Spherical Minkowski content and natural parametrization on the sphere for Schramm-Loewner evolution

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Abstract

Studying SLE_κ on S^2 provides a new and interesting perspective for the conformality of some 2-dimensional physical models. We prove the existence and some basic properties of the spherical Minkowski content of SLE_κ, which is finite and can be viewed as the natural parametrization of SLE_κ on S^2.

1 Introduction

Schramm-Loewner evolution with the positive parameter κ (SLE_κ) is a 2-dimensional stochastic process which has conformal invariance and domain Markovian property as its essential properties. Many planar discrete models in statistical physics have been shown to have SLE_κ as its scaling limit, e.g., the loop-erased random walk for κ = 2 [6], the Ising interface for κ = 3 [14], the Gaussian free field interface for κ = 4 [2] and the percolation interface for κ = 6 [15].

However, the above proofs of convergence ignore the parametrization and hence miss some information. In order to prove the convergence of discrete models in the natural parametrization (e.g., the length in the case of loop-erased random walk), we need the natural parametrization of SLE itself. In [5], the authors achieve this goal and prove that the natural parametrization of SLE_κ is its d-dimensional (d = min{1 + κ/8, 2}) Minkowski content, a concept which generalizes the length of a smooth curve. Up to now, the convergence in the natural parametrization of loop-erased random walk [7] and the percolation interface [3] have been established.

Aiming to deal with the scaling limit of some discrete models in the whole Riemann sphere S^2 (particularly, the Gaussian free field), we consider the corresponding problem in S^2 replacing the Euclidean distance with the spherical distance. Regarding the SLE as a process in S^2 rather than in C has the advantage that we can talk about the Minkowski content of the whole process.
because all subsets of $S^2$ is relatively compact (the Minkowski content of an unbounded subset of $\mathbb{C}$ is always infinity). Also, this perspective is natural when considering some variants of SLE that may pass through both 0 and $\infty$, for instance, the two-sided whole-plane SLE. Note that although we work on the chordal SLE, one can generalize our results to other kinds of SLE using the local equivalence among them.

On the other hand, the term conformality, which is a key point in studying SLE, can be understood in a new sense from the spherical viewpoint. In the Euclidean space, conformality means rotation, scaling and translation locally at every point, while in $S^2$, rotation is 3-dimensional and around a straight line in $\mathbb{R}^3$ passing through $(0,0,0)^T$, scaling at $\eta$ based on a point $\xi$ is moving $\eta$ along the geodesic connecting $\xi$, the anipodal point $-\xi$ and $\eta$ ($\eta$ is fixed if $\eta = \pm \xi$), and translation from $\xi_1$ to $\xi_2$ is a rotation in $\mathbb{R}^3$ making $\xi_1$ go through a geodesic. This new point of view, which we discuss more carefully in section 2.2, may provide an interesting understanding of the conformal invariance of many models in statistical physics.

Now we state our main result.

**Theorem 1.1.** Let $\kappa \in (0,8)$, $d := 1 + \kappa/8$, $D \subset S^2$ be a simply-connected hyperbolic domain and $\gamma$ be a chordal SLE($\kappa$) curve in $D$ (parametrized by capacity) from $w_1$ to $w_2$, where $w_1 \neq w_2$ are prime ends. Then we can define a locally finite random measure $\mu$ on $D$ such that with probability one, $\forall$ domain $V$, $V \subset D$ with $\partial V$ piecewise $C^1$, we have

$$\mu(V \cap \gamma[0,\infty]) = M^S_d(\overline{V} \cap \gamma[0,\infty]).$$

For any Borel set $E \subset D$,

$$\mathbb{E}[\mu(E)] = \int_E \tilde{G}_D(z;w_1,w_2) dS(z),$$

and $\mu(E)$ has finite moments of any finite order providing that $E$ is compact. Here $M^S_d$ is the spherical Minkowski content and $\tilde{G}_D(z;w_1,w_2)$ is the spherical Green’s function.

Furthermore, if $\partial D$ is an analytic Jordan curve, then there exists $\alpha \in (0,1)$ depending only on $\kappa$ such that with probability one, the following holds:

1. $\mu(D) < \infty$.

2. $\forall 0 \leq s < t \leq \infty$, $\forall$ domain $V \subset D$ with $\partial V$ piecewise $C^1$, we have $\mu(V \cap \gamma[s,t]) = \mu(V \cap \gamma[0,t]) - \mu(V \cap \gamma[0,s]) = M^S_d(\overline{V} \cap \gamma[s,t]) = M^S_d\left(\overline{V} \cap (\gamma[0,t] \setminus \gamma[0,s])\right)$.

3. The map $t \mapsto M^S_d(\gamma[0,t])$, $t \in [0,\infty)$ is strictly increasing and Hölder continuous of order $\alpha$ on every bounded interval.

In this case, $\mu(D)$ has finite moments of any finite order.

We call $\mu$ the spherical natural occupation measure of $\gamma$. For the accurate meaning of the terminology in Theorem 1.1, see Section 2.
2 Preliminaries

2.1 Schramm-Loewner Evolution

A domain in \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) is a nonempty, open and connected subset. A domain \( D \) is hyperbolic if and only if there exist two different points in \( \hat{\mathbb{C}} \setminus D \). For such a domain \( D \), we further say \( \partial D \) is an analytic Jordan curve if there is a univalent (which means analytic and injective throughout) map \( F : \{ z \in \mathbb{C} : 1/r < |z| < r \} \rightarrow \hat{\mathbb{C}} \), where \( r \in (1, \infty) \), such that \( F(\{ z \in \mathbb{C} : |z| = 1 \}) = \partial D \).

(The symbol \( \partial \) always denotes the topological boundary in \( \hat{\mathbb{C}} \) and the closure is always taken in \( \hat{\mathbb{C}} \).) Obviously, \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) are hyperbolic domains with analytic Jordan boundaries. Suppose \( (\xi_t : t \in [0, \infty)) \) is a continuous real-valued function. For any \( z \in \mathbb{C} \setminus \xi_0 \), let \( (g_t(z) : z \in [0, T_z)) \) solve the following ODE:

\[
\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = z. \tag{2.1.1}
\]

Here \( T_z \in (0, \infty] \) is the maximal existence time. Let \( K_t = \{ z \in \mathbb{H} : T_z \leq t \} \) and \( H_t = \mathbb{H} \setminus K_t \). Then \( K_t \) is compact and \( H_t \) is simply connected. And \( H_t \) is parametrized by the half-plane capacity in the sense that \( \text{hc}(H_t) = 2t \). For each \( t \), \( g_t \) restricted on \( H_t \) is a conformal map onto \( \mathbb{H} \).

Now we fix a complete probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \) with a complete and right-continuous filtration \( (\mathcal{F}_t : t \in [0, \infty)) \). There is a standard Brownian motion \( (B_t : t \in [0, \infty)) \) starting from 0 satisfying \( \sigma(B_s : s \leq t) = \mathcal{F}_t \) a.s.. Fix \( \kappa \in (0, 8) \). If we replace \( \xi_t \) in (2.1.1) with \( \sqrt{\kappa} B_t \), then the corresponding \( (g_t : t \in [0, \infty)) \) is called chordal SLE\(_\kappa\) in \( \mathbb{H} \) from 0 to \( \infty \). It has been proved that with probability one, \( \gamma_t := g_t^{-1}(\xi_t) \) is continuous and generates \( K_t \) in the sense that \( H_t \) is the unbounded connected component of \( \mathbb{H} \setminus \gamma[0, t] \), for which we call \( \gamma \) the SLE\(_\kappa\) curve. Note that \( \sigma(\gamma_s : s \leq t) = \mathcal{F}_t \) a.s.. We use Riemann mapping theorem to define SLE\(_\kappa\)'s in other simply-connected hyperbolic domains, whose parametrization differ by constant coefficients depending on the choice of conformal isomorphisms. For details about Schramm-Loewner evolution, we refer readers to [4].

From now on, we make a convention that \( \kappa \in (0, 8) \) is fixed, \( a := 2/\kappa \in (\frac{1}{4}, \infty) \), \( d := 1 + \frac{\kappa}{8} \in (1, 2) \) and all constants are allowed to depend on \( \kappa \). Note that \( d \) is the Hausdorff or box dimension of SLE\(_\kappa\) curve.

2.2 Complex Analysis and Riemann Sphere

We first review the well-known Koebe distortion theorem and Koebe-1/4-theorem.

Proposition 2.1 (Koebe distortion theorem). Let \( r \in (0, \infty) \), \( w \in \mathbb{C} \), \( z \in r \mathbb{D} + w \) and \( f \) be a univalent map from \( r \mathbb{D} + w \) to \( \mathbb{C} \). Then

\[
|f'(w)| \frac{|z - w|}{(1 + \frac{|z - w|}{r})^2} \leq |f(z) - f(w)| \leq |f'(w)| \frac{|z - w|}{(1 - \frac{|z - w|}{r})^2}.
\]
**Proposition 2.2** (Koebe 1/4-theorem). Let \( r \in (0, \infty), w \in \mathbb{C}, z \in r\mathbb{D} + w \) and \( f \) be a univalent map from \( r\mathbb{D} + w \) to \( \mathbb{C} \). Then
\[
\frac{r}{4} \left(1 - \frac{|z-w|^2}{r}\right) |f'(z)| \leq df(z) \leq r \left(1 - \frac{|z-w|^2}{r}\right) |f'(z)|
\]
where \( df(z) := \inf\{|f(z) - \zeta| : \zeta \in \partial f(r\mathbb{D} + w)|\} \).

Extremal length, which we will use in the proof, is a powerful tool in geometric function theory. If \( \Gamma \) is a collection of rectifiable curves in \( \mathbb{C} \), then the extremal length of \( \Gamma \) is denoted by \( \mathcal{E}(\Gamma) \). For a univalent function \( f \) taking values in \( \mathbb{C} \), we have \( \mathcal{E}(f(\Gamma)) = \mathcal{E}(\Gamma) \). The extremal length is monotone. That is to say, if \( \forall \sigma_1 \in \Gamma_1, \exists \sigma_2 \in \Gamma_2 \) s.t. \( \sigma_2 \) is a subarc of \( \sigma_1 \), then \( \mathcal{E}(\Gamma_1) \geq \mathcal{E}(\Gamma_2) \). Suppose \( D \subset \mathbb{C} \) is a domain and \( (A, B) \) is a disjoint pair of connected and closed subset of \( \delta D \). (Here and throughout, we use \( \delta D \) to denote the Martin boundary of \( D \), that is, the set of all prime ends of \( D \).) Then we use \( \mathcal{E}_D(A, B) \) to denote the extremal distance between \( A \) and \( B \) in \( D \), that is, the extremal length of \( \{\sigma : \sigma \) is a rectifiable curve defined on \([0,1]\) s.t. \( \sigma(0), \sigma(1) \subset D, \sigma(0+) \in A \) and \( \sigma(1-) \in B\}. \) It is known that when \( D \) is an annulus with radii \( 0 < r < R < \infty \), \( \mathcal{E}_D(A, B) = (2\pi)^{-1} \ln(R/r) \). For details about extremal length, see [1]. We will cite the following lemma.

**Lemma 2.3** ([10] Lemma 2.9). Let \( (A_1, A_2) \) be a disjoint pair of connected and compact subset of \( \overline{\mathbb{H}} \) s.t. both \( A_1 \) and \( A_2 \) intersect \( \mathbb{R} \). Then
\[
\prod_{j=1}^{2} \left(\frac{\text{diam}(A_j)}{\text{dist}(A_1, A_2)} \land 1\right) \leq 144e^{-\pi \mathcal{E}_H(A_1, A_2)}.
\]

Another useful tool is the Beurling estimate.

**Proposition 2.4** ([4] COROLLARY 3.72). Let \( \mathbb{P}^z \) denote the distribution of a planar Brownian motion \( B \) starting from \( z \in \mathbb{C} \). For a domain \( D \), let \( \tau_D \) denote the first exit time from \( D \). There is a \( c \) s.t. if \( 0 < r < R < \infty \) and \( \sigma : [0,1] \rightarrow \mathbb{C} \) is continuous with \( \sigma(0) \in r\mathbb{D} \) and \( \sigma(1) \in R\mathbb{D} \), then
\[
\mathbb{P}^z\{B[0, \tau_{r\mathbb{D}}] \cap \sigma[0,1] = \emptyset\} \leq c(r/R)^{1/2}, \quad |z| \leq r,
\]
\[
\mathbb{P}^z\{B[0, \tau_{r\mathbb{D}}(r\mathbb{D})] \cap \sigma[0,1] = \emptyset\} \leq c(r/R)^{1/2}, \quad |z| \geq R.
\]

Since we will deal with the spherical metric, we recall some basic facts about \( S^2 = \{\xi = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3 : \xi_1^2 + \xi_2^2 + \xi_3^2 = 1\} \). We can define a conformal structure on \( S^2 \) via two local coordinate systems:
\[
(S^2 \setminus (0,0,1)^T, \phi_+), \quad (S^2 \setminus (0,0,-1)^T, \phi_-),
\]
where \( \phi_{\pm} \) are stereographic projections from \((0,0,1)^T\) and \((0,0,-1)^T\) respectively. That is,
\[
\phi_+(\xi_1, \xi_2, \xi_3)^T = \frac{\xi_1 + i\xi_2}{1 - \xi_3}, \quad \phi_-(\xi_1, \xi_2, \xi_3)^T = \frac{\xi_1 - i\xi_2}{1 + \xi_3}.
\]
We identify $S^2$ with $\hat{\mathbb{C}}$ through $\phi_+ ((0, 0, 1)^T$ corresponding to $\infty$) unless otherwise specified, which is equivalent to saying that

$$
x = \frac{x_1}{1 - x_3}, \quad y = \frac{x_2}{1 - x_3}, \quad z = x + iy,
$$

$$
\xi_1 = \frac{2x}{x^2 + y^2 + 1}, \quad \xi_2 = \frac{2y}{x^2 + y^2 + 1}, \quad \xi_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.
$$

The infinitesimal area on $S^2$ (where the Riemann metric is the most natural one induced from $\mathbb{R}^3$) under such parametrization can be written as:

$$
dS(\xi) = \frac{4}{(x^2 + y^2 + 1)^2} dA(z). \quad (2.2.1)
$$

Here and throughout, $A$ and $S$ will denote the Euclidean and spherical area respectively.

