The Number of \( k \)-Dimensional Corner-Free Subsets of Grids

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Abstract

A subset \( A \) of the \( k \)-dimensional grid \( \{1, 2, \ldots, N\}^k \) is said to be \( k \)-dimensional corner-free if it does not contain a set of points of the form \( \{a\} \cup \{a + de_i : 1 \leq i \leq k \} \) for some \( a \in \{1, 2, \ldots, N\}^k \) and \( d > 0 \), where \( e_1, e_2, \ldots, e_k \) is the standard basis of \( \mathbb{R}^k \). We define the maximum size of a \( k \)-dimensional corner-free subset of \( \{1, 2, \ldots, N\}^k \) as \( c_k(N) \). In this paper, we show that the number of \( k \)-dimensional corner-free subsets of the \( k \)-dimensional grid \( \{1, 2, \ldots, N\}^k \) is at most \( 2^{O(c_k(N))} \) for infinitely many values of \( N \). Our main tools for proof are the hypergraph container method and the supersaturation result for \( k \)-dimensional corners in sets of size \( \Theta(c_k(N)) \).

Mathematics Subject Classifications: 05D05

1 Introduction

In 1975, Szemerédi [25] proved that for every real number \( \delta > 0 \) and every positive integer \( k \), there exists a positive integer \( N \) such that every subset \( A \) of the set \( \{1, 2, \ldots, N\} \) with \( |A| \geq \delta N \) contains an arithmetic progression of length \( k \). There has been a plethora of research related to Szemerédi’s theorem mixing methods in many areas of mathematics. Szemerédi’s original proof is a tour de force of involved combinatorial arguments. There have been now alternative proofs of Szemerédi’s theorem by Furstenberg [9] using methods from ergodic theory, and by Gowers [12] using high order Fourier analysis. The case \( k = 3 \) was proven earlier by Roth [20].

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A subset $A$ of the set $\{1,2,\ldots,N\}$ is said to be $k$-AP-free if it does not contain an arithmetic progression of length $k$. We define the maximum size of a $k$-AP-free subset of $\{1,2,\ldots,N\}$ as $r_k(N)$. In 1990, Cameron and Erdős [6] were interested in counting the number of subsets of the set $\{1,2,\ldots,N\}$ which do not contain an arithmetic progression of length $k$ and asked the following question.

**Question 1** (Cameron and Erdős [6]). For every positive integer $k$ and $N$, is it true that the number of $k$-AP free subsets of $\{1,2,\ldots,N\}$ is $2^{(1+o(1))r_k(N)}$?

Until recently, research on how to improve the bounds $r_k(N)$ has been studied by many authors [4, 5, 8, 18, 11, 12]. Despite much effort, the difference between the currently known lower and upper bounds of $r_3(N)$ is still quite large. The upper bound has improved gradually over the years, and the current best upper bound is due to Bloom and Sisask [5]:

$$r_3(N) \leq \frac{N}{(\log N)^{1+c}},$$

where $c > 0$ is an absolute constant.

For a lower bound of $r_3(N)$, the configuration of Behrend [4] shows:

$$r_3(N) = \Omega\left(\frac{N}{2^{2^{\sqrt{\log_2 N}} \cdot \log^{1/2} N}}\right).$$

This has been improved by Elkin’s modification [8] by a factor of $\sqrt{\log n}$.

The currently known lower and upper bounds for $r_k(N)$ are as follows:

Let $m = \lceil \log_2 k \rceil$. For $k \geq 4$, there exist $c_k, c_k' > 0$ such that

$$c_k \cdot N \cdot (\log N)^{1/2m} \cdot 2^{-m2^{(m-1)/2(\log n)^{1/m}}} \leq r_k(N) \leq \frac{N}{(\log \log N)^{c_k'}},$$

where the lower bound is due to O’Bryant [18] and the upper bound is due to Gowers [11, 12].

In 2017, Balogh, Liu, and Sharifzadeh [2] provided a weaker version of Cameron and Erdős’s conjecture [6] that the number of subsets of the set $\{1,2,\ldots,N\}$ without an arithmetic progression of length $k$ is at most $2^{O(r_k(N))}$ for infinitely many values of $N$, which is optimal up to a constant factor in the exponent.

A triple of points in the 2-dimensional grid $\{1,2,\ldots,N\}^2$ is called a corner if it is of the form $(a_1, a_2), (a_1 + d, a_2), (a_1, a_2 + d)$ for some $a_1, a_2 \in \{1,2,\ldots,N\}$ and $d > 0$. In 1974, Ajtai and Szemerédi [1] discovered that for every number $\delta > 0$, there exists a positive integer $N$ such that every subset $A$ of the 2-dimensional grid $\{1,2,\ldots,N\}^2$ with $|A| \geq \delta N^2$ contains a corner. In 1991, Fürstenberg and Katznelson [10] found that their more general theorem implied the result of Ajtai and Szemerédi [1], but did not specify an explicit bound for $N$ as it uses ergodic theory. An easy consequence of their result is the case $k = 3$ of Szemerédi’s theorem, which was first proved by Roth [20] using Fourier analysis. Afterward, in 2003, Solymosi [24] provided a simple proof for Ajtai and Szemerédi [1] theorem using the Triangle Removal Lemma.
A subset $A$ of the 2-dimensional grid $\{1, 2, \ldots, N\}^2$ is called \textit{corner-free} if it does not contain a corner. We define the maximum size of corner-free sets in $\{1, 2, \ldots, N\}^2$ as $c_2(N)$. The problem of improving the bounds for $c_2(N)$ has been studied by many authors [14, 16, 22, 23]. The current best lower bound of $c_2(N)$ is due to Green [14], based on Linial and Shraibman’s construction [16]:

$$\frac{N^2}{2(t_1+o(1))\sqrt{\log N}} \leq c_2(N),$$

where $t_1 \approx 1.822$.

The current best upper bound of $c_2(N)$ is due to Shkredov [22]:

$$c_2(N) \leq \frac{N^2}{(\log \log N)^{t_2}},$$

where $t_2 \approx 0.0137$.

