UPPER LARGE DEVIATIONS BOUND FOR SINGULAR-HYPERBOLIC ATTRACTING SETS

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Abstract. We obtain a exponential large deviation upper bound for continuous observables on suspension semiflows over a non-uniformly expanding base transformation with non-flat singularities and/or discontinuities, where the roof function defining the suspension behaves like the logarithm of the distance to the singular/discontinuous set of the base map. To obtain this upper bound, we show that the base transformation exhibits exponential slow recurrence to the singular set.

The results are applied to semiflows modeling singular-hyperbolic attracting sets of $C^2$ vector fields. As corollary of the methods we obtain result on the existence of physical measure for classes of piecewise $C^{1+}$ expanding maps of the interval with singularities and discontinuities. We are also able to obtain exponentially fast escape rates from subsets without full measure.

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1. Introduction

Arguably one of the most important concepts in Dynamical Systems theory is the notion of physical (or SRB) measure. We say that an invariant probability measure $\mu$ for a flow $X^t$ is physical if the set

$$B(\mu) = \left\{ z \in M : \lim_{t \to \infty} \frac{1}{t} \int_0^t \psi(X^s(z)) \, ds = \int \psi \, d\mu, \forall \psi \in C^0(M, \mathbb{R}) \right\}$$

has non-zero volume, with respect to any volume form on the ambient compact manifold $M$. The set $B(\mu)$ is by definition the basin of $\mu$. It is assumed that time averages of these orbits be observable if the flow models a physical phenomenon.

On the existence of physical/SRB measures for uniformly hyperbolic diffeomorphisms and flows we mention the works of Sinai, Ruelle and Bowen [25, 26, 53, 54, 61]. More recently, Alves, Bonatti and Viana [3] obtained the existence of physical measures for partial hyperbolic diffeomorphism and non-uniformly expanding transformations. Many developments along these lines for non uniformly hyperbolic systems have been obtained; see e.g. [51, 28, 22, 23, 15]. Closer to our setting, Araujo, Pacifico, Pujals and Viana [17] obtained physical measures for singular-hyperbolic attractors.

It is natural to study statistical properties of physical measures, such as the speed of convergence of the time averages to the space average, among many other properties which have been intensely studied recently; see e.g. [39, 65, 66, 24, 2, 5, 27, 44, 43, 31, 36, 41, 43, 10]. The main motivation behind all these results is that the family $\{\psi \circ X^t\}_{t > 0}$ asymptotically behaves like an i.i.d. family of random variables.

One of the ways to quantify this is the volume of the subset of points whose time averages $\frac{1}{T} S_T \psi$ are away from the space average $\mu(\psi)$ by a given amount. More precisely, fixing $\varepsilon > 0$ as the error size, we consider the set

$$B_T = \left\{ x \in M : \left| \frac{1}{T} \int_0^T \psi(X^s(x)) \, ds - \mu(\psi) \right| > \varepsilon \right\}$$

and search for conditions under which the volume of this set decays exponentially fast with $T$. That is, there are constants $C, \xi > 0$ so that

$$\text{Leb}(B_T) \leq C e^{-\xi T}, \text{ for all } T > 0.$$
The decay rate is related to the Thermodynamical Formalism, first developed for hyperbolic diffeomorphisms by Bowen, Ruelle and Sinai; see e.g. [25, 26, 55, 56]. However, in our setting Lebesgue measure is not necessarily an invariant measure and so some tools from the Thermodynamical Formalism are unavailable.

This work extends the first author’s result [6] of upper large deviation estimate for the geometric Lorenz attractor to the singular-hyperbolic attracting setting, encompassing a much more general family of singular three-dimensional flows, not necessarily transitive, with several singularities and with higher dimensional stable direction.

This demanded, first, the construction of a global Poincaré map as in [17] through adapted cross-sections to the flow obtained without assuming transitivity; and, second, to deal with the possible existence of finitely many distinct ergodic physical measures whose convex linear combinations for the set \( E \) of equilibrium states with respect to the log of the central unstable Jacobian. This led us to adapt the strategy of reduction of set of deviations for the flow to a set of deviations for a one-dimensional map while still following the general path presented in [6].

This extension is done through a special choice of adapted cross-sections in the definition of the global Poincaré map which is shown to be always possible for singular-hyperbolic attracting sets of \( C^2 \) flows.

Finally, an technically more delicate, we extended the proof of exponentially slow recurrence from [6] Section 6] for a piecewise expanding one-dimensional interval map with only singular discontinuities (with unbounded derivative) to allow both singularities (with unbounded derivative) and discontinuities (with bounded derivative) as boundary points of the monotonicity intervals of the one-dimensional map. In particular, this result allows us to obtain many statistical properties of a new class of piecewise expanding interval maps. This can be seen as an extension of [30] to Hölder-\( C^1 \) piecewise expanding maps with infinitely many singularities and discontinuities, and also assuming strong interaction between them; see Section 1.4 for more comments.

1.1. The setting: singular-hyperbolicity. We need some preliminary definitions.

From now on \( M \) is a compact boundaryless \( d \)-dimensional manifold; \( \mathcal{X}^2(M) \) is the set of \( C^2 \) vector fields on \( M \), endowed with the \( C^2 \) topology; we fix some smooth Riemannian structure on \( M \) and an induced normalized volume form \( \text{Leb} \) that we call Lebesgue measure, and we write also \( \text{dist} \) for the induced distance on \( M \).

Given \( X \in \mathcal{X}^2(M) \), we write \( X^t, t \in \mathbb{R} \) the flow induced by \( X \) and, for \( x \in M \) and \( [a, b] \subset \mathbb{R} \) we set \( X^{[a,b]}(x) = \{ X^t(x), a \leq t \leq b \} \).

For a compact invariant set \( \Lambda \) for \( X \in \mathcal{X}^2(M) \), we say that it is isolated if we can find an open neighborhood \( U \supset \Lambda \) so that \( \Lambda = \bigcap_{t \in \mathbb{R}} X^t(U) \). If \( U \) above also satisfies \( X^t(U) \subset U \) for \( t > 0 \) then we say that \( \Lambda \) is an attracting set and that \( U \) is a trapping region for \( \Lambda \). The topological basin of the attracting set \( \Lambda \) is \( W^s(\Lambda) = \{ x \in M : \lim_{t \to +\infty} \text{dist}(X_t(x), \Lambda) = 0 \} \).

The invariant set \( \Lambda \) is transitive if it coincides with the \( \omega \)-limit set of a regular \( X \)-orbit: \( \Lambda = \omega_X(p) \) where \( p \in \Lambda \) and \( X(p) \neq 0 \). If \( \sigma \in M \) and \( X(\sigma) = 0 \), then \( \sigma \) is called an equilibrium or singularity.
A point \( p \in M \) is periodic if \( p \) is regular and there exists \( \tau > 0 \) so that \( X^\tau(p) = p \); its orbit \( X^\mathbb{R}(p) = X^{[0,\tau]}(p) \) is a periodic orbit. An invariant set of \( X \) is non-trivial if it is neither a periodic orbit nor a singularity.

An attractor is a transitive attracting set. An attractor is proper if it is not the whole manifold.

**Definition 1.1.** Let \( \Lambda \) be a compact invariant set of \( X \in \mathcal{X}^2(M) \), \( c > 0 \), and \( 0 < \lambda < 1 \). We say that \( \Lambda \) has a \((c,\lambda)\)-dominated splitting if the tangent bundle over \( \Lambda \) can be written as a continuous \( DX^t \)-invariant sum of sub-bundles \( T_\Lambda M = E^1 \oplus E^2 \), (that is, \( DX^t E^i_x = E^i_{X^t x}, \forall t \in \mathbb{R}, i = 1, 2 \)) such that for every \( t > 0 \) and every \( x \in \Lambda \), we have

\[
\| DX^t | E^1_x\| \cdot \| DX^{-t} | E^2_{X^t(x)}\| < c \lambda^t. \tag{1.1}
\]

We say that a \( X \)-invariant subset \( \Lambda \) of \( M \) is partially hyperbolic if it has a \((c,\lambda)\)-dominated splitting, for some \( c > 0 \) and \( \lambda \in (0, 1) \), such that the sub-bundle \( E^1 = E^s \) is uniformly contracting: for every \( t > 0 \) and every \( x \in \Lambda \) we have \( \| DX^t | E^s_x\| < c \lambda^t \).

We assume that \( E^s \) has codimension 2 so that \( E^c \) is two-dimensional.

Let now \( J^c_t(x) \) be the center Jacobian of \( DX^t \) for \( x \in \Lambda \), that is, the absolute value of the determinant of the linear map \( DX^t | E^c_x : E^c_x \to E^c_{X^t(x)} \). We say that the sub-bundle \( E^c_{X^t(x)} \) of the partially hyperbolic invariant set \( \Lambda \) is \((c,\lambda)\)-volume expanding if \( J^c_t(x) \geq c e^{\lambda t} \) for every \( x \in \Lambda \) and \( t \geq 0 \), for some given \( c, \lambda > 0 \).

**Definition 1.2.** Let \( \Lambda \) be a compact invariant set of \( X \in \mathcal{X}^2(M) \) with singularities. We say that \( \Lambda \) is a singular-hyperbolic set for \( X \) if all the singularities of \( \Lambda \) are hyperbolic and \( \Lambda \) is partially hyperbolic with volume expanding central direction.

Singular-hyperbolicity is an extension of the notion of hyperbolic set, which we now recall.

**Definition 1.3.** Let \( \Lambda \) be a compact invariant set of \( X \in \mathcal{X}^2(M) \). We say that \( \Lambda \) is a hyperbolic set for \( X \) if it admits a continuous \( DX^t \)-invariant splitting \( T_\Lambda M = E^s \oplus [X] \oplus E^u \) where \([X] = \mathbb{R} \cdot X\) is the flow direction; \( E^s \) is \((c,\lambda)\) contracting and \( E^u \) is \((c,\lambda)\)-contracting for the inverse flow, for some \((c,\lambda) \in \mathbb{R}^+ \times (0, 1)\).

In particular, every equilibrium point in a hyperbolic set must be isolated in the set. The following result shows that singular-hyperbolicity is a natural extension of notion of hyperbolicity for singular flows.

**Theorem 1.4** (Hyperbolic Lemma). A compact invariant singular-hyperbolic set without singularities is a hyperbolic set.

**Proof.** See [48, Lemma 3] or [16, Proposition 6.2]. \( \square \)

The most representative example of a singular-hyperbolic attractor is the Lorenz attractor; see e.g. [62, 63]. Singular-hyperbolic attracting sets form a class of attracting set sharing similar topological/geometrical features with the Lorenz attractor. For more on singular-hyperbolic attracting sets see e.g. [16].
1.1.1. Lorenz-like singularities. A Lorenz-like singularity is an equilibrium \( \sigma \) of \( X \) contained in a singular-hyperbolic set having index (dimension of the stable direction) equal to \( d - 1 \). Since we are assuming that \( \text{dim } E^c = 2 \), this ensures the existence of the \( DX(\sigma) \)-invariant splitting \( T_\sigma M = E^s_\sigma \oplus F^s_\sigma \oplus F^u_\sigma \) so that

- \( E^c_\sigma = F^s \oplus F^u \) and \( \dim F^s_\sigma = \dim F^u_\sigma = 1 \);
- \( F^u_\sigma \) uniformly expands and \( F^s_\sigma \) uniformly contracts: there exists \( 0 < \tilde{\lambda} < \lambda \) such that \( \|DZ_t|_{F^u_\sigma}\| \geq C^{-1}\lambda^{-t} \) and \( \|DZ_t|_{F^s_\sigma}\| \leq C\tilde{\lambda}^t \) for all \( t \geq 0 \).

Remark 1.5. (1) Partial hyperbolicity implies that the direction of the flow is contained in the center-unstable subbundle \( E^2 = E^{cu} \); see [7, Lemma 5.1].

(2) If the index of singularity \( \sigma \) in a singular-hyperbolic set is not of codimension one, then it must have codimension 2 and there is no orbit of \( \Lambda \) that accumulates \( \sigma \) in the positive time direction.

Indeed, in this case (\( \dim W^s_\sigma = \dim M - 2 \) and \( \dim W^u_\sigma = 2 \)), if \( \sigma \in \omega(z) \) for \( z \in \Lambda \setminus \{\sigma\} \), then there exists \( p \in (W^s_\sigma \setminus \{\sigma\}) \cap \omega(z) \subset \Lambda \) by the local behavior of orbits near hyperbolic saddles. But then \( X(p) \in T_pW^s_\sigma = E^s_p \) in contradiction with the previous item.

1.2. Statement of the results. In [6] Araujo obtained exponential upper large deviations decay for continuous observables on suspension semiflows over a non-uniformly expanding base transformation with non-flat criticalities/singularities, where the roof function defining the suspension grows as the log of the distance to the singular/critical set.

Here we extend this result to a more general class of base transformations which, after constructing a global Poincaré map describing the dynamics of singular-hyperbolic attracting sets and reducing this dynamics to that of a certain semiflow, enables us to obtain the following.

1.2.1. Upper bound for large deviations.

**Theorem A.** Let \( X \) be a \( C^2 \) vector field on a compact manifold exhibiting a connected singular-hyperbolic attracting set on the trapping region \( U \) having some Lorenz-like singularity. Let \( \psi : U \rightarrow \mathbb{R} \) be a bounded continuous function. Then for every \( \varepsilon > 0 \)

\[
\limsup_{T \to \infty} \frac{1}{T} \log \text{Leb} \left\{ z \in U : \inf_{\mu \in \mathcal{E}} \left| \frac{1}{T} \int_0^T \psi(X^t(z)) dt - \mu(\psi) \right| > \varepsilon \right\} < 0,
\]

where \( \mathcal{E} = \{ \mu \in \mathcal{M}_X(U) : h_\mu = \int \log \left| \det(DX^1|_{E^{cu}}) \right| d\mu \} \) is the set of equilibrium states with respect to the central Jacobian and \( \mathcal{M}_X \) is the family of all \( X \)-invariant probability measures supported in \( U \).

Remark 1.6. If the singular-hyperbolic attracting set \( \Lambda \) in \( U \) has several connected components, then each is an attracting set and Theorem A applies to each singular component. Those components which have no singularities, or only non-Lorenz-like equilibria, are necessarily (by Remark 1.5) hyperbolic basic sets to which we can apply known large deviations results [65, 64].
From this result it is easy to deduce escape rates from subsets of the attracting set. Fix $K \subset U$ a compact subset. Given $\varepsilon > 0$ we can find an open subset $W \supset K$ contained in $U$ and a smooth bump function $\varphi : U \to \mathbb{R}$ so that $\text{Leb}(W \setminus K) < \varepsilon$; $0 \leq \varphi \leq 1$; $\varphi|_K \equiv 1$ and $\varphi|_{M \setminus W} \equiv 0$. Then

$$\{ z \in K : X^t(z) \in K, 0 < t < T \} \subset \left\{ z \in U : \frac{1}{T} \int_0^T \varphi(X^t(z)) dt \geq 1 \right\},$$
for all $T > 0$, and using Theorem A we deduce the following.

**Corollary B.** In the same setting of Theorem A let $K$ be a compact subset of $M$ such that $\inf_{\mu \in \Xi} \mu(K) < 1$. Then

$$\limsup_{T \to +\infty} \frac{1}{T} \log \text{Leb} \left( \{ x \in K : X^t(x) \in K, 0 < t < T \} \right) < 0.$$  

1.2.2. **Exponentially slow recurrence to singular set.** In Section 2 we show that each singular-hyperbolic attracting set $\Lambda$ for $X \in X^2(M)$ admits a finite family $\Xi$ of Poincaré sections to $X$ and a global Poincaré map $R : \Xi_0 \to \Xi, R(x) = X^{\tau(x)}(x)$ for a Poincaré return time function $\tau : \Xi_0 \to \mathbb{R}^+$, where $\Xi_0 = \Xi \setminus \Gamma$ and $\Gamma$ is a finite family of smooth hypersurfaces within $\Xi$.

Moreover, by a proper choice of coordinates in $\Xi$ the map $R$ can be written $F : (I \setminus D) \times I \to I \times I, (x,y) = (f(x), g(x,y))$ where $f : I \setminus D \to I$ is piecewise $C^{1+\alpha}$ for some $\alpha > 0$ and uniformly expanding on the connected components of $I \setminus D$, $D$ a finite subset of $I = [0,1]$; and $g : (I \setminus D) \times I \to I$ is a contraction in the second coordinate; see Theorem 2.8.

The main technical result in this work is that such $f$ has exponentially slow recurrence to the set $D$, as follows. Let us write $\Delta_\delta(x) = |\log \text{dist}_\delta(x,D)|$ for the smooth $\delta$-truncated distance of $x$ to $D$, that is, for any given $\delta > 0$ we set

$$\text{dist}_\delta(x,D) = \begin{cases} \text{dist}(x,D) & \text{if } \text{dist}(x,D) \leq \delta; \\ \left(\frac{1-\delta}{\delta}\right) \text{dist}(x,D) + 2\delta - 1 & \text{if } \delta < \text{dist}(x,D) < 2\delta; \\ 1 & \text{if } \text{dist}(x,D) \geq 2\delta. \end{cases}$$

This property is proved for piecewise expanding interval maps possibly with infinitely many branches. To shorten the exposition, we write $(a, a \pm \delta)$ the lateral open intervals around $a$: $(a - \delta, a)$ or $(a, a + \delta)$.

**Theorem C.** Let $f : I \setminus D \to I$ be a $C^{1+\alpha}$ piecewise monotonous one-dimensional map, for some $\alpha > 0$, with monotonous branches on the connected components $\{I_k\}_k$ of $I \setminus D$ so that $\inf |f'| > 0$ whenever defined. In addition, assume that $D$ splits into a pair of disjoint subsets $\mathcal{S}$ and $\mathcal{D} \setminus \mathcal{S}$ satisfying

1. for each $b \in \mathcal{D}$ there exist the lateral limits $f(b^\pm) = \lim_{t \to b^\pm} f(t) \in I$;
2. $\forall b \in \mathcal{S} \exists 0 < \alpha(b) < 1$ so that one (or both) lateral limits $\lim_{t \to b^\pm} |t - b|^{1-\alpha(b)} f'(t)$ exists and are non-zero.

We write $b^\pm \in \mathcal{S}$ whenever each of the lateral limits exists and $b^\pm \in \mathcal{D} \setminus \mathcal{S}$ whenever the corresponding lateral limit $f'(a^\pm) = \lim_{t \to a^\pm} f'(t) \in \mathbb{R}$ exists.
\((3) \exists L \in \mathbb{Z}^+ \forall b^\pm \in \mathcal{D} \setminus \mathcal{S} \exists T = T(b^\pm) \leq L \) so that \(f^T(b^\pm) \in \mathcal{S}\) meaning that we can find \(\varepsilon, \delta > 0\) such that a lateral neighborhood of \(b^\pm\) is diffeomorphically mapped onto a lateral neighborhood of \(f^T(b^\pm)\):

\[
f^T(b^\pm) \in S\]

meaning that we can find \(\varepsilon, \delta > 0\) such that a lateral neighborhood of \(b^\pm\) is diffeomorphically mapped onto a lateral neighborhood of \(f^T(b^\pm)\):

\[
f^T(b^\pm) = (f^T(b^\pm), f^T(b^\pm) + \varepsilon)
\]

(4) \(S\) is nonempty and \(\{f^i(b_k^\pm) : 1 \leq i < T(b_k^\pm), b_k^\pm \in D \setminus \mathcal{S}\}\) is a finite set;

(5) there exist \(-\infty < \beta < \bar{\beta} < 0\) so that \(\beta \leq \alpha(b) - 1 \leq \bar{\beta}\) for all \(b \in S\);

(6) there exists \(\delta_0 > 0\) so that \(|f(I_k)| \geq \delta_0\) for all \(k\) (the “big images” property) where \(|J|\) denotes the length of the interval \(J\).

Then \(f\) has exponentially slow recurrence to the singular/discontinuous subset \(\mathcal{D}\), that is, for each \(\varepsilon > 0\) we can find \(\delta > 0\) so that there exists \(\xi > 0\) satisfying

\[
\limsup_{n \to \infty} \frac{1}{n} \log \lambda \left\{ x \in I : \frac{1}{n} \sum_{j=0}^{n-1} \Delta_\delta(f^j(x)) > \varepsilon \right\} < -\xi
\]

where \(\lambda\) is Lebesgue measure on \(I\).

1.3. Organization of the text. The proof of Theorem A demanded the extension of the construction of adapted cross-sections used in [17] to the singular-hyperbolic attracting case instead of a singular-hyperbolic attractor, in such a way that the boundaries of the cross-sections are contained in stable manifolds of some singularity of the attracting set. Moreover, since we are not assuming the existence of a dense regular orbit, we need to consider the possible existence of singularities in the attracting set which are not Lorenz-like. This construction is presented in Section 2 were a global Poincaré map is built and Theorem 2.8 on the representation of this map as a skew-product over a one-dimensional transformation is proved.

The deduction of Theorem A from the reduction to a one-dimensional transformation in the setting of Theorem C, following the route in [14, 6], is presented in Section 3 assuming the statement of Theorem C. Then Theorem C is proved in Section 4.

1.4. Some comments, corollaries and possible extensions. The construction of adapted cross-sections for general singular-hyperbolic attracting sets provides an extension of the results of [17] in line with the work of [59, 58].

