Rate of Decay of Stable Periodic Solutions of Duffing Equations

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Abstract

In this paper, we consider the second-order equations of Duffing type. Bounds for the derivative of the restoring force are given that ensure the existence and uniqueness of a periodic solution. Furthermore, the stability of the unique periodic solution is analyzed; the

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sharp rate of exponential decay is determined for a solution that is near to the unique periodic solution.

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1 Introduction and statement of main results

This paper is devoted to the existence, uniqueness and stability of periodic solutions of the Duffing-type equation

\[ x'' + cx' + g(t, x) = h(t), \]

where \( g: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a \( T \)-periodic function in \( t \) and continuous in \( x \), \( h(t) \) is a \( T \)-periodic function, and \( c > 0 \) is a positive constant. The existence and multiplicity of periodic solutions of (1.1) or more general types of nonlinear second-order differential equations have been investigated extensively by many authors. However, the stability of periodic solutions is less extensively studied. In [9], R. Ortega studied (1.1) from the stability point of view and obtained stability results by topological index. A.C. Lazer and P.J. McKenna get stability results by converting the equation (1.1) to a fixed-point problem [5]. Recently, more complete results concerning the stability of periodic solutions of (1.1) were obtained by J.M. Alonso and R. Ortega [11 2]. Under the condition that the derivative of the restoring
force is independent of $t$ and positive, they found sharp bounds that guarantee global asymptotic stability. In [2], optimal bounds for stability are obtained. However, the rate of exponential decay of periodic solutions is not known in general. The purpose of this paper is to determine the rate of decay of the periodic solutions. Our results are motivated by careful observation of the following example of a second-order differential equation with constant coefficients:

$$
(1.2) \quad x'' + cx' + kx = h(t),
$$

where $c > 0$. If $k \neq 0$, then (1.2) has a unique $T$-periodic solution $x_0(t)$, and all solutions of (1.2) have the form $x(t) = c_1 e^{\rho_1 t} + c_2 e^{\rho_2 t} + x_0(t)$ where $\rho_1 < \rho_2$ are roots of $\rho^2 + c\rho + k = 0$. Regarding $k$ as a parameter, when $k < 0$, then $\rho_2 > 0$, and the unique periodic solution is unstable; when $k > 0$, the unique periodic solution is stable; keep increasing $k$ till it crosses the critical value $k = \frac{c^2}{4}$, then $\rho_1, \rho_2$ are a pair of conjugates. In this case, every solution other than the unique $T$-periodic solution decays to the periodic solution at the same exponential rate $\frac{c}{2}$ independently of $k$.

The aim of this paper is to show that the above-mentioned $\frac{c}{2}$ decay results can be generalized to (1.1). In order to show that periodic solutions of (1.1) are stable, it is essential to impose conditions on the restoring force that can rule out the existence of additional periodic solutions that are subharmonic of order 2, whereas to determine the rate of decay of periodic solutions,
it is essential to impose conditions on $g(x, t)$ such that the corresponding Dirichlet boundary-value problem does not admit any nontrivial solutions. We shall give existence and uniqueness results, characterizing solutions that are locally asymptotically stable with sharp exponential rate of decay, by a Sturm comparison argument coupled with Floquet theory. Roughly speaking, (1.1) has a unique $T$-periodic solution which is locally asymptotically stable in the sense of Lyapunov, as long as the restoring force is small or the fractional constant is large.

The following notations will be used throughout the rest of the paper.

1. $L_p^T$ T-periodic function $u \in L^p[0, T]$ with $\|u\|_p$ for $1 \leq p \leq \infty$;
2. $C^k_T$ T-periodic function $u \in C^k[0, T]$, $k \geq 0$, with $C^k$-norm;
3. $\alpha(t) \gg \beta(t)$, if $\alpha(t) \geq \beta(t)$ and $\alpha(t) > \beta(t)$ on some positive-measure subset.

Now we state our main results.

**Theorem 1.1.** Assume that $g(t, x) \in C^1(\mathbb{R})$ satisfies the following conditions:

(1) $g'_x(t, x) < \frac{\pi^2}{T^2} + \frac{c^2}{T}$, and

(2) there is an $\alpha(t) \in C_T$ with $\overline{\alpha(t)} \gg \frac{c^2}{T}$ such that $g'_x(t, x) \gg \alpha(t)$, where $\overline{\alpha(t)}$ denotes the the average of $\alpha(t)$ over a period.
Then (1.1) has a unique $T$-periodic solution which is asymptotically stable with sharp rate of decay of $\frac{c}{T}$ for $c > 0$.

