ON POINTWISE AND WEIGHTED ESTIMATES FOR
COMMUTATORS OF CALDERÓN-ZYGMUND
OPERATORS

ANDREI K. LERNER, SHELDY OMBROSI, AND ISRAEL P. RIVERA-RÍOS

ABSTRACT. In recent years, it has been well understood that a
Calderón-Zygmund operator $T$ is pointwise controlled by a finite
number of dyadic operators of a very simple structure (called the
sparse operators). We obtain a similar pointwise estimate for the
commutator $[b, T]$ with a locally integrable function $b$. This result
is applied into two directions. If $b \in BMO$, we improve several
weighted weak type bounds for $[b, T]$. If $b$ belongs to the weighted
$BMO$, we obtain a quantitative form of the two-weighted bound
for $[b, T]$ due to Bloom-Holmes-Lacey-Wick.

1. Introduction

1.1. A pointwise bound for commutators. In the past decade, a
question about sharp weighted inequalities has led to a much better
understanding of classical Calderón-Zygmund operators. In particular,
it was recently discovered by several authors (see [5, 19, 21, 24, 25], and
also [1, 8] for some interesting developments) that a Calderón-Zygmund
operator is dominated pointwise by a finite number of sparse operators
$A_S$ defined by

$$A_S f(x) = \sum_{Q \in S} f_Q \chi_Q(x),$$

where $f_Q = \frac{1}{|Q|} \int_Q f$ and $S$ is a sparse family of cubes from $\mathbb{R}^n$ (the
latter means that each cube $Q \in S$ contains a set $E_Q$ of comparable
measure and the sets $\{E_Q\}_{Q \in S}$ are pairwise disjoint).

In this paper we obtain a similar domination result for the commuta-
tor $[b, T]$ of a Calderón-Zygmund operator $T$ with a locally integrable

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function $b$, defined by
\[
[b, T]f(x) = bTf(x) - T(bf)(x).
\]

Then we apply this result in order to derive several new weighted weak and strong type inequalities for $[b, T]$.

Throughout the paper, we shall deal with $\omega$-Calderón-Zygmund operators $T$ on $\mathbb{R}^n$. By this we mean that $T$ is $L^2$ bounded, represented as
\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy \quad \text{for all } x \not\in \text{supp } f,
\]
with kernel $K$ satisfying the size condition $|K(x, y)| \leq \frac{C_K}{|x-y|^{n}}$, $x \neq y$, and the smoothness condition
\[
|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \omega \left( \frac{|x - x'|}{|x - y|} \right) \frac{1}{|x - y|^n}
\]
for $|x - y| > 2|x - x'|$, where $\omega : [0, 1] \to [0, \infty)$ is continuous, increasing, subadditive and $\omega(0) = 0$.

In [21], M. Lacey established a pointwise bound by sparse operators for $\omega$-Calderón-Zygmund operators with $\omega$ satisfying the Dini condition $[\omega]_{\text{Dini}} = \int_0^1 \omega(t) \frac{dt}{t} < \infty$. For such operators we adopt the notation
\[
C_T = \|T\|_{L^2 \to L^2} + C_K + [\omega]_{\text{Dini}}.
\]

A quantitative version of Lacey’s result due to T. Hytönen, L. Roncal and O. Tapiola [19] states that
\begin{equation}
|Tf(x)| \leq c_n C_T \sum_{j=1}^{3^n} A_{S_j} \left| f \right|(x).
\end{equation}

An alternative proof of this result was obtained by the first author [24].

In order to state an analogue of (1.1) for commutators, we introduce the sparse operator $T_{S, b}$ defined by
\[
T_{S, b}f(x) = \sum_{Q \in S} |b(x) - b_Q| f_Q \chi_Q(x).
\]
Let $T_{S, b}^*$ denote the adjoint operator to $T_{S, b}$:
\[
T_{S, b}^* f(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |b - b_Q| f \right) \chi_Q(x).
\]

Our first main result is the following. Its proof is based on ideas developed in [24].
Theorem 1.1. Let $T$ be an $\omega$-Calderón-Zygmund operator with $\omega$ satisfying the Dini condition, and let $b \in L^1_{\text{loc}}$. For every compactly supported $f \in L^\infty(\mathbb{R}^n)$, there exist $3^n$ dyadic lattices $\mathcal{D}^{(j)}$ and $\frac{1}{2^{3n}}$-sparse families $\mathcal{S}_j \subset \mathcal{D}^{(j)}$ such that for a.e. $x \in \mathbb{R}^n$,

\begin{equation}
\|[b, T]f(x)\| \leq c_n C_T \sum_{j=1}^{3^n} \left( \mathcal{T}_{\mathcal{S}_j,b}|f|(x) + \mathcal{T}_{\mathcal{S}_j,b}^\ast|f|(x) \right).
\end{equation}

Some comments about this result are in order. A classical theorem of R. Coifman, R. Rochberg and G. Weiss [4] says that the condition $b \in BMO$ is sufficient (and for some $T$ is also necessary) for the $L^p$ boundedness of $[b, T]$ for all $1 < p < \infty$. It is easy to see that the definition of $\mathcal{T}_{\mathcal{S}_j,b}$ is adapted to this condition. In Lemma 4.2 below we show that if $b \in BMO$, then $\mathcal{T}_{\mathcal{S}_j,b}$ is of weak type $(1,1)$. On the other hand, C. Pérez [29] showed that $[b, T]$ is not of weak type $(1,1)$. Therefore, the second term $\mathcal{T}_{\mathcal{S}_j,b}^\ast$ cannot be removed from (1.2).

Notice that the first term $\mathcal{T}_{\mathcal{S}_j,b}$ cannot be removed from (1.2), too. Indeed, a standard argument (see the proof of (2.4) in Section 2.2) based on the John-Nirenberg inequality shows that if $b \in BMO$, then

\begin{equation}
\mathcal{T}_{\mathcal{S}_j,b}^\ast f(x) \leq c_n \|b\|_{BMO} \sum_{Q \in \mathcal{S}} \|f\|_{L^1} \log Q \chi_Q(x).
\end{equation}

But it was recently observed [32] that $[b, T]$ cannot be pointwise bounded by an $L \log L$-sparse operator appearing here.

In the following subsections we will show applications of Theorem 1.1 to weighted weak and strong type inequalities for $[b, T]$.

1.2. Improved weighted weak type bounds. Given a weight $w$ (that is, a non-negative locally integrable function) and a measurable set $E \subset \mathbb{R}^n$, denote $w(E) = \int_E w dx$ and

\begin{equation}
w_f(\lambda) = w\{x \in \mathbb{R}^n : |f(x)| > \lambda\}.
\end{equation}

In the classical work [10], C. Fefferman and E.M. Stein obtained the following weighted weak type $(1,1)$ property of the Hardy-Littlewood maximal operator $M$: for an arbitrary weight $w$,

\begin{equation}
w_{Mf}(\lambda) \leq \frac{c_n}{\lambda} \int_{\mathbb{R}^n} |f(x)| Mw(x) dx \quad (\lambda > 0).
\end{equation}

Only forty years after that, M.C. Reguera and C. Thiele [34] gave an example showing that a similar estimate is not true for the Hilbert transform instead of $M$ on the left-hand side of (1.3) (they disproved by this the so-called Muckenhoupt-Wheeden conjecture). On the other hand, it was shown earlier by C. Pérez [28] that an analogue of (1.3)
holds for a general class of Calderón-Zygmund operators but with a slightly bigger Orlicz maximal operator $M_{L(\log L)^{\varepsilon}}$ instead of $M$ on the right-hand side. This result was reproved with a better dependence on $\varepsilon$ in [18]: if $T$ is a Calderón-Zygmund operator and $0 < \varepsilon \leq 1$, then

$$w_{TF}(\lambda) \leq \frac{c(n, T)}{\varepsilon} \int_{\mathbb{R}^n} |f(x)|M_{L(\log L)^{\varepsilon}}w(x)dx \quad (\lambda > 0).$$

A general Orlicz maximal operator $M_{\varphi(L)}$ is defined for a Young function $\varphi$ by

$$M_{\varphi(L)}f(x) = \sup_{Q \ni x} \|f\|_{\varphi,Q},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$, and $\|f\|_{\varphi,Q}$ is the normalized Luxemburg norm defined by

$$\|f\|_{\varphi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|} \int_Q \varphi(|f(y)|/\lambda)dy \leq 1\right\}.$$  

If $\varphi(t) = t\log^\alpha(e+t), \alpha > 0$, denote $M_{\varphi(L)} = M_{L(\log L)^{\alpha}}$.

