Dynamics of compressible displacement in a capillary tube

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Abstract

We examine the interaction between springlike compression of gas and viscous displacement of liquid by studying the steady compression of an air reservoir connected to a liquid-filled capillary tube. Our experiments and modelling reveal complex displacement dynamics that, for large air reservoirs, depend on a single dimensionless compressibility number. We identify two distinct displacement regimes, separated by a critical value of the compressibility number. The high-compressibility regime exhibits burst-like expulsion with applications to fluid, mechanical, or electrical systems.

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The gas-driven displacement of viscous liquid from a confined geometry occurs in a variety of natural and industrial systems, including subsurface storage of carbon dioxide [1, 2], operation of fuel cells [3], reopening of airways [4, 5], and displacement of subretinal blood during eye surgery [6]. Gas–liquid displacement is one of the simplest classes of two-phase flows because the viscosity of the gas is typically negligible, making these flows especially tractable to theoretical analysis. This same feature means that the gas–liquid interface is inherently viscously unstable, leading to complex interfacial structures and serving as an archetype of pattern formation [7]. Injecting gas also poses an inherent complication: relative to liquids, gases are highly compressible, making it difficult to impose a constant volume flow rate. Compression is often neglected or carefully avoided, for instance by withdrawing liquid rather than injecting gas [8] or by performing analysis based on instantaneous flow rates [9]. Nonetheless, compression is unavoidable in settings that require injection and there is growing interest in the nontrivial impacts of compressibility on displacement processes. For example, recent work has identified non-monotonic variations in pressure or invasion rates during displacement in porous media [3, 10], episodic growth of intricate patterns during displacement of granular suspensions [11, 12], or time-dependent growth of fractures during injection of foams into gels [13], despite the constant nominal injection rate in all cases. The nontrivial rheologies, geometries, and pattern-forming features of these systems mean that compressibility is typically treated semi-empirically or phenomenologically. A fundamental understanding of the coupling between gas compression and viscous displacement is lacking.

Here, we consider a model problem: displacement of viscous liquid from a capillary tube by compressing a connected reservoir of air of initial volume $V_i$ at a fixed nominal rate $Q$ [Fig. 1(a)]. Despite the simplicity of this system, we observe strong qualitative and quantitative variations in displacement dynamics depending on $V_i$ and $Q$, the first hint of which is illustrated in Fig. 1(b). When $V_i$ is small, the interface reaches a steady velocity after an initial transient [Fig. 1(b), top]; when $V_i$ is large, the interface accelerates rapidly toward the outlet [Fig. 1(b), bottom]. In this one-dimensional setting, these nontrivial dynamics can be understood in terms of a simple competition between pressurisation due to compression and depressurisation due to viscous flow. We exploit this simplicity to explain the behaviours summarised in Fig. 1(b) in the context of a more general framework for compressible displacements.

Our experimental flow cell comprised a glass capillary tube of length 10 cm and inner...
FIG. 1. Displacement of silicone oil from a capillary tube by the injection of air from a reservoir of initial volume $V_i$ that is compressed at a constant volume rate $Q$ using a syringe pump. (a) Experimental set-up. We measure the displacement of the interface $l(t)$ relative to its initial position ($l(0) = 0$) and the gauge pressure $p_g$ of the gas. The liquid slug has initial length $L$. (b) Experimental timelapses of overlaid frames at equal timesteps $\Delta t = 2.05 \pm 0.02$ s show the motion of the interface; interframe spacing of interfaces is inversely proportional to interface velocity. Top and bottom are for $V_i = 4$ mL and 32 mL, respectively, and $Q = 0.2$ mL/min [$\dot{C} = 0.13$ and 1.0; see also Fig. 2].

radius $R = 0.66 \pm 0.01$ mm [Fig. 1(a)]. One end of the tube was connected to a sealed air reservoir of initial volume $V_i = [4, 8, 16, 32] \pm 0.1$ mL; the other end fed into a bath of silicone oil (viscosity $\mu = 0.096$ Pa s, surface tension $\gamma = 21$ mN m$^{-1}$, and density $\rho = 960$ kg m$^{-3}$ at laboratory temperature $T_{lab} = 22 \pm 1$ ºC; Dow Corning) at fixed hydrostatic pressure $p_{HS}$. Before each experiment, oil was drawn from the bath into the tube, filling an initial length $L = 56 \pm 1$ mm. Experiments were initiated by compressing the air reservoir at a fixed rate $Q = [0.05, 0.1, 0.2, 0.4, 0.8, 1.6]$ mL/min, thus injecting air into the tube and expelling oil into the bath. We used imaging and image processing to measure the motion of the interface, the thickness of thin residual oil films on the tube walls, and the radius of curvature $b$ of the interface. We also measured the gauge pressure $p_g$ of the gas relative to atmospheric pressure $p_{atm} = 101$ kPa, and thus calculated the viscous pressure drop $\Delta p = p_g - 2\gamma/b - p_{HS}$ across the liquid slug. See appendix for more details.

