A CLT in Stein’s distance for generalized Wishart matrices and higher order tensors

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Abstract

We study the convergence along the central limit theorem for sums of independent tensor powers, \( \frac{1}{\sqrt{d}} \sum_{i=1}^{d} X_i^{\otimes p} \). We focus on the high-dimensional regime where \( X_i \in \mathbb{R}^n \) and \( n \) may scale with \( d \). Our main result is a proposed threshold for convergence. Specifically, we show that, under some regularity assumption, if \( n^{2p-1} \gg d \), then the normalized sum converges to a Gaussian. The results apply, among others, to symmetric uniform log-concave measures and to product measures. This generalizes several results found in the literature. Our main technique is a novel application of optimal transport to Stein’s method which accounts for the low dimensional structure which is inherent in \( X_i^{\otimes p} \).

1 Introduction

Let \( \mu \) be an isotropic\(^1\) probability measure on \( \mathbb{R}^n \). For \( 2 \leq p \in \mathbb{N} \), we consider the following tensor analogue of the Wishart matrix,

\[
\frac{1}{\sqrt{d}} \sum_{i=1}^{d} (X_i^{\otimes p} - \mathbb{E}[X_i^{\otimes p}]),
\]

where \( X_i \sim \mu \) are i.i.d. and \( X_i^{\otimes p} \) stands for the symmetric \( p \)'th tensor power of \( X_i \). We denote the law of this random tensor by \( W_{n,d}^p(\mu) \). Such distributions arise naturally as the sample moment tensor of the measure \( \mu \), in which case \( d \) serves as the sample size. For reasons soon to become apparent, we will sometimes refer to such tensors as Wishart tensors.

When \( p = 2 \), \( W_{n,d}^2(\mu) \) is the sample covariance of \( \mu \). If \( X \) is an \( n \times d \) matrix with columns independently distributed as \( \mu \), then \( W_{n,d}^2(\mu) \) may also be realized as the upper triangular part of the matrix

\[
\frac{XX^T - d\text{Id}}{\sqrt{d}}.
\]

Hence, \( W_{n,d}^2(\mu) \) has the law of a Wishart matrix. These matrices have recently been studied in the context of random geometric graphs (\([6–8,22]\)) and one can envision \( W_{n,d}^p(\mu) \) as expressing

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\(^1\)That is, \( \mu \) is centered and its covariance matrix is the identity.
higher order interactions.

For fixed $p, n$, according to the central limit theorem (CLT), as $d \to \infty$, $W_{p,d}^n(\mu)$ approaches a normal law. The aim of this paper is to study the high-dimensional regime of the problem, where we allow the dimension $n$ to scale with the sample size $d$. Specifically, we investigate possible conditions on $n$ and $d$ for the CLT to hold. Observe that this problem may be reformulated as a question about the rate of convergence in the high-dimensional CLT, for the special case of Wishart tensors.

Our starting point is the paper [8], which obtained an optimal bound when $p = 2$, for log-concave product measures. Remark that when $\mu$ is a product measure, then the entries of the matrix $X$ in (1) are all independent. The proof was information-theoretic and made use of the chain rule for relative entropy to account for the low-dimensional structure of $W_{n,d}^2(\mu)$. For now, we denote $\widetilde{W}_{n,d}^2(\mu)$ to be the same law as $W_{n,d}^2(\mu)$, but with the diagonal elements removed (see below for a precise definition).

Theorem 1. Let $\mu$ be a log-concave product measure on $\mathbb{R}^n$ and let $\gamma$ denote the standard Gaussian in $\mathbb{R}^{n^2}$. Then,

1. If $n^3 \gg d$ then $\widetilde{W}_{n,d}^2(\mu) \overset{n \to \infty}{\longrightarrow} \gamma$.

2. If $n^3 \ll d$, then $W_{n,d}^2(\mu)$ remains bounded away from any Gaussian law.

Thus, for log-concave product measures there is a sharp condition for the CLT to hold. Our results, which we now summarize, generalize Point 1 of Theorem 1 in several directions and are aimed to answer questions which were raised in [8].

- We show that it is not necessary for $\mu$ to have a product structure. So, in particular, the matrix $X$ in (1) may admit some dependence between its entries.

- If $\mu$ is a product measure, we relax the log-concavity assumption and show the same result holds for a much larger class of product measures.

- The above results extend to the case $p > 2$, and we propose the new threshold $n^{2p-1} \gg d$.

- We show that Theorem 1 is still true when we take the full symmetric tensor $W_{n,d}^2(\mu)$ and include the diagonal.

Our method is based on a novel application of Stein’s method. Stein’s theory is a prominent set of techniques which was developed in order to answer questions related to convergence rates along the CLT. The method was first introduced in [40, 41] as a way to estimate distances to the normal law. Since then, it had found numerous applications in studying the quantitative central limit theorem, also in high-dimensions (see [38] for an overview).

Na"ively, since $W_{n,d}^p(\mu)$ can be realized as a sum of i.i.d. random vectors, one should be able to employ standard techniques of Stein’s method (such as exchangeable pairs [14], as proposed in [8]) in order to deduce some bounds. However, it turns out that the obtained bounds are sub-optimal. The reason for this sub-optimality is that, while $X \otimes p$ is a random vector in a high-dimensional space, its randomness comes from the lower-dimensional $\mathbb{R}^n$. So, at least on the intuitive level, one must exploit the low-dimensional structure of the random tensor in order to produce better bounds. Our method is particularly adapted to this situation and may be of use in other, similar, settings.
1.1 Definitions

Before stating our results we require some definitions. Let \( \gamma_n \) stand for the normal law on \( \mathbb{R}^n \) with density
\[
d\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-\|x\|^2/2}.
\]
We will sometime omit the subscript, when the dimension is obvious from the context. Fix now a measure \( \mu \) in \( \mathbb{R}^n \). For any \( m > 0 \), the \( m \)-Wasserstein’s distance between \( \mu \) and \( \gamma \) is defined by
\[
W_m(\mu, \gamma) = \min_{\pi} \frac{m}{\sqrt{\int \|x - y\|^2 d\pi(x, y)}},
\]
where the infimum is taken over all couplings \( \pi \), of \( \mu \) and \( \gamma \). If \( \varphi: \mathbb{R}^n \to \mathbb{R}^N \) for some \( N \geq 0 \), we denote \( \varphi_* \mu \), to be the push-forward of \( \mu \) by \( \varphi \).

Our next set of definitions deals with tensor spaces. Fix \( \{e_j\}_{j=1}^n \) to be the standard basis in \( \mathbb{R}^n \).

We identify the tensor space \( (\mathbb{R}^n)^\otimes p \) with \( \mathbb{R}^{n^p} \) where the base is given by
\[
\{e_{j_1}e_{j_2}...e_{j_p}|1 \leq j_1, j_2, ..., j_p \leq n\}.
\]
Under this identification, we may consider the symmetric tensor space \( \operatorname{Sym}^p(\mathbb{R}^n) \subset (\mathbb{R}^n)^\otimes p \) with basis
\[
\{e_{j_1}e_{j_2}...e_{j_p}|1 \leq j_1 \leq j_2 \leq ... \leq j_p \leq n\}.
\]
We will also be interested in the subspace of principal tensors, \( \widetilde{\operatorname{Sym}}^p(\mathbb{R}^n) \subset \operatorname{Sym}^p(\mathbb{R}^n) \), spanned by the basis elements
\[
\{e_{j_1}e_{j_2}...e_{j_p}|1 \leq j_1 < j_2 ... < j_p \leq n\}.
\]
Our main result will deal with the marginal of \( W_{n,d}^p(\mu) \) on the subspace \( \widetilde{\operatorname{Sym}}^p(\mathbb{R}^n) \). We denote this marginal law by \( \tilde{W}_{n,d}^p(\mu) \). Put differently, if \( X_i = (X_{i,1}, ..., X_{i,n}) \) are i.i.d. random vectors with law \( \mu \). Then \( \tilde{W}_{n,d}^p(\mu) \) is the law of a random vector in \( \widetilde{\operatorname{Sym}}^p(\mathbb{R}^n) \) with entries
\[
\left( \frac{1}{\sqrt{d}} \sum_{i=1}^d X_{i,j_1} \cdot X_{i,j_2} \cdot \cdots \cdot X_{i,j_p} \right)_{1 \leq j_1 < ... < j_p \leq n}.
\]
Throughout this paper we use \( C, C', c, c' ... \) to denote absolute positive constants whose value might change between expressions. In case we want to signify that the constant might depend on some parameter \( a \), we will write \( C_a, C'_a \).

