Large Lepton Flavor Mixing and 
E₆-type Unification Models

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Abstract

There are experimental indications of large flavor mixing between \( \nu_\mu \) and \( \nu_\tau \). In the unification models, in which the massless sector includes extra particles beyond the standard model, there possibly appear the mixings between quarks (leptons) and the extra particles. When large mixings occur, lepton flavor mixings can be quite different from quark flavor mixings. By taking the string inspired \( SU(6) \times SU(2)_R \) model with global flavor symmetries, we obtain the neutrino flavor mixing \( \sin \theta_{23} \simeq \lambda = \sin \theta_C \) around the unification scale. It can be expected that due to large Yukawa couplings of neutrinos, the renormalization effect increases \( \sin 2\theta_{23} \) naturally up to \( \sim 1 \) at the electroweak scale. Fermion mass spectra and the CKM matrix of quarks obtained in this paper are also phenomenologically viable.
1 Introduction

The hierarchical patterns of quark-lepton masses and flavor mixings have been one of the outstanding problems in particle physics. From the viewpoint of quark-lepton unification it seems to be plausible that the hierarchical structure of Yukawa couplings for leptons is similar to that for quarks and that lepton flavor mixings are also parallel to quark flavor mixings. Experimentally, however, the large neutrino flavor mixing $\sin^2 \theta_{23}$ has been suggested by the muon neutrino deficit in the atmospheric neutrino flux [1]. This implies that lepton flavor mixings are remarkably different from quark flavor mixings in their hierarchical pattern. A natural question arises as to whether or not the distinct flavor mixings of quarks and leptons are in accord with the quark-lepton unification.

From the viewpoint of unification theory, it is reasonable that the hierarchical structure of Yukawa couplings is attributable to some kinds of the flavor symmetry at the unification scale $M_U$. If there exists the flavor symmetry such as $Z_N$ or $U(1)$ in the theory, it is natural that Froggatt-Nielsen mechanism is at work for the interactions [2]. For instance, the effective Yukawa interactions for up-type quarks are of the form

$$M_{ij}Q_i U_j^c H_u$$

with

$$M_{ij} = c_{ij} \left( \frac{\langle X \rangle}{M_U} \right)^{m_{ij}} = c_{ij} x^{m_{ij}},$$

where subscripts $i$ and $j$ stand for the generation indices and all of the constants $c_{ij}$ are of order $O(1)$ with rank $c_{ij} = 3$. The superfield $X$, which is singlet under the unification gauge group, is an appropriate composite superfield with the canonical normalization. The vacuum expectation value (VEV) of $X$ is supposed to be slightly smaller than $M_U$, where $M_U$ is nearly equal to the reduced Planck scale. For simplicity, we introduce global flavor $U(1)$ symmetry and $Z_2$-symmetry ($R$-parity) at the unification scale. The charge of the superfield $X$ is assumed to be $(-1, +)$ under $U(1) \times Z_2$. Instead of $U(1)$ we may take the $Z_N$-symmetry. In that case the present analysis remains unchanged. Due to the $U(1)$-symmetry the exponents $m_{ij}$ in Eq.(2) are determined according as the $U(1)$-charges of $Q_i$, $U_j^c$ and $H_u$. Here we denote the differences of $U(1)$-charges for $Q_2$-$Q_1$, $Q_3$-$Q_2$, $U_2^c$-$U_1^c$ and $U_3^c$-$U_2^c$ by $\alpha$, $\gamma$, $\beta$ and $\delta$, respectively. We have

$$m_{ij} = m_{33} + \begin{pmatrix} \alpha + \beta + \gamma + \delta & \alpha + \gamma + \delta & \alpha + \gamma \\ \beta + \gamma + \delta & \gamma + \delta & \gamma \\ \beta + \delta & \delta & 0 \end{pmatrix}_{ij},$$

provided that $\alpha$, $\gamma$, $\beta$, $\delta$ and $m_{33}$ are non-negative. The mass matrix of the up-type quarks is described by the matrix $M$ multiplied by $v_u = \langle H_u \rangle$. By taking an ansatz
that only top-quark has a trilinear coupling, i.e.

$$m_{33} = 0,$$

we obtain mass eigenvalues

$$O(v_u x^{\alpha+\beta+\gamma+\delta}), \quad O(v_u x^{\gamma+\delta}), \quad O(v_u),$$

which correspond to $u$-, $c$- and $t$-quarks, respectively. Thus, naively, the mass hierarchy of quarks and leptons, up to the renormalization effects, seems to be controlled only by $U(1)$-charges of the matter fields. However, in a wide class of unification models, the situation is not so simple. This is because the massless sector in the unification theory includes extra particles beyond the standard model and then there may occur extra-particle mixings such as between quarks(leptons) and colored Higgs fields(doublet Higgs fields). In order to study fermion masses and flavor mixings we have to take the effects of the extra-particle mixings into account. In addition, in the neutrino sector we should incorporate the extra-particle mixings with the see-saw mechanism \cite{3}. In Ref.\cite{4} we explained the observed hierarchical structure of the Cabbibo-Kobayashi-Maskawa(CKM) matrix for quarks in the string inspired $SU(6) \times SU(2)_R$ model.