For reference, we first consider an incompressible displacement at a rate $Q$ [dashed lines in Fig. 2]. Neglecting thin films, the interface must advance linearly with time $t$, such that its displacement relative to its initial position is $l = Qt/(\pi R^2)$ [Fig. 2(a)] and its velocity $u = dl/dt = Q/(\pi R^2)$ is constant [Fig. 2(b)]. The experiment ends at breakout time $t_{bo} = \pi R^2 L/Q$, when the interface reaches the outlet (i.e., $l(t_{bo}) = L$). Assuming laminar Hagen-Poiseuille flow (see below), the pressure gradient in the liquid $\Delta p/(L - l) = 8\mu Q/(\pi R^4)$ is constant and the pressure drop $\Delta p$ decreases linearly from its maximum value...
FIG. 2. (a-c) Experimental results (symbols) and numerical solutions to the full model (solid lines; see appendix) for $Q = 0.2$ mL/min and $V_i = 4, 8, 16$ and $32$ mL (i.e., $\hat{C} = 0.13, 0.25, 0.50$ and $1.0$; dark to light). (a) Normalised displacement $\hat{l}$ of the interface as a function of normalised time $\hat{t}$. (b-c) Normalised velocity $\hat{u}$ and normalised pressure drop $\Delta \hat{p}$ as functions of $\hat{l}$. Dashed lines show corresponding incompressible behavior for reference. Arrows indicate increasing $V_i$ and $\hat{C}$.

of $8\mu Q L/(\pi R^4)$ (reached instantaneously after flow starts at $t = 0$) to its breakout value of $\Delta p_{bo} = 0$ [Fig. 2(c)]. We normalise our results below by these incompressible reference values, such that $\hat{l} = l/L$, $\hat{t} = t/(\pi R^2 L/Q)$, $\Delta \hat{p} = \Delta p/(8\mu Q L/(\pi R^4))$ and $\hat{u} = u/(Q/(\pi R^2))$. Hatted quantities below are dimensionless.

The dynamics in our experiments differ significantly from those of an incompressible system [Fig. 2]. In particular, we do not observe a steady interface velocity or pressure gradient. The displacement $l$ evolves nonlinearly with time [Fig. 2(a)], reflecting a monotonically increasing velocity $u$ [Fig. 2(b)]. The pressure drop $\Delta p$ evolves non-monotonically, increasing to a maximum before decreasing as the interface approaches the outlet [Fig. 2(c)]. Ref. [14] made similar observations, but across a more limited range of parameters.

We identify two distinct displacement regimes, as hinted in Fig. 2(b), depending on whether or not $\Delta p$ returns to zero. When $\Delta p$ vanishes at breakout, $\Delta p_{bo} \approx 0$, the near-breakout velocity $u_{nb}$ is similar to the nominal velocity (i.e., $\hat{u}_{nb} \sim 1$). If $\Delta p_{bo} > 0$, however, the near-breakout velocity is substantially larger ($\hat{u}_{nb} \gtrsim 10$). Note that we measure $u_{nb}$ one tube diameter from the outlet to avoid very large values in the latter case (see below). The link between breakout pressure and velocity is straightforward. The velocity is proportional to the pressure gradient $\Delta p/(L - l)$, and the quantity $L - l$ vanishes at breakout by definition. If $\Delta p$ vanishes at the same rate, the pressure gradient and therefore also the velocity remain finite at breakout. If $\Delta p$ is nonzero as $L - l$ vanishes (i.e., overpressure), the pressure
gradient and the velocity diverge at breakout. Below, we explore these two regimes in detail and identify the control parameter that selects between them.

These nontrivial dynamics can be understood in terms of the coupling between pressure build-up due to gas compression and pressure dissipation due to liquid drainage, as is most apparent in the non-monotonic evolution of $\Delta p$ [Fig. 2(c)]. We next introduce a simple model that captures this coupling. We model the air as a fixed mass of ideal gas, assumed to be isothermal. We take the liquid pressure at the outlet to be atmospheric (neglecting $p_{HS}$ since $p_{HS} \ll p_{atm}$), so the initial absolute gas pressure is $p_{atm} + 2\gamma/R$. The gauge pressure of the air is then

$$p_g(t) = (p_{atm} + 2\gamma/R) \frac{V_i}{V(t)} - p_{atm},$$  \hspace{1cm} (1)$$

where $V(t)$ is the current volume of air and $V(0) = V_i$. The syringe pump acts to decrease $V$ at a steady rate $Q$, while the motion of the interface acts to increase $V$ at a rate $\pi R^2 (dl/dt)$; hence,

$$V(t) = V_i - Qt + \pi R^2 l(t).$$  \hspace{1cm} (2)$$

We model the flow of liquid as Hagen-Poiseuille flow. In the absence of thin films, the interface velocity must be equal to the mean liquid velocity, such that

$$\frac{dl}{dt} = \frac{R^2}{8\mu} \left( \frac{\Delta p}{L-l} \right),$$

where the viscous pressure drop along the liquid slug is $\Delta p = p_g - 2\gamma/R$. Substituting Eqs. 1 and 2 into Eq. 3 and introducing $\hat{l}$ and $\hat{t}$ from above yields

$$\frac{d\hat{l}}{d\hat{t}} = \left[ \frac{\hat{p}_0 + 2/\hat{Ca}}{\hat{V}_i + (\hat{t} - \hat{t})} \right] \left( \frac{\hat{t} - \hat{l}}{1 - \hat{t}} \right).$$

Equation 4 is a nonlinear ordinary differential equation containing three independent nondimensional parameters: $\hat{p}_0 = \pi R^4 p_{atm}/(8\mu Q L)$, comparing the compression and viscous pressure scales; the capillary number $\hat{Ca} = 8\mu Q L/(\pi R^3 \gamma)$, comparing the viscous and capillary pressure scales; and $\hat{V}_i = V_i/(\pi R^2 L)$, comparing the initial volumes of air and liquid.

