Hayward black holes in the novel 4D Einstein-Gauss-Bonnet gravity

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Abstract

We present an exact Hayward-like black hole solution in the 4D novel Einstein-Gauss-Bonnet (EGB) gravity and also analyse their thermodynamic properties to calculate exact expressions for the black hole mass, temperature, and entropy. Owing to the magnetic charge $g$ term in the metric, the thermodynamical quantities are corrected, and it is demonstrated that the Hawking-Page phase transition is achievable by showing diverges of the heat capacity at a critical radius $r = r_C$ where incidentally the temperature is maximum. Thus, we have a black hole with Cauchy and event horizons, and it’s evaporation leads to a thermodynamically stable extremal black hole remnant with vanishing temperature. The entropy does not satisfy the usual exact horizon Bekenstein-Hawking area law of general relativity with logarithmic area correction term.

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I. INTRODUCTION

Regular (or non-singular) black holes, widely considered to resolve the singularity problems, are dating back to Bardeen who gave the first regular black hole model by Bardeen [1], according to whom there are horizons but there is no singularity. There has been an enormous advance in the analysis and application of regular black holes and several interesting papers appeared uncovering properties [2–6]. Hayward [4] proposed, Bardeen-like, regular space-times are given that describe the formation of a black hole from an initial vacuum region which has a finite density and pressures, vanishing rapidly at large small and behaving as a cosmological constant at a small distance. The spherically symmetric Hayward’s metric [4] is given by

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{1} \]

with

\[ f(r) = 1 - \frac{2mr^2}{r^3 + 2l^2m}. \]

Here \( m \) is mass and \( l \) is constant. The solution (1), for \( l = 0 \), encompasses well known Schwarzschild black hole and Minkowski for \( m = 0 \). The regularity of the solution (1) is confirmed by the curvature invariants which are well behaved everywhere, including at \( r = 0 \).

The analysis \( f(r) = 0 \), imply a critical mass \( m^* = 3\sqrt{3}l/4 \) and critical radius \( r^* = \sqrt{3}l \), such that a regular extremal black hole with degenerate horizons \( r = r^* \) when \( m = m^* \). When \( m < m^* \), a regular non extremal black hole horizon with both Cauchy and event horizon, corresponding to two roots of \( f(r) = 0 \), and no black hole when \( m > m^* \) [4]. Further, it is shown that the event horizon is located near \( 2m \), while the inner one is close to \( l \) [7, 8]. Thus the metric (1) is a regular black hole, which asymptotically behaves as

\[ f(r) \sim 1 - \frac{2m}{r} \quad \text{as} \quad r \to \infty, \]

where near center

\[ f(r) \sim 1 - \frac{r^2}{l^2} \quad \text{as} \quad r \to 0. \]

which implies at large distance (\( r \)) it reproduces the Schwarzschild metric, while at the origin it is not only regular, but also has de Sitter form. The various property mentioned above makes the metric (1) simple for the analysis is its scaling behaviour [8]. In addition, Hayward black hole can be shown as exact model of general relativity coupled to nonlinear
electrodynamics and hence attracted significant attention in various studies, like Quasinormal modes of the black holes Lin by et al [9], the geodesic equation of a particle by Chiba and Kimura [10], wormholes from the regular black hole [11, 12] with their stability [13], black hole thermodynamics [14, 15] and related properties [16, 17], and strong deflection lensing [18]. The rotating regular Hayward metric has been also studied [19–22] and Hayward black hole solution in EGB has been discussed in [23].

