A Two-Stage Approach to Multivariate Linear Regression with Sparsely Mismatched Data

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Abstract

A tacit assumption in linear regression is that (response, predictor)-pairs correspond to identical observational units. A series of recent works have studied scenarios in which this assumption is violated under terms such as “Unlabeled Sensing” and “Regression with Unknown Permutation”. In this paper, we study the setup of multiple response variables and a notion of mismatches that generalizes permutations in order to allow for missing matches as well as for one-to-many matches. A two-stage method is proposed under the assumption that most pairs are correctly matched. In the first stage, the regression parameter is estimated by handling mismatches as contamination, and subsequently the generalized permutation is estimated by a basic variant of matching. The approach is both computationally convenient and equipped with favorable statistical guarantees. Specifically, it is shown that the conditions for permutation recovery become considerably less stringent as the number of responses $m$ per observation increase. Particularly, for $m = \Omega(\log n)$, the required signal-to-noise ratio does no longer depend on the sample size $n$. Numerical results on synthetic and real data are presented to support the main findings of our analysis.

1 Introduction

Linear regression and its numerous extensions is an object of timeless interest in statistics and related disciplines. Continuous research efforts are being made to increase the range of situations in which it can be applied with success. A specific challenge that has attracted considerable interest recently is regression in the absence of correspondence between predictors and responses, i.e., both are given as separate samples $\mathcal{X} = \{x_i\}_{i=1}^n$ and $\mathcal{Y} = \{y_i\}_{i=1}^n$, but it is not (fully) known a priori which elements from $\mathcal{X}$ and $\mathcal{Y}$ are matching pairs in the sense of belonging to the same observational unit. Motivated by a number of applications in engineering, regression in this setting has been discussed in a series of recent papers [1, 13, 16, 21, 23, 33, 34, 42, 47, 48, 51, 55]. On the other hand, the above setup has a long history in statistics under the term “Broken Sample Problem” dating back to the early 1970s [3, 8, 10, 11, 12, 18, 57] and a related line of research involving record linkage and statistical analysis based on merged data files, e.g., [19, 25, 31, 39, 40] partially motivated by government agencies like the U.S. Census Bureau that routinely combines data from multiple surveys and/or external data to address questions of interest. In this context, the primary interest is in the estimation of parameters (e.g., covariance matrix, regression coefficients, ...) rather than restoration of the correspondence between elements of $\mathcal{X}$ and $\mathcal{Y}$. Instead, the focus is on the adjustment of subsequent analyses for potential mis-matches resulting from errors or ambiguities in record linkage based on quasi-identifiers. In fact, unique identifiers such as the social security number often need to be removed because of privacy concerns. Accordingly, in an alternative perspective on the broken sample problem,
identification of matching pairs in $\mathcal{X}$ and $\mathcal{Y}$ is undesired because $\mathcal{Y}$ contains sensitive data, but an adversary makes the attempt to use external data along with identifying information stored in $\mathcal{X}$ to retrieve matching pieces in $\mathcal{Y}$. Well-known instances of such “linkage attacks” are the identification of the medical history of the former governor of Massachusetts [45] and the partial de-anonymization of Netflix movie rankings with the help of publicly available data in the Internet Movie Database (IMDb) [30]. Broken sample problems thus bear a relationship to data confidentiality; we refer to [14] for a detailed discussion.

**Related Work.** A starting point of recent research on the subject is the work by [51] which studies linear regression in the absence of noise with a scalar response that is observed up to an unknown permutation of the entries, i.e., $y_i = x_{i\pi^*}^T \beta^*$, $i = 1, \ldots, n$, for a permutation $\pi^*$ on $\{1, \ldots, n\}$. The authors of [51] show that $\beta^* \in \mathbb{R}^d$ can be recovered with probability one by exhaustive enumeration over all permutations if $n \geq 2d$ and the entries of $X$ are drawn i.i.d. from a distribution absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}$. Alternative proofs of this result have been obtained in [13, 46]. The paper [34] studies computational and statistical limits of recovering $\pi^*$ for Gaussian $\{x_i\}_{i=1}^n$ and Gaussian additive noise with variance $\sigma^2$. The authors show that least squares estimation recovers $\pi^*$ exactly if the signal-to-noise ratio $\text{SNR} = \|\beta^*\|^2_2/\sigma^2 = n \Omega(1)$ which is also known to be sharp up to a constant factor in the exponent. At the same time, least squares estimation of $\pi^*$ is proved to be NP-hard. The papers [1, 23] shed light on the estimation of $\beta^*$ under similar setups as in [34]. Specifically, [23] establishes that the requirement $\text{SNR} = \Omega(d / \log \log n)$ is necessary to ensure low relative squared $\ell_2$-estimation error which is a dramatic gap compared to the requirement $\text{SNR} = \Omega(d / n)$ if $\pi^*$ is known. The paper [2] proposes Expectation-Maximization (EM) schemes to tackle the least squares problem for estimation of $\pi^*$. A clever initialization strategy for those schemes based on algebraic considerations is developed in [48]. The paper [43] assumes that $\pi^*$ is $k$-sparse, i.e., $\pi^*(i) = i$ except for $k \ll n$ indices, and analyzes a convex formulation for estimating $\beta^*$ in this setting. A similar sparsity assumption is employed in [42] for spherical regression. Order-constrained regression problems with unknown permutation are discussed in [7, 17, 38].

**Contributions.** While several papers have elucidated important aspects of linear regression with unknown permutation for a scalar response, only few papers [33, 44, 59] consider multivariate response, i.e., the $\{y_i\}_{i=1}^n$ are $m$-dimensional, $m > 1$. This case is of independent interest for at least two reasons. First, in the context of record linkage it is natural to assume that both data sets $\mathcal{X}$ and $\mathcal{Y}$ to be merged are multi-dimensional. Second, the availability of multiple responses affected by the same permutation is expected to facilitate estimation as is confirmed by the results herein. Indeed, the requirements on the SNR to achieve permutation recovery can be considerably weaker, with potential drops from $\text{SNR} = n \Omega(1)$ for $m = O(1)$ to $\text{SNR} = \Omega(1)$ for $m = \Omega(\log n)$. Similar benefits are shown in [33, 44, 59]. The results in [33] concern the prediction or denoising error rather than estimation of $\pi^*$. The paper [59] provides information-theoretic lower bounds for permutation recovery; however, the computational scheme therein is only investigated empirically without solid theoretical support. The paper [44] presents a scheme that requires $m \gtrsim d$ to perform well; another downside of the approach is its cubic runtime in $n$. None of [33, 44, 59] contain rigorous results regarding the estimation of the regression parameter. Moreover, we consider a more general notion of faulty correspondence between $\mathcal{X}$ and $\mathcal{Y}$ which goes beyond permutations, specifically allowing for missing matches and one-to-many matches. As in [43], the fraction of mismatches is assumed to be small in order to enable computationally efficient estimation of the regression coefficients and restoration of the correct correspondence based on simple convex optimization problems. The effectiveness of the approach is demonstrated by experiments on synthetic and real data sets.
Outline. In §2, we state the problem and setting under consideration as well as the approach taken, illustrated by a case study. Our main theoretical results are presented in §3. Empirical corroboration based on synthetic and real data is provided in §4. We conclude with a summary and an overview on potential directions of future research in §5.

Notation. The symbol $I$ is used for the indicator function with value one if its argument is true and zero else. For a positive integer $\ell$, $I_\ell$ denotes the $\ell \times \ell$ identity matrix, and $S^{\ell-1}$ denotes the unit sphere in $\mathbb{R}^\ell$. We write $|S|$ for the cardinality of a set $S$. The complement of $S$ with respect to context-dependent base sets is denoted by $S^c$, and $\text{conv} S$ denotes the convex hull of $S$. For a matrix $A$, $\|A\|_2 = \sigma_{\max}(A)$ denotes its spectral norm respectively maximum singular value, $\|A\|_F$ denotes its Frobenius norm, and range$(A)$ denotes the column space of $A$. The $i$-th row of $A$ is denoted by $A_{i,:}$, and is treated as column vector. For an index set $I$ and a vector $v$ of real numbers, $v_I$ denotes the subvector corresponding to $I$. We write $a \lor b = \max\{a, b\}$ and $\land b = \min\{a, b\}$. Positive constants are denoted by $C, e, c_1$ etc. We make use of the usual $\mathcal{O}$ notation in terms of $O, o, \Omega$ and $\Theta$. We often use $a \lesssim b, b \gtrsim a$, and $a \asymp b$ as shortcuts for $a = O(b), b = \Omega(a)$ and $a = \Theta(b)$, respectively.

## 2 Problem statement and proposed approach

We start by fixing the setup under consideration herein before outlining our approach. We then provide a brief case study on a real data set in order to illustrate some of the main challenges and characteristics of the given problem and the proposed approach.

### 2.1 Setup

As stated in the introduction, we assume that we are given two samples $X = \{x_i\}_{i=1}^n$ and $Y = \{y_i\}_{i=1}^n$ taking values in $\mathbb{R}^d$ and $\mathbb{R}^m$, respectively, that are related by the model

$$s_i y_i = B^* x_{\theta^*(i)} + \sigma e_{\theta^*(i)}, \quad 1 \leq i \leq n, \quad \text{(1)}$$

where $\theta^* : \{1, \ldots, n\} \rightarrow \{0, 1, \ldots, n\}$ is a map representing the (unknown) underlying correspondence between observations in $X$ and $Y$, with the convention that $x_0 := 0, e_0 := 0$, and $s_i = I(\theta^*(i) \neq 0)$ indicates whether $y_i$ has a match among $X$, $1 \leq i \leq n$. If $\theta^*(i) = i$ for $1 \leq i \leq n$, the above model reduces to an ordinary multivariate regression model with $m$ responses and $d$ predictor variables, regression coefficients $B^* \in \mathbb{R}^{d \times m}$, and random error variables $\{e_i\}_{i=1}^n$. Model (1) can be expressed equivalently via

$$SY = \Theta^* (XB^* + \sigma E), \quad \text{(2)}$$

where $Y$ and $E$ are $n$-by-$m$ matrices whose rows are given by $\{y_i^\top\}$ and $\{e_i^\top\}$, respectively, $S = \text{diag}(s_1, \ldots, s_n), X$ is an $n$-by-$d$ matrix with rows $\{x_i^\top\}_{i=1}^n$, and $\Theta^* = (\Theta^*_{ij})_{1 \leq i, j \leq n}$ has entries $\Theta^*_{ij} = 1$ if $\theta^*(i) = j$ for $j \neq 0$, and zero otherwise. Observe that by construction, $\Theta^*$ is contained in the following set of matrices

$$\mathcal{M} = \left\{ \Theta \in \mathbb{R}^{n \times n} : \Theta_{ij} \in \{0, 1\}, 1 \leq i, j \leq n, \sum_j \Theta_{ij} \leq 1, 1 \leq i \leq n \right\} \quad \text{(3)}$$

$$\supset \mathcal{P} = \left\{ \Theta \in \mathbb{R}^{n \times n} : \Theta^\top \Theta = I_n, \Theta_{ij} \in \{0, 1\}, 1 \leq i, j \leq n \right\}, \quad \text{(4)}$$

which contains the set of $n$-by-$n$ permutation matrices $\mathcal{P}$ in (4). Model (1) is hence more general compared to existing work in which $\theta^*$ is restricted to be a permutation. In particular, the generalization herein allows for missing matches that correspond to $\theta^*(i) = 0$ and $\Theta_{i,:} = 0$ in which case $y_i$ is not in correspondence to any element in $X$, as well as for one-to-many matches, i.e., more than one element in $Y$ may correspond to the same element
in $X$; cf. Figure 1 for an illustration. Depending on the application, the goals in the setup (1) concern estimation of $B^*$ and/or $\Theta^*$. If $\Theta^*$ is recovered exactly by an estimator $\hat{\Theta}$, i.e., the event $\{\hat{\Theta} = \Theta^*\}$ occurs, estimation of $B^*$ becomes an ordinary regression problem. In post-linkage data analysis, $\Theta^*$ can be used to model error in the file linkage process, caused, e.g., by ambiguities resulting from the use of quasi-identifiers (say, the combination of age, gender, and race), but is typically treated as a nuisance parameter while primary interest concerns $B^*$. By contrast, in the setting of linkage attacks, the adversary aims at leveraging the linear relationship between elements of $\mathcal{X}$ and $\mathcal{Y}$, and hence $B^*$ is only regarded as a means to retrieve $\Theta^*$. In the sequel, we adopt neither viewpoint and consider estimation of both $B^*$ and $\Theta^*$.

