Space-Time Discontinuous Galerkin Solution of Convection Dominated Optimal Control Problems

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Abstract

In this paper, a space-time discontinuous Galerkin finite element method for distributed optimal control problems governed by unsteady diffusion-convection-reaction equations with control constraints is studied. Time discretization is performed by discontinuous Galerkin method with piecewise constant and linear polynomials, while symmetric interior penalty Galerkin with upwinding is used for space discretization. The numerical results presented confirm the theoretically observed convergence rates.

1 Introduction

Optimal control problems (OCPs) governed by diffusion-convection-reaction equations arise in environmental control problems, optimal control of fluid flow and in many other applications. It is well known that the standard Galerkin finite element discretization causes nonphysical oscillating solutions when convection dominates. Stable and accurate numerical solutions can be achieved by various effective stabilization techniques such as the streamline upwind/Petrov Galerkin (SUPG) finite element method [8], the local projection stabilization [3], the edge stabilization [13]. Recently, discontinuous Galerkin (dG) methods gain importance due to their better convergence behaviour, local mass conservation, flexibility in approximating rough solutions on complicated meshes, mesh adaptation and weak imposition of the boundary conditions in OCPs, see, e.g., [14] [15] [26] [27].

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In the recent years much effort has been spent on parabolic OCPs (see for example [1, 17]). There are few publications dealing with OCPs governed by nonstationary diffusion-convection-reaction equation. The local DG approximation of the OCP which is discretized by backward Euler in time is studied in [28] and a priori error estimates for semi-discrete OCP is provided in [21]. In [11, 12], the characteristic finite element solution of the OCP is discussed and numerical results are provided. A priori error estimates for discontinuous Galerkin time discretization for unconstrained parabolic OCPs is proposed in [6]. Crank-Nicolson time discretization is applied to OCP of diffusion-convection equation in [5]. To the best of our knowledge, this is the first study on space-time dG discretization of unsteady OCPs governed by convection-diffusion-reaction equations.

There are two different approaches for solving OCPs: optimize-then-discretize (OD) and discretize-then-optimize (DO). In the OD approach, first the infinite dimensional optimality system is derived containing state and adjoint equation and the variational inequality. Then, the optimality system is discretized by using a suitable discretization method in space and time. In DO approach, the infinite dimensional OCP is discretized and then the finite-dimensional optimality system is derived. The DO and DO approaches do not commute in general for OCPs governed by diffusion-convection-reaction equation [8]. However, commutativity is achieved in the case of SIPG discretization for steady state problems [14, 26]. For dG time discretization, we show that OD and DO approaches commute also for time-dependent problems.

In this paper, we solve the OCP governed by diffusion-convection-reaction equation with control constraints by applying symmetric interior penalty Galerkin (SIPG) method in space and discontinuous Galerkin (dG) discretization in time [7, 9, 10, 22, 25]. In the study of Konstantinos [6], a priori error estimates for continuous in space and discontinuous in time Galerkin discretization for unconstrained parabolic OCPs are derived by decoupling the optimality system. In [12], error analysis concerning the characteristic finite element solution of the OCP with control constraints is discussed. Optimal order of convergence rates for the space-time discretization is confirmed on two numerical examples. Additionally we give numerical results for Crank-Nicolson method and compare them with the DG in time discretization.

The rest of the paper is organized as follows. In Section 2 we define the model problem and then derive the optimality system. In Section 3 discontinuous Galerkin discretization and the semi-discrete optimality system follow. In Section 4 space-time dG methods and state the fully discrete optimality system are presented. In Section 5 numerical results are shown in order to discover the performance of the suggested method. The paper ends with some conclusions.
2 The Optimal Control Problem

We consider the following distributed optimal control problem governed by the unsteady diffusion-convection-reaction equation with control constraints

\[
\begin{align*}
\text{minimize} \quad & J(y,u) := \frac{1}{2} \int_0^T \left( \|y - y_d\|^2_{L^2(\Omega)} + \alpha \|u\|^2_{L^2(\Omega)} \right) \, dt, \\
\text{subject to} \quad & \partial_t y - \epsilon \Delta y + \beta \cdot \nabla y + ry = f + u, \quad (x,t) \in \Omega \times (0,T), \\
& y(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\
& y(x,0) = y_0(x), \quad x \in \Omega,
\end{align*}
\]

where the admissible space of control constraints is given by

\[
U_{ad} = \{ u \in L^2(0,T;U) : u_a \leq u \leq u_b, \text{ a.e. in } \Omega \times (0,T) \}
\]

with the constant bounds \( u_a, u_b \in L^\infty(\Omega) \), i.e., \( u_a < u_b \). We take \( \Omega \) as a bounded open convex domain in \( \mathbb{R}^2 \) with Lipschitz boundary \( \partial\Omega \) and \( I = (0,T) \) as time interval. The source function and the desired state are denoted by \( f \in L^2(0,T;L^2(\Omega)) \) and \( y_d \in L^2(0,T;L^2(\Omega)) \), respectively. The initial condition is also defined as \( y_0(x) \in H^1_0(\Omega) \). The diffusion and reaction coefficients are \( \epsilon > 0 \) and \( r \in L^\infty(\Omega) \), respectively. The velocity field \( \beta \in (W^{1,\infty}(\Omega))^2 \) satisfies the incompressibility condition, i.e. \( \nabla \cdot \beta = 0 \). Furthermore, we assume the existence of constant \( c_0 \equiv c_0(x) \) such that \( r(x) - \frac{1}{2} \nabla \cdot \beta(x) \geq c_0 \) a.e. \( \in \Omega \) so that the well-posedness of the optimal control problem \(2.1\) is guaranteed. The trial and test spaces are

\[
Y = V = H^1_0(\Omega), \quad \forall t \in (0,T].
\]

