TRANSFER OPERATORS AND ATOMIC DECOMPOSITION

ALEXANDER ARBIETO AND DANIEL SMANIA

Abstract. We use the method of atomic decomposition and a new family of Banach spaces to study the action of transfer operators associated to piecewise-defined maps. It turns out that these transfer operators are quasi-compact even when the associated potential, the dynamics and the underlying phase space have very low regularity.

In particular it is often possible to obtain exponential decay of correlations, the Central Limit Theorem and almost sure invariance principle for fairly general observables, including unbounded ones.

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Transfer operators are an almost unavoidable tool to study the ergodic theory of (piecewise) smooth dynamical systems. In the context of expanding maps, we usually have a reference measure, that can be for instance either the volume form on the manifold where the dynamics takes place, or in more general settings some “eigen-measure” $m$ for the dual operator (to find such eigen-measure quite often it is not a trivial matter). The transfer operator describes how finite measures which are absolutely continuous with respect to $m$ are transported by the dynamics. That is, if $\mu = \rho m$, with $\rho \in L^1(m)$ then $\Phi \rho$ is the density of the push-forward $F^* \mu$ with respect to $m$.

We consider new Besov spaces on phase spaces with very mild structure, a finite measure space with a good grid, and we estimate the (essential) spectral radius of transfer operators of piecewise-defined maps acting on them. These spaces often coincide with classical Besov spaces in more familiar settings. On the other hand the assumptions on the regularity of both the map and the phase space are minimal, allowing us to apply the results to new and classical situations alike. We use the atomic decomposition of these Besov spaces to study the action of the transfer operator.

1. Transfer operators and dynamics

We know that even for very regular expanding maps, typical $L^1$ observables do not have good statistical properties as exponential decay of correlations and central limit theorem. Indeed this is related with the bad spectral behaviour of the action on $\Phi$ on $L^1(m)$. The transfer operator acts as a bounded operator on
$L^1(m)$, but its spectrum there is the whole closed unit disc. This obviously have dreadful consequences for the decay of correlations of typical integrable observables.

The most well-behaved linear operators are linear transformations on finite dimensional normed spaces. Its spectrum is just a finite number of eigenvalues with finite dimensional eigen-spaces. The next best thing would be compact operators, for which the spectrum are just a countable number of eigenvalues possibly accumulating at zero. Unfortunately the transfer operator is very rarely a compact operator even in very regular situations. A far more successful approach to obtain good statistical properties of observables in some Banach space of functions $B$, often called functional operator approach, is to show the quasi-compactness of the action of $\Phi$ on $B$, that is, the spectrum near the circle of espectral radius is as of a compact operator, consisting on isolated eigenvalues and finite-dimensional eigen-spaces, and the "weird stuff", the so-called essential spectrum, safely away from it, inside a disc of strictly smaller radius.

One must note that the quasi-compactness of $\Phi$ is not the only difficulty in the functional operator approach, however it is fair to say that finding a proper Banach space of functions and to prove the quasi-compactness of $\Phi$ there is one of the most challenging steps. There are well-know methods on how to use the quasi-compactness property to study the ergodic behaviour of $F$. We list a (purposely vague) description of some of them below. In Section 14 and Section 15 we give precise statements of some consequences of the quasi-compactness of transfer operators action on Besov spaces on measure spaces with a good grid.

Existence of absolutely continuous invariant probabilities. To this end one need to show that 1 is an eigenvalue of $\Phi$ and its eigen-space contains a non-negative function $\rho$. Then the measure $\rho \, dm$ is a finite invariant measure. One can also estimate the number of absolutely continuous ergodic measures by the dimension of this eigen-space if $\Phi$ also satisfies the quite handy Lasota-Yorke inequality for the pair of Banach spaces $(B, L^1(m))$.

Exponential decay of correlations. One needs to show that 1 is an isolated simple eigenvalue and that the rest of the spectrum is contained in a ball centered at zero and with radius strictly smaller than one. Exponential decay of correlation follows for all observables in $B$.

Central Limit Theorem. To show the Central Limit Theorem for a real-valued observable $\phi$ we first show that $\psi \mapsto M_t(\psi) = e^{it\phi} \psi$ is a bounded operator (a multiplier) on $B$, for every $t$ small. Then we consider perturbations of the transfer operator $\Phi_t = \Phi \circ M_t$. Often there is an analytic continuation of the leading eigenvalue for every small $t$. This is closely related with the characterisc function of the observable $\phi$ and a carefully analysis gives the Central Limit Theorem for $\phi$. Note that $\Phi_t$ is also a transfer operator, but with a complex-valued potential.

Analyticity of topological pressure. If $I$ is a compact metric space and $F: I \to I$ is continuous then the spectral radius of the operator $\Phi_g$ is exactly $e^{P_{top}(g)}$. If there is just a single element of the spectrum with maximal modulus, that is a simple
eigenvalue, often this eigenvalue varies analytically under perturbation of $g$, so we get the real analyticity of $P_{top}(g)$ with respect to $g$.

2. Looking for Banach spaces

There is a long history of looking for Banach spaces of observables with good statistical properties. We give below a list of dynamics, potentials and corresponding function spaces where the quasi-compactness of the transfer operator was attained.

The transfer operator (for the potential $\log f'$) appeared in 1956 in Rechard [39] as a tool to find invariant measures of one-dimensional many-to-one transformations. But its impact certainly had a little help of the popular book by Ulam [52], where he asks if one could show a result similar to Perron-Frobenious Theorem for positive matrices.

Results on the spectral theory of the transfer operator (as named by Ruelle but often called Ruelle-Perron-Frobenious operator) started with the seminal work on rigorous statistical mechanics by Ruelle [40], who studied the one-sided shift and the action on Hölder function of the transfer operator associated with potentials in the same class, and in particular got a result analogous to the Perron-Frobenious Theorem in this setting.

The construction of Markov partitions for hyperbolic maps by Sinai [44] allowed to study transfer operators for expanding maps on manifolds [42] and compact sets with the same Banach spaces of functions, since they have Markov partitions that semiconjugate them with subshifts of finite type. See also Ruelle [41], Parry [36], Walters [53], Bowen [9], Bowen and Series [10]. See Bowen [8], Parry and Pollicott [37], Przytycki and Urbanski [38], Zinsmeister [54], as well Craizer [20] for superb expositions on this setting, with different emphasis in its applications.

The next step was given by Lasota and Yorke [32]. They considered piecewise $C^2$ expanding maps on the interval, motivated by a quite concrete problem involving the shape of well drilling bits. Of course any space of continuous functions is not invariant by the transfer operator anymore. Moreover the Markov partition approach is no longer easily adaptable here, once one needs subshifts that are not of finite type. They proved the the action of the transfer operator on the space $BV$ of bounded variation function satisfies what is now called the Lasota-Yorke inequality, that in particular implies the quasicompactness of the action of the transfer operator. With the exception of early results by Gel’fond [23] and Parry [35] on $\beta$-transformations, and Lasota [31], this was the first time one could obtain deep ergodic results for non-markovian maps. Keller and Hofbauer [28][27] pushed these results for bounded variation potentials, and in particular the quasi-compactness of the transfer operator and its consequences. Baladi [2] and Broise [11] are good introductions for these results.

Lasota-Yorke inequality and quasi-compactness became favorite tools to study transfer operators. Keller [29] studied piecewise complex-analytic expanding maps on the plane. The space $BV$ was used in higher dimensions to study piecewise
expanding maps by Góra and Boyarsky [24]. See also Adl-Zarabi [1]. Cowieson [19][18] proved the quasi-compactness of the transfer operator for "generic" piecewise $C^k$ expanding maps. Indeed in dimension larger than one the discontinuities of the dynamics became an even more serious liability. If you pick a piecewise monotone map on the interval whose branches are defined in intervals, its $n$th iteration has monotone branches with the very same property. However, if we iterate a map that is a piecewise expanding map whose domains of the branches are very nice (squares, for instance) then its $n$th iteration may be a piecewise expanding map with branches defined in domains with increasingly more complex geometry and moreover the associated partition may have increasingly complex topology. As a consequence nearly all these results depend either on a priori estimates or hold only for generic maps. The only exceptions are the results by Buzzi [14] and Tsujii [49] on the quasi-compactness of the transfer operator for general piecewise real analytic maps defined in branches with domains whose boundary are piecewise analytic curves. Tsujii used the BV space, while Buzzi used the space introduced by Keller and Saussol result described below. There are also results for piecewise affine maps in the plane by Buzzi [16] (see also Buzzi [13]) and for arbitrary dimension by Tsujii [51]. We note that there are examples by Tsujii [50] and Buzzi [15] of $C^r$-piecewise expanding maps on $\mathbb{R}^n$ without an absolutely continuous invariant probability.

In the late 70's strange attractors attracted the interest of the mathematical community. In particular Lorenz's attractor poses new problems to ergodic theory of expanding maps, since one can reduce many problems on the dynamics of the Lorenz's flow to the study of an one-dimensional expanding map but this map is non-markovian and it has singularities on which the derivative blow-up, so the previous function spaces did not work anymore.

Keller [30] introduced a new space, the spaces of generalized $p$-bounded variation function, that allows him to get the quasi-compactness of the transfer operator with $p$-bounded variation potential $\log F'$ for one-dimensional maps, including Lorenz maps and piecewise $C^{1+\alpha}$-maps. This sparked an intense interest to use the same space to higher dimensional setting, specially given the difficult to deal with BV space in this setting. Saussol [43] result for piecewise Hölder potentials in higher dimension, that depends on an a priori estimate, was applied by Buzzi [14] in his result on piecewise real analytic maps in the plane.

Note that generalized $p$-bounded variation function spaces seems to be ad hoc spaces. Moreover this space is also in $L^\infty$, that it is a constraint given that unbounded observables may be handy sometimes. One may ask if we can get larger and more familiar space to work with. Indeed Thomine [48] obtained a result for Sobolev spaces $H^s_p$, with $0 < s < 1/p$ in the case of $C^s$ piecewise expanding multimodal maps on manifolds (as usual in this setting, the map needs to satisfy an a priori estimate).

There are also recent results for one-dimensional expanding maps by Butterley [12] and Liverani [33] using some spaces of functions. The Liverani's space, in particular, is related with methods to study the transfer operator of hyperbolic maps.
acting on certain anisotropic Banach spaces.

Nakano and Sakamoto [34] recently obtained the quasi-compactness of the transfer operator for smooth expanding maps on manifolds without discontinuities acting on Besov spaces.

See also Baladi and Holschneider [5] for an earlier application of wavelets and multiresolution analysis in the study of transfer operators of smooth expanding maps on manifolds.

If we move away from the functional analytic approach, Eslami [22] studied the decay of correlations for expanding maps on metric spaces. The class of observables under consideration is indeed a cone (rather than a linear subspace) of functions, inspired by the standard pairs developed by Dolgopyat and Chernov (see for instance [21] [17]).

We finish this historic account saying that the development of the functional analytic approach for hyperbolic maps (for instance, Anosov diffeomorphisms) has been very intense in the last years, with many exchange of ideas with results on expanding maps. A fair description of these new developments is beyond the scope of this work. We refer the reader to the works of Blank, Keller and Liverani [7], Guézel and Liverani [26], Baladi and Tsujii [6], as well the survey and the recent book by Baladi [3] [4] and the references therein for more information.

3. WHO NEEDS YET ANOTHER BANACH SPACE?

We offer an "one-fits-all" approach. The Besov spaces on measure spaces with good grids considered here includes many of the Banach spaces of functions considered in the literature on transfers operators. In particular Keller’s spaces of generalised $p$-bounded variation and Sobolev spaces. Moreover one can cover most dynamics already considered before, as Lorenz 1-dimensional maps, piecewise $C^{1+\alpha}$ expanding maps, etc, giving new statistical results for a wider class of observables, including unbounded ones. We can consider Besov spaces (in particular Sobolev spaces) in many settings [46], in particular homogeneous spaces (a quasi-metric space with a doubling measure), as for instance symbolic spaces and hyperbolic Julia sets with an appropriated reference measure. But it also allows us to deal with new situations, as maps with potentials in Besov spaces. In the companion paper [47] we give a long list of applications. Finally this is a elementary approach. Besov spaces on measure spaces with good grids have a fairly elementary definition [45] and it demands simple methods. In particular the atomic decomposition by atoms with discontinuities is embraced from the very beginning, so we do not need to deal with mollifiers, what makes the proofs more transparent and straightforward.

II. PRELIMINARIES.
4. Measure space and good grids

Let $I$ be a measure space with a $\sigma$-algebra $\mathcal{A}$ and $m$ be a measure on $(I, \mathcal{A})$, $m(I) = 1$. We will consider two measure spaces $(I, m)$ and $(J, \mu)$ along this paper, but often we will use $|A|$ for either $m(A)$ or $\mu(A)$, since the measure under consideration will be clear from the context.

A **grid** is a sequence of finite families of measurable sets with positive measure $P = (P_k)_{k \in \mathbb{N}}$, so that at least one of these families is nonempty and

G1. Given $Q \in P_k$, let $\Omega_Q^k = \{P \in P_k : P \cap Q \neq \emptyset\}$.

Then

$$C_1 = \sup_k \sup_{Q \in P_k} \#\Omega_Q^k < \infty.$$ 

Define $||P^k|| = \sup\{|Q| : Q \in P^k\}$.

A $(\lambda_{g1}, \lambda_{g2})$-**good grid**, with $0 < \lambda_{g1} < \lambda_{g2} < 1$, is a grid $P = (P^k)_{k \in \mathbb{N}}$ with the following properties:

G2. We have $P^0 = \{I\}$.

G3. We have $I = \bigcup_{Q \in P^k} Q$ (up to a set of zero $m$-measure).

G4. The elements of the family $\{Q\}_{Q \in P^k}$ are pairwise disjoint.

G5. For every $Q \in P^k$ and $k > 0$ there exists $P \in P^{k-1}$ such that $Q \subset P$.

G6. We have

$$\lambda_{g1} \leq \frac{|Q|}{|P|} \leq \lambda_{g2}$$

for every $Q \subset P$ satisfying $Q \in P^{k+1}$ and $P \in P^k$ for some $k \geq 0$.

G7. The family $\bigcup_k P^k$ generate the $\sigma$-algebra $\mathcal{A}$.

We will often abuse notation replacing $Q \in \bigcup_k P^k$ by $Q \in P$. For every set $\Omega$, let

$$k_0(\Omega, P) = \min\{k \geq 0 : \exists P \in P^k \text{ s.t. } P \subset \Omega\}$$

whenever the set in the r.h.s. is a nonempty set. We will use the simpler notation $k_0(\Omega)$ if the grid under consideration is obvious. Note that $k_0(W) = i$ for every $W \in P^i$.

