A Lax Formulation of a Generalized $q$-Garnier System

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Abstract
Recently, a birational representation of an extended affine Weyl group of type $A_{mn-1}^{(1)} \times A_{m-1}^{(1)} \times A_{m-1}^{(1)}$ was proposed with the aid of a cluster mutation. In this article we formulate this representation in a framework of a system of $q$-difference equations with $mn \times mn$ matrices. This formulation is called a Lax form and is used to derive a generalization of the $q$-Garnier system.

Keywords Affine Weyl group · Discrete Painlevé equation · Garnier system

Mathematics Subject Classification 17B80 · 34M55 · 39A13

1 Introduction

The Garnier system (in $n$ variables) was proposed as a generalization of the sixth Painlevé equation in [1]. It is derived from the isomonodromy deformation of a Fuchsian system of second order with $n + 3$ regular singular points. A $q$-analogue of the Garnier system was proposed from a viewpoint of a connection problem of a system of linear $q$-difference equations in [12]. Afterward it was studied in detail by a Padé method in [7,8].

A group of symmetries for the Garnier system is isomorphic to the affine Weyl group $W(B_{n+3}^{(1)})$. This fact was shown in [5,13]. For the $q$-Garnier system, a symmetry structure was clarified recently. In [9] we formulated a birational representation of an extended affine Weyl group $\widetilde{W}(A_{2n+1}^{(1)} \times A_{1}^{(1)} \times A_{1}^{(1)})$ with the aid of a cluster mutation and derived the $q$-Garnier system as translations. We also generalized this representation to that of $\widetilde{W}(A_{mn-1}^{(1)} \times A_{m-1}^{(1)} \times A_{m-1}^{(1)})$ in [15].

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The aim of this article is to formulate the birational representation of \( \tilde{W}(A_{mn-1}^{(1)} \times A_{m-1}^{(1)} \times A_{m-1}^{(1)}) \) again in a framework of a system of \( q \)-difference equations with \( mn \times mn \) matrices. This formulation is called a Lax form. A Lax form for \( \tilde{W}(A_{2n+1}^{(1)} \times A_{1}^{(1)} \times A_{1}^{(1)}) \) was partially given in a framework of the \( q \)-Drinfeld-Sokolov hierarchy in [14]. Based on this previous work, we give a complete Lax form for \( \tilde{W}(A_{mn-1}^{(1)} \times A_{m-1}^{(1)} \times A_{m-1}^{(1)}) \) in this article.

This article is organized as follows. In Sect. 2 we recall the group of the birational transformations which is isomorphic to \( \tilde{W}(A_{mn-1}^{(1)} \times A_{m-1}^{(1)} \times A_{m-1}^{(1)}) \) given in [15]. In Sect. 3 we introduce a system of \( q \)-difference equations with \( mn \times mn \) matrices, whose compatibility conditions correspond to the group given in Sect. 2. In Sect. 4 we formulate a generalized \( q \)-Garnier system in the case \( (m, n) = (3, 2) \) as an example.

## 2 Birational Representation

Recall that the affine Weyl group \( W(A_{m-1}^{(1)}) \) is generated by the generators \( r_i \) \((i \in \mathbb{Z}_m)\) and the fundamental relations

\[
r_0^2 = r_1^2 = 1,
\]

for \( m = 2 \) and

\[
r_i^2 = 1, \quad r_i r_j r_i = r_j r_i r_j \quad (|i - j| = 1), \quad r_i r_j = r_j r_i \quad (|i - j| > 1),
\]

for \( m \geq 3 \). Here we denote the quotient ring \( \mathbb{Z}/m\mathbb{Z} \) by \( \mathbb{Z}_m \).

Let \( \varphi_{j,i} \) \((j \in \mathbb{Z}_{mn}, i \in \mathbb{Z}_m)\) be dependent variables and \( \alpha_j, \beta_i, \beta'_i \) \((j \in \mathbb{Z}_{mn}, i \in \mathbb{Z}_m)\) be parameters defined by

\[
\alpha_j = \prod_{i=0}^{m-1} \varphi_{j,i}, \quad \beta_i = \prod_{j=0}^{mn-1} \varphi_{j,i}, \quad \beta'_i = \prod_{j=0}^{mn-1} \varphi_{j,i+j}.
\]

We also set

\[
\prod_{j=0}^{mn-1} \alpha_j = \prod_{i=0}^{m-1} \beta_i = \prod_{i=0}^{m-1} \beta'_i = \prod_{j=0}^{mn-1} \prod_{i=0}^{m-1} \varphi_{j,i} = q.
\]

