A note on indecomposable sets of finite perimeter

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Abstract

Boncatto–Pasqualetto–Rajala [6] proved that a decomposition theorem for sets of finite perimeter into indecomposable sets, known to hold in Euclidean spaces, holds also in complete metric spaces equipped with a doubling measure, supporting a Poincaré inequality, and satisfying an isotropicity condition. We show that the last assumption can be removed.

1 Introduction

A set of finite perimeter $E$ in a metric measure space $X$ is said to be indecomposable if it cannot be written as the disjoint union of two non-negligible sets $F, G$ with $P(E, X) = P(F, X) + P(G, X)$. This measure-theoretic notion is similar to the topological notion of connectedness. Properties of indecomposable sets in Euclidean spaces were studied by Ambrosio et al. [2]; in particular, they proved that a set of finite perimeter can always be uniquely decomposed into indecomposable sets.

The theory was generalized to metric measure spaces by Boncatto–Pasqualetto–Rajala [6]. As is common in analysis on metric measure spaces, they assumed the space $(X, d, m)$ to be complete, equipped with a doubling measure, and support a $(1, 1)$-Poincaré inequality. Such a space is called a PI space; we will give definitions in Section 2. Additionally, they assumed that the representation of perimeter by means of the Hausdorff measure satisfies an isotropicity condition. This condition was previously considered e.g. in [4, Section 7] and it is satisfied in Euclidean as well as various other PI spaces. However, it excludes some PI spaces from the theory, see [6, Example 1.27].

The proofs in [6] relied heavily on the isotropicity condition, and the condition was even shown to be necessary for certain results in the theory, but its necessity for the main decomposition theorem remained unclear. In the present paper, we show that the isotropicity assumption can be removed. On the other hand, the assumption of a $(1, 1)$-Poincaré inequality cannot be removed, see [6, Example 2.16].

The following decomposition theorem is [6, Theorem 2.14], except that there isotropicity was assumed.

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Theorem 1.1. Let \((X, d, m)\) be a PI space. Let \(E \subset X\) be a set of finite perimeter. Then there exists a unique (finite or countable) partition \(\{E_i\}_{i \in I}\) of \(E\) into indecomposable subsets of \(X\) such that \(m(E_i) > 0\) for every \(i \in I\) and \(P(E, X) = \sum_{i \in I} P(E_i, X)\), where uniqueness is in the \(m\)-a.e. sense. Moreover, the sets \(\{E_i\}_{i \in I}\) are maximal indecomposable sets, meaning that for any Borel set \(F \subset E\) with \(P(F, X) < \infty\) that is indecomposable there is a (unique) \(i \in I\) such that \(m(F \setminus E_i) = 0\).

First in Section 3 we prove some results on sets of finite perimeter using only elementary methods, mostly involving basic properties of upper gradients in metric spaces. These results may be also of some independent interest.

Then in Section 4 we show how the proofs in [6] can be modified to obtain the decomposition theorem (Theorem 1.1) without the isotropicity assumption. We also give a similar modification for some results of [6] concerning holes. Other results of [6], especially certain ones concerning simple sets, are shown there to be false unless one assumes isotropicity or even a stronger two-sided property. Thus we will not discuss these parts of the theory.

The presentation of Section 3 is essentially self-contained, whereas Section 4 largely consists of describing modifications to [6], so the interested reader is advised to read that paper first.

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2 Notation and definitions

In this section we introduce the basic notation, definitions, and assumptions that are employed in the paper.

Throughout this paper, \((X, d, m)\) is a metric space that is equipped with a metric \(d\) and a Borel regular outer measure \(m\). When a property holds outside a set of \(m\)-measure zero, we say that it holds almost everywhere, abbreviated a.e. We say that \(m\) satisfies a doubling property if there exists a constant \(C_d \geq 1\) such that

\[
0 < m(B(x, 2r)) \leq C_d m(B(x, r)) < \infty
\]

for every ball \(B(x, r) := \{y \in X : d(y, x) < r\}\), \(r > 0\); we understand balls to be open. We will not always assume the doubling property but we will always assume that \(0 < m(B) < \infty\) for every ball \(B\).

We assume \(X\) to be proper, meaning that closed and bounded sets are compact. All functions defined on \(X\) or its subsets will take values in \([-\infty, \infty]\). Given an open set \(W \subset X\), we define \(L^1_{\text{loc}}(W)\) to be the class of functions \(u\) on \(W\) such that \(u \in L^1(W')\) for every open \(W' \subset W\), where the latter notation means that \(W'\) is a compact subset of \(W\). Other local spaces of functions are defined analogously.

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into \(X\). The length of a curve \(\gamma\) is denoted by \(\ell_\gamma\). We assume every curve to be parametrized by arc-length, which can always be done (see e.g.
A nonnegative Borel function $g$ on $X$ is an upper gradient of a function $u$ on $X$ if for all nonconstant curves $\gamma$, we have

$$\vert u(x) - u(y) \vert \leq \int_0^\ell_{\gamma} g(\gamma(s)) \, ds,$$

where $x$ and $y$ are the end points of $\gamma$. We interpret $\vert u(x) - u(y) \vert = \infty$ whenever at least one of $\vert u(x) \vert$, $\vert u(y) \vert$ is infinite. Upper gradients were originally introduced in [9].

The 1-modulus of a family of curves $\Gamma$ is defined by

$$\text{Mod}_1(\Gamma) := \inf \int_X \rho \, dm,$$

where the infimum is taken over all nonnegative Borel functions $\rho$ such that $\int_{\gamma} \rho \, ds \geq 1$ for every curve $\gamma \in \Gamma$. A property is said to hold for 1-almost every curve if it fails only for a curve family with zero 1-modulus. If $g$ is a nonnegative $m$-measurable function on $X$ and (2.2) holds for 1-almost every curve, we say that $g$ is a 1-weak upper gradient of $u$.

