The complete $\mathcal{N} = 3$ Kaluza Klein spectrum of 11D supergravity on $AdS_4 \times N^{010}$

Piet Termonia

Dipartimento di Fisica Teorica, Università di Torino, via P. Giuria 1, I-10125 Torino, Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Torino, Italy

Abstract

We derive the invariant operators of the zero-form, the one-form, the two-form and the spinor from which the mass spectrum of Kaluza Klein of eleven-dimensional supergravity on $AdS_4 \times N^{010}$ can be derived by means of harmonic analysis. We calculate their eigenvalues for all representations of $SU(3) \times SO(3)$. We show that the information contained in these operators is sufficient to reconstruct the complete $\mathcal{N} = 3$ supersymmetry content of the compactified theory. We find the $\mathcal{N} = 3$ massless graviton multiplet, the Betti multiplet and the $SU(3)$ Killing vector multiplet.

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1 Introduction

In order to challenge Maldacena’s conjecture [1] in all kinds of circumstances there is a strong need for solved gauge theories on AdS manifolds. Indeed, they provide one of the two comparison terms in the anti-de Sitter/conformal field theory correspondence [2]. A good deal of information to be checked lies already in the mass spectrum and the supersymmetry multiplet structure of such theories.

A class of eleven-dimensional supergravity [3] solutions which may serve as compactified supersymmetric vacuum backgrounds is given by the Freund-Rubin solutions [4]. They have the form $AdS_4 \times M$, where $M$ is a compact seven-dimensional Einstein manifold. A particular convenient number of manifolds to use for such Kaluza Klein compactifications are the $G/H$ coset spaces. Part of their attraction comes from the fact that it is immediate to read off their isometry groups. Consequently one knows the gauge symmetries of the fundamental particle interactions. Moreover, it is clear how to calculate the mass spectrum of the four-dimensional theory. One can use harmonic analysis. As was shown in [5, 6] and in a series of papers [7], this can be done by exhibiting the group structure of the constituents of the coset rather than laboriously solving the differential equations. Hence the calculation of the mass spectrum gets reduced to a group-theoretical investigation of the coset and the final masses are found upon calculating the eigenvalues of some discrete operators. This is food for computers. In this paper we outline the necessary steps to be performed to reach this goal. Even in the compactifications where the calculations become too gigantic the masses can still successfully be calculated in this way.

The complete list of the seven-dimensional coset spaces that may serve to compactify eleven-dimensional supergravity to four dimensions is known [8]. Also the number of the remaining supersymmetries and their geometries have been intensively studied in the past. It turns out to contain some non-extremal supersymmetric cases that present themselves as promising candidates for the anti-de Sitter/conformal field theory check. This as opposed to Kaluza Klein on the seven-sphere which is related to the extremal $\mathcal{N} = 8$ supergravity theory [9]. There the spectrum can be derived from the short unitary irreducible representation of $Osp(8|4)$ with highest spin two, see [10]. From the perspective of the three-dimensional superconformal theory this means that all the composite primary operators have conformal weight equal to their naive dimensions. Hence no anomalous dimensions are generated. One of such non-trivial compactifications is the one on $AdS \times M_{111}$, where $M_{111}$ is one of Witten’s $M^{pqr}$ spaces [11, 7]. For this manifold the complete spectrum has been calculated and arranged in $\mathcal{N} = 2$ multiplets, see [12] and [13]. This spectrum provides some ideal material for the anti-de Sitter/conformal field theory check which has already successfully been done [14].

In many cases of $G/H$ compactifications the isometry group can be read off directly, being the group $G$. If that is true, then the supersymmetry group is simply the supergroup $Osp(4|\mathcal{N}) \times G'$, where $G = G' \otimes SO(\mathcal{N})$ and $\mathcal{N}$ is determined by the number of Killing spinors that are allowed on the manifold. Yet this becomes slightly more complicated when the normalizer $N$ of $H$ is non zero. Then the isometry group becomes $G \times N$ in stead of $G$. This is for instance the case for eleven-dimensional supergravity on a background of

$$AdS_4 \times N^{010},$$

(1)
\[ N^{010} \equiv \frac{SU(3)}{U(1)}. \] (2)

which was introduced in the paper [15]. It has been proven to yield a \( \mathcal{N} = 3 \) background where the \( SO(3) \) R-symmetry group is the normalizer of the \( U(1) \) in the denominator. Here the identification of the resulting \( SO(3) \) multiplet spectrum is not a straightforward exercise, as can be seen in [16] if one takes the description (2) for the manifold. Still as L. Castellani and L. J. Romans showed in [15], one can elegantly circumvent this difficulty by taking a description which has the normalizer taken into account from the very start. In particular, as they suggested in [13, 16], one can exhibit the fact that the \( \mathcal{N} = 3 \) supersymmetric \( N^{010} \) is equal to

\[ \frac{SU(3) \times SU(2)}{SU(2) \times U(1)} \]

where the \( SU(2) \) in the denominator is diagonally embedded in \( SU(3) \times SU(2) \). In this way the \( SO(3) \approx SU(2) \) is everywhere explicitly present.

The harmonic analysis on \( N^{010} \) has been done partly by Castellani in [16], using the formulation (2). Yet due to lack of computing power his calculation did not cover the complete spectrum. It is the scope of this paper to extend the technique of harmonic analysis to the alternative formulation (3) and derive really the complete spectrum. This choice is motivated by the fact that the field components will then automatically be organized in irreducible representations of \( SO(3) \). This will make the matching of the spectrum with \( \mathcal{N} = 3 \) multiplets possible. We present here a sufficient part of the mass operator eigenvalues that contains enough information to reconstruct the complete spectrum by using the supersymmetry mass relations [7]. This will be used in a subsequent paper [17] to derive the complete multiplet structure of the \( \mathcal{N} = 3 \) \( N^{010} \) Freund-Rubin compactifications.

The importance of the spectrum of eleven-dimensional supergravity on (1) lies in the fact that an anti-de Sitter/conformal field theory comparison in this background is fairly facilitated by the fact that for the four-dimensional theory the structure of the effective Lagrangian is already known to be [18],

\[ \frac{SU(3,8)}{SU(3) \times SU(8) \times U(1)}. \] (4)

Still, as becomes clear from the work presented in this paper, the mass spectrum of the \( \mathcal{N} = 3 \) theory is going to be enough non trivial to provide another powerful check of the anti-de Sitter/conformal field theory duality.

The spectrum that we obtain here has to be organized in \( \mathcal{N} = 3 \) multiplets. A systematic classification of all the possible short \( \mathcal{N} = 3 \) multiplets and their superspace structure does not exist in the literature. Only the short vector multiplet in table [1 has been found explicitly in [19]. As in the case of the \( AdS \times M^{11} \) Kaluza Klein compactification the matching of the mass spectrum will teach us a lot if not everything about the existence of other short multiplets. Following the arguments of [14] it turns out to be necessary to

\footnote{Actually there are many different manifolds of this form due to the present freedom in the choice of the vielbeins on this manifolds. However, as shown in [12, 16] there is a particular choice for which \( \mathcal{N} = 3 \).}
Table 1: The states of the $Osp(3\mid 4)$ vector multiplet representation organized in representations of $SO(2) \times SU(2)$ spin and $SO(3)$ isospin as they appear in Kaluza Klein of eleven-dimensional supergravity. The unitary bound for the ground state is satisfied $E_0 = J$. The names of the fields are chosen as in [7].

decompose the resulting $\mathcal{N} = 3$ multiplets in $\mathcal{N} = 2$ multiplets. Upon doing so the states of the resulting $\mathcal{N} = 3$ multiplets can then be recognized as coming from the on-shell field components of the superfields on the $\mathcal{N} = 2$ superspace that was introduced in [20]. Their superfield constraints are then straightforwardly read off. We postpone this issue to a future publication.

The work that we present here fits in a much wider project that is currently being carried out by the Torino group. The final scope of this $AdS \times N^{010}$ spectrum is to check the anti-de Sitter/conformal field theory correspondence as it has been carried out [14] for $AdS \times M_{11}$. It will be done [21] also for eleven-dimensional supergravity on $AdS \times Q^{ppp}$.

This paper is organized as follows. We start with a short description of the geometry of $N^{010}$. We thereby restrict ourselves to the essentials that we will need further on. We leave a more rigorous treatment to a future publication [22]. Then we will repeat the standard concepts of harmonic analysis and explain how they can be applied to eleven-dimensional supergravity on $AdS_4 \times N^{010}$. Using these techniques we will then compute the zero-form operator $M^{(0)}_3$, the one-form operator $M^{(0)}_2(1)$, the two-form operator $M^{(0)}_1(1)^2$ and the spinor operator $M^{(1/2)}_3$, the notation of these operators being the same as in [3, 4]. We will then present their eigenvalues. We conclude by arguing that the information obtained in this paper is sufficient to calculate the complete spectrum. We will do so by means of a concrete example: the massless graviton multiplet. Moreover, in this way we will prove that the remaining spectrum is indeed $\mathcal{N} = 3$ supersymmetric. Finally, we will identify the series of irreducible representations of $SU(3) \times SU(2)$ with a common field content. For these series we will list the eigenvalues that are present. From these eigenvalues the masses are then obtained by applying the mass formulas from [7].
2 The geometry of $N^{010}$

In this section we introduce the essential concepts of the geometry of $N^{010}$. We restrict ourselves to the elements that serve our purposes. We leave a more elaborated treatment to [22]. We fix some freedom in the choice of the vielbeins. This will ensure that the compactified theory on (1) will have $\mathcal{N} = 3$ supersymmetry.

The $N^{010}$ coset spaces are special manifolds of the class of $N^{pqr}$ coset spaces, $G/H = SU(3) \times U(1)/U(1) \times U(1)$ (5), where the integer numbers $p, q, r$ specify the way in which the two $U(1)$ generators $M$ and $N$ of $H$ are embedded in $G$,

\[
M = -\frac{\sqrt{2}}{RQ} \left( i^3 rp^2 + i^3 rq\lambda_3 - \frac{i}{2}(3p^2 + q^2)Y \right),
\]

\[
N = -\frac{1}{Q} \left( -\frac{i}{2} q\lambda_8 + i^2 p\lambda_3 \right),
\]

with

\[
R = \sqrt{3p^2 + q^2 + 2r^2}, \quad Q = \sqrt{3p^2 + q^2},
\]

where $Y$ is the $U(1)$ generator in $SU(3) \times U(1)$ and the $SU(3)$ generators $\lambda$ are given in appendix [3]. We do not get into details in this paper but for a detailed description of the definition, the geometry and the properties of these spaces we refer the reader to the literature [15, 16, 22]. These spaces are seven dimensional and can be used to make a Freund-Rubin background for eleven-dimensional supergravity,

\[
AdS \times N^{010}.
\]

They are solutions of the field equations of eleven-dimensional supergravity.

In this paper we restrict our attention to the case of $N^{010}$. It has been shown that among these spaces there exists a class of cosets,

\[
\frac{SU(3)}{U(1)}
\]

for $p = 0, r = 0$ which has $SU(3)$ holonomy and yields an $\mathcal{N} = 3$ supergravity. In fact, without loss of generality one can put $q = 1$ in this case. The normalizer of $U(1)$ in $SU(3)$ is $U(1) \times SO(3)$. So the actual isometry is $SU(3) \times SO(3)$ and the resulting isometry supergroup of the $AdS_4 \times N^{010}$ becomes $Osp(4|3) \times SU(3)$. So the extra $SO(3)$ in the normalizer becomes the R symmetry of the supergroup.

In this paper we derive the mass spectrum of the Kaluza Klein theory on $AdS_4 \times N^{010}$ and to this end it is more convenient to use the alternative description which was proposed by Castellani and Romans, where the $SO(3) \approx SU(2)$ is already present form the start. Hence we use

\[
N^{010} = \frac{SU(3) \times SU(2)}{SU(2) \times U(1)}
\]
For the generators of $SU(3)$ we take $\frac{i}{2}\lambda$ and for $SU(2)$ we take $\frac{i}{2}\sigma$, $\lambda$ being the Gell-Mann matrices (see appendix B) and $\sigma$ being the Pauli matrices. The generator of $U(1)$ is given by

$$T_8 = \frac{i}{2}\lambda_8.$$  \hfill (11)

The $SU(2)$ is diagonally embedded in $SU(3) \times SU(2)$ with generators,

$$T_H = \frac{i}{2}(\lambda_a + \sigma_a), \quad a = 1, 2, 3, \quad H = 9, 10, 11.$$  \hfill (12)

where $H = 9$ corresponds to $a = 1$ and so on. We will call it $SU(2)^{\text{diag}}$. For the remaining coset generators we have

$$T_\alpha = \frac{i}{2}(\lambda_1 - \sigma_1, \lambda_2 - \sigma_2, \lambda_3 - \sigma_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$$  \hfill (13)

We need the $SO(7)$ covariant derivative on the coset space. It is defined as

$$\mathcal{D} = d + B^{\alpha\beta} (T^{SO(7)})_{\alpha\beta}$$  \hfill (14)

where the one-form $B^{\alpha\beta}$ is the connection of the coset space and $T^{SO(7)}$ are the generators of $SO(7)$, be it of the vector representation or the spinor representation.

| irrep          | $(T^{SO(7)})$       |
|----------------|---------------------|
| scalar irrep   | $(T^{SO(7)}) = 0$   |
| vector irrep   | $(T^{SO(7)}_{\alpha\beta})^\gamma_\delta = \eta_{\gamma\epsilon}(\delta^{\epsilon}_{[\alpha\delta]})$ |
| spinor irrep   | $(T^{SO(7)}_{\alpha\beta}) = \frac{1}{4}\tau_{[\alpha\beta]}$ |

where the matrices $\tau$ of the $SO(7)$ Clifford algebra are given in the appendix B and $\eta$ is the metric of appendix B.

As already explained, there is some freedom in the choice of the vielbeins, not all of them leading to $\mathcal{N} = 3$.

