Isospin susceptibility in the $O(n)$ sigma-model in the delta-regime

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Abstract: We compute the isospin susceptibility in an effective O($n$) scalar field theory (in $d = 4$ dimensions), to third order in chiral perturbation theory ($\chi$PT) in the delta-regime using the quantum mechanical rotator picture. This is done in the presence of an additional coupling, involving a parameter $\eta$, describing the effect of a small explicit symmetry breaking term (quark mass). For the chiral limit $\eta = 0$ we demonstrate consistency with our previous $\chi$PT computations of the finite-volume mass gap and isospin susceptibility. For the massive case by computing the leading mass effect in the susceptibility using $\chi$PT with dimensional regularization, we determine the $\chi$PT expansion for $\eta$ to third order.
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1 Introduction

In a previous paper [1] we computed the change in the free energy due to a chemical potential coupled to a conserved charge in the non-linear O(n) sigma model with two regularizations, lattice regularization (with standard action) and dimensional regularization (DR) in a general d-dimensional asymmetric $L_s^{d-1} \times L_t$ volume with periodic boundary conditions in all directions. This allowed us, for $d = 4$, to establish two independent relations among the 4-derivative couplings appearing in the effective Langrangians and in turn this enables conversion of results for physical quantities computed by the lattice regularization to those involving scales introduced in DR.

In particular we could convert the computation of the mass gap in a periodic box, by Niedermayer and Weiermann [2] using lattice regularization to a result involving parameters of the dimensionally regularized effective theory, and we verified this result by a direct computation [1]. The proposal to measure the low-lying stable masses in the delta-regime to constrain some of the low energy constants in the effective chiral Langrangian describing low energy pion dynamics of QCD, was first made by Hasenfratz [3]. For two flavors of massless quarks, the relevant $\chi$PT has SU(2)$\times$SU(2)$\simeq$O(4) symmetry.

The so called $\delta$-regime referred to above, is where the system is in a periodic spatial box of sides $L_s$ with $L_t \gg L_s$ and $m_\pi L_s$ is small (i.e. small or zero quark mass) whereas $F_\pi L_s$, ($F_\pi$ the pion decay constant) is large. This regime was first analyzed in a pioneering paper by Leutwyler [4]. He showed that the low lying dynamics in this regime was described by a quantum rotator for the spatially constant modes.

In that paper Leutwyler also obtained the leading order effects of the quark mass on the low-lying spectrum for O(4). The extension to next-to-leading (NLO) order was presented by Weingart [5, 6]. There followed a period of low activity in this field, but recently Matzelle and Tiburzi have, in an interesting paper [7], studied the effect of small symmetry breaking in the QM rotator picture ($N_f = 2$), and extended the results to small non-zero temperatures.

The rotator Hamiltonian given in (2.5) has two parameters, the moment of inertia $\Theta$ and a parameter $\eta$ describing the explicit symmetry breaking O(n) to O(n − 1). These parameters are themselves functions of $F, L_s, M$ where $F$ the pion decay constant in the limit of vanishing quark mass, and $M$ is the mass acquired by the Goldstone bosons in leading order $\chi$PT due to the explicit symmetry breaking.

Within this theoretical framework, in this paper we communicate computations which provide, in our opinion, a non-trivial check on our previous rather technical computations [1], and also gives the NNLO (next-to next-to leading order) $\chi$PT expansion for the parameter $\eta$.

In sect. 2 we obtain the expression (2.7) for the susceptibility for $u = L_t/(2\Theta)$ small, under the assumption that in the $\delta$-regime the low lying spectrum is that of the quantum
rotator. These computations are themselves rather involved and so we defer many details to Appendix A. There we also review the leading order effects of a small symmetry breaking parameter on the low lying spectrum for general $n$, reproducing Leutwyler’s result [4] for $n = 4$.

In sub-section 3.1 we first consider the symmetric case $M = 0 = \eta$. By inserting our expression for $\Theta$ obtained in [1] into (2.4), we obtain the susceptibility for small $u$ as a function of $F, L_s$, $\ell \equiv L_t/L_s$. This can then be compared to the direct $\chi$PT computation of the susceptibility [1] in the $\epsilon$–regime for $\ell \gg 1$. The comparison requires knowledge of the large $\ell$–behavior of shape functions and the sunset integral appearing in the latter; these are discussed in Appendix B. The agreement of the two computations provides the non-trivial check referred to above.

In sub-section 3.2 we compute the susceptibility for small quark masses in the framework of $\chi$PT. Subsequently a comparison to the result (2.7) from the rotator computation determines the term in $\eta$ proportional to $M^2$ as a function of $FL_s$ to NNLO. Various technical aspects are deferred to Appendices C and D.

2 The free energy of the $O(n)$ rotator with isospin chemical potential

We first consider the case with unbroken symmetry and subsequently compute the leading order effect due to the presence of a small $O(n)$ symmetry breaking mass term.

2.1 Symmetric case

The Hamiltonian of the $O(n)$ quantum rotator with a chemical potential coupled to the generator $L_{12}$ of rotations in the $12$–plane is

$$H_0(h) = \frac{\hat{L}^2}{2\Theta} - h\hat{L}_{12}, \quad (2.1)$$

where $\Theta$ is the moment of inertia; to lowest order $\chi$PT one has $\Theta \simeq F^2L_s^3$. The energy levels and the corresponding multiplicities $g^{(n)}_{lm}$ are found in App. A.

The corresponding partition function for Euclidean time extent $L_t$ and its expansion to $O(h^2)$ is given by

$$Z(h; \Theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g^{(n)}_{lm} \exp \left\{ - \left( \frac{C_{n,l}}{2\Theta} - hm \right) L_t \right\} \quad (2.2)$$

where $u = L_t/(2\Theta)$ and

$$C_{n,l} = l(l + n - 2), \quad (2.3)$$

is the eigenvalue of the quadratic Casimir for isospin $l$.

$^1$The notation $l$ is used below because of the analogy with the rotator, here it stands for the isospin.
The coefficients \( z_0(u) \), \( z_1(u) \) and their expansions for small \( u \) are given in App. A. Using these results, the isospin susceptibility of the O(\( n \)) rotator for the unbroken case is given by

\[
\chi = \frac{1}{L_t V_s} \frac{\partial^2}{\partial h^2} \ln Z(h; \Theta) \bigg|_{h=0} = \frac{2\Theta}{n L_s^3} \left[ 1 - \frac{L_t}{6\Theta} (n - 2) + \frac{L_t^2}{180\Theta^2} (n - 2)(n - 4) + \ldots \right].
\]

(2.4)

2.2 Rotator in external field

Consider the effective Lagrangian with a term, breaking the O(\( n \)) symmetry down to O(\( n - 1 \)). In the delta-regime this corresponds to an effective O(\( n \)) rotator in an external “magnetic field”, given by the Hamiltonian (including the isospin chemical potential)

\[
H(h) = H_0(h) + \eta \hat{S}_n = \frac{\hat{L}_t^2}{2\Theta} + \eta \hat{S}_n - h \hat{L}_{12}.
\]

(2.5)

In the effective theory to leading order \( \eta \sim M^2 \Theta \sim M^2 F^2 L_s^3 \). The spectrum and the multiplicities are described in A.1.2

Expanding the partition function in \( h \) and \( \eta \) to NL order one obtains (cf. A.3.2)

\[
Z(h; \Theta, \eta) = \sum_{l=0}^{\infty} \sum_{k=0}^{l} \sum_{m=-k}^{k} g_{km}^{(n-1)} \exp \left\{ - \left( E^{(n)}(l, k) - hm \right) L_t \right\}
\]

\[
= z_0(u) - \eta^2 \Theta L_t z_2(u) + \frac{1}{2} h^2 L_t^2 \left[ z_1(u) - \eta^2 \Theta L_t z_3(u) \right] + \ldots
\]

(2.6)

The expansion of \( z_i(u) \) for small \( u \) is given in Appendix A. The final result for the susceptibility of the rotator in an external magnetic field is

\[
\chi = \frac{2\Theta}{n V_s} \left\{ 1 - \frac{1}{3} (n - 2) u + \frac{1}{45} (n - 2)(n - 4) u^2 - \frac{\eta^2 L_t^2}{n(n + 2)} \left[ 1 - \frac{1}{6} (2n - 5) u + \frac{1}{60} (n^2 - 12n + 17) u^2 \right] + \ldots \right\}.
\]

(2.7)

3 Perturbative computation of the free energy

The expression (2.7) for the susceptibility was obtained under the assumption that the low lying spectrum in the \( \delta \)-regime is that of the quantum rotator. In the next sub-section we first consider the symmetric case and show the consistency of (2.4) with our previous computation of the susceptibility in [1] in the \( \epsilon \)-expansion.

In the \( \epsilon \)-expansion one considers a 4d box of fixed shape, in the case of an \( L_s^3 \times L_t \) volume a fixed aspect ratio \( \ell = L_t/L_s \). After separating the constant mode (the direction of the average magnetization over the 4d volume), integration over the \( p \neq 0 \) modes result in an expansion in powers of \( (F^2 L_t^2)^{-1} \), where by convention \( L = (L_s^3 L_t)^{1/4} \). (For simplicity, here we discuss only the case without explicit symmetry breaking.) The coefficients of the expansion contain the shape coefficients depending on \( \ell \). Usually one assumes \( \ell \sim 1 \). For
a long tube, $\ell \gg 1$, the *spatially constant* modes, $p_0 \neq 0$, $p = 0$, become soft which means that one needs a larger box size $L$ to have a reasonable $\epsilon$–expansion. This is manifested by the fact that the shape coefficients are increasing functions of the aspect ratio $\ell$.

In the $\delta$–expansion one separates the dynamics of the spatially constant modes described by the $O(n)$ rotator, with the energies $O \left( F^{-2} L_s^{-3} \right)$. For $L_l \gg L_s/(2\pi)$ the $p \neq 0$ modes are exponentially suppressed in the partition function, hence the regions of validity of the $\epsilon$– and $\delta$–expansions overlap in this case – they should coincide up to exponentially small corrections $\sim \exp(-c\ell)$. This involves relations between the 4d shape coefficients for $\ell \gg 1$ and the 3d shape coefficients in a cubic box, as given e.g. in (B.24) and (B.28)-(B.30).

In subsection 3.2 we compute the susceptibility for small quark masses (i.e. small $M$) in the framework of $\chi$PT and subsequently use the comparison to the result (2.7) to determine the coefficient of $M^2$ in $\eta$ as function of $F, L_s$ to NNLO.

### 3.1 The symmetric case

From the mass gap calculation in [1] we obtained the moment of inertia:

$$\Theta = F^2 L_s^3 \left[ 1 + \frac{\tilde{\Theta}_1}{F^2 L_s^2} + \frac{\tilde{\Theta}_2}{F^4 L_s^4} + \ldots \right], \quad (3.1)$$

with

$$\tilde{\Theta}_1 = (n - 2) \beta_1^{(3)}, \quad (3.2)$$

$$\tilde{\Theta}_2 = (n - 2) \left[ \beta_1^{(3)} \left( \beta_1^{(3)} + \frac{3}{4} \right) - 2c_w \right] - \frac{5(n - 2)\rho}{12\pi^2} \log(\tau L_s M) - 4\rho (2l_1^n + nl_2^n). \quad (3.3)$$

Here $\beta_1^{(3)} \equiv \beta_1^{(3)}(1)$ is a shape function corresponding to a cubic 3d volume, $\rho = 8\pi^2 \beta_2^{(3)}(1)$ (B.28); for notations undefined here we refer the reader to [8] and [1]. Further $l_i^n$ are renormalized low energy constants (LEC) of Gasser and Leutwyler [9];

$$\ln \tau = -\frac{1}{2} \ln(4\pi) - \gamma_E + 1 = -1.476904292, \quad (3.4)$$

and the constant $c_w$ was determined in [10] as

$$c_w = 0.0986829798, \quad (3.5)$$

differing slightly from the original computation of Hasenfratz [3].