**Assumptions.** Below, we summarize and discuss the main assumptions of our analysis.

- The map $\theta^*$ is said to be $k$-sparse if $\theta^*(i) = i$ except for indices $S_* \subseteq \{1, \ldots, n\}$ with $|S_*| \leq k$ for $k \ll n$. Equivalently, $S_* = \{i : \Theta^*_{ni} \neq 1\}$. Model (2) implies that
  \[ Y = XB^* + \Phi^* + \sigma SE, \]
  where $\Phi^*_{ni} = y_i - B^{*\top}x_i$ if $\theta^*(i) = 0$ and $\Phi^*_{ni} = B^{*\top}x_{\theta^*(i)} - B^{*\top}x_i$; otherwise, $1 \leq i \leq n$; with some slight abuse of notation, we replace $\Theta^*E$ by $SE$, which follows the same distribution. Observe that $k$-sparsity of $\theta^*$ implies that $\Phi^*$ has at most $k$ non-zero rows. Throughout this paper, we shall impose constraints on the size of $k$. As of now, if $k$ is not restricted, no practical estimation scheme with provable guarantees is known even if $\theta^*$ is a permutation.

- The matrix $X$ has i.i.d. Gaussian rows $x_i \sim N(0, \Sigma)$, $1 \leq i \leq n$. Without loss of generality, we assume that $\Sigma = I_d$ as can be ensured by re-defining $B^*$ accordingly.

- Likewise, the matrix $E$ has i.i.d. Gaussian rows $\epsilon_i \sim N(0, I_m)$, $1 \leq i \leq n$, and is independent of $X$.

The second assumption and the first part of the third assumption do not appear to be critical to our approach, but they considerably simplify results and proofs and thus aid presentation. It is expected that the results in this paper continue to hold for $X$ and $E$ with i.i.d. sub-Gaussian rows. Moreover, it is common to assume that the $m$ entries of the noise variables $\{\epsilon_i\}_{i=1}^n$ are correlated. Such extension can be accommodated as well.

Finally, we note that representation (5) is general enough to cover various other scenarios involving mismatched data in regression. For example, it also applies if a subset of the predictors is collected jointly with the response, i.e., we observe samples $D_1 = \{(x^{(1)}_i, y_i)\}_{i=1}^n$ and $D_2 = \{(x^{(2)}_i, y^{(2)}_{\theta^*(i)})\}_{i=1}^n$ with $\{x^{(1)}_i\}_{i=1}^n$ and $\{x^{(2)}_i\}_{i=1}^n$ having dimension $d_1$ and $d_2$, respectively, $d_1 + d_2 = d$, and associated regression model
  \[ y_i = B^{*\top}_{(1)}x^{(1)}_i + B^{*\top}_{(2)}x^{(2)}_{\theta^*(i)} + \sigma \epsilon_i, \quad i = 1, \ldots, n, \]
where $\theta^*$ is a permutation of $\{1, \ldots, n\}$. Model (6) is subsumed by (5) by setting $B^* = \begin{bmatrix} B^*_1 \\ B^*_2 \end{bmatrix}$, $\Phi^*_{i,:} = B^*_2 x_{\theta^*(i)} - B^*_2 x_i$, $1 \leq i \leq n$, and $S = I_n$. The approach and its analysis below applies to this and presumably also to other modifications with slight changes.

### 2.2 Approach

We suggest to tackle estimation of $B^*$ and $\Theta^*$ in a two-stage approach that we motivate as follows. Joint least squares estimation $\min_{\Theta \in \mathcal{M}, B \in \mathbb{R}^{d \times m}} \|Y - \Theta XB\|_F^2$ is $\mathcal{NP}$-hard [34]. However, if $B^*$ is known, least squares estimation of $\Theta^*$ reduces to a tractable optimization problem that decouples along the rows of $Y$:

$$\min_{\Theta \in \mathcal{M}} \|Y - \Theta XB^*\|_F^2 = \sum_{i=1}^n \left\{ \min_{0 \leq j \leq n} \|y_i - B^* x_j\|_2^2 \right\}, \quad (7)$$

where we recall that $x_0 = 0$. Assuming for simplicity that the minimizing indices $\hat{j}(i)$ for the optimization problems inside the curly brackets are unique, we have $\hat{\Theta}_{ij}(i) = 1$ if $\hat{j}(i) \neq 0$, $1 \leq i \leq n$; all other entries of $\hat{\Theta}$ equal zero. Alternatively, if $\theta^*$ is known to be one-to-one (i.e., a permutation), minimization over $\mathcal{M}$ can be replaced by minimization over $\mathcal{P}$ (4). The latter optimization problem reduces to a linear assignment problem [6], a specific linear program that can be solved efficiently by specialized techniques such as the Hungarian Algorithm [24] or the Auction Algorithm [5].

Since $B^*$ is unknown, it has to be replaced by an estimator $\hat{B}$. At this point, our approach makes use of the sparsity assumption for $\theta^*$. In view of relation (5), we consider

$$\min_{B \in \mathbb{R}^{d \times m}, \Xi \in \mathbb{R}^{n \times m}} \frac{1}{2n \cdot m} \|Y - XB - \sqrt{n}\Xi\|_F^2 + \lambda \sum_{i=1}^n \|\Xi_{i,:}\|_2, \quad (8)$$

for a tuning parameter $\lambda > 0$, where $\Xi$ targets $\Xi^* := \Phi^*/\sqrt{n}$ with $\Phi^*$ as in (5), and $\|\Xi_{i,:}\|_2$ being used as a convex surrogate for $I(\|\Xi_{i,:}\|_2 > 0)$, $1 \leq i \leq n$, in order to promote row-wise sparsity of $\Xi$ [15, 29, 58]. The use of the re-scaled quantity $\Xi^*$ in place of $\Phi^*$ is done merely for technical reasons. We note that a variant of (8) for a single response variable has been employed in the context of linear regression with outliers [26, 32, 41]. Optimization problem (8) can be solved efficiently by block coordinate descent as outlined in Algorithm 1 that has performed extremely well throughout our experiments, typically converging after a small number of iterations. Formal convergence results follow immediately from the general framework in [49].

The estimator $\hat{B}$ resulting from (8) can potentially be refined by a least squares re-fitting step after removing data corresponding to $\hat{S}(t) = \{1 \leq i \leq n : \|\hat{\Xi}_{i,:}\|_2 \geq t\}$, where $\hat{\Xi}$ denotes the minimizing $\Xi$ in (8) and $t$ is a suitably chosen threshold. The rationale is to remove mismatches as they hamper parameter estimation. This yields

$$\min_{B \in \mathbb{R}^{d \times m}} \sum_{i \in \hat{S}(t)} \|y_i - B^\top x_i\|_2^2, \quad (9)$$

In summary, this yields the following two-stage (or optionally three-stage) approach for estimating $B^*$ and subsequently $\Theta^*$.

1. Estimate $B^*$ from (8), and optionally refine via (9).

2. Estimate $\Theta^*$ from (7) with $B^*$ replaced by the estimator obtained in 1.
It is worth pointing out that sparsity of $\Theta^*$ is incorporated at step 1. only. Optimization problem (7) can be modified accordingly without affecting computational tractability as long as optimization is over the set $M^1$. Indeed, it is easy to verify that in this case the index set $\{i : \hat{\Theta}_{ii} \neq 1\}$ corresponds to the $k$ largest values among $\{||y_i - B^* x_i||^2_2 - \min_{0 \leq j \leq n} ||y_i - B^* x_j||^2_2\}_{1 \leq i \leq n}$. However, we do not consider this modification in the sequel since it does not change the statistical limits in recovering $\Theta^*$ as stated in Theorem 2 below.

Algorithm 1: Block coordinate descent for minimizing (8)

Compute the QR factorization $X = QR$ of $X$, and initialize $XB^{(0)} = QQ^T Y$, $\Xi^{(0)} = 0$.

1. Update for $\Xi$

$$\Xi^{(t+1)} \leftarrow (1 - \alpha(t))\Xi^{(t)} + \alpha(t)\text{GROUPTHRESHOLD}(Y - XB^{(t)})/\sqrt{n}, \quad \tau := m \cdot \sqrt{n} \cdot \lambda,$$

where for a matrix $A$ with rows $\{a_i\}_{i=1}^n$ and $\eta \geq 0$, $\text{GROUPTHRESHOLD}(A, \eta)$ is defined by

$$a_i \leftarrow a_i \cdot (1- \eta / \|a_i\|_2)_+, \quad i = 1, \ldots, n, \quad (\cdot)_+ := \max\{\cdot, 0\}.$$

2. Update for $XB$:

$$XB^{(t+1)} \leftarrow (1 - \gamma(t))XB^{(t)} + \gamma(t)QQ^T (Y - \sqrt{n}\Xi^{(t+1)}).$$

where the step sizes $\alpha(t), \gamma(t) \subset (0, 1)$ are chosen by back-tracking line search [4].

Illustration. An illustration of the above approach is provided in Figure 2. The data set consists of monthly average temperatures of $n = 46$ U.S. cities as reported in [56]. The data set is broken into two samples $\mathcal{X}$ and $\mathcal{Y}$ with the former containing the temperatures of the odd numbered months (January, March, ..., November) and the latter containing the temperatures of the even numbered months. For a random subset of $k = 10$ cities, we randomly permute matching records in $\mathcal{X}$ and $\mathcal{Y}$. Linear regression is used to predict the $m = 6$ temperatures in $\mathcal{Y}$ from $\mathcal{X}$. Due to high correlations among predictors, we work with the top $d = 3$ principal components as regressors. In the absence of partial data shuffling, this yields a reasonable goodness of fit overall in terms of a coefficient of determination $R^2 \approx 0.73$, apart from poor model fit for several west coast cities (Los Angeles, San Diego, Seattle and San Francisco) with mild winters and small seasonal differences, as well as for cities in desert regions (Las Vegas and Phoenix) with extreme temperatures during summer. After data shuffling, model fit drops to $R^2 \approx 0.4$. The approach outlined above shows some potential in this setting. With the choice of $\lambda = \frac{1}{2} \cdot \tilde{\sigma}_0 / \sqrt{n} \cdot m$, where $\tilde{\sigma}_0$ is the estimated error variance from the regression model in the absence of partial data shuffling, we ensure $R^2 \approx 0.62$. Subsequent restoration of the correct correspondence between $\mathcal{X}$ and $\mathcal{Y}$ is restricted to observations in $\hat{\mathcal{S}} = \{i : \|\hat{\Xi}_{ii}\|_2 \geq \sqrt{2m\tilde{\sigma}_0}\}$; for all other observations, no mismatches are assumed, i.e., $\hat{\Theta}_{ii} = 1, \quad i \notin \hat{\mathcal{S}}$. The results highlight the challenges that are encountered in the estimation of $\Theta^*$. Most crucially, the more an observation is distinct from the rest, the easier it is identified as mismatch and the easier to retrieve its matching counterpart, with Fairbanks here being the most distinct instance. On the other hand, the temperature differences between Milwaukee and Pittsburgh are only marginal, and accordingly this mismatch remains undetected. Moreover, it is hard to disentangle cities affected by shuffling and poor fit of the linear model, respectively. Nevertheless, re-matching succeeds for three cities (Fairbanks, Minneapolis, Tampa) and gets close in case of Phoenix $\rightarrow$ Las Vegas and San Antonio $\rightarrow$ Phoenix.

