On the impact of treewidth in the computational complexity of freezing dynamics

Eric Goles  
Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Santiago, Chile

Pedro Montealegre  
Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Santiago, Chile

Martín Ríos-Wilson  
Departamento de Ingeniería Matemática, FCFM, Universidad de Chile, Santiago, Chile.  
Aix Marseille Univ, Université de Toulon, CNRS, LIS, Marseille, France

Guillaume Theyssier  
Aix-Marseille Université, CNRS, I2M (UMR 7373), Marseille, France

Abstract

An automata network is a network of entities, each holding a state from a finite set and evolving according to a local update rule which depends only on its neighbors in the network’s graph. It is freezing if there is an order on states such that the state evolution of any node is non-decreasing in any orbit. They are commonly used to model epidemic propagation, diffusion phenomena like bootstrap percolation or crystal growth. In this paper we establish how treewidth and maximum degree of the underlying graph are key parameters which influence the overall computational complexity of finite freezing automata networks. First, we define a general model checking formalism that captures many classical decision problems: prediction, nilpotency, predecessor, asynchronous reachability. Then, on one hand, we present an efficient parallel algorithm that solves the general model checking problem in NC for any graph with bounded degree and bounded treewidth. On the other hand, we show that these problems are hard in their respective classes when restricted to families of graph with polynomially growing treewidth. For prediction, predecessor and asynchronous reachability, we establish the hardness result with a fixed set-defiend update rule that is universally hard on any input graph of such families.

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1 Introduction

An automata network is a network of $n$ entities, each holding a state from a finite set $Q$ and evolving according to a local update rule which depends only on its neighbors in the network’s graph. More concisely, it can be seen as a dynamical system (deterministic or not) acting on the set $Q^n$. The model can be seen as a non-uniform generalization of (finite) cellular automata. Automata networks have been used as modelization tools in many areas \cite{23} and they can also be considered as a distributed computational model with various specialized definitions like in \cite{46, 47}.

An automata network is freezing if there is an order on states such that the state evolution of any node is non-decreasing in any orbit. Several models that received a lot of attention in the literature are actually freezing automata networks, for instance: bootstrap percolation which has been studied on various graphs \cite{1, 5, 4, 30}, epidemic \cite{16} or forest
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fire propagation models\(^1\), crystal growth models\([13, 26]\) and more recently self-assembly tilings\([15]\). On the other hand, their complexity as computational models has been studied from various point of view: as language recognizers where they correspond to bounded change or bounded communication models\([14, 38, 10]\), for their computational universality\([37, 20, 6]\), as well as for various associated decision problems\([21, 28, 25, 22]\).

A major topic of interest in automata networks theory is to determine how the network graph affects dynamical or computational properties\([17, 25]\). In the freezing case, it was for instance established that one-dimensional freezing cellular automata, while being Turing universal via Minsky machine simulation, have striking computational limitations when compared to bi-dimensional ones: they are NL-predictable (instead of P-complete)\([37, 29, 44]\), can only produce computable limit fixed points starting from computable initial configurations (instead of non-computable ones starting from finite configurations)\([37]\), and have a polynomial time decidable nilpotency problem (instead of uncomputable)\([37]\).

The present paper aims at understanding what are the key graph parameters which influence the overall computational complexity of finite freezing automata networks. The results mentioned earlier show a gap between paths or rings and bi-dimensional grids. Since Courcelle’s theorem on MSO properties\([12]\), graph parameters like treewidth\([41]\) are often used in parameterized complexity\([14]\). Paths or rings have constant treewidth and it is known that the treewidth of a graph is polynomially related to the size of its largest grid minor\([11]\). Therefore treewidth is a natural parameter for our study. However, the situation is not the same as for MSO model checking since some properties that we are interested in like nilpotency are actually coNP-hard on trees of unbounded degree (see Remark\([11]\) below). We thus focus on the maximum degree of graphs as a second parameter. Finally, thanks to the freezing condition, the alphabet size gives a bound on the maximum number of state change per node in any orbit: it shall be seen as a third parameter in our problems. Our contributions are as follows.

**Model checking formalism and classical dynamical problems.** We first introduce a logical formalism to express many dynamical properties of our model. It consists in first order formulas where variables represent nodes of the network associated to localized trace predicates that represent sets of possible orbits projected on a subset of nodes. From there we define a general model checking problem, that asks whether a given formula is satisfied by a given freezing automata networks. It takes advantage of the sparse orbits of freezing automata networks (a bounded number of changes per node in any orbit) which allow to express properties in the temporal dimension in an efficient way. We show thanks to a kind pumping lemma on orbits (Lemma\([5]\) that it captures many standard problems in automata network theory, among which we consider four ones: prediction\([21, 20, 29]\), nilpotency\([10, 18, 32]\), predecessor\([33, 28]\) and asynchronous reachability\([13]\). Note that since Boolean circuits are easily embedded into freezing automata networks, our framework also includes classical problems on circuit: circuit value problem is a sub-problem of our prediction problem (see Theorem\([25]\) and SAT is a sub-problem of our nilpotency problem (see Remark\([11]\) and Theorem\([23]\)."
problems). Note that our algorithm is uniform in the sense that, besides the graph, both the automata network rule and the formula to test are part of the input and not hidden in an expensive pre-processing step. As suggested above, temporal traces of the evolution of a bounded set of nodes have a space efficient representation. However, it is generally hard to distinguish real orbits projected on a set of nodes from locally valid sequences of states that respect the transition rule for these nodes. Our algorithm exploits bounded treewidth and bounded degree to solve this problem via dynamic programming for any finite set of nodes. In the deterministic case, our algorithm can completely reconstruct the orbit from the initial configuration.

**Hardness results.** On the other hand, we prove that our four problems above are complete in there respective class (NP, coNP, P) when we restrict the input graphs to a constructible family of polynomial treewidth (Theorems 23, 24 and 25). To do so, we rely on an efficient algorithm to embed arbitrary (but polynomially smaller) digraph into our input graph (Lemma 22), which relies on polynomial perfect brambles that can be efficiently found in graphs with polynomially large treewidth [35]. This embedding allows to simulate a precise dynamics on the desired digraph inside the input graph and essentially lifts us from the graph family constraint as soon as the treewidth is large enough. Moreover, for problems prediction (Theorem 26), predecessor and asynchronous reachability (Theorem 24) we achieve the hardness result with a fixed uniform set-defined rule (i.e. a rule that change the state of each node depending only on the set of states seen in the neighborhood) which is not part of the input. This shows that there is a uniform isotropic universally hard rule for these problems, which makes sense for applications like bootstrap percolation, epidemic propagation or crystal growth where models are generally isotropic and spatially uniform.

## 2 Preliminaries

Given a graph $G = (V, E)$ and a vertex $v$ we will call $N(v)$ to the neighborhood of $v$ and $\delta_v$ to the degree of $v$. In addition, we define the closed neighborhood of $v$ as the set $N[v] = N(v) \cup \{v\}$ and we use the following notation $\Delta(G) = \max_{v \in V} \delta_v$ for the maximum degree of $G$. We will use the letter $n$ to denote the order of $G$, i.e. $n = |V|$. Also, if $G$ is a graph and the set of vertices and edges is not specified we use the notation $V(G)$ and $E(G)$ for the set of vertices and the set of edges of $G$ respectively. In addition, we will assume that if $G = (V, E)$ is a graph then, there exist an ordering of the vertices in $V$ from 1 to $n$. During the rest of the text, every graph $G$ will be assumed to be connected and undirected. We define a class or a family of graphs as a set $\mathcal{G} = \{G_n\}_{n \geq 1}$ such that $G_n = (V_n, E_n)$ is a graph and $|V_n| = n$.

**Non-deterministic freezing automata networks.** Let $Q$ be a finite set that we will call an alphabet. We define a non-deterministic automata network in the alphabet $Q$ as a tuple $(G = (V, E), \mathcal{F} = \{F_v : Q^{N(v)} \to \mathcal{P}(Q)|v \in V\})$ where $\mathcal{P}(Q)$ is the power set of $Q$. To every non-deterministic automata network we can associate a non-deterministic dynamics given by the global function $F : Q^n \to \mathcal{P}(Q^n)$ defined by $F(x) = \{x \in Q^n|x_v \in F(x_v)\}$.

> **Definition 1.** Given a a non-deterministic automata network $(G, \mathcal{F})$ we define an orbit of a configuration $x \in Q^n$ at time $t$ as a sequence $(x_s)_{0 \leq s \leq t}$ such that $x_0 = x$ and $x_s \in F(x_{s-1})$. In addition, we call the set of all possible orbits at time $t$ for a configuration $x$ as $\mathcal{O}(x, t)$. Finally, we also define the set of all possible orbits at time $t$ as $\mathcal{O}(A, t) = \bigcup_{x \in Q^n} \mathcal{O}(x, t)$.

We say that a non-deterministic automata network $(G, \mathcal{F})$ defined in the alphabet $Q$ satisfies the freezing property or simply that it is freezing if there exists a partial order $\leq$ in $Q$ such
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that for every \( t \in \mathbb{N} \) and for every orbit \( y = (x_s)_{0 \leq s \leq t} \in O(A, t) \) we have that \( x_s^t \leq x_s^{t+1} \)
for every \( 0 \leq s \leq t \) and for every \( 0 \leq i \leq n \). Let \( y = (x_s)_{0 \leq s \leq t} \) be an orbit for a non-deterministic automata network \((G, F)\) and \( S \subseteq V \) we define the restriction of \( y \) to \( S \) as the sequence \( z \in (Q^t)^{|S|} \) such that \( x_s^v = z_s^v \) for every \( v \in V \).

**Definition 2.** Given a non-deterministic automata network \((G, F)\) and a set \( S \subseteq V \), we define the set of \( S \)-restricted orbits as the set \( T(S, t) = \{ z = (x_s)_{s \leq t} \in Q^{|S|} | \exists y \in O(t) : y|_S = z \} \).

During the rest of the text and we use the notation \( z = x|_S \). Finally, if \( A = (G, F) \) is a non-deterministic freezing automata network such that for every \( v \in V(G) \), \( F_v \in F \) is such that \( |F_v(x)| = 1 \), for all \( x \in Q^{N(v)} \) then, we say that \( A \) is deterministic and view local rules as maps \( F_v : Q^{N(v)} \to Q \) to simplify notations.

Tree decompositions and treewidth. Let \( G = (V, E) \) be a connected graph. A subgraph \( P \) of \( G \) is said to be a path if \( V(P) = \{ v_1, \ldots, v_k \} \) where every \( v_i \) is different and \( E(P) = \{ v_1v_2, v_2v_3, \ldots, v_{k-1}v_k \} \). We define the length of a path \( P \) in \( G \) as the number of edges of \( P \). Given two vertices \( u, v \in V(G) \) we say that \( P \) is a \( u \)-\( v \) path if \( v_1 = v \) and \( v_k = u \) We say that \( P \) is a cycle if \( k \geq 3 \) and \( v_k = v_1 \). We say that \( G \) is a tree-graph or simply a tree if it does not have cycles as subgraphs. Usually, we will distinguish certain node in \( r \in V(G) \) that we will call the root of \( G \). Whenever \( G \) is a tree and there is a fixed vertex \( r \in V(G) \) we will call \( G \) a rooted tree-graph. In addition, we will say that \( v \in V(G) \) is a leaf if \( \delta_v = 1 \). Straightforwardly the choice of \( r \) induces a partial order in the vertices of \( G \) given by the distance (length of the unique path) between a node \( v \in V(G) \) and the root \( r \). We define the height of \( G \) (and we write it as \( h(G) \)) as the longest path between a leaf and \( r \). We say that a node \( v \) is in the \((h(G) - k)\)-th level of a tree-graph \( G \) if the distance between \( v \) and \( r \) is \( k \) and we write \( v \in \mathcal{L}_{h(G)-k} \). We will call the children of a node \( v \in \mathcal{L}_k \) to all \( w \in N(v) \) such that \( w \) is in level \( k - 1 \).

**Definition 3.** Given a graph \( G = (V, E) \) a tree decomposition is pair \( \mathcal{D} = (T, \Lambda) \) such that \( T \) is a tree graph and \( \Lambda \) is a family of subsets of nodes \( \Lambda = \{ X_t \subseteq V | t \in V(T) \} \), called bags, such that:

- Every node in \( G \) is in some \( X_t \), i.e. \( \bigcup_{t \in V(T)} X_t = V \)
- For every \( e = uv \in E \) there exists \( t \in V(T) \) such that \( u, v \in X_t \)
- For every \( u, v \in V(T) \) if \( w \in V(T) \) is in the \( u-v \) path in \( T \), then \( X_u \cap X_v \subseteq X_w \)

We define the width of a tree decomposition \( \mathcal{D} \) as the amount width(\( \mathcal{D} \)) = \( \max_{t \in V(T)} \{|X_t| - 1\} \).

Given a graph \( G = (V, E) \), we define its treewidth as the parameter \( \text{tw}(G) = \min_{\mathcal{D}} \text{width}(\mathcal{D}) \).

In other words, the treewidth is the minimum width of a tree decomposition of \( G \). Note that, if \( G \) is a connected graph such that \( |E(G)| \geq 2 \) then, \( G \) is a tree and if and only if \( \text{tw}(G) = 1 \).

