Algebraic Geometry

A remark on the Abel–Jacobi morphism for the cubic threefold

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A R T I C L E  I N F O

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A B S T R A C T

Let $X$ be a smooth cubic threefold and $J(X)$ be its intermediate Jacobian. We show that there exists a codimension 2 cycle $Z$ on $J(X) \times X$ with $Z_t$ homologically trivial for each $t \in J(X)$, such that the morphism $\phi_Z : J(X) \to J(X)$ induced by the Abel–Jacobi map is the identity. This answers positively a question of Voisin in the case of the cubic threefold.

Version française abrégée

Soit $X$ une variété projective lisse sur $\mathbb{C}$. On a le résultat suivant, obtenu en combinant des résultats de Bloch–Ogus [2], Bloch–Srinivas [3], Merkurjev–Suslin [11] et Murre [12] :

Théorème 0.1. (Voir [12]) Si $CH_0(X)$ est supporté sur une courbe, on a $CH^2(X)_{\text{hom}} = CH^2(X)_{\text{alg}}$ et l’application d’Abel–Jacobi induit un isomorphisme de groupes

$$AJ_X : CH^2(X)_{\text{hom}} \to J(X) := H^3(X, \mathbb{C})/(F^2H^3(X) \oplus H^3(X, \mathbb{Z})).$$

Dans cette Note, nous considérons le cas où $X$ est une variété projective de dimension 3 rationnellement connexe. On a alors $CH_0(X) = \mathbb{Z}$ et $CH^2(X) = CH_1(X)$.

Le morphisme $AJ_X$ est algébrique dans un sens à préciser, car $J(X)$ est une variété algébrique mais $CH_1(X)_{\text{hom}}$ est seulement une limite inductive de quotients de variétés algébriques par une relation d’équivalence. L’algébricité du morphisme $AJ_X$ signifie par définition que pour toute variété projective lisse $Y$ et tout cycle $Z$ de codimension 2 sur $Y \times X$ tel que $Z_y \in CH^2(X)_{\text{hom}}$ pour tout $y \in Y$, le morphisme induit

$$\phi_Z : Y \to J(X), \quad \phi_Z(y) = AJ_X(Z_y)$$

est un morphisme de variétés algébriques, qui sera appelé le morphisme d’Abel–Jacobi.

Une observation importante faite par Voisin est le fait qu’il existe, malgré la similarité entre le Théorème 0.1 et le théorème d’Abel, des différences substantielles entre les 1-cycles sur les variétés de dimension 3 avec petit $CH_0$ et les 0-cycles sur les courbes.

Les deux questions suivantes sont posées dans [13] :

Question 0.2. Soit $X$ une variété porojective lisse de dimension 3 telle que $AJ_X : CH_1(X)_{\text{alg}} \to J(X)$ est surjective. Existe-t-il un cycle $Z$ universel de codimension 2 cycle on $J(X) \times X$, tel que $Z_t \in CH^2(X)_{\text{hom}}$ pour tout $t \in J(X)$ et que le morphisme d’Abel–Jacobi

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\[ \phi_Z : J(X) \to J(X), \quad \phi_Z(t) = Af_X(Z_t) \]
soit l’identité?

Voisin remarque que la Question 0.2 a une réponse positive si la conjecture de Hodge est satisfaite pour les classes de Hodge entières de degré 4 sur \( J(X) \times X \).

**Question 0.3.** Pour quelles variétés projectives lisses \( X \) de dimension 3 la propriété suivante est-elle satisfaite ?

Il existe une variété projective lisse \( Y \) et un cycle \( Z \) de codimension 2 sur \( Y \times X \) tel que \( Z_y \in CH^2(X)_{\text{hom}} \) pour tout \( y \in Y \), et que le morphisme d'Abel–Jacobi \( \phi_Z : Y \to J(X) \) soit surjectif à fibre générale rationnellement connexe.

On sait que la Question 0.3 a une réponse positive pour les cubiques lisses de dimension 3 [9,10] et les intersections de codimension 2 sur \( X \), où la Question 0.3 a une réponse positive pour les cubiques lisses de dimension 3 la propriété suivante est-elle satisfaite ?

Nous donnons dans cette Note une réponse positive à la Question 0.2 pour les cubiques lisses de dimension 3. Pour les propriétés de la jacobienne intermédiaire des cubiques de dimension 3, nous renvoyons à [5] :

**Théorème 0.4.** Soit \( X \) une cubique lisse de dimension 3. Il existe un cycle \( Z \) de codimension 2 sur \( J(X) \times X \) avec \( Z_t \in CH^2(X)_{\text{hom}} \) pour tout \( t \in J(X) \), tel que le morphisme d'Abel–Jacobi induit \( \phi_Z : J(X) \to J(X) \) soit l'identité.