Recently, C. Domingo-Salazar, M. Lacey and G. Rey [9] obtained the following improvement of (1.4): if $C_\varphi = \int_1^\infty \frac{\varphi^{-1}(t)}{t^2 \log(e+t)}dt < \infty$, then

$$w_{TF}(\lambda) \leq \frac{c(n, T)C_\varphi}{\lambda} \int_{\mathbb{R}^n} |f(x)|M_{\varphi(L)^{1+\varepsilon}}w(x)dx.$$  

It is easy to see that if $\varphi(t) = t\log^\alpha(e+t), \alpha > 0$, then $C_\varphi \sim \frac{1}{\varepsilon}$, and thus (1.5) contains (1.4) as a particular case. On the other hand, (1.5) holds for smaller functions than $t\log^\alpha(e+t)$, for instance, for $\varphi(t) = t\log^\alpha(e^\alpha + t), \alpha > 1$. The key ingredient in the proof of (1.5) was a pointwise control of $T$ by sparse operators expressed in (1.1).

Consider now the commutator $[b, T]$ of $T$ with a $BMO$ function $b$. The following analogue of (1.4) was recently obtained by the third author and C. Pérez [31]: for all $0 < \varepsilon \leq 1$,

$$w_{[b,T]f}(\lambda) \leq \frac{c(n, T)}{\varepsilon^2} \int_{\mathbb{R}^n} \Phi(b)\|b\|_{BMO} \frac{|f(x)|}{\lambda} M_{L(\log L)^{1+\varepsilon}}w(x)dx,$$

where $\Phi(t) = t\log(e+t)$. Observe that $\Phi$ here reflects an unweighted $L\log L$ weak type estimate for $[b, T]$ obtained by C. Pérez [29]. Notice also that (1.6) with worst dependence on $\varepsilon$ was proved earlier in [30].

Similarly to the above mentioned improved weak type bound for Calderón-Zygmund operators (1.4), we apply Theorem 1.1 to improve (1.6). Our next result shows that (1.6) holds with $1/\varepsilon$ instead of $1/\varepsilon^2$ and that $M_{L(\log L)^{1+\varepsilon}}$ in (1.6) can be replaced by smaller Orlicz maximal operators.
Theorem 1.2. Let $T$ be an $\omega$-Calderón-Zygmund operator with $\omega$ satisfying the Dini condition, and let $b \in \text{BMO}$. Let $\varphi$ be an arbitrary Young function such that $C_\varphi = \int_1^\infty \frac{\varphi^{-1}(t)}{t \log(e+t)} dt < \infty$. Then for every weight $w$ and for every compactly supported $f \in L^\infty$,

\begin{equation}
 w_{[b,T]}(\lambda) \leq c_n C_T C_\varphi \int_{\mathbb{R}^n} \Phi \left( \frac{\|b\|_{\text{BMO}} |f(x)|}{\lambda} \right) M_{(\Phi \circ \varphi)(L)} w(x) dx,
\end{equation}

where $\Phi(t) = t \log(e+t)$.

By Theorem 1.1, the proof of (1.7) is based on weak type estimates for $T_{S,b}$ and $T_{S,b}^\ast$. The maximal operator $M_{(\Phi \circ \varphi)(L)}$ appears in the weighted weak type $(1,1)$ estimate for $T_{S,b}$. It is interesting that $T_{S,b}^\ast$, being not of weak type $(1,1)$, satisfies a better estimate than (1.7) with a smaller maximal operator than $M_{(\Phi \circ \varphi)(L)}$ (which one can deduce from Lemma 4.5 below).

We mention several particular cases of interest in Theorem 1.2. Notice that if $\varphi(t) \leq t^2$ for $t \geq t_0$, then

$$\Phi \circ \varphi(t) \leq c \varphi(t) \log(e+t) \quad (t > 0).$$

Hence, if $\varphi(t) = t \log(e+t)$, $0 < \varepsilon \leq 1$, then simple estimates along with (1.7) imply

\begin{equation}
 w_{[b,T]}(\lambda) \leq \frac{c(n,T)}{\varepsilon} \int_{\mathbb{R}^n} \Phi \left( \frac{\|b\|_{\text{BMO}} |f(x)|}{\lambda} \right) M_{L(\log L)^{1+\varepsilon}} w(x) dx.
\end{equation}

Similarly, if $\varphi(t) = t \log \log(e^t + e)$, $0 < \varepsilon \leq 1$, then

$$w_{[b,T]}(\lambda) \leq \frac{c(n,T)}{\varepsilon} \int_{\mathbb{R}^n} \Phi \left( \frac{\|b\|_{\text{BMO}} |f(x)|}{\lambda} \right) M_{L(\log L)(\log \log L)^{1+\varepsilon}} w(x) dx.$$

As an application of Theorem 1.2, we obtain an improved weighted weak type estimate for $[b,T]$ assuming that the weight $w \in A_1$. Recall that the latter condition means that

$$[w]_{A_1} = \sup_{x \in \mathbb{R}^n} \frac{M w(x)}{w(x)} < \infty.$$

Also we define the $A_\infty$ constant of $w$ by

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w \chi_Q) dx,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. It was shown in [18] that the dependence on $\varepsilon$ in (1.4) implies the corresponding mixed $A_1-A_\infty$ estimate. In a similar way we have the following.
Corollary 1.3. For every $w \in A_1$,

$$w[b,T]f(\lambda) \leq c_n C_T[w]_{A_1} \Phi([w]_{A_\infty}) \int_{\mathbb{R}^n} \Phi \left( \|b\|_{BMO} \frac{|f(x)|}{\lambda} \right) w(x)dx,$$

where $\Phi(t) = t \log(e + t)$.

This provides a logarithmic improvement of the corresponding bound in [27, 31].

1.3. Two-weighted strong type bounds. Let $w$ be a weight, and let $1 < p < \infty$. Denote $\sigma_w(x) = w^{-\frac{1}{p}}(x)$. Given a cube $Q \subset \mathbb{R}^n$, set

$$[w]_{A_p,Q} = \frac{w(Q)}{|Q|} \left( \frac{\sigma_w(Q)}{|Q|} \right)^{p-1}.$$

We say that $w \in A_p$ if

$$[w]_{A_p} = \sup_Q [w]_{A_p,Q} < \infty.$$

As we have mentioned previously, pointwise bounds by sparse operators were motivated by sharp weighted norm inequalities. For example, (1.1) provides a simple proof of the sharp $L^p(w)$ bound for $T$ (see [19, 24]):

(1.9) \hspace{1cm} \|T\|_{L^p(w)} \leq c_{n,p} C_T[w]_{A_p}^{\max \left(1, \frac{1}{p-1}\right)}. \hspace{1cm} (1 < p < \infty)

In the case of $\omega$-Calderón-Zygmund operators with $\omega(t) = ct^\delta$, (1.9) was proved by T. Hytönen [15] (see also [16, 23] for the history of this result and a different proof).

An analogue of (1.9) for the commutator $[b, T]$ with a $BMO$ function $b$ is the following sharp $L^p(w)$ bound due to D. Chung, C. Pereyra and C. Pérez [3]:

(1.10) \hspace{1cm} \|[b, T]\|_{L^p(w)} \leq c(n, p, T) \|b\|_{BMO} [w]_{A_p}^{2\max \left(1, \frac{1}{p-1}\right)}. \hspace{1cm} (1 < p < \infty)

Much earlier, S. Bloom [2] obtained an interesting two-weighted result for the commutator of the Hilbert transform $H$: if $\mu, \lambda \in A_p, 1 < p < \infty, \nu = (\mu/\lambda)^{1/p}$ and $b \in BMO_\nu$, then

(1.11) \hspace{1cm} \|[b, H]f\|_{L^p(\lambda)} \leq c(p, \mu, \lambda) \|b\|_{BMO_\nu} \|f\|_{L^p(\mu)}.

Here $BMO_\nu$ is the weighted $BMO$ space of locally integrable functions $b$ such that

$$\|b\|_{BMO_\nu} = \sup_Q \frac{1}{\nu(Q)} \int_Q |b - b_Q| dx < \infty.$$
Recently, I. Holmes, M. Lacey and B. Wick [13] extended (1.11) to \( \omega \)-Calderón-Zygmund operators with \( \omega(t) = ct^\delta \); the key role in their proof was played by Hytönen’s representation theorem [15] for such operators. In the particular case when \( \mu = \lambda = w \in A_2 \) the approach in [13] recovers (1.10) (this was checked in [14]; and also, (1.11) was extended in this work to higher-order commutators).

Using Theorem 1.1, we obtain the following quantitative version of the Bloom-Holmes-Lacey-Wick result. It extends (1.11) to any \( \omega \)-Calderón-Zygmund operator with the Dini condition, and the explicit dependence on \([\mu]_{A_p}\) and \([\lambda]_{A_p}\) is found. Also, it can be viewed as a natural extension of (1.10) to the two-weighted setting.

**Theorem 1.4.** Let \( T \) be an \( \omega \)-Calderón-Zygmund operator with \( \omega \) satisfying the Dini condition. Let \( \mu, \lambda \in A_p, 1 < p < \infty \), and \( \nu = (\mu/\lambda)^{1/p} \). If \( b \in BMO_\nu \), then

\[
\| [b, T] f \|_{L^p(\lambda)} \leq c_{n,p} C_T \left( [\mu]_{A_p} [\lambda]_{A_p} \right)^{\max \left( 1, \frac{1}{p-1} \right)} \| b \|_{BMO_\nu} \| f \|_{L^p(\mu)}.
\]

The paper is organized as follows. In Section 2, we collect some preliminary information about dyadic lattices, sparse families and Young functions. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2 and Corollary 1.3 and Section 5 contains the proof of Theorem 1.4.