In this paper we explore the CKM matrix for leptons. Several authors have pointed out that large neutrino flavor mixing can be obtained as a consequence of the cooperation between Dirac and Majorana mass matrices, provided that the Majorana mass matrix has a specific structure \cite{5}. In the quark-lepton unification, however, the Majorana mass matrix is closely linked to the other mass matrices and then it is difficult to expect such a cooperation between the Dirac and the Majorana mass matrices. In this paper, on the basis of $E_6$-type unification models we show that neutrino flavor mixing $\sin \theta_{23}$ is nearly equal to $\lambda = \sin \theta_C$ as an initial condition around the unification scale. Due to large Yukawa couplings of neutrinos the renormalization effect increases $\sin \theta_{23}$ naturally up to $\sim 1$ at the electroweak scale \cite{6}\cite{7}. Although in the present study we take a specific string inspired model, our results are applicable to a wide class of the unification models.

This paper is organized as follows. In section 2 we explain the interrelation between the hierarchical structure of mass matrices and the flavor symmetries based on the string inspired $SU(6) \times SU(2)_R$ model. Section 3 contains the mass spectra and the CKM matrix of quarks, which have been obtained in Ref.\cite{4}. After solving extra-particle mixings including the see-saw mechanism, we derive the CKM matrix of leptons in section 4. In section 5 we discuss the renormalization effect of the CKM matrices from the unification scale to the electroweak scale. Section 6 is devoted to summary.
2 The hierarchical structure of mass matrices

In this study we choose $SU(6) \times SU(2)_R$ as the gauge symmetry at the unification scale, which can be derived from the superstring theory via the flux breaking \cite{8}. Under $SU(6) \times SU(2)_R$ doublet Higgs and color triplet Higgs fields transform differently. This situation is favorable to solve the triplet-doublet splitting problem \cite{9}. The chiral superfields, if we represent them in terms of $E_6$, consist of

$$N_f 27 + \delta (27 + 27^*)$$

except for $E_6$-singlets, where $N_f$ denotes the family number at low energies and $\delta$ is the set number of vector-like multiplets. Let us now consider the case $N_f = 3$ and $\delta = 1$. Under $SU(6) \times SU(2)_R$ chiral superfields in 27 representation of $E_6$ are decomposed into

$$\Phi(15, 1) : Q, L, g, g^c, S,$$

$$\Psi(6^*, 2) : U^c, D^c, N^c, E^c, H_u, H_d,$$

where $g$, $g^c$ and $H_u$, $H_d$ represent colored Higgs and doublet Higgs fields, respectively. $N^c$ stand for the right-handed neutrino superfield and $S$ is an $SO(10)$-singlet. Although $L$ and $H_d$ ($D^c$ and $g^c$) have the same quantum numbers under the standard model gauge group $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$, they belong to different irreducible representations of $SU(6) \times SU(2)_R$. Gauge invariant trilinear couplings are of the forms

$$\left(\Phi(15, 1)\right)^3 = QQg + Qg^cL + g^cgs, \quad (\Phi(15, 1))(\Psi(6^*, 2))^2 = QH_dD^c + QH_uU^c + LH_dE^c + LH_uN^c$$

$$+ SH_uH_d + gN^cD^c + gE^cU^c + g^cU^cD^c.$$

In the present model the massless matter fields are composed of chiral multiplets $\Phi_i$, $\Psi_i$ $(i = 1, 2, 3)$ and one set of vector-like multiplets $\Phi_0$, $\Psi_0$ and $\Phi$, $\Psi$. Supposing that ordinary quarks and leptons are included in chiral multiplets $\Phi_i$, $\Psi_i$ $(i = 1, 2, 3)$, $R$-parity of all $\Phi_i$, $\Psi_i$ $(i = 1, 2, 3)$ are set to be odd. Since light Higgs scalars are even under $R$-parity, light Higgs doublets are bound to reside in $\Psi_0$ and/or $\Psi$. For this reason we assign even $R$-parity to vector-like multiplets. It is expected that the $R$-parity remains unbroken down to the electroweak scale \cite{4}. Hereafter we use the notations $\alpha$, $\beta$, $\gamma$ and $\delta$ defined by

