Anomaly From Quantum Fields’ Equation of Motion, and
Transverse Ward-Takahashi Identities

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Abstract

We exploit the idea that anomaly arises from quantum fields’ equations of motion, and show anomalous conservation laws is nothing but special cases of anomalous equations of motion, thus promote anomaly to cover all kinds of quantum fields’ equations, besides conservation laws. As a straightforward application, it is shown that there is anomaly associated with transverse vector Ward-Takahashi identities. The transverse anomaly is calculated using zeta function regularization, and verified to 1-loop order in perturbation theory. Missing terms in former articles ignoring transverse anomalies in last 20 years are picked out. Role of transverse anomaly is discussed.
I. INTRODUCTION

Equations of motion have always been the foremost thing in physics, since they encode full dynamics of a physical system. Nowadays, quantum fields’ equations of motion (or Dyson-Schwinger equations) become more and more concerned by theorists, especially in the scope of non-perturbative QCD, due to their non-perturbative nature. However, infinite degrees of freedom inherent in field theories produce a lot of singularities in equations of motion, most famous among which must be ultraviolet and infrared divergences. Besides, as we’ll show, anomaly is also a general class of singularities, entirely predictable from quantum fields’ equations of motion, moreover, we shall extend previous definition of anomaly, which is focused to symmetries where anomaly was discovered historically\[1 – 3\], to cover more general equations of motion rather than conservation laws only.

Some efforts have been made towards this assert. Decades ago, \[4, 5\] observed that chiral anomaly can arise from additional terms in quantum equation of motion for fermion field compared to its classical version, and reproduced the same anomaly as that from transformation Jacobian of path integral measure. It’s inspiring to review this in a few lines.

Consider the following fermion field’s equation of motion\[5, 6\] expressed in Green’s function with time-ordered product implying causality required by relativity, with spinor indices shown explicitly (derivatives are outside $T$-product thus act also on terms like $\theta(x^0 - y^0)$ in $T$-product) :

$$\langle Ti/Dx^k \psi(x) \bar{\psi}_n(y) \rangle = m \langle T\psi_k(x) \bar{\psi}_n(y) \rangle + i\delta_{kn}\delta^4(x - y).$$  \hspace{1cm} (1)

To see how chiral anomaly is assembled, contract $(\gamma_5)_{nk}$ in both sides of (1), and take the $y \to x$ limit, then familiar expression $\text{tr}[\gamma_5]\delta^4(x - x)$, a signal for chiral anomaly, appears\[1\] :

$$\langle \bar{\psi}(x)\gamma_5 iD\psi(x) \rangle = m \langle \bar{\psi}(x)\gamma_5\psi(x) \rangle - i\text{tr}[\gamma_5]\delta^4(x - x).$$  \hspace{1cm} (2)

Together with (2)’s Hermitian conjugation (i.e. equation of motion for $\bar{\psi}$) :

$$\langle \bar{\psi}(x)i\bar{D}_5 \gamma_5\psi(x) \rangle = -m \langle \bar{\psi}(x)\gamma_5\psi(x) \rangle + i\text{tr}[\gamma_5]\delta^4(x - x),$$  \hspace{1cm} (3)

anomalous partial conservation equation for axial current $j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma_5\psi$ is obtained \[D_\mu = -

1 Minus sign due to Fermi-statistics, and we drop time-ordered symbol in equal time limit.
\( \partial_{\mu} - igA_{\mu}, \hat{D}_{\mu} = \partial_{\mu} + igA_{\mu} : \)
\[
\begin{align*}
\partial_{\mu} j_{\mu}^\alpha(x) \\
&= -i \left\langle \bar{\psi}(x)i\gamma_\mu \partial_{\mu} \psi(x) \right\rangle + i \left\langle \bar{\psi}(x)i\gamma_5 D_{\mu} \psi(x) \right\rangle \\
&= 2im \left\langle \bar{\psi}(x)i\gamma_5 \psi(x) \right\rangle + 2\text{tr}[\gamma_5] \delta^4(x-x) .
\end{align*}
\]  

This is exactly what Fujikawa got by calculating transformation Jacobian of path integral measure (before regularization). Above example shows chiral anomaly is completely controlled by fermion fields’ equations of motion. Also, it’s easy to generalize to other anomalies such as trace anomaly from conformal symmetry.

Clearly, singular \( i\delta_{kn} \delta^4(x-x) \) serves as the basic element to construct more complicated anomalies. From above discussion, we see that it is originated from delta function term (often called contact term) at the R.H.S. of the equation of motion, for which we call the contact term “anomaly of equations of motion”. And equations with such anomalies, such as (1)(2)(3), may be called “anomalous equations of motion”.

Now that the contact term takes full responsibilities for anomaly, its origination, i.e. derivation of (1), deserves some words. Calculate (1) term by term (for simplicity we take \( A_{\mu} \) to be an external field):
\[
\begin{align*}
\left\langle T iD^\tau_{k\ell} \psi_\ell(x) \bar{\psi}_n(y) \right\rangle \\
= i(\gamma^\mu)_{k\ell} (\partial_{\mu} - igA_{\mu}(x)) \left\{ \theta(x^0 - y^0) \left\langle \psi_\ell(x) \bar{\psi}_n(y) \right\rangle - \theta(y^0 - x^0) \left\langle \bar{\psi}_n(y) \psi_\ell(x) \right\rangle \right. \\
+ \left. i\delta(x^0 - y^0)(\gamma^0)_{k\ell} \left\{ \psi_\ell(x^0, \vec{x}), \bar{\psi}_n(x^0, \vec{y}) \right\} \right\} \\
= \left( \theta(x^0 - y^0) \left\langle m\psi_\ell(x) \bar{\psi}_n(y) \right\rangle - \theta(y^0 - x^0) \left\langle \bar{\psi}_n(y) m\psi_\ell(x) \right\rangle \right) \\
+ i\delta(x^0 - y^0)(\gamma^0)_{k\ell} \delta_{ij} \delta^3(x-y)(\gamma^0)_{jn} \\
= m \left\langle T \psi_\ell(x) \bar{\psi}_n(y) \right\rangle + i\delta_{kn} \delta^4(x-y) .
\end{align*}
\]  

In the first step, we use definition of time-ordered product, then we only distribute the derivative according to Leibniz law in the next step. In the third step, we use \( i\hat{D}\psi = m\psi \) and \( \left\{ \psi_\ell(x^0, \vec{x}), i\psi^\dagger_\ell(x^0, \vec{y}) \right\} = i\delta_{\ell\ell} \delta^3(x-y) \), which are both basic principles in canonical quantization. Equations of motion exist already in classical theory, but anti-commutation relations are totally quantum mechanical. Thus the contact term, or anomaly of equations of motion by our definition, originated from anti-commutation relations, is exactly the difference between classical and quantum mechanics and must be paid some attentions.
However, for (2), there is not any time-ordered product, so presence of $-i\text{tr}[\gamma_5]\delta^4(x - x)$ seems somehow weird. Actually, we have defined $\langle \bar{\psi}(x)\gamma_5 i\mathcal{D}\psi(x) \rangle$ to be $\lim_{y \to x} \langle T\bar{\psi}(y)\gamma_5 i\mathcal{D}\psi(x) \rangle$ in working out (2) thus acquire $\text{tr}[\gamma_5]\delta^4(x - x)$ by taking (1)’s limit, otherwise, naive product of $\bar{\psi}(x)\gamma_5$ and $i\mathcal{D}\psi(x)$ can never produce terms like $\text{tr}[\gamma_5]\delta^4(x - x)$ as long as we admit the operator equation $i\mathcal{D}\psi = m\psi$ which is basic in canonical quantization, thus chiral anomaly cannot be shown explicitly. (In fact, our definition can be seen as a specific regularization scheme since product of two operators is generally ill-defined.) Anyway, by calculating relative Feynman diagrams (e.g. famous triangle diagram [3, 6] for $\langle Tj_\mu j_\nu j_\rho \rangle$) in perturbation theory, we always pick up an anomaly corresponding to (regularized) $\text{tr}[\gamma_5]\delta^4(x - x)$. That’s because the basic element in every Feynman diagram, the Feynman propagator, is itself a result of time-ordered product, thus echoes with our definition rather than definition multiplying two operators naively. Also, in path integral formalism, everything should be a relic of time-ordered product, by definition of path integral [6]. So the appropriate definition for the quantum operator $\bar{\psi}(x)\gamma_5 i\mathcal{D}\psi(x)$, which accommodates with Feynman diagram and path integral that do produce chiral anomaly which is well tested by $\pi^0 \to 2\gamma$ experiment, must lead to (2) rather than that without delta function term, i.e. $\bar{\psi}(x)\gamma_5 i\mathcal{D}\psi(x)$ as a composite operator cannot be naive product of $\bar{\psi}(x)\gamma_5$ and $i\mathcal{D}\psi(x)$.

Above discussion can be summarized as a rule for quantization of composite operators like $\bar{\psi}(x)\gamma_5 i\mathcal{D}\psi(x)$: besides promoting its classical expression to an operator, we must define (or more practically, regularize) the composite operator to be coincidence limit of time-ordered product of single operators and space-time derivatives of single operators should be outside time-ordered product. (Also, derivatives of the whole composite operator should still respect Leibniz law and must be distributed to single operators before above regularization by time-ordered product, corresponding to the first step in (4).) One more word, this rule hasn’t invalidate the equations of motion such as $i\mathcal{D}\psi = m\psi$, but it only reminds us, if we are to use equations of motion inside a composite operator, we must pay attention to possible anomaly from equations of motion arose from coincidence limits of contact terms in time-ordered product. And chiral anomaly in (4) just comes out this way.

There may be questions about difference between anomaly terms from our rule and Schwinger terms [9] as well as seagull terms, since all of them are modification to naive definition of some objects, i.e. to composite operators for our rule, to canonical commutators for Schwinger terms [9, 10], to time-ordered products for seagull terms [10], and they
are all able to work out anomalies in operator level. The most important difference is, our rule gives canonical framework the power to entirely predict an anomaly just like path integral does by watching transformation Jacobian of the measure, but coefficients of possible Schwinger terms and seagull terms must be matched with results from perturbation theories[10] thus no prediction about existence of anomalies can be made without Feynman diagrams. Moreover, Schwinger terms and seagull terms are non-canonical contributions [9, 10] added by hand, however, our scheme is still inside canonical framework, so nothing is essentially “anomalous” thus no needs for peculiar things such as Schwinger and seagull terms that come from unknown.

In this paper, we focus on anomalies from equations of motion in the coincidence limit of the two space-time points, i.e. singular contact terms, and show that all traditional anomalies caused by symmetry can be derived from these singular contact terms (this is trivially established after we show all singular contact terms can be attached to a specific transformation in path integral in Sec. II, since all traditional anomalies are consequences of non-trivial Jacobians for some symmetry transformations). The non-coincidence limits of anomaly of equations of motion, i.e. regular contact terms, are proportional to $\delta^4(x - y)$ thus vanish except for $x = y$, so we actually take all anomalies of equations of motion into consideration. Further more, what[4, 5] mentioned but haven’t specified is, anomalies from equations of motion are richer than that from obstructions to promote a classical symmetry to its quantum version, since all conservation laws are derivable from equations of motion but equations of motion include not only conservation laws (this is also embodied in Sec. II).