Next, we point out the relationship between the Euclidean distance and the spherical one, which is denoted by $\text{dist}$ and $\rho$ respectively. We also introduce the chordal distance $d^\#$, that is, the distance in $\mathbb{R}^3$ restricted to $S^2$. It is easy to show that

$$
d^\#(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}, \quad d^\#(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}, \quad z, w \in \mathbb{C}, \quad (2.2.2)
$$

$$
\rho(z, w) = 2 \arcsin \frac{d^\#(z, w)}{2}, \quad z, w \in \hat{\mathbb{C}}. \quad (2.2.3)
$$

$$
d^\#(z, w) \leq \rho(z, w) \leq \frac{\pi}{2} d^\#(z, w) \leq 2, \quad z, w \in \hat{\mathbb{C}}. \quad (2.2.4)
$$

Although $d^\#$ and $\rho$ can substitute for each other in our arguments, we only focus on $\rho$, because $S^2$ is a Kähler manifold under $\rho$ (precisely, the Riemann metric induced from $\mathbb{R}^3$) and the usual conformal structure. We define $\rho(E, r) \{z \in S^2 : \rho(z, E) < r\}$, $\forall E \subset S^2$ and $r \in (0, \infty)$, while $\bar{\rho}(E, r)$ will denote the “≤” version. Also, $B(E, r)$ and $\overline{B}(E, r)$ are the Euclidean counterparts. When $E = \{z\}$, we write “$z$” for simplicity. Meanwhile, $\rho(E)$ and $\text{diam}(E)$ denote the diameter of $E$ with respect to $\rho$ and $\text{dist}$ respectively.

Suppose that $f : D \rightarrow \hat{\mathbb{C}}$ is a univalent map, where $D \subset \hat{\mathbb{C}}$ is a domain. We define the spherical derivative $f^\#$ as follows.

$$
f^\#(z) := \lim_{w \to z} \frac{\rho(f(z), f(w))}{\rho(z, w)} = \lim_{w \to z} \frac{(1 + |w|^2)|f'(w)|}{1 + |f(w)|^2}.
$$

It is not hard to prove that the limit always exists and belongs to $(0, \infty)$. Here we do not write $(1 + |z|^2)|f'(z)|$ directly in the right hand because $z$, $f(z)$ and $f'(z)$ could all be $\infty$. By using the spherical derivative, we have the following coordinate change formula for infinitesimal
spherical area and univalent \( f \), which is the spherical counterpart of the formula in Euclidean metric.

\[
dS(f(z)) = f^\#(z)^2dS(z).
\]

As we mention in the introduction, conformality has a clear geometric meaning in \( S^2 \). To illustrate the idea, suppose \( D \subset S^2 \) is a domain, \( f : D \to S^2 \) is univalent and let \( f \) map \( \xi \in D \) to \( \zeta \). For simplicity, we only consider the case where \( \zeta = 0 \). Given \( \theta \in \mathbb{R} \) and \( r \in (0, \infty) \), define \( R_\theta \) to be the continuous transformation of \( S^2 \) satisfying \( R_\theta(z) = e^{i\theta}z, \forall z \in \mathbb{C} \), which is a restriction of a rotation on \( \mathbb{R}^3 \) (which we call a 3-dimensional rotation); define \( SC_r \) to be the continuous transformation of \( S^2 \) such that \( SC_r(z) = rz, \forall z \in \mathbb{C} \), which can be viewed as a “scaling” on \( S^2 \). Given \( \eta, \iota \in S^2 \), let \( T_{\eta,\iota} \) denote the 3-dimensional rotation which moves \( \eta \) to \( \iota \) along the geodesic from \( \eta \) to \( \iota \). Note that \( T_{\eta,\iota} \) is uniquely determined by \( (\eta, \iota) \) even if \( \eta = -\iota \), and \( T_{\eta,\iota} = Id \) if \( \eta = \iota \). These can be termed “translations” on \( S^2 \). Note that \( R_\theta \) and \( SC_r \) commute, but \( T_{\eta,\iota} \circ T_{\iota,\eta} \neq T_{\eta,\iota} \) in general. The following proposition shows \( f \) can be approximated by a composition of a translation, a rotation and a scaling locally in the spherical distance. One can compare it with the definition of differential on Euclidean space.

**Proposition 2.5.** Suppose \( D \subset S^2 \) is a domain and \( f : D \to S^2 \) is univalent. Let \( f \) map \( \xi \in D \) to 0. Also let \( \theta \) be the argument of \( (f \circ T_{\eta,\xi})'(0) \), and \( r = \| (f \circ T_{\eta,\xi})'(0) \| = f^\#(\xi) \). Then

\[
\rho(f(\eta), SC_r \circ R_\theta \circ T_{\xi,0}(\eta)) = o(\rho(\eta, \xi)), \quad \eta \to \xi.
\]

**Proof.** By the definition of the spherical derivative, \( |(f \circ T_{\eta,\xi})'(0)| = (f \circ T_{\eta,\xi})^\#(0) \). Since \( T_{\eta,\xi} \) is a 3-dimensional rotation, \( T_{\eta,\xi}^\#(0) = 1 \). Then \( |(f \circ T_{\eta,\xi})'(0)| = f^\#(\xi) \) by the chain rule. Let \( z = T_{\xi,0}(\eta) \) and \( g = f \circ T_{\eta,\xi} \). If \( f(\eta) \neq SC_r \circ R_\theta \circ T_{\xi,0}(\eta) \) (which is equivalent to that \( g(z) \neq g'(0)z \)), then

\[
\frac{\rho(f(\eta), SC_r \circ R_\theta \circ T_{\xi,0}(\eta))}{\rho(\eta, \xi)} = \frac{\rho(g(z), g'(0)z)}{|g(z) - g'(0)z|} \frac{|g(z) - g'(0)z|}{\rho(\eta, \xi)}.
\]

On one hand, \( \frac{\rho(g(z), g'(0)z)}{|g(z) - g'(0)z|} \) is bounded by the definition of \( \rho \). On the other hand,

\[
\frac{|g(z) - g'(0)z|}{\rho(\eta, \xi)} = \frac{|g(z) - g'(0)z|}{\rho(T_{\xi,0}(\eta), T_{\xi,0}(\eta))} = \frac{|g(z) - g'(0)z|}{\rho(z, 0)} \frac{|g(z) - g'(0)z|}{2\arctan|z|} = o(1),
\]

by the differentiability of \( g \) at \( z = 0 \).

\[ \square \]

### 2.3 Green’s Function

The Green’s function plays an important role in studying SLE. First we introduce the notion of conformal radius. If \( D \) is a simply-connected hyperbolic domain of \( \mathbb{C} \), \( z \in D \) and \( \varphi : \mathbb{D} \to D \) is a conformal isomorphism with \( \varphi(0) = z \), then \( \text{crad}_D(z) := |\varphi'(0)| \) is called the conformal radius of \( z \) in \( D \). We have

\[
\frac{1}{4} \text{crad}_D(z) \leq \text{dist}(z, D) \leq \text{crad}_D(z),
\]
by Proposition 2.2. The conformal radius is convenient due to its conformal covariance:

\[ \text{crad}_{f(D)}(f(z)) = |f'(z)| \text{crad}_D(z), \]  

(2.3.1)

where \( f \) is a conformal isomorphism with \( f(D) \subset \mathbb{C} \). We also need to define another notion denoted as \( S_D(z;w_1,w_2) \), where \( w_1, w_2 \in \delta D \) and \( w_1 \neq w_2 \). If \( F : D \to \mathbb{H} \) is a conformal isomorphism with \( F(w_1) = 0, F(w_2) = \infty \), then \( S_D(z;w_1,w_2) := \sin[\arg(F(z))] \). Note that it is conformally invariant. That is,

\[ S_{f(D)}(f(z);f(w_1),f(w_2)) = S_D(z;w_1,w_2), \]  

(2.3.2)

where \( f \) is a conformal isomorphism with \( f(D) \subset \mathbb{C} \). By conformal invariance and direct computation of \( \text{hm}_\mathbb{H} \), the harmonic measure on \( \mathbb{H} \), if \( \partial_1, \partial_2 \) are subarcs of \( \delta D \) cut by \( w_1 \) and \( w_2 \), then

\[ 10^{-1}S_D(z,w_1,w_2) \leq \text{hm}_D(z,\partial_1) \wedge \text{hm}_D(z,\partial_2) \leq 10S_D(z,w_1,w_2). \]  

(2.3.3)

Some typical examples of \( \text{crad}_D(z) \) and \( S_D(z;w_1,w_2) \) are given as follows:

\[ \text{crad}_D(z) = 1 - |z|^2, \quad \text{crad}_\mathbb{H}(z) = 2 \text{Im}(z), \]

\[ S_D(0;1,e^{i2\theta}) = \sin \theta \ (\theta \in (0,\pi)), \quad S_{\mathbb{H}}(z;0,\infty) = \sin[\arg(z)]. \]

Before giving the definition of Green’s function of SLE, note that we have the following proposition.

**Proposition 2.6** ([5] Theorem 2.3). There exist \( \hat{c} \in (0,\infty), u \in (0,2) \) such that if \( \theta \in (0,\pi) \) and \( \gamma \) is a chordal SLE(\( \kappa \)) curve in \( \mathbb{D} \) from 1 to \( e^{i\theta} \), then

\[ \mathbb{P}\{\text{dist}(0,\gamma[0,\infty)) \leq \epsilon\} = \hat{c}(\sin \theta)^{4a-1}\epsilon^{2-\kappa}\{1 + O(\epsilon^u)\}, \quad \epsilon \downarrow 0. \]

Like the case of diffusion process, the Green’s function of SLE is defined to represent a normalized probability of the event that the curve pass a given point.

**Definition 2.7.** Let \( D \) be a simply-connected hyperbolic domain of \( \mathbb{C} \), \( z \in D \) and \( w_1, w_2 \) are two different prime ends. Then the Green’s function of chordal SLE(\( \kappa \)) in \( D \) from \( w_1 \) to \( w_2 \) is defined as follows.

\[ G_D(z;w_1,w_2) := \hat{c} \cdot \text{crad}_D(z)^{d-2}S_D(z;w_1,w_2)^{4a-1}, \]

where \( \hat{c} \) is the constant in Proposition 2.6.

By the conformal covariance of \( \text{crad}_D(z) \) (2.3.1) and invariance of \( S_D(z;w_1,w_2) \) (2.3.2), we have the conformal covariance of the Green’s function. That is

\[ G_{f(D)}(f(z);f(w_1),f(w_2)) = |f'(z)|^{d-2}G_D(z,w_1,w_2). \]  

(2.3.4)

The key one-point estimate used to prove the existence of Euclidean Minkowski content of the SLE curve in [5] is a generalization of Proposition 2.6.
Theorem 2.8 ([5] PROPOSITION 4.3). There exist $C < \infty$ and $q \in (0, 2)$ such that the following holds. Suppose $D \subset \mathbb{C}$ is a simply-connected hyperbolic domain, $z \in D$, $w_1$, $w_2$ are two different prime ends of $D$ and $\gamma$ is a chordal $\text{SLE}(\kappa)$ curve in $D$ from $w_1$ to $w_2$. Then $\forall \epsilon \in (0, \text{dist}(z, \partial D)/2)$,

$$\left| \frac{\mathbb{P}\{\text{dist}(z, \gamma[0, \infty)) \leq \epsilon\}}{\epsilon^{2-d}G_D(z; w_1, w_2)} - 1 \right| \leq C(\epsilon/\text{dist}(z, \partial D))^q.$$ 

In order to establish the corresponding theory with regard to $\rho$, we need to define the spherical Green’s function. Let $D \subset S^2$ be a simply-connected hyperbolic domain and $\varphi : \mathbb{D} \rightarrow D$ be a conformal isomorphism with $\varphi(0) = z$. Then

$$\text{scr}_D(z) := \varphi^\#(0). \quad (2.3.5)$$

Note that if $D \subset \mathbb{C}$, then $\text{scr}_D(z) = \frac{\text{crad}_D(z)}{1 + |z|^2}$. Therefore, we call $\text{scr}_D(z)$ the spherical conformal radius of $z$ in $D$. Similar to (2.3.1), we have the conformal covariance:

$$\text{scr}_{\varphi(D)}(f(z)) = |f^\#(z)|\text{scr}_D(z). \quad (2.3.6)$$

On the other hand, the definition of $S_D(z; w_1, w_2)$ is generalized to all $D \subset S^2$ in a natural way. Now we define the spherical Green’s function for any simply-connected hyperbolic domain of $S^2$.

