On total edge irregularity strength of polar grid graph

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ABSTRACT

For a graph $G$, an edge irregular total $r$-labelling $\pi : V \cup E \rightarrow \{1, 2, 3, \ldots, r\}$ is a labelling for edges and vertices of a graph $G$ in such a way that the weights of any two different edges are distinct. The minimum for which $G$ admits an edge irregular total $r$-labelling is called total edge irregularity strength of $G$, tes($G$). In this paper, the exact value of total edge irregularity strength of the polar grid graph was determined. We have also determined the total edge irregularity strength for a polar grid graph.

1. Introduction

Graph labelling is an extremely useful tool for making a lot of problems in different areas of human life very easy to be handled in a mathematical way. So, it is an important branch in graph theory and has a lot of applications in many fields, for instance, coding theory, astronomy, communication network and optimal circuit layouts.

A labelling of a simple, connected and undirected graph $G(V, E)$ that is defined as a function that assigns some set of elements of a graph $G$ with a set of positive or non-negative integer. According to a domain of a labelling, we have three types of it. The first one vertex labels if the domain is a vertex-set, the second is edge labelling when edge-set is the domain and finally, total labelling that its domain is the union of edge-set and vertex-set.

Bača et al. [1] defined an edge irregular total $r$-labelling of a graph $G$ as a labelling $\theta : V \cup E \rightarrow \{1, 2, 3, \ldots, r\}$ such that every two distinct edges $lm$ and $mm^*$ of a graph $G$ have distinct weights, i.e. $w_\theta (pq) \neq w_\theta (p^*q^*)$. If a graph $G$ admits an edge irregular total $r$-labelling and $r$ is minimum then we say that $G$ has a total edge irregularity strength denoted by tes($G$). Furthermore, in [1], for any graph $G$, a lower bound of tes($G$) is given by

$$\text{tes}(G) \geq \max \left\{ \frac{|E(G)| + 2}{3}, \frac{\Delta G + 1}{2} \right\} \quad (1)$$

Since then, many authors try to find exact values for the total edge irregularity strength of graphs. Ivančo et al. [2] proved that tes($T$) is equal its lower bound where $T$ is an any tree. In [3,4] authors determined the exact value of total edge irregularity strength for complete bipartite graph, complete graph and the corona of the path to path, a star and cycle. Ahmad et al. [5] determined tes($G$) that $G$ is generalized helm graph $H_{n,m}$ with $n \geq 3$, $m \geq 4$. On the other hand, in [6–26] the total edge irregularity strengths for fan graph, wheel graph, triangular Book graph, friendship graph, centralized uniform theta graphs and large graphs are investigated. For definitions, applications and terminology are not mentioned in our paper, see [5,8–26].

In this paper, the exact value of total edge irregularity strength of the polar grid graph was determined.

2. Main results

In this section, we determined the total edge irregularity strength of a polar grid graph $P_{3,n}$, $n \geq 3$, and a polar grid graph $P_{m,3}, m \geq 3$. Finally, the exact value of the total edge irregularity strength of a generalized polar grid graph $P_{m,n}$ was determined.

Theorem 2.1: Let $P_{3,n}$ be a polar grid graph with $3n + 1$ vertices and $6n$ edges, $n \geq 3$. Then

$$\text{tes}(P_{3,n}) = 2n + 1$$

Proof: Since $|V(P_{3,n})| = 3n + 1, |E(P_{3,n})| = 6n$. Then by (1) we have

$$\text{tes}(P_{3,n}) \geq 2n + 1.$$  

For the inverse inequality, we sufficient to show the existence of an edge irregular total labelling with...
\( r = 2n + 1 \). Let \( r = 2n + 1 \) and \( \pi : V \cup E \to \{1, 2, 3, \ldots, r\} \) is a total \( r \)-labelling defined as

\[
\pi(v_0) = 1,
\]

\[
\pi(v_i) = \begin{cases} 
  i & \text{for } 1 \leq i \leq 2n \\
  r & \text{for } 2n + 1 \leq i \leq 3n' 
\end{cases},
\]

