SPECIAL VALUE FORMULA FOR THE TWISTED TRIPLE PRODUCT AND APPLICATIONS

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Abstract. We establish explicit Ichino’s formulae for the central values of the triple product $L$-functions with emphasis on the calculations for the real place. The key ingredient for our computations is Proposition 4.5 for the real place which is the main novelty of this article. As an application we prove the optimal upper bound of a sum of restricted 2-norms of the $L^2$-normalized newforms on certain quadratic extensions with prime level and bounded spectral parameter following the methods in [Blo13].

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1. Introduction and main results

The aim of this article is to establish explicit Ichino’s formulae for the central values of the triple product $L$-functions with emphasis on the calculations for the real place. As an application, we prove the optimal upper bound of a sum of restricted 2-norms of the $L^2$-normalized newforms on certain quadratic extensions with prime level and bounded spectral parameter, following the methods of Blomer in [Blo13]. In this introduction, we state our formulae in the following special cases for the sake of simplicity. Fix a quadratic extension $K$ over $\mathbb{Q}$ with discriminant $D_K = D$. We assume the (narrow) class number of $K$ is 1 according to the sign of $D$. Let $q$ be a prime and $L^2(X_0(q))$ be the $L^2$ space equipped with the inner product

$$\langle f, g \rangle = \int_{X_0(q)} f(\tau) \overline{g(\tau)} \, d\mu(\tau) \quad (d\mu(\tau) := \frac{dx\,dy}{y^2}, \ \tau = x + iy).$$

Here $X_0(q) := \Gamma_0(q) \backslash \mathfrak{H}$. Denote by $\mathcal{B}_q$ (resp. $\mathcal{B}_1$) an orthonormal basis of cuspidal Hecke-Maass newforms for $\Gamma_0(q)$ (resp. $\text{SL}_2(\mathbb{Z})$), with respect to the inner product (1.1). We assume further that $q$ and $D$ are coprime.

Given $f \in \mathcal{B}_q$, we denote by $f_K$ the unique $L^2$-normalized newform on $K$ associated to $f$ via the base change lift (see [22]). Then $f_K$ is an analytic function on $\mathfrak{H}_D$, where $\mathfrak{H}_D$ is the two copies of the upper half plane $\mathfrak{H}$ when $D > 0$, and is the hyperbolic three space when $D < 0$. In any case, the upper half plane $\mathfrak{H}$ sits naturally in $\mathfrak{H}_D$ and we denote by $f_K|\delta$ the restriction of $f_K$ to $\mathfrak{H}$, which is $\Gamma_0(q)$-invariant on the left. Let $\delta_D = 0$ if $D > 0$ and $\delta_D = 1$ if $D < 0$. Then we have the following special case of Theorem 5.2.

Theorem 1.1. Let $f \in \mathcal{B}_q$ and $g \in \mathcal{B}_q \cup \mathcal{B}_1$. Let $c = 0$ or 1 according to $g \in \mathcal{B}_q$ or $g \in \mathcal{B}_1$ respectively. Then

$$|\langle f_K|\delta, g \rangle|^2 = 2^{-1}(8\pi^{-1})^{\delta_D} q^{-2(1 + q^{-1})^{-c}} |D|^{-3/2} \frac{\Lambda(1/2, Asf_K \times g)}{\Lambda(1, Ad^2 f_K)} \Lambda(1, Ad^2 g).$$

Here the $L$-functions are complete $L$-functions.
We should mention here that both sides of the equality in Theorem \[1\] vanish trivially if the root number condition is not fulfilled. Ichino’s formula \[\text{[Ich08]}\] relates the period integrals of triple products of certain automorphic forms on quaternion algebras along the diagonal cycles and the central values of triple \(L\)-functions for \(\text{GL}_2\) together with product of local period integrals for each places. To derive explicit Ichino’s formulae in Theorem \[6.2\] we need to carry out explicit computations of these local period integrals. In the literature, these local period integrals for nonarchimedean places have been computed in \[\text{[Nel11], [NPS14], [Hu17] and [Hsi17]}\]. In this article; however, we pay our attention on the calculations for the real place. In particular, by combining with the results in \[\text{[CCI18]}\], the computations for the real place are completed except the case where the representation of \(\text{GL}_2(\mathbb{C})\) is a principal series, while the representation of \(\text{GL}_2(\mathbb{R})\) is a discrete series. We mention here that some of the remaining case was computed in \[\text{[Che18a]}\]. The key ingredient for our computations and the main novelty of this article is Proposition \[4.5\] for the real place, which reduces the calculations of local period integrals to that of certain local Rankin-Selberg integrals \[\text{[Jac72], [Flis88] and [CCI18]}\]. The proof of Proposition \[4.5\] for the real place occupies a substantial part of this article. We should point out that in \[\text{[Hsi17]}\], he used Proposition \[4.5\] to compute the local period integrals for the nonarchimedean places under quite general settings. The computation for the real place was also treated in \[\text{[Woo16]}\].

Explicit special value formulae for the triple product \(L\)-functions have applications both to the algebraic number theory and the analytic number theory. In addition to the articles mentioned in the previous paragraph, there are also \[\text{[Gar87], [Orl87], [GK92] and [Wat08]}\]. In this article, we give an application to the optimal upper bound of the restricted 2-norm

\[
\|f_K|_B\|^2 := \langle f_K|_B, f_K|_B \rangle
\]

in the level aspect. Indeed, by combining Theorem \[1.1\] with the arguments in \[\text{[Blo13]}\], we immediately obtain:

**Theorem 1.2.** Fix any real number \(T > 1\) and \(\epsilon > 0\). Then

\[
\sum_{f \in \mathcal{B}_q, t_f \ll T} \|f_K|_B\|^2 \ll_{\mathcal{D}, T, \epsilon} q^\epsilon.
\]

Here \(t_f\) is the spectral parameter of \(f\) given by \([2.1]\).

We remark here that the proof Theorem \[1.2\] depends on a bound \(\alpha\) toward the Ramanujan conjecture for \(\text{GL}_2\). By a result of Kim \[\text{[Kim13]}\], one can take \(\alpha = 7/64\). In this article, the bound \(\alpha = 1/6 - \delta\) is enough to obtain Theorem \[1.2\]. Observe that Theorem \[1.2\] implies the best individual bound (in the sense of assuming the Lindelof hypothesis) for all but finitely many \(f \in \mathcal{B}_q\) appearing in the sum. In fact, by Parseval’s identity, the restricted 2-norm of \(f_K\) relates the central values of the twisted triple product \(L\)-functions; by Theorem \[1.1\] one has an equality roughly of the form:

\[
\|f_K|_B\|^2 \approx \frac{1}{q^2} \sum_{g \in \mathcal{B}_q, t_g \ll 1} L(1/2, Asf_K \times g).
\]

Weyl’s law tells us that the sum of the RHS has \(O(q)\) terms, so the Lindelof hypothesis for \(L\)-functions on the RHS would imply \(\|f_K|_B\|^2 \ll q^{-1/2+\epsilon}\). The Lindelof hypothesis for the \(L\)-functions is far beyond reach with present technology; however, we have:

**Corollary 1.3.** Let \(\delta > 0\). The bound \(\|f_K|_B\|^2 \ll_{\mathcal{D}, T, \epsilon} q^{-1/2+\delta}\) holds for all but \(O(q^{1-2\delta+\epsilon})\) of \(f \in \mathcal{B}_q\) occurring in the sum in Theorem \[1.2\].

One of the motivation for bounding \(L^p\)-norms of functions on Riemannian manifolds or their restriction to submanifolds has its roots in quantum chaos. In the arithmetical setting, these manifolds have additional symmetries, such as a commutative algebra of Hecke operators commuting with the Laplacian. Therefore one can consider joint eigenfunctions which may rule out high multiplicity eigenspaces. One searches for bounds of these \((L^2\text{-normalized})\) eigenfunctions in the different aspects, i.e. in the spectral aspect or in the level aspect or in both. In the literature, there are many results on bounding sup-norms of eigenfunctions on the entire domain. In particular, there is a complete list of references in the introduction of \[\text{[Sah17]}\]. In contrast, there are only a small handful results concerning the restricted \(L^p\)-norms \[\text{[BGT07], [LY12], [BM15], [Mar16] and [LLY]}\], all of which are in the spectral aspect. Although our result is somewhat restricted, it seems that its the first time to bound the restricted \(L^2\)-norms in the level aspect.
This article is organized as follows: In §2 we sketch the proof of Theorem 1.2. In §3 we state our special value formulae in terms of adelic language. In §4 we compute the local period integrals. Finally, we prove Proposition 4.5 in §5.

1.1. General notation. We give a list of our most frequently used notation. If $F$ is a number field, let $A_F$ be the ring of adeles of $F$. When $F = \mathbb{Q}$, we simply write $A$ for $A_\mathbb{Q}$. If $w$ is a place of $F$, denote by $F_w$ the completion of $F$ at $w$. Let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$. If $M$ is an abelian group, let $\widehat{M} = M \otimes \mathbb{Z}\widehat{\mathbb{Z}}$.

Let $F$ be a local field of characteristic zero. If $F$ is nonarchimedean, let $\varpi_F$, $\mathcal{O}_F$ and $q_F$ be a uniformizer, the valuation ring and the cardinality of the residue field of $F$, respectively. Let $| \cdot |_F$ be the usual absolute value on $F$. Then we have $|x|_C = x\hat{\pi}$, $|x| = |x|_\mathbb{R} = \max \{x,-x\}$ and $|\varpi_F|_F = q_F$. Define the local zeta functions as usual by

$$\Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s); \quad \Gamma_R(s) = \pi^{-s/2}\Gamma(s/2); \quad \zeta_F(s) = (1-q_F^{-s})^{-1}.$$

If $F$ is a number field, define the Dedekind zeta function and its complete version by

$$\zeta_F(s) = \prod_{w < \infty} \zeta_{F_w}(s) \quad \text{and} \quad \xi_F(s) = \Gamma_R(s)^{-1}\Gamma_C(s)^{1/2}\zeta_F(s).$$

Here $r_1$ (resp. $r_2$) is the number of real (resp. complex) places of $F$.

Let $R$ be a commutative ring with 1. Denote by $R^\times$ the group of invertible elements in $R$ and by $M_2(R)$ the ring consists of $2 \times 2$ matrices with coefficient in $R$. If $n \subset R$ is an ideal, we put

$$K_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R) \mid c \in n \right\}.$$

Let $B(R) \subset \text{GL}_2(R)$ be the subgroup consists of upper triangular matrices, and let $N(R) \subset B(R)$ be the subgroup consists of elements whose diagonal entries are both equal to 1. If $\chi : R^\times \to \mathbb{C}^\times$ is a homomorphism, we still use $\chi$ to denote the homomorphism on $\text{GL}_2(R)$ defined by $\chi(h) := \chi(\det(h))$. If $f$ is a function on $\text{GL}_2(R)$, let $f \otimes \chi$ be a function on $\text{GL}_2(R)$ with $(f \otimes \chi)(h) = f(h)\chi(h)$. Define following elements in $\text{GL}_2(\mathbb{Z})$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $\pi$ be a representation of a group $G$. We often identify the representation space of $\pi$ with $\pi$ itself. The central character (if exist) of $\pi$ is denoted by $\omega_\pi$. If $\chi$ is a character of $G$, we denote by $\pi \otimes \chi$ the representation of $G$ on the same representation space of $\pi$ with the action $\pi \otimes \chi(g) = \pi(g)\chi(g)$. If $K$ is a subgroup of $G$, let $\pi^K$ be the subspace consists of vectors fixed by $K$. When $G = \text{GL}_2(F)$ with $F$ a local field of characteristic zero, and $\pi$ is an admissible representation with finite length, let $\hat{\pi}$ be the admissible dual of $\pi$. When $F$ is archimedean, we will always understand $\pi$ as a Harish-Chandra module.

We use the notation $A \ll x_1, x_2, \ldots, x_n$, $B$ to indicate that there exists a constant $C > 0$, depending at most upon $X_1, X_2, \ldots, X_n$ so that $|A| \leq C|B|$. Finally, if $S$ is a set, let $\mathbb{1}_S$ be the characteristic function on $S$.

2. Proof of Theorem 1.2

We prove Theorem 1.2 in this section. The most important ingredients of the proofs are the special value formulae for the triple product $L$-functions and the Kuznetsov formula [IK04, page 409]. Our proofs are only sketch since the essential parts are already contained in [Blo13]. Indeed, we modify the proofs in [Blo13] so that the proofs are also applicable to our case. It is our policy to point out the differences between these two cases and to give proofs whenever the differences occur.

We follow the notation and the assumption in the introduction, so that $\mathcal{K}$ denote a fixed quadratic extension over $\mathbb{Q}$ whose (narrow) class number is one. Let $\mathcal{O}$ be the ring of integers of $\mathcal{K}$ and $\tau_\mathcal{K}$ be the quadratic Dirichlet character associated to $\mathcal{K}/\mathbb{Q}$. Recall that $\delta_\mathcal{D} \in \{0, 1\}$ satisfies $\mathcal{D} = (-1)^{\delta_{\mathcal{D}}}|\mathcal{D}|$.

2.1. Preliminaries on Maass forms. Let $q > 0$ be a prime. We assume $\mathcal{D}$ and $q$ are coprime so that the implicit constants in the following discussions do not depend on $q$. As usual, we let $\|g\|_2^2 = \langle g, g \rangle$ for every $g \in L^2(X_0(q))$. For any $g \in B_q \cup B_1$ with the Laplacian eigenvalue $\lambda_g$, we denote by

$$t_g := \sqrt{\lambda_g - 1/4} \in \mathbb{R} \cup (-1/2, 1/2)i =: T$$

for
for its spectral parameter, and by $\lambda_g(n)$ for the $n$-th Hecke eigenvalue. We put $\delta_{g} = 0$ or 1 according to $g$ is even or odd. When $g \in B_q$, let $w_g \in \{\pm 1\}$ be the eigenvalue of $g$ under the Atkin-Lehner involution, i.e.

\[(2.2)\quad g(\sigma_0 \cdot \tau) = w_g g(\tau) \quad \text{where} \quad \sigma_0 = \begin{pmatrix} 0 & -\sqrt{q}^{-1} \\ \sqrt{q} & 0 \end{pmatrix}.\]

Note that we have the relation $w_g = -\lambda_g(q)^{1/2}$. For $g \in B_1$, we define $w_g = 1$ and

\[(2.3)\quad g^*(\tau) = \left(1 - \frac{q \lambda_g(q)^{2}}{(q + 1)^2}\right)^{-1/2} \left(g(q\tau) - \frac{q^{1/2} \lambda_g(q)}{q + 1} g(\tau)\right).

Then $g$ and $g^*$ have the same norm and are orthogonal to each other [LS00 Proposition 2.6]. Therefore

\[B_q \cup B_1 \cup B_1^* = \{g^* \mid g \in B_1\}\]

is an orthonormal basis (with respect to [1.11]) of the non-trivial cuspidal spectrum of $L^2(X_0(q))$.

### 2.2. Base change to $K$.

We review the base change lift of $GL_2$ from $\mathbb{Q}$ to $K$ in this subsection [Jac72 Section 20]. From now on, we use $v$ to indicate a place of $\mathbb{Q}$ and $p$ (resp. $\infty$) to denote the finite place (resp. the real place) of $\mathbb{Q}$ corresponds to a prime $p$.

#### 2.2.1. Adelic automorphic forms.

Let $f \in B_q$. For a prime $p \neq q$, let $\{\alpha_p, \alpha_p^{-1}\}$ be the Satake parameter of $f$ at $p$. Then

\[1 - \lambda_f(p) X - p^{-1} X^2 = (1 - p^{-1/2} \alpha_p X)(1 - p^{-1/2} \alpha_p^{-1} X).

We fix $s_p \in \mathbb{C}$ so that $\alpha_p = p^{-s_p}$. The Maass form $f$ determines a cusp form $f$ on $GL_2(\mathbb{A})$ by the formula

\[f(h) = f(h_\infty \cdot \sqrt{-1})\]

for $h = \gamma h_\infty k$ with $\gamma \in GL_2(\mathbb{Q})$, $h_\infty \in GL_2^+(\mathbb{R})$ and $k \in K_0(q\hat{\mathbb{Z}})$. The group $GL_2^+(\mathbb{R})$ acts on $\mathfrak{f}$ by the usual linear fractional transformation, and the action is denoted by $h \cdot \tau$ for $h \in GL_2^+(\mathbb{R})$ and $\tau \in \mathfrak{f}$. Let $\pi = \otimes_v \pi_v$ be the unitary irreducible cuspidal automorphic representation of $PGL_2(\mathbb{A})$ generated by $f$. Then we have

\[\pi = \text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})}(\{ \left| \frac{1}{2} \right| \otimes \left| \frac{1}{2} \right| \} \otimes \text{sgn} \delta); \quad \pi_v = \text{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\{ \left| \frac{s_p}{2} \right| \otimes \left| \frac{-s_p}{2} \right|); \quad \pi_q = \text{St}_{\mathbb{Q}_q} \otimes \chi_q
\]

where $\text{St}_{\mathbb{Q}_q}$ is the Steinberg representation of $GL_2(\mathbb{Q}_q)$ and $\chi_q$ is the unramified quadratic character of $\mathbb{Q}_q^\times$ with $\chi_q(q) = -w_f$. Notice that $\mathfrak{f} \in \pi$ is the unique (up to constants) element such that

\[f(hk_\infty k) = f(h)\]

for $h \in GL_2(\mathbb{A})$, $k_\infty \in SO(2)$ and $k \in K_0(q\hat{\mathbb{Z}})$.

Let $K_v = K \otimes \mathbb{Q}_v$ for each place $v$ of $\mathbb{Q}$. Let $\pi_K = \otimes_v \pi_{K,v}$ be the base change lift of $\pi$ to $K$, which is a unitary irreducible cuspidal automorphic representation of $PGL_2(\mathbb{A}_K)$. Then for $p \neq q$, we have

\[\pi_{K,p} = \text{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\{ \left| \frac{s_p}{2} \right| \otimes \left| \frac{-s_p}{2} \right|).\]

Let $v \in \{\infty, q\}$. If $K_v = \mathbb{Q}_v \oplus \mathbb{Q}_v$, then $\pi_{K,v} = \pi_v \boxtimes \pi_v$. On the other hand, if $K_v$ is a field, then

\[\pi_{K,v} = \text{Ind}_{B(\mathbb{Q}_v)}^{GL_2(\mathbb{Q}_v)}(\{ \left| \frac{1}{2} \right| \otimes \left| \frac{1}{2} \right|); \quad \pi_{K,q} = \text{St}_{\mathbb{Q}_q}\]

By the theory of newform [Cas73] and the $K_\infty$-type of $\pi_{K,\infty}$, there is a unique (up to constants) element $0 \neq f_\infty \in \pi_K$ such that

\[(2.4)\quad f_\infty(hk_\infty k) = f_\infty(h)\]

for $h \in GL_2(\mathbb{A}_K)$, $k_\infty \in K_\infty$ and $k \in K_0(q\hat{\mathbb{O}})$. Here $K_\infty = SO(2) \times SO(2)$ when $D > 0$, and $K_\infty = SU(2)$ when $D < 0$. 

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**Note:** The provided text contains mathematical formulas and notation typical of advanced mathematical exposition, particularly in the fields of number theory and automorphic forms. The content includes advanced concepts such as base change, Hecke operators, and automorphic forms, which are fundamental in the study of modular forms and related areas of mathematics.

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**Reference:** The text references [Jac72] and [LS00], indicating a continuation from previously cited sources for further reading on the topics discussed.
2.2.2. Hyperbolic three space. Recall the hyperbolic three space
\[ \mathcal{H}' = \left\{ \begin{pmatrix} z & -y \\ y & z \end{pmatrix} \mid z \in \mathbb{C}, y \in \mathbb{R}_{>0} \right\}. \]

The group \( SL_2(\mathbb{C}) \) acts transitively on \( \mathcal{H}' \) via
\[
(2.5) \quad h \cdot \tau = \left( t(a) \tau + t(b) \right) \left( t(c) \tau + t(d) \right)^{-1}
\]
where \( h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \in \mathcal{H}' \) and \( t(z) = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \) for \( z \in \mathbb{C} \). Notice that \( w \in \mathcal{H}' \) and the stabilizer of \( w \) is \( SU(2) \), so we may identify the symmetric spaces \( SL_2(\mathbb{C})/SU(2) \cong \mathcal{H}' \). Observe that \( \mathcal{H} \) sits naturally in \( \mathcal{H}' \) via the embedding
\[
(2.6) \quad x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}
\]
which commutes with the \( SL_2(\mathbb{R}) \)-action and the \( SL_2(\mathbb{C}) \)-action. Finally, we have the following \( SL_2(\mathbb{C}) \)-invariant measure \( d\mu(\tau) \) on \( \mathcal{H}' \) \([\text{Hid93 \,(1.4e)}]\)
\[
(2.7) \quad d\mu(\tau) = \frac{|dz \wedge dy \wedge d\bar{z}|}{2y^3}, \quad \tau = \begin{pmatrix} z & -y \\ y & z \end{pmatrix}.
\]

2.2.3. Classical automorphic forms. Put
\[
\mathcal{H}_D = \begin{cases} 
\mathcal{H} \times \mathcal{H} & \text{if } D > 0, \\
\mathcal{H}' & \text{if } D < 0.
\end{cases}
\]

Let \( \sigma \) be the generator of \( \text{Gal}(\mathcal{K}/\mathbb{Q}) \). Note that \( \sigma(z) = \bar{z} \) for \( z \in \mathcal{K} \) when \( D < 0 \). The group \( SL_2(\mathcal{O}) \) acts on \( \mathcal{H}_D \) as follows. Suppose \( D < 0 \). Then \( SL_2(\mathcal{O}) \subset SL_2(\mathbb{C}) \) and the action is given by \([2.3]\). Suppose \( D > 0 \). We embed \( SL_2(\mathcal{O}) \) into \( SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \) via
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix} \right).
\]

The action of \( SL_2(\mathcal{O}) \) on \( \mathcal{H}_D \) is then induced from this embedding and the component-wise action of \( SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \) on \( \mathcal{H} \times \mathcal{H} \).

Now we associate \( f_\mathcal{K} \) with a unique classical automorphic form \( f_\mathcal{K} \) on the symmetric domain \( \mathcal{H}_D \). Let \( GL_2^+(\mathcal{K}_\infty) = GL_2^+(\mathbb{R}) \times GL_2^+(\mathbb{R}) \) when \( D > 0 \), and \( GL_2^+(\mathcal{K}_\infty) = GL_2(\mathbb{C}) \) when \( D < 0 \). Since \( \mathcal{K} \) has the (narrow) class number one, we have
\[
GL_2(\mathcal{K}) = GL_2(\mathcal{K}) GL_2^+(\mathcal{K}_\infty) K_0(q\mathcal{O}).
\]

Let \( \tau \in \mathcal{H}_D \). Choice any \( h \in GL_2^+(\mathcal{K}_\infty) \) so that \( h \cdot i = \tau \) when \( D > 0 \), and \( h \cdot w = \tau \) when \( D < 0 \). Here \( i = (\sqrt{-1}, \sqrt{-1}) \). Then we define
\[
f_\mathcal{K}(\tau) = f_\mathcal{K}(h).
\]

Observe that \( f_\mathcal{K} \) is \( GL_2(\mathcal{K}) \cap GL_2^+(\mathcal{K}_\infty) K_0(q\mathcal{O}) = \Gamma_0(q\mathcal{O}) \) on the left. We assume \( f_\mathcal{K} \) is \( L^2 \)-normalized in the sense that
\[
(2.8) \quad \|f_\mathcal{K}\|_2^2 := \int_{\Gamma_0(q\mathcal{O}) \backslash \mathcal{H}_D} |f_\mathcal{K}(\tau)|^2 d\mu(\tau) = 1.
\]

Here \( d\mu(\tau) \) is the invariant measure given by \([2.7]\) in the case \( D < 0 \), and is given by \( d\mu(\tau) = d\mu(\tau_1)d\mu(\tau_2) \) for \( \tau = (\tau_1, \tau_2) \in \mathcal{H}_D \) when \( D > 0 \). We denote by \( f_\mathcal{K}|_\mathcal{B} \in L^2(X_0(q)) \) the pullback of \( f_\mathcal{K} \) to \( \mathcal{H} \) via the diagonal embedding (resp. the embedding \([2.6]\)) when \( D > 0 \) (resp. \( D < 0 \)). Notice that since \( f \) is an eigenfunction under the Atkin-Lehner involution \([2.2]\), we have
\[
(2.9) \quad f_\mathcal{K}|_\mathcal{B}(\sigma \cdot \tau) = f_\mathcal{K}|_\mathcal{B}(\tau).
\]

The automorphic form \( f_\mathcal{K} \) has the Fourier expansion
\[
(2.10) \quad f_\mathcal{K}(\tau_1, \tau_2) = \rho_{f_\mathcal{K}} \sum_{\theta \neq \alpha \in \mathcal{O}} \lambda_{f_\mathcal{K}}(\alpha)(y_1y_2) \bar{\xi}_{K_{it}}(2\pi|\alpha|D^{-1/2}y_1) \bar{\xi}_{K_{it}}(2\pi|\alpha|D^{-1/2}y_2)e^{2\pi iD^{-1/2}(\alpha_1 - \sigma(\alpha)_x x_2)}.
\]
for $D > 0$, and
\begin{equation}
\tag{2.11}
f_k(\tau) = \rho_{f_k} \sum_{0 \neq \alpha \in \mathcal{O}} \lambda_{f_k}(\alpha) yK_{2it_f}(4\pi|\alpha|^1/2D^{-1/2}y)e^{2\pi i D^{-1/2}(\alpha z - \overline{\tau}z)}
\end{equation}

for $D < 0$. Here $\tau_1 = x_1 + iy_1$, $\tau_2 = x_2 + iy_2$ and $\tau = \left(\begin{smallmatrix} z & -y \\ y & z \end{smallmatrix}\right)$. In the expansions above, we take $\lambda_{f_k}(1) = 1$.

