Application of the Chimera Method to Poisson’s Equation with the Homogeneous Dirichlet Boundary Condition

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Abstract. Establishing variational formulation is an effective way to study the existence and uniqueness of the solution of certain elliptic partial differential equation with boundary condition. For the solution of certain elliptic partial differential equation with boundary condition, we know that the numerical solution obtained by the finite element method approximates the solution of this equation. Moreover, to avoid gridding overly complex domains, we can use the Chimera method to decompose the domain into several overlapping sub-domains. In this paper, we study Poisson’s equation with the homogeneous Dirichlet boundary condition. By analyzing the existence and uniqueness of the solution of the corresponding variational formulation, we know the existence and uniqueness of the solution of Poisson’s equation with the homogeneous Dirichlet boundary condition. We use the Chimera method and the finite element method to deal with Poisson’s equation with the homogeneous Dirichlet boundary condition by constructing two iterative sequences and analyzing their properties.

1. Introduction

Partial differential equations (PDE) are equations that contain unknown functions and their partial derivatives. For some specific partial differential equations with boundary conditions, we can analyze many properties of the solutions, such as the existence, uniqueness, boundedness, and regularity of the solutions. In preliminaries, we introduce Sobolev spaces and the variational formulation to conclude the existence and uniqueness of the solution in $H^1_0(\Omega)$ for Poisson’s equation with the homogeneous Dirichlet boundary condition. We can transform the original problem into a variational formulation problem using Green’s formula, then use Lax-Milgram theorem to analyze the existence and uniqueness of the solution of the variational formulation. Finally, the equivalence between the solution of the variational formulation and the solution of the original equation can be derived by regularity conclusions, which leads to the existence and uniqueness of the solution of the original equation in $H^1_0(\Omega)$ [1].

We are also interested in explicit forms of solutions to partial differential equations, such as Poisson, heat and wave equations under specific boundary conditions [2]. However, we are curious about the explicit form of the solution corresponding to the general boundary conditions. We learned that the finite element method (FEM) can be used to give a numerical solution to Poisson’s equation with the homogeneous Dirichlet boundary condition, so that the numerical solution can be used to approximate the solution of the equation. In Subsection 2.5, we introduce the approximate variational formulation. The solution of the approximate variational formulation can be obtained by computing the corresponding matrix. For Poisson’s equation with the homogeneous Dirichlet boundary condition (in a polyhedral domain), we have the conclusion that the solution of the variational formulation can be approximated by the solution of the approximate variational formulation in $H^1$ by the theory related to the finite element method. When the regularity of the solution of the variational form is better, we can also obtain the error estimate [3].
The Chimera method is used in fluid mechanics to avoid meshing overly complicated objects [4][5]. It solves partial differential equations in a domain by decomposing the domain into several overlapping sub-domains. This allows the meshes of each sub-domain to be independent of each other without the need for an overall mesh.

2. Some preliminaires
In this paper, \( \Omega \) is a domain of \( \mathbb{R}^N \) (bounded or unbounded), with boundary denoted \( \partial \Omega \). Sometimes we also assume that \( \Omega \) is a regular and bounded domain. A regular domain is roughly a domain whose boundary is a regular hypersurface (a manifold of dimension \( N - 1 \)). This domain is locally located on only one side of its boundary. We define the exterior normal to the boundary \( \partial \Omega \) to be the unit vector \( n = (n_i)_{1 \leq i \leq N} \) normal to any point on the tangent plane of \( \Omega \) and pointing to the exterior of \( \Omega \). In \( \Omega \subseteq \mathbb{R}^N \) we note \( dx \) the volume measure, or the Lebesgue measure of dimension \( N \). In \( \partial \Omega \) we denote \( ds \) the surface measure, or the Lebesgue measure of dimension \( N - 1 \) on the submanifold \( \partial \Omega \).

2.1 Poisson’s equation with the homogeneous dirichlet boundary condition and variational formulation
The equal signs in Poisson’s equation are regarded as the equal signs in the sense of almost everywhere, without further special explanation.

We consider Poisson’s equation with the homogeneous Dirichlet boundary condition:

\[
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1)

where \( \Omega \) is a regular and bounded domain in the space \( \mathbb{R}^N \), \( f \in L^2(\Omega) \). We wonder if (1) has a unique solution in \( H_0^1(\Omega) \). We assume that \( u \in H^2(\Omega) \) and then we multiply both sides of the first equation in (1) by \( v \in H_0^1(\Omega) \) and integrate them to get the following formulation:

\[
\text{Find } u \in H_0^1(\Omega) \text{ such that } \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx = \int_{\Omega} f(x)v(x)dx \text{ for all } v \in H_0^1(\Omega).
\]

(2)

Remark 2.1. Different elliptic partial differential equations with boundary conditions correspond to different variational formulations. To get the corresponding variational formulation, we have the following rough approach: we first find a Hilbert space \( (H_0^1(\Omega)) \) in this paper), multiply both sides of the partial differential equation by \( v \in V \) and integrate, then use Green’s formula to reduce the differential order of the integral equation by one order, and finally obtain the following formulation:

\[
\text{Find } u \in V \text{ such that } a(u, v) = L(v) \text{ for all } v \in V.
\]

(3)

For (2.2), we have:

\[
a(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx
\]

(4)

and

\[
L(v) = \int_\Omega f(x)v(x)dx.
\]

(5)

Theorem 2.2. (Lax-Milgram Theorem) Let \( V \) be a real Hilbert space, and

1. \( a(\cdot, \cdot) \) is a continuous bilinear form on \( V \), i.e., there exists \( M > 0 \) such that:

\[
|a(w, v)| \leq M ||w|| ||v|| \text{ for all } w, v \in V.
\]

(6)

2. \( a(\cdot, \cdot) \) is coercive(or elliptic), i.e., there exists \( \alpha > 0 \) such that:

\[
a(v, v) \geq \alpha ||v||^2 \text{ for all } v \in V.
\]

(7)

