Modeling and Analysis of Vector-borne Plant Disease with Two Delays

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Abstract. In this paper, a non-linear mathematical model is formulated and analyzed to understand the dynamics of plant disease in presence of predators. Here predators act as biological control agent that reduce the vectors carrying the disease pathogen. The mathematical model is formulated using system of delay differential equations by considering two delays. The delays are time taken for the plant to become infected after contagion and time taken for the insect to become infected after contagion. The existence and stability of different equilibria of the model are discussed in detail. The basic reproduction number $R_0$ is computed. The Routh-Hurwitz criteria is used to investigate the stability of the disease-free equilibrium and the endemic equilibrium in absence of delay. The stability of endemic equilibrium is also investigated in presence of delay. The occurrence of Hopf-bifurcation is demonstrated by considering the delay as the bifurcation parameter. The critical values of the delays are obtained which preserve the stability of the endemic equilibrium point. The model system shows an oscillatory behavior beyond this critical values of delays. Numerical simulation is also performed to support the analytical results. It is found that biological control has positive impact on reducing the transmission of plant disease.

1. Introduction
Plants are very important as it support life on earth. Almost all terrestrial organisms, including humans are directly or indirectly dependent on plants for food. Human being consumes either plants or other living organisms that consume plants as food. Plants play an important role in maintaining the oxygen level in the atmosphere. Plant disease is the deterioration of the normal state of a plant that interferes with its vital functions over a period of time. Infectious plant diseases are caused by living agents or pathogens. The dynamics of infectious plant diseases can be well investigated though a suitable mathematical model. In fact mathematical modeling and computer simulation are very useful in understanding the dynamics of spread of plant disease and to study the impact of different disease control strategies. In [1], authors have studied a plant disease model by considering virus and host limitation. They assumed that there are two competing viruses and obtained the conditions for displacement and coexistence of viruses. In [2], authors formulated and analyzed mathematical models for plant disease by incorporating both continuous control strategy and impulsive control strategy. In [3], Jackson and Chen-Charpentier formulated a plant virus propagation model with Holling type II functional response and two delays. The first delay is time taken for the plant to become infected and the second delay is time
taken for the insect to become infected. Here explored the model using numerical simulation. In [4], Jackson and Chen-Carpentier extended their model by introducing predator as biological control agent and analyzed it numerically. In [5], authors investigated a mathematical model of plant-virus propagation with two delays by considering Holling type II functional response. Here authors discussed the stability of the positive equilibrium and the existence of Hopf-bifurcation. In [6], authors reconsidered Chen-Carpenter plant disease model in a reduced form. They theoretically and numerically analyzed the model. In [12], authors considered logistic growth of insect vector population and added delay corresponding to the time taken for the plant to become infected and analyzed it theoretically and numerically. In [7], authors considered a three species plant-pest-natural enemy compartmental model with two delays for both pests and natural enemies in a polluted environment and analyzed it. In [8], authors proposed and analyzed a mathematical model for mosaic disease by considering plant and vector populations. Here they also analyzed the dynamics of this disease by considering impulsive system. In [9], authors developed a model to predict the effective period of fungicides for control of Zymoseptoria tritici (septoria leaf blotch) in winter wheat. They examined the efficiency and duration of fungicidal effects of three fungicides under field and laboratory conditions. In [13] authors extended the model in [12] by introducing predator as a biological control agent and analyzed it. Authors further extended this model to stochastic model and compared the results of both deterministic and stochastic models. In [10], authors formulated a mathematical model for the transmission dynamics of vector-borne plant disease by taking incubation period as the time delay factor and analyzed it. In [11], authors proposed a model for the dynamics of a bio-economic plankton with delay and diffusion terms by assuming exploitation of phytoplankton.

In the present study we modify the model in [13] by considering Holling type II functional response for infected plants and infected vectors. Here we also incorporate two delays to the system, one corresponding to the time taken for the plant to become infected and other corresponding to the time taken for the insect to become infected. Here we consider logistic growth of insect population that seems to be more realistic as it is an established fact that the carrying capacity plays a vital role in disease dynamics [14]. The remaining of this paper is organized as follows: Section 2 deals with the formulation of the mathematical model; Section 3 discusses the basic reproduction number and existence of different equilibria of the proposed model; Section 4 deals with stability analysis and existence of Hopf-bifurcation; Section 5 demonstrates the results of numerical simulation; Section 6 finally concludes the paper.

2. Model Formulation

Here the total plant population \( M \) is divided into three disjoint classes namely, susceptible plants \( S(t) \), infected plants \( I(t) \) and recovered plants \( R(t) \). Also the total insect vector \( N(t) \) is divided into two classes, susceptible insect vectors \( X(t) \) and infected insect vectors \( Y(t) \). The virus travel from one plant to another through insect vectors. Here the total plant population is assumed to be a constant. When an infected plant dies due to disease, it is replaced by new plant. The insect population is assumed to grow logistically which is more realistic. When an infected vector feed a susceptible plant, it transmits virus to the plant. Similarly, when a susceptible insect feeds on infected plant, it acquires virus. Keeping in view of the above facts, we formulate our model as
Figure 1. Schematic flow diagram of the plant disease model. Variables are defined in Model formulation.

