Vacuum polarization around a three-dimensional black hole

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Abstract
We calculate the Euclidean propagator for a conformally coupled massless scalar field in the background of the three-dimensional black hole. The expectation value $\langle \phi^2 \rangle$ in the Hartle-Hawking state is obtained in the spacetime.

Recently, the black hole solution to the three-dimensional Einstein equations with a negative cosmological constant has been found [1] and various aspects on the black hole have been examined by many authors [1, 2]. In three-dimensional spacetime, the Einstein equations in vacua with a negative cosmological constant $-\lambda$ are reduced to

$$ R_{\mu\nu} = -2\lambda g_{\mu\nu}. \quad (1) $$

The black hole solution to the equations has been obtained [1] in a simple form. For zero angular momentum, the metric is given by

$$ ds^2 = -(\lambda r^2 - M)dt^2 + \frac{dr^2}{\lambda r^2 - M} + r^2d\theta^2 \quad (2) $$

where $M$ is the mass of the three dimensional black hole [1, 2]. The black hole horizon is located at $r = r_H = (M/\lambda)^{1/2}$.

In this paper, we discuss the vacuum polarization in the non-rotating black hole background in three dimensions. In order to obtain an exact expression for $\langle \phi^2 \rangle$ for a conformally coupled scalar field, we calculate the propagator (two-point function) for the field in the black hole spacetime with Euclidean signature. The computation is done by the mode sum method.

We begin with introducing a new coordinate $\rho$ defined by

$$ r = r_H \sec \rho \quad (0 \leq \rho \leq \pi/2) \quad (3) $$

where \( r_H = (M/\lambda)^{1/2} \). The metric takes the following form in terms of this coordinate:

\[
ds^2 = \lambda^{-1}(\sec \rho)^2(-M\lambda \sin^2 \rho dt^2 + d\rho^2 + M d\theta^2).
\] (4)

Substituting the time coordinate \( t \) by \(-i\tau\), we obtain the Euclidean metric:

\[
ds^2_E = \lambda^{-1}(\sec \rho)^2[dp^2 + \kappa^2 \sin^2 \rho d\tau^2 + M d\theta^2] \quad (5)
\]

where we set \( \kappa^2 = M\lambda \).

One finds that this Euclidean metric is regular at \( \rho = 0 \), which corresponds to the horizon \( r = r_H \), if the Euclidean time \( \tau \) is a periodic coordinate with period \( 2\pi/\kappa \). Thus the Hawking temperature is given by \( \kappa/2\pi = (M\lambda)^{1/2}/2\pi \) for the three-dimensional black hole \([1, 2]\).

Now we consider a scalar field in this spacetime. For a conformally coupled, massless scalar field in three dimensions, the wave equation for the scalar field reads

\[
\Box \phi - \frac{1}{8} R \phi = 0 \quad (6)
\]

where the covariant divergence is defined in terms of the background of the three dimensional black hole. In addition, we must note that the scalar curvature \( R \) takes a constant value, \( R = -6\lambda \), in the spacetime.

The Hartle-Hawking propagator \( G_H \) \([3]\) for this scalar field is the solution to the following equation:

\[
\left( \Box - \frac{1}{8} R \right) G_H(x, x') = -\frac{1}{\sqrt{g}} \delta(x, x'). \quad (7)
\]

Our approach to obtain an explicit form of this propagator is the mode sum method \([4]\). The mode functions are the solutions to the equation (6) with appropriate boundary conditions. The wave equation can be solved through separation of variables:

\[
\varphi_{mn}(x) = u_{mn}(\rho) e^{im\theta} e^{in\kappa \tau} \quad (8)
\]

where \( m \) and \( n \) are integers, by which the correct periodicities with respect to \( \theta \) and \( \tau \) are satisfied.