$^1$If $\hat{\Theta}$ is required to be a permutation, integrality of the corresponding LP relaxation is lost, and as a result, the sparsity-constrained variant of optimization problem (7) is no longer tractable.
Figure 2: Top: Mis-matched subset of the U.S. cities temperatures data set. Bottom: estimated subset of mismatched cities \( \hat{S} \) and estimated correspondence \( \hat{\theta}(\hat{S}) \). Asterisked cities Milwaukee and Pittsburgh did not end up included in \( \hat{S} \) since the misfit resulting from shuffling happened not to be substantial enough. The superscript \( ^\dagger \) refers to cities not affected by shuffling yet included in \( \hat{S} \).

3 Main results

This section provides theoretical results on the approach introduced in the previous section. Theorem 1 quantifies the error in estimating \( B^* \), while recovery of the correct correspondence in terms of \( \Theta^* \) is discussed in a separate subsection.

Theorem 1. Consider model (5) and the minimizer \((\hat{B}, \hat{\Xi})\) of (8) with \( \lambda \geq 2\lambda_0 \), where

\[
\lambda_0 = \frac{\tau_{n,d} \sigma}{\sqrt{n \cdot m}} \left( 1 + \sqrt{\frac{4 \log n}{m}} \right), \quad \tau_{n,d} := \left( \frac{n-d}{n} + \sqrt{24 \log n \over n} \right) \wedge 1,
\]

and suppose \( d/n < 1/4 \). Then for any \( \varepsilon \in (0, 1/3) \), there exists constants \( c_\varepsilon, c'_\varepsilon > 0 \) so that if \( k \leq c_\varepsilon n / \log(n/k) \), it holds that

\[
\left\| \hat{\Xi} - \Xi^* \right\|_F \leq 2\varepsilon^{-2} \cdot \lambda \sqrt{m} \cdot \frac{\lambda + \lambda_0}{\lambda - \lambda_0} \sqrt{k}.
\]

with probability at least \( 1 - 2/3.5 \cdot \exp(-c'_\varepsilon n) \). Furthermore,

\[
\left\| \hat{B} - B^* \right\|_F \leq \frac{1}{1 - \sqrt{4d \vee \log n \over n}} \left( \sigma \sqrt{5(d \vee \log n) \over n} + \left\| \hat{\Xi} - \Xi^* \right\|_F \sqrt{m} \over \sqrt{m} \right)
\]

with probability at least \( 1 - 2 \exp(-\frac{1}{2}(d \vee \log n)) - \exp(-(d \cdot m) \vee \log(n \cdot m)) \).

In order to better understand the consequences of Theorem 1, we spell out essential scalings in \( (n, k, d, m) \) below. We note that the parameter \( \lambda \) should be chosen proportional to \( \lambda_0 \approx 1/\sqrt{n \cdot m} (1 + \sqrt{\log(n)/m}) \) in which case \( \left\| \hat{\Xi} - \Xi^* \right\|_F \sqrt{m} \leq \sqrt{\frac{m}{n}} (1 + \sqrt{\log(n)/m}) \) which are familiar rates for multivariate regression with block sparsity regularization [29]. At the same time, the estimation error for the regression coefficients scales as \( \left\| \hat{B} - B^* \right\|_F \sqrt{m} \leq \sqrt{d/n} + \left\| \hat{\Xi} - \Xi^* \right\|_F \sqrt{m} \).
where the first term equals the usual error in the absence of mismatches while the second term reflects the slack arising from the presence of the latter. The bottom line is that the estimation error is in check as long as the fraction of mismatches \( k/n > 0.3 \). Theorem 1 also indicates a positive influence of the number of response variables \( m \) in that one can choose \( \lambda \asymp \frac{1}{\sqrt{m \cdot n}} \) once \( m \gtrsim \log n \) which in turn eliminates the factor \( \sqrt{\log n} \) in (10). This is a known benefit of block sparsity regularization in comparison to element-wise sparsity regularization [29].

**Restoring correspondence**

In this subsection, we study recovery of \( \Theta^* \). To begin with, we suppose that the regression parameter \( B^* \) is known, and establish one sufficient and one necessary condition for exact recovery of \( \Theta^* \) based on the oracle estimator (7). A crucial quantity in the analysis is

\[
\gamma^2 = \min_{i<j} \frac{\|B^*^\top (x_i - x_j)\|_2^2}{\|B^*\|_F^2},
\]

the minimum squared distance among all pairs of linear predictors scaled by \( \|B^*\|_F^2 \). A lower bound on \( \gamma^2 \) is clearly needed in order to reliably match noisy responses \( \{y_i\}_{i=1}^n \) to the corresponding elements in \( \{B^*^\top x_i\}_{i=1}^n \): if there exists a pair \((i, j)\) such that \( \|B^*^\top (x_i - x_j)\|_2 \) is smaller than the noise level, then there is a good chance that the corresponding responses get swapped. The following two lemmas provide upper and lower bounds on (11).

**Lemma 1.** Let \( \text{srank}(B^*) := \frac{\|B^*\|_2^2}{\|B^*\|_F^2} \) denote the stable rank of \( B^* \), and consider \( \gamma^2 \) as defined in (11). There exists universal constants \( \alpha_0 \in (0, 1) \) and \( \kappa \) such that for any \( \varepsilon > 0 \), with probability at least \( 1 - n^{-2\varepsilon} \), it holds that

\[
\gamma^2 > \min \left\{ 2n^{-\frac{2(1+\varepsilon)}{\kappa \cdot \text{srank}(B^*)}}, \alpha_0 \right\}^2 \tag{12}
\]

The stable rank of \( B^* \) as defined in the lemma crucially governs the scaling of \( \gamma^2 \). It is instructive to consider the extreme case \( \text{srank}(B^*) = 1 \). In the former case, we obtain \( \gamma^2 \gtrsim n^{-C} \) for \( C > 0 \). Results in [43] on the case \( m = 1 \) show that \( \gamma^2 \lesssim n^{-2} \) with constant probability, which indicates sharpness of the above result in this case up to a constant in the exponent of \( n \). On the other hand, if \( \text{srank}(B^*) = m \gtrsim \log n \), we have

\[
2n^{-\frac{2(1+\varepsilon)}{\kappa \cdot \text{srank}(B^*)}} = \exp \left( -\frac{2(1+\varepsilon)}{\kappa \cdot \text{srank}(B^*)} \log(2n) \right) = \Omega(1),
\]

i.e., the lower bound on \( \gamma^2 \) does no longer decay with \( n \). Additional insights can be obtained by considering the special case in which all non-zero singular values of \( B^* \) are equal to \( b_k > 0 \) and thus also \( \text{srank}(B^*) = \text{rank}(B^*) = r \). For \( r = 2(q+1), q \geq 0 \), the quantity (11) then becomes analytically tractable based on a closed form expression for \( \chi^2 \)-random variables with \( e \) even degrees of freedom.

**Lemma 2.** Consider \( \gamma^2 \) as defined in (11) and suppose that \( B^* \) has exactly \( r = 2(q+1), q \in \{0,1,\ldots\} \) non-zero singular values equal to \( b_k > 0 \). Then for all \( \delta > 0 \)

\[
\text{(Lower Bound): } \mathbf{P} \left( \gamma^2 \geq \frac{2}{\varepsilon} \left( n^{-2} \delta^2 \right) \right) \geq 1 - \delta/2.
\]

Moreover, if \( n > 8(r/2)^{r/2} \),

\[
\text{(Upper Bound): } \mathbf{P} \left( \gamma^2 \leq 2 \cdot 8^{2/r} n^{-2/r} \right) \geq 0.75.
\]
Lemma 2 sheds some light on the range of the exponent $\kappa$ in the previous Lemma 1, and provides essentially matching upper and lower bounds on $\gamma^2$, where “essentially” refers to $n^{-4/r} \lesssim \gamma \lesssim n^{-2/r}$, i.e., the match is up to constant factors and a factor 2 in the exponent.

In order to address the case of missing matches, we shall also consider

$$\gamma_0^2 = \min_{1 \leq i \leq n} \|B^T x_i\|_2^2 / \|B^*\|_F^2. \quad (13)$$

The latter quantity exhibits scalings very similar to $\gamma^2$ (11). Since we are primarily interested in lower bounds on those quantities to prove achievability results, it suffices to note that $\gamma_0^2 := 2 \min_{1 \leq i \leq n/2} \|B^T x_i\|_2^2 / \|B^*\|_F^2$ is stochastically larger than $\gamma^2$.

Equipped with Lemmas 1 and 2, we are in better position to interpret the following theorem.

**Theorem 2.** Let $\hat{B} = \hat{B}(X, Y)$ be an estimator of $B^*$, let $\mathcal{M}$ be the set of matrices in (3), and let $\Theta(\hat{B})$ denote the minimizer of

$$\min_{\Theta \in \mathcal{M}} \|Y - \Theta X \hat{B}\|_F^2.$$

Let $\gamma^2$ and $\gamma_0^2$ be as in (11) and (13), respectively, and define the signal-to-noise ratio by $\text{SNR} = \|B^*\|_F^2 / \sigma^2 m$. Consider the event

$$\mathcal{B} = \left\{ \min\{\gamma_0^2, \gamma^2\} \text{SNR} > 36 \max \left\{ \frac{\|\hat{B} - B^*\|_2^2}{\sigma^2 m}, \max_{1 \leq i \leq n} \|x_i\|_2^2, \left( 1 + \sqrt{\frac{4 \log n}{m}} \right)^2, \max_{i : \theta^*(i) = 0} \|y_i\|_2^2 \right\} \right\}.$$

Conditional on $\mathcal{B}$, with probability at least $1 - P(\mathcal{B}^c) - 1/n$, $\{\Theta(\hat{B}) = \Theta^*\}$. Conversely, in the case that $\theta^*(i) \neq 0$ for $1 \leq i \leq n$, the following holds:

- There exists $c > 0$ so that if $\text{SNR} < c \log(n) / m$, $P(\Theta(\hat{B}) \neq \Theta^*) \geq 1/3$.

- If additionally $m = O(1)$, there exists $c' > 0$ so that if $\min\{\gamma_0^2, \gamma^2\} \text{SNR} < c'$, $P(\Theta(\hat{B}) \neq \Theta^*) \geq 1/3$.

The above theorem contains both an achievability result in the form of a sufficient condition for successful recovery of $\Theta^*$ given any estimator of $\hat{B}$, as well as inachievability results concerning failure of recovery in the situation where $B^*$ is known. As explained in more detail below, the above sufficient and necessary conditions agree up to multiplicative constants in certain regimes. To shed more light on the implications of the theorem, it is instructive to consider certain special cases of interest and to make a connection with Theorem 1 and its error bounds for a specific choice of $\hat{B}$ motivated by a sparsity assumption.

The conditions of Theorem 2 involve $\text{SNR}$ as the ratio of the signal energy $\|B^*\|_F^2 / m$ per response variable and noise variance $\sigma^2$. If $\hat{B} = B^*$ and every element of $\mathcal{Y}$ has match in $\mathcal{X}$, the condition of the event $\mathcal{B}$ becomes

$$\min\{\gamma_0^2, \gamma^2\} \text{SNR} \geq 2(1 + \sqrt{\log(n) / m})^2. \quad (14)$$

If $m = O(1)$, the scaling of $\gamma^2$ according Lemmas 1 and 2 imply that the condition $\text{SNR} = \Omega(n^c)$ for a constant $c$ depending on the stable rank of $B^*$ suffices for recovery of $\Theta^*$. The second bullet in Theorem 2 implies that the same condition is also necessary (up to a constant factor in the exponent of $n$). In particular, Theorem 2 qualitatively recovers earlier results in [34] and [43] on $m = 1$. 


Regarding the scaling of \(m\), the threshold case appears to be \(m \asymp \log n \asymp \text{srank}(B^*)\). In this regime, (14) requires only \(\text{SNR} = \Omega(1)\) which is a far less stringent condition compared to the regime of uniformly bounded \(m\). Again, the sufficient condition is matched up to a constant multiplicative factor by the necessary condition stated in the first bullet of Theorem 2. Once \(m\) respectively \(\text{srank}(B^*)\) grow at a faster rate than \(\log n\), the necessary condition of the first bullet is no longer aligned with (14). It remains an open question whether Theorem 2 can be sharpened in this regard.

