ON TILTING COMPLEXES PROVIDING DERIVED EQUIVALENCES THAT SEND SIMPLE-MINDED OBJECTS TO SIMPLE OBJECTS

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Abstract. Given a set of ‘simple-minded’ objects in a derived category, Rickard constructed a complex, which over a symmetric algebra provides a derived equivalence sending the 'simple-minded' objects to simple ones. We characterise in terms of t-structures, when this complex is a tilting complex, show that there is an associated natural t-structure and we provide an alternative construction of this complex in terms of A_\infty-structures. Our approach is similar to that of Keller–Nicolás.

Contents

1. Introduction 1
2. Acknowledgements 2
3. Notations and preliminaries 2
4. Rickard’s construction 3
5. An alternative construction 8
6. Appendix: Finite-dimensional non-positive dg algebras 13
7. References 16

1. Introduction

The module category of a finite dimensional algebra, when seen as an abelian category, has two natural 'generators': a projective generator and the direct sum of a full set of simple modules. Equivalences of abelian categories send progenerators to progenerators and simples to simples. The derived module category of an algebra, when seen as a triangulated category, has two kinds of natural generators: each tilting complex 'generates' the category, and the direct sum of a full set of simple modules does so, too. Equivalences of derived categories send tilting complexes to tilting complexes. It is, however, not clear what happens to simple modules under derived equivalences. For symmetric algebras, Rickard [16] has shown that the 'group' of derived equivalences acts transitively on the class of objects sharing with the simple objects certain obvious conditions. Given such 'simple-minded' objects, he explicitly constructed a tilting complex, and thus a derived equivalence. This has been used extensively in modular representation theory of finite

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1
groups. Rickard’s construction, using Milnor colimits, produces a complex for any algebra, not just a symmetric one. To show that this complex is a tilting complex, the assumption symmetric is used, and in general one cannot expect to get a tilting complex.

This note addresses two questions in this context. First, we characterise in terms of t-structures (Section 3), when Rickard’s construction yields a tilting complex. On the way, we give new proofs of some results by Rickard and by Al-Nofaye [1, 2], who also considered this problem and obtained related results, in particular also extending Rickard’s main result to self-injective algebras, and constructing a t-structure. Similar results are obtained by Keller and Nicolás in [9] in a different context. Secondly, we provide in Section 4 an alternative construction of the same complex, in terms of $A\infty$-categories. This uses work of Keller and Lefèvre [12]. In an appendix we investigate some basic properties of non-positively graded finite-dimensional dg algebras. These properties are used in Section 3 to construct the t-structure and to extend Rickard’s result to self-injective algebras, and used in Section 4 to show that the above results are valid also in the more general setting of finite-dimensional non-positive dg algebras.

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2. Notations and preliminaries

Throughout, $K$ will be an algebraically closed field. All algebras, modules, vector spaces and categories are over the base field $K$. For a category $C$, we denote by $\text{Hom}_C(X, Y)$ the morphism space from $X$ to $Y$, where $X$ and $Y$ are two objects of $C$. We will omit the subscript and write $\text{Hom}(X, Y)$ when it does not cause confusion. By abuse of notation, we will denote by $\Sigma$ the suspension functors of all the triangulated categories appearing in this paper. For a triangulated category $C$ and a set $S$ of objects in $C$, let $\text{thick}(S)$ denote the smallest triangulated subcategory of $C$ containing objects in $S$ and stable for taking direct summands, and let $\text{Add}(S)$ denote the smallest full subcategory of $C$ containing all objects of $S$ and stable for taking coproducts and direct summands.

For a finite-dimensional basic algebra $\Lambda$, let $\text{Mod}\Lambda$ (respectively, $\text{mod}\Lambda$) denote the category of right $\Lambda$-modules (respectively, finite-dimensional right $\Lambda$-modules), and let $\mathcal{D}(\text{Mod}\Lambda)$ (respectively, $\mathcal{D}^b(\text{mod}\Lambda)$, $\mathcal{D}^-(\text{mod}\Lambda)$) denote the derived category of $\text{Mod}\Lambda$ (respectively, bounded derived category of $\text{mod}\Lambda$, bounded above derived category of $\text{mod}\Lambda$). The categories $\mathcal{D}(\text{Mod}(\Lambda))$, $\mathcal{D}^-(\text{mod}\Lambda)$ and $\mathcal{D}^b(\text{mod}\Lambda)$ are triangulated with suspension functor the shift functor. We view $\mathcal{D}^-(\text{mod}\Lambda)$ and $\mathcal{D}^b(\text{mod}\Lambda)$ as triangulated subcategories of $\mathcal{D}(\text{Mod}(\Lambda))$. 
For a differential graded (=dg) algebra $A$, let $\mathcal{D}(A)$ denote the derived category of right dg $A$-modules, cf. [7], and let $\mathcal{D}_{fd}(A)$ denote its full subcategory of dg $A$-modules whose total cohomology is finite-dimensional. They are triangulated categories with suspension functor the shift functor. Let $\text{per}(A) = \text{thick}(A_A)$, i.e. the smallest triangulated subcategory of $\mathcal{D}(A)$ containing the free dg $A$-module of rank 1 and stable for taking direct summands. Let $A$ and $B$ be two dg algebras. Then a triangle equivalence between $\mathcal{D}(A)$ and $\mathcal{D}(B)$ restricts to a triangle equivalence between $\text{per}(A)$ and $\text{per}(B)$ as well as a triangle equivalence between $\mathcal{D}_{fd}(A)$ and $\mathcal{D}_{fd}(B)$. If $A$ is a finite-dimensional algebra viewed as a dg algebra concentrated in degree 0, then $\mathcal{D}(A)$ is exactly $\mathcal{D}(\text{Mod } A)$, $\mathcal{D}_{fd}(A)$ is $\mathcal{D}^b(\text{mod } A)$, and $\text{per}(A)$ is triangle equivalent to the homotopy category of bounded complexes of finitely generated projective $A$-modules.

3. Rickard’s construction

Let $\Lambda$ be a finite-dimensional basic $K$-algebra. In this section we discuss a construction by Rickard [16]. The same construction is studied by Keller–Nicolás [9] in the context of positive dg algebras.