Note that the parameters \( \alpha_j, \beta_i \) and \( \beta'_i \) correspond to multiplicative simple roots for \( W(A_{mn-1}^{(1)}), W(A_{m-1}^{(1)}) \) and \( W(A_{m-1}^{(1)}) \) respectively. We define birational transformations \( r_j \) \((j \in \mathbb{Z}_{mn})\) by
They act on the parameters as

\[ r_j(\varphi_{j-1,i}) = \varphi_{j-1,i} \, \varphi_{j,i+1}, \quad r_j(\varphi_{j,i}) = \frac{1}{\varphi_{j,i+1}} \, \frac{P_{j,i+2}}{P_{j,i+1}}, \quad r_j(\varphi_{j+1,i}) = \varphi_{j,i} \, \varphi_{j+1,i}, \quad r_j(\varphi_{k,i}) = \varphi_{k,i} \quad (k \neq j, j \pm 1), \]  

(2.1)

where

\[ P_{j,i} = \sum_{k=0}^{m-1} \prod_{l=0}^{k-1} \varphi_{j,i+l}. \]

They act on the parameters as

\[ r_j(\alpha_j) = \frac{1}{\alpha_j}, \quad r_j(\alpha_{j+1}) = \alpha_{j+1} \, \alpha_j, \quad r_j(\alpha_k) = \alpha_k \quad (k \neq j, j \pm 1), \]

\[ r_j(\beta_i) = \beta_i, \quad r_j(\beta'_i) = \beta'_i. \]

We also define birational transformations \( s_i, s'_i \ (i \in \mathbb{Z}_m) \) by

\[ s_i(\varphi_{j,i}) = \frac{1}{\varphi_{j+1,i}} \, \frac{Q_{j,i}}{Q_{j+2,i}}, \quad s_i(\varphi_{j,i+1}) = \varphi_{j,i} \, \varphi_{j,i+1} \, \varphi_{j+1,i} \, \varphi_{j+2,i} \, \frac{Q_{j+2,i}}{Q_{j,i}}, \]

\[ s'_i(\varphi'_{j,i}) = \frac{1}{\varphi'_{j+1,i}} \, \frac{Q'_{j,i}}{Q'_{j+2,i}}, \quad s'_i(\varphi'_{j,i+1}) = \varphi'_{j,i} \, \varphi'_{j,i+1} \, \varphi'_{j+1,i} \, \varphi'_{j+2,i} \, \frac{Q'_{j+2,i}}{Q'_{j,i}}, \]

(2.2)

for \( m = 2 \) and

\[ s_i(\varphi_{j-1,i}) = \varphi_{j-1,i} \, \varphi_{j+1,i} \, \varphi_{j+2,i} \, \varphi_{j+1,i} \, \frac{Q_{j+2,i}}{Q_{j+1,i}}, \quad s_i(\varphi_{j,i}) = \frac{1}{\varphi_{j+1,i}} \, \frac{Q_{j,i}}{Q_{j+2,i}}, \]

\[ s_i(\varphi_{j,i+1}) = \varphi_{j,i} \, \varphi_{j,i+1} \, \varphi_{j+1,i} \, \varphi_{j+2,i} \, \frac{Q_{j+2,i}}{Q_{j+1,i}}, \quad s_i(\varphi_{j,k}) = \varphi_{j,k} \quad (k \neq i, i \pm 1), \]

\[ s'_i(\varphi'_{j-1,i}) = \varphi'_{j-1,i} \, \varphi'_{j+1,i} \, \varphi'_{j+2,i} \, \varphi'_{j+1,i} \, \frac{Q'_{j+2,i}}{Q'_{j+1,i}}, \quad s'_i(\varphi'_{j,i}) = \frac{1}{\varphi'_{j+1,i}} \, \frac{Q'_{j,i}}{Q'_{j+2,i}}, \]

\[ s'_i(\varphi'_{j,i+1}) = \varphi'_{j,i} \, \varphi'_{j,i+1} \, \varphi'_{j+1,i} \, \varphi'_{j+2,i} \, \frac{Q'_{j+2,i}}{Q'_{j+1,i}}, \quad s'_i(\varphi'_{j,k}) = \varphi'_{j,k} \quad (k \neq i, i \pm 1), \]