1.2 Main results

Our main contribution is a new approach, detailed in Section 3, to Stein’s method, which allows to capitalize on the fact that a high-dimensional random vector may have some latent low-dimensional structure. Thus, it is particularly well suited to study the CLT for \( W_{n,d}^p(\mu) \). Using this approach, we obtain the following threshold for the CLT: Suppose that \( \mu \) is a “nice” measure. Then, if \( n^{2p-1} \gg d \), \( W_{n,d}^p(\mu) \) is approximately Gaussian, as \( d \) tends to infinity.

We now state several results which are obtained using our method. The first result shows that, under some assumptions (see exact definitions in Section 2.2), the matrix \( \mathbb{X} \) in (1), can admit some dependencies, even when considering higher order tensors.
Theorem 2. Let $\mu$ be an isotropic $L$-uniformly log-concave measure on $\mathbb{R}^n$ which is also unconditional. Denote $\Sigma^{-\frac{1}{2}} = \sqrt{\Sigma_\mu^{-1}}$, where $\Sigma_\mu$ is the covariance matrix of $\tilde{W}_{n,d}(\mu)$. Then, there exists a constant $C_p$, depending only on $p$, such that

$$W_2^2 \left( \Sigma^{-\frac{1}{2}} \tilde{W}^p_{n,d}(\mu), \gamma \right) \leq C_p n^{2p-1} d.$$  

An important remark, which applies to the coming results as well, is that the bounds are formulated with respect to the quadratic Wasserstein distance. However, as will become evident from the proof, the bounds actually hold with a stronger notion of distance. Namely, Stein’s discrepancy (see Section 2 for the definition). We have decided to state our results with the more familiar Wasserstein distance to ease the presentation. Our next result is a direct extension of Theorem 1, as it both applies to a larger class of product measures and to $p > 2$.

Theorem 3. Let $\mu$ be an isotropic product measure on $\mathbb{R}^n$, with independent coordinates. Then, there exists a constant $C_p > 0$, depending only on $p$, such that

1. If $\mu$ is log-concave, then

$$W_2^2 \left( \tilde{W}^p_{n,d}(\mu), \gamma \right) \leq C_p n^{2p-1} d \log(n)^2.$$  

2. If each coordinate marginal of $\mu$ satisfies the $L_1$-Poincaré inequality with constant $c > 0$, then

$$W_2^2 \left( \tilde{W}^p_{n,d}(\mu), \gamma \right) \leq C_p \frac{1}{c^{2p+2}} n^{2p-1} d \log(n)^4.$$  

3. If there exists a uni-variate polynomial $Q$ of degree $k$, such that each coordinate marginal of $\mu$ has the same law as $Q_\gamma \gamma_1$, then

$$W_2^2 \left( \tilde{W}^p_{n,d}(\mu), \gamma \right) \leq C_{Q,p} n^{2p-1} d \log(n)^{2(k-1)},$$  

where $C_{Q,p} > 0$ may depend both on $p$ and the polynomial $Q$.

Observe that, when $\mu$ is an isotropic product measure, then $\tilde{W}^p_{n,d}(\mu)$ is also isotropic, which explains why the matrix $\Sigma^{-\frac{1}{2}}$ does not appear in Theorem 3. As noted above, $W_2^2 \tilde{W}_{n,d}(\mu)$ is the law of a Wishart matrix without its diagonal. Our last result is an extension to Theorem 3 which shows that, sometimes, we may consider subspaces of $(\mathbb{R}^n)^{\otimes p}$ which are strictly larger than $\text{Sym}^p(\mathbb{R}^n)$. We specialize to the case $p = 2$, and show that one may consider the full symmetric matrix $W_{n,d}(\mu)$.

Theorem 4. Let $\mu$ be an isotropic log-concave measure on $\mathbb{R}^n$. Assume that $\mu$ is a product measure with independent coordinates and denote $\Sigma^{-\frac{1}{2}} = \sqrt{\Sigma_2^{-1}}$, where $\Sigma_2$ is the covariance matrix of $W_{n,d}(\mu)$. Then, there exists a universal constant $C > 0$ such that

$$W_2^2 \left( \Sigma^{-\frac{1}{2}} W_{n,d}(\mu), \gamma \right) \leq C n^3 d \log(n)^2.$$  

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1.3 Related work

The study of normal approximations for high-dimensional Wishart tensors was initiated in [7] (see [26] as well, for an independent result), which dealt with the case of $\tilde{W}_{n,d}^2(\gamma)$. The authors were interested in detecting latent geometry in random geometric graphs. In our language, a necessary condition for detection of the latent geometry is that the distance between $\tilde{W}_{n,d}^2(\gamma)$ and the standard Gaussian remains bounded away from 0. The main result of [7] was a particular case of Theorem 1, which gave sharp threshold for detection in the total variation distance. Namely, the total variation between $\tilde{W}_{n,d}^2(\gamma)$ and $\gamma(\frac{1}{d})$ goes to 0, if and only if $n^3 \gg d^3$. As demonstrated in [37], there is a smooth transition between the cases $n^3 \gg d$ and $n^3 \ll d$. The proof of such results was facilitated by the fact that $\tilde{W}_{n,d}^2(\gamma)$ has a tractable density with respect to the Lebesgue measure. This is not the case in general though.

In a follow-up ([8]), as discussed above, the results of [7] were expanded to the relative entropy distance and to Wishart tensors $\tilde{W}_{n,d}^2(\mu)$, where $\mu$ is a log-concave product measure. Specifically, it was shown that one may consider relative entropy in the formulation of Theorem 1, and that

$$\text{Ent}\left(\tilde{W}_{n,d}^2(\mu)||\gamma\right) \leq Cn^3\log(d^2)\frac{d}{d},$$

for a universal constant $C > 0$. The main idea of the proof was a clever use of the chain rule for relative entropy along with ideas adapted from the one-dimensional entropic central limit theorem proven in [4]. We do note the this result is not directly comparable to our results. As remarked, our results holds in Stein’s discrepancy. In general, Stein’s discrepancy and relative entropy are not comparable. However, one may bound the relative entropy by the discrepancy, in some cases. One such case, is when the measure has a finite Fisher information. $\tilde{W}_{n,d}(\gamma)$ is an example of such a measure.

The question of handling dependencies between the entries of the matrix $X$ in (1) was also tackled in [35]. The authors considered the case where the rows of $X$ are i.i.d. copies of a Gaussian measure whose covariance is a symmetric Toeplitz matrix. This is different from Theorem 2, which considers the case where there the dependencies are between the rows of $X$. The paper employed Stein’s method in a clever way, different from our approach which seems less suited to dealing with dependencies between the columns.

Also, note that if the rows of $X$ are independent Gaussian vectors, we may always apply an orthogonal transformation to the rows and obtain a matrix with independent entries which have different variances. Such measures were studied in [22]. Specifically if $\alpha = \{\alpha_i\}_{i=1}^d \subset \mathbb{R}^+$, with $\sum \alpha_i^2 = 1$ and $X_i \sim \gamma$ are independent, then the paper introduced the law, $W_{n,\alpha}^2(\gamma)$, of

$$\sum \alpha_i \left( X_i^{\odot 2} - \mathbb{E} \left[ X_i^{\odot 2} \right] \right).$$

The following variant of 1 was given:

$$\text{Ent} \left( \tilde{W}_{n,\alpha}^2(\gamma)||\gamma(\frac{1}{d}) \right) \leq Cn^3 \sum \alpha_i^4,$$

for a universal constant $C > 0$. When $\alpha_i \equiv \frac{1}{\sqrt{d}}$, this recovers the previous known result. We are not aware of a comparison between the bounds in [22] and [35]. We mention here that our method applies to non-homogeneous sums as well, with the same dependence on $\alpha$. We omit
the details for the sake of simplicity.

The authors of [35] also dealt with Wishart tensors, when the underlying measure is the standard Gaussian. It was shown that for some constant $C_p$, which depends only on $p$,

$$\mathcal{W}_1\left(\tilde{W}^p_{n,d}(\gamma), \gamma(n)\right) \leq C_p \sqrt{\frac{n^{2p-1}}{d}}.$$  

Thus, our results should also be seen as a direct generalization of this bound.

Wishart tensors have recently gained interest in the machine learning community (see [2,39] for recent results and applications). To mention a few examples: In [27] the distribution of the maximal entry of $W^p_{n,d}(\mu)$ is investigated. Using tools of random matrix theory, the spectrum of Wishart tensors is analyzed in [1], while [32] studies the central limit theorem for spectral linear statistics of $W^p_{n,d}(\mu)$. Results of a different flavor are given in [43], where exponential concentration is studied for a class of random tensors.

### 1.4 Organization

The rest of this paper is organized in the following way: In Section 2 we introduce some preliminaries from Stein’s method and concentration of measure, which will be used in our proofs. In Section 3 we describe our method and present the necessary ideas with which we will prove our results. In particular, we will state Theorem 5, which will act as our main technical tool. In Section 4 we introduce a construction in Stein’s theory which will be used in Section 5 to prove Theorem 5. Sections 6, 7 and 8 are then devoted to the proofs of Theorems 2, 3 and 4 respectively.