Nous utilisons pour cela une condition suffisante pour qu'un ouvert de l'espace des faisceaux stables sur une variété projective lisse \( X \) soit fini.

Pour une telle \( X \), le groupe de Grothendieck modulo équivalence numérique \( K_{\text{num}}(X) \) est défini comme \( K(X)/\equiv \), où deux classes \( x \) et \( y \) de \( K(X) \) sont dites numériquement équivalentes (notation \( x \equiv y \)), si la différence \( x - y \) appartient au radical de la forme quadратique

\[ (a, b) \mapsto \chi(a \cdot b) = \int_X \text{ch}(a)\text{ch}(b)t^t d(X) \]

(cf. [7]). Fixons une classe \( c \in K_{\text{num}}(X) \). Soit \( P \) le polynôme de Hilbert associé à \( c \), \( M^3 \) l’espace de modules des faisceaux stables sur \( X \) de polynôme de Hilbert \( P \), et \( M(c)^3 \subset M^3 \) le sous-ensemble localement fermé paramétrant les faisceaux stables de classe numérique \( c \).

**Théorème 0.5.** (Voir [8, Th. 4.6.5.]) Si le PGCD de tous les nombres \( \chi(c \cdot F) \), où \( F \) parcourt une certaine collection des faisceaux cohérents sur \( X \), est égal à 1, il existe un faisceau universel sur \( M(c)^3 \times X \).

**Théorème 0.6.** Soit \( X \) une cubique lisse de dimension 3. Alors l'espace de modules \( M^3_X(2; 0, 2) \) des faisceaux stables de rang 2 sur \( X \) de nombres de Chern \( c_1 = 0, c_2 = 2, c_3 = 0 \) est fini (c'est-à-dire qu'il existe un faisceau universel sur \( M^3_X(2; 0, 2) \times X \)).

**Démonstration.** Soit \( c \) la classe numérique d'un faisceau stable localement libre \( E \) de rang 2 sur \( X \). Rappelons que Pic\((X) = \mathbb{Z} \cdot h \), où \( h \) est la classe d'une section hyperplane de \( X \). Le groupe \( A_1(X) \) des 1-cycles de \( X \) modulo équivalence algébrique est égal à \( \mathbb{Z} \cdot l \), où \( l \) est la classe d'une droite de \( X \). Notons que \( h^2 = 3l \). Comme \( X \) est une variété de Fano, \( CH_0(X) = \mathbb{Z} \cdot \text{pt} \). On a \( \text{ch}(c) \equiv \text{ch}(E) = 2 - 2l \). Comme \( c_1(T_X) = 2h \) et \( c_2(T_X) = 12l \), on a \( t^t d(X) = 1 + h + 2l + pt \). On voit facilement que \( \text{ch}(\mathcal{O}_X(1)) = 1 + h + \frac{3}{2}l \). On trouve alors que \( \chi(c \cdot E) = -4 \) et

\[ \chi(c \cdot \mathcal{O}_X(1)) = \int_X \text{ch}(c) \cdot \text{ch}(\mathcal{O}_X) \cdot t^t d(X) \]

\[ = \int_X (2 - 2l) \cdot 
\left( 1 + h + \frac{3}{2}l \right) \cdot (1 + h + 2l + pt) = 5. \]

Clairement, le PGCD de \( \chi(c \cdot E) \) et \( \chi(c \cdot \mathcal{O}_X(1)) \) est égal à 1. Le Théorème 0.5 entraîne donc qu'il existe un faisceau universel sur \( M^3_X(2; 0, 2) \times X \).
1. Introduction

A classical theorem of Abel states that

**Theorem 1.1.** Let $C$ be a smooth projective complex curve. Then each fiber of the Abel–Jacobi map

$$A_{JC} : \text{Sym}^d C \rightarrow J(C)$$

is a projective space for all $d \geq g(C)$. Moreover, the induced morphism

$$CH_0(C)_{\text{num}} \rightarrow J(C)$$

is an isomorphism.

