On the frontier lines between the regions of invariant solution type for solutions of the Friedmann equation satisfying the Hubble condition

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Abstract

It is well-known that there are four distinct basic types (two BigBang types, Lemaitre and BigCrunch type) for solutions of the general Friedmann equation with positive cosmological constant, where radiation and matter do not couple. In the note it is shown that the ”frontier lines” between BigBang/BigCrunch and BigBang/Lemaitre are given by two smooth function branches, expressing the cosmological constant as unique functions of the matter and radiation density, which satisfy a simple asymptotic relation w.r.t. the matter density. The proof is based on the solution of the equation $\sigma = \sigma_{cr}$, where $\sigma$ is the radiation invariant of the Friedmann equation and $\sigma_{cr}$ the ”critical radiation parameter” (see [3]).

1 Introduction

Einstein’s famous field equation of the general relativity theory, linking together the gravitational field with the stress-energy tensor, is also a basic tool in cosmology to solve problems of the large-scale structure of the cosmos. The idea of the existence of a definite large-scale structure of the cosmos is expressed by the ”cosmological principle” (E.A. Milne 1933). This principle appears as a boundary condition of Einsteins field equation. This boundary condition consists of a special ansatz of the gravitational field given by a so-called Robertson-Walker(RW-)metric, and of the special form of of the stress-energy tensor as that of a perfect fluid (Wald [1,p.69]). Moreover, it is assumed that matter and radiation do not couple. Then Einstein’s field equation reduces to the so-called Friedmann equation

$$\left( \frac{dR}{dt} \right)^2 = \frac{\alpha}{R} + \frac{\sigma}{R^2} + \frac{1}{3} \Lambda c^2 R^2 - \epsilon c^2$$

for the so-called scale-factor $R(t) > 0$, a dimensionless number. The constant parameters are: velocity of light $c$, cosmological constant $\Lambda$, curvature constant $\epsilon = 0, \pm 1$. The matter invariant $\alpha \geq 0$ and the radiation invariant $\sigma \geq 0$ are given by

$$\alpha := \frac{8\pi G}{3} \rho_{mat}(t) R(t)^3,$$
\[ \sigma := \frac{8\pi G}{3} \rho_{\text{rad}}(t) R(t)^4, \]  

(3)

where \( \rho_{\text{mat}}(t), \rho_{\text{rad}}(t) \) denote matter density, radiation density at time \( t \), respectively. The right hand sides of the equations (2) and (3) are constants because matter and radiation do not couple (see Appendix 10.2). The mathematical context of the connection of Einstein’s field equation with the Friedmann equation is briefly explained in Appendix 10.

In the following an initial condition \( t_0, R_0 \) for equation (1) is called \textit{admissible} if the right hand side of (1) is positive for \( R_0 \).

In 1929 Hubble discovered that at present the cosmos expands which means that \( \frac{dR}{dt}(T) > 0 \), where \( T \) denotes the present time. This observation led to a calculation of the so-called \textit{Hubble constant}

\[ H_0 := \left( \frac{1}{R} \frac{dR}{dt} \right)(T). \]

Its actual value is \((74, 3 \pm 2, 1)\text{km}(\text{s} \cdot \text{Mpc})^{-1}\).

Obviously, the cosmic microwave radiation (CMR), a black body radiation of temperature 2,725 K (actual value) is a large-scale property of the cosmos contributing to the radiation density \( \rho_{\text{rad}}(T) \) such that \( \sigma \) is positive. Further new observations in connection with the Supernova Cosmology Project (see e.g. Perlmutter [2]) suggest that the cosmological constant is positive, too.

That is, the real large-scale structure of the cosmos corresponds to a solution of equ. (1) with certain distinct values of the parameters \( \rho_{\text{mat}}(T) > 0, \rho_{\text{rad}}(T) > 0, \Lambda > 0 \), such that this solution realizes the Hubble constant as the present value of the Hubble parameter

\[ t \rightarrow \left( \frac{1}{R} \frac{dR}{dt} \right)(t). \]

In the following this condition is called the \textit{Hubble condition}. As it is well-known, the type of that solution depends strongly on the mutual position of \( R(T) \), the scale-factor at \( T \) given by the Hubble condition, and the positive zeros of the polynomial of the fourth degree

\[ p(R) := \frac{1}{3} \Lambda c^2 R^4 - c^2 R^2 + \alpha R + \sigma. \]

It turns out that there are three cases for these zeros: (i) two different zeros, (ii) one double-zero, (iii) no zero. These cases can be described using the terms \( 9\alpha^2 - 4\epsilon c^4 \), which is essentially the discriminant of the equation of the third degree \( \frac{1}{3} \Lambda c^2 R^3 - c^2 R + \alpha \), and, in the case \( 9\alpha^2 - 4\epsilon c^4 < 0 \), additionally the critical radiation parameter \( \sigma_{cr} \). In this case \( \sigma < \sigma_{cr} \) corresponds to the case (i) and \( \sigma > \sigma_{cr} \) to the case (iii). That is, the equation \( \sigma = \sigma_{cr} \), corresponding to the case (ii), separates the regions corresponding to the cases (i) and (iii). The solution of this equation can be described by two analytic functions (branches) \( \Lambda = \Lambda_{\pm}(\rho_{\text{mat}}(T), \rho_{\text{rad}}(T)) \), which are the frontier lines between regions of constant type, \( \Lambda_{+} \) between BigBang and Lemaitre type and \( \Lambda_{-} \) between BigBang and BigCrunch type. The construction and investigation of these functions, e.g. the derivation of their asymptotic properties, is the topic of this note.
2 The Hubble condition