To see how this goes, let us recall that on the coset there is the invariant

$$L^{-1}dL = e^\alpha T_\alpha + \omega^H T_H,$$  \hfill (16)

where $L$ is coset representative. $H$ is the index running on $H$ and $\alpha$ is the index running on $SO(7)$, see appendix B for conventions. The fields $e^\alpha$ and $\omega^H$ are the $G/H$ vielbein and the $H$ connection. Using the coset vielbein $e^\alpha$ we define the coset connection $B^{\alpha\beta}$,

$$de^\alpha + B^{\alpha\beta} \wedge e^\beta = 0.$$  \hfill (17)

The vielbein is specified up to 7 rescalings of the coset directions, see B, I, L. We take this into account by introducing the seven parameters $r_\alpha$

$$e^\alpha \rightarrow r_\alpha e^\alpha.$$  \hfill (18)
Then the form of the connection is obtained by solving the Maurer-Cartan equation, i.e. applying the external derivative $d$ to (16) and using $d^2 = 0$,
\[
B^\alpha_\beta = -\frac{1}{2} \left( \frac{r_\beta r_\gamma}{r_\alpha} C^\gamma_\alpha - \frac{r_\alpha r_\gamma}{r_\beta} \eta^{\alpha_\phi} \eta_{\phi\beta} C_{\gamma\phi} - \frac{r_\alpha r_\gamma}{r_\beta} \eta^{\alpha_\phi} \eta_{\gamma\beta} C_{\phi\gamma} \right) \eta^{\gamma} - \frac{r_\beta}{r_\alpha} C_{\beta H}^\alpha \omega^H
\] (19)
where the constants $C_{\alpha\beta\gamma}$ are the structure constants of $G$. We take the embedding of $H$ into $SO(7)$ as follows,
\[
T_H = (T_H)^{\alpha\beta} (T^{SO(7)})_{\alpha\beta}.
\] (20)

Now we specify the rescalings $r_\alpha$. As is known [7], only for some well-chosen values of these parameters does the manifold become an Einstein manifold. Moreover, as is clear from [15, 16] not all of the valuable choices for these rescalings necessarily lead to the contemplated number of supersymmetries. To see which of them do, we look at the curvature. The curvature on the coset space is defined by
\[
R^\alpha_\beta = d B^\alpha_\beta + B^\alpha_\gamma \wedge B^\gamma_\beta.
\] (21)

Then we need the following rescalings\(^3\),
\[
\begin{align*}
  r_a &= -2e, \\
  r_A &= 4\sqrt{2}e
\end{align*}
\] (24)
in order to have Ricci curvature
\[
R_{\alpha\beta} = 12 e^2 \eta_{\alpha\beta}.
\] (25)

Following the conventions of the papers [4] we put $e = 1$. Then we will be able to apply the mass relations and mass formulas that were obtained in these papers.

To conclude this section we give the explicit form of the embedding of the $SU(2)_{\text{diag}}$ in $SO(7)$ according to eq. (20). For the vector we get
\[
T_9 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0
\end{pmatrix}
\]

\(^3\) In fact there is the more general class of rescalings
\[
\begin{align*}
  r_a &= \pm 2e, \\
  r_A &= \pm 4\sqrt{2}e
\end{align*}
\] (22)
or
\[
\begin{align*}
  r_a &= \pm \frac{10}{3}e, \\
  r_A &= \pm \frac{4\sqrt{2}}{3}e
\end{align*}
\] (23)
that yield the Einstein curvature (24). However not all of them necessarily lead to an $\mathcal{N} = 3$ theory. As we will show in the last section of this paper, the rescaling (24) that we adopt yields $\mathcal{N} = 3$ supersymmetry. We refer the reader for a detailed study of these rescalings to a future publication [22].
For the matrices $T_{\alpha \beta}$ the order of the indices is $1, 2, 3, 4, 5, 6, 7$. For the spinor representation we have
\[ T_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix} \] (28)

\[ T_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{2} \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2} \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \] (29)

3 Harmonic analysis

In this section we provide the main ingredients of harmonic analysis that we need for the calculation of the masses of the zero-form, the one-form, the two-form and and the spinor. The technique of harmonic analysis has been well established, see papers [6, 7, 16], so we restrict ourselves to the essentials. We focus on its application to our coset \( N^{010} \).

The main idea is that functions on a coset space \( G/H \) can be expanded in terms of the components of the operators of the irreducible representations of \( G \). In particular, a complete set of functions \( \Phi^{(\Lambda)}(L(y)) \) on the coset \( G/H \) is given that transform in an irreducible representation of \( H \) as follows,

\[ \Phi^{(\alpha)}(L(y)h) = \Phi^{(\mu)}(L(y)) (D^{(\mu)})^{\Pi}_{\Lambda}, \] (30)

where \( (\alpha) \) indicates the \( H \)-irreps and the \( \Lambda \) its indices. These functions depend on the coordinates of the coset space via the coset representative \( L(y) \). Then for such functions the expansion is given by,

\[ \Phi^{(\alpha)}(L(y)) = \sum_{(\mu)} c_{G}^{(\mu)} (D^{(\mu)})^{G}_{\Lambda}(L(y)), \] (31)

where \( G \) are the contracted indices of \( G \) and \( (\mu) \) label all the irreducible representations of \( G \) that contain the irreducible representation \( (\alpha) \) under reduction to \( H \) as follows\footnote{In fact \( (\alpha) \) may appear multiple times in the reduction, say \( m \) times. In that case it should be understood that the above sum contains \( (\mu) \) \( m \) times. For a more clear treatment of this see [16]}

\[ (\mu) \rightarrow \ldots + (\alpha) + \ldots, \] (32)

The crucial step now is to use the fact that the covariant derivative \( (14) \) can be expressed in terms of the coset generators plus some additional discrete operators. Indeed,
there is no need to express this covariant derivative as a differential operator. This is a generic feature of coset manifolds [6, 7, 16]. It is extremely useful for our purposes since evaluating the mass terms in the field equation of eleven-dimensional supergravity can then be done without solving differential equations.

To see this, it is sufficient to realize that the harmonic expansion ultimately is an expansion in $L(y)^{-1}$. Then one uses (16) to express the derivatives in terms of the vielbein. Then on a field $Y$ on the coset space that sits in some representation of $SO(7)$ the covariant derivative becomes

$$D_\alpha Y = -r_\alpha T_\alpha Y + \frac{r_\alpha r_\beta}{r_\gamma} C_{\alpha \beta \gamma} (T^{SO(7)})_\gamma^\gamma Y + \frac{1}{2} \eta_{\alpha \delta} \frac{r_\beta r_\gamma}{r_\delta} C_{\beta \gamma \delta} (T^{SO(7)})_\delta^\gamma Y \quad (33)$$

We now show how this reduction is done in the case where the coset is $N^{010}$. In particular we show how the representations of $SU(3) \times SU(2)$ reduce to representations of $SU(2)^{diag} \times U(1)$. We also give the constraint that is to be imposed in order to ensure the right $U(1)$ weight of the harmonic.

Let us look at the index structure of the objects $(D^{(\mu)})_{G \Lambda}(L(y))$ in the expansion (31). Let us start with the indices $G$. Clearly, they have the following structure,

$$G = \begin{array}{cc}
  k_1 \ldots k_{M_2} & l_1 \ldots l_{M_1} \\
  n_1 \ldots n_{M_2} & \end{array} \otimes \begin{array}{c}
m_1 \ldots m_{2J}
\end{array} \quad (34)$$

being the product of a generic $SU(3)$ Young tableau with a generic $SU(2)$ Young tableau. The indices run through

$$k_1, \ldots, k_{M_2}, l_1, \ldots, l_{M_1}, n_1, \ldots, n_{M_2} = 1, 2, 3$$

$$m_1, \ldots, m_{2J} = 1, 2 \quad (35)$$

Let us now see whether the representation $(\mu)$ contains a given representation $(\alpha)$ of $SU(2)^{diag}$, to clarify (32). Hence we look at the indices $\Lambda$ in $(D^{(\mu)})_{G \Lambda}(L(y))$.

By making use of the symmetries of the Young tableaux one can, for fixed values, arrange the indices of $(\mu)$ as follows,

$$G = \begin{array}{cc}
  \underline{1} \begin{array}{c}
    k_1 \ldots k_{2p} \\
    2 \end{array} & l_1 \ldots l_{2q} \underline{3} \\
  \end{array} \otimes \begin{array}{c}
m_1 \ldots m_{2J}
\end{array} \quad (36)$$

where we use the following short-hand notation

$$i_1 \cdots i_p \equiv i_1 \cdots i_p \quad (37)$$

and,

$$\underline{1} \equiv \begin{array}{c}
  1 \ldots 1
\end{array} , \quad \underline{2} \equiv \begin{array}{c}
  2 \ldots 2
\end{array} , \quad \underline{3} \equiv \begin{array}{c}
  3 \ldots 3
\end{array} . \quad (38)$$
We will use this notation henceforth. The indices $k_1, \ldots, k_{2p}, l_1, \ldots, l_{2q}$ in the above Young tableau now get the values 1, 2 only. The parameters $p$ and $q$ get half-integer values. This yields a

$$p \otimes q \otimes J$$

$SU(2)^{diag}$-representation in the reduction (32). From this representation one can extract the irreducible representations by contracting pairs of the indices $k, l, m$ with the $\epsilon$-symbol.

Note however that it is not necessary to contract pairs of a $k$ with an $l$ since then one would be over counting, as can be seen upon using the cyclic identity,

$$\begin{vmatrix} i & j \\ 3 & 2 \end{vmatrix} = \frac{1}{2} \epsilon_{ij} \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}$$

Thus in the Young tableau (36) one takes only those $SU(2)^{diag}$-irrepses that are obtained by contracting pairs of $k$’s and $m$’s and pairs of $l$’s and $m$’s. The remaining indices are then completely symmetrized.

We specify the most generic class of Young tableaux that we will need for the harmonic analysis on $N^{010}$ and introduce the following notation for it,\(^5\)

$$\Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}, m_1 \ldots m_{2u}} (M_1, M_2, J; p)$$

\[\epsilon_{IJ} \equiv \epsilon_{i_1 j_1} \ldots \epsilon_{i_2 j_2} \]

where we have indicated the number of boxes under the Young tableaux. In the above notation for the Young tableaux (42) we do not a priori assume symmetrization of the indices $k_1 \ldots k_{2s}, l_1 \ldots l_{2t}$, although that is what we need in the harmonic expansion. It will be useful for our purposes to consider the more general Young tableaux of (42). It is also clear that in this notation,

$$2s \leq 2p \leq M_2, \quad 2t \leq 2(J - u - p + s + t) \leq M_1,$$

is assumed without notice. Otherwise the Young tableaux simply do not exist. The only free $SU(2)^{diag}$-indices in the above Young tableaux are the indices $k_1 \ldots k_{2s}, l_1 \ldots l_{2t}$ and

\[\otimes \begin{vmatrix} j_1 & \cdots & j_2 (J - u) \\ 2J & m_1 \ldots m_{2u} \end{vmatrix} \]

\[\begin{array}{c} \otimes \begin{vmatrix} j_1 & \cdots & j_2 (J - u) \\ 2J & m_1 \ldots m_{2u} \end{vmatrix} \\ M_2 \\ M_1 \end{array} \]

We use the notation,
An example where this is not possible is the $0$ in an $SU(2)$, then one might make the harmonic expansion in components of the form $\Gamma_{M,2}$ and $m$ components.

However, there is a subset of embeddings where one can restrict oneself to the following $p(U)$ definition. When we will refer to the $SU(2)$ weight, we will refer to the number $\Delta$ henceforth. It is important to realize here that this constraint not only constraints $p$ in terms of $M_1, M_2, J$ but it also implies that the difference $M_2 - M_1$ has to be a multiple of three. Moreover as we will see later on $J$ has to be a positive integer. The parameter $p$ is half-integer.

Now we know the embedding of the representations of $SU(2)^{diag}$ in $SU(3) \times SU(2)$ with a given $SU(2)^{diag}$-spin and $U(1)$-weight.

In general all these Young tableaux are not independent. This can easily be seen using the cyclic identity,

$$\Gamma^{(s,t,u)}_{k_1...k_2s,j_1...j_2t,m_1...m_{2u}} (M_1, M_2, J; p) = \Gamma^{(s,t+1/2,u-1/2)}_{k_1...k_2s,j_1...j_2t,m_2u,m_{1u-1}} (M_1, M_2, J; p + \frac{1}{2})$$

$$- \Gamma^{(s+1/2,t,u-1/2)}_{k_1...k_2s,m_2u,k_1...j_2t,m_{1u-1}} (M_1, M_2, J; p + \frac{1}{2})$$

which is merely a generalization of the identity (45). This identity allows us to reduce the number of Young tableaux that we are using in most of the cases. For instance, if we are considering $SU(3)$ irreducible representations that are big enough, i.e. the values $M_1$ and $M_2$ exceed the numbers,

$$M_2 \geq 2p + 1, \quad M_1 \geq 2(J - u - p + s + t) + 1$$

then one might make the harmonic expansion in components of the form $\Gamma^{(s',t',u-1/2)}$ only. An example where this is not possible is the $0 \otimes 0 \otimes 1$ representation of $SU(2)^{diag}$ embedded in an $SU(3) \times SU(2)$ with

$$M_1 = M_2 = 0, \quad J = 1.$$  

However, there is a subset of embeddings where one can restrict oneself to the following components,
\[
\Gamma_{k_1...k_{2s},l_1...l_{2t}}^{(s,t)} (M_1, M_2, J; p) = \epsilon_{IJ}^{s,t} \left[ \begin{array}{c}
\begin{array}{c}
 \frac{1}{2} k_1 \ldots k_{2s} \\
 3 \ldots 3
\end{array}
\end{array} \right]_{M_2} \left[ \begin{array}{c}
\begin{array}{c}
 i_1 \cdots i_{2(p-s)} \\
 3 \ldots 3
\end{array}
\end{array} \right]_{2s} \left[ \begin{array}{c}
\begin{array}{c}
 i_2(p-s) + 1 \cdots i_{2t} \\
 2(J-p+s) \ldots 2t
\end{array}
\end{array} \right]_{M_1} \left[ \begin{array}{c}
\begin{array}{c}
 l_1 \ldots l_{2t} \\
 3 \ldots 3
\end{array}
\end{array} \right]_{2J} \otimes \left[ \begin{array}{c}
\begin{array}{c}
 j_1 \cdots j_{2t}
\end{array}
\end{array} \right]
\]
\]

where \( u = 0 \) is understood in the notation of \( \Gamma \). In this paper, for the calculation of the eigenvalues of the mass operators we will always assume that \( M_1 \) and \( M_2 \) are “big enough” in order to justify an expansion in the components (52) only. In other words, here we calculate the mass matrix for those \( SU(3) \times SU(2) \) representations where the harmonics sit in the \( s \oplus t \) irreducible representation of the decomposition of the \( s \)-spin times the \( t \)-spin. Afterwards we will argue the eigenvalues thus obtained, seen as functions of the labels \( M_1, M_2, J, \) are also the functions for the eigenvalues to be found in the cases where this assumption fails. To illustrate this we will work out (51). Yet a detailed study of the existing eigenvalues of the mass operators in the cases of short multiplets will become quite a subtle book-keeping exercise. Fortunately it can be implemented in some computer programs. All this will be explained in the last section of this paper.