By only considering curves $\gamma$ in a set $A \subset X$, we can talk about a function $g$ being a (1-weak) upper gradient of $u$ in $A$.

We say that $X$ supports a (1,1)-Poincaré inequality if there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball $B(x,r)$, every $u \in L^1_{\text{loc}}(X)$, and every upper gradient $g$ of $u$, we have

$$\int_{B(x,r)} \vert u - u_{B(x,r)} \vert \, dm \leq C_P \rho \int_{B(x,\lambda r)} g \, dm,$$

where

$$u_{B(x,r)} := \int_{B(x,r)} u \, dm := \frac{1}{m(B(x,r))} \int_{B(x,r)} u \, dm.$$

If $X$ is complete, the measure $m$ satisfies the doubling condition (2.1), and the space supports a (1,1)-Poincaré inequality, then we say that $X = (X, d, m)$ is a PI space.

Let $W \subset X$ be an open set. We let

$$\|u\|_{N^{1,1}(W)} := \|u\|_{L^1(W)} + \inf \|g\|_{L^1(W)},$$

where the infimum is taken over all 1-weak upper gradients $g$ of $u$ in $W$. Then we define the Newton-Sobolev space

$$N^{1,1}(W) := \{ u : \|u\|_{N^{1,1}(W)} < \infty \},$$

which was first introduced in [13].

We understand Newton-Sobolev functions to be defined at every $x \in W$ (even though $\| \cdot \|_{N^{1,1}(W)}$ is then only a seminorm). It is known that for any $u \in N^{1,1}_{\text{loc}}(W)$ there exists a minimal 1-weak upper gradient of $u$ in $W$, always denoted by $g_u$, satisfying $g_u \leq g$ a.e. in $W$ for every 1-weak upper gradient $g \in L^1_{\text{loc}}(W)$ of $u$ in $W$, see e.g. the monograph Björn–Björn [5, Theorem 2.25].
Let \( u, v \in N^{1,1}_{\text{loc}}(W) \). Then we have \( u + v, \min\{u, v\}, \max\{u, v\} \in N^{1,1}_{\text{loc}}(W) \), see [5, Theorem 1.20]. It is easy to see that the minimal 1-weak upper gradients satisfy
\[
g_{u+v} \leq g_u + g_v \quad \text{a.e.} \quad (2.3)
\]
Also,
\[
g_u = g_v \quad \text{a.e. in } \{x \in W : u(x) = v(x)\} \quad (2.4)
\]
by [5, Corollary 2.21]. For the above set, we use the short-hand notation \( \{u = v\} \).

It follows that
\[
g_{\max\{u,v\}} = g_u \chi_{\{u>v\}} + g_v \chi_{\{u\leq v\}}. \quad (2.5)
\]
It also follows that
\[
g_{\min\{u,v\}} + g_{\max\{u,v\}} = g_u \chi_{\{u\leq v\}} + g_v \chi_{\{u>v\}} + g_u \chi_{\{u>v\}} + g_v \chi_{\{u\leq v\}}
= g_u + g_v \quad \text{a.e.} \quad (2.6)
\]

If \( V \subset W \) are open subsets of \( X \) and \( u \in N^{1,1}_{\text{loc}}(W) \), then also \( u \in N^{1,1}_{\text{loc}}(V) \), and we denote the minimal 1-weak upper gradient of \( u \) in \( V \) by \( g_{u,V} \). Note that usually we denote briefly \( g_u = g_{u,W} \). Now by [5, Lemma 2.23] we have
\[
g_{u,V} = g_{u,W} \quad \text{a.e. in } V. \quad (2.7)
\]

Finally we note that \( \text{Lip}_{\text{loc}}(W) \subset N^{1,1}_{\text{loc}}(W) \), see e.g. [5, Proposition 1.14].

Next we present the definition and basic properties of functions of bounded variation on metric spaces, following [12]. See also e.g. [3, 7] for the classical theory in the Euclidean setting. Given an open set \( W \subset X \) and a function \( u \in L^1_{\text{loc}}(W) \), we define the total variation of \( u \) in \( W \) by
\[
\|Du\|(W) := \inf \left\{ \liminf_{i \to \infty} \int_W g_{u_i} \, dm : u_i \in \text{Lip}_{\text{loc}}(W), u_i \to u \text{ in } L^1_{\text{loc}}(W) \right\},
\]
where each \( g_{u_i} \) is again the minimal 1-weak upper gradient of \( u_i \) in \( W \). (In [12], pointwise Lipschitz constants were used in place of upper gradients, but the theory can be developed similarly with either definition.) We say that a function \( u \in L^1(W) \) is of bounded variation, and denote \( u \in \text{BV}(W) \), if \( \|Du\|(W) < \infty \).