The Hubble condition for equ. (1) is a condition for $R(T)$:

$$H_0^2 = \frac{\alpha}{R(T)^3} + \frac{\sigma}{R(T)^4} + \frac{1}{3} \Lambda c^2 - \epsilon \frac{c^2}{R(T)^2},$$

(4)

or, using (2) and (3),

$$H_0^2 = \frac{8\pi G}{3} \rho_{\text{mat}}(T) + \frac{8\pi G}{3} \rho_{\text{rad}}(T) + \frac{1}{3} \Lambda c^2 - \epsilon \frac{c^2}{R(T)^2}.$$  

(5)

That is, the initial condition $T, R(T)$ is admissible. In the special case $\Lambda = 0, \sigma = 0, \epsilon = 0$ it reads

$$H_0^2 = \frac{8\pi G}{3} \rho_{\text{mat}}(T).$$

(6)

This case is the Euclidean Einstein-de Sitter model of 1932. Since that time the term

$$\rho_{\text{cr}} := \frac{3}{8\pi G} H_0^2$$

is called the critical density. Introducing instead of $\rho_{\text{mat}}(T), \rho_{\text{rad}}(T), \Lambda$ the variables

$$x := \frac{\rho_{\text{mat}}(T)}{\rho_{\text{cr}}}, \quad y := \frac{1}{3} \Lambda c^2 H_0^2, \quad z := \frac{\rho_{\text{rad}}(T)}{\rho_{\text{cr}}},$$

(7)

then equ. (5) can be written as

$$1 = x + y + z - \epsilon \frac{c^2}{H_0^2 R(T)^2}.$$  

(8)

and the constants $\alpha, \sigma, \Lambda$ are given by

$$\alpha = x H_0^2 R(T)^3, \quad \sigma = z H_0^2 R(T)^4, \quad \Lambda = y \frac{3 H_0^2}{c^2}.$$  

(9)

If $x + y + z - 1 \neq 0$ then

$$R(T)^2 = \left( \frac{c}{H_0} \right)^2 \frac{\epsilon}{x + y + z - 1},$$

(10)

i.e. in this case the scale-factor $R(T)$ in the present time is uniquely determined by the Hubble condition. Moreover one obtains:

If $x + y + z > 1$ then $\epsilon = +1$, \quad if $x + y + z < 1$ then $\epsilon = -1$.

That is,

$$x + y + z = 1 \text{ iff } \epsilon = 0.$$  

(11)

This means: the Euclidean case is quite singular, realized only on the plane (11) and in this case the scale-factor is not uniquely determined by the Hubble condition. In any case, the geometric structure of the cosmos is uniquely determined by the expression $x + y + z - 1$.

Note that now $\alpha$ and $\sigma$ appear as functions of $x, y, z$. Sometimes the term $\frac{c}{H_0}$ is called the Hubble radius.
3 Discriminant and critical radiation parameter

The right hand side of equation (1) can be written in the form

\[ \frac{\Lambda c^2}{3R^2} \left( Rq(R) + \frac{3}{\Lambda c^2} \sigma \right), \]

where

\[ q(R) := R^3 - \frac{3\epsilon}{\Lambda} R + \frac{3\alpha}{\Lambda c^2}. \]

If \( R(T) \) satisfies the Hubble condition equation (8) then the discriminant \( \Delta \) of \( q \) is given by

\[ \Delta = \frac{1}{4\Lambda^3} \left( \frac{R(T)H_0}{c} \right)^6 D \]

where

\[ D = D(x, y; z) := 27x^2y - 4(x + y + z - 1)^3. \] (12)

If \( D(x, y, z) < 0 \) then necessarily \( \epsilon = 1 \) and we introduce the angle \( \phi, \frac{\pi}{2} < \phi < \pi \) by

\[ \cos \phi = -\frac{\sqrt{27}}{2} \frac{xy^{\frac{1}{2}}}{(x + y + z - 1)^{\frac{3}{2}}}, \quad \frac{\pi}{2} < \phi < \pi. \] (13)

Then \( D = 0 \) corresponds to \( \phi = \pi \).

Note that \( D(x, y; 1) = 0 \) iff \( x = 2y \) and if \( z > 1 \) then \( D(x, y; z) < 0 \) for all \( x \geq 0, y \geq 0 \).

By equation (13) to each \( (x, y) \) with \( x > 0, y > 0 \) there is associated a uniquely determined angle \( \phi, \frac{\pi}{2} < \phi < \pi \). Fixing a parameter \( z = z_0 \geq 0 \) then the algebraic curves \( \phi = \text{const} \) exhaust the region \( D < 0 \) within the first quadrant \( x > 0, y > 0 \).

For \( D < 0 \) the critical radiation parameter \( \sigma_{cr} \) is given by

\[ \sigma_{cr} = \frac{c^4}{H_0^2} \frac{2}{3} y^{-1} \left( \cos \frac{\psi}{3} \right)^2 \left( 2 \left( \cos \frac{\psi}{3} \right)^2 - 1 \right), \quad \cos \psi = \frac{1}{\sqrt{2}} \cos \phi, \quad \frac{3\pi}{4} > \psi > \frac{\pi}{2}, \] (14)

see [3] for details.

The positive roots of the polynomial \( p(\cdot) \) can be described as follows:

(i) If \( D < 0 \) and \( 0 \leq \sigma < \sigma_{cr} \) then there are two positive roots \( R_1 < R_2 \) (if \( x = 0 \) then \( R_1 = 0 \)).

(ii) If \( D = 0 \) and \( \sigma = \sigma_{cr} = 0 \) or \( D < 0 \) and \( \sigma = \sigma_{cr} \) where \( \sigma > 0 \) then there is one positive double-root.