Here, we say that the periodic solution $u_0$ of (1.1) is locally asymptotically stable if there exist constants $C > 0$ and $\alpha > 0$ such that if $u$ is another solution with $|u(0) - u_0(0)| + |u'(0) - u'_0(0)| = d$ sufficiently small, then $|u(t) - u_0(t)| + |u'(t) - u'_0(t)| < Cde^{-\alpha t}$. The super exponent $\alpha$ as above is called the rate of decay of $u_0$.

Remark. The bounds given in Theorem 1.1 are optimal. In fact, the example given at the beginning of the section shows that the lower bound is sharp. An example to show that the upper bound is optimal in the theorem will be given in the following.

Example. Consider the linear differential equation

\begin{equation}
(1.3) \quad x'' + cx' + \frac{1}{4}(1 + c^2 + \epsilon \cos t)x = 0,
\end{equation}

where $c > 0$ and $|\epsilon|$ is small and $\epsilon \neq 0$. By the transformation $y(t) = e^{-\frac{c}{2}t}x(t)$, the damping term can be eliminated and equation (1.3) is equivalent to

\begin{equation}
(1.4) \quad y'' + \frac{1}{4}(1 + \epsilon \cos t)y = 0.
\end{equation}

The following notations are used. Obviously, the nontrivial solution $x(t)$ of (1.3) has rate of decay $\frac{c}{T}$ if and only if the corresponding solution $y(t)$ of
(1.4) is a nontrivial bounded solution. Let \( y_i, i = 1, 2 \), denote the solutions of (1.4) with initial values

\[
y_1(0, \epsilon) = y'_2(0, \epsilon) = 1, \quad y'_1(0, \epsilon) = y_2(0, \epsilon) = 0.
\]

If the discriminant function of (1.3) is denoted by \( \Delta(\epsilon) = y_1(2\pi, \epsilon) + y'_2(2\pi, \epsilon) \), then the Floquet multipliers are the roots of the quadratic

\[
\mu^2 - \Delta(\epsilon) \mu + 1 = 0.
\]

By means of perturbation one can compute that

\[
\Delta(\epsilon) = -2 - \frac{\pi}{64} \epsilon^2 + 0(\epsilon^2)
\]

for \( \epsilon \) small, which shows that the modulus of one of the multipliers is greater than 1 and the modulus of the other multiplier is less than 1. Therefore the multipliers of (1.4) are a pair of distinct real numbers. This implies that the rate of decay of (1.4) is greater than \( \frac{c_2}{2} \). Thus if the derivative of the restoring term crosses over the bound given in Theorem 1.1, the conclusion of Theorem 1.1 does not hold.

Consider the piecewise linear equation

\[
(1.5) \quad x'' + cx' + a(t)x^+ - b(t)x^- = h(t),
\]

where \( x^+ = \max\{x, 0\}, \ x^- = \min\{-x, 0\} \), \( a(t), b(t) \in C_T \). For \( a \) and \( b \) constant, (1.5) is a simple mathematical model for vertical oscillations of
a long-span suspension bridge, which has received great attention after a series of works of Lazer and McKenna [6].

**Theorem 1.2.** Suppose that \( h(t) \in C_T \) has a finite number of zeros in \([0, T]\). Then

1. equation (1.5) has a unique \( T \)-periodic solution if
   \[
   0 \ll a(t), b(t) \ll \frac{(2\pi)^2}{T^2} + \frac{c^2}{4},
   \]

2. the unique \( T \)-periodic solution is asymptotically stable if
   \[
   0 \ll a(t), b(t) \ll \frac{(\pi)^2}{T^2} + \frac{c^2}{4}, \text{ and}
   \]

3. the unique \( T \)-periodic solution has rate of decay \( \frac{c^2}{2} \) if
   \[
   \frac{c^2}{T} \ll a(t), b(t) \ll \frac{(\pi)^2}{T^2} + \frac{c^2}{4}.
   \]

2 The linear periodic problem

In this section we shall recall some basic results about topological methods and prove some stability results for linear periodic systems.