It is well known that, given an arbitrary graph \( G \), and \( k \in \mathbb{N} \), the problem of deciding if \( \text{tw}(G) \leq k \) is \( \text{NP} \)-complete [2]. Nevertheless, if \( k \) is fixed, that is to say, it is not part of the input of the problem then, there exist efficient algorithms that allow us to compute a tree-decomposition of \( G \). More precisely, it is shown that for every constant \( k \in \mathbb{N} \) and a graph \( G \) such that \( \text{tw}(G) \leq k \), there exist a log-space algorithm that computes a tree-decomposition of \( G \) [13]. In addition, in Lemma 2.2 of [2] it is shown that given any tree decomposition of a graph \( G \), there exist a fast parallel algorithm that computes a slightly bigger width binary tree decomposition of \( G \). More precisely, given a tree decomposition of width \( k \), the latter algorithm computes a binary tree decomposition of width at most \( 3k + 2 \). We outline these results in the following proposition:
Proposition 4. Let \( n \geq 2, k \geq 1 \) and let \( G = (V, E) \) with \( |V| = n \) be a graph such that \( tw(G) \leq k \). There exists a CREW PRAM algorithm using \( O(\log^2 n) \) time, \( n^{O(1)} \) processors and \( O(n) \) space that computes a binary treewidth decomposition of width at most \( 3k + 2 \) for \( G \).

3 Localized Trace Properties and Model Checking

In this section we formalize the general model checking problem we consider on our dynamical systems. Freezing automata network have temporally sparse orbits, however the set of possible configurations is still exponential. Our formalism takes this into account by considering properties that are spatially localized but without restriction in their temporal expressive power. We will consider first order formulae where variables represent nodes of the network equipped with any predicate on orbits restricted to these nodes.

Syntax. Given a set \( X \) of variables, an ordered alphabet \( (Q, \leq) \) and an integer \( t \), a \((X,Q,t)\)-sequence is a collection of \( \leq \)-increasing sequences of elements of \( Q \) of length \( t \), indexed by \( X \): formally it is an element of \((Q^t)^X\). We call \((X,Q,t)\)-specification any set of \((X,Q,t)\)-sequences. Then, a \((X,Q,t)\)-formula with set of free variables \( X \) is defined inductively as one of the following:

- a predicate of the form \( T(X,t) \subseteq S \) where \( S \) is a \((X,Q,t)\)-specification,
- a Boolean combination of \((X',Q,t)\)-formulae each such that \( X' \subseteq X \),
- a formula of the form \( \exists y, \phi \) where \( \phi \) is a \((X \cup \{y\},Q,t)\)-formula and \( y \notin X \),
- a formula of the form \( \forall y, \phi \) where \( \phi \) is a \((X \cup \{y\},Q,t)\)-formula and \( y \notin X \).

A \((X,Q,t)\)-formula is in normal form with \( k \) quantifiers if it is in the form of a prefix of \( k \) quantifiers on variables \( y_1, \ldots, y_k \) followed by a \((X \cup \{y_1, \ldots, y_k\},Q,t)\)-formula without quantifiers where \( X \cap \{y_1, \ldots, y_k\} = \emptyset \). A \((Q,t)\)-formula is just a closed formula, i.e. a \((X,Q,t)\)-formula with \( X = \emptyset \).

Semantics. The semantics of our formulae follows from the interpretation of variables as network nodes and corresponds to the intuition that all considered orbits have the desired property when restricted to the interpreted nodes. Formally, a set of orbits \( O \subseteq (Q^t)^V \) for a set of nodes \( V \) satisfies a \((X,Q,t)\)-formula \( \phi \) under interpretation \( \iota : X \to V \) when \( O \) is not empty and:

- or \( \phi = T(X,t) \subseteq S \) where \( S \) is a \((X,Q,t)\)-specification and any \( \iota(X) \)-restricted orbit \( \tau \in (Q^t)^{(X)} \cap O \) verifies \( \forall x \in X, \tau(\iota(x)) = \tau'(x) \) for some \( \tau' \in S \);
- or \( \phi \) is a Boolean combination of \((X,Q,t)\)-formulae and the Boolean combination of satisfaction of each formula is true;
- or \( \phi \) is of the form \( \exists y, \phi \) where \( \phi \) is a \((X \cup \{y\},Q,t)\)-formula and \( y \notin X \), and there exists a node \( v \in V \) such that \( A \) satisfies \( \phi \) under the interpretation \( \iota \cup y \to v \);
- or \( \phi \) is of the form \( \forall y, \phi \) where \( \phi \) is a \((X \cup \{y\},Q,t)\)-formula and \( y \notin X \), and for any node \( v \in V \) it holds that \( A \) satisfies \( \phi \) under the interpretation \( \iota \cup y \to v \).

Note that the interpretation of variables as nodes in this formalism is common to all orbits. We cannot express properties like if the initial configuration contains state \( q_0 \) then state \( q_0 \) is still present somewhere in the network after \( t \) steps. We will often make a slight abuse of notation below using \((S,Q,t)\)-specifications for \( S \) a subset of vertices, to actually represent both a specification and a canonical interpretation of variables as the identity map \( \iota : S \to S \). It is straightforward to check that any formula can be turned into normal form.

Pumping lemma on orbits. The following lemma shows that for all freezing automata networks the set of orbits of any length restricted to a set of nodes is determined by the
set of orbits of fixed (polynomial) length restricted to these nodes. Moreover, if the set of
considered nodes is finite, then the fixed length can be chosen linear.

Lemma 5. Let $Q$ be an alphabet, $V$ a set of nodes with $|V| = n$ and $U \subseteq V$. Let $L = |U||Q|(|Q|n + 1)$. Then if two non-deterministic freezing automata have the same set of
orbits restricted to $U$ of length $L$ then they have the same set of orbits restricted to $U$ of any
length.

Model checking problem. As detailed below, this lemma allows to restrict all orbits in our
problems to a polynomial length in the size $n$ of the considered graph without loss of gener-
ality. In the sequel, the number of possible $(X, Q, t)$-sequences will always be polynomial in
$n$ (with an exponent depending on $|X|$) and therefore a $(X, Q, t)$-specification can be repre-
sented in polynomial space (as a Boolean vector indicating the allowed $(X, Q, t)$-sequences).
Also, in the absence of explicit mention, all the considered graphs will have bounded degree
$\Delta$ by default, so a freezing automata network rule can be represented as bounded size ($|Q|^{Q^\Delta}$) local update rules for each node.

Definition 6. Let $t \in \mathbb{N}$. We say that a $(V, Q, t)$-specification $\mathcal{I}$ is paral-
elizable if it is of
the form $\mathcal{I} = \{s \in (Q^t)^V : s_v \in \mathcal{I}_v, \forall v\}$ where $\mathcal{I}_v$ are $\{(v), Q, t\}$-specifications.

The general model checking problem we consider asks whether a given freezing automata
network verifies a given localized trace property on the set of orbits whose restriction on
each node adheres to a given parallelizable specification.

Problem 7 (General model checking problem).
Parameters: alphabet $Q$, family of graphs $\mathcal{G}$ of max degree $\Delta$, number of quantifiers $k$.
Input:
1. a non-deterministic freezing automata network $\mathcal{A} = (G, F)$ on alphabet $Q$, with set
   of nodes $V$ and $G \in \mathcal{G}$;
2. a parallelizable $(V, Q, t)$-specification $\mathcal{I}$, which specifies a set of orbits $O = O(\mathcal{A}, t) \cap \mathcal{I}$;
3. a $(Q, t)$-formula $\phi$ in normal form with at most $k$ quantifiers.
Question: does $O$ satisfy $\phi$?

Among the various special cases of the general model checking problem above we would
like to put forward the following ones which appeared in the literature.

Four canonical problems. When studying a dynamical system, one is often interested
in determining properties of the future state of the system given its initial state. Various
prediction problems have been studied in the literature where we ask for a property of the
 evolution at a given node.

Problem 8 (Prediction problem).
Parameters: alphabet $Q$, family of graphs $\mathcal{G}$ of max degree $\Delta$
Input:
1. a deterministic freezing automata network $\mathcal{A} = (G, F)$ on alphabet $Q$, with set
   of nodes $V$ with $n = |V|$ and $G \in \mathcal{G}$;
2. an initial configuration $c \in Q^V$; 
3. a node $v \in V$ and a $(\{v\}, Q, t)$-specification $\mathcal{S}_v$ of length $t \in \mathbb{N}$
Question: does the orbit of $c$ restricted to $v$ satisfies specification $\mathcal{S}_v$?

We can always suppose $t \in O(n)$ by Lemma 5 and since an initial configuration can be
specified as a $(\{v\}, Q, t)$-specification for each node $v$, this prediction problem is clearly a
subproblem of the general model checking problem. Note that a specification at a fixed node
allows to ask various questions considered in the literature: what will be the state of the node at a given time \[20, 29\], will the node change its state during the evolution \[21, 23, 22\], or, thanks to Lemma 5, what will be state of the node once a fixed point is reached \[37, section 5\]. Note that the classical circuit value problem for Boolean circuits easily reduces to the prediction problem above when we take \(G\) to be the DAG of the Boolean circuit and choose local rules at each node that implement circuit gates. Theorem 25 below gives a much stronger result using such a reduction where the graph and the rule are independent of the circuit.

We now turn to the classical problem of finding predecessors back in time to a given configuration \[33, 27\].

\[\blacktriangledown\] Problem 9 (Predecessor Problem).
\(\text{Parameters:}\) alphabet \(Q\), family of graphs \(G\) of max degree \(\Delta\)
\(\text{Input:}\)
1. a deterministic freezing automata network \(A = (G, F)\) on alphabet \(Q\), with set of nodes \(V\) with \(n = |V|\) and \(G \in G\);
2. a configuration \(c \in Q^V\)
3. a time \(t \in \mathbb{N}\)
\(\text{Question:}\) \(\exists y \in Q^V : F^t(y) = c?\)

Again, the configuration in the input can be given through the \((\{v\}, Q, t)\)-specification at each node \(v\), so this is a subproblem of our general model checking problem.

Deterministic automata networks have ultimately periodic orbits. When they are freezing, any configuration reaches a fixed point. Nilpotency asks whether there is a unique fixed point whose basin of attraction is the set of all configurations. It is a fundamental problem in finite automata networks theory \[40, 18\] as well as in cellular automata theory where the problem is undecidable for any space dimension \[32\], but whose decidability depends on the space dimension in the freezing case \[37\].

\[\blacktriangledown\] Problem 10 (Nilpotency problem).
\(\text{Parameters:}\) alphabet \(Q\), family of graphs \(G\) of max degree \(\Delta\)
\(\text{Input:}\) a deterministic freezing automata network \(A = (G, F)\) on alphabet \(Q\), with set of nodes \(V\) and \(G \in G\);
\(\text{Question:}\) is there \(t \geq 1\) such that \(F^t(Q^V)\) is a singleton?

To see this problem as a particular case of our general model checking problem, one first uses Lemma 5 to fix \(t = \lambda(n)\) where \(\lambda(n)\) is an appropriate polynomial and then express that \(F^{\lambda(n)}(Q^V)\) is a singleton as the following \((Q, \lambda(n))\)-formula: \(\forall v, \forall q_0 \in Q_T(v, \lambda(n)) \subseteq Q^*q_0\). It is straightforward to reduce coloring problems (does the graph admit a proper coloring with colors in \(Q\)) and more generally tilings problems to nilpotency using an error state that spread across the network when a local condition is not satisfied (note that tiling problem are known to be tightly related to nilpotency in cellular automata \[32\]). Using the same idea one can reduce SAT to nilpotency by choosing \(G\) to be the DAG of a circuit computing the given SAT formula (see Theorem 23 below for a stronger reduction that works on any family of graphs with polynomial treewidth).

\(\blacktriangledown\) Remark 11. If we allow the input automata network to be associated to a graph of unbounded degree (the local rule is then given as a circuit), it is possible to reduce any state instance to an automata network on a star graph with alphabet \(Q = \{0, 1, \epsilon\}\) where the central node simply checks that the Boolean values on leafs represent a satisfying instance of the SAT formula and produces a \(\epsilon\) state that spreads over the network if it is not the case.
The circuit representing the update rule of each node is NC in this case, and the automata network is nilpotent if and only if the formula is not satisfiable.

Given a deterministic freezing automata network of global rule $F: Q^V \rightarrow Q^V$, we define the associated non-deterministic global rule $F^*$ where each node can at each step to apply $F$ or to stay unchanged, formally: $F^*_v(c) = \{F_v(c), c_v \}$. It represents the system $F$ under totally asynchronous update mode.

Problem 12 (Asynchronous reachability Problem).

Parameters: alphabet $Q$, family of graphs $G$ of max degree $\Delta$

Input:
1. a deterministic freezing automata network $A = (G, F)$ on alphabet $Q$, with set of nodes $V$ with $n = |V|$ and $G \in G$;
2. an initial configuration $c_0 \in Q^V$
3. a final configuration $c_1 \in Q^V$

Question: can $c_1$ is reached starting from $c_0$ under $F^*$?

