Dynamics of electrons and explicit solutions of Dirac–Weyl systems

Alexander Sakhrnovich

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090
Vienna, Austria

E-mail: oleksandr.sakhnovych@univie.ac.at

Received 14 September 2016, revised 19 January 2017
Accepted for publication 24 January 2017
Published 10 February 2017

Abstract

Explicit solutions of the Dirac–Weyl system, which are essential in graphene studies, are constructed using our recent approach to the construction of solutions of dynamical systems. The obtained classes of solutions are much wider than the ones which have been considered before. It is proved that neither the constructed potentials nor the corresponding solutions have singularities. Various examples are provided.

Keywords: electrons, graphene, dynamical system, explicit solution, parameter matrices, Dirac–Weyl system, Bäcklund–Darboux transformation

1. Introduction

It is well known that the motion of an electron (in $\mathbb{R}^2$) in the presence of an electrostatic potential is often governed by the Dirac–Weyl system

$$i\hbar v_F \left( \sigma_1 \frac{\partial}{\partial x} + \sigma_2 \frac{\partial}{\partial y} \right) \psi = (U(x, y) - E)\psi,$$

(1.1)

where $\hbar$ is the Planck constant, $v_F$ is the Fermi velocity, $U = \overline{U}$ and $E \in \mathbb{R}$. Here, $\overline{U}$ stands for the complex conjugate of $U$, and $\mathbb{R}$ is the real axis. The matrices $\sigma_1$, $\sigma_2$ and $\sigma_3$ are Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(1.2)

In recent years, interest in graphene has essentially stimulated the study of this equation (see, e.g. [10, 11, 14, 24, 25]). In particular, the important case in which the scalar potential $U$ does not depend on the variable $y$ has recently been studied, for instance, in [10, 11, 14] (see also some references therein). Assuming that $U$ does not depend on $y$ and multiplying both sides of (1.1) by $\frac{1}{i\hbar v_F} \sigma_3$, we rewrite (1.1) in an equivalent form...
\[ \psi_x = i\sigma_3(-\psi_y + iu(x)\sigma_2\psi), \quad \psi_y := \frac{\partial}{\partial x}\psi, \quad u = \frac{E - U}{\hbar \nu_F} = \pi. \]  

The explicit solutions of (1.3) are essential in theory and applications. Some potentials generating explicit solutions are studied in [10, 11, 14] using the separation of variables \( \psi(x,y) = e^{ik\eta}\tilde{\psi}(x) \), which transforms (1.3) into a system depending on one variable. The well-known commutation methods [3, 6, 7, 12, 26] and several versions of the Bäcklund–Darboux transformation (see, e.g. [2, 9, 13, 15, 22, 27]) can be fruitfully used in this case to produce explicit solutions of linear systems depending on one variable. However, the dependence of the solution \( \psi \) on \( y \), which is more complicated than \( \psi(x,y) = e^{ik\eta}\tilde{\psi}(x) \), is of interest, and so we will apply some generalizations of our GBDT version (see [8, 15, 16, 22] and the references therein) of the Bäcklund–Darboux transformation. Such generalizations (in the case of linear systems in several variables) are given, for instance, in the papers [4, 19, 20]. In particular, explicit solutions of non-stationary Dirac systems are constructed in [4]. Yet, taking into account that \( u \) in (1.3) does not depend on \( y \), it seems here more useful to modify our approach to dynamical systems formulated in [20, 21].

In section 2, we present the GBDT for system (2.2), which is somewhat more general than the Dirac–Weyl system (1.3). We give simple conditions under which the constructed (GBDT-transformed) potentials and solutions do not have any singularities. Then, in section 3 we develop a procedure for constructing explicit solutions of systems (2.2) and (1.3) and consider various examples.

As usual, \( \mathbb{N} \) is the set of natural numbers, \( \mathbb{R} \) stands for the set of real numbers, \( \mathbb{C} \) stands for the set of complex numbers and \( \mathbb{C}_+ \) is the open upper half-plane. The notation \( \overline{a} \) stands for the number which is the complex conjugate of \( a \). The notation \( \text{Im} \{ A \} \) stands for the image of the matrix \( A \), \( A^* \) stands for the matrix which is the complex conjugate transpose of \( A \) and \( I_m \) is the \( m \times m \) identity matrix.