As usual, $K_s(z)$ denote the modified Bessel function of order $s$. Recall that for $\text{Re}(z) > 0$, we have the following integral representation
\begin{equation}
\tag{2.12}
K_s(z) = \frac{1}{2} \int_0^\infty \exp \left(-\frac{z}{2}(t + t^{-1})\right) t^{s-1} dt.
\end{equation}

Here $\exp(z) = e^z$ is the exponential function. We compute the constant $\rho_{f_k}$ in following lemma. Let $\text{Ad}^2 f_k$ be the adjoint square lift of $f_k$ to a cusp form on $\text{GL}_2$. \cite{GJ78} and $L(s, \text{Ad}^2 f_k)$ be the associated $L$-function. It has the factorization
\begin{equation}
\tag{2.13}
L(s, \text{Ad}^2 f_k) = L(s, \text{Ad}^2 f) L(s, \text{Ad}^2 f \times \tau_k).
\end{equation}

Define the complete $L$-function
\begin{equation}
\tag{2.14}
\Lambda(s, \text{Ad}^2 f_k) = L_\infty(s, \text{Ad}^2 f_k) L(s, \text{Ad}^2 f_k),
\end{equation}

\begin{equation}
\Lambda_\infty(s, \text{Ad}^2 f_k) = \Gamma_R(s + 2it_f) \Gamma_R(s) \Gamma(s - 2it_f) \Gamma(s + 2it_f + \delta_D) \Gamma(s + \delta_D) \Gamma(s - 2it_f + \delta_D).
\end{equation}

**Lemma 2.1.** We have
\[|\rho_{f_k}|^{-2} = \Lambda(1, \text{Ad}^2 f_k) \cdot (qD)^2 \cdot \left\{\begin{array}{cc} 2^{-3} & \text{if } D > 0, \\ 2^{-4} & \text{if } D < 0. \end{array}\right.\]

**Proof.** We use \cite[Proposition 6]{Wal85} to compute the constant. Let $\psi_Q = \prod_v \psi_v$ be the non-trivial additive character of $\mathbb{Q}/\mathbb{A}$ such that $\psi_Q(x) = e^{2\pi ix}$ for $x \in \mathbb{R}$ and $\psi_{p}(x) = e^{-2\pi ix}$ for $x \in \mathbb{Z}[q^{-1}]$. Put $\psi_{\mathbb{A}} := \psi_Q \circ \text{tr}_{\mathbb{K}/\mathbb{Q}} = \prod_v \psi_{\mathbb{A},v}$. For each place $v$ of $\mathbb{Q}$, let $W(\pi_{\mathbb{A},v}, \psi_{\mathbb{A},v})$ be the Whittaker model of $\pi_{\mathbb{A},v}$ associated to $\psi_{\mathbb{A},v}$ \cite[Theorem 2.14, 5.13, 6.3]{JL70}. Let $W_p \in W(\pi_{\mathbb{A},p}, \psi_{\mathbb{A},p})$ be the newform \cite{Sch02} with
\[W_p \left(\begin{array}{c} D^{-1/2} \\ 0 \\ 0 \end{array}\right) = 1.
\]

For the infinite place, we let $W_\infty \in W(\pi_{\mathbb{A},\infty}, \psi_{\mathbb{A},\infty})$ be the right $K_\infty$-invariant element (see \cite{2.24}) given by
\[W_\infty \left(\begin{array}{c} y \\ 0 \\ 1 \end{array}\right) = \left\{\begin{array}{cc} \text{sgn}(y_1 y_2) & 1/2K_{it_f}(2\pi|y_1|)K_{it_f}(2\pi|y_2|) \\ |y_1|^{1/2}K_{2it_f}(4\pi|y_1|^{1/2}) & \text{if } D > 0, \\ |y_1|^{1/2}K_{2it_f}(2\pi|y_1|) & \text{if } D < 0. \end{array}\right.
\]

Here $y = (y_1, y_2) \in K_\infty = \mathbb{R}^\times \times \mathbb{R}^\times$ if $D > 0$, and $y \in K_\infty = \mathbb{C}^\times$ if $D < 0$. Put $W = \prod_v W_v$. The Fourier expansion
\begin{equation}
\tag{2.15}
f_k^0(h) = \sum_{\alpha \in \mathbb{K}^\times} W \left(\begin{array}{c} \alpha \\ 0 \\ 1 \end{array}\right) h \quad \text{for } h \in \text{GL}_2(\mathbb{A}_K)
\end{equation}

defines a non-zero element in $\pi_{\mathbb{A}}$. Moreover, by the choices of $W_v$, the cusp form $f_k^0$ satisfies \cite{2.24}, and if $f_k^0$ is the classical automorphic form associated to $f_k^0$ as in \cite{2.23} then we have
\begin{equation}
\tag{2.16}
f_k = \rho_{f_k} \cdot |D|^{-1/2} \cdot f_k^0.
\end{equation}

This follows from comparing the expansion \cite{2.15} with \cite{2.10} and \cite{2.11}. By \cite[Proposition 6]{Wal85}, we have
\[\int_{\Lambda_\infty(\mathbb{K}) \backslash \text{GL}_2(\mathbb{K})} |f_k^0(h)|^2 d\mathbb{h}_{\text{Tam}} = 2\pi(2)^{-1}\zeta_{\mathbb{A}}(2)(L(1, \pi_{\mathbb{A}}, \text{Ad}) \prod_v \mathcal{H}_v^0(W_v, W_v),
\]

where $d\mathbb{h}_{\text{Tam}}$ is the Tamagawa measure on $\text{PGL}_2(\mathbb{A}_K)$, and
\[\mathcal{H}_v^0(W_v, W_v) = \frac{\zeta_{\mathbb{K}_v}(2)}{\zeta_{\mathbb{K}_v}(1)L(1, \pi_{\mathbb{K}_v}, \text{Ad})} \int_{\mathbb{K}_v} \left| W_v \left(\begin{array}{c} y_v \\ 0 \\ 1 \end{array}\right) \right|^2 d\mathbb{y}_v.
The measure $d^x y_v$ on $K_v^x$ is given by $d^x y_v = \zeta_{K_v}(1)|y_v|_{K_v}^{-1} d y_v$ where $d y_v$ is the self-dual measure on $K_v$ with respect to $\psi_{K_v}$. A direct computation shows
\begin{equation}
\prod_v \mathcal{H}_v^0(W_v, W_v) = 2^{-2\delta_D} \frac{\zeta_{K_v}(2)}{\zeta_{K_v}(1)}.
\end{equation}

On the other hand, by the choices of measures $d\mu(\tau)$, we have
\begin{equation}
\int_{A_{\infty}^0GL_2(K)\backslash GL_2(A_K)} |f_h^N(h)|^2 d\mu_{\text{Tam}} = \xi_{K}(2)^{-1} q^{-1/2} |D|^{-3/2} (8\pi^{-1})^{\delta_D} \frac{\zeta_{K_v}(2)}{\zeta_{K_v}(1)} \int_{\Gamma_0(q\mathcal{O})\backslash \mathcal{D}} |f_h^N(\tau)|^2 d\mu(\tau).
\end{equation}
The lemma now follows from (2.16), (2.17) and (2.18). \hfill \square

2.3. **Triple product L-functions.** Let $f \in \mathcal{B}_q$ and $f_K$ be its $L^2$-normalized base change to $K$. Let $g \in \mathcal{B}_q \cup \mathcal{B}_1$. By a result of Krishnamurthy [Kri03], the Asai transfer $\pi_K$ of $\pi_{\mathbb{C}}$ is an isobaric automorphic representation of $GL_4(\mathbb{A})$. We write $L(s, As f_K)$ for the associated automorphic $L$-function. The twisted triple product $L$-function $L(s, As f_K \times g)$ [PSR87], [Ike92] associated to $f_K$ and $g$ has the factorization
\[
L(s, As f_K \times g) = L(s, Ad^2 f \times g)L(s, g \times \pi_K).
\]
Observe that if the GL$_2$ $L$-function in our case is twisted by a quadratic character while in [Blo13, Section 3] its not. We recall the complete $L$-functions and the functional equations for these $L$-functions. Let $\epsilon \in \{0, 1\}$ such that $\epsilon \equiv \delta_\mathcal{O} + \delta_D \pmod{2}$. One has
\begin{equation}
L_{\infty}(s, g \times \pi_K) = \Gamma_\mathbb{R}(s + it_\mathcal{O} + \epsilon)\Gamma_\mathbb{R}(s - it_\mathcal{O} + \epsilon),
\end{equation}
\[
\Lambda(s, g \times \pi_K) = L_{\infty}(s, g \times \pi_K)L(s, g \times \pi_K) = (-1)^{\delta_D} w_g(q^c D^2)^{1/2-s} \Lambda(1-s, g \times \pi_K),
\]
where $c = 0$ or $1$, $\delta_D$ is defined in [Blo13, page 1833]. In particular, we have
\[
\Lambda(s, Ad^2 f \times g) = L_{\infty}(s, Ad^2 f \times g)L(s, Ad^2 f \times g) = (-1)^{\delta_D} (q^c D^2)^{1/2-s} \Lambda(1-s, Ad^2 f \times g).
\]
Notice that $L(s, Ad^2 f \times g)$ and hence $L(s, As f_K \times g)$ vanishes at $s = 1/2$ if $g$ is odd. The series expansion of prime to $q$ part $L$-function $L^{(q)}(s, Ad^2 f \times g)$ is given by [Blo13 (3.1)], and we follow the same notation. Similarly the series expansion of $L(s, Ad^2 f \times g)$ is given by [Blo13 (3.2)].

As in [Blo13, page 1833] and [Blo12, page 1392], we need a special type of approximate functional equation for the $L$-functions. More precisely, let $G(u)$ and $G_2(u, t_1, t_2)$ be defined as in [Blo13, page 1833]. On the other hand, we replace $G_1(u, t)$ by
\[
G_1(u, t) = \prod_{\epsilon_1, \epsilon_2 \in \{\pm\}} \prod_{\ell = 0}^A \left( 1 + \epsilon_1 u + \epsilon_2 it + \delta_D + \ell \right).
\]
Here $A \geq 10$ is an integer. Properties of these functions can be found in [Blo13 page 1833]. In particular, we have
\begin{equation}
G_1(0, t), G_2(0, t_1, t_2) \gg 1 \quad \text{for} \quad t, t_1, t_2 \in \mathcal{T}.
\end{equation}

For this lower bound, we need any nontrivial bound towards the Ramanujan conjecture for the infinite place of the GL$_6$ cusp form $Ad^2 f \times g$ [Kim03, LL00]. Let $V_1(y; t)$ and $V_2(y; t_1, t_2)$ be the weight functions defined in the same page. Then [Blo13 Lemma 1] holds for $V_1$ and $V_2$ except the equation (3.5) in that lemma; however, we do not need this property (see 2.5.1).

The proof of the following lemma is standard [IK04 page 98], [Blo12 Section 2].

**Lemma 2.2.** Let $g \in \mathcal{B}_q$ be even. We have

\[
G_1(0, t_g)L(1/2, g \times \pi_K) = (1 - \lambda_g(q)^{1/2}) \sum_{n=1}^\infty \frac{\lambda_g(n)\tau_K(n)}{n^{1/2}} V_1 \left( \frac{n}{q^{1/2} |D|}; t_g \right)
\]
and
\[
G_2(0, t_g, t_f)L(1/2, Ad^2 f \times g) = 2 \sum_{m=0}^\infty \frac{\lambda_{Ad^2 f \times g}(m)}{m^{1/2}} V_2 \left( \frac{m}{q^2}; t_g, t_f \right).
\]
2.4. Parseval’s identity. Let \( f \in \mathcal{B}_0 \) and \( f_K \) be the associated \( L^2 \)-normalized base change to \( K \). From now on we fix \( T > 0 \) and we assume \( t_f \ll T \). The aim of this subsection and the next is to use special value formulae Theorem 1.1 and the Kuznetsov formula to transform the quantity of interest, \( \sum_{\tau} \| f_K\mathbf{g} \|_2^2 \), into character sums, following the methods in [Blo13 Section 5].

The spectrum of \( L^2(X_0(q)) \) consists of the constant function 1, Maass forms, and the Eisenstein series \( \mathcal{E}_\alpha(\tau, 1/2 + it) \) for \( \tau \in \mathfrak{H} \) and \( t \in \mathbb{R} \), corresponding to two \( \Gamma_0(q) \)-equivalent classes of cusp \( \alpha = 0, \infty \). The definition and the Fourier expansion of \( \mathcal{E}_\alpha(\tau, 1/2 + it) \) can be found in [Blo13 page 1830], [Flo09 page 1187-1188]. The Parseval’s identity [Ko13 page 391] gives:

\[
(2.22) \quad \| f_K\mathbf{g} \|_2^2 = \text{Vol}(X_0(q))^{-1} |(f_K\mathbf{g}, 1)|^2 + \sum_{\mathbf{g} \in \mathcal{B}} |(f_K\mathbf{g}, \mathbf{g})|^2 + \sum_{\mathbf{g} \in \mathcal{B}} \frac{1}{4\pi} \int |(f_K\mathbf{g}, \mathcal{E}_{\alpha}(\cdot, 1/2 + it))|^2 dt.
\]

Here \( \text{Vol}(X_0(q)) = \frac{1}{24\pi}(q + 1) \) is the volume of \( X_0(q) \). Notice that by a result of Flicker [Fil88 Section 5], we have

\[
(2.23) \quad (f_{K}\mathbf{g}, 1) = 0.
\]

Also observe that \( |(f_K\mathbf{g}, \mathbf{g})|^2 = 0 \) when \( g \) is odd. Since it relates to the central value \( L(1/2, Asf_K \times g) \) by Theorem 4.2 and we have remarked that the \( L \)-function \( L(s, Asf_K \times g) \) vanishes at \( s = 1/2 \) when \( g \) is odd in the previous subsection. In the rest of this subsection, we show that

\[
(2.24) \quad \| f_K\mathbf{g} \|_2^2 \ll_{\mathcal{D}, T, \varepsilon} \sum_{\mathbf{g} \in \mathcal{B}_1, \mathbf{g} \text{ even}} |(f_K\mathbf{g}, \mathbf{g})|^2 + q^{-1+\varepsilon}.
\]

The arguments; however, are different from that of [Blo13 page 1837-1838].

2.4.1. Contributions of oldforms. To show that \( \sum_{\mathbf{g} \in \mathcal{B}_1, \mathbf{g} \text{ even}} |(f_K\mathbf{g}, \mathbf{g})|^2 \ll_{\mathcal{D}, T, \varepsilon} q^{-1+\varepsilon} \), one may either follow the argument in [Blo13 page 1837], or one can argue as follows. Let \( \mathbf{g} \in \mathcal{B}_1 \). By the funtoriality of the adjoint square lift [GJ78] and the nonexistence of Siegel zeros [Ban99], we have

\[
(2.25) \quad L(1, \text{Ad}^2 f_K) \gg_{\mathcal{D}, T, \varepsilon} q^{-\varepsilon} \quad \text{and} \quad L(1, \text{Ad}^2 g) \gg_{\varepsilon} (1/4 + t_g^2)^{-\varepsilon}
\]

according to [Mo99 Proposition 2.2.2 (iii)] and [HL14]. On the other hand, its a deep result of [KS02] that \( \text{Ad}^2 f \times g \) corresponds to an cusp forms on \( \text{GL}_n \) and hence we have the convexity bound [Ko13 Theorem 5.41]

\[
L(1/2, Asf_K \times g) = L(1/2, \text{Ad}^2 f \times g) L(1/2, g \times \tau_K) \ll_{\mathcal{D}, \varepsilon} q^{1+\varepsilon}(4 + 2T + |t_g|)^{1+\varepsilon}.
\]

As the implicit constants are independence of \( g \), we find by Theorem 1.1 and (2.25) that

\[
(2.26) \quad \sum_{\mathbf{g} \in \mathcal{B}_1} |(f_K\mathbf{g}, \mathbf{g})|^2 \ll_{\mathcal{D}, T, \varepsilon} q^{-1+\varepsilon} \sum_{\mathbf{g} \in \mathcal{B}_1} (1 + t_g^2)^{\varepsilon} (4 + 2T + |t_g|)^{2+\varepsilon} \frac{L(1/2, \text{Ad}^2 f \times g) L(1/2, g \times \tau_K)}{L_\infty(1, \text{Ad}^2 g)} \ll_{\mathcal{D}, T, \varepsilon} q^{-1+\varepsilon}.
\]

Here we have used the fact \( |\Gamma(x + iy)| \sim \sqrt{2\pi}|y|^{-1/2}e^{-\pi|y|/2} \) when \( |y| \to \infty \). Recall that for \( \mathbf{g} \in \mathcal{B}_1 \), we have defined \( g^* \in L^2(X_0(q)) \) by (2.4), and \( \mathcal{B}_1^* = \{ g^* | \mathbf{g} \in \mathcal{B}_1 \} \). Let \( g'(\tau) := g(\sigma_0 \cdot \tau) = g(q\tau) \). By (2.9) and changing the variable, we see that \( |\langle f_K|\mathbf{g}, g^* \rangle| = |\langle f_K|\mathbf{g}, g \rangle| \), and hence the formula in Theorem 1.1 remains the same if we replaced \( g \) by \( g' \) f. Because \( 1 - \frac{q_g|q_g|^2}{(q_g + 1)^2} \) \( \approx 1 \) we have

\[
(2.27) \quad \sum_{\mathbf{g} \in \mathcal{B}_1} |(f_K|\mathbf{g}, g^*)|^2 \ll 2 \sum_{\mathbf{g} \in \mathcal{B}_1} |(f_K|\mathbf{g}, g)|^2 + 2 \sum_{\mathbf{g} \in \mathcal{B}_1} |(f_K|\mathbf{g}, g')|^2 \ll_{\mathcal{D}, T, \varepsilon} q^{-1+\varepsilon}
\]

by (2.26). Therefore we find that \( \sum_{\mathbf{g} \in \mathcal{B}_1, \mathbf{g} \text{ even}} |(f_K|\mathbf{g}, g)|^2 \ll_{\mathcal{D}, T, \varepsilon} q^{-1+\varepsilon} \) as desired.

2.4.2. Contributions of Eisenstein series. In [Blo13 page 1838], the contribution of Eisenstein series relates to the Rankin-Selberg \( L \)-function \( L(s, f \times \bar{f}) \). In our case; however, it relates to the Asai \( L \)-function \( L(s, Asf_K) \). First we observe that by (2.9) and the fact that \( \sigma_0 \) is a scaling matrix for the cusp \( \alpha = 0 \), the contribution of the two cusps is the same. Therefore it suffices to consider the contribution from the \( \infty \) cusp. Set

\[
\Gamma_\mathcal{D}(s) = \begin{cases} \frac{2\pi s^{1/2}}{\Gamma(s)} \Gamma(s/2 + it_f)\Gamma(s/2 - it_f) & \text{if } \mathcal{D} > 0, \\ 2^{-s/2} \pi^{-s/2} \Gamma(s/2 + it_f)\Gamma(s/2 - it_f) & \text{if } \mathcal{D} < 0. \end{cases}
\]
Lemma 2.3. We have

\[ \langle f, \sigma, E_\infty(\cdot, s) \rangle = 2 \rho_{f, \mathcal{O}} |D|^{(s-1)/2} \Gamma^D(s) \cdot \frac{\zeta_{Q_0}(2s)}{L(s, \tau_{q})} \cdot \frac{L(s, \text{As}f, \mathcal{K})}{\zeta_Q(2s)}. \]

Here \( \tau_q \) is the quadratic character of \( \mathbb{Q}_q^\times \) associated to \( \mathcal{K}_q/\mathbb{Q}_q \) by the local class field theory.

Proof. Suppose \( D < 0 \). By the Fourier expansion (2.10) of \( f, \mathcal{K} \) and the fact that \( \lambda_{f, \mathcal{K}}(\alpha) \in \mathbb{R} \) for all \( 0 \neq \alpha \in \mathcal{O}, \) one has \( (\tau = x + iy) \)

\[ \int_{\mathcal{O}(q)} f(x, \sigma, E_\infty(\cdot, s)) d\mu(\tau) = \int_{\mathfrak{m}(\mathcal{O})} f(x, \sigma, E_\infty(\cdot, s)) \frac{dx}{2\pi i} \]

\[ = \rho_{f, \mathcal{O}} \sum_{0 \neq \alpha} \lambda_{f, \mathcal{K}}(\alpha) \int \infty_0 K_{2it}(2\pi|\alpha|^{1/2}D^{-1/2}y)y^{-1}dy \int_0^1 e^{2\pi i|D|^{1/2}y} d\alpha \]

\[ = 2 \rho_{f, \mathcal{O}} |D|^{(s-1)/2} \Gamma^D(s) \sum_{0 \neq n \in \mathbb{Z}} \frac{\lambda_{f, \mathcal{K}}(n)}{n^s} = 2 \rho_{f, \mathcal{O}} |D|^{(s-1)/2} \Gamma^D(s) \cdot \frac{\zeta_{Q_0}(2s)}{L(s, \tau_{q})} \cdot \frac{L(s, \text{As}f, \mathcal{K})}{\zeta_Q(2s)}. \]

The last equality follows from [Asa77, Theorem 2] and the explicit shape \( L_q(s, \text{As}f, \mathcal{K}) = \zeta_{Q_0}(s + 1)L(s, \tau_{q}) \) of the \( L \)-factor at \( q \). The argument for the case \( D > 0 \) is similar. This completes the proof. \( \square \)

Note that \( L(s, \text{As}f, \mathcal{K}) \) has the factorization:

\[ L(s, \text{As}f, \mathcal{K}) = L(s, \text{Ad}^2 f)L(s, \tau_{\mathcal{K}}). \]

By Lemma 2.1 Lemma 2.3 and the convexity bound [IK04, Theorem 5.41] of \( L(s, \text{Ad}^2 f) \) at \( s = 1/2 + it \) for \( t \in \mathbb{R} \) together with the lower bound (2.25) of \( L(1, \text{Ad}^2 f, \mathcal{K}) \), we obtain

\[ \int_\mathbb{R} |\langle f, \sigma, E_\infty(\cdot, 1/2 + it) \rangle|^2 dt \ll_{D,T} q^{-2+\epsilon} \int_\mathbb{R} |\Gamma^{D}(1/2 + it)|^2 \left| \frac{L(1/2 + it, \text{As}f, \mathcal{K})}{\zeta_{Q_0}(1/2 + it)} \right|^2 dt \ll_{D,T} q^{-1+\epsilon}. \]

Now [2.24] follows from (2.23), (2.27) and (2.28).

2.5. The Main Terms. Recall the Weyl’s law \( \{ f \in \mathcal{B}_q \mid |t_f| \leq T \} \ll qT^2 \). It follows from (2.24) that

\[ \sum_{f \in \mathcal{B}_q \atop |t_f| \leq T} \sum_{g \in \mathcal{B}_q \atop g \text{ even}} |\langle f, \sigma, g \rangle|^2 \ll_{D,T} q^t. \]

To complete the proof of Theorem 1.2, we need to show

\[ \sum_{f \in \mathcal{B}_q \atop |t_f| \leq T} |\langle f, \sigma, g \rangle|^2 \ll_{D,T} q^t. \]

Let \( f, g \in \mathcal{B}_q \) with \( g \) even. Notice that \( L(1, \text{Ad}^2 f \times \tau_{\mathcal{K}}) > 0 \) since both \( L(1, \text{Ad}^2 f, \mathcal{K}) \) and \( L(1, \text{Ad}^2 f) \) are positive. Furthermore, we have a lower bound

\[ L(1, \text{Ad}^2 f \times \tau_{\mathcal{K}}) \gg_{D,T} q^{-t}. \]

By Theorem 1.1 and the factorization [2.13] and the lower bound just above, we find that

\[ \sum_{f \in \mathcal{B}_q \atop |t_f| \leq T} \sum_{g \in \mathcal{B}_q \atop g \text{ even}} |\langle f, \sigma, g \rangle|^2 \ll_{D,T} q^{-2+\epsilon} \sum_{f \in \mathcal{B}_q \atop |t_f| \leq T} \sum_{g \in \mathcal{B}_q \atop g \text{ even}} \frac{L(1/2, \text{As}f \times g)}{L(1, \text{Ad}^2 f)L(1, \text{Ad}^2 g)} q^{-2+\epsilon}. \]

\[ \ll_{D,T} q^{-2+\epsilon} \sum_{f \in \mathcal{B}_q \atop |t_f| \leq T} \sum_{g \in \mathcal{B}_q \atop g \text{ even}} \frac{L(1/2, \text{As}f \times g)}{L(1, \text{Ad}^2 f)L(1, \text{Ad}^2 g)} G_1(0, t_g)G_2(0, t_g, t_f) h(t_g). \]

1The term \( e^{-2g|t_f|} \) in [Blo13 page 1838] should be replaced by \( e^{-\pi|t_f|} \); however, this does not affect the proofs.
Here we have inserted artificially the factor \( G_1(0, t_g)G_2(0, t_g, t_f) \) by the positivity \(^{2.21}\) and the nonnegativity of \( L(1/2, Asf_K \times g) \). We also replaced the weight function \( e^{-\pi |t|} \) by the function

\[
    h(t) := \cosh \left( \frac{t}{2A} \right)^{-2\pi A} \prod_{\ell=0}^A \left( t^2 + \left( \frac{1}{2} + \ell \right)^2 \right).
\]

Notice that this function is holomorphic in \( |\text{Im}(t)| < \pi A \) and has zeros at that of \( \prod_{\ell=0}^A \left( t^2 + \left( \frac{1}{2} + \ell \right)^2 \right) \) in this region. Moreover, \( h(t) \gg e^{-\pi |t|} \) for \( t \in \mathcal{T} \). Applying Lemma \(^{2.22}\) we arrive

\[
    (2.30) \quad \sum_{f \in \mathcal{B}_q} \sum_{g \in \mathcal{B}_q, \text{ even}} |\langle f, k \rangle | q^2 \ll_{D, T, \epsilon} q' \sum_{f \in \mathcal{B}_q} q' L(1, Ad^2 f) \sum_{g \in \mathcal{B}_q, \text{ even}} q' L(1, Ad^2 g) (1 - \lambda_g(q)q^{1/2}) S
\]

where \( q' = q^2/(q - 1) \gg q \) and

\[
    S = \sum_{n, nm = 0}^\infty \frac{\lambda_g(n) \tau_K(n) \lambda_{ad^2 f \times g}(m)}{(nm)^{1/2}} V_1 \left( \frac{n}{q^{1/2}|D|}; \xi \right) V_2 \left( \frac{m}{q^2}; \eta, t_g \right) \frac{(a^2 b^2 d^2 m k^2)}{q^2} ; t_g, t_f \right)
\]

Observe that the main difference between our case and the case of \(^{[Blo13] \text{ page 1838}}\) is we have additional factors \( \tau_K(n) \). Fortunately, this difference can be fixed so that the proofs in \(^{[Blo13]}\) work in our case. It will be convenient to remove the terms with \( q \mid nm \) in \( S \). In fact, as argued in \(^{[Blo13] \text{ page 1839}}\), the terms \( n, m \) with \( q \mid n \) and \( q \mid m \) in \( S \) contribute at most \( O(q^{-1/4+\epsilon}) \). Thus by \(^{[Blo13] \text{ (3.1)}}\) and \(^{(2.30)}\), we are left with estimating \( \Sigma(q, g) \) where

\[
    \Sigma_1(q, g) := \sum_{f \in \mathcal{B}_q} \frac{2h(t_f)}{q' L(1, Ad^2 f)} \sum_{g \in \mathcal{B}_q, \text{ even}} \frac{2h(t_g)}{q' L(1, Ad^2 g)} \sum_{q' b'dk} \frac{\mu(d)}{ab^2 d^3 k} \tau_K(n) \lambda_g(n) \lambda_{ad^2 f \times g}(m) \lambda_f(m^2) \lambda_f(m^2)
\]

and

\[
    \Sigma_2(q, g) := q^{1/2} \sum_{f \in \mathcal{B}_q} \frac{2h(t_f)}{q' L(1, Ad^2 f)} \sum_{g \in \mathcal{B}_q, \text{ even}} \frac{2h(t_g)}{q' L(1, Ad^2 g)} \sum_{q' b'dk} \frac{\mu(d)}{ab^2 d^3 k} \tau_K(n) \lambda_g(n) \lambda_{ad^2 f \times g}(m) \lambda_f(m^2) \lambda_f(m^2)
\]

We would like to apply the Kuznetsov formula \(^{[Blo13] \text{ (2.16)-(2.19)}}\) to the spectral sum over \( f \) and \( g \) in \( \Sigma_1(q, g) \) and \( \Sigma_2(q, g) \); however, since the terms involve oldforms and Eisenstein series are missing, we complete the formula by adding and subtracting the missing terms. This gives us \( 8 \) other quantities \( \Sigma(\ast, \ast) = \Sigma_1(\ast, \ast) - \Sigma_2(\ast, \ast) \) with \( \ast, \ast \in \{ q, 1, \mathcal{E} \} \) in analogy with \( \Sigma(q, g) = \Sigma_1(q, g) - \Sigma_2(q, g) \). For example, we define

\[
    \Sigma_1(1, \mathcal{E}) := \sum_{f \in \mathcal{B}_1} \frac{2h(t_f)}{(q + 1)L(1, Ad^2 f)} \int_R h(t) \sum_{q' b'dk} \frac{\mu(d)}{ab^2 d^3 k} \tau_K(n) \lambda_f(k^2) \lambda_f(m^2) \eta(n, t) \eta(a^2 dm, -t)
\]

and

\[
    \Sigma_2(1, \mathcal{E}) := \sum_{f \in \mathcal{B}_1} \frac{2h(t_f)}{(q + 1)L(1, Ad^2 f)} \int_R h(t) \sum_{q' b'dk} \frac{\mu(d)}{ab^2 d^3 k} \tau_K(n) c_2(f, q) \lambda_f(k^2) \lambda_f(m^2) \eta(n, t) \eta(a^2 dm, -t)
\]

and similarly for all the combinations. Here \( \eta(n, t) = \sum_{a \mid n} (a/d)^d \in \mathbb{R} \) are appeared in the Fourier coefficients of the Eisenstein series \(^{[Blo13] \text{ page 1830}}\) and \( q^2 = q^2/(q + 1) \). The terms \( c_1(f, q) , c_2(f, q) \) are
given by \cite[(2.11), (2.12)]{Blo13} with \( q \) replaced by \( f \). Since \( c_1(f, q) \asymp 1 \) and \( c_2(f, q) \ll 1 \), these terms certainly can be ignored in our estimations below. Now the Kuznetsov formula can be applied and we obtain

\[(2.31) \quad \Sigma(q, q) = \sum_{(q, q) \neq (1, 1)} (\Sigma_1(\ast, ?) - \Sigma_2(\ast, ?)) + \sum_{q|abcdmn} \frac{\mu(d)\tau_C(n)}{(nm)^{1/2}ab^2d^3/2k} \sum_{\alpha, \beta, \gamma \in \{1, 2\}} M^2_{\alpha, \gamma} \]

where the terms \( M^2_{\alpha, \gamma} \) are exactly those in \cite[page 1840-1841]{Blo13} with \( \frac{\mu(q)}{q^{1/2}} \) replaced by \( \frac{\mu(q)}{q^{1/2}|D|} \). In the rest of this subsection, we show that the terms on the RHS of (2.31) are all \( O(q^\epsilon) \) so that (2.29) holds.

2.5.1. \textit{Contributions from oldforms and Eisenstein series}. We consider the terms \( \Sigma_1(\ast, ?) \) and \( \Sigma_2(\ast, ?) \) on the RHS of (2.31). As in \cite[Section 6]{Blo13}, the idea is to use inverse Mellin transforms and the convexity of \( L \)-functions. We consider the cases \( \Sigma_1(E, q) \) and \( \Sigma_2(1, E) \) as the other cases require only notational changes.

By an inverse Mellin transform, we find that (see the remark in \cite[page 1833]{Blo13})

\[\Sigma_1(E, q) = \int_{\Re(t_f) > 0} \frac{h(t)}{q'(t_f)} \frac{1}{(1 + 2it)^2} \sum_{g \in B_q} \frac{2h(t_g)}{q(t_g)^{1/2}} \int_{2 - i\infty}^{2 + i\infty} \int_{2 - i\infty}^{2 + i\infty} L(q)^{(1/2 + \epsilon_2it + u, g)} \times \prod_{\epsilon_2 \in \{\pm\}} L(q)^{(1/2 + \epsilon_2it + v, g \times \tau_C)} \prod_{\epsilon_2 \in \{\pm\}} L(q)^{(1/2 + \epsilon_2it + u, Ad^2 f)} \int_{2 - i\infty}^{2 + i\infty} \int_{2 - i\infty}^{2 + i\infty} L(q)^{(1/2 + \epsilon_2it + v, g \times \tau_C)} \prod_{\epsilon_2 \in \{\pm\}} L(q)^{(1/2 + \epsilon_2it + u, Ad^2 f)} \]

We shift both contours to \( \Re(u) = \Re(v) = \epsilon \) and use the convexity bound \cite[Theorem 5.41]{IK} of \( L(s, g) \) and \( L(s, g \times \tau_C) \) in \( \Re(s) > 1/2 \) together with \cite[Lemma 1]{Blo13} to deduce \( \Sigma_1(E, q) \ll_{D, \epsilon} q^{3/4 + \epsilon} \). Similar argument shows

\[\Sigma_2(1, E) = \sum_{f \in B_1} \frac{2c_2(f, q)h(t_f)}{(q + 1)L(1, Ad^2 f)} \int_{\Re(t_f) > 0} \frac{h(t)}{q'(t_f)} \frac{1}{(1 + 2it)^2} \sum_{g \in B_q} \frac{2h(t_g)}{q(t_g)^{1/2}} \int_{2 - i\infty}^{2 + i\infty} \int_{2 - i\infty}^{2 + i\infty} L(q)^{(1/2 + \epsilon_2it + u, Ad^2 f)} \times \prod_{\epsilon_2 \in \{\pm\}} L(q)^{(1/2 + \epsilon_2it + v, g \times \tau_C)} \prod_{\epsilon_2 \in \{\pm\}} L(q)^{(1/2 + \epsilon_2it + u, Ad^2 f)} \]

We also shift both contours to \( \Re(u) = \Re(v) = \epsilon \) to deduce \( \Sigma_2(1, E) \ll_{D, \epsilon} q^{5/2 + \epsilon} \). Observe that unlike the case given in \cite[Section 6]{Blo13}, which he need \( \tilde{V}_1(1/2 \pm it, t) = 0 \) to kill poles of the Riemann zeta function, the \( L \)-function \( L(s, \tau_C) \) in our case is entire. Therefore the above argument still works in our case even our \( V_1 \) does not satisfy \( V_1(1/2 \pm it, t) = 0 \) when \( D < 0 \).