5. Spaces defined by Souza’s atoms

Let $p \in [1, \infty]$, $q \in [1, \infty)$ and $s \in (0, 1)$ satisfying

$$0 < s \leq \frac{1}{p}.$$ 

Let $P = (P^k)_{k \geq 0}$ be a good nested family of partitions. For every $Q \in P$ let $a_Q$ be the function defined by $a_Q(x) = 0$ for every $x \not\in Q$ and

$$a_Q(x) = |Q|^{s-1/p}$$

for every $x \in Q$. The function $a_Q$ will be called a Souza’s canonical atom on $Q$. Let $B^s_{p,q}$ be the space of all complex valued functions $f \in L^p$ that can be represented by an absolutely convergent series on $L^p$

$$f = \sum_{k=0}^{\infty} \sum_{Q \in P^k} s_Q a_Q \tag{5.1}$$
where $s_Q \in \mathbb{C}$ and additionally

\begin{equation}
\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in P_k} |s_Q|^p \right)^{q/p} \right)^{1/q} < \infty.
\end{equation}

By absolutely convergence in $L^p$ we mean that

\begin{equation}
\sum_{k=0}^{\infty} \left( \sum_{Q \in P_k} s_Q a_Q \right)^p < \infty.
\end{equation}

The r.h.s. of (5.1) is called a $B_{p,q}^s$-representation of $f$. Define

\begin{equation}
|f|_{B_{p,q}^s} = \inf \left( \sum_{k=0}^{\infty} \left( \sum_{Q \in P_k} |s_Q|^p \right)^{q/p} \right)^{1/q},
\end{equation}

where the infimum runs over all possible representations of $f$ as in (5.1). We say that $f \in B_{p,q}^s$ is $B_{p,q}^s$-positive if there is a $B_{p,q}^s$-representation of $f$ such that $s_Q \geq 0$ for every $Q \in \mathcal{P}$. The following results were proven in S. [45]. We collect them here for the convenience of the reader.

**Proposition 5.1.** The normed space $(B_{p,q}^s, |\cdot|_{B_{p,q}^s})$ is a complex Banach space and its unit ball is compact in $L^p$.

**Proposition 5.2.** We have that $B_{p,q}^s \subset L^t$ for every $t$ satisfying

\[ p \leq t < \frac{p}{1 - sp}. \]

Moreover this inclusion is continuous, that is, there is $K_t > 0$ such that

\[ |f|_t \leq K_t |f|_{B_{p,q}^s}. \]

for every $f \in B_{p,q}^s$.

There are many alternative definitions for $B_{p,q}^s$. For instance, we can consider far more general atoms. Let

\[ 0 < s < \beta < 1/p \]

Given $P \in \mathcal{P}$, denote by $B_{p,q}(P)$ the set of all function $f \in B_{p,q}$ that has a representation as in (5.1) and (5.2) and additionally $d_Q = 0$ for every $Q \in \mathcal{P}$ that is not contained in $Q$. The norm $|\cdot|_{B_{p,q}^s(P)}$ in $B_{p,q}^s(P)$ is defined as in (5.4), but the infimum is taken over all possible representations satisfying this additional condition.

A $(s, \beta, p, q)$-Besov atom supported on $Q$ is a function $b_Q \in A_{s,\beta,p,q}^b$ satisfying

\begin{equation}
|b_Q|_{B_{p,q}^s(P)} \leq \frac{1}{C_{GBVA}} |Q|^{s-\beta}.
\end{equation}

We denote $A_{s,\beta,p,q}^{bu}$ the set of $(s, \beta, p, q)$-Besov atoms supported on $Q$. The constant $C_{GBVA}$ is chosen such that $a_Q \in A_{s,\beta,p,q}^{bu}$. A $(s, \beta, p, q)$-Besov positive atom supported on $Q$ is a function $b_Q \in A_{s,\beta,p,q}^{bu}$ that has a $B_{p,q}^s(Q)$-representation

\[ b_Q = \sum_{k=0}^{\infty} \sum_{P \subset Q} s_P a_P. \]
with $s_P \geq 0$ and satisfying
\[
\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in P_k} |s_Q|^{\beta/\gamma} \right)^{\gamma/\theta} \right)^{1/\theta} \leq \frac{1}{C_{GBVA}} |Q|^{s-\beta}.
\]
The space $B^s_{p,q}(A^{bv}_{s,\beta,p,q})$ is the space of all functions that can be written as
\[
f = \sum_{k=0}^{\infty} \sum_{Q \in P_k} s_Q b_Q
\]
where $s_Q \in \mathbb{C}$, $b_Q \in A^{bv}_{s,\beta,p,q}$ and additionally (5.2) holds. This is called a $B^s_{p,q}(A^{bv}_{s,\beta,p,q})$-representation of $f$. The norm $|\cdot|_{B^s_{p,q}(A^{bv}_{s,\beta,p,q})}$ is defined as in (5.4), where the infimum is taken instead over all possible $B^s_{p,q}(A^{bv}_{s,\beta,p,q})$-representations of $f$. Quite surprisingly we have

**Proposition 5.3** (From Besov to Souza representation). The Banach spaces $B^s_{p,q}(A^{bv}_{s,\beta,p,q})$ and $B^s_{p,q}$ coincides and its norms are equivalent. Indeed for every $B^s_{p,q}(A^{bv}_{s,\beta,p,q})$-representation
\[
g = \sum_{Q \in P} d_Q b_Q
\]
there is a $B^s_{p,q}(A^{sz}_{s,\beta})$-representation
\[
g = \sum_{Q \in P} z_Q a_Q
\]
such that
\[
\left( \sum_{i} \left( \sum_{Q \in P_i} |z_Q|^{\gamma/\theta} \right)^{\theta/\gamma} \right)^{1/\theta} \leq C_{GBS} \left( \sum_{i} \left( \sum_{W \in P_i} |s_W|^{\gamma/\theta} \right)^{\theta/\gamma} \right)^{1/\theta}.
\]
Moreover if $d_Q \geq 0$ for every $Q \in P$ and every $b_Q$ is a $(s,\beta,p,q)$-Besov positive atom supported on $Q$ then we can choose $z_Q \geq 0$ for every $Q \in P$.

**Proposition 5.4.** The following sets are compact in $L^p$ (and in particular in $L^1$).

A. The set $S^e_C$ of all $f \in B^s_{p,q}$ which have a $B^s_{p,q}$-representation
\[
f = \sum_{k} \sum_{Q \in P_k} d_Q b_Q
\]
satisfying
\[
\sum_{k} \left( \sum_{Q \in P_k} |d_Q|^p \right)^{\theta/p} \leq C.
\]

B. The set $(S^e_C)^+$ of all $f \in B^s_{p,q}$ that have a $B^s_{p,q}$-representation satisfying (5.5) and (5.6) and additionally $d_Q \geq 0$ for every $Q \in P$.

C. The set $S^{ew}_C$ of all $f \in B^s_{p,q}(A^{bw}_{\beta,p,q})$ which has a $B^s_{p,q}(A^{bw}_{\beta,p,q})$-representation
\[
f = \sum_{k} \sum_{Q \in P_k} d_Q b_Q
\]
satisfying
\[
\sum_{k} \left( \sum_{Q \in P_k} |d_Q|^p \right)^{\theta/p} \leq C.
\]
D. The set $(S^{bv}_{\gamma})^+$ of all $f \in \mathcal{B}_{p,q}^{s} \mathcal{A}_{\beta,p,q}^{bv}$ that has a $\mathcal{B}_{p,q}^{s} \mathcal{A}_{\beta,p,q}^{bv}$-representation satisfying (5.7) and (5.8) and additionally $d_Q \geq 0$ for every $Q \in \mathcal{P}$ and $b_Q$ is a $(s,\beta,p,q)$-Besov positive atom.

**Proposition 5.5** (Canonical representation). There is $C_{GC} \geq 1$ with the following property. For every $P \in \mathcal{P}$ there is a linear functional in $(L^1)^*$

$$f \in L^1 \mapsto k^f_P$$

such that if $f \in \mathcal{B}_{p,q}^{s}$ then

$$\sum_k \sum_{P \in \mathcal{P}^k} k^f_P a_P$$

is a $\mathcal{B}_{p,q}^{s}$-representation of $f$ satisfying

$$\left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |k^f_P|^p \right)^{q/p} \right)^{1/q} \leq C_{GC} |f|_{\mathcal{B}_{p,q}^{s}}.$$

**Proposition 5.6** (From Besov to Souza representation). For every $\mathcal{B}_{p,q}^{s} \mathcal{A}_{\beta,p,q}^{bv}$-representation $g = \sum_{Q \in \mathcal{P}} d_Q b_Q$ there is a $\mathcal{B}_{p,q}^{s} \mathcal{A}_{s,p}^{s_z}$-representation $g = \sum_{Q \in \mathcal{P}} z_Q a_Q$

such that

$$\left( \sum_{i} \left( \sum_{Q \in \mathcal{P}^i} |z_Q|^p \right)^{q/p} \right)^{1/q} \leq C_{GS} \left( \sum_{i} \left( \sum_{W \in \mathcal{P}^i} |s_W|^p \right)^{q/p} \right)^{1/q}. $$

Moreover if $d_Q \geq 0$ for every $Q \in \mathcal{P}$ and every atom $b_Q$ is $\mathcal{B}_{p,q}^{s}$-positive then we can choose $z_Q \geq 0$ for every $Q \in \mathcal{P}$.

**Assumption A_1.** From now on we fix measure spaces with good grids $(I,\mathcal{P},m)$, $(J,\mathcal{H},\mu)$, $p \in [1,\infty)$, $q \in [1,\infty]$, $\epsilon > 0$, $\gamma \in [0,1]$ and $s,\beta,\delta \in (0,\infty)$ be such that $0 < s + \epsilon \leq \frac{1}{p}$, $s < \beta < \frac{1}{p}$ and $0 < \delta \leq \max\{s, \epsilon\}$. 

Let

$$t_0 = \frac{p}{1 - sp + \delta p}. $$

Note that

$$p < t_0 < \frac{p}{1 - sp},$$

so in particular $\mathcal{B}_{p,q}^{s} \subset L^{t_0}$. 

Table 1. Constants associated with the geometry of the grid

| Symbol     | Description                                                                 |
|------------|-----------------------------------------------------------------------------|
| $\lambda_{g1} \leq \lambda_{g2}$ | $\mathbf{G}$eometry of the grid                                             |
| $C_{GC}$   | Describes how optimal is the $\mathbf{C}$anonical Souza’s representation     |
| $C_{GBS}$  | Control the conversion of a representation using Besov’s atoms to a         |
|            | representation using Souza’s atoms.                                          |

6. Regular Domains

We say that $\Omega \subset J$ is a $(\alpha, C_2, \lambda_1)$-regular domain if it is possible to find families $\mathcal{F}^k(\Omega) \subset \mathcal{H}^k$, $k \geq k_0(\Omega, \mathcal{H})$, such that

A. We have $\Omega = \cup_{k \geq k_0(\Omega)} \cup Q \in \mathcal{F}^k(\Omega) Q$.
B. If $P, Q \in \cup_{k \geq k_0(\Omega)} \mathcal{F}^k(\Omega)$ and $P \neq Q$ then $P \cap Q = \emptyset$.
C. We have $\lambda_{1} \leq C_{2} \lambda_{2}^{k-1}(\Omega)$.

7. Branches

Let $\hat{I} \subset I$ and $\hat{J} \subset J$ be measurable sets. A

$(s, p, a, \hat{\epsilon}, C_{DGD1}, \lambda_{DGD2}, C_{DC1}, \lambda_{DC2}, \mathcal{G}) - \text{branch}$

is a measurable function $h: \hat{J} \to \hat{I}$ such that $h^{-1}: \hat{I} \to \hat{J}$ is also measurable, $\mathcal{G}$ is a grid on $(\hat{I}, m)$ and

I. We have that $m(Q) = 0$ if and only if $\mu(h^{-1}(Q)) = 0$ for every measurable set $Q \subset J$.

II. (Geometric Distortion Control). For every $Q \subset I$ such that $Q \in \mathcal{G}$ we have that $h^{-1}(Q)$ is a $(1 - sp, C_{DGD1}, \lambda_{DGD2})$-regular domain in $(J, \mathcal{H})$. This property controls how the action of $h^{-1}$ deforms the "shape" of $Q$.

III. (Scaling Control). We have

$$|k_0(Q, \mathcal{G}) - k_0(h^{-1}(Q), \mathcal{H})| \geq a.$$ 

and

$$(\frac{|Q|}{|h^{-1}(Q)|})^{\tilde{\epsilon}/|I|} \leq C_{DC1} \lambda_{DC2}^{k_0(Q, \mathcal{G}) - k_0(h^{-1}(Q), \mathcal{H})}.$$ 

IV. The set $\hat{I}$ is a countable union (up to a set of zero measure) of elements of $\mathcal{P}$.

V. For every $W \in \mathcal{G}$ such that $W \subset \hat{I}$ we have that $h^{-1}(W)$ is a countable union (up to a set of zero measure) of elements of $\mathcal{H}$.
8. Potentials

Let \( h : \hat{J} \to \hat{I} \) be a branch as in Section 7. A \((C_{DRP}, \beta, \epsilon)\)-regular potential, with \( \beta \) such that \( s < \beta \leq 1/p \), associate to \( h \) is a function \( g : \hat{J} \to C \) satisfying

\[
|g \cdot 1_W|_{B^s_{p,q}(W,H(I,W),A_{p,q}^{rs})} \leq C_{DRP} \left( \frac{|Q|}{|h^{-1}Q|} \right)^{\frac{s}{2} - s + \epsilon} |W|^{1/p - \beta}.
\]

for every \( W \in \mathcal{H} \) and \( Q \in \mathcal{G} \) such that \( W \subset \hat{J}, Q \subset \hat{I} \) and \( h(W) \subset Q \).

We say the potential \( g \) is \( B^s_{p,q} \)-positive regular potential if for every \( W \in \mathcal{H} \) such that \( W \subset J \), there is a \( B^s_{p,q} \)-representation of \( g \cdot 1_W \)

\[
g \cdot 1_W = \sum_k \sum_{P \in H^k} c_PA_P
\]
such that \( c_P \geq 0 \) for every \( P \in \mathcal{H} \) and moreover

\[
\left( \sum_k \left( \sum_{P \in H^k} |c_P|^p \right)^{q/p} \right)^{1/q} \leq C_{DRP} \left( \frac{|Q|}{|h^{-1}Q|} \right)^{\frac{s}{2} - s + \epsilon} |W|^{1/p - \beta}.
\]

for every \( W \in \mathcal{H} \) and \( Q \in \mathcal{G} \) such that \( W \subset J, Q \subset I \) and \( h(W) \subset Q \).