Lovelock’s theory of gravitation resembles also with string inspired models of gravity as its action contains, among others, the quadratic Gauss-Bonnet term. This quadratic term naturally arises in the low energy effective action of heterotic string theory [24] and also in and it also appears in six-dimensional Calabi-Yau compactifications of M-theory [25] and the theory is free of ghosts about other exact backgrounds [26]. Indeed, the static spherically symmetric black hole, generalising the Schwarzschild solution, in EGB theory was first discovered by Boulware and Deser [26] to show that the only stable solution the central singularity is still unpreventable which is true to the charged analogous black holes as well [27]. The case 4-dimensional (4D) is special because the Euler-Gauss-Bonnet term becomes a topological invariant that does not contribute to the equations of motion or to the gravitational dynamics. However, a novel EGB gravity formulated in which the Gauss-Bonnet coupling has been rescaled as $\alpha/(D-4)$. The novel 4D EGB theory is defined as the limit $D \to 4$, which preserves the number of degrees of freedom thereby free from the Ostrogradsky instability [28]. Further, this extension of Einstein’s gravity bypasses all conditions of Lovelock’s theorem [29] and is also free from the singularity problem. Such a 4D black hole solutions have been obtained in the semi-classical Einstein’s equations [30], gravity theories with quantum corrections [31], and also recently in Lovelock gravity [32]. A cascade of subsequent interesting work analysed black hole solutions in the novel 4D EGB theory, this includes static, spherically symmetric metrics Reissner–Nordstrom-like solution [33], black hole surrounded by clouds of string [34], Vaidya-like solution [35] and generating black hole solutions [36], and its stability and quasi-normal modes [37, 38]

This paper searches for a static, spherically symmetric Hayward-like solution of the novel 4D EGB gravity namely whose charge is described by a nonlinear electrodynamics (NED), 4D Hayward-EGB black holes. Thus, we investigate a black hole solution in the novel 4D EGB gravity coupled to a NED theory. We study not only the structure of 4D Hayward-EGB black holes but also its thermodynamic properties including stability of the system.
particular, we explicitly bring out how the effect of NED can alter black hole solution and its thermodynamical properties.

II. BASIC EQUATIONS AND BLACK HOLE SOLUTION

We consider fully interacting theory of gravity minimally coupled to nonlinear electrodynamics (NED), whose action in $D$-dimensional spacetime is given by \[ \mathcal{I} = \int d^Dx \sqrt{-g} \mathcal{L} \equiv \mathcal{I}_{\text{gravity}} + \mathcal{I}_{\text{NED}}. \] (2)

The gravity action with the metric $g_{ab}$ is given by

\[ \mathcal{I}_{\text{gravity}} = \frac{1}{16\pi} \int d^Dx \sqrt{-g} \left[ R + \alpha \mathcal{L}_{\text{GB}} \right], \] (3)

where $R$ is Ricci scalar. The action (2) is modification of Einstein-Hilbert action with a quadratic curvature correction Gauss-Bonnet (GB) term, and $\alpha$ is GB coupling constant of dimensions $[\text{length}]^2$ and

\[ \mathcal{L}_{\text{GB}} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}. \]

We assume an NED action given by

\[ \mathcal{I}_{\text{NED}} = \frac{1}{16\pi} \int d^Dx \sqrt{-g} \mathcal{L}(F), \] (4)

with $F = F_{ab}F^{ab}/4$ where $F_{ab}$ is a filed strength tensor, such that $F_{ab} = \partial_a A_b - \partial_b A_a$. $A_a$ is the gauge potential with corresponding tensor field $\mathcal{L}(F)$. By rescaling $\alpha \to \alpha/(D-4)$ and varying the action (2), we get the following equations of motion

\[ G_{ab} + \frac{\alpha}{D-4} H_{ab} = T_{ab} \equiv 2 \left( \mathcal{L}_F F_{ab} F^b_c - \frac{1}{4} g_{ab} \mathcal{L}(F) \right), \] (5)

where, Einstein tensor $G_{ab}$ and Lanczos tensor $H_{ab}$ [39, 40] are

\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \]

\[ H_{ab} = 2 \left( RR_{ab} - 2 R_{ac} R^{c}_b - 2 R_{acbd} R^{cd} - R_{acde} R^{acde} \right) - \frac{1}{2} \mathcal{L}_{\text{GB}} g_{ab}, \] (6)