From the representation of the global Poincaré map as a skew-product given by Theorem 2.8 we can follow [17, Sections 6-8] to obtain

**Theorem 1.7.** Let \(\Lambda\) be a singular-hyperbolic attracting set for a \(C^2\) vector field \(X\) with the open subset \(U\) as trapping region. Then there are finitely many ergodic physical/SRB measures \(\mu_1, \ldots, \mu_k\) supported in \(\Lambda\) such that the union of their ergodic basins covers \(U\) Lebesgue almost everywhere:

\[
\text{Leb} \left( U \setminus \bigcup_{i=1}^{k} B(\mu_i) \right) = 0.
\]

Moreover, for each \(X\)-invariant probability measure \(\mu\) supported in \(\Lambda\) the following are equivalent

(1) \(h_\mu(X_1) = \int \log |\det DX_1|_{E^c} \, d\mu > 0\);

(2) \(\mu\) is a SRB measure, that is, admits an absolutely continuous disintegration along unstable manifolds;
(3) \( \mu \) is a physical measure, i.e., its basin \( B(\mu) \) has positive Lebesgue measure. and the family \( \mathbb{E} \) of all such equilibrium states is the convex hull

\[
\mathbb{E} = \left\{ \sum_{i=1}^{k} t_i \mu_i : \sum_{i} t_i = 1; 0 \leq t_i \leq 1, i = 1, \ldots, k \right\}.
\]

The proof of the last statements, characterizing physical/SRB measures and the set \( \mathbb{E} \) of equilibrium states for the logarithm of the central Jacobian, in the same way as for hyperbolic attracting sets, is presented in Subsection 2.3.

We note that there are many examples of singular-hyperbolic attracting sets, non-transitive and containing non-Lorenz-like singularities; see Figure 1 for an example obtained by conveniently modifying the geometric Lorenz construction, and many others in [47].

![Figure 1. Example of a singular-hyperbolic attracting set, non-transitive and containing non-Lorenz like singularities.](image)

In addition, recent results obtained in [8, 9] depend on Theorem 2.8 (corresponding to [8, Theorem 5] ensuring the application of [8, Theorem A and Proposition 1] to singular-hyperbolic attractors) which now holds without assuming transitivity or that all equilibria are non-resonant Lorenz-like singularities for 3-dimensional vector fields only.

Hence, exponential decay of correlations for the physical measures of the Global Poincaré map together with exact dimensionality and the logarithm law for hitting times for the physical/SRB measures of the flow on \( \Lambda \) [8, Corollaries 1 and 2] are true in the same setting of Theorem A.

1.4.1. Consequences for one-dimensional maps. In the statement of Theorem C we assume that
• each discontinuity point with finite lateral derivative (in $D \setminus S$) admits a lateral neighborhood which is sent to a lateral neighborhood of a singular point (in $S$, a discontinuity point with unbounded lateral derivative) in a finite and uniformly bounded number of iterates, as in assumption (3);
• $S$ is nonempty and the images of $D \setminus S$ until arriving at $S$ form a finite subset, as in assumption (4);
• the set $S$ is non-degenerate in the usual sense from one-dimensional dynamics [29] as in assumption (2), that is, $|f'|$ grows as a power of the distance to $S$ (see also e.g. the conditions on the critical/singular set in [30] for a similar statement in the $C^2$ setting);
• the big image assumption (item (6)) is natural to avoid a counterexample like Rychlik’s [57] and ensure the existence of an absolutely continuous invariant measure, if $D$ is infinite.

Near singular points the rate of expansion is proportional to a power of the distance to the singularity, which simplifies distortion control. The coexistence of singularities and discontinuity points in the same map makes it more difficult to control distortion near the boundaries of the monotonicity intervals. The assumption that each point in $D \setminus S$ is eventually sent in $S$ enables us to adapt the combinatorial method of proof from [6, Section 6] to this setting, which uses partition refinement techniques first developed in the works of Benedicks and Carleson [20, 21] later expanded in [46, 49, 42, 15].

Similar techniques where used by Freitas [32] applied to the quadratic family to obtain exponentially slow approximation to the critical point on Benedicks-Carleson parameters; and by Diaz-Ordaz, Holland and Luzzatto [30] to study one-dimensional $C^2$ maps with finitely many critical points or singularities. In contrast to these works, where only one or finitely many criticalities and/or singularities were allowed and with no interaction between them, here we deal with a H"older-$C^{1}$ map having infinitely many non-degenerate singularities and criticalities and assume a strong interaction between them. We show that the non-degenerate assumptions on the behavior of $f$ near $S$ and the big image assumption enable us to prove the exponentially slow approximation result even with a countably infinite subset $D \setminus S$ of discontinuities and $S$ of singularities.

Applications of this result are given by singular-hyperbolic attracting sets as in Theorem A where we reduce the analysis to a one-dimensional map with a finite singular/discontinuity subset $D$; see Section 3. Coupling with well-known results on non-uniformly expanding maps we obtain results on existence of acip and its statistical properties.

We say that $f$ is non-uniformly expanding if there exists $c > 0$ such that

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |f'(f^j x)| \geq c \quad for \quad \lambda - a.e. x.$$ 

This condition implies in particular that the lower Lyapunov exponent of the map $f$ is strictly positive Lebesgue almost everywhere.
Condition \((\ref{eq1})\) implies that \(S_n\Delta_\delta/n \to 0\) in measure, i.e., the map \(f\) has slow recurrence to \(\mathcal{D}\): for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
\limsup_{n \to \infty} \frac{1}{n} S_n\Delta_\delta(x) \leq \varepsilon \quad \text{for} \quad \lambda - \text{a.e.} x.
\]

These notions suitably generalized to an arbitrary dimensional setting were presented in \cite{[72]} and in \cite{[3, H]} the following result on existence of finitely many absolutely continuous invariant probability measures was obtained.

**Theorem 1.8** (\cite{[3, H]}). Let \(f : M \circlearrowleft\) be a \(C^2\) local diffeomorphism outside a non-degenerate singular set \(\mathcal{D}\). Assume that \(f\) is non-uniformly expanding with slow recurrence to \(\mathcal{D}\). Then there are finitely many ergodic absolutely continuous (in particular physical) probability measures \(\mu_1, \ldots, \mu_k\) whose basins cover the manifold Lebesgue almost everywhere, that is \(B(\mu_1) \cup \cdots \cup B(\mu_k) = M\), \(\lambda -\text{mod} 0\). Moreover the support of each measure contains an open disk in \(M\).

It is clear that \(f\) in the setting of Theorem \(C\) satisfies both the non-uniformly expanding and slow recurrence conditions. Moreover, considering the tail sets \(\mathcal{E}(x) = \min\{N \geq 1 : |(f^n)'(x)| > \sigma^{n/3}, \forall n \geq N\}\) and \(\mathcal{R}(x) = \min\{N \geq 1 : S_n\Delta_\delta < \zeta_n, \forall n \geq N\}\), the exponentially slow recurrence \((\ref{eq1})\) can be translated as: there are constants \(\delta, \zeta, C_1, \xi > 0\) so that

\[
\lambda\left(\left\{x \in I : \mathcal{R}(x) > n\right\}\right) \leq C_1 \cdot e^{-\xi \cdot n} \quad \text{for all} \quad n \geq 1;
\]

and the uniform expanding assumption on \(f\) means that there exist \(\sigma > 1\) and \(N \in \mathbb{Z}^+\) so that \(\{x \in I : \mathcal{E}(x) > n\}\) equals \(I\) except finitely many points, for all \(n > N\). Hence we have

\[
\lambda\left(\left\{x \in I : \mathcal{R}(x) > n \quad \text{and} \quad \mathcal{E}(x) > n\right\}\right) \leq C_1 \cdot e^{-\xi \cdot n} \quad \text{for all} \quad n \geq 1. \quad (\ref{eq4})
\]

This allows us to deduce the following ergodic/statistical properties of \(f\).

**Corollary D.** Let \(f\) be as in the statement of Theorem \(C\). Then

1. (\cite{[3] and \cite{[H]}, Theorem 3}) there are finitely many absolutely continuous invariant probability measures \(\mu_1, \ldots, \mu_k\) such that \(B(\mu_1) \cup \cdots \cup B(\mu_k) = M\), \(\lambda -\text{mod} 0\) and some finite power of \(f\) is mixing with respect to \(\mu_i, i = 1, \ldots, k\);
2. \cite{[33]} there exists an interval \(Y_i\) with a return time function \(R_i : Y_i \to \mathbb{Z}^+\) defining a Markov Tower over \(f\) so that \(\limsup \frac{1}{n} \log \mu_i\{R_i > n\} < 0\) for each \(i = 1, \ldots, k\);
3. \cite{[4] and \cite{[33]}, Theorem 1.1}) there exist constants \(C, \xi > 0\) such that the correlation function \(\text{Corr}_n(\varphi, \psi) = |\int (\varphi \circ g^n) \cdot \psi \, d\mu_i - \int \varphi \, d\mu_i \int \psi \, d\mu_i|\), for Hölder continuous observables \(\varphi, \psi : I \to \mathbb{R}\), satisfy \(\text{Corr}_n(\varphi, \psi) \leq C \cdot e^{-\xi \cdot n}\) for all \(n \geq 1\) and each \(i = 1, \ldots, k\);
4. \cite{[4] Theorem 4}) \(\mu_i\) satisfies the Central Limit Theorem; given a Hölder continuous function \(\phi : I \to \mathbb{R}\) which is not a coboundary (\(\phi \neq \psi \circ f - \psi\) for any continuous \(\psi : I \to \mathbb{R}\)) there exists \(\theta > 0\) such that for every interval \(J \subset \mathbb{R}\) and each
\[ i = 1, \ldots, k \]

\[ \lim_{n \to \infty} \mu_i \left( \left\{ x \in I : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left( \phi(f^j(x)) - \int \phi \, d\mu_i \right) \in J \right\} \right) = \frac{1}{\theta \sqrt{2\pi}} \int_J e^{-t^2/2\theta^2} dt. \]

(5) \[ [44, 34] \]

For each \( i = 1, \ldots, k \) let \( \phi : I \to \mathbb{R} \) be a Hölder observable so that \( \mu_i(\phi) = 0 \).

Then \( \phi \) satisfies the Almost Sure Invariance Principle: there exist \( \varepsilon > 0 \), a sequence \( S_N \) of random variables and a Brownian motion \( W \) with variance \( \sigma^2 > 0 \) such that \( \{ \sum_{j=0}^{N-1} \phi \circ f^j \} =_{d} \{ S_N \} \) and

\[ S_N = W(N) + O(N^{\frac{3}{2} - \varepsilon}) \quad \text{as} \quad N \to \infty \quad \mu_i - \text{almost everywhere}. \]

Remark 1.9. The Almost Sure Invariance Principle implies the Central Limit Theorem and also the functional CLT (weak invariance principle), and the law of the iterated logarithm together with its functional version, and many other results; see e.g. \[ 52 \] for a comprehensive list.

Example with countable infinite \( D \). An example of a transformation with infinitely many monotonicity domains satisfying all the conditions of Theorem C with an enumerable \( D \) and \( S \) a single point, is given by a topologically exact Lorenz transformation in the interval \( [f(0^+), f(0^-)] \) strictly contained in \( J = [-\frac{1}{2}, \frac{1}{2}] \) whose graph we complete as a function \( J \to J \) with affine pieces between points having the same values of \( f \) at some element of the preorbits of the unique singularity at 0; see Figure 2.

We can perform this extension in a way that
- the slope of the affine branches be larger than 2;
- the monotonicity domains form a denumerable partition of \( J \);
- the singularity at 0 is a Lorenz-like singularity which together with the discontinuity points for a non-degenerate singular set;
- every discontinuity point of the map is sent to 0 in finitely many iterates and the orbit of the discontinuities up to arriving at 0 forms a finite subset.
1.4.2. Possible extensions and conjectures. A natural issue is whether it is possible to remove the assumption that $S$ is nonempty or to relax the assumption that discontinuity points are sent to singular points in a uniformly bounded number of iterates.

**Conjecture 1.** Exponential slow recurrence to the singular/discontinuous set still holds in the setting of Theorem C after removing assumptions (3-4).

Extensions of Theorem A to the class of sectional-hyperbolic attracting sets for flows in higher dimensions, with dimension of the central direction higher than two, introduced by Metzger-Morales in [45], seem to involve subtle questions on the smoothness of the stable foliation of these sets which, on the one hand, might prevent the existence of a smooth quotient map of the Poincaré return map over the stable foliation in a natural way and, on the other hand, the dynamics of higher dimensional piecewise expanding maps is not so well understood as its one-dimensional counterpart, where the boundaries of the domains of smoothness have low complexity.

**Conjecture 2.** Large deviations with respect to Lebesgue measure versus physical measures, for continuous observables on a neighborhood of general sectional-hyperbolic attracting sets for $C^2$ flows have exponential upper bound.

Another issue is regularity: what can we say about large deviations for singular-hyperbolic attracting sets of $C^1$ flows?

**Conjecture 3.** The statements of Theorem A and Conjecture 2 are still valid for $C^1$ flows.

Using the existence of Markov towers with exponential tails for the one-dimensional map as in Corollary D it is natural to search for statistical properties for the flows in the setting of Theorem A along the lines of [13, 10, 11]. This will be done in a systematic way in [12].

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2. Existence of adapted cross-sections and construction of global Poincaré map

We let $X \in \mathcal{X}^2(M)$ admit an singular-hyperbolic attracting set $\Lambda = \bigcap_{t > 0} \overline{X_t(U_0)}$ for some open neighborhood $U_0$, with $DX^t$-invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$, $d_s = \dim E^s = d - 2$ and $d_{cu} = \dim E^{cu} = 2$.

2.1. Properties of singular-hyperbolic attracting sets. We extend the stable direction on $\Lambda$ to a $DX^t$-invariant stable bundle over $U_0$ and then integrate these directions into a topological foliation of $U_0$ which admits Hölder-$C^1$ holonomies, combining the following results.
Proposition 2.1. Let Λ be a partially hyperbolic attracting set. The stable bundle $E^s$ over Λ extends to a continuous uniformly contracting $D^s$-invariant bundle $E^s$ over an open neighborhood of Λ.

Proof. See [11, Proposition 3.2].

We assume without loss of generality that $E^s$ extends as in Proposition 2.1 to $U_0$.

Let $D^k$ denote the $k$-dimensional open unit disk endowed with the Euclidean distance induced by the Euclidean norm $\|\cdot\|_2$, and let $\text{Emb}^2(D^k, M)$ denote the set of $C^2$ embeddings $\phi : D^k \to M$ endowed with the $C^2$ distance.

Proposition 2.2. Let Λ be a partially hyperbolic attracting set. There exists a positively invariant neighborhood $U_0$ of Λ, and a constant $\lambda \in (0, 1)$, such that the following are true:

1. For every point $x \in U_0$ there is a $C^2$ embedded $d_s$-dimensional disk $W^s_x \subset M$, with $x \in W^s_x$, such that
   (a) $T_x W^s_x = E^s_x$.
   (b) $X^i(W^s_x) \subset W^s_x$ for all $n \geq 1$.
   (c) $d(X^i(x), X^j(y)) \leq \lambda d(x, y)$ for all $y \in W^s_x$, $n \geq 1$.

2. The disks $W^s_x$ depend continuously on $x$ in the $C^0$ topology: there is a continuous map $\gamma : U_0 \to \text{Emb}^0(D^k, M)$ such that $\gamma(x)(0) = x$ and $\gamma(x)(D^k) = W^s_x$.

   Moreover, there exists $L > 0$ such that $\text{Lip} \gamma(x) \leq L$ for all $x \in U_0$.

3. The family of disks $\{W^s_x : x \in U_0\}$ defines a topological foliation of $U_0$.

Proof. See [11, Theorem 4.2 and Lemma 4.8], where $\text{Lip} \gamma(x)$ is the Lipschitz constant of $\gamma(x)$, given by $\sup\{\frac{\text{dist}(\gamma(x)u, \gamma(x)v)}{\|u-v\|_2} : u \neq v, w, v \in D^k\}$. "

The splitting $T_\Lambda M = E^s \oplus E^{cu}$ extends continuously to a splitting $T_{U_0} M = E^s \oplus E^{cu}$ where $E^s$ is the invariant uniformly contracting bundle in Proposition 2.1 and, in general, $E^{cu}$ is not invariant. Given $a > 0$, we consider the center-unstable cone field

$$C^{cu}_x(a) = \{v = v^s + v^{cu} \in E^s_x \oplus E^{cu}_x : \|v^s\| \leq a \|v^{cu}\|\}, \quad x \in U_0.$$ 

Proposition 2.3. Let Λ be a partially hyperbolic attracting set. There exists $T_0 > 0$ such that for any $a > 0$, after possibly shrinking $U_0$,

$$D^t X^i \cdot C^{cu}_x(a) \subset C^{cu}_{X^t x} (a) \quad \text{for all } t \geq T_0, x \in U_0.$$ 

Proof. See [11, Proposition 3.1].

Proposition 2.4. Let Λ be a singular hyperbolic attracting set. After possibly increasing $T_0$ and shrinking $U_0$, there exist constants $K, \theta > 0$ such that $|\text{det}(D^t X^i E^{cu})| \geq K e^{\theta t}$ for all $x \in U_0$, $t \geq 0$.

Proof. See [12, Proposition 2.10].

2.1. The stable lamination is a topological foliation. The Stable Manifold Theorem [60] ensures the existence of an $X^t$-invariant stable lamination $\mathcal{W}^s_\Lambda$ consisting of smoothly embedded disks $W^s_x$ through each point $x \in \Lambda$. Although not true for general partially hyperbolic attractors, for singular-hyperbolic attractors in our setting $\mathcal{W}^s_\Lambda$ indeed defines a topological foliation in an open neighborhood of Λ.
Theorem 2.5. Let $\Lambda$ be a singular hyperbolic attracting set. Then the stable lamination $W^s_{\Lambda}$ is a topological foliation of an open neighborhood of $\Lambda$.

Proof. See [12, Theorem 5.1] where it is shown that $W^s_{\Lambda}$ coincides with the topological foliation $\{W^s_x : x \in U_0\}$ of item (3) of Proposition 2.2. □

From now on, we refer to $W^s = \{W^s_x : x \in \Lambda\}$ as the stable foliation.

2.1.2. Absolute continuity of the stable foliation. A key fact for us is regularity of stable holonomies. Let $Y_0, Y_1 \subset U_0$ be two smooth disjoint $d_{cu}$-dimensional disks that are transverse to the stable foliation $W^s$. Suppose that for all $x \in Y_0$, the stable leaf $W^s_x$ intersects each of $Y_0$ and $Y_1$ in precisely one point. The stable holonomy $H : Y_0 \to Y_1$ is given by defining $H(x)$ to be the intersection point of $W^s_x$ with $Y_1$.

Theorem 2.6. The stable holonomy $H : Y_0 \to Y_1$ is absolutely continuous. That is, $m_1 \ll H^* m_0$ where $m_i$ is Lebesgue measure on $Y_i$, $i = 0, 1$. Moreover, the Jacobian $JH : Y_0 \to \mathbb{R}$ given by

$$JH(x) = \frac{dm_1}{dH m_0}(Hx) = \lim_{r \to 0} \frac{m_1(H(B(x, r)))}{m_0(B(x, r))}, \quad x \in Y_0,$$

is bounded above and below and is $C^\varepsilon$ for some $\varepsilon > 0$.

Proof. This essentially follows from [18, Theorems 8.6.1 and 8.6.13]; see [12, Theorem 6.3]. □

Hence, we can assume without loss of generality, that there exists a foliation $W^s$ of $U_0$, which continuously extends the stable lamination of $\Lambda$ together with a positively invariant field of cones $(C^s_x)_{x \in U_0}$ on $T_{U_0}M$. Moreover, the Jacobian of holonomies along contracting leaves on cross-sections of singular-hyperbolic attracting sets in our setting is a Hölder function.

2.2. Global Poincaré return map. In [17] the construction of a global Poincaré map for any singular-hyperbolic attractor is carried out based on the existence of “adapted cross-sections” and Hölder-$C^1$ stable holonomies on these cross-sections. With the results just presented this construction can be performed for any singular-hyperbolic attracting set.

This construction was presented in [12, Sections 3 and 4]: we obtain

- a finite collection $\Xi = \Sigma_1 \cup \cdots \cup \Sigma_m$ of cross-sections to $X$ so that
  - each $\Sigma_i$ is diffeomorphically identified with $(-1, 1) \times D^{d_s}$; and
  - the stable boundary $\partial^s \Sigma_i \cong \{\pm 1\} \times D^{d_s}$ consists of two curves contained in stable leaves;
  - each $\Sigma_i$ is foliated by $W^s_x(\Sigma_i) = \bigcup_{t \leq t_0} X^t(W^s_x) \cap \Sigma_i$ for a small fixed $\varepsilon > 0$.
    We denote this foliation by $W^s_x(\Sigma_i), i = 1, \ldots, m$;

- a Poincaré map $R : \Xi \setminus \Gamma \to \Xi$ which is $C^2$ smooth in $\Sigma_i \setminus \Gamma, i = 1, \ldots, m$; preserves the foliation $W^s(\Xi)$ and a big enough time $T > 0$, where $\Gamma = \Gamma_0 \cup \Gamma_1$ is a finite family of stable disks $W^s_x(\Xi)$ so that
- \( \Gamma_0 = \{ x \in \Xi : X^{T+1}(x) \in \bigcup_{\sigma \in S} \gamma_0^s \} \) for \( S = S(X, \Lambda) = \{ \sigma \in \Lambda : X(\sigma) = \bar{0} \} \) and \( \gamma_0^s \) is the local stable manifold of \( \sigma \) in a small fixed neighborhood of \( \sigma \in S \); and
- \( \Gamma_1 = \{ x \in \Xi : R(x) \in \partial^s \Xi = \cup_i \partial^s \Sigma_i \} \)
- and an open neighborhood \( V_0 \) of \( \Xi \) of so that every orbit of a regular point \( z \in U_0 \setminus V_0 \) eventually hits \( \Xi \) or else \( z \in \Gamma \).

