FLOER LASAGNA MODULES FROM LINK FLOER HOMOLOGY

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Abstract. In this paper, we introduce the notion of Floer lasagna modules, which is inspired by the construction of skein lasagna module in [MWW19] by Morrison, Walker and Wedrich. Here we use link Floer homology instead of Khovanov-Rozansky homology. We give a description of the Floer lasagna module for 4-manifolds obtained by adding 2-handles to the 4-ball and we compute some examples.

1. Introduction

In [MWW19], Morrison, Walker and Wedrich introduced the notion of ‘lasagna algebra’, which is a higher dimensional analogue of a planar algebra. Roughly speaking, a lasagna algebra consists of a vector space \( \mathcal{L}(S, L) \) for each pair \((S, L)\) of a link \(L\) in a 3-sphere \(S\), and for each surface \(\Sigma \subset B^4 \setminus \bigcup_i B^4_i\) with \(\partial \Sigma = \Sigma \cap \left( \partial B^4 \cup \bigcup_i \partial B^4_i \right) = L \cup \bigcup_i L_i\), a map

\[ F_{\Sigma} : \bigotimes_i \mathcal{L}(\partial B^4_i, L_i) \to \mathcal{L}(\partial B^4, L), \]

which satisfies some compatibility conditions to give it an operad structure.

Based on this, they proposed a generalization of Khovanov-Rozansky homology to links in the boundary of arbitrary oriented 4-manifold by taking \(\mathcal{L}(S, L)\) to be the Khovanov-Rozansky homology \(\mathcal{K}h\mathcal{R}^N(L)\). The outcome is a triply-graded object

\[ \mathcal{S}^N(W, L) = \bigoplus_{b \in \mathbb{Z}} \bigoplus_{i,j \in \mathbb{Z}} \mathcal{S}_{b,i,j}^N(W; L), \]

where \(i, j\) are as the usual homological and quantum gradings of Khovanov-Rozansky homology, and \(b\), called the blob grading, is a new grading. The \(b = 0\) part \(\mathcal{S}_{0,i,j}^N(W; L)\) consists of equivalent classes of lasagna fillings, where the equivalence relation is induced by applying some operations associated to the lasagna algebra. The resulting \(\mathcal{S}_{0,i,j}^N(W; L)\) is later called skein lasagna module in [MN20], and it could be viewed as a higher dimensional generalization of skein modules of 3-manifolds. The other blob degree part is obtained by applying the construction of blob complex in [MW10].

In the definition, what is important is the functoriality of the invariant \(\mathcal{L}(S, L)\) in \(S^3\), which makes sure the lasagna algebra has the structure of an operad. Khovanov-Rozansky homology is such a functor for links in \(S^3\) and cobordisms in \(S^3 \times I\). Previously, we know the functoriality of Khovanov-Rozansky homology for links in \(\mathbb{R}^3\), see for example [ETW18]. In [MWW19], the authors proved the functoriality for links in \(S^3\), which requires checking that the extra sweep-around move induces identity maps on \(Kh^N\). See [MWW19 Theorem 1.1]. The construction of lasagna algebra, hence the equivalence relations on lasagna fillings, works for any such functors satisfying the functoriality condition.

Another natural candidate of such a functor is the link Floer homology \(\widehat{\mathcal{H}FL}\). In [Juh16], Juhasz gave a definition of cobordism maps on \(\widehat{\mathcal{H}FL}\) using cobordism maps on sutured Floer homology. Later in [Zem19c], Zemke gave a definition of cobordism maps on \(\mathcal{H}\mathcal{L}^{-}(L)\), a version of the minus-favored link Floer homology. In [JZ20], it was shown that both definitions agree when we are setting all the
variables \( U_i \) to 0 in \( HF^- (L) \) and summing over all the \( Spin^c \) structures. In this paper, we will mainly follow the treatment of the cobordism maps from [Zem19c].

We denote by \( L = (L, w, z) \) a multi-based link in a 3-manifold. The main object \( FL(W, L) \) of this paper is the counterpart of the skein lasagna module in the world of link Floer homology, i.e. if we apply the link Floer homology functor instead of the Khovanov-Rozansky functor to lasagna fillings. We call this object the Floer lasagna module; see Definitions 2.2, 2.3, 2.4. It is a doubly-graded object, where the gradings are inherited from the Maslov grading and the collapsed Alexander grading of the link Floer homology, with some adjustments according to the filling surface. It also divides according to the relative homology class of the filling surface in \( H_2(W, L) \). In [MN20], Manolescu and Neithalath gave expressions of \( S^2_{0, i, j}(W; L) \) when \( W \) is obtained by adding 2-handles to \( B^4 \), in terms of the cabled Khovanov homology defined there. See [MN20, Theorem 1.1]. The idea is to find a good representative in each equivalent class of lasagna fillings, such that the filling surface is in some ‘standard’ position.

In this paper, we will give a similar characterization of Floer lasagna modules in terms of the cabled link Floer homology \( cHF(L, K) \) which we will define later.

**Theorem 1.1.** Let \( W \) be the 4-manifold obtained from attaching 2-handles to \( B^4 \) along a framed link \( K \). Let \( L = (L, w, z) \) be a multi-based link in the 3-manifold \( \partial W \) such that \( L \) is disjoint from the surgery loci. Then we have a grading-preserving isomorphism:

\[
\Phi : cHF(L, K) \rightarrow FL(W, L).
\]

This is the main theorem of the paper. The proof is in the same spirit as that in [MN20]. The new feature is that on the filling surfaces \( \Sigma \) of a Floer lasagna filling, there are some dividing arcs \( A \subset \Sigma \) dividing \( \Sigma \) into two subsurfaces \( \Sigma_z \) and \( \Sigma_w \). In search of a good representative, we not only need to make the filling surface in ‘standard’ position, but also the dividing arcs. We call Floer lasagna fillings with the filling surface in ‘standard’ position by almost model Floer lasagna fillings, and we call those such that the dividing arcs are in ‘standard’ position as well model Floer lasagna fillings. See Definitions 3.4 and 3.5. We prove Theorem 1.1 by showing we can find a model Floer lasagna filling in each equivalent class. The ambiguity in choosing such a representative leads to various relations quotiented out in defining the cabled link Floer homology, namely the braid group action, pair-of-pants map and basepoint moving action.

Changing dividing arcs while keeping the filling surface induces a special kind of cobordism maps on link Floer homology, called the quasi-(de)stabilization maps. These first appeared in [MO10], and were further developed in [Zem17], [Zem19c]. We will follow the treatment of this topic in [Zem19c, Section 4].

After proving Theorem 1.1 we give some example calculations of the Floer lasagna module in the case when \( W \) is obtained by attaching a 2-handle along a 0-framed unknot \( U \) to \( B^4 \), i.e. \( W = S^2 \times D^2 \), and a special kind of multi-based links \( L \subset \partial W = S^2 \times S^1 \).

For the empty link \( L \), we obtained a similar result as in the case of skein lasagna module. Compare [MN20, Theorem 1.2]. This is as expected, because for unlinks in \( S^3 \), Juhász and Marengon proved in [JM18] that the functor \( HF \) coincides with the reduced Khovanov homology functor \( \tilde{Kh} \).

**Theorem 1.2.** Let \( W = S^2 \times D^2 \). For each \( \alpha \in H_2(W, \emptyset; \mathbb{Z}) = \mathbb{Z} \), the Floer lasagna module \( FL(W, \emptyset, \alpha) \) of relative homology class \( \alpha \) is given by

\[
FL(W, \emptyset, \alpha) = \bigoplus_{k \in \mathbb{N}} \mathbb{Z}_2,
\]

where there is a copy of \( \mathbb{Z}_2 \) in each Maslov grading \(-k\).

For a multi-based link \( L \) which intersects geometrically a capping disk of the unknot \( U \) once, we get that the Floer lasagna module vanishes:
Theorem 1.3. Suppose $W$ is obtained from $B^4$ by attaching a 2-handle along a 0-framed unknot $K$, and $L = (L, w, z)$ is a multi-based link in $\partial W$ which intersects a capping disk of $K$ geometrically once. Then

\[ FL(W, L) = 0. \]

In general, it is not so easy to compute the cabled link Floer homology. A more general approach would ideally involve some cobordism maps for bordered link Floer homology.

One advantage of link Floer homology compared to Khovanov-Rozansky homology is that we can define cobordism maps for link cobordisms in any $4$-manifolds, not only $S^3 \times I$. Therefore, for a given Floer lasagna filling, we can treat it as an input

\[ \otimes_i v_i \in \otimes_i \widehat{HFL}(S^3_i, \mathbb{L}_i) \cong \widehat{HFL}(\cup_i S^3_i, \cup_i \mathbb{L}_i), \]

together with a link cobordism $(W \setminus (\cup_i S^3_i), \Sigma)$ from $(\cup_i S^3_i, \cup_i \mathbb{L}_i)$ to $(\partial W, \mathbb{L})$. Therefore, one can define the evaluation map:

\[ ev : FL(W, L) \rightarrow \widehat{HFL}(\partial W, \mathbb{L}) \]

In other words:

**Question 1.4.** What are the kernel and cokernel of the map $ev$?

In the case when $W = B^4$, we have $FL(W, L) \cong \widehat{HFL}(S^3, \mathbb{L})$. See Proposition 2.9. In the case of Theorem 1.3, even though $L$ is not null-homologous in $\partial W = S^2 \times S^1$, the would-be link Floer homology should be 0, which agrees with $FL(W, L)$. In this way, we would like to view $FL(W, L)$ as some kind of generalization of $\widehat{HFL}(\partial W, \mathbb{L})$.

Another question is the relation between the Floer lasagna module and the skein lasagna module, especially in the view of the spectral sequence from Khovanov homology to link Floer homology as in [Dow18]:

**Question 1.5.** Is there any relation between $FL(W, L)$ and $S^N_0(W, \mathbb{L})$?

**Organization of the paper.** In Section 2.1 we define the Floer lasagna module, and the gradings on it. In Section 3 we define the cabled link Floer homology $c\widehat{HFL}(L, w, z, K)$, and prove Theorem 1.1. In Section 4 we perform some calculations of Floer lasagna modules, and prove Theorems 1.2, 1.3.

