DENSITY OF POSITIVE LYAPUNOV EXPONENTS FOR SYMPLECTIC COCYCLES

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Abstract. We prove that $Sp(2d, \mathbb{R})$, $HSp(2d)$ and pseudo unitary cocycles with at least one non-zero Lyapunov exponent are dense in all usual regularity classes for non periodic dynamical systems. For Schrödinger operators on the strip, we prove a similar result for density of positive Lyapunov exponents. It generalizes a result of A. Avila in [2] to higher dimensions.

1. Introduction and main result

Let $f : X \to X$ be a homeomorphism of a compact metric space, and $\mu$ a $f$-invariant probability measure on $X$, and $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Suppose $A : X \to SL(n, \mathbb{F})$ or $GL(n, \mathbb{F})$ is a bounded measurable map, we can define the linear cocycle $(f, A)$ acting on $X \times \mathbb{F}^n$ as the following:

$$(x, y) \mapsto (f(x), A(x) \cdot y)$$

Some examples of linear cocycles are the derivative cocycle $(f, Df)$ of a $C^1$-map of arbitrary dimensional torus, the random products of matrices, Schrödinger cocycles, etc.

The main object of interest of linear cocycles is the asymptotic behavior of the products of $A$ along the orbits of $f$, especially the Lyapunov exponents. We consider the following definition.

The iterates of $(f, A)$ have the form $(f^n, A^n)$, where

$$A^n(x) := \begin{cases} A(f^{n-1}(x)) \cdots A(x), & n \geq 1 \\ \text{id}, & n = 0 \\ A(f^n(x))^{-1} \cdots A(f^{-1}(x))^{-1}, & n \leq -1 \end{cases}$$

The top Lyapunov exponent for the cocycle $(f, A)$ is defined by

$$L_1(A) = L(A) = L(f, \mu, A) = \lim_{n \to \infty} \frac{1}{n} \int \ln \|A^n(x)\| d\mu(x)$$

The $k$-th Lyapunov exponent is defined as,

$$L_k(A) := \lim_{n \to \infty} \frac{1}{n} \int \ln \sigma_k(A^n(x)) d\mu(x)$$

where $\sigma_k(A)$ is the $k$-th singular value of $A$. We denote $L^k(A) := \sum_{j=1}^{k} L_j(A)$.

The following remark gives the well-definedness of all the Lyapunov exponents:

Remark 1. For $A \in GL(n, \mathbb{F})$, we can define its natural action, $\Lambda(A)$ on the space $\Lambda^k(\mathbb{F}^n)$.

$$\Lambda^k(A) \cdot v_1 \wedge \cdots \wedge v_k := Av_1 \wedge \cdots \wedge Av_k$$

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As a result, for a cocycle \((f, A)\) acting on \(X \times \mathbb{R}^n\) we can define a cocycle \((f, \Lambda^k(A))\) on \((X, \Lambda^k(\mathbb{R}^n))\). By Oseledec theorem the top Lyapunov exponent of cocycle \((f, \Lambda^k(A))\) is \(L^k(A)\).

We say the Lyapunov exponent of linear cocycle \(A\) is positive if \(L(A) > 0\), the Lyapunov spectrum of \(A\) is simple if

\[ L_1(A) > \cdots > L_n(A). \]

An important problem in the study of dynamical system is whether we can approximate a system by one with hyperbolic behavior. In the setting of linear cocycles, we always assume the base dynamics \((f, \mu)\) is fixed and only the fiber dynamics \(A\) should be allowed to vary. Then we ask whether the given linear cocycle can be approximated by one with positive Lyapunov exponents (or simple spectrum) in some regularity classes.

Basically, the study of Lyapunov exponents of linear cocycles depends on the regularity classes and base dynamics. When the base dynamics has hyperbolicity, and the cocycles are in general position of some higher regularity spaces, they can in most cases "borrow" some hyperbolicity from the base dynamics. When \((f, \mu)\) is a random system, for example for products of random matrices, simplicity of Lyapunov spectrum was investigated in H.Furstenberg [21], H.Furstenberg and H. Kesten [22], Y.Guivarc'h and A.Raugi [25], etc. In particular, when the support of the distribution of random matrices is Zariski dense, we have simple Lyapunov spectrum, see I.Y.Gol’dsheid, G.A.Margulis’s result in [24] for example. And A.Avila and M.Viana [12] gave a criterion of simplicity of Lyapunov spectrum for linear cocycles over Markov map and proved the Zorich-Kontsevich conjecture.

When the ergodic system \((f, \mu)\) is hyperbolic, M.Viana [37] proved for any \(s > 0\), the set of \(C^s\)-cocycles with positive Lyapunov exponents are dense in \(C^s\)-topology. C.Bonatti and M.Viana [13] proved the cocycles of simple Lyapunov spectrum are dense in the space of fiber bunched Hölder continuous cocycles. For weaker hyperbolicity assumption on base dynamics, i.e. partial hyperbolicity, using the techniques of partially hyperbolic systems, A.Avila, J.Santamaria, M.Viana [11] proved there is an open dense subset in the space of fiber bunched Hölder continuous cocycles with positive Lyapunov exponents.

Of course there are many dynamical systems \(f\) without any hyperbolicity. A typical one is quasiperiodic systems. A dynamics system \((f, X, \mu)\) is called quasiperiodic if \((f, X)\) is an irrational rotation on torus preserving the Lebesgue measure \(\mu\). The theory of Schrödinger and \(SL(2, \mathbb{R})\)-cocycles over quasi-periodic systems are extensively studied, see [3], [9], [10], [8], [18], [4] for example. In particular, A.Avila proved the stratified analyticity of Lyapunov exponents and obtained a global theory for one frequency quasiperiodic analytic Schrödinger cocycles (see [4]).

Notice that for the results above in general we need some regularity restriction for the cocycles. For example some bunching condition (maybe non-uniformly) and Hölder continuity are usually necessary for discussing cocycles over hyperbolic base system, and analyticity is a suitable assumption for cocycles over quasi-periodic systems. It is more difficult to study Lyapunov exponents for linear cocycles over general base systems and regularity class. Using a semi-continuity argument and Kotani theory in [29], [30], A.Avila and D.Damanik proved that if the system \((f, \mu)\) is ergodic, then the set of cocycles with positive Lyapunov exponents is dense in \(C(X, SL(2, \mathbb{R}))\). A.Avila in [2] extended this result to all usual regularity classes of
SL(2, \mathbb{R})-cocycles using a local regularization formula proved by complexification method.

1.1. Main result for symplectic cocycles. In this paper, we generalize the result in [2] to symplectic, Hermitian-symplectic and pseudo unitary cocycles.

**Definition 1.** The Symplectic group over \( \mathbb{F} \), denoted by \( \text{Sp}(2d, \mathbb{F}) \), is the group of all matrices \( M \in \text{GL}(2d, \mathbb{F}) \) satisfying

\[
M^TJM = J, \quad J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.
\]

The Hermitian-symplectic group \( \text{HSp}(2d) \) is defined as:

\[
\text{HSp}(2d) = \{ M \in \text{GL}(2d, \mathbb{C}) : M^*JM = J \}.
\]

The pseudo unitary group \( U(d,d) \subset \text{GL}(2d, \mathbb{C}) \) is defined as:

\[
U(d,d) := \{ A : A^* \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} A = \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} \}.
\]

The special pseudo unitary group \( \text{SU}(d,d) \) is defined as:

\[
\text{SU}(d,d) := \{ A \in U(d,d), \det A = 1 \}.
\]

The special Hermitian symplectic group \( \text{SHSp}(2d) \) is defined as:

\[
\text{SHSp}(2d) := \{ A \in \text{HSp}(2d), \det A = 1 \}.
\]

The Lie algebras of these groups are denoted respectively by \( \text{sp}(2d, \mathbb{F}), \text{hsp}(2d), \text{u}(d,d), \text{su}(d,d), \text{shsp}(2d) \).

As in [2], we have the following definition for ample subspace of \( C(X, G) \), where \( G \) is a Lie group. Suppose \( \mathfrak{g} \) is the Lie algebra of \( G \).

**Definition 2.** A topological space \( \mathcal{B} \) continuously included in \( C(X, G) \) is ample if there exists a dense vector space \( \mathfrak{b} \subset C(X, \mathfrak{g}) \), endowed with a finer (than uniform) topological vector space structure, such that for every \( A \in \mathcal{B}, \exp(b)A \in \mathcal{B} \) for all \( b \in \mathfrak{b} \), and the map \( b \mapsto \exp(b)A \) from \( \mathfrak{b} \) to \( \mathcal{B} \) is continuous.

**Remark 2.** If \( X \) is a compact smooth or analytic manifold, then the usual spaces of smooth or analytic maps \( X \to G \) are ample in our sense.

In this paper we prove the following theorem:

**Theorem 1.** Suppose \( f \) is not periodic on \( \text{supp}(\mu) \), and let \( \mathcal{B} \subset C(X, \text{Sp}(2d, \mathbb{R})) \) be ample. Then the set \( \{ A : L(A) > 0 \} \) is dense in \( \mathcal{B} \).

**Corollary 1.** The same result in Theorem 1 holds if we replace \( \text{Sp}(2d, \mathbb{R}) \) by \( \text{SHSp}(2d), \text{SU}(d,d), \text{HSp}(2d) \) and \( U(d,d) \).

1.2. Stochastic Schrödinger operators and Jacobi matrices on the strip.

The most studied (Hermitian) symplectic cocycles are stochastic Schrödinger operator and Jacobi matrices on the strip, coming from the study of solid physics. For earlier studies of stochastic Jacobi matrices and Schrödinger operator on the strip and its relation to the Aubry dual of quasi-periodic Schrödinger operator, see [17], [27], [28], [35] for example.
Consider the following Jacobi matrices on the strip:\footnote{For more general Jacobi matrices with matrices entries, see \cite{34, 35} for example.}

\[ h_\omega : l^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow l^2(\mathbb{Z}, \mathbb{C}^d) \]  
\[ u \mapsto (h_\omega u)(n) = u(n + 1) + u(n - 1) + v_\omega(n) \cdot u(n) \]  
(1.3)

where the potential \( v_\omega(n) \) is a \( d \times d \) Hermitian matrix, when \( d = 1 \), it is 1-dimensional Schrödinger operator (on the line).

In this paper we always assume the potential is dynamically defined, i.e. \( v(\cdot)(n) := v(f^n(\cdot)) \), \( v : X \rightarrow \text{Her}(d) \) or \( \text{Sym}_d \mathbb{R} \) is a bounded measurable map, where \( (f, X, \mu) \) is defined as in the beginning of the paper, and \( \text{Her}(d) \), \( \text{Sym}_d \mathbb{R} \) are respectively the set of Hermitian matrices and symmetric \( d \times d \) matrices over the field \( \mathbb{F} \).

Then for energy \( E \), the corresponding eigen equation is the following:

\[ hu = Eu, \text{ with potential } v(f^n(x)). \]  
(1.4)

Notice that for any \( u : \mathbb{Z} \rightarrow \mathbb{C}^d \) satisfies (1.4), we have

\[ \begin{pmatrix} u(n + 1) \\ u(n) \end{pmatrix} = \begin{pmatrix} E - v(f^n(x)) & -I_d \\ I_d & \end{pmatrix} \cdot \begin{pmatrix} u(n) \\ u(n - 1) \end{pmatrix} \]  
(1.5)

Then the associated linear cocycle \( (f, A^{(E-v)} : X \times \mathbb{C}^n \rightarrow X \times \mathbb{C}^n) \) is defined by

\[ A^{(E-v)}(x) = \begin{pmatrix} (E : I_d - v(x)) & -I_d \\ I_d & \end{pmatrix} \]  
(1.6)

Notice that \( (f, A) \) is a (Hermitian) symplectic cocycle when \( E \in \mathbb{R} \).

As in \cite{2}, we denote \( L(A^{(E-v)}) = L(E - v) \). Using a similar method to the proof of Theorem 1, we prove the following result for (Hermitian) symplectic cocycles related to the stochastic Jacobi matrices on the strip with form in (1.3).

**Theorem 2.** Suppose \( f \) is not periodic on supp(\( \mu \)) and let \( V \subset C(X, \text{Her}(d)) \) or \( C(X, \text{Sym}_d \mathbb{R}) \) be a dense vector space endowed with a finer topological vector space structure. Then for any \( E \in \mathbb{R} \), the set of \( v \) such that \( L(E - v) > 0 \) is dense in \( V \).

Now we study Schrödinger operator on the strip. Suppose \( S \subset \mathbb{Z}^{v-1} \) is a finite connected set, we consider the following operator on \( l^2(\mathbb{Z} \times S) \).

\[ \tilde{u} \mapsto (h_\omega \tilde{u})(\alpha) = \sum_{|\beta - \alpha| = 1, \beta \in \mathbb{Z} \times S} \tilde{u}(\beta) + \tilde{v}_\omega(\alpha) \tilde{u}(\alpha) \]  
(1.7)

where for \( p = (x_1, \ldots, x_v), q = (y_1, \ldots, y_v) \), \( |p - q| := \sum_i |x_i - y_i| \). \( \tilde{v}_\omega \) is a process ergodic under the one-dimensional group of translations.

As in \cite{28}, (1.7) can be viewed as an example of the stochastic Jacobi matrices on the strip with form in (1.3). For example if \( S = \{1, \ldots, d\} \subset \mathbb{Z} \), then the associated Jacobi matrices on strip are as follows: for \( \tilde{u} \) satisfies (1.7), let \( u : Z \rightarrow \mathbb{C}^d \) such that

\[ u(n) = (u_1(n), \ldots, u_d(n)), u_i(n) = \tilde{u}(i, n) \]
then \( u \) satisfies (1.3) with potential

\[
v_\omega(n) = \begin{pmatrix}
\tilde{v}_\omega(n,1) & 1 \\
1 & \tilde{v}_\omega(n,2) & 1 \\
 & 1 & \tilde{v}_\omega(n,3) & 1 \\
 & & & \ddots & \ddots & \ddots \\
 & & & & 1 & \tilde{v}_\omega(n,k-1) & 1 \\
1 & & & & & 1 & \tilde{v}_\omega(n,k)
\end{pmatrix}
\]

For general \( S \), \( v_\omega \) always has entries of \( \tilde{v}_\omega \) as diagonal elements and with non-random off-diagonal elements (be 0 or 1).

For Schrödinger operator on the strip defined in (1.7), if \( \#(S) = d \), and we consider the embedding \( \mathbb{R}^d \hookrightarrow \text{Sym}_d \mathbb{R} \) by identifying a vector with the diagonal elements of a symmetric matrix, then each measurable bounded map \( v \in (X, \mathbb{R}^d) \hookrightarrow (X, \text{Sym}_d \mathbb{R}) \) induces a family of stochastic Schrödinger operator on strip, since the off-diagonal elements of the potential matrix are non-random.

For potential \( v : X \to \mathbb{R}^d \hookrightarrow \text{Sym}_d \mathbb{R} \) and energy \( E \), We denote by \( A^{(E-v)} \) the cocycle associated to the eigenequation \( \hbar u = Eu \) (with potential \( v(f^n(x)) \)) of Schrödinger operator on the strip. Then we have the following similar result to Theorem 2.

**Corollary 2.** Suppose \( f \) is not periodic on \( \text{supp}(\mu) \) and let \( V \subset C(X, \mathbb{R}^d) \) be a dense vector space with a finer topological vector space structure. Then for any \( E \in \mathbb{R} \), the set of \( v \) such that \( L(A^{(E-v)}) = L(E - v) > 0 \) is dense in \( V \).

### 1.3. Main difficulty of the proof, novelty of the paper and some remarks.

The key step in the proof of our result is to prove Theorem 4 in Chapter 4. Basically speaking, it proves that, if the one parameter family of cocycles \( R_\theta A \) satisfies \( L(R_\theta A) = 0 \) for positive Lebesgue measure of \( \theta \), then the function \( m^+ \) can determine \( m^- \), where \( m^+, m^- \) are defined as in Kotani theory and \([9]\), \( R_\theta \) is the higher dimensional rotation.

Before that, we have to prove the monotonicity of fibered rotation function of family \( R_\theta A \), see section 2.4 and Lemma 4.13. To prove it, we consider a cone field on the Lagrangian Grassmannian induced by the invariant cone on Lie algebra of Hermitian type. This idea is inspired by the study of Maslov index, see \([1]\) and \([20]\).

The main difficulty in the proof of Theorem 4 is to generalize Kotani theory to (Hermitian) symplectic cocycles. Basically the generalization of Kotani theory to Schrödinger cocycle on the strip is done by S.Kotani and B. Simon in \([28]\). Our approach is for general symplectic cocycles and is closer to the work of monotonic cocycles in \([9]\). Considering the geometry of Hermitian symmetric spaces and \( \Lambda^d(\mathbb{C}^{2d}) \), Theorem 5 will be further proved by some tricky calculations.

In addition, in Appendix, using techniques of monotonic symplectic cocycles (see \([31]\)) we generalize some results of periodic Schrödinger operators and \( SL(2, \mathbb{R}) \) cocycles in \([5],[2]\) to higher dimensions.

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3 After this paper was completed, I found my proof of monotonicity of fibered rotation function in Chapter 3 partially coincided with the work of Hermann Schulz-Baldes in \([34]\) and the work of Roberta Fabbrì, Russell Johnson and Carmen Núñez in \([19]\). Thanks Qi Zhou for the references.

4 In fact, we can define monotonicity for symplectic cocycles similarly. Using techniques in \([9]\) and this paper, we can get similar results to \([9]\) for symplectic cocycles, see \([31]\).
These ideas are partially inspired by the studies of monotonic cocycles [9], higher dimensional cocycles in [33] and [8], the geometry of Hermitian symmetric space in [15] and [20].

It is natural to ask whether the simplicity of Lyapunov spectrum holds for (Hermitian) symplectic cocycles in a dense set of some regularity set over general base dynamics. The difficulty to prove it is that when only some of the Lyapunov exponents coincides or vanishes we can not get too much dynamical information (for example, deterministic, see [28] or section 5) on the cocycles. It is also difficult to get the density of positive Lyapunov exponents for cocycles taking values in $SL(n, F)$ (except $SL(2, R)$), or even to get the simplicity of Lyapunov spectrum. The basic difficulty lies in the following, Kotani theory, [2] and our paper heavily depend on the fact that the associated groups act biholomorphically on some Hermitian symmetric spaces (see details in section 2.1), which is not true for $SL(n, F)$ unless $SL(2, R)$.

1.4. The structure of the paper. The outline of this paper is as follows: Chapter 2-6 are dedicated to the proof for Theorem 1, in fact the proof can be easily adapted to prove the rest of the results in this paper.

Chapter 2 is a short introduction of the geometry of symplectic action on Siegel upper half plane and its boundary.

Chapter 3 is dedicated to complexification of Lyapunov exponent and monotonicity of fibered rotation function, which implies an important equation in Lemma 16.

