Tropical Cramer Determinants Revisited

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We dedicate this paper to the memory of our friend and colleague Grigory L. Litvinov.

Abstract. We prove general Cramer type theorems for linear systems over various extensions of the tropical semiring, in which tropical numbers are enriched with an information of multiplicity, sign, or argument. We obtain existence or uniqueness results, which extend or refine earlier results of Gondran and Minoux (1978), Plus (1990), Gaubert (1992), Richter-Gebert, Sturmfels and Theobald (2005) and Izhakian and Rowen (2009). Computational issues are also discussed; in particular, some of our proofs lead to Jacobi and Gauss-Seidel type algorithms to solve linear systems in suitably extended tropical semirings.

1. Introduction

1.1. Motivations. The max-plus or tropical semiring $\mathbb{R}_{\text{max}}$ is the set $\mathbb{R} \cup \{-\infty\}$, equipped with the addition $a \oplus b = \max(a, b)$ and the multiplication $a \odot b = a + b$. We refer the reader for instance to [BCOQ92b, KM97, LMS01, IMS07, RGST05] for introductory materials on max-plus or tropical algebra.

We denote by $\mathbb{R}_{\text{max}}^n$ the $n$th-fold Cartesian product of $\mathbb{R}_{\text{max}}$, which can be thought of as the tropical analogue of a finite dimensional vector space. A tropical hyperplane is a subset of $\mathbb{R}_{\text{max}}^n$ of the form

$$ H = \{ x \in \mathbb{R}_{\text{max}}^n \mid \max_{i \in \,[n]} (a_i + x_i) \text{ is attained at least twice} \} , $$

where $a = (a_1, \ldots, a_n)$ is a vector of $\mathbb{R}_{\text{max}}^n$, not identically $-\infty$, and $[n] := \{1, \ldots, n\}$. This definition is motivated by non-archimedean geometry. Indeed, let $\mathbb{K} = \mathbb{C}\{\{t\}\}$ denote the field of complex Puiseux series in the variable $t$, and let $v$ denote the valuation which associates to a series the opposite of its smallest exponent. Consider now a hyperplane of $\mathbb{K}^n$,

$$ H := \{ x \in \mathbb{K}^n \mid \sum_{i \in \,[n]} a_i x_i = 0 \} , $$

where $a = (a_1, \ldots, a_n)$ is a vector of $\mathbb{K}^n$ lifting $a$, meaning that $v(a_i) = a_i$, for all $i \in [n]$. Then it is easily checked that the image of $H$ by the map which applies the valuation $v$ entrywise is precisely the set of vectors of $H$ with rational...
coordinates. This is actually a special case of a result of Kapranov characterizing the non-Archimedean amoeba of a hypersurface \cite{EKL06}.

In the present work, we will extend or refine a series of basic results concerning the intersections of tropical hyperplanes, that we now review. One of these results was established by Richter-Gebert, Sturmfels and Theobald.

**Theorem 1.1** (Tropical Cramer Theorem, “complex” version \cite{RGST05}). Any \(n-1\) vectors of \(\mathbb{R}^n\) in general position are contained in a unique tropical hyperplane.

This statement has also an equivalent dual form: the intersection of \(n-1\) tropical hyperplanes in general position contains a unique vector up to an additive constant.

The parameters \(a_i\) of the hyperplane \(H\) arising in Theorem 1.1 can be obtained by solving the tropical analogue of a square linear system. Its solution turns out to be determined by the tropical analogues of Cramer determinants, which in this context are merely the value of optimal assignment problems. Then the data are said to be in general position if each of the optimal assignment problems arising in this way has a unique optimal solution. The role of the general position notion is made clear by the following result.

**Theorem 1.2** (\cite{RGST05}). A collection of \(n\) vectors of \(\mathbb{R}^n\) is contained in a tropical hyperplane if and only if the matrix having these vectors as columns is tropically singular, meaning that the assignment problem associated with this matrix has at least two optimal solutions.

More information on tropical singularity and related rank notions can be found in \cite{DSS05, AGG09, IR09, AGG12}.

Different but related results were obtained previously by considering tropical numbers with signs. Indeed, in the above results, the tropical semiring is essentially thought of as the image of the field of complex Puiseux series by the valuation. Alternatively, a tropical number with sign may be thought of as the image of a real Puiseux series, i.e., of an element of \(\mathbb{R}\{\{t\}\}\). Ideas of this nature were indeed essential in the development by Viro of the patchworking method (see the references in \cite{Vir01}), as well as in the construction by Plus \cite{Plu90} of the symmetrized tropical (or max-plus) semiring, \(S_{\text{max}}\).

When considering tropical numbers with signs, the notion of equation has to be replaced by a notion of balance \cite{Plu90}: a tropical sum of terms equipped with signs is said to be balanced if the maximum of terms with positive signs coincides with the maximum of the terms with negative signs. Some results of \cite{Plu90, Gau92} concerning systems of linear balances can be interpreted geometrically using the signed variant of tropical hyperplanes, considered by Joswig in \cite{Jos05}. A signed tropical hyperplane is a subset of \(\mathbb{R}_{\text{max}}^n\) of the form

\[
H^{\text{sgn}} = \{x \in \mathbb{R}_{\text{max}}^n \mid \max_{i \in I}(a_i + x_i) = \max_{j \in J}(a_j + x_j)\},
\]

where \([n] = I \cup J\) is a non-trivial partition, and \(a \in \mathbb{R}_{\text{max}}^n\) is a vector non-identically \(-\infty\). Observe that \(H^{\text{sgn}} \subset H\). Consider now a hyperplane of \((\mathbb{R}\{\{t\}\})^n\),

\[
H^{\text{sgn}} := \{x \in (\mathbb{R}\{\{t\}\})^n \mid \sum_{i \in I} a_i x_i = \sum_{j \in J} a_j x_j\},
\]

where \(a\) is any vector of \((\mathbb{R}\{\{t\}\})^n\) lifting \(a\), meaning now that \(v(a_i) = a_i\) and that \(a_i\) is nonnegative (recall that a real Puiseux series is nonnegative if it is zero or if its leading coefficient is positive), for all \(i \in [n]\). Then it can be easily checked that the vectors with rational entries of a signed tropical hyperplane are precisely the
images by the valuation of the nonnegative vectors of the associated hyperplane over the field of real Puiseux series.

With these observations in mind, the following result established by Plus appears to be a “real” analogue of Theorem 1.1.

**Theorem 1.3 (Tropical Cramer theorem, “real” version, Corollary of [Plu90 Th. 6.1]).** Any $n - 1$ vectors of $\mathbb{R}^n_{\max}$ in sign-general position are contained in a unique signed tropical hyperplane.

By comparison with Theorem 1.1, we use here a milder notion of general position. Indeed, the “real” tropical analogue of a determinant consists of the value of an optimal assignment problem, together with the information of all the possible signs of optimal permutations. Then, a tropical determinant is said to be sign-nonsingular if all optimal permutations have the same sign. Finally, $n - 1$ vectors of $\mathbb{R}^n_{\max}$ are said to be in sign-general position if all the associated tropical Cramer determinants are sign-nonsingular. The vector $a$ defining the signed hyperplane $H_{\text{sgn}}$ of Theorem 1.3 is determined by the tropical Cramer determinants, the signs of which provide the sets $I, J$ in the partition.

Actually, a more general result was stated in [Plu90] in the language of systems of balances over the symmetrized tropical semiring $\mathbb{S}^\text{max}$. Theorem 1.3 covers a special case with a more straightforward geometric interpretation. The details of the derivation of the latter theorem from Theorem 6.1 of [Plu90] will be given in Section 7 together with the dual result of Theorem 1.3 concerning intersections of signed tropical hyperplanes.

A result of Gondran and Minoux, that we restate as follows in terms of signed tropical hyperplanes, may be thought of as a “real” analogue of Theorem 1.2.

**Theorem 1.4 (Corollary of [GM78]).** A collection of $n$ vectors of $\mathbb{R}^n_{\max}$ is contained in a signed tropical hyperplane if and only if the matrix having these vectors as columns has a sign-singular tropical determinant.

A dual result, concerning the intersection of $n$ signed tropical hyperplanes, was stated by Plus [Plu90] and proved by Gaubert [Gau92], we shall discuss it in Section 7. As pointed out in [BG99], the notion of sign-nonsingularity arising here is an extension of the notion with the same name arising in combinatorial matrix theory, in particular in the study of the permanent problem of Pólya, see [BS95] for more information.

Whereas the results concerning “balances” involve the extension of the tropical numbers with signs, we note that other extensions have been used more recently. In particular, the incorporation of “phase” (instead of sign) information in tropical constructions has played an important role in the arguments of Mikhalkin [Mik05]. Moreover, Viro introduced a general notion of hyperfield [Vir10] with a multivalued addition, which he used in particular to capture the phase information. The semirings with symmetry introduced by the authors in [AGG09] provide another way to encode the sign or phase. Also, different extensions have been provided by the “supertropical” structures of Izhakian and Rowen [IR10] extending the bi-valued tropical semiring introduced by Izhakian [Izh09]; in the latter, the goal is not to encode “sign” or “phase”, but the fact that the maximum in an expression is achieved twice at least.

### 1.2. Main results

Given the analogy between the “complex” and “real” versions of the tropical Cramer theorem (Theorems 1.1 and 1.3 above), as well as between the unsigned and signed notions of tropical singularity of matrices (Theorems 1.2 and 1.4), one may ask whether all these results may be derived from common principles. One may also ask whether results of this kind are valid for more
In this paper, we answer these questions and deal with related algorithmic issues, by developing a theory of elimination of linear systems over semirings, building on ideas and results of [Plu90, Gau92, AGG09]. This will allow us to show that the earlier results are indeed special instances of general Cramer type theorems, which apply to various extensions of tropical semirings. These theorems are established using axioms allowing one to perform “elimination” of balances, in a way similar to Gaussian elimination. In this way, we will generalize and sometimes refine (handling degenerate cases) earlier results. Also, some of our proofs are based on Jacobi or Gauss-Seidel type iterative schemes, and lead to efficient algorithms. In passing, we revisit some of the results of [Plu90, Gau92], giving their geometric interpretation in terms of signed tropical hyperplanes.

An ingredient of our approach is the introduction (in Section 2) of a rather general notion of extension of the tropical semiring, together with a general “balance” relation $\nabla$, which, depending on the details of the extension, expresses the fact that the maximum is attained at least twice in an expression, or that the maxima of two collections of terms coincide. Our constructions include as a special case the symmetrized tropical semiring of Plus [Pin90] and the bi-valued tropical semiring of Izhakian [Izh09], but also a certain “phase extension” of the tropical semiring, which is a variant of the complex tropical hyperfield of Viro. Our notion also includes certain “supertropical semifields” in the sense of Izhakian and Rowen [IR10]. Then auxiliary combinatorial results are presented in Section 3.

The general affine Cramer system reads $Ax \nabla b$, whereas the homogeneous system reads $Ax \nabla 0$, where $A$ is an $n \times n$ matrix and $b$ is a vector of dimension $n$, both with entries in an extension of the tropical semiring. To be interpreted geometrically, the vector $x$ which is searched will be required to satisfy certain non-degeneracy conditions, typically that the coordinate of $x$ do not belong to the set of balanced non-zero elements of the extension.

In this way, by developing methods of [AGG09], we obtain in Section 4 a general result, Theorem 4.18 concerning the unique solvability of non-singular Cramer system, which includes Theorems 1.1 and 1.3 as special cases.

Then we study in Section 5 the existence problem for the solution of the affine Cramer system $Ax \nabla b$, without making the non-singularity assumptions needed in the previous uniqueness results. Theorem 5.20 below gives a general existence theorem, with a constructive proof based on the idea of the Jacobi algorithm in [Plu90]. Theorem 5.27 gives an alternative Gauss-Seidel type algorithm. These results are valid in a large enough class of semirings, including not only the symmetrized tropical semiring as in [Pin90], but also the phase extension of the tropical semiring.

In Section 6, we deal with the generalization of Theorems 1.2 and 1.4 which concern singular linear systems of $n$ equations in $n$ variables. In Theorem 6.9, we characterize the existence of non-degenerate solutions of $Ax \nabla 0$, recovering Theorems 1.2 and 1.4 as special cases. This extends to more general semirings a theorem of Gaubert [Gau92] dealing with the case of the symmetrized tropical semiring $S_{\max}$. We note however that by comparison with the Jacobi/Gauss-Seidel type results of Section 5 the results of this section hold under more restrictive assumptions on the semiring.

A geometrical interpretation of the previous results, in the case of the symmetrized tropical semiring, is presented in Section 7.
In Section 8, we start to address computational issues. The \( n + 1 \) Cramer determinants of the system \( Ax \nabla b \) correspond to \( n + 1 \) optimal assignment problems. In Section 8.1, we show that our approach based on a Jacobi-type iterative method leads to an algorithm to compute the solution, as well as the Cramer determinants (up to signs), by solving a single (rather than \( n + 1 \)) optimal assignment problem, followed by a single destination shortest path problem. For the sake of comparison, we revisit in Section 8.2 the approach of Richter-Gebert, Sturmfels and Theobald [RGST05], building on results of Sturmfels and Zelevinsky dealing with Minkowski sums of Birkhoff polytopes: it gives a reduction to a different transportation problem. Although the original exposition of [RGST05] is limited to instances in general position, we show that some of their results remain valid even without such an assumption. The computation of the signs of tropical Cramer determinants is finally briefly discussed in Section 9.

2. Semirings with a symmetry and a modulus

2.1. Definitions and first properties.

**Definition 2.1.** A semiring is a set \( S \) with two binary operations, addition, denoted by \(+\), and multiplication, denoted by \( \cdot \) or by concatenation, such that:

- \( S \) is an abelian monoid under addition (with neutral element denoted by \( 0 \) and called zero);
- \( S \) is a monoid under multiplication (with neutral element denoted by \( 1 \) and called unit);
- multiplication is distributive over addition on both sides;
- \( a0 = 0a = 0 \) for all \( a \in S \).

In the sequel, a semiring will mean a non-trivial semiring (different from \( \{0\} \)).

Briefly, a semiring differs from a ring by the fact that an element may not have an additive inverse. The first examples of semirings which are not rings that come to mind are non-negative integers \( \mathbb{N} \), non-negative rationals \( \mathbb{Q}^+ \) and non-negative reals \( \mathbb{R}^+ \) with the usual addition and multiplication. There are classical examples of non-numerical semirings as well. Probably the first such example appeared in the work of Dedekind [Ded94] in connection with the algebra of ideals of a commutative ring (one can add and multiply ideals but it is not possible to subtract them).

**Definition 2.2.** A semiring or an abelian monoid \( S \) is called idempotent if \( a + a = a \) for all \( a \in S \), \( S \) is called zero-sum free or antinegative if \( a + b = 0 \) implies \( a = b = 0 \) for all \( a, b \in S \), and \( S \) is called commutative if \( a \cdot b = b \cdot a \) for all \( a, b \in S \).

An idempotent semiring is necessarily zero-sum free. We shall always assume that the semiring \( S \) is commutative.

An interesting example of an idempotent semiring is the max-plus semiring

\[
\mathbb{R}_{\max} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot),
\]

where \( a \oplus b = \max\{a, b\} \) and \( a \odot b = a + b \). Here the zero element of the semiring is \(-\infty\), denoted by \( \ominus \), and the unit of the semiring is \( 0 \), denoted by \( \mathbb{1} \).

The usual definition of matrix operations carries over to an arbitrary semiring. We denote the set of \( m \times n \) matrices over \( S \) by \( M_{m,n}(S) \). Also we denote \( M_n(S) = M_{n,n}(S) \) and we identify \( S^n \) with \( M_{n,1}(S) \). Note that \( M_n(S) \) is a semiring.

Some of the following notions were introduced in [AGG09], where details and additional properties can be found.
Definition 2.3. Let $S$ be a semiring. A map $\tau : S \to S$ is a symmetry if

\begin{align}
(2.1a) & \quad \tau(a + b) = \tau(a) + \tau(b) \\
(2.1b) & \quad \tau(0) = 0 \\
(2.1c) & \quad \tau(a \cdot b) = a \cdot \tau(b) = \tau(a) \cdot b \\
(2.1d) & \quad \tau(\tau(a)) = a.
\end{align}

Example 2.4. A trivial example of a symmetry is the identity map $\tau(a) = a$. Of course, in a ring, we may take $\tau$ deduce that $\tau$ commutes with all elements of $S$.

Proposition 2.5. A map $\tau$ is a symmetry of the semiring $S$ if and only if there exists $e \in S$ such that $e \cdot e = 1$, and $\tau(a) = e \cdot a = a \cdot e$ for all $a \in S$ (hence $e$ commutes with all elements of $S$).

Proof. Let $\tau$ be a symmetry of $S$, and denote $e = \tau(1)$. By (2.1c), we get that $\tau(a) = \tau(1 \cdot a) = e \cdot a$, and $\tau(a) = \tau(a \cdot 1) = a \cdot e$ for all $a \in S$. By (2.1d), we deduce that $\tau(e) = e = \tau(1) = 1$, and since $\tau(e) = e \cdot e$, we get $e \cdot e = 1$. Conversely, if $\tau(a) = e \cdot a = a \cdot e$ for all $a \in S$, with $e \cdot e = 1$, then $\tau$ satisfies all the conditions in (2.1). □

In the rest of the paper we shall write $-a$ for $\tau(a)$. So, $a - a$ is not zero generally speaking, but is a formal sentence meaning $a + \tau(a)$. Moreover, for any integer $n \geq 0$, $(-1)^n$ will mean the $n$th power of $-1 = \tau(1)$, hence the product of $n$ copies of $-1$. Also $+a$ will mean $a$, in particular in the formula $\pm a$. If the addition of $S$ is denoted by $\oplus$ instead of $+$, then $+a$ and $-a$ will be replaced by $\oplus a$ and $\ominus a$.

Definition 2.6. For any $a \in S$, we set $a^\circ := a - a$, thus $-a^\circ = a^\circ = (-a)^\circ$, and we denote

$$S^\circ := \{a^\circ \mid a \in S\}$$

The elements of this set will be called balanced elements of $S$. Moreover, we define the balance relation $\triangledown$ on $S$ by $a \triangledown b$ if $a - b \in S^\circ$.

Note that $S^\circ$ is an ideal, hence the relation $\triangledown$ is reflexive and symmetric. It may not be transitive. Since $S^\circ$ is an ideal, it contains an invertible element of $S$ if and only if it coincides with $S$. We shall only consider in the sequel symmetries such that $S^\circ \neq S$. This permits the following definition.

Definition 2.7. When $S^\circ \neq S$, we say that a subset $S' \subseteq S$ is thin if $S' \subset (S \setminus S^\circ) \cup \{0\}$ and if it contains 0 and all invertible elements of $S$. When such a set $S'$ is fixed, its elements will be called thin elements.

Note that 0 is both a balanced and a thin element. In the sequel, we shall consider systems of linear “equations” (the equality relation will be replaced by balance), and we will require the variables to be thin. By choosing appropriately the set of thin elements, we shall see that standard Gauss type elimination algorithms carry over. Also, in most applications, only the thin solutions will have simple geometrical interpretations. Note that in [AGG09], we only used the notation $S'$ for the maximal possible thin set, that is $(S \setminus S^\circ) \cup \{0\}$; however, it will be useful to consider also for instance the smallest possible thin set, which is the set of invertible elements completed with 0.

Recall that $(R, \cdot, \leq)$ is an ordered semigroup if $(R, \cdot)$ is a semigroup, and $\leq$ is an order on $R$ such that for all $a, b, c \in R$, $a \leq b$ implies $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. An ordered monoid is a monoid that is an ordered semigroup. An ordered semiring is a semiring $(S, +, 0, \cdot, 1)$ endowed with an order relation $\leq$ such that $(S, +, \leq)$ and $(S, \cdot, \leq)$ are ordered semigroups.
We say that a map which the natural order is total, or for short, a totally ordered idempotent semiring.

We shall write simply \( \preceq \) instead of \( \preceq^S \), and \( \preceq^\circ \) instead of \( \preceq^{S^\circ} \).

These relations are preorders (reflexive and transitive), compatible with the laws of \( S \). They may not be antisymmetric.

**Definition 2.9.** The preorder \( \preceq \) is called the natural preorder on \( S \). A semiring \( S \) is said to be naturally ordered when \( \preceq \) (or equivalently \( \succeq \)) is an order relation, and in that case \( \preceq \) is called the natural order on \( S \), and \( \succeq \) is its opposite order. The notation \( \prec \) and \( \succ \) will be used for the corresponding strict relations.

When \( \preceq \) (or \( \succeq \)) is an order relation, so are \( \preceq^T \), \( \preceq^\circ \), \( \succeq \), and \( \succeq \). An idempotent semiring is necessarily naturally ordered, and a naturally ordered semiring is necessarily zero-sum free. We also have:

\[(2.2) \quad a \preceq^\circ b \text{ or } b \preceq^\circ a \Rightarrow a \nabla b \, .\]

The converse is false in general.

**Definition 2.10.** Let \( S \) be a semiring and \( R \) be an idempotent semiring in which the natural order is total, or for short, a totally ordered idempotent semiring. We say that a map \( \mu : S \rightarrow R \) is a modulus if it is a surjective morphism of semirings. In this case, we denote \( \mu(a) \) by \( |a| \) for all \( a \in S \).

