A GRADIENT FLOW GENERATED BY A NONLOCAL MODEL OF A NEURAL FIELD IN AN UNBOUNDED DOMAIN

SEVERINO H. DA SILVA\textsuperscript{2*} ANTONIO L. PEREIRA\textsuperscript{1†}
E-mail: horacio@mat.ufcg.edu.br and alpereir@ime.usp.br

March 23, 2018

Abstract

In this paper we consider the non local evolution equation

$$\frac{\partial u(x,t)}{\partial t} + u(x,t) = \int_{\mathbb{R}^N} J(x-y) f(u(y,t)) \rho(y) dy + h(x).$$

We show that this equation defines a continuous flow in both the space $C_b(\mathbb{R}^N)$ of bounded continuous functions and the space $C_\rho(\mathbb{R}^N)$ of continuous functions $u$ such that $u \cdot \rho$ is bounded, where $\rho$ is a convenient "weight function". We show the existence of an absorbing ball for the flow in $C_b(\mathbb{R}^N)$ and the existence of a global compact attractor for the flow in $C_\rho(\mathbb{R}^N)$, under additional conditions on the nonlinearity.

We then exhibit a continuous Lyapunov function which is well defined in the whole phase space and continuous in the $C_\rho(\mathbb{R}^N)$ topology, allowing the characterization of the attractor as the unstable set of the equilibrium point set. We also illustrate our result with a concrete example.

2010 Mathematics Subject Classification: 45J05, 37B25.

Keywords: Nonlocal problem, neural field, weighted space, global attractor, Lyapunov functional

1 Introduction

We consider here the non local evolution equation

$$\frac{\partial u(x,t)}{\partial t} + u(x,t) = \int_{\mathbb{R}^N} J(x-y) f(u(y,t)) \rho(y) dy + h(x),$$ (1.1)
where \( f \) is a continuous real function, \( J : \mathbb{R}^N \rightarrow \mathbb{R} \) is a non negative integrable function, \( \rho : \mathbb{R}^N \rightarrow \mathbb{R} \) is a symmetric non negative bounded "weight" function with \( \int_{\mathbb{R}^N} \rho(x) d(x) < \infty \) and \( h \) is a bounded continuous function. Additional hypotheses will be added when needed in the sequel.

We can rewrite equation (1.1) as
\[
\frac{\partial u(x,t)}{\partial t} + u(x,t) = J*_{\rho}(f \circ u)(x,t) + h(x), \quad h \geq 0,
\]
where the \( * \) above denotes convolution product with respect to the measure \( d\mu(y) = \rho(y)dy \), that is
\[
J*_{\rho}(v)(x) := \int_{\mathbb{R}^N} J(x-y)v(y)) \rho(y) d(y).
\]

Equation (1.1) is a variation of the equation derived by Wilson and Cowan, (23), to model neuronal activity. There are also other variations of this model in the literature (see, for example, [1], [3], [5], [6], [7], [8], [9], [10], [12], [13], [16] and [18], [19], [21]). However, their asymptotic behavior have not been fully analyzed in the case of unbounded domains. In particular, the "Lyapunov functional" appearing in the literature is not well defined in the whole phase space, (see [1], [2] and [21]).

There is already a vast literature on the analysis of similar neural field models, (see [1], [2], [3], [5], [6], [7], [8], [9], [11], [12], [13], [14], [16] and [17], [18], [19], [21]). Our proof uses adaptations of the technique used in [6], replacing the compact embedding \( H^1([-l,l]) \hookrightarrow L^2([-l,l]) \) by the compact embedding \( C^1(\mathbb{R}^N) \hookrightarrow C_\rho(\mathbb{R}^N) \), (see also [5], [10], and [20] for related work). In Section 4 motivated by the energy functional from [2], [8], [10], [13], [18], and [24], we exhibit a continuous Lyapunov functional for the flow generated by (1.1), well defined in the whole phase space \( C_\rho(\mathbb{R}^N) \), and use it to prove that the flow is gradient in the sense of [14]. Finally, in Section 5 we present a concrete example to illustrate our results.

This paper is organized as follows. In Section 2 we consider the flow generated by (1.1) in the phase space of continuous bounded functions. In Subsection 2.1 we prove that the Cauchy problem for (1.1) is well posed in this phase space with globally defined solutions, and, in Subsection 2.2 we prove the existence of an absorbing set for the flow generated by (1.1). In Section 3 we consider the problem (1.1) in the phase space \( C_\rho(\mathbb{R}^N) \equiv \{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ continuous with } \| u \|_\rho := \sup_{x \in \mathbb{R}^N} \{|u(x)|,|x| < \infty \} \}, \) where \( \rho \) is a convenient "weight function". In this section, to obtain well-posedness, we impose more stringent conditions on the nonlinearity than in the previous section, (see Subsection 3.1). On the other hand, we obtain stronger results, including existence of a compact global attractor for the corresponding flow. Our proof uses adaptations of the technique used in [6], replacing the compact embedding \( H^1([-l,l]) \hookrightarrow L^2([-l,l]) \) by the compact embedding \( C^1(\mathbb{R}^N) \hookrightarrow C_\rho(\mathbb{R}^N) \), (see also [5], [10], and [20] for related work). In Section 4 motivated by the energy functional from [2], [8], [10], [13], [18], and [24], we exhibit a continuous Lyapunov functional for the flow generated by (1.1), well defined in the whole phase space \( C_\rho(\mathbb{R}^N) \), and use it to prove that the flow is gradient in the sense of [14]. Finally, in Section 5 we present a concrete example to illustrate our results.
2 The flow in the space $C_b(\mathbb{R}^N)$

In this section, we consider the problem (1.1) in the phase space

$$C_b(\mathbb{R}^N) \equiv \{ u : \mathbb{R}^N \to \mathbb{R} \text{ continuous with } \|u\|_\infty := \sup_{x \in \mathbb{R}^N} \{|u(x)|\} < \infty \}.$$ 

After establishing well-posedness, we prove that a ball of appropriate radius is an absorbing set for the corresponding flow.

2.1 Well-posedness

The following estimate will be useful in the sequel. The proof is straightforward and left to the reader.

