Angular and Linear Momentum of Excited Ferromagnets

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The angular momentum vector of a Heisenberg ferromagnet with isotropic exchange interaction is conserved, while under uniaxial crystalline anisotropy the projection of the total spin along the easy axis is a constant of motion. Using Noether’s theorem, we prove that these conservation laws persist in the presence of dipole-dipole interactions. However, spin and orbital angular momentum are not conserved separately. We also define the linear momentum of ferromagnetic textures. We illustrate the general principles with special reference to the spin transfer torques and identify the emergence of a non-adiabatic effective field acting on domain walls in ferromagnetic insulators.

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Mathematics can be very effective in guiding research when physical intuition fails, even in applied sciences such as condensed matter physics. An important tool is Noether’s theorem \cite{1} that helps identifying invariants or continuity equations from the fundamental symmetry properties of a given system. In the field of spintronics, for instance, Noether’s theorem has been used to express the spin current, i.e. the flow of spin angular momentum \cite{2}, in spin-orbit-coupled systems \cite{3}. In metallic ferromagnets spin current is carried by an imbalance between up-spin and down-spin electrons and therefore accompanied long-distance mass motion and strong Joule heating. Spin currents can also be carried by spin waves (magnons), thereby dissipating much less energy in some magnetic insulators with high crystal quality \cite{2}. Magnon mediated spin-current transport in various systems has received some attention in the past years \cite{4–7}. Schütz et al. \cite{8} demonstrated that magnons in a mesoscopic Heisenberg ring generate a persistent spin current under an inhomogeneous magnetic field. In magnetization textures particle-based \cite{9,11} as well as magnonic spin currents \cite{12,13} cause spin transfer torques that induce magnetization dynamics such as a domain wall (DW) motion. Direct imaging of a domain wall motion induced by thermally induced magnonic spin currents has been reported by Jiang et al \cite{10}. The spin transfer torque in magnetic insulators is usually ascribed to conservation of spin angular momentum, implicitly assuming that the exchange interaction is isotropic. However, whereas a negative domain wall velocity, i.e. opposite to the spin wave propagation direction, is the signature of a magnonic spin transfer torque \cite{13,15}, positive domain wall velocities were found in micromagnetic simulations \cite{17,21}. A conclusive explanation of the latter observation is still lacking. Even the spin current and the corresponding continuity equation in Heisenberg magnets has not yet been properly formulated \cite{22}.

Tatara and Kohno \cite{10} predicted domain wall motion by the force or (linear) momentum transfer felt by narrow domain walls at which electron spins are reflected. But Volovik \cite{23} noted that the linear momentum of magnetization dynamics is not invariant under spin rotations \cite{24} and explained this paradox by considering a dynamic equation for the spin degrees of freedom supplemented by a kinetic equation for the underlying incoherent fermionic excitations \cite{23,25}. However, this approach fails for ferromagnetic insulators, illustrating the need for a full understanding of linear and angular momentum transport in ferromagnets.

In this Letter, we formulate the angular and linear momentum of excited ferromagnets based on Noether’s theorem \cite{1}. Starting with the Landau-Lifshitz equation for magnetization dynamics and Maxwell’s equations for dipolar fields, we provide a systematic formulation of the conservation laws for the rotational and translational motion of spin excitations, e.g., magnons or domain walls, based on general symmetry principles. We show that in the presence of magnetic dipole-dipole interactions the spin current is not conserved, only the total angular momentum composed of spin and orbital component is. Noether’s theorem also leads us to a proper formulation of linear momentum in ferromagnets that identifies the non-dissipative linear momentum transfer (“effective field”) mechanism in magnetic textures.

