Optical Resonator Analog of a Two-Dimensional Topological Insulator

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A lattice of optical ring resonators can exhibit a topological insulator phase, with the role of spin played by the direction of propagation of light within each ring. Unlike the system studied by Hafezi \textit{et al.} \textsuperscript{9}, topological protection is achieved without fine-tuning the inter-resonator couplings, which are given the same periodicity as the underlying lattice. The topological insulator phase occurs for strong couplings, when the tight-binding method is inapplicable. Using the transfer matrix method, we derive the bandstructure and phase diagram, and demonstrate the existence of robust edge states. When gain and loss are introduced, the system functions as a diode for coupled resonator modes.

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The idea that photonic modes can have non-trivial topological properties, like topological states of quantum matter, originated with Haldane and Raghu \textsuperscript{1,2}, who predicted that a two-dimensional (2D) photonic crystal with broken time-reversal symmetry can support modes analogous to those of a “zero-field” quantum Hall gas \textsuperscript{3}. This has been confirmed experimentally, using gyro-magnetic photonic crystals operating at microwave frequencies \textsuperscript{14,17}. That system’s most striking feature is the existence of topologically protected one-way photonic edge states, which could be used for on-chip isolation \textsuperscript{4}. However, this is difficult to realize at optical frequencies, where magneto-optic effects are weak. Different systems supporting topological photonic modes have subsequently been proposed \textsuperscript{14,17}. In particular, Hafezi \textit{et al.} \textsuperscript{9} studied a lattice of ring resonators, similar to a 2D version of the CROW (coupled resonator optical waveguide) \textsuperscript{18}, in which the direction of propagation of light within each resonator acts as a two-fold “spin” degree of freedom. In the tight-binding (weak-coupling) regime, coupling waveguides can be used to implement spin-conserving hopping between adjacent resonator modes, and phase shifts in these couplers give rise to an effective vector potential in the tight-binding hopping amplitudes, with opposite signs for the two spins. With a choice of phase shifts implementing the Landau gauge (which is aperiodic in the lattice), the effective magnetic field can be made uniform and non-zero, which yields a photonic analog of the integer quantum Hall effect in each spin sector, with a Hofstadter butterfly spectrum \textsuperscript{19} and topologically protected edge states. Although the system is reciprocal (time-reversal maps the two spin sectors onto each other), and thus cannot be used as a conventional optical isolator, Hafezi \textit{et al.} suggested that the edge states can serve as robust optical delay lines \textsuperscript{9}.

The spin-dependent magnetic field in this system is reminiscent of the topological insulator model of Kane and Mele \textsuperscript{20,21}, which has attracted major theoretical and experimental interest \textsuperscript{22}. However, there is one major difference: the couplings in the Kane-Mele model have the periodicity of the lattice, and decoupling the two spin sectors reduces the model to two zero-field quantum Hall systems \textsuperscript{3}, with zero net magnetic flux through each unit cell. In the system of Hafezi \textit{et al.}, the couplings are aperiodic and decoupled spin sectors act as \textit{integer} quantum Hall systems; the tight-binding analysis seemed to imply that the periodic, zero-field system is topologically trivial \textsuperscript{9}. Aperiodic couplings also impose a practical design challenge, since a variety of different couplers must be used.

In this Letter, we show that the zero-field resonator lattice supports a topological insulator phase. When the inter-resonator couplings are tuned to large values beyond the tight-binding regime, the system exhibits one-way edge states, with non-zero $Z_2$ topological invariant \textsuperscript{21}; if the two spin sectors are decoupled, each acts as a zero-field system, like the Kane-Mele model \textsuperscript{20,21}. The system therefore behaves as a photonic topological insulator. Previously, Khaniakiev \textit{et al.} \textsuperscript{16} have proposed a different photonic topological insulator, which also does not require aperiodic couplings, using linear combinations of polarization states as the spin analog. However, that system relies on the special properties of metamaterials, whereas the present one uses ordinary dielectric materials and is thus considerably more feasible.

Our calculations rely on the transfer matrix method, which has previously been applied to the CROW \textsuperscript{23,24}, and has a wider domain of validity for such systems than the tight-binding method \textsuperscript{21}. This method also lets us easily study the effects of gain and loss, which can produce behaviors not easily obtainable in electronic topological insulators. We focus on the PT (parity/time-reversal) symmetric lattice \textsuperscript{25}, which contains balanced amounts of gain and loss. Theoretical and experimental studies have shown that PT-symmetric lattices possess unusual properties, including bifurcations between real and complex bands \textsuperscript{26,31}. We show that in a PT-symmetric photonic topological insulator, one edge state

\vspace{1cm}

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can be amplified while the back-propagating state of the same spin, on the opposite edge, is damped. The lattice thus acts as a robust optical diode for CROW modes.

