The higher-order heat-type equations via signed Lévy stable and generalized Airy functions

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Abstract
We study the higher-order heat-type equation with first time and $M$th spatial partial derivatives, $M = 2, 3, \ldots$. We demonstrate that its exact solutions for $M$ even can be constructed with the help of signed Lévy stable functions. For $M$ odd the same role is played by a special generalization of the Airy $Ai$ function that we introduce and study. This permits one to generate the exact and explicit heat kernels pertaining to these equations. We examine analytically and graphically the spatial and temporary evolution of particular solutions for simple initial conditions.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The Brownian motion governed by the conventional heat equation [1] has several generalizations. Some of them are related to the Markov processes described by the second order partial differential equation [2–4]. The other ones are connected with the so-called one- and two-sided Lévy stable distribution [5–7]. The last ones are governed by the so-called higher-order heat-type equations (HOHTE)

$$\frac{\partial}{\partial t} F_M(x, t) = \kappa_M \frac{\partial^M}{\partial x^M} F_M(x, t),$$

(1)

which for integer $M > 2$, are associated with a pseudo-Markov processes (signed processes). We choose to normalize $F_M(x, t)$ according to $\int_{-\infty}^{\infty} F_M(x, t) \, dx = 1$. The pseudo-Markov processes were introduced in the 1960s and have been studied in many papers starting from [8–10]. Equation (1) for $M = 4$ is called the biharmonic heat equation [11, 12].
HOHTE of order 3 or 4 (or higher) have been intensively investigated by several authors and display many interesting features [8, 10, 13–15] such as, e.g., an oscillating nature, connection to the arc-sine law and its counterpart, the central limit theorem and so on. HOHTE arise in physical phenomena, e.g., in the fluctuation phenomena in chemical reactions [16, 17] or as a new method for imaginary smoothing based on the biharmonic heat equation [18, 19]. The biharmonic heat equation is used to describe the diffusion on the unit circle [20]. For fixed values of integer $M > 2$ HOHTE can be considered as the composition of Brownian motions or stable processes with Brownian motions [21, 22].

In equation (1) the constants $\kappa_M$ are subject to constraints. Following [13, 15] we choose

$$
\kappa_M = \begin{cases} 
(-1)^{M/2+1}, & M = 2, 4, \ldots, \\
\pm(-1)^{(M/2)+1}, & M = 3, 5, \ldots, 
\end{cases}
$$

(2)

The symbol $[n]$ denotes the integer part of $n$. This choice of equation (2) for $\kappa_M$ warrants, as we will see in section 2, the possibility of obtaining the solution of equation (1) via an appropriate integral transform. Moreover, such a choice of coefficients $\kappa_M$ guarantees that the classical arc-sine law holds for even $M$ [8] and the counterpart to that law in the case of odd $M$ [13].

From mathematical point of view, equation (1), with the initial condition $F_M(x, 0) = f(x)$, is the Cauchy problem. Its formal solution is obtained by using the extension of the evolution operator formalism, introduced by Schrödinger, which gives

$$
F_M(x, t) = \hat{U}_M(t)f(x), \quad \hat{U}_M(t) = \exp\left(\kappa_M\frac{\partial^M}{\partial x^M}\right)
$$

(3)

with $f(x)$ being an infinitely differentiable function, or an appropriate limit of a sequence of such functions, see below.

Below we shall employ equations (3) to solve equation (1) for given initial conditions. For instance, for $f(x) = x^n (n \in \mathbb{N})$ a solution of HOHTE can be expressed with the Hermite–Kampé de Fériet polynomials $H_n^{[M]}(x, y)$ [23–25] as

$$
F_M^{(n)}(x, t) = H_n^{[M]}(x, \kappa_M t) = n! \sum_{r=0}^{[n/M]} \frac{(\kappa_M t)^r x^{n-r}}{r!(n - Mt)!},
$$

(4)

For $M = 2$, $H_n^{[2]}(x, y)$ are known as the heat polynomials [1, 26]. Any initial function in the form of power series, i.e. $f(x) = \sum_{n=0}^{\infty} a_n x^n$, allows $F_M(x, t)$ to be represented as the following expansion

$$
F_M(x, t) = \sum_{n=0}^{\infty} a_n H_n^{[M]}(x, \kappa_M t).
$$

(5)

In the general case the above representation of $F_M(x, t)$ can lead to the divergent series even for the well-defined initial condition $f(x)$. Nevertheless, equation (5) gives the correct asymptotic expansion of $F_M(x, t)$. The formal solution (5) is not effective because it converges for short times only, e.g., for $M = 2$ and $f(x) = \exp[-x^2/(2\sigma^2)]/(\sqrt{2\pi}\sigma)$ the convergence is limited to $t < \sigma^2/(4k)$ [27, 28]. (However, observe that such initial conditions as those before equation (5) are not integrable.) The correct long-time behavior of a solution of the heat equation is provided by the Gauss–Weierstrass transform for $M = 2$ [1]. Therefore we look for an analogous transform for $M = 3, 4, \ldots$. The main purpose of this paper is to find the new type of integral transform which will furnish the long-time behavior of the formal solution (3) for integer $M > 2$. The paper is organized as follows. In section 2 we will develop the operational methods initiated in [29, 31, 32] for generalizing the Gauss–Weierstrass transform. We will show that such a new
Following the example for $M = 3$ developed in [32], we consider the integral

$$p_M(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sxt + \kappa_M t^M} ds$$

(6)

with $s = c + ir$ and $c, \tau \in \mathbb{R}$. The quantity $p_M(x, t)$ of equation (6) will play the role of the higher-order heat kernel needed to represent the solutions of equation (1) via an appropriate integral transform. If $s = |s|e^{i\psi}$, $|s| = \sqrt{c^2 + \tau^2}$ and $\psi = \arctan(\frac{c}{\tau})$, then for $t > 0$, the following relations hold

$$|p_M(x, t)| \leq \int_{-\infty}^{\infty} \exp(\Re(xs + \kappa_M t^M)) d\tau = \int_{-\infty}^{\infty} \exp(x + \kappa_M t|s|^M \cos(M\psi)) d\tau.$$  

(7)