### 2 Preliminaries

In this section we will describe our method and explain how to derive the stated results. We begin with some preliminaries on Stein’s method.

#### 2.1 Stein kernels

For a measure $\mu$ on $\mathbb{R}^n$, we say that a matrix valued map $\tau : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ is a Stein kernel for $\mu$, if the following equality holds, for all smooth test functions, $f : \mathbb{R}^n \to \mathbb{R}^n$,

$$\int \langle x, f(x) \rangle d\mu(x) = \int \langle \tau(x), Df(x) \rangle_{HS} d\mu(x),$$

where $Df$ stands for the Jacobian matrix of $f$, and $\langle \cdot, \cdot \rangle_{HS}$ is the Hilbert-Schmidt inner product. According to Stein’s lemma, ([15, Lemma 2.1]) $\mu = \gamma$ if and only if the function $\tau(x) \equiv \text{Id}$ is a Stein kernel for $\mu$. Thus, in some sense, the deviation of $\tau$ from the identity measures the distance of $\mu$ from the standard Gaussian. Led by this idea we define the Stein discrepancy to the normal distribution as

$$S(\mu) := \inf_{\tau} \sqrt{\int \|\tau(x) - \text{Id}\|_{HS}^2 d\mu(x)},$$

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where the infimum is taken over all Stein kernels of \( \mu \). By the above, \( S(\mu) = 0 \) if and only if \( \mu = \gamma \). Thus, while not strictly being a metric, the Stein discrepancy still serves as some notion of distance to the standard Gaussian. If \( X \) is a random vector in \( \mathbb{R}^n \), by a slight abuse of notation we will also write \( S(X) \) for \( S(\text{Law}(X)) \).

Stein kernels exhibit several nice properties which make their analysis tractable for normal approximations. Suppose that \( A \) is a linear transformation, such that \( A_*\mu \) is well defined. Let \( X \sim \mu \), it is straightforward to verify that if \( \tau_X \) is a Stein kernel of \( X \), then a Stein kernel for \( AX \) is given by (see [19, Section 3] for some examples)

\[
\tau_{AX}(x) = A \mathbb{E}[\tau(X)|AX = x] A^T.
\]

(2)

In particular, if \( X_i \) are i.i.d. copies of \( X \) and \( S_d = \frac{1}{\sqrt{d}} \sum_{i=1}^d X_i \). Then

\[
\tau_{S_d}(x) = \frac{1}{d} \mathbb{E}[\tau_X(X_i)|S_d = x],
\]

(3)
is a Stein kernel for \( S_d \). Now, assume that \( X \) is isotropic. By choosing the test function \( f \) to be linear in the definition of the Stein kernel we may see that

\[
\mathbb{E}[\tau_X(X)] = \text{Cov}(X) = \text{Id}.
\]

(4)

Thus, \( (\tau_X(X_i) - \text{Id}) \) is a centered random variable, and the above observations show

\[
S^2(S_d) \leq \mathbb{E} \left[ \|\tau_{S_d}(S_d) - \text{Id}\|_{HS}^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{d} \sum_{i=1}^d \mathbb{E}[\tau_X(X_i) - \text{Id}|S_d] \right\|_{HS}^2 \right]
\]

\[
\leq \frac{1}{d^2} \mathbb{E} \left[ \left\| \sum_{i=1}^d \tau_X(X_i) - \text{Id} \right\|_{HS}^2 \right] = \frac{1}{d} \mathbb{E} \left[ \|\tau_X(X) - \text{Id}\|_{HS}^2 \right].
\]

By taking the infimum over all Stein kernels, we get

\[
S^2(S_d) \leq \frac{S^2(X)}{d}.
\]

(5)

### 2.1.1 Stein’s discrepancy as a distance

In this section we discuss the relations between Stein’s discrepancy to other, more classical notions of distance. This notion has recently gained prominence in the study of convergence rates along the high-dimensional central limit theorem (see [19, 23, 34] for several examples as well as the book [33]). This popularity stems, among others, from the fact that the Stein’s discrepancy controls several other known distances.

First, by the Kantorovich-Rubinstein ([44]) duality for \( \mathcal{W}_1 \) it is not hard to show that \( \mathcal{W}_1(\mu, \gamma) \leq S(\mu) \). A more remarkable fact is that the same also holds for the quadratic Wasserstein distance ([30, Proposition 3.1])

\[
\mathcal{W}_2^2(\mu, \gamma) \leq S^2(\mu).
\]

(6)
In fact, by slightly changing the definition of the discrepancy, we may bound the Wasserstein distance of any order. This is the content of Proposition 3.1 in [23] which shows that if $\tau$ is a Stein kernel of $\mu$, then

$$W_m(\mu, \gamma) \leq C_m \tau \left( \int \|\tau(x) - \text{Id}\|_{HS}^m d\mu(x) \right),$$

where $C_m > 0$ depends only on $m$. Our methods may then be adapted to deal with the Wasserstein distance of any order.

Another, related, notion of distance is that of relative entropy, which is given by

$$\text{Ent}(\mu||\gamma) = \int \log \left( \frac{d\mu}{d\gamma} \right) d\mu.$$

It is known that relative entropy bounds the quadratic Wasserstein distance to the the Gaussian via Talagrand’s transportation-entropy inequality ([42]) as well as controlling the total variation distance through Pinsker’s inequality ([20]). The so-called HSI inequality from [30] serves as a notable connection with Stein’s discrepancy. According to the inequality

$$\text{Ent}(\mu||\gamma) \leq \frac{1}{2} S^2(\mu) \log \left( 1 + \frac{I(\mu||\gamma)}{S^2(\mu)} \right),$$

where $I(\mu||\gamma)$ is the (relative) Fisher information of $\mu$,

$$I(\mu||\gamma) = \int \left\| \nabla \log \left( \frac{d\mu}{d\gamma} \right) \right\|_2^2 d\mu.$$

So, if we can show that for some large $d$, $\tilde{W}_{n,d}^p(\mu)$ has a finite Fisher information, we could expand our results, and give bounds in relative entropy. Unfortunately, verifying that a measure has finite Fisher information is a non-trivial task in high-dimensions (see Section 5 in [30] for further discussion).

2.2 Smooth measures and concentration inequalities

Our result will mostly apply for measures which satisfy some regularity conditions. We detail here the main examples which will be relevant.

We say that a measure $\mu$ is log-concave, if it has a density, twice differentiable almost everywhere, for which

$$-\text{Hessian} \left( \log \left( \frac{d\mu}{dx} \right) \right) \succeq 0,$$

where the inequality is between positive semi-definite matrices. If instead, for some $\sigma > 0$

$$-\text{Hessian} \left( \log \left( \frac{d\mu}{dx} \right) \right) \succeq \sigma \cdot \text{Id},$$

we say that $\mu$ is $\sigma$-uniformly log-concave.

A measure $\mu$ is said to satisfy the $L_1$-Poincaré inequality with constant $c$ if, for any differentiable function $f$ with 0-median,

$$\int |f| d\mu \leq \frac{1}{c} \int \|\nabla f\|_2^2 d\mu.$$
Remark that the $L_1$-Poincaré inequality is equivalent, up to constants, to the Cheeger’s isoperimetric inequality. That is, if $\mu$ satisfies the $L_1$-Poincaré inequality with constant $c > 0$, then for some other constant $c' > 0$, depending only on $c$, and for every measurable set $B$,

$$\mu^+(\partial B) \geq c' \mu(B) (1 - \mu(B)),$$

where $\mu^+(\partial B)$ is the outer boundary measure of $A$. Moreover, up to universal constants, the $L_1$-Poincaré inequality implies an $L_2$-Poincaré inequality. We refer the reader to [9] for further discussion of those facts.

The above conditions all imply that the measures have sub-exponential tails, in the sense that if $X \sim \mu$ then, for any $m \geq 2$:

$$E[\|X\|_m^m] \leq C_m E[\|X\|_2^2]^{\frac{m}{2}}. \quad (7)$$

In the case of log-concave vectors, the constant $C_m$ and does not depend on the particular law (see [31, Theorem 5.22]). If $\mu$ satisfies the $L_1$ Poincaré inequality with constant $c$, then we can take $C_m = (m!)^2 \left(\frac{1}{c}\right)^m$.

