Hierarchic trees with branching number close to one: noiseless KPZ equation with additional linear term for imitation of 2-d and 3-d phase transitions.

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Abstract

An imitation of 2d field theory is formulated by means of a model on the hierarchic tree (with branching number close to one) with the same potential and the free correlators identical to 2d correlators ones. Such a model carries on some features of the original model for certain scale invariant theories. For the case of 2d conformal models it is possible to derive exact results. The renormalization group equation for the free energy is noiseless KPZ equation with additional linear term.

One of the most fruitful ideas in physics is the idea of universality. In fact it is the only hope due to which rather artificial models of the present theoretical physics can successfully capture the relevant aspects of the nature. We believe, that at the critical point statistical mechanics system omits the secondary details. Usually this concerns the Hamiltonian in the d dimensional Euclidean space. All Hamiltonians from the universality class have the same multi point (2-point, three-point, 4-point..) correlators for fluctuating fields. These variables, as well as the phase structure have deep and universal nature.

We propose to follow another path: by keeping Hamiltonian fixed to simplify the space geometry as much as possible retaining two point correlators and three point (for isosceles triangles) correlators. If the action of original theory consists of the Laplacian and a potential, our model feels the space dimension through the behavior of Green function

\[ G(x, x') \sim \frac{1}{r(x, x')^{d-2}} d \neq 2 \]

\[ G(x, x') \sim \log \frac{1}{r(x, x')}, d = 2 \]  

(1)
The total volume is
\[ \left( \frac{L}{a} \right)^d \]
where \( L \) and \( a \) are the infrared and ultraviolet cutoffs, \( r(x, x') \) is the distance. The Euclidean geometry contains too much constructions. One can rotate a point around some center and write out close circle. Let us now consider some metric space with properties:
A. For every pair of points there is a distance \( r(x, x') \).
B. We have some measure at every point \( d\mu_s(x) \) with the total measure \( \int d\mu_s = R^d \).
C. One can construct a quadratic form with corresponding asymptotics (1) for Green function.

We are going to construct statistical mechanics models on the simplest space, which supports points A-C. We hope, that due to the universality these models will acquire some properties of models in \( d \)-dimensional space. To realize this program we will use certain ideas from the theory of Random Energy Model (REM) [1-5]. In ref. [5] a relation of 2d quantum Liouville model to REM and to the Directed Polymer (DP) on Cayley tree was established.

Our present analysis shows, that the connection with REM is not a specific feature of Liouville model and works well also for other conformal models. Moreover, using similar ideas we intend to construct general 2-d quantum models in the ultrametric space and thereby generalize the above-mentioned connection between the quantum field theoretical models and those defined on the hierarchical lattices.

Let us consider a hierarchic tree with the branching number \( q \). We begin with integer \( q \), then continue the obtained expressions analytically to the point \( q \to 1 \). Instead of \( d \)-d Euclidean space now we have \( q^K \) endpoints, where \( K \) is a number of hierarchic levels. First we define the fields \( f_l \) on branches of a tree. The field \( \phi \) at the endpoint \( x \) is defined as
\[ \phi(x) = f_0 + \sum_l f_l \]

The summation in (3) is made along the trajectory of point \( x \), connecting it with the origin of the tree. We define \( v \) at the hierarchy level \( j \) as
\[ v = \frac{jV}{K} \]
Now determine the kinematic part of the action for the field $\phi(x)$

$$\sum K \frac{f(v, l)}{V} f(v, l)^2$$

(5)

Then the partition under the potential $U(\phi)$ is

$$\int df \exp\{- \sum_{v=0}^V K \frac{f(v, l)}{2V} f(v, l)^2\}$$

$$\exp\{\sum_x U(\phi(x))\}$$

(6)

We have for the correlator

$$< \phi(x)\phi(x') > = v$$

(7)

For usual 2d models with

$$\int d\phi_0 d\phi \exp\{-\frac{1}{8\pi} dx^2 \nabla^2 \phi(x)^2\} \exp\{\int dx U(\phi(x))\}$$