From the definition it follows easily that if \( \{u_i\}_{i=1}^{\infty} \) is a sequence for which \( u_i \to u \) in \( L^1_{\text{loc}}(W) \) as \( i \to \infty \), then
\[
\|Du\|(W) \leq \liminf_{i \to \infty} \|Du_i\|(W). \quad (2.8)
\]

For an arbitrary set \( A \subset X \), we define
\[
\|Du\|(A) := \inf \{ \|Du\|(W) : A \subset W, W \subset X \text{ is open} \}. \quad (2.9)
\]

In general, we understand the expression \( \|Du\|(A) < \infty \) to mean that there exists some open set \( W \supset A \) such that \( u \) is defined in \( W \) with \( u \in L^1_{\text{loc}}(W) \) and \( \|Du\|(W) < \infty \).
If $u \in L^1_{\text{loc}}(W)$ and $\|Du\|(W) < \infty$, then $\|Du\|(\cdot)$ is a Borel regular outer measure on $W$ by [12, Theorem 3.4]. A set $E \subset X$ is said to have finite perimeter in $W$ if $\|D\chi_E\|(W) < \infty$, where $\chi_E$ is the characteristic function of $E$. The perimeter of $E$ in $W$ is also denoted by

$$P(E, W) := \|D\chi_E\|(W).$$

If $P(E, X) < \infty$, we say briefly that $E$ is a set of finite perimeter.

For any set $A \subset X$ and $0 < R < \infty$, the restricted Hausdorff content of codimension one is defined by

$$H_R(A) := \inf \left\{ \sum_{j \in I} \frac{m(B(x_j, r_j))}{r_j} : A \subset \bigcup_{j \in I} B(x_j, r_j), r_j \leq R \right\},$$

where the infimum is taken over finite and countable index sets $I$. The codimension one Hausdorff measure of $A \subset X$ is then defined by

$$\mathcal{H}(A) := \lim_{R \to 0} H_R(A).$$

For any $E \subset X$, the measure-theoretic boundary $\partial^* E$ is defined as the set of points $x \in X$ at which both $E$ and its complement have strictly positive upper density, i.e.

$$\limsup_{r \to 0} \frac{m(B(x, r) \cap E)}{m(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{m(B(x, r) \setminus E)}{m(B(x, r))} > 0.$$

Suppose $(X, d, m)$ is a PI space. For an open set $W \subset X$ and a $m$-measurable set $E \subset X$ with $P(E, W) < \infty$, for any Borel set $A \subset W$ we have

$$P(E, A) = \int_{\partial^* E \cap A} \theta_E \, d\mathcal{H},$$

where $\theta_E : \partial^* E \cap W \to [\alpha, C_d]$ with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [4, Theorem 4.6].

**Definition 2.11.** Let $(X, d, m)$ be a PI space. We say that $(X, d, m)$ is isotropic if for any pair of sets $E, F \subset X$ of finite perimeter with $F \subset E$ it holds that

$$\theta_F(x) = \theta_E(x) \quad \text{for} \ \mathcal{H}\text{-a.e.} \ x \in \partial^* F \cap \partial^* E.$$  

We will never assume isotropicity, but we give the definition for comparison with [6].

In some of our results, we will assume $(X, d, m)$ to be a PI space, but our standing assumptions are merely the following.

*Throughout this paper, $(X, d, m)$ is a proper metric space equipped with the Borel regular outer measure $m$, such that $0 < m(B) < \infty$ for every ball $B$.***
3 Main tools

In this section we prove the following two propositions. These are the main tools that will be needed in order to prove the decomposition theorem without the assumption of the space being isotropic. In these propositions, we do not even need to assume \((X, d, m)\) to be a PI space, though in the decomposition theorem this will be necessary.

**Proposition 3.1.** Let \(F, G \subset X\) with \(m(F \cap G) = 0\), and let \(D \subset X\) be an arbitrary set such that
\[
P(F \cup G, D) = P(F, D) + P(G, D) < \infty.
\]
Let \(F' \subset F\) and \(G' \subset G\) be two other sets with \(P(F', D) + P(G', D) < \infty\). Then
\[
P(F' \cup G', D) = P(F', D) + P(G', D).
\]

**Proposition 3.2.** Let \(W' \subset W \subset X\) be two open sets. Suppose \(F, G \subset X\) with \(m(F \cap G) = 0\) and
\[
P(F \cup G, W) = P(F, W) + P(G, W) < \infty.
\]
Then also
\[
P(F \cup G, W') = P(F, W') + P(G, W').
\]

We start by reciting the following well known fact and its proof.

**Lemma 3.3.** Let \(F, G \subset X\) and \(D \subset X\) be arbitrary sets. Then
\[
P(F \cap G, D) + P(F \cup G, D) \leq P(F, D) + P(G, D).
\]

*Proof.* We can assume that the right-hand side is finite. Let \(\varepsilon > 0\). By the definition of the total variation \((2.9)\), we find an open set \(W \supset D\) such that
\[
P(F, W) < P(F, D) + \varepsilon \quad \text{and} \quad P(G, W) < P(G, D) + \varepsilon. \tag{3.4}
\]
Take sequences of functions \(\{u_i\}_{i=1}^{\infty} \subset \text{Lip}_{loc}(W)\) such that \(u_i \to \chi_F\) and \(v_i \to \chi_G\) in \(L^1_{loc}(W)\), and
\[
\lim_{i \to \infty} \int_W g u_i \, dm = P(F, W) \quad \text{and} \quad \lim_{i \to \infty} \int_W g v_i \, dm = P(G, W).
\]
We also have \(\min\{u_i, v_i\} \to \chi_{F \cap G}\) and \(\max\{u_i, v_i\} \to \chi_{F \cup G}\) in \(L^1_{loc}(W)\). Thus we
get
\[ P(F \cap G, D) + P(F \cup G, D) \]
\[ \leq P(F \cap G, W) + P(F \cup G, W) \]
\[ \leq \liminf_{i \to \infty} \int_{W} g_{\min\{u_{i}, v_{i}\}} \, dm + \liminf_{i \to \infty} \int_{W} g_{\max\{u_{i}, v_{i}\}} \, dm \]
\[ \leq \liminf_{i \to \infty} \left( \int_{W} g_{\min\{u_{i}, v_{i}\}} \, dm + \int_{W} g_{\max\{u_{i}, v_{i}\}} \, dm \right) \]
\[ = \liminf_{i \to \infty} \left( \int_{W} u_{i} \, dm + \int_{W} v_{i} \, dm \right) \text{ by (2.6)} \]
\[ = P(F, W) + P(G, W) \]
\[ \leq P(F, D) + P(G, D) + 2 \varepsilon. \]

