Parity Effects on Electron Tunneling through Small Superconducting Islands

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Abstract

Single electron tunneling into small superconducting islands is sensitive to the gap energy of the excitations created in the process and, hence, depends on the electron number parity. At low temperature the properties of the system are 2e-periodic in the applied gate voltage, turning e-periodic at higher temperature. We evaluate the tunneling rates and determine the probabilities for the even and odd state as well as the cross-over condition from the balance between different processes. Our analysis includes excitations in the leads. They are essential if the leads are superconducting. The influence of parity effects on single electron tunneling, Cooper pair tunneling, and the Andreev reflection in superconducting transistors yields rich structures in the I-V characteristic, which explains recent experimental findings.

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Recent experiments showed electron number parity effects in small superconducting islands [1–3]. In an electron box, if the energy gap is smaller than the charging energy, the voltage shows a characteristic long-short cycle, with a 2e-periodic dependence on an applied gate voltage. This has been interpreted as the even-odd asymmetry predicted by Averin and Nazarov [4]. At higher temperatures both parts of the cycle become more equal in length. Eventually above a cross-over temperature \( T_{cr} \) the e-periodic behaviour typical for normal metal electron boxes is recovered.

The even-odd asymmetry arises since single electron tunneling from an initial state with an even number of electrons on the superconducting island leads to a state with one extra electron - the odd one - in an excited state. In the small island, where charging effects prevent further tunneling, the odd electron does not find another excitation for recombination. As a result the energy of the final state lies above that of the equivalent normal system by the gap energy. Only at larger gate voltages another electron can enter the island, and the system relaxes to the ground state. The observed cross-over at higher temperatures to an e-periodic behavior was explained in Refs. [1–3] by free energy arguments for the island. This approach cannot be extended to describe nonequilibrium excitations in the leads. They are essential, however, if the leads are superconducting and are naturally included in the description presented below.

We evaluate the transition rates due to electron tunneling between a small superconducting island and normal or superconducting lead electrodes. The rates depend on the energy difference before and after the tunneling, which includes the gap energy \( \Delta_i \) of the quasiparticle excitations in the island, and \( \Delta_l \) if the lead is superconducting. At low temperatures processes which cost energy are suppressed. This makes the tunneling of the one excited electron, the odd one, an important process. The rate of this process is smaller by a factor \( 1/N_S \) than the total rate of the other \( N_S \) electrons in the condensate. On the other hand, the excitation energy in the island is regained. The balance between the rates determines the relative probabilities of even and odd number states.

We recover the results of Ref. [2] in the case of a normal-superconducting (NS) electron box. We further provide the extension to the superconducting-superconducting (SS) electron box and superconducting transistors. The single electron tunneling current through SSS and NSN transistors shows at low temperature a 2e-periodic plateau structure (for suitable parameters a double plateau). In NSN transistors with typical parameters this current is several orders of magnitude higher than the cotunneling current studied in Ref. [4]. At \( T > 0 \) both the even and odd states are occupied. Then the Cooper pair tunneling in SSS transistors and the 2e-tunneling via Andreev reflection in NSN transistors also connect two even or two odd states. This results in two e-shifted, 2e-periodic sets of current peaks, recently detected in Refs. [1,5,6].

At higher temperature the odd and the even state are equally populated leading to an e-periodic behaviour. The cross-over temperature \( T_{cr} \) is of the order of the gap energy \( \Delta_i + \Delta_l \), divided by the logarithm of an effective number of electrons. The latter is large, resulting in a substantial reduction.

We first consider an electron box [2], a closed circuit consisting of a small superconducting island connected via a tunnel junction with capacitance \( C_J \) to a lead electrode and via a capacitance \( C_g \) to a voltage source \( U_g \). The charging energy of the system depends on \( Q_g = C_g U_g \) and the number \( n \) of charges on the island.
\[ E_{ch}(n, Q_g) = \frac{(ne - Q_g)^2}{2C}. \]  

The total capacitance is \( C = C_J + C_g \). The normal state conductance of the junction \( 1/R_t = 4\pi e^2 N_i(0)V_i N_i(0)V_i |T|^2/\hbar \) can be expressed by tunneling matrix elements. The island has the normal density of states \( N_i(0) \) (per unit volume) and volume \( V_i \) (similar for the lead). After a tunneling process, starting from an even state, one electron — the odd one — on the island is in an excited state. Therefore, the energy of the final state differs from the initial one not only by the charging energy but also by the energy of the excitations \( \epsilon_i \geq \Delta_i \) (and \( \epsilon_l \geq \Delta_l \) if the lead electrode is superconducting).