Having this, the same arguments from \cite{18} (see \cite{19} Proposition 4.1 and Theorem 4.3) show that \( DR \) contracts \( T_\Xi W^s(\Xi) \) and expands vectors on the unstable cones \( \{ C_x^u(\Xi) = C_{xu}(a) \cap T_x \Xi \}_{x \in \Xi} \). The stable holonomies for \( R \) enable us to reduce its dynamics to a one-dimensional map, as follows.

Let \( \gamma_0, \gamma_1 \subset \Xi \) be two curves such that \( \gamma_i \in C_x^u \), \( i = 0, 1 \) and for all \( x \in \gamma_0 \), the stable leaf \( W_x^s(\Xi) \) intersects each of \( \gamma_0 \) and \( \gamma_1 \) in precisely one point. The (sectional) stable holonomy \( h : \gamma_0 \to \gamma_1 \) is defined by setting \( h(x) \) to be the intersection point of \( W_x^s(\Xi) \) with \( \gamma_1 \).

**Lemma 2.7.** The stable holonomy \( h \) is \( C^{1+\epsilon} \) for some \( \epsilon > 0 \).

**Proof.** See \cite{19} Lemma 7.1 \( \square \)

Following the same arguments in \cite{18} (see also \cite{19} Section 7) we obtain a one-dimensional piecewise \( C^{1+\alpha} \) quotient map over the stable leaves \( f : I \setminus D \to I \) for some \( 0 < \alpha < 1 \) so that \( h \circ R = f \circ h \) and \( |f'| > 2 \), where \( D = h(\Gamma) \) is a finite set of points. In addition, as shown in \cite{19} Proof of Corollary 8.4, \( |f'| \) behaves near singular points \( \mathcal{S} = h(\Gamma_0) \subset D \) as a power of the distance to \( \mathcal{S} \), as in assumption (2) of the statement of \cite{20}.

This construction can be summarized as in \cite{21} Theorem 5 as follows, with adaptations to our more general setting: items (1-5) can be found in \cite{18} but item (6), which is crucial for us, will be obtained in Remark \ref{c16} following Corollary \ref{c15} in Subsection \ref{c43}. This is the only argument where the assumption of connectedness is used; see \cite{11} 12.

In what follows, we say that a function \( \varphi : \Xi_0 = \Xi \setminus \Gamma \to \mathbb{R} \) has logarithmic growth near \( \Gamma \) if there is \( K = K(\varphi) > 0 \) so that \( |\varphi|_{\chi_B(\Gamma, \delta)} \leq K \Delta_\delta \circ p \) for every small enough \( \delta > 0 \).

**Theorem 2.8.** \cite{21} Theorem 5, Section 4, p 1021] For a \( C^2 \) vector field \( X \) on a compact manifold having a connected singular hyperbolic attracting set \( \Lambda \), there exists \( \alpha > 0 \) and a finite family \( \Xi \) of adapted cross-sections and a global Poincaré map \( R : \Xi_0 \to \Xi \), \( R(x) = X^\tau(x)(x) \) such that

1. the domain \( \Xi_0 = \Xi \setminus \Gamma \) is the entire cross-sections with a family \( \Gamma \) of finitely many smooth arcs removed and
   a. \( \tau : \Xi_0 \to [\tau_0, +\infty) \) is a smooth function with logarithmic growth near \( \Gamma \) and bounded away from zero by some uniform constant \( \tau_0 > 0 \);
   b. there exists a constant \( \kappa > 0 \) so that \( |\tau(y) - \tau(w)| < \kappa \) \( \text{dist}(y, w) \) for all points \( w \in W^s(y, \Xi) \) in the stable leaf through a point \( y \) inside a cross-section of \( \Xi \);
    2. We can choose coordinates on \( \Xi \) so that the map \( R \) can be written as \( F : \hat{Q} \to \hat{Q}, F(x, y) = (f(x), g(x, y)) \), where \( Q = I \times I, I = [0, 1] \) and \( \hat{Q} = Q \setminus \Gamma_0 \) with \( \Gamma_0 = D \times I \) and \( D = \{ c_1, \ldots, c_n \} \subset I \) a finite set of points.
(3) The map \( f : I \setminus D \to I \) is \( C^{1+\alpha} \) piecewise monotonic with \( n+1 \) branches defined on the connected components of \( I \setminus D \) and has a finite set of a.c.i.m., \( \mu_f^i \). Also \( \inf |Df| > 2 \) where it is defined, \( 1/|Df| \) has universal bounded \( p \)-variation and then \( d\mu_f^i/dm \) has bounded \( p \)-variation for some \( p > 0 \) (see [38] for the definition of \( p \)-variation).

(4) The map \( g : \tilde{Q} \to I \) preserves and uniformly contracts the vertical foliation \( \mathcal{F} = \{ \{x\} \times I \}_{x \in I} \) of \( Q \): there exists \( 0 < \lambda < 1 \) such that \( \text{dist}(g(x,y_1),g(x,y_2)) \leq \lambda \cdot |y_1 - y_2| \) for each \( y_1, y_2 \in I \).

(5) The map \( F \) admits a finite family of physical probability measures \( \mu_F^i \) which are induced by \( \mu_f^i \) in a standard way. The Poincaré time \( \tau \) is integrable both with respect to each \( \mu_f^i \) and with respect to the two-dimensional Lebesgue area measure of \( Q \).

(6) The set \( D \) splits into \( S \) and \( D \setminus S \), where \( S \) is the nonempty singularity set, satisfying

(a) there exists the lateral limits \( \lim_{t \to b^\pm} f(t) = f(b^\pm), \forall b \in D; \)
(b) \( \forall b^\pm \in D \exists T(b^\pm) \in \mathbb{Z}^*_+ : f^{T(b^\pm)}(b^\pm) \in S; \)
(c) \( b^\pm \in S \iff \exists 0 < \alpha(b^\pm) < 1 : 0 < \lim_{t \to b^\pm} |t - b|^{1-\alpha(b^\pm)} \cdot |Df(t)| < \infty; \)
(d) \( b^\pm \in D \setminus S \iff \exists \lim_{t \to b^\pm} |Df(t)|. \)

Remark 2.9. Due to the dimension and codimension of \( D \) as a submanifold of the quotient \( M = \Xi/W^s(\Xi) \) together with logarithmic growth of \( \tau \) near \( \Gamma \), there exists \( C_d, d > 0 \) such that for all small \( \rho > 0 \)

\[ \text{Leb}\{x \in M : \text{dist}(x,D) < \rho\} \leq C_d \rho^d, \]

that is, the Lebesgue measure of neighborhoods of \( D \) is comparable to a power of the distance to \( D \). In our sectional-hyperbolic case \( d = 1 \).

2.3. Equivalence between SRB/physical measure and equilibrium state. We now prove Theorem 1.7 showing that in singular-hyperbolic attracting sets for a \( C^2 \) smooth flow we can characterize physical/SRB measures in the same way as in hyperbolic attracting sets.

Proof of Theorem 1.7. Let \( \mu \) be a \( X \)-invariant probability measure supported in the singular-hyperbolic attracting set \( \Lambda \).

We start by recalling that from Theorem 2.8 we can follow [17] Sections 6-8] to obtain (1.3).

We show the implication (3) \( \implies \) (1) first.

We note that if the basin \( B(\mu) \) of \( \mu \) has positive Lebesgue measure, then by invariance \( B(\mu) \cap U \) must have positive Lebesgue measure. So we get a Lebesgue modulo zero decomposition \( B(\mu) \cap U = U \cap (\cup_{i=1}^k B(\mu) \cap B(\mu_i)) \). By definition of physical measure, this
means that for each continuous observable \( \varphi : U \to \mathbb{R} \)

\[
\int \varphi \, d\mu = \frac{1}{\text{Leb}(U)} \int_U \int \varphi \left( \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x} \right) \, d\text{Leb}(x) = \sum_{i=1}^{k} \frac{\text{Leb}(B(\mu) \cap B(\mu_i) \cap U)}{\text{Leb}(U)} \int \varphi \, d\mu_i,
\]

where the limit above is in the weak* topology of the probability measures of the manifold. Hence, we obtain \( \mu = \sum_{i=1}^{k} \frac{\text{Leb}(B(\mu) \cap B(\mu_i) \cap U)}{\text{Leb}(U)} \mu_i \) and \( \mu \) is a convex linear combination of the ergodic physical/SRB measures provided by Theorem 1.7 with coefficients \( \alpha_i = \frac{\text{Leb}(B(\mu) \cap B(\mu_i) \cap U)}{\text{Leb}(U)} \). In particular, \( \mu \) is an equilibrium state since

\[
h_\mu(X_1) = \sum_{i=1}^{k} \alpha_i h_{\mu_i}(X_1) = \sum_{i=1}^{k} \alpha_i \int \log |\det D_X_1|_{E^c} \, d\mu_i
\]

\[
= \int \log |\det D_X_1|_{E^c} \, d\mu > 0.
\]

This completes the proof that \((3) \implies (1)\).

Next we prove that \((1) \implies (2) \implies (3)\). The assumption (1) implies, by the work of Ledrappier [30], that \( \mu \) is a SRB measure: \( \mu \) has absolutely continuous disintegration along unstable manifolds \( W^u(x) \) for \( \mu \)-a.e. \( x \in \Lambda \). Hence, by invariance of the unstable manifolds and smoothness of the flow, we see that \( \mu \) has absolutely continuous disintegration along the central-unstable manifolds \( W^{cu}(x) \) for \( \mu \)-a.e. \( x \), where \( W^{cu}(x) = \bigcup_{t \in \mathbb{R}} X_t W^u(x) \). This shows that \((1) \implies (2)\).

Let \( \Lambda_1 \) be a full \( \mu \)-measure subset of \( \Lambda \) where the previous absolutely continuous disintegration property holds.

Now since \( \Lambda \) is a attracting set, then \( \Lambda \) contains all such unstable manifolds and so all the central-unstable manifolds \( W^{cu}(x) \) for \( x \in \Lambda_1 \). We recall that \( W^{cu}(x) \) is tangent to \( E^c(x) \) at \( x \in \Lambda_1 \).

In addition, because \( \Lambda \) has a partially hyperbolic splitting, every point \( y \) of \( \Lambda \) admits a stable manifold \( W^s(y) \) which is tangent to \( E^s(y) \) at \( y \). Thus \( W^s(y) \) are transverse to \( W^{cu}(x) \) for all \( y \in W^{cu}(x) \) and \( x \in \Lambda_1 \).

Moreover, the future time averages of \( z \in W^s(y) \) and \( y \) are the same for all continuous observables. Also, the future time averages of \( y \in A_x \subset W^u(x) \) and \( x \) are the same for all continuous observables and some subset \( A_x \) of \( W^{cu}(x) \) with positive area (by the absolutely continuous disintegration) for each \( x \in \Lambda_1 \). Hence the subset

\[
B = \{ W^s(y) : y \in A_x, x \in \Lambda_1 \}
\]

is contained in the ergodic basin of the measure \( \mu \).

For \( C^2 \) smooth flows it is well-known that the strong-stable foliation \( \{ W^s(x) \}_{x \in \Lambda} \) is an absolutely continuous foliation of \( U \), that is, in particular, the set \( B \) has positive Lebesgue measure. Hence \( \text{Leb}(B(\mu)) \geq \text{Leb}(B) > 0 \) and \( \mu \) is a physical measure.
This shows that \( (2) \implies (3) \) and completes the proof of the equivalence between conditins \( (1)-(3) \). Finally, the characterization of \( \mathbb{E} \) is a consequence of this equivalence together with the argument \( (3) \implies (1) \).

\[ \square \]

2.4. **Density of stable manifolds of singularities.** The following is essential to obtain the property in item (6) of Theorem 2.8.

From Subsection 2.2 the one-dimensional map \( f : I \setminus D \to I \) can be written \( f : \cup I_j \to I := \cup I_j \), where the \( I_j \) are the finitely many connected components of \( I \setminus D \) (that is, open subintervals).

2.4.1. **Topological properties of the dynamics of \( f \).** The following provides the existence of a special class of periodic orbits for \( f \).

**Proposition 2.10.** Let \( f : \cup I_j \to I \) be a piecewise \( C^1 \) expanding map with finitely many branches \( I_1, \ldots, I_l \) such that each \( I_j \) is a non-empty open interval, \(|Df|_{I_j}| \geq \sigma > 2\) and \( I \setminus (\cup I_j) \) is finite.

Then for each small \( \delta > 0 \) there exists \( n = n(\delta) \) such that, for every non-empty open interval \( J \subset \cup I_j \) with \(|J| \geq \delta\), we can find \( 0 \leq k \leq n \), a sub-interval \( \hat{J} \) of \( J \) and \( 1 \leq j \leq l \) satisfying

\[ f^k | \hat{J} : \hat{J} \to I_j \text{ is a diffeomorphism.} \]

In addition, \( f \) has finitely many distinguished periodic orbits \( O(p_1), \ldots, O(p_k) \) contained in \( \cup I_j \), and every non-empty open interval \( J \) admits an open sub-interval \( \hat{J} \), a periodic point \( p_j \) and an iterate \( n \) such that \( f^n | \hat{J} \) is a diffeomorphism onto a neighborhood of \( p_j \).

**Proof.** See [16, Lemma 6.30].

**Remark 2.11.** (1) For the bidimensional map \( F \) this shows that there are finitely many periodic orbits \( O(P_1), \ldots, O(P_k) \) for \( F \) so that \( \pi(O(P_i)) = O(p_i), i = 1, \ldots, k \), where \( \pi : Q \to I \) is the projection on the first coordinate. Moreover, the union of the stable manifolds of these periodic orbits is dense in \( Q \). See [16] Section 6.2] for details.

(2) This also implies that the stable manifolds of the periodic orbits \( P_i \) obtained above are dense in a neighborhood \( U_0 \) of \( \Lambda \).

Indeed, we can write the flow \( X_t \) on a neighborhood of \( \Lambda \) as a suspension flow over \( F \); see [17]. Then the orbit of each \( P_i \) is periodic and hyperbolic and \( W^s_G(P_i) \) is the suspension of \( W^s_F(P_i) \). Therefore, the density of \( \cup W^s_G(P_i) \) in \( Q \) implies the density of \( \cup W^s_G(P_i) \) in a neighborhood \( U_0 \) of \( \Lambda \).

2.4.2. **Ergodic properties of \( f \).** The map \( f \) is piecewise expanding with Hölder derivative which enables us to use strong results on one-dimensional dynamics.

**Existence and finiteness of acims.** It is well known [35] that \( C^1 \) piecewise expanding maps \( f \) of the interval such that \( 1/|Df| \) has bounded variation have absolutely continuous invariant probability measures whose basins cover Lebesgue almost all points of \( I \).
Using an extension of the notion of bounded variation this result was extended in [38] to $C^1$ piecewise expanding maps $f$ such that $g = 1/|f'|$ is $\alpha$-Hölder for some $\alpha \in (0, 1)$. In addition from [38, Theorem 3.3] there are finitely many ergodic absolutely continuous invariant probability measures $\nu_1, \ldots, \nu_l$ of $f$ and every absolutely continuous invariant probability measure $\nu$ decomposes into a convex linear combination $\nu = \sum_{i=1}^l a_i \nu_i$. From [38, Theorem 3.2] considering any subinterval $J \subset I$ and the normalized Lebesgue measure $\lambda_J = (\lambda | J) / \lambda(J)$ on $J$, every weak* accumulation point of $n^{-1} \sum_{j=0}^{n-1} f^n / (\lambda_J)$ is an absolutely continuous invariant probability measure $\nu$ for $f$ (since the indicator function of $J$ is of generalized $1/\alpha$-bounded variation). Hence the basin of the $\nu_1, \ldots, \nu_l$ cover $I$ Lebesgue modulo zero: $\lambda(I \setminus (B(\nu_1) \cup \cdots \cup B(\nu_l))) = 0$.

Note that from [38, Lemma 1.4] we also know that the density $\varphi$ of any absolutely continuous $f$-invariant probability measure is bounded from above.

Absolutely continuous measures and distinguished periodic orbits. Now we relate some topological and ergodic properties.

**Lemma 2.12.** For each distinguished periodic orbit $\mathcal{O}(p_i)$ of $f$ there exists some ergodic absolutely continuous $f$-invariant probability measure $\nu_j$ such that $p_i \in \text{supp} \, \nu_j$, and vice-versa.

**Proof.** Define $E = \{1 \leq i \leq k : \exists 1 \leq j \leq l \text{ s.t. } \mathcal{O}(p_i) \in \text{supp} \, \nu_j \}$. Note that since $\text{int}(\text{supp} \, \nu_1)$ is nonempty, then for an interval $J \subset \text{int}(\text{supp} \, \nu_1)$ we can by Proposition 2.10 find another interval $\hat{J} \subset J$ and $n > 1$ so that $f^n | \hat{J} : \hat{J} \to V(p_i)$ is a diffeomorphism to a neighborhood $V(p_i)$ of $p_i$, for some $1 \leq i \leq k$. The invariance of $\text{supp} \, \nu_1$ shows that $p_i \in \text{int}(\text{supp} \, \nu_1)$ and $E \neq \emptyset$.

We set $B = \{1, \ldots, k\} \setminus E$ and show that $B = \emptyset$. For that, we write $i \to j$ if the preorbit of $\mathcal{O}(p_i)$ accumulates $\mathcal{O}(p_j)$.

**Claim 2.13.** If $i \to j$ and $j \in E$, then $i \in E$.

Hence orbits in $E$ cannot link to orbits in $E$. Since the union of the preorbits of distinguished periodic orbits of $f$ are dense in $I$, then $B$ can only be accumulated by preorbits of elements of $B$. Thus, the union $W$ of the preorbits of the elements of $B$ is $f$-invariant and dense in a neighborhood of the orbits of the elements of $B$. Therefore, $\overline{W}$ is a compact $f$-invariant set with nonempty interior of $I$ and so $\overline{W}$ contains the support of some $\nu_j$. Consequently, the preorbit of some element of $B$ intersects $\text{int}(\text{supp} \, \nu_j)$ and so $B \cap E \neq \emptyset$.

This contradiction proves that $B$ must be empty, except for the proof of the claim.

**Proof of Claim 2.13.** There exists $x_n \xrightarrow{n \to \infty} p_i$ so that $x_n \in \bigcup_{m \geq 0} f^{-m} p_j$ and then we can find $V_n$ neighborhood of $x_n$ and $m_n > 1$ such that $f^{m_n} \mid_{V_n} : V_n \to V(p_i)$ is a diffeomorphism onto a neighborhood $V(p_j)$ of $p_j$.

But $V(p_j) \cap \text{supp} \, \nu_h \neq \emptyset$ for some $1 \leq h \leq l$ and so, by invariance of $\text{supp} \, \nu_h$, there are points of $\text{supp} \, \nu_h$ in $V_n$, for all $n \geq 1$. This shows that $p_i$ is a limit point of $\text{supp} \, \nu_h$, and so $i \in E$. This proves the claim and finishes the proof of the lemma. 

\qed
2.4.3. Stable manifolds of singularities. We are now ready to obtain the property in item (6) of Theorem 2.8.

**Theorem 2.14.** The union of the stable manifolds of the singularities in a connected singular-hyperbolic attracting set is dense in the topological basin of attraction, that is

\[ U_0 \subset \bigcup_{\sigma \in \Lambda \cap S(G)} W^s(\sigma) \]

One important consequence is the possibility of choosing adapted cross-sections with a special feature crucial to obtain item (6) of the statement of Theorem 2.8.

**Corollary 2.15.** Every regular point of a connected singular-hyperbolic attracting set admits cross-sections with arbitrarily small diameter whose stable boundary is formed by stable manifolds of singularities of the set, for every small enough \( \delta > 0 \).

**Proof.** Since there is a dense subset of stable leaves in \( U_0 \) that are part of \( W^s(S) = \bigcup_{\sigma \in S} W^s(\sigma) \), we can choose a cross-section \( \Sigma \) to \( X \) at any point \( x \in U_0 \) with diameter as small as we like having a stable boundary contained in \( W^s(S) \). \( \square \)

**Remark 2.16.** As a consequence of Corollary 2.15, the one-dimensional map \( f \) satisfies item (6) of Theorem 2.8.

Indeed, points in the stable manifold of a singularity \( \sigma \in \Lambda \) are sent in finite positive time by the flow to the local stable manifold of the singularity in a cross-section close to the singularity. For the one-dimensional map the corresponding behavior is precisely given by item (3) of Theorem 2.8.

Items (1-2) are consequences of the properties of \( f \); see [12, Proof of Corollary 8.4] for more details. Moreover, we can assume that all singular points of \( f \) are related to Lorenz-like singularities, after Remark 1.5.

2.4.4. Transversal intersection between unstable manifolds of periodic orbits and stable manifolds of singularities. To finish the proof of Theorem 2.14 we use non-uniform hyperbolic theory.