We shall apply the notations \( \nabla \), \( \preceq^T \), \( \succeq \) (so \( \preceq \), \( \succeq \), \( \preceq^\circ \), \( \succeq^\circ \)), \( |·| \) to matrices and vectors, understanding that the relation holds entrywise. We shall do the same for the notions of “balanced” or “thin” elements.

**Proposition 2.11.** If \( (R, +, 0, \cdot, 1) \) is a totally ordered idempotent semiring, then the only symmetry on \( R \) is the identity. Moreover, any semiring \( (S, +, 0, \cdot, 1) \) with a symmetry and a modulus \( |·| : S \rightarrow R \) satisfies \( |−1| = |1| = 1 \).

**Proof.** Let \( \tau \) be a symmetry of \( R \). Then by Proposition 2.5 there exists \( e \in R \) such that \( e \cdot e = 1 \), and \( \tau(a) = e \cdot a = a \cdot e \) for all \( a \in R \). Since \( R \) is totally ordered, either \( e \preceq 1 \) or \( 1 \preceq e \). Assuming that the inequality \( e \preceq 1 \) holds, then multiplying it by \( e \), we get \( e \cdot e \preceq e \cdot 1 = e \), and since \( e \cdot e = 1 \) and \( \preceq \) is an order relation, we deduce that \( e = 1 \). The same is true when the inequality \( e \preceq 1 \) holds. This shows that \( e = 1 \), hence \( \tau \) is the identity map.

If now \( S \) is a semiring with a symmetry and a modulus, then \( |−1| \cdot |−1| = |−1 \cdot −1| = |1| = 1 \), and by the above arguments, we get that \( |−1| = 1 \). \( \Box \)

**2.2. Tropical extensions of semirings.** In the sequel, we shall consider semirings with a symmetry and a modulus. The following construction allows one to obtain easily such semirings.

**Proposition-Definition 2.12 (Extension of semirings).** Let \( (S, +, 0, \cdot, 1) \) be a semiring and let \( (R, \oplus, 0, \cdot, 1) \) be a totally ordered idempotent semiring. Then the semiring \( S \times R \) is defined as the set \( S \times R \) endowed with the operations

\[
(a, b) \oplus (a', b') = \begin{cases} 
(a + a', b) & \text{if } b = b' \\
(a, b) & \text{if } b \succeq b' \\
(a', b') & \text{if } b \prec b'
\end{cases}
\]

and \( (a, b) \circ (a', b') = (a \cdot a', b \cdot b') \). Its zero element is \( (0, 0) \) and its unit is \( (1, 1) \). For any subset \( A \) of a semiring \( S \), we denote

\[
A^* := A \setminus \{0\} \, .
\]
in particular, $\mathcal{R}^* := \mathcal{R} \setminus \{0\}$. Then the extension of $\mathcal{R}$ by a subset $\mathcal{A}$ of $\mathcal{S}$ is defined by
\begin{equation}
\mathcal{A} \times \mathcal{R} := (\mathcal{A} \times \mathcal{R}^*) \cup \{(0, 0)\}.
\end{equation}
If $\mathcal{A}$ is a subsemiring of $\mathcal{S}$, then $\mathcal{A} \times \mathcal{R}$, and $\mathcal{A} \times \mathcal{R}$ are subsemirings of $\mathcal{S} \times \mathcal{R}$. If in addition $\mathcal{A}$ is zero-sum free and without zero divisors, then
\begin{equation}
\mathcal{A} \times \mathcal{R} := \mathcal{A}^* \times \mathcal{R}
\end{equation}
is a subsemiring of $\mathcal{A} \times \mathcal{R}$.

This construction bears some similarity with a semidirect product, which motivates the notations “$\bar{\times}$”, “$\bar{\times}$” and “$\bar{\times}$”. We shall denote by $0$ and $1$, instead of $(0, 0)$ and $(1, 1)$, the zero and unit of $\mathcal{S} \times \mathcal{R}$.

If $(\mathcal{R}, 1, \leq)$ is a totally ordered monoid, completing $\mathcal{R}$ with a bottom element, denoted by $0$, we get the totally ordered idempotent semiring $\mathcal{R} := \mathcal{R} \cup \{0\}, \max, \ 0, \ 1$). All the semirings $\mathcal{R}$ satisfying the assumptions of Proposition 2.12 are of this form. When $\mathcal{R} = \mathbb{R}$ is equipped with its usual order and addition, we recover the max-plus semiring $\mathbb{R}_{\max}$. We may take more generally for $\mathcal{R}$ any submonoid of $(\mathbb{R}, +)$, or take $\mathbb{R}^d$ equipped with the lexicographic order and entrywise addition.

The intuition of the construction of Proposition 2.12 is best explained by the following example.

**Example 2.13 (Complex extension of the tropical semiring).** Let $\mathbb{C}$ denote the field of complex numbers. Then the semiring $\mathbb{C} \times \mathbb{R}_{\max}$ will be called “complex extension of the tropical semiring”. An element $(a, b) \in \mathbb{C} \times \mathbb{R}_{\max}$ encodes the asymptotic expansion $ae^{-b} + o(e^{-b})$, when $\epsilon$ goes to $0_+$ (when $(a, b) = 0$, this is the identically 0 expansion). Indeed, the “lexicographic” rule in the addition of $\mathbb{C} \times \mathbb{R}_{\max}$ corresponds precisely to the addition of asymptotic expansions, and the entrywise product of $\mathbb{C} \times \mathbb{R}_{\max}$ corresponds to the product of asymptotic expansions. By taking the zero-sum free subsemiring $\mathbb{R}^+ \subset \mathbb{C}$ consisting of the real nonnegative numbers, we end up with the subsemiring $\mathbb{R}_+ \times \mathbb{R}_{\max}$, which encodes the asymptotic expansions $ae^{-b} + o(e^{-b})$ with $a > 0$ and $b \in \mathbb{R}$, together with the identically 0 expansion. The latter semiring was used in [ABC98] under the name of semiring of (first order) jets, to study eigenvalue perturbation problems.

We next list some simple facts concerning extensions of semirings.

**Fact 2.14.** Let $\mathbb{B} := \{0, 1\}$ be the Boolean semiring, i.e., the idempotent semiring with two elements, let $S$ be zero-sum free and without zero divisors. Then $S \times \mathbb{B}$ is isomorphic to $S$. In general $S$ is a subsemiring of $S \times \mathcal{R}$ since the injective map $j : S \to S \times \mathcal{R}$ defined by
\[
\begin{cases}
\ (a, 1) & \text{if } a \neq 0, \\
\ (0, 0) & \text{if } a = 0,
\end{cases}
\]
is a morphism of semirings. So the semirings $S \times \mathcal{R}$ and $S \times \mathcal{R}$ are semiring extensions of $S$. The same is true for $S \times \mathcal{R}$ as soon as $S$ is zero-sum free without zero divisors. Note that one can also consider the map $\gamma : S \times \mathcal{R} \to S$, such that $\gamma(a, b) = a$, for all $(a, b) \in S \times \mathcal{R}$. This map yields a surjective multiplicative morphism from $S \times \mathcal{R}$ (or from $S \times \mathcal{R}$ or $S \times \mathcal{R}$) to $S$, such that the composition $\gamma \circ j$ equals the identity of $S$.

**Fact 2.15.** If $S$ and $\mathcal{R}$ are commutative then so is $S \times \mathcal{R}$. If $S$ is idempotent (resp., naturally ordered) then so is $S \times \mathcal{R}$, and consequently $S \times \mathcal{R}$ and $S \times \mathcal{R}$. The natural preorder on $S \times \mathcal{R}$ is the lexicographic preorder: $(a, b) \preceq (a', b')$ if and only if either $b \preceq b'$, or $b = b'$ and $a \preceq a'$. 

FACT 2.16. Let \( \tau \) be a symmetry of \( S \). We define the map \( \tau' \) from \( S \ltimes R \to R \) by \( \tau'((a,b)) = (\tau(a), b) \) for all \( a \in S \) and \( b \in R \). Then \( \tau' \) is a symmetry of \( S \ltimes R \), which preserves \( S \ltimes R \) and \( S \ltimes R' \). We shall call \( \tau' \) the extension of \( \tau \). Conversely, if \( \tau' \) is a symmetry of \( S \ltimes R \), then it is the extension of some symmetry \( \tau \) of \( S \).

For these symmetries, we have \( (a,b)\tau = (a',b) \) for all \( a \in S \) and \( b \in R \), hence \( (S \ltimes R)\tau = S \ltimes R \), \( (S \ltimes R')^\tau = S \ltimes R = (S \ltimes R) \tau \cap (S \ltimes R'), (S \ltimes R)\tau = (S \ltimes R) \cap (S \ltimes R \tau) \). If \( S \) is an injective and multiplicative morphism, and its image is included in \( S \ltimes R \), \( S \ltimes R' \) is a thin set of \( S \ltimes R \), and \( S \ltimes R' \) is a thin set of \( S \ltimes R \), and also of \( S \ltimes R \).

**Proof.** We only need to show that any symmetry \( \tau' \) of \( S \ltimes R \) is the extension of some symmetry of \( S \), the other properties are immediate. By Proposition 2.5, \( \tau'((a,b)) = e' \circ (a,b) = (a,b) \circ e' \), for all \( (a,b) \in S \ltimes R \), for some \( e' = (e,f) \in S \ltimes R \) such that \( (e,f) \circ (e,f) = (1,1) \). Denoting \( \tau(a) = e \cdot a \) and \( \sigma(b) = f \cdot b \), we get that \( \tau'(a,b) = (\tau(a), \sigma(b)) \). The properties of \( (e,f) \) imply that \( \tau \) is a symmetry of \( S \) and \( \sigma \) is a symmetry of \( R \). Moreover, since \( R \) is totally ordered, then by Proposition 2.11 \( \sigma \) is the identity map, or \( f = 1 \). This shows that \( \tau' \) is necessarily the extension of the symmetry \( \tau \) of \( S \).

**FACT 2.17.** Let \( S \) and \( R \) be as in Proposition 2.12. Then the map \( \mu : S \ltimes R \to R \), \( (a,b) \mapsto b \) is a modulus. Also its restriction to \( S \ltimes \mathbb{R} \) or \( S \ltimes R \) is a modulus. If \( S \) is naturally ordered, then \( \mu \) is order preserving. Moreover, the map \( \iota : R \to S \ltimes R \) defined by

\[
\iota(b) = \begin{cases} (1,b) & \text{if } b \in \mathbb{R}^+, \\ 0 & \text{if } b = 0,
\end{cases}
\]

is an injective and multiplicative morphism, and its image is included in \( S \ltimes R \subset S \ltimes R \). The composition \( \mu \circ \iota \) equals the identity on \( R \), thus the image of \( \iota \) is the set of fixed points of \( \iota \circ \mu \). The map \( \iota \) is a semiring morphism if and only if \( S \) is idempotent. More generally, if \( S \) is naturally ordered, then \( \iota \) is an order preserving map from \( (R, \prec) \) to \( S \ltimes R \) endowed with the natural order \( \preceq \), and it satisfies:

\[
(2.5) \quad \iota(x) \vee \iota(y) \preceq \iota(\max(x,y)) \preceq \iota(x) \odot \iota(y),
\]

where \( \vee \) is the supremum in the natural order of \( S \ltimes R \).

2.3. Symmetrized max-plus semiring. The symmetrized max-plus semiring, which is useful to deal with systems of linear equations over \( \mathbb{R}_{\max} \), was introduced in \[\text{[Pi]}\]. It is also discussed in \[\text{[Ga]}\], \[\text{[BCO]}\] and \[\text{AG}\]. Here we shall adopt a construction of this semiring using the above extension. The resulting semiring is isomorphic to the one of \[\text{[Pi]}\], as shown in \[\text{AG}, \text{Section 5.1}]\.

**DEFINITION 2.18 (Symmetrized Boolean semiring).** Let \( B_s \) be a set with four elements denoted \( 0, 1, \ominus 1 \) and \( I^0 \). Define the laws \( \oplus, \ominus \) on \( B_s \) by:

\[
\begin{array}{cccc}
\oplus & 0 & 1 & \ominus 1 \\
0 & 0 & 1 & \ominus 1 \\
1 & 1 & 1 & 1 \\
\ominus 1 & \ominus 1 & 1 & \ominus 1 \\
I^0 & I^0 & I^0 & I^0 \\
\end{array}
\]

Then \( (B_s, \oplus, \ominus) \) is an idempotent semiring with zero element \( 0 \) and unit element \( 1 \), and the map \( \tau : B_s \to B_s \), such that \( \tau(a) = (\ominus 1) \ominus a \), for all \( a \in B_s \), is a symmetry of \( B_s \), such that \( I^0 = 1 \oplus \tau(1) \).

Since \( B_s \) is idempotent, it is naturally ordered. The order of \( B_s \) satisfies:
The above properties imply that the notation $\mathbb{1}^\circ$ is coherent with the one of Definition 2.6. We have $\mathbb{B}_s^\circ := (\mathbb{B}_s)^* = \{0, \mathbb{1}^\circ\}$, and since $\mathbb{1}$ and $e$ are in $\mathbb{1}$ and $\mathbb{1}^\circ$, the only possible thin set of $\mathbb{B}_s$ is $\mathbb{B}_s^\circ := \{0, \mathbb{1}, \mathbb{1}^\circ\}$.

**Definition 2.19 (Symmetrized max-plus semiring).** The symmetrized max-plus semiring, $\mathbb{S}_{\text{max}}$, is defined to be $\mathbb{B}_s \times \mathbb{R}_{\text{max}}$, according to Proposition 2.12.

Indeed, since $\mathbb{B}_s$ is zero-sum free, with no zero divisors, the extension $\mathbb{B}_s \times \mathbb{R}_{\text{max}}$ is a semiring. By Fact 2.16, the symmetry $\tau$ of $\mathbb{B}_s$ is extended into the symmetry $\tau'$ of $\mathbb{B}_s \times \mathbb{R}_{\text{max}}$. The definition of the symmetrized max-plus semiring given in [Plu90] leads to a structure isomorphic to $\mathbb{B}_s \times \mathbb{R}_{\text{max}}$, which was denoted by $\mathbb{S}_{\text{max}}$ in this reference. In the present paper, $\mathbb{S}_{\text{max}}$ is directly defined as $\mathbb{B}_s \times \mathbb{R}_{\text{max}}$.

By Fact 2.16 we have $\mathbb{S}_{\text{max}}^\circ := (\mathbb{S}_{\text{max}})^* = \mathbb{B}_s^\circ \times \mathbb{R}_{\text{max}} = ((\mathbb{1}^\circ) \times \mathbb{R}) \cup \{0\}$. Moreover $\mathbb{S}_{\text{max}} := \mathbb{B}_s^\circ \times \mathbb{R}_{\text{max}} = ((\mathbb{1}, \mathbb{1}^\circ) \times \mathbb{R}) \cup \{0\}$ is a thin set of $\mathbb{S}_{\text{max}}$, and it is indeed the only possible one. We shall also use the notations $\mathbb{S}_{\text{max}}^\ominus := ((\mathbb{1}) \times \mathbb{R}) \cup \{0\}$ and $\mathbb{S}_{\text{max}}^\odot := ((\mathbb{1}^\circ) \times \mathbb{R}) \cup \{0\}$, thus $\mathbb{S}_{\text{max}}^\circ = \mathbb{S}_{\text{max}}^\odot \cup \mathbb{S}_{\text{max}}^\ominus$. By Fact 2.17, $\tau$ is a morphism, so that we can identify $\mathbb{R}_{\text{max}}$ with $s(\mathbb{R}_{\text{max}}) = \mathbb{S}_{\text{max}}^\odot$. We have $\mathbb{S}_{\text{max}}^\odot = \mathbb{S}_{\text{max}}^\circ \times \mathbb{R}_{\text{max}}^\circ$. In [Plu90], the elements of $\mathbb{S}_{\text{max}}^\circ$ are called *signed*, thus “signed” in this particular semiring is equivalent to “thin”.

**Remark 2.20.** In the idempotent semiring $(\mathbb{B}_s, \oplus, \ominus)$, the elements of $(\mathbb{B}_s^\circ)^*$ are not comparable in the natural order, and $0$ and $\mathbb{1}^\circ$ are respectively the minimal and maximal elements of $\mathbb{B}_s$.

It is natural to extend the tropical semiring by capturing the phase information, rather than the sign. The next construction yields a coarse way to do so.

**Example 2.21 (Tropical extension of the torus, $\mathbb{T} \times \mathbb{R}_{\text{max}}$).** Let $(G, \odot, 1)$ be a group, equip it with the trivial order $\leq$ such that every two elements are incomparable (i.e., $a \leq b$ if and only if $a = b$), and add a minimal and maximal element to $G$, denoted respectively $0$ and $1^\circ$, such that $0$ is absorbing for the multiplication in $G := G \cup \{0, 1^\circ\}$, and $1^\circ$ is absorbing for the multiplication in $G \cup \{1^\circ\}$. Then $(G, \lor, 0, \ominus, 1)$ is an idempotent semiring in which $a \lor b = 1^\circ$ for all $a, b \neq 0$, such that $a \neq b$. Assume that there exists $e \in G \setminus \{1\}$, such that $e \odot e = 1^\circ$ and $e$ commutes with all elements of $G$. Then the map $a \in G \mapsto e \odot a$ is a non trivial symmetry of $G$, and since $a \neq e \odot a$ for all $a \in G$, we get that $G^a = \{0, 1\}$ and that $G^a = G \cup \{0\}$ is the only thin set of $G$. Since $G$ is zero-sum free without zero divisors, one can then construct $G \times \mathbb{R}_{\text{max}}$ with the thin set $G^a \times \mathbb{R}_{\text{max}}$. When $G$ is the group with two elements (of order 2, so isomorphic to the additive group $\mathbb{Z}_2$), we recover the semirings $\mathbb{B}_s$ and $\mathbb{S}_{\text{max}}$. When $G$ is the unit circle $\mathbb{T}$ of $\mathbb{C}$, we obtain a semiring $\mathbb{T} \times \mathbb{R}_{\text{max}}$, with only one possible non trivial symmetry obtained with $e = -1$.

A more powerful semiring than $\mathbb{T} \times \mathbb{R}_{\text{max}}$ is obtained by the following construction which is a variant of the one of the complex tropical hyperfield that Viro [Vir10] made, using a different set of axioms, see Remark 2.23 below.

**Example 2.22 (Phase extension of the tropical semiring).** Let Ph (for “phases”) denote the set of closed convex cones of $\mathbb{C}$ seen as a real 2-dimensional space, that is the set of angular sectors of $\mathbb{C}$ between two half-lines with angle less
or equal to π or equal to 2π, together with the singleton \{0\} (the trivial cone). Consider the following laws on \(\Phi\): the sum \(\Phi + \Phi'\) of two elements \(\Phi, \Phi' \in \Phi\) is the closed convex hull of \(\Phi \cup \Phi'\), and the product \(\Phi \cdot \Phi'\) is the closed convex hull of the set of complex numbers \(a \cdot a'\) with \(a \in \Phi\) and \(a' \in \Phi'\). Then \(\Phi\) is an idempotent semiring. The zero is \(0 := \{0\}\), the unit is the half-line of positive reals, and the invertible elements are the half-lines. Taking \(-\Phi\) equal to the set of \(-a\) for \(a \in \Phi\), we obtain a symmetry of \(\Phi\), which is the only symmetry of \(\Phi\) different from identity. In that case, \(\Phi^0\) is the subset of \(\Phi\) consisting of \(0\), all lines, and the plane. We can consider for the thin set \(\Phi^\vee\) the set of half-lines and \(\emptyset\). We can then construct the semiring \(\Phi \times \mathbb{R}_{\max}\) equipped with the thin set \(\Phi^\vee \times \mathbb{R}_{\max}\). We call this semiring the phase extension of the tropical semiring.

Similarly to Example 2.13 an element \((\Phi, b)\) of \(\Phi \times \mathbb{R}_{\max}\) may be thought of as an abstraction of the set of asymptotic expansions of the form \(ae^{-b} + o(\epsilon^{-b})\), when \(\epsilon\) goes to 0\(+\), where \(a\) is required to belong to the relative interior of \(\Phi\), denoted by relint \(\Phi\). Recall that the relative interior of a convex set is the interior of this set with respect to the topology of the affine space that it generates. For instance, the relative interior of a closed half-line is an open half-line. If \(a \in \text{relint} \Phi\), and if \(a' \in \text{relint} \Phi'\) for some \(\Phi, \Phi' \in \Phi^* = \Phi \setminus \{0\}\), then, it can readily be checked that \(ae^{-b} + o(\epsilon^{-b}) + a'e^{-b'} + o(\epsilon^{-b'}) = a'e^{-b''} + o(\epsilon^{-b''})\), where \(a'' \in \text{relint} \Phi''\) and \((\Phi'', b'') := (\Phi, b) \oplus (\Phi'', b'').\) Similarly, the product of the semiring \(\Phi\) is consistent with the one of asymptotic expansions. Note that when the cone \(\Phi\) is either a line or the whole set \(\mathbb{C}\), 0 is in the relative interior of \(\Phi\). Then the corresponding asymptotic expansion \(ae^{-b} + o(\epsilon^{-b})\) may reduce to \(o(\epsilon^{-b})\), as \(a = 0\) is allowed. The elements of \(\Phi^\vee \times \mathbb{R}_{\max}\) correspond to asymptotic expansions with a well defined information on the angle, whereas an element \((\Phi, b)\) such that \(\Phi\) is a pointed cone (a sector of angle strictly inferior to \(\pi\)) correspond to asymptotic expansions having their leading term in a given angular sector.

