THICK SUBCATEGORIES IN STABLE HOMOTOPY THEORY
(WORK OF DEVINATZ, HOPKINS, AND SMITH).

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In this series of lectures we give an exposition of the seminal work of Devinatz, Hopkins, and Smith which is surrounding the classification of the thick subcategories of finite spectra in stable homotopy theory. The lectures are expository and are aimed primarily at non-homotopy theorists. We begin with an introduction to the stable homotopy category of spectra, and then talk about the celebrated thick subcategory theorem and discuss a few applications to the structure of the Bousfield lattice. Most of the results that we discuss below were conjectured by Ravenel [Rav84] and were proved by Devinatz, Hopkins, and Smith [DHS88, HS98].

1. THE STABLE HOMOTOPY CATEGORY OF SPECTRA

Recall that in homotopy theory one is interested in studying the homotopy classes of maps between CW complexes (spaces that are built in a systematic way by attaching cells): If \( f \) and \( g \) are maps (continuous) between CW complexes \( X \) and \( Y \), we say that they are homotopic if there is a map from the cylinder \( X \times [0, 1] \) to \( Y \) whose restriction to the two ends (top and bottom) of the cylinder gives \( f \) and \( g \) respectively. The homotopy classes of maps between \( X \) and \( Y \) is denoted by \([X, Y]\). In stable homotopy theory one studies a weaker notion of homotopy called stable homotopy – maps \( f \) and \( g \) as above are said to be stably homotopic if \( \Sigma^n f \) and \( \Sigma^n g \) are homotopic for some \( n \). (\( \Sigma \) denotes the reduced suspension functor on the homotopy category of pointed CW complexes.) The notion of stable homotopy is much weaker than homotopy. For example, the obvious quotient map from the torus to the two sphere is not null homotopic but stably null homotopic.

The importance of stable homotopy classes of maps comes from an old result due to Freudenthal which implies that if \( X \) and \( Y \) are finite CW complexes, then the sequence

\[
[X, Y] \rightarrow [\Sigma X, \Sigma Y] \rightarrow [\Sigma^2 X, \Sigma^2 Y] \rightarrow \cdots
\]

eventually stabilises. The stable homotopy classes of maps from \( X \) to \( Y \) is precisely the above colimit. In particular when \( X \) is the \( n \)-sphere \( S^n \), we get the \( n \)-th stable homotopy group of \( Y \), denoted \( \pi_n^s(Y) \). Computing stable homotopy groups is, in general, a more manageable problem than that of homotopy groups. However, it became abundantly clear to homotopy theorists by 1960s that in order to do serious stable calculations efficiently it is absolutely essential to have a nice category in which the objects are stabilised analogue of spaces each of which represent a cohomology theory. The finite objects of such a category can be easily described. This is called the (finite) Spanier-Whitehead category which captures finite stable phenomena, and is defined as follows. The objects are ordered pairs \((X, n)\) where \( X \)
is a finite CW complex and $n$ is an integer, and morphisms between objects $(X, n)$ and $(Y, m)$ are given by

$$
\{(X, n), (Y, m)\} := \text{colim}_k \left[ \Sigma^{n+k} X, \Sigma^{m+k} Y \right].
$$

This category has a formal suspension $\Sigma(X, n) := (X, n + 1)$ which agrees with the geometric suspension, i.e., $(X, n + 1) \cong (\Sigma X, n)$. While there is no geometric desuspension, there is a formal desuspension $\Sigma^{-1}(X, n) = (X, n - 1)$. Thus by passing to the Spanier-Whitehead category we have inverted the suspension functor on CW complexes! Moreover, this category has a tensor triangulated structure: exact triangles are induced by mapping sequences and the product comes from the smash product of CW complexes. Although this category is the right stabilisation of finite CW complexes, it has its limitations. The key point here is that one needs infinite dimensional CW complexes to understand finite CW complexes. For example, the singular cohomology theories on finite CW complexes are represented by Eilenberg-Mac Lane spaces which are infinite dimensional. So naturally one has to enlarge the finite Spanier-Whitehead category so that it has all the desired properties; one at least demands arbitrary coproducts and the Brown representability theorem. Building the “stable category” with all the desired properties is quite challenging. Several stable categories have been proposed; the first satisfactory category was constructed by Mike Boardman in his 1964 Warwick thesis [Boardman], and then by Frank Adams [Adams], followed by several others. All these models share a set of properties which can be taken to be the defining properties of the stable homotopy category. Following Margolis [Margolis], we take this axiomatic approach.

