Fine regularity of Lévy processes and linear (multi)fractional stable motion

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Abstract: We investigate the regularity of Lévy processes within the 2-microlocal analysis framework. A local description of sample paths is obtained, refining the classic spectrum of singularities determined by Jaffard. As a consequence of this result and the properties of the 2-microlocal frontier, we are able to completely characterize the multifractal nature of the linear fractional stable motion (extension of fractional Brownian motion to \( \alpha \)-stable measures) in the case of continuous and unbounded sample paths as well. The regularity of its multifractional extension is also determined, thereby providing an example of a stochastic process with a non-homogeneous and random multifractal spectrum.

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1. Introduction

The study of sample path continuity and Hölder regularity of stochastic processes is a very active field of research in probability theory. The existing literature provides a variety of uniform results on local regularity, especially on the modulus of continuity, for rather general classes of random fields (see e.g. Marcus and Rosen [30], Adler and Taylor [2] on Gaussian processes and Xiao [45] for more recent developments).

On the other hand, the structure of pointwise regularity is often more complex, in particular as it often happens to behave erratically as time passes. Such sample path behaviour was first put into light for Brownian motion by Orey and Taylor [33] and Perkins [34]. They respectively studied fast and slow points, which characterize logarithmic variations of the pointwise modulus of continuity, and proved that the sets of times with a given pointwise regularity have a particular fractal geometry. Khoshnevisan and Shi [25] recently extended the fast points study to fractional Brownian motion.

As exhibited by Jaffard [22], Lévy processes with a jump compound also display an interesting pointwise behaviour. Indeed, for this class of random fields, the pointwise exponent varies randomly inside a closed interval as time passes. This seminal work has been enhanced and extended by Durand [18], Durand and Jaffard [19] and Barral et al. [10]. Particularly, the latter proves that Markov processes have a range of admissible pointwise behaviours much wider than Lévy processes. In the aforementioned works, multifractal analysis happens to be the key concept for the study and the characterisation of local fluctuations of pointwise
regularity. In order to be more specific, let us first recall the definition of the different notions previously outlined.

**Definition 1.** A function \( f : \mathbb{R} \to \mathbb{R}^d \) belongs to \( C^\alpha_t \), with \( t \in \mathbb{R} \) and \( \alpha > 0 \), if there exist \( C > 0 \), \( \rho > 0 \) and a polynomial \( P_t \) of degree less than \( \alpha \) such that

\[
\forall u \in B(t, \rho); \quad \| f(u) - P_t(u) \| \leq C|t - u|^\alpha.
\]

The pointwise Hölder exponent of \( f \) at \( t \) is defined by \( \alpha_{f,t} = \sup\{\alpha \geq 0 : f \in C^\alpha_t\} \), where by convention \( \sup\{\emptyset\} = 0 \).

Multifractal analysis is concerned with the study of the level sets of the pointwise exponent, usually called the isothermal Hölder sets of \( f \),

\[
E_h = \{ t \in \mathbb{R} : \alpha_{f,t} = h \} \quad \text{for every } h \in \mathbb{R}_+ \cup \{+\infty\}. \tag{1.1}
\]

To describe the geometry of the collection \( (E_h)_{h \in \mathbb{R}_+} \), and thereby to determine the arrangement of the Hölder regularity, we are interested in the local spectrum of singularities of \( f \). It is usually denoted \( d_f(h, V) \) and defined by

\[
d_f(h, V) = \dim_H(E_h \cap V) \quad \text{for every } h \in \mathbb{R}_+ \cup \{+\infty\} \text{ and } V \in \mathcal{O}, \tag{1.2}
\]

where \( \mathcal{O} \) denotes the collection of nonempty open sets of \( \mathbb{R} \) and \( \dim_H \) the Hausdorff dimension (by convention \( \dim_H(\emptyset) = -\infty \)).

Although \( (E_h)_{h \in \mathbb{R}_+} \) are random sets, stochastic processes such as Lévy processes [22], Lévy processes in multifractal time [9] and fractional Brownian motion happen to have a deterministic multifractal spectrum. Furthermore, these random fields are also said to be homogeneous as the Hausdorff dimension \( d_X(h, V) \) is independent of the set \( V \) for all \( h \in \mathbb{R}_+ \). When the pointwise exponent is constant along sample paths, the spectrum is described as degenerate, i.e. its support is reduced to a single point (e.g. the Hurst exponent in the case of f.B.m.). Nevertheless, Barral et al. [10] and Durand [17] provided examples of respectively Markov jump processes and wavelet random series with non-homogeneous and random spectrum of singularities.

As stated in Equations (1.1) and (1.2), classic multifractal analysis deals with the study of the variations of pointwise regularity. Unfortunately, it is known that common Hölder exponents (local and pointwise as well) do not give a complete picture of the local regularity (see e.g. the deterministic Chirp function \( t \mapsto |t|^\alpha \sin(|t|^{-\beta}) \) detailed in [20]). Furthermore, they also lack of stability under the action of pseudo-differential operators.

2-microlocal analysis is one natural way to overcome these issues and obtain a more precise description of the local regularity. It has first been introduced by Bony [13] in the deterministic frame to study properties of generalized solutions of PDE. More recently, Herbin and Lévy Véhel [20] and Balança and Herbin [7] developed a stochastic approach based on this framework to investigate the finer regularity of stochastic processes such as Gaussian processes, martingales and stochastic integrals. In order to the study sample path properties in this frame, we need to recall the concept of 2-microlocal space.

**Definition 2.** Let \( t \in \mathbb{R} \), \( s' \leq 0 \) and \( \sigma \in (0, 1) \) such that \( \sigma - s' \notin \mathbb{N} \). A function \( f : \mathbb{R} \to \mathbb{R}^d \) belongs to the 2-microlocal space \( C^\infty_{t,s'} \) if there exist \( C > 0 \), \( \rho > 0 \) and a polynomial \( P_t \) such that

\[
\left\| (f(u) - P_t(u)) - ((f(v) - P_t(v)) \right\| \leq C|u - v|^{\sigma} (|u - t| + |v - t|)^{-s'}, \tag{1.3}
\]

where

\[
\dim_H\{f(t) : t \in [0,1]\} = \sigma/\alpha.
\]
for all $u, v \in B(t, \rho)$.

The time-domain characterisation (1.3) of 2-microlocal spaces has been obtained by Seuret and Lévy Véhel [41]. The original definition given by Bony [13] relies on the Littlewood-Paley decomposition of tempered distributions, and thereby corresponds to a description in the Fourier space. Another characterisation based on wavelet expansion has also been exhibited by Jaffard [21]. The extension of Definition 2 to $\sigma \notin (0, 1)$ relies on the following important property satisfied by 2-microlocal spaces (see Theorem 1.1 in [23]),

$$\forall \alpha > 0; \quad f \in C^{\sigma, s'}_t \iff I^\alpha f \in C^{\sigma + \alpha, s'}_t,$$

where $I^\alpha f$ designates the fractional integral of $f$ of order $\alpha$. As a consequence of (1.4), the application of Equation (1.3) to iterated integrals or differentials of $f$ provides an extension of Definition 2 to $\sigma \in \mathbb{R} \setminus \mathbb{Z}$, which is sufficient for the purpose of this paper.

Similarly to the pointwise Hölder exponent, the introduction of 2-microlocal spaces leads naturally to the definition of regularity tool named the 2-microlocal frontier and given by

$$\forall s' \in \mathbb{R}; \quad \sigma_{f,t}(s') = \sup\{\sigma \in \mathbb{R} : f \in C^{\sigma, s'}_t\}.$$  

2-microlocal spaces enjoy several inclusion properties which imply that the map $s' \mapsto \sigma_{f,t}(s')$ is well-defined and display the following features:

- $\sigma_{f,t}(\cdot)$ is a concave non-decreasing function;
- $\sigma_{f,t}(\cdot)$ has left and right derivatives between 0 and 1.

As a function, the 2-microlocal frontier $\sigma_{f,t}(\cdot)$ offers a more complete description of the local regularity. In particular, it embraces the local Hölder exponent since $\tilde{\alpha}_{f,t} = \sigma_{f,t}(0)$. Furthermore, as stated in [32], if the modulus of continuity of $f$ satisfies $\omega(h) = O\left(1/|\log(h)|\right)$, the pointwise exponent can also be retrieved using the formula $\alpha_{f,t} = -\inf\{s' : \sigma_{f,t}(s') \geq 0\}$. Note that the previous formula can not directly deduced from Equation 1.3 since Definition 2 does not stand when $\sigma = 0$. [32] provides an example of generalized function which does not satisfy this relation.

As observed [20], Brownian motion provides a simple instance of 2-microlocal frontier in the stochastic frame: almost surely for all $t \in \mathbb{R}$, $\sigma_{B,t}$ satisfies

$$\forall s' \in \mathbb{R}; \quad \sigma_{B,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \left(\frac{1}{2}\right).$$

In this paper, the 2-microlocal approach is combined with the classic use of multifractal analysis to obtain a finer description of the regularity of stochastic processes. Following the path of [22, 18, 19], we refine the multifractal description of Lévy processes (Section 2) and observe in particular that the use of the 2-microlocal formalism allows to capture subtle behaviours that can not be characterized by the classic spectrum of singularities.

This finer analysis of sample path properties of Lévy processes happens to be very useful for the study of another class of processes named linear fractional stable motion (LFSM). The LFSM is a common $\alpha$-stable self-similar process with stationary increments, and can be seen as an extension of the fractional Brownian motion to the non-Gaussian frame. Since it also has long range dependence and heavy tails, it is of great interest in modelling. In Section 3, we completely characterize the multifractal nature of the LFSM, and thereby illustrate the fact that 2-microlocal analysis is well-suited to study the regularity of unbounded sample paths as well as continuous ones.
1.1. Statement of the main results

As it is well known, an \( \mathbb{R}^d \)-valued Lévy process \((X_t)_{t \in \mathbb{R}_+}\) has stationary and independent increments. Its law is determined by the Lévy-Khintchine formula (see e.g. [40]): for all \( t \in \mathbb{R}_+ \) and \( \lambda \in \mathbb{R}^d \),

\[
E[e^{i\langle \lambda, X_t \rangle}] = e^{\psi(\lambda)}
\]

where \( \psi \) is given by

\[
\forall \lambda \in \mathbb{R}^d; \quad \psi(\lambda) = i(a, \lambda) - \frac{1}{2} \langle \lambda, Q \lambda \rangle + \int_{\mathbb{R}^d} (e^{i\langle \lambda, x \rangle} - 1 - i\langle \lambda, x \rangle 1_{\{\|x\| \leq 1\}}) \pi(dx).
\]

\( Q \) is a non-negative symmetric matrix and \( \pi \) a Lévy measure, i.e. a positive Radon measure on \( \mathbb{R}^d \setminus \{0\} \) such that \( \int_{\mathbb{R}^d} (1 \wedge \|x\|^2) \pi(dx) < \infty \).

Throughout this paper, it will always be assumed that \( \pi(\mathbb{R}^d) = +\infty \). Otherwise, the Lévy process simply corresponds to the sum of a compound Poisson process with drift and a Brownian motion whose regularity can be simply deduced.

Sample path properties of Lévy processes are known to depend on the growth of the Lévy measure near the origin. More precisely, Blumenthal and Getoor [12] defined the following exponents \( \beta \) and \( \beta' \),

\[
\beta = \inf \left\{ \delta \geq 0 : \int_{\mathbb{R}^d} (1 \wedge \|x\|^\delta) \pi(dx) < \infty \right\}
\quad \text{and} \quad
\beta' = \begin{cases} 
\beta & \text{if } Q = 0; \\
2 & \text{if } Q \neq 0.
\end{cases}
\]

(1.6)

Owing to \( \pi \)'s definition, \( \beta, \beta' \in [0, 2] \). Pruitt [36] proved that \( \alpha_{X,0} \overset{\Delta}{=} 1/\beta \) when \( Q = 0 \). Note that several other exponents have been defined in the literature focusing on Lévy processes sample paths properties (see e.g. [26, 27] for some recent developments).

Jaffard [22] studied the spectrum of singularities of Lévy processes under the following assumption on the measure \( \pi \),

\[
\sum_{j \in \mathbb{N}} 2^{-j} \sqrt{C_j \log(1 + C_j)} < \infty, \quad \text{where} \quad C_j = \int_{2^{-j-1} < \|x\| \leq 2^{-j}} \pi(dx).
\]

(1.7)

Under the Hypothesis (1.7), Theorem 1 in [22] states that the multifractal spectrum of a Lévy process \( X \) is almost surely equal to

\[
\forall V \in \mathcal{O}; \quad d_X(h, V) = \begin{cases} 
\beta h & \text{if } h \in [0, 1/\beta'); \\
1 & \text{if } h = 1/\beta'; \\
-\infty & \text{if } h \in (1/\beta', +\infty].
\end{cases}
\]

(1.8)

Note that Equation (1.8) still holds when \( \beta = 0 \). Durand [18] extended this result to Hausdorff \( g \)-measures, where \( g \) is a gauge function, and Durand and Jaffard [19] generalized the study to multivariate Lévy fields.

We establish in Proposition 1 a new proof of the multifractal spectrum (1.8) which does not require Assumption (1.7). We observe that results obtained in [18] on Hausdorff \( g \)-measure are also extended using this method.

In order to refine the spectrum of singularities (1.8), we focus on the study of the 2-microlocal frontier of Lévy processes. For that purpose, we introduce and study the collections of sets \((E_h)_{h \in \mathbb{R}_+}\) and \((\tilde{E}_h)_{h \in \mathbb{R}_+}\) respectively defined by

\[
\tilde{E}_h = \{ t \in E_h : \forall s' \in \mathbb{R}; \sigma_{X,t}(s') = (h + s') \wedge 0 \} \quad \text{and} \quad \tilde{E}_h = E_h \setminus \tilde{E}_h.
\]
The family \((\tilde{E}_h)_{h \in \mathbb{R}_+}\) represents the set of times at which the 2-microlocal behaviour is rather common (and thus similar the 2-microlocal frontier \((1.5)\) of Brownian motion), whereas at points which belong \((\tilde{E}_h)_{h \in \mathbb{R}_+}\), the 2-microlocal frontier has an unusual form, with in particular a slope different from 1 at \(s' = -h\).

The next statement gathers our main result on the 2-microlocal regularity of Lévy processes.

**Theorem 1.** Sample paths of a Lévy process \(X\) almost surely satisfy

\[
\forall V \in \mathcal{O}; \quad \dim_H(\tilde{E}_h \cap V) = \begin{cases} 
\beta h & \text{if } h \in [0, 1/\beta'); \\
1 & \text{if } h = 1/\beta'; \\
-\infty & \text{if } h \in (1/\beta', +\infty].
\end{cases}
\]  

(1.9)

The collection of sets \((\tilde{E}_h)_{h \in \mathbb{R}_+}\) enjoys almost surely

\[
\forall V \in \mathcal{O}; \quad \dim_H(\tilde{E}_h \cap V) \leq \begin{cases} 
2\beta h - 1 & \text{if } h \in (1/2\beta, 1/\beta'); \\
-\infty & \text{if } h \in [0, 1/2\beta] \cup [1/\beta', +\infty].
\end{cases}
\]  

(1.10)

Furthermore, the 2-microlocal frontier at \(t \in \tilde{E}_h\) satisfies

\[
\sigma_{X,t}(s') \leq \left(\frac{h+s'}{2\beta h}\right) \land \left(\frac{1}{\beta'} + s'\right) \land 0
\]

for all \(s' \geq -1/\beta' - 1\).

