The number of functionally independent invariants of a pseudo–Riemannian metric

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Abstract. The number of functionally independent scalar invariants of arbitrary order of a generic pseudo–Riemannian metric on an $n$–dimensional manifold is determined.

1. Introduction

The goal of this work is to determine the number $i_{n,r}$ of functionally independent differential invariants of order $r$ of a generic pseudo–Riemannian metric $g$ on an $n$–dimensional manifold $N$. The results are as follows: for every $n \geq 1$, $i_{n,0} = i_{n,1} = 0$; for every $r \geq 0$, $i_{1,r} = 0$; $i_{2,2} = 1$, and for every $r \geq 3$, $i_{2,r} = \frac{1}{2}(r + 1)(r - 2)$; finally,

$$i_{n,r} = n + \frac{(r - 1)n^2 - (r + 1)n}{2(r + 1)} \binom{n + r}{r}, \text{ for every } n \geq 3, r \geq 2.$$ 

The theory of metric invariants is classic in both General Relativity and Riemannian Geometry. The standard approach to this topic relies on the definition of an invariant as a polynomial in the $g_{ij}$’s, their partial derivatives up to a certain order, say $\partial^{\alpha}g_{ij}/\partial x^{\alpha}$, $|\alpha| \leq r$, and in $|\text{det}(g_{ij})|^{-1}$, which is “natural” under diffeomorphisms (for example, see [1]). For scalar invariants the above definition does not allow one to pose the question of how many functionally independent invariants there are for each order since some of the functional relationships among invariants may be out of the ring that the above definition prescribes and furthermore, since that ring is not complete standard tools of analysis (as involutiveness, Frobenius theorem, etc.) cannot be applied either. From this point of view, the enumeration of the scalars constructed from the Riemann tensor of the Levi–Cività connection of a pseudo–Riemannian metric by means of covariant differentiation, tensor products and contractions has been discussed in some recent papers: in [5] the number of independent homogeneous scalar monomials of each
order and degree up to order 12 in the derivatives of the metric is determined and in [10] the same number is determined up to order 14. Apart from the interest and complexity of these results specially in relation to the so called Weyl invariants (cf. [4]) for the field theory it is clear that the determination of the number \( i_{n,r} \) is the most relevant fact since it provides the number of essentially different Diff(\( N \))–invariant Langrians of arbitrary order that exist in G.R. It seems thus natural to found the theory on the jet bundle notion of an invariant (cf. [7]) which avoids the aforementioned difficulties of the polynomial notion and translates the naturality condition into an authentic condition of invariance under the action of the group of diffeomorphisms of \( N \) on an appropriate jet bundle.

The plan of this paper is the following: In section 2 we introduce the notion of a metric invariant as well as that of an invariant Lagrangian density although for an oriented ground manifold \( N \) the latter is reduced to the former since the bundle of metrics over \( N \) is endowed with a canonical invariant zero order Lagrangian density, so that the emphasis is put on scalar invariants. The notion of invariance is related to a specific representation of the vector fields of \( N \) into vector fields of the \( r \)–jet bundle of metrics. Section 3 contains the explicit determination of this representation and its formulas are used throughout the paper. In section 4 we prove that on a dense open subset of the \( r \)–jet bundle the metric invariants coincide with the ring of first integrals of an involutive distribution which is obtained by linearizing the basic representation by means of a homomorphism of vector bundles, \( \Phi^r \). The number of invariants \( i_{n,r} \), is thus equivalent to know the rank of \( \Phi^r \). Sections 5, 6 and 7 are devoted to this aim distinguishing the different cases that appear according to the values of the order \( r \) of the jet bundle that we are considering and the dimension \( n \) of the ground manifold. Finally, section 8 contains the calculation of \( i_{n,r} \) and the comparison of \( i_{n,2} \) with the standard procedure (cf. [9]) in order to generate the second order metric invariants.

2. The notion of a metric invariant

Let \( N \) be an \( n \)–dimensional differentiable manifold. Given an integer \( 0 \leq n^+ \leq n \), we shall denote by \( p : \mathcal{M} = M_{n^+}(N) \rightarrow N \) the bundle of pseudo–Riemannian metrics on \( N \) of signature \( (n^+,n^–) \), \( n^– = n – n^+ \) (i.e., the global sections of \( p \) are exactly the pseudo–Riemannian metrics on \( N \) of signature \( (n^+,n^–) \) at each point). Let \( p_r : J^r(\mathcal{M}) \rightarrow N \) be the \( r \)–jet bundle of local sections of \( p \). The \( r \)–jet at a point \( x \in N \) of a metric \( g \) of \( \mathcal{M} \) will be denoted by \( j^r_x(g) \). For every \( r \geq s \), we also have a natural projection \( p_{rs} : J^r(\mathcal{M}) \rightarrow J^s(\mathcal{M}) \), \( p_{rs}(j^r_xg) = j^s_xg \). Let \((U; x_1, \ldots, x_n)\) be an open coordinate domain of \( N \) and let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a multi–index of non–negative integers. We set \( |\alpha| = \sum_1 \alpha_i \). The family of functions \((x_i \circ p_r, y^{j^k}_{\alpha})\), \( j \leq k, |\alpha| \leq r \), defined by \( y^{j^k}_{\alpha}(j^r_xg) = \left( \partial^{|\alpha|} g_{jk}/\partial x^{\alpha} \right)(x) \), where \( g_{jk} = g(\partial/\partial x_j, \partial/\partial x_k) \), constitutes a coordinate chart on \( p_r^{-1}U = J^r(p^{-1}U) \). We shall simply write \( y_{jk} \) instead of \( y^{j^k}_{0} \). Note that the functions \((x_i \circ p_r, y_{jk})\),
1 \leq i \leq n, 1 \leq j \leq k \leq n$, are a coordinate system on $p^{-1}(U)$. We shall also set $y^j_\alpha = y^k_\alpha$ for $j > k$.

Let $f : N \to N'$ be a diffeomorphism. We shall denote by $\mathcal{F} : \mathcal{M} \to \mathcal{M}'$, $\mathcal{M}' = \mathcal{M}_{n+1}(N')$, the natural lift of $f$ to the bundles of pseudo–Riemannian metrics; i.e., $\mathcal{F}(g_x) = (f^{-1})^*g_x$. Hence $p' \circ \mathcal{F} = f \circ p$.

The diffeomorphism $\mathcal{F} : \mathcal{M} \to \mathcal{M}'$ has a natural extension to jet bundles, $J^r(f) : J^r(\mathcal{M}) \to J^r(\mathcal{M}')$, defined as follows: $J^r(f)(j^r_x g) = j^r_{f(x)}(\mathcal{F}^*g \circ f^{-1})$.