\[
\pi(v_0v_i) = i \quad \text{for } 1 \leq i \leq n,
\]

\[
\pi(v_i v_{i+n}) = \begin{cases} 
  n + 1 & \text{for } 1 \leq i \leq n \\
  n + 1 + 2(i + 1 - r) & \text{for } 2n + 1 \leq i \leq 3n - 1 \n\end{cases},
\]

\[
\pi(v_{i+1}) = \begin{cases} 
  1 & \text{for } 1 \leq i < 2n + 1, \\
  n + 1 & \text{for } i \not\in [n, 2n] \\
  2(i + 1 - r) & \text{for } 2n + 1 \leq i \leq 3n - 1 - 1
\end{cases},
\]

\[
\pi(v_0v_1) = n + 1,
\]

\[
\pi(v_{2n}v_{n+1}) = n + 1
\]

\[
\pi(v_{3n}v_{2n+1}) = 2n.
\]

It is clear that the weights for any two different edges in \( P_{3,n} \) are distinct (Figure 1). Therefore, \( \pi \) is an edge irregular total \( r \)-labelling \( f \) \( P_{3,n} \). Hence

\[
tes(P_{3,n}) = 2n + 1
\]

**Theorem 2.2**: Let \( m \geq 3 \) and \( P_{m,3} \) be a polar grid graph with \( 3m + 1 \) vertices and \( 6m \) edges. Then

\[
tes(P_{m,3}) = 2m + 1
\]

**Proof**: Since \( |V(P_{m,3})| = 3m + 1 \), \( |E(P_{m,3})| = 6m \). Then from (1) we have

\[
tes(P_{m,3}) \geq 2m + 1
\]

To prove the invers inequality, we need to show that there exist an edge irregular total \( r \)-labelling, \( r = 2m + 1 \), for a graph \( P_{m,3} \). Suppose that \( r = 2m + 1 \) and \( \pi : V \cup E \to \{1, 2, 3, \ldots, r\} \) is a total \( r \)-labelling defined as

\[
\pi(v_0) = 1,
\]

\[
\pi(v_i) = \begin{cases} 
  i & \text{for } 1 \leq i \leq 2 \n' \\
  r & \text{for } 2n + 1 \leq i \leq 3n' 
\end{cases},
\]

\[
\pi(v_0v_i) = i \quad \text{for } 1 \leq i \leq n,
\]

\[
\pi(v_i v_{i+n}) = \begin{cases} 
  n + 1 & \text{for } 1 \leq i \leq n \\
  n + 1 + 2(i + 1 - r) & \text{for } 2n + 1 \leq i \leq 3n - 1 - 1
\end{cases},
\]

\[
\pi(v_{i+1}) = \begin{cases} 
  1 & \text{for } 1 \leq i < 2n + 1, \\
  n + 1 & \text{for } i \not\in [n, 2n] \\
  2(i + 1 - r) & \text{for } 2n + 1 \leq i \leq 3n - 1 - 1
\end{cases},
\]

\[
\pi(v_0v_1) = n + 1,
\]

\[
\pi(v_{2n}v_{n+1}) = n + 1
\]

\[
\pi(v_{3n}v_{2n+1}) = 2n.
\]

**Proof**: Since \( |V(P_{m,3})| = 3m + 1 \), \( |E(P_{m,3})| = 6m \). Then from (1) we have

\[
tes(P_{m,3}) \geq 2m + 1
\]

To prove the invers inequality, we need to show that there exist an edge irregular total \( r \)-labelling, \( r = 2m + 1 \), for a graph \( P_{m,3} \). Suppose that \( r = 2m + 1 \) and \( \pi : \)
\[ w_\pi(v_{i-2}) = 2i + 2 \quad \text{for } i \in \{3, 6, 9, \ldots, 3m\}, \]

It is easy to check that the edge-weights of edges are pairwise distinct. So,

\[ \text{tes}(P_{m, 3}) = 2m + 1. \]