and $T_{ab}$ is the energy momentum tensor for the NED field. We consider the following $D$ dimensional Lagrangian density of NED field

\[ \mathcal{L}(F) = \frac{(D-1)(D-2)\mu^f}{4g^2} \frac{(2g^2 F)^{\frac{\nu-1}{2}}}{(1 + (\sqrt{2g^2 F})^{\frac{\nu-1}{2}})^2}, \] (7)
where

\[ F = \frac{g^{2(D-3)}}{2r^{2(D-2)}}. \]

The \(D\)-dimensional static, spherically symmetric metric ansatz reads as

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{D-2}, \tag{8} \]

with \(f(r)\) is the metric function to be determined and

\[ d\Omega^2_{D-2} = d\theta^2 + \sum_{i=2}^{D-2} \prod_{j=2}^{i} \sin^2 \theta_{j-1} \] \(d\theta^2, \tag{9} \]

is the line element of a \((D-2)\)-dimensional unit sphere \([41, 42]\). By using metric (8) in Eq. (5), we get the \((r, r)\) equation of motion \([43]\), which in the limit \(D \rightarrow 4\), takes the following form

\[ r^3 f'(r) + \alpha (f(r) - 1) \left( f(r) - 1 - 2rf'(r) \right) + r^2(f(r) - 1) = -\frac{6\mu'g^2r^4}{(r^3 + g^3)^2}, \tag{10} \]

whose integration leads us to the solution

\[ f_\pm(r) = 1 \pm \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + \frac{8M\alpha}{(r^3 + g^3)}} \right), \tag{11} \]

where \(\mu' = M\) is the Arnowitt-Deser-Misner (ADM) mass of the black hole. The \(\pm\) sign in front of the square root term in Eq. (11), corresponds to two different branches of solution. The two branches of the solution (11), in the limit \(\alpha \rightarrow 0\) or GR limit, behaves as

\[ f_-(r) = 1 - \frac{2Mr^2}{r^3 + g^3}, \]
\[ f_+(r) = 1 + \frac{2Mr^2}{r^3 + g^3} + \frac{r^2}{\alpha}. \tag{12} \]

Thus, the \(-ve\) branch corresponds to the Hayward solution (1) \((g^3 = 2ml^2)\) with positive gravitational mass, whereas the \(+ve\) branch reduces to the Hayward ds/AdS with negative gravitational mass. Whereas, in asymptotic large \(r \gg g\) limit, the two branches asymptotically take the form

\[ \lim_{r \rightarrow \infty} f_+(r) = 1 + \frac{2M}{r} + \frac{r^2}{\alpha} + O\left(\frac{1}{r^4}\right), \]
\[ \lim_{r \rightarrow \infty} f_-(r) = 1 - \frac{2M}{r} + O\left(\frac{1}{r^4}\right). \tag{13} \]
Now, the $-$ve branch corresponds to the Schwarzschild black hole solution, on the other hand, the $+$ve branch does correspond to Schwarzschild dS/AdS black hole with negative mass and not physical. Thus, we shall only consider the $-$ve branch of solution (11) can be identified as a static spherically symmetric Hayward-like regular black hole in the novel 4D EGB gravity, which reduces to the in Ref. [28] when one switch off the magnetic monopole charge ($g = 0$). Henceforth, we can call the solution (11) as the 4D Hayward-EGB black hole.

The weak energy condition states that $T_{ab}t^at^b \geq 0$ for all time like vectors $t^a$ [44], i.e., for any observer, the local energy density must not be negative. Hence, the energy conditions require $\rho \geq 0$ and $\rho + P_i \geq 0$, with $P_i = -\rho - \frac{r}{2} \rho'$

$$\rho = \frac{3g^2M}{(r^3 + e^3)^2}$$

$$\rho + P_2 = \rho + P_3 = \rho + P_4 = \frac{9g^2r^3M}{(r^3 + g^3)^2}. \quad (14)$$

Thus, one can notice that the 4D Hayward-EGB black hole satisfies the weak energy conditions.