Remark 2.23 (Viro’s complex tropical hyperfield). A related encoding was proposed by Viro in [Vir10] in a different setting, with his complex tropical hyperfield \(\mathcal{T}\mathcal{C}\). A hyperfield is a set endowed with a multivalued addition and univalued multiplication, that satisfy distributivity and invertibility properties similar to those of semifields. The hyperfield \(\mathcal{T}\mathcal{C}\) is the set of complex numbers \(\mathbb{C}\) endowed with a multivalued addition and the usual multiplication. This allows one to see a non zero complex number \(e^{i\theta}e^{-b}\) as an encoding of asymptotic expansions of the form \(re^{\theta}e^{-b} + o(\epsilon^{-b})\), when \(\epsilon\) goes to 0\(+\), with \(r > 0\). Hence, the phase extension of the tropical semiring and the complex tropical hyperfield provide two abstractions of the arithmetics of asymptotic expansions (or of Puiseux series). The two abstractions differ, however, in the handling of the element \(x - x\). Indeed, if \(x \in \mathcal{T}\mathcal{C}\), then, \(x - x\) is defined to be \(\{y \in \mathbb{C} \mid |y| \leq |x|\}\) in \(\mathcal{T}\mathcal{C}\). This set may be thought of as an encoding of all the expansions in \(\epsilon\) that are \(O(\epsilon^{-|x|})\). If \(x = (\Phi, b) \in \Phi^\vee \times \mathbb{R}_{\max}\) and \(\theta\) is the angle of the half-line \(\Phi\), then \(x - x\) encodes all the asymptotic expansions \(re^{\theta}e^{-b} + o(\epsilon^{-b})\) with \(r \in \mathbb{R}\) (so we get an extra bit of information by comparison with \(O(\epsilon^{-|x|})\)). Note that we may also identify \(\Phi^\vee \times \mathbb{R}_{\max}\) with \(\mathbb{C}\) and \(\Phi \times \mathbb{R}_{\max}\) with subsets of \(\mathbb{C}\), by means of the bijective map \((\Phi, b) \mapsto e^{\theta}e^{b}\) where \(\theta\) is as above, and \((0, 0)\) \(\mapsto 0\). Again, in this identification, the multiplication and addition of \(\mathcal{T}\mathcal{C}\) and \(\Phi^\vee \times \mathbb{R}_{\max}\) coincide except for \(x - x\). Moreover, in this way, \(\Phi^\vee \times \mathbb{R}_{\max}\) is not a hyperfield (since \(0 \notin x - x\)), and \(\mathcal{T}\mathcal{C}\) cannot be put in the form of \(S^\vee \times \mathbb{R}_{\max}\) for some semiring \(S\).

2.4. The bi-valued tropical semiring. Izhakian introduced in [Izh09] an extension of the tropical semiring, which can be cast in the previous general construction. We shall also see that some of the supertropical semifields of Izhakian
and Rowen [IR10] can be reduced to the previous construction. The following presentation is a simplified version of [AGG09].

**Definition 2.24.** Let \( N_2 \) be the semiring which is the quotient of the semiring \( \mathbb{N} \) of non-negative integers by the equivalence relation which identifies all numbers greater than or equal to 2. The bi-valued tropical semiring \( T_2 := N_2 \times \mathbb{R}_{\max} \) in the sense of Proposition 2.12.

Indeed, \( N_2 \) is zero-sum free and without zero divisors, thus \( T_2 = N_2 \rtimes \mathbb{R}_{\max} \) is a subsemiring of \( N_2 \rtimes \mathbb{R}_{\max} \). Moreover \( T_2 \) is isomorphic to the extended tropical semiring defined in [Lzh09], see [AGG09] for details. (In the present paper, we prefer to use the term bi-valued rather than extended since other extensions of the tropical semiring are considered.) Recall that this algebraic structure encodes whether the maximum in an expression is attained once or at least twice. The semiring \( N_2 \) is not idempotent so the injection \( i \) is not a morphism. However, \( N_2 \) is naturally ordered (by the usual order of \( \mathbb{N} \)), so is the semiring \( T_2 \), and \( i \) satisfies (2.5). The only element \( e \) of \( N_2 \) such that \( e \cdot c = 1 \) is equal to 1, thus the only symmetry of \( N_2 \) is the identity map. Since by Fact 2.16, a symmetry of \( T_2 \) is the extension of a symmetry of \( N_2 \), the only symmetry of \( T_2 \) is the identity map. In \( N_2 \), we have \( 2 = 1^o \), \( N_2^1 = \{0, 1^o\} \) and the only possible thin set is \( N_2^1 := \{0, 1\} \). Then \( T_2^1 = (\{1^o\} \times \mathbb{R}) \cup \{e\} \) and \( T_2^2 = ((1) \times \mathbb{R}) \cup \{\emptyset\} = i(\mathbb{R}_{\max}) \) is a thin set of \( T_2 \), which is the only possible one.

**Remark 2.25** (Supertropical semirings as semirings with symmetry). In [IR09], in the particular context of the semiring \( T_2 \), the names “reals” and “ghosts” were given to what we call here “thin” and “balanced” elements. The construction of the bi-valued tropical semiring has been generalized to the notion of supertropical semifield or semiring in [IR10] and of layered semiring in [IKR12]. Supertropical semirings are special cases of semirings with a symmetry and a modulus. Indeed, one can show that the triple \((S, G_0, \nu)\) is a supertropical semiring in the sense of [IR10] if and only if the following conditions hold. (i) \( S \) is a naturally ordered semiring, endowed with the identity symmetry, such that \( S^o \) is a totally ordered idempotent semiring, the map \( \mu : S \rightarrow S^o, a \mapsto a^o \) is a modulus, and \( S \) satisfies the additional properties: \( a + b = b \) if \( |a| \sim |b| \) and \( a + b = |a| \) if \( |a| = |b| \). (ii) \( G_0 \) is an ideal of \( S \) containing \( S^o \), and \( \nu \) is the map from \( S \) to \( G_0 \) such that \( \nu(a) = \mu(a) \). Note that the idempotency of \( \nu \) follows from the fact that \( \mu \) is a morphism. Similarly, a supertropical semiring is a supertropical semifield if the following additional conditions hold. (iii) \( S^+ := S \setminus S^o \) is a multiplicative commutative group such that the map \( \mu \) is onto from \( S^+ \) to \( (S^o)^* \). The latter properties mean that \( S^o \cup \{\emptyset\} \) is the unique thin set \( S^o \) of \( S \), and that any element of \( S^o \) can be written as \( a^o \) with \( a \in S^+ \). (iv) \( G_0 = S^o \) and \( \nu = \mu \). Since Conditions (ii) and (iv) concern only \( G_0 \) and \( \nu \) and do not affect \( S \), we shall use in the sequel the name “supertropical semifield” for any semiring \( S \) satisfying the above conditions (i) and (iii).

**Example 2.26.** The following construction gives the main example of a supertropical semifield. Consider a group \( G \) and its extension \( \hat{G} \) defined as in Example 2.21. Let us consider on \( \hat{G} \) the additive law \( \oplus \) such that \( a \oplus b = 1^o \) for all \( a, b \in G \cup \{1^o\} \). Then \( \hat{G}^o = \{0, 1\} \) and \( \hat{G} \rtimes \mathbb{R}_{\max} \) is a supertropical semifield. More generally, if \( R \) is a totally ordered idempotent semifield, then \( \hat{G} \rtimes R \) is a supertropical semifield.
3. Combinatorial properties of semirings

We next recall or establish some properties of a combinatorial nature, which will be useful when studying Cramer systems over extended tropical semirings.

3.1. Determinants in semirings with symmetry.

Definition 3.1. Let \((S,+,\cdot)\) be a semiring with symmetry and \(A = (A_{ij}) \in \mathcal{M}_n(S)\). We define the determinant \(\det (A)\) of \(A\) to be the element of \(S\) given by the usual formula

\[
\det (A) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) A_{1 \sigma(1)} \cdots A_{n \sigma(n)},
\]

understanding that \(\text{sgn}(\sigma) = \pm 1\) depending on the even or odd parity of \(\sigma\).

In a general semiring, the determinant is not a multiplicative morphism, however the following identities hold.

Proposition 3.2 ([Gau92, AGG09 Section 4]). Let \(S\) be a semiring with symmetry, then for all \(A,B \in \mathcal{M}_n(S)\), we have:

\[
(3.1) \quad \det (AB) \triangleright \det (A) \det (B),
\]

or more precisely:

\[
(3.2) \quad \det (AB) \triangleright\triangleright \det (A) \det (B).
\]

Definition 3.3. Let \(S\) be a semiring. A matrix \(C \in \mathcal{M}_n(S)\) is monomial if it can be written as \(C = DP^\sigma\) where \(D\) is a diagonal matrix \((D_{ij} = 0 \text{ for } i \neq j)\), and \(P^\sigma\) is the matrix of the permutation \(\sigma \in \mathfrak{S}_n\) (that is \(P^\sigma_{ij} = 1\) if \(j = \sigma(i)\) and \(P^\sigma_{ij} = 0\) otherwise).

Note that, in any semiring \(S\), a permutation matrix is invertible since \((P^\sigma)^{-1} = P^{\sigma^{-1}}\). Hence \(C\) can be written as \(C = DP^\sigma\) if and only if it can be written as \(C = P^\sigma D'\), by taking \(D' = (P^\sigma)^{-1}DP^\sigma\). This also implies that \(C = DP^\sigma\) is invertible in \(\mathcal{M}_n(S)\) if and only if all diagonal entries of \(D\) are invertible in \(S\).

The following property is easy to check.

Proposition 3.4. Let \(S\) be a semiring with symmetry, then for all \(A,B \in \mathcal{M}_n(S)\), such that \(A\) or \(B\) is monomial, we have:

\[
(3.3) \quad \det (AB) = \det (A) \det (B).
\]

We introduce now the concept of adjugate matrices.

Definition 3.5. Let \(S\) be a semiring with symmetry. If \(A \in \mathcal{M}_n(S)\), we denote by \(A(i,j)\) the \((n-1) \times (n-1)\) submatrix in which row \(i\) and column \(j\) are suppressed. Define the \(ij\)-cofactor of \(A\) to be \(\text{cof}_{ij}(A) := (-1)^{i+j} \det (A(i,j))\) and the adjugate matrix of \(A\) to be the \(n \times n\) matrix \(A^{\text{adj}}\) with \((i,j)\)-entry:

\[
(A^{\text{adj}})_{ij} := \text{cof}_{ji}(A) = (-1)^{i+j} \det (A(j,i)).
\]

Using Proposition 3.4, we also obtain the following identities.

Lemma 3.6. Let \(S\) be a semiring with symmetry, then for all \(A,B \in \mathcal{M}_n(S)\), such that \(A\) or \(B\) is monomial, we have:

\[
(3.4) \quad (AB)^{\text{adj}} = B^{\text{adj}} A^{\text{adj}}.
\]

Lemma 3.7. Let \(S\) be a semiring with symmetry, and let \(C \in \mathcal{M}_n(S)\) be an invertible monomial matrix. Then \(\det (C)\) is invertible and

\[
(3.5) \quad C^{-1} = (\det C)^{-1} C^{\text{adj}}.
\]
3.2. Diagonal scaling of matrices and Yoeli’s theorem. We next recall some properties concerning the idempotent semiring $\mathcal{R}$ arising as an ingredient of the semiring extension. Although we shall only apply these properties when $\mathcal{R}$ is totally ordered, we state the properties in their full generality as far as possible.

Let $(\mathcal{R}, +, 0, 1)$ be an idempotent semiring equipped with the trivial symmetry and with the natural order. Then $-1 = 1$, so the determinant of a matrix $A$, $\det(A)$ coincides with its permanent, denoted by $\text{per}(A)$. The adjugate matrix $A^{\text{adj}}$ has a useful interpretation in terms of maximal weights of paths. To see this, let us first recall the definition and basic properties of the Kleene star $A^*$ in an idempotent semiring. Recall that to an idempotent semiring equipped with the trivial symmetry and the natural order. Then $\forall i, j \in \mathcal{R}$, $A^{\text{adj}}$ has a useful interpretation in terms of maximal weights of paths. To see this, let us first recall the definition and basic properties of the Kleene star $A^*$ in an idempotent semiring. Recall that to an $n \times n$ matrix $A$ with entries in an idempotent semiring $(\mathcal{R}, +, 0, 1)$, one associates a digraph $G(A)$ with nodes $1, \ldots, n$ and an arc $i \to j$ if $A_{ij} \neq 0$. The weight of a path $p = (i_0, \ldots, i_k)$ in $G(A)$ is defined as $w_A(p) := A_{i_0i_1} \cdots A_{i_{k-1}i_k}$, and its length $l(p)$ is equal to $k$ (the number of arcs). This definition applies in particular to circuits, which are closed paths, meaning that $i_k = i_0$. We denote by $I$ the identity matrix (with diagonal elements equal to 1 and non-diagonal elements equal to 0).

**Proposition-Definition 3.8** [Carré Car71, Gondran Gon75, GM79, p. 72, Th. 1]. Let $(\mathcal{R}, +, 0, 1)$ be any idempotent semiring equipped with the natural order. Let $A \in \mathcal{M}_n(\mathcal{R})$ be a matrix such that every circuit of $A$ has a weight less than or equal to 1. Then

$$A^* := \sum_{i=0}^{\infty} A^i = I + A + A^2 + \cdots + A^{n-1}.$$

When $\mathcal{R}$ is idempotent, the sum is equivalent to the supremum for the natural order of $\mathcal{R}$, hence $A^*_i$ represents the maximal weight of a path from $i$ to $j$. Moreover, when every circuit of $A$ has a weight less than or equal to 1, $A^*_i$ is also equal to the maximal weight of an elementary path from $i$ to $j$ (when $i = j$, this means a path with length 0).

The following result generalizes a theorem of Yoeli [Yoel61], which was stated when the idempotent semiring has a maximal element equal to 1.

**Theorem 3.9** (Compare with Yoeli Yoel61, Theorem 4). Let $(\mathcal{R}, +, 0, 1)$ be an idempotent semiring equipped with the trivial symmetry and the natural order denoted by $\leq$. Let $A = (A_{ij}) \in \mathcal{M}_n(\mathcal{R})$ be a matrix such that $A_{11} = \cdots = A_{nn} = 1$ and $\text{per}(A) = 1$. Then every circuit of $A$ has a weight less than or equal to 1, and

$$A^{\text{adj}} = A^*.$$

**Proof.** Let $A = (A_{ij}) \in \mathcal{M}_n(\mathcal{R})$ be a matrix such that $A_{11} = \cdots = A_{nn} = 1$ and $\text{per}(A) = 1$. Let $c = (i_0, \ldots, i_k = i_0)$ be a circuit of $G(A)$. One can construct the permutation $\sigma \in \mathfrak{S}_n$ containing this circuit and all the circuits with one element not in $c$, that is $\sigma(i_l) = i_{l+1}$ for $l = 0, \ldots, k-1$, and $\sigma(i) = i$ for $i \notin \{i_0, \ldots, i_{k-1}\}$. Since $A_{ii} = 1$, the weight of this permutation $A_{1\sigma(1)} \cdots A_{n\sigma(n)}$ is equal to the weight $w_A(c)$ of the circuit $c$. Since $\text{per}(A) = 1$ is the sum, this is the supremum for the natural order, of the weights of all permutations, we get that $w_A(c) \leq 1$, which shows that every circuit of $A$ has a weight less than or equal to 1.

To prove the last assertion of the theorem, we need to show that $(A^{\text{adj}})_{ij} = A^*_i$ for all $i, j \in [n]$. As remarked after Proposition-Definition 3.8, since all circuits have a weight less than or equal to 1, $A^*_i$ is equal to the maximal weight of an elementary path from $i$ to $j$. In particular, $A^*_1 = 1$, and since $A(1, i)$ has the same properties as $A$ (the diagonal coefficients are equal to 1 and all circuits have a weight less than or equal to 1), we deduce that $(A^{\text{adj}})_{1i} = \text{per}(A(1, i)) = 1 = A^*_1$. 

Assume now that \( i \neq j \). We have

\[
\text{(3.6)} \quad (A^{\text{adj}})_{ij} = \text{per}(A(j, i)) = \sum_{\sigma \in S_n} \prod_{l \in [n] \setminus \{j\}} A_{l\sigma(l)},
\]

where the sum is taken over all bijections from \([n] \setminus \{j\}\) to \([n] \setminus \{i\}\). Since a map \( \sigma : [n] \setminus \{j\} \to [n] \setminus \{i\} \) is a bijection if and only if it can be completed into a permutation of \([n]\) by taking \( \sigma(j) = i \), the above sum can be taken equivalently over all \( \sigma \in S_n \) such that \( \sigma(j) = i \).

We say that a circuit \( c = (i_0, \ldots, i_k = i_0) \) of \( G(A) \) contains the arc \((i, j)\) if there exists \( l = 0, \ldots, k - 1 \) such that \( i = i_l, j = i_{l+1} \). Then \( p = (i_0, \ldots, i_k) \) is an elementary path from \( i_0 = i \) to \( i_k = j \) if and only if \((p, i) = (i_0, \ldots, i_k, i)\) is an elementary circuit, containing the arc \((j, i)\). Let \( p \) be such a path and \( c = (p, i) \). Completing \( c \) as above to a permutation \( \sigma \in S_n \), containing all circuits with one element not in \( c \), we get that \( \sigma(j) = i \), and that \( w_A(p) = w_A(p) \cdot \prod_{l \in \{c\}} A_{ll} = \prod_{l \in [n] \setminus \{j\}} A_{l\sigma(l)} \). Since \( \text{(3.6)} \) holds with a sum over all \( \sigma \in S_n \) such that \( \sigma(j) = i \), we obtain that \( w_A(p) \in (A^{\text{adj}})_{ij} \), and since this holds for all elementary paths from \( i \) to \( j \), we deduce that \( A_{ij}^* \leq (A^{\text{adj}})_{ij} \).

To show the reverse inequality, let \( \sigma \in S_n \) be such that \( \sigma(j) = i \). Decomposing \( \sigma \) into elementary cycles, we get in particular a cycle \( c \) containing the arc \((j, i)\). Let \( p \) be the elementary path from \( i \) to \( j \) such that \( c = (p, i) \), we deduce that \( \prod_{l \in [n] \setminus \{j\}} A_{l\sigma(l)} = w_A(p) \cdot \prod_{l \in c} w_A(c) \), where the last product is taken over all cycles \( c' \) of \( \sigma \) different from \( c \). Since all cycles have weights less than or equal to 1, we get that \( \prod_{l \in [n] \setminus \{j\}} A_{l\sigma(l)} \leq w_A(p) \leq A_{ij}^* \). By applying \( \text{(3.6)} \) with a sum over all \( \sigma \in S_n \) such that \( \sigma(j) = i \), we obtain that \( (A^{\text{adj}})_{ij} \leq A_{ij}^* \), and so the equality holds. This finishes the proof of the theorem.

The following proposition gives a semifield version of a well known duality result concerning the optimal assignment problem. It will allow convenient normalizations.

**Proposition 3.10 (Hungarian scaling).** Let \( C \) be an \( n \times n \) matrix with entries in a totally ordered idempotent semifield \( \mathcal{R} \), and assume that per \( C \neq 0 \). Then there exist two \( n \)-dimensional vectors \( u, v \) with entries in \( \mathcal{R} \setminus \{0\} \) such that

\[
C_{ij} \leq u_i v_j, \quad \forall i, j \in [n]
\]

and \( C_{ij} = u_i v_j \) for all \((i, j)\) such that \( j = \sigma(i) \) for every optimal permutation \( \sigma \), i.e., for every permutation \( \sigma \) such that

\[
\text{per } C = \prod_{i \in [n]} C_{i \sigma(i)}.
\]

In particular,

\[
\text{per } C = \prod_{i \in [n]} u_i \prod_{j \in [n]} v_j
\]

**Proof.** This is a byproduct of the termination of the Hungarian algorithm. We refer the reader to [Sch03] for more information on this algorithm. The latter is usually stated for matrices with entries in the ordered group \((\mathbb{R}, +)\) completed by the \(-\infty\) element, or equivalently for matrices with entries in the ordered group \((\mathbb{R}^+, \times)\) of strictly positive real numbers, completed by the \(0\) element. In the latter context, it allows one to compute \( \max_{\sigma} \prod_{i} C_{i \sigma(i)} \). Let us call a row (resp. column) scaling the operation of multiplying by the inverse of a non-zero number a given row (resp. column) of a matrix. The Hungarian algorithm performs a finite number of row and column scalings, reaching eventually a matrix \( B_{ij} = u_i^{-1} C_{ij} v_j^{-1} \) such that \( B_{ij} \leq 1 \) for all \( i, j \) and \( B_{ij} = 1 \) for all \((i, j)\) in a collection of couples of indices...
among which \( n \) are independent, meaning that none of them belong to the same row or column. Then these independent \((i,j)\) define an optimal permutation, and all the conclusions of the proposition are valid. The algorithm can be readily checked to be valid when the entries of \( C \) belong to any totally ordered semifield. \(\square\)

We note that some generalizations of network flow problems to ordered algebraic structures were studied in \[Zim81\], Chap. 12, the previous result could also be derived from results there.

A variant of the following result has appeared in the work of Butkovic \[But94\]. It allows one to reduce matrices to a “normal form” in which the diagonal consists of unit elements and all other elements are not greater than the unit.