Case 2: \( m \equiv 1 \pmod{3} \).
Define \( \pi \) as follows:

\[ \pi(v_0) = 1, \]

\[ \pi(v_i) = \begin{cases} i & \text{for } 1 \leq i \leq 2m, \\ r & \text{for } 2m + 1 \leq i \leq 3m, \end{cases} \]

\[ \pi(v_0v_i) = i \quad \text{for } 1 \leq i \leq 3, \]

\[ \pi(v_{i-3}v_i) = \begin{cases} 4 & \text{for } 4 \leq i \leq 2m + 1, \\ i - r + 4 & \text{for } 2m + 2 \leq i \leq 2m + 4, \\ 2(i - r) + 1 & \text{for } 2m + 5 \leq i \leq 3m, \end{cases} \]

\[ \pi(v_iv_{i+1}) = \begin{cases} 1 & \text{for } 1 \leq i \leq 2m, \\ i \notin \{3, 6, \ldots, 2m - 2\} & \text{for } 2m + 2 \leq i \leq 3m - 1, \\ 2(i - r) + 2 & \text{for } i \notin \{2m + 4, 2m + 7, \ldots, 3m - 3\}, \end{cases} \]

\[ \pi(v_{i-2}v_i) = \begin{cases} 4 & \text{for } i \in \{3, 6, \ldots, 2m + 1\}, \\ 2(i - r) + 2 & \text{for } i \in \{2m + 4, 2m + 7, \ldots, 3m\}. \end{cases} \]

It is clear that the largest label is \( r \). The edge-weights are given by:

\[ w_\pi(v_0v_i) = 2i + 1 \quad \text{for } 1 \leq i \leq 3, \]

\[ w_\pi(v_{i-3}v_i) = 2i + 1 \quad \text{for } 4 \leq i \leq 3m, \]

\[ w_\pi(v_iv_{i+1}) = 2i + 2 \quad \text{for } 1 \leq i \leq 3m - 1, \\ i \notin \{3, 6, 9, \ldots, 3m - 3\}, \]

\[ w_\pi(v_{i-2}v_i) = 2i + 2 \quad \text{for } i \in \{3, 6, 9, \ldots, 3m\}. \]

It implies that the weights of edges are distinct. Then,

\[ \text{tes}(P_{m, 3}) = 2m + 1. \]

Case 3: \( m \equiv 2 \pmod{3} \).

\[ \pi \] is defined as follows:

\[ \pi(v_0) = 1, \]

\[ \pi(v_i) = \begin{cases} i & \text{for } 1 \leq i \leq 2m, \\ r & \text{for } 2m + 1 \leq i \leq 3m, \end{cases} \]

\[ \pi(v_0v_i) = i \quad \text{for } 1 \leq i \leq 3, \]

\[ \pi(v_{i-3}v_i) = \begin{cases} 4 & \text{for } 4 \leq i \leq 2m + 1, \\ i - r + 4 & \text{for } 2m + 2 \leq i \leq 2m + 4, \\ 2(i - r) + 1 & \text{for } 2m + 5 \leq i \leq 3m, \end{cases} \]

\[ \pi(v_iv_{i+1}) = \begin{cases} 1 & \text{for } 1 \leq i \leq 2m, \\ i \notin \{3, 6, \ldots, 2m - 1\} & \text{for } 2m + 1 \leq i \leq 3m - 1, \\ 2(i - r) + 2 & \text{for } i \notin \{2m + 2, 2m + 5, \ldots, 3m\}; \end{cases} \]

\[ \pi(v_{i-2}v_i) = \begin{cases} 4 & \text{for } i \in \{3, 6, \ldots, 2m - 1\}, \\ 5 & \text{for } i = 2m + 2, \\ 2(i - r) + 2 & \text{for } i \in \{2m + 5, 2m + 8, \ldots, 3m\}. \end{cases} \]