The resonator lattice is shown schematically in Fig. 1(a). A ring resonator occupies each site of a square lattice. Its modes have a two-fold “spin” corresponding to the propagation direction within the ring. As proposed in Ref. [9] and depicted in Fig. 1(c), waveguides can be used to couple these modes to those on neighboring resonators. For our purposes, it is useful to employ a more abstract representation for this coupling. We first assume no spin mixing—modes couple only to other modes of the same spin—and restrict our attention to a single spin. Let \( n \equiv (x_n, y_n) \) denote a lattice site, \( n + x \) the site one unit in the \( +\hat{x} \) direction, etc. We specify the coupling between the resonators at \( n \) and \( n + x \) with complex numbers \( r_{nx}, r'_{nx}, t_{nx}, \) and \( t'_{nx} \); similarly, we specify the coupling between \( n \) and \( n + y \) by \( r_{ny}, r'_{ny}, t_{ny}, \) and \( t'_{ny} \). These relate the wave amplitudes in the resonator—see Fig. 1(a)—according to

\[
S_{nx} \begin{bmatrix} a_n \\ b_{n+x} \end{bmatrix} = \begin{bmatrix} d_{n+x} \\ c_{n+y} \end{bmatrix},
S_{ny} \begin{bmatrix} d_n \\ c_{n+y} \end{bmatrix} = \begin{bmatrix} b_{n+y} \\ a_n \end{bmatrix} e^{-2i\phi},
\]

where

\[
S_{nu} = \begin{bmatrix} r_{nu} & t'_{nu} \\ t_{nu} & r'_{nu} \end{bmatrix}.
\]

Here, and in the following, the dummy index \( \mu \) may stand for \( x \) or \( y \). The parameter \( \phi \) is the phase delay across each quarter of the ring. The \( S_{nu} \)'s, which have the form of scattering matrices, express the most general form of linear spin-conserving coupling between rings. In principle, the coefficients \( \{r_{nu}, t_{nu}, t'_{nu}, r'_{nu}\} \) can be independently varied by tuning the underlying waveguides [32].

In an experimental system, \( \phi \) and the coupling coefficients would depend on frequency, but here we treat them as independent quantities; when calculating the bandstructure, \( \phi \) plays the role of frequency [24].

Consider the special case where the coupling coefficients vary between different sites according to

\[
r_{nu} = r_{\mu} e^{iA_{nu}^x}, \quad t'_{nu} = t'_\mu e^{-iA_{nu}^y}, \quad t_{nu} = t_{\mu}, \quad r'_{nu} = r'_{\mu} e^{-iA_{nu}^x}.
\]

Here, \( A_{nu}^x \) and \( A_{nu}^y \) play the role of a magnetic vector potential. These gauge relations generalize those used in Ref. [9], which involved phase differences in tight-binding hopping amplitudes. Suppose the vector potential corresponds to a uniform rational magnetic flux through each unit cell: \( A_{nu}^x + A_{nu+1}^x = A_{nu+1}^x - A_{nu}^y = 2\pi P/Q, \) where \( P \) and \( Q \) are integers. For \( Q = 1 \), i.e. integer flux through each unit cell, the bandstructure is the same as in the zero-field (\( A_{nu}^x = A_{nu}^y = 0 \)) system. Then the magnetic unit cell coincides with the lattice’s unit cell, and there are solutions of the form [19] [24]

\[
a_{n+\mu} = e^{i(K_\mu + A_{nu}^y)} a_n, \quad b_{n+\mu} = e^{i(K_\mu + A_{nu}^y)} b_n,
\]

where \( K_\mu \) is a Bloch wave-vector. Combining [1]-[4] gives [32]:

\[
e^{-4i\phi} - Be^{-2i\phi} - C = 0,
B = r^2_{\mu} t_{\mu}' e^{iK_x} + r e^{-iK_x} + t x r y e^{iK_y} + t' x r' y e^{-iK_y}
C = (r x t' y - t x t' y)(r y t' y - t y t' y).
\]

As we shall see, for unitary couplings this gives rise to four real bands in the periodic space \( \phi \in [-\pi, \pi] \): two in \([-\pi/2, \pi/2]\) from directly solving [7], and the other two by adding \( \pm \pi \). This result relies crucially on the fact that in Eq. [3] there is no phase variation in \( t_{nu} \) and \( t'_{nu} \). The coupler shown in Fig. 1(c) satisfies this condition if the sum of the phase delays on its two arms is kept constant [32]. For non-integer fluxes (\( Q \neq 1 \)), the current approach gives essentially the same results as Ref. [9]: we could impose the Landau gauge \( A_n^x = (P/Q) y_n \) and \( A_n^y = 0 \), and define a \( 2 \times 1 \) magnetic unit cell for which \( a_{n+Qx} = e^{i(K_x + P y_n)} a_n \) and \( a_{n+Qy} = e^{iK_y} a_n \), and similarly for \( b \). This gives \( 4Q \) bands, analogous to Landau levels.

