LEFSCHETZ DECOMPOSITIONS FOR EIGENFORMS ON A KÄHLER MANIFOLD

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Abstract. We show that the eigenspaces of the Laplacian $\Delta_k$ on $k$-forms on a compact Kähler manifold carry Hodge and Lefschetz decompositions. Among other consequences, we show that the positive part of the spectrum of $\Delta_k$ lies in the spectrum of $\Delta_{k+1}$ for $k < \dim X$.

Given a compact Riemannian manifold $X$ without boundary, let $\Delta_k$ denote the Laplacian on $k$-forms and $\lambda_1^{(k)}$ its smallest positive eigenvalue. We can ask how these numbers vary with $k$. By differentiating eigenfunctions, we easily see that $\lambda_1^{(0)} \geq \lambda_1^{(1)}$. For $k > 1$, the situation is more complicated: Takahashi [T2] has shown that the sign of $\lambda_1^{(k)} - \lambda_1^{(0)}$ can be arbitrary for compact Riemannian manifolds. More generally, Guerini and Savo [G, GS] have constructed examples, where the sequence $\lambda_1^{(2)}, \lambda_1^{(3)}, \ldots \lambda_1^{(\dim X/2)}$ can do just about anything. The goal of this note is to show that when $X$ is compact Kähler, the eigenspaces carry extra structure, and that this imposes strong constraints on the eigenvalues and their multiplicities. For instance, we show that the eigenvalues of $\Delta_k$ occur with even multiplicities when $k$ is odd. We also show that the positive part of the spectrum of $\Delta_k$ is contained in the spectrum of $\Delta_{k+1}$ for all $k < \dim C X$. Therefore the sequence $\lambda_1^{(0)}, \ldots, \lambda_1^{(\dim C X)}$ is weakly decreasing.

After this paper was submitted, it was brought to my attention that Jakobson, Strohmaier, and Zelditch [JSZ] have also studied the spectra of Kähler manifolds, although for rather different reasons.

1. Main theorems

For the remainder of this paper, $X$ will denote a compact Kähler manifold of complex dimension $n$, with Kähler form $\omega$. Let $\Delta = d^* d + dd^*$ be the Laplacian on complex valued forms $\mathcal{E}^*$. Standard arguments in Hodge theory guarantee that the spectrum of $\Delta$ is discrete, and the eigenspaces

$$\mathcal{E}^*_\lambda = \{ \alpha \in \mathcal{E}^* | \Delta \alpha = \lambda \alpha \}$$

are finite dimensional. Since $\Delta$ is positive and self adjoint, the eigenvalues are nonnegative real. We let $\mathcal{E}^k_\lambda$ and $\mathcal{E}^{(p,q)}_\lambda$ denote the intersection of $\mathcal{E}^*_\lambda$ with the space of $k$ forms and $(p,q)$-forms respectively.

The proofs of the following statements will naturally hinge on the Kähler identities [GH, W], which we recall below. We have

$$\Delta = 2(\partial \partial^* + \partial^* \partial) = 2(\partial \partial^* + \partial^* \partial)$$

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which implies that it commutes with the projections $\pi^{pq}, \pi^k : \mathcal{E}^* \to \mathcal{E}^{(p,q)}, \mathcal{E}^k$. The Laplacian $\Delta$ also commutes with the Lefschetz operator $L(-) = \omega \wedge -$ and its adjoint $\Lambda$. An additional set of identities implies that $L, \Lambda$ and $H = \sum (n-k) \pi^k$ together determine an action of the Lie algebra $sl_2(\mathbb{C})$ on $\mathcal{E}^*$.

**Theorem 1.1.** For each $\lambda$, there is a Hodge decomposition

\[
\mathcal{E}^k_{\lambda} = \bigoplus_{p+q=k} \mathcal{E}^{(p,q)}_{\lambda}
\]

(1)

\[
\overline{\mathcal{E}^{(p,q)}_{\lambda}} = \mathcal{E}^{(q,p)}_{\lambda}
\]

(2)

If $i > 0$, there is a hard Lefschetz isomorphism

\[
\omega^i \wedge : \mathcal{E}^{n-i}_{\lambda} \to \mathcal{E}^{n+i}_{\lambda}
\]

(3)

**Proof.** (1) follows from the fact that $\Delta$ commutes with $\pi^{pq}$. Since $\Delta$ and $\lambda$ are real, we obtain (2). The proof of (3) is identical to the usual proof of the hard Lefschetz theorem [GH, pp 118-122]. The key point is that by representation theory, $L_i$ maps $V \cap \mathcal{E}^{n-i}_{\lambda}$ isomorphically to $V \cap \mathcal{E}^{n+i}_{\lambda}$ for any $sl_2(\mathbb{C})$-submodule $V \subset \mathcal{E}^*$. Applying this to the subspace $V = \mathcal{E}_{\lambda}$, which is an $sl_2(\mathbb{C})$-submodule because $L, \Lambda, H$ commute with $\Delta$, proves (3). $\square$