9. Transfer Transformations

Assume

**Assumption A2.** Along this paper we will always assume that

- \( \{I_r\}_{r \in \Lambda} \) and \( \{J_r\}_{r \in \Lambda} \) are families of measurable subsets of \( I \) and \( J \), respectively, with \( \Lambda \subset \mathbb{N} \).
- The maps \( h_r : J_r \to I_r \)
  are \((s,p,a_r,\epsilon_r,C_{DRD1}^r, C_{DC1}^r, \lambda_{DGD2}^r, \lambda_{DC2}^r, \mathcal{G}_r)\)-branches, with \( |\epsilon_r| = \epsilon \) for every \( r \).
- We have that \( \mathcal{A} = \{Q \in \mathcal{G}_r, Q \subset I_r, \text{ for some } r \in \Lambda\} \cup \{Q \in \mathcal{P}, Q \cap I_r = \emptyset, \text{ for every } r \in \Lambda\} \)
  generates the \( \sigma \)-algebra \( \mathcal{A} \).
- We have that \( g_r : J_r \to I_r \)
  are \((C_{DRP}^r, \beta, \epsilon_r)\)-potentials.
- Let \( \lambda_{DRS2}^r = \max\{(\lambda_{DC2}^r)^r, (\lambda_{CDD2}^r)^{1/p}\} < 1 \).
  Then \( \lambda_{DRS2} = \sup_r \lambda_{DRS2}^r < 1 \).

The "\( r \)" in the notation \( C_{DD1}^r, C_{DC1}^r, \lambda_{DGD2}^r, \lambda_{DC2}^r \) and \( C_{DRP}^r \) is just an index, indicating that those constants may depend on \( r \).
The value
\[ \Theta_r = (C_{DC1}^r)^t C_{DRP}^r (C_{DGD1}^r)^{1/p} (\lambda_{DRS2}^r)^{a_r(1-\gamma)} \]
measures how regular is the r-th pair \((h_r, g_r)\).

We want to consider the **transfer transformation**
\[ \Phi(f)(x) = \sum_{r \in \Lambda} g_r(x) f(h_r(x)). \]

Notice that when \(\Lambda\) is an infinity set it is not even clear for which measurable functions \(f\) the operator is well defined. Let
\[ \Phi_r(f)(x) = g_r(x) f(h_r(x)). \]

We also assume

**Assumption A3.** There is \(C_3 \geq 0\) such that for every \(f \in L^{t_0}(m)\)
\[ |\Phi f|_{L^1(\mu)} \leq \sum_r |\Phi_r(f)|_{L^1(\mu)} \leq C_3 |f|_{L^{t_0}(m)}. \]

Section 23 provides some methods to obtain Assumption A3. Note that Assumption A3 implies that \(\Phi: L^{t_0}(m) \mapsto L^1(\mu)\) is a well defined and bounded linear transformation, where \(t_0\) is defined in (5.9). Then
\[ p \leq t_0 < \frac{p}{1-sp} \]
and Proposition 5.2 implies that
\[ \Phi: B_{p,q}^s(I, \mu) \rightarrow L^1(\mu) \]
is a bounded linear transformation.

## 10. Regular dynamical slicing

We want give conditions for \(\Phi\) to be a well-defined linear transformation from \(B_{p,q}^s(I, m, \mathcal{P})\) to \(B_{p,q}^s(J, \mu, \mathcal{H})\) and study its regularity. To this end we need to connect the “local” grid \(\mathcal{G}_r\) on each \((I_r, m)\) with the ”global” good grid \(\mathcal{P}\) in \((I, m)\). This will depend on the data

\[ \mathcal{I} = (s, p, q, \epsilon, \gamma, \{(I_r, J_r, a_r, \lambda_{DGD2}^r, \lambda_{DC2}^r, C_{DC1}^r, C_{DRP}^r, C_{DGD1}^r, \mathcal{G}_r)\}_{r \in \Lambda}) \]

We call \(\mathcal{I}\) a **weighed family of sets**. Let

(10.14) \[ N = \sup_{P \in \mathcal{H}} \#\{r \in \Lambda \text{ s.t. } P \subset J_r\}. \]

We say that the pair
\[ (\mathcal{I}, \sum_k \sum_{Q \in \mathcal{P}_k} d_Q a_Q) \]
has a \(C_{DRS1}\)-**regular slicing** if \(C_{DRS1} \geq 0\) and

i. We have that
\[ f = \sum_k \sum_{Q \in \mathcal{P}_k} d_Q a_Q \]
is a \(B_{p,q}^s(I, m, \mathcal{P})\)-representation of a function \(f \in B_{p,q}^s(I, m, \mathcal{P})\).
ii. For every $r \in \Lambda$ there is a $\mathcal{B}_{p,q}^s (I_r, \mu, \mathcal{G}_r)$-representation of $f \cdot 1_{I_r}$

\begin{equation}
 f \cdot 1_{I_r} = \sum_{Q \in \mathcal{G}_r, Q \subseteq I_r} c^r_Q a_Q,
\end{equation}

satisfying either

\begin{equation}
 \left( \sum_j \left( \sum_r \Theta_r \left( \sum_{Q \in \mathcal{G}_r^j, Q \subseteq I_r} |c^r_Q|^p \right)^{q/p} \right)^{1/q} \right)^{1/q} \\
 \leq C_{DRS1} \left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |d_Q|^p \right)^{q/p} \right)^{1/q},
\end{equation}

or

\begin{equation}
 \left( \sum_j \left( \sum_r \Theta_p \left( \sum_{Q \in \mathcal{G}_r^j, Q \subseteq I_r} |c^r_Q|^p \right)^{q/p} \right)^{1/q} \right)^{1/q} \\
 \leq C_{DRS1} \left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |d_Q|^p \right)^{q/p} \right)^{1/q}.
\end{equation}

Here $N^1 = 1$.

iii. If $d_Q \geq 0$ for every $Q \in \mathcal{P}$ then we can choose $c^r_Q \geq 0$ for every $Q \in \mathcal{G}_r$.

**Table 2.** Constants associated with the Dynamics of the transfer operator

| Symbol | Description |
|--------|-------------|
| $C^r_{DG1}, \lambda_{DG2}$ | Describes the Geometric Deformation of the domains in the grid by the action of branches $h_r$ |
| $a_r, C^r_{DC1}, \lambda_{DC2}$ | Describe the Contracting properties of $h_r$ |
| $C^r_{DRP}$ | Describes the Regularity of the Potentials $g_r$ |
| $C_{DRS1}$ | Controls the Regularity of the dynamical Slicing |
III. STATEMENT OF RESULTS.

11. BOUNDENESS ON $B_{p,q}^s$

Our main technical result is

**Theorem 11.1 (Key Technical Result).** Let

\[
\sum_k \sum_{Q \in P_k} d_Q a_Q
\]

be a $B_{p,q}^s$-representation of a function $f \in B_{p,q}^s (I, m, P)$ such that $(I, \sum_{Q \in P} d_Q a_Q)$ has a $C_{DRS1}$-regular slicing. Define

\[
C_D = \frac{2}{1 - \lambda_{DRS2}^q}.
\]

Then $\Phi(f) \in B_{p,q}^s (J, \mu, H)$ and there is a $B_{p,q}^s$-representation

\[
\sum_k \sum_{Q \in H_k} z_Q a_Q
\]

of $\Phi(f)$ such that

\[
\left( \sum_i \left( \sum_{Q \in H^i} |z_Q|^p \right)^{q/p} \right)^{1/q} \leq C_{GBS} C_D C_{DRS1} \left( \sum_k \left( \sum_{Q \in P_k} |d_Q|^p \right)^q \right)^{1/q}.
\]

Moreover if the potentials $g_r$ are $B_{p,q}^s (J, \mu, H)$-positive then whenever $d_Q \geq 0$ for every $Q$ we can choose $z_Q \geq 0$ for every $Q$.

The proof of the following result is obvious.

**Corollary 11.1.** Let $S$ be a linear subspace of $B_{p,q}^s (I, m, P)$. Suppose that for every $f \in S$ there is a $B_{p,q}^s$-representation

\[
f = \sum_k \sum_{Q \in P_k} d_Q a_Q
\]

such that

\[
\left( \sum_k \left( \sum_{Q \in P_k} |d_Q|^p \right)^{q/p} \right)^{1/q} \leq C |f|_{B_{p,q}^s}
\]

and $(I, \sum_{Q \in P} d_Q a_Q)$ has a $C_{DRS1}$-regular slicing. Then

\[
\Phi: S \rightarrow B_{p,q}^s (J, \mu, H)
\]

is a linear transformation satisfying

\[
|\Phi(f)|_{B_{p,q}^s (J, \mu, H)} \leq C_{GBS} C_D C_{DRS1} C |f|_{B_{p,q}^s (I, m, P)}.
\]

for every $f \in S$. 

12. DYNAMICAL SLICING: HOW TO DO IT

**Definition 12.1.** Let $S$ be a subspace of $B_{p,q}^s(I,m,P)$ and let $I$ be a weighed family of sets, We say that $(S,I)$ has a $(C_{DRSFR},C_{DRSES})$-essential slicing, where $C_{DRSFR},C_{DRSES} \geq 0$ if there is a finite subset $P' \subset P$ such that for every $B_{p,q}^s$-representation

$$\sum_{Q \in P} d_Q a_Q$$

of a function $f \in S$ the pair $(I, \sum_{Q \in P \setminus P'} d_Q a_Q)$ has a $C_{DRSES}$-regular slicing and the pair $(I, \sum_{Q \in P'} d_Q a_Q)$ has a $C_{DRSFR}$-regular slicing. Here FR stands for “Finite-Rank” and ES for “ESsential spectral radius”. We say that $(S,I)$ has a $(C_{DRSFR},C_{DRSES},t)$-essential slicing if $P' = \cup_{k<t} P_k$.

Given $R \subset P$, define the closed subspace $B_{p,q,R}^s \subset B_{p,q}^s$ as

$$B_{p,q,R}^s = \{ f \in B_{p,q}^s : k_P^f = 0 \text{ for every } P \notin R \}.$$

Here $k_P^f$ is as in Proposition 5.5. Note that there is a linear projection

$$\pi_R : B_{p,q}^s \rightarrow B_{p,q,R}^s$$

satisfying $|\pi_R| \leq C_G$ and moreover $f = \pi_R(f) + \pi_{P \setminus R}(f)$ for every $f \in B_{p,q}^s(I,m,P)$. Of course $B_{p,q,R}^s$ has finite dimension when $R$ is finite.

We left unanswered how to obtain a regular dynamical slicing as assumed in our main results on transfer operators, as Theorem 11.1 and Corollary 13.1. This section deals with this question.

**Definition 12.2.** As defined in S. [45], a set $\Omega \subset I$ is $(\alpha,C_4,t)$-strongly regular domain if for each $Q \in P^i$ with $i \geq t$ and $k \geq k_0(Q \cap \Omega)$ there is a family $F^k(Q \cap \Omega) \subset P^k$ such that

i. We have $Q \cap \Omega = \cup_{k \geq k_0(Q \cap \Omega)} \cup_{P \in F^k(Q \cap \Omega)} P$.

ii. If $P, W \in \cup_k F^k(Q \cap \Omega)$ and $P \neq W$ then $P \cap W = \emptyset$.

iii. We have

(12.18) $$\sum_{P \in F^k(Q \cap \Omega)} |P|^{\alpha} \leq C_4 |Q|^{\alpha}.$$ 

in the next four result we will assume

**Assumption PLAIN.** For every $r \in \Lambda$

$$G_r^k = \{ P \in P^k : P \subset I_r \}.$$ 

**Theorem 12.3 (The Core I).** Assume $A_1 - A_3$ and PLAIN. There is $C_{GSN}$, that depends only the good grid $P$, with the following property. Suppose that $\Lambda$ is finite
and there is $t$ such that for every with $r \in \Lambda$ the set $I_r$ is a $(1 - \beta p, C_5, t)$-strongly regular domain. Suppose
\begin{equation}
M = \sup_{P \in P_k} \# \{ r \in \Lambda : I_r \cap P \neq \emptyset \} < \infty.
\end{equation}
Then $(B_{p,q}, I)$ has a $(C_{GSR} C_7, C_{GSR} C_6, t)$-essential slicing, with
\[ C_6 = M C_5^{1/p} \left( \sum_{r \in \Lambda} \Theta_{r}^{p'} \right)^{1/p'}.
\]
and
\[ C_7 = (\# \Lambda) \left( C_5 \lambda_{g_1}^{-t} \right)^{1/p} \left( \sum_{r \in \Lambda} \Theta_{r}^{p'} \right)^{1/p'}.
\]

**Theorem 12.4** (The Core II). Assume $A_1 - A_3$ and PLAIN. There is $C_{GSR}$, that depends only the good grid $P$, with the following property. Suppose that $\Lambda$ is finite and there is $t$ such that for every with $r \in \Lambda$ the set $I_r$ is a $(1 - \beta p, C_5, t)$-strongly regular domain,
\begin{equation}
T = \sup_{Q \in P_k} \sum_{k \geq t} \sum_{Q \cap I_r \neq \emptyset} \Theta_{r} < \infty,
\end{equation}
and
\begin{equation}
N = \sup_{P \in \mathcal{H}} \# \{ r \in \Lambda : P \subset J_r \} < \infty.
\end{equation}
Then $(B_{p,q}, I)$ has a $(C_{GSR} C_9, C_{GSR} C_8, t)$-essential slicing, with
\[ C_8 = N^{1/p'} C_5^{1/p} T,
\]
and
\[ C_9 = N^{1/p'} (\# \Lambda) \left( \sup_{r \in \Lambda} \Theta_{r} \right) (C_5 \lambda_{g_1}^{-t})^{1/p},
\]
with the obvious adaptation when $p = 1$ (in particular we set $N^{1/\infty} = 1$).

**Theorem 12.5** (Tail I). Assume $A_1 - A_3$ and PLAIN. There is $C_{GSR}$, that depends only the good grid $P$, with the following property. Suppose $\{ I_r \}$ is a family of pairwise disjoint $(1 - \beta p, C_{10}, 0)$-strongly regular domains. If
\[ C_{11} = C_{10}^{1/p} \left( \sum_{r \in \Lambda} \Theta_{r} \right),
\]
is finite then $(B_{p,q}, I)$ has a $(C_{GSR} C_{11})$-regular slicing.

**Theorem 12.6** (Tail II). Assume $A_1 - A_3$ and PLAIN. There is $C_{GSR}$, that depends only the good grid $P$, with the following property. Suppose $\{ I_r \}_{r \in \Lambda}$ is a countable family of pairwise disjoint subsets such that the set
\[ \Omega = \bigcup_{r \in \Lambda} I_r
\]
is a $(1 - \beta p, C_{10}, 0)$-strongly regular domain and additionally, if $Q \in \mathcal{P}$ and $Q \subset \Omega$ then $Q \subset I_r$, for some $r \in \Lambda$. Let
\[ N = \sup_{P \in \mathcal{H}} \# \{ r \in \Lambda : P \subset J_r \}.
\]
If
\[
C_{12} = N^{1/p'} C_{10}^{1/p} \left( \sup_{r \in A} \Theta_r \right),
\]
is finite then \((B_{p,q}^s, \mathcal{T})\) has a \((C_{GSR} C_{12})\)-regular slicing. Here we set \(N^{1/\infty} = 1\) even if \(N = \infty\).