We now discuss the situation in which \(B^*\) is replaced by an estimator \(\hat{B}\). In the absence of mismatches, random matrix theory [54] shows that ordinary least squares estimation obeys 
\[
\mathbb{E}[\|\hat{B} - B^*\|_F^2/(\sigma^2m)] \lesssim (d + m)/(n - m)
\]
while \(\max_{1 \leq i \leq n} \|x_i\|_2^2 \lesssim d\) with high probability assuming that \(d \gtrsim \log n\), which implies that the first term in the outer “max” of event \(\mathcal{B}\) is at best of the order \(d^2/(n \cdot m)\). A slightly less favorable condition is obtained when substituting the error bound of the proposed estimator in Theorem 1. In this case,
\[
\|\hat{B} - B^*\|_F^2/(\sigma^2m) \leq (k + d)/n
\]
with the stated probability, and thus Theorem 2 yields the condition \(n \gtrsim d \cdot (k \lor d)\). In summary, the effect of replacing \(B^*\) by the proposed estimator can either be compensated by imposing a more stringent condition on \(\text{SNR}\) or the ratio \(d/n\).

Lastly, the last of the three expressions inside the outer “max” in Theorem 3 implicitly reflects a separability condition involving the norm of points \(y_i\) without match in \(X\). Clearly, if their norms are small enough while at the same time signal energy is large enough, separation between \(\{B^*\top x_i\}_{i=1}^n\) and \(y_i\) without match is guaranteed.

**Identification of Mismatched Data**

In the following, we discuss a simpler task than recovery of \(\Theta^*\), namely recovery of \(S_* = \{1 \leq i \leq n : \theta^*(i) \neq i\}\), or equivalently, \(S_* = \{1 \leq i \leq n : \Xi^*_i \neq 0\}\) with \(\Xi^* = \Phi^* / \sqrt{n}\) as defined in (5). The following statement provides a condition that ensures that we can separate mis-matched data \(S_*\) and correctly matched data \(S^c\) in terms of \(\{|\hat{\Xi}_i|\}_i\) where \(\hat{\Xi}\) is obtained from optimization problem (8) and analyzed in Theorem 1.

**Proposition 1.** Let \(\hat{\Xi}\) be as in Theorem 1, and let \(\gamma_0^2, \gamma^2\), and \(\text{SNR}\) be as in Theorem 2. We then have \(\min_{i \in S_*} |\hat{\Xi}_i| > \max_{i \in S^c} |\hat{\Xi}_i|\) if
\[
\min\{\gamma_0^2, \gamma^2\} \text{SNR} \geq 9 \max \left\{ \frac{\|\hat{\Xi} - \Xi^*\|_F^2}{\sigma^2m}, \frac{\max_{i \in S^c} \|y_i\|_2^2}{\sigma^2m} \right\}.
\]

The practical consequences are as follows: if it holds that \(\min_{i \in S_*} |\hat{\Xi}_i| > \max_{i \in S^c} |\hat{\Xi}_i|\), we can sort the \(\{|\hat{\Xi}_i|\}_i\) and retain the observations corresponding to the \(\lfloor \nu n \rfloor\) smallest elements for \(\nu \in (0, (1 - k/n)]\). Any choice of \(\nu = \Omega(1)\) in that range identifies \(Q \subseteq S^c\) with \(|Q| = \Omega(n)\). The least squares estimator \(\hat{B}\) of \(B^*\) using observations in \(Q\) only, i.e.,
\[
\hat{B} = \arg\min_{B \in \mathbb{R}^{d \times m}} \sum_{i \in Q} \|y_i - B\top x_i\|_2^2
\]
can substantially improve over the estimator \(\hat{B}\) in Theorem 1. We note that the condition of Proposition 1 is easier to satisfy than that for recovery of \(\Theta^*\) in Theorem 2, particularly if the fraction of mismatches \(k/n\) is small as this is the scaling of the term \(\|\Xi - \Xi^*\|_F^2/m\).
4 Experiments

In the sequel, we present empirical evidence supporting central aspects of our analysis. For simplicity, we confine ourselves to the case in which $\Theta^*$ is a permutation matrix, i.e., an element of (4). Accordingly, the minimization in (7) is performed over the set of permutation matrices by means of the Auction Algorithm [5]. We note that this modification does not affect our theoretical results. Specifically, the achievability result in Theorem 2 continues to hold because it asserts recovery over a superset of (4). Similarly, the inachievability results continue to hold if $\Theta^*$ is required to be a permutation.

**Synthetic data.** Data is generated according to the model

\[ y_i = B^* \mathbf{x}_{\theta^*(i)} + \sigma \epsilon_i, \quad i = 1, \ldots, n, \]

where the $\{x_i\}_{i=1}^n$ and $\{\epsilon_i\}_{i=1}^n$ are i.i.d. from $N(0, I_d)$ and $N(0, I_m)$, respectively, $\theta^*$ is a random permutation that shuffles $\{1, \ldots, k\}$ uniformly at random, and is the identity map when restricted to the remaining indices, i.e., $\theta^*(i) = i$ for $i > k$. The matrix $B^*$ is obtained by first generating a $d$-by-$d$ matrix (i.e., $d = m$) with i.i.d. $N(0, 1)$-entries, then computing its singular value decomposition $B^* = U S V^\top$, and replacing the diagonal entries \{$s_1, \ldots, s_d$\} of $S$ according to $s_j \leftarrow j^{-q}$, $1 \leq j \leq d$ for $q \in \{0, 0.05, 0.1, 0.2, 0.5, 1, 2, 5\}$; finally, $B^*$ is re-scaled such that $\|B^*\|_F = m$. This construction ensures that the stable rank $\text{rank}(B^*)$, which has a critical influence on the recovery of $\Theta^*$, varies between $m = d$ (achieved for $q = 0$) and 1 (achieved for $q \to \infty$). In addition, the signal-to-noise ratio then results as $\text{SNR} = \sigma^{-2}$ with $\sigma \in \{0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1, 2\}$. Lastly, the fraction of mismatches $k/n$ varies between 0.05 and 0.4 in steps of 0.05 with $n \in \{200, 500, 1000\}$ and $d/n \in \{0.03, 0.06, 0.12\}$. For each configuration of $(n, d, k, q, \sigma)$, 100 independent replications were performed. The following aspects were investigated in detail.

- **Estimation error**, i.e., $\|\tilde{B} - B^*\|_F$, where $(\tilde{B}, \tilde{S})$ is obtained from (8) with the choice $\lambda = 4\sigma \frac{1}{\sqrt{n} \cdot m}$ which is the lower bound on $\lambda$ suggested by Theorem 1 when treating $\sqrt{4 \log(n)} / m$ simply as 1. For better comparison across experimental configurations, we visualize the following “normalized” estimation error

\[ \sigma^{-1} m^{-1/2} \|\tilde{B} - B^*\|_F - \sqrt{d/n} \lesssim \sqrt{k/n} \] (15)

according to Theorem 1, which only depends on the fraction of mismatches. The above criterion also quantifies the excess error compared to an oracle that knows $\Theta^*$. Selected results are shown in Figure 3. First, the plots confirm that (15) is indeed largely independent of $n$, $d/n$, and $\sigma$. Second, it becomes apparent that improvement of (8) over the naive least squares estimator (ignoring potential mismatches) differ markedly with $\sigma$ and $k/n$. The improvements are particularly pronounced for small $\sigma$ (ranging from a factor of 100 for small $k/n$ to a factor of 10 for $k/n = 0.4$). As $\sigma$ increases to 0.2, improvements are only achieved for small $k/n$, and the error curves of the proposed and naive approach intersect for $k/n \geq 0.35$.

- **Estimation error** of the re-fitting approach (9), cf. also Proposition 1. Assuming that $k$ is known, the set of mismatches $S_*$ is estimated by $\hat{S} = \{1 \leq i \leq n : \|\hat{\xi}_i\|_2 > t_{(n-k)}\}$, where $t_{(i)}$, $1 \leq i \leq n$, denotes the $i$-th order statistic of the $\{\|\hat{\xi}_i\|_2\}_{i=1}^n$. The resulting estimation error $\|\tilde{B} - B^*\|_F$ is expected to scale as $\sigma m^{1/2} \sqrt{d/(n-k)}$ if $\hat{S} = S_*$ as is largely confirmed by Figure 4. The observed errors are within a factor of two of the anticipated errors if $B^*$ has maximum stable rank (left panel) and within a factor of $2^{3/2} \approx 3$ if $B^*$ has minimum stable rank (middle panel); slightly reduced performance
in the latter case is expected in view of Proposition 1. The reductions in estimation error of $\hat{B}$ relative to $\tilde{B}$ can easily attain a factor of five or more, even more than what is suggested by the right panel of Figure 4.

- Recovery of $\Theta^*$ in terms of the normalized Hamming distance $\frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(\theta^*(i) \neq \hat{\theta}(i))$, where $\hat{\Theta}$ is the plug-in estimator $(7)$ (modified to incorporate the constraint that $\Theta^*$ is a permutation) with $B^*$ replaced by $\tilde{B}$. In light of Theorem 2 and Lemmas 1 and 2, recovery of $\Theta^*$ is successful if $\gamma^2 \cdot \text{SNR} \propto n^{-c/\text{rank}(B^*)} \cdot \text{SNR}$ is large enough. We therefore plot the average Hamming distance in dependency of the “normalized” SNR $-c/\text{rank}(B^*) \log(n) - 2 \log(\sigma)$, where the choice $c = 0.7$ was found to ensure a reasonable alignment of the results across different experimental configurations. Figure 5 indicates that recovery of $\Theta^*$ follows a phase transition: if the normalized SNR drops below a certain threshold, the average Hamming distance rises sharply. This observation is in alignment with the inachievability results in Theorem 2. Interestingly, plug-in estimation (lower panel) does not lead to a significant degradation in performance compared to the situation in which $B^*$ is known (upper panel) even if the fraction of mis-matches is large.

![Figure 3](image-url)

**Figure 3:** Normalized estimation errors (15) on a log_{10}-scale. “proposed” refers to (8) (one curve per value of $\sigma$); “naive” refers to the ordinary least squares estimator. The red area marks the region of error of the latter as $\sigma$ varies between 0.01 and 0.2. The dashed line represents $3\sqrt{k/n}$.

**Real data.** We consider three benchmark data sets for multivariate regression as tabulated below. The data sets above are preprocessed versions of their original counterparts. The

| Full Name | Short Name | n   | d   | m   | $R^2$ |
|-----------|------------|-----|-----|-----|-------|
| SARCOS robot arm [37] | sarcos | 44,484 | 10 | 6  | 0.76  |
| Flight Ticket Prices [50] | ftp     | 335  | 30  | 6  | 0.89  |
| Supply Chain Management [50] | scm     | 8,966 | 35 | 16 | 0.58  |

**Table 1:** Overview on the data sets considered in this paragraph. $R^2$ here refers to the coefficient of determination in the absence of shuffling.
Figure 4: Normalized estimation errors $\frac{\sigma^{-1}m^{-1/2}\|B_0 - B^*\|_F}{(d/(n-k))^{1/2}}$ (log$_2$-scale) of the re-fitting approach (9) for $\text{srank}(B^*) = d = m$ (left) and $\text{srank}(B^*) \approx 1$ (middle); curves for different combinations of $n$ and $\sigma$ appear in the same plots. The right panel graphs the ratio of $(\sqrt{k/n} + \sqrt{d/n})$ and $\sqrt{d/(n-k)}$, i.e., the ratio of the estimation rates without (8) and with re-fitting (9).