We now apply this to Kaluza Klein on

\[ AdS_4 \times N^{010} . \] (53)

We write the coordinates of \( AdS_4 \) as \( x \) and the coordinates of \( N^{010} \) as \( y \). A field \( \Phi \) on (53) sits in a representation of \( SO(1,3) \) as well as in a representation of \( SO(7) \), generically being some multiple tensor product of the vector representation and the spinor representation. As will be exemplified further on, these \( SO(7) \) representations decompose in representations \( (\alpha) \) of \( H \),

\[ SO(7) \rightarrow (\alpha_1) \oplus \ldots \oplus (\alpha_n), \] (54)

hence the fields decompose in \( SU(2)^{\text{diag}} \) fragments,

\[ \Phi \rightarrow \left( \begin{array}{c}
\Phi^{(\alpha_1)} \\
\vdots \\
\Phi^{(\alpha_n)}
\end{array} \right) \] (55)

Then, using eq. (51) the fragments \( \Phi^{(\alpha_n)} \) can be expanded as

\[ \Phi^{(\alpha)}_{\Lambda}(x, y) = \sum_{(\mu)} \Phi(x)^{(\mu)} \cdot (D^{(\mu)})_{\Lambda}(y), \] (56)

The dot indicates the contraction of the index \( G \) and

\[ \Lambda = \Gamma_{(k_1...k_s,l_1...l_t)}^{(s,t)} (M_1, M_2, J; p) , \] (57)

which is symmetrized in all indices \( k_1 \ldots k_s, l_1 \ldots l_t \). The linearized field equations on the fragments of the 11-dimensional theory split as follows,

\[ (\Box_x + \Box_y) \Phi^{(\alpha)}_{\Lambda}(x, y) = 0 \] (58)
then $\Box_g \left( \gamma^{(\mu)} \right)_{\Lambda(y)}$ can be evaluated explicitly and provide the mass operators of the four-dimensional theory. The evaluation of these mass matrices and their eigenvalues is the subject of the following sections.

To see how the covariant derivative (33) works on the components of the Young tableaux (12), one first straightforwardly derives the following formulae,

$$\left( \lambda - \tau \right) \alpha \Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s,t)}(p) = \sum_{\mu = 1}^{2s} (\lambda^\alpha)^{m}_\mu \Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s-1/2, t)}(p - \frac{1}{2}) + \sum_{\nu = 1}^{2t} (\lambda^\alpha)^{n}_\nu \Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s,t-1/2)}(p),$$

and

$$\lambda^{\alpha} \Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s,t)}(p) = -(M_2 - 2p) \epsilon^{mn} (\lambda^A)_m^3 \Gamma_{nk_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s+1/2, t)}(p + \frac{1}{2}) + \sum_{\mu = 1}^{2s} \epsilon_{\mu mn} (\lambda^A)_3^m \Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s-1/2, t)}(p - \frac{1}{2}) + 2(s - p)(\lambda^A)_3^m \left( \Gamma_{nk_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s+1/2, t)}(p) - \Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s,t+1/2)}(p + \frac{1}{2}) \right)$$

$$- (M_1 - 2J + 2p - 2s - 2t)(\lambda^A)_3^m \Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s,t+1/2)}(p) + \sum_{\nu = 1}^{2t} \epsilon_{\nu mn} (\lambda^A)_3^m \Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s,t-1/2)}(p)$$

where $m_\mu$ means that $k_\mu$ is replaced by $m$ and $n_\nu$ means that $l_\nu$ is replaced by $n$. The hats on the indices $\hat{k}$ and $\hat{l}$ indicate that these indices are deleted. We have suppressed the labels $M_1, M_2, J$, since they do not change under the transformations. In fact, they do not change under the covariant derivative and the evaluation of the mass operators can be done for fixed $M_1, M_2, J$ in the harmonic expansion (56). For the derivation of the above expressions we used the cyclic identity (19). Mind that in the Young tableaux $\Gamma_{k_1 \ldots k_{2s}, l_1 \ldots l_{2t}}^{(s,t)}(p)$ we have not symmetrized the indices $k_1 \ldots k_{2s}, l_1 \ldots l_{2t}$. Recall that in the notation (22) there is no symmetrization in these indices.

Looking at the expression for the covariant derivative in eq. (33), one sees that the generators $T_\alpha$ are precisely given by the above formulae (59) and (60) multiplied with $\frac{\sqrt{2}}{2}$. Moreover, these formulae can be programmed on a computer and in this way the covariant derivative is easily calculated.

4 The 0-form, the 1-form, the 2-form and the spinor

In the previous section we explained how harmonic analysis is done on the coset $N^{010} = SU(3) \times SU(2)/SU(2) \times U(1)$ when the $SU(2)$ is diagonally embedded in $SU(3)$. In this section we apply this to the fields of eleven-dimensional supergravity.
The fields of eleven-dimensional supergravity are expanded around the Freund-Rubin background. Their fluctuations are the metric $h_{MN}(x,y)$, the three-form $a_{MNP}(x,y)$ and the spinor $\psi_M(x,y)$. They can be expanded in the following harmonics [6, 7]: the scalar $Y(y)$, the transverse vector $Y^{\alpha}(y)$, the transverse 2-form $Y^{\alpha\beta}(y)$, the transverse 3-form $Y^{\alpha\beta\gamma}(y)$, the symmetric transverse traceless tensor $Y^{(\alpha\beta)}(y)$, the spinor $\Xi(y)$ and the irreducible vector spinor $\Xi_\alpha(y)$. We list them in table 2 together with their SO(7) irreducible representation and their invariant field equations written in terms of the covariant derivatives of the coset (33). The following fields are transverse,

$$\mathcal{D}^\alpha Y_\alpha = 0,$$
$$\mathcal{D}^\alpha Y_{[\alpha\beta]} = 0,$$
$$\mathcal{D}^\alpha Y_{[\alpha\beta\gamma]} = 0,$$
$$\mathcal{D}^\alpha Y^{(\alpha\beta)} = 0,$$
$$\mathcal{D}^\alpha \Xi_\alpha = 0.$$  (61)

Since we are dealing with $N = 3$ supersymmetry it will suffice to know the masses for the zero-form, the one-form, the two-form and the spinor only. The rest of the multiplet

| harmonic | SO(7)-irrep | field equation |
|----------|-------------|----------------|
| $Y$      | 1 = (0, 0, 0) | $\mathcal{D}^\alpha \mathcal{D}_\alpha Y = M_{(0)} Y$ |
| $Y^{\alpha}$ | 7 = (1, 0, 0) | $2\mathcal{D}^\alpha \mathcal{D}_{[\alpha} Y_{\beta]} = M_{(1)} Y^{\alpha}$ |
| $Y^{\alpha\beta}$ | 21 = (1, 1, 0) | $3\mathcal{D}^\gamma \mathcal{D}_{[\gamma} Y_{\alpha\beta]} = M_{(2)} Y^{\alpha\beta}$ |
| $Y^{\alpha\beta\gamma}$ | 35 = (1, 1, 1) | $\frac{1}{24} \varepsilon^{\alpha\beta\gamma}_{\mu\nu\rho\sigma} \mathcal{D}_\mu Y_{\nu\rho\sigma} = M_{(3)} Y^{\alpha\beta\gamma}$ |
| $Y^{(\alpha\beta)}$ | 27 = (2, 0, 0) | $\left( (\Box + 56) \delta^{\alpha\beta}_{\gamma\delta} - R^{\alpha\beta}_{\gamma\delta} \right) Y^{(\gamma\delta)} = M_{(4)} Y^{(\alpha\beta)}$ |
| $\Xi$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ | $\tau^\mu \left( \mathcal{D}_\mu - \tau_\mu \right) \Xi = M_{(5)} \Xi$ |
| $\Xi_\alpha$ | $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ | $\tau_\alpha^{\beta\gamma} \nabla_\beta \Xi_\gamma - \frac{5}{7} \tau_\alpha^{\beta\gamma} \nabla_\beta \Xi_\gamma = M_{(6)}^{\frac{1}{2}} \Xi_\alpha$ |

Table 2: The content of harmonics of eleven-dimensional supergravity: their name as in [7], their SO(7) irreducible representations and their field equations.
spectrum can then be determined using supersymmetry.

The decomposition of the $SO(7)$ vector representation into irreducible representations of $SU(2)^{\text{diag}}$ goes as follows

$$(0, 0, 0) \rightarrow (1, 0),$$

$$(1, 0, 0) \rightarrow (3, 0) \oplus (2, 1) \oplus (\bar{2}, -1),$$

$$(1, 1, 0) \rightarrow (4, 1) \oplus (\bar{4}, -1) \oplus (3, 0) \oplus (3, 0) \oplus (2, -1) \oplus (\bar{2}, 1) \oplus (1, 2) \oplus (\bar{1}, -2) \oplus (1, 0),$$

(62)

where the irreducible representations of $SU(2)^{\text{diag}} \times U(1)$ are written as $(2(s + t) + 1, \Delta)$, $\Delta$ being the $U(1)$ weight (47). One may check now that from the formula (48) it follows indeed that $J$ has to be an integer number. Accordingly, an $SO(7)$ one-form and an $SO(7)$ two-form can be expressed in terms of the following $SU(2)^{\text{diag}}$ fragments

$$\Phi^\alpha \rightarrow \begin{pmatrix} C^{(1)}_i \\ C^{(1)j} \\ R^{(1)}_{ij} \end{pmatrix},$$

$$\Phi^{\alpha\beta} \rightarrow \begin{pmatrix} R^{(2)} \\ C^{(2)} \\ C^{(2)s} \\ C^{(2)j} \\ C^{(2)ij} \\ R^{(2)}_{ij} \\ O^{(2)}_{ij} \\ C^{(2)jk} \\ C^{(2)ijk} \end{pmatrix},$$

(63)

where the fields $C^{i_1...i_n}$ are the complex conjugate fields of $C_{i_1...i_n}$ and the symmetrized indices indicate in what representation of $SU(2)^{\text{diag}}$ they sit. The superscript (1) and (2) indicate that these are the fragments of the one-form and the two-form respectively. The fields $R$ are pseudo real, i.e.

$$R^{i_1...i_n} = \epsilon^{i_1j_1} \ldots \epsilon^{i_nj_n} R_{j_1...j_n}. \quad (64)$$

For the choice of the coset generators (13) it is useful to introduce the indices (196) and split

$$\Phi^\alpha = \{ \Phi^\alpha, \Phi^A \},$$

$$\Phi^{\alpha\beta} = \{ \Phi^{ab}, \Phi^{aA}, \Phi^{AB} \}. \quad (65)$$

then one can do the decomposition (63) concretely by using the components of the $\lambda$ matrices,

$$\Phi = R^{(0)},$$

$$\Phi^a = i(\lambda^a)_i^j \epsilon^{ik} R_{jk}^{(1)}.$$
\[ \Phi^A = (\lambda^A)_3^i C^{(1)}_i + (\lambda^A)_3^i C^{(1)i} , \]
\[ \Phi^{ab} = i \varepsilon^{abc} (\lambda^c)_3^j \epsilon^{ik} R^{(2)}_{jk} , \]
\[ \Phi^{\alpha A} = (\lambda^\alpha)_3^i (\lambda^A)_3^k \epsilon^{ik} C^{(2)}_j + (\lambda^\alpha)_3^i (\lambda^A)_3^k \epsilon^{ik} C^{(2)}_j \]
\[ + (\lambda^\alpha)_3^i (\lambda^A)_3^k \epsilon^{ik} C^{(2)}_{jk} + (\lambda^\alpha)_3^i (\lambda^A)_3^k \epsilon^{ik} C^{(2)}_{ij} , \]
\[ \Phi^{AB} = (\lambda^A)_3^i (\lambda^B)_3^j \epsilon^{ij} C^{(2)} + (\lambda^A)_3^i (\lambda^B)_3^j \epsilon^{ij} C^{(2)*} \]
\[ + i(\lambda^A)_3^i (\lambda^B)_3^j \epsilon^{ij} C^{(2)} , \]
\[ (66) \]

This shows how the decomposition (62) is done. The numbers \( \Delta \) in (62) are obtained by applying the matrix \( T_s \) in (27) on the components of \( \Phi^\alpha \) and \( \Phi^{\alpha \beta} \). The scalar \( \Phi \) does not transform under \( T_s \) and hence \( \Delta = 0 \). One may also verify that the above decomposition (66) is in agreement with the \( SU(2)^{\text{diag}} \) matrices (27). The expansion in the harmonics \( D \) then becomes\(^6\)

\[ \mathcal{R}^{(0)}(0,0)(x) \cdot D^{(0,0)(0)}(y) , \]
\[ C^{(1)}_i = AW^{(1/2,0)}_c(x) \cdot D^{(1/2,0)(0)}_i(y) + AW^{(0,1/2)}_c(x) \cdot D^{(0,1/2)(1)}_i(y) , \]
\[ C^{(1)i} = i \varepsilon^{ij} AW^{(1/2,0)}_c(x) \cdot D^{(1/2,0)(-1)}_j(y) + i \varepsilon^{ij} AW^{(0,1/2)}_c(x) \cdot D^{(0,1/2)(-1)}_j(y) , \]
\[ \mathcal{R}^{(1)}_{ij} = AW^{(1,0)}_r(x) \cdot D^{(1,0)(0)}_{ij}(y) + AW^{(1/2,1/2)}_r(x) \cdot D^{(1/2,1/2)(0)}_{ij}(y) + \]
\[ AW^{(0,1)}_r(x) \cdot D^{(0,1)(0)}_{ij}(y) , \]
\[ \mathcal{R}^{(2)}_{ij} = Z^{(1,0)}_c(x) \cdot D^{(1,0)(0)}_{ij}(y) + Z^{(1/2,1/2)}_c(x) \cdot D^{(1/2,1/2)(0)}_{ij}(y) + \]
\[ Z^{(0,1)}_c(x) \cdot D^{(0,1)(0)}_{ij}(y) , \]
\[ C^{(2)}_{ij} = Z^{(1/2,0)}_c(x) \cdot D^{(1/2,0)(0)}_{ij}(y) + Z^{(1/2,0)(0)}_c(x) \cdot D^{(1/2,0)(0)}_{ij}(y) + \]
\[ Z^{(1/2,1/2)}_c(x) \cdot D^{(1/2,1/2)(0)}_{ij}(y) + Z^{(1/2,1/2)}_c(x) \cdot D^{(1/2,1/2)(0)}_{ij}(y) , \]
\[ C^{(2)}_{ijk} = Z^{(3/2,0)}_c(x) \cdot D^{(3/2,0)(1)}_{ijk}(y) + Z^{(1,1/2)}_c(x) \cdot D^{(1,1/2)(1)}_{ijk}(y) + \]
\[ Z^{(1/2,1/2)}_c(x) \cdot D^{(1/2,1/2)(1)}_{ijk}(y) + Z^{(0,3/2)}_c(x) \cdot D^{(0,3/2)(1)}_{ijk}(y) , \]
\[ C^{(2)}_{ijk} = i \varepsilon^{im} \varepsilon^{jn} \epsilon^{kp} Z^{(3/2,0)}_c(x) \cdot D^{(3/2,0)(-1)}_{mnp}(y) + i \varepsilon^{im} \varepsilon^{jn} \epsilon^{kp} Z^{(1,1/2)}_c(x) \cdot D^{(1,1/2)(-1)}_{mnp}(y) + \]
\[ i \varepsilon^{im} \varepsilon^{jn} \epsilon^{kp} Z^{(1/2,1/2)}_c(x) \cdot D^{(1/2,1/2)(-1)}_{mnp}(y) + i \varepsilon^{im} \varepsilon^{jn} \epsilon^{kp} Z^{(0,3/2)}_c(x) \cdot D^{(0,3/2)(-1)}_{mnp}(y) . \]
\[ (67) \]

where the index structure on the harmonics \( D^{(s,t)(\Delta)}_{1_1 \ldots 2_2(\pm \epsilon)} \) is given by the symmetrized Young

