)) + 1 > \Delta$. So, for any $\varphi \in C^\Delta(G - xy)$, every elementary set $X$ with $x \in X$ contains at most one vertex with degree $\leq q$.

Let $z \in N(x) \setminus \{y\}$ be a feasible vertex and $\varphi \in C_z$. By the definition of $Z(\varphi)$, $G[\{x, y, z\} \cup Z(\varphi)]$ contains a simple broom, so $\{x, y, z\} \cup Z(\varphi)$ is elementary with respect to $\varphi$. Consequently, it contains at most one vertex other than $x$ having degree $< q$. Thus, $|C_z(\varphi)| \leq 1$. By considering its dual $\varphi^d$, we have $|C_y(\varphi)| = |C_y(\varphi^d)| \leq 1$. Hence, I holds.

The proofs of II and III are much more complicated. In the remainder of the proof, we let $Z = Z(\varphi)$, $Y = Y(\varphi)$, $C_z = C_z(\varphi)$, $C_y = C_y(\varphi)$, and $T = T(\varphi)$ if the coloring $\varphi$ is clearly referred. Let $R = C_z \cup C_y$ and $\tilde{\varphi}(x, R) = \varphi(x) \setminus R$. A coloring $\varphi \in C_z$ is called optimal if $|C_z| + |C_y|$ is maximum over all feasible colorings.

### 2.1.1 Proof of II.

Suppose to the contrary: $|T| \geq 3$ for every $\varphi \in C_z$. Let $\varphi$ be an optimal feasible coloring and assume, without loss of generality, $\varphi(xz) = 1$.

**Claim A.** For each $i \in \tilde{\varphi}(x, R)$ and $k \in T$, $P_x(i, k, \varphi)$ contains both $y$ and $z$.

**Proof.** We first show that $z \in V(P_x(i, k, \varphi))$. Otherwise, $P_x(i, k, \varphi)$ is disjoint with $P_x(i, k, \varphi)$. Let $\varphi' = \varphi/P_x(i, k, \varphi)$. Since $1 \neq i, k$, $\varphi'$ is also feasible. Since colors in $R$ are unchanged and $d(z_k) < q$, $C_z(\varphi') = C_z \cup \{i\}$ and $C_y(\varphi') \supseteq C_y$, giving a contradiction to the maximality of $|C_y| + |C_z|$. By considering the dual $\varphi^d$, we can verify that $y \in V(P_x(i, k, \varphi))$. $\square$

Since $|T| \geq 3$, there are three colors $k_1, k_2, k_3 \in T$. Let

\[
V_T = \{z_{k_1}, z_{k_2}, z_{k_3}\} \cup \{y_{k_1}, y_{k_2}, y_{k_3}\},
\]

\[
W(\varphi) = \{u \in V_T : \tilde{\varphi}(u) \cap \tilde{\varphi}(x) \subseteq R\},
\]

\[
M(\varphi) = V_T - W(\varphi) = \{u \in V_T : \tilde{\varphi}(u) \cap \tilde{\varphi}(x, R) \neq \emptyset\},
\]

\[
E_T = \{zz_k, zz_k, zy_{k_1}, zy_{k_2}, zy_{k_3}, yz_{k_1}, yz_{k_2}, yz_{k_3}\},
\]

\[
E_W(\varphi) = \{e \in E_T : e \text{ is incident to a vertex in } W(\varphi)\}, \text{ and}
\]

\[
E_M(\varphi) = E_T - E_W(\varphi) = \{e \in E_T : e \text{ is incident to a vertex in } M(\varphi)\}.
\]

For convenience, we let $W = W(\varphi)$, $M = M(\varphi)$, $E_W = E_W(\varphi)$ and $E_M = E_M(\varphi)$. 

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We assume that $|E_W|$ is minimum over all optimal feasible coloring $\varphi$ and all sets of three colors in $T(\varphi)$. For each $v \in M$, pick a color $\alpha_v \in \varphi(v) \cap \varphi(x, R)$. Let $C_M = \{\alpha_v : v \in M\}$. Clearly, $|C_M| \leq |M|$. Note that $\{z_{k_1}, z_{k_2}, z_{k_3}\} \cap \{y_{k_1}, y_{k_2}, y_{k_3}\}$ may be not empty, $|E_W|/2 \leq |W| \leq |E_W|$ and $|E_W|/2 \leq |M| \leq |E_M|$.

**Claim B.** If there exist two vertices $u, v \in V_T$ and a color $\alpha \in \varphi(x) \setminus R$ such that $\alpha \in \varphi(u) \cap \varphi(v)$, then there is an optimal feasible coloring $\varphi^*$ such that $|E_W(\varphi^*)| \leq |E_W|$ and $\{u, v\} \cap M(\varphi^*) \neq \emptyset$. Moreover, if $\varphi(x) \setminus (R \cup C_M) \neq \emptyset$, then $u$ or $v \in M$.

**Proof.** We first note that the condition of $d(x)$ and $q$ gives

$$|\varphi(x)| = \Delta - d(x) + 1 \geq \Delta - 2(\Delta - q) + 5 > 5. \quad (5)$$

If $\{u, v\} \cap M \neq \emptyset$, we are done. Suppose $u, v \in W$. Let $\beta$ be an arbitrary color in $\varphi(x, R)$ with the preference that $\beta \in \varphi(x, R) \setminus C_M$ if the set is not empty. Since $|R| \leq 2$ and (5), such a color $\beta$ exists. Since $u, v \in W$, we have $\beta \in \varphi(u) \cap \varphi(v)$. So, both $u$ and $v$ are endvertices of $(\alpha, \beta)$-chains. Assume without loss of generality $P_u(\alpha, \beta, \varphi)$ is disjoint with $P_x(\alpha, \beta, \varphi)$. We note that $\beta \in \varphi(y) \cap \varphi(z)$ since $\{x, y, z\}$ is an elementary set.

We first consider the case of $\alpha = 1$. In this case, $P_x(\alpha, \beta, \varphi) = P_y(\alpha, \beta, \varphi)$ holds; otherwise, $\varphi/P_x(\alpha, \beta, \varphi)$ would lead a $\Delta$-coloring of $G$. Since $\varphi(xz) = 1, z \in P_x(\alpha, \beta, \varphi)$. So, $P_u(\alpha, \beta, \varphi) \cap \{x, y, z\} = \emptyset$. Hence, coloring $\varphi' = \varphi/P_u(\alpha, \beta, \varphi)$ is feasible, $C_y(\varphi') = C_y$, $C_z(\varphi') = C_z$ and $T(\varphi') = T$. So, $\varphi'$ is also optimal, $u \in M(\varphi')$ and $|E_W(\varphi')| \leq |E_W|$ with that the inequality holds if the other endvertex of $P_u(\alpha, \beta, \varphi)$ is not in $M$ or $\beta \notin R \cup C_M$.