Chapter 4 is the proof for Theorem 4, which is the most difficult part in this paper.

Chapter 5, 6 are the rest of the proof of Theorem 1, based on the arguments (deterministic, semi-continuity, finitely-many-valued cocycles, local regularization formula, etc.) for Schrödinger and $SL(2, R)$ cocycles in [7, 2, 30, 36, 28].

Chapter 7 shows how to adapt the proof of Theorem 1 to Hermitian symplectic group case, pseudo unitary group case and Schrödinger operator on the strip.

The appendix is for some properties of generic periodic (Hermitian) symplectic cocycles which are used in the proof of Theorem 6.

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2. Geometry of the symplectic group action

2.1. Hermitian symmetric space, Bergman Shilov boundary and some notations. Basically speaking, Kotani theory, techniques of monotonic cocycles and Avila’s density result of Schrödinger and $SL(2, R)$-cocycles heavily depend on the fact that the group $SL(2, R)$ (or $SU(1, 1)$) acts biholomorphically on Poincaré upper half plane (or disc) in $C$. In particular, $SL(2, R)$ and $SU(1, 1)$ preserve corresponding Poincaré metric.

The higher dimensional extension of Poincaré upper half plane (or disc) are Hermitian symmetric spaces of non compact type. A Hermitian symmetric space is a Hermitian manifold which at every point has an inversion symmetry preserving
the Hermitian structure. Hermitian symmetric space appears in the theory of automorphic forms and group representations. An example is the Siegel upper half plane and its disc model, our proof heavily depends on the fact that symplectic group can act on it isometrically.

Bergman discovered that, different from the one variable case, for a large class of domain, a holomorphic function of several variables is completely determined by its values on a proper closed subset of the topological boundary of the domain. We call the minimal one the Bergman-Shilov boundary or Shilov boundary.

In this paper, Shilov boundaries of Hermitian symmetric spaces that we are interested in are the set of unitary symmetric matrix \( U_{\text{sym}}(\mathbb{C}^d) \) and unitary group \( U(d) \) which can be identified with real or complex Lagrangian Grassmannian.

We consider the following notations which will be used later.

**Definition 3.** For a pair of complex \( d \times d \) matrices \( M, N \), we denote \( M > N \) if \( (M - N)^* = M - N \) and \( M - N \) is positive definite.

**Definition 4.** For a square matrix \( M \), we denote \( \Im(M) := \frac{M - M^*}{2i} \). For a complex number \( z \), \( \Re(z) := \frac{z + z^*}{2} \), \( \Im(z) := \frac{z - z^*}{2i} \).

**Remark 3.** Later in Chapter 6 we will define \( \Re, \Im \) for an element in a complex Lie algebra \( \mathfrak{g}^\mathbb{C} \) with its real form \( \mathfrak{g} \), which coincides with the definition here when \( \mathfrak{g} = \mathbb{R}, \mathfrak{g}^\mathbb{C} = \mathbb{C} \).

**Definition 5.** We denote by \( \| \cdot \|_{\text{HS}} \) the Hilbert-Schmidt norm of matrix.

### 2.2. The symplectic action on the models of Siegel upper half plane.

We consider Siegel upper half plane and its disc model, which are the generalization of Poincaré upper half plane and Poincaré disc.

**Definition 6.** The Siegel upper half plane \( \text{SH}_d \) is defined as the following:

\[
\text{SH}_d := \{ X + iY \in \text{Sym}_d \mathbb{C}, X, Y \in \text{Sym}_d \mathbb{R}, Y > 0 \}
\]

**Definition 7.** We define \( \text{SD}_d \) as the set

\[
\{ Z \in \text{Sym}_d \mathbb{C}, I_d - ZZ > 0 \}
\]

Notice that \( \text{SD}_d \) is the set of complex \( d \times d \) symmetric matrices with operator norm less than 1.

Now we consider the symplectic action on \( \text{SH}_d \) and \( \text{SD}_d \).

**Lemma 1.** The symplectic group acts on the Siegel upper half plane transitively by the generalized Möbius transformations:

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2d, \mathbb{R}), Z \in \text{SH}_d, M \cdot Z := (AZ + B)(CZ + D)^{-1}
\]

The stabilizer of the point \( i \cdot I_d \in \text{SH}_d \) is \( \text{SO}(2d, \mathbb{R}) \cap \text{Sp}(2d, \mathbb{R}) \).

**Proof.** See [20].

Consider the Cayley element

\[(2.1) \quad C := \frac{1}{\sqrt{2}} \begin{pmatrix} I_d & -i \cdot I_d \\ I_d & i \cdot I_d \end{pmatrix} \]

then for all \( 2d \times 2d \) complex matrix \( A \), we denote \( A := CAC^{-1} \). We have:
LEMMA 2. (1). The map $A \mapsto \hat{A}$ is a Lie group isomorphism from $Sp(2d, \mathbb{R})$ to $U(d,d) \cap Sp(2d, \mathbb{C})$.

(2). The group $U(d,d) \cap Sp(2d, \mathbb{C})$ acts on the set $SD_d$ transitively by the generalized M"obius transformations: suppose $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(d,d) \cap Sp(2d, \mathbb{C})$ then

$$Z \in SD_d, M \cdot Z := (AZ + B)(CZ + D)^{-1}$$

(3). The Cayley element induces a fractional transformation identifying $SH_d$ with $SD_d$, i.e. for $Z \in SH_d, \Phi_C(Z) := (Z - i\cdot I_d)(Z + i\cdot I_d)^{-1}$, we have the following commutative diagram:

$$\begin{array}{ccc}
SH_d & \xrightarrow{A} & SH_d \\
\Phi_C & \downarrow & \Phi_C \\
SD_d & \xrightarrow{\hat{A}} & SD_d
\end{array}$$

Proof. See [20]. \qed

Now we define the projective model for $SH_d$ and $SD_d$. Consider the complex Grassmannian $G_{2d,d,\mathbb{C}}$, the set of all $d$-dimensional subspaces of $\mathbb{C}^{2d}$, and let $M_{2d,d}(\mathbb{C})$ be the space of all full rank $2d \times d$ complex matrices and view the columns of these matrices as a basis of a subspace of $\mathbb{C}^{2d}$.

If we consider the action of $GL(d, \mathbb{C})$ by right multiplication on $M_{2d,d}(\mathbb{C})$, then the Grassmannian is

$$G_{2d,d,\mathbb{C}} = M_{2d,d}(\mathbb{C})/GL(d, \mathbb{C})$$

For each $\begin{pmatrix} A \\ B \end{pmatrix}$, we use $\begin{bmatrix} A \\ B \end{bmatrix}$ to represent the class of $\begin{pmatrix} A \\ B \end{pmatrix}$. The projective model $SPH_d$ of $SH_d$ will be the set of all classes that admit a representative of the type

$$\begin{pmatrix} Z & I_d \end{pmatrix}$$

with $Z \in Sym_d \mathbb{C}, \text{Im}(Z) > 0$

The group action on $SPH_d$ is the left matrix multiplication by a representative of the class:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{bmatrix} Z \\ I_d \end{bmatrix} = \begin{bmatrix} AZ + B \\ CZ + D \end{bmatrix} = \begin{bmatrix} (AZ + B)(CZ + D)^{-1} \\ I_d \end{bmatrix}$$

The map connecting $SH_d$ to $SPH_d$ is

$$SH_d \rightarrow SPH_d$$

$$Z \mapsto \begin{bmatrix} Z \\ I_d \end{bmatrix}$$

Similarly we can define the projective model $SPD_d$ of the disc $SD_d$ as the set of classes in $M_{2d,d}(\mathbb{C})$ that admit a representative of the type:

$$\begin{pmatrix} Z \\ I_d \end{pmatrix}$$

with $Z \in Sym_d \mathbb{C}, \|Z\| < 1$

The symplectic action on $SPD_d$ and the identification between $SPD_d$ and $SD_d$ can be defined similarly.

2.3. The boundaries of different models.
2.3.1. **Stratification of finite and infinite boundaries.** All the properties in this subsection can be found in section 3 of [20].

Consider the boundary of $SD_d$ in $\text{Sym}_d \mathbb{C}$.

$$\partial SD_d = \{ Z^T = Z, \|Z\| = 1 \}$$

The Möbius transform is well-defined on $\partial SD_d$. Moreover, it has a stratification, the strata are, for $1 \leq k \leq d$,

$$\partial_k SD_d = \{ Z \in \partial SD_d : \text{rank}(I - ZZ^T) = d - k \}$$

In particular, $\partial_d SD_d = U_{\text{sym}}(\mathbb{C}^d) = U_d \cap \text{Sym}_d \mathbb{C}$, which is the *Shilov boundary* of $SD_d$, and it is an orbit of $U(d,d) \cap \text{Sp}(2d, \mathbb{C})$–action.

We can also take the closure of the Siegel upper half plane in $\text{Sym}_d \mathbb{C}$,

$$\overline{\mathcal{SH}_d} = \{ Z \in \text{Sym}_d \mathbb{C} : \text{Im}(Z) \geq 0 \}$$

and then map it to $\partial SD_d$ using the extensions of the map $\Phi_C, \Phi_C^{-1}$ defined in Lemma 2. Notice that $\Phi_C^{-1}$ is not defined on the set

$$\{ Z \in \partial SD_d, 1 \in \text{the spectrum of } Z \}$$

We call this set the *infinite boundary* and its complement in $\partial SD_d$ the *finite boundary*.

The finite boundary contains a part of every stratum. We have the following property: the image of the finite part of the stratum $\partial SD_d$ under the extension of $\Phi_C^{-1}$ is

$$\text{fin}(\partial_k \mathcal{SH}_d) = \{ Z \in \text{Sym}_d \mathbb{C} : \text{Im}(Z) \geq 0, \text{rank}(\text{Im}(Z)) = d - k \}$$

2.3.2. **An atlas of Shilov boundary.** Consider $\text{fin}(\partial_d \mathcal{SH}_d) = \text{Sym}_d \mathbb{R}$, then $\Phi_C$ restricted to $\text{Sym}_d \mathbb{R}$ gives a chart of

$$\{ Z \in \partial_d SD_d, 1 \notin \text{the spectrum of } Z \}$$

Similarly, for an element $g \in SL(2, \mathbb{R}), g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, composite with the Cayley element, we get a chart of a dense subset of $\partial_d SD_d$:

$$\Phi_{Cg} : \text{Sym}_d \mathbb{R} \rightarrow \{ Z \in \partial_d SD_d, \frac{a - ic}{a + ic} \notin \text{the spectrum of } Z \}$$

$$Z \mapsto ((a - ic)Z + (b + id))((a + ic)Z + (b + id))^{-1}$$

As a result, if we pick a sequence of $g_k$ such that $\frac{ak - ic_k}{ak + ic_k}$ take more than $d$ different values, then the family $\{ \Phi_{Cg_k} : \text{Sym}_d \mathbb{R} \rightarrow \partial_d SD_d \}$ give an atlas for $\partial_d SD_d = U_{\text{sym}}(\mathbb{C}^d)$.

2.4. **Invariant cone field and partial order.**

2.4.1. **Invariant cone field.** We construct an invariant cone field $\mathcal{C}$ on Shilov boundary. For $\text{fin}(\partial_d \mathcal{SH}_d) = \text{Sym}_d \mathbb{R}$, we consider a cone field $\{ h : h \in T\text{Sym}_d \mathbb{R}, h > 0 \}$. Then using the tangent map of $\Phi_{Cg_k}$ defined in last subsection, we get a cone field $\mathcal{C}$ on $TU_{\text{sym}}(\mathbb{C}^d)$.

It is easy to check that the cone field $\{ h > 0 \}$ is invariant under symplectic action. Therefore the cone field $\mathcal{C}$ is well-defined and invariant under $U(d,d) \cap \text{Sp}(2d, \mathbb{C})$–action.
2.4.2. Partial order defined on the universal covering space. For \( U_{sym}(\mathbb{C}^d) \), consider its universal covering space \( \hat{U}_{sym}(\mathbb{C}^d) \), denote the covering map by \( \Pi: \hat{U}_{sym}(\mathbb{C}^d) \to U_{sym}(\mathbb{C}^d) \). Then we can lift the cone field \( \mathcal{C} \) to a cone field \( \hat{\mathcal{C}} \) on \( \hat{U}_{sym}(\mathbb{C}^d) \), which is invariant under the lift of the \( Sp(2d, \mathbb{C}) \cap U(d, d) \)–action.

Using \( \hat{\mathcal{C}} \), we define a partial order "\(<" on \( U_{sym}(\mathbb{C}^d) \); we say \( \hat{Z}_0 < \hat{Z}_1 \), if there is an \( C^1 \) path \( p: [0, 1] \to \hat{U}_{sym}(\mathbb{C}^d) \) such that

\[
(2.2) \quad p(0) = \hat{Z}_0, p(1) = \hat{Z}_1, p'(t) \in \hat{\mathcal{C}}(p(t))
\]

For the determinant function restricted on \( U_{sym}(\mathbb{C}^d) \), we pick a continuous lift \( \det: \hat{U}_{sym}(\mathbb{C}^d) \to \mathbb{R} \), such that

\[
(2.3) \quad \pi \circ \hat{\det} = \det \circ \Pi
\]

where

\[
(2.4) \quad \pi: \mathbb{R} \to S^1, \pi(x) = e^{ix}
\]

To check "\(<" is actually a partial order and for later use, we have

**Lemma 3.**

1. Suppose \( \hat{Z} \in \hat{U}_{sym}(\mathbb{C}^d) \), for a path \( p: [0, 1] \to \hat{U}_{sym}(\mathbb{C}^d) \) such that \( p(0) = \hat{Z}, p'(0) \in \hat{\mathcal{C}}(Z) \), we have

\[
\left. \frac{d}{dt} \right|_{t=0} \hat{\det}(p(t)) > 0
\]

2. The order "\(<" defined in [2.2] is a strict partial order, i.e. there is no \( \hat{Z}, \hat{Z}_1, \hat{Z}_2 \in \hat{U}_{sym}(\mathbb{C}^d) \) such that \( \hat{Z} < \hat{Z} \) and \( \hat{Z}_1 < \hat{Z}_2 \).

3. For any \( Z \in U_{sym}(\mathbb{C}^d) \), any continuous lift of the path \( \theta \mapsto e^{2i\theta}Z \) is monotonic with respect to the order "\(<".

4. Any lift of the \( Sp(2d, \mathbb{C}) \cap U(d, d) \)–action preserves the order "\(<".

**Proof.** Suppose \( 1 \notin \) the spectrum of \( Z = \Pi(\hat{Z}) \), by computation for \( D\Pi(p'(0)) := H \in \mathcal{C}(Z) \), there is an \( h \in \text{Sym}_d \mathbb{R}, h > 0 \) such that \( H = -i(Z-1)h(Z-1) \). Then

\[
\det(\Pi(p(t))) = \det(Z + tH + o(t)) = \det(Z - it(Z-1)h(Z-1) + o(t)) = \det(Z) \det(1 - it(1 - Z^*)h(Z-1) + o(t))
\]

\[
= \det(Z) \det(1 + it(1 - Z^*)h(1 - Z) + o(t))
\]

notice that \( (1 - Z^*)h(1 - Z) \) is positive definite, lift to the covering space we have

\[
\left. \frac{d}{dt} \right|_{t=0} \hat{\det}(p(t)) > 0
\]

In the case \( 1 \in \) the spectrum of \( Z \), we can get other expression of the tangent vectors in \( \mathcal{C}(Z) \) by \( \Phi_{C_{g1}} \), and the proof is similar. In summary we get the proof of (1).

As a corollary, we have

\[
(2.5) \quad \text{if } \hat{Z}_1 < \hat{Z}_2 \text{ then } \hat{\det}(\hat{Z}_1) < \hat{\det}(\hat{Z}_2)
\]

which implies (2).

For (3), by taking the derivative, we need to prove \( iZ \in \mathcal{C}(Z) \). We only prove it in the case \( 1 \notin \) the spectrum of \( Z \), for other cases the proof is similar.
By computation, $C(Z) = \{-i(Z-1)h(Z-1), h > 0, h \in \text{Sym}_{d}\mathbb{R}\}$. Take $h = -Z(1-Z)^{-2}$, then it can be checked that $h \in \text{Sym}_{d}\mathbb{R}, h > 0$, and $-i(Z-1)h(Z-1) = iZ$, which implies $iZ \in C(Z)$.

(4). is the corollary of invariance of the cone field under the lift of the $Sp(2d, \mathbb{C}) \cap U(d, d)$-action. \hfill \Box

2.5. **Bergman metric and the volume form on $SD_d$**. In this section, we define the Bergman metric on $SD_d$ which is a generalization of Poincaré metric on the Poincaré disc. (see [32] for example) In particular, the symplectic group action preserves the Bergman metric.

**Definition 8.** Let $D$ be a bounded domain of $\mathbb{C}^n$, $d\lambda$ be the Lebesgue measure on $\mathbb{C}^n$, let $L^2(D)$ be the Hilbert space of square integrable functions on $D$, and let $L^2,h(D)$ denote the subspace consisting of holomorphic functions in $D$, the $L^2,h(D)$ is closed in $L^2(D)$.

For every $z \in D$, the evaluation $ev_z : f \mapsto f(z)$ is a continuous linear functional on $L^2,h(D)$. By the Riesz representation theorem, there is a function $\eta_z(\cdot) \in L^2,h(D)$ such that

$$ev_z(f) = \int_D f(\zeta)\bar{\eta}_z(\zeta)d\lambda(\zeta)$$

The Bergman kernel $K$ is defined by $K(z, \zeta) = \eta_z(\zeta)$.

**Definition 9.** Let $D \subset \mathbb{C}^n$ be a domain and let $K(z, w)$ be the Bergman kernel on $D$, consider a Hermitian metric on the tangent bundle of $T_z\mathbb{C}^n$ by

$$g_{ij}(z) := \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, z)$$

for $z \in D$. Then the length of a tangent vector $\xi \in T_z\mathbb{C}^n$ is given by

$$\|\xi\|_{B,z} := \sqrt{\sum_{i,j=1}^n g_{ij}(z)\xi_i \bar{\xi}_j}$$

This metric is called Bergman metric on $D$.

We denoted by $d$ the Bergman metric on $SD_d$. We have the following lemma for $d$.

**Lemma 4.** (1).For $A \in Sp(2d, \mathbb{R})$, $Z_1, Z_2 \in SD_d$,

$$d(AZ_1, AZ_2) = d(Z_1, Z_2).$$

(2).For $t \in (0, 1), t \cdot SD_d := \{tZ, Z \in SD_d\}$ is a bounded precompact set under metric $d$. We have $d(tZ_1, tZ_2) \leq td(Z_1, Z_2)$

**Proof.** (1) is the basic property of Bergman metric, i.e. Bergman metric is invariant under bi-holomorphic map.(2) see Lemma 6 of [15]. \hfill \Box

We give a explicit formula of Lebesgue density $d\lambda$ on $\text{Sym}_{d}\mathbb{C}$. Let $e_{ij}$ denote the matrix such that all the entries are 0 except the one in $i$-th row and $j$-th column is 1. Let $E_{ij} = e_{ii}$ and $E_{ij}(i \neq j) = e_{ij} + e_{ji}$, then $E_{ij}, i \leq j$ form a basis of $\text{Sym}_{d}\mathbb{C}$.