For \( (y,u) \in Y \times U_{ad} \), the variational formulation corresponding to \(2.1\) can be written

\[
\begin{align*}
\text{minimize} \quad & J(y,u) := \frac{1}{2} \int_0^T \left( \|y - y_d\|^2_{L^2(\Omega)} + \alpha \|u\|^2_{L^2(\Omega)} \right) \, dt, \\
\text{subject to} \quad & (\partial_t y,v) + a(y,v) = (f + u,v), \quad \forall v \in V, \quad t \in I, \\
& y(x,0) = y_0(x), \quad x \in \Omega,
\end{align*}
\]

with \( a(y,x) = \int_{\Omega} (\epsilon \nabla y \cdot \nabla v + \beta \cdot \nabla y v + ryv) \, dx \) and \( (u,v) = \int_{\Omega} uv \, dx \). Differentiating the Lagrangian

\[
\mathcal{L}(y,u,p) = \frac{1}{2} \int_0^T \left( \|y - y_d\|^2_{L^2(\Omega)} + \alpha \|u\|^2_{L^2(\Omega)} \right) \, dt + \{ (\partial_t y,p) + a(y,p) - (f + u,p) \} + (y(x,0) - y_0,p(x,0)).
\]
with respect to \( p, y, u \), we obtain the optimality system

\[
\begin{align*}
(\partial_t y, v) + a(y, v) &= (f + u, v), \quad y(x, 0) = y_0, \quad (2.4a) \\
-(\partial_t p, \psi) + a(\psi, p) &= -(y - y_d, \psi), \quad p(x, T) = 0, \quad (2.4b) \\
\int_0^T (\alpha u - p, w - u) dt &\geq 0 \quad \forall w \in U_{ad}. \quad (2.4c)
\end{align*}
\]

It is well known that the functions \((y, u) \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; Y) \times U_{ad}\) solve (2.3) if and only if there is an adjoint \( p \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; Y) \) such that \((y, u, p)\) is the unique solution of the optimality system (2.4) [16, 23].

3 Discontinuous Galerkin Discretization

Let \( \{T_h\}_h \) be a family of shape regular meshes such that \( \overline{\Omega} = \bigcup_{K \in T_h} \overline{K}, \ K_i \cap K_j = \emptyset \) for \( K_i, K_j \in T_h, \ i \neq j \). The diameter of an element \( K \) and the length of an edge \( E \) are denoted by \( h_K \) and \( h_E \), respectively. In addition, the maximum value of element diameter is denoted by \( h = \max_{K \in T_h} h_K \). We only consider discontinuous piecewise linear finite element spaces to define the discrete state and control spaces

\[
V_h = Y_h = U_{ad}^{Hh} = \{ y \in L^2(\Omega) : y|_K \in \mathbb{P}^1(K) \ \forall K \in T_h \} \quad (3.1)
\]

\[
\tilde{U}_h \subset (Y_h \cap U_{ad}), \quad (3.2)
\]

respectively. Here, \( \mathbb{P}^1(K) \) denotes the set of all polynomials on \( K \in T_h \) of degree at most 1.

Remark 3.1. When the state equation (2.1) contains nonhomogeneous Dirichlet boundary conditions, \( Y_h \) and \( V_h \) can still be taken as the same space due to the weak treatment of boundary conditions in dG methods (see for example [15]).

We split the set of all edges \( \mathcal{E}_h \) into the set \( \mathcal{E}_h^0 \) of interior edges and the set \( \mathcal{E}_h^0 \) of boundary edges so that \( \mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^0 \). Let \( n \) denote the unit outward normal to \( \partial \Omega \). We define the inflow boundary

\[
\Gamma^- = \{ x \in \partial \Omega : \beta \cdot n(x) < 0 \}
\]

and the outflow boundary \( \Gamma^+ = \partial \Omega \setminus \Gamma^- \). The boundary edges are decomposed into edges \( \mathcal{E}_h^- = \{ E \in \mathcal{E}_h^0 : E \subset \Gamma^- \} \) that correspond to inflow boundary and edges \( \mathcal{E}_h^+ = \mathcal{E}_h^0 \setminus \mathcal{E}_h^- \) that correspond to outflow boundary. The inflow and outflow boundaries of an element \( K \in T_h \) are defined by

\[
\partial K^- = \{ x \in \partial K : \beta \cdot n_K(x) < 0 \}, \quad \partial K^+ = \partial K \setminus \partial K^-,
\]

where \( n_K \) is the unit normal vector on the boundary \( \partial K \) of an element \( K \).

Let the edge \( E \) be a common edge for two elements \( K \) and \( K^e \). For a piecewise continuous
scalar function $y$, there are two traces of $y$ along $E$, denoted by $y|_E$ from interior of $K$ and $y^e|_E$ from interior of $K^e$. Then, the jump and average of $y$ across the edge $E$ are defined by:

$$[y] = y|_E n_K + y^e|_E n_{K^e}, \quad \{y\} = \frac{1}{2} (y|_E + y^e|_E).$$

(3.3)

Similarly, for a piecewise continuous vector field $\nabla y$, the jump and average across an edge $E$ are given by

$$[\nabla y] = \nabla y|_E \cdot n_K + \nabla y^e|_E \cdot n_{K^e}, \quad \{\nabla y\} = \frac{1}{2} (\nabla y|_E + \nabla y^e|_E).$$

(3.4)

For a boundary edge $E \in K \cap \Gamma$, we set $\{\nabla y\} = \nabla y$ and $[y] = y n$ where $n$ is the outward normal unit vector on $\Gamma$.