**2. Preliminaries**

2.1. **Dyadic lattices and sparse families.** By a cube in \( \mathbb{R}^n \) we mean a half-open cube \( Q = \prod_{i=1}^n [a_i, a_i + h), h > 0 \). Denote by \( \ell_Q \) the side-length of \( Q \). Given a cube \( Q_0 \subset \mathbb{R}^n \), let \( \mathcal{D}(Q_0) \) denote the set of all dyadic cubes with respect to \( Q_0 \), that is, the cubes obtained by repeated subdivision of \( Q_0 \) and each of its descendants into \( 2^n \) congruent subcubes.

A dyadic lattice \( \mathcal{D} \) in \( \mathbb{R}^n \) is any collection of cubes such that

(i) if \( Q \in \mathcal{D} \), then each child of \( Q \) is in \( \mathcal{D} \) as well;

(ii) every two cubes \( Q', Q'' \in \mathcal{D} \) have a common ancestor, i.e., there exists \( Q \in \mathcal{D} \) such that \( Q', Q'' \in \mathcal{D}(Q) \);

(iii) for every compact set \( K \subset \mathbb{R}^n \), there exists a cube \( Q \in \mathcal{D} \) containing \( K \).

For this definition, as well as for the next Theorem, we refer to [25].

**Theorem 2.1.** (The Three Lattice Theorem) For every dyadic lattice \( \mathcal{D} \), there exist \( 3^n \) dyadic lattices \( \mathcal{D}^{(1)}, \ldots, \mathcal{D}^{(3^n)} \) such that

\[
\{ 3Q : Q \in \mathcal{D} \} = \bigcup_{j=1}^{3^n} \mathcal{D}^{(j)}
\]
and for every cube $Q \in \mathcal{D}$ and $j = 1, \ldots, 3^n$, there exists a unique cube $R \in \mathcal{D}^{(j)}$ of sidelength $\ell_R = 3^j \ell_Q$ containing $Q$.

**Remark 2.2.** Fix a dyadic lattice $\mathcal{D}$. For an arbitrary cube $Q \subset \mathbb{R}^n$, there is a cube $Q' \in \mathcal{D}$ such that $\ell_Q/2 < \ell_{Q'} \leq \ell_Q$ and $Q \subset 3Q'$. By Theorem 2.1, there is $P \in \mathcal{D}^{(j)}$, $j = 1, \ldots, 3^n$, such that $Q \subset P$ and $\ell_P \leq 3\ell_Q$. A similar statement can be found in [17, Lemma 2.5].

We say that a family $\mathcal{S}$ of cubes from $\mathcal{D}$ is $\eta$-sparse, $0 < \eta < 1$, if for every $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \geq \eta |Q|$, and the sets $\{E_Q\}_{Q \in S}$ are pairwise disjoint.

A family $\mathcal{S} \subset \mathcal{D}$ is called $\Lambda$-Carleson, $\Lambda > 1$, if for every cube $Q \in \mathcal{D}$, 

$$\sum_{P \in \mathcal{S}, P \subset Q} |P| \leq \Lambda |Q|.$$ 

It is easy to see that every $\eta$-sparse family is $(1/\eta)$-Carleson. In [25, Lemma 6.3], it is shown that the converse statement is also true, namely, every $\Lambda$-Carleson family is $(1/\Lambda)$-sparse. Also, [25, Lemma 6.6] says that if $\mathcal{S}$ is $\Lambda$-Carleson and $m \in \mathbb{N}$ such that $m \geq 2$, then $\mathcal{S}$ can be written as a union of $m$ families $\mathcal{S}_j$, each of which is $(1 + \frac{\Lambda - 1}{m})$-Carleson. Using the above mentioned relation between sparse and Carleson families, one can rewrite the latter fact as follows.

**Lemma 2.3.** If $\mathcal{S} \subset \mathcal{D}$ is $\eta$-sparse and $m \geq 2$, then one can represent $\mathcal{S}$ as a disjoint union $\mathcal{S} = \bigcup_{j=1}^m \mathcal{S}_j$, where each family $\mathcal{S}_j$ is $\frac{m}{m + (1/\eta) - 1}$-sparse.

Now we turn our attention to augmentation. Given a family of cubes $\mathcal{S}$ contained in a dyadic lattice $\mathcal{D}$, we associate to each cube $Q \in \mathcal{S}$ a family $\mathcal{F}(Q) \subseteq \mathcal{D}(Q)$ such that $Q \in \mathcal{F}(Q)$. In some situations it is useful to construct a new family that combines the families $\mathcal{F}(Q)$ and $\mathcal{S}$. One way to build such a family is the following.

For each $\mathcal{F}(Q)$ let $\tilde{\mathcal{F}}(Q)$ be the family that consists of all cubes $P \in \mathcal{F}(Q)$ that are not contained in any cube $R \in \mathcal{S}$ with $R \subsetneq Q$. Now we can define the augmented family $\tilde{\mathcal{S}}$ as 

$$\tilde{\mathcal{S}} = \bigcup_{Q \in \mathcal{S}} \tilde{\mathcal{F}}(Q).$$

It is clear, by construction, that the augmented family $\tilde{\mathcal{S}}$ contains the original family $\mathcal{S}$. Furthermore, if $\mathcal{S}$ and each $\mathcal{F}(Q)$ are sparse families, then the augmented family $\tilde{\mathcal{S}}$ is also sparse. We state this fact more clearly in the following lemma (see [25, Lemma 6.7] and the above
equivalence between the notions of the \( \Lambda \)-Carleson and \( \frac{1}{\Lambda} \)-sparse families).

**Lemma 2.4.** If \( S \subset \mathcal{D} \) is an \( \eta_0 \)-sparse family then the augmented family \( \tilde{S} \) built upon \( \eta \)-sparse families \( \mathcal{F}(Q), Q \in S \), is an \( \frac{\eta_0}{1+\eta} \)-sparse family.

### 2.2. Young functions and normalized Luxemburg norms.

By a Young function we mean a continuous, convex, strictly increasing function \( \varphi: [0, \infty) \to [0, \infty) \) with \( \varphi(0) = 0 \) and \( \varphi(t)/t \to \infty \) as \( t \to \infty \). Notice that such functions are also called in the literature the \( N \)-functions. We refer to [20, 33] for their basic properties. We will use, in particular, that \( \varphi(t)/t \) is also a strictly increasing function (see, e.g., [20, p. 8]).

We will also use the fact that

\[
\|f\|_{\varphi,Q} \leq 1 \Leftrightarrow \frac{1}{|Q|} \int_Q \varphi(|f(x)|) dx \leq 1.
\]

Given a Young function \( \varphi \), its complementary function is defined by

\[
\bar{\varphi}(t) = \sup_{x \geq 0} (xt - \varphi(x)).
\]

Then \( \bar{\varphi} \) is also a Young function satisfying \( t \leq \bar{\varphi}^{-1}(t) \varphi^{-1}(t) \leq 2t \). Also the following Hölder type estimate holds:

\[
\frac{1}{|Q|} \int_Q |f(x)g(x)| dx \leq 2 \|f\|_{\varphi,Q} \|g\|_{\bar{\varphi},Q}.
\]

Recall that the John-Nirenberg inequality (see, e.g., [12, p. 124]) says that for every \( b \in BMO \) and for any cube \( Q \subset \mathbb{R}^n \),

\[
|\{x \in Q : |b(x) - b_Q| > \lambda\}| \leq e|Q|e^{-\frac{\lambda}{\|b\|_{BMO}}} \quad (\lambda > 0).
\]

In particular, this inequality implies (see [12, p. 128])

\[
\frac{1}{|Q|} \int_Q e^{|b(x)|} dx \leq 1.
\]

From this and from (2.1), taking \( \varphi(t) = e^t - 1 \), we obtain

\[
\|b - b_Q\|_{\varphi,Q} \leq c_n \|b\|_{BMO}.
\]

A simple computation shows that in this case \( \bar{\varphi}(t) \approx t \log(e + t) \), and therefore, by (2.2),

\[
\frac{1}{|Q|} \int_Q |(b - b_Q)g| dx \leq c_n \|b\|_{BMO} \|g\|_{L\log L,Q}.
\]
Notice that many important properties of the Luxemburg normalized norms $\|f\|_{\varphi,Q}$ hold without assuming the convexity of $\varphi$. In particular, we will use the following generalized Hölder inequality.