(2.3)

for \( m \geq 3 \), where

\[ Q_{j,i} = \sum_{k=0}^{m-1} \prod_{l=0}^{k-1} \varphi_{j+l,i}, \quad Q'_{j,i} = \sum_{k=0}^{m-1} \prod_{l=0}^{k-1} \varphi'_{j+l,i}. \]
and \( \varphi'_{j,i} = \varphi_{-j,i-j} \). They act on the parameters as

\[
\begin{align*}
s_i(\beta_i) &= \frac{1}{\beta_i}, & s_i(\beta_{i+1}) &= \beta_{i+1} \beta_i^2, & s_i(\alpha_j) &= \alpha_j, & s_i(\beta_k') &= \beta_k', \\
s'_i(\beta_i') &= \frac{1}{\beta_i'}, & s'_i(\beta_{i+1}') &= \beta_{i+1}'(\beta_i')^2, & s'_i(\alpha_j) &= \alpha_j, & s'_i(\beta_k) &= \beta_k,
\end{align*}
\]

for \( m = 2 \) and

\[
\begin{align*}
s_i(\beta_i) &= \frac{1}{\beta_i}, & s_i(\beta_{i+1}) &= \beta_{i+1} \beta_i, & s_i(\beta_k) &= \beta_k \ (k \neq i, i \pm 1), \\
s_i(\alpha_j) &= \alpha_j, & s_i(\beta_k') &= \beta_k', \\
s'_i(\beta_i') &= \frac{1}{\beta_i'}, & s'_i(\beta_{i+1}') &= \beta_{i+1}' \beta_i', \\
s'_i(\beta_k') &= \beta_k' \ (k \neq i, i \pm 1), & s'_i(\alpha_j) &= \alpha_j, & s'_i(\beta_i) &= \beta_i,
\end{align*}
\]

for \( m \geq 3 \).

**Fact 2.1** ([2,6]) If we set

\[ G = \langle r_0, \ldots, r_{mn-1} \rangle, \quad H = \langle s_0, \ldots, s_{m-1} \rangle, \quad H' = \langle s'_0, \ldots, s'_{m-1} \rangle, \]

then the groups \( G, H \) and \( H' \) are isomorphic to the affine Weyl groups \( W(A_{mn-1}^{(1)}) \), \( W(A_{m-1}^{(1)}) \) and \( W(A_{m-1}^{(1)}) \) respectively. Moreover, any two groups are mutually commutative.

**Remark 2.2** In this article we interpret a composition of transformations in terms of automorphisms of the field of rational functions \( \mathbb{C}(\varphi_{j,i}) \). For example, the compositions \( r_0 \ r_1, \ r_1 \ r_0 \) act on the parameter \( \alpha_0 \) as

\[
\begin{align*}
r_0 \ r_1(\alpha_0) &= r_0(\alpha_0) \alpha_1 = r_0(\alpha_0) r_0(\alpha_1) = \alpha_1, \\
r_1 \ r_0(\alpha_0) &= r_1 \left( \frac{1}{\alpha_0} \right) = \frac{1}{r_1(\alpha_0)} = \frac{1}{\alpha_0} \alpha_1.
\end{align*}
\]

In addition, we define birational transformations \( \pi_1, \pi_2 \) by

\[
\begin{align*}
\pi_1(\varphi_{j,i}) &= \varphi_{j+1,i+1}, \\
\pi_2(\varphi_{j,i}) &= \varphi_{j,i+1}.
\end{align*}
\]  

(2.4)  

(2.5)

They act on the parameters as

\[
\begin{align*}
\pi_1(\alpha_j) &= \alpha_{j+1}, & \pi_1(\beta_i) &= \beta_{i+1}, & \pi_1(\beta_i') &= \beta_i', \\
\pi_2(\alpha_j) &= \alpha_j, & \pi_2(\beta_i) &= \beta_{i+1}, & \pi_2(\beta_i') &= \beta_{i+1}'.
\end{align*}
\]

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Fact 2.3 ([15]) The transformations $\pi_1, \pi_2$ satisfy fundamental relations

$$
\pi_1^{mn} = 1, \quad \pi_2^{mn} = 1, \quad \pi_1 \pi_2 = \pi_2 \pi_1,$$

$$
r_j \pi_1 = \pi_1 r_{j-1}, \quad s_i \pi_1 = \pi_1 s_{i-1}, \quad s_i' \pi_1 = \pi_1 s_i',
$$

$$
r_j \pi_2 = \pi_2 r_j, \quad s_i \pi_2 = \pi_2 s_{i-1}, \quad s_i' \pi_2 = \pi_2 s_i',
$$

for $j \in \mathbb{Z}_{mn}$ and $i \in \mathbb{Z}_m$.