We now suppose $\alpha \in \varphi(x) \setminus (R \cup \{1\})$. So, both $\alpha$ and $\beta$ are not in $R \cup \{1\}$. Let $\varphi' = \varphi/P_u(\alpha, \beta, \varphi)$. Then, $\varphi'$ is feasible (for $z$), $C_y(\varphi') = C_y$ and $C_z(\varphi') = C_z$. Thus, $\varphi'$ is still an optimal coloring and $\beta \in \varphi'(u)$. We have $|E_W(\varphi')| \leq |E_W|$ and $u \in M(\varphi')$. By the minimality of $|E_W|$, we have the other endvertex of $P_u(\alpha, \beta, \varphi)$ must be in $M$ and $\beta \in C_M$, which leads a contradiction to the minimality of $|E_W|$ if $\beta \notin R \cup C_M$. \hfill $\square$

**Claim C.** There exist a color $k \in \{k_1, k_2, k_3\}$ and three distinct colors $i, j, \ell$ where $i, j \in \varphi(x, R)$ and $\ell \in \varphi(x, R) \cup \{1\}$ such that $i \in \varphi(z_k), j \in \varphi(y_k)$ and $\ell \in \varphi(z_k) \cup \varphi(y_k)$.

**Proof.** We first note that if there exist $i, j \in \varphi(x, R)$ such that $i \in \varphi(z_k)$ and $j \in \varphi(y_k)$, then $i \neq j$; for otherwise, by Claim A, the path $P_x(i, k, \varphi)$ contains three endvertices $x, z_k$ and $y_k$, a contradiction.

First we show that there exist $i, j \in \varphi(x, R)$ and $k \in \{k_1, k_2, k_3\}$ such that $i \in \varphi(z_k)$ and $j \in \varphi(y_k)$. Suppose not. Then $|E_M| \leq 3$ and $|E_W| \geq 3$, which in turn give $|W| \geq \left\lceil \frac{1}{2} \right\rceil = 2$ and $|M| \leq |E_M| \leq 3$. Let $u, v \in W$. By (5), $|\varphi(x)| \geq 6 > |R| + |M|$. There exists a color $\beta \in \varphi(x, R) \setminus C_M$. Then, $\beta \in \varphi(u) \cap \varphi(v)$ as $u, v \in W$. 

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Since $|R| \leq 2$, we have

$$\left| \varphi(u) \setminus R \right| + \left| \varphi(v) \setminus R \right| + \left| \varphi(x) \right| > 2(\Delta - q - 2) + \Delta - d(x) + 1 \geq \Delta + 1.$$ 

So, there is a color $\alpha$ shared by at least two of these three sets. Since $(\varphi(u) \setminus R) \cap \varphi(x) = \emptyset$ and $(\varphi(v) \setminus R) \cap \varphi(x) = \emptyset$, we have $\alpha \in (\varphi(u) \setminus R) \cap (\varphi(v) \setminus R) \cap \varphi(x)$. By Claim B, there exists an optimal feasible coloring $\varphi'$ such that $|E_W(\varphi')| \leq |E_W|$. Moreover, since $\beta \notin R \cup C_M$, the inequality holds which gives a contradiction to the minimality of $|E_W|$.

We now only need to show that additionally there exists another color $\ell \in \varphi(x, R) \cup \{1\}$ such that $\ell \in \varphi(y_k) \cup \varphi(z_k)$. Suppose on the contrary that there is no such a color $\ell$. Then the following equalities hold.

$$\varphi(z_k) \cap \varphi(x, R) = \{i\} \quad \text{and} \quad \varphi(y_k) \cap \varphi(x, R) = \{j\}$$

Moreover, $1 \notin \varphi(z_k) \cup \varphi(y_k)$. Since $|R| \leq 2$, the following inequalities hold.

$$\left| \varphi(z_k) \setminus (R \cup \{i\}) \right| + \left| \varphi(y_k) \setminus (R \cup \{j\}) \right| + \left| \varphi(x) \setminus (R \cup \{1\}) \right| > 2(\Delta - q - |R| - 1) + \Delta - d(x) + 2 \geq \Delta$$

So, there is color $\alpha$ in two of the three sets. Since $(\varphi(z_k) \setminus (R \cup \{i\})) \cap (\varphi(x) \cap R \cup \{1\}) = \emptyset$ and $(\varphi(y_k) \setminus (R \cup \{j\})) \cap (\varphi(x) \cap R \cup \{1\}) = \emptyset$, $\alpha \in \varphi(z_k) \cap \varphi(y_k) \cap \varphi(x) \setminus (R \cup \{i, j, 1\})$.

Since $|\varphi(x)| = \Delta - d(x) + 1 \geq \Delta - 2(\Delta - q) + 5 > 5$, there exists a color $\beta \in \varphi(x) \setminus (R \cup \{i, j\})$. Then, $\beta \notin \varphi(z_k) \cup \varphi(y_k)$. So, $\beta \in (\varphi(x) \cap \varphi(z_k) \cap \varphi(y_k)) \setminus (R \cup \{i, j\})$.

Applying Claim B with color $\alpha$, we obtain an optimal coloring $\varphi'$ and $|E_W(\varphi')| \leq |E_W|$, but color $\beta$ serves as the required color $\ell$, giving a contradiction. \hfill \Box

Let $k$, $i$, $j$ and $\ell$ be as stated in Claim C. If $\ell \neq 1$, we consider coloring obtained from $\varphi$ by interchange colors 1 and $\ell$ for edges not on the path $P_x(1, \ell, \varphi)$, and rename it as $\varphi$. So we may assume $1 \in \varphi(y_k) \cup \varphi(z_k)$.

We first consider the case of $1 \in \varphi(y_k)$. By Claim A, the paths $P_x(i, k, \varphi)$ and $P_x(j, k, \varphi)$ both contain $y, z$. Since $\varphi(yy_k) = \varphi(zz_k) = k$, these two paths also contain $y_k, z_k$. Since $i \in \varphi(z_k)$, we have $x$ and $z_k$ are the two endvertices of $P_x(i, k, \varphi)$. So, $i \in \varphi(y) \cap \varphi(z) \cap \varphi(y_k)$. Similarly, we have $j \in \varphi(y) \cap \varphi(z) \cap \varphi(y_k)$. We now consider the following sequence of colorings of $G - xy$.

Let $\varphi_1$ be obtained from $\varphi$ by assigning $\varphi_1(yy_k) = 1$. Since 1 is missing at both $y$ and $y_k$, $\varphi_1$ is an edge-$\Delta$-coloring of $G - xy$. Now $k$ is missing at $y$ and $y_k$, $i$ is still missing at $z_k$. Since $G$ is not $\Delta$-colorable, $P_x(i, k, \varphi) = P_y(i, k, \varphi)$; otherwise $\varphi/P_y(i, k, \varphi)$ can be extended to an edge-$\Delta$-coloring of $G$ giving a contradiction. Furthermore, $z_k, y_k \notin V(P_x(i, k, \varphi'))$ since either $i$ or $k$ is missing at these two vertices, which in turn shows that $z \notin V(P_x(i, k, \varphi'))$ since $\varphi_1(zz_k) = k$.\hfill \Box
Let $\varphi_2 = \varphi_1 / P_x(i, k, \varphi_1)$. We have $k \in \varphi_2(x), i \in \varphi_2(y) \cap \varphi_2(z_k)$ and $j \in \varphi_2(x) \cap \varphi_2(y_k)$. Since $G$ is not edge-$\Delta$-colorable, $P_x(i, j, \varphi_2) = P_y(i, j, \varphi_2)$ which contains neither $y_k$ nor $z_k$.

