Abstract Harmonic Analysis on the General Affine Group $GA(n, \mathbb{R})$

Kahar El-Hussein

Department of Mathematics, Faculty of Science, 
Al Furat University, Dear El Zore, Syria and 
Department of Mathematics, Faculty of Arts Science Al Quryyat, 
Al-Jouf University, KSA

E-mail: kumath@ju.edu.sa, kumath@hotmail.com

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Abstract

Let $GL(n, \mathbb{R})$ be the general linear group and let $GA(n, \mathbb{R}) = \mathbb{R}^n \rtimes \rho GL(n, \mathbb{R})$ be its general affine group. Let $GL_+(n, \mathbb{R})$ be the identity component of $GL(n, \mathbb{R})$, which consists of the real $n \times n$ matrices with positive determinant and let $GL_-(n, \mathbb{R})$ be the set of all matrices with negative determinant. Since $GL(n, \mathbb{R})$ is a two copies of $GL_+(n, \mathbb{R})$, i.e. $GL(n, \mathbb{R}) = GL_+(n, \mathbb{R}) \cup GL_+(n, \mathbb{R})$, because $GL_-(n, \mathbb{R})$ has a structure of group isomorphic onto $GL_+(n, \mathbb{R})$, see [13]. Therefore first we consider the group $GA_+(n, \mathbb{R}) = \mathbb{R}^n \rtimes \rho GL_+(n, \mathbb{R})$ to generalize the Fourier transform and to prove the Plancherel theorem for $GA_+(n, \mathbb{R})$. Secondly and since $GA(n, \mathbb{R})$ is a two copies of $GA_+(n, \mathbb{R})$, so we can easily to establish the Plancherel theorem for $GA(n, \mathbb{R})$.

Keywords: Linear Group $GL(n, \mathbb{R})$, Affine General Group $GA(n, \mathbb{R})$, Fourier Transform, Plancherel Theorem

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1 Introduction.

1.1. The general affine group $GA(n, \mathbb{R}) = \mathbb{R}^n \rtimes \rho GL(n, \mathbb{R})$ of the general linear group $GL(n, \mathbb{R})$, which is the semidirect of the real vector $\mathbb{R}^n$ with $GL(n, \mathbb{R})$. Another kind of the affine group is the Poincare group $\mathbb{R}^4 \rtimes O(3,1)$, which is the affine group of the inhomogeneous Lorentz group $O(3,1)$. As well known the affine groups play an important role in physics in cosmology, gauge theory, gravitation, general relativity, etc. The spacetime symmetry of the affine model of gravity is given by the general affine group $GA(n, \mathbb{R})$. The geometrical arena of the theory gravitation and electromagnetism is the modified affine frame bundle over a four dimensional spacetime manifold $M$, the structure group of the frame bundle is the affine group $GA(4, \mathbb{R}) = \mathbb{R}^4 \rtimes GL(4, \mathbb{R})$. The usual definition of phase space coordinates in terms of linear frames and use affine frames instead. This leads from $GL(4, \mathbb{R})$ covariance to $GA(4, \mathbb{R})$ covariance and means that the bundle of linear frames, is replaced by the bundle of affine frames. Also the affine group $GA(2, \mathbb{R})$ has a fundamental role in the visualization. One asks can we do the non commutative Fourier analysis on the group $GA(n, \mathbb{R}) = \mathbb{R}^n \rtimes GL(n, \mathbb{R})$. In fact that the abstract harmonic analysis on the locally compact groups is generally a difficult task. Still now neither the theory of quantum groups nor the representations theory have done to reach this goal. Recently, these problems found a satisfactory solution with the papers [9, 10, 11]. In this paper we will generalize our methods in [12, 13] to define the Fourier transform and to establish the Plancherel theorem for the general affine group on $GA(n, \mathbb{R})$.

2 Notation and Results for the Nilpotent Lie Group $N$.

2.1. The fine structure of the nilpotent Lie groups will help us to do the Fourier transform on a connected and simply connected nilpotent Lie groups $N$. As well known any group connected and simply connected $N$ has the
following form

\[
N = \begin{pmatrix}
1 & x_1^1 & x_2^2 & x_3^3 & \ldots & x_{n-2}^{n-2} & x_{n-1}^{n-1} & x_n^n \\
0 & 1 & x_2^2 & x_3^3 & \ldots & x_{n-2}^{n-2} & x_{n-1}^{n-1} & x_n^n \\
0 & 0 & 1 & x_3^3 & \ldots & x_{n-2}^{n-2} & x_{n-1}^{n-1} & x_n^n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & x_{n-2}^{n-2} & x_{n-1}^{n-1} \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & x_n^n \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}
\]

As shown, this matrix is formed by the subgroup \( \mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^{n-1}, \) and \( \mathbb{R}^n \)

\[
\begin{pmatrix}
x_1^1 \\
1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}, \begin{pmatrix}
x_2^2 \\
1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}, \ldots, \begin{pmatrix}
x_{n-2}^{n-2} \\
x_{n-1}^{n-1} \\
x_n^n \\
\vdots \\
0
\end{pmatrix}, \begin{pmatrix}
x_1^n \\
x_2^n \\
x_3^n \\
\vdots \\
x_n^n
\end{pmatrix}
\]