In the remainder of this paper, we focus on the zero-field (integer flux) system. If the couplings conserve energy, then \( S_{\mu}^t = S_{\mu}^{-1} \). We expect the bandstructure \( \phi(K_x, K_y) \) to be real (for \( K_x \) and \( K_y \) real), and this is easily proven using the parameterization

\[
r_{\mu} = \sin \theta_{\mu} e^{i\chi_{\mu}}, \quad t'_{\mu} = -\cos \theta_{\mu} e^{i(\varphi_{\mu} - \xi_{\mu})},
\]

\[
t_{\mu} = \cos \theta_{\mu} e^{i\xi_{\mu}}, \quad r'_{\mu} = \sin \theta_{\mu} e^{i(\varphi_{\mu} - \chi_{\mu})},
\]

where \( \theta_{\mu} \in [0, \pi/2] \) and \( \chi_{\mu}, \xi_{\mu}, \varphi_{\mu} \in [0, 2\pi] \). Eq. [5] then simplifies to \( e^{-4i\phi} - 2i Y e^{-2i\phi} - 1 = 0 \), where

\[
Y \equiv \sin \theta_{x} \cos \theta_{y} \sin \tilde{K}_x - \cos \theta_{x} \sin \theta_{y} \sin \tilde{K}_y
\]

\[
\tilde{\phi} \equiv \phi + \varphi_{x} + \varphi_{y} / 4,
\]

\[
\tilde{K}_x \equiv K_x + \varphi_{x} / 2 - \chi_{x} + \varphi_{y} / 2 - \xi_{y},
\]

\[
\tilde{K}_y \equiv K_y + \varphi_{y} / 2 - \chi_{y} - \varphi_{x} / 2 + \xi_{x}.
\]
The projected band diagram for a semi-infinite strip can be calculated similarly [32], with results shown in Fig. 2. For \( \theta < \pi/4 \), the system is a trivial insulator; although Fig. 2(a) exhibits edge states for some \( \phi \), these are two-way edge states, and for each \( \phi \) there are states confined to the same edge at different \( K_x \), with positive as well as negative group velocities. For \( \theta > \pi/4 \), the system is a topological insulator. The edge states span the band gaps, and for the given spin (clockwise) there is a positive velocity upper edge state and a negative velocity lower edge state (Fig. 3). In a real system, where the model parameters depend on the frequency \( \omega \), the topologically non-trivial band gaps would correspond to frequencies for which \( \theta(\omega) > \pi/4 \). We have verified, using finite-difference time-domain simulations, that this strong-coupling regime can be achieved with realistic resonator and waveguide designs [32].

It is noteworthy that the topological insulator phase occurs only when the inter-resonator coupling is sufficiently strong. This phase does not appear in the tight-binding analysis, where the zero-field system appears to be topologically trivial [9]. The transfer matrix method, however, accounts for the wave amplitudes at different parts of each ring, which is needed to describe the edge states of the topological insulator phase. Roughly speaking, these edge states move in the same direction in which light propagates inside the upper (lower) half of the uppermost (lowermost) ring resonator of the strip.

Spin mixing can be induced by backscattering within the resonators or waveguides [9]. This lifts the spin degeneracy of the edge states [32], similar to the Rashba term in electronic topological insulators [20, 21]. If the couplings remain unitary and reciprocal (i.e., absent radiative loss and magneto-optic disorder), the states on each edge are Kramers pairs [32]. However, these edge states are not topologically protected against spin-mixing perturbations, because optical wave amplitudes, unlike electrons, are not spin-half objects [20].

We have studied the effects of incorporating gain and loss into the photonic topological insulator, which yields behaviors that are inaccessible in the electronic system [20]. In particular, we consider here the PT-symmetric case, which corresponds to putting “balanced” gain and loss in symmetric regions of the unit cell. PT-symmetric photonic systems have previously been studied experimentally, e.g., using lossy waveguides [23] and optical fiber systems [31]. In the present system, the transfer matrix method can be adapted to include gain and loss simply by making the coupling matrices non-unitary. Specifically, the matrices obey the PT-symmetry relation [33–35]

\[ \mathcal{P} \mathcal{T} S_\mu \mathcal{T}^{-1} = S_\mu^{-1}. \] (10)