For large $\tau$ we have $|s| \approx \tau, \psi \approx \frac{\pi}{2} - \frac{c}{\tau}$ and $\cos(M\psi) \approx \cos(M\frac{\pi}{2}) + M\frac{\pi}{2} \sin(M\frac{\pi}{2})$, where for $M = 2m$, $\cos(2m\psi) \approx (-1)^m$ and for $M = 2m + 1$, $\cos[(2m + 1)\psi] \approx (2m + 1)(-1)^m \frac{\pi}{2}$. For large $\tau$, that gives equation (7) in the form

$$|p_M(x, t)| \leq \int_{-\infty}^{\infty} \exp(x + \kappa_{2m}(-1)^m t^2 \pm 2m) d\tau,$$  

(8)

where $M = 2m$, and

$$|p_M(x, t)| \leq \int_{-\infty}^{\infty} \exp(x + (2m + 1)\kappa_{2m+1}(-1)^{m+1} t^2 \pm 2m) d\tau$$  

(9)

with $M = 2m + 1$. The integral of equation (8) converges only for $\kappa_{2m} = (-1)^{m+1}$, whereas equation (9) converges for two values of $\kappa_M$, i.e. for $c > 0$ we have $\kappa_{2m+1} = (-1)^m$ and for $c < 0$, $\kappa_{2m+1} = (-1)^{m+1}$. That substantiates the conditions specified in equation (2).

Moreover, applying to equation (6) the Cauchy theorem with integration over the rectangle $s = \pm iR, c \pm iR$, it can be shown that in equation (6) we can take $c = 0$. As $|R|$ tends to infinity, the integrals over horizontal sides approach zero. It boils down to two inequalities:

$$\left| \int_{\pm iR} e^{sxt + \kappa_{2m} t^{2m}} ds \right| \leq e^{-R\sigma} \int_0^\infty e^{\sigma} d\sigma,$$  

(10)

for $M = 2m$ and $\tilde{s} = \sigma \pm i\bar{R}$, and

$$\left| \int_{\pm iR} e^{sxt + \kappa_{2m+1} t^{2m+1}} ds \right| \leq \int_0^\infty \exp(x\sigma \mp (2m + 1)\sigma R^{2m}) d\sigma,$$  

(11)

for $M = 2m + 1$, that vanish in the limit of $|R| \to \infty$. Consequently the integral equation (6) converges (for $M = 2m$ absolutely) when $c = 0$. Thus for $t > 0$

$$p_M(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ixt + \kappa_M (it)^M) d\tau,$$  

(12)
which, after substitution of $\kappa_M$ from equation (2) and changing the variable $y = \tau t^{1/M}$, can be expressed in the form
\begin{equation}
    p_M(x, t) = \frac{1}{t^{1/M}} g_M \left( \frac{x}{t^{1/M}} \right),
\end{equation}
(13)

where
\begin{equation}
    g_M(u) = \Re \left[ \frac{1}{\pi} \int_{0}^{\infty} \exp(\imath uy + \kappa_M(uy)^M) \, dy \right].
\end{equation}
(14)

It is clear that $p_M(x, 1) = g_M(x)$. (In [33] the integral kernels of the form of equation (13) have been already treated, but under the restriction $M = \alpha < 1$.) We point out that equations (12), (13) and (14) are defining in an alternative way the formal solution of equation (1) with $M$ challenge: until recently only for a limited number of values of $\alpha < 1$ we obtained explicit and exact forms of $g_M$ ([32]). For the arbitrary integer $M$ we obtain
\begin{equation}
    g_M(x) = \frac{1}{\pi} \int_{0}^{\infty} \exp(\imath uy + \kappa_M(uy)^M) \, dy,
\end{equation}
(15)

where $g_{2m}(u) = g_{2m}(-u)$ is the so-called symmetric Lévy stable signed function [6, 34, 35]. Note that $g_{2m}(u)$ of equation (15) here, are denoted by $g(2m, 0, u)$ in [6]. It should be stressed that the functions $g_{2m+1}(u)$ do not belong to the family of Lévy stable functions considered in [6]. For $M = 2m + 1$
\begin{equation}
    g_{2m+1}^{\pm}(u) = \frac{1}{\pi} \int_{0}^{\infty} \cos(u y \mp y^{2m+1}) \, dy
\end{equation}
(16)
defines the generalization of the Airy function in the sense of [29, 30, 36]. The functions $g_{2m+1}^{\pm}(u)$ are no longer even functions and equation (16) implies $g_{2m+1}^{\pm}(-u) = g_{2m+1}^{\mp}(u)$. Without loss of generality, in this paper we will study the case of $g_{2m+1}^{\mp}(u)$. We shall adopt the notation of [29] and henceforth set $g_{2m+1}^{\mp}(u) \equiv g_{2m+1}(u)$. The notation with $\pm$ will be used when necessary. The explicit and exact form of $g_M(u)$ will be derived in the next two sections.

It seems that obtaining $p_M(x, t)$ for the arbitrary integer $M \geq 2$ constituted a true challenge: until recently only for a limited number of values of $M$, i.e. $M = 2$ [37], see formula 2.3.15.11 on p 344], $M = 3$ [29, 32], $M = 4$ [38], and $M = 6$ [38] were the explicit forms of $p_M(x, t)$ known. The reference [6] provides an explicit solution for all rational values of admissible parameters, which include all integers $M$. We remark that $p_2(x, t)$ and $p_0(x, t)$ play an important role [38] in a theory of energy correlations in the ensembles of Hermitian random matrices [39, 40] and are related to the problem of phase transitions in chiral QCD models [41].

Let us now consider the two-sided (bilateral) Laplace transform [42, 43] of $p_M(x, t)$, compare equation (6). For $t > 0$ and $c \neq 0$ the integral (see equations (8) and (9))
\begin{equation}
    \int_{-\infty}^{\infty} \exp(-sy)p_M(y, t) \, dy = e^{c\imath t^{1/M}}
\end{equation}
(17)
is converging absolutely. The validity of equation (17) is easy to demonstrate for two cases: $M = 2$, where we use formula 2.3.15.11 on p 344 of [37], and $M = 3$, which is proved in [32]. For the arbitrary integer $M \geq 2$ the absolute convergence of equation (17) is ensured by the appropriate asymptotic behavior of $g_M(x)$ at infinity, namely $g_M(x) \approx \exp(-\beta x^{2})$, $\beta \geq 1$, see equations (10) and (11) for $t = 1$.