The higher dimensional analog of a corner in the 2-dimensional grid $\{1, 2, \ldots, N\}^2$ is the following. A subset $A$ of the $k$-dimensional grid $\{1, 2, \ldots, N\}^k$ is called \textit{$k$-dimensional corner} if it is a set of points of the form $\{a\} \cup \{a + de_i : 1 \leq i \leq k\}$ for some $a \in \{1, 2, \ldots, N\}^k$ and $d > 0$, where $e_1, e_2, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$. The following multidimensional version of Ajtai and Szemerédi theorem [1] was proved by Fürstenberg, Katznelson [10], and Gowers [13].

\textbf{Theorem 2} ([10, 13]). For every number $\delta > 0$ and every positive integer $k$, there exists a positive integer $N$ such that every subset $A$ of the $k$-dimensional grid $\{1, 2, \ldots, N\}^k$ with $|A| \geq \delta N^k$ contains a $k$-dimensional corner.

In 1991, Fürstenberg and Katznelson [10] showed that their more general theorem implied Theorem 2, but did not specify an explicit bound as it uses ergodic theory. Later, in 2007, Gowers [13] provided the first proof with explicit bounds and the first proof of Theorem 2 not based on Fürstenberg’s ergodic-theoretic approach. They also proved that Theorem 2 implied the multidimensional Szemerédi theorem.

Another fundamental result in additive combinatorics is the multidimensional Szemerédi theorem, which was demonstrated for the first time by Fürstenberg and Katznelson [9] using the ergodic method, but provided no explicit bounds. In 2007, Gowers [13] yielded a combinatorial proof of the multidimensional Szemerédi theorem by establishing the Regularity and Counting Lemmas for the $r$-uniform hypergraph. This is the first proof to provide an explicit bound. Similar results were obtained independently by Nagle, Rödl, and Schacht [17].

\textbf{Theorem 3} (Multidimensional Szemerédi theorem [9, 13, 17]). For every real number $\delta > 0$, every positive integer $k$, and every finite set $X \subset \mathbb{Z}^k$, there exists a positive integer $N$ such that every subset $A$ of the $k$-dimensional grid $\{1, 2, \ldots, N\}^k$ with $|A| \geq \delta N^k$ contains a subset of the form $a + dX$ for some $a \in \{1, 2, \ldots, N\}^k$ and $d > 0$.

A subset $A$ of the $k$-dimensional grid $\{1, 2, \ldots, N\}^k$ is called \textit{$k$-dimensional corner-free} if it does not contain a $k$-dimensional corner. We define the maximum size of a
A \textit{k-dimensional corner-free subset} of \{1, 2, \ldots, N\}^k \text{ as } c_k(N). In this paper, we study a natural higher dimensional version of the question of Cameron and Erdős, i.e. counting \textit{k-dimensional corner-free sets} in \{1, 2, \ldots, N\}^k as follows.

**Question 4.** For every positive integer \( k \) and \( N \), is it true that the number of \( k \)-dimensional corner-free subsets of the \( k \)-dimensional grid \( \{1, 2, \ldots, N\}^k \) is \( 2^{(1+o(1))c_k(N)} \)?

In addressing this question, we show the following theorem. Similar to the results of Balogh, Liu, and Sharifzadeh [2], despite not knowing the value of the extremal function \( c_k(N) \), we can derive a counting result that is optimal up to a constant factor in the exponent.

**Theorem 5.** The number of \( k \)-dimensional corner-free subsets of the \( k \)-dimensional grid \( \{1, 2, \ldots, N\}^k \) is \( 2^{O(c_k(N))} \) for infinitely many values of \( N \).

Our paper is organized as follows. In Section 2, we provide the two main tools for proof: the hypergraph container theorem and supersaturation results for \( k \)-dimensional corners. In Section 3, we provide proof of the saturation result for \( k \)-dimensional corners in sets of size \( \Theta(c_k(N)) \), which is specified in Section 2. In Section 4, we provide proof of our main result, Theorem 5.

**2 Preliminaries**

**2.1 Hypergraph Container Method**

The hypergraph container method [3, 21] is a very powerful technique for bounding the number of discrete objects avoiding certain forbidden structures. A graph is \( H \)-free if it does not have subgraphs that are isomorphic to \( H \). For example, we use the container method when we count the family of \( H \)-free graphs or the family of sets without \( k \) term arithmetic progression. The \( r \)-uniform hypergraph \( \mathcal{H} \) is defined as the pair \( (V(\mathcal{H}), E(\mathcal{H})) \) where \( V(\mathcal{H}) \) is the set of vertices and \( E(\mathcal{H}) \) is the set of hyperedges that are the \( r \)-subset of the vertices of \( V(\mathcal{H}) \). Let \( \Gamma(\mathcal{H}) \) be a collection of independent sets of hypergraph \( \mathcal{H} \), where the independent set of hypergraph \( \mathcal{H} \) is the set of vertices inducing no hyperedge in \( E(\mathcal{H}) \). For a given hypergraph \( \mathcal{H} \), we define the maximum degree of a set of \( l \) vertices of \( \mathcal{H} \) as

\[
\Delta_l(\mathcal{H}) = \max\{ d_\mathcal{H}(A) : A \subseteq V(\mathcal{H}), |A| = l \},
\]

where \( d_\mathcal{H}(A) \) is the number of hyperedges in \( E(\mathcal{H}) \) containing the set \( A \).

Let \( \mathcal{H} \) be an \( r \)-uniform hypergraph of order \( n \) and average degree \( d \). For any \( 0 < \tau < 1 \), the co-degree \( \Delta(\mathcal{H}, \tau) \) is defined as

\[
\Delta(\mathcal{H}, \tau) = 2^{\left( \frac{r}{2} \right) - 1} \sum_{j=2}^{r} 2^{-j-1} \frac{\Delta_j(\mathcal{H})}{\tau^{j-1}d}.
\]

In this paper, we use the following hypergraph container lemma, which contains accurate estimates for the \( r \)-uniform hypergraph in Corollary 3.6 in [21].
Theorem 6 (Hypergraph Container Lemma [21]). For every positive integer \( r \in \mathbb{N} \), let \( \mathcal{H} \subseteq \binom{V}{r} \) be an \( r \)-uniform hypergraph. Suppose that there exist \( 0 < \epsilon, \tau < 1/2 \) such that

\[
\begin{align*}
&\tau < 1/(200 \cdot r \cdot r!) \\
&\Delta(\mathcal{H}, \tau) \leq \frac{\epsilon}{12r!}.
\end{align*}
\]

Then there exist \( c = c(r) \leq 1000 \cdot r \cdot r!^3 \) and a collection \( \mathcal{C} \) of subsets of \( V(\mathcal{H}) \) such that the following holds:

\[
\begin{align*}
&\text{for every independent set } I \in \Gamma(\mathcal{H}), \text{ there exists } S \in \mathcal{C} \text{ such that } I \subset S, \\
&\log |\mathcal{C}| \leq c \cdot |V| \cdot \tau \cdot \log(1/\epsilon) \cdot \log(1/\tau), \\
&\text{for every } S \in \mathcal{C}, \ e(\mathcal{H}[S]) \leq \epsilon \cdot e(\mathcal{H}),
\end{align*}
\]

where \( \mathcal{H}[S] \) is a subhypergraph of \( \mathcal{H} \) induced by \( S \).