Then we can define the Lebesgue density on $\text{Sym}_{d}\mathbb{C}$, i.e.

\[(2.6) \quad |dE_{11} \wedge d\bar{E}_{11} \wedge \cdots \wedge dE_{ij} \wedge d\bar{E}_{ij} \wedge \cdots \wedge dE_{dd} \wedge d\bar{E}_{dd}|, i \leq j\]
For $Z \in SD_d$, let $V(Z)d\lambda(Z)$ be a volume form on $SD_d$ induced by the Bergman metric on point $Z$. Without loss of generality, we can assume $V(0) = 1$. Then we have:

**Lemma 5.** If $\sigma_i(Z), 1 \leq i \leq d$ are the singular values of $Z$, then

$$V(Z) = \Pi_{1 \leq i \leq d}(1 - \sigma_i(Z)^2)^{-(d+1)}$$

*Proof.* see Lemma 6 in [15] for a computation of Riemann tensor for Bergman metrics for general Hermitian symmetric space. □

3. Fibered rotation function and complexification of Lyapunov exponents

Let us now fix $A \in L^\infty(X,Sp(2d,\mathbb{R}))$. For $\sigma \in \mathbb{R}, t \geq 0, \sigma + it \in \mathbb{C}^+ \cup \mathbb{R}$, we consider the following deformation of $A$:

$$A_{\sigma+it} := \begin{pmatrix}
\cos(\sigma + it) \cdot I_d & \sin(\sigma + it) \cdot I_d \\
-\sin(\sigma + it) \cdot I_d & \cos(\sigma + it) \cdot I_d
\end{pmatrix} \cdot A$$

Notice that $A_{\sigma+it} = \begin{pmatrix}
e^{-t} & e^{it} \\
e^{i\sigma} & e^{-i\sigma}
\end{pmatrix} \cdot A$

The main aim of this chapter is the following theorem, which gives the complexification of $L^d(A)$:

**Theorem 3.** There is a function $\zeta$ defined on $\mathbb{C}^+ \cup \mathbb{R}$ satisfying the following properties:

1. $\zeta$ is a holomorphic on $\mathbb{C}^+$.
2. $\zeta$’s real part $\rho$ is continuous on $\mathbb{C}^+ \cup \mathbb{R}$, non-increasing on $\mathbb{R}$.
3. $-\zeta$’s imaginary part $= L^d(A_{\sigma+it})$, which is subharmonic on $\mathbb{C}^+ \cup \mathbb{R}$.

**Remark 4.** The function $\rho$ defined here is the fibered rotation function (up to multiply $2\pi$) in [9]. It is a generalization of fibered rotation number for a Schrödinger or $SL(2,\mathbb{R})$–cocycle homotopic to identity.

The strategy to prove Theorem 3 is similar to the discussion in section 2 of [9]. But different from the 1–dimensional case, the proof of monotonicity of the fibered rotation function is not trivial. We have to use the partial order defined in the last section.

3.1. Definition of $\zeta$. Denoted by $\Upsilon$ the set

$$\{A \in Sp(2d,\mathbb{R}), \hat{A} \cdot SPD_d \subset SPD_d\}$$

where $SPD_d$ is the projective model of the disc $SD_d$. For a matrix $A \in \Upsilon$, we can define the function $\tau_A : \overline{SD_d} \rightarrow GL(d,\mathbb{C})$ satisfying the following:

$$(3.1) \quad \hat{A} \left( \begin{array}{c} Z \\ 1 \end{array} \right) = \left( \begin{array}{c} \hat{A} \cdot Z \\ 1 \end{array} \right) \tau_A(Z)$$

In fact the Möbius transformation: $\hat{A} \cdot Z$ is well-defined for $A \in \Upsilon, Z \in \overline{SD_d}$: suppose

$$\hat{A} = \begin{pmatrix}
* & * \\
C & D
\end{pmatrix}$$
Since \( \hat{A} \cdot SD_d \subset SD_d \), then \( \begin{bmatrix} \ast & \ast \\ CZ + D \end{bmatrix} \in SD_d \), therefore \( CZ + D \) is invertible. As a result, \( \tau_A(Z) = CZ + D \).

For \( \Upsilon \), denoted by \( \hat{\Upsilon} \), its universal cover considered as a topological semi-group with unity \( \hat{id} \). Then there exists a unique continuous map \( \hat{\tau} \):
\[
(3.2) \quad \hat{\tau} : \hat{\Upsilon} \times SD_d \to C \text{ such that } \hat{\tau}(\hat{id}, Z) = 0 \text{ and } e^{i \hat{\tau}(A,Z)} = \det(\tau_A(Z))
\]
This map satisfies
\[
(3.3) \quad \hat{\tau}(\hat{A}_2 \hat{A}_1, Z) = \hat{\tau}(\hat{A}_2, \hat{\circ}_1 \cdot Z + \hat{\tau}(\hat{A}_1, Z))
\]
and the following lemma:

**Lemma 6.** For any \( \hat{A} \in \hat{\Upsilon} \), and any \( Z, Z' \in SD_d \),
\[
(3.4) \quad \Im \hat{\tau}(A, Z) = -|\ln \det(\tau_A(Z))|
\]
\[
(3.5) \quad |\Re \hat{\tau}(A, Z) - \Re \hat{\tau}(A, Z')| < d\pi
\]

**Proof.** (5.4) is the consequence of (5.2). For (5.5), suppose \( \hat{A} = \begin{pmatrix} C & D \\ \ast & \ast \end{pmatrix} \). Notice that \( \det(\tau_A(Z)) = \det(D) \det(1 + D^{-1}CZ) \), and by Proposition 2.3 of \[33\], \( \|D^{-1}C\| \leq 1 \). Then by well-definedness of Möbius transformation on \( SD_d \), the spectrum of matrix \( 1 + D^{-1}CZ, Z \in SD_d \) is contained in a half plane, which implies (5.5).

Now if \( \gamma : [0, 1] \to \Upsilon \) is continuous, and \( \hat{\gamma} : [0, 1] \to \hat{\Upsilon} \) is a continuous lift, we define \( \delta_\gamma \hat{\tau}(Z_0, Z_1) = \hat{\tau}(\hat{\gamma}(1), Z_1) - \hat{\tau}(\hat{\gamma}(0), Z_0) \); notice that it is independent of the choice of the lift.

For \( x \in X \), consider a path \( \gamma_x(s) := A_{l_{z_0,z_1}(s)}(x), s \in [0, 1], \) where \( l_{z_0,z_1} : [0, 1] \to C^+ \cup R \) is a continuous path such that \( l_{z_0,z_1}(0) = z_0, l_{z_0,z_1}(1) = z_1 \). Then we can define \( \delta_{z_0,z_1} \xi : X \times SD_d \times SD_d \to C \) by \( \delta_{z_0,z_1} \xi(x, Z_0, Z_1) = \delta_\gamma \hat{\tau}(Z_0, Z_1) \). Notice that \( \delta_{z_0,z_1} \xi \) is independent of the choice of \( l_{z_0,z_1} \).

Using the dynamics \( f : X \to X \), we define
\[
\delta_{z_0,z_1} \xi_n : X \times SD_d \times SD_d \to C \ni
\delta_{z_0,z_1} \xi_n(x, Z_0, Z_1) := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{z_0,z_1} \xi(f^k(x), A_{z_0}^k(x) \cdot Z_0, A_{z_1}^k(x) \cdot Z_1)
\]
where \( A^k(x) := A(f^{k-1}(x)) \cdots A(x) \).

We denote \( \delta_\xi \xi \) short for \( \delta_0,2 \xi \). As in [9], we study the limit of \( \delta_\xi \xi_n \):

**Lemma 7.** The limit of \( \Im \delta_\xi \xi_n(x, Z_0, Z_1) \) exists for \( \mu \)-almost every \( x \), all \( z \in C^+ \), and all \( Z_0, Z_1 \in SD_d \). Moreover it is independent of the choice of \( Z_1 \).

**Proof.** The proof of \( d = 1 \) can be found in Lemma 2.3 of [9]. For general \( d \), for any \( Z \in SD_d \) we identify \( Z \) with a vector \( v_1(Z) \wedge \cdots \wedge v_d(Z) \in \Lambda^d(C^{2d}) \), where \( v_i(Z) \) are the column vectors of the matrix \( \begin{pmatrix} Z \\ I_d \end{pmatrix} \). Therefore we define
\[
(3.6) \quad L^d(A, x) := \lim_{n \to \infty} \frac{1}{n} \ln \|\Lambda^d(A)(x)\|
\]
\[
(3.7) \quad L^d(A, x, Z) := \lim_{n \to \infty} \frac{1}{n} \ln \|\Lambda^d(A)(x) \cdot Z\|
\]
By Oseledec theorem, the limit exists for $\mu-$almost every $x \in X$ and all $Z_0 \in SD_d$.

We claim that for $\mu-$almost every $x \in X$, for all $z \in \mathbb{C}^+, Z_0, Z_1 \in SD_d$, (3.8)

$$\lim_{n \to \infty} \mathcal{H}_Z \delta_n(x, Z_0, Z_1) = L_d(A, x, Z_0) - L_d(A_z, x)$$

The proof is basically the same as Lemma 2.3 of [3], we only need to check the following: for $z \in \mathbb{C}^+, Z \in SD_d$, $[Z^T I_d]$ transverses to all the Oseledec stable subspace of $A_z \sigma(x)$.

Consider $A_{\sigma + t}$, $t > 0$. By Lemma 4, $A_{\sigma + t}$ uniformly contracts the Bergman metric of $SD_d$, and there exists a measurable function $m^+(\sigma + it, \cdot) : X \to SD_d$ which is bounded from $\partial SD_d$ and holomorphically depends on $\sigma + it$, such that (3.9)

$$m^+(\sigma + it, f(x)) = \mathcal{A}_{\sigma + it}(x) \cdot m^+(\sigma + it, x)$$

Moreover $m^+(\sigma + it, x) \in \text{Grass}(\mathcal{A}_{\sigma + it})$ in the Grassmannian represents the unstable direction of the cocycle $\mathcal{A}_{\sigma + it}$ (see remark of [33], or section 3 and section 6 of [8]).

In particular, for all $Z \in SD_d, z \in \mathbb{C}^+$, the distance $d_n(A_z(x) \cdot Z, m^+(z, f^n(x)))$ goes to 0 exponentially fast, by Oseledec theorem, $Z$ must transversal to all the Oseledec stable subspace.

**Lemma 8.** For all $z \in \mathbb{C}^+ \cup \mathbb{R}$, $\lim_{n \to \infty} \int_X \Re \delta_n(x, Z_0, Z_1) d\mu(x)$ exists for all $Z_0, Z_1 \in SD_d$. Moreover, it is continuous on $\mathbb{C}^+ \cup \mathbb{R}$ and independent of the choice of $Z_0, Z_1$.

**Proof.** For $z_0, z_1 \in \mathbb{C}^+ \cup \mathbb{R}, Z_0, Z_1 \in SD_d$, let

$$a_n(z_0, z_1, Z_0, Z_1) := \int_X \Re \delta_n(x, Z_0, Z_1) d\mu(x)$$

As in section 2 of [9], by (3.5) and (3.3), we have

(3.10) $$|\Re \delta_n(z_0, z_1, Z_0, Z_1) - \Re \delta_n(z_0, z_1, Z_0', Z_1')| < \frac{2d\pi}{n}$$

Then for any $n, l > 0$, by $f-$invariance of $\mu$ and last equation, (3.11)

$$a_n(z_0, z_1, Z_0, Z_1) = a_l(z_0, z_1, Z_0, Z_1)$$

$$\leq \left| a_n(z_0, z_1, Z_0, Z_1) - a_{n l}(z_0, z_1, Z_0, Z_1) \right| + \left| a_{n l}(z_0, z_1, Z_0, Z_1) - a(l, z_0, z_1, Z_0, Z_1) \right|$$

$$\leq \frac{1}{l} \int_X \sum_{j=0}^{n-l} \left| \Re \delta_j(z_0, z_1, x, Z_0, Z_1) \right|$$

$$+ \frac{1}{n} \int_X \sum_{j=0}^{l-1} \left| \Re \delta_j(z_0, z_1, x, Z_0, Z_1) \right|$$

$$\leq \frac{2d\pi}{n} + \frac{2d\pi}{l}$$
Since \( A \in L^\infty(X, Sp(2d, \mathbb{R})) \), for any \( Z_0, Z_1 \), \( a_n(\cdot, \cdot, Z_0, Z_1) \) is continuous. By (3.11), \( a_n(\cdot, \cdot, Z_0, Z_1) \) converges (uniformly on any bounded subset) to a continuous function on \((\mathbb{C}^+ \cup \mathbb{R})^2\). By (3.10) we know this limit function does not depend on the choice of \( Z_0, Z_1 \). □

**Remark 5.** By the same proof, we can prove for any measurable sections \( Z_0, Z_1: X \to SD_d \)

\[
\lim_{n \to \infty} \int_X \Re \delta_z \xi_n(x, Z_0(x), Z_1(x)) d\mu(x) \text{ exists and does not depend on the choice of } Z_0, Z_1.
\]

Now we can define function \( \zeta \):

\[
\zeta(z) := \lim_{n \to \infty} \int_X \Re \delta_z \xi_n(x, 0, 0) d\mu(x) - iL^d(A_z)
\]

then we claim \( \zeta \) satisfies all conditions of Theorem 3.

### 3.2. Subharmonicity and Holomorphicity

We have the following lemma for the subharmonicity of Lyapunov exponents.

**Lemma 9.** The map \( z \mapsto L^k(A_z) \) is a subharmonic function for \( A \in L^\infty(X, GL(d, \mathbb{C})) \).

**Proof.** Notice that \( z \mapsto L^k(A_z) \) is the limit of the decreasing sequence of subharmonic functions \( z \mapsto \frac{1}{2} \int_X \ln \|A^k(A_z)\|_{HS} d\mu \) (see [2] Lemma 2.3 for example). □

By (3.8) and the subharmonicity of Lyapunov exponents we get \( \zeta \) satisfies (3). of Theorem 3 By Lemma 8 \( \rho \) is a continuous function on \( \mathbb{C}^+ \cup \mathbb{R} \).

Now we prove \( \zeta \) is a holomorphic function on \( \mathbb{C}^+ \). For \( z \in \mathbb{C}^+ \), since \( m^+(z, x) \) depends on \( z \) holomorphically,

\[
\int_X \delta_z \xi_n(x, 0, m^+(z, x)) d\mu(x)
\]

is a sequence of uniformly bounded holomorphic functions of \( z \). By the proof of Lemma 7 and Remark 5

\[
\lim_{n \to \infty} \int_X \delta_z \xi_n(x, 0, m^+(z, x)) d\mu(x) = \lim_{n \to \infty} \int_X \delta_z \xi_n(x, 0, 0) d\mu(x).
\]

Then by Montel theorem, \( \lim_{n \to \infty} \int_X \delta_z \xi_n(x, 0, m^+(z, x)) d\mu(x) \) depends on \( z \) holomorphically. By the definition of \( \zeta \) we get \( \zeta \) is a holomorphic function on \( \mathbb{C}^+ \).

### 3.3. Fibered Rotation Function is Non-Increasing

To prove Theorem 3 we only need to prove \( \rho \) is non-increasing on \( \mathbb{R} \). At first, we give a proof for \( d = 1 \), which gives us the basic idea for the general case.

For all \( z \in \mathbb{S}^1 \), and any lift of \( A \in SL(2, \mathbb{R}) \), we have the following equation:

(3.12)

\[
\circ A \cdot z = e^{-2\sigma(\tilde{\tau}(A, \tilde{z}))} z
\]

Notice that \( \lim_{n \to \infty} \Re \delta_{z_1} \xi_n(x, z_1, z_2) \mu(x) \) does not depend on the choice of \( z_1, z_2 \), we can assume \( z_1, z_2 \in \mathbb{S}^1 \).
Then to prove $\rho$ is non-increasing on $\mathbb{R}$, by definition of $\zeta$ and (3.12) we only need to prove for any $x \in X, z \in S^1, n \in \mathbb{N}$ and any continuous lift of the path $\hat{A}_0(f^n(x)) \cdots \hat{A}_0(x) \cdot z, \theta \in \mathbb{R}$, denoted as

$$\hat{A}_0(f^n(x)) \cdots \hat{A}_0(x) \cdot z$$

is monotonic with respect to $\theta$. Here the lift $\hat{\gamma}$ for a curve $\gamma : \mathbb{R} \rightarrow S^1$ is a continuous function on $\mathbb{R}$ such that $\pi \circ \hat{\gamma} = \gamma$, where $\pi(x) = e^{ix}$.

In fact, for $\theta > 0$ we have

$$\hat{A}(f^n(x)) \hat{A}(f^{n-1}(x)) \cdots \hat{A}(z)$$

$$= e^{2i\theta} \hat{A}(f^n(x))e^{2i\theta} \hat{A}(f^{n-1}(x)) \cdots e^{2i\theta} \hat{A}(x) \cdot z$$

$$> \hat{A}(f^n(x))e^{2i\theta} \hat{A}(f^{n-1}(x)) \cdots e^{2i\theta} \hat{A}(x) \cdot z$$

(the lift of the rotation is a translation)

$$> \hat{A}(f^n(x)) \hat{A}(f^{n-1}(x)) \cdots e^{2i\theta} \hat{A}(x) \cdot z$$

(the lift of the $A$ action preserves the order)

$$> \cdots$$

$$> \hat{A}(f^n(x)) \hat{A}(f^{n-1}(x)) \cdots \hat{A}(x) \cdot z$$

Then $\rho$ is non-increasing when $d = 1$.

For $d > 1$, we have the following lemma to replace (3.12),

**Lemma 10.** For all $Z \in U_{sym}(\mathbb{C}^d)$, and any lift of $A \in Sp(2d, \mathbb{R})$,

(3.13) $\det(\hat{A} \cdot Z) = e^{-2i\text{arg}(\hat{A}, Z)} \det(Z)$

**Proof.** By (3.12), $\hat{\tau}$ behaves well under the iteration, so by Cartan decomposition of $Sp(2d, \mathbb{R})$, we only need to prove (3.13) for

$$\hat{A} = \begin{pmatrix} U & S + S^{-1} \\ (U^{-1})^T & \frac{1}{2}(S - S^{-1}) \end{pmatrix}$$

where $U$ is an arbitrary unitary matrix, $S$ is an arbitrary real non-singular diagonal $d \times d$ matrix.