Given a vector field $X \in \mathfrak{X}(N)$, we shall denote by $\overline{X}$ the natural lift of $X$ to $J^r(\mathcal{M})$. For $r = 0$ we shall simply write $\overline{X}$ instead of $\overline{X}^0$. Note that $\overline{X}$ is the natural lift to the bundle $\mathcal{M}$ of pseudo–Riemannian metrics of the vector field $X$. If $\varphi_t$ is the local flow of $X$, then $J^r(\varphi_t)$ is the local flow of $\overline{X}$. Hence $\overline{X}$ is projectable onto $\overline{X}^{r-1}$, and $\overline{X}$ is projectable onto $X$. The mapping $X \mapsto \overline{X}$, is a $\mathbb{R}$–linear injection and for every $X, Y \in \mathfrak{X}(N)$,

$$[\overline{X}, \overline{Y}] = [X, Y]' .$$

Hence we have a faithful representation of $\mathfrak{X}(N)$ into $\mathfrak{X}(J^r(\mathcal{M}))$.

**Definition 1.** A function $F \in C^\infty(J^r(\mathcal{M}))$ (which may be only defined on an open subset) is said to be a metric differential invariant of order $r$ if for every $X \in \mathfrak{X}(N)$, $\overline{X}^r F = 0$.

**Definition 2.** A function $F \in C^\infty(J^r(\mathcal{M}))$ is said to be a metric invariant of order $r$ if for every diffeomorphism $f : N \to N$, $F \circ J^r(f) = F$.

**Remark 3.** Metric invariants are a subring of the ring of metric differential invariants. In fact, the set of vector fields on $N$ with compact support, $\mathfrak{X}_c(N)$, is a dense ideal of $\mathfrak{X}(N)$ with respect to the $C^\infty$ topology and hence a function $F \in C^\infty(J^r(\mathcal{M}))$ is a metric differential invariant if and only if for every $X \in \mathfrak{X}_c(N)$, $\overline{X}^r F = 0$. This is equivalent to saying that for every $t \in \mathbb{R}$, one has $F \circ J^r(\phi_t) = F$, $\phi_t$ being the one–parameter group generated by $X$, and the last equation evidently holding for a metric invariant.

**Example 4.** Let $\nabla$ be the Levi–Civit\`a connection of a pseudo–Riemannian metric $g$ of $\mathcal{M}$, and $R$ the curvature tensor. Since $R$ is of type $(1, 3)$, for every $r \in \mathbb{N}$, $\nabla^2 R$ is a tensor field of type $(1, 2r + 3)$. Let us choose a sequence of $r + 1$ covariant indices $1 \leq i_0 < \ldots < i_r \leq 2r + 3$, and let us apply to them the isomorphism $g^{ij} : T^*_x(N) \to T_x(N)$, thus obtaining a tensor field $g^{ij} (\nabla^2 R)^{i_0 \ldots i_r}_{\ldots j_r}$ of type $(r+2, r+2)$. If further we choose a permutation $j_1, \ldots, j_{r+2}$ of its covariant indices then we can obtain an scalar by simply setting $S_g = c^{j_1}_{j_1} \cdots c^{j_{r+2}}_{j_{r+2}} (g^{ij} (\nabla^2 R)^{i_0 \ldots i_r})$, where $c^j_k$ stands for the contraction of the $i$–th contravariant index with the $j$–th covariant index. The value of $S_g$ at a point $x \in N$ only depends on $j^r_x g$, since the local coefficients $\Gamma^i_{jk}(x)$ of $\nabla$ only depend on $j^1_x g$, and $R_x$ only depends on $j^2_x g$ (cf. [6], IV.2.4 and III.7.6). Accordingly, we can define a function $F \in C^\infty(J^{2r+2}(\mathcal{M}))$ by imposing that $F(j^{2r+2}_x g) = S_g(x)$, and this function is an invariant. In fact, if $f : N \to N$ is a diffeomorphism and we set
$\bar{g} = (f^{-1})^*g = \bar{f} \circ g \circ f^{-1}$, then the Levi–Civitè connection of $\bar{g}$ is the linear connection $\nabla$ given by $\nabla_X Y = f \cdot (\nabla_{f^{-1} X} f^{-1} \cdot Y)$, as follows from Koszul’s formula ([6], IV.2.3), and consequently for every $r \in \mathbb{N}$, the tensor fields $\nabla^2 R$ and $\nabla^2 \bar{R}$ are $f$–related (cf. [6], VI.1.2); i.e., for every system of vector fields $X_1, \ldots, X_{2r+1} \in \mathfrak{X}(N)$, and every point $x \in N$, $f_* (\nabla^2 R)(X_1)_x, \ldots, (X_{2r+3})_x = (\nabla^2 \bar{R}) ((f \cdot X_1)(f(x)), \ldots, (f \cdot X_{2r+3})(f(x)))$, or else $(\nabla^2 R)(X_1, \ldots, X_{2r+3}) = f^{-1} \cdot (\nabla^2 \bar{R})(f \cdot X_1, \ldots, f \cdot X_{2r+3})$. Hence $S_\bar{g}(x) = S_\bar{g}(f(x))$, and this means $F(j_x^{2r+2}g) = F(j_x^{2r+2}(f)(j_x^{2r+2}g))$.

**Definition 5.** An $r$–th order Lagrangian density is a horizontal $n$–form $\Omega_n$ on $J^r(M)$. An $r$–order Lagrangian density is said to be invariant if for every $X \in \mathfrak{X}(N)$, $L_X \Omega_n = 0$.

**Remark 6.** As $\Omega_n$ is horizontal, locally there exists a function $L \in C^\infty(J^r M)$, such that $\Omega_n = L \, dx_1 \wedge \ldots \wedge dx_n$. Below we shall see that by introducing the factor $\sqrt{(-1)^n \det(y_{ij})}$ in $dx_1 \wedge \ldots \wedge dx_n$ we obtain a globally defined invariant Lagrangian density, thus reducing the problem of determining the invariant Lagrangian densities to that of the scalar invariants. Note that in this case $L$ should be substituted by $F = L/\sqrt{(-1)^n \det(y_{ij})}$.