13. **Essential Spectral Radius of \(\Phi\) acting on \(B_{p,q}^s\)**

Consider the assumption

**Assumption** \(A_4\). We have \((J, \mu, \mathcal{H}) = (I, m, \mathcal{P})\), and \((B_{p,q}^s(I, m, \mathcal{P}), \mathcal{T})\) has a \((C_{DRSFR}, C_{DRSES})\)-essential slicing.

We stress we are not assuming PLAIN anymore.

**Corollary 13.1** (Boundedness and Essential Spectrum Radius of \(\Phi\)). Suppose that \(A_1 - A_4\) hold. Then the transformation \(\Phi\) is a bounded operator acting on \(B_{p,q}^s\) satisfying
\[
|\Phi|_{B_{p,q}^s} \leq C_{GBS} C_D (C_{DRSFR} + C_{DRSES}) C_{GC}
\]
and its essential spectral radius is at most \(C_{GBS} C_D C_{DRSES} C_{GC}\). Indeed
\[
\Phi \circ \pi_{p'} : B_{p,q}^s \to B_{p,q}^s
\]
is a finite-rank linear transformation with norm at most \(C_{GBS} C_D C_{DRSFR} C_{GC}\) and
\[
\Phi \circ \pi_{p \setminus p'} : B_{p,q}^s \to B_{p,q}^s
\]
is a linear transformation with norm at most \(C_{GBS} C_D C_{DRSES} C_{GC}\).

14. **Lasota-Yorke Inequality and its consequences**

Consider

**Assumption** \(A_5\). We have
\[
6 C_{GBS} C_D C_{DRSES} C_{GC} < 1.
\]
Moreover the potentials satisfy \(g_r \geq 0\) \(m\)-almost everywhere, and
\[
\int \Phi(f) \, dm = \int f \, dm
\]
for every \(f \in B_{p,q}^s(I, m, \mathcal{P})\).

Note that Assumption \(A_5\) implies that \(|\Phi(f)|_1 \leq |f|_1\) for every \(f \in B_{p,q}^s\). Since \(B_{p,q}^s\) is dense in \(L^1\) we can extends \(\Phi\) to a bounded linear operator \(\Phi : L^1 \to L^1\). We have

**Theorem 14.1** (Lasota-Yorke Inequality). Suppose that \(A_1 - A_4\) holds,
\[
C_{GBS} C_D C_{DRSES} C_{GC} < 1
\]
and
\[
|\Phi(f)|_1 \leq |f|_1
\]
for every \( f \in \mathcal{B}_{p,q}^s \). Then \( \Phi \) satisfies the Lasota-Yorke inequality

\[
|\Phi^n(f)|_{\mathcal{B}_{p,q}^s} \leq C |f|_1 + (C_{\text{GBS}} C_{\text{D}} C_{\text{BSES}} C_{\text{C}})^n |f|_{\mathcal{B}_{p,q}^s},
\]

for some \( C \geq 0 \) and every \( f \in \mathcal{B}_{p,q}^s \).

**Corollary 14.1.** Suppose that \( A_1 - A_5 \) hold. Then

A. There exists \( \delta \in (0,1) \) such that

\[
E = \sigma_{\mathcal{B}_{p,q}^s}(\Phi) \cap \{ z \in \mathbb{C} : |z| \geq \delta \}
\]

is finite and nonempty and it is contained in \( S^1 \). Every element \( \lambda \in E \) is an eigenvalue with finite-dimensional eigenspace and

\[
(\Phi - \lambda I)^2 f = 0 \text{ implies } (\Phi - \lambda I) f = 0
\]

for every \( f \in \mathcal{B}_{p,q}^s \).

B. For every \( \lambda \in E \) there is a bounded projection \( \pi_{\lambda} \), and there is a linear contraction \( \tilde{\Theta} \), both of them acting on \( \mathcal{B}_{p,q}^s \), such that

\[
\Phi^n = \sum_{\lambda \in E} \lambda^n \pi_{\lambda} + \tilde{\Theta}^n
\]

and

\[
\Phi \pi_{\lambda} = \lambda \pi_{\lambda}, \quad \pi_{\lambda'} \pi_{\lambda} = 0, \quad \tilde{\Theta} \pi_{\lambda} = \pi_{\lambda} \tilde{\Theta} = 0
\]

for every \( \lambda', \lambda \in E, \lambda \neq \lambda' \). In particular \( \sup_{n} |\Phi^n|_{\mathcal{B}_{p,q}^s} < \infty \).

C. There is \( \rho \in \mathcal{B}_{p,q}^s \), with \( \rho \geq 0 \), such that

\[
\int \rho \, dm = 1 \text{ and } \Phi(\rho) = \rho.
\]

D. If \( \lambda \in S^1 \) and \( u \in L^1 \) satisfies \( \Phi u = \lambda u \) then \( u \in \mathcal{B}_{p,q}^s \). Moreover \( \Phi(|u|) = |u| \)

and \( u_k = (\text{sgn } u)^k |u| \) satisfies \( \Phi u_k = \lambda^k u_k \) for every \( k \in \mathbb{Z} \).

E. Every element of the finite set \( E \) is an \( n \)-th root of unit, for some \( n \in \mathbb{N}^* \).

**Corollary 14.2.** Suppose \( A_1 - A_5 \). Then \( T \) has at most \( N \) physical measures, where \( N \) is the dimension of the 1-eigenspace of \( \Phi \). Moreover all these measures are absolutely continuous with respect to \( m \), and the basin of attraction of these measures covers \( I \).

Denote by \( M(\mathcal{B}_{p,q}^s) \subset \mathcal{B}_{p,q}^s \) the set of the functions \( g \) such that the pointwise multiplication

\[
f \mapsto fg
\]

is a bounded operator in \( \mathcal{B}_{p,q}^s \), that is, if \( f \in \mathcal{B}_{p,q}^s \) then \( fg \in \mathcal{B}_{p,q}^s \) and

\[
\sup\{|fg|_{\mathcal{B}_{p,q}^s}, |f|_{\mathcal{B}_{p,q}^s} \leq 1\} < \infty.
\]

**Corollary 14.3** (Almost sure invariant principle). Suppose that \( A_1 - A_5 \) hold, and suppose moreover that \( p > 2 \) and

i. \( E = \{1\} \),

ii. There is \( \rho_0 \in \mathcal{B}_{p,q}^s \) with \( \rho_0 \geq 0 \) that generates

\[
\{\rho \in \mathcal{B}_{p,q}^s \text{ s.t. } \Phi(\rho) = \rho\}.
\]
Then for every real-valued function $v \in \mathcal{B}^{1/p}_{p,\infty} + M(\mathcal{B}^s_{p,q})$ with
\[ \int v \rho_0 \, dm = 0 \]
the sequence
\[ v, v \circ T, v \circ T^2... \]
satisfies the almost sure invariance principle with every error exponent satisfying
\[ \delta > \frac{1}{4} + \frac{1}{4p - 4}. \]

15. Uniqueness and structure of invariant measures, strong a.s.i.p.

The assumption $E = \{1\}$ in Corollary 14.3 is the trickiest one to deal with. The following assumption allows us to understand better the structure of the invariant densities and it makes it easier to check $E = \{1\}$.

**Assumption A_6.** The potentials $g_r$ are $\mathcal{B}^s_{p,q}$-positive.

**Theorem 15.1** (Structure of invariant measures). Suppose $A_1 - A_6$. Then every $\rho \in L^1$, $\rho \geq 0$ that satisfies (14.22) is $\mathcal{B}^s_{p,q}(I, m, \mathcal{P})$-positive. In particular the set
\[ \{x \in I: \rho(x) > 0\} \]
is (except for a set of zero $m$-measure) a countable union of elements of $\mathcal{P}$.

**Assumption A_7.** We have that $T$ is transitive, that is, for every $P, Q \in \mathcal{P}$ there is $n \geq 0$ such that $m(P \cap T^{-n}Q) > 0$.

**Corollary 15.1** (Ergodicity). Suppose $A_1 - A_7$. Then there is an unique $m$-absolutely continuous invariant probability $\mu = \rho_0 \, dm$ for $T$. Moreover
A. The probability $\mu$ is ergodic,
B. We have $\{x \in I: \rho_0(x) > 0\} = I$ (except for a set of zero $m$-measure),
C. The unique function $\rho \in L^1$ that satisfies (14.22) is $\rho_0$,
D. The set $E$ is a cyclic group.

**Corollary 15.2** (Mixing and decay of correlations). Suppose additionally $A_7$. Then
A. There exist $C_{13} \geq 0$ and $\lambda_2 \in (0, 1)$ such that the following holds: If $u \in \mathcal{B}^s_{p,q}$ and $v \in L^{p'}$ then
\[ \left| \int v \circ T^k u \, dm - \int v \rho_0 \, dm \int u \, dm \right| \leq C_{13} \lambda_2^k |u|_{\mathcal{B}^s_{p,q}} |v|_{p'}. \]
B. $E = \{1\}$. 

IV. ACTION ON $B^s_{p,q}$. 

16. Notation

We will use $C_1, C_2, \ldots$ for positive constants, $\lambda_1, \lambda_2, \ldots$ for positive constants smaller than one.

Table 3. List of symbols

| Symbol | Description |
|--------|-------------|
| $A^b_{s,\beta,p,q'}$ | class of $(s, \beta, p, q')$-Besov’s atoms |
| $\mathcal{P}$ | good grid of $I$ |
| $\mathcal{H}$ | good grid of $J$ |
| $\mathcal{G}_r$ | grid of $I_r$ |
| $\mathcal{P}^k$ | partition at the $k$-th level of $\mathcal{P}$ |
| $B^s_{p,q} (A^z_{s,p})$ | $(s,p,q)$-Banach space defined by Souza’s atoms |
| $Q,W$ | elements of grids |
| $\{g_r\}_{r \in \Lambda}$ | family of potentials |
| $| \cdot |_p$ | norm in $L^p$, $p \in [1, \infty]$. |
| $\Phi$ | transfer operator. |
| $p'$ | $1/p + 1/p' = 1$. |

17. Proof of Theorem 11.1

Theorem 11.1 is our main technical result and its proof takes several steps.

17.1. Step 1: Dynamical Slicing. By Section 11 we have that the linear application

$$\Phi: B^s_{p,q}(I, m, \mathcal{P}) \to L^1(\mu)$$

is well defined and bounded, so if $f \in B^s_{p,q}(I, m, \mathcal{P})$ satisfies the assumptions of Theorem 11.1 we need to show that $\Phi(f) \in B^s_{p,q}(J, \mu, \mathcal{H})$ and estimate its norm. By assumption, for every $r \in \Lambda$ there is a $B^s_{p,q}(I_r, m, \mathcal{G}_r)$-representation of $f \cdot 1_{I_r}$

$$f \cdot 1_{I_r} = \sum_k \sum_{Q \in \mathcal{G}_r} c_Q a_Q,$$

satisfying either (10.16) or (10.17).

17.2. Step 2: Applying the transfer transformation.

Claim 1. There is a $B^s_{p,q}(J, \mu, \mathcal{A}^b_{s,\beta,p,q'})$-representation

$$\sum_i \sum_{W \in \mathcal{H}_i} s_W b_W$$

of $\Phi(f)$ satisfying

$$\left( \sum_i \left( \sum_{W \in \mathcal{H}_i} |s_W|^p \right)^{q/p} \right)^{1/q} \leq C_{DRS1} \left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |d_Q|^p \right)^{q/p} \right)^{1/q}. $$
Recall that
\[ \Phi(f) = \sum_r \Phi_r(f \cdot 1_{I_r}). \]
This series converges absolutely on \( L^1(\mu) \), that is
\[ \sum_r |\Phi_r(f \cdot 1_{I_r})|_1 < \infty, \]
so on \( L^1 \) we have
\[ \Phi(f) = \lim_{r_0 \to \infty} \sum_{r \leq r_0} \Phi_r(f \cdot 1_{I_r}). \]
On the other hand, since \( \Phi_r \) is a bounded transformation in \( L^\mu(m) \) we have
\[ \Phi_r(f \cdot 1_{I_r}) = \sum_j \Phi_r(\sum_{Q \in \mathcal{G}^j} c^r_Q a_Q) = \sum_j \sum_{Q \in \mathcal{G}^j} c^r_Q \Phi_r(a_Q), \]
So on \( L^1 \)
\[ \sum_{r \leq r_0} \Phi_r(f \cdot 1_{I_r}) = \lim_{j_0 \to \infty} \sum_{r \leq r_0} \sum_{j \leq j_0} \sum_{Q \in \mathcal{G}^j} c^r_Q \Phi_r(a_Q). \]
Note that all the sums on the r.h.s. are finite. To prove Claim 1. it is enough to show
Figure 2.

Step 2. The image of a fraction of a Souza’s atom $c_Q a_Q$, with $Q \subset I_r$, by $\Phi_r$ is not, in general, a fraction of an atom itself. So we need to cut it in fractions of Besov atoms. In the picture we see the cut above $W$, that is $c_Q^{\Phi_r(a_Q)} 1_W$. We show that this is a fraction $c_Q^{\Phi_r(a_Q)} d_Q^{\Phi_r(a_Q)} b_Q^{\Phi_r(a_Q)}$ of a Besov atom $b_Q^{\Phi_r(a_Q)}$. 

\[ c_Q^{\Phi_r(a_Q)} 1_W \]
Claim 2. For every $r_0$ and $j_0$ we have that
\[ \phi_{i_0,j_0} = \sum_{r \leq r_0} \sum_{j \leq j_0} \sum_{Q \in G_j} c_Q \Phi_r(a_Q) \]
has a $B_{p,q}(J, \mu, H, A_{p,q})$-representation
\[ \phi_{r_0,j_0} = \sum_{k} \sum_{W \in H^k} s_W b_W \]
satisfying \((17.24)\).

Indeed, if Claim 2. holds then $\Phi(f)$ belongs to the $L^1$-closure of $S = \{\phi_{r_0,j_0}\}_{r_0,j_0}$, so Proposition 5.4 implies that Claim 1. holds.