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## 2. Floer lasagna modules

### 2.1. Definition of Floer lasagna modules.

**Definition 2.1.** A multi-based link $\mathbb{L}$ in a 3-manifold $Y$ is a triple $L = (L, w, z) \in Y$, of an oriented link $L$ with two sets of basepoints $w$ and $z$, such that they alternate when we travel along each component of $L$, and there are at least two basepoints on each component of $L$.

The following definition is motivated by the similar one using Khovanov homology instead of link Floer homology. See [MWW19] and [MN20].
Definition 2.2. Let $W$ be a smooth oriented 4-manifold, and $L = (L, w, z)$ a multi-based link in $\partial W$. A Floer lasagna filling $\mathcal{F} = (B_i, L_i, \Sigma, \mathcal{A}, v_i)$ of $(W, L)$ is the following data:

- a disjoint union of finitely many 4-balls $B_i$, called the input balls, embedded in the interior of $W$, with a multi-based link $L_i = (L_i, w_i, z_i) \in B_i$;
- an oriented properly embedded surface $\Sigma$ in $W \setminus (\bigcup_i B_i)$, with $\Sigma \cap \partial W = L$ and $\Sigma \cap \partial B_i = L_i$, whose induced orientations agree with those of $L$ and $L_i$;
- A set of embedded 1-manifolds $\mathcal{A} \in \Sigma$, called dividing arcs, such that the components of $\Sigma \setminus \mathcal{A}$ are partitioned into type-$w$ and type-$z$ sub-surfaces $\Sigma \setminus \mathcal{A} = \Sigma_w \sqcup \Sigma_z$, with $w, w_i \in \Sigma_w$ and $z, z_i \in \Sigma_z$. In diagrams, we will shade the $\Sigma_w$ sub-surfaces, while the $\Sigma_z$ sub-surfaces are left blank. What’s more, the intersection of $\mathcal{A}$ with $L$ divides $L$ into components such that each component contains exactly one basepoint of $w \cup z$; similar condition holds for each $L_i$.
- an input $v_i \in \hat{HF}(S^3, L_i)$ for each $i$, where the link Floer homology are taking with coefficient $\mathbb{F}_2$.

In the language of [Zem19c], this is the same as a decorated link cobordism $(W \setminus \bigcup_i B_i, \Sigma^\sigma)$ from $(\bigcup_i B_i, \bigcup_i L_i)$ to $(\partial W, L)$, with an input $v_i \in \hat{HF}(S^3, L_i)$, where $\sigma$ is the trivial coloring which sends all elements in $w \cup w_i$ to $U$ and all elements in $z \cup z_i$ to $V$.

![Figure 1](image_url)

**Figure 1.** Example of Floer lasagna modules

We define the following equivalence relation on the set of Floer lasagna fillings.

Definition 2.3. Define the following relation $\sim'$ on the set of Floer lasagna fillings. For two Floer lasagna fillings $\mathcal{F} = (B_i, L_i, \Sigma, \mathcal{A}, v_i)$ and $\mathcal{F}' = (B_i', L_i', \Sigma', \mathcal{A}', v_i')$ of $(W, L, w, z)$, we say $\mathcal{F} \sim' \mathcal{F}'$ if there exists an input ball $B_i'$ of $\mathcal{F}'$ which satisfies the following conditions:

- $B_i'$ contains some input balls $\bigcup_{j \in I} B_j$ of $\mathcal{F}$, and is disjoint from other input balls of $\mathcal{F}$;
- up to isotopy relative to boundaries, $(\Sigma', \mathcal{A}')$ is the intersection of $(\Sigma, \mathcal{A})$ with $W \setminus B_i'$;
- up to isotopy, the link $L_i' \in \partial B_i'$ is the intersection $\Sigma \cap \partial B_i'$, and there is one basepoint in each component of $L_i' \setminus \mathcal{A}$, determined by the partition $\Sigma \setminus \mathcal{A} = \Sigma_w \sqcup \Sigma_z$.
the input $v'_i$ equals $F(\otimes v_j)$, where $F$ is the cobordism map induced by the intersection of $(\Sigma, A)$ with $B'_i \setminus \cup_j B_j$, viewed as a decorated link cobordism from $\cup_j (B_j, L_j)$ to $(B'_i, L'_i)$.

Define the equivalence relation $\sim$ to be the symmetric closure of $\sim'$ on the set of Floer lasagna fillings of $(W, L)$.

See [Zem19c, Section 12] for the definition of the cobordism maps induced by decorated link cobordisms. See Figure 2 for an example of equivalent Floer lasagna fillings.

In particular, by taking $B'_i = B_i$, we see that two lasagna fillings $\mathcal{F}, \mathcal{F}'$ are equivalent if there is an isotopy of $W$ relative to $\partial W$ mapping $\mathcal{F}$ to $\mathcal{F}'$.

This equivalence relation essentially says we don’t care about the ‘local’ behaviour of the lasagna fillings, where ‘local’ means everything happens inside a 4-ball in $W$.

Now we define the Floer lasagna module, which is akin to the one in [MWW19] and [MN20], except we apply the Heegaard Floer functor instead of the Khovanov-Rozansky functor.

**Definition 2.4.** For a smooth oriented 4-manifold $W$ and a multi-based link $L = (L, w, z) \in \partial W$, we define the Floer lasagna module $\mathcal{FL}(W, L)$ as the $\mathbb{F}_2$-module generated by the equivalence classes of Floer lasagna fillings of $L$ under the equivalence relation $\sim$, such that linear combinations of lasagna fillings are multilinear in the labels $v_i$.

**2.2. Relative homology classes.** As in the case of skein lasagna modules, we can consider the relative homology class of a Floer lasagna filling $\mathcal{F} = (B_i, L_i, \Sigma, A, v_i)$, which is given by the filling surface $\Sigma$, and takes value in

$$[\Sigma] \in H_2(W; L \cup (\cup_i L_i); \mathbb{Z}) \cong H_2(W, L; \mathbb{Z}),$$

where the last isomorphism could be realized by capping the surface $\Sigma$ along each $L_i \in \partial B_i$ by some surface in $B_i$.

If two Floer lasagna fillings $\mathcal{F} = (B_i, L_i, \Sigma, A, v_i)$ and $\mathcal{F}' = (B'_i, L'_i, \Sigma', A', v'_i)$ are equivalent, then they represent the same relative homology class in $H_2(W, L; \mathbb{Z})$, as $\Sigma$ and $\Sigma'$ are only differed in some $B'_i \setminus \cup_j (B_j)$. Therefore, we can define the relative homology class for each equivalent class of Floer lasagna fillings.

**Definition 2.5.** For each $\alpha \in H_2(W, L; \mathbb{Z})$, define $\mathcal{FL}(W, L, \alpha)$ as the subgroup of $\mathcal{FL}(W, L, w, z, \alpha)$ generated by equivalent classes of Floer lasagna fillings representing the relative homology class $\alpha$. We
have the decomposition
\[ \mathcal{FL}(W, \mathbb{L}) = \bigoplus_{\alpha \in H_2(W, \mathbb{Z})} \mathcal{FL}(W, \mathbb{L}, \alpha). \]

2.3. Gradings. We define a Maslov grading and an Alexander grading on the Floer lasagna module. We will use the collapsed Alexander grading instead of the multi-variable Alexander grading on \( \mathbb{L} \). Otherwise more data is required in the definition of the Floer lasagna fillings, i.e. other choices of the colorings in Zemke’s language. The main point in the definition is to check that it is well-defined under the equivalence relation of Floer lasagna fillings, which is a simple application of grading formulas of the link cobordism maps in [Zem19b].

Remark 2.6. Note that there are two different conventions for the normalization of the absolute Maslov grading, one requires that the top degree generator of \( \widehat{HF}(S^3, w) \) to be of Maslov grading 0, while the other requires it to be \( \frac{1}{2}(|w| - 1) \). We use the first convention in this paper, which leads to some changes of the formulas as presented in [Zem19b].

Definition 2.7. For a lasagna filling \( F = (B_i, L_i, \Sigma, \mathcal{A}, v_i) \) such that each \( v_i \) is a homogeneous element of \( \widehat{HF}(S^3, L_i, w_i, z_i) \), define the **Maslov grading** of \( F \) to be
\[ M(F) = \chi(\Sigma_w) + \sum_i M(v_i), \]
and the **Alexander grading** to be
\[ A(F) = \frac{\chi(\Sigma_w) - \chi(\Sigma_z)}{2} + \sum_i A(v_i), \]
where \( M(v_i) \) is the absolute Maslov grading of \( v_i \in \widehat{HF}(S^3, L_i, w_i, z_i) \) and \( A(v_i) \) is the sum of the multi-Alexander grading of \( v_i \).

Lemma 2.8. The Maslov and Alexander gradings are well-defined for Floer lasagna modules.

Proof. It is enough to show that for \( F = (B_i, L_i, \Sigma, \mathcal{A}, v_i) \), and \( F' = (B'_i, L'_i, \Sigma', \mathcal{A}', v'_i) \) such that \( F \sim F' \) as in Definition 2.3, we have
\[ M(F) = M(F') \text{ and } A(F) = A(F'). \]

Let \( W' = B'_i \cup_{j \in J} B_j \), where \( B'_i \) and \( B_j \) are as defined in Definition 2.3. It follows from [Zem19b], Theorem 1.4]: For a cobordism \( (W, F) : (Y_1, \mathbb{L}) \to (Y_2, \mathbb{L}_2) \),
\[ gr_w(F_{W,z}(x)) - gr_w(x) = c_1(s)^2 - 2\chi(W) - 3\sigma(W) + \chi(\Sigma_w) - \frac{|w_1| + |w_2|}{2}. \]

We apply this to the cobordism \( (W', \Sigma \cap W') : (\cup_{j \in J} B_j, \cup_{j \in J} L_j) \to (B'_i, L'_i) \). Because of the different conventions for normalization, there is a difference with his Maslov grading \( gr_w \) and our \( M \):
\[ gr_w(v'_i) = M(v'_i) + \frac{1}{2}(|w'_i| - 1), \text{ and } gr_w(\otimes_j v_j) = M(\otimes_j v_j) + \sum_{j \in J} \frac{1}{2}(|w'_j| - 1), \]

As \( W' \) has trivial \( H_2(W', \mathbb{Z}) \), it follows that
\[ M(v'_i) = M(\otimes_j v_j) + \chi(\Sigma_w \cap W') - |w'_i|. \]

Since there is exactly one \( w \)-basepoint in \( w'_i \) for each component of \( \Sigma_w \cap \partial B'_i \), and \( \Sigma'_w \) is isotopic to \( \Sigma_w \setminus (\Sigma_w \cap W') \), we have
\[ \chi(\Sigma_w) = \chi(\Sigma_w \cap W') + \chi(\Sigma'_w) - |w'_i|. \]
so

\[ M(\mathcal{F}') - M(\mathcal{F}) = M(v'_i) + \chi(\Sigma'_w) - M(\otimes v_j) - \chi(\Sigma_w) = 0, \]

as required.