**Proposition 7.** Assume $N$ is oriented. Then the bundle of metrics $M$ is endowed with a canonical invariant zero order Lagrangian density $\omega_n$, uniquely defined by the following condition: if $X_1, \ldots, X_n$ is an orthonormal basis for a metric $g$ of $M$, defined on an open subset $U \subset M$, which belongs to the orientation of $N$, then for every $x \in U$, $(\omega_n)_{g_x}(X_1, \ldots, X_n) = 1$. Accordingly, every Lagrangian density $\Omega_n$ on $J^r(M)$ can be uniquely written as $\Omega_n = F \omega_n$, $F \in C^\infty(J^r(M))$, and $\Omega_n$ is invariant if and only if $F$ is a metric differential invariant.

**Proof.** Since $\omega_n$ must be a horizontal $n$–form it is clear that the condition in the statement uniquely determines the desired form. Moreover, we can define a horizontal $n$–form on $M$ by setting for every $Y_1, \ldots, Y_n \in T_{g_x}(M)$, $(\omega_n)_{g_x}(Y_1, \ldots, Y_n) = v_{g_x} (p_* Y_1, \ldots, p_* Y_n)$, where $v_{g_x}$ is the Riemannian volume associated with $g_x$. Since $\nabla_i$ is $p$–projectable onto $X_i$, we have $(\omega_n)_{g_x}(\overrightarrow{X}_1, \ldots, \overrightarrow{X}_n) = v_{g_x}(X_1, \ldots, X_n)$, thus proving that $\omega_n$ satisfies the above condition.

A basis $X_1, \ldots, X_n$ for $T_x(N)$ is said to be orthonormal for the metric $g_x$ if: $g(X_i, X_j) = \delta_{ij}$ for either $1 \leq i \leq n^+$ or $1 \leq j \leq n^-$; $g(X_i, X_j) = -\delta_{ij}$ for $n^+ + 1 \leq i, j \leq n^+ + n^-$; in other words, the matrix of $g_x$ must be

$$
\begin{pmatrix}
I_{n^+} & 0 \\
0 & -I_{n^-}
\end{pmatrix}.
$$

Hence locally we have $\omega_n = \sqrt{(-1)^n \det(y_{ij})} dx_1 \wedge \ldots \wedge dx_n$. Also note that $\omega_n$ cannot be considered as the volume element associated to the canonical metric $G = \sum_{i \leq j} y_{ij} dx_i \otimes dx_j$ on $M$, since $G$ is singular!
We shall now prove that $\omega_n$ is invariant. Given a diffeomorphism $f$ of $N$, with the above notations we have: $(\mathcal{T}^r\omega_n)_{g_x}(Y_1,\ldots,Y_n) = (\omega_n)_{\mathcal{T}(g_x)}(\mathcal{F}_xY_1,\ldots,\mathcal{F}_xY_n) = \mathcal{V}_{\mathcal{T}(g_x)}(p_*(\mathcal{F}_xY_1,\ldots,p_*(\mathcal{F}_xY_n))$, and since $p_* \circ \mathcal{F}_x = p_* \circ p_*$, we obtain

$$(\mathcal{F}_x\omega_n)_{g_x}(Y_1,\ldots,Y_n) = (f^*\mathcal{F}_x\omega_n)_{g_x}(p_*Y_1,\ldots,p_*Y_n) = (f^*(f^{-1})^*\omega_n)(p_*Y_1,\ldots,p_*Y_n) = (\omega_n)_{g_x}(Y_1,\ldots,Y_n).
$$

3. Local expression of the basic representation

**Proposition 8.** Let $X = \sum u_i(\partial/\partial x_i)$, $u_i \in C^\infty(U)$, $1 \leq i \leq n$, be the local expression of a vector field $X \in \mathfrak{X}(N)$ on an open coordinate domain $(U;x_1,\ldots,x_n)$ of $N$. The local expression of the lifting of $X$ to the bundle of pseudo–Riemannian metrics, $\mathcal{X} \in \mathfrak{X}(\mathcal{M})$, in the induced coordinate system $(p^{-1}(U);x_i,y_{jk})$, $1 \leq i \leq n$, $1 \leq j \leq k \leq n$, is given by

$$X = \sum_i u_i \frac{\partial}{\partial x_i} + \sum_{i\leq j} v_{ij} \frac{\partial}{\partial y_{ij}} \quad \text{and} \quad v_{ij} = -\sum_h \frac{\partial u_h}{\partial x_i} y_{ij} - \sum_h \frac{\partial u_h}{\partial x_j} y_{ih}.$$  

**Proof.** First, note that $v_{ij}$ is symmetric with respect to the indices $i,j$, so we shall also write $v_{ij} = v_{ji}$, for $i > j$. As is well–known, the lift $\mathcal{X}$ is the unique vector field on $\mathcal{M}$ which is $p$–projectable onto $X$ and leaves the “canonical metric”, $G = \sum_{i\leq j} y_{ij} dx_i \otimes dx_j$, on the manifold $\mathcal{M}$, invariant; i.e., $\mathcal{X} = \sum_i u_i(\partial/\partial x_i) + \sum_{i\leq j} v_{ij}(\partial/\partial y_{ij})$, for some functions $v_{ij} \in C^\infty(p^{-1}(U))$, and $L_X G = 0$. Hence

$$L_X G = \sum_{i\leq j} [v_{ij} dx_i \otimes dx_j + y_{ij} du_i \otimes dx_j + y_{ij} dx_i \otimes du_j] = 0,$$

and this equation completely determines the unknown functions. \hfill \Box

From the general formulas for the prolongation of vector fields by infinitesimal contact transformations (e.g., see [8]) we then obtain the local expression for $\mathcal{X}^r$; more precisely,

$$X^r = \sum_i u_i \frac{\partial}{\partial x_i} - \sum_{h} \sum_{i \leq j} \sum_{|\alpha|=0}^{r} \left\{ \sum_{\beta \leq \alpha}^{\alpha} \left[ \frac{\partial^{|\beta|+1} u_h}{\partial x^\beta(i)} y_{ij}^{\beta} + \frac{\partial^{|\beta|+1} u_h}{\partial x^\beta(i)} y_{ij}^{\alpha-\beta} \right] y_{ij}^{\alpha} \right\} \frac{\partial}{\partial y_{ij}^\alpha} + \sum_{0 < \beta \leq \alpha}^{\alpha} \left[ \frac{\partial^{|\beta|}}{\partial x^\beta(h)} y_{ij}^{\alpha-\beta} \right] \frac{\partial}{\partial y_{ij}^\beta},$$

where $(i)$ stands for the multi–index $(i) = (0,\ldots,1,\ldots,0)$. The above equations can be obtained by either imposing that: \textit{1st} $X^r$ is $p_r$–projectable onto $X$, and \textit{2nd} $X^r$ leaves the generalized contact differential system $\mathcal{C}$ spanned by the one–forms on $J^r(\mathcal{M})$, $\mathcal{P}_\alpha = dy_{ij}^{\alpha} - \sum_k y_{ij}^{\alpha+(k)} dx_k$, $i,j = 1,\ldots,n$, $|\alpha| < r$ invariant;
i.e., $L_xC \subset C$, or by simply calculating the infinitesimal generator associated to $J^r(\phi_t)$, $\phi_t$ being the local flow of $X$.