In a word, what we want to emphasize in this paper is that, equations of motion like (1), the most fundamental equations in quantum field theories, are able to derive any other types of identities (e.g. transverse Ward-Takahashi identities in this paper), apart from anomalous conservation laws, thus anomalies from equation of motion include anomalies from symmetries and much more.

As an example, we examine anomalies in transverse Ward-Takahashi identities (tWTI)[11–16], which are not consequences of any genuine symmetries, and find non-trivial anomalies in vector tWTI, which is different from previous articles[11, 17–19] that may be results of ignorance of anomaly from equations of motion. To consolidate existence of this transverse anomaly, we employ $\zeta$ function regularization (App.A), Pauli-Villars regularization (App.B) and dimensional regularization (App.C), all of which give consistent results. We also pick
This paper is organized as follows. In Sec.II, we attach anomalies from equations of motion to transformation Jacobian of path integral measure, just as [4, 5] did, but we show this admits anomalies not only from symmetries. In Sec.III, we analyze anomaly in Abelian tWTI. In Sec.IV, we generalize transverse anomaly to non-Abelian case. In Sec.V, we analyze articles ignoring transverse anomaly and point out where transverse anomaly appears in their procedure. In Sec.VI, we conclude, and discuss possible applications of transverse anomaly.

In this paper, space-time metric is $g_{\mu\nu} = \text{diag}(+, -, -, -)$, and $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\epsilon^{0123} = +1$. Define $\Sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

II. ANOMALIES FROM EQUATION OF MOTION, AND PATH INTEGRAL MEASURE

This section is to show that, all identities (perhaps with anomalies) originated from some transformations in path integral, can be derived from equations of motion, and vice versa, as have been showed partly in [4, 5].

To be most general, consider a field $\phi$ of any type with anomalous equation of motion (space-time derivatives in $\frac{\delta S[\phi]}{\delta \phi(x)}$ should be outside time-ordered product): 

$$ \langle T \frac{\delta S[\phi]}{\delta \phi(x)} F(\phi(y)) \rangle = i \langle \frac{\partial F(\phi(x))}{\partial \phi(x)} \rangle \delta^4(x - y). \quad (6) $$

($F(\phi(x))$ is any function of $\phi(x)$.)

(6) can be linked to a transformation with trivial Jacobian, employ $\delta \phi(x) = \alpha(x)$ (where $\alpha(x)$ can be any infinitesimal function irrelevant to $\phi(x)$) in $\int [d\phi] e^{iS[\phi]} F(\phi(y))$, then (6) arises. Non-trivial Jacobians appear only in the $y \to x$ limit. Make the substitution $\phi'(x) = \phi(x) + \alpha(x)F(\phi(x))$ in $\int [d\phi] e^{iS[\phi]}$. Since path integral is unaffected by renaming variables, to first order of $\alpha(x)$, we have ($J = e^{\ln J} \equiv 1 + \ln J$):

$$ 0 = \int [d\phi'] e^{iS[\phi']} - \int [d\phi] e^{iS[\phi]} $$

$$ = \int J [d\phi] e^{iS[\phi] + i \int d^4x \frac{\delta S[\phi]}{\delta \phi(x)} \alpha(x) F(\phi(x))} - \int [d\phi] e^{iS[\phi]} $$

$$ = \int [d\phi] e^{iS[\phi]} \left( \ln J + i \int d^4x \frac{\delta S[\phi]}{\delta \phi(x)} \alpha(x) F(\phi(x)) \right). \quad (7) $$
Where $\ln J$ is logarithmic Jacobian, in first order of $\alpha(x)$ (we use $\ln(1 + x) \cong x$ for $x \to 0$):

$$
\ln J = \ln \text{Det} \left[ \frac{\delta (\phi(x) + \alpha(x) F(\phi(x)))}{\delta \phi(y)} \right] \\
= \text{tr} \left[ \ln \left( 1 + \alpha(x) \frac{\partial F(\phi(x))}{\partial \phi(x)} \delta^4(x - y) \right) \right] \\
= \int d^4 x \text{ tr} \left[ \alpha(x) \frac{\partial F(\phi(x))}{\partial \phi(x)} \right] \delta^4(x - x).
$$

(8)

Throw (8) into (7), and drop $\int d^4 x \alpha(x)$ using arbitrariness of $\alpha(x)$, then we acquire

$$
\langle \delta S[\phi] \delta \phi(x) F(\phi(x)) \rangle = i \langle \partial F(\phi(x)) \rangle \delta^4(x - x),
$$

the same equation with anomaly as that obtained from $y \to x$ limit of equation of motion (6).

Hence, the anomalous equation of motion

$$
\langle \delta S[\phi] \delta \phi(x) F(\phi(x)) \rangle = i \langle \partial F(\phi(x)) \rangle \delta^4(x - x),
$$

is linked with a specific transformation $\delta \phi(x) = \alpha(x) F(\phi(x))$. Moreover, note that anomalous conservation laws, such as that with chiral anomaly and trace anomaly, are all consequences of non-trivial Jacobian of some special forms of $\delta \phi(x) = \alpha(x) F(\phi(x))$, thus automatically included in anomalous equations of motion. But our derivation in this section put no constraints to the form of $\delta \phi(x) = \alpha(x) F(\phi(x))$, thus admitting anomalous equations from non-symmetry transformations, such as tWTI caused by transverse transformations in Sec.III.

III. ANOMALY IN TRANSVERSE WARD-TAKAHASHI IDENTITIES: ABELIAN CASE

Transverse Ward-Takahashi identities (tWTI) for both vector and axial vector current have been proposed in Abelian and non-Abelian case, in this section we focus on Abelian case only and non-Abelian generalization is left to next section.

Abelian vector tWTI is often presented as the following form (without anomaly):

$$
\partial_\mu \langle T_j^{\nu}(x) \psi(y) \bar{\psi}(z) \rangle - \partial_\nu \langle T_j^{\mu}(x) \psi(y) \bar{\psi}(z) \rangle \\
= i \sigma^{\mu \nu} \langle T \bar{\psi}(y) \bar{\psi}(z) \rangle \delta^4(x - y) + i \langle T \bar{\psi}(y) \bar{\psi}(z) \rangle \sigma^{\mu \nu} \delta^4(x - z) \\
+ i \epsilon^{\mu \nu \rho \sigma} \left( \partial_\rho - \partial'_\rho \right) \langle T \bar{\psi}(x') \gamma_\sigma \gamma_5 e^{ig \frac{1}{\hbar} \int^x dx' A(x) \psi(y) \bar{\psi}(z) \rangle \rangle_{x' \to x} \\
+ 2m \langle T \bar{\psi}(x) \sigma^{\mu \nu} \psi(x) \psi(y) \bar{\psi}(z) \rangle
$$

(9)
and axial vector tWTI is:

\[
\partial_\mu \langle T_{\mu5}^\nu(x)\psi(y)\bar{\psi}(z) \rangle - \partial_\nu \langle T_{\nu5}^\mu(x)\psi(y)\bar{\psi}(z) \rangle \\
= i\sigma^{\mu\nu}\gamma_5 \langle T\psi(y)\bar{\psi}(z) \rangle \delta^4(x - y) - i \langle T\psi(y)\bar{\psi}(z) \rangle \sigma^{\mu\nu}\gamma_5 \delta^4(x - z) \\
+ i\epsilon^{\mu\nu\rho\sigma} \left( \partial_\rho - \partial_\rho' \right) \left( T\bar{\psi}(x')\gamma_\rho e^{i} A_\rho x' d_y A_\rho A_\rho^* \psi(y) \bar{\psi}(z) \right) \bigg|_{x' \to x}. 
\]

(10)

These are not (partial) conservation equations for any currents since transformations in (12) leading to tWTIs with \( \alpha(x) = \text{Const.} \) don’t leave Lagrangian or action invariant, even with \( m \to 0 \). So tWTI is a proper example to illustrate the richness of anomalies from equations of motion over anomalies from quantum obstacles to classical symmetries.

Infinitesimal variation of fermion fields that leads to above tWTI have been found\[13\], so it’s not necessary to resort to canonical derivation\[12\] to locate singular equations of motion that contribute to anomaly, since equivalence between anomalies from equation of motion and transformation Jacobian has been proved in Sec[III].

For QED Lagrangian in covariant \( R_\xi \) gauge\[6\]

\[
\mathcal{L}_{\text{QED}} = \frac{1}{2} \bar{\psi}(D \psi) - \frac{1}{2} \bar{\psi} D \psi - m \bar{\psi}\psi - \frac{1}{2\xi} (\partial_\mu A^\mu)^2, 
\]

(11)

infinitesimal variations of fields that lead to vector and axial-vector tWTI are respectively\[13\]

(vector tWTI) \( \delta\psi(x) = + \frac{1}{4} \alpha(x) \epsilon_{\mu\nu}\sigma^{\mu\nu}\psi(x) \),

\( \delta\bar{\psi}(x) = + \frac{1}{4} \alpha(x) \epsilon_{\mu\nu}\bar{\psi}(x)\sigma^{\mu\nu} \).

(axial tWTI) \( \delta\psi(x) = + \frac{1}{4} \alpha(x) \epsilon_{\mu\nu}\sigma^{\mu\nu}\gamma_5\psi(x) \),

\( \delta\bar{\psi}(x) = - \frac{1}{4} \alpha(x) \epsilon_{\mu\nu}\bar{\psi}(x)\sigma^{\mu\nu}\gamma_5 \).

(12)

(\( \epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \) and \( \alpha(x) \) is the infinitesimal parameter)

Transformation Jacobian of (12) can be calculated similar to (8). Clearly, if (properly regularized) \( \text{tr}[\sigma^{\mu\nu}]\delta^4(x - x) \neq 0 \), vector tWTI is going to have an anomaly term \( (-2i)\text{tr}[\sigma^{\mu\nu}]\delta^4(x - x) \) in R.H.S. of (9). However, whether \( \text{tr}[\sigma^{\mu\nu}\gamma_5]\delta^4(x - x) \) vanishes or not\[2\], axial vector tWTI can never have anomaly since Jacobian of \( \delta\psi \) and \( \delta\bar{\psi} \) cancel with each other due to the minus sign in \( \delta\bar{\psi} \).