For clarity, we have neglected thin films in the derivation above. Thin films modify the capillary pressure and the kinematic relationship between liquid and interface velocities. We derive our full model with thin films in appendix. The full model with thin films is in strong quantitative agreement with the experiments with no fitting parameters [Fig. 2, 5.]
suggesting that it captures the key features of this physical system. Note that thin films are only important quantitatively; Equation 4 predicts the same qualitative behaviour.

Our model can be simplified by considering the limit of a much larger initial volume of air than of liquid, $\hat{V}_i \gg 1$, which is the case in our experiments ($\hat{V}_i \approx 60 - 450$). In this limit, Eq. 4 reduces to

$$\frac{d\hat{l}}{dt} \approx \left[ \frac{\hat{p}_0 + 2/\hat{C}_a}{\hat{V}_i} \right] \left( \frac{\hat{t} - \hat{l}}{1 - \hat{l}} \right) \equiv \frac{1}{\hat{C}} \left( \frac{\hat{t} - \hat{l}}{1 - \hat{l}} \right).$$

(5)

The system is then governed by a single non-dimensional ‘compressibility number’,

$$\hat{C} = \frac{8\mu Q V_i}{\pi^2 R^6 p_{atm}} \left( 1 + \frac{2\gamma}{R p_{atm}} \right)^{-1}.$$  (6)

The reduced model, Eq. 5, captures most features of the full model and the experiments, and permits implicit analytical solution (see appendix).

The compressibility number $\hat{C}$ can be interpreted by considering the evolution of $p_g$ in two extreme cases: (i) Compressing a closed container of air at a rate $Q$ would increase pressure at an initial rate $\dot{p}_C = Q p_{atm}/V_i$; and (ii) Draining the viscous slug at a rate $Q$ would decrease pressure at a rate $\dot{p}_I = (\partial p/\partial l)(\partial l/\partial t) = 8\mu Q^2/(\pi^2 R^6)$. The ratio of these two rates is then

$$\frac{\dot{p}_I}{\dot{p}_C} = \frac{8\mu Q V_i}{\pi^2 R^6 p_{atm}}.$$  (7)

Comparing Eqs. 6 and 7 we find that $\hat{C} \approx \dot{p}_I/\dot{p}_C$ when $2\gamma/(R p_{atm}) \ll 1$ (i.e., when the capillary pressure is much less than $p_{atm}$), as is the case for $R \gtrsim 10 \mu m$. Thus, $\hat{C}$ measures the rate of viscous depressurisation relative to compressive pressurisation.

We next use the reduced model to consider the impact of $\hat{C}$ on the evolution of the pressure drop $\Delta \hat{p} = (\hat{t} - \hat{l})/\hat{C}$ [Fig. 3]. As noted above, $\Delta \hat{p}$ is non-monotonic with $\hat{l}$ (and with $\hat{t}$). The maximum in $\Delta \hat{p}(\hat{l})$ coincides with the instant at which the pressure trajectory crosses the incompressible solution [dashed line in Fig. 3]. This feature can be understood directly from Eq. 5. For $\Delta \hat{p} < 1 - \hat{l}$, the liquid flux is less than the nominal flux ($d\hat{l}/d\hat{t} < 1$) and the gas compresses, increasing $\Delta \hat{p}$. Once the system crosses the incompressible solution and $\Delta \hat{p} > 1 - \hat{l}$, the liquid flux exceeds the nominal flux ($d\hat{l}/d\hat{t} > 1$) and the air decompresses, decreasing $\Delta \hat{p}$. For small $\hat{C}$ [e.g., dark blue curve in Fig. 3], the gas pressurises during an initial transient, exceeds the incompressible solution, and then depressurises as it approaches the incompressible solution. For large $\hat{C}$ [e.g., yellow curve in Fig. 3], the gas pressurises
FIG. 3. Pressure drop $\Delta \hat{\rho}$ as a function of $\hat{t}$ from the reduced model (Eq. 5 and inset), for $\hat{C} = 0.05, 0.11, 0.25, 0.56, 1.25, 2.80$ and $6.25$ (dark to light). In our experiments, $\hat{C} = 0.03–8$. The arrow indicates increasing $\hat{C}$. The dashed line is the incompressible behaviour, $\Delta \hat{\rho} = 1 - \hat{t}$.

slowly compared to the timescale of displacement; the pressure significantly overshoots the incompressible solution and terminates at a non-zero value (overpressure). Small values of $\hat{C}$ are characterised by small $\Delta \hat{\rho}_{bo} \approx 0$ and modest $\hat{u}_{nb} \sim 1$, whereas large values of $\hat{C}$ are characterised by significant $\Delta \hat{\rho}_{bo} > 0$ and very large $\hat{u}_{nb} \gtrsim 10$. The full model and experiments exhibit the same behaviour [Fig. 2(c)].