**Remark 1.** The previous statement induces that \(\dim_H(\tilde{E}_h) < \dim_H(\tilde{E}_h)\) for all \(h \in [0, 1/\beta']\). Hence, from a Hausdorff dimension point of view, the majority of the times \(t \in \mathbb{R}_+\) have a rather classic 2-microlocal frontier \(s' \mapsto (\alpha_{X,t} + s') \land 0\).

**Remark 2.** The collection of sets \((\tilde{E}_h)_{h \in \mathbb{R}_+}\) illustrates the fact that 2-microlocal analysis can capture particular behaviours that are not necessarily described by a classic multifractal spectrum.

Examples 1 and 2 constructed in Section 2.3 show that different behaviours may occur, depending on properties of the Lévy measure. The first one provides a class of Lévy processes which satisfy \(\tilde{E}_h = \emptyset\) for all \(h \in [0, 1/\beta']\). On the other hand, in Example 2 is constructed a collection of Lévy measures \((\pi_h)_{h \in [1/2\beta', 1/\beta']}\) such that the related Lévy process almost surely enjoys \(\tilde{E}_h \neq \emptyset\).

It remains an open question to completely characterize the collection \((\tilde{E}_h)_{h \in \mathbb{R}_+}\) in terms of the Lévy measure \(\pi\) (Examples 1 and 2 indeed prove that the Blumenthal-Getoor exponent \(\beta\) is not sufficient).

**Remark 3.** Although sample paths of Lévy processes do not satisfy the condition \(\omega(h) = \mathcal{O}(1/|\log(h)|)\) outlined in the introduction, Theorem 1 nevertheless ensures that the pointwise Hölder exponent can be retrieved from the 2-microlocal frontier at any \(t \in \mathbb{R}_+\) using the formula \(\alpha_{X,t} = -\inf\{s' : \sigma_{X,t}(s') \geq 0\}\).

Since this work extends the study of the classic spectrum of singularities, it is also quite natural to investigate geometrical properties of the sets \((E_{\sigma,s'})_{\sigma,s' \in \mathbb{R}}\) defined by

\[
E_{\sigma,s'} = \{t \in \mathbb{R}_+ : \forall u' > s' ; \ X_t \in C^{\sigma,u'}_t \text{ and } \forall u' < s' ; \ X_t \notin C^{\sigma,u'}_t\}.
\]

Theorem 1 induces the next statement.
Corollary 1. A Lévy process \( X \) satisfies almost surely for any \( \sigma \in [-1, 0] \),

\[
\forall V \in \mathcal{O}; \quad \dim_H(E_{\sigma, s'} \cap V) = \begin{cases} 
\beta s & \text{if } s \in [0, 1/\beta'); \\
1 & \text{if } s = 1/\beta'; \\
-\infty & \text{otherwise.}
\end{cases}
\] (1.11)

where \( s \) denotes the common 2-microlocal parameter \( s = \sigma - s' \). Furthermore, for all \( s' \in \mathbb{R} \), \( E_{0, s'} = E_{-s'} \) and \( E_{\sigma, s'} \) is empty if \( \sigma > 0 \).

Corollary 1 generalizes the multifractal formula (1.8) since the spectrum of singularities corresponds to the case \( \sigma = 0 \). Note that the subtle behaviour exhibited in Theorem 1 is not captured by Equality (1.11). As outlined in the proof (Section 2.2), this property disappears because the sets \( (\hat{E}_h)_h \) are negligible compared to \( (\tilde{E}_h)_h \) in terms of Hausdorff dimension.

Regularity results established in Theorem 1 also happen to be interesting outside the scope of Lévy processes, thanks to the powerful properties satisfied the 2-microlocal frontier. More precisely, it allows to characterize the multifractal nature of the linear fractional stable motion (LFSM). This process is usually defined by the following stochastic integral (see e.g. [39])

\[
X_t = \int_{\mathbb{R}} \left\{ (t - u)^{H-1/\alpha} - (-u)^{H-1/\alpha} \right\} M_{\alpha, \beta}(du),
\] (1.12)

where \( M_{\alpha, \beta} \) is an \( \alpha \)-stable random measure and \( H \in (0, 1) \). Several regularity properties have been determined in the literature. In particular, sample paths are known to be nowhere bounded [29] if \( H < 1/\alpha \), whereas they are Hölder continuous when \( H > 1/\alpha \). In this latter case, Takashima [44], Kôno and Maejima [28] proved that the pointwise and local Hölder exponents satisfy almost surely \( H - 1/\alpha \leq \alpha_{X,t} \leq H \) and \( \tilde{\alpha}_{X,t} = H - 1/\alpha \). In the sequel, we will assume that \( \alpha \in [1, 2) \), which is in particular required to obtain Hölder continuous sample paths \( (H > 1/\alpha) \).

Using an alternative representation of LFSM obtained in Proposition 2, we enhance the aforementioned regularity results and obtain a description of the multifractal spectrum of the LFSM.

Theorem 2. Let \( X \) be a linear fractional stable motion parametrized by \( \alpha \in [1, 2) \) and \( H \in (0, 1) \). It satisfies almost surely for all \( \sigma \in [H - \frac{1}{\alpha} - 1, H - \frac{1}{\alpha}] \),

\[
\forall V \in \mathcal{O}; \quad \dim_H(E_{\sigma, s'} \cap V) = \begin{cases} 
\alpha(s - H) + 1 & \text{if } s \in [H - \frac{1}{\alpha}, H]; \\
-\infty & \text{otherwise.}
\end{cases}
\] (1.13)

where \( s = \sigma - s' \). Furthermore, for all \( s' \in \mathbb{R} \), \( E_{\sigma, s'} \) is empty if \( \sigma > H - \frac{1}{\alpha} \).

Remark 4. In the continuous case \( H > 1/\alpha \), Theorem 2 ensures that the multifractal spectrum \( (\sigma = 0) \) of the LFSM is equal to

\[
\forall V \in \mathcal{O}; \quad d_X(h, V) = \begin{cases} 
\alpha(h - H) + 1 & \text{if } h \in [H - \frac{1}{\alpha}, H]; \\
-\infty & \text{otherwise.}
\end{cases}
\] (1.14)

Spectrum 1.14 and Equation (1.13) clearly extend the aforementioned lower and upper bounds obtained on the pointwise and local Hölder exponents. We also note that as it could be expected, the LFSM is an homogeneous multifractal process.
Remark 5. More generally, we observe that Theorem 2 unifies in terms of regularity the continuous \((H > \frac{1}{2})\) and unbounded \((H < \frac{1}{2})\) cases. Indeed, in both situations, the domain of acceptable 2-microlocal frontiers have the same multifractal structure. When \(H > \frac{1}{2}\), it intersects the \(s'\)-axis, which induces \(\bar{\alpha}_{X,t} > 0\) and therefore the continuity of trajectories owing to properties of the 2-microlocal frontier.

On the contrary, when the domain is located strictly below the \(s'\)-axis, it implies that sample paths are nowhere bounded. Nevertheless, the proof of Theorem 2 ensures in this case the existence of modification of the LFSM such that sample paths are tempered distributions whose 2-microlocal regularity can be studied as well. Figure 1 illustrates this dichotomy.

\[
\dim_{H}(E_{\sigma,s'}) = \alpha(s - H) + 1
\]

(a) Continuous sample paths: \(H = \frac{5}{6}\) and \(\alpha = \frac{3}{2}\)

(b) Unbounded sample paths: \(H = \frac{1}{2}\) and \(\alpha = \frac{3}{2}\)

Figure 1: Domains of admissible 2-microlocal frontiers for the LFSM

An equivalent result is obtained in Proposition 3 for a similar class of processes called fractional Lévy processes (see [11, 31, 15]).

The LFSM admits a natural multifractional extension which has been introduced and studied in [42, 43, 16]. The definition of the linear multifractional stable motion (LMSM) is given by equation (1.12) where the Hurst exponent \(H\) is replaced by a function \(t \mapsto H(t)\). Stoev and Taqqu [42] obtained lower and upper bounds on Hölder exponents which are similar to LFSM results: for all \(t \in \mathbb{R}^+\), \(H(t) - 1/\alpha \leq \alpha_{X,t} \leq H(t)\) and \(\bar{\alpha}_{X,t} = H(t) - 1/\alpha\) almost surely. Ayache and Hamonier [4] recently investigated the existence of optimal local modulus of continuity.

Theorem 2 can be generalized to the LMSM in the continuous case. More precisely, we assume that the Hurst function satisfies the following assumption,

\((\mathcal{H}_0)\) \(H : \mathbb{R} \rightarrow (\frac{1}{\alpha}, 1)\) is \(\delta\)-Hölderian, with \(\delta > \sup_{t \in \mathbb{R}} H(t)\).

Since the LMSM is clearly a non-homogeneous process, it is natural to focus on the study of the spectrum of singularities localized at \(t \in \mathbb{R}^+\), i.e.

\[
\forall t \in \mathbb{R}^+ \quad d_X(h,t) = \lim_{\rho \to 0} d_X(h,B(t,\rho)) = \lim_{\rho \to 0} \dim_{H}(E_h \cap B(t,\rho)).
\]

The next statement correspond to an adaptation of Theorem 2 to the LMSM.
Theorem 3. Let $X$ be a linear multifractional stable motion parametrized by $\alpha \in (1, 2)$ and an $(H_0)$-Hurst function $H$. It satisfies almost surely for all $t \in \mathbb{R}$ and for all $\sigma \in [H(t) - \frac{1}{\alpha} - 1, H(t) - \frac{1}{\alpha})$,

$$\lim_{\rho \to 0} \dim_H(E_{\sigma,s'} \cap B(t, \rho)) = \begin{cases} \alpha(s - H(t)) + 1 & \text{if } s \in [H(t) - \frac{1}{\alpha}, H(t)]; \\ -\infty & \text{otherwise.} \end{cases}$$

(1.15)

where $s = \sigma - s'$. Furthermore, the set $E_{\sigma,s'} \cap B(t, \rho)$ is empty for any $\sigma > H(t) - \frac{1}{\alpha}$ and $\rho > 0$ sufficiently small.

Remark 6. Theorem 3 extends results presented in [42, 43]. In particular, it ensures that the localized multifractal spectrum is equal to

$$\forall t \in \mathbb{R}_+: \quad d_X(h, t) = \begin{cases} \alpha(h - H(t)) + 1 & \text{if } h \in [H(t) - \frac{1}{\alpha}, H(t)]; \\ -\infty & \text{otherwise.} \end{cases}$$

(1.16)

Moreover, we observe that Proposition 2 and Theorem 3 still hold when the Hurst function $H(\cdot)$ is a continuous random process. Thereby, similarly to the works of Barral et al. [10] and Durand [17], it provides a class stochastic processes whose spectrum of singularities, given by equation (1.16), is non-homogeneous and random.

2. Regularity of Lévy processes

In this section, $X$ will designate a Lévy process parametrized by the generating triplet $(a, Q, \pi)$. Lévy-Itô decomposition states that it can represented as the sum of three independent processes $B, N$ and $Y$, where $B$ is a $d$-dimensional Brownian motion, $N$ is a compound Poisson process with drift and $Y$ is a Lévy process characterized by $(0, 0, \pi(dx)1_{\|x\| \leq 1})$.

Without any loss of generality, we restrict the study to the time interval $[0, 1]$. As noticed in [22], the component $N$ does not affect the regularity of $X$ since its trajectories are piecewise linear with a finite number of jumps. Sample path properties of Brownian motion are well-known and therefore, we first focus in the sequel on the study of the jump process $Y$.

We know there exists a Poisson measure $J(dt, dx)$ of intensity $\mathcal{L}^d \otimes \pi$ such that $Y$ is given by

$$Y_t = \lim_{\epsilon \to 0} \left[ \int_{[0,t] \times D(\epsilon,1)} x J(ds, dx) - t \int_{D(\epsilon,1)} x \pi(dx) \right],$$

where for all $0 \leq a < b$, $D(a, b) = \{ x \in \mathbb{R}^d : a < \|x\| \leq b \}$. Moreover, as presented in [40] (Theorem 19.2), the convergence is almost surely uniform on any bounded interval. For any $m \in \mathbb{R}_+$, $Y^m$ will denote the following Lévy process

$$Y_t^m = \lim_{\epsilon \to 0} \left[ \int_{[0,t] \times D(\epsilon,2^{-m})} x J(ds, dx) - t \int_{D(\epsilon,2^{-m})} x \pi(dx) \right].$$

(2.1)

2.1. Pointwise exponent

We extend in this section the multifractal spectrum (1.8) to any Lévy process. To begin with, we prove two technical lemmas that will be extensively used in the sequel.
Lemma 2.1. For any $\delta > \beta$, there exists a constant $C_\delta > 0$ such that for all $m \in \mathbb{R}_+$
\[
P\left(\sup_{t \leq 2^{-m}} \|Y^{m/\delta}_t\|_1 \geq m 2^{-m/\delta}\right) \leq C_\delta e^{-m}.
\]

Proof. Let $\delta > \beta$. We first observe that for any $m \in \mathbb{R}_+$,
\[
\left\{\sup_{t \leq 2^{-m}} \|Y^{m/\delta}_t\|_1 \geq m 2^{-m/\delta}\right\} = \bigcup_{\varepsilon \in \{-1,1\}} \left\{\sup_{t \leq 2^{-m}} \langle \varepsilon, Y^{m/\delta}_t \rangle \geq m 2^{-m/\delta}\right\}
\]
Therefore, it is sufficient to prove that there exists $C_\delta > 0$ such that for any $\varepsilon \in \{-1,1\}^d$,
\[
P\left(\sup_{t \leq 2^{-m}} \langle \varepsilon, Y^{m/\delta}_t \rangle \geq m 2^{-m/\delta}\right) \leq C_\delta e^{-m}.
\]

Let $\lambda = 2^{m/\delta}$ and $M_t = e^{\lambda(\varepsilon, Y^{m/\delta}_t)}$ for all $t \in \mathbb{R}_+$. According to Theorem 25.17 in [40], we have $\mathbb{E}[M_t] = \exp\{t \int_{D(0,2^{-m/\delta})} (e^{\lambda(\varepsilon, x)} - 1 - \lambda(\varepsilon, x)) \pi(dx)\}$. Furthermore, we observe that for all $s \leq t \in \mathbb{R}_+$,
\[
\mathbb{E}[M_t | \mathcal{F}_s] = M_s \exp\left\{(t - s) \int_{D(0,2^{-m/\delta})} (e^{\lambda(\varepsilon, x)} - 1 - \lambda(\varepsilon, x)) \pi(dx)\right\} \geq M_s,
\]
since for any $y \in \mathbb{R}$, $e^y - 1 - y \geq 0$. Hence, $M$ is a positive submartingale, and using Doob’s inequality (Theorem 1.7 in [37]), we obtain
\[
P\left(\sup_{t \leq 2^{-m}} \langle \varepsilon, Y^{m/\delta}_t \rangle \geq m 2^{-m/\delta}\right) = P\left(\sup_{t \leq 2^{-m}} M_t \geq e^m\right) \leq e^{-m} \mathbb{E}[M_{2^{-m}}].
\]

