On blocks with trivial source simple modules

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1. Introduction

1.1. In [3] Danz and Külshammer, investigating the simple modules for the large Mathieu groups, have found two blocks with noncyclic defect groups of order 9 where all the simple modules have trivial sources and whose source algebras are isomorphic to the source algebras of the corresponding blocks of their inertial subgroups [3, Theorems 4.3 and 4.4].

1.2. In their Introduction they note that, in general, any simple module with a trivial source determines an Alperin’s weight [1] — for instance, this follows from [8, Proposition 1.6] — and therefore, in a block with Abelian defect groups and all the simple modules with trivial sources, Alperin’s conjecture in [1] forces a canonical bijection between the sets of isomorphism classes of simple modules of the block and of the corresponding block of its inertial subgroup. From this remark, they raise the question whether, behind this bijection, it should be a true Morita equivalence between both blocks.

1.3. Recently, Zhou proved that, in a suitable inductive context, the answer is in the affirmative [18, Theorem B]; our purpose here is to prove the same fact without any hypothesis on the defect group. In order to explicit our result we need some notation; let $p$ be a prime number, $k$ an algebraically closed field of characteristic $p$, $G$ a finite group, $b$ a primitive idempotent of the center $Z(kG)$ of the group algebra of $G$ — for short, a block of $G$ — and $P$, a defect pointed group of $b$; that is to say, $P$ is a defect group of this block in Brauer’s terms and $\gamma$ is a conjugacy class of primitive idempotents $i$ in $(kGb)^P$ such that $\mathrm{Br}_P(i) \neq 0$; here, $\mathrm{Br}_P$ denotes the usual Brauer homomorphism.

$$\mathrm{Br}_P : (kG)^P \longrightarrow (kG)(P) = (kG)^P / \sum_Q (kG)_Q^P \cong kC_G(P)$$

1.3.1

† As a matter of fact, from [12, Corollary 3.6] one easily may find infinitely many examples of such blocks.
where $Q$ runs over the set of proper subgroups of $P$. Recall that the $P$-interior algebra $(kG)_\gamma = i(kG)i$ is called a source algebra of $b$ and that its underlying $k$-algebra is Morita equivalent to $kGb$ [8, Definition 3.2 and Corollary 3.5].

1.4. If $G'$ is a second finite group and $b'$ a block of $G'$ admitting the same defect group $P$, it follows from [13, Corollary 7.4 and Remark 7.5] that the source algebras of $b$ and $b'$ are isomorphic — as $P$-interior algebras — if and only if the categories of finitely generated $kGb$- and $kG'b'$-modules are equivalent to each other via a $kGb \otimes_k kG'b'$-module admitting a $P \times P$-stable basis, a fact firstly proved by Leonard Scott [17, Lemma]; in this case, we simply say that the blocks $b$ and $b'$ are identical. More generally, we say that the blocks $b$ and $b'$ are stably identical if the categories of finitely generated $kGb$- and $kG'b'$-modules are stably equivalent to each other — namely, equivalent to each other up to projective modules — throughout a $kGb \otimes_k kG'b'$-module admitting a $P \times P$-stable basis.

1.5. Set $N = N_G(P_\gamma)$ — often called the inertial subgroup of $b$ — and denote by $e$ the block of $C_G(P)$ determined by the local point $\gamma$ (cf. 1.3.1). Recall that $e$ is also a block of $N$ and that $kC_G(P)e$ is a simple $k$-algebra, where we set $C_G(P) = C_G(P)/Z(P)$ and denote by $\bar{e}$ the image of $e$ in $kC_G(P)$; then, the action of $N$ on the simple $k$-algebra $kC_G(P)e$ determines a central $k^*$-extension $\bar{E}$ of $E = N/P \cdot C_G(P)$ — often called the inertial quotient of $b$. Setting $\hat{L} = P \rtimes \bar{E}^\gamma$ for a lifting of the canonical homomorphism $E \to \text{Out}(P)$ to $\text{Aut}(P)$, it follows from [11, Proposition 14.6] that the corresponding twisted group algebra $k_\gamma \hat{L}$ is isomorphic to a source algebra of the block $e$ of $N$.

1.6. Recall that a Brauer $(b,G)$-pair $(Q,f)$ is formed by a $p$-subgroup $Q$ of $G$ such that $\text{Br}_Q(b) \neq 0$ and by a block $f$ of $C_G(Q)$ fulfilling $\text{Br}_Q(b)f = f$ [2, Definition 1.6]; note that $f$ is also a block for any subgroup $H$ of $N_G(Q,f)$ containing $C_G(Q)$. Thus, $(P,e)$ is a Brauer $(b,G)$-pair and, as a matter of fact, there is $x \in G$ such that [2, Theorem 1.14]

$$(Q,f) \subset (P,e)^x$$ 1.6.1.

Then, the Frobenius category $\mathcal{F}_{(b,G)}$ of $b$ [16, 3.1] is the category where the objects are the Brauer $(b,G)$-pairs $(Q,f)$ and the morphisms are the homomorphisms between the corresponding $p$-groups induced by the inclusion between Brauer $(b,G)$-pairs and the $G$-conjugation.

1.7. For short, let us say that the block $b$ is inertially controlled whenever the Frobenius categories $\mathcal{F}_{(b,G)}$ and $\mathcal{F}_{\hat{L}}$ are equivalent to each other — note that the unity element is the unique block of $\hat{L}$ and we omit to mention it;
moreover, since $k_\ast \hat{L}$ is isomorphic to a source algebra of the block $e$ of $N$, the Frobenius categories $\mathcal{F}_{e,N}$ and $\mathcal{F}_L$ are always equivalent to each other, so that $e$ is always inertially controlled. Similarly, let us say that $b$ is a block of $G$ with trivial simple modules if all the simple $kGb$-modules have trivial sources.

**Theorem 1.8.** With the notation above, the source algebra $(kG)_\gamma$ of the block $b$ of $G$ is isomorphic to $k_\ast \hat{L}$ if and only if the block $b$ of $G$ is inertially controlled and, for any Brauer $(b,G)$-pair $(Q,f)$ contained in $(P,e)$, $f$ is a block of $C_G(Q)\cdot N_P(Q)$ with trivial source simple modules.

1.9. The main tools in proving this result are the Linckelmann’s Equivalence Criterion on stable equivalences [7, Proposition 2.5], the strict semi-covering homomorphisms that we recall in §3 below, and the general criterion on stable equivalences in [13, Theorem 6.9], which in our context is summarized by the following result.

**Theorem 1.10.** With the notation above, the blocks $b$ of $G$ and $e$ of $N$ are stably identical if and only if, for any nontrivial Brauer $(b,G)$-pair $(Q,f)$ contained in $(P,e)$, the block $f$ of $C_G(Q)$ admits $C_P(Q)$ as a defect $p$-subgroup and a source algebra isomorphic to $k_\ast (C^\gamma_L(Q))$.

1.11. Note that $C^\gamma_E(Q)$ acts faithfully on $C_P(Q)$ since any $(p^\prime)$-subgroup of $C^\gamma_E(Q)$ acting trivially on $C_P(Q)$ still acts trivially on $P$ [5, Ch. 5, Theorem 3.4], and that we actually have

$$C^\gamma_L(Q) \cong C_P(Q) \rtimes C^\gamma_E(Q)$$

Moreover, if the defect group $P$ is Abelian then, for any Brauer $(b,G)$-pair $(Q,f)$ contained in $(P,e)$, $P$ is clearly a defect group of the block $f$ of $C_G(Q)$. Finally, although we only work over $k$, Lemma 7.8 in [10] allows us to lift all the isomorphisms between block source algebras over $k$ above to the corresponding block source algebras over a complete discrete valuation ring $\mathcal{O}$ of characteristic zero having the residue field $k$.

