New Example of Infinite Family of Quiver Gauge Theories

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Abstract

We construct a new infinite family of quiver gauge theories which blow down to the $X^{p,q}$ quiver gauge theories found by Hanany, Kazakopoulos and Wecht. This family includes a quiver gauge theory for the third del Pezzo surface. We show, using Z-minimization, that these theories generically have irrational R-charges. The AdS/CFT correspondence implies that the dual geometries are irregular toric Sasaki-Einstein manifolds, although we do not know the explicit metrics.

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1 Introduction

D3-branes at the tip of Calabi-Yau cones have been extensively studied. The corresponding IIB supergravity solution in the near horizon limit takes the form $AdS_5 \times X_5$, where $X_5$ is a five dimensional Sasaki-Einstein manifold. In dimension five, simply-connected regular Sasaki-Einstein manifolds are classified [1]. Indeed we have $S^5$, $T^{1,1}$ and the total space $S_k$ of the circle bundles $S_k \to dP_k$ ($3 \leq k \leq 8$) where $dP_k$ is a del Pezzo surface with a Kähler Einstein metric. It is known that $S_k$ is diffeomorphic to the $k$-fold connected sum $\#_k(S^2 \times S^3)$. Recently, an infinite family of irregular toric Sasaki-Einstein manifolds $Y_{p,q}$ with topology $S^2 \times S^3$ was constructed [2, 3]. Especially, $Y^{2,1}$ is the horizon of the complex cone over the first del Pezzo surface $dP_1$. Also, the existence of irregular toric Sasaki-Einstein manifolds $X_{p,q} \simeq (S^2 \times S^3)\#(S^2 \times S^3)$ which blow down to $Y_{p,q}^-$ by Higgsing was conjectured in [4]. These are considered to be extension of the Sasaki-Einstein manifold $X^{2,1}$ over the second del Pezzo surface $dP_2$ [5]. For other new Sasaki-Einstein manifolds see [6, 7, 8] and also [9].

The AdS/CFT correspondence states that IIB string theory on $AdS_5 \times X_5$ is dual to a four-dimensional $\mathcal{N} = 1$ quiver gauge theory. Given a toric Sasaki-Einstein manifold, one can determine the corresponding quiver gauge theory by using the brane tiling (dimer) construction [10, 11, 12, 13, 14, 15, 16, 17, 18]. In Figures 1, 2 and 3, we present some data of quiver gauge theories for the del Pezzo surfaces $dP_k$ ($k = 1, 2, 3$), corresponding to $Y^{2,1}, X^{2,1}$ and $S_1$ [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. These quiver theories have been extended to the general $Y_{p,q}^{31, 32}$ and $X_{p,q}$ theories [4, 33, 34].

In this paper we construct an infinite family of $\mathcal{N} = 1$ quiver gauge theories which blow down to $X_{p,q}$. This construction generalizes the $dP_3$ quiver gauge theory, corresponding to $S_1 \equiv Z^{2,1}$. The dual geometries are irregular toric Sasaki-Einstein manifolds which are diffeomorphic to $\#_3(S^2 \times S^3)$. We denote as $Z_{p,q}$ assuming they exist. It should be mentioned that the existence of $Z_{p,q}$ was suggested in a recent paper [35].

In section 2 we describe the brane tiling construction of the $Z_{p,q}$ theories. The explicit examples are given in Figures 5-14. In section 3 we determine the $R$-charges for these theories by using $Z$-minimaization.

2 Un-Higgsing $X_{p,q}$

The brane tiling for $X_{p,q}$ was given in [33]. We use the convention of [33] for the brane tiling. Some data for the $X_{p,q}$ quiver gauge theory are summarized in Figure 4. We take
\( n = 2q - 1, \quad m = p - q, \quad j = 1, 2, \ldots, q - 1 \) and \( k = n + 1, n + 2, \ldots, n + m - 1 \). Then, the brane tiling contains \( 2q - 1 \) hexagons and \( p - q + 1 \) cut hexagons. The \( R_i + R_{i+1} + \cdots \) on edges represent \( R \)-charges: at every vertex the sum of \( R \)-charges is 2 and for every face the sum of \( R \)-charges is equal to the number of edges minus 2.