To examine the transition between these two regimes in more detail, we plot in Figure 4 the breakout time $\hat{t}_{bo}$, near-breakout velocity $\hat{u}_{nb}$, and breakout pressure drop $\Delta \hat{\rho}_{bo}$ against $\hat{C}$ for experiments and the reduced and full models. In the reduced model [thick black curves in Fig. 4], there are two distinct regimes separated by a critical value of compressibility number $\hat{C}_{\text{crit}} = 1/4$ [vertical dashed lines in Fig. 4]. In the low-compressibility regime ($\hat{C} \leq \hat{C}_{\text{crit}}$), breakout occurs at exactly $\hat{t}_{bo} = 1$, meaning that the time taken to drain the liquid is identical to that of an incompressible displacement at the same $Q$ [Fig. 4(a)]. At breakout, the volume displaced by the piston is equal to the volume of liquid expelled, so the air returns to its initial volume and pressure; hence, $\Delta \hat{\rho}_{bo} = 0$ [Fig. 4(c)], the pressure gradient remains finite, and it can be shown that $1 < \hat{u}_{nb} < 2$ [Fig. 4(b) and appendix]. In the high-compressibility regime ($\hat{C} > \hat{C}_{\text{crit}}$), breakout is delayed [$\hat{t}_{bo} > 1$; Fig. 4(a)]. The volume displaced by the piston is greater than the volume of liquid expelled, so the air is compressed at breakout and $\Delta \hat{\rho}_{bo} > 0$ [Fig. 4(b)]. The breakout velocity diverges, indicated by large near-breakout values ($\hat{u}_{nb} \gg 1$ in Fig. 4(c)). See Table S1 for a summary of breakout quantities in each regime (see appendix).
FIG. 4. Comparison between experiments (symbols), the full model with thin films (Eq. 5 red curves), and the reduced model (Eq. 4 thick black curves) for (a) breakout time $\hat{t}_{bo}$, (b) near-breakout velocity $\hat{u}_{nb}$, and (c) breakout pressure drop $\Delta \hat{p}_{bo}$ as functions of $\hat{C}$. Vertical dashed lines indicate the threshold $\hat{C}_{\text{crit}} = 1/4$ between low- and high-compressibility regimes in the reduced model. Arrows and symbols (see legend) indicate increasing $\hat{V}_i$ in the full model and in the experiments, respectively.

The full model (red curves) and the experiments (symbols) are in excellent agreement, especially for $\hat{t}_{bo}$ and $\hat{u}_{nb}$ [Figs. 4(a,b)]. For $\Delta \hat{p}_{bo}$ [Fig. 4(c)], experimental measurements somewhat exceed the model predictions in the high-$\hat{C}$ regime due to insufficient experimental resolution close to breakout, but the two agree qualitatively. The near-collapse of the experimental data in Fig. 4 suggests that, to leading order, the dynamics are captured by the single parameter $\hat{C}$. The full model further captures the spread in the data via the
additional parameter $\dot{V}_i$.

Like the reduced model, the full model and the experiments exhibit two distinct displacement regimes. For $\hat{C} < \hat{C}_{\text{crit}}$, we observe $\Delta p_{bo} \approx 0$ to within experimental uncertainty; for $\hat{C} > \hat{C}_{\text{crit}}$, we observe a marked rise in both $\Delta \hat{p}_{bo}$ and $\hat{u}_{nb}$. Although inertia (neglected in our modelling) prevents ‘divergent’ velocities, we observe strong agreement between experiments and the full model for near-breakout velocities up to $\hat{u}_{nb} \approx 20$. The experiments and full model differ most significantly from the reduced model in terms of $\hat{t}_{bo}$. While $\hat{t}_{bo} \geq 1$ for all $\hat{C}$ in the reduced model, we observe $\hat{t}_{bo} < 1$ for $\hat{C} \lesssim 1/2$ in the experiments and the full model. This early breakout is due to thin films, which allow the mean velocity to exceed the nominal velocity since not all of the liquid is expelled. We nonetheless observe minima in $\hat{t}_{bo}$ around $\hat{C}_{\text{crit}}$ and delayed breakout times $\hat{t}_{bo} > 1$ for $\hat{C} \gtrsim 1/2$, a result unique to compressible systems.