### 2. GBDT for Dirac systems

In [21, section 7] we constructed the GBDT (generalized Bäcklund–Darboux transformation) for the dynamical system

\[ z_x = J(-H_1(x)z_t + H_0(x)z), \quad J = -J^*, \quad H_1 = H_1^*, \quad H_0 = H_0^*. \]  

The systems (2.1) include dynamical Dirac systems as a special case. Here, we substitute into (2.1) the variables \( \psi \) and \( y \) instead of \( z \) and \( t \), respectively. Since the equality \( H_0 = -H_0^* \) (instead of \( H_0 = H_0^* \)) holds in our case, we refuse the requirement \( H_0 = H_0^* \) in (2.1). This change is essential in further considerations. We set \( J = i\sigma_3 \) and \( H_1 = I_2 \). Although the equality \( H_0(x) = i\nu(x)\sigma_2 \) \( (\nu = \pi) \) corresponds to the case (1.3), in this section we do not require \( \nu = \pi \) and consider a more general system than (1.3). Namely, we consider the system

\[ \psi_x = i\sigma_3(-\psi_y + V(x)\psi), \quad V(x) = \begin{bmatrix} 0 & u(x) \\ -\overline{u(x)} & 0 \end{bmatrix}. \]  

Further assume that \( 2 \times 2 \) matrix functions \( Q_1(x) \) and \( Q_2(x) \) are given and are locally summable on some interval \( \mathcal{I} \) such that \( 0 \in \mathcal{I} \). (In our case \( Q_1 \) and \( Q_2 \) are the matrix coefficients in (2.2). More precisely, we have \( Q_1 \equiv -i\sigma_3 \) and \( Q_2(x) = -i\sigma_2V(x) \).) The following auxiliary result, which is also used in [21] (see [21, (2.10)])], is a special case among some much more
general GBDT-formulas \cite{16, 22}. We note that the GBDT is based on the method of operator identities (see relations (2.3) and (2.24), and various references in \cite{22, 23}).

**Proposition 2.1.** Fix \( n \in \mathbb{N} \) and five parameter matrices; that is to say, fix three \( n \times n \) matrices \( A_1, A_2 \) and \( S(0) \), and two \( n \times 2 \) matrices \( \Pi_1(0) \) and \( \Pi_2(0) \), such that the equality

\[ A_1 S(0) - S(0) A_2 = \Pi_1(0) \Pi_2(0)^* \]  

holds. For \( x \in I \), introduce the matrix functions \( \Pi_1(x) \), \( \Pi_2(x) \) and \( S(x) \) via their values at \( x = 0 \) (i.e. via the values \( \Pi_1(0), \Pi_2(0) \) and \( S(0) \)) and via equations

\[ \Pi_1' = A_1 \Pi_1 Q_1 + \Pi_1 Q_0, \quad (\Pi_2)' = -A_2^* \Pi_2 Q_1^* - \Pi_2 Q_0, \quad S' = \Pi_1 Q_1 \Pi_2^*, \]  

where \( \Pi_i(x) = \left( \frac{d}{dx} \Pi_i \right)(x) \).

Then, in the points of the invertibility of \( S(x) \), we have

\[ (\Pi_1 S^{-1})' = -Q_1 \Pi_2 S^{-1} A_1 - \bar{Q}_0 \Pi_2 S^{-1}, \]  

where

\[ \bar{Q}_0 := Q_0 - (Q_1 X - X Q_1), \quad X := \Pi_2 S^{-1} \Pi_1. \]

**Remark 2.2.** Further in the text, speaking about \( S(x)^{-1} \), we mean \( S(x)^{-1} \) in the points where \( S(x) \) is invertible. The invertibility of \( S(x) \) is discussed, in particular, in proposition 2.4 and in remark 2.5.