Figure 5: Averaged Hamming distance between $\hat{\Theta}(B^*)$ and $\Theta^*$ (upper panel) and between $\hat{\Theta}(\hat{B})$ and $\Theta^*$ (lower panel) in dependence of the normalized SNR $-c/\text{srank}(B^*) \log(n) - 2\log(\sigma)$. Different curves to different values of the parameter $q$ controlling $\text{srank}(B^*)$.

columns of the matrices $X$ and $Y$ were centered, and $X$ was subsequently reduced to an adequate number of principal components since due to (almost) linearly independent predictors the oracle least squares estimator (here assigned the role of $B^*$) would (essentially) not be defined. For sarcos, one of the response variables was removed to improve goodness of fit, and hence to observe a better contrast in performance with an increasing fraction of mismatches. Likewise, two outliers with Cook’s distance $> 0.7$ were removed from ftp. None of these is originally affected by mismatches, hence in order to mimic this scenario, we randomly permute varying fractions (between 0.05 and 0.4) of the rows of $Y$, and in-
vestigate to what extent the proposed approach is able to restore the goodness-of-fit (in terms of the coefficient of determination $R^2$) and the regression coefficients of the least squares estimator in the complete absence of mismatches that here takes the role of $B^*$. The performance of the proposed approach is compared to naive least squares based on the permuted data. For each data set, we consider 20 independent random permutations for each value of $k/n$. Performance with regard to permutation recovery is assessed via $\|\hat{\Theta} - \Theta^*\|_F/\|I_n - \Theta^*\|_F$, i.e., via the relative reduction in error induced by random shuffling. This is a somewhat less stringent metric than the Hamming distance reported for synthetic data. The change in metric is motivated by the fact that exact permutation recovery cannot be expected for the data sets under consideration given that separability in terms of (11) relative to the noise level is poor. Approach (8) is run with the choice $\lambda = M \cdot \frac{\hat{\sigma}_0}{\sqrt{n-m}}$ for $M \in \{0.25, 0.5, 1, 2\}$ and $\hat{\sigma}_0$ denoting the root mean square error of the least squares estimator in the absence of shuffling. As can be seen from Figure 6, the results are not sensitive to the choice of the multiplier $M$. The proposed approach consistently improves over naive least squares once the fraction of mismatches exceeds 0.2, and yields more pronounced improvements as that fraction increases. Two-stage estimation of $\Theta^*$ yields noticeable reductions of the error $\|I_n - \Theta^*\|_F$ induced by shuffling.

![Graphs showing goodness of fit and estimation error](image_url)

**Figure 6:** Top: Goodness of fit in terms of the coefficient of determination $R^2 = \|Y - X\hat{B}\|_F^2/\|Y\|_F^2$. Middle: Relative estimation errors $\|\hat{B} - B^*\|_F/\|B^*\|_F$, where $B^*$ here refers to the oracle least squares estimator equipped with knowledge of $\Theta^*$. Bottom: Performance in approximate recovery of $\Theta^*$ evaluated in terms of $\|\hat{\Theta} - \Theta^*\|_F/\|I_n - \Theta^*\|_F$. Each of the black lines corresponds to one specific value of the multiplier $M$ in $\lambda = M\hat{\sigma}_0/\sqrt{n-m}$. 
5 Conclusion

In this paper, we have presented a computationally appealing two-stage approach to multivariate linear regression in the presence of a small to moderate number of mismatches. The proposed approach can be used to safeguard against a potentially dramatic increase in the estimation error that can be incurred when ignoring the possibility of mismatches, as demonstrated in terms of statistical analysis and supported by a series of empirical results. Moreover, under certain conditions involving “separability” of pairs of data points and the signal-to-noise ratio, it is shown that the true correspondence between those pairs can be perfectly recovered. A key result in this paper asserts that the availability of multiple, linearly independent response variables (as measured by the stable rank of the regression coefficients) considerably simplifies the problem as it increases separability.

A limitation of the proposed approach is that it imposes a stringent limit on the allowed fraction of mismatches. In fact, as long as a sufficiently large superset of correctly matched data (of size \( \Omega(n) \)) can be identified, the regression parameter can still be estimated at the usual rate. Accordingly, the given problem does not appear hopeless even for significantly larger fraction of mismatches, say, up to \( 1 - \delta \) for \( \delta \) bounded away from zero. Closing this gap is a worthwhile endeavor for future research. A second direction of future work concerns extension of the setup beyond classical linear models, specifically more flexibility regarding the range of the response variables (binary, mixed discrete/continuous etc.).

A Proof of Theorem 1

(I) **Bound on** \( \|\Xi^* - \hat{\Xi}\|_F \).

A crucial observation is that the joint optimization problem (8) in \( B \) and \( \Xi \) can be decomposed into two optimization problems involving only \( B \) and \( \Xi \), respectively, as stated in the following Lemma.

**Lemma A.1.** Consider optimization problem (8) with solution \( (\hat{B}, \hat{\Xi}) \) and denote by \( P_{\Xi}^\perp \) the projection on the orthogonal complement of range(\( X \)). Then, if \( n \geq d \), with probability one

\[
\hat{\Xi} \in \mathcal{X}, \quad \mathcal{X} := \arg\min_{\Xi} \frac{1}{2n \cdot m} \|P_{\Xi}^\perp (Y - \sqrt{n}\Xi)\|_2^2 + \lambda \sum_{i=1}^{n} \|\Xi_{i,:}\|_2, \tag{16}
\]

\[
\hat{B} \in \left\{ \left( \frac{X^\top X}{n} \right)^{-1} \frac{X^\top (Y - \sqrt{n}\Xi)}, \quad \Xi \in \mathcal{X} \right\}. \tag{17}
\]

The proof is along the lines of the proof of Lemma 1 in [43], and is hence omitted. Note that \( P_{\Xi}^\perp Y = P_{\Xi}^\perp (\sqrt{n}\Xi^* + \sigma \tilde{E}) \) with \( \tilde{E} = SE \). The optimization problem in (16) thus becomes

\[
\min_{\Xi} \frac{1}{2n \cdot m} \|P_{\Xi}^\perp (\sqrt{n}\Xi^* + \sigma \tilde{E} - \sqrt{n}\Xi)\|_2^2 + \lambda \sum_{i=1}^{n} \|\Xi_{i,:}\|_2 \tag{18}
\]

In the sequel, we study an equivalent vectorized problem. Accordingly, we define

\[
\xi^* = [(\Xi^*_{1,1})^\top; \ldots; (\Xi^*_{n,m})^\top] \in \mathbb{R}^{n \cdot m}, \quad \tilde{e} = [\tilde{E}^\top_{1,1}; \ldots; \tilde{E}^\top_{n,m}] \]

\[
P_{\Xi}^\perp \otimes I_m = P_{\Xi}^\perp = \begin{pmatrix}
P_{\Xi}^\perp & 0 & \ldots & 0 \\
0 & P_{\Xi}^\perp & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & P_{\Xi}^\perp
\end{pmatrix}, \tag{19}
\]
with \( \otimes \) denoting the Kronecker product, the subscripts \( .j \) refer to the \( j \)-th column, \( j = 1, \ldots, m \), and ";" here means row-wise concatenation. Moreover, for any \( v \in \mathbb{R}^{n \cdot m} \), we let
\[
v[i] = (v_j)_{j \in G_i}, \quad i = 1, \ldots, n, \quad G_i = \{i, i + n, \ldots, i + (m - 1) \cdot n\}.
\]
With this in place, the \((2, q)\)-norm with respect to \( G_1, \ldots, G_n \) is defined by
\[
\|v\|_2,q := \left( \frac{1}{q} \sum_{i=1}^{n} \|v[i]\|_2^q \right)^{1/q}, \quad 1 \leq q < \infty, \quad \text{and} \quad \|v\|_2,\infty := \max_{1 \leq i \leq n} \|v[i]\|_2,
\]
where the latter is not a norm; it counts the number of non-zero groups of components, with each of the \( \{G_i\}_{i=1}^n \) forming a group. Note that \( \|\xi^*[i]\|_2 \leq k \) with support
\[
S_* = \{1 \leq i \leq n : \Theta_n^i \neq 1\} = \{1 \leq i \leq n : \|\xi^*[i]\|_2 > 0\}.
\]
We also observe that for all \( v, w \in \mathbb{R}^{n \cdot m} \)
\[
\|v\|_2,2 = \|v\|_2, \quad |\langle v, w \rangle| = \sum_{i=1}^{n} \|v[i]\|_2 \cdot \|w[i]\|_2 \leq \|v\|_2,1 \cdot \|w\|_2,\infty
\]
by the inequalities of Cauchy-Schwarz and Hölder.

After these preparations, we are in position to state another Lemma. First note that optimization problem (18) can be expressed in vectorized form as
\[
\min_{\xi} \frac{1}{2n \cdot m} \|P_X^{\perp}(\sqrt{n}\xi^* + \sigma \tilde{e}) - P_X^{\perp} \xi \sqrt{n}\|_2^2 + \lambda \sum_{i=1}^{n} \|\xi[i]\|_2,
\]
Letting \( \hat{\delta} = \xi^* - \tilde{\xi} \), where \( \tilde{\xi} \) is a minimizer of (23), we have the following basic inequality
\[
\frac{1}{2n \cdot m} \|P_X^{\perp} \sqrt{n}\hat{\delta}\|_2^2 + \lambda \sum_{i=1}^{n} \|\xi[i]\|_2 \leq \frac{1}{\sqrt{n} \cdot m} |\langle P_X^{\perp} \hat{\delta}, \sigma \tilde{e} \rangle| + \lambda \sum_{i \in S_*} \|\xi^*[i]\|_2,
\]
which is obtained by evaluating (23) at \( \xi = 0 \), expanding squares and re-arranging.

**Lemma A.2.** Consider \( \hat{\delta} \) in (24) and Let \( \lambda_0 \) be a number such that
\[
\frac{1}{\sqrt{n} \cdot m} \|P_X^{\perp} \sigma \tilde{e}\|_2,\infty \leq \lambda_0.
\]
Then for any \( \lambda \geq 2\lambda_0 \), it holds that either \( \hat{\delta} = 0 \) or \( \hat{\delta}/\|\hat{\delta}\|_2 \in 2 \text{conv}(B_0(k')) \cap \mathbb{S}^{n \cdot m - 1} \), where for \( r \geq 0 \), \( B_0(r) = \{v \in \mathbb{R}^{n \cdot m} : \|v\|_2,0 \leq r, \|v\|_2 \leq 1\} \) according to (20) and \( k' = \left(1 + \frac{\lambda + \lambda_0}{\lambda_0}\right)^2 k \leq 16k \).