For the first case,

$$\det(\hat{A} \cdot Z) = \det(UZU^T) = \det(U)^2 \det(Z) = e^{-2i\text{arg}(\hat{A}, Z)} \det(Z)$$

For the second case,

$$\det(\hat{A} \cdot Z)$$

$$= \det((S + S^{-1})Z + (S - S^{-1})) \det((S - S^{-1})Z + (S + S^{-1}))^{-1}$$

$$= \det((S + S^{-1}) + (S - S^{-1})Z) \det((S - S^{-1})Z + (S + S^{-1}))^{-1} \cdot \det(Z)$$

(since $Z \in U_{sym}(\mathbb{C}^d), Z^{-1} = \overline{Z}$)

$$= e^{-2i\text{arg}(\hat{A}, Z)} \det(Z)$$

(since $S$ is a real matrix)

$$= e^{-2i\text{arg}(\hat{A}, Z)} \det(Z)$$

$\square$
Come back to the proof of the non-increasing property of \( \rho \). As in the case \( d = 1 \), by \( 3.13 \), we have to prove for all \( x \in X, Z \in U_{sym}(\mathbb{C}^d) \), any continuous lift of the path \( \det(\tilde{A}_\theta(f^n(x)) \cdots \tilde{A}_\theta(x) \cdot Z), \theta \in \mathbb{R} \) is monotonic with respect to \( \theta \).

In other words, we need to prove for any continuous lift of the path \( \tilde{A}_\theta(f^n(x)) \cdots \tilde{A}_\theta(x) \cdot Z \), denoted as \( \tilde{A}_\theta(f^n(x)) \cdots \tilde{A}_\theta(x) \cdot Z \), we have that \( \det(\tilde{A}_\theta(f^n(x)) \cdots \tilde{A}_\theta(x) \cdot Z) \) is monotonic with respect to \( \theta \), where \( \det \) is defined in \( 2.3 \).

By (3), (4) of Lemma \([3]\) using the order defined in the subsection \( 2.4.2 \) as the 1-dimensional case, we have for \( \theta > 0 \),

\[
\tilde{A}_\theta(f^n(x)) \cdots \tilde{A}_\theta(x) \cdot Z > \tilde{A}(f^n(x)) \cdots \tilde{A}(x) \cdot Z
\]

But by (1) of Lemma \([3]\) \( \det \) is monotonic with respect to the order “\( \leq \)”. Combining with last equation, we have that the function \( \det(\tilde{A}_\theta(f^n(x)) \cdots \tilde{A}_\theta(x) \cdot Z) \) is monotonic with respect to \( \theta \), which completes the proof of Theorem \([3]\)

4. A Kotani theoretic estimate

The main aim of this chapter is Theorem \([4]\) which is a higher dimensional generalization of Lemma 2.6 in \([9]\). We introduce the concept of \( m^- \)–function firstly.

**4.1. The \( m^- \)-function.** By Lemma \([4]\) \( \tilde{A}_{\sigma-it}, t > 0 \) contracts the Bergman metric uniformly on \( SD_d \). We can define \( m^-(\sigma - it, \cdot) \in SD_d, t > 0 \) which depends on \( \sigma - it \) holomorphically, such that

\[
m^-(\sigma - it, f(x)) = \tilde{A}_{\sigma-it}(x) \cdot m^-(\sigma - it, x)
\]

For later use, we consider the following property of \( m^- \): for \( t > 0 \), by (4.1) and the definition of function \( \tau(\cdot)(\cdot) \), there exists a function \( \tau_{A_{\sigma-it}}(m^-)(\sigma - it, x) \in GL(d, \mathbb{C}) \) such that

\[
\tilde{A}_{\sigma-it} \left( m^-(\sigma - it, x) \begin{pmatrix} I_d \\ m^-(\sigma - it, f(x)) \end{pmatrix} \right) = \left( m^-(\sigma - it, f(x)) \begin{pmatrix} I_d \\ m^-(\sigma - it, f(x)) \end{pmatrix} \right) \tau_{A_{\sigma-it}}(m^-)(\sigma - it, x)
\]

Moreover we have:

**LEMMA 11.**

\[
\tilde{A}_{\sigma+it} \left( m^- \begin{pmatrix} I_d \\ m^-(\sigma - it, x) \end{pmatrix} \right) = \left( m^- \begin{pmatrix} I_d \\ m^-(\sigma - it, f(x)) \end{pmatrix} \right) \tau_{A_{\sigma-it}}(m^-)(\sigma - it, x)
\]

**Proof.** We denote \( A \) for \( A_{\sigma+it}, A_- \) for \( A_{\sigma-it}, m^- \) for \( m^- (\sigma - it, x) \) \( \tilde{m}^- \) for \( m^- (\sigma - it, f(x)) \), \( \tau_- \) for \( \tau_{A_{\sigma-it}}(m^-)(\sigma - it, x) \). Recall that \( C \) is the Cayley element defined in \([2.1]\), then by (4.1) we have

\[
\tilde{A}_- \left( m^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) = \left( \tilde{m}^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) \tau_-
\]

\[
CA_+ C^{-1} \left( m^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) = \left( \tilde{m}^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) \tau_-(\text{by definition of } \tilde{A}_-)
\]

\[
A_- C^{-1} \left( m^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) = C^{-1} \left( \tilde{m}^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) \tau_-
\]

\[
\tilde{A}_+ \left( m^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) = \left( \tilde{m}^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) \tau_-
\]

\[
A_+ C^{-1} \left( m^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) = C^{-1} \left( \tilde{m}^- \begin{pmatrix} I_d \\ m^- \end{pmatrix} \right) \tau_-
\]
Take complex conjugate for both sides of last equation, we have

\[ AC^{-1} \left( \begin{array}{c} m^- \\ I_d \end{array} \right) = C^{-1} \left( \begin{array}{c} m^- \\ I_d \end{array} \right) \tau \]

\[ AC^{-1}(CC^{-1} m^-) = C^{-1} \left( \begin{array}{c} m^- \\ I_d \end{array} \right) \tau \]

\[ CAC^{-1}(CC^{-1} m^-) = CC^{-1} \left( \begin{array}{c} m^- \\ I_d \end{array} \right) \tau \]

\[ \overset{\circ}{A}(CC^{-1} m^-) = CC^{-1} \left( \begin{array}{c} m^- \\ I_d \end{array} \right) \tau \]

Notice that \( CC^{-1} = \left( \begin{array}{cc} 0 & I_d \\ I_d & 0 \end{array} \right) \), we have

\[ \overset{\circ}{A} \left( \begin{array}{c} I_d \\ m \end{array} \right) = \left( \begin{array}{c} I_d \\ m \end{array} \right) \tau \]

Now we can state Theorem 4.

**Theorem 4.** For almost every \( \sigma_0 \in \mathbb{R} \) such that \( L(A_{\sigma_0}) = 0 \), we have that:

1. \[ \limsup_{t \to 0^+} \int_X \frac{1}{1 - \|m^+ (\sigma_0 + it, x)\|^2} d\mu(x) + \int_X \frac{1}{1 - \|m^- (\sigma_0 - it, x)\|^2} d\mu(x) < \infty \]

2. \[ \liminf_{t \to 0^+} \int_X \|m^+ (\sigma_0 + it, x) - m^- (\sigma_0 - it, x)\|^2 d\mu(x) = 0 \]

To prove Theorem 4 we introduce the following concepts.

### 4.2. \( q \)-function and Lyapunov exponents.

**Definition 10.** Consider the derivative of the holomorphic map \( Z \mapsto A_{\sigma + it}(x) \cdot Z \) at point \( m^+(\sigma + it, x) \), denote \( q_{\sigma + it}(x) \) as the Jacobian of the derivative with respect to the volume form induced by the Bergman metric.

By the discussion before Lemma 3 we have the following expression of \( q \)-function:

**Lemma 12.**

\[ q_{\sigma + it}(x) = \left| \frac{dm^+(\sigma + it, f(x))}{dm^+(\sigma + it, x)} \right| \frac{V(m^+(\sigma + it, f(x)))}{V(m^+(\sigma + it, x))} \]

where \( \left| \frac{dm^+(\sigma + it, f(x))}{dm^+(\sigma + it, x)} \right| \) is the Jacobian of the map \( Z \mapsto A_{\sigma + it}(x) \cdot Z \) at point \( m^+(\sigma + it, x) \) with respect to the Lebesgue measure \( d\lambda \) defined in (2.6).

The following lemma gives a explicit formula of \( \left| \frac{dm^+(\sigma + it, f(x))}{dm^+(\sigma + it, x)} \right| \).

**Lemma 13.** \[ \left| \frac{dm^+(\sigma + it, f(x))}{dm^+(\sigma + it, x)} \right| = | \det(r_{A_{\sigma + it}(x)}) |^{-2(d+1)} \]
Proof. We only need to prove the following: for an arbitrary $Z \in \text{Sym}_d \mathbb{C}$, \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2d, \mathbb{C}) \) such that $CZ + D$ is invertible, the map

\[
\text{Sym}_d \mathbb{C} \rightarrow \text{Sym}_d \mathbb{C} \\
X \mapsto (AX + B)(CX + D)^{-1}
\]

at point $Z$ has Jacobian (with respect to $d\lambda$) \( \frac{1}{\det(CZ + D)} \).

At first, by computation the tangent map of $X \rightarrow (AX + B)(CX + D)^{-1}$ at point $Z$ is:

\[
\text{Sym}_d \mathbb{C} \rightarrow \text{Sym}_d \mathbb{C} \\
H \mapsto (A - (AZ + B)(CZ + D)^{-1}C) \cdot H \cdot (CZ + D)^{-1}
\]

We need the following equation for symplectic group:

**Lemma 14.** For an arbitrary $Z \in \text{Sym}_d \mathbb{C}$, \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2d, \mathbb{C}) \) such that $CZ + D$ is invertible: we have that

\[
(A - (AZ + B)(CZ + D)^{-1}C) = (CZ + D)^{-1T}
\]

**Proof.** Since \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2d, \mathbb{C}) \), we have that

\[
A^T C, B^T D \text{ are symmetric. } A^T D - C^T B = 1
\]

Moreover, for all $Z \in \text{Sym}_d \mathbb{C}$ such that $CZ + D$ is invertible, we have that

\[
(AZ + B)(CZ + D)^{-1} \text{ is symmetric.}
\]

then

\[
(AZ + B)(CZ + D)^{-1} = (D^T + ZC^T)^{-1}(B^T + ZA^T)
\]

By \(4.7\), to prove Lemma 13 we have to prove:

\[
(A - (D^T + ZC)^{-1}(B^T + ZA^T)C)(CZ + D)^T = I_d
\]

Multiply by $D^T + ZC^T$ from the left to both sides, we need to prove

\[
(D^T + ZC^T)A(CZ + D)^T = (B^T + ZA^T)C(CZ + D)^T + (D^T + ZC^T)
\]

which is the consequence of \(4.6\). \(\square\)

Come back to the proof of Lemma 13 by last lemma the tangent map of $X \rightarrow (AX + B)(CX + D)^{-1}$ at point $Z$ is

\[
\text{Sym}_d \mathbb{C} \rightarrow \text{Sym}_d \mathbb{C} \\
H \mapsto (CZ + D)^{-1T} \cdot H \cdot (CZ + D)^{-1}
\]

So Lemma 13 is the consequence of the following lemma:

**Lemma 15.** Suppose $g \in GL(d, \mathbb{C})$, the linear map

\[
\text{Sym}_d \mathbb{C} \rightarrow \text{Sym}_d \mathbb{C} \\
H \rightarrow g^T H g
\]

has Jacobian $|\det g|^{2(d+1)}$ with respect to the density $d\lambda$ on $\text{Sym}_d \mathbb{C}$. 
Proof. The Jacobian behaves well under the multiplication on $GL(d, \mathbb{C})$. By the polar decomposition of $GL(d, \mathbb{C})$, we only need to prove the lemma in the case $g$ is diagonal or $g$ is contained in the unitary group. When $g$ is diagonal, the lemma can be verified by computation directly. Notice the Jacobian of the map gives a homomorphism from $GL(d, \mathbb{C})$ to $(\mathbb{R}^+, \times)$. So it maps the unitary group to the unique compact subgroup of $(\mathbb{R}^+, \times)$: the identity. □

By our construction of $m^+$, for $t > 0$, \[
\left[ \begin{array}{c} m^+ \\ I_d \end{array} \right]
\] represents the unstable direction of the cocycle $\Lambda^d(A)$. As a result, we have

\[
L^d(A_{\sigma + it}) = \int_X \ln |\det \tau_{A_{\sigma + it}(x)}(m^+ (\sigma + it, x))| d\mu(x)
\]

Combining (4.11) and Lemma 12, 13 we get

\[
L^d(A_{\sigma + it}) = \frac{1}{2(d + 1)} \int_X - \ln q_{\sigma + it}(x) d\mu(x)
\]

4.3. Boundary behavior of Lyapunov exponents. As in Kotani theory and [9], using the results of Theorem 3, we get the following lemma for boundary behavior of Lyapunov exponents.

**Lemma 16.** For almost every $\sigma_0 \in \mathbb{R}$ such that $L(A_{\sigma_0}) = 0$,

\[
\lim_{t \to 0^+} \frac{L^d(A_{\sigma_0 + it})}{t} = \frac{\partial L^d(A_{\sigma_0 + it})}{\partial t} = 0
\]

**Proof.** We follow the proof of Theorem 2.5 in [9]. By upper semi-continuity of $L^d$, for every $\sigma_0 \in \mathbb{R}$ such that $L(A_{\sigma_0}) = 0$, we have

\[
\lim_{t \to 0^+} L^d(A_{\sigma_0 + it}) = 0
\]

Then

\[
\lim_{t \to 0^+} \frac{L^d(A_{\sigma_0 + it})}{t} = \frac{\lim_{t \to 0^+} L^d(A_{\sigma_0 + it}) - L^d(A_{\sigma_0 + i0^+})}{t} = \frac{\int_0^t \frac{\partial L^d(A_{\sigma_0 + it})}{\partial t} dt}{t}
\]

To prove Lemma 16 we only need to prove the following limit exists for almost every $\sigma_0 \in \mathbb{R}$.

\[
\lim_{t \to 0^+} \frac{\partial L^d(A_{\sigma_0 + it})}{\partial t}
\]

By Cauchy-Riemann equations,

\[
\frac{\partial L^d(A_{\sigma_0 + it})}{\partial t} = - \frac{\partial \rho}{\partial \sigma}(\sigma_0 + it)
\]

By Theorem [8] since the map $\rho$ is harmonic on $\mathbb{C}^+$, continuous on $\mathbb{C}^+ \cup \mathbb{R}$, non-increasing on $\mathbb{R}$, one can say that for Lebesgue almost every $\sigma_0 \in \mathbb{R}$, (see Theorem 2.5 of [9])

\[
\lim_{t \to 0^+} \frac{\partial \rho}{\partial \sigma}(\sigma_0 + it) = \frac{d}{d\sigma}(\rho(\sigma_0))
\]
Since $\rho$ is non-increasing, the derivative of $\rho$ on $\mathbb{R}$ exists almost every where, which implies the limit in (4.15) exists for almost every $\sigma_0$. □

4.4. **Proof of Theorem 4.** Now we come back to the proof of theorem. By Lemma 16 for almost every $\sigma_0 \in \mathbb{R}$ such that $L(A_{\sigma_0}) = 0$, we have (4.13) holds, lim$_{t \to 0^+} \frac{V(A_{\sigma_0} + it)}{it}$ exists and is finite.

We claim that for these $\sigma_0$, equations (1), (2) of Theorem 4 hold. From now to the end of the proof of Theorem 4, we denote for simplicity $L$ by last lemma, we get

By Lemma 12 and the definition of $\Pi^d$, we have $\frac{\partial L^d(A_{\sigma_0} + it)}{it}$ exists for almost every $\sigma_0 \in \mathbb{R}$ such that $L(A_{\sigma_0}) = 0$, we have (4.13)

4.4.1. **Proof of Theorem 4.** Notice that $\sigma_0 + it = \left(\begin{array}{c} e^{-t} \\ e^{t} \end{array}\right)$, we have an expression of $q$ by the singular values of $\tilde{A}$.

**Lemma 17.**

(4.18) $q^{-1} = e^{-2t(d^2 + d)} \cdot \Pi_{i=1}^d \left( \frac{e^{4t}(1 - \sigma_i(\tilde{m}^+)^2)}{1 - e^{4t}\sigma_i(\tilde{m}^+)^2} \right)^{d+1}$

**Proof.** By Lemma 13 and the definition of $q$,

$$q^{-1} = \frac{V(m^+)}{V(\tilde{m}^+)} \cdot \frac{\left| dm^+ \right|}{\left| d\tilde{m}^+ \right|} = \frac{V(e^{2t}\tilde{m}^+)}{V(\tilde{m}^+)} \cdot \frac{V(m^+)}{V(e^{2t}\tilde{m}^+)} \cdot \frac{\left| dm^+ \right|}{\left| d\tilde{m}^+ \right|} \cdot e^{2t(d^2 + d)} \cdot e^{2t(d^2 + d)}$$

( since $SD_d$ has $d^2 + d$ real dimension)

$$= \frac{V(e^{2t}\tilde{m}^+)}{V(\tilde{m}^+)} \cdot e^{2t(d^2 + d)}$$

( since $m \to e^{2t}\tilde{m}^+$ is an isometry for Bergman metric )

$$= e^{-2t(d^2 + d)} \cdot \Pi_{i=1}^d \left( \frac{e^{4t}(1 - \sigma_i(\tilde{m}^+)^2)}{1 - e^{4t}\sigma_i(\tilde{m}^+)^2} \right)^{d+1} \text{ (by Lemma 5}$$

□

Using that for $r > 0, 0 \leq s < e^{-r}$ we have

(4.19) $\ln\left( \frac{e^r(1 - s)}{1 - e^r s} \right) \geq \frac{r}{1 - s}$

by last lemma, we get

$$\ln q^{-1} \geq -2t(d^2 + d) + \sum_{i=1}^d (d + 1) \cdot \frac{4t}{1 - \sigma_i(\tilde{m}^+)^2}$$

$$= 2(d + 1) \sum_{i=1}^d \frac{1 + \sigma_i(\tilde{m}^+)^2}{1 - \sigma_i(\tilde{m}^+)^2}$$

By (4.12), since $L^d = \frac{1}{2(d+1)} \int_X \ln q^{-1} d\mu$, we have

(4.20) $L^d \geq t \int_X \sum_{i=1}^d \frac{1 + \sigma_i(\tilde{m}^+)^2}{1 - \sigma_i(\tilde{m}^+)^2} d\mu$
An analogous argument yields

\[ L^d \geq t \int X \sum_{i=1}^{d} \frac{1 + \sigma_i(m)^2}{1 - \sigma_i(m)^2} d\mu \]

Then we conclude that

\[ \frac{L^d}{t} \geq \frac{1}{2} \int \sum_{i=1}^{d} \left( \frac{1 + \sigma_i(m)^2}{1 - \sigma_i(m)^2} + \frac{1 + \sigma_i(m)^2}{1 - \sigma_i(m)^2} \right) d\mu \]

By our assumption of \( \sigma_0 \), we get the proof of (1).