\(^2\) In fact, since \( \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = -2i\sigma^{\mu\nu}\gamma_5 \), \( \text{tr}[\sigma^{\mu\nu}\gamma_5]\delta^4(x - x) = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\text{tr}[\sigma_{\rho\sigma}]\delta^4(x - x) \).
According to our calculation in App\[A\], i.e. \([A1]\), \(\text{tr}[\sigma^{\mu \nu}]\delta^4(x-x)\) is not zero, but acquires the finite form (after \(\zeta\) function regularization) which may be called transverse anomaly:

\[
\left\{ \text{tr}[\sigma^{\mu \nu}]\delta^4(x-x) \right\}_\zeta = \frac{i}{2} \left( -\frac{gm^2}{2\pi^2} F^{\mu \nu} - \frac{g}{12\pi^2} \partial^\rho \partial_\rho F^{\mu \nu} \right) - \frac{g}{12\pi^2} \partial_\rho \partial^\rho \left( \partial^\alpha \partial_\alpha F^{\rho \nu} - \partial^\nu \partial_\nu F^{\rho \mu} \right). \tag{13}
\]

So the anomaly corrected Abelian vector tWTI is:

\[
\partial^\mu \left\{ T j^\nu(x)\psi(y)\bar{\psi}(z) \right\} - \partial^\nu \left\{ T j^\mu(x)\psi(y)\bar{\psi}(z) \right\} = i\sigma^{\mu \nu} \left\{ T \psi(y)\bar{\psi}(z) \right\} \delta^4(x-y) + i \left\{ T \psi(y)\bar{\psi}(z) \right\} \sigma^{\mu \nu} \delta^4(x-z)
\]

\[
+ i e^{\mu \nu \rho \sigma} \left( \partial_\rho \bar{\psi} - \partial_\sigma \psi \right) \left\{ T \bar{\psi}(x')\gamma_\sigma \gamma_5 e^{ig} \int_\varepsilon dx' A^\nu(x')\psi(y)\bar{\psi}(z) \right\}_{x' \rightarrow x}
\]

\[
+ 2m \left\{ T \bar{\psi}(x)\sigma^{\mu \nu} \psi(x)\bar{\psi}(y) \right\} + \frac{gm^2}{2\pi^2} \left\{ TF^{\mu \nu}(x)\psi(y)\bar{\psi}(z) \right\} - \frac{g}{12\pi^2} \partial^\rho \partial_\rho \left\{ TF^{\mu \nu}(x)\psi(y)\bar{\psi}(z) \right\}.
\tag{14}
\]

Further more, with the help of Bianchi identity (already imposed in \([13]\)) and gauge fields’ equation of motion\[^3\]:

\[
\partial^\mu \left\{ T F^{\mu \nu}(x)\psi(y)\bar{\psi}(z) \right\} + \frac{1}{\zeta} \partial^\nu \partial^\mu \left\{ T A^\mu(x)\psi(y)\bar{\psi}(z) \right\} = -g \left\{ T j^\nu(x)\psi(y)\bar{\psi}(z) \right\}, \tag{15}
\]

it’s legitimate to absorb the second term of transverse anomaly into coefficient of \(\partial^\mu j^\nu\) (the gauge dependent \(\frac{1}{\zeta}\) term in \([15]\) cancels due to anti-symmetrization):

\[
\left( 1 - \frac{g^2}{12\pi^2} \right) \left\{ T j^\nu(x)\psi(y)\bar{\psi}(z) \right\} - \partial^\nu \left\{ T j^\mu(x)\psi(y)\bar{\psi}(z) \right\}
\]

\[
= i\sigma^{\mu \nu} \left\{ T \psi(y)\bar{\psi}(z) \right\} \delta^4(x-y) + i \left\{ T \psi(y)\bar{\psi}(z) \right\} \sigma^{\mu \nu} \delta^4(x-z)
\]

\[
+ i e^{\mu \nu \rho \sigma} \left( \partial_\rho \bar{\psi} - \partial_\sigma \psi \right) \left\{ T \bar{\psi}(x')\gamma_\sigma \gamma_5 e^{ig} \int_\varepsilon dx' A^\nu(x')\psi(y)\bar{\psi}(z) \right\}_{x' \rightarrow x}
\]

\[
+ 2m \left\{ T \bar{\psi}(x)\sigma^{\mu \nu} \psi(x)\bar{\psi}(y) \right\} - \frac{gm^2}{2\pi^2} \left\{ TF^{\mu \nu}(x)\psi(y)\bar{\psi}(z) \right\}. \tag{16}
\]

IV. NON-ABELIAN TRANSVERSE ANOMALY

Generalization of transverse anomaly to non-Abelian case (with gauge group \(SU(N)\)) is quite easy. Employ the Lagrangian in covariant \(R_\zeta\) gauge with gauge-fixing terms and

\[^3\] From Sec\[II\] this equation of motion doesn’t carry anomaly unless we insert additional \(A^\mu(x)\) or \(F^{\mu \nu}(x)\) into Green’s function.
In path integral framework, apply the variation of fermion fields (this is different from \[13\] and we will explain later)

\[
\delta \psi(x) = \frac{1}{4} \epsilon_{\mu\nu} \alpha_a(x) t_a \sigma^{\mu\nu} \psi(x), \quad \delta \bar{\psi}(x) = \frac{1}{4} \epsilon_{\mu\nu} \alpha_a(x) \bar{\psi}(x) t_a \sigma^{\mu\nu}.
\]  

(18)

Take transformation Jacobian (i.e. the transverse anomaly) into account (see \[\text{App.A}\]), then we get vector tWTI in \(SU(N)\) gauge theory:

\[
\langle TD^{\mu} j_\nu(x) \psi(y) \bar{\psi}(z) \rangle - \langle TD^{\nu} j_\mu(x) \psi(y) \bar{\psi}(z) \rangle = i \sigma^{\mu\nu} t_a \langle T \bar{\psi}(x) \gamma_\sigma \gamma_5 \left\{ e^{igt_a f'_{\rho\sigma}} d_\rho A_\nu, t_a \right\} \psi(y) \bar{\psi}(z) \rangle_{x' \to x} \]

\[
+ 2m \langle T \bar{\psi}(x) \sigma^{\mu\nu} t_a \psi(x) \psi(y) \bar{\psi}(z) \rangle
+ \frac{g m^2}{4\pi^2} \langle TF^{\mu\nu}_a(x) \psi(y) \bar{\psi}(z) \rangle - \frac{g}{24\pi^2} \langle TD^{\mu}_\rho F^{\nu\mu}_a(x) \psi(y) \bar{\psi}(z) \rangle
- \frac{g^2}{8\pi^2} C_{abc} \langle TF^{\nu\mu}_b(x) F^{\mu\nu}_c(x) \psi(y) \bar{\psi}(z) \rangle.
\]  

(19)

(All derivatives should be outside time-ordered product.)

With the aid of equation of motion for gauge field (no anomaly with the same reason as Abelian case):

\[
\langle TD^{\mu}_\rho F^{\nu\mu}_a(x) \psi(y) \bar{\psi}(z) \rangle + g C_{abc} \langle T (\partial^{\nu}_x \bar{c}_b(x)) c_c(x) \psi(y) \bar{\psi}(z) \rangle
+ \frac{1}{\xi} \partial^{\nu}_x \partial^{\rho}_\mu \langle T A^{\mu}_a(x) \psi(y) \bar{\psi}(z) \rangle = -g \langle T j^{\nu}_a(x) \psi(y) \bar{\psi}(z) \rangle,
\]  

(20)

and Bianchi identity (see \[\text{A11}\] in \[\text{App.A}\]), we can absorb part of the second term of non-
Abelian transverse anomaly into coefficient of $D^[\mu j\nu]$:

\[
\left(1 - \frac{g^2}{24\pi^2}\right) \left[\langle T D^\mu_x j^\nu_a(x) \psi(y) \bar{\psi}(z) \rangle - \langle T D^\nu_x j^\mu_a(x) \bar{\psi}(y) \psi(z) \rangle\right]
\]

\[
= i\sigma^\mu t_a \langle T \psi(y) \bar{\psi}(z) \rangle \delta^4(x - y) + i \langle T \psi(y) \bar{\psi}(z) \rangle t_a \sigma^{\mu \nu} \delta^4(x - z)
\]

\[
+ \frac{i}{2} \epsilon^{\mu \nu \rho \sigma} \left(\partial^\rho_x - \partial^\rho_y\right) \left\{ \epsilon_{igtb} f^g_x f^d y \cdot d A_b, t_a \right\} \psi(y) \bar{\psi}(z)\big|_{x' \rightarrow x}
\]

\[
+ 2m \langle T \bar{\psi}(x) \sigma^{\mu \nu} t_a \psi(x) \psi(y) \bar{\psi}(z) \rangle
\]

\[
\frac{g^2}{4\pi^2} \left(\langle T F^\mu_\nu(x) \psi(y) \bar{\psi}(z) \rangle - \frac{g^2}{24\pi^2} C_{abc} \langle T F^\mu_\nu F^\nu_{cp}(x) \psi(y) \bar{\psi}(z) \rangle \right)
\]

\[
+ \frac{g^2}{24\pi^2} \left(\langle T D^\mu_x \left[ C_{abc} \left(\partial^\nu \bar{c}_b(x) \right) c_c(x) \psi(y) \bar{\psi}(z) \right] \rangle - \langle T D^\nu_x \left[ C_{abc} \left(\partial^\mu c_b(x) \right) c_c(x) \psi(y) \bar{\psi}(z) \right] \rangle \right)
\]

\[
+ \frac{g}{24\pi^2} \frac{1}{\xi} \left(\langle T D^\mu_x \left[ \partial^\nu \left(\partial^\nu \bar{A}_a(x) \right) \psi(y) \bar{\psi}(z) \right] \rangle - \langle T D^\nu_x \left[ \partial^\mu \left(\partial^\nu \bar{A}_a(x) \right) \psi(y) \bar{\psi}(z) \right] \rangle \right).\]

(21)

Apart from anomaly terms, non-Abelian tWTI is complicated by gauge-fixing terms from equation of motion.

[13] (without consideration about anomaly) presented another derivation of non-Abelian, where the variation of fields is

\[
\delta \psi = \frac{1}{4} g \epsilon^{\mu \nu} \omega c_a t^a \sigma_{\mu \nu} \psi,
\]

\[
\delta \bar{\psi} = \frac{1}{4} g \epsilon^{\mu \nu} \bar{\psi} \sigma_{\mu \nu} t^a \omega c_a,
\]

\[
\delta A_{\mu a} = \omega \epsilon^{\mu \nu} \left(D^\nu_c \right)_a.
\]

(\omega is a Grassmann number)

[13]’s derivation is based on BRST transformation [21, 22] thus entangled with ghost field $c_a$. Obviously there is a nontrivial Jacobian with $\delta A_{\mu a} = \omega \epsilon^{\mu \nu} \left(D^\nu_c \right)_a$, however, this does not affect final results since in Eq.(40) of [13], gauge fields’ equation of motion which does
have anomalous contribution is used (to make 13’s results cleaner):

\[
\langle T\partial^\lambda F_{a\lambda\mu}(x) (D_\nu c(x))_a \bar{c}_e(0)\psi(y)\bar{\psi}(z) \rangle \\
+ gC_{abc} \langle TF_{b\mu\lambda}(x) A_c^\lambda(x) (D_\nu c(x))_a \bar{c}_e(0)\psi(y)\bar{\psi}(z) \rangle \\
+ g \langle Tj_{a\mu}(x) (D_\nu c(x))_a \bar{c}_e(0)\psi(y)\bar{\psi}(z) \rangle \\
+ \frac{1}{\xi} \langle T\partial_\mu (\partial^\lambda A_{a\lambda}(x)) (D_\nu c(x))_a \bar{c}_e(0)\psi(y)\bar{\psi}(z) \rangle \\
+ gC_{abc} \langle T (\partial_\mu \bar{c}_b(x)) c_c(x) (D_\nu c(x))_a \bar{c}_e(0)\psi(y)\bar{\psi}(z) \rangle \\
= \text{regular contact terms} \\
+ \text{anomalous contribution not taken into account in 13.}
\]

(23)

Since \((D_\nu c(x))_a = \partial_\nu c_a(x) + gC_{abc} A_{b\nu}(x) c_c(x)\) contains \(A_{b\nu}(x)\), from our discussion in Sec II (23)’s anomalous contribution to tWTI is \(- \langle T \frac{\partial (D_{\nu c(x)})}{\partial \partial_a(x)} \bar{c}_e(0)\psi(y)\bar{\psi}(z) \rangle \delta^4(x - x)\) and cancels exactly the contribution of Jacobian of \(\delta A_{a\mu}\) which is \(\langle T \frac{\partial (D_{\nu c(x)})}{\partial \partial_a(x)} \bar{c}_e(0)\psi(y)\bar{\psi}(z) \rangle \delta^4(x - x)\). So our result coincides with 13 in anomaly level 4.