2. Notation and quoted results

2.1. Let $A$ be a finitely dimensional $k$-algebra; we denote by $1_A$ the unity element of $A$ and by $A^\ast$ the multiplicative group of $A$. An algebra homomorphism $f$ from $A$ to another finitely dimensional $k$-algebra $A'$ is not necessarily unitary and we say that $f$ is an embedding whenever

$$\text{Ker}(f) = \{0\} \quad \text{and} \quad \text{Im}(f) = f(1_A)A'f(1_A)$$
Following Green, a $G$-algebra is a finitely dimensional $k$-algebra $A$ endowed with a $G$-action; recall that, for any subgroup $H$ of $G$, a point $\alpha$ of $H$ on $A$ is an $(A^H)^*$-conjugacy class of primitive idempotents of $A^H$ and the pair $H_\alpha$ is called a pointed group on $A$ [8, 1.1]; we denote by $A(H_\alpha)$ the simple quotient of $A^H$ determined by $\alpha$. A second pointed group $K_\beta$ on $A$ is contained in $H_\alpha$ if $K \subset H$ and, for any $i \in \alpha$, there is $j \in \beta$ such that [8, 1.1]
\[ ij = j = ji \]

2.2. Following Broué, for any $p$-subgroup $P$ of $G$ we consider the Brauer quotient and the Brauer homomorphism
\[ \text{Br}_P^A : A^P \rightarrow A(P) = A^P / \sum_Q A^P_Q \]
where $Q$ runs over the set of proper subgroups of $P$ and $A^P_Q$ is the ideal formed by the sums $\sum_u a^u$ where $a$ runs over $A^Q$ and $u \in P$ over a set of representatives for $P/Q$; we call local any point $\gamma$ of $P$ on $A$ not contained in $\text{Ker}(\text{Br}_P^A)$ [8, 1.1]. Let us say that $A$ is a $p$-permutation $G$-algebra if a Sylow $p$-subgroup of $G$ stabilizes a basis of $A$; in this case, recall that if $P$ is a $p$-subgroup of $G$ and $Q$ a normal subgroup of $P$ then the corresponding Brauer homomorphisms induce a $k$-algebra isomorphism [2, Proposition 1.5]
\[ (A(Q))(P/Q) \cong A(Q) \]
Obviously, the group algebra $A = kG$ is a $p$-permutation $G$-algebra and the composition of the inclusion $kC_G(Q) \subset A^Q$ with $\text{Br}_P^A$ is an isomorphism which allows us to identify $kC_G(Q)$ with $A(Q)$; then any local point $\delta$ of $Q$ on $kG$ determines a block $b_\delta$ of $kC_G(Q)$ such that $b_\delta \text{Br}_Q^{kG}(\delta) = \text{Br}_Q^{kG}(\delta)$.

2.3. We are specially interested in the $G$-algebras $A$ endowed with a group homomorphism $\rho : G \rightarrow A^*$ inducing the action of $G$ on $A$ — called $G$-interior algebras. In this case, for any pointed group $H_\alpha$ on $A$ and any $i \in \alpha$, the subalgebra $A_\alpha = iA$ has a structure of $H$-interior algebra mapping $y \in H$ on $\rho(y)i = i\rho(y)$; moreover, setting $x:a-y = \rho(x)a\rho(y)$ for any $a \in A$ and any $x, y \in G$, a $G$-interior algebra homomorphism from $A$ to another $G$-interior algebra $A'$ is a $G$-algebra homomorphism $f : A \rightarrow A'$ fulfilling
\[ f(x \cdot a - y) = x \cdot f(a) - y \]
We also consider the mixed situation of an $H$-interior $G$-algebra $B$ where $H$ is a subgroup of $G$ and $B$ is a $G$-algebra endowed with a compatible $H$-interior algebra structure, in such a way that the $kG$-module $B \otimes_{kH} kG$ endowed with the product
\[ (a \otimes x), (b \otimes y) = abx^{-1} \otimes xy \]
for any $a, b \in B$ and any $x, y \in G$, and with the group homomorphism mapping $x \in G$ on $1_B \otimes x$ becomes a $G$-interior algebra — simply noted $B \otimes_H G$. For instance, for any $p$-subgroup $P$ of $G$, $A(P)$ is a $C_G(P)$-interior $N_G(P)$-algebra.
2.4. In particular, if $H_\alpha$ and $K_\beta$ are two pointed groups on $A$, we say that an injective group homomorphism $\varphi: K \to H$ is an $A$-fusion from $K_\beta$ to $H_\alpha$ whenever there is a $K$-interior algebra embedding

$$f_\varphi: A_\beta \longrightarrow \text{Res}^H_K(A_\alpha)$$

such that the inclusion $A_\beta \subset A$ and the composition of $f_\varphi$ with the inclusion $A_\alpha \subset A$ are $A^*$-conjugate; we denote by $F_A(K_\beta, H_\alpha)$ the set of $H$-conjugacy classes of $A$-fusions from $K_\beta$ to $H_\alpha$ and we write $F_A(H_\alpha)$ instead of $F_A(H_\alpha, H_\alpha)$. If $A_\alpha = i_A i$ for $i \in \alpha$, it follows from [9, Corollary 2.13] that we have a group homomorphism

$$F_A(H_\alpha) \longrightarrow N_{A^*_\alpha}(H \cdot i)/H \cdot (A^H_\alpha)^*$$

such that the inclusion $A_\beta \subset A$ and the composition of $f_\varphi$ with the inclusion $A_\alpha \subset A$ are $A^*$-conjugate; we denote by $F_A(H_\alpha)$ the set of $H$-conjugacy classes of $A$-fusions from $K_\beta$ to $H_\alpha$ and we write $F_A(H_\alpha)$ instead of $F_A(H_\alpha, H_\alpha)$. If $A_\alpha = i_A i$ for $i \in \alpha$, it follows from [9, Corollary 2.13] that we have a group homomorphism

$$F_A(H_\alpha) \longrightarrow N_{A^*_\alpha}(H \cdot i)/H \cdot (A^H_\alpha)^*$$

2.5. Let $b$ be a block of $G$; then $\alpha = \{b\}$ is a point of $G$ on $kG$ and we let $P_\gamma$ be a local pointed group contained in $G_\alpha$ which is maximal with respect to the inclusion of pointed groups; namely $P_\gamma$ is a defect pointed group of $b$. Note that, for any $p$-subgroup $Q$ of $G$ and any subgroup $H$ of $N_G(Q)$ containing $Q$, we have

$$\text{Br}_Q((kG)^H) = (kC_G(Q))^H$$

thus, we have an injection from the set of points of $H$ on $kC_G(Q)$ to the set of points of $H$ on $kG$ such that the corresponding points $\beta$ and $\beta$ fulfill $\text{Br}_Q^G(\beta) = \text{Br}_Q^{kC_G(Q)}(\beta')$; moreover, this injection preserves the localness and the inclusion of pointed groups [16, 1.19]. In particular, if $P$ is Abelian and $Q_\delta$ is a local pointed group on $kG$ contained in $P_\gamma$, a point $\mu$ of $C_G(Q)$ on $kG$ fulfilling

$$Q_\delta \subset P_\gamma \subset C_G(Q)_\mu$$

is the unique point determined by the block $b_\delta$ of $C_G(Q)$ and therefore $P$ is a defect group of this block (cf. 1.8).