**Proof.** As an immediate consequence of (24) and the triangle inequality, we obtain that
\[
\lambda \sum_{i \in S_*} \|\hat{\delta}[i]\|_2 \leq \frac{1}{\sqrt{n} \cdot m} |\langle P_X^{\perp} \hat{\delta}, \sigma \tilde{e} \rangle| + \lambda \sum_{i \in S_*} \|\hat{\delta}[i]\|_2 \leq \lambda_0 \|\hat{\delta}\|_2,1 + \lambda \sum_{i \in S_*} \|\hat{\delta}[i]\|_2,
\]
where the second inequality is a result of (22) and (25). If \( k = 0, S_* = \emptyset \), we must have \( \hat{\delta} = \tilde{\xi} = \xi^* = 0 \) as the above inequality would be violated otherwise, and the claim of the
the above chain of inequalities yields

\[
\lambda \sum_{i \in S} \|\hat{\delta}^{[i]}\|_2 \leq \lambda_0 \|\hat{\delta}\|_{2,1} + \lambda \sum_{i \in S} \|\hat{\delta}^{[i]}\|_2 = \lambda_0 \left( \sum_{i \in S} \|\hat{\delta}^{[i]}\|_2 + \sum_{i \in S} \|\hat{\delta}^{[i]}\|_2 \right) + \lambda \sum_{i \in S} \|\hat{\delta}^{[i]}\|_2
\]

\[
\Rightarrow \sum_{i \in S} \|\hat{\delta}^{[i]}\|_2 \leq \frac{\lambda + \lambda_0}{\lambda - \lambda_0} \sum_{i \in S} \|\hat{\delta}^{[i]}\|_2
\]

\[
\Rightarrow \|\hat{\delta}\|_{2,1} \leq \left( 1 + \frac{\lambda + \lambda_0}{\lambda - \lambda_0} \right) \sum_{i \in S} \|\hat{\delta}^{[i]}\|_2 \leq \left( 1 + \frac{\lambda + \lambda_0}{\lambda - \lambda_0} \right) \sqrt{k} \|\hat{\delta}\|_2
\]  \quad (26)

The assertion then follows from Lemma E.1 provided in a separate section below. \[\square\]

As in the above Lemma, under event (25), inequality (24) implies

\[
\frac{1}{2n \cdot m} \mathbb{P}_X \log \sqrt{\|\delta\|_2^2} \leq \left( \frac{\lambda_0}{\lambda - \lambda_0} + \lambda \right) \sqrt{k} \|\delta\|_2 = \lambda \left( \frac{\lambda + \lambda_0}{\lambda - \lambda_0} \right) \sqrt{k} \|\delta\|_2
\]  \quad (27)

by following the steps leading to (26). We now lower bound the l.h.s. of (27). Let \( \Lambda = \{(\lambda_s)_{s=1}^N \subset \mathbb{R}_+ : N \in \{1, 2, \ldots, \}, \sum_{s=0}^N \lambda_s \leq 2 \} \). In light of Lemma A.2, we have

\[
\min_{\sum_s \lambda_s v_s \in \mathbb{R}^{n-1}} \|\mathbb{P}_X^{1/2} \sum_s \lambda_s v_s\|_2^2
\]

Structuring each \( v_s \) into sub-vectors \( v_s^{(l)} \in \mathbb{R}^n, l = 1, \ldots, m \), we obtain

\[
\min_{\sum_s \lambda_s v_s^{(l)} \in \mathbb{R}^{n-1}} \|\mathbb{P}_X^{1/2} \sum_s \lambda_s v_s^{(l)}\|_2^2
\]

Since each \( v_s \) is \( k' \)-group sparse according to the partitioning defined by \( \{G_i\}_{i=1}^n \), each \( v_s^{(l)} \) is at most \( k' \)-sparse in the ordinary sense, i.e., having at most \( k' \) non-zero entries. Letting \( B_0(k') = \{v \in \mathbb{R}^n : \|v\|_0 \leq k'\} \) denote the usual \( k' \)-sparsity ball in \( \mathbb{R}^n \), we have

\[
\min_{\sum_s \lambda_s v_s^{(l)} \in \mathbb{R}^{n-1}} \sum_{l=1}^m \|\mathbb{P}_X^{1/2} \sum_s \lambda_s v_s^{(l)}\|_2^2 = \min_{\sum_s \gamma \lambda_s v_s^{(l)} \in \mathbb{R}^{n-1}} \sum_{l=1}^m \|\mathbb{P}_X^{1/2} \gamma \sum_s \lambda_s v_s^{(l)}\|_2^2 = \min_{u \in \text{conv}(B_0(k')) \cap \mathbb{S}^{n-1}} \|\mathbb{P}_X^{1/2} u\|_2^2
\]  \quad (28)

In order to lower bound this squared distance, we apply Gordon’s Theorem (cf. Lemma E.3 below) with \( K = 2\text{conv}(B_0(k')) \cap \mathbb{S}^{n-1} \) and \( V = \text{range}(X) \) noting that the latter random subspace follows a uniform distribution on the Grassmannian \( G(n, d) \), thus we identify \( p = n, p - q = d \Leftrightarrow q = n - d \). It is well-known that \( \nu_r = \sqrt{r^{2/(r+1)}} = (1 - O(1/\sqrt{r})) \sqrt{r} \sim \sqrt{r} \)
as \( r \to \infty \); to simplify our argument, we henceforth replace \( \nu_r \) by \( \sqrt{r} \). Translated to the setting under consideration, the condition \( w(K) < (1 - \varepsilon) \nu_q - \varepsilon \nu_r \) in Lemma E.3 reads

\[
\frac{1}{1 - \varepsilon} w(2 \text{conv}(B_0(k')) \cap S^{n-1}) < \sqrt{n - d} - \frac{\varepsilon}{1 - \varepsilon} \sqrt{n}.
\]

(29)

Invoking the assumption \( d/n \leq 1/4 \), the r.h.s. of (29) evaluates as \((\sqrt{3}/2 - \frac{\varepsilon}{1 - \varepsilon})\sqrt{n}\). Regarding the l.h.s. of (29), it follows from standard results (cf. [36], Lemma 2.3) that the Gaussian width \( w(2 \text{conv}(B_0(k')) \cap S^{n-1}) \leq 2 \sqrt{k/\log(en/k')} \). It thus follows that for any \( \varepsilon \in (0, 1/3) \), there exists \( c_\varepsilon, c'_\varepsilon > 0 \) so that if \[ k \leq c_\varepsilon \cdot n / \log(n/k) \]

inequality (29) is satisfied, so that with probability at least \( 1 - 3.5 \cdot \exp(-c'_\varepsilon n) \), (28) is lower bounded by \( \varepsilon^2 \). Combining (27) and this lower bound on (28), we conclude that

\[
m^{-1/2} \|\mathbb{E} - \Xi^*\|_F = m^{-1/2} \|\hat{\delta}\|_2 \leq \varepsilon^{-2} \cdot 2\lambda \sqrt{m} \cdot \frac{\lambda + \lambda_0}{\lambda - \lambda_0} \sqrt{k}.
\]

The lemma below elaborates on the choice of \( \lambda_0 \), which completes the proof of the bound on \( m^{-1/2} \|\mathbb{E} - \Xi^*\|_F \).

**Lemma A.3.** With probability at least \( 1 - 2/n \), it holds that

\[
\frac{1}{\sqrt{n} \cdot m} \|P_X^\perp \sigma e\|_{2,\infty} \leq \lambda_0 \quad \text{with} \quad \lambda_0 = \frac{\tau_{n,d} \sigma}{\sqrt{n} \cdot m} \left( 1 + \sqrt{\frac{4 \log n}{m}} \right), \quad \tau_{n,d} := \left( \frac{n-d}{n} + \sqrt{\frac{4 \log n}{n}} \right) \wedge 1.
\]

**Proof.**

\[
\frac{1}{\sqrt{n} \cdot m} \|P_X^\perp \sigma e\|_{2,\infty} = \frac{\sigma}{\sqrt{n} \cdot m} \max_{1 \leq i \leq n} \|E^T S^T P_X^\perp e_i\|_{2,\infty},
\]

where \( \{e_i\}_{i=1}^n \) is the canonical basis of \( \mathbb{R}^n \). Observe that conditional on \( P_X^\perp \), \( E^T S^T P_X^\perp e_i \) is a zero mean-Gaussian random vector with covariance matrix \( \|S^T P_X^\perp e_i\|_2^2 \cdot I_m \), \( 1 \leq i \leq n \). Since \( \|S\|_2 \leq 1 \) and since \( P_X^\perp \) is a random projection in the sense of DasGupta and Gupta [9], it follows from results therein that for all \( \eta > 0 \)

\[
P \left( \max_{1 \leq i \leq n} \|S^T P_X^\perp e_i\|_2^2 \geq \frac{n-d}{n} (1 + \eta) \wedge 1 \right) \leq n \exp \left( -\left( n-d \right) \frac{\eta^2}{12} \right).
\]

In particular, with the choice \( \eta = \sqrt{\frac{2 \log n}{n-d}} =: c_1 \),

\[
P \left( \max_{1 \leq i \leq n} \|S^T P_X^\perp e_i\|_2^2 \geq \tau_{n,d} \right) \leq 1/n, \quad \tau_{n,d} := \left( \frac{n-d}{n} + \sqrt{\frac{4 \log n}{n}} \right) \wedge 1
\]

Combining this result with Lemma E.2 with \( r = m \), \( L = n \), \( \max_{1 \leq \ell \leq L} \sigma_\ell = \tau_{n,d} \), we have

\[
\|P_X^\perp \sigma e\|_{2,\infty} \leq \tau_{n,d} \sigma \sqrt{m + 2 \sqrt{\log n}}
\]

with probability at least \( 1 - 2/n \). This finally yields the choice

\[
\lambda_0 = \frac{\tau_{n,d} \sigma}{\sqrt{n} \cdot m} \left( 1 + \sqrt{\frac{4 \log n}{m}} \right).
\]
(II) Bound on $\|B^* - B\|_F$. Let $\sigma_{\min}(\cdot)$ and $\sigma_{\max}(\cdot)$ denote the minimum and maximum singular value functional, respectively. Invoking Lemma A.1, we bound

$$\|\tilde{B} - B^*\|_F \leq \sqrt{\frac{n}{\sqrt{n}}} \left\| \left( \frac{X^T X}{n} \right)^{-1} \frac{X^T}{\sqrt{n}} SE + \frac{\sqrt{n}(\Xi^* - \hat{\Xi})}{\sigma_{\min}(X/\sqrt{n})} \right\|_F$$

where we have used that $\left( \frac{X^T X}{n} \right)^{-1} \frac{X^T}{\sqrt{n}} = \left( \frac{X}{\sqrt{n}} \right)^\dagger$, with $\dagger$ denoting the Moore-Penrose pseudo-inverse, and $\sigma_{\max} \left( \left( \frac{X}{\sqrt{n}} \right)^\dagger \right) = \sigma_{\min}^{-1}((X/\sqrt{n})^\dagger)$. Consider $\Gamma = S \frac{X}{\sqrt{n}} (\frac{X^T X}{n})^{-2} S^\dagger \frac{X^T}{\sqrt{n}}$, and let $\Gamma^O = I_m \otimes \Gamma$. We then can write

$$\left\| \left( \frac{X^T X}{n} \right)^{-1} \frac{X^T}{\sqrt{n}} SE \right\|_F^2 = \|\Gamma^O e\|_2^2,$$

where $e$ is a standard Gaussian random vector of dimension $n \cdot m$. By straightforward adaptations of Lemma 3 in [43] that is based on a concentration result for quadratic forms in [22], we obtain that

$$\mathbb{P} \left( \left\| \left( \frac{X^T X}{n} \right)^{-1} \frac{X^T}{\sqrt{n}} E \right\|_F > \frac{\sqrt{5(d \cdot m \vee \log(n \cdot m))}}{\sigma_{\min}(X/\sqrt{n})} \right) \leq \exp(-(d \cdot m) \vee \log(n \cdot m))$$

The proof is then completed by appealing to concentration results (e.g., Corollary 5.35 in [52]) to lower bound $\sigma_{\min}(X/\sqrt{n})$ with $X$ having i.i.d. standard Gaussian entries.

## B Proofs of Lemmas 1 and 2

Lemma 1 is an immediate consequence of the following result.

**Lemma B.1.** (Proposition 2.6 in [27])

Let $g \sim N(0, I_d)$. There exist universal constants $\alpha_0 \in (0, 1)$ and $\kappa > 0$ such that for any $\alpha \in (0, \alpha_0)$

$$\mathbb{P} \left( \|B^* g\|_2 \leq \alpha \|B^*\|_F \right) \leq \exp(\kappa \log(\alpha) \text{srank}(B^*)) .$$

Lemma 1 is obtained by applying Lemma B.1 with $g = \frac{x_i - x_j}{\sqrt{2}}$, and then using a union bound over pairs, i.e., $\{\min_{i < j} \|B^*(x_i - x_j)\|_2 \leq \delta \} \subseteq \bigcup_{i < j} \{\|B^*(x_i - x_j)\|_2 \leq \delta \}$ for any $\delta > 0$. We then choose $\alpha$ as the term inside the curly brackets in (12) to conclude the result.