Let $\varphi_3 = \varphi_2 / P_x(i, j, \varphi_2)$. Then $k \in \varphi_3(x)$ and $j \in \varphi_3(y) \cap \varphi_3(y_k)$.

Let $\varphi_4$ be obtained from $\varphi_3$ by recoloring $yy_i$ by $j$. Then $1 \in \varphi_4(y), \varphi_4(xz) = 1, k \in \varphi_4(x), \varphi_4(z_{z_k}) = k$. Since $\varphi_4(xz) = 1 \in \varphi_4(y), \varphi_4$ is feasible. Since $i, j, k \notin R = C_y \cup C_z$, the colors in $R$ are unchanged during this sequence of re-colorings, so $C_y(\varphi_4) \supseteq C_y$ and $C_z(\varphi_4) \supseteq C_z$. Since $\varphi_4(z_{z_k}) = k \in \varphi_4(x)$ and $d(z_k) < q$, we have $k = \varphi_4(z_{z_k}) \in C_z(\varphi_4)$.

So, $C_z(\varphi_4) \supseteq C_z \cup \{k\}$. We therefore have $|C_y(\varphi_4)| + |C_z(\varphi_4)| \geq |C_y| + |C_z| + 1$, giving a contradiction.

For the case of $1 \in \varphi(z_k)$, we consider the dual coloring $\varphi^d$ of $G - xz$ obtained from $\varphi$ by uncoloring $xz$ and coloring $xy$ with color 1. Following the exact same argument above, we can reach a contradiction to the maximum of $|C_y| + |C_z|$. This completes the proof of \textit{II}. \hfill \qed

### 2.1.2 Proof of III.

Denote by $Z$ the set of all feasible vertices. For a coloring $\varphi \in C_\Delta(G - xy)$, let $Z(\varphi) = \{z \in N(x) : \varphi(xz) \in \varphi(y)\}$ and $S(\varphi) = \{z \in N(x) \setminus Z(\varphi) : d(y_{\varphi(xz)}) < q\}$, where $y_j \in N(y)$ with $\varphi(yy_j) = j$ for any color $j$. We call vertices in $S(\varphi)$ \textit{semi-feasible} vertices of $\varphi$.

**Claim 2.1.** For any coloring $\varphi \in C_\Delta(G - xy)$, the following two inequalities hold.

- **a.** $|Z(\varphi) \cup S(\varphi)| \geq \Delta - \sigma_q(x, y) - 1$;

- **b.** With one possible exception, for all $z \in S(\varphi)$ there exists a coloring $\varphi^* \in C_\Delta(G - xy)$ such that $\varphi^*(xz) \in \varphi^*(y)$.

**Proof.** Let $\varphi \in C_\Delta(G - xy)$. Since $G$ is edge-$\Delta$-critical, $xy$ is an edge of $G$, it is easy to see that $\varphi(y) \subseteq \varphi(x)$ and $\varphi(x) \subseteq \varphi(y)$. Divide $\varphi(y)$ into two subsets:

$$\varphi(y, \geq q) = \{i \in \varphi(y) : d(y_i) \geq q\} \quad \text{and} \quad \varphi(y, < q) = \{i \in \varphi(y) : d(y_i) < q\}.$$  

Clearly, $\sigma_q(x, y) = |\varphi(y, \geq q)|$ and $|\varphi(y)| + |\varphi(y, < q)| = \Delta - \sigma_q(x, y)$. Since $\varphi(y) \subseteq \varphi(x)$, to prove a, we only need to show $|\varphi(x) \cap \varphi(y, < q)| \leq 1$. Since edge $xy$ and the edges incident to $y$ with colors in $\varphi(x)$ form a Vizing fan $F$, the vertex set $V(F)$ is elementary with respect to $\varphi$. Since $d(x) + 2q < 2\Delta$, $V(F) \setminus \{x\}$ contains at most one vertex with degree $< q$. So $|\varphi(x) \cap \varphi(y, < q)| \leq 1$ holds.
To prove b, we show that for any two distinct vertices $z_k, z_\ell \in S(\varphi)$, there is a coloring $\varphi^* \in C^\Delta(G - xy)$ such that at least one of $\varphi^*(xz_k)$ and $\varphi^*(xz_\ell)$ is in $\bar{\varphi}^*(y)$. We assume $\varphi(xz_k) = k$ and $\varphi(xz_\ell) = \ell$. Let $y_k, y_\ell \in N(y) \setminus \{x\}$ such that $\varphi(yy_k) = k$ and $\varphi(yy_\ell) = \ell$.

By the definition of $S(\varphi)$, we have $d(y_k) < q$ and $d(y_\ell) < q$. Since $\Delta/2 < q \leq \Delta - d(x)/2 - 2$, the following inequality holds.

$$|\bar{\varphi}(x)| + |\bar{\varphi}(y_k)| + |\bar{\varphi}(y_\ell)| > \Delta$$  \hspace{1cm} (6)

We claim that there exists a coloring $\varphi^* \in C^\Delta(G - xy)$ such that keeping the property $\varphi^*(xz) = 1 \in \bar{\varphi}^*(y)$ and having the following property.

$$\bar{\varphi}^*(x) \cap (\bar{\varphi}^*(y_k) \cup \bar{\varphi}^*(y_\ell)) \neq \emptyset$$  \hspace{1cm} (7)

Otherwise, by (6), there exists $r \in \varphi(x) \cap \bar{\varphi}(y_k) \cap \bar{\varphi}(y_\ell)$. Choose a color $i \in \varphi(x)$. Since at least one of colors $i$ and $r$ is missing at each of $x$, $y_k$ and $y_\ell$, we may assume $P_{y_k}(i, r, \varphi)$ is disjoint with $P_x(i, r, \varphi)$. Then, in coloring $\varphi/P_{y_k}(i, r, \varphi)$, color $i$ is missing at both $x$ and $y_k$, giving a contradiction.

