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On the genera of semisimple groups defined over an integral domain of a global function field

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Résumé. Soit $K = \mathbb{F}_q(C)$ un corps de fonctions global, i.e. le corps des fonctions d’une courbe projective lisse $C$ définie sur un corps fini $\mathbb{F}_q$. L’anneau des fonctions régulières sur $C - S$, où $S \neq \emptyset$ est un ensemble fini de points fermés sur $C$, est un domaine de Dedekind $\mathcal{O}_S$ de $K$. Étant donné un $\mathcal{O}_S$-groupe $G$ semisimple dont le groupe fondamental $\pi_1$ est lisse, on aimerait décrire l’ensemble des genres de $G$ et encore (dans le cas où le groupe $G \otimes \mathcal{O}_S$ $K$ est isotrope à $S$) son genre principal en termes des groupes abéliens ne dépendant que de $\mathcal{O}_S$ et de $\pi_1$. Ceci conduit à une condition nécessaire et suffisante pour que le principe local-global de Hasse soit valable pour certains groupes $G$. Nous l’utilisons aussi pour exprimer le nombre de Tamagawa $\tau(G)$ d’un $K$-groupe semisimple $G$ par l’invariant d’Euler–Poincaré et faciliter le calcul de $\tau(G)$ pour les $K$-groupes tordus.

Abstract. Let $K = \mathbb{F}_q(C)$ be the global function field of rational functions over a smooth and projective curve $C$ defined over a finite field $\mathbb{F}_q$. The ring of regular functions on $C - S$ where $S \neq \emptyset$ is any finite set of closed points on $C$ is a Dedekind domain $\mathcal{O}_S$ of $K$. For a semisimple $\mathcal{O}_S$-group $G$ with a smooth fundamental group $\pi_1$, we aim to describe both the set of genera of $G$ and its principal genus (the latter if $G \otimes \mathcal{O}_S$ $K$ is isotropic at $S$) in terms of abelian groups depending on $\mathcal{O}_S$ and $\pi_1$ only. This leads to a necessary and sufficient condition for the Hasse local-global principle to hold for certain $G$. We also use it to express the Tamagawa number $\tau(G)$ of a semisimple $K$-group $G$ by the Euler–Poincaré invariant. This facilitates the computation of $\tau(G)$ for twisted $K$-groups.

1. Introduction

Let $C$ be a projective algebraic curve defined over a finite field $\mathbb{F}_q$, assumed to be geometrically connected and smooth. Let $K = \mathbb{F}_q(C)$ be the global field of rational functions over $C$, and let $\Omega$ be the set of all closed points of $C$. For any point $p \in \Omega$, let $v_p$ be the induced discrete valuation on $K$, $\mathcal{O}_p$ the complete valuation ring with respect to $v_p$, and $\kappa_p = \mathcal{O}_p / v_p$ its fraction field and residue field at $p$, respectively. Any Hasse set of $K$, namely,
a non-empty finite set \( S \subset \Omega \), gives rise to an integral domain of \( K \) called a Hasse domain:

\[
\mathcal{O}_S := \{ x \in K : v_p(x) \geq 0 \ \forall \ p \notin S \}.
\]

This is a regular and one dimensional Dedekind domain. Group schemes defined over \( \text{Spec} \mathcal{O}_S \) are underlined, being omitted in the notation of their generic fibers.

Let \( G \) be an affine, smooth and of finite type group scheme defined over \( \text{Spec} \mathcal{O}_S \). We define \( H^1_{\text{ét}}(\mathcal{O}_S, G) \) to be the set of isomorphism classes of \( G \)-torsors over \( \text{Spec} \mathcal{O}_S \) relative to the étale or the flat topology (the classification for the two topologies coincide when \( G \) is smooth; cf. [2, VIII, Cor. 2.3]). The sets \( H^1(K, G) \) and \( H^1_{\text{ét}}(\mathcal{O}_{p}, G_{p}) \), for every \( p \notin S \), are defined similarly. All these three sets are naturally pointed: the distinguished point of \( \mathcal{O}_{p} \)-torsors over \( \text{Spec} \mathcal{O}_S \) relative to the étale or the flat topology (the classification for the two topologies coincide when \( G \) is smooth; cf. [2, VIII, Cor. 2.3]). The sets \( H^1(K, G) \) and \( H^1_{\text{ét}}(\mathcal{O}_{p}, G_{p}) \), for every \( p \notin S \), are defined similarly. All these three sets are naturally pointed: the distinguished point of \( \mathcal{O}_{p} \)-torsors over \( \text{Spec} \mathcal{O}_S \) relative to the étale or the flat topology (the classification for the two topologies coincide when \( G \) is smooth; cf. [2, VIII, Cor. 2.3]). The sets \( H^1(K, G) \) and \( H^1_{\text{ét}}(\mathcal{O}_{p}, G_{p}) \), for every \( p \notin S \), are defined similarly. All these three sets are naturally pointed: the distinguished point of \( \mathcal{O}_{p} \)-torsors over \( \text{Spec} \mathcal{O}_S \) relative to the étale or the flat topology (the classification for the two topologies coincide when \( G \) is smooth; cf. [2, VIII, Cor. 2.3]).

Let \( G \) be an affine, smooth and of finite type group scheme defined over \( \text{Spec} \mathcal{O}_S \). We define \( H^1_{\text{ét}}(\mathcal{O}_S, G) \) to be the set of isomorphism classes of \( G \)-torsors over \( \text{Spec} \mathcal{O}_S \) relative to the étale or the flat topology (the classification for the two topologies coincide when \( G \) is smooth; cf. [2, VIII, Cor. 2.3]). The sets \( H^1(K, G) \) and \( H^1_{\text{ét}}(\mathcal{O}_{p}, G_{p}) \), for every \( p \notin S \), are defined similarly. All these three sets are naturally pointed: the distinguished point of \( H^1_{\text{ét}}(\mathcal{O}_S, G) \) (resp., \( H^1(K, G) \), \( H^1_{\text{ét}}(\mathcal{O}_{p}, G_{p}) \)) is the class of the trivial \( G \)-torsor \( G \) (resp. trivial \( G \)-torsor \( G \), trivial \( G_{p} \)-torsor \( G_{p} \)). There exists a canonical map of pointed-sets (mapping the distinguished point to the distinguished point):

\[
(1.1) \quad \lambda : H^1_{\text{ét}}(\mathcal{O}_S, G) \to H^1(K, G) \times \prod_{p \notin S} H^1_{\text{ét}}(\mathcal{O}_{p}, G_{p})
\]

which is defined by mapping a class in \( H^1_{\text{ét}}(\mathcal{O}_S, G) \) represented by \( X \) to the class represented by \( (X \otimes_{\mathcal{O}_S} \text{Spec} K) \times \prod_{p \notin S} X \otimes_{\mathcal{O}_S} \text{Spec} \mathcal{O}_{p} \). Let \([\xi_0] := \lambda([G])\). The principal genus \( G \) is then \( \lambda^{-1}([\xi_0]) \), i.e., the classes of \( G \)-torsors over \( \text{Spec} \mathcal{O}_S \) that are generically and locally trivial at all points of \( \mathcal{O}_S \). More generally, a genus of \( G \) is any fiber \( \lambda^{-1}([\xi]) \) where \([\xi] \in \text{Im}(\lambda)\).

The set of genera of \( G \) is then:

\[
\text{gen}(G) := \{ \lambda^{-1}([\xi]) : [\xi] \in \text{Im}(\lambda) \},
\]

whence \( H^1_{\text{ét}}(\mathcal{O}_S, G) \) is a disjoint union of its genera.

Given a representative \( P \) of a class in \( H^1_{\text{ét}}(\mathcal{O}_S, G) \), by referring also to \( G \) as a \( G \)-torsor acting on itself by conjugations, the quotient of \( P \times_{\mathcal{O}_S} G \) by the \( G \)-action \( (p, g) \mapsto (ps^{-1}, sgs^{-1}) \) is an affine \( \mathcal{O}_S \)-group scheme \( P_G \), called the twist of \( G \) by \( P \). It is an inner form of \( G \), thus is locally isomorphic to \( G \) in the étale topology, namely, every fiber of it at a prime of \( \mathcal{O}_S \) is isomorphic to \( G_{p} := G \otimes_{\mathcal{O}_S} \mathcal{O}_{p} \) over some finite étale extension of \( \mathcal{O}_{p} \). The map \( G \mapsto P_G \) defines a bijection of pointed-sets \( H^1_{\text{ét}}(\mathcal{O}_S, G) \to H^1_{\text{ét}}(\mathcal{O}_S, P_G) \) (e.g., [31, §2.2, Lem. 2.2.3, Ex. 1 and 2]).