For all $y \in [-1,1]$, we note that $e^y - 1 - y \leq y^2$. Thus, for any $m \in \mathbb{R}_+$,
\[
\mathbb{E}[M_{2^{-m}}] \leq \exp\left\{2^{-m} \int_{D(0,2^{-m/\delta})} \lambda^2 \langle \varepsilon, x \rangle^2 \pi(dx)\right\} \leq \exp\left\{2^{-m} \int_{D(0,2^{-m/\delta})} \lambda^2 \|x\|^2 \pi(dx)\right\}.
\]

If $\beta < 2$, let $\gamma > 0$ be such that $\beta < \gamma < 2$ and $\gamma < \delta$. Then, we obtain
\[
2^{-m} \int_{D(0,2^{-m/\delta})} \lambda^2 \|x\|^2 \pi(dx) = 2^{-m(1-2/\delta)} \int_{D(0,2^{-m/\delta})} \|x\|^\gamma \cdot \|x\|^{2-\gamma} \pi(dx)
\leq 2^{-m(1-2/\delta)} 2^{-m/\delta(2-\gamma)} \int_{D(0,1)} \|x\|^\gamma \pi(dx)
= 2^{-m(1-\gamma/\delta)} \int_{D(0,1)} \|x\|^\gamma \pi(dx) \leq \int_{D(0,1)} \|x\|^\gamma \pi(dx),
\]
since $\gamma < \delta$. If $\beta = 2$, we simply observe that
\[
2^{-m} \int_{D(0,2^{-m/\delta})} \lambda^2 \|x\|^2 \pi(dx) \leq 2^{-m(1-2/\delta)} \int_{D(0,1)} \|x\|^2 \pi(dx) \leq \int_{D(0,1)} \|x\|^2 \pi(dx),
\]
as $\delta > 2$. Therefore, there exists $C_\delta > 0$ such that for all $m \in \mathbb{R}_+$, $\mathbb{E}[M_{2^{-m}}] \leq C_\delta$, which proves the lemma. \qed
Lemma 2.2. For any $\delta > \beta$, there exists a constant $C_\delta > 0$ such that for all $m \in \mathbb{R}_+$

$$
\mathbb{P}\left(\sup_{u,v \in [0,1], |u-v| \leq 2^{-m}} \left\| Y_u^{m/\delta} - Y_v^{m/\delta} \right\|_1 \geq 3m2^{-m/\delta}\right) \leq C_\delta e^{-Dm},
$$

where $D$ is positive constant independent of $\delta$.

Proof. We note that for any $m \in \mathbb{R}_+$ and all $\delta > \beta$,

$$
\left\{ \sup_{u,v \in [0,1], |u-v| \leq 2^{-m}} \left\| Y_u^{m/\delta} - Y_v^{m/\delta} \right\|_1 \geq 3m2^{-m/\delta} \right\}
\subseteq \bigcup_{k=0}^{2^m-1} \left\{ \sup_{t \leq 2^{-m}} \left\| Y_t^{m/\delta} - Y_{t+k2^{-m}}^{m/\delta} \right\|_1 \geq m2^{-m/\delta} \right\}.
$$

Therefore, the stationarity of Lévy processes and Lemma 2.1 yield

$$
\mathbb{P}\left(\sup_{u,v \in [0,1], |u-v| \leq 2^{-m}} \left\| Y_u^{m/\delta} - Y_v^{m/\delta} \right\|_1 \geq 3m2^{-m/\delta}\right) \leq 2^m C_\delta e^{-m} = C_\delta e^{-Dm},
$$

where $D = 1 - \log(2)$.

Let us recall the definition of the collection of random sets $(A_\delta)_{\delta>0}$ introduced in [22]. For every $\omega \in \Omega$, $S(\omega)$ denotes the countable set of jumps of $Y_\cdot(\omega)$. Moreover, for any $\varepsilon > 0$, let $A_\varepsilon$ be

$$
A_\varepsilon = \bigcup_{t \in S(\omega)} [t - \| \Delta Y_t \|^\delta, t + \| \Delta Y_t \|^\delta].
$$

Then, the random set $A_\delta$ is defined by $A_\delta = \lim \sup_{\varepsilon \to 0} A_\varepsilon$. As noticed in [22], if $t \in A_\delta$, we necessarily have $\alpha_{Y,t} \leq \frac{1}{\delta}$. The other side inequality is obtained in the next statement, extending the proof of Proposition 2 from [22].

Proposition 1. Let $\delta > \beta$. Almost surely for all $t \in [0,1] \setminus S(\omega)$, $Y$ satisfies

$$
t \notin A_\delta \implies \alpha_{Y,t} \geq \frac{1}{\delta}.
$$

Proof. Using Lemma 2.2 and Borel-Cantelli lemma, we know that almost surely, there exists $M(\omega)$ such that for any $m \in \mathbb{N} \geq M(\omega)$,

$$
\forall u, v \in [0,1] \text{ such that } |u-v| \leq 2^{-m}; \quad \left\| Y_u^{m/\delta} - Y_v^{m/\delta} \right\|_1 \leq Cm2^{-m/\delta}, \quad (2.2)
$$

where $C$ is a positive constant. Furthermore, as $t \notin A_\delta$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, $t \notin A_\varepsilon$. Then, for all $u$ in the neighbourhood of $t$, we have

$$
\int_{[t,u] \times D(\varepsilon_0,1)} x J(ds, dx) - (t-u) \int_{D(\varepsilon_0,1)} x \pi(dx) = -(t-u) \int_{D(\varepsilon_0,1)} x \pi(dx).
$$

Since a linear component does not contribute to the pointwise exponent, we only have to consider the remaining part of the Lévy process to characterize the regularity.
Let \( u \in [0, 1] \) and \( m \in \mathbb{N} \) such that \( 2^{-m-1} \leq |t - u| < 2^{-m} \). \( m \) can be supposed large enough to satisfy \( m \geq M(\omega) \). Let \( \varepsilon_1 = 2^{-m/\delta} \). For any jump \( \Delta Y_s \) whose norm is in the interval \([\varepsilon_1, \varepsilon_0]\), we have \( \|\Delta Y_s\| \delta \geq \varepsilon_1^\delta > |t - u| \).

Therefore, there is no such jumps \( \Delta Y_s \) in the interval \([t, u] \) and \( \int_{[t,u] \times D(\varepsilon_1, \varepsilon_0)} x J(ds, dx) = 0 \). Let now distinguish two different cases.

1. If \( \delta \geq 1 \), we obtain

\[
\left\| (t - u) \int_{D(\varepsilon_1, \varepsilon_0)} x \pi(dx) \right\| \leq |t - u| \int_{D(\varepsilon_1, \varepsilon_0)} \|x\|^{\delta} \cdot \|x\|^{1-\delta} \pi(dx)
\]

\[
\leq |t - u| \cdot \varepsilon_1^{1-\delta} \int_{D(\varepsilon_1, \varepsilon_0)} \|x\|^{\delta} \pi(dx) \leq C|t - u|^{1/\delta}.
\]

Furthermore, \( \|Y_t^{m/\delta} - Y_u^{m/\delta}\| \leq C \log(|t - u|) \cdot |t - u|^{1/\delta} \) according to equation (2.2).

Hence, these two inequalities imply \( \alpha_{Y,t} \geq \frac{1}{\delta} \).

2. If \( \delta < 1 \) (and thus \( \beta < 1 \)), we have \( (t - u) \int_{D(\varepsilon_1, \varepsilon_0)} x \pi(dx) = (t - u) \int_{D(0, \varepsilon_0)} x \pi(dx) - (t - u) \int_{D(\varepsilon_1, \varepsilon_0)} x \pi(dx) \). The component \( (t - u) \int_{D(0, \varepsilon_0)} x \pi(dx) \) is linear in the neighbourhood of \( t \), and therefore can be ignored. For the second integral, we similarly observe that

\[
\left\| (t - u) \int_{D(0, \varepsilon_1)} x \pi(dx) \right\| \leq |t - u| \cdot \varepsilon_1^{1-\delta} \int_{D(0, \varepsilon_1)} \|x\|^{\delta} \pi(dx) \leq C|t - u|^{1/\delta}.
\]

This last inequality and Equation (2.2) prove that \( \alpha_{Y,t} \geq \frac{1}{\delta} \).

\[\square\]

Proposition 1 ensures that almost surely

\[\forall h > 0; \ E_h = \left( \bigcap_{\delta < 1/h} A_\delta \right) \setminus \left( \bigcup_{\delta > 1/h} A_\delta \right) \setminus S \quad \text{and} \quad E_0 = \left( \bigcap_{\delta > 0} A_\delta \right) \cup S. \quad (2.3)\]

Furthermore, since the estimate of the Hausdorff dimension obtained in [22] does not rely on Assumption (1.7), \( Y \) satisfies almost surely

\[\forall V \in \mathcal{O}; \quad \dim_R(E_h \cap V) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta]; \\ -\infty & \text{else.} \end{cases}\]

### 2.2. 2-microlocal frontier: proof of Theorem 1

Let us now study the 2-microlocal frontier of Lévy processes. According to Theorem 3.13 in [32], for all \( t \in [0, 1] \) and any \( s' > -\alpha_{Y,t} \), the sample path \( Y_t(\omega) \) almost surely belongs to the 2-microlocal space \( C_{t,s}' \). Furthermore, owing to the density of the set of jumps \( S(\omega) \) in \([0, 1] \), we necessarily have \( Y_t(\omega) \notin C_{t,s}' \) for any \( \alpha > 0 \) and \( s' \in \mathbb{R} \). Hence, since the 2-microlocal frontier is a concave function with left- and right-derivatives between 0 and 1, we obtain that almost surely, for all \( t \in [0, 1] \)

\[\forall s' \in \mathbb{R}_+; \quad \sigma_{Y,t}(s') \geq (\alpha_{Y,t} + s') \wedge 0 \quad \text{and} \quad \sigma_{Y,t}(s') \leq 0.\]
Therefore, we consider in the sequel on the negative component of the 2-microlocal frontier of \( Y \). As outlined in the introduction and according to Definition 2 of 2-microlocal spaces, we need to study the increments around \( t \) of the integral of the process \( Y \), i.e.

\[
\forall u, v \in B(t, \rho); \quad \left\| \int_b^v Y_s \, ds - P_t(v) - \int_b^u Y_s \, ds + P_t(u) \right\|,
\]

where \( b < t \) is fixed. The form of the polynomial \( P_t \) depends on \( \beta \). If \( \beta \geq 1 \), we only need to remove a linear component equal to \( \int_b^t Y \, ds \), and therefore the increment simply corresponds to \( \int_b^u (Y_s - Y_t) \, ds \). On the other hand, if \( \beta < 1 \), the proof of Proposition 1 induces that we also need to subtract the compensation term \( s \mapsto (s - t) \int_{D(0,1)} x \pi(dx) \). Then, in this case, we have to study

\[
\forall u, v \in B(t, \rho); \quad \left\| \int_u^v \left\{ Y_s - Y_t - (s - t) \int_{D(0,1)} x \pi(dx) \right\} ds \right\|
\]

For sake of readability, we divide the proof of Theorem 1 and its corollary in different technical lemmas. We begin by obtaining a global upper bound of the frontier.

**Lemma 2.3.** Almost surely for all \( t \in [0, 1] \), the 2-microlocal frontier \( \sigma_{Y,t} \) satisfies

\[
\forall s' \in \mathbb{R} \quad \text{s.t} \quad \sigma_{Y,t}(s') \in [-1, 0]; \quad \sigma_{Y,t}(s') \leq \left( \frac{1}{\beta} + s' \right) \land 0.
\]

**Proof.** Let \( m \in \mathbb{N} \), \( \varepsilon > 0 \), \( \alpha = \beta(1 - 2\varepsilon) \) and \( \gamma = \beta(1 + 4\varepsilon) \). As stated in [22], to prove that \( \sigma_{Y,t} \leq 1/\beta \), it is sufficient to exhibit for any \( \varepsilon > 0 \) a sequence \( (t_n)_{n \in \mathbb{N}} \) such that \( \|\Delta Y_n\| = 2^{-n} \) and \( |t - t_n| \leq 2^{-m\alpha} \). In order to extend this inequality to the 2-microlocal frontier and obtain equation (2.5), we need to reinforce the previous statement and show that in the neighbourhood of \( t_n \) there is no other jump of similar size.

More precisely, let consider \( [2^{m\alpha}] \) consecutive intervals \( I_j \) of size \( 2^{-m\alpha} \). The family \( (I_j)_j \) forms a cover of \([0, 1]\). Each \( I_j \) can be divided into at least \( 2^{m(\gamma - \alpha)} \) disjoint of intervals \( I_{i,j} \) of size \( 2^{-m\gamma} \). Finally, let \( I_{i,j} = I^1_{i,j} \cup I^2_{i,j} \cup I^3_{i,j} \) be the three consecutive intervals of same size inside \( I_{i,j} \). We investigate the probability of the following event

\[
A_{i,j} = \{ J(I^1_{i,j}, D(2^{-m}, 1)) = 0 \} \cap \{ J(I^3_{i,j}, D(2^{-m}, 1)) = 0 \} \cap
\{ J(I^2_{i,j}, D(2^{-m}, 1)) = 1 \} \cap \{ J(I_{i,j}, D(2^{-m(1+\varepsilon)}, 2^{-m})) = 0 \},
\]

Since \( J \) is a Poisson measure, \( A_{i,j} \) corresponds to the intersection of independent events and we have

\[
\mathbb{P}(A_{i,j}) = \frac{2^{-m\gamma}}{3} \pi(D(2^{-m}, 1)) \cdot \exp \left( -2^{-m\gamma} \pi(D(2^{-m}, 1)) + 2^{-m\gamma} \pi(D(2^{-m(1+\varepsilon)}, 2^{-m})) \right).\]

As described in [12], \( \beta \) can also be defined by \( \beta = \inf \{ \delta > 0 : \lim \sup_{r \rightarrow 0} r^\delta \pi(B(r, 1)) < \infty \} \).

Therefore, there exists \( r_0 > 0 \) such that for all \( r \in (0, r_0] \), \( \pi(B(r, 1)) \leq r^{-\beta(1+\varepsilon)} \). Then, for all \( m \in \mathbb{N} \) large enough, we obtain

\[
\mathbb{P}(A_{i,j}) \geq 2^{-m\gamma - 2} \pi(D(2^{-m}, 1)) \cdot \exp \left( -2 \cdot 2^{-m\beta\varepsilon} \right) = 2^{-m\gamma - 2} \pi(D(2^{-m}, 1)) \cdot (1 + o_m(1))\]
According to the definition of $\beta$, there also exists an increasing sequence $(m_n)_{n \in \N}$ such that for all $n \in \N$, $\pi(D(2^{-m_n}, 1)) \geq 2^{m_n(\beta(1+\varepsilon))}$. Therefore, along this sub-sequence, we have $\mathbb{P}(A_{i,j}) \geq 2^{-5m_n\beta}\varepsilon^{-2} \cdot (1 + o_n(1))$ for any $n \in \N$.

