STUDIES OF NORMALIZED SOLUTIONS TO SCHRÖDINGER EQUATIONS WITH SOBOLEV CRITICAL EXponent AND COMBINED NONLINEARITIES

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ABSTRACT. We consider the Sobolev critical Schrödinger equation with combined nonlinearities

\[
\begin{cases}
\Delta u = \lambda u + |u|^{2^* - 2}u + \mu |u|^{q - 2}u, & x \in \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^2 dx = a,
\end{cases}
\]

where \(N \geq 3, \mu > 0, \lambda \in \mathbb{R}, a > 0\) and \(q \in (2, 2^*)\). We prove in this paper

(1) Multiplicity and stability of solutions for \(q \in (2, 2 + \frac{4}{N})\) and \(\mu a^{\frac{(1 - \gamma q)}{2}} \leq (2K)^{\frac{2^* - 2}{2^*}}\) with \(\gamma_q := \frac{N}{2} - \frac{N}{q}\) and \(K\) being some positive constant. This result extends the results obtained in Jeanjean et al. [5] and Jeanjean and Le [6] for the case \(\mu a^{\frac{(1 - \gamma q)}{2}} < (2K)^{\frac{2^* - 2}{2^*}}\).

(2) Nonexistence of ground states for \(q = 2 + \frac{4}{N}\) and \(\mu a^{\frac{(1 - \gamma q)}{2}} \geq \bar{a}_N\) with \(\bar{a}_N\) being some positive constant. We give a new proof to this result different with Wei and Wu [13].

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the standing waves to the Sobolev critical Schrödinger equation with combined nonlinearities

\[
i\partial_t \psi + \Delta \psi + |\psi|^{2^* - 2}\psi + \mu |\psi|^{q - 2}\psi = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,
\]

where \(N \geq 3, \mu > 0, 2^* := \frac{2N}{N - 2}\) and \(q \in (2, 2^*)\). Starting from the fundamental contribution by Tao, Visan and Zhang [12], the nonlinear Schrödinger equation with combined nonlinearities attracted much attention, see for example [1, 4, 7, 8, 15].

Standing waves to (1.1) are solutions of the form \(\psi(t, x) = e^{-i\lambda t}u(x)\), where \(\lambda \in \mathbb{R}\) and \(u : \mathbb{R}^N \to \mathbb{C}\). Then \(u\) satisfies the equation

\[
-\Delta u = \lambda u + |u|^{2^* - 2}u + \mu |u|^{q - 2}u, \quad x \in \mathbb{R}^N.
\]

When looking for solutions to (1.2) one choice is to fix \(\lambda < 0\) and to search for solutions to (1.2) as critical points of the action functional

\[
J(u) := \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla u|^2 - \frac{\lambda}{2} |u|^2 - \frac{1}{2^*} |u|^{2^*} - \frac{\mu}{q} |u|^q\right) dx,
\]

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see for example [2, 9] and the references therein. Another choice is to fix the $L^2$-norm of the unknown $u$, that is, to consider the problem

\[
-\Delta u = \lambda u + |u|^{p-2}u + \mu|u|^{q-2}u, \quad x \in \mathbb{R}^N,
\]

\[
u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^2dx = a \tag{1.3}
\]

with fixed $a > 0$ and unknown $\lambda \in \mathbb{R}$. In this direction, define on $H^1(\mathbb{R}^N)$ the energy functional

\[
E(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx.
\]

It is standard to check that $E \in C^1$ and a critical point of $E$ constrained to

\[
S_a := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = a\} \tag{1.4}
\]

gives rise to a solution to (1.3). Such solution is usually called a normalized solution of (1.2) on $S_a$, which is the aim of this paper. In studying normalized solutions to the Schrödinger equation

\[
- \Delta u = \lambda u + |u|^{p-2}u + |u|^{q-2}u, \quad x \in \mathbb{R}^N \tag{1.5}
\]

with $2 < q < p \leq 2^*$, the so-called $L^2$-critical exponent $2 + \frac{4}{N}$ plays an important role. A very complete analysis of the various cases that may happen for (1.5)-(1.4), depending on the values of $(p, q)$, has been provided recently in [5, 6, 10, 11, 13]. For future reference, we recall

**Definition 1.1.** We say that $u$ is a normalized ground state to (1.2) on $S_a$ or a ground state to (1.3) if

\[
E(u) = c_a^0 := \inf \{E(v) : v \in S_a, \ (E|_{S_a})'(v) = 0\}.
\]

The set of the ground states to (1.3) will be denoted by $G_a$.

**Definition 1.2.** $G_a$ is orbitally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that, for any $\psi_0 \in H^1(\mathbb{R}^N)$ with $\inf_{v \in G_a} \|\psi_0 - v\|_{H^1} < \delta$, we have

\[
\inf_{v \in V_a} \|\psi(t, \cdot) - v\|_{H^1} < \epsilon \quad \text{for any} \ t > 0,
\]

where $\psi(t, x)$ denotes the solution of (1.1) with initial value $\psi_0$.

