A Uniqueness Property for $H^\infty$ on Coverings of Projective Manifolds

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Abstract

Let $Y$ be a regular covering of a complex projective manifold $M \hookrightarrow \mathbb{CP}^N$ of dimension $n \geq 2$. Let $C$ be intersection with $M$ of at most $n - 1$ generic hypersurfaces of degree $d$ in $\mathbb{CP}^N$. The preimage $X$ of $C$ in $Y$ is a connected submanifold. Let $H^\infty(Y)$ and $H^\infty(X)$ be the Banach spaces of bounded holomorphic functions on $Y$ and $X$ in the corresponding supremum norms. We prove that the restriction $H^\infty(Y) \longrightarrow H^\infty(X)$ is an isometry for $d$ large enough. This answers the question posed in [L].

1. Introduction and Formulation of the Result.

1.1. An Extension Theorem. Let $M$ be a complex projective manifold of dimension $n \geq 2$ with a Kähler form $\omega$ and let $L$ be a positive line bundle on $M$ with canonical connection $\nabla$ and curvature $\Theta$ in a hermitian metric $h$. Let $C$ be the common zero locus of holomorphic sections $s_1, \ldots, s_k$, $k < n$, of $L$ over $M$ which, in a trivialization, can be completed to a set of local coordinates at each point $C$. Then $C$ is a (possibly disconnected) $k$-dimensional submanifold of $M$ which will be referred to as an $L$-submanifold of $M$. Let $\pi : Y_G \longrightarrow M$ be a regular covering of $M$ with a transformation group $G$ and $X_G = \pi^{-1}(C)$. We denote the pullbacks to $Y_G$ of $\omega$ and $\Theta$ by the same letters.

Example 1.1 If $L$ is very ample, then it is pullback of the hyperplane bundle by an embedding of $M$ into some projective space $\mathbb{CP}^N$. Further, zero loci of holomorphic sections of $L$ are hyperplane sections of $M$. By Bertini’s theorem, the generic linear subspace of codimension $n-k$, $k < n$, intersects $M$ transversely in a smooth manifold $C$ of dimension $k$, and by the Lefschetz hyperplane theorem, $C$ is connected and

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the induced homomorphism $\pi_1(C) \to \pi_1(M)$ of fundamental groups is surjective. Thus in this case $X_G \subset Y_G$ is a connected submanifold.

Further, let $\text{dist}(\cdot,\cdot)$ be the distance on $Y_G$ induced by $\omega$. Consider a function $\phi : Y_G \to \mathbb{R}$ such that $d\phi$ is bounded, i.e.

$$|\phi(x) - \phi(y)| \leq a \cdot \text{dist}(x,y) \quad \text{for some } \ a > 0.$$ 

By $\mathcal{O}_\phi(X_G)$ we denote the vector space of holomorphic functions on $X_G$ such that $|f|^2 e^{-\phi}$ is integrable on $X_G$ with respect to the volume form of the induced Kähler metric on $X_G$. This is a Hilbert space with respect to the inner product

$$(f,g) \mapsto \int_{X_G} f g e^{-\phi} \omega^k .$$

We define $\mathcal{O}_\phi(Y_G)$ similarly. By $| \cdot |_{\phi,X_G}$ and $| \cdot |_{\phi,Y_G}$ we denote the corresponding norms. It was shown in [L] that the restriction determines a continuous linear map

$$\rho : \mathcal{O}_\phi(Y_G) \to \mathcal{O}_\phi(X_G), \quad f \mapsto f|_{X_G} .$$

The following remarkable result was proved by Lárusson [L, Th.1.2].

**Theorem 1.2** Suppose

$$\Theta \geq i\partial\bar{\partial}\phi + \epsilon\omega$$

for some $\epsilon > 0$ in the sense of Nakano. Then $\rho$ is an isomorphism.