**Theorem 1.** There is a category $\mathcal{S}$ called the stable homotopy category (whose objects are called spectra) which has the following properties.

1. $\mathcal{S}$ is a triangulated category which admits arbitrary set indexed coproducts.
2. $\mathcal{S}$ has a unital, commutative and associative smash product which is compatible with the triangulation.
3. The sphere spectrum is a graded weak generator: $\pi_*(X) = 0$ implies $X = 0$.
4. The full subcategory of compact objects of $\mathcal{S}$ is equivalent to the Spanier-Whitehead category of finite CW complexes.

Note that there are a lot of categories in the literature which satisfy the first three properties. It is the property (4) that makes the theorem very unique, important and non-trivial. It is also worth pointing out that the study of spectra is equivalent to that of generalised homology theories on CW complexes (theories which satisfy all the Eilenberg-Steenrod axioms except the dimension axiom.) Some standard examples of such theories are the singular homology, complex K-theory and Complex bordism which are represented by the Eilenberg-Mac Lane spectrum, the K-theory spectrum $BU$ and the Thom spectrum $MU$ respectively. The study of these two subjects is in turn essentially equivalent to the study of infinite loop spaces. To get a better picture of the strong connections between spectra, generalised homology theories and infinite loop spaces, we refer the reader to Adams excellent account [Adams].

The stable homotopy category is very rich in its structural complexity, and one of the goals of the subject is to understand the global structure of this category. Doug Ravenel in the late 70s suspected some deep and interesting structure in this category (which was inspired by his algebraic calculations) and has formulated
seven conjectures \cite{Rav84} on the structure of $S$. All but one of them have been solved by 1986, due largely to the seminal work of Devinatz, Hopkins, and Smith \cite{DHSS, HS98}. We discuss some of these conjectures which are surrounding the thick subcategory theorem.

To start, let $f : X \to Y$ be a map between spectra. Then we can ask several questions, the first one is when is $f$ null-homotopic? Detecting null homotopy of maps is an extremely difficult problem. A long standing conjecture of Peter Freyd \cite{Fre66} called Generating Hypothesis says that if $X$ and $Y$ are finite spectra, then $f$ is null homotopic if $\pi_*(f)$ is zero. Some partial results are known due to Devinatz \cite{Dev90}, the conjecture remains open; see \cite{Fre66} for some very interesting consequences of this conjecture. The second question is when is $f$ nilpotent under composition. The nilpotence theorem which was conjectured by Ravenel gives an answer to this question when the spectra in question are finite. This theorem is very deep and its proof involves some hard homotopy theory. It generalises a well-known theorem of Nishida which tells that every positive degree self map of the sphere spectrum is nilpotent.

**Theorem 2.** \cite{DHSS} (Nilpotence theorem) There is a generalised homology theory known as $MU_*(-)$ (complex bordism) such that a map $f : X \to Y$ between finite spectra is nilpotent if and only if $MU_*(f)$ is nilpotent.

A much sharper view of the stable homotopy theory is obtained when one localises at the prime $p$ and studies the $p$-local stable category whose objects are spectra whose homotopy groups are $p$-local, i.e., $\pi_*(X) \cong \pi_*(X) \otimes \mathbb{Z}(p)$. It is a very standard practise in stable homotopy theory to localise at a prime $p$. When this is done, there are distinguished field objects known as Morava K-theories $K(n)$ (with the prime $p$ suppressed) which play a key role in the $p$-local stable category. We now begin discussing these objects which also play an important role in the thick subcategory theorem.

2. Morava K-theories and the thick subcategory theorem.

To set the stage, let $\mathcal{F}$ denote the category of compact objects in the $p$-local stable homotopy category $S$. There are many naturally arising properties of spectra called generic properties which are properties that are preserved under cofibrations, retraction and suspensions. Recall that a subcategory is thick precisely when it is closed under these operations. Thus one is naturally led to the study the lattice of thick subcategories of $\mathcal{F}$.

The lattice of thick subcategories of $\mathcal{F}$ is determined by the Morava K-theories. For each $n \geq 1$ there is a spectrum $K(n)$ called the $n$-th Morava K-theory whose coefficient ring $K(n)_*$ is isomorphic to $\mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$. We also set $K(0)$ to be the rational Eilenberg-Mac Lane spectrum and $K(\infty)$ the mod-$p$ Eilenberg-Mac Lane spectrum. These theories have the following pleasant properties.