A standing wave $e^{-i\lambda t}u$ is strongly unstable if for every $\epsilon > 0$ there exists $\psi_0 \in H^1(\mathbb{R}^N)$ such that $\|\psi_0 - u\|_{H^1} < \epsilon$, and $\psi(t, x)$ blows up in finite time.

We first consider the case $q \in (2, 2 + \frac{4}{N})$. Recently, [5] and [6] studied the multiplicity and stability of solutions to (1.3) under the condition $\mu a^{\frac{4(1-q)}{2}} < (2K)^{\frac{4q-2^*}{q-2}}$, which is obtained as follows. By using the Sobolev inequality (see [3])

\[
S\|u\|_2^2 \leq \|\nabla u\|_2^2 \quad \text{for all} \ u \in D^{1,2}(\mathbb{R}^N) \tag{1.6}
\]

and the Gagliardo-Nirenberg inequality (see [14])

\[
\|u\|_q \leq C_{N,q}\|\nabla u\|_2^\gamma \|u\|_2^{1-\gamma_q} \quad \text{for all} \ u \in H^1(\mathbb{R}^N), \gamma_q := \frac{N}{2} - \frac{N}{q}, \tag{1.7}
\]

we have for any $u \in S_a$,

\[
E(u) \geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2^*} \left( S^{-1}\|\nabla u\|_2^{2^*} \right) - \frac{\mu}{q} C_{N,q}^q \|\nabla u\|_2^{q\gamma_q} \|u\|_2^{q(1-\gamma_q)} = \|\nabla u\|_2^2 f_{\mu,a}(\|\nabla u\|_2^2),
\]
Then by using the concentration compactness principle, 

\[ f_{\mu,a}(\rho) := \frac{1}{2} - \frac{1}{2^*} S^{\frac{2^*}{2}} \rho^{2^* - 1} - \frac{\mu}{q} C_{N,q} a^{\frac{q(1-\gamma_q)}{2}} \rho^{\frac{2q}{2^*} - 1}, \quad \rho \in (0, \infty). \]

Direct calculations give that

\[ \max_{\rho > 0} f_{\mu,a}(\rho) = f_{\mu,a}(\rho_{\mu,a}) \begin{cases} < 0, & \text{if } \mu a^{\frac{q(1-\gamma_q)}{2}} > (2K)^{\frac{q-2}{2^*-2}}, \\ = 0, & \text{if } \mu a^{\frac{q(1-\gamma_q)}{2}} = (2K)^{\frac{q-2}{2^*-2}}, \\ > 0, & \text{if } \mu a^{\frac{q(1-\gamma_q)}{2}} < (2K)^{\frac{q-2}{2^*-2}}. \end{cases} \]

where

\[ \rho_{\mu,a} = \left( \frac{(2 - q\gamma_q)2^* S^{\frac{2^*}{2}} C_{N,q} a^{\frac{q(1-\gamma_q)}{2}}}{q(2^* - 2)} \right)^{\frac{1}{q - \gamma_q}} \]

and

\[ K = \frac{2^* - q\gamma_q}{2^*(2 - q\gamma_q) S^{\frac{2^*}{2}}} \left( \frac{2^* S^{\frac{2^*}{2}}(2 - q\gamma_q) C_{N,q} a^{\frac{q(1-\gamma_q)}{2}}}{q(2^* - 2)} \right)^{\frac{2^* - 2}{q - \gamma_q}}. \]

Thus, the domain \( \{ (\mu, a) \in \mathbb{R}^2 : \mu > 0, \ a > 0 \} \) is divided into three parts \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) by the curve \( \mu a^{\frac{q(1-\gamma_q)}{2}} = (2K)^{\frac{q-2}{2^*-2}} \) with

\[ \Omega_1 = \{ (\mu, a) \in \mathbb{R}^2 : \mu > 0, \ a > 0, \ \mu a^{\frac{q(1-\gamma_q)}{2}} < (2K)^{\frac{q-2}{2^*-2}} \}, \]

\[ \Omega_2 = \{ (\mu, a) \in \mathbb{R}^2 : \mu > 0, \ a > 0, \ \mu a^{\frac{q(1-\gamma_q)}{2}} = (2K)^{\frac{q-2}{2^*-2}} \} \]

and

\[ \Omega_3 = \{ (\mu, a) \in \mathbb{R}^2 : \mu > 0, \ a > 0, \ \mu a^{\frac{q(1-\gamma_q)}{2}} > (2K)^{\frac{q-2}{2^*-2}} \}. \]

For fixed \( \mu > 0 \), define \( a_0 \) such that

\[ \mu a_0^{\frac{q(1-\gamma_q)}{2}} = (2K)^{\frac{q-2}{2^*-2}}. \]