Definition 2.9. Let $D$ be a simply-connected hyperbolic domain of $S^2$, $z \in D$ and $w_1, w_2$ are two different prime ends. Then the spherical Green’s function of chordal $\text{SLE}(\kappa)$ in $D$ from $w_1$ to $w_2$ is defined as follows.

$$\tilde{G}_D(z; w_1, w_2) := 2^{d-2}\hat{c} \cdot \text{scr}_D(z)^{d-2}S_D(z; w_1, w_2)^{4a-1},$$

where $\hat{c}$ is the constant in Proposition 2.6.

The definition of $\tilde{G}$ is inspired by (2.2.2), (2.2.3) and Theorem 2.8, since when $D \subset \mathbb{C}$,

$$\tilde{G}_D(z; w_1, w_2) = \left(\frac{1 + |z|^2}{2}\right)^{2-d}G_D(z; w_1, w_2).$$

Likewise, we have the conformal covariance for the spherical Green’s function, which is

$$\tilde{G}_{\varphi(D)}(f(z); f(w_1), f(w_2)) = [f^\#(z)]^{d-2}\tilde{G}_D(z; w_1, w_2). \quad (2.3.7)$$

We also need a spherical version of Theorem 2.8.

Theorem 2.10. Let $q \in (0, 2)$ be the constant in Theorem 2.8. There exists $C < \infty$ such that the following holds. Suppose $D \subset S^2$ is a simply-connected hyperbolic domain, $z$ is an interior
point whose antipode is not in $D$, $w_1$, $w_2$ are two different prime ends of $D$ and $\gamma$ is a chordal SLE($\kappa$) curve in $D$ from $w_1$ to $w_2$. Then $\forall \epsilon > 0$ and $\epsilon \leq \frac{1}{4} \rho(z, \partial D) < \frac{\pi}{4}$, we have

$$
\left| \frac{\mathbb{P}\{\rho(0, \gamma[0, \infty]) \leq \epsilon\}}{e^{2-\delta}G_D(z; w_1, w_2)} - 1 \right| \leq C(\epsilon/\rho(z, \partial D))^q.
$$

**Proof.** By a rotation in $\mathbb{R}^3$, we can without loss of generality assume $z = 0$ and $D \subset \mathbb{C}$. Then $\forall w \in \mathbb{C}$, $\rho(0, w) = 2 \arctan(|w|)$ by (2.2.2) and (2.2.3). To simplify, we write $G := G_D(0; w_1, w_2), \tilde{G} := \tilde{G}_D(0; w_1, w_2), B := \rho(0, \partial D) = 2 \arctan(\text{dist}(0, \partial D))$ and $P := \mathbb{P}\{\rho(0, \gamma(0, \infty)) \leq \epsilon\} = \mathbb{P}\{\text{dist}(0, \gamma(0, \infty)) \leq \tan(\epsilon/2)\}$ (note that $\tilde{G} = 2^{d-2}G$). Then

$$
\frac{|\epsilon^{d-2\tilde{G}^{-1}P - 1}|}{(\epsilon/B)^q} \leq \left| \frac{\tan(\epsilon/2)^{d-2\tilde{G}^{-1}P - 1}}{(\epsilon/B)^q} \right| \leq \left| \frac{\tan(\epsilon/2)^{d-2\tilde{G}^{-1}P - 1}}{(\epsilon/B)^q} \right| + \left| \frac{\tan(\epsilon/2)^{d-2\tilde{G}^{-1}P - 1}}{(\epsilon/B)^q} \right| \leq \left| \frac{\tan(\epsilon/2)^{d-2\tilde{G}^{-1}P - 1}}{(\epsilon/B)^q} \right| + \left| \frac{\tan(\epsilon/2)^{d-2\tilde{G}^{-1}P - 1}}{(\epsilon/B)^q} \right|
$$

First, it is easy to show that $\tan(2x) \geq 2\tan x \quad \forall x \in [0, \pi/4)$, which implies $\text{dist}(0, \partial D) = \tan\left(\rho(0, \partial D)/2\right) \geq \tan(2\epsilon) \geq 2\tan(\epsilon) > 2\tan(\epsilon/2)$. So we can use Theorem 2.8 to bound $\frac{|\tan(\epsilon/2)^{d-2\tilde{G}^{-1}P - 1}|}{(\tan(\epsilon/2)^{d-2\tilde{G}^{-1}P - 1})/\text{dist}(0, \partial D)^q}$.

Second, $\frac{2\tan(\epsilon/2)}{\epsilon} = \frac{\arctan(\text{dist}(0, \partial D))}{\text{dist}(0, \partial D)} = \frac{2\arctan(\text{dist}(0, \partial D))}{\text{dist}(0, \partial D)}$, $\frac{\tan(\epsilon/2)}{\epsilon}$ and $B$ are bounded.

Third, using the Taylor expansion $2\tan(\epsilon/2) = \epsilon + O(\epsilon^3)$, we can get a bound for $\frac{|\frac{2\tan(\epsilon/2)}{\epsilon}|^{2-\delta} - 1}{\epsilon^q}$. \hfill \square

For ease, we let $\tilde{G}$ denote the spherical Green’s function in $\mathbb{H}$ from 0 to $\infty$ and $\forall E \in \mathcal{B}(\mathbb{H})$, $\tilde{G}(E) := \int_E \tilde{G}(z) dS(z)$, similarly for $G$ and $G(E)$.

In [11] and [10], authors show the existence of the multi-point Green’s function of the SLE curve and give up-to-constant bounds. We state some needed results about this. For $x, y \geq 0$,
define
\[ P_y(x) := \begin{cases} y^{(4a-1)-(2-d)}x^{2-d}, & x \leq y; \\ x^{4a-1}, & x \geq y. \end{cases} \]

Since \( 4a - 1 > 2 - d > 0 \), \( \forall 0 \leq x_1 \leq x_2, y \geq 0 \), we have \( \frac{x_1^{4a-1}}{x_2^{4a-1}} \leq \frac{P_y(x_1)}{P_y(x_2)} \leq \frac{x_2^{2-d}}{x_1^{2-d}} \).

**Theorem 2.11** ([11] Theorem 1.1). Fix a positive integer \( n \). Let \( z_0, z_1, \ldots, z_n \) be distinct points on \( \mathbb{H} \) with \( z_0 = 0 \). Let \( y_k := \text{Im}(z_k) \) and \( l_k := \text{dist}(z_k, \{z_j : 0 \leq j < k\}) \), \( 1 \leq k \leq n \). Let \( r_1, \ldots, r_n > 0 \). Then there exists \( C_n \) such that
\[ \mathbb{P}\{\text{dist}(\gamma[0, \infty], z_k) \leq r_k, 1 \leq k \leq n\} \leq C_n \prod_{k=1}^{n} \frac{P_{y_k}(r_k \land l_k)}{P_{y_k}(l_k)}. \]

**Proposition 2.12.** We use the notation in Theorem 2.11. Then
\[ r^{n(d-2)}\mathbb{P}\{\text{dist}(\gamma[0, \infty], z_k) \leq r, 1 \leq k \leq n\} \leq C_n \prod_{k=1}^{n} r_k^{d-2}, \quad \forall r \in (0, \infty). \]

And \( f(z_1, \ldots, z_n) := \prod_{k=1}^{n} l_k^{d-2} \) is integrable on \( D^n \) if \( D \) is a bounded Borel set of \( \mathbb{H} \).

**Proof.** Suppose \( r \) is smaller than or equal to \( l_{j_1}, \ldots, l_{j_m} \) and bigger than the others. By Theorem 2.11,
\[ \text{LHS} \leq r^{n(d-2)}C_n \prod_{k=1}^{n} \frac{P_{y_k}(r \land l_k)}{P_{y_k}(l_k)} = r^{n(d-2)}C_n \prod_{s=1}^{m} \frac{P_{y_{j_s}}(r)}{P_{y_{j_s}}(l_{j_s})} \leq r^{n(d-2)}C_n \prod_{s=1}^{m} \frac{r^{2-d}}{l_{j_s}^{2-d}} \leq \text{RHS}. \]

Next, for fixed \( k \in [1, n] \),
\[ \int_D l_k^{d-2}dA(z_k) \leq \sum_{j=0}^{k-1} \int_D |z_k - z_j|^{d-2}dA(z_k) \leq k \int \int_{|w_k| \leq 2diam(D \cup \{0\})} |w_k|^{d-2}dA(w_k) < \infty. \]

\( \square \)

### 2.4 Minkowski Content

The Minkowski content in Euclidean space is a generalization of length, area, volume, etc. The spherical Minkowski content is defined to imitate the Euclidean one.
Definition 2.13. For a bounded set $E \subset \mathbb{C}$, and $n \in (0, 2)$ we define the $n$-dimensional upper and lower Minkowski content as follows (recall that $A$ is the area in $\mathbb{R}^2$).

$$M^n_+(E) := \lim_{\epsilon \to 0^+} \frac{A\{z \in \mathbb{C} : \text{dist}(z, E) \leq \epsilon\}}{\epsilon^{2-n}}, \quad M^n_-(E) := \lim_{\epsilon \to 0^+} \frac{A\{z \in \mathbb{C} : \text{dist}(z, E) \leq \epsilon\}}{\epsilon^{2-n}}.$$ 

If $M^n_+(E) = M^n_-(E)$, then we call $M^n(E) := M^n_+(E)$ as the $n$-dimensional Minkowski content of $E$.

For any set $E \subset \widehat{\mathbb{C}}$, we define its $n$-dimensional upper and lower spherical Minkowski content as follows ($S$ is the area in $S^2$).

$$M^n_S(\partial E) := \lim_{\epsilon \to 0^+} \frac{S\{z \in S^2 : \rho(z, E) \leq \epsilon\}}{\epsilon^{2-n}}, \quad M^n_S(\hat{E}) := \lim_{\epsilon \to 0^+} \frac{S\{z \in S^2 : \rho(z, \hat{E}) \leq \epsilon\}}{\epsilon^{2-n}}.$$ 

If $M^n_S(\partial E) = M^n_S(\hat{E})$, then we call $M^n_S(E) := M^n_S(\partial E)$ as the $n$-dimensional spherical Minkowski content of $E$. (We allow both $M^n(E)$ and $M^n_S(E)$ to be $\infty$.)

Note that $A\{z \in \mathbb{C} : \text{dist}(z, E) \leq \epsilon\}$ is $\infty$ for all $\epsilon$ when $E$ is unbounded, so it is of no use to discuss the Minkowski content for unbounded sets in $\mathbb{C}$. But all subsets of $S^2$ are bounded (i.e. relatively compact), which is more convenient. Here are some basic properties of the Minkowski content.

Proposition 2.14. Let $E, F, \{E_n\}_{n=1}^{\infty}$ be subsets of $S^2$.

1. $M^n_S(E) = M^n_S(\hat{E})$, $M^n_S(\partial E) = M^n_S(\hat{E})$.
2. $M^n_S(E \cup F) \leq M^n_S(E) + M^n_S(F)$.
3. If $\rho(E, F) > 0$, then $M^n_S(E \cup F) = M^n_S(E) + M^n_S(F)$, $M^n_S(E \cap F) = M^n_S(E \cup F)$.
4. If $\forall n$, $M^n_S(E_n)$ exists and

$$\lim_{n \to \infty} [M^n_S(E \setminus E_n)] + M^n_S(E_n \setminus E) = 0,$$

then

$$M^n_S(E) = \lim_{n \to \infty} M^n_S(E_n).$$

5. If $E$ is a domain with a piecewise $C^1$ boundary, then $M^n_S(\partial E) = 0$. And

$$M^n_S(E \cap F) = M^n_S(E \cap F) = \lim_{\epsilon \to 0^+} \frac{S\{z \in E : \text{dist}(z, F) \leq \epsilon\}}{\epsilon^{2-d}}.$$ 

Similar claims hold for the lower spherical Minkowski content.
If $T$ is a 3-dimensional rotation, then $\overline{M^d_S(T(E))} = \overline{M^d_S(E)}$, $\underline{M^d_S(T(E))} = \underline{M^d_S(E)}$.

The Euclidean version of (1)–(5) hold for bounded sets of $\mathbb{C}$.