$$a_2 - a_1 \equiv \alpha, \quad b_2 - b_1 \equiv \beta, \quad a_3 - a_2 \equiv \gamma, \quad b_3 - b_2 \equiv \delta,$$

where $a_i$ and $b_i$ $(i = 1, 2, 3)$ represent the global $U(1)$-charges of matter fields $\Phi_i$ and $\Psi_i$, respectively. All of $\alpha$, $\beta$, $\gamma$ and $\delta$ are taken to be positive.
As shown in Ref. [4], under an appropriate condition the gauge symmetry $SU(6) \times SU(2)_R$ is spontaneously broken in two steps at the scales $\langle \Phi_0 \rangle = \langle \overline{\Phi} \rangle$ ($\langle S_0 \rangle = \langle S \rangle$) and $\langle \Psi_0 \rangle = \langle \overline{\Psi} \rangle$ ($\langle N_0^c \rangle = \langle N \rangle$) as

$$SU(6) \times SU(2)_R \xrightarrow{\langle S_0 \rangle} SU(4)_{PS} \times SU(2)_L \times SU(2)_R \quad (10)$$

$$\xrightarrow{\langle N_0^c \rangle} SU(3)_c \times SU(2)_L \times U(1)_Y, \quad (11)$$

where $SU(4)_{PS}$ stands for the Pati-Salam $SU(4)$ [10]. Further, when $N_0^c N$ has the same $U(1)$-charge as $(S_0 S)^k$ for a positive integer $k$, we are led to the relation $(\langle S_0 \rangle/M_U)^k \approx (\langle N_0^c \rangle/M_U)^k$. We will assume that Froggatt-Nielsen mechanism is at work and that the superfields $X$ in Eq. (14) is expressed in terms of $\Phi_0$, $\overline{\Phi}$ and/or moduli fields. In our scheme this assumption is natural. The $SU(6) \times SU(2)_R$ invariant and the global $U(1) \times Z_2$ invariant interactions which contribute to the mass matrices of quarks and leptons are classified into six types of the superpotential term

$$\Phi_i \Phi_j \Phi_0 X_{z_{ij}}, \quad \Phi_i \Psi_j \Phi_0 X^{m_{ij}}, \quad \Psi_i \Psi_j \Phi_0 X^{h_{ij}}$$

$$\Phi_i \overline{\Phi}_j (\Phi_j \overline{\Psi}) X^{s_{ij}}, \quad (\Phi_i \overline{\Phi}_j (\overline{\Psi}_j \overline{\Psi})) X^{t_{ij}}, \quad (\Psi_i \overline{\Psi}_j (\overline{\Psi}_j \overline{\Psi})) X^{n_{ij}}$$

in $M_U$ units with $i, j = 1, 2, 3$. The exponents are expressed as

$$z_{ij} - z_{33} = s_{ij} - s_{33} = \begin{pmatrix} 2\alpha + 2\gamma & \alpha + 2\gamma & \alpha + \gamma \\ \alpha + 2\gamma & 2\gamma & \gamma \\ \alpha + \gamma & \gamma & 0 \end{pmatrix}_{ij}, \quad (14)$$

$$m_{ij} = t_{ij} - t_{33} = \begin{pmatrix} \alpha + \beta + \gamma + \delta & \alpha + \gamma + \delta & \alpha + \gamma \\ \beta + \gamma + \delta & \gamma + \delta & \gamma \\ \beta + \delta & \delta & 0 \end{pmatrix}_{ij}, \quad (15)$$

$$h_{ij} - h_{33} = n_{ij} - n_{33} = \begin{pmatrix} 2\beta + 2\delta & \beta + 2\delta & \beta + \delta \\ \beta + 2\delta & 2\delta & \delta \\ \beta + \delta & \delta & 0 \end{pmatrix}_{ij}, \quad (16)$$

provided that $z_{33}, s_{33}, t_{33}, h_{33}, n_{33} > m_{33} = 0$. At energies below the scale $\langle N_0^c \rangle(\equiv M_{PS} = M_U y)$ we have six types of bilinear terms

$$M_U Z_{ij} \Phi_i \Phi_j, \quad M_U G_{ij} \Phi_i \Psi_j, \quad M_U H_{ij} \Psi_i \Psi_j, \quad (17)$$

$$M_U S_{ij} \Phi_i \Phi_j, \quad M_U T_{ij} \Phi_i \Psi_j, \quad M_U N_{ij} \Psi_i \Psi_j \quad (18)$$

with

$$Z = y_Z \Gamma_1 Z_0 \Gamma_1, \quad G = y \Gamma_1 G_0 \Gamma_2, \quad H = y_H \Gamma_2 H_0 \Gamma_2, \quad (19)$$

$$S = y_S \Gamma_1 S_0 \Gamma_1, \quad T = y_T \Gamma_1 T_0 \Gamma_2, \quad N = y_N \Gamma_2 N_0 \Gamma_2, \quad (20)$$