Equation (17) is a crucial formula of our paper because by making the substitution $s = \partial$, we obtain
\begin{equation}
    \hat{U}_M(t) = \int_{-\infty}^{\infty} \exp\left(-\imath y \frac{\partial}{\partial x}\right) p_M(y, t) \, dy,
\end{equation}
(18)
where \( \exp(-y \frac{\partial}{\partial x}) \) is the shift operator. That gives the integral representation of the evolution operator of equation (3) and, as a consequence, the general form of \( F_M(x,t) \):

\[
F_M(x,t) = \int_{-\infty}^{\infty} \exp \left( -y \frac{\partial}{\partial x} \right) p_M(y,t) f(x) \, dy = \int_{-\infty}^{\infty} p_M(y,t) f(x-y) \, dy. \tag{19}
\]

Equation (19) for \( M = 2 \) is the Gauss–Weierstrass transform, see [1], and for \( M = 3 \) it is the Airy \( \text{Ai} \) transform, see [24, 29, 32]. Several examples of \( f(x) \) such that equation (19) can be evaluated analytically are given in section 5.

3. The signed Lévy stable laws

This section is devoted to the exact and explicit forms of \( g_{2m}(u) \) for \( m = 1, 2, 3, \ldots \), see equation (15), and thereafter we look more closely at their asymptotic behavior at infinity and the associated Hamburger moment problem\(^4\).

In the spirit of references [5, 6] and [38] we shall provide the exact expression of \( g_{2m}(u) \) using the Mellin transform.

We start by supposing that for certain values of complex \( s \) the Mellin transform of \( g_{2m}(u) \) exists:

\[
g_{2m}^s(s) = \mathcal{M}[g_{2m}(u), s] = \int_0^\infty u^{s-1} g_{2m}(u) \, du, \tag{20}
\]

and \( g_{2m}(u) = \mathcal{M}^{-1}[g_{2m}^s(s), u] \). Then, using equation (15) and [37, see formulas 2.5.3.10 on p 387 and 2.3.3.1 on p 322] we have

\[
g_{2m}^s(s) = \frac{1}{2\pi i} \Gamma(s) \Gamma \left( \frac{1-s}{2m} \right) \cos \left( \frac{s\pi}{2m} \right). \tag{21}
\]

With the help of the second Euler reflection formula we express the cosine via the gamma function. Inverting the Mellin transform of \( g_{2m}^s(s) \) we obtain

\[
g_{2m}(u) = \mathcal{M}^{-1}[g_{2m}^s(s), u] = \frac{1}{2\pi i} \int_L u^{-s} \Gamma(s-1) \Gamma \left( 1 - \frac{s-1}{2m} \right) \, ds, \tag{22}
\]

with the contour \( L \) lying between the poles of \( \Gamma(s-1) \) and those of \( \Gamma \left( 1 - \frac{s-1}{2m} \right) \). After applying the Gauss–Legendre multiplication formula to the gamma functions in equation (22) we can express \( g_{2m}(u) \) in terms of the Meijer G functions \( G_{p,q}^{m,n}(\cdot) \) \([44]\):

\[
g_{2m}(u) = \sqrt{\frac{m}{\pi}} \frac{1}{u^{2m}} G_{2m,2m+1}^{1,2m+1} \left( \frac{(2m)^{2m}}{u^{2m}} \Delta(2m,0), \Delta(m,0) \right), \tag{23}
\]

where \( \Delta(k,a) = \frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k-1}{k} \) is a special list of \( k \) elements. Furthermore, it turns out that \( g_{2m}(u) \) is a finite sum of \( m \) generalized hypergeometric functions of type \( pF_q \):

\[
g_{2m}(u) = \sum_{j=1}^{m} c_j(m) u^{2j-2} \, \Gamma(2m, 2j) \left( \frac{1, 1 + \frac{2j-1}{2m}}{\Delta(2m, 2j)} \right)^z \tag{24}
\]

with \( z = (-1)^m u/(2m) \) and coefficients \( c_j(m) \) read

\[
c_j(m) = \sqrt{\frac{m/\pi}{(2m)^{2j-1} \pi^m}} \Gamma \left( 1 + \frac{2j-1}{2m} \right) \left[ \prod_{i=1}^{2j-1} \Gamma \left( \frac{2i-1}{2m} \right) \right] \left[ \prod_{i=2}^{2m} \sin \left( \pi \frac{m-1}{2m} \right) \right]^{-1}. \tag{25}
\]

The formulas equation (24) and (25) follow from equation (8.2.2.3) of [44] and are in agreement with equation (2.4) of [45]. Here, we have used the compact notation of the \( pF_q \) functions.

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\(^4\) For the purpose of this paper we use this terminology for signed functions.
where the upper (lower) list of parameters corresponds to the first (second) list of parameters in standard notation. We remark that in the lists of parameters of $2F_{2m}$ two cancellations of the same terms appear due to the obvious identity $pF_{q+r}(\frac{(α_j)_r}{(β_j)_r}) = pF_{q}(\frac{α_j}{β_j})$, where $(β_j)$ is an arbitrary sequence of $r$ parameters. Thus equation (24) finally reads as the sum of $m$ generalizations of hypergeometric functions of type $aF_{2m-2}(\frac{(β_j)_r}{z})$ [45].

Formulas (24) and (25) reconstruct the explicitly known cases presented in [6] and give an unlimited number of new exact solutions $g_{2m}(u)$, e.g. for $m = 4$

$$g_4(u) = c_4(4)u^6\; _0F_6\left(\frac{9}{8},\frac{5}{8},\frac{11}{8},\frac{3}{8},\frac{13}{8},\frac{7}{8};\frac{u^8}{8^8}\right) + c_2(4)u^4\; _0F_6\left(\frac{3}{4},\frac{7}{8},\frac{9}{8},\frac{5}{8},\frac{11}{8},\frac{3}{8};\frac{u^8}{8^8}\right)$$

$$+ c_2(4)u^4\; _0F_6\left(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{7}{8},\frac{9}{8},\frac{5}{8};\frac{u^8}{8^8}\right) + c_1(4)u^2\; _0F_6\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{7}{8};\frac{u^8}{8^8}\right) .
$$

(26)

The coefficients $c_j(4)$ for $j = 1, \ldots, 4$ in equation (26) are equal to $\sqrt{2}\cos\left(\frac{π}{8}\right)/\left[4\Gamma\left(\frac{3}{4}\right)\right]$, $-\sqrt{2}\sin\left(\frac{3π}{8}\right)/\left[8\Gamma\left(\frac{3}{2}\right)\right]$, $\sqrt{2}\sin\left(\frac{π}{8}\right)\sin\left(\frac{3π}{8}\right)/\left(96π\right)$, and $-\sqrt{2}\sin\left(\frac{5π}{8}\right)\sin\left(\frac{7π}{8}\right)/\left(2880π\right)$, respectively.