follows:

\[
\begin{align*}
\dot{S} & = \mu (M - S) + \eta I - \frac{\beta Y}{1 + \alpha Y} S \\
\dot{I} & = \frac{\beta Y}{1 + \alpha Y} S - (\gamma + \mu + \eta) I \\
\dot{X} & = \left( b - \frac{ar}{K} N \right) N - \left\{ d + (1 - a) \frac{r}{K} N \right\} X - \frac{\beta_1 X}{1 + \alpha_1 I} - \alpha_2 XP \\
\dot{Y} & = \frac{\beta_1 X}{1 + \alpha_1 I} - \left\{ d + (1 - a) \frac{r}{K} N \right\} Y - \alpha_3 YP \\
\dot{P} & = \theta \alpha_2 XP + \theta \alpha_3 YP - \delta P - \epsilon P^2 \\
\dot{R} & = \gamma I - \mu R
\end{align*}
\]

\( S(0) > 0, \ I(0) \geq 0, \ R(0) \geq 0, \ X(0) > 0, Y(0) \geq 0, P(0) \geq 0. \)

The schematic flow diagram of our model is shown in Figure 1. Since \( X + Y = N \), the above
model can be written as follows:
\[
\begin{align*}
\dot{S} &= \mu(M - S) + \eta I - \frac{\beta Y}{1 + \alpha Y} S,
\dot{I} &= \frac{\beta Y}{1 + \alpha Y} S - (\gamma + \mu + \eta) I,
\dot{N} &= r N \left(1 - \frac{N}{K}\right) - \alpha_2(N - Y) P - \alpha_3 Y P,
\dot{Y} &= \frac{\beta_1 I (N - Y)}{1 + \alpha_1 I} - \{d + (1 - a) \frac{r}{K} N\} Y - \alpha_3 Y P,
\dot{P} &= \theta \alpha_2 (N - Y) P + \theta \alpha_3 Y P - \delta P - \epsilon P^2,
\dot{R} &= \gamma I - \mu R,
\end{align*}
\]  

(2.1)

where \(S + I + R = M\) and \(M\) is the total plant population. Now introducing the time taken for the plant to become infected as delay \(\tau_1\) and time taken for the insect to become infected as delay \(\tau_2\), the above model (2.1) can be modified as follows:

\[
\begin{align*}
\dot{S} &= \mu(M - S) + \eta I - \frac{\beta Y}{1 + \alpha Y} S(t - \tau_1),
\dot{I} &= \frac{\beta Y}{1 + \alpha Y} S(t - \tau_1) - (\gamma + \mu + \eta) I(t - \tau_1),
\dot{N} &= r N \left(1 - \frac{N}{K}\right) - \alpha_2(N - Y) P - \alpha_3 Y P(t - \tau_2),
\dot{Y} &= \frac{\beta_1 I (t - \tau_2)}{1 + \alpha_1 I(t - \tau_2)} \{N(t - \tau_2) - Y(t - \tau_2)\} - \{d + (1 - a) \frac{r}{K} N\} Y(t - \tau_2) - \alpha_3 Y P(t - \tau_2) - \epsilon P^2(t - \tau_2),
\dot{P} &= \theta \alpha_2 (N - Y) P(t - \tau_2) + \theta \alpha_3 Y P(t - \tau_2) - \delta P(t - \tau_2) - \epsilon P^2(t - \tau_2).
\end{align*}
\]

(2.2)

Here it can be noted that we have not considered the differential equation corresponding to recovered class \(R\) as the total plant population \(M\) is a constant. The initial conditions for the above system are given by \(S(\theta) = \phi_1(\theta)\), \(I(\theta) = \phi_2(\theta)\), \(N(\theta) = \phi_3(\theta)\), \(Y(\theta) = \phi_4(\theta)\), \(P(\theta) = \phi_5(\theta)\), \(\phi_1(\theta) \geq 0\), \(\phi_2(\theta) \geq 0\), \(\phi_3(\theta) \geq 0\), \(\phi_4(\theta) \geq 0\), \(\phi_5(\theta) \geq 0\), where \(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta), \phi_5(\theta) \in C([\tau, 0], \mathbb{R}^5_+)\), the Banach space of continuous functions mapping the interval \([-\tau, 0]\) into \(\mathbb{R}^5_+\), where \(\mathbb{R}^5_+ = \{(x_1, x_2, x_3, x_4, x_5) : x_i \geq 0, i = 1, 2, 3, 4, 5\}\).

3. The Basic Reproduction Number and Existence of Equilibria

The basic reproduction number \(R_0\) of our proposed model is computed by following the next generation matrix methods as discussed in [16, 17] and is given below:

\[
R_0 = \sqrt[\rho(K\alpha_2 + \epsilon r)(\epsilon d - \alpha_3 \delta) + (\alpha_2 \delta + \epsilon r)(1 - a)\epsilon + \alpha_3 K\theta \alpha_2],
\]

where \(\rho = (\gamma + \mu + \eta)\).