In particular, the geometry of the fibers of the Abel–Jacobi map for curves is well understood. For higher dimensional varieties, the work of Bloch–Ogus [2], Bloch–Srinivas [3], Merkurjev–Suslin [11] and Murre [12] leads to the following theorem, which can be regarded as the higher dimensional generalization of the second assertion of Theorem 1.1:

**Theorem 1.2.** (See [12].) Let $X$ be a smooth projective complex variety such that $CH_0(X)$ is supported on a curve. Then $CH^2(X)_{\text{hom}} = CH^2(X)_{\text{alg}}$ and the Abel–Jacobi map induces an isomorphism

$$A_{JX} : CH^2(X)_{\text{hom}} \rightarrow J(X) := H^3(X, \mathbb{C})/(F^2 H^3(X) \oplus H^3(X, \mathbb{Z})).$$

In the present note, we will consider the case where $X$ is a rationally connected threefold, so that $CH_0(X) = \mathbb{Z}$ is supported on a point and $CH^2(X) = CH_1(X)$.

Since the group $CH_1(X)_{\text{alg}}$ does not have the structure of an algebraic variety, one has to be careful when stating that $A_{JX}$ is algebraic. In fact, $CH_1(X)_{\text{alg}}$ is an inductive limit of quotients of algebraic varieties by an equivalence relation, and to say that the morphism $A_{JX}$ is algebraic means by definition that for any smooth projective variety $Y$ and any codimension 2 cycle $Z$ on $Y \times X$ with $Z_y \in CH^2(Y)_{\text{hom}}$ for any $y \in Y$, the induced morphism

$$\phi_Z : Y \rightarrow J(X), \quad \phi_Z(y) = A_{JX}(Z_y)$$

is a morphism of algebraic varieties, which will be called the **Abel–Jacobi morphism**.

An important observation made by Voisin is that, despite the similarity between Theorem 1.2 and Abel’s Theorem 1.1, there are substantial differences between 1-cycles on threefolds with small $CH_0$ and 0-cycles on curves, which she relates to the geometry of the fibers of the Abel–Jacobi morphisms.

In fact, the following two questions are proposed in [13]:

**Question 1.3.** Let $X$ be a smooth projective threefold such that $A_{JX} : CH_1(X)_{\text{alg}} \rightarrow J(X)$ is surjective. Is there a codimension 2 cycle $Z$ on $J(X) \times X$ with $Z_t \in CH^2(X)_{\text{hom}}$ for any $t \in J(X)$ such that the Abel–Jacobi morphism

$$\phi_Z : J(X) \rightarrow J(X), \quad \phi_Z(t) = A_{JX}(Z_t)$$

is the identity?

As remarked by Voisin, Question 1.3 has a positive answer if the Hodge conjecture holds true for degree 4 integral Hodge classes on $J(X) \times X$.

**Question 1.4.** For which threefolds $X$ is the following property satisfied?

There exist a smooth projective variety $Y$ and a codimension 2 cycle $Z$ on $Y \times X$ with $Z_y \in CH^2(Y)_{\text{hom}}$ for any $y \in Y$, such that the Abel–Jacobi morphism $\phi_Z : Y \rightarrow J(X)$ is surjective with rationally connected general fiber.
It is known that Question 1.4 has a positive answer for smooth cubic threefolds \([9,10]\) and smooth complete intersections of two quadrics in \(\mathbb{P}^3\) \([4]\). It was proved in \([13]\) that if Question 1.4 has a positive answer for \(X\) and the intermediate Jacobian \(J(X)\) admits a 1-cycle \(\Gamma\) such that \(\Gamma \in CH_2(J(X))\) with \(\deg \Gamma = g \cdot J(X)\), where \(g = \dim J(X)\), then Question 1.3 also has a positive answer for \(X\). In particular, if the intermediate Jacobian of \(X\) is isomorphic to the Jacobian of a curve, then Question 1.3 has a positive answer for \(X\) if Question 1.4 does. Therefore, Question 1.3 has a positive answer for smooth complete intersections of two quadrics in \(\mathbb{P}^3\).

Unfortunately, there are very few rationally connected threefolds whose intermediate Jacobians are not Jacobians and for which the existence of a cycle \(\Gamma\) as above is known (this is a very classical question in the case of general cubic threefolds, and is equivalent in this case to the algebraicity of the so-called minimal class \(\Theta_4\) which is an integral Hodge class on \(J(X)\)). Question 1.3 needs therefore other approaches.

In this Note we give a positive answer to Question 1.3 for any smooth cubic threefold. For the properties of the intermediate Jacobian of the cubic threefold, see \([5]\). The key point of our proof lies in the observation that the moduli space of stable sheaves of rank 2 with Chern numbers \(c_1 = 0, c_2 = 2, c_3 = 0\) on a smooth cubic threefold is fine.

We will work over the complex number field \(\mathbb{C}\).

