On calculation of cross sections in Lorentz violating theories

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Abstract

We develop a systematic approach to the calculation of scattering cross sections in theories with violation of the Lorentz invariance taking into account the whole information about the theory Lagrangian. As an illustration we derive the Feynman rules and formulas for sums over polarizations in spinor electrodynamics with Lorentz violating operators of dimensions four and six. These rules are applied to compute the probabilities of several astrophysically relevant processes. We calculate the rates of photon decay and vacuum Cherenkov radiation along with the cross sections of electron-positron pair production on background radiation and in the Coulomb field. The latter process is essential for detection of photon-induced air showers in the atmosphere.

1 Introduction

A number of approaches to quantum gravity suggest that Lorentz invariance (LI) may be not exact and breaks down at high energies, see [1] and references therein (see also [2]). Within the effective field theory approach the deviations from LI are described by higher dimension operators suppressed by the putative scale $M$ of quantum gravity. This scale is supposed to coincide with the Planck mass, $M_P \approx 10^{19}$ GeV, in most approaches, but can lie a few orders of magnitude below in certain scenarios [3]. Important constraints
on the high-energy violation of LI (Lorentz violation (LV) for short) have been obtained from considerations of various astrophysical phenomena [4, 5]. Indeed, in the astrophysical processes the elementary particles often reach energies that vastly exceed those attained in the accelerator experiments. Therefore these processes provide a unique probe of the particle dynamics at very high energy. The extreme energies ever observed are reached by ultra-high-energy cosmic rays (UHECR). The power of UHECR physics in constraining Planck-suppressed LV has been extensively discussed in the literature [7, 8, 9, 10, 11, 12].

Most of these studies concentrate on the kinematical effects of LV: appearance of new reactions that are kinematically forbidden in the LI case and the shift of energy thresholds of the known processes. Clearly, more information can be gained by considering the dynamical consequences of LV, i.e., the effect of LV on various reaction rates. This is particularly important in the case of reactions that do not possess a threshold such as photon splitting into several photons [13] and neutrino splitting [11] (see also [14]). Another example is provided by the Bethe–Heitler process — production of electron-positron pairs by a photon in the Coulomb field of a nucleus — that plays the key role in the detection and identification of UHECR photons through their interaction with the Earth atmosphere [15]. The dynamical analysis requires developing a technique for evaluation of the Feynman diagrams in LV theories with higher order operators. So far, no systematic treatment of this issue has been performed. The aim of the present paper is to fill this gap.

LV affects the rate of a given process in three ways: (i) through modification of the phase space integrals; (ii) different wave-functions of the external states; (iii) changes in the vertices and propagators. The effects (ii) and (iii) lead to modification of the expression for the matrix element of the process compared to the standard LI case. We are going to see that all three effects are of the same order and must be taken into account simultaneously to obtain the correct result. As an illustration of our technique we will present calculations of the rates of several astrophysically relevant processes in spinor quantum electrodynamics (QED) with LV operators of dimension four and six. The results will be compared with the estimates based on the kinematical considerations.

Let us mention several works relevant for our study. The modifications of the phase space and the external-states wavefunctions were discussed in [4] in the context of QED with dimension-five LV terms. Spin sums over external states for the model similar to the one considered in this paper were derived in [16]. Ref. [17] considers QED with LV restricted to dimension four operators and calculates the rates of the vacuum Cherenkov radiation and photon decay into electron-positron pair in this theory. The necessity to take all effects (i)
(iii) into account was recently stressed in [18, 19] in the context of theories with LV in the neutrino sector.

The paper is organized as follows. In Sec. 2 we introduce the model, derive the formulas for the spin sums over external states and the Feynman rules. In Sec. 3 we apply these rules to calculate the rates of several processes: photon decay, vacuum Cherenkov radiation, pair production in two-photon collision and pair-production by a photon in the Coulomb field. Sec. 4 is devoted to the discussion of our results.

2 The model: spin sums and Feynman rules

We are going to consider QED with LV operators of dimension up to 6 assuming that the gauge invariance is preserved. To simplify the analysis we impose several additional restrictions:

(a) The theory is required to be invariant under rotations in the three-dimensional space.

(b) We impose the CPT and P invariance. The CPT symmetry is physically essential as it forbids dangerous dimension 3 operators that would lead to an unacceptably large LV at low energies [20]. On the other hand, the requirement of spatial parity is purely technical and is invoked just to further reduce the number of LV structures. It can be easily dropped in a more general treatment.

(c) We include in the Lagrangian only operators that cannot be removed by a field and/or coordinate redefinition.

(d) According to the general logic of the effective field theory, the higher-dimensional operators are equivalent if they coincide on the equations of motion obtained from the lower-dimensional part of the Lagrangian. We consider only operators that are different with respect to this identification.

(e) Above the electroweak scale $M_{EW}$ the model must be embedded into the full Standard Model. The chiral structure of the latter forbids CPT-even LV operators of dimension 5 [21]. This means that even though such terms can be generated below $M_{EW}$, the dimensionless coefficients in front of them will be suppressed by the ratio $M_{EW}/M$ making their contribution negligible. We do not consider dimension 5 operators below.
We include only operators that contain parts quadratic in the fields meaning that they contribute into the free-particle Lagrangian. The rationale behind this requirement is purely pragmatic: to have a minimal framework where LV affects both kinematics and dynamics of the theory.

The above conditions lead to the following Lagrangian, cf. [22],

\[ \mathcal{L} = i\bar{\psi} \gamma^\mu D_\mu \psi - m\bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^i D_i \psi + \frac{ig}{M^2} D_j \bar{\psi} \gamma^i D_i D_j \psi + \frac{\xi}{4M^2} F_{kj} \partial_i^2 F^{kj}, \]  

(1)

where

\[ D_\mu \psi = (\partial_\mu + i e A_\mu) \psi. \]

is the standard covariant derivative. The Greek indices \( \mu, \nu, \ldots \) run from 0 to 3 and are raised/lowered using the Minkowski metric with the signature \((+, -, -, -)\). The Latin indices \( i, j, \ldots \) are purely space-like and take the values 1, 2, 3. Notice that we distinguish the lower and upper space indices: \( \gamma^i = -\gamma_i \), etc. Summation over repeated indices is understood. The last three terms in (1) describe LV. The first of them has dimension 4 while the other two have dimension 6 and are suppressed by the LV scale \( M \). The dimensionless parameters \( \kappa, g, \xi \) characterize the strength of LV (one of the parameters \( g, \xi \) can be absorbed into redefinition of the scale \( M \) but we prefer to keep them explicitly). From the viewpoint of the effective theory, the dimension 4 operator should be treated as the leading LV term. However, the corresponding coefficient \( \kappa \) is experimentally constrained to be extremely small\(^1\), \( |\kappa| < 10^{-15} \) [20], implying that for astrophysically relevant processes the effects of the dimension 6 operators can be comparable or even dominant\(^2\).