By (7), we may assume that there exists a color $i \in \bar{\varphi}(x) \cap \bar{\varphi}(y_k)$. Since $G$ is not edge-$\Delta$-colorable, $P_x(i, 1, \varphi) = P_y(i, 1, \varphi)$. So, $P_{y_k}(i, 1, \varphi)$ is disjoint with $P_x(i, 1, \varphi)$. If $1 \in \bar{\varphi}(y_k)$, let $\varphi' = \varphi/P_{y_k}(1, i, \varphi)$. For coloring $\varphi'$, we have $1 \in \bar{\varphi}'(y_k)$ and $\varphi'(x) = \varphi(x), \varphi'(y) = \varphi(y)$. Thus we can assume $1 \in \bar{\varphi}(y_k)$. Let $\varphi^*$ be a coloring obtained from $\varphi$ by recoloring $yy_k$ with color 1. Then, $\varphi^*(xz_k) = k \in \bar{\varphi}^*(y)$. This completes the proof of III. \hfill \Box

3 Proof of Theorem 1

Let $G$ be an edge-$\Delta$-critical graph with $n$ vertices and $m$ edges. Clearly, $\bar{d}(G) = 2m/n$. We assume $\Delta \geq 56$. Let $q := \min\{2\sqrt{2}\Delta - 2, \frac{3}{4}\Delta - 2\}$, that is, $q = \frac{2\sqrt{2}\Delta - 2}{2\sqrt{2} + 1}$ if $\Delta \geq 66$ and $q = \frac{3}{4}\Delta - 2$ if $56 \leq \Delta \leq 65$. We initially assign to each vertex $x$ of $G$ a charge $M(x) = d(x)$ and redistribute the charge according to the following rule:

- **Rule of Discharge**: each ($> q$)-vertex $y$ distributes its surplus charge of $d(y) - q$ equally among all ($< q$)-neighbors of $y$.

Denote by $M'(x)$ the resulting charge on each vertex $x$. Clearly, $\sum_{x \in V(G)} M'(x) = \sum_{x \in V(G)} M(x) = 2m$. Let $X_1 = \{x \in V(G) : d(x) \leq 3q - 2\Delta\}$. We show that $M'(x) \geq 2 + 2(\Delta - q)$ for all vertices in $X_1$ and $M'(x) \geq q$ for all other vertices, which gives $\bar{d}(G) \geq q - (3q - 2\Delta - 2)\frac{|X_1|}{n}$. We then show that $|X_1|/n$ is small in order to complete our proof.
Since \( q = \min\{\frac{2\sqrt{2}(\Delta-1)-2}{2\sqrt{2}+1}, \frac{3}{4}\Delta - 2\} \) and \( \Delta \geq 56 \), we have \( \frac{\Delta+2}{2} < q < \frac{3\Delta}{4} \). Thus \( q > \Delta - q + 2 > 3q - 2\Delta \).

**Claim 3.1.** If \( d(x) \leq \Delta - q + 2 \), then \( M'(x) \geq d(x) + 2(\Delta - q) \). Consequently, \( M'(x) \geq d(x) + 2(\Delta - q) \) for each \( x \in X_1 \).

**Proof.** Let \( y \) be an arbitrary neighbor of \( x \). Since \( 2\Delta - d(x) - d(y) + 2 \geq \Delta - d(x) + 2 \geq q \), we have \( \sigma_q(x, y) \geq \sigma(x, y) \). We will use lower bounds of \( \sigma(x, y) \) to estimate \( \sigma_q(x, y) \). Following the definition \( p_{\min}(x) = \min_{v \in N(x)} \sigma(x, v) - \Delta + d(x) - 1 \) and \( p(x) = \min\{p_{\min}(x), \lfloor \frac{d(x)}{2} \rfloor - 1\} \), we have the following inequalities.

\[
1 \leq d_{<q}(y) \leq d(y) - \sigma(x, y) \leq d(y) - (\Delta - d(x) + p(x) + 1)
\]

By Lemma 3, \( x \) has at least \( d(x) - p(x) - 1 \) neighbors \( y \) for which \( \sigma(x, y) \geq \Delta - p(x) - 1 \), so for these neighbors \( y \) the following inequalities hold.

\[
1 \leq d_{<q}(y) \leq d(y) - \sigma(x, y) \leq d(y) - (\Delta - p(x) - 1)
\]

We first consider the case \( p(x) \geq 1 \). In this case, we have \( q \leq \Delta - d(x) + 2 \leq \Delta - d(x) + p + 1 \). Since \( \frac{d(y) - q}{d(y) - p(x) - 1} \) with \( a \leq b \) is a decreasing function of \( d(y) \), for each \( y \in N(x) \), \( x \) receives charge at least

\[
\frac{d(y) - q}{d(y) - (\Delta - p(x) - 1)} \geq \frac{\Delta - q}{d(x) - p(x) - 1},
\]

And there are at least \( d(x) - p(x) - 1 \) neighbors \( y \) of \( x \) giving \( x \) at least

\[
\frac{d(y) - q}{d(y) - (\Delta - p(x) - 1)} \geq \frac{\Delta - q}{p(x) + 1},
\]

where the inequality holds because \( q \leq \Delta - d(x) + 2 \leq \Delta - p(x) - 1 \) as \( 1 \leq p(x) \leq \lfloor \frac{d(x)}{2} \rfloor - 1 \).

Thus \( x \) receives at least

\[
(d(x) - p(x) - 1) \frac{\Delta - q}{p(x) + 1} + (p(x) + 1) \frac{\Delta - q}{d(x) - p(x) - 1} = (\theta + \theta^{-1})(\Delta - q) \geq 2(\Delta - q),
\]

where \( \theta = \frac{d(x) - p(x) - 1}{p(x) + 1} \). It follows that \( M'(x) \geq M(x) + 2(\Delta - q) = d(x) + 2(\Delta - q) \).

We now consider the case \( p(x) = \min\{p_{\min}(x), \lfloor \frac{d(x)}{2} \rfloor - 1\} = 0 \). If \( d(x) = 2 \), then by (8) for every neighbor \( y \) of \( x \) we have \( d_{<q}(y) = 1 \) and \( d(y) = \Delta \), thus \( M'(x) \geq M(x) + 2(\Delta - q) = d(x) + 2(\Delta - q) \). If \( d(x) \geq 3 \), then by (9) for at least \( d(x) - 1 \) neighbors \( y \) of \( x \), we have \( d_{<q}(y) = 1 \) and \( d(y) = \Delta \). Thus \( M'(x) \geq M(x) + (d(x) - 1)(\Delta - q) \geq d(x) + 2(\Delta - q) \).
Claim 3.2. For each $x \in V(G) - X_1$, $M'(x) \geq q$ holds.

Proof. Let $x \in V(G) - X_1$, i.e., $d(x) > 3q - 2\Delta$. If $d(x) \geq q$, then $M'(x) = M(x) - \frac{d(x)-q}{d_{\sigma_q}(x)}d_{<q}(x) = q$. If $3q - 2\Delta < d(x) \leq \Delta - q + 2$, then by Claim 3.1, we have $M'(x) \geq d(x) + 2(\Delta - q) > q$. So we only need to consider the case $\Delta - q + 2 < d(x) < q$.

Since $G$ is edge-$\Delta$-critical and $xy \in E(G)$, there exists a coloring $\varphi \in \mathcal{C}\Delta(G - xy)$. Let $Z_q = \{z \in N(x) : d(z) > q\}$, $Z_y = \{z \in N(x) \setminus \{y\} : \varphi(xz) \in \varphi(y)\}$ and $Z_y = Z_q \cap Z_y$. Clearly, for each $z \in Z_q$, $x$ receives at least $\frac{d(z)-q}{d(z)-\sigma_q(x,z)}$ charge. Thus $M'(x) \geq d(x) + \sum_{z \in Z_q} \frac{d(z)-q}{d(z)-\sigma_q(x,z)}$. We consider the following three cases to complete the proof.