The coordinate singularity of the metric Eq. (8) at $f(r) = 0$, implies that the black hole horizons exist. Thus, for the given values of $g$ and $\alpha$, the radii of the horizons are the zeros of

$$(r_H^3 + g^3)(r_H^2 + \alpha) - 2Mr_H^4 = 0. \quad (15)$$

The analysis of Eq. (15) leads us to find, for a given set of values $g$ and $M$, a maximum allowed value of Gauss-Bonnet coupling constant $\alpha = \alpha_0$ (cf. Fig. 1) such that for $\alpha < \alpha_0$, we get a black hole solution with double horizons, say $r_\pm$, where $r_-$ and $r_+$, respectively, denote the Cauchy and the event horizon. For $\alpha = \alpha_0$, we get an extremal black hole solution with degenerate horizon $r_+ = r_- \equiv r_E$. One can also find a maximum (or minimum) allowed value $g = g_0$ (or $M = M_0$) for the given values of $M$ and $\alpha$ (or $g$ and $\alpha$). From Fig. 1, it is evident that the maximum allowed value of $\alpha$ (or $g$), decreases as we increase the value of $g$ (or $\alpha$), whereas, minimum allowed value of $M$ increases with $\alpha$. We summarize the Cauchy and the event horizon radii of 4D Hayward-EGB black hole for various values of $\alpha$, $g$ and $M$, in Table. I. It is evident that the Cauchy (event) horizon radius increases (decreases) as we increase $\alpha$ and $g$ (cf. Table. I), but, this trend reverses for increasing $M$. 

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III. BLACK HOLE THERMODYNAMICS

The conclusion by Wheeler [45, 46] that any system consists of a black hole violates the non-decreasing entropy law necessitated to assign temperature and entropy to a black hole. This association of thermodynamical quantities with black holes led the study of black holes
as a thermodynamical system. The black hole mass, in terms of the horizon radius \( r_+ \) by solving equation \( f(r_+) = 0 \), reads

\[
M_+ = \frac{r_+}{2} \left[ (1 + \frac{\alpha}{r_+^2}) (1 + \frac{g^3}{r_+^2}) \right], \tag{16}
\]

by taking \( g = 0 \), one obtains mass for 4D EGB black holes \([28, 47, 48]\)

\[
M_+ = \frac{r_+}{2} \left[ 1 + \frac{\alpha}{r_+^2} \right], \tag{17}
\]

which further goes over to mass of Schwarzschild black holes \([15, 49]\) in GR limits (\( \alpha \to 0 \)), we have \( M_+ = r_+/2 \).

The Hawking temperature associated with horizon radius \( r_+ \), can be obtained through the relation, \( T_+ = \kappa/2\pi [50–52] \), where

\[
\kappa = \sqrt{-\frac{1}{2} \nabla_\mu \chi^\nu \nabla^\mu \chi^\nu} \equiv \frac{1}{2} \frac{\partial f(r)}{\partial r} \bigg|_{r=r_+},
\]