Assuming the mode function (8), we find that the wave equation (6) is reduced to a differential equation for the radial function:

\[
\cot \rho \frac{d}{d\rho} \tan \rho \frac{d}{d\rho} u_{mn}(\rho) - \left[ \frac{n^2}{\sin^2 \rho} + \frac{m^2}{M} \right] u_{mn}(\rho) = 0. \quad (9)
\]

We find that the general solution to this equation can be written by

\[
u_{mn}(\rho) = (\cos \rho)^{1/2}\left[ \alpha P_{-1/2+im/\sqrt{M}}(\cos \rho) + \beta Q_{-1/2+im/\sqrt{M}}(\cos \rho) \right] \quad (10)
\]

where \( P_{\nu}^m(z) \) and \( Q_{\nu}^m(z) \) are the Legendre functions and \( \alpha \) and \( \beta \) are arbitrary constants.
The propagator $G_H$ is constructed from these mode functions with the boundary condition that requires the regularity at $\rho = 0$ and $\pi/2$. The normalization is determined by the Fourier coefficients of the delta functions and the Wronskian condition with respect to the radial function. Consequently, the expression for the propagator by mode summation turns out to be

$$G_H(\rho, \tau, \theta; \rho', \tau', \theta') = \frac{\kappa^2}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\kappa(\tau-\tau')} \sum_{m=-\infty}^{\infty} e^{im(\theta-\theta')} (\cos \rho)^{1/2} \times \frac{(\cos \rho')^{1/2} (-1)^n}{M} \frac{P_n^{1/2+im}/\sqrt{M}}{(\cos \rho_<)^{1/2}} \left(\cos \rho_\to + \sin \rho \sin \rho' \cos \kappa(\tau - \tau')\right)$$

where $\rho_\to = \rho$ and $\rho_\to' = \rho'$ if $\rho < \rho'$, while $\rho_\to = \rho'$ and $\rho_\to' = \rho$ if $\rho > \rho'$.

This expression can be simplified by use of the addition theorem [5]. We can rewrite the sum over $n$ and then we get

$$G_H(\rho, \tau, \theta; \rho', \tau', \theta') = \frac{1}{4\pi^2 r_H} \sum_{n=-\infty}^{\infty} e^{im(\theta-\theta')} (\cos \rho)^{1/2} (\cos \rho')^{1/2} \times Q_{-1/2+im}/\sqrt{M}(\cos \rho \cos \rho' + \sin \rho \sin \rho' \cos \kappa(\tau - \tau'))$$

where $Q_\nu(z) = Q_{\mu=0}(z)$.

We can further simplify the expression with the help of the integral representation of the conical function [6]. The sum over $m$ yields delta functions, so the representation becomes fairly simple. We find

$$G_H = \frac{(\cos \rho)^{1/2} (\cos \rho')^{1/2}}{16\sqrt{2}\pi^2 r_H} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\cos m(\theta - \theta') \cos (m\phi/\sqrt{M})\right] d\phi$$

$$\times \left[\cosh \sqrt{\lambda}(\cos \rho)^{1/2} (\cos \rho')^{1/2} (4\sqrt{2}\pi)^{-1}\right]$$

$$= \sum_{k=-\infty}^{\infty} \left[\sqrt{\lambda}(\cos \rho)^{1/2} (\cos \rho')^{1/2} (4\sqrt{2}\pi)^{-1}\right]$$

$$\times \left[\cosh \sqrt{\lambda}(\theta - \theta + 2\pi k - \cos \rho \cos \rho' - \sin \rho \sin \rho' \cos \kappa(\tau - \tau'))\right]^{-1/2}$$

Now we calculate the vacuum polarization in the black hole spacetime. The quantum effects around the black hole can be calculated from the propagator. We first compute the vacuum expectation value $\langle \varphi^2 \rangle$ for a conformally coupled real scalar field, as the simplest example.

In the Hartle-Hawking vacuum [3], we take the vacuum polarization $\langle \varphi^2 \rangle$ as the coincidence limit of the Hartle-Hawking propagator $G_H(x, x)$ obtained above with appropriate regularization [7, 8, 9, 10, 11]. The regularization is done by a similar method adopted by Frolov et al. [9].

We find that the propagator takes the form, in the case that the separation between two points is restricted to the radial direction (i.e., when $\theta = \theta'$ and

1There is a misprint in 8.12.4 in their book [6], the cosh in the second line should read cos.
\( \tau = \tau' \):

\[
G_H(\rho, \rho') = \frac{\sqrt{\lambda} (\cos \rho)^{1/2} (\cos \rho')^{1/2}}{8 \pi \sin((\rho - \rho')/2)} + \sum_{k=1}^{\infty} \frac{\sqrt{\lambda} (\cos \rho)^{1/2} (\cos \rho')^{1/2}}{2 \sqrt{2\pi} \sqrt{\cosh 2\pi \sqrt{Mk} - \cos(\rho - \rho')}}.
\]

(14)

Note that the latter summation part in the RHS of (14) does not include divergent contribution in the coincidence limit \( \rho' \to \rho \).