The first part of the theorem can be rephrased as saying that $\mathcal{E}^k_{\lambda}$ is a real Hodge structure of weight $k$. We define the multiplicities $h^{pq}_{\lambda} = \dim \mathcal{E}^{(p,q)}_{\lambda}$ and $b^k_{\lambda} = \dim \mathcal{E}^k_{\lambda}$. When $\lambda = 0$, these are the usual Hodge and Betti numbers. In general, they depend on the metric. These numbers share many properties of ordinary Hodge and Betti numbers:

**Corollary 1.2.** For each $\lambda$,

(a) $b^k_{\lambda} = \sum_{p+q=k} h^{pq}_{\lambda}

(b) $h^{pq}_{\lambda} = h^{qp}_{\lambda}$

(c) $b^k_{\lambda}$ is even if $k$ is odd.

(d) $h^{2n-k}_{\lambda} = b^k_{\lambda}$.

(e) if $k < n$, $b^k_{\lambda} \leq b^{k+2}_{\lambda}$

(f) if $p + q < n$, $h^{pq}_{\lambda} = h^{n-p,n-q}_{\lambda}$.

(g) if $p + q < n$, $h^{pq}_{\lambda} \leq h^{p+1,q+1}_{\lambda}$

**Proof.** The first four statements are immediate. For (e) we use the fact that $\omega \wedge : \mathcal{E}^k_{\lambda} \to \mathcal{E}^{k+2}_{\lambda}$ is an injection by (3). For (f) and (g), we use (3), and observe that $\omega^i \wedge -$ shifts the bigrading by $(i, i)$. $\square$

The above results can be visualized in terms of the geometry of the “Hodge diamond”. When $\lambda > 0$, there are some new patterns as well. We start with a warm up.

**Lemma 1.3.** If $\lambda$ is a positive eigenvalue of $\Delta_0$, then $h^{0,1}_{\lambda} \geq h^{0,0}_{\lambda} = b^0_{\lambda}$ and $b^1_{\lambda} \geq 2b^0_{\lambda}$.

**Proof.** The map $\bar{\partial} : \mathcal{E}^0_{\lambda} \to \mathcal{E}^{0,1}_{\lambda}$ is injective, because the kernel consists of global holomorphic eigenfunctions which are necessarily constant and therefore 0. This implies the first inequality, which in turn implies the second. $\square$

We will give an extension to higher degrees, but first we start with a lemma.
Lemma 1.4. If \( \lambda > 0 \),
\[
E^{k}_\lambda = dE^{k-1}_\lambda \oplus d^* E^{k+1}_\lambda
\]
and
\[
E^{(p,q)}_\lambda = \bar{\partial}E^{(p,q-1)}_\lambda \oplus \bar{\partial}^* E^{(p,q+1)}_\lambda
\]

Proof. By standard Hodge theory \[GH, W\], we have the decompositions
\[
E^k = E_0 \oplus dE^{k-1} \oplus d^* E^{k+1}
\]
These can be combined to yield the decomposition
\[
E^k = E^k_0 \oplus \bigoplus_{i=1}^{\infty} dE^{k-1}_{\lambda, i} \oplus \bigoplus_{i=1}^{\infty} d^* E^{k+1}_{\lambda, i}
\]
Since \(dE^{k-1}_\lambda, d^* E^{k+1}_\lambda \subset E^k_\lambda\), the first part of the lemma
\[
E^k_\lambda = dE^{k-1}_\lambda \oplus d^* E^{k+1}_\lambda
\]
follows immediately. The proof of the second part is identical. \(\square\)

Theorem 1.5. Suppose that \( \lambda > 0 \).

(a) For all \( k \), \( b^k_\lambda \leq b^{k-1}_\lambda + b^{k+1}_\lambda \)

(b) If \( p + q < n \), then \( h^{pq}_\lambda \leq h^{p+1,q}_\lambda + h^{p,q+1}_\lambda \).

(c) If \( k < n \), then \( b^k_\lambda \leq b^{k+1}_\lambda \)

Proof. The first statement is an immediate consequence of lemma 1.4.

By lemma 1.4, we have a direct sum \( E^{(p,q)}_\lambda = E^{(p,q)}_{\im \bar{\partial}, \lambda} \oplus E^{(p,q)}_{\im \bar{\partial^*}, \lambda} \) of the \( \bar{\partial} \)-exact \( E^{(p,q)}_{\im \bar{\partial}, \lambda} \) and \( \bar{\partial}^* \)-coexact \( E^{(p,q)}_{\im \bar{\partial^*}, \lambda} \) parts. We denote the dimensions of these spaces by \( h^{pq}_{\im \bar{\partial}, \lambda} \) and \( h^{pq}_{\im \bar{\partial^*}, \lambda} \) respectively.