A group scheme defined over \( \text{Spec} \mathcal{O}_S \) is said to be reductive if it is affine and smooth over \( \text{Spec} \mathcal{O}_S \), and each geometric fiber of it at a prime \( p \) is (connected) reductive over \( k_p \) ([15, Exp. XIX, Def. 2.7]). It is semisimple if it is reductive, and the rank of its root system equals that of its lattice of weights ([15, Exp. XXI, Def. 1.1.1]). Suppose \( G \) is semisimple and that its
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fundamental group $F$ is of order prime to $\text{char}(K)$. Being finite, of multiplicative type ([15, XXII, Cor. 4.1.7]), commutative and smooth, $F$ decomposes into finitely many factors of the form $\text{Res}_{R/O_S}(\mu_m)$ or $\text{Res}_{R/O_S}^{(1)}(\mu_m)$ where $\mu_m := \text{Spec} \, O_S[t]/(t^m - 1)$ and $R$ is some finite (possibly trivial) étale extension of $O_S$. Consequently, $H^r_{\text{ét}}(O_S, F)$ are abelian groups for all $r \geq 0$. The following two $O_S$-invariants of $F$ will play a major role in the description of $H^1_{\text{ét}}(O_S, G)$:

**Definition 1.1.** Let $R$ be a finite étale extension of $O_S$. We define:

$$i(F) := \begin{cases} \text{Br}(R)[m] & F = \text{Res}_{R/O_S}(\mu_m) \\ \ker(\text{Br}(R)[m] \xrightarrow{N(2)} \text{Br}(O_S)[m]) & F = \text{Res}_{R/O_S}^{(1)}(\mu_m) \end{cases}$$

where $N(2)$ is induced by the norm map $N_{R/O_S}$ and for a group $\ast, \ast[m]$ stands for its $m$-torsion part. For $F = \prod_{k=1}^r F_k$ where each $F_k$ is one of the above, $i(F)$ is the direct product $\prod_{k=1}^r i(F_k)$.

We also define for such $R$:

$$(1.2) \quad j(F) := \begin{cases} \text{Pic}(R)/m & F = \text{Res}_{R/O_S}(\mu_m) \\ \ker(\text{Pic}(R)/m \xrightarrow{N(1)/m} \text{Pic}(O_S)/m) & F = \text{Res}_{R/O_S}^{(1)}(\mu_m) \end{cases}$$

where $N(1)$ is induced by $N_{R/O_S}$, and again $j(\prod_{k=1}^r F_k) := \prod_{k=1}^r j(F_k)$.

**Definition 1.2.** We call $F$ admissible if it is a finite direct product of the following factors:

1. $\text{Res}_{R/O_S}(\mu_m)$,
2. $\text{Res}_{R/O_S}^{(1)}(\mu_m), [R : O_S]$ is prime to $m$,

where $R$ is any finite étale extension of $O_S$.

After computing in Section 2 the cohomology sets of some related $O_S$-groups, we observe in Section 3 Proposition 3.1, that if $F$ is admissible then there exists an exact sequence of pointed sets:

$$1 \to \text{Cl}_S(G) \hookrightarrow H^1_{\text{ét}}(O_S, G) \xrightarrow{w_G} i(F) \to 1.$$ 

We deduce in Corollary 3.2 that $\text{gen}(G)$ bijects to $i(F)$. In Section 4, Theorem 4.1, we show that $\text{Cl}_S(G)$ surjects onto $j(F)$. If $G_S := \prod_{s \in S} G(\hat{K}_s)$ is non-compact, then this is a bijection. This leads us to formulate in Corollary 4.4 a necessary and sufficient condition for the *Hasse local-global principle* to hold for $G$. In Section 5, we use the above results to express in Theorem 5.2 the Tamagawa number $\tau(G)$ of an almost simple $K$-group $G$ with an admissible fundamental group $F$, using the (restricted) Euler–Poincaré characteristic of some $O_S$-model of $F$ and a local invariant, and show how this new description facilitates the computation of $\tau(G)$ when $G$ is a twisted group.
2. Étale cohomology

2.1. The class set. Consider the ring of $S$-integral adèles $\mathbb{A}_S := \prod_{p \in S} \hat{K}_p \times \prod_{p \notin S} \hat{O}_p$, being a subring of the adèles $\mathbb{A}$. The $S$-class set of an affine and of finite type $\mathcal{O}_S$-group $G$ is the set of double cosets:

$$\text{Cl}_S(G) := \frac{G(\mathbb{A}_S)}{G(\mathbb{A}) / G(K)}$$

(when over each $\hat{O}_p$ the above local model $G_p$ is taken). It is finite (cf. [8, Prop. 3.9]), and its cardinality, called the $S$-class number of $G$, is denoted by $h_S(G)$. According to Nisnevich ([26, Thm. I.3.5]) if $G$ is smooth, the map $\lambda$ introduced in (1.1) applied to it forms the following exact sequence of pointed-sets (when the trivial coset is considered as the distinguished point in Cl$_S(G)$):

$$(2.1) \quad 1 \rightarrow \text{Cl}_S(G) \rightarrow H^1_{\text{ét}}(\mathcal{O}_S, G) \xrightarrow{\lambda} H^1(K, G) \times \prod_{p \notin S} H^1_{\text{ét}}(\hat{O}_p, G_p).$$

The left exactness reflects the fact that Cl$_S(G)$ can be identified with the principal genus of $G$.

If, furthermore, $G$ has the property:

$$(2.2) \quad \forall \ p \notin S : \ H^1_{\text{ét}}(\hat{O}_p, G_p) \hookrightarrow H^1_{\text{ét}}(\hat{K}_p, G_p),$$

then sequence (2.1) is simplified to (cf. [26, Cor. 3.6]):

$$(2.3) \quad 1 \rightarrow \text{Cl}_S(G) \rightarrow H^1_{\text{ét}}(\mathcal{O}_S, G) \xrightarrow{\lambda_K} H^1(K, G),$$

which indicates that any two $G$-torsors share the same genus if and only if they are $K$-isomorphic. If $G$ has connected fibers, then by Lang’s Theorem $H^1_{\text{ét}}(\hat{O}_p, G_p)$ vanishes for any prime $p$ (see [30, Ch. VI, Prop. 5] and recall that all residue fields are finite), thus $G$ has property (2.2).

Remark 2.1. The multiplicative $\mathcal{O}_S$-group $\mathbb{G}_m$ admits property (2.2) thus sequence (2.3), in which the rightmost term vanishes by Hilbert 90 Theorem. Hence the class set Cl$_S(\mathbb{G}_m)$, being finite as previously mentioned, is bijective as a pointed-set to $H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m)$, which is identified with Pic$(\mathcal{O}_S)$ (cf. [24, Ch. III, §4]) thus being finite too. This holds true for any finite étale extension $R$ of $\mathcal{O}_S$.

Remark 2.2. If $G$ (locally of finite presentation) is disconnected but its connected component $G^0$ is reductive and $G/G^0$ is a finite representable group, then it admits again property (2.2) (see the proof of Proposition 3.14 in [12]), thus sequence (2.3) as well. If, furthermore, for any $[G'] \in \text{Cl}_S(G)$, the map $G'(K) \rightarrow (G'/(G')^0)(K)$ is surjective, then Cl$_S(G) = \text{Cl}_S(G^0)$ (cf. [5, Lem. 3.2]).
Lemma 2.3. Let $G$ be a smooth and affine $\mathcal{O}_S$-group scheme with connected fibers. Suppose that its generic fiber $G$ is almost simple, simply connected and $G_S$ is non-compact. Then $H^1_{\mathrm{ét}}(\mathcal{O}_S, G) = 1$.

Proof. The proof, basically relying on the strong approximation property related to $G$, is the one of Lemma 3.2 in [4], replacing $\{\infty\}$ by $S$. □

2.2. The fundamental group: the quasi-split case. The following is the Shapiro Lemma for the étale cohomology:

Lemma 2.4. Let $f : R \to S$ be a finite étale extension of schemes and $\Gamma$ a smooth $R$-module. Then $\forall p : H^p_{\mathrm{ét}}(S, \mathrm{Res}_{R/S}(\Gamma)) \cong H^p_{\mathrm{ét}}(R, \Gamma)$.