We can now give dG discretizations of the state equation (2.1) in space for fixed control $u$. The dG method proposed here is based on the upwind discretization of the convection term and on the SIPG discretization of the diffusion term [19]. This leads to the following (bi-)linear forms applied to $y_h \in H^1(0,T; Y_h)$ for $\forall t \in (0,T]$,

$$(\partial_t y_h, v_h) + a^s_h(y_h, v_h) + b_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, \quad t \in (0,T],$$

(3.5)

where

$$a^d(y, v) = \sum_{K \in T_h} \int_K \epsilon \nabla y \cdot \nabla v \, dx$$

$$- \sum_{E \in T_h} \int_E \left( \{\epsilon \nabla y\} \cdot [v] + \{\epsilon \nabla v\} \cdot [y] - \frac{\sigma \epsilon}{h_E} [y] \cdot [v] \right) ds$$

(3.6)

and

$$a^s_h(y, v) = a^d(y, v) + \sum_{K \in T_h} \int_K (\beta \cdot \nabla y v + r y v) \, dx$$

$$+ \sum_{K \in T_h} \int_{\partial K \setminus \Gamma^-} \beta \cdot n (y^e - y) v \, ds - \sum_{K \in T_h} \int_{\partial K \cap \Gamma^-} \beta \cdot n y v \, ds,$$

(3.7)

$$b_h(u, v) = - \sum_{K \in T_h} \int_K u v \, dx$$

(3.8)

with a constant interior penalty parameter $\sigma > 0$. We choose $\sigma$ to be sufficiently large, independent of the mesh size $h$ and the diffusion coefficient $\epsilon$ to ensure the stability of the dG discretization as described in [18, Sec. 2.7.1] with a lower bound depending only on the polyno-
mial degree. Large penalty parameters decrease the jumps across element interfaces, which can affect the numerical approximation [2].

3.1 Semi-discrete Formulation of The Optimal Control Problem

Let \( f_h, y^d_h \) and \( y_0^h \) be approximations of the source function \( f \), the desired state function \( y_d \) and initial condition \( y_0 \), respectively. Then, the semi-discrete approximation of the optimal control problem (2.4) can be defined as follows:

\[
\min_{u_h \in U_{ad}^h} \int_0^T \left( \frac{1}{2} \sum_{K \in T_h} \|y_h - y^d_h\|_{L^2(K)}^2 + \frac{\alpha}{2} \sum_{K \in T_h} \|u_h\|_{L^2(K)}^2 \right) dt, \tag{3.9a}
\]

subject to \( (\partial_t y_h, v_h) + a_h^s(y_h, v_h) + b_h(u_h, v_h) = (f_h, v_h), \tag{3.9b} \)

\( y_h(x, 0) = y_0^h, \)

\( t \in (0, T], v_h \in V_h, (y_h, u_h) \in Y_h \times U_{ad}^h. \)

The semi-discrete optimality system is written as follows:

\[
(\partial_t y_h, v_h) + a_h^s(y_h, v_h) + b(u_h, v_h) = (f_h, v_h), \quad y_h(x, 0) = y_0^h,
\]

\[
- (\partial_t p_h, \psi_h) + a_h^a(p_h, \psi_h) = -(y_h - y^d_h, \psi_h), \quad p_h(x, T) = 0, \tag{3.10}
\]

\[
\int_0^T (\alpha u_h - p_h, w_h - u_h) dt \geq 0, \quad \forall w_h \in U_{ad}^h,
\]

where

\[
a_h^a(p, \psi) = \sum_{K \in T_h} \int K \epsilon \nabla p \cdot \nabla \psi \ dx
\]

\[
- \sum_{E \in E_h} \int (\|\epsilon \nabla p\| \cdot \psi \| + \|\epsilon \nabla \psi\| \cdot \|p\| - \frac{\sigma \epsilon}{k_E} \|p\| \cdot \|\psi\|) ds
\]

\[
+ \sum_{K \in T_h} \int K \beta \cdot \nabla p \psi ds
\]

\[
- \sum_{K \in T_h} \int_{\partial K} \beta \cdot n(p^e - p) \psi ds + \sum_{K \in T_h} \int_{\partial K \cap \Gamma^+} \beta \cdot n \psi ds.
\]

4 Time Discretization of The Optimal Control Problem

In this section, we derive the fully-discrete optimality system, by using \( \theta \)-method and discontinuous Galerkin method. We compare the resulting optimality systems based on two approaches, i.e. optimize-then-discretize (OD) and discretize-then-optimize (DO). Let \( 0 = t_0 < t_1 < \cdots < t_{N_T} = T \) be a subdivision of \( I = (0, T) \) with time intervals \( I_m = (t_{m-1}, t_m] \) and time steps \( k_m = t_m - t_{m-1} \)
for \( m = 1, \ldots, N_T \) and \( k = \max_{1 \leq m \leq N_T} k_m \).