3. \( L(\cdot) \) is a continuous linear form on \( V \), i.e., there exists \( C > 0 \) such that:

\[
|L(v)| \leq C ||v|| \text{ for all } v \in V.
\]

(8)

then the variational formulation (2.3) has a unique solution in \( V \).
Using the Lax-Milgram theorem, we can analyze the existence and uniqueness of the solution of (2). Since $H^1_0(\Omega)$ is a Hilbert space, we only need to verify (6)-(8) for $a(u, v)$ and $L(v)$ in the form of (4), (5). From Cauchy-Schwarz inequality, we have:

1. For $u, v \in H^1_0(\Omega)$
   \[ |a(u, v)| = |\int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx| \leq \| \nabla u \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)} \leq \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)}. \]  

2. For $v \in H^1_0(\Omega)$
   \[ a(v, v) = \int_{\Omega} |\nabla v(x)|^2 \, dx \geq \frac{1}{2} \| v \|_{H^1_0(\Omega)}^2. \]  

3. For $v \in H^1_0(\Omega)$
   \[ |L(v)| = |\int_{\Omega} f(x)v(x) \, dx| \leq \| f \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} \leq C \| v \|_{H^1(\Omega)}. \]

Then we can deduce that there exists a unique solution in $H^1_0(\Omega)$ of (2). Since we have the existence and uniqueness of the solution in $H^1_0(\Omega)$ of (2), we want to show that the solution of (2) solves (1). We can assume that the solution of (2) satisfies $\Delta u = \nabla \cdot \nabla u$ exists in the weak sense and $\Delta u \in L^2(\Omega)$. Then by the definition of the weak divergence, (2) can be transformed into:

\[ \int_{\Omega} (\Delta u + f) \phi \, dx = 0 \quad \forall \phi \in C_c^\infty(\Omega). \]

Then by the density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$, we can deduce that

\[ -\Delta u = f \text{ almost everywhere in } \Omega. \]

### 2.2 Finite Element Method

The finite element method is a numerical method for the calculation of solutions of boundary problems. Given a Hilbert space $V$, a continuous and coercive bilinear form $a(u, v)$, and a continuous linear form $L(v)$, we consider the following variational formulation:

**Find** $u \in V$ such that $a(u, v) = L(v) \quad \forall v \in V$.  

The internal approximation consists of replacing the Hilbert space $V$ with a finite dimensional subspace $V_h$, i.e., in search of the solution of:

**Find** $u_h \in V_h$ such that $a(u_h, v_h) = L(v_h) \quad \forall v \in V$.  

Then we have the following lemma:

**Lemma 2.3.** Let $V$ be a real Hilbert space, and $V_h$ a finite dimensional subspace. Let $a(u, v)$ be a continuous and coercive bilinear form on $V$, and $L(v)$ a continuous linear form on $V$. Then the internal approximation (15) has a unique solution. Moreover, this solution can be obtained by solving a linear system of positive definite matrix (and symmetric if $a(u, v)$ is symmetric).

**Proof.** For finite dimensional space $V_h$, we have a finite basis $(\phi_i)_{1 \leq i \leq N_h}$. We write $u_h = \sum_{j=1}^{N_h} u_j \phi_j$, let $U_h = (u_1, \ldots, u_{N_h})$ be the vector in $\mathbb{R}^{N_h}$ of the coordinates of $u_h$. Then (2.27) is equivalent to

Find $U_h \in \mathbb{R}^{N_h}$ such that $a\left(\sum_{i=1}^{N_h} u_i \phi_j, \phi_i\right) = L(\phi_i) \quad \forall 1 \leq i \leq N_h$.

which is written as a linear system:

\[ K_h U_h = b_h \]

**Definition 2.4.** Given a mesh $\mathcal{T}_h$ of a connected and open polyhedral $\Omega$, the finite element method $P_k$, or triangular Lagrangian finite elements of order $k$, associated with this mesh is defined by the discrete space:

\[ V_h = \{ v \in C(\bar{\Omega}) \text{ such that } v|_{K_i} \in P_k \text{ for all } K_i \in \mathcal{T}_h \}. \]

We also define the subspace $V_{0h}$ by:

\[ V_{0h} = \{ v \in V_h \text{ such that } v = 0 \text{ on } \partial \Omega \}. \]

**Theorem 2.5.** Let $(\mathcal{T}_h)_{h>0}$ be a sequence of regular meshes of $\Omega$. Let $u \in H^1(\Omega)$ be the solution of the Poisson’s equation with the homogeneous Dirichlet boundary condition (13), and $u_h \in V_{0h}$ be the solution of its internal approximation (15) by the $P_k$ finite element method. Then the finite element method $P_k$ converges, i.e.
Moreover, if $u \in H^{k+1}(\Omega)$ and if $k + 1 > N/2$, then we have the error estimate
\begin{equation}
\lim_{h \to 0} \| u - u_h \|_{H^1(\Omega)} = 0.
\end{equation}

where $C$ is a constant independent of $h$ and $u$.

**Proposition 2.6** Let $(\mathcal{T}_h)_{h>0}$ be a sequence of regular meshes of $\Omega$. We assume that $k + 1 > N/2$. Then, for any $v \in H^{k+1}(\Omega)$, the interpolation $r_h v = \sum_{i=1}^{n_{di}} v(\bar{a}_i) \phi_i(x)$ is well defined, and there exists a constant $C$, independent of $h$ and $v$, such that:
\begin{equation}
\| v - r_h v \|_{H^1(\Omega)} \leq C h^k \| v \|_{H^{k+1}(\Omega)},
\end{equation}

where $(\phi_i)_{1 \leq i \leq n_{di}}$ is the basis for $V_{th}$ of the finite element method $\mathcal{P}_k$ (see definition 2.29).