Let $r$ be the rank of the Grothendieck group of $\text{mod } \Lambda$. Following [11] we say that a set of objects $X_1, \ldots, X_r$ in the bounded derived category $\mathcal{D}^b(\text{mod } \Lambda)$ are simple-minded if the following conditions hold

1. $\text{Hom}(X_i, \Sigma^m X_j) = 0$, $\forall$ $m < 0$,
2. $\text{Hom}(X_i, X_j) = \begin{cases} K & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases}$
3. $X_1, \ldots, X_r$ generates $\mathcal{D}^b(\text{mod } \Lambda)$, i.e. $\mathcal{D}^b(\text{mod } \Lambda) = \text{thick}(X_1, \ldots, X_r)$.

In [16] Rickard constructed from $X_1, \ldots, X_r$ a set of objects $T_1, \ldots, T_r$ as follows.

Set $X_i^{(0)} = X_i$. Suppose $X_i^{(n-1)}$ is constructed. For $i, j = 1, \ldots, r$ and $m < 0$, let $B(j, m, i)$ be a basis of $\text{Hom}(\Sigma^m X_j, X_i^{(n-1)})$. Put

$$Z_i^{(n-1)} = \bigoplus_{j} \bigoplus_{B(j, m, i)} \Sigma^m X_j$$

and let $\alpha_i^{(n-1)} : Z_i^{(n-1)} \to X_i^{(n-1)}$ be the map whose component corresponding to $f \in B(j, m, i)$ is exactly $f$.

Let $X_i^{(n)}$ be a cone of $\alpha_i^{(n-1)}$ and form the corresponding triangle

$$X_i^{(n-1)} \xrightarrow{\alpha_i^{(n-1)}} X_i^{(n-1)} \xrightarrow{\beta_i^{(n-1)}} X_i^{(n)} \to \Sigma Z_i^{(n-1)}.$$

Inductively we obtain a sequence of morphisms in $\mathcal{D}(\text{Mod } \Lambda)$:

$$X_i^{(0)} \xrightarrow{\beta_i^{(0)}} X_i^{(1)} \to \cdots X_i^{(n-1)} \xrightarrow{\beta_i^{(n-1)}} X_i^{(n)} \to \cdots$$
Let $T_i$ be the Milnor colimit of this sequence. Up to isomorphism, $T_i$ is determined by the following triangle
\[
\bigoplus_{n \geq 0} X_i^{(n)} \xrightarrow{id - \beta} \bigoplus_{n \geq 0} X_i^{(n)} \xrightarrow{T_i} \Sigma \bigoplus_{n \geq 0} X_i^{(n)}.
\]

3.1. Properties of $T_i$’s. The following properties of $T_i$’s were proved in [16] for symmetric algebras $\Lambda$, but the proofs there also work in general.

**Lemma 3.2.**

a) ([16, Lemma 5.4]) For $1 \leq i, j \leq r$, and $m \in \mathbb{Z}$,
\[
\text{Hom}(X_j, \Sigma^m T_i) = \begin{cases} 
K & \text{if } i = j \text{ and } m = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

b) ([16, Lemma 5.5]) For each $1 \leq i \leq r$, $T_i$ is isomorphic to a bounded complex of finitely generated injectives. From now on we assume that $T_i$ is such a complex.

c) ([16, Lemma 5.8]) Let $C$ be an object of $D^{-}(\text{mod } \Lambda)$. If $\text{Hom}(C, \Sigma^m T_i) = 0$ for all $m \in \mathbb{Z}$ and all $1 \leq i \leq r$, then $C = 0$.

Let $\nu$ be the Nakayama functor, and $\nu^{-1}$ the inverse Nakayama functor (cf. [6, Chapter 1, Section 4.6]). They are quasi-inverse triangle equivalences between the triangulated subcategories $\text{per}(\Lambda)$ and $\text{thick}(D(\Lambda))$ of $D(\text{Mod } \Lambda)$, where $D = \text{Hom}_K(?, K)$ is the duality functor. The following is a consequence of Lemma 3.2 and the property of the Nakayama functor.

**Lemma 3.3.**

a) For $1 \leq i, j \leq r$, and $m \in \mathbb{Z}$,
\[
\text{Hom}(\nu^{-1} T_i, \Sigma^m X_j) = \begin{cases} 
K & \text{if } i = j \text{ and } m = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

b) For each $1 \leq i \leq r$, $\nu^{-1} T_i$ is a bounded complex of finitely generated projectives.

c) Let $C$ be an object of $D^{-}(\text{mod } \Lambda)$. If $\text{Hom}(\nu^{-1} T_i, \Sigma^m C) = 0$ for all $m \in \mathbb{Z}$ and all $1 \leq i \leq r$, then $C = 0$.

We put $T = \bigoplus_{i=1}^{r} T_i$ and $\nu^{-1} T = \bigoplus_{i=1}^{r} \nu^{-1} T_i$.

**Lemma 3.4.** We have
\[
\text{Hom}(\nu^{-1} T, \Sigma^m T) = 0
\]
for $m < 0$. Equivalently,
\[
\text{Hom}(\nu^{-1} T, \Sigma^m \nu^{-1} T) = \text{Hom}(T, \Sigma^m T) = 0
\]
for $m > 0$.

**Proof.** Same as the proof of [16 Lemma 5.2], with the $T_i$ in the first entry of $\text{Hom}$ there replaced by $\nu^{-1} T_i$. \hfill $\Box$
Theorem 3.5 ([16] Theorem 5.1). When $\Lambda$ is a symmetric algebra, $T = \nu^{-1}T$ is a tilting complex.

Proof. By Lemma 3.3, $\nu^{-1}T$ is a compact generator of $D(\text{Mod } \Lambda)$. Moreover, when $\Lambda$ is symmetric, the Nakajama functor is isomorphic to the identity. The desired result follows from Lemma 3.4.

In general, we may ask when $\nu^{-1}T$ is a tilting complex. If this is the case, then by Rickard’s Morita’s theorem for derived categories (cf. [15]) we have a triangle equivalence

$$D(\text{Mod } \Lambda) \simeq D(\text{Mod } \Gamma),$$

which takes $X_1, \ldots, X_r$ to a complete set of non-isomorphic simple $\Gamma$-modules, where $\Gamma = \text{Hom}(\nu^{-1}T, \nu^{-1}T)$. Conversely, assume there is a finite-dimensional algebra $\Gamma$ with an equivalence $F : D(\text{Mod } \Lambda) \simeq D(\text{Mod } \Gamma)$ sending $X_1, \ldots, X_r$ to a complete set of non-isomorphic simple $\Gamma$-modules. It follows from Lemma 3.3 that for $1 \leq i, j \leq r$ and $m \in \mathbb{Z}$, we have

$$\text{Hom}_{D(\text{Mod } \Gamma)}(F \nu^{-1}T_i, \Sigma^m F X_j) = \begin{cases} K & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This means that $F \nu^{-1}T_i$ is the projective cover of $FX_i$, and hence $F \nu^{-1}T = \Gamma$ is the free $\Gamma$-module of rank 1. Thus $\nu^{-1}T$ is a tilting complex.