Suppose that \( \alpha \in E^{(p,q)}_{\im \bar{\partial^*}, \lambda} \) then we can write \( \alpha = \bar{\partial}^* \beta \). We have \( \bar{\partial} \alpha \in E^{(p,q+1)}_\lambda \) because \( \bar{\partial} \) and \( \Delta \) commute. Suppose that \( \bar{\partial} \alpha = 0 \). Then
\[
\alpha = \frac{1}{\lambda} \Delta \alpha = \frac{2}{\lambda} (\bar{\partial} \bar{\partial} + \bar{\partial} \bar{\partial}^*) \alpha = \frac{2}{\lambda} \bar{\partial} \bar{\partial}^* \alpha = \frac{2}{\lambda} \bar{\partial} (\bar{\partial}^*)^2 \beta
\]
This is zero, because \( \langle (\bar{\partial}^*)^2 \beta, \xi \rangle = \langle \beta, \bar{\partial}^* \xi \rangle = 0 \) for any \( \xi \). Thus the map
\[
\bar{\partial} : E^{(p,q)}_{\im \bar{\partial^*}, \lambda} \hookrightarrow E^{(p,q+1)}_{\im \bar{\partial}, \lambda}
\]
is injective. Although, we will not need it, it is worth noting that it also surjective because
\[
E^{(p,q+1)}_{\im \bar{\partial}, \lambda} = \bar{\partial} (E^{(p,q)}_{\im \bar{\partial}, \lambda} \oplus E^{(p,q)}_{\im \bar{\partial^*}, \lambda}) = \bar{\partial} E^{(p,q)}_{\im \bar{\partial^*}, \lambda}
\]
Therefore
\[
\tag{4} h^{p,q}_{\im \bar{\partial^*}, \lambda} = h^{p+1,q}_{\im \bar{\partial}, \lambda}
\]
for all \( p, q \). We now assume that \( p + q < n \). We will also establish an inequality
\[
\tag{5} h^{p,q}_{\im \bar{\partial}, \lambda} \leq h^{p+1,q}_{\im \bar{\partial}, \lambda}
\]
Let $\alpha = \bar{\partial}\beta \in \mathcal{E}_{\im\bar{\partial},\lambda}^{(p,q)}$ be a nonzero element. The previous theorem shows that $\gamma = \omega \wedge \alpha$ is a nonzero element of $\mathcal{E}_{\lambda}$. The form $\bar{\partial}^* \gamma \neq 0$, since otherwise

$$\gamma = \frac{1}{\lambda} \Delta \gamma = \frac{2}{\lambda} \bar{\partial}^2 (\omega \wedge \beta) = 0$$

Thus we have proved that the map

$$\mathcal{E}_{\im\bar{\partial},\lambda}^{(p,q)} \rightarrow \mathcal{E}_{\im\bar{\partial}^*,\lambda}^{(p+1,q)}$$

given by $\alpha \mapsto \bar{\partial}^* (\omega \wedge \alpha)$ is injective. Equation (5) is an immediate consequence.

Adding (4) and (5) yields

$$h_{\lambda}^{pq} \leq h_{\im\bar{\partial},\lambda}^{p,q+1} + h_{\im\bar{\partial}^*,\lambda}^{p+1,q}$$

which implies (b). Equation (6) also implies

$$h_{\lambda}^{k} = h_{\lambda}^{0,k} + h_{\lambda}^{1,k-1} + \ldots$$

Let $\lambda_1^{(k)}$ denote the first strictly positive eigenvalue of $\Delta_k = \Delta|_{E^k}$. 

**Corollary 1.6.** The positive spectrum of $\Delta_k$ is contained in the union of the spectra of $\Delta_{k-1}$ and $\Delta_{k+1}$.

Let $\lambda_1^{(k)}$ denote the first strictly positive eigenvalue of $\Delta_k = \Delta|_{E^k}$.

**Corollary 1.7.** If $k < n$, the positive spectrum of $\Delta_k$ is contained in the positive spectrum of $\Delta_{k+1}$. Consequently, $\lambda_1^{(0)} \geq \lambda_1^{(1)} \geq \ldots \geq \lambda_1^{(n)}$.

We can show that the positive spectra of $\Delta_k$ coincide for certain values of $k$.

**Corollary 1.8.** The positive spectra of $\Delta_{n-1}$, $\Delta_n$ and $\Delta_{n+1}$ coincide. In particular when $n = 1$, the positive spectra of all the Laplacians coincide.

**Proof.** If $\lambda > 0$, then the inequalities

$$b_{\lambda}^{n-1} = b_{\lambda}^{n+1}$$

follow from theorems 1.1 and 1.5. These imply the corollary.

The spectra are difficult to calculate in general, although there is at least one case where it is straightforward.

**Example 1.9.** Let $L \subset \mathbb{C}^n$ be a lattice with dual lattice $L^*$ with respect to the Euclidean inner product. The spectrum of each $\Delta_k$ on the flat torus $\mathbb{C}^n/L$ is easily calculated to be the same set $\{4\pi^2 ||v||^2 | v \in L^*\}$, cf. [BGM, pp 146-148].
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