(2)

Each \( \mathbb{R}^i \) is a subgroup of \( N \) of dimension \( i \), \( 1 \leq i \leq n \), put \( d = n + (n - 1) + \ldots + 2 + 1 = \frac{n(n+1)}{2} \), which is the dimension of \( N \). According to [9], the group \( N \) is isomorphic onto the following group

\[
(((R^n \rtimes_{\rho_n} R^{n-1}) \rtimes_{\rho_{n-1}} R^{n-2} \rtimes_{\rho_{n-2}} \ldots) \rtimes_{\rho_2} R^2) \rtimes_{\rho_1} R
\]

(3)

That means
\[ N \cong (((\mathbb{R}^n \times \rho_n) \mathbb{R}^{n-1}) \times \rho_{n-1}) \mathbb{R}^{n-2} \times \rho_{n-2} \ldots) \times \rho_4 \mathbb{R}^3 \times \rho_3 \mathbb{R}^2 \times \rho_2 \mathbb{R} \tag{4} \]

2.2. Denote by \( L^1(N) \) the Banach algebra of \( N \) that consists of all complex valued functions on the group \( N \), which are integrable with respect to the Haar measure of \( N \) and multiplication is defined by convolution on \( N \) as follows:

\[ g \ast f(X) = \int_N f(Y^{-1}X)g(Y)dY \tag{5} \]

for any \( f \in L^1(N) \) and \( g \in L^1(N) \), where \( X = (X^1, X^2, X^3, \ldots, X^{n-2}, X^{n-1}, X^n) \in N, Y = (Y^1, Y^2, Y^3, \ldots, Y^{n-2}, Y^{n-1}, Y^n) \in N, X^1 = x_1^1, X^2 = (x_1^2, x_2^2), X^3 = (x_1^3, x_2^3, x_3^3), \ldots, X^{n-2} = (x_1^{n-2}, x_2^{n-2}, x_3^{n-2}, \ldots, x_{n-2}^{n-2}), X^{n-1} = (x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, \ldots, x_{n-1}^{n-1}, x_{n-1}^{n-1}), X^n = (x_1^n, x_2^n, x_3^n, \ldots, x_{n-2}^n, x_{n-1}^n, x_n^n) \) and \( dY = dY^1dY^2dY^3, \ldots, dY^{n-2}dY^{n-1}dY^n \) is the Haar measure on \( N \) and \( \ast \) denotes the convolution product on \( N \). We denote by \( L^2(N) \) its Hilbert space. We refer to [12] to define the Fourier transform on \( N \).

**Definition 2.1.** For \( 1 \leq i \leq n \), let \( F^i \) be the Fourier transform on \( \mathbb{R}^i \) and \( 0 \leq j \leq n - 1 \), let \( \prod_{0 \leq i \leq j} \mathbb{R}^{n-i} = (\ldots(((\mathbb{R}^n \times \rho_n) \mathbb{R}^{n-1}) \times \rho_{n-1}) \mathbb{R}^{n-2} \times \rho_{n-2}) \ldots \times \rho_{n-j} \mathbb{R}^{n-j} \), and let \( \prod_{0 \leq i \leq j} F^{n-i} = F^n F^{n-1} F^{n-2} \ldots F^{n-j} \), we can define the Fourier transform on \( \prod_{0 \leq i \leq j} \mathbb{R}^{n-i} = \mathbb{R}^n \times \rho_n \mathbb{R}^{n-1} \times \rho_{n-1} \mathbb{R}^{n-2} \times \rho_{n-2} \ldots \times \rho_4 \mathbb{R}^3 \times \rho_3 \mathbb{R}^2 \times \rho_2 \mathbb{R} \) as

\[
F^n F^{n-1} F^{n-2} \ldots F^2 F^1 f(\lambda^n, \lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda^2, \lambda^1)
= \int_N f(X^n, X^{n-1}, \ldots, X^2, X^1)e^{-i(\lambda^n, \lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda^2, \lambda^1)(X^n, X^{n-1}, \ldots, X^2, X^1)}
dX^n dX^{n-1} \ldots dX^2 dX^1 \tag{6}
\]

for any \( f \in L^1(N) \), where \( X = (X^n, X^{n-1}, \ldots, X^2, X^1) \), \( dX = dX^ndX^{n-1} \ldots dX^2 dX^1 \), \( \lambda = (\lambda^n, \lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda^2, \lambda^1) \), and

\[
\langle (\lambda^n, \lambda^{n-1}), (X^n, X^{n-1}), \ldots, (\lambda^2, \lambda^1), (X^2, X^1) \rangle = \sum_{i=1}^{n} X_i^n \lambda_i^n + \sum_{j=1}^{n-1} X_{i}^{n-j} \lambda_{i}^{n-j} +
\]

\[
.. + \sum_{i=1}^{2} X_i^2 \lambda_i^2 + X^1 \lambda^1
\]
Plancherel’s theorem 2.1. For any function $f \in L^1(N)$, we have

$$\int_{N} |\mathcal{F}^d f(\xi, X^{n-1}, X^{n-2}, \ldots, X^2, X^1)|^2 dX dX^{n-1} dX^{n-2} \ldots dX^2 dX^1$$

$$\int_{N} |\mathcal{F}^d f(\xi, \lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda^2, \lambda^1)|^2 d\xi d\lambda^{n-1} d\lambda^{n-2} \ldots d\lambda^2 d\lambda^1 \quad (7)$$

Proof: To prove this theorem, we refer to [12].