\[
\begin{array}{cccccc}
\hline
\ & M = 0.5, \ g = 0.3 & & M = 0.5, \ g = 0.4 & \\
\hline
\alpha & r_- & r_+ & \delta & \alpha & r_- & r_+ & \delta \\
0.05 & 0.288 & 0.919 & 0.631 & 0.01 & 0.332 & 0.926 & 0.594 \\
0.10 & 0.358 & 0.843 & 0.485 & 0.04 & 0.398 & 0.873 & 0.475 \\
\alpha_0 = 0.176 & 0.597 & 0.597 & 0 & \alpha_0 = 0.107 & 0.644 & 0.644 & 0 \\
\hline
\ & M = 0.6, \ \alpha = 0.1 & & M = 0.6, \ \alpha = 0.2 & \\
\hline
\delta & g & r_- & r_+ & \delta & g & r_- & r_+ & \delta \\
0.25 & 0.268 & 1.107 & 0.839 & 0.10 & 0.225 & 1.012 & 0.787 \\
0.40 & 0.432 & 1.044 & 0.612 & 0.25 & 0.365 & 0.982 & 0.617 \\
g_0 = 0.54 & 0.797 & 0.797 & 0 & g_0 = 0.43 & 0.753 & 0.753 & 0 \\
\hline
\ & g = 0.4, \ \alpha = 0.1 & & g = 0.4, \ \alpha = 0.2 & \\
\hline
\delta & M & r_- & r_+ & \delta & M & r_- & r_+ & \delta \\
0.80 & 0.352 & 1.513 & 1.161 & 0.80 & 0.425 & 1.442 & 1.017 \\
0.65 & 0.406 & 1.172 & 0.766 & 0.65 & 0.527 & 1.046 & 0.519 \\
M_0 = 0.493 & 0.6515 & 0.6515 & 0 & M_0 = 0.58 & 0.722 & 0.722 & 0 \\
\hline
\end{array}
\]
is the surface gravity. Then the temperature, on using (11), yields

$$T_+ = \frac{1}{4\pi r_+} \left[ \frac{r_+^2 - \alpha - 2g^2(r_+^2 + 2\alpha)}{(1 + \frac{g^2}{r_+^2})(r_+^2 + 2\alpha)} \right]. \quad (18)$$

The expression of temperature for 4D EGB black holes [28, 47, 48], we keep $g = 0$

$$T_+ = \frac{1}{4\pi r_+} \left[ \frac{r_+^2 - \alpha}{r_+^2 + 2\alpha} \right], \quad (19)$$

further, in GR limits ($\alpha \to 0$), one gets the temperature of Schwarzschild black holes [15, 49]

$$T_+ = \frac{1}{4\pi r_+}. \quad (20)$$

We analyse the behaviour of the Hawking temperature of 4D Hayward-EGB black hole through temperature $T_+$ vs horizon radius $r_+$ plots shown in Fig. 2. When 4D Hayward-EGB black hole undergoes Hawking evaporation, initially its temperature increases as the
event horizon shrinks, to reach a maximum value before decreasing rapidly to vanish at some particular value of horizon radius. One can notice that the Hawking temperature, at a particular value of horizon radius, $r^c_+$ (critical radius), possesses a local maximum, which means the first derivative of temperature vanishes leading to the divergence of specific heat. The numerical results of critical radius $r^c_+$ with corresponding maximum temperature $T^{Max}_+$ have been summarized in Table. II, from which it can notice that an increment in magnetic monopole charge $g$ or the Gauss-Bonnet coupling constant $\alpha$ results in the increased and decreased values of $r^c_+$ and $T^{Max}_+$, respectively.

By following the first law of black hole thermodynamics [53, 54]

$$dM_+ = T_+dS_+,$$

we get the expression for the entropy of the 4D Hayward-EGB black hole as

$$S_+ = \frac{A}{4} \left[ 1 - \frac{2g^3}{3r^5_+} (3r^2_+ + 2\alpha) + \frac{4\alpha}{r^2_+} \log(r_+) \right],$$

which does not follows the usual entropy-area law ($S_+ = A/4$) [51], where $A = 4\pi r^2_+$ is the black hole event horizon area, due to the presence of $\alpha$ and $g$. In the limit of $g = 0$, it goes over to

$$S_+ = \frac{A}{4} \left[ 1 + \frac{4\alpha}{r^2_+} \log(r_+) \right],$$

entropy of the 4D EGB black holes [47]. Further in GR limits ($\alpha = 0$), we get the entropy of the Schwarzschild black holes obeying area law. [23, 48].