3.6. A $t$-structure. Let $\mathcal{C}$ be a triangulated category. A $t$-structure on $\mathcal{C}$ ([5]) is a pair $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ of strictly full subcategories of $\mathcal{C}$ such that

- $\Sigma \mathcal{C}^{\leq 0} \subseteq \mathcal{C}^{\leq 0}$ and $\Sigma^{-1} \mathcal{C}^{\geq 0} \subseteq \mathcal{C}^{\geq 0}$;
- $\text{Hom}_{\mathcal{C}}(M, \Sigma^{-1} N) = 0$ for $M \in \mathcal{C}^{\leq 0}$ and $N \in \mathcal{C}^{\geq 0}$,
- for each $M \in \mathcal{C}$ there is a triangle $M' \to M \to M'' \to \Sigma M'$ in $\mathcal{C}$ with $M' \in \mathcal{C}^{\leq 0}$ and $M'' \in \Sigma^{-1} \mathcal{C}^{\geq 0}$.

The heart $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ is always abelian. The $t$-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is said to be bounded if

$$\bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{C}^{\leq 0} = \mathcal{C} = \bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{C}^{\geq 0}.$$ 

A typical example of a $t$-structure is the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ for the derived category $\mathcal{D} = D(\text{Mod } A)$ of an (ordinary) algebra $A$, where $\mathcal{D}^{\leq 0}$ consists of complexes with vanishing cohomologies in positive degrees, and $\mathcal{D}^{\geq 0}$ consists of complexes with vanishing cohomologies in negative degrees. This $t$-structure restricts to a bounded $t$-structure of $\mathcal{D}^b(\text{mod } A)$.

Assume $\Lambda$, $X_1, \ldots, X_r$, $T$ as in the preceding subsection. Recall that by Lemma 3.3, $\nu^{-1}T$ is a compact generator of $D(\text{Mod } \Lambda)$. The following proposition is an immediate consequence of Lemma 3.4 and the definition of the $t$-structure.

Proposition 3.7. The following assertions are equivalent:
(i) $\nu^{-1} T$ is a tilting complex,
(ii) $\nu^{-1} T$ is in the heart of some t-structure.

There are two natural t-structures related to the set $X_1, \ldots, X_r$. Let $X \leq 0$ be the smallest full subcategory of $D(\text{Mod} \Lambda)$ containing $X_1, \ldots, X_r$ and stable for taking suspensions, extensions and coproducts. By [3, Proposition 3.2], the pair $(X \leq 0, \Sigma(X \leq 0)^{\perp})$ is a t-structure of $D(\text{Mod} \Lambda)$, where $(X \leq 0)^{\perp}$ is the full subcategory of $D(\text{Mod} \Lambda)$ of objects $M$ such that $\text{Hom}(N, M) = 0$ for any object $N$ of $X \leq 0$. Dually so is $(\Sigma^{-1}(X \geq 0), X \geq 0)$, where $X \geq 0$ is defined as the smallest full subcategory of $D(\text{Mod} \Lambda)$ which contains $X_1, \ldots, X_r$ and which is stable for taking cosuspensions, extensions and products, and $(X \geq 0)$ is the full subcategory of $D(\text{Mod} \Lambda)$ of objects $M$ such that $\text{Hom}(M, N) = 0$ for any object $N$ of $X \geq 0$.

Yet there is a third natural t-structure. Let $\tilde{\Gamma}$ be the dg endomorphism algebra of $\nu^{-1} T$. Precisely, the degree $n$ component of $\tilde{\Gamma}$ consists of those $\Lambda$-linear maps from $\nu^{-1} T$ to itself which are homogeneous of degree $n$, and the differential of $\tilde{\Gamma}$ takes a homogeneous map $f$ of degree $n$ to $d \circ f - (-1)^n f \circ d$, where $d$ is the differential of $\nu^{-1} T$. We have $H^m(\tilde{\Gamma}) = \text{Hom}(\nu^{-1} T, \Sigma^m \nu^{-1} T)$ for any integer $m$. The dg algebra $\tilde{\Gamma}$ is finite-dimensional by Lemma 3.3 b), and it has cohomology concentrated in non-positive degrees by Lemma 3.3.

It follows that the derived category $D(\tilde{\Gamma})$ carries a natural t-structure $(D \leq 0, D \geq 0)$, where $D \leq 0$ is the full subcategory of $D(\tilde{\Gamma})$ consisting of dg $\tilde{\Gamma}$-modules $M$ with $H^m(M) = 0$ for $m > 0$, and $D \geq 0$ is the full subcategory of $D(\tilde{\Gamma})$ consisting of dg $\tilde{\Gamma}$-modules $M$ with $H^m(M) = 0$ for $m < 0$, and the heart $D \leq 0 \cap D \geq 0$ is equivalent to $\text{Mod} \Gamma$, where $\Gamma = H^0(\tilde{\Gamma})$, see the appendix. This t-structure restricts to a t-structure of $D_{fd}(\tilde{\Gamma})$, denoted by $(D_{fd} \leq 0, D_{fd} \geq 0)$, whose heart is equivalent to $\text{mod} \Gamma$.

The complex $\nu^{-1} T$ has a natural dg $\tilde{\Gamma}$-$\Lambda$-bimodule structure. By [7, Lemma 6.1 (a)], we have a triangle equivalence

$$L_{\tilde{\Gamma}} \nu^{-1} T : D(\tilde{\Gamma}) \xrightarrow{\sim} D(\text{Mod} \Lambda).$$

This equivalence takes $\tilde{\Gamma}$ to $\nu^{-1} T$, takes a complete set of non-isomorphic simple $\Gamma$-modules to $X_1, \ldots, X_r$, and restricts to a triangle equivalence between $D_{fd}(\tilde{\Gamma})$ and $D_{fd}(\Lambda) = D^b(\text{mod} \Lambda)$. The image of the t-structure $(D \leq 0, D \geq 0)$ under the triangle equivalence $L_{\tilde{\Gamma}}$ of $\nu^{-1} T$ is a t-structure of $D(\text{Mod} \Lambda)$, which we still denote by $(D \leq 0, D \geq 0)$. The image of the t-structure $(D_{fd} \leq 0, D_{fd} \geq 0)$ is exactly the t-structure $(C \leq 0, C \geq 0)$ in [2].