\(^6\) We expand the complex conjugates in the conjugate representations. Upon doing so there appear some irrelevant signs in the front of the harmonics. We have absorbed these signs in the \( x \) fields.
gives the masses for the spin-2 fields $A$ and $\Sigma$ representation of four dimensional $x$ equations of table 2 to the harmonics in (67) which will then determine the masses of the

In order to determine the mass spectrum we have to apply the invariant Laplace Beltrami tableau

$$
\Gamma^{(s,t)}_{(i_{1},...;i_{2s},i_{2s+1}...i_{2s+t})}(M_{1}, M_{2}, J; p)
$$

with $p$ constrained as in (18). We see that we have summed in the harmonic expansion a multiple of times over the same irreducible representations of $G$ when a given irreducible representation of $SU(2)^{diag}$ appears multiple times in the reduction $G \rightarrow H$. The names of the $x$-fields refer to names that were used in the papers [4]. For instance the $x$-field $h^{(0,0)}$ gives the masses for the spin-2 fields $h_{\mu\nu}$, the $x$-fields $AW$ will contribute to the vectors $A$ and $W$, and $Z$ will contribute to the vector $Z$. So now, for a given representation of $G$, labeled by $M_{1}, M_{2}, J$ we have the following basis of four-dimensional fields

$$
h \equiv (h^{(0,0)} cr) (x), \quad AW \equiv \begin{pmatrix} AW_{(0,1)}^{(0,1)} \\
AW_{(1/2,1/2)}^{(1/2,1/2)} \\
AW_{(1,0)}^{(1,0)} \\
AW_{(0,1/2)}^{(0,1/2)} \\
\overline{AW}_{(1,2/0)}^{(0,1/2)} \\
\overline{AW}_{(1/2,0)}^{(1/2,0)} \end{pmatrix} (x), \quad Z \equiv \begin{pmatrix} Z_{c}^{(0,3/2)} \\
Z_{s}^{(1,2/1)} \\
Z_{c}^{(1,1/2)} \\
Z_{c}^{(3/2,0)} \\
Z_{c}^{(0,3/2)} \\
\tilde{Z}_{c}^{(1/2,1)} \\
\tilde{Z}_{c}^{(1,1/2)} \\
\tilde{Z}_{c}^{(3/2,0)} \\
Z_{o}^{(0,1)} \\
\tilde{Z}_{o}^{(1/2,1/2)} \\
Z_{o}^{(1,0)} \\
Z_{o}^{(0,1)} \\
Z_{r}^{(1/2,1/2)} \\
Z_{r}^{(1,0)} \\
Z_{r}^{(0,1/2)} \\
Z_{c}^{(1,2/0)} \\
Z_{c}^{(0,1/2)} \\
\tilde{Z}_{c}^{(1/2,0)} \\
\tilde{Z}_{c}^{(0,0)} \\
Z_{c}^{(0,0)} \\
\tilde{Z}_{r}^{(0,0)} \end{pmatrix} (x). \quad (69)
$$

In order to determine the mass spectrum we have to apply the invariant Laplace Beltrami equations of table 2 to the harmonics in (67) which will then determine the masses of the four dimensional $x$ fields by means of the eq. (58). Clearly, since we have a 1-dimensional, a 7-dimensional and a 21-dimensional basis in (69) for the zero-form, the one-form and the two-form respectively the matrices $M_{(0,3)}$, $M_{(1,0,0)}$ and $M_{(1,2,0)}$ will be $1 \times 1$, $7 \times 7$ and $21 \times 21$. Notice that for the two-form we will already need to handle a $21 \times 21$ matrix. Had we not restricted ourselves to representations made with $s$-spin and $t$-spin only, then we would have had to handle a $41 \times 41$ matrix for the two-form. Moreover we would be over counting in the harmonic expansion. We then calculate the eigenvalues of these operators and the masses are then found by using the mass formulas of [4].

In order to obtain the complete multiplet spectrum by the method of paper [13], it is most convenient to have the eigenvalues of the spinor also. The spinor $\Xi$ is also decomposed in irreducible representations of $SU(2)^{diag}$ as follows,

$$
(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \rightarrow (3, 0) \oplus (2, -1) \oplus (2, 1) \oplus (1, 0)
$$

(70)
Concretely, when using the $SO(7)$ $\tau$-matrices for the Clifford algebra as we are advocating in the appendix, then one can see from (28) that its components are

$$\Xi = \begin{pmatrix} \phi \\ \xi_i \\ \zeta_i \\ \psi_{ij} \end{pmatrix} \quad (71)$$

Moreover the Majorana condition

$$C\Xi^* = \Xi, \quad (72)$$

with the conjugation matrix as in the appendix, is translated into (pseudo) reality conditions on its components,

$$\phi^* = \phi, \quad \xi^i = \epsilon^{ij} \xi_j, \quad \zeta^i = -\epsilon^{ij} \zeta_j, \quad \psi_{ij} = \epsilon^{ik} \epsilon^{jl} \psi_{kl}. \quad (73)$$

The numbers $\Delta$ are read off from (29). then similarly as for the forms, the harmonic expansion acquires the form,

$$\phi = \chi^{(0,0)}(x) \cdot D^{(0,0)(0)}(y), \quad \xi_i = \chi^{(1/2,0)}_\xi(x) \cdot D^{(1/2,0)(-1)}_i(y) + \chi^{(0,1/2)}_\xi(x) \cdot D^{(0,1/2)(-1)}_i(y),$$
$$\zeta_i = \chi^{(1/2,0)}_\zeta(x) \cdot D^{(1/2,0)(1)}_i(y) + \chi^{(0,1/2)}_\zeta(x) \cdot D^{(0,1/2)(1)}_i(y),$$
$$\psi_{ij} = \chi^{(1,0)}(x) \cdot D^{(1,0)(0)}_{ij} + \chi^{(1/2,1/2)}(x) \cdot D^{(1/2,1/2)(0)}_{ij} + \chi^{(0,1)}(x) \cdot D^{(0,1)(0)}_{ij}. \quad (74)$$

So we conclude that for the $x$-fields of the spinor we need the basis

$$\chi \equiv \begin{pmatrix} \chi^{(0,1)} \\ \chi^{(1/2,1/2)} \\ \chi^{(1,0)} \\ \chi^{(0,1/2)} \\ \chi^{(1/2,0)}_\xi \\ \chi^{(0,1/2)}_\zeta \\ \chi^{(1/2,0)}_\zeta \\ \chi^{(0,0)} \end{pmatrix}(x) \quad (75)$$

for which we should find an $8 \times 8$ matrix $M_{(2)^2(0)}$.

At this stage we have all the ingredients to calculate the matrices $M_{(0)^3}$, $M_{(0)^2(1)}$, $M_{(0)(1)^2}$ and $M_{(1/2)^3}$. As explained before, it is possible to restrict oneself on the terms in the expansion for a given irreducible representation of $G$. Such representation is characterized by the labels $M_1, M_2, J$, see (34). To do the calculation, we need to perform the following steps:

1. Invert (66). We know how the mass operators work on the the fields $\Phi$. They are given in table 2. To see how they work on the fragments we invert (66).
2. For all of the fragments in (63) and (71) draw the Young tableaux of $D^{(s,t)(\Delta)}$. Check whether it exists. Solve the constraint (48) and then check the bounds (43) and (44).

3. Apply the operators of table 2, by evaluating the covariant derivative (33) on the Young tableaux of the fragments. Recall that these are not differential operators. All one has to do is to evaluate the operators $T_\alpha$ and do the multiplication with the structure constants. To this end one can use the formulae (59) and (60).

All the above steps can be programmed in say Mathematica. We assume that the bounds (43) and (44) are satisfied and calculate the four contemplated operators. Because of this assumption these matrices are the matrices for the long $\mathcal{N} = 3$ multiplets. Indeed, since all the possible fragments are present that implies that all the field components of the multiplets are present. We list the matrices in appendix A.

We finish this section by presenting the eigenvectors of the operators $M_{(0)}^3$, $M_{(0)^2(1)}$, $M_{(0)(1)^2}$ and $M_{(1/2)^3}$, which is what we ultimately need for the masses. We use the notation

$$H_0 \equiv \frac{16}{3} \left( 2(M_1^2 + M_2^2 + M_1 M_2 + 3M_1 + 3M_2) - 3J(J+1) \right).$$

(76)

We write the subscript 0 to remember that this is the eigenvalue of the zero-form operator. We denote the eigenvalues by $\lambda^{(\mu)}_{(f)}$, where $f = 0, 1, 2, s$ refers to the zero-form, the one-form, the two-form and the spinor respectively and $\mu$ enumerates its different eigenvalues. Then,

- the zero-form eigenvalues:

$$\lambda^{(0)} = H_0 .$$

(77)

- the one-form eigenvalues:

$$\lambda^{(1)}_1 = H_0 - 32J - 8 - 4\sqrt{H_0 - 32J + 4},$$

(78)

$$\lambda^{(1)}_2 = H_0 - 32J - 8 + 4\sqrt{H_0 - 32J + 4},$$

(79)

$$\lambda^{(1)}_3 = H_0 + 24 - 4\sqrt{H_0 + 36},$$

(80)

$$\lambda^{(1)}_4 = H_0 + 24 + 4\sqrt{H_0 + 36},$$

(81)

$$\lambda^{(1)}_5 = H_0 + 32J + 24 - 4\sqrt{H_0 + 32J + 36},$$

(82)

$$\lambda^{(1)}_6 = H_0 + 32J + 24 + 4\sqrt{H_0 + 32J + 36},$$

(83)

$$\lambda^{(1)}_7 = H_0 .$$

(84)

- the two-form eigenvalues:

$$\lambda^{(2)}_1 = H_0 + 64J ,$$

(85)

$$\lambda^{(2)}_2 = H_0 + 32J + 32 ,$$

(86)

$$\lambda^{(2)}_3 = H_0 + 32J + 32 ,$$

(87)

$$\lambda^{(2)}_4 = H_0 + 32 ,$$

(88)

$$\lambda^{(2)}_5 = H_0 + 32 ,$$

(89)
\[
\begin{align*}
\lambda_6^{(2)} &= H_0 + 32, \\
\lambda_7^{(2)} &= H_0 - 32J, \\
\lambda_8^{(2)} &= H_0 - 32J, \\
\lambda_9^{(2)} &= H_0 - 64J - 64, \\
\lambda_{10}^{(2)} &= H_0 + 48 + 8\sqrt{H_0 + 36}, \\
\lambda_{11}^{(2)} &= H_0 + 48 - 8\sqrt{H_0 + 36}, \\
\lambda_{12}^{(2)} &= H_0 + 32J + 48 + 8\sqrt{H_0 + 32J + 36}, \\
\lambda_{13}^{(2)} &= H_0 + 32J + 48 - 8\sqrt{H_0 + 32J + 36}, \\
\lambda_{14}^{(2)} &= H_0 - 32J + 16 + 8\sqrt{H_0 - 32J + 4}, \\
\lambda_{15}^{(2)} &= H_0 - 32J + 16 - 8\sqrt{H_0 - 32J + 4}, \\
\lambda_{16}^{(2)} &= H_0 - 32J - 8 - 4\sqrt{H_0 - 32J + 4}, \\
\lambda_{17}^{(2)} &= H_0 - 32J - 8 + 4\sqrt{H_0 - 32J + 4}, \\
\lambda_{18}^{(2)} &= H_0 + 24 - 4\sqrt{H_0 + 36}, \\
\lambda_{19}^{(2)} &= H_0 + 24 + 4\sqrt{H_0 + 36}, \\
\lambda_{20}^{(2)} &= H_0 + 32J + 24 - 4\sqrt{H_0 + 32J + 36}, \\
\lambda_{21}^{(2)} &= H_0 + 32J + 24 + 4\sqrt{H_0 + 32J + 36}.
\end{align*}
\]  

- the spinor eigenvalues:

\[
\begin{align*}
\lambda_1^{(s)} &= -6 - \sqrt{H_0 - 32J + 4}, \\
\lambda_2^{(s)} &= -6 + \sqrt{H_0 - 32J + 4}, \\
\lambda_3^{(s)} &= -6 - \sqrt{H_0 + 36}, \\
\lambda_4^{(s)} &= -6 + \sqrt{H_0 + 36}, \\
\lambda_5^{(s)} &= -6 - \sqrt{H_0 + 32J + 36}, \\
\lambda_6^{(s)} &= -6 + \sqrt{H_0 + 32J + 36}, \\
\lambda_7^{(s)} &= -10 - \sqrt{H_0 + 36}, \\
\lambda_8^{(s)} &= -10 + \sqrt{H_0 + 36}.
\end{align*}
\]  

The eigenvalues \(\lambda_7^{(1)}\) and \(\lambda_{16}^{(2)}, \ldots, \lambda_{21}^{(2)}\) are the unphysical longitudinal eigenvalues. For instance \(\lambda_7^{(1)}\) is the longitudinal one-form mode that comes from the zero-form and \(\lambda_{16}^{(2)}, \ldots, \lambda_{21}^{(2)}\) are the longitudinal two-form modes that come from the one-form. For an explanation, see [6, 7, 13].

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5 Shortening: the mechanism

The mass matrices that we have calculated and presented in appendix [A] are the mass matrices for the long $\mathcal{N} = 3$ multiplets. In this section we clarify the mechanism by which for certain representations of $G$, some of the components of the multiplet decouple, hence multiplet shortening. We first describe the general mechanism and we will illustrate this then by means of some concrete examples: the graviton multiplet, the Killing vector multiplet and the Betti multiplet. Will see that the graviton multiplet has three gravitini in the fundamental of $SO(3)$ and this will prove that the theory has indeed $\mathcal{N} = 3$. Moreover, the case of the graviton multiplet will illustrate how to calculate the masses for representations with non-zero $u$-spin.