**Proof.** We only give the proof of the spherical case. (1)–(3) and (6) are obvious from the definition. To get (4), note that

$$\{z \in S^2 : \rho(z, E_n) \leq \epsilon\} \setminus \{z \in S^2 : \rho(z, E \setminus E_n) \leq 2\epsilon\} \subset \{z \in S^2 : \rho(z, E \setminus E_n) \leq 2\epsilon\}.$$

Next we prove (5). For the closed piecewise $C^1$ curve $\partial E$, $M^d_1(\partial E) < \infty$ (which is equivalent to saying that $\partial E$ has a finite length), so $d > 1$ implies $\overline{M^d_S(\partial E)} = 0$. $F \cap E = F \cap (E \cup \partial E) = (F \cap E) \cup (F \cap \partial E) \subset (F \cap E) \cup \partial E$, so $\overline{M^d_S(F \cap E)} \leq \overline{M^d_S(F \cap E)} + 0$, which implies $\overline{M^d_S(E \cap F)} = \overline{M^d_S(E \cap F)}$. For the second equality in (5), note that

$$\{z \in S^2 : \rho(z, E \cap F) \leq \epsilon\} \setminus \{z \in S^2 : \rho(z, \partial E) \leq 2\epsilon\} \subset \{z \in D : \rho(z, F) \leq \epsilon\} \cup \{z \in S^2 : \rho(z, \partial E) \leq 2\epsilon\}, \quad \overline{M^d_S(\partial E)} = 0.$$

The case of the lower content is similar. \hfill \box

We will use some results about $M^d$ directly to get the spherical counterpart. Therefore, the following observation is helpful.

**Proposition 2.15.** For any $R \in (0, \infty)$, there exists constants $C_1, C_2 < \infty$ depending on $R$ such that $\forall E \subset \{z \in \mathbb{C} : |z| \leq R\}$,

$$C_1 \overline{M^d(E)} \leq \overline{M^d_S(E)} \leq C_2 \overline{M^d(E)},
\quad C_1 \underline{M^d(E)} \leq \underline{M^d_S(E)} \leq C_2 \underline{M^d(E)}.$$

**Proof.** This is immediate from (2.2.1) and Definition 2.13. \hfill \box

## 3 Proof of the Main Result

In this section, we give part of the proof of the main result. Consider the case in which $\gamma$ is an $\text{SLE}(\kappa)$ curve in $\mathbb{H}$ from 0 to $\infty$. Let

$$\tau_\epsilon(z) := \inf\{t \geq 0 : \rho(\gamma(t), z) \leq \epsilon\},$$

$$J_\epsilon(z) := \epsilon^{d-2}1\{\tau_\epsilon(z) < \infty\}, \quad J_\epsilon(E) := \int_E J_\epsilon(z) dS(z),$$

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\[ Q_{\epsilon,\lambda}(z) := J_{\epsilon}(z) - J_{\lambda\epsilon}(z), \quad Q_{\epsilon,\lambda}(E) := J_{\epsilon}(E) - J_{\lambda\epsilon}(E), \]
\[ \forall \epsilon > 0, \lambda \in (0, 1), z \in \mathbb{H}, E \in \mathcal{B}([0, \infty)). \]

Note that
\[ \lim_{\epsilon \to 0^+} J_{\epsilon}(V) = M_S^d(V \cap \gamma[0, \infty]), \quad \lim_{\epsilon \to 0^+} J_{\epsilon}(V) = M_S^d(V \cap \gamma[0, \infty]) \quad (3.0.1) \]
if \( V \) is a domain with a piecewise \( C^1 \) boundary, by Proposition 2.14 (5). We also have a simple observation: \( \forall r \in (0, 1), z \in \mathbb{H}, \) we have
\[ J_{r\epsilon}(z) \leq r^{d-2} J_{\epsilon}(z). \]
Particularly, \( \forall z \in \mathbb{H}, \epsilon \in (0, 0.01), \lambda \in (0, 1), \) if \( \lambda^n \leq \epsilon < \lambda^{n-1} \) for some positive integer \( n, \) then
\[ \lambda^{2-d}J_{\lambda^n}(z) \leq J_{\epsilon}(z) \leq \lambda^{d-2}J_{\lambda^{n-1}}(z). \quad (3.0.2) \]
Fix a nonnegative integer \( n. \) For any \( E \subset S^2, \partial E := \{z \in \mathbb{H} : \rho(z, \partial E) \leq 2^{-n}\}. \) In addition, \( \mathcal{I}_n := \{\Gamma_n(j, k) := (j2^{-n}, (j + 1)2^{-n}) \times (k2^{-n}, (k + 1)2^{-n}) : j, k \text{ are integers and } k \geq 0\}, \)
\[ \mathcal{I}_n^+ := \{\Gamma_n(j, k) \in \mathcal{I}_n : k > 0\}, \mathcal{I} := \bigcup_{n=1}^\infty \mathcal{I}_n \text{ and } \mathcal{I}^+ := \bigcup_{n=1}^\infty \mathcal{I}_n^+. \] Note that \( \mathcal{I} \) is countable.

The next key theorem is the counterpart of THEOREM 3.2 in [5]. We delay its proof to section 4.

**Theorem 3.1.** There exists \( \beta \in (0, \infty), \beta^\prime \in (0, 2) \) such that \( \forall \) compact set \( E \subset \mathbb{H}, \exists c = c(E) \in (0, \infty), \forall \epsilon \in (0, \rho(E, \partial\mathbb{H})/4), \lambda \in (0.1, 1), z, w \in E, \) we have
\[ \left| \mathbb{E}[Q_{\epsilon,\lambda}(z)Q_{\epsilon,\lambda}(w)] \right| \leq ce^{\beta} \rho(z, w)^{\beta^\prime-2}. \quad (3.0.3) \]

### 3.1 Existence of the Spherical Natural Occupation Measure

**Lemma 3.2.** For any compact set \( E \subset \mathbb{H}, \) there exists \( c = c(E) < \infty \) such that \( \forall \epsilon \in (0, \rho(E, \partial\mathbb{H})/4), \lambda \in (0.1, 1), \) we have
\[ \mathbb{E}[Q_{\epsilon,\lambda}^2(E)] \leq ce^\beta, \]
where \( \beta \) is the constant in Theorem 3.1.

**Proof.** It is immediate by integrating (3.0.3) in \((z, w) \in E^2. \)

**Proposition 3.3.** For any \( \Gamma \in \mathcal{I}^+ \), \( J_{\epsilon}(\Gamma) \) converges in \( L^2 \) to a random variable \( J_0(\Gamma). \) Moreover, with probability 1: \( \forall \Gamma \in \mathcal{I}^+, \lim_{\epsilon \to 0} J_{\epsilon}(\Gamma) = J_0(\Gamma) = M_S^d(\Gamma \cap \gamma[0, \infty]). \)
Proof. Fix $\Gamma \in \mathcal{I}^+$. $J_\epsilon := J_\epsilon(\Gamma)$, $Q_{\epsilon,\lambda} := Q_{\epsilon,\lambda}(\Gamma)$.

We first deal with the $L^2$ convergence. Choose $\lambda = 0.5$ and $\forall \epsilon, \delta \in (0, \rho(\Gamma, \partial \mathbb{H})/4)$ with $\delta < \epsilon$, suppose $\lambda^{n+1} \leq \epsilon < \lambda^n$, $\lambda^{n+p+1} \leq \delta < \lambda^{n+p}$ for some nonnegative integers $n$ and $p$.

$$||J_\epsilon - J_\delta||_2 \leq ||J_\lambda^n - J_\lambda^{n+p}||_2 + ||J_\lambda^n - J_\epsilon||_2 + ||J_\lambda^{n+p} - J_\delta||_2.$$  

By Lemma 3.2,

$$||J_\lambda^n - J_\lambda^{n+p}||_2 \leq \sum_{k=n}^{n+p-1} ||Q_{\lambda, k}||_2 \leq \sum_{k=n}^{n+p-1} c\lambda^{k\beta/2} \to 0, \ n \to \infty, \ \forall p.$$  

Since $\epsilon/\lambda^n \geq \lambda^{n+1}/\lambda^n = \lambda > 0.1$ and $\lambda^{n+1}/\lambda^n \leq \epsilon/\lambda = 2\epsilon < \rho(\Gamma, \partial \mathbb{H})/4$, by Proposition 3.2,

$$||J_\lambda^n - J_\epsilon||_2 = ||Q_{\lambda^n, \epsilon/\lambda^n}||_2 \leq c\lambda^{n\beta/2} \to 0, \ n \to \infty.$$  

In a similar way, $||J_\lambda^{n+p} - J_\delta||_2 \to 0, \ n \to \infty, \ \forall p$. So $J_\epsilon \xrightarrow{L^2} J_0(\Gamma)$ (also denoted by $J_0$), $\epsilon \to 0$, by Cauchy’s criterion.

Next, \( \forall \lambda \in (0,1) \cap \mathbb{Q} \), Chebyshev’s inequality shows that

$$\mathbb{P}\{ |Q_{\lambda^n, \lambda} \geq \lambda^{\beta/4} \} \leq \frac{c\lambda^{n\beta}}{\lambda^{\beta/2}} = c\lambda^{\beta/2}, \ \text{n sufficiently large s.t. } \lambda^n < \rho(\Gamma, \partial \mathbb{H})/4.$$  

Then, by Borel-Cantelli lemma, w.p.1.:

$$|J_{\lambda^{n+1}} - J_{\lambda^n}| \leq \lambda^{n\beta/4} \text{ for all but finitely many } n.$$  

This implies \( \{J_{\lambda^n}\} \) is a Cauchy sequence. Together with the $L^2$ convergence, we have proven that w.p.1.: \( \forall \lambda \in (0,1) \cap \mathbb{Q}, \lim_{n\to\infty} J_{\lambda^n} = J_0. \)

By (3.0.2), w.p.1.: \( \forall \lambda \in (0,1) \cap \mathbb{Q}, \)

$$\lambda^{2-d} J_0 \leq \lim_{\epsilon \to 0} J_\epsilon \leq \lim_{\epsilon \to 0} J_{\epsilon/\lambda^n} \leq \lambda^{d-2} J_0.$$  

Letting $\lambda$ tends to 1, we get $\lim_{\epsilon \to 0} J_\epsilon = J_0$. Finally, $J_0(\Gamma) = M_d(\Gamma \cap \gamma[0,\infty])$ by Proposition 2.14 (5).

\[ \square \]

Lemma 3.4. With probability 1, \( \forall \Gamma \in \mathcal{I}^+, \lim_{n\to\infty} J_{2^{-n}}(\partial_m \Gamma) \leq 2^{-m/2} \) for all but finitely many \( m. \)

Proof. Fix $\Gamma \in \mathcal{I}^+$. Theorem 2.10 shows for $n, m$ sufficiently large s.t. $2^{-m} \vee 2^{-n} < \rho(\Gamma, \partial \mathbb{H})/8$,

$$\mathbb{E}[J_{2^{-n}}(\partial_m \Gamma)] = 2^{n(2-d)} \int_{\partial_m \Gamma} \mathbb{P}\{ \tau_{2^{-n}}(z) < \infty \} dS(z) = \int_{\partial_m \Gamma} \tilde{G}(z) [1 + O(2^{-nu})] dS(z) \leq M \cdot S(\partial_m \Gamma) \leq M \cdot 2^{-m},$$

where $M$ is a constant depending on $\Gamma$. Therefore, by Chebyshev’s inequality,

$$\mathbb{P}[J_{2^{-n}}(\partial_m \Gamma) \geq 2^{-m/2}] \leq \mathbb{E}[J_{2^{-n}}(\partial_m \Gamma)]/2^{-m/2} \leq M \cdot 2^{-m/2}.$$  

Then the proof is done by using the Borel-Cantelli lemma. \[ \square \]
For any positive integer \( n \), define \( \mu_n \) to be the finite random measure on \( \mathbb{H} \) by letting its Radon-Nikodym derivative about \( S \) be \( J_{2^{-n}}(z) \).

**Theorem 3.5.** With probability 1, \( \mu_n \) converges vaguely to a finite random measure \( \mu \) satisfying 
\[
\mu(\Gamma) = M^d_S(\Gamma \cap \gamma[0, \infty]), \forall \Gamma \in \mathcal{I}^+.
\]

And 
\[
\mathbb{E}[\mu(E)] = \tilde{G}(E), \forall E \in \mathcal{B}(\mathbb{H}), \quad \mathbb{E}[\mu(\mathbb{H})] < \infty.
\]

**Proof.** It is easy to show that \( \{\mu_n\} \) is relatively compact in the vague topology by using the compactness criterion of the vague topology. Let \( \mu \) be one of the limit points and a subsequence \( \{\mu_{n_j}\} \) converge to it. (Note that \( \{\mu_j\} \) may be stochastic.) By the vague convergence and Lemma 3.4, \( \forall \Gamma \in \mathcal{I}^+ \), for large \( m \), 
\[
\mu(\text{int}(\partial_m \Gamma)) \leq \lim_{j \to \infty} \mu_{n_j}(\text{int}(\partial_m \Gamma)) \leq 2^{-m/2}, \text{ so } \mu(\partial \Gamma) = 0.
\]

Hence, again by the vague convergence, 
\[
\mu(\Gamma) = \lim_{j \to \infty} \mu_{n_j}(\Gamma) = M^d_S(\Gamma \cap \gamma[0, \infty]).
\]

This shows that all limit points of \( \{\mu_n\} \) have the same value on \( \mathcal{I}^+ \). Since \( \mathcal{B}(\mathbb{H}) = \sigma(\mathcal{I}^+) \), we can prove any limit point of \( \{\mu_n\} \) is \( \mu \) by a monotone class argument. Therefore, \( \mu_n \) converges in the vague topology to \( \mu \).