(See [2, VIII, Cor. 5.6] in which the Leray spectral sequence for $R/S$ degenerates, whence the edge morphism $H^p_{\mathrm{ét}}(S, \mathrm{Res}_{R/S}(\Gamma)) \to H^p_{\mathrm{ét}}(R, \Gamma)$ is an isomorphism.)

Remark 2.5. As $C$ is smooth, Spec $\mathcal{O}_S$ is normal, i.e., is integrally closed locally everywhere, thus any finite étale covering of $\mathcal{O}_S$ arises by its normalization in some separable unramified extension of $K$ (e.g., [22, Thm. 6.13]).

Assume $\mathcal{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\mu_m)$, $R$ is finite étale over $\mathcal{O}_S$. Then the Shapiro Lemma (2.4) with $p = 2$ gives $H^2_{\mathrm{ét}}(\mathcal{O}_S, \mathcal{F}) \cong H^2_{\mathrm{ét}}(R, \mu_m)$. Étale cohomology applied to the Kummer sequence over $R$

\begin{equation}
1 \to \mu_m \to \mathbb{G}_m \xrightarrow{x \mapsto x^m} \mathbb{G}_m \to 1
\end{equation}

gives rise to the exact sequences of abelian groups:

\begin{equation}
1 \to H^0_{\mathrm{ét}}(R, \mu_m) \to R^\times \xrightarrow{x \mapsto (x^m)} (R^\times)^m \to 1,
\end{equation}

\begin{equation}
1 \to R^\times / (R^\times)^m \to H^1_{\mathrm{ét}}(R, \mu_m) \to \mathrm{Pic}(R)[m] \to 1,
\end{equation}

in which as above $\mathrm{Pic}(R)$ is identified with $H^1_{\mathrm{ét}}(R, \mathbb{G}_m)$, and the Brauer group $\mathrm{Br}(R)$ (classifying Azumaya $R$-algebras) is identified with $H^2_{\mathrm{ét}}(R, \mathbb{G}_m)$ (cf. [24, Ch. IV, §2]).

2.3. The fundamental group: the non quasi-split case. The group $\mathcal{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\mu_m)$ fits into the short exact sequence of smooth $\mathcal{O}_S$-groups (recall $\mu_m$ is assumed to be smooth as $m$ is prime to $\text{char}(K)$):

\begin{equation}
1 \to \mathcal{F} \to \mathrm{Res}_{R/\mathcal{O}_S}(\mu_m) \xrightarrow{N_{R/\mathcal{O}_S}} \mu_m \to 1
\end{equation}
which yields by étale cohomology together with Shapiro's isomorphism the long exact sequence:

\[(2.6) \quad \cdots \to H^r_{\text{ét}}(\mathcal{O}_S, F) \xrightarrow{I^{(r)}} H^r_{\text{ét}}(R, \mu_m) \xrightarrow{N^{(r)}} H^r_{\text{ét}}(\mathcal{O}_S, \mu_m) \to H^{r+1}_{\text{ét}}(\mathcal{O}_S, F) \to \cdots.\]

**Notation 2.6.** For a group homomorphism \(f : A \to B\), we denote by \(f/m : A/m \to B/m\) and \(f[m] : A[m] \to B[m]\) the canonical maps induced by \(f\).

**Lemma 2.7.** If \([R : \mathcal{O}_S]\) is prime to \(m\), then \(N^{(r)}, N^{(r)}[m]\) and \(N^{(r)}/m\) are surjective for all \(r \geq 0\). In particular, if \(F = \text{Res}_{R/\mathcal{O}_S}(\mu_m)\), then sequence (2.6) induces an exact sequence of abelian groups for every \(r \geq 0\):

\[(2.7) \quad 1 \to H^r_{\text{ét}}(\mathcal{O}_S, F) \xrightarrow{I^{(r)}} H^r_{\text{ét}}(R, \mu_m) \xrightarrow{N^{(r)}} H^r_{\text{ét}}(\mathcal{O}_S, \mu_m) \to 1.\]

**Proof.** The composition of the induced norm \(N_{R/\mathcal{O}_S}\) with the diagonal morphism coming from the Weil restriction

\[(2.8) \quad \mu_{m, \mathcal{O}_S} \to \text{Res}_{R/\mathcal{O}_S}(\mu_{m,R}) \xrightarrow{N_{R/\mathcal{O}_S}} \mu_{m, \mathcal{O}_S}\]

is the multiplication by \(n := [R : \mathcal{O}_S]\). It induces for every \(r \geq 0\) the maps:

\[(2.9) \quad H^r_{\text{ét}}(\mathcal{O}_S, \mu_m) \to H^r_{\text{ét}}(R, \mu_m) \xrightarrow{N^{(r)}} H^r_{\text{ét}}(\mathcal{O}_S, \mu_m)\]

whose composition is again the multiplication by \(n\) on \(H^r_{\text{ét}}(\mathcal{O}_S, \mu_m)\), being an automorphism when \(n\) is prime to \(m\). Hence \(N^{(r)}\) is surjective for all \(r \geq 0\).

Replacing \(\mu_m\) with \(\mathbb{G}_m\) in sequence (2.8) and taking the \(m\)-torsion subgroups of the resulting cohomology sets, we get the group maps:

\[H^r_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m)[m] \to H^r_{\text{ét}}(R, \mathbb{G}_m)[m] \xrightarrow{N^{(r)}[m]} H^r_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m)[m]\]

whose composition is multiplication by \(n\) on \(H^r_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m)[m]\), being an automorphism again as \(n\) is prime to \(m\), whence \(N^{(r)}[m]\) is an epimorphism for every \(r \geq 0\). The same argument applied to \(N^{(r)}/m\) shows it is surjective for every \(r \geq 0\) as well. \(\square\)

Back to the general case ([\([R : \mathcal{O}_S]\) does not have to be prime to \(m\)], applying the Snake lemma to the exact and commutative diagram of abelian groups:

\[(2.10)\]

\[
\begin{array}{cccccc}
1 & \longrightarrow & \text{Pic}(R)/m & \longrightarrow & H^2_{\text{ét}}(R, \mu_m) & \xrightarrow{i_0} \text{Br}(R)[m] & \longrightarrow & 1 \\
& & \downarrow{N^{(1)}/m} & & \downarrow{N^{(2)}} & & \downarrow{N^{(2)}/m} \\
1 & \longrightarrow & \text{Pic}(\mathcal{O}_S)/m & \longrightarrow & H^2_{\text{ét}}(\mathcal{O}_S, \mu_m) & \longrightarrow & \text{Br}(\mathcal{O}_S)[m] & \longrightarrow & 1
\end{array}
\]
yields an exact sequence of $m$-torsion abelian groups:

\[(2.11) \quad 1 \rightarrow \ker(\text{Pic}(R)/m \quad N^{(1)/m} \rightarrow \text{Pic}(\mathcal{O}_S)/m) \rightarrow \ker(N^{(2)}) \]

\[\xrightarrow{i'_*} \ker(\text{Br}(R)[m] \quad N^{(2)/m}[m] \rightarrow \text{Br}(\mathcal{O}_S)[m]) \]

\[\rightarrow \text{coker}(\text{Pic}(R)/m \quad N^{(1)/m} \rightarrow \text{Pic}(\mathcal{O}_S)/m),\]

where $i'_*$ is the restriction of $i_*$ to $\ker(N^{(2)})$. Together with the surjection $I^{(2)} : H^2_{\text{ét}}(\mathcal{O}_S, F) \twoheadrightarrow \ker(N^{(2)})$ coming from sequence (2.6), we get the commutative diagram:

\[
\begin{array}{cccc}
& & & \\
& & & \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
\ker(\text{Pic}(R)/m \rightarrow \text{Pic}(\mathcal{O}_S)/m) & \xrightarrow{I^{(2)}} & H^2_{\text{ét}}(\mathcal{O}_S, F) & \xrightarrow{i_'} \ker(\text{Br}(R)[m] \rightarrow \text{Br}(\mathcal{O}_S)[m]) \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]

**Proposition 2.8.** If $[R : \mathcal{O}_S]$ is prime to $m$, then there exists a canonical exact sequence of abelian groups

\[1 \rightarrow \ker(\text{Pic}(R)/m \rightarrow \text{Pic}(\mathcal{O}_S)/m) \rightarrow \ker(N^{(2)}) \]

\[\xrightarrow{i'_*} \ker(\text{Br}(R)[m] \rightarrow \text{Br}(\mathcal{O}_S)[m]) \rightarrow 1.\]

**Proof.** This sequence is the column in diagram (2.12) since Lemma 2.7 shows the surjectivity of $N^{(1)/m}$, which in turn implies the surjectivity of $i'_*$ by the exactness of sequence (2.11). \qed

Recall the definition of $i(F)$ (Definition 1.1), and of the maps $i_*$ and $i_*/^{(1)}$ (sequences (2.5) and (2.12)).