1. For every spectrum $X$, $K(n) \wedge X$ has the homotopy type of a wedge of suspensions of $K(n)$.
2. Künneth isomorphism: $K(n)_*(X \wedge Y) \cong K(n)_* X \otimes_{K(n)} K(n)_* Y$. In particular $K(n)_*(X \wedge Y) = 0$ if and only if either $K(n)_* X = 0$ or $K(n)_* Y = 0$.
3. If $X \neq 0$ and finite, then for all $n >> 0$, $K(n)_* X \neq 0$.
4. For each $n$, $K(n+1)_* X = 0$ implies $K(n)_* X = 0$. 
(5) (Nilpotence theorem) Morava K-theories detect ring spectra: If \( R \) is a non-trivial ring spectrum, then there exists an \( n \) \((0 \leq n \leq \infty)\) such that \( K(n)_* R \neq 0 \).

The first three properties can be easily derived from the fact that every graded module over \( K(n)_* \) is a direct sum of suspensions of \( K(n) \). The third property is proved in [Rav84], and the last property can be derived from the \( MU \)-version of the nilpotence theorem stated above; see [HS98] for a proof of this implication. In view of the above properties, one prefers to work with \( K(n)_* \) as opposed to \( MU \) because it is easier to do computations with \( K(n) \).

Let \( C_0 = \mathcal{F} \), and for \( n \geq 1 \), let \( C_n := \{ X \in \mathcal{F} : K(n-1)_* X = 0 \} \), and finally let \( C_\infty \) denote the subcategory of contractible spectra. We can now state the celebrated thick subcategory theorem.

**Theorem 3.** [HS98] (Thick subcategory theorem) A subcategory \( \mathcal{C} \) of \( \mathcal{F} \) is thick if and only if \( \mathcal{C} = C_n \) for some \( n \). Further these subcategories form a nested decreasing filtration of \( \mathcal{F} \):

\[
C_\infty \subseteq \cdots \subseteq C_{n+1} \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_1 \subseteq C_0
\]

We say that a spectrum \( X \) is of type-\( n \) if it belongs to \( C_n - C_{n+1} \), and we write \( \text{type}(X) = n \). For example the sphere spectrum is of type 0 and the mod-p Moore spectrum is of type 1. The existence of type-\( n \) spectra was first proved by Mitchell [Mit85].

It is not hard to prove the above theorem using the nilpotence theorem. The proof we sketch below is following Rickard. The only other tool that we need is finite localisation.

**Theorem 4.** [Mil92] (Finite Localisation) Let \( \mathcal{C} \) be a thick subcategory of \( \mathcal{F} \), and let \( \mathcal{D} \) denote the localising subcategory generated by \( \mathcal{C} \). Then there is a localisation functor \( \mathcal{L}_n^\mathcal{C} : \mathcal{S} \rightarrow \mathcal{S} \) called "finite localisation away from \( \mathcal{C} \)" which has the following properties.

1. For \( X \) finite, \( \mathcal{L}_n^\mathcal{C} X = 0 \) if and only if \( X \) belongs to \( \mathcal{F} \).
2. For \( X \) arbitrary, \( \mathcal{L}_n^\mathcal{C} X = 0 \) if and only if \( X \) belongs to \( \mathcal{D} \).
3. \( \mathcal{L}_n^\mathcal{C} \) is a smashing localisation functor, i.e., \( \mathcal{L}_n^\mathcal{C} X \cong \mathcal{L}_n^\mathcal{C} S^0 \wedge X \), where \( S^0 \) is the \( p \)-local sphere spectrum.

The idea involved in the construction of such a localisation functor is well-known to homotopy theorists. For instance, it shows up in the proof of the Brown representability theorem. A very good treatment of this construction is also given by Rickard [Ric97] where he constructs idempotent modules in the stable module category using finite localisation.

We should mention at this point that a version of Ravenel’s telescope conjecture states that every smashing localisation functor (a localisation functor that satisfies property (3) above) on \( \mathcal{S} \) is isomorphic to \( \mathcal{L}_n^\mathcal{C} \) for some integer \( n \). This is the only conjecture of Ravenel that is still open; some experts believe that it is false, see [Rav92].