2.6. Set $e = b$, and $N = N_G(P_\gamma)$; thus, $e$ is a block of $N$, it determines a point $\nu$ of $N$ on $kG$ (cf. 2.5) and $P$ is a defect group of this block; moreover, we have (cf. 1.3.1)

$$(kN)(P) \cong kC_N(P) = kC_G(P) \cong (kG)(P)$$

there is a local point $\gamma$ of $P$ on $kN \subset kG$ such that $\text{Br}_P(\gamma) = \text{Br}_P(\gamma)$ and it follows from [4, Proposition 4.10] that, for any $i \in \gamma$ and any $\ell \in \nu$, the idempotent $i\ell$ belongs to $\gamma$ and that the multiplication by $\ell$ defines a unitary $P$-interior algebra homomorphism (cf. 1.5)

$$k, \hat{\gamma}\hat{L} \cong (kN)_\gamma \longrightarrow (kG)_\gamma$$

which is actually a direct injection of $k(P \times P)$-modules.
2.7. For any pair of local pointed groups $Q_δ$ and $R_ε$ on $kG$, we denote by $E_G(R_ε, Q_δ)$ the set of $Q$-conjugacy classes of group homomorphisms

$$\varphi : R \to Q$$

induced the conjugation by some $x \in G$ fulfilling $R_ε \subset (Q_δ)^x$, and write $E_G(Q_δ)$ instead of $E_G(Q_δ, Q_δ)$; it follows from [9, Theorem 3.1] that

$$E_G(R_ε, Q_δ) = F_{kG}(R_ε, Q_δ)$$

2.7.1

and if $P_γ$ contains $Q_δ$ and $R_ε$ then they can be considered as local pointed groups on $(kG)_γ$ and it follows from [9, Proposition 2.14] that

$$E_G(R_ε, Q_δ) = F_{kG}(R_ε, Q_δ) = F_{(kG)_γ}(R_ε, Q_δ)$$

2.7.2.

In particular, it is clear that $N_G(Q_δ)/(Q \cdot C_G(Q)) \cong E_G(Q_δ)$ and the action of $N_G(Q_δ)$ on the simple $k$-algebra $(kG)(Q_δ)$ (cf. 2.1) determines a central $k^*$-extension $\bar{E}_G(Q_δ)$ of $E_G(Q_δ)$.

2.8. Recall that a Brauer $(b, G)$-pair $(Q, f)$ is called selfcentralizing if, setting $C_G(Q) = C_G(Q)/Z(Q)$ and denoting by $f$ the image of $f$ in $kC_G(Q)$, the $k$-algebra $kC_G(Q)f$ is simple [14, 1.6], so that $kC_G(Q)f \cong (kG)(Q_δ)$ for a local point $δ$ of $Q$ on $kG$ clearly determined by $f$; we also say that $Q_δ$ is a selfcentralizing pointed group on $kG$; thus we have a bijection, which preserves inclusion and $G$-conjugacy, between the sets of selfcentralizing pointed groups on $kGb$ and of selfcentralizing Brauer $(b, G)$-pairs. Moreover, according to [14, Theorem A.9], an essential pointed group on $kG$ is a selfcentralizing pointed group $Q_δ$ on $kG$ fulfilling the following condition

2.8.1 $E_G(Q_δ)$ admits a proper subgroup $M$ such that $p$ divides $|M|$ and does not divide $|M \cap M^σ|$ for any $σ \in E_G(Q_δ) - M$.

Then, from [14, Corollary A.12] and [16, Corollary 5.14], it is not difficult to prove that the block $b$ of $G$ is inertially controlled (cf. 1.7) if and only if there are no essential pointed groups on $kGb$; thus, if the defect group $P$ is Abelian the block $b$ of $G$ is inertially controlled.

Lemma 2.9. With the notation above, the block $b$ of $G$ is inertially controlled if and only if, for any nontrivial Brauer $(b, G)$-pair $(Q, f)$ contained in $(P, e)$, the block $f$ of $C_G(Q)$ admits $C_P(Q)$ as a defect group and it is inertially controlled.

Proof: Firstly assume that $b$ is inertially controlled; let $(Q, f)$ be a Brauer $(b, G)$-pair contained in $(P, e)$ and choose a maximal Brauer $(f, Q \cdot C_G(Q))$-pair $(R, g)$; since $(Q, f)$ is also a Brauer $(f, Q \cdot C_G(Q))$-pair, $(R, g)$ necessarily contains $(Q, f)$ (cf. 1.6.1) and therefore it is also a Brauer $(b, G)$-pair; hence, there is $x \in G$ such that (cf. 1.6.1)

$$(Q, f)^x \subset (R, g)^x \subset (P, e)$$

2.9.1
and therefore we get $x = zn$ for suitable $z \in C_G(Q)$ and $n \in N$; so that the maximal Brauer $(f, Q; C_G(Q))$-pair $(R, g)^z$ is contained in $(P, e)$.

Moreover, if $(T, h)$ is a Brauer $(f, C_G(Q))$-pair, it is clear that $(Q; T, h)$ is a Brauer $(b, G)$-pair; conversely, by the argument above, $(C_P(Q), g^x)$ is a maximal Brauer $(f, C_G(Q))$-pair; then, if $(C_P(Q), g^x)$ contains $(T, h)$ and $(T, h)^z$ with $z \in C_G(Q)$, it is easily checked that $(P, e)$ contains $(Q; T, h)$ and $(Q; T, h)^z$ and therefore we still get $z = wn$ for suitable $w \in C_G(Q; T)$ and $n \in N$, so that $n$ actually belongs to $C_N(Q)$; consequently, since we have $N/C_G(P) \cong L/C_L(P)$, the block $f$ of $C_G(Q)$ is inertially controlled.

Conversely, arguing by contradiction, assume that $Q_\delta$ is an essential pointed group contained in $P_\gamma$. According to [10, Lemma 3.10], we may assume that the image of $N_P(Q)$ is a Sylow $p$-subgroup of $E_G(Q_\delta)$ and, since a proper subgroup $M$ of $E_G(Q_\delta)$ fulfilling condition 2.8.1 above contains a Sylow $p$-subgroup of $E_G(Q_\delta)$, we still may assume that $M$ contains the image of $N_P(Q)$. Moreover, it follows again from [10, Lemma 3.10] that there is a local pointed group $R_e$ containing and normalizing $Q_\delta$ such that its image in $E_G(Q_\delta)$ is not contained in $M$; then, $R$ centralizes some nontrivial subgroup $Z$ of $Z(Q)$ and, denoting by $f$ the unique block of $H = C_G(Z)$ such that $(P, e)$ contains $(Z, f)$, it follows from our hypothesis that $H \cap P$ is a defect group of this block.

Consequently, denoting by $h$ the block of $C_G(H \cap P)$ such that $(P, e)$ contains $(H \cap P, h)$, this pair is a maximal Brauer $(f, H)$-pair; moreover, $H$ contains $R$ and $C_G(Q)$, and in particular we have

$$ (kH)(Q) \cong (kG)(Q) $$

so that $Br_Q(\delta)$ determines a local point $\delta$ of $Q$ on $kH$ fulfilling

$$ E_H(Q_\delta) \subset E_G(Q_\delta) $$

then, applying again [10, Lemma 3.10], we may assume that the image of $N_H; P(Q)$ in the intersection $E_H(Q_\delta) \cap M$ is a Sylow $p$-subgroup of $E_H(Q_\delta)$, whereas this intersection does not contain the image of $R$; hence, $Q_\delta$ is an essential pointed group on $kHf$, which contradicts our hypothesis. We are done.