**Lemma 2.1.** If $u \in C_b(\mathbb{R}^N)$ then

$$\|J^*(\rho u)\|_\infty \leq \|J\|_{L^1(\mathbb{R}^N)} \|\rho\|_\infty \|u\|_\infty,$$

where $J^*(\rho u)$ is given by (1.2).

**Definition 2.2.** If $E$ and $F$ are normed spaces, we say that a function $F : E \to F$ is locally Lipschitz continuous (or simply locally Lipschitz) if for any $x_0 \in E$, there exists a constant $C$ and a ball $B = \{ x \in E : \|x - x_0\| < b \}$ such that, if $x$ and $y$ belong to $B$ then $\|F(x) - F(y)\| \leq C\|x - y\|$. We say that $F$ is Lipschitz continuous on bounded sets if the ball $B$ in the previous definition can be chosen as any bounded ball in $E$.

**Remark 2.3.** The two definitions in (2.2) are equivalent if the normed space $E$ is locally compact.

**Proposition 2.4.** If $f$ is continuous, then the map $F : C_b(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$, given by

$$F(u) = -u + J^*(\rho f \circ u) + h,$$

is well defined. If $f$ is locally Lipschitz, then $F$ Lipschitz in bounded sets.

**Proof.** The first assertion is immediate. Now, from triangle inequality and Lemma 2.1 it follows that

$$\|F(u) - F(v)\|_\infty \leq \|v - u\|_\infty + \|J^*(\rho f \circ u) - J^*(\rho f \circ v)\|_\infty$$

$$\leq \|v - u\|_\infty + \|J\|_{L^1(\mathbb{R}^N)} \|\rho\|_\infty \|f \circ u - (f \circ v)\|_\infty.$$ 

If $\|u\|_\infty, \|v\|_\infty \leq R$ then $|(f \circ u)(x) - (f \circ v)(x)| \leq k_R |u(x) - v(x)|$, where $k_R$ is a Lipschitz constant for $f$ in the interval $[-R, R]$. It follows that

$$\|F(u) - F(v)\|_\infty \leq (1 + k_R \|J\|_{L^1(\mathbb{R}^N)} \|\rho\|_\infty) \|u - v\|_\infty,$$

which concludes the proof.

**Theorem 2.5.** If $f$ is locally Lipschitz, the Cauchy problem for (1.1) is well posed in $C_b(\mathbb{R}^N)$ with globally defined solutions.

**Proof.** It follows from Proposition 2.4 and well-known results (see [1] or [15], Theorems 3.3.3 and 3.3.4).
2.2 Existence of an absorbing set

In this section, we denote by $T(t)$ the flow generated by (1.1) in $C_b(\mathbb{R}^N)$. Under some additional hypotheses on the nonlinearity, we prove here the existence of an absorbing bounded ball $B \subset C_b(\mathbb{R}^N)$ for $T(t)$.

We recall that a set $B \subset C_b(\mathbb{R}^N)$ is an absorbing set for the flow $T(t)$ if, for any bounded set $C \subset C_b(\mathbb{R}^N)$, there is a $t_1 = t_1(C) > 0$ such that $T(t)C \subset B$ for any $t \geq t_1$, (see [22]).

**Lemma 2.6.** Suppose that $f$ is locally Lipschitz and satisfies the dissipative condition

$$|f(x)| \leq \eta|x| + K, \text{ for any } x \in \mathbb{R}. \quad (2.3)$$

with $\eta\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty < 1$. Then, if $\eta\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty < \delta < 1$, the ball in $C_b(\mathbb{R}^N)$, centered at the origin with radius $R = \frac{\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty K + \|h\|_\infty}{\delta - \eta\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty}$, is an absorbing set for the flow $T(t)$.

**Proof.** Let $u(x,t)$ be the solution of (1.1) with initial condition $u(\cdot, 0) = u_0$. Then, by the variation of constants formula,

$$u(x,t) = e^{-t}u_0(x) + \int_0^t e^{s-t}[J\rho(f \circ u)(x,s) + h]ds.$$

From (2.3), there exists a constant $K$ such that $|f(x)| \leq \eta|x| + K$, for any $x \in \mathbb{R}$.

Hence, using Lemma 2.4 and (2.3), we obtain

$$|u(x,t)| \leq e^{-t}|u_0(x)| + \int_0^t e^{s-t}[|J\rho(f \circ u)(x,s)| + |h(x)|]ds$$

$$\leq e^{-t}|u_0|_\infty + \int_0^t e^{s-t}[|J\rho(f \circ u)(\cdot,s)|_\infty + |h|_\infty]ds$$

$$\leq e^{-t}|u_0|_\infty + \int_0^t e^{s-t}[\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty \|f \circ u(\cdot,s)\|_\infty + |h|_\infty]ds$$

$$\leq e^{-t}|u_0|_\infty + \int_0^t e^{s-t}[\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty \eta \|u(\cdot,s)\|_\infty + \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty K + |h|_\infty]ds.$$

Suppose $\|u(\cdot,s)\|_\infty \geq \delta - \frac{1}{\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty \eta}$, for $0 \leq t \leq T$. Then,

$$e^t|u(x,t)| \leq |u_0|_\infty + \delta \int_0^t e^s\|u(\cdot,s)\|_\infty ds \quad \text{for any } x \in \mathbb{R}^N.$$

Taking the supremum on the left side, it follows that

$$e^t\|u(x,\cdot)\|_\infty ds \leq |u_0|_\infty + \delta \int_0^t e^s\|u(\cdot,s)\|_\infty.$$

From Gronwall’s inequality, it then follows that $e^t\|u(\cdot,t)\|_\infty \leq |u_0|_\infty e^{\delta t}$ and, therefore

$$\|u(\cdot,t)\|_\infty \leq |u_0|_\infty e^{\delta t}, \quad \text{for } t \in [0,T]. \quad (2.4)$$

It follows that there exists $T_0 \leq \frac{1}{(1-\delta)} \ln \left(\frac{\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty K + |h|_\infty}{|u_0|_\infty (\delta - \frac{1}{\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty \eta})}\right)$ such that
3 The flow in the space \( C_\rho(\mathbb{R}^N) \)

In this section, we consider the problem (1.1) in the phase space

\[
C_\rho(\mathbb{R}^N) \equiv \{ u : \mathbb{R}^N \to \mathbb{R} \text{ continuous with } \| u \|_\rho := \sup_{x \in \mathbb{R}^N} \{|u(x)|\rho(x)\} < \infty \}.
\]

We will need to impose more stringent conditions on the nonlinearity than in the previous section, to obtain well-posedness. On the other hand, we will obtain stronger results, including existence of a compact global attractor for the corresponding flow.