We can un-Higgs \( X^{p,q} \) to \( Z^{p,q} \) by cutting the \( i \)-th hexagon horizontally \( (1 \leq i \leq n) \). This procedure introduces one new edge with weight \(-w^{-1}\) which connects the \((i + 2)\)-th white node and the \( i \)-th black node. If we replace 0 at \((i + 2, i)\) element of the Kasteleyn matrix for \( X^{p,q} \) with \(-w^{-1}\), we obtain the Kasteleyn matrix for \( Z^{p,q} \). The determinant is modified to contain the term \( \pm w^{-1}z \), which corresponds to an additional node (see next section for details):

\[
V_6 = (1, w_6), \quad w_6 = (2, 1).
\] 

The perfect matchings which correspond to the node \((2, 1)\) are given by

\[
P^{(i)}_6 = \begin{pmatrix}
1 & \cdots & i & i+1 & i+2 & i+3 & \cdots & n+m+1 \\
1 & 0 \\
1 & 0 \\
\vdots & \ddots & \ddots \\
1 & 0 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
1 & 0 \\
-1 \\
0 \\
0 \\
\vdots \\
1 \\
0
\end{pmatrix}.
\] 

The matrices \( P^{(i)}_6 \) \((i = 1, 2, \ldots, n)\) have non-zero elements \( K_{ab} \) at the edges \( a \rightarrow b \) connected by bold lines (see Figures 7, 10, 12 and 14), and the values \( K_{ab} \) are decided according to a rule of the Kasteleyn matrix. The charge \( R_6 \) is added to the corresponding edges.

The contents of the minimal toric phase are summarized as follows:

|        | \( N_g \) | \( N_f \) | \( N_W \) |
|--------|----------|----------|----------|
| \( Y^{p,q} \) | \( 2p \)  | \( 4p + 2q \) | \( 2p + 2q \) |
| \( X^{p,q} \) | \( 2p + 1 \) | \( 4p + 2q + 1 \) | \( 2p + 2q \) |
| \( Z^{p,q} \) | \( 2p + 2 \) | \( 4p + 2q + 2 \) | \( 2p + 2q \) |

Here \( N_g \) represents the number of gauge groups, \( N_f \) the number of bifundamental
fields, $N_W$ the number of the interaction terms in the superpotential. These correspond to the numbers of faces, edges and nodes on the brane tiling, respectively.

In the following we describe the brane tiling construction of the $Z^{3,1}$ and $Z^{3,2}$ theories explicitly.

2.1 $Z^{3,1}$

The brane tiling for $X^{3,1}$ is given by Figure 5. In this case, there is only one hexagon that we can put a cut in. We have drawn the resulting brane tiling for $Z^{3,1}$ in Figure 6 together with the corresponding quiver diagram and toric diagram. The perfect matching in Figure 7 is given by the matrix

$$P_6^{(1)}(Z^{3,1}) = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 \\
3 & -w^{-1} & 0 & 0 \\
4 & 0 & 0 & 1
\end{pmatrix}. \tag{2.3}$$

2.2 $Z^{3,2}$

The brane tiling for $X^{3,2}$ is given by Figure 8. In this case we have three types of cuttings corresponding to three hexagons. These lead to different brane tilings and quiver diagrams for $Z^{3,2}$, although their toric diagrams are equivalent, as shown in Figures 8-14.