Inserting (3.1) into (2.4) one obtains

$$\chi = \frac{2}{n} F^2 \left[ 1 + \frac{1}{F^2 L_s^2} \left( \tilde{\Theta}_1 - \frac{\ell}{6}(n - 2) \right) + \frac{1}{F^4 L_s^4} \left( \tilde{\Theta}_2 + \frac{\ell^2}{180}(n - 2)(n - 4) \right) + \ldots \right]. \quad (3.6)$$
On the other hand the susceptibility calculated by \( \chi \) PT is given by

\[
\chi = \frac{2}{n} F^2 \left[ 1 + \frac{\tilde{R}_1}{F^2 L_s^2} + \frac{\tilde{R}_2}{F^4 L_s^4} + \ldots \right],
\]

(3.7)

with

\[
\tilde{R}_1 = -\frac{(n-2)}{4\pi} (\gamma_2 - 1),
\]

(3.8)

and

\[
\tilde{R}_2 = -\frac{(n-2)}{16\pi^2} \left[ (\gamma_2 - 1)^2 + 8\pi (\gamma_2 - 1) \beta_1 + 2\bar{W} - \frac{(n-2)}{\ell} \gamma_3 \right]
\]
\[
+ \frac{(n-2)}{24\pi^2} \left[ \frac{3n-7}{\ell} - 5 \left( \gamma_1 - \frac{1}{2} \right) \right] \log (\ell LM_\pi)
\]
\[
- 2 \left( 2\ell_r^1 + nl_r^2 \right) \left( \gamma_1 - \frac{1}{2} \right) + 4 [(n-1)\ell_r^1 + \ell_r^2] \frac{1}{\ell},
\]

(3.9)

The next-to-leading (NLO) and NNLO terms in (3.6) and (3.7) agree for large \( \ell \) provided that

\[
\tilde{R}_1 + \frac{\ell}{6} (n-2) \simeq \tilde{\Theta}_1,
\]

(3.10)

\[
\tilde{R}_2 - \frac{\ell^2}{180} (n-2)(n-4) \simeq \tilde{\Theta}_2.
\]

(3.11)

These involve relations between the 4d shape functions in the long cylinder (\( \ell \gg 1 \)) to the shape functions for a 3d cube. Derivations of the relations required are supplied in App. B.

Firstly at NLO, using the large \( \ell \) behavior of \( \gamma_2 \) given in (B.29), one indeed verifies (3.10).

Next, the three relations (B.24), (B.28), (B.30) for the 1-loop shape functions together with a fourth relation (B.45) referring to the 2-loop sunset diagram, ensure the validity of the NNLO consistency condition (3.11). This agreement provides, in our opinion, a highly non-trivial check on the various rather technical computations.

3.2 Perturbative computation of the isospin susceptibility for \( \mathcal{O}(n) \) with a mass term using dimensional regularization

Here we restrict attention to a \( L_s^{d-1} \times L_\ell, d = 4 \) volume with periodic boundary conditions in all directions. For perturbative computations we employ dimensional regularization and add further \( D-4 \) dimensions of size \( \hat{L} = L_s \). In ref. [1] we did not fix the value of \( \ell \equiv \hat{L}/L_s \); confirming independence of physical quantities on \( \ell \) serves as an additional useful check of the computations.

3.2.1 The effective action with mass terms

The effective action is given by

\[
A = A_0 + A_M,
\]

(3.12)
where \( A_M \) vanishes in the chiral limit \( M = 0 \), with
\[
A_0 = \sum_{r=1} A_{2r},
\]
\[
A_M = \sum_{r=1} A_{M;2r}.
\]

The lowest order actions are (\( S(x)^2 = 1 \))
\[
A_2 = \frac{1}{2g_0^2} \int_x \sum_{\mu} (\partial_{\mu} S(x) \partial_{\mu} S(x)) ,
\]
\[
A_{M;2} = -\frac{M^2}{g_0^2} \int_x S_n(x) .
\]

There are two independent 4–derivative terms in the massless case:
\[
A_4 = \sum_{i=2,3} \frac{g_4^{(i)}}{4} A_4^{(i)},
\]
with the \( A_4^{(i)} \) given in App. C.1.

Terms involving the mass parameter in the next order are
\[
A_{M;4} = \sum_{i=1,2} L_i A_{M;4}^{(i)},
\]
with
\[
A_{M;4}^{(1)} = M^4 \int_x S_n(x)^2 ,
\]
\[
A_{M;4}^{(2)} = \frac{1}{2} M^2 \int_x S_n(x) \sum_{\mu} (\partial_{\mu} S(x) \partial_{\mu} S(x)) .
\]

Actually one of the terms above is redundant for physical quantities. To see this consider the infinitesimal change of variables (preserving \( S^2 = 1 \)):
\[
S \rightarrow S + \epsilon \delta(S),
\]
with
\[
\delta(S_n) = -\sum_{j=1}^{n-1} S_j^2 ,
\]
\[
\delta(S_i) = S_i S_n , \quad i \neq n .
\]
Under this change the leading \( \chi \)PT action changes according to
\[
\delta (A_2 + A_{M;2}) = \frac{1}{M^2 g_0^2} \left[ -A_{M;4}^{(1)} + 2A_{M;4}^{(2)} \right] .
\]

Rearranging \( A_{M;4} \)
\[
L_1 A_{M;4}^{(1)} + L_2 A_{M;4}^{(2)} = \frac{1}{2} [2L_1 + L_2] A_{M;4}^{(1)} + \frac{1}{2} L_2 \left[ -A_{M;4}^{(1)} + 2A_{M;4}^{(2)} \right] ,
\]
we see that physical quantities should just depend on the combination \( 2L_1 + L_2 \); this will serve as a check on the computed contributions coming from the \( A_{M;4}^{(i)} \).
3.2.2 The chemical potential with mass terms

The chemical potential $h$ is introduced by

$$ \partial_\theta \to \partial_\theta - hQ, $$

(3.26)

where $(QS)_1 = iS_2$, $(QS)_2 = -iS_1$, $(QS)_3 = \ldots = 0$. This gives the $h$-dependent part of the action:

$$ A_h + A_M h, $$

where the massless part has contributions

$$ A_h = A_{2h} + A_{4h} + \ldots $$

(3.27)

Here we only consider terms up to and including order $h^2$:

$$ A_{2h} = ihB_2 + h^2C_2 + \ldots, \quad A_{4h} = ihB_4 + h^2C_4 + \ldots, $$

(3.28)

The expressions for $B_2, C_2, B_4^{(i)}, C_4^{(i)}$ are given in App. C.1.

To lowest order for the mass term

$$ A_{Mh} = ihB_{M;4} + h^2C_{M;4} + \ldots, $$

(3.29)

with

$$ B_{M;4} = -M^2L^2 \int_x S_n(x) j_0(x), $$

(3.30)

$$ C_{M;4} = \frac{1}{2}M^2L^2 \int_x S_n(x)[QS(x)]^2, $$

(3.31)

where $j_\mu(x)$ is defined in (C.3).

The $h$-dependent part of the free energy $f_h$ is given by ($V_D = L_tL_s^{D-1}$):

$$ e^{-V_D f_h} = \frac{Z(h)}{Z(0)} = \frac{\langle e^{-A_h - A_M - A_{Mh}} \rangle}{\langle e^{-A_M} \rangle} $$

$$ = 1 + \left[ 1 + \langle A_M \rangle + \langle A_M \rangle^2 - \frac{1}{2} \langle A_M^2 \rangle \right] \left[ -\langle A_M \rangle + \frac{1}{2} \langle A_M^2 \rangle \right] $$

$$ + [1 + \langle A_M \rangle] \left[ \langle A_h A_M \rangle - \frac{1}{2} \langle A_h^2 A_M \rangle - \langle A_M h \rangle + \langle A_h A_{Mh} \rangle \right] $$

$$ - \frac{1}{2} \langle A_h A_M^2 \rangle + \frac{1}{4} \langle A_h^2 A_M^2 \rangle $$

$$ + \langle A_M A_{Mh} \rangle - \langle A_h A_M A_{Mh} \rangle + \frac{1}{2} \langle A_{Mh}^2 \rangle + O(M^6, h^3). $$

(3.32)
where, $\langle \ldots \rangle$ denotes correlation functions with the massless action, and we have only kept terms up to order $M^4$ and $h^2$. So for the free energy we have

$$-V_D f_h [1 + \langle A_M \rangle + \langle A_M \rangle^2 - \frac{1}{2} \langle A_M^2 \rangle] \left[ -\langle A_h \rangle + \frac{1}{2} \langle A_h^2 \rangle \right]$$

$$+ [1 + \langle A_M \rangle] \left[ \langle A_h A_M \rangle - \frac{1}{2} \langle A_h^2 A_M \rangle - \langle A_{Mh} \rangle + \langle A_M A_{Mh} \rangle \right]$$

$$- \frac{1}{2} \langle A_h A_M^2 \rangle + \frac{1}{2} \langle A_h^2 A_M^2 \rangle + \langle A_M A_{Mh} \rangle - \langle A_h A_M A_{Mh} \rangle + \frac{1}{2} \langle A_M^2 \rangle$$

$$+ \langle A_h \rangle [\langle A_h A_M \rangle - \langle A_{Mh} \rangle] - \frac{1}{2} \langle A_h \rangle^2 \left[ 1 + 2 \langle A_M \rangle + 3 \langle A_M^2 \rangle - \langle A_M^4 \rangle \right]$$

$$+ \langle A_h \rangle \left[ 2 \langle A_M \rangle \langle A_h A_M \rangle - 2 \langle A_M \rangle \langle A_{Mh} \rangle - \frac{1}{2} \langle A_h A_M^2 \rangle + \langle A_M A_{Mh} \rangle \right]$$

$$- \frac{1}{2} \left[ \langle A_h A_M \rangle - \langle A_{Mh} \rangle \right]^2 + O(M^6, h^3). \quad (3.33)$$

Noting that

$$\langle A_h \rangle = O(h^2), \quad (3.34)$$

and that correlation functions with an odd number of spins $S$ vanish:

$$\langle A_{M;2} \rangle = 0 = \langle A_{M;A}^{(2)} \rangle, \quad (3.35)$$

many contributions drop out so that to the order we are considering we have

$$-V_D f_h = \left[ 1 + L_1 \langle A_M^{(1)} \rangle - \frac{1}{2} \langle A_M^{(2)} \rangle \right] \left[ -\langle A_h \rangle + \frac{1}{2} \langle A_h^2 \rangle \right]$$

$$- \frac{h^2}{2} \langle C_2 A_{M;2}^2 \rangle - \frac{h^2}{4} \langle B_2 A_{M;2}^2 \rangle$$

$$+ h^2 L_1 \langle C_2 A_{M;4}^{(1)} \rangle - \frac{h^2}{2} \langle C_4 A_{M;4}^2 \rangle - h^2 L_2 \langle C_2 A_{M;2} A_{M;4}^{(2)} \rangle + h^2 \langle A_{M;2} C_{M;4} \rangle$$

$$- \frac{h^2}{2} \langle B_2 A_{M;4}^{(1)} \rangle - \frac{h^2}{2} \langle B_2 A_{M;2} A_{M;4}^{(2)} \rangle$$

$$- \frac{h^2}{2} \langle B_2 B_{M;2} A_{M;4}^{(2)} \rangle + h^2 \langle B_2 A_{M;2} B_{M;4} \rangle + O(M^6, h^3) + h^4, \quad (3.36)$$

where “ho” stands for higher order terms in the chiral expansion. In particular we drop terms $O(M^4 g_0^6)$.

The terms appearing in the averages above are not $O(n)$ invariant, and so before starting the perturbative computations we set $S(x) = \Omega(x)$ and average over the rotations $\Omega$. The resulting expressions are given in App. C.2.

The free energy has a small $h$ expansion

$$f_h = -\frac{1}{2} h^2 \chi(M) + O(h^4). \quad (3.37)$$

Using the results in App. C.2 the uniform susceptibility (at $h = 0$) is given by

$$\chi(M) = \chi_0 + \frac{M^4 V_D^2}{g_0^2} \chi_1 + O(M^6), \quad (3.38)$$
where $\chi_0$ is the susceptibility for $M = 0$ which we computed previously [1]:

$$\chi_0 = \frac{2}{ng_0^2} - \frac{4}{n(n-1)g_0^2} \langle W \rangle - \frac{2}{V_D} \langle C_4 \rangle,$$  \tag{3.39}

where $W$ is defined in (C.11) and $\langle C_4^{(2)} \rangle$ in (C.12) and (C.13).

Next, collecting the terms in App. C.2,

$$\chi_1 = \frac{1}{2n} \left[ -1 - P_1 + \frac{2}{V_D} g_0^3 L_1 + g_0^2 L_2 P_4 \right] \chi_0$$

$$+ \frac{1}{n(n-1)(n+2)g_0^2} \left[ n - 1 + (n-3)P_1 - 2P_2 - \frac{1}{(n-2)} P_3 \right]$$

$$+ \frac{1}{n(n+2)V_D} \left[ -2g_0^2 L_1 - n\langle C_4 \rangle - L_2 \int_z \langle (\partial_\mu R(z) \partial_\mu R(z)) \rangle + 2g_0^2 L_2 \right] + O(g_0^4), \tag{3.40}
$$

where $P_1, P_2, P_3, P_4$ are given in (C.17), (C.21), (C.25), (C.30) respectively.