Regarding Lemma 2, we first prove the lower bound. We observe that under the assumption of $B^*$ having constant non-zero singular values, $\|B^*(x_i - x_j)\|_2^2 \sim 2b_i^2 \chi^2(r)$, where $\chi^2(\nu)$ denotes the Chi-Square distribution with $\nu \in \{1, 2, \ldots\}$ degrees of freedom. It is easy to verify that for $r = 2(q+1)$, $q \in \{0, 1, \ldots\}$,

$$\mathbb{P}(\chi^2(r) \leq z) = 1 - \exp(-z/2) \sum_{s=0}^{q} \frac{(z/2)^s}{s!}, \quad z \geq 0. \quad (31)$$
Combining (31) with a union bound over pairs \( i < j \), we obtain

\[
\mathbf{P}\left( \min_{i<j} \|B^*\mathbf{x}_i - x_j\|_2^2 \leq 2b_*^2 z \right) \leq \left( \frac{n}{2} \right) \left( 1 - \exp\left( -\frac{z}{2} \sum_{s=0}^{q} \frac{(z/2)^s}{s!} \right) \right) \tag{32}
\]

Below, \( z \) is chosen s.t. the r.h.s. of the above inequality is upper bounded by \( \delta \). We have

\[
\left( \frac{n}{2} \right) \left( 1 - \exp\left( -\frac{z}{2} \sum_{s=0}^{q} \frac{(z/2)^s}{s!} \right) \right) = \left( \frac{n}{2} \right) \left( \exp\left( -\frac{z}{2} \sum_{s=0}^{\infty} \frac{(z/2)^s}{s!} \right) \right) \leq \left( \frac{n}{2} \right) \left( \frac{z}{2} \right)^q \left( q + 1 \right)^q \tag{33}
\]

where the inequality follows from a Taylor expansion with Lagrange form of the remainder:

\[
\exp\left( \frac{z}{2} \right) = \sum_{s=0}^{q} \frac{(z/2)^s}{s!} + \exp(\xi) \frac{(z/2)^{q+1}}{(q+1)!} \text{ for some } \xi \in \left[ 0,\frac{z}{2} \right]
\]

\[
\Rightarrow \exp\left( \frac{z}{2} \right) - \sum_{s=0}^{q} \frac{(z/2)^s}{s!} = \sum_{s=0}^{\infty} \frac{(z/2)^s}{s!} \leq \frac{\exp(\xi)}{(q+1)!} (z/2)^{q+1} \leq \exp\left( \frac{z}{2} \right) \left( \frac{z}{2} \right)^q \left( q + 1 \right)^q.
\]

Using that \( \frac{1}{(q+1)!} \leq \left( (q+1)/e \right)^{-(q+1)} \), (33) can be upper bounded as

\[
\left( \frac{n}{2} \right) \exp\left( -(q+1) \log\left( \frac{2(q+1)}{z \cdot e} \right) \right) \leq \frac{n^2}{2} \left( \frac{2(q+1)}{z \cdot e} \right)^{-(q+1)}
\]

Choosing \( z = \frac{2}{e} (q+1) \cdot (n^{-2}\delta)^{1/(q+1)} \) ensures that the probability in (32) is bounded by \( \frac{\delta}{2} \).

We turn to the upper bound in Lemma 2. Let \( n_2 = \left\lfloor \frac{n}{3} \right\rfloor \). We first use that for any \( z \geq 0 \)

\[
\mathbf{P}\left( \min_{i<j} \|B^\top(x_i - x_j)\|_2^2 < z \right) \geq \mathbf{P}\left( \min_{1 \leq i \leq n/2} \|B^*\mathbf{x}_{2i} - x_{2i-1}\|_2^2 < z \right) = 1 - \mathbf{P}(\chi^2(r) > z/2b_*^2)^{n_2}, \tag{34}
\]

where we have used that \( \{\|B^\top(x_{2i} - x_{2i-1})\|_2^2\}_{i=1}^{n_2} \) i.i.d. \( \sim 2b_*^2 \chi^2(r) \). Using (31) and setting \( z = c \cdot 4b_*^2 \) in (34) for \( c > 0 \) to be determined below, we obtain that

\[
\mathbf{P}\left( \min_{i<j} \|B^\top(x_i - x_j)\|_2^2 < z \right) \geq 1 - \left( \sum_{s=0}^{q} \frac{c^s}{s!} \exp(-c) \right)^{n_2} = 1 - \left( 1 - \sum_{s=q+1}^{\infty} \frac{c^s}{s!} \exp(-c) \right)^{n_2} \geq 1 - \left( 1 - \frac{c^{q+1}}{(q+1)!} \exp(-c) \right)^{n_2} \tag{35}
\]

Choosing \( c = \theta^{1/(q+1)} n^{-1/(q+1)} (q+1) \) and using that \( (q+1)! < (q+1)^{q+1} \), we obtain the following lower bound on (35)

\[
1 - \left( \left( 1 - \frac{\theta}{n} \exp(-c) \right)^n \right)^{1/2} \geq 1 - \exp\left( -\frac{(z/2)^q}{\exp(-c)} \right)
\]

as long as \( n \geq \theta \). Setting \( \theta = 8 \), the above probability is lower bounded by 0.75 if \( n > 8(q+1)^{q+1} \). Combining this with the choice of \( z = c \cdot 4b_*^2 \) in (34) yields the assertion.
C Proof of Theorem 2

We first show that $\hat{\Theta}(\hat{B})_{i:} = \Theta_{i:}^* = 0$ for $i \in \mathcal{N} = \{1 \leq i \leq n : \theta^*(i) = 0\}$. We have

$$\bigcap_{i \in \mathcal{N} \cap 1 \leq j \leq n} \left\{ \|y_i\|_2^2 \leq \|y_i - \hat{B}^\top x_j\|_2^2 \right\} \supseteq \bigcap_{i \in \mathcal{N} \cap 1 \leq j \leq n} \left\{ 2\langle y_i, \hat{B}^\top x_j \rangle \leq \|\hat{B}^\top x_j\|_2 \right\}$$

$$\supseteq \bigcap_{i \in \mathcal{N} \cap 1 \leq j \leq n} \left\{ \sqrt{2}\|y_i\|_2 \leq \|\hat{B}^\top x_j\|_2 \right\}$$

$$\supseteq \bigcap_{i \in \mathcal{N} \cap 1 \leq j \leq n} \left\{ \sqrt{2}\|y_i\|_2 + \|(\hat{B} - B^*)^\top x_j\|_2 \leq \|B^*^\top x_j\|_2 \right\}$$

$$\supseteq \left\{ \frac{\sqrt{2}\max_{i \in \mathcal{N}}\|y_i\|_2}{\min_{1 \leq j \leq n}\|B^*^\top x_j\|_2} + \frac{\|(\hat{B} - B^*)^\top x_j\|_2}{\min_{1 \leq j \leq n}\|B^*^\top x_j\|_2} \leq 1 \right\} \tag{36}$$

Now observe that

$$\min_j \|B^*^\top x_j\|_2 = \gamma_0\|B^*\|_F = \gamma_0\sqrt{m}\text{SNR}^{1/2}. \tag{37}$$

Under the conditions of the Theorem, the left hand side of (36) is upper bounded by $(1 + \sqrt{2})/6 < 1$, hence that event holds.

Next, we show that $\hat{\Theta}(\hat{B})_{i:} \neq 0$ if $i \in \mathcal{N}^c$. This leads to

$$\bigcap_{i \in \mathcal{N}^c} \left\{ \|y_i - \hat{B}^\top x_{\theta^*(i)}\|_2 \leq \|y_i\|_2^2 \right\} = \bigcap_{i \in \mathcal{N}^c} \left\{ \|\hat{B}^\top x_{\theta^*(i)}\|_2^2 \leq 2\langle \hat{B}^\top x_{\theta^*(i)}, y_i \rangle \right\}$$

$$= \bigcap_{i \in \mathcal{N}^c} \left\{ \|\hat{B}^\top x_{\theta^*(i)}\|_2 \leq 2\langle \hat{B}^\top x_{\theta^*(i)}, (B^* - \hat{B})^\top x_{\theta^*(i)} + \hat{B}^\top x_{\theta^*(i)} + \sigma\epsilon_{\theta^*(i)} \rangle \right\}$$

$$= \bigcap_{i \in \mathcal{N}^c} \left\{ \|\hat{B}^\top x_{\theta^*(i)}\|_2 \leq 2\langle \hat{B}^\top x_{\theta^*(i)}, \hat{B}^\top x_{\theta^*(i)} \rangle \right\}$$

$$= \bigcap_{i \in \mathcal{N}^c} \left\{ \|\hat{B}^\top x_{\theta^*(i)}\|_2 \leq \|\hat{B}^\top x_{\theta^*(i)}\|_2 \right\} \tag{38}$$

Consider the event

$$\left\{ \sigma \max_{1 \leq i \leq n} \|\epsilon_i\|_2 \leq \sigma\sqrt{m} + 2\sqrt{\log n} \right\}. \tag{39}$$

By Lemma E.2, event (39) holds with probability at least $1 - 1/n$. Arguing similarly as for (36),(37), we conclude that under the conditions of the theorem, the left hand side in (38) is upper bounded by $5/6$ with the stated probability.

Finally, we show that for $i \in \mathcal{N}^c$, it holds that $\hat{\Theta}(\hat{B})_{i\theta^*(i)} = 1$ which then in conjunction
with the two previous results implies that $\widehat{\Theta}(\widehat{B}) = \Theta^*$. For this purpose, we consider

$$\bigcap_{i \in \mathcal{N}_c} \bigcap_{1 \leq j \leq n, j \neq \theta^*(i)} \left\{ \|y_i - \widehat{B}^T x_{\theta^*(i)}\|_2^2 \leq \|y_i - \widehat{B}^T x_{j}\|_2^2 \right\}$$

$$= \bigcap_{i \in \mathcal{N}_c} \bigcap_{1 \leq j \leq n, j \neq \theta^*(i)} \left\{ \|B^* - \widehat{B}\|^T x_{\theta^*(i)}\|_2^2 \leq \|B^* x_{\theta^*(i)} - \widehat{B}^T x_{j} + \sigma \epsilon_{\theta^*(i)}\|_2^2 \right\}$$

$$= \bigcap_{i \in \mathcal{N}_c} \bigcap_{1 \leq j \leq n, j \neq \theta^*(i)} \left\{ \|B^* - \widehat{B}\|^T x_{\theta^*(i)}\|_2^2 \leq \|B^* x_{\theta^*(i)} - \widehat{B}^T x_{j} + \sigma \epsilon_{\theta^*(i)}\|_2^2 \right\}$$

$$= \bigcap_{i \in \mathcal{N}_c} \bigcap_{1 \leq j \leq n, j \neq \theta^*(i)} \left\{ \|B^* - \widehat{B}\|^T x_{\theta^*(i)}\|_2^2 \leq \|B^* x_{\theta^*(i)} - \widehat{B}^T x_{j} + \sigma \epsilon_{\theta^*(i)}\|_2^2 \right\}$$

$$\geq \bigcap_{i \in \mathcal{N}_c} \bigcap_{1 \leq j \leq n, j \neq \theta^*(i)} \left\{ \frac{\|B^* - \widehat{B}\|^T x_{\theta^*(i)}\|_2^2}{\|B^* x_{\theta^*(i)} - \widehat{B}^T x_{j}\|_2^2} \right\}$$

$$\geq \left\{ \frac{\|B^* - \widehat{B}\|^T x_{\theta^*(i)}\|_2^2}{\|B^* x_{\theta^*(i)} - \widehat{B}^T x_{j}\|_2^2} \right\}$$

$$\geq \left\{ \frac{\|B^* - \widehat{B}\|^T x_{\theta^*(i)}\|_2^2}{\|B^* x_{\theta^*(i)} - \widehat{B}^T x_{j}\|_2^2} \right\}$$

Similarly to (37), we have that

$$\min_{i \neq j} \|B^* x_{j} - x_{i}\|_2 = \gamma \|B^*\|_F = \gamma \sigma \sqrt{m \text{SNR}}^{1/2}. \quad (41)$$

Plugging (41) into (40) and (39), it is easy to verify that under the conditions of the theorem the left hand side of the event in (41) is upper bounded by $1/36 + 1/3 + 1/3 + 1/9 < 1$ with the stated probability.