3.1 Well-posedness

The following result is the analogous of Lemma 2.1. The proof is again straightforward and left to the reader.

**Lemma 3.1.**

If \( u \in C_\rho(\mathbb{R}^N) \) then \( \| J \ast \rho u \|_\rho \leq \| J \|_{L^1(\mathbb{R}^N)} \| \rho \|_\infty \| u \|_\rho \).

**Proposition 3.2.** If \( f \) is globally Lipschitzian, then the map \( F : C_\rho(\mathbb{R}^N) \to C_\rho(\mathbb{R}^N) \), given by

\[
F(u) = -u + J \ast \rho (f \circ u) + h,
\]

is well defined and globally Lipschitzian.

**Proof.** Suppose \( |f(x) - f(y)| \leq k|x - y| \), for any \( x, y \in \mathbb{R} \). Then, in particular, \( |f(x)| \leq k|x| + M \), where \( M = f(0) \) for any \( x \in \mathbb{R} \). It follows that \( \| f \circ u \|_\rho \leq k \| u \|_\rho + M \| \rho \|_\infty \). From Lemma 3.1, we then obtain

\[
\| F(u) \|_\rho \leq \| u \|_\rho + \| J \ast \rho (f \circ u) \|_\rho \\
\leq \| u \|_\rho + \| J \|_{L^1(\mathbb{R}^N)} \| \rho \|_\infty \| f \circ u \|_\rho \\
\leq \| u \|_\rho + \| J \|_{L^1(\mathbb{R}^N)} \| \rho \|_\infty (k \| u \|_\rho + M \| \rho \|_\infty),
\]

so \( F \) is well defined. Furthermore,

\[
\| F(u) - F(v) \|_\rho \leq \| u - v \|_\rho + \| J \ast \rho (f \circ u) - J \ast \rho (f \circ v) \|_\rho \\
\leq \| u - v \|_\rho + \| J \|_{L^1(\mathbb{R}^N)} \| \rho \|_\infty \| (f \circ u) - (f \circ v) \|_\rho \\
\leq \| u - v \|_\rho + \| J \|_{L^1(\mathbb{R}^N)} \| \rho \|_\infty k \| u - v \|_\rho \\
= (1 + k \| J \|_{L^1(\mathbb{R}^N)} \| \rho \|_\infty) \| u - v \|_\rho.
\]

Therefore \( F \) is globally Lipschitz in \( C_\rho(\mathbb{R}^N) \).
Theorem 3.3. If \( f \) is globally Lipschitzian, the Cauchy problem for (1.1) is well posed in \( C_\rho(\mathbb{R}^N) \) with globally defined solutions.

Proof. It follows from Proposition 2.4 and well-known results (see [4] or [15], Theorems 3.3.3 and 3.3.4).

3.2 Existence of an absorbing set

In this section, we denote by \( T(t) \) the flow generated by (1.1) in \( C_\rho(\mathbb{R}^N) \). Under some additional hypotheses on the nonlinearity, we prove the existence of a bounded ball \( B \subset C_\rho(\mathbb{R}^N) \) which is an absorbing set for \( T(t) \).

Lemma 3.4. Suppose that \( f \) is globally Lipschitz and satisfies the dissipative condition

\[
|f(x)| \leq \eta |x| + K, \quad \text{for any } x \in \mathbb{R}. \tag{3.5}
\]

with \( \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_{\infty}\eta < 1 \). Then, if \( \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_{\infty}\eta < \delta < 1 \), the ball in \( C_\rho(\mathbb{R}^N) \), centered at the origin with radius \( R = \frac{\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_{\infty}\eta K + \|h\|_\rho}{\delta - \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_{\infty}\eta} \), is an absorbing set for the flow \( T(t) \).

Proof. Let \( u(x,t) \) be the solution of (1.1) with initial condition \( u(\cdot,0) = u_0 \). Then, by the variation of constants formula,

\[
u(x,t) = e^{-t}u_0(x) + \int_0^t e^{s-t}[J\ast f(u)(x,s) + h]ds.
\]

From (3.5) and Lemma 3.1 we obtain

\[
|u(x,t)\rho(x)| \leq e^{-t}|u_0(x)\rho(x)| + \int_0^t e^{s-t}|[J\ast f(u)(x,s)\rho(x)| + |h(x)\rho(x)|]ds
\]

\[
\leq e^{-t}\|u_0\|_\rho + \int_0^t e^{s-t}\|[J\ast f(u)/(x,s)]\|_\rho + \|h\|_\rho]ds
\]

\[
\leq e^{-t}\|u_0\|_\rho + \int_0^t e^{s-t}\|[J\|_{L^1(\mathbb{R}^N)}|\rho|_{\infty}|(f(u)/(x,s))|_\rho + \|h\|_\rho]ds
\]

\[
\leq e^{-t}\|u_0\|_\rho + \int_0^t e^{s-t}\|[J\|_{L^1(\mathbb{R}^N)}|\rho|_{\infty}|u(\cdot,s)|_\rho + \|J\|_{L^1(\mathbb{R}^N)}|\rho|_{\infty}K + \|h\|_\rho]ds.
\]

Suppose \( \|u(\cdot,s)\|_{\infty} \geq \frac{\|J\|_{L^1(\mathbb{R}^N)}|\rho|_{\infty}K + \|h\|_\rho}{\delta - \|J\|_{L^1(\mathbb{R}^N)}|\rho|_{\infty}\eta} \), for \( 0 \leq t \leq T \). Then for \( t \in [0,T] \), we obtain

\[
e^t|u(x,t)\rho(x)| \leq \|u_0\|_\rho + \delta \int_0^t e^s|u(\cdot,s)|_\rho ds \quad \text{for any } x \in \mathbb{R}^N.
\]