**Proposition 3.8.** The above three t-structures $(X \leq 0, \Sigma(X \leq 0)^{\perp})$, $(\Sigma^{-1}(X \geq 0), X \geq 0)$ and $(D \leq 0, D \geq 0)$ coincide.

**Proof.** If suffices to prove that $X \leq 0 = D \leq 0$ and $X \geq 0 = D \geq 0$. We only prove the first statement, and the second statement is dual. Let $Y \leq 0$ be the image of $X \leq 0$ under a quasi-inverse of $L_{\tilde{\Gamma}} \nu^{-1} T$, i.e. $Y \leq 0$ is the smallest full subcategory of $D(\tilde{\Gamma})$ containing the simple
Γ-modules and stable for taking suspensions, extensions and coproducts. We shall prove the equivalent statement $Y_{\leq 0} = D_{\leq 0}$ in $D(\tilde{\Gamma})$.

Let $M$ be a dg $\tilde{\Gamma}$-module whose cohomology is concentrated in non-positive degrees. Then the graded module $H^\ast(M)$ over the non-positively graded algebra $H^\ast(\tilde{\Gamma})$ admits an $\text{Add}(\Sigma^m H^\ast(\tilde{\Gamma})|m \geq 0)$-resolution. It follows from [7, Theorem 3.1 (c)] that $M$ belongs to the smallest full subcategory of $D(\tilde{\Gamma})$ containing $\tilde{\Gamma}$ and stable for taking suspensions, extensions and coproducts. Therefore, this latter category coincides with $D_{\leq 0}$. But it is clear that $Y_{\leq 0}$ contains $\tilde{\Gamma}$, and hence $Y_{\leq 0}$ contains $D_{\leq 0}$. The inclusion in the other direction is obvious.

An abelian category is a length category if every object in it has finite length. Two sets of simple-minded objects are equivalent if they have the same closure under extensions. The following is a counterpart of [9, Corollary 11.5].

**Corollary 3.9.** There is a bijection between the set of bounded $t$-structures of $D^b(\text{mod } \Lambda)$ whose heart is a length category with finite many simple objects (up to isomorphism) and the set of equivalence classes of families of simple-minded objects of $D^b(\text{mod } \Lambda)$. In particular, the heart of a bounded $t$-structure of $D^b(\text{mod } \Lambda)$ is a length category if and only if it is equivalent to $\text{mod } \Gamma$ for some finite-dimensional algebra $\Gamma$.

We remind the reader that the heart of a $t$-structure of $D^b(\text{mod } \Lambda)$ is not always a length category. For example, the derived category of the path algebra of the Kronecker quiver has a $t$-structure whose heart is the category of coherent sheaves over the projective line, which is not a length category.

### 3.10. The case of self-injective algebras.

Al-Noayee in [1] extended Rickard’s result Theorem 3.5 to the case when $\Lambda$ is a self-injective algebra; then $T = \nu^{-1}T$ is a tilting complex. This result can now be derived again.

Let $\Lambda$ be a finite-dimensional self-injective algebra. In this case, the two categories $\text{per}(\Lambda)$ and $\text{thick}(D(\Lambda \Lambda))$ coincide. The Nakayama functor $\nu$ and its quasi-inverse $\nu^{-1}$ can be extended to auto-equivalences of $D^-(\text{mod } \Lambda)$ because each object in $D^-(\text{mod } \Lambda)$ admits a projective resolution which is bounded above and whose components are finite generated.

Let $X_1, \ldots, X_r$ be a set of simple-minded objects in $D^b(\text{mod } \Lambda)$ stable under the Nakayama functor $\nu$. Let $T$ be constructed as in Section 3.

**Proposition 3.11 ([1] Theorem 4).** The complex $T$ is a tilting complex.

**Proof.** The Nakayama functor $\nu$ induces a permutation on the set $\{1, \ldots, r\}$, also denoted by $\nu$, given by $X_{\nu(i)} = \nu(X_i)$ for $i = 1, \ldots, r$. Applying $\nu$ to the formula Lemma 3.3 a), we
obtain for $1 \leq i, j \leq r$, and $m \in \mathbb{Z}$,

$$\text{Hom}(T_i, \Sigma^m X_{\nu(j)}) = \begin{cases} K & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applying a quasi-inverse $G$ of the triangle equivalence $L : \mathcal{D}(\overline{\Gamma}) \to \mathcal{D}(\Lambda)$, we obtain

$$\text{Hom}(GT_i, \Sigma^m GX_{\nu(j)}) = \begin{cases} K & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $GX_{\nu(1)}, \ldots, GX_{\nu(r)}$ is a complete set of simple $\overline{\Gamma}$-modules, it follows from Section 5.5 (the appendix) that $GT_1, \ldots, GT_r$ sum up to the free module $\overline{\Gamma}$. Recall that $G(\nu^{-1}T) \cong \overline{\Gamma}$. As a consequence, we have $T \cong \nu^{-1}T$. Now the desired result follows from Lemma 3.4.

4. AN ALTERNATIVE CONSTRUCTION

In this section we will give another construction of $T$ using the $A_\infty$-version of Morita’s theorem for triangulated categories (cf. [12]). Let us first recall the definition and basic properties of $A_\infty$-algebras and $A_\infty$-modules.

4.1. $A_\infty$-algebras and $A_\infty$-modules. We follow [12], [8] and [13] are also nice references.

An $A_\infty$-algebra is a graded vector space $A$ endowed with a family of homogeneous maps

$$m_n : A^\otimes n \to A, n \geq 1$$

of degree $2 - n$ satisfying the equations

$$\sum_{j+k+l=n} (-1)^{j+l} m_{j+1+l}(id^\otimes j \otimes m_k \otimes id^\otimes l) = 0, n \geq 1.$$

These $m_n$ are called the multiplications of $A$. The $A_\infty$-algebra $A$ is minimal if $m_1 = 0$. We say that $A$ is strictly unital if $A$ has a strict unit, i.e. a homogeneous element $1_A$ of degree 0 such that for $n \neq 2$ the multiplication $m_n$ has value zero if one of its $n$ arguments equals $1_A$, and

$$m_2(1_A \otimes a) = a = m_2(a \otimes 1_A)$$

for all $a$ in $A$.