2.5.2. \textit{Estimating character sums}. We estimate the terms \( M^2_{\alpha, \gamma} \) on the RHS of (2.31). We explain how to modify the proofs in \cite[Section 8]{Blo13} so that it can be applied to our case. The bound

\[\sum_{q|abcdmn} \frac{1}{(nm)^{1/2}ab^2d^3/2k} \bigl(|M_{1,1}^{2,1}| + |M_{1,1}^{1,2}| + |M_{1,2}^{2,1}|\bigr) \ll_{D, \epsilon} q^\epsilon \]

in \cite[page 1845]{Blo13} certainly holds in our case. For the terms \( M_{1,1}^{2,2}, M_{1,2}^{1,2} \) and \( M_{2,2}^{2,2} \), the idea is to replace \( \tau_C \) by any smooth function \( \tau_C^\ast \) on \( \mathbb{R} \) such that \( \tau_C^\ast(n) \equiv \tau_C(n) \) for every \( n \in \mathbb{Z} \). Then the proofs in \cite[Section 8]{Blo13} work in our case. We illustrate this by considering the term \( M_{1,2}^{2,2} \). Recall that we need to estimate

\[\sum_{q|abcdmn} \frac{\mu(d)\tau_C(n)M_{1,2}^{2,2}}{ab^2d^3/2m^3/2n^{1/2}} = q^{1/2} \sum_{q|abcdmn} \sum_{q \in \mathcal{Q}} \frac{\mu(d)\tau_C(n)S(qn, a^2dmq, c)}{ab^2cd^3/2m^3/2n^{1/2}} \mathcal{W}_{1,2}

\]

As argued in \cite[page 1845]{Blo13}, we need to bound

\[\frac{1}{q^{1/2}N^{1/2}} \sum_{q|abcdmn} \frac{w_2(a/C)w_3(c/C)}{ab^2cd^3/2m^3/2} \sum_n S(qn, a^2dmq, c)\tau_C(n)w_1(n/N)\mathcal{W}_{1,2}

\]

where \( w_1, w_2 \) and \( w_3 \) are smooth weight functions with supports in \([1, 2]\) and we assume

\[a\sqrt{dmq} \leq q^{7/4 + \epsilon}, \quad A \leq \frac{q^{1+\epsilon}}{b^2(dm)^{3/2}}, \quad C \leq \frac{A\sqrt{dmN}}{q^{1/2-\epsilon}}\]
by the decay property [Blo13, Lemma 5] of \( W^{1,2} \). By [Blo13, Lemma 2] with \( \alpha = a^2 dmq \) and \( \gamma = c \), the n-sum is

\[
\leq \sum_{h \neq 0} \left| \int_0^\infty w_1^* \left( \frac{x}{N} \right) W^{1,2} \left( \frac{x}{\sqrt{1/2|D|}} \frac{a^2b^2d^2m^3}{q^2} \frac{a\sqrt{dmx}}{cq^{1/2}} \right) \exp \left( -2\pi i \frac{xh}{c} \right) dx \right|.
\]

Here \( w_1(x) := w_1(x)\tau_E^x(Qx) \) is again a smooth function with support in \([1, 2]\). Therefore the bound [Blo13 (7.4)] is still applicable (with \( \rho_1 = \frac{1}{q^{1/4|D|}} \)) and the rest of the proof is identical to that of [Blo13 page 1846].

The modifications of the proofs for the terms \( M_1^{1,2} \) and \( M_2^{1,2} \) are similar. In this way, the proof of (2.29) and hence Theorem 1.2 is completed.

3. Special value formulae

In this section, we will state our special value formulae of the triple product \( L \)-functions in terms of adelic language. To avoid drowning the reader in notation, we restrict our attention to the case \( F = \mathbb{Q} \). The case \( F \) is a totally real number field was treated by the author in his thesis [Che18b].

3.1. Notation and definitions. Let \( F \) be a number field or a local field of characteristic zero. If \( w \) is a place of a number field \( F \) and \( R \) is a \( F \)-algebra, let \( R_w = R \otimes_F F_w \).

3.1.1. Quaternion algebras. Let \( D \) be a quaternion algebra over \( F \). We denote \( D(R) = D \otimes_F R \). Therefore \( D = D(F) \). When \( F \) is nonarchimedean and \( D \) is division, we write \( \mathcal{O}_D \) for the maximal order in \( D \). If \( F \) is a number field, let \( \Sigma_D \) denote the ramification set of \( D \) and we define a positive integer \( C_D \) associated to \( D \) by

\[
C_D = \prod_{w \in \Sigma_D, w < \infty} (q_{F_w} - 1).
\]

We understand that \( C_{M_2} = 1 \). More generally, if \( R = F_1 \times F_2 \times \cdots \times F_r \), where \( F_j \) are finite extensions over \( F \) for \( 1 \leq j \leq r \), we set

\[
C_{D(R)} = \prod_{j=1}^r C_{D(F_j)}.
\]

3.1.2. Étale cubic algebras. Let \( E \) be an étale cubic \( F \)-algebra. Then we have (i) \( E \) is a cubic extension over \( F \); (ii) \( E = F' \times F \) for some quadratic extension \( F' \) over \( F \); (iii) \( E = F \times F \times F \). Define \( c_E = 1, 2 \) and \( 3 \) if \( E \) is of the form (i), (ii) and (iii), respectively. Denote by \( \mathcal{O}_E \) the maximal order of \( E \) when \( F \) is a number field or a nonarchimedean local field. If \( F \) is a local field and \( \Pi \) is an irreducible admissible generic representation of \( \text{GL}_2(E) \), then \( \Pi = \pi' \boxtimes \pi \) when \( c_E = 2 \) and \( \Pi = \pi_1 \boxtimes \pi_2 \boxtimes \pi_3 \) when \( c_E = 3 \), where \( \pi' \) is an irreducible admissible generic representation of \( \text{GL}_2(F') \) and \( \pi, \pi_1, \pi_2 \) and \( \pi_3 \) are irreducible admissible generic representations of \( \text{GL}_2(F) \). Suppose the central character of \( \Pi \) is trivial on \( F^\times \). We define the local root number \( \epsilon(\Pi) \in \{ \pm 1 \} \) associated to \( \Pi \) by

\[
\epsilon(\Pi) = 1 \iff \text{Hom}_{\text{GL}_2(F)}(\Pi, \mathbb{C}) \neq \{ 0 \}.
\]

According to the results of [Pra90], [Pra92] and [Lok01], the Hom space is at most one-dimensional, and if \( \epsilon(\Pi) = -1 \), then the Jacquet-Langlands lift \( \Pi^D \) of \( \Pi \) to the unique quaternion division \( F \)-algebra \( D \) is nonzero, and moreover, there is a unique nonzero \( D^\times \)-invariant form on \( \Pi^D \).

Suppose \( F \) is nonarchimedean and the central character of \( \Pi \) is trivial. The conductor \( \epsilon_H \) of \( \Pi \) is the unique ideal [Cas73] \( \epsilon_H \subset \mathcal{O}_E \) such that

\[
\dim_{\mathbb{C}} \Pi^D_{\epsilon(\Pi)} = 1.
\]

We say \( \epsilon_H \) is square free if (i) \( c_E = 1 \) and \( \epsilon_H = \tau_E^a \mathcal{O}_E \) with \( a \leq 1 \); (ii) \( c_E = 2 \) and \( \epsilon_H = (\tau_E^a, \tau_E^b) \mathcal{O}_E \) with \( a, b \leq 1 \); (iii) \( c_E = 3 \) and \( \epsilon_H = (\tau_E^a, \tau_E^b, \tau_E^c) \mathcal{O}_E \) with \( a, b, c \leq 1 \).
3.1.3. Algebraic representations of $GL_2(\mathbb{C})$. We write $\rho_n$ for the $n$-th symmetric power of the standard two-dimensional representation of $GL_2(\mathbb{C})$. We realize $\rho_n$ as $(\rho, L_n(\mathbb{C}))$ with

$$L_n(\mathbb{C}) = \bigoplus_{j=0}^{n} CX^jY^{n-j}$$

and $\rho(g)P(X,Y) = P((X,Y)g)$.

for $P(X,Y) \in L_n(\mathbb{C})$ and $g \in GL_2(\mathbb{C})$. Let $\langle \cdot, \cdot \rangle_n : L_n(\mathbb{C}) \times L_n(\mathbb{C}) \to \mathbb{C}$ be the non-degenerated bilinear pairing on $L_n(\mathbb{C})$ defined by

$$\langle X^jY^{n-j}, X^lY^{n-l} \rangle_n = \begin{cases} (-1)^j \binom{n}{j}^{-1} & \text{if } j + l = n, \\ 0 & \text{if } j + l \neq n, \end{cases}$$

One check that $\langle \rho(g)P(X,Y), \rho(g)Q(X,Y) \rangle_n = \det^n(g)\langle P(X,Y), Q(X,Y) \rangle_n$, for $g \in GL_2(\mathbb{C})$ and $P, Q \in L_n(\mathbb{C})$. In particular, $\langle \cdot, \cdot \rangle_n$ defines a $SU(2)$-invariant bilinear pairing on $L_n(\mathbb{C})$. We also let $\langle \cdot, \cdot \rangle : L_n(\mathbb{C}) \times L_n(\mathbb{C}) \to \mathbb{C}$ be the $SU(2)$-invariant hermitian pairing given by

$$\langle P, Q \rangle_n = \langle P, \rho\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \overline{Q} \rangle_n$$

where $\overline{Q} = \sum_{j=0}^{n} c_j X^jY^{n-j}$ if $Q = \sum_{j=0}^{n} c_j X^jY^{n-j}$.

In the rest of this section, we assume $E$ is an étale cubic $\mathbb{Q}$-algebra, and we write $A_E = \mathbb{A} \otimes \mathbb{Q} E$. As in the previous section, we use $v$ to indicate a place of $\mathbb{Q}$ and $p$ (resp. $\infty$) to denote the finite place (resp. the real place) of $\mathbb{Q}$ corresponds to a prime $p$. We call the case $E_\infty \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ the real case, and the case $E_\infty \cong \mathbb{C} \times \mathbb{R}$ the complex case.

3.2. Global settings and assumptions. Let $\Pi = \otimes'_{\nu} \Pi_\nu$ be a unitary irreducible cuspidal automorphic representation of $GL_2(A_E)$ with trivial central character. Here $\Pi_\nu$ is a unitary irreducible admissible generic representation of $GL_2(E_v)$. Note that $\Pi_\nu = \pi_\nu \boxtimes \pi_\nu$ when $c_{E_v} = 2$ and $\Pi_\nu = \pi_{1,v} \boxtimes \pi_{2,v} \boxtimes \pi_{3,v}$ when $c_{E_v} = 3$.

3.2.1. Assumptions. We make the following assumptions on $\Pi$ throughout this section.

- We assume $\pi_\infty$ and one of $\pi_{j,\infty}$ ($j = 1, 2, 3$) is a principal series representation of $GL_2(\mathbb{R})$.
- We assume $c_{\Pi_\nu}$ is square free for all $\nu$.
- We assume the global root number $\epsilon(\Pi) := \prod_\nu \epsilon(\Pi_\nu) = 1$, where $\nu$ runs over all places of $\mathbb{Q}$.

Remark 3.1. By the results of [Pra90] and [Pra92], the local root number $\epsilon(\Pi_\nu) = 1$ for almost all $\nu$. Therefore the infinite product is defined. Notice that our first assumption implies $\epsilon(\Pi_\infty) = 1$ [Pra90 Theorem 9.5], [Lok01 Theorem 1.3]. The real case with $\pi_{1,\infty} \boxtimes \pi_{2,\infty} \boxtimes \pi_{3,\infty}$ are (limit of) discrete series representations of $GL_2(\mathbb{R})$ is treated in [CCT18]. The complex case with $\pi_\infty$ is a discrete series is treated in [Che18a].

3.2.2. The quaternion algebra. Our assumption on the global root number determines a unique indefinite quaternion $\mathbb{Q}$-algebra. Precisely, let $D$ be the quaternion algebra over $\mathbb{Q}$ such that

$$p \in \Sigma_D \iff \epsilon(\Pi_\nu) = -1.$$
3.3. Automorphic forms and raising elements. In this subsection, we introduce distinguished automorphic forms and raising elements which appear in the global period integrals of our special value formulae. By the results of [Pra90], [Pra92], [Lok01] and [GJ79], the global Jacquet-Langlands lift $\Pi^D = \otimes_v^\prime \Pi^D_v$ of $\Pi$ to $D^\times(\mathbb{A}_E)$ exists, where $\Pi^D_v$ is the local Jacquet-Langlands lift of $\Pi_v$ to $D^\times(\mathbb{E}_v)$ for every place $v$. Note that the central character of $\Pi^D$ is also trivial.

3.3.1. An Eichler order. Let $\mathcal{O}_{D(E)}$ be the maximal order of $D(E)$ (w.r.t. the isomorphisms $\nu_v$ for $v \notin \Sigma_D$). We define an Eichler order $\mathcal{R}_{\Pi^D} \subset \mathcal{O}_{D(E)}$ as follows.

- For $p \notin \Sigma_D$, we require
  \[ \mathcal{R}_{\Pi^D} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_p} \cong M_2(\mathcal{O}_p). \]

- For $p \in \Sigma_D$ with $c_{E_p} \neq 2$, we require
  \[ \mathcal{R}_{\Pi^D} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_p} \cong \mathcal{O}_{D(E_p)}. \]

Here $\mathcal{O}_{D(E_p)}$ is the maximal order of $D(E_p)$.

- For $p \in \Sigma_D$ and $c_{E_p} = 2$, we require
  \[ \mathcal{R}_{\Pi^D} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_p} \cong M_2(c_{E_p}) \times \mathcal{O}_{D_p}. \]

Here $c_{E_p}$ is the conductor of $\pi^D_p$.

3.3.2. Distinguished automorphic form: the real case. We fix a unique (up to nonzero constants) automorphic form $f^D \in \Pi^D$. We have $\Pi^\infty = \Pi^\infty_1 \boxtimes \pi_2 \boxtimes \pi_3$. Let $k_j \geq 0$ be the minimal $SO(2)$-type of $\pi_j$ for $j = 1, 2, 3$. For $b \in \mathbb{R}$, let

\[ k(b) = \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix} \in SO(2). \]

Let $f^D : D^\times(\mathbb{A}_E) \to \mathbb{C}$ be the unique automorphic form in $\Pi^D$ characterized by

\[ f^D(z\gamma h u_\infty u) = \omega_H(z)f^D(h)e^{i(k_1b_1+k_2b_2+k_3b_3)} \]

for $z \in \mathbb{A}_E^\times$, $\gamma \in D^\times(\mathbb{A}_E)$, $u \in \tilde{\mathcal{R}}^\times_{\Pi^D}$ and $u_\infty = (k(b_1), k(b_2), k(b_3)) \in SO(2) \times SO(2) \times SO(2)$.

3.3.3. Distinguished automorphic form: the complex case. For the complex case, we need to consider vector-valued automorphic forms. We have $\Pi^\infty = \Pi^\infty_1 \boxtimes \pi^\prime_2 \boxtimes \pi_3$. Let $\rho_{k'}$ (resp. $k \in \{0, 1\}$) be the minimal $SU(2)$-type (resp. $SO(2)$-type) of $\pi_{k'}$ (resp. $\pi_k$). Denote by $\bar{\rho}_{k'}$ the conjugate representation of $\rho_{k'}$. Elements in $\Pi^D \boxtimes \bar{\rho}_{k'}$ (as a $SU(2)$-module) can be regarded as $\mathcal{L}_{k'}(\mathbb{C})$-valued automorphic forms on $D^\times(\mathbb{A}_E)$. Fix a unique (up to nonzero constants) vector-valued automorphic form $\tilde{f}^D \in \Pi^D \boxtimes \bar{\rho}_{k'}$ such that

\[ f^D(z\gamma h u_\infty u) = \omega_H(z)\bar{\rho}_{k'}(u')^{-1}f^D(h)e^{ikb} \]

for $z \in \mathbb{A}_E^\times$, $\gamma \in D^\times(\mathbb{A}_E)$, $u \in \tilde{\mathcal{R}}^\times_{\Pi^D}$ and $u_\infty = (u', k(b)) \in SU(2) \times SO(2)$. We also need a scalar-valued automorphic form. For $0 \leq j \leq k'$, we set

\[ v_j = (X + iY)^j(X - iY)^{k'-j} \in \mathcal{L}_{k'}(\mathbb{C}). \]

Write $\pi_{\infty} = \text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})}(\mu_{\infty} \boxtimes \nu_{\infty})$ and $\pi'_{\infty} = \text{Ind}_{B(\mathbb{C})}^{GL_2(\mathbb{C})}(\mu'_{\infty} \boxtimes \nu'_{\infty})$, and let $\epsilon \in \{0, 1\}$ so that $\nu'_{\infty} \mu_{\infty}(-1) = (-1)^\epsilon$. Put

\[ m_{\Pi^\infty} = m = \frac{k' + k}{2} + \epsilon \frac{1 + (-1)^k}{2}. \]

The scalar-valued automorphic form $f^D \in \Pi^D$ is then defined by

\[ f^D(h) := (v_m, f^D(h))_{k'} \text{ for } h \in D^\times(\mathbb{A}_E). \]
3.3.4. **Raising element.** Let

\[
\tilde{V}_+ = \left( -\frac{1}{8\pi} \right) \cdot V_+ \quad \text{with} \quad V_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sqrt{-1} \in \mathbb{U}_\mathbb{R}
\]

be the (normalized) weight raising element [JL76] Lemma 5.6. We use $\mathbb{U}_F$ to denote the universal enveloping algebra of $\text{Lie}(\text{GL}_2(F)) \otimes \mathbb{R}$ where $F = \mathbb{R}$ or $\mathbb{C}$. In any case, we write $\text{Id}$ for the identity element in $\mathbb{U}_F$. We introduce an element $t = (t_v)_v$ as follows.

- Suppose we are in the real case. Let $k_1, k_2, k_3$ as in [3.3.2] Re-index if necessarily we may assume $k_1 \geq k_2 \geq k_3$. Note that $k_3 \leq 1$ by our assumption. If $k_1 \geq 2$, let $\ell = \frac{k_1 - k_2 - k_3}{2} \geq 0$ and we set

\[
t_{\infty} = (\text{Id} \otimes \text{Id} \otimes \tilde{V}_+, (I_2, I_2, J)).
\]

If $k_1 \leq 1$, then $\pi_{j, \infty} = \text{Ind}_{B(\mathbb{R})}^{\text{GL}_2(\mathbb{R})}(\mu_{j, \infty} \boxtimes \nu_{j, \infty})$ are principal series for $j = 1, 2, 3$. In this case, we set

\[
t_{\infty} = \begin{cases} (\text{Id} \otimes \text{Id} \otimes \text{Id}, (I_2, I_2, I_2)) & \text{if } k_1 = k_2 = k_3 = 0 \text{ and } \mu_{1, \infty} \mu_{2, \infty} \mu_{3, \infty}(-1) = 1, \\
(\text{Id} \otimes \tilde{V}_+ \otimes \tilde{V}_+, (I_2, I_2, J)) & \text{if } k_1 = k_2 = k_3 = 0 \text{ and } \mu_{1, \infty} \mu_{2, \infty} \mu_{3, \infty}(-1) = -1, \\
(\text{Id} \otimes \text{Id} \otimes \text{Id}, (I_2, I_2, J)) & \text{if } k_1 = k_3 = 1 \text{ and } k_2 = 0. 
\end{cases}
\]

- Suppose we are in the complex case. Let $\pi'_{\infty}, k'$ and $\pi_{\infty}, k$ as in [3.3.3] Define

\[
t_{\infty} = \left\{ \begin{array}{ll}
(\text{Id} \otimes \text{Id}, (I_2, I_2)) & \text{if } k' \geq 0 \text{ is even and } \mu_{\infty} \nu_{\infty}(-1) = 1, \\
(\text{Id} \otimes \tilde{V}_+, (I_2, J)) & \text{if } k' \geq 2 \text{ is even and } \mu_{\infty} \nu_{\infty}(-1) = -1, \\
(\text{Id} \otimes \text{Id}, (I_2, J)) & \text{if } k' \geq 0 \text{ is odd and } \mu_{\infty} \nu_{\infty}(-1) = 1, \\
(V \otimes \text{Id}, (I_2, I_2)) & \text{if } k' = 0 \text{ and } \mu_{\infty} \nu_{\infty}(-1) = -1,
\end{array} \right.
\]

where $V = \frac{1}{2\pi} \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix} \otimes \sqrt{-1} \in \text{Lie}(\text{GL}_2(\mathbb{C})) \otimes \mathbb{C}$ in the last case.

- Let $p$ such that $c_{E_p} = 1$ and $p \notin \Sigma_D$. If $E_p / \mathbb{Q}_p$ is ramified and $\Pi_p$ is a unramified special representation, put

\[
t_p = \begin{pmatrix} \infty_{E_p}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(E_p).
\]

For all other cases, let $t_p \in GL_2(E_p)$ be the identity element.

- Let $p$ such that $c_{E_p} = 2$ and $p \notin \Sigma_D$. If $F_p / \mathbb{Q}_p$ is ramified and one of $\pi'_p$, $\pi_p$ is unramified, while another is a unramified special representation, put

\[
t_p = \left\{ \begin{array}{ll}
\left( \begin{array}{cc} \infty_{E_p}^{-1} & 0 \\ 0 & 1 \end{array} \right), I_2 & \text{if } \pi'_p \text{ is unramified,} \\
I_2, \left( \begin{array}{cc} p^{-1} & 0 \\ 0 & 1 \end{array} \right) & \text{if } \pi_p \text{ is unramified.}
\end{array} \right.
\]

For all other cases, let $t_p \in GL_2(E_p)$ be the identity element.

- Let $p$ such that $c_{E_p} = 3$ and $p \notin \Sigma_D$. If $\pi_{1,p}, \pi_{2,p}$ are unramified and $\pi_{3,p}$ is a unramified special representation, put

\[
t_p = \left( I_2, \left( \begin{array}{cc} p^{-1} & 0 \\ 0 & 1 \end{array} \right), I_2 \right) \in \text{GL}_2(E_p).
\]

For all other cases, let $t_p \in GL_2(E_p)$ be the identity element.

- If $p \in \Sigma_D$, let $t_p \in D^\times(E_p)$ be the identity element.

3.4. **The formulae.** We are ready to state our special value formulae. To unify the statement, we also write $f^D = f^D$ in the real case. Define the norm $\mathcal{H}(f^D, f^D)$ of $f^D$ by

\[
\mathcal{H}(f^D, f^D) = \left\{ \begin{array}{ll}
\int_{A_E^\times \ldots (E)} |f^D(h)|^2 dh_{\text{Tam}} & \text{if } E_\infty = \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \\
\int_{A_E^\times \ldots (E)} \left| f^D(h) \right|^2 dh'_{\text{Tam}} & \text{if } E_\infty = \mathbb{C} \times \mathbb{R}.
\end{array} \right.
\]

Here $dh_{\text{Tam}}$ is the Tamagawa measure on $A_E^\times \ldots (A_E)$. Let $\nu(H)$ be a number of finite place $p$ such that

- $c_{E_p} = 1$ and $\Pi_p$ is a unramified special representation.
\begin{itemize}
  \item \(c_{E_p} = 2,\; F_p'/\mathbb{Q}_p\) is unramified and both \(\pi'_p\) and \(\pi_p\) are unramified special representations.
  \item \(c_{E_p} = 2,\; F_p'/\mathbb{Q}_p\) is ramified, \(\pi'_p\) is unramified and \(\pi_p\) is a unramified special representation.
  \item \(c_{E_p} = 3\) and all \(\pi_1,p,\; \pi_2,p,\; \pi_3,p\) are unramified special representations.
\end{itemize}

For each \(p\), we also define \(c_p = 0\) or 1 according to \(\epsilon_{I_p} = \mathcal{O}_{E_p}\) or \(\epsilon_{I_p} \subset \mathcal{O}_{E_p}\), respectively. Notice that \(c_p = 0\) for almost all \(p\). Let \(N := \prod_{p \notin \Sigma_D} p^{c_p}\) and \(R_N \subset \mathcal{O}_D\) be the Eichler order of level \(N\). We put

\[
C(\Pi) = \prod_p [\mathbb{Z}_p : p^{\nu_p} \mathbb{Z}_p] \times [\mathcal{O}_{E_p} : \epsilon_{I_p}]^{-1}.
\]

Notice that the RHS of (3.11) is 1 for almost all \(p\) so that the product is defined. Recall that \(C_D\) and \(C_D(E)\) are given by (6.1) and (6.2), respectively. Let \(L(s,\Pi, r)\) and \(L(s,\Pi, \text{Ad})\) be the triple product \(L\)-function and the adjoint \(L\)-function defined in [Ich08].

**Theorem 3.2.** We have

\[
\left| \int_{\mathbb{A}^\times \backslash D^\times(A) \mathbb{A}^\times} \Pi^D(t) f^D(h) dh^\text{Tam} \right|^2 = 2^{-c_E + \nu(\Pi)} C(\Pi) \cdot \frac{C_D(\Pi_{D(E)} : \hat{R}_N^\times)}{C_D(\hat{\mathcal{O}}_D : \hat{R}_N^\times)^2} \cdot I^*_{\text{her}}(\Pi_{\infty}, t_{\infty})
\]

\[
\times \frac{\xi_E(2)}{\xi_0(2)^2} \cdot \frac{L(1/2, \Pi, r)}{L(1, \Pi, \text{Ad})}.
\]

Here \(dh^\text{Tam}\) is the Tamagawa measure on \(\mathbb{A}^\times \backslash D^\times(A)\), and \(I^*_{\text{her}}(\Pi_{\infty}, t_{\infty})\) is given by Proposition 4.12 and Proposition 4.14, respectively, depending on \(E_{\infty}\) and \(\Pi_{\infty}\).

**Remark 3.3.** Observe that the LHS of (3.12) is independent of the choice of \(t^D\); however, in the complex case, it does depend on the choice of the hermitian pairing on \(\mathcal{L}_v(\mathbb{C})\) as well as the particular element \(v_m \in \mathcal{L}_v(\mathbb{C})\).

**Proof.** For each place \(v\) of \(\mathbb{Q}\), let \(dh_v\) be the Haar measure on \(F_v^\times \backslash D_v^\times\) given in (3.25) depending on \(\mathbb{Q}_v\) and \(D_v\). By [Ip] Lemma 6.1, we have \(dh^\text{Tam} = C_D^{-1} \xi_0(2)^{-1} \prod_v dh_v\). By Ichino’s formula [Ich08] Theorem 1.1 and Remark 1.3, we find that

\[
\left| \int_{\mathbb{A}^\times \backslash D^\times(A) \mathbb{A}^\times} \Pi^D(t) f^D(h) dh^\text{Tam} \right|^2 = 2^{-c_E} C_D^{-1} \cdot \frac{\xi_E(2)}{\xi_0(2)^2} \cdot \frac{L(1/2, \Pi, r)}{L(1, \Pi, \text{Ad})} \prod_v I^*_{\text{her}}(\Pi_{\infty}^D, t_v).
\]

Here \(I^*_{\text{her}}(\Pi_{\infty}^D, t_v)\) are the local period integrals defined in the next section. By Corollary [Ip] and the results in [CC18] Sections 4, 5, we have

\[
\prod_p I^*_{\text{her}}(\Pi_{\infty}^D, t_p) = 2^{\nu(\Pi)} C(\Pi) C_D^{-1} C_D(E) \left[ \hat{\mathcal{O}}_{D(E)} : \hat{R}_N^\times \right] \left[ \hat{\mathcal{O}}_D : \hat{R}_N^\times \right]^{-2}.
\]

This proves the theorem. \(\square\)

Following corollary is a direct consequence of Theorem 3.2.

**Corollary 3.4.** The central values \(L(1/2, \Pi, r)\) are nonnegative real numbers.

4. LOCAL CALCULATIONS

The purpose of this section is to compute the local period integrals \(I^*_{\text{her}}(\Pi_{\infty}^D, t_v)\) appeared in our special value formulae. For the ease of notation, we suppress the subscript \(v\). Also, since the calculations are valid for any local field, we assume \(F\) is a local field of characteristic zero throughout this section. Let \(E\) be an étale cubic \(F\)-algebra and \(D\) be the quaternion division algebra over \(F\). We follow the notation and the conventions in [3.1.2] and [3.2.2].

