A complex Feynman-Kac formula via linear backward stochastic differential equations *

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Abstract. A complex notion of backward stochastic differential equation (BSDE) is proposed in this paper to give a probabilistic interpretation for linear first order complex partial differential equation (PDE). By the uniqueness and existence of regular solutions to complex BSDE, we deduce that there exists a unique classical solution \{U(t, x)\} to complex PDE and \{U(t, x)\} is analytic in x for each t. Thus we extend the well known real Feynman-Kac formula to a complex version. It is stressed that our complex BSDE corresponds to a linear PDE without the second order term.

Key words. Backward stochastic differential equation; complex stochastic analysis; Feynman-Kac formula; partial differential equation

AMS subject classifications. 60H10; 35A10

1 Introduction

The Feynman-Kac formula, named after Richard Feynman and Mark Kac, establishes a link between PDEs and stochastic differential equations (SDEs). It offers a method of solving certain PDEs by simulating random paths of a stochastic process. Linear real-valued Feynman-Kac formula was studied early in Kac (1949, 1951) as a formula for determining the distribution of certain Wiener functionals. Then by the theory of BSDE, Pardoux and Peng (1992) and Peng (1991, 1992) generalized them to nonlinear versions. They also derived some stochastic versions (Pardoux and Peng 1992, Peng 1992). There are many other papers in this direction, however we don’t list them all here. The present paper propose a complex notion of BSDE and deduce a complex Feynman-Kac formula for linear first order complex PDE.

Let us be more precise. We first introduce a one dimensional complex BSDE:

\[ Y_{s} = h(X_{T}^{t,x}) + \int_{s}^{T} g(r, X_{r}^{t,x}, Y_{r}^{t,x}, Z_{r}^{t,x}, T_{r}^{t, x}) \, dr - \int_{s}^{T} Z_{r}^{t,x} \, dB_{r} - \int_{s}^{T} T_{r}^{t, x} \, dB_{r}, \quad t \leq s \leq T. \tag{1.1} \]

where \((B_{s})_{s \in[0,T]}\) is a complex Brownian motion with \((\overline{B}_{s})_{s \in[0,T]}\) its conjugate counterpart, \( (X_{t}^{l,x}) \) is defined as

\[ X_{s}^{t,x} = x + \int_{t}^{s} \sigma (r) \, dB_{r} + \int_{t}^{s} \gamma (r) \, dB_{r}, \quad t \leq s \leq T. \tag{1.2} \]

We next want to find a triple of adapted processes \( \{(Y_{s}^{t,x}, Z_{s}^{t,x}, T_{s}^{t, x}); t \leq s \leq T\} \) with values in \( \mathbb{C} \times \mathbb{C} \times \mathbb{C} \) which solves uniquely \eqref{1.1}. We finally show that under some analytic conditions on

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coefficients, \( \{U(t, x) = \mathbb{V}_{t}^{t,x}; 0 \leq t \leq T, x \in \mathbb{C}\} \) is the unique solution of the following complex PDE:

\[
\begin{align*}
U_t(t, x) &= -g(t, x, U(t, x), \sigma_t U_x(t, x), \gamma_t U_x(t, x)) , \\
U(T, x) &= h(x), \quad 0 \leq t \leq T.
\end{align*}
\]

(1.3)

where \( U \) takes values in \( \mathbb{C} \) and is analytic with respect to \( x \) for each \( t \). Note that not like the real Feynman-Kac formula, BSDE (1.1) gives a probabilistic interpretation to PDEs without the second order term, which is essentially due to the complex Itô’s formula involving analytic functions. We refer to Ubøe (1987), Davis (1979) and Varopoulos (1981) for complex stochastic analysis.

This paper is organized as follows. In section 2, we make some preliminaries. Section 3 proves existence, uniqueness and regularity for the solutions of complex BSDEs. In section 4, we establish a link between a class of linear PDEs and complex BSDEs.

2 Preliminaries

Let \( (B_t) = (B^1_t, B^2_t)_{t \geq 0} \) be a standard Brownian motion in \( \mathbb{R}^2 \) on a probability space \( (\Omega, \mathcal{F}, P) \) and \( (\mathbb{B}_t)_{t \geq 0} \) be its complex counterpart, i.e. \( \mathbb{B}_t = B^1_t + iB^2_t \) where \( i = \sqrt{-1} \) is the imaginary unit. Let \( (\mathcal{F}^t_s)_{s\geq t} \) be the augmented Brownian filtration generated by the Brownian motion \( (\mathbb{B}_s)_{s\geq t} \) from time \( t \). \( T < 0 \) is a fixed time. Throughout the paper we will work within the time interval \([0, T]\). For \( x \in \mathbb{R}^2 \), \( x \) means a column vector \( \left( \begin{array} {c} x^1 \\
 x^2 \end{array} \right) \). We write \( x = x^1 + i x^2 \) as its complex counterpart in \( \mathbb{C} \), the set of complex numbers. For \( x \in \mathbb{R}^2 \) or \( x \in \mathbb{C} \), We define a common Euclid norm \( |x| = \sqrt{|x^1|^2 + |x^2|^2} \). For \( x, y \in \mathbb{R}^2 \), we denote by \( \langle x, y \rangle \) the scalar product of \( x, y \).