A measure is said to be unconditional, if its density satisfies

$$\frac{d\mu}{dx}(\pm x_1, \ldots, \pm x_n) = \frac{d\mu}{dx}(|x_1|, \ldots, |x_n|),$$

where in the left side of the equality we consider all possible sign patterns. Note that, in particular, if $X = (X_{(1)}, \ldots, X_{(n)})$ is isotropic and unconditional, then, for any choice of distinct indices $j_1, \ldots, j_k$ and powers $n_2, \ldots, n_k$,

$$E[X_{(j_1)} \cdot X_{(j_2)}^{n_2} \cdot X_{(j_3)}^{n_3} \cdot \ldots \cdot X_{(j_k)}^{n_k}] = 0. \quad (8)$$

### 3 The method

With the above results, the following theorem is our main tool, with which we may prove CLTs for $W_{n,d}^p(\mu)$.

**Theorem 5.** Let $X \sim \mu$ be an isotropic random vector in $\mathbb{R}^n$ and let $G \sim \mathcal{N}(0, \text{Id})$ stand for the standard Gaussian. Assume that $X \overset{\text{law}}{=} \varphi(G)$ for some $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, which is locally-Lipschitz and let $A : \text{Sym}^p(\mathbb{R}^n) \to V$ be a linear transformation with $V \subset \text{Sym}^p(\mathbb{R}^n)$, such that $A_* W_{n,d}^p(\mu)$ is isotropic. Then, for any $2 \leq p \in \mathbb{N},$

$$S^2 \left(A_* W_{n,d}^p(\mu)\right) \leq 2 \|A\|_{op}^2 p^4 \cdot \frac{n}{d} \sqrt{E[\|X\|_2^{2(p-1)}]} \sqrt{E[\|D\varphi(G)\|_{op}^8]} + \frac{2np^4}{d}.$$

Some remarks are in order concerning the theorem. We first discuss the role of the matrix $A$. In order to use the sub-additive property (5) of the Stein discrepancy, the random vectors need to be normalized. To avoid effects of small eigenvalues in the covariance matrix we will sometimes project the vectors into a subspace, such as $\overline{\text{Sym}}^p(\mathbb{R}^n)$, where the covariance matrix is easier to control. Thus, $A$ should be thought as a product of a projection matrix with the inverse of a covariance matrix on the projected space. For our applications we will make sure
that \( \|A\|_{op} = O(1) \).

Concerning the other terms in the stated bound, there are two terms which we will need to control, \( \sqrt{\mathbb{E} \left[ \|X\|_2^{8(p-1)} \right]} \) and \( \sqrt{\mathbb{E} \left[ \|D\varphi(G)\|_{op}^8 \right]} \). Since we are mainly interested in measures with sub-exponential tails, the first term will be of order \( n^{2p-2} \) and we will focus on the second term. Thus, in some sense, our bounds are meaningful mainly for measures which can be transported from the standard Gaussian with low distortion. Still, the class of measures which can be realized in such a way is rather large and contains many interesting examples.

A map \( \psi \) is said to transport \( G \) to \( X \) if \( \psi(G) \) has the same law as \( X \). To apply the result we must realize \( X \) by choosing an appropriate transport map. It is a classical fact (\[5\]) that, whenever \( \mu \) has a finite second moment and is absolutely continuous with respect to \( \gamma \), there is a distinguished map which transports \( G \) to \( X \). Namely, the Brenier map which minimizes the quadratic distance,

\[
\varphi_{\mu} := \inf_{\psi: \psi(G) \sim X} \mathbb{E} \left[ \|G - \psi(G)\|^2 \right].
\]

The Brenier map has been studied extensively (see \[10, 11, 16, 29\] for example). Here, we will concern ourselves with cases where one can bound the derivative of \( \varphi_{\mu} \). The celebrated Caffarelli’s log-concave perturbation theorem (\[12\]) states that if \( \mu \) is \( L \)-uniformly log-concave, then \( \varphi_{\mu} \) is \( \frac{1}{L} \)-Lipschitz. In particular, \( \varphi \) is differentiable almost everywhere with

\[
\|D\varphi_{\mu}(x)\|_{op} \leq \frac{1}{L}.
\]

In this case we get

\[
\sqrt{\mathbb{E} \left[ \|D\varphi_{\mu}(G)\|_{op}^8 \right]} \leq \frac{1}{L^{\frac{1}{4}}}. \tag{9}
\]

Theorem 2 will follow from this bound. The reason why the theorem specializes to unconditional measures is that, in light of the dependence on the matrix \( A \) in Theorem 5, we need to have some control over the covariance structure of \( \tilde{W}_{n,d}^p(\mu) \). It turns out, that for unconditional log-concave measures the covariance of \( \tilde{W}_{n,d}^p(\mu) \) is well behaved. The result might be extended to uniformly log-concave measures which are not necessarily centrally symmetric as long as we allow the bound to depend on the minimal eigenvalue of the covariance matrix of \( \tilde{W}_{n,d}^p(\mu) \).

There are more examples of measures for which the Brenier map admits bounds on the Lipschitz constant. In \[17\] it is shown that for measures \( \mu \) which are bounded perturbation of the Gaussian, including radially symmetric measures, \( \varphi_{\mu} \) is Lipschitz. The theorem may thus be applied to those measures as well.

One may also consider cases where the transport map is only locally-Lipschitz in a well behaved way. For example, consider the case where \( X = (X_{(1)}, \ldots, X_{(n)}) \sim \mu \) is a product measure. That is, for \( i \neq j \), \( X_{(i)} \) is independent from \( X_{(j)} \). Suppose that for \( i = 1, \ldots, n \), there exist functions \( \varphi_i : \mathbb{R} \to \mathbb{R} \) such that, if \( G^i \) is a standard Gaussian in \( \mathbb{R} \), then \( \varphi_i(G^i) \overset{law}{=} X_{(i)} \) and that \( \varphi \) has polynomial growth. Meaning, that for some constants \( \alpha, \beta \geq 0 \),

\[
\varphi_i'(x) \leq \alpha(1 + |x|^\beta).
\]
Since $\mu$ is a product measure, it follows that the map $\varphi = (\varphi_1, \ldots, \varphi_n)$ transports $G$ to $X$ and that
\[ \| D\varphi(x) \|_{op} \leq \alpha (1 + \| x \|_\infty^\beta). \]
Thus, for product measures, we can translate bounds on the derivative of one-dimensional transport maps into multivariate bounds involving the $L_\infty$ norm. Theorem 3 will be proved by using these ideas and known estimates on the one-dimensional Brenier map (also known as monotone rearrangement). Results like Theorem 4 can then be proven by bounding the covariance matrix of $W^2_{n,d}(\mu)$. Indeed, this is the main ingredient in the proof of the Theorem.

One may hope that Theorem 5 could be applied to general log-concave measures. However, this would be a highly non-trivial task. Indeed, if we wish to use Theorem 5 in order to verify the threshold $n^{2p-1} \gg d$, up to logarithmic terms, we should require that for any isotropic log-concave measure $\mu$, there exists a map $\psi_\mu$ such that $\psi_\mu(G) \sim \mu$ and $\mathbb{E} \left[ \| D\psi_\mu(G) \|_{op}^8 \right] \leq \log(n)^\beta$, for some fixed $\beta \geq 0$. Then, by applying the Gaussian $L_2$-Poincaré inequality to the function $\| \cdot \|_2$, we would get,
\[
\text{Var} \left( \| \psi_\mu(G) \|_2 \right) \leq \mathbb{E} \left[ \| D \left( \| \psi_\mu(G) \|_2 \right) \|_2^2 \right] \\
\leq \mathbb{E} \left[ \| D \| \psi_\mu(G) \|_2 \|_2^2 \cdot \| D\psi_\mu(G) \|_{op}^2 \right] \\
= \mathbb{E} \left[ \| D\psi_\mu(G) \|_2^\beta \right] \leq \log(n)^\beta.
\]
This would, up to logarithmic factors, verify the thin-shell conjecture (see [3]), and, through the results of [21], also the KLS conjecture ( [28]). These two conjectures are both famous long-standing open problems in convex geometry.

### 3.1 High-level idea

We now present the idea behind the proof of Theorem 5 and detail the main steps. We first explain why standard techniques fail to give optimal bounds. We may treat $W^p_{n,d}(\mu)$ as a sum of independent random vectors and invoke Theorem 7 from [14] (similar results will encounter the same difficulty). So, if $X \sim \mu$, optimistically, the theorem will give
\[ \mathcal{W}_1(W^p_{n,d}(\mu), \gamma) \leq \frac{\mathbb{E} \left[ \| X^{\odot p} \|_2^3 \right]}{\sqrt{d}}, \]
where we take the Euclidean norm of $X^{\odot p}$ when considered as a vector in $\text{Sym}^p(\mathbb{R}^n)$. Since $\dim(\text{Sym}^p(\mathbb{R}^n)) = \binom{n+p-1}{p}$, it is not reasonable to expect that $\mathbb{E} \left[ \| X^{\odot p} \|_2^3 \right]$ will be of order $\sqrt{n^{2p-1}}$, which is the bound achieved by Theorem 5.