3. Strict semicovering homomorphism

3.1. Let $P$ be a finite $p$-group, $B$ and $\hat{B}$ two $P$-algebras and $g: B \to \hat{B}$ a unitary $P$-algebra homomorphism; we say that $g$ is a strict semicovering if, for any subgroup $Q$ of $P$, we have $\text{Ker}(g)^Q \subset J(B^Q)$ and the image $g(j)$ of a primitive idempotent $j$ of $B^Q$ is still primitive in $B^Q$ [6, 3.10]; namely if $g$ induces a homomorphism from the maximal semisimple quotient of $B^Q$ to the maximal semisimple quotient of $\hat{B}^Q$, mapping primitive idempotents on primitive idempotents.
3.2. In other words, \( g \) is a strict semicovering if and only if, for any subgroup \( Q \) of \( P \), it induces a surjective map from the set of points of \( Q \) on \( B \) to the set of points of \( Q \) on \( \hat{B} \) and, for any pair of mutually corresponding such points \( \delta \) and \( \hat{\delta} \), it induces a \( k \)-algebra embedding \([6, 3.10]\)

\[
g(Q_\delta) : B(Q_\delta) \rightarrow \hat{B}(Q_\delta)
\]

3.3. Explicitly, if \( g \) is a strict semicovering then, for any pointed group \( Q_\delta \) on \( B \), there is a unique point \( \hat{\delta} \) of \( Q \) on \( \hat{B} \) fulfilling \( g(\delta) \subset \hat{\delta} \); moreover, this correspondence preserves inclusion and localness \([6, \text{Proposition } 3.15]\). The composition of strict semicoverings is clearly a strict semicovering but, more precisely, the strictness provides a converse \([6, \text{Proposition } 3.6]\).

**Proposition 3.4.** With the notation above, let \( \hat{\gamma} : \hat{B} \rightarrow \hat{\hat{B}} \) a second unitary \( P \)-algebra homomorphism. Then, \( \hat{\gamma} \circ g \) is a strict semicovering if and only if \( \hat{\gamma} \) and \( g \) are so.

3.5. The fact for a \( P \)-algebra homomorphism of being a strict semicovering is essentially of “local” nature as it shows the following result \([6, \text{Theorem } 3.16]\).

**Theorem 3.6.** With the notation above, the unitary \( P \)-algebra homomorphism \( g \) is a strict semicovering if and only if, for any \( p \)-subgroup \( Q \) of \( P \), the \( \{1\} \)-algebra homomorphism

\[
g(Q) : B(Q) \rightarrow \hat{B}(Q)
\]

induced by \( g \) is a strict semicovering.

3.7. Here, we may restrict ourselves to consider the following situation. Let \( G \) be a finite group, \( H \) a normal subgroup of \( G \) such that \( G/H \) is a \( p \)-group, \( P \) a \( p \)-subgroup of \( G \) and \( Z \) a subgroup of \( Q = H \cap P \) normal in \( G \) and central in \( H \); set \( \bar{G} = G/Z \) and \( \bar{P} = P/Z \).

**Proposition 3.8.** With the notation above, the canonical \( P \)-algebra homomorphism \( kH \rightarrow k\bar{G} \) is a semicovering.

**Proof:** For any subgroup \( \bar{Q} = Q/Z \) of \( \bar{P} \), we have (cf. 1.3.1)

\[
(kH)(\bar{Q}) \cong kC_H(Q) \quad \text{and} \quad (k\bar{G})(\bar{Q}) \cong kC_{\bar{G}}(\bar{Q})
\]

thus, a \( p' \)-subgroup \( K \) of the converse image of \( C_{\bar{G}}(\bar{Q}) \) centralizes \( Q \) \([5 \text{ Ch. 5, Theorem } 3.2]\) and therefore it is contained in \( C_H(Q) \); that is to say, setting \( \bar{C}_H(Q) = C_H(Q)/Z \), the quotient \( C_{\bar{G}}(\bar{Q})/\bar{C}_H(\bar{Q}) \) is a \( p \)-group.
Then, it follows from Lemma 3.9 below that any simple $kC_G(\bar{Q})$-module $M$ has the form

$$M \cong \text{Ind}_{kC_G(\bar{Q})N}^{kC_G(\bar{Q})}(\hat{N})$$

where $N$ is a simple $kCH(Q)$-module, $kC_G(\bar{Q})N$ the stabilizer in $kC_G(\bar{Q})$ of the isomorphism class of $N$ and $\hat{N}$ the extended $kC_G(\bar{Q})$-module. Moreover, any simple $kCH(Q)$-module is also a simple $kC_H(Q)$-module and it appears in some simple $kC_G(\bar{Q})$-module. All this amounts to saying that the canonical $\{1\}$-algebra homomorphism

$$kCH(Q) \rightarrow kC\bar{G}(\bar{Q})$$

induces a homomorphism between the corresponding semisimple quotients preserving primitivity and then it suffices to apply Theorem 3.6.

**Lemma 3.9.** Let $X$ be a finite group and $Y$ a normal subgroup of $X$ such that $X/Y$ is a $p$-group. Then, any simple $kY$-module $N$ can be extended to the stabilizer $X_N$ in $X$ of the isomorphism class of $N$ and $\hat{N}$ the extended $kX_N$-module, $\text{Ind}_{X_N}^{X}(\hat{N})$ is a simple $kX$-module. Moreover, all the simple $kX$-modules have this form.

**Proof:** Straightforward.

**Corollary 3.10.** With the same notation, let $\alpha = \{b\}$ be a point of $G$ on $kH$ and assume that $P_\gamma$ is a defect pointed group of $G_\alpha$; denote by $\bar{b}$ and $\bar{\gamma}$ the respective images in $k\bar{G}$ of $b$ and $\gamma$. Then, $b$ and $\bar{b}$ are respective blocks of $G$ and $\bar{G}$, $\gamma$ and $\bar{\gamma}$ are respectively contained in local points $\hat{\gamma}$ and $\bar{\hat{\gamma}}$ of $P$ and $\bar{P}$ on $kG$ and $k\bar{G}$, and moreover $P_\gamma$ and $\bar{P}_{\bar{\gamma}}$ are respective defect pointed groups of these blocks. In particular, setting $Q = H \cap P$, $\bar{H} = H/Z$ and $\bar{Q} = Q/Z$, the respective $P$- and $\bar{P}$-interior algebras

$$(kH)_\gamma \otimes_Q P = \bigoplus_u (kH)_\gamma \cdot u \quad \text{and} \quad (k\bar{H})_{\bar{\gamma}} \otimes_{\bar{Q}} \bar{P} = \bigoplus_{\bar{u}} (k\bar{H})_{\bar{\gamma}} \cdot \bar{u}$$

where $u \in P$ runs over a set of representatives for $P/Q$ and $\bar{u}$ is the image in $\bar{P}$ of $u$, are respective source algebras of these blocks.

**Proof:** Since any block of $G$ is a $k$-linear combination of $p'$-elements of $G$, $kH$ contains all the blocks of $G$ and therefore $b$ is primitive in $Z(kG)$; moreover, it is easily checked that $(kH)_G$ maps surjectively onto $(k\bar{H})_{\bar{G}}$ and therefore $\bar{\alpha} = \{\bar{b}\}$ is also a point of $\bar{G}$ on $k\bar{H}$, so that $\bar{b}$ is a block of $\bar{G}$.

Moreover, it follows from Propositions 3.4 and 3.8 that the canonical $\bar{P}$-algebra homomorphisms

$$kH \rightarrow kG \quad \text{and} \quad kH \rightarrow k\bar{G}$$
are strict semicovering; hence, $\gamma$ is contained in a local point $\hat{\gamma}$ of $P$ on $kG$ and $\bar{\gamma}$ in a local point $\hat{\bar{\gamma}}$ of $P$ on $k\bar{G}$; we claim that $P_{\hat{\gamma}}$ and $\bar{P}_{\hat{\bar{\gamma}}}$ are maximal local pointed groups on $kG$ and $k\bar{G}$ respectively.