Letting \( \varepsilon \to 0 \), we get the result. \( \square \)

For any sequence of \( m \)-measurable sets \( \{E_{i}\}_{i=1}^{\infty} \), we have that \( \chi_{\bigcup_{i=1}^{j} E_{i}} \to \chi_{\bigcup_{i=1}^{\infty} E_{i}} \) in \( L_{1}^{1}(X) \) as \( j \to \infty \), and so by using (2.8) and then Lemma 3.3, we get
\[ P\left( \bigcup_{i=1}^{\infty} E_{i}, X \right) \leq \liminf_{j \to \infty} P\left( \bigcup_{i=1}^{j} E_{i}, X \right) \leq \sum_{i=1}^{\infty} P(E_{i}, X). \quad (3.5) \]

Next we give the following result concerning approximating sequences that are “almost optimal” for a given set of finite perimeter.

**Lemma 3.6.** Let \( E' \subset E \subset X \) be two sets that have finite perimeter in an open set \( W \subset X \). Let \( \varepsilon \geq 0 \) and let \( \{w_{i}\}_{i=1}^{\infty} \) be a sequence in \( \text{Lip}_{\text{loc}}(W) \) such that \( w_{i} \to \chi_{E} \) in \( L_{1}^{1}(W) \) and
\[ \liminf_{i \to \infty} \int_{W} g_{w_{i}} \, dm \leq P(E, W) + \varepsilon. \]

Then there exists a sequence \( \{w'_{i}\}_{i=1}^{\infty} \) in \( \text{Lip}_{\text{loc}}(W) \) such that \( w'_{i} \leq w_{i}, w'_{i} \to \chi_{E'} \) in \( L_{1}^{1}(W) \), and
\[ \liminf_{i \to \infty} \int_{W} g_{w'_{i}} \, dm \leq P(E', W) + \varepsilon. \]

**Proof.** We choose a sequence \( \{v_{i}\}_{i=1}^{\infty} \) in \( \text{Lip}_{\text{loc}}(W) \) such that \( v_{i} \to \chi_{E'} \) in \( L_{1}^{1}(W) \) and
\[ \lim_{i \to \infty} \int_{W} g_{v_{i}} \, dm = P(E', W). \quad (3.7) \]

Note that
\[ \max\{w_{i}, v_{i}\} \to \chi_{E} \text{ in } L_{1}^{1}(W), \]
and so
\[ P(E, W) \leq \liminf_{i \to \infty} \int_{W} g_{\max\{w_{i}, v_{i}\}} \, dm. \quad (3.8) \]
By assumption and by (3.7), we have
\[ P(E, W) + P(E', W) \]
\[ \geq \liminf_{i \to \infty} \left[ \int_W g_{w_i} \, dm + \int_W g_{v_i} \, dm \right] - \varepsilon \]
\[ = \liminf_{i \to \infty} \left[ \int_W g_{\min\{w_i, v_i\}} \, dm + \int_W g_{\max\{w_i, v_i\}} \, dm \right] - \varepsilon \quad \text{by (2.6)} \]
\[ \geq \liminf_{i \to \infty} \int_W g_{\min\{w_i, v_i\}} \, dm + P(E, W) - \varepsilon \quad \text{by (3.8)} \]

It follows that
\[ P(E', W) \geq \liminf_{i \to \infty} \int_W g_{\min\{w_i, v_i\}} \, dm - \varepsilon. \]

Now \( w'_i := \min\{w_i, v_i\} \to \chi_{E'} \) in \( L^1_{\text{loc}}(W) \), giving the result.

Proposition 3.1 will follow almost directly from the following version that considers open sets \( W \subset X \).

**Lemma 3.9.** Let \( W \subset X \) be an open set, let \( F, G \subset X \) have finite perimeter in \( W \) with \( m(F \cap G) = 0 \), let \( \varepsilon \geq 0 \), and suppose that
\[ P(F \cup G, W) + \varepsilon \geq P(F, W) + P(G, W). \]

Let \( F' \subset F \) and \( G' \subset G \) be two other sets with finite perimeter in \( W \). Then also
\[ P(F' \cup G', W) + 2\varepsilon \geq P(F', W) + P(G', W). \]

**Proof.** Choose sequences \( \{u_i\}_{i=1}^\infty \) and \( \{v_i\}_{i=1}^\infty \) in \( \text{Lip}_{\text{loc}}(W) \) such that \( u_i \to \chi_F \) and \( v_i \to \chi_G \) in \( L^1_{\text{loc}}(W) \), and
\[ \lim_{i \to \infty} \int_W g_{u_i} \, dm = P(F, W) \quad \text{and} \quad \lim_{i \to \infty} \int_W g_{v_i} \, dm = P(G, W). \]

Passing to subsequences (not relabeled), we can also get \( u_i \to \chi_F \) and \( v_i \to \chi_G \) a.e. in \( W \). We have
\[ \limsup_{i \to \infty} \int_W g_{\max\{u_i, v_i\}} \, dm \leq \limsup_{i \to \infty} \left[ \int_W g_{u_i} \, dm + \int_W g_{v_i} \, dm \right] \quad \text{by (2.5)} \]
\[ = P(F, W) + P(G, W) \]
\[ \leq P(F \cup G, W) + \varepsilon \quad \text{by assumption}. \]

