Enumeration of Words that Contain the Pattern 123 Exactly Once

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Abstract. Enumeration problems related to words avoiding patterns as well as permutations that contain the pattern 123 exactly once have been studied in great detail. However, the problem of enumerating words that contain the pattern 123 exactly once is new and will be the focus of this paper. Previously, Zeilberger provided a shortened version of Burstein’s combinatorial proof of Noonan’s theorem which states that the number of permutations with exactly one 321 pattern is equal to \( \frac{2}{n} \binom{2n}{n} \). Surprisingly, a similar method can be directly adapted to words. We are able to use this method to find a formula enumerating the words with exactly one 123 pattern. Further inspired by Shar and Zeilberger’s work on generating functions enumerating 123-avoiding words with \( r \) occurrences of each letter, we examine the algebraic equations for generating functions for words with \( r \) occurrences of each letter and with exactly one 123 pattern.

Mathematics Subject Classification. 05A05, 05A15.

Keywords. Generating functions, Pattern in words, Enumeration.

1. Introduction

Recall that a word \( w = w_1 \cdots w_k \) is an ordered list of letters on some alphabet. We say that a word contains a pattern (a certain permutation of \( \{1, \ldots, m\} \)) \( \sigma \) if there exist \( 1 \leq i_1 < i_2 < \cdots < i_m \leq k \) such that the subword \( w_{i_1} \cdots w_{i_m} \) is order isomorphic to \( \sigma \) (for example, 246 is order isomorphic to 123). A word avoids the pattern \( \sigma \) if it does not contain \( \sigma \).

For a lucid history on the study of forbidden patterns, the readers are welcome to refer to the introduction of Shar and Zeilberger’s paper [6]. We also found the organization of [6] suitable for this paper, which we will more or less follow.

In the present article, we say that a word \( w \) on an alphabet \( \{a_1, a_2, \ldots, a_n\} \) \( (a_1 < a_2 < \cdots < a_n) \) is associated with the list \( [l_1, \ldots, l_n] \) if \( w \) has \( l_i \) many \( a_i \)’s in it, for \( i \) from 1 to \( n \). For example, 231113233 is a word associated with the
list [3, 2, 4], and 223344 is a word associated with the list [2, 2, 2]. When not specified, our default alphabet will be \{1, \ldots, n\} for some \(n \geq 1\).

In the second section, we will generalize Zeilberger’s bijective proof \cite{8} (a shortened version of Burstein’s elegant combinatorial proof \cite{2}) that the number of permutations of \{1, \ldots, n\} that contain the pattern 321 exactly once equals \(\frac{3}{n} \binom{2n}{n+3}\) and apply it to words. Although no closed-form formula was found, we have a summation whose summands are expressions involving the enumeration of 123-avoiding words (for details, see Theorem 2.1).

In the third section, we will study, using ideas from the second section, how to extend Shar and Zeilberger’s work \cite{6} on generating functions enumerating 123-avoiding words (with \(r\) occurrences of each letter) to words (with \(r\) occurrences of each letter) having exactly one pattern 123. More precisely, for every positive integer \(r\), Shar and Zeilberger found an algorithm for finding the defining algebraic equation for the ordinary generating function enumerating 123-avoiding words of length \(rn\) where each of the \(n\) letters of \{1, 2, \ldots, n\} occurs exactly \(r\) times.

We will present an algorithm for finding an algebraic equation for the ordinary generating function enumerating words of length \(rn\) with exactly one pattern 123, where each of the \(n\) letters of \{1, 2, \ldots, n\} occurs exactly \(r\) times. As in Shar and Zeilberger’s paper \cite{6}, we used Buchberger’s algorithm for finding Gröbner bases, and our computer (running Maple) found the defining algebraic equation for \(r = 2\):

\[
x^4 (x + 4)^2 F^4 + 2 x^3 (x + 4) (11 x + 23) F^3
- 4 x (3 x^4 - 10 x^3 - 97 x^2 - 146 x + 1) F^2
+ (-168 x^4 - 840 x^3 - 744 x^2 + 336 x - 24) F + 144 x^3 (x + 2) = 0.
\]

This took about a second. The minimal algebraic equation for \(r = 3\) has 12 as the highest power for \(F\) and the computation took about 20 seconds. Interested readers can find it on the website accompanying this paper: http://sites.math.rutgers.edu/~my237/One123. The case when \(r = 4\) already took too long to compute (more than a month).