(iii) If \( D > 0 \) and \( \sigma \geq 0 \) or \( D = 0 \) and \( \sigma > 0 \) or \( D < 0 \) and \( \sigma > \sigma_{cr} \) then there is no positive root.
In the case \( D < 0 \) and \( \sigma = 0 \), i.e. \( z = 0 \), one obtains the explicit expressions

\[
R_1(x, y; 0) := \frac{c}{H_0} \sqrt{\frac{2}{3}} y^\frac{1}{2} \cos \left( \frac{1}{3} (2\pi - \phi) \right), \quad R_2(x, y; 0) := \frac{c}{H_0} \sqrt{\frac{2}{3}} y^\frac{1}{2} \cos \frac{1}{3} \phi. \tag{15}
\]

Since in the case (i) one has \( p(R) < 0 \) for \( R \in (R_1, R_2) \), this is an forbidden interval for initial conditions \( t, R \). However, according to the Hubble condition, \( R(T) = R(x, y; z_0) \) given by equation (10) is admissible, i.e. one obtains that either

\[
R(x, y; z_0) > R_2(x, y; z_0)
\tag{16}
\]

or

\[
R(x, y; z_0) < R_1(x, y; z_0).
\tag{17}
\]

## 4 The equation \( \sigma = \sigma_{cr} \)

As already mentioned in Sec.1 the equation \( \sigma = \sigma_{cr} \) separates the regions corresponding to the cases (i) and (iii). That is, the regions for these cases can be determined by the solution of this equation. In the following \( z \) is considered as a parameter and and the equation is considered in the first quadrant \( x \geq 0, y \geq 0 \). The terms \( \sigma \) and \( \sigma_{cr} \) are functions of \((x, y)\) and of the parameter \( z \). The function \( \sigma_{cr} \) is given by (14).

According to equations (9) and (11) one obtains for

\[
\sigma = \frac{c^4}{H_0^2} \frac{z}{(x + y + z - 1)^2}. \tag{18}
\]

Then the equation \( \sigma = \sigma_{cr} \) reads

\[
\frac{4zy}{(x + y + z - 1)^2} = \frac{8}{3} \left( \cos \frac{\psi}{3} \right)^2 \left( 2 \left( \cos \frac{\psi}{3} \right)^2 - 1 \right), \tag{19}
\]

where \( \psi \) is a function of \( x, y \) and \( z \), according to equ. (13) and (14). Solution of equation (19) means the construction of functions \( x \rightarrow Y(x; z_0), x \geq 0 \), for every parameter \( z_0 > 0 \), which satisfy this equation. As a function of \( \phi \), the right hand side of equ. (19) has a simple structure. In the following we put

\[
\frac{8}{3} \left( \cos \frac{\psi}{3} \right)^2 \left( 2 \left( \cos \frac{\psi}{3} \right)^2 - 1 \right) =: F(\phi), \quad \frac{\pi}{2} \leq \phi \leq \pi. \tag{20}
\]

One obtains \( F(\frac{\pi}{2}) = 1, F(\pi) = 0 \) and \( F(\cdot) \) is strongly monotonically decreasing.

In the following the proofs for the solution of equ. (19) are considered separately for the parameter regions \( 0 \leq z < 1, z = 1 \) and \( z > 1 \). The case \( z = 1 \) is considered first. On the one hand, it is a rather singular case, but on the other hand it is explicitly soluble.
5 The case $z = 1$

5.1 Result

The solution of equ. (19) consists of two functions (branches)

$$(0, \infty) \ni x \rightarrow Y_\pm(x),$$

given by the parameter representation

$$x := x_\pm(\phi), \quad Y_\pm(x) := \mu_\pm(\phi)x_\pm(\phi), \quad \frac{\pi}{2} < \phi < \pi,$$

where

$$\mu_\pm(\phi) := 3\nu_\pm(\phi) - 1, \quad \nu_+(\phi) := \frac{\cos \frac{1}{3}\phi}{\cos \phi}, \quad \nu_-(\phi) := \frac{\cos \frac{1}{3}(2\pi - \phi)}{\cos \phi}$$

and

$$x_\pm(\phi) := \frac{4}{9} F(\phi) \left( \frac{3}{\nu_+(\phi)} - \frac{1}{\nu_-(\phi)^2} \right).$$

The solutions $Y_\pm(\cdot)$ have the properties

$$Y_-(x) < \frac{1}{2} x < Y_+(x), \quad x > 0,$$

$$\lim_{x \to 0^+} Y_+(x) = 4, \quad \lim_{x \to 0^-} Y_-(x) = 0,$$

$$\lim_{x \to \infty} \frac{Y_\pm(x)}{x} = \frac{1}{2}.$$  

5.2 Proof

First note that

$$\mu_+(\phi) \to \infty \quad \text{and} \quad \mu_-(\phi) \to 0 \quad \text{for} \quad \phi \to \frac{\pi}{2},$$

further

$$\mu_-(\phi) < \frac{1}{2} < \mu_+(\phi), \quad \frac{\pi}{2} < \phi < \pi; \quad \mu_\pm(\phi) \to \mu_\pm(\pi) = \frac{1}{2} \quad \text{for} \quad \phi \to \pi.$$

The equation for the $\phi$-curves reads

$$4(x + y)^3(\cos \phi)^2 = 27x^2y, \quad \frac{\pi}{2} < \phi < \pi.$$
In the case $\phi = \pi$ for the left hand side of (27) one has $4(1+\mu)^3 - 27\mu = (2\mu - 1)^2(\mu + 4)$. Moreover one obtains

$$27\mu - 4(1 + \mu)^3(\cos \phi)^2 > 0, \quad \mu_-(\phi) < \mu < \mu_+(\phi).$$

Therefore, the $\phi$-curve consists of the two rays

$$y = \mu_\pm(\phi)x, \ x > 0, \ \frac{\pi}{2} < \phi < \pi.$$  \hspace{1cm} (29)

In the case $\phi = \pi$ there is only one ray and for $\phi = \frac{\pi}{2}$ the limit rays are the half axes $x = 0$ and $y = 0$.