Consider the periodic boundary-value problem

\[
\begin{cases}
  x' = F(t, x), \\
  x(0) = x(T).
\end{cases}
\]  

(2.1)

where \( F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function that is \( T \)-periodic in \( t \). In order to use a homotopic method to compute the degree, we assume
that $h: [0, T] \times \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$ is a continuous function such that

$$h(t, x, 1) = F(t, x),$$
$$h(t, x, 0) = G(x),$$

where $G(x)$ is continuous. The following theorem is due to J. Mawhin [8].

**Lemma 2.1.** Let $\Omega \subset C_T$ be an open bounded set such that the following conditions are satisfied.

1. There is no $x \in \partial \Omega$ such that

   $$x' = h(t, x, \lambda) \quad \forall \lambda \in [0, 1].$$

2. $\deg(g, \Omega \cap \mathbb{R}^n, 0) \neq 0$.

Then (2.1) has at least one solution.

Next we consider the above system for $n = 2$. We denote by $x(t, x_0)$ the initial-value solution of (2.1) and introduce the Poincaré map $P: x_0 \rightarrow x(T, x_0)$. It is well known that $x(t, x_0)$ is a $T$-periodic solution of the system (2.1) if and only if $x_0$ is a fixed point of $P$. If $x$ is an isolated $T$-periodic solution of (2.1), then $x_0$ is an isolated fixed point of $P$. Hence the Brouwer index is defined by

$$\text{ind}[P, x_0] = \deg(I - P, B_{\varepsilon}(0), 0).$$

**Definition.** A $T$-periodic solution $x$ of (2.1) will be called a nondegenerate $T$-periodic solution if the linearized equation

$$(2.2) \quad y' = f_x(t, x)y$$

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does not admit any nontrivial $T$-periodic solutions.

Let $M(t)$ be the fundamental matrix of (2.2) and $\mu_1$ and $\mu_2$ the eigenvalues of the matrix $M(T)$. Then $x(t, x_0)$ is asymptotically stable if and only if $|\mu_i| < 1, i = 1, 2$: otherwise, if there is an eigenvalue of $M(T)$ with modulus greater than one, then $x(t, x_0)$ is unstable.

Before giving some results concerning stability of linear periodic equations, we consider the following eigenvalue problem:

$$
\begin{align*}
&x'' + \lambda x = 0, \\
&x(0) = x(T) = 0, \quad \text{sgn } x'(0) = \text{sgn } x'(T).
\end{align*}
$$

(2.3)

It is easy to verify that $\lambda_n = \frac{(2n\pi)^2}{T^2}$, the $n$-th eigenvalue of (2.3) with the eigenfunction $\varphi_n(t) = \sin \frac{2n\pi}{T} t$. Now for

$$
\begin{align*}
&x'' + q(t)x = 0, \\
&x(0) = x(T) = 0, \quad \text{sgn } x'(0) = \text{sgn } x'(T),
\end{align*}
$$

(2.4)

we have the following result.

**Lemma 2.2.** Assume that $q(t) \ll \frac{(2\pi)^2}{T^2}$. Then equation (2.4) does not admit any nontrivial solutions.

**Proof.** Suppose that $x(t)$ is a nontrivial solution of (2.4). Then $x$ is an eigenfunction. There is an $n$ such that $\lambda_n(q) = 0$. Since $q(t) \ll \frac{(2\pi)^2}{T^2}$, by a theorem concerning comparison of eigenvalues, we have that $\lambda_n(q) > \lambda_n \left( \frac{(2\pi)^2}{T^2} \right) = \lambda_n - \lambda_1 > 0$, a contradiction. \qed
Consider the homogeneous periodic equation

\[(2.5)\quad L_\alpha x := x'' + cx' + \alpha(t)x = 0,\]

where \(c \in \mathbb{R}\) is constant and \(\alpha(t) \in L^\infty_T\).

**Lemma 2.3.** Assume \(\alpha(t) \in L^\infty_T\) satisfies the following conditions: \(\alpha(t) \ll \frac{(2\pi)^2}{T^2} + \frac{c^2}{T}\) and \(\overline{\alpha(t)} > 0\).