The semiclassical dynamics of a ferromagnet is described by the Landau-Lifshitz equation

\[
\frac{\partial \mathbf{M}}{\partial t} = -\mathbf{M} \times \mathbf{H}_{\text{eff}},
\]

where \(\mathbf{M} = (M_x, M_y, M_z)\) is the magnetization vector with modulus \(M_0 = |\mathbf{M}|\) and \(\mathbf{H}_{\text{eff}} = -\delta E/\delta \mathbf{M}\) is the effective field expressed as the variational derivative of the energy \(E = \int \mathcal{H} dV\). The energy density

\[
\mathcal{H} = \frac{J}{2} (\nabla \mathbf{M})^2 + f (M_z) - \mathbf{M} \cdot \mathbf{h} - \frac{\hbar^2}{8\pi}
\]

consists of the exchange interaction, the magnetic dipole-interaction expressed by the field \(\mathbf{h}\), and we chose here an easy uniaxial anisotropy \(f\) along the \(z\)-axis. \(\mathbf{h}\) obeys Maxwell’s equations but for slow modulations considered here the magnetostatic approximation suffices \cite{24}:

\[
\nabla \times \mathbf{h} = 0; \quad \nabla \cdot (\mathbf{h} + 4\pi \mathbf{M}) = 0.
\]

We can write the Lagrangian density of the system as

\[
\mathcal{L} = -M_z \phi - \mathcal{H},
\]
where \( \phi = \arctan \left( \frac{M_y}{M_x} \right) \) is the azimuthal angle of \( \mathbf{M} \). With \( \mathbf{h} = \nabla \psi \), the first of Eqs. (5) is satisfied identically, while Eq. (1) and the second equation of (5) are the Euler-Lagrange equations

\[
\frac{\partial}{\partial x_i} \left( \frac{\partial \mathcal{L}}{\partial (\partial_q / \partial x_i)} \right) = \frac{\partial \mathcal{L}}{\partial q},
\]

where \( q = M_z, \phi, \psi; i = \{1, \ldots, 4\} \); \( x_{1,2,3} = x, y, z \); \( x_4 = t \).

We can now construct field invariants, i.e., a combination of the fields and their derivatives as functions of time and space that is conserved in time \([27]\). According to Noether’s theorem any continuous transformation of coordinates under which the variation of the action vanishes generates a definite invariant. We employ the global symmetries to obtain conservation laws for a closed system containing a magnetization texture and the associated dipolar field.

**Translation symmetry.** Spatial translational invariance leads to the conservation of linear momentum while a time translation symmetry gives rise to energy conservation. Application of Noether’s theorem leads to the continuity equation \( \partial T_{ik} / \partial x_k = 0 \) for the energy-momentum tensor

\[
T_{ik} = \left( \frac{\partial q}{\partial x_i} \frac{\partial}{\partial q} \partial (\partial q / \partial x_k) - \delta_{ik} \right) \mathcal{L},
\]

which can be derived from the invariance of the action under the spatiotemporal translation transformations \( \delta x_i = \delta q \delta x_i \) and \( \delta M_z = \delta \phi = \delta \psi = 0 \), where \( \delta_{ij} \) is the Kronecker function and \( \delta x_i \) are infinitesimal translations. \( T_{44} \) is the energy density and \( T_{ik} \) the linear momentum density

\[
p_i = T_{ik} = -M_z \frac{\partial \phi}{\partial x_i}.
\]

Hence, the total energy \( E = \int T_{44} dV \) and linear momentum

\[
P_i = \int T_{ik} dV
\]

are conserved.

This conservation law is complicated by the non-differentiability of the azimuthal angle \( \phi \) at the north \((\theta = 0)\) or south poles \( (\theta = \pi) [28] \). By parameterizing the spin variables in terms of \( \{M_x, M_y, M_z\} \) the momentum density \( (1 - \cos \theta) \frac{\partial}{\partial \theta} \) can be written as \( \mathbf{A} \cdot \nabla \), where \( \mathbf{A} = \left( M_y \mathbf{e}_z - M_z \mathbf{e}_y \right) / M_0 (M_0 + M_z) \) diverges on the line specified by the equations \( M_x = M_z = 0 \) and \( M_z = -M_0 \) (south pole). The singularity can be removed by employing an arbitrary in the Lagrangian \([4]\) that can be written as \( \mathcal{L} = (C - M_z) \phi - \mathcal{H} \), where the choice of the constant \( C \) does not affect the dynamics. While a given \( C \) cannot remove the singularities at two poles simultaneously.