Example 9. For $r = 1$, the above formula reads as follows:

$$X^1 = \sum_i u_i \frac{\partial}{\partial x_i} - \sum_h \sum_{i \leq j} \left( \frac{\partial u_h}{\partial x_j} y_{ih} \frac{\partial}{\partial y_{ij}} \right) - \sum_{h,k} \sum_{i \leq j} \left( \frac{\partial^2 u_h}{\partial x_i \partial x_k} y_{ih} \frac{\partial}{\partial y_{ij}} + \frac{\partial u_h}{\partial x_j} y_{ih} \frac{\partial}{\partial y_{ij}} + \frac{\partial u_h}{\partial x_i} y_{ih} \frac{\partial}{\partial y_{ij}} + \frac{\partial u_h}{\partial x_k} y_{ih} \frac{\partial}{\partial y_{ij}} \right) \frac{\partial}{\partial y_{ij}}.$$  \hspace{1cm} (4)

4. The fundamental distribution

Theorem 10. With the above hypotheses and notations we have:

(i) $\overline{X}_{j_xg}$ only depends on $j_x^{r+1}(X)$.

(ii) There exists a unique homomorphism of vector bundles over $J^r(M)$,

$$\Phi^r : p^r_*J^{r+1}(TN) \rightarrow T(J^rM),$$

such that for every $X \in \mathfrak{X}(N)$, $j_x^r g \in J^r(M)$,

$$\Phi^r(j_x^r g, j_x^{r+1} X) = \overline{X}_{j_xg}^r.$$

(iii) On a dense open subset $O^r \subset J^r(M)$, the image of $\Phi^r$ defines an involutive distribution $\mathcal{D}^r$, such that for every $j_x^r g \in O^r$, $X \in \mathfrak{X}(N)$,

$$\mathcal{D}^r_{j_xg} = \left\{ \overline{X}_{j_xg}^r, X \in \mathfrak{X}(N) \right\} \subset T_{j_xg}(J^rM).$$

(iv) A function $F \in C^\infty(O^r)$ is a metric differential invariant if and only if $F$ is a first integral of $\mathcal{D}^r$.

Proof. Evaluating $\overline{X}^r$ at the point $j_x^r g$, from formula (3) we obtain:

$$\overline{X}_{j_xg}^r(y_{ij}^0) =$$

$$- \sum_h \left( \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \frac{\partial^{|\beta|+1} u_h}{\partial x^{\beta+1}}(x) \frac{\partial^{|\alpha-\beta|} g_{ih}}{\partial x^{\alpha-\beta}}(x) + \frac{\partial^{|\beta|+1} u_h}{\partial x^{\beta+1}}(x) \frac{\partial^{|\alpha-\beta|} g_{ih}}{\partial x^{\alpha-\beta}}(x) \right) +$$

$$\sum_{0 < \beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \frac{\partial^{|\beta|} u_h}{\partial x^{\beta}}(x) \frac{\partial^{|\alpha-\beta|+1} g_{ij}}{\partial x^{\alpha-\beta+1}}(x),$$

where $\alpha, \beta$ are multi-indices.
thus proving (i). Accordingly, we can define a unique map \( \Phi^r \), by setting:

\[
\Phi^r(j^r_x g, j^r_x + 1 X) = X_{j^r_x g}.
\]

Since the map \( X \mapsto X^r \) is \( \mathbb{R} \)-linear, it is clear that \( \Phi^r \) is a homomorphism of vector bundles as stated in (ii).

Let us define the subset \( \mathcal{O}^r \) as follows: a point \( j^r_x g \) belongs to \( \mathcal{O}^r \) if and only if it has a neighbourhood \( \mathcal{N}_{j^r_x g} \) such that the rank of \( \Phi^r|_{\mathcal{N}_{j^r_x g}} \) is constant.

From the very definition, \( \mathcal{O}^r \) is an open subset and the rank of \( \Phi^r|_{\mathcal{O}^r} \) is locally constant. Next we prove that \( \mathcal{O}^r \) is dense in \( J^r(\mathcal{M}) \). Let \( \mathcal{U} \) be a non-empty open subset of \( J^r(\mathcal{M}) \). Since the rank of \( \mathcal{D}^r \) only takes a finite number of values, there exists a point \( j^r_x g \in \mathcal{U}, \) such that for every \( j^r_x g' \in \mathcal{U}, \) \( \text{rk.} \mathcal{D}^r_{j^r_x g'} \leq \text{rk.} \mathcal{D}^r_{j^r_x g} \), and since the rank of a homomorphism of vector bundles is a lower semicontinuous function, \( j^r_x g \in \mathcal{O}^r. \) In order to prove that \( \mathcal{D}^r \) is involutive we proceed as follows. Given a point \( j^r_x g \in \mathcal{O}^r \), let \( X_1, ..., X_k \) be vector fields on \( N \) such that \( (X_1)^r_{j^r_x g}, ..., (X_k)^r_{j^r_x g} \) is a basis for \( \mathcal{D}^r_{j^r_x g}. \) Then, there exists an open neighbourhood \( \mathcal{N}_{j^r_x g} \) such that \( X_1, ..., X_k \) is a basis of \( \mathcal{D}^r_{j^r_x g'} \), for every \( j^r_x g' \in \mathcal{N}_{j^r_x g}. \) Accordingly any two vector fields \( \xi, \xi', \) belonging to \( \mathcal{D}_{\mathcal{N}_{j^r_x g}} \) can be written as \( \xi = \sum_i f_i(X_i^r), \xi' = \sum_j f'_j(X_j^r) \), and from formula (1) we obtain

\[
[\xi, \xi'] = \sum_{i,j=1}^k \left\{ f_i X_i^r(f'_j) X_j^r - f'_j X_j^r(f_i) X_i^r + f_i f'_j [X_i, X_j] \right\},
\]

thus showing that \( [\xi, \xi'] \) also belongs to \( \mathcal{D}^r \), and finishing the proof of (iii). Part (iv) follows directly from the definitions.