**Corollary 3.11** (\[But94\], Th. 3.1). Let \( C \) be an \( n \times n \) matrix with entries in a totally ordered idempotent semifield \( R \), and assume that \( \text{per} C \neq 0 \). Then there exist two diagonal matrices \( D \) and \( D' \) with invertible diagonal entries, and a permutation matrix \( \Sigma \), such that \( B = \Sigma CD\Sigma' \) satisfies

\[
B_{ij} \leq 1, \quad \forall i,j \in [n], \quad B_{ii} = 1, \quad \forall i \in [n].
\]

**Remark 3.12.** Corollary 3.11 shows that in the special case of a totally ordered idempotent semifield, as soon as \( \text{per} C \neq 0 \), we may reduce \( C \) by diagonal scaling and permutation to a matrix \( A \) satisfying the assumptions of Yoeli’s theorem.

### 4. Elimination in semirings and Cramer theorem

#### 4.1. Elimination in semirings with symmetry

In the sequel, we shall consider a semiring \( S \) with symmetry and a thin set \( S' \) satisfying the following properties, which will allow us to eliminate variables in order to solve tropical linear systems.

**Property 4.1.** For \( x, y \in S' \), we have that \( x \nmax y \) implies \( x = y \).

**Property 4.2.** The set of non-zero thin elements \((S')^*\) is closed under multiplication. So is \( S' \), a fortiori.

**Property 4.3 (Weak transitivity of systems of balances).** For all \( n,p \geq 1 \), \( a \in S', \ C \in M_{n,p}(S) \), \( b \in S^p \), and \( d \in S^n \), we have

\[
(x \in (S')^p, \ ax \nmax b \text{ and } Cx \nmax d) \implies Cb \nmax ad.
\]

**Definition 4.4.** If a semiring \( S \) with a symmetry and a thin set \( S' \) satisfies Properties \[4.2\] and \[4.3\] we will say that it allows weak balance elimination. If it satisfies also Property \[4.1\] then we will say that it allows strong balance elimination.

It was pointed out in \[Plu90\] (see also \[AGG09\] Section 6) that the symmetrized max-plus semiring \( S_{\max} \) satisfies Properties \[4.2\] and \[4.3\]. It was also observed in \[AGG09\] that so does the bi-valued tropical semiring \( T_2 \). Note that in the latter reference, Property \[4.2\] was replaced by the stronger property that the set \((S')^* = S \setminus S^2 \) is exactly the set of all invertible elements in \( S \), but this stronger property will not always be needed. In \[AGG09\] a proof of Property \[4.3\] was given specially for \( S_{\max} \), we shall give now some sufficient conditions for the above properties to hold, which allow one to check them easily for the semirings \( B_\mathbb{R} \) and \( N_2 \) and deduce them for \( S_{\max} \) and \( T_2 \). In particular, we shall also consider the following properties:

**Property 4.5 (Weak transitivity of balances).** For all \( b, d \in S \), we have

\[
(x \in S', b \nmax x \text{ and } x \nmax d) \implies b \nmax d.
\]
**Property 4.6.** (Weak transitivity of scalar balances). For all $b, c, d \in S$, we have

\[(x \in S^\vee, \ x \nabla b \text{ and } cx \nabla d) \implies cb \nabla d,\]

which is (4.1) for $n = p = 1$ and $a = 1$.

**Property 4.7.** $(S^\vee)^* = S \setminus S^o$.

**Property 4.8.** The set $S$ is additively generated by $S^\vee$, which means that any element of $S$ is the sum of a finite number of elements of $S^\vee$.

**Lemma 4.9.** $S$ allows weak balance elimination if and only if Properties 4.2 and 4.6 hold together.

**Proof.** Since Property 4.3 implies in particular Property 4.6, we get the “only if” part of the assertion of the lemma. For the “if” part, let us assume that Properties 4.5, 4.2, and 4.8 hold. Then (4.1) holds when $C = (c_1, \ldots, c_p) \in M_{n,p}(S)$, $b \in S^p$, $d \in S^o$, and $x \in (S^\vee)^p$ be such that $ax \nabla b$ and $cx \nabla d$. Let us show that $C \nabla ad$. Since, by Property 4.2, $S^\vee$ is stable under product, $ax = (ax_i)_{i \in [p]} \in (S^\vee)^p$. Since $S^o$ is an ideal of $S$, multiplying the equation $C \nabla d$ by $a$, we get that $C \nabla ad$. Then it remains to show the above implication for $1, ax$ and $ad$ instead of $a, x$ and $d$ respectively. Without loss of generality we can assume that $a = 1$. Moreover, since $C \nabla d$ is equivalent to $C_1 \nabla d$, for all $i \in [n]$, where $C_i$ denotes the $i$th row of $C$, it is sufficient to prove the above implication for each row of $C$ instead of $C$. We can thus assume that $n = 1$.

Let $C = (c_1, \ldots, c_p) \in M_{1,p}(S)$, and assume that $x \nabla b$ and $C \nabla d$. The relation $C \nabla d$ is equivalent to $c_1 \cdot x_1 \nabla d = \cdots = c_p \cdot x_p$. From (4.2), which holds by Property 4.6 and $x_1 \nabla b_1$, we deduce that $c_1 \cdot b_1 \nabla d = \cdots = c_p \cdot x_p$. Now exchanging the sides of $c_1 \cdot b_1$ and $c_2 \cdot x_2$, and applying (4.2) with $x_2 \nabla b_2$, we can replace $x_2$ by $b_2$ in the previous balance equation. Doing this inductively on all $x_i$, we obtain $C \nabla d$, which concludes the proof.

**Lemma 4.10.** Properties 4.4 and 4.7 together imply Property 4.5.

**Proof.** Assume Property 4.4 holds. Let $b, d \in S$, $x \in S^\vee$ be such that $b \nabla x$ and $x \nabla d$. We need to show that $b \nabla d$. Since $x \in S^\vee$, if $b \in S^\vee$ then Property 4.4 implies that $b = x$, so that $b \nabla d$. Similarly, $b \nabla d$ if $d \in S^\vee$. Otherwise, $b$ and $d \notin S^\vee$, which implies that $b$ and $d \notin S^o$ by Property 4.7. Hence $b \nabla d$. We can thus assume that $b \nabla d$.

**Lemma 4.11.** Properties 4.4, 4.2 and 4.8 all together imply Property 4.6, hence, Property 4.5, and that $S$ allows weak balance elimination.

**Proof.** Assume Properties 4.5, 4.2, and 4.8 hold. Then (4.1) holds when $n = p = 1$ and $a = C = 1$. Let us show that (4.1) also holds when $n = p = 1$, $a = 1$, and $C \in S$. This will mean that Property 4.6 holds and will imply by Lemma 4.9 that Property 4.3 holds. So let $C, b, d \in S$, and $x \in S^\vee$ be such that $x \nabla b$ and $C \nabla d$, and let us show that $C \nabla d$. If $C \in S^\vee$, then by Property 4.2, $Cx \in S^\vee$. Multiplying $x \nabla b$ by $C$, we get that $Cx \nabla Cb$, and since $Cx \nabla d$ and $Cx \in S^\vee$, Property 4.5 implies $C \nabla d$. This shows that (4.1) holds when $n = p = 1$, $a = 1$, and $C \in S^\vee$. Assume now that $C \notin S^\vee$. Then by Property 4.8, there exist $E_1, \ldots, E_k \in S^\vee$ such that $C = E_1 + \cdots + E_k$. The relation $Cx \nabla d$ is then equivalent to $E_1 \cdot x \nabla E_2 \cdot x \cdots E_k \cdot x + d$. Since $E_1 \in S^\vee$, $x \in S^\vee$ and $x \nabla b$, applying (4.1) with $n = p = 1$, $a = 1$, $E_1 \in S^\vee$ instead of $C$ and $-E_2 \cdot x \cdots - E_k \cdot x + d$ instead of $d$ (the implication (4.1) is already known to hold in that case), we get that $E_1 \cdot b \nabla -E_2 \cdot x \cdots - E_k \cdot x + d$. Now exchanging the sides of $E_1 \cdot b$ and $E_2 \cdot x$, and applying (4.1) with $n = p = 1$, $a = 1$, $E_2 \in S^\vee$ instead of $C$, we can replace
$E_2 \cdot x$ by $E_2 \cdot b$ in the previous balance equation. Doing this inductively, we obtain $Cb \nabla d$, hence (4.1) holds when $n = p = 1$, $a = 1$, and for all $C \in \mathcal{S}$. □

**Corollary 4.12.** Properties (4.3), (4.4), and (4.5) together with either Property (4.9) or (4.6) imply that $\mathcal{S}$ allows strong balance elimination.

**Proposition 4.13.** Let $(\mathcal{S}, +, \cdot)$ be a semiring with symmetry, let $\mathcal{S}^\gamma$ be a thin set of $\mathcal{S}$, and let $(\mathcal{R}, \oplus, 0, \cdot, 1)$ be a totally ordered idempotent semiring.

Denote by $\mathcal{S}_e$ any of the semiring extensions $\mathcal{S} \times \mathcal{R}$, $\mathcal{S} \odot \mathcal{R}$, or $\mathcal{S} \odot \mathcal{R}$ defined in Proposition 2.13 endowed with the extension of the symmetry of $\mathcal{S}$ defined in Fact 2.10. Here, $\mathcal{S}_e$ is assumed to be a subsemiring of $\mathcal{S} \times \mathcal{R}$.

Consider the thin set $\mathcal{S}_e^\gamma = \mathcal{S}^\gamma \times \mathcal{R}$ if $\mathcal{S}_e = \mathcal{S} \times \mathcal{R}$ or $\mathcal{S}_e = \mathcal{S} \odot \mathcal{R}$, and $\mathcal{S}_e^\gamma = ((\mathcal{S}^\gamma)^* \times \mathcal{R}) \cup \{(0)\}$ if $\mathcal{S}_e = \mathcal{S} \odot \mathcal{R}$. Then the following properties hold:

(a) $\mathcal{S}_e$ satisfies Property (4.1) (resp. (4.2), resp. (4.6) if and only if $\mathcal{S}$ does.

(b) $\mathcal{S}_e$ allows weak balance elimination if and only if $\mathcal{S}$ does.

(c) $\mathcal{S}_e$ allows strong balance elimination if and only if $\mathcal{S}$ does.

**Proof.** Assertions (b) and (c) follow from Assertion (a), Lemma 4.9, and the definition of strong balance elimination.

1. Let us first prove the “only if” part of Assertion (a) of the proposition. Assume that $\mathcal{S}$, $\mathcal{S}^\gamma$, $\mathcal{S}_e$, $\mathcal{S}_e^\gamma$ are as in the statement. By Fact 2.13, $\mathcal{S}$ is isomorphic by $j$ to a subsemiring of $\mathcal{S}_e$. Moreover, by definition, the map $j$ is compatible with the symmetries of $\mathcal{S}$ and $\mathcal{S}_e$, hence $j(\mathcal{S}^\gamma) = \mathcal{S}_e^\gamma \odot j(\mathcal{S})$ on $\mathcal{S}$ the balance relation of $\mathcal{S}$ coincides with the one of $\mathcal{S}_e$. Also, by definition of $\mathcal{S}_e^\gamma$, we have $\mathcal{S}_e^\gamma \odot j(\mathcal{S}) = \mathcal{S}_e \odot j(\mathcal{S}^\gamma)$, and $(\mathcal{S}_e^\gamma)^* \odot j(\mathcal{S}) = j((\mathcal{S}^\gamma)^*)$. From this, we deduce that Property (4.1) (resp. (4.2), resp. (4.6)) for $\mathcal{S}$ implies the same for $\mathcal{S}_e$.

2. Let us now show the “if” part of the Assertion (a). By definition, $(\mathcal{S}_e^\gamma)^* = (\mathcal{S}^\gamma)^* \times \mathcal{R}$ or $(\mathcal{S}^\gamma)^* \times \mathcal{R}$. Since $\mathcal{R}$ and $\mathcal{R}$ are closed by multiplication, it is clear that Property (4.1) (resp. (4.2), resp. (4.6)) for $\mathcal{S}$ implies that the same property is valid for $\mathcal{S}_e$.

Assume now that Property (4.1) holds for $\mathcal{S}$ and let us prove it for $\mathcal{S}_e$. Remark that for any semiring $\mathcal{S}$ with symmetry, the assertion of Property (4.1) is equivalent to the same assertion with $\mathcal{S}^\gamma$ replaced by $(\mathcal{S}^\gamma)^*$. Indeed, when $x, y \in \mathcal{S}^\gamma$ with $x = 0$ or $y = 0$, the equation $x \nabla y$ implies that $x, y \in \mathcal{S}^\gamma$. Thus $x, y \in \mathcal{S}^\gamma \odot \mathcal{S}^\gamma = \{(0)\}$. Hence $x = y = 0$. Then it is sufficient to prove the assertion of Property (4.1) for $(\mathcal{S}^\gamma)^*$ instead of $\mathcal{S}_e$. Let $x = (a, b)$ and $y = (a', b') \in (\mathcal{S}^\gamma)^*$ be such that $x \nabla y$. Since the equation $x \nabla y$ is equivalent to $x = y \in \mathcal{S}_e^\gamma$ and since $x \nabla y = x$ if $b > b'$, $x - y = y$ if $b < b'$, and $x \nabla y = (a - a', b)$ if $b = b'$, we deduce that $b = b'$ and $(a - a', b) \in \mathcal{S}_e^\gamma$. Since $\mathcal{S}_e^\gamma = \mathcal{S}_e \odot (\mathcal{S} \times \mathcal{R})$, we deduce that $a \nabla a'$. Since $(\mathcal{S}^\gamma)^* \times \mathcal{R}$, we also get that $a, a' \in (\mathcal{S}^\gamma)^*$. Hence if Property (4.1) holds for $\mathcal{S}$, the above properties imply $a = a'$, so $x = y$, which shows that Property (4.1) also holds for $\mathcal{S}_e$.

Assume now that Property (4.6) holds for $\mathcal{S}$ and let us prove it for $\mathcal{S}_e$. Let $x = (a, b) \in (\mathcal{S}^\gamma)^*$, and $y = (a', b')$, $z = (a'', b'')$, $w = (a''', b''') \in \mathcal{S}_e$ be such that $x \nabla y$ and $wx \nabla z$. We need to show that $wy \nabla z$. It is easy to show that this holds when $x = 0$, since then $y, z \in \mathcal{S}_e^\gamma$ so that $wy = z \in \mathcal{S}_e^\gamma$. Also this holds when $y = 0$, since then $x \in \mathcal{S}_e^\gamma$ and thus again $x = 0$ (since $x \in \mathcal{S}_e^\gamma$), and when $w = 0$ since then $wx = wy$. So we can assume that $x \in (\mathcal{S}^\gamma)^*$, and $y, w \in (\mathcal{S}_e)^*$. Recall that $(\mathcal{S}_e^\gamma)^* = (\mathcal{S}^\gamma)^* \times \mathcal{R}$ when $\mathcal{S}_e = \mathcal{S} \times \mathcal{R}$ and $(\mathcal{S}_e^\gamma)^* = (\mathcal{S}^\gamma)^* \times \mathcal{R}$ otherwise. It follows that $a \in (\mathcal{S}^\gamma)^*$. Moreover, we have $b \in \mathcal{R}$ when $\mathcal{S}_e = \mathcal{S} \times \mathcal{R}$ or $\mathcal{S} \odot \mathcal{R}$. Now $w = (a''', 1) \odot (1, b''')$. Since $w \neq 0$, and $(\mathcal{S}_e^\gamma)^* \subset \mathcal{S} \times \mathcal{R}$, we get that $b''' \neq 0$ when $\mathcal{S}_e = \mathcal{S} \times \mathcal{R}$ or $\mathcal{S} \odot \mathcal{R}$. Then in all cases $x' := (1, b''') \odot x = (a, b''') \cdot b \in (\mathcal{S}_e^\gamma)^*$. Multiplying both terms of the relation $x \nabla y$ by $(1, b''')$, and replacing $x$ by $x'$, $y$ by $y' = (1, b''') \odot y$ and $w$ by $w' := (a'''')$, we are reduced to the case where $b''' = 1$. Then by the same arguments as above we obtain from $x \nabla y$ that either $b < b'$
and \( y - x = y \in S_\varepsilon^c \), or \( b' = b \) and \( a' \triangledown a \). Similarly, from \( wx \triangledown z \), we obtain that either \( b < b' \) and \( z \in S_\varepsilon^c \), or \( b = b' \) and \( a'' \triangledown a'' \), or \( b > b' \) and \( wx \in S_\varepsilon^c \) so that \( a'' \triangledown a \in S_\varepsilon \).

When \( b < b' \) and \( b < b'' \), we get that \( y, z \in S_\varepsilon^c \), so \( wy - z \in S_\varepsilon^c \) and \( wy \triangledown z \). When \( b < b' \) and \( b > b'' \), we get that \( wy - z = wy \in S_\varepsilon^c \), and again \( wy \triangledown z \). When \( b = b' \) and \( b < b'' \), we get that \( wy - z = wy = (a'' \triangledown a', b) \). Since \( a \in (S')^\ast \), \( a' \triangledown a \) and \( a'' \triangledown a'' \), we deduce from Property 4.6 for \( S \), that \( a'' \triangledown a' \triangledown a'' \), hence \( wy - z \in S_\varepsilon^c \) again. When \( b = b' = b'' \), we get that \( a' \triangledown a \) and \( a'' \triangledown a'' \). Since we also have \( a \in (S')^\ast \), we deduce from Property 4.5 for \( S \), that \( a'' \triangledown a' \triangledown a'' \). This implies again \( wy \triangledown z \). Since in all cases \( wy \triangledown z \), this shows Property 4.6 for \( S \).

\[ \Box \]

**Remark 4.14.** Note that one can also prove similar equivalences to the ones of item (a) of Proposition 4.13 for Properties 4.5 and 4.7. Also, one can do the same for Property 4.8 if \( S = S \times R \). We do not detail these equivalences here, since we are interested mostly in “balance elimination”.

We are now able to give examples of semirings allowing balance elimination, as a consequence of Proposition 4.13 Lemmas 4.10 and 4.11 and Corollary 4.12.

**Fact 4.15.** The semirings \( S = B_\varepsilon \) and \( S = N_2 \) satisfy Properties 4.7.1.1.2. and 4.8. Then these semirings allow strong balance elimination. Hence so do \( S_{\max} = B_\varepsilon \times R_{\max} \) and \( T_2 = N_2 \times R_{\max} \). The semiring \( S = \Phi \) of Example 2.22 i.e., the set of closed convex cones of \( C \), does not satisfy Property 4.7 but it does satisfy Properties 4.7.1.2. and 4.8. So, \( \Phi \) allows strong balance elimination, and so does \( \Phi \times R_{\max} \) (the phase extension of the tropical semiring).

Note that for the same semirings \( S \), the semirings \( S \times R_{\max} \) and \( S \times R_{\max} \) also allow strong balance elimination.

**Fact 4.16.** Let \( S \) be an integral domain (a ring without zero divisors). Since \( S^0 = \{0\} \), the balance relation reduces to the equality relation. Taking \( S' = S \), we get that \( S \) allows trivially strong balance elimination. Hence, the semirings \( S \times R \) and \( S \times R \) also allow strong balance elimination. However \( S \times R \) is not a semiring, since \( S \) is not zero sum free.

Recall that when \( S = C \) or \( S = R \), an element \( (a, b) \in S \times R \) is equivalent to the asymptotic expansion \( ae^{-b} + o(e^{-b}) \), when \( e \) goes to \( 0^+ \) (Example 2.13).

**Fact 4.17.** Let \( S \) be a supertropical semifield (see Remark 2.25). Then Properties 4.5.4.2 and 4.8 hold, thus \( S \) allows weak balance elimination, by Lemma 4.11. However, if the map \( \mu \) from \( S' \) to \( S' \) is not injective, then Property 4.1 does not hold, thus \( S \) does not allow strong balance elimination.

**Proof.** The first assertion can be checked easily. For the second one, if \( \mu \) is not injective, there exist \( a, b \in S' \), \( a \neq b \) such that \( |a| = |b| \). Since \( a - b = a + b = |a| \in S^0 \), we get that \( a \triangledown b \), thus Property 4.1 does not hold.

\[ \Box \]

### 4.2. Cramer formulae in semirings allowing balance elimination

We state here a general Cramer theorem. In the special case of the symmetrized max-plus semiring, this was established by Plu [Plu90], see also [Gau92] and [AGG09] Theorem 6.4. It is remarked in the latter reference (see the paragraph before Theorem 6.6 of [AGG09]) that the proof of [AGG09] Theorem 6.4] is valid in any semiring satisfying Properties 4.1.4.3 and the property that \( (S')^\ast = S \setminus S^0 \) is the set of invertible elements. However looking at the proof of [AGG09] Theorem 6.4] more carefully, one see that the latter property can be replaced by Property 4.2.
and that the first part does not use Property 4.1. This leads to the following general result.

**Theorem 4.18** (Cramer theorem, compare with [Plu90] Theorem 6.1, [Gau92] Chap. III, Theorem 3.2.1 and Proposition 3.4.1 and [AGG09] Theorem 6.4] for $S_{\text{max}}$ and [AGG09] Theorem 6.6] for $T_2$). Let $S$ be a semiring with a symmetry and a thin set $S^{\vee}$, allowing weak balance elimination (Definition 4.4). Let $A \in M_n(S)$ and $b \in S^n$, then

1. Every thin solution $x$ (such that $x \in (S^{\vee})^n$) of the linear system $Ax \nabla b$ satisfies the relation
   $$(\det A)x \nabla A^{\text{adj}}b.$$  

2. Assume also that $S$ allows strong balance elimination (Definition 4.4), that the vector $A^{\text{adj}}b$ is thin and that $\det A$ is invertible in $S$. Then $\hat{x} := (\det A)^{-1}A^{\text{adj}}b$ is the unique thin solution of $Ax \nabla b$.