We now turn to the converse statement in the regime $m = O(1)$ (second bullet); the converse statement without restriction on $m$ is given subsequently. Define $i^* \in \{1, \ldots, n\}$ by

$$\min_{1 \leq i \leq n} \|B^* x_i\|_2 = \|B^* x_{i^*}\|_2 = \gamma \|B^*\|_F. \quad (39)$$

Note that the condition $\Theta(B^*)_{i^*} \neq 0$ for $1 \leq i \leq n$ becomes

$$\bigcap_{1 \leq i \leq n} \left\{ -2 \langle \sigma \epsilon_{\theta^*(i)}, B^* x_{\theta^*(i)} \rangle \leq \|B^* x_{\theta^*(i)}\|_2^2 \right\} \leq \left\{ -2 \langle \sigma \epsilon_{i^*}, B^* x_{i^*} \rangle \leq \|B^* x_{i^*}\|_2^2 \right\}$$

$$= \left\{ -2 \langle \epsilon_{i^*}, \frac{B^* x_{i^*}}{\|B^* x_{i^*}\|_2} \rangle \leq \gamma \|B^*\|_F = \sqrt{\gamma \sigma \sqrt{m \text{SNR}}}^{1/2} \right\} \quad (42)$$

The above inequalities hold for $1 \leq i \leq n$.
Note that conditional on \(x_i\), the left hand side follows a \(N(0, 4\sigma^2)\)-distribution. It is easy to show that if \(g \sim N(0, 1)\), \(P(|g| \leq \delta) \leq \delta\) and thus \(P(g > \delta) \geq \frac{1}{2}(1 - \delta)\) for all \(\delta > 0\). Hence if
\[
\gamma_0 \text{SNR}^{1/2} < \frac{2}{3 \sqrt{m}} \iff \gamma_0^2 \text{SNR} < \frac{4}{9m} =: c, \tag{43}
\]
\(\hat{\Theta}(B^\ast) \neq \Theta^\ast\) with probability at least 1/3. Similarly, let \((i_0, j_0)\) denote the pair of indices such that \(\min_{i < j} \|B^\ast^T(x_i - x_j)\|_2^2 = \gamma^2 \|B^\ast\|_{F}^2\). Supposing for simplicity that \(i_0^\ast = \theta^{-1}(i_0) \neq \emptyset\), for the event \(\{\hat{\Theta}(B^\ast) = \Theta^\ast\}\) to hold it is required that
\[
\|y_{i_0} - B^\ast^T x_{i_0}\|_2^2 \leq \|y_{j_0} - B^\ast^T x_{j_0}\|_2^2 \iff 2\langle \sigma e_{i_0}, B^\ast^T (x_{j_0} - x_{i_0}) \rangle \leq \|B^\ast^T (x_{i_0} - x_{j_0})\|_2^2
\]
With an argument parallel to (42), (43), one shows that if \(\gamma^2 \text{SNR} < \frac{4}{9m}\), \(\hat{\Theta}(B^\ast) \neq \Theta^\ast\) with probability at least 1/3.

Regarding the converse statement without restriction on \(m\) (first bullet), it follows by first conditioning on \(\{x_i\}_{i=1}^n\) and then using standard concentration arguments for the maximum of a collection of Gaussian random variables (cf. [28], p. 79)
\[
P \left( \max_{1 \leq i \leq n} 2\langle \sigma e_i, B^\ast^T x_i / \|B^\ast^T x_i\|_2 \rangle < 2c_0 \sqrt{\log n} \right) \leq 2/5. \tag{44}
\]
for a constant \(c_0 > 0\). At the same time, concentration of Lipschitz functions of Gaussian random variables yields
\[
P(\|B^\ast^T x_i\|_2^2 \geq (1 + t)^2 \|B^\ast\|_{F}^2) \leq \exp \left( -\frac{t^2 \|B^\ast\|_{F}^2}{2 \|B^\ast^T x_i\|_2^2} \right) \leq \exp \left( -\frac{t^2}{2} \right), \ t \geq 0, 1 \leq i \leq n. \tag{45}
\]
Let \(i_{\text{max}}\) be the index such that
\[
\langle e_{i_{\text{max}}}, B^\ast^T x_{i_{\text{max}}} / \|B^\ast^T x_{i_{\text{max}}}\|_2 \rangle = \max_{1 \leq i \leq n} \langle e_i, B^\ast^T x_i / \|B^\ast^T x_i\|_2 \rangle
\]
Since \(\langle e_i, B^\ast^T x_i / \|B^\ast^T x_i\|_2 \rangle\), \(\|B^\ast^T x_i\|_2^2\) are pairs of independent random variables, \(1 \leq i \leq n\), we combine (44) and (45) to conclude that the event
\[
\left\{ 2\sigma \langle e_{i_{\text{max}}}, B^\ast^T x_{i_{\text{max}}} / \|B^\ast^T x_{i_{\text{max}}}\|_2 \rangle > 2c_0 \sigma \sqrt{\log n} \right\} \cap \{ \|B^\ast^T x_{i_{\text{max}}}\|_2 \leq 3\|B^\ast\|_F \}
\]
occurs with probability at least 1/3. Combining (42) and the previous display then yields that \(\hat{\Theta}(B^\ast) \neq \Theta^\ast\) with the stated probability if
\[
\text{SNR} < \frac{4}{9c_0^2 \log n} =: c_1 \log n. \tag{46}
\]

D Proof of Proposition 1

By the triangle inequality and the fact that \(\hat{\Xi}_{i_i}^\ast = 0\) for all \(i \in S^\ast\), we have
\[
\min_{i \in S^\ast} \|\hat{\Xi}_{i_i} - \Xi_{i_i}\|_2 \geq \min_{i \in S^\ast} \|\hat{\Xi}_{i_i} - \Xi_{i_i}\|_2 - 2 \max_{1 \leq i \leq n} \|\hat{\Xi}_{i_i} - \Xi_{i_i}\|_2
\]
\[
\geq \min_{i \in S^\ast} \|\hat{\Xi}_{i_i} - \Xi_{i_i}\|_2 - 2\|\hat{\Xi} - \Xi^\ast\|_F. \tag{46}
\]
In the sequel, we derive a lower bound on \(\min_{i \in S} \|\Xi_{i}^\ast\|_2\) in a fashion similar to the previous proof. For \(i\) with \(\theta^\ast(i) = 0\), we have
\[
\|\Xi_{i}^\ast\|_2 = \|y_i - B^\ast x_i\|_2 \geq \gamma_0 \|B^\ast\|_F - \|y_i\|_2 = \gamma_0 \cdot \sigma \sqrt{\text{SNR}} \sqrt{m} - \|y_i\|_2. \tag{47}
\]
On the other hand, for \(i\) with \(\theta^\ast(i) \notin \{0, i\}\), we have
\[
\|\Xi_{i}^\ast\|_2 = \|B^\ast x_{\theta^\ast(i)} - B^\ast x_i\|_2 \geq \gamma \|B^\ast\|_F = \gamma \cdot \sigma \sqrt{\text{SNR}} \sqrt{m} \tag{48}
\]
Combining (46), (47) and (48) yields the assertion.

### E Auxiliary Results

**Lemma E.1.** For any \(r \geq 1\), we have the inclusion
\[
\{ v \in \mathbb{R}^{n \cdot m} : \|v\|_2 \leq 1, \|v\|_{2,1} \leq \sqrt{r} \} \subset 2 \text{conv} B_0(r), \tag{49}
\]
with \(\|\cdot\|_{2,1}\) and \(B_0(r)\) are defined in (20) and Lemma A.2, respectively.

**Proof.** The proof is an adaptation of a standard argument in the sparsity literature, cf. Lemma 3.1 in [35]. Pick an arbitrary element \(v\) contained in the left hand side in (49), and consider subsets \(T_\ell \subset \{1, \ldots, n\}\), \(|T_\ell| \leq r\), and corresponding vectors \(v(T_\ell) \in B_0(r)\) such that
\[
(v(T_\ell))_j = \begin{cases} v_j & \text{if } j \in \bigcup_{i \in T_\ell} G_i, \\ 0 & \text{else}. \end{cases}
\]
and such that \(T_1\) contains the \(r\) indices of \(\{1, \ldots, n\}\) corresponding to the \(r\) largest norms among \(\{\|v^{[i]}\|_2\}_{i=1}^{n}\), \(T_2\) contains the \(r\) indices corresponding to the next \(r\) largest norms among \(\{\|v^{[i]}\|_2\}_{i=1}^{n}\), and so forth. Observe that \(v = \sum_\ell v(T_\ell)\) and that for any \(\ell\)
\[
\|v(T_{\ell+1})\|_{2,\infty} = \max_{i \in T_{\ell+1}} \|v^{[i]}\|_2 \leq \frac{1}{r} \sum_{i \in T_\ell} \|v^{[i]}\|_2 = \frac{1}{r} \|v(T_\ell)\|_{2,1}
\]
As a result,
\[
\|v(T_{\ell+1})\|_2 \leq \sqrt{r} \|v(T_{\ell+1})\|_{2,\infty} = \frac{1}{\sqrt{r}} \|v(T_\ell)\|_{2,1}.
\]
Consequently,
\[
\sum_\ell \|v(T_\ell)\|_2 = \|v(T_1)\|_2 + \sum_{\ell \geq 2} \|v(T_\ell)\|_2 \\
\leq 1 + \frac{1}{\sqrt{r}} \sum_{\ell \geq 1} \|v(T_\ell)\|_{2,1} \\
\leq 1 + \frac{1}{\sqrt{r}} \sum_{\ell \geq 1} \sum_{i \in T_\ell} \|v_{G_i}\|_2 \\
\leq 1 + \frac{1}{\sqrt{r}} \|v\|_{2,1} \leq 2.
\]
In conclusion, we have demonstrated that
\[
v = \sum_{\ell} \frac{v(T_\ell)}{\|v(T_\ell)\|_2} \|v(T_\ell)\|_2, \quad \sum_{\ell} \lambda_\ell \leq 2,
\]
and thus \(v \in 2 \text{conv} B_0(r)\). Since \(v\) was an arbitrary element of the left hand side in (49), the proof is complete. \(\Box\)
Lemma E.2. Let $g_{\ell} \sim N(0, \sigma_{\ell}^2 I_r)$, \(\ell = 1, \ldots, L\), be isotropic Gaussian random vectors. Then:

$$P\left( \max_{1 \leq \ell \leq L} \|g_{\ell}\|_2 > \max_{1 \leq \ell \leq L} \sigma_{\ell} \left( \sqrt{r} + 2 \sqrt{\log L} \right) \right) \leq 1/L.$$  

Proof. We note that $E[\|g_\ell\|] \leq \sigma_\ell \sqrt{r}, \, \ell = 1, \ldots, L$, and that the map $x \mapsto \|x\|_2$ is 1-Lipschitz. By concentration of measure of Lipschitz functions of Gaussian random vectors, we hence have

$$P(\|g_\ell\|_2 \geq \sigma_\ell(\sqrt{r} + 2\sqrt{\log L})) \leq \exp(-2 \log L), \, \ell = 1, \ldots, L.$$  

The result then follows from a union bound over $\{1, \ldots, L\}$. □

Lemma E.3. (Gordon’s Escape through a Mesh theorem [20]) Let $K$ be a closed subset of the unit sphere in $\mathbb{R}^p$, let $\nu_r = E_{g \sim N(0,I_r)}[\|g\|_2]$, and let $\varepsilon > 0$. If the Gaussian width (cf. §7.5 in [53]) of $K$ obeys $w(K) < (1-\varepsilon)\nu_q - \varepsilon \nu_p$, then a $(p-q)$-dimensional subspace $V$ drawn uniformly from the Grassmannian $G(p,p-q)$ satisfies

$$P(\text{dist}(K, V) > \varepsilon) \geq 1 - \frac{7}{2} \exp\left( -\frac{1}{2} \left( \frac{(1-\varepsilon)\nu_q - \varepsilon \nu_p - w(K)}{3 + \varepsilon + \varepsilon \nu_p/\nu_q} \right)^2 \right).$$

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