**1.2. Formulation of the Main Result.** An important example of a function $\phi$ as above is obtained by smoothing the distance $\delta$ from a fixed point $o$ in $Y_G$. By a result of Napier [N], there is a smooth function $\tau$ on $Y_G$ such that

(A) $c_1 \delta \leq \tau \leq c_2 \delta + c_3$ for some $c_1, c_2, c_3 > 0,$

(B) $d\tau$ is bounded, and

(C) $i\partial\bar{\partial}\tau$ is bounded.

Furthermore, by (A) and since the curvature of $Y_G$ is bounded below, there is $c > 0$ such that $e^{-c\tau}$ is integrable on $Y_G$. Then $e^{-c\tau}$ is also integrable on $X_G$. We set

$$A := \frac{cc_2}{c_1} . \quad (1.1)$$

Let $\tilde{L}$ be any positive line bundle on $M$ with curvature $\tilde{\Theta}$. By (C) there is a non-negative integer $m_0$ such that for any integer $m > m_0$

$$m\tilde{\Theta} > i\partial\bar{\partial}(A\tau) . \quad (1.2)$$

We set $L := \tilde{L}^\otimes m$. Then Lárusson’s theorem holds for coverings $X_G := \pi^{-1}(C) \subset Y_G$ of $L$-submanifolds $C \subset M$ with $\phi := A\tau$ and with $\phi := c\tau$ (because $A \geq c$).

Let $H^\infty(Y_G)$, $H^\infty(X_G)$ be the Banach spaces of bounded holomorphic functions on $Y_G$ and $X_G$ in the corresponding supremum norms.
Theorem 1.3 The map $\rho : H^\infty(Y_G) \to H^\infty(X_G)$, $f \mapsto f|_{X_G}$, is an isometry.

1.3. Corollaries and Examples. Let $X$ be a complex manifold and $H^\infty(X)$ be the Banach algebra (in the supremum norm) of bounded holomorphic functions on $X$. The maximal ideal space $M = \mathcal{M}(H^\infty(X))$ is the set of all nontrivial linear multiplicative functionals on $H^\infty(X)$. The norm of any $\phi \in M$ is $\leq 1$ and so $M$ is embedded into the unit ball of the dual space $(H^\infty(X))^\ast$. Then $M$ is a compact Hausdorff space in the weak $*$ topology induced by $(H^\infty(X))^\ast$ (i.e., the Gelfand topology). Further, there is a continuous map $i : X \to M$ taking $x \in X$ to the evaluation homomorphism $f \mapsto f(x)$. This map is an embedding if $H^\infty(X)$ separates points of $X$. Recall also that the complement to the closure of $i(X)$ in $M$ is called the corona. The corona problem is to determine these $X$ for which the corona is empty. For example, according to Carleson’s celebrated Corona Theorem [C] this is true for $X$ being the open unit disk $\mathbb{D} \subset \mathbb{C}$. Also there are non-planar Riemann surfaces for which the corona is non-trivial (see e.g., [JM], [G], [BD] and references therein). The general problem for planar domains is still open, as is the problem in several variables for the ball and polydisk. In [L, Th. 2.1] Lárusson discovered simplest examples of Riemann surfaces with big corona. Namely, using his Theorem 1.2 he proved that if $Y_G \subset \mathbb{C}^n$ is a bounded domain and $X_G \subset Y_G$ is a Riemann surface satisfying assumptions of Theorem 1.3 then the natural map $i : X_G \hookrightarrow \mathcal{M}(H^\infty(X_G))$ extends to an embedding $Y_G \hookrightarrow \mathcal{M}(H^\infty(X_G))$. The next statement is an extension of his result.

Corollary 1.4 Under the assumptions of Theorem 1.3, the transpose map $\rho^* : \mathcal{M}(H^\infty(X_G)) \to \mathcal{M}(H^\infty(Y_G))$, $\phi \mapsto \phi \circ \rho$, is a homeomorphism.