3. **Application of the Chimera Method**

Chimera method is used in Fluid Mechanics to avoid meshing overly complicated objects. The Chimera method aims at solving partial differential equations in a domain $\Omega$ by decomposition into subdomains $\Omega_i$ with overlap. This allows each of the subdomains $\Omega_i$ to be meshed independently of each other without the need for a global mesh of $\Omega = \bigcup \Omega_i$. In this section, we will use the Chimera method to deal with the Poisson’s equation with the homogeneous Dirichlet boundary condition. To simplify the process, we divide the entire domain into two overlapping domains.

![Figure 1. $\Omega$ decomposed into two overlapping subdomains $\Omega_1$ and $\Omega_2$.](image)

Let $\Omega \subset \mathbb{R}^2$ be a regular and bounded domain. For a function $f \in L^2(\Omega)$, consider the following problem:

\begin{equation}
\text{Find } u \in H_0^1(\Omega) \text{ such that: } a(u, v) = \int_{\Omega} f(x)v(x)dx, \quad \forall v \in H_0^1(\Omega),
\end{equation}

where
\begin{equation}
a(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x)dx.
\end{equation}

It is assumed that $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1, \Omega_2$ two open sets such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. We note $\Gamma_i = \partial \Omega_i \setminus \partial \Omega$ for $i = 1, 2$, and it is assumed throughout the problem that $\Gamma_1 \cap \Gamma_2 = \emptyset$ (see Figure 1 for example).

### 3.1 Iterative Algorithm Analysis

In this subsection, we will construct two iterative sequences, then prove the weak limit of these sequences relate to the solution of the variational formulation (22). We expand the function $v_i \in H_0^1(\Omega_i)$ with $i = \{1, 2\}$. We define $\tilde{v}_i$ by:
\begin{equation}
\tilde{v}_i = \begin{cases} v_i & \text{almost everywhere in } \Omega_i, \\ 0 & \text{almost everywhere in } \Omega \setminus \Omega_i. \end{cases}
\end{equation}

In the sequel, when there is no ambiguity, we will continue to note $v_i$ the extension $\tilde{v}_i$.

Let $b(\cdot, \cdot)$ be a continuous coercive bilinear form on $L^2(\Omega)$. Now we consider the following iterative algorithm:

- Let $u_0^1 \in H_0^1(\Omega_1) \cap H^2(\Omega_1), u_0^2 \in H_0^1(\Omega_2) \cap H^2(\Omega_2)$;
- For $n \geq 0$, we assume $u^n_i \in H_0^1(\Omega_i)(i = 1, 2)$, then we solve the problems:
Find $u_{i}^{n+1} \in H_{0}^{1}(\Omega_{i}), i = \{1,2\}$, such that:

$$b(u_{i}^{n+1}, v_{1,i}) + a(u_{i}^{n+1}, u_{2,i}, v_{1,i}) = \int_{\Omega} f(x) v_{1,i}(x) dx, \quad \forall v_{1,i} \in H_{0}^{1}(\Omega_{i,2})$$

(23)

And we have the following conclusion for this iterative algorithm:

**Proposition 3.1.** The equations (23) define the sequences $(u_{1}^{n})_{n \geq 0}$ and $(u_{2}^{n})_{n \geq 0}$ uniquely.

**Proof.** We can rewrite equation (23) as follows:

Find $u_{i}^{n+1} \in H_{0}^{1}(\Omega_{i}), i = \{1,2\}$, such that:

$$b(u_{i}^{n+1}, v_{1,i}) + a(u_{i}^{n+1}, u_{2,i}, v_{1,i}) = \int_{\Omega} f(x) v_{1,i}(x) dx + b(u_{i}^{n+1}, v_{1,i}) - a(u_{2,i}^{n}, v_{1,i}), \quad \forall v_{1,i} \in H_{0}^{1}(\Omega_{i,2})$$

By noting:

$$c(u, v) = b(u, v) + a(u, v),$$

$$l_{i}^{n}(v) = \int_{\Omega} f(x) v(x) dx + b(u_{i}^{n}, v) - a(u_{2,i}^{n}, v)$$

this is the same as solving the problem:

Find $u_{i}^{n+1} \in H_{0}^{1}(\Omega_{i}), i = \{1,2\}$, such that:

$$c(u_{i}^{n+1}, v_{1}) = \ell_{i}^{n}(v_{1}), \quad \forall v_{1} \in H_{0}^{1}(\Omega_{i}).$$

$$c(u_{i}^{n+1}, v_{2}) = \ell_{i}^{n}(v_{2}), \quad \forall v_{2} \in H_{0}^{1}(\Omega_{i}).$$

For $i = \{1,2\}$ we will therefore apply the Lax-Milgram theorem to the Hilbert space $H_{0}^{1}(\Omega_{i})$ equipped with the $\|\cdot\|_{H_{0}^{1}(\Omega_{i})}$ norm.

- The bilinear form $c$ is continuous on $H_{0}^{1}(\Omega_{i})$: if $u_{i}, v_{i} \in H_{0}^{1}(\Omega_{i})$,

$$|c(u_{i}, v_{i})| \leq C_{b} \|u_{i}\|_{L^{2}(\Omega_{i})} \|v_{i}\|_{L^{2}(\Omega_{i})} + \|u_{i}\|_{H_{0}^{1}(\Omega_{i})} \|v_{i}\|_{H_{0}^{1}(\Omega_{i})} \leq (C_{b}C_{p}^{2} + 1) \|u_{i}\|_{H_{0}^{1}(\Omega_{i})} \|v_{i}\|_{H_{0}^{1}(\Omega_{i})},$$

where we used the continuity of the bilinear form $b$, the Cauchy-Schwarz inequality and Poincaré inequality.

- The bilinear form $c$ is coercive on $H_{0}^{1}(\Omega_{i})$: if $u_{i} \in H_{0}^{1}(\Omega_{i})$,

$$c(u_{i}, u_{i}) = b(u_{i}, u_{i}) + \|u_{i}\|_{L^{2}(\Omega_{i})}^{2} \geq \alpha_{b} \|u_{i}\|_{L^{2}(\Omega_{i})}^{2} + \|u_{i}\|_{H_{0}^{1}(\Omega_{i})}^{2} \geq \|u_{i}\|_{H_{0}^{1}(\Omega_{i})}^{2},$$

where we used the coercivity of the bilinear application $b$.