Here, \( \mathcal{P} \) and \( \mathcal{T} \) are parity and time-reversal operators. We choose \( \mathcal{P} = [0, 1; 1, 0] \) and \( \mathcal{T} \) to be the complex conjugation operator; for the coupler shown in Fig. 1(c), setting \( \psi_2 = \psi_1^* \) satisfies Eq. (10). The \( S_\mu \)'s can then be parameterized by \( r = |r| e^{i\phi} \), \( t' = -|t| e^{i(\phi - \phi')} \), \( t = |t| e^{i(\phi + \phi')} \), and \( r' = |r| e^{i\phi} \), where \( |rr'| + |t|^2 = 1 \).
Transmission B topological protection. (c,d) Transmittance leftward from A to A onator lattice. (a) Schematic. (b) Transmittance from port FIG. 5. Transmittance across a disordered PT-symmetric resonator lattice. Γ is the gain-loss parameter, defined in Eq. (11). All other parameters are the same as in Fig. 2 with θ = 0.2π. Both edge states acquire the same value of Im[Kx], so one is damped and the other amplified. Inset: intensity profiles for the lower edge state (filled circles) and upper edge state (open circles) at Γ = 0.2.

FIG. 4. (color online) Amplification and damping of edge states in the PT-symmetric resonator lattice. Γ is varied for the x couplings, while the y couplings are kept unitary (Γ = 0). For the bulk bands, Eq. (11) causes Kx to be replaced by Kx − iΓ in the solution [9], so that the bands are real for Kx = mπ, m ∈ Z, and complex otherwise. The edge states on opposite edges of the semi-infinite strip, which have opposite velocities, acquire the same imaginary component to Kx, and are respectively amplified and damped. This has a simple interpretation. The upper edge state’s wave amplitude is multiplied by r each time it hops one ring to the right; for Γ > 0, |r| > 0 and hence the state is amplified. Likewise, the lower edge state is damped by r each leftward hop. Previous studies of PT-symmetric waveguides have shown that modes with different transverse profiles can be selectively amplified and damped, but in those waveguides each amplified (damped) mode has a counter-propagating partner which is amplified (damped) by an equal amount. Here, the edge states have no counter-propagating partners of the same spin.

Fig. 5 shows the transmittance between waveguides coupled to opposite ends of the finite PT-symmetric lattice. Left-to-right transmission is amplified, while transmission in the opposite direction is damped. Within the band gaps, the transmission is insensitive to disorder, due to the topological protection on the edge states. In Fig. 3(b), we test the effect of removing this topological protection by performing the calculation with the lattice width reduced to a single unit cell; the resulting transmission is considerably less stable, varying by an order of magnitude for the same values of φ [38]. In terms of the underlying waveguides, the system is reciprocal, but it can nonetheless serve as a diode element for CROW modes. Such modes are susceptible to backscattering, even in the absence of spin mixing [39]; this is a particular problem in slow-light applications [39]. A photonic topological insulator can offset the effects of backscattering loss by robustly amplifying forward modes and damping backward modes. Unlike the PT-symmetric diode of Ref. [39], this device does not require optical nonlinearity.

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Appendix A: Inter-loop Coupling Coefficients

In Eq. (1) of the main text, we abstract away the couplings between resonators into a set of “reflection coefficients” and “transmission coefficients”, reproduced here for convenience:

\[
S_{nx} = \begin{bmatrix} a_n \\ b_{n+x} \end{bmatrix} = \begin{bmatrix} d_{n+z} \\ c_n \end{bmatrix}, \\
S_{ny} = \begin{bmatrix} d_n \\ c_{n+y} \end{bmatrix} = \begin{bmatrix} b_{n+y} \\ a_n \end{bmatrix} e^{-2i\phi}, \\
S_{n\mu} = \begin{bmatrix} r_{n\mu} \\ t'_{n\mu} \end{bmatrix} = \begin{bmatrix} t'_{n\mu} \\ r_{n\mu} \end{bmatrix}. 
\]

(A1) (A2) (A3)

In this section, we discuss how these coefficients can be related to the parameters of the underlying waveguide-coupling mechanism, such as that discussed in Ref. 3. As shown in Fig. 1(c), the resonator amplitudes \(a, b, c, \) and \(d\) (we will omit the redundant subscripts for simplicity) are coupled to waveguide amplitudes \(\alpha, \beta, \gamma, \) and \(\delta,\) by coupling relations 24:

\[
\begin{bmatrix} \tau & ik \kappa \\ ik \kappa & \tau \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{\tau} e^{i\phi_1} & -\kappa' \kappa e^{i\phi_2} \\ -\kappa \kappa' e^{i\phi_2} & \frac{1}{\tau'} e^{i\phi_1} \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix}. 
\]

(A4) (A5)

Without loss of generality, \(\tau, \tau', \kappa, \) and \(\kappa'\) may be taken to be real. If the phase delays along the two arms of the waveguide are \(\psi_1\) and \(\psi_2\) respectively, then

\[
\alpha = e^{i\psi_2} \delta, \quad \beta = e^{i\psi_1} \gamma. 
\]