\( L^2 \) convergence in Proposition 3.3 implies 
\[
\mathbb{E}[\mu(\Gamma)] = \mathbb{E}[\lim_{n \to \infty} \mu_n(\Gamma)] = \lim_{n \to \infty} \mathbb{E}[\mu_n(\Gamma)], \forall \Gamma \in \mathcal{I}^+.
\]

Then, by Theorem 2.10 and the dominated convergence theorem, 
\[
\mathbb{E}[\mu(\Gamma)] = \tilde{G}(\Gamma), \forall \Gamma \in \mathcal{I}^+.
\]

Note that both sides of the above equality define a measure on \( \mathbb{H} \) when regarding \( \Gamma \) as the variable and \( \mathcal{B}(\mathbb{H}) = \sigma(\mathcal{I}^+) \). We conclude that 
\[
\mathbb{E}[\mu(E)] = \tilde{G}(E), \forall E \in \mathcal{B}(\mathbb{H}).
\]

Lastly, we compute \( \mathbb{E}[\mu(\mathbb{H})] \).

\[
\mathbb{E}[\mu(\mathbb{H})] = \int_{\mathbb{H}} \tilde{G}(z) dS(z) = \int_{\mathbb{R}^2} \frac{4}{(x^2 + y^2 + 1)^2} \tilde{G}(x + iy) dx dy
\]
\[
= c \int_{\mathbb{R}^2} \frac{(x^2 + y^2 + 1)^{-2} y^{d-2} \left( \frac{y}{\sqrt{x^2 + y^2}} \right)^{4a-1} dx dy}
\]
\[
= c \int_{\mathbb{R}^2} \frac{y^{4a+d-3}}{(x^2 + y^2 + 1)^d} dxdy = c \int_{0}^{\pi} r^{d-1} \sin^{4a+d-3} \theta dr d\theta
\]
\[
= c \int_{0}^{\pi} r^{d-1} \left( \frac{1}{1 + r^2} \right)^d dr \int_{0}^{\pi} \sin^{4a+d-2} \theta d\theta < \infty, \quad \left( \frac{8}{\kappa} + \frac{\kappa}{8} - 2 > 0 \right)
\]

where \( c \) is a constant depending only on \( \kappa \). Consequently, \( \mu(\mathbb{H}) < \infty \), a.s. \( \square \)
3.2 Further Properties

Next, we need the reversibility of chordal SLE (cf. [8] and [16]). That is, if \( \sigma \) is a chordal SLE curve in a hyperbolic simply-connected domain \( D \) from \( a \) to \( b \), then \( \{ \sigma(t^{-1}) \}_{t \in [0, \infty)} \) has the same distribution (modulo the parametrization) as the SLE curve in \( D \) from \( b \) to \( a \). Note that \( \exists \) a 3-dimensional rotation \( T \) s.t. \( T(\{ z \in \mathbb{H} : |z| \leq 1 \}) = \{ z \in \mathbb{H} : |z| \geq 1 \} \). Then by conformal invariance and reversibility, \( \gamma[0, \infty] \cap \{ z \in \mathbb{H} : |z| \leq 1 \} \) \( \text{distribution} = \gamma[0, \infty] \cap \{ z \in \mathbb{H} : |z| \geq 1 \} \).

**Lemma 3.6.** There exists \( \alpha \in (0, 1) \) such that the following holds with probability one.

1. \( \forall T \in (0, \infty), \exists N, c \in (0, \infty) \) s.t. \( \forall t \in [0, T], n \geq N \), we have
   \[
   \frac{M_{S}^{d}(\gamma[t, t + 2^{-n}])}{c 2^{-n\alpha}}.
   \]

2. \( \lim_{t \to 0} \frac{M_{S}^{d}(\gamma[0, t])}{c} = 0, \lim_{t \to \infty} \frac{M_{S}^{d}(\gamma[t, \infty])}{c} = 0. \)

3. If \( t \in [0, \infty) \) and \( u \in (0, \infty) \), then
   \[
   \lim_{n \to \infty} \frac{M_{S}^{d}(\gamma[t + u, \infty] \cap \partial_{n}H_{t})}{c} = 0.
   \]

4. If \( t \in [0, \infty) \), then
   \[
   \lim_{n \to \infty} \frac{M_{S}^{d}(\gamma[t, \infty] \cap \partial_{n}H_{t})}{c} = 0.
   \]

Moreover, \( \mathbb{E}\left[ \frac{M_{S}^{d}(\gamma[0, \infty])}{c} \right] < \infty, \forall m \in \mathbb{N}. \)

**Proof.** (1) It is immediate by using (31) in [5] and Proposition 2.15.

(2) The first equality is a direct corollary of (1), and the second follows from the reversibility.

(3) For \( t \) fixed, we choose \( T \in (0, \infty) \) and \( \epsilon \in (0, 1) \) s.t. \( \forall s \geq T, \rho(\gamma(s), K_{t}) \geq \epsilon \). By (32) in [5] and Proposition 2.15 (noting that \( \gamma[t + u, T] \cap \partial_{n}H_{t} \) has a positive distance from \( \infty \) for \( n \) large enough),
   \[
   \lim_{n \to \infty} \frac{M_{S}^{d}(\gamma[t + u, T] \cap \partial_{n}H_{t})}{c} = 0.
   \]

On the other hand, the reversibility, LEMMA 3.7 in [5] and Proposition 2.15 yield
   \[
   \lim_{n \to \infty} \frac{M_{S}^{d}(\gamma[T, \infty] \cap \partial_{n}H_{t})}{c} = 0.
   \]

(4) This is a corollary of (3) and the first equality in (2).

Finally, Theorem 1.2 in [11] and Proposition 2.15 imply
   \[
   \mathbb{E}\left[ \frac{M_{S}^{d}(\gamma[0, \infty] \cap \{ z \in \mathbb{H} : |z| \leq 1 \})}{c} \right] < \infty.
   \]
Then we get
\[ \mathbb{E}\left[ M^d_\mathbb{H}(\gamma[0, \infty] \cap \{z \in \mathbb{H} : |z| \geq 1\})^m \right] < \infty \]
by reversibility.

**Lemma 3.7.** Suppose \( E \) is a compact subset of \( \mathbb{H} \) and \( 2 \leq n \) is a positive integer. Then there exists \( c = c(E, n) < \infty \) s.t. \( \forall m \in \mathbb{N}, \Gamma \in \mathcal{I}_m, \) if \( \Gamma \subset E, \) then
\[ \mathbb{E}\left[ M^d_\mathbb{H}(\Gamma \cap \gamma[0, \infty])^m \right] \leq c 2^{-(n-1)d+2|m}. \]

**Proof.** By Proposition 2.15, it suffices to prove the lemma for \( M^d(\Gamma) \). Fix \( m \in \mathbb{N}, \Gamma \in \mathcal{I}_m, \Gamma \subset E. \) By Proposition 2.12, \( \forall z_1, \ldots, z_n \in \Gamma, \) we have
\[ \sup_{0 < r < 1} \left\{ r^{n(d-2)} \mathbb{P}\left[ \text{dist}(\gamma, z_k) \leq r, \ 1 \leq k \leq n \right] \right\} \leq c \prod_{k=1}^{n} l_k^{d-2}, \]
where \( l_k \) is defined in Theorem 2.11. Since \( \text{dist}(\Gamma, \mathbb{R}) = 0, c^{-1} \prod_{k=1}^{n} l_k^{d-2} \leq s(z_1, \ldots, z_n) := \prod_{k=2}^{n} \min\{ |z_k - z_1|, \ldots, |z_k - z_{k-1}| \}^{d-2} \leq c \prod_{k=1}^{n} l_k^{d-2}. \) Also Proposition 2.12, \( \prod_{k=1}^{n} l_k^{d-2} \) is integrable over all \( D^n \) where \( D \) is a compact subset of \( \mathbb{H}, \) and hence so is \( s(z_1, \ldots, z_n). \) Using Fatou’s lemma and Fubini’s theorem
\[ \mathbb{E}\left[ M^d(\gamma \cap \Gamma)^n \right] = \mathbb{E}\left[ \left( \lim_{r \to 0} r^{d-2} \int_{\Gamma^n} 1\{ \text{dist}(z, \gamma) \leq r \} dA(z) \right)^n \right] \]
\[ = \mathbb{E}\left[ \lim_{r \to 0} r^{n(d-2)} \int_{\Gamma^n} \prod_{k=1}^{n} 1\{ \text{dist}(z^k, \gamma) \leq r \} dA(z_1) \ldots dA(z_n) \right] \]
\[ \leq \lim_{r \to 0} \mathbb{E}\left[ r^{n(d-2)} \int_{\Gamma^n} \prod_{k=1}^{n} 1\{ \text{dist}(z^k, \gamma) \leq r \} dA(z_1) \ldots dA(z_n) \right] \]
\[ = \lim_{r \to 0} \int_{\Gamma^n} r^{n(d-2)} \mathbb{P}\left[ \text{dist}(\gamma, z_k) \leq r, \ 1 \leq k \leq n \right] dA(z_1) \ldots dA(z_n) \]
\[ \leq \lim_{r \to 0} \int_{\Gamma^n} r^{n(d-2)} \mathbb{P}\left[ \text{dist}(\gamma, z_k) \leq r, \ 1 \leq k \leq n \right] dA(z_1) \ldots dA(z_n) \]
\[ \leq \int_{\Gamma^n} \lim_{r \to 0} r^{n(d-2)} \mathbb{P}\left[ \text{dist}(\gamma, z_k) \leq r, \ 1 \leq k \leq n \right] dA(z_1) \ldots dA(z_n) \]
\[ \leq c \int_{\Gamma^n} s(z_1, \ldots, z_n) dA(z_1) \ldots dA(z_n) \]
\[ = c \int_{\Gamma^n} s(z_1, \ldots, z_n) \left( \sum_{k=0}^{m} w_k = 2^{m} z_k \right) \int_{\Gamma^n} \left( 2^{m} w_{1} \ldots, 2^{-m} w_{n} \right) dA(z_1) \ldots dA(z_n) \]
\[ \left( 2^{n} r \right)^n \]
\[ \leq c 2^{-(n-1)d+2|m}. \]
\[
= c \, 2^{-[(n-1)d+2]m} \int s(w_1, \ldots, w_n) dA(w_1) \ldots dA(w_n) \\
= c \, 2^{-[(n-1)d+2]m} \int_{[1,2]^n} s(w_1, \ldots, w_n) dA(w_1) \ldots dA(w_n) \leq c \, 2^{-[(n-1)d+2]m}.
\]

\[\square\]

**Proposition 3.8.** With probability 1, the following holds. If \(\xi : [0, 1] \to \mathbb{H}\) is piecewise \(C^1\), then \(\lim_{m \to \infty} M_S^d(\gamma \cap \mathbb{H} \cap \partial_m \xi[0,1]) = 0\); in particular, if \(D \subset \mathbb{H}\) is a domain with \(\partial D\) piecewise \(C^1\), then \(\mu(D) = M_S^d(\gamma \cap \mathbb{H} \cap D)\).

**Proof.** Fix a positive integer \(n\) s.t. \(n < (n-1)d\). \(\forall k \in \mathbb{N}, Z_k := \{z \in \mathbb{H} : \rho(z, \partial \mathbb{H}) > 1/k\}\).

For all \(k, m\), \(\text{card}\{\Gamma \in \mathcal{I}_m : \Gamma \subset Z_k\} \leq 2^{2m}A_k\), where \(A_k\) is the Euclidean area of \(Z_k\). Using Lemma 3.7 and Chebyshev’s inequality, we get that \(\forall k\),

\[
\sum_{m \geq 1} \mathbb{P}\left(\exists \Gamma \in \mathcal{I}_m, \Gamma \subset Z_k \text{ s.t. } M_S^d(\gamma \cap \Gamma) \geq \frac{2^{-m}}{m^2}\right) \leq \sum_{m \geq 1} \sum_{\Gamma \in \mathcal{I}_m} \mathbb{P}\left(M_S^d(\gamma \cap \Gamma) \geq \frac{2^{-m}}{m^2}\right)
\]

\[
\leq \sum_{m \geq 1} A_k 2^{2m} \mathbb{P}\left(M_S^d(\gamma \cap \Gamma) \geq \frac{2^{-m}}{m^2}\right) \leq \sum_{m \geq 1} A_k 2^{2m} m^{2m} 2^{m} 2^{(n-1)d+2]m} c 2^{-(n-1)d+2]m}
\]

\[
\leq \sum_{m \geq 1} c m^{2m} 2^{(n-1)d]m} < \infty,
\]

where \(c = c(k, n) < \infty\). By Borel-Cantelli Lemma, w.p.1., \(\forall k, \exists M_k < \infty\), s.t. \(\forall m \geq M_k, \Gamma \in \mathcal{I}_m\), if \(\Gamma \subset Z_k\), then \(M_S^d(\gamma \cap \Gamma) \leq \frac{2^{-m}}{m^2}\).