To understand shortening we recall that the labels $M_1, M_2, J, p, s, t, u$ of a given Young tableau are subject to the bounds (43) and (44) and to the constraint (48). The constraint ensures the right $U(1)$-weight for the $G$-irrep. The bounds ensure the existence of a Young tableau that corresponds to them. The components of a multiplet that sits in a $G$-irrep labeled by $M_1, M_2, J$ are provided by the terms in the harmonic expansion with these labels. There will be a priori a multiple of them for different values of $s, t, u$. However, a given term in the harmonic expansion with particular labels $s, t, u$ will only be there if the bounds are satisfied. If not, it will mean that its corresponding component in one of the multiplets decouples. Thus it is useful to reformulate the bounds (43) and (44) in such a way as to make the above mechanism more transparent.

To this end, remark that for a given $s, t, u$ and $\Delta$, one can express the bounds (43) and (44) as bounds on $J$ in terms of $M_1, M_2$. This is useful to understand shortening since what we are after is to find a saturation of the unitary bound to get shortening. We have summarized this in table [3] and in table [4] for all the values $s, t, u$ and $\Delta$ that appear in the decompositions (62) and (70). From these tables one can read off the bounds on $J$. These bounds only depend on the numbers $s, t, u$ and $\Delta$. The notation for the entries should be understood as

$$\Delta^{(s, t, u)}. \quad (114)$$

To see which of the fragments survive the bounds we now apply the two table [3] and [4] as if they were two sieves. We now illustrate how to use these table by some concrete examples,

1. Let us consider the $G$-irrep with

$$M_1 = M_2 = 0, \quad J = 0. \quad (115)$$

Sifting the fragments with table [3] we see that none of the fragments of the decomposition survive the bounds except for $1^{(0, 1, 0)}, 1^{(0, 1, 0)}, 0^{(1, 1, 0)}, 0^{(0, 0, 0)}, -1^{(1, 1, 0)}, -1^{(1, 1, 0)}$. Then sifting the fragments with table [4] we see the of these fragments only

$$0^{(0, 0, 0)} \quad (116)$$

survives. This implies that in the harmonic expansion (57) only the harmonic $D^{(0, 0)(0)}$ appears.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$J \geq$ & $\frac{M_2 - M_1}{3} - 2$ & $\frac{M_2 - M_1}{3} - 1$ & $\frac{M_2 - M_1}{3}$ & $\frac{M_2 - M_1}{3} + 1$ & $\frac{M_2 - M_1}{3} + 2$ \\ \hline
$\frac{M_1 - M_2}{3} - 2$ & & & & $-1^{(0, \frac{3}{2}, 0)}$ & \\
\hline
$\frac{M_1 - M_2}{3} - 1$ & & & $-2^{(0, 0, 0)}$ & $-1^{(0, \frac{1}{2}, 0)}$ & $-1^{(0, 0, \frac{1}{2})}$ \\
\hline
$\frac{M_1 - M_2}{3}$ & $-1^{(1, \frac{1}{2}, 0)}$ & $-1^{(0, 0, 0)}$ & $0^{(0, 0, 0)}$ & $0^{(0, 0, \frac{1}{2})}$ & $1^{(0, 1, \frac{1}{2})}$ \\
\hline
$\frac{M_1 - M_2}{3} + 1$ & $-1^{(1, \frac{1}{2}, 0)}$ & $0^{(1, 0, 0)}$ & $0^{(1, \frac{1}{2}, 0)}$ & $1^{(0, 0, \frac{1}{2})}$ & $1^{(0, 0, \frac{3}{2})}$ \\
\hline
$\frac{M_1 - M_2}{3} + 2$ & $1^{(0, 0, 0)}$ & $1^{(1, 0, \frac{1}{2})}$ & $1^{(0, \frac{1}{2}, 0)}$ & $1^{(0, 0, \frac{3}{2})}$ & \\
\hline
\end{tabular}
\caption{The two lower bounds as in (43) and (44). The rows represent the lower bound in (43) and the columns represent the lower bound in (44).}
\end{table}
| $J \leq$ | $\frac{2M_1 + M_2}{3} - 2$ | $\frac{2M_1 + M_2}{3} - 1$ | $\frac{2M_1 + M_2}{3}$ | $\frac{2M_1 + M_2}{3} + 1$ | $\frac{2M_1 + M_2}{3} + 2$ |
|-----------|-----------------------------|----------------------------|-----------------------------|----------------------------|-----------------------------|
| $\frac{2M_1 + M_1}{3} - 2$ | $-1^{(\frac{3}{2},0,0)}$ | $-1^{(1,\frac{1}{2},0)}$ | $-1^{(0,\frac{3}{2},0)}$ | $-1^{(0,\frac{1}{2},0)}$ | $-1^{(0,\frac{3}{2},0)}$ |
| $\frac{2M_1 + M_1}{3} - 1$ | $1^{(\frac{3}{2},0,0)}$ | $1^{(1,\frac{1}{2},0)}$ | $0^{(1,0,0)}$ | $-1^{(1,0,\frac{1}{2})}$ | $-2^{(0,0,0)}$ |
| $\frac{2M_1 + M_1}{3}$ | $1^{(1,0,\frac{1}{2})}$ | $1^{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ | $0^{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ | $-1^{(\frac{1}{2},0,1)}$ | $-1^{(0,\frac{1}{2},\frac{1}{2})}$ |
| $\frac{2M_1 + M_1}{3} + 1$ | $2^{(0,0,0)}$ | $1^{(\frac{1}{2},0,1)}$ | $1^{(0,\frac{1}{2},\frac{1}{2})}$ | $0^{(0,0,1)}$ | $-1^{(0,0,\frac{3}{2})}$ |
| $\frac{2M_1 + M_1}{3} + 2$ | $1^{(0,0,\frac{3}{2})}$ |

Table 4: the two upper bounds as in (43) and (44). The rows represent the upper bound in (43) and the columns represent the upper bound in (44).
2. Let us consider
\[ M_1 = M_2 = 0, \quad J = 1. \] (117)

Then from table 4 we see that only \(1^{(0, \frac{3}{2})}, 0^{(0,0,1)}, -1^{(0, \frac{3}{2})}\) satisfy the upper bounds. From table 3 we see that of these only
\[ 0^{(0,0,1)} \] (118)
satisfies the lower bounds.

3. Let us also consider the \(G\)-irrep with
\[ M_1 = M_2 = 1, \quad J = 0. \] (119)

Then we read off from table 3 and table 4 that only
\[ 0^{(1, \frac{1}{2}, 0)}, 1^{(0, \frac{1}{2}, 0)}, -1^{(1, \frac{1}{2}, 0)}, 0^{(0,0,0)} \] (120)
satisfy the upper and the lower bounds.

This illustrates how, given a \(G\)-irrep, the fragments in the \(G \rightarrow H\) that survive the bounds can be obtained. In this way, one can study shortening for any such \(G\)-irrep given. Yet, this is not sufficient to study shortening for the complete spectrum. To this end we need to identify the series of \(G\)-irrepses that have the same content. Before doing so, let us restrict our attention to some of the above examples.

By means of the first example we show that the spectrum that we have obtained now is really the complete spectrum in spite of the fact that we have only been considering harmonics with \(s\) and \(t\) indices. This we argued was valid for certain \(SU(3) \times SU(2)\) representations where \(M_1\) and \(M_2\) are big enough. Here we show that the matrices that we have calculated in the previous section can actually be used to obtain the eigenvalues for all the other cases where \(M_1\) and \(M_2\) are not big enough. We will do this by means of a concrete and particular relevant example: the massless graviton multiplet. Moreover, the subsequent material will also serve as a proof of the fact that the compactification on \(AdS \times N^{10}\) formulated as \(SU(3) \times SU(2)/SU(2) \times U(1)\) and with the rescalings chosen as in (24) has indeed \(N = 3\) supersymmetry.

Clearly, the massless graviton multiplet contains the graviton field which is to sit in the representation \(M_1 = M_2 = J = 0\). Applying the mass formula
\[ m_h^2 = M_{(0)^2}, \] (121)
we find that it has mass zero. The gravitini and the graviphoton correspond to case we already mentioned, \(M_1 = M_2 = 0\) and \(J = 1\),
\[ 1 \otimes \begin{array}{c} \hline \hline \end{array} \] (122)
This is a case where \(M_1\) and \(M_2\) are not big enough to express everything in terms of \(SU(3) \times SU(2)\) Young tableaux with \(s\)-spin and \(t\)-spin only, as we have assumed in all the previous discussion. We illustrate that the eigenvalues of the previous section can still be used to derive the masses of the graviphoton and the gravitino.
Let us first consider the one-form. We show that it contains the graviphoton in its harmonic expansion. It sits in in the fundamental of $SO(3)$. Using the information in (118) we see that in the reduction $G \rightarrow H$, the representations $(2, 0)$ and $(\bar{2}, 1)$ do not appear. Indeed, one can not put one $SU(2)$ index $i$ in (122). The fragment $(3, 1)$ appears as (118), namely for the fragment $R_{ij}$ there is $1 \otimes \begin{array}{c} i \\ j \end{array}$. This can be seen as the Young tableau

$$
\Gamma_{ij}^{(0,0,1)}(0,0,1;0)
$$

(123)

where $p = 1$ is given by (118). Let us now formally apply the generalized cyclic identity (49) to this. We write,

$$
\Gamma_{ij}^{(1,0)}(0,0,1;1) - 2 \Gamma_{ij}^{(1/2,1/2)}(0,0,1;1) + \Gamma_{ij}^{(0,1)}(0,0,1;1).
$$

(124)

Clearly none of the above terms exists as a Young tableaux, whereas (123) does. In order to obtain the mass matrices for the one-form we should in principle derive the formulae of the type (59) and (60) for the case $u \neq 0$. We argue now that such work can be avoided by using the information that we have already obtained.

Since all these formulas are linear one does not have to derive these formulae on the Young tableau (123), but one introduces the objects $\Gamma$ even if they do not correspond to an irreducible representation of $SU(3) \times SU(2)$ with the definition that they transform as in the transformation rules (53) and (64) for the case $u \neq 0$. We argue now that such work can be avoided by using the information that we have already obtained.

In order to exhibit the results for the matrices that we have already obtained in the previous section, we consider $M_{(0)^2(1)}$ for

$$
M_1 = M_2 = 0, \quad J = 1.
$$

(126)

Since the fragments $C_{i}(1)$ and $C^{(1)i}$ do not get a contribution, we have to consider $M_{(0)^2(1)}$ on the basis

$$
AW = \begin{pmatrix}
AW_{r}^{(0,1)} \\
AW_{r}^{(1/2,1/2)} \\
AW_{r}^{(1,0)}
\end{pmatrix}
$$

(127)

only. Hence, it is a $3 \times 3$ matrix

$$
M_{(0)^2(1)} = \begin{pmatrix}
0 & -24 & 0 \\
-96 & -48 & -96 \\
0 & -24 & 0
\end{pmatrix}
$$

(128)
It is useful to realize that we can make change of basis in the harmonic expansion (67) 

$$AW' = (P^{-1})^T AW,$$

$$R_{ij}^{(1)} = AW^T P^{-1} \cdot P D_{ij}.$$  \hspace{1cm} (129)

where $P$ is a $3 \times 3$ invertible matrix. Let us take

$$P = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & -1 \\ -1 & -4 & -1 \end{pmatrix},$$ \hspace{1cm} (130)

Then the first term in the harmonic expansion (129) gets the form (125). The matrix (128) in the basis $AW'$ gets the form,

$$M'_{(0)^2(1)} = (P^{-1})^T M_{(0)^2(1)} P = \begin{pmatrix} 48 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -96 \end{pmatrix}.$$ \hspace{1cm} (131)

So we see that the eigenvalue of $M_{(0)^2(1)}$ for component $AW'(0,0,1)$ in (125) of the graviphoton is given by the first entry of the above matrix,

$$M_{(0)^2(1)} = 48.$$ \hspace{1cm} (132)

Mind that the matrix $P$ diagonalizes the matrix $M_{(0)^2(1)}$.

As is known from [7], the mass of the vector $A$ is given by the following mass formula,

$$m_A^2 = M_{(0)^2(1)} + 48 - 12 \sqrt{M_{(0)^2(1)} + 16}.$$ \hspace{1cm} (133)

So we find a massless vector indeed.

So far we have found only the vectors and the graviton of the massless graviton multiplet. Still it is crucial to check that we have the right number of gravitini in this multiplet. That will prove that we have indeed $\mathcal{N} = 3$. Hence, let us now consider the spinor. We are interested in the representation (122). Again there is only a contribution for the fragment $(3,0)$ in the decomposition (70) and the treatment is the same as for the $(3,1)$ of the one-form. Indeed, there only appears the fragment $\psi_{ij}$ in the expansion,

$$\psi_{ij} = \chi^{(0,0,1)}(x) \cdot \left(-D_{ij}^{(1,0)} + 2D_{ij}^{(1/2,1/2)} - D_{ij}^{(0,1)}\right).$$ \hspace{1cm} (134)

Consequently, from the $8 \times 8$ matrix $M_{(1/2)^3}$ there is only the upper $3 \times 3$ matrix

$$M_{(1/2)^3} = \begin{pmatrix} -4 & -2 & 0 \\ -8 & -8 & -8 \\ 0 & -2 & -4 \end{pmatrix}.$$ \hspace{1cm} (135)

relevant for $M_1 = M_2 = 0$ and $J = 1$. We can diagonalize this matrix similarly by means of the matrix $P$ in (130) and we get,

$$M'_{(1/2)^3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -12 \end{pmatrix}.$$ \hspace{1cm} (136)

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Only the first zero entry of this matrix is physically relevant. Using the fact,

\[ m_\chi = M_{(1/2)^3}, \] (137)

we conclude that there are three massless spin-\(3/2\) fields in the fundamental of \(SO(3)\). They provide the gravitini of the \(\mathcal{N} = 3\) massless graviton multiplet.

So we have found the massless graviton, gravitini and graviphoton of the \(\mathcal{N} = 3\) \((2,3(3/2),3(1),1/2)\) graviton multiplet.