Taking the supremum on the left side, it follows that

\[
e^t\|u(x,\cdot)\|_\rho ds \leq \|u_0\|_\rho + \delta \int_0^t e^s|u(\cdot,s)|_\rho.
\]

From Gronwall’s inequality, it then follows that \( e^t|u(\cdot,t)|_\rho \leq \|u_0\|_\rho e^{\delta t} \) and hence

\[
u(\cdot,t)|_{\infty} \leq \|u_0\|_{\infty}e^{(\delta - 1)t}, \quad \text{for } t \in [0,T]. \tag{3.6}
\]
Therefore, there exists $T_0 \leq \frac{1}{(1-\delta)} \ln \left( \frac{\|J\|_{L^1(\mathbb{R}^N)} \rho \| \infty K + \|h\|_{\infty} u_0 \|_{\infty} (\delta - \|J\|_{L^1(\mathbb{R}^N)} \rho \|_{\infty} \eta) }{\delta - \|J\|_{L^1(\mathbb{R}^N)} \rho \|_{\infty} \eta} \right)$ such that

$$
\|u(\cdot, T_0)\|_{\rho} \leq \frac{\|J\|_{L^1(\mathbb{R}^N)} \rho \| \infty K + \|h\|_{\rho}}{\delta - \|J\|_{L^1(\mathbb{R}^N)} \rho \|_{\infty} \eta}
$$

Also, we must have $\|u(\cdot, T_0)\|_{\rho} \leq \frac{\|J\|_{L^1(\mathbb{R}^N)} \rho \| \infty K + \|h\|_{\rho}}{\delta - \|J\|_{L^1(\mathbb{R}^N)} \rho \|_{\infty} \eta}$, for any $t \geq T_0$, since $\|u(\cdot, t)\|_{\rho}$ decreases (exponentially) if the opposite inequality holds by (3.6).

**Remark 3.5.** From (3.6), it follows that the ball $B_{\rho}(0, R')$ of radius $R'$ in $C_{\rho}(\mathbb{R}^N)$ is positively invariant under the flow $T(t)$ if $R' \geq R$.

### 3.3 Existence of a global attractor

We denote below by $C^1_b(\mathbb{R}^N)$, the subspace of functions in $C_b(\mathbb{R}^N)$ with bounded derivatives.

**Lemma 3.6.** The inclusion map $i : C^1_b(\mathbb{R}^N) \rightarrow C_{\rho}(\mathbb{R}^N)$ is compact.

**Proof.** Let $C$ be a bounded set in $C^1_b(\mathbb{R}^N)$. For any $l > 0$, let $\varphi : \mathbb{R}^N \rightarrow [0, 1]$ be a smooth function satisfying

$$
\varphi(x) = \begin{cases} 
0, & \text{if } \|x\| \geq l, \\
1, & \text{if } \|x\| \leq \frac{l}{2}.
\end{cases}
$$

Let $C^0(B_l)$ denote the space of continuous functions defined in the ball of $\mathbb{R}^N$ with radius $l$ and center at the origin, which vanish at the boundary. Consider the subset $C_l$ of functions in $C^0(B_l)$ defined by

$$
C_l := \{\varphi u|_{B_l} \text{ with } u \in C\}.
$$

Then $C_l$ is a bounded subset of $C^1_b(B_l)$ and, therefore, a precompact subset of $C^0(B_l)$, by the Arzel-Ascoli theorem. Let now $E_1$ be the subset of $C^1_{\rho}(\mathbb{R}^N)$ given by

$$
G_l := \{E(u) \text{ with } u \in C_l\}.
$$

where $E(u)$ is the extension by zero outside $B_l$. Since $E$ is continuous as an operator from $C^0(B_l)$ into $C_{\rho}(\mathbb{R}^N)$, it follows that $\overline{G_l}$ is a compact subset of $C_{\rho}(\mathbb{R}^N)$.

Let now

$$
G^c_l := \{(1 - \varphi) u \text{ with } u \in C\}.
$$

Let $R$ be such that $\|u\|_{\infty} \leq R$, for any $u \in C$. Then, for any $\epsilon > 0$, we may find $l$ such that $0 < \rho(x) < \frac{\epsilon}{R}$ if $\|x\| \geq l/2$. Then, it follows that $\|u\|_{\rho} \leq \epsilon$, for any $u \in G^c_l$, that is, $G^c_l$ is contained in the ball of radius $\epsilon$ around the origin.

Since $G_l$ is precompact, it can be covered by a finite number of balls of radius $\epsilon$. Since any function $u$ in $C$ can be written as $u = u_1 + u_2$, with $u_1 = \varphi u \in G_l$ and $u_2 = (1 - \varphi) u \in G^c_l$, it follows that $C$ can be covered by a finite number of balls with radius $2\epsilon$, for any $\epsilon > 0$. Thus $C$ is precompact as a subset of $C_{\rho}(\mathbb{R}^N)$.

**Lemma 3.7.** In addition the hypotheses of Lemma 3.3 suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is bounded and $h$ has bounded derivative. Let $C$ be a bounded set in $C_{\rho}(\mathbb{R}^N)$ Then for any $\eta > 0$, there exists $t_\eta$ such that $T(t_\eta)C$, has a finite covering by balls with radius smaller than $\eta$. 


Proof. Let \( u(x, t) \) be the solution of (1.1) with initial condition \( u_0 \in C \). We may suppose that \( C \) is contained in the ball \( B_R \) of radius \( R \), centered at the origin. By the variation of constants formula

\[
T(t)u_0(x) = e^{-t}u_0(x) + \int_0^t e^{s-t}[J_\rho(f \circ u)(x, s) + h(x)]ds.
\]

Write

\[
(T_1(t)u_0)(x) = e^{-t}u_0(x)
\]

and

\[
(T_2(t)u_0)(x) = \int_0^t e^{-(t-s)}[J_\rho(f \circ u)(x, s) + h(x)]ds.
\]

Let \( \eta > 0 \) given. Then there exists \( t(\eta) > 0 \), uniform for \( u_0 \in C \), such that if \( t \geq t(\eta) \) then \( \|T_1(t)u_0\|_\rho \leq \frac{\eta}{2} \). In fact,

\[
|(T_1(t)u_0)(x)|_{\rho(x)} = e^{-t}|u_0(x)|_{\rho(x)}.
\]

Thus

\[
\|T_1(t)u_0\|_\rho = e^{-t}\|u_0\|_\rho.
\]

Hence, for \( t > t_\eta = \ln\left(\frac{2R}{\eta}\right) \), we have \( \|T_1(t)u_0\|_\rho \leq \frac{\eta}{2} \), for any \( u_0 \in C \), that is, \( T_1(t)C \) is contained in the ball of radius \( \frac{\eta}{2} \) around the origin.