This follows from the fact that $\rho : H^\infty(Y_G) \to H^\infty(X_G)$ is an isometry of Banach algebras. □

Example 1.5 (1) (The references for this example are in [L, Sect. 4].) Let $M$ be a projective manifold covered by the unit ball $\mathbb{B} \subset \mathbb{C}^n$ with a positive line bundle $L$ with curvature $\Theta$, and $X \subset \mathbb{B}$ be the preimage of an $L$-submanifold $C \subset M$. Let $\delta$ be the distance from the origin in the Bergman metric of $\mathbb{B}$. By $\omega$ we denote the Kähler form of the Bergman metric. It was shown in [L, Sect. 4] that there is a nonnegative function $\tau$ on $\mathbb{B}$ such that $i\partial\bar{\partial}\tau = \omega$, $d\tau$ is bounded, and

$$\sqrt{n+1}\delta \leq \tau \leq \sqrt{n+1}\delta + (n+1)\log 2.$$  

Moreover,

$$\int_{\mathbb{B}} e^{-ct} \omega^n < \infty \quad \text{if and only if} \quad c > \frac{2n}{n+1}.$$  

Applying Theorem 1.3 (with $c_2 = c_1 = \sqrt{n+1}$) we obtain that $\rho : H^\infty(\mathbb{B}) \to H^\infty(X)$ is an isometry if $\Theta > \frac{2n}{n+1}\omega$. This holds for instance if $L = K^{\otimes m}$ with $m \geq 2$ where $K$ is the canonical bundle of $M$.

(2) Let $S$ be a compact complex curve of genus $g \geq 2$ and $\mathbb{CT}$ be a one-dimensional complex torus. Consider an $L$-curve $C$ in $M := S \times \mathbb{CT}$ with a very ample $L$ satisfying the assumptions of Theorem 1.3. Let $\pi : \mathbb{D} \times \mathbb{C} \to M$ be the universal
Then there is a constant \( b > 0 \) depending on \( U \) and \( V \) only such that
\[
\max_{V} |f| \leq b \sqrt{B}.
\] (2.1)

The proof of the lemma is the consequence of the following facts:
(a) after the identification of \( U_{i} \) with the closed unit polydisk \( D \) and of \( V_{j} \) with a compact subset \( D_{j} \subset D \), the volume form \( \omega^{n} \) restricted to each \( U_{i} \) is equivalent to the Euclidean volume form \( do := dz_{1} \wedge d\overline{z}_{1} \wedge \ldots \wedge dz_{n} \wedge d\overline{z}_{n} \);
(b) the Bergman inequality (see [GR, Ch.6, Th.1.3])
\[
\max_{D_{j}} |f| \leq \frac{(\sqrt{n})^{n}}{(\sqrt{n\pi})^{n}} \left( \int_{D} |f|^{2}do \right)^{1/2},
\]

where \( d \) is the Euclidean distance from \( D_{j} \) to the boundary of \( D \).

We leave the details to the reader. \( \square \)

Let \( \text{dist}(\cdot, \cdot) \) be the distance on \( Y_{G} \) in the metric induced by \( \omega \). In particular, \( \delta(x) := \text{dist}(x, o) \). Since \( \omega \) is invariant with respect to the action of \( G \) we also have \( \text{dist}(g(x), g(y)) = \text{dist}(x, y) \) for any \( g \in G \). From inequalities (A) for \( \tau \) and the triangle inequality for the distance we obtain
\[
\tau(g(x)) \geq c_{1} \text{dist}(g(x), o) \geq c_{1} [\text{dist}(g(x), g(o)) - \text{dist}(g(o), o)] = c_{1} [\text{dist}(x, o) - \text{dist}(g(o), o)] + c_{1} \delta(g(o)).
\] (2.2)

Further, if \( x \in K \) then
\[
a_{1} \leq \tau(x) \leq a_{2} \text{ for some } a_{1}, a_{2} > 0.
\] (2.3)