Corollary 11. On a neighbourhood of each point \( j^r_x g \in \mathcal{O}^r \), the number of functionally independent metric differential invariants is \( \dim J^r(\mathcal{M}) - \text{rk.} \Phi^r_{j^r_x g}. \)

Proof. This follows from Theorem 7 and the Frobenius theorem.

Our next goal is to determine the rank of \( \Phi^r \). In doing this we shall use normal coordinates which will always be assumed to be metric (i.e., associated with an orthonormal frame) and defined on a convex open neighbourhood of a given point \( x \in N \) ([6], III.§8, IV.§3). The expansion of the metric in a normal coordinate system starts as follows (cf. [3]):

\[
g_{ij} = g_{ij}(x) + \frac{1}{6} \sum_{k,l=1}^n (R_{iklj}(x) + R_{jkil}(x)) x_k x_l + \cdots , \tag{5}
\]

where \( R_{ijkl} \) are the components of the curvature tensor, i.e.,

\[
R_{ijkl} = g(R(\partial/\partial x_k, \partial/\partial x_l)\partial/\partial x_i, \partial/\partial x_j).
\]

Taking derivatives in (5), we obtain

\[
\frac{\partial g_{ij}}{\partial x_k}(x) = 0 \quad 1 \leq i, j, k \leq n , \tag{6}
\]
and again taking derivatives,
\[
\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(x) = \frac{1}{3} (R_{iklj}(x) + R_{ilkj}(x)).
\]

(7)

5. The rank of \( \Phi^1 \)

Theorem 12. With the same notations as in Proposition 8, \( \Phi^1(j_x^2 X) = X_j^g = 0 \) if and only if in a normal coordinate system \( x_1, ..., x_n \) centred at \( x \in N \), the following conditions hold true: For every \( i, j, k = 1, ..., n \),

\[
u_i(x) = 0 \quad g_{ii}(x) \frac{\partial u_i}{\partial x_j}(x) + g_{jj}(x) \frac{\partial u_j}{\partial x_i}(x) = 0 \quad \frac{\partial^2 u_i}{\partial x_j \partial x_k}(x) = 0.
\]

(8)

Hence \( \Phi^1 : J^2_x(TN) \to T_{j_x^g}(J^1M) \) is surjective at each point \( j_x^g \in J^1M \).

Proof. From formula (4) we obtain

\[
u_i(x) = 0 \quad g_{ii}(x) \frac{\partial u_i}{\partial x_j}(x) + g_{jj}(x) \frac{\partial u_j}{\partial x_i}(x) = 0,
\]

(9)

\[
g_{jj}(x) \frac{\partial^2 u_j}{\partial x_i \partial x_k}(x) + g_{ii}(x) \frac{\partial^2 u_i}{\partial x_j \partial x_k}(x) + \sum_h \left( \frac{\partial u_h}{\partial x_i}(x) \frac{\partial g_{ih}}{\partial x_k}(x) + \frac{\partial u_h}{\partial x_k}(x) \frac{\partial g_{ji}}{\partial x_h}(x) \right) = 0.
\]

(10)

By applying (6), equation (10) becomes

\[
g_{jj}(x) \frac{\partial^2 u_j}{\partial x_i \partial x_k}(x) + g_{ii}(x) \frac{\partial^2 u_i}{\partial x_j \partial x_k}(x) = 0.
\]

(11)

By permuting \((i \ k) \mapsto (k \ i)\) in (11), we have

\[
g_{jj}(x) \frac{\partial^2 u_j}{\partial x_k \partial x_i}(x) + g_{kk}(x) \frac{\partial^2 u_k}{\partial x_j \partial x_i}(x) = 0.
\]

(12)

Comparing (11) and (12), we obtain

\[
g_{kk}(x) \frac{\partial^2 u_k}{\partial x_i \partial x_j}(x) = g_{ii}(x) \frac{\partial^2 u_i}{\partial x_j \partial x_k}(x).
\]

(13)

From (13) and (11), and again applying (13) after making the permutation \((i \ j \ k) \mapsto (j \ i \ k)\), we obtain

\[
g_{kk}(x) \frac{\partial^2 u_k}{\partial x_i \partial x_j}(x) = g_{ii}(x) \frac{\partial^2 u_i}{\partial x_j \partial x_k}(x) = -g_{jj}(x) \frac{\partial^2 u_j}{\partial x_i \partial x_k}(x) = -g_{kk}(x) \frac{\partial^2 u_k}{\partial x_j \partial x_i}(x).
\]
Accordingly,

\[ \frac{\partial^2 u_k}{\partial x_i \partial x_j}(x) = 0, \]

thus finishing the proof of (8). Hence

\[ \text{rk.} \Phi^1 = \dim J^2_x(TN) - \frac{n(n - 1)}{2} = \frac{n(n^2 + 2n + 3)}{2} = \dim T_{j_2^g(J^1M)}. \]

\[ \therefore \]

6. The rank of \( \Phi^2 \)

**Lemma 13.** With the same notations as in Proposition 8 and Theorem 12, \( \Phi^2(j^2 X) = X_{j_2^g} = 0 \) if and only if in addition to equations (8) the following conditions hold true: For every \( i, j, k, l = 1, \ldots, n, \)

\[ \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}(x) = 0, \]  \[ \text{(14)} \]

\[ \sum_{h=1}^{n} \left( \frac{\partial u_h}{\partial x_i}(x) R_{hkji}(x) + \frac{\partial u_h}{\partial x_j}(x) R_{hlki}(x) + \frac{\partial u_h}{\partial x_k}(x) R_{hijl}(x) + \frac{\partial u_h}{\partial x_l}(x) R_{hijkl}(x) \right) = 0. \]  \[ \text{(15)} \]

**Proof.** It follows from formula (3) that \( X_{j_2^g} = 0 \) if and only if equations (8) hold and furthermore for every \( i, j, k, l = 1, \ldots, n, \)

\[ g_{ii}(x) \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}(x) + g_{jj}(x) \frac{\partial^3 u_j}{\partial x_i \partial x_k \partial x_l}(x) + \lambda_{ijkl} = 0, \]  \[ \text{(16)} \]

where we have set:

\[ \lambda_{ijkl} = \sum_{h=1}^{n} \left( \frac{\partial u_h}{\partial x_i}(x) \frac{\partial^2 g_{ij}}{\partial x_h \partial x_l}(x) + \frac{\partial u_h}{\partial x_j}(x) \frac{\partial^2 g_{ij}}{\partial x_h \partial x_k}(x) + \frac{\partial u_h}{\partial x_k}(x) \frac{\partial^2 g_{ij}}{\partial x_h \partial x_l}(x) + \frac{\partial u_h}{\partial x_l}(x) \frac{\partial^2 g_{ij}}{\partial x_h \partial x_k}(x) \right). \]