The high-level plan of our proof is to use the fact that $X^{\odot p}$ has some low-dimensional structure. We will construct a map which transports the standard Gaussian $G$, from the lower dimensional space $\mathbb{R}^n$ into the law of $X^{\odot p}$ in the higher dimensional space $\text{Sym}^p(\mathbb{R}^n)$. In some sense, the role of this transport map is to preserve the low-dimensional randomness coming from $\mathbb{R}^n$. The map can be constructed in two steps, first use a transport map $\varphi$, such that $\varphi(G) \overset{\text{law}}{=} X$, and then take its tensor power $\varphi(G)^{\odot p}$. We will use this map in order to construct a Stein kernel and show that tame tails of the map’s derivative translate into small norms for the Stein kernel.
4 From transport maps to Stein kernels

We now explain how to construct a Stein kernel from a given transport map. For the rest of this section let $\nu$ be a measure on $\mathbb{R}^N$ and $Y \sim \nu$. Recall the definition of a Stein kernel: A matrix-valued map, $\tau : \mathbb{R}^N \to M_N(\mathbb{R})$, is a Stein kernel for $\nu$, if for every smooth $f : \mathbb{R}^N \to \mathbb{R}^N$,

$$\mathbb{E} \left[ \langle Y, f(Y) \rangle \right] = \mathbb{E} \left[ \langle \tau(Y), Df(Y) \rangle_{HS} \right].$$

Our construction is based on differential operators which arise naturally when performing analysis in Gaussian spaces. We incorporate into this construction the idea of considering transport measures between spaces of different dimensions. For completeness, we give all of the necessary details, but see [24, 33] for a rigorous treatment. The construction is based on ideas which have appeared implicitly in the literature, at least as far as [13] (see [33] for a more modern point of view).

4.1 Analysis in finite dimensional Gaussian space

We let $\gamma$ stand for the standard Gaussian measure in $\mathbb{R}^N$ and consider the Sobolev subspace of weakly differentiable functions,

$$W^{1,2}(\gamma) := \{ f \in L^2(\gamma) \mid f \text{ is weakly differentiable, and } \mathbb{E}_\gamma \| Df \|^2 < \infty \},$$

where $D : W^{1,2}(\gamma) \to L^2(\gamma, \mathbb{R}^N)$ is the natural (weak) derivative operator. The divergence $\delta$ is defined to be the formal adjoint of $D$, so that

$$\mathbb{E}_\gamma \left[ \langle Df, g \rangle \right] = \mathbb{E}_\gamma \left[ f \delta g \right],$$

where $g$ is a vector-valued function. $\delta$ is given explicitly by the relation

$$\delta f(x) = \langle x, f(x) \rangle - \text{div}(f(x)).$$

The Ornstein-Uhlenbeck (OU) operator is now defined by $L := -\delta \circ D$. On functions, $L$ operates as $Lf(x) = -xDf(x) + \Delta f(x)$. The operator $L$ also serves as the infinitesimal generator of the OU semi-group ( [34, Proposition 1.3.6]). That is,

$$L = \frac{d}{dt} P_t \bigg|_{t=0},$$

where

$$P_t f(x) := \mathbb{E}_{N \sim \gamma} \left[ f(e^{-t}x + \sqrt{1-e^{-2t}}N) \right].$$

The following fact, which may be proved by the Hermite decomposition of $L^2(\gamma)$, will be useful; There exists an operator, denoted $L^{-1}$ such that $LL^{-1} f = f$. In particular, on the subspace of functions whose Gaussian expectation vanishes, $L^{-1}$ is the inverse of $L$ ( [33, Proposition 2.8.11]).

We now introduce a general construction for Stein kernels. By a slight abuse of notation, even when working in different dimensions, we will refer to the above differential operators as the same, as well as extending them to act of vector and matrix valued functions. Note that, in particular, if $f$ is a vector-valued function and $g$ is matrix-valued of compatible dimensions, then

$$\mathbb{E}_\gamma \left[ \langle Df, g \rangle_{HS} \right] = \mathbb{E}_\gamma \left[ \langle f, \delta g \rangle \right].$$
Lemma 1. Let $\gamma_m$ be the standard Gaussian measure on $\mathbb{R}^m$ and let $\varphi : \mathbb{R}^m \to \mathbb{R}^N$. Set $\nu = \varphi_* \gamma_m$ and suppose that $\int_{\mathbb{R}^N} x d\nu = 0$. Then, when the following expectation is well defined,

$$
\tau_\varphi(x) := \mathbb{E}_{y \sim \gamma_m} \left[ ( -DL^{-1}) \varphi(y) (D\varphi(y))^T | \varphi(y) = x \right],
$$

is a Stein kernel of $\nu$.

**Proof.** Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be a smooth function and set $Y \sim \nu, G \sim \gamma_m$. We then have

$$
\mathbb{E} \left[ (Df(Y), \tau_\varphi(Y))_{HS} \right] = \mathbb{E} \left[ (Df(Y), \mathbb{E} \left[ ( -DL^{-1}) \varphi(G) (D\varphi(G))^T | \varphi(G) = Y \right] )_{HS} \right] 
$$

$$
= \mathbb{E} \left[ (Df(\varphi(G)) D\varphi(G), ( -DL^{-1}) \varphi(G))_{HS} \right] 
$$

$$
= \mathbb{E} \left[ (f \circ \varphi(G), ( -DL^{-1}) \varphi(G))_{HS} \right] 
$$

$$
= \mathbb{E} \left[ (f \circ \varphi(G), LL^{-1} \varphi(G)) \right] 
$$

$$
= \mathbb{E} \left[ (f \circ \varphi(G), \varphi(G)) \right] 
$$

$$
= \mathbb{E} \left[ (f(Y), Y) \right]. 
$$

The above formula suggests that one might control the kernel $\tau_\varphi$ by controlling the gradient of the transport map, $\varphi$. This will be the main step in proving Theorem 5. The following formula from [33, Proposition 29.3] will be useful:

$$
-DL^{-1} \varphi = \int_0^\infty e^{-t} P_t D\varphi dt. \quad (11)
$$

We thus have the corollary:

**Corollary 6.** With the same notations as in Lemma 1,

$$
\tau_\varphi(x) = \int_0^\infty e^{-t} \mathbb{E}_{y \sim \gamma_m} \left[ D\varphi(y) P_t (D\varphi(y))^T | \varphi(y) = x \right].
$$

As a warm up we present a simple case in which we can show that the Stein kernel obtained from the construction is bounded almost surely.

**Lemma 2.** Let the notations of Lemma 1 prevail and suppose that $\|D\varphi(x)\|_{op} \leq 1$ almost surely. Then

$$
\|\tau_\varphi(x)\|_{op} \leq 1,
$$

almost surely.

**Proof.** From the representation (10) and by Jensen’s inequality, $P_t$ is a contraction. That is, for any function $h$

$$
\mathbb{E}_{y \sim \gamma_m} [h(y) P_t (h(y))] \leq \mathbb{E}_{y \sim \gamma_m} [h(y)^2].
$$
So, from Corollary 6 and since \( \|D \varphi(x)\|_{op} \leq 1 \), we get

\[
\|\tau_\varphi(x)\|_{op} \leq \int_0^\infty e^{-t} \|\mathbb{E}_{y \sim \gamma_m} [D \varphi(y) P_t (D \varphi(y))] | \varphi(y) = x]\|_{op} \leq \int_0^\infty e^{-t} \|\mathbb{E}_{y \sim \gamma_m} [\|D \varphi(y)\|_{op} P_t (\|D \varphi(y)\|_{op})] | \varphi(y) = x]\|_{op} \leq \int_0^\infty e^{-t} \|\mathbb{E}_{y \sim \gamma_m} [\|D \varphi(y)\|^2_{op} | \varphi(y) = x]\|_{op} \leq 1.
\]

\[
\square
\]

5 Proof of Theorem 5

Let \( A : (\mathbb{R}^n)^\otimes p \to V \) be any linear transformation such that \( A (X^\otimes p - \mathbb{E} [X^\otimes p]) \) is isotropic, and let \( \tau \) be a Stein kernel for \( X^\otimes p - \mathbb{E} [X^\otimes p] \). In light of (2), we know that

\[
S^2 \left( A (X^\otimes p - \mathbb{E} [X^\otimes p]) \right) \leq \mathbb{E} \left[ \|A \varphi (X^\otimes p - \mathbb{E} [X^\otimes p]) A^T - \text{Id}\|^2_{HS} \right] \leq 2 \mathbb{E} \left[ \|A \varphi (X^\otimes p - \mathbb{E} [X^\otimes p]) A^T\|^2_{HS} \right] + 2 \|\text{Id}\|^2_{HS} \leq 2 \|A\|^2_{op} \mathbb{E} \left[ \|\tau (X^\otimes p - \mathbb{E} [X^\otimes p])\|^2_{HS} \right] + 2 \dim(V).
\]

Thus, by combining the above with (5), Theorem 5 is directly implied by the following lemma.