Consider the free-particle states. The LV terms modify the dispersion relations for photons and electrons / positrons,

\[ E_\gamma^2 = k^2 + \frac{\xi k^4}{M^2}, \]  

(2)

\[ E_e^2 = m^2 + p^2 \left(1 + \kappa + \frac{g p^2}{M^2}\right)^2 \approx m^2 + p^2 \left(1 + 2\kappa\right) + \frac{2g p^4}{M^2}. \]  

(3)

We observe two types of modifications. First, at \( p \ll M \) the velocity of electrons is different from that of photons. Second, at large energies the dispersion relations receive contributions that are quartic in the particle momenta. Note that with our conventions the velocity of the low-energy photons is equal to 1.

\(^1\)We do not address the naturalness issues related to the smallness of \( \kappa \), see e.g. the discussion in [5].

\(^2\)For example, for UHECR energies \( E \sim 10^{19} \) eV and \( M = 10^{16} \) GeV we obtain \( (E/M)^2 \sim 10^{-12} \gg \kappa \).
Our immediate task is determination of the particles’ wavefunctions together with the sums over polarizations needed for the calculation of the inclusive cross sections. First, consider photon. In the frame where its three-momentum \( k^i \) is directed along the third axis the two polarization vectors are

\[
\varepsilon_{\mu}^{(1)} = (0, 1, 0, 0), \quad \varepsilon_{\mu}^{(2)} = (0, 0, 1, 0).
\]

This gives the standard formula

\[
\sum_{a=1,2} \varepsilon_{\mu}^{(a)} \varepsilon_{\nu}^{(a)} = \text{diag}(0, 1, 1, 0).
\]

We want to cast it into the form that preserves three-dimensional rotational invariance. To do this we use the gauge invariance that ensures that photon couples to a conserved current. Thus the Ward identities remain valid and one can add to the above expression any combination of the form \( n_{\mu} k_{\nu} + k_{\mu} n_{\nu} \) with an arbitrary vector \( n_{\mu} \) without affecting the matrix element. A proper choice of \( n_{\mu} \) allows to cast the above sum into the form,

\[
\sum_{a=1,2} \varepsilon_{\mu}^{(a)} \varepsilon_{\nu}^{(a)} \simeq \text{diag}(-E^2/k^2, 1, 1, 1).
\]

This reduces to the standard expression

\[
\sum_{a=1,2} \varepsilon_{\mu}^{(a)} \varepsilon_{\nu}^{(a)} \simeq -\eta_{\mu\nu}
\]

in the case of the LI dispersion relation \( E_\gamma = k \).

Let us turn now to the spinor wave-functions. Decomposing the spinor field into positive and negative frequency components,

\[
\psi(x, t) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E}} \left( e^{-iEt+i}px u^s(p) a_s(p) + e^{iEt-ixp} v^s(p) b_s^+(p) \right),
\]

we find the solutions of the modified Dirac equation:

\[
u^s(p) = \begin{pmatrix} \sqrt{E - (\sigma^i p^i) \left( 1 + \kappa + \frac{p^2}{M^2} \right) \chi^s} \\ \sqrt{E + (\sigma^i p^i) \left( 1 + \kappa + \frac{p^2}{M^2} \right) \chi^s} \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{E - (\sigma^i p^i) \left( 1 + \kappa + \frac{p^2}{M^2} \right) \zeta^s} \\ -\sqrt{E + (\sigma^i p^i) \left( 1 + \kappa + \frac{p^2}{M^2} \right) \zeta^s} \end{pmatrix}
\]

where \( \sigma^i \) are the Pauli matrices and \( \chi^s \) and \( \zeta^s \), \( s = 1, 2 \) are two-component basis spinors. Choosing the latter to be orthogonal with unit norm we obtain the usual normalization conditions,

\[
(u^s(p))^\dagger u^s(p) = (v^s(p))^\dagger v^s(p) = 2E \delta^{rs}, \quad (u^s(p))^\dagger v^s(-p) = 0.
\]
This leads to the spin sums:

\[
\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \gamma^0 E - \gamma^i p^i \left( 1 + \kappa + \frac{gp^2}{M^2} \right) + m ,
\]

(5a)

\[
\sum_{s=1,2} v^s(p)\bar{v}^s(p) = \gamma^0 E - \gamma^i p^i \left( 1 + \kappa + \frac{gp^2}{M^2} \right) - m .
\]

(5b)

One can check using these expressions and the standard creation–annihilation algebra

\[
\{a_s(p), a^+_r(q)\} = \{b_s(p), b^+_r(q)\} = (2\pi)^3 \delta_{sr} \delta(p-q)
\]

that the spinor operators satisfy the canonical commutation relations

\[
\{\psi(x, t), \psi^+(y, t)\} = \delta(x - y) .
\]

It is convenient to introduce the notations

\[
p^0 = E , \quad \tilde{p}^i = p^i \left( 1 + \kappa + \frac{gp^2}{M^2} \right)
\]

(6)

that allows to write (5) in a more compact form

\[
\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \gamma^\mu \tilde{p}_\mu + m , \quad \sum_{s=1,2} v^s(p)\bar{v}^s(p) = \gamma^\mu \tilde{p}_\mu - m .
\]

The formulas (4), (5) were previously derived in [16].

To compute the cross sections we need the Feynman rules for the theory (1). In terms of the vector \(\tilde{p}^\mu\) the fermion propagator is given by the formally standard expression,

\[
\frac{i(\gamma^\mu \tilde{p}_\mu + m)}{\tilde{p}_\mu \tilde{p}^\mu - m^2 + i\epsilon} .
\]

To obtain the photon propagator\(^3\) we have to choose the gauge. A convenient gauge fixing term, that eliminates non-diagonal contributions in the photon Lagrangian, is

\[
\mathcal{L}_{GF} = -\frac{1}{2} \left( \partial_0 A_0 - \left( 1 - \frac{\xi}{M^2} \Delta \right) \partial_i A_i \right) \left[ 1 - \frac{\xi}{M^2} \Delta \right]^{-1} \left( \partial_0 A_0 - \left( 1 - \frac{\xi}{M^2} \Delta \right) \partial_i A_i \right) ,
\]

(7)

where \(\Delta \equiv \partial_i \partial_i\) is the spatial Laplacian. The resulting propagator has the form,

\[
\frac{i}{k^2} = \frac{i}{E^2 - k^2 \left( 1 + \frac{\xi k^2}{M^2} \right) + i\epsilon} \left[ E^2 - k^2 \left( 1 + \frac{\xi k^2}{M^2} \right) + i\epsilon \right]^{-1} \text{diag} \left( - \left( 1 + \frac{\xi k^2}{M^2} \right), 1, 1, 1 \right) ,
\]

\(^3\)We will not use the photon propagator in the rest of the paper and present it here only for the sake of completeness.
Note that the gauge-fixing term (7) is non-local in space. However, one does not expect any problems related to that, at least within the perturbation theory. As an alternative, one can consider local gauge fixing and work with an off-diagonal photon propagator.