Now, let \(a_r(n)\) be the number of words of length \(rn\) with exactly one pattern 123, where each of the \(n\) letters of \{1, 2, \ldots, n\} occurs exactly \(r\) times. In the last section of the present article, we will use the Maple package \textsc{SCHUTZENBERGER} to derive recurrence relations for our sequences. Having obtained the defining algebraic equations of the generating functions for \(a_r(n)\) in the cases \(r = 2\) and \(r = 3\), Manuel Kauers kindly helped us in finding the asymptotics for our sequences \(a_2(n)\) and \(a_3(n)\) (thanks to Kauers, the constants in the following formulas were computed via a step-by-step procedure; for details, please refer to \cite{4}):

\[
a_2(n) = \frac{3(13 - \sqrt{21})}{49} \cdot \frac{1}{\sqrt{\pi}} \cdot 12^n \cdot n^{-3/2} \cdot (1 + O(n^{-1})),
\]

\[
a_3(n) = \frac{-7 + 6\sqrt{7}}{56} \cdot \frac{1}{\sqrt{\pi}} \cdot 32^n \cdot n^{-3/2} \cdot (1 + O(n^{-1})).
\]
2. Words with One Pattern 123

In a paper published in 2011, Burstein [2] gave an elegant combinatorial proof of Noonan’s theorem [5] that the number of permutations of \(\{1, \ldots, n\}\) that contain the pattern 321 exactly once equals \(\frac{3n}{n!}\). Zeilberger [8] was able to shorten Burstein’s proof using a bijection between a permutation with exactly one pattern 321, denoted \(\pi_1\sigma_2\sigma_3\pi_4\) \((a < b < c)\), and a pair \((\pi_3b\pi_2a, c\pi_3b\pi_4)\) where \(\pi_1\sigma_2a\) is a 321-avoiding permutation of \(\{1, \ldots, b\}\) and \(c\pi_3b\pi_4\) is a 321-avoiding permutation of \(\{b, \ldots, n\}\). The readers are encouraged to read Zeilberger’s proof as a motivation and warm-up. Below we will see how we can use the same idea and apply it to words.

**Theorem 2.1.** Let \(A(l_1, \ldots, l_n)\) be the number of 123-avoiding words associated with the list \([l_1, \ldots, l_n]\). Let \(B(l_1, \ldots, l_n)\) be the number of words associated with list \([l_1, \ldots, l_n]\) that contain the pattern 123 exactly once. Then we have

\[
B(l_1, \ldots, l_n) = \sum_{b=2}^{n-1} \sum_{j=0}^{l_b-1} (A(l_1, \ldots, l_{b-1}, j+1) - A(l_1, \ldots, l_{b-1}, j))
\]

\[
\cdot (A(l_b - j, l_{b+1}, \ldots, l_n) - A(l_b - j - 1, l_{b+1}, \ldots, l_n)).
\]

Before we start the proof, we define a good pair of words. Fix any \(2 \leq b \leq n - 1\) and a list \([l_1, \ldots, l_n]\). For any \(0 \leq j \leq l_b - 1\), a pair of words \((\sigma_1, \sigma_2)\) is called good if \(\sigma_1\) is a 123-avoiding word on \(\{1, \ldots, b\}\) associated with the list \([l_1, \ldots, l_{b-1}, j+1]\) that does not start with \(b\) and \(\sigma_2\) is a 123-avoiding word on \([b, \ldots, n]\) associated with the list \([l_b - j, l_{b+1}, \ldots, l_n]\) that does not end with \(b\). For example, if \([l_1, \ldots, l_n] = [2, 2, 2, 2]\), and \(b = 2\), then \((112, 422433)\) is a good pair. We will also say that \(\sigma_i\) \((i = 1, 2)\) is good if it belongs to a good pair \((\sigma_1, \sigma_2)\).