**Lemma 2.5.** Let $A$, $B$ and $C$ be non-negative, continuous, strictly increasing functions on $[0, \infty)$ satisfying $A^{-1}(t)B^{-1}(t) \leq C^{-1}(t)$ for all $t \geq 0$. Assume also that $C$ is convex. Then

$$\|fg\|_{C,Q} \leq 2 \|f\|_{A,Q}\|g\|_{B,Q}. \tag{2.5}$$

This lemma was proved by R. O’Neil [26] under the assumption that $A$, $B$ and $C$ are Young functions but the same proof works under the above conditions. Indeed, by homogeneity, it suffices to assume that $\|f\|_{A,Q} = \|g\|_{B,Q} = 1$. Next, notice that the assumptions on $A$, $B$ and $C$ easily imply that $C(xy) \leq A(x) + B(y)$ for all $x, y \geq 0$. Therefore, using the convexity of $C$ and (2.1), we obtain

$$\frac{1}{|Q|} \int_Q C(|fg|/2)dx \leq \frac{1}{2} \left( \frac{1}{|Q|} \int_Q A(|f|)dx + \frac{1}{|Q|} \int_Q B(|g|)dx \right) \leq 1,$$

which, by (2.1) again, implies (2.5).

Given a dyadic lattice $\mathcal{D}$, denote

$$M^\mathcal{D}_f(x) = \sup_{Q \ni x, Q \in \mathcal{D}} \|f\|_{\varphi,Q}.$$

The following lemma is a generalization of the Fefferman-Stein inequality (1.3) to general Orlicz maximal functions, and it is apparently well-known. We give its proof for the sake of completeness.

**Lemma 2.6.** Let $\Phi$ be a Young function. For an arbitrary weight $w$,

$$w \{ x \in \mathbb{R}^n : M^\mathcal{D}_f(x) > \lambda \} \leq 3^n \int_{\mathbb{R}^n} \Phi \left( \frac{9^n|f(x)|}{\lambda} \right) Mw(x)dx.$$

**Proof.** By the Calderón-Zygmund decomposition adapted to $M^\mathcal{D}_f$ (see [6, p. 237]), there exists a family of disjoint cubes $\{Q_i\}$ such that

$$\{ x \in \mathbb{R}^n : M^\mathcal{D}_f(x) > \lambda \} = \bigcup_i Q_i$$

and $\lambda < \|f\|_{\Phi,Q_i} \leq 2^n \lambda$. By (2.1), we see that $\|f\|_{\Phi,Q_i} > \lambda$ implies $\int_{Q_i} \Phi(|f(x)|/\lambda) > |Q_i|$. Therefore,

$$w \{ x \in \mathbb{R}^n : M^\mathcal{D}_f(x) > \lambda \} = \sum_i w(Q_i)$$

$$< \sum_i w(Q_i) \int_{Q_i} \Phi(|f(x)|/\lambda)dx \leq \int_{\mathbb{R}^n} \Phi(|f(x)|/\lambda) Mw(x)dx.$$
Now we observe that by the convexity of $\Phi$ and Remark 2.2 there exist $3^n$ dyadic lattices $D^{(j)}$ such that

$$M\Phi f(x) \leq 3^n \sum_{j=1}^{3^n} M\Phi^{D^{(j)}} f(x).$$

Combining this estimate with the previous one completes the proof. $\Box$

Remark 2.7. Suppose that $\Phi(t) = t \log(e + t)$. It is easy to see that for all $a, b \geq 0$,

$$\Phi(ab) \leq 2\Phi(a)\Phi(b).$$

From this and from Lemma 2.6

$$w\{x \in \mathbb{R}^n : M_{L\log L}f(x) > \lambda\} \leq c_n \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) Mw(x)dx.$$

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is a slight modification of the argument in [24]. Although some parts of the proofs here and in [24] are almost identical, certain details are different, and hence we give a complete proof. We start by defining several important objects.

Let $T$ be an $\omega$-Calderón-Zygmund operator with $\omega$ satisfying the Dini condition. Recall that the maximal truncated operator $T^*$ is defined by

$$T^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|y-x| > \varepsilon} K(x,y) f(y) dy \right|.$$

Define the grand maximal truncated operator $M_T$ by

$$M_T f(x) = \sup_{Q \ni x} \sup_{\xi \in Q} |T(f\chi_{R^n\setminus 3Q})(\xi)|,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$.

Given a cube $Q_0$, for $x \in Q_0$ define a local version of $M_T$ by

$$M_{T,Q_0} f(x) = \sup_{Q \ni x, Q \subset Q_0} \sup_{\xi \in Q} |T(f\chi_{3Q_0\setminus 3Q})(\xi)|.$$

The next lemma was proved in [24].

Lemma 3.1. The following pointwise estimates hold:

(i) for a.e. $x \in Q_0$,

$$|T(f\chi_{3Q_0})(x)| \leq c_n \|T\|_{L^1 \rightarrow L^1} |f(x)| + M_{T,Q_0} f(x);$$

(ii) for all $x \in \mathbb{R}^n$,

$$M_T f(x) \leq c_n (\|\omega\|_{\text{Dini}} + C_K)M f(x) + T^* f(x).$$
An examination of standard proofs (see, e.g., [12 Ch. 8.2]) shows that
\[ \max(\|T\|_{L^1 \rightarrow L^1}, \|T^*\|_{L^1 \rightarrow L^{1, \infty}}) \leq c_n C_T. \]

By part (ii) of Lemma 3.1 and by (3.1),
\[ \|M\|_{L^1 \rightarrow L^{1, \infty}} \leq c_n C_T. \]

Proof of Theorem 1.1. By Remark 2.2 there exist 3n dyadic lattices \( \mathcal{G}^{(j)} \) such that for every \( Q \subset \mathbb{R}^n \), there is a cube \( R = R_Q \in \mathcal{G}^{(j)} \) for some \( j \), for which \( 3Q \subset R_Q \) and \( |R_Q| \leq 9^n|Q| \).

Fix a cube \( Q_0 \subset \mathbb{R}^n \). Let us show that there exists a \( \frac{1}{2} \)-sparse family \( F \subset \mathcal{D}(Q_0) \) such that for a.e. \( x \in Q_0 \),
\[ \frac{1}{2} \]
\[ \sum_{Q \in F} \left( |\langle b(x) \rangle - b_{R_{Q_0}} |f|_{3Q} + |(b - b_{R_{Q_0}})f|_{3Q} \right) \chi_Q(x). \]

It suffices to prove the following recursive claim: there exist pairwise disjoint cubes \( P_j \in \mathcal{D}(Q_0) \) such that \( \sum_j |P_j| \leq \frac{1}{2}|Q_0| \) and
\[ \frac{1}{2} \]
\[ + \sum_j |\langle b, T \rangle(f\chi_{3P_j})(x) \chi_{P_j}. \]

Indeed, iterating this estimate, we immediately get (3.3) with \( F = \{P_j^k\}, k \in \mathbb{Z}_+ \), where \( \{P_j^0\} = \{Q_0\}, \{P_j^1\} = \{P_j\} \) and \( \{P_j^k\} \) are the cubes obtained at the \( k \)-th stage of the iterative process.

Next, observe that for arbitrary pairwise disjoint cubes \( P_j \in \mathcal{D}(Q_0) \),
\[ \frac{1}{2} \]
\[ \leq \frac{1}{2} \]
\[ + \sum_j |\langle b, T \rangle(f\chi_{3P_j})(x) \chi_{P_j}. \]

Hence, in order to prove the recursive claim, it suffices to show that one can select pairwise disjoint cubes \( P_j \in \mathcal{D}(Q_0) \) with \( \sum_j |P_j| \leq \frac{1}{2}|Q_0| \) and such that for a.e. \( x \in Q_0 \),
\[ \frac{1}{2} \]
\[ \leq \frac{1}{2} \]
\[ \leq c_n C_T \left( |\langle b(x) \rangle - b_{R_{Q_0}} |f|_{3Q_0} + |(b - b_{R_{Q_0}})f|_{3Q_0} \right). \]
Using that \([b, T] f = [b - c, T] f\) for any \(c \in \mathbb{R}\), we obtain
\begin{equation}
(b, T)(f \chi_{3Q_0}) \chi_{Q_0 \cup \cup_j P_j} + \sum_j \big| [b, T](f \chi_{3Q_0 \setminus 3P_j}) \chi_{P_j}
\end{equation}

\[\leq |b - b_{RQ_0}| \left( |T(f \chi_{3Q_0})| \chi_{Q_0 \cup \cup_j P_j} + \sum_j \big| T(f \chi_{3Q_0 \setminus 3P_j}) \chi_{P_j} \big| \right)\]

\[+ |T((b - b_{RQ_0}) f \chi_{3Q_0})| \chi_{Q_0 \cup \cup_j P_j} + \sum_j \big| T((b - b_{RQ_0}) f \chi_{3Q_0 \setminus 3P_j}) \chi_{P_j} \big|.\]

By (3.2), one can choose \(\alpha_n\) such that the set \(E = E_1 \cup E_2\), where
\[E_1 = \{ x \in Q_0 : |f| > \alpha_n |f|_{3Q_0} \} \cup \{ x \in Q_0 : \mathcal{M}_{T,Q_0} f > \alpha_n C_T |f|_{3Q_0} \}\]
and
\[E_2 = \{ x \in Q_0 : \big((b - b_{RQ_0}) f\big) > \alpha_n \big((b - b_{RQ_0}) f\big)_{3Q_0} \}
\cup \{ x \in Q_0 : \mathcal{M}_{T,Q_0} (b - b_{RQ_0}) f > \alpha_n C_T (b - b_{RQ_0}) f_{3Q_0} \},\]
will satisfy \(|E| \leq \frac{1}{2n+1} |Q_0|\).