It remains to obtain the expressions for the interaction vertices. The fourth and fifth terms in (1) modify the photon–fermion interaction that now takes the form

\[ V_1^{\mu} \equiv \frac{-ie\gamma^\mu - ie\delta^\mu_i \left[ \gamma^i(p_1 \cdot \gamma) + \gamma^i(p_2 \cdot \gamma) - (p_1 \cdot p_2)\gamma^i \right]}{M^2} \],

(8)

where the two fermion momenta \( p_1, p_2 \) are assumed to flow out of the vertex and dot stands for the scalar product of three-dimensional vectors, \( (p_1 \cdot \gamma) = p^i_1 \gamma^i, \) etc. Besides, the fifth term in (1) introduces new vertices involving two and three photons,

\[ V_2^{\mu\nu} \equiv \frac{ige^2}{M^2} \left[ \delta^\mu_i \delta^\nu_j \right] \gamma^i, \]

(9)

\[ V_3^{\mu\nu\lambda} \equiv \frac{-2ige^3}{M^2} \left[ \delta^\mu_i \delta^\nu_j \delta^\lambda_k + \delta^\nu_i \delta^\mu_j \delta^\lambda_k + \delta^\lambda_i \delta^\mu_j \delta^\nu_k \right] \gamma^i, \]

(10)

where the momenta \( p_1 \) and \( p_2 \) are again flowing out of the vertex. Note that the 2-photon interaction (9) is antisymmetric under the exchange of the electron and positron momenta. It is worth stressing that the modification of the 1-photon interaction and the presence of the multi-photon vertices (9), (10) are required by the gauge invariance and ensure that the Ward identities are satisfied.

We are now going to apply the above rules to compute the rates of several reactions.

3 Processes and rates

We start with the elementary processes of photon decay and vacuum Cherenkov radiation. From the viewpoint of the astrophysical applications the exact expressions for the rates in
these cases are unnecessary. Indeed, these processes are extremely fast, once kinematically allowed, and for all practical purposes may be considered as happening instantaneously. However, the study of these simple reactions is instructive and sets the stage for the more involved calculations in the next subsections.

3.1 Photon decay

The photon decay

$$\gamma \rightarrow e^+ e^-$$

(11)
can become allowed in the presence of LV above certain threshold, see e.g. Ref. [8] for the discussion of the kinematics of this reaction. We are interested in computing the rate of the process well above the threshold. In this regime the masses of the electron and positron can be neglected which considerably simplifies the calculation. The matrix element has the form

$$\mathcal{M} = \bar{u}(p_1) V_{1\gamma}^{\mu} v(p_2) \epsilon_{\mu},$$

with the vertex given by Eq. (8). The inclusive rate is obtained by taking the square of this expression, summing over the spins of the final states and averaging over the photon polarizations. Using (4), (5), where we neglect the fermion mass, and keeping only up to linear terms in the LV parameters we obtain,

$$|\mathcal{M}|^2 = 4e^2(1 + 3\kappa)(E_1 E_2 - (p_1 \cdot p_2))$$

$$- \frac{2e^2}{M^2} k^2(E_1 E_2 + (p_1 \cdot p_2)) + \frac{4e^2 g}{M^2} (p_1 \cdot p_2)^2 + \frac{4e^2 g}{M^2} E_1 E_2(p_1^2 + p_2^2 - 3(p_1 \cdot p_2)).$$

(12)
The first line here is the standard matrix element of the Lorentz invariant QED multiplied by an overall $\kappa$-dependent factor. We will see shortly that in the leading approximation the dependence of such overall factors on the LV parameters can be safely neglected. On the other hand, the second line represents the genuine LV correction to the matrix element.

The decay width is given by the textbook formula,

$$\Gamma = \frac{1}{2E_\gamma} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \sqrt{(2\pi)^4 \delta(E_\gamma - E_1 - E_2) \delta(k - p_1 - p_2) |\mathcal{M}|^2},$$

that remains valid in the presence of LV (its derivation does not make any use of LI). One chooses the photon momentum to be directed along the first axis and parameterizes the momenta of the fermions as follows,

$$p_1^i = (k(1 + x)/2, p_\perp, 0), \quad p_2^i = (k(1 - x)/2, -p_\perp, 0).$$
Clearly, up to rotations, this is the most general parameterization satisfying the momentum conservation. The conservation of energy in the zeroth order in the LV parameters requires $x$ to lie in the range $-1 \leq x \leq 1$. In the ultrarelativistic regime, that is of interest to us, we have $p_\perp \ll k$. This allows to expand

$$E_\gamma = k + \frac{\xi k^3}{2M^2}, \quad E_{1,2} = \frac{k}{2}(1 + x)(1 \pm x) + \frac{gk^3}{8M^2}(1 \pm x)^3 + \frac{p_\perp^2}{k(1 \pm x)},$$

which yields

$$\Gamma = \frac{1}{2k} \int \frac{dx dp_\perp}{8\pi k(1 - x^2)} \delta \left( \omega_{LV}(x) - \frac{2p_\perp^2}{k(1 - x^2)} \right) |\mathcal{M}|^2,$$  \hspace{1cm} (13)

where we have introduced the notation

$$\omega_{LV}(x) = -\kappa k + \frac{\xi k^3}{2M^2} - \frac{gk^3}{4M^2}(1 + 3x^2).$$  \hspace{1cm} (14)