**Definition 2.9.** Let $F$ be one of the basic factors of an admissible fundamental group (see Def. 1.2). The map $\overline{i}_* : H^2_{\text{ét}}(\mathcal{O}_S, F) \rightarrow i(F)$ is defined as:

\[
\overline{i}_* := \begin{cases}
  i_* & F = \text{Res}_{R/\mathcal{O}_S}(\mu_m), \\
  i_*/^{(1)} & F = \text{Res}_{R/\mathcal{O}_S}^{(1)}(\mu_m) \quad \text{and} \quad ([R : \mathcal{O}_S], m) = 1.
\end{cases}
\]

More generally, if $F = \prod_{k=1}^r F_k$ where each $F_k$ is one of the above, we set it to be the composition:

\[
\overline{i}_* : H^2_{\text{ét}}(\mathcal{O}_S, F) \xrightarrow{\sim} \bigoplus_{k=1}^r H^2_{\text{ét}}(\mathcal{O}_S, F_k) \xrightarrow{\oplus_{k=1}^r (\overline{i}_*)_k} i(F) = \prod_{k=1}^r i(F_k).
\]
Corollary 2.10. If $F$ is admissible, then there exists a short exact sequence
\begin{equation}
1 \to j(F) \to H^2_{ét}(\mathcal{O}_S, F) \xrightarrow{i_*} i(F) \to 1.
\end{equation}

Proof. If $F = \text{Res}_{R/\mathcal{O}_S}(\mu_m)$ then the sequence of the corollary is simply a restatement of the last sequence in (2.5) by the definitions of $i(F)$ and $j(F)$ (see Definition 1.1). On the other hand, if $F = \text{Res}^{(1)}_{R/\mathcal{O}_S}(\mu_m)$ with $[R : \mathcal{O}_S]$ is prime to $m$, then $I(2)$ induces an isomorphism of abelian groups $H^2_{ét}(\mathcal{O}_S, F) \cong \ker(N(2))$ by the exactness of (2.7) for $r = 2$. Thus the sequence of the corollary is isomorphic to the sequence in Proposition 2.8 again by the definitions of $j(F)$ and $i(F)$. The two cases considered above suffice to establish the corollary by the definition of admissible (see Definition 1.2) and the definition of $i_*$ (see Definition 2.9). $\square$

**Definition 2.11.** Let $X$ be a constructible sheaf defined over $\text{Spec} \mathcal{O}_S$ and let $h_i(X) := |H^i_{ét}(\mathcal{O}_S, X)|$. The (restricted) Euler–Poincaré characteristic of $X$ is defined to be (cf. [25, Ch. II, §2]):
\[ \chi_S(X) := \prod_{i=0}^2 h_i(X)(-1)^i. \]

**Definition 2.12.** Let $R$ be a finite étale extension of $\mathcal{O}_S$. We define:
\[ \ell(F) := \begin{cases} 
\frac{|R^\times[m]|}{|R^\times : (R^\times)^m|} & F = \text{Res}_{R/\mathcal{O}_S}(\mu_m) \\
\frac{|\ker(N^{(0)}[m])|}{|\ker(N^{(0)}/m)|} & F = \text{Res}^{(1)}_{R/\mathcal{O}_S}(\mu_m).
\end{cases} \]

As usual, for $F = \prod_{k=1}^r F_k$ where each $F_k$ is one of the above, we put $l(F) = \prod_{k=1}^r l(F_k)$.

**Lemma 2.13.** If $F$ is admissible then $\chi_S(F) = l(F) \cdot |i(F)|$.

Proof. It is sufficient to check the assertion for the two basic types of (direct) factors:
Suppose $F = \text{Res}_{R/\mathcal{O}_S}(\mu_m)$. Then sequences (2.5) together with Shapiro’s Lemma give
\[ h_i(F) = |H^i_{ét}(R, \mu_m)| = \begin{cases} 
|R^\times[m]|, & i = 0 \\
|R^\times : (R^\times)^m| \cdot |\text{Pic}(R)[m]|, & i = 1 \\
|\text{Pic}(R)/m| \cdot |\text{Br}(R)[m]|, & i = 2.
\end{cases} \]

So as $\text{Pic}(R)$ is finite (see Remark 2.1), $|\text{Pic}(R)[m]| = |\text{Pic}(R)/m|$ and we get:
\[ \chi_S(F) := \frac{h_0(F) \cdot h_2(F)}{h_1(F)} = \frac{|R^\times[m]| \cdot |\text{Pic}(R)/m| \cdot |\text{Br}(R)[m]|}{|R^\times : (R^\times)^m| \cdot |\text{Pic}(R)[m]|} = l(F) \cdot |i(F)|. \]
Now suppose $F = \text{Res}_{R/O_S}^{(1)}(\mu_m)$ such that $[R : O_S]$ is prime to $m$. By Lemma 2.7 $N^{(r)}, N^{(r)}[m]$ and $N^{(r)}/m$ are surjective for all $r \geq 0$, so the long sequence (2.6) is cut into short exact sequences:

\begin{equation}
\forall r \geq 0 : 1 \to H^1_{\text{et}}(O_S, F) \xrightarrow{I^{(r)}} H^1_{\text{et}}(R, \mu_m) \xrightarrow{N^{(r)}} H^1_{\text{et}}(O_S, \mu_m) \to 1
\end{equation}

from which we see together with sequence (2.14) that:

\begin{equation}
h_0(F) = \left|\ker(R^×[m] \xrightarrow{N^{(0)[m]}} O^×_S[m])\right|.
\end{equation}

The Kummer exact sequences for $\mu_m$ defined over both $O_S$ and $R$ yield the exact diagram:

\begin{equation}
\begin{array}{cccc}
1 & \longrightarrow & R^×/(R^×)^m & \longrightarrow & H^1_{\text{et}}(R, \mu_m) \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & O^×_S/(O^×_S)^m & \longrightarrow & H^1_{\text{et}}(O_S, \mu_m) \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{cccc}
& & \text{Pic}(R)[m] & \longrightarrow & 1 \\
& & \downarrow & & \downarrow \\
& & \text{Pic}(O_S)[m] & \longrightarrow & 1
\end{array}
\end{equation}

from which we see together with sequence (2.14) that:

\begin{equation}
h_1(F) = \left|\ker(N^{(1)})\right| = \left|\ker(R^×/(R^×)^m \xrightarrow{N^{(0)/m}} O^×_S/(O^×_S)^m)\right| \\
\quad \cdot \left|\ker(\text{Pic}(R)[m] \xrightarrow{N^{(1)[m]}} \text{Pic}(O_S)[m])\right|.
\end{equation}

Similarly, by sequence (2.14) and Proposition 2.8 we find that:

\begin{equation}
h_2(F) = \left|\ker(N^{(2)})\right| = \left|\ker(\text{Pic}(R)/m \xrightarrow{N^{(1)/m}} \text{Pic}(O_S)/m)\right| \\
\quad \cdot \left|\ker(\text{Br}(R)[m] \xrightarrow{N^{(2)[m]}} \text{Br}(O_S)[m])\right|.
\end{equation}

Altogether we get:

\begin{equation}
\chi_S(F) = \frac{h_0(F) \cdot h_2(F)}{h_1(F)} = \frac{\left|\ker(N^{(0)[m]})\right| \cdot \left|\ker(N^{(1)[m]})\right|}{\left|\ker(N^{(0)/m})\right| \cdot \left|\ker(N^{(1)/m})\right|} \cdot \left|\ker(N^{(2)[m]})\right|.
\end{equation}

The group of units $R^×$ is a finitely generated abelian group (cf. [29, Prop. 14.2]), thus the quotient $R^×/(R^×)^m$ is a finite group. Since $\text{Pic}(R)[m]$ is also finite, $\ker(N^{(1)})$ in diagram (2.16) is finite, thus $\left|\ker(N^{(1)})[m]\right| = \left|\ker(N^{(1)}/m)\right|$, and we are left with:

\begin{equation}
\chi_S(F) = \frac{\left|\ker(N^{(0)[m]})\right| \cdot \left|\ker(N^{(2)[m]})\right|}{\left|\ker(N^{(0)/m})\right|} = l(F) \cdot |i(F)|. \quad \square
\end{equation}

**Remark 2.14.** The computation of $l(F)$, for specific choices of $R$, $O_S$ and $m$, is an interesting (and probably open) problem. For example, when $F$ is not quasi-split, the denominator of this number is the order of the group of units of $R$ whose norm down to $O_S$ is an $m$-th power of a unit in $O_S$. 