In our case, that is, in the case of system (2.2), we set

\[ A = A_1 = A_2^*, \quad S(0) = S(0)^*, \quad \Pi(0) = \Pi_i(0) = i \Pi_2(0), \]  

so that (2.3) takes on the form

\[ AS(0) - S(0) A^* = i \Pi(0) \Pi(0)^* \quad (S(0) = S(0)^*). \]

As mentioned above, we put

\[ Q_i = -i \sigma_3, \quad Q_0(x) = -i \sigma_3 V(x), \]

which yields \( Q_i^* = -Q_i \) (\( i = 0, 1 \)). Therefore, since \( A = A_1 = A_2^* \), the equations on \( \Pi_1 \) and \( \Pi_2 \) in (2.4) coincide. Hence, taking into account \( \Pi(0) = \Pi_i(0) = i \Pi_2(0) \) we can introduce \( \Pi(x) \) by the equalities \( \Pi_1(x) = \Pi_1(0) = i \Pi_2(0) \). Next, we rewrite formulas (2.4)–(2.6) (from proposition 2.1) in the form

\[ \Pi' = A \Pi Q_1 + \Pi Q_0, \quad S' = i \Pi Q_1 \Pi^*; \]

\[ (\Pi S^{-1})' = -Q_1 \Pi S^{-1} A - \bar{Q}_0 \Pi S^{-1}; \]

\[ \bar{Q}_0 := Q_0 - i (Q_1 X - X Q_1), \quad X := \Pi S^{-1} \Pi. \]

Since \( S(0) = S(0)^* \) and \( S'(x) = S'(x)^* \), we have

\[ S(x) = S(x)^*, \quad X(x) = X(x)^*. \]
Recall that $A$ and $S(0)$ are $n \times n$ matrices and $\Pi(0)$ is an $n \times 2$ matrix. The GBDT-transformed system of the form (2.2) is determined by the initial system (2.2) and a triple of the matrices $\{A, S(0), \Pi(0)\}$ such that (2.8) holds. More precisely, $\Pi(x)$, $S(x)$ and the GBDT-transformed system are determined by $\{A, S(0), \Pi(0)\}$ via relations (2.9), (2.10) and (2.12). The explicit solutions of the GBDT-transformed system are expressed using $\Pi(x)$ and $S(x)$.

**Theorem 2.3.** Let the initial system (2.2) and a triple of the matrices $\{A, S(0), \Pi(0)\}$ (such that (2.8) holds) be given. Then, the vector functions

$$\tilde{\psi}(x, y) = \Pi(x)^* S(x)^{-1} e^{-iA h} , \quad h \in \mathbb{C}^n ,$$

(2.14)

where $\Pi(x)$ and $S(x)$ are defined by $\Pi(0)$, $S(0)$ and the relations (2.10) and (2.9), satisfy the GBDT-transformed system

$$\tilde{\psi}_i = i \sigma_3 (-\tilde{\psi}_i + \tilde{V}(x)\tilde{\psi}) , \quad \tilde{V} := V + i(\sigma_3 A \sigma_3 - X).$$

(2.15)

Here, $X = \Pi^* S^{-1} \Pi$ and $\tilde{V}$ has the same structure as $V$, that is, $\tilde{V} = \begin{bmatrix} 0 & \tilde{a} \\ -\tilde{a} & 0 \end{bmatrix}$.

**Proof.** The structure of $\tilde{V}$ follows from the second equalities in (2.2), (2.12), (2.13) and (2.15). Moreover, in view of (2.9), (2.12) and the second equality in (2.15), we have

$$i \sigma_3 \tilde{V} = -\tilde{Q}_0 \quad \text{and} \quad i \sigma_3 \tilde{V} \tilde{\psi} = -\tilde{Q}_0 \tilde{\psi} .$$

(2.16)

Clearly, (2.14) yields

$$-i \sigma_3 \tilde{V} y = -Q \Pi(x)^* S(x)^{-1} A e^{-iA h} .$$

(2.17)