We now show that \( T_2(t)C_\rho(\mathbb{R}^N) \) lies in a bounded ball of \( C^1_\rho(\mathbb{R}^N) \).

In fact, using Lemma 2.1 we have, for any \( u_0 \in C_\rho(\mathbb{R}^N) \),

\[
\|T_2(t)u_0\|_\infty \leq \int_0^t e^{s-t}[\|J_\rho(f \circ u)(\cdot, s)\|_\infty + \|h\|_\infty]ds
\]

\[
\leq \int_0^t e^{s-t}[\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\|f \circ u\|_{L^1(\mathbb{R}^N)} + \|h\|_\infty]ds
\]

\[
\leq (M\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty + \|h\|_\infty) \int_0^t e^{s-t}ds
\]

\[
\leq M\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty + \|h\|_\infty,
\]

where \( M = \|f\|_\infty < \infty \), and

\[
\left\|\frac{\partial}{\partial x}T_2(t)u_0\right\|_\infty \leq \int_0^t e^{s-t}[\|J_\rho'(f \circ u)(\cdot, s)\|_\infty + \|h'\|_\infty]ds
\]

\[
\leq \int_0^t e^{s-t}[\|J'\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\|f \circ u\|_{L^1(\mathbb{R}^N)} + \|h'\|_\infty]ds
\]

\[
\leq (M\|J'\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty + \|h'\|_\infty) \int_0^t e^{s-t}ds
\]

\[
\leq M\|J'\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty + \|h'\|_\infty.
\]

Then, for \( t \geq 0 \) and any \( u_0 \in C_\rho(\mathbb{R}^N) \), \( \frac{\partial}{\partial x}T_2(t)u_0 \) is bounded by a constant independent of \( t \) and \( u \).

Therefore, by Lemma 3.6 it follows that \( \{T_2(t)\}C_\rho(\mathbb{R}^N) \), is compact as a subset of \( C_\rho(\mathbb{R}^N) \) and, therefore it can be covered by a finite number of balls with radius \( \frac{\eta}{2} \).
Therefore, since
\[ T(t)C = T_1(t)C + T_2(t)C, \]
we obtain that \( T(t)C \), can be covered by a finite number of balls of radius \( \eta \), as claimed. \( \square \)

In what follows we denote by \( \omega(B_\rho(0, R)) \) the \( \omega \)-limit set of the ball \( B_\rho(0, R) \).

Then as consequence from Lemma 3.7 we have the following result:

**Theorem 3.8.** Assume the same hypotheses of Lemma 3.7. Then \( \mathcal{A} = \omega(B_\rho(0, R)) \), is a global attractor for the flow \( T(t) \) generated by (1.1) in \( B_\rho(0, R) \) which is contained in the ball of radius \( B_\rho(0, R) \).

**Proof.** From Lemma 3.7 it follows that, for any \( \eta > 0 \), there exists \( t_\eta > 0 \) such that \( T(t_\eta)B_\rho(0, R) \) can be covered by a finite number of ball of radius \( \eta \). Since \( B_\rho(0, R) \) is positively invariant, (see Remark 3.3) we have, for any \( t \geq t_\eta \), \( T(t)B_\rho(0, R) = T(t_\eta)T(t - t_\eta)B_\rho(0, R) \subset T(t_\eta)B_\rho(0, R) \) and thus, \( \bigcup_{t \geq t_\eta} T(t)B_\rho(0, R) \subset T(t_\eta)B_\rho(0, R) \), can also be covered by a finite number of ball with radius \( \eta \).

Therefore
\[ \mathcal{A} := \omega(B_\rho(0, R)) = \bigcap_{t_0 \geq 0} \bigcup_{t \geq t_0} T(t)B_\rho(0, R) = \bigcap_{t_0 \geq 0} \overline{T(t)B_\rho(0, R)}, \]
can be covered by a finite number of balls of radius arbitrarily small radius and is closed, so it is a compact set. From the positive invariance of \( B_\rho(0, R) \) (Lemma 2.6), it is clear that \( \mathcal{A} \subset B_\rho(0, R) \).

It remains to prove that \( \mathcal{A} \) attracts bounded sets of \( C_\rho(\mathbb{R}^N) \). It is enough to prove that it attracts the ball \( B_\rho(0, R) \). Suppose, for contradiction, that there exist \( \epsilon > 0 \) and sequences \( t_n \to \infty \), \( x_n \in B_\rho(0, R) \), with \( d(T(t_n)(x_n), \mathcal{A}) > \epsilon \).

Now, the set \( \{ T(t_n)(x_n) \, : \, n \geq n_0 \} \) is contained in \( T(t_{n_0})B_\rho(0, R) \). Thus for, any \( \eta > 0 \), it can be covered by balls with radius \( \eta \) if \( n_0 \) is big enough. Since the remainder of the sequence is a finite set, the same happens with the whole sequence. It follows that the sequence \( \{ T(t_n)(x_n) \, : \, n \in \mathbb{N} \} \) is a precompact set and so, passing to a subsequence, it converges to a point \( x_0 \in B_\rho(0, R) \). But then \( x_0 \) must belong to \( \mathcal{A} = \omega(B_\rho(0, R)) \) and we reach a contradiction. This concludes the proof. \( \square \)

## 4 Existence of a Lyapunov functional

Energy-like Lyapunov functional for models of neural fields are well known in the literature, (see for example, [2], [8], [9], [10], [13], [18] and [24]. However, when dealing with unbounded domains, these functionals are frequently not well defined in the whole phase space, since they can assume the value \( \infty \), at some points (see, for example, [10], [18]).