**Proof of Theorem 2.1.** Any word \(w\) associated with the list \([l_1, l_2, \ldots, l_n]\) with exactly one pattern 123 can be written as \(\pi_1a\pi_2b\pi_3c\pi_4\) \((a < b < c)\), where \(abc\) is the unique 123 pattern. All entries to the left of \(b\), except for \(a\), must be greater than or equal to \(b\), and all the entries to the right of \(b\), except for \(c\), must be smaller than or equal to \(b\). Also, \(\pi_2\) and \(\pi_3\) must not contain any \(b\)'s, otherwise there will be another 123 pattern. Observe that \(a\pi_3b\pi_4\) is a word on \(\{1, 2, \ldots, b\}\) avoiding the pattern 123 and does not start with \(b\) and \(\pi_1b\pi_2c\) is a word on \(\{b, b+1, \ldots, n\}\) avoiding the pattern 123 and does not end with \(b\). Therefore \((a\pi_3b\pi_4, \pi_1b\pi_2c)\) is a good pair.
We now verify that there is indeed a bijection from the set of words having exactly one pattern 123 to the set of good pairs $(\sigma_1, \sigma_2)$ for $2 \leq b \leq n - 1$ and $0 \leq j \leq b - 1$.

Fix $b$ and $j$ $(2 \leq b \leq n - 1, 0 \leq j \leq b - 1)$. Given a word $\pi_1a\pi_2b\pi_3c\pi_4$ that has exactly one 123 pattern, we can easily map it to a unique good pair $(a\pi_3b\pi_4, \pi_1b\pi_2c)$ by first determining $a$ and $c$. This is easy since we have only one 123 pattern. Conversely, given a good pair $(\sigma_1, \sigma_2)$, we take the first letter of $\sigma_1$ as “$a$” and the leftmost occurrence of $b$ as “$b$” and get $\pi_3$ and $\pi_4$ ($\sigma_1 = a\pi_3b\pi_4$). Similarly, we take the last letter of $\sigma_2$ as “$c$” and the rightmost occurrence of $b$ as “$b$” and get $\pi_1$ and $\pi_2$ ($\sigma_2 = \pi_1b\pi_2c$). Putting everything together we get a unique expression $\pi_1a\pi_2b\pi_3c\pi_4$.

Now, for any $b$ and $j$, the number of good $\sigma_1$ is $A(l_1, \ldots, l_{b-1}, j+1) - A(l_1, \ldots, l_{b-1}, j)$ and the number of good $\sigma_2$ is $A(l_b - j, l_{b+1}, \ldots, l_n) - A(l_b - j - 1, l_{b+1}, \ldots, l_n)$. Therefore, the number of words $\pi_1a\pi_2b\pi_3c\pi_4$ with exactly one pattern 123 is

$$
(A(l_1, \ldots, l_{b-1}, j+1) - A(l_1, \ldots, l_{b-1}, j)) \\
\cdot (A(l_b - j, l_{b+1}, \ldots, l_n) - A(l_b - j - 1, l_{b+1}, \ldots, l_n)).
$$

Summing over all $b$ and $j$, we get the desired result. \qed

**Corollary 2.2.** $B(l_1, \ldots, l_n) = B(l_n, \ldots, l_1)$.

**Proof.** By Theorem 2.1, we have

$$
B(l_1, \ldots, l_n) = \sum_{b=2}^{n-1} \sum_{j=0}^{l_{b-1}-1} (A(l_1, \ldots, l_{b-1}, j+1) - A(l_1, \ldots, l_{b-1}, j)) \\
\cdot (A(l_b - j, l_{b+1}, \ldots, l_n) - A(l_b - j - 1, l_{b+1}, \ldots, l_n))
$$

(2.1)

and

$$
B(l_n, \ldots, l_1) = \sum_{b=2}^{n-1} \sum_{j=0}^{l_{n-b+1}-1} (A(l_n, \ldots, l_{n-b+2}, j+1) - A(l_n, \ldots, l_{n-b+2}, j)) \\
\cdot (A(l_{n-b+1} - j, l_{n-b}, \ldots, l_1) - A(l_{n-b+1} - j - 1, l_{n-b}, \ldots, l_1)).
$$

(2.2)

Taking $b = k$ $(2 \leq k \leq n - 1)$ in (2.1), the inner sum of (2.1) becomes

$$
\sum_{j=0}^{l_k-1} (A(l_1, \ldots, l_{k-1}, j+1) - A(l_1, \ldots, l_{k-1}, j)) \\
\cdot (A(l_k - j, l_{k+1}, \ldots, l_n) - A(l_k - j - 1, l_{k+1}, \ldots, l_n)).
$$