Let us consider a \((k + 1)\)-uniform hypergraph \( \mathcal{G} \) encoding the set of all \( k \)-dimensional corners in the \( k \)-dimensional grid \([n]^k\). It means that \( V(\mathcal{G}) = [n]^k \) and the edge set of \( \mathcal{G} \) consists of all \((k + 1)\)-tuples forming \( k \)-dimensional corners. Note that the independent set in \( \mathcal{G} \) is the \( k \)-dimensional corner-free set in \([n]^k\). Applying the Hypergraph Container Lemma to the hypergraph \( \mathcal{G} \) gives the following theorem, which is an important result to prove our main result, Theorem 5.

Theorem 7. For every positive integer \( k \in \mathbb{N} \), let \( \mathcal{G} \) be a \((k + 1)\)-uniform hypergraph encoding the set of all \( k \)-dimensional corners in \([n]^k\). Suppose that there exists \( 0 < \epsilon, \tau < 1/2 \) satisfying that

\[
\begin{align*}
&\tau < 1/(200 \cdot (k + 1) \cdot (k + 1)!^2) \\
&\Delta(\mathcal{G}, \tau) \leq \frac{\epsilon}{12(k+1)!^2}.
\end{align*}
\]

Then there exist \( c = c(k + 1) \leq 1000 \cdot (k + 1) \cdot (k + 1)!^3 \) and a collection \( \mathcal{C} \) of subsets of \( V(\mathcal{G}) \) such that the following holds:

\[
\begin{align*}
&(i) \text{ every } k\text{-dimensional corner-free subset of } [n]^k \text{ is contained in some } S \in \mathcal{C}, \\
&(ii) \log |\mathcal{C}| \leq c \cdot |V(\mathcal{G})| \cdot \tau \cdot \log(1/\epsilon) \cdot \log(1/\tau), \\
&(iii) \text{ for every } S \in \mathcal{C}, \text{ the number of } k\text{-dimensional corners in } S \text{ is at most } \epsilon \cdot e(\mathcal{G}).
\end{align*}
\]
2.2 Supersaturation Results

In this section, we present the supersaturation result for $k$-dimensional corners, which is the second main ingredient for proof of our main result. A supersaturation result says that sufficiently dense subsets of a given set contain many copies of certain structures. For the arithmetic progression, the supersaturation result concerned only sets of size linear in $n$ was first demonstrated by Varnavides [26] by showing that any subset of $[n]$ of size $\Omega(n^2)$ has $\Omega(n^2/\log^k n)$ $k$-APs. In 2008, Green and Tao [15] obtained the supersaturation result by proving that any subset of $\mathbf{P}_{\leq n}$ of size $\Omega(\lfloor \mathbf{P}_{\leq n} \rfloor)$ has $\Theta(n^2/\log k n)$ $k$-APs, where $\mathbf{P}_{\leq n}$ is the set of prime numbers up to $n$. Later, Croot and Sisask [7] provided a quantitative version of Varnavides [26] by proving that for every $1 \leq M \leq n$, the number of 3-AP in $A$ is at least

$\left( |A| - \frac{r_3(M) + 1}{M} \right) \cdot \frac{n^2}{M^4}$.

To prove Theorem 5, we need the supersaturation result of the minimum value of the number of $k$-dimensional corners for any set $A$ in the $k$-dimensional grid $[n]^k$ of size $\Theta(c_k(N))$. To explain the supersaturation results, we introduce the following definitions.

Recall that we define the maximum size of a $k$-dimensional corner-free subset of the $k$-dimensional grid $[n]^k$ as $c_k(n)$. Let $\Gamma_k(A)$ denote the number of $k$-dimensional corners in the set $A \subseteq [n]^k$. The following theorem shows that the number of $k$-dimensional corners in any set $A \subseteq [n]^k$ of size constant factor times larger than $c_k(n)$ is superlinear in $n$. In Section 3, we provide proof of Theorem 8.

**Theorem 8.** For the given $k \geq 3$, there exist $C' := C'(k)$ and an infinite sequence $\{n_i\}_{i=1}^{\infty}$ such that the following holds. For all $n \in \{n_i\}_{i=1}^{\infty}$ and any set $A$ in the $k$-dimensional grid $[n]^k$ of size $C' \cdot c_k(n)$, we have

$\Gamma_k(A) \geq \log^{(3k+1)} n \cdot \left( \frac{n^k}{c_k(n)} \right)^k \cdot n^{k-1} = \Upsilon(n) \cdot n^k$.

where $\Upsilon(n) = \frac{\log^{2k+1} n}{n} \cdot \left( \frac{n^k}{c_k(n)} \right)^k$.

2.2.1 Supersaturation Lemmas

In this section, we present more supersaturation results for the minimum value of the number of $k$-dimensional corners to obtain a superlinear bound in Theorem 8. First, we provide the following simple supersaturation result using the greedy algorithm.

**Lemma 9.** For the positive integer $k \geq 2$, let $A$ be any set in the $k$-dimensional grid $[n]^k$ of size $K \cdot c_k(n)$, where $K \geq 2$ is a constant. Then we get

$\Gamma_k(A) \geq (K - 1) \cdot c_k(n)$.

**Proof.** We use the greedy algorithm to determine the minimum value of the number of $k$-dimensional corners in a set $A$ of size $K \cdot c_k(n)$, where $K \geq 2$. We consider the following
process iteratively. As $|A| > c_k(n)$, there exists a $k$-dimensional corner $C$ in the set $A$. It then updates the set $A$ by removing an arbitrary element from $C$. By repeating this process $(K - 1) \cdot c_k(n)$ times, we have

$$\Gamma_k(A) \geq (K - 1) \cdot c_k(n).$$

Next, we use Lemma 9 to give the following improved supersaturation result.