Then \( \Omega_1 \) can also be expressed as \( \Omega_1 = \{ (\mu, a) \in \mathbb{R}^2, \mu > 0, 0 < a < a_0 \} \). Define

\[ \rho_0 := \rho_{\mu,a_0} = \left( \frac{2^*(2 - q\gamma_q) S^{\frac{2^*}{2}}}{2^*(2^* - 2)} \right)^{\frac{2^*-2}{q - \gamma_q}}, \]

\[ B_{\rho_0} := \{ u \in H^1(\mathbb{R}^N) : \| \nabla u \|_2^2 < \rho_0 \}, \quad V_a := S_a \cap B_{\rho_0}. \]

Under the condition \( (\mu, a) \in \Omega_1 \), it was proved in \([5]\), Lemma 2.4) that

\[ m_a := \inf_{u \in V_a} E(u) < 0 \leq \inf_{u \in \partial V_a} E(u). \]

Then by using the concentration compactness principle, \([5]\) obtained the following results:

**Theorem A.** Let \( N \geq 3, \ q \in (2, 2 + \frac{4}{N}), \mu > 0 \) and \( 0 < a < a_0 \). Then (1.3) has a ground state \( u \), which is a minimizer of \( E \) on \( V_a \). In addition, any ground state of (1.3) is a minimizer of \( E \) on \( V_a \). Moreover, \( G_\alpha \) is compact, up to translation, and it is orbitally stable.

Noting that

\[ \Omega_1 \cup \Omega_2 = \{ (\mu, a) \in \mathbb{R}^2, \mu > 0, 0 < a \leq a_0 \} = \{ (\mu, a) \in \mathbb{R}^2 : \mu > 0, \ a > 0, \ \mu a^{\frac{q(1-\gamma_q)}{2}} \leq (2K)^{\frac{q-2}{2^*-2}} \}, \]
we find that the results in Theorem A can be extended to the case \((\mu, a) \in \Omega_1 \cup \Omega_2\) by repeating word by word the proof of the paper [5]:

**Theorem 1.3.** Let \(N \geq 3, q \in (2, 2 + \frac{4}{N}), \mu > 0, a > 0\) and \(\mu a^{\frac{(1 - \gamma a)}{2}} \leq (2K)^{\frac{q - 2^*}{2^* - 2}}\). Then \((1.3)\) has a ground state \(u\), which is a minimizer of \(E\) on \(V_a\).

In addition, any ground state of \((1.3)\) is a minimizer of \(E\) on \(V_a\). Moreover, \(G_a\) is compact, up to translation, and it is orbitally stable.

In addition, under the condition \(q \in (2, 2 + \frac{4}{N})\) and \((\mu, a) \in \Omega_1\), for any \(u \in S_a\), the fiber map

\[
\Psi_u(\tau) := E(u_\tau) = \frac{1}{2} \tau^2 \| \nabla u \|^2_2 - \frac{1}{2} \tau^{2^*} \| u \|^2_{2^*} - \frac{\mu}{q} \tau^{2^* \gamma_a} \| u \|^q_q
\]

with

\[
u_\tau(x) := \tau^\frac{N}{2} u(\tau x), \ x \in \mathbb{R}^N, \ \tau > 0
\]

has exactly two critical points, one is a local minimum point at negative level and the other one is a global maximum point at positive level. Consequently, the Pohožaev set

\[
P_a := \{ u \in S_a : P(u) = 0 \} \text{ with } P(u) := \| \nabla u \|^2_2 - \| u \|_{2^*}^{2^*} - \mu |\gamma_a| \| u \|^q_q
\]

admits the decomposition into the disjoint union \(P_a = P_{a,+} \cup P_{a,-}\), where

\[
P_{a,+} := \{ u \in P_a : E(u) < 0 \} \text{ and } P_{a,-} := \{ u \in P_a : E(u) > 0 \},
\]

see Lemma 2.4 in [6]. Moreover, \(0 < \inf_{u \in P_{a,-}} E(u) < \frac{1}{N} S^\frac{2^*}{2}\), see Propositions 1.15 and 1.16 in [6] for \(N \geq 4\) and Lemma 3.1 and Remark 3.1 in [13] for \(N \geq 3\). By using these results and the mountain pass lemma, [6] obtained the following results (The case \(N = 3\) was complemented by [13]):

**Theorem B.** Let \(N \geq 3, q \in (2, 2 + \frac{4}{N}), \mu > 0, a > 0\) and \(\mu a^{\frac{(1 - \gamma a)}{2}} \leq (2K)^{\frac{q - 2^*}{2^* - 2}}\). Then there exists a mountain pass type solution \(v\) to \((1.3)\) with \(\lambda < 0\) and

\[
0 < E(v) = \inf_{u \in P_{a,-}} E(u) < m_a + \frac{S^\frac{2^*}{2}}{N}.
\]

Moreover, the associated standing wave \(e^{-i\lambda t}v(x)\) is strongly unstable.