In this section, under appropriate assumptions on \( f \), we exhibit a continuous Lyapunov functional for the flow of (1.1), which is well defined in the whole phase space \( C_\rho(\mathbb{R}^N) \), and used it to prove that this flow has the gradient property, in the sense of [14].

Suppose that \( f \) is strictly increasing. Motivated by the energy functionals appearing in [2], [13], [18], and [24], we define the functional \( F : C_\rho(\mathbb{R}^N) \to \mathbb{R} \) by
\[
F(u) = \int_{\mathbb{R}^N} \left[ -\frac{1}{2} f(u(x)) \int_{\mathbb{R}^N} J(x - y)f(u(y))\rho(y)dy + \int_0^{f(u(x))} f^{-1}(r)dr - hf(u(x)) \right] \rho(x)dx.
\] (4.7)
Equivalently, with \( d\mu(x) = \rho(x)dx \), we can rewrite (4.7) as

\[
F(u) = \int_{\mathbb{R}^N} \left[ -\frac{1}{2} f(u(x)) \int_{\mathbb{R}^N} J(x - y) f(u(y)) d\mu(y) + \int_{0}^{f(u(x))} f^{-1}(r) dr - hf(u(x)) \right] d\mu(x).
\]

We can then prove the following result:

**Proposition 4.1.** In addition to the hypotheses of Theorem 3.8, assume that \( f : \mathbb{R} \to \mathbb{R} \) is strictly increasing. Then the functional given in (4.7) satisfies \( |F(u)| < \infty \), for all \( u \in C^\rho(\mathbb{R}^N) \).

**Proof.** We start by noting that

\[
F(u) = F_1(u) + F_2(u) - F_3(u),
\]

where

\[
F_1(u) = -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(u(x)) J(x - y) f(u(y)) \rho(y) \rho(x) dydx,
\]

\[
F_2(u) = \int_{\mathbb{R}^N} \left[ \int_{0}^{f(u(x))} f^{-1}(r) dr \right] \rho(x) dx
\]

and

\[
F_3(u) = \int_{\mathbb{R}^N} h f(u(x)) \rho(x) dx.
\]

Let

\[
G_1(x, y) := f(u(x)) J(x - y) f(u(y)) \rho(y) \rho(x)
\]

denote the integrand of \( F_1(u) \) Then, since \( M = \|f \circ u\|_\infty < \infty \), we obtain

\[
|G_1(x, y)| \leq M^2 J(x - y) \rho(y) \rho(x)
\]

and, therefore

\[
|F_1(u)| \leq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M^2 J(x - y) \rho(y) \rho(x) dydx
\]

\[
\leq \frac{1}{2} M^2 \|J\|_{L^1(\mathbb{R}^N)} \|\rho\|_\infty \int_{\mathbb{R}^N} \rho(x) dx
\]

\[
\leq \frac{1}{2} M^2 \|J\|_{L^1(\mathbb{R}^N)} \|\rho\|_\infty \|\rho\|_{L^1(\mathbb{R}^N)} ,
\]

(4.9)

Let now

\[
G_2(x) := \int_{0}^{f(u(x))} f^{-1}(r) dr \rho(x)
\]

denote the integrand of \( F_2(u) \). Then,

\[
|G_2(x)| \leq \int_{0}^{M} |f^{-1}(r)| dr \rho(x)
\]

(4.10)
and
\[
|F_2(u)| \leq \int_{\mathbb{R}^N} \left[ \int_0^M \left| f^{-1}(r) \right| \, dr \right] \rho(x) \, dx \\
\leq \int_{\mathbb{R}^N} \mathcal{L} \rho(x) \, dx \\
\leq \mathcal{L} \left\| \rho \right\|_{L^1(\mathbb{R}^N)},
\tag{4.11}
\]
where \( \mathcal{L} \) is the integral of the continuous function \( f^{-1} \) in the (finite) interval \([0, M] \).

Finally let
\[
G_3(x) := h(x)f(u(x))\rho(x)
\tag{4.12}
\]
denote the integrand of \( F_u(u) \). Then
\[
|G_3(x)| \leq M \left\| h \right\|_\infty \rho(x)
\]
and
\[
|F_3(u)| \leq \int_{\mathbb{R}^N} M \left\| h \right\|_\infty \rho(x) \, dx \\
\leq M \left\| h \right\|_\infty \left\| \rho \right\|_{L^1(\mathbb{R}^N)}.
\tag{4.13}
\]

\[\blacksquare\]

**Theorem 4.2.** Suppose \( f \) satisfies the same hypotheses of Proposition 4.1. Then the functional given in (4.7) is continuous in the topology of \( C_\rho(\mathbb{R}^N) \).

**Proof.** Write \( F(u) = F_1(u) + F_2(u) - F_3(u) \) as in the proof of the Proposition 4.1.

Let \( u_n \) be a sequence of functions converging to \( u \) in \( C_\rho(\mathbb{R}^N) \).

Let also
\[
G_1(x, y), G_2(x), G_3(x) \quad \text{as in (4.8), (4.10), (4.12)} \quad \text{and}
\]
\[
G_1^n(x, y), G_2^n(x), G_3^n(x) \quad \text{as in (4.8), (4.10), (4.12)} \quad \text{with} \ u \ \text{replaced by} \ u_n.
\]

Then
\[
F_1(u_n) = -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G_1^n(x, y) \, dy \, dx
\]
\[
F_2(u_n) = \int_{\mathbb{R}^N} G_2^n(x) \, dx
\]
and
\[
F_3(u_n) = \int_{\mathbb{R}^N} G_3^n(x) \, dx.
\]

By (4.8), (4.10), (4.12) and (4.9), (4.11), (4.13); the integrands \( G_1^n(x, y), G_2^n(x), G_3^n(x) \) are all bounded by integrable functions independent of \( n \). Also from the pointwise convergence of \( u_n \) to \( u \) and the continuity of the functions \( f, \rho \) and \( h \), it follows that \( G_1^n(x, y) \to G_1(x, y), G_2^n(x) \to G_2(x) \) and \( G_3^n(x) \to G_3(x) \), for all \( x, y \in \mathbb{R}^N \).