Let \( M, N \) be two fields of (real or complex) numbers. We denote by \( \mathcal{H}^2_x(t, T; N) \) the space of all \( \mathcal{F}^t_s \)-progressively measurable \( N \)-valued processes \( \{ (\varphi_s); t \leq s \leq T \} \) s.t. \( \mathbb{E} \left[ f_t^T |\varphi_s|^2 ds \right] < \infty \) and by \( \mathcal{S}^2_x(t, T; N) \) the set of continuous and progressively measurable \( N \)-valued processes \( \{ (\psi_s); t \leq s \leq T \} \) s.t. \( \mathbb{E} \left[ \sup_{t \leq s \leq T} |\psi_s|^2 \right] < \infty \). \( C^k(M, N), C^k_x(M, N), C^k_y(M, N) \) will denote respectively the set of functions of class \( C^k \) from \( M \) to \( N \), the set of those functions of class \( C^k \) whose partial derivatives of order less than or equal to \( k \) are bounded, and the set of those functions of class \( C^k \) which, together with all their partial derivatives of order less than or equal to \( k \), grow at most like a polynomial function of the variable \( x \) at infinity.

For a single-variable function \( f \), \( f’ \) denotes its derivative, for a multi-variable function \( f \), we sometimes denote by \( f_x \) its partial derivative w.r.t \( x \) variable.

We now announce a result about \( 2 \times 2 \) matrices.

**Definition 2.1** A real \( 2 \times 2 \) matrix \( A \) is of class \( \mathbb{C}_L \) if and only if it has the form

\[
\begin{pmatrix}
 a & -b \\
 b & a
\end{pmatrix}
\]

(2.1)

where \( a, b \in \mathbb{R} \).

By a direct calculation, we have

**Lemma 2.1** Class \( \mathbb{C}_L \) is an exchangeable semigroup, i.e.: let \( 2 \times 2 \) matrices \( A, B \) be of class \( \mathbb{C}_L \), then \( A + B, A \cdot B = B \cdot A, \lambda A \) (\( \lambda \in \mathbb{R} \)) are all of class \( \mathbb{C}_L \). If \( A^{-1} \) exists, then \( A^{-1} \) also belongs to class \( \mathbb{C}_L \).

**Proof.** The following calculation leads to the above results:
\[
\begin{bmatrix}
a & -b \\
b & a \\
\end{bmatrix} + \begin{bmatrix}
c & -d \\
d & c \\
\end{bmatrix} = \begin{bmatrix}
a + c & -b - d \\
b + d & a + c \\
\end{bmatrix}; \\
\begin{bmatrix}
a & -b \\
b & a \\
\end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix}
a & b \\
-b & a \\
\end{bmatrix}. \\
\]

3 Complex BSDE: existence, uniqueness and regularity

Let \( X^{t,x} \) satisfy (1.2) with \( \sigma(\cdot), \gamma(\cdot) \) being deterministic and square-integrable functions. We introduce the following complex BSDE:

\[
Y^{t,x}_s = h(X^{t,x}_r) + \int_s^T \left[ g(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r, T^{t,x}_r) dr - \int_s^T Z^{t,x}_r dB_r - \int_s^T T^{t,x}_r dB_r, \right] t \leq s \leq T \quad (3.1)
\]

where \( h : \mathbb{C} \to \mathbb{C}, \ g = \overline{g}(r, x) + \alpha(r) y + \beta(r) z + \theta(r) \gamma, \ \overline{g} : [0, T] \times \mathbb{C} \to \mathbb{C}. \)

We assume the following

(H1). (Polynomial growth) \( \overline{g}(s, \cdot) \in C^3_p(\mathbb{C}, \mathbb{C}) \) and \( h \in C^3_p(\mathbb{C}, \mathbb{C}). \)

(H2). (Bounded derivatives) \( \alpha, \beta, \theta \) are \( \mathbb{C} \)-valued bounded and deterministic functions

(H3). (Analyticity) \( h, \overline{g} \) are analytic w.r.t spatial variable. e.g. for \( x = x^1 + ix^2 \), \( \frac{\partial h}{\partial x^1} = \frac{\partial h}{\partial x^2} \),

\[
\frac{\partial h}{\partial x^1} = -\frac{\partial h}{\partial x^2}. 
\]

Remark 3.1 From the Maximum Modulus Principle, for a nonlinear complex function \( g(r, X, Y, Z, T) \), the derivatives of \( g \) w.r.t \( Y, Z, T \) are bounded and analytic on the whole complex plane means that \( g \) is linear in \( Y, Z, T \).