**Lemma 10.** For the positive integer $k \geq 2$, let $A$ be any set in the $k$-dimensional grid $[n]^k$ of size at least $K \cdot c_k(n)$, where $K \geq 2$ is a constant. Then we obtain

$$\Gamma_k(A) \geq \left(\frac{K}{2}\right)^{k+1} \cdot c_k(n).$$

**Proof.** Let $A$ be any set of $[n]^k$ and have a size greater than equal to $K \cdot c_k(n)$. We consider the set $S$, which is one of all subsets of $A$ of size $2 \cdot c_k(n)$. With Lemma 9, we have $\Gamma_k(S) \geq c_k(n)$ for every $S$. Therefore we get

$$\left(\frac{|A|}{2 \cdot c_k(n)}\right) \cdot c_k(n) \leq \sum_{S \subseteq A, |S| = 2 \cdot c_k(n)} \Gamma_k(S) \leq \Gamma_k(A) \cdot \left(\frac{|A| - k - 1}{2 \cdot c_k(n) - k - 1}\right).$$

Then we conclude that

$$\Gamma_k(A) \geq \frac{|A|}{2 \cdot c_k(n)} \cdot c_k(n) \geq \left(\frac{|A|}{2 \cdot c_k(n)}\right)^{k+1} \cdot c_k(n) \geq \left(\frac{K}{2}\right)^{k+1} \cdot c_k(n).$$

Note that the bounds of Lemma 9 and Lemma 10 are linear in the set $A$ of $[n]^k$. In the following lemma, we provide a superlinear bound for the minimum value of the number of $k$-dimensional corners by applying Lemma 10 to the set of carefully chosen $k$-dimensional corners with prime common differences. The following lemma is an important result for proving the supersaturation result for $k$-dimensional corners in sets of size $\Theta(c_k(N))$ with superlinear bounds, which is specified in Theorem 8.

**Lemma 11.** For the positive integer $k \geq 2$, let $A$ be any set in the $k$-dimensional grid $[n]^k$ such that there exists a positive constant $M$ satisfying $\frac{|A|}{n^k} \geq \frac{8K\cdot c_k(M)}{M^{k+1}n^{k-1}}$, where $K \geq 2$ is a constant. Then we obtain

$$\Gamma_k(A) \geq \frac{|A|^2}{2^{2k+1}} \cdot \frac{(K)^{k+1} \cdot c_k(M)}{M^{k+1}n^{k-1} \log^2 n}.$$
Proof. Given the set $A$ of $[n]^k$, we let $x = \frac{|A|}{2^{k+1}Mn^{k-1}}$ which is sufficiently large. Let $G_d$ be the set of $M \times \cdots \times M$ grids in $[n]^k$, whose consecutive layers are of distance $d$ apart, for a prime $d \leq x$. Let us consider $G = \bigcup_{d \leq x} G_d$. For any $k$-dimensional corner $C = \{a\} \cup \{a + d'e_i : 1 \leq i \leq k\}$ for some $a \in [n]^k$ and $d' > 0$, where $e_1, e_2, \ldots, e_k$ are the standard bases of $\mathbb{R}^k$, we consider a grid $G \in G_d$ containing $C$. This means that $d$ must be a prime divisor of $d'$. The number of prime divisors of $d'$ is at most $\log d' \leq \log n$, so the number of these choices is at most $\log n$. Since every corner can occur in at most $(M-1)^k$ grids from each fixed $G_d$ and the length of the corner has at most $\log n$ distinct prime factors, we get

$$\Gamma_k(A) \geq \frac{1}{M^k \cdot \log n} \sum_{G \in G} \Gamma_k(A \cap G).$$

(1)

Let us consider $\mathcal{R} \subseteq G$ consisting of all $G \in G$ such that $|A \cap G| \geq K \cdot c_k(M)$, where $K \geq 2$ is a constant. Applying Lemma 10 to $A \cap G$ gives:

$$\Gamma_k(A \cap G) \geq \left(\frac{K}{2}\right)^{k+1} \cdot c_k(M).$$

(2)

for all $G \in \mathcal{R}$. Combining the inequalities (1) and (2), we obtain

$$\Gamma_k(A) \geq \frac{1}{M^k \cdot \log n} \sum_{G \in G} \Gamma_k(A \cap G)$$

$$= \frac{1}{M^k \cdot \log n} \left( \sum_{G \in \mathcal{R}} \Gamma_k(A \cap G) + \sum_{G \in G/\mathcal{R}} \Gamma_k(A \cap G) \right)$$

$$\geq |\mathcal{R}| \cdot \left(\frac{K}{2}\right)^{k+1} \cdot \frac{c_k(M)}{M^k \cdot \log n}.$$  

(3)

Next, let us prove the lower bound for $|\mathcal{R}|$. For a prime number $d \leq x = \frac{|A|}{2^{k+1}Mn^{k-1}}$, we define $\zeta_d := [(M-1)d + 1, n - (M-1)d]^k$. Then we get the following inequality:

$$|A \cap \zeta_d| \geq |A| - 2^k Mn^{k-1}$$

$$\geq |A| - 2^k Mn^{k-1} - \frac{|A|}{2^{k+1}Mn^{k-1}} = \frac{|A|}{2}.$$

Note that the number of primes less than or equal to $x$ is at least $\frac{x}{\log x}$ and at most $\frac{2x}{\log x}$ by the Prime Number Theorem. Since every $z \in \zeta_d$ appears exactly in the $M^k$ members of $G_d$, we derive that

$$\sum_{G \in G} |A \cap G| = \sum_{d \leq x} \sum_{G \in G_d} |A \cap G|$$

$$\geq M^k \sum_{d \leq x} |A \cap \zeta_d| \geq M^k \cdot \frac{x}{\log x} \cdot \frac{|A|}{2}.$$  

(4)
Obviously the inequality $|G_d| \leq n^k$ is held for each prime number $d \leq x$. Then we get the following equation:

$$|G| = \left| \bigcup_{d \leq x} G_d \right| \leq \frac{2x}{\log x} \cdot n^k. \quad (5)$$