The superpotentials contain the terms with the following degree:

|     | cubic | quartic | quintic | sextic |
|-----|-------|---------|---------|--------|
| case 1 | 5     | 4       | 1       | 0      |
| case 2 | 6     | 2       | 2       | 0      |
| case 3 | 6     | 3       | 0       | 1      |

These provide an example of toric duality and are connected by Seiberg duality [19, 24, 25, 27].
3 Z-minimization

The toric diagram for $Z^{p,q}$ can be obtained by adding one vertex to the toric diagram for $X^{p,q}$, as shown in Figures 15 and 16. The six vectors $V_i = (1, w_i)$ for $Z^{p,q}$ are written as

$$
\begin{align*}
    w_1 &= (1, p), & w_2 &= (0, p-q+1), & w_3 &= (0, p-q), \\
    w_4 &= (1, 0), & w_5 &= (2, 0), & w_6 &= (2, 1),
\end{align*}
$$

(3.1)

where $p$ and $q$ are integers with $0 < q < p$. Let us determine the Reeb vector $b = (3, x, y)$ for $Z^{p,q}$ using the method of [36], Z-minimization. Then, we obtain $x = 3$ and the remaining component $y$ is given by a root of the polynomial

$$
P(y) = 2y^3 - 9pqy^2 - 9p^2(2p - 3q)y + 27p^3(p - q).
$$

(3.2)

This polynomial has three real roots $y = y_i$ ($i = 1, 2, 3$). By $P(0) = 27p^3(p - q) > 0$ and $P(3p) = -27p^3(p + q - 2) < 0$ they satisfy $y_1 < 0 < y_2 < 3p < y_3$. We find that the correct root is a middle one $y_2$, which is explicitly given by

$$
y_2 = \frac{3pq}{2} - 3p\sqrt{q^2 - 2q + (4/3)p\cos \theta + \frac{\pi}{3}},
$$

(3.3)

where the angle $\theta$ ($0 \leq \theta \leq \pi/2$) is calculated as

$$
\tan \theta = \frac{\sqrt{D}}{(q - 1)(q^2 - 2q + 2p)}
$$

(3.4)

with

$$
D = \frac{64}{27}p^3 + \frac{4}{3}p^2(q + 1)(q - 3) - 4pq(q - 2) - q^2(q - 2)^2.
$$

(3.5)

The R-charges can be computed from volumes of certain calibrated submanifolds $\Sigma_i$ in $Z^{p,q}$:

$$
R_i = \frac{\pi \text{vol}(\Sigma_i)}{3\text{vol}(Z^{p,q})},
$$

(3.6)

where

$$
\text{vol}(\Sigma_i) = \frac{2(V_{i-1}, V_i, V_{i+1})\pi^2}{(b, V_{i-1}, V_i)(b, V_i, V_{i+1})}, \quad \text{vol}(Z^{p,q}) = \frac{\pi}{6} \sum_{i=1}^{6} \text{vol}(\Sigma_i).
$$

(3.7)

Explicitly we have the following volumes

$$
\text{vol}(\Sigma_1) = \frac{2(p + q - 2)\pi^2}{(3p - y_2)^2}, \quad \text{vol}(\Sigma_2) = \text{vol}(\Sigma_6) = \frac{2\pi^2}{3(3p - y_2)},
$$

$$
\text{vol}(\Sigma_3) = \text{vol}(\Sigma_5) = \frac{2\pi^2}{3y_2}, \quad \text{vol}(\Sigma_4) = \frac{2(p - q)\pi^2}{y_2^2}
$$

(3.8)
and
\[ \text{vol}(Z^{p,q}) = \frac{9p^3 - 9p^2q + 6pqy_2 - 2y_2^2}{3y_2^2(3p - y_2)^2} \pi^3. \] (3.9)

It should be noticed that
\[ 1 > \frac{\text{vol}(Z^{p,1})}{\pi^3} > \frac{\text{vol}(Z^{p,2})}{\pi^3} > \cdots. \] (3.10)

This implies the inequality of the central charge
\[ a = \frac{\pi^3 N^2}{4 \text{vol}(Z^{p,q})} > \frac{N^2}{4}, \] (3.11)
where \( N \) is the number of D3-branes and \( N^2/4 \) the central charge of \( \mathcal{N} = 4 \) Yang-Mills theory.