Let $A$ be a strictly unital $A_\infty$-algebra. A (right) $A_\infty$-module over $A$ is a graded vector space $M$ endowed with a family of homogeneous maps

$$m_n^M : M \otimes A^\otimes n-1 \to M, n \geq 1$$

of degree $2 - n$ such that

$$\sum_{j+k+l=n} (-1)^{j+l} m_{j+1+l}(id^\otimes j \otimes m_k \otimes id^\otimes l) = 0.$$
An $A_\infty$-module $M$ minimal if $m_1^M = 0$, and is strictly unital if one of $a_2, \ldots, a_n$ equals $1_A$ implies

$$m_n^M (m \otimes a_2 \otimes \cdots \otimes a_n) = 0$$

for all $n \geq 3$, and $m_2^M (m \otimes 1_A) = m$ for all $m$ in $M$.

Let $M$ and $M'$ be two strictly unital $A_\infty$-modules over $A$. An $A_\infty$-morphism $f : M \to M'$ is a family of homogeneous maps

$$f_n : M \otimes A^{\otimes n} \to M', n \geq 1$$

of degree $1 - n$ satisfying the following identity for all $n \geq 1$

$$\sum (-1)^{jk+l} f_{j+1+l} (id^{\otimes j} \otimes m_k \otimes id^{\otimes l}) = \sum m_{s+1} (f_r \otimes id^{\otimes s}),$$

where $j + k + l = n$ and $r + s = n$. In particular, $f_1$ is a chain map of complexes. The $A_\infty$-morphism $f$ is a quasi-isomorphism if $f_1$ induces identities on all cohomologies. $f$ is strictly unital if one of $a_2, \ldots, a_n$ equals $1_A$ implies

$$f_n (m \otimes a_2 \otimes \cdots \otimes a_n) = 0$$

for all $n \geq 2$. $f$ is strict if $f_n = 0$ for all $n \geq 2$.

Let $\text{Mod}_\infty(A)$ be the category of strictly unital $A_\infty$-modules over $A$ with strictly unital $A_\infty$-morphisms as morphisms. The derived category $D(A)$ is the category obtained from $\text{Mod}_\infty(A)$ by formally inverting all quasi-isomorphisms. The category $D(A)$ is a triangulated category whose suspension functor $\Sigma$ the shift functor. For a strictly unital $A_\infty$-module $M$ over $A$ and an integer $i$, we have

$$\text{Hom}_{D(A)} (A, \Sigma^i M) = H^i M.$$
taking $X$ to $A$. It follows that the underlying graded algebra of $A$ is the graded endomorphism algebra $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_C(X, X^i \cdot X)$. Moreover, $m_n(a_1 \otimes \cdots \otimes a_n) = 0$ for all $n \neq 2$ if one of $a_j$ is the identity morphism of a direct summand of $X$.

4.5. Minimal positive $A_\infty$-algebras. Let $A$ be a strictly unital minimal positive $A_\infty$-algebra, i.e. $A$ is strictly unital and minimal and satisfies

- $A^i = 0$ for all negative integers $i$,
- $A^0$ is the product of $r$ copies of the base field $K$ for some positive integer $r$,
- $m_n(a_1 \otimes \cdots \otimes a_n) = 0$ if one of $a_1, \ldots, a_n$ belongs to $A^0$.

Put $A^{>0} = \bigoplus_{i > 0} A^i$. Then $A^{>0}$ is an $A_\infty$-ideal of $A$: the multiplication $m_n$ takes value in $A^{>0}$ if one of the $n$ arguments belongs to $A^{>0}$. It has vanishing higher multiplications, and is isomorphic to the product of $r$ copies of $K$.

Let $1 = e_1 + \ldots + e_r$ be the unique (up to reordering) decomposition of the identity of $A^0$ into the sum of primitive orthogonal idempotents. Then each $P_i = e_i A$ is an $A_\infty$-submodule of the free module of rank 1:

$$m_n(e_i a \otimes a_2 \otimes \cdots \otimes a_n) = -(-1)^n e_i m_n(a \otimes a_2 \otimes \cdots \otimes a_n).$$

The subspace $e_i A^{>0}$ is an $A_\infty$-submodule of $P_i$, and the quotient $A_\infty$-module $S_i = P_i / e_i A^{>0}$ is 1-dimensional with basis the class of $e_i$. We call $S_1, \ldots, S_r$ simple modules over $A$. Viewed as an $A_\infty$-module over $A$, $\bar{A}$ is isomorphic to the direct sum of $S_1, \ldots, S_r$. The following two lemmas are also proved in [9] (in the form of dg algebras and dg modules).

**Lemma 4.6.** Let $A$ be a strictly unital minimal positive $A_\infty$-algebra. Then

$$\text{Hom}_{D(A)}(\bar{A}, \Sigma^m \bar{A}) = 0$$

for positive integers $m$.

**Proof.** The graded module $\bar{A}$ over the positively graded algebra $A$ admits an $\text{Add}(\Sigma^{-m} A|m \geq 0)$-resolution. Now the desired result follows from an $A_\infty$-version of [7, Theorem 3.1 (c)] (one can obtain this, for example, by going to the enveloping dg algebra).

**Lemma 4.7.** Let $A$ be a strictly unital minimal positive $A_\infty$-algebra. Then the triangulated category $D_{fd}(A)$ is generated by $\bar{A}$.

**Proof.** By Proposition 4.3 it suffices to prove that a finite-dimensional strictly unital minimal $A_\infty$-module $M$ over $A$ is generated by $\bar{A}$. Up to shift we may assume that $M^i = 0$ for all negative integers $i$ and $M^0 \neq 0$. Put $M^{>0} = \bigoplus_{i > 0} M^i$. Then $M^{>0}$ is an $A_\infty$-submodule of $M$, and we have a triangle in $D(A)$

$$M^{>0} \longrightarrow M \longrightarrow \bar{M} \longrightarrow \Sigma M^{>0}.$$
Here $\tilde{M}$ is the $A_\infty$-quotient module $M/M^{>0}$, and is concentrated in degree 0. Its structure of an $A_\infty$-module comes from its structure of an $\tilde{A}$-module, and hence is generated by $\tilde{A}$. Now by induction on the dimension of $M$ we finish the proof.