It is instructive to compute the contributions of the first (Lorentz invariant) and second (Lorentz violating) lines of (12) into the decay rate separately. For the contribution of the LI part of the matrix element we have,

$$\Gamma_1 = \frac{\alpha(1 + 3x)}{k} \int \frac{dx dp_\perp}{k(1 - x^2)} \delta \left( \omega_{LV}(x) - \frac{2p_\perp^2}{k(1 - x^2)} \right) \left[ \frac{\kappa k^2}{2}(1 - x^2) + \frac{gk^4}{8M^2}(1 - x^4) + \frac{2p_\perp^2}{1 - x^2} \right]$$

$$= \alpha k \int dx \left[ -\frac{\kappa}{4}(1 + x^2) + \frac{k^2}{M^2} \left( \frac{\xi}{4} - \frac{g}{16} - \frac{3gx^2}{8} - \frac{gx^4}{16} \right) \right],$$  \hspace{1cm} (15)

where $\alpha = e^2/4\pi$ is the fine structure constant and passing to the second line we performed the integration of $p_\perp$. We see that the rate is suppressed by the parameters describing LV. Notice, however, that the suppression does not come from the phase space integration, that reduces to the integral over $x$. Instead, the suppression is due to the smallness of the LI matrix element for the nearly collinear kinematics realized in the decay. For the contribution of the terms in the second line of (12) we obtain,

$$\Gamma_2 = \alpha k \int dx \frac{k^2}{M^2} \left( -\frac{\xi}{8} + \frac{\xi x^2}{8} + \frac{gx^2}{8} - \frac{gx^4}{8} \right).$$

One observes that it is of the same order as (15). Combining the two contributions we obtain the total decay rate,

$$\Gamma_{\gamma \rightarrow e^+e^-} = \frac{\alpha}{4} \int dx \,(1 + x^2) \omega_{LV}(x).$$  \hspace{1cm} (16)
The domain of integration in $x$ is determined by the condition that $p_2^\perp$ found from the \(\delta\)-function in (13) is positive. This implies

$$\omega_{LV}(x) \geq 0.$$ 

One distinguishes several cases:

(a) $0 \leq 2g \leq \xi_*$ or $g \leq 0, \ g/2 \leq \xi_*$, where $\xi_* = \xi - 2M^2\kappa/k^2$. In this case the integration is over the whole range $-1 \leq x \leq 1$ yielding

$$\Gamma_{\gamma\rightarrow e^+e^-} = \alpha k \left[ -\frac{2\kappa}{3} + \frac{k^2}{M^2} \left( \frac{\xi}{3} - \frac{11g}{30} \right) \right].$$

For $\xi = g = 0, \ \kappa < 0$ this coincides with the result of\(^4\) [17].

(b) $0 < g, \ g/2 < \xi_* < 2g$. The integration range is

$$-\frac{1}{\sqrt{3}} \sqrt{\frac{2\xi_*}{g}} - 1 \leq x \leq \frac{1}{\sqrt{3}} \sqrt{\frac{2\xi_*}{g}} - 1$$

which gives

$$\Gamma_{\gamma\rightarrow e^+e^-} = \frac{\alpha k^3}{30\sqrt{3}M^2} \sqrt{\frac{2\xi_*}{g}} - 1 \left( 13\xi_* - 7g + \frac{2\xi_*^2}{g} \right).$$

Note that in this case the fraction of the total energy carried by each fermion is bounded from below by a strictly positive number.

(c) $2g < \xi_* < g/2 < 0$. The integration is performed over two disjoint regions

$$-1 \leq x \leq -\frac{1}{\sqrt{3}} \sqrt{\frac{2\xi_*}{g}} - 1 \quad \text{and} \quad \frac{1}{\sqrt{3}} \sqrt{\frac{2\xi_*}{g}} - 1 \leq x \leq 1$$

implying that the electron and positron momenta are necessarily different. This corresponds to the regime of the the so-called “asymmetric threshold” [23]. In this case we have

$$\Gamma_{\gamma\rightarrow e^+e^-} = \alpha k^3 \left[ \left( \frac{\xi_* - 11g}{10} \right) - \frac{1}{30\sqrt{3}} \sqrt{\frac{2\xi_*}{g}} - 1 \left( 13\xi_* - 7g + \frac{2\xi_*^2}{g} \right) \right].$$

\(^4\)In [17] the rate is calculated in the model where the electron/positron have unit velocities while the photon velocity differs from one. This is related to our setup by a rescaling of the space coordinates and therefore is physically equivalent.
3.2 Cherenkov radiation

Another elementary process that can become allowed in the presence of LV is the vacuum Cherenkov radiation — spontaneous emission of a photon by a fast moving electron,

\[ e^- \rightarrow \gamma e^- . \]

This is the cross-channel of the reaction (11). As before, we consider the rate well above the threshold and thus neglect the electron mass. Thus the corresponding matrix element is obtained from (12) by replacing the positron energy and momentum with the (minus) energy and momentum of the incoming electron,

\[ E_2 \mapsto -E , \quad p^i_2 \mapsto -p^i , \]

and by flipping the overall sign. This yields

\[
|\mathcal{M}|^2 = 4e^2(1 + 3\kappa)(EE' - (p \cdot p'))
- \frac{2e^2\xi}{M^2} k^2 (EE' + (p \cdot p')) - \frac{4e^2g}{M^2} (p \cdot p')^2 + \frac{4e^2g}{M^2} EE'(p^2 + p'^2 + 3(p \cdot p')) ,
\]

where \(E', p^i\) are the energy and momentum of the outgoing electron. Assuming that the incoming electron propagates along the first axis, we write

\[ p^i = (p(1 - x), p'_L, 0) , \quad k^i = (px, -p'_L, 0) , \]

where now \(0 \leq x \leq 1\) is the fraction of momentum carried away by the photon. Adopting the ultrarelativistic kinematics to expand the energies of the particles, we obtain for the rate of the process,

\[
\Gamma_{e^- \rightarrow \gamma e^-} = \frac{\alpha}{2p} \int \frac{dx dp'_L}{px(1 - x)} \delta \left( \omega'_{LV}(x) - \frac{p'^2}{2px(1 - x)} \right)
\left[ 2\kappa p^4(1 - x) + \frac{gp^4(1 - x)(2 - 2x + x^2)}{M^2} + \frac{p'^2}{2(1 - x)} \right.
- \left. \frac{\xi p^4}{M^2} x^2(1 - x) - \frac{gp^4}{M^2} (1 - x)^2 + \frac{gp^4}{M^2} (1 - x)(5 - 5x + x^2) \right] ,
\]

where

\[
\omega'_{LV}(x) = \kappa px - \frac{\xi p^3x^3}{2M^2} + \frac{gp^3(3x - 3x^2 + x^3)}{M^2} .
\]