3 Notation and Results for the Solvable Lie Group $AN$.

3.1. Let $G = SL(n, \mathbb{R})$ be the real semi-simple Lie group and let $G = KAN$ be the Iwasawa decomposition of $G$, where $K = SO(n, \mathbb{R})$, and

$$N = \begin{pmatrix}
1 & \ast & \ldots & \ast \\
0 & 1 & \ast & \ast \\
\vdots & \vdots & \ddots & \ast \\
0 & 0 & \ldots & 1
\end{pmatrix}, \quad (8)$$

$$A = \begin{pmatrix}
a_1 & 0 & 0 & 0 \\
0 & a_2 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_n
\end{pmatrix}, \quad (9)$$

where $a_1, a_2, \ldots, a_n = 1$ and $a_i \in \mathbb{R}^*_+$. The product $AN$ is a closed subgroup of $G$ and is isomorphic (algebraically and topologically) to the semi-direct product of $A$ and $N$ with $N$ normal in $AN$.

Then the group $AN$ is nothing but the group $S = N \rtimes \tau A$. So the product of two elements $X$ and $Y$ by

$$(x, a)(m, b) = (x \tau(a)y, ab) \quad (10)$$
for any \( X = (x, a_1, a_2, ..., a_n) \in S \) and \( Y = (m, b_1, b_2, ..., b_n) \in S \). Let \( dnda = dmda_1 da_2 .. da_{n-1} \) be the right haar measure on \( S \) and let \( L^2(S) \) be the Hilbert space of the group \( S \). Let \( L^1(S) \) be the Banach algebra that consists of all complex valued functions on the group \( S \), which are integrable with respect to the Haar measure of \( S \) and multiplication is defined by convolution on \( S \) as

\[
g * f = \int_S f((m, b)^{-1}(n, a))g(m, b)dmdb
\]

where \( dmdb = dmdb_1 db_2 .. db_{n-1} \) is the right Haar measure on \( S = N \rtimes \rho A \).

Let \( \Lambda = N \times A \times A \) be the group with law

\[
(x, b, a)(y, c, d) = (x.\tau(a)y, bc, ad)
\]

for any \((x, b, a) \in \Lambda\), and \((y, c, d) \in \Lambda\).

**Definition 3.1.** For every function \( f \) defined on \( S \), one can define a function \( \tilde{f} \) on \( \Lambda \) as follows:

\[
\tilde{f}(n, a, b) = f(\rho(a)n, ab)
\]

for all \((n, a, b) \in \Lambda\). So every function \( \psi(n, a) \) on \( S \) extends uniquely as an invariant function \( \tilde{\psi}(n, b, a) \) on \( \Lambda \).

**Remark 3.1.** The function \( \tilde{f} \) is invariant in the following sense:

\[
\tilde{f}(\rho(s)n, as^{-1}, bs) = \tilde{f}(\rho(s)n, s^{-1}a, bs) = \tilde{f}(n, a, b)
\]

for any \((n, a, b) \in \Lambda\) and \( s \in A \).

**Lemma 3.1.** For every function \( f \in L^1(S) \) and for every \( g \in L^1(S) \), we have

\[
g * f(n, a, b) = g *_{c} \tilde{f}(n, a, b)
\]

\[
\int_{\mathbb{R}^{n-1}} \mathcal{F}^{n-1}_A \mathcal{F}^d (g * \tilde{f})(\lambda, \mu, \nu) d\nu = \mathcal{F}^{n-1}_A \mathcal{F}^d \tilde{f}(\lambda, \mu, 0) \mathcal{F}^{n-1}_A \mathcal{F}^d \bar{g}(\lambda, \mu)
\]

for every \((n, a, b) \in \Lambda\), where \( * \) signifies the convolution product on \( S \) with respect the variables \((n, b)\) and \( *_c \) signifies the commutative convolution product on the commutative group \( B = N \times A \) with respect the variables \((n, a)\), where \( \mathcal{F}^{n-1}_A \) is the Fourier transform on \( A \).
Plancherel theorem for $S$. 3.1. For any function $\Psi \in L^1(S)$, we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^n} |\mathcal{F}^d \mathcal{F}_A^{-1} \Psi(\lambda, \mu)|^2 d\lambda d\mu = \int_{\mathcal{A}N} |\Psi(X, a)|^2 dX da$$ \hspace{1cm} (17)

Go back to [12], for the prove of this theorem.