4.4.2. \textit{map \( \Lambda \) and basis \( \mathcal{B}(\cdot, \cdot) \).} To prove (2), we consider the following map:

\textbf{Definition 11.} Let \( \text{Mat}_{2d,d}(\mathbb{C}) \) be the space of all \( 2d \times d \) complex matrices, we can define the map:

\[ \Lambda : \text{Mat}_{2d,d}(\mathbb{C}) \rightarrow \Lambda^d(\mathbb{C}^{2d}) \]

\[ X \mapsto x_1 \wedge \cdots \wedge x_d \]

where \( \{ x_i, 1 \leq i \leq d \} \) are the column vectors of \( X \).

The following lemma lists some properties of \( \Lambda \) we will use later. Recall that for \( A \in \text{GL}(2d, \mathbb{C}) \), \( \Lambda^k(A) \) is the natural action induced by \( A \) on \( \Lambda^k(\mathbb{C}^{2d}) \). For arbitrary two \( 2d \times d \) matrices \( X, Y \), denote

\[ DA(X)(Y) := \lim_{t \to 0} \frac{\Lambda(X + tY) - \Lambda(X)}{t} \]

\textbf{Lemma 18.} For \( A \in \text{GL}(2d, \mathbb{C}), B \in \text{GL}(d, \mathbb{C}), X, Y \in \text{Mat}_{2d,d}(\mathbb{C}) \), suppose that \( X = (x_1, \ldots, x_d), Y = (y_1, \ldots, y_d) \), where \( \{ x_i, 1 \leq i \leq d \}, \{ y_i, 1 \leq i \leq d \} \) are the column vectors of \( X, Y \) respectively, then we have the following equations:

\[ \Lambda^d(A) \cdot \Lambda(X) = \Lambda(A \cdot X) \]

\[ DA(X)(Y) = \sum_{i=1}^{d} x_1 \wedge \cdots \wedge x_{i-1} \wedge y_i \wedge x_{i+1} \wedge \cdots \wedge x_d \]

\[ DA(AX)(AY) = \Lambda^d(A) \cdot DA(X)(Y) \]

\[ \Lambda(X \cdot B) = \det(B) \Lambda(X) \]

\textbf{Proof.} By computation directly. \( \square \)

From now on we identify \( \Lambda^{2d}(\mathbb{C}^{2d}) \) with \( \mathbb{C} \) as the following:

\textbf{Identification} If \( \varpi \in \Lambda^{2d}(\mathbb{C}^{2d}) = c(\varpi) \cdot e_1 \wedge \cdots \wedge e_{2d} \), then we identify \( \varpi \) with \( c(\varpi) \). Here \( e_i \) are standard basis of \( \mathbb{C}^{2d} \).

Now we define a collection of basis of \( \Lambda^d(\mathbb{C}^{2d}) \) for later use.

\textbf{Definition 12.} Suppose \( X, Y \in \text{Mat}_{2d,d}(\mathbb{C}) \) are with rank \( d \), and the column vectors \( \{ x_i, 1 \leq i \leq d \}, \{ y_i, 1 \leq i \leq d \} \) of \( X, Y \) are linearly independent, then the following subset in \( \Lambda^d(\mathbb{C}^{2d}) \) forms a basis of \( \Lambda^d(\mathbb{C}^{2d}) \):

\[ \{ x_i \wedge \cdots \wedge x_{i|I|} \wedge y_{j|J|} \wedge \cdots \wedge y_{j|J|} : I, J \subset \{ 1, \ldots, d \}, |I| + |J| = d, i_1 < i_2 < \ldots, j_1 < j_2 < \ldots \} \]

denoted by \( \mathcal{B}(X, Y) \).
For any element $\omega \in \Lambda^d(C^{2d})$, the coefficient of $x_1 \wedge \cdots \wedge x_d$ for the expansion of $\omega$ with respect to the basis $\mathfrak{B}(X,Y)$ is
\begin{equation}
\frac{\omega \wedge (y_1 \wedge \cdots \wedge y_d)}{x_1 \wedge \cdots \wedge x_d \wedge y_1 \wedge \cdots \wedge y_d}
\end{equation}
Here we use the identification above.

We will use the following lemma later.

**Lemma 19.** Suppose $X,Z \in \text{Mat}_{2d,d}(C)$ are with rank $d$, and the column vectors $\{x_i, 1 \leq i \leq d\}, \{z_i, 1 \leq i \leq d\}$ of $X,Z$ are linearly independent, then the coefficient of $\Lambda(X)$ for the expansion of $D\Lambda(X)(Y)$ with respect to the basis $\mathfrak{B}(X,Z)$ is the trace of the matrix $(X,Z)^{-1} \cdot Y$. (the trace of a $2d \times d$ matrix is the sum of the diagonal entries)

**Proof.** Suppose $(X,Z)^{-1} \cdot Y = \left(\begin{array}{c} a_{ij} \\ b_{ij} \end{array}\right)_{1 \leq i \leq d, 1 \leq j \leq d}$, then
\begin{equation}
Y = \left(\cdots \sum_{k=1}^{d} (a_{ki}x_k + b_{ki}z_k) \cdots \right)_{1 \leq i \leq d}
\end{equation}
By (4.25) we get that
\begin{align*}
D\Lambda(X)(Y) &= \sum_{i=1}^{d} x_1 \wedge \cdots \wedge x_{i-1} \wedge y_i \wedge x_{i+1} \wedge \cdots \wedge x_d \\
&= \sum_{i=1}^{d} x_1 \wedge \cdots \wedge x_{i-1} \wedge \left(\sum_{k=1}^{d} (a_{ki}x_k + b_{ki}z_k)\right) \wedge \cdots \wedge x_d \\
&= \sum_{i=1}^{d} a_{ii}x_1 \wedge \cdots \wedge x_d + \text{other term in } \mathfrak{B}(X,Z) \\
&= (\text{trace of } (X,Z)^{-1} \cdot Y)\Lambda(X) + \text{other terms in } \mathfrak{B}(X,Z)
\end{align*}
which gives the proof. \hfill \Box

4.4.3. **Proof of (2). of Theorem 4.** Come back to the proof of (2). At first,
\begin{equation}
\circ \bigg( \begin{array}{c} m^+ \\ I_d \end{array} \bigg) = \bigg( \begin{array}{c} \bar{m}^+ \\ I_d \end{array} \bigg) \tau
\end{equation}
Take the inverse,
\begin{equation}
\circ^{-1} \bigg( \begin{array}{c} \bar{m}^+ \\ I_d \end{array} \bigg) = \bigg( \begin{array}{c} m^+ \\ I_d \end{array} \bigg) \tau^{-1}
\end{equation}
Let the operator $\Lambda$ acting on both sides of (4.31), we get:
\begin{equation}
\Lambda \circ^{-1} \bigg( \begin{array}{c} \bar{m}^+ \\ I_d \end{array} \bigg) = \Lambda \bigg( \begin{array}{c} m^+ \\ I_d \end{array} \bigg) \tau^{-1}
\end{equation}
Then differentiate with respect to $t$,
\begin{equation}
\frac{\partial}{\partial t} \Lambda \circ^{-1} \bigg( \begin{array}{c} \bar{m}^+ \\ I_d \end{array} \bigg) = \frac{\partial}{\partial t} \frac{1}{\text{det } \tau} \Lambda \bigg( \begin{array}{c} m^+ \\ I_d \end{array} \bigg) (\text{by (4.27)})
\end{equation}
Using Lemma 18 to compute the derivative, we get
\[
\text{left of (4.33)} = D\Lambda \left( A^{-1} \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \right) \left( \frac{\partial}{\partial t} A^{-1} \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \right)
\]
\[=
D\Lambda \left( A^{-1} \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \right) \left( -A^{-1} \frac{\partial}{\partial t} A \right) \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix}
\]
\[+
A \begin{pmatrix} I_d \\ 0 \end{pmatrix} \frac{\partial \tilde{m}^+}{\partial t}
\]
\[=
-\Lambda^d (\tilde{A}^{-1}) \cdot D\Lambda \left( \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \right) \left( \begin{pmatrix} -\tilde{m}^+ \\ I_d \end{pmatrix} \right) - \begin{pmatrix} I_d \\ 0 \end{pmatrix} \frac{\partial \tilde{m}^+}{\partial t}
\]
where we use (4.26) and \(\frac{\partial}{\partial t} A = \begin{pmatrix} -I_d \\ I_d \end{pmatrix} \frac{\partial}{\partial t} A\) in the last equality.

\[
\text{right of (4.33)} = -\frac{1}{(\det \tau)^2} \frac{\partial \det \tau}{\partial t} \Lambda \left( \begin{pmatrix} m^+ \\ I_d \end{pmatrix} \right)
\]
\[+
\frac{1}{\det \tau} D\Lambda \left( \begin{pmatrix} m^+ \\ I_d \end{pmatrix} \right) \left( \begin{pmatrix} I_d \\ 0 \end{pmatrix} \frac{\partial m^+}{\partial t} \right)
\]

Notice that
\[
\Lambda^d (\tilde{A}) \cdot \Lambda \left( \begin{pmatrix} m^+ \\ I_d \end{pmatrix} \right) = \Lambda \left( A \begin{pmatrix} m^+ \\ I_d \end{pmatrix} \right)
\]
\[= \Lambda \left( \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \tau \right)
\]
\[= \det \tau \cdot \Lambda \left( \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \right)
\]

Applying \(-\Lambda^d (\tilde{A})\) to both sides of (4.33), by previous discussion we have the key equation
\[
D\Lambda \left( \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \right) \left( \begin{pmatrix} -\tilde{m}^+ \\ I_d \end{pmatrix} \right) - \frac{1}{\det \tau} \frac{\partial \det \tau}{\partial t} \Lambda \left( \begin{pmatrix} m^+ \\ I_d \end{pmatrix} \right) \left( \begin{pmatrix} I_d \\ 0 \end{pmatrix} \frac{\partial m^+}{\partial t} \right) =
\]
\[
\frac{1}{\det \tau} \frac{\partial \det \tau}{\partial t} \Lambda \left( \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \right) - \frac{1}{\det \tau} \Lambda^d (\tilde{A}) \cdot D\Lambda \left( \begin{pmatrix} m^+ \\ I_d \end{pmatrix} \right) \left( \begin{pmatrix} I_d \\ 0 \end{pmatrix} \frac{\partial m^+}{\partial t} \right)
\]

4.4.4. the key equation. To analyse each term of the key equation, for \(\begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix}, \begin{pmatrix} I_d \\ \tilde{m}^- \end{pmatrix} \in Mat_{2d, d}(\mathbb{C})\), we consider the basis \(\mathfrak{B} \left( \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix}, \begin{pmatrix} I_d \\ \tilde{m}^- \end{pmatrix} \right)\). This is actually a basis since \(\|\tilde{m}^+\|, \|\tilde{m}^-\| < 1, \det \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \begin{pmatrix} I_d \\ \tilde{m}^- \end{pmatrix} \neq 0\).

For the key equation, the following lemmas give the coefficients of \(\Lambda \left( \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix} \right)\) for the expansion of each term with respect to the basis \(\mathfrak{B} \left( \begin{pmatrix} \tilde{m}^+ \\ I_d \end{pmatrix}, \begin{pmatrix} I_d \\ \tilde{m}^- \end{pmatrix} \right)\).
LEMMA 20. The coefficient of $\Lambda\left(\frac{\hat{m}^+}{I_d}\right)$ for the expansion of $DA\left(\frac{\hat{m}^+}{I_d}\right)\left(\frac{-\hat{m}^+}{I_d}\right)$ with respect to the basis $B\left(\frac{\hat{m}^+}{I_d},\frac{I_d}{\hat{m}^-}\right)$ is the trace of $(I_d - \overline{m}^+\hat{m}^+)^{-1}(I_d + \overline{m}^-\hat{m}^+)$. 

Proof. By Lemma [19] to prove lemma [20] we only need to compute 

\[ (\hat{m}^+ \frac{I_d}{\hat{m}^-})^{-1} = (I_d + \hat{m}^+(I_d - \overline{m}^+\hat{m}^+)^{-1}\overline{m}^- - \hat{m}^+(I_d - \overline{m}^-\hat{m}^+)^{-1}) \]

Then we get 

\[ (\hat{m}^+ \frac{I_d}{\hat{m}^-})^{-1} \cdot (-\hat{m}^+ \frac{I_d}{\hat{m}^-}) = ((I_d - \overline{m}^-\hat{m}^+)^{-1}(I_d + \overline{m}^-\hat{m}^+)) \]

By Lemma [19] we get the proof of Lemma [20].

LEMMA 21. The coefficient of $\Lambda\left(\frac{\hat{m}^+}{I_d}\right)$ for the expansion of $DA\left(\frac{\hat{m}^+}{I_d}\right)\left(\frac{I_d}{0}\right)\frac{\partial\hat{m}^+}{\partial t}$ with respect to the basis $B\left(\frac{\hat{m}^+}{I_d},\frac{I_d}{\hat{m}^-}\right)$ is 

\[ \det\left(\hat{m}^+ \frac{I_d}{\hat{m}^-}\right) \cdot DA\left(\frac{\hat{m}^+}{I_d}\right)\left(\frac{I_d}{0}\right)\frac{\partial\hat{m}^+}{\partial t} \wedge \Lambda\left(\frac{I_d}{\hat{m}^-}\right) \]

Proof. Using (4.28).

LEMMA 22. The coefficient of $\Lambda\left(\frac{\hat{m}^+}{I_d}\right)$ for the expansion of 

\[ \frac{1}{\det \tau} \Lambda^d(\hat{m}^+) \cdot DA\left(\frac{\hat{m}^+}{I_d}\right)\left(\frac{I_d}{0}\right)\frac{\partial\hat{m}^+}{\partial t} \]

with respect to the basis $B\left(\frac{\hat{m}^+}{I_d},\frac{I_d}{\hat{m}^-}\right)$ is 

\[ \det\left(\hat{m}^+ \frac{I_d}{\hat{m}^-}\right) \cdot DA\left(\frac{\hat{m}^+}{I_d}\right)\left(\frac{I_d}{0}\right)\frac{\partial\hat{m}^+}{\partial t} \wedge \Lambda\left(\frac{I_d}{\hat{m}^-}\right) \]

Proof. Let $X = \left(\frac{m^+}{I_d}\right), W_1 = \left(\frac{I_d}{0}\right)\frac{\partial m^+}{\partial t}, W_2 = \left(\frac{I_d}{m^-}\right)$. By (4.28) we get the coefficient 

\[ = \frac{1}{\det \tau} \Lambda^d(\hat{m}^+) \cdot DA(X)(W_1) \wedge \Lambda\left(\frac{I_d}{m^-}\right) \cdot \det\left(\hat{m}^+ \frac{I_d}{\hat{m}^-}\right)^{-1} \]

\[ = \frac{1}{\det \tau \cdot \det m^-} \Lambda^d(\hat{m}^+) \cdot DA(X)(W_1) \wedge (\Lambda \cdot \Lambda(W_2)) \]

\[ \cdot \det\left(\hat{m}^+ \frac{I_d}{\hat{m}^-}\right)^{-1} \text{ (use Lemma [11])} \]

\[ = \frac{1}{\det \tau \cdot \det m^-} \Lambda^{2d}(\hat{m}^+) \cdot DA(X)(W_1) \wedge \Lambda(W_2) \cdot \det\left(\hat{m}^+ \frac{I_d}{\hat{m}^-}\right)^{-1} \]
To prove Lemma 22, we only need to prove the following equation:

\[
\det (A) \det \left( \begin{array}{c|c} \bar{m}^+ & I_d \\ \hline I_d & \bar{m}^- \end{array} \right) = \det \tau \det \left( \begin{array}{c|c} \bar{m}^+ & I_d \\ \hline I_d & \bar{m}^- \end{array} \right)
\]

which is just a corollary of (4.30) and Lemma 11.

Now come back to the key equation. By Lemma 20,21,22 taking the coefficient of \( \Lambda(\bar{m}^-) \) in the key equation and integrating with respect to the measure \( \mu \), we have

\[
\int_X \tau \det A \left( \begin{array}{c|c} \bar{m}^+ & I_d \\ \hline I_d & \bar{m}^- \end{array} \right) d\mu = \int_X \frac{1}{\det \tau} \frac{\partial \det \tau}{\partial t} d\mu
\]

Consider the real part, which gives

\[
\int_X R \left( \tau \det A \left( \begin{array}{c|c} \bar{m}^+ & I_d \\ \hline I_d & \bar{m}^- \end{array} \right) \right) d\mu = \frac{\partial L^d}{\partial t}
\]

4.4.5. A trace inequality and the rest of the proof. By 1422 and Lemma 16, we have that

\[
\lim_{t \to 0^+} \int_X \frac{1}{2} \sum_{i=1}^d \frac{1 + \sigma_i(\bar{m}^+)^2 + 1 + \sigma_i(\bar{m}^-)^2}{1 - \sigma_i(\bar{m}^+)^2 + 1 - \sigma_i(\bar{m}^-)^2} - R(\tr((I_d - \bar{m}^-)\bar{m}^+)^{-1}(I_d + \bar{m}^-\bar{m}^+))) d\mu
\]

\[
\leq \lim_{t \to 0^+} \frac{L^d(A_{\sigma_0 + t})}{t} - \frac{\partial L^d(A_{\sigma_0 + t})}{\partial t}
\]

\[
= 0
\]

Compare with (2). of Theorem 14 to finish the proof, we only need to prove the following inequality:

**Lemma 23.**

\[
\frac{1}{2} \sum_{i=1}^d \frac{1 + \sigma_i(\bar{m}^+)^2 + 1 + \sigma_i(\bar{m}^-)^2}{1 - \sigma_i(\bar{m}^+)^2 + 1 - \sigma_i(\bar{m}^-)^2} - R(\tr((I_d - \bar{m}^-)\bar{m}^+)^{-1}(I_d + \bar{m}^-\bar{m}^+)))
\]

\[
\geq \|\bar{m}^+ - \bar{m}^-\|^2_{HS}
\]

**Proof.** Notice that for \( ||x|| < 1, \frac{1 + x}{1 + x} = 2(1 - x)^{-1} - 1 = -1 + 2\sum_{k=0}^\infty x^k \), and \( \bar{m}^- = (\bar{m}^-)^* \) We have that:

left of Lemma 23

\[
\sum_{i=1}^d \frac{1}{1 - \sigma_i(\bar{m}^+)^2} + \frac{1}{1 - \sigma_i(\bar{m}^-)^2} - 2R(\tr((I_d - \bar{m}^-)\bar{m}^+)^{-1})
\]

\[
= \sum_{k=0}^\infty \sum_{i=1}^d \sigma_i(\bar{m}^+)^{2k} + \sigma_i(\bar{m}^-)^{2k} - 2R(\tr((\bar{m}^-)^*\bar{m}^+)^k)
\]

\[
= \sum_{k=0}^\infty \tr(((\bar{m}^+)^*\bar{m}^+)^k) + \tr(((\bar{m}^-)^*\bar{m}^-)^k) - 2R(\tr((\bar{m}^-)^*\bar{m}^+)^k)
\]
Then the proof of Lemma 23 is the consequence of the following matrix inequalities: for arbitrary \( d \times d \) complex matrices \( X, Y, k > 1, \)
\[
\text{tr}((X^* X)^k + (Y^* Y)^k) \geq 2^k \text{tr}((X^* Y)^k)
\]
\[
\text{tr}(X^* X) + \text{tr}(Y^* Y) - 2^k \text{tr}(X^* Y) = \|X - Y\|_{HS}^2
\]
\[\square\]

5. Density of positive Lyapunov exponents for continuous symplectic cocycle

5.1. Herglotz function. We recall that \( m \) is called a Herglotz (matrix valued) function if \( m \) is an analytic matrix valued function defined on \( \mathbb{C}^+ \) and \( \text{Im}(m(z)) \) is a positive definite Hermitian matrix for all \( z \in \mathbb{C}^+ \), we list some basic properties we will use (see [23]).