Since $\mathcal{R} \subseteq \mathcal{G}$ consists of all $G \in \mathcal{G}$ such that $|A \cap G| \geq K \cdot c_k(M)$, using the equation (5) we get

$$\sum_{G \in \mathcal{G}} |A \cap G| = \sum_{G \in \mathcal{R}} |A \cap G| + \sum_{G \in \mathcal{G} \setminus \mathcal{R}} |A \cap G|$$

$$\leq M^k|\mathcal{R}| + K \cdot c_k(M) \cdot |\mathcal{G} \setminus \mathcal{R}|$$

$$\leq M^k|\mathcal{R}| + K \cdot c_k(M) \cdot |G|$$

$$\leq M^k|\mathcal{R}| + K \cdot c_k(M) \cdot \frac{2x}{\log x} \cdot n^k. \quad (6)$$

Using the equations (4) and (6), we obtain

$$|\mathcal{R}| \geq \frac{1}{M^k} \left( \sum_{G \in \mathcal{G}} |A \cap G| - K \cdot c_k(M) \cdot \frac{2x}{\log x} \cdot n^k \right)$$

$$\geq \frac{1}{M^k} \cdot \left( M^k \cdot \frac{x}{\log x} \cdot \frac{|A|}{2} - K \cdot c_k(M) \cdot \frac{2x}{\log x} \cdot n^k \right)$$

$$= \frac{x}{\log x} \cdot \left( \frac{|A|}{2} - \frac{K \cdot c_k(M)}{M^k} \cdot \frac{2x}{\log x} \cdot n^k \right).$$

From the condition $\frac{|A|}{n^k} \geq \frac{8Kc_k(M)}{M^k}$, we have

$$|\mathcal{R}| \geq \frac{x}{\log x} \cdot \left( \frac{|A|}{2} - \frac{2K \cdot c_k(M)}{M^k} \cdot n^k \right)$$

$$\geq \frac{x}{\log x} \cdot \left( \frac{|A|}{2} - \frac{1}{4} \cdot \frac{|A|}{n^k} \right)$$

$$\geq \frac{x}{\log x} \cdot \left( \frac{|A|}{4} \geq \frac{|A|}{4} - \frac{|A|}{2^{k+1}Mn^{k-1}} \right) \cdot \frac{1}{\log n}. \quad (7)$$

Using the equations (3) and (7), we conclude that

$$\Gamma_k(A) \geq \frac{|\mathcal{R}|}{(2^{k+1}Mn^{k-1})} \cdot \frac{c_k(M)}{M^k \log n}$$

$$\geq \frac{|A|^2}{4} \cdot \frac{1}{2^{k+1}Mn^{k-1}} \cdot \frac{1}{\log n} \cdot (\frac{K}{2})^{k+1} \cdot \frac{c_k(M)}{M^k \log n}$$

$$= \frac{|A|^2}{2^{2k+4}} \cdot \frac{(K^{k+1} \cdot c_k(M))}{M^{k+1}n^{k-1}\log^2 n}. \quad \square$$
3 Proof of Theorem 8

The supersaturation result of \(k\)-dimensional corners in sets of size \(\Theta(c_k(N))\), which is specified in Theorem 8, is the main tool for proof of Theorem 5. In this section, we prove Theorem 8 using Lemma 11 and the following relationship between \(f(n_i)\) and \(f(\Lambda(n_i))\) for some infinite sequence \(\{n_i\}_{i=1}^{\infty}\).

For every \(n \in \{n_i\}_{i=1}^{\infty}\), we define the following functions:

\[
\Lambda(n) = \frac{n}{\log^{3k+3} n} \cdot \left(\frac{c_k(n)}{n^k}\right)^{k+3}, \quad f(n) = \frac{c_k(n)}{n^k},
\]

where \(c_k(n)\) is the maximum size of a \(k\)-dimensional corner-free subset of \([n]^k\).

**Lemma 12.** For the given \(k \geq 3\), there exist \(b := b(k) > 2^{2k}\) and an infinite sequence \(\{n_i\}_{i=1}^{\infty}\) such that

\[
bf(n_i) \geq f(\Lambda(n_i))
\]

for all \(i \geq 1\).

First, we give the following relationship between \(f(n)\) and \(f(m)\) for any \(m < n\), which is what we need to get Lemma 12.

**Lemma 13.** For every \(m < n\), we obtain 

\[
f(n) < 2^k \cdot f(m).
\]

**Proof.** For every \(m < n\), we divide the \(k\)-dimensional grid \([n]^k\) into consecutive grids of size \(m^k\) because the corner-free property is invariant under translation. Since any given \(k\)-dimensional corner free subset of \([n]^k\) contains at most \(c_k(m)\) elements in each grid of size \(m^k\), for any \(m < n\) we have

\[
c_k(n) \leq \left\lceil \frac{n}{m} \right\rceil^k c_k(m).
\]

Since \(\frac{1}{n^k} \cdot \left\lceil \frac{n}{m} \right\rceil^k < \frac{2^k}{m^k}\) for every \(m < n\), we conclude that

\[
f(n) = \frac{c_k(n)}{n^k} \leq \left\lceil \frac{n}{m} \right\rceil^k \cdot \frac{c_k(m)}{m^k} < \frac{2^k}{m^k} \cdot c_k(m) = 2^k \cdot f(m).
\]

This completes the proof of Lemma 13. \(\square\)

To get Lemma 12, we also need a lower bound on \(c_k(n)\), which follows from Rankin [19]’s result that is a generalization of Behrend [4]’s construction of dense 3-AP-free subset of integers to the case of arbitrary \(k \geq 3\).

**Lemma 14.** For the given \(k \geq 2\), there exists \(\alpha_k\) such that

\[
\frac{c_k(n)}{n^k} > 2^{-\alpha_k (\log n)^{\beta_k}}
\]

for all sufficiently large \(n\), where \(\alpha_k\) is a positive absolute constant that depends only on \(k\) and \(\beta_k = \frac{1}{\lceil \log k \rceil}\).
**Proof.** Let us first consider the case when \( k = 2 \). Let \( A \) be the 3-AP-free subset of \([n]\) with size \( n \cdot 2^{1 - \alpha \sqrt{\log n}} \) from Behrend [4]'s construction. We construct a dense 2-dimensional corner-free subset \( B \) of \([n]^2\) of size \( \Omega(|A|n) \) as follows: Let \( L \) be the collection of all lines of the form \( y = x + a \) for every \( a \in A \), and \( B \) be the intersection of \( L \) and \([n]^2\). It is easy to see that \( |B| = \Omega(|A|n) \). It remains to prove that \( B \) is 2-dimensional corner-free. Let us assume otherwise, i.e. there exists a 2-dimensional corner in the set \( B \), say \((x, y), (x + d, y), (x, y + d)\). Then, depending on the configuration, the three elements \( y - x = a_1, y - (x + d) = a_2 \), and \((y + d) - x = a_3\) are all in the set \( A \) forming 3-AP with \( a_2 + a_3 = 2a_1 \). This is a contradiction. Since the case of \( k \geq 3 \) is similar, the result of Rankin [19] is used instead, so details are omitted.