4.1. Definition of local period integrals. We define the local period integrals appeared in our special value formulae. Let \(\Pi\) be a unitary irreducible admissible generic representation of \(\text{GL}_2(E)\). We assume \(\omega_{\Pi}\) is trivial on \(F^\times\).
4.1.1. Assumptions. We make the following assumptions on $\Pi$ in the rest of this section.

- When $F$ is archimedean, we assume $F = \mathbb{R}$.
- When $F = \mathbb{R}$, we assume $\pi$ and one of $\pi_j$ ($j = 1, 2, 3$) is a principal series representation of $\text{GL}_2(\mathbb{R})$.
- When $F$ is nonarchimedean, we assume $\omega_H$ is trivial and $I$ is square free.
- We assume $\Lambda(I) < 1/2$.

Here $\Lambda(I)$ is the nonnegative real number associated to $I$ defined in [Ch08, page 285].

**Remark 4.1.** Observe that these assumptions come from our global assumptions [3.2.1] except the cases when $F = \mathbb{R}$, for which we merely assume $\omega_H$ is trivial on $\mathbb{R}^\times$.

Now we are in the position to define the local period integrals. We have two cases according to $\epsilon(I) = \pm 1$. Recall that $\epsilon(I)$ is characterized by [3.3.4].

4.1.2. Case $\epsilon(I) = 1$. In this case, we have $\text{Hom}_{\text{GL}_2(F) \times \text{GL}_2(F)}(\Pi \otimes \tilde{\Pi}, \mathbb{C}) \neq 0$. Notice that $\epsilon(I) = 1$ when $F = \mathbb{R}$ by our assumption [Pra90, Lok01]. Let $\mathcal{B}_H : \Pi \times \tilde{\Pi} \to \mathbb{C}$ (resp. $\mathcal{H}_H : \Pi \times \tilde{\Pi} \to \mathbb{C}$) be a nontrivial $\text{GL}_2(E)$-equivariant bilinear (resp. hermitian) pairing on $\Pi$. We first introduce elements $\phi_H$ as follows. Suppose $E = \mathbb{C} \times \mathbb{R}$ so that $\Pi = \pi' \boxtimes \pi$. Let $k' \geq 0$ (resp. $k = 0, 1$) be the minimal $\text{SU}(2)$-type (resp. $\text{SO}(2)$-type) of $\pi'$ (resp. $\pi$). Let $\phi_{\pi'}$ be a nonzero element in the one-dimensional space $(\pi' \boxtimes \rho_{k'})^\dagger$, and $\phi_\pi \in \pi$ be a nonzero weight $k$ element. Put $\phi_H = \phi_{\pi'} \otimes \phi_\pi$. Suppose $E$ is three copies of $\mathbb{R}$. Then $\Pi = \pi_1 \boxtimes \pi_2 \boxtimes \pi_3$. Let $k_j \geq 0$ be the minimal $\text{SO}(2)$-type of $\pi_j$, and $\phi_{\pi_j} \in \pi_j$ be a nonzero weight $k_j$ element for $j = 1, 2, 3$. Put $\phi_H = \phi_{\pi_1} \otimes \phi_{\pi_2} \otimes \phi_{\pi_3}$. Finally, suppose $F$ is nonarchimedean. Let $\phi_{\hat{H}}$ be a nonzero element in the one-dimensional space $\mathcal{H}(\mathbb{C})$. Similarly, we define a nonzero element $\phi_{\hat{H}}$ for $\tilde{\Pi}$.

We now define the local period integrals for the case $\epsilon(I) = 1$. Let $t$ be the elements attached to $\Pi$ given in [3.3.4] with obvious modifications. For example, $p$ is replaced by $\varpi_F$ and so on. When $E$ is three copies of $\mathbb{R}$, we may assume $k_1 \geq k_2 \geq k_3$ after re-index. When $E = \mathbb{C} \times \mathbb{R}$, let $v_m \in \mathcal{L}(\mathcal{E})$ be the element given by [3.8] and [3.9], and let $\varphi_H = \langle v_m, \phi_H \rangle_{K_v}$ and $\varphi_{\hat{H}} = \langle v_m, \phi_{\hat{H}} \rangle_{K_v}$. Put $\mathcal{B}_{\Pi \mathcal{E}_{\rho_{k_3}}}(\cdot, \cdot) = \mathcal{B}_{\Pi}(\cdot, \cdot) \otimes \langle \cdot, \cdot \rangle_{K_v}$ and $\mathcal{H}_{\Pi \mathcal{E}_{\rho_{k_3}}}(\cdot, \cdot) = \mathcal{H}_{\Pi}(\cdot, \cdot) \otimes \langle \cdot, \cdot \rangle_{K_v}$. To unify the statement, we also write $\phi_H = \varphi_H$, $\phi_{\hat{H}} = \varphi_{\hat{H}}$ and $\mathcal{B}_H = \mathcal{B}_{\Pi \mathcal{E}_{\rho_{k_3}}}$, $\mathcal{H}_H = \mathcal{H}_{\Pi \mathcal{E}_{\rho_{k_3}}}$ when $F$ is nonarchimedean or $E$ is three copies of $\mathbb{R}$. Define

$$I^*(\Pi, t) = \frac{\zeta_F(2)}{\zeta_E(2)} \cdot \frac{\mathcal{L}(1, \Pi, \text{Ad})}{\mathcal{L}(1/2, \Pi, r)} \cdot \mathcal{I}(\Pi, t); \quad \mathcal{I}(\Pi, t) = \int_{F^\times \setminus \text{GL}_2(E)} \frac{\mathcal{B}_H(\mathcal{H}(\mathcal{E})) \varphi_H, \mathcal{H}(t) \varphi_{\hat{H}}}{\mathcal{B}_{\Pi \mathcal{E}_{\rho_{k_3}}}((\mathcal{J}_E) \phi_H, \phi_{\hat{H}})} \, dh. \tag{4.1}$$

Here $\mathcal{J}_E \in \text{GL}_2(E)$ is the identity element if $F$ is nonarchimedean, and $\mathcal{J}_E = (J, \mathcal{J}, J)$ if $E = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and $\mathcal{J}_E = (I_2, \mathcal{J})$ if $E = \mathbb{C} \times \mathbb{R}$. Similarly we define $\mathcal{I}_{\text{her}}^*(\Pi, t)$ by replacing $\mathcal{B}_H$, $\mathcal{B}_{\Pi \mathcal{E}_{\rho_{k_3}}}$, $\mathcal{H}(\mathcal{J}_E) \mathcal{H}_H(\mathcal{E}) \mathcal{H}_H(\mathcal{E}) \mathcal{H}_H(\mathcal{E})$, $\varphi_H$ and $\varphi_{\hat{H}}$ by $\mathcal{H}_{\Pi \mathcal{E}_{\rho_{k_3}}}$, $\mathcal{H}_H$, $\phi_H$ and $\varphi_H$, respectively. These integrals converge absolutely by the assumption $\Lambda(I) < 1/2$ [Ch08, Lemma 2.1]. For the definitions of the $L$-factors, see [Ch08, Introduction].

4.1.3. Case $\epsilon(I) = -1$. Note that by our assumption, $F$ is nonarchimedean in this case. Also, the Jacquet-Langlands lift $\Pi^D$ of $\Pi$ to $D^\times(E)$ is nonzero and we have $\text{Hom}_{D^\times(E) \times D^\times(F)}(\Pi^D \boxtimes \tilde{\Pi}, D^\times(E) \boxtimes \tilde{\Pi}, \mathbb{C}) \neq 0$. Let $\mathcal{B}_{\Pi^D} : \Pi^D \times \tilde{\Pi} \to \mathbb{C}$ (resp. $\mathcal{H}_{\Pi^D} : \Pi^D \times \tilde{\Pi} \to \mathbb{C}$) be a nontrivial $D^\times(E)$-equivariant bilinear (resp. hermitian) pairing on $\Pi^D$. As in the previous case, we first introduce elements $\phi_{\Pi^D}$. The assumption that $\Pi$ is square free implies $\Pi^D$ is one-dimensional when $E = F \times F$ or $E$ is a field. In these cases, let $\phi_{\Pi^D} \in \Pi^D$ be any nonzero elements. Assume $E = F \times F$. Then $D^\times(E) = \text{GL}_2(F) \times D^\times(F)$, $\Pi = \pi' \boxtimes \pi$ and $\Pi^D = \pi' \boxtimes \pi^D$. Since $\Pi$ is square free, we have $\pi^D$ is one-dimensional. Let $\phi_{\pi^D} \in \pi^D$ be any nonzero element, and $\phi_{\pi'} \in \pi'$ be a nonzero newform, i.e. $\pi' = \ker(\mathcal{E}) = \mathbb{C} \phi_{\pi'}$. Put $\phi_{\Pi^D} = \phi_{\pi'} \otimes \phi_{\pi^D}$. Similarly we can define $\phi_{\Pi^D}$ for $\Pi^D$.

We now define the local period integrals for the case $\epsilon(I) = -1$. Let $t$ be the elements attached to $\Pi^D$ given in [3.3.4]. Fix an embedding $\iota_D : D \hookrightarrow M_2(F)$ as in [CC18, Section 5.2]. Define

$$I^*(\Pi^D, t) = \frac{\zeta_F(2)}{\zeta_E(2)} \cdot \frac{\mathcal{L}(1, \Pi, \text{Ad})}{\mathcal{L}(1/2, \Pi, r)} \cdot \mathcal{I}(\Pi^D, t); \quad \mathcal{I}(\Pi^D, t) = \int_{F^\times \setminus D^\times(F)} \frac{\mathcal{B}_{\Pi^D}(\Pi^D(\mathcal{H}(\mathcal{E})) \phi_{\Pi^D}, \mathcal{H}(t) \phi_{\Pi^D})}{\mathcal{B}_{\Pi^D}(\phi_{\Pi^D}, \phi_{\Pi^D})} \, dh. \tag{4.2}$$

Similarly we define $I^*_{\text{her}}(\Pi^D, t)$ by replacing $\mathcal{B}_{\Pi^D}$ and $\phi_{\Pi^D}$ by $\mathcal{H}_{\Pi^D}$ and $\phi_{\Pi^D}$, respectively. Certainly the integrals converge absolutely.
Remark 4.2. Clearly, $I'(\Pi, t)$ and $I^*(\Pi^D, t)$ are independent of the choices of the elements $\phi_H$ and $\phi_{\bar{H}}$. Its also independent of the choice of the bilinear pairing $B_H$ and the model we used to realize the various representations. But it does depend on the choice of the pairing $\langle \cdot, \cdot \rangle_{\nu}$ and the elements $v_m$, as well as the embedding $\iota_D$ and Haar measure $dh$. Similar remarks apply to $I^*_{\mathrm{her}}(\Pi, t)$ and $I^*_{\mathrm{her}}(\Pi^D, t)$.

Following corollary is a consequence of Lemma 4.3 and our choice of $t$.

**Corollary 4.3.** We have $I'(\Pi, t) = I^*_{\mathrm{her}}(\Pi, t)$ and $I^*(\Pi^D, t) = I^*_{\mathrm{her}}(\Pi^D, t)$ when $F$ is nonarchimedean or $E = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. If $E = \mathbb{C} \times \mathbb{R}$, we have $I^*_{\mathrm{her}}(\Pi, t) = \omega_{\nu}(\sqrt{-1})(-1)^m I'(\Pi, t)$ with $m$ given by (3.3).

### 4.2. Preliminaries for the calculations.

We give some preparations for the computations of the local period integrals in this subsection. Let $K \subset \mathrm{GL}_2(F)$ be the compact subgroup given by

$$K = \mathrm{SU}(2); \quad K = \mathrm{SO}(2); \quad K = \mathrm{GL}_2(O_F),$$

according to $F = C$ or $F = R$ or $F$ is nonarchimedean, respectively.

#### 4.2.1. An identity between invariant forms.

We state Proposition 4.3, which is the key ingredient for our calculations. Let $K$ be an étale quadratic $F$-algebra. When $K = F \times F$, we identify $F$ with a subfield of $K$ via the diagonal embedding. Let $z \mapsto \sigma(z)$ denote the non-trivial $F$-automorphism of $K$. Then $|z|_K = |z \cdot \sigma(z)|_F$. Fix an element $\delta \in K^\times$ satisfies $\sigma(\delta) = -\delta$ and put $\Delta = \delta^2 \in F^\times$. Fix a non-trivial additive character $\psi$ of $F$ and let $\psi_K$ be an additive character of $K$ defined by $\psi_K(z) = \psi(z + \sigma(z))$. Let $\pi_K$ (resp. $\pi$) be a unitary irreducible admissible generic representation of $\mathrm{GL}_2(K)$ (resp. $\mathrm{GL}_2(F)$). We assume $\omega_{\pi_K} \cdot \omega_{\pi}$ is trivial on $F^\times$ and

$$\pi = \mathrm{Ind}^{\mathrm{GL}_2(F)}_{\mathrm{B}_2(F)}(\mu \boxtimes \nu)$$

is a principal series representation of $\mathrm{GL}_2(F)$.

Let $W(\pi_K, \psi_K)$ be the Whittaker model of $\pi_K$ with respect to $\psi_K$ and $B(\mu, \nu)$ be the underlying space of $\pi$. Recall that $B(\mu, \nu)$ consists of right $K$-finite $\mathbb{C}$-valued functions $f$ on $\mathrm{GL}_2(F)$ which satisfy the following rule:

$$f \left( \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} h \right) = \mu(y_1)\nu(y_2) \left| \frac{y_1}{y_2} \right|^{1/2}_F f(h),$$

for $y_1, y_2 \in F^\times$, $x \in F$ and $h \in \mathrm{GL}_2(F)$. Let $B_{\pi_K}$ be the $\mathrm{GL}_2(K)$-equivalent bilinear pairing between $W(\pi_K, \psi_K)$ and $W(\pi_K, \psi_K)$ defined by (4.11) with $F^\times$ replaced by $K^\times$. Let $B_\pi$ be the $\mathrm{GL}_2(F)$-equivalent bilinear pairing between $B(\mu, \nu)$ and $B(\mu^{-1}, \nu^{-1})$ given by

$$B_\pi(f, \tilde{f}) = \int_K f(k)\tilde{f}(k)dk,$$

for $f \in B(\mu, \nu)$ and $\tilde{f} \in B(\mu^{-1}, \nu^{-1})$. Define the local Ichino integral by

$$\mathcal{I}(W \otimes f; \tilde{W} \otimes \tilde{f}) = \int_{F^\times \backslash \mathrm{GL}_2(F)} B_{\pi_K}(\rho(h)W, \tilde{W})B_{\pi}(\rho(h)f, \tilde{f})dh.$$

Here $\rho$ is the right translation. On the other hand, we also define the local Rankin-Selberg integrals by

$$\mathcal{R}_\delta(W \otimes f) = \int_{F^\times N(F) \backslash \mathrm{GL}_2(F)} W \left( \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} h \right) f(h)dh,$$

and

$$\hat{\mathcal{R}}_\delta(W \otimes \tilde{f}) = \int_{F^\times N(F) \backslash \mathrm{GL}_2(F)} \tilde{W} \left( \begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix} h \right) \tilde{f}(h)dh.$$

Haar measures on various groups are chosen as follows. On $F$, we let $dx$ be the self-dual Haar measure with respect to $\psi$. Similarly we choose the self-dual Haar measure $dz$ (with respect to $\psi_K$) on $K$. The Haar measure $d^\times x$ on $F^\times$ is $d^\times x = \zeta_F(1)|x|^\frac{1}{2}dx$ and the Haar measure $d^\times z$ on $K^\times$ is $d^\times z = \zeta_K(1)|z|^\frac{1}{2}dz$. We identify $N(F)$ with $F$ so that the Haar measure $dn$ on $N(F)$ is also defined. The choices of the measures on $F$ and $F^\times$ uniquely determine a left Haar measure $db$ on $B(F)$ [MV10, Section 3.1.5]. We take any Haar measure $dk$ on $K$. Then the Haar measure $dh$ on $\mathrm{GL}_2(F)$ is given by $dh = dbdk$. Similarly the Haar measure $dh$ on $\mathrm{PGL}_2(F)$ is $dh = dbdk$, where $db$ is now the quotient measure on $F^\times \backslash B(F)$. Finally, the measure $dn$ on $F^\times N(F) \backslash \mathrm{GL}_2(F)$ is the quotient measure induced from $dn$ on $\mathrm{PGL}_2(F)$ and $dn$ on $N(F)$.

It was shown by Ichino [Ich18, Lemma 2.1] that the integral (4.5) converges absolutely when $\Lambda(II) < 1/2$, where $II = \pi_K \boxtimes \pi$. This also holds for the local Rankin-Selberg integrals.
Lemma 4.4. Suppose \( \Lambda(\Pi) < 1/2 \). Then the integrals \( \mathcal{E}_\delta(W \otimes f) \) and \( \hat{\mathcal{E}}_\delta(W \otimes \hat{f}) \) converge absolutely.

Proof. By the Iwasawa decomposition,

\[
\mathcal{E}_\delta(W \otimes f) = \int_K f(k) \int_{F^\times} W \left( \begin{pmatrix} \delta y & 0 \\ 0 & 1 \end{pmatrix} k \right) \mu(y)|y|_{F}^{-\frac{1}{2}} d^\times ydk.
\]

Similar equation holds for \( \hat{\mathcal{E}}_\delta(W \otimes \hat{f}) \). By symmetry, we only need to verify the assertion for \( \Psi_\delta(W \otimes f) \).

By equation (4.8) and the fact that \( f \) is identity when \( s = 1 \), we have

\[
W \left( \begin{pmatrix} \delta y & 0 \\ 0 & 1 \end{pmatrix} k \right) \mu(y)|y|_{F}^{-\frac{1}{2}} \ll_{\Pi,W,\epsilon} |y|_{F}^{-\frac{1}{2} - \Lambda(\Pi) - \epsilon} \Phi(\delta y).
\]

for every \( y \in F \) and \( k \in K \). Here \( \Phi \) is a continuous function on \( F \) which decreasing rapidly as \( |y|_{F} \to \infty \). This shows the lemma.

Proposition 4.5. Suppose \( F \neq \mathbb{C} \) and \( \Lambda(\Pi) < 1/2 \). We have

\[
\mathcal{I}(W \otimes f; W \otimes \hat{f}) = |\Delta|_{F}^{-\frac{1}{2}} \cdot \frac{\zeta_c(1)}{\zeta_F(1)} \cdot \mathcal{E}_\delta(W \otimes f) \cdot \hat{\mathcal{E}}_\delta(W \otimes \hat{f}).
\]

Proof. We will prove this proposition in [5].

Remark 4.6. Proposition 4.5 is motivated by the work of [MV10] Lemma 3.4.2, in which they proved this identity when \( \mathcal{K} = F \times F \) under tempered assumption. In [HS17] Proposition 5.1, he gave a different and more elementary proof for the case \( \mathcal{K} = F \times F \) and \( F \) is nonarchimedean. Moreover, Hsieh replaced the temperedness assumption by a much weaker hypothesis \( \Lambda(\Pi) < 1/2 \). Following the method in [HS17], I. Ishikawa in his thesis [Ish17] Theorem 5.1 proved Proposition 4.5 when \( \mathcal{K} \) is a field and \( F \) is nonarchimedean. In this article, we adopt the same ideal to the case \( F = \mathbb{R} \); however, the proof is much more involved. We expect the same ideal should also apply to the case \( F = \mathbb{C} \).

4.2.2. Godement section and intertwining map. Let \( S(F^2) \) be the space of \( \mathbb{C} \)-valued Bruhat-Schwartz functions on \( F^2 \) and let \( S(F^2, \psi) \) be the subspace of \( S(F^2) \) given by \( S(\mathbb{C}^2, \psi) = \mathbb{C}[x, y, x, y] e^{-\pi(a)_{1/2}(x^2 + y^2)} \), and \( S(\mathbb{R}^2, \psi) = \mathbb{C}[x, y] e^{-\pi(a)_{1/2}(x^2 + y^2)} \) and \( S(F^2, \psi) = S(F^2) \) when \( F \) is nonarchimedean. Here \( a \) is the non-zero number such that \( \psi(x) = e^{2\pi ia x} \) when \( F = \mathbb{R} \) and \( \psi(x) = e^{2\pi ia x + \bar{a} \bar{x}} \) when \( F = \mathbb{C} \). Recall the Fourier transform \( \hat{\Phi} \) of \( \Phi \in S(F^2) \) is given by

\[
\hat{\Phi}(x, y) = \int_F \int_F \Phi(u, v) \psi(uy - vx) dudv.
\]

Here \( du, dv \) are the Haar measures on \( F \) which are self-dual with respect to \( \psi \). Observe that the subspace \( S(F^2, \psi) \) is invariant under the Fourier transform.

Following facts can be found in [LL70] and [GJ79] Section 4. Let \( \pi = \text{Ind}_{H(F)}^{GL_2(F)}(\mu \boxtimes \nu) \) be a principal series representation of \( GL_2(F) \). For \( s \in \mathbb{C} \), \( \Phi \in S(F^2, \psi) \) and \( h \in GL_2(F) \), define

\[
\Phi^s(h) = \mu(h)|h|^s_F \int_{F^\times} \Phi((0, t)h) \mu^{-1}(t)|t|^{2s} d^\times t;
\]

\[
\hat{\Phi}^s(h) = \nu(h)|h|^s_F \int_{F^\times} \Phi((0, t)h) \mu^{-1}(t)|t|^{2s} d^\times t.
\]

These two integrals converge absolutely for \( \text{Re}(s) \gg 0 \) and have meromorphic continuations to the whole complex plane. Moreover, whenever they are defined, we have \( \Phi^s \in \mathcal{B}(\mu \cdot |\cdot|_{F}^{-1/2}, \nu \cdot |\cdot|_{F}^{1/2 - s}) \) and \( \hat{\Phi}^s \in \mathcal{B}(\nu \cdot |\cdot|_{F}^{-1/2}, \mu \cdot |\cdot|_{F}^{1/2 - s}) \). Since \( \pi \) is a principal series, both of \( \Phi^s \) and \( \hat{\Phi}^s \) are defined at \( s = 1/2 \), and we put \( \Phi := \Phi^{1/2} \) and \( \hat{\Phi} := \hat{\Phi}^{1/2} \). Then the linear map \( \Phi \mapsto \Phi \) (resp. \( \Phi \mapsto \hat{\Phi} \) from \( S(F^2, \psi) \) to \( \mathcal{B}(\mu, \nu) \) (resp. \( \mathcal{B}(\nu, \mu) \)) is surjective. Define the normalized intertwining operator

\[
M^s_{\Phi}(\mu, \nu, s) \Phi^s(h) = \gamma(2s - 1, \mu^{-1}, \psi) \int_F \Phi^s \left( w \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} h \right) dx.
\]

Here \( dx \) is the Haar measure on \( F \) which is self-dual with respect to \( \psi \) and \( \gamma(s, \mu^{-1}, \psi) \) is the Tate \( \gamma \)-factor attached to \( \mu^{-1} \) and \( \psi \). This integral converges absolutely for \( \text{Re}(s) \gg 0 \) and admits a meromorphic continuation to whole complex plane. Moreover, we have \( M^s_{\Phi}(\mu, \nu, s) \Phi^s = \hat{\Phi}^{1/2 - s} \) in the sense of the meromorphic...
continuations. We define the $GL_2(F)$-isomorphism
\begin{equation}
M_\psi^t(\mu, \nu) : B(\mu, \nu) \xrightarrow{\sim} B(\nu, \mu); \quad f_\Phi \mapsto \tilde{f}_\Phi.
\end{equation}

Notice that $M_\psi^t(\mu, \nu)$ is well-defined (i.e. independent of the choice of $\Phi$).

Let $\mathcal{K}$ and $\pi_\mathcal{K}$ be as in \cite{Jac72}. Let $\pi_\mathcal{K}$ be either the Asai transfer to $GL_4(F)$ \cite{Kri03} or the Rankin-Selberg product \cite{Jac72}, according to $\mathcal{K}$ is a field or not. In both cases, $\pi_{\mathcal{K}}$ is an irreducible admissible generic representation of $GL_4(F)$ \cite{Kri03, Ram04}. Let $\mu$ be a character of $F^\times$. Denote by $L(s, \pi_{\mathcal{K}} \otimes \mu)$ and $\epsilon(s, \pi_{\mathcal{K}} \otimes \mu, \psi)$ the $L$- and the $\epsilon$-factor attached to $As \pi_{\mathcal{K}} \otimes \mu$ and $\psi$ defined in \cite{Tat79} (3.6.4). Following corollary is a generalization of \cite{Hsi17} corollary 5.2.

**Corollary 4.7.** Let assumptions be as in Proposition \ref{4.3}. Let $W \in \mathcal{W}(\pi_{\mathcal{K}}, \psi_{\mathcal{K}})$, $f \in B(\mu, \nu)$ and put $\tilde{W} = W \otimes \omega_\pi^{-1} \in \mathcal{W}(\tilde{\pi}_{\mathcal{K}}, \psi_{\mathcal{K}})$, $\tilde{f} = M_\psi^t(\mu, \nu) f \otimes \omega_\pi^{-1} \in B(\mu^{-1}, \nu^{-1})$. Then we have
\[
\mathcal{I}(W \otimes f; \tilde{W} \otimes \tilde{f}) = \mu(-\Delta) |\Delta|_{F^-}^{1/2} \cdot \lambda_{K/F}(\psi)^{-1} \cdot \frac{\zeta(1)}{\zeta_F(1)} \cdot \gamma(1/2, As \pi_{\mathcal{K}} \otimes \mu, \psi) \cdot \mathcal{R}_\delta(W \otimes f)^2
\]
where
\[
\gamma(s, As \pi_{\mathcal{K}} \otimes \mu, \psi) = \epsilon(s, As \pi_{\mathcal{K}} \otimes \mu, \psi) \frac{L(1-s, As \pi_{\mathcal{K}} \otimes \mu^{-1})}{L(s, As \pi_{\mathcal{K}} \otimes \mu)}
\]
is the gamma factor \cite{CCT18} and $\lambda_{K/F}(\psi)$ is the Langlands constant given in \cite{JL70} Lemma 1.2.

**Proof.** The proof is similar to that of \cite{Hsi17} Corollary 5.2; however, when $\pi_{\mathcal{K}}$ is a field, instead of using the results of \cite{Jac72} and \cite{Hsi17} Proposition 5.1, one needs to use the results of \cite{CCT18} and Proposition 4.3. \hfill \Box

### 4.2.3. Additive characters and Haar measures.
We fix choices of additive characters and Haar measures on various groups in the rest of this section. If $F = \mathbb{R}$, let $\psi(x) = e^{2\pi i x}$. If $F = \mathbb{C}$, let $\psi(x) = e^{2\pi i (x+\bar{x})}$. If $F$ is nonarchimedean, we require $\psi$ to be trivial on $O_F$, but not on $\mathbb{R}^{-1}O_F$.

Haar measures on $F$, $F^\times$ and $GL_2(F)$ are those described in \ref{4.2.1}. Therefore according to the choices of $\psi$, we see that the measure $dx$ on $F$ is the usual Lebesgue measure when $F = \mathbb{R}$, and is twice of the usual Lebesgue measure when $F = \mathbb{C}$, and is characterized by $\text{Vol}(O_F, dx) = 1$ when $F$ is nonarchimedean. On $GL_2(F)$, we need to specify the choice of the measure $dk$ on $K$. In any case, we choose $dk$ so that the total volume of $K$ is 1. Suppose $D$ is the division algebra and $F$ is nonarchimedean, we choose $dh$ so that $\text{Vol}(O_D, dh) = 1$. Finally, the measures on the quotient spaces $F^\times/\text{GL}_2(F)$ and $F^\times/\text{N}(F)/\text{GL}_2(F)$ are also those given in \ref{4.2.1}, while the measure $dh$ on $F^\times/\text{D}(F)$ is the quotient measure induced from the measure on $F^\times$ and the measure on $D^\times(F)$.

### 4.2.4. Whittaker functions on $GL_2$.
We describe various Whittaker functions on $GL_2$ when evaluate at diagonal elements. These facts can be deduced from \cite{JL70} and \cite{Jac72}. For proofs, one can see \cite{Che88} Appendix A.