**Lemma 24.** The function \( m(\cdot) \) has a finite normal limit \( m(\sigma + i0^+) = \lim_{t \to 0^+} m(\sigma + it) \) for a.e. \( \sigma \in \mathbb{R} \). Moreover if two Herglotz function \( m_1, m_2 \) have the same limit on a positive measure set on \( \mathbb{R} \), then \( m_1 = m_2 \).

Notice that \( \Phi_C^{-1} \cdot m^+(\cdot, x), \Phi_C^{-1} \cdot m^-(\cdot, x) \) are Herglotz functions.

5.2. \( M \)-function. Consider the following definition of \( M \)-function, which is introduced in [7].

**Definition 13.** For \( A \in L^\infty(X, \text{Sp}(2d, \mathbb{R})) \), we denote
\[
(5.1) \quad M(A) := \text{the Lebesgue measure of } \{ \theta \in [0, 2\pi], L(A_\theta) = 0 \}
\]

We hope to prove for generic \( A, M(A) = 0 \). At first, we prove it for a family of symplectic cocycles taking finitely many values.

5.3. Symplectic cocycles taking finitely many values. We introduce the following definition of deterministic, which is similar to the definition for Schödinger operator in [36] and [30].

**Definition 14.** For \( A \in L^\infty(X, \text{Sp}(2d, \mathbb{R})) \), we say \( A \) is deterministic if \( A(f^n(x)), n \geq 0 \) is a.e. a measurable function of \( \{ A(f^n(x)), n < 0 \} \).

As [30], we have the following theorem for the \( M \)-function for symplectic cocycles.

**Theorem 5.** Suppose \( A \in L^\infty(X, \text{Sp}(2d, \mathbb{R})) \) such that
(1) \( A(x), x \in X \) only takes finitely many values.
(2) \( A(f^n(x)), n \in \mathbb{Z}, \) is not periodic for almost every \( x \in X \).
(3) If \( A(x) \neq A(y), x, y \in X, \) then \( A(x)^{-1}(0) \neq A(y)^{-1}(0) \).
We have \( M(A) = 0 \).

**Proof.** We know that for almost every \( x \in X, A(f^n(x)), n \geq 0 \) can determine the function \( m^-(x) \). In fact, for \( z \) such that \( \Im(z) < 0, A_z(x)^{-1} \) uniformly contracts the Bergman metric on \( S D_d \), so like the property of \( m^- \) function in Kotani theory, we have that
\[
(5.2) \quad m^-(z, x) = \lim_{n \to \infty} A_z(x)^{-1} \cdots A_z(f^n(x))^{-1} \cdot 0
\]
But we also have the following lemma for the inverse problem:
Lemma 25. If a cocycle $A \in L^\infty(X, \text{Sp}(2d, \mathbb{R}))$ satisfies (1), (3) of Theorem [5], then the function $m^-(z, \cdot), z \in \mathbb{C}^+$ determines $\{A(f^n(\cdot)), n \geq 0\}$ in the sense that if $x, y \in X$ such that $A(f^n(x)), A(f^n(y)), n \geq 0$ are bounded, and $m^-(\cdot, x) = m^-(\cdot, y)$, then $A(f^n(x)) = A(f^n(y)), n \geq 0$.

Proof. Let $z$ tends to $\infty$ along the line $\{\Re(z) = 0, \Im(z) < 0\}$ in (5.2) we get

$$\lim_{\Re(z) = 0, \Im(z) \to -\infty} m^-(z, x) = \hat{A}(x)^{-1}(0) \tag{5.3}$$

By (3) of Theorem [5] we know that $m^-(\cdot, x)$ can determine $\hat{A}(x)$, by

$$\hat{A}_x(x) \cdot m^-(z, x) = m^-(z, f(x)) \tag{5.4}$$

it implies $m^-(\cdot, x)$ can determine $m^-(\cdot, f(x))$, using the same method again, we can determine $\hat{A}(f(x))$. Repeat this process, we determine all $\{A(f^n(x)), n \geq 0\}$. \hfill \Box

Come back to the proof of Theorem [5]. Suppose $M(A) > 0$, we claim that under the assumptions (1), (3), $A$ must be deterministic. Then by Kotani’s argument in [30], $A$ must be periodic, which contradicts the assumption (2).

In fact, the set $\{A(f^n(x)), n < 0\}$ determines $m^+(\cdot, x)$. If $M(A) > 0$, by (2) of Theorem [4] $m^+(\cdot, x)$ determines $m^-(\cdot, x)$ on a full measure subset of $\{\theta : L(A\theta) = 0\}$.

By Lemma 24, since $\Phi_{21} \cdot m^+(\cdot, x), \Phi_{21} \cdot m^-(\cdot, x)$ are Herglotz functions, $m^+(\cdot, x)$ determines $m^-(\cdot, x)$ on all of $\mathbb{C}^+$. By Lemma 25 $\{A(f^n(x)), n \geq 0\}$ is determined by $\{A(f^n(x)), n < 0\}$. That means $A$ is deterministic. \hfill \Box

5.4. Continuous symplectic cocycles.

Theorem 6. Suppose $f$ is not periodic on $\text{supp}(\mu)$, then the set of $A$ such that $L(A) > 0$ is dense in $C(X, \text{Sp}(2d, \mathbb{R}))$.

Proof. At first we consider the following lemma:

Lemma 26. Suppose $f : (X, \mu) \to (X, \mu)$ $(f \neq \text{id})$ is ergodic, then there is a residual subset of cocycles $A$ in $C(X, \text{Sp}(2d, \mathbb{R}))$ such that $M(A) = 0$.

Proof. We follow the proof in [7]. At first we consider the following lemma:

Lemma 27. There exists a dense subset $Z$ of $L^\infty(X, \text{Sp}(2d, \mathbb{R}))$ satisfying all conditions of Theorem [5].

Proof. By Lemma 2 of [7], the cocycles in $L^\infty(X, \text{Sp}(2d, \mathbb{R}))$ satisfying the first two conditions of Theorem [5] are dense in $L^\infty(X, \text{Sp}(2d, \mathbb{R}))$. But for each cocycle $A$ satisfying the first two condition of Theorem [5] we can find a new cocycle $A'$ satisfying all conditions in Theorem [5] and arbitrarily close to $A$. \hfill \Box

Lemma 28. For every $r > 0$, the map

$$(L^1(X, \text{Sp}(2d, \mathbb{R})) \cap B_r(L^\infty(X, \text{Sp}(2d, \mathbb{R}))), \| \cdot \|_1) \to \mathbb{R}, \quad A \mapsto M(A)$$

is upper semi-continuous.

Proof. The proof is the same as the $SL(2, \mathbb{R})$ case, since we have the formula in [33] to replace the Herman-Avila-Bochi formula in [6] for $SL(2, \mathbb{R})$ case. And by Theorem [3] $L^d(A_z)$ is harmonic for $z \in \mathbb{C}^+$ and subharmonic on $\mathbb{C}^+ \cup \mathbb{R}$, we can move the proof for $SL(2, \mathbb{R})$ case in [7] to here. \hfill \Box
LEMMA 29. For \( A \in C(X, Sp(2d, \mathbb{R})) \), \( \epsilon > 0, \delta > 0 \), there is an \( A' \in C(X, Sp(2d, \mathbb{R})) \) such that \( \| A - A' \|_\infty < \epsilon, M(A) < \delta \).

**Proof.** The proof is almost the same as Lemma 3 of [7], we only need to use the set \( Z \) in Lemma 26 and Theorem 6 to replace the set \( Z \) and Kotani result in Lemma 3 of [7].

Come back to the proof of Lemma 26 for \( \delta > 0 \), we define

\[ M_\delta = \{ A \in C(X, Sp(2d, \mathbb{R}) : M(A) < \delta \} \]

By Lemma 28, \( M_\delta \) is open, and by Lemma 29, \( M_\delta \) is dense. It follows that

\[ \{ A \in C(X, Sp(2d, \mathbb{R}) : M(A) = 0 \} = \cap_{\delta > 0} M_\delta \]

is residual. \( \square \)

Come back to the proof of Theorem 6. Let \( P \subset X \) be the set of periodic orbits of \( f \). If \( \mu(P) < 1 \), then using Lemma 26, we get the proof.

Assume \( \mu(P) = 1 \), we follow the argument of Lemma 3.1 in [2]. Let \( P_k \subset X \) be the set of periodic orbits of period \( k \geq 1 \). Since \( f \) is not periodic on \( supp(\mu) \), \( P_n = \cup_{k \leq n} P_k \neq supp(\mu) \) for every \( n \geq 1 \). Thus there are arbitrarily large \( n \) such that \( \mu(P_n \setminus P_{n-1}) > 0 \).

Choose such a large \( n \), and take \( x \in supp(\mu) \setminus P_n \setminus P_{n-1} \). We can approximate any \( A \in C(X, Sp(2d, \mathbb{R})) \) by some \( A' \) which is constant in a compact neighborhood \( K \) of \( \{ f^k(x) \}_{k=0}^{n-1} \). The details of the following argument can be found in the Appendix [A] here we only give an outline. We will prove that there is a constant \( C \) independent of \( n, f \) such that for generic \( \{ A'(f^k(x)) \} \), there exist \( \theta \in (-\frac{\pi}{n}, \frac{\pi}{n}) \) with \( L(A'_\theta(x)) > 0 \). (Since \( A' \) is locally constant near the orbit of \( x \in supp(\mu) \), we have \( L(A'_\theta) > 0 \).)

Otherwise there is an open interval \( I \) contained \( 0 \) and \( |I| > O(\frac{1}{n}) \), such that for all \( \theta \in I \), all the eigenvalues of \( A^{(n)}_{\theta}(x) \) are norm 1, where \( |I| \) is the length of \( I \). But in the Appendix [A] we will prove for any interval \( I' \) such that

\[ \forall \theta \in I', \text{ all the eigenvalues of } A^{(n)}_{\theta}(x) \text{ are simple and norm 1} \]

we have

\[ |I'| \leq O(\frac{1}{n}) \]

As a result, there are two intervals \( I_1, I_2 \subset I \) satisfying (5.5) and sharing common boundary point \( \theta_0 \) such that \( A^{(n)}_{\theta_0}(x) \) has repeated eigenvalues with norm 1. But this can not happen for generic choice of \( \{ A'(f^k(x)) \} \).

\( \square \)

6. The Proof of Theorem 1

By Theorem 6 as in the \( SL(2, \mathbb{R}) \)-case, to prove Theorem 1, we need a local regularization formula similar to Theorem 7 in [2].

At first we need the following lemma:

LEMMA 30. Suppose \( A \in C(X, Sp(2d, \mathbb{R})) \), \( \Omega \subset C \) is a domain. Suppose an analytic \( Sp(2d, \mathbb{C}) \)-valued map \( B \) is defined on \( \Omega \) such that for all \( z \in \Omega, x \in X, B^0(z) : SD_d \subset SD_d \), then the Lyapunov exponent \( L^d(B(z)A) \) harmonically depends on \( z \in \Omega \).
Proof: The proof is basically the same as the discussion in Chapter 3 for holomorphicity of \( \zeta \)-function. See the remark at page 7 of [33], and section 3 and 6 of [8].

As in [2], let \( \| \cdot \| \) denote the sup norm in the space \( C(X, \mathfrak{sp}(2d, \mathbb{R})) \) and \( C(X, \mathfrak{sp}(2d, \mathbb{C})) \). And for \( r > 0 \), let \( B_r(x), \mathcal{B}_r^C \) be the corresponding \( r \)-ball. For \( A \in C(X, \text{Sp}(2d, \mathbb{R})) \), \( a, b \in C(X, \mathfrak{sp}(2d, \mathbb{R})) \), we define the following function

\[
\Phi_\epsilon(A, a, b) := \int_{-1}^{1} \frac{1 - t^2}{|t^2 + 2it + 1|^2} L^d(e^{\epsilon(tb + (1-t^2)a)} A) dt
\]

The following local regularization formula is the main result of this chapter.

**Theorem 7.** There exists \( \eta > 0 \) such that if \( b \in C(X, \mathfrak{sp}(2d, \mathbb{R})) \) is \( \eta \)-close to \( (0, I_d^-) \), then for every \( \epsilon > 0 \), and every \( A \in C(X, \text{Sp}(2d, \mathbb{R})) \),

\[
e^{\epsilon zb + (1-z^2)a} \cdot \overline{SD_d} \subset SD_d
\]

when

1. \( z \in \{ |z| = 1 \} \cap \mathcal{I}(z) > 0 \) or \( z = (\sqrt{2} - 1)i, a \in \mathcal{B}_r^C(\eta) \),
2. \( z \in \{ |z| < 1 \} \cap \mathcal{I}(z) > 0, a \in \mathcal{B}_r(\eta) \).

Moreover

\[
a \mapsto \Phi_\epsilon(A, a, b)
\]

is a continuous function of \( a \in \mathcal{B}_r(\eta) \) and depends continuously (as an analytic function) on \( A \).

**Proof.** In fact we only need to prove (6.2), (6.3) is the consequence of (6.2) and Lemma 30 see Theorem 7 of [2].

To prove (6.2), we claim there exists a positive number \( \eta > 0 \) such that for every point \( Z \in \partial SD_d, \{ Z^T = Z, \| Z \| = 1 \} \), for \( \epsilon > 0 \) small, the path \( Z_\epsilon := e^{\epsilon(zb + (1-z^2)a)} \cdot Z \) is contained in \( SD_d \) for \( z \) and \( a \) in either case (1) or (2). This implies there exists \( \epsilon_0 > 0 \) small, for all \( \epsilon < \epsilon_0 \), \( e^{\epsilon(zb + (1-z^2)a)} \cdot \overline{SD_d} \subset SD_d, \) By iteration, \( e^{\epsilon(zb + (1-z^2)a)} \) takes \( \overline{SD_d} \) into \( SD_d \) for every \( \epsilon > 0 \).

At first, by the "left-oriented" Zassenhaus formula (see [14], for example), we have the following equation for exponential map of matrix when \( \epsilon \) is small, \( \| X \|, \| Y \| \leq 2 \).

\[
e^{\epsilon(X+Y)} = e^{O(\epsilon^2 \| X \| \| Y \|)} e^{X} e^{Y}
\]

which means there exist a vector \( W \) in the Lie algebra with norm less than \( O(\epsilon^2 \| X \| \| Y \|) \), such that \( e^{\epsilon(X+Y)} = e^{W} e^{X} e^{Y} \).

In addition, we need some notations for a real Lie algebra \( \mathfrak{g} \) and its complexification \( \mathfrak{g}^C = \mathfrak{g} \oplus i\mathfrak{g} \). For an element \( c \in \mathfrak{g}^C, a, b \in \mathfrak{g} \) such that \( c = a + ib \), we denote

\[
\Re(c) = a, \Im(c) = b
\]

From now to the end of this chapter, we always consider \( \mathfrak{g} \) is the Lie algebra of \( U(d, d) \cap \text{Sp}(2d, \mathbb{C}) \) or \( \mathbb{R} \). Then \( \mathfrak{g}^C \) is \( \mathfrak{sp}(2d, \mathbb{C}) \) or \( \mathbb{C} \).
Now we denote \( R(a, b, z) = \mathcal{H}(z^0 b + (1 - z^2)\tilde{a}) = \mathcal{H}(z^0 b + \mathcal{H}((1 - z^2)\tilde{a})) \) and \( I(a, b, z) = \mathcal{I}(z^0 b + (1 - z^2)\tilde{a}) = \mathcal{I}(z^0 b + \mathcal{I}((1 - z^2)\tilde{a})) \).

Let \( \eta \) be small, then for \( z, a \) in either case (1) or (2) we have the following equations:

\[
\begin{align*}
(6.6) & \quad Z_e = e^{(zb + (1 - z^2)a)} \cdot Z = e^{(R + iI)} \cdot Z \\
(6.7) & \quad e^{R} \cdot Z \in \partial SD_d \\
(6.8) & \quad I(a, b, z) = \mathcal{I}(z)(\frac{i}{-i}) + O(\eta) \\
(6.9) & \quad \|R(a, b, z)\|_* \leq 2 \\
(6.10) & \quad \|I(a, b, z)\|_* \leq 2\mathcal{I}(z)
\end{align*}
\]

Here (6.8) comes from the fact that \( \|\mathcal{I}((1 - z^2)\tilde{a})\|_* \leq O(\eta \mathcal{I}(z)) \) holds for either case (1) or (2).

Denote \( Z' = e^{R}Z \). Then by (6.7) we know \( \|Z'\| = 1 \), and we have:

\[
\begin{align*}
Z_e &= e^{(R + iI)} \cdot Z \\
&= e^{O(\|R\|_* \|I\|_*)} e^{i\mathcal{I}} e^{R} \cdot Z \quad \text{by (6.4) (6.9) (6.10)} \\
&= e^{O(\mathcal{I}(z))} e^{i\mathcal{I}} \cdot Z' \quad \text{by (6.9) (6.10)} \\
&= e^{O(\mathcal{I}(z))} e^{\mathcal{I}(z)(\frac{-1}{1}) + O(\eta)} \cdot Z' \quad \text{by (6.8)} \\
&= e^{O(\mathcal{I}(z))} e^{O(\|\mathcal{I}(z)\|)} e^{O(\|\mathcal{I}(z)\|)} (e^{-2\mathcal{I}(z)} Z') \quad \text{by (6.4)} \\
&= e^{O(\mathcal{I}(z))} (e^{-2\mathcal{I}(z)} Z') \quad \text{since } \eta \text{ is small.}
\end{align*}
\]

If \( \epsilon \) is small enough, since the action on the equation above is a Möbius transformation, then by computation we have

\[
\|Z_e\| \leq \|e^{O(\mathcal{I}(z))} (e^{-2\mathcal{I}(z)} Z')\| \leq e^{-2\mathcal{I}(z)}
\]

for \( \epsilon \) small, which implies \( Z_e \in SD_d \). Then we get the proof of Theorem 7. \( \square \)

To finish the proof of Theorem 1 we need the following short lemma.