The root \( y_2 \) is generically an irrational number and hence \( Z^{p,q} \) are irregular Sasaki-Einstein manifolds. As a special case we have a rational number \( y_2 = 3p/2 \) for \( q = 1 \). For the toric diagram with six external lines, the third homology is given by \( H_3(Z^{p,q}) = \mathbb{Z} \). It turns out that from the work of Smale \( ^{37} \) \( Z^{p,q} \) must be diffeomorphic to \( \sharp 3(S^2 \times S^3) \). \(^1\)

The cone of a Sasaki-Einstein manifold is Ricci-flat Kähler, i.e., Calabi-Yau. As described in \( ^{36} \), the cone metric in the symplectic coordinates is given by a symplectic potential. We found that for \( Y^{p,q} \) the symplectic potential takes very simple form \( ^{39} \). At present the Sasaki-Einstein metrics on \( X^{p,q} \) and \( Z^{p,q} \) are not constructed or even proved to exist for generic integers \( p \) and \( q \). It is expected that the symplectic approach can be useful to study these metrics.

Before concluding this paper, we comment on the the generating function \( f \) for the single-trace gauge invariant operators \( ^{40} \). In the notation of \( ^{40} \), the toric diagram for \( Z^{p,q} \) is obtained by adding the point \( (1,1,1) \) to the one for \( X^{p,q} \). One more triangle with vertices \( (1,0,1) \), \( (1,1,1) \) and \( (0,p,1) \) gives a new term to the generating function (see eq.(3.22) and the next equation of \( ^{40} \)),
\[ f(x, y, z; Z^{p,q}) = f(x, y, z; X^{p,q}) + \frac{1}{\left(1 - \frac{x^p y}{z^p}\right) \left(1 - \frac{z^p}{x}\right) \left(1 - \frac{y}{x^{p-1}}\right)}. \] (3.12)

Taking the limit \( ^{41} \)
\[ V(b) = \lim_{t \to 0} t^3 f(e^{-bt}, e^{-bt}, e^{-bt}; Z^{p,q}) \] (3.13)

\(^*\)We calculate as \( \text{vol}(Z^{p,1})/\pi^3 = 8(2p-1)/(27p^2) < 1 \), and the successive inequalities can be evaluated by perturbation.

\(^\dagger\)In \( ^{38} \) it has been shown that \( \sharp 3(S^2 \times S^3) \) admits an infinite family of non-regular Sasaki-Einstein structures. We do not know whether \( Z^{p,q} \) are equivalent to those.
we have
\[ V(b) = \frac{-(p-1)b_1 + pb_3}{b_1 b_2 (b_1 - b_3)((p-1)b_1 + b_2 - pb_3)} \]
\[ - \frac{(p-1)b_1 + pb_3}{b_1 (b_1 + b_3)(((p-q)b_1 + b_2)((q-1)b_1 - b_2 + pb_3)}, \]
(3.14)
which reproduces the volume \( \text{vol}(Z^{p,q})/\pi^3 \) for the Reeb vector \((b_1, b_2, b_3) = (0, y_2, 3)\).

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References

[1] H. Baum, T. Friedrich, R. Grunewald and I. Kath, “Twistors and Killing Spinors on Riemannian Manifolds”, Teubner-Texte für Mathematik, vol. 124, Teubner, Stuttgart, Leipzig, 1991.

[2] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Sasaki-Einstein metrics on \( S^2 \times S^3 \)”, Adv. Theor. Math. Phys. 8 (2004)711-734 [arXiv:hep-th/0403002].

[3] D. Martelli and J. Sparks, “Toric Sasaki-Einstein metrics on \( S^2 \times S^3 \)”, Phys. Lett. B 621(2005)208-212 [arXiv:hep-th/0505027].

[4] A. Hanany, P. Kazakopoulos and B. Wecht, “A New Infinite Class of Quiver Gauge Theories”, JHEP 0508(2005)054 [arXiv:hep-th/0503177].

[5] A. Futaki, H. Ono and G. Wang, “Transverse Kähler Geometry of Sasaki Manifolds and Toric Sasaki-Einstein Manifolds”, arXiv:math.DG/0607586.

[6] M. Cvetic, H. Lü, Don N. Page and C.N. Pope, “New Einstein-Sasaki Spaces in Five and Higher Dimensions”, Phys. Rev. Lett. 95 (2005)071101 [arXiv:hep-th/0504225].