Indeed, since the canonical homomorphism $kH \rightarrow kG$ is a semicovering, a local pointed group $P'_{\hat{\gamma}'}$ on $kG$ containing $P_{\hat{\gamma}}$ comes from a local pointed group $P'_{\gamma'}$ on $kH$ and it is easily checked that $P'_{\gamma'} \subset (P_{\gamma})^x$ for a suitable $x \in G$, which forces $P'_{\gamma'} = P_{\gamma}$; since $\bar{\alpha}$ is a point of $\bar{G}$ on $k\bar{H}$, the same argument proves that $\bar{P}_{\hat{\bar{\gamma}}}$ is a maximal local pointed group on $k\bar{G}$.

The proof of the last statement is straightforward. We are done.

4. Stable embeddings: the proof of Theorem 1.10

4.1. Let $G$ be a finite group and $A$ a $G$-interior algebra; we say that a point $\beta$ of $H$ on $A$ is projective if it is contained in $A_1^H$ or, equivalently, if it has a trivial defect group. Let $\hat{A}$ be a second $G$-interior algebra and $f: \hat{A} \rightarrow A$ a $G$-interior algebra homomorphism; following [13, 6.4], we say that $f$ is a stable embedding if $\ker(f)$ and $f(1_{\hat{A}})A f(1_{\hat{A}})/f(\hat{A})$ are projective $k(G \times G)$-modules or, equivalently, if the classe of the $k(G \times G)$-module homomorphism

$$f : \hat{A} \rightarrow f(1_{\hat{A}})A f(1_{\hat{A}})$$

in the stable category of $k(G \times G)$-modules is an isomorphism.

4.2. In this case, if $f$ is unitary, the exact sequence of $k(G \times G)$-modules

$$0 \rightarrow \ker(f) \rightarrow \hat{A} \rightarrow A \rightarrow A/f(\hat{A}) \rightarrow 0$$

is split [13, 6.4.1] and therefore, for any subgroup $H$ of $G$, $f$ induces a $C_G(H)$-interior $N_G(H)$-algebra isomorphism

$$\hat{A}_1^H / \hat{A}_1^H \cong A^H / A_1^H$$

in particular, $f$ induces a bijection between the sets of nonprojective points of $H$ on $\hat{A}$ and on $A$ and, for any pair of corresponding nonprojective points $\hat{\beta}$ and $\beta$, we have $N_G(H_{\hat{\beta}}) = N_G(H_{\beta})$, $f$ induces a $C_G(H)$-interior $N_G(H_{\beta})$-algebra isomorphism [13, 4.6.2]

$$f(H_{\beta}) : \hat{A}(H_{\hat{\beta}}) \cong A(H_{\beta})$$

and this isomorphism determines a central $k^*$-extension isomorphism

$$\hat{f}(H_{\beta}) : \hat{N}_G(H_{\hat{\beta}}) \cong \hat{N}_G(H_{\beta})$$

Moreover, this correspondence preserves inclusion, localness and fusions.
4.3. We are ready to prove Theorem 1.10; thus, if $b$ is a block of $G$, $P$, $\gamma$ is a defect pointed group of $b$, we set $N = N_G(P, \gamma)$, $e$ is the corresponding block of $N$, $\nu$ is the point of $N$ on $kG$ determined by $e$, $\hat{\gamma}$ is the local point of $P$ on $kN$ fulfilling $\Br_P(\hat{\gamma}) = \Br_P(\gamma)$ and we denote by (cf. 2.6.2)

$$g : (kN)_{\hat{\gamma}} \longrightarrow (kG)_{\gamma}$$

4.3.1

the unitary $P$-interior algebra homomorphism determined as above by the multiplication by $\ell \in \nu$; note that the restriction throughout $g$ induces a functor from the category of $kGb$-modules to the category of $kNe$-modules which actually coincides with the functor determined by the $k(N \times G)$-module $\ell(kG)$. Firstly, we prove a stronger form of the converse part.

**Proposition 4.4.** With the notation above, assume that the blocks $b$ of $G$ and $e$ of $N$ are stably identical. Then, for any nontrivial Brauer $(b, G)$-pair $(Q, f)$ contained in $(P, e)$, $N_P(Q)$ is a defect group of the block $f$ of $C_G(Q)$-$N_P(Q)$ and a source algebra of this block is isomorphic to $k_4(C_L(Q) \cdot N_P(Q))$ via an isomorphism inducing a $C_P(Q)$-interior algebra isomorphism from a source algebra of the block $f$ of $C_G(Q)$ onto $k_4C_L(Q)$.

**Proof:** We can apply Theorem 6.9 and Corollary 7.4 in [13] to the Morita stable equivalences between $b$ and $e$, and between $e$ and $b$; in our present situation, $b$ and $e$ have the same defect group $P$ and, with the notation in [13], we may assume that $\tilde{P} = P$ and then $\tilde{S} = k$ is the trivial $P$-interior algebra and $\sigma = \sigma' = \text{id}_P$. Consequently, it follows from [13, 7.6.6] that the block $b$ of $G$ is inertially controled and, for any nontrivial subgroup $Q$ of $P$, from [13, 6.9.1] we get $C_P(Q)$-$N_P(Q)$-algebra embeddings

$$(kG)_{\gamma}(Q) \longrightarrow (kN)_{\hat{\gamma}}(Q) \quad \text{and} \quad (kN)_{\hat{\gamma}}(Q) \longrightarrow (kG)_{\gamma}(Q)$$

4.4.1

so that both are isomorphisms.

Now, we have $C_P(Q)$-$N_P(Q)$-algebra isomorphisms

$$(kG)_{\gamma}(Q) \cong (kN)_{\hat{\gamma}}(Q) \cong k_4C_L(Q)$$

4.4.2

and therefore the unity element is primitive in the $k$-algebra

$$(kG)_{\gamma}(Q)^C_{P(Q)} \cong (kN)_{\hat{\gamma}}(Q)^C_{P(Q)}$$

4.4.3

thus, denoting by $f$ the block of $C_G(Q)$ such that $(Q, f) \subset (P, e)$ [2, Theorem 1.8], it is quite clear that the $C_P(Q)$-interior algebra $(kG)_{\gamma}(Q)$ is a source algebra of this block and it is indeed isomorphic to $k_4C_L(Q)$.

Moreover, it follows from Corollary 3.10 above, applied to the groups $C_G(Q)$-$N_P(Q)$ and $C_G(Q)$, that $N_P(Q)$ is a defect group of the block $f$ of $C_G(Q)$-$N_P(Q)$ and that the $N_P(Q)$-interior algebra

$$(kG)_{\gamma}(Q) \otimes_{C_P(Q)} N_P(Q) = \bigoplus_{u} (kG)_{\gamma}(Q) \cdot u$$

4.4.4
where \( u \in N_P(Q) \) runs over a set of representatives for \( N_P(Q)/C_P(Q) \), is a source algebra of this block; thus, according to isomorphisms 4.4.2, this \( N_P(Q) \)-interior algebra is isomorphic to \( k_*(C_G(Q)_N(P)) \). We are done.

**Theorem 4.5.** With the notation above, for any nontrivial Brauer \((b, G)\)-pair \((Q, f)\) contained in \((P, e)\), assume that \( C_P(Q) \) is a defect group of the block \( f \) of \( C_G(Q) \) and that a source algebra of this block is isomorphic to \( k^*(C_G(Q)_N(P)) \). Then, \( g \) is a stable embedding.