The Calderón-Zygmund decomposition applied to the function \(\chi_E\) on \(Q_0\) at height \(\lambda = \frac{1}{2n+1}\) produces pairwise disjoint cubes \(P_j \in \mathcal{D}(Q_0)\) such that
\[\frac{1}{2n+1} |P_j| \leq |P_j \cap E| \leq \frac{1}{2} |P_j|\]
and \(|E \setminus \cup_j P_j| = 0\). It follows that \(\sum_j |P_j| \leq \frac{1}{2} |Q_0|\) and \(P_j \cap E^c \neq \emptyset\). Therefore,
\[\text{ess sup}_{\xi \in P_j} |T(f \chi_{3Q_0 \setminus 3P_j})(\xi)| \leq c_n C_T |f|_{3Q_0}\]
and
\[\text{ess sup}_{\xi \in P_j} |T((b - b_{RQ_0}) f \chi_{3Q_0 \setminus 3P_j})(\xi)| \leq c_n C_T (b - b_{RQ_0}) f_{3Q_0}\]

Also, by part (i) of Lemma 3.1 and by (3.1), for a.e. \(x \in Q_0 \setminus \cup_j P_j\),
\[|T(f \chi_{3Q_0})(x)| \leq c_n C_T |f|_{3Q_0}\]
and
\[|T((b - b_{RQ_0}) f \chi_{3Q_0})(x)| \leq c_n C_T (b - b_{RQ_0}) f_{3Q_0}\]

Combining the obtained estimates with (3.5) proves (3.4), and therefore, (3.3) is proved.

Take now a partition of \(\mathbb{R}^n\) by cubes \(Q_j\) such that \(\text{supp} (f) \subset 3Q_j\) for each \(j\). For example, take a cube \(Q_0\) such that \(\text{supp} (f) \subset Q_0\) and cover \(3Q_0 \setminus Q_0\) by \(3^n - 1\) congruent cubes \(Q_j\). Each of them satisfies \(Q_0 \subset 3Q_j\). Next, in the same way cover \(9Q_0 \setminus 3Q_0\), and so on. The union of resulting cubes, including \(Q_0\), will satisfy the desired property.
Having such a partition, apply (3.3) to each $Q_j$. We obtain a $\frac{1}{2}$-sparse family $F_j \subset D(Q_j)$ such that (3.3) holds for a.e. $x \in Q_j$ with $|Tf|$ on the left-hand side. Therefore, setting $F = \cup_j F_j$, we obtain that $F$ is a $\frac{1}{2}$-sparse family, and for a.e. $x \in \mathbb{R}^n$,

\begin{equation}
(3.6) \quad |[b, T]f(x)| \leq c_n C_T \sum_{Q \in F} (|b(x) - b_{R_Q}| |f|_{3Q} + |(b - b_{R_Q})f|_{3Q}) \chi_{Q}(x).
\end{equation}

Since $3Q \subset R_Q$ and $|R_Q| \leq 3^n |3Q|$, we obtain $|f|_{3Q} \leq c_n |f|_{R_Q}$. Further, setting $S_j = \{R_Q \in \mathcal{D}^{(j)} : Q \in F\}$, and using that $F$ is $\frac{1}{2}$-sparse, we obtain that each family $S_j$ is $\frac{1}{2^9}$-sparse. It follows from (3.6) that

\begin{equation}
[b, T]f(x) \leq c_n C_T \sum_{j=1}^{3^n} \sum_{R \in S_j} (|b(x) - b_R| f_R + |(b - b_R)f_R|) \chi_R(x),
\end{equation}

and therefore, the proof is complete. \hfill \Box

4. Proof of Theorem 1.2 and Corollary 1.3

Fix a dyadic lattice $\mathcal{D}$. Let $S \subset \mathcal{D}$ be a sparse family. Define the $L \log L$ sparse operator by

\begin{equation}
A_{S,L \log L}f(x) = \sum_{Q \in S} \|f\|_{L \log L,Q} \chi_Q(x).
\end{equation}

It follows from (2.4) that

\begin{equation}
|T^s b_{S}f(x)| \leq c_n \|b\|_{\text{BMO}} A_{S,L \log L}f(x).
\end{equation}

Let $\Phi(t) = t \log(e + t)$. Given a Young function $\varphi$, denote

\[ C_\varphi = \int_1^\infty \frac{\varphi^{-1}(t)}{t^2 \log(e + t)} dt. \]

By Theorem 1.1 combined with (1.1), Lemma 2.3 and a submultiplicative property of $\Phi$ expressed in (2.6), Theorem 1.2 is an immediate consequence of the following two lemmas.

Lemma 4.1. Suppose that $S$ is $\frac{31}{32}$-sparse. Let $\varphi$ be a Young function such that $C_\varphi < \infty$. Then for an arbitrary weight $w$,

\[ w A_{S,L \log L}f(\lambda) \leq c C_\varphi \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) M_{(\Phi \circ \varphi)(L)} w(x)dx \quad (\lambda > 0), \]

where $c > 0$ is an absolute constant.
Lemma 4.2. Let $b \in BMO$. Suppose that $S$ is $\frac{7}{8}$-sparse. Let $\varphi$ be a Young function such that $C_\varphi < \infty$. Then for an arbitrary weight $w$,
\[
w_{T_{b, S} f}(\lambda) \leq \frac{c_n C_\varphi \|b\|_{BMO}}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_{(\varphi \circ \Phi)(L)} w(x) dx \quad (\lambda > 0).
\]

In the following subsection we separate a common ingredient used in the proofs of both Lemmas 4.1 and 4.2.

4.1. The key lemma. Assume that $\Psi$ is a Young function satisfying
\[(4.2) \quad \Psi(4t) \leq \Lambda \Psi(t) \quad (t > 0, \Lambda \geq 1).
\]

Given a dyadic lattice $\mathcal{D}$ and $k \in \mathbb{N}$, denote
\[F_k = \{Q \in \mathcal{D} : 4^{k-1} < \|f\|_{\Psi, Q} \leq 4^k\}.
\]

The following lemma in the case $\Psi(t) = t$ was proved in [9]. Our extension to any Young function satisfying (4.2) is based on similar ideas. Notice that the main cases of interest for us are $\Psi(t) = t$ and $\Psi(t) = \Phi(t)$.

Lemma 4.3. Suppose that the family $F_k$ is $\left(1 - \frac{1}{2\Lambda \Psi}\right)$-sparse. Let $w$ be a weight and let $E$ be an arbitrary measurable set with $w(E) < \infty$. Then, for every Young function $\varphi$,
\[
\int_E \left(\sum_{Q \in F_k} \chi_Q\right) w dx \leq 2^k w(E) + \frac{4\Lambda \Psi}{\varphi^{-1}((2\Lambda \Psi)^{2k})} \int_{\mathbb{R}^n} \Psi(4^k |f|) M_{\varphi(L)} w dx.
\]

Proof. By Fatou’s lemma, one can assume that the family $F_k$ is finite. Split $F_k$ into the layers $F_{k, \nu}$, $\nu = 0, 1, \ldots$, where $F_{k, 0}$ is the family of the maximal cubes in $F_k$ and $F_{k, \nu+1}$ is the family of the maximal cubes in $F_k \setminus \bigcup_{t=0}^\nu F_{k, t}$.

Denote $E_Q = Q \setminus \bigcup_{Q' \in F_{k, \nu+1}} Q'$ for each $Q \in F_{k, \nu}$. Then the sets $E_Q$ are pairwise disjoint for $Q \in F_k$.

For $\nu \geq 0$ and $Q \in F_{k, \nu}$ denote
\[A_k(Q) = \bigcup_{Q' \in F_{k, \nu+1}, Q' \subseteq Q} Q'.\]

Observe that
\[Q \setminus A_k(Q) = \bigcup_{t=0}^{2^k-1} \bigcup_{Q' \in F_{k, \nu+1}, Q' \subseteq Q} E_{Q'}.
\]
Using the disjointness of the sets $E_Q$, we obtain
\[
\sum_{Q \in F_k} w\left(E \cap (Q \setminus A_k(Q))\right) \leq \sum_{\nu=0}^{\infty} \sum_{Q \in F_{k,\nu}} \sum_{l=0}^{2^{k-1}} \sum_{Q' \subseteq Q} w(E \cap E_{Q'}) \leq 2^{k} \sum_{Q \in F_k} w(E \cap E_Q) \leq 2^{k} w(E).
\]

(4.3)

Now, let us show that
\[
1 \leq \frac{2\Lambda_{\Psi}}{|Q|} \int_{E_Q} \Psi(4^k|f(x)|) dx \quad (Q \in S_k).
\]

(4.4)

Fix a cube $Q \in F_{k,\nu}$. Since $4^{-k-1} < \|f\|_{\Psi,Q}$, by (2.1) and by (4.2),
\[
1 < \frac{1}{|Q|} \int_Q^{\Psi(4^{k+1}|f|)} \leq \frac{\Lambda_{\Psi}}{|Q|} \int_Q \Psi(4^k|f|).
\]

(4.5)

On the other hand, for any $P \in F_k$ we have $\|f\|_{\Psi,P} \leq 4^{-k}$, and hence, by (2.1),
\[
1 \leq \frac{1}{|P|} \int_P \Psi(4^k|f|) \leq 1.
\]

Using also that, by the sparseness condition, $|Q \setminus E_Q| \leq \frac{1}{2\Lambda_{\Psi}} |Q|$, we obtain
\[
\frac{1}{|Q|} \int_Q \Psi(4^k|f|) = \frac{1}{|Q|} \int_{E_Q} \Psi(4^k|f|) + \frac{1}{|Q|} \sum_{Q' \subseteq Q_{k+1}} \int_{Q'} \Psi(4^k|f|)
\]
\[
\leq \frac{1}{|Q|} \int_{E_Q} \Psi(4^k|f|) + \frac{|Q \setminus E_Q|}{|Q|} \leq \frac{1}{|Q|} \int_{E_Q} \Psi(4^k|f|) + \frac{1}{2\Lambda_{\Psi}},
\]

which, along with (4.5), proves (4.4).