5
where the matrices $\Gamma_1$ and $\Gamma_2$ are defined by

$$
\Gamma_1 = \text{diag}(x^{\alpha+\gamma}, x^\gamma, 1),
$$

and all elements of the rank 3 matrices $Z_0, G_0$ and so on are of order $O(1)$. We suppose that there is no fine-tuning among elements of each matrix $Z_0$, etc. The normalization constants $y_Z, y_H, y_S, y_T$ and $y_N$ are expressed in terms of VEVs of $G_{SM}$-neutral fields divided by $M_U$. We have the relation $G = y M$. In what follows we direct our attention to the case

$$
M_U y, M_U y_Z, M_U y_H = 10^{16} \sim 10^{17} \text{GeV},
$$

$$
M_U y_S, M_U y_T, M_U y_N = 10^{11} \sim 10^{12} \text{GeV}.
$$

3 The CKM matrix of quarks

In this section we briefly review the results obtained in Ref.\[4\]. The mass matrix $M$ of the up-type quarks can be diagonalized by a bi-unitary transformation as

$$
V_u^{-1}M U_u.
$$

Since we have the form

$$
M = \Gamma_1 G_0 \Gamma_2,
$$

$V_u$ and $U_u$ become

$$
V_u = \begin{pmatrix}
1 - O(x^{2\alpha}) & O(x^\alpha) & O(x^{\alpha+\gamma}) \\
O(x^\alpha) & 1 - O(x^{2\alpha}) & O(x^\gamma) \\
O(x^{\alpha+\gamma}) & O(x^\gamma) & 1 - O(x^{2\gamma})
\end{pmatrix},
$$

$$
U_u = \begin{pmatrix}
1 - O(x^{2\beta}) & O(x^\beta) & O(x^{\beta+\delta}) \\
O(x^\beta) & 1 - O(x^{2\delta}) & O(x^\delta) \\
O(x^{\beta+\delta}) & O(x^\delta) & 1 - O(x^{2\delta})
\end{pmatrix}.
$$

The mass eigenvalues are given in Eq.(5).

For down-type quarks there appear the mixings between $g^c$ and $D^c$ at energies below the scale $M_{PS} = \langle N_0^5 \rangle$. An early attempt of explaining the CKM matrix via $D^c$-$g^c$ mixings has been made in Ref.\[12\], in which a SUSY $SO(10)$ model was considered. The mass matrix of down-type colored fields is written as

$$
\hat{M}_d = \begin{pmatrix} g^c & D^c \\
g & D \\
Z & G \\
0 & \rho_d M
\end{pmatrix}.
$$
in $M_U$ units, where we used the notation $\rho_d = \langle H_d \rangle / M_U = v_d / M_U$. Note that $G = yM$. Since $\rho_d$ is a very small number ($\sim 10^{-16}$), the left-handed light quarks consist almost only of $D$-components of the quark doublet $Q$. On the other hand, the mixing between $g^c$ and $D^c$ can be sizeable depending on the ratio $y_G / y$. The mixing matrix $V_d$ of the $SU(2)_L$-doublet light quarks $D$ is determined such that $V_d^{-1}(A_d^{-1} + B_d^{-1})^{-1}V_d$ becomes diagonal, where $A_d$ and $B_d$ stand for $ZZ^\dagger$, $GG^\dagger$, respectively. From the expression

$$(A_d^{-1} + B_d^{-1})^{-1} = \Gamma_1 \left( y_{Z}^{-2}(Z_0 \Gamma_1 Z_0^\dagger)^{-1} + y^{-2}(M_0 \Gamma_2 M_0^\dagger)^{-1} \right)^{-1} \Gamma_1,$$

we find that corresponding elements of the matrices $V_d$ and $V_u$ are of the same order of magnitude. Let us assume that the $(1,1)$ elements of $A_d^{-1}$ and $B_d^{-1}$ are of the same order, i.e.

$$y_Z x^{\alpha+\gamma} \simeq y x^{\beta+\delta}.$$ (31)

In this case the coefficients of the leading term in off-diagonal elements of $V_u$ and $V_d$ become different because of large mixing between $D^c$ and $g^c$. Consequently, the CKM matrix for quarks is given by

$$V^{Q}_{CKM} = V_u^{-1}V_d = \begin{pmatrix} 1 & O(x^2) & O(x^\alpha) \\ O(x^\gamma) & 1 & O(x^\alpha+\gamma) \\ O(x^\alpha+\gamma) & O(x^\gamma) & 1 - O(x^2\gamma) \end{pmatrix}.$$ (32)