The equation $\sigma = \sigma_{cr}$ reads

$$4y = (x + y)^2 F(\phi).$$

For a point $(x, y)$ on a $\phi$-ray (29) this means

$$4\mu_\pm(\phi)x = F(\phi)(1 + \mu_\pm(\phi))^2 x^2.$$  \hspace{1cm} (30)

The solution of this equation is given by (22) which implies $x_-(\phi) < x_+(\phi)$ and

$$x_\pm(\phi) \to 0 \quad \text{if} \quad \phi \to \frac{\pi}{2}, \quad x_\pm(\phi) \to \infty \quad \text{if} \quad \phi \to \pi.$$  \hspace{1cm} (31)

Then $\mu_-(\phi)x_-(\phi) \to 0$ and $\mu_+(\phi)x_+(\phi) \to 4$ for $\phi \to \frac{\pi}{2}$. This proves (24). If $x = x_+(\phi_+) = x_-(\phi_-)$ then $x \to \infty$ iff $\phi_+ \to \pi$. In this case one has $Y_\pm(x) = \mu_\pm(\phi_\pm)x$ and $\mu_+(\phi_+) > \frac{1}{2} < \mu_-(\phi_-)$ because of (26). This proves (23). Finally, if $x \to \infty$ then $\mu_\pm(\phi_\pm) \to \mu_\pm(\pi) = \frac{1}{2}$. This proves (25).

6 The case $z > 1$

6.1 Result

The solution of equ. (19) consists of two functions (branches)

$$(0, \infty) \ni x \to Y_\pm(x)$$

given by the parameter representation

$$x = x(\phi, \rho_\pm(\phi)), \quad Y_\pm(x) = \rho_\pm(\phi)x(\phi, \rho_\pm(\phi)), \quad \frac{\pi}{2} < \phi < \pi,$$  \hspace{1cm} (32)

where

$$x(\phi, \mu) := (z - 1) - \frac{2^{2/3}(\cos \phi)^{2/3}}{3\mu^{1/3} - 2^{2/3}(\cos \phi)^{2/3}(\mu + 1)}, \quad \mu_-(\phi) < \mu < \mu_+(\phi),$$  \hspace{1cm} (33)

and the terms $\mu := \rho_\pm(\phi)$ are uniquely determined solutions of the equation

$$(\cos \phi)^{2/3}\mu^{1/3}(3\mu^{1/3} - 2^{2/3}(\cos \phi)^{2/3}(\mu + 1)) = \frac{3^2}{2^{8/3}} \frac{z - 1}{z} F(\phi),$$  \hspace{1cm} (34)
where
\[ \mu_-(\phi) < \rho_-(\phi) < \rho_+(\phi) < \mu_+ (\phi). \] (33)

The solutions \( Y_\pm (\cdot) \) have the properties
\[ Y_-(x) < Y_+(x), \quad x > 0, \] (34)
\[ \lim_{x \to 0} Y_\pm (x) = (1 \pm \sqrt{2})^2, \] (35)
\[ \lim_{x \to \infty} \frac{Y_\pm (x)}{x} = \frac{1}{2}. \] (36)

6.2 Proof

For convenience put \( a := z - 1, a > 0 \). The equation for the \( \phi \)-curves reads
\[ 4(x + y + a)^3 (\cos \phi)^2 = 27x^2y, \quad \frac{\pi}{2} < \phi < \pi. \] (37)

Note that in this case the limit case \( \phi = \pi \) is excluded because \( D < 0 \) everywhere. The limit case \( \phi = \frac{\pi}{2} \) corresponds to the half axes \( x = 0, y = 0 \). The parameter representation for the \( \phi \)-curve using \( \mu \)-rays \( y = \mu x, \mu > 0 \) yields the term \( x = x(\phi, \mu) \) given by equ. (31). According to equations (27) and (28) one has \( x \to \infty \) for \( \mu \to \mu_\pm (\phi) \). That is, in this case the \( \phi \)-curve (37) has only a single branch, where \( x \geq a \frac{1}{|\cos \phi|} \) for fixed \( \phi \). Moreover, the expression \( 3\mu^{1/3} - 2^{2/3}(1 + \mu)(\cos \phi)^{2/3} \) takes its maximum at \( \mu := \frac{1}{2(\cos \phi)^{2/3}} \), i.e. for any \( x > a \frac{|\cos \phi|}{1 - |\cos \phi|} \) there are parameters \( \mu_+, \mu_- \) such that
\[ \mu_- < \frac{1}{2(\cos \phi)^{2/3}} < \mu_+ \]
and \( x = x(\phi, \mu_-) = x(\phi, \mu_+) \). The corresponding values for \( y \) are \( y_\pm (x) = \mu_\pm x \), i.e. one obtains
\[ \lim_{x \to \infty} \frac{y_\pm (x)}{x} = \mu_\pm (\phi). \] (38)