Therefore, \( F(u_n) \to F(u) \), by Lebesgue Dominated Convergence Theorem

This completes the proof. \[\blacksquare\]
Theorem 4.3. Suppose that $f$ satisfies the same hypotheses of Proposition [3.1] and that $|f'(x)| \leq (|x| + c)\rho^3(x)$, for all $x \in \mathbb{R}^N$ and some positive constant $c$. Let $u(\cdot, t)$ be a solutions of (1.1). Then $F(u(\cdot, t))$ is differentiable with respect to $t$ and

$$\frac{dF}{dt} = -\int_{\mathbb{R}^N} [-u(x, t) + J_\rho(f \circ u)(x, t) + h]^2 f'(u(x, t))d\mu(x) \leq 0.$$

Proof. Let

$$\varphi(x, s) = -\frac{1}{2}f(u(x, s)) \int_{\mathbb{R}^N} J(x - y)f(u(y, s))\rho(y)dy + \int_0^{f(u(x, s))} f^{-1}(r)dr - hf(u(x, s)).$$

Using the hypotheses on $f$ and the fact that $|f'(x)| \leq (|x| + c)\rho^3(x)$, it is easy to see that $\|\frac{\partial \varphi(x, s)}{\partial s}\|_{L^1(\mathbb{R}^N, d\mu(x))} < \infty$, for all $s \in \mathbb{R}_+$. Hence, derivating under the integration sign, we obtain

$$\frac{d}{dt}F(u(\cdot, t)) = \int_{\mathbb{R}^N} \left[-\frac{1}{2} \frac{\partial f(u(x, t))}{\partial t}\int_{\mathbb{R}^N} J(x - y)f(u(y, t))d\mu(y)ight]$$

$$- \frac{1}{2} f(u(x, t)) \int_{\mathbb{R}^N} J(x - y)\frac{\partial f(u(y, t))}{\partial t}d\mu(y)$$

$$+ f^{-1}(f(u(x, t))) \frac{\partial f(u(x, t))}{\partial t} - h \frac{\partial f(u(x, t))}{\partial t} d\mu(x)$$

$$= -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x - y)f(u(y, t))\frac{\partial f(u(x, t))}{\partial t}d\mu(y)d\mu(x)$$

$$- \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x - y)f(u(x, t))\frac{\partial f(u(y, t))}{\partial t}d\mu(y)d\mu(x)$$

$$+ \int_{\mathbb{R}^N} [u(x, t) - h] \frac{\partial f(u(x, t))}{\partial t} d\mu(x).$$

Since

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x - y)f(u(y, t))\frac{\partial f(u(x, t))}{\partial t}d\mu(y)d\mu(x) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x - y)f(u(x, t))\frac{\partial f(u(y, t))}{\partial t}d\mu(y)d\mu(x),$$

It follows that

$$\frac{d}{dt}F(u(\cdot, t)) = -\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} [u(x, t) - h] \frac{\partial f(u(x, t))}{\partial t} d\mu(x)$$

Using that $f$ is strictly increasing, the result follows. □
Remark 4.4. From Theorem 4.3 follows that, if $F(T(t)u_0) = F(u_0)$ for $t \in \mathbb{R}$, then $u_0$ is an equilibrium point for $T(t)$.

4.1 Gradient property

We recall that a semigroup, $T(t)$, is gradient if each bounded positive orbit is precompact and there exists a continuous Lyapunov Functional for $T(t)$, (see [14]).

Proposition 4.5. Assume the same hypotheses from Theorems 4.3 and 3.8. Then the flow generated by equation (1.1) is gradient.

Proof. The precompacity of the orbits follows from existence of the global attractor. From Proposition 4.1, Theorem 4.2, Theorem 4.3 and Remark 4.4 follows that the functional given in (4.7) is a continuous Lyapunov functional.\[\square\]

As consequence of the Proposition 4.5 we have the convergence of the solutions of (1.1) to the equilibrium point set of $T(t)$ (see [14] - Lemma 3.8.2)

Corollary 4.6. For any $u \in C_p(\mathbb{R})$, the $\omega$-limit set, $\omega(u)$, of $u$ under $T(t)$ belongs to $E$. Analogously the $\alpha$-limit set, $\alpha(u)$, of $u$ under $T(t)$ belongs to $E$.

Also as a consequence of the Proposition 4.5 we have that the global attractor given in the Theorem 3.8 has the following characterization (see [14] - Theorem 3.8.5).

Theorem 4.7. Under the same hypotheses from Theorem 4.3, the attractor $A$ is the unstable set of the equilibrium point set of $T(t)$, that is,

$$A = W^u(E),$$

where $E = \{u \in B_p(0, R) : u(x) = J*\rho(f \circ u)(x) + h\}$.

Proof. Let $u \in A$. Then, there exists a complete orbit through $u$ which is contained in $A$. Since $A$ is compact, the $\alpha$-limit set, $\alpha(u)$, of $u$ under $T(t)$ is nonempty. By Lemma 4.8 it belongs to $E$ and, therefore, $u \in W^u(E)$.

Conversely, suppose $u \in W^u(E)$ and let $E^\delta$ denote a $\delta$-neighborhood of $E$. Then, for any $\delta > 0$, there exists $\bar{t}$ such that $T(-\bar{t})u \in E^\delta$, for any $t \geq \bar{t}$. Thus, $u \in T(t)(E^\delta)$, for any $t \geq \bar{t}$. It follows that $u$ is arbitrarily close to $A$, so it must belong to $A$.

This concludes the proof. $\square$

5 An example

Motivated by the example given in [7], we consider the one dimensional case of (1.1), with $f(x) = (1 + e^{-x})^{-1}$,

$$J(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

that is, we consider the equation

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{x-1}^{x+1} e^{\frac{-1}{(x-y)^2}} (1 + e^{-u(y)})^{-1} \sqrt{(1 + y^2)^{-1}} dy + h.$$  (5.14)

It is easy to see that the function $J$ meet all the hypotheses assumed in introduction, that is, $J$ is an even non negative function of class $C^1(\mathbb{R})$. Furthermore, we have:
Remark 5.1. The function $f$ satisfies the condition (2.3), with $\eta = 1$ and $K = \frac{1}{2}$.