Note that no bound is given in the problem for the time needed to reach the target configuration. However, Lemma 5 ensures that $c_1$ can be reached from $c_0$ if and only if it can be reach in a polynomial number of steps (in $n$). Thus this problem can again be seen as a sub-problem of our general model checking problem. This bound on the maximum time needed to reach the target ensures that the problem is NP (a witness of reachability is an orbit of polynomial length). Note that the problem is PSPACE-complete for general automata networks: in fact it is PSPACE-complete even when the networks considered are one-dimensional (network is a ring) cellular automata (same local rule everywhere) [13].

4 A fast-parallel algorithm for the General Model Checking Problem

In this section we present a fast-parallel algorithm for solving General model checking problem when the input graph is restricted to the family of graphs with bounded degree and treewidth. More precisely, we show that the problem can be solved by a CREW PRAM that runs in the time $O(\log^2(n))$ where $n$ is the amount of nodes of the network. Thus, restricted to graphs of bounded degree and bounded treewidth, General model checking problem belongs to the class NC. Roughly speaking, in order to achieve this aim, our main strategy consists in expressing certain instance of the latter problem as a boolean combination of a polynomial amount of sub-instances of certain problem, that is, in some way, easier to study. Generally speaking, this decision problem consists in answering if given a parallel specification and a specification related to a fixed constant size set of vertices, the automata network is capable of exhibiting orbits satisfying both specifications or not. More precisely, it is the case in which we consider General model checking problem with $\phi$ as a $(S, Q, t)$-specification with $S \subseteq V$ is a constant size set of vertices. We call this problem Specification Checking problem. Note also that it is possible to think in Specification Checking problem as a generalized non-deterministic version of Prediction problem. In general terms, the parallelizable specification $I$ plays the role of the initial condition and the specification $E$ plays the role of the verification on the final state of certain vertex $v$ at time $t$. In addition, it is easy to check that in the case in which $A$ is a deterministic freezing automata network, we can, given a configuration $x$, define an specification $I(x)$ such that the problem with $E$ being simply the elements of $(Q^V)^t$ such that its final state is not given by $x_v$, correspond to Prediction problem with $x$ as a initial condition.
To explain how our main algorithm solve the latter problem, we will divide it in a number of sub-routines, that can be executed efficiently in parallel. Then, we will present an NC algorithm for Specification Checking problem as a combination of this sub-routines. Finally, we present our algorithm for solving General model checking problem in terms of these latter results. We begin fixing sets $Q$, $G$, and natural numbers $\Delta$ and $k$. Let $A = (G,F)$, $t$, $I$, $S$ and $E$ be an instance of the Specification Checking Problem, that we consider for the following definitions.

**Definition 13.** A locally-valid trace of a node $v \in V$ is a $(N[v],Q,t)$-sequence $\alpha : N[v] \rightarrow Q^t$ such that:
1. $\alpha(v)_{s+1} \in F_v((\alpha(u)_s)_{u \in N[v]})$ for all $0 \leq s < t$,
2. $\alpha(v)$ belongs to $I_v$.
We call the set of all locally-valid traces of $v$ as $LVT(v)$

Roughly speaking, a locally-valid trace of a vertex $v$ is a sequence of state-transitions of all the vertices in $N[v]$ which are consistent with local rule of $v$, but not necessarily consistent with the local-rules of the vertices in $N(v)$. We also ask that the state-transitions of $v$ satisfy the $((v),Q,t)$-specification $I_v$.

Given two finite sets $A,B$, and a function $f : A \rightarrow B$. We define the restriction function of $f$ to a subset $A' \subseteq A$ as the function $f|_{A'} : A' \rightarrow B$ such that, for all $v \in A'$ we have that $f|_{A'}(v) = f(v)$.

**Definition 14.** Let $U \subseteq V$ be such that $S \subseteq U$. A partially-valid trace of a set of nodes $U \subseteq V$ is a $(N[U],Q,t)$-sequence $\beta : N[U] \rightarrow Q^t$ such that:
1. $\beta|_{N[U]}$ belongs to $LVT(v)$ for each $v \in U$,
2. $\beta|_S$ belongs to $E$.
We call the set of all partially-valid traces of $U$ as $PVT(U)$

Roughly, a partially-valid trace for a set $U$ containing $S$, is a sequence of state-transition of all the vertices in $N[U]$, which are consistent with the local rules of all vertices in $U$, but not necessarily consistent with the local-rules of the vertices in $N(U)$. We also ask that the state-transitions of the vertices in $S$ satisfy the $(S,Q,t)$-specification $E$.

Let $(W,F,\{U_w : w \in W\})$ be a rooted binary-tree-decomposition of graph $G$ with root $r$, that we assume that has width at most $(3 \text{tw}(G)+2)$. Call now $T = (W,F,\{X_w \cup S : w \in W\})$ the binary-tree-decomposition of width $(3 \text{tw}(G)+2+k)$, where we include on each bag the set $S$. For $w \in W$ let us call $X_w = U_w \cup S$ the bag associated to $w$. We call $T_w$ the set of all the descendents of $w$, including $w$.

Our algorithm consists in a dynamic programming scheme over the bags of the tree. First, we assume that $PVT(X_w)$ is nonempty for all bags $w \in W$, otherwise the answer of the Specification Checking problem is false. For each bag $w \in T$ and $\beta^w \in PVT(X_w)$ we call $\text{Sol}_w(\beta^w)$ the partial answer of the problem on the vertices contained bags in $T_w$, when the locally-valid traces of the vertices in $X_w$ are induced by $\beta^w$. We say that $\text{Sol}_w(\beta^w) = \textbf{accept}$ when it is possible to extend $\beta^w$ into a partially-valid trace of all the vertices in bags of $T_w$, and $\textbf{reject}$ otherwise. More precisely, if $w$ is a leaf of $T$, we define $\text{Sol}_w(\beta^w) = \textbf{accept}$ for all $\beta^w \in PVT(X_w)$. For the other bags, $\text{Sol}_w(\beta^w) = \textbf{accept}$ if and only if exists a $\beta \in PVT(\bigcup_{z \in T \setminus \{w\}} X_z)$ such that $\beta(u) = \beta^w(u)$, for all $u \in X_w$. Observe that the instance of the Specification Checking problem is accepted when there exists a $\beta^r \in PVT(X_r)$ such that $\text{Sol}_r(\beta^r) = \textbf{accept}$. The following lemma is the core of our dynamic programming scheme:
Lemma 15. Let \( w \) be a bag of \( T \) that is not a leaf and \( \beta^w \in PVT(X_w) \). Then \( \text{Sol}_w(\beta^w) = \text{accept} \) and only if for each child \( v \) of \( w \) in \( T_w \) there exists a \( \beta^v \in PVT(X_v) \) such that
1. \( \beta^v(u) = \beta^w(u) \) for all \( u \in N[X_w] \cap N[X_v] \).
2. \( \text{Sol}_v(\beta^v) = \text{accept} \)

In order to solve our problem efficiently in parallel, we define a data structure that allows us efficiently encode locally-valid traces and a partially-valid traces. More precisely, in \( N[v] \) there are at most \( |Q|^2 \) possible state transitions. Therefore, when \( t \) is comparable to \( n \), most of the time the vertices in \( N[v] \) remain in the same state. Then, in order to efficiently encode a trace, it is enough to keep track of only the time-steps on which some state-transition occurs. We are now ready to give our algorithm solving the Specification Checking problem.

Lemma 16. Specification Checking problem can be solved by an CREW PRAM algorithm running in time \( O(\log^2 n) \) and using \( n^{O(1)} \) processors on graphs of bounded treewidth.

Remark 17. Note that latter lemma not only computes the answer of Specification Checking problem but also gives the coding of the \( S \)-restricted orbits satisfying specifications \( I \) and \( E \) simultaneously.

Now that we have an algorithm for the Specification Checking problem, we are ready to tackle the General model checking problem. First, consider the following technical lemma, which essentially states that General model checking problem can be reduced to a polynomial number of instances of the Specification Checking problem.

Lemma 18. Let \( k \in \mathbb{N} \) and \( \mathcal{A} \) a freezing non-deterministic automata network. For each \( (Q,t) \)-formula \( \phi \) with \( k \) quantifiers there exist \( r = n^{O(k)} \), a boolean combination \( \psi = \psi(\phi) : \{0,1\}^r \rightarrow \{0,1\} \), a collection of sub sets \( S_1, \ldots, S_r \) such that \( |S_i| = O(k) \) and a sequence of \((S_i,Q,t)\)-specifications \( E_i \) for \( i = 1, \ldots, r \) such that \( \mathcal{A} \) satisfies \( \phi \) if and only if \( \mathcal{A} \) satisfies the boolean combination \( \psi \) of every satisfiability value in each specification \( E_1 \ldots E_r \). Moreover, there is an CREW PRAM algorithm that given \( \phi \) computes the sets \( S_1, \ldots, S_r \) and \( E_i \) using \( n^{O(1)} \) processors and time \( O(\log n) \).

Now we are ready to give the main result of this section. Roughly, as a direct consequence of the latter lemma, we will show that decide General model checking problem can be done by deciding a polynomial amount of sub-instances of Specification Checking problem and thus, we show that there exist a NC-Turing reduction.

Theorem 19. Restricted to graphs of bounded treewidth, we have that
General model checking problem \( \leq^\text{NC} \) Specification Checking problem

Corollary 20. Restricted to graphs of bounded treewidth, General model checking problem is decidable in \( \text{NC} \).

Remark 21. We would like to remark that in the case in which the freezing automata network \( \mathcal{A} = (G, \mathcal{F}) \) is deterministic, we can say a lot more using latter algorithms. Giving \( t \) and an initial condition \( x \in Q^n \), we are actually capable of testing any global dynamic property in \( \text{NC} \) provided that this property has \( F^t(x) \) as input and it is decidable in \( \text{NC} \). In fact, note that given an initial condition \( x \in Q^n \), there is only one possible orbit for each node \( v \in V(G) \). Therefore, as a consequence of Remark 17 we are able to calculate the global evolution of the system in time \( t \) starting from \( x \).
5 Hardness results for polynomial treewidth networks

We say a family of graphs $\mathcal{G}$ has polynomial treewidth if there is a polynomial map $p_\mathcal{G}$ such that for any $G = (V, E) \in \mathcal{G}$ it holds $\text{tw}(G) \geq p_\mathcal{G}(|V|)$. Moreover, we say the family is constructible if there is a polynomial time algorithm that given $n$ produces a connex graph $G_n \in \mathcal{G}$ with $n$ nodes. The following lemma is based on a polynomial time algorithm to find large perfect brambles in graphs [35]. This structure allows to embed any digraph in an input graph with sufficiently large treewidth via path routing while controlling the maximum number of intersections per node of the set of paths.

- **Lemma 22** (Subgraph routing lemma). For any family $\mathcal{G}$ of graphs with polynomial treewidth, there is a polynomial map $p$ and a deterministic polynomial time algorithm that, given any graph $G = (V, E) \in \mathcal{G}$ and any digraph $D = (V', E')$ of maximum (in/out) degree $\Delta$ and size at most $p(|V|)$, outputs:
  1. a mapping $\mu : V' \rightarrow V$ such that, for each $v \in V$, $\mu^{-1}(v)$ contains at most two elements,
  2. a collection $C = (\rho_{c'})_{c' \in E'}$ of paths connecting $\mu(v'_1)$ to $\mu(v'_2)$ for each $(v'_1, v'_2) \in E'$, and such that any node in $V$ belongs to at most $4\Delta$ paths from $C$.

- **Theorem 23.** For any family $\mathcal{G}$ of constructible graphs of polynomial treewidth, the problem nilpotency is coNP-complete. When giving an automata network as input, the description of the local functions depends on the underlying graph (and in particular the degree of each node). However, some local functions are completely isotropic and blind to the number of neighbors and therefore can be described once for all graphs. This is the case of local functions that only depend on the set of states present in the neighborhood. Indeed, given a map $\rho : Q \times 2^Q \rightarrow Q$ and any graph $G = (V, E)$, we define the automata network on $G$ with local functions $F_v : Q^{N(v)} \rightarrow Q$ such that $F_v(c) = \rho(c(v), \{c(v'_1), \ldots, c(v'_k)\})$ where $N(v) = \{v_1, \ldots, v_k\}$ is the neighborhood of $v$ which includes $v$. We then say that the automata network is set defined by $\rho$. We will prove the next two hardness results with a fixed set defined rule, showing that there is a uniform and universally hard rule on graphs of polynomial treewidth for predecessor and asynchronous reachability problems. The proof below uses again Lemma 22 to embed arbitrary circuits like in theorems above, but the difference here is that the circuit embedding is written in the configuration and is not hardwired into the local rule. Moreover, the reduction also uses $L(1, 1)$ graph coloring [9] to deal with communication routing in a set defined rule in a similar way as in a radio network.

- **Theorem 24.** There exists a map $\rho : Q \times 2^Q \rightarrow Q$ such that for any family $\mathcal{G}$ of constructible graphs of polynomial treewidth and bounded degree, the problems predecessor and asynchronous reachability are both NP-complete when restricted to $\mathcal{G}$ and automata networks set-defined by $\rho$.