The verification for the well-definedness of Alexander grading follows similarly from the formula for the Alexander grading change of link cobordism maps in [Zem19b, Theorem 1.4] applied to the cobordism \((W', \Sigma \cap W')\), which says

\[ A(F(x)) - A(x) = \frac{\chi(\Sigma_w) - \chi(\Sigma_z)}{2}, \]

when the cobordism \(W'\) has trivial \(H_2(W'; \mathbb{Z})\). Note that

\[ \chi(\Sigma_w) - \chi(\Sigma_z) = \chi(\Sigma_w \cap W') - \chi(\Sigma_z \cap W') + \chi(\Sigma'_w) - \chi(\Sigma'_z), \]

as \(|w'_i| = |z'_i|\), so

\[ A(F') - A(\mathcal{F}) = A(v'_i) + \frac{\chi(\Sigma'_w) - \chi(\Sigma'_z)}{2} - A(\otimes v_j) - \frac{\chi(\Sigma_w) - \chi(\Sigma_z)}{2} = 0, \]

as required. \(\square\)

Let us see an example of the Floer lasagna modules, in the simplest case when \(W = B^4\). From this, we can view the Floer lasagna module as an extension of the link Floer homology.

**Lemma 2.9.** If \(W = B^4\), and \(L = (L, w, z)\) a multi-based link in \(\partial W = S^3\), then

\[ \mathcal{FL}(B^4, L) \cong \widehat{HFL}(S^3, L), \]

as doubly-graded \(\mathbb{F}_2\)-modules, with the multi-Alexander grading on \(\widehat{HFL}\) collapsed to the single Alexander grading on \(\mathcal{FL}(B^4, L, w, z)\).

**Proof.** For any equivalent class of Floer lasagna fillings \([\mathcal{F}]\), we can always take a particular Floer lasagna filling \(\mathcal{F}' = (B', L', \Sigma', \mathcal{A}', v')\) as a representative, such that \(B' = W \setminus \text{nbhd}(\partial W)\) where \(\text{nbhd}(\partial W)\) is a collar neighborhood of \(\partial W\), and \((W \setminus B', \Sigma', \mathcal{A}')\) is the identity cobordism with \(\Sigma' = L' \times I, \mathcal{A}' = C \times I\) for some set of points \(C\) on \(L\), one between each pair of adjacent basepoints \(w \cup z\). This defines a map

\[ \Psi : \mathcal{FL}(B^4, L) \to \widehat{HFL}(S^3, L), \]

\[ \Psi([\mathcal{F}]) = v'. \]

It is easy to write down the inverse map \(\Phi\) of \(\Psi\) such that for any \(v \in \widehat{HFL}(S^3, L)\), \(\Phi(v) = [(B, L, \Sigma, \mathcal{A}, v)]\) is the particular representative we have picked as before for each equivalence class of Floer lasagna fillings. \(\square\)

3. **Adding 2-handles and the cabled link Floer homology**

In this section, we study the effect of adding 2-handles to the Floer lasagna modules. We will see that we can always find some good representative of an equivalence class of Floer lasagna fillings of \((W, L)\), such that one component of the filling surface \(\Sigma\) is \(L \times I\) with dividing arcs \(C \times I\), and each other connected component is a 2-disck, with a diameter as the dividing arc. Then it could be represented by an element in the so-called cabled link Floer homology \(cHFL\), which we will define later. The treatment is very close to that in [MN20], with some extra attention paid to the dividing arcs. The main proposition of this section is the following:
**Proposition 3.1.** Let $W$ be the 4-manifold obtained from attaching 2-handles to $B^4$ along a framed link $K$. Let $L = (L, w, z)$ be a multi-based link in the 3-manifold $\partial W$ such that $L$ is disjoint from the surgery loci. Then we have a grading-preserving isomorphism:

$$\Phi : \widetilde{cHF}(L, K) \rightarrow FL(W, L).$$

We begin by defining the cabled link Floer homology $\widetilde{cHF}(L, K)$.

### 3.1. Cabled link Floer homology

Let $K$ be a framed link in $S^3$. Fix a parametrization $f_i : S^1 \times D^2 \to \text{nhbd}(K_i)$ for each component $K_i$ of $K$, where $S^1 = \mathbb{R}/\mathbb{Z}$. For two $n$-tuples $k^+ = (k^+_1, \ldots, k^+_n)$ and $k^- = (k^-_1, \ldots, k^-_n)$, define $\hat{K}(k^+, k^-)$ as the link obtained from $K$ by replacing each component $K_i$ with $k^+_i$ many positively oriented and $k^-_i$ many negatively oriented parallel copies along the framing of $K_i$, e.g. for each $i$, picking distinct points $x_{i,1}, \ldots, x_{i, k^+_i}, x_{i, k^-_i}, \ldots, x_{i,1}$ in $D^2$ which lie on a fixed diameter of $D^2$, and let

$$K(k^+, k^-) = \bigcup_i f_i(S^1 \times \{x^+_{i,1}, \ldots, x^+_{i, k^+_i}, x^-_{i,1}, \ldots, x^-_{i, k^-_i}\}).$$

Add two basepoints

$$(3.1)\quad w_{i,j}^\pm = f_i(0, x^\pm_{i,j}), \quad z_{i,j}^\pm = f_i\bigg(\frac{1}{2}, x^\pm_{i,j}\bigg)$$

on each component of $K(k^+, k^-)$. Let $w' = \cup_{i,j} w_{i,j}$ and $z' = \cup_{i,j} z_{i,j}$. We denote the link Floer homology of $(S^3, L \cup K(k^+, k^-), w \cup w', z \cup z')$ by $\widetilde{HF}(L, K, k^+, k^-)$.

There are three maps between $\widetilde{HF}(L, K, k^+, k^-)$ that we will consider: An action of some subgroup of the braid group switching parallel components of the same orientation, a pair-of-pants map combining two parallel components of opposite orientations, and a basepoint moving map on each component. We describe them more carefully.

We start with the braid group action. For each $i$, consider the subgroup $B_{k^-_i, k^+_i} = \phi^{-1}(S_{k^-_i} \times S_{k^+_i}) \subset B_{k^-_i + k^+_i}$, where $\phi : B_{k^-_i + k^+_i} \to S_{k^-_i + k^+_i}$ is the natural map to the symmetric group $S_{k^-_i + k^+_i}$, and $S_{k^-_i} \times S_{k^+_i}$ is viewed as a subgroup of $S_{k^-_i + k^+_i}$. Then each element $\tau$ of $B_{k^-_i, k^+_i}$ could be viewed as a map

$$\tau : \bigcup_{j=1, \ldots, k^-_i + k^+_i} I \to D^2 \times I,$$

which is an embedding of $(k^-_i + k^+_i)$-many intervals into $D^2 \times I$. Multiplying this embedding by $S^1$ and composing with $f_i$ gives a cobordism $f_i \circ \tau : \cup_j S^1 \times I \to f_i(S^1 \times D^2 \times I) \subset S^3 \times I$ from parallel copies of $K_i$ to itself. Denote the image of $f_i \circ \tau$ by $\Sigma$. Add dividing arcs $A$ which are the image $f_i \circ \tau(\cup_j \{1/4, 3/4\} \times I)$, which divides each cylinder into two halves. Call the half containing $w$ basepoints $S^1_w$ and the other half $\Sigma_w$. Take the union of this link cobordism with the identity cobordism over other components of $L \cup K(k^+, k^-)$. This cobordism gives a map

$$F_\tau : \widetilde{HF}(L, K, k^+, k^-) \to \widetilde{HF}(L, K, k^+, k^-).$$

This defines an action of the subgroup $B_{k^-_i, k^+_i}$ for each $i$, by the functoriality of the link cobordism map as proved in [Zem19c]. See Figure 3 for the corresponding cobordism of a single crossing in $B_{k^-_i, k^+_i}$.

Now we look at the pair-of-pants map. Two oppositely oriented parallel copies of $K_i$ bounds a cylinder $P$ in $S^3$. Isotope $P$ into $S^3 \times I$ such that it intersects with $S^3 \times \{0\}$ at a trivial knot $U$ in $S^3 \times S^3$. Remove the disk in $P$ bounded by $U$ gives a cobordism from $U$ to two oppositely oriented parallel copies of $K_i$, which is a pair-of-pants. Give the following dividing arc on this pair-of-pants as
Figure 3. Decorated link cobordism for $\tau_{i,i+1} \in B_n$

in Figure 4. In particular, $U = (U, w', z')$ has exactly two basepoints on it. Together with the identity cobordism over other components, it gives a map

$$F_p : HFL(L \cup U, K, k^+, k^-) \to HFL(L, K, k^+ + e_i, k^- + e_i),$$

where $e_i$ is the $n$-tuple with 1 in the $i$th slot and 0 elsewhere.

Figure 4. Pair-of-pants cobordism

The pair-of-pants map is the composition of a quasi-stabilization map with a band surgery map. See [Zem19c, Section 4, 6] for details of the definition.

Note that

$$HFL(L \cup U, K, k^+, k^-) \cong HFL(L, K, k^+, k^-) \otimes V,$$

where $V = F_2(T, B)$ is a 2-dimensional $\mathbb{F}_2$-vector space generated by two elements $T, B$, with $M(T) = 0, M(B) = -1,$ and $A(T) = A(B) = 0$. The death cobordism $D$ capping the unknot $U \cup D : HFL(L \cup U, K, k^+, k^-) \to HFL(L, K, k^+, k^-)$ acts by

$$D(v \otimes B) = v, \quad D(v \otimes T) = 0.$$  

See Figure 5 for the decorated link cobordism inducing $D$.