The equation (19) for a point \((x, y) = (x, \mu x)\) on a fixed \( \phi \)-curve leads the equation (32). The left hand side of this equation is positive for parameters \( \mu \) satisfying \( \mu_-(\phi) < \mu < \mu_+(\phi) \) and vanishes for \( \mu = \mu_\pm (\phi) \). It takes its maximal value in this interval at
\[ \mu_{\max}^{1/3} := \frac{21/6}{|\cos \phi|^{2/3}} \cos \frac{\psi}{3}, \]
where the angle \( \psi \) is given by equ. (23). The corresponding maximum is given by \( 2^{1/3} \cos \frac{\psi}{3} (2(\cos \frac{\psi}{3})^3 - 3 \cos \psi) \). Now the inequality
\[ 2^{1/3} \cos \frac{\psi}{3} \left( 2 \left( \cos \frac{\psi}{3} \right)^3 - 3 \cos \psi \right) > \frac{3^2}{2^{8/3}} F(\phi) \]
is true, it is equivalent with the inequality \( \frac{1}{2} \sqrt{3} > \cos \frac{\psi}{3} \) and \( \frac{1}{2} \sqrt{3} \) is the maximal value of \( \cos \frac{\psi}{3} \) in the admissible interval for \( \psi \) (see equ. (14)), which is taken at the limit case \( \phi = \frac{\pi}{2} \). The consequence is that for every \( \phi \in (\frac{\pi}{2}, \pi) \) there are exactly two
solutions $\mu := \rho_{\pm}(\phi)$ of equ. (32), where the inequality (33) is satisfied. This proves (30). The relation (34) is obvious. The relations (35) can be obtained by solving equ. (19) directly for $x = 0$ which implies $\phi = \frac{\pi}{2}$. Concerning relation (36) note that $x \to \infty$ on a fixed $\phi$-curve corresponds to $\mu \to \mu_{\pm}(\phi)$ and for each $\phi < \pi$ there is a solution $x = x(\phi, \mu)$ with $\mu := \rho_{\pm}(\phi)$, according to equ. (30). Taking the limit $\phi \to \pi$ then with equ. (38) one obtains (36).

7 The case $0 < z < 1$

7.1 Result

In this case every $\phi$-curve, $\frac{\pi}{2} < \phi < \pi$ has two branches, a lower one $x \to y_-(x; \phi)$ for $x \geq a$, where $a := 1 - z$, starting at $(a, 0)$, and an upper one $x \to y_+(x; \phi)$ for $x \geq 0$, starting at $(0, a)$. The branches $y_{\pm}(\cdot; \pi)$ of the $\pi$-curve form the boundary of the region defined by $D < 0$. It corresponds to $D = 0$ (cf. Sec. 5). The solution of equ. (19) consists of two functions (branches)

$$(0, \infty) \ni x \to Y_+(x), \quad (a, \infty) \ni x \to Y_-(x).$$

The branch $Y_+(\cdot)$ is given by the parameter representation

$$x = x(\phi, \rho_+(\phi)), \quad Y_+(x) = \rho_+(\phi)x(\phi, \rho_+(\phi)), \quad \frac{\pi}{2} < \phi < \pi,$$  \hspace{1cm} (39)

where

$$x(\phi, \mu) := (1 - z)\frac{2^{2/3}(\cos \phi)^{2/3}}{2^{2/3}(\cos \phi)^{2/3}(\mu + 1) - 3\mu^{1/3}}, \quad \mu > \mu_+(\phi),$$  \hspace{1cm} (40)

and the term $\mu := \rho_+(\phi)$ is the uniquely determined solution of the equation

$$(\cos \phi)^{2/3}\mu^{1/3}(2^{2/3}(\cos \phi)^{2/3}(\mu + 1) - 3\mu^{1/3}) = \frac{3^2}{2^{8/3}}\frac{1 - z}{z}F(\phi),$$  \hspace{1cm} (41)

where $\rho_+(\phi) > \mu_+(\phi)$. This means: every $\phi$-curve has exactly one intersection with the branch $Y_+(\cdot)$, realized by the upper branch $y_+$ of the $\phi$-curve.

In contrast to this property of $Y_+(\cdot)$ the branch $Y_-(\cdot)$ has either exactly two intersections with a $\phi$-curve or there is no intersection. More precisely:

To every $a, 0 < a < 1$, there is an angle $\phi(a), \frac{\pi}{2} < \phi(a) < \pi$, such that the parameter representation of $Y_-(\cdot)$ is given by

$$x = x(\phi, \rho_{\pm}(\phi)), \quad Y_-(x) = \rho_{\pm}(\phi)x(\phi, \rho_{\pm}(\phi)), \quad \phi(a) < \phi < \pi,$$  \hspace{1cm} (42)

where

$$0 < \rho_-(\phi) < \mu_{\text{max}} < \rho_+(\phi) < \mu_-(\phi),$$  \hspace{1cm} (43)

and the left hand side of equ. (50) takes its maximum at $\mu_{\text{max}}$. Note that for $\phi := \phi(a)$ the terms $\rho_-(\phi(a))$ and $\rho_+(\phi(a))$ coincide, $\rho_+(\phi(a)) = \rho_-(\phi(a))$. If $\phi < \phi(a)$ then there is no intersection with this $\phi$-curve.

The solutions $Y_{\pm}(\cdot)$ have the properties

$$Y_-(x) < y_-(x; \pi) < \frac{1}{2}x < y_+(x; \pi) < Y_+(x), \quad x \geq a,$$  \hspace{1cm} (44)
\[
\lim_{x \to 0} Y_-(x) = 0, \\
\lim_{x \to 0} Y_+(x) = (1 + \sqrt{2})^2, \\
\lim_{x \to \infty} \frac{Y_+(x)}{x} = \frac{1}{2}.
\]

### 7.2 Proof

The equation for the \( \varphi \)-curve reads

\[
4(x + y - a)^3(\cos \varphi)^2 = 27x^2y, \quad \frac{\pi}{2} < \varphi < \pi.
\]

The limit case \( \varphi = \frac{\pi}{2} \) corresponds to the half axes \( x = 0, y \geq a \) and \( y = 0, x \geq a \).