Our proposed model exhibits four equilibria, namely (i) the plant-only equilibrium \(E_0 = (M, 0, 0, 0, 0)\), (ii) the equilibrium \(E_1 = (M, 0, K, 0, 0)\), where both disease and predator are absent, (iii) the disease-free equilibrium \(E_2 = \left(M, 0, \hat{N}, 0, \hat{P}\right)\), where \(\hat{N} = \frac{K(\alpha_2 \delta + \epsilon r)}{K\alpha_2 + \epsilon r}\) and \(\hat{P} = \frac{1}{\epsilon} \left[K\theta \alpha_2 \left\{\frac{\alpha_2 \delta + \epsilon r}{K\alpha_2 + \epsilon r}\right\} - \delta\right]\) and (iv) the endemic equilibrium \(E_3 = (S^*, I^*, N^*, Y^*, P^*)\), where

\[
S^* = \frac{\mu M (1 + \alpha Y)}{\beta Y (\gamma + \mu) + \mu (1 + \alpha Y)\rho}, I^* = \frac{\beta Y S^*}{(1 + \alpha Y)\rho}, P^* = \frac{r N^* (1 - N^* K)}{\alpha_2 (N^* Y^*) + \alpha_3 Y^*}.
\]
Table 1. Description of parameters

| Parameter | Description                                      | Value   |
|-----------|--------------------------------------------------|---------|
| $M$       | Total plant host population                      | 140     |
| $\beta$  | Infection rate of plants due to vectors          | 0.01    |
| $\beta_1$| Infection rate of vectors due to plants          | 0.02    |
| $\alpha$ | Saturation constant of plants due to vectors     | 0.2     |
| $\alpha_1$| Saturation constant of vectors due to plants     | 0.05    |
| $\alpha_2$| contact rate between predators and healthy insects | 0.05 |
| $\alpha_3$| contact rate between predators and infected insects | 0.05 |
| $\mu$    | Natural death rate of plants                     | 0.01    |
| $\gamma$ | Recovery rate of plants                          | 0.01    |
| $\eta$   | Death rate of infected plants due to the disease | 0.005   |
| $\theta$ | Conversion rate of predators due to insects      | 0.1     |
| $\delta$ | Natural death rate of predators                  | 0.05    |
| $\epsilon$ | Competition constant between predators         | 0.015   |
| $b - d = r$ | Intrinsic growth rate of insects           | 0.4     |
| $a$      | Convex combination constant                      | 0.3     |
| $b$      | Natural birth rate constant for insect population | 0.5    |
| $d$      | Natural death rate constant for insect population | 0.1    |
| $K$      | Carrying capacity                                | 150     |

\[ N^* = \frac{\beta \beta_1 \mu M \rho Y^* + \alpha Y^{*2} + \{C_1 Y^{*2} + C_2 Y^* + C_3\} \{C_4 Y^* + C_5\}}{\beta \beta_1 \mu M \rho [1 + \alpha Y^*] - \{C_6 Y^{*2} + C_7 Y^* + C_8\}}, \]

and

\[
C_1 = \frac{\beta \alpha (\gamma + \mu) \rho + \mu \alpha^2 \rho^2 + \beta \beta_1 \mu \alpha M \omega, C_2 = \beta \rho (\gamma + \mu) + 2 \mu \alpha \rho^2 + \beta \alpha \mu M \rho, C_3 = \mu \rho^2}
\]

\[
C_4 = \frac{\theta \alpha^2}{\epsilon} - \frac{\theta \alpha_2 \alpha_3}{\epsilon}, C_5 = d - \frac{\delta \alpha_3}{\epsilon},
\]

\[
C_6 = \left( \beta \rho (\gamma + \mu) + \mu \alpha^2 \rho^2 + \beta \alpha_1 \mu \rho M \right) \left( (1 - a) \frac{r}{K} + \frac{\theta \alpha_2 \alpha_3}{\epsilon} \right)
\]

\[
C_7 = \left( \beta \rho (\gamma + \mu) + 2 \mu \alpha \rho^2 + \beta \alpha_1 \mu \rho M \right) \left( (1 - a) \frac{r}{K} + \frac{\theta \alpha_2 \alpha_3}{\epsilon} \right),
\]

\[
C_8 = \mu \rho^2 \left( (1 - a) \frac{r}{K} + \frac{\theta \alpha_2 \alpha_3}{\epsilon} \right).
\]

Here $Y^*$ is the positive root of the following polynomial equation:

\[ L_1 Y^6 + L_2 Y^5 + L_3 Y^4 + L_4 Y^3 + L_5 Y^2 + L_6 Y + L_7 = 0, \]

where,

\[
C_9 = -C_6, C_{10} = -C_7 + \beta \beta_1 \mu \alpha M \rho, C_{11} = -C_8 + \beta \beta_1 \mu M \rho, D_1 = \theta (\alpha_3 - \alpha_2),
\]

\[
D_2 = (K \theta \alpha_2 + \epsilon), D_3 = (\alpha_3 - \alpha_2), L_1 = B_1^2 D_2 + C_1^2 K D_1 D_3 + K \alpha_2 D_1 B_1 C_9 + K \theta \alpha_2 D_3 B_1 C_9
\]

\[
L_2 = 2 B_1 B_2 D_2 + 2 C_9 C_{10} K D_1 D_3 - C_9 K D_3 \delta + B_1 C_{10} K \alpha_2 D_1 + K \alpha_2 D_1 B_2 C_9 - K \delta \alpha_2 B_1 C_9
\]

\[
- \epsilon r K B_1 C_9 + K \theta \alpha_2 D_3 B_1 C_9 + K \theta \alpha_2 B_2 C_9 D_3
\]