BSDE (3.1) is equivalent to the following 2-dimensional BSDE: for \( s \in [0, T] \)

\[
\begin{bmatrix}
Y^1_s \\
Y^2_s \\
\end{bmatrix} = \begin{bmatrix}
h^1(X^1_r, X^2_r) \\
h^2(Y^1_r, Y^2_r) \\
\end{bmatrix} + \int_s^T \begin{bmatrix}
g^1(r, X^1_r, X^2_r, Y^1_r, Y^2_r, Z^1_r, Z^2_r, T^1_r, T^2_r) \\
g^2(r, X^1_r, X^2_r, Y^1_r, Y^2_r, Z^1_r, Z^2_r, T^1_r, T^2_r) \\
\end{bmatrix} dr \\
- \int_s^T \begin{bmatrix}
Z^1_r - Z^2_r \\
Z^2_r - Z^1_r \\
\end{bmatrix} \left( dB^1_r - dB^2_r \right) - \int_s^T \begin{bmatrix}
\Gamma^1_r \\
\Gamma^2_r \\
\end{bmatrix} \left( dB^1_r - dB^2_r \right), \quad (3.2)
\]

or the following

\[
\begin{bmatrix}
Y^1_s \\
Y^2_s \\
\end{bmatrix} = \begin{bmatrix}
h^1(X^1_r, X^2_r) \\
h^2(Y^1_r, Y^2_r) \\
\end{bmatrix} + \int_s^T \begin{bmatrix}
f^1(r, X_r, Y_r, Z_r) \\
f^2(r, X_r, Y_r, Z_r) \\
\end{bmatrix} dr \\
- \int_s^T \begin{bmatrix}
Z^1_r - Z^2_r \\
Z^2_r - Z^1_r \\
\end{bmatrix} \left( dB^1_r - dB^2_r \right), \quad (3.3)
\]

where \( X = (X^1_r, X^2_r), \ Y_s = (Y^1_s, Y^2_s), \ Z = (Z^1, Z^2) = \begin{bmatrix}
Z^1 \\
Z^2 \\
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} (Z^1 + Z^2) \\
\frac{1}{2} (Z^1 - Z^2) \\
\end{bmatrix} \begin{bmatrix}
\Gamma^1 + \Gamma^2 \\
\Gamma^2 - \Gamma^1 \\
\end{bmatrix}, \quad (3.4)
\]

\[
f^1(r, X_r, Y_r, Z_r) = g^1(r, X^1_r, X^2_r, Y^1_r, Y^2_r, Z^1_r + Z^2_r, \frac{1}{2} (Z^1 + Z^2), \frac{1}{2} (Z^1 - Z^2)), \\
\frac{1}{2} (Z^1 - Z^2), \frac{1}{2} (Z^1 + Z^2)), \quad (3.4)
\]

\[
f^2(r, X_r, Y_r, Z_r) = g^2(r, X^1_r, X^2_r, Y^1_r, Y^2_r, Z^1_r + Z^2_r, \frac{1}{2} (Z^1 + Z^2), \frac{1}{2} (Z^1 - Z^2)), \quad (3.5)
\]
One can check that \( f = (f^1, f^2) \) is analytic in \((X, Y, Z)\), e.g. \( \frac{\partial f^1}{\partial Z^1} = \frac{\partial f^2}{\partial Z^2}, \frac{\partial f^1}{\partial Z^2} = -\frac{\partial f^2}{\partial Z^1} \), and satisfies the Lipschitz condition in \((Y, Z)\), thus there is a unique pair \((Y^s_x, Z^s_x) \in S^2_T (t, T; \mathbb{R}^2) \times \mathcal{H}^2_T(t, T; \mathbb{R}^2)\) which solves the real BSDE (3.3) (see Pardoux and Peng 1990). Therefore there is a unique triple \((Y^s_t, X^s_t, Z^s_t) \in S^2_T (t, T; \mathbb{R}^2) \times \mathcal{H}^2_T(t, T; \mathbb{R}^2) \times \mathcal{H}^2_T(t, T; \mathbb{R}^2)\) for BSDE (3.1).

**Remark 3.2** The analyticity is not used for the existence and uniqueness of the solutions for BSDE (3.1). It is just useful when we derive the Feynman-Kac formula.

**Theorem 3.1** \( \{Y^t_{s,x}; t \leq s \leq T, x \in \mathbb{C}\} \) is analytic in \( x \) and continuous in \((s, t)\).