We also have
\[ \lim_{i \to \infty} \left[ \int_W g_{u_i} \, dm + \int_W g_{v_i} \, dm \right] \]
\[ = P(F, W) + P(G, W) \]
\[ \leq P(F \cup G, W) + \varepsilon \]
\[ \leq \liminf_{i \to \infty} \int_W g_{\max\{u_i, v_i\}} \, dm + \varepsilon \quad \text{since } \max\{u_i, v_i\} \to \chi_{F \cup G} \text{ in } L^1_{\text{loc}}(W) \]
\[ = \liminf_{i \to \infty} \left[ \int_{\{u_i > v_i\}} g_{u_i} \, dm + \int_{\{u_i \leq v_i\}} g_{v_i} \, dm \right] + \varepsilon \quad \text{by (2.5)}. \]
Thus necessarily (note that \(g_u\) are \(g_v\) are still the minimal 1-weak upper gradients in \(W\), even though we integrate over a smaller set)

\[
\limsup_{i \to \infty} \left[ \int_{\{u_i \leq v_i\}} g_{u_i} \, dm + \int_{\{u_i > v_i\}} g_{v_i} \, dm \right] \leq \varepsilon.
\]

Thus by the analog of (2.5) for min, we get

\[
\limsup_{i \to \infty} \int_W g_{\min\{u_i, v_i\}} \, dm = \limsup_{i \to \infty} \left[ \int_{\{u_i \leq v_i\}} g_{u_i} \, dm + \int_{\{u_i > v_i\}} g_{v_i} \, dm \right] \leq \varepsilon. \tag{3.11}
\]

Since \(m(F \cap G) = 0\), the functions

\[w_i := \max\{u_i, v_i\} - \min\{u_i, v_i\}\]

still converge to \(\chi_{F \cup G}\) in \(L^1_{\text{loc}}(W)\), with the following “almost optimality”:

\[
\limsup_{i \to \infty} \int_W g_{w_i} \, dm \leq \limsup_{i \to \infty} \int_W g_{\max\{u_i, v_i\}} \, dm + \limsup_{i \to \infty} \int_W g_{\min\{u_i, v_i\}} \, dm \quad \text{by (2.3)}
\]

\[
\leq P(F \cup G, W) + 2\varepsilon \quad \text{by (3.10) and (3.11).} \tag{3.12}
\]

Recall that \(u_i, v_i \in \text{Lip}_{\text{loc}}(W)\) and so also \(w_i \in \text{Lip}_{\text{loc}}(W)\). Each \(\{w_i > 0\}\) consists of two disjoint open sets \(F_i := \{u_i > v_i\}\) and \(G_i := \{v_i > u_i\}\). Since we had \(u_i \to \chi_F\) and \(v_i \to \chi_G\) a.e. in \(W\), it follows that

\[\chi_{F_i} \to 0 \text{ a.e. in } G \text{ and } \chi_{G_i} \to 0 \text{ a.e. in } F. \tag{3.13}\]

By Lemma 3.3, we have \(P(F' \cup G', W) < \infty\). Now by Lemma 3.6 we find a sequence \(\{w_i\}_{i=1}^{\infty}\) in \(\text{Lip}_{\text{loc}}(W)\) with \(w'_i \to \chi_{F' \cup G'}\) in \(L^1_{\text{loc}}(W)\), \(w'_i \leq w_i\), and

\[
\liminf_{i \to \infty} \int_W g_{w'_i} \, dm \leq P(F' \cup G', W) + 2\varepsilon. \tag{3.14}
\]

Passing to a subsequence (not relabeled), we can also get \(w'_i \to \chi_{F' \cup G'}\) a.e. in \(W\).

By truncating, we can assume that \(w'_i \geq 0\) (then we still have \(w'_i \leq \max\{0, w_i\}\)). Now each set \(\{w'_i > 0\} \subset \{w_i > 0\}\) is also contained in the union of the disjoint open sets \(F_i\) and \(G_i\). It follows that \(w'_i = w'_i \chi_{F_i} + w'_i \chi_{G_i}\). We have \(w'_i = 0\) on \(\partial F_i \cap W\), and so \(w'_i \chi_{F_i}\) is in \(\text{Lip}_{\text{loc}}(W) \subset N^{1,1}_{\text{loc}}(W)\). Similarly, \(w'_i \chi_{G_i}\) is in \(\text{Lip}_{\text{loc}}(W)\). By (2.4), now

\[
g_{w'_i} = g_{w'_i} \chi_{F_i} + g_{w'_i} \chi_{G_i} = g_{w'_i} \chi_{F_i} + g_{w'_i} \chi_{G_i} \quad \text{a.e. in } W. \tag{3.15}
\]

Moreover, since \(w'_i \to \chi_{F' \cup G'}\) a.e. in \(W\) and using (3.13), we get

\[w'_i \chi_{F_i} \to \chi_{F'} \text{ a.e. in } W \quad \text{and similarly} \quad w'_i \chi_{G_i} \to \chi_{G'} \text{ a.e. in } W. \tag{3.16}\]
By Lebesgue’s dominated convergence theorem, we have these convergences also in $L^1_{\text{loc}}(W)$. It now follows that

$$P(F', W) + P(G', W) \leq \lim \inf_{i \to \infty} \left[ \int_W g_{w_i'} \chi_{F_i} \, dm + \int_W g_{w_i'} \chi_{G_i} \, dm \right]$$

$$= \lim \inf_{i \to \infty} \int_W g_{w_i'} \, dm \quad \text{by (3.15)}$$

$$\leq P(F' \cup G', W) + 2\varepsilon \quad \text{by (3.14).}$$

\[\square\]