In the remaining of this section, we focus on the prediction problem for families of graphs with polynomial treewidth. In particular, we are interested in deriving an analogous of Theorem 24 for prediction problem. Nevertheless, as a log-space or a NC reduction of some P-complete problem is required, most of the latter results that worked for Theorem 24 are not necessarily valid in this context as we only know that there exist polynomial time algorithms that compute certain needed structures. In order to face this task, our approach is based in slightly modifying the input of our prediction problem and then show that we can efficiently compute paths in a polynomial treewidth graph $G$. The latter will allow us to show that we have an analogous of subgraph routing lemma (Lemma 22). In particular, as it is not clear
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if the perfect bramble structures used in order to obtain Lemma 22 are calculable in NC or in log-space, we need to modify the problem in order to show that there exists a log-space reduction or an NC reduction of circuit value problem (CVP) in this particular variation of prediction, and thus that it is P-complete. More precisely, we add a perfect bramble of polynomial size to the input of Prediction problem. We call this modified version of prediction Routed Prediction problem. Now, having this latter problem in mind, we slightly modify the definition of a constructible family of graphs \(G\) of polynomial treewidth introduced at the beginning of this section: we define a routed collection of graphs of polynomial treewidth to the set \(\mathcal{G} = \{ (G_n, B_n) \}_{n \in \mathbb{N}}\) such that \(G_n\) is an undirected connected graph of order \(n\) and treewidth \(\text{tw}(G_n) \geq p(n)\) and \(B_n\) is a perfect bramble such that \(|B_n| \geq p'(n)\) where \(p\) and \(p'\) are polynomials. We say that a routed collection of graphs of polynomial treewidth \(G\) is log-constructible if there is a log-space algorithm that given \(n\) produce the tuple \((G_n, B_n)\in \mathcal{G}\).

As we will be working with a log-constructible collection of routed graphs, we would like to say that we could have the result of Lemma 22 in order to show the main result of this section. Nevertheless, in order to do that, we need to have a log-space or a NC algorithm computing the paths that we will be using for the proof of the main result. More precisely, we need to compute the function \(\mu\) and the collection of paths \(C\). Fortunately, in [39, Theorem 5.3] it is shown that there exist a log-space algorithm that accomplish this task. Finally, as in the proof of Theorem 24 we need a proper coloring of the square graph \(G^2\) in order to broadcast information through the paths in the collection \(C\) without encountering problems in the nodes that are in different paths at the same time. Fortunately, we can do this in NC as it is stated in [19, Theorem 3]. We are now in condition of showing our main result concerning routed prediction problem:

\[\textbf{Theorem 25.} \quad \text{There exists a map } \rho : Q \times 2^Q \to Q \text{ such that Routed Prediction problem is } P\text{-complete restricted to any family } \mathcal{G} \text{ of log-constructible routed collection of graphs of polynomial treewidth.}\]

6 Discussion

In this paper, we established the key role of treewidth and maximum degree in the computational complexity of freezing automata networks. We believe that our results can be extended in several ways.

First, our algorithm for the general model checking problem is not as efficient as known algorithms for specific sub-problems [8] and it would be interesting to establish hardness results in the NC hierarchy to make this gap more precise. In the same vein, our algorithm doesn’t yield fixed parameter tractability results for any of the parameters (treewidth, degree, alphabet), and we believe that hardness results in the framework of parameterized complexity [14] could be established here to clarify the situation. We could also consider intermediate treewidth classes (non-constant but sub-polynomial). Concerning these complexity questions, we think that considering other (more restrictive) parameters like pathwidth could definitely help to obtain better bounds.

Besides, one might wonder whether the set of dynamical properties that are efficiently decidable on graphs of bounded degree and treewidth could be in fact much larger than what gives our model checking formalism. This question remains largely open, but we can already add ingredients in our formalism (for instance, a relational predicate representing the input graph structure). We however conjecture that there are NP-hard properties for freezing automata network on trees of bounded degree that can be expressed in the following language: first order quantification on configurations together with a reachability predicate.
configuration $y$ can be reached from $x$ in the system).

Finally, we think that we can push our algorithm further and partly release the constraint on maximum degree (for instance allowing a bounded number of nodes of unbounded degree). This can however not work in the general model checking setting as shown in Remark

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A Auxiliary decision problems

▷ Problem 26 (Specification Checking problem).

Parameters: alphabet $Q$, family of graphs $G$ of max degree $\Delta$, a natural number $k$.

Input:
1. a non-deterministic freezing automata network $\mathcal{A} = (G, F)$ on alphabet $Q$, with set of nodes
2. a natural number $t$;
3. a parallelizable $(V, Q, t)$-specification $\mathcal{I}$, which specifies a set of orbits
   \[
   \mathcal{O} = \{ o \in \mathcal{O}(\mathcal{A}, t) : \forall v, o_v \in \mathcal{I}_v \};
   \]
4. a $k$-vertex set $S$ and a $(S, Q, t)$-specification $\mathcal{E}$.

Question: does there exist $o \in \mathcal{O}$ such that $o|_S \in \mathcal{E}$?

▷ Problem 27 (Routed prediction problem).

Parameters: alphabet $Q$, family of graphs $G$ of max degree $\Delta$

Input:
1. a deterministic freezing automata network $\mathcal{A} = (G, F)$ on alphabet $Q$, with set of nodes $V$ with $n = |V|$ and $G \in G$;
2. an initial configuration $c \in Q^V$
3. a node $v \in V$ and a $(\{v\}, Q, t)$-specification $\mathcal{S}_v$ of length $t \in \mathbb{N}$
4. A perfect bramble $B = (B_1, \ldots, B_p)$ in with $p = n^{O(1)}$ in $G$

Question: does the orbit of $c$ restricted to $v$ satisfies specification $\mathcal{S}_v$?
B Fast-parallel sub-routines

\[\text{Proposition 28 (Prefix-sum algorithm, [31]). Then the following problem can be solved by a CREW PRAM machine with } p = O(n) \text{ processors in time } O(\log n): \text{ Given } A = \{x_1, \ldots, x_n\} \text{ be a finite set, } k \leq n \text{ and } \oplus \text{ be a binary associative operation in } A, \text{ compute } \oplus_{i=1}^k x_i.\]

\[\text{Proposition 29 ([39 Theorem 5.3]). Let } n \in \mathbb{N}. \text{ The following problem can be solved in space } O(\log n): \text{ given an undirected graph } G = (V, E) \text{ with } |V| = n, s, t \in V \text{ find a path from } s \text{ to } t \text{ and if there exists such a path, return the path as an output.}\]

\[\text{Proposition 30 ([19 Theorem 3]). Let } \Delta \in \mathbb{N}. \text{ The following problem can be solved in time } O(\Delta \log(\Delta + \log^* n)) \text{ by an EREW PRAM: given a graph } G = (V, E) \text{ such that } \Delta(G) \leq \Delta \text{ finding a } \Delta + 1 \text{ coloring of } G.\]

C Proofs of main results

C.1 Localized Trace Properties and Model Checking

\[\text{Lemma 5. Let } Q \text{ be an alphabet, } V \text{ a set of nodes with } |V| = n \text{ and } U \subseteq V. \text{ Let } L = |U||Q|(|Q|n + 1). \text{ Then if two non-deterministic freezing automata have the same set of orbits restricted to } U \text{ of length } L \text{ then they have the same set of orbits restricted to } U \text{ of any length.} \]

\[\text{Proof. Any orbit restricted to } U \text{ of any length can be seen as a sequence of elements of } Q^U \text{ and, since the considered automata network is freezing, there are at most } |U||Q| \text{ changes in this sequence so that it can be written } p_1^i p_2^i \cdots p_m^i \text{ with } m \leq |U||Q|. \text{ Let } p_i \in Q^U \text{ and } t_i \in \mathbb{N}. \text{ The key observation is that } p_1^1 p_2^1 \cdots p_m^1 \text{ is a valid restricted orbit if and only if } p_1^1 p_2^1 \cdots p_m^1 \text{ is a valid restricted orbit for all } T \geq |Q|n + 1: \text{ this is because any sequence of } |Q|n + 1 \text{ configurations in any orbit must contain two consecutive identical configurations since } |Q|n \text{ is the maximal total number of possible state changes. From this we deduce that it is sufficient to know all the restricted orbits of the form } p_1^1 p_2^1 \cdots p_m^1 \text{ with } t_i \leq |Q|n + 1 \text{ and } m \leq |U||Q| \text{ to know all restricted orbits of any length. The lemma follows.} \]

D A fast-parallel algorithm for the General Model Checking Problem

\[\text{Lemma 15. Let } w \text{ be a bag of } T \text{ that is not a leaf and } \beta^w \in PVT(X_w). \text{ Then } \text{Sol}_w(\beta^w) = \text{accept} \text{ and only if for each child } v \text{ of } w \text{ in } T_w \text{ there exists a } \beta^v \in PVT(X_v) \text{ such that } 1. \beta^w(u) = \beta^v(u) \text{ for all } u \in N[X_w] \cap N[X_v], 2. \text{Sol}_v(\beta^v) = \text{accept} \]

\[\text{Proof. First, let us assume that } \text{Sol}_w(\beta^w) = \text{True} \text{ and let } v \text{ be on of the children of } w \text{ in } T_w. \text{ This implies that there exists a partially-valid trace } \beta \in PVT(\bigcup_{z \in T_w} X_z) \text{ such that } \beta(u) = \beta^w(u), \text{ for all } u \in X_w. \text{ Observe that } N[\bigcup_{z \in T_w} X_z] \subseteq N[\bigcup_{z \in T_w} X_z]. \text{ Since } \beta \text{ is defined over } N[\bigcup_{z \in T_w} X_z], \text{ we can define } \beta^v \text{ and } \beta^T \text{ as the restrictions of } \beta \text{ to the sets } N[X_v] \text{ and } N[\bigcup_{z \in T_w} X_z], \text{ respectively. Observe that } \beta^v \text{ satisfies the condition (1) and (2) because, by definition, } \beta(v) \text{ and } \beta^w(u) \text{ are both equal to } \beta(u) \text{ for all } u \in N[X_w] \cap N[X_v]. \text{ Moreover, } \text{Sol}_w(\beta^w) = \text{accept} \text{ because } \beta^T \text{ is a partially-valid trace of } \bigcup_{\in T_w} X_z \text{ such that } \beta^w(u) = \beta(u) = \beta^v(u) \text{ for each } u \in X_v. \]

Conversely, suppose that we have that conditions (1), (2) for each child of } w. \text{ If } w \text{ is a leaf the proposition is trivially true. Suppose then that } w \text{ is not a leaf. For each child } v \text{ of } w, \text{ let}
be the partially-valid trace of $X_u$ satisfying that $\text{Sol}_u(\beta^v) = \text{accept}$ and $\beta^v(u) = \beta^w(u)$ for each $u \in N[X_u] \cap N[X_v]$. Since $\text{Sol}_u(\beta^v) = \text{accept}$ we know that $\beta^v$ can be extended into a partially-valid trace of $\bigcup_{x \in T_u} X_z$, that we call $\beta^{T_u}$. Let us call $v_1$ and $v_2$ the children of $w$. We define then the function $\beta : N[\bigcup_{x \in T_u} X_z] \rightarrow Q^t$.

$$
\beta(u) = \begin{cases} 
\beta^w(u) & \text{if } u \in N[X_w] \\
\beta^{T_{v_1}}(u) & \text{if } u \in N[\bigcup_{x \in T_{v_1}} X_z] \\
\beta^{T_{v_2}}(u) & \text{if } u \in N[\bigcup_{x \in T_{v_2}} X_z]
\end{cases}
$$

We claim that there is no ambiguity in the definition of $\beta$. First, we claim that $N[\bigcup_{x \in T_{v_1}} X_v] \cap N[\bigcup_{x \in T_{v_2}} X_v]$ is contained in $N[X_w]$. Indeed, let $u$ be a vertex in $N[\bigcup_{x \in T_{v_1}} X_z] \cap N[\bigcup_{x \in T_{v_2}} X_z]$. There are three possibilities:

- $u$ belongs to a bag in $T_{v_1}$ and to another bag in $T_{v_2}$. In this case necessarily $u \in X_w$, because otherwise the bags containing $u$ would not induce a (connected) subtree of $T$.
- $u$ is not contained in a bag of $T_{v_1}$. Since $u$ belongs to $N[\bigcup_{x \in T_{v_1}} X_v]$, there exists a vertex $\tilde{u}$ adjacent to $u$ and contained in a bag of $T_{v_1}$. Note that $X_w$ contains $\tilde{u}$, because otherwise all the bags containing $\tilde{u}$ would be in $T_{v_1}$. Then, no bag would contain both $u$ and $\tilde{u}$. That contradicts the property of a tree-decomposition that states that for each edge of the graph $G$, there must exist a bag containing both endpoints. We deduce $\tilde{u}$ is contained in $X_w$ and then $u$ is contained in $N[X_w]$.
- $u$ is not contained in a bag of $T_{v_2}$. This case is analogous to the previous one.