**Lemma 31.** Let \( A \in C(X, Sp(2d, \mathbb{R})) \), \( a, b \in C(X, \mathfrak{sp}(2d, \mathbb{R})) \) and \( \epsilon > 0 \), if \( L^d(e^{\epsilon a}) > 0 \), then \( \text{det}(A, a, b) > 0 \).

**Proof.** As in the proof of Lemma 3.2 and Lemma 2.3 in [2], the map

\[
\gamma : t \mapsto L^d(e^{t(b + (1 - t^2)a)} [A]
\]

is a subharmonic function.

By subharmonicity, if \( \gamma(t) = 0 \) for almost every \( t \in (-1, 1) \), then \( \gamma(t) = 0 \) for all \( t \in (-1, 1) \). As a result, if \( \gamma(0) > 0 \), \( \text{det}(A, a, b) > 0 \) must be positive. \( \square \)

Now we can prove Theorem 1. We follow the argument in [1]. For all \( \delta > 0 \), \( A \in \mathfrak{B} \subset C(X, Sp(2d, \mathbb{R})) \), where \( \mathfrak{B} \) is ample, we need to prove there is a \( v \in \mathfrak{b} \subset C(X, \mathfrak{sp}(2d, \mathbb{R})) \) such that \( \|v\|_\mathfrak{b} < \delta, L^d(e^{\epsilon A}) > 0 \).
Choose a positive number \( \eta \) satisfying the conditions in Theorem 7 and take \( b \in \mathfrak{b} \) such that \( b \) is \( \eta \)-close to \( \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \). Let \( \epsilon > 0 \) such that \( \epsilon \|b\|_b < \frac{\delta}{2} \). By Theorem 6 there is an element \( a \in B_\epsilon(\eta) \subset C(X,\mathfrak{sp}(2d,\mathbb{R})) \) such that \( L^d(e^{\epsilon a}A) > 0 \). By Lemma 11 we have \( \Phi_c(A,a,b) > 0 \) for any biholomorphic transformation group for (non compact) Hermitian symmetric space.

Since \( \mathfrak{b} \) is dense in \( C(X,\mathfrak{sp}(2d,\mathbb{R})) \), and by Theorem 7 we know the map in (6.3) is continuous, there is an element \( a' \in B_\epsilon(\eta) \cap \mathfrak{b} \) such that \( \Phi_c(A,a',b) > 0 \).

By Theorem 4 the map \( \gamma' : s \mapsto \Phi_c(A,sa',b) \) is an analytic function of \( s \in [-1,1] \). Since \( \gamma'(1) > 0 \), we can choose

\[
0 < s < \min\{1, \frac{\delta}{2\epsilon \|a\|_b} \}
\]

such that \( \gamma'(s) > 0 \). Then there exists \( t \in (-1,1) \) such that

\[
L^d(e^{\epsilon(tb+(1-t^2)sa')}A) > 0
\]

Let \( v = \epsilon(tb+(1-t^2)sa') \), then \( v \in \mathfrak{b} \), \( \|v\|_b < \epsilon(\|b\|_b+s\|a'\|_b) < \delta \) and \( L^d(e^vA) > 0 \).

7. Proof of the rest of results

7.1. Proof of Corollary 11. SHSp\((2d)\) and \( SU(d,d) \) cocycles. The proof of Corollary 11 for \( SHSp(2d) \) and \( SU(d,d) \) cocycles is similar to Theorem 11. Consider the following correspondences between the concepts used in symplectic cocycle and special Hermitian symplectic cocycle. Almost all the following concept also works for any biholomorphic transformation group for (non compact) Hermitian symmetric space.

- \( Sp(2d,\mathbb{R}) \leftrightarrow SHSp(2d) \)
- \( U(d,d) \cap Sp(2d,\mathbb{C}) \leftrightarrow SU(d,d) \)
- \( SHd \leftrightarrow \{ Z = X + iY, X,Y \in Her(d), Y > 0 \} \)
- \( SD_d \leftrightarrow \{ Z, I_d - Z^*Z > 0 \} \)
- Cayley element \( \frac{1}{\sqrt{2}} \begin{pmatrix} I_d & -i \cdot I_d \\ I_d & i \cdot I_d \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} I_d & -i \cdot I_d \\ I_d & i \cdot I_d \end{pmatrix} \)
- \( A \mapsto A := CAC^{-1} \leftrightarrow A \mapsto A := CAC^{-1} \)
- \( Sp(2d,\mathbb{R}) \cong U(d,d) \cap Sp(2d,\mathbb{C}) \leftrightarrow SHSp(2d) \cong SU(d,d) \)
- Möbius transformation \( \leftrightarrow \) Möbius transformation
- \( \Phi_c : SHd \rightarrow SD_d \leftrightarrow \Phi_c : \{ Z = X + iY, X,Y \in Her(d), Y > 0 \} \rightarrow \{ Z, I_d - Z^*Z > 0 \} \)
- \( \partial SD_d \leftrightarrow \partial\{ Z, I_d - Z^*Z > 0 \} = \{ \|Z\| = 1 \} \)
- \( \partial_b SD_d \leftrightarrow \{ \|Z\| = 1, \text{rank}(1 - Z^*Z) = d - k \} \)
- \( \partial_b SD_d = U_{\text{sym}}(\mathbb{C}^d) \leftrightarrow U(d) \)
- \( \text{fin}\partial_b SHd = Sym_d \mathbb{R} \leftrightarrow \text{her}(d) \)
- \( \Phi_C \) gives chart \( : Sym_d \mathbb{R} \leftrightarrow \Phi_C : \text{her}(d) \)
- \( \rightarrow \{ \det(Z - 1) \neq 0 \} \cap \partial_b SD_d \rightarrow \{ Z, \|Z\| = 1, \det(Z - 1) \neq 0 \} \)
atlas $\Phi_{C,sk}$ ↔ $\Phi_{C,sk}$ be defined similarly.

Bergman metric, volume form ↔ Bergman metric, volume form on $\{\|Z\| < 1\}$

\[ V(Z) \leftrightarrow V(Z) = \prod_{1 \leq i \leq d} (1 - \sigma_i(Z)^2)^{-2d} \]

Consider $A_{\sigma + it}, A_{\sigma + it}$ defined as in the symplectic case, we hope to prove the same result as Theorem 3 firstly. Obviously we can define $\tau_{A}, \hat{\tau}$ as in the symplectic case, and we also have similar results to Lemma 6, 8.

By [15], Lemma 4 holds for all (non compact) bounded Hermitian symmetric space, then we can define $m^{+}, m^{-}$ as in the symplectic case. So Lemma 7 also works for special pseudo unitary group. So we can define function $\zeta$ which satisfies condition 1 and 3 in Theorem 3.

To prove statement in Theorem 3 for $SHSp(2d)$, we need to prove the fibered rotation function, the real part of $\zeta$ is non-increasing on $\mathbb{R}$.

Define the cone field $\{h > 0\}$ on $\text{her}(d)$, and then use the tangent map of $\Phi_{C,sk}$ to map it to $TU(d)$. As in the symplectic case, it gives well-defined cone fields $\hat{C}, \hat{\hat{C}}$ on $U(d)$ and $\hat{U}(d)$. Using $\hat{\hat{C}}$ we give a partial order on $\hat{U}(d)$ as the $U_{sym}(\mathbb{C}^d)$ case.

By Cartan decomposition of $SHSp(2d)$ and identity

\[ \det(1 + XY) = \det(1 + YX) \]

we can get the same equation as in Lemma 10. Then we can prove the non-increasing property of the fibered rotation function as Theorem 3.

To prove the same result as Theorem 4. We can use the following equation to replace Lemma 11.

Suppose $t > 0$, $m^{-} = m^{-}(\sigma - it, x)$, $\tilde{m}^{-} = m^{-}(\sigma - it, f(x))$ satisfying $\tilde{m}^{-} = A_{\sigma + it}(x) \cdot m^{-}$. There exists $\tau^{-} \in GL(d, \mathbb{C})$ such that

\[ (7.1) \]

As in the symplectic case, we can define the $q^{-}$ function, we have the following properties for $q^{-}$ function for $SHSp(2d)$ and $SU(d, d)$ to replace Lemma 13 and (4.12).

\[ (7.2) \]

\[ (7.3) \]

Now we can prove Theorem 4 as the following. By non-increasing property of fibered rotation function, Lemma 10 holds. And by the same proof as Lemma 14 we have

\[ (7.4) \]

then we have the same inequality for $\frac{d^d}{dt}$ as (4.22). For the estimate of $\frac{\partial L^d}{\partial t}$ for Hermitian symplectic case, we use $\left( \frac{I_d}{m^{-}} \right)$ to replace $\left( \frac{I_d}{m^{+}} \right), \tau^{-}$ defined in (7.1).
to substitute \( \tau \) defined in Lemma 11, use (7.1) to replace Lemma 11, we can get the following equation

\[
\int_X \mathcal{H}(\text{tr}(I_d - \tilde{m}^{-} \tilde{m}^{+})^{-1}(I_d + \tilde{m}^{-} \tilde{m}^{+}))d\mu = \frac{\partial L_d}{\partial t}
\]

then the rest of the proof of Corollary 1 for \( SHSp(2d), SU(d,d) \) cocycles is the same as the proof of Theorem 1.

7.2. **The proof of Corollary 1.** \( HSp(2d) \) and \( U(d,d) \) cocycles. To prove corollary 1 for \( HSp(2d) \) and \( U(d,d) \) cocycles, we consider the following lemma which similar to Lemma 27:

**Lemma 32.** There exists a dense subset \( Z \) of \( L^\infty(X, HSp(2d)) \) satisfying all conditions of Theorem 5. As a result, for all \( A \in Z, M(A) = 0 \).

**Proof.** Consider a \( HSp(2d) \)–cocycle \( A \) taking finitely many values, Notice that there are cocycles \( B, C \) also taking finitely many values and satisfying

\[
B(x) \in SHSp(2d), C(x) = c(x) \cdot I_{2d}, A(x) = B(x)C(x)
\]

And the statement of Lemma 27 also holds for the space of \( SHSp(2d) \)–cocycle. So we can find a \( SHSp(2d) \)–cocycle \( B' \), \( L^\infty \)–close to \( B \) satisfying all conditions of Theorem 5.

As a result, the \( HSp(2d) \)–cocycle \( A' := B'C \) is \( L^\infty \)–close to \( A \) and satisfying all conditions of Theorem 5. In particular \( M(A') = 0 \).

The rest of the proof is the same as the part after Lemma 27 for the proof of Theorem 1.

**Remark 6.** In general, for a \( HSp(2d) \)–cocycle \( A(x) \), there is no cocycles \( B, C \) in the same regularity class as \( A \) and satisfying (7.6). So we can not use the result of \( SHSp(2d) \) and \( SU(d,d) \) cocycles directly to get the proof for \( HSp(2d) \) and \( U(d,d) \) cocycles. Moreover, since Lemma 10 does not holds for general scalar matrix \( A \), we can not prove corollary 7 for \( HSp(2d) \) and \( U(d,d) \) cocycles by mimicking the proof of Theorem 1 step by step.

7.3. **Proof of Theorem 2 and Corollary 2.** Firstly we consider the following classical result for stochastic Jacobi matrices on the strip in (1.3) proved by B.Simon and S.Kotani (see [28]).

**Definition 15.** For potential \( v \) we define the following function,

\[
M(v) := \text{the Lebesgue measure of } \{ E \in \mathbb{R}, L(A^{E-v}) = 0 \}.
\]

And we say \( v \) is deterministic if for \( n \geq 0 \), \( v(f^n(x)) \) is a measurable function of \( \{ v(f^k(x), k < 0) \} \).

**Lemma 33.** Suppose \( M(v) > 0 \), then \( v \) is deterministic.

5see appendix A for a discussion corresponds to the final part of the proof of Theorem 1 for special Hermitian symplectic cocycles.

6see Appendix A for a discussion corresponds to the final part of Theorem 1 for Hermitian symplectic cocycles.
Combining Lemma 33 with semi-continuity argument in subsection 5.4 (the proof is the same as in [7], using harmonicity of $L^d$ on upper half plane), we can prove result similar to Theorem 6 for stochastic Schrödinger operators and Jacobi matrices on the strips since non-periodic potentials which taking finitely many values are dense in $L^\infty(X, \text{Her}(d)), L^\infty(X, \text{Sym}_d \mathbb{R})$ and $L^\infty(X, \mathbb{R}^d)$.

Then using local regularization formula in [2] (the proof is similar to Theorem 6.2), we can get the proof of Theorem 2 and Corollary 2.

Appendix A. Generic periodic (Hermitian) symplectic cocycles

In this section we finish the proof of Theorem 6 (also for Hermitian symplectic cocycles) by considering the generic periodic cocycles. Without loss of generality, we consider the dynamics $(f, X, \mu)$ as the following:

$$f(x) = x + 1 \quad \text{for} \quad x \in X := \mathbb{Z}/n\mathbb{Z},$$

and for any $Z \subset X$, $\mu(Z) = \frac{1}{n} #(Z)$.

It is easy to see there is a set $O \subset C(X, HSp(2d)) = HSp(2d)^n$ such that $O, O \cap Sp(2d, \mathbb{R}), O \cap SHSp(2d)$ are residual sets in $HSp(2d), Sp(2d, \mathbb{R}), SHSp(2d)$ respectively and satisfy the following property: for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the geometric and algebraic multiplicities of eigenvalues of $A^{(n)}(\theta)$ are 1 and at most 2 respectively.

And there are only finite $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that $A^{(n)}(\theta)$ has repeated eigenvalues.

We only need to prove the following lemma:

**Lemma 34.** There is a constant $C$ independent of $n$ such that for $A \in O$, there exists $\theta \in (-C/n, C/n)$ such that $L(A^{(n)}(\theta)) > 0$.

**Proof.** Without loss of generality, we only need to consider those $A$ with determinant equal to 1. We fix such an $A \in O$. Suppose there is an open interval $I$ containing 0 and $|I| > O(\frac{1}{n})$, such that

$$\forall \theta \in I, \sigma(A^{(n)}(\theta)) \subset \{|z| = 1\}$$

where $|I|$ is the length of $I$. We prove the following lemma:

**Lemma 35.** For any interval $I'$ such that

$$\forall \theta \in I', \text{all eigenvalues of } A^{(n)}(\theta) \text{ are simple and norm 1}$$

we have

$$|I'| \leq O(\frac{1}{n})$$

**Proof.** Suppose $I' = (a, b)$, for $\theta \in I'$, we conjugate it by $A^{(n)}(\theta)$ to the matrix with form

$$\text{diag}(e^{i\rho_1(\theta)}, \ldots, e^{i\rho_d(\theta)}, e^{-i\rho_1(\theta)}, \ldots, e^{-i\rho_d(\theta)})$$

where $\rho, \rho'$ depend continuously on $\theta$, $\sum_{i=1}^d \rho_i = \sum_{i=1}^d \rho'_i$ and $\rho_i + \rho_j' \notin 2\pi\mathbb{Z}$.

\[7\] in that case where $\mu-$almost all points are periodic, see Appendix B.
Therefore \( \rho'_{i}(b-) + \rho_{i}(b-) - \rho_{i}(a+) - \rho'_{i}(a+) < 4\pi \). As a result, by the definition and property of fibered rotation function \( \rho \), we have

\[
(A.3) \quad \rho(a) - \rho(b) = \frac{1}{n} \sum_{i=1}^{d} \rho'_{i}(b-) - \frac{1}{n} \sum_{i=1}^{d} \rho_{i}(a+)
\]

\[
= \frac{1}{2n} \sum_{i=1}^{d} \rho'_{i}(b-) + \rho_{i}(b-) - \rho'_{i}(a+) - \rho_{i}(a+)
\]

\[
\leq \frac{2d\pi}{n}
\]

But as in the proof of Lemma 16 and (4.22), for almost all \( \theta \in I' \), we have

\[
(A.4) \quad -\frac{d\rho}{d\theta} = \lim_{t \to 0^{+}} \frac{\partial L^{d}(A_{\theta+it})}{\partial t} = \lim_{t \to 0} \frac{L^{d}(A_{\theta+it})}{t} \geq d
\]

Combined with (A.3), we get the bound of \( |I'| \). \[\square\]

As a result, there are two intervals \( I_{1}, I_{2} \subset I \) satisfying (A.3) and sharing common boundary point \( \theta_{0} \) such that \( \det(\lambda I_{d} - A_{\theta_{0}}^{(n)}(x)) \) has a double root \( e^{i\rho(\theta_{0})} \). Denote the corresponding general eigenspace \( V \) by \( V_{\theta_{0}} \).

To state the next lemma and for later use we introduce some basic concepts of Hermitian symplectic geometry on \( \mathbb{C}^{2d} \) (see for example [26]).

**Definition 16.** The two-form \( \langle \cdot, \cdot \rangle \) is linear in the second argument and conjugate linear in the first argument, is a hermitian symplectic form if

\[ \langle \phi, \psi \rangle = -\langle \psi, \phi \rangle \]

Let \( e_{i} \) be the standard basis of \( \mathbb{C}^{2d} \), then we can define a classical Hermitian symplectic form \( \langle \cdot, \cdot \rangle \) such that \( \langle e_{i}, e_{d+i} \rangle = 1 \) for \( 1 \leq i \leq d \) and \( \langle e_{i}, e_{j} \rangle = 0 \) if \( |i - j| \neq d \). The Hermitian symplectic groups are all transformation preserving this form. A basis satisfying the same relation is called a canonical basis. A subspace \( V \) of \( \mathbb{C}^{2d} \) is called a Hermitian symplectic subspace if \( \langle \cdot, \cdot \rangle | V \) is non-degenerate. If an even dimensional Hermitian symplectic subspace \( V \) admits a canonical basis then we say \( V \) is canonical. A subspace \( N \) is called isotropic if \( N \subset N^\perp := \{v, \langle v, w \rangle = 0, \forall w \in N\} \). For a \( 2l \)–dimensional Hermitian symplectic subspace \( V \), if there is an \( l \)–dimensional isotropic subspace \( N \subset V \), then we call \( N \) is a Lagrange plane of \( V \). It can be proved that a Hermitian symplectic subspace \( V \) contains a Lagrange plane if and only if it is canonical (see [26]).