Let now consider the event $B_{n,j}$ defined by

$$B_{n,j} = \bigcap_{i=1}^{\lceil 2^{m_n(\gamma-\alpha)} \rceil} A_{i,j}.$$  

Since the events $(A_{i,j})_i$ are identical and independent, we obtain

$$\mathbb{P}(B_{n,j}) = \mathbb{P}(A_{1,j}^{\lceil 2^{m_n(\gamma-\alpha)} \rceil}) \leq \left(1 - 2^{-5m_n\beta}\varepsilon^{-2} \cdot (1 + o(1))\right)^{2^{m_n(\gamma-\alpha)} - 1},$$

which leads to $\log(\mathbb{P}(B_{n,j})) \leq 2^{m_n(\gamma-\alpha)} - 1 \log \left(1 - 2^{-5m_n\beta}\varepsilon^{-2} \cdot (1 + o(1))\right) = \frac{-2m_n\beta}{3} \cdot (1 + o(1))$, since $\gamma - \alpha = 6\beta\varepsilon$. Therefore, for all $n \in \N$, $\mathbb{P}(B_{n,j}) \leq \exp \left(-\frac{2m_n\beta}{3} \cdot (1 + o(1))\right)$.

Finally, let $B_n = \bigcup_{j=1}^{2^{m_n}} B_{n,j}$. For all $n \in \N$, it satisfies

$$\mathbb{P}(B_n) \leq 2^{m_n+1} \cdot \exp \left(-\frac{2m_n\beta}{3} \cdot (1 + o(1))\right) \leq \exp \left(Cm_n - 2\frac{m_n\beta}{3} \cdot (1 + o(1))\right),$$

where $C$ is a positive constant. Hence, using Borel-Cantelli lemma, there exists an event $\Omega_0$ of probability 1 such that for any $\omega \in \Omega_0$, there exists $N(\omega)$ and for all $n \geq N(\omega)$,

$$\omega \in B_n^c(\omega) = \bigcap_{j=1}^{2^{m_n}} \bigcup_{i=1}^{\lceil 2^{m_n(\gamma-\alpha)} \rceil} A_{i,j}.$$  

Let now $t \in [0, 1]$, $\omega \in \Omega_0$ and $n \geq N(\omega)$. There exist $i, j \in \N$ such that $t \in I_j$ and $\omega \in A_{i,j}$. Hence, according to the definition of the event $A_{i,j}$, there is $t_n \in I_{i,j}$ such that $\|\Delta Y_{t_n}\| \geq 2^{-m_n}$ and $|t - t_n| \leq 2^{-m_n\alpha}$. Furthermore, there is no other jump of size greater than $2^{-m_n(1+\varepsilon)}$ in the ball $B(t_n, \rho_n)$, where $\rho_n = 2^{-m_n\gamma}/3$. Since $\Delta Y_{t_n} = (Y_{t_n} - Y_t) + (Y_t - Y_{t_n})$, we necessarily have $|Y_{t_n} - Y_t| \geq 2^{-m_n-1}$ or $|Y_t - Y_{t_n}| \geq 2^{-m_n-1}$. Without any loss of generality, we assume in the sequel that $|Y_{t_n} - Y_t| \geq 2^{-m_n-1}$ is satisfied.

As previously outlined, we need to study the increments described in equation (2.4). Specifically, we observe on the interval $[t_n, t_n + \rho_n]$ that

$$\left\| \int_{t_n}^{t_n + \rho_n} (Y_s - Y_{t_n}) \, ds \right\| = \left\| \rho_n \cdot (Y_{t_n} - Y_t) + \int_{t_n}^{t_n + \rho_n} (Y_s - Y_{t_n}) \, ds \right\| \geq \rho_n \cdot \|Y_{t_n} - Y_t\| - \int_{t_n}^{t_n + \rho_n} \|Y_s - Y_{t_n}\| \, ds.$$

Let obtain an upper bound for the second term. There is no jump of size greater than $2^{-m_n(1+\varepsilon)}$ in the interval $[t_n, t_n + \rho_n]$. Therefore, for all $s \in (t_n, t_n + \rho_n)$, we have $Y_s - Y_{t_n} = Y_{s - \rho_n - t_n}^{m_n(1+\varepsilon)} - Y_{\rho_n - t_n}^{m_n(1+\varepsilon)} - (s - t_n) \int_{D(2^{-m_n(1+\varepsilon)}, 1)} x \pi(dx)$. Lemma 2.2 provides an upper bound for the first term. Indeed, let $m_n = m_n(1 + \varepsilon)\delta$, where $\delta = \beta(1 + \varepsilon) > \beta$. Using Borel-Cantelli lemma, we know that for any $n \in \N$ sufficiently large, for all $u, v \in [0, 1]$ such that $|u - v| \leq 2^{-m_n}$, $\left| Y_u^{m_n(1+\varepsilon)} - Y_v^{m_n(1+\varepsilon)} \right| \leq Cm_n 2^{-m_n(1+\varepsilon)}$. Moreover, we observe that
\[ |s - t_n| \leq \rho_n = \frac{1}{2} 2^{-m_n} \leq 2^{-m_n(1+\varepsilon)^2} = 2^{-\tilde{m}_n \delta}. \] Therefore, using the previous inequalities, we obtain that almost surely for all \( n \geq N(\omega) \)
\[
\forall s \in [t_n, t_n + \rho_n]; \quad \|Y_s^{m_n(1+\varepsilon)} - Y_{t_n}^{m_n(1+\varepsilon)}\| \leq C m_n 2^{-m_n(1+\varepsilon)}.
\] (2.6)

To investigate the second integral term, we have distinguished the two different cases introduced above.

1. If \( \beta \geq 1 \), the integral of the linear drift corresponds to
\[
\int_{t_n}^{t_n + \rho_n} (s - t_n) \, ds \int_{D(2^{-m_n(1+\varepsilon)}, 1)} x \pi(dx) = \frac{\rho_n^2}{2} \int_{D(2^{-m_n(1+\varepsilon)}, 1)} x \pi(dx).
\]
The norm of the previous expression is bounded above by
\[
\rho_n^2 \int_{D(2^{-m_n(1+\varepsilon)}, 1)} \|x\| \pi(dx) = \rho_n^2 \int_{D(2^{-m_n(1+\varepsilon)}, 1)} \|x\|(1+\varepsilon)^{\varepsilon} \cdot \|x\|^{1-(1+\varepsilon)^{\varepsilon}} \pi(dx)
\leq \rho_n^2 \cdot 2^{-m_n(1+\varepsilon)(1-(1+\varepsilon)^{\varepsilon})} \int_{D(0,1)} \|x\|(1+\varepsilon)^{\varepsilon} \pi(dx)
\leq C \rho_n \cdot 2^{-m_n(1+\varepsilon)},
\] (2.7)
as \( \rho_n = 2^{-m_n \gamma} \).

2. If \( \beta < 1 \), we know that the drift \( u \int_{D(0,1)} x \pi(dx) \) can be removed from the Lévy process. Therefore, we only have to consider the following quantity
\[
\int_{t_n}^{t_n + \rho_n} (s - t_n) \, ds \int_{D(0,2^{-m_n(1+\varepsilon)})} x \pi(dx) = \frac{\rho_n^2}{2} \int_{D(0,2^{-m_n(1+\varepsilon)})} x \pi(dx).
\]
Since it can be assumed that \( (1+\varepsilon)\beta \leq 1 \), we similarly get
\[
\left\| \rho_n^2 \int_{D(0,2^{-m_n(1+\varepsilon)})} x \pi(dx) \right\| \leq \rho_n^2 \int_{D(0,2^{-m_n(1+\varepsilon)})} \|x\|(1+\varepsilon)^{\varepsilon} \cdot \|x\|^{1-(1+\varepsilon)^{\varepsilon}} \pi(dx)
\leq C \rho_n^2 \cdot 2^{-m_n(1+\varepsilon)(1-(1+\varepsilon)^{\varepsilon})} \leq C \rho_n \cdot 2^{-m_n(1+\varepsilon)}.
\] (2.8)

Inequalities (2.6), (2.7) and (2.8) yield
\[
\left\| \int_{t_n}^{t_n + \rho_n} Y_s \, ds - P_t(t_n + \rho_n) + P_t(t_n) \right\| \geq \rho_n \cdot \|Y_{t_n} - Y_t\| - C m_n \rho_n \cdot 2^{-m_n(1+\varepsilon)}
\geq \rho_n \cdot 2^{-m_n - 1} - C m_n \rho_n \cdot 2^{-m_n(1+\varepsilon)} \geq \tilde{C} \rho_n \cdot 2^{-m_n}.
\]
where \( C \) and \( \tilde{C} \) are positive constants independent of \( n \). Finally, since \( |t - t_n| \leq 2^{-m_n \alpha} \),
\[
\left\| \int_{t_n}^{t_n + \rho_n} Y_s \, ds - P_t(t_n + \rho_n) + P_t(t_n) \right\| \geq \tilde{C} 2^{-m_n(1+\varepsilon + \beta(1+4\varepsilon))} \geq \tilde{C} |t - t_n|^{-m_n \lambda}.
\] (2.9)
where \( \lambda = \frac{1+\varepsilon + \beta(1+4\varepsilon)}{\alpha(1-2\varepsilon)}. \)
Hence, according to Definition 2 of the 2-microlocal spaces, this last equation (2.9) ensures that
\[ \forall s' \in \mathbb{R} \quad \text{s.t.} \quad \sigma_{Y,t}(s') \in [-1,0]: \quad \sigma_{Y,t}(s') \leq (\lambda - 1 + s') \wedge 0. \]
Since the inequality holds for any \( \varepsilon \in \mathbb{Q} > 0 \) and \( \lambda \to_{\varepsilon \to 0+} 1 + \frac{1}{\beta} \), we obtain the expected result. \( \square \)

The following simple lemma will be used in the sequel to obtain the 2-microlocal frontier when \( \alpha_{Y,t} < 1/\beta \).

**Lemma 2.4.** Let \( \alpha > \beta, \varepsilon > 0 \) and \( k \in \mathbb{N} \) such that
\[ \frac{\alpha}{\beta} > \frac{(1 + 3\varepsilon) \cdot (k + 1)}{k}. \] (2.10)
For all \( m \in \mathbb{N} \), let \( (I_{j,m})_{j \in \mathbb{N}} \) be the collection of successive subintervals of \([0,1]\) of size \( 2^{-\alpha m} \).
Then, there exists almost surely \( M(\omega) \in \mathbb{N} \) such that for all \( m \geq M \) and for any interval \( I_{j,m} \), there are at most \( k \) jumps of size greater than \( 2^{-m(1+\varepsilon)} \) in \( I_{j,m} \).

**Proof.** For any \( m \in \mathbb{N} \) and \( j \in \mathbb{N} \), we know that \( J(I_{m,j}, D(2^{-m(1+\varepsilon)}, 1)) \) is Poisson variable. Hence, we have
\[
\mathbb{P}\left(\bigcup_{j=1}^{2^{m\alpha}} \left\{ J(I_{m,j}, D(2^{-m(1+\varepsilon)}, 1)) > k \right\} \right) \leq 2^{m\alpha} \cdot \exp(-\lambda_m) \left\{ \sum_{i=k+1}^{+\infty} \frac{\lambda_i^i}{i!} \right\},
\]
where \( \lambda_m = 2^{-m\alpha} \pi(D(2^{-m(1+\varepsilon)}, 1)) \). According to the definition of \( \beta \), we know that for all \( m \) sufficiently large, \( \lambda_m \leq 2^{-m(\alpha-\beta(1+3\varepsilon))} \). Therefore,
\[
\mathbb{P}\left(\bigcup_{j=1}^{2^{m\alpha}} \left\{ J(I_{m,j}, D(2^{-m(1+\varepsilon)}, 1)) > k \right\} \right) \leq C 2^{m\alpha} \cdot 2^{-m(k+1)(\alpha-\beta(1+3\varepsilon))}(1 + o(1)) = 2^{-m\delta}(1 + o(1)),
\]
where \( \delta \) is a positive constant, according to the assumption made on \( \alpha, \varepsilon \) and \( k \). Borel-Cantelli lemma concludes the proof. \( \square \)

In the next lemma, we study the behaviour of the 2-microlocal frontier of \( Y \) at points \( t \in [0,1] \) where \( \alpha_{Y,t} \in [0, 1/2\beta] \).

**Lemma 2.5.** Almost surely, for all \( h \in [0,1/2\beta) \), we have \( \hat{E}_h = E_h \) and \( \hat{E}_h = \emptyset \), i.e. for all \( t \in E_h \)
\[ \forall s' \in \mathbb{R}; \quad \sigma_{Y,t}(s') = (\alpha_{Y,t} + s') \wedge 0. \]

**Proof.** Let \( \varepsilon \in \mathbb{Q} > 0 \) and \( \alpha \in \mathbb{Q} > 2\beta + 7\varepsilon \). Inequality (2.10) is satisfied for \( k = 1 \). Therefore, Lemma 2.4 ensures that there exists almost surely \( M(\omega) \) such that for any \( m \geq M(\omega) \), the distance between two consecutive jumps of size larger than \( 2^{-m(1+\varepsilon)} \) is at least \( 2^{-m\alpha} \).

Let \( h \in [0,1/2\beta] \) and \( t \in E_h \cap [0,1] \setminus S(\omega) \) (\( t \) is not a jump time). According to characterisation (2.3) of the set \( E_h \), there exist sequences \( (m_n)_{n \in \mathbb{N}} \) and \( (t_n)_{n \in \mathbb{N}} \) such that
\[ \forall n \in \mathbb{N}; \quad t_n \in B(t, 2^{-m_n/h(1+\varepsilon)}) \quad \text{and} \quad \|\Delta Y_{t_n}\| = 2^{-m_n}. \]
As previously, we shall assume that $||Y_t - Y_{t_n}|| \geq 2^{-m-n-1}$ and investigate the increment $\int_{t_n + \rho_n}^{t_n + \rho_n} (Y_s - Y_{t_n}) \, ds$, where $\rho_n = 2^{-m-n}$. As stated in Lemma 2.3, for all $n \geq N(\omega)$,

$$\forall s \in [t_n, t_n + \rho_n]; \quad ||Y_s^{m_n(1+\varepsilon)} - Y_{t_n}^{m_n(1+\varepsilon)}|| \leq C m_n 2^{-m_n(1+\varepsilon)}.$$ 

Furthermore, the remaining integral also satisfies the following inequality.

1. If $\beta \geq 1$, the norm of $\rho_n^2 \int_{D(2^{-m_n(1+\varepsilon)}, 1)} x \, \pi(dx)$ is upper bounded by

$$\rho_n^2 \int_{D(2^{-m_n(1+\varepsilon)}, 1)} ||x|| \, \pi(dx) \leq \rho_n^2 \cdot 2^{-m_n(1+\varepsilon)(1-(1+\varepsilon)\beta)} \int_{D(0,1)} ||x||^{(1+\varepsilon)\beta} \, \pi(dx) \leq C \rho_n \cdot 2^{-m_n(1+\varepsilon)},$$

since $\alpha > 2\beta + 7\varepsilon$.