We need the Schwinger-de Witt expansion of the propagator (two-point function) with respect to the powers of the geodesic distance between the two points for subtraction of the divergence in the coincidence limit [4, 9].

The divergent and constant contributions in the Schwinger-De Witt expansion near the horizon is written by use of the geodesic distance as [8, 9]

\[
G_{H}^{\text{div}}(x, x') = \frac{1}{4\pi \sqrt{2\sigma(x, x')}}
\]

(15)

where \( \sigma = s^2/2 \) and \( s(x, x') \) is the geodesic distance between \( x \) and \( x' \). For the radial separation, \( s \) is given by

\[
s(\rho, \rho') = \frac{1}{\sqrt{\lambda}} \int_{\rho}^{\rho'} \frac{dy}{\cos y} = \left[ \frac{1}{2\sqrt{\lambda}} \ln \left( \frac{1 + \sin y}{1 - \sin y} \right) \right]_{y=\rho}^{y=\rho'}.
\]

(16)

Using (14, 15 and 16), we get the final result:

\[
\langle \varphi^2 \rangle(\rho) = \lim_{\rho' \to \rho} (G_H(\rho, \rho') - G_{H}^{\text{div}}(\rho, \rho')) = \sum_{k=1}^{\infty} \frac{\sqrt{\lambda} \cos \rho}{4\pi \sin \pi \sqrt{Mk}} = \langle \varphi^2 \rangle(0) \cos \rho.
\]

(17)

Here the divergent and constant parts of \( G_H \) in the first part of the RHS of (14) have been exactly cancelled by those of \( G_{H}^{\text{div}} \). At the edge of the universe, \( \rho = \pi/2 \), \( \langle \varphi^2 \rangle \) vanishes as \( \cos \rho \). In terms of the original coordinate \( r \) in (2), it is found that \( \langle \varphi^2 \rangle \) is proportional to \( 1/r \).

We can calculate the propagator and \( \langle \varphi^2 \rangle \) for a twisted scalar field which obeys the boundary condition [12]:

\[
\varphi(\theta + 2\pi) = \varphi(\theta).
\]

(18)

The calculation for the twisted field around the three-dimensional black hole can be done similarly to the previous untwisted case. We only show the result for the vacuum expectation value \( \langle \varphi^2 \rangle_{\text{twisted}} \):

\[
\langle \varphi^2 \rangle_{\text{twisted}}(\rho) = \sum_{k=1}^{\infty} \frac{\sqrt{\lambda} (-1)^k \cos \rho}{4\pi \sin \pi \sqrt{Mk}} = \langle \varphi^2 \rangle_{\text{twisted}}(0) \cos \rho.
\]

(19)

The numerical results in shown in Figure 1. For large \( M \), the absolute value of \( \langle \varphi^2 \rangle \) on the horizon is dumped as \( \exp(-\pi \sqrt{M}) \), while in the limit of \( M \to 0 \), \( \langle \varphi^2 \rangle \) diverges.
In this paper, we have calculated the Hartle-Hawking propagator for a conformally coupled massless scalar field in three-dimensional black hole spacetime with Euclidean signature. Using the exact form of the propagator, we have obtained the vacuum value \( \langle \varphi^2 \rangle \) for untwisted and twisted scalar fields in the Hartle-Hawking vacuum. Its dependence on the radial coordinate has been found as \( \langle \varphi^2 \rangle \approx \cos \rho \approx 1/r \). The mass dependence of \( \langle \varphi^2 \rangle(0) \) has been numerically evaluated and shown in Figure 1.

In the limit of small mass, the amount of the vacuum fluctuation becomes unlimitedly large, according to our result. We must consider the back reaction to the metric in such a case. The effect of the vacuum fluctuation may have much importance on the final stage of the black hole evaporation.

We discussed only the conformally coupled massless scalar field. One may wish to extend the analyses in the present paper to the general couplings and masses. The rotation as well as the charge of the black hole in three dimensions will change the behaviour of the quantum fields. These are interesting subjects worth studying. The thermodynamics of the three-dimensional black holes and the effect of the back reaction due to the quantum effects should also be investigated in the future.

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