**Proof:** Since \( g \) is a direct injection of \( k(P \times P) \)-modules (cf. 2.6), we have \( \text{Ker}(g) = \{0\} \) and the quotient

\[
M = (kG)_\gamma / g((kN)_\gamma) \quad \text{4.5.1}
\]

is a direct summand of \( (kG)_\gamma \) as \( k(P \times P) \)-modules; hence, since \( (kG)_\gamma \) is a permutation \( k(P \times P) \)-module, it suffices to prove that \( M(W) = \{0\} \) for any nontrivial subgroup \( W \) of \( P \times P \). Actually, we have \( (kG)(W) = \{0\} \) unless

\[
W = \Delta \varphi(Q) = \{(u, \varphi(u))\}_{u \in Q} \quad \text{4.5.2}
\]

for some subgroup \( Q \) of \( P \) and some group homomorphism \( \varphi : Q \to P \) induced by the conjugation by some \( x \in G \).

More precisely, choosing \( i \in \gamma \), the multiplication by \( x \) on the right determines a \( k \)-linear isomorphism

\[
(kG)_\gamma (\Delta \varphi(Q)) \cong (i(kG)ix)(Q) \quad \text{4.5.3}
\]

thus, denoting by \( f \) the block of \( C_G(Q) \) such that \( (P, e) \) contains \( (Q, f) \) or, equivalently, such that \( f\text{Br}_Q(i) \neq 0 \), if we have \( (kG)_\gamma (\Delta \varphi(Q)) \neq \{0\} \), we still have \( f\text{Br}_Q(i^2) \neq 0 \) or, equivalently, \( (P, e)^2 \) contains \( (Q, f) \) which amounts to saying that \( \varphi : Q \to P \) is an \( F_{(b,G)} \)-morphism (cf. 2.9). Hence, it suffices to prove that, for any nontrivial subgroup \( Q \) of \( P \) and any \( F_{(b,G)} \)-morphism \( \varphi : Q \to P \), we have \( M(\Delta \varphi(Q)) = \{0\} \); but, always since \( g \) is a direct injection of \( k(P \times P) \)-modules, \( g \) induces an injective homomorphism

\[
g(\Delta \varphi(Q)) : (kN)_\gamma (\Delta \varphi(Q)) \to (kG)_\gamma (\Delta \varphi(Q)) \quad \text{4.5.4}
\]

consequently, it suffices to prove that

\[
\dim((kN)_\gamma (\Delta \varphi(Q))) = \dim((kG)_\gamma (\Delta \varphi(Q))) \quad \text{4.5.5}
\]

and we argue by induction on \( |P:Q| \).
Since we have a $P$-interior algebra isomorphism $k_*\hat{L} \cong (kN)_\gamma$, we still have
\[(k_*\hat{L})(\Delta_P(Q)) \cong (kN)_\gamma(\Delta_P(Q)) \quad 4.5.6;\]
moreover, it is clear that $N_P(Q)$ centralizes a nontrivial subgroup $Z$ of $Z(Q)$ and then, according to our hypothesis, the $C_P(Z)$-interior algebra $k_*C_L(Z)$ is isomorphic to a source algebra of the block $h$ of $C_G(Z)$ such that $(P,e)$ contains $(Z,h)$; in particular, setting $H = C_G(Z)$, $(Q,f)$ is also a Brauer $(h,H)$-pair, we have $C_H(Q) = C_G(Q)$ and $N_P(Q)$ remains a defect group of the block $f$ of $C_H(Q)\cdot N_P(Q)$. Consequently, it easily follows from Proposition 4.4 above, applied to the block $h$ of $H$, that a source algebra of the block $f$ of $C_G(Q)\cdot N_P(Q)$ is isomorphic to $k_*(C_L(Q)\cdot N_P(Q))$.

At this point, we claim that in $(kG)_\gamma(Q)^{N_P(Q)}$ the unity element is primitive; since the point $\gamma$ is local, it follows from isomorphism 2.2.2 that there is a primitive idempotent $\bar{\ell}$ of $(kG)_\gamma(Q)^{N_P(Q)}$ determining a local point of $N_P(Q)$ on $(kG)_\gamma(Q)$; but, according to our induction hypothesis, for any subgroup $R$ of $N_P(Q)$ strictly containing $Q$ we may assume that that
\[(kN)_\gamma(R) \cong (kG)_\gamma(R) \quad (\text{cf. } 4.5.4) \quad \text{and, since } Br_{\bar{R}}^{(kG)_\gamma(Q)}(\bar{\ell}) \neq 0 \quad \text{where we set } \bar{R} = R/Q \quad (\text{cf. isomorphism 2.2.2}), \]
we necessarily have
\[\text{Br}_{\bar{R}}^{(kG)_\gamma(Q)}(1_{(kG)_\gamma(Q)} - \bar{\ell}) = 0 \quad 4.5.7;\]
thus, the idempotent $1_{(kG)_\gamma(Q)} - \bar{\ell}$ belongs to $[2, \text{Lemmas 1.11 and 1.12}]$
\[\bigcap_{R} \text{Ker}(\text{Br}_{\bar{R}}^{(kG)_\gamma(Q)}) = ((kG)_\gamma(Q))^{N_P(Q)} = \text{Br}_Q\left(((kG)_\gamma)^P\right) \quad 4.5.8\]
where $R$ runs over that set of subgroups of $N_P(Q)$ strictly containing $Q$; but 0 is the unique idempotent in $((kG)_\gamma)^P$; hence, we get $1_{(kG)_\gamma(Q)} = \bar{\ell}$, proving the claim.

Consequently, it follows from Corollary 3.10 above, applied to the groups $C_G(Q)\cdot N_P(Q)$ and $C_G(Q)$, that the $N_P(Q)$-interior algebra (cf. 2.3)
\[(kG)_\gamma(Q) \otimes_{C_P(Q)} N_P(Q) = \bigoplus_u (kG)_\gamma(Q) \cdot u \quad 4.5.9,\]
where $u \in N_P(Q)$ runs over a set of representatives for $N_P(Q)/C_P(Q)$, is a source algebra of the block $f$ of $C_G(Q)\cdot N_P(Q)$; hence, according to our hypothesis, we have a $N_P(Q)$-interior algebra isomorphism
\[k_*(C_L(Q)\cdot N_P(Q)) \cong (kG)_\gamma(Q) \otimes_{C_P(Q)} N_P(Q) \quad 4.5.10;\]
now, according to isomorphism 4.5.6 and equality 4.5.9, we actually get
\[\dim((kN)_\gamma(Q)) = \dim(k_*(C_L(Q))) = \dim((kG)_\gamma(Q)) \quad 4.5.11\]
and therefore $g(Q)$ is an isomorphism.
In particular, the interior $C_P(Q)$-algebra $(kG)_\gamma(Q) \cong k_\ast C_L(Q)$ is actually a source algebra of the block $f$ of $C_G(Q)$ and therefore, since we have (cf. 1.11.1)
$$C_L(Q) \cong C_P(Q) \times C_{E}(Q)$$
it follows from equalities 2.7.2 that there is no essential pointed groups on $kC_G(Q)f$, so that the block $f$ of $C_G(Q)$ is inertially controlled (cf. 2.9); hence, it follows from Lemma 2.9 and from our hypothesis that the block $b$ of $G$ is also inertially controlled.

Consequently, the $\mathcal{F}(b,G)$-morphism $\varphi: Q \to P$ above is induced by some element $n \in N$ and therefore there is an inversible element $a \in (kG)^P$ fulfilling $i^n = ia$, so that the multiplication by $na^{-1}$ on the right still determines a $k$-linear isomorphism
$$(kG)_\gamma(\Delta_\varphi(Q)) \cong (kG)_\gamma(Q)$$
similarly, we also get
$$(kN)_\gamma(\Delta_\varphi(Q)) \cong (kN)_\gamma(Q)$$
finally, equality 4.5.5 follows from these isomorphisms and equality 4.5.11.