Abelian and non-Abelian tWTIs with transverse anomaly, (14)(16)(19)(21) are our central results.

V. ANALYSES OF FORMER ARTICLES IGNORING TRANSVERSE ANOMALY

There are papers [11, 17, 18] about anomalies of tWTI, while all of them didn’t pick up anomaly for vector tWTI. Also, [19] verified vector tWTI to 1-loop order and found no anomaly. [11, 17, 18] are analyzed in this section, and App[C] deals with the missing key in [19].

In [11], the author identified transverse vector transformation (12) as “local Lorentz transformation”\(^5\), thus denied the possibility of anomaly in vector tWTI by Lorentz invariance. However, Lorentz transformation of Dirac fermion mismatches with transverse transformation (12) in signs. Lorentz boost of fermion (only spinor part) is \(\delta \psi(x) = +\frac{i}{4}\alpha(x)\epsilon_{\mu\nu}\sigma^{\mu\nu}\psi(x), \delta \bar{\psi}(x) = -\frac{i}{4}\alpha(x)\epsilon_{\mu\nu}\bar{\psi}(x)\sigma^{\mu\nu}\), where Jacobians of \(\psi\) and \(\bar{\psi}\) cancel with each other, regardless of \(\text{tr}[\sigma^{\mu\nu}]\delta^4(x - x)\) vanishing or not. From our derivation, \(\text{tr}[\sigma^{\mu\nu}]\delta^4(x - x)\) is not zero, thus transverse transformation, both sign of which are positive, cannot be protected by Lorentz symmetry to be free of anomalies.

\(^4\) Of course, 13 threw more fields into Green’s function than \(\psi(y)\bar{\psi}(z)\).

\(^5\) In fact, only spinor part.
Point-splitting method was used in [17]. A spurious transverse axial anomaly was proposed but corrected in [18]. Meanwhile [17] gave an expression for “vanishing” transverse vector anomaly, however, following this formulation, we can get a non-vanishing result.

Eq. (12) of [17] is

\[
\partial^\mu j^\nu(x) - \partial^\nu j^\mu(x) = \lim_{x' \to x} i \left( \partial^\mu_x - \partial^\mu_{x'} \right) \epsilon^{\lambda\mu\rho\sigma} \bar{\psi}(x') \gamma_\lambda \gamma_5 U_P(x', x) \psi(x) \\
+ \text{Symm lim}_{\epsilon \to 0} \{ \bar{\psi}(x + \epsilon/2) [-ig(\gamma^\nu F^{\mu\rho}(x) - \gamma^\mu F^{\nu\rho}(x)) \epsilon^\rho] \psi(x - \epsilon/2) \}. 
\]

using

\[
\langle \psi(x) \bar{\psi}(y) \rangle \propto \frac{\gamma^\sigma(x-y)_\sigma}{(x-y)^4},
\]

and [17] Symm lim_{\epsilon \to 0} \{ \frac{\epsilon_{\mu\sigma}}{\epsilon^2} \} = \frac{1}{2} g^{\mu\nu} \text{we can finish calculation of the last term as follows:}

\[
\text{Symm lim}_{\epsilon \to 0} \{ \bar{\psi}(x + \epsilon/2) [-ig(\gamma^\nu F^{\mu\rho}(x) - \gamma^\mu F^{\nu\rho}(x)) \epsilon^\rho] \psi(x - \epsilon/2) \} \\
\propto \text{Symm lim}_{\epsilon \to 0} \text{tr} \left[ (\gamma^\nu F^{\mu\rho}(x) - \gamma^\mu F^{\nu\rho}(x)) \gamma^\sigma \right] \frac{\epsilon_{\sigma\rho}}{\epsilon^4} \\
\propto \lim_{\epsilon \to 0} \left( g^{\nu\rho} F^{\mu\rho}(x) - g^{\mu\rho} F^{\nu\rho}(x) \right) \frac{1}{\epsilon^2} \\
\propto \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} F^{\mu\nu}(x) \neq 0.
\]

Moreover, since above result is quadratically divergent, we need to expand \( F^{\mu\nu}(x \pm \epsilon/2) \) in intermediate steps (see App. D) to \( O(\epsilon^4) \) to extract finite contributions, which means (25) is incomplete.

In a word, [17] actually worked out part of transverse anomaly. It’s a pity [17] omitted the non-vanishing result (25).

Pauli-Villars regularization [23] (PV) was devoted to calculating transverse anomalies in [18]. Unfortunately [18] forgot a vital procedure in PV thus missed transverse anomaly. This step is: express any amplitude with its regularized form so anomalies may appear from WTI with mass terms [23, 24], which is the case of vector tWTI (14). Consider any WTI with the form

\[
A = mB + C,
\]

where \( m \) is some particle’s mass. PV requires [24] regularized W-T identities to be made up of regularized amplitudes

\[
f^{\text{phys}} = \lim_{M \to \infty} f_m - f_M,
\]

\[6\] \( U_P(x', x) = \exp(-ig \int_x^{x'} dy \cdot A) \) distinguishes with ours in sign, since [17] assigned \( \overrightarrow{D}_\mu = \overrightarrow{\partial}_\mu + igA_\mu \). \[7\] However, even if we go to \( O(\epsilon^4) \), arbitrariness in coefficient of \( \partial^\mu \partial_\mu F^{\mu\nu} \) that originates from arbitrariness of \( a \in \mathbb{R} \) in \( \bar{\psi}(x + (a + 1)\epsilon) \gamma^\mu \psi(x + a\epsilon) \) prevents point-splitting method to work for transverse anomaly, see App. D.
where \( f \) stands for any amplitude, and \( f_m, f_M \) are amplitudes calculated with physical mass \( m \) or regulated mass \( M \), respectively. Then if we proved the bare WTI

\[
A_m - A_M = mB_m + C_m - MB_M - C_M,
\]  

(28)

the regularized W-T identity may acquire an anomaly

\[
A^{\text{phys}} = A_m - A_M = mB_m + C_m - MB_M - C_M
\]

\[-mB^{\text{phys}} + C^{\text{phys}} + (m - M)B_M.\]

(29)

Indeed, based on proof of bare tWTI in \cite{18}, we worked out transverse anomaly which \cite{18} has ignored, see App.\[B\].

VI. CONCLUSION AND DISCUSSION

We exploit the concept that anomalies from equations of motion are more universal than anomalies from symmetry transformations. Based on calculation of transverse anomaly in Abelian \cite{14,16} and non-Abelian \cite{19,21} vector tWTI, we extend anomalies to cover identities which are not results of symmetry transformations. Three methods including \( \zeta \) function regularization, dimensional regularization and Pauli-Villars regularization, all give consistent results, which we think are enough to overturn the conclusion of \cite{11,17–19} saying non-existence of transverse anomaly in vector tWTI.

So far, anomaly in all kinds of local linear transformation of Fermion fields\[8\] (not all symmetry transformations) has been exhausted. There are only three non-zero anomalies, see Table\[I\].

| anomaly type          | “bare” expression | in 4-dim SU\((N)\) QCD                                                                 |
|-----------------------|-------------------|--------------------------------------------------------------------------------------|
| trace anomaly\[20, 25\] | \( \text{tr} [1] \delta^4(x-x) \) | \( \frac{g^2}{48\pi^2} F^\mu\nu_a F_a^{\mu\nu} \)                                      |
| transverse anomaly    | \( \text{tr} \{ t_a [\gamma^\mu, \gamma^\nu] \} \delta^4(x-x) \) | \( -\frac{g m^2}{4\pi^2} F^\mu\nu_a - \frac{g}{24\pi^2} D^\rho D_\rho F^\mu\nu_a - \frac{g^2}{8\pi^2} C_{abc} F_b^{\mu\rho} F_c^{\nu\rho} \) |
| chiral anomaly\[7, 25\] | \( \text{tr} [t_a \gamma_5] \delta^4(x-x) \) | \( \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} F_b^{\mu\nu} F_c^{\rho\sigma} \text{tr} [t_a t_b t_c] \) |

\[8\] Local and linear transformation of \( \psi(x), \bar{\psi}(x) \) must be \( \delta \psi(x) = \alpha(x) \Omega \psi(x), \delta \bar{\psi}(x) = \alpha(x) \bar{\psi}(x) \bar{\Omega} \), where \( \Omega \) and \( \bar{\Omega} \) are linear combination of \( \gamma \) matrices hence of \( 1, \gamma^\mu, [\gamma^\mu, \gamma^\nu], \gamma^\mu \gamma_5, \gamma_5 \). But trace of odd number \( \gamma \) matrices is zero, \( \text{tr} [\gamma^\mu] \delta^4(x-x) \) and \( \text{tr} [\gamma^\mu, \gamma_5] \delta^4(x-x) \) is zero even after regularization.
As can be seen from Table I, transverse anomaly possesses much more kinds of operators than trace anomaly and chiral anomaly. Apart from modification to coefficient of tWTI from $D_\rho D_\rho F_{\mu \nu}^a$, the other brand new two operators, $F_{\mu \nu}^a$ and $C_{abc} F_{b\rho}^{\mu \rho} F_{c\nu}^{\nu}$, may bring us unexpected results, especially in structure of quark-gluon vertex, since longitudinal and transverse WTIs make the vertex completely constrained\[13, 14\]. However, in present schemes\[14–16\] to make use of Abelian tWTI, transverse anomalies have no places to plug in, since the general method \[15\] is to contract $\epsilon_{\alpha \mu \nu \beta} t_\alpha q_\beta, \epsilon_{\alpha \mu \nu \beta} \gamma_\alpha q_\beta$ to vector tWTI in momentum space\[9\]:

$$q_\mu \Gamma_\nu(k, p) - q_\nu \Gamma_\mu(k, p) = S^{-1}(p) \sigma_{\mu \nu} + \sigma_{\mu \nu} S^{-1}(k)$$

$$+ 2im \Gamma_{\mu \nu}(k, p) + t_\lambda \epsilon_{\lambda \mu \nu \rho} \Gamma_A^\rho(k, p) + A_{\mu \nu}^V(k, p),$$

(30)

such that identically vanishing L.H.S. and the contracted R.H.S. serves as constraints for axial vertex $\Gamma_A^\rho(k, p)$ to be solved. So correction to coefficient of $q_\mu \Gamma_\nu$ and additional term\[10\] $q_\mu \tilde{A}_\nu(k, p)$ all vanish after contraction with $\epsilon_{\alpha \mu \nu \beta} q_\alpha$ since $\epsilon_{\alpha \mu \nu \beta} q_\alpha q_\mu = 0$. Nevertheless, non-Abelian transverse anomaly has contribution from $C_{abc} F_{b\rho}^{\mu \rho} F_{c\nu}^{\nu}$ survival in this scheme. Unfortunately, Abelian approximation (i.e. $\Gamma_{\mu}^a$ (non-Abelian) $\approx t_a \Gamma_{\mu}^a$ (Abelian)) is still the backbone in present stage\[15, 20, 27\]. But once we begin to attack non-Abelian quark-gluon vertex directly using tWTI \[19\] \[21\], transverse anomaly will take some responsibilities. Further more, other possible applications to transverse anomaly are still under research.