(8)

the total surface area is equal to $R^2$, and the correlators read as

$$< \phi(x)\phi(x') > = \ln \frac{L^2}{r^2}$$

(9)

It is possible to take n component fields in Eq. (8) instead of the one-component field $\phi(x)$. We can determine the distance from the equality $V = \ln r^2$. Then our correlators coincide. What is the advantage of representation (6)? We are in a position to calculate the partition function by means of iterations. This is well known for models on hierarchical lattices [6]. Let us take some large number K and derive

$$I_1(x) = \sqrt{\frac{K}{2V\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{K}{2V}y^2 + U(x + y)\} dy$$

$$I_{i+1}(x) = \sqrt{\frac{K}{2V\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{K}{2V}y^2\} [I_i(x + y)] q dy$$

$$Z = \lim_{K \to \infty} [I_K(0)]^q$$

(10)
As for the determination of partition function, we need only the equation (10) and can define our model for any value of \( q \) consideration of the analytical continuation of (10). Let us consider the limit

\[ q \to 1, K \to \infty \]

\[ q^K = e^V \]  

(11)

Using the small factor \((q - 1)\), it is possible to introduce continuous measures \( d\mu_x, d\mu_t \) construct perturbative field theory on this ultrametric space, and calculate diagrams. In reality we need in expressions for the propagator (7), as well as the total volume measure inside the sphere with maximal hierarchic distance \( v \) given by equality

\[ \int d\mu_x = e^v - 1 \]  

(12)

For finite or large values of \( q \) considered in [5] it is impossible (or too difficult) to define the perturbative regime.

Let us consider carefully equation (10) at the limit (11). We introduce a variable \( w(v, x) = I_{Kv}(x) \) and consider the limit \( \frac{V}{K} \ll 1 \). For the differential \( dv \) we have the expression \( \frac{V}{K} \). Let us take also

\[ q - 1 = \frac{V}{K} \equiv dv \]  

(13)

Using expression \( x^q \approx x(1 + \log x(q - 1)) \) it is easy to obtain

\[ \frac{dw}{dv} = w \ln w + \frac{1}{2} \Delta w \]  

(14)

After the replacement \( w = \exp(u(t, x)) \) we arrive at

\[ \frac{du}{dv} = \frac{1}{2} \Delta u + \frac{1}{2} (\nabla u)^2 + u \]

\[ u(0, x) = U(x) \]  

(15)

where \( U(x) \) is a potential in Eq.(8). The dimension \( n \) of the space where this equation is formulated is equal to the number of different fields \( \phi(x) \) in Eq. (8). Having an expression for \( u(v, x) \) we obtain for the free energy

\[ \ln Z = u(V, 0) \]  

(16)
For the free energy $u(v, x)$ we have the noiseless KPZ equation (15) with an additional linear term. There are two interesting solutions of Eq. (15) at large values $v$. If the couplings in the polynomial potential are $O(1)$, it is reasonable to consider the solution at large values of $v$ (and far from the renormalization group fixed points):

$$u(v, x) = \text{const} \exp(v)$$  \hspace{1cm} (17)

If one considers the couplings $\sim \frac{1}{\exp(v)}$ in the potential $U(x)$, then a solution:

$$u(v, x) = \text{const} \exp(v) + u_s(x), u_s(x) \sim 1.$$  \hspace{1cm} (18)

$$\frac{1}{2} \Delta u_s + \frac{1}{2}(\nabla u_s)^2 + u_s = 0$$  \hspace{1cm} (19)

corresponds to the perturbative regime. This equation gives the efficient potential at the stable point of renormalization group. One can rewrite Eq. (19) in another form for the $z \equiv \frac{du_s}{dv}$:

$$\frac{dz}{du_s} + z + \frac{2}{z} u_s$$  \hspace{1cm} (20)

