Nikolskii inequality and functional classes on compact Lie groups

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Abstract. In this note we study Besov, Triebel–Lizorkin, Wiener, and Beurling function spaces on compact Lie groups. A major role in the analysis is played by the Nikolskii inequality.

1. Introduction. The classical Nikolskii inequality for trigonometric polynomials $T_L$ of degree up to $L$ can be written as (\[Nik51\]):
$$
\|T_L\|_{L^q(T)} \leq 2L^{1/p-1/q-1/q} \|T_L\|_{L^p(T)},
$$
where $1 \leq p < q \leq \infty$. The Nikolskii inequality plays an important role in the analysis of different function spaces (for example, see \[Tri83\]) and in the approximation theory (for example, see \[DT05\]).

In the Euclidean case, for functions $f \in L^p(\mathbb{R}^n)$ such that supp$(\hat{f})$ is compact (cf. \[NW78\]) we have
$$
\|f\|_{L^q(\mathbb{R}^n)} \leq \left( C(p) \mu(\text{conv}\{\text{supp}(\hat{f})\}) \right)^{1/p-1/q} \|f\|_{L^p(\mathbb{R}^n)},
$$
where $1 \leq p \leq q \leq \infty$, $\mu(E)$ is the Lebesgue measure of $E$, and conv$[E]$ is the convex hull of $E$. Inequalities of the form \[1\] are often called Plancherel–Polya–Nikolskii inequalities.

Recently, in \[Pes09\] Pesenson obtained the Bernstein–Nikolskii inequality on symmetric spaces of noncompact type, and in \[Pes08\] on compact homogeneous spaces.

Let $G$ be a compact Lie group of dimension $\dim G$ and let $\hat{G}$ be its unitary dual. If we fix bases in representation spaces we can work with matrix representations $\xi : G \to \mathbb{C}^{d_\xi \times d_\xi}$ of dimensions $d_\xi$. By the Peter–Weyl theorem the system \{\(\sqrt{d_\xi} \xi_{ij} : [\xi] \in \hat{G}, 1 \leq i, j \leq d_\xi\}\} is an orthonormal basis in $L^2(G)$ with respect to the normalized Haar measure on $G$. All the integrals below and the spaces $L^p(G)$ will be always considered with respect to this normalized bi-invariant Haar measure on $G$.

For $f \in C^\infty(G)$ we define its Fourier coefficient at $\xi \in [\xi] \in \hat{G}$ by
$$
\hat{f}(\xi) = \int_G f(x)\xi(x)^*dx.
$$
Thus, we have $\hat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$. The Fourier series of a function $f$ takes the form
$$
f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\hat{f}(\xi)\xi(x)).
$$

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For $[\xi] \in \hat{G}$ by $\langle \xi \rangle$ we denote the eigenvalue of the operator $(1 - \mathcal{L}_G)^{1/2}$ corresponding to the representation class $[\xi] \in \hat{G}$, where $\mathcal{L}_G$ is the Laplacian on $G$, see, for example [Ste70, Chapter 1.7].

In [RT10] the following Lebesgue spaces $\ell^p(\hat{G})$ on $\hat{G}$ were defined as follows: using the Fourier coefficients of $f$, we set

$$\|\hat{f}\|_{\ell^p(\hat{G})} = \left( \sum_{[\xi] \in \hat{G}} d_{\xi}^p \frac{1}{\|\hat{f}(\xi)\|_{HS}^p} \right)^{1/p}, \quad 1 \leq p \leq \infty,$$

and

$$\|\hat{f}\|_{\ell^\infty(\hat{G})} = \sup_{[\xi] \in \hat{G}} d_{\xi}^{-\frac{1}{2}} \|\hat{f}(\xi)\|_{HS},$$

where $\|\hat{f}(\xi)\|_{HS} = \text{Tr}(\hat{f}(\xi)\hat{f}(\xi)^*)^{1/2}$. For these spaces the following Hausdorff–Young inequalities are valid:

$$\|\hat{f}\|_{\ell^p(\hat{G})} \leq \|f\|_{L^p(G)}, \quad \|f\|_{L^p(G)} \leq \|\hat{f}\|_{\ell^p(\hat{G})}, \quad 1 \leq p \leq 2, \quad \frac{1}{p'} + \frac{1}{p} = 1.$$

Let $N(L)$ be the Weyl eigenvalue counting function for the elliptic pseudo-differential operator $(1 - \mathcal{L}_G)^{1/2}$, denoting the number of its eigenvalues $\leq L$ counted with multiplicities. Then

$$N(L) = \sum_{\langle \xi \rangle \leq L} d_{\xi}^2.$$

For sufficiently large $L$ the Weyl asymptotic formula says that

$$N(L) \sim C_0 L^n, \quad C_0 = (2\pi)^{-n} \int_{\sigma_1(x,\omega) < 1} dx d\omega,$$

where $n = \text{dim} G$, and the integral is taken with respect to the canonical measure on the cotangent bundle $T^*(G)$ induced by the canonical symplectic form. Here $\sigma_1$ is the principal symbol of the operator $(1 - \mathcal{L}_G)^{1/2}$, see e.g. [Shu01].