**Theorem 2.17.** Let \( \mu \) be an ergodic \( f \)-invariant hyperbolic probability measure supported in a connected singular-hyperbolic attracting set \( \Lambda \).

Let us assume that \( \mu \) is a u-Gibbs state, that is, for \( \mu \)-a.e. \( x \) the unstable manifold \( W^u_x \) is well-defined and Lebesgue-a.e. \( y \in W^u_x \) is \( \mu \)-generic: \( \frac{1}{T} \int_0^T \delta_{X,y} ds \xrightarrow{T \to \infty} \mu \).

Then there exists \( \sigma \in S \) such that \( W^u(p) \cap W^s(\sigma) \neq \emptyset \) for every \( p \in \text{supp} \mu \).

We can now present

**Proof of Theorem 2.14.** This theorem really is a corollary of Theorem 2.17 since we already know that the stable manifolds of the distinguished periodic orbits are dense in a neighborhood \( U_0 \) of \( \Lambda \); see Remark 2.11(2). The transverse intersection provided by Theorem 2.17 ensures, through the Inclination Lemma, that each of these stable manifolds is accumulated the stable manifold of some singularity, and the statement of Theorem 2.14 follows. \( \square \)
The proof of Theorem 2.17 is based on a few results.

**Lemma 2.18.** In the setting of the statement of Theorem 2.17, fix \( p_0 \in \text{Per}(X) \cap \text{supp} \mu \) and let \( J = [a,b] \) be an arc on a connected component of \( W^{su}_J(p_0) \setminus \{p_0\} \) with \( a \neq b \). Then \( H = \bigcup_{t>0} X^t(J) \) contains some singularity of \( \Lambda \).

**Proof.** It is well-known from the non-uniform hyperbolic theory (Pesin’s Theory) that the support of a non-atomic hyperbolic ergodic probability measure \( \mu \) is contained in a homoclinic class of a hyperbolic periodic orbit \( \mathcal{O}(p) \); see e.g. [37] Appendix or [19].

Hence, for \( \mu \)-a.e. \( x \) we have \( W^u_x \subset \text{supp} \mu \) (since \( \mu \) is a \( u \)-Gibbs measure) and \( W^u_x \cap W^s(\mathcal{O}(p)) \neq \emptyset \). Thus by the Inclination Lemma (see [51]) we have \( W^u(p) \subset W^u(x) \subset \text{supp} \mu \).

Since every periodic point \( p_0 \in \text{supp} \mu \) is homoclinically related to \( p \) (that is, \( W^s(p) \cap W^u(p_0) \neq \emptyset \)), we also have \( W^{su}(p_0) \subset W^u(p_0) \subset \text{supp} \mu \).

Note that \( H \subset W^u(p_0) \subset \text{supp} \mu \) and \( H \) is a compact invariant set by construction, where \( W^u(p_0) \) is the connected component of \( W^u(p_0) \setminus \mathcal{O}(p_0) \) containing \( J \). In addition, \( H \) is clearly connected, since \( H \) is also the closure of the orbit of the connected set \( J \) under a continuous flow.

If \( H \) has no singularities, then \( H \) is a compact connected hyperbolic set and so contains the strong-unstable manifolds through any of its points, since every point in \( H \) is accumulated by forward iterates of the arc \( J \). This means that \( H \) is an attracting set and so \( H = \Lambda \) by connectedness, and \( H \) contains all singularities of \( \Lambda \). This contradiction proves that \( H \) must contain a singularity. \( \square \)

Fix \( p_0 \) and \( \sigma \in S \cap H \) as in the statement of Lemma 2.18. We have shown that there exists \( \sigma \in \text{supp} \mu \cap S \) so that \( \sigma \in \overline{W^u(p_0)} \). We assume that \( J \) is a fundamental domain for \( W^u(p_0) \), that is, \( b = X^T(a) \) with \( T > 0 \) the first return time of the orbit of \( a \) to \( W^{su}(p_0) \), i.e., \( X^t(a) \notin W^{su}(p_0) \) for all \( 0 < t < T \). We now argue just as in [16] Section 6.3.2, pp 199-202] and show that there exists some singularity whose stable manifold transversely intersects \( J \).

This is enough to conclude the proof of Theorem 2.17. Indeed, since all periodic orbits in \( \text{supp} \mu \) are homoclinically related, it is enough to obtain \( W^u(p_0) \cap W^s(\sigma) \neq \emptyset \) for a distinguished periodic point \( p_0 \in \text{supp} \mu \).

To complete the argument, since in [16] Section 6.3.2] it was assumed that \( \Lambda \) was either a singular-hyperbolic attractor or attracting set with dense periodic orbits for a 3-vector field, we state [16] Lemma 6.49 in our setting.

**Lemma 2.19.** Let \( \tilde{\Sigma} \) be a cross-section of \( X \) containing a compact \( cu \)-curve \( \zeta \), which is the image of a regular parametrization \( \zeta : [0,1] \rightarrow \Sigma \), and assume that \( \zeta \) is contained in \( \text{supp} \mu \). Let \( \Sigma \) be another cross-section of \( X \). Suppose that \( \zeta \) falls off \( \Sigma \), that is

1. the positive orbit of \( \zeta(t) \) visits \( \text{int}(\Sigma) \) for all \( t \in [0,1] \);
2. and the \( \omega \)-limit of \( \zeta(1) \) is disjoint from \( \Sigma \).

Then \( \zeta(1) \) belongs either to the stable manifold of some periodic orbit \( p \) in \( \text{supp} \mu \), or to the stable manifold of some singularity.
Proof. Just follow the same arguments in the proof of [16, Lemma 6.49] since the proof assumes that stable manifolds of the flow intersected with cross-sections disconnect the cross-sections (that is, the transverse intersection is a hypersurface inside the cross-section); and either the existence of a dense regular orbit, or the denseness of periodic orbits, each of which is true in the invariant subset supp $\mu$ in our setting. □

3. Dimensional reduction of large deviations subset

Here we explain how to use the representation of the global Poincaré map obtained in Subsection 2.2 to reduce the problem of estimating an upper bound for the large deviations subset of the flow to a similar problem for a expanding quotient map on the base dynamics of a suspension semiflow, in the setting of Theorem 2.8 assuming exponentially slow recurrence to the subset $\mathcal{D}$ as in Theorem C.

We start by representing the flow as a suspension semiflow over the global Poincaré map constructed in Section 2 to reduce the large deviations subset of a continuous bounded observable to a similar large deviations subset of an induced observable for the dynamics of $F$ and its quotient $f$ over stable leaves. Then we use the uniform expansion of $f$ and assume exponentially slow recurrence to a singular subset to deduce exponential decay of large deviations for continuous observables on a neighborhood of the attracting set.

3.1. Reduction to the global Poincaré map and quotient along stable leaves. Let $\phi^t : \Xi^\tau_0 \to \Xi^\tau_0$ denote the suspension semiflow with roof function $\tau$ and base dynamics $F$, where $F$ and $\tau$ satisfying the properties stated in Theorem 2.8.

More precisely, we assume that

\[(P1) \quad \tau \text{ grows as } |\log \text{dist} (\cdot, \mathcal{D})|: \text{ the roof function } \tau \text{ has logarithmic growth near } \mathcal{D}; \]

is uniformly bounded away from zero $\tau \geq \tau_0 > 0$;

and set $\Xi^\tau = \{(x, y) \in \Xi \times [0, +\infty) : 0 \leq y < \tau(x)\}$ and for $x = x_0 \in \Xi$ denote by $x_n$ the $n$th iterate $F^n(x_0)$ for $n \geq 0$. Then for each pair $(x_0, s_0) \in \Xi^\tau$ and $t > 0$ there exists a unique $n \geq 1$ such that $S_n\tau(x_0) \leq s_0 + t < S_{n+1}\tau(x_0)$ and we define $\phi^t(x_0, s_0) = (x_n, s_0 + t - S_n\tau(x_0))$.

For each $F$-invariant physical measure $\mu^i_F, i = 1, \ldots, k$ from Theorem 2.8 we denote by $\mu^i = \mu^i_F \times \lambda$ the natural $\phi^t$-invariant extension of $\mu^i_F$ to $\Xi^\tau$ and by $\lambda^\tau$ the natural extension of Leb induced on $\Xi$ to $\Xi^\tau$, i.e. $\lambda^\tau = \text{Leb} \times \lambda$, where $\lambda$ is one-dimensional Lebesgue measure on $\mathbb{R}$: for any subset $A \subset \Xi^\tau$ and $\chi_A$ its characteristic function

\[
\mu^i(A) = \frac{1}{\mu^i_F(\tau)} \int d\mu^i_F(x) \int_0^{\tau(x)} ds \chi_A(x, s), \quad \text{and} \quad \lambda^\tau(A) = \frac{1}{\text{Leb}(\tau)} \int d\text{Leb}(x) \int_0^{\tau(x)} ds \chi_A(x, s).
\]

From the construction of $F$ from the proof of Theorem 2.8 we see that the map $\Psi : \Xi^\tau \to M, (x, s) \mapsto X^s(x)$ is a finite-to-1 locally $C^2$ smooth semiconjugation $\Psi \circ \phi^t = X^t \circ \Psi$ for all $t > 0$ so that we can naturally identify $\Psi_*(\mu^i) = \mu_i$, where $\mu_i$ are the physical measures.
supported on the singular-hyperbolic attracting set given by Theorem 1.7. In particular we get \( \Psi_*(\lambda^\tau) \leq \ell \cdot \text{Leb} \) where \( \ell \) is the maximum number of preimages of \( \Psi \).

3.1.1. *The quotient map.* Let \( \Xi \) be a compact metric space, \( \Gamma \subset \Xi \) and \( F : (\Xi \setminus \Gamma) \to \Xi \) be a measurable map. We assume that there exists a partition \( \mathcal{F} \) of \( \Xi \) into measurable subsets, having \( \Gamma \) as an element, which is

\[
\begin{align*}
\mathbf{(P2)} & \text{ invariant: the image of any } \xi \in \mathcal{F} \text{ distinct from } \Gamma \text{ is contained in some element } \eta \text{ of } \mathcal{F}; \\
\mathbf{(P3)} & \text{ contracting: the diameter of } F^n(\xi) \text{ goes to zero when } n \to \infty, \text{ uniformly over all the } \xi \in \mathcal{F} \text{ for which } F^n(\xi) \text{ is defined.}
\end{align*}
\]

We denote \( p : \Xi \to \mathcal{F} \) the canonical projection, i.e. \( p \) assigns to each point \( x \in \Xi \) the atom \( \xi \in \mathcal{F} \) that contains it. By definition, \( A \subset \mathcal{F} \) is measurable if and only if \( p^{-1}(A) \) is a measurable subset of \( \Xi \) and likewise \( A \) is open if, and only if, \( p^{-1}_\Sigma(A) \) is open in \( \Xi \). The invariance condition means that there is a uniquely defined map

\[
f : (\mathcal{F} \setminus \{\Gamma\}) \to \mathcal{F} \quad \text{such that} \quad f \circ p = p \circ F.
\]

Clearly, \( f \) is measurable with respect to the measurable structure we introduced in \( \mathcal{F} \). We assume from now on that the leaves are sufficiently regular so that

\[
\mathbf{(P4)} \text{ regular quotient: the quotient } M = \Xi/\mathcal{F} \text{ is a compact finite dimensional manifold with the topology induced by the natural projection } p \text{ and } \lambda = p, \text{ Leb is a finite Borel measure.}
\]

It is well-known (see e.g. \cite[Section 6]{17}) that each \( F \)-invariant probability measure \( \mu_F \) is in one-to-one correspondence with the \( f \)-invariant probability measure \( \mu_f \) by \( p_* \mu_f = \mu_f \) and this map preserves ergodicity. We also need

\[
\mathbf{(P5)} \text{ uniform expansion and non-degenerate singular set: the quotient map } f \text{ is uniformly expanding: there are } \sigma > 2 \text{ and } q \in \mathbb{Z}^+, q \geq 2 \text{ so that } f \text{ is expanding with rate } \|Df^{-1}\| < \sigma^{-1} \text{ and number of pre-images of a point (degree) bounded by } q; \text{ also } p(\Gamma) \text{ is a non-degenerate singular set for } f.
\]

\[
\mathbf{(P6)} \text{ integrability: } \tau \text{ satisfies conditions (1b) and (5) from the statement of Theorem 2.8;}
\]

\[
\mathbf{(P7)} \text{ measure of singular neighborhoods: we have the statement of Remark 2.9 with } \lambda \text{ and } p(\Gamma) \text{ is the place of Leb and } \mathcal{D} \text{ respectively.}
\]

Moreover, we identify the equilibrium states \( \mathbb{E} \) for \( \log J \) with \( \Psi_\tau \mathbb{E} \). In addition, the ergodic physical/SRB measures that are the extremes points of \( \mathbb{E} \) are naturally induced uniquely by ergodic physical measures for \( F \) which, in turn, are also related to a unique absolutely continuous ergodic invariant probability measure for \( f \). We denote in what follows \( \mathbb{E}_F \) and \( \mathbb{E}_f \) to be the convex hull of these ergodic measures with respect to \( F \) and \( f \), respectively; and note that \( p_* \mathbb{E}_F = \mathbb{E}_f \).

3.1.2. *Exponentially slow recurrence for the suspension flow.* In the rest of this section we prove the following.
Theorem 3.1. Let $\phi^t : \Xi^0_0 \rightarrow \Xi^0_0$ be the suspension semiflow with roof function $\tau$ and base dynamics $F$, where $F$ and $\tau$ satisfy conditions (P1)-(P7) stated above. Let the quotient map $\phi$ have exponentially slow recurrence to the finite subset $D$; set $\mathbb{E}$ to be the family of all measures that are sent into equilibrium states of $X$ for $\log J$ on $\Lambda$; and let $\psi : \Xi^r \rightarrow \mathbb{R}$ be a bounded uniformly continuous observable. Then, for any given $\epsilon > 0$

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \log \lambda^r \left\{ z \in \Xi^r : \inf_{\mu \in \mathbb{E}} \frac{1}{T} \int_0^T \psi(\phi^t(z)) dt - \mu(\psi) < \epsilon \right\} < 0;$$

This result proves Theorem A as soon as we prove exponentially slow recurrence to $D$ for the quotient base map $f$; this is Theorem C to be proved in Section 4.

Remark 3.2. We assumed that $D$ is finite in several places along the following argument. This assumption is not needed in Theorem C. However this is a natural assumption for the quotient map induced from singular-hyperbolic attracting sets.

The proof of this result is based on the observation that, for a continuous function $\psi : \Xi^r \rightarrow \mathbb{R}, T > 0, z = (x, s) \in \Xi^r$ we have

$$\int_0^T \psi(\phi^t(z)) dt = \int_s^{\tau(x)} \psi(\phi^t(x, 0)) dt + \sum_{j=1}^{n-1} \int_0^{\tau(F^j(x))} \psi(\phi^t(F^j(x), 0)) dt + \int_0^{T+s-S_n \tau(x)} \psi(\phi^t(F^n(x), 0)) dt$$

where $n = n(x, s, T)$ is the lap number so that $0 \leq T + s - S_n^F \tau(x) < \tau(F^n(x))$. So setting $\varphi(x) = \int_0^{\tau(x)} \psi(\phi^t(x, 0)) dt$ we obtain

$$\frac{1}{T} \int_0^T \psi(\phi^t(z)) dt = \frac{1}{T} S_n^F \varphi(x) + I(x, s, T)$$

where

$$I = I(x, s, T) = \frac{1}{T} \left( \int_0^{T+s-S_n \tau(x)} \psi(\phi^t(F^n(x), 0)) dt - \int_0^s \psi(\phi^t(x, 0)) dt \right)$$

can be bounded as follows, with $\|\psi\| = \sup |\psi|

$$I \leq \left( 2 \frac{s}{T} + \frac{S_{n+1}^F \tau(x) - S_n^F \tau(x)}{T} \right) \cdot \|\psi\|.$$
Assuming that \( \psi \neq 0 \) (otherwise we consider only the right hand side of (3.2)) we estimate the \( \lambda^r \)-measure of each subset in (3.2) showing that they are deviations sets for the dynamics of \( F \).

We note that assumption (P6), more precisely item (1b) from Theorem 2.8 ensures that
\[
I(x, s, T) \leq \frac{1}{T}(2s + \tau(F^n x))\|\psi\| \leq \frac{1}{T}(2s + \tau(f^n(p(x))))\|\psi\| + c \frac{1}{T} \sup |\psi|
\]
\[
\leq \frac{S_{n+1}^f \tau - S_n^f \tau}{T} \circ p(x) + \frac{2s + c \sup |\psi|}{T};
\]
which shows that \( I(x, s, T) \) is bounded by an expression depending essentially on the dynamics of \( f \).

Now the left hand side subset of (3.2) is contained in
\[
\left\{ (x, s) \in \Xi' : \inf_{\mu \in \mathbb{E}} \left| n \frac{S_n^f \varphi}{n} - \frac{\mu(\varphi)}{\mu} \right| > \frac{\omega}{4} \right\} \cup \left\{ (x, s) \in \Xi' : \inf_{\mu \in \mathbb{E}} \left| \frac{n}{T} - \frac{1}{\mu(\tau)} \right| > \frac{\omega}{4} \right\}
\]
which we denote by \( (3.4) \) since for each \( \mu \in \mathbb{E} \) we have
\[
\left| \frac{1}{T} S_n^f \varphi - \frac{\mu(\varphi)}{\mu} \right| \leq \left| n \frac{S_n^f \varphi}{n} - n \frac{\mu(\varphi)}{\mu} \right| + \left| \frac{n}{T} \mu(\varphi) - \frac{\mu(\varphi)}{\mu} \right| \leq \left| n \frac{1}{T} S_n^f \varphi - \mu(\varphi) \right| + \mu(\varphi) \left| \frac{n}{T} - \frac{1}{\mu(\tau)} \right|,
\]
and the lap number \( n = n(x, s, T) \) satisfies
\[
\frac{S_n^f \tau(p(x))}{n} - c \leq \frac{S_n^f \tau(x)}{n} \leq \frac{T + s}{n} \leq \frac{S_{n+1}^f \tau(x)}{n} \leq \frac{S_{n+1}^f \tau(p(x))}{n} + \frac{n + 1}{n} c.
\]
Therefore, bounds involving \( n(x, s, T)/T \) can be replaced by others involving ergodic sums \( \frac{S_n^f \tau(p(x))}{n} \) and hence we reduce its study to the dynamics of the one-dimensional map \( f \). We deal with the sums \( S_n^f \varphi \) in the next Subsection 3.3 and with the sums \( S_n^f \tau \) in Subsection 3.3.

3.2. Reduction to the quotiented base dynamics. Here we use the contracting foliation that covers the cross-sections \( \Xi \) to show that large deviations of an induced observable for the dynamics of \( F \) can be reduced to a similar property for the dynamics of the quotient map \( f \). Then we show how this large deviation bound for \( f \) follows assuming exponentially slow recurrence to \( \mathcal{D} \).

Proposition 3.3. Let \( \varepsilon > 0 \) and a continuous and bounded \( \psi : U \to \mathbb{R} \) be given on the trapping neighborhood \( U \) of \( \Lambda \) and set \( \varphi : \Xi_0 \to \mathbb{R} \) as \( \varphi(z) = \int_0^{\tau(z)} \psi(X^t(z)) \, dt \), where \( \tau(z) \) is the Poincaré time of \( z \in \Xi_0 \). Let \( \mu \) be a measure on \( \Xi \) such that \( \int |\varphi| \, d\mu < \infty \). If we assume that there are \( \sigma > 2 \) and \( q \in \mathbb{Z}^+ \) so that
\begin{itemize}
  \item the quotient map \( f : M \setminus \mathcal{D} \to M \) is a \( C^1 \) local diffeomorphism away from the finite subset \( \mathcal{D} \) of the finite-dimensional compact manifold \( M \),
  \item \( f \) is expanding with rate \( \|Df^{-1}\| < \sigma^{-1} \) and degree \( (\text{number of pre-images of a point}) q \),
\end{itemize}
then there exist \( N, k \in \mathbb{Z}^+, \delta > 0, \) a constant \( \gamma > 0 \) depending only on \( \psi \) and the flow, and a continuous function \( \xi : M \setminus \bigcup_{j=0}^{k-1} f^{-j}D \to \mathbb{R} \) with logarithmic growth near \( D_k = \bigcup_{j=0}^{k-1} f^{-j}D \) such that, for all \( n > N \)

\[
\left\{ \frac{1}{n} S_n f^k \varphi - \mu(\varphi) \right\} > 3 \varepsilon \right) \subset \mathcal{P}^{-1} \left( \left\{ \frac{1}{n} S_n f^k \Delta_\delta > \frac{\varepsilon}{\gamma} \right\} \cup \left\{ \frac{1}{n} S_n f^k \xi - \mu(\xi) \right\} > \varepsilon \right) .
\]

This shows that it is enough to obtain an exponential decay for large deviations for observables with logarithmic growth near \( D \) if we are able to obtain such exponential decay for a power of \( f \) together with exponentially slow recurrence to \( D \).