Now we can begin with the perturbative computations proceeding as usual by first separating the zero mode and then changing to $\vec{\pi}$ variables according to $R = (g_0\vec{\pi}, \sqrt{1 - g_0^2\pi^2})$. Details of the perturbative computations to NNLO are given in App. C.3.

Summing the terms (for $d = 4$) one obtains for $\chi_0$ the result (3.7), (note $F^2 = 1/g_0^2$).

For $\chi_1$ it is shown that it has a perturbative expansion of the form

$$\chi_1 = -\frac{2}{n^2(n+2)} F^2 \left[ 1 + \frac{\tilde{\chi}_1^{(1)}}{F^2 L_s^2} + \frac{\tilde{\chi}_1^{(2)}}{F^4 L_s^4} + \ldots \right]. \tag{3.41}
$$

Here the NLO coefficient is

$$\tilde{\chi}_1^{(1)} = (2n-1)\beta_1 + \frac{1}{2\pi} (\gamma_2 - 1). \tag{3.42}
$$

Using the behavior of the shape functions $\beta_1$ and $\gamma_2$ given in (B.24) and (B.28) we have for large $\ell$:

$$\tilde{\chi}_1^{(1)} \simeq (2n-3)\beta_1^{(3)}(1) - \frac{1}{12} (2n - 5)\ell. \tag{3.43}
$$

At the NNLO we first show that in order to cancel the $1/(D - 4)$ pole terms we must have

$$L_1 + \frac{1}{2} L_2 = \frac{(n-3)}{32\pi^2} \left[ \frac{1}{D-4} + \ln(\bar{\pi} A_3) \right]. \tag{3.44}
$$

This agrees with the result of Gasser and Leutwyler [9] for $n = 4$ if we set $L_1 = -l_3, L_2 = 0$. Then

$$\tilde{\chi}_1^{(2)} = \tilde{R}_2 + \frac{1}{64\pi^2} \left[ 4n(\gamma_2 - 1)^2 + 32(2n-1)(n+1)\pi^2 \beta_1^2 + 32(2n-1)\pi \beta_1 (\gamma_2 - 1) + \frac{2}{\ell} \left\{ (4n^2 - 5n + 3)\alpha_2 - 4n(n-1)\gamma_3 - 2(n-3) \ln(\bar{\pi} A_3 L_s) \right\} \right.

\left. + (5n - 4n^2 - 3)\frac{1}{\ell^2} \right]. \tag{3.45}
$$
Using the expansions (B.20) for \( s = 2, d = 4 \) and (B.30) we have for large \( \ell \):

\[
\tilde{\chi}_1^{(2)} \simeq \tilde{\Theta}_2 + \frac{1}{2} (2n - 3)(n - 1)\beta_1^{(3)}(1)^2 \\
- \frac{1}{12} (2n - 5)(n - 1)\beta_1^{(3)}(1)\ell + \frac{1}{240}(n^2 - 12n + 17)\ell^2 \\
+ \frac{(n - 3)}{16\pi^2\ell} \left\{ \frac{1}{3} - \ln(\pi\Lambda_3 L_s) - \frac{1}{2} \theta_0^{(3)}(1) \right\} .
\]

Agreement of the rotator result (2.7) with the perturbative result for large \( \ell \) above requires that the parameter \( \eta \) in the rotator Hamiltonian (2.5) has a \( \chi \)PT expansion of the form:

\[
\eta = M^2 F^2 L_s^3 \left[ 1 + \frac{Z_1}{F^2 L_s^2} + \frac{Z_2}{F^4 L_s^4} + \ldots \right] + O(M^3) ,
\]

with

\[
Z_1 = \frac{1}{2}(n - 1)\beta_1^{(3)}(1) ,
\]

\[
Z_2 = -\frac{1}{8}(n - 1)(n - 3)\beta_1^{(3)}(1)^2 .
\]

4 A final comment

It is at present not known at which order \( \chi \)PT the rotator spectrum is modified. Concerning this question we remark that the consistency of the matching between the \( \chi \)PT and the rotator results on the susceptibility can (under reasonable assumptions) give quite strong constraints on possible corrections to the spectrum.

As an example, assume that the spectrum of the transfer matrix has a correction proportional to the square of the quadratic Casimir eigenvalue:

\[
\Delta E_l \sim \frac{1}{L_s^2} \frac{C_{n,l}^2}{(F^2 L_s^2)^k} .
\]

The corresponding contribution to the isospin susceptibility is then

\[
\frac{\Delta \chi}{\chi} \sim \frac{L_s}{L_l} \frac{1}{(F^2 L_s^2)^{k-2}} .
\]

Since the computations for the susceptibility obtained in the two expansions, as reported here, match to order \( (F^2 L_s^2)^{-2} \) for large \( \ell \) up to exponentially small corrections, it practically follows that such a deviation from the rotator spectrum can occur first for \( k \geq 5 \).

The corresponding correction can be obtained along the lines indicated in (A.43), (A.46).

A The effective \( O(n) \) rotator

A.1 Spectrum and multiplicities

A.1.1 Symmetric case

The energy levels of the Hamiltonian (2.1) are

\[
E_{lm}^{(n)} = \frac{1}{2\Theta} C_{n,l} - hm .
\]
Here $m$ is the quantum number associated with rotation in the 12-plane, with values
$m = -l, \ldots, l$.

The subspace of states corresponding to a given energy level can be decomposed ac-
cording to their transformation properties under the O($n-1$) transformations affecting the
first $n-1$ components.

The multiplicity of the energy level (A.1), $g_{lm}^{(n)}$ satisfies the recursion relation

$$g_{lm}^{(n)} = \sum_{k=|m|}^{l} g_{km}^{(n-1)}, \quad (A.2)$$

For $n = 3, 4$ one has (assuming $|m| \leq l$)

$$g_{lm}^{(3)} = 1, \quad g_{lm}^{(4)} = l - |m| + 1. \quad (A.3)$$

The solution of (A.2), (A.3) for general $n$ is given by

$$g_{lm}^{(n)} = \frac{(l - |m| + n - 3)!}{(n-3)!l!} \cdot (2l + n - 2). \quad (A.4)$$

The total multiplicity of states with a given $l$ is

$$g_{l}^{(n)} = \sum_{m=-l}^{l} g_{lm}^{(n)} = \frac{(l + n - 3)!}{(n-2)!l!} (2l + n - 2). \quad (A.5)$$

A.1.2 Rotator in external field

Consider now the O($n$) rotator in a small external magnetic field, with Hamiltonian given
in (2.5). For simplicity we take here $h = 0$ since its effect is simply adding $-hm$ to the
energy levels, as in (A.1).

The external field splits the energy levels $E^{(n)}(l) = C_{n,l}/(2\Theta)$ corresponding to a given
O($n$) isospin $l$ into levels characterized by the O($n-1$) isospin $k$, where $k = 0, \ldots, l$.

To determine the expansion of the energy levels for small breaking parameter $\eta$ one
needs the transition matrix elements between the corresponding eigenstates $^3$. These are

$$\epsilon_{lk}^{(n)} = \langle l+1,k,\alpha|\hat{S}_n|l,k,\alpha\rangle = \sqrt{\frac{(l+1-k)(l+n-2+k)}{(2l+n-2)(2l+n)}}, \quad 0 \leq k \leq l. \quad (A.6)$$

Here $\alpha$ denotes the remaining quantum numbers besides $k$ characterizing an O($n-1$)
eigenvector. (For $n = 3$ one has $k = |m|$ and $\alpha$ is the sign of $m$.) The leading order PT
gives

$$E^{(n)}(l,k) = E^{(n)}(l) + \epsilon^{(n)}(l,k)\eta^2\Theta + O(\eta^4\Theta^3), \quad (A.7)$$

where

$$\epsilon^{(n)}(l,k) = \rho^{(n)}(l-1,k) - \rho^{(n)}(l,k), \quad (A.8)$$

$^3$ unit normalized $\langle l,k,\alpha|l,k,\alpha\rangle = 1$
\[ \rho^{(n)}(l, k) = \frac{2(l + 1 - k)(l + n - 2 + k)}{(2l + n - 2)(2l + n - 1)(2l + n)}. \tag{A.9} \]

Accordingly the energy of the ground state becomes
\[ E^{(n)}(0, 0) = -\frac{2}{n(n - 1)}\eta^2\Theta, \tag{A.10} \]
while the \( n \)–plet of \( O(n) \) is split into a singlet and \( (n - 1) \)–plet under \( O(n - 1) \):
\[
\begin{align*}
E^{(n)}(1, 0) &= \frac{n - 1}{2\Theta} - \frac{2(n - 7)}{(n + 2)(n^2 - 1)}\eta^2\Theta, \\
E^{(n)}(1, 1) &= \frac{n - 1}{2\Theta} - \frac{2}{(n + 1)(n + 2)}\eta^2\Theta.
\end{align*}
\tag{A.11}
\]

The lower one is the \( (n - 1) \)–plet. For \( n = 4 \) one obtains for the mass gap
\[ E^{(1)}(1, 1) - E^{(0)}(0, 0) = \frac{3}{(2\Theta)^{1/2}}[1 + \eta^2\Theta^2/15 + \ldots], \]
in agreement with \([4, 5]\).

A useful check for the transition matrix element and the multiplicity is to calculate the trace of \( \hat{S}_n^2 \) in the \( O(n) \) invariant subspace with given \( l \) which should give \( g^{(n)}_l/n \).

\[
\text{Tr}(\hat{S}_n^2 P_l) = \sum_{k=0}^{l} \sum_{\alpha} \langle l, k, \alpha | \hat{S}_n^2 | l, k, \alpha \rangle
= \sum_{k=0}^{l} g^{(n-1)}_k \left[ \left( v^{(n)}(l, k) \right)^2 + \left( v^{(n)}(l - 1, k) \right)^2 \right] = \frac{1}{n} g^{(n)}_l.
\tag{A.12}
\]

### A.2 The transition matrix elements

The squared angular momentum operator in \( n \) dimensions, \( \hat{L}_2 \) is given (up to the sign) by the angular Laplacian \( \Delta_{S^{n-1}} \) on the unit sphere \( S^{n-1} \). The eigenfunctions of \( \hat{L}_2 \) corresponding to the eigenvalue \( C_{n,l} \) can be written as
\[
\Psi_l(\phi_1, \ldots, \phi_{n-2}, \theta) = f_{lk}(\theta)g_k(\phi_1, \ldots, \phi_{n-2})
\tag{A.13}
\]
where \( g_k(\phi_1, \ldots, \phi_{n-2}) \) is an eigenfunction of the \( n - 1 \) dimensional squared angular momentum operator \( \sum_{i=1}^{n-1} \hat{L}_i^2 \) with eigenvalue \( k(k + n - 3) \) and \( f_{lk}(\theta) \) satisfies the differential equation
\[
(1 - x^2)y'' - (n - 1)xy' + \left[ C_{n,l} - \frac{k(k + n - 3)}{1 - x^2} \right] y = 0,
\tag{A.14}
\]
where \( x = \cos \theta \) and \( f_{lk}(\theta) = y(\cos \theta) \).