For the down-type light quarks their masses squared are described as the eigenvalues of the matrix

$$\epsilon_d^2 \times (A_d^{-1} + B_d^{-1})^{-1},$$ (33)

with $\epsilon_d = \rho_d / y$. Thus, we have the mass eigenvalues

$$O(v_d x^{\alpha+\beta+\gamma+\delta}), \quad O(v_d x^{\beta+\gamma+\delta}), \quad O(v_d x^{-\alpha+\beta+\delta})$$ (34)

at the scale $M_{PS}$, which correspond to $d$-, $s$- and $b$-quarks, respectively. In this model $v_u / v_d = \tan \beta \sim 1$ is preferable. The masses of the down-type heavy quarks are $O(M_U y)$.

4 The CKM matrix of leptons

We now proceed to study the flavor mixing of leptons. In the lepton sector large $L$-$H_d$ mixing possibly occurs at energies below the scale $M_{PS}$. It is worth emphasizing that both $L$ and $H_d$ are $SU(2)_L$-doublets. This situation is in contrast to the $D^c$-$g^c$ mixing. For charged leptons we obtain a $6 \times 6$ mass matrix

$$\tilde{M}_l = \begin{pmatrix} H_u^+ & E^c+ \\ H_d^- & \begin{pmatrix} H & 0 \\ G & \rho_d M \end{pmatrix} \end{pmatrix}.$$ (35)
The unitary matrices $\hat{M}_l$ can be diagonalized by a bi-unitary transformation as

$$\hat{V}_l^{-1} \hat{M}_l \hat{U}_l.$$  \hspace{1cm} (36)

The unitary matrices $\hat{V}_l$ and $\hat{U}_l$ have the forms

$$\hat{V}_l \simeq \begin{pmatrix} H W_l (\Lambda_l^{(0)})^{-1} & -(H^\dagger)^{-1} V_l A_l^{(2)} \\ G W_l (\Lambda_l^{(0)})^{-1} & (G^\dagger)^{-1} V_l A_l^{(2)} \end{pmatrix},$$  \hspace{1cm} (37)

$$\hat{U}_l \simeq \begin{pmatrix} W_l & -\epsilon_d (A_l + B_l)^{-1} B_l V_l \\ \epsilon_d B_l (A_l + B_l)^{-1} W_l & V_l \end{pmatrix}$$  \hspace{1cm} (38)

in the $\epsilon_d$ expansion, where $A_l = H^\dagger H$ and $B_l = G^\dagger G$. $W_l$ and $V_l$ are unitary matrices which diagonalize $(A_l + B_l)$ and $(A_l^{-1} + B_l^{-1})^{-1}$ as

$$W_l^{-1}(A_l + B_l)W_l = (\Lambda_l^{(0)})^2, \quad V_l^{-1}(A_l^{-1} + B_l^{-1})^{-1} V_l = (\Lambda_l^{(2)})^2,$$  \hspace{1cm} (39)

where $\Lambda_l^{(0)}$ and $\Lambda_l^{(2)}$ are diagonal. Equation (38) implies that the light $SU(2)_L$-singlet charged leptons are mainly $E^c$-components. From Eq.(37), the mass eigenstates of light $SU(2)_L$-doublet charged leptons are given by

$$\tilde{L}^- = \Lambda_l^{(2)} V_l^{-1} (-H^{-1} H_d^- + G^{-1} L^-).$$  \hspace{1cm} (40)

Consequently, when the elements of $H^{-1}$ and $G^{-1}$ are comparable to each other, there occurs large mixing between $H_d^-$ and $L^-$. By inspecting the expression

$$(A_l^{-1} + B_l^{-1})^{-1} = \Gamma_2 \left( y_H^{-2} (H_0^\dagger \Gamma_2 H_0)^{-1} + y^{-2} (M_0^\dagger \Gamma_2 M_0)^{-1} \right)^{-1} \Gamma_2,$$  \hspace{1cm} (41)

it is easy to see that $V_l$ has the same hierarchical pattern as $U_l$. Let us take an additional ansatz

$$y_H x^\beta + \delta \simeq y x^{\alpha + \gamma + \xi}$$  \hspace{1cm} (42)

with $\beta - \alpha \leq \xi > 0$. This implies that the large mixing occurs between $H_d^-$ and $L^-$. Thus we find that

$$\Lambda_l^{(2)} \simeq y x^\gamma \times \text{diag}(x^{\alpha + \beta + \delta + \xi}, x^{\alpha + \delta}, 1) = y x^\gamma \Gamma_2 \Gamma_3,$$  \hspace{1cm} (43)

where

$$\Gamma_3 = \text{diag}(x^{\alpha + \xi}, x^{\alpha}, 1).$$  \hspace{1cm} (44)