(A6)

By combining Eqs. (A4)-(A6), we obtain

\[
\begin{bmatrix} d \\ c \end{bmatrix} = \frac{1}{1 - \tau \tau' e^{i(\psi_1 + \psi_2)}} \begin{bmatrix} -\kappa \kappa' e^{i\phi_1} & \tau' - \tau e^{i(\psi_1 + \psi_2)} \\ \tau - \tau' e^{i(\psi_1 + \psi_2)} & -\kappa \kappa' e^{i\phi_2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. 
\]

(A7)

Comparison with (A1)-(A3) immediately yields the corresponding values of \(r, r', t, \) and \(t'.\)

As indicated in Ref. 9, a synthetic magnetic vector potential can be implemented by altering the phase delay \(\psi_1\) at different sites, while keeping \(\psi_1 + \psi_2\) fixed. This leaves \(t\) and \(t'\), which are given by the off-diagonal matrix elements in (A7), independent of \(\psi_1,\) in agreement with Eq. (3). The relevant component of the resulting vector potential is \(A = \psi_1.\)

A realization of this coupling scheme, using optical ring resonators, is shown in Fig. 6(a). Each ring has width 200 nm and refractive index \(n = 3\) (with \(n = 1\) in the surrounding space). The rings at lattice sites have inner radius \(R_1 = 1.6 \mu m,\) and the rings serving as couplers have inner radius \(R_2 = 1.653 \mu m.\) The gap between each ring is fixed at 100 nm. As discussed in the main text, the coupling parameters entering into the \(S\) matrices depend implicitly on the operating frequency. Using finite-difference time-domain (FDTD) simulations of the structure shown in Fig. 6(b), we extract the parameter \(\theta\) as a function of frequency, via the relation \(\theta = \sin^{-1}(\sqrt{I_{\text{out}}/I_{\text{in}}})\) where \(I_{\text{in}}\) and \(I_{\text{out}}\) are the intensities at the input and output ports. The result is shown in Fig. 6(c). The strong-coupling regime \(\theta > \pi/4\) is found to be achievable, e.g. within the frequency range 0.669–0.676 \(\mu m^{-1}.\) In Fig. 7 we show the results of an FDTD simulation of a finite lattice at frequency 0.6725 \(\mu m^{-1},\) demonstrating that the system indeed behaves as a photonic topological insulator. We thus conclude that on-chip realizations of the photonic topological insulator are quite feasible. More detailed simulation studies of such systems will be presented in a subsequent paper.

Appendix B: Gauge Transformations

In this section, we provide additional details about the gauge structure of the resonator lattice. The coupling relations between resonator amplitudes, given by (A1)-(A3), can be simplified by eliminating the \(c\) and \(d\) variables, which yields

\[
e^{-2i\phi} b_n - r_{n-y,y} b_{n-y} - t'_{n-x-y,x} b_{n-x} = r_{n-y,y} r_{n-x-y,x} a_{n-x-y} + t'_{n-y,y} t_{n-x} a_n \\
e^{-2i\phi} a_n - t_{n-y} r_{n-x} a_{n-x} - r'_{n-y} t_{n+y,x} a_{n+y} = r'_{n-y,y} r'_{n+y,x} b_{n+y} + t_{n+y} t'_{n-x} b_n.
\]

(B1)
When the coupling coefficients obey the gauge relations (3), the system of equations (B1) simplifies to

\[
\begin{align*}
e^{-2i\phi}b_n - t_x' r_y e^{iA^y_n} b_{n-y} - r_x' t_y e^{-iA^y_n} b_{n+x} &= r_x r_y e^{i(A^x_{n-x-y} + A^y_{n-y})} a_{n-x-y} + t_x t_y' a_n \\
e^{-2i\phi} a_n - r_x t_y e^{iA^x_n} a_{n-x} - t_x r_y e^{-iA^x_n} a_{n+y} &= r_x' r_y' e^{-i(A^x_{n+y} + A^y_n)} b_{n+x+y} + t_x' t_y' b_n.
\end{align*}
\]

We look for solutions to these two equations, using the Bloch ansatz (4). When writing down the amplitudes for negative displacements and/or composed displacements using this ansatz, some care is needed to ensure that the \(A\)'s are evaluated at lattice positions consistent with (4). For example,

\[
\begin{align*}
a_{n-x} &= e^{-i(K_x + A^x_{n-x})} a_n \\
a_{n+x} &= e^{i(K_x + K_y + A^x_{n-x} + A^y_n)} a_n \\
a_{n-x-y} &= e^{-i(K_x + K_y + A^x_{n-x-y} + A^y_{n-y})} a_n, \text{ etc.}
\end{align*}
\]

FIG. 6. (a) Realization of the resonator lattice system, using ring resonators as couplers. (b) Simulation cell for calculating the coupling parameter \(\theta\). (c) Plot of \(\theta\) versus frequency.