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**Genera of semisimple groups**

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modulo \((R^\times)^m\). Such computations are hard to find in the literature, if they exist at all.

3. The set of genera

From now and on we assume \(G\) is semisimple and that its fundamental group \(\mathcal{F}\) is of order prime to \(\text{char}(K)\), thus smooth. Étale cohomology applied to the universal covering of \(G\)

\[
1 \to \mathcal{F} \to \mathcal{G}^\text{sc} \to \mathcal{G} \to 1,
\]

gives rise to the exact sequence of pointed-sets:

\[
H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{G}^\text{sc}) \to H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{G}) \xrightarrow{\delta_G} H^2_{\text{ét}}(\mathcal{O}_S, \mathcal{F})
\]
in which the co-boundary map \(\delta_G\) is surjective, as the domain \(\mathcal{O}_S\) is of Douai-type, implying that \(H^2_{\text{ét}}(\mathcal{O}_S, \mathcal{G}^\text{sc}) = 1\) (see [17, Def. 5.2 and Ex. 5.4(iii)]).

**Proposition 3.1.** There exists an exact sequence of pointed-sets:

\[
1 \to \text{Cl}_S(G) \xrightarrow{h} H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{G}) \xrightarrow{w_G} i(\mathcal{F})
\]
in which \(h\) is injective. If \(\mathcal{F}\) is admissible, then \(w_G\) is surjective.

**Proof.** It is shown in [26, Thm. 2.8 and proof of Thm. 3.5] that there exist a canonical bijection \(\alpha_G : H^1_{\text{Nis}}(\mathcal{O}_S, \mathcal{G}) \cong \text{Cl}_S(G)\) and a canonical injection \(i_G : H^1_{\text{Nis}}(\mathcal{O}_S, \mathcal{G}) \hookrightarrow H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{G})\) of pointed-sets (as Nisnevich’s covers are étale). Then the map \(h\) of the statement is the composition \(i_G \circ \alpha_G^{-1}\).

Assume \(\mathcal{F} = \text{Res}_{R/\mathcal{O}_S}(\mu_m)\). The composition of the surjective map \(\delta_G\) from (3.2) with Shapiro’s isomorphism and the surjective morphism \(i_*\) from (2.5), is a surjective \(R\)-map:

\[
w_G : H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{G}) \xrightarrow{\delta_G} H^2_{\text{ét}}(\mathcal{O}_S, \mathcal{F}) \xrightarrow{\sim} H^2_{\text{ét}}(R, \mu_m) \xrightarrow{i_*} \text{Br}(R)[m].
\]

On the generic fiber, since \(G^\text{sc} := \mathcal{G}^\text{sc} \otimes_{\mathcal{O}_S} K\) is simply connected, \(H^1(K, G^\text{sc})\) vanishes due to Harder (cf. [19, Satz A]), as well as its other \(K\)-forms (this would not be true, however, if \(K\) were a number field with real places). So Galois cohomology applied to the universal \(K\)-covering

\[
1 \to \mathcal{F} \to \mathcal{G}^\text{sc} \to \mathcal{G} \to 1
\]
yields an embedding of pointed-sets \(\delta_G : H^1(K, G) \hookrightarrow H^2(K, \mathcal{F})\), which is also surjective as \(K\) is of Douai-type as well. The extension \(R\) of \(\mathcal{O}_S\) arises from an unramified Galois extension \(L\) of \(K\) by Remark 2.5, and Galois cohomology applied to the Kummer exact sequence of \(L\)-groups

\[
1 \to \mu_m \to \mathbb{G}_m \xrightarrow{x \mapsto x^m} \mathbb{G}_m \to 1
\]
yields, together with Shapiro’s Lemma \( H^2(K, F) \cong H^2(L, \mu_m) \) and Hilbert 90 Theorem, the identification \((i_*)_L : H^2(K, F) \cong \text{Br}(L)[m] \), whence the composition \((i_*)_L \circ \delta_G\) is an injective \( L\)-map:

\[
w_G : H^1(K, G) \xrightarrow{\delta_G} H^2(K, F) \xrightarrow{(i_*)_L} \text{Br}(L)[m].
\]

Now we know due to Grothendieck that \( \text{Br}(R) \) is a subgroup of \( \text{Br}(L) \) (see [18, Prop. 2.1] and [24, Ex. 2.22, case (a)]). Altogether we retrieve the commutative diagram of pointed-sets:

\[
\begin{array}{ccc}
H^1_{\text{ét}}(O_S, G) & \xrightarrow{w_G} & \text{Br}(R)[m] \\
\downarrow{\lambda_K} & & \downarrow{j} \\
H^1(K, G) & \xrightarrow{w_G} & \text{Br}(L)[m],
\end{array}
\]

from which, together with sequence (2.3) (recall \( G \) has connected fibers), we may observe that:

\[
\text{Cl}_S(G) = \text{ker}(\lambda_K) = \text{ker}(w_G).
\]

More generally, if \( F \) is a direct product of such basic factors, then as the cohomology sets commute with direct products, the target groups of \( w_G \)
Corollary 3.2. There is an injection of pointed sets $w'_G : \text{gen}(G) \mapsto \text{i}(F)$. If $F$ is admissible then $w'_G$ is a bijection. In particular if $F$ is split, then $|\text{gen}(G)| = |F||S|-1$.

Proof. The commutativity of diagrams (3.5) and (3.7) and the injectivity of the map $j$ in them show that $w_G$ is constant on each fiber of $\lambda_K$, i.e., on the genera of $G$. Thus $w_G$ induces a map (see Proposition 3.1):

$$w'_G : \text{gen}(G) \to \text{Im}(w_G) \subseteq \text{i}(F).$$

These diagrams commutativity together with the injectivity of $w_G$ imply the injectivity of $w'_G$.

If $F$ is admissible then $\text{Im}(w_G) = \text{i}(F)$. In particular if $F$ is split, then

$$\text{gen}(G) \cong \prod_{i=1}^r \text{Br}({\mathcal{O}}_S)[m_i].$$

It is shown in the proof of [4, Lem. 2.2] that

$$\text{Br}({\mathcal{O}}_S) = \ker \left( \frac{Q}{Z} \xrightarrow{\sum_{p \in S} \text{Cor}_p} \frac{Q}{Z} \right)$$

where $\text{Cor}_p$ is the corestriction map at $p$. So $|\text{Br}({\mathcal{O}}_S)[m_i]| = m_i|S|-1$ for all $i$ and the last assertion follows. □

The following table refers to absolutely almost simple and adjoint $\mathcal{O}_S$-groups whose fundamental group is split. The right column is Corollary 3.2:

| Type of $G$ | $F$ | $\# \text{gen}(G)$ |
|-------------|-----|-------------------|
| $^{1}A_{n-1}$ | $\mu_n$ | $n_i|S|-1$ |
| $B_n, C_n, E_7$ | $\mu_2$ | $2|S|-1$ |
| $^{1}D_n$ | $\mu_4$, $n = 2k + 1$ or $\mu_2 \times \mu_2$, $n = 2k$ | $4|S|-1$ |
| $^{1}E_6$ | $\mu_3$ | $3|S|-1$ |
| $E_8, F_4, G_2$ | 1 | 1 |

Lemma 3.3. Let $G$ be a semisimple and almost simple $\mathcal{O}_S$-group not of (absolute) type $A$, then $H^1_{\text{et}}(\mathcal{O}_S, G)$ bijects as a pointed-set to the abelian group $H^2_{\text{et}}(\mathcal{O}_S, F)$.

Proof. Since $G^{\text{sc}}$ is not of (absolute) type $A$, it is locally isotropic everywhere ([9, 4.3 and 4.4]), whence $\ker(\delta_G) \subseteq H^1_{\text{et}}(\mathcal{O}_S, G^{\text{sc}})$ vanishes due to Lemma 2.3. Moreover, for any $G$-torsor $P$, the base-point change: $G \mapsto P G$
defines a bijection of pointed-sets: \( H^1_{\text{ét}}(\mathcal{O}_S, G) \to H^1_{\text{ét}}(\mathcal{O}_S, P^G) \) (see in Section 1). But \( P^G \) is an inner form of \( G \), thus not of type A as well, hence also \( H^1_{\text{ét}}(\mathcal{O}_S, (P^G)^{\text{sc}}) = 1 \). We get that all fibers of \( \delta_G \) in (3.2) are trivial, which together with the surjectivity of \( \delta_G \) amounts to the asserted bijection. □

In other words, the fact that \( G \) is not of (absolute) type A guarantees that not only \( G^{\text{sc}} \), but also the universal covering of the generic fiber of inner forms of \( G \) of other genera are locally isotropic everywhere. This provides \( H^1_{\text{ét}}(\mathcal{O}_S, G) \) the structure of an abelian group.