For every $Q \subset I_r$, with $Q \in G_I$ we have that $h_r^{-1}(Q)$ is a $\lambda_{DGD}^1$-regular domain of $(J, \mu, H)$, so we can consider the corresponding families $F_k(h_r^{-1}(Q)) \subset H^k$, $k \geq k_0(h_r^{-1}(Q), H)$. Since
\[ 1_{h_r^{-1}(Q)} = \sum_{k \geq k_0(h_r^{-1}(Q), H)} \sum_{W \in F_k(h_r^{-1}(Q))} 1_W. \]

Claim 3. We have the following limit on $L^1(\mu)$
\[ \Phi_r(a_Q) = \lim_{K \to \infty} \sum_{k \leq K} \sum_{W \in F_k(h_r^{-1}(Q))} \Phi_r(a_Q)1_W. \]

Note that this limit holds pointwise almost everywhere since $\Phi_r(a_Q)$ vanishes outside $h_r^{-1}(Q)$. Furthermore since $\Phi_r(a_Q) \in L^1(\mu)$ and the elements of
\[ \bigcup_{k \geq k_0(h_r^{-1}(Q), H)} F_k(h_r^{-1}(Q)) \]
are pairwise disjoint we have
\[ \sum_{k \geq k_0(h_r^{-1}(Q), H)} \sum_{W \in F_k(h_r^{-1}(Q))} |\Phi_r(a_Q)1_W|_1 \leq |\Phi_r(a_Q)|_1 < \infty. \]

In particular the sequence in Claim 3 converges absolutely in $L^1(\mu)$ to $\Phi_r(a_Q)$. This proves Claim 3. So again by Proposition 5.4, we reduce the proof of Claim 2. to the following claim

Claim 4. For every $r_0$ and $j_0$ we have that
\[ \phi_{i_0,j_0,K} = \sum_{r \leq r_0} \sum_{j \leq j_0} \sum_{Q \in G_I, Q \subset I_r} \sum_{k \leq K} \sum_{W \in F_k(h_r^{-1}(Q), H)} c_Q \Phi_r(a_Q)1_W. \]
has a $B_{p,q}(A^{bw}_{p,q})$-representation
\[ \phi_{r_0,j_0,K} = \sum_{k} \sum_{W \in P^k} s_W b_W \]
satisfying \((17.24)\).
Note that for every pair \((Q, W)\) such that \(W \subset h_r^{-1}(Q)\), we have
\[
\Phi_r(aQ)1_W = g \cdot aQ \circ h_r1_W = g \cdot |Q|^{s-1/p}1_Q \circ h_r1_W = g|Q|^{s-1/p}1_W.
\]
So by (8.12)
\[
|\Phi_r(aQ)1_W|_{b^{\beta,p,q}_{r,*}(W;|W|,\mathcal{A}_{\beta,p,q})} \leq C_{DRP}^r \left( \frac{|Q|}{|h_r^{-1}(Q)|} \right)^{1/p-s+\epsilon_r} |W|^{1/p-\beta} |Q|^{s-1/p}.
\]
If \(W \in \mathcal{F}^k(h_r^{-1}(Q))\), \(Q \subset I_r\) and \(c_Q^r \neq 0\), define
\[
b_{Q,W}(x) = \frac{1}{C_{DRP}^r} \left( \frac{|h_r^{-1}(Q)|}{|Q|} \right)^{1/p-s+\epsilon_r} |Q|^{1/p-s} |W|^{s-1/p} \Phi_r(aQ)1_W,
\]
and
\[
d_{Q,W} = C_{DRP}^r \frac{|Q|}{|h_r^{-1}(Q)|} \frac{|W|^{1/p-s}}{|Q|^{1/p-s}}.
\]
Otherwise define \(d_{Q,W} = 0\) and \(b_{Q,W}(x) = 0\) everywhere.

**Claim 5.** \(b_{Q,W}\) is a \(A^{\beta,p,q}_{r,*}\)-atom supported on \(W\).

Indeed
\[
|b_{Q,W}|_{b^{\beta,p,q}_{r,*}(W;|W|,\mathcal{A}_{\beta,p,q})} \leq \left( \frac{|h_r^{-1}(Q)|}{|Q|} \right)^{1/p-s+\epsilon_r} \left( \frac{|Q|}{|h_r^{-1}(Q)|} \right)^{1/p-s+\epsilon_r} |W|^{1/p-\beta} |W|^{s-1/p}
\]
\[
\leq |W|^{1/p-\beta} |W|^{s-1/p} = |W|^{s-\beta}.
\]
This proves Claim 5. We have
\[
c_Q^r \Phi_r(aQ)1_W = c_Q^r d_{Q,W} b_{Q,W}.
\]
For every \(W \in \mathcal{H}^k\) and \(r \in \Lambda\) and \(j \in \mathbb{N}\) there exists at most one set \(Q_j^r(W) \subset I_r\) such that \(Q_j^r(W) \in \mathcal{G}_r^j, W \subset h_r^{-1}(Q_j^r(W))\) and \(W \in \mathcal{F}^k(h_r^{-1}(Q_j^r(W)))\). If such set does not exist define \(Q_j^r(W) = \emptyset\). We have
\[
\Phi_{i_0,j_0,K} = \sum_{r \leq r_0, j \leq j_0} \sum_{Q \in \mathcal{D}^j, Q \subset I_r} \sum_{k \leq K} \sum_{h_r \circ (h_r^{-1}(Q)) \in \mathcal{F}^k(h_r^{-1}(Q))} c_Q^r \Phi_r(aQ)1_W
\]
\[
= \sum_{r \leq r_0, j \leq j_0} \sum_{Q \in \mathcal{D}^j, Q \subset I_r} \sum_{k \leq K} \sum_{h_r \circ (h_r^{-1}(Q)) \in \mathcal{F}^k(h_r^{-1}(Q))} c_Q^r d_{Q,W} b_{Q,W}
\]
\[
= \sum_{r \leq r_0, j \leq j_0} \sum_{W \in \mathcal{H}^k} \sum_{k \leq K} c_{Q_j^r(W)}^r d_{Q_j^r(W)} b_{Q_j^r(W),W}
\]
\[
= \sum_{W \in \mathcal{H}^k} \sum_{r \leq r_0} \sum_{j \leq j_0} c_{Q_j^r(W)}^r d_{Q_j^r(W)} b_{Q_j^r(W),W}
\]
Define
\[
s_W = \sum_{r \leq r_0} \sum_{j \leq j_0} |c_{Q_j^r(W)}^r d_{Q_j^r(W)},W|
\]
Note that these sums have a finite number of terms. Then

\[ b_W = \frac{1}{s_W} \sum_{r \leq r_0} \sum_{j \leq j_0} c_{Q^r_j}^r(W) d_{Q^r_j}^r(W) b_{Q^r_j}^r(W), \]

is a \( \mathcal{A}_{\beta,p,q}^b \)-atom supported on \( W \), since it is a convex combination of \( \mathcal{A}_{\beta,p,q}^b \)-atoms.

We obtained a \( B_{p,q}^s(J, \mu, \mathcal{H}, \mathcal{A}_{\beta,p,q}^b) \)-representation

\[ \phi_{i_0,j_0,K} = \sum_{k \leq K} \sum_{W \in \mathcal{H}^k} s_W b_W. \]

Now it remains to prove (17.24). Recall that \( \lambda_{DRS}^r \max\{(\lambda_{DC2}^r\)^\gamma, (\lambda_{DGD2}^r\)^{1/p}\} \).

**Claim 6.** If \( Q \in \mathcal{P}^i \) and \( W \in \mathcal{F}^i(h^{-1}_r(Q)) \) then

\[ (17.25) \quad (\lambda_{DRS2}^r)^{|j-k_0(h^{-1}_r(Q))|}(\lambda_{DRS2}^r)^{|i-k_0(h^{-1}_r(Q))|} \leq (\lambda_{DRS2}^r)^\gamma |j-i| + (1-\gamma) a_r. \]

Indeed, note that if \( i, j, k \in \mathbb{N} \), with \( |j - k| \geq a \geq 0 \) and \( i \geq k \) then

\[ i - k + |j - k| = |i - k| + |j - k| = \gamma(|i - k| + |j - k|) + (1 - \gamma)(|i - k| + |j - k|) \geq \gamma |k - j| + (1 - \gamma) a. \]

In particular since

\[ |j - k_0(h^{-1}_r(Q))| = |k_0(Q) - k_0(h^{-1}_r(Q))| \geq a_r \]

and \( W \in \mathcal{F}^i(h^{-1}_r(Q)) \) implies \( i \geq k_0(h^{-1}_r(Q)) \) we have (17.25). This proves Claim 6.
Since $h^{-1}(Q_j^r(W))$ is a $(1 - sp, C_{DG_{D1}}, \lambda_{DG_{D2}})$-regular set by (6.10) and (7.11), if (10.16) holds then for every $i \leq K$

\[
\left( \sum_{i} \left( \sum_{W \in \mathcal{H}^i} |s_W|^p \right)^{1/p} \right)^{1/q} \leq \frac{2}{1 - \lambda_{DRS_2}} \left( \sum_{j} \left( \sum_{r \leq r_0} (C_{DG_{D1}})^{r} C_{DRP} (C_{DG_{D1}})^{1/p} \lambda_{DRS_2}^{1 - sp} (1 - \gamma) a_{r} \left( \sum_{Q \in \mathcal{G}_{h}^j} |c_Q|^p \right)^{1/p} \right) \cdot \left( \sum_{k \in \mathcal{P}_{k}} |d_Q|^p \right)^{q/p} \right)^{1/q}.
\]
The case \( q = \infty \) is similar. On the other hand, if (10.17) holds then for every \( i \leq K \)

\[
\left( \sum_{W \in \mathcal{H}^i} |s_W|^p \right)^{1/p} \leq \left( \sum_{W \in \mathcal{H}^i} \left( \sum_{r \leq r_0} \sum_{j \leq j_0} \left| c_{Q_j}^r(W) d_{Q_j}^r(W) \right|^p \right)^{1/p} \right.
\]

\[
\leq \sum_{j \leq j_0} \left( \sum_{W \in \mathcal{H}^i} \left( \sum_{r \leq r_0} \left| c_{Q_j}^r(W) d_{Q_j}^r(W) \right|^p \right)^{1/p} \right.
\]

\[
\leq N^{1/p} \sum_{j \leq j_0} \left( \sum_{W \in \mathcal{H}^i} \sum_{r \leq r_0} (C_{DRP}^r)^p \left( \frac{|Q_j^r(W)|}{|h_r^{-1}(Q_j^r(W))|} \right)^{1-s_p} \left| c_{Q_j}^r(W) \right|^p \right)^{1/p}
\]

\[
\leq N^{1/p} \sum_{j \leq j_0} \left( \sum_{W \in \mathcal{H}^i} \sum_{r \leq r_0} (C_{DC}^r)^p (C_{DRP}^r)^p \left( \frac{|W|}{|h_r^{-1}(Q_j^r(W))|} \right)^{1-s_p} \left| c_{Q_j}^r(W) \right|^p \right)^{1/p}
\]

\[
\leq N^{1/p} \sum_{j \leq j_0} \left( \sum_{r \leq r_0} \sum_{Q \in \mathcal{Q}^j_{DC}} (C_{DC}^r)^p (C_{DRP}^r)^p \left( \frac{|W|}{|h_r^{-1}(Q_j^r(W))|} \right)^{1-s_p} \left| c_{Q_j}^r(W) \right|^p \right)^{1/p}
\]

\[
\leq N^{1/p} \sum_{j \leq j_0} \left( \sum_{r \leq r_0} \left( (C_{DC}^r)^p (C_{DRP}^r)^p \sum_{Q \in \mathcal{Q}^j_{DC}} (\lambda_{DC}^r)^{1-\alpha_0(h_r^{-1}(Q_j^r(W)))} \sum_{i \geq k_0(h_r^{-1}(Q_j^r(W)))} |c_{Q_j}^r(W)| \right)^{1/p} \right)^{1/p}
\]

\[
\leq N^{1/p} \sum_{j \leq j_0} \left( \sum_{r \leq r_0} \left( (C_{DC}^r)^p (C_{DRP}^r)^p \sum_{Q \in \mathcal{Q}^j_{DC}} (\lambda_{DC}^r)^{1-\alpha_0(h_r^{-1}(Q_j^r(W)))} \sum_{i \geq k_0(h_r^{-1}(Q_j^r(W)))} |c_{Q_j}^r(W)| \right)^{1/p} \right)^{1/p}
\]

This is again a convolution, so for \( q \in [1, \infty) \) we have

\[
\left( \sum_{i \leq j_0} \left( \sum_{W \in \mathcal{H}^i} |s_W|^p \right)^{q/p} \right)^{1/q} \leq N^{1/p} \sum_{j \leq j_0} \left( \sum_{r \leq r_0} \left( (C_{DC}^r)^p (C_{DRP}^r)^p \sum_{Q \in \mathcal{Q}^j_{DC}} (\lambda_{DC}^r)^{1-\alpha_0(h_r^{-1}(Q_j^r(W)))} \sum_{i \geq k_0(h_r^{-1}(Q_j^r(W)))} |c_{Q_j}^r(W)| \right)^{q/p} \right)^{1/q}
\]

\[
\leq N^{1/p} \left( \sum_{j \leq j_0} \left( \sum_{r \leq r_0} \left( (C_{DC}^r)^p (C_{DRP}^r)^p \sum_{Q \in \mathcal{Q}^j_{DC}} (\lambda_{DC}^r)^{1-\alpha_0(h_r^{-1}(Q_j^r(W)))} \sum_{i \geq k_0(h_r^{-1}(Q_j^r(W)))} |c_{Q_j}^r(W)| \right)^{q/p} \right)^{1/q} \right)^{1/p}
\]

\[
\sum_{k \leq k_0} \left( \sum_{Q \in \mathcal{Q}^j_k} |d_Q|^p \right)^{q/p} \right)^{1/q}
\]

and the case \( q = \infty \) is similar.

17.3. **Step 3. Going back to \( B_{p,q}^s \).** By Proposition 5.6 there is a \( B_{p,q}^s \)-representation

\[
\Phi(f) = \sum_{Q \in \mathcal{H}^s} \sum_{k \leq k_0} z_Q a_Q
\]
such that

\[
\left( \sum_i \left( \sum_{Q \in H} |z_Q|^p \right)^{q/p} \right)^{1/q} \leq C_{GBS} \left( \sum_i \left( \sum_{W \in H} |s_W|^p \right)^{q/p} \right)^{1/q} \\
\leq \frac{2C_{GBS}C_{DRS}}{1 - \lambda_{DRS}^2} \left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |d_Q|^p \right)^{q/p} \right)^{1/q}
\]

This completes the proof of Theorem 11.1.