In conclusion, we have studied the gas-driven displacement of viscous liquid from a capillary tube. A simple model captures the experimental observations over the full range of observed behaviours. In the limit of much more air than liquid, the dynamics are governed by a single compressibility number $\hat{C}$ that measures the importance of compression relative to viscous displacement. This limit is of practical significance, for instance in the displacement of mucus plugs in pulmonary airways via coughing [15] or invasive ventilation [16]. We showed that this problem can be reduced to a relatively simple dynamical system, in which spring-like energy storage competes with a decreasing viscous resistance, as encapsulated in the reduced model. Despite its simplicity, this system exhibits a sharp dynamical transition between two distinct displacement regimes. The low-compressibility regime ($\hat{C} \leq 1/4$) is comparable to incompressible displacement after an initial transient, including the same breakout time and the same mean velocity. The high-compressibility regime ($\hat{C} > 1/4$) is associated with delayed breakout leading to rapid, burst-like expulsion. A recent study of foam-driven fracture in gels hinted at similar flow regimes, with fractures growing nearly linearly in time for low injection rates and strongly nonlinearly for high rates [13]; our model captures the transition between these regimes and reveals the relevant control parameter. Moreover, our work serves as a starting point for studying more complex flows. For example, our work generalises directly to any confined domain involving gas in a tube, including series of liquid slugs separated by bubbles. More generally, the high-compressibility regime offers a passive means of generating short bursts of large flux,
which could be exploited in fluid systems and also in analogous electrical and mechanical systems; for example, a capacitor-membristor circuit incorporates the essential ingredients of energy storage and reducing resistance, as would a mechanism coupling elastic components with vanishing frictional contacts. Alternatively, our results suggest that any pressurised container of liquid and gas (take the humble yoghurt pot) will seep liquid if punctured while ‘under-compressed’ or spurt liquid if punctured while ‘over-compressed’, terms with concrete meaning within our framework.

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Appendix A: Experimental methods

Experiments were performed in a flow cell comprising a 10 cm long glass capillary tube (Cole-Parmer) of inner radius $R = 0.66 \pm 0.01$ mm, partially inserted horizontally through a hole in the side of an acrylic box of internal dimensions $5 \times 5 \times 10$ cm$^3$. The end of the tube external to the box was connected to a sealed reservoir of air, comprising stiff polyurethane tubing (of internal volume $3.05 \pm 0.05$ mL), short lengths of flexible tubing (Tygon; Saint-Gobain) and Gastight syringes (Hamilton). The total initial volume $V_i = [4, 8, 16, 32]$ mL of the air reservoir was set by adjusting the plunger of a ‘passive’ 50 mL syringe (1050TTL; Hamilton), while an ‘active’ 1 mL syringe (initial volume 1.00 mL; 1001TTL; Hamilton) was used to inject. The box was filled beyond the level of the capillary tube with silicone oil (Dow Corning) of dynamic viscosity $\mu = 0.096$ Pa s, surface tension $\gamma = 21$ mN m$^{-1}$, and density $\rho = 960$ kg m$^{-3}$ at laboratory temperature $T_{\text{lab}} = 22 \pm 1$ °C. Before each experiment, oil was drawn from the box into the tube, filling an initial length $L = 56 \pm 1$ mm; the system was allowed to equilibrate, such that the air-oil interface was stationary. The initial volume of liquid in the tubing, $\pi R^2 L$, was much less than the volume of the oil bath, such that filling or draining the tube produced a negligible change in the level of the oil. Experiments were initiated by injecting air into the tube to displace oil. Injection was driven at a fixed nominal volumetric rate $Q = [0.05, 0.1, 0.2, 0.4, 0.8, 1.6]$ mL min$^{-1}$ by depressing the plunger of the active syringe with a syringe pump (AL-4000; WPI). Variations in air temperature over the course of each experiment produced small anomalous volume fluxes $\sim 1 \mu$L min$^{-1} \ll Q$. The motion of the air-oil interface was recorded by a CMOS camera (ACA4096-30UM; Basler) at a spatial resolution of $28.9 \ \mu$m pixel$^{-1}$ and at a rate of 6.16-197 frames per second (fps), depending on $Q$. The capillary tube was back-lit using an LED array diffused through opalescent acrylic. We used image processing techniques (MATLAB 2020b) to extract from each frame the position of the advancing interface and the thickness of the oil films deposited on the walls of the tube.

To aid our analysis, we devised a method of measuring the viscous pressure drop $\Delta p$ over the length of the liquid column. The pressure at the tube outlet was assumed to be fixed at hydrostatic pressure $p_{HS} \approx 90$ Pa set by the oil bath level in the box. The gauge pressure of the air reservoir $p_g$, assumed to be uniform in space, was measured with a differential pressure transducer ($\pm 50$ mm Hg; 40PC001B3A; Honeywell) attached upstream of the inlet, from
which readings were taken approximately 50 times per second. Before injection, the initial
gauge pressure was \( p_{g0} = p_{HS} + 2\gamma/R \) due to the Laplace pressure jump across the interface.
During injection, the gauge pressure was \( p_g = p_{HS} + \Delta p + 2\gamma/b \). The radius of curvature \( b \)
of the interface during injection is less than the tube radius \( R \), due to thin films, and was measured from experimental images. The viscous pressure drop \( \Delta p = p_g - p_{g0} + 2\gamma(1/R - 1/b) \) can therefore be calculated indirectly from air pressure measurements, image analysis and material properties. The exact value of \( p_{HS} \) is unimportant, as long as \( p_{HS} \ll p_{atm} \).

Each experiment was repeated twice. Fig. 5(a) shows data for two experimental repeats overlaid (red squares and magenta dots), demonstrating the strong reproducibility observed. For clarity, we plot data from a single experiment for dynamical results, such as \( l(t) \), \( p(l) \), etc. For breakout metrics, such as \( p_{bo} \), \( u_{bo} \) and \( t_{bo} \), the symbols and error bars plotted correspond to the mean and standard deviation, respectively, of each pair. The main source of uncertainty in our analysis stems from the radius of the tube (known to within 10 µm), which strongly affects viscous pressures and the compressibility number.