| $\alpha$ | $g = 0.3$ | $g = 0.4$ | $\alpha = 0.1$ | $\alpha = 0.2$ |
| --- | --- | --- | --- | --- |
| $r^c_+$ | $T^{Max}_+$ | $r^c_+$ | $T^{Max}_+$ | $r^c_+$ | $T^{Max}_+$ | $r^c_+$ | $T^{Max}_+$ |
| 0.1 | 0.9656 | 0.0540 | 0.8293 | 0.0599 | 0.2 | 0.8293 | 0.0599 | 1.0931 | 0.0442 |
| 0.3 | 1.3862 | 0.0353 | 1.5021 | 0.0282 | 0.4 | 1.1293 | 0.0477 | 1.3282 | 0.0389 |
| 0.5 | 1.7165 | 0.0283 | 1.8032 | 0.0272 | 0.6 | 1.4964 | 0.0373 | 1.6573 | 0.0327 |

TABLE II: The critical radius ($r^c_+$) with corresponding maximum temperature ($T^{Max}_+$) of 4D Hayward-EGB black hole for different values of $\alpha$ and $g$. 
A. Stability Analysis

Next, we find the regions of local and global stability of the 4D Hayward-EGB black holes, respectively, via the behaviour analysis of the specific heat \( C_+ \) and Gibb’s free energy \( F_+ \) of the black holes. The local thermodynamical stability of thermodynamical systems is governed by the behaviour of their specific heat \( C_+ \), such that thermodynamical systems with \( C_+ > 0 \) and \( C_+ < 0 \), respectively, are locally stable and unstable. We calculate the specific heat of 4D Hayward-EGB black holes by using [43, 54]

\[
C_+ = \frac{\partial M_+}{\partial T_+} \equiv \left( \frac{\partial M_+}{\partial r_+} \right) \left( \frac{\partial r_+}{\partial T_+} \right),
\]

which reads

\[
C_+ = -2\pi r_+^2 \left[ \frac{(1 + \frac{g}{r_+^3})^2(r_+^2 + 2\alpha)^2 \left(1 - \frac{\alpha}{r_+^2} - 2\frac{g}{r_+^5}(r_+^2 + 2\alpha)\right)}{r_+^4 - \alpha(5r_+^2 + 2\alpha) - C\frac{g^2}{r_+^4} - D\frac{g^6}{r_+^8}} \right],
\]

with

\[
C = 2 \left( 5r_+^4 + 2\alpha(10r_+^2 + 7\alpha) \right) \quad \text{and} \quad D = 2(r_+^2 + 2\alpha)^2.
\]

By switching off magnetic monopole charge \((g = 0)\), one gets the specific heat of 4D EGB black holes [48]

\[
C_+ = -2\pi r_+^2 \left[ \frac{(r_+^2 + 2\alpha)^2 \left(1 - \frac{\alpha}{r_+^2}\right)}{r_+^4 - \alpha(5r_+^2 + 2\alpha)} \right],
\]

which further in GR limits \((\alpha = 0)\) retain the following value

\[
C_+ = -2\pi r_+^2,
\]

which is the expression of specific heat for the Schwarzschild black hole [15, 49]. We depict the numerical results of the specific heat of 4D Hayward-EGB black holes for various values of parameters in Fig. 3. It is evident that the black holes with very small horizon radius, \( r_+ < r_* \), having negative specific heat \((C_+ < 0)\) are locally unstable. The diverging specific heat of 4D Hayward-EGB black holes flips its sign from positive to negative at critical radius, \( r^c_+ \), confirming the existence of second-order phase transition [55, 56] between locally stable and unstable black holes. Hence, the black in the region, \( r_* < r_+ < r^c_+ \), are locally stable, whereas the black holes with radius \( r_+ < r_* \) and \( r_+ > r^c_+ \) are locally unstable (cf. Fig. 3).