Now we use Lemma 13 and Lemma 14 to prove Lemma 12.

**Proof of Lemma 12.** Fix \( b := b(k) > 2^{2k} \) a large enough constant. Let us assume otherwise, i.e. there exists \( n_0 \) for all \( n \geq n_0 \) satisfying

\[
f(n) < b^{-1}f(\Lambda(n)). \quad (8)
\]

Using Lemma 14, there exists \( \alpha_k \) such that \( f(n) > 2^{-\alpha_k(\log n)^{\beta_k}} \) for every sufficiently large \( n \), where \( \beta_k = \frac{1}{\log k} \) and \( \alpha_k \) is a positive absolute constant depending only on \( k \). Using these \( \alpha_k \) and \( \beta_k \), for all \( x \geq 1 \), we define the decreasing function \( g(x) \) as

\[
g(x) = 2^{-(k\alpha_k + 3\alpha_k + 1)(\log x)^{\beta_k}}.
\]

Then we get the following inequality for every \( n \geq n_0 \):

\[
\Lambda(n) = \frac{n}{\log^{3k+3} n} \cdot \left( \frac{c_k(n)}{n^k} \right)^{k+3} = \frac{n}{\log^{3k+3} n} \cdot (f(n))^{k+3} > n \cdot 2^{-(k\alpha_k + 3\alpha_k + 1)(\log n)^{\beta_k}} = n \cdot g(n).
\]

From the equation (9), if we apply Lemma 13 to \( \Lambda(n) \) and \( n \cdot g(n) \) then we derive

\[
f(n) < b^{-1}f(\Lambda(n)) \leq 13 b^{-2k} \cdot f(n \cdot g(n)) = \left( \frac{b}{2k} \right)^{-1} f(n \cdot g(n)), \quad (10)
\]

for all \( n \geq n_0 \).

To prove Lemma 12, we need the following claim.

**Claim 15.** Let us write \( t = \left\lfloor \frac{1}{2} (\log n)^{\beta_k} \right\rfloor \) with \( \alpha_k \) satisfying \( f(n) > 2^{-\alpha_k(\log n)^{\beta_k}} \). Then for all \( n > n_0^{1/(1 - \beta_k)} \) we obtain that

\[
f(n) < \left( \frac{b}{2k} \right)^{-j} f \left( n \cdot (g(n))^j \right)
\]
for all $1 \leq j \leq t$.

**Proof of Claim 15.** We proceed by induction on $j$. The base case $j = 1$ is done by the equation (10). Assume that the statement of Claim 15 holds for every $1 \leq j < t$. Now we consider $n' = n \cdot (g(n))^j$ for all $1 \leq j < t$. Since $g(n)$ is a decreasing function, for each $j < t$ we have

$$n' = n \cdot (g(n))^j > n \cdot (g(n))^t = n \cdot 2^{-(k\alpha_k+3\alpha_k+1)(\log n)^{\beta_k} \cdot \frac{1}{2} \frac{(\log n)^{\beta_k}}{2^{k\alpha_k+3\alpha_k+t} + 1}}$$

$$= n \cdot \left( \frac{1}{2} \right)^{(k\alpha_k+3\alpha_k+1)(\log n)^{\beta_k} \cdot \frac{1}{2} \frac{(\log n)^{\beta_k}}{2^{k\alpha_k+3\alpha_k+t} + 1}}$$

$$\geq n \cdot \left( \frac{1}{2} \right)^{(k\alpha_k+3\alpha_k+1)(\log n)^{\beta_k} \cdot \frac{1}{2} \frac{(\log n)^{\beta_k}}{2^{k\alpha_k+3\alpha_k+t} + 1}}$$

$$\geq n \cdot \frac{1}{2}^{(\log n)^{2\beta_k}} = n \cdot 2^{-\frac{1}{2} \frac{(\log n)^{2\beta_k}}{2^{k\alpha_k+3\alpha_k+1}}} = n^{1-\beta_k} > n_0, \quad (11)$$

for all $n > n_0^{1/(1-\beta_k)} > n_0$.

Note that $n' > n_0$ in the equation (11). Then we use the equation (10) to get

$$f(n') < \left( \frac{b}{2^k} \right)^{-1} \cdot f(n' \cdot g(n')). \quad (12)$$

Since $n' < n$ and $g(n)$ is a decreasing function, we have $n' \cdot g(n') > n' \cdot g(n)$. Applying Lemma 13 to $n' \cdot g(n')$ and $n' \cdot g(n)$ gives:

$$f(n' \cdot g(n')) < 2^k f(n' \cdot g(n)). \quad (13)$$

Using the equations (12) and (13), we obtain that

$$f(n') < \left( \frac{b}{2^k} \right)^{-1} \cdot f(n' \cdot g(n')) < \left( \frac{b}{2^k} \right)^{-1} \cdot 2^k f(n' \cdot g(n)) = \left( \frac{b}{2^k} \right)^{-1} f(n' \cdot g(n)), \quad (14)$$

for all $n > n_0^{1/(1-\beta_k)}$. According to the inductive hypothesis, for every $1 \leq j < t$, we get

$$f(n) < \left( \frac{b}{2^k} \right)^{-j} \cdot f(n \cdot (g(n))^j). \quad (15)$$
From the equations (14) and (15), for every $1 \leq j < t$, we observe that

$$f(n) < (\frac{b}{2^k})^{-j} \cdot f(n \cdot (g(n))^j)$$

$$= (\frac{b}{2^k})^{-j} \cdot f(n')$$

$$< (\frac{b}{2^k})^{-j} \cdot (\frac{b}{2^k})^{-1} f(n' \cdot g(n))$$

$$= (\frac{b}{2^k})^{-j-1} \cdot f(n \cdot (g(n))^{j+1}),$$

(16)

when $n > n_0^{1/(1-\beta_k)}$.

