Natural frequency of functionally graded plates resting on elastic foundation using finite element method

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Abstract

This paper deals with the numerical solution for free vibration analysis of simply supported functionally graded material (FGMs) plates on elastic foundation. The displacement filed based on the higher order shear deformation theory (HSDT) with 13 degree of freedom is used. The foundation is described by the Pasternak (two-parameter) model. A power law distribution is used to explain the variation of mechanical and physical properties through the thickness of the FGMs plate. The effect of volume fraction index, geometric configuration and foundation parameter on the natural frequency of FGMs plate is investigated. The results reveal that the Pasternak (shear) elastic foundation significantly affects the natural frequency.

Keywords: Functionally graded material, volume fraction index, foundation parameter, geometric configuration

1. Introduction:

Functionally graded materials (FGMs) are microscopically inhomogeneous composites usually made from a mixture of metals and ceramics. These high technological materials are fabricated by varying the percentage content of two or more materials such that the new materials have the desired gradient property in spatial directions. The concept of FGM was proposed in 1984 by materials scientists in the Sendai area as a means of preparing thermal barrier materials [1-2]. However, since last several decades, the application of FGMs has been reported in various fields like aerospace, nuclear, civil, biomechanical, electronic, and mechanical industries.

Due to its wide range of applications, numerous researches about functionally graded material in the field of structural application has been carried out over the past three decades. For instance, Praveen and Reddy [3] investigated the vibration characteristics of functionally graded materials (FGMs) plate by using finite element method. The transverse shear strains and rotary inertia are considered during the analysis. In the continuation, Reddy [4] has presented solutions for FGMs rectangular plates based on his third-order shear deformation plate theory.
Vibration response of FGMs simply-supported and clamped rectangular thin plates were obtained by Abrate [5] using the Classical Plate theory.

Main and Spencer [6] studied the exact three dimensional (3D) solutions for FGMs plates with essential traction-free boundary condition. Khoma [7] established a generalized solution of the equilibrium equations for heterogeneous transversely isotropic plates with elastic moduli that depend linearly on the transverse co-ordinate. Kashtalyan [8] used Plevako general solution to obtain a 3D elasticity solution for a FGMs plate. Vel and Batra [9] also presented a three-dimensional solution for free and forced vibrations of simply supported rectangular plates.

Qian et al. [10] investigated the free and forced vibrations and static deformations of an FG thick simply-supported square plate by using a higher-order shear and normal deformable plate theory and a meshless local Petrov–Galerkin method. Based on the first and third-order shear deformation plate theories, an analysis of free vibrations of FG plates has been presented by Ferreira et al. [11]. Zhao et al. [12] used element-free kp-Ritz method on the basis of the first order shear deformation theory (FSDT) to get the vibration response of FGMs square and skew plates with different boundary conditions. Matsunaga [13] developed a 2-D higher-order deformation theory to investigate the free vibration problem of isotropic elastic plates. Roque et al. [14] investigated the free vibration of FG plates with different combinations of boundary conditions by the multiquadric radial basis function method and the HSDT. Talha and Singh [15-19] considered HSDT to investigate the effects of different parameters like volume fraction, thickness ratio etc. on the vibration characteristics of FGMs plate.

Plates resting on elastic foundations have found significant applications in structural engineering problems. Several civil or mechanical engineering problems, such as building infrastructures, tanks or silos foundations, aerospace engineering, etc. are well-known direct applications of these kinds of plates. Some research has been reported on the aforementioned topic, for example Yang and Shen [20] investigated the effect of Pasternak elastic foundation on the vibration and transient response of initially stressed FGMs rectangular thin plates subjected to impulsive lateral loads. Malekzadeh [21] used the 3D elasticity theories to find the free-vibration response of simply-supported FGMs plates resting on elastic foundations. Thai et al. [22] has suggested a refined shear deformation theory for bending, buckling, and vibration of plates on elastic foundation. This theory was based on the assumption that the in-plane and transverse displacements consist of bending and shear components.

Hashemi et al. [23] used first order shear deformation theory to find the analytical solutions for free vibration analysis of moderately thick rectangular FGMs plates resting on Winkler or Pasternak elastic foundations. Sheikholeslami et al. [24] used higher order shear and normal deformation theory to investigate the free vibration behaviour of FGMs plate on elastic foundation. Ramu and Mohanti [25] applied Hamilton’s principle to formulate the vibrational response of FGMs plates on elastic foundation exposed to biaxial in plane periodic load. Gupta and Talha [26] presented a detailed review on recent development in modeling and analysis of functionally graded materials and structures.