Now suppose \(\xi : [0, 1] \to \mathbb{H}\) is piecewise \(C^1\). Assume that \(\xi[0,1] \subset Z_{k+1}\). Then for large \(m\), \(\text{card}\{\Gamma \in \mathcal{I}_m : \Gamma \cap \partial_m \xi \neq \emptyset\} \leq 10L(\xi)2^m \leq 2^m\), where \(L(\xi)\) is the Euclidean length of \(\xi\), and if \(\Gamma \in \mathcal{I}_m\) and \(\Gamma \cap \partial_m \xi \neq \emptyset\), then \(\Gamma \subset Z_k\). So

\[
\lim_{m \to \infty} M_S^d(\gamma \cap \partial_m \xi) \leq \lim_{m \to \infty} \sum_{\Gamma \in \mathcal{I}_m, \Gamma \cap \partial_m \xi \neq \emptyset} M_S^d(\gamma \cap \Gamma) \leq \lim_{m \to \infty} m^{2m} \frac{2^{-m}}{m^2} = 0.
\]

Lastly, for a domain \(D \subset \mathbb{H}\) with \(\partial D\) piecewise \(C^1\), we will show that \(M_S^d(\gamma \cap \overline{D})=\mu(D)\).

For any \(k, D \cap Z_k\) is also a domain with a piecewise \(C^1\) boundary, so we may hence forth assume \(\overline{D} \subset \mathbb{H}\) by Lemma 3.6 (4) \((t=0)\) and Proposition 2.14 (4). If \(D\) is a finite union of elements of \(\mathcal{I}^+\), then the conclusion is obvious by Theorem 3.5. For general \(D\), we can choose a sequence \(\{D_m\}\) increasing to \(D\) s.t. each \(D_m\) is a finite union of elements of \(\mathcal{I}^+\) and \(D \setminus D_m \subset \partial_m D\). Because \(\lim_{m \to \infty} M_S^d(\gamma \cap \partial_m D) = 0\), the proof is complete by using Proposition 2.14 (4). \[\square\]
**Theorem 3.9.** With probability 1, \( \forall 0 \leq s < t \leq \infty, \forall \text{ domain } W \subset \mathbb{H} \) with \( \partial W \) piecewise \( C^1 \), \( \mu(W \cap \gamma[s,t]) = \mu(W \cap \gamma[0,t]) - \mu(W \cap \gamma[0,s]) = M_S^d(W \cap \gamma[s,t]) = M_S^d(W \cap (\gamma[0,t] \setminus \gamma[0,s])) \).

**Proof.** Lemma 3.6 (4) imply that \( \overline{M_S^d}(\gamma[s,\infty] \cap \partial H_s) = 0 \). Since \( \gamma[s,t] = (\gamma[0,t] \setminus \gamma[0,s]) \cup (\gamma[s,t] \cap \partial H_s) \), the last equality follows immediately.

Next, we prove \( \mu(W \cap (\gamma[0,t] \setminus \gamma[0,s])) \) \( \mu(W \cap \gamma[s,t]) \). By Lemma 3.6 (2) and Proposition 2.14 (4), we can assume \( t < \infty \). Let \( m \) be a positive integer which we will determine later and depends only on \( K_t \). For any positive integer \( n \) large enough, \( V_n(s,t) := \{ \Gamma \in \mathcal{I}_n : \Gamma \subset H_s \setminus \partial H_s \text{ and } \gamma[s,t] \cap \Gamma \neq \emptyset \} \) and \( O_n(s,t) := \gamma[0,\infty] \cap (\bigcup_{\Gamma \in V_n(s,t)} \Gamma) \). Note that \( V_n(s,t) \) is finite and \( \rho \) and \( \text{dist} \) are equivalent on a neighborhood of \( K_t \). Then we can choose \( m \) large enough to satisfy

\[
\gamma[0,t] \setminus \partial_{n-m} H_s \subset O_n(s,t) \subset (\gamma[0,t] \setminus \gamma[0,s]) \cup (O_n(s,t) \setminus \gamma[0,t]),
\]

for all large \( n \). Because \( \forall z \in \mathbb{H}, 1\{O_n(s,t) \setminus \gamma[0,s]\}(z) \to 0 \) as \( n \to \infty \) and \( \gamma[0,t] \setminus \partial_{n-m} H_s \uparrow \gamma[0,t] \setminus \gamma[0,s] \) as \( n \to \infty \), \( \mu(W \cap O_n(s,t)) \to \mu(W \cap (\gamma[0,t] \setminus \gamma[0,s])) \). On the other hand, we can choose \( m \) large enough such that

\[
O_n(s,t) \cap \gamma[s,t] \subset \gamma[t,\infty] \cap \partial_{n-m} H_s, \quad \gamma[s,t] \setminus O_n(s,t) \subset \gamma[s,\infty] \cap \partial_{n-m} H_s,
\]

for all large \( n \). So \( \overline{M_S^d}(W \cap (\gamma[0,t] \setminus \gamma[0,s])) \to 0 \) as \( n \to \infty \) by Lemma 3.6 (3). By Proposition 3.8, \( M_S^d(W \cap O_n(s,t)) \to \mu(W \cap (\bigcup_{\Gamma \in V_n(s,t)} \Gamma)) \to \mu(W \cap O_n(s,t)) \) \( \mu \) is supported on \( \gamma[0,\infty] \cap \mathbb{H} \). Then the conclusion follows by Proposition 2.14 (4).

To prove the first equality, it suffices to show \( \mu(\mathbb{H} \cap \gamma[s,\infty] \cap \partial H_s) = 0 \). It is equivalent to proving that \( \forall \epsilon > 0, \mu(\mathbb{H} \cap \gamma[s,\infty] \cap \partial H_s) < \epsilon \). By Lemma 3.6 (4), we can choose large \( N \) s.t. \( \overline{M_S^d}(\gamma[s,\infty] \cap \partial H_s) < \epsilon \). For such \( N \), we can find finite \( \Gamma_1, \Gamma_2, \ldots, \Gamma_m \in \mathcal{I} \) s.t.

\[
\partial H_s \subset \Gamma := \bigcup_{k=1}^m \Gamma_k \subset \partial N H_s.
\]

Then

\[
\mu(\mathbb{H} \cap \gamma[s,\infty] \cap \partial H_s) = \mu(\mathbb{H} \cap \gamma[0,\infty] \cap \partial H_s) - \mu(\mathbb{H} \cap \gamma[0,s] \cap \partial H_s) = \mu(\mathbb{H} \cap \gamma[0,\infty] \cap \partial H_s) - M_S^d(\gamma[0,s]) \leq \mu(\gamma[0,\infty] \cap \Gamma) - M_S^d(\gamma[0,s]) = M_S^d(\gamma[0,\infty] \cap \Gamma) - M_S^d(\gamma[0,s]) \leq M_S^d(\gamma[0,s]) + \overline{M_S^d}(\gamma[s,\infty] \cap \partial N H_s) - M_S^d(\gamma[0,s]) < \epsilon.
\]

To deal with other domains, we need the following lemma.
Lemma 3.10. Let $f : \mathbb{H} \rightarrow D$ be a conformal isomorphism, $\nu$ be a locally finite Borel measure on $\mathbb{H}$ and $E$ be a relatively closed subset of $\mathbb{H}$. If $\forall$ domain $V \subset \mathbb{H}$ with $\partial V$ piecewise $C^1$ and $\overline{V} \subset \mathbb{H}$, $\nu(V) = \mathcal{M}^d(E \cap V)$, then $\forall$ domain $W \subset D$ with $\partial W$ piecewise $C^1$ and $\overline{W} \subset D$, we have

$$\mathcal{M}^d(f(E) \cap W) = \int_{f^{-1}(W)} f^\#(z)^d d\nu(z).$$

Proof. With a 3-dimensional rotation whose spherical derivative is identically 1, we can without loss of generality assume $D \subset \mathbb{C}$. First, we state the following version of Koebe distortion theorem: If $g : r\mathbb{D} \rightarrow \mathbb{C}$ ($r \in (0, \infty)$) is univalent and $g(0) = 0$, then $\forall z \in r\mathbb{D}$, we have

$$\frac{g^\#(0)\tan \frac{\rho(z,0)}{2}}{(1 + \frac{1}{r}\tan \frac{\rho(z,0)}{2})^2} \leq \tan \frac{\rho(g(z),0)}{2} \leq \frac{g^\#(0)\tan \frac{\rho(z,0)}{2}}{(1 - \frac{1}{r}\tan \frac{\rho(z,0)}{2})^2}.$$  \hfill (3.2.1)

The proof is immediate by using Koebe distortion theorem and the definition of $\rho$ and $g^\#$. Let $R := \frac{\rho(E \cap f^{-1}(W), \partial \mathbb{H}) \wedge \rho(f(E) \cap W, \partial D)}{100}$. Fix $\epsilon, \lambda \in (0, 0.01)$ and choose a positive integer $N = N(\epsilon, \lambda)$ to be determined. Also let $\{ \Gamma \in \mathcal{I}_N : \Gamma \cap f^{-1}(W) \cap E \neq \emptyset \} = \{ \Gamma_1, \ldots, \Gamma_L \}$ ($L < \infty$). For $1 \leq j \leq L$, $K_j := \Gamma_j \cap f^{-1}(W) \cap E$, $a_j := \inf_{z \in \rho(K_j, 2^{-N})} f^\#(z)$, $b_j := \sup_{z \in \rho(K_j, 2^{-N})} f^\#(z)$.

Choose $\delta_1 \in (0, 1)$ s.t. $\forall x \in (0, \delta_1)$, it holds that $\tan x < (1 + \lambda)x$; choose $\delta_2 \in (0, R)$ s.t. $\forall z \in \bar{\rho}(f^{-1}(W), R)$, it holds that

$$\frac{0.5\epsilon}{1 + \nu(f^{-1}(W))} + \frac{\inf_{w \in \rho(z, \delta_2)} f^\#(w)^2}{\sup_{w \in \rho(z, \delta_2)} f^\#(w)^{2-d}} \geq f^\#(z)^d \geq \frac{\sup_{w \in \rho(z, \delta_2)} f^\#(w)^2}{\inf_{w \in \rho(z, \delta_2)} f^\#(w)^{2-d}} - \frac{0.5\epsilon}{1 + \nu(f^{-1}(W))}. \hfill (3.2.2)$$

Now we choose $N$ s.t. $2^{-N} < 0.01(\delta_1 \wedge \delta_2)$. Then we conclude that (note that $\nu$ is supported by $E$)

$$\epsilon + \int_{f^{-1}(W)} f^\#(z) d\nu(z) \geq \sum_{j=1}^L \nu(K_j) \frac{a_j^2}{b_j^2} \geq \int_{f^{-1}(W)} f^\#(z) d\nu(z) - \epsilon. \hfill (3.2.2)$$

On the other hand, $\forall r \in (0, \inf_{z \in \rho(f^{-1}(W), R)} f^\#(z) \cdot 2^{-N-10}$, we use (3.2.1) for $g = f^{-1}$. Then:

(1) $\forall z \in E \cap f^{-1}(W), w \in \mathbb{H}$, s.t. $\rho(z, w) \leq 2^{-N}$, if $\rho(f(z), f(w)) \leq r$, then

$$\rho(z, w) \leq \tan \frac{\rho(z, w)}{2} < \frac{f^\#(z)^{-1}\tan \frac{\rho(f(z), f(w))}{2}}{1 - \frac{1}{r}\tan \frac{\rho(f(z), f(w))}{2}} \leq \frac{f^\#(z)^{-1}(1 + \lambda)}{1 - (1 + \lambda)^r}.$$

That is,

$$\bar{\rho}(f(z), r) \subset \left[ \rho \left( z, \frac{r f^\#(z)^{-1}(1 + \lambda)}{1 - (1 + \lambda)^r} \right) \right].$$
So $\forall 1 \leq j \leq L$, 
\[
\bar{\rho}(f(K_j), r) \subset f\left[\bar{\rho}\left(K_j, \frac{ra_j^{-1}(1 + \lambda)}{[1 - (1+\lambda)r^2/2R]^2}\right)\right].
\] 
(3.2.3)

(2) $\forall z \in E \cap f^{-1}(W)$, $w \in \mathbb{H}$, s.t. $\rho(z, w) \leq 2^{-N}$, if $\rho(f(z), f(w)) > r$, then 
\[
\frac{f^\#(z)^{-1}r}{(1 + r^2R)^2} < \frac{f^\#(z)^{-1}\tan\frac{\rho(f(z), f(w))}{2}}{1 + \frac{1}{R}\tan\frac{\rho(f(z), f(w))}{2}} < \frac{\rho(z, w)}{2} < \frac{1 + \lambda}{2}\rho(z, w).
\]
That is, 
\[
f\left[\bar{\rho}\left(z, \frac{rf^\#(z)^{-1}}{(1 + \lambda)(1 + r^2R)^2}\right)\right] \subset \bar{\rho}(f(z), r).
\]
So $\forall 1 \leq j \leq L$, 
\[
f\left[\bar{\rho}\left(K_j, \frac{rb_j^{-1}}{(1 + \lambda)(1 + r^2R)^2}\right)\right] \subset \bar{\rho}(f(K_j), r).
\] 
(3.2.4)