### 4.1 Time Discretization Using \theta\-method

We start with OD approach by discretizing the semi-discrete optimality system (3.10) using \theta\-method.

\[
(y_{h,m+1} - y_{h,m}, v) + k\alpha_h^k ((1 - \theta) y_{h,m} + \theta y_{h,m+1}, v) = \\
k((1 - \theta) f_{h,m} + \theta f_{h,m+1}) + k((1 - \theta) u_{h,m} + \theta u_{h,m+1}, v), \ m = 0, \ldots, N - 1,
\]

\[
y_{h,0}(x, 0) = y_0
\]

\[
(p_{h,m} - p_{h,m+1}, q) + k\alpha_h^k ((\theta p_{h,m} + (1 - \theta)p_{h,m+1}, q) = \\
- k \left( \theta (y_{h,m} - y_{h,m}^d, q) + (1 - \theta) (y_{h,m+1} - y_{h,m+1}^d, q) \right), \ m = N - 1, \ldots, 0,
\]

\[
p_{h,N} = 0,
\]

\[
(\alpha u_{h,m} - p_{h,m}, w - u_{h,m}) \geq 0 \quad m = 0, 1, \ldots, N.
\]

We proceed with DO approach. To do this, we approximate the first part of the cost functional by the rectangle rule, the second part of it by the trapezoidal rule and discretize the state equation using \theta\-method in time. We use the rectangle rule to approximate the first part so that the value of the adjoint at the final time becomes zero as in [20].

\[
\minimize_{u_h \in U_h} k \sum_{m=0}^{N-1} (y_{h,m} - y_{h,m}^d)^T M (y_{h,m} - y_{h,m}^d) + \frac{\alpha}{2} \left( \frac{1}{2} u_{h,0}^T M u_{h,0} + \sum_{m=1}^{N-1} u_{h,m}^T M u_{h,m} + \frac{1}{2} u_{h,N}^T M u_{h,N} \right)
\]

subject to

\[
(y_{h,m+1} - y_{h,m}, v) + k\alpha_h^k ((1 - \theta) y_{h,m} + \theta y_{h,m+1}, v) = \\
k((1 - \theta) f_{h,m} + \theta f_{h,m+1}) + k((1 - \theta) u_{h,m} + \theta u_{h,m+1}, v), \ m = 0, \ldots, N - 1,
\]

\[
(y_{h,0}, v) = (y_0, v),
\]

where \( M \) is the mass matrix.
Now we construct the discrete Lagrangian

\[ \mathcal{L}(y_{h,1}, \ldots, y_{h,N}, p_{h,0}, \ldots, p_{h,N}, u_{h,0}, \ldots, u_{h,N}) \]

\[ = \frac{k}{2} \sum_{m=0}^{N-1} \left( y_{h,m} - y_{h,m}^d \right)^T M \left( y_{h,m} - y_{h,m}^d \right) \]

\[ + \frac{\alpha k}{2} \left( \frac{1}{2} u_{h,0}^T M u_{h,0} + \sum_{m=1}^{N-1} u_{h,m}^T M u_{h,m} + \frac{1}{2} u_{h,N}^T M u_{h,N} \right) + (y_{h,0} - y_{0}, p_{h,0}) \]

\[ + \sum_{m=0}^{N-1} \left( (y_{h,m+1} - y_{h,m}, p_{h,m+1}) + k \alpha u_h^s ((1 - \theta) y_{h,m} + \theta y_{h,m+1}, p_{h,m+1}) \right) \]

\[ - k((1 - \theta) f_{h,m} + \theta f_{h,m+1}) + k((1 - \theta) u_{h,m} + \theta u_{h,m+1}, p_{h,m+1})) \].

(4.2)

By differentiating Lagrangian (4.2), we derive the fully-discrete optimality system

\[ (y_{h,m+1} - y_{h,m}, v) + k \alpha u_h^s ((1 - \theta) y_{h,m} + \theta y_{h,m+1}, v) = \]

\[ k((1 - \theta) f_{h,m} + \theta f_{h,m+1}) + k((1 - \theta) u_{h,m} + \theta u_{h,m+1}, v), \quad m = 0, \ldots, N - 1 \]

\[ y_{h,0}(x, 0) = y_0 \]

\[ (q, p_{h,N}) + k \alpha u_h^s (q, \theta p_{h,N}) = 0, \]

\[ (p_{h,m} - p_{h,m+1}, q) + k \alpha u_h^s (q, \theta p_{h,m} + (1 - \theta) p_{h,m+1}) = \]

\[ -k(y_{h,m} - y_{h,m}^d, q), \quad m = N - 1, \ldots, 1, \]

\[ (q, p_{h,0} - p_{h,1}) + k \alpha u_h^s (q, (1 - \theta) p_{h,1}) = -k(y_{h,0} - y_{h,0}^d, q), \]

\[ \left( \frac{\alpha}{2} u_{h,0} - (1 - \theta) p_{h,1}, w - u_{h,0} \right) \geq 0, \]

\[ \left( \alpha u_{h,m} - (\theta p_{h,m} + (1 - \theta) p_{h,m+1}), w - u_{h,m} \right) \geq 0, \quad m = 1, \ldots, N - 1, \]

\[ \left( \frac{\alpha}{2} u_{h,N} - \theta p_{h,N}, w - u_{h,N} \right) \geq 0. \]

(4.3)

In the case of backward Euler method (\( \theta = 1 \)), the value \( u_{h,0} \) is not needed as we observe from (4.3). As we mentioned before, the approximation of the first integral in the cost functional by using the rectangle rule leads to \( p_{h,N} = 0, u_{h,N} = 0 \), as we see from (4.4). For the SIPG we obtain \( a_h^s(\psi_\delta, p_\delta) = a_h^s(p_\delta, \psi_\delta) \) and therefore (4.1) and (4.3) gives the same variational formulation.