The first three terms in the square brackets come from the LI part of the matrix element (the first line in (17)), while the rest of the integrand corresponds to the LV part: the second line
in (17). One again observes that the two contributions are of the same order. Performing
the integral over \( p'_{\perp} \) we obtain the differential rate

\[
\frac{d\Gamma_{e^-\rightarrow\gamma e^-}}{dx} = \alpha \left( \frac{2}{x} - 2 + x \right) \omega'_{LV}(x) .
\]

To obtain the total rate one has to integrate over the values of \( x \) satisfying the condition

\[
\omega'_{LV}(x) \geq 0 .
\]

The range of the integration depends on the hierarchies among the LV parameters. Classifying all possibilities is not relevant for our discussion. In the simplest case \( 0 \leq \kappa, 0 \leq g, \xi \leq 2g \) the integral is over the whole range \( 0 \leq x \leq 1 \) giving

\[
\Gamma_{e^-\rightarrow\gamma e^-} = \alpha p \left[ \frac{4\kappa}{3} + \frac{p^2}{M^2} \left( \frac{157g}{60} - \frac{11\xi}{60} \right) \right] . \tag{18}
\]

A useful characteristic of the process is the rate of the energy loss by the electron

\[
\frac{dE}{dt} = -\int dx \, px \, \frac{d\Gamma_{e^-\rightarrow\gamma e^-}}{dx} = -\alpha p^2 \left[ \frac{7\kappa}{12} + \frac{p^2}{M^2} \left( \frac{11g}{12} - \frac{2\xi}{15} \right) \right] . \tag{19}
\]

Taking the ratio of (19) and (18) we observe that on average the emitted photon takes away an order-one fraction of the electron energy. In the case \( g = \xi = 0 \) the expressions (18), (19) reproduce those of Ref. [17].

### 3.3 Pair production

We now turn to more complicated reactions containing two particles in the initial state. The first reaction is production of an electron – positron pair in the collision of a high-energy photon with a soft photon from an astrophysical background,

\[
\gamma\gamma \rightarrow e^+e^- .
\]

Unlike the two reactions considered in the previous subsections, this process is kinematically allowed in the LI case. Our goal is to find how its cross section is affected by LV.

The diagrams contributing to the required matrix element are shown in Fig. 1. Note that the third diagram with the two-photon vertex is absent in the standard case. Denoting the
three contributions by $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ we write,

$$
\mathcal{M}_1 = \bar{u}(p_1) V_{1\gamma}^{\mu}(p_1, p_2 - q) \frac{i(\gamma^\lambda (\tilde{q} - \tilde{p}_2)_\lambda + m)}{(\tilde{q} - \tilde{p}_2)_\rho (\tilde{q} - \tilde{p}_2)^\rho - m^2} V_{1\nu}^{\nu}(q - p_2, p_2) v(p_2) \varepsilon_\mu(k) \varepsilon_\nu(q) ,
$$

$$
\mathcal{M}_2 = \bar{u}(p_1) V_{1\gamma}^{\mu}(p_1, q - p_1) \frac{i(\gamma^\lambda (\tilde{p}_1 - \tilde{q})_\lambda + m)}{(\tilde{p}_1 - \tilde{q})_\rho (\tilde{p}_1 - \tilde{q})^\rho - m^2} V_{1\nu}^{\nu}(p_1 - q, p_2) v(p_2) \varepsilon_\mu(q) \varepsilon_\nu(k) ,
$$

$$
\mathcal{M}_3 = \bar{u}(p_1) V_{2\gamma}^{\mu\nu}(p_1, p_2) v(p_2) \varepsilon_\mu(k) \varepsilon_\nu(q) ,
$$

where $V_{1\gamma}^{\mu}$ and $V_{2\gamma}^{\mu\nu}$ are given by (8), (9) and the four-vectors with tildes are defined in (6).

We consider the following kinematical configuration,

$$
k^i = (k, 0, 0) ,
q^i = (q_x, q_y, 0) ,
$$

$$
p_1^i = \left( \frac{k + q_x}{2} (1 + x), \frac{q_y}{2} (1 + x) + p_y, p_z \right) ,
q_2^i = \left( \frac{k + q_x}{2} (1 - x), \frac{q_y}{2} (1 - x) - p_y, -p_z \right) .
$$

The photon with the momentum $q^i$ is assumed to be soft, $q_x, q_y \ll k$. This allows to neglect any LV in the corresponding dispersion relation and write

$$
q^0 = \omega , \ q_x = \omega \cos \theta , \ q_y = \omega \sin \theta ,
$$

where $\theta$ is the collision angle ($\theta = \pi$ corresponds to the head-on collision).

Derivation of the complete expression for the square of the matrix element in this case would be too cumbersome and not illuminating. Instead, our strategy will be to compute the matrix element in the leading approximation expanding in the small ratio $\omega/k$. To this end we need to determine the orders of magnitude of various quantities appearing in the calculation. The phenomenologically interesting case is when the LV corrections in the particle dispersion relations (2), (3) are of the same order as the square of the relativistic invariant mass $s = 2k\omega(1 - \cos \theta)$. In other words, we shall treat the LV corrections $\propto k^2$, $gk^4/M^2$, $\xi k^4/M^2$ as being of the same order as $k\omega$. Besides, in the standard LI case the
perpendicular components of the electron and positron momenta are also determined by the invariant mass,

\[ p_y^2, p_z^2 \sim k \omega. \]

We will assume this estimate to hold in the presence of LV, as will be verified by the explicit calculation below.

We now observe that the denominators of the propagators in the amplitudes (20a), (20b) are of order \( k \omega \),

\[ (\tilde{q} - \tilde{p}_2)_\lambda(\tilde{q} - \tilde{p}_2)^\lambda - m^2 \approx -k(\omega - q_x)(1 - x), \quad (\tilde{p}_1 - \tilde{q})_\lambda(\tilde{p}_1 - \tilde{q})^\lambda - m^2 \approx -k(\omega - q_x)(1 + x), \]

where we have used \( \tilde{p}_1^\lambda \tilde{p}_1^\lambda = \tilde{p}_2^\lambda \tilde{p}_2^\lambda = m^2 \).