The terms from the right-hand side of system (2.15) are rewritten in (2.16) and (2.17). Now, using (2.11) and (2.14) we see that the left-hand side of (2.15) coincides (for our $\tilde{\psi}$) with the right-hand side of (2.15), that is, $\tilde{\psi}$ given by (2.14) satisfies the system (2.15). $\square$

Equality (2.8) and equations (2.10) yield the identity

$$A S(x) - S(x) A^* = i \Pi(x) \Pi(x)^* \quad (x \in \mathcal{I}) .$$

(2.18)

Indeed, in view of $Q^* = -Q$, the direct differentiation shows that the derivatives of both sides of (2.18) coincide for all $x \in \mathcal{I}$, and according to (2.8) the identity (2.18) is valid at $x = 0$. Thus, (2.18) holds for all $x \in \mathcal{I}$.

The next proposition on the invertibility of $S$ and on $\sigma(A)$ (where $\sigma(A)$ denotes the spectrum of $A$) easily follows from (2.18).

**Proposition 2.4.** Let the inequality $S(0) > 0$ and the conditions of theorem 2.3 hold. Then, we have $S(x) > 0$ (and so $S(x)$ is invertible) for all $x \in \mathcal{I}$. Moreover, we have $\sigma(A) \subset (\mathbb{C}_+ \cup \mathbb{R})$.

**Proof.** The second equality in (2.10) and the identity (2.18) imply the inequalities

$$e^{-iA S(x) e^{iA h}} = e^{-iA (S(x) - i(AS(x) - S(x) A^*)) e^{iA h}}$$

$$= e^{-iA (\Pi(x)(\sigma_3 + I_2) \Pi(x)^* e^{iA h})} \geq 0 ;$$

(2.19)

$$e^{iA S(x) e^{-iA h}} = e^{iA (S(x) + i(AS(x) - S(x) A^*)) e^{-iA h}}$$

$$= e^{iA (\Pi(x)(\sigma_3 - I_2) \Pi(x)^* e^{-iA h})} \leq 0 .$$

(2.20)
By virtue of (2.19) we derive
\[ e^{-iAx}S(x)e^{iBx} \geq S(0) > 0 \quad \text{for} \quad x \geq 0. \] (2.21)

On the other hand, by virtue of (2.20) we derive
\[ e^{iAx}S(x)e^{-iBx} \geq S(0) > 0 \quad \text{for} \quad x \leq 0. \] (2.22)

It is immediate (from (2.21) and (2.22)) that \( S(x) > 0 \) everywhere on \( I \).

Rewriting (2.18) at \( x = 0 \) as
\[ S(0)^{-1/2}AS(0)^{1/2} - (S(0)^{-1/2}AS(0)^{1/2})^* = iS(0)^{-1/2}P(0)P(0)^*S(0)^{-1/2}, \] (2.23)
we see that \( S(0)^{-1/2}P(0)P(0)^*S(0)^{-1/2} \geq 0 \). Hence, for the case when \( c \) and \( f \) are an eigenvalue and a corresponding eigenfunction of \( S(0)^{-1/2}AS(0)^{1/2} \), formula (2.23) yields
\[ -i \{ S(0)^{-1/2}AS(0)^{1/2} - (S(0)^{-1/2}AS(0)^{1/2})^* \} f = i(c - c)f^* f > 0. \]

The relation above means that \( i(c - c) \geq 0 \) or, equivalently, that \( c \in (C_+ \cup R) \). In other words, \( \sigma(S(0)^{-1/2}AS(0)^{1/2}) \subset (C_+ \cup R) \), and so \( \sigma(A) \subset (C_+ \cup R) \).

**Remark 2.5.** We note (see [15, 16, 22]) that the identities of the form (2.18) appear in the GBDT for skew-self-adjoint spectral Dirac systems, whereas the identities
\[ AS(x) - S(x)A^* = i\Pi(x)\sigma_i\Pi(x)^* \quad (x \in I) \] (2.24)
appear in the self-adjoint case. For nonlinear integrable equations with an auxiliary linear Dirac or, equivalently, ZS-AKNS systems (e.g. for nonlinear Schrödinger and mKdV equations) the identity (2.18) appears in the case of multisoliton-type potentials, whereas the identity (2.24) corresponds to potentials with singularities [5, 8, 18]. A discussion on multisoliton-type potentials and on the connections between the system (1.1) and mKdV is also contained in [11].