Since the behaviour of the system is \( 2e \)-periodic in \( Q_g \) it is sufficient to consider \( 0 \leq Q_g \leq 2e \). We further assume that for \( 0 \leq Q_g \leq e \) the superconducting island is the even state. The transition rates due to single electron tunneling between the states with \( n = 0 \) and \( n = 1 \) electron charges (up or down) on the island are \([7,8]\)

\[ \Gamma^\pm = \frac{1}{e} \frac{1}{I_t(\delta E^\pm(Q_g))} \frac{1}{\exp[\delta E_{ch}(Q_g)/T] - 1}, \]  

This is the total rate due to the tunneling of one of the many electrons in the junction. It depends on the difference in the charging energy \( \delta E_{ch}(Q_g) = \pm[E_{ch}(1, Q_g) - E_{ch}(0, Q_g)] \). The minimum energy difference is \( \delta E^\pm = \pm\delta E_{ch}(Q_g) + \Delta_i + \Delta_l \). The energy of the excitations created in the tunneling process enter through the density of states into the well known classical quasiparticle tunneling characteristic \( I_t(eV) \) \([9]\). The asymptotic form of the tunneling rates for \( T \ll \Delta_i \) in the SN junctions is

\[ \Gamma(\delta E) = \begin{cases} \frac{\sqrt{\pi \Delta_i T/2}}{e^2 R_t} \exp\left(-\frac{\delta E}{T}\right) & \text{for } \delta E > 0, \\ \frac{1}{e^2 R_t} \sqrt{\delta E(\delta E - 2\Delta_i)} & \text{for } \delta E < 0, \end{cases} \]  

For SS junctions we have for \( |\delta E| > T \) near the onset

\[ \Gamma(\delta E) = \begin{cases} \frac{\pi (\Delta_i \Delta_l)^{1/2}}{2e^2 R_t} \exp\left(-\frac{\delta E}{T}\right) & \text{for } \delta E > 0, \\ \frac{\pi (\Delta_i \Delta_l)^{1/2}}{2e^2 R_t} & \text{for } \delta E < 0, \end{cases} \]  

The expressions \([3,4]\) describes single electron tunneling from a state with even electron number to one with an odd number (more precisely from an initial state, where no electron is in an excited state). On the other hand, if we consider the tunneling from an odd electron number state (where one electron is in an excited state) we have to take into account two types of transitions:

(i) One of the many electrons in the condensate of the island can tunnel back. The rate is also given by eq. \([3]\). The minimum energy difference is \( \delta E^-(Q_g) = -\delta E_{ch}(Q_g) + \Delta_i + \Delta_l \). Also in this transition we create two excitations, one on the island and one in the electrode. Hence \( \delta E^-(Q_g) \) is positive in the interesting range of \( Q_g \), and at low temperature the transition rate is exponentially small. After such a process the two excitations in the island can recombine fast and return the island to the ground state. The extra excitation in the electrode will diffuse rapidly further into the leads and recombine there. Nevertheless, energy was needed to create the two excitations, which enters into the transition rate.
Accordingly, the tunneling rate of the odd electron into any one of the states in the lead is
\[
\gamma(Q_g) = 2\pi |T|^2 N_l(0) V_i \int_0^\infty d\epsilon_i \int_{-\infty}^{\infty} d\epsilon f(\epsilon_i) = 1.
\]
Here, \(N_l(0) = \Theta(|\epsilon| - \Delta_i)\) are the reduced BCS densities of states. The rate is substantially different when the odd electron can tunnel from the lowest energy state \(\epsilon_i = \Delta_i\) and when it has to be in a higher excited state (with exponentially low probability). Accordingly,
\[
\gamma(Q_g) = \begin{cases} 
\frac{1}{2\pi R_i l(0)V_i} \sqrt{\frac{\Delta_i + \delta E_{ch}}{\Delta_i - \delta E_{ch}}} \exp \left(-\frac{\Delta_i - \Delta_i - \delta E_{ch}}{T}\right) & \text{for } \delta E_{ch} - \Delta_i + \Delta_i > T \\
\frac{\Delta_i - \delta E_{ch}}{2\pi R_i l(0)V_i} \sqrt{\frac{\Delta_i - \delta E_{ch}}{\Delta_i - \delta E_{ch}}} \exp \left(-\frac{\Delta_i - \Delta_i - \delta E_{ch}}{T}\right) & \text{for } \Delta_i - \Delta_i - \delta E_{ch} > T
\end{cases}
\]