Therefore to get the leading-order result we need to calculate the numerator in the expression for \( |M_1 + M_2|^2 \) to order \( O((\omega/k)^2) \). On the other hand, \( M_3 \) is already suppressed by the first power of the LV parameters. Thus we can neglect its square while in the interference terms \( M_3M_1^\star, M_3M_2^\star \) it is enough to take only the LI part of \( M_1, M_2 \). After a rather tedious but straightforward calculation we obtain

\[
|M|^2 = e^4 \left[ 4 \frac{1 + x^2}{1 - x^2} - \frac{32p_2^4}{k(\omega - q_x)(1 - x^2)^2} + \frac{64p_1^4}{k^2(\omega - q_x)^2(1 - x^2)^3} - \frac{16\omega_{LV}(x)p_1^4}{k(\omega - q_x)^2(1 - x^2)} \right],
\]

(22)

where \( p_2^2 = p_y^2 + p_z^2 \) and \( \omega_{LV}(x) \) is defined in (14). In deriving (22) we have neglected the electron mass, which is justified well above the threshold\(^5\). It is worth mentioning that in the calculation leading to (22) one finds a large amount of cancellations between the contributions from various products of the amplitudes (20). Thus all terms quadratic in the LV parameters \( \kappa, g, \xi \) disappear. Besides, the two interference terms containing \( M_3 \) cancel each other. The latter is an artifact of the massless approximation: one can check that the contribution of the two-photon vertex does not vanish if the finite electron mass is taken into account.

The cross section is given by the formula\(^6\)

\[
\sigma_{\gamma\gamma \rightarrow e^+e^-} = \frac{1}{32\pi k\omega(1 - \cos \theta)} \int dx \frac{dp_1^2}{k(1 - x^2)} \delta\left(\omega - q_x + \omega_{LV}(x) - \frac{2p_1^2}{k(1 - x^2)}\right)|M|^2. \tag{23}
\]

\(^5\)We will need to take the mass into account later in order to cut off the logarithmic divergence in the phase space integral.

\(^6\)The combination \((1 - \cos \theta)\) in the denominator of the prefactor comes from the projection of the relative velocity of the colliding photons on the \( x \)-axis, \(|v_{k,x} - v_{q,x}|\). This combination enters in the relativistic definition of the invariant cross section. Though in our case the relativistic invariance is absent, we prefer
Substituting \( q_x = \omega \cos \theta \) and integrating over \( p_\perp \) we obtain

\[
\sigma_{\gamma\gamma \to e^+e^-} = \frac{\alpha^2 \pi}{2k\omega(1 - \cos \theta)} \int dx \frac{1 + x^2}{1 - x^2} \left[ 1 + \left( 1 + \frac{2\omega_{LV}(x)}{\omega(1 - \cos \theta)} \right)^2 \right].
\]

The domain of integration in \( x \) is determined by the condition,

\[
\omega(1 - \cos \theta) + \omega_{LV}(x) > 0. \tag{24}
\]

In what follows we restrict to the case when it covers the whole range \(-1 \leq x \leq 1\). Then the integral logarithmically diverges at the end-points. These correspond to the kinematical configuration when the total energy is carried away by one of the fermions, while the second fermion stays with zero energy. Clearly, such configuration is possible only for strictly massless fermions and the divergence is cut off once we take into account the finite electron mass. This is achieved by substituting

\[
p_\perp^2 \mapsto p_\perp^2 + m^2 \tag{25}
\]

in the argument of the \( \delta \)-function in (23). Consequently the condition (24) is replaced by

\[
p_\perp^2 > 0 \implies 1 - x^2 > \frac{2m^2}{k(\omega(1 - \cos \theta) + \omega_{LV})},
\]

implying that \( |x| \) is strictly less than one. This yields the total cross section with the logarithmic accuracy,

\[
\sigma_{\gamma\gamma \to e^+e^-} = \frac{\alpha^2 \pi}{k\omega(1 - \cos \theta)} \left[ 1 + \left( 1 + \frac{2\omega_{LV}(1)}{\omega(1 - \cos \theta)} \right)^2 \right] \log \left[ \frac{k(\omega(1 - \cos \theta) + \omega_{LV})}{m^2} \right], \tag{26}
\]

where \( \omega_{LV} \) is taken at \( x = 1 \). This formula is valid as long as

\[
k(\omega(1 - \cos \theta) + \omega_{LV}) \gg m^2.
\]

We see that the effect of LV on the cross section is governed by the ratio

\[
r = \frac{\omega_{LV}}{\omega(1 - \cos \theta)}.
\]

The condition (24) implies \( r > -1 \).

---

To stick to the textbook definition to facilitate comparison with the standard results. Note that in the LI case the above prefactor turns into relativistic invariant inversly proportional to the square of the invariant mass \( s \).
One observes that the cross section can be significantly enhanced if \( r \) is larger than one. However, in this case \( \omega_{LV} \) is positive implying that the photon decay is kinematically allowed. The latter will dominate the pair production in most astrophysical circumstances. From this viewpoint, the case of \( r \) belonging to the interval \(-1 < r < 0\) is more interesting. Then the photon decay is forbidden, but the pair production on the background still takes place. In this case the cross section (26) differs from the standard relativistic expression by a factor of order one.

### 3.4 Pair production in the Coulomb field

The last reaction we consider is pair production by a high-energy photon in the Coulomb field of a nucleus,

\[
\gamma Z \rightarrow Ze^+e^-.
\]

Here \( Z \) denotes the charge of the nucleus in the units of the elementary charge. This reaction does not have a threshold and in the standard LI case dominates the interaction of the UHECR photons with the Earth atmosphere giving rise to the electromagnetic showers that are used for the detection and identification of such photons [15]. The analysis of the changes induced by LV in the cross section of this reaction is therefore crucial to determine the detection efficiency for UHECR photons in LV models.

The process is conveniently represented as a collision of the high-energy photon with a soft virtual photon from the nucleus’ Coulomb field. Thus it is described by the same diagrams shown in Fig. 1, as the two-photon collision of the previous subsection. For the momenta of the particles taking part in the reaction we use the parameterization (21) and evaluate the square of the matrix element to the leading order in the small quantity \( q_x/k \).

The LV contributions appearing in the dispersion relations (2), (3) will be assumed to be of order \( k q_x \).

There are several simplifications compared to the case of the previous subsection. First, the virtual photon has purely time-like polarization, \( \varepsilon_\mu(q) = \delta_\mu^0 \), and thus the contribution of the third diagram in Fig. 1 vanishes identically, see (9). Second, it has vanishing energy, \( q^0 = 0 \), which reduces the number of terms appearing in the calculations. Moreover, it turns out that the leading contribution in the numerator of the square of the matrix element is of order \( O(k q_x) \) (as opposed to \( O((k q_x)^2) \) in the case of collision of two real photons). This means that it is sufficient to evaluate the numerator up to linear approximation in the LV parameters.
On the other hand, unlike the case of the previous subsection, one cannot assume the components \( q_x, q_y \) of the virtual photon momentum to be of the same order, as the calculation of the cross section involves integration over all possible values. Instead, we are going to find that the integral is saturated at

\[
q_y^2 \sim kq_x \quad \iff \quad q_y \gg q_x .
\]

Thus we have to keep all terms with \( q_y \) up to second power.