In analogy to Eqs. (10), (14) it is also possible to derive the correlators. To calculate the correlator $\langle \exp(i \alpha \phi(x) - \alpha \phi(y)) \rangle$, where the hierarchic distance between points $x, y$ is $v_0$, one has to consider also an equation

$$\frac{df(v, x, \alpha)}{dv} = f \ln w + \frac{1}{2} \Delta f$$

$$f(0, x, \alpha) = \exp(U(x) + i \alpha x)$$  \hspace{1cm} (21)

Then for the generating function $f_0(v, x)$ of correlator one should solve again Eq. (21) with the boundary conditions at the point $v_0$

$$f_0(v_0, x) = f(v_0, x, \alpha)f(v_0, x, -\alpha)/w(v_0, x)$$  \hspace{1cm} (22)

For the correlator we obtain an expression:

$$\langle \exp(i \alpha \phi(x) - \alpha \phi(y)) \rangle = \frac{f_0(\infty, 0)}{w(\infty, 0)}$$  \hspace{1cm} (23)

Let us use the same approach for the case of $d > 2$. For the volume in d-d space one has $\sim a^d$. If we identify it with our $q^L$, then derive $a = q^{\frac{d}{4}}$. The
fields \( f_i \) are defined on the branches of tree, \( f_0, f_1 \) at the origin. Let us define the free field action.

\[
\phi(x) = f_0 + f_1 + \sum_{vl} f(v, l)
\]  

(24)

The summation in (24) is along the trajectory of point X. Now determine the kinematic part of the action for the field \( \phi(x) \)

\[
A = \frac{1}{2} |f_1|^2 + \sum_{vl} \exp(-\alpha v) f(v, l)^2 / \alpha
\]  

(25)

If one takes \( \alpha = \frac{d-2}{d} \) for the combined field, then

\[
\langle \phi(x) \phi(x') \rangle = \exp(\alpha v) \sim \frac{L}{r(x, x')}^{(d-2)},
\]  

(26)

where L is the maximal distance in the model (the infrared cutoff). Now (10) transforms into:

\[
I_1(x) = \sqrt{\frac{K}{2V\alpha \pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{K}{2V\alpha} y^2 + U(x + y)\right\} dy
\]

\[
I_{i+1}(x) = \sqrt{\frac{K}{2V\alpha e^{\alpha V(K-i+1)/K} \pi}} \int_{-\infty}^{\infty} e^{-\frac{K}{2V\alpha \exp[\alpha V(K-i+1)/K]} y^2} [I_i(x + y)]^q dy
\]

\[Z = \lim_{K \to \infty} \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2} y^2 + U(y)\right\} dy[I_K(y)]^q(27)\]

To calculate \( I_K(x) \equiv w(V, x) \equiv \exp(u(V, x)) \) we should solve the equation like (15)

\[
\frac{du}{dv} = \frac{1}{2} \alpha \exp[\alpha v] \Delta u + \frac{1}{2} (\nabla u)^2 + u
\]

\[u(0, x) = U(x)\]  

(28)

We gave a simplified, approximate method for the 2d field theoretical models. We hope, that the bulk structure, the two and three point correlators (for isosceles triangles) are the same, as in 2d models. We checked, that three point correlators are the same, as those in 2d case. According to [5], the model with \( U(x) = \exp(kx) \) on such hierarchic lattices at large values of \( q \) exactly is equivalent to Liouville model. According to results of [8] the
thermodynamic limit of this model is independent of $q$. Thus at least for these case models on our tree (branching number $q$ is close to one) are equivalent to those in 2-d. It is possible to check our hypothesis about equivalence of models on our trees with some segment of dd field theory by means of direct numerical calculation of Eq. (15),(28), for example for field version of 3d Ising model with proper choice of potential $U$.

I am grateful to ISTC grant A-102 for partial financial support, C. Lang and W. Janke for invitation to Graz and Leipzig and discussions, P. Grassberger for useful remark.

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