4 Fourier Transform and Plancherel Theorem on $GL(n, \mathbb{R})$.

4.1. In the following we use the Iwasawa decomposition of $G = SL(n, \mathbb{R})$, to define the Fourier transform and to get Plancherel theorem on $G = SL(n, \mathbb{R})$. We denote by $L^1(G)$ the Banach algebra that consists of all complex valued functions on the group $G$, which are integrable with respect to the Haar measure of $G$ and multiplication is defined by convolution on $G$,

$$\phi * f(g) = \int_G f(h^{-1}g)\phi(g)dg$$ \hspace{1cm} (18)

Let $G = SL(n, \mathbb{R}) = KNA$ be the Iwasawa decomposition of $G$. The Haar measure $dg$ on $G$ can be calculated from the Haar measures $dn$, $da$ and $dk$ on $N$, $A$ and $K$; respectively, by the formula

$$\int_G f(g)dg = \int_A \int_N \int_K f(ank)dndadk$$ \hspace{1cm} (19)

Keeping in mind that $a^{-2\rho}$ is the modulus of the automorphism $n \rightarrow ana^{-1}$ of $N$ we get also the following representation of $dg$

$$\int_G f(g)dg = \int_A \int_N \int_K f(ank)dndadk = \int_N \int_A \int_K f(nak)a^{-2\rho}dndadk$$ \hspace{1cm} (20)

where $\rho = \dim N = \frac{n(n+1)}{2} = 1 + 2 + 3 + \ldots + n - 2 + n - 1 + n$. Furthermore, using the relation $\int_G f(g)dg = \int_G f(g^{-1})dg$, we receive

$$\int_N \int_A \int_K f(nak)a^{-2\rho}dndadk = \int_N \int_A \int_K f(kan)a^{2\rho}dndadk$$ \hspace{1cm} (21)
Definition 4.1. The Fourier transform of a function $f \in C^\infty(K)$ is defined as

\[ Tf(\gamma) = \int_K f(x)\gamma(x^{-1})dx \] (22)

where $T$ is the Fourier transform on $K$, and $\gamma \in \hat{K}$ ($\hat{K}$ is the set of irreducible unitary representations of $K$)

Theorem (A. Cerezo) 4.1. Let $f \in C^\infty(K)$, then we have the inversion of the Fourier transform

\[ f(x) = \sum_{\gamma \in \hat{K}} d_{\gamma} tr[Tf(\gamma)\gamma(x)] \] (23)

and the Plancherel formula

\[ \|f(x)\|_2^2 = \int |f(x)|^2 dx = \sum_{\gamma \in \hat{K}} d_{\gamma} \|Tf(\gamma)\|_{H.S}^2 \] (24)

for any $f \in L^1(K)$, see [2, P.562 – 563], where $\|Tf(\gamma)\|_{H.S}^2$ is the norm of Hilbert-Schmidt of the operator $Tf(\gamma)$.

Definition 4.2. For any function $f \in D(G)$, we can define a function $\Upsilon(f)$ on $G \times K$ by

\[ \Upsilon(f)(g, k_1) = \Upsilon(f)(kna, k_1) = f(gk_1) = f(knak_1) \] (25)

for $g = kna \in G$, and $k_1 \in K$. The restriction of $\Upsilon(f) \ast \psi(g, k_1)$ on $K(G)$ is $\Upsilon(f) \ast \psi(g, k_1) \downarrow_{K(G)} = f(nak_1) \in D(G)$, and $\Upsilon(f)(g, k_1) \downarrow_K = f(g, I_K) = f(kna) \in D(G)$.

Remark 4.1. The function $\Upsilon(f)$ is invariant in the following sense

\[ \Upsilon(f)(gh, h^{-1}k_1) = f(gk_1) = f(knak_1) \] (26)

Definition 4.3. Let $f$ and $\psi$ be two functions belong to $D(G)$, then we can define the convolution of $\Upsilon(f)$ and $\psi$ on $G \times K$ as

\[ \Upsilon(f) \ast \psi(g, k_1) = \int_G \Upsilon(f)(gg_2^{-1}, k_1)\psi(g_2)dg_2 \]

\[ = \int_K \int_N \int_A \Upsilon(f)(knaa_2r1n_2^{-1}k^{-1}k_1)\psi(k_2n_2a_2)dk_2dn_2da_2 \]
So we get

\[ \Upsilon(f) \ast \psi(g, k_1) \downarrow_{K(G)} = \Upsilon(f) \ast \psi(I_K n a, k_1) \]
\[ = \int_K \int_N \int_A f(na a^{-1} n^{-1} a^{-1} k_1) \psi(k_2 n a_2) dk_2 dn_2 da_2 \]

where \( T \) is the Fourier transform on \( K \), and \( I_K \) is the identity element of \( K \).

**Definition 4.4.** If \( f \in \mathcal{D}(G) \) and let \( \Upsilon(f) \) be the associated function to \( f \), we define the Fourier transform of \( \Upsilon(f)(g, k_1) \) by

\[ T \mathcal{F} \Upsilon(f)(I_K, \xi, \lambda, \gamma) = T \mathcal{F} \Upsilon(f)(I_K, \xi, \lambda, \gamma) \]
\[ = \int_K \int_N \int_A \sum_{\delta \in \hat{K}} \int_K \Upsilon(f)(k_{na, k_1}) \delta(k^{-1}) dk a^{-i \lambda} e^{-i \langle \xi, n \rangle} \gamma(k_1^{-1}) da dn dk \]
\[ = \int_N \int_A \int_K \Upsilon(f)(I_K n a, k_1) a^{-i \lambda} e^{-i \langle \xi, n \rangle} \gamma(k_1^{-1}) da dn dk \]
\[ = \int_N \int_A \int_K f(n a k_1) a^{-i \lambda} e^{-i \langle \xi, n \rangle} \gamma(k_1^{-1}) da dn dk \] (27)