Applying the sparseness assumption again, we obtain $|A_k(Q)| \leq (1/2\Lambda_{\Psi})^{2^k}|Q|$. From this and from Hölder’s inequality (2.2),
\[
\frac{w(A_k(Q))}{|Q|} \leq 2\|\chi_{A_k(Q)}\|_{\varphi,Q} \|w\|_{\varphi,Q} = \frac{2}{\varphi^{-1}(|Q|/|A_k(Q)|)} \|w\|_{\varphi,Q}
\]
\[
\leq \frac{2}{\varphi^{-1}((2\Lambda_{\Psi})^{2^k})} \|w\|_{\varphi,Q}.
\]

Combining this with (4.3) yields
\[
w(A_k(Q)) \leq \frac{4\Lambda_{\Psi}}{\varphi^{-1}((2\Lambda_{\Psi})^{2^k})} \int_{E_Q} \Psi(4^k|f|) M_{\varphi(L)} w dx.
\]
Hence, by the disjointness of the sets $E_Q$,
\[
\sum_{Q \in F_k} w(A_k(Q)) \leq \frac{4\Lambda \Psi}{\varphi^{-1}\left((2\Lambda \Psi)^2\right)} \int_{\mathbb{R}^n} \Psi(4^k |f|) M_{\varphi(L)} w dx,
\]
which, along with (4.3), completes the proof. □

4.2. **Proof of Lemmas 4.1 and 4.2.** We first mention another common ingredient used in both proofs.

**Proposition 4.4.** Let $\Psi$ be a Young function. Assume that $G$ is an operator such that for every Young function $\varphi$,
\[
(4.6) \quad w_{Gf}(\lambda) \leq \left( \int_1^{\infty} \frac{\varphi^{-1}(t)}{t^2} dt \right) \int_{\mathbb{R}^n} \Psi \left( \frac{|f(x)|}{\lambda} \right) M_{\varphi(L)} w(x) dx.
\]
Then
\[
(4.7) \quad w_{Gf}(\lambda) \leq c C \varphi \int_{\mathbb{R}^n} \Psi \left( \frac{|f(x)|}{\lambda} \right) M_{(\Phi \circ \varphi)(L)} w(x) dx,
\]
where $c > 0$ is an absolute constant, and $C \varphi = \int_1^{\infty} \varphi^{-1}(t) \frac{1}{t^2 \log(e + t)} dt$.

Indeed, this follows immediately by setting $\Phi \circ \varphi$ instead of $\varphi$ in (4.6) and observing that $(\Phi \circ \varphi)^{-1} = \varphi^{-1} \circ \Phi^{-1}$ and
\[
\int_1^{\infty} \frac{\varphi^{-1}(t) \Phi^{-1}(t)}{t^2} dt = \int_{\Phi^{-1}(1)}^{\infty} \frac{\varphi^{-1}(t)}{\Phi(t)^2} \Phi'(t) dt \leq c C \varphi.
\]

Turn to Lemma 4.1. We actually obtain a stronger statement, namely, we will prove the following.

**Lemma 4.5.** Suppose that $S$ is $\frac{31}{32}$-sparse. Let $\varphi$ be a Young function such that
\[
K \varphi = \int_1^{\infty} \frac{\varphi^{-1}(t) \log \log(e^2 + t)}{t^2 \log(e + t)} dt < \infty.
\]
Then for an arbitrary weight $w$,
\[
w_{A_{S,L \log L} f}(\lambda) \leq c K \varphi \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) M_{\varphi(L)} w(x) dx \quad (\lambda > 0),
\]
where $c > 0$ is an absolute constant.

Since $K \varphi \leq \int_1^{\infty} \frac{\varphi^{-1}(t)}{t^2} dt$, Proposition 4.4 shows that Lemma 4.1 follows from Lemma 4.5.

**Proof of Lemma 4.3.** Consider the set
\[
E = \{ x \in \mathbb{R}^n : A_{S,L \log L} f(x) > 4, M_{L \log L} f(x) \leq 1/4 \}.
\]
By homogeneity combined with Remark 2.7, it suffices to prove that
\begin{equation}
(4.8) \quad w(E) \leq cK_{\varphi} \int_{\mathbb{R}^n} \Phi(|f(x)|) M_{\varphi(L)} w(x) \, dx.
\end{equation}

One can assume that \(w(E) < \infty\) (otherwise, one could first obtain (4.8) for \(E \cap K\) instead of \(E\), for any compact set \(K\)).

Denote
\[ S_k = \{ Q \in S : 4^{-k-1} < \| f \|_{L^{\log L} Q} \leq 4^{-k} \} \]
and set
\[ T_k f(x) = \sum_{Q \in S_k} \| f \|_{L^{\log L} Q} \chi_Q(x). \]

If \(E \cap Q \neq \emptyset\) for some \(Q \in S\), then \(\| f \|_{L^{\log L} Q} \leq 1/4\). Therefore, for \(x \in E\),
\begin{equation}
(4.9) \quad A_{S, L^{\log L} f}(x) = \sum_{k=1}^{\infty} T_k f(x).
\end{equation}

Now we apply Lemma 4.3 with \(\Psi = \Phi\) and \(F_k = S_k\). Notice that, by (2.6), one can take \(\Lambda_{\Psi} = 16\) in (4.2) and \(\Phi(4^k |f|) \leq c k 4^k \Phi(|f|)\). Combining this with \(T_k f(x) \leq 4^{-k} \sum_{Q \in S_k} \chi_Q\), by Lemma 4.3 we obtain
\[ \int_E (T_k f)w dx \leq 2^{-k} w(E) + \frac{c k}{\varphi^{-1}(2^{2k})} \int_{\mathbb{R}^n} \Phi(|f(x)|) M_{\varphi(L)} w(x) \, dx. \]

Combining (4.9) with the latter estimate implies,
\[ w(E) \leq \frac{1}{4} \int_E (A_{S, L^{\log L} f})w dx \leq \frac{1}{4} \sum_{k=1}^{\infty} \int_E (T_k f)w dx \]
\[ \leq \frac{1}{4} w(E) + c \left( \sum_{k=1}^{\infty} \frac{k}{\varphi^{-1}(2^{2k})} \right) \int_{\mathbb{R}^n} \Phi(|f(x)|) M_{\varphi(L)} w(x) \, dx. \]

From this,
\[ w(E) \leq c \left( \sum_{k=1}^{\infty} \frac{k}{\varphi^{-1}(2^{2k})} \right) \int_{\mathbb{R}^n} \Phi(|f(x)|) M_{\varphi(L)} w(x) \, dx. \]