We wonder what will happen if \(\mu a^{\frac{(1 - \gamma a)}{2}} = (2K)^{\frac{q - 2^*}{2^* - 2}}\). By examining the proof of Theorem B, we find that \(\inf_{u \in P_{a,-}} E(u) \geq 0\) and there are two possibilities:

1. If \(\inf_{u \in P_{a,-}} E(u) = 0\), then by using \((1.6)\) and \((1.7)\), we can show that there is no \(u \in P_a\) such that \(E(u) = 0\);

2. If \(\inf_{u \in P_{a,-}} E(u) > 0\), we can certainly obtain the same results as in Theorem B.

At first, we try to show that \(\inf_{u \in P_{a,-}} E(u) = 0\) by choosing some functions and find that it is a difficult task. Then we turn to study the properties satisfied by \(\{u_n\}\) with \(\{u_n\} \subset P_{a,-}\) and \(E(u_n) \to 0\), and find that such \(\{u_n\}\) does not exist. Hence, \(\inf_{u \in P_{a,-}} E(u) > 0\) (see Lemma 2.5) and we obtain the following result:

**Theorem 1.4.** Let \(N \geq 3, q \in (2, 2 + \frac{4}{N}), \mu > 0, a > 0\) and \(\mu a^{\frac{(1 - \gamma a)}{2}} \leq (2K)^{\frac{q - 2^*}{2^* - 2}}\). Then there exists a mountain pass type solution \(v\) to \((1.3)\) with \(\lambda < 0\) and

\[
0 < E(v) = \inf_{u \in P_{a,-}} E(u) < m_a + \frac{1}{N} S^\frac{2^*}{2}.
\]
Moreover, the associated standing wave \( e^{-i\lambda t}v(x) \) is strongly unstable.

Now we consider the case \( q = 2 + \frac{4}{N} \). In this case, [11] recently obtained that (1.3) admits a ground state if \( \mu a^{\frac{q(1-\gamma)}{2}} < \tilde{a}_N \), where \( \tilde{a}_N := \frac{q}{2C_{N,q}} \) and \( C_{N,q} \) is defined in 1.7, while if \( \mu a^{\frac{q(1-\gamma)}{2}} \geq \tilde{a}_N \), [13] obtained that (1.3) has no ground states by showing that \( c^\text{po} := \inf_{v \in \mathcal{P}_a} E(v) = 0 \). Precisely, [13] obtained:

**Theorem 1.5.** Let \( N \geq 3 \), \( q = 2 + \frac{4}{N} \), \( \mu > 0 \), \( a > 0 \) and \( \mu a^{\frac{q(1-\gamma)}{2}} \geq \tilde{a}_N \). Then \( c^\text{po} = 0 \). Moreover, \( c^\text{po} \) cannot be attained and (1.3) has no ground states.

In the proof of Theorem 1.5, a key step is to show that \( c^\text{po} = 0 \). In [13], the proof of \( c^\text{po} = 0 \) depends on the monotonicity of \( c^\text{po} \) with \( \mu \) (see Lemma 3.3 in [13]). In this paper, we will give a different proof by choosing appropriate functions but without using the monotonicity of \( c^\text{po} \) (see Lemma 3.2). Our methods used to prove \( c^\text{po} = 0 \) has some similarities with the methods used to prove \( \inf_{u \in \mathcal{P}_a} E(u) > 0 \) in Theorem 1.4 (see Lemmas 2.5 and 3.2). It seems to be two sides of a question.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.4. Section 3 is devoted to the proof of Theorem 1.5.

**Notation:** In this paper, it is understood that all functions, unless otherwise stated, are complex valued, but for simplicity we write \( L^r(\mathbb{R}^N) \), \( H^1(\mathbb{R}^N) \) and \( D^{1,2}(\mathbb{R}^N) \). For \( 1 \leq r < \infty \), \( L^r(\mathbb{R}^N) \) is the usual Lebesgue space endowed with the norm \( \|u\|_r := \int_{\mathbb{R}^N} |u|^r dx \), \( H^1(\mathbb{R}^N) \) is the usual Sobolev space endowed with the norm \( \|u\|_{H^1} := \|\nabla u\|_2^2 + \|u\|_2^2 \), and \( D^{1,2}(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \|\nabla u\|_2 < \infty\} \). \( H^1_0(\mathbb{R}^N) \) denotes the subspace of functions in \( H^1(\mathbb{R}^N) \) which are radially symmetric with respect to zero. \( S_{N,a} := S_a \cap H^1_0(\mathbb{R}^N) \).

## 2. Proof of Theorem 1.4

We use the strategy of [6] to prove Theorem 1.4. Since many results obtained in [6] can be extended to the case \( \mu a^{\frac{q(1-\gamma)}{2}} \leq (2K) \frac{\gamma q - 2}{2q} \), directly, we just list them without proofs and concentrate on proving the new result (Lemma 2.5).