Permuting the indices \( i, k, \) in (16), and subtracting we obtain

\[ g_{ii}(x) \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}(x) - g_{kk}(x) \frac{\partial^3 u_k}{\partial x_i \partial x_j \partial x_l}(x) = \lambda_{kjil} - \lambda_{ijkl}, \]

and permuting the indices \( j, k, \)

\[ g_{ii}(x) \frac{\partial^3 u_i}{\partial x_k \partial x_j \partial x_l}(x) - g_{jj}(x) \frac{\partial^3 u_j}{\partial x_k \partial x_i \partial x_l}(x) = \lambda_{jkil} - \lambda_{ikjl}. \]  \[ \text{(17)} \]
By adding (16) and (17), we obtain

\[ 2g_{ii}(x) \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}(x) = \lambda_{jkil} - \lambda_{ikjl} - \lambda_{ijkl}. \]

Formula (7) then yields
If \( \dim \) for each 

**Theorem 15.**

(i) If \( \dim \) for every \( j, k, l \), permuting \( j \) and \( l \), and equating the corresponding right hand sides we obtain (15), thus also showing that the right hand side of (18) vanishes and hence (18) reduces to (14). The proof is thus complete. \( \square \)

**Remark 14.** Equation (15) is invariant under the group of order 8 generated by the permutations \( \gamma : (ijkl) \mapsto (jikl) \), and \( \tau : (ijkl) \mapsto (ilkj) \). Note that \( \gamma \circ \tau = \tau \circ \gamma^2 \).

Accordingly, in examining (15) we only need to consider the following three cases: 

1. \( \text{st} \) \( i \leq j \leq k \leq l \), 
2. \( \text{nd} \) \( i \leq k < j \leq l \), 
3. \( \text{rd} \) \( i \leq j \leq l < k \).

(ii) If \( \dim N = n = 2 \), for every \( j^2g \in J^2(\mathcal{M}) \), \( \Phi^2_{j^2g} \) is bijective. 

(iii) For each \( n \geq 3 \), there exists a dense open subset \( \mathcal{O}^{n,2} \subset J^2(\mathcal{M}) \), such that for every \( j^2g \in \mathcal{O}^{n,2} \), \( \Phi^2_{j^2g} \) is injective.

**Proof.** (i) From (8) and (14) it follows that \( \Phi^2 \) is injective. Hence \( \text{rk} \Phi^2_{j^2g} = \dim J^3_x(TN) = 4 = \dim T^2_{j^2g}(J^2N) \).

(ii) Using the above remark it is not difficult to check that equation (15) is identically satisfied if \( n = 2 \). Accordingly, from (8) and (14) it follows that a 3-jet \( j^3_xX \in \text{Ker} \Phi^2_{j^2g} \) is completely determined by \( (\partial u_2/\partial x_1)(x) \). Hence the kernel of \( \Phi^2 \) is a vector subbundle of rank 1, and therefore \( \text{rk} \Phi^2_{j^2g} = \dim J^3_x(TN) - 1 = 20 - 1 = 19 \).

(iii) Let \( r = \sum_{i,j} r_{ij}dx_i \otimes dx_j \) be the Ricci tensor of \( g \); i.e., \( r(X, Y) = \text{trace of } Z \mapsto R(Z, X)(Y) \). Then we have: \( r_{ij}(x) = \sum_{k} g_{kh}(x) R_{hj'i}(x) \). Since \( r \) is symmetric there exists a unique endomorphism \( A : T_x(N) \rightarrow T_xN \), such that for every \( X, Y \in T_x(N) \),

\[
r(X, Y) = g(A(X), Y) = g(X, A(Y)) . \tag{19}
\]

Moreover, let \( B \) be the endomorphism given by: \( B = \sum_{i,j} (\partial u_i/\partial x_j)(x)dx_i \otimes (\partial/\partial x_j)x \). From the second equation of (8) we deduce that for every \( X, Y \in T_x(N) \),

\[
g(B(X), Y) + g(X, B(Y)) = 0 . \tag{20}
\]

Let \( \mathcal{O}^{n,2} \) be the set of points \( j^2g \) such that the eigenvalues of \( A \) in \( T_x(N) \otimes \mathbb{C} \) are pairwise different. We shall prove that on \( \mathcal{O}^{n,2} \) the unique solution of (15) is the
trivial one. Let us denote by $\langle X, Y \rangle$ the bilinear form induced by $g_x$ on the complex vector space $T_x(N) \otimes \mathbb{C}$, so that (19) and (20) imply

$$< A(Z), W >= < Z, A(W) > < B(Z), W > + < Z, B(W) >= 0 ,$$

for every $Z, W \in T_x(N) \otimes \mathbb{C}$. Letting $h = k$ in (15), multiplying by $g_{kk}(x)$, and using the second equation of (8), we obtain

$$\sum_{h=1}^{n} \left( r_{jh}(x) \frac{\partial u_h}{\partial x_i}(x) + r_{ih}(x) \frac{\partial u_h}{\partial x_j}(x) \right) = 0 ,$$

or equivalently,

$$< AB(Z), W > + < Z, AB(W) >= 0 ,$$

for every $Z, W \in T_x(N) \otimes \mathbb{C}$. Let $A(Z_i) = \lambda_i Z_i$ be the eigenvalues (and the eigenvectors) of $A$. From the above equations we then obtain: $< AB(Z_i), Z_j > + < Z_i, AB(Z_j) >= 0$, or equivalently, $< B(Z_i), A(Z_j) > + < A(Z_i), B(Z_j) >= 0$; i.e., $\lambda_j < B(Z_i), Z_j > + \lambda_i < Z_i, B(Z_j) >= 0$. Hence $(\lambda_i - \lambda_j) < Z_i, B(Z_j) >= 0$. By virtue of the hypothesis this implies $B(Z_j) = 0$, and since $Z_1, ..., Z_n$ is a basis of the complex tangent space, we have $B = 0$. \hfill \Box

7. The rank of $\Phi^r$, $r \geq 3$

Theorem 16. For each $r \geq 3$, there exists a dense open subset $O^{n,r} \subset J^r(\mathcal{M})$, such that for every $j_x^3 g \in O^{n,r}$, $\Phi_{j_x^3 g}^r$ is injective.