FIG. 7. FDTD simulation of a photonic topological insulator lattice. All lattice parameters are the same as in Fig. 6 and the operating frequency is 0.6725 \(\mu m^{-1}\), in the middle of the bandgap indicated by Fig. 6(c). An edge state is observed, as predicted by the transfer matrix theory.
Plugging these relations into (B2) yields
\[
\left[ e^{-2i\phi} - r'_x t'_y e^{iK_x} - t'_x r_y e^{-iK_y} \right] b_n = \left[ r_x r_y e^{-i(K_x + K_y)} + t_x t'_y \right] a_n
\]
\[
\left[ e^{-2i\phi} - r_x t_y e^{-iK_x} - t_x r'_y e^{iK_y} \right] a_n = \left[ r'_x r'_y e^{i(K_x + K_y)} + t'_x t'_y \right] b_n.
\]
(B4)

The resulting dispersion relation (φ versus \(K_x\) and \(K_y\)) is independent of \(A^x_n\) and \(A^y_n\).

### Appendix C: Projected band structure

The calculation of the projected band diagram for the resonator lattice is much like the calculation of the CROW’s dispersion relation by Yariv et al. [24]. Suppose we have a semi-infinite strip, whose width is \(N\) lattice sites in the \(y\) direction and infinite in the \(x\) direction, with the coupling matrices independent of the \(x\) coordinate. First, assume as before that the spins are decoupled and consider a single spin sector (clockwise). We seek solutions of the form

\[
[a, b, c, d]_{n+x} = e^{iK_x} [a, b, c, d]_n.
\]

Let us consider a single column of the lattice, and replace the site index \(n\) with \(j \in \{1, 2, \cdots, N\}\) denoting the \(y\) coordinate. From (A1),

\[
M^x_j \begin{bmatrix} c_j \\ a_j \end{bmatrix} = e^{iK_x} \begin{bmatrix} b_j \\ d_j \end{bmatrix} \quad \text{where} \quad M^x_j = \frac{1}{r'_{jx}} \begin{bmatrix} 1 & -t'_{jx} \\ r'_{jx} & \det[S_{jx}] \end{bmatrix}.
\]

(C1)

From (A2),

\[
M^y_j(\phi) \begin{bmatrix} d_j \\ b_{j+1} \end{bmatrix} = \begin{bmatrix} a_j \\ c_{j+1} \end{bmatrix} \quad \text{where} \quad M^y_j(\phi) = \frac{1}{r'_{jy}} \begin{bmatrix} e^{-2i\phi} \det[S_{jy}] & r'_{jy} \\ -r'_{jy} & e^{-2i\phi} \end{bmatrix},
\]

(C2)

for \(j = 1, \cdots, N - 1\). At the edges of the lattice, we have the boundary conditions (refer to Fig. 1 of the paper):

\[
c_1 = e^{-2i\phi} b_1, \quad a_N = e^{2i\phi} d_N.
\]

(C3)

Combining (C1)-(C3) yields the eigenvalue equation

\[
M_A M_B \begin{bmatrix} b_1 \\ d_1 \\ \vdots \\ b_N \\ d_N \end{bmatrix} = e^{iK_x} \begin{bmatrix} b_1 \\ d_1 \\ \vdots \\ b_N \\ d_N \end{bmatrix},
\]

(C4)

where \(M_A\) and \(M_B\) are the following \(2N \times 2N\) matrices:

\[
M_A = \begin{bmatrix} M^x_1 & M^x_2 & \cdots & M^x_N \\ M^x_2 & M^x_3 & \cdots & M^x_N \\ \vdots & \vdots & \ddots & \vdots \\ M^x_N & M^x_1 & \cdots & M^x_{N-1} \end{bmatrix}, \quad M_B = \begin{bmatrix} e^{-2i\phi} & \cdots & \cdots & e^{-2i\phi} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ e^{2i\phi} & \cdots & \cdots & e^{2i\phi} \end{bmatrix}.
\]

(C5)

To generate the projected band diagram, we sweep \(\phi\) through the desired range (usually \([-\pi/2, \pi/2]\) due to the \(\pi\)-periodicity of the band structure), solving the eigenvalue problem (C4) at each \(\phi\). The eigenvalues which do not have unit modulus, which correspond to evanescent modes, are discarded; the arguments of the rest are the value(s) of \(K_x\) for that value of \(\phi\). This yields the plots shown in Fig. 2 of the paper.