Finally there is a basepoint moving map, which is induced by the decorated link cobordism as in Figure 6 on the $j$th parallel copy of $K_i$, $j = 1, \ldots, k_i^- + k_i^+$, and the identity cobordism on the rest components. We denote it by

$$F_{m_{i,j}} : HFL(L, K, k^+, k^-) \to HFL(L, K, k^+, k^-).$$

See [Zem17] and [Zem19c, Section 4] for the definition of the basepoint moving maps.
Now we give the definition of the cabled link Floer homology, which is the $\mathbb{F}_2$-vector space generated by $\widehat{HF}(L, K, k^+, k^-)$ for all possible $n$-tuples $k^+, k^-$, quotienting out the relations given by maps $F_\tau, F_P, F_{m_{i,j}}$.

**Definition 3.2.** The **cabled link Floer homology** of $(L, K)$ is defined as

$$\widehat{cHF}(L, K) = \bigoplus_{k^+, k^- \in \mathbb{N}^n} \widehat{HF}(L, K, k^+, k^-) \left[|k^+| + |k^-|\right] / \sim,$$

where $[l]$ shifts the Maslov grading up by $l$, $|k^+|, |k^-|$ are the sum of entries of $k^+, k^-$ respectively, and $\sim$ is the equivalence relation which is the transitive and linear closure of the following relations:

- $v \sim F_\tau(v)$;
- $v \sim F_P(v \otimes B), 0 \sim F_P(v \otimes T)$;
- $v \sim F_{m_{i,j}}(v)$.

**Remark 3.3.** Note the equivalence relation preserves both the Maslov grading and the Alexander grading, so they descend to gradings on $\widehat{cHF}(L, K)$. For the grading shift of the pair-of-pants cobordism, see [Zem19b].

### 3.2. Almost model and model Floer lasagna fillings

In this subsection, we give the definition of almost model and model Floer lasagna fillings, which are some particular representatives in each equivalent class of Floer lasagna fillings, which we will use later in the proof of Proposition 3.1. This is motivated by the proof of the similar statement as Proposition 3.1 in the skein lasagna module in [MN20].
Definition 3.4. Suppose \( W \) is a 4-manifold obtained from attaching 2-handles to \( B^4 \) along a framed link \( K, \mathbb{L} = (L, w, z) \) is a multi-based link in \( \partial W \) away from the surgery loci, and \( v \in HF(L, K, k^+, k^-) \).

A model Floer lasagna filling \( \mathcal{F} = (B, \mathcal{L} \cup \mathcal{K}(k^+, k^-), \Sigma, A, v) \) of \( (W, \mathbb{L}) \) is a Floer lasagna filling such that:

- Parameterize each 2-handle by \( f_i : D^2 \times D^2 \to W \), such that the framing of \( K_i \) is given by \( f_i(\partial D^2 \times \{0\}) \). Let \( W' = W\setminus(\bigcup_i f_i(B_i(0) \times D^2)) \),

where \( B_i(0) \) is a small neighborhood of \( 0 \in D^2 \), i.e. we delete a small neighborhood of each cocore of the 2-handle. Let \( B = W' \setminus \text{nbhd}(\partial W') \), where \( \text{nbhd}(\partial W') \cong (0, 1) \times \partial W' \) is a collar neighborhood of \( \partial W' \). By construction, \( B \) is a 4-ball.

- \( L' = \{1\} \times L \subset \partial B \subset (0, 1) \times \partial W' \) is a copy of \( L \) and \( K'(k^+, k^-) = \bigcup_i f_i(B_i(0) \times \{x^+_{i,1}, ..., x^+_{i,k_i^+}, x^-_{i,1}, ..., x^-_{i,k_i^-}\}) \).

are copies of \( K(k^+, k^-) \) which are on the boundary \( \partial B_i(0) \times D^2 \subset \partial B \) instead of \( \partial D^2 \times D^2 \). See Equation 3.1.

- \( w', z' \) consist of basepoints on \( L' \) which correspond to \( w, z \) on \( L \), and basepoints \( w_{i,j}^+, z_{i,j}^\pm \) on \( K'(k^+, k^-) \), two on each components. See Equation 3.2. \( L' \) and \( \mathcal{K}(k^+, k^-) \) are the corresponding multi-based links.

- \( \Sigma \) is the disjoint union of the following surface: There is a union of cylinders \( L \times I \subset \text{nbhd}(\partial W') \subset W \setminus B \) connecting \( L \subset \partial W \) and \( L' \subset \partial B \). For each parallel copy \( f_i(\partial B_i(0) \times x_{i,j}^\pm) \) of \( K_i \), there is a disk \( f_i(B_i(0) \times x_{i,j}^\pm) \) capping it, whose orientation is induced by the boundary orientation. Let \( \Sigma = (L \times I) \cup \bigcup_{i,j} f_i(B_i(0) \times x_{i,j}^\pm) \).

- The dividing arcs \( A \) on \( L \times I \) is given by \( C \times I \), where \( C \) is a set of points, one in each connected component of \( L \setminus (w \cup z) \). On the disk \( f_i(\partial B_i(0) \times x_{i,j}^\pm) \), the dividing arc is given by \( f_i([-\epsilon, \epsilon] \times \{0\} \times x_{i,j}^\pm) \). We partition \( \Sigma \setminus A = \Sigma_w \cup \Sigma_z \) depending on whether it contains \( w' \)-basepoints or \( z' \)-basepoints.

- The input \( v \in HF(L, K, k^+, k^-) \) could be viewed as an element of \( HF(L' \cup K'(k^+, k^-)) \) naturally.

See Figure 3.1 for an example of a model Floer lasagna filling.

Definition 3.5. If we loose the restriction on the dividing arcs on disks \( f_i(B_i(0) \times x_{i,j}^\pm) \), and hence the number of basepoints on \( K'(k^+, k^-) \) of a model Floer lasagna filling, we call such a Floer lasagna filling as an almost model Floer lasagna filling.

In our proof of Proposition 3.1, we will associate a model Floer lasagna filling for each element \( v \in HF(L, K, k^+, k^-) \), and proves this gives a well-defined map from the cabled link Floer homology \( cHF(L, K) \) to the Floer lasagna module \( \mathcal{FLL}(W, L) \). To show it is a bijection, we will define the inverse map, which amounts to find a model Floer lasagna filling in each equivalent class of Floer lasagna fillings. Roughly speaking, this is given by taking \( B \) as in the definition of the model Floer lasagna filling, and use the equivalence relation on Floer lasagna fillings. The problem is that we don’t have control of the dividing arcs on disks of the form \( f_i(B_i(0) \times x_{i,j}^\pm) \), so we will get almost model Floer lasagna fillings, instead of model ones. In the rest of the subsection, we will present how to obtain
represent an almost model Floer lasagna filling as a linear combination of model Floer lasagna fillings, viewed as elements in $\mathcal{FL}(W, L)$.

The main tool is quasi-(de)stabilization map on link Floer homology. It deals with what happens to link Floer homology when we add/delete basepoints on the link. It first appeared in [MO10], and was further illustrated in [Zem17], [Zem19c]. The main reference of the following discussion is [Zem19c, Section 4].

**Definition 3.6.** Let $L = (L, w, z)$ be a multi-based link in $S^3$. Suppose $w', z'$ are two new basepoints of on some component of $L$, such that $w', z'$ lie in the same connected component of $L \setminus (w \cup z)$, and $w'$ follows $z'$ with respect to the orientation of $L$. We define the following four cobordism maps

$$S^+, T^+: \widehat{HFL}(L, w, z) \to \widehat{HFL}(L, w \cup w', z \cup z'),$$

$$S^-, T^-: \widehat{HFL}(L, w \cup w', z \cup z') \to \widehat{HFL}(L, w, z),$$

induced by the following decorated link cobordisms $(S^3 \times I, L \times I, A_i)$ for $i = 1, 2, 3, 4$.

**Figure 8.** Different quasi-(de)stabilization cobordisms
Lemma 3.7. Suppose \((L, w, z)\) is a multi-based link in \(S^3\) of \(m\) components, with \(|w| = |z| = n\). Let \(w' \subset w, z' \subset z\) be subsets of the basepoints such that on each component of \(L\), there is exactly one basepoint in \(w'\) and exactly one basepoint in \(z'\). Then
\[
\widehat{HF}(L, w, z) \cong \widehat{HF}(L, w', z') \otimes V^{\otimes(n-m)},
\]
where \(V = \mathbb{F}_2(\theta^w, \xi^w)\) is a bigraded 2-dimensional vector space over \(\mathbb{F}_2\), where
\[
M(\theta^w) = 0, \quad M(\xi^w) = -1, \quad A(\theta^w) = \frac{1}{2}, \quad A(\xi^w) = -\frac{1}{2}.
\]

Proof. It follows from [Zem17, Proposition 5.3] by setting all the \(U\) and \(V\) variables equal to 0, and induction on \(n\). Note the difference in the Maslov grading conventions.

In this formalism, we can write down the maps \(S^\pm, T^\pm\) explicitly as follows:
\[
\begin{align*}
S^+(v) &= v \otimes \theta^w, & S^-(v \otimes \theta^w) &= 0, & S^-(v \otimes \xi^w) &= v \\
T^+(v) &= v \otimes \xi^w, & T^-(v \otimes \theta^w) &= v, & T^-(v \otimes \xi^w) &= 0,
\end{align*}
\]
for any \(v \in \widehat{HF}(L, w, z)\).

From these descriptions, it is easy to see the following lemmas:

Lemma 3.8. Let \((L, w, z)\) and \((L, w', z')\) be as in Lemma 3.7. For any element \(v \in \widehat{HF}(L, w', z')\) and \(u \in V^{\otimes(n-m)}\), there exists a decorated link cobordism \((S^3 \times I, K \times I, \mathcal{A})\), such that
\[
F(v) = v \otimes u,
\]
where \(F\) is the map induced by the decorated link cobordism on link Floer homology.

Proof. We concatenate \((S^3 \times I, L \times I, \mathcal{A}_1)\) and \((S^3 \times I, L \times I, \mathcal{A}_2)\), one for each pair of extra basepoints in \((w \setminus w') \cup (z \setminus z')\), depending on whether the corresponding component of \(V^{\otimes(n-m)}\) is \(\theta^w\) or \(\xi^w\). The result then follows from Equation (3.4).

Lemma 3.9. If \((S^3 \times I, L \times I, \mathcal{A})\) is a decorated link cobordism from \((L, w, z)\) to itself, such that there exists one connected component of either the subsurfaces \(\Sigma_w\) or \(\Sigma_z\) which is a closed disk not intersecting the boundary \(L \times \{0, 1\}\), then the induced map \(F\) on link Floer homology is 0.