### 2. The main result

In this section, we state and prove the main result of this Note:

**Theorem 2.1.** Let \(X\) be a smooth cubic threefold. Then there exists a codimension 2 cycle \(Z\) on \(J(X) \times X\) with \(Z_t \in CH^2(J(X))\) for any \(t \in J(X)\), such that the induced Abel–Jacobi morphism \(\phi_Z : J(X) \to J(X)\) is the identity.

We will need a sufficient condition for an open subset of the moduli space of stable sheaves on a smooth projective variety to be fine.

Let \(X\) be a smooth projective variety. Recall that the Grothendieck group modulo numerical equivalence \(K_{\text{num}}(X)\) is defined to be \(K(X)/\equiv\), where two classes \(x\) and \(y\) in \(K(X)\) are said to be numerically equivalent (notation \(x \equiv y\)), if the difference \(x - y\) is contained in the radical of the quadratic form

\[
(a, b) \mapsto \chi(a \cdot b) = \int_X \text{ch}(a) \text{ch}(b) \, t \, d(X)
\]

(cf. \([7]\)). Now fix a class \(c \in K_{\text{num}}(X)\). Let \(P\) be the associated Hilbert polynomial, \(M^i\) be the moduli space of stable sheaves on \(X\) and \(M(c) \subseteq M^i\) be the open and closed part parametrizing stable sheaves of numerical class \(c\).

**Theorem 2.2.** (See \([8, \text{Theorem 4.6.5}]\)) If the greatest common divisor of all numbers \(\chi(c \cdot F)\), where \(F\) runs through some collection of coherent sheaves on \(X\), is equal to 1, then there is a universal sheaf on \(M(c)^i \times X\).

**Theorem 2.3.** Let \(X\) be a smooth cubic threefold. Then the moduli space \(M_X^c(2; 0, 2)\) of stable sheaves of rank 2 with Chern numbers \(c_1 = 0, c_2 = 2, c_3 = 0\) on \(X\) is fine. Equivalently, there exists a universal sheaf on \(M_X^c(2; 0, 2) \times X\).

**Proof.** Let \(c\) be the numerical class of a locally free, stable sheaf \(E\) of rank 2 on \(X\). Recall that \(\text{Pic}(X) = \mathbb{Z} \cdot h\), where \(h\) is the class of a hyperplane section of \(X\). The Chow group of 1-cycles on \(X\) modulo algebraic equivalence \(A_1(X) = \mathbb{Z} \cdot l\), where \(l\) is the class of a line on \(X\). Note that \(h^2 = 3l\). Since \(X\) is Fano, \(\text{CH}_0(X) = \mathbb{Z} \cdot \text{pt}\). Then \(\text{ch}(c) = \text{ch}(E) = 2 - 2l\). Since \(c_1(T_X) = 2h\) and \(c_2(T_X) = 12l\), \(t d(X) = 1 + h + 2l + pt\). It is easy to see that \(\text{ch}(O_X(1)) = 1 + h + \frac{3}{2}l\). Then we compute that \(\chi(c \cdot E) = -4, \chi(c \cdot O_X(1)) = \int_X \text{ch}(c) \cdot \text{ch}(O_X) \cdot t \, d(X) = \int_X (2 - 2l) \cdot \left(1 + h + \frac{3}{2}l\right) \cdot (1 + h + 2l + pt) = 5\).

Obviously, the greatest common divisor of \(\chi(c \cdot E)\) and \(\chi(c \cdot O_X(1))\) is equal to 1. Then Theorem 2.2 implies that there exists a universal sheaf on \(M_X^c(2; 0, 2) \times X\).

**Proof of Theorem 2.1.** It is shown in \([6,9]\) (see also \([1]\)) that the morphism \(\phi : \text{Hilb}^n(X) \to J(X)\) factorizes through the birational morphism \(c_2 : M_X(2; 0, 2) \to J(X)\). Moreover, letting \(M_0\) be the open subset of \(M_X(2; 0, 2)\) parametrizing locally free stable sheaves, the restricted morphism \(M_0 \to J(X)\) is an open immersion. Now we regard \(M_0\) as an open subset of \(J(X)\). Let \(Z\) be the closure in \(J(X) \times X\) of a global section of \(E|_{M_0(2; 0, 2) \times X}\), where \(M_0\) is an open subset of \(M_0\) over which such a transverse section exists, and \(Z = Z' \setminus J(X) \times C_0\), where \(C_0\) is a quintic elliptic curve on \(X\). Then the induced Abel–Jacobi morphism \(\phi_Z : J(X) \to J(X)\) is the identity, since by construction it induces the natural inclusion on \(M_0\).
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