Indeed, for \( n = k\ell + m \) with \( 0 \leq m < k \) and all big enough \( \ell \in \mathbb{Z}^+ \)

\[
\left\{ \frac{S_n f^k \varphi - \mu(\varphi)}{k\ell + m} > (4k + 1)\varepsilon \right\} \subset \mathcal{P}^{-1} \left( \left\{ \frac{S_i f^k \varphi}{\ell} \circ F^{m+i} - \mu(\varphi) \right\} > 3\varepsilon \right) \cup \left\{ \frac{S_m f^k \varphi}{k\ell + m} > \varepsilon \right\} .
\]

Then the Lebesgue measure of the right hand side subset can be bounded using that:

\[
\text{Leb} \left( \left\{ \frac{S_m f^k \varphi}{k\ell + m} > \varepsilon \right\} \leq \sum_{i=0}^{k-1} \left( \frac{q}{\sigma^d} \right)^i \frac{e^{-kd\varepsilon}}{K} \leq \frac{\sigma^d}{q-1} \left( \frac{q}{\sigma^d} \right)^k e^{-kd\varepsilon},
\]

where \( \sigma > 1 \) is the least expansion rate of \( f \), \( q \) is the maximum number of pre-images of the map \( f \) and \( d \) is the dimension of the quotient manifold \( M \). For the remaining union of subsets we obtain

\[
\text{Leb} \left( \left\{ \frac{S_i f^k \varphi}{\ell} \circ F^{m+i} - \mu(\varphi) \right\} > 3\varepsilon \right) \leq k \left( \frac{q}{\sigma^d} \right)^{2k} \lambda \left( \left\{ \frac{S_n f^k \varphi}{k\ell + m} \right\} > 3\varepsilon \right) .
\]

Thus, from the statement of Proposition 3.3 we are left to study upper large deviations for continuous observables with logarithmic growth near \( D \) and exponentially slow recurrence to \( D \) for a power of \( f \).

**Proof of Proposition 3.3.** First note that \( \varphi \) is continuous on \( \Xi_0 \) and \( \psi \) is bounded on \( U \) we get \( \varphi(x) \leq \tau(x) \cdot \sup |\psi| \leq K \Delta_\delta(p(x)) \cdot \sup |\psi| \) for \( x \in B(D, \delta) \), some small enough \( \delta > 0 \) and \( K = K(\varphi) > 0 \), since the return time function has logarithmic growth near the singular set \( \Gamma \).

From the assumptions (P1)-(P7) we can write \( F \) as a skew-product as in Theorem 2.8 and so \( \text{dist}(F^k(x, y), F^k(x', y')) < \lambda^k \) for all \( 1 \leq k \leq n \) and points in the same stable leaf of \( F \), where \( n \) is the first time the points visit the singular lines \( \Sigma \). These times \( n \) are given by \( f^n x \in D \) and since \( X_0 = \bigcup_{n \geq 1} f^{-n}D \) is enumerable the set of points which can be iterated indefinitely by \( F \) has full Lebesgue measure in \( \Xi \). Moreover, we can write \( \text{dist}(X^t(x, y), X^t(x', y')) \leq \kappa |y - y'|, \forall t > 0 \) since stable leaves of \( F \) are contained in central stable leaves of the flow, by construction of \( W^c(x, \Xi) \).
This ensures the bound
\[
\left| \frac{1}{n} \sum_{j=0}^{n-1} (\varphi(F^j(x, y)) - \varphi(F^j(x, y'))) \right| \leq \frac{1}{n} \sum_{j=0}^{n-1} \tilde{\varphi}_j(x)
\]
where \( \tilde{\varphi}_j(x) = \sup_{y,y' \in W^s(x, \Xi)} |\varphi(F^j(x, y)) - \varphi(F^j(x, y'))| \).

For each \( \varepsilon > 0 \) there exist \( \delta, \eta > 0 \) such that \(-K\eta \log \delta < \varepsilon/3 \) and \( \eta \sup |\psi| < \varepsilon/3 \) and, using uniform continuity, we can also find \( \zeta > 0 \) satisfying \( \sup \frac{|\psi(x, y') - \psi(x, y)|}{\zeta} < \eta \). Hence, by the uniform contraction of stable leaves by \( F \), there exists \( j_0 = j_0(\eta) \in \mathbb{Z}^+ \) so that \( |F^j(x, y) - F^j(x, y')| \leq \frac{\zeta}{\eta}, \forall j \geq j_0 \). Thus, by the previous choices together with item (1b) from Theorem 2.8, we get
\[
\varphi_j(x) \leq K \max \{ \Delta_\delta(x_j), \log \delta^{-1} \} \sup_{0 < t < T_\Delta(x_j)} |\psi(X^t(F^j(x, y))) - \psi(X^t(F^j(x, y')))|
\]
\[
+ |\tau(F^j(x, y)) - \tau(F^j(x, y'))| \cdot \sup |\psi|
\]
\[
\leq K \max \{ \Delta_\delta(x_j), \log \delta^{-1} \} \eta + \eta \sup |\psi| \leq K \Delta_\delta(x_j) \eta + \frac{2}{3} \varepsilon. \tag{3.5}
\]

Now take a continuous function \( \xi : M \setminus \mathcal{D} \to \mathbb{R} \) such that for some \( 0 < a < \varepsilon/3 \)
- \( \int \xi \circ p \ d\mu = \int \varphi \ d\mu; \)
- \( \min_{y \in W^s(x, \Xi)} \varphi(F^{j_0}(y)) - a \leq \xi(x) \leq a + \max_{y \in W^s(x, \Xi)} \varphi(F^{j_0}(y)). \)

This is possible since \( \varphi \) is \( \mu \)-integrable and disintegrating \( \mu \) on the measurable partition of \( \Xi \) given by the stable leaves we obtain the family \( \{ \mu_x \}_{x \in M} \) of conditional probabilities and we set \( \xi_0(x) = \int \varphi \ d\mu_x. \) Then we approximate \( \xi_0 \) by a continuous function \( \xi_1 \) satisfying \( \int |\xi_0 - \xi_1| \ d\mu_p < \varepsilon/3 \) and so for some \( b \in (-\varepsilon/3, \varepsilon/3) \) the function \( \xi = \xi_1 + b \) satisfies the above items.

Now note that \( \xi \) also has logarithmic growth near \( \bigcup_{i=0}^{j_0-1} f^{-i} \mathcal{D} \). In addition, for \( n \in \mathbb{Z}^+ \) using (3.5) and \( f \circ p = p \circ F \) and summing over orbits of \( F^{j_0} \) and \( F^{j_0} \) we get
\[
|S_n^{F^{j_0}}(\xi \circ p) - S_n^{F^{j_0}}\varphi|(x, y) \leq |\xi \circ p - \varphi|(x, y) + |S_n^{F^{j_0}}(\xi \circ p - \varphi)|(x, y)
\]
\[
\leq \sup |\psi| K \eta \Delta_\delta(x) + \frac{2 \varepsilon}{3} + a + \sum_{i=1}^{n-1} \left( K \eta \Delta_\delta(f^{j_0}(x)) + a + \frac{2 \varepsilon}{3} \right)
\]
\[
\leq n \varepsilon + K \eta (1 + \sup |\psi|) \cdot S_n^{F^{j_0}} \Delta_\delta(x) \tag{3.6}
\]
We finally observe that
\[
\left\{ \left| \frac{S_n^{F^{j_0}} \varphi}{n} - \mu(\varphi) \right| > 3 \varepsilon \right\} \subseteq \left\{ \left| \frac{S_n^{F^{j_0}}(\xi \circ p) - S_n^{F^{j_0}}\varphi}{n} \right| > 2 \varepsilon \right\} \cup \left\{ \frac{1}{n} |S_n^{F^{j_0}}(\xi \circ p) - \mu(\varphi)| > \varepsilon \right\}. \tag{3.7}
\]
and by (3.6) we obtain
\[
\left\{ \frac{1}{n} (S_n^{F^{j_0}}(\xi \circ P) - S_n^{F^{j_0}}\varphi) > 2 \varepsilon \right\} \subseteq p^{-1} \left\{ \frac{1}{n} S_n^{F^{j_0}} \Delta_\delta > \frac{\varepsilon}{K \eta (1 + \sup |\psi|)} \right\}
\]
which together with (3.7) completes the proof of the proposition with \( k = j_0 \). \( \square \)
3.2.1. Large deviations for observables with logarithmic growth near singularities. This is based in [6, Section 3] adapted to the setting where there might be several equilibria for the the potential \( \log J = \log |\det DX|^1 \).

The main bound on large deviations for suspension semiflows over a non-uniformly expanding base will be obtained from the following large deviation statement for non-uniformly expanding transformations assuming exponentially slow recurrence to the singular/discontinuous set.

**Theorem 3.4.** Let \( f : M \setminus D \rightarrow M \) be a regular \( C^{1+\alpha} \) local diffeomorphism, where \( D \) is a non-flat critical set and \( \alpha \in (0, 1) \). Assume that \( f \) is a non-uniformly expanding map with exponentially slow recurrence to the singular/discontinuous set \( D \) and let \( \varphi : M \setminus D \rightarrow \mathbb{R} \) be a continuous map which has logarithmic growth near \( D \). Moreover, assume that there are at most finitely many ergodic equilibrium states \( \mu_1, \ldots, \mu_k \) with respect to \( \log J \) which are absolutely continuous. Then for any given \( \omega > 0 \)

\[
\limsup_{n \to +\infty} \frac{1}{n} \log \text{Leb} \left\{ x \in M : \inf_{\mu \in \mathcal{E}} \left| \frac{1}{n} S_n \varphi(x) - \mu(\varphi) \right| \geq \omega \right\} < 0,
\]

where \( \mathcal{E} \) is the family of all equilibrium states with respect to \( \log J \).

**Remark 3.5.** Since we assume in Theorem 3.1 that \( f \) has exponentially slow recurrence to the non-degenerate singular set \( D \) and is also expanding, then \( f \) is in particular non-uniformly expanding with slow recurrence to \( D \).

This finishes the reduction of the estimate of the Lebesgue measure of the large deviation subset (3.1) to obtaining exponentially slow recurrence to \( D \) as in Theorem C, through the inclusion (3.2), Proposition 3.3 and Theorem 3.4.

**Proof of Theorem 3.4.** Fix \( \varphi : M \setminus S \rightarrow \mathbb{R} \) as in the statement, \( \varepsilon_0 > 0 \) and \( c \in \mathbb{R} \).

By assumption we may choose \( \varepsilon_1, \delta_1 > 0 \) small enough such that the exponential slow recurrence condition (1.2) is true for the pair \( (\varepsilon_1, \delta_1) \), \( |\varphi \chi_{B(S,\delta_1)}| \leq K(\varphi) \Delta_{\delta_1} \) and \( K(\varphi) \cdot \varepsilon_1 \leq \varepsilon_0 \), where \( K(\varphi) \) is the constant given by the assumption of logarithmic growth of \( \varphi \) near \( S \).

Let \( \varphi_0 : M \rightarrow \mathbb{R} \) be the continuous extension of \( \varphi \mid_{B(S,\delta_1)} \) given by the Tietze Extension Theorem, that is

- \( \varphi_0 \) is continuous; \( \varphi_0 \mid_{M \setminus B(S,\delta_1)} = \varphi \mid_{M \setminus B(S,\delta_1)} \), and
- \( \sup_{x \in M} |\varphi_0(x)| = \sup_{x \in M \setminus B(S,\delta_1)} |\varphi(x)| \).

We may choose \( K \geq K(\varphi) \) big enough so that and \( |(\varphi - \varphi_0) \chi_{B(S,\delta_1)}| \leq K \Delta_{\delta_1} \). Then for all \( n \geq 1 \) we have

\[
S_n \varphi_0 - S_n |\varphi - \varphi_0| \leq S_n \varphi = S_n \varphi_0 + S_n (\varphi - \varphi_0) \leq S_n \varphi_0 + S_n |\varphi - \varphi_0|.
\]

and deduce the following inclusions

\[
\left\{ \frac{1}{n} S_n \varphi > c \right\} \subseteq \left\{ \frac{1}{n} S_n \varphi_0 > c - \varepsilon_0 \right\} \cup \left\{ \frac{1}{n} S_n \Delta_{\delta_1} \geq \varepsilon_1 \right\},
\]

(3.8)
where in (3.8) we use the assumption that $\varphi$ is of logarithmic growth near $S$ and the choices of $K, \varepsilon_1, \delta_1$. Analogously we get with opposite inequalities

$$\left\{ \frac{1}{n} S_n \varphi < c \right\} \subseteq \left\{ \frac{1}{n} S_n \varphi_0 < c + \varepsilon_0 \right\} \cup \left\{ \frac{1}{n} S_n \Delta \delta_1 \geq \varepsilon_1 \right\}; \quad (3.9)$$

see [6, Section 4, pp 352] for the derivation of these inequalities.

From (3.8) and (3.9) we see that to get the bound for large deviations in the statement of Theorem 3.4 it suffices to obtain a large deviation bound for the continuous function $\varphi_0$ with respect to the same transformation $f$ and to have exponentially slow recurrence to the singular set $S$.

To obtain this large deviation bound, we use the following result from [14].

Theorem 3.6. [14, Theorem B] Let $f: M \setminus D \to M$ be a local diffeomorphism outside a non-flat singular set $D$ which is non-uniformly expanding and has slow recurrence to $D$. For $\omega_0 > 0$ and a continuous function $\varphi_0: M \to \mathbb{R}$ there exists $\varepsilon, \delta > 0$ arbitrarily close to 0 such that, writing

$$A_n = \left\{ x \in M : \frac{1}{n} S_n \Delta \delta(x) \leq \varepsilon \right\} \quad \text{and} \quad B_n = \left\{ x \in M : \inf_{\mu \in E} \left\{ \frac{1}{n} S_n \varphi_0(x) - \mu(\varphi_0) \right\} > \omega_0 \right\},$$

we get

$$\limsup_{n \to +\infty} \frac{1}{n} \log \text{Leb} (A_n \cap B_n) < 0.$$ (3.10)

Recall that $E = E_{\varepsilon, \delta} = \{ \nu \in M_f : h_{\nu}(f) = \nu(\log J) \text{ and } \nu(\Delta) < \varepsilon \}$ is the set of all equilibrium states of $f$ with respect to the potential $\log J$ which have slow recurrence to $D$. From [6, Theorem 5.1] we have that $E$ is a non-empty compact convex subset of the set of invariant probability measures, in the weak* topology.

Note that exponentially slow recurrence implies $\limsup_{n \to +\infty} \frac{1}{n} \text{Leb}(M \setminus A_n) < 0$. Under this assumption Theorem 3.6 ensures that for $(\varepsilon, \delta)$ close enough to $(0, 0)$ we get

$$\limsup_{n \to +\infty} \frac{1}{n} \log \text{Leb}(B_n) < 0.$$ (3.11)

Now in Theorem 3.6 we take $\omega, \varepsilon_0 > 0$ small, choose $\varphi_0$ as before and $\omega_0 = \omega + \varepsilon_0$. Hence $\{ \mu(\varphi_0) : \mu \in E \}$ is a compact interval of the real line.

In (3.8) set $c = \inf_{\mu \in E} \mu(\varphi_0) - \omega$ and in (3.9) set $c = \sup_{\mu \in E} \mu(\varphi_0) + \omega$. Then we have the inclusion

$$\left\{ \inf_{\mu \in E} \left\{ \frac{1}{n} S_n \varphi - \mu(\varphi) \right\} > \omega \right\} \subseteq B_n \cup \left\{ \frac{1}{n} S_n \Delta \delta_1 \geq \varepsilon_1 \right\}. \quad (3.10)$$

By Theorem 3.6 we may find $\varepsilon, \delta > 0$ small enough so that the exponentially slow recurrence holds also for the pair $(\varepsilon, \delta)$ and hence

$$\limsup_{n \to +\infty} \frac{1}{n} \log \text{Leb} \left\{ \inf_{\mu \in E} \left\{ \frac{1}{n} S_n \varphi_0 - \mu(\varphi_0) \right\} > \omega_0 \right\} < 0.$$ (3.11)

Finally the choice of $\varepsilon_1, \delta_1$ according to the condition on exponential slow recurrence to $D$ ensures that the Lebesgue measure of the right hand subset in (3.10) is also exponentially small when $n \to \infty$. This together with (3.11) concludes the proof of Theorem 3.4. □
3.3. **The roof function and the induced observable as observables over the base dynamics.** We now proceed with the estimate of the Lebesgue measure of the sets in (3.2) using the results from the previous Subsection 3.2, assuming exponentially slow recurrence to $D$ under the dynamics of $f$ and also that $\mathcal{D}$ is finite: we write $\# \mathcal{D}$ for the number of elements of $D$ in what follows.

To estimate the Lebesgue measure of the right hand side subset in (3.2) we take a sufficiently large $N \in \mathbb{Z}^+$ so that $N \sup |\psi| > 2$ and note that for $\omega > 0$ by using (3.3) and a large $T > 0$

$$\lambda \{ \tau > \omega T \} = \int d \text{Leb}(x) \int_0^{\tau(x)} ds \left( \chi_{(\omega/2, +\infty)} \circ I \right)(x, s, T)$$

$$\leq \text{Leb}\{ \tau > \omega T \} + \omega T \sum_{i=0}^{[T/\tau_0]+1} \lambda \left\{ \frac{|S_i^f \tau - S_i^f|}{T} > \frac{\omega}{4} \right\},$$

where $\tau_0 = \inf \tau > 0$. Because $\tau$ has logarithmic growth near the finite subset $\mathcal{D}$

$$\lambda \{ \tau > \omega T \} = \lambda \{ x \in I : d(x, \mathcal{D}) \leq e^{-\omega T} \} \leq C_d e^{-\frac{d(x, \mathcal{D})}{K} \# \mathcal{D}}.$$}

On the other hand, since $T \geq S_i \tau(x) \geq \tau_0 i$ we obtain for each $i = 0, \ldots, [T/\tau_0] + 1$

$$\lambda \left\{ \frac{|S_i^f \tau - S_i^f|}{T} > \frac{\omega}{4} \right\} \leq \sum_{j=0,1} \lambda \left\{ \inf_{\mu \in \mathbb{E}_f} \left| \frac{1}{i+j} S_i^f r - \mu(\tau) \right| > \frac{\omega \tau_0}{4} \right\} \leq 2C_0 e^{-\beta i}$$

for some constants $C_0, \beta > 0$. This follows from Theorem 3.6 assuming exponentially slow recurrence for $f$. Hence (3.12) is bounded from above by

$$C_d e^{-\frac{d(x, \mathcal{D})}{K}} \# \mathcal{D} + \omega T 2C_0 \sum_{i=0}^{[T/\tau_0]+1} e^{-\beta i} \leq C_d C_0 \omega T (e^{-\frac{d(x, \mathcal{D})}{K}} + e^{-\beta T/\tau_0})$$

for all big enough $T > 0$. Hence we have proved

$$\limsup_{T \to \infty} \frac{1}{T} \log \lambda \{ I > \frac{\omega}{2} \} < 0. \quad (3.14)$$

3.3.1. **Using $\varphi$ as an observable for the $f$ dynamics.** Now we consider the measures of the subsets in (3.4). For the right hand side subset in (3.4) we can bound its Lebesgue measure by

$$\text{Leb} \left\{ (x, s) \in \Xi^\tau : \inf_{\mu \in \mathbb{E}} \left| \frac{n}{T} - \frac{1}{\mu(\tau)} \right| > \frac{\omega}{4|\mu(\varphi)|} \& \tau \leq T \right\} + \text{Leb}\{ (x, s) \in \Xi^\tau : \tau > T \}$$

$$\leq T \sum_{i=0}^{[T/\tau_0]+1} \sum_{j=0,1} \text{Leb} \left\{ \left| \frac{i}{S_i^f \tau} - \frac{1}{\mu(\tau)} \right| > \frac{\omega}{|\mu(\varphi)|} \right\} + \int_{\{\tau > T\}} \tau \, d \text{Leb}$$

(3.15)
Since $\tau$ has logarithmic growth near $D$ and $D$ is finite, we get for $T$ large enough so that $i > [T]$ implies $C_d(i + 1)e^{-id/K} \cdot \#D < e^{-\gamma i}$ for some $\gamma > 0$

$$\int_{\{\tau > T\}} \tau \, d\text{Leb} \leq \sum_{i \geq [T]} \int_i^{i+1} \tau \, d\text{Leb} \leq \sum_{i \geq [T]} (i + 1) \text{Leb}\{\tau > i\}$$

$$\leq \sum_{i \geq [T]} (i + 1) C_d e^{-id/K} \#D \leq \sum_{i \geq [T]} e^{-\gamma i} \leq e^{-\gamma T} \frac{1}{1 - e^{-\gamma}}. \tag{3.16}$$

For the double summation (3.15) we use again large deviations for $f$ on the observable $\tau$ as in (3.13) to get the upper bound $C_1 e^{-\beta T/\tau_0}$ for a constant $C_1$ depending only on $f, C_0, C_d, \beta$ and $\tau_0$.

This shows that the Lebesgue measure of the right hand side subset of (3.14) decays exponentially fast as $T \nearrow \infty$.

Finally, for a small $\hat{\omega} > 0$ the left hand side subset of (3.14) is contained in the union

$$\left\{ \inf_{\mu \in E} \left| \frac{T}{n} - \mu(\tau) \right| > \hat{\omega} \right\} \cup \left\{ \inf_{\mu \in E_F} \left| \frac{S_n \varphi}{n} - \mu(\varphi) \right| > \frac{\hat{\omega}}{4} \left( \hat{\omega} + \inf_{\mu \in E_F} \mu(\tau) \right) \right\}. \tag{3.17}$$

The left hand side subset of (3.17) has Lebesgue measure which decays exponentially fast as $T \nearrow \infty$ following the same arguments as in (3.15). For the right hand side subset, we again use a large deviation bound for $\varphi$ with respect to the dynamics of $f$ as in (3.13). Putting all together we conclude the proof of Theorem 3.1.