**Proof.** The proof follows exactly the same lines as the proof of [AGG09] Theorem 6.4], so we do not reproduce it. Let us just remark that it relies on an elimination argument, in which “equations” involving balances rather than equalities are considered. □

The above uniqueness part can be reformulated equivalently in the following homogeneous form. For a matrix $A \in M_{n,m}(S)$ over a semiring $S$, and $k \in [m]$, we denote by $A_{(k)}$ the $n \times (m-1)$ matrix obtained from $A$ by deleting the $k$th column, and by $A_k$ the $k$th column of $A$.

**Corollary 4.19.** Let $S$ be a semiring allowing strong balance elimination. Let $A \in M_{n,n+1}(S)$, and let $\hat{x} \in S^{n+1}$ be such that $\hat{x}_k = (\oplus 1)^{n-k} \det A_{(k)}$, for all $k \in [n+1]$. Then if $\hat{x}$ is thin and has at least one invertible entry (which is the case when $\hat{x}$ is non-zero and $(S^{\vee})^n$ is exactly the set of invertible elements of $S$), then any thin solution of $Ax \nabla 0$ is a thin multiple of $\hat{x}$.

**Proof.** Let $\hat{x}$ be as in the statement of the theorem and assume that $\hat{x}$ is thin and has at least one invertible entry, for instance $\hat{x}_{n+1}$ is invertible. Let $x$ be a thin solution of $Ax \nabla 0$. Then taking $M := A_{[n+1]}$ and $b = \oplus x_{n+1}A_{n+1}$, we get that $\det M = \hat{x}_{n+1}$ is invertible and $(M^{\text{adj}}b)_i = x_{n+1}\hat{x}_i$ is thin for all $i \in [n]$. Let $y \in S^n$ be such that $y_i = x_i$ for $i \in [n]$. Then $y$ is a thin solution of $My \nabla b$, and applying the uniqueness part of Theorem 4.18 to this system, we get that $x_i = y_i = (\det M)^{-1}(M^{\text{adj}}b)_i = (\hat{x}_{n+1})^{-1}x_{n+1}\hat{x}_i$, for all $i \in [n]$. Hence $x = \lambda \hat{x}$, with $\lambda = (\hat{x}_{n+1})^{-1}x_{n+1} \in S^{\vee}$. We can show that the same conclusion holds for any $i \in [n+1]$, such that $\hat{x}_i$ is invertible, which implies the corollary. □

5. **Existence of solutions of tropical linear systems**

In this section we consider a semiring $(\mathcal{T}, \oplus, 0, \ominus, 1)$ with a symmetry, a thin set $\mathcal{T}^{\vee}$, and a modulus taking its values in a totally ordered semiring $\mathcal{R}$. For instance $\mathcal{T} = S \times \mathcal{R}$ where $S$ is a zero-sum free semiring with a symmetry and without zero divisors, and $\mathcal{T}^{\vee} = S^{\vee} \times \mathcal{R}$. We shall study the square affine systems

$$Ax \nabla b$$

where $A$ is an $n \times n$ matrix and $b$ is a vector of dimension $n$, all with entries in $\mathcal{T}$. We shall look for the solutions $x \in \mathcal{T}^n$ with thin entries.
5.1. Monotone algorithms in semirings with symmetry. The existence results that we shall state in the next sections extend the ones proved in [Plu90] for $S_{\text{max}}$. As the latter results, they will be derived as a byproduct of the convergence of an iterative Jacobi-type algorithm to solve the system $Ax \nabla b$. Recall that the usual Jacobi algorithm constructs a sequence which is known to converge to the solution of a linear system under a strict diagonal dominance property.

Here we shall first transform the initial system to meet a diagonal dominance property. Then we shall construct a monotone sequence of thin vectors satisfying balance relations which are similar to the equations used in the definition of the usual Jacobi algorithm. To show that these thin vectors do exist and that the resulting sequence does converge, we shall however need some new definitions and properties concerning the semiring $T$, which are somehow more technical than the properties used to establish the Cramer theorem, Theorem 4.18.

**Definition 5.1.** We define on $T$ the relation $|\nabla|$, which is finer than the balance relation $\nabla$:

$$x |\nabla| y \iff x \nabla y \text{ and } |x| = |y| .$$

Note that when $T$ is a semiring extension as in Proposition 2.12 and $\gamma$ is as in Fact 2.14, we have:

(5.1) \[ x |\nabla| y \iff |x| = |y| \text{ and } \gamma(x) \nabla \gamma(y) \text{ (in } S) . \]

**Property 5.2.** $T$ is naturally ordered and for all $x, y \in T$ we have:

$$(x \in T^\vee \text{ and } x \leq y) \implies \exists z \in T^\vee \text{ such that } x \leq z \leq y \text{ and } z |\nabla| y .$$

**Property 5.3.** $T$ is naturally ordered and for all $a \in R$ any totally ordered subset of $T_a^\vee = \{x \in T^\vee \mid |x| = a\}$ is finite.

The following property is stronger.

**Property 5.4.** $T$ is naturally ordered and for all $x, y \in T$ we have:

$$(x, y \in T^\vee, x \leq y \text{ and } |x| = |y|) \implies x = y .$$

**Property 5.5.** For all $d \in T^\vee$ such that $|d|$ is invertible in $R$, there exists $\tilde{d} \in T$ such that for all $x, y \in T$,

$$dx |\nabla| y \iff x |\nabla| \tilde{d} y .$$

**Definition 5.6.** Let $T$ be a semiring with a symmetry, a thin set $T^\vee$, and a modulus taking its values in a totally ordered semiring $R$. We shall say that $T$ allows the construction of monotone algorithms if it satisfies Properties 5.2 and 5.5. If it satisfies in addition Property 5.3, we shall say that it allows the convergence of monotone algorithms.

From (5.1) and the property that $S$ can be seen as a subsemiring of $S \times R$, we obtain easily the following results.

**Lemma 5.7.** Let $S$ be a naturally ordered semiring with a symmetry and without zero-divisors. Then $S \times R$ satisfies Property 5.2 if and only if the following holds for all $a, a' \in S$

$$(a \in S^\vee \text{ and } a \nabla a' \iff \exists a'' \in S^\vee \text{ such that } a \nabla a'' \nabla a' .$$

Moreover, $S \times R$ satisfies Property 5.4 if and only if the following holds for all $a, a' \in S$

$$(a, a' \in (S^\vee)^* \text{ and } a \nabla a') \implies a = a' .$$

Also $S \times R$ satisfies Property 5.3 if and only if any totally ordered subset of $S^\vee$ is finite. \[\square\]
LEMMA 5.8. Let \( \mathcal{S} \) be as in Lemma 5.7. Then \( \mathcal{S} \times \mathcal{R} \) satisfies Property 5.5 if and only if for all \( d \in (\mathcal{S}^\vee)^* \) there exists \( d \in \mathcal{S} \) such that for all \( a, a' \in \mathcal{S} \)

\[
d a \vee a' \iff a \vee da'
\]

\( \square \)

The conditions of Lemma 5.7 are easily satisfied when \( \mathcal{S} = \mathbb{B}_n \) and \( \mathcal{S} = \mathbb{N}_2 \). For instance, for the first condition of Lemma 5.7 one can take \( a'' = a \) if \( a \neq 0 \), \( a'' = a' \) if \( a = 0 \) and \( a' \in \mathcal{S}^\vee \), and any \( a'' \in (\mathcal{S}^\vee)^* \) otherwise. Then by Lemma 5.7 \( \mathcal{T} = \mathcal{S}_{\max} \) and \( \mathcal{T} = \mathbb{T}_2 \) satisfy Properties 5.2 - 5.4. The same holds for \( \mathcal{P} \mathcal{h} \) and \( \mathcal{P} \mathcal{h} \times \mathcal{R}_{\max} \) as in Example 2.22 (the phase extension of the tropical semiring).

The condition of Lemma 5.8 holds as soon as \((\mathcal{S}^\vee)^*\) is the set of invertible elements of \( \mathcal{S} \), since then \( d = d^{-1} \) is a solution (\( \mathcal{S}^\circ \) is an ideal). This is the case for \( \mathcal{S} = \mathbb{B}_n \), \( \mathcal{S} = \mathbb{N}_2 \), and for the semiring \( \mathcal{P} \mathcal{h} \) of Example 2.22. Then by Lemma 5.8 the semirings \( \mathcal{T} = \mathcal{S}_{\max} \), \( \mathcal{T} = \mathbb{T}_2 \), and \( \mathcal{P} \mathcal{h} \times \mathcal{R}_{\max} \) satisfy Property 5.5. Note that similarly Property 5.5 holds as soon as \((\mathcal{T}^\vee)^*\), or at least the set of elements \( d \in \mathcal{T}^\vee \) such that \(|d|\) is invertible in \( \mathcal{R} \), is the set of invertible elements of \( \mathcal{T} \).

The above properties also hold for each of the extensions \( \mathcal{G} \) of a group \( G \) defined in Examples 2.21 and 2.26 (with a nontrivial symmetry in the first case and the identity symmetry in the second one). Thus for each of these examples, \( \mathcal{G} \times \mathcal{R}_{\max} \) satisfies Properties 5.2 - 5.5. This is in particular the case for the tropical extension of the torus \( \mathbb{T} \times \mathcal{R}_{\max} \) of Example 2.21 and the supertropical semifield of Example 2.26. We can also prove directly that Properties 5.2 - 5.5 hold for any supertropical semifield.

These examples can be summarized as follows.

FACT 5.9. All the following semirings allow the convergence of monotone algorithms: the bi-valued tropical semifield \( \mathbb{T}_2 \), the symmetrized max-plus semifield \( \mathcal{S}_{\max} \), the phase extension of the tropical semifield \( \mathcal{P} \mathcal{h} \times \mathcal{R}_{\max} \) (Example 2.22), the tropical extension of the torus \( \mathbb{T} \times \mathcal{R}_{\max} \) or that of any group with a non trivial symmetry \( \mathcal{G} \times \mathcal{R}_{\max} \) (Example 2.21), and any supertropical semifield (see Remark 2.25).

We also have:

PROPOSITION 5.10. Let \( \mathcal{T} \) satisfy Properties 5.2 and 5.4. Then for all \( x, y \in \mathcal{T} \) we have:

\[
(y \in \mathcal{T}, x \leq y \text{ and } |x| = |y|) \implies x = y
\]

PROOF. Let \( x, y \in \mathcal{T} \) be such that \( y \in \mathcal{T}, x \leq y \) and \(|x| = |y|\). Then by Property 5.2 applied to 0 and \( x \), there exists \( x' \in \mathcal{T} \), such that \( x' \leq x \), \( x \vee x' \) and \(|x'| = |x|\). Then \( x' \leq y \) and \(|x'| = |y|\). By Property 5.4 we get that \( x' = y \). Since \( x' \leq x \leq y \), we deduce that \( x = y \). \( \square \)

5.2. Existence theorems. The following result shows that the existence part of Theorem 4.18 does not require the condition that all the Cramer determinants are thin.

THEOREM 5.11 (Compare with [Plu90 Th. 6.2]). Let \( \mathcal{T} \) be a semiring allowing the convergence of monotone algorithms, see Definition 5.6, let \( A \in \mathcal{M}_n(\mathcal{T}) \), and assume that \(|\det A|\) is invertible in \( \mathcal{R} \) (but possibly \( \det A \nabla 0 \)). Then for every \( b \in \mathcal{T}^n \) there exists a thin solution \( x \) of \( Ax \nabla b \), which can be chosen in such a way that \(|x| = |\det A|^{-1}|A^{\text{adj}}b|\).

This result will be proved in Section 5.4 as a corollary of Theorem 5.20 below, which builds the solution using a Jacobi-type algorithm.

Applying Theorem 5.11 to the tropical extension \( \mathcal{S}_{\max} \) (in which \(|x|\) is invertible if and only if \(|x| \neq 0 \) or equivalently \( x \neq 0 \)), we recover the statement of [Plu90 Th. 6.2].
Corollary 5.12 (Plu90 Th. 6.2). Let $A \in \mathcal{M}_n(S_{\text{max}})$. Assume that $\det A \neq 0$ (but possibly $\det A \triangleright 0$). Then for every $b \in S_{\text{max}}^n$ there exists a thin solution $x$ of $Ax \triangleright b$, which can be chosen in such a way that $|x| = |\det A|^{-1}|A^{\text{adj}}b|$. \hfill $\Box$

A sketch of the proof of the latter result appeared in Plu90; the complete proof appeared in Gau92. The proof of Theorem 5.11 that we next give generalizes the former proof to the present setting. We also derive as a corollary the following analogous result in the bi-valued tropical semiring $T_2$.

Corollary 5.13. Let $A \in \mathcal{M}_n(T_2)$, and assume that $\det A \neq 0$ (but $\det A$ may be balanced). Then for every $b \in T_2^n$ the thin vector ("Cramer solution")

$$x := i(|\det A|^{-1}|A^{\text{adj}}b|)$$

satisfies $Ax \triangleright b$.

Proof. An element of $T_2$ is such that $|x|$ is invertible if and only if $|x| \neq 0$ or equivalently $x \neq 0$. It is easy to see that an element $x \in T_2$ is thin if and only if $x = i(|x|)$. Hence, the only possible thin vector $x$ such that $|x| = |\det A|^{-1}|A^{\text{adj}}b|$ is given by $x = i(|\det A|^{-1}|A^{\text{adj}}b|)$. Applying Theorem 5.11 to the tropical extension $T_2$ we get the corollary. \hfill $\Box$

The previous results can be reformulated equivalently in the following homogeneous forms.

Theorem 5.14. Let $\mathcal{T}$ be a semiring allowing the convergence of monotone algorithms, see Definition 5.6, and $A \in \mathcal{M}_{n,n+1}(\mathcal{T})$. Let $\hat{x} \in \mathcal{T}^{n+1}$ be such that $\hat{x}_k = (\oplus 1)^{n-k+1} \det A_k$ for all $k \in [n+1]$. Assume that either $\hat{x} = 0$, or at least one entry of $|\hat{x}|$ is invertible in $\mathcal{R}$. Then there exists a thin solution $x$ of $Ax \triangleright 0$ such that $|x| = |\hat{x}|$.

Proof. Let $\hat{x}$ be as in the statement of the theorem and let us show that there exists $x \in (\mathcal{T}^\n)^{n+1}$, such that $Ax \triangleright 0$ and $|x| = |\hat{x}|$. If all the entries of $\hat{x}$ are 0, then $x = 0$ satisfies trivially the above conditions. Hence, assume without loss of generality that $|\hat{x}_{n+1}|$ is invertible in $\mathcal{R}$. We set $M := A_{[n+1]}$ and $b = \ominus A_{n+1}$. By applying Theorem 5.11 to the system $My \triangleright b$, we get a thin solution $y \in \mathcal{T}^n$ of $My \triangleright b$ such that $|y_i| = |\det M|^{-1}|(M^{\text{adj}}b)_i| = |\hat{x}_{n+1}|^{-1}|\hat{x}_i|$ for all $i \in [n]$. Let $x \in \mathcal{T}^{n+1}$ be such that $x_{n+1} = i(|\hat{x}_{n+1}|)$ and $x_i = x_{n+1} \ominus y_i$ for $i \in [n]$. Then $|x_i| = |\hat{x}_i|$ for all $i \in [n+1]$ and multiplying the equation $My \triangleright b$ by $x_{n+1}$, we get that $Ax \triangleright 0$. \hfill $\Box$

Note that when $\mathcal{R}$ is a semifield, $\hat{x}$ always satisfies the condition of Theorem 5.14. The uniqueness of the solution is obtained from Corollary 4.19.

In the particular case of the bi-valued tropical semiring $T_2$ we obtain more precise result.

Corollary 5.15. Let $A \in \mathcal{M}_{n,n+1}(T_2)$. Let $\hat{x} \in T_2^{n+1}$ be such that $\hat{x}_k = \det A_k$ for all $k \in [n+1]$. Then the thin vector $x = i(|\hat{x}|) \in T_2^{n+1}$ satisfies $Ax \triangleright 0$. Moreover, if $\hat{x}$ is thin and non-zero, then for any thin vector $\hat{x}$ which is a solution of $Ax \triangleright 0$, we have that $x$ is a thin multiple of $\hat{x}$.

Proof. An element $x \in T_2$ is thin if and only if $x = i(|x|)$ and the set $(T_2^\n)^*$ is exactly the set of invertible elements of $T_2$. Then applying Theorem 5.14 to $T_2$, we get the first assertion. The second assertion follows similarly from Corollary 4.19. \hfill $\Box$

Remark 5.16. The special case of the existence result, Theorem 5.11 concerning $S = T_2$ or $S_{\text{max}}$ could be derived alternatively from the existence and uniqueness result in the Cramer theorem, Theorem 4.18. To do this a perturbation argument
can be used since the matrix \( A \) and the vector \( b \) can always be “approximated” by matrices satisfying the condition of Item (2) of the latter theorem.

### 5.3. The tropical Jacobi algorithm

The thin solution \( x \) in Theorem \[ \text{5.11} \]
will be established constructively by means of the Jacobi algorithm of \[ \text{Plu90} \].

The following notion of diagonal dominance is inspired by the notion with the same name which is classically used in numerical analysis. A real matrix \( A = (A_{ij}) \) is classically said to have a dominant diagonal if \( A_{ii} \geq \sum_{j \neq i} |A_{ij}| \) holds for all \( i \in [n] \).

The tropical analogue of this condition is
\begin{equation}
A_{ii} \geq \max_{j \neq i} |A_{ij}|, \quad \forall i \in [n].
\end{equation}

We shall use the following related condition.

**Definition 5.17.** We shall say that \( A = (A_{ij}) \in \mathcal{M}_n(T) \) has a dominant diagonal if
\[
|\det A| = |A_{11} \cdots A_{nn}| \quad \text{and is invertible in} \quad \mathcal{R}.
\]

Corollary \[ \text{5.11} \] implies that, when \( \mathcal{R} \) is a semifield, any matrix which has a dominant diagonal in this sense is diagonally similar to a matrix satisfying the tropical analogue \[ \text{(5.2)} \] of the usual condition of diagonal dominance, in particular, Definition \[ \text{5.17} \] is weaker than \[ \text{(5.2)} \].

The following decomposition is similar to the one used in the classical (relaxed) Jacobi algorithm.

**Proposition 5.18.** Let \( T \) be a semiring satisfying Property \[ \text{5.2} \]. Then any matrix \( A \in \mathcal{M}_n(T) \) with a dominant diagonal can be decomposed into the sum
\[
A = D \oplus N
\]
of matrices \( D \) and \( N \in \mathcal{M}_n(T) \) such that \( D \) is a diagonal matrix with diagonal entries in \( T \), and \( |\det D| = |\det A| \). The latter decomposition will be called a Jacobi-decomposition of \( A \).

**Proof.** Let \( A = (A_{ij}) \in \mathcal{M}_n(T) \) be a matrix with a dominant diagonal. Let \( i \in [n] \). From Property \[ \text{5.2} \] applied to \( x = 0 \) and \( y = A_{ii} \), there exists \( \delta_i \in T^{\vee} \) such that \( 0 \leq \delta_i \leq A_{ii} \) and \( \delta_i \uplus A_{ii} \). Since \( \delta_i \leq A_{ii} \), there exists \( \delta_i' \in T \) such that \( \delta_i \oplus \delta_i' = A_{ii} \). Taking for \( D \) the diagonal matrix such that \( D_{ii} = \delta_i' \) and for \( N \) the matrix such that \( N_{ii} = \delta_i' \) and \( N_{ij} = A_{ij} \) for all \( i \neq j \), we get that \( A = D \oplus N \). Moreover, \( |D_{ii}| = |A_{ii}| \) for all \( i \in [n] \), so \( |\det D| = |A_{11} \cdots A_{nn}| \). Since \( A \) has a dominant diagonal, we obtain \( |\det D| = |\det A| \).

**Remark 5.19.** We may always reduce the problem \( Ax \nabla b \) to the case where \( |\det A| = |A_{11} \cdots A_{nn}| \). Indeed, since \( \mu : x \mapsto |x| \) is a morphism and \( |0| = 1 \), we have \( |\det A| = |\det A| \) for all \( A \in \mathcal{M}_n(T) \). Since \( \mathcal{R} \) is a totally ordered idempotent semiring, computing the permanent of the matrix \( |\det A| \) is equivalent to solving an optimal assignment problem, which furnishes an optimal permutation. Permuting the rows of the matrix \( A \), we can transform the system into a system \( A'x \nabla b' \) such that the optimal permutation of \( A' \) is the identity, which implies \( |\det A'| = |\det A| = |A_{11} \cdots A_{nn}| = |A_{11}' \cdots A_{nn}'| \). Since \( |\det A'| = |\det A| \), then \( A' \) has a dominant diagonal as soon as \( |\det A| \) is invertible in \( \mathcal{R} \). Moreover, since \( \mu \) is a morphism, we get that \( |A_{11}'b| = |A_{11}|b \) and \( |\det A| = |\det (A)| \). Using Lemmas \[ \text{5.7} \text{ and } \text{5.6} \] we deduce that proving Theorem \[ \text{5.11} \] for \( Ax \nabla b \) is equivalent to proving it for \( A'x \nabla b' \).