- The linear form $\ell_{i}^{n}$ is continuous on $H_{0}^{1}(\Omega_{i})$: let $v_{i} \in H_{0}^{1}(\Omega_{i})$ and $j \in \{1,2\}, i \neq j$

$$|\ell_{i}^{n}(v_{j})| \leq \left( \|f\|_{L^{2}(\Omega_{j})} + C_{b} \|u_{i}^{n}\|_{L^{2}(\Omega_{j})} \|v_{j}\|_{L^{2}(\Omega_{j})} + \|u_{j}\|_{H_{0}^{1}(\Omega_{j})} \|v_{j}\|_{H_{0}^{1}(\Omega_{j})} \right) \leq \left( C_{p} \|f\|_{L^{2}(\Omega_{j})} + C_{b} \|u_{i}^{n}\|_{L^{2}(\Omega_{j})} + \|u_{j}\|_{H_{0}^{1}(\Omega_{j})} \|v_{j}\|_{H_{0}^{1}(\Omega_{j})} \right),$$

Note that we have used the Cauchy-Schwarz inequality, the continuity of the bilinear form $b$, and Poincaré inequality.

For $i \in \{1,2\}$ fixed, we can then use the Lax-Milgram theorem on the Hilbert space $H_{0}^{1}(\Omega_{i})$. Note that $u_{1}^{2} \in H_{0}^{1}(\Omega_{1}), u_{2}^{2} \in H_{0}^{1}(\Omega_{2})$, there exists a unique $u_{1}^{n+1} \in H_{0}^{1}(\Omega_{1})$ such that:

$$c(u_{1}^{n+1}, v_{i}) = \ell_{i}^{n}(v_{i}), \quad \forall v_{i} \in H_{0}^{1}(\Omega_{i}).$$
Now we decompose $u \in H^1_0(\Omega)$. We construct $\chi \in C^\infty(\Omega)$ that satisfies the following properties:

- $\chi = 0$ on $\partial \Omega_1 \setminus \partial \Omega = \Gamma_1$ and $\chi = 1$ on $\partial \Omega_2 \setminus \partial \Omega = \Gamma_2$.
- $\chi = 0$ in $\Omega_2 \setminus \Omega_1$ and $\chi = 1$ in $\Omega_1 \setminus \Omega_2$.

Then we have the following decomposition:

$$w_1 = \chi u \in H^1_0(\Omega_1), \quad w_2 = (1-\chi)u \in H^1_0(\Omega_2), \quad u = w_1 + w_2. \quad (24)$$

**Remark 3.2.** We observe that the function $\chi$ which verify the conditions (3.3) is not unique. By example, we can choose $\chi \in C^\infty(\Omega)$ such that:

$$0 \leq \chi \leq 1, \quad \chi = 0 \text{ in } \Omega_2 \setminus \Omega_1, \quad \chi = 1 \text{ in } \Omega_1 \setminus \Omega_2.$$ 

We notice that $\chi^2$ verifies the same conditions and thus the decomposition $u = w_1 + w_2$ is not unique.

Let $u = w_1 + w_2$ be a decomposition of the solution $u$ of problem (3.1) as shown in (24). For $i = 1, 2$, we set $w_i^n = u_i^n - w_i$. We want to find a formulation verified by each $w_i^{n+1}$, $i = \{1, 2\}$, $n \geq 0$ but not $u_i^n$, so we deal with the original problem as follows:

Since $u = w_1 + w_2$ is a solution of the variational problem (3.1), for all $v \in H^1_0(\Omega)$ we have:

$$a(w_1 + w_2, v) = \int_\Omega f(x)v(x)dx.$$ 

Let $i = \{1, 2\}$ be fixed and $v_i \in H^1_0(\Omega_i)$. Then $\tilde{v}_i \in H^1_0(\Omega)$ and noting $v_i$ the extension $\tilde{v}_i$, we can write:

$$a(w_1 + w_2, v_i) = \int_\Omega f(x)v_i(x)dx.$$ 

By subtracting this equation from the following equation (3.2):

$$b\left(u_i^{n+1} - u_i^n, v_i\right) + a\left(u_i^{n+1} + u_j^n, v_i\right) = \int_\Omega f(x)v_i(x)dx.$$ 

we obtain:

$$b\left(u_i^{n+1} - u_i^n, v_i\right) + a\left(w_i^{n+1} + w_j^n, v_i\right) = 0.$$ 

which can be rewritten as:

$$b\left(u_i^{n+1}, v_i\right) + a\left(w_i^{n+1}, v_i\right) = b\left(u_i^n, v_i\right) - a\left(w_i^n, v_i\right).$$ 

Subtracting the term $b\left(w_i, v_i\right)$ on each side, we finally obtain:

$$b\left(w_i^{n+1}, v_i\right) + a\left(w_i^{n+1}, v_i\right) = b\left(w_i^n, v_i\right) - a\left(w_i^n, v_i\right),$$ 

and the variational formulation verified by each $w_i^{n+1}$ is written as follows:

Find $w_i^{n+1} \in H^1_0(\Omega_i), i = \{1, 2\}$, such that:

$$b\left(w_i^{n+1}, v_i\right) + a\left(w_i^{n+1}, v_i\right) = b\left(w_i^n, v_i\right) - a\left(w_i^n, v_i\right), \quad \forall v_i \in H^1_0(\Omega_i). \quad (25)$$

By writing the variational formulation verified by each $w_i^{n+1}, i = \{1, 2\}$, $n \geq 0$, we give a formulation that does not show $u_i^n$.