### Appendix D: Transfer matrix analysis for both spins

It is straightforward to generalize the above calculations to include both spins, and to include spin-mixing processes. In Fig. 8, we show the \(x\)-couplings (the \(y\)-couplings are dealt with similarly), which have been augmented to include
the counter-clockwise amplitudes \( \{ a', b', c', d' \} \). These amplitudes, together with the original clockwise counterparts, are related by the \( S \) matrix

\[
S_x \begin{bmatrix}
  a_n \\
  b_{n+x} \\
  d_n' \\
  c_n'
\end{bmatrix} = \begin{bmatrix}
  a'_n \\
  b'_{n+x} \\
  d_n \\
  c_n
\end{bmatrix}.
\] (D1)

This \( 4 \times 4 \) \( S \) matrix is symmetric, due to optical reciprocity. If the coupling conserves spin, then the \( S \) matrix can be expressed in terms of the old \( 2 \times 2 \) \( S \) matrices (A1), as

\[
S_x = \begin{bmatrix}
  0 & S_x^T \\
  S_x & 0
\end{bmatrix}.
\] (D2)

If, on the other hand, there are spin-mixing processes, then the \( S \) matrix has a more complicated form, but it remains unitary and symmetric so long as the system remains reciprocal (i.e. there are no nonlinear or magneto-optic materials present).

Suppose the coupling is unitary (no gain or loss) as well as reciprocal. If there exists a set of wave amplitudes \( \{ a, b, c, d, a', b', c', d' \} \) which satisfies (D3), then

\[
S_x \begin{bmatrix}
  a_n' \\
  b_{n+x}' \\
  d_{n+x} \\
  c_n'
\end{bmatrix} = \begin{bmatrix}
  a_n^* \\
  b_{n+x}^* \\
  d_{n+x}^* \\
  c_n^*
\end{bmatrix}.
\] (D3)

This is the time-reversed solution, obtained by replacing \( a \) with \( a'^* \), \( a' \) with \( a^* \), and so on.

The calculation of the projected band structure proceeds along the same lines as the previous section. We look for solutions

\[
[a, b, c, d, a', b', c', d']_{n+x} = e^{iK_x} [a, b, c, d, a', b', c', d']_n.
\]

We can easily re-arrange the above \( S \) matrices to obtain transfer matrices, satisfying

\[
\begin{align*}
M_j^x & = e^{iK_x} \\
& = \begin{bmatrix}
  b_j & d_j \\
  d_j & b_j + 1
\end{bmatrix} \\
M_j(y(\phi)) & = \begin{bmatrix}
  a_j & a_j' \\
  c_j + 1 & c_j + 1
\end{bmatrix}.
\end{align*}
\] (D4)

The projected band structure is then obtained by solving

\[
\begin{bmatrix}
  M_1^x \\
  M_2^x \\
  \vdots
\end{bmatrix} \begin{bmatrix}
  e^{-2i\phi} & e^{2i\phi} \\
  e^{2i\phi} & e^{-2i\phi}
\end{bmatrix} \begin{bmatrix}
  M_N^y \\
  M_N^{y-1} \\
  \vdots
\end{bmatrix} = e^{iK_x} \begin{bmatrix}
  b_1 \\
  b_1' \\
  \vdots
\end{bmatrix}.
\] (D5)

FIG. 8. Schematic of inter-resonator couplings, with wave amplitudes for both spins included.
In Fig. 9, we show the projected band structure for a resonator lattice when spin mixing is included. Here, we model the inter-resonator couplings by

$$ S_{\mu} = e^{iA} \begin{bmatrix} 0 & S_{\mu}^T \\ S_{\mu} & 0 \end{bmatrix} e^{iA^T}, $$

where $S_{\mu}$ is the $2 \times 2$ coupling matrix for spin-conserving couplings (using the same parameters as in Fig. 2 of the paper, with $\theta = 0.4\pi$), and $A$ is a Hermitian spin-mixing matrix. The above form of the coupling matrix is designed to preserve reciprocity and unitarity. For illustrative purposes, we pick the spin-mixing matrix $A$ from the Gaussian unitary ensemble of $4 \times 4$ random Hermitian matrices, multiplied by a scale factor:

$$ A = 0.1 \times \begin{bmatrix} -0.4257 & 0.3271 + 0.9193i & -0.4521 + 0.6750i & 0.3534 + 0.4516i \\ 0.3271 - 0.9193i & 0.8429 & -0.1835 + 0.1817i & -0.1301 + 1.7570i \\ -0.4521 - 0.6750i & -0.1835 - 0.1817i & 1.7557 & 1.1908 - 0.7575i \\ 0.3534 - 0.4516i & -0.1301 - 1.7570i & 1.1908 + 0.7575i & 0.2727 \end{bmatrix}. $$

As in the Kane-Mele model [20][21], the introduction of spin mixing lifts the two-fold degeneracy of the edge states. Each band gap contains two pairs of edge states; each pair is confined to a single edge, and consists of two Kramers partners which are related by the time-reversal operation detailed above. However, because optical wave amplitudes are not spin-half objects, the time-reversal operation does not satisfy $T^2 = -1$, and Kane and Mele’s result [20] that the edge states are protected against spin-nonconserving perturbations does not hold. The edge states therefore receive no protection against spin-mixing perturbations.