The objective of the present study is to investigate the influence of volume fraction index, and foundation parameters on the vibration behaviour of FGMs plates resting on two parameter Pasternak foundation. To examine this, the author used earlier proposed theory by Talha and Singh [15] with a suitable C0 continuous isoparametric finite element with 13 degrees of freedom (DOFs) per node. The Novelty of the present approach is that it accounts both, transverse normal strain and transverse shear strain. The material properties of functionally graded material plate are assumed to vary continuously through the thickness of the plate, according to a power-law distribution of the volume fraction of the constituents. Convergence and validation studies of the present numerical solutions have been presented to ensure the accuracy and efficacy of the present approach used in the investigation.

2. Fundamental equations of functionally graded plates

Introducing a FGMs plate with dimension (a x b x h) with the Cartesian coordinate system as shown in Fig 1. The displacement components in a plate are expressed as [15]:

\[
\begin{bmatrix}
\bar{u} \\
\bar{v} \\
\bar{w}
\end{bmatrix} =
\begin{bmatrix}
1 & z & z^2 & z^3 \\
1 & z & z^2 & z^3 \\
0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
u_0 & \psi_x & \xi_x & \rho_x \\
\nu_0 & \psi_y & \xi_y & \rho_y \\
w_0 & \psi_z & \xi_z & \rho_z
\end{bmatrix},
\]

(1)

Where, \(\bar{u}, \bar{v}, \text{ and } \bar{w}\) represents the displacements of a point along the (x, y, z) coordinates. \(u_0, v_0, \text{ and } w_0\) are
corresponding displacements of a point on the mid-plane. \( \psi_x \) and \( \psi_y \) are the rotations of normal to the mid-plane about the y-axis and x-axis respectively. The functions \( \xi_x \), \( \xi_y \), \( \rho_x \) and \( \rho_y \) are the higher order terms in the Taylor series expansion defined in the mid-plane of the plate. After implementing stress free conditions at top and bottom surface of the plate, displacement field becomes:

\[
[u] = \begin{bmatrix} g_1(z) & g_2(z) & g_3(z) & g_4(z) \end{bmatrix} u_0 \begin{bmatrix} \psi_x & \phi_x & \chi_x & \theta_x \end{bmatrix}^T
\]

\[
[v] = [1 \ g_1(z) \ g_2(z) \ g_3(z) \ g_4(z)] v_0 \begin{bmatrix} \psi_y & \phi_y & \chi_y & \theta_y \end{bmatrix}^T, \quad [w] = [1 \ g_5(z) \ g_6(z)] w_0 \begin{bmatrix} \psi_z & \phi_z \end{bmatrix}^T
\]

Where, \( g_i(z) = C_i z - C_z z^3 \) \( g_i(z) = -C_i z - C_i z^3 \) \( g_i(z) = -C_i z^3 \). \( C_i (z) = C_i \), \( g_i(z) = C_i z^3 \). \( C_i = 1/2 \cdot C_i = 1/3 \cdot C_i = 1/3 \cdot C_i = 2/3 \cdot C_i = 3 \cdot C_i = 4/3 \cdot h^2 \)

It is evident from Eq (2) that the basic field variables are defined mathematically as,

\[
\{\mathbf{u} \ v \ w \ \psi_x \ \psi_y \ \psi_z \ \phi_x \ \phi_y \ \phi_x \ \chi_x \ \chi_y \ \chi_x \ \chi_y \ \theta_x \ \theta_y \}\n\]

The linear strains corresponding to the displacement field as given in Eq. (2) is represented as:

\[
\begin{bmatrix} e_{xx} \ e_{yy} \ e_{zz} \ \gamma_{yz} \ \gamma_{xz} \ \gamma_{xy} \end{bmatrix}^T = \begin{bmatrix} e_1^0 \ e_2^0 \ e_3^0 \ e_4^0 \ e_5^0 \ e_6^0 \end{bmatrix} \begin{bmatrix} \lambda_1^1 \ \lambda_2^1 \ \lambda_3^1 \ \lambda_4^1 \ \lambda_5^1 \ \lambda_6^1 \ \lambda_1^2 \ \lambda_2^2 \ 0 \ \lambda_4^2 \ \lambda_5^2 \ \lambda_6^2 \ \lambda_3^3 \ \lambda_2^3 \ 0 \ 0 \ 0 \ \lambda_6^3 \end{bmatrix}
\]