2. If $\beta < 1$, we similarly obtain

$$\left\| \int_{t_n}^{t_n + \rho_n} (s - t_n) \, ds \int_{D(0,2^{-m_n(1+\varepsilon)})} x \, \pi(dx) \right\| \leq C \rho_n \cdot 2^{-m_n(1+\varepsilon)}.$$ 

Hence, previous inequalities yield

$$\left\| \int_{t_n}^{t_n + \rho_n} Y_s \, ds - P_t(t_n + \rho_n) + P_t(t_n) \right\| \geq \rho_n \cdot ||Y_t - Y_{t_n}|| - C m_n \rho_n \cdot 2^{-m_n(1+\varepsilon)} \geq C \rho_n \cdot 2^{-m_n}. \quad (2.11)$$

If $h < 1/2\beta$, there exists $\alpha \in Q > 2\beta$ such that $h^{(1+\varepsilon)} < 1/\alpha$. We observe that $|t_n + \rho_n - t| \leq 2\rho_n$ and the previous expression is lower bounded by $C|t_n + \rho_n - t|^{1+1/\alpha}$. Hence, the 2-microlocal frontier of $Y$ at $t$ enjoys $\sigma_{Y,t}(s') \leq (1/\alpha + s') \land 0$ for all $s' \in R$ such that $\sigma_{Y,t}(s') \in [-1,0]$. Since this inequality is obtained for any $\alpha \in Q$ such that $h^{(1+\varepsilon)} < 1/\alpha$ and $\alpha > 2\beta + 5\varepsilon$, we get the expected formula and $E_h = E_h$.

In the second case $h = 1/\beta$, we know that we can assume $|t_n + \rho_n - t| \leq 2|t - t_n|$. Thus, we have $\rho_n \cdot 2^{-m_n} \geq |t_n + \rho_n - t|^{[2\beta/(1+\varepsilon)] + 2/\alpha(1+\varepsilon)}$. Similarily to the previous case and as $\varepsilon \to 0$, we obtain the expected upper bound of the 2-microlocal frontier.

To conclude this proof, let us consider the case $t \in S(\omega)$. We observe that for all $u \geq t$,

$$\int_t^u Y_s \, ds = (u - t)Y_t + \int_t^u (Y_s - Y_t) \, ds \quad \text{with} \quad \left\| \int_t^u (Y_s - Y_t) \, ds \right\| = o(|t - u|),$$

since $Y$ is right-continuous. Similarly, for all $u \leq t$, $\int_u^t Y_s \, ds = (t - u)Y_t + o(|t - u|)$. Therefore, since $\Delta Y_t = Y_t - Y_{t_-} \neq 0$, there does not exist a polynomial $P_t$ that can cancel both terms $(u - t)Y_t$ and $(t - u)Y_{t_-}$, which proves that $\sigma_{Y,t}(s') = s' \land 0$ for all $s' \in R$. 

For this last technical lemma, we focus on the particular case $\alpha_{Y,t} \in (1/2\beta, 1/\beta)$.

**Lemma 2.6.** Almost surely, for all $h \in [1/2\beta, 1/\beta]$, $Y$ satisfies

$$\forall V \in \mathcal{O}; \quad \dim_H(\tilde{E}_h \cap V) = \beta h \quad \text{and} \quad \dim_H(\hat{E}_h \cap V) \leq 2\beta h - 1 \quad (< \beta h). \quad (2.12)$$

Furthermore, for any $t \in \tilde{E}_h$, we have $\sigma_{Y,t}(s') \leq \frac{h + s'}{2\beta h}$ for all $s' \in R$. 

Proof. According to Lemma 1 in [22], for any \( m \in \mathbb{N} \) and \( \varepsilon > 0 \),
\[
\mathbb{P}\left( J([0, 1], D(2^{-m(1+\varepsilon)}, 1)) \geq 2\pi (D(2^{-m(1+\varepsilon)}, 1)) + 2m \right) \leq e^{-m}.
\]
Hence, using Borel-Cantelli lemma, we know that almost surely, for any \( \varepsilon \in \mathbb{Q} > 0 \) and \( m \) sufficiently large, there are at most \( N_m = 2\pi (D(2^{-m(1+\varepsilon)}, 1)) + 2m \) jumps of size greater than \( 2^{-m(1+\varepsilon)} \) on the interval \([0, 1]\).

For any \( m \in \mathbb{N} \), the process \( t \mapsto \int_{[0, t]} x \cdot J(ds, dx) \) is compound Poisson process, and therefore differences between successive jumps are i.i.d. exponential random variables \((\varepsilon_{i,m})_{i\in\mathbb{N}}\). According to the previous calculus, we only have to consider the first \( N_m \) r.v. \((\varepsilon_{i,m})_{i\in\mathbb{N}}\).

Let \( \varepsilon \in \mathbb{Q} > 0 \), \( \alpha \in (\beta, 2\beta) \cap \mathbb{Q} \) and for all \( m \in \mathbb{N} \), \( Y_m^\alpha \) be the number of variables \((\varepsilon_{j,m})_{j\leq N_m}\), which are smaller than \( 2^{-m\alpha} \). \( Y_m^\alpha \) follows the binomial distribution of parameters \( p_m^\alpha = 1 - \exp(2^{-m\alpha} \cdot \pi (D(2^{-m(1+\varepsilon)}, 1))) \) and \( N_m = \pi (D(2^{-m(1+\varepsilon)}, 1)) + 2m \). According to Markov inequality,
\[
\mathbb{P}(Y_m^\alpha \geq 2^{m(2\beta(1+4\varepsilon)-\alpha)}) \leq 2^{-m(2\beta(1+4\varepsilon)-\alpha)} \cdot p_m^\alpha N_m \\
\leq 2^{-m(2\beta(1+4\varepsilon)-\alpha)} \cdot 2^{-m\alpha} 2^{m\beta(1+3\varepsilon)} = 2^{-m\beta \varepsilon}.
\]
Therefore, using Borel-Cantelli lemma, almost surely for any \( \alpha \in (\beta, 2\beta) \cap \mathbb{Q} \), \( Y_m^\alpha \leq 2^{m(2\beta(1+4\varepsilon)-\alpha)} \) for all \( m \geq M^\alpha(\omega) \). For any \( \alpha \in (\beta, 2\beta) \) and \( m \in \mathbb{N} \), let \( S^\alpha(\omega) \) designates the following set
\[
S_m^\alpha = \bigcup_{j \times j \leq 2^{-m\alpha}} \left\{ \left[ t^l_{j,m} - 2^{-m\alpha}, t^l_{j,m} + 2^{-m\alpha} \right] \cup \left[ t^r_{j,m} - 2^{-m\alpha}, t^r_{j,m} + 2^{-m\alpha} \right] \right\},
\]
where \( t^l_{j,m} \) and \( t^r_{j,m} \) are respectively the left and right points which define the r.v. \( \varepsilon_{j,m} \). Moreover, let \( S^\alpha(\omega) \) denote \( \limsup_{m \to \infty} S_m^\alpha(\omega) \). To obtain an upper for the Hausdorff dimension, we observe that for any \( M \) sufficiently large and \( \gamma > 0 \),
\[
\sum_{m=M}^{+\infty} Y_m^\alpha \cdot (2^{-m\alpha+4})^{\gamma} \leq C \sum_{m=M}^{+\infty} 2^{m(2\beta(1+4\varepsilon)-\alpha(1+\gamma))}.
\]
The series converges if \( \gamma < 2\beta(1+4\varepsilon)/\alpha - 1 \). Since this property is satisfied for any \( \varepsilon > 0 \) and the definition of \( S^\alpha \) does not depend on \( \varepsilon \), it ensures that almost surely for any \( \alpha \in (\beta, 2\beta) \cap \mathbb{Q} \), \( \dim_H(S^\alpha) \leq 2\beta/\alpha - 1 \). Finally, since the family \( (S^\alpha)_{\alpha} \) is decreasing, the inequality stands for any \( \alpha \in (\beta, 2\beta) \).

Let now consider \( h \in (1/2\beta, 1/\beta) \) and \( F_h \) be \( \cap_{\alpha<1/h} S^\alpha \). One readily verifies that \( \dim_H(F_h) \leq 2\beta h - 1 \). Let \( t \in E_h \setminus F_h \). For any \( \varepsilon > 0 \), there exists \( M \) such that for all \( m \geq M \), \( t \not\in S_m^\alpha \), where \( \alpha = \frac{1}{1+\varepsilon} \). Furthermore, as \( t \in E_h \), there exist sequences \((m_n)_{n \in \mathbb{N}}\) and \((t_n)_{n \in \mathbb{N}}\) such that \( v_n \in \mathbb{N} \), \( t_n \in B(t, 2^{-m_n}) \) and \( \|\Delta Y_{m_n}\| = 2^{-m_n} \). Without any loss of generality, we can assume that \( m_n \geq M \) for all \( n \in \mathbb{N} \). Then, since \( t \not\in S_m^\alpha \), we know that for all \( n \in \mathbb{N} \), there is no jump of size larger than \( 2^{-m_n(1+\varepsilon)} \) in \( B(t_n, 2^{-m_n}) \).

Therefore, a reasoning similar to Lemma 2.5 yields
\[
\left\| \int_{t_n}^{t_n+\rho_n} Y_s ds - P_t(t_n+\rho_n) + P_t(t_n) \right\| \geq C \rho_n \cdot 2^{-m_n} \geq C |t-t_n|^{1+1/\alpha},
\]
where \( \rho_n = 2^{-m_n \alpha} \). Since this inequality is satisfied for any \( \varepsilon > 0 \) and \( \alpha = 1/h(1 + \varepsilon) \), the 2-microlocal frontier enjoys \( \sigma_{Y,t}(s') \leq (h + s') \wedge 0 \) for all \( s' \in \mathbb{R} \) such that \( \sigma_{Y,t}(s') \in [-1, 0] \). Hence, we have proved that \( E_h \setminus F_h \subseteq \tilde{E}_h \) and \( \tilde{E}_h \subseteq F_h \), and since \( \dim_H(F_h) \leq 2/3h - 1 \) and \( \dim_H(E_h \setminus F_h) = \beta h \), we obtain the expected estimates.

To conclude this lemma, we obtain an upper bound of the 2-microlocal frontier in the case \( t \in \tilde{E}_h \). Let \( \gamma > 2\beta + 7\varepsilon \), \( s' < -h \) and \( \rho_n = 2^{-m_n \gamma} \). Equation (2.11) obtained in the previous lemma still holds

\[
\left\| \int_{t_n}^{t_n + \rho_n} Y_s \, ds - P_t(t_n + \rho_n) + P_t(t_n) \right\| \geq C \rho_n \cdot 2^{-m_n}
\]

\[
\geq C \rho_n |t - t_n|^{-s'(1+\varepsilon)} \cdot |t - t_n|^{(h+s')(1+\varepsilon)}
\]

\[
\geq C \rho_n^{1+(h+s')(1+\varepsilon)\alpha/\gamma} \cdot |t - t_n|^{-s'(1+\varepsilon)}.
\]

This inequality ensures that for all \( \varepsilon \in \mathbb{Q} > 0 \) and \( s' < -h \), \( \sigma_{Y,t}(1+\varepsilon) \leq (h+s')(1+\varepsilon)\alpha/\gamma \). The limit \( \varepsilon \to 0 \) leads to the expected upper bound. \( \square \)

Before finally proving Theorem 1 and its corollary on the 2-microlocal frontier of Lévy processes, we recall the following result on the increments of a Brownian motion. The proof can be found in [1] (inequality (8.8.26)).

**Lemma 2.7.** Let \( B \) be a \( d \)-dimensional Brownian motion. There exists an event \( \Omega_0 \) of probability one such that for all \( \omega \in \Omega_0, \varepsilon > 0 \), there exists \( h(\omega) > 0 \) such that for all \( \rho \leq h(\omega) \) and \( t \in [0,1] \), we have

\[
\sup_{u,v \in B(t,\rho)} \{\|B_u - B_v\|\} \geq \rho^{1/2+\varepsilon}.
\]

**Proof of Theorem 1.** We use the notations introduced at the beginning of the section. As previously said, the compound Poisson process \( N' \) can be ignored since it does not influence the final regularity.

If \( Q = 0 \), and therefore \( B = 0 \) and \( \beta' = \beta \), Lemmas 2.3, 2.5 and 2.6 on the component \( Y \) yield Theorem 1.

Otherwise, the Lévy process \( X \) corresponds to the sum of the Brownian motion \( B \) and the jump component \( Y \). Still using Lemmas 2.3, 2.5 and 2.6, it is sufficient to prove that almost surely for all \( t \in [0,1] \), \( \sigma_{X,t} = \sigma_{B,t} \wedge \sigma_{Y,t} \). Owing to the definition of 2-microlocal frontier, we already know that \( \sigma_{X,t} \geq \sigma_{B,t} \wedge \sigma_{Y,t} \). Furthermore, when \( \sigma_{B,t}(s') \neq \sigma_{Y,t}(s') \), it is straightforward to get \( \sigma_{X,t}(s') = \sigma_{B,t}(s') \wedge \sigma_{Y,t}(s') \). Therefore, to obtain Theorem 1, we have to prove that almost surely for all \( t \in [0,1] \), \( \sigma_{X,t} \leq \sigma_{B,t} = s' \mapsto (1/2 + s') \wedge 1/2 \).

1. If \( \beta' = \beta = 2 \), we only need to slightly modify the proof of Lemma 2.3. More precisely, let consider the same constructed sequence \( (t_n)_{n \in \mathbb{N}} \). We observe that since \( B \) is almost surely continuous, \( \|\Delta X_{t_n}\| \geq 2^{-m_n} \) and thus, we can still assume that \( \|X_{t_n} - X_t\| \geq 2^{-m_n} \).

Then, we have

\[
\int_{t_n}^{t_n + \rho_n} (X_s - X_t) \, ds = \rho_n \cdot (X_{t_n} - X_t) + \int_{t_n}^{t_n + \rho_n} (Y_s - Y_{t_n}) \, ds + \int_{t_n}^{t_n + \rho_n} (B_s - B_{t_n}) \, ds.
\]
Since $B$ is a Brownian motion, we know there exists $C(\sigma) > 0$ such that for all $u, v \in [0, 1]$, \( \|B_u - B_v\| \leq C|u - v|^{1/2 - \varepsilon} \). Hence, the last term in the previous equation satisfies
\[
\left\| \int_{t_n}^{t_n + \rho_n} (B_s - B_{t_n}) \, ds \right\| \leq C \int_{t_n}^{t_n + \rho_n} (s - t_n)^{1/2 - \varepsilon} \, ds = C \rho_n^{3/2 - \varepsilon} \leq \rho_n \cdot 2^{-m_n(1+2\varepsilon)},
\]
where we recall that $\rho_n = 2^{-m_n \beta (1+4\varepsilon)}/3$. This term is negligible in front of $\rho_n (X_{t_n} - X_t)$, and the rest of the proof of Lemma 2.3 ensures that $\sigma_{X,t}(s') \leq (1/2 + s') \wedge 0$, for all $s' \in \mathbb{R}$ such that $\sigma_{X,t}(s') \in [-1,0]$, which is sufficient to obtain Theorem 1.

2. If $\beta < 2$, let $\alpha = 2$ and $\varepsilon > 0$. According to Lemma 2.4, there exist $k \in \mathbb{N}$ and $M(\omega) \in \mathbb{N}$ such that for all $m \geq M$, there are at most $k$ jumps of size greater than $2^{-m(1+3\varepsilon)}$ in any interval $I_{j,m}$. Hence, for any $j \in \mathbb{N}$, there exists a subinterval $I_{m,j}$ of size $2^{-m\varepsilon}/k$ and with no such jump inside.