Hence we can regard the semi-direct product $\langle G, H, H' \rangle \rtimes \langle \pi_1, \pi_2 \rangle$ as an extended affine Weyl group $\tilde{W}(A_{mn-1}^{(1)} \times A_m^{(1)} \times A_m^{(1)})$. On the other hand, the group $\tilde{W}(A_{mn-1}^{(1)} \times A_m^{(1)} \times A_m^{(1)})$ contains an abelian normal subgroup generated by translations. Combining them, we can provide a class of generalized $q$-Garnier systems as translations. Recall that the $q$-Garnier system was obtained as the case $m = 2$ in [9].

3 Lax Form

Let us introduce an independent variable $z$ satisfying

$$
r_j(z) = z \quad (j \in \mathbb{Z}_{mn}), \quad s_i(z) = z, \quad s_i'(z) = z \quad (i \in \mathbb{Z}_m), \quad \pi_1(z) = z,
$$

and

$$
\pi_2(z) = q^{1/m} z. \quad (3.1)
$$

We also set

$$
\zeta = z \prod_{j=1}^{mn-1} \prod_{i=0}^{m-2} \alpha_j^{(mn-j)/m} \beta_i^{(i+1)/m}.
$$

Then the birational transformations act on $\zeta$ as

$$
r_0(\zeta) = \alpha_0^{n} \zeta, \quad r_1(\zeta) = \frac{1}{\alpha_1^{m}} \zeta, \quad r_k(\zeta) = \zeta \quad (k \neq 0, 1),
$$

$$
s_{m-2}(\zeta) = \frac{1}{\beta_{m-2}} \zeta, \quad s_{m-1}(\zeta) = \beta_{m-1} \zeta, \quad s_k(\zeta) = \zeta \quad (k \neq m-2, m-1),
$$

$$
s_i'(\zeta) = \zeta, \quad \pi_1(\zeta) = \frac{\beta_{m-1}}{\alpha_1^{n}} \zeta, \quad \pi_2(\zeta) = \beta_{m-1} \zeta.
$$

We first give a Lax form for the birational transformations $\pi_1, \pi_2$. Let $E_{j_1,j_2}$ be a $mn \times mn$ matrix with 1 in $(j_1, j_2)$-th entry and 0 elsewhere. Consider $mn \times mn$ matrices

$$
\Pi_1 = \zeta^{-\log_q \alpha_1} \left( \sum_{j=1}^{mn-1} \prod_{k=0}^{j-1} \frac{1}{\varphi_{1,k}} E_{j,j+1} + \frac{1}{\alpha_1^{n} \prod_{i=0}^{m-2} \beta_i} \zeta E_{mn,1} \right).
$$
and
\[
\Pi_2 = \sum_{j=1}^{mn} \prod_{k=1}^{j-1} \varphi_{k,j-1} E_{j,j} + \sum_{j=1}^{mn-1} E_{j,j+1} + \frac{1}{\prod_{i=0}^{m-2} \beta_i} \zeta E_{mn,1}.
\]

We also set
\[
M = \pi_{2}^{m-1}(\Pi_2) \pi_{2}^{m-2}(\Pi_2) \ldots \pi_2(\Pi_2) \Pi_2,
\]
and \(T_{q,z} = \pi_z^m\).

**Theorem 3.1** Under transformations (2.5) and (3.1), the compatibility condition of a system of linear q-difference equations

\[
T_{q,z}(\psi) = M \psi, \quad \pi_1(\psi) = \Pi_1 \psi,
\]

with a fundamental relation

\[
\pi_2(\Pi_1) \Pi_2 = \beta_{m-1}^{-\log q \alpha_1} \varphi_{1,0} \pi_1(\Pi_2) \Pi_1, \quad (3.2)
\]

is equivalent to transformation (2.4).

**Proof** If we assume transformation (2.4), then we can show system (3.2) by a direct calculation. It follows that

\[
T_{q,z}(\Pi_1) = \pi_{2}^{m-1}\pi_2(\Pi_1)
\]

\[
= \beta_{m-2}^{-\log q \alpha_1} \varphi_{1,m-1} \pi_{2}^{m-1}\pi_1(\Pi_2) \pi_{2}^{m-1}(\Pi_1) \pi_{2}^{m-1}(\Pi_2^{-1})
\]

\[
= \beta_{m-3}^{-\log q \alpha_1} \beta_{m-2}^{-\log q \alpha_1} \varphi_{1,m-2} \varphi_{1,m-1} \pi_{2}^{m-1}\pi_1(\Pi_2) \pi_{2}^{m-2}\pi_1(\Pi_2)
\]