The dynamic part of Lagrangian \([4]\) in terms of \( \mathbf{M} \) and \( \partial \mathbf{M} / \partial t \) coincides with that of a charged particle in a non-singular magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} = -\mathbf{M}/M_0^3 \) for \( M_0 \neq 0 \), in terms of a vector potential \( \mathbf{A} \). Varying \( C \) is therefore equivalent to a gauge transformation. The linear momentum \( p_i = (C - M_z) \partial \phi / \partial x_i \) is not invariant under these gauge transformations, but the difference between momenta of distinguishable states is. We illustrate this notion by the momentum of a 180° magnetic domain wall that is determined by path integrals of the form \( M_0 \int \mathbf{A} \cdot d\mathbf{M} \) along a trajectory connecting \( M_z = -M_0 \) and \( M_z = +M_0 \) [see \( l_1 \) or \( l_2 \) in Fig. 1]. The difference between the momenta is governed by the integral \( M_0 \int \mathbf{A} \cdot d\mathbf{M} \) along a closed contour, i.e., path \( l_2 l_1 \) of its plane. According to Stoke’s theorem, the integral in question can be represented as the flux of the vector \( \mathbf{B} = \nabla \times \mathbf{A} \) through the surface bounded by this contour

\[
\Delta P_{2-1} = M_0 \int_{S_{2-1}} \mathbf{B} \cdot d\mathbf{S},
\]

which is now gauge invariant. The spin electromotive force for electrons in moving magnetization textures is expressed by a similar contour integral \([29]\). Here we express the DW momentum as

\[
P_{DW} = M_0 \int \sin \theta d\theta d\phi
\]

where the momentum of a fully in-plane DW \((\phi = 0)\) defines the zero. When the DW plane is not twisted, the above equation leads to \( P_{DW} = 2 \phi M_0 \). The linear momentum carried by a rigid DW only depends on the tilt angle \( \phi \) of its plane. The conclusion that even a completely static domain wall has a finite linear momentum is counter-intuitive but can be rationalized in terms of the persistent angular momentum current generated by this topological defect.
Let us now consider the reflection of a spin wave with wave vector $\mathbf{k}$ by a planar domain wall as illustrated in Fig. 2(a). According to Eq. (8), the total linear momentum should be conserved during the scattering process

$$0 = \frac{d\mathbf{P}}{dt} = \frac{d}{dt} (\mathbf{P}_{\text{SW}} + \mathbf{P}_{\text{DW}}). \quad (11)$$

where $\mathbf{P}_{\text{SW}}$ and $\mathbf{P}_{\text{DW}}$ are the momenta of spin wave and domain wall, respectively. Therefore, we have

$$\mathbf{F}_{\text{DW}} = 2M_0 \alpha \mathbf{e}_z, \quad (12)$$

where $\mathbf{F}_{\text{DW}} = d\mathbf{P}_{\text{DW}}/dt$ is the force transferred by the spin wave to the domain wall by reflection. We find that spin wave reflection is possible only under simultaneous rotation of the domain wall. *Vice versa*, linear momentum transfer does not lead to linear motion of the domain wall, but a rotation of the domain wall plane. Linear momentum transfer is thereby shown to be equivalent to an “effective” Zeeman magnetic field. Note that when the axial symmetry is broken, spin wave reflection without coherent rotation becomes possible.

**FIG. 2:** (Color online) (a) Illustration of spin waves (wavy lines with arrows, red for incoming, aqua for reflection, and gray for transmission) scattered by a Bloch wall. $\mathbf{k}_i$ and $\mathbf{k}_f$ stand for the wave vectors parallel and perpendicular to the domain wall plane, respectively. $\Delta$ is the domain wall width. (b) Spin wave reflection by a DW in 3D with $\mathbf{k}_i = (\sqrt{2}/\Delta) (\mathbf{e}_x + \mathbf{e}_z)$ and $\mathbf{k}_f = 0.45 \times (2\pi/\Delta) \mathbf{e}_z$, obtained by solving the linearized Landau-Lifshitz equation coupled with Maxwell’s equation [32]. The anisotropy energy is $J (M_z - M_0) \frac{\partial \phi}{\partial x}$, where $\delta\phi = \delta M_z = \delta\psi = 0$, and $\delta\phi = -\delta\epsilon$, where $\delta\epsilon$ is the infinitesimal rotation parameter around $z$-axis. Defining the $z$-component of the angular momentum current density as