**Lemma 2.1.** (Lemma 2.4, [6]) Let the assumptions in Theorem 1.4 hold. Then for every \( u \in \mathcal{S}_a \), the function \( \Psi_u(\tau) \) defined in (1.8) has exactly two critical points \( \tau^+_u \) and \( \tau^-_u \) with \( 0 < \tau^+_u < \tau^-_u \). Moreover:

1. \( \tau^+_u \) is a local minimum point for \( \Psi_u(\tau), E(u_{\tau^+_u}) < 0 \) and \( u_{\tau^+_u} \in V_a \).
2. \( \tau^-_u \) is a global maximum point for \( \Psi_u(\tau), \Psi'_u(\tau) < 0 \) for \( \tau > \tau^-_u \) and
   \[
   E(u_{\tau^-_u}) \geq \inf_{u \in \partial V_a} E(u) > 0.
   \]

In particular, if \( \mu a^{\frac{q(1-\gamma)}{2}} < (2K) \frac{\gamma q - 2}{2q} \), then \( \inf_{u \in \partial V_a} E(u) > 0 \).

3. \( \Psi''_u(\tau^-_u) < 0 \) and the maps \( u \in S_a \mapsto \tau^-_u \in \mathbb{R} \) is of class \( C^1 \).

**Lemma 2.2.** (Lemma 2.6, [5]) Let \( N \geq 3, q \in (2,2+\frac{4}{N}) \), \( \mu > 0 \), \( a \in (0,a_0) \). Then for any \( a_1 \in (0,a) \), we have \( m_a \leq m_{a_1} + m_{a-a_1} \), and if \( m_{a_1} \) or \( m_{a-a_1} \) is reached then the inequality is strict.

Now we set
\[
M_r(a) := \inf_{g \in \Gamma_r(a)} \max_{t \in [0,\infty)} E(g(t)),
\]
where
\[ \Gamma_r(a) := \{ g \in C([0, \infty), S_{a,r}) : g(0) \in \mathcal{P}_{a,+}, \exists t_g \text{ s.t. } g(t) \in E_{2m_n} \text{ for } t \geq t_g \} \]
with \( E_c := \{ u \in H^1(\mathbb{R}^N) : E(u) < c \} \). Then we have

**Lemma 2.3.** (Proposition 1.10, [6]) Let the assumptions in Theorem 1.4 hold. Then there exists a Palais-Smale sequence \( \{ u_n \} \subset S_{a,r} \) for \( E|_{S_a} \) at level \( M_r(a) \), with \( P(u_n) \to 0 \) as \( n \to \infty \).

Next we study the value of \( M_r(a) \). For this aim, we set
\[ M(a) := \inf_{g \in \Gamma(a)} \max_{t \in [0, \infty)} E(g(t)), \]
where
\[ \Gamma(a) := \{ g \in C([0, \infty), S_a) : g(0) \in V_a \cap E_0, \exists t_g \text{ s.t. } g(t) \in E_{2m_n}, t \geq t_g \}. \]

**Lemma 2.4.** (Proposition 1.15 and Remark 5.1, [6]) Let the assumptions in Theorem 1.4 hold. Then \( M_r(a) = M(a) = \inf_{\mathcal{P}_{a,-}} E(u) = \inf_{\mathcal{P}_{a,-} \cap H^1(\mathbb{R}^N)} E(u) \).

**Lemma 2.5.** Let the assumptions in Theorem 1.4 hold. Then \( \inf_{u \in \mathcal{P}_{a,-}} E(u) > 0 \).

**Proof.** In view of Lemma 2.1, we just need to show that \( \inf_{u \in \mathcal{P}_{a,-}} E(u) \neq 0 \) for the case \( \mu a^{\frac{q(1-\gamma)}{2}} = (2K)^{-\frac{2q}{2q-2}} \). Suppose by contradiction that \( \inf_{u \in \mathcal{P}_{a,-}} E(u) = 0 \).

By Lemma 2.4, \( \inf_{\mathcal{P}_{a,-}} E(u) = \inf_{\mathcal{P}_{a,-} \cap H^1(\mathbb{R}^N)} E(u) \), which combined with Lemma 2.1 yields that there exists \( \{ u_n \} \subset S_a \cap H^1(\mathbb{R}^N) \) such that \( P(u_n) = 0 \) and \( E(u_n) = A_n \), where \( A_n \geq 0 \) and \( A_n \to 0 \) as \( n \to \infty \). By using \( E(u_n) = A_n, P(u_n) = 0, \|u_n\|_2^2 = a, (1.6) \) and (1.7), we obtain that
\[
\left\{
\begin{align*}
\|\nabla u_n\|_2^2 &\leq \frac{2(2^*-q\gamma)}{2(2-q\gamma)} \mu C_{N,q}^\mu a^{\frac{q(1-\gamma)}{2}} \|\nabla u_n\|_{2^*}^{q\gamma} + C_1 A_n, \\
\|\nabla u_n\|_2^2 &\leq \frac{2(2^*-q\gamma)}{2(2-q\gamma)} \|\nabla u_n\|_{2^*}^{q\gamma} + C_2 A_n \leq \frac{2(2^*-q\gamma)}{2(2-q\gamma)} S^{\frac{2^*-q\gamma}{2}} \|\nabla u_n\|_2^2 - C_2 A_n,
\end{align*}
\right.
\]
where \( C_1 \) and \( C_2 \) are some positive constants. Consequently, \( \lim_{n \to \infty} \|\nabla u_n\|_2^2 > 0 \) and
\[
\left\{
\begin{align*}
\|\nabla u_n\|_2^{2-q\gamma} &\leq \frac{2(2^*-q\gamma)}{2(2-q\gamma)} \mu C_{N,q}^\mu a^{\frac{q(1-\gamma)}{2}} + o_n(1) = \left( \frac{2(2^*-q\gamma)}{2(2-q\gamma)} S^{\frac{2^*-q\gamma}{2}} \right)^{\frac{2-q\gamma}{2}} + o_n(1), \\
\|\nabla u_n\|_2^{2^*-2} &\geq \frac{2(2^*-q\gamma)}{2(2-q\gamma)} S^{\frac{2^*-q\gamma}{2}} + o_n(1),
\end{align*}
\right.
\]
which implies that
\[
\left\{
\begin{align*}
\|\nabla u_n\|_2^2 &\leq \rho_0 + o_n(1), \\
\|\nabla u_n\|_2^2 &\geq \rho_0 + o_n(1).
\end{align*}
\]