Following an analogous argument, we deduce that $N[\bigcup_{x \in T_{v_1}} X_z] \cap N[X_w]$ is contained in $N[X_{v_1}]$ and that $N[\bigcup_{x \in T_{v_2}} X_z] \cap N[X_w]$ is contained in $N[X_{v_2}]$. We conclude that $\beta$ is well defined. Moreover, $\beta$ is a partially-valid trace of $\bigcup_{x \in T_u} X_z$ which restricted to $N[X_w]$ equals $\beta^w$. We conclude that $\text{Sol}_w(\beta^w) = \text{accept}$.

Let $U$ be a set of vertices of $G$, and consider a $(U, Q, t)$-sequence $S$. Remember that $S$ is a function $S : U \rightarrow Q^t$ such that the sequence $S(u)$ is increasing, for all $u \in U$. For each $0 \leq s \leq t$ let us call $S_s$ the sequence $(S(u))_{u \in U} \in [Q]^{\lceil|U|\rceil}$. Let $\text{Times}(S) = (t_0, t_1, \ldots, t_\ell)$ be the increasing sequence of maximum length satisfying that $S_s = S_{s_i}$ for each $t_i \leq s < t_{i+1}$ and each $0 \leq i < \ell$. Observe that $t_0 = 0$ and $\ell = \ell(S) \leq |Q|^{\lceil|U|\rceil}$. For a natural numbers $m$ and $\ell$, let us call $\langle m \rangle_\ell$ the binary representation of $m$ using $\ell$ bits, padded with $\ell - \lceil \log m \rceil$ zeros when $\ell > \lceil \log m \rceil$.

**Definition 31.** Let $S$ be a $(U, Q, t)$-sequence. A succinct representation of $S$, that we call $\epsilon(S)$ is a pair $(\text{Times}(S), \text{States}(S))$ such that:

- $\text{Times}(S)$ is a list of elements of $\{0, 1\}^{\lceil \log(t+1) \rceil}$ of length $|Q|^{\lceil|U|\rceil}$, such that

$$
\text{Times}(S)_i = \begin{cases} 
\langle t_i \rangle^{\lceil \log(t+1) \rceil} & \text{if } i \leq \ell(S) \\
\langle t \rangle^{\lceil \log(t+1) \rceil} & \text{if } i > \ell(S)
\end{cases}
$$

- $\text{States}(S)$ is a matrix of elements of $\{0, 1\}^{\lceil \log|U| \rceil}$ of dimensions $|Q|^{\lceil|U|\rceil} \times |U|$, such that, if we call $u_1, \ldots, u_{\lceil|U|\rceil}$ the vertices of $U$ sorted by their labels, then:

$$
\text{States}(S)_{i,j} = \begin{cases} 
\langle S(u_j) \rangle_{\lceil \log|U| \rceil} & \text{if } i \leq \ell(S) \\
\langle S(u_j) \rangle_{\lceil \log|U| \rceil} & \text{if } i > \ell(S)
\end{cases}
$$

We also call $\#(U, t) = |Q|^{\lceil|U|\rceil} \lceil \log(t+1) \rceil + |U||Q|^{\lceil|U|\rceil} \lceil \log|Q| \rceil$.

Observe that $\epsilon(S)$ can be written using exactly $N = \#(U, t)$ bits. In other words, all succinct representations of $(U, Q, t)$-sequences can be stored in the same number of bits,
which is $O(|U| \log t)$. Therefore, there are at most $2^N = 2^g(|U|)$ possible $(U,Q,t)$-sequences, for some function $g$ exponential in $|U|$. Moreover, we identify the succinct representation of $(U,Q,t)$-sequence $S$ with a number $x \in \{0, \ldots, 2^N\}$, such that $\epsilon(S) = (x)_N$.

**Definition 32.** Let $U$ be a set of vertices an let us call $N = \#(U,t)$. A succinct representation of a $(U,Q,t)$-specification $X$ is a Boolean vector $\epsilon(X)$ of length $2^N$ such that $\epsilon(X)_i = \text{True}$ when $i$ represents the succinct representation of a $(U,Q,t)$-sequence contained in $X$.

Remember that we are assuming that all specifications have polynomial size in the size of the input graph $n$ and therefore can be stored in polynomial space. Next lemma states that the succinct representation of a $(U,Q,t)$-specification can be computed by fast parallel algorithms.

**Lemma 33.** For each set of vertices $U$, there are CREW PRAM algorithms performing the following tasks in time $O(|Q|^{[U]} \log n)$ using $n^{O(1)}$ processors:

- Given a $(U,Q,t)$-sequence $S$ as a $t \times |U|$ table of states in $Q$, compute $\epsilon(S)$
- Given a $(U,Q,t)$-specification $X$ as a list of $(U,Q,t)$-sequences, compute $\epsilon(X)$

**Proof.**

The algorithm first computes $\text{Times}(S)$. Then, it constructs the list $\text{Times}(S)$ and the matrix $\text{States}(S)$ copying the lines of $S$ given in $\text{Times}(S)$.

The algorithm starts reserving $N = \#(U,t)$ bits of memory for in the list $\text{Times}$ and the matrix $\text{States}$, and $t + 1$ bits of memory represented in a vector $\text{Indices}$. The vector $\text{Indices}$ stores the time-steps on which that belong to $\text{Times}(S)$.

For each $i \in \{1, \ldots, t\}$ the algorithm initializes a processor $P_i$ and assigns the $i$-th bit of $\text{Indices}$ to it. Processor $P_i$ looks at the $i$-th and $i-1$-th lines of $S$. If $S_i \neq S_{i-1}$ then processor writes a 1 in $\text{Indices}_i$. Otherwise, the processor writes a 0 in $\text{Indices}_i$. Then $P_i$ stops. All this process can be done in time $O(|U| \log |Q| + \log t)$ per processor.

Then, the algorithm computes the vector $p$ of length $t$ such that $p_j = \sum_{i=1}^{j} \text{Indices}_j$, for each $j \in \{1, \ldots, t\}$. This process can be done in time $O(\log t)$ using $O(t)$ processors using the prefix sum algorithm given by Proposition 28. Observe that if $\text{Indices}_i = 1$ for some index $i$, then $i = \text{Times}(S)_{p_i}$. Moreover, $p_i = \ell(S)$.

Once every processor $(P_i)_{0 \leq i \leq t}$ stops, the algorithms reinitialize them. For each $0 < i \leq t$, each processor $P_i$ looks at $\text{Indices}_i$. If $p_i < p_i$ and $\text{Indices}_i = 0$ then processor $P_i$ stops.

If $p_i = p_i = p_i$ the processor stops. If $p_i \neq p_i$ and $\text{Indices}_i = 1$, then the algorithm writes $(i)_{\log(t+1)}$ in $\text{Times}_{p_i}$, and for each $u \in \{1, \ldots, |U|\}$ writes $(S_{t,u})_{\log(|Q|)}$ in $\text{States}_{p_i,u}$. If $p_i = p_i$ and $p_i = p_i$, then the processor $P_i$ writes $(i)_{\log(t+1)}$ in $\text{Times}_{j}$ and writes $(S_{t,u})_{\log(|Q|)}$ in $\text{States}_{j,u}$ for each $p_i \leq j \leq |Q|^{[U]}$ and for each $u \in \{1, \ldots, |U|\}$. The algorithm writes $\text{Times}_0 = (0)_{\log(t+1)}$ and writes $(S_{0,u})_{\log(|Q|)}$ in $\text{States}_{0,u}$ for each $u \in \{1, \ldots, |U|\}$. All this process can be done in time $O(|Q|^{[U]} \log t)$ per processor.

The algorithm returns $\epsilon(S) = (\text{Times}, \text{States})$. The whole process takes time $O(|Q|^{[U]} \log t)$ and $O(t)$ processors.

The algorithm initializes $\epsilon(X)$ as $2^N$ bits of memory bits, initially all in 0. Then, it assigns one processor $P_S$ to each $(U,Q,t)$-sequence $S$ in $X$. For each $S \in X$, processor $P_S$ uses the previous algorithm to compute $y = \epsilon(S)$. Then processor $P_S$ writes $\epsilon(S)_y = 1$. Once every processor has finished, the algorithm returns $\epsilon(X)$. The whole process takes time $O(\log |X| |Q|^{[U]} \log t)$ and uses $|X|^{O(1)}$ processors. As we are consider only specifications
Lemma 34. Let \( \mathcal{S} \) be a set containing \( S \). Observe that if \( \beta \) is a partially-valid trace of \( U \), then in particular \( \beta \) is a \((N[U], Q, t)\)-sequence. Therefore, there exists \( x \leq 2^N \) with \( N = \#(N[U], t) \), such that \( \epsilon(\beta) = x \). In the following lemma we show how to characterize the values on \( x \leq 2^N \) that are the encoding of some partially-valid trace of \( U \). We need the following definition. Let \( U \) be a set of vertices and let \( x \in \{0, \ldots, 2^N\} \), with \( N = \#(U, t) \). Then we call \( \text{TIMES}(x) \) and \( \text{STATES}(x) \) the vector and matrix such that \( x = (\text{TIMES}(x), \text{STATES}(x)) \). More precisely:

- \( \text{TIMES}(x) \) are the first \( |Q|^{|U|} \) bits of \( x \) interpreted as sequence of elements of \( \{0, 1\}^{\log(t+1)} \) of length \( |Q|^{|U|} \).
- \( \text{STATES}(x) \) are the rest of the bits of \( x \) interpreted as the matrix of elements of \( \{0, 1\}^{\log |Q|} \) of dimensions \( |Q|^{|U|} \times |U| \).

\textbf{Lemma 34.} There is a sequential algorithm which given a succinct representation \( \epsilon(S) \) of a \((U, Q, t)\)-sequence and a set \( Z \subseteq U \), computes \( \epsilon(S)[Z] \) in time linear in the size of \( \epsilon(S) \).

\textbf{Proof.} Let \( \kappa = |Q| \). The computes algorithm \( \epsilon(S)[Z] \) checking each pair of lines of \( \text{STATES} \) and verifying if the columns of \( Z \) differ on any coordinate, keeping only the lines on which some of the vertices in \( Z \) switches states for the first time. More precisely, let \( u_1, \ldots, u_\kappa \) be the set \( U \) ordered by their labels. Let \( J = \{j_1, \ldots, j_\kappa\} \) be the set of indices of vertices of \( Z \) (i.e., \( u_j \in Z \) for all \( q \in \{1, \ldots, |Z|\} \)). The algorithm computes the set \( L \) of indices \( i \leq |Q|^\kappa \) such that \( i \in L \) if and only if there exists \( q \in J \) such that \( \text{STATES}_{i,j_q} \neq \text{STATES}_{i-1,j_q} \). Let \( \{i_1, \ldots, i_L\} \) the indices in \( L \). Observe that \( |L| \leq |Q|^{|Z|} \). Then for each \( p \leq |Q|^{|Z|} \) and \( q \in |Z| \),

\[
\text{TIMES}[Z]_{p} = \begin{cases} 
\text{TIMES}_{i_p} & \text{if } p \leq |L| \\
\langle t \rangle & \text{if } i > |L| 
\end{cases}
\]

\[
\text{STATES}[Z]_{p,q} = \begin{cases} 
\text{STATES}_{i_p,j_q} & \text{if } p \leq |L| \\
\text{STATES}_{i_q,j_q} & \text{if } p > |L| 
\end{cases}
\]

The algorithm returns \( (\text{TIMES}[Z], \text{STATES}[Z]) \).

\textbf{Lemma 35.} Let \( U \) be a set of vertices containing \( S \), and let \( N = \#(N[U], t) \). There is a sequential algorithm which given \( x \geq 0 \), \( \epsilon(E) \) and \( \epsilon(I_u) \) for each \( u \in V(G) \), decides in time \( f(|U|) \log n \) whether \( x \) is a succinct representation of a partially-valid trace of \( U \), where \( f \) is an exponential function.

\textbf{Proof.} Let \( U \) be a set of vertices containing \( S \) and let \( x \in \{0, \ldots, 2^N\} \), with \( N = \#(N[U], t) \). Let \( \{u_1, \ldots, u_\kappa\} \) be the vertices of \( N[U] \) ordered by label. The algorithm first verifies that \( x \leq 2^N \) and rejects otherwise. Then, the algorithm verifies that the pair \( (\text{TIMES}, \text{STATES}) = (\text{TIMES}(x), \text{STATES}(x)) \) satisfies \( \text{TIMES}_0 = 0 \) and that \( \text{TIMES} \) and each column of \( \text{STATES} \) are increasing. Otherwise, the algorithm rejects because \( x \) is not a succinct representation of a \((N[U], Q, t)\)-sequence. If the algorithm passes this test we assume that \( x = \epsilon(S) \) for some \((N[U], Q, t)\)-sequence \( S \). For a subset of vertices \( Z \), let us call \( (\text{TIMES}[Z], \text{STATES}[Z]) = \epsilon(S)[Z] \). Consider now the following conditions:

1. \( \text{STATES}_{i,j} \in F_{u_j}(\text{STATES}[N[u_j]])_j \) for each \( i \in \{0, \ldots, |Q|^\kappa\} \) and \( j \in \{1, \ldots, \kappa\} \) such that \( u_j \in U \).
2. \((\text{TIMES}[\{u\}], \text{STATES}[\{u\}])\) belongs to \(I_u\) for each \(u \in U\).
3. \((\text{TIMES}[S], \text{STATES}[S])\) belongs to \(E\).