Lemma 2.2 proves that $M(\omega)$ can be chosen such that for all $m \geq M(\omega)$,
\[
\forall u, v \in [0, 1] \ \text{s.t.} \ |u - v| \leq 2^{-m\varepsilon}; \quad \|Y_u^{m(1+3\varepsilon)} - Y_v^{m(1+3\varepsilon)}\| \leq C m 2^{-m(1+3\varepsilon)}.
\]
Let $t \in [0, 1]$ and $j \in \mathbb{N}$ such that $t \in I_{m,j}$. According to Lemma 2.7, there exist $u, v \in I_{m,j}$ such that $\|B_u - B_v\| \geq 2^{-m(1+2\varepsilon)}$. Then, we know that
\[
\|X_u - X_v\| \geq \|B_u - B_v\| - \|Y_u^{m(1+3\varepsilon)} - Y_v^{m(1+3\varepsilon)}\| - |u - v| \cdot \left| \int_{D(2^{-m(1+3\varepsilon)},1)} x \pi(dx) \right|,
\]
where $C$ can be chosen such that $|u - v| \cdot \left| \int_{D(2^{-m(1+3\varepsilon)},1)} x \pi(dx) \right| \leq C 2^{-m(1+3\varepsilon)}$.

Thus, $\|X_u - X_v\| \geq 2^{-m(1+2\varepsilon) - 1}$ and therefore, for any $m$ sufficiently large, there exists $t_m \in I_{m,j}$ such that $\|X_{t_m} - X_t\| \geq 2^{-m(1+2\varepsilon) - 2}$. Furthermore, $t_m$ can be chosen such that there are no jumps of size greater than $2^{-m(1+3\varepsilon)}$ in $B(t_m, \rho_n)$, where $\rho_n = 2^{-m\varepsilon}/3k$.

Then, using a reasoning similar to the previous point, we obtain for any $m \geq M(\omega)$,
\[
\left\| \int_{t_m}^{t_m + \rho_m} (X_s - X_t) \, ds \right\| \geq C \rho_m \cdot 2^{-m(1+2\varepsilon)} \geq C |t - t_m|^{3/2 + \varepsilon}.
\]
This inequality implies that for all $s' \in \mathbb{R}$ such that $\sigma_{X,t}(s') \in [-1,0]$, $\sigma_{X,t}(s') \leq 1/2 + s'$, which concludes the proof.

\[ \square \]

\textbf{Proof of Corollary 1.} Owing to Theorem 1, the case $\sigma = 0$ corresponds to the classic spectrum of singularity. Hence, let $\sigma \in [-1,0)$. We recall that $s$ denotes the parameter $\sigma - s'$.

Hence, we suppose $\sigma \in [-1,0]$ and $s \in [0,1/\beta]$. We note that $E_{\sigma,s'} = \{ t \in \mathbb{R}_+ : \sigma_{X,t}(s') = \sigma \}$, since the negative component of the 2-microlocal frontier of $X$ can not be constant. Thus, $E_{\sigma,s'}$ satisfies
\[
\forall s \in [0,1/\beta); \quad \tilde{E}_s \subseteq E_{\sigma,s'} \subseteq \tilde{E}_s \cup \bigcup_{h<s} \tilde{E}_h.
\] (2.13)
Using notations introduced in the proof of Lemma 2.6, we know that for all \( h < s \), \( \hat{E}_h \subseteq F_s \), where \( \dim_H(F_s) \leq 2\beta s - 1 < \beta s \). Therefore, this inequality and equation (2.13) ensures that \( \dim_H(E_{s', \sigma} \cap V) = \beta s \).

2.3. Examples of Lévy measures

As previously outlined, the collection of sets \( (\hat{E}_h)_{h \in \mathbb{R}_+} \) considered in Theorem 1 gathers times at which the 2-microlocal regularity is unusual (the slope of the frontier is not equal to 1). In this section, we present examples of Lévy processes which show that for a fixed Blumenthal-Getoor exponent, different situations may occur, depending on the form of the Lévy measure. It is assumed in the sequel that \( d = 1 \).

Example 1. Let \( \pi \) be a Lévy measure such that \( \pi((-\infty, 0)) = 0 \). Then, the Lévy process \( Y \) with generating triplet \((0, 0, \pi)\) almost surely satisfies,

\[
\forall t \in \mathbb{R}_+, \ s' \in \mathbb{R}; \quad \sigma_{Y, t}(s') = (\alpha_{Y, t} + s') \land 0.
\]

Proof. According to Theorem 1, we have to prove that for all \( h \in \mathbb{R}_+, \hat{E}_h = \emptyset \). For that purpose, we extend Lemma 2.5 to any \( h \in [0, 1/\beta) \). Let \((t_n)_{n \in \mathbb{N}}\) still be a sequence such that \( |\Delta Y_{t_n}| \geq 2^{-m_n} \) and \( \rho_n = t_n - t \) for all \( n \in \mathbb{N} \). We first assume that \( \beta \geq 1 \). Similarly to Lemma 2.5, we obtain

\[
|\int_{D(2^{-m_n(1+\varepsilon)}, 1)} x \pi(dx)| \leq C 2^{-m_n(1+\varepsilon)} \quad \text{and} \quad |Y_{u_{m_n}} - Y_{t_{m_n}}(1+\varepsilon)| \leq C m_n 2^{-m_n(1+\varepsilon)} \quad \text{for all} \quad u \in [t, t_n + \rho_n].
\]

Furthermore, owing to the definition of \( \pi \), there are only positive jumps in the interval \([t, t_n + \rho_n]\). Hence, if we consider the contribution to the increment \( Y_u - Y_t \) of jumps of size greater than \( 2^{-m_n(1+\varepsilon)} \), it is positive and larger than \( 2^{-m_n} \) for all \( u \geq t_n \). Therefore,\[ \forall n \in \mathbb{N}; \quad \left| \int_{t_n + \rho_n}^{t_n + \rho_n} (Y_s - Y_t) \, ds \right| \geq C \rho_n 2^{-m_n}. \]

The rest of the proof of Lemma 2.5 holds and ensures the equality. The case \( \beta < 1 \) is treated similarly.

The 2-microlocal frontier obtained in the previous example is proved to be classic at any \( t \in \mathbb{R}_+ \). This behaviour is due to the existence of only positive jumps which can not locally compensate each other. Similarly, the proof of Lemma 2.5 focuses on times where the distance between two consecutive jumps of close size is always sufficiently large so that there is no compensation phenomena.

Hence, in order to exhibit some unusual 2-microlocal regularity, we consider in the next example points where jumps are locally compensated at any scale.

Example 2. For any \( \beta \in (0, 1) \) and all \( h \in (\frac{1}{2\beta}, \frac{1}{3}) \), there exists a Lévy measure \( \pi_h \) such that its Blumenthal-Getoor exponent is \( \beta \) and the Lévy process \( Y \) with generating triplet \((0, 0, \pi_h)\) almost surely satisfies \( \hat{E}_h \neq \emptyset \).

Proof. Let \( \beta \in (0, 1) \) and \( \alpha, \delta, \gamma \in (\beta, 2\beta) \) such that \( \alpha < \delta < \gamma \). Let the measure \( \pi_h \), where \( h = 1/\delta \), be

\[
\pi_h(dx) = \sum_{n=1}^{+\infty} 2^{jn_{n+1}} (\delta_{2^{-j_n}}(dx) + \delta_{2^{-2j_n}}(dx)) \quad \text{where} \quad \forall n \in \mathbb{N}; \quad j_{n+1} = \frac{j_n \delta + 1}{2\beta - 2\gamma + \alpha}.
\]
α and δ are supposed to satisfy $2\beta - 2\gamma + \alpha > 0$. We note that $j_n \to_n +\infty$ since $\delta > \beta$ and $2\beta - 2\gamma + \alpha < \beta$. One readily verifies that the Blumenthal-Getoor exponent of $\pi_{h}$ is equal to $\beta$. Moreover, since the measure $\pi_{h}$ is symmetric, $Y$ is a pure jump process with no linear component.

The construction of the example is divided in two parts. We first define a random time set $K(\omega)$ and prove it is not empty with a positive probability. Then, we determine the 2-microlocal frontier of points from $K(\omega)$ in order to exhibit unusual behaviours.

More precisely, let define an inhomogeneous Galton-Watson process $(T_n)_{n \in \mathbb{N}}$ that is used to construct the random set $K(\omega)$. Every individual of generation $n - 1$ represents an interval $I_{n-1}$ of size $2^{-j_{n-1}\delta}$ and the distribution of its offspring is denoted $L_{n}$.

There exist at least $m_n = \lfloor 2^{-j_{n-1}\delta + j_{n}\alpha - 2} \rfloor$ intervals $(\tilde{I}_{i,n})_{i}$ of size $3 \cdot 2^{-j_{n}\alpha}$ inside $I_{n}$. Then, for every $i$, let consider an interval $\tilde{I}_{i,n}$ of size $2^{-j_{n}\gamma}$ centered inside $\tilde{I}_{i,n}$.

Let $p_n$ denotes the probability of the existence of two jumps of size $2^{-j_{n}}$, but with different signs inside an interval $\tilde{I}_{i,n}$ and the absence of $2^{-j_{n}}$ jumps in the rest of $\tilde{I}_{i,n}$. It is equal to

$$p_n = 2^{-2j_{n}(\gamma-\beta)+1} \exp\left(-2^{-j_{n}(\gamma-\beta)+1}\right) \cdot \exp\left(-2^{j_{n}\beta}(3 \cdot 2^{-j_{n}\alpha} - 2^{-j_{n}\gamma})\right).$$

The distribution of the offspring $L_{n}$ is defined as the number of intervals $\tilde{I}_{i,n}$ which satisfy such a configuration. It follows a Binomial distribution $B(m_n, p_n)$ with the following mean,

$$\mathbb{E}[L_{n}] = m_n p_n \sim_{n} 2^{-j_{n-1}\delta + j_{n}(2\beta + \alpha - 2\gamma) - 1} \sim_{n} 2,$$

owing to the definition of $(j_n)_{n \in \mathbb{N}}$. Then, to every $I_{n-1}$ is associated a family of subintervals $(\tilde{I}_{i,n})_{i \in L_{n}}$ such that for all $i \in L_{n}$, $I_{i,n}$ is a subinterval of $\tilde{I}_{i,n}$ of size $2^{-j_{n}\delta}$ and the distance between $\tilde{I}_{i,n}$ and $I_{i,n}$ is equal to $2^{-j_{n}\delta}$.

Let now define the random set $K(\omega)$ as following:

$$K(\omega) = \bigcap_{n \in \mathbb{N}} \bigcup_{i \in L_{n}} I_{i,n}.$$

We observe that $K(\omega)$ is nonempty if and only if the Galton-Watson tree $(T_n)_{n \in \mathbb{N}}$ survives. According to Proposition 1.2 in [14], if $(T_n)_{n \in \mathbb{N}}$ satisfies the conditions

$$\sup_{n \in \mathbb{N}} \mathbb{E}[L_{n}^{2}] < +\infty, \quad \inf_{n \in \mathbb{N}} \mathbb{E}[L_{n}] > 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}[T_{n}]^{1/n} > 1$$

then the survival probability $\lim_{n \to \infty} \mathbb{P}(T_{n} > 0)$ is strictly positive. One readily verifies that $\mathbb{E}[L_{n}] \leq 3m_n p_n$ and $\mathbb{E}[T_{n}] = \prod_{p=1}^{n} \mathbb{E}[L_{p}]$ for all $n \in \mathbb{N}$, proving that $\mathbb{P}(K(\omega) \neq \emptyset) > 0$.

Let set $\omega \in \{K \neq \emptyset\}$, $t \in K(\omega)$ and determine the regularity of $Y_{\omega}(t)$ at $t$. Owing to the construction of $K(\omega)$, there exists a sequence $(t_{n})_{n \in \mathbb{N}}$ which converges to $t$ and such that

$$\forall n \in \mathbb{N}; \quad 2^{-j_{n}\delta} \leq |t - t_{n}| \leq 2 \cdot 2^{-j_{n}\delta} \quad \text{and} \quad |\Delta Y_{t_{n}}| = 2^{-j_{n}}.$$

It clearly proves that $t$ belongs to $A_{\delta'}$ for all $\delta' > \delta$. Since we also know that the distance between $t$ and a jump of size $2^{-j_{n}}$ is at least $2^{-j_{n}\delta}$, $t \notin A_{\delta'}$ for all $\delta' < \delta$. Hence $t \in E_{h}$ and the pointwise Hölder exponent $\alpha_{Y,t}$ is equal to $h$.

Let now study the 2-microlocal frontier of $Y$ at $t$ and set $u \in B(t, \rho)$ with $\rho$ sufficiently small. There exists $n \in \mathbb{N}$ such that $2^{-j_{n}\delta} \leq |t - u| < 2^{-j_{n-1}\delta}$. Two different cases have to be distinguished.
1. If $2^{-j_n^\alpha} \leq |t-u| \leq 2^{-j_n^\beta}$, there exist at most two jumps of size $2^{-j_n}$ in the interval $[t,u]$ (corresponding to the jumps inside $\mathcal{I}_{j_n}$). Let first estimate the contribution $|Y_u^{j_n+1} - Y_v^{j_n+1}|$ to the increment. For any $\delta' \in (\beta,1)$, Lemma 2.2 ensures that almost surely for all $n \in \mathbb{N}$

$$\sup_{s,v \in [0,1], |s-v| \leq 2^{-j_n+1}\delta'} |Y_s^{j_n+1} - Y_v^{j_n+1}| \leq 3j_n 2^{-j_n+1}$$  \hspace{1cm} (2.14)

Then, dividing $[t,u]$ in subintervals of size $2^{-j_n+1}\delta'$, we obtain for any $s \in [t,u]$

$$|Y_s^{j_n+1} - Y_t^{j_n+1}| \leq 2^{j_n+1}\delta' |t-u| \cdot 3j_n 2^{-j_n+1}$$

$$\leq C |t-u| \cdot j_n 2^{-j_n}(\delta(1-\delta')/(2\delta-2\gamma+\alpha))$$

$$\leq C \log(|t-u|^{-1}) \cdot |t-u|^{1/\delta + (\delta(1-\delta')/(2\delta-2\gamma+\alpha))}.$$ 

If we consider the exponent at the limit $\alpha, \gamma \to \delta$ and $\delta' \to \beta$, it is equal to $\delta + \delta(1-\beta)/(2\beta-\gamma)$. One easily verifies that it is strictly greater than 1 for all $\beta \in (0,1)$ and $\delta \in (\beta,2\beta)$. Hence, owing to the continuity of the fraction, we can assume there exists $\varepsilon > 0$ such that $\delta/(2\beta-2\gamma+\alpha) \geq 1 + \varepsilon$ when $\alpha, \gamma$ and $\delta'$ are sufficiently close to $\delta$ and $\beta$. It ensures that for $s \in [t,u]$, $|Y_s^{j_n+1} - Y_t^{j_n+1}| \leq C |t-u|^{(1+\varepsilon)/\delta}$.

Since the distance between the two jumps of size $2^{-j_n}$ in the interval $[t,u]$ is at most $2^{-j_n\gamma}$,

$$\int_t^u (Y_s - Y_t) \, ds \leq \int_t^u (Y_s^{j_n+1} - Y_t^{j_n+1}) \, ds + 2^{-j_n(1+\gamma)}$$

$$\leq C |t-u|^{(1+\varepsilon)/\delta + 1} + |t-u|^{1/\delta + \gamma/\delta} \leq C |t-u|^{(1+\varepsilon)/\delta + 1},$$  \hspace{1cm} (2.15)

as $\gamma > \delta$.