Case 1. $\Delta - q + 2 < d(x) < q$ and $x$ has a neighbor $y$ such that $d(y) \leq q$.

By Lemma 8 (3), for each vertex $z \in Z_q^*$, we have

$$\sigma_q(x,z) \geq 2\Delta - d(x) - d(y) + 1 - \left[ \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \right] \geq 2\Delta - d(x) - d(y),$$

where we used the inequality $\left\lfloor \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \right\rfloor \leq 1$ following $d(x) < q, d(y) \leq q, d(z) \leq \Delta$ and $q < \frac{3}{4}\Delta$. Thus $\sigma_q(x,z) \geq 2\Delta - d(x) - d(y)$. So, $M'(x) \geq M(x) + \sum_{z \in Z_q^*} \frac{d(z)-q}{d(z)-\sigma_q(x,z)}$. By Lemma 8 (2), we have $\sum_{z \in Z_q^*} (d(z) - q) \geq (\Delta - d(y) + 1)(\Delta - q) - d(x) - d(y) + \Delta + 2$. Thus

$$\sum_{z \in Z_q^*} \frac{d(z)-q}{d(z) - (2\Delta - d(x) - d(y))} \geq \frac{(\Delta - q)(\Delta - d(y) + 1) - (d(x) + d(y) - \Delta - 2)}{d(x) + d(y) - \Delta} \geq \frac{(\Delta - q)(\Delta - d(y) + 1) + 2}{d(x) + d(y) - \Delta} - 1.$$

So, we have the following inequality.

$$M'(x) \geq M(x) + \sum_{z \in Z_q^*} \frac{d(z)-q}{d(z) - (2\Delta - d(x) - d(y))} \geq d(x) + \frac{(\Delta - q)(\Delta - d(y) + 1) + 2}{d(x) + d(y) - \Delta} - 1$$

$$\geq d(x) + q - \Delta + \frac{(\Delta - q)(\Delta - q + 1) + 2}{d(x) + q - \Delta} - 1 - q + \Delta$$

$$\geq 2\sqrt{(\Delta - q)(\Delta - q + 1) + 2 + \Delta - q - 1}$$

$$\geq 3(\Delta - q) \geq q.$$
Case 2. $2(\Delta - q) - 4 < d(x) < q$ and $d(y) > q$ for every neighbor $y$ of $x$.

Let $y \in N(x)$ such that $d(y) := \min \{d(u) : u \in N(x)\}$. By Lemma 8 (3), for each vertex $z \in Z_q^*$ we have

$$
\sigma_q(x, z) \geq 2\Delta - d(x) + 1 - \left[ \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \right]
$$

$$
\geq 2\Delta - d(x) - d(y) - 1,
$$

where we used the inequality $\left\lfloor \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \right\rfloor \leq 2$ when $d(x) < q$ and $q < \frac{3}{4}\Delta$. By Lemma 8 (2), we have

$$
\sum_{z \in Z_q^*} \frac{d(z) - q}{d(z) - \sigma_q(x, y)} \geq \frac{(\Delta - d(y) + 1)(\Delta - q) - d(x) - d(y) + \Delta + 2}{d(x) + d(y) - \Delta + 1} = q - \Delta - 1 + \frac{(\Delta - q)(d(x) + 2) + 3}{d(x) + d(y) - \Delta + 1}.
$$

By Lemma 1, for each neighbor $u$ of $x$ we have $\sigma_{\Delta}(x, u) \geq \Delta - d(x) + 1$. Since $d(u) \geq d(y)$ for each $u \in N(x)$ and $q \geq \Delta - d(x) + 1$, we have

$$
\frac{d(u) - q}{d(u) - (\Delta - d(x) + 1)} \geq \frac{d(y) - q}{d(y) - (\Delta - d(x) + 1)}.
$$

So,

$$
\sum_{u \in N(x) \setminus Z_q^*} \frac{d(u) - q}{d(u) - (\Delta - d(x) + 1)} \geq |N(x) \setminus Z_q^*| \cdot \frac{d(y) - q}{d(y) - (\Delta - d(x) + 1)} = \frac{(d(x) - (\Delta - d(y) + 1))(d(y) - q)}{d(y) - (\Delta - d(x) + 1)} = d(y) - q.
$$

Thus

$$
M'(x) \geq d(x) + q - \Delta - 1 + \frac{(\Delta - q)(d(x) + 2) + 3}{d(x) + d(y) - \Delta + 1} + d(y) - q
$$

$$
= d(x) + d(y) - \Delta + 1 + \frac{(\Delta - q)(d(x) + 2) + 3}{d(x) + d(y) - \Delta + 1} - 2
$$

$$
\geq 2\sqrt{(\Delta - q)(d(x) + 2) + 3} - 2
$$

$$
> 2\sqrt{2(\Delta - q)(\Delta - q - 1)} - 2 \geq q.
$$
Case 3. $\Delta - q + 2 < d(x) \leq 2(\Delta - q) - 4$ and $d(y) > q$ for every neighbor $y$ of $x$.

Since $\Delta \geq 56$, we have $\Delta - q + 2 < 2(\Delta - q) - 4$, so this case occurs. Since the notation $p(x, q)$ will be used heavily in this proof, we let $p' := p(x, q)$ for convenience. So, $p' = \min\{p_{\min}(x, q), \lfloor \frac{d(x)}{2} \rfloor - 3\}$, where $p_{\min}(x, q) := \min_{y \in N(x)} \sigma_q(x, y) - \Delta + d(x) - 1$.

Following this definition, for every $y \in Z_q$, $\sigma_q(x, y) \geq \Delta - d(x) + p' + 1$, which in turn gives

$$\frac{d(y) - q}{d(y) - \sigma_q(x, y)} \geq \frac{d(y) - q}{d(y) - (\Delta - d(x) + p' + 1)}.$$ 

So, if $q \leq \Delta - d(x) + p' + 1$, then

$$\frac{d(y) - q}{d(y) - \sigma_q(x, y)} \geq \frac{\Delta - q}{d(x) - p' - 1}.$$ 

By Lemma 5, $x$ has at least $d(x) - p' - 3$ neighbors $y$ for which $\sigma_q(x, y) \geq \Delta - p' - 5$. For such neighbors $y$, since $q \leq \Delta - \frac{d(x)}{2} - 2 \leq \Delta - p' - 5$, we have

$$\frac{d(y) - q}{d(y) - \sigma_q(x, y)} \geq \frac{d(y) - q}{d(y) - (\Delta - p' - 5)} \geq \frac{\Delta - q}{p' + 5}.$$ 