\[
L_3 = B_2^2 D_2 + (C_3^2 + 2 C_9 C_{11}) K D_1 D_3 - 2 C_9 C_{10} K D_3 \delta + (K \alpha_2 D_1 + K \theta \alpha_2 D_3)(B_1 C_{11} B_2 C_{10} + B_3 C_9)
\]

\[
- D_2 (B_1 C_{10} + B_2 C_9)
\]

\[
L_4 = D_2 (2 B_2 B_3 + 2 B_1 B_4) + 2 K D_1 D_3 C_{10} C_{11} + (K \alpha_2 D_1 + K \theta \alpha_2 D_3)(B_2 C_{11} + B_3 C_{10} + B_4 C_9)
\]
\[-KD_3 \delta (C_{10}^2 + 2C_9 C_{11}) - (K \delta \alpha_2 + \epsilon r K) (B_1 C_{11} + B_2 C_{10} + B_3 C_9) \]
\[L_5 = B_3^2 D_2 + C_{11}^2 K D_1 D_3 - 2C_9 C_{11} K D_3 \delta + (K \alpha_2 D_1 + K \delta \alpha_2 D_3) (B_3 C_{11} + B_4 C_{10}) \]
\[\quad - (K \delta \alpha_2 + \epsilon r K) (B_2 C_{11} + B_3 C_{10} + B_4 C_9) \]
\[L_6 = 2B_3 B_4 D_2 - C_{11}^2 K D_3 \delta + (K \alpha_2 D_1 + K \delta \alpha_2 D_3) B_4 C_{11} - (K \delta \alpha_2 + \epsilon r K) (B_3 C_{11} + B_4 C_{10}) \]
\[L_7 = D_2 B_3^2 - (K \delta \alpha_2 + \epsilon r K) B_4 C_{11}. \]

4. Stability and Existence of Hopf-bifurcation

The characteristic polynomial for the delay differential equation is given by

\[
\det \left( J + J_1 e^{-\lambda_1} + J_2 e^{-\lambda_2} - \lambda I \right) = 0,
\]

where the following matrices \( J, J_1 \) and \( J_2 \) are evaluated at the corresponding equilibrium point.

\[
J = \begin{pmatrix} -\mu & \eta & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & -\alpha_3 Y \\ 0 & 0 & \theta \alpha_2 P & -\theta \alpha_3 & a_{55} \end{pmatrix},
J_1 = \begin{pmatrix} -b_{21} & 0 & 0 & -b_{24} & 0 \\ b_{21} & 0 & 0 & b_{24} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

where

\[
a_{33} = r - \frac{2r N}{K} - \alpha_2 P, a_{34} = (\alpha_2 - \alpha_3) P, a_{35} = -[\alpha_2 (N - Y) + \alpha_3 Y], a_{43} = -(1 - a) \frac{r}{K} Y,
\]

\[
a_{44} = -\{d + (1 - a) \frac{r}{K} N\} - \alpha_3 P, a_{55} = \theta \alpha_2 (N - Y) + \theta \alpha_3 Y - \delta - 2 \epsilon P - \lambda, b_{21} = \frac{\beta Y}{1 + \alpha Y},
\]

\[
b_{24} = \frac{\beta S}{(1 + \alpha Y)^2}, c_{42} = \frac{\beta_1 (N - Y)}{(1 + \alpha_1 I)^2}, c_{43} = \frac{\beta_1 I}{1 + \alpha_1 I}.
\]

The above equation can be written as follows:

\[
P(\lambda) + Q(\lambda) e^{-\lambda_1} + R(\lambda) e^{-\lambda_2} + H(\lambda) e^{-\lambda(\tau_1 + \tau_2)} = 0,
\]

where \( P(\lambda), Q(\lambda), R(\lambda), H(\lambda) \), are polynomials in \( \lambda \) and are given by,

\[
P(\lambda) = \left( \lambda^5 + A_1 \lambda^4 + A_2 \lambda^3 + A_3 \lambda^2 + A_4 \lambda + A_5 \right), Q(\lambda) = \left( B_1 \lambda^4 + B_2 \lambda^3 + B_3 \lambda^2 + B_4 \lambda + B_5 \right),
\]