By (3.2.3), 
\[
S\left[\bar{\rho}(f(K), r)\right] \leq \sum_{j=1}^{L} b_j^2 S\left[\bar{\rho}\left(K_j, \frac{(1 + \lambda)ra_j^{-1}}{[1 - (1+\lambda)r^2/2R]^2}\right)\right].
\]
Combining this (letting $r \to 0$) with the first inequality in (3.2.2), we get 
\[
\overline{M}_d^d(f(K)) \leq \sum_{j=1}^{L} \frac{b_j^2(1 + \lambda)^{2-d}}{a_j^{2-d}} M_d^d(K_j) \leq (1 + \lambda)^{2-d} \int_{f^{-1}(W)} f^\#(z)^d d\nu(z) + \epsilon. 
\] 
(3.2.5)

By (3.2.4), 
\[
S\left[\bar{\rho}(f(K), r)\right] \geq \sum_{j=1}^{L} a_j^2 S\left[\bar{\rho}\left(K_j, \frac{rb_j^{-1}}{(1 + \lambda)(1 + r^2R)^2}\right)\right] - 2 \sum_{j \neq k} S\left[\bar{\rho}(\Gamma_j \cap \Gamma_k, r)\right].
\]
Combining this (letting $r \to 0$ and noting $M_d^d(\Gamma_j \cap \Gamma_k) = 0$) with the second inequality in (3.2.2), we get 
\[
\underline{M}_d^d(f(K)) \geq \sum_{j=1}^{L} \frac{a_j^2}{b_j^{2-d}(1 + \lambda)^{2-d}} M_d^d(K_j) \geq (1 + \lambda)^{d-2} \int_{f^{-1}(W)} f^\#(z)^d d\nu(z) - \epsilon. 
\] 
(3.2.6)

Finally, we complete the proof by letting $\epsilon, \lambda \to 0$ both in (3.2.5) and (3.2.6).
With the above lemma, for a conformal isomorphism $f: \mathbb{H} \to D$, we can prove all claims about the general domain $D$ (no need for the analyticity of $\partial D$) in Theorem 1.1 by defining the spherical natural occupation measure of $f(\gamma)$ to be $(f^\# \circ f^{-1})^d \cdot (\mu \circ f^{-1})$.

Finally, we deal with the property of $\Theta_t := M_2^d(\gamma[0,t])$ seemed as an increasing function on $[0, \infty)$.

**Proposition 3.11.** Let $\alpha \in (0, 1)$ be the constant in lemma 3.6. Then with probability 1, $\{\Theta_t, t \in [0, \infty)\}$ is strictly increasing and Hölder continuous of order $\alpha$ on every bounded interval.

**Proof.** The Hölder continuity is immediate from lemma 3.6 (1). For the strict monotonicity, we will use the following version of domain Markovian property of SLE: For $t \in (0, \infty)$, $\{g_t(\gamma(t + s)) - g_t(\gamma(t))\}_{s \in (0, \infty)}$ is independent of $\mathcal{F}_t$. From this Markovian property, lemma 3.10, theorem 3.9 and lemma 3.6 (4), we deduce $\forall 0 < t_1 < t_2 < \cdots < t_n < \infty$, one has that $\{\Theta_{t_1} = 0\}$, $\{\Theta_{t_2} - \Theta_{t_1} = 0\}$, $\cdots$, $\{\Theta_{t_n} - \Theta_{t_{n-1}} = 0\}$ are mutually independent. Also, by the conformal invariance, lemma 3.10, theorem 3.9 and lemma 3.6 (4), $\forall 0 < s < t < \infty$, one has $P\{\Theta_t - \Theta_s = 0\} = P\{\Theta_{t-s} = 0\}$. And the scaling property, lemma 3.10 and lemma 3.6 (4) imply $\forall t \in (0, \infty)$, one has $P\{\Theta_t = 0\} = P\{\Theta_1 = 0\}$.

From the above claims, we get

$$P\{\Theta_\infty = 0\} = \lim_{n \to \infty} P\{\Theta_1 = 0, \Theta_2 - \Theta_1 = 0, \ldots, \Theta_n - \Theta_{n-1} = 0\} = \lim_{n \to \infty} P\{\Theta_1 = 0\}^n$$

So $P\{\Theta_\infty = 0\} \in \{0, 1\}$. On the other hand, $E(\Theta_\infty) = G(\mathbb{H}) > 0$. Hence, $P\{\Theta_\infty = 0\} = 0$. And

$$0 = P\{\Theta_\infty = 0\} = \lim_{n \to \infty} P\{\Theta_n = 0\} = \lim_{n \to \infty} P\{\Theta_1 = 0\} = P\{\Theta_1 = 0\}$$

Then $\forall 0 < s < t < \infty$, one has $P\{\Theta_t - \Theta_s = 0\} = P\{\Theta_{t-s} = 0\} = P\{\Theta_1 = 0\} = 0$. Therefore, w.p.1., $\forall 0 < s < t < \infty$ and $s, t \in \mathbb{Q}$, one has $\Theta_t > \Theta_s$. Then we finish the proof by using the fact that $\mathbb{Q}^+$ is dense in $[0, \infty)$. \hfill \qed

We have completed the proof of Theorem 1.1 for $D = \mathbb{H}$. To extend it to all domains with analytic boundaries, we can just use the following lemma.

**Lemma 3.12** ([9] Proposition 3.1). Let $f: \mathbb{D} \to D$ be a conformal isomorphism and $\partial D$ be a Jordan curve. Then $\partial D$ is analytic if and only if $f$ is univalent in $r\mathbb{D}$ for some $r \in (1, \infty)$.

## 4 Proof of Theorem 3.1

This section basically follows the idea in [5], but we simplify some proofs.

**Lemma 4.1.** Suppose $E$ is a compact subset of $\mathbb{H}$, $s \in (0,1)$ and $u \in (0, s)$. Then $\exists \delta = \delta(E, s, u) \in (0,1)$, $c = c(E) \in (0, \infty)$ s.t. $\forall z, w \in E, \epsilon \in (0, \delta)$, if $\rho(z, w) \geq \epsilon u$, then

$$P\{\tau_{\epsilon u}(w) < \tau_{\epsilon u}(z) < \tau_{\epsilon u}(w) < \infty\} \leq c\epsilon^{2(2-\delta)+\frac{1}{2}(s-u)(4a-1)}.$$
Proof. \( \tau := \tau_\epsilon(z) \). Note that \( \rho \) and \( \text{dist} \) are equivalent on \( \{ \zeta : \rho(\zeta, E) \leq \frac{1}{2}\rho(E, \partial \mathbb{H}) \} \). Choose \( \delta \) so small that \( 100\delta < 10\delta < \delta u < \rho(E, \partial H) \leq \text{dist}(E, \partial \mathbb{H}) \). Then for \( \epsilon \in (0, \delta) \),

\[
\mathbb{P}\{ \tau_{\epsilon}(w) < \tau_\epsilon(z) < \tau_\epsilon(w) < \infty \} = \mathbb{E}\left[ 1\{ \tau_{\epsilon}(w) < \tau < \tau_\epsilon(w) \} \mathbb{P}\{ \tau_\epsilon(w) < \infty | \mathcal{F}_\tau \} \right]
\]

\[
\leq \sum_{n=1}^{[(s-1)\log\epsilon] + 1} \mathbb{E}\left[ 1\{ \tau_{10^n\epsilon}(w) \leq \tau < \tau_{10^{n-1}\epsilon}(w) \} \mathbb{P}\{ \tau_\epsilon(w) < \infty | \mathcal{F}_\tau \} \right]. \tag{4.0.1}
\]

By Theorem 2.11 for \( w \) and \( z \),

\[
\mathbb{P}\left\{ \tau_{10^n\epsilon}(w) \leq \tau < \tau_{10^{n-1}\epsilon}(w) \right\} \leq \mathbb{P}\left\{ \tau_{10^n\epsilon}(w) < \infty, \ \tau_\epsilon(z) < \infty \right\} \leq c10^{(2-d)n}e^{2(2-d)}. \tag{4.0.2}
\]

On \( \{ \tau_{10^n\epsilon}(w) \leq \tau < \tau_{10^{n-1}\epsilon}(w) \} \), define \( \iota := \inf\{ t \geq 0 : \rho(w, \gamma(t)) = \rho(w, \gamma[0, \tau]) \} \) and \( r := \rho(w, \gamma[0, \tau]) \). Then \( \tau_{10^n\epsilon} \leq \iota < \tau \) and \( 10^{n-1}\epsilon \leq r \leq 10^n\epsilon \). Also let \( A \) denote the closed arc of \( \delta H_\tau \) with boundary points \( \gamma(\tau) \) and \( \infty \) which does not intersect \( \bar{\rho}(w, r) \). (See Figure 1.)

(1) \( n = 1 \). By considering \( \bar{\rho}(w, r) \) and \( \bar{\rho}[w, \rho(\gamma(\tau), w)] \), we get \( \mathcal{E}_{H_\tau}(A, \bar{\rho}(w, r)) \geq (2\pi)^{-1}\ln\left( \frac{e^{u-1}}{c} \right) \).

Let \( g : H_\tau \to \mathbb{H} \) be a conformal isomorphism with \( g(\gamma(\tau)) = 0 \), \( g(\infty) = \infty \) and \( g(\gamma(t)) > 0 \). (Here \( g(\gamma(i)) \) means the limit of \( g(\xi) \) as \( \xi \) moves from \( w \) to \( \gamma(i) \) along the geodesic.) Applying Lemma 2.3 to \( B := g(\bar{\rho}(w, r)) \) and \( [-2\text{dist}(0, B), 0] \), we get \( \frac{\text{diam}(B)}{\text{dist}((-\infty, 0], B)} \leq \frac{\epsilon}{c} \).
\[ cc^{1/2}(1-u). \] Then we use conformal Markovian property at \( \tau \) and Theorem 2.11 for \( g(\gamma(t)) \) to get
\[
\mathbb{P}\{\tau_{c}(w) < \infty | \mathcal{F}_\tau \} \leq cc^{1/2}(1-u)(4a-1). \tag{4.0.3}
\]

(2) Suppose \( n > 1 \). By applying Proposition 2.4 to \( \bar{\rho}(w, r) \) and \( \bar{\rho}([w, \rho(\gamma(\tau), w]) \), we get \( \text{hm}_{H_\tau}(w, A) \leq c10^n e^{1/2}(1-u) \), and hence \( S_{H_\tau}(w; \gamma(\tau), \infty) \leq c10^n e^{1/2}(1-u) \) by (2.3.3). Then by Theorem 2.10 for \( w, \)
\[
\mathbb{P}\{\tau_\epsilon(w) < \infty | \mathcal{F}_\tau \} \leq c10^{-(2-d)n} + c^{1/2}(4a-1)n . \tag{4.0.4}
\]
Combining (4.0.1), (4.0.2), (4.0.3) and (4.0.4),
\[
\mathbb{P}\{\tau_{e^\epsilon}(w) < \tau_{e}(z) < \tau_{e}(w) < \infty \} \leq cc^{2(2-d)+1/2(1-u)(4a-1)}
\]
\[
+ c(2-d) + c^{1/2}(1-u)(4a-1) \cdot \sum_{n=2}^{\lceil (s-1)c \epsilon \rceil + 1} 10^{1/2}(4a-1)n \leq cc^{2(2-d)+1/2(1-u)(4a-1)}
\]
\[
+ c(2-d) + c^{1/2}(1-u)(4a-1) \cdot \epsilon^{1/2}(s-1)(4a-1) \leq cc^{2(2-d)+1/2(s-u)(4a-1)}.
\]

Lemma 4.2. There exist \( s \in (0, 1), u \in (0, s) \) and \( \theta \in (0, \infty) \) such that \( \forall \) compact set \( E \subset \mathbb{H} \), \( \exists c = c(E) \in (0, \infty) \) and \( \delta = \delta(E) \in (0, 1) \) s.t. if \( \epsilon \in (0, \delta), \lambda \in (0, 1) \) and \( z, w \in E \) with \( \rho(z, w) \geq \epsilon^u \), then
\[
\left| \mathbb{E}[Q_{c, \lambda}(z) Q_{c, \lambda}(w); \tau_{c}(z) < \tau_{e^\epsilon}(w) < \tau_{c}(w) < \infty] \right| \leq cc^\theta.
\]

Proof. Let \( s \in (0, 3/4) \) be a small positive number which we will choose later and \( u := s/10 \). Note that \( \rho \) and \( \text{dist} \) are equivalent on \( \{ \zeta : \rho(\zeta, E) \leq \frac{1}{2}\rho(E, \partial \mathbb{H}) \} \). Choose \( \delta \) so small that \( 1000\delta < 100\delta^{3/4} < 10\delta^s < \delta^u < \frac{\rho(E, \partial \mathbb{H}) \land \text{dist}(E, \mathbb{R})}{10} \). Define \( \tau := \tau_{c}(z), \rho_1 := \rho(z, \epsilon) \) and \( \rho_{3/4} := \rho(z, \epsilon^{3/4}) \). Let \( V \) denote the component of \( \rho_{3/4} \cap H_\tau \) containing \( z \). \( \partial V \cap H_\tau \) is a disjoint union of open arcs in \( \partial \rho_{3/4} \) and there is a unique arc, which we denote by \( l \), such that \( z \) and \( \infty \) are in different components of \( H_\tau \backslash l \). Define \( \sigma := \inf \{ t \geq \tau : \gamma(t) \in \mathbb{I} \} \). (See Figure 2.) Let \( J \) be the indicator function of the event that \( \tau < \tau_{e^\epsilon}(w) \) and \( w, \infty \) are in different components of \( H_\tau \backslash l \) and \( K \) be the indicator function of the event that \( \tau < \tau_{e^\epsilon}(w) \) and \( w, \infty \) are in the same component of \( H_\tau \backslash l \).