Now we give a proof (due to Rickard) of the thick subcategory theorem. Let \( \mathcal{C} \) be a non-zero thick subcategory of \( \mathcal{F} \). Then define

\[
n := \max \{ l : \mathcal{C} \subseteq \mathcal{C}_l \}.
\]

From property (3) of the Morava K-theories we infer that \( n \) is a well-defined non-negative integer. We claim that \( \mathcal{C} = \mathcal{C}_n \). Note that we only have to show that
In showing this inclusion we use the finite localisation functors $L_C^l$. So let $X$ in $C_n$. Now to show that $X$ is in $C$, it is enough to show that $L_C^l X = 0$. But since these functors are smashing we have to show $X \wedge L_C^l S^0 = 0$. Since every finite spectrum $(X)$ is Bousfield equivalent to a ring spectrum $(X \wedge D X)$, we can assume without loss of generality that $X$ is a ring spectrum. Then by property (5) it suffices to show that for all $0 \leq l \leq \infty$, $K(l)_* (X \wedge L_C^l S^0) = 0$. Further by property (2) we have to show that for each $l$, either $K(l)_* X = 0$ or $K(l)_* (L_C^l S^0) = 0$. Since $X$ is in $C_n$, the former holds for all $0 \leq l < n$ by property (4). So we have to show that the latter holds for $n \leq l \leq \infty$. Now by the definition of $n$, we have for each $n \leq l \leq \infty$, a spectrum $X_l$ in $C$ such that $K(l)_* (X_l) \neq 0$. Since $X_l$ is in $C$, we have $L_C^l X_l = X_l \wedge L_C^l S^0 = 0$. So clearly $K(l)_* L_C^l X_l = 0$, but since $K(l)_* (X_l) \neq 0$, we must have for all $n \leq l \leq \infty$, that $K(l)_* L_C^l S^0 = 0$ as desired.

Note that this proof highlights the key properties of Morava K-theories which are used in proving the thick subcategory theorem, and therefore it can be adapted easily to the other algebraic settings such as derived categories of rings and stable module categories of group algebras. The role played by the Morava K-theories in the former are the residue fields and in the latter are the kappa modules; see [HPS97] for a thick subcategory theorem in an axiomatic stable homotopy category.

We now illustrate how one can use the thick subcategory theorem. Suppose $P$ is some generic property of spectra and we want to identify the subcategory of finite spectra which satisfy $P$. If we can find a type-$k$ spectrum which satisfies $P$ and a type-$(k - 1)$ spectrum which does not satisfy $P$, that forces the subcategory in question to be $C_k$. For example, consider the generalised homology theory $BP\langle n \rangle$ whose coefficient ring is given by $\mathbb{Z}_p[v_1, v_2, \cdots v_n]$ with $|v_1| = 2(p^i - 1)$. Using the above strategy one can easily show that the full subcategory of finite spectra which have bounded $BP\langle n \rangle$ homology (spectra $X$ such that $BP\langle n \rangle, X = 0$ for $i >> 0$) is precisely $C_{n+1}$.

### 3. Bousfield classes of finite spectra

There are several interesting applications of the thick subcategory theorem. We focus on its applications to the Bousfield lattice – an important lattice which encapsulates the gross structure of stable homotopy theory. This was introduced by Ravenel in [Bou79a, Bou79b]. Given a spectrum $E$, define its Bousfield Class $\langle E \rangle$ to be the collection of all spectra which are invisible to the $E$-homology theory, i.e., spectra $X$ such that $E_* (X) = 0$ or equivalently $E \wedge X = 0$. Then we say that spectra $E$ and $F$ are Bousfield equivalent if $\langle E \rangle = \langle F \rangle$. It is a result by Ohkawa that there is only a set of Bousfield classes. With the partial order given by reverse inclusion, the set of Bousfield classes form a lattice which is called the Bousfield lattice and will be denoted by $B$. One can perform various operations on $B$. The two important ones being the wedge ($\vee$) : $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and the smash ($\wedge$) : $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. In this lattice, the Bousfield class of the sphere spectrum is the largest element and that of the trivial spectrum is the smallest. This lattice plays an important role in the study of modern stable homotopy theory. While much of the current knowledge about the $B$ is only conjectural, the thick subcategory theorem completes determines the Bousfield classes of finite spectra. We describe them in the next theorem which was conjectured by Ravenel.
**Theorem 5.** [HS98] (Class-invariance theorem) Let $X$ and $Y$ be finite $p$-local spectra, then $\langle X \rangle \leq \langle Y \rangle$ if and only if $\text{type}(X) \geq \text{type}(Y)$.

Although this theorem follows as an immediate corollary to the thick subcategory theorem, it is a very non-trivial statement about finite spectra. It says that the Bousfield class of a finite spectrum is completely determined by its type.