Before proceeding to the proof, we first state a useful corollary:

**Corollary 3.1** For any \( t \in [0, T]\), the mapping \( x \to Y^t_{s,x} \) is analytic, the function and its partial derivatives of one and two being continuous in \((t, x)\).

**Proof of Theorem 4.1.** It suffice to prove the following

Step 1. \( \{Y^t_{s,x}\} \) is continuous in \((s, t)\), \( \{\frac{\partial Y^t_{s,x}}{\partial s}\} \) is continuous in \( x \).

Step 2. Cauchy-Riemann equations: \( \frac{\partial Y^t_{s,x}}{\partial s} = \frac{\partial Y^t_{s,x}}{\partial t} = -\frac{\partial Y^t_{s,x}}{\partial x} \).

Step 1 follows immediately the fact that

\[
E \left[ \sup_{t \leq s \leq T} |Y^t_{s,x} - Y^t_{s,x}'|^p \right] \leq c_p (1 + |x|^q) (|x - \hat{x}|^p + |t - \hat{t}|^p)
\]

\[
E \left[ \sup_{t \leq s \leq T} |\Delta_h Y^t_{s,x} - \Delta_h Y^t_{s,x}'|^p \right] \leq c_p (1 + |x|^q + |\hat{x}|^q + |h|^q + |\hat{h}|^q) (|x - \hat{x}|^p + |h - \hat{h}|^p + |t - \hat{t}|^p)
\]

where \( \Delta_h Y^t_{s,x} = (Y^t_{s+x+he^1} - Y^t_{s,x})/h, h \in \mathbb{R}\backslash\{0\}, \{e^1, e^2\} \) is an orthogonal basis of \( \mathbb{R}^2 \).

We now prove the Cauchy-Riemann equations. Pardoux and Peng (1992, eq. 13) says that,

\[
\nabla Y^t_{s,x} = \int^T_s f^t_x (r, X^{l,x}_r, Y^{l,x}_r, Z^{l,x}_r) \nabla X^{l,x}_r + f^t_y (r, X^{l,x}_r, Y^{l,x}_r, Z^{l,x}_r) \nabla Y^{l,x}_r + f^t_z (r, X^{l,x}_r, Y^{l,x}_r, Z^{l,x}_r) \nabla Z^{l,x}_r dr + \int^T_s \langle (\nabla Z^{l,x}_r)^*, dB_r \rangle, s \in [t, T].
\]

(3.6)

where \( \nabla Y^t_{s,x} \) is the matrix of first order partial derivatives of \( Y^t_{s,x} \) \((x \) denotes the initial condition of SDE (2.1)). \( \nabla X^{l,x}_r \) and \( \nabla Z^{l,x}_r \) are defined analogously. Let \( (M^t_s)_{t \leq s \leq T} \) be the solution of the following matrix-valued SDE:

\[
dM_s = M_s f^t_y ds + \langle M_s f^t_z, dB_s \rangle, t \leq s \leq T.
M_t = I
\]

(3.7)

which is given by

\[
M_t = \exp \left[ \int^T_t \left[ f^t_y - \frac{1}{2} f^t_z \cdot (f^t_z)^* \right] dr + \int^T_t \langle f^t_z, dB_r \rangle \right], t \leq s \leq T.
\]

(3.8)

Since \( f^t_y, f^t_z \) is of class \( \mathbb{C}_L \), \( M^t_s \) is also of class \( \mathbb{C}_L \).
Then applying a 4-dimensional version of Itô’s formula (see Øksendal 2005, Th.4.2.1) to \( M_s \nabla Y^{t,x}_s \), we deduce that

\[
d(M_s \nabla Y^{t,x}_s) = M_s d(\nabla Y^{t,x}_s) + (dM_s) \nabla Y^{t,x}_s + (dM_s)_s d(\nabla Y^{t,x}_s)
\]

\[
= -M_s f'_y \nabla Y^{t,x}_s + f'_z \nabla Z^{t,x}_s + f'_x \nabla X^{t,x}_s ds + \langle M_s \nabla Z^{t,x}_s, dB_s \rangle + M_s f'_y \nabla Y^{t,x}_s ds + \langle M_s f'_x \nabla Y^{t,x}_s, dB_s \rangle + M_s f'_z \nabla Z^{t,x}_s ds, s \in [t, T].
\]

Therefore

\[
M_s \nabla Y^{t,x}_s = M_T \nabla Y^{t,x}_T + \int_s^T M_r f'_x \nabla X^{r,x}_r dr - \int_s^T \langle M_r (f'_y \nabla Y^{r,x}_r + \nabla Z^{r,x}_r)^*, dB_r \rangle,
\]

then

\[
\nabla Y^{t,x}_s = E \left[ M_T^T h' \left( X^{t,x}_T \right) \nabla X^{t,x}_T + \int_s^T M_r^T f'_x \nabla X^{r,x}_r dr \mid F_r \right].
\]