Next, using that \(\varphi^{-1}(t) \varphi^{-1}(t) \approx t\), we obtain
\[ \sum_{k=1}^{\infty} \frac{k}{\varphi^{-1}(2^{2k})} \leq c \sum_{k=1}^{\infty} \int_{2^{2k-1}}^{2^{2k}} \frac{\log \log (e^2 + t)}{\varphi^{-1}(t) \log (e + t)} dt \leq c K_{\varphi}, \]
which, along with the previous estimate, yields (4.8), and therefore, the proof is complete.
Proof of Lemma 4.2. Denote
\[ E = \{ x : |T_{b,S}f(x)| > 8, Mf(x) \leq 1/4 \} . \]
By the Fefferman-Stein estimate (1.3) and by homogeneity, it suffices to assume that \( \|b\|_{BMO} = 1 \) and to show that in this case,
\[ w(E) \leq cC_\varphi \int_{\mathbb{R}^n} |f| M_{(\Phi \varphi)(L)} \, wd.x. \]
Let \( S_k = \{ Q \in S : 4^{-k-1} < |Q| \leq 4^{-k} \} \) and for \( Q \in S_k \), set \( F_k(Q) = \{ x \in \partial Q : |b(x) - b_Q| > (3/2)^k \} \).
If \( E \cap Q \neq \emptyset \) for some \( Q \in S \), then \( \|f\|_Q \leq 1/4 \). Therefore, for \( x \in E \),
\[ |T_{b,S}f(x)| \leq \sum_{k=1}^{\infty} \sum_{Q \in S_k} |b(x) - b_Q| |f| \chi_Q(x) \]
and for \( Q \in S_k \), set \( F_k(Q) = \{ x \in \partial Q : |b(x) - b_Q| > (3/2)^k \} \).
If \( E_i = \{ x \in E : T_i f(x) > 4 \}, i = 1, 2 \). Then
(4.10) \[ w(E) \leq w(E_1) + w(E_2). \]
Lemma 4.3 with \( \Psi(t) = t \) yields (with any Young function \( \varphi \))
\[ \int_{E_1} (T_1 f) \,wdx \leq \left( \sum_{k=1}^{\infty} (3/4)^k \right) w(E_1) + 16 \left( \sum_{k=1}^{\infty} \frac{(3/2)^k}{\varphi^{-1}(2^{2^k})} \right) \int_{\mathbb{R}^n} |f| M_{\varphi(L)} \, wdx. \]
This estimate, combined with \( w(E_1) \leq \frac{1}{4} \int_{E_1} (T_1 f) \,wdx \), implies
\[ w(E_1) \leq 16 \left( \sum_{k=1}^{\infty} \frac{(3/2)^k}{\varphi^{-1}(2^{2^k})} \right) \int_{\mathbb{R}^n} |f| M_{\varphi(L)} \, wdx. \]
Since \( \varphi^{-1}(t) \varphi^{-1}(t) \approx t \), we obtain
\[ \sum_{k=1}^{\infty} \frac{(3/2)^k}{\varphi^{-1}(2^{2^k})} \leq c \sum_{k=1}^{\infty} \int_{2^{2^k-1}}^{2^{2^k}} \frac{1}{t} \frac{dt}{t} \leq c \int_{1}^{\infty} \frac{\varphi^{-1}(t)}{t^2} \, dt. \]
Hence,
\[ w(E_1) \leq c \left( \int_{1}^{\infty} \frac{\varphi^{-1}(t)}{t^2} \, dt \right) \int_{\mathbb{R}^n} |f| M_{\varphi(L)} \, wdx, \]
which by Proposition 4.4 yields

\[ w(E_1) \leq cC \varphi \int_{\mathbb{R}^n} |f| M(\Phi \circ \varphi)(L) w \, dx. \]  

Turn to the estimate of \( w(E_2) \). Exactly as in the proof of Lemma 4.3, for \( Q \in \mathcal{S}_k \) define disjoint subsets \( E_Q \). Then, by (4.4),

\[ |f|_Q \leq \frac{8}{|Q|} \int_{E_Q} |f| \, dx. \]

Hence,

\[ w(E_2) \leq \frac{1}{4} \| T f \|_{L^1} \]

\[ \leq 2 \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \left( \frac{1}{|Q|} \int_{F_k(Q)} |b - b_Q| w \, dx \right) \int_{E_Q} |f|. \]

Now we apply twice the generalized Hölder inequality. First, by (2.4),

\[ \int_{F_k(Q)} |b - b_Q| w \, dx \leq c_n \| w \chi_{F_k(Q)} \|_{L^1, \log, L, Q}. \]

Second, we use (2.5) with \( C(t) = \Phi(t), B(t) = \Phi \circ \varphi(t) \) and \( A \) defined by

\[ A^{-1}(t) = \frac{C^{-1}(t)}{B^{-1}(t)} = \frac{\Phi^{-1}(t)}{\varphi^{-1}(t)}. \]

Since \( \varphi(t)/t \) and \( \Phi \) are strictly increasing functions, \( A \) is strictly increasing, too. Hence, by (2.5), we obtain

\[ \| w \chi_{F_k(Q)} \|_{L^1, \log, L, Q} \leq 2 \| w \chi_{F_k(Q)} \|_{A, Q} \| w \|_{(\Phi \circ \varphi), Q} \]

\[ = \frac{2}{A^{-1}(\|F_k(Q)\|)} \| w \|_{(\Phi \circ \varphi), Q}. \]

By the John-Nirenberg inequality (2.3), \( |F_k(Q)| \leq \alpha_k |Q| \), where \( \alpha_k = \min(1, e^{-(3/2)^k e^{-1}}) \). Combining this with (4.13) and (4.14) yields

\[ \frac{1}{|Q|} \int_{F_k(Q)} |b - b_Q| w \, dx \leq \frac{c_n}{A^{-1}(1/\alpha_k)} \| w \|_{(\Phi \circ \varphi), Q}. \]

From this and from (4.12) we obtain

\[ w(E_2) \leq c_n \sum_{k=1}^{\infty} \frac{1}{A^{-1}(1/\alpha_k)} \sum_{Q \in \mathcal{S}_k} \| w \|_{(\Phi \circ \varphi), Q} \int_{E_Q} |f| \]

\[ \leq c_n \left( \sum_{k=1}^{\infty} \frac{1}{A^{-1}(1/\alpha_k)} \right) \int_{\mathbb{R}^n} |f| M(\Phi \circ \varphi)(L) w(x) \, dx. \]
Further, the sum on the right-hand side can be estimated as follows:

\[
\sum_{k=1}^{\infty} \frac{1}{A^{-1}(1/\alpha_k)} \leq c \sum_{k=1}^{\infty} \int_{1/\alpha_{k-1}}^{1/\alpha_k} \frac{1}{t \log(e + t)} dt \\
\leq c \int_{1}^{\infty} \frac{\varphi^{-1} \circ \Phi^{-1}(t)}{t \log(e + t)} dt \leq c \int_{1}^{\infty} \frac{\varphi^{-1} \circ \Phi^{-1}(t)}{t^2} dt.
\]

Therefore, by (4.7),

\[
w(E_2) \leq 2n C \varphi \int_{\mathbb{R}^n} |f| M(\Phi \circ \varphi)(L) w(x) dx,
\]

which, along with (4.10) and (4.11), completes the proof. \(\square\)

4.3. Proof of Corollary 1.3. The proof follows the same scheme as in the proof of [18, Corollary 1.4], and hence we outline it briefly.

Using that \(\log t \leq t^{\alpha} / \alpha\) for \(t \geq 1\) and \(\alpha > 0\), we obtain

\[
M_L |\log L|^{1+\varepsilon} w(x) \leq \frac{c}{\alpha^{1+\varepsilon}} M_{L^{1+(1+\varepsilon)\alpha}} w(x).
\]

Next we use that for \(r_n = 1 + \frac{1}{cn[w]_{A_\infty}}\), \(M_{L^{r_n}} w(x) \leq 2M w(x)\). Hence, if \(\alpha\) is such that \((1 + \varepsilon)\alpha = \frac{1}{cn[w]_{A_\infty}}\), then

\[
\frac{1}{\varepsilon} M_{L |\log L|^{1+\varepsilon}} w(x) \leq \frac{c}{\varepsilon} [w]_{A_\infty}^{1+\varepsilon} M w(x) \leq \frac{c}{\varepsilon} [w]_{A_\infty}^{1+\varepsilon} [w]_{A_1} w(x).
\]

This estimate with \(\varepsilon = 1/ \log(e + [w]_{A_\infty})\), along with (1.8), completes the proof of Corollary 1.3.

5. Proof of Theorem 1.4

The main role in the proof is played by the following lemma. Denote by \(\Omega(b; Q)\) the standard mean oscillation,

\[
\Omega(b; Q) = \frac{1}{|Q|} \int_{Q} |b - b_Q| dx.
\]

Lemma 5.1. Let \(\mathcal{D}\) be a dyadic lattice and let \(\mathcal{S} \subset \mathcal{D}\) be a \(\gamma\)-sparse family. Assume that \(b \in L^{1}_{\text{loc}}\). Then there exists a \(\frac{\gamma}{2(1+\gamma)}\)-sparse family \(\tilde{\mathcal{S}} \subset \mathcal{D}\) such that \(\mathcal{S} \subset \tilde{\mathcal{S}}\) and for every cube \(Q \in \tilde{\mathcal{S}}\),

\[
|b(x) - b_Q| \leq 2^{n+2} \sum_{R \in \tilde{\mathcal{S}}, R \subseteq Q} \Omega(b; R) \chi_R(x)
\]

for a.e. \(x \in Q\).