The case \( \varphi = \pi \) corresponds to \( D = 0 \). The parameter representation using \( \mu \)-rays \( y = \mu x, \mu > 0 \), yields the term

\[
x = x(\varphi, \mu) = a \frac{2^{2/3} (\cos \varphi)^{2/3}}{2^{2/3} (\cos \varphi)^{2/3} (\mu + 1) - 3 \mu^{1/3}}, \quad \mu > \mu_+(\varphi), \mu < \mu_-(\varphi)
\]

for the \( x \)-coordinate of the point of the \( \varphi \)-curve. Note equations (27) and (28). The parameter values \( \mu > \mu_+(\varphi), \mu < \mu_-(\varphi) \) describe the upper and lower branch \( y_\pm(\cdot; \varphi) \) of the \( \varphi \)-curve, respectively. Inserting (49) and \( y = \mu x \) into equ. (19) then the resulting equ. (41) is the condition for those \( \mu \), where the intersection point of the \( \mu \)-ray with the \( \varphi \)-curve is simultaneously a solution point of equ. (19). The left hand side of equ. (41) vanishes for \( \mu := \mu_+(\varphi) \), it tends to infinity for \( \mu \to \infty \) and it is strongly monotonically increasing for \( \mu > \mu_+(\varphi) \). If \( \frac{\pi}{2} < \varphi < \pi \) then \( 0 < F(\varphi) < 1 \) and \( 0 < \frac{1}{2}aF(\varphi) < \infty \) because of \( 0 < z < 1 \). That is, for every pair \( \{\varphi, z\} \) there is exactly one solution \( \rho_+(\varphi) \) of equ. (41) and one has \( \mu_+(\varphi) < \rho_+(\varphi) \).

For the investigation of solutions of equ. (41) in the interval \((0, \mu_-(\varphi))\) we write this equation in the form

\[
(1-a)(\cos \varphi)^{2/3}\mu_+^{1/3}(2^{2/3}(\cos \varphi)^{2/3}(\mu + 1) - 3\mu^{1/3}) = \frac{3^2}{2^{8/3}}aF(\varphi).
\]

The extrema of the left hand side of equ. (50) as a function of \( \mu \) are

\[
\mu_+^{1/3} := \frac{2^{1/6}}{|\cos \varphi|^{1/3}} \cos \frac{\psi}{3}, \quad \mu_-^{1/3} := \frac{2^{1/6}}{|\cos \varphi|^{1/3}} \cos \frac{1}{3}(2\pi - \psi).
\]

First one obtains \( \mu_-(\varphi) < \mu_{\text{min}} < \mu_+(\varphi) \). This term has been used for the solution of equ. (19) in the case \( z > 1 \). Further, \( 0 < \mu_{\text{max}} < \mu_-(\varphi) \). At this term the left hand side takes its maximum. Inserting this value into the left hand side one gets

\[
(1-a)G(\varphi) = 3 \cdot 2^{1/3} \left( \cos \frac{1}{3}(2\pi - \psi) \right)^2 \left( 1 - 2 \left( \cos \frac{1}{3}(2\pi - \psi) \right)^2 \right).
\]

Recall equ. (14) which implies \( \frac{\pi}{4} > \frac{\psi}{3} > \frac{\pi}{6} \) and \( \frac{5}{12} \pi < \frac{1}{3}(2\pi - \psi) < \frac{\pi}{2} \). The function \( \varphi \to G(\varphi), \frac{\pi}{2} < \varphi < \pi \), is monotonically increasing and

\[
G(\frac{\pi}{2}) = 0, \quad G(\pi) = 3 \cdot 2^{1/3} \left( \cos \frac{5}{12} \pi \right)^2 \left( 1 - 2 \left( \cos \frac{5}{12} \pi \right)^2 \right) > 0.
\]
Comparing the maximum value \( G(\phi) \) of the left hand side of equ. (50) with the right hand side \( a \cdot 3^{2}2^{-8/3}F(\phi) \) and taking into account the monotony properties of \( G(\cdot) \) and \( F(\cdot) \), further (52) and \( F(\pi) = 1 \), \( F(\pi) = 0 \) then one obtains: There is exactly one angle \( \phi(a), \frac{\pi}{2} < \phi(a) < \pi \), such that

\[
(1 - a)G(\phi(a)) = a \cdot 3^{2}2^{-8/3}F(\phi(a)).
\]

Further, if \( \frac{\pi}{2} < \phi < \phi(a) \) then the left hand side of equ. (50) at the maximum point \( \mu_{max} \) is smaller than the right hand side at \( \phi \); however, if \( \phi(a) < \phi < \pi \) then the left hand side of equ. (50) at the maximum point is larger than the right hand side at \( \phi \). This implies: In the case \( \frac{\pi}{2} < \phi < \phi(a) \) there is no solution \( \mu \) of equ. (50). If \( \phi(a) < \phi < \pi \) then there are exactly two solutions \( \rho^{\pm}(\phi) \) of equ. (50) such that equ. (43) is true.

The relation (44) is obvious. From equ. (49) it follows that \( \lim_{\mu \to 0} x(\phi, \mu) = a \) for all \( \phi, \frac{\pi}{2} < \phi < \pi \), i.e. the lower branch of the \( \phi \)-curve starts at \( (a,0) \). Then equ. (45) follows from (44).