Hence, \( \|\nabla u_n\|_2^2 \to \rho_0 \) as \( n \to \infty \), which combined with (2.1) gives that
\[
\left\{
\begin{align*}
\|u_n\|_{2^*}^2 &\to C_{N,q}^\mu \|u_n\|_{2^*}^{q\gamma(1-\gamma)} \|\nabla u_n\|_{2^*}^{q\gamma}, \\
\|u_n\|_{2^*}^2 &\to S^{-\frac{2^*}{2}} \|\nabla u_n\|_2^{2^*}
\end{align*}
\right.
\]
as \( n \to \infty \). That is, \( \{ u_n \} \subset H^1(\mathbb{R}^N) \) is a minimizing sequence of
\[
\frac{1}{C_{N,q}^\mu} := \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_{2^*}^{q\gamma} \|u\|_{2^*}^{q(1-\gamma)}}{\|u\|_q^2} \quad (2.2)
\]
and
\[
S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}.
\] (2.3)

Since \(\{u_n\} \subset H^1(\mathbb{R}^N)\) is bounded, there exists \(u_0 \in H^1(\mathbb{R}^N)\) \(\setminus \{0\}\) such that \(u_n \rightharpoonup u_0\) weakly in \(H^1(\mathbb{R}^N)\), \(u_n \rightarrow u_0\) strongly in \(L^t(\mathbb{R}^N)\) with \(t \in (2, 2^*)\) and \(u_n \rightarrow u_0\) a.e. in \(\mathbb{R}^N\). By the weak convergence, we have \(\|u_0\|_2^2 \leq \|u_n\|_2^2\) and \(\|\nabla u_0\|_2^2 \leq \|\nabla u_n\|_2^2\).

Consequently, \(u_0\) is a minimizer of (2.2) and \(u_n \rightarrow u_0\) strongly in \(H^1(\mathbb{R}^N)\). By Theorem B in [14], \(u_0\) is the ground state of the equation
\[
\frac{(q-2)N}{4} \Delta u - \left(1 + \frac{(q-2)(2-N)}{4}\right) u + |u|^{q-2}u = 0.
\] (2.4)

By using (2.3) and \(u_n \rightarrow u_0\) strongly in \(H^1(\mathbb{R}^N)\), we obtain that \(u_0\) is a minimizer of \(S\). So \(u_0\) is of the form
\[
u_0 = C \left(\frac{b}{b^2 + |x|^2}\right)^{\frac{N-2}{4}},
\] (2.5)
where \(C > 0\) is a fixed constant and \(b \in (0, \infty)\) is a parameter, see [3]. (2.5) contradicts to (2.4). Thus, \(\inf_{u \in \mathcal{P}_{a,-}} E(u) > 0\).

**Lemma 2.6.** (Lemma 3.1 and Remark 3.1, [13]) Let the assumptions in Theorem 1.4 hold. Then
\[
\inf_{u \in \mathcal{P}_{a,-}} E(u) < m_a + \frac{1}{N} S^N.
\]

**Lemma 2.7.** (Proposition 1.11, [6]) Assume the assumptions in Theorem 1.4 hold. Let \(\{u_n\} \subset S_{a,r}\) be a Palais-Smale sequence for \(E|_{S_a}\) at level \(c\), with \(P(u_n) \rightarrow 0\) as \(n \rightarrow \infty\). If
\[
0 < c < m_a + \frac{1}{N} S^N,
\]
then up a subsequence, \(u_n \rightarrow u\) strongly in \(H^1(\mathbb{R}^N)\), and \(u\) is a radial solution to (1.3) with \(E(u) = c\) and some \(\lambda < 0\).