**Theorem 4.2. (Plancherel’s Formula for the Group \( G = SL(n, \mathbb{R}) \)).** For any function \( f \in L^1(G) \cap L^2(G) \), we get

\[ \int |f(g)|^2 dg = \int_K \int_N \int_A |f(k_{na})|^2 da dk \]
\[ = \sum_{\gamma \in \hat{K}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \|T \mathcal{F} f(\lambda, \xi, \gamma)\|_{H,S}^2 d \lambda d \xi \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \hat{K}} d \gamma \|T \mathcal{F} f(\lambda, \xi, \gamma)\|_{H,S}^2 d \lambda d \xi \] (28)

where \( I_A, I_N \) and \( I_K \) are the identity elements of \( A, N \) and \( K \) respectively, \( \mathcal{F} \) is the Fourier transform on \( AN \) and \( T \) is the Fourier transform on \( K \).

For the proof of this theorem see [13]

**New Group.** Let \( GL(n, \mathbb{R}) \) be the general linear group consisting of all matrices of the form

\[ GL(n, \mathbb{R}) = \{ X = (a_{ij}), a_{ij} \in \mathbb{R}, \ 1 < i < n, \ j < n, \ and \ det A \neq 0 \} \] (29)
As a manifold, $GL(n, \mathbb{R})$ is not connected but rather has two connected components: the matrices with positive determinant and the ones with negative determinant which is denoted by $GL_-(n, \mathbb{R})$. The identity component, denoted by $GL_+(n, \mathbb{R})$, consists of the real $n \times n$ matrices with positive determinant. This is also a Lie group of dimension $n^2$; it has the same Lie algebra as $GL(n, \mathbb{R})$.

The group $GL(n, \mathbb{R})$ is also noncompact. The maximal compact subgroup of $GL(n, \mathbb{R})$ is the orthogonal group $O(n)$, while the maximal compact subgroup of $GL_+(n, \mathbb{R})$ is the special orthogonal group $SO(n)$. As for $SO(n)$, the group $GL_+(n, \mathbb{R})$ is not simply connected.

**Theorem 4.3.** $GL_-(n, \mathbb{R})$ is group isomorphic onto $GL_+(n, \mathbb{R})$

For the proof of this theorem see [13].

As well known the group $GL_+(n, \mathbb{R})$ is isomorphic onto the direct product of the two groups $SL(n, \mathbb{R})$ and $\mathbb{R}_+^*$, i.e $GL_+(n, \mathbb{R}) = SL(n, \mathbb{R}) \times \mathbb{R}_+^*$ and $GL(n, \mathbb{R}) = GL_-(n, \mathbb{R}) \cup GL_+(n, \mathbb{R}) = (SL(n, \mathbb{R}) \times \mathbb{R}_+^*) \cup (SL(n, \mathbb{R}) \times \mathbb{R}_+^*)$.

Our aim result is

**Plancherel theorem 4.4.** Let $\mathcal{F}_+^*$ be the Fourier transform on $GL_+(n, \mathbb{R})$ the we get

$$
\int_{GL_+(n, \mathbb{R})} |f(g, t)|^2 dg \frac{dt}{t} = \int_{\mathbb{R}^*_+} \int_K \int_N \int_A |f(kna, t)|^2 d\alpha nk \frac{dt}{t}
$$

$$
= \sum_{\gamma \in \hat{K}} d_{\gamma} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \|\mathcal{F}_+^* T \mathcal{F} f(\lambda, \xi, \gamma, \eta)\|_{H.S}^2 d\lambda d\xi d\eta
$$

$$
= \int_{\mathbb{R}} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \hat{K}} d_{\gamma} \|\mathcal{F}_+^* T \mathcal{F} f(\lambda, \xi, \gamma, \eta)\|_{H.S}^2 d\lambda d\xi d\eta
$$

for any $f \in L^1(GL_+(n, \mathbb{R}) \cap L^2(GL_+(n, \mathbb{R})$. For the proof of this theorem see [13].
Corollary 4.1. Let $f$ be a function belongs $L^1(GL(n, \mathbb{R}) \cap L^2(GL(n, \mathbb{R})$

$$\int_{GL(n, \mathbb{R})} |f(g, t)|^2 \, dg \frac{dt}{t}$$

$$= \int_{GL^-(n, \mathbb{R}) \cup GL^+(n, \mathbb{R})} |f(g, t)|^2 \, dg \frac{dt}{t} = 2 \int_{GL^+(n, \mathbb{R})} |f(g, t)|^2 \, dg \frac{dt}{t}$$

$$= 2 \int \int \int \int |f(kna, t)|^2 \, d\gamma d\eta d\xi$$

Remark 2.1. this corollary explains the Fourier transform and Plancherel formula on the non connected Lie group $GL(n, \mathbb{R})$.