For \( k = 0 \) the solution is given by the \( O(n) \) Legendre polynomials
\[
P_{n0} = \frac{(-1)^l\Gamma \left( \frac{n-1}{2} \right)}{2\Gamma \left( l + \frac{n-1}{2} \right)} (1 - x^2)^{-(n-3)/2}\left( \frac{d}{dx} \right)^l (1 - x^2)^{l+(n-3)/2}.
\tag{A.15}
\]
The coefficient is chosen to satisfy the normalization condition \( P_{n0}(1) = 1 \).
For general \( k \) the solution of the differential equation (A.14) is given by the associated \( O(n) \) Legendre functions:

\[
P_{nlk} = (1 - x^2)^{k/2} \left( \frac{d}{dx} \right)^k P_{nl0}(x).
\] (A.16)

Two of these functions corresponding to different values of \( l \) (and same \( n, k \)) are orthogonal with the weight

\[
\int_{-1}^{1} dx \ (1 - x^2)^{(n-3)/2} \ldots
\] (A.17)

For convenience we introduce the notation

\[
\langle f(x) \rangle = \int_{-1}^{1} dx \ (1 - x^2)^{(n-3)/2} f(x).
\] (A.18)

The transition matrix element (A.6) is then given by

\[
\frac{\langle xP_{nlk}(x)P_{nl+1,k}(x) \rangle}{\sqrt{\langle P_{nlk}(x)^2 \rangle \langle P_{nl+1,k}(x)^2 \rangle}} = \sqrt{\frac{(l+1-k)(l+n-2+k)}{(2l+n-2)(2l+n)}}.
\] (A.19)

A.3 Partition function of the \( O(n) \) rotator

A.3.1 Symmetric case

Here we consider the partition function (2.2) of the \( O(n) \) rotator without an external magnetic field. Expanding it in the chemical potential, the expansion coefficients are given by

\[
z_0(u) = \sum_{l=0}^{\infty} w_0(l) e^{-u c_{nl}; l} = \frac{\Gamma \left( (n-1)/2 \right)}{\Gamma (n-1)} u^{-(n-1)/2}
\]

\[
\times \left( 1 + \frac{u}{6} (n-1)(n-2) + \frac{u^2}{360} (n-1)(n-2)(5n^2 - 17n + 18) + \ldots \right),
\] (A.20)

and

\[
z_1(u) = \sum_{l=0}^{\infty} w_1(l) e^{-u c_{nl}; l} = 2 \frac{\Gamma \left( (n+1)/2 \right)}{\Gamma (n+1)} u^{-(n+1)/2}
\]

\[
\times \left( 1 + \frac{u}{6} (n-2)(n-3) + \frac{u^2}{360} (n-2)(n-5)(5n^2 - 17n + 18) + \ldots \right).
\] (A.21)

Here the weights \( w_i(l) \) are (cf. (A.5))

\[
w_0(l) \equiv g^{(n)}_l,
\] (A.22)

\[
w_1(l) = \sum_{m=-l}^{l} g^{(n)}_{lm} m^2 = 2 \frac{(l + n - 2)!}{n!(l-1)!} [2l + n - 2].
\] (A.23)

The coefficients appearing in (A.20), (A.21) can be determined numerically. The sum converges fast enough to separate the different powers of \( u \) and then obtain the corresponding polynomials in \( n \), “beyond a reasonable doubt”. An exact calculation of the expansions is given in Appendix A.4.
A.3.2 Rotator in external field

For calculating $z_i(u)$ to order $\eta^2$ and $h^2$ one also needs the weights

$$w_2(l) = \sum_{k=0}^{l} \epsilon^{(n)}(l, k) g_k^{(n-1)}$$

$$= -\frac{2(n-3)(2l+n-2)}{n(2l+n-3)(2l+n-1)} \frac{(l+n-3)!}{l!(n-2)!},$$

(A.24)

and

$$w_3(l) = \sum_{k=0}^{l} \sum_{m=-k}^{k} \epsilon^{(n)}(l, k) g_{km}^{(n-1)} m^2$$

$$= -\frac{4(n-1)(2l+n-2)}{(n+2)(2l+n-3)(2l+n-1)} \frac{(l+n-2)!}{(l-1)!n!},$$

(A.25)

With these one obtains

$$z_2(u) = \sum_{l=0}^{\infty} w_2(l) e^{-u c_{n; l}} = -2\frac{\Gamma \left( (n+1)/2 \right)}{\Gamma (n+1)} u^{-(n-3)/2}$$

$$\times \left( 1 + \frac{u}{6} (n-1)(n-3) + \frac{u^2}{360} (n-1)(n-3)(5n^2 - 22n + 18) + \ldots \right),$$

(A.26)

and

$$z_3(u) = \sum_{l=0}^{\infty} w_3(l) e^{-u c_{n; l}} = -4\frac{\Gamma \left( (n+3)/2 \right)}{\Gamma (n+3)} u^{-(n-1)/2}$$

$$\times \left( 1 + \frac{u}{6} (n-1)(n-5) + \frac{u^2}{360} (n-1)(n-7)(5n^2 - 22n + 18) + \ldots \right).$$

(A.27)

For the partition function these give

$$Z(h; \Theta, \eta) = z_0(u) \left[ 1 - \eta^2 \Theta L_t r_2(u) + \frac{1}{2} h^2 L_t^2 \left( r_1(u) - \eta^2 \Theta L_t r_3(u) \right) + \ldots \right],$$

(A.28)

where $r_i(u) = z_i(u)/z_0(u)$.

Finally, one obtains for the corresponding susceptibility

$$\chi = \frac{L_t}{V_3} \left[ r_1(u) - \eta^2 \Theta L_t [r_3(u) - r_1(u)r_2(u)] + \ldots \right],$$

(A.29)

from which (2.7) follows.

A.4 Expansion of $z_i(u)$ for small $u$

Consider the sum

$$f_\nu(u) = \sum_{k=1}^{\infty} k^\nu e^{-uk^2}.$$

(A.30)

A useful representation to obtain the behavior as $u \to 0$ is

$$f_\nu(u) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dt \, u^{-t} \Gamma(t) \zeta(2t - \nu)$$

(A.31)
where \( \sigma > (\nu + 1)/2 \) and \( \zeta(s) \) is the Riemann zeta-function

\[
\zeta(s) = \sum_{k=1}^{\infty} k^{-s}.
\]  

(A.32)\( \zeta(s) \) has a pole at \( s = 1 \) with residue 1. Also we will need

\[
\zeta(0) = -\frac{1}{2}, \quad \zeta(-n) = -\frac{B_{n+1}}{n+1}, \quad n \geq 1,
\]

(A.33)\( \) where \( B_n \) are the Bernoulli numbers

\[
B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \ldots
\]

(A.35)\( \) We will use the convention \( B_1 = \frac{1}{2} \) to have \( \zeta(0) = -B_1 \).

Also we note the Gamma function \( \Gamma(t) \) has poles at \( t = -n, \quad n = 0, 1, 2, \ldots \) with residue

\[
\text{Res}_{t=-n} \Gamma(t) = \frac{(-1)^n}{n!}.
\]

(A.36)\( \) Shifting the integration contour to the left we pick up the residues of the poles, the first one at \( t = (\nu + 1)/2 \) from the \( \zeta \) function and then poles at \( t = 0, -1, \ldots \) from the Gamma function. In this way we get (with the convention \( B_1 = 1/2 \)) for small \( u \)

\[
f_{\nu}(u) \sim \frac{1}{2} \Gamma \left( \frac{\nu + 1}{2} \right) u^{-(\nu+1)/2} - \sum_{k=0}^{\infty} \frac{(-1)^k u^k}{k! (2k + \nu + 1)} B_{2k+\nu+1}.
\]

(A.37)Explicitly

\[
f_{\nu}(u) \sim \frac{1}{2} \Gamma \left( \frac{\nu + 1}{2} \right) u^{-(\nu+1)/2} - \frac{1}{\nu + 1} B_{\nu+1} + \frac{u}{\nu + 3} B_{\nu+3} - \frac{u^2}{2!(\nu + 5)} B_{\nu+5} + \ldots
\]

(A.38)For even \( \nu \) the function \( f_{\nu}(u) \) is related to the Jacobi theta-function and its derivatives. In particular

\[
S(u) = 1 + 2 f_0(\pi u) = u^{-1/2} \left( 1 + 2 e^{-\pi/u} + \ldots \right).
\]

(A.39)In this case there are no power corrections from the sum in (A.37), except the \(-1/2\) term for \( k = \nu = 0 \).

A.4.1 Even \( n \) values

Consider (A.22) first for general integer \( n \) introducing the variable \( q = l + n/2 - 1 \) and

\[
\tilde{w}_0(q) = w_0(l) = \frac{2q}{(n-2)!} (q-n/2+2)(q-n/2+3) \ldots (q+n/2-2).
\]

(A.40)It is even/odd according to the parity of \( n, \tilde{w}_0(-q) = (-1)^n \tilde{w}_0(q) \). Expanding in powers of \( q \) one has

\[
\tilde{w}_0(q) = \sum_{r=1}^{[n/2-1]} \tilde{\gamma}_{0,r} q^{n-2r}
\]

(A.41)
where \( |x| = \text{floor}(x) \), with the leading coefficients

\[
\tilde{\gamma}_{0,1} = \frac{2}{(n-2)!}, \quad \tilde{\gamma}_{0,2} = -\frac{1}{12(n-5)!}, \quad \tilde{\gamma}_{0,3} = \frac{(5n-8)}{2880(n-7)!}
\]

(A.42)

For the case when \( n \) is even one has

\[
z_0(u) = \sum_{l=0}^{\infty} w_0(l)e^{-uC_{n,l}} = e^{u(n/2-1)^2} \sum_{k=n/2-1}^{\infty} w_0(k-n/2+1)e^{-uk^2}
\]

\[
= e^{u(n/2-1)^2} \sum_{k=1}^{\infty} \tilde{w}_0(k)e^{-uk^2} = e^{u(n/2-1)^2} \sum_{r=1}^{n/2-1} \tilde{\gamma}_{0,r} f_{n-2r}(u).
\]

(A.43)

Here we extended the summation range from \( k = n/2-1, \ldots, \infty \) to \( k = 1, \ldots, \infty \) observing that \( \tilde{w}_0(k) = 0 \) for \( k = 1, 2, \ldots, n/2 - 2 \). Finally, inserting (A.37) and (A.42) we obtain the expansion (A.20). The calculation goes the same way for \( z_i(u), i = 1, 2, 3 \) yielding the results stated in (A.21), (A.26) and (A.27).

It is interesting to note that the properties listed above for \( \tilde{w}_0(g) \) are essential for having a proper power series for \( z_i(u) \), otherwise one would obtain in the small-\( u \) expansion a mixture of integer and half-integer powers.

### A.4.2 Odd \( n \) values

Define

\[
\tilde{f}_\nu(u) = \sum_{k=0}^{\infty} \left( k + \frac{1}{2} \right)^\nu e^{-u(k+1/2)^2}
\]

(A.44)

One has

\[
\tilde{f}_\nu(u) \sim \frac{1}{2} \Gamma \left( \frac{\nu + 1}{2} \right) u^{-(\nu+1)/2} + (1 - 2^{\nu-1}) \frac{1}{\nu + 1} B_{\nu+1}
\]

\[
- (1 - 2^{-\nu-2}) \frac{u}{\nu + 3} B_{\nu+3} + (1 - 2^{-\nu-4}) \frac{2u^2}{2!(\nu + 5)} B_{\nu+5} + \ldots
\]

(A.45)

In this case

\[
z_0(u) = \sum_{l=0}^{\infty} w_0(l)e^{-uC_{n,l}} = e^{u(n/2-1)^2} \sum_{l=0}^{\infty} w_0(l)e^{-u(l+(n-3)/2+1/2)^2}
\]

\[
= e^{u(n/2-1)^2} \sum_{k=0}^{\infty} w_0(k+(n-3)/2+1/2)e^{-u(k+1/2)^2}
\]

(A.46)

\[
= e^{u(n/2-1)^2} \sum_{k=0}^{\infty} \tilde{w}_0(k+1/2)e^{-u(k+1/2)^2} = e^{u(n/2-1)^2} \sum_{r=1}^{(n-1)/2} \tilde{\gamma}_{0,r} f_{n-2r}(u).
\]

This yields the same analytic form (A.20) as obtained for even \( n \) values.
B Shape coefficients for long 4d tube

In this appendix we calculate the behavior of the 1– and 2–loop shape functions (appearing in our perturbative computations) in \(d = 4\) dimensions for a long “tube”, \(L_3^s \times L_t\) with \(\ell \equiv L_t/L_s \gg 1\). They are related to the shape functions defined for a 3–dimensional cube.

We first recall some useful relations involving the Jacobi theta–function and some properties of the free massless propagator in an asymmetric periodic volume.