### 3. Explicit solutions of Dirac–Weyl systems

Explicit solutions usually appear in Bäcklund–Darboux transformations when we choose trivial initial systems. Indeed, setting in (2.2) \( V \equiv 0 \) we rewrite the equation on \( \Pi \) in (2.10) as
\[ \Pi' = -iA\Pi\sigma_3, \]
which yields
\[ \Pi(x) = [A_1(x) \quad A_2(x)], \quad A_1(x) = e^{-iAx}A_1(0), \quad A_2(x) = e^{iAx}A_2(0), \] (3.1)
where \( A_k \) \( (k = 1, 2) \) are the columns of \( \Pi \). Explicit expressions for \( \Pi \) provide explicit expressions for \( S, \tilde{V} \) and \( \tilde{\psi} \). In particular, we have
\[ S(x) = S(0) + \int_0^x \Pi(r)\sigma_3\Pi(r)^*dr \quad (x > 0), \] (3.2)
\[ S(x) = S(0) - \int_x^0 \Pi(r)\sigma_3\Pi(r)^*dr \quad (x < 0) \] (3.3)
with \( \Pi \) determined in (3.1). The corresponding explicit solutions \( \tilde{\psi} \) are given in the next corollary of theorem 2.3.

**Corollary 3.1.** Let some triple of the parameter matrices \( \{ A, S(0) > 0, \Pi(0) \} \) (such that (2.8) holds) be fixed. Introduce the matrix functions \( \Pi(x) \) and \( S(x) \) by the relations (3.1)–(3.3). Then, the vector functions \( \tilde{\psi} \) of the form (2.14) satisfy (are explicit solutions of) the Dirac system

\[
\tilde{\psi}_x = i\sigma_3(-\tilde{\psi}_x + \tilde{V}(x)\tilde{\psi}); \quad \tilde{V} := i(\sigma_3 \Lambda \sigma_3 - A), \quad \Lambda^* = \Pi S^{-1} \Pi \tag{3.4}
\]
on the real axis \(-\infty < x < \infty\). The matrix function \( S(x) \) in (2.14) and (3.4) is positive definite (i.e. \( S(x) > 0 \)), and so invertible, everywhere on \( \mathbb{R} \). Thus, \( \tilde{V} \) and \( \tilde{\psi} \) are well-defined everywhere on \( \mathbb{R} \).

Under the additional conditions that the entries of the matrices \( iA \) and \( S(0) \) and of the vectors \( i\Lambda_1(0) \) and \( \Lambda_2(0) \) are all real-valued, the potential \( \tilde{V} \) is real-valued as well, and we can rewrite (3.4) in the form of the Dirac–Weyl system (1.3):

\[
\tilde{\psi}_x = i\sigma_3(-\tilde{\psi}_x + i\tilde{\psi}(x)\sigma_2\tilde{\psi}); \quad \tilde{\psi} := -2i\Lambda_1^*S^{-1} \Lambda_2 = \tilde{u} \tag{3.5}
\]

**Proof.** The fact that \( \tilde{\psi} \) given by (2.14) satisfies (3.4) is immediate from theorem 2.3. The positive definiteness of \( S(x) \) follows from proposition 2.4. Finally, ‘additional conditions’ in the corollary 3.1 and relations (3.1)–(3.3) imply that the entries of \( i\Lambda_1(x) \), \( \Lambda_2(x) \) and \( S(x) \) are real-valued. Hence, \( \tilde{u} \) given by the second equality in (3.5) is real-valued. For \( \tilde{V} \) introduced in (3.4) and the real-valued \( \tilde{u} \) from (3.5), we have the equalities

\[
V = \begin{bmatrix} 0 & \tilde{u} \\ -\tilde{u} & i\tilde{u} \end{bmatrix} = i\tilde{u}\sigma_2 \tag{3.6}
\]

and, substituting (3.6) into the Dirac system (3.4), we obtain the Dirac–Weyl system (3.5).