Full proofs of our results below will appear in [NRT15].

2. Nikolskii inequality. Let $T$ be a trigonometric polynomial on a compact Lie group $G$, i.e. a function with only finitely many non-zero Fourier coefficients. Let $D$ be the Dirichlet kernel, i.e. the function $D \in C^\infty(G)$ such that

$$\hat{D}(\xi) := I_{d_{\xi}} \quad \text{for} \quad \langle \xi \rangle \leq L,$$

and zero otherwise. Here $I_{d_{\xi}} \in \mathbb{C}^{d_{\xi} \times d_{\xi}}$ denote the identity matrix.

**Theorem 1.** Let $0 < p < q \leq \infty$. For $0 < p \leq 2$ set $\rho := 1$, and for $2 < p < \infty$ let $\rho$ be the smallest integer $\geq p/2$. Then

$$\|T\|_{L^q(G)} \leq \left( \sum_{\langle \xi \rangle \neq 0} d_{\xi}^{\frac{1}{p} - \frac{1}{q}} \right)^{\frac{1}{q} - \frac{1}{q}} \|T\|_{L^p(G)}.$$

Moreover, this inequality is sharp for $p = 2$ and $q = \infty$, and the equality is attained at $T = D$.

We note that for the classical trigonometric polynomials of several variables the Nikolskii inequality is well known ([Nik51]).
Remark 2. Note that if $T = T_L$, i.e. if $\widehat{T}(\xi) = 0$ for $\langle \xi \rangle > L$, then $\sum \widehat{T}(\xi) \neq 0 d_\xi^2 \leq N(L)$ and, therefore,

$$\|T_L\|_{L^q(G)} \leq N(pL)^{\frac{1}{p} - \frac{1}{2}}\|T_L\|_{L^p(G)} \asymp (pL)^{\frac{n}{2p} - \frac{1}{2}}\|T_L\|_{L^p(G)}.$$  

For a partial sum of the Fourier series of $f$:

$$S_L f(x) = \sum_{\langle \xi \rangle \leq L} d_\xi \text{ Tr}(\widehat{f}(\xi)\xi(x))$$

one can prove the following result.

Corollary 3. Let $G$ be a compact Lie group and let $1 \leq p < q \leq \infty$ be such that $\frac{1}{p} > \frac{1}{q} + \frac{1}{2}$. Then we have

$$\left( \sum_{k=1}^{\infty} \left( \frac{k^{1-1/p+1/q}}{N(L)^k} \sup_{N(L) \geq k} \|S_L f\|_{L^q(G)} \right)^p \right)^{1/p} \leq C\|f\|_{L^p(G)}$$

for all $f \in L^p(G)$. In particular, we have $N(L)^{\frac{1}{2} - \frac{1}{p}}\|S_L f\|_{L^q(G)} = o(1)$ as $L \to \infty$.

3. Embeddings of functional classes. Here we investigate embedding theorems and interpolation properties of several classes of functions on a compact Lie group $G$. Using the definition \[2\] of the Fourier series, we can defined Sobolev, Besov, and Triebel–Lizorkin spaces, respectively, as follows:

$$H^r_p(G) = \left\{ f \in \mathcal{D}'(G) : \|f\|_{H^r_p} := \| (1 - \mathcal{L}_G)^{r/2} f \|_p < \infty \right\},$$

$$B^r_{p,q}(G) = \left\{ f \in \mathcal{D}'(G) : \|f\|_{B^r_{p,q}} := \left( \sum_{s=0}^{\infty} 2^{srq} \left( \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \text{ Tr}(\widehat{f}(\xi)\xi(x)) \right)^q \right)^{1/q} < \infty \right\},$$

$$F^r_{p,q}(G) = \left\{ f \in \mathcal{D}'(G) : \|f\|_{F^r_{p,q}} := \left( \sum_{s=0}^{\infty} 2^{srq} \left( \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \text{ Tr}(\widehat{f}(\xi)\xi(x)) \right)^q \right)^{1/q} \right\}_p < \infty \right\}.$$  

Then we have the following result:

Theorem 4. Let $G$ be a compact Lie group of dimension $n$. Then

1. $B^r_{p_1,q} \hookrightarrow B^r_{p_2,q}$, $0 < p_1 \leq p_2 \leq \infty$, $0 < q \leq \infty$, $r_2 = r_1 - n(\frac{1}{p_1} - \frac{1}{p_2})$;
2. $B^r_{p,\min\{p,2\}} \hookrightarrow H^r_p \hookrightarrow B^r_{p,\max\{p,2\}}$, $r \in \mathbb{R}$, $1 < p < \infty$;
3. $B^r_{p,q} \hookrightarrow L_q$, $1 < p < q < \infty$, $r = n(\frac{1}{p} - \frac{1}{q})$;
4. $B^r_{p,1} \hookrightarrow L_\infty$, $0 < p \leq \infty$, $r = \frac{n}{p}$;
5. $B^r_{p,\min\{p,q\}} \hookrightarrow F^r_{p,q} \hookrightarrow B^r_{p,\max\{p,q\}}$, $1 < p < \infty$, $0 < p < \infty$, $0 < q \leq \infty$;
6. $(B^r_{p_0,\beta_0}, B^r_{p_1,\beta_1})_{\theta,q} = (H^r_{p_0}, H^r_{p_1})_{\theta,q} = (F^r_{p_0,\beta_0}, F^r_{p_1,\beta_1})_{\theta,q} = B^r_{p,\beta}, 0 < r_1 < r_0 < \infty, 0 < \beta_0, \beta_1 \leq \infty, 1 < p < \infty, r = (1 - \theta)r_0 + \theta r_1, 0 < \theta < 1$.

For functions on the torus the corresponding results can be found, for example, in the book [Tri83].

Consequently, using norms \[3\] and \[4\], we can investigate the embeddings between Wiener and Beurling classes defined as follows:

$$A^\beta(G) = \left\{ f \in \mathcal{D}'(G) : \|f\|_{A^\beta} := \|\widehat{f}\|_{L^\beta(G)} = \left( \sum_{|\xi| \in \widehat{G}} d_\xi^\beta \|\widehat{f}(\xi)\|^\beta \right)^{1/\beta} < \infty \right\}.$$
and

\[ A^{*, \beta}(\hat{G}) = \left\{ f : \|f\|_{A^{*, \beta}(\hat{G})} := \left( \sum_{s=0}^{\infty} 2^{n_s} \left( \sup_{2^s \leq \langle \xi \rangle} d_{e_1}^{-1/2} \|\hat{f}(\xi)\|_{HS} \right)^\beta \right)^{1/\beta} < \infty \} , \]

where \( 0 < \beta < \infty \). For periodic functions, i.e. for \( G = \mathbb{T}^n \), we have \( d_{e_1} \equiv 1 \), \( \hat{G} \simeq \mathbb{Z}^n \), and \( \|\hat{f}(\xi)\|_{HS} = |\hat{f}(\xi)| \), and such spaces have been investigated, for example, in [BLT97] and [TB04 , Ch. 6].

**Theorem 5.** Let \( G \) be a compact Lie group of dimension \( n \).

(A). Let \( \alpha > 0 \) and \( \frac{1}{\beta} = \frac{n}{\alpha} + \frac{1}{p} \). Then

\[ \|f\|_{A^\beta} \leq C\|f\|_{B_{p, \beta}^\alpha}, \quad 1 < p \leq 2; \]

\[ \|f\|_{B_{p, \beta}^\alpha} \leq C\|f\|_{A^\beta}, \quad 2 \leq p < \infty. \]  

(B). Let \( 0 < \beta < \infty \) \( p \geq 2 \). Then

\[ C_1\|f\|_{B_{p, \beta}^\alpha} \leq \|f\|_{A^{*, \beta}} \leq C_2\|f\|_{B_{1, \beta}^{n(1/\beta - 1/2)}}. \]

As a consequence, we obtain

\[ C_1\|f\|_{A^\beta} \leq \|f\|_{B_{2, \beta}^{1/n(1/\beta - 1/2)}} \leq C_2\|f\|_{A^{*, \beta}}. \]

The left inequality is an analogue of Bernstein theorem on the absolute convergence of Fourier series. It strengthens the following inequality proved by Faraut in [Far08] for groups \( G \) of unitary matrices: if \( f \in C^k(G) \) for an even \( k > \frac{\dim G}{2} \) then \( \hat{f} \in \ell^1(\hat{G}) \), i.e. \( f \in A(G) \). For periodic functions of several variables inequality (5) follows from the results in [MS47].

Finally, we look at the following Beurling-type spaces:

\[ A^{*, \beta}_{1/\beta} = \left\{ f : \|f\|_{A^{*, \beta}_{1/\beta}} := \left( \sum_{s=0}^{\infty} 2^{n_s} \left( \sup_{2^s \leq \langle \xi \rangle} d_{e_1}^{-1/2} \|\hat{f}(\xi)\|_{HS} \right)^\beta \right)^{1/\beta} < \infty \} . \]

Note that \( A^{*, \beta}_{1/\beta} = A^{*, \beta} \). These spaces are interpolation spaces in the following sense:

**Theorem 6.** Let \( 0 < r_1 < r_0 < \infty \), \( 0 < \beta_0, \beta_1, q \leq \infty \), \( r = (1 - \theta)r_0 + \theta r_1 \), \( 0 < \theta < 1 \). Then

\[ (A^{*, \beta_0}_{r_0}, A^{*, \beta_1}_{r_1})_{\theta, q} = A^{*, \beta}_{r}. \]

In particular,

\[ (A^{*, 1/r_0}, A^{*, 1/r_1})_{\theta, 1/\beta} = A^{*, 1/r} . \]

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