5 Fourier Transform and Plancherel Theorem for $GA(n, \mathbb{R})$

5.1. We begin by the group $GA_+(n, \mathbb{R}) = \mathbb{R}^n \rtimes_{\rho} GL_+(n, \mathbb{R})$ to define the Fourier transform and to prove the Plancherel formula on the group $GA(n, \mathbb{R})$, where $\rho$ is the group homomorphism from $GL_+(n, \mathbb{R})$ into the group $Aut(\mathbb{R}^n)$ of all automorphisms of the real vector group $\mathbb{R}^n$. Let $H_+ = \mathbb{R}^n \times GL(n, \mathbb{R})_+ \times GL_+(n, \mathbb{R})$ be the Lie group with the following law

$$(A, X, Y)(B, P, Q) = (A + \rho(Y)B, XB, YQ)$$

for all $(A, X, Y) \in H_+$, $(C, D, Y) \in H_+$. Denote by $L_+ = \mathbb{R}^n \times GL_+(n, \mathbb{R})$ the lie group which is direct product of the two groups $\mathbb{R}^n$ and $GL_+(n, \mathbb{R})$. In this case the group $GA_+(n, \mathbb{R})$ can be identified with the closed subgroup $\mathbb{R}^n \times \{0\} \times \rho GL_+(n, \mathbb{R})$ of $H_+$ and $L_+$ with the subgroup $\mathbb{R}^n \times GL_+(n, \mathbb{R}) \times \{0\}$ of $H_+$

Definition 5.1. For every function $f$ defined on $GA_+(n, \mathbb{R})$, one can define a function $\tilde{f}$ on $H_+$ as follows:

$$\tilde{f}(A, X, Y) = f(\rho(X)A, XY)$$

(33)
for all \((A, X, Y) \in H_+\), where \(X = (k_1 n_1 a_1, t_1), Y = (k_2 n_2 a_2, t_2), (k_1, k_2) \in K \times K, (n_1, n_2) \in N \times N, \) and \((t_1, t_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*\). So every function \(\psi(A, Y) \in GA_+(n, \mathbb{R})\) extends uniquely as an invariant function \(\tilde{\psi}(A, X, Y)\) on \(H_+\).

**Remark 5.1.** The function \(\tilde{f}\) is invariant in the following sense:

\[
\tilde{f}(\rho(T)A, XT^{-1}, TY) = \tilde{f}(A, X, Y)
\]

for any \((A, X, Y) \in H_+\) and \(T \in GL_+(n, \mathbb{R})\)

**Lemma 5.1.** For every function \(f \in L^1(GA_+(n, \mathbb{R}))\) and for every \(g \in L^1(GA_+(n, \mathbb{R}))\), we have

\[
g * \tilde{f}(A, X, Y) = \tilde{f} * g(A, X, Y)
\]

for every \((A, X, Y) \in H_+\), where \(*\) signifies the convolution product on \(GA_+(n, \mathbb{R})\) with respect the variables \((A, Y)\) and \(*_c\) signifies the convolution product on \(L^1 = \mathbb{R}^n \times GL_+(n, \mathbb{R})\) with respect the variables \((A, X)\).

**Proof:** In fact for any \(f\) and \(g\) belong \(L^1(GA_+(n, \mathbb{R}))\) we have

\[
g * \tilde{f}(A, X, Y) = \int \int_{\mathbb{R}^n GL_+(n, \mathbb{R})} \tilde{f}(B, Q)^{-1}(A, X, Y)) g(B, Q) dBdQ
\]

\[
= \int \int_{\mathbb{R}^n GL_+(n, \mathbb{R})} \tilde{f} [\rho(Q^{-1})(-B), Q^{-1}(A, X, Y)] g(B, Q) dBdQ
\]

\[
= \int \int_{\mathbb{R}^n GL_+(n, \mathbb{R})} \tilde{f} [\rho(Q^{-1})(A - B), X, Q^{-1}Y] g(B, Q) dBdQ
\]

\[
= \int \int_{\mathbb{R}^n GL_+(n, \mathbb{R})} \tilde{f} [A - B, XQ^{-1}, Y] g(B, Q) dBdQ
\]

\[
= \tilde{f} * g(A, X, Y)
\]

**Definition 5.1.** If \(f \in L^1(GA_+(n, \mathbb{R}))\), one can define the Fourier transform of \(f\) as

\[
\mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL_+} f(\mu, (\gamma, \xi, \lambda, \eta))
\]

\[
= \int \int_{\mathbb{R}^n} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} f(A, ank, t) e^{-i(\mu, A) t} e^{-i(\gamma, \eta) t} e^{-i(\xi, \lambda) a} \gamma(k^{-1}) dAdadnk dt
\]

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here $\mathcal{F}_{GL^+}$ denotes the Fourier transform on $GL^+(n, \mathbb{R})$ and let $\mathcal{F}_{\mathbb{R}^n}$ be the Fourier transform on $\mathbb{R}^n$

**Corollary 5.1.** For any function $f \in L^1(GA^+(n, \mathbb{R}))$, we have

$$\int \int \int \sum_{\delta \in \mathcal{K}} d\delta tr[\mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL^+}(g \ast \tilde{f})(\mu, (\gamma, \xi, \lambda, \eta), (\delta, \nu, \sigma, \omega)) dv d\sigma d\omega$$

$$= [\mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL^+}(\tilde{f})(\mu, (\gamma, \xi, \lambda, \eta), I_{GL^+}) \mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL^+}(g)(\mu, (\gamma, \xi, \lambda, \eta))] \quad (37)$$