Masses of light charged leptons turn out to be

$$O(v_d x^{\alpha + \beta + \gamma + \delta + \xi}), \quad O(v_d x^{\alpha + \gamma + \delta}), \quad O(v_d x^\gamma)$$  \hspace{1cm} (45)

at the scale $M_{PS}$, which correspond to $e$-, $\mu$- and $\tau$-leptons, respectively.
In the neutral lepton sector we have a $15 \times 15$ mass matrix

$$
\tilde{M}_{NS} = \begin{pmatrix}
H_u^0 & H_d^0 & L^0 & N^c & S \\
H_u^0 & 0 & H & G^T & 0 & \rho_d M^T \\
H_d^0 & H & 0 & 0 & 0 & \rho_u M^T \\
L^0 & G & 0 & 0 & \rho_u M & 0 \\
N^c & 0 & 0 & \rho_u M^T & N & T^T \\
S & \rho_d M & \rho_u M & 0 & T & S
\end{pmatrix}
$$

(46)

in $M_U$ units, where $\rho_u = v_u / M_S$. The $6 \times 6$ submatrix

$$
\tilde{M}_M = \begin{pmatrix}
N & T^T \\
T & S
\end{pmatrix}
$$

(47)

represents the Majorana mass terms. In the study of lepton flavor mixing, instead of the matrix (46) we may consider the $12 \times 12$ submatrix

$$
\tilde{M}_N = \begin{pmatrix}
H_u^0 & H_d^0 & L^0 & N^c \\
H_u^0 & 0 & H & G^T & 0 \\
H_d^0 & H & 0 & 0 & 0 \\
L^0 & G & 0 & 0 & \rho_u M \\
N^c & 0 & 0 & \rho_u M^T & N \\
S & \rho_d M & \rho_u M & 0 & T & S
\end{pmatrix}
$$

(48)

We will show shortly that the result remains unchanged. By recalling the above study in the charged lepton sector, it is easy to see that the unitary matrix $\hat{U}_N$ which diagonalizes $\tilde{M}_N$ is of the form

$$
\hat{U}_N \simeq \begin{pmatrix}
\mathcal{W}_l & 0 & 0 \\
0 & \hat{V}_l & 0 \\
0 & 0 & \mathcal{U}_N
\end{pmatrix} \times \begin{pmatrix}
1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\
1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\
0 & 0 & \mathcal{V} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(49)

Note that $\hat{V}_l$ is a $6 \times 6$ matrix Eq.(37) and that $\mathcal{W}_l$ is defined in Eq.(39). The unitary matrix $\mathcal{U}_N$ diagonalizes the Majorana mass matrix $N$ as

$$
\mathcal{U}_N^{-1} N \mathcal{U}_N \simeq M_{UYN} (\Gamma_2)^2 = M_{UYN} \times diag(x^{\beta+2\delta}, x^{2\delta}, 1).
$$

(50)

This means that the Majorana masses have also hierarchical structure. By examining the unitary matrix $\hat{U}_N$, it turns out that the light neutrino mass eigenstates are

$$
\bar{L}^0 = \mathcal{V}^{-1} \Lambda_l^{(2)} \bar{V}_l^{-1} (-H^{-1} H_d^0 + G^{-1} L^0).
$$

(51)
Comparing these eigenstates $\tilde{L}^0$ with those of the light charged leptons $\tilde{L}^-$ given by Eq.(40), we find that $V$ is the CKM matrix for leptons. $V$ is determined such that

$$V^{-1}(\Lambda_l^{(2)}V_l^{-1}N^{-1}V_l\Lambda_l^{(2)})V$$

becomes diagonal. Using Eq.(43), we obtain

$$\Lambda_l^{(2)}V_l^{-1}N^{-1}V_l\Lambda_l^{(2)} \simeq \frac{y^2}{y_N} x^{2\gamma} \Gamma_3 (\Gamma_2 V_l^{-1} \Gamma_2^{-1} N_0^{-1} \Gamma_2^{-1} V_l \Gamma_2) \Gamma_3$$

$$= \frac{y^2}{y_N} x^{2\gamma} \times \begin{pmatrix} O(x^{2\alpha+2\xi}) & O(x^{2\alpha+\xi}) & O(x^{\alpha+\xi}) \\ O(x^{2\alpha+\xi}) & O(x^{2\alpha}) & O(x^{\alpha}) \\ O(x^{\alpha+\xi}) & O(x^{\alpha}) & O(1) \end{pmatrix}. \quad (53)$$