Appendix B: Data analysis

1. Pressure data

Figure 5(b) shows raw experimental data (red circles) for gauge pressure \( p_g \) as a function of time \( t \). To smooth out the electronic noise of the signal, we take a moving average of the data over a time-window covering one percent of the experiment’s duration, from \( t = 0 \) until breakout. In the main text, we exclusively report moving-average pressure data [blue dots in Fig. 5(b)]. All data beyond the instant of breakout (i.e., \( l = L \)) is discounted.

2. Image analysis

Image processing was performed in MATLAB 2020b to extract the interface displacement \( l(t) \) and the radius of curvature \( b \) of the air-oil interface. The main steps and results are illustrated in Figs. 5(c-f). Fig. 5(a) shows a typical frame from an experiment performed at \( Q = 0.4 \) mL/min and \( V_i = 8 \) mL. The \( x \)-coordinate is measured along the axis of the tube and normalised such that \( \hat{x} = 0 \) and \( \hat{x} = 1 \) are the initial position of the interface and the end of the tube, respectively. To locate the tip of the finger, we take advantage of
FIG. 5. Illustrations of our data analysis methods for an experiment performed at $Q = 0.4$ mL/min and $V_i = 8$ mL. (a) Experimental data for normalised interface displacement $\hat{l}$ as a function of normalised time $\hat{t}$. Red squares and magenta dots show, respectively, the first and second experimental repeats at these parameters. The solid and dashed lines show, respectively, the full model with and without films (Eqs. C5 and C1). (b) Raw gauge pressure $p_g$ for the same experiment (red squares) as a function of time $t$. The blue dots are the processed moving-average data. The dashed line indicates the instant of breakout. (c) A frame recorded part-way through the same experiment. The normalised axial coordinate $\hat{x}$ is indicated, as is the detected normalised interface displacement $\hat{l}$. (d-e) Sum pixel intensity within the tube $\Sigma$ as a function of $\hat{x}$ is plotted in (d). The red box indicates the detected vicinity of the tip of the air finger, which is shown close-up in (e); here, circles are the measured values of $\Sigma$, the solid line is the fitted profile (Eq. B1), and the dashed line is the detected interface displacement $\hat{l}$. (f) Normalised radius of curvature $\hat{b}$ of the air-oil interface as a function of $\hat{x}$ (solid line). The dashed line is $1 - \hat{b}$, intended to illustrate the shape of the detected air finger.
the approximately circular dark region at the end of the air finger (where light is strongly refracted). We first measure $\Sigma$, which is the sum of pixel intensities over each column of pixels between the tube inner walls as a function of $\hat{x}$ [Fig. 5(d)]. We then locate the region of $\hat{x}$ where $\Sigma$ exhibits the strongest monotonic increase with increasing $\hat{x}$, assumed to be the vicinity of the tip. This region is shown in detail in Fig. 5(e), where circles indicate the value of $\Sigma$ at each column of pixels over this range of $\hat{x}$. We fit a function of the form

$$\Sigma = \Sigma_0 - 0.5 \left[ 1 - \tanh \left( S(\hat{x} - \hat{l}) \right) \right] \sqrt{R_{\text{int}}^2 - (\hat{x} - \hat{l} + R_{\text{int}})^2}, \quad (B1)$$

where $\Sigma_0$, $\hat{l}$, $S$ and $R_{\text{int}}$ are fitting parameters. Eq. (B1) approximately matches between a region of near-uniform intensity $\Sigma_0$ for $\hat{x} > \hat{l}$ (ahead of the interface) and a region dimmed by an approximately spherical occlusion (the interface) for $\hat{x} < \hat{l}$ (behind the interface). This ad hoc method produces reliable tracking of the interface displacement $\hat{l}(\hat{t})$. The thick blue line in Fig. 5(c) shows the fitted value of $\hat{l}$.

To measure the radius of curvature $b$ of the interface, we assume that the residual films do not evolve significantly over experimental timescales. In each frame at time $t$, we subtract a background image of the experiment immediately prior to injection in order to highlight the position of the interface. We then use edge detection algorithms to locate the pixels on the interface. The radius of the air finger at each column of pixels (i.e., each value of $\hat{x}$) between $x = 0$ and $x = l(t) - 2R$ is stored in an array; we only measure up to $x = l(t) - 2R$ as we assume the films are static once the tip of the interface has advanced a sufficient distance (one tube diameter). To reduce pixel noise, the measured values of $b$ at each position $\hat{x}$ are averaged over all recorded frames. The resulting profile of $\hat{b}(\hat{x})$ for the experiment performed at $Q = 0.4$ mL/min and $V_i = 8$ mL is shown in Fig. 5(f). There is more pixel noise closer to the outlet at $\hat{x} = 1$ because these films form later, meaning there are less recorded frames over which to average the measured values of $b$. For reference, we also plot $1 - \hat{b}$ (dashed line) to give a sense of the shape of the air finger. Nonmonotonic variations in $\hat{b}$ suggest the uncertainty of this method is around $\pm 10 \mu m$. We expect refractive distortions to be weak due to the similar refractive indices of the glass tube and the silicone oil, as well as due to the small thickness of the tube walls (0.3 mm).
3. Interface velocity

The interface velocity \( u = \frac{dl}{dt} \) is measured from experimental data by taking least-squares linear fits to segments of \( l(t) \) data. The value of \( u \) at time \( t \) is taken to be the gradient \( m \) of the linear fit \( l_i = mt_i + c \) for data \( l_i \) and \( t_i \) in the range \( (t - \Delta t) < t_i < (t + \Delta t) \), where \( \Delta t \) is 0.143% of the total duration of the experiment. At lower values of \( Q \), there is evidence of stick-slip behaviour causing non-smooth motion of the syringe pump, though the effect was not severe enough to alter or cloud the conclusions of our study.