**Corollary 3.4.** If \( G \) is not of (absolute) type A, then all its genera share the same cardinality.

**Proof.** The map \( w_G \) factors through \( \delta_G \) (see (3.3) and (3.6)) which is a bijection of pointed-sets in this case by Lemma 3.3. So writing: \( w_G = \pi_G \circ \delta_G \), we get due to Proposition 3.1 the exact sequence of pointed-sets (a-priory, abelian groups):

\[
1 \to \text{Cl}_S(G) \to H^2_{\text{ét}}(\mathcal{O}_S, F) \xrightarrow{\pi_G} i(F)
\]

in which all genera, corresponding to the fibers of \( w_G \), are of the same cardinality. □

Following E. Artin in [1], we shall say that a Galois extension \( L \) of \( K \) is **imaginary** if no prime of \( K \) is decomposed into distinct primes in \( L \).

**Remark 3.5.** If \( G \) is of (absolute) type A, but \( S = \{\infty\} \), \( G \) is \( \hat{K}_\infty \)-isotropic, and \( F \) splits over an imaginary extension of \( K \), then \( H^1_{\text{ét}}(\mathcal{O}_S, G) \) still bijects as a pointed-set to \( H^2_{\text{ét}}(\mathcal{O}_S, F) \).

**Proof.** As aforementioned, removing one closed point of a projective curve, the resulting Hasse domain has a trivial Brauer group. Thus \( \text{Br}(\mathcal{O}_S = \mathcal{O}_{\{\infty\}}) = 1 \), and as \( F \) splits over an imaginary extension \( L = \mathbb{F}_q(C') \), corresponding to an étale extension \( R = \mathbb{F}_q[C' - \{\infty'\}] \) of \( \mathcal{O}_{\{\infty\}} \) (see Remark 2.5) where \( \infty' \) is the unique prime of \( L \) lying above \( \infty \), \( \text{Br}(R) \) remains trivial. This implies by Corollary 3.2 that \( G \) has only one genus, namely, the principal one, in which the generic fibers of all representatives (being \( K \)-isomorphic to \( G \)) are isotropic at \( \infty \). Then the resulting vanishing of \( \ker(\delta_G) \subseteq H^1_{\text{ét}}(\mathcal{O}_S, G^{\text{sc}}) \) due to Lemma 2.3 is equivalent to the injectivity of \( \delta_G \). □

The following general framework due to Giraud (see [10, §2.2.4]), gives an interpretation of the \( G \)-torsors which may help us describe \( w_G \) more concretely.
Proposition 3.6. Let $R$ be a scheme and $X_0$ be an $R$-form, namely, an object of a fibered category of schemes defined over $R$. Let $\text{Aut}_{X_0}$ be its $R$-group of automorphisms. Let $\mathfrak{forms}(X_0)$ be the category of $R$-forms that are locally isomorphic for some topology to $X_0$ and let $\mathfrak{Tors}(\text{Aut}_{X_0})$ be the category of $\text{Aut}_{X_0}$-torsors in that topology. The functor

$$\varphi : \mathfrak{forms}(X_0) \to \mathfrak{Tors}(\text{Aut}_{X_0}) : X \mapsto \text{Iso}_{X_0,X}$$

is an equivalence of fibered categories.

Example 3.7. Let $(V, q)$ be a regular quadratic $O_S$-space of rank $n \geq 3$ and let $G$ be the associated special orthogonal group $\text{SO}_q$ (see [13, Def. 1.6]). It is smooth and connected (cf. [13, Thm. 1.7]), and its generic fiber is of type $B_n$ if rank$(V)$ is odd, and of type $1D_n$ otherwise. In both cases $F = \mu_2$, so we assume char$(K)$ is odd. Any such quadratic regular $O_S$-space $(V', q')$ of rank $n$ gives rise to a $G$-torsor $P$ by

$$V' \mapsto P = \text{Iso}_{V,V'}$$

where an isomorphism $A : V \to V'$ is a proper $q$-isometry, i.e., such that $q' \circ A = q$ and det$(A) = 1$. So $H^1(\text{et}, G)$ properly classifies regular quadratic $O_S$-spaces that are locally isomorphic to $(V, q)$ in the étale topology. Then $\delta_G([P])$ is the second Stiefel–Whitney class of $P$ in $H^2(\text{et}, \text{SO}_q, \mu_2)$, classifying $O_S$-Azumaya algebras with involutions (see [6, Def. 1, Rem. 3.3 and Prop. 4.5]), and

$$w_G([\text{SO}_q]) = \begin{cases} [C_0(q') - C_0(q)] \in \text{Br}(O_S)[2] & n \text{ is odd} \\ [C(q') - C(q)] \in \text{Br}(O_S)[2] & n \text{ is even} \end{cases}$$

where $C(q)$ and $C_0(q)$ are the Clifford algebra of $q$ and its even part, respectively.

Example 3.8. Let $G = \text{PGL}_n$ for $n \geq 2$. It is smooth and connected ([14, Lem. 3.3.1]) with $F = \mu_n$, so we assume (char$(K), n) = 1$. For any projective $O_S$-space of rank $n$, by the Skolem–Noether Theorem for unital rings (see [21, p. 145]) $\text{PGL}(V) = \text{Aut}(\text{End}_{O_S}(V))$. It is an inner form of $G$ obtained for $V = O_{S}^n$. So the pointed-set $H^1(\text{et}, (O_S, G))$ classifies the projective $O_S$-modules of rank $n$ up to invertible $O_S$-modules. Given such a projective $O_S$-module $V$, the Azumaya $O_S$-algebra $A = \text{End}_{O_S}(V)$ of rank $n^2$ corresponds to a $G$-torsor by (see [16, V, Rem. 4.2]):

$$A \mapsto P = \text{Iso}_{M_n,A}$$

where $M_n$ is the $O_S$-sheaf of $n \times n$ matrices. Here $w_G([P]) = [A]$ in $\text{Br}(O_S)[n]$. 


4. The principal genus

In this section, we study the structure of the principal genus $\text{Cl}_S(G)$.

**Theorem 4.1.** If $F$ is admissible then there exists a surjection of pointed-sets

$$\psi_G : \text{Cl}_S(G) \twoheadrightarrow j(F),$$

being a bijection provided that $G_S$ is non-compact (e.g., $G$ is not anisotropic of type $A$).

**Proof.** Combining the two epimorphisms, $w_G$ defined in Prop. 3.1 and $\delta_G$ described in Section 3, together with the exact sequence (2.13), yields the exact and commutative diagram:

$$
\begin{array}{ccc}
1 & \longrightarrow & H^1_{\text{ét}}(O_S, G) \\
\downarrow & & \downarrow \delta_G \\
1 & \longrightarrow & \text{Cl}_S(G)
\end{array}
\quad
\begin{array}{ccc}
\downarrow \psi_G & & \downarrow \wedge \\
H^2_{\text{ét}}(O_S, F) & \longrightarrow & j(F) \\
\downarrow i^* & & \downarrow i \\
1 & \longrightarrow & 1
\end{array}
$$

in which $\ker(w_G) = \text{Cl}_S(G)$. We imitate the Snake Lemma argument (the diagram terms are not necessarily all groups): for any $[H] \in \text{Cl}_S(G)$ one has $i_*([\delta_G([H]]) = [0]$, i.e., $\delta_G([H])$ has a $\partial$-preimage in $j(F)$ which is unique as $\partial$ is a monomorphism of groups. This constructed map denoted $\psi_G$ gives rise to an exact sequence of pointed-sets:

$$
1 \rightarrow \mathfrak{A} \rightarrow \text{Cl}_S(G) \xrightarrow{\psi_G} j(F) \rightarrow 1.
$$

If $G_S$ is non-compact, then for any $[H] \in \text{Cl}_S(G)$ the generic fiber $H$ is $K$-isomorphic to $G$ thus $H_S$ is non-compact as well, thus $\ker(H^1_{\text{ét}}(O_S, H) \xrightarrow{\delta_H} H^2_{\text{ét}}(O_S, F)) \subseteq H^1_{\text{ét}}(O_S, H^{\text{sc}})$ vanishes by Lemma 2.3. This means that $\delta_G$ restricted to $\text{Cl}_S(G)$ is an embedding, so $\mathfrak{A} = 1$ and $\psi_G$ is a bijection. □

**Remark 4.2.** The description of $\text{Cl}_S(G)$ in Theorem 4.1 holds true also for a disconnected group $G$ (where $F$ is the fundamental group of $G^0$), under the hypotheses of Remark 2.2.