**Lemma 3.** Let \( X \) be an isotropic random vector in \( \mathbb{R}^n \) and let \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) be differentiable almost everywhere, such that \( \varphi(G) \overset{law}{=} X \), where \( G \) is the standard Gaussian in \( \mathbb{R}^n \). Then, for any integer \( p \geq 2 \), there exists a Stein kernel \( \tau \) of \( X^\otimes p - \mathbb{E} [X^\otimes p] \), such that

\[
\mathbb{E} \left[ \|\tau (X^\otimes p - \mathbb{E} [X^\otimes p])\|^2_{HS} \right] \leq p^4 n \sqrt{\mathbb{E} \left[ \|X\|_2^{8(p-1)} \right]} \sqrt{\mathbb{E} \left[ \|D \varphi(G)\|^8_{op} \right]}.
\]

**Proof.** Consider the map \( v \to \varphi(v)^\otimes p - \mathbb{E} [X^\otimes p] \), which transports \( G \) to \( X^\otimes p - \mathbb{E} [X^\otimes p] \). Corollary 6 shows that the function defined by

\[
\tau(v^\otimes p) : = \int_0^\infty e^{-t} \mathbb{E} \left[ P_t (D(\varphi(G)^\otimes p)) \cdot D(\varphi(G)^\otimes p)^T | \varphi(G)^\otimes p = v^\otimes p \right] dt
\]

\[
= \int_0^\infty e^{-t} \mathbb{E} \left[ P_t (D(\varphi(G)^\otimes p)) \cdot D(\varphi(G)^\otimes p)^T | \varphi(G) = (\pm 1)^p v \right] dt,
\]

is a Stein kernel for \( X^\otimes p - \mathbb{E} [X^\otimes p] \). Here \( v \in \mathbb{R}^n \) and \( \tau \) vanishes on tensors which have rank higher than 1. Note that for any two matrices \( A, B \),

\[
\|AB\|_{HS} \leq \sqrt{\text{rank}(A)} \|A\|_{op} \|B\|_{op}.
\]
Thus, using Jensen’s inequality, we have the bound

\[
\mathbb{E} \left[ \| \tau(X^\otimes p) \|_{HS}^2 \right] \leq \int_0^\infty e^{-t} \mathbb{E} \left[ \text{rank}(D(\varphi(G)^\otimes p)) \| D(\varphi(G)^\otimes p) \|_{op}^2 P_t \left( \| D(\varphi(G)^\otimes p) \|_{op}^2 \right) \right]
\]

\[
\leq n \int_0^\infty e^{-t} \mathbb{E} \left[ \| D(\varphi(G)^\otimes p) \|_{op}^2 P_t \left( \| D(\varphi(G)^\otimes p) \|_{op}^2 \right) \right] .
\]

To see why the second inequality is true, observe that \( v \to \varphi(v)^\otimes p \) is a map from \( \mathbb{R}^n \) to \( \mathbb{R}^{np} \), hence \( D(\varphi(G)^\otimes p) \) is an \( np \times n \) matrix, which leads to \( \text{rank}(D(\varphi(G)^\otimes p)) \leq n \). We now use the fact that \( P_t \) is a contraction, so that for every \( t > 0 \),

\[
\mathbb{E} \left[ \| D(\varphi(G)^\otimes p) \|_{op}^2 P_t \left( \| D(\varphi(G)^\otimes p) \|_{op}^2 \right) \right] \leq \mathbb{E} \left[ \| D(\varphi(G)^\otimes p) \|_{op}^4 \right] .
\]

So,

\[
\mathbb{E} \left[ \| \tau(X^\otimes p) \|_{HS}^2 \right] \leq n \mathbb{E} \left[ \| D(\varphi(G)^\otimes p) \|_{op}^4 \right] .
\]

We may realize the map \( v \to \varphi(v)^\otimes p \) as the \( p \)-fold Kronecker power (the reader is referred to [36] for the relevant details concerning the Kronecker product of \( \varphi(v) \)). With \( \otimes \) now standing for the Kronecker product, the following Leibniz law holds for the Jacobian:

\[
D (\varphi(x)^\otimes p) = \sum_{i=1}^p \varphi(x)^\otimes i-1 \otimes D \varphi(x) \otimes \varphi(x)^\otimes p-i .
\]

The Kronecker product is multiplicative with respect to singular values, and for any \( A_1, \ldots, A_p \) matrices,

\[
\| A_1 \otimes \ldots \otimes A_p \|_{op} = \prod_{i=1}^p \| A_i \|_{op} .
\]

We then have

\[
\mathbb{E} \left[ \| \tau(X^\otimes p) \|_{HS}^2 \right] \leq n \mathbb{E} \left[ \| D(\varphi(G)^\otimes p) \|_{op}^4 \right] = n \mathbb{E} \left[ \left( \sum_{i=1}^p \varphi(x)^\otimes i-1 \otimes D \varphi(x) \otimes \varphi(x)^\otimes p-i \right) \|_{op}^4 \right] \\
\leq n \mathbb{E} \left[ \left( \sum_{i=1}^p \| \varphi(x)^\otimes i-1 \otimes D \varphi(x) \otimes \varphi(x)^\otimes p-i \|_{op} \right)^4 \right] \\
\leq np^2 \mathbb{E} \left[ \| \varphi(G) \|_{2}^{4(p-1)} \right] D \varphi(G) \|_{op}^4 \right] \\
\leq np^2 \sqrt{\mathbb{E} \left[ \| \varphi(G) \|_{2}^{8(p-1)} \right]} \sqrt{\mathbb{E} \left[ \| D \varphi(G) \|_{op}^{8} \right]} \\
= np^2 \sqrt{\mathbb{E} \left[ \| X \|_{2}^{8(p-1)} \right] \mathbb{E} \left[ \| D \varphi(G) \|_{op}^{8} \right]} ,
\]

where the last inequality is Cauchy-Schwartz.
6 Unconditional log-concave measures; Proof of Theorem 2

We now wish to apply Theorem 5 to unconditional measures which are uniformly log-concave. In this case, we begin by showing that the covariance of $\tilde{W}_{n,d}^p(\mu)$ is well conditioned.

**Lemma 4.** Let $\mu$ be an unconditional log-concave measure on $\mathbb{R}^n$ and let $\tilde{\Sigma}_p(\mu)$ denote the covariance matrix $\tilde{W}_{n,d}^p(\mu)$. Then, there exists a constant $c_p > 0$, depending only on $p$, such that if $\tilde{\lambda}_{\min}$ stands for the smallest eigenvalue of $\tilde{\Sigma}_p(\mu)$, then

$$c_p \leq \tilde{\lambda}_{\min}.$$  

**Proof.** We write $X = (X_1, ..., X_n)$ and observe that $\Sigma_p(\mu)$ is diagonal. Indeed, if $1 \leq j_1 < j_2 < ... < j_p \leq n$ and $1 \leq j'_1 < j'_2 < ... < j'_p \leq n$ are two different sequences of indices then the covariance between $X_{(j_1)} \cdot ... \cdot X_{(j_p)}$ and $X_{(j'_1)} \cdot ... \cdot X_{(j'_p)}$ can be written as

$$E \left[ X_{(i_1)} \cdot X_{(i_2)}^n \cdot ... \cdot X_{(i_k)}^n \right],$$

where $p + 1 \leq k \leq 2p$ and for every $i = 2, ..., k, n_i \in \{1, 2\}$. By (8), those terms vanish. Thus, in order to prove the lemma, it will suffice to show that for every set of distinct indices $j_1, ..., j_p$,

$$c_p \leq E \left[ (X_{(j_1)} \cdot ... \cdot X_{(j_p)})^2 \right],$$

for some constant $c_p > 0$, which depends only on $p$. If we consider the random isotropic and log-concave vector $(X_{j_1}, ..., X_{j_p})$ in $\mathbb{R}^p$, the existence of such a constant is assured by the fact that the density of this vector is uniformly bounded from below on some ball around the origin (see [31, Theorem 5.14]).

We now prove Theorem 2.