Finally, in the matrix element we will provisionally neglect the electron mass. The conditions for the validity of this approximation will be discussed below. Then the direct computation yields,

\[
|M|^2 = \frac{2Z^2 e^2}{(q_x^2 + q_y^2)^2} \left[ \frac{1-x}{1+x} \cdot \frac{4p_y^2 + 4p_z^2 + 4p_y q_y(1+x) + q_y^2(1+x)^2 - k\omega_{LV}(x)(1-x)^2}{(2p_y q_y + q_y^2 x - kq_x(1-x))^2} \right. \\
+ \left. \frac{1-x}{1+x} \cdot \frac{4p_y^2 + 4p_z^2 - 4p_y q_y(1-x) + q_y^2(1-x)^2 - k\omega_{LV}(x)(1-x)^2}{(2p_y q_y + q_y^2 x + kq_x(1+x))^2} \right] \\
+ \frac{2(4p_y^2 + 4p_z^2 + 4p_y q_y x - k\omega_{LV}(x)(1-x)^2)}{(2p_y q_y + q_y^2 x - kq_x(1-x))(2p_y q_y + q_y^2 x + kq_x(1+x))}. \tag{28}
\]

Note the factor

\[
\frac{Z^2 e^2}{(q_x^2 + q_y^2)^2}
\]

describing the density of virtual photons in the Coulomb field. In deriving (28) we have summed over the spins of the electron and positron and averaged over the polarizations of the high-energy photon.

The formula for the cross section in the case of scattering on a fixed scattering center reads,

\[
\sigma_{\gamma Z \to e^+ e^-} = \frac{1}{2k} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi) \delta(E_\gamma - E_1 - E_2) \frac{|M|^2}{|M|^2}.
\]

It is convenient to trade the integration variable \( p_2^i \) for \( q^i = (k - p_1 - p_2)^i \). Then, using the axial symmetry of the problem, we can perform integration over the direction of \( q^i \) in the \( yz \)-plane. This gives a factor \( 2\pi \) and leaves us with the integrals over \( q_x \) and \( q_y \). The first is removed with the help of the \( \delta \)-function using the expansion

\[
E_\gamma - E_1 - E_2 \approx \omega_{LV}(x) - \frac{2(p_y^2 + p_z^2)}{k(1-x^2)} - \frac{q_y^2}{2k} - q_x .
\]

Notice that this restricts the integral to the portion of the phase space where

\[
q_x \sim |\omega_{LV}| , \quad q_y^2, p_y^2, p_z^2 \sim k|\omega_{LV}| .
\]

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In particular, (27) is indeed fulfilled. Neglecting \( q_x \) in the Coulomb propagator\(^7\) (29) and splitting the integral over \( p_1^i \) into the longitudinal and transverse parts we arrive at

\[
\sigma_{\gamma Z \rightarrow Ze^+e^-} = \int dx \, dp_y \, dp_z \, dq_y \frac{2Z^2 \alpha_0^6}{(2\pi)^4 q_y^3} \left[ \frac{4p_y^2 + (2p_y + q_y(1 + x))^2 - k\omega_{LV}(x)(1 - x^2)^2}{D_1^2} \right. \\
+ \left. \frac{4p_y^2 + (2p_y - q_y(1 - x))^2 - k\omega_{LV}(x)(1 - x^2)^2}{D_2^2} \right. \\
- \left. \frac{2(4p_y^2 + 4p_y^2 q_y x - k\omega_{LV}(x)(1 - x^2)^2)}{D_1 D_2} \right],
\]

(31)

where

\[
D_1 = 4p_y^2 + (2p_y + q_y(1 + x))^2 - 2k\omega_{LV}(x)(1 - x^2),
\]

(32a)

\[
D_2 = 4p_y^2 + (2p_y - q_y(1 - x))^2 - 2k\omega_{LV}(x)(1 - x^2).
\]

(32b)

In what follows we will restrict to the case \( \omega_{LV}(x) < 0 \), so that the denominators (32) never vanish. Physically, this corresponds to the parameter region where spontaneous photon decay is forbidden.

The next step is to integrate over \( p_y, p_z \). Note that the contribution of each term in the square brackets in (31), when considered separately, is logarithmically divergent at \( p_y, p_z \rightarrow \infty \). However, these divergences cancel in the sum. We obtain,

\[
\sigma_{\gamma Z \rightarrow Ze^+e^-} = \alpha_0^3 Z^2 \int dx \, k |\omega_{LV}(x)| \int \frac{dy}{y^2} \left[ \frac{y + 1 - x^2}{\sqrt{y(y + 2(1 - x^2))}} \log \left[ \frac{\sqrt{y + 2(1 - x^2)} + \sqrt{y}}{\sqrt{y + 2(1 - x^2)} - \sqrt{y}} \right] - 1 \right],
\]

where

\[
y = \frac{q_y^2}{k |\omega_{LV}(x)|}.
\]

The \( y \)-integral logarithmically diverges at \( y \rightarrow 0 \). This is an artifact of neglecting \( q_x \) in the Coulomb propagator. Given that \( q_x \) is of order \( |\omega_{LV}| \), we conclude that the \( y \)-integral must be cut off at

\[
y_0 \sim \frac{|\omega_{LV}|}{k}.
\]

(33)

This leads to the expression

\[
\sigma_{\gamma Z \rightarrow Ze^+e^-} = \frac{2Z^2 \alpha_0^3}{3} \int dx \, k |\omega_{LV}(x)| \frac{1 + x^2}{1 - x^2} \left[ \log \frac{1 - x^2}{y_0} + \frac{13 - 6 \log 2}{6} \right].
\]

(34)

\(^7\)This approximation breaks down in a certain corner of the phase space, see below.
Here we have expanded the integrand under the assumption
\[ y_0 \ll (1 - x^2) , \] (35)
whose validity will be checked shortly. The above integral is again logarithmically divergent at \( x \to \pm 1 \). As in the previous subsection, to cut off this divergence we have to take into account the finite electron mass. This is achieved by the substitution (25) in the argument of the energy-conservation \( \delta \)-function. From (30) we find that the mass can be neglected as long as
\[ 1 - x^2 \gg \frac{m^2}{|k \omega_{LV}(x)|} . \] (36)
This is satisfied in most of the integration domain provided the hierarchy
\[ |k \omega_{LV}| \gg m^2 , \] (37)
which represents the condition for the validity of our approximation.