[7] M. Cvetic, H. Lü, Don N. Page and C.N. Pope, “New Einstein-Sasaki and Einstein Spaces from Kerr-de Sitter”, arXiv:hep-th/0505223.
[8] W. Chen, H. Lü, and C.N. Pope, “General Kerr-NUT-AdS Metrics in All Dimensions”, Class. Quant. Grav. 23(2006)5323-5340 [arXiv:hep-th/0604125].

[9] C.P. Boyer and K. Galicki, “Sasakian Geometry, Hypersurface Singularities and Einstein Metrics”, Rendiconti del Circolo Matematico di Palermo (2), Suppl. No. 75 (2005)57-87 [arXiv:math.DG/0406627].

[10] A. Hanany, and K.D. Kennaway, “Dimer models and toric diagrams”, [arXiv:hep-th/0503149].

[11] S. Franco, A. Hanany, K.D. Kennaway, D. Vegh and B. Wecht, “Brane Dimers and Quiver Gauge Theories”, JHEP 0601(2006)096 [arXiv:hep-th/0504110].

[12] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, “Gauge Theories from Toric Geometry and Brane Tilings”, JHEP 0601(2006)128 [arXiv:hep-th/0505211].

[13] A. Hanany and D. Vegh, “Quiver, Tilings Branes and Rhombi”, [arXiv:hep-th/0511063].

[14] B. Feng, Y-H. He, K.D. Kennaway and C. Vafa, “Dimer Models from Mirror Symmetry and Quivering Amoebae”, [arXiv:hep-th/0511287].

[15] A. Hanany, C.P. Herzog and D. Vegh, “Brane Tilings and Exceptional Collections”, JHEP 0607(2006)001 [arXiv:hep-th/0602041].

[16] S. Franco and D. Vegh, “Moduli spaces of gauge theories from dimer models: Proof of the correspondence”, [arXiv:hep-th/0601063].

[17] Y. Imamura, “Anomaly Cancellations in Brane Tilings”, [arXiv:hep-th/0605097].

[18] Y. Imamura, “Global symmetries and ‘t Hooft anomalies in brane tilings,” [arXiv:hep-th/0609163].

[19] B. Feng, A. Hanany and Y-H. He, “D-brane gauge theories from toric singularities and toric duality”, Nucl.Phys. B 595, (2001)165-200 [arXiv:hep-th/0003085].

[20] S. Franco, A. Hanany, F. Saad and A.M. Uranga, “Fractional Branes and Dynamical Supersymmetry Breaking”, JHEP 0601(2006)011 [arXiv:hep-th/0505040].
[21] S. Franco, A. Hanany and A.M. Uranga, “Multi-Flux Warped Throats and Cascading Gauge Theories”, JHEP 0509(2005)028 [arXiv:hep-th/0502113].

[22] C. Beasley, B.R. Green, C.I. Lazaroiu and M.R. Plesser, “D3-branes on partial resolutions of abelian quotient singularities of Calabi-Yau threefolds”, Nucl. Phys. B 566(2000)599-640 [arXiv:hep-th/9907186].

[23] M. Bertolini, F. Bigazzi and A.L. Cotrone, “New checks and subtleties for AdS/CFT and a-maximaization”, JHEP 0412(2004)024 [arXiv:hep-th/0411249].

[24] B. Feng, A. Hanany and Y.-H. He, “Phase Structures of D-brane Gauge Theories and Toric Duality”, JHEP 0108(2001) 040 [arXiv:hep-th/0104259].

[25] C.E. Beasley and M.R. Plesser, “Toric Duality Is Seiberg Duality”, JHEP 0212(2002) 076 [arXiv:hep-th/0109053].

[26] B. Feng, A. Hanany, Y.-H. He and A.M. Uranga, “Toric Duality as Seiberg Duality and Brane Diamonds”, JHEP 0112 (2001) 035 [arXiv:hep-th/0109063].

[27] B. Feng, S. Franco, A. Hanany and Y.-H. He, “Symmetries of Toric Duality”, JHEP 0212(2002) 076 [arXiv:hep-th/0205144].