**Proof of Theorem 2.** Set $P : (\mathbb{R}^n)^{\otimes p} \to \text{Sym}^p(\mathbb{R}^n)$ to be the linear projection operator and $\tilde{\Sigma}_p(\mu)$ to be as in Lemma 4. Denote $A = \sqrt{\tilde{\Sigma}_p^{-1}(\mu)} P$. Then, $A (X^{\otimes p} - E[X^{\otimes p}])$ is isotropic and has the same law as $\sqrt{\tilde{\Sigma}_p^{-1}(\mu)} \tilde{W}_{n,d}^p(\mu)$. The lemma implies

$$\|A\|_{op}^2 \leq \frac{1}{c_p}.$$  

As $X$ is log-concave and isotropic, from (7), we get

$$\sqrt{E \left[ \|X\|_2^{8(p-1)} \right]} \leq C_p n^{2p-2}.$$  

$X$ is also $L$-uniformly log-concave. So, as in (9), if $\varphi_\mu$ is the Brenier map, sending the standard Gaussian $G$ to $X$,

$$\sqrt{E \left[ \|D\varphi_\mu(G)\|_{op}^8 \right]} \leq \frac{1}{L^4}.$$  

Combining the above displays with Theorem 5, gives the desired result. □
7 Product measures; Proof of Theorem 3

As mentioned in Section 3, when \( \mu \) is a product measure, transport bounds on the marginals of \( \mu \) may be used to construct a transport map \( \varphi \) whose derivative satisfies an \( L_\infty \) bound of the form,

\[
\|D\varphi(x)\|_{op} \leq \alpha (1 + \|x\|_\infty^\beta).
\]

(12)

for some \( \alpha, \beta \geq 0 \). Such conditions can be verified for a wide variety of product measures. For example, it holds, a fortiori, when the marginals of \( \mu \) are polynomials of the standard one-dimensional Gaussian with bounded degrees. Furthermore, we mention now two more cases where the one-dimensional Brenier map is known to have tame growth. Those estimates will serve as the basis for the proof of Theorem 3.

In [18] it is shown that if \( \mu \) is an isotropic log-concave measure in \( \mathbb{R} \), and \( \varphi_\mu \) is its associated Brenier map, then for some universal constant \( C > 0 \),

\[
\varphi_\mu'(x) \leq C (1 + |x|).
\]

(13)

If, instead, \( \mu \) satisfies an \( L_1 \)-Poincaré inequality with constant \( c_\ell > 0 \), then for another universal constant \( C > 0 \)

\[
\varphi_\mu'(x) \leq C \frac{1}{c_\ell} (1 + x^2).
\]

Thus, for log-concave product measures (12) holds with \( \beta = 1 \) and for products of measures which satisfy the \( L_1 \)-Poincaré inequality it holds with \( \beta = 2 \). Using these bounds, Theorem 3 becomes a consequence of the following lemma.

**Lemma 5.** Let \( X \) be a random vector in \( \mathbb{R}^n \) and let \( G \) stand for the standard Gaussian. Suppose that for some \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \), \( \varphi(G) \overset{\text{law}}{=} X \), and that \( \varphi \) is differentiable almost everywhere with

\[
\|D\varphi(x)\|_{op} \leq \alpha (1 + \|x\|_\infty^\beta),
\]

for some \( \beta, \alpha > 0 \). Then, there exists a constant \( C_\beta \), depending only on \( \beta \), such that

\[
\mathbb{E} \left[ \|D\varphi(x)\|_{op}^8 \right] \leq C_\beta \alpha^8 \log(n)^{4\beta}.
\]

**Proof.** For any \( x, y \geq 0 \), the following elementary inequality holds,

\[
(x + y)^8 \leq 2^7 (x^8 + y^8).
\]

Thus, we begin the proof with,

\[
\mathbb{E} \left[ \|D\varphi(G)\|_{op}^8 \right] \leq \alpha^8 \mathbb{E} \left[ (1 + \|G\|_\infty^\beta)^8 \right] \leq 256 \alpha^8 \mathbb{E} \left[ \|G\|_{\infty}^{8\beta} \right].
\]

Observe that the density function of \( \|G\|_\infty \) is given by \( n \psi \cdot \Phi^{n-1} \), where \( \psi \) is the density of the standard Gaussian in \( \mathbb{R} \) and \( \Phi \) is its cumulative distribution function. Since the product of log-concave functions is also log-concave, we deduce that \( \|G\|_\infty \) is a log-concave random variable. From (7), we thus get

\[
\mathbb{E} \left[ \|G\|_{\infty}^{8\beta} \right] \leq C'_\beta \mathbb{E} \left[ \|G\|_\infty^2 \right]^{4\beta},
\]

where \( C'_\beta \) depends only on \( \beta \). The proof is concluded by applying known estimates to \( \mathbb{E} \left[ \|G\|_\infty^2 \right] \).
Proof of Theorem 3. We first observe that, since $\mu$ is an isotropic product measure, $\tilde{W}_{n,d}^p(\mu)$ is an isotropic random vector in $\tilde{\Sym}_p^{n,d}(\mathbb{R}^n)$. Thus, the matrix $A$ in Theorem 5, reduces to a projection matrix and $\|A\|_{op} = 1$.

Let $X \sim \mu$. For the first case, we assume that $X$ is log-concave. Since it is also isotropic, from (7) there exists a constant $C_p'$ depending only $p$, such that

$$\sqrt{\mathbb{E} \left[ \|X\|_2^{8(p-1)} \right]} \leq C_p' n^{2p-2}. \quad (14)$$

We now let $\varphi_\mu$ stand for the Brenier map between the standard Gaussian $G$ and $X$. Since $X$ has independent coordinates it follows from (13) that for some absolute constant $C > 0$.

$$\|D\varphi(x)\|_{op} \leq C(1 + \|x\|_\infty).$$

In this case, Lemma 5 gives:

$$\sqrt{\mathbb{E} \left[ \|D\varphi(G)\|_8^{8} \right]} \leq C' \log(n)^2, \quad (15)$$

where $C' > 0$ is some other absolute constant. Plugging these estimates into Theorem 5 and taking $C_p = 2C' \cdot C_p' \cdot p^4$ shows Point 1 of the corollary.

The proof of Point 2 is almost identical and we omit it. For Point 3, when each marginal of $\mu$ is a polynomial of the standard Gaussian, observe that the map $\tilde{Q} : \mathbb{R}^n \to \mathbb{R}^n$,

$$\tilde{Q}(x_1, ..., x_n) = (Q(x_1), ..., Q(x_n)),$$

is by definition a transport map between $G$ and $X$. Since $Q$ is a degree $k$ polynomial, there exists some constant $C_Q$, such that

$$Q'(x) \leq C_Q(1 + |x|^{k-1}).$$

So, from Lemma 5, there is some constant $C_{Q,p}'$, such that

$$\sqrt{\mathbb{E} \left[ \|D\tilde{Q}(G)\|_{op}^{8} \right]} \leq C_{Q,p}' \log(n)^{2(k-1)}.$$

Moreover, using hypercontractivity (see [25, Theorem 5.10]), since $X$ is given by a degree $k$ polynomial of the standard Gaussian, we also have the following bound on the moments of $X$:

$$\sqrt{\mathbb{E} \left[ \|X\|_2^{8(p-1)} \right]} \leq (8p)^{2kp} \mathbb{E} \left[ \|X\|_2^{2} \right]^{2p-2} = (8p)^{2kp} n^{2p-2}.$$

Using the above two displays in Theorem 5 finishes the proof.

8  Extending Theorem 3; Proof of Theorem 4

We now fix $X \sim \mu$ to be an unconditional isotropic log-concave measure on $\mathbb{R}^n$ with independent coordinates. If $\Sigma_2(\mu)$ stands for the covariance matrix of $W_{n,d}^2(\mu)$, then, using the same arguments as in the proof of Theorem 3, it will be enough to show that $\Sigma_2(\mu)$ is bounded uniformly from below. Towards that, we first prove:
Lemma 6. Let $Y$ be an isotropic log-concave random variable in $\mathbb{R}$. Then

\[ \text{Var}(Y^2) \geq \frac{1}{100}. \]

Proof. Denote by $\rho$ the density of $Y$. We will use the following 3 facts, pertaining to isotropic log-concave densities in $\mathbb{R}$ (see Section 5.2 in [31]).

- $\rho$ is uni-modal. That is, there exists a point $x_0 \in \mathbb{R}$, such that $\rho$ is non-decreasing on $(-\infty, x_0)$ and non-increasing on $(x_0, \infty)$.
- $\rho(0) \geq \frac{1}{8}$ and $\rho(x) \leq 1$, for every $x \in \mathbb{R}$.
- $\int_{|x| \geq 2} \rho(x) dx \leq \frac{1}{e}$.

The first observation is that either $\rho\left(\frac{1}{9}\right) \geq \frac{1}{10}$, or $\rho\left(-\frac{1}{9}\right) \geq \frac{1}{10}$. Indeed, if not, then as $\rho$ is uni-modal and $\rho(0) \geq \frac{1}{10}$,

\[
\int_{-2}^{-\frac{1}{9}} \rho(x) dx \leq \int_{-2}^{-\frac{1}{9}} \rho(x) dx + \frac{4}{10} \leq \frac{2}{9} + \frac{4}{10} < 1 - \frac{1}{e},
\]

which is a contradiction. We assume, without loss of generality, that $\rho\left(\frac{1}{9}\right) \geq \frac{1}{10}$. Similar considerations then show

\[
\text{Var}(Y^2) = \int_{\mathbb{R}} (x^2 - 1)^2 \rho(x) dx \geq \frac{1}{10} \int_{0}^{\frac{1}{9}} (x^2 - 1)^2 dx \geq \frac{1}{100}.
\]

Using the lemma, we now prove Theorem 4.