In the case of Crank-Nicolson method (\( \theta = 1/2 \)), we observe that some differences occur in the adjoint equation. In (4.1), the right-hand side of the adjoint equation is evaluated at two successive points, while it is evaluated at just one point in (4.3). Additional differences are seen in the variational inequalities (4.1) and (4.3), too. Thus, \( OD \) and \( DO \) approaches lead to different weak formulations. In [1], the optimal control of the heat equation is concerned by applying continuous Galerkin discretization. For \( DO \) approach, the cost functional is discretized by using the midpoint rule. On the other hand, for \( OD \) approach, the semi-discrete state equation is
discretized by using the midpoint rule and a variation of the trapezoidal rule is applied to the semi-discrete adjoint equation to obtain the fully discrete optimality system. Then OD and DO approaches commute.

### 4.2 Time Discretization Using Discontinuous Galerkin Method

We derive the fully discrete optimality system by employing discontinuous Galerkin time discretization to the semi-discrete optimality system \((3.10)\). We define the space-time finite element space of piecewise discontinuous functions for state and control as

\[
V_h^{k,q} = Y_h^{k,q} = \left\{ v \in L^2(0,T;L^2(\Omega)) : v|_{I_m} = \sum_{s=0}^{q} t^s \phi_s, t \in I_m, \phi_s \in V_h, m = 1, \ldots, N \right\},
\]

\[
\widetilde{U}_h^{k,q} \subset (Y_h^{k,q} \cap U_{ad}).
\]

We define the temporal jump of \(v \in V_h^{k,q}\) as \([v]_m = v^m_+ - v^m_-\), where \(w^m_{\pm} = \lim_{\varepsilon \rightarrow 0_{\pm}} v(t_m \pm \varepsilon)\). Let \(f_\delta\) and \(y^d_\delta\) be approximations of the source function \(f\) and the desired state function \(y^d\) on each interval \(I_m\). Then, the fully-discrete optimal control problem is written as

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \int_0^T \sum_{K \in T_h} \left( \|y_\delta - y^d_\delta\|_{L^2(K)}^2 + \alpha \|u_\delta\|_{L^2(K)}^2 \right) dt, \\
\text{subject to} & \quad \int_0^T \left( (\partial_t y_\delta, v_\delta) + a^*_h(y_\delta, v_\delta) \right) dt + \sum_{m=1}^{N_T} ([y_\delta]_{m-1}, v_{\delta,+}^{m-1}) \\
& \quad = \int_0^T (f_\delta + u_\delta, v_\delta) dt, \quad y_{\delta,0} = (y_0)_\delta. \quad \text{(4.4b)}
\end{align*}
\]

The OCP \((4.4)\) has a unique solution \((y_\delta, u_\delta)\) and that pair \((y_\delta, u_\delta)\) \(\in V_h^{k,q} \times \widetilde{U}_h^{k,q}\) is the solution of \((4.4)\) if and only if there is an adjoint \(p_\delta \in V_h^{k,q}\) such that \((y_\delta, u_\delta, p_\delta) \in V_h^{k,q} \times \widetilde{U}_h^{k,q} \times V_h^{k,q}\) is the unique solution of the fully-discrete optimality system

\[
\begin{align*}
\int_0^T ( (\partial_t y_\delta, v_\delta) + a^*_h(y_\delta, v_\delta) ) dt + \sum_{m=1}^{N_T} ([y_\delta]_{m-1}, v_{\delta,+}^{m-1}) & = \int_0^T (f_\delta + u_\delta, v_\delta) dt, \\
\int_0^T ( - (\partial_t p_\delta, \psi_\delta) + a^*_h(p_\delta, \psi_\delta) ) dt - \sum_{m=1}^{N_T} ([p_\delta]_{m}, \psi_{\delta,-}^m) & = - \int_0^T (y_\delta - y^d_\delta, \psi_\delta) dt, \quad p_{\delta,N}^+ = 0, \quad \text{(4.5b)}
\end{align*}
\]

\[
\int_0^T (\alpha u_\delta - p_\delta, w_\delta - u_\delta) dt \geq 0 \quad \forall w_\delta \in \widetilde{U}_h^{k,q}. \quad \text{(4.5c)}
\]

We note that \((4.5)\) is obtained by discretizing \((2.4)\), that is, we employ OD approach.
Finally, we define the auxiliary problem which is needed for a priori error analysis

\[ (J'_\delta(u), v - u) = \int_0^T (\alpha u - p^u_\delta, v - u) dt, \]  

subject to

\[ \int_0^T ((\partial_t y^u_\delta, v_\delta) + a^u_h(y^u_\delta, v_\delta)) dt + \sum_{m=1}^{N_T} ([y^u_\delta]_{m-1}, v^{m-1}_\delta) = \int_0^T (f_\delta + u, v_\delta) dt, \]

\[ y^{u,0}_\delta = (y_0)_\delta, \]

\[ \int_0^T (-(\partial_t p^u_\delta, \psi_\delta) + a^u_h(p^u_\delta, \psi_\delta)) dt - \sum_{m=1}^{N_T} ([p^u_\delta]_m, \psi^{m}_\delta, -) = -\int_0^T (y^u_\delta - y^d_\delta, \psi_\delta) dt, \]

\[ p^{u,0, 0}_\delta = 0. \]  