**Proof of Proposition 3.1.** By Lemma 3.3, we know that $P(F' \cup G', D) < \infty$. Let $\varepsilon > 0$. By the definition of the total variation (2.9), there exists an open set $W \supset D$ with

$$P(F' \cup G', W) \leq P(F' \cup G', D) + \varepsilon \quad (3.17)$$

as well as $P(F', W) < \infty$, $P(G', W) < \infty$, and

$$P(F, W) < P(F, D) + \varepsilon/2 \quad \text{and} \quad P(G, W) < P(G, D) + \varepsilon/2. \quad (3.18)$$

Then

$$P(F \cup G, W) \geq P(F \cup G, D)$$

$$= P(F, D) + P(G, D) \quad \text{by assumption}$$

$$\geq P(F, W) + P(G, W) - \varepsilon \quad \text{by (3.18)}$$

and so

$$P(F' \cup G', D) \geq P(F' \cup G', W) - \varepsilon \quad \text{by (3.17)}$$

$$\geq P(F', W) + P(G', W) - 2\varepsilon \quad \text{by Lemma 3.9}$$

$$\geq P(F', D) + P(G', D) - 2\varepsilon.$$

Letting $\varepsilon \to 0$, we get $P(F' \cup G', D) \geq P(F', D) + P(G', D)$. Since the opposite inequality always holds by Lemma 3.3, we get the result. \[\square\]

**Proof of Proposition 3.2.** Take sequences $\{u_i\}_{i=1}^\infty$ and $\{v_i\}_{i=1}^\infty$ in $\text{Lip}_{\text{loc}}(W)$ such that $u_i \to \chi_F$ in $L^1_{\text{loc}}(W)$ and $v_i \to \chi_G$ in $L^1_{\text{loc}}(W)$, as well as

$$\lim_{i \to \infty} \int_W g_{u_i} \, dm = P(F, W) \quad \text{and} \quad \lim_{i \to \infty} \int_W g_{v_i} \, dm = P(G, W).$$

Using (3.10) with $\varepsilon = 0$, we get

$$P(F \cup G, W) \geq \limsup_{i \to \infty} \int_W g_{\max\{u_i, v_i\}} \, dm,$$

and then necessarily

$$P(F \cup G, W) = \lim_{i \to \infty} \int_W g_{\max\{u_i, v_i\}} \, dm. \quad (3.19)$$
Define
\[ W_t' := \{ x \in W' : d(x, X \setminus W') > t \}, \quad t > 0. \]
Since \( P(F \cup G, \cdot) \) is a Borel outer measure on \( W \), for all but at most countably many \( t > 0 \) we have
\[ P(F \cup G, \partial W_t') = 0 \quad \text{and also} \quad m(\partial W_t') = 0. \]
For all such \( t > 0 \), the fact that \( \max\{u_i, v_i\} \to \chi_{F \cup G} \) in \( L^1(W_t') \) implies
\[ P(F \cup G, W_t') \leq \liminf_{i \to \infty} \int_{W_t'} g_{\max\{u_i, v_i\}} \, dm \]
(note that \( g_{\max\{u_i, v_i\}} \) still denotes the minimal 1-weak upper gradient in \( W \); recall (2.7)) and
\[ P(F \cup G, W \setminus W_t') \leq \liminf_{i \to \infty} \int_{W \setminus W_t'} g_{\max\{u_i, v_i\}} \, dm, \]
and thirdly
\[ P(F \cup G, W_t') + P(F \cup G, W \setminus W_t') = P(F \cup G, W) \]
\[ = \lim_{i \to \infty} \int_{W_t'} g_{\max\{u_i, v_i\}} \, dm \quad \text{by (3.19)}. \]
Note that generally, if for nonnegative numbers \( a, b, \{a_i\}_{i=1}^{\infty}, \{b_i\}_{i=1}^{\infty} \) we have
\[ a \leq \liminf_{i \to \infty} a_i \quad \text{and} \quad b \leq \liminf_{i \to \infty} b_i \quad \text{and} \quad a + b = \lim_{i \to \infty} (a_i + b_i), \]
then necessarily \( \lim_{i \to \infty} a_i = a \) and \( \lim_{i \to \infty} b_i = b \). Hence we get
\[ P(F \cup G, W_t') = \lim_{i \to \infty} \int_{W_t'} g_{\max\{u_i, v_i\}} \, dm. \]
By (3.11) with \( \varepsilon = 0 \), we have
\[ \limsup_{i \to \infty} \int_{W_t'} g_{\min\{u_i, v_i\}} \, dm \leq \limsup_{i \to \infty} \int_{W} g_{\min\{u_i, v_i\}} \, dm = 0. \quad (3.20) \]
Thus
\[ P(F \cup G, W_t') = \lim_{i \to \infty} \int_{W_t'} g_{\max\{u_i, v_i\}} \, dm \]
\[ = \lim_{i \to \infty} \left[ \int_{W_t'} g_{u_i} \, dm + \int_{W_t'} g_{v_i} \, dm - \int_{W_t'} g_{\min\{u_i, v_i\}} \, dm \right] \quad \text{by (2.6)} \]
\[ = \lim_{i \to \infty} \left[ \int_{W_t'} g_{u_i} \, dm + \int_{W_t'} g_{v_i} \, dm \right] \quad \text{by (3.20)} \]
\[ \geq P(F, W_t') + P(G, W_t') \]
since \( u_i \to \chi_F \) and \( v_i \to \chi_G \) in \( L^1_{\text{loc}}(W_t') \). Letting \( t \to 0 \), we get
\[ P(F \cup G, W_t') \geq P(F, W_t') + P(G, W_t'). \]
Since we also have the opposite inequality by Lemma 3.3, the result follows. \( \square \)
Remark 3.21. In [6, Section 2], results similar to this section were proved by relying on the representation \((2.10)\) of perimeter with respect to the Hausdorff measure, as well as isotropicity (Definition 2.11). Especially the first fact is very commonly used when studying BV functions, see e.g. [2, 4, 10, 11], to mention just a few examples. In this section, we have instead used only quite elementary methods, mostly relying on the definition of perimeter and the basic properties of weak upper gradients, which hold in much more general metric measure spaces than just PI spaces. Thus our results and methods may be also of some independent interest.