B.1 Some properties Jacobi theta–function

The Jacobi theta–function is defined by

\[
S(u, z) = \sum_{n=-\infty}^{\infty} e^{-\pi u (n+z)^2} = u^{-1/2} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/u} \cos(2\pi nz). \tag{B.1}
\]

The first sum above converges quickly for \(u \geq 1\) while the second for \(0 < u \leq 1\). For small and large \(u\) it is given by

\[
S(u, z) = \begin{cases} 
  u^{-1/2} + O(e^{-\pi/u}), & \text{for } u \to 0, \\
  e^{-\pi z^2 u} + O(e^{-\pi u/4}), & \text{for } u \to \infty, |z| \leq 1/2. 
\end{cases} \tag{B.2}
\]

Defining

\[
S(u) \equiv S(u,0), \tag{B.3}
\]

one has

\[
\int_{-1/2}^{1/2} \mathrm{d}z \ S\left(\frac{1}{u}, z\right) = \sqrt{u}, \tag{B.4}
\]

\[
\int_{-1/2}^{1/2} \mathrm{d}z \ S\left(\frac{1}{u}, z\right) S\left(\frac{1}{v}, z\right) = \sqrt{uv} S(u + v), \tag{B.5}
\]

and

\[
\int_{-1/2}^{1/2} \mathrm{d}z \ S\left(\frac{1}{u}, z\right) S\left(\frac{1}{v}, z\right) = 4\pi \sqrt{uv} S'(u + v). \tag{B.6}
\]

Some relations which are also used in the following:

\[
\int_0^\infty \mathrm{d}t \ t^{a-1} (S^d(t) - 1) = \int_0^\infty \mathrm{d}t \ t^{a-1} \left[ S^d(t) \right]_{\sub} - \frac{2}{d-2a} - \frac{1}{a} \tag{B.7}
\]

\[
= \alpha_a^{(d)}(1) - \frac{d}{a(d-2a)},
\]

since

\[
\int_0^1 \mathrm{d}t \ t^{a-1} (t^{-d/2} - 1) = -\frac{2}{d-2a} - \frac{1}{a}. \tag{B.8}
\]

In particular

\[
\int_0^\infty \mathrm{d}t \ t^{a-1} (S^3(t) - 1) = \alpha_a^{(3)}(1) - \frac{3}{a(3-2a)} = \alpha_{3/2-a}^{(3)}(1) - \frac{3}{a(3-2a)}. \tag{B.9}
\]
For $D \sim 4$, $a = 3/2$ one obtains a pole term
\[
\int_0^\infty dt \ t^{1/2} \left( S^{D-1}(t) - 1 \right) = - \frac{2}{D - 4} + o_D(1) - \frac{2}{3} + O(D - 4). \tag{B.10}
\]

### B.2 Some properties of the free massless propagator

In this paper we employ dimensional regularization and add $q = D - 4$ extra dimensions of length $\hat{L} = L_k$. We introduce $L_0 = L_t$ and $L_\mu = L_s$, $\mu \geq 1$ and the volume $V_D = \bar{V}_D L_t$, $\bar{V}_D = L_s^{D-1}$.

The massless propagator with periodic boundary conditions in all directions is given by
\[
G(x) = \frac{1}{V_D} \sum_p e^{i p x} \frac{1}{p^2}, \tag{B.11}
\]
where the sum is over momenta $p = 2\pi(n_0/L_0, \ldots, n_{D-1}/L_{D-1})$, $n_k \in \mathbb{Z}$, and the prime on the sum means that the zero momentum is omitted: $\sum'_p = \sum_{p \neq 0}$.

For $G(x)$ and $\ddot{G}(x)$ we have the representations \cite{10} (3.55)
\[
L_s^{D-2} G(x) = \frac{1}{4\pi} \int_0^\infty du \left\{ u^{-D/2} \dot{S} \left( \frac{u^2}{u}, z_0 \right) \prod_{\mu=1}^{D-1} S \left( \frac{1}{u}, z_\mu \right) - \frac{1}{V_D} \right\} \bigg|_{z_\nu = x_\nu/L_\nu}, \tag{B.12}
\]
and
\[
L_s^D \ddot{G}(x) = \frac{1}{4\pi \ell^2} \int_0^\infty du \ u^{-D/2} \ddot{S} \left( \frac{\ell^2}{u}, z_0 \right) \prod_{\mu=1}^{D-1} S \left( \frac{1}{u}, z_\mu \right) \bigg|_{z_\nu = x_\nu/L_\nu}, \tag{B.13}
\]
where $\ddot{S}(u,z) = \partial_z^2 S(u,z)$.

To study the large $\ell$ behavior it is convenient to separate the 1d propagator with periodic b.c. from $G(x)$:
\[
G(x) = G_1(x) + R(x), \tag{B.14}
\]
where
\[
G_1(x) = \frac{L_t}{V_D} \Delta_1 \left( \frac{x_0}{L_t} \right), \quad (|x_0| \leq L_t/2), \tag{B.15}
\]
with the 1d propagator with pbc on the interval $z \in [-1/2, 1/2]$,
\[
\Delta_1(z) = - \frac{1}{2} |z| + \frac{1}{2} z^2 + \frac{1}{12} = \frac{1}{4\pi} \int_0^\infty du \ u^{-1/2} \dot{S} \left( \frac{1}{u}, z \right) - 1. \tag{B.16}
\]

Next
\[
R(x) = \sum_{m=-\infty}^{\infty} R(x_0 + m L_t, x), \tag{B.17}
\]
with (see eq. (5.8) in \cite{2})
\[
R(x) = \frac{1}{2 V_D} \sum_{p \neq 0} \frac{1}{\omega_p} e^{-\omega_p |x_0|} e^{i p x}, \quad (\omega_p = |p|). \tag{B.18}
\]
The function $R(x)$ is defined in (B.18) for all $x \in \mathbb{R}^D \setminus 0$; in particular for $|x_0| \to \infty$ the function $R(x)$ falls exponentially. The singularity at $x = 0$ is regularized dimensionally through an alternative representation in terms of the Jacobi theta function $S(u, z)$:

$$L_s^{D-2} R(x) = \frac{1}{4\pi} \int_0^\infty \frac{du}{u} u^{-x_0^2/(L_s^2 u)} \left\{ u^{-(D-1)/2} \prod_{\mu=1}^{D-1} S \left( \frac{1}{u} \frac{x_\mu}{L_s} \right) - 1 \right\}. \tag{B.19}$$

**B.3 1-loop shape functions**

We start with the 1–loop functions $\alpha_s^{(d)}(\ell), \beta_s(\ell), \gamma_s(\ell)$; for notations undefined here we again refer the reader to [8] and [1].

Neglecting terms decreasing exponentially fast with $\ell$ one has

$$\alpha_s^{(d)}(\ell) = \frac{1}{\ell} \int_0^\infty dt \; t^{1-s} \left[ S \left( \frac{t}{\ell^2} \right) S^{d-1}(t) \right]_{\text{sub}}$$

$$\equiv \frac{1}{\ell} \int_0^\infty dt \; t^{1-s} \left[ S \left( \frac{t}{\ell^2} \right) S^{d-1}(t) - \ell t^{-d/2} \right] + \frac{1}{\ell} \int_1^\infty dt \; t^{1-s} \left[ S \left( \frac{t}{\ell^2} \right) S^{d-1}(t) - 1 \right]$$

$$\simeq \int_0^\infty dt \; t^{s-3/2} \left[ S^{d-1}(t) \right]_{\text{sub}} + \frac{1}{\ell} \int_1^\infty dt \; t^{1-s} \left[ S \left( \frac{t}{\ell^2} \right) - 1 \right]$$

$$= \alpha_{s-1/2}^{(d-1)}(1) + 2\pi^{-s} \Gamma(s) \zeta(2s) \ell^{2s-1} - \frac{2}{2s-1} + \frac{1}{s} \ell. \tag{B.20}$$

The pole contributions at $s = 0$ and $s = 1/2$ cancel and one obtains in the corresponding limits

$$\alpha_{1/2}^{(d)}(\ell) \simeq \alpha_{0}^{(d-1)}(1) + 2\ln(\ell) + \gamma_E - \ln(4\pi) + \frac{2}{\ell}, \tag{B.21}$$

$$\alpha_{0}^{(d)}(\ell) \simeq \alpha_{-1/2}^{(d-1)}(1) + 2 - 2\frac{\ln(\ell)}{\ell} + \left[ \gamma_E - \ln(4\pi) \right] \frac{1}{\ell}. \tag{B.22}$$

Note that $\alpha_{s-1/2}^{(d-1)}(1) = \alpha_{d/2-s}^{(d-1)}(1)$.

Now

$$\beta_k^{(d)}(\ell) = \left( -\frac{1}{4\pi} \right)^k \left[ \alpha_k^{(d)}(\ell) + \frac{2}{2k-d} - \frac{1}{k\ell} \right], \quad (k \neq 0, d/2). \tag{B.23}$$

From this and (B.20) we get for $d = 4, k = 1$:

$$\beta_1^{(4)}(\ell) + \frac{\ell}{12} \simeq \beta_1^{(3)}(1). \tag{B.24}$$

This relation holds numerically extremely well already at $\ell = 4$ where it gives $-0.10754837390 \simeq -0.10754837389$, illustrating the fact that the approach to the $\ell = \infty$ limit is exponentially fast. The same comment will apply to similar relations obtained in the following.

\[^4\text{Note } \beta_s \text{ is only defined for integer values of } s.\]
For $\gamma_s(\ell)$ we obtain
\begin{align*}
\gamma_s^{(d)}(\ell) &= -\frac{2}{\ell^3} \int_0^\infty \mathrm{d}t \, t^{s-1} \left[ S^{d-1}(t) S'(\frac{t}{\ell^2}) \right]_{\sub}
\equiv -\frac{2}{\ell^3} \int_0^1 \mathrm{d}t \, t^{s-1} \left[ S^{d-1}(t) S'(\frac{t}{\ell^2}) \right] + \frac{1}{2} \ell^3 t^{-d/2-1} \nonumber
- \frac{2}{\ell^3} \int_1^\infty \mathrm{d}t \, t^{s-1} S^{d-1}(t) S'(\frac{t}{\ell^2})
\simeq \int_0^\infty \mathrm{d}t \, t^{s-3/2} \left[ S^{d-1}(t) \right]_{\sub} - 2\ell^{2s-3} \int_1^{\infty} \mathrm{d}t \, t^{s-1} S'(t)
= \alpha_s^{(d-1)}(1) - 2\ell^{2s-3} \int_1^{\infty} \mathrm{d}t \, t^{s-1} S'(t),
\end{align*}
where $\ell = 1$.

Note that $\alpha_s^{(d-1)}(1) = \alpha_{d/2-s+1}^{(d-1)}(1)$. For $s = 1$ this gives
\begin{align*}
\gamma_1^{(d)}(\ell) &\simeq \alpha_{-1/2}^{(d-1)}(1) + 2 - \frac{2}{\ell} = \alpha_{d/2}^{(d-1)}(1) + 2 - \frac{2}{\ell},
\end{align*}
and for $s > 3/2$
\begin{align*}
\gamma_s^{(d)}(\ell) &\simeq \alpha_{d/2-s+1}^{(d-1)}(1) - \frac{2}{2s-3} + 4\pi^{1-s} \Gamma(s) \zeta(2s-2) \ell^{2s-3}.
\end{align*}

Note since $\zeta(0) = -1/2$ this also reproduces (B.26) for $s = 1$.

For $d = 4$ the above relations imply for $s = 1$:
\begin{align*}
\frac{1}{2} \left( \gamma_1^{(4)}(\ell) - \frac{1}{2} \right) + \frac{1}{\ell} &\simeq \rho \equiv 8\pi^2 \beta_2^{(3)}(1) = \frac{1}{2} \alpha_2^{(3)}(1) + \frac{3}{4},
\end{align*}
for $s = 2$:
\begin{align*}
- \frac{1}{4\pi} \left( \gamma_2^{(4)}(\ell) - 1 \right) + \frac{\ell}{6} &\simeq \beta_1^{(3)}(1),
\end{align*}
and for $s = 3$:
\begin{align*}
\gamma_3^{(4)}(\ell) - \frac{4\pi^2}{45} \ell^3 &\simeq \alpha_0^{(3)}(1) - \frac{2}{3}.
\end{align*}

### B.4 The sunset diagram for large $\ell$

The sunset diagram for the susceptibility is (cf. [2] (4.1) and (4.36))
\begin{align*}
\Psi(\ell, \hat{\ell}) &= L_{s D}^{2D-4} \int_{V_D} \mathrm{d}x \, \hat{G}(x) \hat{G}^2(x)
\quad = -\frac{1}{48\pi^2(D-4)} \left( 10 \hat{g}(0; \ell, \hat{\ell}) - \frac{1}{V_D} \right) - \frac{1}{16\pi^2} \overline{\Psi}(\ell) + O(D - 4),
\end{align*}
where $\ell D = \hat{\ell} D^4 = \ell$.