In figure 1 $g_8(u)$ (I; red line) and $g_2(u)$ (II; black line) are presented. The tails of the symmetric function $g_8(u)$ oscillate. The amplitude of these oscillations decreases with increasing values of $|u|$. Analogous behavior is observed in the other examples of $g_{2m}(u)$, for instance see figure 1 in [38] where the functions $g_4(u)$ and $g_6(u)$ are exhibited. Figures 1 and 2 in [38] illustrate well the considerations presented in [14].

3.1. Asymptotics of $g_{2m}(u)$.

The asymptotics of $g_4(u)$ and $g_6(u)$ for large $u$ are presented in [34, 38], whereas the general formula of $g_{2m}^r(u)$ ($m = 1, 2, \ldots$) for large $u$ is given in [15, 35]. Here, following [15, 35], we
Figure 2. Plot of the functions $A_{i}^{(2m+1)}(u)$ for $m = 1, 2, 3$; i.e. $A_{3}^{(3)}(u) = 3^{-1/3}A_{3}(-u/3^{-1/3})$ (I; black line) and $A_{3}^{(5)}(u)$ (II; blue line) given in equation (35), and $A_{3}^{(7)}(u)$ (III; red line) calculated from equations (33) and (34) for $m = 3$. Note that $A_{3}^{(3)}(u)$ is not oscillating for $u < 0$.

just present their compact form:

$$g_{2m}^{2}(u) \sim \frac{\sqrt{2}}{2m - 1} \frac{\pi}{\sqrt{(2m - 1)\pi}} u^{m-1} e^{-\frac{\pi}{2(2m - 1)}} \cos \left\{ \cos \left[ \frac{\pi}{2(2m - 1)} \right] Z - \frac{m - 1}{2m - 1} \right\},$$

$Z = (2m - 1)\left|\frac{u}{2m}\right|^{2m}$, which guarantees the absolute convergence of (17). For small $u$, the Lévy signed function can be represented by the series

$$g_{2m}^{2}(u) \sim \frac{1}{2m\pi} \sum_{r=0}^{\infty} \frac{(-1)^{r}}{(2r)!} \Gamma \left( \frac{r}{m} + \frac{1}{2m} \right) u^{2r},$$

which has an infinite radius of convergence.

3.2. The Hamburger moment problem for signed $g_{2m}(u)$.

Let us consider the following integral defining Hamburger moments of $g_{M}(u)$

$$h_{M}(\mu) = \int_{-\infty}^{\infty} u^{\mu} g_{M}(u) \, du,$$  \hspace{1cm} (28)

for $M = 2, 3, \ldots$ and arbitrary real $\mu$. In this section we will look closer at the case of even $M = 2m$, whereas equation (28) for odd $M = 2m + 1$ will be studied in section 4.

Since $g_{M}(u)$ is an even function for $M = 2m$, the integral (28) vanishes by symmetry for all odd $\mu = (2n + 1)$, regardless of the value of $m (m = 1, 2, \ldots)$. The calculation of $h_{2m}(2n)$ is a somewhat subtler problem which has been carefully analyzed in [34]. Decomposing $h_{2m}(\mu)$
function (see equations (33) and (34)). For the expression for $A_i$ we express equation (28) for $M = 2m$, $(m = 1, 2, \ldots)$, in the form, compare equation (20):

$$ h_{2m}(\mu) = [1 + (-1)^m]g_{2m}(\mu + 1) = \frac{1}{2m}[1 + (-1)^m] \frac{\Gamma(1 + \mu)}{\Gamma(1 + \frac{\mu}{m})} \sin \left( \frac{\pi \mu}{m} \right). \tag{29} $$

The moments $h_{2m}(\mu)$ vanish for $\mu = 2(mp + r)$, $p = 1, 2, \ldots$, $r = 1, 2, \ldots, m - 1$. The only non-zero terms of $h_{2m}(\mu)$ occur for $\mu = 2mp$ for which

$$ \lim_{\mu \to 2mp} h_{2m}(\mu) = (-1)^{(1+m)p} \frac{(2mp)!}{p!}. \tag{30} $$

We see that $h_{2m}(0) = 1$, that is $g_{2m}(a)$ is normalized to unity. The first few non-zero terms of $h_{2m}(\mu)$ for $m = 1, 2, 3$ and $\mu = 0, 1, \ldots, 12$ are presented in Table 1.

### Table 1. The values of the integrals (28) for $M = 2, 4, 6$ and $\mu = 0, 1, \ldots, 12.$

| $\mu$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| $h_2(\mu)$ | 1 | 2 | 12 | 120 | 1680 | 30 240 | 665 280 |
| $h_4(\mu)$ | 1 | -24 | 20 160 | -79 833 600 |
| $h_6(\mu)$ | 1 | 720 | 239 500 800 |

We observe that $h_{2m}(0) = 1$. For $m = 1, 2, 3$, the cancellation of the same two terms occurs. That gives the finite sum of the generalized hypergeometric functions of type $\,_{1}F_{0}$ for $\frac{\mu}{m} = 1, 2, 3$.

#### 4. Generalized Airy functions

To derive the exact and explicit form of $\text{Ai}^{(2m+1)}(u)$ we apply the method of section 3 with the Mellin transform $[\text{Ai}^{(2m+1)}(s)]^s = \mathcal{M}[\text{Ai}^{(2m+1)}(u); s]$ for complex $s$. Then, using equation (16) and formulas 2.5.3.10 on p 387 and 2.3.3.1 on p 322 of [37], we get

$$ [\text{Ai}^{(2m+1)}(s)]^s = \frac{\Gamma(s)\Gamma \left( \frac{1-s}{2m+1} \right)}{\pi(2m+1)} \cos \left( \frac{\pi(2ms+1)}{2(2m+1)} \right). \tag{31} $$

and performing the Mellin inversion as in equation (22), we obtain the exact and explicit expression for $\text{Ai}^{(2m+1)}(u)$ in terms of the Meijer G function:

$$ \text{Ai}^{(2m+1)}(u) = \sqrt{\frac{2m+1}{2\pi}} u^{\frac{1}{2m+1}} \sum_{\alpha = 0}^{2m-1} \left( \frac{\alpha}{u} \right)^{2m-\alpha} \frac{\Delta(2m+1, 0), \Delta(m, 0)}{\Delta(2m+1, 1, 0)}, \tag{32} $$

with $z = (-1)^m [u/(2m+1)]^{2m+1}$. Furthermore, the Meijer G function is converted to a finite sum of the generalized hypergeometric functions:

$$ \text{Ai}^{(2m+1)}(u) = \sum_{j=1}^{2m} \frac{b_j(m)}{u^{1-j}} F_{2m+1} \left( \begin{array}{c} 1, 1 + \frac{j}{2m+1} \\ 2m+1 \end{array} ; \Delta(2m+1, 1, 1 + j) \right) \tag{33} $$

with

$$ b_j(m) = \sqrt{\frac{(2m+1)}{(2m+1)}} \Gamma \left( 1 + \frac{j}{2m+1} \right) \left[ \prod_{\ell=1}^{2m-1} \Gamma \left( \frac{i-j-1}{2m+1} \right) \prod_{\ell=m+1}^{2m+1} \Gamma \left( \frac{i-j-1}{2m+1} \right) \right] \left[ \prod_{\ell=0}^{m} \sin \left( \frac{\pi}{m+1} \ell \right) \right]^{-1}. \tag{34} $$

In obtaining equations (33) and (34) we have again used the formula 8.2.2.3 of [44]. We point out that similarly to the case of the Lévy signed functions of the previous section, in the list of parameters $F_{2m+1}$ the cancellation of the same two terms occurs. That gives the finite sum of the $2m$ hypergeometric functions of type $\,_{0}F_{m-1}(\frac{\alpha}{m+1}) z$. Equations (33) and (34) are in agreement with the case of $t = 1$ of the formula for $u_{2m+1}(x, t)$ on p 2 of [46].

The formulas (33) and (34) reproduce the well-known case of the conventional Airy Ai function (see equations (33) and (34) for $m = 1$) and give the exact and explicit form of the
generalized Airy Ai functions \( \text{Ai}^{(2m+1)}(u) \), which were only numerically obtained in [36]. Without loss of generality, below we write out \( \text{Ai}^{(2m+1)}(u) \) for \( m = 2 \):

\[
\text{Ai}^{(5)}(u) = b_1(2) \ _0F_3 \left( \begin{array}{c} \frac{1}{5}, \frac{3}{5}, \frac{4}{5} \\ \frac{u^5}{5^5} \end{array} \right) + b_2(2) \ _0F_3 \left( \begin{array}{c} \frac{1}{5}, \frac{4}{5} \\ \frac{u^5}{5^5} \end{array} \right)
\]

\[
+ b_3(2) \ _0F_3 \left( \begin{array}{c} \frac{4}{5}, \frac{6}{5}, \frac{7}{5} \\ \frac{u^5}{5^5} \end{array} \right) + b_4(2) \ _0F_3 \left( \begin{array}{c} \frac{6}{5}, \frac{7}{5}, \frac{8}{5} \\ \frac{u^5}{5^5} \end{array} \right).
\]

(35)

The coefficients \( b_j(2) \), \( j = 1, \ldots, 4 \), are equal to \( \sqrt{5A}/[10\Gamma(\frac{3}{5}) \sin \left( \frac{\pi}{10} \right)] \), \( -\sqrt{5B}/[10\Gamma(\frac{3}{5}) \sin \left( \frac{\pi}{10} \right)] \), \( -\sqrt{5A}/[10\Gamma(\frac{3}{5})/(2\pi)] \), and \( \sqrt{5A}/[10\Gamma(\frac{3}{5})/(6\pi)] \), where \( A = \sin \left( \frac{3\pi}{10} \right) / \sin \left( \frac{\pi}{10} \right) \) and \( B = \sin \left( \frac{\pi}{10} \right) / \sin \left( \frac{\pi}{10} \right) \), respectively.

In figure 2 the functions \( \text{Ai}^{(2m+1)}(u) \) are presented from equations (33) and (34) for \( m = 1 \) (I; black line), \( m = 2 \) (II; blue line), and \( m = 3 \) (III; red line). For these generalized Airy Ai functions we observe the oscillation for positive and negative \( u \), which constitutes a good illustration of the considerations presented in [15].

The comparison between the functions \( g_{2m}(u) \) and \( \text{Ai}^{(2m+1)}(u) \) for fixed values of \( m \) is shown in figure 3; in figure 3(a) it is done for \( m = 8 \), whereas figure 3(b) is for \( m = 50 \). It turns out that for large \( m \) the difference between \( g_{2m}(u) \) and \( \text{Ai}^{(2m+1)}(u) \) is negligible.

4.1. Asymptotics of \( \text{Ai}^{(2m+1)}(u) \).

The asymptotic expansions of the generalized Airy Ai functions for large negative and positive values of \( u \) are given as

\[
\text{Ai}^{(2m+1)}_u(u) \sim \frac{1}{\sqrt{m\pi}} \left( 2m + 1 \right)^{-\frac{1}{2m+1}} u^{\frac{1}{2m+1}} e^{-\sin \left( \frac{\pi}{2m} \right) |Z|} \cos \left( \frac{\pi}{2m} \right) \left( \frac{\pi}{2m} \right) |Z| - m - \frac{\pi}{4},
\]

for \( u \to -\infty \), and

\[
\text{Ai}^{(2m+1)}_u(u) \sim \frac{1}{\sqrt{m\pi}} \left( 2m + 1 \right)^{-\frac{1}{2m+1}} u^{\frac{1}{2m+1}} e^{-\cos \left( \frac{\pi}{2m} \right) \left( \frac{\pi}{2m} \right) |Z| - m - \frac{\pi}{4},}
\]

for \( u \to \infty \).
for \( u \to \infty \), \( \bar{Z} = 2m[u/(2m + 1)]^{2m+1} \), see proposition 2 in [15]. The symbol \( \mathcal{A}_{2m+1}^{(2m+1)}(u) \) \( [\mathcal{A}^{(2m+1)}_{2m+1}(u)] \) denotes the asymptotic estimate for large negative (positive) \( u \). For \( m = 1 \) equations (36) and (37) lead to the known formulas on the asymptotic behavior of \( \mathcal{A}^{(2)}(u) \), see e.g. [32]. For \( m = 2 \) we have

\[
\mathcal{A}^{(2m+1)}_{2m+1}(u) \sim \frac{5^{-1/8}}{\sqrt{2\pi}} |u|^{-3/8} e^{-\bar{Z}} \sin \left( \frac{\sqrt{2}}{2} |\bar{Z}| + \frac{3\pi}{8} \right),
\]

\[
\mathcal{A}^{(2m+1)}_{2m+1}(u) \sim \frac{5^{-1/8}}{\sqrt{2\pi}} u^{-3/8} \sin \left( \bar{Z} + \frac{\pi}{4} \right),
\]

with \( \bar{Z} = 4(u/5)^{5/4} \). We point out that for large values of \( m \), \( \mathcal{A}^{(2m+1)}_{2m+1}(u) \) approaches \( \mathcal{A}^{(2m+1)}_{2m+1}(u) \).