### 4.3 Commutativity Properties of Space-Time dG Method

In the case of time-dependent OCP, the difference between the optimality system arising from \textit{OD} and \textit{DO} is caused by nonsymmetric nature of the bilinear form or the inconsistency of the final condition of the adjoint equation with the optimality system. In the \textit{DO} approach, we construct the discrete Lagrangian

\[ \mathcal{L}(y_\delta, u_\delta, p_\delta) = \frac{1}{2} \int_0^T \left( \sum_{K \in \mathcal{T}_h} (\|y_\delta - y^d_\delta\|_{L^2(K)}^2 + \alpha \|u_\delta\|_{L^2(K)}^2) + \sum_{m=1}^{N_T} \right) \]

\[ \int_{I_m} (\partial_t v_\delta, \psi_\delta) dt + \left( [y^h_{\delta,m-1}, v^{m-1}_{\delta,1}] + ([p^h_{\delta,m}, \psi^{m}_{\delta,1}, -] + (y_0)_\delta - y^{d,0}_\delta, p^{d,0}_\delta). \]

Differentiating \( \mathcal{L} \) with respect to \( y_\delta \) and applying integration by parts, we obtain

\[ \sum_{m=1}^{N_T} \int_{I_m} (\psi_\delta, -\partial_t p_\delta) + a^u(h(\psi_\delta, p_\delta)) dt + \sum_{m=1}^{N_T-1} (\psi^{m, -}_{\delta,m, -}, [p^m_\delta, m] + (q^m_{\delta,N_T}, p^{m,N_T}_{\delta, -}) \]

\[ = -\sum_{m=1}^{N_T} \int_{I_m} (y^u_\delta - y^d_\delta, \psi_\delta) dt, \forall \psi_\delta \in V^k_A. \]  

(4.8)
Now, we add and subtract \((\psi_{\delta,N_T}^-, p_{\delta,N_T}^+)\) to (4.8) and obtain
\[
\sum_{m=1}^{N_T} \int_{I_m} \left( - (\partial_t p_\delta, \psi_\delta) + a_h^*(\psi_\delta, p_\delta) \right) dt - \sum_{m=1}^{N_T} ([p_\delta]_m, \psi_{m,\delta}^-) + (q_{\delta,N_T}^-, p_{\delta,N_T}^+) = - \sum_{m=1}^{N_T} \int_{I_m} (y_\delta - y_d^d, \psi_\delta) dt, \quad \forall \psi_\delta \in V_h^{k,q}. \tag{4.9}
\]
On each subinterval \(I_m\), the adjoint equation reads as
\[
\int_{I_m} \left( - (\partial_t p_\delta, \psi_\delta) + a_h^*(\psi_\delta, p_\delta) \right) dt - ([p_\delta]_m, \psi_{m,\delta}^-) = - \int_{I_m} (y_\delta - y_d^d, \psi_\delta) dt.
\]
However, \((q_{\delta,N_T}^-, p_{\delta,N_T}^+)\) does not match the right-hand side of (4.9), so it is set to zero, i.e. \(p_{\delta,N_T}^+ = 0\). Now, we use \(a_h^*(\psi_\delta, p_\delta) = a_h^*(p_\delta, \psi_\delta)\). Thus, we arrive at (4.5). Therefore, OD and DO approaches commute.

5 Numerical Results

In this section, we present numerical results. The state, the adjoint, and the control variables are discretized using the piecewise linear polynomials in space. The discretized control problem are solved by the primal dual active set (PDAS) algorithm [4]. In order to measure the error in the state and adjoint approximation in terms of \(L^\infty(0,1; L^2(\Omega))\) norm, the error in the control approximation in terms of \(L^2(0,1; L^2(\Omega))\) norm. In all numerical examples, we have taken \(h = k\).

We note that, in the case of dG(0) method, the approximating polynomials are piecewise constant in time and the resulting scheme is a version of the backward Euler method with a modified right-hand side [22, Chapter 7]
\[
(M + k A^s) y_{h,m} = My_{h,m-1} + \frac{k}{2} (f_{h,m} + f_{h,m-1}) + \frac{k}{2} M (u_{h,m} + u_{h,m-1}),
\]
\[
(M + k A^a) p_{h,m-1} = Mp_{h,m} - \frac{k}{2} M (y_{h,m} + y_{h,m-1}) + \frac{k}{2} (y_d^d + y_d^d_{h,m-1}).
\]
For dG(1) method, we use piecewise linear polynomials in time. The resulting linear system for the state on each time step is given as follows [22 Chapter 7]:
\[
\begin{pmatrix}
M + k A^s & M + \frac{k}{2} A^s \\
\frac{k}{2} A^s & \frac{1}{2} M + \frac{k}{3} A^s
\end{pmatrix}
\begin{pmatrix}
Y_0 \\
Y_1
\end{pmatrix} =
\begin{pmatrix}
My_{h,m-1} + \frac{k}{2} (f_{h,m} + f_{h,m-1}) + \frac{k}{2} M (u_{h,m} + u_{h,m-1}) \\
\frac{k}{2} (f_{h,m} + M u_{h,m})
\end{pmatrix},
\]
where \(A^s, M\) are the stiffness matrix of the state equation and the mass matrix, respectively.
We derive the solution at the time step \(t_m\) as \(y_{h,m} = Y_0 + Y_1\). For the adjoint equation, we have
Table 1: Example 1 by dG(0) and backward Euler (in parenthesis) method.