Proof. First we prove the theorem for $r = 3$. We distinguish two cases:

(i) $\dim N = n \neq 2$. For the sake of simplicity we write $O^{1,2} = J^2(\mathcal{M})$. From (i) and (iii) in Theorem 15 we know that $\Phi^2$ is injective on $O^{n,2}$. We set $O^{n,3} = p_{j_x^3}^{-1}(O^{n,2})$. Assume $j_x^3 g \in O^{n,3}$. Since $\Phi^2_{|O^{n,2}}$ is injective from formula (3) we have that: $j_x^4(X) \in \ker \Phi_{j_x^3 g}^{3}$ if and only if $j_x^3(X) = 0$, and furthermore for every $i, j, k, l, m = 1, ..., n$,

$$\frac{\partial^4 u_{ij}}{\partial x_i \partial x_k \partial x_l \partial x_m}(x) g_{jj}(x) + \frac{\partial^4 u_{ij}}{\partial x_j \partial x_k \partial x_l \partial x_m}(x) g_{ii}(x) = 0 . \quad (21)$$

Permuting $i$ and $k$ in (21) and subtracting we have

$$g_{ii}(x) \frac{\partial^4 u_i}{\partial x_j \partial x_k \partial x_l \partial x_m}(x) - g_{ii}(x) \frac{\partial^4 u_i}{\partial x_j \partial x_k \partial x_l \partial x_m}(x) = 0 ,$$

and permuting the indices $j, k$, and adding the equation thus obtained to (21) we have

$$\frac{\partial^4 u_i}{\partial x_j \partial x_k \partial x_l \partial x_m}(x) = 0 , \text{ for every } i, j, k, l, m = 1, ..., n .$$

Hence $j_x^4(X) = 0$, and accordingly $\Phi_{|O^{n,3}}^3$ is injective.
(ii) \( \dim N = n = 2 \). From formula (3) we conclude that \( j^1_2(X) \) belongs to the kernel of \( \Phi(j^1_2) \), if and only if in addition to equations (8) and (14) the following conditions hold true: For every \( |\alpha| = 3, i, j = 1, 2 \),

\[
\sum_{k=1}^{2} \left( \frac{\partial u_k}{\partial x_i}(x) \frac{\partial^3 g_{ij}}{\partial x^\alpha}(x) + \frac{\partial u_k}{\partial x_j}(x) \frac{\partial^3 g_{ji}}{\partial x^\alpha}(x) \right) + \sum_{k=1}^{2} \alpha_k \frac{\partial u_k}{\partial x_k}(x) \frac{\partial^3 g_{ij}}{\partial x^\alpha-(k)+(i)}(x) + g_{jj}(x) \frac{\partial^4 u_j}{\partial x^{\alpha+(i)}}(x) + g_{ii}(x) \frac{\partial^4 u_i}{\partial x^{\alpha+(j)}}(x) = 0. \tag{22}
\]

Recall that equation (15) is identically satisfied if \( n = 2 \). Moreover the expansion given in (5) yields (cf. [3]):

\[
\frac{\partial^3 g_{ij}}{\partial x_k \partial x_i \partial x_m}(x) = \frac{1}{6} \frac{\partial}{\partial x_k}(R_{ijkt} + R_{iktj})(x) + \frac{1}{6} \frac{\partial}{\partial x_i}(R_{imjk} + R_{ikjm})(x) + \frac{1}{6} \frac{\partial}{\partial x_m}(R_{ijkt} + R_{iktj})(x).
\]

From the above equation and (22) we then obtain:

\[
\begin{align*}
(\alpha = (3, 0), i = j = 1) : & \quad \frac{\partial^4 u_1}{\partial x_1^4}(x) = 0, \\
(\alpha = (3, 0), i = 1, j = 2) : & \quad g_{22}(x) \frac{\partial^4 u_2}{\partial x_1^2}(x) + g_{11}(x) \frac{\partial^4 u_1}{\partial x_1^2 \partial x_2}(x) = 0, \\
(\alpha = (3, 0), i = j = 2) : & \quad \frac{\partial u_2}{\partial x_1}(x) \frac{\partial R_{1212}}{\partial x_2}(x) + 2g_{22}(x) \frac{\partial^4 u_2}{\partial x_1^4}(x) = 0, \\
(\alpha = (2, 1), i = j = 1) : & \quad \frac{\partial^4 u_1}{\partial x_1^4 \partial x_2}(x) = 0, \\
(\alpha = (2, 1), i = 1, j = 2) : & \quad -\frac{1}{3} \frac{\partial u_2}{\partial x_1}(x) \frac{\partial R_{1212}}{\partial x_2}(x) + g_{11}(x) \frac{\partial^4 u_1}{\partial x_1^3 \partial x_2^2}(x) + g_{22}(x) \frac{\partial^4 u_2}{\partial x_1^4 \partial x_2^2}(x) = 0, \\
(\alpha = (2, 1), i = j = 2) : & \quad -\frac{1}{3} \frac{\partial u_2}{\partial x_1}(x) \frac{\partial R_{1212}}{\partial x_1}(x) + 2g_{11}(x) \frac{\partial^4 u_2}{\partial x_1 \partial x_2^2}(x) = 0, \\
(\alpha = (1, 2), i = j = 1) : & \quad \frac{1}{3} \frac{\partial u_2}{\partial x_1}(x) \frac{\partial R_{1212}}{\partial x_2}(x) + 2g_{11}(x) \frac{\partial^4 u_1}{\partial x_1 \partial x_2^2}(x) = 0, \\
(\alpha = (1, 2), i = 1, j = 2) : & \quad \frac{1}{3} g_{22}(x) \frac{\partial u_2}{\partial x_1}(x) \frac{\partial R_{1212}}{\partial x_1}(x) + \frac{\partial u_2}{\partial x_1}(x) \frac{\partial^4 u_2}{\partial x_1 \partial x_2^2}(x) + g_{11}(x) g_{22}(x) \frac{\partial^4 u_2}{\partial x_1 \partial x_2^2}(x) = 0, \\
(\alpha = (1, 2), i = j = 2) : & \quad \frac{\partial^4 u_2}{\partial x_1 \partial x_2^2}(x) = 0,
\end{align*}
\]
\[ (\alpha = (0, 3), i = j = 1) : -g_{22}(x) \frac{\partial u_2}{\partial x_1} \frac{\partial R_{1212}}{\partial x_1}(x) + 2 \frac{\partial^4 u_1}{\partial x_1 \partial x_2^3}(x) = 0, \]
\[ (\alpha = (0, 3), i = 1, j = 2) : g_{22}(x) \frac{\partial^4 u_2}{\partial x_1 \partial x_2^2}(x) + g_{11}(x) \frac{\partial^4 u_1}{\partial x_1^4}(x) = 0, \]
\[ (\alpha = (0, 3), i = j = 2) : \frac{\partial^4 u_2}{\partial x_2^4}(x) = 0. \]

It is not difficult to check that the above system is equivalent to saying that for every \( i, j, k, l, m = 1, 2, \)
\[
\frac{\partial^4 u_i}{\partial x_j \partial x_k \partial x_l \partial x_m}(x) = 0,
\]
and

\[
\frac{\partial u_2}{\partial x_1}(x) \frac{\partial R_{1212}}{\partial x_1}(x) = 0 \quad \frac{\partial u_2}{\partial x_1}(x) \frac{\partial R_{1212}}{\partial x_2}(x) = 0.
\]

Hence in the dense open subset \( O_{2,3} \) of the 2–jets of metrics of a surface whose curvature satisfies \( \| \nabla R(x) \| > 0, \) we have \( j^4_{X}(X) = 0, \) and therefore \( \Phi_{O_{2,3}}^3 \) is injective.