From the equation (16), we see that the statement of Claim 15 also holds for $j + 1$. By the Induction axiom, the statement of Claim 15 holds for every $1 \leq j \leq t$. This completes the proof of Claim 15.

Let $t = \lfloor \frac{1}{2^k \alpha_k + 3 \alpha_k + 3} \rfloor$ be an integer when $\alpha_k$ satisfies the inequality $f(n) > 2^{-\alpha_k (\log n)^{\beta_k}}$.

Assume that $n > n_0^{1/(1-\beta_k)} \geq n_0$. Applying Claim 15, we get

$$f(n) < (\frac{b}{2^k})^{-t} f(n \cdot (g(n))^t).$$

(17)

Note that $n \cdot (g(n))^t \geq n^{1-\beta_k}$ from the equation (11). Applying Lemma 13 to $n \cdot (g(n))^t$ and $n^{1-\beta_k}$ gives:

$$f(n \cdot (g(n))^t) < 2^k \cdot f(n^{1-\beta_k}).$$

(18)

Using the equations (17) and (18), we draw the following conclusion.

$$f(n) \overset{(17)}{<} (\frac{b}{2^k})^{-t} f(n \cdot (g(n))^t)$$

$$\overset{(18)}{<} (\frac{b}{2^k})^{-t} \cdot 2^k \cdot f(n^{1-\beta_k})$$

$$\leq (\frac{b}{2^k})^{-t} \cdot 2^k$$

$$= 2^k \cdot \left(\frac{b}{2^k}\right)^{-\left\lfloor \frac{1}{2^k \alpha_k + 3 \alpha_k + 3}\right\rfloor} < 2^{-\alpha_k (\log n)^{\beta_k}},$$

(19)

where $b := b(k) > 2^{2k}$ is a sufficiently large constant. The equation (19) contradicts the definition of $\alpha_k$. This completes the proof of Lemma 12.

Now we use Lemma 11 and Lemma 13 to provide a proof of Theorem 8.
Proof of Theorem 8. Let \( b(k) \) and an infinite sequence \( \{n_i\}_{i=1}^\infty \) obtained from Lemma 12. For all \( n \in \{n_i\}_{i=1}^\infty \), we let \( A \) be any set in the \( k \)-dimensional grid \([n]_k^k\) of size \( 8K \cdot b(k) \cdot c_k(n) \).

Using Lemma 12, we get
\[
\frac{|A|}{n^k} = \frac{8K \cdot b(k) \cdot c_k(n)}{n^k} \geq \frac{8K \cdot c_k(\Lambda(n))}{(\Lambda(n))^k},
\]
and
\[
\frac{|A|}{2^{k+1} \cdot \Lambda(n) \cdot n^{k-1}} = \frac{8K \cdot b(k) \cdot c_k(n)}{2^{k+1} \cdot \Lambda(n) \cdot n^{k-1}} \geq 8K \cdot b(k) \cdot \left( \frac{\log^2 n}{2} \right)^{k+1},
\]
where \( \Lambda(n) = \frac{n}{\log^{3k} n} \cdot \left( \frac{c_k(n)}{n^k} \right)^{k+3} \). Applying Lemma 13 to the inequality \( \Lambda(n) \leq n \) gives:
\[
\frac{c_k(n)}{n^k} < 2^k \cdot \frac{c_k(\Lambda(n))}{(\Lambda(n))^k}.
\]
From the inequality \( \sqrt{n} \leq \Lambda(n) \), we get
\[
\frac{n}{\Lambda(n)^2} \leq 1.
\]
From the equations (20) and (21), we can apply Lemma 11 with \( M = \Lambda(n) \) and derive that
\[
\Gamma_k(A) \geq \frac{|A|^2}{2^{2k+4} \cdot \left( \frac{(K)^{k+1} \cdot c_k(\Lambda(n))}{(\Lambda(n))^k \cdot n^{k-1} \cdot \log^2 n} \right)} = \frac{8^2 \cdot K^2 \cdot (b(k))^2 \cdot (c_k(n))^2 \cdot c_k(\Lambda(n))}{(\Lambda(n))^k \cdot \log^2 n} \cdot \frac{(K)^{k+1}}{n^{k-1} \cdot 2^{2k+4}},
\]
where \( |A| = 8K \cdot b(k) \cdot c_k(n) \). The following conclusion is drawn using the equations (21), (22), (23), and (24):
\[
\Gamma_k(A) \geq \log^{3k+1} n \cdot \left( \frac{n^k}{c_k(n)} \right)^{k+2} \cdot n^{k-1} \cdot 8^2 \cdot K^2 \cdot (b(k))^2 \cdot \left( \frac{\log^3 n}{2} \right)^{2k+2} \cdot \frac{(K)^{k+1}}{2^{k+2}} \geq \log^{3k+1} n \cdot \left( \frac{n^k}{c_k(n)} \right)^{k} \cdot n^{k-1} = \mathcal{Y}(n) \cdot n^k.
\]
This completes the proof of Theorem 8. □
4 Proof of Theorem 5

In this section, we prove the main result Theorem 5 using the hypergraph container method (Theorem 7) and supersaturation result for \( k \)-dimensional corners in sets of size \( \Theta(c_k(N)) \) (Theorem 8).

Proof of Theorem 5. Let \( b(k) \) and the infinite sequence \( \{n_i\}_{i=1}^{\infty} \) obtained from Lemma 12. For every \( n \in \{n_i\}_{i=1}^{\infty} \), we define the following functions:

\[
\Upsilon(n) = \frac{\log^{3k+1} n}{n} \cdot \left( \frac{n^k}{c_k(n)} \right)^k,
\]

\[
\Psi(n) = \frac{c_k(n)}{n^k} \cdot \frac{1}{\log^3 n},
\]

where \( c_k(n) \) is the maximum size of a \( k \)-dimensional corner-free subset of \( [n]^k \). For sufficiently large \( n \), we have

\[
\Psi(n) < \frac{1}{200 \cdot (k + 1)^2} < \frac{1}{200 \cdot ((k + 1)!)^2 \cdot (k + 1)},
\]

and

\[
\Upsilon(n) \cdot n \cdot \Psi(n)^k = \frac{\log^{3k+1} n}{n} \cdot \left( \frac{n^k}{c_k(n)} \right)^k \cdot n \cdot \left( c_k(n) \cdot \frac{1}{\log^3 n} \right)^k
\]

\[
= \log n
\]

\[
> (k + 1)^{3(k+1)},
\]