It is also noteworthy that the value of a critical radius \( r^c_+ \) increases with \( g \) as well as \( \alpha \).
Further, we are going to examine the global stability of black hole via Gibb’s free energy, the reason for this is that even if the black hole is thermodynamically stable, it could be globally unstable or vice-versa [15]. The thermodynamical systems with negative Gibb’s free energy ($F_+ < 0$) are globally stable, whereas, on the other hand, those with positive free energy ($F_+ > 0$) are globally unstable [15]. The Gibb’s free energy of a black hole can be defined as [15, 57, 58]

$$F_+ = M_+ - T_+ S_+$$  \hfill (28)

to get Gibb’s free energy for 4D Hayward-EGB black holes

$$F_+ = -\frac{r_+}{4} \frac{1}{(1 + \frac{g^3}{r_+^3})(\frac{r_+^2}{r_+^2} + 2\alpha)} \left[-2(1 + \frac{g^3}{r_+^3})^2(1 + \frac{\alpha}{r_+^2})(\frac{r_+^2}{r_+^2} + 2\alpha)\right]$$

$$+ (r_+^2 - \alpha) \left(1 + \frac{4\alpha}{r_+^2} \log(r_+)\right) - A\frac{g^3}{r_+^3} + B\frac{g^6}{r_+^6}, \hfill (29)$$
with

\[ A = \frac{2}{3} \left[ 6r_+^2 + \frac{\alpha}{r_+^2} (1 + r_+^2 (5 + 12 \log(r_+)) + 2\alpha (-1 + 12 \log(r_+))) \right], \]
\[ B = \frac{4}{3} \left( 3r_+^2 + 4\alpha (2 + \frac{\alpha}{r_+^2}) \right), \]

which turns into the expression of free energy of 4D EGB black hole, when we take \( g = 0 \)

\[ F_+ = -\frac{r_+}{4(r_+^2 + 2\alpha)} \left[ -2(1 + \frac{\alpha}{r_+^2})(r_+^2 + 2\alpha) + (r_+^2 - \alpha) \left( 1 + \frac{4\alpha}{r_+^2} \log(r_+) \right) \right], \quad (30) \]

further, we take GR limits \((\alpha = 0)\), to get the Gibb’s free energy of Schwarzschild black holes \([15, 48],\)

\[ F_+ = \frac{r_+}{4}. \quad (31) \]

The behaviour of Gibb’s free energy with varying horizon radius of 4D Hayward-EGB black

\[ \text{FIG. 4: Gibb’s free energy } F_+ \text{ vs horizon } r_+ \text{ of 4D Hayward-EGB black hole for various values of } \alpha \text{ and } g. \]
holes has been depicted in Fig. 4. It is well known that the regions of parametric space with negative Gibb’s free energy are preferable because the black holes are globally stable there. The Gibb’s free energy behaviour analysis of 4D Hayward-EGB black holes lead us to find that the black holes with a smaller horizon radius with $F_+ < 0$ are globally stable, whereas those with larger horizon radius having positive Gibb’s ($F_+ > 0$) free energy are globally unstable. It is noteworthy that the black holes with a larger value of Gauss-Bonnet coupling constant $\alpha$ or magnetic monopole charge $g$ have a larger region of global stability.

B. Black hole remnant

The black hole undergoing the Hawking evaporation results in either an absolutely stable or a long-lived localized stage, which is known as black hole remnant. The candidature of black hole remnant as a source of dark matter [59] made the study of the black hole remnant
\[ g = 0.3 \]
\[ g = 0.4 \]
\[ \alpha = 0.1 \]
\[ \alpha = 0.2 \]

| \( \alpha \) | \( M_0 \) | \( r_0 \) | \( M_0 \) | \( r_0 \) | \( g \) | \( M_0 \) | \( r_0 \) | \( M_0 \) | \( r_0 \) |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.423 | 0.538 | 0.493 | 0.643 | 0.1 | 0.324 | 0.342 | 0.450 | 0.471 |
| 0.3 | 0.605 | 0.695 | 0.656 | 0.792 | 0.3 | 0.423 | 0.538 | 0.493 | 0.643 |
| 0.5 | 0.745 | 0.817 | 0.785 | 0.921 | 0.5 | 0.571 | 0.757 | 0.652 | 0.826 |
| 0.7 | 0.864 | 0.918 | 0.900 | 0.987 | 0.7 | 0.737 | 0.982 | 0.803 | 1.071 |