To complete the treatment of (126), we also consider the eigenvalues of the two-form for this representation of \(G\). As in the case of the one-form, the only fragments that contribute are the ones with \(u\)-spin 1, i.e. for the representation (126) the \(21 \times 21\) matrix works on the basis,

\[
\begin{pmatrix}
Z_{a}^{(0,1)} \\
Z_{a}^{(1/2,1/2)} \\
Z_{a}^{(1,0)} \\
Z_{r}^{(0,1)} \\
Z_{r}^{(1/2,1/2)} \\
Z_{r}^{(1,0)} 
\end{pmatrix}
\] (138)

only. Hence, it gets the form of a \(6 \times 6\) matrix,

\[
M_{(0)(1)^2} = \begin{pmatrix}
32 & -16 & 0 & -32 & 0 & 0 \\
-64 & 0 & -64 & 0 & -32 & 0 \\
0 & -16 & 32 & 0 & 0 & -32 \\
-16 & 0 & 0 & 32 & -24 & 0 \\
0 & -16 & 0 & -96 & -16 & -96 \\
0 & 0 & -16 & 0 & -24 & 32 
\end{pmatrix}
\] (139)

Following the same line of reasoning as for the one-form and the spinor, we introduce the diagonalization matrix

\[
P = \begin{pmatrix}
2 & -4 & 2 & 1 & -2 & 1 \\
-1 & 2 & -1 & 1 & -2 & 1 \\
-\sqrt{2} & 0 & \sqrt{2} & -1 & 0 & 1 \\
\sqrt{2} & 0 & -\sqrt{2} & -1 & 0 & 1 \\
-1 + \sqrt{3} & 4 \left(-1 + \sqrt{3}\right) & -1 + \sqrt{3} & 1 & 4 & 1 \\
-1 - \sqrt{3} & -4 \left(1 + \sqrt{3}\right) & -1 - \sqrt{3} & 1 & 4 & 1 
\end{pmatrix}
\] (140)

Thus diagonalizing we find,

\[
M_{(0)(1)^2}' = (P^{-1})^T M_{(0)(1)^2} P^T = \begin{pmatrix}
48 & 0 & 0 & 0 & 0 & 0 \\
0 & 96 & 0 & 0 & 0 & 0 \\
0 & 0 & -16 \left(-2 + \sqrt{2}\right) & 0 & 0 & 0 \\
0 & 0 & 0 & 16 \left(2 + \sqrt{2}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & -16 \left(3 + \sqrt{3}\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 16 \left(-3 + \sqrt{3}\right) 
\end{pmatrix}
\] (141)
From this matrix we have to discard the eigenvalues
\begin{equation}
-16 \left( -2 + \sqrt{2} \right), 16 \left( 2 + \sqrt{2} \right),
-16 \left( 3 + \sqrt{3} \right), 16 \left( -3 + \sqrt{3} \right)
\end{equation}
as being unphysical. Only the first two eigenvectors in $P$ correspond to the two $0^{(0,0,1)}$ that we have found in the decomposition $G \to H$ in (62). So we see that the first eigenvector (corresponding to the first row of the matrix $P$) is the longitudinal unphysical mode that corresponds to the one-form eigenvector that we obtained above. It has the same eigenvector $M_{(0)(1)^2} = 48$. Furthermore, there is the eigenvalue $M_{(0)(1)^2} = 96$ whose eigenvector is an extra massive vector $Z$ with mass
\begin{equation}
m_Z^2 = M_{(0)(1)^2} = 96.
\end{equation}

As will be explained in [17] this vector is part of a massive gravitino multiplet.

To come back to our example $M_1 = M_2 = J = 0$, let us also consider the two-form for this case. This yields the vector field of the massless Betti multiplet. Indeed, Then the operator $M_{(0)(1)^2}$ acts on the component $Z_i^{(0,0)}$ only in which case is has eigenvalue zero. So we find the vector of the massless Betti multiplet.

At this stage we have elucidated the examples (115) and (117). This yielded the vector field of the massless Betti multiplet. It is also instructive to consider example (119) a bit more closely, since it is in this irreducible representation of $G$ that we find the massless vector multiplet which contains the Killing vector of the $SU(3)$ isometry.

To this end we have to consider the one-form. Acting on the fragments with (120) it it becomes the $3 \times 3$ matrix,
\begin{equation}
\begin{pmatrix}
80 & -16 i \sqrt{2} & -16 i \sqrt{2} \\
24 i \sqrt{2} & 96 & 0 \\
24 i \sqrt{2} & 0 & 96
\end{pmatrix}
\end{equation}
which has eigenvalues
\begin{equation}
M_{(0)^2(1)} = 48, 96, 128.
\end{equation}
The first eigenvalue 48 according to [133], gives rise to a massless vector $A$. This is the Killing vector of the $SU(3)$ isometry.

### 6 The series

In the previous section we explained how the masses of the $\mathcal{N} = 3$ theory can be calculated from the matrices of appendix A when $M_1, M_2, J$ are given. Yet for a systematic study of the complete spectrum we need to do this for all the irreducible representations of $G$. In order to do so it is useful to arrange the representations that are allowed by the bounds (43) and (44) and the constraint (48), into series that give rise to the same content of fragments. This can be done most easily by using the tables 3 and 4.

First of all, notice that it is sufficient to consider the cases where
\begin{equation}
M_2 \geq M_1.
\end{equation}
Indeed, the irreducible representations with $M_2 < M_1$ are the conjugate representations of $SU(3)$ and hence correspond to the complex conjugate multiplets in the $\mathcal{N} = 3$ theory. Reflecting on the two tables 3 and 4 we conclude that we should make a distinction among the following series $R$ and $E_n$,

$$R : (M_1 \geq 4) ; \quad \begin{cases} J \geq \frac{1}{3} (M_2 - M_1) + 2 \\ J \leq \frac{1}{3} (2M_1 + M_2) - 2 \end{cases}, \quad (147)$$

$$E_1 : (M_1 \geq 3) ; \quad \begin{cases} J \geq \frac{1}{3} (M_2 - M_1) + 2 \\ J = \frac{1}{3} (2M_1 + M_2) - 1 \end{cases}, \quad (148)$$

$$E_2 : (M_1 \geq 2) ; \quad \begin{cases} J \geq \frac{1}{3} (M_2 - M_1) + 2 \\ J = \frac{1}{3} (2M_1 + M_2) \end{cases}, \quad (149)$$

$$E_3 : (M_1 \geq 1) ; \quad \begin{cases} J \geq \frac{1}{3} (M_2 - M_1) + 2 \\ J = \frac{1}{3} (2M_1 + M_2) + 1 \end{cases}, \quad (150)$$

$$E_4 : (M_1 \geq 0) ; \quad \begin{cases} J \geq \frac{1}{3} (M_2 - M_1) + 2 \\ J = \frac{1}{3} (2M_1 + M_2) + 2 \end{cases}, \quad (151)$$

$$E_5 : (M_1 \geq 3) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) + 1 \\ J \leq \frac{1}{3} (2M_1 + M_2) - 2 \end{cases}, \quad (152)$$

$$E_6 : (M_1 = 2) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) + 1 \\ J = \frac{1}{3} (2M_1 + M_2) - 1 \end{cases}, \quad (153)$$

$$E_7 : (M_1 = 1) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) + 1 \\ J = \frac{1}{3} (2M_1 + M_2) \end{cases}, \quad (154)$$

$$E_8 : (M_1 = 0) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) + 1 \\ J = \frac{1}{3} (2M_1 + M_2) + 1 \end{cases}, \quad (155)$$

$$E_9 : (M_1 \geq 2) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) \\ J \leq \frac{1}{3} (2M_1 + M_2) - 2 \end{cases}, \quad (156)$$

$$E_{10} : (M_1 = 1) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) \\ J = \frac{1}{3} (2M_1 + M_2) - 1 \end{cases}, \quad (157)$$

$$E_{11} : (M_1 = 0) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) \\ J = \frac{1}{3} (2M_1 + M_2) \end{cases}, \quad (158)$$

$$E_{12} : (M_1 \geq 1) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) - 1 \\ J \leq \frac{1}{3} (2M_1 + M_2) - 2 \end{cases}, \quad (159)$$

$$E_{13} : (M_1 = 0) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) - 1 \\ J = \frac{1}{3} (2M_1 + M_2) - 1 \end{cases}, \quad (160)$$

$$E_{14} : (M_1 \geq 0) ; \quad \begin{cases} J = \frac{1}{3} (M_2 - M_1) - 2 \\ J \leq \frac{1}{3} (2M_1 + M_2) - 2 \end{cases}. \quad (161)$$

The series $R$ is the regular series and contains the long multiplets. For the calculation of the masses of the components, one takes the matrices in the appendix A. The series $E$ are the exceptional series. One may verify that in these exceptional series some columns of the tables 3 and 4 are absent. In order to see which rows survive the bounds it is useful to make a distinction between the cases

$$M_2 = M_1 + 3j, \quad (162)$$

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for \( j \) a non-negative integer. Hence, the series (148) ... (161) have a different content depending on the parameter \( j \). We will denote them as \( E_j^n \). One may verify that it is sufficient to make a distinction between

\[
j = 0, \quad j = 1, \quad j \geq 2.
\] (163)

We write \( E_0^0, E_1^1 \) and \( E_2^n, \ldots \).

In order to determine the content of the series one can take a representative set of number \( M_1, M_2, J \) for that series and do the following:

1. Use table 3 and table 4 to determine which of the fragments are present. Call this set \( F \).

2. Consider each of the fragments \( \Delta^{(s,t,u)} \) in \( F \) with non-zero \( u \)-spin. If both the fragments \( \Delta^{(s,t+1/2,u-1/2)} \) and \( \Delta^{(s+1/2,t,u-1/2)} \) are present as well then eliminate \( \Delta^{(s,t,u)} \). This has to be done because of the cyclic identity (19). Otherwise we are overly expanding. Call the remaining set of fragments \( F_c \).

3. Consider the fragments of \( F_c \) with non-zero \( u \)-spin that are left. They can not be eliminated. In fact they span a basis for the harmonics. However, in the spirit of the previous section, they have to be expressed in terms of the objects \( \Gamma \) (that do not correspond to Young tableaux). All of this has to be done as explained in the example of (117). We recall that this is to avoid unnecessary work. This determines an unnormalized row vector of the conjugation matrix \( P \). See for example (124) and (125), where the row vector is

\[
(-1 \quad 2 \quad -1).
\] (164)

4. For all the remaining fragments with zero \( u \)-spin, add the unit vector to \( P \).

5. Fill the rest of \( P \) in such a way as to get an invertible matrix. The choice of the vectors is completely free as long as the matrix \( P \) becomes invertible.

6. Apply \( P \) as in (131).

7. Delete the unphysical row and columns of the resulting matrix. The rows and columns that have to be deleted are the ones for which we had to insert the vectors in step 4.

8. Calculate the eigenvalues of this matrix. They have to be in the lists (77), (78)...(84), (85)...(105) and (106)...(113). Since we are doing this for specific values of \( M_1, M_2, J \) we have to plug in these values also in these lists.

Remark that after step 2 we have found a complete basis of fragments. Given some values for \( M_1, M_2, J \), all the above steps can be implemented in a Mathematica program. So it is sufficient to choose one representative \( M_1, M_2, J \) for the series in (148)...(161) and let the computer perform these steps.

We now list the results of this. For the zero-form, the one-form, the two-form and the spinor we list which of the eigenvalues are present and we give the masses of the fields (see [3, 4, 13] for conventions concerning names)

\[
h, \chi, A, W, Z, \lambda, \Sigma, S
\] (165)

that can be calculated from them.
6.1 The zero-form

The eigenvalue $\lambda^{(0)}$ of the zero-form (77) is only present in the series

\[ R, \ E_1^0, E_2^0, E_5^0, E_6^0, E_7^0, E_9^0, E_{10}^0, E_{11}^0, \ E_1^1, E_2^1, E_5^1, E_6^1, E_7^1, E_9^1, E_{10}^1, E_{11}^1, \ E_1^2, E_2^2, E_5^2, E_6^2, E_7^2, E_9^2, E_{10}^2, E_{11}^2. \] (166)

From the zero-form eigenvalue $\lambda^{(0)}$ we can obtain the mass of the spin-2 field $h$ and the spin-0 fields $\Sigma$ and $S$ [6, 7, 13]:

\[ m_h^2 = \lambda^{(0)}, \]
\[ m_\Sigma^2 = \lambda^{(0)} + 176 + 24 \sqrt{\lambda^{(0)}} + 36, \]
\[ m_S^2 = \lambda^{(0)} + 176 - 24 \sqrt{\lambda^{(0)}} + 36. \] (167)

So for each $G$-irrep in the series (166) there is a field $h$, a fields $\Sigma$ and a field $S$ with masses as given in (167).

6.2 The one-form

We list which of the eigenvalues are present for the one-form. We use the notation of (78)...(84). In the following tables we suppress the superscript (1).

For the series $R$ all seven eigenvalues are present:

\[ R \ | \ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \] (168)

For the exceptional series with $j = 0$ we have,

\[
\begin{array}{|c|c|}
\hline
E_1^0 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_2^0 & \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_3^0 & \lambda_6 \\
E_4^0 & \text{none} \\
E_5^0 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_6^0 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_7^0 & \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_8^0 & \lambda_6 \\
E_9^0 & \lambda_1, \lambda_2, \lambda_7 \\
E_{10}^0 & \lambda_1, \lambda_2, \lambda_7 \\
E_{11}^0 & \text{none} \\
E_{12}^0 & \text{empty} \\
E_{13}^0 & \text{empty} \\
E_{14}^0 & \text{empty} \\
\hline
\end{array}
\] (169)

Mind that there are no values for $M_1, M_2, J$ that match the series $E_{12}^0, E_{13}^0, E_{14}^0$. Indeed, if $j = 0$ then (159), (160) and (161) would imply that $J = -1$ or $J = -2$. We have indicated this in the table with empty. For the series $E_9^0$ there exist values of $M_1, M_2, J$. However, no eigenvalues $\lambda^{(1)}$ are present. We have indicated that with none.
For the exceptional series with \( j = 1 \):

\[
\begin{array}{|c|c|}
\hline
E_i^1 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^2 & \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^3 & \lambda_5, \lambda_6 \\
E_i^4 & \text{none} \\
E_i^5 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^6 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^7 & \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^8 & \lambda_5, \lambda_6 \\
E_i^9 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7 \\
E_i^{10} & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7 \\
E_i^{11} & \lambda_3, \lambda_4, \lambda_7 \\
E_i^{12} & \lambda_1, \lambda_2 \\
E_i^{13} & \lambda_1, \lambda_2 \\
E_i^{14} & \text{empty} \\
\hline
\end{array}
\]

(170)

For the exceptional series with \( j \geq 2 \):

\[
\begin{array}{|c|c|}
\hline
E_i^2 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^3 & \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^4 & \lambda_5, \lambda_6 \\
E_i^5 & \text{none} \\
E_i^6 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^7 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^8 & \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \\
E_i^9 & \lambda_5, \lambda_6 \\
E_i^{10} & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7 \\
E_i^{11} & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7 \\
E_i^{12} & \lambda_3, \lambda_4, \lambda_7 \\
E_i^{13} & \lambda_1, \lambda_2 \\
E_i^{14} & \text{none} \\
\hline
\end{array}
\]

(171)

From the one-form eigenvalue \( \lambda^{(1)} \) we can obtain the masses of the vectors \( A \) and \( W \),

\[
\begin{align*}
m_A^2 &= \lambda^{(1)} + 48 - 12 \sqrt{\lambda^{(1)} + 16}, \\
m_W^2 &= \lambda^{(1)} + 48 + 12 \sqrt{\lambda^{(1)} + 16}.
\end{align*}
\]

(172)

So for each entry \( \lambda \) in the above tables there is both a vector \( A \) and a vector \( W \) whose mass can be obtained from \( \lambda \) by the formulas (172).