**Theorem 5.20** (Tropical Jacobi Algorithm, compare with \[ \text{Plu90} \] Th. 6.3) and \[ \text{Gau92} \] III, 6.0.2) for \( S_{\max} \). Let \( T \) be a semiring allowing the construction of monotone algorithms, see Definition \[ \text{5.6} \]. Let \( A \in \mathcal{M}_n(T) \) have a dominant diagonal, and let \( A = D \oplus N \) be a Jacobi-decomposition. Then
(1) One can construct a sequence \( \{x^k\} \) of thin vectors satisfying:
   (a) \( \emptyset = x^0 \preceq x^1 \preceq \ldots \preceq x^k \preceq \ldots; \)
   (b) \( DX^{k+1} \nabla \ominus Nx^k \oplus b. \)

(2) The sequence \( \{x^k\} \) is stationary after at most \( n \) iterations, meaning that
    \( |x^k| = |x^n| \) for all \( k \geq n \), and we have
    \[ |x^n| = |\det A|^{-1}A^{\text{adj}}b| . \]

(3) When \( T \) allows the convergence of monotone algorithms (Definition \( 5.6 \)),
    the sequence \( x^k \) is stationary, meaning that there exists \( m \geq 0 \) such that
    \( x^k = x^m \) for all \( k \geq m \). Moreover, the limit \( x^m \) is a solution of \( Ax \nabla b \).

(4) When \( T \) satisfies also Property \( 5.3 \), one can take \( m = n \) in the previous assertion.

Applying the previous result to the tropical extension \( S_{\text{max}} \) (in which \( |x| \) is
invertible if and only if \( |x| \neq \emptyset \) or equivalently \( x \neq \emptyset \)), we recover [Plu90, Th. 6.3].
Applying the same result to the case of the bi-valued tropical semiring \( T = \mathbb{T}_2 \),
and using the same arguments as for Corollary \( 5.13 \) of Theorem \( 5.11 \) we obtain
immediately the following result.

**Corollary 5.21 (Jacobi Algorithm in the Bi-Valued Tropical Semiring).** Let \( A \in M_n(\mathbb{T}_2) \) have a dominant diagonal and let \( A = D \oplus N \) be a Jacobi-decomposition.
Then the sequence \( \{x^k\} \) of thin vectors defined by \( x^0 = 0, \)
\[ x^{k+1} = \nu(|D|^{-1}|N_{x^k} \oplus b|) \]
is stationary after at most \( n \) iterations, meaning that \( x^n = x^{n+1} \). Moreover, \( x^n \) is a thin solution of \( Ax \nabla b \) and
\[ x^n = \nu(|\det A|^{-1}A^{\text{adj}}b|) . \]

Before proving Theorem \( 5.20 \) we illustrate it by an example.

**Example 5.22.** We take \( T = S_{\text{max}} \), and apply the tropical Jacobi algorithm
in the linear system
\[
\begin{align*}
5 & \ominus 0 & 3 \\
1 & 3 & \ominus 1 \\
3 & \ominus 2 & 1^\circ
\end{align*}
\begin{align*}
x_1 \\
x_2 \\
x_3
\end{align*}
\begin{align*}
\nabla & \ominus 1 \\
4^\circ \\
0
\end{align*}
\]

Denoting by \( A \) the matrix of the system and by \( b \) its right-hand side, we get:
\[ |\det A|^{-1}A^{\text{adj}}b| = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} . \]

Let us choose the Jacobi-decomposition \( A = D \oplus N \) with:
\[
D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Applying the Jacobi algorithm, we get the following sequence starting from \( x^0 = 0 \):
\[
\begin{align*}
5x_1^2 & \nabla 0x_2^2 \ominus 3x_3^2 \ominus 1 = \ominus 1 \\
3x_2^2 & \nabla 1x_1^2 \ominus 1x_3^2 \ominus 4^\circ = 4^\circ \Rightarrow x_2^1 = 1 \text{ or } 2, \text{ we choose } x_2^1 := 1 \\
1x_3^2 & \nabla 3x_1^2 \ominus 0x_2^2 \ominus 1x_3^2 \ominus 0 = 0 \Rightarrow x_3^1 = -1
\end{align*}
\]
\[
\begin{align*}
5x_1^2 & \nabla 0x_2^2 \ominus 3x_3^2 \ominus 1 = \ominus 2 \\
3x_2^2 & \nabla 1x_1^2 \ominus 1x_3^2 \ominus 4^\circ = 4^\circ \text{ and } x_2 \geq x_1 \Rightarrow \begin{cases} x_2^1 = \ominus 1 \\ x_3^1 = 1 \\ x_3^2 = 1 \\ x_3^3 = 2 \end{cases}
\end{align*}
\]
Hence that I | \equal{} 1

From Theorem 3.9 we get that the weight of a circuit for \( D_{13} \) and \( N_{33} \) would have lead to the other possible solutions \( x^3 = (0, 1, \circ2)^T \), and \( x = (\circ0, \circ1, 2)^T \).

5.4. Proof of Theorems 5.20 and 5.11. For the proof of Theorem 5.20 we shall need the following lemmas which are derived from the results of Section 4.2 and from Theorem 3.9.

**Lemma 5.23.** Let \( A \in M_n(\mathbb{T}) \) be a matrix with a dominant diagonal and let \( A = D \oplus N \) be a Jacobi-decomposition. Then \( |D| \) is an invertible diagonal matrix which coincides with the diagonal submatrix of \( |A| \): \( |D_{ii}| = |A_{ii}| \) for all \( i \in [n] \).

**Proof.** Since \( A = D \oplus N \) is a Jacobi-decomposition and \( \mu \) is a morphism, we have \( \per |D| = |\det D| = |\det A| \). This implies that \( \per |D| \) is invertible in \( \mathbb{R} \), since \( A \) has a dominant diagonal. \(|D|\) is a diagonal matrix. Thus it is necessarily a monomial matrix with entries in \( \mathbb{R} \). From \( |D_{11}| \cdots |D_{nn}| = |D| \) and the property that \( \per |D| \) is invertible in \( \mathbb{R} \), we get that all diagonal entries \( |D_{ii}| \) of \( |D| \) are invertible in \( \mathbb{R} \). Thus \(|D|\) is invertible in \( M_n(\mathbb{R}) \). From \( |D| \oplus |N| = |A| \) we deduce that \( |D_{ii}| \leq |A_{ii}| \) for all \( i \in [n] \).

Let us show the reverse inequalities, which will imply the equalities. Since \( A \) has a dominant diagonal, we have \( |D_{11}| \cdots |D_{nn}| = |D| = |\det A| = |A_{11}| \cdots |A_{nn}| = |A_{11}| \cdots |A_{nn}| \). Hence all the \( |A_{ii}| \) are invertible in \( \mathbb{R} \). We have for instance \( |A_{11}| \cdots |A_{nn}| = |D_{11}| \cdots |D_{nn}| \leq |D_{11}| |A_{22}| \cdots |A_{nn}| \). Since \( |A_{22}|, \ldots, |A_{nn}| \) are invertible, we deduce that \(|A_{11}| \leq |D_{11}| \). The same argument shows that \( |A_{ii}| \leq |D_{ii}| \) holds for all \( i \in [n] \). □

**Lemma 5.24.** Let \( A \in M_n(\mathbb{T}) \) be a matrix with a dominant diagonal and let \( A = D \oplus N \) be a Jacobi-decomposition. Then \( \per(|D|^{-1} |N|) \leq \per(|D|^{-1} |A|) \).

Hence every circuit of \( |D|^{-1} |N| \) has a weight less than or equal to \( \per |A| \).

**Proof.** By Lemma 5.23 \(|D|\) is invertible. Let us show that \( \per(|D|^{-1} |A|) = \per |A| \).

Since \(|D|\) is a monomial matrix of \( M_n(\mathbb{R}) \), we have \( \per(|D|^{-1} |A|) = (\per |D|)^{-1} \per |A| \). Since \( \per |D| = |\det A| = \per |A| \), we obtain that \( \per(|D|^{-1} |A|) = \per |A| \).

Let us denote \( M = |D|^{-1} |N| \). Since \( |D|^{-1} |A| = I \oplus M \), we get that \( M \leq |D|^{-1} |A| \) for the natural order of \( \mathbb{R} \). This implies that \( \per M \leq \per(|D|^{-1} |A|) = \per |A| \) and that the weight of a circuit for \( M \) is less than or equal to its weight for \( |D|^{-1} |A| \).

From Theorem 3.9 we get that the weight of a circuit of \( |D|^{-1} |A| \) is less than or equal to \( \per |A| \), which implies the same for \( M \). □

**Lemma 5.25.** Let \( A \in M_n(\mathbb{T}) \) be a matrix with a dominant diagonal and let \( A = D \oplus N \) be a Jacobi-decomposition. Then \( (|D|^{-1} |N|)^* |D|^{-1} = (\per |A|)^{-1} |A|^{\text{adj}} \).

**Proof.** Let \( M = |D|^{-1} |N| \) and \( Q = |D|^{-1} |A| \). Since \( \mu \) is a morphism, we have easily \( (\per |A|)^{-1} |A|^{\text{adj}} = |\det A|^{-1} |A|^{\text{adj}} \). Since \( A = D \oplus N \), we also have \( Q = I \oplus M \). Then \( M^* = Q^* \) and it remains to show that \( Q^* |D|^{-1} = (\per |A|)^{-1} |A|^{\text{adj}} \).

By Lemma 5.24 \( \per(Q) = \per |A| \). Since by Lemma 5.23 \(|D|\) is the diagonal of \(|A|\), the diagonal entries of \( Q \) are equal to \( \per |A| \). Thus \( Q \) satisfies the conditions of Theorem 3.9. Hence \( Q^* = Q^{\text{adj}} \). Since \(|D|\) is a monomial matrix, Lemmas 3.6 and 3.7 imply that \( Q^{\text{adj}} = |A|^{\text{adj}} |(D|^{-1} |A|^{\text{adj}} = |A|^{\text{adj}} (\per |D|^{-1} |D| = (\per |A|^{-1} |A|^{\text{adj}}) |D| \), thus \( Q^* |D|^{-1} = Q^{\text{adj}} |D|^{-1} = (\per |A|)^{-1} |A|^{\text{adj}} \), which finishes the proof. □
Proof of Theorem 5.20 We first prove Item (1). Since $A = D \oplus N$ is a Jacobi-decomposition, $D$ is a diagonal matrix with entries in $T^\vee$. By Lemma 5.23, $\det D$ is an invertible diagonal matrix, which implies that all its diagonal entries are invertible in $R$. Hence, one can construct from Property 5.3 the diagonal $n \times n$ matrix $\hat{D}$ such that $\hat{D}_{ii} = D_{ii}$. Then Condition (1b) of Theorem 5.11 is equivalent to:

$$x^{k+1} \ni \hat{D}(\oplus N x^k \oplus b) \ .$$

Let us show by induction on $k$ that there exist thin $n$-dimensional vectors $0 = x^0 \preceq x^1 \preceq \cdots \preceq x^k \preceq x^{k+1}$ satisfying (5.5), together with

$$x^{k+1} \preceq \hat{D}(\oplus N x^k \oplus b) \ .$$

When $k = 0$, the above conditions are equivalent to $x^1 \in (T^\vee)^n$, $0 \preceq x^1 \preceq \hat{D}b$ and $x^1 \triangledown \hat{D}b$. Thus $x^1$ can be constructed by applying Property 5.2 entrywise. Hence, the properties of the induction hold for $k = 0$.

Assume now that these properties hold for some $k \geq 0$. Since the map $x \mapsto \hat{D}(\oplus N x \oplus b)$ is non-decreasing and $x^k \preceq x^{k+1}$, we have from (5.6) that $x^{k+1} \preceq \hat{D}(\oplus N x^{k+1} \oplus b)$. Then applying Property 5.2 entrywise, we construct $x^{k+2} \in (T^\vee)^n$ such that $x^{k+1} \preceq x^{k+2} \preceq \hat{D}(\oplus N x^{k+1} \oplus b)$ and $x^{k+2} \triangledown \hat{D}(\oplus N x^{k+1} \oplus b)$. This shows that the induction hypothesis holds for $k+1$. Thus Item (1) of the theorem is proven.

We next prove Item (2). Denote $\hat{x}^k := |x^k|$. Since $\mu$ is a morphism, Condition (1b) of Theorem 5.20 implies that $|D|\hat{x}^{k+1} = |N|\hat{x}^k \oplus |b|$. Since $|D|$ is invertible in $\mathcal{M}_n(R)$, we obtain that $x^{k+1} = M\hat{x}^k \oplus |D|^{-1}|b|$ for $M = |D|^{-1}|N|$. Hence, for all $k \geq 0$ we get that

$$\hat{x}^{k+1} = (I \oplus M \oplus \cdots \oplus M^k)|D|^{-1}|b| \ .$$

Lemma 5.24 allows us to apply the theorem of Carré and Gondran (Proposition-Definition 3.8) to the matrix $M$. It follows that

$$|x^k| = \hat{x}^k = M^*|D|^{-1}|b| \quad \text{for all } k \geq n \ .$$

The last assertion of Item (2) follows from the previous equation and Lemma 5.25.

We now prove Item (3). Since all $x^k$ are thin, $|x^k| = |x^n|$ for all $k \geq n$, and $x^n \preceq x^{n+1} \preceq \cdots$. Applying Property 5.3 entrywise implies that $x^n$ is stationary after some finite time $m$. Then by Condition (1b) of the theorem we get that $Dx^n \triangledown N x^n \oplus b$, which is equivalent to $Ax^n \triangledown b$.

We finally prove Item (4). When $T$ satisfies also Property 5.4 the above properties imply that $x^n = x^{n+1}$. Thus one can take $m = n$ in the previous conclusions.

Proof of Theorem 5.11 Applying Theorem 5.20 after the transformation described in Remark 5.19 and using Proposition 5.18, we deduce Theorem 5.11.

5.5. The tropical Gauss-Seidel algorithm. We now introduce a Gauss-Seidel type algorithm. It is a variant of the Jacobi algorithm in which the information is propagated more quickly.

Proposition 5.26. Let $T$ be a semiring satisfying Property 5.2. Then any matrix $A \in \mathcal{M}_n(T)$ with a dominant diagonal can be decomposed into the sum

$$A = D \oplus L \oplus U$$

of matrices $D$, $L$ and $U \in \mathcal{M}_n(T)$ such that $D$ is a diagonal matrix with diagonal entries in $T^\vee$, $|\det D| = |\det A|$, all the entries on the main diagonal and above the main diagonal of $L$ equal to 0, and all the entries below the main diagonal of $U$ equal to 0. The latter decomposition will be called a Gauss-Seidel-decomposition of $A$. 
Proof. By Proposition 5.18 there exists a Jacobi-decomposition $A = D \oplus N$. Taking for $L$ the (strict) lower diagonal part of $N$ and for $U$ the upper diagonal and diagonal part of $N$, we get the result. \( \square \)

Theorem 5.27. Let $T$ be a semiring allowing the construction of monotone algorithms, see Definition 5.6. Let $A \in M_n(T)$ have a dominant diagonal and let $A = D \oplus L \oplus U$ be a Gauss-Seidel-decomposition. Then

1. One can construct a sequence $\{x^k\}$ of thin vectors satisfying:
   (a) $0 = x^0 \preceq x^1 \preceq \ldots \preceq x^k \preceq \ldots$;
   (b) $Dx^{k+1} \preceq Lx^{k+1} \preceq Ux^k \oplus b$;
   (c) $|x^{k+1}| = |\det A|^{-1} |(D \oplus L)^{ad}| \ominus Ux^k \oplus b$.

2. The sequence $|x^k|$ is stationary after at most $n$ iterations, meaning that
   $|x^k| = |x^n|$ for all $k \geq n$, and we have
   $|x^n| = |\det A|^{-1} |A^{ad}|_b$.

3. When $T$ allows the convergence of monotone algorithms (Definition 5.6), the sequence $x^k$ is stationary, meaning that there exists $m \geq 0$ such that $x^m = x^{m+1}$. Moreover, the limit $x^m$ is a solution of $Ax \nabla b$.

4. When $T$ satisfies also Property 5.4 one can choose $m = n$ in the previous assertion.

Example 5.28. We consider the system of Example 5.22 with the Gauss-Seidel decomposition $A = D \oplus L \oplus U$ such that $L \oplus U = N$ with $D$ and $N$ as in (5.4). Then starting from $x^0 = 0$ we obtain the following sequence:

\[
\begin{align*}
5x_1 - 2 & \preceq 3x_2 \ominus 1 = \ominus 1 \quad \Rightarrow \quad x_1 = 0 - 4 \\
3x_2 & \ominus 1 \ominus 4 = 4 \quad \Rightarrow \quad x_2^0 = 1 \ominus 1, \text{ we choose } x_2 = 1 \\
1x_3 & \ominus 3x_1 \ominus 2x_2 \ominus 1 = 3 \quad \Rightarrow \quad x_3^0 = 2
\end{align*}
\]

\[
\begin{align*}
5x_1 - 2 & \preceq 3x_2 \ominus 1 = 0 \quad \Rightarrow \quad x_1 = 2 \\
3x_2 & \ominus 1 \ominus 4 = 4 \quad \Rightarrow \quad x_2^0 = 1 \ominus 1, \text{ and } x_2 \geq x_1 \Rightarrow x_2 = 1 \quad \Rightarrow \quad x_3 = 2
\end{align*}
\]

We find the solution $(0, 1, 2)^T$ after 2 iterations only, whereas the Jacobi algorithm required 3 iterations.

Lemma 5.29. Let $A \in M_n(T)$ be a matrix with a dominant diagonal and let $A = D \oplus L \oplus U$ be a Gauss-Seidel-decomposition. Let $M = |\det A|^{-1} |(D \oplus L)^{ad}| \in M_n(T)$. Then $|D|^{-1} |L|$ has no circuit (all circuits have a zero weight) and we have $M = |\det A|^{-1} |L||D|^{-1}$.

Proof. Since the entries of $L$ are 0 on and above the diagonal, the graph of $|D|^{-1} |L|$ has no circuit. Let $A' = D \oplus L$. Then $|\det A'| = |\det D| = |\det A|$. Taking $D' = D$ and $N' = L$, we get a Jacobi-decomposition $D' \oplus N'$ of $A'$. Applying Lemma 5.25 to it, we get the last assertion of the lemma. \( \square \)

Lemma 5.30. Let $A \in M_n(T)$ be a matrix with a dominant diagonal. Assume $A = D \oplus L \oplus U$ is a Gauss-Seidel-decomposition. Let $M = |\det A|^{-1} |(D \oplus L)^{ad}| \in M_n(T)$. Then

(a) Every circuit of $M[U]$ has a weight less than or equal to 1.

(b) $(M[U])^* = |\det A|^{-1} |A^{ad}|$.

Proof. We start by proving item (b). Indeed, adding a top element to $R$ allows one to define the Kleene star $B^*$ of a matrix $B$ with entries in $R$. Then every circuit of $B$ has a weight less than or equal to 1 if and only if $B^*$ has all its entries in $R$ (meaning that they are all different from the top element).
Let $N = L \oplus U$. It is easy to see that $D \otimes N$ is a Jacobi-decomposition of $A$. Thus by Lemma 5.24, every circuit of $|D|^{-1}|N|$ has a weight less than or equal to 1. By Lemma 5.25, we have $(|D|^{-1}|N|)^*|D|^{-1} = |\det \, A^{-1}|A^{adj}|$.

Denote $L' = |D|^{-1}|L|$ and $U' = |D|^{-1}|U|$. Then by Lemma 5.29, we have $M = (L')^*|D|^{-1}$, so $M[U] = (L')U'$, and $|D|^{-1}|N| = L' \oplus U'$. This implies that the assertion of item (b) is equivalent to the equality

$$(5.7) \quad ((L')^*U')^* = (L' \oplus U')^*,$$

which is indeed a well known unambiguous rational identity (by expanding the Kleene star and products in both expressions, we arrive at the sum of all words in the letters $L'$ and $U'$).

(a) Let us note that the identity (5.7) also shows that $(M[U])^* = ((L')^*U')^* \leq (L' \oplus U')^*$ and since every circuit of $|D|^{-1}|N|$ has a weight less than or equal to 1, the latter expression has all its entries in $R$, so has $(M[U])^*$. This implies that every circuit of $M[U]$ has a weight less than or equal to 1.