| $k$ | $\|y - y_d\|$ | Rate | $\|p - p_d\|$ | Rate | $\|u - u_d\|$ | Rate |
|-----|----------------|------|----------------|------|----------------|------|
| $1/4$ | 4.41e-2(2.45e-2) | (-) | 8.77e-2(3.39e-2) | (-) | 4.37e-2(2.43e-2) | (-) |
| $1/10$ | 2.22e-2(6.84e-3) | 0.99(1.84) | 4.53e-2(1.42e-2) | 0.95(1.26) | 8.63e-3(2.67e-3) | 1.03(1.10) |
| $1/20$ | 1.18e-2(6.84e-3) | 0.99(1.53) | 2.35e-2(6.88e-3) | 0.95(1.04) | 8.63e-3(2.67e-3) | 1.03(1.10) |
| $1/40$ | 6.20e-3(6.84e-3) | 0.93(1.17) | 1.20e-2(3.45e-3) | 0.96(1.00) | 4.28e-3(1.34e-3) | 1.01(1.00) |

\[
\begin{pmatrix}
M + kA^a \\ \frac{k}{2} A^a \\
\frac{k}{2} M + \frac{k}{2} A^a
\end{pmatrix}
\begin{pmatrix}
P_0 \\ P_1
\end{pmatrix} =
\begin{pmatrix}
M_{ph,m} - \frac{k}{2} M (y_{h,m} + y_{h,m-1}) + \frac{k}{2} (y_{d,h,m} + y_{d,h,m-1}) \\
-\frac{k}{2} (M y_{h,m-1} + y_{d,h,m-1})
\end{pmatrix},
\]

where $A^a$ is the stiffness matrix for the adjoint equation. We obtain the adjoint at the time step $t_{m-1}$ as $p_{h,m-1} = P_0 + P_1$. We apply block-partitioning to these linear systems once in order to solve the systems for each time interval.

The main drawback of the dG time discretization is the solution of large coupled linear systems in block form. Several solvers are suggested to overcome this especially for nonlinear problems [24]. Because we are using constant time steps, the coupled matrices on the right-hand side of (5.1) and (5.2) have to decomposed (LU block factorization) at the begin of the integration. Then the the state and adjoint equations are solved at each time step by forward elimination and back substitution using the block factorized matrices.

**Example 1**: We consider the problem in [12] with the following parameters by adding the reaction term

\[
Q = (0, 1] \times \Omega, \quad \Omega = (0, 1)^2, \quad \epsilon = 10^{-5}, \quad \beta = (1, 0)^T, \quad r = 1, \quad \alpha = 1 \text{ and } u \geq 0.
\]

The source function $f$, the desired state $y_d$ and the initial condition $y_0$ are computed from (2.4) using the following exact solutions of the state, adjoint and control, respectively,

\[
y(x,t) = \exp(-t) \sin(2\pi x_1) \sin(2\pi x_2),
\]

\[
p(x,t) = \exp(-t)(1 - t) \sin(2\pi x_1) \sin(2\pi x_2),
\]

\[
u(x,t) = \max\left(0, -\frac{1}{\alpha} p\right).
\]

In Table 1 errors and converge rates for dG(0) and backward Euler method are shown. For dG(0) and backward Euler method leads the first order convergence, due to the dominance of temporal errors, which is optimal in time.
In Table 2, errors and converge rates for Crank-Nicolson method obtained by OD and DO approaches are shown. For Crank-Nicolson method, OD approach optimal second order convergence is achieved. But in the DO approach, the discretization of the right-hand side of the adjoint by a one-step method is reflected affects the numerical results and the optimal order of convergence is not achieved.

Table 2: Example 1 by Crank-Nicolson method OD and DO approach (in parenthesis).

| $k$ | $\|y - y_\delta\|$ | Rate | $\|p - p_\delta\|$ | Rate | $\|u - u_\delta\|$ | Rate |
|-----|----------------------|------|----------------------|------|----------------------|------|
| 1/3 | 5.38e-2(5.31e-2)     | -    | 3.22e-2(4.16e-1)     | -    | 2.18e-2(4.33e-2)     | -    |
| 1/10| 1.35e-2(1.36e-2)     | 1.99(1.97)| 8.19e-3(1.90e-1)     | 1.98(1.13)| 3.68e-3(1.24e-2)     | 2.57(1.80) |
| 1/20| 3.41e-3(3.43e-3)     | 1.99(1.98)| 2.07e-3(9.10e-2)     | 1.98(1.06)| 9.34e-4(4.38e-3)     | 1.98(1.50) |
| 1/40| 8.58e-4(8.65e-4)     | 1.99(1.99)| 5.02e-4(4.45e-2)     | 2.05(1.03)| 2.13e-4(1.63e-3)     | 2.13(1.42) |

In Table 3, results for dG(1) time discretization is shown and indicates that the second order convergence is achieved. The error in the state is smaller than for Crank-Nicolson method with OD approach, while the errors in adjoint and the control are close for both discretizations.