Obviously, the greatest label is \( r \). Also, the weights of the edges of the graph \( P_{m, 3} \) are given by:

\[ w_\pi(v_0v_i) = 2i + 1 \quad \text{for } 1 \leq i \leq 3, \]

\[ w_\pi(v_{i-3}v_i) = 2i + 1 \quad \text{for } 4 \leq i \leq 3m, \]

\[ w_\pi(v_iv_{i+1}) = \begin{cases} 2r + 5 & \text{for } i = 2m + 2, \\ 2i + 2 & \text{for } i \in \{3, 6, \ldots, 2m - 1, \}
 & \quad \{2m + 5, \ldots, 3m - 3\}. \end{cases} \]

From the above equations it is clear that the edge-weights are distinct. So \( \pi \) is an edge irregular total \( r \)-labelling, \( r = 2m + 1 \). Hence

\[ \text{tes}(P_{m, 3}) = 2m + 1. \]

**Theorem 2.3:** If \( P_{m,n} \) is a polar grid graph with \( nm + 1 \) vertices and \( 2mn \) edges, \( m \geq 3, n \geq 3 \). Then

\[ \text{tes}(P_{m,n}) = \frac{2mn + 2}{3}. \]

**Proof:** Since \( |V(P_{m,n})| = nm + 1, |E(P_{m,n})| = 2mn \), then by (1)

\[ \text{tes}(P_{m,n}) \geq \frac{2mn + 2}{3}. \]

To prove the inverse of previous inequality, it is necessary to show that there exists an edge irregular total \( r \)-labelling for \( P_{m,n} \) with \( r = \frac{2mn + 2}{3} \) as follows:
Let \( r = \frac{2mn+2}{3} \) and a total \( r \)-labelling \( \eta : V \cup E \rightarrow \{1, 2, 3, \ldots, r\} \) is defined as:

\[
\eta(v_0) = 1,
\]

\[
\eta(v_i) = \begin{cases} 
  i & \text{for } 1 \leq i \leq \frac{2mn+2}{3} \\
  r & \text{for } \frac{2mn+2}{3} + 1 \leq i \leq mn 
\end{cases},
\]

\[
\eta(v_0 v_i) = \begin{cases} 
  n + 1 & \text{for } n + 1 \leq i \leq \frac{2mn+2}{3} \\
  i - r + n + 1 & \text{for } \frac{2mn+2}{3} + 1 \leq i \leq \frac{2mn+2}{3} + n - 1 \\
  2(i-r) + 1 & \text{for } \frac{2mn+2}{3} + n \leq i \leq mn 
\end{cases}
\]

\[
\eta(v_{i-n} v_i) = \begin{cases} 
  n + 1 & \text{for } i \in \{n, 2n, 3n, \ldots, k\}, k < r - 1 \\
  mn & \text{for } i = k + sn, s \in \{1, 2, \ldots, l\}, k \geq mn \\
  s \frac{m}{3} & \text{for } k + ln = mn 
\end{cases}
\]

Obiously, the greatest label is \( r = \frac{2mn+2}{3} \). The weights of the edges of \( P_{m,n} \) expressed as:

\[
w_\eta(v_0 v_i) = 2i + 1 \text{ for } 1 \leq i \leq n,
\]

\[
w_\eta(v_{i-n} v_i) = 2i + 1 \text{ for } n + 1 \leq i \leq mn,
\]

\[
w_\eta(v_{i-n} v_i) = \begin{cases} 
  2i + 2 & \text{for } i \in \{n, 2n, \ldots, k\}, k < r - 1 \\
  k - r + 1 & \text{for } i = k + sn, s \in \{1, 2, \ldots, l\}, k \geq mn \\
  2r + s \frac{m}{3} & \text{for } k + ln = mn 
\end{cases}
\]

Upon checking, it was found that the weights of any two different edges are different. Hence \( \eta \) is an edge irregular total \( r \)-labelling for being \( r = \frac{2mn+2}{3} \), i.e.

\[
tes(P_{m,n}) \geq \frac{2mn+2}{3}.
\]

### 3. Conclusion

In this paper, we have determined the total edge irregularity strength for a polar grid graph \( P_{m,n} \), \( n \geq 3 \), and a polar grid graph \( P_{m,3}, m \geq 3 \). Finally, the exact value of total edge irregularity strength for a generalized polar grid graph \( P_{m,n} \) was determined.

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