Proof. Suppose there exists such a connected component of \(\Sigma_w\) as in the Lemma, then by shrinking this disk via isotopy, we can decompose \(F\) as
\[
F = F_1 \circ S^- \circ S^+ \circ F_2,
\]
for some \(F_1\) and \(F_2\). Now
\[
S^- \circ S^+ = 0,
\]
so \(F = 0\). The situation for disks in \(\Sigma_z\) is the same by replacing \(S\) with \(T\).

Now suppose \(F = (B, L' \cup \mathcal{K}'(k^+, k^-), \Sigma, \mathcal{A}, v)\) is a almost model Floer lasagna filling.

To simplify notation, we assume \(W\) is obtained by attaching a single 2-handle along the framed knot \(K\) and \(L = \emptyset\), as all the operations we are going to do are concentrated near the cocore of each 2-handle. We assume further that \(k^+ = 1, k^- = 0\), as we can do the following operation to each disk \(f_i(B_{r(0)} \times x_{i,j})\) separately. Then \(K'(k^+, k^-)\) is just a copy of the knot \(K\), denoted as \(K'\). Pick two basepoints \(w_0, z_0\) on \(K'\) among \(w', z'\). By Lemma 3.8, we can write \(v \in \widehat{HF}(K', w', z')\) as
\[
v = \sum_{i=1}^k u_i \otimes u_i,
\]
with \( v_i \in \widehat{HFL}(K', w_0, z_0) \), \( u_i \in V^{\otimes(|w'|-1)} \).

To get a model Floer lasagna filling, we can apply some isotopy \( \mathcal{I} \) of \( \Sigma \) relative to the boundary, such that some dividing arc in \( \mathcal{A} \) passes through the center of the disk. Then \( \mathcal{F} \) is equivalent to the model Floer lasagna filling \( \mathcal{F}_\mathcal{I} = (B, \mathcal{K}', \mathcal{A}_\mathcal{I}, \mathcal{F}_\mathcal{I}(v)) \), where \( \mathcal{K}' = (K, w_0, z_0) \), \( \mathcal{A}_\mathcal{I} \) is a neighborhood of the center of \( \Sigma \), \( \mathcal{A}_\mathcal{I} = \mathcal{I}(\mathcal{A}) \cap \mathcal{A}' \), and \( \mathcal{F}_\mathcal{I} \) is the induced map by the decorated link cobordism \((S^3 \times I, \Sigma', \mathcal{I}(\mathcal{A}) \setminus \mathcal{A}')\).

Of course, if we apply a different isotopy \( \mathcal{I}' \) instead of \( \mathcal{I} \), we will get another model Floer lasagna filling \( \mathcal{F}_{\mathcal{I}'} = (B, \mathcal{K}', \mathcal{A}_{\mathcal{I}'}, \mathcal{F}_{\mathcal{I}'}(v)) \) defined similarly. See Figure 9 for an illustration. Note all the data in \( \mathcal{F}_{\mathcal{I}} \) and \( \mathcal{F}_{\mathcal{I}'} \) are the same except \( \mathcal{F}_{\mathcal{I}}(v) \) and \( \mathcal{F}_{\mathcal{I}'}(v) \). We would like to see how different they could be.

It seems a bit complicated to analyze the effect of applying isotopy to a decorated disk with some random dividing arcs on it. Instead, we will show first that such a Floer lasagna filling could be replaced by an equivalent one, which is a linear combination of Floer lasagna fillings such that the dividing arc on each filling disk is just a single line segment. It is done by concatenating \( \Sigma \) with \( K \times I \), as mentioned in Lemma 3.8. For such Floer lasagna fillings, the effect of applying isotopy to the dividing arcs is the same as applying the basepoint moving maps on link Floer homology. The explicit statement is given in Lemma 3.11.

![Figure 9](image.png)

**Figure 9.** Different isotopies \( \mathcal{I}, \mathcal{I}' \) to \( \Sigma \)

**Lemma 3.10.** For a model Floer lasagna fillings \( \mathcal{F} = (B, K', w', z', \Sigma, \mathcal{A}, v) \), we can find model Floer lasagna fillings \( \mathcal{F}_i = (B, K', w_0, z_0, \Sigma'_i, \mathcal{A}_i, v_i) \) for \( i = 1, \ldots, n \), such that

\[
[\mathcal{F}] = \sum_{i \in \mathcal{J}} [\mathcal{F}_i]
\]

in the Floer lasagna module \( \mathcal{FL}(W, \emptyset) \). Here \( \mathcal{J} \) is a subset of \( \{1, \ldots, n\} \), such that for each \( i \in \mathcal{J} \), the decorated disk \( (\Sigma_i, \mathcal{A}_i) \) is isotopic to the following standard one in Figure 10.

**Proof.** It follows from the previous lemmas. By Lemma 3.8 for each element \( u_i \in V^{\otimes(|w'|-1)} \) as in Equation 3.5 we can find some decorated link cobordism \((S^3 \times I, K \times I, \mathcal{A}_i')\), such that

\[
F_i(v_i) = v_i \otimes u_i,
\]

where \( F_i \) is the induced map on link Floer homology. Therefore, if we define \( (\Sigma'_i, \mathcal{A}_i) \) by concatenating \( (\Sigma, \mathcal{A}) \) with \( (K \times I, \mathcal{A}_i') \), and let \( \mathcal{F}_i = (B, K, w_0, z_0, \Sigma'_i, \mathcal{A}_i, v_i) \), for \( i = 1, \ldots, n \), we get

\[
[\mathcal{F}] = \sum_{i=1}^{n} [\mathcal{F}_i]
\]
in the Floer lasagna module $\mathcal{FL}(W, \emptyset)$, by the equivalence relation and the linearity.

By Lemma 3.9 among all the $[\mathcal{F}_i]$’s, those that contain a closed disk as a connected components of $\Sigma_w$ or $\Sigma_z$ are actually 0 in $\mathcal{FL}(W, \emptyset)$. See Figure 11 for an illustration. Define $\mathcal{J} \subset \{1, \ldots, n\}$ as the subset consisting of some indices $i$ such that $[\mathcal{F}_i] \neq 0$.

![Figure 11](image.png)

**Figure 11.** Different concatenations according to $u_i$

Take an index $i \in \mathcal{J}$. Each component of $\Sigma^i_w$ and $\Sigma^i_z$ intersects the boundary $\partial\Sigma^i$ non-trivially, by the definition of $\mathcal{J}$. Since there are only two basepoints $w_0$ and $z_0$ on $\partial\Sigma^i$, and there is a basepoint in each connected component of $\Sigma^i_w \cap \partial\Sigma^i$ and $\Sigma^i_z \cap \partial\Sigma^i$, each of $\Sigma^i_w$ and $\Sigma^i_z$ has exactly one connected component, which intersects $\partial\Sigma^i$ once. If $\Sigma^i_w$ is not topologically a disk, then $\Sigma^i_z = \Sigma^i \setminus \overline{\Sigma^i_w}$ would have more than one connected component. A similar statement holds for $\Sigma^i_z$ as well. So both $\Sigma^i_w$ and $\Sigma^i_z$
are topologically a disk, and after applying some isotopy to $\Sigma_i$ relative to the boundary, it will be of the form as in Figure 10.

Now we can compare the difference between applying different isotopies $\mathcal{I}$ and $\mathcal{I}'$. More explicitly:

**Lemma 3.11.** For a almost model Floer lasagna fillings $\mathcal{F} = (B, K', w', z', \Sigma, \mathcal{A}, v)$ of $(W, \emptyset)$ as in the Lemma 3.10. Let

$$[\mathcal{F}] = \sum_{i \in J} [\mathcal{F}_i],$$

where $\mathcal{F}_i = (B, K, w_0, \Sigma^i, \mathcal{A}_i, v_i)$ is as defined in Lemma 3.10. Suppose $\mathcal{I}, \mathcal{I}'$ are two isotopies of $\Sigma$ which moves one of the dividing arcs passing the origin. For each $i \in J$, extend each of the isotopies $\mathcal{I}, \mathcal{I}'$ to an isotopy on $\Sigma^i$ by identity on $K \times I$, still denoted as $\mathcal{I}, \mathcal{I}'$. Let $\Sigma'$ be a small neighborhood of the center of $\Sigma$, such that the dividing arcs $\mathcal{I}(\mathcal{A}) \cap \Sigma'$ and $\mathcal{I}'(\mathcal{A}) \cap \Sigma'$ are some line segments. Consider the induced map

$$F_i, F'_i : \widehat{HF}_L(K, w_0, z_0) \to \widehat{HF}_L(K, w_0, z_0)$$

by the decorated link cobordism $(S^3 \times I, \Sigma^i \setminus \Sigma', \mathcal{I}(\mathcal{A}_i) \cap (\Sigma^i \setminus \Sigma'))$ and $(S^3 \times I, \Sigma^i \setminus \Sigma', \mathcal{I}'(\mathcal{A}_i) \cap (\Sigma^i \setminus \Sigma'))$ respectively, then

$$F'_i = \phi^{k_i} \circ F_i,$$

for some $k_i \in \mathbb{Z}$, where $\phi$ is the basepoint moving map on $\widehat{HF}_L(K, w_0, z_0)$ induced by the decorated link cobordism $(S^3 \times I, K \times I, tw)$ as in Figure 6.