By Lemma 2.1, It is known that \( \nabla Y^{t,x}_s \) is of class \( C_L \), thus the Cauchy-Riemann equations hold true. The proof is complete. □

**Remark 3.3** Let \( Z^{t,x}_s = (Z^1_s, Z^2_s) \) \( (\nabla Z^{t,x}_s)^* = (A, B), (f'_z)^* = \left( \begin{array}{c} A \\ B \end{array} \right) \),

\[
\nabla Z^{t,x}_s = \left( \begin{array}{c} C \\ D \end{array} \right), (\nabla Z^{t,x}_s)^* = (C, D), \text{ where } A = \left( \begin{array}{cc} \frac{\partial f^1}{\partial y^1} & \frac{\partial f^1}{\partial y^2} \\ \frac{\partial f^2}{\partial y^1} & \frac{\partial f^2}{\partial y^2} \end{array} \right), B = \left( \begin{array}{cc} \frac{\partial f^1}{\partial z^1} & \frac{\partial f^1}{\partial z^2} \\ \frac{\partial f^2}{\partial z^1} & \frac{\partial f^2}{\partial z^2} \end{array} \right).
\]

The Jacobian matrix is of class \( C_L \).

Since \( f \) is analytic in \( Z = Z^1 + i \cdot Z^2 \), one can check that \( A, B \) are of class \( C_L \).

For BSDE (3.3), Pardoux and Peng (1992) proved that: for any \( 0 \leq t \leq s \leq T, x \in \mathbb{R}^n \), \( (Z^{t,x}_s)^* = \nabla Y^{t,x}_s (\nabla X^{t,x}_s)^{-1} \sigma^*_s \) and particularly \( (Z^{t,x}_s)^* = \nabla Y^{t,x}_s \sigma^*_t \). Note that \( \nabla X^{t,x}_s = I \) in this paper, for BSDE (3.1), we have the following results.

**Proposition 3.1** For \( 0 \leq t \leq s \leq T, x \in \mathbb{C} \),

\[
(Z^{t,x}_s)^* = \frac{dZ^{t,x}_s}{dx} \sigma_s,
\]

\[
(T^{t,x}_s)^* = \frac{dZ^{t,x}_s}{dx} \gamma_s,
\]

**Proof.** By Pardoux and Peng (1992, Lemma 2.5), we get that

\[
(Z^{t,x}_s)^* = \left( \begin{array}{c} Z^1 + \Gamma^1 \\ \Gamma^2 \end{array} \right) \frac{\partial (\sigma^1 + \gamma^1)}{\partial x} + \left( \begin{array}{c} \partial \gamma^1 \\ \partial \gamma^2 \end{array} \right) \left( \begin{array}{cc} \gamma^1 + \sigma^1 & \gamma^2 - \sigma^1 \\ \sigma^1 & \sigma^1 - \gamma^1 \end{array} \right).
\]

From the above equation, we have that

\[
\begin{align*}
Z^1 + \Gamma^1 &= \frac{\partial Y^1}{\partial x^1} (\sigma^1 + \gamma^1) + \frac{\partial Y^1}{\partial x^2} (\sigma^2 + \gamma^2), \\
Z^1 - \Gamma^1 &= \frac{\partial Y^1}{\partial x^1} (\sigma^1 - \gamma^1) + \frac{\partial Y^1}{\partial x^2} (\sigma^2 - \gamma^2),
\end{align*}
\]

(3.13)
and

\[
\begin{align*}
\Gamma^2 - Z^2 &= \frac{\partial Y^2}{\partial x^1}(\gamma^1 - \sigma^1) + \frac{\partial Y^2}{\partial x^2}(\gamma^2 - \sigma^2), \\
\Gamma^2 + Z^2 &= \frac{\partial Y^2}{\partial x^1}(\gamma^1 + \sigma^1) + \frac{\partial Y^2}{\partial x^2}(\gamma^2 + \sigma^2).
\end{align*}
\]  
(3.14)

Therefore,

\[
\begin{align*}
Z^1 &= \frac{\partial Y^1}{\partial x^1} \sigma^1 + \frac{\partial Y^1}{\partial x^2} \sigma^2, \\
\Gamma^1 &= \frac{\partial Y^1}{\partial x^1} \gamma^1 + \frac{\partial Y^1}{\partial x^2} \gamma^2,
\end{align*}
\]  
(3.15)

and

\[
\begin{align*}
Z^2 &= \frac{\partial Y^2}{\partial x^1} \sigma^1 + \frac{\partial Y^2}{\partial x^2} \sigma^2, \\
\Gamma^2 &= \frac{\partial Y^2}{\partial x^1} \gamma^1 + \frac{\partial Y^2}{\partial x^2} \gamma^2.
\end{align*}
\]  
(3.16)