If $q \leq \Delta - d(x) + p' + 1$, then

$$M'(x) \geq d(x) + (d(x) - p' - 3)\frac{\Delta - q}{p' + 5} + (p' + 3)\frac{\Delta - q}{d(x) - p' - 1} \geq \Delta - q + 2 + (\Delta - q)(2 - \frac{8(d(x) + 2)}{(\Delta - q + 2)(d(x) + 4)}) \geq q,$$

where we used the inequality $\theta + \theta^{-1} \geq 2 \ (\theta = \frac{d(x) - p' - 3}{p' + 5})$ to show the following

$$\frac{d(x) - p' - 3}{p' + 5} + \frac{p' + 3}{d(x) - p' - 1} > 2 - \frac{2d(x) + 4}{(\frac{\Delta - q + 2}{2})^2} \geq 2 - \frac{8(d(x) + 2)}{(\Delta - q + 2)(d(x) + 4)}.$$ 

Suppose $q > \Delta - d(x) + p' + 1$, i.e., $p' < d(x) + q - \Delta - 1$. So, $\frac{d(x) - p' - 3}{p' + 5} > \frac{\Delta - q - 2}{d(x) + q - \Delta + 4}$, which gives

$$M'(x) \geq d(x) + \frac{\Delta - q - 2}{d(x) + q - \Delta + 4}(\Delta - q) \geq (d(x) + q - \Delta + 4) + \frac{(\Delta - q)(\Delta - q - 2)}{d(x) + q - \Delta + 4} - (q - \Delta + 4) \geq 2\sqrt{(\Delta - q)(\Delta - q - 2)} + \Delta - q - 4 \geq 3(\Delta - q) - 8 \geq q.$$ 

\[\square\]
Claim 3.3. $d(y) > q$ for each $y \in N(X_1)$ and $|N(X_1)| \geq 2|X_1|$ where $N(X_1) = \cup_{x \in X_1} N(x)$.

Proof. Since $G$ is edge-$\Delta$-critical, for each edge $xy \in E(G)$ we have $d(x) + d(y) \geq \Delta + 2$. Since $q < \frac{\Delta}{2}$ and $d(x) \leq 3q - 2\Delta$ for each $x \in X_1$, we have $d(y) \geq \Delta + 2 - (3q - 2\Delta) > q$ for each $y \in N(x)$. Thus the vertices in $N(X_1)$ does not receive charges from any other vertices. As the vertices in $X_1$ receive charges only from the vertices in $N(X_1)$, we have

$$\sum_{x \in X_1} M'(x) + \sum_{y \in N(X_1)} M'(y) \leq \sum_{x \in X_1} M(x) + \sum_{y \in N(X_1)} M(y) \leq \sum_{x \in X_1} d(x) + \Delta|N(X_1)|. \quad (10)$$

Also, by Claims 3.1 and 3.2, we have $M'(x) \geq d(x) + 2(\Delta - q)$ for each $x \in X_1$ and $M'(y) \geq q$ for each $y \in N(X_1)$. Thus we have

$$\sum_{x \in X_1} M'(x) + \sum_{y \in N(X_1)} M'(y) \geq \sum_{x \in X_1} d(x) + 2(\Delta - q)|X_1| + q|N(X_1)|. \quad (11)$$

Combining (10) with (11), we have $|N(X_1)| \geq 2|X_1|$.

For each edge $xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G - xy)$, let $Y(x, \varphi) = \{w \in N(y) \setminus \{x\} : \varphi(yw) \in \varphi(x)\}$, $Y_1(x, \varphi) = Y(x, \varphi) \cap N(X_1)$ and $Y_2(x, \varphi) = Y(x, \varphi) - (X_1 \cup N(X_1))$. Clearly, $|Y(x, \varphi)| = \Delta - d(x) + 1$. Note that with respect to the coloring $\varphi$, $\{x, y\} \cup Y(x, \varphi)$ forms a Vizing fan, so it is elementary.

Claim 3.4. For each $y \in N(X_1)$ and $x \in N(y) \cap X_1$, $|Y_2(x, \varphi)| \geq \Delta - 2d(x) + 3$.

Proof. Recall that $\{x, y\} \cup Y(x, \varphi)$ is elementary. Then for each vertex in $w \in Y(x, \varphi)$ we have $|\varphi(w)| + |\varphi(x)| + |\varphi(y)| \leq \Delta$, it follows that $d(w) \geq |\varphi(x)| + |\varphi(y)| > 3q - 2\Delta$. Thus we have $Y(x, \varphi) \cap X_1 = \emptyset$. If $|Y_1(x, \varphi)| \leq d(x) - 2$, then $|Y_2(x, \varphi)| = |Y(x, \varphi) - Y_1(x, \varphi) - (Y(x, \varphi) \cap X_1)| \geq \Delta - d(x) + 1 - (d(x) - 2) \geq \Delta - 2d(x) + 3$. So the Claim 3.4 is equivalent to show that $|Y_1(x, \varphi)| \leq d(x) - 2$.

Subclaim 3.4.1. If $w \in Y_1(x, \varphi)$, then for each neighbor $z$ of $w$ in $X_1$, we have $\varphi(wz) \in \varphi(x) \cap \varphi(y)$.

Proof. If there exists a neighbor of $w$ in $X_1$, say $z$, such that $\varphi(wz) \notin \varphi(x) \cap \varphi(y)$. Then $\varphi(wz) \in \varphi(x) \cup \varphi(y)$. Thus $\{x, y, w, z\}$ forms a Kierstead path. By Lemma 6, we have $|\varphi(z) \cap (\varphi(x) \cup \varphi(y))| \leq 1$, it follows that $d(z) \geq (\Delta - d(x) + 1) + (\Delta - d(y) + 1) - 1 > 3q - 2\Delta$, this contradicts with the fact that $z \in X_1$. So Subclaim 3.4.1 holds.
For each color \( j \in \varphi(x) \cap \varphi(y) \), set \( Y_j = \{ w \in Y^1(x, \varphi) : j \in \varphi(w) \} \) and \( Z_j = \{ z \in X_1 : \text{there exists a vertex } w \in Y_j \text{ such that } \varphi(wz) = j \} \). Clearly, \( \sum_{j \in \varphi(x) \cap \varphi(y)} |Z_j| \geq |Y^1(x, \varphi)| \). Since \( |\varphi(x) \cap \varphi(y)| = \Delta - (\Delta - d(x) + 1) - (\Delta - d(y) + 1) \leq d(x) - 2 \), to show that \( |Y^1(x, \varphi)| \leq d(x) - 2 \), we only need to prove that \( |Z_j| \leq 1 \) for each \( j \). Let \( |Z_j| = t \) and \( Z_j = \{ z_{a_1}, \ldots, z_{a_t} \} \), where for each \( z_{a_i} \) there exists \( y_{a_i} \in Y_j \) such that \( \varphi(y_{a_i}) = \alpha_i \) and \( \varphi(y_{a_i}z_{a_i}) = j \). Clearly, \( \alpha_i \in \bar{\varphi}(x) \) for each \( 1 \leq i \leq t \).