4. Exponentially slow recurrence

As explained in Section 3, we are now left to prove Theorem C to complete the proof of Theorem A

Let $f : I \setminus D \to I$ be a $C^{1+\alpha}$ piecewise monotonous one-dimensional map, for some $\alpha > 0$, with monotonous branches on the connected components of $I \setminus D$ so that $\sigma = \inf |f'| > 1$ whenever defined. We assume that $D$ splits into a pair of disjoint subsets $S$ and $D \setminus S$ satisfying conditions (1-6) of the statement of Theorem C.

Then $D = \{b_n\}_n$ where we may assume that the sequence is strictly monotonous and we distinguish between the left and right hand side behavior of the point. We consider the middle points $c_n = (b_n + b_{n+1})/2$ for all applicable indexes $n$ and define a partition $P_0$ of $I$ as follows.

4.1. Initial partition. We partition $(b_n, c_n)$ and $(c_n, b_{n+1})$ into subintervals according to whether $b_n^+ \in S$ or $b_n^- \in D \setminus S$ as explained in item (2) in the statement of Theorem C. The Lebesgue modulo zero partition to be constructed consists of small intervals whose length is exponentially small with respect to the distance to $D$.

We denote $b_n^+$ by $a_{2n}$ and $b_n^-$ by $a_{2n-1}$ in what follows.
4.1.1. Near a singular point. If \( a_{2n} = b_n^+ \in S \) then we partition \((b_n, c_n)\) into (see Figure 3)

\[
M(2n, p) = (b_n + d_{2n}e^{-p}, b_n + d_{2n}e^{-(p-1)}), \\
\text{where } d_{2n} = c_n - b_n, p \geq \rho(2n)
\]

and \( \rho(2n) \in \mathbb{Z}^+ \) is big enough, to be defined shortly.

Analogously, if \( a_{2n-1} = b_n^- \in S \) then we partition \((c_{n-1}, b_n)\) into

\[
M(2n-1, p) = (b_n - d_{2n-1}e^{-(p-1)}, b_n - d_{2n-1}e^{-p}), \\
\text{with } d_{2n-1} = b_n - c_{n-1}, p \geq \rho(2n-1)
\]

and \( \rho(2n-1) \in \mathbb{Z}^+ \) is big enough.

In addition, from item (2) of the statement of Theorem C there are \( \epsilon < \bar{c} \) so that

\[
\epsilon \leq |f'x| \cdot |x - b_n|^{1-\alpha(b_n)} \leq \bar{c} \text{ when } x \text{ is near } b_n^+.
\]

Hence using item (5) of the statement of Theorem C we can find \( B > 0 \) so that

\[
|f'x| \geq \frac{1}{B}d(x, S)^\bar{\alpha} \geq \frac{1}{B}d(x, S)^{-\bar{\alpha}}, \quad x \in (c_{n-1}, b_n) \text{ or } (b_n, c_n) \quad (4.1)
\]

according to the side of the singularity, where \( B \) does not depend on the singular point.

Note that since \( f' \) is \( \alpha \)-Hölder we also get

\[
\left| \log \frac{|f'y|}{|f'x|} \right| \leq \frac{|f'y| - |f'x|}{|f'x|} \leq B \frac{|x - y|^{\alpha}}{d(x, S)^{\bar{\alpha}}}, \quad x \in (c_{n-1}, b_n) \text{ or } (b_n, c_n). \quad (4.2)
\]

4.1.2. Near a discontinuity point. From assumption (3) in the statement of Theorem C we can map a lateral neighborhood of \( b_n^+ \in \mathcal{D} \setminus \mathcal{D} \) (a neighborhood of \( a_{2n-1} \) or \( a_{2n} \)) into a corresponding lateral neighborhood of some \( b_m^+ \in S \) in finitely many \( T = T(b_n^+) \) iterates of \( f \).

Hence, if \( a_{2n-1} = b_n^- \in \mathcal{D} \setminus \mathcal{D} \) there exist \( \rho(2n-1) \in \mathbb{Z}^+, \delta > 0 \) so that

\[
M(2n-1, p) = (f^T | (b_n - \delta, b_n))^{-1}(M(2m^+, p)), \quad p \geq \rho(2n-1)
\]

is well defined, where \( 2m^* \) denotes \( 2m \) or \( 2m - 1 \) whether \( a_{2m} \) or \( a_{2m-1} \) corresponds to \( f^T(a_{2m-1}) \).

Analogously, if \( a_{2n} = b_n^+ \in \mathcal{D} \setminus \mathcal{S} \) there exist \( \rho(2n) \in \mathbb{Z}^+, \delta > 0 \) so that

\[
M(2n, p) = (f^T | (b_n, b_n + \delta))^{-1}(M(2m^*, p)), \quad p \geq \rho(2n)
\]

is well defined, where \( 2m^* \) denotes \( 2m \) or \( 2m - 1 \) whether \( a_{2n} = b_m^+ \) or \( a_{2m-1} = b_m^- \) corresponds to \( f^T(a_{2m}) \). In addition, from item (2) of the statement of Theorem C we get

\[
\frac{1}{B}d(x, \mathcal{D})^{-\bar{\alpha}} \leq |f'x| \leq Bd(x, \mathcal{D})^\bar{\alpha}, \quad x \in (c_{n-1}, b_n) \text{ or } (b_n, c_n) \quad (4.3)
\]
we obtain, using the above equality and (4.6) on the other hand
\[\sigma >\]

Properties (4.1), (4.2) and (4.3) together show that the singular set \(\mathcal{D}\) is non-flat/non-degenerate similarly to the assumptions on \([3]\).

We denote by \(\mathcal{P}\) the family of all intervals \(\{M(k,p) : p \geq \rho(k)\}\) defined up to this point.

4.1.3. Global initial partition. For each element \(\eta\) of \(\mathcal{P}\) denote by \(\eta^+\) the interval obtained by joining \(\eta\) with its two neighbors in \(\mathcal{P}\). Moreover, for a subset \(A\) of \(I\) we denote by \(|A|\) the Lebesgue measure of \(A\).

From the definition in Subsection 4.1.1 we have
\[|M(k,p)| \leq 9|M(k,p)| = 9d_k e^{-p}(e-1) \leq 9d_k e^{-p/2}, \quad p \geq \rho(k),\] (4.4)

for atoms of \(\mathcal{P}_0\) near a singular point, if we fix a big enough \(\rho(k) \in \mathbb{Z}^+\).

For atoms near a discontinuity point \(a_k\) we note that, by assumption (4) on the statement of Theorem [C] we can choose \(\rho_0 \in \mathbb{Z}^+\) big enough and find \(\delta_0 > 0\) so that for \(1 \leq j < T = T(a_k)\)
\[d(f^j(a_k), D) = d(f^j(a_k), a_n) \implies d(f^j(a_k), a_n) > e^{-\rho_0}d_n \geq \delta_0.\] (4.5)

Hence, besides \(|M(k,p)| \leq 9|M(k,p)|\) we also get, from the Mean Value Theorem
\[\sigma^T|M(k,p)| \leq |M(k^*,p)| = |(f^T)'(\xi_{k,p})| \cdot |M(k,p)| \leq \kappa_0|M(k,p)|\] (4.6)
for some \(\xi_{k,p} \in M(k,p)\), where \(k^*\) denotes the index of the singular point connected to \(b_k\), and \(|(f^T)'(\xi_{k,p})|\) is uniformly bounded from below by \(\sigma^T\) and from above by a constant \(\kappa_0 \geq \sigma^T > 1\) for all \(p \geq \rho(k) \geq \rho_0\) and discontinuity points \(b_k\). Therefore we arrive at
\[d_k e^{-p} e^{-1} \leq |M(k,p)| \leq d_k e^{-p} e^{-1} \sigma^T \leq d_k e^{-p/2}, \quad p \geq \rho(k) - 1\] (4.7)

since \(e^{-1} \leq e - 1\).

From (4.4) and (4.7) we can relate distance to \(\mathcal{D}\) with the length of the atoms of \(\mathcal{P}\).

First \(d(M(k,p), D) = d_k e^{-p} = \frac{|M(k,p)|}{e^{-1}}\) for \(a_k \in \mathcal{S}\). Then, on the one hand, for \(a_k \in \mathcal{D} \setminus \mathcal{S}\) we obtain, using the above equality and (4.6)
\[d(M(k,p), D) \leq \frac{d(f^T(M(k,p)), \mathcal{D})}{\sigma^T} = \frac{|M(k^*,p)|}{\sigma^T(e-1)} \leq \frac{\kappa_0|M(k,p)|}{\kappa_0(e-1)}.\]

On the other hand
\[d(M(k,p), D) \geq \frac{d(f^T(M(k,p)), \mathcal{D})}{\kappa_0} = \frac{|M(k^*,p)|}{\kappa_0(e-1)} \geq \frac{\sigma^T|M(k,p)|}{\kappa_0(e-1)}.\]

Hence, since \(\sigma > 1\), in all cases we have the relation
\[\frac{|M(k,p)|}{\kappa_0(e-1)} \leq d(M(k,p), D) \leq \frac{\kappa_0}{(e-1)}|M(k,p)|, \quad p \geq \rho(k) \geq \rho_0.\] (4.8)

Let now \(\mathcal{P}_0\) be formed by the collection of all intervals \(M(n,p)\) for all \(n\) and every \(p \geq \rho_0\) together with the connected components of \(M \setminus \left( \bigcup_{n,p \geq \rho_0} M(n,p) \right)\), which will be denoted by \(M(n,\rho_0 - 1)\) whenever they intersect \(M(n,\rho_0)^+\); see Figure [4].
We denote by $M(k, \rho_0 - 1)^+$ the union of $M(k, \rho_0 - 1)$ with its neighboring intervals in $P_0$.

### 4.2. Dynamical refinement of the partition and bounded distortion

Following [6, Section 6], the partition $P_0$ is dynamically refined so that any pair $x, y$ of points in the same atom of the $n$th refinement $P_n$, i.e. $y \in P_n(x)$, belong to the same element $\eta^+$ during the first consecutive $n$ iterates: there are $\eta_i \in P_0$ so that $f^i(x), f^i(y) \in \eta_i^+$ for $i = 0, \ldots, n$. Moreover, $P_n$ is a collection of intervals for each $n \geq 1$ and $f^{n+1} | \omega : \omega \rightarrow f^{n+1}\omega$ is a diffeomorphism for every interval $\omega \in P_n$.

We define the refinement algorithm inductively, assuming that $P_n$ is already defined and, for each $\omega \in P_n$, there are sets $R_n(\omega) = \{r_1 < \cdots < r_s\}$ (with $r_1 \geq 1$ and $r_s \leq n$) of splitting times and $D_n(\zeta) = \{(k_1, p_1), \ldots, (k_s, p_s)\}$ whose pairs give the corresponding splitting depths, to be defined below.

First for $\omega = M(k, p) \in P_0$ we set $R_0(\omega) = \{0\}$ and $D_0(\omega) = \{(k, p)\}$. Then, for each $n \geq 1$ we assume that $P_n$ is defined and for each $\omega \in P_n$ that $R_n(\omega), D_n(\omega)$ are also defined. Then we analyze $f^{n+1}\omega$:

- if $f^{n+1}(\omega)$ intersects three or fewer elements of $P_0$, then we set $\omega \in P_{n+1}$, $R_{n+1}(\omega) = R_n(\omega)$ and $D_{n+1}(\omega) = D_n(\omega)$. For the points in $\omega$, the iterate $n + 1$ is called a free time.

- otherwise, $f^{n+1}(\omega)$ covers at least one atom of $P_0$, the iterate $n + 1$ is then a return time for the points in $\omega$, and we consider the subsets $\eta_{k,p} = (f^{n+1} | \omega)^{-1}(M(k, p))$ of the interval $\omega$ for all elements $M(k, p)$ of $P_0$ which intersect $f^{n+1}(\omega)$; see Figure 5.

This family $Q = \{\eta_{k,p}\}$ is a partition of $\omega$ Lebesgue modulo zero such that $f^{n+1}(\eta_{k,p})$ is...

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**Figure 4.** Exponentially small partition around the points of $S$.

**Figure 5.** Refinement of $\omega \in P_n$. 

...
– either equal to \( M(k, p) \); or
– strictly contained in \( M(k, p) \).

In the latter case, \( f^{n+1}(\eta_{k,p}) \) is necessarily at an extreme of the interval \( f^{n+1}(\omega) \) and we join \( \eta_{k,p} \) with its neighbor in \( Q \).

In this way we construct a new partition \( Q' = \{ \eta_{k,p} \} \) of \( \omega \) which satisfies

\[
M(k, p) \subseteq f^{n+1}(\eta_{k,p}) \subseteq M(k, p^+) \quad \text{for all} \quad (k, p) \text{ with } p \geq \rho_0 - 1 \text{ so that } \eta_{k,p} \in Q';
\]

and we set

\[
\eta_{k,p} \in P_{n+1}, \quad R_{n+1}(\eta_{k,p}) = R_n(\eta_{k,p}) \cup \{ n + 1 \} \quad \text{and}
D_{n+1}(\eta_{k,p}) = D_n(\eta_{k,p}) \cup \{ (k, p, T(b_k)) \}
\]

where \( T(b_k) = 0 \) if \( b_k \) is a singularity, or \( T(b_k) \) is the connecting time to a singularity if \( b_k \) is a discontinuity. The cases where \( \eta_{k,\rho_0-1} \in Q' \) deserve special attention in what follows.

To finish the refining algorithm, we repeat the procedure for each \( \omega \in P_n \) completing the construction of \( P_{n+1} \) from \( P_n \) for \( n \geq 1 \).

Clearly, since the atoms of the initial partition \( P_0 \) are intervals, then this construction shows that all the atoms of \( P_n \) are intervals, for all \( n \geq 1 \).

Remark 4.2. Note that if \( \eta_{k,\rho_0-1} \in Q' \), that is, we have an escape time, then we necessarily get

\[
M(k, \rho_0 - 1) \subset f^{n+1}(\eta_{k,\rho_0-1}) \subset M(k, \rho_0 - 1^+);
\]

see Figure 6. Moreover, denoting by \( U_0 = \bigcup_{k \in P_{\geq \rho_0}} M(k, p) \) the return region and setting \( U_1 = \bigcup_{k \in P_{\geq \rho_0+1}} M(k, p) \), we have that if \( \eta_{k,\rho_0} \in Q' \) then we may have

\[
f^{n+1} \eta_{k,\rho_0} \cap (I \setminus U_0) \neq \emptyset \neq f^{n+1} \eta_{k,\rho_0} \cap U_1.
\]

Figure 6. Different kinds of positions of \( f^{n+1} \omega \) and the partition elements.

4.2.1. Bounded distortion. Slightly more general than in [6, Section 6.3] (where this was only stated for atoms of \( P_n \) in \( n \) iterates while it is also valid for atoms of \( P_{n-1} \)), uniform expansion implies bounded distortion on atoms of the partition \( P_{n-1} \).

Indeed, for \( \omega \in P_{n-1} \) for some \( n \geq 1 \) and \( x, y \in \omega \), since \( f^i | \omega \) is a \( \alpha \)-Hölder diffeomorphism for \( i = 1, \ldots, n \) and distances are expanded at a minimum rate of \( \sigma \), then there are
constants $C, D > 0$ so that
\[
\log \left| \frac{f^n(x)}{f^n(y)} \right|\left( \frac{f^n(x)}{f^n(y)} \right) = \sum_{i=0}^{n-1} \left| \log |f^i(x)| - \log |f^j(y)| \right| \leq \sum_{i=0}^{n-1} C \cdot \frac{|f^i(x) - f^j(y)|}{\max\{|f^i(f^i(x))|, |f^j(f^j(y))|\}} \leq \frac{C}{\sigma} \sum_{i=0}^{n-1} \sigma^{i-n} \cdot |f^n(x) - f^n(y)|^\alpha \leq D,
\]
where $D$ depends only on $\sigma$ which, in turn, depends only on $Df$.

4.3. Measure of atoms of $\mathcal{P}_n$ as a function of return depths. Here we estimate the measure of $\omega \in \mathcal{P}_n$ using $R_n(\omega)$ and $D_n(\omega)$ in a similar way to [6, Section 6.4].

We start by fixing $n \in \mathbb{Z}^+$, $u \in \{1, \ldots, n\}$ and taking $\omega_0 \in \mathcal{P}_0$. Let $\omega \in \mathcal{P}_n$ be such that $\omega \subset \omega_0$ and $u_n(\omega) = u$ and $0 = t_0 < t_1 < \cdots < t_u \leq n$ be the return situations of $\omega$.

For $m = 1, \ldots, u$ we write $\omega^m = \omega((k_1, p_1), \ldots, (k_m, p_m)) \in \mathcal{P}_{t_m}$ the subset of $\omega_0$ satisfying
\[
f^i(\omega^m) \subset M(k_i, p_i)^+, \quad i = 1, \ldots, m - 1 \quad \text{and} \quad M(k_m, p_m) \subset f^m(\omega^m) \subset M(k_m, p_m)^+,
\]
by the definition of the sequence of partitions $\mathcal{P}_n$. Note that we get a nested sequence of sets
\[\omega^0 \supseteq \omega^1 \supseteq \cdots \supseteq \omega^u = \omega.\]
We define $\mathcal{T} = \{\omega \in \mathcal{P}_n : \omega \subset \omega^0, u_n(\omega) = u\}$. Now we define by induction a sequence of partitions of $\mathcal{T}$ which will enable us to determine the estimates we need.

Start by putting $\mathcal{V}_0 = \cup\{\omega \in \mathcal{T}\}$. We define for $1 \leq i \leq u$ the subset
\[\mathcal{V}_i = \bigcup\{\omega^i : \omega \in \mathcal{T} \& (4.10) \text{ holds with } m = i\}.\]
Now we compare $|\mathcal{V}_j|$ with $|\mathcal{V}_{j-1}|$, $j = 1, \ldots, u$. We start by defining
\[E = \left\{ k : \frac{9^{1-\beta/2}k_0^\beta BD^2(M(k, \rho_0 - 1)^+)\beta/2}{(e - 1)^\beta} > 1 \right\}.\]

Remark 4.3. By definition, $E$ is a finite set of indexes of partition elements of $\mathcal{P}_0$ which does not depend on $\rho_0$. 
Lemma 4.4. If \( \rho_0 \in \mathbb{Z}^+ \) big enough so that \( \rho(k) \geq \rho_0 \) for all points of \( \mathcal{D} \) and \( p \geq \rho(k) \geq \rho_0 \) satisfies

\[
\frac{9^{1-\beta/2} \kappa_0 \beta BD^2 |M(k, p)|^{\beta/2}}{e^{1-1/2}} \leq 1, \forall k; \tag{4.11}
\]

\[
\xi_0 = \frac{e^{-\rho_0}(e-1)}{2(1-e^{-\rho_0}) + 2e^{-\rho_0}(e-1)} < \frac{e^{-\rho_0}}{1+2e^{-\rho_0}} < D^2; \quad \text{and} \tag{4.12}
\]

\[
\frac{9^{1-\beta/2} \kappa_1 \beta BD^2 |M(k, p)|^{\beta/2}}{(e-1)^{2\beta}} \leq 1, \forall k; \tag{4.13}
\]

where \( \kappa_1 = \sup_{k,k' \in E} 9 d_k^{1-\beta} d_k (2(1-e^{-\rho_0}))^\beta \); then for \( m = 1, \ldots, u \)

\[
|\mathcal{V}_m| \leq \frac{e^{-\xi_0}}{\kappa} |\mathcal{V}_{m-1}|, \quad \text{if } p_i = \rho_0 - 1 \text{ and } k_i \in E \text{ for } i = m, m - 1; \tag{4.14}
\]

\[
|\mathcal{V}_m| \leq \frac{1}{\sigma_{c}^{m}_{t} - m - 1} |M(k_{m-1}, p_{m-1})|^{1-\beta} |\mathcal{V}_{m-1}|, \quad \text{if } M(k_{m}, p_{m}) \text{ satisfies (4.11)}; \tag{4.15}
\]

\[
|\mathcal{V}_m| \leq \frac{\kappa_0 \beta BD^2}{(e-1)^{2\beta} \sigma_{c}^{m}_{t} - m - 1} |M(k_{m}, \rho_0 - 1)|^{1-\beta} |\mathcal{V}_{m-1}|, \quad \text{otherwise.} \tag{4.16}
\]

These estimates provide the general bound

\[
|\mathcal{V}_u| \leq |\omega|^{0 |\beta| e^{-\xi_0 (n-u)}} \prod_{k_i \in E \text{ or } p_i \geq \rho_0} |M(k_i, p_i)|^{\beta/2}. \tag{4.17}
\]

Assuming the lemma, we consider the subset

\[
A^u = A^u_{(k_1, p_1), \ldots, (k_u, p_u)}(n) = \{ x \in M : u_n(x) = u \text{ and } f^{i}(x) = M(k_i, p_i), i = 1, \ldots, u \}
\]

where \( p_i \geq \rho_0, i = 1, \ldots, u \) and \( u_n(x) \) for \( x \in \omega \in \mathcal{P}_n \) denotes the number of returns of \( \omega \) until the \( n \)th iterate. Then from (4.17) we get

\[
|A^u_{(k_1, p_1), \ldots, (k_u, p_u)}(n)| \leq \sum_{\omega \in \mathcal{P}_n \cap A^u} e^{-\xi_0 (n-u)} \prod_{k \in E \text{ or } p_i \geq \rho_0} |M(k_i, p_i)|^{\beta/2} \cdot |\omega|^{0 |\beta|} \leq e^{-\xi_0 (n-u)} \prod_{k \in E \text{ or } p_i \geq \rho_0} |M(k_i, p_i)|^{\beta/2} \sum_{\omega_0 \in \mathcal{P}_0} |\omega_0|^{\beta} \leq C_0 e^{-\xi_0 (n-u)} \prod_{k \in E \text{ or } p_i \geq \rho_0} |M(k_i, p_i)|^{\beta/2}, \tag{4.18}
\]

for a constant \( C_0 > 1 \) depending only on \( \beta, \rho_0 \) and \( \mathcal{P}_0 \).