---

![Figure 12](image.png)

**Figure 12.** Different isotopies differed by basepoint moving
Proof. See Figure 12 for an illustration of the proof. By Lemma 3.10, the dividing arc $\mathcal{I}(\mathcal{A}_i)$ divides the filling disk $\Sigma^i$ into $\Sigma^i_w$ and $\Sigma^i_z$, both of which are disks. Then after deleting $\Sigma^i$, which is a small disk centered at some point on $\mathcal{I}(\mathcal{A}_i)$, each of the subsurfaces $\mathcal{S}_i^w \cap (\Sigma^i \setminus \Sigma^i)$ and $\mathcal{S}_i^z \cap (\Sigma^i \setminus \Sigma^i)$ is homeomorphic to $I \times I$, with $I \times \{0\} \subset \partial \Sigma^i$ and $I \times \{1\} \subset \partial \Sigma^i$. Let $[p^i_2] \in \pi_1(\Sigma^i \setminus \Sigma^i, \partial(\Sigma^i \setminus \Sigma^i)) = \mathbb{Z}$ be the homotopy class of the path corresponding to $\{1/2\} \times I$ in $\Sigma^i_w \cap (\Sigma^i \setminus \Sigma^i)$. Now, up to isotopy on the annulus $\Sigma^i \setminus \Sigma^i$ relative to the boundary, such decorated link cobordism is classified by the class $[p^i_2] \in \mathbb{Z}$. Suppose we start with a different isotopy $\mathcal{I}'$. Then we have, up to isotopy relative to the boundary,

$$(S^3 \times I, \Sigma^i \setminus \Sigma^i, \mathcal{I}'(\mathcal{A}) \cap (\Sigma^i \setminus \Sigma^i)) = (S^3 \times I, K \times I, tw)^{\varphi} \circ (S^3 \times I, \Sigma^i \setminus \Sigma^i, \mathcal{I}(\mathcal{A}) \cap (\Sigma^i \setminus \Sigma^i))$$

where $\circ$ is concatenating decorated link cobordisms, and $k = [p^i_2] - [p^i_2]$. Therefore,

$$F'_i = \varphi^{\varphi_i} \circ F_i.$$  

□

Finally, we begin the proof of Proposition 3.1.

3.3. Proof of Proposition 3.1

Proof of Proposition 3.1 This follows closely the treatment as in [MN20, Section 4], except for the dividing arcs on Floer lasagna fillings. We begin by defining the map $\Phi : c\text{HF}(L, K) \to \mathcal{F}(W, L)$. For every $k^+, k^- \in \mathbb{N}$ and every $v \in \text{HF}(L, K, k^+, k^-)$, we define the Floer lasagna filling $\hat{\Phi}(v) = (B, L' \cup \mathbb{R}(k^+, k^-), \Sigma, A, v)$ to be the model Floer lasagna filling as in Definition 3.4.

Now we check that equivalent elements in $\text{HF}(L, K, k^+, k^-)$ give equivalent Floer lasagna fillings. For the braid group action and the basepoint moving maps, it is easy to see that

$$\hat{\Phi}(v) = \hat{\Phi}(F_{\tau}(v)), \quad \hat{\Phi}(v) = \hat{\Phi}(F_{m_{i,j}}(v)) \quad \text{in } \mathcal{F}(W, L)$$

by gluing the corresponding cobordism to the interior of $B$, and using functoriality of the link cobordism maps on link Floer homology.

![Figure 13. Relations given by the pair-of-pants map](image)

For the relation involving the pair-of-pants map, note that we can isotopy the death cobordism over an unknot to the composition of the pair-of-pants cobordism with two capping disks over the newly added oppositely oriented parallel copies of $K_i$. See Figure 13 for an illustration. On the other hand,
we can evaluate the death cobordism as in Equation \ref{eq:3.3} by the equivalence relation of Floer lasagna fillings. This gives the relations involving \(F_p\), and we see \(\Phi\) descends to a well-defined map

\[
\Phi : cHF(L, K) \to \mathcal{FL}(W, L).
\]

It is easy to see \(\Phi\) preserves both the Maslov grading and the Alexander grading, as we have done shifts in the Maslov grading appropriately in the definition of the Maslov gradings on Floer lasagna module.

To show \(\Phi\) is a bijection, we define the inverse map

\[
\Psi : \mathcal{FL}(W, L) \to cHF(L, K)
\]

as follows. Given a Floer lasagna filling \(\mathcal{F} = (B_i, L_i, \Sigma, A, \nu_i)\), we can always take some small neighborhood \(f_i(B_i(0) \times D^2)\) of each cocore such that the intersection of \(\Sigma\) with \(f_i(B_i(0) \times D^2)\) could be isotoped to parallel copies of disks \(f_i(B_i(0) \times x_{i,j}^\pm)\), which are transverse to the cocore disk \(f_i(\{0\} \times D^2)\). Let \(W' = W \setminus \cup_i f_i(B_i(0) \times D^2)\). Take a small neighborhood \(nbhd(\partial W')\) of \(\partial W'\) such that the components of \(\Sigma \cap nbhd(\partial W')\) which have boundary \(L \times \{0, 1\}\) could be isotoped to \(L \times I\), and the dividing arcs on \(L \times I\) is given by \(C \times I\) for some set of points \(C \subset L \setminus W \cup z\). Take \(B' = W' \setminus nbhd(\partial W')\), which contains all the \(B_i\). Define \(L', K', w', z'\) as in the second and third bullet point of the definition of \(\Phi\).

Let \(\Sigma' = \Sigma \cap (\cup_i f_i(B_i(0) \times D^2))\), \(A' = A \cap \Sigma'\), and \(\nu' = F'(\otimes \nu_i)\), where \(F'\) is the map induced by the decorated link cobordism \((B \setminus (\cup_i B_i), \Sigma \setminus \Sigma', A', A')\). By the equivalence relation on Floer lasagna fillings, we get

\[
[\mathcal{F}] = [\mathcal{F}'], \quad \text{where } \mathcal{F}' = (B', L', \Sigma' \cap \Sigma, A', A', \nu')
\]

\(\mathcal{F}'\) is almost of the form \(\Phi(v)\) for some \(v \in cHF(L, K)\). We get an almost model Floer lasagna filling instead of a model one. To resolve this, we take the following procedures:

Apply an isotopy \(I_{i,j}\) relative the boundary to each disk \(f_i(B_i(0) \times x_{i,j}^\pm)\), such that the set of dividing arcs on it passes through the point \(f_i(\{0\} \times x_{i,j}^\pm)\), i.e. intersecting with the cocore. Extend each isotopy to an isotopy \(I_{i,j}\) on \(W\), which is identity outside some small neighborhood of \(f_i(B_i(0) \times x_{i,j}^\pm)\). Denote the composition of all the isotopies \(I_{i,j}\) by \(I\). Since there are finitely many such disks, we can take some smaller neighborhood \(\Sigma'' = \cup_i f_i(B_i(0) \times D^2)\) of the cocores, such that the set of dividing arcs on \(I_{i,j} f_i(B_i(0) \times x_{i,j}^\pm)\) \(\cap \Sigma''\) is given by a single line segment. In particular, the \(\Sigma_w\) and \(\Sigma_z\) part of \(I_{i,j} f_i(B_i(0) \times x_{i,j}^\pm) \cap \Sigma''\) are both half-disks, hence there are exactly 2 basepoints on the boundary link \(\partial((I)_{i,j} f_i(B_i(0) \times x_{i,j}^\pm)) \cap \Sigma''\), for each \(i, j\).

Now replace \(B'\) by a slightly larger ball \(B''\), which is \(W \setminus \Sigma''\) minus some collar neighborhood of its boundary. Define \(L'', K''(k^+, k^-)\) as before. Let \(w'', z''\) consists of basepoints on \(L''\) which correspond to \(w, z\) on \(L\), and two basepoints on each component of \(K''(k^+, k^-)\). Let \(\Sigma'' = I(\Sigma) \cap \Sigma'', A'' = I(A) \cap \Sigma'',\) and \(\nu'' = F''(\nu')\), where \(F''\) is the map induced by the decorated link cobordism \((B'' \setminus B', I(\Sigma) \setminus \Sigma'', I(A) \setminus A'')\). Then \(F'' = (B'', L'' \cup K''(k^+, k^-), w'', z'', \Sigma'', A'', v'')\) is a model Floer lasagna filling, hence an element of the form \(F'' = \Phi(\nu'')\). Define \(\Psi([\mathcal{F}]) = [\nu'']\), the equivalence class represented by \(\nu''\) in \(cHF(L, K)\).

Now we check \(\Psi\) is well-defined. Equivalent Floer lasagna fillings give the same value under \(\Psi\) by the functoriality of link cobordism maps, as the equivalence amounts to applying the link cobordism map corresponding to \((B'' \setminus (\cup_i B_i), \Sigma \cap (B'' \setminus (\cup_i B_i)))\), where \(B''\) is away from all the cocores. Hence they return the same result after the above the procedure.

In the process of defining \(\Psi\), we have made the following choices:

- An isotopy which makes the part of the filling surface \(\Sigma \cap f_i(B_i(0))\) transverse to the cocore.
- An isotopy for each copy of the disk \(f_i(B_i(0) \times x_{i,j}^\pm)\) which moves the dividing arcs passing through the cocore.
The ambiguity in the choice of the second isotopy leads to quotienting out the basepoints moving maps, as explained in Lemma 3.11.

The ambiguity in the choice of the first isotopy leads to quotienting out the braid group action and the pair-of-pants maps, as explained in the proof of Theorem 1.1 in [MN20 Section 4]. Here is an outline of the argument. Suppose we have chosen another isotopy of $W$ relative to $\partial W$, connect these two by a $[0, 1]$-family of isotopies $\Lambda_t$. Consider the image of $\Sigma$ under this family of isotopies. When $\Lambda_t(\Sigma)$ stays transverse to the cocore disks, this gives cobordism corresponding to the braid group action as in Figure 3. Then the output of $\Phi$ is changed from $v$ to $F(v)$, where $F$ is the map induced by the cobordism of the form as in Figure 3 except we don’t know the decoration on the surface.

There are finitely many times $t_k \in [0, 1]$ at which the image of $\Sigma$ stops being transverse to the core disk. In that case, there will be a creation or cancellation of two intersection points of $\Lambda_t(\Sigma)$ with the cocore disks. This corresponds to the birth/death cobordisms of an unknot. Suppose at time $t_k$ two new intersection points are created, then before time $t_k$, the map $\Phi$ gives some element

$$v \in \widehat{HFL}(L \cup U, K, k^+, k^-),$$

and after time $t_k$, the map $\Phi$ gives the element

$$F(v) \in \widehat{HFL}(L \cup U, K, k^+ + e_1, k^- + e),$$

where $F$ is the map induced by a pair-of-pants, which is the capping disk of $U$ cutting out two core disks. See Figure 13 for an illustration.