Let $\pi$ be an irreducible admissible generic representation of $GL_2(F)$. Let $W(\pi, \psi)$ be the Whittaker model of $\pi$ with respect to $\psi$. If $\pi$ is unitary, then the following integral, which converges absolutely, defines a $GL_2(F)$-equivariant bilinear pairing between $W(\pi, \psi)$ and $W(\tilde{\pi}, \psi)$:
\begin{equation}
\mathcal{B}_\pi(W, \tilde{W}) = \int_{F^\times} W \left( \begin{array}{cc} y & 0 \\ 0 & 1 \end{array} \right) \tilde{W} \left( \begin{array}{cc} -y & 0 \\ 0 & 1 \end{array} \right) d^*y.
\end{equation}
for $W \in W(\pi, \psi)$ and $\tilde{W} \in W(\tilde{\pi}, \psi)$. There is also a $GL_2(F)$-equivariant hermitian pairing on $W(\pi, \psi)$
\begin{equation}
\mathcal{H}_\pi(W, W') = \int_{F^+} W \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \overline{W'} \left( \begin{array}{cc} y & 0 \\ 0 & 1 \end{array} \right) d^*y.
\end{equation}

When $F$ is achimedean, we assume $\pi$ is the constitute of $\text{Ind}_{\text{B}^2(F)}^{GL_2(F)}(\mu \boxtimes \nu)$ with $\mu^{-1}(r) = |r|^2_s$ for some $s \in \mathbb{C}$ for every $r \in \mathbb{R}_{>0}$, and let $k \geq 0$ be the minimal $\text{SU}(2)$-type (resp. $\text{SO}(2)$-type) of $\pi$ when $F = \mathbb{C}$ (resp. $F = \mathbb{R}$). We introduce a distinguished element $W_s \in W(\pi, \psi)$, which is unique up to nonzero constants. Suppose $F = \mathbb{C}$. Then $\pi$ is a principal series. By \cite{JL70} Lemma 6.1, we have
\[
(W(\pi, \psi) \otimes \mathcal{L}_k(C))^{\text{SU}(2)} = \mathbb{C} \cdot W_\pi.
\]
Observe that $W_\pi$ is a $L_k(\mathbb{C})$-valued Whittaker function on $GL_2(\mathbb{C})$ and we have $W_\pi(hu) = \rho_k(u^{-1})W_\pi(h)$ for $h \in GL_2(\mathbb{C})$ and $u \in SU(2)$. For $0 \leq j \leq k$, we set $W_j(h) = \langle W_\pi(h), X^jY^{k-j} \rangle_k$. Then $W_j \in W(\pi, \psi)$ for all $j$, and one has

\begin{equation}
W_\pi(h) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} W_j(h) X^{k-j}Y^j.
\end{equation}

By the uniqueness of the Whittaker model, we may assume $\mu^{-1}(e^{iy}) = e^{ik\theta}$. Then form [Jac72, Section 18], one can take

\begin{equation}
W_j \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = 4\mu^{-1}(-1) \left( \sqrt{-1} \right)^j \cdot \mu(y) y^{k-j} |y|^{-\frac{k+s}{2}-\frac{k}{4}} \cdot K_{\frac{k}{2}+2s-j} \left( 4\pi |y|^{\frac{1}{2}} \right).
\end{equation}

Suppose $F = \mathbb{R}$. We assume $\text{Re}(s) \geq 0$ by the uniqueness of the Whittaker model. We let $W_\pi$ be a weight $k$ element, i.e. $W_\pi(he^{it}) = e^{ik\theta}W_\pi(h)$ for $h \in GL_2(\mathbb{R})$. If $k \geq 2$, then $\pi$ is a discrete series and we take

\begin{equation}
W_\pi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = \mu(y) y^{\frac{k}{4}} e^{-2\pi y} \cdot I_{\infty} \left( y \right).
\end{equation}

If $k \leq 1$, then $\pi$ is a principal series and we take

\begin{equation}
W_\pi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = \sum_{\ell=0}^{k} \binom{k}{\ell} \mu(y) y^{k-\ell} |y|^{-\frac{k+s}{2}-\frac{k}{4}} K_{\frac{k}{2}+2s-\ell} \left( 2\pi |y| \right).
\end{equation}

Recall that $V_+ \in \mathfrak{nil}$ is the weight raising element given by (3.10). If $k = 0$, we have

\begin{equation}
\rho(V_+)W_\pi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = -2\pi \mu(y) |y|^2 \cdot K_{s+1} \left( 2\pi |y| \right) - 4\pi \mu(y) |y|^2 \cdot \text{sgn}(y) K_s \left( 2\pi |y| \right)
\end{equation}

Finally, suppose $F$ is nonarchimedean. We let $W_\pi$ be the newform with $W_\pi(I_2) = 1$ [Sch02, Summary].

We need to compute the norm for $W_\pi$ when $F$ is archimedean. More precisely, suppose $F$ is archimedean and $\pi$ is unitary. Define $\mathcal{B}(W_\pi) = \mathcal{B}_\pi(\rho(J)W_\pi, W_\pi \otimes \omega^{-1})$ when $F = \mathbb{R}$, and

\begin{equation}
\mathcal{B}(W_\pi) = \int_{\mathbb{C}^x} \left( W_\pi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right), W_\pi \left( \begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix} \right) \right)_k \omega^{-1}(-y) d^c y
\end{equation}

when $F = \mathbb{C}$. Observe that $W_\pi \otimes \omega^{-1} = W_\pi$ by our definition.

**Lemma 4.8.** Notation be as above.

(1) If $F = \mathbb{R}$, then

\begin{equation}
\mathcal{B}(W_\pi) = \begin{cases}
2^{-2k} \pi^{-k} \Gamma(k) & \text{if } k \geq 2, \\
2^{-2k} \pi^{-k} \Gamma \left( \frac{k+1}{2} + s \right) \Gamma \left( \frac{k+1}{2} - s \right) & \text{if } k \leq 1.
\end{cases}
\end{equation}

(2) If $F = \mathbb{C}$, then

\begin{equation}
\mathcal{B}(W_\pi) = \left( \sqrt{-1} \right)^k \mu(-1) 2^{-k+1} \pi^{-2k} \Gamma \left( \frac{k}{2} + 2s + 1 \right) \Gamma \left( \frac{k}{2} - 2s + 1 \right).
\end{equation}

**Proof.** Note that $\omega = \mu \nu$. Suppose $F = \mathbb{R}$. If $k \geq 2$, then $\mu^{-1}(y) = |y|^{k-1} \text{sgn}(y)$ by our assumption. Now $\mathcal{B}(W_\pi)$ for this case follows immediately from this observation and (4.14). If $k \leq 1$, then $\mu^{-1}(y) = |y|^2 \text{sgn}(y)$ and from (4.14), we have

\begin{equation}
\mathcal{B}(W_\pi) = 2^k \int_{\mathbb{R}^x} \left| y \right| K_{s+\frac{1}{2}} \left( 2\pi |y| \right) K_{s-\frac{1}{2}} \left( 2\pi |y| \right) d^c y = 2^{-2k} \pi^{-k} \Gamma \left( \frac{k+1}{2} + s \right) \Gamma \left( \frac{k+1}{2} - s \right).
\end{equation}

The last equality follows from [1076.4], 6.576.4).

Suppose $F = \mathbb{C}$. We need a combinatorial identity, which follows easily from the induction on a nonnegative integer $N$. Let $z, w$ be two complex numbers. Then

\begin{equation}
\sum_{j=0}^{N} \binom{N}{j} \Gamma(z+j) \Gamma(w-j) = \frac{\Gamma(z) \Gamma(w-N) \Gamma(z+w)}{\Gamma(z+w-N)}.
\end{equation}
By (4.15), (4.13), (4.14), (4.17), 6.576.4] and the combinatorial identity, we have

\[ \mathcal{B}(W_\pi) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathcal{B}_\pi (W_j, W_{k-j} \otimes \omega^{-1}_\pi) \]

\[ = (\sqrt{-1})^k \mu(-1)^{-k+1} \pi^{-k+2} \Gamma \left( \frac{k}{2} + 2s + 1 \right) \Gamma \left( \frac{k}{2} - 2s + 1 \right) \sum_{j=0}^{k} \binom{k}{j} \Gamma(k + 1 + j) \Gamma(1 + j) \]

\[ = (\sqrt{-1})^k \mu(-1)^{-k+1} \pi^{-k+2} \Gamma \left( \frac{k}{2} + 2s + 1 \right) \Gamma \left( \frac{k}{2} - 2s + 1 \right). \]

This completes the proof. \qed

4.2.5. Matrix coefficients for $GL_2$. Let notation be as in (4.2.3). Suppose $\pi$ is unitary. We define $\mathcal{H}(W_\pi)$ similar to that of $\mathcal{B}(W_\pi)$ as follows. If $F = \mathbb{R}$, set $\mathcal{H}(W_\pi) = \mathcal{H}_\pi(W_\pi, W_\pi)$. If $F = \mathbb{C}$, let

\[ \mathcal{H}(W_\pi) = \int_{\mathbb{C} \times (W_\pi \otimes (y \ 0 \ 1), W_\pi \left( \begin{array}{l} y \\ 0 \\ 1 \end{array} \right))} \mathcal{d}^\times y. \]

We now define matrix coefficients associated to $\pi$. Let $t \in GL_2(F)$ or $t \in \mathbb{U}_\mathbb{R} \times O(2)$ or $t \in \mathbb{U}_\mathbb{C} \times U(2)$ according to $F$ is nonarchimedean or $F = \mathbb{R}$ or $F = \mathbb{C}$, respectively. Suppose $F$ is nonarchimedean or $F = \mathbb{R}$. Put

\[ \Phi_\pi(h; t) = \frac{\mathcal{B}_\pi(\rho(ht)W_\pi, \rho(t)W_\pi)}{\mathcal{B}_\pi(W_\pi, W_\pi)} \quad \text{and} \quad \Phi_\pi^{\text{her}}(h; t) = \frac{\mathcal{H}_\pi(\rho(ht)W_\pi, \rho(t)W_\pi)}{\mathcal{H}(W_\pi, W_\pi)} \]

as functions in $h \in GL_2(F)$. Suppose $F = \mathbb{C}$. Let $v_j \in \mathcal{L}_k(\mathbb{C})$ be the element given by (5.8) with $k'$ replaced by $k$ and $W_\pi^{(j)}(h) = (v_j, W_\pi(h))$ for $h \in GL_2(\mathbb{C})$. We define

\[ \Phi_\pi^{(j)}(h; t) = \frac{\mathcal{B}_\pi(\rho(ht)W_\pi^{(j)}, \rho(t)(W_\pi^{(j)} \otimes \omega^{-1}_\pi))}{\mathcal{B}(W_\pi)} \quad \text{and} \quad \Phi_\pi^{(j), \text{her}}(h; t) = \frac{\mathcal{H}_\pi(\rho(ht)W_\pi^{(j)}, \rho(t)W_\pi^{(j)})}{\mathcal{H}(W_\pi)} \]

as functions in $h \in GL_2(\mathbb{C})$.

Let $F = \mathbb{R}$ or $\mathbb{C}$. For $h \in GL_2(F)$ and $X \in \text{Lie}(GL_2(F))$, we denote by $\text{Ad}_h(X) = hXh^{-1}$. This action can be extend to $\mathbb{K}_F$, and we will use the same notation to indicate this action. We have following relations between these matrix coefficients.

Lemma 4.9. Write $t = (Z, u)$ when $F$ is archimedean. Suppose $\text{Ad}_J Z = \tilde{Z}$ and $u$ commutes with $J$.

1. If $F = \mathbb{C}$, then

\[ \Phi_\pi^{(j), \text{her}}(h; t) = \omega_\pi(\sqrt{-1})(-1)^j \Phi_\pi^{(j)}(\mathcal{J}h; t). \]

2. If $F = \mathbb{R}$, then

\[ \Phi_\pi^{\text{her}}(h; t) = \Phi_\pi(Jh; t). \]

3. If $F$ is nonarchimedean, then

\[ \Phi_\pi^{\text{her}}(h; t) = \Phi_\pi(h; t). \]

Proof. Notice that since $\pi$ is unitary, we have $\pi \cong \pi$ where $\pi$ is the conjugate representation of $\pi$. It then follows from the uniqueness of Whittaker model that $\mathcal{W}(\pi, \psi) = \mathcal{W}(\pi, \psi)$. Furthermore, its clear that $\mathcal{W}(\pi, \psi)$ consists of functions $W''(h) := W'(J^h)$ for $W \in \mathcal{W}(\pi, \psi)$.

The assertion is clear when $F$ is nonarchimedean. Indeed, its easy to see that $W''_\pi \in \mathcal{W}(\pi, \psi)$ is a nonzero newform, and hence $W''_\pi = cW_\pi$ for some $c \neq 0$. We consider the case when $F$ is archimedean. Recall that $W''_\pi \otimes \omega^{-1}_\pi = W_\pi$. Suppose $F = \mathbb{R}$. Its clear that both $\rho(J)W''_\pi$ and $W_\pi$ are nonzero weight $k$ elements in $\mathcal{W}(\pi, \psi)$. Therefore there is a nonzero constant $c$ such that $W''_\pi(h) = cW_\pi(J_hJ)$ for $h \in GL_2(\mathbb{R})$. It follows that if $t$ satisfies the assumption above, then one has $\rho(t)W''_\pi(h) = \rho(t)W_\pi(J_hJ)$. This shows (2). Finally, suppose $F = \mathbb{C}$. Consider the function

\[ W'_\pi(h) := W''_\pi(J_hw^{-1}) = \rho_k(w)W_\pi(J_hh). \]

Then both $W'_\pi$ and $W_\pi$ are nonzero elements in the one-dimensional space $(\mathcal{W}(\pi, \psi) \otimes \mathcal{L}_k(\mathbb{C}))^{SU(2)}$, and hence there is $c \neq 0$ so that $W'_\pi = cW''_\pi$. It follows that

\[ (4.19) \quad \mathcal{B}(W_\pi) = c \omega_\pi(-1) \mathcal{H}(W_\pi). \]
On the other hand, if we set $W^{(j)}(h) = \langle v_j, W(h) \rangle_k$ and $W^{(j)}(h) = \langle v_j, W(h) \rangle_k$ for $h \in \text{GL}_2(\mathbb{C})$, then we have the relation

$$W^{(j)}(f h) = c(-1)^j \rho(u_0) W^{(j)}(h), \quad u_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = (-i) \cdot f \in \text{SU}(2).$$

It follows that if $t$ satisfies the assumption above, then $\rho(t) W^{(j)}(f h) = c(-1)^j \rho(u_0) \rho(t) W^{(j)}(h)$. The first assertion then follows from this observation and (4.19). This completes the proof. \(\square\)

### 4.3. Calculations of local period integrals

In this section, we are devoted to compute the local period integrals defined in (4.11). By our assumptions, we only need to dual with the cases when $F = \mathbb{R}$. Indeed, the local period integrals for the nonarchimedean case were computed in [CC18, Sections 4, 5]. Combining these results with Corollary 4.19, we obtain $I^\star_{\text{per}}(I, t)$ and $I^\star_{\text{per}}(I^D, t)$ for the nonarchimedean case.

From now on, we assume $F = \mathbb{R}$. We first review some facts about a principal series representation $\pi = \text{Ind}_{B(\mathbb{R})}^{\text{GL}_2(\mathbb{R})}(\mu \boxtimes \nu)$ of $\text{GL}_2(\mathbb{R})$. Let $k \in \{0, 1\}$ be the minimal $\text{SO}(2)$-type of $\pi$ and $n$ be an integer with $n \equiv k (\text{mod} \ 2)$. We write $f^{(n)}_\mu \in \mathcal{B}(\chi, \eta)$ for the weight $n$ element with $f^{(n)}_\mu(k(\theta)) = e^{in\theta}$. When $n = k$, we simply denote $f_\pi = f^{(k)}_\pi$. Suppose $m\nu^{-1}(r) = r^{2s}$ for some $s \in \mathbb{C}$ for all $r > 0$. Then by [JJ70, Lemma 5.6], we have

$$\rho\left(\tilde{V}_\pi^t\right) f^{(n)}_\mu = (-1)^{\ell} 2^{-2\ell} \pi^{\frac{1}{2}} \left(\frac{\Gamma(s + \frac{n+1}{2} + \ell)}{\Gamma(s + \frac{n+1}{2})}\right) f^{(n+2\ell)}_\mu$$

for every integer $\ell \geq 0$.

#### Lemma 4.10

Let $\tilde{\Phi} = \left(\text{Ind}_{\text{SO}(2)}^{\text{GL}_2(\mathbb{R})}(\mu \boxtimes \nu)\right)$ be the intertwining map defined by (4.10). Then

$$\mathcal{B}_\pi(\rho(f_\pi)) = \mu(-1)^{2s} \cdot \Gamma\left(\frac{k+1}{2} - s\right) \Gamma\left(\frac{k+1}{2} + s\right)^{-1}.$$

**Proof.** Let $\Phi(x, y) = (x + iy)^k e^{-\pi(x^2 + y^2)} \in S(\mathbb{R}^2, \psi)$, and $f_\Phi \in \mathcal{B}(\chi, \eta)$ be the Godement section attached to $\Phi$ defined in (4.2.2). One check that $f_\Phi$ is the weight $k$ element with $f_\Phi(I_2) = \left(\sqrt{-1}\right)^k \Gamma_\mathbb{R}(1 + k + 2s)$. It follows that $f_\pi = \left(\sqrt{-1}\right)^{-k} \Gamma_\mathbb{R}(1 + k + 2s)^{-1} f_\Phi$. Since $\tilde{\Phi} = \Phi$, we find that

$$\tilde{\pi}(I_2) = \left(\sqrt{-1}\right)^{-k} \Gamma_\mathbb{R}(1 + k + 2s)^{-1} f_\Phi(I_2) = \pi^{2s} \Gamma_\mathbb{R}(1 + k - 2s) \Gamma_\mathbb{R}(1 + k + 2s)^{-1}.$$

By definition, we have

$$\mathcal{B}_\pi(\rho(f_\pi)) = \int_{\text{SO}(2)} \rho(f_\pi(k)) dk = \mu(-1)^{2s} \Gamma\left(\frac{k+1}{2} - s\right) \Gamma\left(\frac{k+1}{2} + s\right)^{-1}.$$

This completes the proof. \(\square\)

We would like to use the results in (4.2.1) and (4.2.2) to compute the local period integrals for the real case. In order to consist with the notation given there, let $K = \mathbb{R} \times \mathbb{R}$ or $K = \mathbb{C}$. When $K = \mathbb{R} \times \mathbb{R}$, set $\pi_\mathbb{R} = \pi_1 \boxtimes \pi_2$ and $\pi_\mathbb{C} = \pi_3$. When $K = \mathbb{C}$, let $\pi_\mathbb{C} = \pi'$. In any case, our assumption implies $\pi = \text{Ind}_{\text{SO}(2)}^{\text{GL}_2(\mathbb{R})}(\mu \boxtimes \nu)$ is a principal series with minimal $\text{SO}(2)$-type $k \in \{0, 1\}$ ($k = k_3$ if $K = \mathbb{R} \times \mathbb{R}$). We then realize

$$\Pi = W(\pi_\mathbb{C}, \psi_\mathbb{C}) \boxtimes (\mu, \nu) \quad \text{and} \quad \tilde{\Pi} = W(\tilde{\pi}_\mathbb{C}, \tilde{\psi}_\mathbb{C}) \boxtimes (\mu, \nu)^{-1}.$$

Put $W_{\pi_\mathbb{C}} = W_{\pi_1} \otimes W_{\pi_2}$ when $K = \mathbb{R} \times \mathbb{R}$. In the case $K = \mathbb{C}$, we also need scalar-valued Whittaker functions $W^\nu_{\pi_\mathbb{C}} := \langle v_m, W_{\pi_\mathbb{C}} \rangle_k$ and $W^\nu_{\pi_\mathbb{C}} := \langle v_m, W_{\pi_\mathbb{C}} \rangle_k$ with $v_m$ given by (3.8) and (3.9). Let $\tilde{\pi}_\mathbb{C} = (M^0_\mathbb{C}(\mu, \nu) f_\pi) \boxtimes (\mu, \nu)^{-1}$. Then $\tilde{f}_\pi \in \mathcal{B}(\mu, \nu)^{-1}$ is a nonzero weight $k$ element.

We write $t = (Z_1 \otimes Z_2, (u_1, u_2, u_3))$ and put $t_j = (Z_j, u_j) \in \mathcal{U}_\mathbb{R} \times O(2)$ for $j = 1, 2, 3$ when $K = \mathbb{R} \times \mathbb{R}$, and we write $t = (Z', Z') \otimes (u', u')$ and put $t' = (Z', Z') \in \mathcal{U}_\mathbb{C} \times O(2)$ when $K = \mathbb{C}$. Let $u_0 \in \text{GL}_2(F)$ with $u_0 = (u_1, u_2, u_3)$ when $K = \mathbb{R} \times \mathbb{R}$ and $u_0 = (u', u)$ when $K = \mathbb{C}$. Set

$$\begin{cases}
W = \rho(t_1) W_{\pi_1} \otimes \rho(t_2) W_{\pi_2} \quad \text{and} \quad f = \rho(t_3) f_\pi \quad \text{if} \ K = \mathbb{R} \times \mathbb{R}, \\
W = \rho(t') W_{\pi_3} \quad \text{and} \quad f = \rho(t') f_\pi \quad \text{if} \ K = \mathbb{C}.
\end{cases}$$
Recall that we have defined and computed the norms $\mathcal{B}(W_{\nu'})$ and $\mathcal{B}(W_{\tau'})$ in \[ \textcolor{red}{[1,2,3]} \] We set $\mathcal{B}(\pi\tau_\nu) = \mathcal{B}(\pi_\nu)\mathcal{B}(\pi_\tau)$ when $\mathcal{K} = \mathbb{R} \times \mathbb{R}$, and $\mathcal{B}(\pi_\nu) = \mathcal{B}((\rho(J)f_\pi, \tilde{f}_\pi))$, which has been calculated in Lemma \[ \textcolor{red}{[4,10]} \] In the following, we pick $\delta$ in Proposition \[ \textcolor{red}{[4,3]} \] to be

\[
\delta = \begin{cases} 
(-1,1) & \text{if } \mathcal{K} = \mathbb{R} \times \mathbb{R}, \\
\sqrt{-1} & \text{if } \mathcal{K} = \mathbb{C}.
\end{cases}
\]

Then $I^*(\Pi, t)$ can be simplified as follow.

**Corollary 4.11. Notation be as above. We have**

\[
I^*(\Pi, t) = C_K \cdot \omega_{\Pi}(u_0)^{-1} \cdot \epsilon(1/2, As\pi_\mathcal{K} \otimes \mu, \psi) \cdot \frac{L(1, \Pi, Ad)}{\mathcal{B}(\pi_\mathcal{K})} \cdot \frac{\mathcal{R}_{\delta}(W \otimes f)^2}{L(1/2, As\pi_\mathcal{K} \otimes \mu)^2}.
\]

Here $C_K = \mu(-1)^2$ if $\mathcal{K} = \mathbb{R} \times \mathbb{R}$ and $C_K = -2\pi\sqrt{-1}$ if $\mathcal{K} = \mathbb{C}$.

**Proof.** We realize $\phi_{\Pi} = W_{\pi_\mathcal{K}} \otimes f_\pi$ and $\phi_{R} = (W_{\pi_\mathcal{K}} \otimes \omega_{\pi_\mathcal{K}}^{-1}) \otimes \tilde{f}_\pi$. Then this corollary is a direct consequence of Corollary \[ \textcolor{red}{[4,4]} \] and the following observations. By our choice of $t$, we have

\[
\rho(t) \left( (W_{\pi_\mathcal{K}} \otimes \omega_{\pi_\mathcal{K}}^{-1}) \otimes \tilde{f}_\pi \right) = \omega_{\Pi}(u_0)^{-1} \cdot (W \otimes M^*_\nu(\mu, \nu)f) \otimes \omega_{\Pi}^{-1},
\]

and since $\omega_{\Pi}$ is trivial on $\mathbb{R}^\times$, one has $L(1/2, As\tilde{\pi}_{\mathcal{K}} \otimes \mu^{-1}) = L(1/2, As\pi_\mathcal{K} \otimes \nu)$. This finishes the proof. $\square$

Form Corollary \[ \textcolor{red}{[4,11]} \] we see that it suffices to compute the local Rankin-Selberg integral $\mathcal{R}_{\delta}(W \otimes f)$. This is what we are going to do in the rest of this subsection.

### 4.3.1. Case $E = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

We compute the local period integrals for the case $E = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Let $j = 1, 2, 3$. Recall that $k_j \geq 0$ is the minimal SO(2)-type of $\pi_j$, and we have assumed $k_1 \geq k_2 \geq k_3$. Also by our convention $\pi_3 = \pi = \text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})}(\mu \boxtimes \nu)$ is a principal series. For the convenience, we also write $\mu_3 = \mu$ and $\nu_3 = \nu$. Its no harm to assume that $\pi_j$ is the constitute of $\text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})}(\mu_j \boxtimes \nu_j)$, with $\mu_j \nu_j^{-1} = | \cdot |^{s_j} \text{sgn}^{k_j}$ for some $s_j \in \mathbb{C}$, $\text{Re}(s_j) \geq 0$. Note that $\omega_{\Pi} = \prod_{j=1}^{3} \mu_j \nu_j$ is trivial on $\mathbb{R}^\times$ implies $k_1 + k_2 + k_3$ is even and $\mu_1 \mu_2 \mu_3 = | \cdot |^{s_1 + s_2 + s_3} \text{sgn}^{k}$ for some $\lambda \in \{0, 1\}$.

**Proposition 4.12. We have**

1. Suppose $\pi_1, \pi_2$ and $\pi_3$ are principal series. Then $I^*(\Pi, t) = I^*_{\text{het}}(\Pi, t) = 1$.
2. Suppose $\pi_1$ is a discrete series and $\pi_2, \pi_3$ are principal series. Then $I^*(\Pi, t) = I^*_{\text{het}}(\Pi, t) = 2^{-k_1 + 2k_2 + 2k_3 - 1}$.
3. Suppose $\pi_1, \pi_2$ are discrete series and $\pi_3$ is a principal series. Then $I^*(\Pi, t) = I^*_{\text{het}}(\Pi, t) = 2^{-k_1 + k_2 + 3k_3 - 1}$.

**Proof.** By Corollary \[ \textcolor{red}{[4,3]} \] we only need to compute $I^*(\Pi, t)$. Suppose we are in the case (1). Note that $\epsilon(1/2, As\pi_\mathcal{K} \otimes \mu, \psi) = \left( \sqrt{-1} \right)^{k_1 + k_2 + k_3}$. This case have the following three sub-cases:

(a) $k_1 = k_2 = k_3 = 0$ and $\mu_1 \mu_2 \mu_3 (-1) = 1$;
(b) $k_1 = k_2 = k_3 = 0$ and $\mu_1 \mu_2 \mu_3 (-1) = -1$;
(c) $k_1 = k_3 = 1$ and $k_2 = 0$.

We only compute sub-case (b) since the other two sub-cases are similar. We have $W = W_{\pi_1} \otimes \rho(\tilde{V}_+)W_{\pi_2}$ and $f = \rho(J)\rho(\tilde{V}_+)f_\pi$. 


Observe that $W \otimes f$ is $\text{SO}(2)$-invariant on the right. By (4.16), (4.17), (4.20) and (4.107) [6.576.4] we find that
\[
\mathcal{R}_S(W \otimes f) = -\mu(-1) \left( \frac{2s_3 + 1}{8\pi} \right) \int_{\mathbb{R}^*} |y|^{s_3 + \frac{3}{2}} K_{s_1}(2\pi|y|) K_{s_2}(2\pi|y|) d^x y
\]
\[
= -\mu(-1) 2^{-2\pi s_3 - \frac{3}{2}} \Gamma \left( \frac{s_1 + s_2 + s_3}{2} + \frac{3}{4} \right) \Gamma \left( \frac{s_1 - s_2 + s_3}{2} + \frac{3}{4} \right) \Gamma \left( \frac{s_1 + s_2 + s_3}{2} + \frac{3}{4} \right) \Gamma \left( \frac{s_1 - s_2 + s_3}{2} + \frac{3}{4} \right)
\]
\[
= -\mu(-1) 2^{-2\pi s_3 + \frac{3}{2}} \frac{L \left( \frac{1}{2}, \text{As} \pi_K \otimes \mu \right)}{\Gamma \left( s_3 + \frac{1}{2} \right)}.
\]

Suppose we are in the case (2). In this case, the $L$- and the $e$-factor is given by
\[
L \left( 1/2, \text{As} \pi_K \otimes \mu \right) = 2^{-k_1 - 2s_3 + 2} \pi^{-k_1 - 2s_3} \Gamma \left( k_1/2 + s_2 + s_3 \right) \Gamma \left( k_1/2 - s_2 + s_3 \right),
\]
\[
\epsilon \left( 1/2, \text{As} \pi_K \otimes \mu, \psi \right) = (-1)^{k_1}.
\]

This case also has three sub-cases:
(a) $k_1 \equiv 0 \pmod{2}$ and $k_2 = k_3 = 0$;
(b) $k_1 \equiv 0 \pmod{2}$ and $k_2 = k_3 = 1$;
(c) $k_1 \equiv 1 \pmod{2}$ and $k_2 = 0, k_3 = 1$.

Note that in any sub-case,
\[
W = W_{\pi_1} \otimes W_{\pi_2} \quad \text{and} \quad f = \rho(r_\pi) \rho(\hat{V}_+ V_+) f_{\pi_3},
\]
and the element $W \otimes f$ is $\text{SO}(2)$-invariant on the right. We only compute sub-case (c) as the other two sub-cases are similar. By (4.15), (4.16), (4.20) and (4.107) [6.621.3] one has
\[
\mathcal{R}_S(W \otimes f) = \mu(-1) \left( -1 \right) \frac{k_1 - 2 - k_1 + 1}{2} \frac{\Gamma \left( s_3 + \frac{k_1 + 1}{2} \right)}{\Gamma \left( s_3 + 1 \right)} \int_0^\infty y^{k_1 - 2s_3 - 1} e^{-2\pi y} K_{s_2}(2\pi y) dy
\]
\[
= \mu(-1) \left( -1 \right) \frac{k_1 - 2 - k_1 + 1}{2} \frac{\Gamma \left( k_1 + k_2 - 1 \right) + \Gamma \left( k_1 - k_2 + 1 \right)}{\Gamma \left( s_3 + 1 \right)}.
\]

Suppose we are in the case (3). In this case, the $L$- and the $e$-factor is given by
\[
L \left( 1/2, \text{As} \pi_K \otimes \mu \right) = 2^{-k_1 - 2s_3 + 2} \pi^{-k_1 - 2s_3} \Gamma \left( k_1/2 + s_2 + s_3 \right) \Gamma \left( k_1/2 - s_2 + s_3 \right),
\]
\[
\epsilon \left( 1/2, \text{As} \pi_K \otimes \mu, \psi \right) = (-1)^{k_1}.
\]

Also, by definition
\[
W = W_{\pi_1} \otimes W_{\pi_2} \quad \text{and} \quad f = \rho(r_\pi) \rho(\hat{V}_+ V_+) f_{\pi_3},
\]
Note that the element $W \otimes f$ is $\text{SO}(2)$-invariant on the right. By (4.15) and (4.20), we have
\[
\mathcal{R}_S(W \otimes f) = \mu(-1) \left( -1 \right) \frac{k_1 - 2 - k_3}{2} \frac{\Gamma \left( k_1 + k_2 - 1 \right) + \Gamma \left( k_1 - k_2 + 1 \right)}{\Gamma \left( s_3 + 1 \right)} \int_0^\infty y^{k_1 - 2k_3 - 1} e^{-4\pi y} dy
\]
\[
= \mu(-1) \left( -1 \right) \frac{k_1 - 2 - k_3}{2} \frac{\Gamma \left( k_1 + k_2 - 1 \right) + \Gamma \left( k_1 - k_2 + 1 \right)}{\Gamma \left( s_3 + 1 \right)}.
\]

The proposition now follows from these calculations together with Lemma 4.8, Lemma 4.10 and Corollary 4.11.