Where,

\[
E_1 = \partial u_{0,x}, \quad e_0^0 = \partial u_{0,y}, \quad e_3^0 = \partial u_{0,z}, \quad e_2^0 = \psi_x + \partial w_{0,y}, \quad e_4^0 = \psi_y + \partial w_{0,z}, \quad e_0^0 = \partial u_{0,y} + \partial v_{0,z}, \quad \lambda_1^1 = \psi_{x,y}, \quad \lambda_1^2 = \psi_{x,y},
\]

\[
\lambda_2^1 = 2 \phi_x, \quad \lambda_3^1 = \partial \phi_x, \quad \lambda_5^1 = \partial \phi_y, \quad \lambda_6^1 = \partial \phi_z, \quad \lambda_4^1 = \partial \psi_x, \quad \lambda_1^2 = \partial \psi_y, \quad \lambda_5^2 = \partial \psi_z, \quad \lambda_1^3 = \partial \psi_x + \partial \psi_y, \quad \lambda_2^3 = \partial \psi_x + \partial \psi_y, \quad \lambda_5^3 = \partial \psi_z, \quad \lambda_6^3 = \partial \psi_x + \partial \psi_y.
\]

3. Constitutive Relationship and mechanical properties:

The material properties of FG plates are assumed to vary continuously through the thickness with a desired variation of the volume fractions of the two materials in between the two surfaces. The volume fraction composition is defined using a power law function as

\[
E_i(Z) = \left[ E_i - E_m \right] \left( \frac{2Z + h}{2h} \right)^n + E_m, \quad V_c = \left( \frac{2Z + h}{2h} \right)^n (0 \leq n \leq \infty)
\]

Where \( E_i \) represents the effective material property (Young’s modulus), subscripts ‘m’ and ‘c’ represent the metal and ceramic constituents, respectively, and ‘\( n \)’ is the volume fraction exponent.

\[
\begin{bmatrix} \sigma_{xx} \ \\ \sigma_{yy} \ \\ \sigma_{zz} \ \\ \tau_{yz} \ \\ \tau_{xz} \ \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 1 & \psi_x & \phi_x & \chi_x & \theta_x \ \\ \psi_x & 1 & \phi_x & \chi_x & \theta_x \ \\ \phi_x & \psi_x & 1 & \chi_x & \theta_x \ \\ \chi_x & \phi_x & \chi_x & 1 & \theta_x \ \\ \theta_x & \chi_x & \theta_x & \theta_x & 1 \end{bmatrix} \begin{bmatrix} e_{xx} \ e_{yy} \ e_{zz} \ \gamma_{yz} \ \gamma_{xz} \ \gamma_{xy} \end{bmatrix}
\]

Where, \( \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{yz}, \tau_{xz}, \tau_{xy}\} \) and \( \{e_{xx}, e_{yy}, e_{zz}, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\} \) are stress and strains vector with respect to plate
4. Solution Methodology

This section includes the finite element formulation of the displacement field, strain and kinetic energy variation and generalized equation of motion of the plate. The displacement vector and element geometry of the model are given as,

$$\{ R \} = \sum_{i=1}^{NN} N_i \{ R \}_i,$$

$$x = \sum_{i=1}^{NN} N_i x_i, \quad y = \sum_{i=1}^{NN} N_i y_i \quad (5)$$

Where $N_i$ and $R_i$ is the shape function and the displacement vector of $i^{th}$ node respectively. $NN$ is the number of nodes per element and $x_i$ and $y_i$ are the Cartesian coordinate of the $i^{th}$ node. A nine-noded isoparametric element is used for finite element modeling, the shape function used for analysis is as follows,

$$N_i = \frac{1}{4} (\xi^2 - \xi)(\eta^2 - \eta), \quad N_2 = \frac{1}{4} (1-\xi^2)(\eta^2 - \eta), \quad N_3 = \frac{1}{4} (\xi^2 + \xi)(\eta^2 - \eta), $$

$$N_4 = \frac{1}{4} (\xi^2 - \xi)(1-\eta^2), \quad N_5 = \frac{1}{4} (1-\xi^2)(1-\eta^2), \quad N_6 = \frac{1}{4} (\xi^2 + \xi)(1-\eta^2), \quad N_7 = \frac{1}{4} (\xi^2 - \xi)(\eta^2 + \eta),$$

$$N_8 = \frac{1}{4} (\xi^2 + \xi)(\eta^2 + \eta)$$

4.1 Strain energy of the plate

If $S(e)$ is the elemental strain energy and $NE$ is the number of elements used for meshing the plate, the strain energy of the plate is given as,

$$S^e = \frac{1}{2} \int \{ \varepsilon \}^T \{ D \} \{ \varepsilon \} dA + \int \{ \varepsilon \}^T \{ D \} \{ \varepsilon \} dA, \quad S = \sum_{e=1}^{NE} S^{(e)} , \quad S = \int \frac{1}{2} \sum_{e=1}^{NE} \{ R \}^{(e)T} \{ K \}^{(e)} \{ R \}^{(e)}$$