**Definition 4.3.** We say that the **local-global Hasse principle** holds for $G$ if $h_S(G) = 1$.

This property means (when $G$ is connected) that a $G$-torsor is $O_S$-isomorphic to $G$ if and only if its generic fiber is $K$-isomorphic to $G$. Recall the definition of $j(F)$ from Definition 1.1.

**Corollary 4.4.** Suppose $F \cong \prod_{i=1}^t \text{Res}_{R_i/O_S}(\mu_{m_i})$ where $R_i$ are finite étale extensions of $O_S$. If $G_S$ is non-compact, then the Hasse principle holds for $G$ if and only if $\forall i: (|\text{Pic}(R_i)|, m_i) = 1$. Otherwise ($G_S$ is compact), this principle holds for $G$ only if $\forall i: (|\text{Pic}(R_i)|, m_i) = 1$. More generally, if $F$ is
admissible and $G_S$ is non-compact, then this principle holds for $G$ provided that for each factor of the form $\text{Res}_{R/O_S}(\mu_m)$ or $\text{Res}_{R/O_S}^{(1)}(\mu_m)$ one has: $\left(|\text{Pic}(R)|, m\right) = 1$.

Example 4.5. If $C^{af}$ is an affine non-singular $\mathbb{F}_q$-curve of the form $y^2 = x^3 + ax + b$, i.e., obtained by removing some $\mathbb{F}_q$-rational point $\infty$ from an elliptic (projective) $\mathbb{F}_q$-curve $C$, then $\text{Pic}(C^{af}) = \text{Pic}(\mathcal{O}(\infty)) \cong C(\mathbb{F}_q)$ (cf. e.g., [4, Ex. 4.8]). Let again $G = \text{PGL}_n$ such that $(\text{char}(K), n) = 1$. As $|S| = 1$ and $F$ is split, $G$ admits a single genus (Corollary 3.2), which means that all projective $\mathcal{O}(\infty)$-modules of rank $n$ are $K$-isomorphic. If $G$ is $K$-isotropic, according to Theorem 4.1, there are exactly $|C^{af}(\mathbb{F}_q)/2|$ $\mathcal{O}(\infty)$-isomorphism classes of such modules, so the Hasse principle fails for $G$ if and only if $|C^{af}(\mathbb{F}_q)|$ is even. This occurs exactly when $C^{af}$ has at least one $\mathbb{F}_q$-point on the $x$-axis (thus of order 2).

On the other hand, take $\mathcal{O}_S = \mathbb{F}_3[t, t^{-1}]$ obtained by removing $S = \{t, t^{-1}\}$ from the projective $\mathbb{F}_3$-line, and $G = \text{PGL}_n$ to be rationally isotropic over $\mathcal{O}_S$: for example for $n = 2$, it is isomorphic to the special orthogonal group of the standard split $\mathcal{O}_S$-form $q_3(x_1, x_2, x_3) = x_1x_2 + x_3^2$. Then as $q_3$ is rationally isotropic over $\mathcal{O}_S$ (e.g., $q_3(1, 2, 1) = 0$) and $\mathcal{O}_S$ is a UFD, according to Corollary 4.4 the Hasse-principle holds for $G$ and there are two genera as $|F| = |S| = 2$ (see Cor. 3.2).

Example 4.6. Let $(V, q)$ be an $\mathcal{O}_S$-regular quadratic form of even rank $n = 2k \geq 4$ and let $G = \text{Res}_{R/O_S}(\text{SO}_q)$ where $R$ is finite étale over $\mathcal{O}_S$. Then $F = \text{Res}_{R/O_S}(\mu_2)$, whence according to Corollary 3.2, $\text{gen}(G) \cong \text{Br}(R)[2]$. As $G$ and its twisted $K$-forms are $K$-isotropic (e.g., [28, p. 352]), each genus of $q$ contains exactly $\text{Pic}(R)/2$ elements.

Example 4.7. Let $C'$ be an elliptic $\mathbb{F}_q$-curve and $(C')^{af} := C' - \{\infty'\}$. Then $R := \mathbb{F}_q[(C')^{af}]$ is a quadratic extension of $\mathcal{O}(\infty) = \mathbb{F}_q[x]$ where $\infty = (1/x)$ and $\infty'$ is the unique prime lying above $\infty$, thus $L := R \otimes_{\mathcal{O}(\infty)} K$ is imaginary over $K$. Let $G = \text{Res}_{R/O_S}(\text{PGL}_m)$, $m$ is odd and prime to $q$. Then $F = \text{Res}_{R/O_S}^{(1)}(\mu_m)$ is smooth, and $G$ is smooth and quasi-split as well as its generic fiber, thus is $K$-isotropic. By Remark 3.5 and sequence (2.13), we get (notice that $\mathcal{O}(\infty)$ is a PID and that $\text{Br}(R) = 1$):

$$\text{Cl}_S(G) = H^1_{\text{ét}}(\mathcal{O}(\infty), G) \cong H^2_{\text{ét}}(\mathcal{O}(\infty), F) \cong \ker(\text{Pic}(R)/m \to \text{Pic}(\mathcal{O}(\infty))/m) = \text{Pic}(R)/m.$$ 

Hence the Hasse-principle holds for $G$ if and only if $|\text{Pic}(R)| = |C'(\mathbb{F}_q)|$ is prime to $m$. 
5. The Tamagawa number of twisted groups

In this section we start with the generic fiber. Let $G$ be a semisimple group defined over a global field $K = \mathbb{F}_q(C)$ with fundamental group $F$. The Tamagawa number $\tau(G)$ of $G$ is defined as the covolume of the group $G(K)$ in the adelic group $G(\mathbb{A})$ (embedded diagonally as a discrete subgroup), with respect to the Tamagawa measure (see [33]). T. Ono has established in [27] a formula for the computation of $\tau(G)$ in case $K$ is an algebraic number field, which was later proved by Behrend and Dhillon in [3, Thm. 6.1] also in the function field case:

\begin{equation}
\tau(G) = \frac{|\hat{F}^g|}{|\mathfrak{III}^1(\hat{F})|}
\end{equation}

where $\hat{F} := \text{Hom}(F \otimes K^*, \mathbb{G}_m)$, $\mathfrak{g}$ is the absolute Galois group $\text{Gal}(K^*/K)$, and $\mathfrak{III}^1(\hat{F})$ is the first Shafarevitch–Tate group assigned to $\hat{F}$ over $K$. As a result, if $F$ is split, then $\tau(G) = |F|$. So our main innovation, based on the above results and the following ones, would be simplifying the computation of $\tau(G)$ in case $F$ is not split, as may occur when $G$ is a twisted group.

The following construction, as described in [7] and briefly revised here, expresses the global invariant $\tau(G)$ using some local data. Suppose $G$ is almost simple defined over the above $K = \mathbb{F}_q(C)$, not anisotropic of type $A$, such that $(|F|, \text{char}(K)) = 1$. We remove one arbitrary closed point $\infty$ from $C$ and refer as above to the integral domain $\mathcal{O}_S = \mathcal{O}_{\{\infty\}}$. At any prime $p \neq \infty$, we consider the Bruhat–Tits $\mathcal{O}_p$-model of $G_p$ corresponding to some special vertex in its associated building. Patching all these $\mathcal{O}_p$-models along the generic fiber results in an affine and smooth $\mathcal{O}_{\{\infty\}}$-model $G$ of $G$ (see [7, §5]). It may be locally disconnected only at places that ramify over a minimal splitting field $L$ of $G$ (cf. [9, 4.6.22]).