By induction on \( r \) we now prove the general statement of the theorem. For every \( r \geq 3, n \geq 1, \)
we set \( O_{n,r} = p_{r-1}(O^{n,3}). \) Assume \( j^{r+1}_{x}(X) \in \text{Ker} \Phi_{j^{r}g}, \)
with \( j^{r}g \in O^{n,r}. \) Since \( X_{j^{r}g} \) projects onto \( X_{j^{r-1}g}, \) it follows from the induction hypothesis that \( j^{r}_{x}(X) = 0, \) and from formula (3) we thus deduce
\[
\frac{\partial^{r+1} u_j}{\partial x^{\alpha+(i)}(x)} g_{jj}(x) + \frac{\partial^{r+1} u_i}{\partial x^{\alpha+(i)}(x)} g_{ii}(x) = 0,
\]
for every \( i, j = 1, ..., n, |\alpha| = r. \) Let \( k \) be an index such that \( \alpha_k > 0. \) We set \( \beta = \alpha - (k), \) so that the above equation reads as follows
\[
g_{ii}(x) \frac{\partial^{r+1} u_i}{\partial x_j \partial x_k \partial x^\beta}(x) + g_{jj}(x) \frac{\partial^{r+1} u_j}{\partial x_i \partial x_k \partial x^\beta}(x) = 0. \tag{23}
\]

Permuting the indices \( i, k, \) in (23), and subtracting we have
\[
g_{ii}(x) \frac{\partial^{r+1} u_i}{\partial x_j \partial x_k \partial x^\beta}(x) - g_{kk}(x) \frac{\partial^{r+1} u_k}{\partial x_i \partial x_j \partial x^\beta}(x) = 0,
\]
and permuting the indices \( j, k, \) and adding the above equation to (23), we obtain
\[
\frac{\partial^{r+1} u_i}{\partial x_j \partial x_k \partial x^\beta}(x) = 0,
\]
thus proving that \( j^{r+1}_{x}(X) = 0, \) and finishing the proof of the theorem. \( \square \)

8. Calculating \( i_{n,r} \)

**Theorem 17.** On a dense open subset of \( J^r(M), \) the number \( i_{n,r} \) of functionally independent metric differential invariants is the following:
(i) For every \( n \geq 1 \), \( i_{n,0} = i_{n,1} = 0 \),

(ii) For every \( r \geq 0 \), \( i_{1,r} = 0 \),

(iii) \( i_{2,2} = 1 \), and for every \( r \geq 3 \), \( i_{2,r} = \frac{1}{2}(r+1)(r-2) \),

(iv) For every \( n \geq 3 \), \( r \geq 2 \),

\[
i_{n,r} = n + \frac{(r-1)n^2 - (r+1)n}{2(r+1)} \binom{n+r}{r}.
\]

**Proof.** First of all we confine ourselves to the dense open subset \( O^r \) prescribed in Theorem 10–(iii) where we know that the metric differential invariants of order \( r \) coincide with the ring of first integrals of the fundamental distribution.

(i) It follows from formula (2) that the vector fields \( \mathbf{X}_{g}^{\lambda} \) span \( T_{g}^{\lambda} \), hence \( i_{n,0} = 0 \). Since \( \Phi^{1} \) is surjective (Theorem 12) from Corollary 11 we conclude that \( i_{n,1} = 0 \).

(ii) From Theorem 16 we know that \( \Phi^{r} \) is injective (in fact \( O^{1,r} = J^{r}(M) \) in this case). From Corollary 11 we thus have \( i_{1,r} = \dim J^{r}(M) - \text{rk.} \Phi_{g}^{r} = \dim J^{r}(M) - \dim J_{x}^{r+1}(TN) = (r+2) - (r+2) = 0 \).

(iii) That \( i_{2,2} = 1 \), follows directly from part (ii) of Theorem 15. Moreover, the formula for \( r \geq 3 \), is a particular case of the formula in (iv).

(iv) From Theorem 15–(iii), Theorem 16 and Corollary 11 we have

\[
i_{n,r} = \dim J^{r}(M) - \dim J_{x}^{r+1}(TN) = n + \frac{(r-1)n^2 - (r+1)n}{2(r+1)} \binom{n+r}{r}.
\]

**Remark 18.** For \( n \geq 3 \), there is a classical procedure in order to obtain second order metric invariants, the so–called curvature invariants ([9], p. 146). In the generic case there is an essentially unique frame reducing \( g \) and its Ricci tensor to a canonical form. The invariants are the components of the Weyl tensor on that frame plus the \( n \) eigenvalues of the Ricci tensor. Let us calculate the dimension of the space of Weyl tensors. Following the same notations as [2], 1.105–116, we have that the space \( CE \) of curvature tensors (here \( E = T_{x}^{r}(N) \)), breaks into three irreducible subspaces under the natural action of the orthogonal group, \( CE = UE \oplus ZE \oplus WE \), where \( \dim UE = 1 \), \( \dim ZE = \frac{n(n+1)}{2} \) (traceless symmetric 2–tensors), and \( WE \) are the Weyl tensors. Hence:
\[
\dim \mathcal{W}E = \dim \mathcal{E} - \frac{n(n + 1)}{2} \\
= \dim S^2 \bigwedge^2 E - \dim \bigwedge^4 E - \frac{n(n + 1)}{2} \\
= \frac{n^4 - 7n^2 - 6n}{2}.
\]

Hence the number of curvature invariants is

\[
n + \frac{n^4 - 7n^2 - 6n}{2} = \frac{n^4 - 7n^2 + 6n}{2} = i_{n, 2}.
\]

Accordingly, this shows that the number of functionally independent curvature invariants are exactly the number of functionally independent second order metric invariants.

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