**Corollary 4.6.** With the notation above, for any nontrivial Brauer $(b,G)$-pair $(Q,f)$ contained in $(P,e)$, assume that $C_P(Q)$ is a defect group of the block $f$ of $C_G(Q)$ and that a source algebra of this block is isomorphic to $k_\ast(C_L(Q))$. Then, the restriction throughout $g$ induces a stable equivalence between the categories of $(kG)_\gamma$- and $(kN)_\gamma$-modules. In particular, the blocks $b$ of $G$ and $e$ of $N$ are stably identical.

**Proof:** With the notation in 4.3 above, the indecomposable $k(N \times G)$-module $\ell(kG)$ defined by the left-hand and the right-hand multiplication has the $p$-group $\Delta(P) = \{(u,u)|u \in P\}$ as a vertex and the trivial $k\Delta(P)$-module $k$ as a source. Then this corollary follows from Theorem 4.5 above and [13, Theorem 6.9] applied to the case where $\bar{M} = \ell(kG)$, $b = e$, $b' = b$, $P_\gamma = P_\gamma$, $P'_\gamma = P_\gamma$ and $\bar{S} = k$.

5. An inductive context: the proof of Theorem 1.8

5.1. Let $G$ be a finite group, $b$ a block of $G$ and $P_\gamma$ a defect pointed group of $b$; with the notation in 1.5 above, consider the following condition

5.1.1. The block $b$ of $G$ is inertially controlled and, for any Brauer $(b,G)$-pair $(Q,f)$ contained in $(P,e)$, $f$ is a block of $C_G(Q)$-$NP(Q)$ with trivial source simple modules.

First of all, we claim that if the block $b$ of $G$ fulfills this condition then, for any Brauer $(b,G)$-pair $(R,h)$ contained in $(P,e)$, the block $h$ of the group $H = C_G(R)$ fulfills the corresponding condition.
5.2. Indeed, it follows from Lemma 2.9 that the block \( b \) of \( H \) is inertly controlled and that \( T = C_P(R) \) is a defect group of the block \( h \) of \( H \); thus, denoting by \( \ell \) the block of \( C_G(RT) \) such that \( (RT, \ell) \subset (P, e) \) [2, Theorem 1.8], \( (T, \ell) \) is a maximal Brauer \( (h, H) \)-pair and, if \( (Q, f) \) is a Brauer \( (b, G) \)-pair contained in \( (T, \ell) \), \( (RQ, f) \) is a Brauer \( (b, G) \)-pair still contained in \( (RT, \ell) \subset (P, e) \) and therefore \( f \) is a block of \( C_G(RQ) \cdot N_P(RQ) \) with trivial source simple modules. Then, since \( C_H(Q) \cdot N_T(Q) \) is clearly subnormal in \( C_G(RQ) \cdot N_P(RQ) \), it follows from Lemma 3.9, possibly applied more than once, that \( f \) is still a block of \( C_H(Q) \cdot N_C(G)(Q) \) with trivial source simple modules.

5.3. At this point, assuming that the block \( b \) of \( G \) fulfills condition 5.1.1 and that, for any nontrivial Brauer \( (b, G) \)-pair \( (Q, f) \) contained in \( (P, e) \), we have \( C_G(Q) \neq G \), it suffices to argue by induction on \(|G|\) to get the hypothesis of Theorem 4.5, namely to get that, for any nontrivial Brauer \( (b, G) \)-pair \( (Q, f) \) contained in \( (P, e) \), \( C_P(Q) \) is a defect group of the block \( f \) of \( C_G(Q) \) (cf. Lemma 2.9) and that a source algebra of this block is isomorphic to \( k_s(C_L(Q)) \).

5.4. In this situation, it follows from this theorem and from [13, Theorem 6.9] that the blocks \( b \) of \( G \) and \( e \) of \( N \) are stably identical (cf. 1.4); more precisely, if \( M \) is a simple \( kGb \)-module of vertex \( Q \subset P \) and \( f \) is the block of \( C_G(Q) \) such that \( (P, e) \) contains \( (Q, f) \), on the one hand it follows from [8, Proposition 1.6] that the Brauer \( (b, G) \)-pair \( (Q, f) \) is selfcentalizing, so that \( C_P(Q) = Z(Q) \) [16, 4.8 and Corollary 7.3] and, on the other hand, it easily follows from Theorem 4.5 that the simple modules are indecomposable \[7, \text{Theorem } 2.1\], has also vertex \( Q \); moreover, since we are assuming that the trivial \( kQ \)-module \( k \) is a source of \( M \), it is clear that the trivial \( kQ \)-module \( k \) is also a source of \( \ell \cdot M \).

5.5. Then, it follows again from [8, Proposition 1.6] applied to the \( N \)-interior algebra \( \text{End}_k(\ell \cdot M) \), that the quotient \( N_N(Q)/Q \cdot C_N(Q) \) [15, Theorem 3.6], admits blocks of defect zero — namely, with trivial defect groups — which forces [16, 1.19]

\[ \mathcal{O}_P(N_N(Q)/Q \cdot C_N(Q)) = \{1\} \tag{5.5.1} \]

but we have [11, Proposition 14.6]

\[ C_P(Q) = Z(Q) \quad \text{and} \quad (kN)_{\delta} = k_s(\hat{L} = k_s(\hat{P} \times \hat{E}) \tag{5.5.2} \]

hence, denoting by \( \hat{\delta} \) the unique local point of \( Q \) on \( kNe \) such that \( P_{\hat{\delta}} \) contains \( Q_{\hat{\delta}} \) (cf. 2.8), it follows from 2.7.2 and from the isomorphism in 5.5.2 that, as in 1.11.1, we get [5, Ch. 5, Theorem 3.4]

\[ N_N(Q)/Q \cdot C_N(Q) \cong E_N(Q_{\hat{\delta}}) \]

\[ = F(kN)_{\delta}(Q_{\hat{\delta}}) \cong (N_P(Q)/Q) \times N_{E^{\circ}}(Q) \tag{5.5.3} \]
and, since \( O_p \left( N_N(Q)/Q\cdot C_N(Q) \right) = \{1\} \), we still get \( N_P(Q) = Q \) which forces \( P = Q \). In conclusion, \( \ell \cdot M \) admits \( P \) as a vertex and it has a trivial source, so that it is a simple \( kNe \)-module according again to isomorphism 5.5.2.

5.6. Finally, since the stable equivalence induced by the restriction throughout \( g \) (cf. 4.3.1) sends any simple \( kGb \)-module to a simple \( kNe \)-module, it follows from [7, Proposition 2.5] that the restriction throughout \( g \) actually induces an equivalence of categories; moreover, since this equivalence is defined by a \( k(G \times N) \)-module admitting a \( P \times P \)-stable basis (cf. 4.3), it follows from [13, Corollary 7.4 and Remark 7.5] that the source algebras of the blocks \( b \) of \( G \) and \( e \) of \( N \) are isomorphic.

5.7. Assume now that the block \( b \) of \( G \) fulfills condition 5.1.1 and that there is an Abelian subgroup \( Z \) of \( P \) such that \( G = C_G(Z) \); we are in the situation considered in 3.7 above with \( H = G \); hence, it follows from Corollary 3.10 that \( \bar{b} \) is a block of \( \bar{G} \) and that \( \bar{\gamma} \) is contained in a local point \( \bar{\bar{\gamma}} \) of \( \bar{G} \) such that \( \bar{P}_\bar{\gamma} \) is a defect pointed group of \( \bar{b} \); denote by \( \bar{e} \) the block of \( C_G(\bar{P}) \) determined by the point \( \bar{\bar{\gamma}} \).