Denote $\mathbb{A}_{\infty} := \mathbb{A}_{\{\infty\}} = \hat{K}_\infty \times \prod_{p \neq \infty} \hat{O}_p \subset \mathbb{A}$. Then $G(\mathbb{A}_{\infty})G(K)$ is a normal subgroup of $G(\mathbb{A})$ (cf. [32, Thm. 3.2]). The set of places $\text{Ram}_G$ that ramify in $L$ is finite, thus by the Borel density theorem (e.g., [11, Thm. 2.4, Prop. 2.8]), $G(\mathcal{O}_{\{\infty\}} \cup \text{Ram}_G)$ is Zariski-dense in $\prod_{p \in \text{Ram}_G \setminus \{\infty\}} G_p$. This implies that $G(\mathbb{A}_{\infty})G(K) = G^0(\mathbb{A}_{\infty})G(K)$, where $G^0$ is the connected component of $G$.

Since all fibers of the natural epimorphism

$$\varphi : G(\mathbb{A})/G(K) \to G(\mathbb{A})/G(\mathbb{A}_{\infty})G(K)$$
are isomorphic to \( \ker(\varphi) = G(\mathbb{A}_\infty)G(K)/G(K) \), we get a bijection of measure spaces

\[
(5.2) \ G(\mathbb{A})/G(K) \cong \text{Im}(\varphi) \times \ker(\varphi)
\]

\[
= (G(\mathbb{A})/G(\mathbb{A}_\infty)G(K)) \times (G(\mathbb{A}_\infty)/G(\mathbb{A}_\infty) \cap G(K))
\]

\[
= (G^0(\mathbb{A})/G^0(\mathbb{A}_\infty)G(K)) \times (G^0(\mathbb{A}_\infty)/G(\mathbb{A}_\infty) \cap G(K))
\]

\[
\cong \text{Cl}_{\infty}(G^0) \times (G^0(\mathbb{A}_\infty)/G^0(\mathbb{A}_\infty) \cap G(K))
\]

in which the left factor cardinality is the finite index \( h_\infty(G) := h_{\{\infty\}}(G^0) \) (see Section 2), and in the right factor \( G^0(\mathbb{A}_\infty) \cap G(K) = G^0(\mathcal{O}_{\{\infty\}}) \). Due to the Weil conjecture stating that \( \tau(G^{\text{sc}}) = 1 \), as was recently proved in the function field case by Gaistgory and Lurie (see [23, (2.4)]), applying the Tamagawa measure \( \tau \) to the Weil conjecture stating that \( \chi \) where \( \chi \) is the (restricted) Euler–Poincaré characteristic (cf. Definition 2.11), \( i(E) \) and \( l(E) \) are as in Definitions 1.1 and 2.12, respectively, and the right factor is a local invariant.

**Theorem 5.1.** Let \( g_\infty = \text{Gal}(\hat{K}_\infty^*/K_\infty) \) be the Galois absolute group, \( F_\infty := \ker(G_\infty^{\text{sc}} \to G_\infty) \), \( E := \ker(G_\infty^{\text{sc}} \to G) \) whose order is prime to \( \text{char}(K) \), and \( F_{\infty} := \text{Hom}(F_\infty \otimes \hat{K}_\infty^*, \mathbb{G}_{m, K_\infty}) \). Then

\[
\tau(G) = h_\infty(G) \cdot \frac{t_\infty(G)}{j_\infty(G)},
\]

where \( t_\infty(G) = |\hat{F}_\infty^{g_\infty}| \) is the number of types in one orbit of a special vertex, in the Bruhat–Tits building associated to \( G_\infty(\hat{K}_\infty) \), and \( j_\infty(G) = h_1(E)/h_0(E) \).

We adopt Definition 1.2 of being admissible to \( F \), with a Galois extension \( L/K \) replacing \( R/\mathcal{O}_S \). If \( G \) is not of (absolute) type A and \( F \) is admissible, then due to the above results Theorem 5.1 can be reformulated involving the fundamental group data only:

**Theorem 5.2.** Let \( G \) be an almost-simple group not of (absolute) type A defined over \( K = \mathbb{F}_q(C) \) with an admissible fundamental group \( F \) whose order is prime to \( \text{char}(K) \). Then for any choice of a prime \( \infty \) of \( K \) one has:

\[
\tau(G) = \frac{\chi(\infty)(E)}{|i(E)|} \cdot \frac{|\hat{F}_\infty^{g_\infty}|}{|F_\infty^{g_\infty}|} = l(E) \cdot |\hat{F}_\infty^{g_\infty}|,
\]

where \( \chi(\infty)(E) \) is the (restricted) Euler–Poincaré characteristic (cf. Definition 2.11), \( i(E) \) and \( l(E) \) are as in Definitions 1.1 and 2.12, respectively, and the right factor is a local invariant.

**Proof.** If \( G \) is not of (absolute) type A, according to Corollary 3.4 all genera of \( G \) have the same cardinality. By Lemma 3.3 and Corollary 3.2 \( (E) \) is admissible as \( F \) is, see Remark 2.5) we then get

\[
h_\infty(G) = |\text{Cl}_{\infty}(G)| = \frac{|H^1_{\text{et}}(\mathcal{O}_{\{\infty\}}, G)|}{|\text{gen}(G)|} = \frac{h_2(E)}{|i(E)|}.
\]
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Now the first asserted equality follows from Theorem 5.1 together with Definition 2.11:

\[ \tau(G) = \frac{1}{j_\infty(G)} \cdot h_\infty(G) \cdot t_\infty(G) = \frac{h_0(F)}{h_1(F)} \cdot \frac{h_2(E)}{|i(E)|} \cdot |F_\infty^{g_\infty}| = \frac{\chi_\infty(E)}{|i(E)|} \cdot |F_\infty^{g_\infty}|. \]

The rest is Lemma 2.13. \( \square \)

**Remark 5.3.** By the geometric version of Čebotarev’s density theorem (see in [20]), there exists a closed point \( \infty \) on \( C \) at which \( G_\infty \) is split. We shall call such a point a splitting point of \( G \).

**Corollary 5.4.** Let \( G \) be an adjoint group defined over \( K = \mathbb{F}_q(C) \) with fundamental group \( F \) whose order is prime to \( \text{char}(K) \) and whose splitting field is \( L \). Choose some splitting point \( \infty \) of \( G \) on \( C \) and let \( R \) be a minimal \( \acute{e} \text{tale} \) extension of \( \mathcal{O}_\{\infty\} := \mathbb{F}_q[C - \{\infty\}] \) such that \( R \otimes_{\mathcal{O}_\{\infty\}} K = L \). Let

\[ N^{(0)} : R^\times \to \mathcal{O}_\{\infty\}^\times \] be the induced norm. Then:

1. If \( G \) is of type \( 2D_{2k} \) then \( \tau(G) = \frac{|R^\times[2]|}{|R^\times: (R^\times)^2|} \cdot |F| \).
2. If \( G \) is of type \( 3,6D_4 \) or \( 2E_6 \) then \( \tau(G) = \frac{|\ker(N^{(0)}[m])|}{|\ker(N^{(0)})/m|} \cdot |F| \) (see Notation 2.6).

In both cases if \( L \) is imaginary over \( K \), then \( \tau(G) = |F| \).

**Proof.** All groups under consideration are almost simple. When \( G \) is adjoint of type \( 2D_{2k} \) then \( F \) is quasi-split, and when it is adjoint both of type \( 3,6D_4 \) or \( 2E_6 \) then \( F = \text{Res}_{L/K}^{(1)}(\mu_m) \) where \( m \) is prime to \( [L : K] \) (e.g., [28, p. 333]), thus \( F \) is admissible. So the assertions (1), (2) are just Theorem 5.2 in which as \( F_\infty \) splits, \( |F_\infty^{g_\infty}| = |F_\infty| = |F| \).

As \( C \) is projective, removing a single point \( \infty \) from it implies that \( \mathcal{O}_\{\infty\}^\times = \mathbb{F}_q^\times \) (an element of \( \mathcal{O}_\{\infty\} \) is regular at \( \infty^{-1} \), thus its inverse is irregular there, hence not invertible in \( \mathcal{O}_\{\infty\} \), unless it is a unit). If \( L \) is imaginary, then in particular \( R = \mathbb{F}_q[C' - \{\infty'\}] \) where \( C' \) is a finite \( \acute{e} \text{tale} \) cover of \( C \) and \( \infty' \) is the unique point lying over \( \infty \), thus still \( R^\times = \mathbb{F}_q^\times \) being finite, whence \( |R^\times[2]| = |R^\times: (R^\times)^2| \). In the cases \( F \) is not quasi-split the equality \( R^\times = \mathcal{O}_\{\infty\}^\times = \mathbb{F}_q^\times \) means that \( N^{(0)} \) is trivial, and we are done. \( \square \)

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