4.3.2. Case $E = \mathbb{C} \times \mathbb{R}$. We compute the local period integrals for the case $E = \mathbb{C} \times \mathbb{R}$. Recall that $k' \geq 0$ is the minimal $\text{SU}(2)$-type of $\pi'$ and $K = \mathbb{C}$ by our convention. Let $\pi' = \text{Ind}^{\text{GL}_2(\mathbb{C})}(\mu' \boxtimes \nu')$ be a principal series representation of $\text{GL}_2(\mathbb{C})$. Let $s' \in \mathbb{C}$ so that $\mu' \nu'^{-1} (r e^{i\theta}) = |r|^{2s' + k} e^{ik' \theta}$ for $r, \theta \in \mathbb{R}$ with $r > 0$. Suppose $\mu' \nu^{-1} = |r|^{2s} \text{sgn}^k$ for some $s \in \mathbb{C}$ with $\text{Re}(s) \geq 0$. Note that our assumption on $\omega_{11}$ implies $k' + k$ is even and $\mu' |_{\text{Re}} = \lambda \cdot |r|^{2s' + \text{sgn}}$ for some $\lambda \in \{0, 1\}$.

To compute the local Rankin-Selberg integrals $\mathcal{R}_S(W \otimes f)$ we need a lemma. Consider the following analogue of the Tate integral. Let $W' \in W(\pi_K, \psi_K)$. Define
\[
\zeta(W', \mu) = \int_{\mathbb{R}^*} W' \left( \begin{array}{cc} iy & 0 \\ 0 & 1 \end{array} \right) \mu(y)v(y)^{-\frac{1}{2}} dy.
\]
The convergence of this integral is guaranteed by the assumption $\Lambda(\Pi) < 1/2$ according to Lemma 4.4.

Recall that $W_j$ is given by (4.14) with $s$ and $k$ replaced by $s'$ and $k'$, respectively, for $0 \leq j \leq k'$.

**Lemma 4.13.** Let $0 \leq j \leq k'$. If $\nu' \mu(-1) = (-1)^j$, then

$$\zeta(W_j, \mu) = 2\mu'((\sqrt{-1})^{k'}(2\pi)^{-s}2^{-k'-\frac{3}{2}}\Gamma\left(s' + \frac{s}{2} + \frac{k'}{2} - \frac{j}{2} + \frac{1}{4}\right) \Gamma\left(-s' + \frac{s}{2} + \frac{j}{2} + \frac{1}{4}\right).$$

If $\nu' \mu(-1) \neq (-1)^j$, then the integrals vanish.

**Proof.** By (4.14), we have

$$\zeta(W_j, \mu) = 2\mu'((\sqrt{-1})^{k'}(2\pi)^{-s}2^{-k'-\frac{3}{2}}\Gamma\left(s' + \frac{s}{2} + \frac{k'}{2} - \frac{j}{2} + \frac{1}{4}\right) \Gamma\left(-s' + \frac{s}{2} + \frac{j}{2} + \frac{1}{4}\right).$$

In particular, if

$$\mu' \mu(-1)(1)^k = \mu' \mu(-1)\mu' \nu(-1)(1)^j = \nu' \mu(-1)(1)^j = -1,$$

then the integral vanishes. Assume $\nu' \mu(-1) = (-1)^j$. By [G107, 6.561.16]

$$\zeta(W_j, \chi) = 8(\sqrt{-1})^{k'}(2\pi)^{-s}2^{-k'-\frac{3}{2}}\Gamma\left(s' + \frac{s}{2} + \frac{k'}{2} - \frac{j}{2} + \frac{1}{4}\right) \Gamma\left(-s' + \frac{s}{2} + \frac{j}{2} + \frac{1}{4}\right).$$

This finishes the proof. \(\square\)

We now compute the local period integrals for the case $E = \mathbb{C} \times \mathbb{R}$.

**Proposition 4.14.** Let $\epsilon \in \{0, 1\}$ so that $\nu' \mu(-1) = (-1)^\epsilon$. Suppose either $k' > 0$ or $\epsilon = 0$. Then

$$I^*(\Pi, t) = \mu(-1)\left(\sqrt{-1}\right)^k2^{k'-2\epsilon} \quad \text{and} \quad I^*_{\text{nor}}(\Pi, t) = 2^{k'-2\epsilon}.$$

Suppose $k' = 0$ and $\epsilon = 1$. Then

$$I^*(\Pi, t) = -\mu(-1) \quad \text{and} \quad I^*_{\text{nor}}(\Pi, t) = 1.$$

**Proof.** We have four cases:

1. $k' \geq 0$ is even and $\epsilon = 0$;
2. $k' \geq 2$ is even and $\epsilon = 1$;
3. $k' > 0$ is odd and $\epsilon = 0$;
4. $k' = 0$ and $\epsilon = 1$.

Note that when $k'$ is odd, the two cases $\epsilon = 0, 1$ are essentially the same. Indeed, if $\nu' \mu(-1) = -1$, then $\mu' \mu(-1) = \mu' \nu^2 \mu(-1) = -\nu' \mu(-1) = 1$. As the local period integrals only depend on the isomorphism class of $\Pi$, we can replace $\nu'$ by Ind_{GL_2}(\mathbb{C})^{\nu' \otimes \mu'}$.

By Corollary 4.3, it suffices to compute $I^*(\Pi, t)$, and by Corollary 4.11 we need to calculate $\mathcal{R}_f(W \otimes f)$. The calculations for the first three cases are similar, so we only demonstrate the computation for the third case. In contrast, the last case needs more reasoning. Suppose we are in the case (3). The $L$- and the $\epsilon$-factor is given by

$$L(1/2, \text{As}_{\pi_K} \otimes \mu) = 2^{-s}5^{-s}2^{-2s}2^{k'-3}2\Gamma\left(s' + \frac{s}{2} + \frac{3}{4}\right) \Gamma\left(-s' + \frac{s}{2} + \frac{1}{4}\right) \Gamma\left(s + \frac{k'}{2} + \frac{1}{2}\right),$$

$$\epsilon(1/2, \text{As}_{\pi_K} \otimes \mu, \psi) = \left(\sqrt{-1}\right)^{k'+2}.$$

Also, by definition

$$W = W_{\pi'} \quad \text{and} \quad f = \rho(\pi_K)f_\pi.$$
Notice that $W \otimes f$ is SO(2)-invariant on the right. By \[4.18\] and Lemma \[4.13\], we find that
\[
\mathcal{R}_\delta(W \otimes f) = \sqrt{-1} \sum_{j=0}^{k'-1} \left( \frac{k'-1}{2} \right) \zeta(W_{2j}, \mu)
\]
\[
= 2\mu' (\sqrt{-1})^{k'+1} (2\pi)^{-s+k'\mu + \frac{1}{4}} \sum_{j=0}^{k'-1} \left( \frac{k'-1}{2} \right) \Gamma \left( s' + \frac{3}{2} + \frac{k'}{2} + 1 - j \right) \Gamma \left( -s' + \frac{s}{2} + \frac{k'}{2} + 1 + j \right)
\]
\[
= 2\mu' (\sqrt{-1})^{k'+1} (2\pi)^{-s+k'\mu + \frac{1}{4}} \Gamma \left( s' + \frac{3}{2} + \frac{k'}{2} + 1 - j \right) \Gamma \left( -s' + \frac{s}{2} + \frac{k'}{2} + 1 + j \right)
\]
The third case now follows from this computation together with Lemma \[4.8\], Lemma \[4.10\] and Corollary \[4.11\]. Suppose we are in the last case. The $L$- and the $\epsilon$-factor is given by
\[
L(1/2, \pi_K \otimes \mu) = 2^{-s+k'\mu - 2s - 2} \Gamma \left( s' + \frac{s}{2} + \frac{3}{4} \right) \Gamma \left( -s' + \frac{s}{2} + \frac{3}{4} \right) \Gamma \left( s + \frac{1}{2} \right),
\]
and we have
\[
W = \rho(V)W'_{\pi_K} \quad \text{and} \quad f = f_\pi.
\]
We first show that $W$ is SO(2)-invariant. To prove this, we consider a vector-valued Whittaker function $W'_{\pi_K}$, which generates the one-dimensional space
\[
(W(\pi_K, \psi_K) \otimes \mathcal{L}_2(\mathbb{C}))^{SU(2)}.
\]
Notice that by definition, $W'_{\pi_K}(hu) = \rho_2(u)^{-1} W_{\pi_K}(h)$ for all $h \in \text{GL}_2(\mathbb{C})$ and $u \in \text{SU}(2)$. This function $W'_{\pi_K}$ can be constructed as follows. Let $\tilde{\Phi}$ be the element in $\mathcal{S}(\mathbb{C}^2) \otimes \mathcal{L}_2(\mathbb{C})$ given by
\[
\tilde{\Phi}(z, w) = (-\sqrt{-1}) \Phi_1(z, w) X^2 - \Phi_2(z, w) Y^2 + \sqrt{-1} \Phi_3(z, w) Y^2,
\]
with
\[
\Phi_1(z, w) = \bar{z}\bar{w}, \quad \Phi_2(z, w) = -\left( z\bar{z} - w\bar{w} + \frac{1}{2\pi} \right) \Phi_0(z, w), \quad \Phi_3(z, w) = zw\Phi_0(z, w),
\]
where
\[
\Phi_0(z, w) = e^{-2\pi (z\bar{z} + w\bar{w})}.
\]
Then we can take
\[
W'_{\pi_K}(h) = \mu' (h)[h]_{\mathbb{C}}^{\frac{1}{4}} \int_{\mathbb{C}^\times} r_{\psi_K}(h) \tilde{\Phi}(t, t^{-1}) \mu' \mu' \mu(t) d^\times t, \quad h \in \text{GL}_2(\mathbb{C}).
\]
Here $(r_{\psi_K}, \mathcal{S}(\mathbb{C}^2))$ denote the Weil representation of $\text{GL}_2(\mathbb{C})$ given in \[JL\] Section 1, and we use the same notation $r_{\psi_K}$ to indicate the representation $(r_{\psi_K} \otimes 1, \mathcal{S}(\mathbb{C}^2) \otimes \mathcal{L}_2(\mathbb{C}))$ of $\text{GL}_2(\mathbb{C})$. All these facts can be deduced from \[Jac\] Section 18. For any element $\Phi$ in $\mathcal{S}(\mathbb{C}^2)$, we define a $\mathbb{C}$-valued function $W_\Phi$ on $\text{GL}_2(\mathbb{C})$ by the same integral with $\tilde{\Phi}$ replaced by $\Phi$. With this notation, we have $W_{\pi_K} = W_{\Phi_{\pi_0}}$.

Consider an element in $W(\pi_K, \psi_K)$ defined by
\[
W'(h) = (\bar{W}_\pi(h), (X^2 + Y^2))_2, \quad h \in \text{GL}_2(\mathbb{C}).
\]
Evidently, $W'$ is SO(2)-invariant and we have
\[
W'(h) = (-\sqrt{-1}) W_{\Phi_1}(h) + (\sqrt{-1}) W_{\Phi_3}(h).
\]
On the other hand, one check easily that
\[
r_{\psi_K}(V)\Phi_0 = (-\sqrt{-1}) \Phi_1 + (\sqrt{-1}) \Phi_3.
\]
It follows that
\[
W = \rho(V)W_{\pi_K} = \rho(V)W_{\Phi_0} = W_{r_{\psi_K}(V)\Phi_0} = W'.
\]
This shows our claim.
By [2.12], we find that
\[
W_{\Phi_1} \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = -4\mu(y) \overline{y} |y|_C^{\frac{1}{2}} K_{2s'} \left( 4\pi |y|_C^2 \right) ; \quad W_{\Phi_3} \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = 4\mu(y) \overline{y} |y|_C^{\frac{1}{2}} K_{2s'} \left( 4\pi |y|_C^2 \right).
\]
Using the formula [CH07, 6.561.16], we obtain
\[
\mathcal{B}_0(W \otimes f) = \left( -\sqrt{-1} \right) \zeta(W_{\Phi_1}, \mu) + \left( \sqrt{-1} \right) \zeta(W_{\Phi_3}, \mu) \\
= (-4)\mu (\sqrt{-1}) (2\pi)^{-\left(s' + \frac{3}{4}\right)} \Gamma \left( s' + \frac{8}{2} \right). \\
\]
The last case now follows from these computations together with Lemma 4.8, Lemma 4.10, and Corollary 4.11. This completes the proof.

5. Proof of Proposition 4.3

The purpose of this section is to prove Proposition 4.3 when \( F = \mathbb{R} \). We need some preparations; however, its safe to ship the first two subsections and go directly to the proofs.

5.1. Majorization of Whittaker functions. Let \( F = \mathbb{R} \) or \( \mathbb{C} \) and let \( \psi = \psi_F \) be the additive character described in [2.3]. Let \( dt \) be the measure on \( F \) which is self-dual with respect to \( \psi \) and let \( d^x t = |t|_F^{-1} dt \) be the measure on \( F^\times \). Let \( \text{Ind}_{B(F)}^{GL_2(F)}(\mu \boxtimes \nu) \) be an induced representation of \( GL_2(F) \) with underlying space \( \mathcal{B}(\mu, \nu) \). Let \( S(F^2, \psi) \) be the spaces defined in [4.22] and \( (r_\psi, S(F^2)) \) be the Weil representation of \( GL_2(F) \) given in [IL70, Section 1]. For \( \Phi \in S(F^2) \), we define
\[
W_\Phi(g) = \mu(g)|g|_F^{\frac{1}{2}} \int_{F^\times} r_\psi(t) \Phi(t^{-1}) \mu^{-1}(t) d^x t, \quad g \in GL_2(F).
\]
This integral converges absolutely. Moreover, if \( W(\mu, \nu; \psi) \) is the space spanned by \( W_\Phi \) with \( \Phi \in S(F^2) \), then we have \( W(\mu, \nu; \psi) = W(\nu, \mu; \psi) \) and \( \mathcal{B}(\mu, \nu) \simeq W(\mu, \nu) \) as \( GL_2(F) \)-modules whenever \( |\mu^{-1}(y)| = |y|_F^r \) with \( r > -1 \). Let \( \mathcal{K}(\mu, \nu; \psi) \) be the space which consists of the functions \( y \mapsto W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \) for \( W \in W(\mu, \nu; \psi) \). We want to describe the space \( \mathcal{K}(\mu, \nu; \psi) \). Notice that if \( a \in F^\times \) and \( \psi_a \) is the character given by \( \psi_a(x) = \psi(ax) \), then the space \( \mathcal{K}(\mu, \nu; \psi_a) \) consists of the functions \( y \mapsto \varphi(ay) \) with \( \varphi \in \mathcal{K}(\mu, \nu; \psi) \). Therefore there is no loss of generality to choose \( \psi \) as above.

Lemma 5.1. Notation be as above.

1. Let \( F = \mathbb{R} \) and \( \mu(y) = |y|_R^{s_1} \text{sgn}^{m_1}(y), \nu(y) = |y|_R^{s_2} \text{sgn}^{m_2}(y) \) for some \( s_1, s_2 \in \mathbb{C} \) and \( m_1, m_2 \in \{0, 1\} \). If \( \mu^{-1}(-1) = (-1)^m \) for some integer \( m \), then the space \( \mathcal{K}(\mu, \nu; \psi) \) is spanned by the functions of the form
\[
y \mapsto \text{sgn}^{m_1}(y) \cdot y^a \cdot |y|_R^{\frac{a+b}{2} - \frac{a+b}{2}} \cdot K_{\frac{s_1+s_2}{2}}(2\pi |y|_R), \quad a, b \in \mathbb{Z}_{\geq 0}, \quad a - b \equiv m \pmod{2}.
\]

2. Let \( F = \mathbb{C} \) and \( \mu(y) = |y|_C^{\frac{m_1+m_1-n_1}{2}} \cdot y^{m_1-n_1}, \nu(y) = |y|_C^{\frac{m_2+n_2}{2}} \cdot y^{m_2+n_2} \) for some \( s_1, s_2 \in \mathbb{C} \) and non-negative integers \( m_1, m_2, n_1, n_2 \) with \( m_1 n_1 = m_2 n_2 = 0 \). If \( \mu^{-1}(y) = y^m \psi_n \) for some non-negative integers \( m, n \) with \( mn = 0 \) for all \( y \in \mathbb{C} \) with \( |y|_C = 1 \), then the space \( \mathcal{K}(\mu, \nu; \psi) \) is spanned by the functions of the form
\[
y \mapsto y^{a+m_1} \cdot \psi^{b+n_1} \cdot |y|_C^{\frac{a+b}{2} - \frac{a+b}{2}} \cdot K_{\frac{n_1-n_2}{2}}(2\pi |y|_C^x),
\]
where \( a, b, c, d \) are non-negative integers satisfying \( a - c + m = b - d + n \).

Proof. It suffices to compute \( W_\Phi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \) with \( \Phi(x_1, x_2) = x_1^a x_2^b e^{-\pi(x_1^2+x_2^2)} \) or \( \Phi(x_1, \bar{x}_1, x_2, \bar{x}_2) = x_1^{a+b+c} x_2^{a+b+d} e^{-2\pi(x_1 \bar{x}_1 + x_2 \bar{x}_2)} \) according to \( F = \mathbb{R} \) or \( F = \mathbb{C} \). Here \( a, b, c, d \) are non-negative integers. Recall that
\[
r_\psi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \Phi(x_1, x_2) = \Phi((x_1 y, x_2)), \quad \Phi \in S(F^2).
\]
The lemma now follows immediately from [5.1]. The constraints on \( a, b, c \) and \( d \) are to avoid \( W_\Phi = 0 \). \( \square \)
Our next step is to estimate the functions in $K(\mu, \nu; \psi)$. We need a lemma.

**Lemma 5.2.** Let $s \in \mathbb{C}$ with $|\text{Re}(s)| = r$. Let $r_1 > 0$ such that $r_1 \geq r$. Let $\varphi$ be a function on $\mathbb{R}^+$ defined by

$$\varphi(y) = y^{r_1}K_s(|y|).$$

Then we have

$$\varphi(y) \ll_{r,r_1} \exp \left( \frac{1}{2} |y| \right).$$

**Proof.** Since $K_s(z) = K_{-s}(z)$ and $K_s(|y|) \ll K_{\text{Re}(s)}(|y|)$, we may assume $s = r$. The lemma follows immediately from the asymptotic form of $K_r(|y|)$. Indeed, when $|y| \to \infty$, we have

$$K_r(|y|) \sim \sqrt{\frac{\pi}{2|y|}} e^{-|y|},$$

while when $0 < |y| \ll 1$, we have

$$K_r(|y|) \sim \begin{cases} -\ln \left( \frac{|y|}{2} \right) - \gamma & \text{if } r = 0, \\ \frac{\Gamma(r)}{2|y|^r} \left( \frac{2}{|y|} \right)^r & \text{if } r > 0. \end{cases}$$

Here $\gamma$ stands for the Euler's constant. This finishes the proof. \hfill \Box

There are several corollaries we should need. Let $\pi$ be a unitary irreducible admissible generic representation of $\text{GL}_2(F)$, which we assume to be the constituent of $\text{Ind}^{\text{GL}_2(F)}_{\text{B}_1(F)}(\mu \boxtimes \nu)$ in the next three corollaries. Let $0 \leq s(\pi) < 1/2$ be the real number attached to $\pi$ defined in [IC08, page 284]. Let $K$ be the compact subgroup given by [4.3].

**Corollary 5.3.** Let $\varepsilon > 0$ and $W \in W(\pi, \psi)$. Then

$$W \left( \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} \right) \ll_{\pi, W, \varepsilon} |y|^{\frac{1}{2} - s(\pi) - \varepsilon} \exp \left(-d|y|^{\frac{1}{2}}\right),$$

for every $k \in K$. Here $d = [F : \mathbb{R}]$.

**Proof.** By the right $K$-finiteness, it suffices to prove the assertion when $k$ is the identity element. To do this, we use Lemma 5.1 and Lemma 5.2. By Lemma 5.1, we may assume

$$W \left( \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} \right) = y^{a+m_1} \cdot \tilde{y}^{b+n_1} \cdot |y|^{\frac{1}{2} + \frac{a+c}{2} - \frac{b+d}{2} - \frac{m_1+n_1}{2}} \cdot K_{s_1 - s_2 + \frac{a-c}{2} + \frac{b+d}{2}} \left( 4\pi |y|^{\frac{1}{2}} \right),$$

with $a, b, c, d, \in \mathbb{Z}_{>0}$. Notice that since $\pi$ is unitary, we have $\text{Re}(s_1 + s_2) = 0$ and $\text{Re}(s_1 - s_2) = \pm 2s(\pi)$. Let $r = \frac{a+b}{2} - \left\{ \frac{a-c}{2} + \frac{b+d}{2} \right\}$ then $r \geq 0$. Since

$$\left| \text{Re}(s_1 - s_2) + \frac{a-c}{2} + \frac{b+d}{2} \right| < \left| \text{Re}(s_1 - s_2) \right| + \left| \frac{a-c}{2} + \frac{b+d}{2} \right| + r = 2s(\pi) + \left| \frac{a-c}{2} + \frac{b+d}{2} \right| + r,$$

we have, by Lemma 5.2 that

$$K_{s_1 - s_2 + \frac{a-c}{2} + \frac{b+d}{2}} \left( 4\pi |y|^{\frac{1}{2}} \right) \ll_{\pi, W, \varepsilon} |y|^{-s(\pi) - \frac{1}{2}} \exp \left(-2\pi |y|^{\frac{1}{2}}\right).$$

It follows that

$$W \left( \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} \right) \ll_{\pi, W, \varepsilon} |y|^{\frac{1}{2} - s(\pi) - \frac{1}{2} - \frac{a+c}{2} + \frac{b+d}{2}} \exp \left(-2\pi |y|^{\frac{1}{2}}\right).$$

The case $F = \mathbb{R}$ can be proved in the similar way. \hfill \Box

**Corollary 5.4.** Suppose $F = \mathbb{C}$. Let $y \in \mathbb{C}^\times$ and $k \in \text{SU}(2)$. Given $W \in W(\pi, \psi)$, we define a function $\varphi = \varphi_{y,k}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ by

$$\varphi(\alpha, \beta) = W \left( \begin{bmatrix} (\alpha + i\beta)y & 0 \\ 0 & 1 \end{bmatrix} \right).$$

Then for $\varepsilon > 0$,

$$\partial_\alpha \varphi(\alpha, \beta) \ll_{\pi, W, \varepsilon} |y|^{\frac{1}{2} - s(\pi) - \varepsilon} \exp \left(-|\alpha y|^{\frac{1}{2}} - |\beta y|^{\frac{1}{2}}\right).$$
Proof. By the right SU(2)-finiteness, we may assume $k$ is the identity element. By Lemma 5.1, it suffices to prove this corollary for $W$ with

$$W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = y^{a + m_1} y^{b + n_1} |y|_{C}^{\frac{b - d - m_1}{2}} \left(4\pi|y|_{C}^{\frac{1}{2}}\right),$$

with $a, b, c, d, m_1, n_1 \in \mathbb{Z}_{\geq 0}$. Note that $\text{Re}(s_1 + s_2) = 0$ and $\text{Re}(s_1 - s_2) = \pm 2\lambda(\pi)$. For brevity, we put

$$u = \frac{1}{2} s_1 + s_2 = \frac{a - c}{4} - \frac{b - d}{4} \frac{m_1 + n_1}{2}; \quad v = s_1 - s_2 + \frac{a - c}{2} + \frac{b - d}{2}.$$

According to our definition,

$$\varphi(\alpha, \beta) = (\alpha + i\beta)^{a + m_1}(\alpha - i\beta)^{b + n_1}(\alpha^2 + \beta^2)^u y^{a + m_1} y^{b + n_1} |y|_{C}^{\frac{u}{2}} K_{v} \left(4\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right).$$

Using the equation

$$(5.2) \quad \partial_{s} K_{s}(z) = -\frac{1}{2} \{K_{s-1}(z) + K_{s+1}(z)\},$$

and by a direct computation, we find that

$$\partial_{\alpha} \varphi(\alpha, \beta) = C_{1} (\alpha + i\beta)^{a + m_1 - 1}(\alpha - i\beta)^{b + n_1}(\alpha^2 + \beta^2)^u y^{a + m_1} y^{b + n_1} |y|_{C}^{\frac{u}{2}} K_{v} \left(4\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right)$$

and by a direct computation, we find that

$$\partial_{\beta} \varphi(\alpha, \beta) = C_{2} (\alpha + i\beta)^{a + m_1}(\alpha - i\beta)^{b + n_1 - 1}(\alpha^2 + \beta^2)^u y^{a + m_1} y^{b + n_1} |y|_{C}^{\frac{u}{2}} K_{v} \left(4\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right)$$

and by a direct computation, we find that

$$\partial_{\alpha} \varphi(\alpha, \beta) = C_{3} (\alpha + i\beta)^{a + m_1}(\alpha - i\beta)^{b + n_1 + 1}(\alpha^2 + \beta^2)^u y^{a + m_1} y^{b + n_1} |y|_{C}^{\frac{u}{2}} K_{v} \left(4\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right)$$

and by a direct computation, we find that

$$\partial_{\beta} \varphi(\alpha, \beta) = C_{4} (\alpha + i\beta)^{a + m_1}(\alpha - i\beta)^{b + n_1 + 1}(\alpha^2 + \beta^2)^u y^{a + m_1} y^{b + n_1} |y|_{C}^{\frac{u}{2}} K_{v-1} \left(4\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right)$$

and by a direct computation, we find that

$$\partial_{\alpha} \varphi(\alpha, \beta) = C_{5} (\alpha + i\beta)^{a + m_1}(\alpha - i\beta)^{b + n_1 + 1}(\alpha^2 + \beta^2)^u y^{a + m_1} y^{b + n_1} |y|_{C}^{\frac{u}{2}} K_{v+1} \left(4\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right).$$

Here $C_{1}, C_{2}, \ldots, C_{5}$ are constants whose absolute values depend at most upon $\pi$ and $W$. We estimate, for example, the last term on the RHS of (5.3). Let $r = (a + b) - (\frac{a - c}{2} + \frac{b - d}{2} + |\frac{a - c}{2} + \frac{b - d}{2}|) + 2\epsilon$. Then $r \geq 2\epsilon > 0$. Since

$$|v + 1| = \left|\text{Re}(s_1 - s_2) + \frac{a - c}{2} + \frac{b - d}{2} + 1\right| < 2\lambda(\pi) + \left|\frac{a - c}{2} + \frac{b - d}{2}\right| + (r + 1),$$

we can apply Lemma 5.2 to obtain

$$K_{v+1} \left(4\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right) \ll_{\pi, W, \epsilon} \left\{|y|_{C}(\alpha^2 + \beta^2)\right\}^{-\frac{1}{2} - \lambda(\pi) - \epsilon} \exp \left(-2\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right).$$

Hence the last term is bounded by

$$C_{\pi, W, \epsilon} |\alpha|(\alpha^2 + \beta^2)^{-\frac{1}{2} - \lambda(\pi) - \epsilon} \left|y|_{C}^{-\lambda(\pi)} \exp \left(-2\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right)\right),$$

where $C_{\pi, W, \epsilon}$ is a constant depend at most upon $\pi$, $W$ and $\epsilon$. Since $|\alpha|(\alpha^2 + \beta^2)^{-\frac{1}{2} \leq 1$ and

$$\exp \left(-2\pi|y|_{C}^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}\right) \leq \exp \left(-\sqrt{2}\pi|y|_{C}^{\frac{1}{2}}(|\alpha| + |\beta|)\right) \leq \exp \left(-|y|_{C}^{\frac{1}{2}}(|\alpha| + |\beta|)\right),$$

our assertion holds for the last term. One can use similar arguments to estimate each of other terms and obtain the same result. Then the corollary follows. \qed

Corollary 5.5. Suppose $F = \mathbb{R}$. Let $0 < \xi < 1/2$, $y \in \mathbb{R}^{\times}$ and $k \in \text{SO}(2)$. Given $W \in W(\pi, \psi)$, we define a function $\varphi = \varphi_{y,k}$ on the interval $(-\xi, \xi)$ by

$$\varphi(\alpha) = W\left(\begin{pmatrix} (-1 + \alpha)y & 0 \\ 0 & 1 \end{pmatrix}k\right).$$