Let us consider \((k+1)\)-uniform hypergraph \( G \) encoding the set of all \( k \)-dimensional corners in \([n]^k\). For a given hypergraph \( G \), the maximum degree of a set of \( j \) vertices of \( G \) is \( \Delta_j(G) = \max \{ d_G(A) : A \subset V(G), |A| = j \} \), where \( d_G(A) \) is the number of hyperedges in \( E(G) \) containing the set \( A \). Then the co-degree of a \((k+1)\)-uniform hypergraph \( G \) of order \( n \) and average degree \( d \) is written as

\[
\Delta(G, \Psi) = 2^{(k+1)-1} \sum_{j=2}^{k+1} 2^{-(j-1)} \Psi(n)^{-j-1} \cdot \frac{\Delta_j(G)}{d}
\]

\[
= 2^{(k+1)-1} \sum_{j=2}^{k+1} \beta_j \cdot \frac{\Delta_j(G)}{d},
\]

where \( \beta_j = 2^{-(j-1)} \Psi(n)^{-j-1} \) for all \( 2 \leq j \leq k+1 \). Since \( \Psi(n) < \frac{1}{200 \cdot (k+1)^{2(k+1)}} < 2^{-3(k+1)} \), we have

\[
\frac{\beta_j}{\beta_{j+1}} = \frac{2^{(j)}}{2^{(j-1)}} \Psi(n)^j = 2^{j-1} \Psi(n) < 2^{(k+1)} \Psi(n) < 1,
\]

\[
(28)
\]
for all $2 \leq j \leq k - 1$. For the case $j = k$, we obtain the following inequality:

$$
(k - 1)(k + 1)^2 \cdot \frac{\beta_k}{\beta_{k + 1}} = (k - 1)(k + 1)^2 \cdot 2^{k - 1} \Psi(n) < 1. 
$$

(29)

Using the equations (26), (28) and (29), we derive that

$$
\Delta(G, \Psi) = 2 \frac{(k + 1)^2}{2} \sum_{j=2}^{k+1} \beta_j \Delta_j(G)
\leq 2 \frac{(k + 1)^2}{2} \left( \sum_{j=2}^{k} \beta_j \frac{(k + 1)^2}{d} + \frac{\beta_{k + 1}}{d} \right)
\leq 2 \frac{(k + 1)^2}{2} \left( (k - 1) \cdot \beta_k \cdot \frac{(k + 1)^2}{d} + \frac{\beta_{k + 1}}{d} \right)
\leq 2 \frac{(k + 1)^2}{2} \left( \frac{2 \beta_{k + 1}}{d} \right) = \frac{2^k}{d \cdot (\Psi(n))^k}
\leq \frac{(k + 1)^{k + 1}}{n \cdot (\Psi(n))^k} \leq \frac{\Upsilon(n)}{12 \cdot (k + 1)!}. 
$$

(30)

From the equations (25) and (30), we can apply the Hypergraph Container Lemma (Theorem 7) on the hypergraph $G$ with $\epsilon = \Upsilon(n)$, $\tau = \Psi(n)$ as a function of $n$ to get the collection $C$ of containers such that all $k$-dimensional corner-free subsets of the $k$-dimensional grid $[n]^k$ are contained in some container in $C$. Using Theorem 7, there exist $c = c(k + 1) \leq 1000 \cdot (k + 1) \cdot ((k + 1)!)^3$ and a collection $C$ of containers such that the followings hold:

- for every $k$-dimensional corner free subset of the $k$-dimensional grid $[n]^k$ is contained in some container in $C$,
- $\log |C| \leq c \cdot n \cdot \Psi(n) \cdot \log \frac{1}{\Upsilon(n)} \cdot \log \frac{1}{\Psi(n)}$,
- for every container $A \in C$ the number of $k$-dimensional corners in $A$ is at most $\Upsilon(n) \cdot n^k$.

The definitions of $\Upsilon(n)$ and $\Psi(n)$ give the following inequality:

$$
\log \frac{1}{\Upsilon(n)} \cdot \log \frac{1}{\Psi(n)} = \log \left( \frac{n}{\log^{3k+1} n} \cdot \left( \frac{c_k(n)}{n^k} \right)^k \right) \cdot \log \left( \frac{n^k}{c_k(n) \cdot \log^3 n} \right)
\leq \log n \cdot ((k + 3) \log n) = (k + 3) (\log n)^2. 
$$

(31)
Using the equation (31) for the collection $\mathcal{C}$ of containers gives:

$$\log |\mathcal{C}| \leq c \cdot n \cdot \Psi(n) \cdot \log \frac{1}{\Upsilon(n)} \cdot \log \frac{1}{\Psi(n)}$$

$$\leq 1000 \cdot (k + 1) \cdot ((k + 1)!)^3 \cdot n \cdot \Psi(n) \cdot \log \frac{1}{\Upsilon(n)} \cdot \log \frac{1}{\Psi(n)}$$

$$\leq 1000 \cdot (k + 1) \cdot ((k + 1)!)^3 \cdot n \cdot \frac{c_k(n)}{n^k} \cdot \frac{1}{\log^3 n} \cdot (k + 3) (\log n)^2 = o(c_k(n)). \quad (32)$$

Note that for every container $A \in \mathcal{C}$, the number of $k$-dimensional corners in $A$ is at most $\Upsilon(n) \cdot n^k$. Now applying Theorem 8 gives:

$$|A| < C' \cdot c_k(n), \quad (33)$$

for every container $A \in \mathcal{C}$. Since every $k$-dimensional corner free subset of the $k$-dimensional grid $[n]^k$ is contained in some container in $\mathcal{C}$, we conclude that the number of $k$-dimensional corner free subsets of $[n]^k$ is at most

$$\sum_{A \in \mathcal{C}} 2^{|A|} \leq |\mathcal{C}| \cdot \max_{A \in \mathcal{C}} 2^{|A|}$$

$$\leq (32) \cdot (33) \leq 2^{o(c_k(n))} \cdot 2^{C' \cdot c_k(n)} = 2^{O(c_k(n))},$$

using the equations (32) and (33). This completes the proof of Theorem 5. \hfill \Box

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