Now we state the lemma:

---

8A general eigenspace of eigenvalue \( \lambda \) of linear transform \( A \) is defined by \( V_{\lambda} := \{v : \exists n, (A - \lambda)^n \cdot v = 0\} \)
By a suitable choice of canonical basis, we can assume $A_{\theta_0}^{(n)} \cdot A_{\theta_0}^{(n)} |_{V_{\theta_0}}$ to be

$$
\begin{pmatrix}
e^{i\rho(\theta_0)} & \beta(\theta_0) \\
*(d-1) \times (d-1) & e^{i\rho(\theta_0)} \\
*(d-1) \times (d-1) & *(d-1) \times (d-1)
\end{pmatrix}
$$

(A.5)

and

$$
\begin{pmatrix}
e^{i\rho(\theta_0)} & \beta(\theta_0) \\
0 & e^{i\rho(\theta_0)} \\
0 & 0
\end{pmatrix}
$$

(A.6)

respectively, where $\beta(\theta_0) \neq 0$.

Proof. Notice that different general eigenspaces are mutually orthogonal (in the Hermitian symplectic sense). So $V_{\theta_0}$ is a Hermitian symplectic subspace of $\mathbb{C}^{2d}$.

Moreover consider the only one eigenvector $v \in V_{\theta_0} - \{0\}$ and $w \in V_{\theta_0} - \mathbb{C} \cdot v$, then

$$A \cdot v = e^{i\rho(\theta_0)} v, A \cdot w - e^{i\rho(\theta_0)} w \in \mathbb{C} \cdot v - \{0\}$$

Therefore computing $\langle Av, Aw \rangle - \langle v, w \rangle$ we get

$$\langle v, v \rangle = 0$$

Since $V_{\theta_0}$ contains a Lagrange plane, it is a canonical Hermitian symplectic subspace of $\mathbb{C}^{2d}$. Then we can pick a vector $v' \in V_{\theta_0} - \mathbb{C} \cdot v$ such that $\langle v', v' \rangle = 0$ and by further normalization we can assume $\langle v, v' \rangle = 1$.

It is easy to prove that $V_{\theta_0}$ is also a canonical Hermitian symplectic subspace. Then we can extend $v, v'$ to a canonical basis of $\mathbb{C}^{2d}$. Then $A_{\theta_0}^{(n)} |_{V_{\theta_0}}$ have the form in (A.6) with respect to this canonical basis.

Choose a contour $\Gamma$ enclosing $e^{i\rho(\theta_0)}$ but no other points of $\sigma(A_{\theta_0}^{(n)})$, then for $\theta$ close to $\theta_0$, $\Gamma$ encloses exactly two points of $\sigma(A_{\theta_0}^{(n)})$. Therefore there is a 2−dimensional invariant space $V_\theta$ for eigenvalues of $A_{\theta_0}^{(n)}$ contained in the bounded region with boundary $\Gamma$.

Consider the spectral projection $\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{z-A_{\theta_0}^{(n)}} dz$ of $A_{\theta_0}^{(n)}$ onto $V_\theta$. Since $\sigma(A_{\theta_0}^{(n)}) \subset \{|z| = 1\}$, we have (by suitable choice of branch of $z \mapsto \sqrt{z}$):

$$\frac{\text{tr}(\frac{1}{2\pi i} \int_{\Gamma} \frac{z}{z-A_{\theta_0}^{(n)}} dz)}{\sqrt{\text{tr}(\Lambda^{2}(\frac{1}{2\pi i} \int_{\Gamma} \frac{z}{z-A_{\theta_0}^{(n)}} dz))}} \leq 2$$

(A.9)

and

$$\frac{\text{tr}(\frac{1}{2\pi i} \int_{\Gamma} \frac{z}{z-A_{\theta_0}^{(n)}} dz)}{\sqrt{\text{tr}(\Lambda^{2}(\frac{1}{2\pi i} \int_{\Gamma} \frac{z}{z-A_{\theta_0}^{(n)}} dz))}} = 2$$

(A.10)

Therefore

$$\frac{d}{d\theta} \bigg|_{\theta = \theta_0} \frac{\text{tr}(\frac{1}{2\pi i} \int_{\Gamma} \frac{z}{z-A_{\theta_0}^{(n)}} dz)}{\sqrt{\text{tr}(\Lambda^{2}(\frac{1}{2\pi i} \int_{\Gamma} \frac{z}{z-A_{\theta_0}^{(n)}} dz))}} = 0$$

(A.11)
Then we get

\[
(A.12) \quad 2 \frac{d}{d\theta} |_{\theta=\theta_0} \text{tr} \left( \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right) \cdot \text{tr} \left( \Lambda^2 \left( \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right) \right) = \text{tr} \left( \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right) \cdot \frac{d}{d\theta} |_{\theta=\theta_0} \text{tr} \left( \Lambda^2 \left( \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right) \right)
\]

By (A.6), we have

\[
(A.13) \quad \text{tr} \left( \Lambda^2 \left( \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right) \right) = e^{2i\rho(\theta_0)}
\]

\[
(A.14) \quad \text{tr} \left( \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right) = 2e^{i\rho(\theta_0)}
\]

Since \( \frac{d}{d\theta} |_{\theta=\theta_0} A_{\theta_0}^{(n)} A_{\theta_0}^{(n)-1} \in \mathfrak{sl}(2d) \), we can assume

\[
(A.15) \quad \frac{d}{d\theta} |_{\theta=\theta_0} A_{\theta_0}^{(n)} A_{\theta_0}^{(n)-1} = \begin{pmatrix} X & Y \\ Z & -X^* \end{pmatrix}
\]

where \( Y = Y^*, Z = Z^* \) and \( \text{tr}(X - X^*) = 0 \). Let \( X, Y, Z \) be \( (x_{ij})_{i,j=1}^d, (y_{ij})_{i,j=1}^d, (z_{ij})_{i,j=1}^d \) respectively.

By computation we get,

\[
(A.16) \quad \frac{d}{d\theta} |_{\theta=\theta_0} \text{tr} \left( \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right) = \text{tr} \left( \frac{d}{d\theta} |_{\theta=\theta_0} A_{\theta_0} \cdot \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right)
\]

Notice that \( \frac{1}{2\pi i} \int \frac{z}{(z - A_{\theta_0}^{(n)})^2} dz \) is the spectral projection of \( A_{\theta_0}^{(n)} \) onto \( V_{\theta_0} \), we have

\[
(A.17) \quad \frac{d}{d\theta} |_{\theta=\theta_0} \text{tr} \left( \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right) = \text{tr} \left( \frac{d}{d\theta} |_{\theta=\theta_0} A_{\theta_0}^{(n)} A_{\theta_0}^{(n)-1} \cdot A_{\theta_0}^{(n)} |_{V_{\theta_0}} \right) = e^{i\rho(\theta_0)}(x_{11} - \frac{1}{2} x_{11}) + \beta(\theta_0) z_{11}
\]

Denote \( A_{\theta_0}^{(n)} |_{V_{\theta_0}} \) by \( B_0 = (b_{i,j}(\theta))_{1 \leq i,j \leq d} \), then

\[
(A.18) \quad \frac{d}{d\theta} |_{\theta=\theta_0} \text{tr} \left( \Lambda^2 \left( \frac{1}{2\pi i} \int \frac{z}{z - A_{\theta_0}^{(n)}} dz \right) \right) = \frac{d}{d\theta} |_{\theta=\theta_0} \text{tr} \left( A^2 |_{V_{\theta_0}} \right) = \frac{d}{d\theta} |_{\theta=\theta_0} \text{tr} \left( A^2 |_{B_0} \right) = \sum_{i<j} | b'_{i,j}(\theta_0) b_{i,j}(\theta_0) | + | b_{i,j}(\theta_0) b_{j,i}(\theta_0) | = | b'_{1,1}(\theta_0) b_{1,1}(\theta_0) | + | b_{1,1}(\theta_0) b_{1,1}(\theta_0) | ( \text{since } b_{i,j}(\theta_0) = 0 \text{ except } \{i, j\} \subset \{1, d+1\} )
\]
By (A.5), (A.9), (A.15) we have

\[
\begin{pmatrix}
b_{1,1}(\theta_0) & b_{1,d+1}(\theta_0) \\
b_{d+1,1}(\theta_0) & b_{d+1,d+1}(\theta_0)
\end{pmatrix} = 
\begin{pmatrix}
e^{ip(\theta_0)} & \beta(\theta_0) \\
e^{ip(\theta_0)} & e^{ip(\theta_0)}
\end{pmatrix}
\]

Then combining with (A.18) we get

\[
(A.19) \quad \frac{d}{d\theta}|_{\theta=\theta_0} \text{tr}(\Lambda^2) = e^{2ip(\theta_0)}(x_{11} - \bar{x}_{11})
\]

Combining (A.19), (A.17), (A.13), (A.14), (A.12) we get

\[
(A.20) \quad z_{11} = 0
\]

Now we define the *monotonic* special Hermitian symplectic cocycles: consider a cone \( C \) on \( \mathfrak{shsp}(2d, \mathbb{R}) \) defined by

\[
(A.21) \quad C := \{ W \in \mathfrak{shsp}(2d, \mathbb{R}) : J \cdot W \text{ is negative definite} \}
\]

where \( J \) is defined in Definition 1. Obviously for any \( W_1, W_2 \in C \), \( W_1 + W_2 \in C \).

**Definition 17.** Let \( \theta \to D_\theta \in C(X, SHSp(2d, \mathbb{R})) \) be a one parameter family of continuous symplectic cocycles and \( C^1 \) in \( \theta \). We say it is monotonic at \( \theta_0 \) if

\[
(A.22) \quad \frac{d}{d\theta}|_{\theta=\theta_0} D_\theta(x)D_\theta^{-1}(x) \in C
\]

for any \( x \in X \).

We have:

**Lemma 37.** If the one parameter family of symplectic cocycle \( D_\theta \) is monotonic at \( \theta_0 \), then for any \( n > 0 \), \( D^{(n)}_\theta \) is also monotonic at \( \theta_0 \).

**Proof.** First of all we prove the invariance of the cone \( C \) under inner automorphism.

**Lemma 38.** For any \( g \in SHSp(2d, \mathbb{R}) \), \( gCg^{-1} = C \)

**Proof.** Notice that for any \( g \in SHSp(2d, \mathbb{R}) \), \( Jg = (g^{-1})^*J \). Then for any \( W \in C \),

\[
(A.23) \quad JgWg^{-1} = (g^{-1})^*JWg^{-1}
\]

is negative definite (since \( g \) is invertible and \( JW \) is negative definite). Therefore we have for any \( g \), \( gCg^{-1} \subset C \). It is easy to prove that the equality holds. \( \square \)

Suppose \( D_\theta \) is monotonic at \( \theta_0 \), then for any \( x \in X \), using Lemma 38 we have

\[
(A.24) \quad \frac{d}{d\theta}|_{\theta=\theta_0} D^{(n)}_\theta(x)D^{(n)}_{\theta_0}^{-1}(x)
\]

\[= \sum_{i=0}^{n} D_{\theta_0}(f^{n-1}(x)) \cdots D_{\theta_0}(f^{1}(x)) \cdot \left( \frac{d}{d\theta}|_{\theta=\theta_0} D_{\theta}(f^{i}(x))D_{\theta_0}(f^{i}(x))^{-1} \right)
\]

\[\cdot (D_{\theta_0}(f^{n-1}(x)) \cdots D_{\theta_0}(f^{1}(x)))^{-1} \in C \]

which implies \( D^{(n)}_\theta \) is also monotonic at \( \theta_0 \). \( \square \)
Come back to the discussion of periodic cocycle $A_\theta$. It is easy to check that the one parameter family cocycles $\theta \mapsto A_\theta$ is monotonic on $\mathbb{R}$, then by Lemma 37 $\theta \mapsto A_\theta^{(n)}$ is monotonic at $\theta_0$. As a result, $J \cdot \frac{d}{d\theta}|_{\theta=\theta_0} A_\theta^{(n)} A_\theta^{(n)-1}$ is negative definite, using (A.15) we get $Z$ is negative definite, which contradicts with (A.20). In summary, $I_1, I_2$ cannot have common boundary point.

**APPENDIX B. GENERIC PERIODIC SCHRODINGER OPERATOR AND JACOBI MATRICES ON THE STRIP**

For periodic Schrödinger operator and Jacobi matrices on the strip, consider the corresponding (Hermitian) symplectic cocycle:

$$A^{(E-v)} : \mathbb{Z}/n\mathbb{Z} \to Sp(2d, \mathbb{R})$$

where $A^{(v)}(x) := \begin{pmatrix} v(x) & -I_d \\ I_d & -v(x) \end{pmatrix}$, $v(x) \in \mathbb{R}^d \hookrightarrow Sym_d \mathbb{R}, Sym_d \mathbb{R}$ or her($d$) is the corresponding potential. As in Appendix A, we hope to prove the following lemma:

**Lemma 39.** There is a constant $C$ independent of $n$ such that for generic potential $v$ in $(\mathbb{R}^d)^n$, $(Sym_d \mathbb{R})^n$ or her($d$)$^n$, there exists $E \in (-\frac{C}{n}, \frac{C}{n})$ such that $L(A^{(E-v)}) > 0$.

**Proof.** At first we prove the following fact:

**Lemma 40.** For any Jacobi matrices on the strip, the corresponding one parameter cocycle $E \mapsto (A^{(E-v)})^{(2)}$ is monotonic on $\mathbb{R}$.

**Proof.** By computation

$$\frac{d}{dE}|_{E=E_0} (A^{(E_0-v)}(f(x))A^{(E_0-v)}(x)) \cdot (A^{(E_0-v)}(x))^{-1}(A^{(E_0-v)}(f(x))^{-1}$$

$$= \begin{pmatrix} -I_d & E - v(f(x)) \\ E - v(f(x)) & -I_d - (E - v(f(x)))^2 \end{pmatrix}$$

$$= - \begin{pmatrix} I_d & I_d \\ -(E - v(f(x))) & I_d \end{pmatrix} \cdot \begin{pmatrix} I_d & -(E - v(f(x))) \\ I_d & I_d \end{pmatrix}$$

which is negative definite, by definition of monotonicity in Appendix A we get the proof. □

Denote $B_E(x) := (A^{(E-v)})^{(2)}$. Since $E \mapsto B_E(x)$ is monotonic, mimic the proof of Theorem 39 we can define the $m-$function (taking values in $\{Z, \|Z\| < 1\}$), fibered rotation function (which is non increasing) and complexify the Lyapunov exponent $E + it \mapsto (L^d + i\rho)(B_{E+it})$. In fact these properties are already known for Jacobi matrices on the strip with form in [L3], see [28], [34] for example or [31] for general monotonic symplectic cocycles.

The proof of Lemma 39 is basically the same as the discussion in Appendix A.

At first we prove the following Lemma similar to Lemma 40

**Lemma 41.** For any interval $I'$ such that

$$\forall E \in I', \text{ all eigenvalues of } B_E^{(n)}(x) \text{ are simple and norm } 1$$

we have

$$|I'| \leq O\left(\frac{1}{n}\right)$$
Proof. As in the proof of Lemma \[33\] we only need to prove for almost all \( E \in I' \), we have

\begin{equation}
(B.2) \quad \limsup_{t \to 0^+} \frac{L^d(B_{E+it})}{t} \geq c(d)
\end{equation}

where \( c(d) \) is a constant independent of \( n \). Let

\[ \hat{B}_{E+it}(x) := C(f^2(x))B_{E+it}(x)C(x)^{-1}. \]

where \( C(x) := \begin{pmatrix} I_d & E - v(f^{-1}(x)) \end{pmatrix} \). Then all the dynamical properties of cocycles \( B_{E+it} \) and \( \hat{B}_{E+it} \) are the same. Moreover by computation,

\begin{equation}
(B.3) \quad \frac{\partial \hat{B}_{E+it}(x)}{\partial t} = (-i \cdot I_d \cdot i \cdot I_d) \cdot \hat{B}_{E+it}(x)
\end{equation}

Therefore we have

\begin{equation}
(B.4) \quad \hat{B}_{E+it}(x) = \left( \begin{array}{c} e^{-t} \\ e^t \end{array} \right) + o(t) \cdot \hat{B}_E(x)
\end{equation}

where \( \hat{B}(x) := C \hat{B}_E(x)C^{-1}, C \) is the Cayley element defined in \(2.1\).

Without loss of generality, we fix an \( E \) such that \( \lim_{t \to 0^+} \Phi_{\hat{C}^{-1}} \cdot \hat{m}(E + it, x) \) exists and be finite, where \( \hat{m}(E + it, \cdot) \) is the associated \( m \)-function of cocycle \( \hat{B}_{E+it} \). Since \( \Phi_{\hat{C}^{-1}} \cdot \hat{m} \) is the classical \( m \)-function taking values in \( \{ Z, \text{Im}(Z) > 0 \} \) for Jacobi matrices on the strip (see \[28\] for example), by Lemma \[24\] we know that almost every \( E \in \mathbb{R} \) satisfies our condition.

As Definition \[10\] we define by \( \hat{q}_{E+it}(x) \) the Jacobian (with respect to the volume form induced by the Bergman metric) of the map

\[ Z \mapsto \hat{B}_{E+it}(x) \cdot Z \]

at point \( Z = \hat{m}(E + it, x) \). Then similar to the case of symplectic cocycles, we have

\begin{equation}
(B.5) \quad L^d(\hat{B}_{E+it}) = \frac{1}{4d} \int_X \ln \hat{q}^{-1}_{E+it} d\mu.
\end{equation}

Moreover as the proof of Lemma \[17\] we have

\begin{equation}
(B.6) \quad \hat{q}^{-1}_{E+it}(x) = \frac{V(e^{2it} \hat{m}(E + it, f^2(x)) + o(t))}{V(\hat{m}(E + it, f^2(x)))} \cdot e^{4td^2 + o(t)}
\end{equation}

By our choice of \( E \), \( \lim_{t \to 0^+} \sigma_1^2(\hat{m}(E + it, f^2(x))) \) exists and not greater than \( 1 \). Therefore using the inequality \( [4,19] \), we have

\begin{equation}
(B.7) \quad \frac{1 - \sigma_1^2(\hat{m}(E + it, f^2(x)))}{1 - (e^{4t\sigma_1^2(\hat{m}(E + it, f^2(x))) + o(t))} \geq 1, \text{ when } t \text{ is small}
\end{equation}

Combining with \( B.6, B.5 \) we get,

\begin{equation}
(B.8) \quad L^d(\hat{B}_{E+it}) \geq dt + o(t), \text{ when } t \text{ is small}
\end{equation}

for almost all \( E \in I' \), which implies our lemma. \( \square \)
As a result, to prove Lemma [39], we only need to prove for generic potential $v$, two intervals $I_1, I_2$ satisfying the condition of Lemma [41] cannot share a common boundary point. But in the corresponding part of Appendix A, we only use the condition that the one parameter family $\theta \mapsto A_\theta$ is monotonic, then by Lemma [40] the proof is the same here.

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