[28] B. Feng, S. Franco, A. Hanany and Y.-H. He, “Unhiggsing the del Pezzo”, JHEP 0308 (2003) 058 [arXiv:hep-th/0209228].

[29] S. Franco and A. Hanany, “Geometric dualities in 4d field theories and their 5d interpretation”, JHEP 0304 (2003) 043 [arXiv:hep-th/0207006].

[30] K. Intriligator and B. Wecht, “Baryon Charges in 4d Superconformal Field Theories and Their AdS Duals”, Commun.Math.Phys. 245 (2004) 407-424 [arXiv:hep-th/0305046].

[31] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, “An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals”, JHEP 0506(2005)064 [arXiv:hep-th/0411264].

[32] A. Butti, D. Forcella and A. Zaffaroni, “The dual superconformal theory for $L^{p,q,r}$ manifolds”, JHEP 0509(2005)018 [arXiv:hep-th/0505220].

[33] A. Butti and A. Zaffaroni, “R-charges from Toric Diagrams and the Equivalence of a-maximization and Z-minimization”, JHEP 0511(2005)019 [arXiv:hep-th/0506232].
[34] A. Butti and A. Zaffaroni, “From Toric Geometry to Quiver Gauge Theory: the Equivalence of a-maximization and Z-minimization”, Fortsch. Phys. 54(2006)309-316 [arXiv:hep-th/0512240].

[35] R. Argurio, M. Bertolini, C. Closset and S. Cremonesi, “On Stable Non-Supersymmetric Vacua at the Bottom of Cascading Theories”, [arXiv:hep-th/0606175].

[36] D. Martelli, J. Sparks and S.T. Yau, “The Geometric Dual of a-maximisation for Toric Sasaki-Einstein Manifolds”, [arXiv:hep-th/0503183].

[37] S. Smale, “On the structure of 5-manifolds”, Ann. Math. 75(1962)38-46.

[38] C.P. Boyer, K. Galicki and M. Nakamaye, “On the geometry of Sasaki-Einstein 5-manifolds”, Math. Ann. 325 (2003)485-524.

[39] T. Oota and Y. Yasui, “Toric Sasaki-Einstein manifolds and Heun equations”, Nucl. Phys. B742 (2006)275-294 [arXiv:hep-th/0512124].

[40] S. Benvenuti, B. Feng, A. Hanany and Y.-H. He, “Counting BPS Operators in Gauge Theories - Quivers, Syzygies and Plethystics”, [arXiv: hep-th/0608050].

[41] D. Martelli and J. Sparks and S.T. Yau, “Sasaki-Einstein Manifolds and Volume Minimisation”, [arXiv:hep-th/0603021].
Figure 1: Toric diagram, quiver diagram and brane tiling for $dP_1$. 
Figure 2: Toric diagram, quiver diagram (model II) and brane tiling (model II) for $dP_2$.

Figure 3: Toric diagram, quiver diagram (model I) and brane tiling (model I) for $dP_3$. 
Figure 4: Brane tiling for $X^{p,q}$. 
Figure 5: Toric diagram, quiver diagram and brane tiling for $X^{3,1}$. 
Figure 6: Toric diagram, quiver diagram and brane tiling for $Z^{3,1}$. 
Figure 7: The perfect matching which corresponds to the node $V_6$ and the $R$-charge assignment for $Z^{3,1}$. 
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Figure 9: Toric diagram, quiver diagram and brane tiling for $Z^{3,2}$ (case 1).
Figure 10: The perfect matching and the $R$-charge assignment for $Z^{3,2}$ (case 1).
Figure 11: Toric diagram, quiver diagram and brane tiling for $Z^{3,2}$ (case 2).
Figure 12: The perfect matching and the $R$-charge assignment for $Z_3^{3,2}$ (case 2).
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Figure 14: The perfect matching and the $R$-charge assignment for $Z^{3,2}$ (case 3).
Figure 15: Toric diagram for $X^{p,q}$.

Figure 16: Toric diagram for $Z^{p,q}$.