Thus

\[
Z^1 + i \cdot Z^2 = \left( \frac{\partial Y^1}{\partial x^1} \sigma^1 + \frac{\partial Y^1}{\partial x^2} \sigma^2 \right) + i \left( \frac{\partial Y^2}{\partial x^1} \sigma^1 + \frac{\partial Y^2}{\partial x^2} \sigma^2 \right) = \left( \frac{\partial Y^1}{\partial x^1} + i \frac{\partial Y^2}{\partial x^1} \right) (\sigma^1 + i \sigma^2),
\]

\[
\Gamma^1 + i \cdot \Gamma^2 = \left( \frac{\partial Y^1}{\partial x^1} \gamma^1 + \frac{\partial Y^1}{\partial x^2} \gamma^2 \right) + i \left( \frac{\partial Y^2}{\partial x^1} \gamma^1 + \frac{\partial Y^2}{\partial x^2} \gamma^2 \right) = \left( \frac{\partial Y^1}{\partial x^1} + i \frac{\partial Y^2}{\partial x^1} \right) (\gamma^1 + i \gamma^2),
\]

that is

\[
\begin{align*}
(Z^1)^{t,x}_s &= \frac{dY^1_t}{dx} \sigma_s, \\
(\Gamma^1)^{t,x}_s &= \frac{dY^1_t}{dx} \gamma_s.
\end{align*}
\]

\[
\square
\]

By Proposition 3.1 we know that, if \( \sigma = 0 \) (resp. \( \gamma = 0 \)), then \( Z = 0 \) (resp. \( T = 0 \)). Since there is a unique solution \((Y, Z, T)\) for BSDE (3.1), we have the following results:

Corollary 3.2 Under conditions \((H1) \sim (H3)\), there is a unique solution in \(S^2(t, T; \mathbb{R}^2) \times \mathcal{H}^2(t, T; \mathbb{R}^2 \times \mathbb{R}^2)\) respectively for the following two real forward-backward SDEs:

\[
\begin{align*}
\begin{cases}
(X^1_s, X^2_s) = (x^1, x^2) + \int_t^s \begin{pmatrix} \sigma^1_r & \sigma^2_r \\ \sigma^1_r & \sigma^2_r \end{pmatrix} \begin{pmatrix} dB^1_r \\ dB^2_r \end{pmatrix}, & s \in [t, T], \\
(Y^1_s, Y^2_s) = (h^1(X^1_r, X^2_r), h^2(X^1_r, X^2_r)) + \int_t^s \begin{pmatrix} g^1(r, X^1_r, X^2_r, Y^1_r, Y^2_r, Z^1_r, Z^2_r, \Gamma^1_r, \Gamma^2_r) \\ g^2(r, X^1_r, X^2_r, Y^1_r, Y^2_r, Z^1_r, Z^2_r, \Gamma^1_r, \Gamma^2_r) \end{pmatrix} dr & s \in [t, T], \\
- \int_t^s \begin{pmatrix} Z^1_r \\ Z^2_r \end{pmatrix} \begin{pmatrix} dB^1_r \\ dB^2_r \end{pmatrix}, & s \in [t, T],
\end{cases}
\end{align*}
\]  
(3.17)
and

\[
\begin{cases}
(X_1^1) = \left( \frac{x^1}{x^2} \right) + \int_t^s \left( \begin{array}{c}
\gamma_r^1 \\
\gamma_r^2 \\
\end{array} \right) \begin{array}{c}
\frac{dB^1_r}{dB^2_r} \\
\end{array} , \\
(Y_1^1) = \left( \begin{array}{c}
h^1 (X_1^1, X_2^1) \\
+h^2 (X_1^1, X_2^1) \\
\end{array} \right) + \int_s^t \left( \begin{array}{c}
g^1 (r, X_1^1, X_2^1, Y_1^1, Y_2^1, Z_1^1, Z_2^1, \Gamma^1_r, \Gamma^2_r) \\
g^2 (r, X_1^1, X_2^1, Y_1^1, Y_2^1, Z_1^1, Z_2^1, \Gamma^1_r, \Gamma^2_r) \\
\end{array} \right) dr \\
- \int_s^T \left( \begin{array}{c}
\Gamma^1_r \\
\Gamma^2_r \\
\end{array} \right) \begin{array}{c}
\frac{dB^1_r}{dB^2_r} \\
\end{array} , \\
\end{cases}
\]

\(s \in [t, T],\)

**Remark 3.4** The above two BSDEs are real-valued BSDEs with \(Z\)-constraints. Usually there are no solutions for constrained BSDEs.