$$j_{\mu} = J \left( M_0^2 - M_z^2 \right) \frac{\partial \phi}{\partial x} - J \left( M_0^2 - M_z^2 \right) \frac{\partial \phi}{\partial x} \mathbf{r} \times \nabla \phi \times \mathbf{e}_z$$

$$+ \frac{1}{4\pi} \frac{\partial \phi}{\partial x} \mathbf{r} \times \nabla \phi \times \mathbf{e}_z + \epsilon_{\mu
u\lambda\tau} x_{\nu} L, \quad (13)$$

where $\epsilon_{\mu
u\lambda\tau}$ is the Levi-Civita symbol and $\mu = (1, 2, 3)$, as well as

$$j_{z4} = M_z - M_0 (\mathbf{r} \times \nabla \phi)_z, \quad (14)$$

Noether’s theorem leads us to the conservation law for the angular momentum along the $z$-axis

$$\frac{\partial j_{z4}}{\partial t} + \frac{\partial j_{z4}}{\partial x} = 0, \quad (15)$$

The first term in Eq. (14) is the spin angular momentum density [13], while the second one can be identified as an orbital angular momentum density since it can be written as $(\mathbf{r} \times \mathbf{p})$, where $\mathbf{p} = -M_0 \nabla \phi$ is the linear momentum density obtained before [Eq. (7)]. Noether’s theorem states that...
angular momentum Eq. (16) reads
\[ \nabla \mathbf{H} = \mathbf{j} \times \mathbf{H} \]
are related by the equations
\[ \hbar \partial_t \mathbf{J} \equiv \nabla \times \mathbf{H} = \mathbf{j} \times \mathbf{H} \]
where \[ \{ \text{the eigenvalue of the operator} \]
on the right-hand side are the spin
\[ \mathbf{J} \]
its and the orbital angular momentum
\[ \mathbf{L} = \int \epsilon_{ijk} \mathbf{J}_i \mathbf{x}_j \mathbf{T} dV, \quad i = \{1, 2, 3\} \]
does not vanish. The \( z \)-projection of the integrand, i.e. the orbital angular momentum density, agrees with the second term of Eq (14). Since \[ \mathbf{L}_z = \int \epsilon_{ijk} \mathbf{J}_i \mathbf{x}_j \mathbf{T} dV \]
non-zero \( \mathbf{L}_z \) is then not conserved. We note the analogy with the coupling of spins by the spin-orbit interaction of electrons in the weakly relativistic limit, a role that is played here by the dipole-dipole interaction. We note that the angular momentum density of the electromagnetic field \[ \mathbf{J} = \mathbf{r} \times (\mathbf{e} \times \mathbf{B}) \]
is negligibly small in the regime where the magnetostatic approximation [as in Eq. (3)] holds, in which the electric field \( \mathbf{e} \) plays no role whatsoever [26].