First, suppose \( J = 1 \). Since \( w \notin \bar{\rho}(z, \epsilon^{1/2}) \), there is a unique subarc \( l' \) of \( \partial V \cap H_\tau \) s.t. \( z, w \) are in different components of \( H_{l'} \). Because \( w, \infty \) are in different components of \( H_\tau \backslash l \), \( l \neq l' \). By Proposition 2.4, the probability that a Brownian motion starting from \( w \) reaches \( l' \) before leaving \( H_\tau \) is smaller than \( cc^{0.5(3/4-u)} \) (considering \( \rho_{3/4} \) and \( \bar{\rho}(z, \rho(w, z)) \)), and hence \( S_{H_\tau}(w; \gamma(\tau), \infty) \leq cc^{0.5(3/4-u)} \) by (2.3.3). Therefore, by Theorem 2.10,
\[
\mathbb{P}\{\tau_{e}(w) < \infty | \mathcal{F}_\tau \} \leq cc^{(1-u)(2-d)+0.5(3/4-u)(4a-1)}.
\]

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On the other hand, $\mathbb{E}[J] \leq \mathbb{P}\{\tau < \infty\} \leq c\epsilon^{2-d}$. Hence,

$$
\mathbb{E}\left[|Q_{\epsilon,\lambda}(z)Q_{\epsilon,\lambda}(w)| \cdot J; \tau_\epsilon(w) < \infty\right] \leq c\epsilon^{2(d-2)}\mathbb{E}[J; \tau_\epsilon(w) < \infty] =
$$

$$
c\epsilon^{2(d-2)}\mathbb{E}\left[J; \mathbb{P}\{\tau_\epsilon(w) < \infty | F_{\tau}\} \right] \leq c\epsilon^{2(d-2)} \cdot \epsilon^{2-d} \cdot \epsilon^{(1-s)(2-d)+0.5(3/4-u)(4a-1)}
$$

$$
= c\epsilon^{0.5(3/4-u)(4a-1)-s(2-d)}.
$$

So we choose $s$ to satisfy $s < \frac{3(4a-1)}{8[0.1(4a-1) + (2-d)]}$. (u = 0.1s)

Second, suppose $K = 1$. Define $\overline{Q} := \epsilon^{d-2}1\{\tau_\epsilon(z) < \sigma\} - (\lambda\epsilon)^{d-2}1\{\tau_{\lambda\epsilon}(z) < \sigma\}$ and $Q := Q_{\epsilon,\lambda}(z)$. It suffices to show that

$$
\left|\mathbb{E}[\overline{Q} \cdot Q_{\epsilon,\lambda}(w)K]\right| \leq c\epsilon^3 \quad (4.0.5)
$$
and
\[ \mathbb{E}[|\tilde{Q} - Q| | Q_{c,\lambda}(w) | \mathcal{K}] \leq ce^q \]  
(4.0.6)

To establish (4.0.5), we first use Theorem 2.10 to get
\[ \left| e^{-2} \mathbb{P}\{ \tau(w) < \infty | \mathcal{F}_\sigma \} - \tilde{G}_{H_\sigma}(w; \gamma(\sigma), \infty) \right| \leq cG_{H_\sigma}(w; \gamma(\sigma), \infty)(\epsilon/\rho(w, \partial H_\sigma))^q \]
and
\[ \left| (\lambda e)^{-2} \mathbb{P}\{ \tau_{\lambda}(w) < \infty | \mathcal{F}_\sigma \} - \tilde{G}_{H_\sigma}(w; \gamma(\sigma), \infty) \right| \leq cG_{H_\sigma}(w; \gamma(\sigma), \infty)(\epsilon/\rho(w, \partial H_\sigma))^q. \]

Since \( w \) and \( \infty \) are in the same component of \( H_\tau \setminus l, \rho(w, \gamma[0, \sigma]) \geq \rho(w, \gamma[0, \tau]) \) \( \wedge (\rho(z, w) - \epsilon^{3/4}) \geq \epsilon^s \wedge (\epsilon^{s} - \epsilon^{3/4}) = \epsilon^s \). So
\[ \mathbb{E}(Q_{c,\lambda}(w) | \mathcal{F}_\sigma) \leq cG_{H_\sigma}(w; \gamma(\sigma), \infty)(\epsilon/\rho(w, \partial H_\sigma))^q \leq ce^{s(d-2)+q(1-s)}. \]

Hence
\[ \left| \mathbb{E}[\tilde{Q} \cdot Q_{c,\lambda}(w) | \mathcal{K}] \right| = \left| \mathbb{E}[\tilde{Q} \cdot \mathcal{K} \cdot \mathbb{E}(Q_{c,\lambda}(w) | \mathcal{F}_\sigma)] \right| \leq \mathbb{E}\left| \tilde{Q} \cdot \mathcal{K} \cdot \mathbb{E}(Q_{c,\lambda}(w) | \mathcal{F}_\sigma) \right| \]
\[ \leq ce^{-2}e^{(d-2)s+q(1-s)}\mathbb{E}[\mathcal{K}] \leq ce^{(s+1)(d-2)+q(1-s)}\mathbb{P}\{ \tau < \infty \} \leq ce^{(d-2)+q(1-s)}. \]

So we need to let \( s < \frac{q}{q + 2 - d} \). As for (4.0.6), note that \( \{ \tilde{Q} \neq Q \} \subset \{ \sigma < \tau_{\lambda}(z) < \infty \} \). Then
\[ \mathbb{E}[|\tilde{Q} - Q| | Q_{c,\lambda}(w) | \mathcal{K}; \sigma < \tau_{\lambda}(z) < \infty] \leq ce^{-2}\mathbb{E}[|Q_{c,\lambda}(w) | \mathcal{K}; \sigma < \tau_{\lambda}(z) < \infty] \]
\[ = ce^{-2}\mathbb{E}\left[ \mathcal{K} \cdot 1\{ \sigma < \tau_{\lambda}(z) \} \mathbb{E}(Q_{c,\lambda}(w) | \mathcal{F}_\sigma) \right] \]
\[ \leq ce^{2(d-2)}\mathbb{E}\left[ \mathcal{K} \cdot 1\{ \sigma < \tau_{\lambda}(z) \} \mathbb{P}\{ \tau_{\lambda}(w) < \infty, \tau_{\lambda}(z) < \infty | \mathcal{F}_\sigma \} \right] \]
(4.0.7)

Define \( \iota := \inf\{ t \geq \tau : \rho(z, \gamma(t)) = \rho(z, \gamma[\tau, \sigma]) \} \) and \( r := \rho(z, \gamma[\tau, \sigma]) \). Also let \( A \) denote the closed arc of \( \delta H_\sigma \) with boundary points \( \gamma(\sigma) \) and \( \infty \) which does not intersect \( \partial(z, r) \). Let \( g : H_\sigma \rightarrow \mathbb{H} \) be a conformal isomorphism s.t. \( g(\gamma(\sigma)) = 0, g(\infty) = \infty \) and \( g(\gamma(\iota)) < 0 \). (Here \( g(\gamma(\iota)) \) means the limit of \( g(\xi) \) as \( \xi \) moves from \( z \) to \( \gamma(\iota) \) along the geodesic.) By considering \( \overline{\rho} \) and \( \overline{\rho}_{3/4} \), we get \( E_{H_\sigma}(A, \rho(z, r)) \geq (2\pi)^{-1}\ln(\epsilon^{1/4}) \). Applying Lemma 2.3 to \( B := g(\rho(z, r)) \) and \( [0, 2\text{dist}(0, B)] \), we get
\[ \frac{\text{diam}(B)}{\text{dist}(\{0, \infty\}, B)} \leq ce^{1/8}. \]
(4.0.8)

Using Koebe 1/4-theorem and Koebe distortion theorem on \( B(w, \epsilon^s) \) for \( g \), we get:
\[ \frac{1}{4}\epsilon^s|g'(w)| \leq \text{dist}\left[ g(w), \partial g(B(w, \epsilon^s)) \right] \leq \min\{|g(w)|, |g(w) - g(z)|\}. \]
\[ |g(\zeta) - g(w)| \leq \frac{|g'(w)||\zeta - w|}{\left(1 - \left|\frac{\zeta - w}{\epsilon s}\right|\right)^2}, \quad \forall \zeta \in \overline{B}(w, \epsilon). \]

Hence
\[ \frac{\max\left\{|g(w) - g(\zeta) : \zeta \in \overline{B}(w, \epsilon)\}\right\}}{\min\left\{|g(w)|, |g(z) - g(w)|\right\}} \leq \epsilon^{1-s}. \quad (4.0.9) \]

By Theorem 2.11 (letting \( z_1 := g(\gamma(\nu)), z_2 := g(w) \)),
\[ \mathbb{P}\{\tau_\epsilon(w) < \infty, \tau_{\lambda\epsilon}(z) < \infty | \mathcal{F}_\sigma \} \leq c(\frac{\text{diam}(B)}{\text{dist}([0, \infty), B)})^{4a-1} \left(\frac{\max\{ |g(w) - g(\zeta) : \zeta \in \overline{B}(w, \epsilon)\}}{\min\{ |g(w)|, |g(z) - g(w)|\}}\right)^{2-d} \]

Combining (4.0.7), (4.0.8) and (4.0.9),
\[ \mathbb{E}\left[|Q - Q| | Q_{\epsilon, \lambda}(w) | \kappa ; \sigma < \tau_{\lambda\epsilon}(z) < \infty \right] \leq ce^{2(d-2)} \epsilon^{(4a-1)/8 + (1-s)(2-d)} \mathbb{E}\left[\kappa \cdot 1 \{ \sigma < \tau_{\lambda\epsilon}(z) \}\right] \]
\[ \leq ce^{(4a-1) - (s+1)(2-d)} \mathbb{E}[\tau < \infty] \leq ce^{(4a-1)/8 - (s+1)(2-d)} \epsilon^{2-d} = ce^{(4a-1)/8 - s(2-d)}. \]

So we have to choose \( s \) so small that \( s < \frac{4a - 1}{8(2 - d)}. \)

**Proof of Theorem 3.1.** Let \( s, u \) and \( \theta \) be real numbers in Lemma 4.2. By Theorem 2.11, if \( \rho(z, w) \leq \epsilon^u \), then
\[ \mathbb{E}\left[Q_{\epsilon, \lambda}(z)Q_{\epsilon, \lambda}(w) ; \tau_\epsilon(z) < \tau_\epsilon(w) < \infty \right] \leq \mathbb{E}\left[|Q_{\epsilon, \lambda}(z)Q_{\epsilon, \lambda}(w)|\right] \]
\[ \leq ce^{2(d-2)} \mathbb{P}\{\tau_\epsilon(z) < \infty, \tau_\epsilon(w) < \infty \} \leq c\rho(z, w)^{d-2} \leq c\rho(z, w)^{(d-1)-2} \epsilon^u. \]

On the other hand, if \( \rho(z, w) > \epsilon^u \), then by Lemma 4.1
\[ \mathbb{E}\left[Q_{\epsilon, \lambda}(z)Q_{\epsilon, \lambda}(w) ; \tau_\epsilon^e(w) < \tau_\epsilon(z) < \tau_\epsilon(w) < \infty \right] \leq \mathbb{E}\left[|Q_{\epsilon, \lambda}(z)Q_{\epsilon, \lambda}(w)|\right] \]
\[ \leq ce^{2(d-2)} \mathbb{P}\{\tau_\epsilon(z) < \infty, \tau_\epsilon(w) < \tau_\epsilon(z) < \tau_\epsilon(w) < \infty \} \]
\[ \leq ce^{\frac{1}{2}(s-u)(4a-1)} \leq ce^{s-u(4a-1)} \rho(z, w)^{(d-1)-2}. \]

The last inequality is deduced by noting that \( \rho(z, w) \leq 2\pi \). Similarly, by Lemma 4.2,
\[ \mathbb{E}\left[Q_{\epsilon, \lambda}(z)Q_{\epsilon, \lambda}(w) ; \tau_\epsilon(z) < \tau_\epsilon(w) < \tau_\epsilon(w) < \infty \right] \leq ce^{\theta} \leq c\rho(z, w)^{(d-1)-2}. \]

Finally, if we let \( \beta := \min\{u, 0.5(s - u)(4a - 1), \theta\} \) and \( \beta' := d - 1 \), then
\[ \mathbb{E}\left[Q_{\epsilon, \lambda}(z)Q_{\epsilon, \lambda}(w) ; \tau_\epsilon(z) < \tau_\epsilon(w) < \infty \right] \leq ce^{\beta} \rho(z, w)^{(d-1)-2}. \]

Now Theorem 3.1 follows by symmetry. \( \square \)
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