6.3 The two-form

We list which of the eigenvalues are present for the two-form. We use the notation of (85)...(105). Mind that there are the multiplicities of the eigenvalues

\[
\lambda_2^{(2)} = \lambda_3^{(2)}
\]

32
\[
\begin{align*}
\lambda_4^{(2)} &= \lambda_5^{(2)} = \lambda_6^{(2)} \\
\lambda_7^{(2)} &= \lambda_8^{(2)}
\end{align*}
\] (173)

In the following tables we will suppress the superscript \((2)\).
The series \(R\) contains all eigenvalues:

\[
\begin{array}{|c|c|}
\hline
R & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
\hline
\end{array}
\] (174)

The exceptional series \(E\) with \(j = 0\) contain:

\[
\begin{array}{|c|c|}
\hline
E_1^0 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_2^0 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_3^0 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_4^0 & none \\
E_5^0 & \lambda_{12}, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_6^0 & \lambda_{12}, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_7^0 & \lambda_{12}, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_8^0 & \lambda_{12}, \lambda_{21} \\
E_9^0 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_{10}^0 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_{11}^0 & empty \\
E_{12}^0 & empty \\
E_{13}^0 & empty \\
\hline
\end{array}
\] (175)

The exceptional series \(E\) with \(j = 1\) contain:

\[
\begin{array}{|c|c|}
\hline
E_1^1 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_2^1 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_3^1 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_4^1 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_5^1 & \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_6^1 & \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_7^1 & \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_8^1 & \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_9^1 & \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_{10}^1 & \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21} \\
E_{11}^1 & empty \\
\hline
\end{array}
\] (176)
The exceptional series $E$ with $j \geq 2$ contain:

| $E^\geq_1$ | $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21}$ |
| $E^\geq_2$ | $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21}$ |
| $E^\geq_3$ | $\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{20}, \lambda_{21}$ |
| $E^\geq_4$ | $\lambda_1$ |
| $E^\geq_5$ | $\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21}$ |
| $E^\geq_6$ | $\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21}$ |
| $E^\geq_7$ | $\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21}$ |
| $E^\geq_8$ | $\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21}$ |
| $E^\geq_9$ | $\lambda_4, \lambda_5, \lambda_6, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}$ |
| $E^\geq_{10}$ | $\lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_{10}, \lambda_{11}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}$ |
| $E^\geq_{11}$ | $\lambda_4, \lambda_5, \lambda_6, \lambda_{10}, \lambda_{11}, \lambda_{18}, \lambda_{19}$ |
| $E^\geq_{12}$ | $\lambda_7, \lambda_8, \lambda_{10}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}$ |
| $E^\geq_{13}$ | $\lambda_7, \lambda_8, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}$ |
| $E^\geq_{14}$ | $\lambda_9$ |

Mind that besides the multiplicities in (173), there may be occasional multiplicities that arise for certain values of $M_1, M_2, J$. If that occurs, then we have indicated it in the above tables by writing the eigenvalues $\lambda$ with multiple indices. For instance, $\lambda_{1,2}$ in $E^\geq_0$ means that for the values $M_1 = M_2 \geq 3, J = 1$ we have $\lambda^{(2)}_1 = \lambda^{(2)}_2 = \lambda^{(2)}_3$ in (105).

From this one can obtain the masses of the vector field $Z$,

$$m_Z^2 = \lambda^{(2)}.$$  \hspace{1cm} (178)

So for each entry in the above tables there is a vector $Z$ with mass (178).

### 6.4 The spinor

We list which of the eigenvalues are present for the spinor. We use the notation of (106)...(113). In the following tables we will suppress the superscript ($s$).

For the series $R$ we find all of the eight eigenvalues:

| $R$ | $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8$ |

(179)
For the exceptional series with \( j = 0 \):

| \( E_n^0 \) | \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \) |
|---|---|
| \( E_1^0 \) | \( \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \) |
| \( E_2^0 \) | \( \lambda_6 \) |
| \( E_3^0 \) | \( \text{none} \) |
| \( E_4^0 \) | \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \) |
| \( E_5^0 \) | \( \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \) |
| \( E_6^0 \) | \( \lambda_6 \) |
| \( E_7^0 \) | \( \lambda_1, \lambda_2, \lambda_7, \lambda_8 \) |
| \( E_8^0 \) | \( \lambda_7 \) |
| \( E_9^0 \) | \( \text{empty} \) |
| \( E_{12}^0 \) | \( \text{empty} \) |
| \( E_{13}^0 \) | \( \text{empty} \) |

\[ (180) \]

For the exceptional series with \( j = 1 \):

| \( E_n^1 \) | \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \) |
|---|---|
| \( E_1^1 \) | \( \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \) |
| \( E_2^1 \) | \( \lambda_5, \lambda_6 \) |
| \( E_3^1 \) | \( \text{none} \) |
| \( E_4^1 \) | \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \) |
| \( E_5^1 \) | \( \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \) |
| \( E_6^1 \) | \( \lambda_5, \lambda_6 \) |
| \( E_7^1 \) | \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7, \lambda_8 \) |
| \( E_8^1 \) | \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7, \lambda_8 \) |
| \( E_9^1 \) | \( \lambda_3, \lambda_4, \lambda_7, \lambda_8 \) |
| \( E_{12}^1 \) | \( \lambda_1, \lambda_2 \) |
| \( E_{13}^1 \) | \( \lambda_1, \lambda_2 \) |
| \( E_{14}^1 \) | \( \text{empty} \) |

\[ (181) \]
For the exceptional series with \( j \geq 2 \):

\[
\begin{array}{|c|c|}
\hline
E^\geq_1 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \\
E^\geq_2 & \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \\
E^\geq_3 & \lambda_5, \lambda_6 \\
E^\geq_4 & \text{none} \\
E^\geq_5 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \\
E^\geq_6 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \\
E^\geq_7 & \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \\
E^\geq_8 & \lambda_5, \lambda_6 \\
E^\geq_9 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7, \lambda_8 \\
E^\geq_{10} & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7, \lambda_8 \\
E^\geq_{11} & \lambda_3, \lambda_4, \lambda_7, \lambda_8 \\
E^\geq_{12} & \lambda_1, \lambda_2 \\
E^\geq_{13} & \lambda_1, \lambda_2 \\
E^\geq_{14} & \text{none} \\
\hline
\end{array}
\]

From this one can obtain the masses of the spin-\( \frac{3}{2} \) field and the spin-\( \frac{1}{2} \) field \( \lambda_L \),

\[
m_\chi = \lambda^{(s)}, \\
m_{\lambda L} = -\left(\lambda^{(s)} + 16\right).
\]

So for each entry in the above tables there is the field \( \chi \) and the field \( \lambda_L \) with masses (183).

7 Conclusions and outlook

To conclude this paper, we have calculated the operators \( M_{(0)^3}, M_{(0)^2(1)}, M_{(1/2)^2} \) and \( M_{(0)(1)^2} \). For the long multiplets, they are listed in appendix A. We have found the massless graviton multiplet with massless gravitini in the fundamental of \( SO(3) \), hence we have found the \( \mathcal{N} = 3 \) graviton multiplet. This proves that the result is indeed \( \mathcal{N} = 3 \).

We have also found the Betti multiplet and the massless vector multiplet whose vectors gauge the \( SU(3) \) isometry. Moreover, this serves as a check on our results. We explained that the operators \( M_{(0)^3}, M_{(0)^2(1)}, M_{(1/2)^2} \) and \( M_{(0)(1)^2} \) that are listed in the appendix contain the complete information. We explained the mechanism for shortening and we showed how the masses of the representations that contain u-spin can be obtained. We identified the series of \( G \)-irrepses with a common field content. We have formulated the procedure to calculate the masses for all the shortened series. Finally we listed the results of this.

Hence we have done everything that can be done concerning the harmonic analysis. The next step is to draw group theory into the analysis to uncover the structure of the \( \mathcal{N} = 3 \) multiplets. This will yield a small zoo of new \( \mathcal{N} = 3 \) multiplets beside Freedman and Nicolai’s vector multiplet in table [4]. Then it will be easy to recognize these multiplets as superfields on the \( \mathcal{N} = 2 \) superspace of [20]. The invariant constraints that characterize these multiplets can then be read off directly. All of this will be done in a forthcoming publication [17].
The subsequent step is to check the anti-de Sitter correspondence

\[ \text{AdS}_4 \times N^{010}/\text{CFT}_3 \]  

(184)

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A The mass matrices

In this appendix we list the operators $M_{(0)^3}$, $M_{(0)^2(1)}$, $M_{(1/2)^2}$ and $M_{(0)(1)^2}$ for representations with s-spin and t-spin only, as output of the Mathematica programs.

The eigenvalue of the operator $M_{(0)^3}$ is,

$$M_{(0)^3} = H_0 = \frac{16}{3} \left( +2 (M_1^2 + M_2^2 + M_1 M_2 + 3 M_1 + 3 M_2) - 3 J (J + 1) \right)$$  \hspace{1cm} (185)$$

The matrix $M_{(0)^2(1)}$ of the one-form is

$$\begin{array}{|c|c|c|}
\hline
AW_r^{(0,1)} & AW_r^{(1/2,1/2)} & AW_r^{(1,0)} \\
\hline
32 + H_0 - 16 M_1 + 16 M_2 & 0 & 0 \\
-16 (3 + 3 J - M_1 + M_2) & -8 (3 J + M_1 - M_2) & -16 (3 + 3 J + M_1 - M_2) \\
0 & -16 + H_0 & H_0 + 16 (2 + M_1 - M_2) \\
\frac{16}{3} \sqrt{2} (3 + 3 J - M_1 + M_2) & -8 (3 J - M_1 + M_2) & \frac{16}{3} \sqrt{2} (3 + M_1 + 2 M_2) \\
0 & \frac{8}{3} \sqrt{2} (6 + M_1 + 2 M_2) & 0 \\
\frac{16}{3} \sqrt{2} (3 + 2 M_1 + M_2) & \frac{8}{3} \sqrt{2} (3 J + M_1 - M_2) & \frac{16}{3} \sqrt{2} (3 + 3 J + M_1 - M_2) \\
0 & \frac{8}{3} \sqrt{2} (6 + 2 M_1 + M_2) & 0 \\
\hline
\end{array}$$

$$\begin{array}{|c|c|}
\hline
AW_r^{(0,1/2)} & AW_r^{(1/2,0)} \\
\hline
\frac{16}{3} \sqrt{2} (3 J + M_1 - M_2) & 0 \\
\frac{-16}{3} \sqrt{2} (2 M_1 + M_2) & \frac{16}{3} \sqrt{2} (3 + 3 J + M_1 - M_2) \\
0 & \frac{-16}{3} \sqrt{2} (3 + 2 M_1 + M_2) \\
H_0 - \frac{16}{3} (M_1 - M_2) & \frac{-16}{3} (3 + 3 J + M_1 - M_2) \\
\frac{-16}{3} (3 J - M_1 + M_2) & H_0 + \frac{16}{3} (3 + M_1 - M_2) \\
0 & 0 \\
\hline
\end{array}$$

$$\begin{array}{|c|c|}
\hline
\tilde{AW}_r^{(0,1/2)} & \tilde{AW}_r^{(1/2,0)} \\
\hline
\frac{-16}{3} \sqrt{2} (3 + M_1 + 2 M_2) & 0 \\
\frac{16}{3} \sqrt{2} (3 + 3 J - M_1 + M_2) & \frac{-16}{3} \sqrt{2} (M_1 + 2 M_2) \\
0 & \frac{16}{3} \sqrt{2} (3 J - M_1 + M_2) \\
0 & 0 \\
H_0 - \frac{16}{3} (-3 + M_1 - M_2) & \frac{-16}{3} (3 J + M_1 - M_2) \\
\frac{-16}{3} (3 + 3 J - M_1 + M_2) & H_0 + \frac{16}{3} (M_1 - M_2) \\
0 & 0 \\
\hline
\end{array}$$

(186)
The matrix $M_{(1/2)}^{3}$ of the spinor is

$$
\begin{array}{ccc}
\chi^{(0,1)} & \chi^{(1/2,1/2)} & \chi^{(1,0)} \\
-4(3+M_{1}-M_{2}) & -2(3J+M_{1}-M_{2}) & 0 \\
-4(3+3J^{3}-M_{1}+M_{2})/3 & 3 & -4(3+3J+M_{1}-M_{2}) \\
0 & -8 & 3 \\
4\sqrt{2}(3+2M_{1}+M_{2})/3 & 2\sqrt{2}(3J+M_{1}-M_{2})/3 & 0 \\
0 & 2\sqrt{2}(6+2M_{1}+M_{2})/3 & 4\sqrt{2}(3+J+M_{1}-M_{2})/3 \\
-4\sqrt{2}(3+3J-M_{1}+M_{2})/3 & -2\sqrt{2}(6+M_{1}+2M_{2})/3 & 0 \\
0 & -2\sqrt{2}(3J-M_{1}+M_{2})/3 & 2(3+3J+M_{1}-M_{2})/3 \\
-2(3+3J-M_{1}+M_{2})/3 & 2(3+3J-M_{1}+M_{2})/3 & 0 \\
\end{array}
$$

\[
\chi_{\xi}^{(0,1/2)}
\begin{array}{cc}
4\sqrt{2}(3+M_{1}+2M_{2})/3 & 0 \\
-4\sqrt{2}(3+3J-M_{1}+M_{2})/3 & 4\sqrt{2}(M_{1}+2M_{2})/3 \\
0 & -4\sqrt{2}(3J-M_{1}+M_{2})/3 \\
4(-6+M_{1}-M_{2})/3 & 4(3J+M_{1}-M_{2})/3 \\
4(3+3J-M_{1}+M_{2})/3 & -4(3+3J-M_{1}-M_{2})/3 \\
0 & 0 \\
0 & 2\sqrt{2}(6+M_{1}+2M_{2})/3 \\
2\sqrt{2}(3+3J-M_{1}+M_{2})/3 & 0 \\
\end{array}
\]

\[
\chi_{\xi}^{(1/2,0)}
\begin{array}{cc}
4\sqrt{2}(3J+M_{1}-M_{2})/3 & 0 \\
-4\sqrt{2}(2M_{1}+M_{2})/3 & 4\sqrt{2}(3+3J+M_{1}-M_{2})/3 \\
0 & -4\sqrt{2}(3+2M_{1}+M_{2})/3 \\
0 & 4\sqrt{2}(3+3J+M_{1}-M_{2})/3 \\
0 & 0 \\
0 & -4(6+M_{1}-M_{2})/3 \\
4(-3+M_{1}-M_{2})/3 & 4\sqrt{2}(2M_{1}+M_{2})/3 \\
4(3J-M_{1}+M_{2})/3 & 4\sqrt{2}(M_{1}+2M_{2})/3 \\
2\sqrt{2}(6+2M_{1}+M_{2})/3 & 2\sqrt{2}(3+3J+M_{1}-M_{2})/3 \\
2\sqrt{2}(6+3J+M_{1}-M_{2})/3 & -16 \\
\end{array}
\]