[1] F. D. M. Haldane and S. Raghu, Phys. Rev. Lett. 100, 013904 (2008).
[2] S. Raghu and F. D. M. Haldane, Phys. Rev. A 78, 033834 (2008).
[3] F. D. M. Haldane, Phys. Rev. Lett. 61, 15 (1988).
[4] Z. Wang, Y. D. Chong, J. D. Joannopoulos, and M. Soljačić, Phys. Rev. Lett. 100, 013905 (2008).
[5] Z. Wang, Y. D. Chong, J. D. Joannopoulos, and M. Soljačić, Nature 461, 772 (2009).
[6] J.-X. Fu, R.-J. Liu, and Z.-Y. Li, Appl. Phys. Lett. 97, 041112 (2010).
[7] J.-X. Fu, J. Lian, R.-J. Liu, L. Gan, and Z.-Y. Li, Appl. Phys. Lett. 98, 211104 (2011).
[8] J. Koch, A. A. Houck, K. Le Hur, and S. M. Girvin, Phys. Rev. A 82, 043811 (2010).
[9] M. Hafezi, E. A. Demler, M. D. Lukin, and J. M. Taylor, Nature Phys. 7, 907 (2011).
[10] T. Ochiai, Phys. Rev. B 86, 075152 (2012).
[11] K. Fang, Z. Yu, and S. Fan, Nature Phot. 6, 782 (2012).
[12] J. Koch, A. A. Houck, K. Le Hur, and S. M. Girvin, Phys. Rev. A 82, 043811 (2010).
[13] A. Petrescu, A. A. Houck, and K. Le Hur, Phys. Rev. A 86, 053804 (2012).
[14] I. Carusotto and C. Ciuti, arXiv:1205.6500.
[15] V. Yannopapas, N. J. Phys. 14, 113017 (2012).
[16] A. B. Khanikaev, S. H. Mousavi, W.-K. Tse, M. Kargar-
ian, A. H. MacDonald, and G. Shvets, Nature Materials, doi:10.1038/nmat3520 (2012).

[17] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, S. Nolte, M. Segev, and A. Szameit, arXiv:1212.3146

[18] A. Yariv, Y. Xu, R. K. Lee, and A. Scherer, Opt. Lett. 24, 711 (1999).

[19] D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976).

[20] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005).

[21] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).

[22] J. E. Moore, Nature 464, 194 (2010).

[23] A. Yariv, IEEE Phot. Tech. Lett. 14, 483 (2002).

[24] J. K. S. Poon, J. Scheuer, S. Mookherjea, G. T. Paloczi, Y. Huang, and A. Yariv, Opt. Ex. 12, 90 (2004).

[25] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).

[26] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Phys. Rev. Lett. 100, 103904 (2008); Phys. Rev. A 81, 063807 (2010).

[27] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, Phys. Rev. Lett. 100, 030402 (2008); J. Phys. A 41, 244019 (2008).

[28] A. Guo et al., Phys. Rev. Lett. 103, 093902 (2009).

[29] C. E. Ruter et al., Nature Phys. 6, 192 (2010).

[30] A. Szameit, M. C. Rechtsman, O. Bahat-Treidel, and M. Segev, Phys. Rev. A 84, 021806(R) (2011).

[31] A. Regensburger, C. Bersch, M.-A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, Nature 488, 167 (2012).

[32] See Supplementary Material.

[33] H. Schomerus, Phys. Rev. Lett. 104, 233601 (2010).

[34] Y. D. Chong, L. Ge, and A. D. Stone, Phys. Rev. Lett. 106, 093902 (2011).

[35] L. Ge, Y. D. Chong and A. D. Stone, Phys. Rev. A 85, 023802 (2012).

[36] Adding gain/loss to the $y$ couplings, while keeping the $x$ couplings unitary, causes the projected bandstructure to remain strictly real, including the edge states.

[37] This parameterization differs from that of Ref. [35], which considered symmetric (reciprocal) scattering matrices.

[38] F. Morichetti et al. Appl. Phys. Lett. 96, 081112 (2010).

[39] H. Ramezani, T. Kottos, R. El-Ganainy, and D. N. Christodoulides, Phys. Rev. A 82, 043803 (2010).