Below by \( | \cdot |_{\infty, X_{G}} \) and \( | \cdot |_{\text{cr}, X_{G}} \) we denote the corresponding \( H^{\infty} \)-norms. Let \( f \in H^{\infty}(X_{G}) \). Then \( f \in O_{Ar}(X_{G}) \cap O_{ct}(X_{G}) \) and there is \( a_{3} > 0 \) such that
\[
\max \{|f|_{Ar, X_{G}}, |f|_{ct, X_{G}}\} \leq a_{3} \sup_{X_{G}} |f| := a_{3} |f|_{\infty, X_{G}}.
\]
By Theorem 1.2, there is a unique \( \tilde{f} \in \mathcal{O}_{\mathcal{A}}(Y_G) \cap \mathcal{O}_{\mathcal{C}}(Y_G) \) such that \( \tilde{f}|_{X_G} = f \) and
\[
\max\{|\tilde{f}|_{\mathcal{A}_Y, Y_G}, |\tilde{f}|_{\mathcal{C}_Y, Y_G}\} \leq a_4 \max\{||f|_{\mathcal{A}_X, X_G}, |f|_{\mathcal{C}_X, X_G}\} \quad \text{for some } a_4 > 0.
\]
Combining these inequalities with (2.3) and (2.1) yields
\[
\max_K |\tilde{f}| \leq a_5 |f|_{\infty, X_G},
\]
with some \( a_5 > 0 \) depending on \( X_G, Y_G \) only. For a fixed \( g \in G \) consider \( (g^*f)(z) := f(g(z)) \). As above, there is a unique function \( \tilde{f}_g \in \mathcal{O}_{\mathcal{A}}(Y_G) \cap \mathcal{O}_{\mathcal{C}}(Y_G) \), \( \tilde{f}_g|_{X_G} = g^*f \), such that
\[
\max_K |\tilde{f}_g| \leq a_5 |f|_{\infty, X_G},
\]
But according to (2.2) and (1.1), function \( (g^*\tilde{f})(z) := \tilde{f}(g(z)) \) belongs to \( \mathcal{O}_{\mathcal{A}}(Y_G) \) and \( (g^*\tilde{f} - \tilde{f}_g)|_{X_G} \equiv 0 \). Thus by Theorem 1.2 we have \( \tilde{f}_g = g^*\tilde{f} \). Since \( K \) is the fundamental compact, the above inequality for each \( \tilde{f}_g \) implies that
\[
|\tilde{f}|_{\infty, Y_G} \leq a_5 |f|_{\infty, X_G}. \tag{2.4}
\]
We will prove now that \( a_5 = 1 \) which gives the required statement. Indeed, the same arguments as above show that for any integer \( n \geq 1 \) the function \( (\tilde{f})^n \) is the unique extension of \( f^n \) satisfying (2.4):
\[
|(\tilde{f})^n|_{\infty, Y_G} \leq a_5 |f^n|_{\infty, X_G}.
\]
Thus
\[
|\tilde{f}|_{\infty, Y_G} \leq \lim_{n \to \infty} (a_5)^{1/n} |f|_{\infty, X_G} = |f|_{\infty, X_G}
\]
The proof of the theorem is complete. \( \square \)

References

[BD] D. E. Barett and J. Diller, A new construction of Riemann surfaces with corona. J. Geom. Anal. 8 (1998), 341-347.

[C] L. Carleson, Interpolation of bounded analytic functions and the corona problem. Ann. of Math. 76 (1962), 547-559.

[G] T. W. Gamelin, Uniform algebras and Jensen measures. London Math. Soc. Lecture Notes Series 32, Cambridge University Press, 1978.

[GR] H. Grauert and R. Remmert, Theorie der Steinschen Räume. Springer-Verlag, Berlin, 1977.

[JM] P. Jones and D. Marshall, Critical points of Green’s functions, harmonic measure and the corona theorem. Ark. Mat. 23 no.2 (1985), 281-314.

[L] F. Lárusson, Holomorphic functions of slow growth on nested covering spaces of compact manifolds. Canad. J. Math. 52 (5) (2000), 982-998.

[N] T. Napier, Convexity properties of coverings of smooth projective varieties. Math. Ann. 286 (1990), 433-479.