2. Let now assume $2^{-j_n^\alpha} \leq |t-u| \leq 2^{-j_n^\beta}$. We first note that there are no jumps of size greater than $2^{-j_n}$ inside the interval $[t,u]$. Hence, if $|t-u| \leq 2^{-j_n^\delta'}$, equation (2.14) ensures that

$$|Y_u - Y_t| \leq 3j_n 2^{-j_n} \leq C \log(|t-u|^{-1}) \cdot |t-u|^{1/\alpha} \leq C |t-u|^{(1+\varepsilon)/\delta},$$  \hspace{1cm} (2.16)

since we can assume that $\alpha(1+\varepsilon) < \delta$. If $|t-u| \geq 2^{-j_n^\delta'}$, we have

$$|Y_u - Y_t| \leq 2^{j_n^\delta'} |t-u| \cdot 3j_n 2^{-j_n}$$

$$\leq C \log(|t-u|^{-1}) |t-u|^{1/\delta + \delta'/2\beta - 2\gamma + \alpha}.$$  \hspace{1cm} (2.17)

The exponent at the limit $\alpha, \gamma \to \delta$ and $\delta' \to \beta$ is equal to $1 + 1/\beta - \beta/(2\beta-\gamma)$. Similarly to the previous case, one verifies that it is strictly greater than $1/\delta$ for all $\beta \in (0,1)$ and $\delta \in (\beta,2\beta)$, and therefore $\alpha, \gamma$ and $\delta'$ can be chosen such that $|Y_u - Y_t| \leq C |t-u|^{(1+\varepsilon)/\delta}$.

Hence, equations (2.16) and (2.17) yield

$$\int_t^u (Y_s - Y_t) \, ds \leq C |t-u|^{(1+\varepsilon)/\delta + 1},$$  \hspace{1cm} (2.18)

where the constant $C$ is independent of $n$. 

Owing to equations (2.15) and (2.18), the 2-microlocal frontier $\sigma_{Y,t}$ of $Y$ at $t$ is greater than $s' \mapsto \frac{s'+h}{1+ch}$ and $0$. Hence, for any $\omega \in \Omega$, $K(\omega) \subseteq \tilde{E}_h(\omega)$ and thus $\mathbb{P}(\tilde{E}_h \neq \emptyset) > 0$. Furthermore, using Kolmogorov $0-1$ law, we also know that $\mathbb{P}(\tilde{E}_h \neq \emptyset) \in \{0,1\}$, which ends the proof. 

Example 2 justifies the distinction made in Theorem 1 between the normal behaviour $(\tilde{E}_h)_h$ and the exceptional one $(\tilde{E}_h)_h$. Nevertheless, if the previous examples prove that different regularities can be obtained, depending on the form of the Lévy measure, it remains an open problem to characterize completely the 2-microlocal frontier for any Lévy process.

3. Regularity of linear (multi)fractional stable motion

The linear fractional stable motion (LFSM) is a stochastic process that has been considered by several authors: Maejima [29], Takashima [44], Kôno and Maejima [28], Samorodnitsky [39], Ayache et al. [6], Ayache and Hamonier [4]. Its general integral form is defined by

$$X_t = \int_\mathbb{R} \left\{ a^+ \left[ (t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha} \right] + a^- \left[ (t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha} \right] \right\} M_{\alpha,\beta}(du),$$

where $H \in (0,1)$, $(a^+, a^-) \in \mathbb{R}^2 \setminus (0,0)$ and $M_{\alpha,\beta}$ is an $\alpha$-stable random measure on $\mathbb{R}$ with Lebesgue control measure $\lambda$ and skewness intensity $\beta(\cdot) \in [-1,1]$. Throughout this paper, it is assumed that $\beta$ is constant, and equal to zero when $\alpha = 1$. In this context, for any Borel set $A \subset \mathbb{R}$, the characteristic function of $M_{\alpha,\beta}(A)$ is given by

$$\mathbb{E}[e^{i\theta M_{\alpha,\beta}(A)}] = \begin{cases} \exp\left\{ -\lambda(A) |\theta|^\alpha (1 - i\beta \text{sign}(\theta) \tan(\alpha \pi/2)) \right\} & \text{if } \alpha \in (0,1) \cup (1,2); \\ \exp\left\{ -\lambda(A) |\theta| \right\} & \text{if } \alpha = 1. \end{cases}$$

For sake of readability, we consider in the rest of the section the particular case $(a^+, a^-) = (1,0)$ (even though as stated [39], the law of the process depends on values $(a^+, a^-)$ chosen).

To begin with, let us obtain in the next statement an alternative representation for the two-parameter field $(t, H) \mapsto X(t, H) = \int_\mathbb{R} \left\{ (t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha} \right\} M_{\alpha,\beta}(du)$.

**Proposition 2.** For all $t \in \mathbb{R}$ and $H \in (0,1)$, the random variable $X(t, H)$ satisfies

$$X(t, H) \overset{\text{a.s.}}{=} \begin{cases} CH \int_\mathbb{R} \left\{ (t-u)^{H-1/\alpha-1} - (-u)^{H-1/\alpha-1} \right\} du & \text{if } H \in \left[ \frac{1}{\alpha}, 1 \right]; \\ Lt & \text{if } H = \frac{1}{\alpha}; \\ CH \int_\mathbb{R} \left\{ (L_u - Lt)(t-u)^{H-1/\alpha-1} - Lu(-u)^{H-1/\alpha-1} \right\} du & \text{if } H \in \left( 0, \frac{1}{\alpha} \right], \end{cases}$$

where $C_H = H - 1/\alpha$ and $L$ is an $\alpha$-stable Lévy process defined by

$$\forall t \in \mathbb{R}_+ \quad Lt = M_{\alpha,\beta}([0,t]) \quad \text{and} \quad \forall t \in \mathbb{R}_- \quad Lt = -M_{\alpha,\beta}([t,0]).$$
Proof. Let \( t \in \mathbb{R} \) and \( H \in (0, 1) \). Since \( (L_t)_{t \in \mathbb{R}} \) is an \( \alpha \)-stable Lévy process, it has càdlàg sample paths. According to \cite{3} (chap. 4.3.4), the theory of the stochastic integration based \( \alpha \)-stable Lévy processes coincide integrals with respect to \( \alpha \)-stable random measure. Therefore, the r.v. \( X(t, H) \) is almost surely equal to \( \int_{\mathbb{R}} \{(t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha}\} \, dL_u \). Let \( \varepsilon > 0 \) and \( b < t \). Using a classic integration by parts formula, we obtain

\[
L_{t-}\varepsilon^{H-1/\alpha} - L_{\alpha}(t-b)^{H-1/\alpha} = \int_b^{t-\varepsilon} (t-u)^{H-1/\alpha} \, dL_u - \left( H - \frac{1}{\alpha} \right) \int_b^{t-\varepsilon} L_u(t-u)^{H-1/\alpha-1} \, du. \tag{3.3}
\]

1. If \( H \in \left( \frac{1}{\alpha}, 1 \right), H - 1/\alpha > 0 \). Hence, \( \int_b^{t-\varepsilon} L_u(t-u)^{H-1/\alpha-1} \, du \) almost surely converges to \( \int_b^t L_u(-u)^{H-1/\alpha-1} \, du \) when \( \varepsilon \to 0 \). Similarly, \( \int_b^{t-\varepsilon} (t-u)^{H-1/\alpha} \, dL_u \) converges in \( L^\alpha(\Omega) \). Therefore, using equation (3.3) with \( t = 0 \) and \( b < 0 \), we obtain almost surely

\[
\int_b^t \{(t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha}\} \, dL_u = C_H \int_b^t L_u \{(t-u)^{H-1/\alpha-1} - (-u)^{H-1/\alpha-1}\} \, du - L_b \{(t-b)^{H-1/\alpha} - (-b)^{H-1/\alpha}\}.
\]

When \( b \to -\infty \), the left-term clearly converges to \( X(t, H) \) in \( L^\alpha(\Omega) \). According to \cite{36}, we know that almost surely for any \( \varepsilon > 0 \), \( \limsup_{u \to -\infty} u^{1/\alpha+\varepsilon} |L_u| = 0 \). Furthermore, we also have \( (t-u)^{H-1/\alpha-1} - (-u)^{H-1/\alpha-1} \sim_{-\infty} (-u)^{H-1/\alpha-2} \) and \( (t-b)^{H-1/\alpha} - (-b)^{H-1/\alpha} \sim_{-\infty} (-b)^{H-1/\alpha-1} \). Therefore, as \( H < 1 \) and using the dominated convergence theorem, the right-term almost surely converges to the expected integral.

2. If \( H \in (0, \frac{1}{\alpha}) \), we observe that equation (3.3) can be slightly transformed into

\[
(L_{t-}\varepsilon - L_{\alpha})(t-b)^{H-1/\alpha} = \int_b^{t-\varepsilon} (t-u)^{H-1/\alpha} \, dL_u - \left( H - \frac{1}{\alpha} \right) \int_b^{t-\varepsilon} (L_u - L_{\alpha})(t-u)^{H-1/\alpha-1} \, du.
\]

According to \cite{36}, \( \alpha \gamma^t \overset{as}{=} 1/\alpha \). Therefore, up to an extracted sequence, the previous expression almost surely converges when \( \varepsilon \to 0 \) and using a similar formula for \( t = 0 \), we obtain

\[
\int_b^t \{(t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha}\} \, dL_u
\]

\[
= C_H \int_b^t \{(L_u - L_{\alpha})(t-u)^{H-1/\alpha-1} - L_u(-u)^{H-1/\alpha-1}\} \, du
- L_b \{(t-b)^{H-1/\alpha} - (-b)^{H-1/\alpha}\} + L_{\alpha}(t-b)^{H-1/\alpha}.
\]

The property \( \limsup_{u \to -\infty} u^{1/\alpha+\varepsilon} |L_u| = 0 \) and the previous equivalents yield equation (3.2).
To end this proof, let consider the integral representation in the particular case \( H = 1/\alpha \). In fact, equation (3.2) is a slightly misuse since the expression does not exist. Nevertheless, we prove that it converges almost surely to \( X(t, 1/\alpha) = L_t \) when \( H \to 1/\alpha \).

Let first assume that \( H \nearrow 1/\alpha \) and rewrite \( X(t, H) \) as

\[
X(t, H) = C_H \int_{\mathbb{R}} \left\{ (L_u - L_t)_{1_{u \geq b}}(t-u)^{H-1/\alpha-1} - L_u(u)^{H-1/\alpha-1} \right\} \, du + L_t(t-b)^{H-1/\alpha},
\]

The first component of the expression converges to zero since \( C_H \, H \to 1/\alpha \to 0 \) and \( \alpha \). As the second part simply converges to \( L_t \), we get the expected limit. The case \( H \searrow 1/\alpha \) is treated similarly.

Picard [35] determined a similar representation for fractional Brownian motion. We now use this statement to prove Theorem 2.

**Proof of Theorem 2.** Let \( H \in (0, 1) \) and \( \alpha \in [1, 2) \).

1. If \( H > 1/\alpha \), we note that the representation obtained in Proposition 2 is defined almost surely for all \( t \in \mathbb{R} \). Therefore, let set \( \omega \in \Omega_0 \) and \( t \in \mathbb{R} \). As previously, we can assume that \( t \in [0, 1] \). Then, we observe that

\[
X_t = C_H \int_{b}^{t} L_u(t-u)^{H-1/\alpha-1} \, du + C_H \int_{b}^{0} L_u(u)^{H-1/\alpha-1} \, du
\]

\[
+ C_H \int_{-\infty}^{b} L_u \left\{ (t-u)^{H-1/\alpha-1} - (-u)^{H-1/\alpha-1} \right\} \, du,
\]

where \( b < 0 \) is fixed. The second term is simply a constant that does not influence the regularity. Similarly, using the dominated convergence theorem, we note that the third one is a \( C^\infty \) function on the interval \([0, 1]\), and therefore has no impact on the 2-microlocal frontier. Finally, the first term corresponds to a fractional integral of order \( H - 1/\alpha \) of the process \( L \). According to the properties satisfied by the 2-microlocal spaces (see e.g. Theorem 1.1 in [24]), we know that almost surely for all \( t \in \mathbb{R} \), the 2-microlocal frontier \( \sigma_{X,t} \) of \( X(\cdot, H) \) simply corresponds to a translation of \( L \)'s frontier \( \sigma_{L,t} \).

2. If \( H < 1/\alpha \), we observe that \( \dim_H \{ t \in \mathbb{R} : \alpha_{L,t} \leq 1/\alpha - H \} \) < 1 owing to Proposition 1 and since \( H > 0 \). Hence, for every \( \omega \in \Omega_0 \), formula (3.2) is well-defined almost everywhere on \( \mathbb{R} \). Anywhere else, we may simply assume that \( X(t, H) \) is set to zero. Similarly to the previous case \( H > 1/\alpha \), the regularity of \( X \) only depends on properties of the component

\[
t \mapsto C_H \int_{b}^{t} (L_u - L_t)(t-u)^{H-1/\alpha-1} \, du.
\]

One might recognize a Marchaud fractional derivative (see e.g. [38]). Let us modify this expression to exhibit a more classic form of fractional derivative. For almost all \( s \in [0, 1] \)
and \( \varepsilon > 0 \), we have
\[
\int_b^{s-\varepsilon} L_u(s-u)^{H-1/\alpha} \, du = C_H \int_b^{s-\varepsilon} L_u \, du \int_{u+\varepsilon}^{s} (v-u)^{H-1/\alpha-1} \, dv + \varepsilon^{H-1/\alpha} \int_b^{s-\varepsilon} L_u \, du \\
= C_H \int_b^{s-\varepsilon} du \int_{u+\varepsilon}^{s} (L_u - L_v)(v-u)^{H-1/\alpha-1} \, dv \\
+ \varepsilon^{H-1/\alpha} \int_b^{s-\varepsilon} L_u \, du + C_H \int_b^{s-\varepsilon} du \int_{u+\varepsilon}^{s} L_v(v-u)^{H-1/\alpha-1} \, dv.
\]

The last two terms are equal to
\[
\varepsilon^{H-1/\alpha} \int_b^{s-\varepsilon} L_u \, du - \varepsilon^{H-1/\alpha} \int_b^{s+\varepsilon} L_u \, dv + \int_{b+\varepsilon}^{s} L_v(v-b)^{H-1/\alpha} \, dv,
\]
which converges to \( \int_b^{s} L_v(v-b)^{H-1/\alpha} \, dv \) as \( \varepsilon \to 0 \). Similarly, using the dominated convergence theorem, the first term converges to \( C_H \int_b^{s} dv \int_b^{v} (L_u-L_v)(v-u)^{H-1/\alpha-1} \, du \), and therefore
\[
\int_b^{s} L_u(s-u)^{H-1/\alpha} \, du = C_H \int_b^{s} dv \int_b^{v} (L_u-L_v)(v-u)^{H-1/\alpha-1} \, du + \int_b^{s} L_v(v-b)^{H-1/\alpha} \, dv.
\]

According to classic real analysis results, the previous expression is differentiable almost everywhere on the interval \([0,1]\), and therefore
\[
C_H \int_b^{t} (L_u-L_t)(t-u)^{H-1/\alpha-1} \, du \xrightarrow{a.e.} \frac{d}{dt} \int_b^{t} L_u(t-u)^{H-1/\alpha} \, du - L_t(t-b)^{H-1/\alpha},
\]
for almost all \( t \in [0,1] \). The last two formulas ensure that \( X(\cdot, H) \in L^1_{\text{loc}}(\mathbb{R}) \) a.s., and thus, \( X(\cdot, H) \) is a tempered distribution whose 2-microlocal regularity can be determined as well.