For the analogous diagram in the infinite strip one had (cf. [2] (5.11) and (5.61))
\begin{align*}
\overline{\Psi}(\ell) &= L_{s D}^{2D-4} \int_{V_{\infty}} \mathrm{d}x \, \hat{R}(x) R^2(x) = -\frac{1}{48\pi^2(D-4)} 10 \hat{R}(0; \ell) - c_w + O(D - 4).
\end{align*}
Further, for $\ell \gg 1$ one has up to exponentially small corrections (cf. [2](5.20))

$$
\tilde{g}(0; \ell, \ell) - \frac{1}{V_D} \simeq \tilde{R}(0; \ell).
$$

(B.33)

Since $R(x) \simeq R(x)$ for $\ell \gg 1$ we have

$$
\Delta \Psi \equiv -L_s^{2D-4} \int_{V_D} dx \left[ \tilde{G}(x)G^2(x) - \tilde{R}(x)R^2(x) \right] \simeq -\Psi(\ell, \ell) + \Psi(\ell).
$$

(B.34)

Writing $G(x) = G_1(x) + R(x)$ we have (noting $\int dx R(x) = 0$)

$$
\Delta \Psi = -L_s^{2D-4} \int_{V_D} dx \left\{ \tilde{G}_1(x)G^2(x) + 2G_1(x)R(x)R(x) \right\} = \Delta \Psi_1 + \Delta \Psi_2.
$$

(B.35)

The first term here is

$$
\Delta \Psi_1 = -L_s^{2D-4} \int_{V_D} dx \tilde{G}_1(x)G^2(x) = \Delta \Psi_{1a} + \Delta \Psi_{1b}
$$

$$
= L_s^{D-3} \int_{V_D} dx G^2(0, x) - \frac{1}{\ell} L_s^{D-4} \int_{V_D} dx G^2(x).
$$

(B.36)

### B.4.1 Calculating $\Delta \Psi_1$

Since $\Delta \Psi_{1a}$ is regular at $D = 4$ we can calculate it in 4-dimensions. Using (B.6) one has

$$
\Delta \Psi_{1a} = L_s \int_{V_s} dx G^2(0, x)
$$

$$
= \frac{1}{16\pi^2} \int_{V_s} dx \int_0^\infty dv d\mu \left( (uv)^{-2} S\left( \frac{\ell^2}{u} \right) S\left( \frac{\ell^2}{v} \right) \prod_{\mu=1}^3 S\left( \frac{1}{u} \frac{\mu}{L_s} \right) S\left( \frac{1}{v} \frac{\mu}{L_s} \right) 

- \frac{1}{\ell} u^{-2} S\left( \frac{\ell^2}{u} \right) \prod_{\mu=1}^3 S\left( \frac{1}{u} \frac{\mu}{L_s} \right) - \frac{1}{\ell} v^{-2} S\left( \frac{\ell^2}{v} \right) \prod_{\mu=1}^3 S\left( \frac{1}{v} \frac{\mu}{L_s} \right) + \frac{1}{\ell^2} \right)
$$

$$
= \frac{1}{16\pi^2} \int_0^\infty dv d\mu (uv)^{-1/2} S\left( \frac{\ell^2}{u} \right) S\left( \frac{\ell^2}{v} \right) [S^3(u + v) - 1]

+ \frac{\ell^2}{16\pi^2} \left\{ \int_0^\infty d\mu (S(u) - 1) \right\}^2
$$

$$
= \frac{1}{16\pi} \left( \alpha_1^3(1) - 3 \right) + \frac{\ell^2}{144}.
$$

From [2] (3.74),(3.30) one has

$$
\Delta \Psi_{1b} = -\frac{1}{V_D} L_s^{D-4} \int_{V_D} dx G^2(x) = \frac{1}{8\pi^2(D-4)V_D} - \frac{1}{16\pi^2\ell} \left( \alpha_2(\ell) - \frac{1}{2\ell} \right)
$$

$$
\simeq \frac{1}{8\pi^2(D-4)V_D} - \frac{1}{720} \ell^2 - \frac{1}{16\pi^2\ell} \left( \frac{\alpha_3(1)}{3} - \frac{2}{3} \right).
$$

(B.38)
Combining the results we get
\[ \Delta \Psi_1 \simeq \frac{1}{8\pi^2(D - 4)V_D} - \frac{1}{16\pi^2 \ell} \left( \alpha_0^{(3)}(1) - \frac{2}{3} \right) + \frac{1}{16\pi} \left( \alpha_1^{(3)}(1) - 3 \right) + \frac{\ell^2}{180}. \] (B.39)

**B.4.2 Calculating \( \Delta \Psi_2 \)**

\[ \Delta \Psi_2 = -2L_s^{2D-4} \int_{V_D} dx \ G_1(x) \tilde{R}(x) \mathcal{R}(x) = \Delta \Psi_{2a} + \Delta \Psi_{2b} \]

\[ = -2L_s^{2D-4} \int_{V_D} dx \ G_1(x) \tilde{G}(x)G(x) + 2L_s^{2D-4} \int_{V_D} dx \ \tilde{G}_1(x)G_1^2(x). \] (B.40)

We have

\[ \Delta \Psi_{2a} = -2L_s^{2D-4} \int_{V_D} dx \ G_1(x) \tilde{G}(x)G(x) = -2 \frac{1}{16\pi^2} \int_{-1/2}^{1/2} dz_0 \Delta_1(z_0) \]

\[ \times \int_{0}^{\infty} du \nu^{-1/2} \bar{S} \left( \frac{\nu^2}{u}, z_0 \right) \left\{ v^{-1/2} \bar{S} \left( \frac{\nu^2}{v}, z_0 \right) S^{D-1}(u + v) - 1 \right\} \]

\[ = -2 \frac{1}{16\pi^2} \int_{-1/2}^{1/2} dz_0 \Delta_1(z_0) \]

\[ \times \int_{0}^{\infty} du \nu^{-1/2} \bar{S} \left( \frac{\nu^2}{u}, z_0 \right) \left\{ v^{-1/2} \bar{S} \left( \frac{\nu^2}{v}, z_0 \right) \right\} \]

\[ \simeq -2 \frac{1}{16\pi^2} \int_{-1/2}^{1/2} dz_0 \Delta_1(z_0) \left\{ \int_{0}^{\infty} du \nu^{-1/2} \left( -2\pi \nu^2 u^{-1} + 4\pi^2 \nu^4 u^{-2} z_0^2 \right) \right\} \]

\[ \times e^{-\pi z_0^2 \nu (1/u + 1/v)} \left[ S^{D-1}(u + v) - 1 \right] - 2\ell^2 \int_{-1/2}^{1/2} dz_0 \Delta_1(z_0) \Delta_1^2(z_0). \]

The second integral cancels with \( \Delta \Psi_{2b} \) in (B.40).

For \( \ell^2(1/u + 1/v) \gg 1 \) one has

\[ \int_{-1/2}^{1/2} |z_0|^r \exp \left( -\pi z_0^2 \ell^2(1/u + 1/v) \right) \simeq \frac{\Gamma \left( \frac{r+1}{2} \right)}{\pi^{(r+1)/2}} \left( \frac{uv}{u + v} \right)^{(r+1)/2}. \] (B.42)

Integrating over \( z_0 \) one obtains \(^5\)

\[ \Delta \Psi_2 \simeq -2 \frac{1}{16\pi^2} \int_{0}^{\infty} du \nu^{-1/2} \left[ S^{D-1}(u + v) - 1 \right] \]

\[ \times \left\{ \frac{\pi}{6(u+v)^{3/2}} \ell + \frac{\sqrt{v(u-v)}}{\sqrt{u(u+v)}} \frac{u(u-2v)}{2(u+v)^{5/2}} \frac{1}{\ell} \right\} \]

\[ = \frac{1}{16\pi^2} \left\{ \frac{\pi}{3} \left( \alpha_0^{(3)}(1) - 3 \right) \ell + \frac{\pi}{2} \left( \alpha_1^{(3)}(1) - 3 \right) \right\} + \frac{1}{(D - 4)V_D} \left[ 2 \left( \alpha_0^{(3)}(1) - \frac{2}{3} \right) \left( \alpha_1^{(3)}(1) - 3 \right) \right] \ell. \] (B.43)

\(^5\) Note \( \int_{0}^{\infty} du \nu^{-1/2} f(u,v)g(u,v) = \frac{1}{2} \int_{0}^{\infty} dt f(t) \int_{-1}^{1} d\eta g((t + \eta)/2, (t - \eta)/2). \) The integral over \( \eta \) in our case gives a power of \( t. \)
Finally using (B.23) one gets
\[
\Delta \Psi \simeq \frac{3}{16\pi^2 (D-4)V_D} - \frac{1}{12\beta_1} (1) \left( \ell + \frac{9}{2} \right) \\
- \frac{1}{16\pi^2} \left( \frac{3}{2} \alpha_0^{(3)} (1) - 1 \right) \frac{1}{\ell} + \frac{\ell^2}{180}.
\] (B.44)

Comparing this with (B.34) one obtains the desired relation
\[
\frac{1}{16\pi^2} V(\ell) - \frac{1}{180} \ell^2 + \frac{1}{12\beta_1} (1) \left( \ell + \frac{9}{2} \right) + \frac{1}{16\pi^2} \left( \frac{3}{2} \alpha_0^{(3)} (1) - 1 \right) \frac{1}{\ell} \simeq c_w. \] (B.45)

The lhs converges exponentially; e.g. for \( \ell = 4 \) it is evaluated as 0.0986829793 which agrees to 8 significant figures with (3.5). In fact already at \( \ell = 2 \) the lhs is very close to \( c_w \) as one can check using the result eq.(4.45) in [2].

C Details of the perturbative perturbative computation in subsect. 3.1

C.1 Contributions to the action

The four derivative terms in (3.17) are given by \( \int \cdots = \int_{V_D} dx \cdots \):
\[
A_4^{(2)} = \int_x \sum_{\mu\nu} (\partial_x S(x) \partial_x S(x)) (\partial_x S(x) \partial_x S(x)), \] (C.1)
\[
A_4^{(3)} = \int_x \sum_{\mu\nu} (\partial_x S(x) \partial_x S(x)) (\partial_x S(x) \partial_x S(x)). \] (C.2)

Terms appearing in (3.28) are given by the following:
\[
B_2 = -\frac{1}{g_0^2} \int_x j_0(x), \quad j_\mu(x) = S_2(x) \partial_x S_1(x) - S_1(x) \partial_x S_2(x), \] (C.3)
\[
C_2 = \frac{1}{2g_0^2} \int_x [Q S(x)]^2. \] (C.4)

For the operator 2:
\[
B_4^{(2)} = -4 \int_x (\partial_x S(x) \partial_x S(x)) j_0(x), \] (C.5)
\[
C_4^{(2)} = -2 \int_x \left\{ (\partial_x S(x) \partial_x S(x)) \left[ S_1(x)^2 + S_2(x)^2 \right] + 2 \left[ j_0(x) \right]^2 \right\}, \] (C.6)

and for the operator 3:
\[
B_4^{(3)} = -4 \int_x (\partial_x S(x) \partial_x S(x)) j_\mu(x), \] (C.7)
\[
C_4^{(3)} = -2 \int_x \left\{ (\partial_x S(x) \partial_x S(x)) \left[ S_1(x)^2 + S_2(x)^2 \right] + 2 \left[ j_0(x) \right]^2 + \left[ j_k(x) \right]^2 \right\}. \] (C.8)
C.2 Averages over rotations

In this section we set \( S(x) = \Omega R(x) \) in the correlation functions in (3.36) and average over the rotations.

