Comment on “Removable singularities for solutions of coupled Yang-Mills-Dirac equations” [J. Math. Phys. 47, 103502 (2006)]

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Abstract

A lemma from elliptic theory is used to improve a recent result by Li concerning the removability of an isolated point singularity from solutions of the coupled Yang-Mills-Dirac equations. MSC2000: 35J60, 70S15

Recently, Wei Li proved [2] that a smooth solution of the coupled Yang-Mills-Dirac equations in a punctured Euclidean 4-disk $B\setminus\{0\}$ is equivalent via a continuous gauge transformation to a smooth solution in all of $B$ whenever $F \in L^2(B)$ and $\phi \in L^{8/3}(B)$. Here $F$ denotes the Yang-Mills field and $\phi$ is the spinor field. This represents a technical improvement of a well known theorem by Parker [4], in that no condition is placed on the derivative of the spinor field in [2]. An important feature of Li’s proof is that the fields are estimated by purely analytic arguments, away from the singularity, in terms of conformally invariant norms; an application of Uhlenbeck’s broken Hodge gauges [7] completes the proof.

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In fact the $L^p$ hypothesis on $\phi$ in [2] can be weakened by an argument which is also purely analytic. A lemma due to L. M. Sibner [6] can be used to show the sufficiency of the assumption $\phi \in L^p(B), p > 2$, provided the other hypotheses of [2] are retained.

**Lemma 1** (L. M. Sibner). Let the nonnegative scalar function $u$ be $C^\infty$ in the punctured $n$-disk $B'\{0\}$, for $n > 2$, and satisfy there the subelliptic inequality

$$\Delta u + g(x)u \geq 0$$

for a function $g \in L^{n/2}(B)$. If for $1/2 < q_0 < q$ we have $u \in L^{2q_0/(n-2)}(B) \cap L^{2q}(B)$, then $\nabla (u^q) \in L^2(B)$ and in a sufficiently small $n$-disk $\tilde{B}$,

$$\int_{\tilde{B}} \eta^2 |\nabla (u^q)|^2 d(vol) \leq C \int_{\tilde{B}} |\nabla \eta|^2 u^{2q} d(vol)$$

for a positive constant $C$ and all $\eta \in C_0^\infty (\tilde{B})$.

The proof of Lemma 1 in [6] depends on the use of a delicate test function introduced by Serrin ([5], see also [1], Sec. 3), but is otherwise elementary.

Thus we have:

**Theorem 2.** Let the pair $(F, \phi)$ smoothly satisfy the coupled Yang-Mills-Dirac equations in the punctured Euclidean 4-disk $B'\{0\}$. If $F \in L^2(B)$ and $\phi \in L^p(B)$ for some $p > 2$, then $F$ and $\phi$ are equivalent via a continuous gauge transformation to a smooth solution over all of $B$.

Proof. In Lemma 1, choose $u = |\phi|$ and $g = k|F|$ for constant $k$. Then (11) is satisfied, by Proposition 3.3 of [2]. Choose $q_0 = 1/ (2-\varepsilon)$, for $0 < \varepsilon < 2$, and $q = n/(2 - \varepsilon) (n-2)$. Lemma 1 implies that $\nabla (|\phi|^q) \in L^2(B)$. Applying the Sobolev Theorem, we conclude that $|\phi| \in L^r(B)$ for $r = 2 [n/(n-2)]^2 / (2-\varepsilon)$. If $n = 4$, then $r$ exceeds $8/3$. Now apply Theorem 4.1 of [2]. This completes the proof of Theorem 2.

The $L^p$ estimate of $\phi$ in the proof of Theorem 2 does not use any of the properties of a geometric object. In particular, it does not use the conformal weight of $\phi$. Iterating this estimate will show that $\phi$ lies in the space $L^r$ for any finite value of $r$ (see [3], Proposition 3.7); but the arguments of [2] make such an iteration unnecessary.

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