(2.3)

Meanwhile, taking $b = n - k + 1$ $(2 \leq k \leq n - 1)$ in (2.2), the inner sum of (2.2) becomes

$$
\sum_{j=0}^{l_k-1} (A(l_n, \ldots, l_{k+1}, j+1) - A(l_n, \ldots, l_{k+1}, j)) \\
\cdot (A(l_k - j, l_{k-1}, \ldots, l_1) - A(l_k - j - 1, l_{k-1}, \ldots, l_1)).
$$

(2.4)
We only need to show (2.3)=(2.4) in order to show (2.1)=(2.2). Now notice that when \( j = t \) (0 \( \leq t \leq l_k - 1 \)), the summand of (2.3) is
\[
(A(l_1, \ldots, l_{k-1}, t + 1) - A(l_1, \ldots, l_{k-1}, t))
\cdot (A(l_k - t, l_{k+1}, \ldots, l_n) - A(l_k - t - 1, l_{k+1}, \ldots, l_n)),
\]
and when \( j = l_k - 1 - t \) (the “symmetric counterpart” of \( j = t \)), the summand of (2.4) is
\[
(A(l_n, \ldots, l_{k+1}, l_k - t) - A(l_n, \ldots, l_{k+1}, l_k - t - 1))
\cdot (A(t + 1, l_{k-1}, \ldots, l_1) - A(t, l_{k-1}, \ldots, l_1)).
\]
Because \( A(l_1, \ldots, l_n) \) is symmetric in its arguments (this is not true in general for \( B(l_1, \ldots, l_n) \); for details of this result, see [6, Page 4]), we see that (2.5) = (2.6). Therefore, as \( j \) ranges from 0 to \( l_k - 1 \), we have (2.3) = (2.4). And as \( b \) ranges from 2 to \( n - 1 \), we have (2.1) = (2.2).

**Corollary 2.3.** Fix a list \( L = [l_1, \ldots, l_n] \). The number of words associated with \( L \) that contain exactly one pattern 123 (i.e., \( B(l_1, \ldots, l_n) \)) is equal to the number of words associated with \( L \) that contain exactly one pattern 321.

**Proof.** Let \( S_1 \) be the set of words associated with \([l_1, \ldots, l_n]\) that contain exactly one pattern 123 and \( S_2 \) be the set of words associated with \([l_n, \ldots, l_1]\) that contain exactly one pattern 321. For any \( w_1 \in S_1 \), we can map it to a word \( w_2 \) associated with \([l_n, \ldots, l_1]\) by mapping letter \( i \) to letter \( n - i + 1 \) (for all \( i \) from 1 to \( n \)). For example, the word 121322 is mapped to 323122. Observe that \( w_2 \) must contain exactly one pattern 321, which occurs at the same location in \( w_2 \) as the location of the 123 pattern in \( w_1 \). Therefore, \( w_2 \in S_2 \). Clearly this is a bijection from \( S_1 \) to \( S_2 \). So \(|S_1| = |S_2|\). This along with Corollary 2.2 gives Corollary 2.3. \( \square \)

**Remark 2.4.** One may wonder if the number of words associated with \([l_1, \ldots, l_n]\) that contain exactly one pattern 123 is equal to the number of words associated with \([l_1, \ldots, l_n]\) that contain exactly one pattern 132. This is not the case (if this were the case, we would have an analogue of the result that the 123-avoiding words associated with \([l_1, \ldots, l_n]\) are equinumerous with the 132-avoiding words associated with \([l_1, \ldots, l_n]\), see [7]). For example, the number of permutations of \( \{1, 2, \ldots, n\} \) \((n \geq 1)\) that contain exactly one pattern 123 is \( \frac{2}{n} \binom{2n}{n+3} \) [8] while the number of permutations of \( \{1, 2, \ldots, n\} \) \((n \geq 1)\) that contain exactly one pattern 132 is \( \binom{2n-3}{n-3} \) [1].