For the mass independent terms we have

\[
\langle A_{2h} \rangle = \frac{-\hbar^2 V_D}{n g_0^2}, \tag{C.9}
\]

and (using (D.14)):

\[
-\frac{1}{2} \langle A_{2h}^2 \rangle = \frac{2\hbar^2 V_D}{n(n-1)g_0} \langle W \rangle, \tag{C.10}
\]

where

\[
W = \frac{1}{V_D} \int_{xy} (\partial_\theta R(x)\partial_\theta R(y)) \left[ (R(x)R(y)) - 1 \right]. \tag{C.11}
\]

For the 4-derivative terms:

\[
\langle C^{(2)}_4 \rangle = -\frac{4}{n(n-1)} \int_x ((n-1)(\partial_\mu R(x)\partial_\mu R(x)) + 2(\partial_\theta R(x)\partial_\theta R(x))), \tag{C.12}
\]

\[
\langle C^{(3)}_4 \rangle = -\frac{4}{n(n-1)} \int_x ((\partial_\mu R(x)\partial_\mu R(x)) + n(\partial_\theta R(x)\partial_\theta R(x))). \tag{C.13}
\]

Turning now to the mass terms we get

\[
\langle A^{(1)}_{M;4} \rangle = \frac{1}{n} M^4 V_D, \tag{C.14}
\]

and

\[
\langle A^{2}_{M;2} \rangle = \frac{M^4}{n g_0} \int_{yz} \langle (R(y)R(z)) \rangle \tag{C.15}
\]

\[
= \frac{M^4 V_D^2}{n g_0^2} [1 + P_1], \tag{C.16}
\]

where

\[
P_1 = \frac{1}{V_D^2} \int_{yz} \langle (R(y)R(z)) - 1 \rangle. \tag{C.17}
\]

Using (D.13):

\[
\langle C_2 A^{2}_{M;2} \rangle = -\frac{M^4}{g_0} \int_{xyz} \langle S_1(x)^2 S_n(y)S_n(z) \rangle \tag{C.18}
\]

\[
= -\frac{M^4}{g_0^2} \frac{1}{n(n-1)(n+2)} \times \int_{yz} \langle (n+1)V_D(R(y)R(z)) - 2 \int_x (R(x)R(y))(R(x)R(z)) \rangle \tag{C.19}
\]

\[
= -\frac{M^4 V_D^3}{g_0^2} \frac{1}{n(n-1)(n+2)} [n - 1 + (n - 3)P_1 - 2P_2], \tag{C.20}
\]

where

\[
P_2 = \frac{1}{V_D^2} \int_{xyz} \langle [(R(x)R(y)) - 1][(R(x)R(z)) - 1] \rangle. \tag{C.21}
\]
Next, using (D.16):

\[ \langle B_2^2 A_{M;2}^3 \rangle = \frac{M^4}{g_0^2} \int_{xyz} \langle j_0(w) j_0(x) S_n(y) S_n(z) \rangle \]  \tag{C.22}

\[ = \frac{2M^4V_0^3}{n(n-1)(n-2)(n+2)g_0^2} P_3, \]  \tag{C.23}

where

\[ P_3 = \frac{1}{V_0^3 g_0^2} \int_{xyz} \langle n(R(y)R(z)) [R(w)R(x)](\partial_0 R(w) \partial_0 R(x)) - (R(w) \partial_0 R(x))(R(x) \partial_0 R(w)) \rangle \\
+ 2(R(x)R(z)) [R(w) \partial_0 R(x)(R(y) \partial_0 R(w)) - (\partial_0 R(w) \partial_0 R(x))(R(w)R(y))] \\
+ 2(R(z) \partial_0 R(x))(R(x) \partial_0 R(y))(R(w)R(y)) - (R(w)R(x))(R(y) \partial_0 R(w)) \rangle. \]  \tag{C.24}

By partial integration this simplifies to

\[ P_3 = \frac{2}{V_0^3 g_0^2} \int_{xyz} \langle 2n \langle (R(y)R(z)) - 4(R(x)R(z))(R(w)R(y)) \rangle \rangle \]  \tag{C.25}

\[ = \frac{2n}{g_0^2} (W + \overline{P}_3), \]  \tag{C.26}

where \( W \) is given in (C.11) and

\[ \overline{P}_3 = \frac{2}{V_0^3 g_0^2} \int_{xyz} \langle \langle n\{ (R(y)R(z)) - 1 \} \{ (R(w)R(x)) - 1 \} - 4 \{ (R(x)R(z)) - 1 \} \{ (R(w)R(y)) - 1 \} \rangle \rangle \times (\partial_0 R(w) \partial_0 R(x))) \rangle. \]  \tag{C.27}

Next

\[ \langle A_{M;2} A_{M;4}^{(2)} \rangle = -\frac{M^4}{2g_0^2} \int_{xy} S_n(x)S_n(y)(\partial_\mu S(y)\partial_\mu S(y)) \]  \tag{C.28}

\[ = -\frac{M^4V_0^3}{2ng_0^2} P_4, \]  \tag{C.29}

where

\[ P_4 = \frac{1}{V_0^3} \int_{xy} \langle R(x)R(y)))(\partial_\mu R(y)\partial_\mu R(y) \rangle, \]  \tag{C.30}

and

\[ \langle C_2 A_{M;4}^{(1)} \rangle = -\frac{M^4}{g_0^2} \int_{xy} \langle S_1(x)^2 S_n(y)^2 \rangle \]  \tag{C.31}

\[ = -\frac{M^4}{g_0^2} \frac{1}{n(n-1)(n+2)} \int_{y} \langle (n+1)V_B - 2 \int_{x}(R(x)R(y))^2 \rangle \]  \tag{C.32}

\[ = -\frac{M^4V_0^3}{g_0^2} \frac{1}{n(n-1)(n+2)} [n - 1 + (n - 3)P_1 - 2P_3] \]  \tag{C.33}
where
\[ P_5 = \frac{1}{V_0^2} \int_{xy} \langle (\mathbf{R}(x)\mathbf{R}(y)) - 1 \rangle^2 = O(g_0^4). \] (C.34)

Proceeding to the terms \( \langle C_4^{(i)} A_{M;2}^2 \rangle \), first
\[ \langle C_4^{(2)} A_{M;2}^2 \rangle = -\frac{4M^4}{g_0^4} \left[ Q^{(2A)} + Q^{(2B)} \right], \] (C.35)

with
\[ Q^{(2A)} = \int_{xyz} \langle (\partial_\mu \mathbf{S}(x)\partial_\mu \mathbf{S}(x))S_1(x)^2S_n(y)S_n(z) \rangle \]
\[ = \frac{1}{n(n-1)(n+2)} \times \int_{xyz} \langle (\partial_\mu \mathbf{R}(x)\partial_\mu \mathbf{R}(x))[(n+1)(\mathbf{R}(y)\mathbf{R}(z)) - 2(\mathbf{R}(x)\mathbf{R}(y))(\mathbf{R}(x)\mathbf{R}(z))] \rangle \]
\[ = \frac{V_0^2}{n(n+2)} \int_x \langle (\partial_\mu \mathbf{R}(x)\partial_\mu \mathbf{R}(x)) \rangle + O(g_0^4), \] (C.36)

and
\[ Q^{(2B)} = \int_{xyz} \langle j_0(x)^2S_n(y)S_n(z) \rangle, \]
\[ = \frac{2V_0^2}{n(n-1)(n-2)(n+2)P_5}, \] (C.37)

where
\[ P_6 = \frac{1}{V_0^2} \int_{xyz} \langle n(\mathbf{R}(y)\mathbf{R}(z))\partial_0 \mathbf{R}(x)\partial_0 \mathbf{R}(x) \rangle - 2(\mathbf{R}(x)\mathbf{R}(z))(\mathbf{R}(x)\mathbf{R}(y)\partial_0 \mathbf{R}(x))(\mathbf{R}(y)\partial_0 \mathbf{R}(x)) \rangle \]
\[ = (n-2) \int_x \langle (\partial_0 \mathbf{R}(x)\partial_0 \mathbf{R}(x)) \rangle + O(g_0^4). \] (C.38)

So
\[ Q^{(2A)} + Q^{(2B)} = -\frac{V_0^2}{4(n+2)}\langle C_4^{(2)} \rangle + O(g_0^4). \] (C.39)

Similarly
\[ \langle C_4^{(3)} A_{M;2}^2 \rangle = -\frac{4M^4}{g_0^4} \left[ Q^{(3A)} + Q^{(3B)} + Q^{(3C)} \right], \] (C.40)

with
\[ Q^{(3A)} = \int_{xyz} \langle (\partial_0 \mathbf{S}(x)\partial_0 \mathbf{S}(x))S_1(x)^2S_n(y)S_n(z) \rangle \]
\[ = \frac{V_0^2}{n(n+2)} \int_x \langle (\partial_0 \mathbf{R}(x)\partial_0 \mathbf{R}(x)) \rangle + O(g_0^4), \] (C.41)

\[ Q^{(3C)} = \frac{1}{2} \int_{xyz} \langle j_k(x)^2S_n(y)S_n(z) \rangle \]
\[ = \frac{V_0^2}{n(n-1)(n+2)} \int_x \langle (\partial_k \mathbf{R}(x)\partial_k \mathbf{R}(x)) \rangle + O(g_0^4). \] (C.42)
\[ Q^{(3A)} + Q^{(3B)} + Q^{(3C)} = -\frac{V_0^2}{4(n+2)} \langle C_4^{(3)} \rangle + O(g_0^4) . \] (C.49)

Together we have simply

\[ \langle C_4 A_{M;2}^2 \rangle = \frac{M^4 V_0^2}{(n+2)g_0^4} \langle C_4 \rangle + O(M^4 g_0^0) . \] (C.50)

The next two averages in (3.36) to be taken into account are

\[ \langle C_2 A_{M;2}^{(2)} A_{M;4} \rangle = \frac{M^4}{2g_0^4} \int_{xyz} \langle S_1(x)^2 S_n(y) S_n(z) (\partial_\mu S(z) \partial_\mu S(z)) \rangle \] (C.51)

and

\[ \langle A_{M;2} C_{M;4} \rangle = \frac{M^4}{g_0^4} L_2 \int_{xy} \langle S_n(y) S_n(x) S_1(x)^2 \rangle \] (C.54)

With similar considerations we can show that the last 4 terms in (3.36) can be neglected to the order of interest:

\[ \langle B_2 A_{M;4}^{(1)} \rangle, \langle B_2 A_{M;2} A_{M;4}^{(2)} \rangle, \langle B_2 B_4^{(2)} A_{M;2}^{2} \rangle, \langle B_2 B_4^{(3)} A_{M;2}^{2} \rangle = O(M^4 g_0^0) . \] (C.57)

### C.3 Perturbative computations

After separating the zero mode and changing to \( \vec{\pi} \) variables according to \( \mathbf{R} = (g_0 \vec{\pi}, \sqrt{1 - g_0^2 \vec{\pi}^2}) \) the effective action is given by

\[ A_{\text{eff}}[\vec{\pi}] = A[\vec{\pi}] + A_{\text{zero}}[\vec{\pi}] , \] (C.58)

where with DR we have dropped the measure term, and

\[ A_{\text{zero}}[\vec{\pi}] = -(n-1) \ln \left( \frac{1}{V_D} \int_x (1 - g_0^2 \vec{\pi}(x)^2)^{\frac{1}{2}} \right) . \] (C.59)

The effective action has a perturbative expansion

\[ A_{\text{eff}} = A_0 + g_0^2 A_1 + g_0^4 A_2 + \ldots \] (C.60)
where

\[ A_0 = \frac{1}{2} \int_x \partial_\mu \bar{\pi}(x) \partial_\mu \bar{\pi}(x), \]  

(C.61)

and

\[ A_1 = A_1^{(a)} + A_1^{(b)}, \]  

(C.62)

with

\[ A_1^{(a)} = \frac{1}{8} \int_x \partial_\mu [\bar{\pi}(x)^2] \partial_\mu [\bar{\pi}(x)^2], \]  

(C.63)

\[ A_1^{(b)} = \frac{1}{2} \frac{(n-1)}{V_D} \int_x \bar{\pi}(x)^2. \]  

(C.64)

For the details of the perturbative computation of \( \chi_0 \) we refer the reader to [1].

\( P_j \) have perturbative expansions (to the order we need)

\[ P_j = P_j^{(1)} g_0^2 + P_j^{(2)} g_0^4 + \ldots, j = 1, 2, 3. \]  

(C.65)

The expansion of \( \chi_1 \) is then

\[
\chi_1 = -\frac{1}{2n} \left[ 1 + P_1 \right] \chi_0 + \frac{1}{n(n-1)(n+2)g_0^2} \left[ n - 1 + (n - 3)P_1 - 2P_2 - \frac{1}{(n-2)} P_3 \right] - \frac{1}{(n+2)V_D} \langle C_4 \rangle + \frac{2g_0^2}{n^2(n+2)V_D} \left[ 2L_1 + L_2 \right] + O(g_0^4) \\
= -\frac{2}{n^2(n+2)g_0^2} \left[ 1 + \chi_1^{(1)} g_0^2 + \chi_1^{(2)} g_0^4 + \ldots \right].
\]  

(C.66)

(C.67)

with

\[ \chi_1^{(1)} = \frac{1}{(n-1)} \left[ (2n-1)P_1^{(1)} + nP_2^{(1)} + \frac{n}{2(n-2)} P_3^{(1)} - \frac{2(n-1)}{(n-2)} R_1 \right], \]  

(C.68)

\[ \chi_1^{(2)} = \frac{1}{(n-1)} \left[ (2n-1)P_1^{(2)} + nP_2^{(2)} + \frac{n}{2(n-2)} P_3^{(2)} - \frac{2(n-1)}{(n-2)} R_2^{(a)} \right] - \frac{1}{2n} R_1 P_1^{(1)} + R_2^{(b)} - \frac{1}{V_D} [2L_1 + L_2].\]  

(C.69)

Here \( R_1, R_2^{(a)}, R_2^{(b)} \) appear in the perturbative expansions of \( \langle W \rangle \) and \( \langle C_4 \rangle \) [1]:

\[ \frac{1}{g_0^2} \langle W \rangle = \frac{1}{2(n-1)} \left[ g_0^2 R_1 + g_0^4 R_2^{(a)} + \ldots \right], \]  

(C.70)

\[ \langle C_4 \rangle = -\frac{V_D}{n} g_0^2 R_2^{(b)} + \ldots. \]  

(C.71)

In the following we will need the expansions

\[ (R(x)R(y)) = 1 - \frac{1}{2} g_0^2 |\bar{\pi}(x) - \bar{\pi}(y)|^2 - \frac{1}{8} g_0^4 |\bar{\pi}(x)^2 - \bar{\pi}(y)^2|^2 + \ldots, \]  

(C.72)

\[ (\partial_\mu R(x) \partial_\mu R(y)) = g_0^2 (\partial_\mu \bar{\pi}(x) \partial_\mu \bar{\pi}(y)) + \frac{1}{4} g_0^4 \partial_\mu [\bar{\pi}(x)^2] \partial_\mu [\bar{\pi}(y)^2] + \ldots. \]  

(C.73)
C.3.1 NL order

From (C.72) we have

\[ P^{(1)}_1 = -\frac{1}{V_D^2} \int_{yz} \langle \bar{\pi}(y)^2 \rangle_0 \]
\[ = -(n-1)T_{10}. \]  

(C.74)

Here \( T_{10} \) is a particular case of the dimensionally regularized sums

\[ T_{nm} = \frac{1}{V_D} \sum_p \frac{p^2_m}{(p^2)^n}, \]  

(C.76)

which are discussed in [2].