We now illustrate our results [Eq. (14)] for a uniform ferromagnetic nanocylinder \( (\mathbf{M} = M_0 \mathbf{n}) \) with uniaxial anisotropy \( f(\mathbf{M}) = -\frac{1}{2} K M^2 \) and spin wave excitation \( \mathbf{m} = (m_z, m_x, 0) \)
\( (\mathbf{M} = M_0 \mathbf{e}_z + \mathbf{m}) \). The Hamiltonian to leading order in \( \mathbf{m} \) is diagonalized by the Bogoliubov transformation
\[ m_+ = m_z + \text{i} m_y = \sqrt{\frac{2\hbar M_0}{V}} \sum_k \left[ a_k^+ a_k^* e^{i(k \mathbf{r} - \omega_k t)} + a_k^* a_k^+ e^{-i(k \mathbf{r} - \omega_k t)} \right], \]
where \( a_k^+ \) and \( a_k \) are Bose creation and annihilation operators while the coefficients \( u_k \) and \( v_k \) and the frequency \( \omega_k \) are related by the equations \( A_k u_k + B_k' v_k = \omega_k u_k, \quad B_k u_k + A_k' v_k = -\omega_k v_k \) and \( |u_k|^2 - |v_k|^2 = 1 \), with \( A_k = i M_0 k^2 + K M_0 + 2\pi M_0 (k_1^2 + k_2^2) k^2 \), \( B_k = 2\pi M_0 (k_1 + ik_2)^2 / k^2 \) and \( \mathcal{H} = \mathcal{H}_0 + \sum_k \hbar \omega_k a_k^* a_k^+ \) with \( \omega_k = \sqrt{A_k^2 - |B_k|^2} \). The total angular momentum Eq. (16) reads
\[ J_z - J_\phi = -\hbar \sum_k a_k^* a_k^+ - i \hbar \sum_k a_k^* \left( \mathbf{k} \times \nabla k \right) a_k, \]
where \( \nabla k \) is the gradient in \( \mathbf{k} \)-space. The first/second terms on the right-hand side are the spin/orbital angular momenta of the magnon excitations. By transforming from Cartesian to cylindrical coordinates \( \rho, \phi, \) and \( n \), and by \( a_k = \sum_n a_{\rho, \phi, \rho, n}^* \)
where \( \rho = \sqrt{k_1^2 + k_2^2} \) and \( n \) is the azimuthal quantum number
\[ \{ \text{the eigenvalue of the operator} \]
plane. Different \( n \) correspond to different wave-front shapes of the helical (vortex) spin wave modes. The dipole-exchange spin waves in cylindrical ferromagnetic nanowires display a rich wave pattern in the cross-section of the nanowire [34]. Vortex modes with high orbital angular momentum have been achieved in photonic [35] and electronic [36] wave guides using spiral phase-plates, computer-generated holograms, etc. It should be very interesting to generate helical spin wave modes and realize the conversion of angular momentum from orbit to spin experimentally. Spin waves with high orbital angular momenta would be efficient drivers of domain wall motion in axially symmetric nanocylinders that have been successfully fabricated and imaged recently [37]. The natural magnon mode in a cylindrical domain carries an orbital angular momentum \( n \hbar \). Spin waves propagating through a domain wall accumulate phase shifts [13] corresponding to an orbital angular momentum \( n' \hbar \) of transmitted waves \( (n \neq n' \) for complex wall structures in the presence of dipole-dipole interactions). The transfer of orbital angular momentum is enhanced for large \( |n - n'| \), leading to efficient domain wall motion.

To conclude, we formulate the conservation laws of linear and angular momenta in ferromagnetic textures in the presence of magnetic dipole-dipole interactions based on Noether’s theorem. We derive a well-defined linear momentum for insulating ferromagnets without involving incoherent fermionic excitations, thereby resolving a paradox raised by Volovik [23]. Mathematics helps to correct misguided physical intuition that naïvely associates linear momentum to domain wall translational motion. Instead, we show that linear momentum transfer of spin waves reflected at magnetic domain walls induces an effective field and steady rotation of the domain wall plane rather than translation. Only in the presence of dissipation this leads to domain wall propagation. Besides the usual spin angular momentum, we identify an orbital angular momentum for spin waves that is linked to the shape of their wave fronts. We expect to stimulate experiments on the preparation and manipulation of spin waves thereby opening a new research direction in the field of magnonics.

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[1] E. Noether, Nachr. Ges. Wiss. Göttingen 235 (1918).
[2] S. Maekawa, S.O. Valenzuela, E. Saitoh, and T. Kimura (eds), Spin Current (Oxford University Press, Oxford, 2012).
[3] Y. Wang, K. Xia, Z.B. Su, and Z.S. Ma, Phys. Rev. Lett. 96, 066601 (2006).
[4] M. Takigawa, N. Motoyama, H. Eisaki, and S. Uchida, Phys. Rev. Lett. 76, 4612 (1996).
[5] X. Zotos, Phys. Rev. Lett. 82, 1764 (1999).
[6] J.V. Alvarez and C. Gros, Phys. Rev. Lett. 88, 077203 (2002).
[7] F. Meier and D. Loss, Phys. Rev. Lett. 90, 167204 (2003).
[8] F. Schütz, M. Kollar, and P. Kopietz, Phys. Rev. Lett. 91,
5