### 4 Complex BSDE and associated PDE

Consider the following complex PDE:

\[
\begin{align*}
\frac{\partial u}{\partial t} (t, x) &= -g(t, x, U(t, x), \sigma_t U_x(t, x), \gamma_t U_x(t, x)), \\
U(T, x) &= h(x), \quad 0 \leq t \leq T.
\end{align*}
\]

(4.1)

where \(U : \mathbb{R}^+ \times \mathbb{C} \to \mathbb{C}, \sigma = \sigma^1 + i\sigma^2, \gamma = \gamma^1 + i\gamma^2, \sigma^1 \gamma^1 = \sigma^2 \gamma^2.\)

**Theorem 4.1** Let \(h, g\) satisfy (H1) \(\sim\) (H3). If for each \((t, x), \{U(t, x); t \leq s \leq T, x \in \mathbb{C}\}\) is analytic with respect to \(x\) and continuous in \(t\), and satisfies PDE (4.1), then

\[
(U(t, x), \sigma_t U_x(t, x), \gamma_t U_x(t, x))_{0 \leq t \leq T} = \left( \begin{array}{c}
Y_{t}^{t, x} \\
Z_{t}^{t, x} \\
T_{t}^{t, x}
\end{array} \right),
\]

solves BSDE (3.1). Furthermore,

\[
U(t, x) = Y_{t}^{t, x} = E \left[ h \left( X_{t}^{t, x} \right) + \int_t^T g \left( r, X_{r}^{t, x}, Y_{r}^{t, x}, Z_{r}^{t, x}, T_{r}^{t, x} \right) dr \right].
\]

Before proving the above theorem, we need an Itô's lemma.

**Lemma 4.1** Let \(X_t \in \mathcal{S}_t^{2} (0, T; \mathbb{C}), b_t, \sigma_t, \gamma_t \times \mathcal{H}_t^{2} (0, T; \mathbb{C}),\) such that \(\sigma^1 \gamma^1 = \sigma^2 \gamma^2\) for almost all \(t\) and

\[
dX_t = b_t dt + \sigma_t dB_t + \gamma_t dB^*_t, \quad t \geq 0.
\]

If \(F(t, x) = u(t, x^1, x^2) + iv(t, x^1, x^2)\) is an analytic function w.r.t the complex variable \(x\) and continuous in \(t\), i.e. for any \(t, \forall x \in \mathbb{C}, \) \(F\) satisfies the Cauchy-Riemann equations:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \bar{x}}, \quad \frac{\partial u}{\partial \bar{x}} = -\frac{\partial v}{\partial x},
\]

then

\[
dF(t, X_t) = \frac{\partial F}{\partial t} (t, X_t) dt + \frac{\partial F}{\partial X} (t, X_t) dX_t
\]

\[
= \frac{\partial F}{\partial t} (t, X_t) dt + \frac{\partial F}{\partial X} (t, X_t) b_t dt + \frac{\partial F}{\partial X} (t, X_t) \sigma_t dB_t + \frac{\partial F}{\partial X} (t, X_t) \gamma_t dB^*_t, \quad t \geq 0.
\]

(4.3)

where \(\frac{\partial F}{\partial X} (t, x)\) is the complex partial derivative of \(F\) w.r.t \(x\).
Proof.
\[d\mathbb{F}(t, \mathcal{X}_t) = du(t, X^1_t, X^2_t) + iv(t, X^1_t, X^2_t)\]
\[= \frac{\partial u}{\partial x^1} dx^1_t + \frac{\partial u}{\partial x^2} dx^2_t + \frac{\partial u}{\partial t} dt + \frac{1}{2} \left( \frac{\partial^2 u}{\partial (x^1)^2} d < X^1 >_t + \frac{\partial^2 u}{\partial (x^2)^2} d < X^2 >_t \right)\]
\[+ i \left[ \frac{\partial v}{\partial x^1} dx^1_t + \frac{\partial v}{\partial x^2} dx^2_t + \frac{\partial v}{\partial t} dt + \frac{1}{2} \left( \frac{\partial^2 v}{\partial (x^1)^2} d < X^1 >_t + \frac{\partial^2 v}{\partial (x^2)^2} d < X^2 >_t \right) \right]\]
\[= \left( \frac{\partial u}{\partial x^1} + i \frac{\partial v}{\partial x^1} \right) (dx^1_t + idX^2_t) + \left( \frac{\partial u}{\partial t} + i \frac{\partial v}{\partial t} \right) dt\]
\[= \frac{\partial \mathbb{F}}{\partial t}(t, \mathcal{X}_t) dt + \frac{\partial \mathbb{F}}{\partial x}(t, \mathcal{X}_t) d\mathcal{X}_t\]

where we have used the conjugate harmonicity of function \(u, v\) and the condition \(\sigma^1\gamma^1 = \sigma^2\gamma^2\) \(\Box\)