There are no contributions from \( P_2, T_3 \) to the NL order:

\[ P^{(1)}_2 = 0 = T^{(1)}_3. \]  

(C.77)

So

\[ \chi^{(1)}_1 = -(2n-1)T_{10} + 4T_{21}. \]  

(C.78)

Recalling for \( D = 4 \):

\[ T_{10} = -\frac{1}{L_s^2} \beta_1, \]  

(C.79)

\[ T_{21} = \frac{1}{8\pi L_s^2} (\gamma_2 - 1), \]  

(C.80)

we obtain (3.42).

C.3.2 NNL order

First we have

\[ P^{(2)}_1 = P^{(2a)}_1 + P^{(2b)}_1 + P^{(2c)}_1, \]  

(C.81)

with

\[ P^{(2a)}_1 = \frac{1}{V_D} \int_y \langle \bar{\pi}(y)^2 A_1^{(a)} \rangle_0 \]
\[ = (n-1) \int_{xy} \{ \partial_\mu \partial_\nu \{ G(y-z)G(y-x)G(x-z) \} \} z=x \]
\[ = (n-1) \left[ T_{10}^2 - \frac{1}{V_D} T_{20} \right], \]  

(C.82)

\[ P^{(2b)}_1 = \frac{1}{V_D} \int_y \langle \bar{\pi}(y)^2 A_1^{(b)} \rangle_0 \]
\[ = (n-1)^2 \frac{1}{V_D} T_{20}, \]  

(C.83)

\[ P^{(2c)}_1 = -\frac{1}{8V_D^2} \int_{yz} \langle [\bar{\pi}(y)^2 - \bar{\pi}(z)^2]^2 \rangle_0 \]
\[ = -\frac{1}{2}(n-1) \left[ T_{10}^2 - \frac{1}{V_D} T_{20} \right]. \]  

(C.84)
So
\[ P_1^{(2)} = \frac{1}{2} (n-1) \left[ \mathcal{T}_1^{(2)} + (2n-3) \frac{1}{V_D} \mathcal{T}_2^{(0)} \right]. \]  
(C.89)

Next
\[ P_2^{(2)} = \frac{1}{4V_D^2} \int_{xyz} \langle [\bar{\pi}(x) - \bar{\pi}(y)]^2|\bar{\pi}(x) - \bar{\pi}(z)]^2 \rangle_0 \]
\[ = \frac{1}{2} (n-1) \left[ (2n-1) \mathcal{T}_1^{(2)} + \frac{3}{V_D} \mathcal{T}_2^{(0)} \right]. \]  
(C.90)

Finally
\[ \mathcal{T}_3^{(2)} = -\frac{2}{V_D^2} \int_{wxy} \langle [n\bar{\pi}(y)^2(\bar{\pi}(w)\bar{\pi}(x)) + \bar{\pi}(w)^2\bar{\pi}(x)^2] (\partial_0 \bar{\pi}(w)\partial_0 \bar{\pi}(x)) \rangle_0 \]
\[ = -2n(n-1)(n-2) \left[ (n-1) \mathcal{T}_1^{(2)} + \frac{4}{V_D} \mathcal{T}_3^{(0)} \right] - 4(n-1)(n-2)\bar{\mathcal{W}}, \]  
where we have used
\[ \int_x G(x) \partial_0 G(x) \partial_0 G(x) = \frac{1}{2} \bar{\mathcal{W}}. \]  
(C.94)

So we obtain
\[ \chi_1^{(2)} = R_2 + \frac{1}{2} (2n-1)(n+1) \mathcal{T}_1^{(2)} + 4n \mathcal{T}_2^{(2)} - 4(2n-1) \mathcal{T}_1^{(2)} \mathcal{T}_2^{(1)} \]
\[ + \frac{1}{V_D} \left[ -8n(n-1) \mathcal{T}_3^{(1)} + \frac{1}{2} (4n^2 - 5n + 3) \mathcal{T}_2^{(0)} - 2L_1 - L_2 \right]. \]  
(C.95)

Note the result depends only on the combination $2L_1 + L_2$ as anticipated from our discussion in subsect. 3.2.

Recalling for $D \sim 4$
\[ \mathcal{T}_2^{(0)} = \frac{1}{8\pi} \left[ \ln L_s - \frac{1}{D-4} + \frac{1}{2} \alpha_2 - \frac{1}{4\ell} \right] + O(D-4), \]  
(C.96)
\[ \mathcal{T}_3^{(1)} = \frac{1}{32\pi^2} \left[ \ln L_s - \frac{1}{D-4} + \frac{1}{2} \gamma_3 \right] + O(D-4), \]  
(C.97)
we see that to cancel the pole terms the combination $L_1 + \frac{1}{2} L_2$ must be of the form given in (3.44). Finally combining the results we obtain (3.45).

D Integrals over $O(n)$ matrices

Here we consider integrals of the form
\[ H^{(r)}_{i_1\ldots i_{2r}, j_1\ldots j_{2r}} \int d\Omega \prod_{a=1}^{2r} \Omega_{i_a j_a}, \]  
(D.1)
with $\Omega$ a real orthogonal $n \times n$ matrix:
\[ \sum_{k=1}^{n} \Omega_{i k} \Omega_{j k} = \delta_{i j} = \sum_{k=1}^{n} \Omega_{k i} \Omega_{k j}, \]  
(D.2)
and $d\Omega$ the $O(n)$ invariant Haar measure with normalization

$$
\int d\Omega = 1. \tag{D.3}
$$

For $r \geq 1$ the integrals are evaluated as sums over $r$ terms:

$$
H_{i_1 \ldots i_{2r}; j_1 \ldots j_{2r}}^{(r)} = \sum_{s=1}^{r} \left\{ \prod_{b=1}^{r} \delta_{i_{2b-1}i_{2b}} \right\} T_{j_1 \ldots j_{2r}}^{(r)(s)} + (N_r - 1) \text{perms} \right\}, \tag{D.4}
$$

where the $T^{(r)(s)}$ are $n$-dependent coefficients times sums of products of $\delta$-functions involving only the $j_a$ which are invariant under interchanges $j_a \to j_{\sigma(a)}$, $a = 1, \ldots, 2r$, where $\sigma$ is any permutation such that

$$
\prod_{b=1}^{r} \delta_{i_{2b-1}i_{2b}} = \prod_{b=1}^{r} \delta_{i_{\sigma(2b-1)}i_{\sigma(2b)}}. \tag{D.5}
$$

The sum over permutations in (D.4) gives the sum over the $N_r = (2r)!/(2^r r!)$ distinct pairings of the labels $i_1, \ldots, i_{2r}$.

In our computation we only need the integrals for $r = 1, 2, 3$. First for $r = 1$:

$$
H_{i_1i_2; j_1j_2}^{(1)} = \frac{1}{n} \delta_{i_1i_2} \delta_{j_1j_2}, \tag{D.6}
$$

For $r = 2$ and $n > 1$:

$$
T_{j_1j_2; j_3j_4}^{(2)(1)} = \frac{(n+1)}{n(n-1)(n+2)} \delta_{j_1j_2} \delta_{j_3j_4}, \tag{D.7}
$$

$$
T_{j_1j_2; j_3j_4}^{(2)(2)} = -\frac{1}{n(n-1)(n+2)} \left[ \delta_{j_1j_2} \delta_{j_3j_4} + \delta_{j_1j_4} \delta_{j_2j_3} \right]. \tag{D.8}
$$

Finally for $r = 3$ and $n > 2$:

$$
T_{j_1j_2j_3; j_4j_5j_6}^{(3)(1)} = \frac{(n^2 + 3n - 2)}{n(n-1)(n-2)(n+2)(n+4)} \delta_{j_1j_2} \delta_{j_3j_4} \delta_{j_5j_6}, \tag{D.9}
$$

$$
T_{j_1j_2j_3; j_4j_5j_6}^{(3)(2)} = -\frac{1}{n(n-1)(n-2)(n+4)} \left\{ \delta_{j_1j_2} \left[ \delta_{j_3j_5} \delta_{j_4j_6} + \delta_{j_3j_6} \delta_{j_4j_5} \right] + \delta_{j_1j_4} \left[ \delta_{j_2j_5} \delta_{j_3j_6} + \delta_{j_2j_6} \delta_{j_3j_5} \right] + \delta_{j_1j_5} \left[ \delta_{j_2j_3} \delta_{j_4j_6} + \delta_{j_2j_6} \delta_{j_3j_4} \right] + \delta_{j_1j_6} \left[ \delta_{j_2j_3} \delta_{j_4j_5} + \delta_{j_2j_5} \delta_{j_3j_4} \right] \right\}, \tag{D.10}
$$

$$
T_{j_1j_2j_3; j_4j_5j_6}^{(3)(3)} = \frac{2}{n(n-1)(n-2)(n+2)(n+4)} \times \\
\times \left\{ \delta_{j_1j_3} \left[ \delta_{j_2j_5} \delta_{j_4j_6} + \delta_{j_2j_6} \delta_{j_4j_5} \right] + \delta_{j_1j_4} \left[ \delta_{j_2j_5} \delta_{j_3j_6} + \delta_{j_2j_6} \delta_{j_3j_5} \right] + \delta_{j_1j_5} \left[ \delta_{j_2j_3} \delta_{j_4j_6} + \delta_{j_2j_6} \delta_{j_3j_4} \right] + \delta_{j_1j_6} \left[ \delta_{j_2j_3} \delta_{j_4j_5} + \delta_{j_2j_5} \delta_{j_3j_4} \right] \right\}. \tag{D.11}
$$
It follows for averages over \( O(n) \) vectors \( a, b, c, d, e, f \):
\[
\langle a_1 b_1 \rangle = \frac{1}{n} \langle a \cdot b \rangle, \tag{D.12}
\]
\[
\langle a_1 b_1 c_2 d_2 \rangle = \frac{1}{n(n-1)(n+2)} \langle (n+1)(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) \rangle, \tag{D.13}
\]
\[
\langle (a_2 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1) \rangle = \frac{2}{n(n-1)} \langle (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) \rangle, \tag{D.14}
\]
\[
\langle a_1 b_1 c_2 d_2 e_3 f_3 \rangle = \frac{1}{n(n-1)(n-2)(n+2)(n+4)} \langle (n^2 + 3n - 2)(a \cdot b)(c \cdot d)(e \cdot f) - (n + 2) \langle (a \cdot b)[(c \cdot e)(d \cdot f) + (c \cdot f)(d \cdot e)] + (c \cdot d)[(a \cdot e)(b \cdot f) + (a \cdot f)(b \cdot e)] + (a \cdot e)[(b \cdot c)(d \cdot f) + (b \cdot d)(c \cdot f)] \rangle \rangle \langle (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) \rangle
\]
\[
\times \langle (n \cdot e \cdot f)[(a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)] + (a \cdot e)[(b \cdot c)(d \cdot f) + (b \cdot d)(c \cdot f)] - (b \cdot c)[(a \cdot e)(d \cdot f) + (a \cdot f)(d \cdot e)] - (b \cdot d)[(a \cdot e)(c \cdot f) + (a \cdot f)(c \cdot e)] \rangle. \tag{D.15}
\]

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