### 3. Algebraic Equations for Generating Functions

In their beautiful paper, Shar and Zeilberger [6] developed methods for finding the algebraic equations for the ordinary generating functions enumerating 123-avoiding words of length \( rn \), where each of the \( n \) letters of \( \{1, 2, \ldots, n\} \) occurs exactly \( r \) times. First, we present some definitions and results in [6].

For \( 0 \leq i \leq j \leq r - 1 \) and \( n \geq 0 \), let \( W_r^{(i,j)}(n) \) be the set of 123-avoiding words of length \( rn + i + j \) on the alphabet \( \{1, 2, \ldots, n, n+1, n+2\} \), with \( i \)
occurrences of the letter 1, \( j \) occurrences of \( n + 2 \), and exactly \( r \) occurrences of the other \( n \) letters. Let \( W_r^{(i,j)} \) be the union of \( W_r^{(i,j)}(n) \) over all \( n \geq 0 \). Let \( g_r^{(i,j)}(x) \) be the weight enumerator for \( W_r^{(i,j)} \), with respect to the weight \( w \to x^{\text{length}(w)} \). Note that the weight enumerators will remain the same if the letters 1 and \( n + 2 \) are exchanged with any two distinct letters in the alphabet \( \{1, 2, \ldots, n, n + 1, n + 2\} \). For more detailed explanation, see [6, Pages 3–4].

Shar and Zeilberger were able to find a system of \( (r + 1) \) equations for \( g_r^{(i,j)}(x) \) \((0 \leq i \leq j \leq r - 1)\), with the convention that if \( s > k \) then \( g_r^{(s,k)} = g_r^{(k,s)} \):

\[
g_r^{(i,j)}(x) = \delta_{i,0}\delta_{j,0} + x \sum_{t=0}^{r-1} g_r^{(i,t)}(x)g_r^{(r-t \mod r, (j-1 \mod r))(x)} + \sum_{m=0}^{i-1} x^{m+1}g_r^{(i-m,j-1)}(x),
\]

where

\[
\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}
\]

For example, in the case when \( r = 2 \), we would get the following system of equations:

\[
g_2^{(0,0)}(x) = 1 + xg_2^{(0,0)}(x)g_2^{(0,1)}(x) + xg_2^{(0,1)}(x)g_2^{(1,1)}(x),
\]

\[
g_2^{(0,1)}(x) = xg_2^{(0,0)}(x)^2 + xg_2^{(0,1)}(x)^2,
\]

\[
g_2^{(1,1)}(x) = xg_2^{(0,0)}(x)g_2^{(0,1)}(x) + xg_2^{(0,1)}(x)\left(1 + g_2^{(1,1)}(x)\right).
\]

Solving this system of equations in the three unknowns \( g_2^{(0,0)}(x) \), \( g_2^{(0,1)}(x) \), \( g_2^{(1,1)}(x) \), we get the weight enumerators for \( W_2^{(0,0)} \), \( W_2^{(0,1)} \) and \( W_2^{(1,1)} \).

Once we have the weight enumerators, we can easily get the corresponding generating functions by doing a little operation. For example, because we have an explicit expression for \( g_2^{(0,0)}(x) \) \((g_2^{(0,0)}(x) = 1 + x^2 + 6x^4 + 43x^6 + 352x^8 + 3114x^{10} + \cdots\)\), the corresponding generating function is \( f_2^{(0,0)}(x) = 1 + x + 6x^2 + 43x^3 + 352x^4 + 3114x^5 + \cdots \) (that is, \( f_2^{(0,0)}(x) = g_2^{(0,0)}(x^{1/2})\)).

Let \( V_r(n) \) be the set of words on the alphabet \( \{1, \ldots, n\} \) with exactly \( r \) occurrences of each letter, and with exactly one pattern 123. Let \( V_r = \bigcup_{n=0}^{\infty} V_r(n) \). Let \( h_r(x) \) be the weight enumerator for \( V_r \) (as always, with weight \( w \to x^{\text{length}(w)} \)) and let \( f_r(x) \) be the corresponding generating function. We will follow the framework of [6], with two warm-up cases leading to the general case.