The rate \(\gamma\) is reduced by a factor \(1/N_l(0)V_i\) compared to the transition rate \(\Gamma(2)\). On the other hand, \(\gamma\) may still be larger than \(\Gamma\) since it depends on the difference of the gaps \(\delta E = \Delta_i - \Delta_i - \delta E_{ch}\), whereas the sum of the gaps enters into \(\Gamma\). The three transitions between different states are visualized in the energy scheme in Fig. 1. Three states appear to play a role, but since the state with 2 excitations relaxes quickly to the ground state by quasiparticle recombination, effectively only one even state and the odd state need to be considered. However, the transition rate \(\Gamma^{eo}\) from the odd to the even state is the sum of the two relevant rates
\[
\Gamma^{eo}(Q_g) = \Gamma[\delta E^+] + \gamma.
\]

The master equation for the occupation probabilities of these states \(W_e(Q_g)\) and \(W_o(Q_g)\) is
\[
dW_e(Q_g)/dt = -\Gamma^{eo}(Q_g)W_e(Q_g) + \Gamma^{eo}(Q_g)W_o(Q_g) \quad \text{with } W_e(Q_g) + W_o(Q_g) = 1.
\]
The equilibrium solutions are
\[
W_{e(o)}(Q_g) = \frac{\Gamma^{eo}(Q_g)}{\Gamma^{eo}(Q_g) + \Gamma^{eo}(Q_g)}.
\]
where \(\Gamma^\Sigma(Q_g) = \Gamma^{eo}(Q_g) + \Gamma^{eo}(Q_g)\). For \(\Gamma^{eo} \gg \Gamma^{eo}\) we have \(W_e(Q_g) \approx 1\), i.e. the system occupies the even state, while for \(\Gamma^{eo} \gg \Gamma^{eo}\) the island is in the odd state. The cross-over occurs at \(W_e \approx W_o\), i.e.
\[
\Gamma^{eo}(Q_{cr}) \approx \Gamma^{eo}(Q_{cr}).
\]

At low temperatures the odd electron tunneling rate \(\gamma\) is large compared to \(\Gamma[\delta E^-(Q_g)]\). In this regime we find the even-odd asymmetric, 2\(e\)-periodic behaviour. At high temperature \(\gamma\) can be neglected and the cross-over between even and odd states occurs at the symmetry points \(Q_{cr} = e/2, 3e/2, \ldots\), leading to an \(e\)-periodic behaviour in \(Q_g\). The cross-over temperature \(T_{cr}\) between both regimes can be found from the condition \(\gamma(e/2) \approx \Gamma[\delta E^+(e/2)]\).

Up to numerical coefficients inside the logarithm we find for \(T \ll \Delta_i\)
\[ T_{cr} = \begin{cases} \Delta_i + \Delta_l - e & \text{for } \Delta_i < \Delta_l \\ 2\Delta_l / \ln N_S & \text{for } \Delta_i < \Delta_l \end{cases} \]  

where \( N(T) = N_0(T) \) for \( \Delta_l \ll T \ll \Delta_i \) and \( N(T) = N_S \) otherwise. The numbers \( N_0(T) = N_i(0)V_i \sqrt{2\pi \Delta_i T} \) and \( N_S = \pi N_i(0)V_i \Delta_i \) are the number of states available for quasiparticles near the gap and the effective number of superconducting electrons in the island, respectively.

The second half of the parameter range \( e \leq Q_g \leq 2e \) can be treated analogously. The tunneling now connects the states \( n = 2 \) and \( 1 \). The symmetry implies \( \Gamma_{eo/oe}(Q_g) = \Gamma_{eo/oe}(2e - Q_g) \). This leads to the following picture for \( T < T_{cr} \): For \( -Q_{cr} < Q_g < Q_{cr}(T) < e \) the system is in the even state, then it switches to the odd state where it stays for \( Q_{cr} < Q_g < 2e - Q_{cr} \), and periodic beyond.

The switching point is temperature dependent. For \( \Delta_l < \Delta_i \) we find for \( T < T_l \equiv 2\Delta_l / \ln N_S \)

\[ Q_{cr}(T) = \frac{C \Delta_i}{e} + \frac{e^2}{2e} - \frac{CT}{e} \ln N(T) \]  

whereas for \( T_l < T < T_{cr} \)

\[ Q_{cr}(T) = \frac{C}{e} (\Delta_i + \Delta_l) + \frac{e^2}{2e} - \frac{CT}{e} \ln N(T). \]  

For \( \Delta_l > \Delta_i \) the result (11) with \( N(T) = N_S \) is valid at any temperature \( T < T_{cr} \).