Relation (46) can be obtained by solving equ. (19) directly for \( x = 0 \) which implies \( \phi = \frac{\pi}{2} \). Note that in this case the second (formal) solution \((1 - \sqrt{z})^2\) is excluded because of \((1 - \sqrt{z})^2 < 1 - z \) which implies \( D(0, (1 - \sqrt{z})^2, z) > 0 \), i.e. the pair \((0, (1 - \sqrt{z})^2)\) belongs to the region \( D > 0 \).

Concerning relation (47) note that - according to the equations (27) and (28) - it follows from (49) that for \( \mu > \mu_{+}(\phi) \), \( \mu \to \mu_{+}(\phi) \) or \( \mu < \mu_{-}(\phi) \), \( \mu \to \mu_{-}(\phi) \) one has \( x(\phi, \mu) \to \infty \). This means that for the two branches \( y_{\pm}(\cdot; \phi) \) of the \( \phi \)-curves one has

\[
\lim_{x \to \infty} \frac{y_{\pm}(x; \phi)}{x} = \mu_{\pm}(\phi).
\] (53)

However, for each \( \phi, \phi(a) < \phi < \pi \), there is a solution \( x := x(\phi, \mu) \) of equation (19), either with \( \mu := \rho_{+}(\phi) \) according to (39) or with \( \mu := \rho_{\pm}(\phi) \) according to (42) and (43). Taking the limit \( \phi \to \pi \) then with equ. (53) one obtains (47).

### 7.3 The case \( z = 0 \)

In this case one has \( \sigma = 0 \), i.e. the solution branches of the equation \( \sigma = \sigma_{cr} \) coincide with the branches \( y_{\pm}(\cdot; \pi) \) of the \( \pi \)-curve

\[
4(x + y - 1)^3 = 27x^2y,
\]

i.e. they coincide with the curve where \( D = 0 \).

### 8 The regions \( G_{\pm}(z) \)

In the case \( D(x, y; z) < 0 \) these regions are defined for \( z \geq 1 \) by

\[
G_{+}(z) := \{(x, y) : x > 0, y > Y_{+}(x; z)\}, \quad G_{-}(z) := \{(x, y) : x > 0, y < Y_{-}(x; z)\},
\]

and for \( 0 \leq z < 1 \) by

\[
G_{+}(z) := \{(x, y) : x > 0, y > Y_{+}(x; z)\}, \quad G_{-}(z) := \{(x, y) : x > 1 - z, y < Y_{-}(x; z)\},
\]
such that the region $D(x,y;z) < 0$ is the union $G_+(z) \cup G_-(z)$.

These regions can be also characterized by the mutual position of $R(T) = R(x,y;z)$ and the roots $R_1(x,y;z) < R_2(x,y;z)$ of the polynomial $p(\cdot)$. Actually one obtains

$$G_+(z) = \{(x,y) : R(x,y;z) > R_2(x,y;z)\} \quad G_-(z) := \{(x,y) : R(x,y;z) < R_1(x,y;z)\}.$$ 

To prove this assertion first note that, according to equations (10) and (15), in the case $z_0 = 0$ one has to show that

$$(x + y - 1)^{-1} > \frac{4}{3} y^{-1} \left(\cos \frac{\phi}{3}\right)^2, \quad (x,y) \in G_+,$$

and

$$(x + y - 1)^{-1} < \frac{4}{3} y^{-1} \left(\cos \frac{1}{3}(2\pi - \phi)\right)^2, \quad (x,y) \in G_-.$$ 

These inequalities can be verified by a direct estimation.

The corresponding property is also valid for $z > 0$. First there are special points satisfying this property, for example $(0,y)$, where $y > (1 + \sqrt{z})^2$, for $G_+(z)$, and, on the other hand, $(x,y)$, where $x > 1$ large and $0 < y < 1$, for $G_-(z)$. Further both regions are simply connected, i.e. an arbitrary point of a region can be smoothly joined within the region with such a special point. Choose a smooth path joining the special point $(x_0,y_0)$, say, of $G_+(z)$ with an arbitrary point $(x_1,y_1)$ of this region. Consider the function $R(x,y;z) - R_2(x,y;z)$ for points of the path. At $(x_0,y_0)$ it is positive. Assume that $R(x_1,y_1;z) < R_1(x_1,y_1;z)$, then the value of the function at this point is smaller than $R_1(x_1,y_1;z) - R_2(x_1,y_1;z) < 0$, i.e. then there is a point on the path such that value of the function at this point is zero which is a contradiction.

9 References

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10 Appendix

10.1 Robertson-Walker metric

The cosmological principle requires the homogeneity and isotropy for the whole time-space manifold $M$, equipped with the quadratic form
of the gravitational field, where \( \{g_{i,j}\} \) is the fundamental metric tensor of signature \(+--\). This means:

(i) \( M \) is the union of a congruence of geodesics \( x(\tau) \) with unique proper time \( \tau \) and it is the foliation of the 3-manifolds \( M_{\tau} \), consisting of all points with equal proper time \( \tau \). For each time \( \tau \) all points of \( M_{\tau} \) are equivalent.

(ii) \( M_{\tau} \) and \( dx(\tau)/d\tau \) are always orthogonal, i.e. for every fixed \( \tau \) all spacelike directions are equivalent.