In fact, since $f'(x) = (1 + e^{-x})^{-2} e^{-x} > 0$, it follows that $1 < (1 + e^{-x})^2 \leq 4$, $\forall x \in \mathbb{R}^N$. Thus
\[
\frac{1}{4} \leq (1 + e^{-x})^{-2} < 1.
\] (5.15)

Then, since $f''(x) = 2(1 + e^{-x})^{-3} e^{-2x} - (1 + e^{-x})^{-2} e^{-x}$, follows that $|f''(x)| < 3$, $\forall x \in \mathbb{R}^N$. Hence $f'$ is locally Lipschitz. Furthermore, follows from (5.15) that
\[
|f(x) - f(y)| = |(1 + e^{-x})^{-1} - (1 + e^{-y})^{-1}| \leq |x - y|.
\]

In particular, using that $f(0) = \frac{1}{2}$, results
\[
|f(x)| \leq |x| + \frac{1}{2}, \; \forall x \in \mathbb{R}^N.
\]

Remark 5.2. With $\rho(x) = \sqrt{(1 + x^2)^{-1}}$, the hypothesis that $\int_{\mathbb{R}^N} \rho(x)dx < \infty$ is easily verified and $|\rho(x)| \leq 1$, for all $x \in \mathbb{R}$. Furthermore, we also have
\[
f'(x) = (1 + e^{-x})^{-2} e^{-x} \leq (1 + |x|)(1 + x^2)^{-1} = (1 + |x|)\rho^3(x).
\]

Remark 5.3. The hypotheses in the Theorem 4.3 are also satisfied.

In fact, note that $0 < |(1 + e^{-x})^{-1}| < 1$ and $f^{-1}(x) = -\ln(\frac{1-x}{x})$. Thus it is easy to see that, for $0 \leq s \leq 1$,
\[
\left| \int_{0}^{s} -\ln(\frac{1-x}{x})dx \right| \leq \ln 2.
\]

Therefore the results of the preview sections are valid for the flow generated by equation (5.14).

References

[1] Amari, S., Dynamics of pattern formation in lateral-inhibition type neural fields, Biol. Cybernetics, 27 (1977), 77-87.

[2] Amari S., Dynamics stability of formation of cortical maps, In: M. A. Arbib and S. I. Amari (Eds), Dynamic Interactions in Neural Networks: Models and Data, Springer-Verlag, New York, 1989, pp. 15-34.

[3] Chen, F., Traveling waves for a neural network, Electronic Journal Differential Equations, 2003 (2003), no. 13, 1-14.

[4] Daleckii, J. L., Krein, M. G., Stability of Solutions of Differential Equations in Banach Space; American Mathematical Society Providence, Rhode Island, 1974,

[5] da Silva, S. H., Pereira, A.L., Global attractors for neural fields in a weighted space. Matemática Contemporanea, 36 (2009) 139-153.

[6] da Silva, S. H., Existence and upper semicontinuity of global attractors for neural fields in an unbounded domain. Electronic Journal of Differential Equations, 2010, no. 138, (2010) 1-12.
[7] da Silva, S. H., Existence and upper semicontinuity of global attractors for neural network in a bounded domain. Differential Equations and Dynamical Systems, 19, no.1,(2011) 87-96.

[8] da Silva, S.H., Properties of an equation for neural fields in a bounded domain, Electronic Journal of Differential Equations, 2012, no. 42, (2012) 1-9.

[9] da Silva, S. H., Lower Semicontinuity of Global Attractors for a Class of Evolution Equations of Neural Fields Type in a Bounded Domain Differential Equations and Dynamical Systems, (2015). doi:10.1007/s12591-015-0258-6

[10] da Silva, S. H., Silva, M. B.: Asymptotic Behavior of Neural Fields in an Unbounded Domain Differential Equations and Dynamical Systems, 23, no. 4, (2015).

[11] Ermentrout, G. B., McLeod, J. B., Existence and uniqueness of travelling waves for a neural network, Proceedings of the Royal Society of Edinburgh, 123A (1993), 461-478.

[12] Ermentrout, G.B., Jalics, J.Z., Rubin, J.E., Stimulus-driven travelling solutions in continuum neuronal models with general smooth firing rate functions. SIAM, J. Appl. Math, 70 (2010) 3039-3064.

[13] French, D. A., Identification of a Free Energy Functional in an Integro-Differential Equation Model for Neuronal Network Activity. Applied Mathematics Letters, 17 (2004) 1047-1051.

[14] Hale, J. K., Asymptotic Behavior of dissipative Systems, American Surveys and Monographs, N. 25, 1988.

[15] Henry, D., Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics 840, Springer Verlag, 1981.

[16] Kishimoto,K., Amari, S., Existence and Stability of Local Excitations in Homogeneous Neural Fields, J. Math. Biology, 07 (1979), 303-1979.

[17] Krisner, E. P., The link between integral equations and higher order ODEs, J. Math. Anal. Appl., 291 (2004), 165-179.

[18] Kubota, S., Aihara, K., Analyzing Global Dynamics of a Neural Field Model, Neural Processing Letters, 21 (2005) 133-141.

[19] Laing, C. R., Troy, W. C., Gutkin, B., Ermentrout, G. B., Multiplos Bumps in a Neural Model of Working Memory, SIAM J. Appl. Math., 63 (2002), no. 1, 62-97.

[20] Pereira, A. L., Global attractor and nonhomogeneous equilibria for a non local evolution equation in an unbounded domain, J. Diff. Equations, 226 (2006), 352-372.

[21] Rubin, J. E., Troy, W. C., Sustained spatial patterns of activity in neural populations without recurrent Excitation, SIAM J. Appl. Math., 64 (2004), 1609-1635.

[22] Teman, R., Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer, 1988.
[23] Wilson, H. R., Cowan, J. D., *Excitatory and inhibitory interactions in localized populations of model neurons*, Biophys. J., **12** (1972), 1-24.

[24] Wu, S., Amari, S., Nakahara, H., *Population coding and decoding in a neural field: a computational study*, Neural Computation, **14** (2002), 999-1026.