18. Controlling the Essential Spectral Radius

Proof of Corollary 13.1. Given \( R \subset \mathcal{P} \), Proposition 5.5 tells us that \( \pi_G(f) = \sum_k \sum_{P \in \mathcal{P}^k, P \in R} k^f_P a_P \) is a \( B_{p,q}^s(I, m, \mathcal{P}) \)-representation of \( \pi_R(f) \) satisfying

\[
\left( \sum_k \left( \sum_{P \in \mathcal{P}^k, P \in R} |k^f_P|^p \right)^{q/p} \right)^{1/q} \leq C_{GC} |f|_{B_{p,q}^s}.
\]

If the pair \( (I, \sum_k \sum_{P \in \mathcal{P}^k, P \in G} k^f_P a_P) \),

has a \((C, \gamma)\)-regular slicing then by Theorem 11.1 we have that

\[
|\Phi(\pi_R(f))|_{B_{p,q}^s} \leq C_{GBS} C_D C_{GC} |f|_{B_{p,q}^s}.
\]

Applying this inequality for \( R = \mathcal{P}' \) and \( R = \mathcal{P} \setminus \mathcal{P}' \) we conclude the proof. \( \square \)

V. Positive Transfer Operators.

19. Lasota-Yorke Inequality and the Dynamics of \( \Phi \)

Proof of Theorem 14.1. Note that

\[
\pi_{\mathcal{P}'}(f) = \sum_k \sum_{P \in \mathcal{P}^k, P \in \mathcal{P}'} k^f_P a_P
\]

is a \( B_{p,q}^s \)-representation of \( \pi_{\mathcal{P}'}(f) \). Moreover \( f \mapsto k^f_P \) is a bounded linear functional in \((L^1)^*\). Denote its norm by \( |k^f_P|_{(L^1)^*} \). So since \( \mathcal{P}' \) is finite

\[
\left( \sum_k \left( \sum_{P \in \mathcal{P}^k, P \in \mathcal{P}'} |k^f_P|^p \right)^{q/p} \right)^{1/q} \leq \left( \sum_k \left( \sum_{P \in \mathcal{P}^k, P \in \mathcal{P}'} |k^f_P|_{(L^1)^*}^p \right)^{q/p} \right)^{1/q} |f|_1.
\]

The pair

\[
(I, \sum_k \sum_{P \in \mathcal{P}^k, P \in G} k^f_P a_P),
\]
has a \((C_{DRSFR}, \gamma)\)-regular slicing so by Theorem 11.1 we have that
\[
|\Phi(\pi_P(f))|_{B_{p,q}} \leq C_{GBS} C_{D_{DRSFR}} |f|_1.
\]
Consequently Corollary 13.1 gives
\[
|\Phi(f)|_{B_{p,q}} \leq |\Phi(\pi_P(f))|_{B_{p,q}} \leq C_{GBS} C_{D_{DRSFR}} |f|_{B_{p,q}}.
\]
Using that \(|\Phi(f)|_1 \leq |f|_1\) and \(C_{GBS} C_{D_{DRSFR}} < 1\) one can easily get the Lasota-Yorke inequality for \(\Phi^n\).

**Proof of Corollary 14.1.** The methods we are going to use here are sort of standard, however we provided them for the sake of compactness.

**Proof of A.** Since the essential spectral radius of \(\Phi\) is at most
\[
C_{GBS} C_{D_{DRSES}} < 1
\]
we have that every
\[
\lambda \in \sigma_{B_{p,q}}(\Phi)
\]
satisfying \(|\lambda| \geq (C_{GBS} C_{D_{DRSES}})^{1/2}\) is an isolated point of the spectrum that is an eigenvalue with finite-dimensional generalized eigenspace. We claim that the spectral radius \(r_{B_{p,q}}\) of \(\Phi\) is 1. Note that Lasota-Yorke inequality implies
\[
\sup_n |\Phi^n(f)|_{B_{p,q}} < \infty
\]
so \(r_{B_{p,q}} \leq 1\). On the other hand if \(r_{B_{p,q}} < 1\) then
\[
\lim_n \int |\Phi^n(f)| \ dm = 0
\]
for every \(f \in B_{p,q}^s\), but this is impossible since
\[
\int |\Phi^n(1)| \ dm = 1
\]
for every \(n\). Moreover if \((\Phi - \lambda I)^2 f = 0\) but \((\Phi - \lambda I)f \neq 0\), with \(|\lambda| = 1\), then \(|\Phi^n f|_{B_{p,q}}\) diverges to infinity. This it is impossible. In particular if \(\delta \in (0, 1)\) is close enough to 1 we have that \(E\) is finite, non-empty and contained in \(\mathbb{S}^1\).

**Proof of B.** This follows easily from A. using arguments with spectral projections, since the spectral projections on the generalized eigenspace of \(\lambda \in E\) is indeed a projection on the eigenspace of \(\lambda\).

**Proof of C.** Let \(f \geq 0\) with \(f \in B_{p,q}^s\). The Lasota-Yorke inequality implies that there is \(N\) such that
\[
|\Phi^n(f)|_{B_{p,q}} \leq 2C |f|_1
\]
for every \(n \geq N\). The ball of center 0 and radius \(2C\) in \(B_{p,q}^s\) is compact in \(L^1\), so we can find a convergent subsequence in \(L^1\)
\[
\rho = \lim_k \frac{1}{n_k} \sum_{n < n_k} \Phi^n(f),
\]
with \( \rho \in B_{p,q}^s \). Of course the positivity of \( \Phi \) implies \( \rho \geq 0 \). Note also that \( \Phi(\rho) = \rho \) and \( \int \rho \ dm = \int f \ dm \). We conclude the proof choosing \( f = 1 \).

**Proof of D.** Let \( u \in L^1 \) such that \( \Phi(u) = \lambda u \), with \( |\lambda| = 1 \). Since \( B_{p,q}^s \) is dense in \( L^1 \) one can choose \( u_i \in B_{p,q}^s \) such that \( u_i \) converges to \( u \) in the topology of \( L^1 \). Due the Lasota-Yorke inequality for every large \( i \) there is \( n_i \geq i \) such that

\[
|\Phi^n(u_i)|_{B_{p,q}^s} \leq 2C|u|_1
\]

for \( n \geq n_i \). Here \( C \) depends only on \( \Phi \). Note also that

\[
\lim_i |\Phi^{n_i}(\frac{1}{\lambda^{n_i}} u_i) - u|_1 = 0.
\]

The ball of center 0 and radius \( 2C|u|_1 \) in \( B_{p,q}^s \) is compact in \( L^1 \), so \( u \in B_{p,q}^s \) and \( |u|_{B_{p,q}^s} \leq 2C|u|_1 \).

Finally note that \( \Phi(|u|)(x) \geq |u|(x) \) almost everywhere. On the other hand

\[
\int \Phi(|u|) \ dm = \int |u| \ dm,
\]

so \( \Phi(|u|) = |u| \) almost everywhere. Denote

\[
s(x) = (sgn \ u)(x) \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0. \end{cases}
\]

Note that

\[
\sum_r g_r(x)s(h_r(x))|u|(h_r(x)) = \lambda s(x)|u|(x).
\]

and

\[
\sum_r g_r(x)|u|(h_r(x)) = |u|(x).
\]

So

\[
|\sum_r g_r(x)s(h_r(x))|u|(h_r(x))| = \sum_r g_r(x)|u|(h_r(x)).
\]

which implies that \( s(h_r(x)) = s(h_{r'}(x)) \) for every \( r, r' \) such that \( x \in J_r \cap J_{r'} \) and \( g_r(x)g_{r'}(x) \neq 0 \) and consequently \( s(h_r(x)) = \lambda s(x) \) for \( x \in J_r \) satisfying \( g_r(x) \neq 0 \).

In particular \( s^k(h_r(x)) = \lambda^k s^k(x) \) under the same conditions, with \( k \in \mathbb{Z} \) (here we define \( s^0(x) = x \) whenever \( s(x) = 0 \), and it is easy to see that \( \Phi(s^k|u|) = \lambda^k s^k|u| \).

Of course \( u_k = s^k|u| \in L^1 \), so \( u_k \in B_{p,q}^s \).

**Proof of E.** If \( \lambda \in E \) is not a root of unit then \( \lambda^k \neq \lambda^{k'} \) for \( k \neq k' \). So \( \{\lambda^k\}_{k \in \mathbb{N}} \) is an infinite set. But by D. this set is contained in \( E \), that is finite. This is a contradiction.

**Lemma 19.1.** Let \( \mu \) be a finite invariant measure of \( T \) such that \( \mu \) is absolutely continuous with respect to \( m \). Let \( \Omega_\mu = \{x: \ \rho(x) > 0\} \), where \( \rho \) is the density of \( \mu \) with respect to \( m \). Then there is ergodic probability measure \( \hat{\mu} \), absolutely continuous with respect to \( m \), such that \( \Omega_\mu \subseteq \Omega_{\hat{\mu}} \).

**Proof.** Suppose that such \( \hat{\mu} \) does not exist. Then it is easy to construct an infinite sequence of subsets \( \Omega_\mu = A_0 \supset A_1 \supset A_2 \supset \cdots \) such that \( m(A_{i+1}) < m(A_i) \) and \( \mu_i(S) = \mu(S \cap (A_i \setminus A_{i+1})) \) is a (no vanishing) finite invariant measure for \( T \). Note that \( \Omega_{\mu_i} \cap \Omega_{\mu_{i+1}} = 0 \), so if \( \mu_i = \rho_i \) then by Corollary 14.1.D we have that \( \{\rho_i\}_i \)
is linearly independent family of functions in $B_{p,q}^s$, satisfying $\Phi(\rho_i) = \rho_i$, so the $1$-eigenspace has infinite dimension. This contradicts Corollary 14.1.A.

\textbf{Proof of Corollary 14.2.} If $\mu_i = \rho_i$ m, with $\rho_i \geq 0$, $i = 1, \ldots, n$, are distinct ergodic invariant probabilities of $T$ then by Corollary 14.1.D we have that $\rho_i \in B_{p,q}^s$. Since they are ergodic and distinct we have that the sets $\Omega_i = \{x \in I: \rho_i(x) > 0\}$ are pairwise disjoint, so $\rho_i$ are linearly independent on $B_{p,q}^s$. Since these functions belong to the $1$-eigenspace of $\Phi$ we have that $n$ is bounded by the (finite) dimension of this eigenspace. Form now one let $\mu_i$, $i = 1, \ldots, n$ be the list of all distinct ergodic invariant probabilities of $T$ absolutely continuous with respect to $m$. We claim that $\Omega^c = ( \cup_i \cup_j \geq 0 T^{-j}(\Omega_i))^c$ satisfies $m(\Omega^c) = 0$. Indeed, otherwise consider $u = 1_{\Omega^c}$. Now we use an argument similar to the proof of Corollary 14.1.D. Since $B_{p,q}^s$ is dense in $L^1$ one can choose $u_i \in B_{p,q}^s$ such that $|u_i^1 - u|_1 \leq \frac{1}{i}$

Due the Lasota-Yorke inequality for every large $i$ there is $n_i$ such that

$$|\Phi^{n_i}(u_i)|_{B_{p,q}^s} \leq 2C|u|_1$$

for $n \geq n_i$, so there is $N_i$ such that for $N \geq N_i$ we have

$$\frac{1}{N} \sum_{n \leq N} \Phi^{n}(u_i)|_{B_{p,q}^s} \leq 3C|u|_1$$

Here $C$ depends only on $\Phi$. The ball of center 0 and radius $3C|u|_1$ in $B_{p,q}^s$ is compact in $L^1$, so we can use the Cantor diagonal argument to show that there is a sequence $M_k$ and $v_i \in B_{p,q}^s$ such that $|v_i|_{B_{p,q}^s} \leq 3C|u|_1$ satisfying

$$\lim_k \frac{1}{M_k} \sum_{n \leq M_k} \Phi^{n}(u_i) - v_i|_1 = 0$$

for every $i$, and $\Phi(v_i) = v_i$, $v_i \geq 0$. Using a similar argument we can assume that $\lim_i v_i = v$ on $L^1$, with $v \in B_{p,q}^s$. Note also that

$$\frac{1}{M_k} \sum_{n \leq M_k} \Phi^{n}(u_i) - \frac{1}{M_k} \sum_{n \leq M_k} \Phi^{n}(u)|_{L^1} \leq \frac{1}{i}$$

so we conclude that

$$\lim_i \frac{1}{M_k} \sum_{n \leq M_k} \Phi^{n}(u) - v|_1 = 0.$$
20. Positivity, invariant measures and decay of correlations

Proposition 20.1. Assume $A_1 = A_6$. Then there is $C_{14}$ that depends only on $\mathcal{P}'$ such that the following holds. Suppose that $f \in \mathcal{B}_{p,q}$ has a $\mathcal{B}_{p,q}$-representation

$$f = \sum_{Q \in \mathcal{P}} d_Q^0 a_Q$$

such that $d_Q^0 \geq 0$ for every $Q \in \mathcal{P}$. Then $\Phi^i(f)$ has a $\mathcal{B}_{p,q}$-representation

$$\sum_{Q \in \mathcal{P}} d_Q^i a_Q$$

with $d_Q^i \geq 0$ satisfying

$$\left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |d_Q^0|^{|q/p|} \right)^{1/q} \right)^{1/q} \leq C_{GBS} C_D C_{DRSFR} C_{14} |f|_1 + (C_{GBS} C_D C_{DRSFR})^i \left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |d_Q^0|^{|q/p|} \right)^{1/q} \right)^{1/q}.$$

Proof. This proof is similar to the proof of Theorem 14.1. Consider the function $f_1 \in \mathcal{B}_{p,q}$ given by $\mathcal{B}_{p,q}$-representation

$$f_1 = \sum_k \sum_{P \in \mathcal{P}^k, P \in \mathcal{P}'} d_P^0 a_P,$$

Note that for $P \in \mathcal{P}'$

$$|d_P^0| = |P|^{s-1/p+1} \int d_P^0 a_P \ dm \leq |P|^{s-1/p+1} \int f_1 \ dm = |P|^{s-1/p+1} |f_1|_1.$$

So since $\mathcal{P}'$ is finite there is $C_{14}$, that depends only on $\mathcal{P}'$, such that

$$\left( \sum_k \left( \sum_{P \in \mathcal{P}^k, P \in \mathcal{P}'} |d_P^0|^{|q/p|} \right)^{1/q} \right)^{1/q} \leq C_{14} |f_1|_1.$$

The pair

$$(\mathcal{I}, \sum_k \sum_{P \in \mathcal{P}^k, P \in \mathcal{P}'} d_P^0 a_P),$$

has a $(C_{DRSFR}, \gamma)$-regular slicing so by Theorem 11.1 there is a $\mathcal{B}_{p,q}$-representation

$$\Phi(g) = \sum_k \sum_{Q \in \mathcal{P}} d_Q^i a_Q$$

with $d_Q^i \geq 0$ for every $Q$, such that

$$\left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |d_Q^i|^{|q/p|} \right)^{1/q} \right)^{1/q} \leq C_{GBS} C_D C_{DRSFR} C_{14} |f_1|_1.$$

Moreover consider $f_2 \in \mathcal{B}_{p,q}$ with $\mathcal{B}_{p,q}$-representation

$$f_2 = \sum_k \sum_{P \in \mathcal{P}^k, P \in \mathcal{P} \setminus \mathcal{P}'} d_P^0 a_P.$$

The pair

$$(\mathcal{I}, \sum_k \sum_{P \in \mathcal{P}^k, P \in \mathcal{P} \setminus \mathcal{P}'} d_P^0 a_P),$$
has a \((C_{\text{DRSES}}, \gamma)\)-regular slicing so by Theorem 11.1 there is a \(B_{p,q}^s\)-representation
\[
\Phi(f) = \sum_k \sum_{Q \in \mathcal{P}} d_Q'' a_Q
\]
with \(d_Q'' \geq 0\) for every \(Q\), such that
\[
\left( \sum_k \left( \sum_{Q \in \mathcal{P}_k} |d_Q'''|^{p'/p} \right)^{q/p} \right)^{1/q} \leq C_{\text{GBS}} C_{D_{\text{DRSES}}} \left( \sum_k \left( \sum_{Q \in \mathcal{P}} |d_Q^0|^{p'/p} \right)^{q/p} \right)^{1/q}
\]
and
\[
\left( \sum_k \left( \sum_{Q \in \mathcal{P} \setminus \mathcal{P}_k} |d_Q^0|^{p'/p} \right)^{q/p} \right)^{1/q} \leq C_{\text{GBS}} C_{D_{\text{DRSES}}} \left( \sum_k \left( \sum_{Q \in \mathcal{P}} |d_Q^0|^{p'/p} \right)^{q/p} \right)^{1/q}.
\]
Then \(\Phi(f)\) as a \(B_{p,q}^s\)-representation
\[
\Phi(f) = \sum_k \sum_{Q \in \mathcal{P}_k} d_Q^0 a_Q
\]
where \(d_Q = d_Q'' + d_Q'''\) satisfy
\[
\left( \sum_k \left( \sum_{Q \in \mathcal{P}_k} |d_Q^1|^{p'/p} \right)^{q/p} \right)^{1/q} \leq C_{\text{GBS}} C_{D_{\text{DRSF}}} C_{14} |f|_1 + C_{\text{GBS}} C_{D_{\text{DRSES}}},
\]
and the conclusion of the proposition easily follows by an induction argument on \(i\) with the above inequality and that fact that \(|\Phi(f)|_1 \leq |f|_1\) for every \(f \in L^1\).