When these conditions are satisfied, we can deduce that \(x = \epsilon(\beta)\) for some partially-valid trace \(\beta\) of \(U\). Indeed, as \(x\) represents a succinct representation of \(S\), then the vertices in \(N[U]\) only have state-transitions of the time-steps given in \(\text{TIMES}[S]\). Therefore, condition (1.) and (2.) imply that \(S_{\lfloor N[u] \rfloor}\) is a locally-valid trace of \(u\). On the other hand, condition (3.) implies that \(S_{\lfloor u \rfloor}\) is in \(E\).

To verify condition (1.) - (3.) we use the algorithm of Lemma 15 to \(\text{STATES}[Z]\) for a given set of vertices \(Z \subseteq N[U]\). Observe that the algorithm computes \(\text{TIMES}[Z]\) and \(\text{STATES}[Z]\) in time \(O(\kappa |Q|^{\kappa} \log n)\). The algorithm computes \(\epsilon(S_u) = (\text{TIMES}[S], \text{STATES}[S])\) and verifies whether \(S_{\lfloor u \rfloor}\) belongs to \(E\) by looking at the \(\epsilon(S_{\lfloor u \rfloor})\)-element of the table \(\epsilon(E)\). Similarly, the algorithm verifies (2.) computing \(\epsilon(S) = (\text{TIMES}[\{u\}], \text{STATES}[\{u\}])\) verifying whether \(S_{\lfloor u \rfloor}\) belongs to \(I_u\) for each \(u \in U\). Finally, the algorithm checks (1.) by looking at each row of \(\text{STATES}[N[u]]\) and the column corresponding to vertex \(u\), and the table of \(F_u\) given in the input. All these processes take time \(O(|U|\kappa |Q|^{\kappa} \log n)\). Overall the whole algorithm takes time \(O(|U|\kappa |Q|^{\kappa} \log n) = f(|U|) \log n\).

Lemma 16. Restricted to graphs of bounded treewidth, Specification Checking problem can be solved by an CREW PRAM algorithm running in time \(O(\log^2 n)\) and using \(n^{O(1)}\) processors.

Proof. Our algorithm consists in an implementation of the dynamic programming scheme explained at the beginning of this section. Our algorithm starts computing a rooted binary-tree decomposition \((W, F, \{U_w : w \in W\})\) of the input graph using the logarithmic-space algorithm given by Proposition 14. Then, in parallel for each \(w \in W\) we put the set \(S\) on each bag of the tree-decomposition, obtaining a tree decomposition \(T = (W, F, \{X_w : w \in W\})\), with \(X_w = U_w \cup S\) for every \(w \in W\). The algorithm also computes the succinct representations of \(E\) and \(I_u\) for each \(v \in V\) using Lemma 15.

Then, the algorithm preforms the dynamic programming scheme over \(T\). Let \(r\) be the root of \(T\). The for a bag \(w \in W\), we define the level of \(w\) denoted by \(\text{LEVEL}(w)\), as the distance between \(w\) and the root \(r\). There is a fast-parallel algorithm computing the level of each vertex of a tree by a EREW PRAM running in time \(O(\log n)\) and using \(O(n)\) processors [31]. Using a prefix-sum algorithm we can compute the maximum level \(M\) of a vertex, which correspond to the leaves of the binary-tree \(T\). For each \(i \in \{0, \ldots, M\}\), let \(L_i\) the set of bags \(w\) such that \(\text{LEVEL}(w) = M - i\).

For each \(w \in W\), we represent the values of the function \(\text{Sol}_w\) as a table \(S_w^w\) indexed as a table of size \(2^{N}\), with \(N = O((|Q|^{\Delta(3w(G) + 2 + k)} \log n))\) greater that \(|N[X_w], t|\) for all \(w \in W\). Each \(x \in \{0, \ldots, 2^{N}\}\) is interpreted as a potentially succinct encoding of a partially-valid trace \(\beta\). Initially \(S_w^w = 0^{2^{N}}\), which meaning that a priori we reject all \(x \in \{0, \ldots, 2^{N}\}\). Then, our algorithm iterates in a reverse order over the levels of the tree, starting from \(L_0\) until reaching the the root \(r \in L_M\). In the \(i\)-th iteration, we compute for each \(w \in L_i\) the set of all \(x \in \{0, \ldots, 2^{N}\}\) that represent partially-valid traces \(\beta_w^w \in \text{PVT}(X_w)\) such that \(\text{Sol}_w(\beta_w^w) = \text{accept}\). To do so, the algorithm uses the calculations done on the bags in \(L_{i-1}\), and use Lemma 15. The algorithm saves the answer of each partial solution in a variable \(\text{out}\) consisting in \(|W|\) bits, such that, and the end of the algorithm \(\text{out} = 1^{W}\) if and only the instance of the Specification Checking problem is accepted.

At the first iteration, for each \(w \in L_0\) the algorithm sets in parallel \(S_w^w = 1\) for all \(x\) representing a partially-valid trace of \(w\), because \(\text{Sol}_w(\beta_w^w)\) is defined to \(\text{accept}\) for all partially-valid trace of a leaf of \(T\). Therefore, in parallel for all bag \(w \in L_0\), the algorithm
runs $2^N$ parallel instances of the algorithm of Lemma \ref{lem:ntt} one for each $x \in \{0, \ldots, 2^N\}$, and for each one that is accepted, the algorithm writes $S^w_x = 1$. Once every parallel verification finishes, the algorithm sets $\text{out}_w = 1$. We now detail the algorithm on the $i$-th iteration, assuming that we have computed $S^w$ for all bag $w \in L_{i-1}$.

Let $w$ be a vertex in $L_i$ and let us call $w_L$ and $w_R$ the children of $w$, which belong to $L_{i-1}$. Roughly, as we know the partial solutions restricted to the subtrees rooted at $w_1$ and $w_2$, the algorithm will try to extend it to a partial solution of $w$ according to the gluing procedure given by Lemma \ref{lem:glue} testing all possible combinations. More precisely, we initialize a set $\{L_i\}$ processors $\{P^w\}_{w \in L_i}$, one assigned each bag in $L_i$. Each processor $P^w$ verifies if $\text{out}_{w_L} = \text{out}_{w_R} = 1$, or stops and writes $\text{out}_w = 0$. Otherwise, processor $P^w$ initializes a set of $2^N$ processors, that we call $\{P^w\}_{z \in \{1, \ldots, 2^N\}}$, and reserves $2^N$ bits of memory $S^w \in \{0, 1\}^{2^N}$.

For each $z \in 2^N$, processor $P^w_z$ verifies if $z$ is a succinct representation of a partially-valid trace of $X_w$ using Lemma \ref{lem:succinct}. If its not the case then $P^w_z$ stops and writes a $0$ in $S^w_z$. Otherwise, processor $P^w_z$ initializes $(2^N)^2$ processors $\{P^w_{z,z_R,z_L} : z_R, z_L \in \{1, \ldots, 2^N\}\}$ and reserves $2^N \times 2^N$ bits of memory $(S^w_z) \in \{0, 1\}^{2^N \times 2^N}$.

If $S^w_{w_L} = 0$ or $S^w_{w_R} = 0$ the processor $P^w_{z,z_R,z_L}$ stops and writes a $0$ in $S^w_{z,z_R,z_L}$. Otherwise, the processor $P^w_{z,z_R,z_L}$ interprets $z, z_R$ and $z_L$ as $\beta^{w_L}(z), \epsilon(\beta^{w_L})$ and $\epsilon(\beta^{w_L})$, for partially-valid traces $\beta^w$ and $\beta^{w_R}$ of $X_w, X_{w_L}$ and $X_{w_R}$, respectively. Which means that $\beta^w$ belongs to $PTV(X_w)$ and $Sol_w(\beta^{w_R}) = Sol_w(\beta^{w_L}) = \text{accept}$. Therefore $\beta^w$ is a partially-valid trace of $X_w$ and $\beta^{w_L}$ and $\beta^{w_R}$ verify the condition (2) of Lemma \ref{lem:glue} Up to this point, all verifications can be done in time $O(N) = O(|Q(\Delta(\text{tw}(G))+2+k)\log n)$ because we are just looking at the given tables.

Then, the processor $P^w_{z,z_R,z_L}$ computes sets $Y_L = N[X_w] \cap N[X_{w_L}]$ and using the algorithm of Lemma \ref{lem:ntt} computes $\epsilon(\beta^w|Y_L)$ and $\epsilon(\beta^{w_L}|Y_L)$. If $\epsilon(\beta^w|Y_L) = \epsilon(\beta^{w_L}|Y_L)$ the processor deduces that $\beta^w(u) = \beta^{w_L}(u)$, for all $u \in N[X_w] \cap N[X_{w_L}]$. Then $P^w_{z,z_R,z_L}$ computes sets $Y_R = N[X_w] \cap N[X_{w_R}]$ and using the algorithm of Lemma \ref{lem:ntt} computes $\epsilon(\beta^w|Y_R)$ and $\epsilon(\beta^{w_R}|Y_R)$. Then, if $\epsilon(\beta^w|Y_R) = \epsilon(\beta^{w_R}|Y_R)$ the processor deduces that $\beta^w(u) = \beta^{w_R}(u)$, for all $u \in N[X_w] \cap N[X_{w_R}]$. If both verifications are satisfied, processor $P^w_{z,z_R,z_L}$ stops and writes a $1$ in $S^w_{z,z_R,z_L}$. Otherwise, the processor $P^w_{z,z_R,z_L}$ stops and writes a $0$ in $S^w_{z,z_R,z_L}$. All of these verifications can be executed by $P^w_{z,z_R,z_L}$ in time $O(N)$.

Once that all processors in $\{P^w_{z,z_R,z_L} : z_R, z_L \in \{1, \ldots, 2^N\}\}$ finished, processor $P^w$ runs a prefix-sum algorithm in $S^w$, simply summing the elements of the vector to verify if some instance was accepted. If the result is different than 0, processor $P^w$ writes a $1$ in $S^w$, and writes a $0$ otherwise. When every processor $\{P^w\}_{z \in \{1, \ldots, 2^N\}}$ finishes, we obtain that $S^w$ is the table representing function $Sol_w$. Then processor $P^w$ runs a prefix-sum algorithm on $S^w$ to verify that there exists a partial solution for bag $X_w$. If the result of the prefix sum equals zero, processor $P^w$ stops and writes a $\text{out}_w = 0$. Otherwise, it writes $\text{out}_w = 1$ and stops.

After all processors $\{P^w\}_{w \in L_i}$ have finished, the algorithm continues with the next level. When the last level is reached, before halting processor $P^r$ decides if $\text{out}_w = 1$ for all $w \in W$ using a prefix-sum algorithm. If the answer is affirmative the algorithm accepts the input, and otherwise rejects. On each level, the algorithm takes time $O(\Delta|Q(\Delta(\text{tw}(G))+2+k)\log n)$ and uses $O(|Q(\Delta(\text{tw}(G))+2+k)|)$ processors. Proposition\ref{prop:ntt} provides a construction of a binary-tree-decomposition $T$ of depth $O(\log n)$. This means that $M = O(\log n)$, and implies that the whole takes time $O(\Delta|Q(\Delta(\text{tw}(G))+2+k)\log^2 n) = O(\log^2 n)$ and $n^O(|Q(\Delta(\text{tw}(G))+2+k)|) = n^O(1)$ processors. The correctness of the algorithm is given by Lemmas \ref{lem:ntt}, \ref{lem:glue}, \ref{lem:ntt} and \ref{lem:ntt}.
\{0,1\}^r \to \{0,1\}, a collection of sub sets \(S_1, \ldots, S_r\) such that \(|S_i| = O(k)\) and a sequence of 
\((S_i, Q, t, l)\)-specifications \(E_i\) for \(i = 1, \ldots, r\) such that \(A\) satisfies \(\phi\) if and only if \(A\) satisfies

the boolean combination \(\psi\) of every satisfiability value in each specification \(E_1, \ldots, E_r\). Moreover,

there is an CREW PRAM algorithm that given \(\phi\) computes the sets \(S_1, \ldots, S_r\) and \(E_i\) using

\(n^{O(1)}\) processors and time \(O(\log n)\).