Consequently, the lepton flavor mixing matrix is of the form

$$V_{CKM}^{L} = V = \begin{pmatrix} 1 - O(x^{2\xi}) & O(x^{\xi}) & O(x^{\alpha+\xi}) \\ O(x^{\xi}) & 1 - O(x^{2\alpha}) & O(x^{\alpha}) \\ O(x^{\alpha+\xi}) & O(x^{\alpha}) & 1 - O(x^{2\alpha}) \end{pmatrix} \quad (54)$$

at the scale $M_{PS}$. It should be emphasized that we obtain an interesting relation between the lepton mixing angle and the quark mixing angle

$$\sin \theta_{23}^L \simeq \sin \theta_{12}^Q \quad (55)$$

at the scale $M_{PS}$. The light neutrino masses are given by the eigenvalues of the matrix Eq.(53) multiplied by $(\rho_u/y)^2$. Thus we have

$$m_{\nu_i} \simeq \frac{v_u^2}{M_U y_N} x^{2\gamma} (\Gamma_3)^2 = \frac{v_u^2}{M_U y_N} x^{2\gamma} \times (x^{2\alpha+2\xi}, x^{2\alpha}, 1). \quad (56)$$

In the case of the matrix Eq.(46), $N^{-1}$ in Eq.(52) should be replaced by

$$\left( \begin{array}{cc} 1 & -yH^{-1}M^T \\ -yM(H^\dagger)^{-1} & M^{-1} \end{array} \right). \quad (57)$$

Since this quantity has essentially the same form as $N^{-1} \propto \Gamma_2^{-1}N_0^{-1} \Gamma_2^{-1}$, we obtain the same mixing matrix Eq.(54).

5 The CKM matrices at the electroweak scale

In the above study we obtain the CKM matrices for quarks and leptons at the scale $M_{PS}$. Although we derive a remarkable relation, i.e. Eq.(55) at the scale $M_{PS}$, we should correctly relate them to the values at low energies where they are measured.
Since matter fields of the third generation have large Yukawa couplings, it is important to investigate the renormalization effects of the flavor mixings $\sin \theta_{23}^Q$ and $\sin \theta_{23}^L$ from the scale $M_{PS}$ to the electroweak scale. Here $M_{PS}$ is supposed to be $10^{16} \sim 10^{17}$ GeV \[4\]. In Refs.\[6\] \[7\], it has been shown in the minimal supersymmetric standard model that the neutrino flavor mixing $\sin \theta_{23}^L$ increases significantly with running down to the electroweak scale by the renormalization group equation (RGE). From the scale $M_{PS}$ to the Majorana mass scale $M_U y_N$ we manipulate the RGE for Yukawa couplings. Below the scale $M_U y_N$ we study the RGE of the neutrino mass operator $\kappa$ \[6\], which is given by

$$\kappa = y^{-2} \times \Lambda_i^{(2)} \nu_i^{-1} N^{-1} \nu_i \Lambda_i^{(2)}.$$ \hspace{1cm} (58)

Since effective Yukawa couplings of the first and the second generations are very small, the renormalization effect of $\sin \theta_{12}^L$ is also rather small compared with that of $\sin \theta_{23}^L$. In the lepton sector the $SU(3)_c$ gauge interaction does not contribute to the one-loop RGE of Yukawa couplings. Therefore, the renormalization effect of $\sin \theta_{23}^L$ becomes more significant compared with that of $\sin \theta_{23}^Q$. From rough estimations, we can expect that $\sin \theta_{23}^L$ is enhanced by a factor 2 in running down to the electroweak scale from the scale $M_{PS}$. In this case we have

$$\sin \theta_{23}^L(M_Z) \sim 2 \sin \theta_{23}^L(M_{PS}) \simeq 2 \sin \theta_{12}^Q(M_{PS}) \simeq 2 \lambda$$ \hspace{1cm} (59)

with $\lambda = \sin \theta_C \simeq 0.22$. This means that $2 \theta_{23}^L(M_Z) = 0.8 \sim 0.9$. This is consistent with the atmospheric neutrino data \[4\]. The detailed study of the renormalization effect of $\sin \theta_{23}$ will be presented elsewhere.

Confronting the CKM matrix for quarks obtained here with the observed one, it is feasible for us to take a simple parametrization

$$x^\alpha = \lambda = 0.22, \quad \alpha = \delta : \beta = \gamma : \xi = 1 : 2.5 : 1.5.$$ \hspace{1cm} (60)