**First warm-up: \( r = 1 \)**

**Claim.** \( h_1(x) = (g_1^{(0,0)}(x) - xg_1^{(0,0)}(x) - 1)^2 / x. \)

**Proof.** Recall that \( g_1^{(0,0)}(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \cdots \) is the weight enumerator for 123-avoiding permutations on \( \{1, 2, \ldots\} \). We prove this
claim by showing that the coefficient of $x^n$ ($n \geq 0$) on the right-hand side is exactly the number of good pairs $(a\pi_3b\pi_4, \pi_1b\pi_2c)$ $(2 \leq b \leq n - 1)$, which equals $B(1, 1, \ldots, 1)$ (with $n$ 1’s) (by Zeilberger’s proof [8]).

For any fixed $b$ $(2 \leq b \leq n - 1)$, a good $a\pi_3b\pi_4$ would be a 123-avoiding permutation on $\{1, \ldots, b\}$ that does not start with $b$. Similarly, a good $\pi_1b\pi_2c$ would be a 123-avoiding permutation on $\{b, \ldots, n\}$ that does not end with $b$.

Note that the coefficient of $x^b$ in $g_1^{(0,0)}(x) - xg_1^{(0,0)}(x) - 1$ is exactly the number of good $a\pi_3b\pi_4$. (The $x$ in front of $g_1^{(0,0)}(x)$ corresponds to having $b$ in front of a permutation, and the $-1$ corresponds to an empty permutation. We do not want either of these.)

Similarly, the coefficient of $x^{n-b+1}$ in $g_1^{(0,0)}(x) - xg_1^{(0,0)}(x) - 1$ is the number of good $\pi_1b\pi_2c$. Multiplying the two, we have that the coefficient of $x^{n+1}$ $(= x^b \cdot x^{n-b+1})$ in

$$
(g_1^{(0,0)}(x) - xg_1^{(0,0)}(x) - 1)^2
$$

is the number of good pairs $(a\pi_3b\pi_4, \pi_1b\pi_2c)$ ($b$ ranges from 2 to $n-1$). Dividing by $x$, we find that the coefficient of $x^n$ in

$$
(g_1^{(0,0)}(x) - xg_1^{(0,0)}(x) - 1)^2 / x
$$

is the number of good pairs $(a\pi_3b\pi_4, \pi_1b\pi_2c)$.

\[\square\]

**Second warm-up:** $r = 2$

**Claim.** $h_2(x) = 2 \cdot (g_2^{(0,0)}(x) - xg_2^{(0,1)}(x) - 1)(g_2^{(0,1)}(x) - xg_2^{(0,0)}(x))/x$.

**Proof.** Recall that $g_2^{(0,0)}(x)$ $(= 1 + x^2 + 6x^4 + 43x^6 + \cdots)$ is the weight enumerator for 123-avoiding words on $\{1, 1, 2, 2, \ldots\}$ (or equivalently, 123-avoiding words associated with $[2, 2, \ldots]$) and $g_2^{(0,1)}(x)$ $(= x + 3x^3 + 19x^5 + 145x^7 + \cdots)$ is the weight enumerator for 123-avoiding words on $\{1, 2, 2, 3, 3, \ldots, n, n, \ldots\}$. As in the first warm-up, we prove this claim by showing that the coefficient of $x^{2n}$ ($n \geq 0$) on the right-hand side is exactly the number of good pairs $(a\pi_3b\pi_4, \pi_1b\pi_2c)$ $(2 \leq b \leq n - 1)$, which equals $B(2, 2, \ldots, 2)$ (with $n$ 2’s) by the proof of Theorem 2.1.

For any $b$ $(2 \leq b \leq n - 1)$, we have the following two cases: either $\pi_4$ contains one $b$ and $\pi_1$ contains no $b$ or the other way around.

**Case 1.** $\pi_4$ contains one $b$ and $\pi_1$ contains no $b$.

Then a good $a\pi_3b\pi_4$ would be a 123-avoiding word on $\{1, 1, \ldots, b, b\}$ that does not start with $b$. Similarly, a good $\pi_1b\pi_2c$ would be a 123-avoiding word on $\{b, b + 1, b + 1, \ldots, n, n\}$ that does not end with $b$.

Note that the coefficient of $x^{2b}$ in $g_2^{(0,0)}(x) - xg_2^{(0,1)}(x) - 1$ is exactly the number of good $a\pi_3b\pi_4$. (The $x$ in front of $g_2^{(0,1)}(x)$ corresponds to having $b$ in front of a word, and the $-1$ corresponds to an empty word. We do not want either of these.)