The asymptotic behavior for small values of \( u \) is obtained by using the Taylor expansion of \( \mathcal{A}^{(2m+1)}_{2m+1}(u) \) or formula 2.5.21.6 on p 430 of [37]. That gives

\[
\mathcal{A}^{(2m+1)}_{2m+1}(u) \sim \frac{\pi^{-1}}{(2m + 1)} \sum_{r=0}^{\infty} \frac{u^r}{r!} \Gamma \left( \frac{1 + r}{2m + 1} \right) \cos \left( \frac{1 - 2mr}{4m + 2} \pi \right). \tag{39}
\]

The series in equation (39) has the infinite radius of convergence. For a given \( m \) the summation can be carried out and it agrees with the general formula of equation (33).

4.2. The Hamburger moment problem for \( \mathcal{A}^{(2m+1)}_{2m+1}(u) \).

Now, we look more closely at the Hamburger moment problem of \( \mathcal{A}^{(2m+1)}_{2m+1}(u) \). For \( M = 2m+1 \), equation (28) decomposes into two integrals according to the sign of \( u \). That gives the sum

\[
h_{2m+1}(\mu) = g_{2m+1}^{\ell, \ell^*}(\mu + 1) + (-1)^{\mu} g_{2m+1}^{\ell^*, \ell}(\mu + 1), \tag{40}
\]

where \( g_{2m+1}(\mu) \) denotes the Mellin transform of \( \mathcal{A}_{2m+1}(u) \) given in equation (16). Considering separately the case of even and odd moments, after using equations (20), (16) and formula 2.3.3.1 on p 322 of [37] we have

\[
h_{2m+1}(2n) = \frac{(2n-1)!}{\Gamma \left( \frac{2n}{2m+1} \right)} \sin (\pi n) \sin \left( \frac{\pi n}{2m+1} \right), \tag{41}
\]

\[
h_{2m+1}(2n+1) = -\frac{(2n)!}{\Gamma \left( \frac{2n+1}{2m+1} \right)} \cos \left( \frac{\pi n}{2m+1} \right). \tag{42}
\]

Let us analyze the properties of equations (41) and (42). At first we see that \( h_{2m+1}(2n) \) vanish for integer \( n \) except where \( n \) is the multiple of \( 2m+1 \). For \( n = (2m+1)k, k = 0, 1, \ldots \) the ratio of sines in \( h_{2m+1}(2n) \) is finite and goes to \( (2m+1) \). The only non-zero terms can be written in the form

\[
\lim_{n \to (2m+1)k} h_{2m+1}(2n) = \frac{(2m+1)!(2k)!}{(2k)!}, \tag{43}
\]

It is obvious that \( h_{2m+1}(0) = 1 \) for \( k = 0 \). An analogous situation to the case above appears for \( h_{2m+1}(2n + 1) \), for which for \( n = (2m+1)k + m \) the ratio of sine and cosine goes to \( (-1)^m(2m+1) \). That gives

\[
\lim_{n \to (2m+1)k+m} h_{2m+1}(2n+1) = (-1)^m \frac{(2m+1)(2k+1)!}{(2k+1)!}. \tag{44}
\]

The first few values of the non-vanishing terms of \( h_{2m+1}(\mu) \) for \( m = 1, 2 \) and 3 are presented in table 2.
5. Specific examples

The content of this section concerns the formal deliberations on the relations between $F_M(x, t)$, for a fixed initial condition, expressed by equations (5) and (19), and the integral transform approach.

(A) First we observe that equation (19) with $f(x) = x^n$ and the kernel given in equation (13) lead to

$$F^{(n)}_M(x, t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n-k} h_M(k),$$

where $h_M(k)$ are defined in (28) and for the initial condition we have used the Newton binomial theorem. Without loss of generality, let us look at the first few terms of \( (45) \) for $M = 3$. Using for that purpose $h_3(k)$ exhibited in table 2 we get: $F^{(0)}_3(x, t) = 1$, $F^{(1)}_3(x, t) = x$, $F^{(2)}_3(x, t) = x^2$, and $F^{(3)}_3(x, t) = x^3 + 6t$. That reproduces the Hermite–Kampé de Fériet polynomials $H^{(3)}_n(x, t)$ for $n = 0, \ldots, 3$. In the general case, from tables 1 and 2 it emerges that only \([n/M]th\) terms of $h_M(\mu)$ are different from zero. After changing the summation index in equation (45) to $k = Mp (p = 0, 1, \ldots, \lfloor n/M \rfloor)$ and employing formulas (30), (43) and (44) we convert equation (45) into equation (4). Making a similar consideration for the initial condition formally written in the form of the power series we can express the representation of $F_M(x, t)$ in the form of equation (5).

Our approach suggests a new way of looking at the Hermite–Kampé de Fériet polynomials, which according to equation (45) can be defined by the operational form

$$H^{(M)}_n(x, \kappa_M t) = (x - t^{1/M} \hat{h}_M)^\kappa \phi_0,$$

with

$$\hat{h}_M \phi_0 = h_M(k),$$

where $\phi_0$ is the ‘vacuum’ and $h_M(k)$ are given by equations (30), (43) and (44). See also [47] for related considerations in the context of lacunary Laguerre polynomials.