For details see Wald [1, p.91]. The ansatz of Robertson-Walker to satisfy the cosmological principle requires the existence of a distinct coordinate system \( \{\tau, \xi\} \), where \( \xi := \{\chi, \Theta, \phi\} \) the spacelike coordinates, such that the quadratic form (2) is given by

\[
ds^2 = c^2d\tau^2 - R(\tau)^2(d\chi^2 + f(\chi)^2d\Theta^2 + f(\chi)^2(sin \Theta)^2d\phi^2), \tag{55}\]

where

\[
f(\chi) = \begin{cases} 
    \sin \chi, \epsilon = +1, 0 \leq \chi \leq \pi, 0 \leq \Theta \leq 2\pi, 0 \leq \phi \leq 2\pi, \\
    \sinh \chi, \epsilon = -1, 0 \leq \chi < \infty, 0 \leq \Theta \leq 2\pi, 0 \leq \phi \leq 2\pi,
\end{cases}
\]

and where \( R(\tau) > 0 \) is the scale-factor. This means

\[
g_{0,0} = c^2, \ g_{1,1} = -R^2, \ g_{2,2} = -R^2f(\chi)^2, \ g_{3,3} = -R^2f(\chi)^2(sin \Theta)^2
\]

and \( g_{i,j} = 0 \) for \( i \neq j \). The RW-ansatz satisfies the conditions (i) and (ii). Moreover, the 3-manifolds \( M_{\tau} \) are spherical (\( \epsilon = +1 \)), hyperbolic (\( \epsilon = -1 \)) or euclidean (\( \epsilon = 0 \)).

### 10.2 Einstein’s field equation, RW-boundary condition and Friedmann equation

In the following Einstein’s field equation is used in the form

\[
(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S) - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa := \frac{8\pi G}{c^2}, \tag{56}
\]

where \( R_{\mu\nu} \) is the Ricci tensor, \( S := \sum_{\mu=0}^{3} g^{\mu\mu}R_{\mu\mu} \) the Ricci scalar and \( G \) the gravitational constant. As mentioned in the introduction, the ansatz for the stress-energy tensor corresponding to the perfect fluid is given by

\[
T_{\mu\nu} := \frac{1}{c^2} \left( \rho + \frac{P}{c^2} \right) u_{\mu}u_{\nu} - g_{\mu\nu}P, \tag{57}
\]

where \( \rho \) is the energy density, \( P \) the radiation pressure and \( u_{\mu} := g_{\mu\nu}u^{\nu} \). Because of the homogeneity \( \rho \) and \( P \) depend only on the time \( \tau \). For details see Wald [1,p.96].
the RW-ansatz (3) for the metric implies that only the terms \( R_{0,0}, R_{1,1} \) and \( T_{0,0}, T_{1,1} \) are essential and the non-diagonal terms of \( R_{\mu\nu} \) and \( T_{\mu\nu} \) vanish. One obtains

\[
R_{0,0} = -3 \frac{d^2 R}{d\tau^2} \frac{1}{R}, \quad R_{1,1} = \frac{1}{c^2} \left( R \frac{d^2 R}{d\tau^2} + 2 \left( \frac{dR}{d\tau} \right)^2 \right) + 2\epsilon,
\]

\[
S = -3 \frac{d^2 R}{d\tau^2} \frac{1}{R^3} - \frac{3}{R^2} R_{1,1},
\]

\[
T_{0,0} = \rho c^2, \quad T_{1,1} = \frac{R^2}{c^2} P.
\]

Therefore, equ. (56) reduces to the two equations

(58) \[
\left( \frac{dR}{d\tau} \right)^2 + \epsilon c^2 - \frac{1}{3} \Lambda c^2 R^2 = \frac{8\pi G}{3} \rho R^2, \quad 0 - 0 \text{ component},
\]

(59) \[
-\frac{2}{c^2} R \frac{d^2 R}{d\tau^2} - \frac{1}{c^2} \left( \frac{dR}{d\tau} \right)^2 - \epsilon + \Lambda R^2 = \frac{8\pi G}{c^4} P, \quad 1 - 1 \text{ component}.
\]

Combining the equations (58) and (59) one obtains after some calculations

(60) \[
\frac{d\rho}{d\tau} = -3 \frac{dR}{d\tau} \frac{1}{R} \left( \rho + \frac{P}{c^2} \right).
\]

Note that the equations (58) and (59) are equivalent with the equations (58) and (60). There are two special cases:

(i) There is no radiation, i.e. \( P = 0 \) and \( \rho = \rho_{\text{mat}} \) (i.e. the cosmos is matter dominated),

(ii) There is only radiation, i.e. \( \rho_{\text{mat}} = 0 \) and \( P = \frac{1}{3} c^2 \rho_{\text{rad}} \) (i.e. the cosmos is radiation dominated).

If matter and radiation do not couple then (60) is satisfied separately for \( \rho_{\text{mat}} \) and \( \rho_{\text{rad}} \). This implies that \( \rho_{\text{mat}}(\tau)R(\tau)^3 \) and \( \rho_{\text{rad}}(\tau)R(\tau)^4 \) are constants, i.e. independent of the time \( \tau \). We put

(61) \[
\alpha := \frac{8\pi G}{3} \rho_{\text{mat}}(\tau)R(\tau)^3, \quad \text{matter invariant},
\]

(62) \[
\sigma := \frac{8\pi G}{3} \rho_{\text{rad}}(\tau)R(\tau)^4, \quad \text{radiation invariant}.
\]

Further one has

(63) \[
\rho = \rho_{\text{mat}} + \rho_{\text{rad}}.
\]

Then the equations (61), (62) and (63) imply (60) for the total density \( \rho \). Inserting equ. (63) into equ. (58) one obtains the Friedmann equation (1). Its solutions are also solutions of the equations (58) and (59), hence the corresponding metric tensor of the RW-metrics satisfies equation (56). If \( \tau \rightarrow R(\tau) \) is a solution of equ. (1) then also \( \tau \rightarrow R(\tau + \tau_0) \) is a solution for all \( \tau_0 \in \mathbb{R} \).