**Proof.** Without loose of generality, we can write \(\phi = Q \phi'\) where \(Q\) is a sequence of quantifiers and \(\phi'\) is a boolean formula without quantifiers. There is a polynomial number \(r\) of

assignment \(i : (x_1, \ldots, x_k) \in X \to (v_1, \ldots, v_k) \in V\) for \(\phi\) since the amount of quantifiers is \(k\). We enumerate all the possible choices for \(i\) as \(\{i_1, \ldots, i_r\}\). For each such assignment \(i\), we define \(E_i\) as the \((S_i, Q, t)\)-specification made of all traces that satisfy \(\phi'\) for vertices in \(S_i = i(X)\). Then it is sufficient to take the boolean combination of depth \(k\) with \(r\) inputs

indexed by the choice of \(i\) and defined according to the sequence of quantifiers of \(\phi\) in the

following way: to each \(\exists\) on variable \(x_j\) we associate a \(\lor\) operation that tests all possible

choices for \(i(x_j)\) and to each \(\forall\) we associate an \(\land\) operation in the same way. For instance,

if \(\phi = \exists x_1 \forall x_2 \phi'\), we get the formula

\[
\psi(y_1, \ldots, y_r) = \bigvee_{v_1 \in V} \bigwedge_{v_2 \in V} y_i \text{ with } i \text{ such that } i_1(x_1) = v_1 \text{ and } i_2(x_2) = v_2.
\]

Additionally we identify for each \(1 \leq i \leq r\) the specification \(E_i\), with a \(\{0,1\}\) variable that takes the value 0 if the specification \(E_i\) is empty and 1 in the complementary case. It is clear

that \(A\) satisfies \(\phi\) if and only if \(\psi(E_1, \ldots, E_r) = 1\). We conclude that the previous lemma

holds.

Now we explain how to compute the corresponding \((S_i, Q, t)\)-specifications \(E_i\), for each

\(i \in \{1, \ldots, r\}\). Each set \(S_i\) is represented by a constant length list of vertices. Then, we use

Lemma [24] in order to compute in parallel \(PVT(S_i)\). Note that we thus obtain for every

\(\beta \in PVT(S_i)\) a succinct representation \(\epsilon(\beta)\). Now we use the fact that \(\text{Boolean formula value problem}\) is decidable in log-space (see for example [8] in which they show that the problem

is even decidable in \(\text{ALOGLTIME}\)). We initialize for each \(\beta \in PVT(S_i)\) a processor \(P^S_{\beta}\) that decides in parallel logarithmic time if \(\beta\) satisfies \(\phi'\) and writes a bit in memory cell

\(M^{\beta}_{\epsilon(\beta)}\). Let \(\epsilon_i(\phi')\) be a boolean vector of length \(2^{N_i}\) where \(N_i = \#(S_i, t)\) and is such that

\(\epsilon_i(\phi')(\ell(\beta)) = M^{\beta}_{\epsilon(\beta)}\) where \(\ell(\beta) \in \{0, \ldots, 2^{N_i} - 1\}\) represents \(\beta\). Because of the latter checks computed by the algorithm we have that, for every \(1 \leq i \leq r\), the vector \(\epsilon_i(\phi)\) is the succinct

representation of a certain \((S_i, Q, t)\)-specification that we call \(E_i\) representing all the possible

valid traces of \(S_i\) satisfying \(\phi'\). Summarizing the time resources used in order to perform these tasks, latter algorithm runs in a CREW PRAM in time \(O(\log)\) and \(n^{O(1)}\) processors and thus, the result holds.

**Theorem [19].** Restricted to graphs of bounded treewidth, we have that

General model checking problem \(\leq^{NC^2}_{T}\text{ Specification Checking problem}\)

**Proof.** First, note that a CREW PRAM algorithm with \(n^{O(1)}\) processors and time \(O(\log^k n)\)

can be decided in \(\text{NC}^{k+1}\) (see [88] for more details). By Lemma [18] there exists a boolean combination \(\psi = \psi(\phi)\) and set of specifications \(E_1, \ldots, E_r\) with \(r = n^{\Omega(k)}\) such that, \(A\) satisfies \(\phi\) if and only if the evaluation by \(\psi\) of the values of satisfiability of each specification \(E_i\) for

\(i = 1, \ldots, r\) is 1. More over it provides a CREW algorithm that computes the sets \(S_i\) and the specifications \(E_i\) for \(i = 1, \ldots, r\) in \(O(\log n)\). Thus, by the last observation regarding the class \(\text{NC}\) and the PRAM model, this task can be computed by a uniform family of circuits \(\alpha'_n\) such that \(\text{deph}(\alpha'_n) = O(\log^2 n)\) and \(\text{size}(\alpha'_n) = n^{O(1)}\). Using these latter observations,
we claim that there exist a Specification Checking problem-oracle uniform circuit \( \{ \alpha_n \}_{n \geq 1} \) family deciding General model checking problem such that \( \text{deph}(\alpha_n) = O(\log^2 n) \). In fact, for fixed instance of General model checking problem and corresponding \( n \), \( \alpha_n \) computes first computes the output of \( \alpha' \) in its first \( \text{deph}(\alpha' = O(\log n) \) layers. Then, we define a layer of \( r \) oracle gates computing solutions for Specification Checking problem for the subinstances given by specifications \((T, \mathcal{E}_i)\). Subsequently, a third part of the circuit is given by \( O(\log n) \) layers that compute \( \psi \) for the outputs of the oracle gates. Finally, given an instance \( z \in \{0,1\}^n \) of General model checking problem, by Lemma 5.3 we have that \( \alpha_n(z) = 1 \) if and only if \( z \) is a yes-instance of General model checking problem. We conclude that \( \{ \alpha_n \}_{n \geq 1} \) is a Specification Checking problem-oracle uniform circuit \( \{ \alpha_n \}_{n \geq 1} \) family with \( \text{deph}(\alpha_n) = O(\log^2 n) \) and \( \text{size}(\alpha_n) = n^{O(1)} \) that decides General model checking problem. Therefore, the \( \text{NC}^2 \)-Turing reduction holds.

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Hardness results for polynomial treewidth networks

**Lemma** [22] [Subgraph routing lemma] [22] For any family \( \mathcal{G} \) of graphs with polynomial treewidth, there is a polynomial map \( p \) and a deterministic polynomial time algorithm that, given any graph \( G = (V, E) \in \mathcal{G} \) and any digraph \( D = (V', E') \) of maximum (in/out) degree \( \Delta \) and size at most \( p(|V|) \), outputs:

- a mapping \( \mu : V' \rightarrow V \) such that, for each \( v \in V \), \( \mu^{-1}(v) \) contains at most two elements,
- a collection \( \mathcal{C} = \{ e \circ e_{C} \in E' \} \) of paths connecting \( \mu(e_{1}) \) to \( \mu(e'_{2}) \) for each \( e_{1}, e'_{2} \in E' \), and such that any node in \( V \) belongs to at most 4\( \Delta \) paths from \( \mathcal{C} \).

**Proof.** By [35] Theorem 5.3] there exists a polynomial map \( p_{1} \) and a polynomial time algorithm that given a graph \( G = (V, E) \in \mathcal{G} \) finds a perfect bramble \( \mathcal{B} = (B_{1}, \ldots, B_{k}) \) with \( k \geq p_{1}(p_{2}(|V|)) \), i.e. a list of connected subgraphs \( B_{i} \subseteq V \) such that:

1. \( B_{i} \cap B_{j} \neq \emptyset \) for all \( i \) and \( j \),
2. for all \( v \in V \) there are at most two elements of \( \mathcal{B} \) that contain \( v \).

We set the polynomial map of the lemma to be \( p = p_{1} \circ p_{2} \) and consider any digraph \( D = (V', E') \) of maximum (in/out) degree \( \Delta \) and size at most \( p(|V|) \). We suppose \( k = |V'| \) (by forgetting some elements of \( \mathcal{B} \)) and reindex the element of \( \mathcal{B} \) by \( V' \). The map \( \mu : V' \rightarrow V \) is constructed by picking some element \( \mu(v') \in B_{i} \) for all \( v' \in V' \). The fact that any vertex \( v \in V \) is contained in at most two elements of the bramble \( \mathcal{B} \) ensures the first condition of the lemma on \( \mu \). Now, for each \( (v'_{1}, v'_{2}) \in E' \) we define a path from \( \mu(v'_{1}) \) to \( \mu(v'_{2}) \) as follows: let \( v \in B_{i} \cap B_{j} \) (first property of perfect brambles) then choose a path from \( \mu(v'_{1}) \) to \( v \) inside \( B_{i} \) (which is connected) followed by a path from \( v \) to \( \mu(v'_{2}) \) inside \( B_{j} \). The collection of paths \( \mathcal{C} \) thus defined is such that there are at most 2\( \Delta \) paths that start or end in \( \mu(v') \) for any \( v' \in E \). Moreover, for any \( v \in V \), there are at most two elements of \( \mathcal{B} \) that contain \( v \), let’s say \( B_{i} \) and \( B_{j} \) Then the only paths from \( \mathcal{C} \) that can go through \( v \) are those starting or ending at either \( \mu(v'_{1}) \) or \( \mu(v'_{2}) \), so they are at most 4\( \Delta \) in total.

**Theorem** [23] For any family \( \mathcal{G} \) of constructible graphs of polynomial treewidth, the problem nilpotency is coNP-complete.

**Proof.** First, by Lemma 5.3 a freezing automata networks with \( n \) nodes is nilpotent if and only if \( F^{\lambda(n)}(n) \) is constant where \( \lambda(n) = O(n) \) is the concrete computable bound from the lemma. The nilpotency problem is therefore clearly coNP.

We now describe a reduction from problem SAT. Given a formula with \( n \) variables seen as a Boolean circuit of maximum input/output degree 2 (of size polynomial in \( n \)), we first construct \( G = (V, E) \in \mathcal{G} \) such that the DAG \( G' = (V', E') \) associated to the circuit is of
size at most $p(|V|)$ where $p$ is the polynomial map of Lemma 22. Then, using Lemma 22 we have a map $\mu : V' \to V$ and a collection $C$ of paths in $G$ that represent an embedding of $G'$ inside $G$. The lemma gives a bound 8 on the number of paths visiting a given node $v \in V$. Then each node will hold 8 Boolean values, each one corresponding either a node $v' \in V'$ of the Boolean circuit or an intermediate node of a path from the collection $C$. The alphabet is then $Q = \{0, 1\}^8 \cup \{\bot\}$ where $\bot$ is a special error state. In any configuration $c \in Q^V$, a node can be either in error state $\bot$, or it holds 8 Boolean components. We then construct the local rule at each node $v \in V$ that give a precise fixed role to each such component: it either represent a node $v' \in V'$ such that $\mu(v') = v$, or an intermediate node in one of the paths from $C$, or is unused (because not all vertices of $V$ have 8 paths from $C$ visiting them).

The local rule at $v \in V$ is as follows:

- if in state $\bot$ or if some neighbors is in state $\bot$, it stays in or changes to $\bot$;
- it then make the following checks and let the state unchanged if they all succeed or changes to $\bot$ if at least one test fails:
  1. check for any component corresponding to a node $v' \in V'$ that it holds the Boolean value $g(x, y)$ where $g$ is the Boolean gate associated to $v'$ in the the circuit and $x$ and $y$ are the Boolean values of the components corresponding to the vertex just before $v$ in the two paths $\rho_{e_1}$ and $\rho_{e_2}$ in $C$ that arrive at $\mu(v') = v$. In the case where $g$ is a 'not' gate, there is only one input and in the case where $v'$ is an input of the circuit, there is no input and nothing is checked;
  2. moreover, if the gate corresponding to $v'$ is the output gate of the circuit, check that its Boolean value is 1;
  3. check for any component corresponding to an intermediate node in some path from $C$ that the Boolean value it holds is the same as that of the component corresponding to the predecessor in the path.

We claim that $F$ is not nilpotent if and only if the formula represented by the Boolean circuit is satisfiable. Indeed the configuration everywhere equal to $\bot$ is always a fixed point. It should be clear that if the formula is satisfiable then one can build a configuration corresponding to a valid computation of the circuit on a valid input which is a fixed point not containing state $\bot$. In this case we have two distinct fixed points and the automata network is not nilpotent. Conversely, suppose the the automata network is not nilpotent. Then it must possess a fixed point $c$ distinct from the all $\bot$ one. Indeed, all configurations of $X = F^t(Q^V)$ are fixed points for $t$ large enough (by the freezing condition) and if $F^t$ is not a constant map then $X$ must contain at least two elements. Moreover, the fixed point $c$ do not contain state $\bot$, because otherwise it would contain a state from $Q \setminus \{\bot\}$ at some node which has a neighbor in state $\bot$, which would contradict the fact that it is a fixed point according to the local rule. Then $c$ is a configuration where all checks made by the local rules are correct: said differently, $c$ contains the simulation of a valid computation of the Boolean circuit that outputs 1. Therefore the Boolean formula is satisfiable and the reduction follows.

**Theorem 24** There exists a map $\rho : Q \times 2^Q \to Q$ such that for any family $G$ of constructible graphs of polynomial treewidth and bounded degree, the problems predecessor and asynchronous reachability are both NP-complete when restricted to $G$ and automata networks set-defined by $\rho$.