Below, we give several examples (and classes of examples) of the triples \( A, S(0) \) and \( \Pi(0) \) of the parameter matrices which satisfy all the conditions of the corollary 3.1. For simplicity, we set \( S(0) = I_n \) in these examples. We set \( A = i\mathcal{A} \), where \( \mathcal{A} \) is a matrix with real-valued entries. GBDT allows us to consider the non-diagonal matrices \( A \) (in which case, explicit solutions with interesting properties appear; see, e.g. [1, 17]), and in example 3.3 we calculate \( \tilde{u} \) and \( \tilde{\psi} \) for the case when the matrix \( A \) is similar to a \( 2 \times 2 \) Jordan cell.

**Example 3.2.** If \( A = i\mathcal{A} \) is a scalar and \( S(0) = 1 \) (i.e. the order \( n \) of \( A \) and \( S \) equals 1), then \( \Lambda_1(0) \) and \( \Lambda_2(0) \) are scalars as well and the identity (2.8) takes on the form \( 2\mathcal{A} = |\Lambda_1(0)|^2 + |\Lambda_2(0)|^2 \). All values \( \mathcal{A} \) and \( |\Lambda_k(0)| \) \( (k = 1, 2) \), which satisfy this equality, generate the corresponding parameters \( A = i\mathcal{A} \) and \( \Pi(0) = [\pm i|\Lambda_1(0)| \pm \pm |\Lambda_2(0)|] \) or \( \Pi(0) = [\pm i|\Lambda_1(0)| \mp |\Lambda_2(0)|] \) satisfying the conditions of the corollary 3.1.

**Example 3.3.** In the case \( n = 2 \), we consider the example: \( S(0) = I_2 \),

\[
A = i\mathcal{A} = i\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \Pi(0) = [\Lambda_1(0) \quad \Lambda_2(0)] = \frac{1}{\sqrt{2}} \begin{bmatrix} 2i & 0 \\ i & \sqrt{3} \end{bmatrix} \tag{3.7}
\]
Clearly, (2.8) holds for this example. Simple calculations show that
\[ e^{\pm iA} = e^{\pm A} = e^{\pm x} e^{\pm (A - i I)} = e^{\pm x} \begin{bmatrix} 1 & 0 \\ \pm x & 1 \end{bmatrix}. \] (3.8)

Using (3.1), (3.7) and (3.8) we obtain
\[ \Lambda_1(x) = \frac{1}{\sqrt{2}} e^{x/2} \begin{bmatrix} 2i/(2x + 1) \\ i(2x + 1) \end{bmatrix}, \quad \Lambda_2(x) = \frac{1}{\sqrt{2}} e^{-x} \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}. \] (3.9)

\[ \Pi(x)\sigma_3 \Pi(x)^* = \frac{1}{2} \left( e^{2x} \begin{bmatrix} 2 \\ 2x + 1 \end{bmatrix} [2x + 1] - 3e^{-2x} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right). \] (3.10)

Integrating (3.10) and substituting the result into (3.2) and (3.3), we calculate \( S \) and easily derive from the expression for \( S(x) \) that \( \det S(x) = \frac{1}{2}(e^{4x} + 3) \) and
\[ S(x)^{-1} = \frac{4}{e^{4x} + 3} \begin{bmatrix} \frac{1}{4}((4x^2 + 1)e^{2x} + 3e^{-2x}) - xe^{2x} \\ -xe^{2x} \\ e^{2x} \end{bmatrix}. \] (3.11)

It is immediate from (3.5), (3.9) and (3.11) that
\[ \bar{u}(x) = -2i\Lambda_1(x)^* S(x)^{-1} \Lambda_2(x) = -\frac{4\sqrt{3}}{e^{4x} + 3} e^{2x}. \] (3.12)