The proof of Lemma 4.4 is contained in the remaining of this subsection.
4.3.1. Between escapes. Now we start the proof of Lemma 4.4. Let us assume that \( m \in \{1, \ldots, u\} \) and \( t_m, t_m - 1 \) are escapes, that is, \( p_m = p_{m-1} = \rho_0 - 1 \) and both \( k_m, k_m - 1 \in E \). Then, by the partition algorithm

\[
\frac{|\omega^{m-1} \setminus \omega^m|}{|\omega^{m-1}|} \geq \frac{1}{D^2} \cdot \frac{|f_{t_m}(\omega^{m-1} \setminus \omega^m)|}{|f_{t_m}(\omega^{m-1})|} \geq \frac{|M(k_m, \rho_0)|}{D^2|\omega^{m-1}|(\rho_0 - 1)^+} = \frac{1}{D^2} \cdot \frac{e^{-\rho_0}(e - 1)}{2(1 - e^{-\rho_0}) + 2e^{-\rho_0}(e - 1)} = \frac{\xi_0}{D^2} \in (0, 1),
\]

by the condition (4.12) on \( \rho_0 \); recall Figure 6 and note that this does not depend on \( k \). This proves (4.14) since

\[
|\mathcal{V}_m| = \sum_{\omega^m \in \mathcal{V}_m} |\omega^m| = \sum_{\omega^m \in \mathcal{V}_m} \frac{|\omega^{m-1} \setminus (\omega^{m-1} \setminus \omega^m)|}{|\omega^{m-1}|} |\omega^{m-1}|
\leq \left(1 - \frac{\xi_0}{D^2}\right) \sum_{\omega^m \in \mathcal{V}_m} |\omega^{m-1}| \leq e^{-\zeta} |\mathcal{V}_{m-1}|, \text{ for } \zeta = -\log(1 - \xi_0/D^2).
\]

Remark 4.5. (1) Note that \( \zeta = \zeta(\rho_0) \xrightarrow{\rho_0 \to \infty} 0. \)

(2) Since \( 1 \geq |f_{t_m}(\omega^{m-1})| \geq \sigma^{t_m-t_{m-1}}|f_{t_m}(\omega^{m-1})| \geq \sigma^{t_m-t_{m-1}}|M(k_{m-1}, \rho_0 - 1)| \) we deduce \( t_m - t_{m-1} \leq -\log|M(k_{m-1}, \rho_0 - 1)|/\log \sigma \leq L_0 \) for a uniform constant \( L_0 \) depending only on the minimum length of \( M(k, \rho_0 - 1) \) for \( k \in E \). This shows that the number of iterates between consecutive escape times is uniformly bounded by a constant which does not depend on \( \rho_0 \).

4.3.2. Between returns which are not consecutive escapes. We fix \( m \in \{1, \ldots, u\} \). On the one hand, note that if \( t_{m-1} \) is a return with \( p_{m-1} \geq \rho(k) \geq \rho_0 \), then by the partition algorithm, either \( b = b_{k_{m-1}} \) is a singularity and we get the bound

\[
\frac{|\omega^m|}{|\omega^{m-1}|} \leq \frac{|\omega^m|}{|\omega^{m-1}|} \leq D^2 \cdot \frac{|f_{t_m}(\omega^m)|}{|f_{t_m}(\widehat{\omega}^{m-1})|}, \text{ where } \widehat{\omega}^{m-1} = \omega^{m-1} \cap f_{t_m}(U_0)
\leq D^2 \cdot \frac{|M(k_m, p_m)^+|}{\sigma^{t_m-t_{m-1}}|f'(\xi)| \cdot |f_{t_m}(\widehat{\omega}^{m-1})|} \text{ for some } \xi \in f_{t_m}^{-1} \widehat{\omega}^{m-1};
\]

or \( b \) is a discontinuity and we use the bound

\[
|f_{t_m} \widehat{\omega}^{m-1}| \geq \sigma^{t_m-t_{m-1}-T(b)-1} |f'(\zeta)| \cdot |f_{t_m-T(b)}(\widehat{\omega}^{m-1})| \geq \sigma^{t_m-t_{m-1}-1} |f'(\zeta)| \cdot |f_{t_m-M} \widehat{\omega}^{m-1}|
\]

for some \( \zeta \in f_{t_m-M} \widehat{\omega}^{m-1} \). Note that in the case that \( b \) is a singularity, by the choice of \( \widehat{\omega}^{m-1} \)

\[
|f'(\xi)| \geq B^{-1} (d_{m-1} e^{-p_{m-1}})^{-\beta} = B^{-1} (e - 1)^{\beta} |M(k_{m-1}, p_{m-1})|^{-\beta} \quad (4.19)
\]
where $M(k_{m-1}, p_{m-1}) \subset f^{m-1} \omega^{m-1} \subset f^{m-1} \omega^{m-1}$ by definition of return time. Otherwise, $b$ is a discontinuity and by the construction of the partition

$$|f'(\zeta)| \geq B^{-1} (d_{m-1} e^{-p_{m-1}})^{-\beta} = \frac{(e - 1)^\beta}{B} |M(k_{m-1}, p_{m-1})|^{-\beta}$$

$$= \frac{(e - 1)^\beta}{B} |f^T M(k_{m-1}, p_{m-1})|^{-\beta} \geq \frac{(e - 1)^\beta}{B k_0^\beta} |M(k_{m-1}, p_{m-1})|^{-\beta}$$

where $b_{m-1} = f^T b$ is the singularity to which $b$ is connected.

On the other hand, if $p_{m-1} = \rho_0 - 1$, then $f^{m-1} \omega^{m-1} \subset M(k_{m-1}, \rho_0 - 1)$ and we replace (4.19) by

$$|f'(\zeta)| \geq B^{-1} d_{m-1}^{-\beta} = B^{-1} [2(1 - e^{-\rho_0})]^\beta [2 d_{m-1} (1 - e^{-\rho_0})]^{-\beta}$$

$$= B^{-1} [2(1 - e^{-\rho_0})]^\beta |M(k_{m-1}, \rho_0 - 1)|^{-\beta}$$

and since $2(1 - e^{-\rho_0}) > e - 1$ from (4.11), in any of the three cases above we arrive at

$$\frac{\omega^m}{|\omega^{m-1}|} \leq \frac{\kappa_0^2 B D^2}{(e - 1)^\beta \sigma^{t_{m-1} - 1}} \cdot \frac{|M(k_m, p_m)|}{|M(k_{m-1}, p_{m-1})|^{1-\beta}}.$$  (4.21)

Now if $t_m$ is such that either $p_m \geq \rho_0$ or $|M(k_{m-1}, \rho_0 - 1)|$ satisfies (4.11), then we can bound the expression by

$$\frac{\kappa_0^2 B D^2 |M(k_m, p_m)|^{1/2}}{(e - 1)^\beta \sigma^{t_{m-1} - 1}} \cdot \frac{|M(k_m, p_m)|^{1-\beta/2}}{|M(k_{m-1}, p_{m-1})|^{1-\beta}} \leq \frac{|M(k_m, p_m)|^{1-\beta/2}}{\sigma^{t_{m-1} - 1} |M(k_{m-1}, p_{m-1})|^{1-\beta}}.$$ 

Otherwise we keep (4.21).

Now we can obtain (4.15), where we write $\Delta t_m = t_m - t_{m-1}$

$$|\mathcal{V}_m| = \sum_{\omega^m \in \mathcal{V}_m} |\omega^m| = \sum_{\omega^m \in \mathcal{V}_m \cap \mathcal{V}_{m-1}} \frac{|\omega^m|}{|\omega^{m-1}|} |\omega^{m-1}| \leq \frac{|M(k_m, p_m)|^{1-\beta/2} \sigma^{1-\Delta t_m}}{|M(k_{m-1}, p_{m-1})|^{1-\beta}} |\mathcal{V}_{m-1}|.$$ 

Analogously we obtain (4.16) using the bound (4.21).

This completes the proof of all but the last estimate in Lemma 4.4.

4.3.3. Probability of a given sequence of returns. Now we apply the estimates (4.14), (4.15) and (4.16) to prove (4.17). Consider $s_0 = 0 < 1 \leq r_1 < s_1 < r_2 < s_2 \leq \cdots < r_h < s_h \leq u < r_{h+1} = u + 1$ the indexes marking the beginning of sequences of returns in $t_1 < t_2 < \cdots < t_u$ whose depths $(k_i, p_i)$ consecutively satisfy condition (4.11) from $t_r_i$ to $t_s_i$ and do not satisfy (4.11) from $t_s_i$ to $t_{r_{i+1} - 1}$.

In other words, $r_i$ marks the beginning of a sequence of returns satisfying (4.11) and $s_i$ marks the beginning of a sequence of returns not satisfying (4.11).

More precisely, for each $j = 0, \ldots, h$ and each $s_j \leq i < r_{j+1}$ we have a return time (escape) $t_i$ with $p_i = \rho_0 - 1$ and $M(k_i, \rho_0 - 1)$ not satisfying (4.11); and each $r_j \leq i < s_j$ corresponds to a return time $t_i$ with $M(k_i, p_i)$ satisfying (4.11).

In this way
• every pair of consecutive escape times introduces a factor $e^{-\zeta}$ as in (4.14);
• following (4.16), the extremes of these sequences of escapes introduce quotients $\frac{|M(k,\rho_0-1)|}{|M(k',\rho_0-1)|} \leq \kappa_1$, for $k, k' \in E$; and
• the remaining factors introduce quotients as in (4.15).

We get from the previous claims, assuming first that $r_1 = 1$ and $s_h = u$, and writing $s = \sum_{j=1}^{h}(r_j - s_{j-1} - 1)$ the number of consecutive pairs of escapes

$$|\mathcal{V}_u| = |\mathcal{V}_0| \prod_{i=1}^{u} \frac{|\mathcal{V}_i|}{|\mathcal{V}_{i-1}|}$$

$$\leq e^{-\zeta s} \frac{|M(k_u, p_u)|^{1-\beta/2} \cdot |\omega|^0}{\sigma^{\Delta t_u - 1} |\omega|^0} \prod_{r_k < i < u} \frac{|M(k_i, p_i)|^{\beta/2}}{\sigma^{\Delta t_i - 1}}$$

$$\cdot \prod_{j=1}^{h-1} \frac{\kappa_0^{2\beta} B^2 D^4}{(e-1)^{2\beta} \rho \sigma^{\Delta t_j + \Delta t_{j-1} - 2}} \frac{|M(k_{s_j}, \rho_0 - 1)^+| \cdot |M(k_{r_j}, p)_{r_j}^+|}{|M(k_{r_j-1}, \rho_0 - 1)|^{1-\beta}} \prod_{r_j < i < s_j} \frac{|M(k_i, p_i)|^{\beta/2}}{\sigma^{\Delta t_i - 1}}$$

$$\leq |M(k_u, p_u)|^{1-\beta/2} \cdot \frac{|\omega|^0 e^{-\zeta s}}{\sigma^{\Delta t_u - 1}} \prod_{j=1}^{h} \frac{1}{\sigma^{\Delta t_j}} \prod_{r_j \leq i < s_j} \frac{|M(k_i, p_i)|^{\beta/2}}{\sigma^{\Delta t_i - 1}}$$

(4.23)

where we used $|\mathcal{V}_0| \leq |\mathcal{V}_0|$, the property (4.13) and the definition of $\kappa_1$. Moreover, if $u = s_h$ then $f^{t_u+i} \omega^u$ has no returns for $j = 1, \ldots, n - t_u$ and so

$$1 \geq |f^n \omega^u| = |f^{n-t_u-1}(f^{t_u+1} \omega^u)| \geq \sigma^{n-t_u-1} |f^{t_u} \omega^u|^{1-\beta/2}$$

where we used (4.19) for the last inequality. This implies

$$|M(k_u, p_u)|^{1-\beta/2} = |M(k_u, p_u)|^{\beta/2} |M(k_u, p_u)|^{1-\beta} \leq |M(k_u, p_u)|^{\beta/2} \sigma^{t_u-n+1}$$

and then (4.23) becomes, writing $R = \sum_{j=1}^{h}(t_{s_j} + 1 - t_{r_j})$ and $r = \sum_{j=1}^{h}(s_j - r_j)$

$$|\mathcal{V}_u| \leq |\omega|^{\beta e^{-\zeta s} \sigma^{t_u-n+1-R+r}} \prod_{k \notin E \text{ or } p_i \geq \rho_0} |M(k_i, p_i)|^{\beta/2}.$$  

(4.24)

Otherwise, we might have $1 < r_1$ or $s_h < u$, and so there is an initial and/or final sequence of consecutive returns which are escape times. Since

$$|\mathcal{V}_u| = |\mathcal{V}_0| \prod_{i=1}^{r_1-1} \frac{|\mathcal{V}_i|}{|\mathcal{V}_{i-1}|} \prod_{i=r_1}^{s_h} \frac{|\mathcal{V}_i|}{|\mathcal{V}_{i-1}|} \prod_{i=s_h+1}^{u} \frac{|\mathcal{V}_i|}{|\mathcal{V}_{i-1}|}$$
we can bound the middle product by the expression (4.23) with \(|M(k_{r_1-1}, \rho_0 - 1)|^{\beta-1}\) in the place of \(|\omega^0|^{\beta}\) and \(s_h\) in the place of \(u\). So we obtain a bound
\[
\frac{|V_{r_1}|}{|V_{r_1-1}|} \prod_{i=r_1+1}^{s_h} \frac{|V_i|}{|V_{i-1}|} \leq \frac{\kappa_0^\beta BD^2(e - 1)^{-\beta}|M(k_{r_1-1}, p_{r_1})^+|}{\sigma^{\Delta t_{r_1-1}}|M(k_{r_1-1}, \rho_0 - 1)|^{1-\beta}} \frac{\kappa_0^\beta BD^2(e - 1)^{-\beta}|M(k_{s_h}, \rho_0 - 1)|}{\sigma^{\Delta t_{s_h-1}}|M(k_{s_h-1}, p_{s_h-1})|^{1-\beta}}.
\]
\[
e^{-\zeta s} \prod_{r_{h+1} \leq i < s_h} \frac{|M(k_i, p_i)|^{\beta/2}}{\sigma^{\Delta t_{i-1}}} \prod_{j=2}^{s_h/2} \frac{1}{\sigma^{\Delta t_{s_j}}} \prod_{r_j \leq i < s_j} \frac{|M(k_i, p_i)|^{\beta/2}}{\sigma^{\Delta t_{i-1}}} \leq e^{-\zeta s} \sigma^{-R+r} \prod_{k \notin E \text{ or } p_i \geq p_0} |M(k_i, p_i)|^{\beta/2}.
\]
Finally, the first factor is bounded according to Subsection 4.3.1
\[
|V_0| \prod_{i=1}^{r_1-1} \frac{|V_i|}{|V_{i-1}|} \leq e^{-\zeta (r_1-2)}|V_1| \leq e^{-\zeta (r_1-2)}|\omega^0| = e^{-\zeta (r_1-s_0-2)}|\omega^0|
\]
and likewise the last factor is bounded by \(e^{-\zeta (u-s_h-1)} = e^{-\zeta (r_{h+1}-s_h-2)}\). Hence writing \(\bar{s} = s + r_{h+1} - s_h - 1 + r_1 - s_0 - 1\) arrive at
\[
|V_0| \leq |\omega^0|^{\beta} e^{-\zeta \bar{s}} \sigma^{-R+r} \prod_{k \notin E \text{ or } p_i \geq p_0} |M(k_i, p_i)|^{\beta/2}.
\]
In addition, by Remark 4.5(2) the total number of iterates between consecutive escape times is related to the number of escapes times by
\[
\bar{s} L_0 \geq \sum_{j=0}^{h} (t_{r_{j+1}} - t_{s_{j+1}}) = n - R,
\]
so \(e^{-\zeta \bar{s}} \leq e^{-\frac{\zeta L_0}{n} (n-R)}\) and, since we can assume without loss of generality that \(e^\zeta < \sigma\), we get
\[
e^{-\zeta \bar{s}} \sigma^{-R+r} \leq e^{-\zeta_0 (n-R)} \leq e^{-\zeta_0 (n-R-\bar{s})} \leq e^{-\zeta_0 (n-u)} \text{ setting } \zeta_0 = \zeta / L_0.
\]
This provides the general bound (4.17) completing the proof of Lemma 4.4.

4.4. Distance to the singular set between returns. We show that the distance to the singular set for iterates \(t_i < j < t_{i+1}\) between return times of a given \(\omega \in \mathcal{P}_n\) is controlled by the depth of the last return time, as follows.

In the same setting since the beginning of this section, by construction of the refinement of the partition we can write
\[
M(k_i, p_i) \subset f^{t_i} \omega^i \subset M(k_i, p_i)^+\], where \(p_i \geq \rho_0 - 1\)
since \(t_i\) is a return. We also have that there are \((k^j, p^j)\) so that \(f^{j} \omega^i \subset M(k^j, p^j)^+, p^j \geq \rho_0 - 1\) for each \(j = t_i + 1, \ldots , t_{i+1} - 1\). These are called the host intervals at iterate \(t_i < j < t_{i+1}\).
Moreover, we can compare the distance to the singular set with the length of the host interval following (4.8) and noting that \( d(M(k, \rho_0 - 1)^{+}, D) = d(M(k, \rho(k)), D) \).

We now divide the iterates \( j = t_i, \ldots, t_{i+1} - 1 \) into sequences of consecutive visits near a singularity and connection iterates between a discontinuity and a singularity, as follows.

If \( a_{k_i} \in D \setminus S \) (a discontinuity near \( f^{t_i} \omega^i \)) then, as a consequence of (4.5), we can assume without loss of generality that \( \delta_0 > \delta \) so that \( \text{dist}_{\delta}(f^{j} \omega^j, D) = 1 \) for \( t_i < j < t_i + T \) where \( T = T(a_{k_i}) \). By construction of the partition \( f^{t_i+1}(\omega^i) \supset M(k_i^*, p) \) and so \( |f^{t_i+1}(\omega^i)| \geq \sigma^T |f^{t_i} \omega^i| \geq |M(k_i, p_i)| \) which implies from the relation (4.8)

\[
\sum_{j=t_i}^{t_i+T} \log \text{dist}_{\delta}(f^{j} \omega^j, D) \leq -2 \log \text{dist}_{\delta}(f^{t_i} \omega^i, D) \leq -2 \log \frac{|M(k_i, p_i)|}{\kappa_0(e - 1)}. 
\] (4.25)

**Remark 4.6.** We can assume without loss of generality that \( p_i \geq \rho_0 \), since

1. In the particular case \( p_i = \rho_0 - 1 \) and \( k_i \in E \) (that is, an escape iterate), we can assume without loss of generality that \( \delta < d(M(k, \rho_0 - 1), D), \forall k \in E \). Consequently, we have that \( M(k_i, \rho_0 - 1)^{+} \supset f^{t_i} \omega^i \) is \( \delta \)-away from \( D \), so all the iterates \( t_i \leq j < t_i + 1 \) are truncated: \( \sum_{j=t_i}^{t_i+1} \log \text{dist}_{\delta}(f^{j} \omega^j, D) = 0. \)

2. In addition, if \( p_i = \rho_0 - 1 / k \notin E \), then we can also assume without loss of generality that \( |f(M(k_i, \rho_0 - 1))| \geq \delta_0 / 2 > \delta \), by letting \( \rho_0 \) be big enough and using the big images property (item (6) of Theorem 3). Hence, we can again ensure that \( f^{j} \omega^j \) for \( t_i < j < t_i - 1 \) is \( \delta \)-away from \( D \) and \( \sum_{j=t_i+1}^{t_i+1} \log \text{dist}_{\delta}(f^{j} \omega^j, D) = 0. \)

By the Mean Value Theorem and the properties of \( f \) near the singularities we get

\[
\frac{|f^{j+1} \omega^j|}{|f^j \omega^j|} = |f'(\xi_j)| \geq \frac{d(\xi_j, S)^{-\beta}}{B}, \quad \text{for some } \xi_j \in f^{j} \omega^i \text{ if } a_{k^j_i} \text{ is a singularity.}
\]

On the one hand, if \( (k^j, p^j) \) is such that \( a_{k^j} \in S \) and \( p^j \geq \rho_0 \), we obtain

\[
\frac{|f^{j+1} \omega^j|}{|f^j \omega^j|} \geq \frac{(d_{k^j} e^{-p^j - 1})^{-\beta/2}}{B(e - 1)^{\beta/2}} |M(k^j, p^j + 1)|^{-\beta/2} \geq |M(k^j, p^j + 1)|^{-\beta/2}
\]

by the conditions (4.11) on \( \rho_0 \). On the other hand, if \( a_{k^j} \in S \) again but \( p^j = \rho_0 - 1 \) and \( k^j \in E \), then we can use the big images property as in Remark 4.6 to truncate all the distances to \( D \) for all \( j + 1, \ldots, t_{i+1} - 1 \).