4 Decomposition theorem

In this section we show how to prove the decomposition theorem as well as some other results of [6] without assuming the space to be isotropic. Many of the definitions and results below are directly from [6], apart from the fact that \((X, d, m)\) is not assumed to be an isotropic PI space.

First we note a small technical point: until now we have understood sets of finite perimeter to be \(m\)-measurable, but in [6] a set of finite perimeter is always understood to be Borel. For consistency, in this section we also adopt the latter as part of the definition of sets of finite perimeter.

Definition 4.1. Let \(E \subset X\) be a set of finite perimeter. Given any Borel set \(D \subset X\), we say that \(E\) is decomposable in \(D\) provided there exists a partition \(\{F, G\}\) of \(E \cap D\) into sets of finite perimeter such that \(m(F), m(G) > 0\) and \(P(E, D) = P(F, D) + P(G, D)\). On the other hand, we say that \(E\) is indecomposable in \(D\) if it is not decomposable in \(D\). For brevity, we say that \(E\) is decomposable (resp. indecomposable) provided it is decomposable in \(X\) (resp. indecomposable in \(X\)).

The following lemma is [6, Lemma 2.8], except that there \((X, d, m)\) was assumed to be an isotropic PI space.

Lemma 4.2. Let \(E \subset X\) be a set of finite perimeter and let \(D \subset X\) be a Borel set. Suppose that \(\{F, G\}\) is a Borel partition of \(E\) such that \(P(E, D) = P(F, D) + P(G, D)\). Then \(P(A, D) = P(A \cap F, D) + P(A \cap G, D)\) for every set \(A \subset E\) of finite perimeter.

Proof. By Lemma 3.3, we have \(P(A \cap F, D) < \infty\) and \(P(A \cap G, D) < \infty\). Let \(F' := A \cap F\) and \(G' := A \cap G\). Now the claim follows from Proposition 3.1. \(\square\)

Next, we note that Corollary 2.9 of [6] holds without the isotropicity condition, since this condition is used in the proof only via the application of [6, Lemma 2.8].

We also get the following

Proposition 4.3. Let \(E \subset X\) be a set of finite perimeter. Let \(\{E_n\}_{n=1}^\infty\) be an increasing sequence of indecomposable sets such that \(E = \bigcup_{n=1}^\infty E_n\). Then \(E\) is an indecomposable set.

Proof. The proof is verbatim the same as that of [6, Proposition 2.10], except that instead of [6, Lemma 2.8] we apply Lemma 4.2. \(\square\)
The following lemma is [6, Lemma 2.11], except that there the sets $W', W$ were assumed to be Borel, and $(X, d, m)$ was assumed to be an isotropic PI space.

**Lemma 4.4.** Let $E \subset X$ be a set of finite perimeter and let $W', W \subset X$ be open sets. Suppose that $E \subset W' \subset W$ and that $E$ is indecomposable in $W'$. Then $E$ is indecomposable in $W$.

**Proof.** If $E$ is decomposable in $W$, then there exists a Borel partition $\{F, G\}$ of $E$ such that $m(F) > 0$ and $m(G) > 0$, and

$$P(E, W) = P(F, W) + P(G, W).$$

Now by Proposition 3.2, we get

$$P(E, W') = P(F, W') + P(G, W'),$$

and thus $E$ is decomposable in $W'$.

Next, [6, Proposition 2.13] also holds without isotropicity (though the PI space assumption is now needed), since this condition is used in the proof only via the application of [6, Corollary 2.9].

**Proof of Theorem 1.1.** We can follow the proof of [6, Theorem 2.14] almost verbatim, since isotropicity is only used via the application of [6, Proposition 2.13], [6, Lemma 2.11] in open sets (given by Lemma 4.4), and finally [6, Proposition 2.10] (Proposition 4.3).

**Remark 4.5.** Example 2.16 in [6] shows that in Theorem 1.1, the assumption of a $(1, 1)$-Poincaré inequality cannot be removed or even replaced by any $(1, p)$-Poincaré inequality with $p > 1$. In this sense, our assumption that $(X, d, m)$ is a PI space is close to optimal.

**Definition 4.6.** Let $(X, d, m)$ be a PI space. Let $E \subset X$ be a set of finite perimeter. Then we denote by

$$\mathcal{CC}^e(E) := \{E_i\}_{i \in I}$$

the decomposition of $E$ given by Theorem 1.1. (By relabelling, we can assume that either $I = \{1, \ldots, n\}$ or $I = \mathbb{N}$.) The sets $E_i$ are called the **essential connected components** of $E$.