We now discuss the Boolean algebra conjecture of Ravenel which identifies the Boolean subalgebra generated by the Bousfield classes of finite $p$-local spectra. But first we have to introduce some important non-nilpotent maps of finite spectra called $v_n$-self maps. A self map $f : \Sigma^j X \to X$ is a $v_n$-self map ($n \geq 1$) if $K(n)_*(f)$ is an isomorphism and $K(m)_*(f)$ is zero for $m \neq n$. For example, the degree $p$ map on the sphere spectrum is a $v_0$-self map, and the Adams map [Ada66] on the Moore spectrum: $\Sigma^j M(p) \to M(p)$ which induces isomorphism in complex $K$-theory is a $v_1$-self map. These $v_n$-self maps are important because give rise to periodic families in the stable homotopy groups of spheres. For example, one can iterate Adams maps and get a periodic family in $\pi_*(- S^0)$ called the $\alpha$-family. Showing the existence of such maps is highly non-trivial. A deep result of Hopkins and Smith called the periodicity theorem produces a wealth of such maps. More precisely:

**Theorem 6.** [HS98] (Periodicity Theorem)

1. Every type-$n$ spectrum admits an asymptotically unique $v_n$-self map $\phi_X : \Sigma^j X \to X$
2. If $h : X \to Y$ is a map between type-$n$ spectra, then there exits integers $i$ and $j$ such that the follow diagram commutes:

$$
\begin{array}{ccc}
\Sigma^j X & \xrightarrow{\Sigma^j h} & \Sigma^j Y \\
\phi_X & \downarrow & \phi_Y \\
X & \xrightarrow{h} & Y
\end{array}
$$

Using this periodicity theorem it is not hard to show that the full subcategory of finite $p$-local spectra admitting $v_n$-self map is precisely $C_n$. So this theorem gives another characterisation of the thick subcategories of $F$.

For every positive integer $n$, let $F(n)$ denote some spectrum of type-$n$. Note that the Bousfield class of $F(n)$ is well-defined by the class-invariance theorem. Now the periodicity theorem says that $F(n)$ admits an essentially unique $v_n$-self map. So let $T(n)$ denote the mapping telescope of this $v_n$-self map. It follows that the Bousfield classes of $T(n)$ is also well-defined.

A Bousfield class $\langle E \rangle$ is said to be complemented if there exists another class $\langle F \rangle$ such that $\langle E \rangle \wedge \langle F \rangle = \langle 0 \rangle$ and $\langle E \rangle \lor \langle F \rangle = \langle S^0 \rangle$. The collection of all complemented Bousfield classes forms a Boolean algebra with respect to the smash and wedge operations and will be denoted by $\text{BA}$. Bousfield [Bon79b] showed that every possibly infinite wedge of finite spectra belongs to $\text{BA}$. A pleasant consequence of the thick subcategory theorem is the classification of the Boolean subalgebra generated by the finite $p$-local spectra and their complements in the $p$-local sphere spectrum.

**Theorem 7.** [Rav84] (Boolean Algebra Theorem) Let $\text{FBA}$ denote the Boolean subalgebra generated by the Bousfield classes of the finite $p$-local spectra and their complements in $\langle S^0 \rangle$. Then $\text{FBA}$ is the free (under complements, finite unions
Boolean algebra generated by the Bousfield classes of the telescopes $\langle T(n) \rangle$ for $n \geq 0$.

So by this theorem one can identify $\mathbf{FBA}$ with the Boolean algebra of finite and cofinite subsets of non-negative integers: the Bousfield class $\langle T(n) \rangle$ corresponds to the subset $\{n\}$, and $\langle F(n) \rangle$ corresponds to the subset $\{n, n+1, n+2, \cdots \}$. By the way, the Boolean algebra conjecture of Ravenel uses $K(\langle X \rangle)$ instead of $T(n)$ in the above theorem; according to telescope conjecture (which is still open) these two spectra are Bousfield equivalent.

There are several other interesting sublattices of $\mathbf{B}$ which have been studied. For example there is a distributive lattice $\mathbf{DL}$ which consists of the Bousfield classes $\langle X \rangle$ such that $\langle X \rangle \land \langle X \rangle = \langle X \rangle$ which has some nice properties. A good discussion on the structure of the Bousfield lattice can be found in [HP99]. These authors use lattice theoretic methods to explore the structure of the Bousfield lattice. They also pose a number of interesting conjectures and study their implications.

We end by mentioning briefly one other application of the thick subcategory theorem. Thomason has given a brilliant K-theory recipe [Tho97] which refines the thick subcategory theorem and gives a classification of the triangulated subcategories of finite spectra. This recipe amounts to computing the Grothendieck groups of the thick subcategories of the finite $p$-local spectra. We refer the reader to [Che06] where we use this recipe to study the lattice of triangulated subcategories of finite spectra.

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