\[
\times \pi_{2}^{m-2}(\Pi_1) \pi_{2}^{m-2}(\Pi_2^{-1}) \pi_{2}^{m-2}(\Pi_2^{-1}) \pi_{2}^{m-2}(\Pi_2^{-1})
\]

\[
= \ldots
\]

\[
= q^{-\log q \alpha_1} \alpha_1 \pi_{2}^{m-1}\pi_1(\Pi_2) \ldots \pi_2(\Pi_2) \pi_1(\Pi_2) \Pi_1 \Pi_2^{-1}
\]

\[
\times \pi_{2}(\Pi_2^{-1}) \ldots \pi_{2}^{m-1}(\Pi_2^{-1})
\]

\[
= \pi_1(M) \Pi_1 M^{-1}.
\]

Inversely, we can restore transformation (2.4) by going back the way we came.

We next give a Lax form for the other birational transformations. Let \(I\) be the identity matrix. Consider \(mn \times mn\) matrices

\[\text{Springer}\]
\[ R_0 = \zeta^{\log_q \alpha_0} \left( \sum_{j=1}^{mn} \frac{P_{0,j-1} \prod_{k=0}^{j-2} \varphi_{0,k}}{P_{0,0}} E_{j,j} + \frac{q (1 - \alpha_0)}{\varphi_{0,m-1}} \frac{1}{P_{0,0}} \zeta E_{1,mn} \right), \]

\[ R_1 = \zeta^{-\log_q \alpha_1} \left( \sum_{j=1}^{mn} \frac{\varphi_{1,1} P_{1,2}}{P_{1,1}} E_{j,j} + \frac{1 - \alpha_1}{P_{1,1}} E_{2,1} \right), \]

\[ R_j = I + \frac{(1 - \alpha_j)}{P_{j,j}} \prod_{k=1}^{j-1} \varphi_{k,j-1} E_{j+1,j} \quad (j = 2, \ldots, mn - 1), \]

and

\[ S_i = I + \sum_{k=0}^{n-1} \left( \frac{Q_{mk+i+2,i} \prod_{l=2}^{mk+i+1} \varphi_{l,i}}{Q_{2,i}} - 1 \right) E_{mk+i+1,mk+i+1} \]

\[ + \sum_{k=0}^{n-1} \left( \frac{Q_{1,i}}{Q_{mk+i+2,i} \prod_{l=1}^{mk+i+1} \varphi_{l,i}} - 1 \right) E_{mk+i+2,mk+i+2} \]

\[ + \sum_{k=0}^{n-1} \frac{\beta_i - 1}{\varphi_{1,i}} E_{mk+i+1,mk+i+2} \quad (i = 0, \ldots, m - 2), \]

\[ S_{m-1} = I + \sum_{k=1}^{n} \left( \frac{Q_{mk+1,m-1} \prod_{l=2}^{mk} \varphi_{l,m-1}}{Q_{2,m-1}} - 1 \right) E_{mk,mk} \]

\[ + \sum_{k=1}^{n} \left( \frac{Q_{1,m-1}}{Q_{mk+1,m-1} \prod_{l=1}^{mk} \varphi_{l,m-1}} - 1 \right) E_{mk+1,mk+1} \]

\[ + \sum_{k=1}^{n} \frac{\beta_{m-1} - 1}{\varphi_{1,m-1}} E_{mk,mk+1} + \frac{\beta_{m-1} - 1}{\varphi_{1,m-1} \prod_{l=0}^{m-2} \beta_l} \zeta E_{mn,1}. \]

\[ S'_i = I + \sum_{k=0}^{n-1} \left( \frac{\varphi'_{0,i}}{Q'_{0,i}} \frac{Q'_{1,i}}{Q'_{0,i}} - 1 \right) E_{mk+i+2,mk+i+2} \quad (i = 0, \ldots, m - 2). \]

We also set

\[ S'_{m-1} = s'_{m-1} (\Pi_2^{-1}) \pi_2 (s'_{m-2}) \Pi_2. \]

The explicit formula of \( S'_{m-1} \) is not given here.

**Remark 3.2** Strangely the matrix \( S'_{m-1} \) is rational in \( \zeta \), is not diagonal and hence is much more complicated than the others. The cause has not been clarified yet. However, this is not a serious matter. We can avoid using the matrix \( S'_{m-1} \), or equivalently the transformation \( s'_{m-1} \), when we define the group of translations.