Note that with the above notation, for any index set $J \subset I$ we have

$$P(E, X) \leq P\left(\bigcup_{i \in J} E_i, X\right) + P\left(\bigcup_{i \in I \setminus J} E_i, X\right)$$

by Lemma 3.3

$$\leq \sum_{i \in J} P(E_i, X) + P\left(\bigcup_{i \in I \setminus J} E_i, X\right)$$

by (3.5)

$$\leq \sum_{i \in I} P(E_i, X)$$

by (3.5)

$$= P(E, X),$$
and hence necessarily
\[ P\left( \bigcup_{i \in J} E_i, X \right) = \sum_{i \in J} P(E_i, X). \] (4.7)

For any disjoint index sets \( J_1, J_2 \subset I \), this implies
\[ P\left( \bigcup_{i \in J_1 \cup J_2} E_i, X \right) = P\left( \bigcup_{i \in J_1} E_i, X \right) + P\left( \bigcup_{i \in J_2} E_i, X \right). \] (4.8)

**Proposition 4.9.** Let \((X, d, m)\) be a PI space. Let \( E \subset X \) be an indecomposable set, and let \( CC^e(X \setminus E) = \{ G_i \}_{i \in I} \). Then \( E \cup \bigcup_{i \in J} G_i \) for any \( J \subset I \) is also indecomposable, with
\[ P\left( E \cup \bigcup_{i \in J} G_i, X \right) \leq P(E, X). \]

**Proof.** Fix an index set \( J \subset I \). We have either \( J = \{ i_1, \ldots, i_n \} \) or a sequence \( J = \{ i_1, i_2, \ldots \} \). First consider \( G_{i_1} \). By Lemma 3.3, \( E \cup G_{i_1} \) has finite perimeter. Suppose \( E \cup G_{i_1} \) is decomposable. Since \( E \) and \( G_{i_1} \) are indecomposable sets, necessarily one essential connected component (by its maximality property) of \( E \cup G_{i_1} \) contains \( E \) and another contains \( G_{i_1} \) (up to \( m \)-negligible sets). Thus the essential connected components of \( E \cup G_{i_1} \) are \( E \) and \( G_{i_1} \), and so
\[ P(E \cup G_{i_1}, X) = P(E, X) + P(G_{i_1}, X). \] (4.10)

Now since \( X \setminus E = G_{i_1} \cup \bigcup_{i \in I \setminus \{ i_1 \}} G_i \), we get
\[ P(E, X) = P\left( G_{i_1} \cup \bigcup_{i \in I \setminus \{ i_1 \}} G_i, X \right) \]
\[ = P(G_{i_1}, X) + P\left( \bigcup_{i \in I \setminus \{ i_1 \}} G_i, X \right) \] by (4.8)
\[ = P(G_{i_1}, X) + P(E \cup G_{i_1}, X) \]
\[ = 2P(G_{i_1}, X) + P(E, X) \] by (4.10),

implying \( P(G_{i_1}, X) = 0 \), a contradiction. Thus \( E \cup G_{i_1} \) is indecomposable. Also,
\[ P\left( E \cup G_{i_1}, X \right) = P\left( \bigcup_{i \in I \setminus \{ i_1 \}} G_i, X \right) \]
\[ \leq \sum_{i \in I \setminus \{ i_1 \}} P(G_i, X) \] by (3.5)
\[ \leq \sum_{i \in I} P(G_i, X) \]
\[ = P\left( \bigcup_{i \in I} G_i, X \right) \]
\[ = P(E, X). \]
Now clearly \( \{G_i\}_{i \in I \setminus \{i_1\}} \) is a partition of \( \bigcup_{i \in I \setminus \{i_1\}} G_i \) into indecomposable subsets of \( X \) with nonzero \( m \)-measure, and (4.7) gives

\[
P\left( \bigcup_{i \in I \setminus \{i_1\}} G_i, X \right) = \sum_{i \in I \setminus \{i_1\}} P(G_i, X).
\]

Since such a decomposition is unique by Theorem 1.1, necessarily \( CC^c(X \setminus (E \cup G_{i_1})) = \{G_i\}_{i \in I \setminus \{i_1\}} \). Thus we can repeat the first step and inductively obtain the result for any finite index set \( J \). Finally, if \( J \) is infinite, the result is obtained by Proposition 4.3 and the lower semicontinuity of perimeter.

**Definition 4.11.** Let \((X, d, m)\) be a PI space such that \( m(X) = \infty \). Let \( E \subset X \) be an indecomposable set. Then any essential connected component of \( X \setminus E \) with finite \( m \)-measure is a hole of \( E \).

**Definition 4.12.** Let \((X, d, m)\) be a PI space such that \( m(X) = \infty \). Given an indecomposable set \( E \subset X \), we define its saturation \( sat(E) \) as the union of \( E \) and its holes. We say that \( E \) is saturated provided that \( m(E \Delta sat(E)) = 0 \).

**Proposition 4.13.** Let \((X, d, m)\) be a PI space such that \( m(X) = \infty \). Let \( E \subset X \) be an indecomposable set. Then the following properties hold:

(i) Any hole of \( E \) is saturated.

(ii) The set \( sat(E) \) is indecomposable and saturated. In particular, \( sat(sat(E)) = sat(E) \).

(iii) It holds that \( \mathcal{H}(\partial^* sat(E) \setminus \partial^* E) = 0 \). In particular, \( P(sat(E), X) \leq P(E, X) \).

(iv) If \( F \subset X \) is an indecomposable set with \( m(E \setminus sat(F)) = 0 \), then \( m(sat(E) \setminus sat(F)) = 0 \).

**Proof.** We can follow the proof of [6, Proposition 3.12] almost verbatim, with the following changes. In the proof of (i), we need to apply Proposition 4.9 in place of [6, Proposition 2.18, Proposition 2.10]. In the proof of (iii), we note that [6, Remark 2.17] is true also without the assumption of isotropicity, and we get the second claim of (iii) from Proposition 4.9.

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