**Proof.** These problems are clearly NP. For clarity of exposition we will construct a distinct map $\rho$ for each of the two problems. Then, by taking the disjoint union of the alphabets and merging the two rules with the additional condition that any node that sees both alphabets
We then build configuration \( c \) with 
\[
\begin{align*}
\text{To implement the routing of information along paths of } C \text{ variables seen as a Boolean circuit of maximum input/output degree } 2 \text{ (of size polynomial in } n)\text{, we first construct } G = (V,E) \in \mathcal{G} \text{ such that the DAG } G' = (V',E') \text{ associated to the circuit is of size at most } p(|V|) \text{ where } p \text{ is the polynomial map of Lemma 22. Then, using Lemma 22 we have a map } \mu : V' \to V \text{ and a collection } \mathcal{C} \text{ of paths in } G \text{ that represent an embedding of } G' \text{ inside } G. \text{ The lemma gives a bound } 8 \text{ on the number of paths visiting a given node } v \in V. \text{ Let’s compute a vertex coloring } \chi : V \to \{1, \ldots, k\} \text{ of the square of } G \text{ with } k = \deg(G)^2 + 1 \text{ colors, i.e. a vertex coloring of } G \text{ such that no pair of neighbors of a given node has the same color (this can be done in polynomial time by a greedy algorithm). To implement the routing of information along paths of } \mathcal{C} \text{ and the circuit simulation by } \rho_1, \text{ the alphabet } Q_1 \text{ holds } 8k \text{ state components, and we will use configurations where each node } v \in V \text{ uses only components } 8\chi(v) \text{ to } 8\chi(v) + 7. \text{ These components can be seen as communication channels. Indeed, in such configurations, a node can distinguish the information going through up to } 8 \text{ distinct paths coming from each neighbor individually just by looking at the set of states present in the neighborhood (because no pair of neighbors can use the same channel). Apart from the routing of information through paths, the rule } \rho_1 \text{ implements each gate } v' \in V' \text{ of the Boolean circuits inside node } \mu(v') \text{ of } G. \text{ We think of paths from } \mathcal{C} \text{ as being part of the circuit with nodes that implement the identity map. For that purpose each state component in a node is associated to a descriptor that gives the type of gate to implement (input, identity, not, or, and, output) and the component numbers corresponding to input(s) of the gate (gates of type 'input' have no input). Formally, a state component is given by } S_1 = \{0, 1, \text{ok}, \text{off}\} \text{ where 0 and 1 are Boolean values, } \text{off} \text{ means that the component is unused and } \text{ok} \text{ is a special transitory state used to check correctness of computations (see below). A descriptor component is given by } D_1, \text{ a finite set used to code any possible combination of gate type and input component numbers } (|D_1| = 6(8k)^2 \text{ is enough). Then the state set of } \rho_1 \text{ is } Q_1 = (S_1 \times D_1)^{8k}. \text{ In a given configuration, we say that a given node } v \in V \text{ reads value } x \in \{0, 1\} \text{ on channel } i \text{ if there is a unique state in the neighborhood with a state component } i \text{ which is not } \text{off}, \text{ and if this state component contains value } x. \text{ In any other case, the value read on channel } i \text{ is undefined. The rule } \rho_1 \text{ does the following:}
\begin{itemize}
\item the } D_1 \text{ component are never changed;}
\item state components in } \text{off} \text{ stay unchanged;}
\item any state component in } \text{ok} \text{ becomes } \text{off};
\item any state component in state } x \in \{0, 1\} \text{ checks that } x \text{ is the correct output value of its gate type applied to the values read on the input channels given by its corresponding descriptor (in particular these input values must be defined). If it is the case, it becomes } \text{ok}, \text{ otherwise } \text{off}. \text{ The only exception to this rule is the case of the gate of type “output” where we only change state to } \text{ok} \text{ if } x = 1 \text{ and the computation check is correct, and change to } \text{off} \text{ in any other case.}
\end{itemize}
We then build configuration } c \in Q_1^\mathcal{C} \text{ for the predecessor problem as follows:}
\begin{itemize}
\item input component numbers and gate types in } D \text{ components are set according to the Boolean circuit and the path collection } \mathcal{C};
all unused state components are marked as off; 
all used state components are marked ok.

We claim that \( c \) has a predecessor in one step (i.e. \( F_{\rho_1}(y) = c \) for some \( y \in Q_1^V \)) if and only if the SAT formula represented by the Boolean circuit is satisfiable. Indeed, the only possible predecessor configurations of \( c \) are such that all used state component hold a Boolean value equal to the output value of the gate they code applied to their corresponding input Boolean values, and that the output gate holds value 1.

We now describe \( \rho_2 : Q_2 \times 2^{Q_2} \rightarrow Q_2 \) that set defines automata networks which have a NP-complete asynchronous reachability problem when restricted to \( G \). The construction is almost identical to \( \rho_1 \) and the reduction is again from SAT problems, but with the following modifications:

- the state component is now \( S_2 = \{ ?, 0,1, ok, off \} \) where the new state \( ? \) represents a pre-update standby state; in each state component, the possible state sequences are subsequences of either \( ? \rightarrow \{ 0,1 \} \rightarrow ok \rightarrow off \) or \( ? \rightarrow \{ 0,1 \} \rightarrow off \);
- to each input gate of the Boolean circuit is attached a pre-input gate that serve as non-deterministic choice for input gates using asynchronous updates; the set \( D_2 \) is a modification of \( D_1 \) taking into account this new type of gates; the alphabet is then \( Q_2 = (S_2 \times D_2)^{8k} \);
- the behavior of each state component depending on its type is as follows:
  - pre-input components become \( ok \) if previously in state \( ? \) and \( off \) in any other case;
  - input components in state \( ? \) become either 0 or 1 depending on whether there corresponding pre-input component is in state \( ? \) or not;
  - any other state component in state \( ? \) become \( x \in \{ 0,1 \} \) the output value of its gate type applied to the values read on the input channels given by its corresponding descriptor (in particular these input values must be defined). If in a state from \( \{ 0,1 \} \), it becomes \( ok \), and in any other case it becomes \( off \). The only exception to this rule is the case of gates of type “output” where we only change state to \( ok \) if the current value is 1, and change to \( off \) if the current value is 0.

When then define source configuration \( c_0 \) and destination configuration \( c_1 \) for the asynchronous reachability problem as follows. They both use the same circuit embedding like in \( c \) above but with a pre-input attached to each input. In \( c_0 \) all unused components are in state \( off \) and all used state components (including pre-inputs) are in state \( ? \). In \( c_1 \) all unused components are in state \( off \) and all used state components are in state \( ok \). It should be clear that \( c_1 \) can be reached from \( c_0 \) if the formula associated to the Boolean circuit is satisfiable since either 0 or 1 can be produced at each input depending on whether the associated pre-input is update before the input update or not. Suppose now that \( c_1 \) can be reached from \( c_0 \) with some asynchronous update. First, all used state components except pre-inputs must follow either the sequence \( ? \rightarrow 0 \rightarrow ok \) or \( ? \rightarrow 1 \rightarrow ok \). Therefore we can associate to each such component a unique Boolean value (0 or 1 respectively) and the rule \( \rho_2 \) ensures that the Boolean value of each such component is the output value of its corresponding circuit gate applied on the Boolean value of its corresponding inputs. Moreover the output gate must have Boolean value 1 so we deduce that the simulated circuit outputs 1 on the particular choice of Boolean values of inputs. The reduction from SAT follows.

\[\triangleright\] Problem 36 (Routed prediction problem).

**Parameters:** alphabet \( Q \), family of graphs \( G \) of max degree \( \Delta \).
Input:
1. a deterministic freezing automata network \( \mathcal{A} = (G, F) \) on alphabet \( Q \), with set of
   nodes \( V \) with \( n = |V| \) and \( G \in G \);
2. an initial configuration \( c \in Q^V \)
3. a node \( v \in V \) and a \( ((v), Q, l) \)-specification \( S_v \) of length \( l \in \mathbb{N} \)
4. A perfect bramble \( B = (B_1, \ldots, B_p) \) in with \( p = n^{\Theta(1)} \) in \( G \)

Question: does the orbit of \( c \) restricted to \( v \) satisfies specification \( S_v \)?

**Proposition 29.** [30] Theorem 5.3] Let \( n \in \mathbb{N} \). The following problem can be solved in
space \( O(|\log n|) \): given an undirected graph \( G = (V, E) \) with \( |V| = n \), \( s, t \in V \) find a path
from \( s \) to \( t \) and if there exists such a path, return the path as an output.

**Proposition 30.** [19] Theorem 3] Let \( \Delta \in \mathbb{N} \). The following problem can be solved in
time \( O(\Delta \log(\Delta + \log^* n)) \) by an EREW PRAM: given a graph \( G = (V, E) \) such that
\( \Delta(G) \leq \Delta \) finding a \( \Delta + 1 \) coloring of \( G \).

**Theorem 25.** There exists a map \( \rho : Q \times 2^Q \to Q \) such that routed prediction problem
is \( \mathbb{P} \)-complete restricted to any family \( G \) of log-constructible routed collection of graphs of
polynomial treewidth.

**Proof.** We start by observing that prediction problem is in \( \mathbb{P} \). We also recall that in order
to show the \( \mathbb{P} \)-hardness of prediction problem, it suffices to show that there exist a \( \mathbb{N} \)
reduction for the alternating monotone 2 fan-in 2 fan-out circuit value problem (AM2CVP),
more precisely \( \text{AM2CVP} \leq_{\mathbb{N}C^2} \text{PRED}_G \) (see [24] Theorem 4.2.2 and Lemma 6.1.2). Let
\( n, l \in \mathbb{N} \), \( C : \{0, 1\}^n \to \{0, 1\} \) a monotone alternating 2 fan in 2 fan out circuit, \( x \in \{0, 1\}^n \)
and \( o \in \{0, \ldots, l - 1\} \) a fixed output of \( C \). We call \( C' = (V', E') \) to the underlying DAG
defining \( C \) and we fix \( G \in G \) where \( G \) is a log-constructible family of graphs with polynomial
treewidth. We note that, by definition we can compute \( G \) and a perfect bramble of size
\( p = n^{\Theta(1)} \) in log-space and thus we can do the latter computations in \( \mathbb{N}C^2 \). Now, we use
\( B \) and Proposition 29 in order to compute a mapping \( \mu : V' \to V \) and a collection of paths
\( \mathcal{C} \) as in Lemma 22. As we did in Theorem 24 we use Proposition 30 in order to compute a
k-proper coloring \( \chi : V \to \{1, \ldots, k\} \) of \( G^2 \) in \( \mathbb{N}C^2 \) with \( k = \Delta^2 + 1 \) for \( \Delta \in \mathbb{N} \) such that
\( \Delta(G) = \Delta \). From here we construct \( \rho \) analogously as we did for \( \rho_1 \) and \( \rho_2 \) in the proof of
Theorem 24 but observing that now we have only 5 type of gates as the circuit is monotone.
We also consider state component \( S = \{0, 1, \text{wait}, \text{off}\} \). Remember that the descriptor
component assures that there won’t be overlappings of the channels during broadcasting.
We map \( x \) into a configuration \( y \in Q = \{S \times D\}^{3b} \) in the following way:

- The \( D \) component is assigned according to the structure of \( C' \).
- For every input we assign a boolean value given by \( x \).
- For every unused node we assign the state \text{off}.
- For every other node we assign the state \text{wait}.

The rule \( \rho \) is defined in the following way:

- Every node in state \text{off}, 0 or 1 is fixed and does not change its state.
- Every node in state \text{wait} reads the information of its neighbors and do the following
depending on its type of gate:
  - identity will take the value of its input
  - AND will read its inputs: if both inputs are in 1 it will change to 1 and it will change
to 0 if it reads one neighbor in 0. In any other case it will remain in \text{wait}
  - OR will read its inputs: if both inputs are in 0 it will change to 0 and it will change
to 1 if it reads one neighbor in 1. In any other case it will remain in \text{wait}.
In order to show the desired result, it will suffices to show the following simulation property: there exists $t \in \mathbb{N}$, $t = n^{O(1)}$ such that for every output $o \in V'$ we have $C(x)_o = (F^t(y)_{\mu(o)})|_S$ where $y \in Q^V$ is the configuration computed from $x$ as explained above. In fact, if we have the latter property, for some fixed output $o$, we define $v = \mu(o)$ and $S_v$ be a $\langle \{v\}, Q, t \rangle$-Specification such that $S_v = \{z \in \{0, 1\} : z \neq y_v$ and $(F^t(y)^t|_v)|_S = z\}$ and then we can answer if the orbit of $y$ in time $t$ given by $F^t(y)$ satisfies $S_v$ if and only if we can answer if $C(x)_o = 1$ (and thus $AM2CVP \leq_m NC^2$ $PRED$). We now show that the latter simulation property holds. In order to do that, we inductively check, that eventually, the orbit of $y$ will evaluate every layer of the circuit. We start by the input. Note that in one time step all the information is broadcasted through the different channels and through the paths given by $C$. In a maximum of $L = n^{O(1)}$ time steps (given by the longest path of $C'$) the last signal will arrive to a gate in the first layer. Note that, with the gates described above, signals arriving at different times do not change its output value as gates have a monotone behaviors on states with order $\text{wait} \leq 0 \leq 1$. Iteratively, we have maximum arriving times for signals of $L$ time steps for each layer and then, defining $t = L \times \text{deph}(C) = n^{O(1)}$ and observing that output nodes will will remain constant once they have done a computation (when they change to a boolean value), we get the desire result. Therefore, $AM2CVP \leq_m NC^2$ $PRED$ holds and then, $PRED_{\mathbb{G}}$ is $\mathbb{P}$-complete.