**Theorem 3.3** Under transformations (2.5) and (3.1), the compatibility condition of a system of linear \( q \)-difference equations
with fundamental relations

\[ \pi_2(R_j) \Pi_2 = r_j(\Pi_2) R_j \quad (j \in \mathbb{Z}_{mn}), \]
\[ \pi_2(S_{i-1}) \Pi_2 = s_i(\Pi_2) S_i, \quad \pi_2(S'_{i-1}) \Pi_2 = s'_i(\Pi_2) S'_i \quad (i \in \mathbb{Z}_m). \]

(3.3)
is equivalent to transformations (2.1), (2.2) and (2.3).

**Proof** We prove the formulae for the transformations \( s_i \). If we assume transformations (2.2) and (2.3), then we can show the second equation of (3.3) by a direct calculation with

\[ \phi_{j,i} Q_{j+1,i} = Q_{j,i} + \beta_j + 1. \]

It follows that

\[
S_i = s_i(\Pi_2^{-1}) \pi_2(S_{i-1}) \Pi_2 \\
= s_i(\Pi_2^{-1}) \pi_2 s_{i-1}(\Pi_2^{-1}) \pi_2^2(S_{i-2}) \pi_2(\Pi_2) \Pi_2 \\
= \ldots \\
= s_i(\Pi_2^{-1}) \pi_2 s_{i-1}(\Pi_2^{-1}) \ldots \pi_2^{m-1} s_{i+1}(\Pi_2^{-1}) \\
\times \pi_2^m(S_i) \pi_2^{m-1}(\Pi_2) \ldots \pi_2(\Pi_2) \Pi_2 \\
= s_i(M^{-1}) T_{q,z}(S_i) M.
\]

Recall that \( \pi_2 s_{i-1} = s_i \pi_2 \). Inversely, we can restore transformations (2.2) and (2.3) by going back the way we came.

We can prove the formulae for the other transformations in a similar manner by using

\[ \phi_{j,i} P_{j,i+1} = P_{j,i} + \alpha_j + 1 \quad (j \in \mathbb{Z}_{mn}, \ i \in \mathbb{Z}_m), \]

and

\[ Q'_{j,i} + \phi'_{j+1,i} Q'_{j+2,i} = (1 + \phi'_{j,i}) Q'_{j+1,i} \quad (j \in \mathbb{Z}_{mn}, \ i \in \mathbb{Z}_m). \]

We don’t state its detail here. \( \square \)

**Remark 3.4** The above definition of the matrices is suggested by [14], in which we propose a \( q \)-analogue of the Drinfeld-Sokolov hierarchy of type \( A^{(1)}_{mn-1} \) corresponding to the partition \((n, \ldots, n)\) of \( mn \). A relationship between our Lax form and the cluster algebra is not completely understood. It is a future problem.

**Remark 3.5** The transformations \( r_j \) were given by gauge transformations of the \( q \)-KP hierarchy in [3]. Besides, the transformations \( s_i \) (or \( s'_i \)) were given with the aid of a factorization of a matrix and permutations of the factors in [3,4,10,11].
In the previous section we obtained a Lax form for the extended affine Weyl group \( \langle G, H, H' \rangle \simeq \langle \pi_1, \pi_2 \rangle \). Hence we can derive a class of generalized \( q \)-Garnier systems together with their Lax pairs from the group of translations systematically. Since the case \( m = 2 \) has been already studied in detail in \([9,15]\), we consider another case as an example in this section.

**Remark 4.1** In Remark 4.2 of \([9]\), we conjectured that the translation \( s_1 r_1 \ldots r_{2n-1} \pi_1 \) for \( m = 2 \) implies a variation of the \( q \)-Garnier system given in §3.2.4 of \([8]\). We can show that this conjecture is true by using the Lax form. Since the proof can be given in a similar manner as that given in §6 of \([9]\), we omit it here.