\[
R(\lambda) = \left( C_1 \lambda^4 + C_2 \lambda^3 + C_3 \lambda^2 + C_4 \lambda + C_5 \right), H(\lambda) = \left( D_1 \lambda^3 + D_2 \lambda^2 + D_3 \lambda + D_4 \right).
\]
where,
\[
\begin{align*}
a &= a_{33} + a_{55} + a_{44}, \quad b = -a_{33}a_{55} - a_{33}a_{44} - a_{44}a_{55} + a_{45}a_{54} + a_{34}a_{43} - a_{35}a_{53}, \\
c &= a_{55} + a_{33} + a_{34}, \quad e = -a_{33}a_{55} - a_{34}a_{55} - a_{35}a_{54} - a_{35}a_{53}, \quad f = \mu - a_{22}, \\
g &= -a_{22} - a_{12}, \quad h = -\mu a_{22}, \\
A_1 &= f - a, \quad B_1 = b_2, \quad C_1 = c_{43}, \quad A_2 = -b - af + h, \quad B_2 = -ab_2 + gb_2, \quad C_2 = fc_{43} - cc_{43}, \\
D_1 &= c_{43}b_2 - b_2c_{42}, \quad A_3 = -d - bf - ha, \quad B_3 = -bb_2 - agb_2, \\
C_3 &= -ec_{43} - fcc_{43} + hc_{43} - hec_{43}, \\
D_2 &= -ec_{43}b_2 + gc_{43}b_2 - \mu b_{24}c_{42} + b_{24}a_{33}c_{42} + b_{24}c_{42}a_{55} \\
A_4 &= -fd - hb, \quad B_4 = -db_2 - gb_2, \quad C_4 = -fe_{c43} - hex_{43}, \\
D_3 &= -ec_{43}b_2 - cc_{43}b_2 + \mu b_{24}a_{33}c_{42} + \mu b_{24}c_{42}a_{55} - b_{24}c_{42}a_{33}a_{55} + b_{24}a_{33}a_{55} + b_{24}a_{33}c_{42}c_{42} - gec_{43}b_2 \\
A_5 &= -hd, \quad B_5 = -dgb_2, \quad C_5 = -hec_{43}, \quad D_4 = -\mu b_{24}a_{33}c_{42}a_{55} + \mu b_{24}c_{42}a_{33}c_{42} - gec_{43}b_2 \\
\end{align*}
\]

We have following results for the stability of different equilibria of the model system (2.2)

**Theorem 4.1** The plant-only equilibrium point \(E_0 = (M, 0, 0, 0, 0)\) is always unstable.

**Proof:** The proof is too trivial to be reported here.

**Theorem 4.2** The disease and predator free equilibrium \(E_1 = (M, 0, K, 0, 0)\) is locally asymptotically stable when \(K < \frac{\delta}{\theta} \left( \frac{\beta_2 M K}{\rho (d + (1-a) r)} \right) < 1\) in the absence of delay. This equilibrium remains stable for all delay once it is stable without delay.

**Proof:** The proof is too trivial to be reported here.

**Theorem 4.3** For the model system without delay, the disease-free equilibrium \(E_2 = (M, 0, N, 0, \hat{P})\) is locally asymptotically stable provided the Routh-Hurwitz criteria \((T_i > 0\) for \(i = 1, 2, 3, 4\) and \(T_1T_2 - T_3 > 0, T_1T_2T_3 - T_2^2T_4 - T_3^2 > 0\)) for the following biquadratic equation are satisfied:

\[
\lambda^4 + T_1\lambda^3 + T_2\lambda^2 + T_3\lambda + T_4 = 0,
\]

where

\[
\begin{align*}
T_1 &= \rho + \frac{r\hat{N}}{K} + \epsilon \hat{P} + \left( d + (1-a) \frac{r}{K} \right) + \alpha_3 \hat{P} \\
T_2 &= \rho \frac{r\hat{N}}{K} + \rho \epsilon \hat{P} + \rho \left[ \left( d + (1-a) \frac{r}{K} \right) + \alpha_3 \hat{P} \right] + \frac{r\hat{N}}{K} \epsilon \hat{P} + \frac{r\hat{N}}{K} \left[ \left( d + (1-a) \frac{r}{K} \right) + \alpha_3 \hat{P} \right] \\
T_3 &= \rho \frac{r\hat{N}}{K} \hat{P} + \rho \frac{r\hat{N}}{K} \left[ \left( d + (1-a) \frac{r}{K} \right) + \alpha_3 \hat{P} \right] + \rho \left[ \left( d + (1-a) \frac{r}{K} \right) + \alpha_3 \hat{P} \right] \epsilon \hat{P} \\
&\quad + \rho \alpha_2 \hat{N} \alpha_2 \hat{P} + \rho \frac{r\hat{N}}{K} \left[ \left( d + (1-a) \frac{r}{K} \right) + \alpha_3 \hat{P} \right] \epsilon \hat{P} + \rho \alpha_2 \hat{N} \alpha_2 \hat{P} \left[ \left( d + (1-a) \frac{r}{K} \right) \right] \\
&\quad + \alpha_2 \hat{N} \alpha_2 \hat{P} \alpha_3 \hat{P} + \beta \frac{r\hat{N}}{K} + \epsilon \hat{P} \\
T_4 &= \rho \left[ \left( d + (1-a) \frac{r}{K} \right) + \alpha_3 \hat{P} \right] \left[ \frac{r\hat{N}}{K} \epsilon \hat{P} + \alpha_2 \hat{N} \alpha_2 \hat{P} \right] + \beta \frac{r\hat{N}}{K} + \frac{r\hat{N}}{K} \epsilon \hat{P}.
\end{align*}
\]
Furthermore, the equilibrium point $E_2(M, 0, \hat{N}, 0, \hat{P})$ is locally asymptotically stable for all delay $\tau_1$ and $\tau_2$ provided $\left[ \alpha_2 \theta \alpha_2 \hat{P} + \frac{r \hat{N}}{K} \hat{P} \right]^2 \left[ \rho^2 \left( (d + (1 - a) \frac{r}{K} \hat{N}) + \alpha_3 \hat{P} \right)^2 - (\beta \beta_1 M \hat{N})^2 \right] < 0$ which corresponds to $R_0 < 1$.