ACKNOWLEDGMENTS

The work of Q. Wang was supported in part by National Key Research and Development Program of China (Grant No.2017YFA0402200), the National Natural Science Foundation of China (Grant No. 11475092).

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\[9\] Eq.(4) in \[14\], in Euclidean metric; $q \equiv k - p, t \equiv k + p$.

\[10\] $\tilde{A}_\mu(k, p)$ is $\langle T F_{\mu \nu}(x) \psi(y) \bar{\psi}(z) \rangle$ in momentum space.
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Appendix A: Calculation of transverse anomaly from path integral measure

In this appendix, we calculate the most important object of our paper, $\text{tr}[t_a \sigma^{\mu\nu}] \delta^4(x - x)$, by $\zeta$ function regularization. Since $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, it’s enough to calculate $\text{tr}\{t_a[\gamma^\mu, \gamma^\nu]\} \delta^4(x - x)$.

Combination of Fujikawa’s approach\[7\] with $\zeta$ function regularization\[24, 28\] leads to:

\[
\begin{align*}
\left\{\text{tr}\{t_a[\gamma^\mu, \gamma^\nu]\} \delta^4(x - x)\right\}_\zeta &= \frac{d}{ds}\left|_{s=0} \frac{s}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \left\{ \text{tr}\left[e^{-(\not{D}^2 + m^2)\tau} t_a[\gamma^\mu, \gamma^\nu]\right] \right\} \right|_{y \to x} \\
&= \frac{d}{ds}\left|_{s=0} \frac{s}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-m^2\tau} \frac{1}{(\sqrt{\tau})^4} \int \frac{d^4k}{(2\pi)^4} \right. \\
&\left. \times \text{tr}\left[t_a[\gamma^\mu, \gamma^\nu]\right] \text{exp}\left(k^2 + 2ik \cdot D\sqrt{\tau} - D^2\tau + \frac{i}{4} gt_b[\gamma^\rho, \gamma^\sigma] F_{b\rho\sigma\tau}\right) \right].
\end{align*}
\]

(A1)

(in the last step we have used $\not{D}^2 = D^2 - \frac{i}{4}igt_a[\gamma^\mu, \gamma^\nu] F_{a\mu\nu}$ and rescaled $k \to k/\sqrt{\tau}$)

Then we must expand the exponential

\[
\exp\left(k^2 + 2ik \cdot D\sqrt{\tau} - D^2\tau + \frac{i}{4} gt_b[\gamma^\rho, \gamma^\sigma] F_{b\rho\sigma\tau}\right) = e^{k^2} \sum_{n=0}^\infty \frac{1}{n!} \left(2ik \cdot D\sqrt{\tau} - D^2\tau + \frac{i}{4} gt_b[\gamma^\rho, \gamma^\sigma] F_{b\rho\sigma\tau}\right)^n,
\]

(A2)

and work out the trace term by term.

Noting that

\[
\begin{align*}
\frac{d}{ds}\left|_{s=0} \frac{s}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-m^2\tau} \frac{1}{(\sqrt{\tau})^4}\right. \\
&\left. \times \left(2ik \cdot D\sqrt{\tau} - D^2\tau + \frac{i}{4} gt_b[\gamma^\rho, \gamma^\sigma] F_{b\rho\sigma\tau}\right)^n\right|_{y \to x} \\
&= \frac{d}{ds}\left|_{s=0} \frac{s}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-m^2\tau} \frac{1}{(\sqrt{\tau})^4}\right. \\
&\left. \times \left(2ik \cdot D\sqrt{\tau} - D^2\tau + \frac{i}{4} gt_b[\gamma^\rho, \gamma^\sigma] F_{b\rho\sigma\tau}\right)^n\right|_{y \to x} \\
&= \left\{ \begin{array}{ll}
\frac{m^4}{2}, & n = 0; \\
-m^2, & n = 2; \\
1, & n = 4; \\
0, & \text{otherwise}. 
\end{array} \right.
\]

(A3)
So the only contributing terms in (A2) are those proportional to \((\sqrt{\tau})^0, (\sqrt{\tau})^2, (\sqrt{\tau})^4\).

It’s easy to see \((\sqrt{\tau})^0\) term is zero since \(\text{tr}\{[\gamma^\mu, \gamma^\nu]\} = 0\).

And \((\sqrt{\tau})^2\) term is (after finishing \(\frac{d}{ds}\big|_{s=0}\))

\[
(-m^2)\text{tr}[t_at_b] \text{tr}\{[\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma]\} \int \frac{d^4k}{(2\pi)^4} e^{ik^2} \frac{i}{4} g F_{b\rho\sigma}.
\]

(A4)

And \((\sqrt{\tau})^4\) term is

\[
\text{tr}[t_at_bt_c] \text{tr}\{[\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma][\gamma^\alpha, \gamma^\beta]\} \int \frac{d^4k}{(2\pi)^4} e^{ik^2} \frac{1}{2!} \left(\frac{i}{4}\right)^2 g(-1)\text{tr}[t_a(D^2 t_b F_{b\rho\sigma} + t_b F_{b\rho\sigma} D^2)]
\]

(A5)

\[
+ \text{tr}\{[\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma]\} \int \frac{d^4k}{(2\pi)^4} e^{ik^2} \frac{1}{3!} \frac{i}{4} g(2i)^2 
\times \text{tr}[t_a((k \cdot D)^2t_b F_{b\rho\sigma} + (k \cdot D)t_b F_{b\rho\sigma}(k \cdot D) + t_b F_{b\rho\sigma}(k \cdot D)^2)].
\]

Using following integral results (after Wick rotation) and identities in gauge theories (where \(D_\mu X_a = \partial_\mu X_a + gC_{abc}A_{b\mu}X_c\) is covariant derivative for any field \(X_a\) in adjoint representation of gauge group):

\[
\int \frac{d^4k}{(2\pi)^4} e^{ik} = \frac{i}{16\pi^2}, \quad \int \frac{d^4k}{(2\pi)^4} e^{ik^2} k^\mu k^\nu = \frac{-i}{32\pi^2} g^{\mu\nu},
\]

(A6)

\[
[D_\rho, t_a F_{a\mu\nu}] = t_a D_\rho F_{a\mu\nu}, \quad [D_\rho, t_a D_\sigma F_{a\mu\nu}] = t_a D_\rho D_\sigma F_{a\mu\nu},
\]

and results for trace of \(\gamma\) matrices and group generator \(t_a\):

\[
\text{tr}\{[\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma]\} = 16(-g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}), \quad \text{tr}[t_at_b] = \frac{1}{2} \delta_{ab},
\]

(A7)

\[
\text{tr}[t_at_b t_c] \text{tr}\{[\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma][\gamma^\alpha, \gamma^\beta]\} F_{b\rho\sigma} F_{c\alpha\beta} = -64iC_{abc} F_{b}^{\mu\nu} F_{c\rho}^{\nu},
\]

it’s easy to work out the final result:

\[
\{\text{tr}[t_a[\gamma^\mu, \gamma^\nu]]\} \delta^4(x - x)\} = -\frac{gm^2}{4\pi^2} F_{a\mu\nu} - \frac{g}{24\pi^2} \partial_\rho D_\rho F_{a\mu\nu} - \frac{g^2}{8\pi^2} C_{abc} F_{b}^{\mu\nu} F_{c\rho}^{\nu}.
\]

(A8)

The only difference between Abelian and non-Abelian case is the use of \(\text{tr}[t_at_b] = \frac{1}{2} \delta_{ab}\), which Abelian case doesn’t need. So Abelian result is

\[
\{\text{tr}[\gamma^\mu, \gamma^\nu] \delta^4(x - x)\} = -\frac{gm^2}{2\pi^2} F_{\mu\nu} - \frac{g}{12\pi^2} \partial_\rho \partial_\rho F_{\mu\nu},
\]

(A9)

\[11\] Also, note that \(\text{tr}[t_b] = 0\) for \(SU(N)\), so there is no contribution in \(\text{tr}[\gamma^\mu, \gamma^\nu] \delta^4(x - x)\) from non-Abelian fields by observation on (A4) (A5) with \(t_a\) stripped away, using \(\text{tr}[t_at_c] \text{tr}\{[\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma][\gamma^\alpha, \gamma^\beta]\} F_{b\rho\sigma} F_{c\alpha\beta} = 0\).
where \( F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \) is the \( U(1) \) gauge field.

Moreover, we can obtain some further expression for transverse anomaly by Bianchi identity
\[
\mathcal{D}^\rho F^\mu_\rho + \mathcal{D}^\nu F^\rho_\mu + \mathcal{D}^\mu F^\nu_\rho = 0. \tag{A10}
\]
together with \([\mathcal{D}_\mu, \mathcal{D}_\nu] F^\rho_\sigma = g C_{abc} F^\rho_{b\mu} F^\sigma_{c\nu} \), which is:

(Abelian) \[
\left\{ \text{tr} [\gamma^\mu, \gamma^\nu] \delta^4 (x - x') \right\} \zeta
= - \frac{g m^2}{2\pi^2} F^{\mu\nu} - \frac{g}{12\pi^2} \partial^\rho \partial_\rho F^{\mu\nu},
\]

(non-Abelian) \[
\left\{ \text{tr} \{ t^a [\gamma^\mu, \gamma^\nu] \} \delta^4 (x - x') \right\} \zeta
= - \frac{g m^2}{4\pi^2} F^\mu_\alpha - \frac{g}{24\pi^2} \mathcal{D}_\rho \mathcal{D}^\rho F^\mu_\alpha - \frac{g^2}{8\pi^2} C_{abc} F^{\mu\rho}_{b\nu} F^{\nu}_{c\rho},
\]
\[
= - \frac{g m^2}{4\pi^2} F^\mu_\alpha - \frac{g}{24\pi^2} (\mathcal{D}^\mu \mathcal{D}_\rho F^\rho_\alpha - \mathcal{D}^\nu \mathcal{D}_\rho F^\rho_\mu + g C_{abc} F^{\rho\mu}_{b\nu} F^{\nu}_{c\rho}). \tag{A11}
\]

The form of (A11) makes it easier to employ equation of motion for gauge fields, see (15) and (20).