4.8. The alternative construction. Let $\Lambda$ be a finite-dimensional basic $K$-algebra. Let $S_1, \ldots, S_r$ be a complete set of representatives of simple $\Lambda$-modules.

By Theorem 4.4 there is a strictly unital minimal positive $A_\infty$-algebra

$$\mathcal{S} = \bigoplus_{m \in \mathbb{Z}} \text{Hom}(\bigoplus_i S_i, \Sigma^m \bigoplus_i S_i)$$

(the $A_\infty$-Koszul dual of $\Lambda$) and a triangle equivalence

$$\Phi : \mathcal{D}^b(\text{mod } \Lambda) \longrightarrow \text{per}(\mathcal{S})$$

taking $S_j$ ($j = 1, \ldots, r$) to $P_j = \bigoplus_{m \in \mathbb{Z}} \text{Hom}(\bigoplus_i S_i, \Sigma^m S_j)$.

The indecomposable injective $\Lambda$-modules $I_1, \ldots, I_r$ are characterized by the property

$$\text{Hom}(S_i, \Sigma^m I_j) = \begin{cases} K & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So their images $\Phi(I_1), \ldots, \Phi(I_r)$ under the equivalence $\Phi$ are characterized by the property

$$\text{Hom}(P_i, \Sigma^m \Phi(I_j)) = \begin{cases} K & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\Phi(I_1), \ldots, \Phi(I_r)$ are precisely the indecomposable direct summands of $\tilde{S}$. In other words, the equivalence $\Phi$ restricts to a triangle equivalence

$$\Phi| : \text{thick}(\mathcal{D}(\Lambda \Lambda)) = \text{thick}(I_1, \ldots, I_r) \longrightarrow \text{thick}(\tilde{S}) = \mathcal{D}_{fd}(S),$$

where the last equality follows from Lemma 4.6.

Let $X_1, \ldots, X_r \in \mathcal{D}^b(\text{mod } \Lambda)$ be a set of simple-minded objects, i.e. they satisfy the following conditions

(1) $\text{Hom}(X_i, \Sigma^m X_j) = 0$, $\forall m < 0$,

(2) $\text{Hom}(X_i, X_j) = \begin{cases} K & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$

(3) $X_1, \ldots, X_r$ generates $\mathcal{D}^b(\text{mod } \Lambda)$.

On the graded algebra $\bigoplus_m \text{Hom}(\bigoplus_i X_i, \Sigma^m \bigoplus_i X_i)$ there is a strictly unital minimal $A_\infty$-algebra structure. We will denote this $A_\infty$-algebra by $\mathcal{X}$. The conditions (1) and (2) imply that $\mathcal{X}$ is positive, while it follows from condition (3) that there is a triangle equivalence

$$\mathcal{D}(S) \longrightarrow \mathcal{D}(\mathcal{X}).$$
This equivalence restricts to triangle equivalences
\[ \Psi : \text{per}(S) \to \text{per}(X), \]
\[ \Psi : \text{D}_{fd}(S) \to \text{D}_{fd}(X). \]
Thus we have the following commutative triangles of triangle equivalences
\[ D_b(\text{mod } \Lambda) \xrightarrow{\Phi} \text{thick}(I_1, \ldots, I_r) \]
\[ \xrightarrow{\Psi \circ \Phi} \text{D}_{fd}(S) = \text{thick}(S) \]
\[ \xrightarrow{\Psi} \text{per}(X) \]
\[ \xrightarrow{(\Psi \circ \Phi)} \text{D}_{fd}(X) = \text{thick}(\bar{X}). \]

Associated with \( X_1, \ldots, X_r \) there is the decomposition \( 1 = id_{X_1} + \ldots + id_{X_r} \) of the identity of \( \mathcal{A}^0 \) into the sum of primitive orthogonal idempotents. Let \( Y_1, \ldots, Y_r \) be corresponding simple modules over \( X \), and let \( T_1, \ldots, T_r \) be their images under a quasi-inverse of the equivalence \((\Psi \circ \Phi)\). Put \( T = \bigoplus_i T_i \).

**Lemma 4.9** (Lemma 3.2 and Lemma 3.4).

a) \( T \) generates \( \text{thick}(I_1, \ldots, I_r) \). 

b) For \( 1 \leq i, j \leq r \), and \( m \in \mathbb{Z} \),
\[ \text{Hom}(X_j, \Sigma^m T_i) = \begin{cases} 
K & \text{if } i = j \text{ and } m = 0, \\
0 & \text{otherwise}.
\end{cases} \]
c) \( T \) is isomorphic to a bounded complex of finitely generated injectives.
d) Let \( C \) be an object of \( \mathcal{D}^-(\text{mod } \Lambda) \). If \( \text{Hom}(C, \Sigma^m T) = 0 \) for all \( m \in \mathbb{Z} \), then \( C = 0 \).
e) \( \text{Hom}(T, \Sigma^m T) = 0 \) for \( m > 0 \).

**Proof.** a) b) c) hold because they hold after applying the triangle equivalence \( \Psi \circ \Phi \). c) is trivial. d) follows from a).

**Remark 4.10.** From the appendix we see that, from the viewpoint of derived categories, finite-dimensional dg algebras (whose cohomology is) concentrated in non-positive degrees behave like ordinary finite-dimensional algebras. The construction of \( T \) and Lemma 4.9 can be easily generalized to this more general setting, namely, the setting that \( \Lambda \) is a finite-dimensional dg algebra (whose cohomology is) concentrated in non-positive degrees. In the statement of d) one replaces \( \mathcal{D}^-(\text{mod } \Lambda) \) by the full subcategory of \( \mathcal{D}(\Lambda) \) of dg \( \Lambda \)-modules \( M \) such that \( H^m(M) \) vanishes for sufficiently large \( m \) and each \( H^m(M) \) \((m \in \mathbb{Z})\) is finite-dimensional.