Restricting the domain of integration in (34) according to (36) we obtain the total cross section with logarithmic accuracy,
\[ \sigma_{\gamma Z \to Z e^+ e^-} = \frac{4Z^2 \alpha^3}{3k|\omega_{LV}|} \left[ \log \frac{k}{|\omega_{LV}|} - \frac{1}{2} \log \frac{k|\omega_{LV}|}{m^2} \right] \log \frac{k|\omega_{LV}|}{m^2} , \] (38)
where \( \omega_{LV} \) is taken at \( x = 1 \).

In the realistic situation the nucleus is surrounded by atomic electrons that screen its Coulomb field at large distances. Thus \( q_y \) is bounded from below by the inverse size of the atom, \( q_y \gtrsim 1/a \). In the mean-field atomic model one finds (see e.g. [24]),
\[ a \sim \frac{1}{\alpha Z^{1/3}m} . \]
If the momentum \( 1/a \) is larger than \( |\omega_{LV}| \), we have to replace (33) by
\[ y_0 \sim \alpha^2 Z^{2/3} \frac{m^2}{|k \omega_{LV}|} . \] (39)
The rest of the analysis goes as before and yields
\[ \sigma_{\gamma Z \to Z e^+ e^-} = \frac{4Z^2 \alpha^3}{3k|\omega_{LV}|} \left[ 2 \log \frac{1}{\alpha Z^{1/3}} + \frac{1}{2} \log \frac{k|\omega_{LV}|}{m^2} \right] \log \frac{k|\omega_{LV}|}{m^2} . \] (40)
It remains to check the assumption (35). Comparing (36) with (33), (39) we find that it is equivalent to the requirement \( |\omega_{LV}| \ll m \) (in the case of no screening), or \( \alpha Z^{1/3} \ll 1 \) (with
screening). These conditions are satisfied for real nuclei and phenomenologically interesting values of the LV parameters.

The expressions (38), (40) must be compared to the standard result [25]

\[
\sigma_{\gamma Z \rightarrow Z e^+ e^-}^{LI} = \frac{28Z^2 \alpha^3}{9m^2} \times \begin{cases}
\log \frac{24}{m} - \frac{109}{42} & \text{no screening} \\
\log \frac{183}{Z^{7/3}} - \frac{1}{42} & \text{with screening}
\end{cases}
\]

We see that in the regime (37) LV strongly suppresses the cross section of pair production on nuclei. Besides, the LV cross section is dominated, according to (34), by the configurations when one of the produced fermions carries most of the energy (\(|x| \approx 1\)). This is in contrast to the standard QED where the energy distribution of the pair is smooth over the whole range \(-1 \leq x \leq 1\). Physical consequences of the above effects for registration of UHECR photons will be reported elsewhere [26].

4 Discussion

We have systematically derived the Feynman rules for a model of LV QED and have applied them to calculate the rates of several astrophysically relevant processes. Our analysis demonstrates that to find the precise result one must take into account both kinematical and dynamical aspects of LV whose effects on the rates are of the same order. The first — kinematical — class of effects amounts to the change of the phase-space integrals due to modified dispersion relations of the particles. While the dynamical effects include the modifications of the matrix elements due to the changes in the particles’ wavefunctions and in the interactions vertices. It is worth stressing that the gauge invariance relates the structure of the interaction vertices to the dispersion relations. Thus the modification of the vertices is unavoidable in gauge theories with LV.

The Feynman rules formulated in this paper can be straightforwardly applied to calculation of other cross sections in QED with LV. An interesting process from the astrophysical viewpoint is the photon splitting \(\gamma \rightarrow 3\gamma\) that becomes kinematically allowed whenever the parameter \(\xi\) in the photon dispersion relation (2) is positive. However, technically the calculation of the corresponding rate appears very challenging. Indeed, the process goes via the fermion loop and apart from the standard box diagram will involve the graphs with the insertions of the multi-photon vertices (9), (10). This drastically increases the number of topologically distinct contributions. The issues related to the treatment of the divergences appearing in the loop integrals further complicate the task.
A technically more promising direction is extension of the approach put forward in this paper to include the electroweak and hadronic processes. Of particular interest is an accurate analysis of the LV effects on the rates of various processes involving cosmogenic neutrinos.

As the concluding remark we note that the results of this paper provide justification of a heuristic method often used in the literature to make order-of-magnitude estimates of LV effects [27]. The reasoning goes as follows. One observes from the dispersion relations (2), (3) that LV introduces effective momentum-dependent masses for the photon and electron,

\[ m_\gamma^2(k) \equiv E_\gamma^2 - k^2 = \frac{\xi k^4}{M^2}, \]
\[ m_e^2(p) \equiv E_e^2 - p^2 = m^2 + 2\kappa p^2 + \frac{2\kappa p^4}{M^2}. \]

These masses set the scale of the characteristic energy-momentum transfer in various processes and can be used to estimate the corresponding cross sections on dimensional grounds. For example, consider the photon decay. Treating the photon as massive, let us perform a boost into its rest frame \(^8\). In that frame the energy released in the photon decay is of the order \(m_\gamma\), which gives for the width in this frame

\[ \Gamma_{\gamma \rightarrow e^+ e^-}^{\text{rest}} \sim \alpha m_\gamma. \]

To obtain the width in the original frame, we have to perform the reverse Lorentz transformation. This introduces a time-dilation factor \(m_\gamma/k\). In this way one obtains

\[ \Gamma_{\gamma \rightarrow e^+ e^-} \sim \alpha m_\gamma^2(k)/k. \]

The exact formula (16) is more complicated and depends also on \(m_e^2\) in a non-trivial way. However, for \(m_e \sim m_\gamma\) the simple derivation outlined above gives the correct order-of-magnitude estimate. Similarly, for pair production in the Coulomb field the momentum transfer between the nucleus and the photon is of order \(2m_e(k)\). This gives on dimensional grounds,

\[ \sigma_{\gamma Z \rightarrow Z e^+ e^-} \sim \frac{Z^2 \alpha^3}{m_e^2(k)}, \]

where we have included the appropriate powers of the fine-structure constant and the nucleus’ charge. For \(m_\gamma^2 \lesssim m_e^2\) this estimate coincides with the exact expressions (38), (40) up to the logarithmic factors.

\(^8\)Of course, this reasoning applies only to the case \(m_\gamma^2 > 0\)
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