Then \((2.5)\) does not admit any nontrivial \(T\)-periodic solutions.

**Proof.** Suppose on the contrary that \((2.5)\) admits a nontrivial \(T\)-periodic solution \(x(t)\). We claim that \(x(t)\) vanishes at some \(t_0 \in [0, T]\). If not, then \(x(t) \neq 0\) for all \(t\) in \(\mathbb{R}\). By the periodic boundary conditions, we have \(x'(T) = x'(0)\) and \(\frac{x'(T)}{x(T)} = \frac{x'(0)}{x(0)}\). Dividing \((2.5)\) by \(x(t)\) and integrating by parts gives that

\[
\int_0^T \frac{x'(t)^2}{x(t)^2} \, dt + \int_0^T \alpha(t) \, dt = 0,
\]

which contradicts the hypothesis of the lemma. So \(x(t)\) has a zero in \([0, T]\).

We may assume that \(x(0) = 0\) so that \(x(0) = x(T) = 0\). By the transformation \(y(t) = e^{\frac{c}{2}t}x(t)\), \(y(t)\) is a nontrivial solution of \((2.5)\) with \(q(t) = \alpha(t) - \frac{c^2}{T}\), and if the first condition of the lemma holds, \(q(t) \ll \frac{(2\pi)^2}{T^2}\). Then according to Lemma 2.2, \(y(t) \equiv 0\), hence \(x(t) \equiv 0\), a contradiction. \(\Box\)

The following simple lemma, given in [4], will be used in proving the existence and the uniqueness of periodic solutions.
Lemma 2.4. Suppose that \( \alpha(t) \), \( \alpha_1(t) \) and \( \alpha_2(t) \) \( \in L_T^\infty \) such that \( \alpha_1(t) \), \( \alpha_2(t) \), and \( \alpha(t) \) are all \( \ll \frac{(2\pi)^2}{T^2} + \frac{c^2}{4} \). Then

1. the possible \( T \)-periodic solution \( x \) of equation (2.5) is either trivial or different from zero for each \( t \in \mathbb{R} \);

2. \( L_{\alpha_i} x = 0 \) (\( i = 1, 2 \)) cannot admit nontrivial \( T \)-periodic solutions simultaneously if \( \alpha_1(t) \ll \alpha_2(t) \); and

3. \( L_{\alpha} x = 0 \) has no nontrivial \( T \)-periodic solution, if \( \alpha(t) \gg 0 \) (resp., \( \alpha(t) \ll 0 \)).

The following lemma is essential for determining the rates of decay of periodic solutions of (1.1).

Lemma 2.5. Assume that \( \alpha(t) \ll \frac{\pi^2}{T^2} + \frac{c^2}{4} \) and that \( \overline{\alpha(t)} > \frac{c^2}{4} \). Then (2.5) does not admit real Floquet multipliers.

Proof. If the conclusion of the lemma does not hold, then there is a real Floquet multiplier \( \rho \) and a nontrivial solution \( x(t) \) such that \( x(t+T) = \rho x(t) \). Introduce the transformation \( x = e^{-\frac{1}{T} \rho t} u \). Then \( u \) solves the equation

\[
(2.6) \quad u'' + \left[ \alpha(t) - \frac{c^2}{4} \right] u = 0
\]

with Floquet multiplier \( \chi = e^{-\frac{cT}{2}} \rho \). If \( x \) does not change sign in \( \mathbb{R} \), neither does \( u \). Dividing equation (2.6) by \( u \) and integrating by parts, and noting
that \( \frac{u'(T)}{u(T)} = \frac{u'(0)}{u(0)} \), we have that

\[
\int_0^T \frac{u'(t)^2}{u(t)^2} \, dt + \int_0^T \left[ \alpha(t) - \frac{c^2}{4} \right] \, dt = 0,
\]

which contradicts the assumption that \( \overline{\alpha(t)} > \frac{c^2}{4} \). Therefore \( x(t) \) vanishes at some \( t_0 \in [0, T] \), and the conditions are such that \( x(t_0 + T) = \rho x(t_0) = 0 \). Thus the corresponding \( u \) is a nontrivial solution of the following Dirichlet boundary-value problem:

\[
(2.7) \quad u'' + \left[ \alpha(t) - \frac{c^2}{4} \right] u = 0, \quad u(t_0) = u(t_0 + T) = 0.
\]