One issue that is not covered there is that we need to argue that it is enough to consider decorated link cobordisms with decorations of the forms we used in defining the cabled link Floer homology, as in Figure 3 and 4. The reason is as follows:

For any almost model Floer lasagna fillings, we can replace it by a linear combination of model ones as in Lemma 3.10. Then consider the corresponding cobordisms, starting from and ending with these model Floer lasagna fillings. For braid group actions, if the decoration is not of the form as in Figure 3 then it divides into two cases. If either $\Sigma_\omega$ or $\Sigma_w$ contains a closed disk component, then by Lemma 3.9 the Floer lasagna filling is 0, so the relation trivially holds. Otherwise, with similar proof as in Lemma 3.11, we can always compose some basepoint moving cobordism as in Figure 6 such that it becomes the one in Figure 3.

For the relations involving the pair-or-pants map $F_P$, we start with a standard decorated disk $D$ as in Figure 10 capping the unknot $U$, and cut out two standard decorated capping disks $D_1, D_2$ of $K'$. The remaining region $D \setminus (D_1 \cup D_2)$ induces the pair-of-pants cobordism. If we allow isotopy of the remaining surface $D \setminus (D_1 \cup D_2)$ which could move the boundaries, then we can isotopy the decoration on $D \setminus (D_1 \cup D_2)$ to one of the two decorations as drawn in Figure 14. These two decorations are different by the ordering of the two outputs, i.e. if we travel along the boundary of $\Sigma_w$ clockwise, whether we go through $U, K^+, K^-$, or $U, K^-, K^+$. This amounts to the following difference: when we add the two new basepoints $w', z'$ to $U$ during quasi-stabilization, whether it is $w'$ followed by $z'$, or $z'$ followed by $w'$ when we travel along the orientation of $U$. At the level of Heegaard splitting description of the quasi-stabilization map, the difference is when we isotope $U$ to intersect the Heegaard splitting surface at two more points during the quasi-stabilization, whether the extra arc lies in the $\alpha$-handle body or the $\beta$-handle body in the Heegaard splitting. These two moves give the same quasi-stabilization map. See [Zem19c Section 4.1] for more discussion. Therefore, these two decorated link cobordism give the same cobordism map. Isotopies which move the boundary correspond to basepoint moving maps. Hence, by composing basepoint moving cobordisms on each boundary component if necessary, we can assume the map induced by the pair-of-pants cobordism is the $F_P$ used in the definition of the cabled link Floer homology.
Hence $\Psi$ is well-defined. From the definition of $\Psi$ and $\Phi$, it is easy to see that $\Phi$ and $\Psi$ are inverse to each other. Hence $\Phi$ gives a bijection between $cHFL(\mathbb{L}, K)$ and $FL(W, L)$. □

4. Example calculations for 0-surgery on unknot

In this section, we give some example calculations the Floer lasagna module, using the description in terms of the cable Floer homology.

Suppose $K$ is the 0-framed unknot in $S^3$, and let $W$ be the 4-manifold of attaching $D^2 \times D^2$ to $B^4$ along $K$, i.e., $W = S^3 \times D^2$. We will compute $FL(W, \emptyset)$ and $FL(W, L)$ when $L = (L, w, z)$ is a link which geometrically intersects a capping disk of $K$ in $S^3$ once. We will see that $FL(W, \emptyset)$ is very similar to the skein lasagna modules of $(W, \emptyset)$, and $FL(W, L)$ turns out to be 0.

4.1. Calculation of $FL(W, \emptyset)$. In this case, all we need to compute the cabled link Floer homology $cHFL(\mathbb{L}, \emptyset)$ are link Floer homology of unlinks in $S^3$, and elementary cobordism maps between them in $S^3 \times I$. As first noticed in [JM18], it is closely related to the reduced Khovanov homology. More explicitly, by marking a component of an unlink $U^n$, one can find a canonical basis of $\widehat{HFL}(S^3, U^n) \cong V^{\otimes(n-1)} \cong \tilde{Kh}(U^n)$, where $V = (T, B)$ is a 2-dimensional $\mathbb{F}_2$-vector space. If we compute the action of the braid group and pair-of-pants maps, they coincide with the corresponding maps induced on reduced Khovanov homology. See [JM18 Theorem 7.6]. What’s more, the basepoint moving map acts trivially on $\widehat{HFL}(S^3, U^n)$ for unlinks. Therefore, the computation of the cabled Floer homology is very similar to that of the cabled Khovanov homology, except that we need to pick a marked component. If the braid group action $F_{i, i+1}$ involves the marked component, then the corresponding map on link Floer homology is trivial. Therefore, the computation is exactly the same as in [MN20 Section 5]. The role of quantum grading there is taken place by Maslov grading here. We translate their result in the following proposition:

**Proposition 4.1.** Let $W = S^2 \times D^2$. For each $\alpha \in H_2(W, \emptyset; \mathbb{Z}) = \mathbb{Z}$, the Floer lasagna module $FL(W, \emptyset, \alpha)$ of relative homology class $\alpha$ is given by

$$FL(W, \emptyset, \alpha) = \bigoplus_{k \in \mathbb{N}} \mathbb{F}_2,$$

where there is a copy of $\mathbb{F}_2$ in each Maslov grading $-k$. 

Figure 14. Different decorations on the pair-of-pants
4.2. Calculation of $\mathcal{FL}(W, L)$ when the geometric intersection number is 1. In this subsection, we compute $\mathcal{FL}(W, L)$ when $L = (L, w, z)$ intersects a capping disk of the unknot $K$ geometrically once. As we will see, the result turns out to be that $\mathcal{FL}(W, L)$ vanishes in this case, due to some grading constraint on the pair-of-pants map.

**Proposition 4.2.** Suppose $W$ is obtained from $B^4$ by attaching a 2-handle along a 0-framed unknot $K$, and $L = (L, w, z)$ is a multi-based link in $\partial W$ which intersects a capping disk of $K$ geometrically once. Then

$$\mathcal{FL}(W, L) = 0.$$ 

We will use the following result on the effect of connected sum of link cobordism maps by Zemke:

**Proposition 4.3.** [Zem19a, Proposition 5.2] Suppose that $(W_1, \Sigma_1)$ and $(W_2, \Sigma_2)$ are two link cobordisms, with chosen points $y_1 \in A_1 \subset W_1$ and $y_2 \in A_2 \subset W_2$, as well as an embedding of $(S^0 \times D^4, N_0)$, centered at $\{y_1, y_2\}$, which is orientation preserving and maps type-$w$ regions to type-$w$ regions and maps type-$z$ regions to type-$z$ regions. Then

$$F_{W_1 \# W_2, \Sigma_1 \# \Sigma_2, a_1 \# a_2} \cong F_{W_1 \cup W_2, \Sigma_1 \cup \Sigma_2, a_1 \cup a_2},$$

where the connected sum is taken at the 4-balls centered at $y_1, y_2$.

**Proof of Proposition 4.2.** Suppose $U = (U, w_2, z_2)$ is another unknot with 2 basepoints not linking with $L$, and we want to study the map

$$F_p : \widehat{HFL}(L \cup U, K, k^+, k^-) \to \widehat{HFL}(L, K, k^+ + 1, k^- + 1),$$

induced by the pair-of-pants cobordism. See Section 3.1 for the definitions of $\widehat{HFL}(L, K, k^+, k^-)$ and $F_p$.

Because of the condition of the geometric intersection number being 1, we can write the link $L \cup K(k^+, k^-) \cup U$ as the connected sum $L_1 \# (L \cup K(k^+, k^-))$, and the link $L \cup K(k^+ + 1, k^- + 1)$ as the connected sum $L_2 \# (L \cup K(k^+, k^-))$, where $L_1$, $L_2$ are the following 2 specific links in Figure 15:

![Figure 15. Links $L_1, L_1'$ and $L_2$](image)

By the connected sum formula of link Floer homology, we have

$$\widehat{HFL}(L \cup K(k^+, k^-) \cup U) \cong \widehat{HFL}(L_1) \otimes \widehat{HFL}(L \cup K(k^+, k^-)),$$

$$\widehat{HFL}(L \cup K(k^+ + 1, k^- + 1)) \cong \widehat{HFL}(L_2) \otimes \widehat{HFL}(L \cup K(k^+, k^-)).$$

The pair-of-pants cobordism from $L \cup K(k^+, k^-) \cup U$ to $L \cup K(k^+ + 1, k^- + 1)$ could be written as a connected sum of a pair-of-pants cobordism from $L_1$ to $L_2$, with an identity cobordism from $\widehat{HFL}(L \cup K(k^+, k^-))$ to itself. Therefore, by Proposition 4.3, we have

$$F_p \cong F'_{p} \otimes id,$$
where $F_p'$ is the map induced by the pair-of-pants cobordism from $L_1$ to $L_2$. The pair-of-pants relation in the definition of the cabled link Floer homology (See Definition 3.2) can be written in the form

\[(4.1) \quad v \sim F_p'(B) \otimes v, \quad 0 \sim F_p'(T) \otimes v,\]

for any $v \in \widehat{HF}(L \cup K(k^+, k^-))$.

It is enough to compute $F_p'$, which could be written as the composition of a quasi-stabilization map $T +$ with a band surgery map $F_B^z$. See Figure 16 for an illustration.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{decorated_link_cobordisms.pdf}
\caption{Decorated link cobordisms of $T^+$ and $F_B^z$.}
\end{figure}

The quasi-stabilization map $T^+$ is mentioned in Definition 3.6. See [Zem19c, Section 6] for the definition of the band surgery map $F_B^z$.

For the following discussion, we are going to use the multi-Alexander grading $(A_1, A_2)$ on links, such that $A_1$ is induced by the basepoint $w_1, z_1$, and $A_2$ is induced by the basepoint $w_2, w_3, z_2, z_3$. The link $L_1$ is an unlink with 2 components, which has link Floer homology

\[
\widehat{HF}(L_1) \cong \mathbb{F}_2 \cong \langle T, B \rangle,
\]

with $M(T) = 0$, $M(B) = -1$, $A_i(T) = A_i(B) = 0$, for $i = 1, 2$.

The link $L_1'$ is obtained from $L_1$ by adding two basepoints on $U$, so the link Floer homology of $L_1'$ is

\[
\widehat{HF}(L_1') \cong \mathbb{F}_2 \cong \langle T, B \rangle \otimes \langle \theta^{w}, \xi^{w} \rangle,
\]

with $M(\theta^{w}) = 0$, $M(\xi^{w}) = -1$, $A_1(\theta^{w}) = A_1(\xi^{w}) = 0$, $A_2(\theta^{w}) = \frac{1}{2}$, $A_2(\xi^{w}) = -\frac{1}{2}$.