**Proof:** The proof is too trivial to be reported here.

**Theorem 4.4** For the model system without delay, the endemic equilibrium $E_3 = (S^*, I^*, N^*, Y^*, P^*)$ is locally asymptotically stable under the following conditions obtained from Routh-Hurwitz criteria:

\[
\begin{array}{c|ccc|ccc|c}
E_1 > 0, & E_1 & E_3 & E_5 & 0 & E_1 & E_3 & E_5 & 0 \\
1 & E_2 & E_4 & 0 & 1 & E_2 & E_4 & 0 & 1 \\
0 & E_1 & E_3 & 0 & 0 & E_1 & E_3 & 0 & 0 \\
0 & 0 & E_2 & E_4 & 0 & 0 & E_2 & E_4 & 0 \\
0 & 0 & E_3 & E_5 & 0 & 0 & E_3 & E_5 & 0 \\
\end{array}
\]

where expressions for $E_i$'s for $i = 1, 2 \ldots 5$ are given in the proof of the theorem.

Furthermore, when the delay $\tau_1 > 0$ and $\tau_2 = 0$, the equilibrium point $E_3 = (S^*, I^*, N^*, Y^*, P^*)$ is locally asymptotically stable under above mentioned condition for all delay $\tau_1 < \tau_1^*$ if $h_{25} < 0$ where the critical value of the delay $\tau_1^*$ is given by

\[
\tau_1^* = \frac{1}{\omega_1} \arccos \left( \frac{g_1 g_2 + g_3 g_4}{g_1^2 + g_4^2} \right) + \frac{2j\pi}{\omega_1}
\]

for $j = 0, 1, 2 \ldots$. The expressions for $h_{25}, g_1, g_2, g_3, g_4$ and $\omega_1$ are given in the proof of this theorem. The equilibrium $E_3$ is stable for $\tau_1 \in [0, \tau_1^*)$ and Hopf-bifurcation occurs when $\tau_1 = \tau_1^*$.

**Proof:** First we consider the case when $\tau_1 = 0, \tau_2 = 0$. In this case the characteristic equation (4.1) reduces to

\[
\lambda^5 + E_1 \lambda^4 + E_2 \lambda^3 + E_3 \lambda^2 + E_4 \lambda + E_5 = 0,
\]

where $E_1 = A_1 + B_1 + C_1$, $E_2 = A_2 + B_2 + C_2 + D_1$, $E_3 = A_3 + B_3 + C_3 + D_2$, $E_4 = A_4 + B_4 + C_4 + D_3$, $E_5 = A_5 + B_5 + C_5 + D_4$ and $A_i$'s, $B_i$'s, $C_i$'s and $D_i$'s are evaluated at the equilibrium point $E_3$. Clearly, this equilibrium point $E_3$ is locally asymptotically stable if the Routh-Hurwitz criteria mentioned in the statement of the theorem are satisfied. Next we consider the case when $\tau_1 > 0, \tau_2 = 0$. In this case the characteristic equation (4.1) reduces to

\[
(\lambda^5 + F_1 \lambda^4 + F_2 \lambda^3 + F_3 \lambda^2 + F_4 \lambda + F_5) + (B_1 \lambda^4 + F_6 \lambda^3 + F_7 \lambda^2 + F_8 \lambda + F_9)e^{-\lambda \tau_1} = 0.
\]

where, $F_1 = (A_1 + C_1)$, $F_2 = (A_2 + C_2)$, $F_3 = (A_3 + C_3)$, $F_4 = (A_4 + C_4)$, $F_5 = (A_5 + C_5)$, $F_6 = (B_2 + D_1)$, $F_7 = (B_3 + D_2)$, $F_8 = (B_4 + D_3)$, $F_9 = (B_5 + D_4)$. Let $\lambda = i \omega_1 (\omega_1 > 0)$ be the root of (4.2) then we have,

\[
F_1 \omega_1^4 - F_3 \omega_1^2 + F_5 = (\omega_1^2 F_7 - B_1 \omega_1^4 - F_9) \cos \omega_1 \tau_1 + (-F_8 \omega_1 + F_6 \omega_1^3) \sin \omega_1 \tau_1
\]

\[
\omega_1^5 - F_2 \omega_1^3 + F_4 \omega_1 = (-F_7 \omega_1^2 + B_1 \omega_1^4 + F_9) \sin \omega_1 \tau_1 - (-F_8 \omega_1 + F_6 \omega_1^3) \cos \omega_1 \tau_1
\]

Squaring and adding above equations, we get

\[
\omega_1^{10} + h_{21} \omega_1^8 + h_{22} \omega_1^6 + h_{23} \omega_1^4 + h_{24} \omega_1^2 + h_{25} = 0,
\]

where

\[
h_{21} = F_1^2 - 2F_2 - B_1^2, h_{22} = F_2^2 + 2F_4 + 2B_1 F_1 - F_6^2 - 2F_1 F_3,
\]

\[
h_{23} = 2F_1 F_2 + 2B_1 F_2 - F_6 F_9 - 2F_4 F_7,
\]

\[
h_{24} = 2F_2^2 + 2F_6 F_9 + 2F_4 F_7 - F_6^2 F_9 - 2F_4 F_7 - 2F_1 F_3,
\]

\[
h_{25} = 0.
\]
\[ h_{23} = F_3^2 + 2F_1F_5 - 2F_2F_4 - F_7^2 - 2B_1F_9 + 2F_8F_6, \]
\[ h_{24} = F_4^2 + 2F_7F_9 - F_8^2 - 2F_3F_6, \]
\[ h_{25} = F_5^2 - F_9^2. \]