Since \( \alpha(t) - \frac{c^2}{4} \ll \frac{\pi^2}{T^2} \), multiplying \((2.7)\) by \( u \) and integrating we have that

\[
\int_{t_0}^{T+t_0} u'^2 \, dt = \int_{t_0}^{T+t_0} \left[ \alpha(t) - \frac{c^2}{4} \right] u^2 \, dt < \frac{\pi^2}{T^2} \int_{t_0}^{T+t_0} u^2 \, dt,
\]

which contradicts the Poincaré inequality. Therefore \((2.5)\) does not admit real multipliers.

\[\square\]

**Lemma 2.6.** Under the conditions of Lemma \((2.5)\) the rate of decay of any nontrivial solution of \((2.5)\) is \( \frac{\pi}{T} \).

**Proof.** Consider the corresponding system

\[
(2.8) \quad X'(t) = A(t) X(t),
\]

where the column vector function \( X(t) = (x(t), x'(t))^T \) and \( A(t) \) is the matrix function

\[
A(t) = \begin{pmatrix}
0 & 1 \\
-p(t) & -c
\end{pmatrix}.
\]
Let $M(t)$ be a fundamental matrix solution of (2.8). It is well known that $M(t)$ has the form

\begin{equation}
M(t) = P(t)e^{Bt},
\end{equation}

where $P(t)$ and $B$ are $2 \times 2$ matrices, $P(t) = P(t + T)$, and $B$ is a constant matrix. Let $\rho_1 = e^{T\lambda_1}$, $\rho_2 = e^{T\lambda_2}$ be the Floquet multipliers and $\lambda_1$ and $\lambda_2$ the Floquet exponents associated with $\rho_1$ and $\rho_2$. Let $x_1$ and $x_2$ be the eigenvector components of the matrix $e^{TB}$. It follows from Lemma 2.5 that $\rho_1$ and $\rho_2$ are a pair of conjugates. Thus the eigenvectors associated with different eigenvalues are linearly independent. Therefore $y_i = p_i(t)e^{\lambda_i t}$ (for $i = 1, 2$) form the fundamental solution system of equation (2.8). On the other hand, by applying the Jacobi–Liouville formula, we have

\begin{equation}
|\rho_1|^2 = \rho_1\rho_2 = e^{-\int_0^T c dt} = e^{-cT}
\end{equation}

and

\begin{equation}
\text{Re} \lambda_1 = \text{Re} \lambda_2 = \frac{1}{2} \text{Re} (\lambda_1 + \lambda_2) = \frac{1}{2T} \ln(\rho_1\rho_2) = -\frac{c}{2}.
\end{equation}

Since every solution is a linear combination of $y_1(t)$ and $y_2(t)$, $p_i(t)$ is $T$-periodic, hence it is bounded. Therefore every nonzero solution of equation (2.5) decays at the same exponential rate of $\frac{c}{2}$. \qed
3 Proof of main results

Now we prove our main results.

3.1 Proof of Theorem 1.1

We begin with the following existence result, dividing the proof into two steps.

Step 1. Existence.

Define \( F: C^2_T \rightarrow C_T \) by

\[
F(x(t)) := x'' + cx' + g(t, x(t)).
\]

We have the following.

**Theorem 3.1.** Assume that \( g(x, t) \in C^1(\mathbb{R} \times \mathbb{R}) \), and that \( g(x, t) \) is \( T \)-periodic in \( t \). If in addition \( g \) satisfies the two conditions

1. \( g(t, x)/x \ll \frac{(2\pi)^2}{4} + \frac{c^2}{4} \forall x \in \mathbb{R} \) and
2. \( \exists \) a \( T \)-periodic function \( \beta(t) \in C_T \) such that \( \beta(t) > 0 \) and \( g(t, x)/x \gg \beta(t) \) for all \( x \in \mathbb{R} \),

then the differential equation \( (1.1) \) has a \( T \)-periodic solution.