The expressions for \( Q_{cr} \) were derived for \( \Delta_i < E_C \equiv e^2/2C \). In the opposite case, \( \Delta_i > E_C \), for \( T < T_0 \) tunneling of 2 electrons becomes more favourable and the two-state model is not sufficient anymore. Here we defined

\[ T_0 = \begin{cases} (\Delta_i + \Delta_l - E_C) / \ln N(T) & \text{for } \Delta_l < \Delta_i - E_C \\ 2(\Delta_i - E_C) / \ln N_S & \text{for } \Delta_l > \Delta_i - E_C \end{cases} \]  

On the other hand, for \( T > T_0 \) the results given above remain valid.

In the case where the lead electrode is normal \( \Delta_l = 0 \) we reproduce the results of Refs. 11, which were derived from considerations of the free energy of the superconducting island only. If the electrode is superconducting the energy of excitations in the lead \( \epsilon_i \geq \Delta_l \) needs to be taken into account, since the transition rates depend on the total energy difference. In contrast to the excitation which is trapped in the island, the excitations in the lead diffuse away on a time scale fast in comparison to inverse tunneling rates. Hence, their energy cannot be regained. This behavior involving different time scales is not accounted for in an equilibrium free energy description.

The analysis presented above can also be extended to describe even-odd effects in NSN and SSS transistors. In this system the charging energy depends also on the transport voltage and on the number of electrons transported through the transistor. The total capacitance of the island \( C = C_g + C_l + C_r \) defines \( E_C = e^2/2C \). The energy differences for tunneling in the left and right junctions are

\[ \delta E_{ch,l/r} = E_C - \frac{e(Q_g \pm Q_{tr}/2)}{C} \]  

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For definiteness we assume \( C_l = C_r \) and \( eU_{tr} < 2E_C \), and we define \( Q_{tr} = CU_{tr} \). It is again sufficient to consider only one even and one odd state. The transitions in this system are described by a master equation with the rates

\[
\Gamma^{eo} \rightarrow \Gamma^{eo}_l + \Gamma^{eo}_r
\]  

(15)

which depend on the energy differences \( \delta E_l \) and \( \delta E_r \). They again are the sum of \( \delta E_{ch,l/r} \) and the energies of the excitations created in the island and electrodes. A similar relation holds for the transition from the odd state to the even state. After the substitution (15) the stationary solutions for the occupation probabilities are given by eq. (8), and the cross-over gate voltage \( \{Q_g\}_{cr} \equiv Q_{cr} \) follows from (9). It depends on the transport voltage, which opens the possibility to tune the cross-over condition. The effect becomes visible only for \( T < eU_{tr} \). Then the two dominant tunneling processes are tunneling in the left junction with rate \( \Gamma_l^{eo} \) followed by a tunneling process in the right junction \( \Gamma_r^{eo} \). Both balance each other in equilibrium. The cross-over temperature is defined by the condition

\[
Q_{cr}(T) \approx \frac{e}{2}.
\]

(19)

A second current peak or plateau exists in the window \( 3e/2 - \Delta C/e - Q_{tr}/2 < Q_g < 2e - Q_{cr} \). These plateaus merge forming a \( 2e \)-periodic single plateau structure. Note that the current (19) is much larger (by a factor \( 10^2 \) for parameters of ref. [6]) than the cotunneling current [4]. It can explain the presence of a constant current \( 80 \) \( fA \) of ”unknown origin” detected in Ref. [5].

The occupation probabilities \( W_e \) and \( W_o \), which are regulated by the single electron tunneling processes, also influence the supercurrent through SSS transistors and the Andreev reflection in NSN systems. If the SSS transistors is in the even state Cooper pairs can tunnel through the system at \( Q_g = \pm e, \pm 3e, ... \) for small transport voltages. This leads to a set of \( 2e \)-periodic sharp resonant peaks in the I-V curves. The amplitude of this peak is reduced below the \( T = 0 \) result \( I_e = I(T = 0)W_e(e) \). Analogously, if the system is in the odd state Cooper pair tunneling through the system occurs at \( Q_g = 0, \pm 2e, ... , \) with current peak amplitude \( I_o = I(T = 0)W_o(0) \). These probabilities are

\[
W_e(e) = \frac{e^{-2\Delta/T} + 1/N_S}{e^{-2(\Delta-E_C)/T} + 1/N_S} \quad W_o(0) = \frac{e^{-(2\Delta+E_C)/T}}{e^{-(2\Delta-E_C)/T} + 1/N_S}.
\]