The quasi-stabilization map $T^+$ is such that

\[
T^+(T) = T \otimes \xi^{w}, \quad T^+(B) = B \otimes \xi^{w}.
\]

For the link $L_2$, it is easy to get its link Floer homology, e.g. noting that it is an alternating link, and using the multi-variable Alexander polynomial

\[
\Delta(L_2) = x_1^{1/2} - x_1^{-1/2},
\]

where $x_i$ is the variable associated to link component with basepoints $w_i$ and $z_i$. Then, by [OS08, Theorem 1.3],

\[
\widehat{HF}(L_2, w, z, h) = \mathbb{F}_2^{\otimes |a_h|} [ |h| - 1 ]
\]
where
\[
\left(x_{1/2} - x_{1/2}^{-1/2}\right) \prod_{i=1}^{3} \left(x_{i/2}^{-1} - x_{i/2}\right) = \sum_{h=(h_1,h_2,h_3) \in \mathcal{H}} a_h x_1^{h_1} x_2^{h_2} x_3^{h_3}.
\]
\[\mathcal{H} = \{-1/2,1/2\} \times \{-1/2,1/2\} \times \{-1,0,1\}, \quad |h| = h_1 + h_2 + h_3,\]
and \(F \otimes |a_h|^2 [|h| - 1] \) means \(|a_h|^2\) copies of \(F\) with Maslov grading \(|h| - 1\). In particular, \(\widehat{HF}(L_2,w,z)\) is a 16-dimensional vector space over \(\mathbb{F}_2\).

\[
\theta_w \mu_w \omega_w \omega_2 \omega_3 \omega_1 \theta_2 \theta_3 \theta_1 (a) \quad B \quad B \quad \omega_1 \omega_2 \omega_3 \theta_1 \theta_2 \theta_3 \omega_1 \omega_2 \omega_3 (b)
\]

**Figure 17.** Heegaard diagrams of \(L'_1\) and \(L_2\)

We draw the following Heegaard diagrams for \(L'_1\) and \(L_2\) separately in \((a),(b)\) of Figure 17. Note that there are 4 and 16 generators in the corresponding Floer chain complexes, hence there are no non-trivial differential in these two chain complexes, and each generator represents a different homology class in the link Floer homology \(\widehat{HF}(L'_1)\) and \(\widehat{HF}(L_2)\) respectively. Recall \(A_1\) is the Alexander grading with respect to the basepoints \(w_1, z_1\), and \(A_2\) is the collapsed Alexander grading with respect to the basepoints \(w_2, w_3, z_2, z_3\). Then we can write
\[
\widehat{HF}(L_2) \cong \langle a, b, c, d \rangle \otimes \langle x, y, u, v \rangle
\]
such that the Alexander gradings are as given in Figure 18.

The multi-Alexander grading of a generator is the sum of the corresponding multi-Alexander grading, e.g.
\[
A_1(du) = A_1(d) + A_1(u) = 1/2 + 1/2 = 1, \quad A_2(du) = A_2(d) + A_2(u) = 1/2 - 1/2 = 0.
\]

The element in \(\widehat{HF}(L_2)\) with highest Maslov grading is \(dy\), with \(M(dy) = 1\), which is computed by ignoring the type-\(z\) basepoints and consider it as a multi-pointed Heegaard diagram of \(S^3\). The Maslov gradings of other generators could be computed by the relative grading formula of Maslov grading.

The band surgery map \(F_B^z\) is defined by counting holomorphic triangles which avoid both type-\(w\) and type-\(z\) basepoints in the triple Heegaard diagram when we draw all the red, blue and green curves
in Figure 17 together in one diagram. If we ignore the type-z basepoints in the diagrams, then both diagrams are Heegaard diagrams for \( S^3 \) with 3 basepoints. We define

\[
G^*_B : \widehat{CF}_a(S^3, w) \to \widehat{CF}_b(S^3, w)
\]

by counting holomorphic triangles which avoid only type-w basepoints, then \( G^*_B \) is a composition of handle-sliding maps, which induces isomorphism on \( \widehat{HF}(S^3, w) \). Here \( \widehat{CF}_a(S^3, w) \) and \( \widehat{CF}_b(S^3, w) \) refer to the specific Heegaard chain complex generated by diagram (a) and (b) in Figure 17. See [OS04] for details.

Now we focus on the element \( G^*_B(B \otimes \xi^w) \).

If we denote by \( \partial_w : \widehat{CF}(S^3, w) \to \widehat{CF}(S^3, w) \) the differential on complex in (b) of Figure 17, then \( G^*_B(B \otimes \xi^w) \in \ker(\partial_w) \), and the Maslov grading of \( G^*_B(B \otimes \xi^w) \) should be the same as the Maslov grading of \( B \otimes \xi^w \), as we are counting holomorphic triangles of Maslov index 0 which avoid type-w basepoints, so

\[
M(G^*_B(B \otimes \xi^w)) = M(B \otimes \xi^w) = -2.
\]

Elements in \( \ker(\partial_w) \) of the right Maslov grading are \( av, bu + bx \) and \( cv \), which have the corresponding multi-Alexander gradings as follows:

\[
A_1(av) = -1, \quad A_2(av) = 0, \quad A_1(bu) = A_1(cv) = 0, \quad A_2(bu) = A_2(cv) = -1, \quad A_1(bx) = -1, A_2(bx) = 0
\]

Recall what we are actually looking for is \( F^*_B(B \otimes \xi^w) \), which is given by counting holomorphic triangles avoiding both type-w and type-z basepoints. Following [Zem19b] Lemma 7.2, we know the change of the multi-Alexander gradings of the band surgery map \( F^*_B \), therefore

\[
A_1(F^*_B(B \otimes \xi^w)) = A_1(B \otimes \xi^w) = 0, \quad A_2(F^*_B(B \otimes \xi^w)) = A_2(B \otimes \xi^w) + \frac{1}{2} = 0.
\]

Now as each term of \( G^*_B(B \otimes \xi^w) \) has multi-Alexander gradings different from the expected multi-Alexander gradings of \( F^*_B(B \otimes \xi^w) \), it means that every holomorphic triangle that contributes to \( G^*_B(B \otimes \xi^w) \) actually passes through some type-z basepoints. In other words, there is no contribution from holomorphic triangles that avoid both type-w and type-z basepoints to \( G^*_B(B \otimes \xi^w) \). That is to say,

\[
F^*_B(B \otimes \xi^w) = 0.
\]

Hence,

\[
F^*_B(B) = (F^*_B \circ T^+)(B) = F^*_B(B \otimes \xi^w) = 0.
\]
Then, by Equation 4.1, we get

\[ v \sim 0 \]

for any \( v \in \widehat{HFL}(L, K, k^+, k^-) \), which says the cabled link Floer homology \( cHFL(L, K) \) vanishes in this case. By Proposition 4.1, the Floer lasagna module \( \mathcal{FL}(W, L) \) vanishes as well.

\[ \Box \]

**Remark 4.4.** If we naively compute the link Floer homology of the link \( S^1 \times \{pt\} \subset S^1 \times S^2 \) by the complex represented in the following Heegaard diagram, then there are two generators with a holomorphic bigon connecting them, so we get trivial homology as well. This calculation of the Floer lasagna module \( \mathcal{FL}(W, L) = 0 \) in Proposition 4.2 indicates that it could work as some generalization of link Floer homology for non-homologous links in general 3-manifolds.

**Remark 4.5.** For \( \mathcal{FL}(W, L) \) with \( W \) the 4-manifold obtained by attaching a 2-handle along the 0-framed unknot \( K \) and general \( L \), as all the cobordisms we need to define the cabled link Floer homology \( cHFL(L, K) \) could be written as a connected sum of a cobordism near a capping disk of \( K \) and the identity cobordism on the rest part, we expect that using the language of bordered knot Floer homology, one can compute the cabled link Floer homology for general \( L \).

**References**

[Dow18] Nathan Dowlin, *A spectral sequence from Khovanov homology to knot Floer homology*, arXiv preprint arXiv:1811.07848 (2018).

[ETW18] Michael Ehrig, Daniel Tubbenhauer, and Paul Wedrich, * Functoriality of colored link homologies*, Proceedings of the London Mathematical Society **117** (2018), no. 5, 996–1040.

[JM18] András Juhász and Marco Marengon, *Computing cobordism maps in link Floer homology and the reduced Khovanov TQFT*, Selecta Mathematica **24** (2018), no. 2, 1315–1390.

[Juh16] András Juhász, *Cobordisms of sutured manifolds and the functoriality of link Floer homology*, Advances in Mathematics **299** (2016), 940–1038.

[JZ20] András Juhász and Ian Zemke, *Contact handles, duality, and sutured Floer homology*, Geometry & Topology **24** (2020), no. 1, 179–307.

[MN20] Ciprian Manolescu and Ikshu Neithalath, *Skew lasagna modules for 2-handlebodies*, arXiv preprint arXiv:2009.08520 (2020).

[MO10] Ciprian Manolescu and Peter Ozsváth, *Heegaard Floer homology and integer surgeries on links*, arXiv preprint arXiv:1011.1317 (2010).

[MW10] Scott Morrison and Kevin Walker, *The blob complex*, arXiv preprint arXiv:1009.5025 (2010).

[MWW19] Scott Morrison, Kevin Walker, and Paul Wedrich, *Invariants of 4-manifolds from Khovanov-Rozansky link homology*, arXiv preprint arXiv:1907.12194 (2019).

[OS04] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Annals of Mathematics (2004), 1027–1158.
[OS08] Daren Chen, Holomorphic disks, link invariants and the multi-variable Alexander polynomial, Algebraic & Geometric Topology 8 (2008), no. 2, 615–692.

[Zem17] Ian Zemke, Quasistabilization and basepoint moving maps in link Floer homology, Algebraic & geometric topology 17 (2017), no. 6, 3461–3518.

[Zem19a] Ian Zemke, Connected sums and involutive knot Floer homology, Proceedings of the London Mathematical Society 119 (2019), no. 1, 214–265.

[Zem19b] Ian Zemke, Link cobordisms and absolute gradings on link Floer homology, Quantum Topology 10 (2019), no. 2, 207–323.

[Zem19c] Ian Zemke, Link cobordisms and functoriality in link Floer homology, Journal of Topology 12 (2019), no. 1, 94–220.