Similarly to the previous case, the term \( t \to \int_b^{t} L_u(t-u)^{H-1/\alpha-1} \, du \) is a Riemann-Liouville fractional integral of order \( H-1/\alpha+1 > 0 \). Hence, at any \( t \in \mathbb{R} \), the 2-microlocal frontier of the distribution \( \frac{d}{dt} \int_b^{t} L_u(t-u)^{H-1/\alpha} \, du \) is equal to \( \sigma_{Y,t} + H - 1/\alpha \).

Since the regularity of the second component locally corresponds to spectrum of \( L \), it does not have any influence. Therefore, almost surely for all \( t \in \mathbb{R} \), the 2-microlocal frontier \( \sigma_{X,t} \) of \( X(\cdot, H) \) corresponds to a translation of \( L \)'s frontier, i.e. \( \sigma_{X,t} = \sigma_{L,t} + H - 1/\alpha \).

In both cases that regularity of \( X \) can be deduced from \( L \)'s 2-microlocal frontier. Hence, using Corollary 1 and since \( L \) is an \( \alpha \)-stable Lévy process, we obtain the spectrum described in equation (1.13).

Another class of processes similar to the LFSM has been introduced and studied in [11, 31, 15]. Named fractional Lévy processes, it is defined by
\[
X_t = \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left\{ (t-u)^d_+ - (-u)^d_+ \right\} L(du),
\]
where \( d \in (0,1/2) \) and \( L \) is a Lévy process enjoying \( Q = 0 \) (no Brownian component), \( E[L(1)] = 0 \) and \( E[L(1)^2] < +\infty \). Owing to this last assumption on \( L \), LFSMs are not fractional Lévy processes. Nevertheless, their multifractal regularity can be determined as well.
Proposition 3. Let $X$ be a fractional Lévy process parametrized by $d \in (0,1/2)$. It satisfies almost surely for all $\sigma \in [d-1,d]$,

$$\forall V \in \mathcal{O}; \quad \dim_H(E_{\sigma,s'} \cap V) = \begin{cases} \beta(s-d) & \text{if } s \in [d,d+\frac{1}{\beta}] \setminus \mathbb{N} \\ -\infty & \text{otherwise.} \end{cases}$$

(3.4)

where $\beta$ designates the Blumenthal-Getoor exponent of the Lévy process. Furthermore, for all $s' \in \mathbb{R}$, $E_{\sigma,s'}$ is empty if $\sigma > d$.

Proof. Marquardt [31] established (Theorem 3.3) a representation of fractional Lévy processes equivalent to Proposition 2. Based on this result, an adaptation of Theorem 2 proof yields equation (3.4).

Similarly to the LFSM, this statement refines regularity results established in [11, 31] and proves that the multifractal spectrum of a fractional Lévy process is equal to

$$\forall V \in \mathcal{O}; \quad d_X(h,V) = \begin{cases} \beta(h-d) & \text{if } h \in [d,d+\frac{1}{\beta}] \\ -\infty & \text{otherwise.} \end{cases} \quad (3.5)$$

Let us finally conclude this section with the proof of Theorem 3.

Proof of Theorem 3. Let $(X_t)_{t \in \mathbb{R}}$ be a linear multifractional stable motion with $\alpha \in (1,2)$ and Hurst function $H(\cdot) \in (1/\alpha,1)$. According to the representation obtained in Proposition 2, $X_t$ is almost surely equal to $X(t,H(t))$. Let $t \in \mathbb{R}$ and $\rho > 0$. For all $u,v \in B(t,\rho)$, we investigate the increment

$$X_u - X_v = (X(u,H(u)) - X(v,H(u))) + (X(v,H(u)) - X(v,H(v)))$$

Using the dominated convergence theorem, we know that the field $(t,H) \mapsto X(t,H)$ is differentiable on the variable $H$. Therefore, the mean value theorem ensures that the second term is upper bounded by $C|H(u) - H(v)|$. Furthermore, the first component satisfies

$$X(u,H(u)) - X(v,H(u)) = X(u,H(t)) - X(v,H(t)) + \int_{H(t)}^{H(u)} (\partial_H X(u,h) - \partial_H X(v,h)) dh.$$ 

The increment $X(u,H(t)) - X(v,H(t))$ corresponds to the increment a LFSM with Hurst index $H(t)$ whereas the second term can be upper bounded by $C|H(u) - H(t)||u - v|^h$, where $h$ satisfies $1/\alpha < h < \min_{B(t,\rho)} H(\cdot)$.

Since $H$ satisfies the $H_d$ assumption, the proof of Theorem 2 and the previous inequalities ensure that,

$$\forall s' \in \mathbb{R}; \quad \sigma_{X,t}(s') \geq (\alpha_{L,t} + H(t) - \frac{1}{\alpha} + s') \wedge (H(t) - \frac{1}{\alpha}).$$

Furthermore, if $t \in \mathcal{E}_h$, we also obtain $\sigma_{X,t}(s') \leq \alpha_{L,t} + H(t) - \frac{1}{\alpha} + s'$, proving in particular that $\alpha_{X,t} = \alpha_{L,t} + H(t) - 1/\alpha$.

Finally, as the set of jumps $S(\omega)$ of $L$ is dense in $[0,1]$, for all $\rho, \varepsilon > 0$, there exists $v \in B(t,\rho)$ such that $\alpha_{X,v} \leq H(t) - 1/\alpha + \varepsilon$. Hence, the 2-microlocal frontier almost surely satisfies for all $t \in [0,1]$

$$\forall s' \in \mathbb{R}; \quad \sigma_{X,t}(s') = (\alpha_{L,t} + H(t) - \frac{1}{\alpha} + s') \wedge (H(t) - \frac{1}{\alpha}).$$
Therefore, Theorem 1 and the continuity of $H(\cdot)$ ensure Equality (1.15) for all $\sigma \in [0, H(t) - 1/\alpha)$.

**Remark 7.** The representation of the LMSM derived from Proposition 2 provides answers to different points raised in [42].

Firstly, if the Hurst function is continuous and satisfies $H(\cdot) \in (1/\alpha, 1]$, then the corresponding LMSM $t \mapsto X(t, H(t))$ is almost surely continuous. Indeed, according to Proposition 2, the random field $X(t, H)$ is continuous on the domain $\mathbb{R} \times (1/\alpha, 1]$. Hence, the composition with the Hurst function $H(\cdot)$ also enjoys this property.

Moreover, if the continuous Hurst function satisfies the weaker condition $H(\cdot) \in [1/\alpha, 1]$, then the LMSM $t \mapsto X(t, H(t))$ is continuous if and only if $\mathcal{L}^1\left(\mathcal{H}^{-1}\left(\frac{1}{\alpha}\right)\right) = 0$. Otherwise, the process almost surely has càdlàg sample paths. Indeed, still according to Proposition 2, the random field $X(t, H)$ is càdlàg on the domain $\mathbb{R} \times [1/\alpha, 1]$ and its jumps coincide with the jumps of $L$. Hence, $X(t, H(t))$ is continuous if and only if $\mathcal{H}^{-1}\left(\frac{1}{\alpha}\right) \cap S(\omega) = \emptyset$, i.e. iff $J(\mathcal{H}^{-1}\left(\frac{1}{\alpha}\right), (0, +\infty)) \approx 0$. Since $J$ is a Poisson measure of intensity $\mathcal{L}^1 \otimes \pi$, it occurs if and only if we have $\mathcal{L}^1\left(\mathcal{H}^{-1}\left(\frac{1}{\alpha}\right)\right) = 0$.

**Remark 8.** In the case $H(\cdot)$ does not satisfy the assumption $\delta > \sup_{t \in \mathbb{R}} H(t)$, the proof of Theorem 3 can be modified to extend the statement and generalize results obtained in [42]. This complete study is made in [8] for the multifractional Brownian motion. For sake of clarity, we prefer to focus in this work on $(H_0)$-Hurst functions and LMSM’s multifractal nature obtained in this case.

**Remark 9.** Although it is assumed all along this section that $H(\cdot)$ is deterministic, owing to the representation exhibited in Proposition 2, Theorems 2 and 3 still hold if $H(\cdot)$ is a continuous random process. Hence, based on these results, a class of random processes with random and non-homogeneous multifractal spectrum can be easily exhibited. A similar extension of the multifractional Brownian motion has been introduced and studied by Ayache and Taqqu [5].

**References**

[1] R. J. Adler. *The geometry of random fields*. John Wiley & Sons Ltd., Chichester, 1981. Wiley Series in Probability and Mathematical Statistics.

[2] R. J. Adler and J. E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.

[3] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.

[4] A. Ayache and J. Hamonier. Linear Multifractional Stable Motion: fine path properties. *Preprint*, 2013. arXiv:1302.1670.

[5] A. Ayache and M. S. Taqqu. Multifractional processes with random exponent. *Publ. Mat.*, 49(2):459–486, 2005.

[6] A. Ayache, F. Roueff, and Y. Xiao. Linear fractional stable sheets: wavelet expansion and sample path properties. *Stochastic Process. Appl.*, 119(4):1168–1197, 2009.

[7] P. Balança and E. Herbin. 2-microlocal analysis of martingales and stochastic integrals. *Stochastic Process. Appl.*, 122(6):2346–2382, 2012.
[8] P. Balança and E. Herbin. Sample paths properties of irregular multifractional Brownian motion. *In preparation*, 2013.
[9] J. Barral and S. Seuret. The singularity spectrum of Lévy processes in multifractal time. *Adv. Math.*, 214(1):437–468, 2007.
[10] J. Barral, N. Fournier, S. Jaffard, and S. Seuret. A pure jump Markov process with a random singularity spectrum. *Ann. Probab.*, 38(5):1924–1946, 2010.
[11] A. Benassi, S. Cohen, and J. Istas. On roughness indices for fractional fields. *Bernoulli*, 10(2):357–373, 2004.
[12] R. M. Blumenthal and R. K. Getoor. Sample functions of stochastic processes with stationary independent increments. *J. Math. Mech.*, 10:493–516, 1961.
[13] J.-M. Bony. Second microlocalization and propagation of singularities for semilinear hyperbolic equations. In *Hyperbolic equations and related topics (Katata/Kyoto, 1984)*, pages 11–49. Academic Press, Boston, MA, 1986.
[14] E. Broman and R. Meester. Survival of inhomogeneous Galton-Watson processes. *Adv. in Appl. Probab.*, 40(3):798–814, 2008.
[15] S. Cohen, C. Lacaux, and M. Ledoux. A general framework for simulation of fractional fields. *Stochastic Process. Appl.*, 118(9):1489–1517, 2008.
[16] M. Dozzi and G. Shevchenko. Real harmonizable multifractional stable process and its local properties. *Stochastic Process. Appl.*, 121(7):1509–1523, 2011.
[17] A. Durand. Random wavelet series based on a tree-indexed Markov chain. *Comm. Math. Phys.*, 283(2):451–477, 2008.
[18] A. Durand. Singularity sets of Lévy processes. *Probab. Theory Related Fields*, 143(3-4):517–544, 2009.
[19] A. Durand and S. Jaffard. Multifractal analysis of Lévy fields. *Probab. Theory Related Fields*, 111, 2011.
[20] E. Herbin and J. Lévy Véhel. Stochastic 2-microlocal analysis. *Stochastic Process. Appl.*, 119(7):2277–2311, 2009.
[21] S. Jaffard. Pointwise smoothness, two-microlocalization and wavelet coefficients. *Publ. Mat.*, 35(1):155–168, 1991. Conference on Mathematical Analysis (El Escorial, 1989).
[22] S. Jaffard. The multifractal nature of Lévy processes. *Probab. Theory Related Fields*, 114(2):207–227, 1999.
[23] S. Jaffard and Y. Meyer. Wavelet methods for pointwise regularity and local oscillations of functions. *Mem. Amer. Math. Soc.*, 123(587):x+110, 1996.
[24] S. Jaffard and Y. Meyer. On the pointwise regularity of functions in critical Besov spaces. *J. Funct. Anal.*, 175(2):415–434, 2000.
[25] D. Khoshnevisan and Z. Shi. Fast sets and points for fractional Brownian motion. In *Séminaire de Probabilités*, XXXIV, volume 1729 of *Lecture Notes in Math.*, pages 393–416. Springer, Berlin, 2000.
[26] D. Khoshnevisan and Y. Xiao. Level sets of additive Lévy processes. *Ann. Probab.*, 30(1):62–100, 2002.
[27] D. Khoshnevisan, N.-R. Shieh, and Y. Xiao. Hausdorff dimension of the contours of symmetric additive Lévy processes. *Probab. Theory Related Fields*, 140(1-2):129–167, 2008.
[28] N. Kôno and M. Maejima. Hölder continuity of sample paths of some self-similar stable processes. *Tokyo J. Math.*, 14(1):93–100, 1991.
[29] M. Maejima. A self-similar process with nowhere bounded sample paths. *Z. Wahrsch.*
[30] M. B. Marcus and J. Rosen. *Markov processes, Gaussian processes, and local times*, volume 100 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[31] T. Marquardt. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli*, 12(6):1099–1126, 2006.

[32] Y. Meyer. *Wavelets, vibrations and scalings*, volume 9 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1998.

[33] S. Orey and S. J. Taylor. How often on a Brownian path does the law of iterated logarithm fail? *Proc. London Math. Soc. (3)*, 28:174–192, 1974.

[34] E. Perkins. On the Hausdorff dimension of the Brownian slow points. *Z. Wahrsch. Verw. Gebiete*, 64(3):369–399, 1983.

[35] J. Picard. Representation formulæ for the fractional Brownian motion. *Séminaire de Probabilités*, XLIII:3–70, 2011.

[36] W. E. Pruitt. The growth of random walks and Lévy processes. *Ann. Probab.*, 9(6): 948–956, 1981.

[37] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[38] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, Yverdon, 1993.

[39] G. Samorodnitsky and M. S. Taqqu. *Stable non-Gaussian random processes*. Stochastic Modeling. Chapman & Hall, New York, 1994. Stochastic models with infinite variance.

[40] K.-i. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.

[41] S. Seuret and J. Lévy Véhel. A time domain characterization of $2$-microlocal spaces. *J. Fourier Anal. Appl.*, 9(5):473–495, 2003.

[42] S. Stoev and M. S. Taqqu. Stochastic properties of the linear multifractional stable motion. *Adv. in Appl. Probab.*, 36(4):1085–1115, 2004.

[43] S. Stoev and M. S. Taqqu. Path properties of the linear multifractional stable motion. *Fractals*, 13(2):157–178, 2005.

[45] Y. Xiao. Uniform modulus of continuity of random fields. *Monatsh. Math.*, 159(1-2): 163–184, 2010.