**Corollary 20.1.** Suppose that \(f \in B_{p,q}^s\) has a \(B_{p,q}^s\)-representation
\[
f = \sum_{Q \in \mathcal{P}} d_Q^0 a_Q
\]
such that \(d_Q^0 \geq 0\) for every \(Q \in \mathcal{P}\). Then the set
\[
\{ \frac{1}{n+1} \sum_{i=0}^n \Phi^i(f) \}_{n \in \mathbb{N}}
\]
is pre-compact in \(L^p\) and every accumulation point \(\rho\) of this sequence belongs to \(B_{p,q}^s\) and it has a \(B_{p,q}^s\)-representation
\[
\rho = \sum_{Q \in \mathcal{P}} d_Q^\infty a_Q
\]
satisfying \(d_Q^\infty \geq 0\) for every \(Q \in \mathcal{P}\) and
\[
\left( \sum_k \left( \sum_{Q \in \mathcal{P}_k} |d_Q^\infty|^{p'/p} \right)^{q/p} \right)^{1/q} \leq C_{\text{GBS}} C_{D_{\text{DRSF}}} C_{14} |f|_1.
\]
Moreover
\[
\int \rho \, dm = \int f \, dm \quad \text{and} \quad \Phi(\rho) = \rho.
\]

**Proof.** The proof is quite similar to the proof of Corollary 14.1.C. \(\Box\)
21. Almost sure invariance principle.

Proof of Corollary 14.3. By [45] we have that $e^{iv_1(x)t} \in B_{p,\infty}^{1/p}$ whenever $v_1 \in B_{p,\infty}^{1/p}$ and the pointwise multiplier $M_t(w) = e^{iv_1(x)t}w$ is a bounded operator in $B_{p,q}^s$. We have that for $t$ small enough

$$\|M_tf\|_{B_{p,q}^s} = |(e^{ivt} - 1)f|_{B_{p,q}^s} + |f|_{B_{p,q}^s}$$

$$\leq \left| e^{ivt} - 1 \right|_{B_{p,\infty}^{1/p}} + |e^{ivt} - 1|_{\infty} + 1 |f|_{B_{p,q}^s}$$

$$\leq \left( \frac{|t||v_1|_{B_{p,\infty}^{1/p}}}{1 - \lambda_{G_2}^{-p}} + 2 |f|_{B_{p,q}^s} \right)$$

$$\leq \left( \frac{|t||v_1|_{B_{p,\infty}^{1/p}}}{1 - \lambda_{G_2}^{-p}} + 2 |f|_{B_{p,q}^s} \right)$$

$$\leq 3 |f|_{B_{p,q}^s}.$$

On the other hand if $v_2 \in M(B_{p,q}^s)$ then there is $C$ such that

$$|v_2f|_{B_{p,q}^s} \leq C |f|_{B_{p,q}^s},$$

and using the power series of $e^{iv_2t}$ one can easily see that $e^{iv_2t} \in M(B_{p,q}^s)$ and for $t$ small enough

$$e^{itv_2f} \in M(B_{p,q}^s).$$

Consequently for every $v \in B_{p,\infty}^{1/p} + M(B_{p,q}^s)$ and $t$ small enough

$$e^{itv} \in M(B_{p,q}^s).$$

Consider the operator $\Phi_t(f) = \Phi(e^{it}f)$. Note that

$$|\Phi_t(f)|_{1} \leq |f|_{1}.$$

By Theorem 14.1 we have that for $t$ small

$$|\Phi_t(f)|_{B_{p,q}^s} \leq C|f|_{1} + 6C_{GBS}C_{DRSES}C_{GC}|f|_{B_{p,q}^s},$$

for some $C \geq 0$ and every $f \in B_{p,q}^s$. So if $6C_{GBS}C_{DRSES}C_{GC} < 1$ we have

$$|\Phi^n_t(f)|_{B_{p,q}^s} \leq \frac{C}{1 - 6C_{GBS}C_{DRSES}C_{GC}} |f|_{1} + (6C_{GBS}C_{DRSES}C_{GC})^n |f|_{B_{p,q}^s},$$

for $t$ small enough and every $n$. In particular

$$|\Phi^n_t|_{B_{p,q}^s} \leq \frac{CK_0}{1 - 6C_{GBS}C_{DRSES}C_{GC}} + 1$$

for $t$ small enough and every $n$. Here $K_p$ is as in Proposition 5.2. By Corollary 14.1.B and the assumptions of Corollary 14.3 we have that the sequence $v, v \circ T, \ldots$ satisfies the assumptions (I) of Theorem 2.1 in Gouëzel [25]. This completes the proof. □
VI. HOW TO DO IT.

22. Dynamical Slicing.

Proposition 22.1. There is $C_{GSR}$, that depends only the good grid $\mathcal{P}$, with the following property. Let $\{I_r\}_{r \in \Lambda}$ be a countable family of pairwise disjoint $(1 - \beta p, C_4, t)$-strongly regular domains in $(I, m, \mathcal{P})$ and $\alpha_r > 0$, for every $r \in \Lambda$. Let

$$
g = \sum_{k \geq t} \sum_{Q \in \mathcal{P}^k} d_Q a_Q
$$

be a $\mathcal{B}_{p,q}$-representation, where $a_Q$ is the standard $(s, p)$-Souza’s atom supported on $Q$. Assume that

$$
T = \sup_{Q \in \mathcal{P}^k} \sum_{Q \cap I_r \neq \emptyset} \alpha_r.
$$

Then for every $r \in \Lambda$ there is a $\mathcal{B}_{p,q}$-representation

$$(22.27) \quad g \cdot 1_{I_r} = \sum_{k} \sum_{Q \in \mathcal{P}^k} c_r^Q a_Q,$$

satisfying

$$(22.28) \quad \left( \sum_{k} \left( \sum_{Q \in \mathcal{P}^k} |c_r^Q|^q \right)^{p/q} \right)^{1/q} \leq C_{GSR} T C_4^{1/p} \left( \sum_{k} \left( \sum_{Q \in \mathcal{P}^k} |d_Q|^q \right)^{p/q} \right)^{1/q}.
$$

Proof. If we apply Proposition 18.9 in S. [45] for the family of functions $\alpha_r 1_{I_r}$ we conclude that there is $C_{GSR}$, that depends only on the good grid $\mathcal{P}$, with the following property. There is a $\mathcal{B}_{p,q}$-representation

$$
g \cdot \alpha_r 1_{I_r} = \sum_{k} \sum_{Q \in \mathcal{P}^k} e_r^Q a_Q
$$

satisfying

$$
\left( \sum_{k} \left( \sum_{Q \in \mathcal{P}^k} |e_r^Q|^q \right)^{p/q} \right)^{1/q} \leq C_{GSR} T C_4^{1/p} \left( \sum_{k} \left( \sum_{Q \in \mathcal{P}^k} |d_Q|^q \right)^{p/q} \right)^{1/q}
$$

and if $e_r^Q \neq 0$ then $Q \subset I_r$, for some $r \in \Lambda$. Such $r$ in our case must be unique, since the sets in the family $\{I_r\}$ are pairwise disjoint. So if $Q \subset I_r$ define $c_r^Q = e_r^Q/\alpha_r$. It is easy to see that (22.27) and (22.28) hold.

Proof of Theorem 12.3. Let

$$
f = \sum_{k} \sum_{Q \in \mathcal{P}^k} d_Q a_Q
$$
be a $B_{p,q}^s$-representation, where $a_Q$ is the standard $(s,p)$-Souza’s atom supported on $Q$. Consider also the $B_{p,q}^s$-representations

$$f_1 = \sum_{k \geq t} \sum_{Q \in P^k} d_Q a_Q.$$

$$f_2 = \sum_{k < t} \sum_{Q \in P^k} d_Q a_Q.$$

**Step I.** We can apply Proposition 22.1 to the family $\{I_r\}_{r \in \Lambda}$, taking $g = f_1$, $\alpha_r = 1$ and $T = M$. So for each $r \in \Lambda$ there is a $B_{p,q}^s$-representation

$$f_1 \cdot 1_{I_r} = \sum_k \sum_{Q \in P^k} c_Q a_Q$$

satisfying

$$\left(\sum_k \left(\sum_{r \in \Lambda} \left(\sum_{Q \in p^k} |c_Q|^p\right)^{q/p} \right) \right)^{1/q} \leq C_{GSR} M C_5^{1/p} \left(\sum_k \left(\sum_{Q \in P^k} |d_Q|^p\right)^{q/p} \right)^{1/q}.$$

Note that

$$\left(\sum_j \left(\sum_{r \in \Lambda} \left(\sum_{Q \in P^k} |c_Q|^p\right)^{q/p} \right) \right)^{1/q} \leq \left(\sum_{r \in \Lambda} \Theta_{r'} \left(\sum_j \left(\sum_{Q \in P^t} |c_Q|^p\right)^{q/p} \right) \right)^{1/q}$$

$$\leq \left(\sum_{r \in \Lambda} \Theta_{r'} \right)^{1/p'} \left(\sum_j \left(\sum_{r \in \Lambda} \sum_{Q \in P^t} |c_Q|^p\right)^{q/p} \right)^{1/q}$$

$$\leq \left(\sum_{r \in \Lambda} \Theta_{r'} \right)^{1/p'} C_{GSR} M C_5^{1/p} \left(\sum_k \left(\sum_{Q \in P^k} |d_Q|^p\right)^{q/p} \right)^{1/q}.$$ 

So

$$(I_r, \sum_k \sum_{Q \in P^k} d_Q a_Q)$$

has a $C_{GSR} C_6$-slicing.

**Step II.** Note that since $I_r$, $r \in \Lambda$, is a $(1 - \beta p, C_5, t)$-strongly regular domain then $I_r$ is also a $(1 - \beta p, C_{15}, 0)$-strongly regular domain, where

$$C_{15} = C_3 \lambda_{G_1}^{-t}$$

depends only on the grid $P$, $C_5$ and $t$. Now apply Proposition 22.1 to the family $\{I_r\}_{r \in \Lambda}$, with $g = f_2$ and taking $\alpha_r = 1$ and $T = \# \Lambda$. We conclude that for every $i \in \Lambda$ there exists a $B_{p,q}^s$-representation

$$f_2 \cdot 1_{I_r} = \sum_k \sum_{Q \in P^k} c_Q a_Q$$

satisfying

$$\left(\sum_k \left(\sum_{r \in \Lambda} \left(\sum_{Q \in P^k} |c_Q|^p\right)^{q/p} \right) \right)^{1/q} \leq (\#\Lambda) C_{GSR} C_5^{1/p} \left(\sum_k \left(\sum_{Q \in P^k} |d_Q|^p\right)^{q/p} \right)^{1/q}.$$
The same argument as in Step I gives
\[
\left( \sum_j \left( \sum_{r \in \Lambda} \Theta_r \left( \sum_{Q \in P_j} |c_Q|^p \right)^{1/p} \right)^q \right)^{1/q} \\
\leq \left( \sum_{r \in \Lambda} \Theta_r^{p'} \left( |r| \right)^{1/p} C_{GSR} C_{15}^{1/p} \left( \sum_k \left( \sum_{Q \in P_k} |d_Q|^p \right)^{q/p} \right)^{1/q} \right),
\]
so we conclude that
\[
(\mathcal{I}, \sum_{k \geq t} \sum_{Q \in P_k} d_Q a_Q)
\]
has a $C_{GSR} C_7$-slicing. \[\square\]

**Proof of Theorem 12.4.** Let
\[
f = \sum_k \sum_{Q \in P_k} d_Q a_Q
\]
be a $\mathcal{B}_{p,q}$-representation, where $a_Q$ is the standard $(s,p)$-Souza’s atom supported on $Q$. Consider also the $\mathcal{B}_{s,p}$-representations
\[
f_1 = \sum_{k \geq t} \sum_{Q \in P_k} d_Q a_Q.
\]
\[
f_2 = \sum_{k < t} \sum_{Q \in P_k} d_Q a_Q.
\]

**Step I.** We can apply Proposition 22.1 to the family $\{I_r\}_{r \in \Lambda}$, taking $g = f_1$, $\alpha_r = \Theta_r$. So for each $r \in \Lambda$ there is a $\mathcal{B}_{p,q}$-representation
\[
f_1 \cdot 1_{I_r} = \sum_k \sum_{Q \subseteq I_r} c_Q a_Q
\]
satisfying
\[
N^{1/p'} \left( \sum_j \left( \sum_{r \in \Lambda} \Theta_r \left( \sum_{Q \in P_j} |c_Q|^p \right)^{1/p} \right)^q \right)^{1/q} \\
\leq N^{1/p'} TC_{GSR} C_5^{1/p} \left( \sum_k \left( \sum_{Q \in P_k} |d_Q|^p \right)^{q/p} \right)^{1/q}.
\]
So
\[
(\mathcal{I}, \sum_{k \geq t} \sum_{Q \in P_k} d_Q a_Q)
\]
has a $C_{GSR} C_8$-slicing.