This lemma is based on several known ideas. The first idea is an estimate by oscillations over a sparse family (see [11, 16, 22]) and the second idea is an augmentation process (see Section 2.1).
Proof. Fix a cube $Q \in \mathcal{D}$. Let us show that there exists a (possibly empty) family of pairwise disjoint cubes $\{P_j\} \in \mathcal{D}(Q)$ such that $\sum_j |P_j| \leq \frac{1}{2}|Q|$ and for a.e. $x \in Q$,

\begin{equation}
|b(x) - b_Q| \leq 2^{n+2}\Omega(b; Q) + \sum_j |b(x) - b_{P_j}|\chi_{P_j},
\end{equation}

Consider the set

$$E = \left\{ x \in Q : M_Q^d(b - b_Q)(x) > 2^{n+2}\Omega(b; Q) \right\},$$

where $M_Q^d$ is the standard dyadic local maximal operator restricted to a cube $Q$. Then $|E| \leq \frac{1}{2^{n+2}}|Q|$. If $E = \emptyset$, then (5.2) holds trivially with the empty family $\{P_j\}$. Suppose that $E \neq \emptyset$. The Calderón-Zygmund decomposition applied to the function $\chi_E$ on $Q$ at height $\lambda = \frac{1}{2^{n+1}}$ produces pairwise disjoint cubes $P_j \in \mathcal{D}(Q)$ such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|$$

and $|E \setminus \cup_j P_j| = 0$. It follows that $\sum_j |P_j| \leq \frac{1}{2}|Q|$ and $P_j \cap E^c \neq \emptyset$.

Therefore,

\begin{equation}
|b_{P_j} - b_Q| \leq \frac{1}{|P_j|} \int_{P_j} |b - b_Q| dx \leq 2^{n+2}\Omega(b; Q)
\end{equation}

and for a.e. $x \in Q$,

$$|b(x) - b_Q|\chi_{Q \cup \cup_j P_j} \leq 2^{n+2}\Omega(b; Q).$$

From this,

$$|b(x) - b_Q|\chi_Q \leq |b(x) - b_Q|\chi_{Q \cup \cup_j P_j}(x) + \sum_j |b_{P_j} - b_Q|\chi_{P_j}$$

$$+ \sum_j |b(x) - b_{P_j}|\chi_{P_j}$$

$$\leq 2^{n+2}\Omega(b; Q) + \sum_j |b(x) - b_{P_j}|\chi_{P_j},$$

which proves (5.2).

We now observe that if $P_j \subseteq R$, where $R \in \mathcal{D}(Q)$, then $R \cap E^c \neq \emptyset$, and hence $P_j$ in (5.3) can be replaced by $R$, namely, we have

$$|b_R - b_Q| \leq 2^{n+2}\Omega(b; Q).$$
Therefore, if \( \bigcup_j P_j \subset \bigcup_i R_i \), where \( R_i \in \mathcal{D}(Q) \), and the cubes \( \{R_i\} \) are pairwise disjoint, then exactly as above,

\[
|b(x) - b_Q| \leq 2^{n+2} \Omega(b; Q) + \sum_i |b(x) - b_{R_i}|\chi_{R_i}.
\]

(5.4)

Iterating \((5.2)\), we obtain that there exists a \( \frac{1}{2} \)-sparse family \( \mathcal{F}(Q) \subset \mathcal{D}(Q) \) such that for a.e. \( x \in Q \),

\[
|b(x) - b_Q|\chi_Q \leq 2^{n+2} \sum_{P \in \mathcal{F}(Q)} \Omega(b; P)\chi_P.
\]

We now augment \( S \) by families \( \mathcal{F}(Q) \), \( Q \in S \). Denote the resulting family by \( \tilde{S} \). By Lemma 2.4, \( \tilde{S} \) is \( \frac{\gamma}{2(1+\gamma)} \)-sparse.

Let us show that \((5.1)\) holds. Take an arbitrary cube \( Q \in \tilde{S} \). Let \( \{P_j\} \) be the cubes appearing in \((5.2)\). Denote by \( \mathcal{M}(Q) \) the family of the maximal pairwise disjoint cubes from \( \tilde{S} \) which are strictly contained in \( Q \). Then, by the augmentation process, \( \bigcup_j P_j \subset \bigcup_{P \in \mathcal{M}(Q)} P \). Therefore, by \((5.4)\),

\[
|b(x) - b_Q|\chi_Q \leq 2^{n+2} \sum_{P \in \mathcal{M}(Q)} \Omega(b; P)\chi_P + \sum_{P \in \mathcal{M}(Q)} |b(x) - b_P|\chi_P(x).
\]

(5.5)

Iterating this estimate completes the proof. Indeed, split \( \tilde{S}(Q) = \{P \in \tilde{S} : P \subset Q\} \) into the layers \( \tilde{S}(Q) = \bigcup_{k=0}^{\infty} \mathcal{M}_k \), where \( \mathcal{M}_0 = Q \), \( \mathcal{M}_1 = \mathcal{M}(Q) \) and \( \mathcal{M}_k \) is the family of the maximal elements of \( \mathcal{M}_{k-1} \). Iterating \((5.5)\) \( k \) times, we obtain

\[
|b(x) - b_Q|\chi_Q \leq 2^{n+2} \sum_{P \in \tilde{S}(Q)} \Omega(b; P)\chi_P + \sum_{P \in \mathcal{M}_k} |b(x) - b_P|\chi_P(x).
\]

(5.6)

Now we observe that since \( \tilde{S} \) is \( \frac{\gamma}{2(1+\gamma)} \)-sparse,

\[
\sum_{P \in \mathcal{M}_k} |P| \leq \frac{1}{(k+1)} \sum_{i=0}^k \sum_{P \in \mathcal{M}_i} |P| \leq \frac{1}{(k+1)} \sum_{P \in \tilde{S}(Q)} |P| \leq \frac{2(1+\gamma)}{\gamma(k+1)} |Q|.
\]

Therefore, letting \( k \to \infty \) in \((5.6)\), we obtain \((5.1)\). \( \square \)

Recall the well-known (see [7] or [25] for a different proof) bound for the sparse operator \( \mathcal{A}_S \), where \( S \) is \( \gamma \)-sparse:

\[
\| \mathcal{A}_S \|_{L^p(w)} \leq c_{\gamma, n, p}[w]_{A_p}^{(1-p^{-1})} \quad (1 < p < \infty).
\]

(5.7)
Proof of Theorem 1.4. By Theorem 1.1 and by duality,

\[ (5.8) \quad \| [b, T] \|_{L^p(\mu) \to L^p(\lambda)} \]

\[ \leq c_n C_T \sum_{j=1}^{3^n} (\| T_{S_j,b} \|_{L^p(\mu) \to L^p(\lambda)} + \| T_{S_j,b}^* \|_{L^p(\mu) \to L^p(\lambda)}) \]

\[ = c_n C_T \sum_{j=1}^{3^n} (\| T_{S_j,b}^* \|_{L^{p'}(\sigma_\lambda) \to L^{p'}(\sigma_\mu)} + \| T_{S_j,b}^* \|_{L^p(\mu) \to L^p(\lambda)}) , \]

where \( S_j \subset \mathcal{D}^{(j)} \) is \( \frac{1}{3^n} \)-sparse.

By Lemma 5.1 there are \( \frac{1}{8 \cdot 3^n} \)-sparse families \( \tilde{S}_j \) containing \( S_j \), and also, for every cube \( Q \in \tilde{S}_j \),

\[ \int_Q \| b(x) - b_Q \| |f| \leq c_n \sum_{R \in \tilde{S}_j, R \subseteq Q} \Omega(b; R) \int_R |f| \]

\[ \leq c_n \| b \|_{BMO_v} \sum_{R \in \tilde{S}_j, R \subseteq Q} |f|_{R \nu(R)} \leq c_n \| b \|_{BMO_v} \int_Q (A_{\tilde{S}_j}|f|) \nu dx. \]

Therefore,

\[ T_{S_j,b}^* |f|_v(x) \leq c_n \| b \|_{BMO_v} A_{\tilde{S}_j}((A_{\tilde{S}_j}|f|)\nu)(x). \]

Hence, applying (5.7) twice yields

\[ (5.9) \quad \| T_{S_j,b}^* \|_{L^p(\mu) \to L^p(\lambda)} \leq c_{n,p}\| b \|_{BMO_v} \| A_{\tilde{S}_j} \|_{L^p(\lambda)} \| A_{\tilde{S}_j} \|_{L^p(\mu)} \]

\[ \leq c_{n,p} (\| \lambda \|_{A_p} \| \mu \|_{A_p})^{\max\left(1, \frac{1}{p-1}\right)} \| b \|_{BMO_v}. \]

From this and from the facts that \( \nu = (\mu/\lambda)^{1/p} = (\sigma_\lambda/\sigma_\mu)^{1/p'} \) and \([\sigma_\lambda]_{A_p} = [\sigma_\mu]_{A_p} = [\mu]_{A_p}^{-1} \), we obtain

\[ \| T_{S_j,b}^* \|_{L^{p'}(\sigma_\lambda) \to L^{p'}(\sigma_\mu)} \leq c_{n,p'} (\| \sigma_\mu \|_{A_p} \| \lambda_\mu \|_{A_p})^{\max\left(1, \frac{1}{p'-1}\right)} \| b \|_{BMO_v} \]

\[ = c_{n,p'} (\| \mu \|_{A_p} \| \lambda \|_{A_p})^{\max\left(1, \frac{1}{p'-1}\right)} \| b \|_{BMO_v}. \]

It remains to combine this estimate with (5.8) and (5.9), and to observe that \( T_{S_j,b}^* |f(x)| \leq T_{\tilde{S}_j,b}^* |f(x)| \).

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