**Remark 4.11.** The \( A_{\infty} \)-algebra \( \mathcal{X} \) can be computed as a minimal model of the dg endomorphism algebra of the direct sum of projective resolutions of \( X_1, \ldots, X_r \). In fact, it is Koszul dual to the dg algebra \( \tilde{\Gamma} \) introduced in Section 3.6. Thus knowing that \( \tilde{\Gamma} \) is finite-dimensional a priori one can construct it from \( \mathcal{X} \) using the dual bar construction, and vice versa. In particular, if the restriction of the \( A_{\infty} \)-structure of \( \mathcal{X} \) in degrees 0, 1 and 2 is
known, it is not hard to work out the precise structure of $\Gamma = H^0 \tilde{\Gamma}$. However, this does not help us to understand when $\tilde{\Gamma}$ has cohomology concentrated in degree 0.

5. Appendix: Finite-dimensional non-positive dg algebras

Let $K$ be a field. Let $A$ be a finite-dimensional non-positive dg $K$-algebra (associative with 1), i.e. $A = \bigoplus_{i \leq 0} A^i$ with each $A^i$ finite-dimensional $K$-space and $A^i = 0$ for $i \ll 0$.

Let $\mathcal{C}(A)$ denote the category of (right) dg modules over $A$, $\mathcal{D}(A)$ denote the derived category, $\mathcal{D}^{fd}(A)$ denote the finite-dimensional derived category, and $\text{per}(A)$ denote the perfect derived category.

The 0-th cohomology $\bar{A} = H^0(A)$ of $A$ is an ordinary $K$-algebra. Let $\text{Mod} \bar{A}$ and $\text{mod} \bar{A}$ denote the category of (right) modules over $\bar{A}$ and its subcategory consisting of those finite-dimensional modules. Let $\pi : A \rightarrow \bar{A}$ be the canonical projection. We view $\text{Mod} \bar{A}$ as a subcategory of $\mathcal{C}(A)$ via $\pi$.

The total cohomology $H^*(A)$ of $A$ is a finite-dimensional graded algebra with multiplication induced from the multiplication of $A$. Let $M$ be a dg $A$-module. Then the total cohomology $H^*(M)$ carries a graded $H^*(A)$-module structure, and hence a graded $\bar{A}$-module structure. In particular, a stalk dg $A$-module concentrated in degree 0 is an $\bar{A}$-module.

5.1. The standard $t$-structure. We follow [4] and [10], where the dg algebra is not necessarily finite-dimensional.

Let $M = \ldots \rightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \rightarrow \ldots$ be a dg $A$-module. We define the truncation functors $\tau_{\leq 0}$ and $\tau_{\geq 1}$ as follows:

$$
\tau_{\leq 0} M = \ldots \rightarrow M^{-2} \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} \ker d^0 \rightarrow 0 \rightarrow \ldots
$$

$$
\tau_{\geq 1} M = \ldots \rightarrow 0 \rightarrow M^1/\text{im} d^0 \xrightarrow{d^1} M^2 \xrightarrow{d^2} M^3 \rightarrow \ldots
$$

Thanks to the assumption that $A$ is non-positive, $\tau_{\leq 0} M$ and $\tau_{\geq 1} M$ are again dg $A$-modules. Moreover we have a distinguished triangle in $\mathcal{D}(A)$

$$
\tau_{\leq 0} M \rightarrow M \rightarrow \tau_{\geq 1} M \rightarrow \Sigma \tau_{\leq 0} M.
$$

These two functors define a $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}(A)$, where $\mathcal{D}^{\leq 0}$ is the subcategory of $\mathcal{D}(A)$ consisting of dg $A$-modules with vanishing cohomology in positive degrees, and $\mathcal{D}^{\leq 0}$ is the subcategory of $\mathcal{D}(A)$ consisting of dg $A$-modules with vanishing cohomology in negative degrees.

Immediately from the definition of the $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, we see that the heart $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ consists of those dg $A$-modules whose cohomology are concentrated in
degree 0. Thus the functor $H^0$ induces an equivalence

$$H^0: \mathcal{H} \rightarrow \text{Mod} \, \bar{A}.$$  

$$M \mapsto H^0(M)$$

The $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $\mathcal{D}(A)$ restricts to a $t$-structure of $\mathcal{D}_{fd}(A)$ with heart equivalent to $\text{mod} \, \bar{A}$. It is easy to see that $\mathcal{D}_{fd}(A)$ is generated by this heart, and hence generated by the simple $\bar{A}$-modules.

5.2. **Morita reduction.** Let $d$ be the differential of $A$. Then $d(A^0) = 0$.

Let $e$ be an idempotent of $A$. For degree reasons, $e$ must belong to $A^0$, and the graded subspace $eA$ of $A$ is a dg submodule: $d(ea) = d(e)a + ed(a) = ed(a)$. Therefore for each decomposition $1 = e_1 + \ldots + e_n$ of unity into the sum of primitive orthogonal idempotents, we have a direct sum decomposition $A = e_1 A \oplus \ldots \oplus e_n A$ of $A$ into indecomposable dg $A$-modules. Moreover, if $e$ and $e'$ are two idempotents of $A$ such that $eA \cong e'A$ as ordinary modules over the ordinary algebra $A$, then this isomorphism is also an isomorphism of dg modules. Indeed, there are two elements of $A$ such that $fg = e$ and $gf = e'$. Again for degree reasons, $f$ and $g$ belong to $A^0$. So they induce isomorphisms of dg $A$-modules: $eA \rightarrow e'A$, $a \mapsto ga$ and $e'A \rightarrow eA$, $a \mapsto fa$. It follows that the above decomposition of $A$ into the direct sum of indecomposable dg modules is essentially unique. Namely, if $1 = e'_1 + \ldots + e'_n$ is another decomposition of the unity into the sum of primitive orthogonal idempotents, then $m = n$ and up to reordering, $e_1 A \cong e'_1 A$, $\ldots$, $e_n A \cong e'_n A$.

Let $A$ and $A'$ be two finite-dimensional non-positive dg algebras. If $A$ and $A'$ are Morita equivalent as ordinary algebras, then $\mathcal{C}(A)$ and $\mathcal{C}(A')$ are equivalent.

5.3. **The perfect derived category.** Since $A$ is finite-dimensional (thus has finite-dimensional total cohomology), it follows that $\text{per}(A)$ is a triangulated subcategory of $\mathcal{D}_{fd}(A)$.

We assume, as we may, that $A$ is basic. Let $1 = e_1 + \ldots + e_n$ be a decomposition of 1 in $A$ into the sum of primitive orthogonal idempotents. Since $d(x) = \lambda_1 e_1 + \ldots + \lambda_n e_n$ implies that $d(e_i x) = \lambda_j e_{i_j}$, it follows that the intersection of the space with basis $e_1, \ldots, e_n$ with the image of the differential $d$ has a basis consisting of some $e_i$’s, say $e_{r+1}, \ldots, e_n$. It is easy to see that $e_{r+1}A, \ldots, e_n A$ are homotopic to zero.