Table 3: Example 1 by dG(1) method.

| $k$ | $\|y - y_\delta\|$ | Rate | $\|p - p_\delta\|$ | Rate | $\|u - u_\delta\|$ | Rate |
|-----|----------------------|------|----------------------|------|----------------------|------|
| 1/3 | 3.65e-2              | -    | 5.36e-2              | -    | 4.34e-2              | -    |
| 1/10| 8.59e-3              | 2.09 | 3.35e-3              | 2.02 | 1.56e-3              | 2.10 |
| 1/20| 2.14e-3              | 2.00 | 1.25e-3              | 2.04 | 3.61e-4              | 2.11 |
| 1/40| 5.36e-4              | 2.00 | 8.16e-4              | 2.04 | 3.61e-4              | 2.11 |

In Table 3, results for dG(1) time discretization is shown and indicates that the second order convergence is achieved. The error in the state is smaller than for Crank-Nicolson method with OD approach, while the errors in adjoint and the control are close for both discretizations.

**Example 2:** We consider the problem in [11] with the following parameters by adding the reaction term

$$Q = (0, 1] \times \Omega, \ \Omega = (0, 1)^2, \ \epsilon = 10^{-5}, \ \beta = (0.5, 0.5)^T, \ r = 1, \ \alpha = 1 \ \text{and} \ 0 \leq u \leq 0.5.$$  

The source function $f$, the desired state $y_d$ and the initial condition $y_0$ are computed from (2.4) using the following exact solutions of the state, adjoint and control, respectively,

\[
p(x, t) = \sin(\pi t) \sin(2\pi x_1) \sin(2\pi x_2) \exp\left(\frac{-1 + \cos(t_x)}{\sqrt{\epsilon}}\right),
\]

\[
y(x, t) = p \left(\frac{1}{2\sqrt{\epsilon}} \sin(t_x) + 8\epsilon \pi^2 + \frac{\sqrt{\epsilon}}{2} \cos(t_x) - \frac{1}{2} \sin^2(t_x)\right)
\]

\[
- \pi \cos(\pi t) \sin(2\pi x_1) \sin(2\pi x_2) \exp\left(\frac{-1 + \cos(t_x)}{\sqrt{\epsilon}}\right)
\]

\[
u(x, t) = \max\left(0, \min\left(-\frac{1}{\alpha} p, 0.5\right)\right).
\]

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As opposed to the previous example, the exact solution of PDE constrained depends on the diffusion explicitly and the problem is highly convection dominated. This example cannot be solved properly by using dG(0) and backward Euler method for time discretization. Therefore, we present errors for Crank-Nicolson method in Table 4 where the differences between OD and DO can be seen clearly. DO approach causes order reduction for adjoint and control. We observe that DO approach produces oscillations in the control Figure 2 whereas the approximate solutions for the control are smooth for OD Figure 3. However, due to the convection dominated nature of the problem, the optimal order of convergence cannot be achieved in the OD approach in contrast to the Example 1.

| $k$ | $\| y - y_\delta \|$ | Rate | $\| p - p_\delta \|$ | Rate | $\| u - u_\delta \|$ | Rate |
|-----|-----------------|------|-----------------|------|-----------------|------|
| $\frac{1}{4}$ | 2.32(2.32) | -(-) | 3.17e-1(3.21e-1) | -(-) | 1.51e-1(1.43e-1) | -(-) |
| $\frac{1}{10}$ | 1.05(1.05) | 1.14(1.14) | 1.25e-1(1.26e-1) | 1.34(1.35) | 5.09e-2(4.93e-2) | 1.57(1.54) |
| $\frac{1}{20}$ | 3.72e-1(3.74e-1) | 1.50(1.50) | 6.35e-2(7.47e-2) | 0.97(0.76) | 2.45e-2(3.36e-2) | 1.05(0.55) |
| $\frac{1}{40}$ | 1.09e-1(1.10e-1) | 1.77(1.76) | 2.17e-2(3.57e-2) | 1.55(1.07) | 8.31e-3(2.07e-2) | 1.56(0.70) |

Table 4: Example 2 by Crank-Nicolson method OD and DO approach (in parenthesis).

In Table 5 numerical results for dG(1) discretization is shown. As opposed to the results in Table 4, the error in state, adjoint and control are smaller than in case of CN and the optimal quadratic convergence is achieved.

In Figure 4 we present the exact and the approximate solution at $t = 0.5$ showing that the problem is approximated accurately.
Table 5: Example 2 by dG(1) method.

| $k$ | $\| y - y_h \|$ | Rate | $\| p - p_h \|$ | Rate | $\| u - u_h \|$ | Rate |
|-----|------------------|------|------------------|------|------------------|------|
| $\frac{1}{5}$ | 2.25e+0 | - | 3.30e-1 | - | 1.48e-1 | - |
| $\frac{1}{10}$ | 6.15e-1 | 1.87 | 5.50e-2 | 2.58 | 2.38e-2 | 2.63 |
| $\frac{1}{20}$ | 1.34e-1 | 2.20 | 1.45e-2 | 1.92 | 8.01e-3 | 1.57 |
| $\frac{1}{40}$ | 2.65e-2 | 2.34 | 3.13e-3 | 2.22 | 2.27e-3 | 1.82 |

6 Conclusions

For dG in time discretization, the numerical results confirm convergence rates and DO, OD approaches commute. In a future work, we will study derivation of the optimal convergence rates under lower regularity assumptions and we will apply space-time adaptivity for convection dominated problems with boundary or interior layers.

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Figure 3: Example 2: Approximate solutions at $t=0.5$ with Crank-Nicolson $OD$ approach.

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Figure 4: Example 2: Approximate solutions at $t=0.5$ with dG(1) method.

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