**Proof of Theorem 5.27.** The proof follows the same lines as the one of Theorem 5.20. In particular, constructing the same matrix $\widetilde{D}$ as in this proof, we get that Condition (1b) of Theorem 5.27 is equivalent to $x^{k+1} \mathbf{v} \widetilde{D}(\oplus Lx^{k+1} \ominus Ux^k \oplus b)$, which corresponds to the system:

$$x_i^{k+1} \mathbf{v} \widetilde{D}_{i1}(\oplus U_{1i}x_i^k \ominus U_{1n}x_n^k \oplus b_i)$$

$$\vdots$$

$$x_i^{k+1} \mathbf{v} \widetilde{D}_{ih}(\oplus L_{i1}x_1^k \cdots \ominus L_{i,i-1}x_{i-1}^k \ominus U_{i1}x_i^k \cdots \ominus U_{in}x_n^k \oplus b_i)$$

$$\vdots$$

$$x_n^{k+1} \mathbf{v} \widetilde{D}_{nn}(\oplus L_{n1}x_1^k \cdots \ominus L_{n,n-1}x_{n-1}^k \ominus U_{nn}x_n^k \oplus b_n).$$

Using Property 5.2 for each $k$ and each $i \in [n]$ one chooses $x_i^{k+1}$ such that it satisfies the $i$th equation of this system, together with the conditions $x_i^k \leq x_i^{k+1}$ and

$$x_i^{k+1} \leq \widetilde{D}_{i1}(\oplus U_{1i}x_i^k \cdots \ominus U_{1n}x_n^k \oplus b_i).$$

Then the sequence satisfies Conditions (1a) and (1b) of the theorem. In particular it satisfies $|x^{k+1}| = |D|^{-1}|L||x^k| \oplus |D|^{-1}|Ux^k \oplus b|$. Since $|D|^{-1}|L|$ has no circuit, the theorem of Carré and Gondran (Proposition-Definition 3.8) implies that $|x^{k+1}| = (|D|^{-1}|L|)^*|D|^{-1}|Ux^k \oplus b|$, which by Lemma 5.29 is equivalent to Condition (1c) of the theorem.

Again the latter condition implies that $|x^{k+1}| = M[U]|x^k| \oplus M[b]$ with $M$ as in Lemma 5.29. Then by Lemma 5.30 and the theorem of Carré and Gondran, we get that $|x^k| = (M[U])^*M[b]$ for all $k \geq n$. Moreover from the second assertion of Lemma 5.30 we have $|x^k| = |\det \, A^{-1}|A^{adj}|b| = |\det \, A^{-1}|A^{adj}b|$ which shows Item (2) of the theorem.

Items (3) and (4) of the theorem are obtained by the same arguments as for Theorem 5.20.

The particular case of Theorem 5.27 concerning the tropical extension $S_{max}$ was obtained in [Gau92]. One can also apply the same result to the case of the bi-valued tropical semiring $T = T_2$. This leads to the same solution $x = i(|\det \, A^{-1}|A^{adj}b)$ as in Corollary 5.21 by using the Gauss-Seidel algorithm instead of the Jacobi algorithm.
6. Homogeneous systems: the generalized Gondran-Minoux theorem

The following result was stated in [Plu90].

**Theorem 6.1** ([Plu90, Th. 6.5]). Let $A \in \mathcal{M}_n(S_{\text{max}})$. Then there exists $x \in (S_{\text{max}}^\vee)^n \setminus \{0\}$ such that $Ax \nabla 0$ if and only if $\det A \nabla 0$.

The special case in which $A \in \mathcal{M}_n(\mathbb{R}_{\text{max}})$ is equivalent to the theorem of Gondran and Minoux quoted in the introduction (Theorem 1.1).

The "only if" part is obtained by taking $b = 0$ in the first assertion of Theorem 4.18. The "if" part was proved in [Gau92]. An analogous result was proved by Izhakian and Rowen [IR09], when the symmetrized tropical semiring $S_{\text{max}}$ is replaced by the bi-valued tropical semiring $T_2$. We next provide a general result which includes Theorem 6.1 as well as the result of [IR09] as special cases.

Let $\mathcal{T}$ be a semiring with a symmetry, a thin set $\mathcal{T}^\vee$, and a modulus taking its values in a totally ordered semiring $\mathcal{R}$. We shall need the following additional properties.

**Property 6.2.** $\mathcal{R}$ is an idempotent semifield and the thin set $\mathcal{T}^\vee$ is such that the set of invertible elements of $\mathcal{T}$ is $(\mathcal{T}^\vee)^*$ and that it coincides with $\mathcal{T} \setminus \mathcal{T}^\circ$.

This property is satisfied when $\mathcal{T} = S \times \mathcal{R}$, with the thin set $\mathcal{T}^\vee = S^\vee \times \mathcal{R}$, $\mathcal{R}$ is an idempotent semifield and $(S^\vee)^* = S \setminus S^\circ$ is the set of invertible elements.

**Property 6.3.** For all $x \in \mathcal{T}$ we have $x = 0 \Leftrightarrow |x| = 0$.

This property is satisfied when $\mathcal{T} = S \times \mathcal{R}$.

**Property 6.4.** For all $x_1, \ldots, x_k \in \mathcal{T}$ such that $x := x_1 \oplus \cdots \oplus x_k \in \mathcal{T}^\circ$, either there exists a single index $i \in [k]$ such that $x_i \in \mathcal{T}^\circ$ and $|x_i| = |x|$, or there exist two different indices $i, j \in [k]$ such that $x_i \oplus x_j \in \mathcal{T}^\circ$ and $|x_i| = |x_j| = |x|$.

This property is satisfied when $\mathcal{T} = S \times \mathcal{R}$, with the thin set $\mathcal{T}^\vee = S^\vee \times \mathcal{R}$, and for all $a_1, \ldots, a_k \in S$ such that $a := a_1 + \cdots + a_k \in S^\circ$, either there exists a single index $i \in [k]$ such that $a_i \in S^\circ$, or there exist two different indices $i, j \in [k]$ such that $a_i + a_j \in S^\circ$.

**Property 6.5.** If $x \in \mathcal{T}^\circ$ and $|y| \leq |x|$ then $x \oplus y \in \mathcal{T}^\circ$.

This property is satisfied when Property 6.2 and the result of Proposition 5.10 hold. We can then obtain the following assertion.

**Fact 6.6.** All the following semirings satisfy Properties 6.2, 6.3, the symmetrized max-plus semiring $S_{\text{max}}$, the bi-valued tropical semiring $T_2$, the tropical extension of the torus $\mathbb{T} \times \mathbb{R}_{\text{max}}$ or that of any group with a non trivial symmetry $G \times \mathbb{R}_{\text{max}}$ (Example 2.21), and any supertropical semifield (see Remark 2.25). However the phase extension of the tropical semiring $\mathbb{P} \times \mathbb{R}_{\text{max}}$ (Example 2.22) does not satisfy Properties 6.2, 6.3, nor 6.5.

We note the following consequences, the first one being easy.

**Proposition 6.7.** Let $\mathcal{T}$ satisfy Property 6.2. For all $x, y \in \mathcal{T}$ if $xy \in \mathcal{T}^\circ$ then $x \in \mathcal{T}^\circ$ or $y \in \mathcal{T}^\circ$.

**Proposition 6.8.** Let $\mathcal{T}$ satisfy Properties 6.2 and 6.5. If $1 \oplus xy \in \mathcal{T}^\circ$ where $|x| \leq 1$ and $|y| \leq 1$, then $1 \oplus x \in \mathcal{T}^\circ$ or $1 \oplus y \in \mathcal{T}^\circ$.

**Proof.** We have $(1 \oplus x)(1 \oplus y) = 1 \oplus xy \oplus x \oplus y \in \mathcal{T}^\circ$ by Property 6.5. By Proposition 6.7 we must have $1 \oplus x \in \mathcal{T}^\circ$ or $1 \oplus y \in \mathcal{T}^\circ$. 

Theorem 6.9 (Homogeneous balances). Let $\mathcal{T}$ be a semiring allowing weak balance elimination (Definition 4.4) and the convergence of monotone algorithms (Definition 5.6), and satisfying Properties 5.2–6.5. Let $A \in \mathcal{A}_n(\mathcal{T})$. Then there exists $x \in (T^\kappa)^n \setminus \{0\}$ such that $Ax \nabla 0$ if and only if $\det A \nabla 0$.

Proof. The necessity of the condition $\det A \nabla 0$ follows by taking $b = 0$ in the first assertion of Theorem 4.18. Indeed, $Ax \nabla 0$ and $x \in (T^\kappa)^n$ imply $(\det A)x \nabla 0$. At least one entry of $x$ belongs to $(T^\kappa)^*$. By Property 6.2 this entry is invertible and so $\det A \nabla 0$.

Let us prove that $Ax \nabla 0$ implies the existence of $x \in (T^\kappa)^n \setminus \{0\}$ such that $Ax \nabla 0$.

Case 1: We first deal with the degenerate case in which $\det A = 0$. Then per $|A| = |\det A| = 0$, so applying the Frobenius-König’s theorem to $|A|$, and using Property 6.3, there exists a reordering of rows and columns such that the matrix $A$ has the following form:

$$A = \begin{bmatrix} 0_{p \times q} & B \\ C & D \end{bmatrix},$$

where $p + q = n + 1$, $C \in \mathcal{T}^{(n-p)\times q}$, and $0_{p \times q}$ denotes the $p \times q$ zero matrix. It suffices to look for a solution $x$ such that $x_i = 0$ for all $q + 1 \leq i \leq n$. Denoting by $y$ the vector with entries $x_1, \ldots, x_q$, it remains to solve the system $Cy \nabla 0$ which has $q$ unknowns and $n - p = q + 1$ equations. Thus if this system has a non-zero $(q - 1) \times (q - 1)$-minor, an application of Theorem 5.14 provides a non-zero thin solution of the system $Cy \nabla 0$. Otherwise, we may assume by induction that the sufficiency in the theorem (or at least its restriction to the case $\det A = 0$) is already proved for systems of lower dimension. Then we apply the induction to a square subsystem $C'y \nabla 0$ obtained by setting to zero one coordinate of $y$. This completes the treatment of the degenerate case.

Case 2: We now assume that $\det A \neq 0$, so per $|A| = |\det A| \neq 0$. By Corollary 3.11 applied to the matrix $C = |A|$, there exist two diagonal matrices $D$ and $D'$ with invertible diagonal entries in $\mathcal{R}$ and a permutation matrix $\Sigma$, such that $C' = \Sigma D C D'$ satisfies $C'_{ij} \leq 1$ and $C'_{ii} = 1$ for all $i, j \in [n]$. Then applying the injection $i$ to the matrices $D$, $D'$ and $\Sigma$, and using the fact that $i$ is a multiplicative morphism, we obtain a matrix $A' = i(\Sigma)(D)A(i)D'(\Sigma')$ such that $|A'| = C'$ and so satisfies the above properties. Since a diagonal scaling of $A$ does not change the balanced character of the determinant, nor the existence of a thin solution of $Ax \nabla 0$, we may always assume that $A = A'$. Thus $A$ satisfies:

$$|A_{i,j}| \leq 1, \quad |A_{i,i}| = 1, \quad \forall i, j \in [n].$$

Subcase 2.1: We shall first consider the subcase in which there is a permutation $\sigma$ such that

$$\bigcup_{i \in [n]} A_{\sigma(i)} \nabla 0 \text{ and per } |A| = |\det A| = \bigcup_{i \in [n]} |A_{\sigma(i)}| = \prod_{i \in [n]} |A_{\sigma(i)}|. \quad (6.2)$$

Assume, possibly after permuting the rows of $A$, that $\sigma$ is the identity permutation (this does not change Property 6.2). Since $\bigcup_{i \in [n]} A_{ii} \nabla 0$, by Proposition 6.7 we must have $A_{jj} \nabla 0$ for some $j \in [n]$ and we may always assume that $j = 1$. Then $A$ can be written in block form as

$$A = \begin{pmatrix} A_{11} & c \\ b & F \end{pmatrix},$$

We set $x_1 := 1$ and define $y := (x_2, \ldots, x_n)^T$ to be a thin solution of $b \oplus F y \nabla 0$ provided by Theorem 5.11. Thus $|y| = |F^{\oplus j} b| = |F| b$, and so, $|y| \leq 1$ for all $j$. Since $A_{11} \nabla 0$ and $|A_{11}| = 1$, it follows from Property 6.5 and 6.2 that $A_{11} \oplus cy \nabla 0$.

Hence, $x := (x_1, y_1, \ldots, y_{n-1})^T$ is a non-zero thin solution of $Ax \nabla 0$. 
Let us decompose $\pi$ Property 6.5 that $Ax$ and (6.5) and from Property 6.5 that $(G\zeta)_i = |Ax|, \zeta \in [n]$.

Let $z$ be a solution of the form $Ax$, and we observe that $z$ is a solution of the form $Ax$ with all the above properties, and such that $A_{i\pi} = 1$ for all $i \in [n]$. Thus

$$1 \oplus 1 = \prod_{i \in [n]} A_{i\pi} \nabla 0.$$  

Let us decompose $\pi$ as a product of disjoint cycles $c^1, \ldots, c^k$, with supports $I_1, \ldots, I_k$ of cardinalities $p_1, \ldots, p_k$, respectively. Then

$$1 \oplus \bigoplus_{m \in [k]} (\bigoplus_{i \in I_m} A_{i,c^m(i)} \bigoplus A_{i,c^m(i)} \nabla 0.$$  

It follows from (6.1) and Proposition 6.8 that there exists a cycle $c^m$ such that

$$1 \oplus (\bigoplus_{i \in I_m} A_{i,c^m(i)} \bigoplus A_{i,c^m(i)} \nabla 0.$$  

We may assume, without loss of generality, that $I_m = \{1, \ldots, p_m\}$, with $c^m(1) = 2, \ldots, c^m(p_m - 1) = p_m, c^m(p_m) = 1$. Then we define inductively the entries $z_{p_m}, \ldots, z_1$ of the vector $z \in (T^c)^{p_m}$ by

$$z_{p_m} = 1, \quad z_{p_m-1} = \bigoplus A_{p_m-1,p_m} z_{p_m} \nabla 0, \quad \ldots \quad z_1 = \bigoplus A_{1,2} z_2 \nabla 0.$$  

Since the permutation $\pi$ does not satisfy (6.2), but satisfies $|\bigcup_{i \in [n]} A_{i\pi}| = |Ax|$, the entries $A_{p_m-1,p_m}, \ldots, A_{1,2}$ are all invertible in $T$. Hence, the former relations define the vector $z$ uniquely. Actually,

$$z_{p_m} = 1, \quad z_{p_m-1} = \bigoplus A_{p_m-1,p_m} z_{p_m}, \quad \ldots, \quad z_1 = \bigoplus A_{1,2} z_2$$

and we observe that $|z_i| = 1$ for all $i \in [p_m]$. Moreover, from (6.3) and (6.4) we deduce that

$$z_{p_m} \oplus A_{p_m,1} z_1 \nabla 0.$$  

Let $G$ denote the $p_m \times p_m$ top-left submatrix of $A$. It follows from Formulas (6.4) and (6.5) and from Property 6.5 that $(G\zeta)_i \nabla 0$ holds for all $i \in [p_m]$.

Let us now write $A$ in the block form

$$A = \begin{pmatrix} G & * \\ V & F \end{pmatrix}$$

where $V$ and $F$ are of sizes $q \times p_m$ and $q \times q$, respectively, with $q := n - p_m$, and look for a solution of the form $x = (z_1, \ldots, z_{p_m}, y_1, \ldots, y_q)^\top$. Then we may choose for $y = (y_1, \ldots, y_q)^\top$ a thin vector solution of $V z \nabla 0$ given by Theorem 5.11. Thus $|y| = \|F^{\text{adj}} V z\| = |F|^* |V z|$. Hence, $|y_j| \leq 1$ for all $j \in [q]$. It follows from Property 6.5 that $Ax \nabla 0$.  \[\square\]
COROLLARY 6.10. Let $\mathcal{T}$ be a semiring allowing strong balance elimination (Definition 4.4) and the convergence of monotone algorithms (Definition 5.6), and satisfying Properties 6.2–6.5. Let $A \in \mathcal{M}_n(\mathcal{T})$ such that $det A$ is invertible and let $b \in \mathcal{T}^\circ$. Then the system $Ax \nabla b$ has a unique thin solution if and only if $A^{adj}b$ is thin.

PROOF. The sufficient condition follows from the second assertion of Theorem 4.18. For the necessary condition, let us assume that $A^{adj}b$ is not thin, and prove that there exist at least two different solutions. Since $A^{adj}b$ is not thin, there exists $i \in [n]$ such that $(A^{adj}b)_{i} \in \mathcal{T}^\circ \setminus \{0\}$. Assume without loss of generality that $i = n$.

From Theorem 5.11, since $det A \neq 0$, there exists a solution $x'$ of $Ax \nabla b$ such that $|x'| = |A^{adj}b|$, thus $|x'| = |(A^{adj}b)_{n}| \neq 0$ hence $x'_n \neq 0$. Now let us construct a solution $x'' \neq 0$ such that $x''_n = 0$. Let $B$ be the block matrix $B = (A_{[n]} \circ b)$. We have $det B = (A^{adj}b)_{n} \nabla 0$, so by Theorem 6.9 there exists $y \in (\mathcal{T}^\circ)^n \setminus \{0\}$ such that $By \nabla 0$. Let us show that $y_n \neq 0$. Indeed, if $y_n = 0$ then $y$ yields a thin solution of $Ax \nabla 0$, and since $det A$ is invertible, this implies that $y = 0$, a contradiction. So $y_n \neq 0$ and since $y_n \in \mathcal{T}^\circ$, $y_n$ is invertible. So multiplying $y$ by $y_n^{-1}$ we obtain a solution of $By \nabla 0$ such that $y_n = 1$. Then the vector $x'' = (y_1, \ldots, y_{n-1}, 0)^\top$ is a solution of $Ax \nabla b$ such that $x''_n = 0$. This shows that the system $Ax \nabla b$ has at least two non-zero solutions.

Remark 6.11. Proposition 8.8 of [AGG09] gives a $6 \times 7$ matrix with entries in $\mathbb{R}_{\max}$, such that there is no signed no-zero row vector $x$ such that $xA \nabla 0$, but all maximal determinants taken from $A$ are balanced. This shows that Theorem 6.9 cannot be extended to the rectangular case.

7. Systems of balances and intersections of signed hyperplanes

We now give a geometrical interpretation of the previous results. We first consider, as in Section 3 a semiring $\mathcal{S}$ with symmetry and a thin set $\mathcal{S}^\circ$. We call hyperplane of $\mathcal{S}^n$ a set of the form

$$H = \{ x \in (\mathcal{S}^\circ)^n \mid \bigoplus_{i \in [n]} a_i x_i \nabla 0 \}$$

where $a \in (\mathcal{S}^\circ)^n$ is a non-zero vector.

Example 7.1. When $\mathcal{S}$ is the bi-valued tropical semiring $\mathcal{T}_2$, $\mathcal{S}^\circ$ can be identified to $\mathbb{R}_{\max}$, then $H \cap \mathbb{R}^n$ coincides with the tropical hyperplane (1.1).

Example 7.2. Assume now that $\mathcal{S}$ is the symmetrized tropical semiring $\mathcal{S}_{\max}$, so that $a \in (\mathcal{S}_{\max}^\circ)^n$, with $\mathcal{S}_{\max}^\circ = \mathcal{S}_{\max}^0 \cup \mathcal{S}_{\max}^1$. Identifying $\mathcal{S}_{\max}^\circ$ with $\mathbb{R}_{\max}$, and setting $I := \{ i \in [n] \mid a_i \in \mathcal{S}_{\max}^0 \}$ and $J := [n] \setminus I$, it is readily seen that $H \cap \mathbb{R}_{\max}^n = H^{\text{sgn}}$ is the signed tropical hyperplane defined in (1.2).

The following result is a simple consequence of Theorem 4.18. We say that $n-1$ vectors $v^1, \ldots, v^{n-1}$ of $(\mathcal{S}^\circ)^n$ are in general position if every $(n-1) \times (n-1)$ minor of the $n \times (n-1)$ matrix $M$ with columns $v^1, \ldots, v^{n-1}$ is thin and non-zero ($\in (\mathcal{S}^\circ)^n$). Similarly, we say that $n-1$ hyperplanes of $\mathcal{S}^n$ are in general position if the vectors of parameters of these hyperplanes are in general position.

Theorem 7.3 (Geometric form of Cramer theorem). Let $\mathcal{S}$ be a semiring with a thin set $\mathcal{S}^\circ$, allowing strong balance elimination (Definition 4.4). Assume that $(\mathcal{S}^\circ)^*$ is the set of invertible elements of $\mathcal{S}$. Then

Primal. Any $n-1$ vectors of $(\mathcal{S}^\circ)^n$ in general position are contained in a unique hyperplane.
Dual. Any \( n - 1 \) hyperplanes of \((S^\lor)^n\) in general position contain a non-zero vector which is unique up to an invertible scalar multiple.

**Proof.** We prove the primal statement (the dual statement follows along the same lines). Assume that the vectors \( v^1, \ldots, v^{n-1} \) are included in the hyperplane \( H \) of \((7.1)\). Then the vector \( a \) of parameters of this hyperplane, thought of as a row vector, satisfies \( aM \nabla 0 \) where \( M \) is as above. Up to a transposition, and to the replacement of \( n \) by \( n + 1 \), this system is of the type considered in Corollary 4.19, and the conclusion follows from the latter corollary. \( \square \)

It follows from Examples 7.1 and 7.2 that Theorems 1.1 and 1.3 stated in the introduction can be re-obtained by specializing the primal form in Theorem 7.3 to \( S = T_2 \) or \( S = S_{\text{max}} \).

Similarly, Theorem 6.9 admits the following geometric interpretation. The derivation is straightforward.