Let \( \tau_1 = \omega_1^2 \). Then we can write equation (4.3) as
\[ r_1^5 + h_{21}r_1^4 + h_{22}r_1^3 + h_{23}r_1^2 + h_{24}r_1 + h_{25} = 0 \tag{4.4} \]
If \( h_{25} < 0 \) holds, then equation (4.4) will have at least one positive root. If \( \omega_1^2 \) is the smallest positive root of (4.4) then \( \omega_1 \) will be a purely imaginary root, \( \pm \omega_1 \) corresponding to the delay \( \tau_1 \). By Buttlar’s lemma, the endemic equilibrium \( E_3 \) remains stable for \( \tau_1 < \tau_{i1j}^* \), where \( \tau_{i1j}^* \) is computed as follows:
\[ \tau_{i1j}^* = \frac{1}{\omega_1} \arccos \left( \frac{g_1g_2 + g_3g_4}{g_1^2 + g_2^2} \right) + \frac{2j\pi}{\omega_1}, \text{ where } j = 0, 1, 2, \ldots \]
and \( g_1 = (\omega_1^2 - B_1\omega_1^2 - F_5), g_2 = (F_1\omega_1^2 - F_3\omega_1^2 + F_5), g_3 = (F_2\omega_1^2 - \omega_1^2 - F_4\omega_1), g_4 = (F_8\omega_1 - F_6\omega_1^2). \)

The equilibrium point \( E_3 \) is stable for \( \tau_1 \in [0, \tau_{i1j}^*] \) and Hopf-bifurcation occurs at \( \tau_1 = \tau_{i1j}^* \). We now verify the transversality condition, for the existence of Hopf-bifurcation. Differentiating (4.2) with respect to \( \tau_1 \) we get,
\[
(5\lambda^4 + 4F_1\lambda^3 + 3F_2\lambda^2 + 2F_3\lambda + F_4) + e^{-\lambda\tau_1} (4B_1\lambda^3 + 3F_6\lambda^2 + 2F_7\lambda + F_8) - \tau_1 e^{-\lambda\tau_1} (B_1\lambda^4 + F_6\lambda^3 + F_7\lambda^2 + F_8\lambda + F_9) \frac{d\lambda}{d\tau_1} - \lambda e^{-\lambda\tau_1} (B_1\lambda^4 + F_6\lambda^3 + F_7\lambda^2 + F_8\lambda + F_9) = 0
\]
which implies,
\[
\left( \frac{d\lambda}{d\tau_1} \right)^{-1} = \frac{(5\lambda^4 + 4F_1\lambda^3 + 3F_2\lambda^2 + 2F_3\lambda + F_4)}{\lambda e^{-\lambda\tau_1} (B_1\lambda^4 + F_6\lambda^3 + F_7\lambda^2 + F_8\lambda + F_9) + \frac{(4B_1\lambda^3 + 3F_6\lambda^2 + 2F_7\lambda + F_8)}{\lambda(B_1\lambda^4 + F_6\lambda^3 + F_7\lambda^2 + F_8\lambda + F_9)} - \frac{\tau_1}{\lambda}} - \frac{(4\lambda^5 + 3F_1\lambda^4 + 2F_2\lambda^3 + F_3\lambda^2 - F_5)}{-\lambda^2(\lambda^5 + F_1\lambda^4 + F_2\lambda^3 + F_3\lambda^2 + F_4\lambda + F_5)} \frac{(3B_1\lambda^4 + 3F_6\lambda^3 + F_7\lambda^2 - F_9)}{\lambda^2(B_1\lambda^4 + F_6\lambda^3 + F_7\lambda^2 + F_8\lambda + F_9)} - \frac{\tau_1}{\lambda}
\]
\[
\frac{(5\lambda^5 + 4F_1\lambda^4 + 3F_2\lambda^3 + 2F_3\lambda^2 + F_4\lambda)}{-\lambda^2(\lambda^5 + F_1\lambda^4 + F_2\lambda^3 + F_3\lambda^2 + F_4\lambda + F_5)} + \frac{(4B_1\lambda^4 + 3F_6\lambda^3 + 2F_7\lambda^2 + F_8\lambda)}{\lambda^2(B_1\lambda^4 + F_6\lambda^3 + F_7\lambda^2 + F_8\lambda + F_9)} - \frac{\tau_1}{\lambda}
\]
Therefore,
\[
E = \text{sign} \left\{ \text{Re} \left( \frac{(4\lambda^5 + 3F_1\lambda^4 + 2F_2\lambda^3 + F_3\lambda^2 - F_5)}{-\lambda^2(\lambda^5 + F_1\lambda^4 + F_2\lambda^3 + F_3\lambda^2 + F_4\lambda + F_5)} + \frac{(3B_1\lambda^4 + 3F_6\lambda^3 + F_7\lambda^2 - F_9)}{\lambda^2(B_1\lambda^4 + F_6\lambda^3 + F_7\lambda^2 + F_8\lambda + F_9)} - \frac{\tau_1}{\lambda} \right) \right\}
\]
\[
= \text{sign} \left\{ \text{Re} \left( \frac{(3F_1\omega_1^4 - F_3\omega_1^2 - F_5) + i(4\omega_1^5 - 2F_2\omega_1^3)}{\omega_1^2[(F_1\omega_1^4 - F_3\omega_1^2 + F_5) + i(F_4\omega_1 + \omega_1^2 - F_2\omega_1^2)]} \right) \right\}
\]
Here it can be noted that when \( \tau \) the critical value of delay keeping the second delay stable. This fact is demonstrated in Figure 2(b). Next we explore the impact of delay \( \omega \). Therefore the transversality condition holds and hence Hopf bifurcation occurs at \( \beta_1, \beta \). As numerator of the last bracketed term is positive by equation (4.3), we get
\[
\frac{d \text{Re}(\omega_1)}{d \tau_1} > 0|_{\omega_1 = \omega, \tau_1 = \tau_1^*}
\]
Therefore the transversality condition holds and hence Hopf bifurcation occurs at \( \omega_1^* = \omega_1, \tau_1 = \tau_1^* \).
The case when \( \tau_1, \tau_2 > 0, \tau_1 \neq \tau_2 \) is explored numerically in the following section.