**Appendix B: 1-loop calculation by Pauli-Villars regularization**

This appendix calculates missing transverse anomaly in [18] by Pauli-Villars regularization which is also the method used by [18]. Same as [18], we work with an external field \( A_\mu \), then Lagrangian is simply
\[
\mathcal{L} = \bar{\psi} \left( i \slashed{\partial} - m \right) \psi, \quad \slashed{\partial} \equiv \partial + ig \bar{A} \tag{B1}
\]

So the vector tWTI we must verify becomes
\[
\partial^\mu \left\langle j^\nu (x) \right\rangle - \partial^\nu \left\langle j^\mu (x) \right\rangle
= i \left( \partial^\mu \psi - \partial^\mu \bar{\psi} \right) \left\langle \bar{\psi} (x') \partial^\rho \gamma_\sigma \gamma_5 \varepsilon^{\rho\nu\sigma \gamma} j^\gamma A^\rho (x') \right\rangle_{x' \to x} \tag{B2}
+ 2m \left\langle \bar{\psi} (x) \sigma^{\mu\nu} \psi (x) \right\rangle - \frac{g m^2}{2\pi^2} F^{\mu\nu}(x) - \frac{g}{12\pi^2} \partial^\rho \partial_\rho F^{\mu\nu}(x).
For convenience, we define some Fourier transformed objects:

\[
\Gamma_\mu(q) \equiv \int d^4 x \ e^{-i q \cdot x} \left\langle j_\mu(x) \right\rangle,
\]

\[
N^{\mu\nu}(q) \equiv \int d^4 x \ e^{-i q \cdot x} i \left( \partial_\rho - \partial_\rho' \right) \left\langle \bar{\psi}(x') e^{i\sigma \gamma_5 e^{ig} \int_{x'}^x dy \cdot A(x)} \psi(x) \right\rangle_{x' \to x},
\]

\[
T^{\mu\nu}(q) \equiv \int d^4 x \ e^{-i q \cdot x} \left\langle \bar{\psi}(x) \sigma^{\mu\nu} \psi(x) \right\rangle.
\]

The following “bare” identity has been verified in \[18\],

\[
i q^\mu \left( \Gamma_\nu^\mu_m + \Gamma_\nu^\mu_{M_1} - 2\Gamma_\nu^\mu_{M_2} \right) - i q^\nu \left( \Gamma_\mu^\nu_m + \Gamma_\mu^\nu_{M_1} - 2\Gamma_\mu^\nu_{M_2} \right)
= N^{\mu\nu}_m + N^{\mu\nu}_{M_1} - 2 N^{\mu\nu}_{M_2} + 2 \left( m T^{\mu\nu}_m + M_1 T^{\mu\nu}_{M_1} - 2 M_2 T^{\mu\nu}_{M_2} \right). \tag{B4}
\]

\( \Gamma_\mu^\mu_m \) indicates this amplitude is calculated with fermion mass \( m \); There are twice subtraction since leading divergence is quadratic; \( M_1 = \sqrt{m^2 + 2\Lambda^2}, M_2 = \sqrt{m^2 + \Lambda^2} \), where \( \Lambda \) serves as an effective cut-off, same as \[18\].

However, in the spirit of Pauli-Villars regularization\[23, 24\], WTI should be expressed in regularized “physical” amplitude which for any amplitude \( f \) is defined to be:

\[
f_{\text{phys}} \equiv \lim_{\Lambda \to \infty} f_m + r f_{M_1} - 2 s f_{M_2}. \tag{B5}
\]

(Where \( r \) and \( s \) should be chosen to cancel all divergences in \( f \); For \( T^{\mu\nu} \), it’s easy to see \( r = m/M_1, s = m/M_2 \) by direct analysis of diagram in lowest order of \( A_\mu(x) \).)

Then “bare” identity gets an extra term after assembling every amplitude into its regularized form:

\[
i q^\mu \Gamma_{\text{phys}}^\nu - i q^\nu \Gamma_{\text{phys}}^\mu
= i q^\mu \left( \Gamma_\nu^\mu_m + \Gamma_\nu^\mu_{M_1} - 2\Gamma_\nu^\mu_{M_2} \right) - i q^\nu \left( \Gamma_\mu^\nu_m + \Gamma_\mu^\nu_{M_1} - 2\Gamma_\mu^\nu_{M_2} \right)
= N^{\mu\nu}_m + N^{\mu\nu}_{M_1} - 2 N^{\mu\nu}_{M_2} + 2 m \left( T^{\mu\nu}_m + \frac{m}{M_1} T^{\mu\nu}_{M_1} - \frac{2m}{M_2} T^{\mu\nu}_{M_2} \right)
+ 2 \left( M_1 - \frac{m^2}{M_1} \right) T^{\mu\nu}_{M_1} - 4 \left( M_2 - \frac{m^2}{M_2} \right) T^{\mu\nu}_{M_2}
\equiv N^{\mu\nu}_{\text{phys}} - 2m T^{\mu\nu}_{\text{phys}} + \mathcal{A}^{\mu\nu}. \tag{B6}
\]

We will show that

\[
\mathcal{A}^{\mu\nu} \equiv 2 \left( M_1 - \frac{m^2}{M_1} \right) T^{\mu\nu}_{M_1} - 4 \left( M_2 - \frac{m^2}{M_2} \right) T^{\mu\nu}_{M_2}, \tag{B7}
\]

is exactly the anomaly we obtained in App[A] up to a quadratic divergence term which corresponds to photon mass that should be subtracted.
$T_{m}^{\mu \nu}$ is represented by the following Feynman graph\(^{12}\):

\[ T_{m}^{\mu \nu} = \cdots + \frac{\sigma^{\mu \nu}(\not{\phi} + \not{q})}{(2\pi)^4} \left[ (p - q_{1})^2 - m^2 \right] \left[ (p - q_{2})^2 - m^2 \right] \cdots \left[ (p - q_{2n+1})^2 - m^2 \right] \left( \not{A} - \not{q} \right) + \cdots \]

(B8)

\[ A^{2n+1} \text{ term is:} \]

\[ (-i g)^{2n+1} (= 2^{n+1} n_{2n+2} \int \frac{d^4 p}{(2\pi)^4} \frac{\sigma^{\mu \nu}(\not{\phi} + \not{q})}{(p - q_{1})^2 - m^2} \left[ (p - q_{2})^2 - m^2 \right] \cdots \left[ (p - q_{2n+1})^2 - m^2 \right] \left( \not{A} - \not{q} \right) + \cdots \rangle \]

(B9)

where $\tilde{A}_{k} \equiv \int d^4 x \ e^{-i (\not{q}_{k} - \not{q}_{k+1}) x} A(x)$. Since $\langle T \bar{\psi}(x) \sigma^{\mu \nu} \psi(x) j^{\rho_{1}}(x_{1}) \cdots j^{\rho_{2n+1}}(x_{2n+1}) \rangle$ is gauge invariant, $A_{\mu}(x)$ in each diagram must appears as gauge invariant objects such as $F^{\mu \nu}(x), \partial^{\mu} \partial_{\nu} F^{\mu \nu}(x)$. Thus there is at least one $\not{q}$ for one $\tilde{A}$ survives the trace. So numerator of (B9) is $(q \tilde{A})^{2n+1} m$ or $(q \tilde{A})^{2n+1} q$ (since $\int d^4 p \frac{\not{p}}{p^{2n+2}} = 0$). And integral of the denominator is of order $1/m^{4n}$ from dimensional counting if $m \gg q_{k}^2, \forall k$. In summary, $A^{2n+1}$ term is at most of order $1/m^{4n-1}$, so contribution to $\mathcal{J}^{\mu \nu}$, which is at most of order $m/m^{4n-1}$, comes from $n = 0$ term only.

Then we can calculate $\mathcal{J}^{\mu \nu}$ from lowest order diagram:

\[ \mathcal{J}^{\mu \nu} = 2 \left( M_1 - \frac{m^2}{M_1} \right) T_{M_1}^{\mu \nu} - 4 \left( M_2 - \frac{m^2}{M_2} \right) T_{M_2}^{\mu \nu} \]

\[ = 2 \left( M_1 - \frac{m^2}{M_1} \right) i g \int \frac{d^4 p}{(2\pi)^4} \frac{\sigma^{\mu \nu}(\not{\phi} + \not{q})}{(p - q_{1})^2 - m^2 - 2\Lambda^2} \left[ (p - q_{2})^2 - m^2 - 2\Lambda^2 \right] \cdots \left[ (p - q_{2n+1})^2 - m^2 - 2\Lambda^2 \right] \left( \not{A} - \not{q} \right) + \cdots \]

\[ = -16 g \Lambda^4 \int \frac{d^4 p}{(2\pi)^4} \frac{[p^2 + (p - q)^2 - 2m^2 - 3\Lambda^2]}{[(p - q)^2 - m^2 - 2\Lambda^2]} \left( q^\mu \tilde{A}^\nu - q^\nu \tilde{A}^\mu \right) \]

\[ = -16 g \Lambda^4 \left( q^\mu \tilde{A}^\nu - q^\nu \tilde{A}^\mu \right) \times 3! \int_0^1 dx \int_0^x dy \int_0^y dz \int \frac{d^4 p}{(2\pi)^4} \]

\[ \times \frac{2p^2 + y^2 + (1 - y)^2}{[p^2 + y(1 - y)q^2 - m^2 - 2\Lambda^2]} \left[ (1 + x - y + z)\Lambda^2 \right]^4 \]

\[ = \left( i q^\mu \tilde{A}^\nu - i q^\nu \tilde{A}^\mu \right) \int_0^1 dx \int_0^x dy \int_0^y dz \]

\[ \times \frac{4 g \Lambda^4}{\pi^2} \left[ (m^2 + (1 + x - y + z)\Lambda^2 - y(1 - y)q^2) ^2 \right] + \frac{2m^2 + 3\Lambda^2 - [y^2 + (1 - y)^2]q^2}{[m^2 + (1 + x - y + z)\Lambda^2 - y(1 - y)q^2]^2} \].

(B10)

\(^{12}\) Recall that $C$ parity of $\bar{\psi} \sigma^{\mu \nu} \psi$ is odd, so there is only $A^{2n+1}$ terms.)
Take the limit $\Lambda \to \infty$, then we get
\[
\mathcal{A}^{\mu\nu} = \left( \frac{g \ln 2}{\pi^2} \Lambda^2 - \frac{g}{2\pi^2} m^2 + \frac{g}{12\pi^2} q^2 \right) \left( iq^{\mu} \tilde{A}^\nu - iq^{\nu} \tilde{A}^{\mu} \right), \quad \tilde{A}^{\mu}(q) \equiv \int d^4x \, e^{-iq \cdot x} A^{\mu}(x).
\]

This is what we get in App. A in momentum space, up to a quadratic divergent term. However, it’s well-known\cite{23, 29} that quadratic divergence in $\langle j^{\mu} \rangle$ in Pauli-Villars scheme corresponds to an infinite photon mass, which must be subtracted to insure gauge invariance.

As long as we subtract quadratic divergence in both sides of\cite{13}
\[
iq^{\mu} \Gamma_{\text{phys}}^{\nu} - iq^{\nu} \Gamma_{\text{phys}}^{\mu} = N_{\text{phys}}^{\mu\nu} - 2mT_{\text{phys}}^{\mu\nu} + \mathcal{A}_{\text{phys}}^{\mu\nu},
\]
nothing is bothered by quadratic divergence.

**Appendix C: 1-loop calculation by dimensional regularization**

As continuation of Sec. V, this appendix calculates missing anomaly in\cite{19},\cite{19} verified vector tWTI (14) to 1-loop order, however,\cite{19} ignored a contributing Feynman graph thus missed transverse anomaly. Here we show by explicit calculation that when that graph is taken into account, the 1-loop results for transverse anomaly is exactly what we get by $\zeta$ function regularization in App. A (and with no quadratic divergence).