**Theorem 7.4 (Singular matrices).** Let \( T \) be a semiring allowing weak balance elimination (Definition 4.4) and the convergence of monotone algorithms (Definition 5.6), and satisfying Properties 6.2–6.5. Then

Primal. A collection of \( n \) vectors \( v^1, \ldots, v^n \) of \( T^n \) is contained in a hyperplane if and only if the determinant of the matrix \((v^1, \ldots, v^n)\) is balanced;

Dual. A collection of \( n \) hyperplanes of \( T^n \)

\[ H^j = \{ x \in (T^\lor)^n | \bigoplus_{i \in [n]} a^j_i x_i \nabla 0 \}, \quad j \in [n] \],

contains a non-zero vector if and only if the determinant of the matrix \((a^j_i)_{i,j \in [n]}\) is balanced. \( \square \)

When \( S \) or \( T \) are equal to \( S_{\text{max}} \), the dual statements in Theorems 7.3 and 7.4 turn out to have a geometric interpretation which can be stated elementarily, without introducing the symmetrized tropical semiring.

This interpretation relies on the notion of sign-transformation of a signed hyperplane. Such a transformation is specified by a sign pattern \( \epsilon \in \{ \pm 1 \}^n \), it corresponds, in loose terms, to putting variables \( x_i \) such that \( \epsilon_i = -1 \) on the other side of the equality. Formally, the sign-transformation of pattern \( \epsilon \) transforms the signed-hyperplane \( H^{\text{sgn}} \) to

\[ H^{\text{sgn}}(\epsilon) = \{ x \in R^n_{\text{max}} | \max_{i \in I, \epsilon_i = 1 \text{ or } j \in J, \epsilon_j = -1} (a_i + x_i) = \max_{j \in J, \epsilon_j = 1 \text{ or } j \in I, \epsilon_j = -1} (a_j + x_j) \}. \]

Figure 1 gives an illustration of this notion. The following theorem follows readily from the dual statement in Theorem 7.3, it can also be derived from [Plu90, Th. 6.1].

\[ x_1 = \max(x_2, x_3) \quad x_2 = \max(x_1, x_3) \quad x_3 = \max(x_1, x_2) \]

**Figure 1.** Sign-transformation of a signed hyperplane
THEOREM 7.5. Given \( n - 1 \) signed tropical hyperplanes \( H_1^{\text{sgn}}, \ldots, H_{n-1}^{\text{sgn}} \), in general position, there is a unique sign pattern \( \epsilon \) such that the transformed hyperplanes \( H_1^{\text{sgn}}(\epsilon), \ldots, H_{n-1}^{\text{sgn}}(\epsilon) \), meet at a non-zero vector. Moreover, such a vector is unique up to an additive constant.

PROOF. Let \( H \) be the hyperplane defined by (7.1), and let \( H^{\text{sgn}} = H \cap \mathbb{R}_n^{\max} \) as in Example 7.2. It is easily seen that a vector \( x \in (S_n^{\max})^n \) belongs to \( H \) if and only if the vector \( |x| \) belongs to the transformed signed tropical hyperplane \( H^{\text{sgn}}(\epsilon) \) where \( \epsilon \) is the sign vector of \( x \). Then the theorem follows from the dual form of Theorem 7.3. \( \square \)

Figure 2. Illustration of the dual form of the Cramer theorem in the symmetrized tropical semiring (Theorem 7.5)

The interpretation of the dual form of Theorem 7.4, which could also be derived from the result of [Plu90, Th. 6.5] proved in [Gau92, Chap. 3, Th. 9.0.1], can be stated as follows.

THEOREM 7.6. Given \( n \) signed hyperplanes \( H_1^{\text{sgn}}, \ldots, H_n^{\text{sgn}} \), there exists a sign pattern \( \epsilon \) such that the transformed hyperplanes \( H_1^{\text{sgn}}(\epsilon), \ldots, H_n^{\text{sgn}}(\epsilon) \) contain a common non-zero vector if and only if the matrix having as rows the vectors of parameters of \( H_1^{\text{sgn}}, \ldots, H_n^{\text{sgn}} \) has a balanced determinant.

PROOF. Argue as in the proof of Theorem 7.5. \( \square \)

8. Computing all Cramer Permanents: tropical Jacobi versus transportation approach

8.1. Computing all Cramer permanents by the tropical Jacobi algorithm. The present approach via the tropical Jacobi algorithms leads to an algorithm to compute all the Cramer permanents.

COROLLARY 8.1 (Computing all the Cramer permanents). Let \( A \in \mathcal{M}_n(\mathbb{R}_{\max}) \) and \( b \in \mathbb{R}^n_{\max} \). Assume that \( \text{per} A \neq 0 \). Then the vector \( A^{\text{adj}}b \), the entries of which are the \( n \) Cramer permanents of the system with matrix \( A \) and right-hand side \( b \), can be computed by solving a single optimal assignment problem, followed by a multiple origins-single destination shortest path problem.

PROOF. Let \( \sigma \) denote an optimal permutation for the matrix \( A \). After permuting the rows of \( A \) and \( b \), we may assume that this permutation is the identity. Dividing every row of \( A \) and \( b \) by \( A_{ii} \), we may assume that \( A_{ii} = 1 \) for all \( i \in [n] \). Then using Yoeli’s theorem we get \( A^{\text{adj}}b = A^*b \). Computing the latter vector is equivalent to solving a shortest path problem from all origins to a single destination. \( \square \)
Remark 8.2. The Hungarian algorithm of Kuhn runs in time $O(n(m + n \log n))$, where $m$ is the number of finite entries of the matrix $A$. The subsequent shortest path problem can be solved for instance by the Ford-Bellman algorithm, in time $O(mn)$. Hence, we arrive at the strongly polynomial bound $O(n(m + n \log n)) \leq O(n^3)$, for the time needed to compute all Cramer permanents. Alternative (non-strongly polynomial) optimal assignment algorithms may be used [BDM09], leading to incomparable bounds.

8.2. The transportation approach of [RGST05]. Richter-Gebert, Sturmfels, and Theobald developed in [RGST05] a different approach, based on earlier results of Sturmfels and Zelevinsky [SZ93]. It provides an alternative method in which all the Cramer determinants are obtained by solving a single transportation problem. A beauty of this approach is that it directly works with homogeneous coordinates, preserving the symmetry which is broken by the Jacobi approach, of an “affine” nature. So, we next revisit the method of [RGST05], in order to compare the results obtained in this way with the present ones. In passing, we shall derive some refinements of results in [RGST05] concerning the case in which the data are not in general position. First, we observe that one of the results of [RGST05] can be recovered as a corollary of the present elimination approach.

Definition 8.3. Let $A \in M_{n-1,n}(\mathbb{R}_{\max})$. The tropical Cramer permanent $\text{per}_{A|_{k}}$ is the permanent of the matrix obtained from $A$ by deleting the $k$'th column.

Theorem 8.4 (Compare with [RGST05], Corollary 5.4). Assume that $A \in M_{n-1,n}(\mathbb{R}_{\max})$ is such that at least one of the tropical permanents $\text{per}_{A|_{k}}$ is finite. Then the vector $x = (x_k)$ with $x_k = \text{per}_{A|_{k}}$ is such that in the expression

$$Ax = \bigoplus_{k \in [n]} A_kx_k$$

the maximum is attained at least twice in every row. Moreover, if all the tropical Cramer permanents $\text{per}_{A|_{k}}$ are non-singular, then the vector $x$ having the latter property is unique up to an additive constant.

Proof. This is a special case of Theorem 5.15 in which the matrix $A$ is thin.

Corollary 5.4 of [RGST05] gives an explicit construction of the solution $x_k$ when the tropical Cramer permanents are non-singular. Then it proceeds by showing that the solution of a certain transportation problem is unique. Here the uniqueness of the solution of the tropical equations is obtained by the elimination argument used in the proof of Theorem 4.18. We now present the method of [RGST05] in some detail.

Let $T_{n-1,n}$ denote the transportation polytope consisting of all nonnegative $(n-1) \times n$ matrices $y = (y_{ij})$ such that

$$\sum_{j \in [n]} y_{ij} = n, \quad i \in [n-1], \quad \sum_{i \in [n-1]} y_{ij} = n-1, \quad j \in [n].$$

For $k \in [n]$ define $\Pi^k_{n-1,n}$ to be the Birkhoff polytope obtained as the convex hull of the $(n-1) \times n$ matrices with $0, 1$-entries, representing matchings between $[n-1]$ and $[n] \setminus \{k\}$. Define the Minkowski sum:

$$\Pi_{n-1,n} := \bigoplus_{k \in [n]} \Pi^k_{n-1,n}.$$ 

Here (partial) matchings refer to the complete bipartite graph $K_{n-1,n}$ with $n-1$ nodes in one class (corresponding to the rows of $y$) and $n$ nodes in the other class.
(corresponding to the columns of $y$). Every vertex of $\Pi_{n-1,n}$ can be written as $y = y^1 + \cdots + y^n$, where for all $k \in [n]$, $y^k \in \mathcal{M}_{n-1,n}([0,1])$ represents a matching between $[n-1]$ and $[n] \setminus \{k\}$, meaning that the set of edges $\{(i,j) \mid y^k_{ij} = 1\}$ constitutes a matching between $[n-1]$ and $[n] \setminus \{k\}$.

More generally, Sturmfels and Zelevinsky [SZ93] considered the Newton polytope $\Pi_{m,n}$ of the product of maximal minors of any $m \times n$ matrix such that $m \leq n$. The entries of this matrix are thought of as pairwise distinct indeterminates. The former polytope is obtained by taking $m = n - 1$. They showed that $T_{n-1,n} = \Pi_{n-1,n}$, see [SZ93] Th. 2.8, but that $\Pi_{m,n}$ is no longer a transportation polytope when $m < n - 1$. They also showed that the vertices of $\Pi_{n-1,n}$ are in bijective correspondence with combinatorial objects called linkage trees. A linkage tree is a tree with set of nodes $[n]$, the edges of which are bijectively labeled by the integers $1, \ldots, n - 1$. Given a linkage tree, we associate to every $k \in [n]$ the matching between $[n-1]$ and $[n] \setminus \{k\}$, such that $j \in [n] \setminus \{k\}$ is matched to the unique $i \in [n-1]$ labeling the edge adjacent to $j$ in the path connecting $j$ to $k$ in this tree. Let $y^k$ denote the matrix representing this matching. Then the vertex of $\Pi_{n-1,n}$ that corresponds to this linkage tree is $y^1 + \cdots + y^n$.

We now associate to every $(n-1) \times n$ matrix $y$ a subgraph $G(y)$ of $K_{n-1,n}$, consisting of the edges $(i,j)$ such that $y_{ij} \neq 0$. If $y$ is a vertex of $\Pi_{n-1,n}$, then the previous characterization in terms of linkage trees implies that $G(y)$ is a spanning tree of $K_{n-1,n}$.

To simplify the exposition, we shall assume first that $c_{ij} \in \mathbb{R}$, $i \in [n-1]$, $j \in [n]$. Following the idea of Richter-Gebert, Sturmfels, and Theobald [RGST05], we consider the transportation problem $\mathcal{P}$

$$\begin{align*}
(\mathcal{P}) & \quad \max \sum_{i \in [n-1], j \in [n]} c_{ij}y_{ij}; \quad y = (y_{ij}) \in T_{n-1,n}.
\end{align*}$$

and its dual

$$\begin{align*}
(\mathcal{D}) & \quad \min \left( n \left( \sum_{i \in [n-1]} u_i \right) + (n-1) \left( \sum_{j \in [n]} v_j \right) \right); \quad u = (u_i) \in \mathbb{R}^{n-1}, v = (v_j) \in \mathbb{R}^n,
\end{align*}$$

$$\begin{align*}
& \quad c_{ij} \leq u_i + v_j, \quad i \in [n-1], \ j \in [n].
\end{align*}$$

The values of these problems will be denoted by $\text{val} \mathcal{P}$ and $\text{val} \mathcal{D}$, respectively.

Recall the complementary slackness condition: a primal feasible solution $y$ and a dual feasible solution $(u,v)$ are both optimal if and only if $y_{ij}(u_i + v_j - c_{ij}) = 0$ holds for all $i \in [n-1]$ and $j \in [n]$.

The case in which $c_{ij} = -\infty$ holds for some $i, j$ can be dealt with by adopting the convention that $(-\infty) \times 0 = 0$ in the expression of the primal objective function. Equivalently, it can be considered as an ordinary linear programming problem by adding the constraints $y_{ij} = 0$ for all $(i,j)$ such that $c_{ij} = -\infty$, and ignoring all $(i,j)$ such that $c_{ij} = -\infty$ in the formulation of the dual problem and in the complementary slackness conditions.

We now give a slight refinement of a result of [RGST05], allowing one to interpret the optimal dual variable in terms of tropical Cramer determinants. Corollary 5.4 of [RGST05] only deals with the case where the matrix $C$ is generic, whereas Theorem 8.5 shows that this assumption is not needed for the optimal solution of the dual problem $(\mathcal{D})$ to be unique, up to a transformation by an additive constant. Genericity is only needed for the uniqueness of the optimal solution of the primal problem, which we do not use here.

**Theorem 8.5 (Compare with [RGST05], Corollary 5.4).** The primal problem $(\mathcal{P})$ is feasible if and only if all the tropical Cramer permanents of the matrix $C$ are finite. When this is the case, the optimal solution $(u,v) \in \mathbb{R}^{n-1} \times \mathbb{R}^n$ of the dual
transportation problem \((\mathcal{D})\) is unique up to a modification of the vector \((u, -v)\) by an additive constant, and

\[
\text{per} \ C_{(k)} = \sum_{i \in [n-1]} u_i + \sum_{j \in [n] \setminus \{k\}} v_j , \quad \text{for all } k \in [n].
\]

**Proof.** We first consider the case in which all \(c_{ij}\) are finite. As the initial step, we show the announced uniqueness result for the optimal solution \((u, v)\) of the dual problem.

Let \(y\) denote an optimal solution of the primal problem, which we choose to be a vertex of \(T_{n-1,n} = \Pi_{n-1,n}\), so that \(G(y)\) is a spanning tree of \(K_{n-1,n}\). Using the complementary slackness condition, we have

\[
c_{ij} = u_i + v_j \quad \text{for all } (i, j) \in G(y).
\]

Using these relations and the fact that a spanning tree is connected, we see that all the values of the variables \(u_i\) and \(v_j\), \(i \in [n-1]\) and \(j \in [n]\), are uniquely determined by the value of a single variable \(u_\ell, \ell \in [n-1]\) (or dually, by the value of a single variable \(v_k, k \in [n]\)). Moreover, an increase of \(u_\ell\) by a constant increases every other variable \(u_i\) by the same constant and decreases every variable \(v_j\) by the same constant. This establishes the announced uniqueness result.

Since \(c_{ij} \leq u_i + v_j\) holds for all \(i \in [n-1]\) and \(j \in [n]\), we deduce that for all \(k \in [n]\) and for all bijections \(\sigma\) from \([n-1]\) to \([n] \setminus \{k\}\)

\[
\sum_{i \in [n-1]} c_{i\sigma(i)} \leq \sum_{i \in [n-1]} u_i + \sum_{j \in [n] \setminus \{k\}} v_j .
\]

(8.3) Considering the maximum over all such bijections \(\sigma\), we deduce that per \(C_{(k)}\) is bounded from above by the right-hand side of the latter inequality.

Now, using again the fact \(T_{n-1,n} = \Pi_{n-1,n}\), we can write \(y = y^1 + \cdots + y^n\) where every \(y^k\) represents a matching between \([n-1]\) and \([n] \setminus \{k\}\), to which we associate a bijection \(\sigma^k\) from \([n-1]\) to \([n] \setminus \{k\}\). By the complementary slackness condition we have \(c_{i\sigma^k(i)} = u_i + v_{\sigma^k(i)}\) for all \(i \in [n-1]\). It follows that \(\sigma = \sigma^k\) achieves the equality in (8.3), showing that per \(C_{(k)}\) is given by the right-hand side of 8.3.

We now deal with the case in which some coefficients \(c_{ij}\) can take the \(-\infty\) value. The reward function \(y \rightarrow \sum_{i \in [n-1]} c_{i\sigma(i)} y_{j\sigma(i)}\) over \(T_{n-1,n}\), with the convention \((-\infty) \times 0 = 0\), now takes its value in \(\mathbb{R} \cup \{-\infty\}\). It is upper semicontinuous and concave, so it attains its maximum at a vertex of \(T_{n-1,n} = \Pi_{n-1,n}\). It follows that the value of the primal problem is finite if and only if every tropical Cramer permanent is finite. Then the remaining arguments of the proof above are easily checked to carry over, by working with the modified linear programming formulation, in which the constraints \(y_{ij} = 0\) for all \((i, j)\) such that \(c_{ij} = -\infty\), are added. \(\square\)

9. Computing determinants

To compute determinants over \(\mathbb{T}_2\) or \(\mathbb{S}_{\max}\), we need to recall how tropical singularity can be checked. Let \(A \in \mathcal{M}_n(\mathbb{R}_{\max})\). We observed in Proposition 3.10 that, as soon as per \(A \neq 0\), the Hungarian algorithm gives scalars \(u_i, v_j \in \mathbb{R}\), for \(i, j \in [n]\), such that

\[
a_{ij} \leq u_i v_j , \quad \text{and} \quad \text{per} \ A = \prod_i u_i \prod_j v_j.
\]

The optimal permutations \(\sigma\) are characterized by the condition that \(a_{i\sigma(i)} = u_i v_{\sigma(i)}\), for all \(i \in [n]\). After multiplying \(A\) by a permutation matrix, we may always assume that the identity is a solution of the optimal assignment problem. Then we define
the digraph \(G\) with nodes 1,\ldots,n, and an arc from \(i\) to \(j\) whenever \(a_{ij} = u_{ij}\). Butković proved two results which can be formulated equivalently as follows.

**Theorem 9.1** (See [But94] and [But95]). Let \(A \in \mathcal{M}_n(\mathbb{R}_{\text{max}})\), and assume that \(\det A \neq 0\). Then checking whether the optimal assignment problem has at least two optimal solutions reduces to finding a cycle in the digraph \(G\), whereas checking whether \(G\) has at least two optimal solutions of a different parity reduces to finding an even cycle in \(G\).

By exploiting the proof technique of Theorem 9.1 we obtain the following result.

**Corollary 9.2.** Assume, \(A \in \mathcal{M}_n(\mathbb{T}_2)\) or \(A \in \mathcal{M}_n(\mathbb{S}_{\text{max}})\). Then the determinant \(\det A\) can be computed in polynomial time.

**Proof.** Assume first that \(A \in \mathcal{M}_n(\mathbb{T}_2)\). We set \(B = |A|\), meaning that \(B_{ij} = |A_{ij}|\) for all \(i, j\), and compute \(B\) together with a permutation \(\sigma\) solving the optimal assignment problem for the matrix \(B\). If \(\det B = 0\), then \(\det A = 0\), so we next assume that \(\det B \neq 0\). After multiplying \(B\) by the inverse of the matrix of the optimal permutation \(\sigma\), we may assume that this permutation is the identity. If one of the diagonal coefficients of \(B\) belongs to \(\mathbb{T}_2\), then we conclude that \(\det A = (\det B)^\circ\). Otherwise, we define the digraph \(G\) as above, starting from the matrix \(B\) instead of \(A\). If \(G\) has a cycle, then this cycle can be completed by loops (cycles of length one) to obtain an optimal permutation for \(B\) distinct from the identity and so \(\det A = (\det B)^\circ\). Otherwise, \(\det A = \det B\).

To compute \(\det A\) when \(A \in \mathcal{M}_n(\mathbb{S}_{\text{max}})\), we consider firstly the case \(A \in \mathcal{M}_n(\mathbb{R}_{\text{max}})\), where \(\mathbb{R}_{\text{max}}\) is thought of as a subsemiring of \(\mathbb{S}_{\text{max}}\). We define again \(B = |A|\) and assume that the identity is an optimal permutation for the matrix \(B\) and we define \(G\) as above. Then \(\det A = \det B\) if \(G\) has no even cycle. Otherwise \(\det A = (\det B)^\circ\) since any even cycle can be completed by loops to get an optimal permutation for \(B\) of odd parity.

We assume finally that \(A \in \mathcal{M}_n(\mathbb{S}_{\text{max}})\) and make the same assumptions and constructions. If one diagonal entry of \(A\) is in \(\mathbb{S}_{\text{max}}\), we conclude that \(\det A = (\det B)^\circ\). Otherwise, for every \((i, j)\) in \(G\) such that \(a_{ij} \in \mathbb{S}_{\text{max}}\), we check whether \((i, j)\) belongs to a cycle of \(G\) (not necessarily an even one). If this is the case, we conclude that \(\det A = (\det B)^\circ\). Otherwise, the elements \(a_{ij}\) such that \(a_{ij} \in \mathbb{S}_{\text{max}}\) do not contribute to the optimal permutations of the matrix \(B\), and we may replace them by 0 without changing the value of \(\det A\). Then we may write \(A = A^+ \oplus A^-\) where \(A^+, A^- \in \mathcal{M}_n(\mathbb{R}_{\text{max}})\). Recall that for all matrices \(C, D\) with entries in a commutative ring the following block formula for the determinant holds

\[
\begin{vmatrix}
C & D \\
I & I
\end{vmatrix} = \det(C - D),
\]

where \(I\) is the identity matrix. Moreover, when viewing the entries of \(C\) and \(D\) as independent indeterminates and expanding the expressions at left and at right of the latter equality, we see that the same monomials appear (each with multiplicity one) on each side of the equality. Therefore, the equality is valid over an arbitrary symmetrized semiring. In particular,

\[
\det A = \begin{vmatrix}
A^+ & A^- \\
I & I
\end{vmatrix},
\]

which reduces the computation of \(\det A\) to the computation of the determinant of a matrix with entries in \(\mathbb{R}_{\text{max}}\), which is already solved thanks to Theorem 9.1
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