The proof of this theorem results immediately from lemma 4.1

**Plancherel Theorem 5.1.** For any function $f \in L^1(GA^+(n, \mathbb{R}))$, we have

$$= \int \int \int_{R^n} |f(B, Q)|^2 dBdQ$$

$$= \int_{GA^+(n, \mathbb{R})} |f(A, g, t)|^2 dB dg \frac{dt}{t}$$

$$= \int \int \int_{K} \int_{N} \int_{A} |f(A, kna, t)|^2 dB da dn \frac{dt}{t}$$

$$= \sum_{\gamma \in \mathcal{K}} d_{\gamma} \int \int \int \int_{R^n} \|\mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL^+} f(\mu, (\gamma, \xi, \lambda, \eta))\|_{H.S}^2 d\mu d\lambda d\xi d\eta$$

$$= \int \int \int \int_{R^n} \sum_{\gamma \in \mathcal{K}} d_{\gamma} \|\mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL^+} f(\mu, (\gamma, \xi, \lambda, \eta))\|_{H.S}^2 d\mu d\lambda d\xi d\eta \quad (38)$$

**Proof:** Let $f$ the function defined as

$$\tilde{\tilde{\nu}} f(A, X, Y) = \tilde{\tilde{\nu}} f(\rho(X)A, XY) = f[(\rho(X)A, XY)^{-1}] \quad (39)$$
Then we have

\[
\begin{align*}
  f * f(0, I_{GL+}, I_{GL+}) & = \int_{\mathbb{R}^n \times GL_+(n, \mathbb{R})} \tilde{\nu}(f(B, Q)^{-1}(0, I_{GL+}, I_{GL+})) f(B, Q) dBdQ \\
  & = \int_{\mathbb{R}^n \times GL_+(n, \mathbb{R})} \tilde{\nu}(f([-B, Q^{-1}I_{GL+}], Q^{-1}I_{GL+}) f(B, Q) dBdQ \\
  & = \int_{\mathbb{R}^n \times GL_+(n, \mathbb{R})} \tilde{\nu}(\rho(Q^{-1})(B, Q) f(B, Q) dBdQ \\
  & = \int_{\mathbb{R}^n \times GL_+(n, \mathbb{R})} \tilde{\nu}(\rho(Q^{-1})(B, Q) f(B, Q) dBdQ \\
  & = \int_{\mathbb{R}^n \times GL_+(n, \mathbb{R})} \tilde{\nu}(f(B, Q)^{2} dBdQ \\
  & = \int_{\mathbb{R}^n \times GA_+(n, \mathbb{R})} |f(B, g, t)|^2 dBd\frac{g dt}{t} \\
  & = \int_{\mathbb{R}^n \times GA_+(n, \mathbb{R})} |f(B, ank, t)|^2 dBd\frac{ank dt}{t} \quad \text{(40)} \\
  & = \int_{\mathbb{R}^n \times GA_+(n, \mathbb{R})} |f(B, kna, t)|^2 dBd\frac{ank dt}{t} \quad \text{(41)}
\end{align*}
\]

where \(I_{GL+}\) is the identity element of \(GL_+(n, \mathbb{R})\). In other hand, we get
\[ f \ast f(0, I_{GL_+}, I_{GL_+}) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} \sum_{\gamma \in \hat{K}} d\gamma \sum_{\delta \in \hat{K}} d\delta \text{tr}[\mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL_+}(f \ast f)(\mu, (\gamma, \xi, \lambda, \eta), (\delta, \nu, \sigma, \omega)]
\]

Calculate the formula \( \mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL_+}(f)(\mu, (\gamma, \xi, \lambda, \eta), I_{GL_+}) \)
\[
\begin{align*}
&= \int_\mathbb{R}^n \int_\mathbb{R}^d \int_\mathbb{R}^{n-1} \mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{\mathbb{G}_L^+} (\tilde{f})(\mu, (\gamma, \xi, \lambda, \eta), I_{\mathbb{G}_L^+}) \\
&= \int_\mathbb{R}^n \int_\mathbb{R}^d \int_\mathbb{R}^{n-1} \mathcal{F}_{\mathbb{R}^n} \int_{\mathbb{R}^+} \int_K \int_N \int_A \tilde{f}(A, (kna, t), I_{\mathbb{G}_L^+}) \\
&\quad \times e^{-i(\mu, A)t} \gamma(k^{-1}) dAdkdnda \frac{dt}{t} d\mu d\lambda d\xi d\eta \\
&= \int_\mathbb{R}^n \int_\mathbb{R}^d \int_\mathbb{R}^{n-1} \mathcal{F}_{\mathbb{R}^n} \int_{\mathbb{R}^+} \int_K \int_N \int_A \tilde{f}(\rho(ank, t)A, (ank.t)) \\
&\quad \times e^{-i(\mu, A)t} \gamma(k^{-1}) dkdAdadn \frac{dt}{t} d\mu d\lambda d\xi d\eta \\
&= \int_\mathbb{R}^n \int_\mathbb{R}^d \int_\mathbb{R}^{n-1} \mathcal{F}_{\mathbb{R}^n} \int_{\mathbb{R}^+} \int_K \int_N \int_A \tilde{f}(\rho(ank, t)A, (ank.t))^{-1}) \\
&\quad \times e^{-i(\mu, A)t} \gamma(k^{-1}) dkdAdadn \frac{dt}{t} d\mu d\lambda d\xi d\eta \\
&= \int_\mathbb{R}^n \int_\mathbb{R}^d \int_\mathbb{R}^{n-1} \mathcal{F}_{\mathbb{R}^n} \int_{\mathbb{R}^+} \int_K \int_N \int_A \tilde{f}((-A, (k^{-1}n^{-1}a^{-1}, t^{-1})) \\
&\quad \times e^{-i(\mu, A)t} \gamma(k^{-1}) dkdAdadn \frac{dt}{t} d\mu d\lambda d\xi d\eta \\
&= \int_\mathbb{R}^n \int_\mathbb{R}^d \int_\mathbb{R}^{n-1} \mathcal{F}_{\mathbb{R}^n} \int_{\mathbb{R}^+} \int_K \int_N \int_A \tilde{f}((-A, (kna, t)) \\
&\quad \times e^{-i(\mu, A)t} \gamma(k^{-1}) dkdAdadn \frac{dt}{t} d\mu d\lambda d\xi d\eta \\
&= \int_\mathbb{R}^n \int_\mathbb{R}^d \int_\mathbb{R}^{n-1} \mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{\mathbb{G}_L^+} (\tilde{f})(\mu, (\gamma^*, \xi, \eta)) d\mu d\lambda d\xi d\eta
\end{align*}
\]