**Proof of Theorem 1.4.** The existence of a mountain pass type solution is a direct result of Lemmas 2.3, 2.4, 2.5, 2.6 and 2.7. The strong instability of the associated standing wave is the same as Theorem 1.9 in [6].

3. PROOF OF THEOREM 1.5

Let \(\Psi_\nu(\tau)\) be defined in (1.8) and define
\[
\Phi_\nu(\tau) := P(u_\nu) = \tau^2 \|\nabla u\|_2^2 - \tau^{2^*} \|u\|_2^{2^*} - \mu \gamma_\nu \tau^q \|u\|_q^q.
\]

**Lemma 3.1.** Let \(N \geq 3\), \(q = 2 + \frac{4}{N}\), \(\mu > 0\), \(a > 0\) and \(\mu a^{\frac{q(1-\gamma_a)}{2}} \geq a_N\). Then we have the following results:

1. If \(u \in S_a\) such that \(\|\nabla u\|^2_2 > \mu \gamma_\nu \|u\|^2_q\), then there exists a unique \(\tau_u \in (0, \infty)\) such that \(P(u_\tau) = 0\). \(\tau_u\) is the unique critical point of \(\Psi_\nu(\tau)\), and is a maximum point at positive level. Moreover, \(P(u) \leq 0 \iff \tau_u \leq 1\).

2. If \(u \in S_a\) such that \(\|\nabla u\|^2_2 \leq \mu \gamma_\nu \|u\|^2_q\), then there does not exist \(\tau \in (0, \infty)\) such that \(P(u_\tau) = 0\), and \(\Psi_\nu(\tau)\) is negative and is strictly decreasing on \((0, \infty)\).

Proof. The proof of (1) is similar to Lemma 6.2 in [10], and the proof of (2) is a direct result of the expressions of \(P(u_\tau)\) and \(\Psi_\nu(\tau)\). \(\square\)
Lemma 3.2. Let $N \geq 3$, $q = 2 + \frac{4}{N}$, $\mu > 0$, $a > 0$ and $\mu a^q \geq \bar{a}_N$. Then
\begin{align*}
\nu_{q,p}^* := \inf_{v \in \mathcal{P}_N} E(v) = 0.
\end{align*}

Proof. By Lemma 3.1, for any $u \in S_a$ with $\|\nabla u\|^2_2 > \mu \gamma \|u\|^q_2$, there exists a unique
\begin{align*}
\tau_u \in (0, \infty) \text{ such that } P(u, \tau_u) = 0.
\end{align*}
Hence, $\tau_u$ satisfies
\begin{align*}
(\tau_u)^2 (\|\nabla u\|^2_2 - \mu \gamma \|u\|^q_2) = (\tau_u)^2 \|u\|^q_2.
\end{align*}
and
\begin{align*}
E(u, \tau_u) = \left(1 - \frac{1}{2^q}\right) \frac{\|\nabla u\|^2_2 - \mu \gamma \|u\|^q_2}{\|u\|^q_2}.
\end{align*}
Consequently, to prove Lemma 3.2, it is enough to find $\{u_n\} \subset S_a$ with
\begin{align}
\|\nabla u_n\|^2_2 > \mu \gamma \|u_n\|^q_2 \quad \text{and} \quad \frac{\|\nabla u_n\|^2_2 - \mu \gamma \|u_n\|^q_2}{\|u_n\|^q_2} \to 0. \quad (3.1)
\end{align}