What\cite{19} has verified is in fact:
\[
iq^{\mu} \Gamma_{V,1PI}^{\nu}(p_1, p_2) - iq^{\nu} \Gamma_{V,1PI}^{\mu}(p_1, p_2)
= S_F^{-1}(p_1)\sigma^{\mu\nu} + \sigma^{\mu\nu} S_F^{-1}(p_2) + 2mT_{T,1PI}^{\mu\nu}(p_1, p_2)
+ (p_{1\lambda} + p_{2\lambda})\epsilon^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2)
- \int \frac{d^4k}{(2\pi)^4} 2k^{\lambda} \epsilon^{\lambda\mu\nu\rho} \Gamma_{A\rho,1PI}(p_1, p_2; k),
\]
where Fourier transformed vertex is\cite{19}:
\[
\int d^4x d^4y d^4z \, e^{-iq \cdot x + ip_1 \cdot y - ip_2 \cdot z} \langle T J(x) \psi(y) \bar{\psi}(z) \rangle
=(2\pi)^4 \delta^4(p_1 - p_2 - q) i S_F(p_1) \Gamma(p_1, p_2) i S_F(p_2),
\]
\[
\int d^4x d^4x' d^4y d^4z \, e^{i(p_2 - k) \cdot x - i(p_2 - k + q) \cdot x'} + ip_1 \cdot y - ip_2 \cdot z
\times \langle T \bar{\psi}(x') \gamma_\rho \gamma_5 \psi \sigma^{\mu\nu} \epsilon^{\mu\nu\rho\sigma} f_{A\rho}^\sigma(x) \psi(y) \bar{\psi}(z) \rangle
=(2\pi)^4 \delta^4(p_1 - p_2 - q) i S_F(p_1) \Gamma_{A\rho}(p_1, p_2; k) i S_F(p_2),
\]

\cite{13} Above equation holds so that quadratic divergences in both sides are equal.
(Where \(J(x), \Gamma(p_1, p_2)\) stands for \(j^\mu, \Gamma_V^\mu; j_5^\mu, \Gamma_A^\mu; \bar{\psi}\sigma^{\mu\nu}\psi, \Gamma_T^{\mu\nu};\) And \(iS_F(p)\) is fermion’s propagator in momentum space.)

By definition (C2), \(\Gamma_V^\mu(p_1, p_2), \Gamma_T^{\mu\nu}(p_1, p_2), \Gamma_A^\mu(p_1, p_2; k)\) all include the following two kinds (non-1PI part and 1PI (1-particle-irreducible) part) of graphs in 1-loop order (\(\Gamma_A^\mu(p_1, p_2), S_F^{-1}(p)\) only have 1PI contribution):

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
p_1 \\
p_2
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
p_1 \\
p_2
\end{array}
\end{array}
\]

(C3)

But [19] only included the second graph, i.e. the 1PI part, and acquired the above identity (C4). We will show that the other non-1PI part is just what contributes to transverse anomaly.

For convenience, define non-1PI part of \(\Gamma_V^\mu(p_1, p_2), \Gamma_T^{\mu\nu}(p_1, p_2), \Gamma_A^\mu(p_1, p_2; k)\) to be \(\tilde{\Gamma}_V^\mu(p_1, p_2), \tilde{\Gamma}_T^{\mu\nu}(p_1, p_2), \tilde{\Gamma}_A^\mu(p_1, p_2; k)\).

We will show by explicit calculation that \((q \equiv p_1 - p_2)\):

\[
\begin{align*}
 iq^\mu \tilde{\Gamma}^\mu_V(p_1, p_2) & = -2m \tilde{\Gamma}_T^{\mu\nu}(p_1, p_2) - \int \frac{d^4k}{(2\pi)^4} 2k_\lambda \epsilon^{\lambda\mu\nu\rho} \tilde{\Gamma}_A^{\mu\rho}(p_1, p_2; k) \\
 & + \left( \frac{g^2 m^2}{2\pi^2 q^2} + \frac{g^2}{12\pi^2} \right) (i\eta^\mu \eta^\nu - i\eta^\nu \eta^\mu).
\end{align*}
\]

(C4)

According to general procedure of dimensional regularization [24, 30], calculation of above three terms is as follows. For \(\tilde{\Gamma}_V^\mu(p_1, p_2),\)

\[
\begin{align*}
\tilde{\Gamma}_V^\mu(p_1, p_2) & = (-) \int \frac{d^4p}{(2\pi)^4} \frac{d^{n-4}P}{(2\pi)^{n-4}} \text{ tr } \left[ \gamma^\mu i \left( \not{p} + \not{P} - \not{q} + m \right) i\gamma^\nu i \left( \not{p} + \not{P} + m \right) \right] (-i) q^\mu \\
 & = \frac{-ig^2}{q^2} \gamma_\nu \int_0^1 dx \int \frac{d^4p}{(2\pi)^4} \frac{d^{n-4}P}{(2\pi)^{n-4}} \frac{4g^{\mu\nu}}{[-p^2 - x(1-x)q^2 + m^2]^{n-4} \Gamma \left( \frac{n-4}{2} \right) \Gamma \left( \frac{n}{2} \right)} \\
 & - \frac{(4m^2 - 2p^2) g^{\mu\nu} - 4x(1-x) (2q^\mu q^\nu - q^2 g^{\mu\nu}) \Gamma \left( \frac{2 - n}{2} \right)}{[-p^2 - x(1-x)q^2 + m^2]^{n-4} \Gamma \left( \frac{n-4}{2} \right) \Gamma \left( \frac{n}{2} \right)} \\
 & = i \frac{g^2}{q^2} \int_0^1 \frac{dx}{(2\pi)^4 (2\sqrt{\pi})^{n-4}} x(1-x) \left[ m^2 - x(1-x)q^2 \right]^{\frac{n}{2} - 2} \Gamma \left( 2 - \frac{n}{2} \right).
\end{align*}
\]

(C5)
For $\widetilde{\Gamma}_{\mu\nu}^\mu\nu (p_1, p_2)$,

$$\widetilde{\Gamma}_{\mu\nu}^\mu\nu (p_1, p_2)$$

$= (-) \int \frac{d^4 p}{(2\pi)^4} \frac{d^{n-4} P}{(2\pi)^n} \text{tr} [\sigma^{\mu\nu} i (p + P - \hat{a} + m) ig^{\rho\nu} i (p + P + m)] (-i)ig^{\gamma_\rho}$

$= -\frac{g^2}{q^2} m \int_0^1 dx \int \frac{d^4 p}{(2\pi)^4} \frac{d^{n-4} P}{(2\pi)^n} \frac{q^{\mu\nu} - q^{\rho\mu}}{(2\pi)^4 [ -p^2 + p^2 + x(1-x)q^2 - m^2]^2}$

$= -\frac{g^2}{q^2} m \int_0^1 dx \int \frac{d^4 p}{(2\pi)^4} \frac{q^{\mu\nu} - q^{\rho\mu}}{2\pi^{n-4}} \frac{\Gamma (2 - \frac{n-4}{4})}{(2\sqrt{\pi})^{n-4}}$

$= -\frac{4g^2}{q^2} m (iq^{\mu\nu} - iq^{\nu\mu}) \int_0^1 dx \frac{2\pi^2}{\pi^{2\frac{n-4}{4}}} \left[ m^2 - x(1-x)q^2 \right]^{\frac{n-4}{2}} \Gamma \left( 2 - \frac{n}{2} \right)$.

(C6)

For $(-) \int \frac{d^4 k}{(2\pi)^4} 2k_\lambda \epsilon^{\lambda\mu\nu\rho} \widetilde{\Gamma}_\rho(p_1, p_2; k)$,

$= \int \frac{d^4 k}{(2\pi)^4} 2k_\lambda \epsilon^{\lambda\mu\nu\rho} \widetilde{\Gamma}_\rho(p_1, p_2; k)$

$= (-) \int \frac{d^4 p}{(2\pi)^4} \frac{d^{n-4} P}{(2\pi)^n} \text{tr} [\sigma^{\mu\nu} i (p + P - \hat{a} + m) ig^{\rho\nu} i (p + P + m)] (-i)ig^{\gamma_\rho}$

$= \frac{4g^2}{q^2} \int_0^1 dx \int \frac{d^4 p}{(2\pi)^4} \frac{d^{n-4} P}{(2\pi)^n} \frac{p^2 (q^{\mu\nu} - q^{\nu\mu})}{(2\pi)^4 [-p^2 + p^2 + x(1-x)q^2 - m^2]^2}$

$= \frac{4g^2}{q^2} \int_0^1 dx \int \frac{d^4 p}{(2\pi)^4} \frac{p^2 (q^{\mu\nu} - q^{\nu\mu})}{2\pi^{n-4}} \frac{\Gamma (2 - \frac{n-4}{4})}{(2\sqrt{\pi})^{n-4}}$

$= -\frac{4g^2}{q^2} (iq^{\mu\nu} - iq^{\nu\mu}) \int_0^1 dx \frac{2\pi^2}{\pi^{2\frac{n-4}{4}}} \left[ m^2 - x(1-x)q^2 \right]^{\frac{n-4}{2}} \Gamma \left( 1 - \frac{n}{2} \right) \right) \cdot (C7)$

Collect above results, then we get

$$iq^{\mu} \widetilde{\Gamma}_{\nu}^\mu (p_1, p_2) - iq^{\nu} \widetilde{\Gamma}_{\nu}^\mu (p_1, p_2) - 2m \widetilde{\Gamma}_{\nu}^\mu (p_1, p_2) + \int \frac{d^4 k}{(2\pi)^4} 2k_\lambda \epsilon^{\lambda\mu\nu\rho} \widetilde{\Gamma}_\rho(p_1, p_2; k)$$

$$= \frac{4g^2}{q^2} (iq^{\mu\nu} - iq^{\nu\mu}) \int_0^1 dx \frac{2\pi^2}{\pi^{2\frac{n-4}{4}}}$$

$$\times \left[ -x(1-x)q^2 + m^2 + \frac{m^2 - x(1-x)q^2}{1 - \frac{n}{2}} \right] \left[ m^2 - x(1-x)q^2 \right]^{\frac{n-4}{2}} \Gamma \left( 2 - \frac{n}{2} \right)$$

$$= \frac{4g^2}{q^2} (iq^{\mu\nu} - iq^{\nu\mu}) \int_0^1 dx \frac{2\pi^2}{\pi^{2\frac{n-4}{4}}}$$

$$\times \left[ m^2 - x(1-x)q^2 \right]^{\frac{n}{2}} \left[ 1 - \frac{n}{2} \right] \Gamma \left( 2 - \frac{n}{2} \right) \left[ m^2 - x(1-x)q^2 \right]^{\frac{n-2}{4}}$$

$$= -\frac{4g^2}{q^2} (iq^{\mu\nu} - iq^{\nu\mu}) \int_0^1 dx \frac{2\pi^2}{\pi^{2\frac{n-4}{4}}} \left( m^2 - x(1-x)q^2 \right)$$

$$= \left( -\frac{g^2 m^2}{2\pi^2 q^2} + \frac{g^2}{12\pi^2} \right) (iq^{\mu\nu} - iq^{\nu\mu}) \cdot (C8)$$

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Thus our result for transverse anomaly (14) is verified by dimensional regularization.

Appendix D: Failure of point-splitting method

Similar to chiral anomaly, point-splitting method can also give results for transverse anomaly. However, dependence on splitting ratio prevent this method to work for transverse anomaly.