Finally, we get
\[
f \ast f(0, I_{GL+}, I_{GL+})
= \int_{R^n} \int_{GL^+(n,R)} |f(A, Q)|^2 \, dA \, dQ \\
= \int_{R^n} \int_{GA^+(n,R)} |f(A, g, t)|^2 \, dA \, dg \frac{dt}{t} \\
= \int_{R^n} \int_{GA^+(n,R)} |f(A, ank, t)|^2 \, dA \, dn \, dk \frac{dt}{t} \\
= \int_{R^n} \int_{GA^+(n,R)} |f(A, kna, t)|^2 \, dA \, nd \, ad \frac{dt}{t} \\
= \int_{R^n} \int_{R^n} \int_{R^n} \int_{R^n} \int_{\hat{K}} d_\gamma \text{tr} \left[ \mathcal{F}_{R^n} \mathcal{F}_{GL+} (f) \left( \mu, (\gamma, \xi, \lambda, \eta), I_{GL+} \right) \mathcal{F}_{R^n} \mathcal{F}_{GL+} f(\mu, \gamma, \xi, \lambda, \eta) \right] \\
\sum_{\gamma \in \hat{K}} d_\mu d_\lambda d_\xi d_\eta \\
= \int_{R^n} \int_{R^n} \int_{R^n} \int_{R^n} \int_{\hat{K}} d_\delta \text{tr} \left[ \mathcal{F}_{R^n} \mathcal{F}_{GL+} (f) \left( \mu, (\gamma^*, \xi, \lambda, \eta) \right) \mathcal{F}_{R^n} \mathcal{F}_{GL+} f(\mu, (\gamma, \xi, \lambda, \eta)) \right] \\
\sum_{\gamma \in \hat{K}} d_\mu d_\lambda d_\xi d_\eta \\
= \int_{R^n} \int_{R^n} \int_{R^n} \int_{R^n} \int_{\hat{K}} d_\delta \| \mathcal{F}_{R^n} \mathcal{F}_{GL+} f(\mu, (\gamma, \xi, \lambda, \eta)) \|^2_{H.S} d_\mu d_\lambda d_\xi d_\eta \\
\sum_{\gamma \in \hat{K}} (43)
\]

Hence the proof of our theorem. Now we can state our final result.
Theorem 5.2. For any function \( f \in L^1(GA(n, \mathbb{R})) \), we get

\[
\begin{align*}
&= \int_{\mathbb{R}^n} \int_{GL(n, \mathbb{R})} |f(A, Q)|^2 dA dQ \\
&= 2 \int_{\mathbb{R}^n} \int_{GL^+(n, \mathbb{R})} |f(A, Q)|^2 dA dQ \\
&= 2 \int_{GA_+(n, \mathbb{R})} |f(A, g, t)|^2 dAdg \frac{dt}{t} \\
&= 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} \int_{K} \int_{N} \int_{A} |f(A, kna, t)|^2 dAdadnk \frac{dt}{t} \\
&= 2 \sum_{\gamma \in \hat{K}} d_\gamma \int_{\mathbb{R}^n} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \|\mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL_+} f(\mu, (\gamma, \xi, \lambda, \eta))\|_{H.S}^2 d\mu d\lambda d\xi d\eta \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} 2 \sum_{\gamma \in \hat{K}} \|\mathcal{F}_{\mathbb{R}^n} \mathcal{F}_{GL_+} f(\mu, (\gamma, \xi, \lambda, \eta))\|_{H.S}^2 d\mu d\lambda d\xi d\eta \tag{45}
\end{align*}
\]

Conclusion. I believe that the results of this paper will be a guideline to study the Fourier analysis on non connected locally compact Lie groups.

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