Next, we note that
\[ e^{-xA} = e^{-iy} e^{x(A - i I)} = e^{-iy} \begin{bmatrix} 1 & 0 \\ -iy & 1 \end{bmatrix}. \] (3.13)

Finally, relations (3.7), (3.9), (3.11) and (3.13) yield the equalities:
\[ \Pi(x) S(x)^{-1} e^{-yA} h = \frac{2\sqrt{2}}{e^{4x} + 3} e^{-iy} \begin{bmatrix} \frac{1}{2}((2x - 1)e^{3x} - 3e^{-x}) - ie^{3x} \\ -\sqrt{3} xe^x \\ \sqrt{3} e^x \end{bmatrix} \times \begin{bmatrix} 1 \\ -iy \\ 1 \end{bmatrix} h, \quad h \in \mathbb{C}^2. \] (3.14)

Thus, we obtain the following corollary.

**Corollary 3.4.** The vector functions \( \tilde{\psi} \) of the form (3.14) are solutions of the Dirac–Weyl system (3.5) where \( \bar{u} \) is given by (3.12).

Now, we introduce some wide classes of parameter triples \( \{A, S(0), \Pi(0)\} \) (satisfying the conditions of the corollary 3.1) with an arbitrary fixed number \( n \in \mathbb{N} \). As before, we assume that \( S(0) = I_n \) and \( A = iA \), where the entries of the \( n \times n \) matrix \( A \) are real-valued.

**Example 3.5.** Set
\[ A = A_0 + h_1 h_1^* + h_2 h_2^* \quad (A_0^\dagger = -A_0), \quad \Pi(0) = \sqrt{2} [ih_1 \ h_2]. \] (3.15)
where \( h_k \in \mathbb{R}^n \) for \( k = 1, 2 \), and the entries of \( A_0 \) are real-valued. Then, all the conditions of the corollary 3.1 are fulfilled.

**Example 3.6.** Set \( \Pi(0) = [h_1 \ h_2] \), where \( h_k \in \mathbb{R}^n \) for \( k = 1, 2 \), and choose a triangular matrix \( A \) such that \( A + A^t = \Pi(0)\Pi(0)^t \). Then, all the conditions of the corollary 3.1 are fulfilled.

**Remark 3.7.** We note that the ‘additional conditions’ in corollary 3.1 are only sufficient (not necessary) for \( \tilde{u} \) to be real-valued. Clearly, we may multiply \( \Pi(0) \) (and so \( \Pi(x) \)) by the scalar value \( e^{iu} \) such that \( |e^{iu}| = 1 \), and obtain the same potential \( \tilde{u} \) as without multiplication. Moreover, we may rewrite the formula for \( \tilde{u} \) in (3.5) as

\[
\tilde{u}(x) = -2i\Lambda_1(0)^t(e^{-ixe}S(x)e^{-ixe^t})^{-1}\Lambda_2(0).
\]

Thus, in the case of degenerate matrices \( A \), we can use the fact that \( S(x) \) is self-adjoint in order to construct real-valued potentials \( \tilde{u} \). However, the case of degenerate matrices \( A \) is of less importance.

It is our hypothesis that the sufficient conditions (i.e. the ‘additional conditions’ given in corollary 3.1) for \( \tilde{u} \) to be real-valued are close to the necessary ones. It would be of interest to prove this hypothesis (or to find some physically meaningful cases when a real-valued potential \( \tilde{u} \) can be constructed without the additional conditions but cannot be constructed with them), and we plan to study this problem.

**Acknowledgments**

This research was supported by the Austrian Science Fund (FWF) under grant no. P29177. The author is grateful to an Editorial Board Member (referee) for the interesting comments, which led to remark 3.7.