Proof of Theorem 4. First, as in Lemma 4, the product structure of $\mu$ implies that $\Sigma_2(\mu)$, the covariance matrix of $W_{n,d}(\mu)$, is diagonal. Write $X = (X_{(1)}, \ldots, X_{(n)})$. There are two types of elements on the diagonal:

- The first corresponds to elements of the form $\text{Var}(X_{(i)} X_{(j)})$. For those elements, by independence, $\text{Var}(X_{(i)} X_{(j)}) = 1$.
- The other type of elements are of the form $\text{Var}(X_{(i)}^2)$. By Lemma 6, $\text{Var}(X_{(i)}^2) \geq \frac{1}{100}$.

So, if $P : (\mathbb{R}^n)^\otimes 2 \to \text{Sym}^2(\mathbb{R}^n)$ is the projection operator, we have that

\[
\left\| \Sigma^{-\frac{1}{2}} P \right\|_{op}^2 \leq 100.
\]

The estimates (14) and (15) are valid here as well. Thus, Theorem 5 implies the result.
References

[1] Ambainis, A., Harrow, A. W., and Hastings, M. B. Random tensor theory: extending random matrix theory to mixtures of random product states. *Comm. Math. Phys.* 310, 1 (2012), 25–74.

[2] Anandkumar, A., Ge, R., Hsu, D., Kakade, S. M., and Telgarsky, M. Tensor decompositions for learning latent variable models. *J. Mach. Learn. Res.* 15 (2014), 2773–2832.

[3] Anttila, M., Ball, K., and Perissinaki, I. The central limit problem for convex bodies. *Trans. Amer. Math. Soc.* 355, 12 (2003), 4723–4735.

[4] Artstein, S., Ball, K. M., Barthe, F., and Naor, A. On the rate of convergence in the entropic central limit theorem. *Probab. Theory Related Fields* 129, 3 (2004), 381–390.

[5] Brenier, Y. Décomposition polaire et réarrangement monotone des champs de vecteurs. *C. R. Acad. Sci. Paris Sér. I Math.* 305, 19 (1987), 805–808.

[6] Brennan, M., Bresler, G., and Nagaraj, D. Phase transitions for detecting latent geometry in random graphs. *arXiv preprint arXiv:1910.14167* (2019).

[7] Bubeck, S., Ding, J., Eldan, R., and Rácz, M. Z. Testing for high-dimensional geometry in random graphs. *Random Structures Algorithms* 49, 3 (2016), 503–532.

[8] Bubeck, S., and Ganguly, S. Entropic CLT and phase transition in high-dimensional Wishart matrices. *Int. Math. Res. Not. IMRN*, 2 (2018), 588–606.

[9] Buser, P. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)* 15, 2 (1982), 213–230.

[10] Caffarelli, L. A. A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math.* (2) 131, 1 (1990), 129–134.

[11] Caffarelli, L. A. The regularity of mappings with a convex potential. *J. Amer. Math. Soc.* 5, 1 (1992), 99–104.

[12] Caffarelli, L. A. Monotonicity properties of optimal transportation and the FKG and related inequalities. *Comm. Math. Phys.* 214, 3 (2000), 547–563.

[13] Chatterjee, S. Fluctuations of eigenvalues and second order Poincaré inequalities. *Probab. Theory Related Fields* 143, 1-2 (2009), 1–40.

[14] Chatterjee, S., and Meckes, E. Multivariate normal approximation using exchangeable pairs. *ALEA Lat. Am. J. Probab. Math. Stat.* 4 (2008), 257–283.

[15] Chen, L. H. Y., Goldstein, L., and Shao, Q.-M. *Normal approximation by Stein’s method*. Probability and its Applications (New York). Springer, Heidelberg, 2011.

[16] Colombo, M., and Fathi, M. Bounds on optimal transport maps onto log-concave measures. *arXiv preprint arXiv:1910.09035* (2019).
[17] Colombo, M., Figalli, A., and Jhaeri, Y. Lipschitz changes of variables between perturbations of log-concave measures. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 17*, 4 (2017), 1491–1519.

[18] Courtade, T. A., Fathi, M., and Pananjady, A. Quantitative stability of the entropy power inequality. *IEEE Trans. Inform. Theory 64*, 8 (2018), 5691–5703.

[19] Courtade, T. A., Fathi, M., and Pananjady, A. Existence of Stein kernels under a spectral gap, and discrepancy bounds. *Ann. Inst. Henri Poincaré Probab. Stat. 55*, 2 (2019), 777–790.

[20] Cover, T. M., and Thomas, J. A. *Elements of information theory*, second ed. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2006.

[21] Eldan, R. Thin shell implies spectral gap up to polylog via a stochastic localization scheme. *Geom. Funct. Anal. 23*, 2 (2013), 532–569.

[22] Eldan, R., and Mikulincer, D. Information and dimensionality of anisotropic random geometric graphs. *arXiv preprint arXiv:1609.02490* (2016).

[23] Fathi, M. Stein kernels and moment maps. *Ann. Probab. 47*, 4 (2019), 2172–2185.

[24] Huang, Z.-Y., and Yan, J.-A. *Introduction to infinite dimensional stochastic analysis*, chinese ed., vol. 502 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht; Science Press Beijing, Beijing, 2000.

[25] Janson, S. *Gaussian Hilbert spaces*, vol. 129 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997.

[26] Jiang, T., and Li, D. Approximation of rectangular beta-Laguerre ensembles and large deviations. *J. Theoret. Probab. 28*, 3 (2015), 804–847.

[27] Jiang, T., and Xie, J. Limiting behavior of largest entry of random tensor constructed by high-dimensional data. *Journal of Theoretical Probability* (2019), 1–21.

[28] Kannan, R., Lovász, L., and Simonovits, M. Isoperimetric problems for convex bodies and a localization lemma. *Discrete Comput. Geom. 13*, 3-4 (1995), 541–559.

[29] Kolesnikov, A. V. Global Hölder estimates for optimal transportation. *Mat. Zametki 88*, 5 (2010), 708–728.

[30] Ledoux, M., Nourdin, I., and Peccati, G. Stein’s method, logarithmic Sobolev and transport inequalities. *Geom. Funct. Anal. 25*, 1 (2015), 256–306.

[31] Lovász, L., and Vempala, S. The geometry of logconcave functions and sampling algorithms. *Random Structures Algorithms 30*, 3 (2007), 307–358.

[32] Lytova, A. Central limit theorem for linear eigenvalue statistics for a tensor product version of sample covariance matrices. *J. Theoret. Probab. 31*, 2 (2018), 1024–1057.

[33] Nourdin, I., and Peccati, G. *Normal approximations with Malliavin calculus: from Stein’s method to universality*, vol. 192. Cambridge University Press, 2012.
[34] NOURDIN, I., PECCATI, G., AND SWAN, Y. Entropy and the fourth moment phenomenon. *J. Funct. Anal.* 266, 5 (2014), 3170–3207.

[35] NOURDIN, I., AND ZHENG, G. Asymptotic behavior of large gaussian correlated wishart matrices. *arXiv preprint arXiv:1804.06220* (2018).

[36] PETERSEN, K. B., AND PEDERSEN, M. S. The matrix cookbook, 2012.

[37] RÁCZ, M. Z., AND RICHEY, J. A smooth transition from Wishart to GOE. *J. Theoret. Probab.* 32, 2 (2019), 898–906.

[38] ROSS, N. Fundamentals of Stein’s method. *Probab. Surv.* 8 (2011), 210–293.

[39] SHI, X., QIU, R., HE, X., CHU, L., LING, Z., AND YANG, H. Anomaly detection and location in distribution network: A data-driven approach. *arXiv preprint arXiv:1801.01669* (2018).

[40] STEIN, C. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory* (1972), pp. 583–602.

[41] STEIN, C. *Approximate computation of expectations*, vol. 7 of *Institute of Mathematical Statistics Lecture Notes—Monograph Series*. Institute of Mathematical Statistics, Hayward, CA, 1986.

[42] TALAGRAND, M. Transportation cost for Gaussian and other product measures. *Geom. Funct. Anal.* 6, 3 (1996), 587–600.

[43] VERSHYNIN, R. Concentration inequalities for random tensors. *arXiv preprint arXiv:1905.00802* (2019).

[44] VILLANI, C. *Optimal transport: old and new*, vol. 338. Springer Science & Business Media, 2008.