Now we assumed that the form $b(\cdot, \cdot) = \beta(\cdot, \cdot)_{L^2(\Omega)}$, for a constant $\beta > 0$. Then we have the following result:

**Proposition 3.3.** By verifying the following formula:

$$\frac{\beta}{2} \left(\|w_i^{n+1}\|_{L^2(\Omega)}^2 - \|w_i^n\|_{L^2(\Omega)}^2 + \|w_i^{n+1} - w_i^n\|^2_{L^2(\Omega)}ight)$$

$$+ \|w_i^{n+1}\|_{L^2(\Omega)}^2 - \|w_i^n\|_{L^2(\Omega)}^2 + \|w_i^{n+1} - w_i^n\|^2_{L^2(\Omega)})$$

$$+ a(w_i^{n+1}, w_i^{n+1}) + a(w_i^n, w_i^{n+1}) + a(w_i^n, w_i^{n+1}) + a(w_i^{n+1}, w_i^{n+1}) = 0,$$
we can deduce the following inequality:
\[
\|w_1^{n+1}\|_{L^2(\Omega)}^2 + \|w_2^{n+1}\|_{L^2(\Omega)}^2 + \|w_1^{n+1} - w_2^{n+1}\|_{L^2(\Omega)}^2 + \frac{1}{\beta} (a(w_1^{n+1}, w_1^{n+1}) + a(w_2^{n+1}, w_2^{n+1})) \\
\leq \|w_1^n\|_{L^2(\Omega)}^2 + \|w_2^n\|_{L^2(\Omega)}^2 + \frac{1}{\beta} (a(w_1^n, w_1^n) + a(w_2^n, w_2^n)).
\]

**Proof.** We choose \(v_i = w_i^{n+1}\) in the variational formulation obtained in (25), then for \(i, j = \{1,2\}, i \neq j:\)
\[
\beta (w_i^{n+1} - w_i^n, w_i^{n+1}) + a(w_i^{n+1}, w_i^{n+1}) + a(w_j^n, w_i^{n+1}) = 0.
\]
We observe that \(y(y - x) = \frac{1}{2}(y^2 - x^2 + (x - y)^2)\), then we obtain:
\[
\frac{\beta}{2} \left( \|w_i^{n+1}\|_{L^2(\Omega)}^2 - \|w_i^n\|_{L^2(\Omega)}^2 + \|w_i^{n+1} - w_i^n\|_{L^2(\Omega)}^2 \right) + a(w_i^{n+1}, w_i^{n+1}) + a(w_j^n, w_i^{n+1}) = 0.
\]
By summing over \(i = \{1,2\}\), we obtain the expected formula.

To obtain the inequality, we use the Cauchy-Schwartz inequality to get:
\[
-a(w_i^n, w_i^{n+1}) = -\int_\Omega \nabla w_i^n(x) \cdot \nabla w_i^{n+1}(x) dx \leq \|w_i^n\|_{H^1_0(\Omega)} \|w_i^{n+1}\|_{L^2(\Omega)} \\
\leq \frac{1}{2} \|w_i^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w_i^{n+1}\|_{L^2(\Omega)}^2 = \frac{1}{2} a(w_i^n, w_i^n) + \frac{1}{2} a(w_i^{n+1}, w_i^{n+1}).
\]

The same calculation on the term \(a(w_i^n, w_i^{n+1})\) gives the expected inequality.

From the above inequality, we have the following result:

**Lemma 3.4.** the sequences \((u_i^n)_{n \geq 0}(i = \{1,2\})\) are bounded in \(H_0^1(\Omega_i)\) and furthermore:
\[
\sum_{n=0}^{+\infty} \left( \|u_i^{n+1} - u_i^n\|_{L^2(\Omega)}^2 + \|u_i^{n+1} - u_i^n\|_{L^2(\Omega)}^2 \right) < \infty.
\]

**Proof.** Let \(N \in \mathbb{N}^*\). By summing the inequality obtained in the previous proposition for \(n = \{0, \cdots, N\}\), we obtain:
\[
\|w_1^{N+1}\|_{L^2(\Omega)}^2 + \|w_2^{N+1}\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N} \left( \|w_1^{n+1} - w_1^n\|_{L^2(\Omega)}^2 + \|w_2^{n+1} - w_2^n\|_{L^2(\Omega)}^2 \right) \\
+ \frac{1}{\beta} \left( a(w_1^{n+1}, w_1^{n+1}) + a(w_2^{n+1}, w_2^{n+1}) \right) \\
\leq \|w_1^0\|_{L^2(\Omega)}^2 + \|w_2^0\|_{L^2(\Omega)}^2 + \frac{1}{\beta} \left( a(w_1^0, w_1^0) + a(w_2^0, w_2^0) \right) + \frac{1}{\beta} \left( a(w_1^0, w_2^0) + a(w_2^0, w_1^0) \right).
\]

Thus, for \(i = \{1,2\}\) we have:
\[
\|w_i^{N+1}\|_{H^1_0(\Omega_i)}^2 = \left( a(w_i^{n+1}, w_i^{n+1}) \right) \\
\leq \beta \left( \|w_i^0\|_{L^2(\Omega)}^2 + \|w_i^0\|_{L^2(\Omega)}^2 \right) + a(w_1^0, w_1^0) + a(w_2^0, w_2^0) \\
\leq 2\beta \left( \|u_i^0\|_{H^1_0(\Omega)}^2 + \|u_i^0\|_{H^1_0(\Omega)}^2 + \|w_i^0\|_{L^2(\Omega)}^2 + \|w_i^0\|_{L^2(\Omega)}^2 \right) \\
+ 2 \left( \|u_i^0\|_{H^1_0(\Omega)}^2 + \|u_i^0\|_{H^1_0(\Omega)}^2 + \|w_i^0\|_{H^2(\Omega_i)}^2 + \|w_i^0\|_{H^2(\Omega_i)}^2 \right),
\]
which shows that the sequences \((u_i^n)_{n \geq 0}(i = \{1,2\})\) are well bounded in \(H_0^1(\Omega_i)\).