(187)
The matrix $M_{(0)(1)^2}$ of the two-form is

$$
\begin{array}{c|c|c}
 & Z_c^{(0,3/2)} & Z_c^{(1/2,1)} \\
\hline
H_0 & -64 (M_1 - M_2) & \frac{-64 (3+3J+M_1 - M_2)}{9} \\
-64 (3+3J-M_1+M_2) & -64 (9+M_1-M_2) & -128 (3+3J+M_1-M_2) \\
\frac{3}{9} & \frac{9}{9} & \frac{9}{9} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
32 \sqrt{2} (6+2M_1+M_2) & 32 \sqrt{2} (3J+M_1-M_2) & 0 \\
-16 \sqrt{2} (6+2M_1+M_2) & -16 \sqrt{2} (3J+M_1-M_2) & 64 \sqrt{2} (3+3J+M_1-M_2) \\
\frac{3}{9} & \frac{9}{9} & \frac{9}{9} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-16 (3+3J-M_1+M_2) & 32 (M_1-M_2) & 16 (3+3J+M_1-M_2) \\
9 & \frac{27}{27} & \frac{27}{27} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
$$

(188)
\[
\begin{array}{|c|c|c|}
\hline
Z_c^{(3/2,0)} & \tilde{Z}_c^{(0,3/2)} & \tilde{Z}_c^{(1/2,1)} \\
\hline
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{-64 (6+3J_1+M_1-M_2)}{3} & \frac{64 (-3+M_1-M_2)}{3} & \frac{-64 (6+3J_1+M_1-M_2)}{3} \\
H_0 + \frac{64 (3+M_1-M_2)}{3} & \frac{64 (6+3J_1+M_1-M_2)}{3} & \frac{64 (6+3J_1+M_1-M_2)}{3} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{32 \sqrt{2} (6+3J_1+M_1-M_2)}{3} & \frac{32 \sqrt{2} (6+3J_1+M_1-M_2)}{3} & \frac{32 \sqrt{2} (6+3J_1+M_1-M_2)}{3} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{-16 \sqrt{2} (6+3J_1-M_1+M_2)}{3} & \frac{-16 \sqrt{2} (6+3J_1-M_1+M_2)}{3} & \frac{-16 \sqrt{2} (6+3J_1-M_1+M_2)}{3} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{16 (6+3J_1-M_1+M_2)}{9} & \frac{16 (6+3J_1-M_1+M_2)}{9} & \frac{16 (6+3J_1-M_1+M_2)}{9} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{-32 (-3+M_1-M_2)}{27} & \frac{-32 (-3+M_1-M_2)}{27} & \frac{-32 (-3+M_1-M_2)}{27} \\
\frac{16 (3+3J_1-M_1+M_2)}{27} & \frac{16 (3+3J_1-M_1+M_2)}{27} & \frac{16 (3+3J_1-M_1+M_2)}{27} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

(189)
\[
\begin{array}{ccc}
\tilde{Z}_c^{(1,1/2)} & \tilde{Z}_c^{(3/2,1)} & Z_0^{(0,1)} \\
0 & 0 & \frac{8 \sqrt{2} (M_1+2 M_2)}{3} - \frac{8 \sqrt{2} (3+3 J-M_1+M_2)}{3} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-128 (3 J+M_1-M_2) & \frac{64 (-9+M_1-M_2)}{9} - \frac{64 (3 J-M_1+M_2)}{9} & \frac{64 (-6+M_1-M_2)}{3} \\
H_0 + \frac{64 (-9+M_1-M_2)}{9} - \frac{64 (3 J-M_1+M_2)}{9} & \frac{64 (3 J+M_1-M_2)}{9} - \frac{64 (3+3 J-M_1+M_2)}{9} & \frac{32 (-6+M_1-M_2)}{3} \\
\frac{64 \sqrt{2} (9+M_1+2 M_2)}{9} - \frac{64 \sqrt{2} (3 J-M_1+M_2)}{9} & \frac{32 \sqrt{2} (6+M_1+2 M_2)}{3} - \frac{32 \sqrt{2} (3+3 J-M_1+M_2)}{3} & \frac{32 (-6+M_1-M_2)}{3} \\
-16 \sqrt{2} (3 J-M_1+M_2) & -16 \sqrt{2} (6+M_1+2 M_2) & -16 \\
-16 (3 J-M_1-M_2) & -16 (3+3 J-M_1-M_2) & 0 \\
-32 (M_1-M_2) & -32 (3 J-M_1-M_2) & -8 \sqrt{2} (3+3 J-M_1+M_2) \\
27 & 9 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

(190)
| $Z_{o}^{(1/2,1/2)}$ | $Z_{o}^{(1,0)}$ | $Z_{r}^{(0,1)}$ |
|-----------------|----------------|----------------|
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $0$ | $0$ | $\frac{-8 \sqrt{2} (M_{1}+2 M_{2})}{3}$ |
| $8 \sqrt{2} (-3+M_{1}+2 M_{2})$ | $8 \sqrt{2} (-6+M_{1}+2 M_{2})$ | $8 \sqrt{2} (3+3 J+M_{1}+M_{2})$ |
| $\frac{3}{3}$ | $\frac{3}{3}$ | $\frac{-8 \sqrt{2} (3 J+M_{1}+M_{2})}{3}$ |
| $0$ | $0$ | $\frac{-8 \sqrt{2} (3 M_{1}+M_{2})}{3}$ |
| $-8 \sqrt{2} (3 J+M_{1}+M_{2})$ | $-8 \sqrt{2} (3+3 J+M_{1}+M_{2})$ | $0$ |
| $\frac{3}{3}$ | $\frac{3}{3}$ | $0$ |
| $8 \sqrt{2} (-3+2 M_{1}+M_{2})$ | $8 \sqrt{2} (2 M_{1}+M_{2})$ | $\frac{-32 (3+3 J+M_{1}+M_{2})}{3}$ |
| $\frac{3}{3}$ | $\frac{3}{3}$ | $\frac{32 (6+M_{1}+M_{2})}{3}$ |
| $0$ | $0$ | $0$ |
| $-16 (3 J+M_{1}+M_{2})$ | $\frac{32 (6+M_{1}+M_{2})}{3}$ | $-32$ |
| $\frac{3}{3}$ | $\frac{3}{3}$ | $0$ |
| $0$ | $0$ | $H_{0}$ |
| $-16 (3 J+M_{1}+M_{2})$ | $0$ | $-16 (3+3 J+M_{1}+M_{2})$ |
| $0$ | $0$ | $0$ |
| $0$ | $-16$ | $0$ |
| $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ |
\[
\begin{array}{ccc}
Z_r^{(1/2,1/2)} & Z_r^{(1,0)} & Z_c^{(0,1/2)} \\
\hline
0 & 0 & 0 \\
-8 \sqrt{2} (-3+M_1+2M_2) & -8 \sqrt{2} (-6+M_1+2M_2) & -8 \sqrt{2} (-3+3J-M_1+J_2) \\
8 \sqrt{2} (3J-M_1+M_2) & 8 \sqrt{2} (-3+3J-M_1+M_2) & 0 \\
0 & 0 & 0 \\
0 & 0 & 8 \sqrt{2} (-3+3J+M_1-M_2) \\
8 \sqrt{2} (3J+M_1-M_2) & 8 \sqrt{2} (3J+M_1-M_2) & -8 \sqrt{2} (2M_1+M_2) \\
-8 \sqrt{2} (-3+2M_1+M_2) & -8 \sqrt{2} (2M_1+M_2) & 0 \\
0 & 0 & -16 \sqrt{2} (3M_1+2M_2) \\
0 & 0 & 16 \sqrt{2} (3J-M_1+M_2) \\
-32 & -32 & 32 \sqrt{2} (3+J-M_1+M_2) \\
0 & 0 & 32 \sqrt{2} (3+J-M_1+M_2) \\
16 + H_0 & -16 (3+3J+M_1-M_2) & 0 \\
-8 (3J+M_1-M_2) & H_0 + 16 (4+M_1-M_2) & 0 \\
16 \sqrt{2} (3J+M_1-M_2) & 0 & 16 \sqrt{2} (3+3J-M_1+M_2) \\
16 \sqrt{2} (6+2M_1+M_2) & 32 \sqrt{2} (3+3J+M_1-M_2) & 0 \\
16 \sqrt{2} (6+M_1+2M_2) & 9 & 32 \sqrt{2} (3+M_1+2M_2) \\
16 \sqrt{2} (3J+M_1-M_2) & 9 & 8 \sqrt{2} (2M_1+M_2) \\
0 & 0 & 8 i \sqrt{2} (3+3J-M_1+M_2) \\
0 & 0 & 8 i \sqrt{2} (3+3J-M_1+M_2) \\
0 & 0 & 8 i \sqrt{2} (3+3J-M_1+M_2)
\end{array}
\]

(192)
\[
\begin{array}{|c|c|c|}
\hline
Z_c^{(1/2,0)} & \tilde{Z}_c^{(0,1/2)} & \tilde{Z}_c^{(1/2,0)} \\
\hline
0 & -8 (3 J + M_1 - M_2) & 0 \\
0 & 16 (M_1^3 - M_2) & -8 (3 J + M_1 - M_2) \\
0 & 8 (3 J - M_1 + M_2) & 8 (M_1^3 - M_2) \\
0 & 3 & 8 (3 J - M_1 + M_2) \\
0 & 0 & 8 (3 J - M_1 + M_2) \\
0 & 0 & 8 (3 J - M_1 + M_2) \\
0 & 0 & 8 (3 J - M_1 + M_2) \\
0 & 0 & 8 (3 J - M_1 + M_2) \\
8 (3 J + M_1 - M_2) & 0 & 8 (3 J - M_1 + M_2) \\
-16 (M_1^3 - M_2) & 0 & 8 (3 J - M_1 + M_2) \\
-8 (3 J + M_1 - M_2) & 0 & 8 (3 J - M_1 + M_2) \\
16 \sqrt{2} (3 J + M_1 - M_2) & 0 & 8 (3 J - M_1 + M_2) \\
\frac{3}{3} & 0 & 8 (3 J - M_1 + M_2) \\
-16 \sqrt{2} (M_1 + 2 M_2) & 0 & 8 (3 J - M_1 + M_2) \\
16 \sqrt{2} (3 J + M_1 - M_2) & 0 & 8 (3 J - M_1 + M_2) \\
\frac{3}{3} & 0 & 8 (3 J - M_1 + M_2) \\
32 \sqrt{2} (3 J + M_1 - M_2) & 0 & 8 (3 J - M_1 + M_2) \\
-32 \sqrt{2} (3 J + M_1 - M_2) & 0 & 8 (3 J - M_1 + M_2) \\
32 \sqrt{2} (3 J + M_1 - M_2) & 0 & 8 (3 J - M_1 + M_2) \\
\frac{9}{9} & 0 & 8 (3 J - M_1 + M_2) \\
0 & 16 (18 + 5 M_1 - 5 M_2) & 8 \sqrt{2} (3 J + M_1 - M_2) \\
0 & 8 \sqrt{2} (3 J - M_1 + M_2) & 8 \sqrt{2} (3 J - M_1 + M_2) \\
0 & 8 \sqrt{2} (6 + M_1 + 2 M_2) & 8 \sqrt{2} (6 + M_1 + 2 M_2) \\
0 & 8 i \sqrt{2} (6 + M_1 + 2 M_2) & 8 i \sqrt{2} (6 + M_1 + 2 M_2) \\
\hline
\end{array}
\]

(193)
| $Z_c^{(0,0)}$ | $\bar{Z}_c^{(0,0)}$ | $Z_f^{(0,0)}$ |
|-------------|----------------|-------------|
| 0           | 0              | 0           |
| 0           | 0              | 0           |
| 0           | 0              | 0           |
| 0           | 0              | 0           |
| 0           | 0              | 0           |
| 0           | 0              | 0           |
| $-16 \sqrt{2} (3+3 J+M_1-M_2)$ | $\frac{16 \sqrt{2} (6+M_1+2 M_2)}{3}$ | $\frac{8i \sqrt{2}}{3} (3 J - M_1 + M_2)$ |
| $\frac{16 \sqrt{2} (6+2 M_1+M_2)}{3}$ | $\frac{-8i \sqrt{2}}{3} (2 M_1 + M_2)$ | $\frac{8i \sqrt{2}}{3} (3 J + M_1 - M_2)$ |
| $H_0$       | $0$            | $H_0$       |
| 0           | $H_0$          |              |

(194)
B conventions

The indices $\alpha, \beta$ are the $SO(7)$ Lorentz indices, we have negative definite metric

$$\eta_{\alpha\beta} = (- - - - - -).$$  \hfill (195)

The indices on the generators of the coset $G/H$: \hfill \hfill (196)

$$a, b, c = 1, 2, 3$$

$$A, B, C = 4, 5, 6, 7$$

The indices on $H$: \hfill \hfill (197)

$$m, n, p, q = 9, 10, 11$$

$$N = 8$$

If written on the structure constants of $SU(3)$ we use $m, n, p, q$ instead of $a, b, c$.\hfill \hfill (198)

The generators of $SU(3)$ are given by $\frac{i}{2} \lambda_i$, where $\lambda_i$ are the Gell–Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$  \hfill (199)

The $SU(2)$ generators are: $\frac{i}{2} \sigma_a$

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \hfill (200)

The structure constants of $SU(3)$ are given by $f_{ijk} = f_{[ijk]}, [\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k$

$$f_{123} = \frac{1}{2}, \quad f_{147} = \frac{1}{2}, \quad f_{156} = \frac{1}{2}, \quad f_{246} = \frac{1}{2}, f_{257} = \frac{1}{2}, \quad f_{345} = \frac{1}{2}, \quad f_{367} = \frac{1}{2}$$

$$f_{458} = \frac{1}{2}, \quad f_{678} = \frac{1}{2}$$
C the $SO(7)$ spinor representation

For our purposes it is convenient to take the following $SO(7)$ Clifford algebra,

$$\tau_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & \frac{-i}{2} \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & -i & 0 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 0 & 0 & i & 0 & 0 & -\frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \end{pmatrix}$$

$$\tau_5 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_6 = \begin{pmatrix} 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & i & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tau_7 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The conjugation matrix,

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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