We say that a dg $A$-module $M$ is strictly perfect if its underlying graded module is of the form $\bigoplus_{j=1}^N R_j$, where each $R_j$ is isomorphic to a shifted copy of some $e_i A$ ($1 \leq i \leq n$), and if its differential is of the form $d_{int} + \delta$, where $d_{int}$ is the direct sum of the differential of the $R_j$’s, and $\delta$, as a degree 1 map from $\bigoplus_{j=1}^N R_j$ to itself, is a strictly upper triangular matrix whose entries are in $A$. It is minimal perfect if in addition no $R_j$ is isomorphic to any shifted copy of $e_{r+1}A, \ldots, e_n A$, and the entries of $\delta$ are in the radical of $A$, cf. [14].

**Lemma 5.4.** Let $M$ be a dg $A$-module belonging to $\text{per}(A)$. Then $M$ is quasi-isomorphic to a minimal perfect dg $A$-module.
Proof. Bearing in mind that \(e_1 A, \ldots, e_r A\) have local endomorphism algebras and \(e_{r+1} A, \ldots, e_n A\) are homotopic to zero, we prove the assertion as in [14].

5.5. Simple modules. We assume that \(A\) is basic and that \(K\) is algebraically closed.

According to the preceding subsection, we may assume that there is a decomposition 
\(1 = e_1 + \ldots + e_r + e_{r+1} + \ldots + e_n\) of the unity of \(A\) into a sum of primitive orthogonal idempotents such that 
\(1 = \bar{e}_1 + \ldots + \bar{e}_r\) is a decomposition of \(1\) in \(\bar{A}\) into a sum of primitive orthogonal idempotents.

Let \(S_1, \ldots, S_r\) be a complete set of representatives of isomorphism classes of simple \(\bar{A}\)-modules. Then
\[
\text{Hom}_A(e_i A, S_j) = \begin{cases} 
K & \text{if } i = j, \\
0 & \text{otherwise}.
\end{cases}
\]

Here for two dg \(A\)-modules \(M\) and \(N\), \(\text{Hom}_A(M, N)\) denotes the complex whose degree \(p\) component of consists of those \(A\)-linear maps from \(M\) to \(N\) which are homogeneous of degree \(p\), and whose differential takes a homogeneous map \(f\) of degree \(p\) to \(d_N \circ f - (-1)^p f \circ d_M\). Therefore we have
\[
\text{Hom}_{D(A)}(e_i A, \Sigma^m S_j) = \begin{cases} 
K & \text{if } i = j \text{ and } m = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Moreover, \(\{e_1 A, \ldots, e_r A\}\) and \(\{S_1, \ldots, S_r\}\) characterize each other by this property. On the one hand, if \(M\) is a dg \(A\)-module such that for some integer \(1 \leq j \leq r\)
\[
\text{Hom}_{D(A)}(e_i A, \Sigma^m M) = \begin{cases} 
K & \text{if } i = j \text{ and } m = 0, \\
0 & \text{otherwise},
\end{cases}
\]
then \(M\) is isomorphic in \(D(A)\) to \(S_j\). On the other hand, let \(M\) be an object of \(\text{per}(A)\) such that for some integer \(1 \leq i \leq r\)
\[
\text{Hom}_{D(A)}(M, \Sigma^m S_j) = \begin{cases} 
K & \text{if } i = j \text{ and } m = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Then by replacing \(M\) by its minimal perfect resolution (Lemma 5.4), we see that \(M\) is isomorphic in \(D(A)\) to \(e_i A\).

Further, recall from Section 5.4 that \(D_{fd}(A)\) admits a standard \(t\)-structure whose heart is equivalent to \(\text{mod} \bar{A}\). This implies that the simple modules \(S_1, \ldots, S_r\) form a set of simple-minded objects in \(D_{fd}(A)\).

5.6. The Nakayama functor. For a complex \(M\) of \(K\)-vector spaces, we define its dual as \(D(M) = \text{Hom}_K(M, K)\), where the last \(K\) is considered as a complex concentrated in degree 0. One checks that \(D\) defines a duality between finite-dimensional dg \(A\)-modules and finite-dimensional \(A^{op}\)-modules.
Let $e$ be an idempotent of $A$ and $M$ a dg $A$-module. Then we have a canonical isomorphism
\[ \mathcal{H}om_A(eA, M) \cong Me. \]
If in addition each component of $M$ is finite-dimensional, we have canonical isomorphisms
\[ \mathcal{H}om_A(eA, M) \cong Me \cong D\mathcal{H}om_A(M, D(Ae)). \]

We define the Nakayama functor $\nu : \mathcal{C}(A) \to \mathcal{C}(A)$ by $\nu(M) = D\mathcal{H}om_A(M, A)$ [7, Section 10]. We have canonical isomorphisms
\[ D\mathcal{H}om_A(M, N) \cong \mathcal{H}om_A(N, \nu M) \]
for strictly perfect dg $A$-module $M$ and any dg $A$-module $N$. We have $\nu(eA) = D(Ae)$ for an idempotent $e$ of $A$, and the functor $\nu$ induces a triangle equivalences between the subcategories $\text{per}(A)$ and $\text{thick}(\mathcal{D}(A))$ of $\mathcal{C}(A)$ with quasi-inverse given by $\nu^{-1}(M) = \mathcal{H}om_A(D(A), M)$. Let $e_1, \ldots, e_r$ and $S_1, \ldots, S_r$ be as in the preceding subsection. Then we have
\[ \mathcal{H}om_A(S_j, D(Ae_i)) \cong D\mathcal{H}om_A(e_iA, S_j) = \begin{cases} K & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \]
That is,
\[ \mathcal{H}om_{\mathcal{D}(A)}(S_j, \Sigma^m D(Ae_i)) = \begin{cases} K & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases} \]
Moreover, $\{D(Ae_1), \ldots, D(Ae_r)\}$ and $\{S_1, \ldots, S_r\}$ characterize each other in $\mathcal{D}(A)$ by this property. This follows from the arguments in the preceding subsection by applying the functors $\nu$ and $\nu^{-1}$.

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