In particular, if \( j = t_i + T(a_{k_i}) \) satisfies \( p_i \geq \rho_0 \) (the other cases are truncated by Remark 4.6) then, by the construction of the refinement

\[
|f^{t_i+T+1} \omega^j| \geq \frac{(d_{k_i} e^{-p_i - 1})^{-\beta/2}}{B(e - 1)^{\beta/2}} |M(k_i^*, p_i)|^{-\beta/2} |f^{t_i+T} \omega^j| \geq |M(k_i^*, p_i)|^{1-\beta/2} > |M(k_i, p_i)|^{1-\beta/2}
\]

where we have used condition (4.11) on \( \rho_0 \).

Finally, if \( a_{k_i} \in D \setminus S \) and \( p_i \geq \rho_0 \), then we know from (4.5) that there are no returns during times \( j + 1, \ldots, j + T \) where \( T = T(a_{k_i}) \) and so

\[
\frac{|f^{j+T+1} \omega^j|}{|f^j \omega^j|} \geq \sigma^T \frac{(d_{k_i} e^{-p_i - 1})^{-\beta/2}}{B(e - 1)^{\beta/2}} |M(k^j, p^j + 1)|^{-\beta/2} > |M(k^j, p^j + 1)|^{-\beta/2}
\]
where \( M(k^j, p^j)^+ \) is the host interval of \( f^{j+T} \omega_i \). Hence, denoting by \( t_i < \ell_1 < \cdots < \ell_s < t_{i+1} \) the free iterates that have an host interval near a discontinuity and \( T_m = T(f^m a_{k_m}^m), m = 1, \ldots, s \), we obtain the relation

\[
|f^{t_i+1} \omega_i| = |f^{t_i+T} \omega_i| \prod_{j=t_i+T}^{t_i} \prod_{j=t_i}^{j=m} \left( \prod_{j=m}^{j=1} |f^{j+1} \omega_i| \right) \prod_{j=m+1}^{j=\ell_m} |f^{j+1} \omega_i| \prod_{j=m+1}^{j=\ell_s+1} |f^{j+1} \omega_i| |M(k^j, p^j + 1)|^{-\beta/2}.
\]

Since \( |f^{t_i+1} \omega_i| \leq 1 \) we deduce that \(- (1 - \beta/2) \log |M(k^j, p^j)|\) is bounded from below by

\[
\log \sigma \sum_{m=1}^{t_i} T_m - \frac{\beta}{2} \sum_{t_i+T_s < \ell < t_i+1} \log |M(k^j, p^j + 1)|
\]

\[
\geq \sum_{m=1}^{s} \left( T_m \log \sigma - \frac{\beta}{2} \log(\kappa_0 |M(k^{\ell_m}, p^{\ell_m} + 1)|) \right) - \frac{\beta}{2} \sum_{t_i+T_s < \ell < t_i+1} \log |M(k^j, p^j + 1)|
\]

\[
\geq -3\beta \sum_{m=1}^{s} \log |M(k^{\ell_m}, p^{\ell_m} + 1)| - \frac{\beta}{2} \sum_{t_i+T_s < \ell < t_i+1} \log |M(k^j, p^j + 1)|
\]

\[
\geq -\frac{\beta}{2} \left( \sum_{m=1}^{s} \log |M(k^{\ell_m}, p^{\ell_m} + 1)| + \sum_{t_i+T_s < \ell < t_i+1} \log |M(k^j, p^j + 1)| \right)
\]

(4.26)

where we used \( |M(k^{\ell_m+T_m}, p^{\ell_m} + 1)| \leq \kappa_0 |M(k^{\ell_m}, p^{\ell_m} + 1)| \) in the second inequality above and let \( \rho_0 \in \mathbb{Z}^+ \) big enough so that

\[
|M(k^j, p^j)|^{-\beta/2} > \kappa_0^2 (e - 1) \quad \text{for} \ p^j \geq \rho_0 \text{ and all} \ k.
\]

(4.27)
Now using (4.8) we get
\[
\sum_{j=t_i+T}^{t_{i+1}-1} \log \text{dist}_\delta(f^j \omega_i, D) \geq \sum_{m=1}^{s} \log \frac{|M(k^{\ell_m}, p^{\ell_m}) + 1|}{\kappa_0(e - 1)} + \sum_{j \text{ is free and } a_{kj} \in S} \log \frac{|M(k^j, p^j) + 1|}{\kappa_0(e - 1)}
\]
\[
\geq \frac{3}{2} \left( \sum_{m=1}^{s} \log |M(k^{\ell_m}, p^{\ell_m}) + 1| + \sum_{j \text{ is free and } a_{kj} \in S} \log |M(k^j, p^j)| \right)
\]
\[
\geq \frac{3}{2} \left( \sum_{m=1}^{s} (T_m \log \sigma + 2 \log |M(k^{\ell_m}, p^{\ell_m}) + 1|) + \sum_{j \text{ is free and } a_{kj} \in S} \log |M(k^j, p^j)| \right)
\]
\[
\geq 3 \left( \sum_{m=1}^{s} \log |M(k^{\ell_m}, p^{\ell_m}) + 1| + \sum_{j \text{ is free and } a_{kj} \in S} \log |M(k^j, p^j)| \right)
\]
where we used \(|M(k^{\ell_m+T_m}, p^{\ell_m} + 1|) \geq \sigma^{T_m} |M(k^{\ell_m}, p^{\ell_m} + 1)|\) in the third inequality above.
Hence we conclude using the last inequality together with (4.26), (4.8) and (4.25)
\[
- \sum_{j=t_i}^{t_{i+1}-1} \log \text{dist}_\delta(f^j \omega_i, D) \leq -\frac{6}{\beta} \left( 1 - \frac{\beta}{2} \right) \log |M(k_i, p_i)| - 2 \log \text{dist}_\delta(f^{t_i} \omega_i, D)
\]
\[
\leq -\frac{2 + 3 \beta}{2 \beta} \log \text{dist}_\delta(f^{t_i} \omega_i, D) \quad (4.28)
\]
Finally, if \(a_{k_i} \in S\), then we need not use the initial sequence of \(T\) iterates, so we obtain a similar inequality to (4.28) with a different constant.

Consequently, we obtain the following upper bound for the recurrence frequency to \(D\) of the orbits of points of \(\omega \in \mathcal{P}_n\) in this setting
\[
- \sum_{j=0}^{n} \log \text{dist}_\delta(f^j \omega, D) \leq -C_0 \sum_{j=0}^{n} \sum_{k_i \in E \text{ or } p_i \geq \rho_0} \log |M(k_i, p_i)| \quad (4.29)
\]
for a constant \(C_0 > 0\) not depending on \(\omega\) or \(n\); and for all \(\delta\) satisfying (4.6), that is,
\(0 < \delta < \min\{\delta_0/2, d(M(k, \rho_0 - 1), D) : k \in E\}\)

4.5. Expected value of splitting depths and exponentially slow recurrence. Here we complete the proof of Theorem C. We estimate the expected value of deep returns up to \(n\) iterates of the dynamics, proving the following statement, where we denote
\[
D_n^\delta(x) = \sum_{j=0}^{n} \sum_{k_i \in E \text{ or } p_i \geq \rho_0} \log |M(k_i, p_i)|
\]
for $\delta = e^{-\Theta}$ with $\Theta \in \mathbb{Z}^+$ big enough (to be defined below) and $x \in \omega \in \mathcal{P}_n$ with $u = u_n(\omega) \leq n$ the return depths $(k_1, p_1), \ldots, (k_u, p_u)$.

**Lemma 4.7.** Given $\varepsilon_0 > 0$ there exists a big enough $\rho_0 \in \mathbb{Z}^+$ such that for any $0 < z < \beta/8$ and all sufficiently big $\Theta \in \mathbb{Z}^+$ and $n \geq 1$ we have $\int e^{zD_n^s(x)} \, dx \leq e^{\varepsilon_0 n}$.

This result provides the exponential slow recurrence bound (1.4) as follows.

### 4.5.1. Measure of bad recurrence

This is now a direct consequence of Tchebishev’s inequality together with (4.29) and Lemma 4.7: given $\varepsilon > 0$ we can choose $0 < \varepsilon_0 < \varepsilon/C_0$ and find $\Theta > \rho_0$ big enough so that

$$\{ x \in M : -\frac{1}{n} \sum_{j=0}^{n-1} \log \text{dist}_\delta (f^j(x), D) \geq \varepsilon \} \subseteq \left\{ x : \frac{D_n^s(x)}{n} \geq \frac{\varepsilon}{C_0} \right\} = \left\{ x : e^{zD_n^s(x)} \geq e^{n\varepsilon/C_0} \right\}.$$  

and so

$$\lambda \left\{ x \in M : -\frac{1}{n} \sum_{j=0}^{n-1} \log \text{dist}_\delta (f^j(x), D) \geq \varepsilon \right\} \leq e^{-n\varepsilon/C} \int e^{zD_n^s} \, dx = e^{-n(\varepsilon/C_0 - \varepsilon_0)}$$

which is exponentially small with rate $\xi = \varepsilon/C_0 - \varepsilon_0$.

This concludes the proof of Theorem C, except for the proof of Lemma 4.7, which comprises the rest of this section.

### 4.5.2. Probability of deep returns

We fix $n \in \mathbb{Z}^+$, $\omega \in \mathcal{P}_n$, and $u \in \{1, \ldots, n\}$ but also $v \in \{1, \ldots, u\}$ and assume that there exists a sequence of deep returns for $\omega$ at indexes $1 \leq r_1 < \cdots < r_v \leq u$ between the return times $0 = t_0 < t_1 < \cdots < t_u \leq n$. That is, for $i = 1, \ldots, v$

$$t_{r_i} \text{ satisfies } f^{r_{r_i}}(x) \in M(\eta_i, v_i) \text{ for all } x \in \omega \text{ and } |M(\eta_i, v_i)| \leq e^{-\Theta}$$

for $\Theta \in \mathbb{Z}^+$, $\Theta > \rho_0$ to be defined below.

We let $u_n(x), v_n(x)$ for $x \in \omega \in \mathcal{P}_n$ be the number of returns and deep returns of $\omega$ until the nth iterate. Then we define

$$A^{u,v} = A_{(k_1, p_1), \ldots, (k_u, p_u)}^{u,v}(n) = \{ x \in M : u_n(x) = u, v_n(x) = v \text{ and } (k_i, p_i) = (k_i, p_i), i = 1, \ldots, v \}$$

the set of points which in $n$ iterates have $u$ returns and $v$ deep returns among these, with the specified depths $(k_1, p_1), \ldots, (k_u, p_u)$.

From (4.18) we know how to estimate the probability of a certain sequence of returns and have seen that the estimate does not depend on the particular order of the returns nor on their return times, but only on theirs depths. Using (4.18) we estimate the measure of $A^{u,v}$ considering all possible combinations of the events $Y_u$ which are included in $A^{u,v}$. Note that for any given $v \leq u$ there are $\binom{u}{v}$ ways of having $v$ deep returns among $u$
return situations and \( \binom{n}{u} \) choices of the times of the return situations. Hence the Lebesgue measure of \( A_{u,v}^{(n_1,p_1),\ldots,(n_u,p_u)}(n) \) is bounded from above by

\[
|A_{u,v}^{(n)}| \leq \binom{n}{u} \sum_{\omega \in P_n \cap A_{u,v}^{(n)}} \left( \frac{u}{v} \right)^v \prod_{i=1}^u |M(\kappa_i, p_i)|^{\beta/2} \cdot e^{-\xi_0(n-u)} \prod_{i=1}^v |M(\kappa_j, p_j)|^{\beta/2} \cdot |\omega^0|^\beta
\]

\[
\leq \binom{n}{u} e^{-\xi_0(n-u)} \left( \frac{u}{v} \right)^v \prod_{i=1}^u |M(\kappa_i, p_i)|^{\beta/2} \sum_{\omega^0 \in P_0} |\omega^0|^\beta
\]

\[
\leq \binom{n}{u} e^{-\xi_0(n-u)} \left( \frac{u}{v} \right)^v \prod_{i=1}^u |M(\kappa_i, p_i)|^{\beta/4}
\]

(4.30)

as long as \( \Theta \) is sufficiently large to compensate the constant \( C_0 \).

4.5.3. Probability of no (deep) returns. We consider the special cases of intervals in \( P_n \) which return only a given number of times for all \( n \) or have no deep returns.

Finitely many returns. We show that no interval \( \omega \in P_n \) is such that \( \omega \in P_{n+k} \) for all \( k \geq 1 \), that is, it is not possible that \( \omega \) has only finitely many returns.

Indeed, let \( \omega \in P_n \) be so that \( \omega \in P_{n+k} \) and \( u_n(\omega) = u \leq n \) for all \( k \geq 1 \). Let \( t_u \leq n-1 \) be the last return and note that \( \omega = \omega^u \). Then

\[
1 \geq |f^{n+k}_u| \geq |f^{n+k-t_u}(f_tu)\omega| \geq \sigma^{u+k-t_u}|f^{t_u}\omega|
\]

and so \( |f^{t_u}\omega| \leq \sigma^{t_u-n-k}, \forall k \geq 1 \) which is impossible if \( \omega = \omega^u \) is an interval.

No returns. Consider now the family of atoms of \( P_0 \) which have no returns during the first \( n \) iterates. Then by uniform expansion of lengths at a rate \( \sigma \), this family has total measure given by

\[
\sum_{\omega^0 \in P_0} |\omega^0| = \sum_{\omega^0 \in P_0, |\omega^0| \leq \delta_k} |\omega^0| + \sum_{\omega^0 \in P_0, |\omega^0| > \delta_k} |\omega^0| \leq \sum_{d_k < \sigma^{-n/2}} |\omega^0| + \sum_{d_k \sigma^{n/2} \geq 1} |\omega^0| \leq |B(S, \sigma^{-n/2})| + \sum_{\substack{p > \log (e^{-1}) \sigma^{n/2} \geq e^{-p}}} e^{-p} \leq Ce^{-cn}
\]

(4.31)

for some constants \( C, c > 0 \) using \( d_k \leq 1 \) and the expression for the length of the atoms of \( P_0 \) together with a condition on discreetness of \( S \): \( |B(S, \varepsilon)| \leq \kappa \varepsilon^d \) for some \( \kappa, d > 0 \) and all small enough \( \varepsilon > 0 \).

No deep returns. We can also estimate the measure of the family of all intervals \( \omega \in P_n \) which have no deep returns.

Indeed, let us consider the family \( A_u(n) \) of all intervals \( \omega \in P_n \) having \( 1 \leq u < n \) returns at times \( t_1 < \cdots < t_u < n \) none of those are deep. This means that the depths are given by
\((\kappa_1, \rho_1), \ldots, (\kappa_u, \rho_u) \in Q(\Theta)\) where \(Q(\Theta) = \{(\kappa, \rho) : |M(k, \rho)| \geq e^{-\Theta}, p \geq \rho_0\}\). Considering all the possible combinations of sequences of returns we get

\[
|A^n_u(n)| \leq \sum_{u=1}^{n-1} \binom{n}{u} \sum_{(\kappa_i, \rho_i) \in Q(\Theta)} \left| A^n_{(\kappa_1, \rho_1), \ldots, (\kappa_u, \rho_u)}(n) \right|
\]

\[
\leq \sum_{u=1}^{n-1} \binom{n}{u} \sum_{(\kappa_i, \rho_i) \in Q(\Theta)} C_0 e^{-\zeta_0(n-u)} \prod_{i=1}^{u} |M(\kappa_i, \rho_i)|^{\beta/2}
\]

\[
\leq C_0 \sum_{u=1}^{n-1} \binom{n}{u} e^{-\zeta_0(n-u)} \left( \sum_{(\kappa, \rho) \in Q(\Theta)} |M(\kappa, \rho)| \right)^{\beta/2}
\]

\[
\leq C_0 \sum_{u=1}^{n-1} \binom{n}{u} e^{-\zeta_0(n-u)} \left( \sum_{k=p \geq \rho_0} d_k e^{-p} \right)^{\beta/2}
\]

\[
\leq C_0 \sum_{u=1}^{n-1} \binom{n}{u} e^{-\zeta_0(n-u)} e^{-\rho_0} \left( \frac{1}{1-e^{-\rho_0}} \right)^{\beta/2}
\]

\[
\leq C_0 \left( e^{-\zeta_0} + \left( \frac{e^{-\rho_0}}{1-e^{-\rho_0}} \right)^{\beta/2} \right) n
\]

where we can make

\[
\xi_1 = e^{-\zeta/L} + \left( \frac{e^{-\rho_0}}{1-e^{-\rho_0}} \right)^{\beta/2} < 1
\]

by choosing a bigger \(\rho_0 \in \mathbb{Z}^+\) if needed.

4.5.4. Expected value of deep return depths. We can now complete the proof of Lemma 4.7. By definition

\[
\int e^{zD_n^u(x)} \, dx = \sum_{\omega \in \mathcal{P}_n} e^{zD_n^u(\omega)} \cdot |\omega| \leq \sum_{\omega \in \mathcal{P}_n} |\omega_0| + \sum_{\omega \in \mathcal{P}_n} |\omega_0|
\]

\[
+ \sum_{0 < v_n(\omega) \leq u_n(\omega) < n} e^{zD_n^u(\omega)} |A^n_{(\kappa_1, \rho_1), \ldots, (\kappa_u, \rho_u)}(n)|
\]

where after the inequality

- in the first sum we are considering all \(\omega \in \mathcal{P}_n\) having no returns up to the \(n\)th iterate;
- in the second sum we are considering all \(\omega \in \mathcal{P}_n\) having no deep returns;
- in the third sum, we are considering all possible combinations of return depths and of deep returns among all the splitting times.
The first two sums are exponentially small in $n$ according to the estimates from Subsection 4.5.3. For the last sum (4.33), we use (4.30) to get the bound
\[
\sum_{0<v \leq u<n} \binom{n}{u} e^{-\zeta_0(n-u)} \sum_{i=1}^{v} \binom{u}{v} e^{(z-\beta/4)\sum_{i=(\rho_i)}} L(h, v) e^{(z-\beta/4)h} u
\]
where $\eta_i = \lfloor -\log d_{\xi_i} \rfloor$ and $h = \sum_i (\eta_i + \rho_i)$, $\ell(\delta)$ is an integer such that every pair $(k, \rho)$ satisfying $d_k e^{-p} < \delta = e^{-\Theta}$ also satisfies $\eta_k + \rho > \ell(\delta)$, and
\[
L(h, v) = \# \left\{ \left( (\eta_i, \rho_i) \right)_{i=1}^{v} \in \mathbb{Z}_{2u}^{2u} : \sum_{i=1}^{v} (\eta_i + \rho_i) = h \text{ with } \rho_i \geq \Theta \right\}
\]
counts the number of possible sums of $2v$ integers equal to a given integer $h$. Since by a standard application of Stirling’s Formula we have can find a constant $a > 0$ independent of the other variables so that
\[
L(h, v) \leq \# \left\{ (h_i) \in \mathbb{Z}_{2u}^{2u} : \sum_{i=0}^{2v} h_i = h \right\} = \binom{h + 2v - 1}{2v - 1} \leq e^{\beta h/8}
\]
and the last inequality follows by $h \geq \Theta v$. Hence if $0 < z < \beta/8$, then (4.33) can be bounded by
\[
\sum_{u=2}^{n} \binom{n}{u} e^{-\zeta_0(n-u)} \sum_{v=1}^{u} \binom{u}{v} e^{(z-\beta/8)h} \leq \sum_{u=2}^{n} \binom{n}{u} e^{-\zeta_0(n-u)} \left( 1 + \frac{e^{(z-\beta/8)\ell(\delta)}}{1 - e^{z-\beta/8}} \right)^{-u}
\]
where
\[
\xi(\delta) = \log \left( 1 + e^{-\zeta_0} + \frac{e^{(z-\beta/8)\ell(\delta)}}{1 - e^{z-\beta/8}} \right)
\]
and for any given $\varepsilon > 0$, we can find $\Theta > \rho_0$ so that for $0 < \delta < e^{-\Theta}$ we have $0 < \xi(\delta) < \varepsilon_0$, by Remark 4.5(1) and because $\ell(\delta) = \ell(e^{-\Theta}) \xrightarrow{\Theta \to \infty} 0$. The proof of Lemma 4.7 is complete.

**Remark 4.8.** (1) The estimates depend on a finite number of constraints on $\rho_0$ given by (4.11), (4.12), (4.13), (4.27) and (4.32) which are easily seen to be satisfied for a sufficiently big $\rho_0 \in \mathbb{Z}_{+}$. (2) The value of $\rho_0$ ultimately depends on $\sigma, \beta, D, \kappa_0, B$ which are given by the assumptions (1-6) of Theorem 4.8. Thus, a family of maps satisfying these assumptions with
uniform values of these constants also satisfies the exponentially slow recurrence condition with a uniform value for the decay rate $\xi$.

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