(18)
At $T > T^* = 2\Delta / \ln(N_S)$ both current peaks are equal in height and the behaviour of the system is e-periodic. At lower temperature the two peaks are distinct. A behaviour of this type has recently been observed in experiments [10,11], the $T$-dependence is in qualitative agreement with the results of Ref [1].

If we consider NSN transistors we have to distinguish two cases. For $\Delta < E_C$ the single electron tunneling is crucial as discussed above. In the other limit $\Delta > E_C$ the mechanism of Andreev reflection transferring 2 electrons becomes important [10,11]. Close to $Q_g = \pm e$, $\pm 3e$, ... at $T = 0$ the shape of the current resonance

$$I_{res}(\delta U_g, U_{tr}) = G(\delta U_g, U_{tr}) \left( U_{tr} - \frac{4C_0^2}{U_{tr}C^2} (\delta U_g)^2 \right)$$

(19)

has been calculated in Ref. [11] for tunneling between even states. At finite temperature, due to single electron tunneling, there exists a finite probability for the odd state. This leads to an additional set of peaks due to 2e tunneling between two odd states. The amplitudes of the even-even and odd-odd current resonances then are $I_{ee} = I_{res}W_e(U_g - 2E_C, U_{tr})$ for $Q_g \approx e$ and $I_{oo} = I_{res}W_o(U_g, U_{tr})$ for $Q_g \approx 0$, where

$$W_e(Q_g \approx e) = \frac{e^{-(\Delta + E_C)/T} + 1/[N_0(T)] \cosh \frac{e\delta Q_g}{C^2}}{e^{-(\Delta - E_C)/T} + 1/[N_0(T)] \cosh \frac{e\delta Q_g}{C^2}}$$

(20)

$$W_o(Q_g \approx 0) = \frac{e^{-(\Delta + E_C - e|U_g|)/T}}{e^{-(\Delta - E_C + e|U_g|)/T} + 1/[N_0(T)] \cosh \frac{eU_g}{2T}}$$

(21)

Here $\delta Q_g = Q_g - e, E_C^\pm = E_C \pm eU_{tr}/2$. The two e-shifted peaks acquire equal height and the picture becomes e-periodic above a cross-over temperature $T^*_A = [\Delta + E_C]/ \ln N_0(T) \geq T_{cr}$. The presence of the odd peaks $I_{oo}$ has been clearly demonstrated in recent experiments [3].

For metals with a short elastic mean free path $l$ at low temperatures and transport voltages the effective conductance $G$ in eq. (19) near the maximum of the current is [12]

$$G(0, U_{tr}) = \frac{12d}{e^2 l R_t^2 p_S} \frac{\Delta^2}{\Delta^2 - (E_C^+)^2} \left[ \frac{\arctan \sqrt{\frac{\Delta + E_C^+}{\Delta - E_C^+}}}{(E_C^+)^2} \right].$$

(22)

Here $S$ is the junction area, $d$ the size of the normal leads which have resistance small compared to $R_t$. In contrast to the result quoted in Ref. [13], the conductance (22) remains finite at $T = eU_{tr} = 0$. For larger temperature or voltage $\delta = \max(T, eU_{tr}/2) > E_d = v_Fl/3d^2$ the conductance $G$ is smaller than (22) by a factor of order $(E_d/\delta)^{1/2}$ and roughly agrees with the result of Ref. [13] in this limit. The result (22) is consistent with the height of the experimentally observed current peaks [3].

In conclusion, we have developed a theory of parity effects in small superconducting islands by analyzing the rates of electron tunneling. Nonequilibrium excitations in the superconducting leads generated by the tunneling processes turn out to be essential. Our theory explains a large number of recent experiments in the electron box or transistors and can easily be extended to more general situations.

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Figure Captions :

Fig. 1 : Energy of different states in the NS electron box. Shown are the regular single electron tunneling transitions $\Gamma^{\pm}$ and the transition $\gamma$ due to the tunneling of the odd electron. The state with two excitations and energy $E > 2\Delta$ relaxes quickly to the ground state by quasiparticle recombination.