Then for $\epsilon > 0$,

$$\partial_{\alpha} \varphi(\alpha) \ll_{\pi, W, \epsilon} |y|_{C}^{\frac{1}{2} - \lambda(\pi) - \epsilon}.$$
Lemma 5.6. We have
\[ W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = \text{sgn}^{m_1}(y) y |y|^{\frac{\alpha}{2} + \frac{\alpha+b}{2}} \frac{\alpha+b}{2} K_{\frac{\alpha+b}{2}, \frac{\alpha+b}{2}} \left( 2\pi |y| \right), \]
with \( a, b, m_1 \in \mathbb{Z}_{\geq 0}. \) Note that \( \text{Re}(s_1 + s_2) = 0 \) and \( \text{Re}(s_1 - s_2) = \pm 2\lambda(\pi). \) By definition, we have
\[ \varphi(\alpha) = (-1)^{m_1 + a} \text{sgn}^{m_1}(y) y |y|^{\frac{\alpha}{2} + \frac{\alpha+b}{2} - \frac{\alpha+b}{2}} \left( 1 - \alpha \right)^{\frac{\alpha+b}{2} + \frac{\alpha+b}{2} + \frac{\alpha+b}{2}} K_{\frac{\alpha+b}{2}, \frac{\alpha+b}{2}} \left( 2\pi(1 - \alpha) |y| \right), \quad |\alpha| < \xi. \]
By (5.2) and a direct calculation, one finds that
\[ \partial_\alpha \varphi(\alpha) = C_1 \varphi(\alpha) \left( 1 - \alpha \right)^{\frac{\alpha+b}{2} + \frac{\alpha+b}{2} + \frac{\alpha+b}{2}} K_{\frac{\alpha+b}{2}, \frac{\alpha+b}{2}} \left( 2\pi(1 - \alpha) |y| \right) \]
\[ + C_2 \varphi(\alpha) \left( 1 - \alpha \right)^{\frac{\alpha+b}{2} + \frac{\alpha+b}{2} + \frac{\alpha+b}{2}} K_{\frac{\alpha+b}{2}, \frac{\alpha+b}{2} - 1} \left( 2\pi(1 - \alpha) |y| \right) \]
\[ + C_3 \varphi(\alpha) \left( 1 - \alpha \right)^{\frac{\alpha+b}{2} + \frac{\alpha+b}{2} + \frac{\alpha+b}{2}} K_{\frac{\alpha+b}{2}, \frac{\alpha+b}{2} - 1} \left( 2\pi(1 - \alpha) |y| \right). \]
Here \( C_1, C_2 \) and \( C_3 \) are constants whose absolute values depend at most upon \( \pi \) and \( W. \) We now estimate, for example, the second term on the right hand side of (5.4) as the proofs for the other terms are similar. Let \( r = \frac{a+b}{2} + |\frac{a+b}{2}| + \epsilon. \) Then \( r \geq \epsilon > 0. \) Since
\[ \left| \frac{s_1 - s_2}{2} + \frac{a-b}{2} - 1 \right| < \left| \frac{s_1 - s_2}{2} \right| + \left| \frac{a-b}{2} \right| + (r + 1), \]
we have, by Lemma 5.2, that
\[ K_{\frac{\alpha+b}{2}, \frac{\alpha+b}{2}-1} \left( 2\pi(1 - \alpha) |y| \right) \ll_{\pi, W, \epsilon} \left( 2\pi(1 - \alpha) |y| \right)^{-\lambda(\pi)-|\frac{a+b}{2}|+(r+1)} \exp\left( -\pi(1 - \alpha) |y| \right) \ll_{\pi, W, \epsilon} |y|^{-\lambda(\pi)-|\frac{a+b}{2}|+(r+1)}. \]
For the least inequality, we have used \( (1 - \alpha)^{\pm} \leq 2 \) and \( \exp(-\pi(1 - \alpha) |y|) \leq 1. \) It follows that the second term is bounded by \( C |y|^{\frac{1}{2} - \lambda(\pi) + |\frac{a+b}{2}| - |\frac{a+b}{2}| - r} = C |y|^{\frac{1}{2} - \lambda(\pi) - r}. \) Here \( C \) is a constant depends at most upon \( \pi, \)
\( W \) and \( \epsilon. \) This concludes the proof. \( \square \)

5.2. Some lemmas. Let \( n \in \mathbb{N}. \) We introduce a function \( \phi_n : \mathbb{R}^\times \setminus \text{GL}_2(\mathbb{R}) \to \{0, 1\}, \) which will be used later. The idea of defining these functions comes from [HS17, Proposition 5.1]. Define
\[ \phi_n \left( k \begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix} k' \right) = \begin{cases} 1 & \text{if } n^{-1} \leq |y_1/y_2| \leq n, \\
0 & \text{otherwise}, \end{cases} \]
where \( k, k' \in \text{SO}(2). \) Its clear from the definition that \( \phi_n \) are compact support functions on \( \mathbb{R}^\times \setminus \text{GL}_2(\mathbb{R}), \) as well as \( \text{SO}(2) \)-invariant on both sides. Moreover, we have \( \lim_{n \to \infty} \phi_n(g) = 1_{\mathbb{R}^\times \setminus \text{GL}_2(\mathbb{R})}(g) \) pointwisely. Following lemma tells us the support of \( \phi_n \) on \( B(\mathbb{R}). \)

Lemma 5.6. We have
\[ \phi_n \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = 1 \iff \frac{|x|^2}{|y|} + |y| + |y|^{-1} \leq n + n^{-1}. \]

Proof. Let \( g \in \text{GL}_2(\mathbb{R}). \) We denote \( g^t \) to be the transport of \( g. \) Evidently, we have \( \phi_n(g) = 1 \) if and only if \( \phi_n(\gamma g^t) = 1. \) Now let \( g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}. \) Then
\[ gg^t = \begin{pmatrix} x^2 + y^2 & x \\ x & 1 \end{pmatrix} = zk \begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix} k', \]
for some \( z > 0, r \geq 1 \) and \( k \in \text{SO}(2). \) Taking trace and norm we find that \( x^2 + y^2 + 1 = z(r^2 + 1) \) and \( y^2 = z^2 r^2. \) This implies
\[ \frac{|x|^2}{|y|} + |y| + |y|^{-1} = r + r^{-1}. \]
The function \( f(x) = x + x^{-1} \) with \( x > 0 \) has the following properties:
- \( f(x) \) is strictly increasing on the interval \( [1, \infty); \)
- \( f(x) \) is decreasing on the interval \( (0, 1]. \)
\[ f(x) = f(x^{-1}). \]

Notice that these imply \( f(x) \) is strictly decreasing on the interval \((0, 1)\) and hence has the minimal 2 at \( x = 1 \).

Since \( \phi_n(zg') = 1 \) if and only if \( r \leq n \), the sufficient condition follows from \((5.6)\) and the fact \( f(r) \leq f(n) \).

Conversely, suppose \( \frac{1}{|y|} + |y| + |y|^{-1} \leq n + n^{-1} \). Since the left hand side of the inequality above is \( \geq 2 \), there exists \( 1 \leq r \leq n \) such that the \((5.6)\) holds. Let \( z > 0 \) such that \( |y| = zr \). We then have \( z^2 + y^2 + 1 = z(r^2 + 1) \) by the equation \((5.6)\). It follows that the two eigenvalues of \( gg' \) are \( z \) and \( zr^2 \). Hence \( \phi_n(zg') = 1 \). \( \square \)

**Remark 5.7.** One can define \( \phi_n \) and prove Lemma \((5.4)\) for \( \phi_n \) when \( F = \mathbb{C} \) similarly [Che18b, Section 6], which might be used to prove Proposition \((4.5)\) for the complex field case.

Recall that \( \int_0^\infty \sin x dx = \frac{\pi}{2} < \infty \). Next lemma is simple, but important to us.

**Lemma 5.8.** There exists \( M > 0 \) such that \( \left| \int_a^b \sin x dx \right| \leq M \) for any \(-\infty \leq a \leq b \leq \infty\).

**Proof.** Since the integrand is an even function, its enough to prove the assertion for \( 0 \leq a \leq b \leq \infty \).

Let \( N > 0 \) so that \( \left| \int_c^\infty \sin x dx \right| \leq 1 \), for every \( c \geq N \). Then for any \( b \geq N \), we have
\[
\left| \int_N^b \sin x dx \right| = \left| \int_N^\infty \sin x dx - \int_0^b \sin x dx \right| \leq 2.
\]

On the other hand, let \( \Omega = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid 0 \leq a \leq b \leq N\} \) and consider the continuous function \( f(a,b) \) on \( \Omega \) defined by \( f(a,b) = \int_a^b \sin x dx \). As \( \Omega \) is compact, there exists \( M' > 0 \) so that \( |f(a,b)| \leq M' \) on \( \Omega \). Then for \( M \geq M' + 2 \), we have \( \left| \int_a^b \sin x dx \right| \leq M \), for every \( 0 \leq a \leq b \leq \infty \). This proves the lemma. \( \square \)

In the next two lemmas, we let \( \{r_n\}_{n \in \mathbb{N}} \) be a sequence of positive real numbers such that \( \lim_{n \to \infty} r_n = \infty \).

**Lemma 5.9.** Let \( \alpha, \beta > 0 \) and \( 0 \leq a < b \). For every \( f \in C^1([a,b]) \), we have
\[
\lim_{n \to \infty} \int_a^b f(x) \frac{\sin r_n x}{\beta x} dx = \begin{cases} 0 & \text{if } a > 0, \\ \frac{\pi}{2\beta} f(0) & \text{if } a = 0. \end{cases}
\]

**Proof.** One can see, for example [Sne51, Theorem 5]. \( \square \)

**Lemma 5.10.** Let \( \alpha, \beta > 0 \). Let \( f \in L^1(\mathbb{R}) \) and let \( D_f \subset \mathbb{R} \) be the set consisting all the discontinuous points of \( f \). Suppose \( D_f \) is a finite set and \( f \in C^1(\mathbb{R} \setminus D_f) \). Then for every \( 0 < b \) with \( a, b \notin D_f \), we have
\[
\lim_{n \to \infty} \int_{-\infty}^a f(x) \frac{\sin r_n x}{\beta x} dx = \lim_{n \to \infty} \int_b^\infty f(x) \frac{\sin r_n x}{\beta x} dx = 0.
\]

**Proof.** We first note that it suffices to show \( \lim_{n \to \infty} \int_b^\infty f(x) \frac{\sin r_n x}{\beta x} dx = 0 \). Indeed, if we put \( f^-(x) = f(-x) \), then \( f^– \) satisfies the same conditions as \( f \) does. Moreover, we have \( -a \notin D_f \). Therefore, we find that
\[
\lim_{n \to \infty} \int_{-\infty}^a f(x) \frac{\sin r_n x}{\beta x} dx = \lim_{n \to \infty} \int_a^{-\infty} f^{-1}(x) \frac{\sin r_n x}{\beta x} dx = 0.
\]
Let \( D_{f,b} = D_f \cap (b, \infty) = \{y_1, y_2, \ldots, y_m\} \) for some \( m \geq 0 \). We assume \( y_1 < y_2 < \cdots < y_m \). Let \( \epsilon > 0 \) be given. Choice \( y_{m+1} > y_m \) large so that
\[
\int_{y_{m+1}}^\infty \left| \frac{f(x)}{\beta x} \right| dx < \epsilon.
\]
Let \( \xi > 0 \) be small so that \( b < y_1 - \xi \) and \( y_j < y_{j+1} - \xi \) for \( j = 1, 2, \ldots, m \). Moreover, we have \( \int_{y_{j+1}}^{y_j} f(x) \frac{\sin r_n x}{\beta x} dx \) can be arbitrary small as \( n \to \infty \) for \( j = 1, 2, \ldots, m \). It follows that
\[
\int_b^\infty f(x) \frac{\sin r_n x}{\beta x} dx \leq (m + \epsilon) \epsilon
\]
for all \( n \) sufficiently large. This completes the proof. \( \square \)

5.3. **Proof of Proposition 4.5** We start the proof for Proposition 4.5. We follow the notation in (1.2.1). By Remark 4.6, we only need to dual with the case \( F = \mathbb{R} \), and hence \( \mathcal{K} = \mathbb{R} \times \mathbb{R} \) or \( \mathcal{K} = \mathbb{C} \). Let \( \psi \) be any nontrivial additive character of \( \mathbb{R} \). Haar measures on various groups are those described in (12.1). We will see that Proposition 4.5 follows immediately from Lemma 5.12, whose proof will be occurred in the next subsection. To state the lemma, we need some notation. Let \( W \in \mathcal{W}(\pi \mathcal{K}, \psi \mathcal{K}) \) and \( \tilde{W} \in \mathcal{W}(\tilde{\pi} \mathcal{K}, \psi \mathcal{K}) \). Note that when \( \mathcal{K} = \mathbb{R} \times \mathbb{R} \), we have \( \pi \mathcal{K} = \pi_1 \boxtimes \pi_2 \), where \( \pi_j \) are unitary irreducible admissible generic representations of \( GL_2(\mathbb{R}) \) for \( j = 1, 2 \). We assume \( W = W_1 \otimes W_2 \) and \( \tilde{W} = W_1 \otimes \tilde{W}_2 \) when \( \mathcal{K} = \mathbb{R} \times \mathbb{R} \), where \( W_j \in \mathcal{W}(\pi_j, \psi) \)
and $\tilde{W}_j \in \mathcal{W}(\bar{\pi}_j, \psi)$. Let $\mu$ be a character of $\mathbb{R}^\times$. For $n \in \mathbb{N}$ and $k_1, k_2 \in \text{SO}(2)$, let $\phi_n$ be the function defined by (5.6) and we define the following integrals. If $K = \mathbb{C}$, we put

$$I_n(k_1, k_2) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \mathcal{B}_\nu \left( \rho \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k_1 \right) W, \rho(k_2) \tilde{W} \right) \mu(y) |y|^{-\frac{1}{2}} \phi_n \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) dx dy.$$

If $K = \mathbb{R} \times \mathbb{R}$, we put

$$I_n(k_1, k_2) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \psi(x) \mathcal{B}_\nu \left( \rho \left( \begin{pmatrix} y_1 & x \\ 0 & 1 \end{pmatrix} k_1 \right) W_1, \rho \left( \begin{pmatrix} y_2 & 0 \\ 0 & 1 \end{pmatrix} k_2 \right) \tilde{W}_1 \right) W_2 \left( \begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix} k_1 \right) dx_1 dy_1.$$  

Finally, we put, for both cases

$$R_n(k_1, k_2) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} W \left( \begin{pmatrix} \delta y_1 & 0 \\ 0 & 1 \end{pmatrix} k_1 \right) \tilde{W} \left( \begin{pmatrix} -\delta y_2 & 0 \\ 0 & 1 \end{pmatrix} k_2 \right) \mu(y_1 y_2^{-1}) |y_1 y_2|^{-\frac{1}{2}} \phi_n \left( \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} \right) dx_1 dy_1 dy_2 y.$$  

To study the convergence of the integrals we just defined, we associate a nonnegative real number $\Lambda(\pi_K)$ to $\pi_K$, which is analogy to $\Lambda(\Pi)$ defined in [LCh83, page 285]. Put

$$\Lambda(\pi_K) = \begin{cases} 2\lambda(\pi_K) & \text{if } K = \mathbb{C}, \\ \lambda(\pi_1) + \lambda(\pi_2) & \text{if } K = \mathbb{R} \times \mathbb{R}. \end{cases}$$

Here $\lambda(*)$ is defined by [LCh83, page 284]. Let $\lambda \in \mathbb{R}$ such that $|\mu(y)| = |y|^\lambda$.

**Lemma 5.11.** Suppose $\Lambda(\pi_K) + |\lambda| < 1/2$. Then the integrals $I_n(k_1, k_2)$ and $R_n(k_1, k_2)$ converge absolutely.

**Proof.** The convergence of the integral $R_n(k_1, k_2)$ follows from the proof of Lemma 5.4. Therefore it suffices to prove that the integral $I_n(k_1, k_2)$ converges absolutely. We first note that if $\pi$ is a unitary irreducible admissible generic representation of $\text{GL}_2(F)$ with $F = \mathbb{C}$ or $F = \mathbb{R}$ and $W \in \mathcal{W}(\pi, \psi)$, $\tilde{W} \in \mathcal{W}(\bar{\pi}, \psi)$, then for $\epsilon > 0$ small, we have

$$\mathcal{B}_\pi \left( \rho \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k_1 \right) W, \rho(k_2) \tilde{W} \right) \ll_{\pi, \psi, \epsilon} |y|^{-\lambda(\pi) - \epsilon} \quad (\epsilon = \pm 1),$$

for every $k_1, k_2 \in K$, $x \in F$ and $y \in F^\times$. This follows from Corollary 5.3 and the fact that the pairing $\mathcal{B}_\pi$ is $\text{GL}_2(F)$-invariant.

Define

$$S_n = \begin{cases} \{(x, y) \in \mathbb{R} \times \mathbb{R} | n^{-1} \leq |y| \leq n, |x|^2 \leq n|y|\} & \text{if } K = \mathbb{C}, \\ \{(x, y_1, y_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} | n^{-1} \leq y_1 y_2^{-1} \leq n, |x|^2 \leq n|y_1 y_2| \} & \text{if } K = \mathbb{R} \times \mathbb{R}. \end{cases}$$

Then

$$\phi_n \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = 1 \implies (x, y) \in S_n,$$

when $K = \mathbb{C}$. Similar statement holds for $K = \mathbb{R} \times \mathbb{R}$. Let $\epsilon > 0$ so that $\kappa := 1/2 - \Lambda(\pi_K) - |\lambda| - 2\epsilon > 0$. Let $\varphi(x, y; k_1, k_2)$ and $\varphi(x, y_1, y_2; k_1, k_2)$ be the integrand of the equations (5.7) and (5.8), respectively, without the term $\phi_n$. Here $x \in \mathbb{R}$, $y, y_1, y_2 \in \mathbb{R}^\times$ and $k_1, k_2 \in \text{SO}(2)$. Combining (5.8) and Corollary 5.3, we find that when $K = \mathbb{C}$,

$$\varphi(x, y; k_1, k_2) \ll_{\pi, \mu, \psi, \epsilon} |y|^\kappa,$$

and when $K = \mathbb{R} \times \mathbb{R}$,

$$\varphi(x, y_1, y_2; k_1, k_2) \ll_{\pi, \mu, \psi, \epsilon} |y_1|^\kappa |y_2|^{-\kappa - 2\lambda(\pi_2) - 2\epsilon} \Phi(y_1) \tilde{\Phi}(y_2).$$

Here $\Phi$ and $\tilde{\Phi}$ are some continuous functions on $\mathbb{R}$ which are rapidly decreasing when $|y_j| \to \infty$ for $j = 1, 2$. It follows that when $K = \mathbb{C}$,

$$I_n(k_1, k_2) \ll_{\pi, \mu, \psi, \epsilon} \int_{n^{-1} \leq |y| \leq n} \int_{|x|^2 \leq n|y|} |y|^\kappa dx dy \ll_{\pi, \mu, \psi, \epsilon, dx} n^{1/2} \int_{n^{-1} \leq |y| \leq n} |y|^{\frac{1}{2} + \kappa} dy.$$
and that when $\mathcal{K} = \mathbb{R} \times \mathbb{R}$,
\[
I_n(k_1, k_2) \ll \pi_{K, W, W} n^{1/2} \int_{y_{1}^{-1} \leq |y_1|^{-1} \leq n} \int_{|x|^2 \leq n|y_1 y_2|} |y_1|^\kappa |y_2|^{-\kappa - 2\lambda(\pi_\sigma) - 2\varepsilon} \Phi(y_1) \Phi(y_2) dx dy_1 dy_2
\]
(5.14)
\[
\ll \pi_{K, W, W, \sigma, dx} n^{1/2} \int_{y_{1}^{-1} \leq |y_1|^{-1} \leq n} \int_{|x|^2 \leq n|y_1 y_2|} |y_1|^\kappa |y_2|^{-\kappa + \lambda(\pi_\sigma) - 2\varepsilon} \Phi(y_1) \Phi(y_2) dx dy_1 dy_2.
\]

The last inequality in the equation (5.14) follows from changing the variable $y_1 y_2^{-1} \rightarrow y$, together with the fact that $\Phi$ is bounded. This completes the proof. \hfill \Box

Following lemma is the core of the proof of Proposition 4.5, whose proof will occur in the next subsection.

Lemma 5.12. Suppose $\Lambda(\pi_K) + |\lambda| < 1/2$. We have
\[
\lim_{n \to \infty} I_n(k_1, k_2) = |\Delta|^{-1/2} \cdot \frac{\zeta_K(1)}{\zeta(1)} \cdot R_\delta(k_1, k_2),
\]
(5.15)
uniformly on $SO(2) \times SO(2)$.

Remark 5.13.

(1) Observe that the right hand side of the equation (5.15) is independent of the choice of $\delta$. Indeed if $\delta$ is replaced by $\delta^\prime = \alpha \delta$ for some $\alpha \in \mathbb{R}^\times$, then $R_{\delta^\prime}(k_1, k_2) = |\alpha| \cdot R_\delta(k_1, k_2)$. We also note that if Lemma 5.12 holds for a particular choice of $\psi$, then it holds for all non-trivial additive character of $F$.

In fact, if $\psi$ changes to $\psi_{\alpha}$, where $\psi_{\alpha}(x) = \psi(\alpha x)$, then both sides of the equation will be multiplied by $|\alpha|^2$. The outcome of these observations is that we can choose $\delta$ and $\psi$ for our convenience when we prove Lemma 5.12.

(2) The Lemma 5.12 can be proved formally by interchanging the limit and the integrals, and then using the Fourier inversion formula. However, interchanging the limit and the integrals is NOT justified due to the appearance of the factor $n^{1/2}$ in the equations (5.13) and (5.14).

Taking Lemma 5.12 for granted, we now prove Proposition 4.5.

Proof. First notice that when $\mathcal{K} = \mathbb{R} \times \mathbb{R}$, there is no loss of generality to assume $W = W_1 \otimes W_2$ and $\tilde{W} = \tilde{W}_1 \otimes \tilde{W}_2$. For $n \in \mathbb{N}$, we put
\[
\mathcal{I}_n(W \otimes f; \tilde{W} \otimes \tilde{f}) = \int_{\mathbb{R}^\times \setminus GL_2(\mathbb{R})} \mathcal{B}_{\pi_{K}}(\rho(h) \tilde{W}, \tilde{W}) B_{\pi}(\rho(h) f, \tilde{f}) \phi_n(h) dh.
\]
Clearly we have
\[
\lim_{n \to \infty} \mathcal{I}_n(W \otimes f; \tilde{W} \otimes \tilde{f}) = \mathcal{I}(W \otimes f; \tilde{W} \otimes \tilde{f})
\]
by the Lebesgue dominant convergent theorem.

On the other hand, we claim that
\[
\mathcal{I}_n(W \otimes f; \tilde{W} \otimes \tilde{f}) = \int_{SO(2)} \int_{SO(2)} f(k_1) \tilde{f}(k_2) I_n(k_1, k_2) dk_1 dk_2.
\]
(5.16)

Note that by Lemma 5.11 the RHS of (5.16) converges absolutely. If (5.16) holds, then by Lemma 5.12
\[
\mathcal{I}(W \otimes f; \tilde{W} \otimes \tilde{f}) = \lim_{n \to \infty} \mathcal{I}_n(W \otimes f; \tilde{W} \otimes \tilde{f}) = \int_{SO(2)} \int_{SO(2)} \lim_{n \to \infty} f(k_1) \tilde{f}(k_2) I_n(k_1, k_2) dk_1 dk_2
\]
\[
= |\Delta|^{-1/2} \cdot \frac{\zeta_K(1)}{\zeta(1)} \int_{SO(2)} \int_{SO(2)} f(k_1) \tilde{f}(k_2) R_\delta(k_1, k_2) dk_1 dk_2
\]
\[
= |\Delta|^{-1/2} \cdot \frac{\zeta_K(1)}{\zeta(1)} \cdot \mathcal{R}(W \otimes f) \cdot \tilde{R}_\delta(\tilde{W} \otimes \tilde{f}).
\]
Therefore Proposition 4.5 is proved. It remains to prove equation (5.16). Suppose $K = \mathbb{C}$. We have

$$\mathcal{I}_n(W \otimes f; \tilde{W} \otimes \tilde{f}) = \int_{\mathbb{R}^x \times GL_2(\mathbb{R})} \mathcal{B}_{\pi}(\rho(h)W_i, \tilde{W}_i) \mathcal{B}_{\pi_2}(\rho(h)W_2, \tilde{W}_2) \mathcal{B}_{\pi_1}(\rho(h)f, \tilde{f}) \phi_n(h) dh$$

$$= \int_{\mathbb{R}^x \times GL_2(\mathbb{R})} \mathcal{B}_{\pi}(\rho(h)W_i, \tilde{W}_i) \mathcal{B}(\rho(h)F, \tilde{F}) \phi_n(h) dh$$

$$= \int_{\mathbb{R}^x \times GL_2(\mathbb{R})} \mathcal{B}_{\pi_1}(\rho(h)W_1, \tilde{W}_1) F(g \tilde{F}) \phi_n(h) dg dh$$

$$= \int_{\mathbb{R}^x \times GL_2(\mathbb{R})} \mathcal{B}_{\pi_1}(\rho(h)W_1, \tilde{W}_1) F(h \tilde{F}) \phi_n(g^{-1}h) dh dg$$

Here we have used change of the variable $k_2h \rightarrow h$, and the fact that $\mathcal{B}_{\pi_1}$ is $GL_2(\mathbb{C})$-invariant as well as $\phi_n$ is $SO(2)$-invariant on both sides. Now we use little trick here. Put

$$F(h) = W_2(h) f(h) \quad \text{and} \quad \tilde{F}(h) = \tilde{W}_2 \left( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} h \right) \tilde{f}(h),$$

for $h \in GL_2(\mathbb{R})$, and we define

$$\mathcal{B}(F, \tilde{F}) = \int_{\mathbb{R}^x \times GL_2(\mathbb{R})} F(h) \tilde{F}(h) dh.$$

This integral converges absolutely. In fact, one checks easily that $\mathcal{B}(F, \tilde{F}) = \mathcal{B}_{\pi_2}(W_2, \tilde{W}_2) \mathcal{B}_{\pi}(f, \tilde{f})$ by the decomposition (4.8). It follows that

$$\mathcal{I}_n(W \otimes f; \tilde{W} \otimes \tilde{f}) = \int_{\mathbb{R}^x \times GL_2(\mathbb{R})} \mathcal{B}_{\pi_1}(\rho(h)W_1, \tilde{W}_1) \mathcal{B}_{\pi_2}(\rho(h)W_2, \tilde{W}_2) \mathcal{B}_{\pi_1}(\rho(h)f, \tilde{f}) \phi_n(h) dh$$

$$= \int_{\mathbb{R}^x \times GL_2(\mathbb{R})} \mathcal{B}_{\pi_1}(\rho(h)W_1, \tilde{W}_1) \mathcal{B}(\rho(h)F, \tilde{F}) \phi_n(h) dh$$

$$= \int_{\mathbb{R}^x \times GL_2(\mathbb{R})} \mathcal{B}_{\pi_1}(\rho(h)W_1, \tilde{W}_1) F(g \tilde{F}) \phi_n(h) dg dh$$

$$= \int_{\mathbb{R}^x \times GL_2(\mathbb{R})} \mathcal{B}_{\pi_1}(\rho(h)W_1, \tilde{W}_1) F(h \tilde{F}) \phi_n(g^{-1}h) dh dg$$

In the last equation, the first two integrals are over $\mathbb{R}^x \times GL_2(\mathbb{R})$. Using the decomposition (4.8) again and the fact that $\phi_n$ is $SO(2)$-invariant on the both sides, we find that

$$\mathcal{I}_n(W \otimes f; \tilde{W} \otimes \tilde{f}) = \int_{SO(2)} \int_{SO(2)} f(k_1) \tilde{f}(k_2) I_n(k_1, k_2) dk_1 dk_2.$$

Note that in both cases, our formal computations are justified by the final expressions. □

5.4. Proof of Lemma 5.12 This subsection is devoted to prove Lemma 5.12. As we have mentioned in Remark 5.13, we can choose $\delta$ and $\psi$ for our convenience. We take $\psi(x) = e^{2\pi i x}$. The self-dual Haar measure on $\mathbb{R}$ is then the usual Lebesgue measure. Let $\delta$ be given by (4.21). Then $|\Delta| = 1$. Let $n \in \mathbb{N}$. Define two functions $r_n(\beta, y)$, $r_n(y)$ by

$$r_n(\beta, y) = \left\{ \begin{array}{ll}
|\beta y| (n + n^{-1} - |\beta y^{-1}| - |\beta y^{-1}||) & \text{for } \beta, y \in \mathbb{R}^x
\end{array} \right.$$
Lemma 5.12 is not hard to prove, but the proof is quite lengthy. There are three steps. The first step is to rewrite $I_n(k_1, k_2)$ in a much more manageable form. Suppose $K = \mathbb{C}$. By (5.7) and Lemma 5.6, we have

$$I_n(k_1, k_2) = \int_{\mathbb{R}^x} \int_{|x| \leq r_n(y)} B_{\pi^1} \left( \rho \left( \begin{array}{c} \beta \\ 0 \\ 1 \end{array} \right) k_1 \right) W_1, \rho \left( \begin{array}{c} y \\ 0 \\ 1 \end{array} \right) k_2 \right) \mu(y) |y|^{-\frac{1}{2}} dx dy$$

$$= \int_{\mathbb{R}^x} \int_{|x| \leq r_n(y)} \psi_C(zx) W \left( \begin{array}{c} zy \\ 0 \\ 1 \end{array} \right) k_1 \right) \bar{W} \left( \begin{array}{c} -z \\ 0 \\ 1 \end{array} \right) k_2 \right) \mu(y) |y|^{-\frac{1}{2}} dz dx dy$$

$$= \int_{\mathbb{R}^x} \int_{\mathbb{C}^x} W \left( \begin{array}{c} zy \\ 0 \\ 1 \end{array} \right) k_1 \right) \bar{W} \left( \begin{array}{c} -z \\ 0 \\ 1 \end{array} \right) k_2 \right) \mu(y) |y|^{-\frac{1}{2}} \left\{ \int_{|x| \leq r_n(y)} \psi_C(zx) dx \right\} dx dy.$$

Let $z = \alpha + i \beta$ with $\alpha, \beta \in \mathbb{R}$. Then

$$\int_{|x| \leq r_n(y)} \psi_C(zx) dx = \frac{\sin 4\pi r_n(y) \alpha}{2\pi \alpha}.$$

Recall that $d^\times z = \zeta_C(1)|z|^{-1} dz$. It follows that $I_n(k_1, k_2)$ is equal to $2\zeta_C(1)$ times

$$\int_{\mathbb{R}^x} \mu(y) |y|^{-\frac{1}{2}} \int_{\mathbb{R}^x} W \left( \begin{array}{c} \alpha + i \beta y \\ 0 \\ 1 \end{array} \right) k_1 \right) \bar{W} \left( \begin{array}{c} -\alpha - i \beta y \\ 0 \\ 1 \end{array} \right) k_2 \right) \mu(y) |y|^{-\frac{1}{2}} \frac{\sin 4\pi r_n(y) \alpha}{2\pi \alpha} \frac{d\alpha d\beta}{(\alpha^2 + \beta^2)} d^\times y.$$

For brevity, we set, for $\beta, y \in \mathbb{R}^x$ and $\alpha \in \mathbb{R}$

$$\varphi(\alpha, \beta, y; k_1, k_2) = W \left( \begin{array}{c} \alpha + i \beta y \\ 0 \\ 1 \end{array} \right) \bar{W} \left( \begin{array}{c} -\alpha - i \beta y \\ 0 \\ 1 \end{array} \right) k_2 \right) \mu(y) |y|^{-\frac{1}{2}} \frac{\sin 4\pi r_n(y) \alpha}{\pi \alpha} \frac{d\alpha d\beta}{(\alpha^2 + \beta^2)}.$$

With this notation, we have

$$I_n(k_1, k_2) = \zeta_C(1) \int_{\mathbb{R}^x} \int_{\mathbb{R}^x} \int_{|x| \leq r_n(\beta, y)} \varphi(\alpha, \beta, y; k_1, k_2) \mu(y) |y|^{-\frac{1}{2}} \frac{\sin 4\pi r_n(y) \alpha}{\pi \alpha} \frac{d\alpha d\beta}{(\alpha^2 + \beta^2)} d^\times y.$$
which we now describe. In the case \( \mathcal{K} = \mathbb{C} \),

\[
I_n^{(3)}(k_1, k_2; \xi) = \zeta_\mathcal{C}(1) \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \int_{|\alpha| \geq \xi} \varphi(\alpha, \beta, y; k_1, k_2)\mu(y)|y|^{-\frac{1}{2}} \frac{\sin 4\pi r_n(y)\alpha}{\pi \alpha} \, d\alpha \, d\beta \, d^\times y,
\]

while in the case \( \mathcal{K} = \mathbb{R} \times \mathbb{R} \), we have

\[
I_n^{(3)}(k_1, k_2; \xi) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \int_{|\alpha| \geq \xi} \varphi(\alpha, \beta, y; k_1, k_2)\mu(\beta y^{-1})|\beta y|^{-\frac{1}{2}} \frac{\sin 2\pi r_n(\beta, y)\alpha}{\pi \alpha} \, d\alpha \, d\beta \, d^\times y.
\]

To describe the first two terms, we need some notation. Let \( \alpha \in \mathbb{R}, \beta, y \in \mathbb{R}^\times \) and put

\[
g_n(\alpha, y) = \int_{-\xi}^{\alpha} \frac{\sin 4\pi r_n(y)t}{\pi t} \, dt \quad \text{and} \quad g_n(\alpha, \beta, y) = \int_{-\xi}^{\alpha} \frac{2\sin 2\pi r_n(\beta, y)t}{\pi t} \, dt,
\]

according to \( \mathcal{K} = \mathbb{C} \) and \( \mathcal{K} = \mathbb{R} \times \mathbb{R} \), respectively. Here \( dt \) is the usual Lebesgue measure on \( \mathbb{R} \). Its important to notice that

\[
g_n(\alpha, y), g_n(\alpha, \beta, y) \ll 1.
\]

In fact, if \( |y| \geq n \), then \( g_n(\alpha, y) = 0 \) and \[5.24\] holds trivially. On the other hand, if \( |y| < n \), then \( r_n(y) \neq 0 \) and we can change the variable to obtain \( g_n(\alpha, y) = \frac{1}{\pi} \int_{-\xi}^{\alpha} \frac{4\pi r_n(y)\alpha}{t} \, dt \). Our assertion now follows from Lemma \[5.8\]. Similar argument applies to \( g_n(\alpha, \beta, y) \).