Under this parametrization the hierarchical fermion masses at the scale $M_{PS}$ become

$$v_u \times (O(\lambda^7), \ O(\lambda^{3.5}), \ O(1))$$ \hspace{1cm} (61)

for up-type quarks,

$$v_d \times (O(\lambda^7), \ O(\lambda^6), \ O(\lambda^{2.5}))$$ \hspace{1cm} (62)

for down-type quarks,

$$v_d \times (O(\lambda^{8.5}), \ O(\lambda^{4.5}), \ O(\lambda^{2.5}))$$ \hspace{1cm} (63)

for charged leptons. These results are consistent with the observed mass spectra of quarks and leptons. Further, the neutrino masses are

$$\frac{v_u^2}{M_U y_N} \times (O(\lambda^{10}), \ O(\lambda^7), \ O(\lambda^5))$$ \hspace{1cm} (64)
at the scale $M_{PS}$. Taking $M_U y_N = 10^{11.5}$ GeV as a typical Majorana mass scale, apart from the renormalization effect, we obtain

$$m_{\nu_e} \simeq 2 \times 10^{-4} \text{eV}, \quad m_{\nu_\mu} \simeq 4 \times 10^{-3} \text{eV}, \quad m_{\nu_\tau} \simeq 7 \times 10^{-2} \text{eV}. \quad (65)$$

Provided that the renormalization effects of $\sin \theta^q_{23}$ and $\sin \theta^L_{23}$ are roughly about a factor $2 \sim \lambda^{-0.5}$, the CKM matrices at the electroweak scale are expressed as

$$V^Q_{CKM} \sim \begin{pmatrix} 1 - O(\lambda^2) & O(\lambda) & O(\lambda^3) \\ O(\lambda) & 1 - O(\lambda^2) & O(\lambda^2) \\ O(\lambda^3) & O(\lambda^2) & 1 - O(\lambda^4) \end{pmatrix} \quad (66)$$

for quarks and

$$V^L_{CKM} \sim \begin{pmatrix} 1 - O(\lambda^3) & O(\lambda^{1.5}) & O(\lambda^2) \\ O(\lambda^{1.5}) & 1 - O(\lambda) & O(\lambda^{0.5}) \\ O(\lambda^2) & O(\lambda^{0.5}) & 1 - O(\lambda) \end{pmatrix} \quad (67)$$

for leptons. The hierarchical structure of $V^Q_{CKM}$ is also consistent with the experimental data. Furthermore, it is worthy of note that neutrino masses and mixings obtained in this paper roughly accommodate both the small angle MSW solution to the solar neutrino problem \cite{13} \cite{14} as well as the $\nu_\mu$-$\nu_\tau$ oscillation solution to the muon neutrino deficit in the atmospheric neutrino flux. However, the present model cannot explain the LSND results which, if confirmed, indicate $\Delta m^2_{LSND} \sim 1 \text{eV}^2$ and $\sin^2 2\theta \sim 10^{-3}$ \cite{15}.

6 Summary

Large neutrino flavor mixing $\sin \theta^L_{23}$ has been suggested by the muon neutrino deficit in the atmospheric neutrino flux. This means that lepton flavor mixings are remarkably different from quark flavor mixings in their hierarchical pattern. The difference of the structure of the CKM matrix between quarks and leptons is explainable in the $SU(6) \times SU(2)_R$ model and in the $E_6$-type unification models. In the models the massless sector includes extra particles beyond the minimal supersymmetric standard model. Concretely, we have triplet Higgs and doublet Higgs fields in each generation. As a consequence of such matter contents, there appear the mixings between the down-type quarks $D^c$ and triplet Higgses $g^c$ and between doublet leptons $L$ and doublet Higgses $H_d$. These mixings, if large, turn the mass pattern of down-type quarks and leptons out of that of up-type quarks. Further, it is expected that the hierarchical structure of Yukawa couplings comes from Froggatt-Nielsen mechanism under some kind of the flavor symmetry. Combining the unification models with this
mechanism, we obtained the CKM matrices $V_{CKM}^Q$ and $V_{CKM}^L$ which are remarkably different from each other in the hierarchical pattern. In particular, the relation

$$\sin \theta_{23}^L \simeq \sin \theta_{12}^Q$$

holds at the scale $M_{PS}$. Due to large Yukawa couplings of the third generation the renormalization effect of flavor mixing $\sin \theta_{23}$ is sizeable. It can be expected that neutrino flavor mixing $\sin \theta_{23}^L$ increases by a factor about 2 with running down to the electroweak scale by the RGE. Thus we obtain

$$\sin \theta_{23}^L \sim 2\lambda = 0.4 \sim 0.5,$$

which is consistent with the data. Fermion mass spectra and the CKM matrix of quarks obtained here are also phenomenologically viable. The hierarchical structure of fermion masses and the CKM matrices of quarks and leptons provides an important clue to the matter contents and the flavor symmetries at the unification scale.

**Acknowledgements**

The authors thank Prof. S. Kitakado for careful reading of the manuscript. This work is supported in part by the Grant-in-aid for Scientific Research, Ministry of Education, Science and Culture, Japan (No. 08640366).
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