Where $\{ D \}$ and $\{ \varepsilon \}$ are the elastic stiffness matrix and linear strain vector, respectively. $\{ K \}^{(e)}$ and $\{ R \}^{(e)}$ are the effective stiffness matrix and displacement vector for the element, respectively.

4.2 Kinetic energy of the plate

Kinetic energy of vibrating plate for total number of element $NE$ may be given as

$$\Delta = \sum_{e=1}^{MN} T^{(e)}, \quad where \Delta = \frac{1}{2} \int_{A(e)} \{ R \}^{(e)T} \{ m \}^{(e)} \{ R \}^{(e)} dA$$

Here $\{ m \}^{(e)}$ is the inertia matrix of element.

4.3 Strain energy due to foundation

The strain energy due to elastic foundation having a shear deformable layer is expressed as

$$S_f = \frac{1}{2} \int \left( K_e w^2 + K_s \left[ (w_{x})^2 + (w_{y})^2 \right] \right) dA$$

$$K_e^s = K_s \int \left( \frac{\partial N}{\partial x} \frac{\partial N}{\partial x} \left( \frac{\partial N}{\partial y} \frac{\partial N}{\partial y} + \left( \frac{\partial N}{\partial y} \frac{\partial N}{\partial y} \right) \right), K_e^s = \frac{1}{2} \int \left( \frac{\partial N}{\partial x} \frac{\partial N}{\partial x} \left( \frac{\partial N}{\partial y} \frac{\partial N}{\partial y} + \left( \frac{\partial N}{\partial y} \frac{\partial N}{\partial y} \right) \right) \right)$$

Where $K_e^s$ and $K_s$ are the Winkler’s foundation and shear layer element stiffness matrix, respectively.

4.4 Governing Equation

The governing equation of motion for free vibration of FGMs plate can be developed through the variational principle, which is the generalization of the principle of virtual displacement. It may be given as follows,

$$\left[ K_{ef} \right] \{ q \} = \mu \left[ M \right] \{ q \}$$

With $\mu = \omega^2$, where $\omega$ is defined as frequency of natural vibration. Where $\left[ M \right]$, $\left[ K_{ef} \right]$, and $\{ q \}$ are global mass
matrix, global stiffness matrix which includes linear stiffness and effective foundation stiffness, and global displacement, respectively.

5. Numerical results and discussion

5.1 Convergence and validation:

The Natural frequency of FGMs plates with simply supported edges is analyzed. In order to verify the accuracy of the present results, convergence and validation study is shown through some examples by comparing the results from the available results in the open literature.

Example 1: This example deal with the convergence study of the present solution. The upper surface of FGMs plate is ceramic-rich (Alumina (Al₂O₃)) and the lower surface is metal-rich (aluminium (Al)). The plate is simply supported at all four edges and non-dimensional frequency parameter is defined as \( \bar{\omega} = \omega h \sqrt{\rho_m / E_m} \). Mechanical properties of Al are considered as, Young’s modulus \( (E_m) = 70 \text{GPa} \) and density \( (\rho_m) = 2702 \text{Kg/m}^3 \) whereas for Alumina, Young’s modulus \( (E_c) \) and density \( (\rho_c) \) are 380 GPa and 3800 Kg/m³ respectively. The Poisson’s ratio of the plate is assumed to be constant through the thickness of the plate and is taken equal to 0.3. Frequency parameter is calculated for various mesh density and volume fraction ratio, keeping side to thickness ratio equals to 10. It is clear from the fig. 2 that the present approach shows a good convergence as the mesh density increases.

Example 2: In the second example a square FGMs plate made of aluminium (Al) and Alumina (Al₂O₃) is considered to investigate the vibration characteristic. The natural frequency parameter \( \bar{\omega} = \omega h \sqrt{\rho_m / E_m} \) is computed for various values of thickness ratio and foundation parameter which is compared with Sheikholeslami, et al [24] as shown in fig. 3. It can be observed that a good agreement exists between the results. Mechanical properties are considered same as in case of example 1.