5.8. We claim that the block \( \bar{b} \) of \( \bar{G} \) fulfills the corresponding condition 5.1.1. Indeed, if \( (\bar{Q}, \bar{f}) \) is a Brauer \((\bar{b}, \bar{G})\)-pair contained in \((\bar{P}, \bar{e})\) and \( Q \) is the converse image of \( Q \) in \( G \), the image of \( C_G(Q) \) in \( C_G(\bar{Q}) \) is a normal subgroup and, once again, the corresponding quotient is a \( p \)-group [5, Ch. 5, Theorem 3.4]; hence, it follows again from Corollary 3.10 that \( \bar{f} \) is the image in \( kC_G(\bar{Q}) \) of a block \( \bar{f} \) of the converse image \( C \) of \( C_G(\bar{Q}) \) in \( G \) and then, since \( C_G(Q) \) is normal in \( C \), it is quite clear that \( f = \text{Tr}_{C}^{C}(\bar{f}) \) for a suitable block \( \bar{f} \) of \( C_G(Q) \) where \( C_t \) denotes the stabilizer of \( \bar{f} \) in \( C \).

5.9. More precisely, we claim that we can choose \( \bar{f} \) in such a way that \( (\bar{P}, \bar{e}) \) contains \((\bar{Q}, \bar{f})\); indeed, since \((\bar{P}, \bar{e})\) contains \((\bar{Q}, \bar{f})\), there is a local point \( \bar{\delta} \) of \( \bar{Q} \) on \( k\bar{G} \) such that we have \( b_{\delta} = \bar{f} \) and that \( \bar{P}_{\bar{\delta}} \) contains \( Q_{\bar{\delta}} \); then, it follows easily from Proposition 3.8 and from the obvious commutative diagram

\[ \begin{array}{ccc}
 k\bar{G}\bar{P} & \longrightarrow & k\bar{G}\bar{Q} \\
 \uparrow & & \uparrow \\
 kG\bar{P} & \longrightarrow & kG\bar{Q}
\end{array} \]

5.9.1

that there is a point \( \delta \) of \( Q \) on \( kG \) such that \( P_{\gamma} \) contains \( Q_{\delta} \) and that the image \( \delta \) of \( \delta \) in \( k\bar{G} \) is contained in \( \bar{\delta} \), which forces \( \delta \) to be local; at this point, it is easily checked that we can choose \( \bar{f} = b_{\delta} \).

5.10. Now, for any \( \bar{x} \in \bar{G} \) such that \( (\bar{Q}, \bar{f})^{\bar{x}} \subset (\bar{P}, \bar{e}) \), the same argument proves that we have \((Q, f)^{cx} \subset (P, e)\) for some \( x \in G \) lifting \( \bar{x} \) and a suitable element \( c \) of \( C \); then, since the block \( b \) of \( G \) is inertially controlled, there are \( n \in N \) and \( z \in C_G(Q) \) fulfilling \( cx = zn \) (cf. 1.7) and therefore we get
\[ z = c^{-1} \tilde{z} \bar{n} \] where \( c, \tilde{z} \) and \( \bar{n} \) denote the respective images of \( c, z \) and \( n \) in \( \tilde{G} \), \( \tilde{c}^{-1} \tilde{z} \) centralizes \( \bar{Q} \) and \( \bar{n} \) normalizes \( (P, \tilde{e}) \). This proves that the block \( \bar{b} \) of \( \tilde{G} \) is also inertially controlled.

5.11. Moreover, since \((Q, \tilde{f})\) is a Brauer \((b, G)\)-pair contained in \((P, e)\), according to our hypothesis \( \tilde{f} \) is a block of \( C_{\tilde{G}}(Q) \cdot N_P(Q) \) with trivial source simple modules; but, since the block \( b \) of \( G \) is inertially controlled, we have

\[ E_G(Q, \tilde{f}) \cong (N_P(Q)/Q) \cong N_{E_K}(Q) \]  

and therefore \( C_{\tilde{f}} \) is contained in \( C_G(Q) \cdot N_P(Q) \); hence, since we have \([2, Theorem 1.8]\)

\[ C_{\tilde{f}} \cdot N_P(Q) = C_G(Q) \cdot N_P(Q) \quad \text{and} \quad C_{\tilde{f}} \cap N_P(Q) = C \cap N_P(Q) \]  

we clearly have \([13, 2.6.4]\)

\[ k(C \cdot N_P(Q)) \tilde{f} \cong \text{Ind}^{C \cdot N_P(Q)}_{C_{\tilde{G}}(Q) \cdot N_P(Q)}(k(C_{\tilde{G}}(Q) \cdot N_P(Q)) \tilde{f}) \]  

and therefore \( f \) is also a block of \( C \cdot N_P(Q) \) with trivial source simple modules. Finally, since the \( k \)-algebra \( k(C_{\tilde{G}}(Q) \cdot N_P(Q)) \tilde{f} \) is the image of \( k(C \cdot N_P(Q)) \tilde{f} \), \( \tilde{f} \) is a block of \( C_G(Q) \cdot N_P(Q) \) with trivial source simple modules too.

5.12. Consequently, setting \( \hat{L} = \tilde{L}/Z \), it follows from our induction hypothesis that the source algebra of the block \( \bar{b} \) of \( \tilde{G} \) is isomorphic to \( k, \hat{L} \) and, in particular, we have

\[ \dim((kG)_{\tilde{\gamma}}) = |L|/|Z| \]  

but, since the point \( \tilde{\gamma} \) contains the image of \( \gamma \), we may assume that \( (kG)_{\tilde{\gamma}} \) is the image of \( (kG)_{\gamma} \), or, equivalently, that

\[(kG)_{\tilde{\gamma}} \cong k \otimes_{kZ} (kG)_{\gamma}\]  

and, in particular, we get

\[ \dim((kG)_{\gamma}) = |Z| \dim((kG)_{\tilde{\gamma}}) = |L| \]  

hence, the unitary \( P \)-interior algebra homomorphism 2.6.2 is actually an isomorphism

\[ k, \hat{L} \cong (kN)_{\tilde{\gamma}} \cong (kG)_{\gamma} \]  

5.13. Conversely, assume that the source algebra \((kG)_{\gamma}\) is isomorphic to \( k, \hat{L} \), so that the unitary \( P \)-interior algebra homomorphism 2.6.2 is an isomorphism; then, it follows from equalities 2.7.2 applied to the blocks \( b \) of \( G \) and \( \{1\} \) of \( \hat{L} \) that there are no essential pointed groups on \( kGb \) (cf. 2.8) and therefore the block \( b \) of \( G \) is inertially controlled (cf. 2.9).
5.14. For any Brauer \((b, G)\)-pair \((Q, f)\) contained in \((P, e)\), since we have (cf. 1.3.1 and 1.11.1)
\[
(kG)(Q) \cong kC_G(Q) \quad \text{and} \quad (kG)_\gamma(Q) \cong (k, \hat{L})(Q) \cong k, C_L(Q)
\]
\[= k, (C_P(Q) \rtimes C_E(Q))
\]
the \(C_P(Q)\)-interior algebra \((kG)_\gamma(Q)\) is a source algebra of the block \(f\) of \(C_G(Q)\); then, it follows from Corollary 3.10 that a source algebra of the block \(f\) of \(C_G(Q)\cdot N_P(Q)\) is isomorphic to the \(N_P(Q)\)-interior algebra
\[
(kG)_\gamma(Q) \otimes_{C_P(Q)} N_P(Q)
\]
finally, according to isomorphisms 5.14.1, this \(N_P(Q)\)-interior algebra is isomorphic to
\[
(k, \hat{L})(Q) \otimes_{C_P(Q)} N_P(Q) \cong k, N_L(Q) = k, (N_P(Q) \rtimes N_E(Q))
\]
which clearly has trivial source simple modules. We are done.

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Abstract. Motivated by an observation in [3], we determine the source algebra, and therefore all the structure, of the blocks without essential Brauer pairs where the simple modules of all the Brauer corespondents have trivial sources.