**Step II.** As in Step II. of the proof of Proposition 12.3 we have that $I_r$, $r \in \Lambda$, is a $(1 - \beta p, C_{16}, 0)$-strongly regular domain, where
\[
C_{16} = C_5 \lambda_{G_1}^{-t}
\]
depends only on the grid \( P, C_5 \) and \( t \). Now apply Proposition 22.1 to the family \( \{ I_r \} \) for \( r \in \Lambda \), with \( g = f_2 \) and taking \( \alpha_r = \Theta_r \). We conclude that for every \( i \in \Lambda_1 \) there exists a \( B^*_{p,q} \)-representation

\[
 f_2 \cdot 1_{I_r} = \sum_k \sum_{Q \subset I_r} c_Q a_Q
\]
satisfying

\[
 N^{1/p'} \left( \sum_k \left( \sum_r \Theta_r \sum_{Q \subset I_r} \left| c_Q \right|^p \right)^{q/p} \right)^{1/q} \leq N^{1/p'} \left( \#\Lambda \right) (\sup_{r \in \Lambda} \Theta_r) C_{GSR} C_{16}^{1/p} \left( \sum_k \left( \sum_{Q \subset I_r} \left| d_Q \right|^p \right)^{q/p} \right)^{1/q},
\]

so we obtain that

\[
 (I, \sum_{k < t} \sum_{Q \subset I_r} d_Q a_Q)
\]
has a \( C_{GSR} C_9 \)-slicing.

**Proof of Theorem 12.5.** For every \( r \in \Lambda \) we apply (as usual) Proposition 22.1, this time for the family with a unique element \( \{ I_r \} \), with \( g = f \) and taking \( \alpha_r = 1 \) and \( T = 1 \). We conclude that for every \( r \in \Lambda \) there exists a \( B^*_{p,q} \)-representation

\[
 f \cdot 1_{I_r} = \sum_k \sum_{Q \subset I_r} c_Q a_Q
\]
satisfying

\[
 \left( \sum_k \left( \sum_{r \in \Lambda} \left( \sum_{Q \subset I_r} \left| c_Q \right|^p \right)^{q/p} \right)^{1/q} \right) \leq C_{GSR} C_{16}^{1/p} \left( \sum_k \left( \sum_{Q \subset I_r} \left| d_Q \right|^p \right)^{q/p} \right)^{1/q}.
\]

Of course

\[
 \left( \sum_k \left( \sum_{r \in \Lambda} \Theta_r \left( \sum_{Q \subset I_r} \left| c_Q \right|^p \right)^{1/p} \right)^{q/p} \right)^{1/q} \leq \sum_r \Theta_r \left( \sum_k \left( \sum_{Q \subset I_r} \left| c_Q \right|^p \right)^{q/p} \right)^{1/q} \leq C_{GSR} C_{16}^{1/p} \left( \sum_r \Theta_r \left( \sum_k \left( \sum_{Q \subset I_r} \left| d_Q \right|^p \right)^{q/p} \right)^{1/q} \right).
\]

**Proof of Theorem 12.6.** We apply again Proposition 22.1, this time for the family with a unique element \( \{ \Omega \} \), with \( \Omega = \bigcup_r I_r \), \( g = f \) and taking \( T = 1 \). We conclude that there exists a \( B^*_{p,q} \)-representation

\[
 f \cdot 1_{\Omega} = \sum_k \sum_{Q \subset I_r} c_Q a_Q
\]
satisfying
\[
\left( \sum_k \left( \sum_{Q \in P^k} |c_Q|^p \right)^{q/p} \right)^{1/q} \leq C_{GSN} C_{10}^{1/p} \left( \sum_k \left( \sum_{Q \in P^k} |d_Q|^p \right)^{q/p} \right)^{1/q}.
\]

if \( c_Q \neq 0 \) then \( Q \subset \Omega \), so by assumption \( Q \subset I_r \), for some \( r \in \Lambda \). Such \( r \) must be unique, since the sets in the family \( \{I_r\} \) are pairwise disjoint. So if \( Q \subset I_r \) define \( c_r Q = c_Q \).

Then
\[
f \cdot 1_{I_r} = \sum_k \sum_{Q \subset I_r} c_Q a_Q
\]
for every \( r \) and
\[
N^{1/p'} \left( \sum_k \left( \sum_r \Theta_r \sum_{Q \in P^k} |c_r Q|^p \right)^{q/p} \right)^{1/q}
\leq N^{1/p'} \left( \sup_r \Theta_r \right) \left( \sum_k \left( \sum_r \sum_{Q \in P^k} |c_r Q|^p \right)^{q/p} \right)^{1/q}
\leq N^{1/p'} \left( \sup_r \Theta_r \right) C_{GSN} C_{10}^{1/p} \left( \sum_k \Theta_r \left( \sum_r \sum_{Q \in P^k} |d_Q|^p \right) \right)^{1/q}.
\]

\[\Box\]

23. Boundness on Lebesgue spaces

We have that \( \Phi \), under very mild conditions, defines a bounded transformation from \( L^{t_0}(m) \) to \( L^1(\mu) \).

Corollary 23.1 (Boundeness on Lebesgue spaces). Let \( \epsilon' \) be such that \( \delta = \epsilon - \epsilon' \) and \( \epsilon, \epsilon' > 0 \). Suppose that
\[
\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3,
\]
with \( \Lambda_i \cap \Lambda_k = \emptyset \), for \( i \neq k \), such that
i. If \( i, j \in \Lambda_1 \), with \( i \neq j \), then \( I_i \cap I_j = \emptyset \). Moreover there is \( C_{17} \geq 0 \) such that for every \( i \in \Lambda_1 \) and \( Q \in P \) such that \( Q \subset I_i \)
\[
(23.29) \quad \sup(|g|, h_i^{-1}Q) \leq C_{17} \frac{|Q|}{|h_i^{-1}Q|}
\]
ii. We have
\[
(23.30) \quad \sup(|g|, h_r^{-1}Q) \leq C_{18} \left( \frac{|Q|}{|h_r^{-1}Q|} \right)^{1/p-s+\epsilon}
\]
for every \( Q \in P \) such that \( Q \subset I_r \), with \( r \in \Lambda_2 \cup \Lambda_3 \),
iii. We have
\[
(23.31) \quad \sum_{r \in \Lambda_2} \lambda |c_r|_{2/r} < \infty.
\]
iv. If \( i, j \in \Lambda_3 \), with \( i \neq j \), then \( I_i \cap I_j = \emptyset \). Moreover
\[
(23.32) \quad \sum_{r \in \Lambda_3} \lambda |c_r|_{2/r} < \infty,
\]
where \( t_0' \) satisfies \( 1/t_0' + 1/t_0 = 1 \).
For every \( f \in L^1 \) and \( r \in \Lambda \) consider the measurable functions \( \Phi_r(f) : I \rightarrow \mathbb{C} \) defined by

\[
\Phi_r(f) = g_r(x) f(h_r(x))
\]

if \( x \in J_r \) and \( \Phi_r(f)(x) = 0 \) otherwise. Then

A. For every \( r \in \Lambda_1 \) and \( f \in L^1(m) \) we have that \( \Phi_r(f) \) belongs to \( L^1(\mu) \),

\[
\Phi_r : L^1(m) \rightarrow L^1(\mu)
\]

is a bounded linear transformation and

\[
|\Phi_r(f)|_1 \leq C_{17} |f|_1.
\]

In particular

\[
\sum_{r \in \Lambda_1} |\Phi_r(f)|_1 \leq C_{17} |f|_1.
\]

B. For every \( r \in \Lambda_2 \cup \Lambda_3 \) and \( f \in L^{t_0}(m) \) we have that \( \Phi_r(f) \) belongs to \( L^{t_0}(\mu) \), and

\[
\Phi_r : L^{t_0}(m) \rightarrow L^{t_0}(\mu)
\]

is a bounded linear transformation and

\[
|\Phi_r(f)|_{t_0} \leq C_{18} C_{\dc}^{1|c'|} \lambda_{\dc}^{a_r|c'|} |f|_{t_0}.
\]

In particular

\[
\sum_{r \in \Lambda_2} |\Phi_r(f)|_1 \leq C_{18} C_{\dc}^{1|c'|} \left( \sum_{r \in \Lambda_2} \lambda_{\dc}^{a_r|c'|} \right) |f|_{t_0}
\]

and

\[
\sum_{r \in \Lambda_3} |\Phi_r(f)|_1 \leq C_{18} C_{\dc}^{1|c'|} \left( \sum_{r \in \Lambda_3} \lambda_{\dc}^{a_r|c'|} \right)^{1/t_0} |f|_{t_0}.
\]

C. In particular \( \Phi f \) is well defined and belongs to \( L^1(\mu) \) for every \( f \in L^{t_0}(m) \) and

\[
\Phi : L^{t_0}(m) \rightarrow L^1(\mu)
\]

is a continuous linear transformation, with

\[
|\Phi f|_1 \leq \sum_r |\Phi_r(f)|_1 \leq C_{19} |f|_{t_0}
\]

for some \( C_{19} \geq 0 \).

Proof. Recall

\( \mathcal{A} = \{ Q \in \mathcal{P}, \ Q \subset I_r, \) for some \( r \in \Lambda \} \cup \{ Q \in \mathcal{P}, \ Q \cap I_r = \emptyset, \) for every \( r \in \Lambda \} \).

If \( f \) is a (finite) linear combinations of characteristic functions of sets in \( \mathcal{A} \) we can write

\[
f1_{I_r} = \sum_{i \leq n} c_i 1_{Q_i},
\]

with \( Q_i \subset I_r \). We can assume that \( \{Q_i\}_i \) is a family of pairwise disjoint sets. In particular

\[
|f1_{I_r}|_{t_0}^0 = \sum_{i \leq n} |c_i|^{t_0} |Q_i|,
\]

and

\[
\Phi_r(f) = \sum_{i \leq n} c_i g^{1_{h_r^{-1}} Q_i},
\]
Then for every $r \in \Lambda_2 \cup \Lambda_3$

\[
\left| \sum_{i \leq n} c_i g \chi_{h_{r^{-1}}Q_i} \right|_{t_0} = \int \left| \sum_{i \leq n} c_i g \chi_{h_{r^{-1}}Q_i} \right|_{t_0} \, dm \\
= \int \sum_{i \leq n} \left| c_i \right|_{t_0} \left| g \right|_{t_0} \chi_{h_{r^{-1}}Q_i} \, dm \\
\leq C_{18}^{t_0} \sum_{i \leq n} \left| c_i \right|_{t_0} \left( \frac{|Q_i|}{|h_{r^{-1}}Q_i|} \right)^{(1/p' + \epsilon) t_0} |h_{r^{-1}}Q_i| \\
\leq C_{18}^{t_0} \sum_{i \leq n} \left| c_i \right|_{t_0} \left( \frac{|Q_i|}{|h_{r^{-1}}Q_i|} \right)^{1 + t_0'} |h_{r^{-1}}Q_i| \\
\leq C_{18}^{t_0} \sum_{i \leq n} \left| c_i \right|_{t_0} \left( \frac{|Q_i|}{|h_{r^{-1}}Q_i|} \right)^{t_0'} |Q_i| \\
\leq C_{18}^{t_0} C_{DC1}' \lambda_{DC2}^{a_r, \epsilon} |f|_{t_0} \\
\leq C_{18}^{t_0} C_{DC1}' \lambda_{DC2}^{a_r, \epsilon} |f|_{t_0}.
\]

Since linear combinations of characteristic functions of sets in $A$ are dense in $L^{t_0}$, we conclude that for every $f \in L^{t_0}$ and $r \in \Lambda_2 \cup \Lambda_3$ we have

\[
|\Phi_r(f)|_{t_0} \leq C_{18}^{t_0} C_{DC1}' \lambda_{DC2}^{a_r, \epsilon} |f|_{t_0}.
\]

So

\[
\sum_{r \in \Lambda_2} |\Phi_r(f)|_1 \leq \sum_{r \in \Lambda_2} |\Phi_r(f)|_{t_0} \leq C_{18}^{t_0} C_{DC1}' \sum_{r \in \Lambda_2} \lambda_{DC2}^{a_r, \epsilon} |f|_{t_0} \\
\leq C_{18}^{t_0} C_{DC1}' \left( \sum_{r \in \Lambda_2} \lambda_{DC2}^{a_r, \epsilon} \right) |f|_{t_0},
\]

and

\[
\sum_{r \in \Lambda_3} |\Phi_r(f)|_1 \leq \sum_{r \in \Lambda_3} |\Phi_r(f)|_{t_0} \leq C_{18}^{t_0} C_{DC1}' \sum_{r \in \Lambda_3} \lambda_{DC2}^{a_r, \epsilon} |f|_{t_0} \\
\leq C_{18}^{t_0} C_{DC1}' \left( \sum_{r \in \Lambda_3} \lambda_{DC2}^{a_r, \epsilon} \right)^{1/t_0} \left( \sum_{r \in \Lambda_3} |f|_{t_0} \right)^{1/t_0} \\
\leq C_{18}^{t_0} C_{DC1}' \left( \sum_{r \in \Lambda_3} \lambda_{DC2}^{a_r, \epsilon} \right)^{1/t_0} \left( \sum_{r \in \Lambda_3} |f|_{t_0} \right)^{1/t_0} \\
\leq C_{18}^{t_0} C_{DC1}' \left( \sum_{r \in \Lambda_3} \lambda_{DC2}^{a_r, \epsilon} \right)^{1/t_0} |f|_{t_0}.
\]
If \( r \in \Lambda_1 \) we have

\[
\sum_{i \leq n} c_i g_{1_{h^{-1}_r Q_i}} = \int \left| \sum_{i \leq n} c_i g_{1_{h^{-1}_r Q_i}} \right| \, dm
\]

\[
= \int \sum_{i \leq n} |c_i| g_{1_{h^{-1}_r Q_i}} \, dm
\]

\[
\leq C_{17} \sum_{i \leq n} |c_i| \left| Q_i \right| \left| h^{-1}_r Q_i \right|
\]

\[
\leq C_{17} \sum_{i \leq n} |c_i| \left| Q_i \right|
\]

\[
\leq C_{17} |f_{1_r}|_1.
\]

Since linear combinations \( f \) of characteristic functions of sets in \( G \) are dense in \( L^1 \), we conclude that for every \( f \in L^1 \) and \( r \in \Lambda_1 \)

\[
|\Phi_r(f)|_1 \leq C_{17} |f_{1_r}|_1,
\]

so

\[
\sum_{r \in \Lambda_1} |\Phi_r(f)|_1 \leq C_{17} \sum_{r \in \Lambda_1} |f_{1_r}|_1 \leq C_{17} |f|_1 \leq C_{17} |f|_{t_0}.
\]

\[ \square \]

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**INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, P. O. BOX 68530, 21945-970 RIO DE JANEIRO, BRAZIL.**

**DEPARTAMENTO DE MATEMÁTICA, INSTITUTO DE CIÊNCIAS MATEMÁTICAS E DA COMPUTAÇÃO-UNIVERSIDADE DE SÃO PAULO (ICMC/USP), CAIXA POSTAL 668, SÃO CARLOS-SP, BRAZIL.**

**E-mail address:** arbieto@im.ufrj.br
**E-mail address:** smania@icmc.usp.br
**URL:** http://conteudo.icmc.usp.br/pessoas/smania/