Similarly, the coefficient of $x^{2(n-b)+1}$ in $g_2^{(0,1)}(x) - xg_2^{(0,0)}(x)$ is the number of good $\pi_1b\pi_2c$. Letting $b$ range from 2 to $n - 1$, we see that the coefficient of $x^{2n+1}$ $(= x^{2b} \cdot x^{2(n-b)+1})$ in
\[(g_2^{(0,0)}(x) - xg_2^{(0,1)}(x) - 1)(g_2^{(0,1)}(x) - xg_2^{(0,0)}(x))\]
is the number of good pairs \((a\pi_3b\pi_4, \pi_1b\pi_2c)\) if the additional \(b\) is in \(\pi_4\). Dividing by \(x\), we get that the coefficient of \(x^{2n}\) in

\[(g_2^{(0,0)}(x) - xg_2^{(0,1)}(x) - 1)(g_2^{(0,1)}(x) - xg_2^{(0,0)}(x))/x\]
is the number of good pairs \((a\pi_3b\pi_4, \pi_1b\pi_2c)\) if the additional \(b\) is in \(\pi_4\).

**Case 2.** \(\pi_1\) contains one \(b\) and \(\pi_4\) contains no \(b\).

Then a good \(a\pi_3b\pi_4\) would be a 123-avoiding word on \(\{1, 1, \ldots, b-1, b-1, b\}\) that does not start with \(b\). A good \(\pi_1b\pi_2c\) would be a 123-avoiding word on \(\{b, b, \ldots, n, n\}\) that does not end with \(b\).

Now, the coefficient of \(x^{2b-1}\) in \(g_2^{(0,1)}(x)-xg_2^{(0,0)}(x)\) is exactly the number of good \(a\pi_3b\pi_4\). Similarly, the coefficient of \(x^{2(n-b)+2}\) in \(g_2^{(0,0)}(x)-xg_2^{(0,1)}(x)-1\) is the number of good \(\pi_1b\pi_2c\). It follows that the coefficient of \(x^{2n+1} (= x^{2b-1} + x^{2(n-b)+2})\) in

\[(g_2^{(0,1)}(x) - xg_2^{(0,0)}(x))(g_2^{(0,0)}(x) - xg_2^{(0,1)}(x) - 1)\]
is the number of good pairs \((a\pi_3b\pi_4, \pi_1b\pi_2c)\) \((b\) ranges from 2 to \(n-1\)) if the additional \(b\) is in \(\pi_1\). Dividing by \(x\), we obtain that the coefficient of \(x^{2n}\) in

\[(g_2^{(0,1)}(x) - xg_2^{(0,0)}(x))(g_2^{(0,0)}(x) - xg_2^{(0,1)}(x) - 1)/x\]
is the number of good pairs \((a\pi_3b\pi_4, \pi_1b\pi_2c)\) if the additional \(b\) is in \(\pi_1\).

Therefore, the coefficient of \(x^{2n}\) in

\[2 \cdot (g_2^{(0,0)}(x) - xg_2^{(0,1)}(x) - 1)(g_2^{(0,1)}(x) - xg_2^{(0,0)}(x))/x\]
is the number of good pairs \((a\pi_3b\pi_4, \pi_1b\pi_2c)\), which is equal to \(B(2, 2, \ldots, 2)\) (with \(n\) 2’s). Note that the coefficient of \(x^{2n+1}\) in

\[2 \cdot (g_2^{(0,0)}(x) - xg_2^{(0,1)}(x) - 1)(g_2^{(0,1)}(x) - xg_2^{(0,0)}(x))/x\]
is 0 because \(g_2^{(0,1)}(x)\) has only odd powers of \(x\) and \(g_2^{(0,0)}(x)\) has only even powers of \(x\). So we have shown the weight enumerator for \(V_2\) is as claimed. To get the generating function \(f_2(x)\) for \(V_2\), we simply let \(f_2(x) = h_2(x^{1/2})\). \(\square\)