Appendix C: Mathematical modelling

1. Full model with thin films

The mathematical model describing compressible displacement in the absence of thin residual films is given by Eq. 4 of the main text,

\[
\frac{d\hat{l}}{d\hat{t}} = \left[ \hat{p}_0 + \frac{2}{\hat{C}_a} \hat{V}_i \left( \hat{t} - \hat{l} \right) \right] \left( \hat{t} - \hat{l} \right) - \hat{t},
\]

where we use dimensionless variables for clarity. In experiments, the tube walls are coated in a residual liquid film of thickness \( \hat{t}_{\text{film}} = t_{\text{film}}/R \). We assume that \( \hat{t}_{\text{film}} \) depends on the local capillary number \( \hat{C}_a_l = \mu Q / (\pi R^2 \gamma) \) and the dimensionless velocity \( \hat{u} = \frac{dl}{dt} \), via

\[
\hat{t}_{\text{film}} = \frac{A \left( \hat{C}_a_l \hat{u} \right)^{2/3}}{1 + B \left( \hat{C}_a_l \hat{u} \right)^{2/3}}.
\]

The form of Eq. (C2), proposed by Aussillous and Quéré [9], reduces to the classical Bretherton [17] law in the limit of low \( \hat{C}_a_l \), while saturating at a finite thickness at high \( \hat{C}_a_l \) due to the confining geometry of the tube. We choose to use the values \( A = 1.34 \) and \( B = 2.79 A \) derived by Klaseboer et al. [18] which produce excellent agreement with our experimental data. Thin films effectively lead the air to propagate in a narrower tube of radius

\[
\hat{b} = 1 - \hat{t}_{\text{film}}.
\]

This has two key effects on the dynamics: (i) The tip of the air finger is squeezed, leading to a greater capillary pressure drop \( p_c = 2\gamma/b \) across the interface; (ii) The kinematic
boundary condition is modified by volume conservation such that the interface velocity
\[ \hat{u} = \frac{d\hat{l}}{dt} = \hat{U}/\hat{b}^2 \]
is greater than the mean velocity \( \hat{U} = \Delta\hat{p}/(1 - \hat{l}) \) of the liquid slug
downstream. The full model with thin films is then
\[
\frac{d\hat{l}}{dt} = \left[ 2\hat{V}_i/\hat{C}_a \left( 1 - \frac{1}{b} \right) + \left( \hat{p}_0 + \frac{2}{\hat{C}_a} \right) \left( \hat{t} - \hat{V}_b \right) \right] \frac{1}{\hat{b}^2 (1 - \hat{l})} = \frac{\Delta\hat{p}}{\hat{b}^2 (1 - \hat{l})}, \tag{C4a}
\]
and
\[
\frac{d\hat{V}_b}{dt} = \hat{b}^2 \frac{d\hat{l}}{dt}. \tag{C4b}
\]
where \( \hat{V}_b \) is the dimensionless volume of liquid expelled at time \( \hat{t} \).

To avoid the divergent breakout velocities present in the high-compressibility regime, we
recast the full model in terms of the pressure drop \( \Delta\hat{p} \) versus interface position \( \hat{l} \), which
gives
\[
\frac{d (\Delta\hat{p})}{d\hat{l}} = \frac{2}{\hat{b}^2 \hat{C}_a} \frac{d\hat{b}}{d\hat{l}} + \left[ \hat{V}_i \left( \hat{p}_0 + \frac{2}{\hat{C}_a} \right) \right] \left( \frac{\hat{b}^2 (1 - \hat{l})}{\Delta\hat{p}} - \frac{d\hat{V}_b}{d\hat{l}} \right), \tag{C5a}
\]
\[
\frac{d\hat{b}}{d\hat{l}} = \frac{2}{3A} \left( \hat{C}_a \hat{u} \right)^{2/3} \frac{d\hat{u}}{d\hat{l}} \left( \frac{B \left( \hat{C}_a \hat{u} \right)^{2/3}}{1 + B \left( \hat{C}_a \hat{u} \right)^{2/3} - 1} \right), \tag{C5b}
\]
where
\[
\hat{t} = \hat{V}_b + \hat{V}_i \left( \frac{\Delta\hat{p} + \frac{2}{\hat{b} \hat{C}_a} - \frac{2}{\hat{C}_a}}{\Delta\hat{p} + \frac{2}{\hat{b} \hat{C}_a} + \hat{p}_0} \right), \tag{C6a}
\]
and
\[
\frac{d\hat{u}}{d\hat{l}} = \frac{1}{\hat{b}^2 (1 - \hat{l})} \left[ \frac{d(\Delta\hat{p})}{d\hat{l}} - \left( \frac{2\Delta\hat{p}}{\hat{b}} \right) \frac{d\hat{b}}{d\hat{l}} \frac{\Delta\hat{p}}{1 - \hat{l}} \right]. \tag{C6b}
\]
Equation C5 is a system of nonlinear implicit ordinary differential equations, which we solve
with the built-in function ODE15I in MATLAB 2020b, subject to the initial conditions
\( \hat{l}(0) = 0, \Delta\hat{p}(0) = 10^{-10}, \hat{b}(0) = 1, \hat{V}_b(0) = 0, \) and
\( (d\hat{V}_b/d\hat{l})(0) = 1. \) To reduce the number of
steps needed for the solver to converge onto the solution, we first solve Eq. C4 with \( \hat{u}(0) = 0, \)
then take the values of \( d(\Delta\hat{p})/d\hat{l} \) and \( d\hat{b}/d\hat{l} \) at the first nonzero timestep to be the initial
values for Eq. C5. In Fig. 5(a), we plot \( \hat{l}(\hat{t}) \) from the full model both with films (solid line; 
Eq. C5) and without films (dashed line; Eq. C1) for comparison. Both produce qualitatively
similar dynamics, but the model with films gives much better quantitative agreement with
experiments.
2. Reduced model: Analytical implicit solution