**Subclaim 3.4.2.** Let \( k \) be a color in \( \bar{\varphi}(x) \). Then the followings hold.

1. For each \( k \notin \{ \alpha_1, \ldots, \alpha_t \} \), at least \( t - 1 \) vertices of \( Z_j \) have the color \( k \).
2. For each \( k \in \{ \alpha_1, \ldots, \alpha_t \} \), at least \( t - 2 \) vertices of \( Z_j \) have the color \( k \).

**Proof.** First suppose that \( k \notin \{ \alpha_1, \ldots, \alpha_t \} \). We consider the path \( P_z(j, k, \varphi) \), and \( w \) is the other end vertex of this path. We will show that the color \( k \) seen by each vertex in \( Z_j \setminus \{ w \} \). For otherwise, we assume \( k \notin \varphi(z) \) for some \( z \in Z_j \setminus \{ w \} \), say \( z = z_{a_1} \), then \( P_{z_{a_1}}(j, k, \varphi) \) is disjoint from \( P_z(j, k, \varphi) \), thus let \( \varphi' = \varphi/P_{z_{a_1}}(j, k, \varphi) \) be the new coloring which \( \varphi'(y_{a_1}) = \varphi(y_{a_1}) = \bar{\varphi}'(x) \) and \( \varphi'(y_{a_1}z_{a_1}) = k \in \bar{\varphi}'(x) \). Thus \( \{ x, y, y_{a_1}, z_{a_1} \} \) forms a Kierstead path. So by Lemma 6 we have \( |\bar{\varphi}(z_{a_1}) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y))| \leq 1 \), it follows that \( d(z_{a_1}) \geq (\Delta - d(x) + 1) + (\Delta - d(y) + 1) - 1 > 3q - 2\Delta \) as \( d(x) \leq 3q - 2\Delta \), this contradicts with the fact that \( z_{a_1} \in X_1 \).

Then suppose that \( k \in \{ \alpha_1, \ldots, \alpha_t \} \). We may assume that \( k = \alpha_t \). Clearly, \( k \notin \{ \alpha_1, \ldots, \alpha_{t-1} \} \). Let \( Z'_j = Z_j \setminus \{ z_k \} \). By (1), we have at least \( |Z'_j| - 1 \) vertices of \( Z'_j \) has the color \( k \), that is, at least \( t - 2 \) vertices of \( Z_j \) has the color \( k \).

By Subclaim 3.4.2 and the definition of \( X_1 \), we have

\[
(\Delta - d(x) + 1 - t)(t - 1) + t(t - 2) \leq \sum_{z \in Z_j} d(z) \leq t(3q - 2\Delta).
\]

Since \( d(x) \leq 3q - 2\Delta \) and \( q < \frac{3}{4}\Delta \), we have

\[
t \leq \frac{\Delta - d(x) + 1}{3\Delta - 3q - d(x)} \leq 1 + \frac{3q - 2\Delta + 1}{5\Delta - 6q} < 2.
\]

Since \( t \) is an integer, we have \( t \leq 1 \). Then Claim 3.4 holds.

Let \( c \) be a positive integer, set \( Z_1(c) = \{ z \in V(G) - (X_1 \cup N(X_1)) : d(z) \geq \Delta - c \} \) and \( Z_2(c) = \{ z \in V(G) - (X_1 \cup N(X_1)) : d(z) < \Delta - c \} \).

**Claim 3.5.** \( |Z_1(c)| \geq \frac{(5c+2)\Delta-(6c+3)q+3c+2}{c\Delta}|N(X_1)| \).
Proof. For each \( y \in N(X_1), x \in N(y) \cap X_1 \) and \( \varphi \in \mathcal{C}^\Delta(G - xy) \), let \( Y_{< c} = \{ z \in Y^2(x, \varphi) : d(z) < \Delta - c \} \). Since \( \{ x, y \} \cup Y(x, \varphi) \) is elementary and \( Y^2(x, \varphi) \subseteq Y(x, \varphi) \), we have \( \Delta - d(x) + 1 + \Delta - d(y) + 1 + c|Y_{< c}| < \sum_{v \in \{ x, y \} \cup Y(x, \varphi)} \varphi(v) \leq \Delta \). Thus \( |Y_{< c}| < \frac{d(x) - 2}{c} \). By Claim 3.4, we have \( |Y^2(x, \varphi) - Y_{< c}| > \Delta - 2d(x) + 3 - \frac{d(x) - 2}{c} \), that is, for each \( y \in N(X_1) \) we have \( d_{Z_1(c)}(y) \geq \Delta - 2d(x) + 3 - \frac{d(x) - 2}{c} \). Hence,

\[
(\Delta - 2d(x) + 3 - \frac{d(x) - 2}{c})|N(X_1)| \leq |E(N(X_1), Z_1(c))| \leq \Delta|Z_1(c)|,
\]

where \( E(N(X_1), Z_1(c)) \) are the edges with one vertex in \( N(X_1) \) and the other endvertex in \( Z_1(c) \). Since \( d(x) \leq 3q - 2\Delta \), solving the above inequalities we have

\[
|Z_1(c)| \geq \frac{(5c + 2)\Delta - (6c + 3)q + 3c + 2}{c\Delta}|N(X_1)|.
\]

\[\square\]

By Claims 3.1 and 3.2, we have

\[
M'(x) \geq \begin{cases}
2 + 2(\Delta - q) & x \in X_1, \\
q & x \in V(G) - X_1.
\end{cases}
\]

And by the definitions of \( Z_1(c) \) and \( Z_2(c) \), we get the following two lower bounds of \( \sum_{x \in V(G)} M'(x) \).

\[
b_1 = (2 + 2(\Delta - q))|X_1| + q|N(X_1)| + (\Delta - c)|Z_1(c)| + (3q - 2\Delta)|Z_2(c)|
\]

\[
b_2 = (2 + 2(\Delta - q))|X_1| + (n - |X_1|)q
\]

We now divide into a few cases to estimate the lower bound of \( \max\{b_1, b_2\} \).

First we consider the case \( \Delta - q - c > 0 \). For fixed value \( |X_1| + |N(X_1)|, |Z_1(c)| + |Z_2(c)| \) is a constant. Since \( \Delta - c > q > 3q - 2\Delta \), \( \max\{((\Delta - c)|Z_1(c)| + (3q - 2\Delta)|Z_2(c)|, q(|Z_1(c)| + |Z_2(c)|)\} \) takes minimum when \( (\Delta - c)|Z_1(c)| + (3q - 2\Delta)|Z_2(c)| = q(|Z_1(c)| + |Z_2(c)|) \), that is \( |Z_2(c)| = \frac{1}{2} - \frac{c}{2\Delta - 2q} |Z_1(c)| \). So by Claims 3.3 and 3.5, \( |Z_1(c)| + |Z_2(c)| \geq (3 - \frac{c}{\Delta - q})(\frac{5c + 2)\Delta - (6c + 3)q + 3c + 2}{c\Delta}|X_1| \). Let \( f(c) = (3 - \frac{c}{\Delta - q})(\frac{5c + 2)\Delta - (6c + 3)q + 3c + 2}{c\Delta}\). So \( n = |X_1| + |N(X_1)| + |Z_1(c)| + |Z_2(c)| \geq (3 + f(c))|X_1| \). Hence, \( \sum_{x \in V(G)} M'(x) \geq \max\{b_1, b_2\} \geq qn + (2 + 2\Delta - 3q)|X_1| \geq (q + \frac{2 + 2\Delta - 3q}{3 + f(c)})n \). So \( d(G) \geq q + \frac{2 + 2\Delta - 3q}{3 + f(c)} \).
Let \( q^* = \frac{2\sqrt{2}\Delta}{2\sqrt{2}+1} \) and \( a = 1 + \frac{1}{2\sqrt{2}+1} \) if \( q = \frac{2\sqrt{2}(\Delta-1)-2}{2\sqrt{2}+1} \), and \( q^* = \frac{3\Delta}{4} \), \( a = 2 \) if \( q = \frac{3}{4}\Delta - 2 \). So \( q = q^* - a \) and we have