**Proof of Theorem 4.1.** It suffices to show that
\[
\{U(s, X^t_s, \sigma_s U_x(s, X^t_s), \gamma_s U_x(s, X^t_s); t \leq s \leq T)\}
\]
solves BSDE (3.1). Applying the complex Itô formula to \(U(s, X^t_s)\) between \(s = t\) and \(s = T\), we get that
\[
Y^t_s = h \left( X^t_s \right) + \int_t^T \left[ g \left( r, X^t_r, U(r, X^t_r), \sigma_r U_x(r, X^t_r), \gamma_r U_x(r, X^t_r) \right) \right] dr
\]
\[- \int_t^T \sigma_r U_x(r, X^t_r) d\mathbb{B}_r - \int_t^T \gamma_r U_x(r, X^t_r) d\mathbb{B}_r, \quad t \leq s \leq T.
\]

Thus \((Y^t_s, Z^t_s, \Upsilon^t_s) = \{U(s, X^t_s), \sigma_s U_x(s, X^t_s), \gamma_s U_x(s, X^t_s)\}\) solves BSDE (3.1). \(\Box\)

We now show the converse of Theorem 4.1.

**Theorem 4.2** Let \(h, g\) satisfy (H1) \(\sim\) (H3). Let \(\{(Y^t_s); t \leq s \leq T\}\) be the solution of BSDE (3.1). Then \((U(t, x))_{0 \leq t \leq T} = (Y^t_t)_{0 \leq t \leq T}\) is the unique classical solution of backward PDE (4.1) and \(U(t, x)\) is analytic in \(x\) for each \(t\).

**Proof.** Uniqueness follows from Theorem 4.1. We now prove that \((Y^t_t)\) is a solution to PDE (4.1). Let \(\delta > 0\) s.t. \(t + \delta \leq T\). Clearly \(Y^{t+\delta}_t = Y^{t+\delta}_t\). Hence by the complex Itô formula and the analyticity of \(U(t, x)\) in \(x\) and BSDE (3.1), we have
\[
U(t + \delta, x) - U(t, x) = \left[ U(t + \delta, x) - U(t, X^t_{t+\delta}) \right] + \left[ U(t + \delta, X^t_{t+\delta}) - U(t, x) \right]
\]
\[= - \int_t^{t+\delta} g \left( r, X^t_r, Y^t_r, Z^t_r, \Upsilon^t_r, \sigma^t_r U_x(r, X^t_r) \right) dr
\]
\[\quad - \int_t^{t+\delta} \sigma_r U_x(t + \delta, X^t_r) d\mathbb{B}_r
\]
\[\quad - \int_t^{t+\delta} \gamma_r U_x(t + \delta, X^t_r) d\mathbb{B}_r
\]
\[\quad + \int_t^{t+\delta} Z^t_r d\mathbb{B}_r + \int_t^{t+\delta} \Upsilon^t_r d\bar{\mathbb{B}}_r,
\]
Let $t = t_0 \leq t_1 \leq \ldots \leq t_n = T$, we get

$$h(x) - U(t, x) = -\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} g(r, x, X_r^{t,x}, \gamma_r^{t,x}, Z_r^{t,x}, T_r^{t,x}) \, dr$$

$$+ \int_{t_i}^{t_{i+1}} \left[ Z_r^{t,x} - \sigma_r U_x(r, x) \right] \, dB_r$$

$$+ \int_{t_i}^{t_{i+1}} \left[ T_r^{t,x} - \gamma_r U_x(r, x) \right] \, d\bar{B}_r. \tag{4.4}$$

Let the mesh size $\sup_{0 \leq i \leq n-1} (t_{i+1} - t_i) \to 0$, we obtain the limit

$$U(t, x) = h(x) + \int_s^T g(r, x, U(r, x), \sigma_r U_x(r, x), \gamma_r U_x(r, x)) \, dr, \ t \leq s \leq T. \quad \Box$$

**Remark 4.1** Viscosity solution for PDE (4.1) is not involved in the present paper because, the analyticity of parameters of PDE (4.1) leads to existence of the first and the second order derivatives.

The well-known Cauchy–Kovalevski theorem states a local existence and uniqueness of solution for partial differential equations whose coefficients are analytic functions, associated with Cauchy initial value problems. A special case was proven by Cauchy in 1842, and the full result by Kowalevski (1875). Theorem 4.2 extends the first order Cauchy–Kovalevski theorem to the case of global solutions.

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