Point-splitting method chooses a special regularization for $j^\mu$ ($a$ is a real number):

$$j^\mu(x) \rightarrow \bar{\psi}(x + (a + 1)\epsilon)\gamma^\mu e^{ig \int_{x + a\epsilon}^{x + (a + 1)\epsilon} dy \cdot A(y)} \psi(x + a\epsilon).$$  \hspace{1cm} (D1)

Usually $a = -1/2$ so that $x$ is midpoint of the two split points. However, there is no principle that demands $a$ to be $-1/2$, so if this method is to make sense, final results must be independent of $a$, as is the case of chiral anomaly (looking into concrete process of calculating chiral anomaly, it’s easy to see chiral anomaly only needs expansion to $O(\epsilon)$ so that $a$ dependence is of the form $(a + 1) - a = 1$). We’ll see soon point-splitting method cannot be devoted to calculating transverse anomaly due to its non-trivial dependence on $a$.

It’s useful to work out the following integral :

$$\int_{x + a\epsilon}^{x + (a + 1)\epsilon} dy \cdot A(y)$$

$$= \int_{a\epsilon}^{(a + 1)\epsilon} ds \frac{e^\rho}{\epsilon} \left( A_\rho(x) + y^\mu \partial_\mu A_\rho(x) + \frac{y^{\mu_1} y^{\mu_2}}{2!} \partial_{\mu_1} \partial_{\mu_2} A_\rho(x) + \cdots \right)$$

$$= e^\rho A_\rho(x) + \frac{(a + 1)^2 - a^2}{2} e^\sigma e^\lambda \partial_\sigma A_\rho(x) + \frac{(a + 1)^3 - a^3}{6} e^\rho e^\sigma e^\lambda \partial_\sigma \partial_\lambda A_\rho(x) + O(\epsilon^4).$$  \hspace{1cm} (D2)

Then we can calculate curl of $j^\mu$. First, use $[\gamma^\rho, \frac{1}{2} \sigma^{\mu\nu}] = ig^{\mu\rho} \gamma^\nu - ig^{\nu\rho} \gamma^\mu$ to rewrite $\partial^{[\mu} j^{\nu]}$ as :

$$\partial^{[\mu} j^{\nu]}(x) = -i \partial^{\rho} \left( \bar{\psi}(x + (a + 1)\epsilon) \left[ \gamma^\rho, \frac{1}{2} \sigma^{\mu\nu} \right] e^{ig \int_{x + a\epsilon}^{x + (a + 1)\epsilon} dy \cdot A(y)} \psi(x + a\epsilon) \right).$$  \hspace{1cm} (D3)

Second, use $[\gamma^\rho, \frac{1}{2} \sigma^{\mu\nu}] = \gamma^\rho \sigma^{\mu\nu} - \left\{ \gamma^\rho, \frac{1}{2} \sigma^{\mu\nu} \right\} = -\sigma^{\mu\nu} \gamma^\rho + \{ \gamma^\rho, \frac{1}{2} \sigma^{\mu\nu} \}$ and $\{ \gamma^\rho, \frac{1}{2} \sigma^{\mu\nu} \} = \frac{1}{2} \epsilon_{\mu\nu} \gamma^\rho$.
\[-\epsilon_{\mu
u\rho\sigma}\gamma_\sigma \gamma_5\] to expand \([\gamma^\rho, \frac{1}{2}\sigma^{\mu\nu}]\):

\[
\partial^\mu j^\nu(x) - \partial^\nu j^\mu(x)
= i \left( \bar{\psi}(x + (a + 1)\epsilon) \left( \overleftrightarrow{\partial}_\rho - \overleftrightarrow{\partial}_\rho \right) \epsilon_{\mu
u\rho\sigma}\gamma_\sigma \gamma_5 \psi(x + a\epsilon) \right) e^{ig \int_{x+\alpha}^{x+(a+1)\epsilon} dy A(y)}
- i \left( \bar{\psi}(x + (a + 1)\epsilon) \gamma^\rho \sigma^{\mu\nu} \psi(x + a\epsilon) \right) e^{ig \int_{x+\alpha}^{x+(a+1)\epsilon} dy A(y)}
+ i \left( \bar{\psi}(x + (a + 1)\epsilon) \sigma^{\mu\nu\rho} \gamma_\rho \psi(x + a\epsilon) \right) e^{ig \int_{x+\alpha}^{x+(a+1)\epsilon} dy A(y)}
+ g \bar{\psi}(x + (a + 1)\epsilon) \left[ \gamma^\rho, \frac{1}{2}\sigma^{\mu\nu} \right] e^{ig \int_{x+\alpha}^{x+(a+1)\epsilon} dy A(y)} \psi(x + a\epsilon)
\times \left( \epsilon^\sigma \partial_\rho A_\sigma(x) + \frac{(a + 1)^2}{2} - a^2 \epsilon^\lambda \partial_\lambda \partial_\rho A_\sigma(x) + \frac{(a + 1)^3}{6} - a^3 \epsilon^\lambda \epsilon^\kappa \partial_\lambda \partial_\kappa \partial_\rho A_\sigma(x) + \mathcal{O}(\epsilon^4) \right)
\quad \text{(D4)}
\]

Then, use equations of motion for massless (for simplicity) fermion \(\overleftrightarrow{D}\psi(x) = 0, \bar{\psi}(x) \overleftrightarrow{D} = 0\):

\[
\partial^\mu j^\nu(x) - \partial^\nu j^\mu(x)
= i \left( \bar{\psi}(x + (a + 1)\epsilon) \left( \overleftrightarrow{\partial}_\rho - \overleftrightarrow{\partial}_\rho \right) \epsilon_{\mu
u\rho\sigma}\gamma_\sigma \gamma_5 \psi(x + a\epsilon) \right) e^{ig \int_{x+\alpha}^{x+(a+1)\epsilon} dy A(y)}
- g \bar{\psi}(x + (a + 1)\epsilon) A(x + (a + 1)\epsilon) \gamma^\rho \sigma^{\mu\nu} \psi(x + a\epsilon) e^{ig \int_{x+\alpha}^{x+(a+1)\epsilon} dy A(y)}
- g \bar{\psi}(x + (a + 1)\epsilon) \sigma^{\mu\nu\rho} \gamma_\rho A(x + a\epsilon) \psi(x + a\epsilon) e^{ig \int_{x+\alpha}^{x+(a+1)\epsilon} dy A(y)}
+ g \bar{\psi}(x + (a + 1)\epsilon) \left[ \gamma^\rho, \frac{1}{2}\sigma^{\mu\nu} \right] e^{ig \int_{x+\alpha}^{x+(a+1)\epsilon} dy A(y)} \psi(x + a\epsilon)
\times \left( \epsilon^\sigma \partial_\rho A_\sigma(x) + \frac{(a + 1)^2}{2} - a^2 \epsilon^\lambda \partial_\lambda \partial_\rho A_\sigma(x) + \frac{(a + 1)^3}{6} - a^3 \epsilon^\lambda \epsilon^\kappa \partial_\lambda \partial_\kappa \partial_\rho A_\sigma(x) + \mathcal{O}(\epsilon^4) \right)
\quad \text{(D5)}
\]
Next, expand $A_\mu$ at $x$ to $\mathcal{O}(\epsilon^4)$:

$$\partial^\mu j^\nu(x) - \partial^\nu j^\mu(x)$$

$$= i \left( \bar{\psi}(x + (a + 1)\epsilon) \left( \overrightarrow{D}_\rho - \overleftarrow{D}_\rho \right) \epsilon^{\mu\nu\rho\sigma} \gamma_\sigma \gamma_5 \psi(x + a\epsilon) \right) e^{ig \int_{x+a\epsilon}^{x+(a+1)\epsilon} dy \cdot A(y)}$$

$$- g \bar{\psi}(x + (a + 1)\epsilon) \gamma^\rho \sigma^{\mu\nu} \psi(x + a\epsilon) e^{ig \int_{x+a\epsilon}^{x+(a+1)\epsilon} dy \cdot A(y)}$$

$$\times \left( (a + 1)\epsilon^\rho \partial_\rho A_\mu(x) + \frac{(a + 1)^2}{2} \epsilon^\rho \partial_\rho \partial_\sigma A_\mu(x) + \frac{(a + 1)^3}{6} \epsilon^\rho \epsilon^\lambda \partial_\lambda \partial_\rho \partial_\sigma A_\mu(x) + \mathcal{O}(\epsilon^4) \right)$$

$$- g \bar{\psi}(x + (a + 1)\epsilon) \sigma^{\mu\nu} \gamma^\rho \psi(x + a\epsilon) e^{ig \int_{x+a\epsilon}^{x+(a+1)\epsilon} dy \cdot A(y)}$$

$$\times \left( \epsilon^\rho \partial_\rho A_\sigma(x) + \frac{(a + 1)^2 - a^2}{2} \epsilon^\rho \epsilon^\lambda \partial_\lambda \partial_\rho A_\sigma(x) + \frac{(a + 1)^3 - a^3}{6} \epsilon^\rho \epsilon^\lambda \epsilon^\kappa \partial_\lambda \partial_\kappa \partial_\rho A_\sigma(x) + \mathcal{O}(\epsilon^4) \right).$$

$$\text{(D6)}$$

Finally, we must take $\epsilon \to 0$ limit. From [6, 17], we have [14]:

$$\langle \psi(x + a\epsilon) \bar{\psi}(x + (a + 1)\epsilon) \rangle = \frac{i}{2\pi^2} \frac{\gamma^\alpha \epsilon^\alpha}{\epsilon^4} + \mathcal{O}(A^1), \quad \lim_{\epsilon \to 0} \frac{\epsilon^\mu \epsilon^\nu}{\epsilon^4} = \frac{1}{24} (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}).$$

$$\text{(D7)}$$

So the final result is (Bianchi identity is used and be careful of Fermi-statistics)

$$\partial^\mu j^\nu(x) - \partial^\nu j^\mu(x)$$

$$= i \left( \bar{\psi}(x) \left( \overrightarrow{D}_\rho - \overleftarrow{D}_\rho \right) \epsilon^{\mu\nu\rho\sigma} \gamma_\sigma \gamma_5 \psi(x) \right)$$

$$+ g \frac{1}{2\pi^2} \epsilon^2 F^{\mu\nu} + g \frac{[(a + 1)^3 - a^3]}{72\pi^2} (2\partial_\rho \partial_\sigma F^{\mu\nu} + 2\partial^\rho \partial_\sigma F_{\nu\sigma} - 2\partial_\sigma \partial_\rho F^{\mu\nu})$$

$$= i \epsilon^{\mu\nu\rho\sigma} \left( \partial^\rho - \partial^{\rho'} \right) \left( \bar{\psi}(x') \gamma_\sigma \gamma_5 e^{ig \int_{x'}^{y} dy \cdot A(y)} \right)_{x' \to x}$$

$$+ g \frac{1}{2\pi^2} \epsilon^2 F^{\mu\nu} + g \frac{[(a + 1)^3 - a^3]}{18\pi^2} \partial^\rho \partial_\rho F^{\mu\nu}.$$

$$\text{(D8)}$$

This result is not only bothered by quadratic divergence, but also dependent on $a$ non-trivially! So point splitting method is not suitable for transverse anomaly.

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14 Here we only need $\mathcal{O}(A^0)$ of $\langle \psi(x + a\epsilon) \bar{\psi}(x + (a + 1)\epsilon) \rangle$ since $C$ parity of $j^\mu$ and $A^\mu$ are both odd, and $\mathcal{O}(A^2)$ of $\langle \psi(x + a\epsilon) \bar{\psi}(x + (a + 1)\epsilon) \rangle$ is of $\mathcal{O}(\epsilon)$ thus doesn’t contribute.