The readers are welcome to compare \(h_2(x)\) with the earlier formula in the case when \(l_i = 2\) \((1 \leq i \leq n)\):

\[
\sum_{b=2}^{n-1} \sum_{j=0}^{1} \left( A(j + 1, 2, 2, \ldots, 2) - A(j, 2, 2, \ldots, 2) \right) \\
\cdot \left( A(2 - j, 2, 2, \ldots, 2) - A(1 - j, 2, 2, \ldots, 2) \right)
\]

The following theorem is concerned with the general case.
Theorem 3.1. For $r \geq 1$,

$$h_r(x) = \frac{1}{x} \sum_{i=1}^{r} \left( g_r^{(0, i \mod r)}(x) - xg_r^{(0, i-1)}(x) - \delta(i \mod r, 0) \left( g_r^{(0, (r+1-i) \mod r)}(x) - xg_r^{(0, r-i)}(x) - \delta((r+1-i) \mod r, 0) \right) \right).$$

The general case is derived using the exact same idea as for the warm-up cases. Instead of having two cases as in the second warm-up, here we have $r$ cases. The interested readers are welcome to verify the formula for $r = 3$, and the general case should be apparent after this verification. As before, to get the generating function for $V_r$, we simply let $f_r(x) = h_r(x^{1/r})$.

4. Using Maple Packages

As noted in Shar and Zeilberger's paper [6, Page 7], we know that $f_r(x)$ is algebraic and the sequence $a_r(n)$ satisfies some homogeneous linear recurrence equation with polynomial coefficients.

Using the $\text{algtorec}$ procedure in the $\text{SCHUTZENBERGER}$ package written by Zeilberger (available at \url{http://www.math.rutgers.edu/~zeilberg/tokhniot/SCHUTZENBERGER.txt}), we are able to find (rigorously) recurrences (in operator notation) for our sequences when $r = 1$ and $r = 2$ ($r = 3$ took too long to compute).

For $r = 1$, we get

$$(2n(2n+1) - (n+4)(n-2) N)a_1(n) = 0,$$

which agrees with the already known formula $a_1(n) = \frac{3}{n} \left( \frac{2n}{n+3} \right)$.

In the case when $r = 2$, $\text{algtorec}$ returned an operator of degree 8, but it can be reduced to a minimal operator of degree 4 (thanks to Manuel Kauers for pointing it out), that is,

\[
\begin{align*}
(36(1+n)(2+n)(1+2n)(3+2n)(18154800 + 23101940n + 10635771n^2 + 2093616n^3 + 147833n^4) \\
+ 12(2+n)(3+2n)(1283329440 + 3700267618n + 4200957553n^2 \\
+ 2408049238n^3 + 735936616n^4 + 113774584n^5 + 6948151n^6)N \\
+ (282564806400 + 1066356868608n + 1704365727480n^2 \\
+ 1511140337906n^3 + 814587362081n^4 + 273775889012n^5 \\
+ 56080140110n^6 + 6405068474n^7 + 312371129n^8)N^2 \\
- 2(4+n)(11939685120 + 40890299130n + 56943840213n^2 \\
+ 41794221496n^3 + 17488032270n^4 + 4183030930n^5 + 531527997n^6 \\
+ 27792604n^7)N^3 \\
+ 8(1+n)(4+n)(5+n)(11+2n)(3742848 + 7519914n + 5241921n^2 \\
+ 1502284n^3 + 147833n^4)N^4)a_2(n) = 0.
\end{align*}
\]
Here are some initial terms of \( f_2(x) \) (i.e., the generating function for \( a_2(n) \)):

\[
f_2(x) = 12 x^3 + 174 x^4 + 2064 x^5 + 23082 x^6 + 252966 x^7 + 2755332 x^8 + 30001026 x^9 + 327381492 x^{10} + \cdots.
\]

(We can easily get more terms.)

Everything in this paper is implemented (with explanation) in the Maple packages \texttt{Words123New} and \texttt{PW123}, which are available at \url{http://sites.math.rutgers.edu/~my237/One123}, where one can also find some sample input and output files.

**Acknowledgements**

The author thanks Doron Zeilberger for bringing this project to her attention and continued conversation. The author also would like to thank Matthew Russell for offering suggestions that improved the exposition of this paper, Andrew Lohr and Alejandro Ginory for proofreading the draft, and Manuel Kauers for his help in finding the asymptotics.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

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Received: 23 January 2018.
Accepted: 23 May 2018.