**Proof.** Without loss of generality, we may assume that \( g(0, t) = 0 \), for otherwise we can reduce both sides of equation \( (1.1) \) by \( g(0, t) \). Consider the
parametrized equation

\[ F_\lambda := x'' + cx' + \lambda g(t, x) + (1 - \lambda)ax = \lambda h(t) \]

for some \( a \in \left(0, \frac{(2\pi)^2}{T^2} + \frac{c^2}{T} \right) \). We claim that there is an \( R > 0 \) such that equation (3.1) has no solution on \( \partial B_R \) for any \( \lambda \in [0, 1] \). If there is not such an \( R \), let \( x_n \) be a sequence of solutions such that \( \|x_n\| \to \infty \) and \( \lambda_n \in [0, 1] \). Denote by \( z_n := \frac{x_n}{\|x_n\|} \). Dividing (3.1) by \( \|x_n\| \), then multiplying by \( \varphi(t) \in C^2_T \) and integrating by parts, we have that

\[ (3.2) \quad \int_0^T z_n \varphi'' - cz_n \varphi' + g(t, x_n)\varphi/\|x_n\| \, dt = \lambda_n \int_0^T \varphi h_n/\|x_n\| \, dt. \]

The conditions of Theorem 3.1 imply that \( \{[\lambda_n g(t, x_n) + (1 - \lambda_n)ax_n/\|x_n\|] \) is bounded. It is pre-compact in the weak topology in \( L^1[0, T] \). Thus there are subsequences such that \( g(t, x_n)/x_n \to q(t) \) and \( \lambda_n \to \lambda \). Taking the limit in the equation (3.2), one obtains that

\[ (3.3) \quad \int_0^T \{z\varphi'' - cz\varphi' + w(t)\varphi z\} \, dt = 0, \]

where \( w(t) = \lambda q(t) + (1 - \lambda)a \). Evidently, \( w(t) \geq \lambda \alpha(t) + (1 - \lambda)a > 0 \), and \( w(t) < \frac{(2\pi)^2}{T^2} + \frac{c^2}{T} \), which satisfies the conditions of Lemma 2.2 in [4], and since here \( z(t) \) is a \( T \)-periodic solution, it follows from Lemma 2.3 that \( z(t) \equiv 0 \), which contradicts the fact that, by construction, \( \|z(t)\| = 1 \). Next, by applying the homotopic invariance property, we have that

\[ \deg(F_0, B_R, 0) = \deg(F_1, B_R, 0) = 1. \]
According to Lemma 2.1 equation (1.1) then has a $T$-periodic solution. This completes the proof of existence.

Now, we will finish the proof of Theorem 3.1 by verifying uniqueness. Let $x_1$ and $x_2$ be two distinct $T$-periodic solutions of the equation (1.1), and let $u = x_1 - x_2$. Then $u$ satisfies the equation

$$u'' + cu' + p(t)u = 0,$$

where $p(t) = \frac{[g(t, x_1(t)) - g(t, x_1(t))]}{[x_1 - x_2]}$. If $u \neq 0$ identically, it will imply that $u$ is an eigenfunction associated with a Floquet multiplier equal to one. Again, Lemma 2.3 rules out this possibility. Therefore $u \equiv 0$.

Step 2. Rate of decay.

To show that every solution of the nonlinear equation (1.1) locally decays at the rate of $\frac{c}{2}$ to the unique $T$-periodic solution, we need the following $C^1$ version of the Hartman–Grobman theorem [3].

**Lemma 3.1.** Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ function with $f(0) = 0$ such that $f_x'(0): \mathbb{R}^n \to \mathbb{R}^n$ is a contracting mapping. Then $f$ is $C^1$ conjugate equivalent to $f_x'(0)$.

Consider the planar system associated with equation (1.1),

$$\begin{cases}
    x' = cx - y, \\
    y' = h(t) - g(t, x).
\end{cases}$$

(3.5)
Let \( X_0(t) = (x_0(t), y_0(t)) \) be the unique \( T \)-periodic solution determined by the initial condition \( X_0(0) = (x_0, y_0) \). Then \( X_0 \) corresponds to the unique fixed point of the Poincaré mapping \( PX = U(T, X) \), where \( U(t, X) \) is the initial-value solution of (3.5) with \( U(0, X) = X \). Let \( M(t) \) be the fundamental matrix solution of the linearization

\[
(3.6) \quad X' = A(t)X
\]

of (3.5), where

\[
A(t) = \begin{pmatrix}
    c & -1 \\
    -q(t) & 0
\end{pmatrix}.
\]