Let \( m = 3 \) and \( n = 2 \). Then the matrix \( M \) is described as

\[
M = \begin{pmatrix}
1 & P_{1,1} & P_{1,2}^* & 1 & 0 & 0 \\
0 & \alpha_1 & \varphi_{1,0} P_{2,0} & \varphi_{1,1} P_{3,1} & 1 & 0 \\
0 & 0 & \varphi_{1,0} P_{1,1} & \varphi_{1,1} P_{2,1} & \varphi_{1,2} P_{3,2} & 1 \\
\zeta \frac{\rho_0}{\beta_0} & 0 & 0 & \alpha_1 \alpha_2 \alpha_3 & \alpha_1 \alpha_2 \alpha_3 \alpha_4 & \alpha_1 \alpha_2 \alpha_3 \alpha_4 \\
\frac{\varphi_{1,0} \varphi_{2,0} \varphi_{3,0} \varphi_{4,0} P_{5,0}^* \zeta}{\rho_0 \beta_0} & \zeta \frac{\beta_1}{\beta_1} & 0 & 0 & \alpha_1 \alpha_2 \alpha_3 \alpha_4 & \alpha_1 \alpha_2 \alpha_3 \alpha_4 \\
\frac{\varphi_{1,1} \varphi_{2,1} \varphi_{3,1} \varphi_{4,1} P_{5,1}^* \zeta}{\rho_0 \beta_0 \varphi_{1,2} \varphi_{2,2} \varphi_{3,2} \varphi_{4,2} \varphi_{4,1}} & \frac{\varphi_{1,1} \varphi_{2,1} \varphi_{3,1} \varphi_{4,1} P_{5,1}^* \zeta}{\rho_0 \beta_0 \varphi_{1,2} \varphi_{2,2} \varphi_{3,2} \varphi_{4,2} \varphi_{4,1}} & \zeta & 0 & 0 & \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \\
\end{pmatrix}
\]

where

\[
P_{j,i} = 1 + \varphi_{j,i} + \varphi_{j,i} \varphi_{j,i+1}, \quad P_{j,i}^* = 1 + \varphi_{j,i} + \varphi_{j,i} \varphi_{j,i+1,i}.
\]

Note that the dependent variables and the parameters satisfy a periodic condition

\[
\varphi_{j,i} = \varphi_{j+6,i} = \varphi_{j,i+3}, \quad \alpha_j = \alpha_{j+6}, \quad \beta_i = \beta_{i+3}, \quad \beta_i' = \beta_{i+3}'.
\]

We consider a translation

\[
\tau = s_0 s_1 s_0' s_1' \pi_2,
\]

which acts on the parameters as

\[
\tau(\alpha_j) = \alpha_j, \quad \tau(\beta_i) = q^{-\delta_{i,0}+\delta_{i,2}} \beta_i, \quad \tau(\beta_i') = q^{-\delta_{i,0}+\delta_{i,2}} \beta_i'.
\]

where \( \delta_{i,k} \) stands for the Kronecker’s delta. Then the action of \( \tau \) on the dependent variables \( \varphi_{j,i} \) is derived from the compatibility condition of a Lax pair \( T_q(z)(T) = \tau(M) T \), \( T = s_1 s_0 s_1' \pi_2 (s_0) s_0' s_1' \pi_2 (s_1) s_1' \pi_2 (s_0') \pi_2 (s_1') \Pi_2 \). (4.1)

System (4.1) can be regarded as a system of meromorphic \( q \)-difference equations of eighth order; see Remark 4.3 below. We don’t give its explicit formula here.
In the following, we introduce a particular solution of system (4.1) in terms of a linear $q$-difference equation. Assume that

$$P_{1,1} = P_{1,2}^* = P_{4,1} = P_{4,2}^* = 0,$$

which contains a constraint between parameters

$$\alpha_1^2 \alpha_2 \alpha_4^2 \alpha_5 = \frac{\beta_0 \beta_2'}{\beta_2 \beta_0'}.$$

Also assume that

$$\beta_0' \beta_2' = 1.$$

Then there exist two invariants

$$\tau(\varphi_{0.0} \varphi_{0.2} \varphi_{1.0}) = \varphi_{0.0} \varphi_{0.2} \varphi_{1.0}, \quad \tau(\varphi_{1.2} \varphi_{2.0} \varphi_{2.2}) = \varphi_{1.2} \varphi_{2.0} \varphi_{2.2}.$$

Now we normalize that

$$\varphi_{0.0} \varphi_{0.2} \varphi_{1.0} = c, \quad \varphi_{1.2} \varphi_{2.0} \varphi_{2.2} = -\frac{(c - 1) \alpha_2 \beta_0 \beta_2'}{c \alpha_1 \alpha_2 \alpha_4 \alpha_5 - \beta_0 \beta_2'},$$

where $c$ is an arbitrary constant. Then we obtain a system of meromorphic $q$-difference equations with one independent variable $\beta_2'$, three dependent variables $(1 + \varphi_{0.0}) \varphi_{0.2}, \varphi_{0.2}, (1 + \varphi_{3.0}) \varphi_{2.0}$ and six parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta_0 \beta_2'$. We don’t give its explicit formula here.