5. Numerical Simulation
The model (2.2) is simulated for the following set of parameters without delay:
\[
\mu = 0.01, \gamma = 0.01, \eta = 0.01, M = 100, K = 150, \beta = 0.005, \beta_1 = 0.02, d = 0.3,
\]
\[
a = 0.3, b = 0.5, \alpha_1 = 0.1, \alpha_1 = 0.05, \alpha_2 = 0.05, \alpha_3 = 0.05, \theta = 0.1, \delta = 0.05, \epsilon = 0.015.
\]
For this set of parameters \( R_0 = 3.71 \) and we get stable non-trivial equilibrium point as (43.04, 28.48, 20.37, 6.58, 3.45). Here Figure 2(a) is demonstrating this fact. Next we change our \( \beta \) and \( \beta_1 \) as follows: \( \beta = 0.0005, \beta_1 = .01 \) and keep other parameters as mentioned above. For this set of parameters we get \( R_0 = 0.83 \) and in this case disease-free equilibrium point is stable. This fact is demonstrated in Figure 2(b). Next we explore the impact of delay \( \tau_1 \) by keeping the second delay \( \tau_2 = 0 \) for the set of parameters given in the Table 1. Here we see that the critical value of delay \( \tau_1^* \) is 19.9. When delay value is less than 19.9, the equilibrium point \( E_3 \) is locally asymptotically stable. At \( \tau_1 = 19.9 \) Hopf-bifurcating periodic oscillation occurs. This fact is demonstrated in Figure 3. Further, we consider the following set of parameters
\[
\mu = 0.01, \gamma = 0.01, \eta = 0.005, M = 140, K = 150, \beta = 0.01, \beta_1 = 0.03, d = 0.1, a = 0.3,
\]
\[
b = 0.5, \alpha = .02, \alpha_1 = 0.05, \alpha_2 = 0.05, \alpha_3 = .05, \theta = 0.1, \delta = 0.05, \epsilon = 0.015.
\]
Here we take delays as \( \tau_1 = 19 \) and \( \tau_2 = 5 \). The simulation results are demonstrated in Figure 4. Here it can be noted that when \( \tau_1 \neq \tau_2 \) then too we get Hopf-bifurcation.
(a) Variation of $S, I, N, Y$ and $P$ with time when $R_0 = 3.71$.

(b) Variation of $S, I, N, Y$ and $P$ with time when $R_0 = 0.83$.

**Figure 2.** Stability of the equilibria $E_2$ and $E_3$ for different values of $R_0$.

(a) Susceptible plants with time.  
(b) Infected plants with time.  
(c) Susceptible insects with time.  
(d) Infected insects with time.  
(e) Predators with time.

**Figure 3.** Existence of Hopf-bifurcation when $\tau_1 > 0$ and $\tau_2 = 0$ when the critical value of delay $\tau_1 = \tau_1^* = 19.9$.

6. Conclusion

In this paper, we propose and investigate a plant disease model by considering saturated type incidence and two delays. Here we compute the basic reproduction number and all possible equilibria of the model. The stability of the non-trivial equilibrium point for the model with
Figure 4. Existence of Hopf-bifurcation when \(\tau_1 \neq \tau_2\).

delay is discussed in detail. The critical value of the delay \(\tau_1 = \tau_1^*\) is computed when the delay \(\tau_2 = 0\) analytically and is demonstrated through numerical simulation. It is found that as long as delay \(\tau_1\) is less than this critical value \(\tau_1^*\), the non-trivial equilibrium point is stable. At \(\tau_1 = \tau_1^*\), Hopf-bifurcation occurs leading to periodic oscillation. The case when \(\tau_1, \tau_2 > 0\), & \(\tau_1 \neq \tau_2\) also leads to Hopf-bifurcation. This fact is demonstrated through numerical simulation. The condition for existence of Hopf-bifurcation is also obtained. It is observed that delay is not causing much impact on predator population but it causes periodic oscillations in plants and insect populations. Certainly, this is not a desirable situation as in this situation it is difficult to predict the impact of any control strategy. One need to choose parameter values in such a way that the Hopf-bifurcation does not exist for reasonable range of delays. And in this case, impact of predators as biological control agent can be accessed properly.

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