By the differentiability of \( X(t) \) with respect to the initial value, the Poincaré mapping can be expressed in terms of the initial value \( X \) by the following formula:

\[
(3.7) \quad PX - X_0 = M(T)(X - X_0) + o(X - X_0).
\]

Referring to Lemma 2.5, \( M(T) \) has a pair of conjugate eigenvalues \( \lambda \), \( \tilde{\lambda} \) with \( |\lambda| = e^{-cT/2} \). Thus \( P(X) \) is a contracting mapping. According to Lemma 3.1, there is a \( C^1 \) diffeomorphism \( \varphi \) which is near enough to the identity that \( PX - X_0 \) is conjugate equivalent to \( M(T) \). There is an invertible constant matrix \( C \) such that

\[
C^{-1}M(T)C = \begin{pmatrix}
    \lambda & 0 \\
    0 & \tilde{\lambda}
\end{pmatrix} = D(\lambda),
\]
and we may suppose that

\( 2 |X - X_0| < |\varphi(X) - \varphi(X_0)| < 2 |X - X_0| \)

for \( X - X_0 \) small, since \( \varphi \) is near the identity. Therefore, the Liapunov exponent is given by

\[
\mu_x = \lim_{n \to \infty} \frac{1}{nT} \ln |P^n X - X_0|
\]

\[
= \lim_{n \to \infty} \frac{1}{nT} \ln |\varphi \circ M(T^n \circ \varphi^{-1}(X) - \varphi \circ M(T^n \circ \varphi^{-1}(X_0))|
\]

\[
= \lim_{n \to \infty} \frac{1}{nT} \ln |D(\lambda)^n \circ C^{-1} [\varphi^{-1}(X) - \varphi^{-1}(X_0)]| = -\frac{c}{2}.
\]

The second equality follows from (3.8). Hence, the rate of decay of the solution to the unique \( T \)-periodic solution is \( c/2 \), independently of the initial value \( X \).

**Remark.** The above result shows that the Lyapunov exponent is invariant under a \( C^1 \) conjugate transformation. From the proof, the conclusion is still true if \( \varphi \) is a bi-Lipschitz mapping.

### 3.2 Proof of Theorem 1.2

The existence and uniqueness of \( T \)-periodic solutions of (1.5) can be obtained by the same argument in the proof of Theorem 1.1. But for the stability of the periodic solution, the \( C^1 \) regularity of the solutions of (1.5) with respect to the initial value is needed. The following result is due to Lazer and McKenna.
**Lemma 3.2.** Let $u(t, \xi, \eta)$ be the solution of the initial-value problem

$$
\begin{align*}
\begin{cases}
u'' + cu' + a(t)u^+ - b(t)u^- &= h(t), \\
u(0) &= \xi, \\
u'(0) &= \eta.
\end{cases}
\end{align*}
$$

Assume that $a(t), b(t) \in C_T$, and that $h \in C_T$ has a finite number of zeros.

Then, for $t \in [0, \bar{t}]$, the partial derivatives of $u$ and $u'$ with respect to $\xi, \eta$ exist and are continuous. Moreover, if

$$
X(t) = \begin{bmatrix}
\frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\
\frac{\partial u'}{\partial \xi} & \frac{\partial u'}{\partial \eta}
\end{bmatrix},
$$

then

$$
X'(t) = A(t)X(t), \quad X(0) = \text{Id},
$$

where

$$
A(t) = \begin{bmatrix}
0 & 1 \\
-p(t) & -c
\end{bmatrix}
$$

and $p(t) = a(t)\chi_+ + b(t)\chi_-$, where $\chi_\pm$ denotes the characteristic function of the set $\{t \in [0, T], u(t) \geq 0\}$.

**Remark.** Though Lemma 3.2 was originally stated only for both $a$ and $b$ constant, by carefully examining the proof in the appendix of [7], one may assert that Lemma 3.2 holds when $a(t)$ and $b(t)$ are continuous.

Now that $u(t)$ and $u'(t)$ are known to be $C^1$ with respect to the initial value, by reasoning along the same lines as in the proof of Theorem 1.1, the stability and rate of decay of the periodic solutions can be obtained.
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