\[
\frac{2 + 2\Delta - 3q}{3 + f(c)} = \frac{c\Delta(2 + 2\Delta - 3q)}{3c\Delta + (3 - \frac{c}{\Delta - q})(5c + 2)\Delta -(6c + 3)q + 3c + 2} \]

\[
= \frac{(2c\Delta - 3cq^*)\Delta + f_1f_2\Delta - f_1f_2\Delta + 2c\Delta + 3ca\Delta}{(18c + 6)\Delta - (18c + 9)q^* + f_2} \]

\[
= f_1\Delta + \frac{-f_1f_2\Delta + 2c\Delta + 3ca\Delta}{(18c + 6)\Delta - (18c + 9)q^* + f_2},
\]

where \( f_1 = \frac{2c\Delta - 3cq^*}{(18c + 6)\Delta - (18c + 9)q^*} \) and \( f_2 = 9c + 6 + (18c + 9)a - \frac{(5c^2 + 2c)\Delta - (6c^2 + 3c)q + 3c^2 + 2c}{\Delta - q} \).

Clearly, \( f_1 = \frac{2c\Delta - 3cq^*}{(18c + 6)\Delta - (18c + 9)q^*} = \frac{2\Delta - 3q^*}{18\Delta - 18q^* + \frac{6\Delta - 9q^*}{c}} \) is an increasing function of \( c \). To make \( f_1 \) as large as possible when \( \Delta \geq l \), where \( l \) is a positive integer, we choose \( c \) such that \( c = \min\{\lfloor \Delta - q \rfloor : \Delta \geq l \} \). If \( l \) is large enough, \( c \) is large too and we can see that the value of \( f_1 \) will approximate to \( \frac{2\Delta - 3q^*}{18\Delta - 18q^*} \) and \( d(G) \) will approximate to 0.69277Δ. Note that \( q = \frac{2\sqrt{2}(\Delta-1)-2}{2\sqrt{2}+1} \) if \( \Delta \geq 66 \). Let \( l = 66 \). Then we have \( c = 18 \). So we have \( \Delta - q - c > 0 \) if \( \Delta \geq 65 \). Plugging \( c = 18 \) and the value of \( q^* \) into \( f_2 \) and \( \frac{-f_1f_2\Delta + 2\Delta + 3ca\Delta}{(18c + 6)\Delta - (18c + 9)q^* + f_2} \), we have \( \frac{-f_1f_2\Delta + 2\Delta + 3ca\Delta}{(18c + 6)\Delta - (18c + 9)q^* + f_2} > 0 \) and \( f_2 < 0 \). Thus

\[
\frac{2 + 2\Delta - 3q}{3 + f(c)} \geq f_1\Delta + \frac{2c\Delta + 3ca\Delta - f_1f_2\Delta}{(18c + 6)\Delta - (18c + 9)q^*}.
\]

If \( \Delta \geq 66 \), then \( q = \frac{2\sqrt{2}(\Delta-1)-2}{2\sqrt{2}+1} \). Plugging \( c = 18 \) into the inequality (12), we get \( f_1 \geq -0.04638 \), \( f_2 \geq -244.43905 \) and

\[
\frac{2 + 2\Delta - 3q}{3 + f(c)} \geq -0.04638\Delta + 1.10463.
\]

Thus

\[
d(G) \geq q - 0.04638\Delta + 1.10463 \geq 0.69241\Delta - 0.15658.
\]

If \( \Delta = 65 \), then \( q = \frac{3}{4}\Delta - 2 \). Plugging \( c = 18 \) and \( \Delta = 65 \) into the inequality (12), we get \( f_1 \geq -0.05608 \), \( f_2 \geq -1.15069 \) and \( \frac{2 + 2\Delta - 3q}{3 + f(c)} \geq -0.05608\Delta + 1.79358 \). It follows that

\[
d(G) \geq q - 0.05608\Delta + 1.79358 \geq 0.69392\Delta - 0.20642.
\]
Now we consider the case $\Delta - q - c \leq 0$. It is easy to see that $b_2 > b_1$ and $\Delta \leq 64$. Thus \( \sum_{x \in V(G)} M'(x) \geq qn - (3q - 2\Delta - 2)|X_1| \). So \( \bar{d}(G) \geq q - (3q - 2\Delta - 2)\frac{|X_1|}{n} \). By Claims 3.3 and 3.5, we have

\[
n \geq |X_1| + |N(X_1)| + |Z_1(c)| \geq (3 + f'(c))|X_1|,
\]

where \( f'(c) = 2^{\frac{5c+2}{c\Delta} - \frac{6c+3}{9\Delta} + \frac{c+2}{3+f'(c)}} \).

Plugging \( c = 18 \) and \( q = \frac{3}{4}\Delta - 2 \) into \( f'(c) \), we have \( f'(c) = \frac{8.75\Delta + 278}{9\Delta} \). Since \( \frac{|X_1|}{n} \leq \frac{1}{3+f'(c)} \), we have \( (3q - 2\Delta - 2)\frac{|X_1|}{n} \leq \frac{2.25\Delta^2 - 72\Delta}{35.75\Delta + 278} = \frac{9\Delta}{143} + \frac{695.94484}{35.75\Delta + 278} = 2.50339 \), thus \( \bar{d}(G) \geq \frac{393\Delta}{572} - \frac{695.94484}{35.75\Delta + 278} + 0.50339 \). It is easy to check that if \( \Delta \geq 56 \) then \( \bar{d}(G) \geq \frac{2}{3}(\Delta + 2) \), which improve Woodall’s result in [15]. If \( \Delta \geq 56 \), we have

\[\bar{d}(G) \geq 0.68706\Delta + 0.19815.\]

Hence,

\[\bar{d}(G) \geq \begin{cases} 0.69241\Delta - 0.15658 & \text{if } \Delta \geq 66, \\ 0.69392\Delta - 0.20642 & \text{if } \Delta = 65, \text{ and} \\ 0.68706\Delta + 0.19815 & \text{if } 56 \leq \Delta \leq 64. \end{cases}\]

This completes the proof of Theorem 1.

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