Let $x_0, x_1, x_2, x_3$ be dependent variables such that

$$\frac{\tau(x_0)}{x_0} = \frac{\beta_0' - \alpha_1}{\alpha_1 \beta_0'} - \frac{c \alpha_1 \alpha_2 \alpha_4 \alpha_5 (\alpha_4 - 1) + \alpha_1 \alpha_2 \alpha_4^2 \alpha_5 - \beta_0 \beta_2'}{(c - 1) \alpha_1^2 \alpha_2^2 \alpha_4^2 \alpha_5} \varphi_{2.0} \varphi_{3.2} + \frac{c (\alpha_1 \alpha_2 \alpha_4 \varphi_{2.0} - \beta_0 \beta_2')}{(c - 1) \alpha_1 \alpha_2 \beta_0 \beta_2'} (1 + \varphi_{3.0}) \varphi_{2.0},$$

and

$$\frac{x_1}{x_0} = \frac{1}{c - 1} \frac{1}{1 + \varphi_{0.0} \varphi_{0.2} - \frac{c (\alpha_0 - 1) + \alpha_0 (\alpha_1 - 1)}{(c - 1) (\alpha_0 \alpha_1 - 1)}},$$

$$\frac{x_2}{x_0} = \frac{c \alpha_1 \alpha_2 \alpha_4 \alpha_5 - \beta_0 \beta_2'}{c - 1} \varphi_{2.0} \varphi_{3.2},$$

$$\frac{x_3}{x_0} = \frac{c \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 (\alpha_4 - 1) + \beta_0 \beta_2' (\alpha_3 - 1)}{(c - 1) \alpha_1 \alpha_2 \alpha_4 \alpha_5 (\alpha_3 \alpha_4 - 1)} \frac{\varphi_{2.0} \varphi_{3.2}}{c - 1} (1 + \varphi_{3.0}) \varphi_{2.0}.$$

Recall that $\alpha_0 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 = q$. 

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Proposition 4.2 A vector of variables $x = \{x_0, x_1, x_2, x_3\}$ satisfies a system of linear $q$-difference equation

$$\tau(x) = (A_0 + \beta_2 A_1) x,$$

with $4 \times 4$ matrices

$$A_0 = \begin{pmatrix}
\frac{1}{\alpha_1} & 0 & -\frac{(\alpha_4-1)(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5-\beta_0\beta_5)}{\alpha_1^2\alpha_2^2\alpha_3^2\alpha_4\alpha_5\beta_0\beta_5(\alpha_3\alpha_4-1)} & \frac{\alpha_1\alpha_2\alpha_4\alpha_5-\beta_0\beta_5}{\alpha_1\alpha_2\beta_0\beta_5}\alpha_3\alpha_4-1) \\
0 & \alpha_0 & \frac{\alpha_1\alpha_2\alpha_3\alpha_4}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) \\
0 & 0 & \frac{1}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) \\
0 & 0 & 0 \\
\end{pmatrix},$$

$$A_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\frac{(\alpha_1-1)(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2-\beta_0\beta_5)}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & \frac{\alpha_1\alpha_2\alpha_4\alpha_5-\beta_0\beta_5}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & \frac{1}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & 0 \\
\frac{(\alpha_1-1)(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2-\beta_0\beta_5)}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & \frac{\alpha_1\alpha_2\alpha_4\alpha_5-\beta_0\beta_5}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & \frac{1}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & 0 \\
\frac{(\alpha_1-1)(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2-\beta_0\beta_5)}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & \frac{\alpha_1\alpha_2\alpha_4\alpha_5-\beta_0\beta_5}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & \frac{1}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & 0 \\
\frac{(\alpha_1-1)(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2-\beta_0\beta_5)}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & \frac{\alpha_1\alpha_2\alpha_4\alpha_5-\beta_0\beta_5}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & \frac{1}{\alpha_1\alpha_2}\alpha_3\alpha_4-1) & 0 \\
\end{pmatrix}.$$

We can prove this proposition by a direct calculation.

Remark 4.3 In a continuous limit $q \to 1$, system (4.2) reduces to the rigid system with the spectral type $[22, 211, 111]$ whose solution is expressed in terms of Simpson’s even four hypergeometric function. Moreover, system (4.1) reduces to the isomonodromy deformation equation of eighth order with the spectral type $[31, 31, 31, 22, 111]$.

Remark 4.4 Another generalization of the $q$-Garnier system is proposed together with its particular solution expressed in terms of the $q$-hypergeometric function $F_{N,M}$ in [10,11]. We have not clarified a relationship between our result and that of [10,11] yet. It is a future problem.

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