Back to our description for \[5.21\]. In the case \( \mathcal{K} = \mathbb{C} \), the function \( I_n(k_1, k_2) \) is equal to \( I_n^{(3)}(k_1, k_2; \xi) \) plus the term

\[
\zeta_\mathcal{C}(1) \int_{\mathbb{R}^\times} \mu(y)|y|^{-\frac{1}{2}} \int_{\mathbb{R}^\times} \int_{|\alpha| \leq \xi} \varphi(\alpha, \beta, y; k_1, k_2) \frac{\sin 4\pi r_n(y)\alpha}{\pi \alpha} \, d\alpha \, d\beta \, d^\times y.
\]

Integration by parts we find that

\[
\int_{|\alpha| \leq \xi} \varphi(\alpha, \beta, y; k_1, k_2) \frac{\sin 4\pi r_n(y)\alpha}{\pi \alpha} \, d\alpha = \varphi(\xi, \beta, y; k_1, k_2) g_n(\xi, y) - \int_{-\xi}^{\alpha} \partial_\alpha \varphi(\alpha, \beta, y; k_1, k_2) g_n(\alpha, y) d\alpha.
\]

Therefore equation \[5.25\] is equal to

\[
I_n^{(1)}(k_1, k_2; \xi) := \zeta_\mathcal{C}(1) \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \int_{|\alpha| \leq \xi} \partial_\alpha \varphi(\alpha, \beta, y; k_1, k_2) g_n(\alpha, y) \mu(y)|y|^{-\frac{1}{2}} \, d\alpha \, d\beta \, d^\times y
\]

subtract

\[
I_n^{(2)}(k_1, k_2; \xi) := \zeta_\mathcal{C}(1) \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \int_{|\alpha| \leq \xi} \partial_\alpha \varphi(\alpha, \beta, y; k_1, k_2) g_n(\alpha, y) \mu(y)|y|^{-\frac{1}{2}} \, d\alpha \, d\beta \, d^\times y
\]

A similar process can apply to the case \( \mathcal{K} = \mathbb{R} \times \mathbb{R} \). We put

\[
I_n^{(1)}(k_1, k_2; \xi) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \varphi(\xi, \beta, y; k_1, k_2) g_n(\xi, \beta, y) \mu(\beta y^{-1})|\beta y|^{-\frac{1}{2}} \, d\alpha \, d\beta \, d^\times y,
\]

and

\[
I_n^{(2)}(k_1, k_2; \xi) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \int_{|\alpha| \leq \xi} \partial_\alpha \varphi(\alpha, \beta, y; k_1, k_2) g_n(\alpha, \beta, y) \mu(\beta y^{-1})|\beta y|^{-\frac{1}{2}} \, d\alpha \, d\beta \, d^\times y.
\]

Then we have the equality \[5.21\]. This completes the second step.

In the last step, we prove following assertions:

1. \( \lim_{n \to \infty} I_n^{(3)}(k_1, k_2; \xi) = 0 \) uniformly on \( \text{SO}(2) \times \text{SO}(2) \).
2. There are two positive constants \( c_1, c_2 \), which are independent of \( n, \xi \) such that \( \left| I_n^{(2)}(k_1, k_2; \xi) \right| \leq c_1 \xi^{c_2} \) for all \( k_1, k_2 \in \text{SO}(2) \).
3. \( I^{(1)}(k_1, k_2; \xi) := \lim_{n \to \infty} I_n^{(1)}(k_1, k_2; \xi) \) exists and uniformly on \( \text{SO}(2) \times \text{SO}(2) \). Moreover, we have \( \lim_{\xi \to 0+} I^{(1)}(k_1, k_2; \xi) = \frac{\zeta_\mathcal{C}(1)}{\zeta_\mathcal{C}(1)} \cdot R_\mathcal{C}(k_1, k_2) \) uniformly on \( \text{SO}(2) \times \text{SO}(2) \).
Suppose for the moment that all these assertions are satisfied. Then Lemma 5.12 follows immediately. For we
first choose $\xi$ so that both $I_n^{(2)}(k_1, k_2; \xi)$ and $I^{(1)}(k_1, k_2; \xi) = \frac{\zeta(1)}{\zeta(1)} \cdot R_0(k_1, k_2)$ are small on SO(2) × SO(2). Then we take $N$ so that both $I_n^{(1)}(k_1, k_2; \xi) = I^{(1)}(k_1, k_2; \xi)$ and $I_n^{(3)}(k_1, k_2; \xi)$ are also small on SO(2) × SO(2) for all $n > N$.

We note that by the right SO(2)-finiteness of the Whittaker functions, it suffices to prove all these claims
for $k_1 = k_2 = e := I_2$. To save notation, we let $\varphi(\alpha, \beta, y) = \varphi(\alpha, \beta, y; e, e)$ for both cases. We also fix $\epsilon > 0$ so that

$$\kappa := \frac{1}{2} - \lambda(\pi_K) - |\lambda| - 2\epsilon > 0.$$  

By Corollary 5.3, we have following estimations, which will be used later. When $K = \mathbb{C}$,

$$\varphi(\alpha, \beta, y)\mu(y)|y|^{-\frac{x}{2}} \ll_{\pi_K, \mu, \overline{W}, e} |\beta|((\alpha^2 + \beta^2)^{-\frac{1}{2} + \kappa + |\lambda|}) \cdot \exp(-|\alpha y| - |\beta y| - |\alpha| - |\beta|).$$

When $K = \mathbb{R} \times \mathbb{R}$,

$$\varphi(\alpha, \beta, y)\mu(\beta y^{-1})|\beta y|^{-\frac{1}{2}} \ll_{\pi_K, \mu, \overline{W}, e} |\beta|^{\kappa}|y|^{|\kappa + 2|\lambda|} \cdot (1 - \alpha)^{-2\lambda(\pi_2)} \cdot \exp(-|1 - \alpha|\beta) - |\beta| - |1 - \alpha|y - |y|).$$

We first consider assertion (1). Let

$$f_n(\beta, y; \xi) = \left\{ \begin{array}{ll}
\int_{|\alpha| \geq |\xi|} \varphi(\alpha, \beta, y)\mu(y)|y|^{- \frac{x}{2} \sin 4\pi \pi_0(y) \alpha} d\alpha & \text{if } K = \mathbb{C}, \\
\int_{|\alpha| \geq |\xi|} \varphi(\alpha, \beta, y)\mu(\beta y^{-1})|\beta y|^{-\frac{x}{2} \sin 2\pi \pi_0(y) \alpha} d\alpha & \text{if } K = \mathbb{R} \times \mathbb{R}.
\end{array} \right.$$  

Then one sees that

$$I_n^{(3)}(e, e; \xi) = \frac{\zeta(1)}{\zeta(1)} \int_{\mathbb{R} \times \mathbb{R}} f_n(\beta, y; \xi) d^2 \beta d^2 y,$$

in both cases. By Lemma 5.10, we have $\lim_{n \to \infty} f_n(\beta, y; \xi) = 0$. To prove (1), it suffices to show that $f_n(\beta, y; \xi)$ is bounded by some integrable function on $\mathbb{R} \times \mathbb{R}$ so that we can apply the Lebesgue dominate converge theorem to pass the limit. Applying (5.30), we find that when $K = \mathbb{C}$,

$$f_n(\beta, y; \xi) \ll_{\pi_K, \mu, \overline{W}, e} |\beta|^\kappa |\beta| \cdot \exp(-|\beta y| - |\beta|) \cdot \int_{|\alpha| \geq |\xi|} (\alpha^2 + \beta^2)^{-\frac{1}{2} + \kappa + |\lambda|} \cdot |\alpha|^{-1} \cdot \exp(-|\alpha y| - |\alpha|) d\alpha$$

Suppose $K = \mathbb{R} \times \mathbb{R}$. We use (5.31) to obtain

$$f_n(\beta, y; \xi) \ll_{\pi_K, \mu, \overline{W}, e} |\beta|^\kappa |\beta|^{-1} \cdot \exp(-|\beta y| - |\beta|) \cdot \int_{|\alpha| \geq |\xi|} (\alpha^2 + \beta^2)^{-\frac{1}{2} + \kappa + |\lambda|} \cdot |\alpha|^{-1} \cdot \exp(-|\alpha y| - |\alpha|) d\alpha.$$

This shows the first assertion.

Now we prove (2). We first consider the case $K = \mathbb{C}$. By Corollary 5.3 and Corollary 5.4, one has

$$\partial_\alpha \varphi(\alpha, \beta, y) \ll_{\pi_K, \mu, \overline{W}, e} |\beta|^{1 - 2\lambda(\pi_K)} \cdot |\beta|^{-1} \cdot (\alpha^2 + \beta^2)^{-\frac{1}{2} - 2\lambda(\pi_K) - 2e} \cdot \exp(-|\alpha y|).$$

Since $g_n(\alpha, y) \ll 1$, we find that

$$I_n^{(2)}(e, e; \xi) \ll_{\pi_K, \mu, \overline{W}, e} \int_{\mathbb{R} \times \mathbb{R}} \int_{|\alpha| \leq |\xi|} |\beta|^{\kappa} (\alpha^2 + \beta^2)^{-1 + \kappa + |\lambda|} \cdot \exp(-|\alpha y|) d^2 y d^2 \beta d\alpha.$$
We estimate the RHS of the inequality above. First consider

\[
I_n^{(2)}(e, e; \xi) := \int_{|\alpha| \leq \xi} \int_{y \in \mathbb{R}^\times} \int_{|\beta| \leq |\alpha|} |y|\beta|\alpha^2 + \beta^2|^{-1+\kappa+|\lambda|} \exp(-|\alpha|) d^\times \beta d^\times y d\alpha
\]

\[
\leq \int_{|\alpha| \leq \xi} \int_{y \in \mathbb{R}^\times} |y|\beta \exp(-|\alpha| y) \int_{|\beta| \leq |\alpha|} |\beta|^{-1+2\kappa+2|\lambda|} d^\times \beta d^\times y d\alpha
\]

\[
\ll \pi_{\kappa, \mu, e} \int_{|\alpha| \leq \xi} \int_{y \in \mathbb{R}^\times} |y|\beta \exp(-|\alpha|) |\alpha|^{-1+2\kappa+2|\lambda|} d^\times y d\alpha
\]

\[
\ll \pi_{\kappa, \mu, e} \left( \int_{y \in \mathbb{R}^\times} |y|^\kappa \beta |\alpha|^\kappa \right) \left( \int_{|\alpha| \leq \xi} |\alpha|^{-1+\kappa+2|\lambda|} d\alpha \right)
\]

The remaining part is

\[
I_n^{(2)}(e, e; \xi) := \int_{|\alpha| \leq \xi} \int_{y \in \mathbb{R}^\times} \int_{|\beta| \leq |\alpha|} |y|\beta|\alpha^2 + \beta^2|^{-1+\kappa+|\lambda|} \exp(-|\alpha|) d^\times \beta d^\times y d\alpha.
\]

Changing the variable \(\beta \mapsto \alpha \beta\), we find that

\[
I_n^{(2)}(e, e; \xi) \ll \pi_{\kappa, \mu, W, \tilde{W}, e} \left( \int_{|\alpha| \leq \xi} \int_{y \in \mathbb{R}^\times} |y|^\kappa (1 + \beta^2)^{-1+\kappa+|\lambda|} \exp(-|\alpha|) |\alpha|^{-1+2\kappa+2|\lambda|} d\alpha d^\times y d\alpha \right)
\]

\[
\ll \pi_{\kappa, \mu, e} \left( \int_{|\alpha| \leq \xi} \int_{y \in \mathbb{R}^\times} |y|^\kappa \beta |\alpha|^{\kappa+2|\lambda|} \right).
\]

as above. It follows that

\[
I_n^{(2)}(e, e; \xi) \ll \pi_{\kappa, \mu, W, \tilde{W}, e} I_n^{(2)}(e, e; \xi) + J_n^{(2)}(e, e; \xi) + \pi_{\kappa, \mu, W, \tilde{W}, e} \Lam_{\kappa+2|\lambda|}.
\]

This shows (2) for the case \(\mathcal{K} = \mathbb{C}\). Next we consider the case \(\mathcal{K} = \mathbb{R} \times \mathbb{R}\), which is easier. By Corollary \(\text{[5.3]}\) and Corollary \(\text{[5.4]}\) we have

\[
\partial_\beta \varphi(\alpha, \beta, y) \ll \pi_{\kappa, \mu, W, \tilde{W}, e} |\beta y|^{1-\Lambda(\kappa)-2\epsilon} \exp(-|\beta| - |y|).
\]

Since \(g_n(\alpha, \beta, y) \ll 1\), we find that

\[
I_n^{(2)}(e, e; \xi) \ll \pi_{\kappa, \mu, W, \tilde{W}, e} \left( \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \int_{|\alpha| \leq \xi} |\beta| |y|^{\kappa+2|\lambda|} \exp(-|\beta| - |y|) d\alpha d^\times y \right) \ll \pi_{\kappa, \mu, e} \xi.
\]

This finishes the proof of the second assertion.

It remains to prove (3). We show that the integrand of \(I_n^{(3)}(e, e; \xi)\) is bounded by integrable function on \(\mathbb{R}^\times \times \mathbb{R}^\times\) so that we can interchange the integrals and the limit. By (5.30) and (5.31), and the fact that \(g_n(\xi, y) \ll 1\) and \(g_n(\xi, \beta, y) \ll 1\), we find that when \(\mathcal{K} = \mathbb{C}\),

\[
\varphi(\xi, \beta, y) \mu(y) |y|^{-\frac{1}{2}} g_n(\xi, y) \ll \pi_{\kappa, \mu, W, \tilde{W}, e} |\beta| |y|^{\kappa} (1 + \beta^2)^{-\frac{1}{2}+\kappa+|\lambda|} \exp(-|\beta| - |y|) \exp(-\xi - |\beta|) \exp(-|\beta| - |y|)
\]

\[
\ll \pi_{\kappa, \mu, W, \tilde{W}, e} ||y|^{\kappa+2|\lambda|} \exp(-|\beta| - |y|) |\beta|^{-\frac{1}{2}} g_n(\xi, y) |y|^{-\frac{1}{2}} g_n(\xi, y) d\alpha d^\times \beta d^\times y d\xi.
\]

while when \(\mathcal{K} = \mathbb{R} \times \mathbb{R}\),

\[
\varphi(\xi, \beta, y) \mu(y) |y|^{-\frac{1}{2}} g_n(\xi, y) \ll \pi_{\kappa, \mu, W, \tilde{W}, e} |\beta| |y|^{\kappa+2|\lambda|} \exp(-|\beta| - |y|).
\]

Its important to observe that by (5.32) and (5.33) we can bound the integrands of \(I_n^{(1)}(e, e; \xi)\) by integrable functions on \(\mathbb{R}^\times \times \mathbb{R}^\times\) times some constants which are independent of \(\xi\). This follows from the assumption that \(0 < \xi < 1/2\), so that we have \(2^{-1} < (1 - \xi)^{-1} < 2\), as well as the fact that \(\exp(-a) \leq 1\) for every \(a \geq 0\).

Taking \(f = 1\) in Lemma \(\text{[5.3]}\) we see that \(\lim_{n \to \infty} g_n(\xi, y) = \lim_{n \to \infty} g_n(\xi, \beta, y) = 1\). Therefore when \(\mathcal{K} = \mathbb{C}\),

\[
I_n^{(1)}(e, e; \xi) := \lim_{n \to \infty} I_n^{(1)}(e, e; \xi) = \zeta(\frac{1}{2}) \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \lim_{n \to \infty} \varphi(\xi, \beta, y) \mu(y) |y|^{-\frac{1}{2}} g_n(\xi, y) d\alpha d^\times \beta d^\times y d\xi
\]

\[
= \zeta(\frac{1}{2}) \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \varphi(\xi, \beta, y) \mu(y) |y|^{-\frac{1}{2}} d^\times \beta d^\times y d\xi.
\]
Similarly we have
\[ f^{(1)}(e, e; \xi) := \lim_{n \to \infty} f_n^{(1)}(e, e; \xi) = \int_{\mathbb{R} \times \mathbb{R}} \varphi(\xi, \beta, y) \mu(\beta y^{-1}) |\beta y|^{-\frac{d}{2}} d^\beta d^\xi y, \]
if \( K = \mathbb{R} \times \mathbb{R} \). Equations (5.32) and (5.33) tell us that we can interchange the integrals and \( \lim_{\xi \to 0^+} \). It follows that
\[ \lim_{\xi \to 0^+} f^{(1)}(e, e; \xi) = \zeta_C(1) \int_{\mathbb{R} \times \mathbb{R}} \varphi(0, \beta, y) \mu(y) |y|^{-\frac{d}{2}} d^\beta d^\xi y, \]
when \( K = \mathbb{C} \), and
\[ \lim_{\xi \to 0^+} f^{(1)}(e, e; \xi) = \int_{\mathbb{R} \times \mathbb{R}} \varphi(0, \beta, y) \mu(\beta y^{-1}) |\beta y|^{-\frac{d}{2}} d^\beta d^\xi y, \]
when \( K = \mathbb{R} \times \mathbb{R} \). Finally, by (5.17) and (5.19), our proof for the last assertion, and hence the proof for Lemma 5.12 is now complete.

REFERENCES

[A] Tetsuya Asai. On certain Dirichlet series associated with Hilbert modular forms and Rankin’s method. Mathematische Annalen, 226:81-94, 1977.
[Ba] William D. Banks. Twisted symmetric-square \( L \)-functions and the nonexistence of Siegel zeros on GL(3). Duke Mathematical Journal, 87(2):343-353, 1997.
[BGT] N. Burq, P. Gerard, and N. Tzvetkov. Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. Duke Mathematical Journal, 138(3):445-486, 2007.
[BI] Conrey J. B. and H. Iwaniec. The cubic moment of central values of automorphic \( L \)-functions. Annals of Mathematics, 151:1175-1216, 2000.
[Blo12] V. Blomer. Subconvexity for twisted \( L \)-functions on GL(3). American Journal of Mathematics, 134(5):1385-1421, 2012.
[Blo13] Valentin Blomer. On the 4-norm of an automorphic form. Journal of European Mathematical Society, 15:1825-1852, 2013.
[BM] V. Blomer and P. Maga. The sup-norm problem for PGL(4). International Mathematics Research Notices, (14):5311-5332, 2015.
[Cas73] W. Casselman. On some results of Atkin and Lehner. Mathematische Annalen, 201:301-314, 1973.
[CC18] S. Y. Chen and Y. Cheng. On Deligne’s conjecture for central values of certain automorphic \( L \)-functions on GL(3) \( \times \) GL(2). 2018. arXiv:1806.09767v2.
[CC18a] S. Y. Chen. Pullback formulae for nearly holomorphic saito-kurokawa lifts. 2018. arXiv:1806.09467v2.
[Che18] Y. Cheng. Special value formulae for triple product \( L \)-functions and applications. PhD thesis, National Taiwan University, 2018.
[FL] Y. Flicker. Twisted tensor and Euler products. Bull. Soc. Math. France, 116(3):295-313, 1988.
[Gar87] P. Garrett. Decomposition of Eisenstein series: Rankin Triple Products. Annals of Mathematics, 125(2):209-235, 1987.
[Gr] I.S. Gradshteyn and I.M.Ryzhik. Table of integrals, series, and products. Academic Press, seventh edition, 2007.
[GJ78] S. Gelbart and H. Jacquet. A relation between automorphic representations of GL(2) and GL(3). Ann. scient. Ecole Norm. Sup., 11(4):471-542, 1978.
[GJ79] S. Gelbart and H. Jacquet. Forms of GL(2) from the analytic point of view. In Automorphic forms, representations, and \( L \)-functions, volume 33 of Proceedings of Symposia in Pure Mathematics, pages 213-251, 1979. part 1.
[GP92] B. Gross and S. Kudla. Heights and the central critical values of triple product \( L \)-functions. Compositio Mathematica, 81(2):143-209, 1992.
[Hid93] Haruzo Hida. \( p \)-ordinary cohomology groups for SL(2) over number fields. Duke Mathematical Journal, 69(2):259-314, 1993.
[HL] J. Hoffstein and P. Lockhart. Coefficients of Maass forms and the Siegel zero. Annals of Mathematics, 140(1):161-176, 1994.
[Hsi17] M. L. Hsieh. Hida families and \( p \)-adic triple \( L \)-functions. 2017. arXiv:1705.02717.
[Hu17] Y. K. Hu. Triple product formula and the subconvexity bound of Triple product \( L \)-function in level aspect. American Journal of Mathematics, 139(1):125-159, February 2017.
[Ich08] A. Ichino. Trilinear forms and the central values of triple product \( L \)-functions. Duke Mathematical Journal, 145(2):281-307, 2008.
[IK] Henryk Iwaniec and Emmanuel Kowalski. Analytic Number theory, volume 53. American Mathematical Society Colloquium Publications, 2004.
[Ike92] T. Ikeda. On the location of poles of the triple \( L \)-functions. Compositio Mathematica, 83(2):187-237, 1992.
[IWS00] H. Iwaniec, W. Luo, and P. Sarnak. Low lying zeros of families of \( L \)-functions. Publ. Math. Inst. Hautes Etudes Sci., 91:55-131, 2000.
[IP] A. Ichino and K. Prasanna. Period of quaternionic Simura varieties. 1. Memoirs of the American Mathematical Society, to appear.
[Ish17] Isao Ishikawa. On the construction of twisted triple product \( p \)-adic \( L \)-functions. PhD thesis, Kyoto University, 2017.
SPECIAL VALUE FORMULA FOR THE TWISTED TRIPLE PRODUCT AND APPLICATIONS

[Jac72] H. Jacquet. Automorphic Forms on GL(2) II, volume 278. Springer-Verlag, Berlin and New York, 1972.

[Jac70] H. Jacquet and R. Langlands. Automorphic Forms on GL(2), volume 114 of Lecture Notes in Mathematics. Springer-Verlag, Berlin and New York, 1970.

[Kim03] H. Kim. Functoriality for the exterior square of GL4 and the symmetric fourth of GL2. Journal of American mathematics society, 16(1):139–183, 2003. appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[Kri03] M. Krishnamurthy. The Asai Transfer to GL4 via the Langlands-Shahidi Method. International Mathematics Research Notices, (41):2221–2254, 2003.

[KS02] H. Kim and F. Shahidi. Functorial product for GL2 × GL3 and symmetric cube for GL2. Annals of Mathematics, 155:837–893, 2002. With an appendix by Colin J. Bushell and Guy Henniart.

[Li03] X. Li, S. C. Liu, and M. Young. The L2 restriction norm of a Maass form on SL(n+1(Z)). Mathematische Annalen. To appear.

[LY12] Xiaoqing Li and Matthew Young. The L2 restriction norm of a GL3 Maass form. Compositio Mathematica, 148:675–717, 2012.

[Mar16] S. Marshall. Lp norms of higher rank eigenfunctions and bounds for spherical functions. Journal of european mathematical society, 18:1437–1493, 2016.

[Mol99] G. Molteni. L-functions: Siegel type theorems and structure theorems. 1999. PhD thesis.

[MV10] P. Michel and A. Venkatesh. The subconvexity problem for GL2. Publ. Math. Inst. Hautes Etudes Sci., (111):171–271, 2010.

[Nel11] P. Nelson. Equidistribution of cusp forms in the level aspect. Duke Mathematical Journal, 160(3):467–501, 2011.

[NPS14] P. Nelson, A. Pitale, and A. Saha. Bound for Rankin-Selberg integrals and quantum unique ergodicity for powerful levels. Journal of American mathematics society, 27(1):147–191, 2014.

[Orl87] T. Orloff. Special values and mixed weigh triple products (with appendix by don blasius). Inventiones mathematicae, 90:169–180, 1987. with appendix by Don Blasius.

[Pra90] D. Prasad. Trilinear forms for representations of GL(2) and local -factors. Compositio Mathematica, 75(1):1–46, 1990.

[Pra92] D. Prasad. Invariant linear forms for representations of GL(2) over a local field. American Journal of Mathematics, 114:1317–1363, 1992.

[PSR87] I. Piatetski-Shapiro and S. Rallis. Rankin triple L-functions. Compositio Mathematica, 64(1):31–115, 1987. MR 911357.

[Ram00] D. Ramakrishnan. Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2). Annals of Mathematics, 152(4):45–111, 2000.

[Sah17] A. Saha. On sup-norms of cusp forms of powerful level. Journal of european mathematical society, 19(11):35493573, 2017.

[Sch02] R. Schmidt. Some remarks on local newforms for GL(2). Journal of Ramanujan Mathematical Society, 17(2):115–147, 2002.

[Sne51] Ian N. Sneddon. Fourier transforms. McGraw-Hill Company, 1951.

[Tat79] J. Tate. Number theoretic background. In Automorphic forms, representations, and -functions, volume 33 of Precedings of Symposia in Pure Mathematics, pages 3–26, 1979. part 2.

[Wal85] J-L. Waldspurger. Sur les valeurs de certaines fonctions automorphes en leur centre de symetrie. Compositio Mathematica, 54(2):173–242, 1985.

[Wat08] T. Watson. Rankin triple products and quantum chaos. 2008. Ph.D. dissertation, Princeton University, Princeton, N.J., 2002, arXiv:0810.0425v3 [math.NT].

[Woo16] M. Woodbury. On the triple product formula: Real local calculations. In Jan Hendrik Bruinier and Winfried Kohnen, editors, L-functions and automphic forms, volume 10 of Contributions in Mathematical and Computational Sciences., pages 275–298. Springer, Chem, 2016.

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