**Case 1** ($\mu a^q \geq \bar{a}_N$). Since in this case we have $\mu \gamma C_{q,a}^q a^q = \bar{a}_N = 1$, so for any $u \in S_a$, we get that
\begin{align*}
\mu \gamma \|u\|^q_2 \leq \mu \gamma C_{q,a}^q a^q \|\nabla u\|^2_2 = \|\nabla u\|^2_2
\end{align*}
and the equality holds if and only if $u$ is a minimizer of
\begin{align}
\frac{1}{C_{q,a}^q} := \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|^2 q \|u\|^{q(1-\gamma)}_q}{\|u\|^q_q}, \quad (3.2)
\end{align}
or equivalently, $u$ is the ground state of the equation
\begin{align*}
\frac{(q - 2)N}{4} \Delta u - \left(1 + \frac{(q - 2)(2 - N)}{4}\right) u + |u|^{q-2} u = 0.
\end{align*}
Now let $u \in S_a$ be a minimizer of (3.2) and $\varphi(x) \in \mathcal{C}_c^\infty (\mathbb{R}^N)$ be a cut off function satisfying: (a) $0 \leq \varphi(x) \leq 1$ for any $x \in \mathbb{R}^N$; (b) $\varphi(x) \equiv 1$ in $B_1$; (c) $\varphi(x) \equiv 0$ in $\mathbb{R}^N \setminus B_2$. Here, $B_s$ denotes the ball in $\mathbb{R}^N$ of center at origin and radius $s$. Define
\begin{align*}
v_n(x) := \varphi(x) u_n(x), \quad u_n(x) := a^{1/2} \|v_n\|_2^{-1} v_n(x).
\end{align*}
Then $v_n \to u$ strongly in $H^1(\mathbb{R}^N)$,
\begin{align*}
\|u_n\|^2_2 = a, \quad \|u_n\|^2_q = (a^{1/2} \|v_n\|_2^{-1})^2 \|v_n\|^q_2,
\|\nabla u_n\|^2_2 = (a^{1/2} \|v_n\|_2^{-1})^2 \|\nabla v_n\|^2_2, \quad \|u_n\|^q_2 = (a^{1/2} \|v_n\|_2^{-1})^q \|v_n\|^q_2.
\end{align*}
Next we show that $\{u_n\} \subset S_a$ satisfies (3.1). Since $u_n$ is not a minimizer of (3.2), we deduce that $\|\nabla u_n\|^2_2 > \mu \gamma \|u_n\|^q_2$. Noting that $u \in S_a$ is a minimizer of (3.2), by direct calculations, we obtain that
\begin{align*}
\frac{\|\nabla u_n\|^2_2 - \mu \gamma \|u_n\|^q_2}{\|u_n\|^q_2} = \frac{(a^{1/2} \|v_n\|_2^{-1})^2 \|\nabla v_n\|^2_2 - \mu \gamma (a^{1/2} \|v_n\|_2^{-1})^q \|v_n\|^q_2}{(a^{1/2} \|v_n\|_2^{-1})^q \|v_n\|^q_2} \to 0.
\end{align*}
Thus $\{u_n\} \subset S_a$ satisfies (3.1).
Case 2 \((\mu a^{\frac{1-\gamma_1}{2}} > \bar{a}_N)\). Define 
\[
f(u) := \frac{\|\nabla u\|_2^{q\gamma_q} \|\theta\|_2^{q(1-\gamma_1)}}{\|u\|_q^q}.
\]
By (3.2), for any \(M > \frac{1}{C_{N,q}}\), there exists \(u \in H^1(\mathbb{R}^N) \setminus \{0\}\) such that \(f(u) = M\). For any \(\alpha, \beta > 0\), we define \(\bar{u}(x) := \alpha u(\beta x)\). By direct calculations, we have 
\[
\|\bar{u}\|_2 = \alpha \beta^{-N/2} \|u\|_2, \quad \|\bar{u}\|_q = \alpha \beta^{-N/q} \|u\|_q, \
\|\nabla \bar{u}\|_2 = \alpha \beta^{-N/2} \|\nabla u\|_2
\]
and 
\[
f(\bar{u}) = \frac{(\alpha \beta^{-N/2} \|\nabla u\|_2)^{q\gamma_q} (\alpha \beta^{-N/2} \|u\|_2)^{q(1-\gamma_1)}}{(\alpha \beta^{-N/q} \|u\|_q)^q} = M.
\]
So we can choose \(\alpha = \frac{1}{\|u\|_q} \left(\frac{\|u\|_2}{\|u\|_q}\right)^{N/2}\) and \(\beta = \left(\frac{\|u\|_2}{\|u\|_q}\right)^{q/2}\) such that \(\|\bar{u}\|_2 = a\) and \(\|\bar{u}\|_q = 1\).

Under the assumption \(\mu a^{\frac{1-\gamma_1}{2}} > \bar{a}_N\), we have \(\mu \gamma_q a^{\frac{1-\gamma_1}{2}} > \frac{1}{C_{N,q}}\). Thus, there exists \(\{A_n\} \subset \mathbb{R}\) with \(A_n > 0\) and \(A_n \to 0\) as \(n \to \infty\) such that 
\[
M_n := (\mu \gamma_q + A_n) a^{\frac{1-\gamma_1}{2}} > \frac{1}{C_{N,q}}.
\]
For such chosen \(M_n\), we choose \(\{u_n\} \subset H^1(\mathbb{R}^N)\) such that \(\|u_n\|_2 = a, \|u_n\|_q = 1\) and \(f(u_n) = M_n\). Then we have 
\[
\|\nabla u_n\|_2^2 = (\mu \gamma_q + A_n) \|u_n\|_q^2 > \mu \gamma_q \|u_n\|_q^2, 
\]
1 = \(\|u_n\|_q^2 \leq \|u_n\|_2^{1-\theta} \|u_n\|_2^\theta = a^{\frac{1-\theta}{2}} \|u_n\|_2^\theta\) with \(\frac{1}{q} = \frac{1}{2} - \frac{1}{2\theta}\), and 
\[
\|\nabla u_n\|_2^2 - \mu \gamma_q \|u_n\|_q^2 = A_n \|u_n\|_2^\theta \|u_n\|_2^{2-\theta} \to 0 \text{ as } n \to \infty.
\]
That is, \(\{u_n\} \subset S_q\) satisfies (3.1). The proof is complete. \(\square\)

Proof of Theorem 1.5. In view of Lemma 3.2, the proof of Theorem 1.5 is the same to Proposition 3.2 (2) in [13].

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