The reduced model [Eq. 5 in the main text],

$$\frac{d\hat{l}}{dt} = \frac{1}{\hat{C}} \left( \frac{\hat{t} - \hat{l}}{1 - \hat{l}} \right),$$  \hspace{1cm} (C7)

permits the implicit analytical solution

$$\ln \left[ \left( \hat{t} - 1 \right)^2 + \frac{\left( \hat{t} - \hat{l} \right) \left( \hat{t} - 1 \right)}{\hat{C}} \right] + \frac{2}{\sqrt{4\hat{C}} - 1} \left[ \arctan \left( \frac{1}{\sqrt{4\hat{C}} - 1} \right) - \arctan \left( \frac{2\hat{t} - \hat{l} - 1}{\left( \hat{t} - 1 \right) \sqrt{4\hat{C}} - 1} \right) \right] = 0 \hspace{1cm} (C8)$$

when $\hat{C} > 1/4$, or

$$\ln \left[ \left( \hat{t} - 1 \right)^2 + \frac{\left( \hat{t} - \hat{l} \right) \left( \hat{t} - 1 \right)}{\hat{C}} \right] + \frac{2}{\sqrt{1 - 4\hat{C}}} \left[ \text{artanh} \left( \frac{1}{\sqrt{1 - 4\hat{C}}} \right) - \text{artanh} \left( \frac{2\hat{t} - \hat{l} - 1}{\left( \hat{t} - 1 \right) \sqrt{1 - 4\hat{C}}} \right) \right] = 0 \hspace{1cm} (C9)$$

when $\hat{C} \leq 1/4$. The breakout time $\hat{t}_{bo} > 1$ in the high-compressibility regime ($\hat{C} > 1/4$) can be found by substituting $\hat{l} = 1$ into Eq. (C8), which implies that the breakout velocity $\hat{u}_{bo}$ is divergent in this regime (note that $u_{bo}$ is measured at breakout, as opposed to the near-breakout velocity $u_{nb}$ discussed in the main text). Similarly, the breakout velocity in the low-compressibility regime ($\hat{C} \leq 1/4$) can be found by taking $\hat{l} = 1 - \epsilon$ and $t_{bo} = 1 + \epsilon(\hat{C}\hat{u}_{bo} - 1)$ (from Eq. (C7) in the limit $\epsilon \to 0$ in Eq. (C9), which gives $1 \leq \hat{u}_{bo} \leq 2$. The finite breakout velocities in the low compressibility regime imply that the breakout time $t_{bo} = 1$. Finally, the breakout pressure is given by $p_{bo} = \left( t_{bo} - 1 \right)/\hat{C}$. Table I summarises these results.

TABLE I. Analytical results for the times, velocities, and pressures and breakout in the high and low compressibility regimes.

|          | $\hat{C} \leq 1/4$ | $\hat{C} > 1/4$ |
|----------|--------------------|----------------|
| $\hat{t}_{bo}$ | $1$ | $1 + \sqrt{\hat{C}} \exp \left[ \frac{-1}{\sqrt{4\hat{C} - 1}} \left( \frac{\pi}{2} + \arctan \left( \frac{1}{\sqrt{4\hat{C} - 1}} \right) \right) \right]$ |
| $\hat{u}_{bo}$ | $\frac{1}{2\sqrt{\hat{C}}} \left( 1 - \sqrt{1 - 4\hat{C}} \right)$ | $\infty$ |
| $\hat{p}_{bo}$ | $0$ | $\frac{1}{\sqrt{\hat{C}}} \exp \left[ \frac{-1}{\sqrt{4\hat{C} - 1}} \left( \frac{\pi}{2} + \arctan \left( \frac{1}{\sqrt{4\hat{C} - 1}} \right) \right) \right]$ |