In the same way we have:
\[
\sum_{n=0}^{N} \left( \| w_1^{n+1} - w_1^{n} \|_{L^2(\Omega)}^2 + \| w_2^{n+1} - w_2^{n} \|_{L^2(\Omega)}^2 \right) \\
\leq 2 \left( \| u_1^0 \|_{L^2(\Omega)}^2 + \| u_2^0 \|_{L^2(\Omega)}^2 + \| w_1 \|_{L^2(\Omega)}^2 + \| w_2 \|_{L^2(\Omega)}^2 \right) \\
+ \frac{2}{\beta} \left( \| u_1^0 \|_{H_0^1(\Omega)}^2 + \| u_2^0 \|_{H_0^1(\Omega)}^2 + \| w_1 \|_{H_0^1(\Omega)}^2 + \| w_2 \|_{H_0^1(\Omega)}^2 \right) .
\]

Observe that \( w_i^{n+1} - w_i^{n} = (u_i^{n+1} - w_i) - (u_i^{n} - w_i) = u_i^{n+1} - u_i^{n} , \) we obtain:

\[
\sum_{n=0}^{N} \left( \| u_1^{n+1} - u_1^{n} \|_{L^2(\Omega)}^2 + \| u_2^{n+1} - u_2^{n} \|_{L^2(\Omega)}^2 \right) \leq C ,
\]

so the series of partial sums is increased and the series is convergent.

From Lemma 3.4, we can deduce the following result:

**Lemma 3.5.** The sequence \( (u_i^{n+1} - u_i^{n})_{n \geq 0} (i = 1, 2) \) converges to 0 weakly in \( H_0^1(\Omega_i) \) and strongly in \( L^2(\Omega_i) \).

**Proof.** From Lemma 3.4 we know that the sequence \( (u_i^{n+1} - u_i^{n})_{n \geq 0} \) is bounded in \( H^1(\Omega_i) \). So from Theorem 2.5 and Rellich–Kondrachov theorem, we can extract a sub-sequence that converges weakly in \( H^1(\Omega_i) \) and strongly in \( L^2(\Omega_i) \) to some limit \( u_\infty \in H^1(\Omega_i) \). Again from (3.5), we know that \( \lim_{n \to \infty} \| u_i^{n+1} - u_i^{n} \|_{L^2(\Omega_i)} = 0 \) and thus the entire sequence \( (u_i^{n+1} - u_i^{n})_{n \geq 0} \) converges strongly to 0 in \( L^2(\Omega_i) \). Thus, we have a unique limit point and the whole sequence \( (u_i^{n+1} - u_i^{n})_{n \geq 0} \) converges weakly to 0 in \( H_0^1(\Omega_i) \).

Now we can introduce the main result here:

**Theorem 3.6.** We take \( u_0^1 \in H_0^1(\Omega_1) \cap H^2(\Omega_1) , u_0^2 \in H_0^1(\Omega_2) \cap H^2(\Omega_2) \). The iterative sequence \( (u_1^n,u_2^n)_{n\geq0} \) defined by

\[
b(u_1^{n+1} - u_1^n,v_1) + a(u_1^{n+1} + u_1^n,v_1) = \int_\Omega f(x)v_1(x)dx , \quad \forall v_1 \in H_0^1(\Omega_1) .
\]

\[
b(u_2^{n+1} - u_2^n,v_2) + a(u_2^{n+1} + u_2^n,v_2) = \int_\Omega f(x)v_2(x)dx , \quad \forall v_2 \in H_0^1(\Omega_2) ,
\]

converges weakly to a limit \( (u_1^*,u_2^*) \) in \( H_0^1(\Omega_1) \times H_0^1(\Omega_2) \), where

\[
a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)dx ,
\]

\[
b(u,v) = \beta \int_\Omega u(x)v(x)dx \text{ with } \beta > 0 .
\]

Moreover, the solution in \( H_0^1(\Omega) \) of the variational formulation:

\[
\int_\Omega \nabla u(x) \cdot \nabla v(x)dx = \int_\Omega f(x)v(x)dx , \quad \forall v \in H_0^1(\Omega) ,
\]

satisfies \( u = u_1^* + u_2^* \).

**Proof.** From Lemma 3.4, the sequence \( (u_1^n,u_2^n)_{n \geq 0} \) is bounded in \( H_0^1(\Omega_1) \times H_0^1(\Omega_2) \), then by Theorem 2.5, we can extract a sub-sequence which converges weakly to a limit \( (u_1^*,u_2^*) \) in \( H_0^1(\Omega_1) \times H_0^1(\Omega_2) \).

Since the functions \( u_1^* \) and \( u_2^* \) were uniquely defined independently of the decomposition of \( u \) into \( w_1 + w_2 \), we have the uniqueness of the limit \( (u_1^*,u_2^*) \). We observe that the sequence \( (u_i^n,u_i^n)_{n \geq 0} \) admits a unique limit point, so we can conclude the weak convergence of the whole sequence \( (u_1^n,u_2^n)_{n \geq 0} \) to \( (u_1^*,u_2^*) \) in \( H_0^1(\Omega_1) \times H_0^1(\Omega_2) \).
We write the variational formulation (35) for the sequence \((u^{n_1}_1, u^{n_2}_2)\) for \(i, j = 1, 2, i \neq j\). We consider \(v_i \in H^1_0(\Omega_i)\) and we observe that \(v_i = 0\) in \(\Omega \setminus \Omega_i\). With respect to the first term, knowing that \(\lim_{n \to +\infty} \| u^{n_1+1}_i(x) - u^n_i(x) \|_{L^2(\Omega_i)} = 0\), we have:

\[
\int_{\Omega_i} (u^{n_1+1}_i(x) - u^n_i(x)) \cdot v_i(x) dx \leq \| u^{n_1+1}_i - u^n_i \|_{L^2(\Omega_i)} \| v_i \|_{L^2(\Omega_i)} \to 0.
\]