(B) The next example, which shows the validity of the method based on the integral transform (19), we consider the generalization of the Glaisher formula whose original form reads:

$$F_2(x, t) = \exp \left( k_2 \frac{\partial^2}{\partial x^2} \right) e^{-ax^2} = \exp \left( -\frac{ax^2}{1 + 4\alpha^2 t} \right).$$

for $t > -1/(4\alpha^2)$ and $k_2 = 1$. For that purpose we take as the initial condition $f(x) = g_M(\alpha x)$, $\alpha > 0$. For such a choice, we get

$$F_M(x, t) = \exp \left( \kappa_M \frac{\partial^M}{\partial x^M} \right) g_M(\alpha x) = \left( 1 + \alpha^M t \right)^{-1/M} g_M \left( \frac{\alpha x}{(1 + \alpha^M t)^{1/M}} \right),$$

$$-\infty < x < \infty \text{ and } t > -\alpha^M.$$ The constant $\kappa_M$ is hidden in the definition of $g_M(u)$ given in equation (14). Equation (49) neatly illustrates the scaling character of the time
In the last two examples, we choose, rather arbitrarily, the initial condition given by the FM, we are able to find the exact and explicit form of \( F_{gM} \) of the evolution operator on the initial condition first by using the integral transform of equation (19) and then by the study of the action the validity of equations (48), (49), and (50) are always satisfied for \( t > 0 \), compare with equation (12). Below, we will sketch the proof of equation (49) in two independent ways: first by using the integral transform of equation (19) and then by the study of the action of the evolution operator on the initial condition \( g_M(\alpha x) \). First, we use equation (19):

\[
F_M(x, t) = \frac{1}{\Gamma(1/M)} \int_{-\infty}^{\infty} g_M \left( \frac{y}{\Gamma(1/M)} \right) g_M(\alpha (x - y)) \, dy.
\]  

(51)

Substituting equation (14) into equation (51) we have three integrals to calculate, which after changing the order of integration, can be written as

\[
F_M(x, t) = \Re \left\{ \int_0^\infty \exp(\kappa M (iy_1)^M) \frac{dy_1}{\pi} \int_0^\infty \exp(\kappa M (iy_2)^M + \alpha \kappa x y_2) \frac{dy_2}{\pi} \right. \\
\left. \times \int_{-\infty}^{\infty} \exp \left( i \left( \frac{y_1}{\Gamma(1/M)} - \alpha y_2 \right) y \right) \, dy \right\}. \tag{52}
\]

The integral over \( y \) in equation (52) is equal to \( 2\pi \delta(y_1 - \alpha^{1/M} y_2) \). That simplifies the integration over \( y_1 \) in equation (52) and, in consequence, gives formula (49). Otherwise, \( F_M(x, t) \) can be obtained by employing equations (3) and (14), illustrated below:

\[
F_M(x, t) = \Re \left\{ \frac{1}{\pi} \int_0^\infty \exp \left( \kappa M \frac{dy}{\Gamma(1/M)} \right) e^{i\alpha \kappa x y} e^{i\kappa y_2} \, dy \right\} \\
= \Re \left\{ \frac{1}{\pi} \int_0^\infty \exp(i\alpha \kappa x y + \kappa M (iy)^M (1 + \alpha^{M} t)) \, dy \right\}. 
\]

After introducing \( y = u(1 + \alpha^{M} t)^{-1/M} \) we will recover equation (49).

(C) In the last two examples, we choose, rather arbitrarily, the initial condition given by the Cauchy distribution \( f(x) = \frac{1}{\pi \alpha^2 + x^2}, \alpha > 0 \). For that choice of \( f(x) \) and for even \( M \) we are able to find the exact and explicit form of \( F_M(x, t) \). Using equation (19) with the integral kernel \( p_{2m}(y, t) \) defined in equations (13) and (14), we obtain

\[
F_{2m}(x, t) = \Re \left\{ \int_{-\infty}^{\infty} \frac{du}{\pi^2 (\alpha^2 + u^2)} \left\{ \int_0^\infty \exp \left( i(x - u)z - z^{2m} \right) \frac{dz}{\pi^2} \right\} \right\}. \tag{53}
\]

where \( u = (x - y) \). Calculating first the integral over \( u \) and thereafter using formula 2.3.2.13 of [37], we get

\[
F_{2m}(x, t) = \Re \left\{ \int_0^\infty \exp \left( \frac{\alpha - iz}{\Gamma(1/(2m))} \right) \frac{dz}{\pi z} \right\} \\
= \frac{1}{2\alpha \pi} \Re \left\{ \sum_{j=0}^{2m} \frac{(-1)^{j-1}}{(j - 1)!} \frac{\Gamma(j)}{\Gamma(1/(2m))} \left( \frac{\alpha - i z}{\Gamma(1/(2m))} \right)^{j-1} \right\} \\
\times \frac{1}{\Delta(2m, j)} \left( \frac{(\alpha - i z)^{2m}}{t(2m)^{2m}} \right) \right\}. \tag{54}
\]
For odd $M$ the function $F_{2m+1}(x, t)$ given by

$$F_{2m+1}(x, t) = \frac{1}{\pi \alpha^{1/(2m+1)}} \int_0^\infty \exp \left( -\frac{z\alpha}{t^{1/(2m+1)}} \right) \cos \left( \frac{xz}{t^{1/(2m+1)}} - \frac{z^2}{2} + \frac{1}{2m+1} \right) dz$$

(55)
can be calculated only numerically.

For $m = 3$ the functions $F_6(x, t)$ given in equation (54) for $\alpha = 1$ and $t = 0.125, 0.415, 3$ and $10$ are illustrated in figure 4. The calculations indicate that for $\alpha = 1$ there exists a border time $t_1 \cong 0.415$ for which $F_6(x, t), t < t_1$, is positive, see line I in figure 4. For $t > t_1$ the function $F_6(x, t)$ is negative, see lines III and IV in figure 4. Line II in figure 4 presents the border case between the two previous situations. The function $F_6(x, t_1)$ is positive with the roots at the points $x_1 \equiv \pm 4.839$. The roots of $F_6(x, t > t_1)$ and the depth of the negative parts of $F_6(x, t > t_1)$ depend on the values of the parameter $\alpha$ and they can be minimized for an appropriate choice of $\alpha$ for given $t$.

In figure 5 for $m = 1$ the function $F_3(x, t)$ is presented, as numerically calculated from equation (55) for $\alpha = 2$ and $t = 0.5, 0.926, 2$ and $4$. For a given value of $\alpha$, we can also find the border time $t_1$ for which $F_3(x, t_1)$ ceases to be strictly positive and starts to have roots. For $\alpha = 2$ the border time $t_1$ is equal to 0.926. The strictly positive function $F_3(x, t < t_1)$ is shown as line I in figure 5, $F_3(x, t_1)$ is presented in line II, whereas two signed functions $F_3(x, t > t_1)$ are illustrated in lines III and IV. The existence of the negative parts in figure 5 are relics of the oscillations of the integral kernel $p_3(x, t)$ and they can be minimized by a suitable choice of $\alpha$ for fixed $t$.