TABLE III: The remnant radius \((r_0)\) and the remnant mass \((M_0)\) of 4D Hayward-EGB black hole for different values of \((\alpha)\) and \(g\).

very interesting as well as important. To analyse the emitted feature of 4D Hayward-EGB black holes we plot the metric function \((11)\) vs varying radius of the extremal 4D Hayward-EGB black hole in Fig. 5. The numerical analysis of \(f'(r)|_{r=r_E} = 0\) leads us to find a minimum allowed value, \(M_0\) (remnant mass), of black hole mass with corresponding horizon radius \(r_0\) (remnant size), for the existence of black hole solution such that for \(M < M_0\), no black hole solution does exist. The numerical results of remnant mass \((M_0)\) and remnant size \((r_0)\) tabulated in Table. III lead us to conclude that the values of \(M_0\) and \(r_0\) increase with \(\alpha\) and \(g\).

IV. CONCLUSION

It is widely believed that the general relativity requires modification in the regions where the spacetime curvature becomes high and theory which is ultraviolet (UV) complete. It has been established that the addition of the higher order in curvature terms, can improve the UV properties of the Einstein gravity. EGB gravity, with quadratic curvature, has a number of additional nice properties than Einstein’s general relativity and also free from ghosts [26] Glavan and Lin [Phys. Rev. Lett. 124, 081301 (2020)] reformulated EGB gravity by rescaling of Gauss-Bonnet coupling constant to \(\alpha/(D-4)\) and taking limit \(D \rightarrow 0\) at the level field equations makes a non-trivial contribution to the gravitational dynamics even in \(D = 4\) [28]. The theory preserves the number of degrees of freedom and remains free from Ostrogradsky instability [28]. Hence, we have obtained an exact 4D Hayward-EGB black hole metric, characterized by three parameters, mass \((M)\), the Gauss-Bonnet coupling
constant ($\alpha$) and magnetic monopole charge ($g$), and it regains the 4D EGB [28] metric as a special case in the absence of magnetic charge ($g = 0$).

The analysis of horizon structure leads us to get a maximum allowed value of Gauss-Bonnet coupling constant, $\alpha_0$, for fixed values of $M$ and $g$ such that for $\alpha > \alpha_0$ no black hole solution. Whereas, $\alpha < \alpha_0$ and $\alpha = \alpha_0$, respectively, corresponded black hole with double and single degenerate horizon radius. We computed the Hawking temperature ($T_+$), entropy ($S_+$), heat capacity ($C_+$) and Gibb’s free energy ($F_+$) associated with the horizon radius of the black hole. In turns, we in details analysed the specific heat and found that there existed second-order phase transition with diverging $C_+$ at critical radius, $r_+^c$. The black holes with $r_+ < r_+^c$ found to be locally thermodynamical stable and on the other hand black holes with $r_+ < r_*^c$ and $r_+ > r_+^c$ found to be locally unstable. While the analysis of Gibb’s free energy $F_+$ leads us to find that 4D Hayward-EGB black holes with smaller horizon radius are globally stable with $F_+ < 0$, whereas those with the larger value of horizon radius having $F_+ > 0$ are globally unstable. Finally, we have also shown that the 4D Hayward-EGB black holes evaporation results in a stable black hole remnant with zero temperature $T_+ = 0$ and positive specific heat $C_+ > 0$.

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