For the second term, we observe that:

\[
\int_{\Omega_i} (\nabla u^{n_1+1}_i(x) + \nabla u^n_i(x)) \cdot \nabla v_i(x) dx = \int_{\Omega_i} \nabla (u^{n_1+1}_i(x) - u^n_i(x)) \cdot \nabla v_i(x) dx + \int_{\Omega_i} (\nabla u^n_i(x) + \nabla v^n_i(x)) \cdot \nabla v_i(x) dx.
\]

By Lemma 3.5, we know that the sequence \((u^{n_1+1}_i - u^n_i)\) converges weakly to 0 in \(H^1_0(\Omega_i)\), so we can pass to the limit in the first term which tends to 0. The weak convergence of the sequence \((u^n_i, u^n_2)\) to \((u^*_i, u^*_2)\) in \(H^1_0(\Omega_i) \times H^1_0(\Omega_2)\) gives:

\[
\int_{\Omega_i} (\nabla u^{n_1+1}_i(x) + \nabla u^n_i(x)) \cdot \nabla v_i(x) dx \to \int_{\Omega_i} (\nabla u^*_i(x) + \nabla u^*_i(x)) \cdot \nabla v_i(x) dx,
\]

so

\[
\int_{\Omega} \nabla (u^*_i + u^*_2(x)) \cdot \nabla v_i(x) dx = \int_{\Omega} f(x)(v_i(x) dx, \quad \forall v_i \in H^1_0(\Omega_i).
\]

Let \(v \in H^1_0(\Omega)\), then by the decomposition, there exists \((v_1, v_2) \in H^1_0(\Omega_1) \times H^1_0(\Omega_2)\) such that \(v = v_1 + v_2\). By choosing these \(v_i\) as test functions in the previous equation and summing over \(i \in \{1, 2\}\), we obtain:

\[
\int_{\Omega} \nabla (u^*_i + u^*_2(x)) \cdot \nabla v(x) dx = \int_{\Omega} \nabla (u^*_i + u^*_2(x)) \cdot \nabla (v_1 + v_2) dx = \int_{\Omega} f(x)(v_1(x) + v_2(x) dx = \int_{\Omega} f(x)v(x) dx.
\]

By the uniqueness of the solution of the variational formulation (3.1), we can deduce that \(u = u^*_1 + u^*_2\).

By Theorem 3.6, we know that the two iterative sequences constructed are weakly convergent and that the sum of their weak limits is equal to the solution of the variational formulation in the entire domain. But according to this theorem, we cannot directly use these two iterative sequences for numerical approximation because the convergences here are weak convergences. So we need to combine the finite element method for numerical approximation, which is introduced in the next subsection.

### 3.2 Finite Element Discretization

Now we assume that the domains \(\Omega_1\) and \(\Omega_2\) are polygonal. Let \(\mathcal{T}_h^1\) be a regular family of triangular meshes of \(\Omega_1\) and \(\mathcal{T}_h^2\) a regular family of triangular meshes of \(\Omega_2\) (see Figure 2 for example).

**Figure 2.** Meshes \(\mathcal{T}_h^1\) of \(\Omega_1\) and \(\mathcal{T}_h^2\) of \(\Omega_2\).
We can write the explicit subspaces $V_{i,h}$ of $H_0^1(\Omega_i) (i = \{1,2\})$ constructed from $\mathcal{T}_h^i$ triangulations and Lagrange finite elements of order $k$ as follows:

$$V_{i,h} = \left\{ v_{i,h} \in C_0(\Omega_i) : v_{i,h}|_T \in P_k, \forall T \in \mathcal{T}_h^i \text{ and } v_{i,h} = 0 \text{ on } \partial \Omega_i \right\}.$$ 

For $u_{i,0}^n \in V_{i,h} (i = \{1,2\})$, the discrete version of problem (35) is written as:

$$b(u_{i,0}^{n+1}, v_{i,h}) + a(u_{i,0}^{n+1}, u_{i+1,0,n}^n, v_{i,0,h}) = \int_\Omega f(x) v_{i+1,0,0}(x) dx, \forall v_{i,0,h} \in V_{i,0,h}(28)$$

It can be shown by the same arguments as before that equations (40) define the sequences $(u_{i,0}^n)_{n \geq 0}$ and $(u_{2,0}^n)_{n \geq 0}$ in a unique way. Moreover, the entire sequence $(u_{1,0}^n, u_{2,0}^n)_{n \geq 0}$ converges strongly in $V_{i,0} \times V_{2,0}$ to a limit $(u_{1,0}, u_{2,0})$ because weak convergence and strong convergence are equivalent in finite dimensional spaces.

To prove the error estimate result, we introduce the following conclusion:

**Proposition 3.7.** Verify the following formula:

$$\int_\Omega \nabla (u_{1,0}^* - u_1^* + u_{2,0}^* - u_2^*)(x) \cdot \nabla (v_{1,0} + v_{2,0})(x) dx = 0, \quad \forall v_{1,0} \in V_{1,0}, v_{2,0} \in V_{2,0}.$$

**Proof.** By passing to the limit in equations (40) in the same way as for the continuous case, we obtain:

$$\int_\Omega \nabla (u_{1,0}^* + u_{2,0}^*)(x) \cdot \nabla v_{1,0}(x) dx = \int_\Omega f(x) v_{1,0}(x) dx, \quad \forall v_{1,0} \in V_{1,0}.$$

Moreover, we have:

$$\int_\Omega \nabla (u_{1,0}^* + u_{2,0}^*)(x) \cdot \nabla v_{1,0}(x) dx = \int_\Omega f(x) v_{1,0}(x) dx, \quad \forall v_{1,0} \in H_0^1(\Omega_{1,2}) \quad (29)$$

Let $v_{1,0} \in V_{1,0}, v_{2,0} \in V_{2,0}$. Now $V_{i,h} \subset H_0^1(\Omega_i)$, so we can choose $v_i = v_{i,0}$ as the test function in (29) which gives:

$$\int_\Omega \nabla (u_{1,0}^* + u_{2,0}^*)(x) \cdot \nabla (v_{1,0} + v_{2,0})(x) dx = \int_\Omega f(x) (v_{1,0} + v_{2,0})(x) dx,$$

and

$$\int_\Omega \nabla (u_{1,0}^* + u_{2,0}^*)(x) \cdot \nabla (v_{1,0} + v_{2,0})(x) dx = \int_\Omega f(x) (v_{1,0} + v_{2,0})(x) dx.$$

By subtracting these two equations we obtain the desired formula.