**References**

[1] Ablowitz M J, Chakravarty S, Trubatch A D and Villarroel J 2000 A novel class of solutions of the non-stationary Schrödinger and the Kadomtsev–Petviashvili I equations Phys. Lett. A 267 132–46

[2] Ciesielski J L 2009 Algebraic construction of the Darboux matrix revisited J. Phys. A: Math. Theor. 42 404003

[3] Deift P A 1978 Applications of a commutation formula Duke Math. J. 45 267–310

[4] Fritzsche B, Kirstein B, Roitberg I and Sakhnovich A L 2015 Pseudo-exponential-type solutions of wave equations depending on several variables SIGMA 11 010

[5] Fritzsche B, Kirstein B and Sakhnovich A L 2006 Completion problems and scattering problems for Dirac type differential equations with singularities J. Math. Anal. Appl. 317 510–25

[6] Gesztesy F 1993 A complete spectral characterization of the double commutation method J. Funct. Anal. 117 401–46

[7] Gesztesy F and Teschl G 1996 On the double commutation method Proc. Am. Math. Soc. 124 1831–40

[8] Gohberg I, Kaashoek M A and Sakhnovich A L 1998 Canonical systems with rational spectral densities: explicit formulas and applications Math. Nachr. 194 93–125

[9] Gu C, Hu H and Zhou Z 2005 Darboux Transformations in Integrable Systems. Theory and their Applications to Geometry (Mathematical Physics Studies vol 26) (Dordrecht: Springer)

[10] Hartmann R R and Portnoi M E 2014 Quasi-exact solution to the Dirac equation for the hyperbolic-secant potential Phys. Rev. A 89 012101
[11] Ho C-L and Roy P 2015 mKdV equation approach to zero energy states of graphene Europhys. Lett. 112 47004
[12] Kostenko A, Sakhnovich A and Teschl G 2012 Commutation methods for Schrödinger operators with strongly singular potentials Math. Nachr. 285 392–410
[13] Matveev V B and Salle M A 1991 Darboux Transformations and Solitons (Berlin: Springer)
[14] Midya B and Fernandez D J 2014 Dirac electron in graphene under supersymmetry generated magnetic fields J. Phys. A: Math. Theor. 47 285302
[15] Sakhnovich A L 1994 Dressing procedure for solutions of nonlinear equations and the method of operator identities Inverse Problems 10 699–710
[16] Sakhnovich A L 2001 Generalized Bäcklund–Darboux transformation: spectral properties and nonlinear equations J. Math. Anal. Appl. 262 274–306
[17] Sakhnovich A L 2003 Matrix Kadomtsev–Petviashvili equation: matrix identities and explicit nonsingular solutions J. Phys. A: Math. Gen. 36 5023–33
[18] Sakhnovich A L 2003 Dirac type system on the axis: explicit formulas for matrix potentials with singularities and soliton-positon interactions Inverse Problems 19 845–54
[19] Sakhnovich A L 2011 The time-dependent Schrödinger equation of dimension k + 1: explicit and rational solutions via GBDT and multinoles J. Phys. A: Math. Theor. 44 475201
[20] Sakhnovich A L 2017 Dynamical canonical systems and their explicit solutions Discrete and Continuous Dyn. Syst. A 37 1679–89
[21] Sakhnovich A L 2016 Hamiltonian systems and Sturm–Liouville equations: Darboux transformation and applications (arXiv:1608.02348)
[22] Sakhnovich A L, Sakhnovich L A and Roitberg I 2013 Inverse Problems and Nonlinear Evolution Equations, Solutions, Darboux Matrices and Weyl–Titchmarsh Functions (De Gruyter Studies in Mathematics vol 47) (Berlin: De Gruyter)
[23] Sakhnovich L A 1999 Spectral theory of canonical differential systems, method of operator identities (Operator Theory Advances and Applications vol 107) (Basel: Birkhäuser)
[24] Schmidt K M and Umeda T 2015 Schnol’s theorem and spectral properties of massless Dirac operators with scalar potentials Lett. Math. Phys. 105 1479–97
[25] Stauber T 2014 Plasmonics in Dirac systems: from graphene to topological insulators J. Phys.: Condens. Matter 26 123201
[26] Teschl G 1998 Deforming the point spectra of one-dimensional Dirac operators Proc. Am. Math. Soc. 126 2873–81
[27] Zakharov V E and Mikhailov A V 1980 On the integrability of classical spinor models in two-dimensional space-time Commun. Math. Phys. 74 21–40