(D) The formalism developed so far shows wide flexibility. It can be applied to solve a large class of partial differential equations in the form

$$\frac{\partial}{\partial t} \tilde{F}_M(x, t) = \hat{\mathcal{O}} M \tilde{F}_M(x, t)$$

(56)

Figure 4. Plot of $F_6(x, t)$ given by equation (54) for $\alpha = 1$ and a fixed value of $t$; line I is for $t = 0.125$ (red line), line II is for $t = 0.415$ (green line), line III is for $t = 3$ (yellow line) and line IV (blue line) is for $t = 10$. 

For odd $M$ the function $F_{2m+1}(x, t)$ given by

$$F_{2m+1}(x, t) = \frac{1}{\pi \alpha^{1/(2m+1)}} \int_0^\infty \exp \left( -\frac{z\alpha}{t^{1/(2m+1)}} \right) \cos \left( \frac{xz}{t^{1/(2m+1)}} - \frac{z^2}{2} + \frac{1}{2m+1} \right) dz$$

(55)
can be calculated only numerically.

For $m = 3$ the functions $F_6(x, t)$ given in equation (54) for $\alpha = 1$ and $t = 0.125, 0.415, 3$ and $10$ are illustrated in figure 4. The calculations indicate that for $\alpha = 1$ there exists a border time $t_1 \cong 0.415$ for which $F_6(x, t), t < t_1$, is positive, see line I in figure 4. For $t > t_1$ the function $F_6(x, t)$ is negative, see lines III and IV in figure 4. Line II in figure 4 presents the border case between the two previous situations. The function $F_6(x, t_1)$ is positive with the roots at the points $x_1 \equiv \pm 4.839$. The roots of $F_6(x, t > t_1)$ and the depth of the negative parts of $F_6(x, t > t_1)$ depend on the values of the parameter $\alpha$ and they can be minimized for an appropriate choice of $\alpha$ for given $t$.

In figure 5 for $m = 1$ the function $F_3(x, t)$ is presented, as numerically calculated from equation (55) for $\alpha = 2$ and $t = 0.5, 0.926, 2$ and $4$. For a given value of $\alpha$, we can also find the border time $t_1$ for which $F_3(x, t_1)$ ceases to be strictly positive and starts to have roots. For $\alpha = 2$ the border time $t_1$ is equal to 0.926. The strictly positive function $F_3(x, t < t_1)$ is shown as line I in figure 5, $F_3(x, t_1)$ is presented in line II, whereas two signed functions $F_3(x, t > t_1)$ are illustrated in lines III and IV. The existence of the negative parts in figure 5 are relics of the oscillations of the integral kernel $p_3(x, t)$ and they can be minimized by a suitable choice of $\alpha$ for fixed $t$.
Figure 5. Plot of $F_3(x, t)$ given in equation (55) for $\alpha = 2$ and fixed value of $t$; line I is for $t = 0.5$ (red line), line II is for $t = 0.926$ (green line), line III is for $t = 2$ (brown line) and line IV (blue line) is for $t = 4$.

with $\hat{O}_M$ the differential operator of order $M$ being the function of $x$ and $\frac{\partial}{\partial x}$ and with the initial condition $\tilde{F}(x, 0) = \tilde{f}(x)$. According to the technique proposed here the formal solution of equation (56) can be expressed by

$$\tilde{F}_M(x, t) = \int_{-\infty}^{\infty} \tilde{p}_M(y, t) e^{-y\hat{O}_M} \tilde{f}(x) \, dy,$$

(57)

where the kernel $\tilde{p}_M(y, t)$ has to be adapted to a precise form of the operator $\hat{O}_M$.

Equation (56) encompasses a large class of Fokker–Planck type operators, see [28] for related considerations. Furthermore the method can be shown to be applicable to the solution of problems where fractional evolution differential equations occur, as in the case of anomalous diffusion [28] and relativistic quantum mechanics [56, 57].

6. Conclusions

In this paper we have shown that the formalism of the evolution equation and associated integral transforms is a very efficient tool to deal with evolution problems involving generalizations of the heat equations through the introduction of higher-order derivatives. We have seen how the formalism is capable of including popular transforms such as Gauss–Weierstrass and Airy via the so-called signed Lévy stable and generalized Airy Ai functions.

The key result of the paper is the construction of a new technique for solving the HOHTE which furnishes the long-time behavior of equation (3). We have also shown that our technique reconstructs the Hermite–Kampé de Férié polynomials being the formal solution of equation (1) with the initial condition $f(x) = x^n$. For the initial condition given by the Lévy signed function and the generalized Airy Ai function we observe the scaling character of the time evolutions, which are a natural extension of Glaisher-type relations. The next interesting result is related to the existence the border time in which the time evolution calculated for the Cauchy distribution begins to possess negative parts.

Most of the formalism developed in the paper can be applied to non-standard forms of evolution equations which are encountered in physical problems concerning anomalous diffusion and quantum mechanical relativistic effects. Regarding the first point, we note
that many problems concerning the anomalous transport (in particular sub-diffusive) can be treated using HOHTE with not necessarily integer derivatives. Fractional transport is within the capabilities of the present formalism, which potentially offers the possibility of treating in a unified way the different phenomena occurring in economics [49, 50], population mobility [51, 52], infectious disease propagation [53], metastatic cancer spread [54, 55] etc.

Regarding the more genuine physical aspects, we believe that the methods we have explored may certainly help to illuminate old problems in relativistic quantum mechanics, such as some of its non-local features, occurring, e.g., in the analysis of the relativistic Schrödinger equation (see [56, 57], where some aspects of the underlying problems have started to be explored.) Before closing the paper we want to mention the possibility of looking at old problems with fresh eyes. The present formalism may yield unique tools to merge two aspects of anomalous diffusion and non-local quantum mechanics through the emergent Lévy generators [58], non-local in nature, which are naturally suited to provide a bridge between anomalous transport and pseudo-differential evolution in semi-relativistic quantum mechanics. It will be discussed in a forthcoming investigation.

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