So for all $v_{1,0} \in V_{1,0}, v_{2,0} \in V_{2,0}:

$$\int_\Omega \nabla (u_{1,0}^* + u_{2,0}^*)(x) \cdot \nabla (v_{1,0} + v_{2,0})(x) dx = \int_\Omega \nabla (u_{1,0}^* + u_{2,0}^*)(x) \cdot \nabla (v_{1,0} + v_{2,0})(x) dx,$$

that is:

$$a(u_{1,0}^* + u_{2,0}^*, v_{1,0} + v_{2,0}) = a(u_{1,0}^* + u_{2,0}^*, v_{1,0} + v_{2,0}).$$

Since the bilinear form $a$ is symmetric, we have:

$$\langle u_{1,0}^* + u_{2,0}^*, v_{1,0} + v_{2,0} \rangle = \min_{(v_{1,0}, v_{2,0}) \in V_{1,0} \times V_{2,0}} \langle v_{1,0} + v_{2,0} \rangle$$

which can be written as:
\[
\frac{1}{2} \int_{\Omega} |\nabla (u_{1,h}^* - u_1 + u_{2,h} - u_2)|^2 (x) \, dx - \frac{1}{2} \int_{\Omega} |\nabla (u_1^* + u_2^*)|^2 (x) \, dx
= \min_{(v_1,h,v_2,h) \in V_{1,h} \times V_{2,h}} \left( \frac{1}{2} \int_{\Omega} |\nabla (v_1,h + v_2,h - u_1^* - u_2^*)|^2 (x) \, dx - \frac{1}{2} \int_{\Omega} |\nabla (u_1^* + u_2^*)|^2 (x) \right).
\]

Since the term \(\frac{1}{2} \int_{\Omega} |\nabla (u_1^* + u_2^*)|^2 \) is independent of \((v_1,h,v_2,h)\), then we obtain:
\[
\int_{\Omega} \left| \nabla (u_{1,h}^* - u_1^* + u_{2,h} - u_2^*) \right|^2 (x) \, dx
= \min_{(v_1,h,v_2,h) \in V_{1,h} \times V_{2,h}} \left( \int_{\Omega} |\nabla (v_1,h + v_2,h - u_1^* - u_2^*)|^2 (x) \, dx \right).
\]

Then by the conclusion of finite element method and proposition 3.7, we can deduce the the error estimate result:

**Theorem 3.8.** We assume that the domains \(\Omega_1\) and \(\Omega_2\) are polygonal. Let \((\mathcal{T}_h^1)\) be a regular family of triangular meshes of \(\Omega_1\) and \((\mathcal{T}_h^2)\) a regular family of triangular meshes of \(\Omega_2\). Let \(u \in H^k(\Omega)\) be the solution of the variational formulation (22), and \(u_{i,h} \in V_{i,h}, i = \{1,2\}\) be solutions of (29) by the finite element method \(\mathbb{P}_k\). Moreover, we assume that \(u_1^* \in H^{k+1}(\Omega_1)\) and \(u_2^* \in H^{k+1}(\Omega_2)\). Then we have the following error estimate:

\[
\|u - u_{1,h}^* - u_{2,h}^*\|_{H^1(\Omega)} \leq C h^k \left( \|u_1^*\|_{H^{k+1}(\Omega_1)} + \|u_2^*\|_{H^{k+1}(\Omega_2)} \right),
\]

where \(h = \max(h_1,h_2)\) and \(h_i, i = \{1,2\}\) is the pitch of the \(\mathcal{T}_h^i\) mesh.

**Proof.** For \(i = \{1,2\}\), let us consider the interpolation operator \(r_i^k; H_0^2(\Omega_i) \rightarrow V_{i,h}\). For all \(k \geq 1\), we have \(k + 1 > N/2 = 1\). According to Proposition 3.7, we have:

\[
\|u_{1,h}^* - u_1^* + u_{2,h}^* - u_2^*\|_{H_0^2(\Omega)} \leq \|u_1^* - r_1^k u_1^* + u_2^* - r_2^k u_2^*\|_{H_0^2(\Omega)}.
\]

Then, we can obtain:

\[
\|u_{1,h}^* - u_1^* + u_{2,h}^* - u_2^*\|_{H_0^2(\Omega)} \leq C_1 h^k \|u_1^*\|_{H^{k+1}(\Omega_1)} + C_2 h^k \|u_2^*\|_{H^{k+1}(\Omega_2)}.
\]

where \(h_i\) is the pitch of the \(\mathcal{T}_h^i\) mesh.

We note that \(h = \max(h_1,h_2)\) and use Poincaré inequality, we finally obtain:

\[
\|u - u_{1,h}^* - u_{2,h}^*\|_{H^1(\Omega)} \leq C h^k \left( \|u_1^*\|_{H^{k+1}(\Omega_1)} + \|u_2^*\|_{H^{k+1}(\Omega_2)} \right).
\]

By Theorem 3.8, when we assume the regularity of the two weak limits in Theorem 3.6, the sum of the two limits of the two convergent sequences obtained by the finite element method can approximate the solution of the variational formulation in \(H^1\) norm, and we can also give the corresponding error estimate. This result shows that for Poisson’s equation with the homogeneous Dirichlet boundary condition defined on a complex domain, if we can decompose the domain into two overlapping polygonal domain, then we can construct two iterative sequences obtained by the finite element method. Moreover, if we assume some regularity properties, we can give the numerical solution of Poisson’s equation with the homogeneous Dirichlet boundary condition. The advantage of this method is that it avoids gridding complex domains directly.

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