017205 (2003); Phys. Rev. B 69, 035313 (2004).
[9] S.S.P. Parkin, M. Hayashi, and L. Thomas, Science 320, 190 (2008).
[10] G. Tatara and H. Kohno, Phys. Rev. Lett. 92, 086601 (2004).
[11] S. Zhang and Z. Li, Phys. Rev. Lett. 93, 127204 (2004).
[12] A.V. Mikhailov and A.I. Yaremchuk, JETP Lett. 39, 354 (1984).
[13] P. Yan, X.S. Wang, and X.R. Wang, Phys. Rev. Lett. 107, 177207 (2011).
[14] D. Hinze and U. Nowak, Phys. Rev. Lett. 107, 027205 (2011).
[15] A.A. Kovalev and Y. Tserkovnyak, Europhys. Lett. 97, 67002 (2012).
[16] W.J. Jiang, P. Upadhyaya, Y.B. Fan, J. Zhao, M.S. Wang, L.T. Chang, M.R. Lang, K.L. Wong, M. Lewis, Y.T. Lin, J.S. Tang, S. Cherepov, X.Z. Zhou, Y. Tserkovnyak, R.N. Schwartz, and K.L. Wang, Phys. Rev. Lett. 110, 177202 (2013).
[17] D.S. Han, S.K. Kim, J.Y. Lee, S.J. Hermasdorfer, H. Schultheiss, B. Leven, and B. Hillebrands, Appl. Phys. Lett. 94, 112502 (2009).
[18] M. Jamali, H. Yang, and K.J. Lee, Appl. Phys. Lett. 96, 242501 (2010).
[19] S.M. Seo, H.W. Lee, H. Kohno, K.J. Lee, Appl. Phys. Lett. 98, 012514 (2011).
[20] J.S. Kim, M. Stärk, M. Klüüi, J. Yoon, C.Y. You, L. Lopez-Diaz, and E. Martinez, Phys. Rev. B 85, 174428 (2012).
[21] X.G. Wang, G.H. Guo, Y.Z. Nie, G.F. Zhang, and Z.X. Li, Phys. Rev. B 86, 054445 (2012).
[22] F. Schütz, P. Kopietz, and M. Kollar, Eur. Phys. J. B 41, 557 (2004).
[23] G.E. Volovik, J. Phys. C 20, L83 (1987).
[24] F.D.M. Haldane, Phys. Rev. Lett. 57, 1488 (1986).
[25] C.H. Wong and Y. Tserkovnyak, Phys. Rev. B 80, 184411 (2009).
[26] D.D. Stancil, Theory of Magnetostatic Waves (Springer-Verlag, New York, 1993).
[27] N.N. Bogolyubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience, New York, 1959).
[28] N. Papanicolaou and T.N. Tomaras, Nucl. Phys. B 360, 425 (1991).
[29] S.E. Barnes and S. Maekawa, Phys. Rev. Lett. 98, 246601 (2007).
[30] N.L. Schryer and L.R. Walker, J. Appl. Phys. 45, 5406 (1974).
[31] P. Yan and G.E.W. Bauer, Phys. Rev. Lett. 109, 087202 (2012); IEEE Trans. Magn., in press (2013).
[32] Detailed calculations will be presented elsewhere.
[33] L. Allen, S.M. Barnett, and M.J. Padgett, Optical Angular Momentum (Institute of Physics Publishing, Bristol, 2003).
[34] R. Arias and D.L. Mills, Phys. Rev. B 66, 149903 (2002).
[35] X.L. Cai, J.W. Wang, M.J. Strain, B.J. Morris, J.B. Zhu, M. Sorel, J.L. O’Brien, M.G. Thompson, and S.Y. Yu, Science 338, 363 (2012).
[36] B.J. McMorran, A. Agrawal, I.M. Anderson, A.A. Herzing, H.J. Lezec, J.J. McClelland, J. Unguris, Science 331, 192 (2011).
[37] N. Biziere, C. Gatel, R.L. Balier, M.C. Clochard, J.E. Wegrowe, and E. Snoeck, Nano Lett. 13, 2053 (2013).