5.2 Parametric Study

The square FGMs plate composed of stainless steel (SUS 304) and Silicon nitride (Si₃N₄) resting on two parameter Pasternak elastic foundation is considered. Non dimension frequency parameter \( \bar{\omega} = \omega h \sqrt{\rho_m / E_m} \) and non-dimensional foundation parameters \( (\phi_m = k_w a^4 / D, \phi_s = k_s a^2 / D, D = Eh^3 / 12(1-\nu^2)) \) are considered during the analysis of natural frequencies. D is the flexural rigidity of the plate. The influence of various values of foundation parameter, i.e. Winkler parameter \( (K_w) \) and shear parameter \( (K_s) \), side to thickness ratio \( (a/h) \) and volume fraction index ‘n’ on the frequency parameter are investigated in the following section.

Table 1 illustrates the variation of frequency parameters for different values of the volume fraction index and foundation parameter keeping side to thickness ratio \( (a/h) \) as 2. It is observed that when a shear parameter \( (K_s) \) is zero, then the Winkler parameter \( (K_w) \) has a significant effect on the frequency parameter. But when shear parameter \( (K_s) \) increased to 100, then it overshadows the effect of Winkler foundation. It is clear from the table that for a constant shear parameter, the frequency parameter shows insignificant effect on the variation of Winkler parameter. It is also observed that the frequency parameter decrease as the volume fraction index increases from 0 to
a higher value. This is due to fact that the larger volume fraction index means the plate has a smaller ceramic component and hence the stiffness is reduced.

Fig. 2: Convergence of non-dimensional frequency parameter for various values of mesh density for square plate

Fig 3: Comparison of non-dimensional parameter for various values of thickness ratio (h/a) from present analysis and Sheikholeslami et al. [24].

Similar trend can be observed in table 2 and 3 in which side to thickness ratio (a/h) is considered as 5 and 10 respectively. It is also notable that as side to thickness ratio increase, the effect of the Winkler parameter increase. Hence it can be concluded that for thin plates, the effect of shear parameter of elastic foundation on the natural frequency is remarkable as compared to the effect of Winkler parameter. However, as the side to thickness ratio decreases, this effect tends to vanish and there will be almost no change in frequencies for thick plates.

Table 1 Variation of non-dimensional frequency parameter with volume fraction index ‘n’ for different values of elastic foundation parameters (Kw and Ks) (a/b=1, a/h=2)

| Kw  | Ks  | Volume fraction index ‘n’ |
|-----|-----|---------------------------|
|     | 0   | 1 | 5 | 100 |
| 0   |     | 1.9285 | 1.1668 | 0.9312 | 0.8412 |
| 10  | 0   | 2.0264 | 1.2381 | 0.9974 | 0.9060 |
| 100 |     | 2.3735 | 1.4941 | 1.1547 | 1.0334 |
Table 2 Variation of non-dimensional frequency parameter with volume fraction index ‘n’ for different values of elastic foundation parameters (Kw and Ks) (a/b=1, a/h=5)

| Kw | Ks | Volume fraction index ‘n’ |
|----|----|--------------------------|
|    |    | 0                        |
|    |    | 1                        |
|    |    | 5                        |
|    |    | 100                      |
| 0  | 10 | 0.4430                   |
| 10 | 0  | 0.5438                   |
| 100| 0  | 0.6176                   |
| 1000|0  | 0.7728                   |

Table 3 Variation of non-dimensional frequency parameter with volume fraction index ‘n’ for different values of elastic foundation parameters (Kw and Ks) (a/b=1, a/h=10)

| Kw | Ks | Volume fraction index ‘n’ |
|----|----|--------------------------|
|    |    | 0                        |
|    |    | 1                        |
|    |    | 5                        |
|    |    | 100                      |
| 0  | 10 | 0.1231                   |
| 10  | 0  | 0.1653                   |
| 100 | 0  | 0.2980                   |
| 1000|0  | 0.2980                   |

6. Conclusion:
The higher-order shear deformable plate theory of Talha and Singh [18] has been used to determine the natural frequencies of simply supported functionally graded square plate resting on elastic foundation. The influence of volume fraction index, geometric configuration and foundation parameter on the vibration behaviour of FGMs plate is investigated. The results show that the Pasternak elastic foundation has more significant role in increasing the natural frequency. The shear parameter has a dominant influence on the natural frequency of the FGMs plate compared to the Winkler foundation parameter. It is also concluded that for thin plates or moderately thick plate, the effect of shear parameter of elastic foundation on the natural frequency is significant as compared to thick plate.

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