Locally Finite Associative Algebras and Their Lie Subalgebras

HASAN M. SHLAKA

Department of Mathematics, Faculty of Computer Science and Mathematics, University of Kufa, Iraq
Email address: hasan.shlaka@uokufa.edu.iq

Abstract. An infinite dimensional associative algebra $\mathcal{A}$ over a field $\mathbb{F}$ is called locally finite associative algebra if every finite set of elements is contained in a finite dimensional subalgebra of $\mathcal{A}$. Given any associative algebra $\mathcal{A}$ over field $\mathbb{F}$ of any characteristic. Consider a new multiplication on $\mathcal{A}$ called the Lie multiplication which defined by $[a,b] = ab - ba$ for all $a, b \in \mathcal{A}$, where $ab$ is the associative multiplication in $\mathcal{A}$. Then $L = \mathcal{A}(-)$ together with the Lie multiplication form a Lie subalgebra of $\mathcal{A}$. It is natural to expect that the structures of $L$ and $\mathcal{A}$ are connected closely. In this paper, we study and discuss the structure of infinite dimensional locally finite Lie and associative algebras. The relation between them, their ideals and their inner ideals is considered. A brief discussion of the simple associative algebras and simple Lie algebras is also be provided.

1. Introduction

Throughout this paper, unless otherwise stated, $\mathbb{F}$ is an algebraically closed field of characteristic $p$, $\mathcal{A}$ is an infinite dimensional locally finite associative algebra over $\mathbb{F}$ and $L$ is an infinite dimensional locally finite Lie algebra over $\mathbb{F}$.

In 2004, Bahturin, Baranov and Zalesski [1] studied simple locally finite Lie subalgebra of the locally finite associative ones. A locally finite (Lie or Associative) algebra $\mathcal{A}$ is a algebra in which for every finite set of elements of $\mathcal{A}$ is contained in a finite dimensional subalgebra $P$ of $\mathcal{A}$. The Lie structure of associative rings or algebras were investigated by the American Mathematician Herstein in 1954 (see [20] and [21]) after defining a new multiplication called the Lie Multiplication by

$$[x,y] := xy - yx \quad \text{for all} \quad x,y \in \mathcal{A},$$

(1.1)

where $xy$ is the usual associative multiplication in the simple associative ring $\mathcal{A}$ over its centre $Z(\mathcal{A})$. Then $\mathcal{A}(-)$ together with the multiplication in (1.1) form a Lie algebra over $Z(\mathcal{A})$. We denote by $\mathcal{A}(-)$ to be the Lie subalgebra of $\mathcal{A}(-)$ together with the multiplication defined in (1.1). Moreover, if an involution $*$ is defined on $\mathcal{A}$, then for any subalgebra $\mathcal{U}$ of $\mathcal{A}$

$$\text{skew}(\mathcal{U}) := \{a \in \mathcal{U} : a^* = -a\}$$

(1.2)

form a Lie algebra with the Lie multiplication that defined as (1.1). Recall that an involution $\star : \mathcal{A} \rightarrow \mathcal{A}$ is an anti-automorphism, defined by $\star (a) = a^*$, satisfy the following conditions $\star (a + b) = a^* + b^*$, $\star (ab) = b^*a^*$ and $\star (a) = \bar{a}$ for all $a, b \in \mathcal{A}$. Involutions of the first kind only is considered in this paper, that is, involutions with the following property: $\star ((aa)) = aa^*$. 

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Baxter [11] focused on the study of the Lie algebras come from simple associative rings with involution in 1958 and Ericson [17] studied the Lie subalgebras of prime rings with involutions in 1972. A revision to Herstein’s Lie theory was giving by Martindale [22] 1986. All of these studies focused on the structure of the Lie ideals and Lie subalgebras that obtained from simple associative rings or algebras. Recall that a subspace \( I \) of \( L \) is called a subalgebra of \( L \) if \( I^{(1)} \subseteq I \) and \( \alpha \) ideal if \([I, L] \subseteq I\). Although simple Lie algebras have no ideals except themselves and the trivial ones, it has been proved in [12] that all simple Lie algebras of classical type have non-zero inner ideals.

In 1976, the American mathematician Georgia Benkart introduced the notion inner ideals of Lie algebras. An inner ideal is a vector subspace \( B \) of \( L \) which satisfies the property \([B, [B, L]] \subseteq B\). By the definition of the Lie ideals, one can see that every ideal is an inner ideal. Moreover, inner ideals are more difficult to be studied as some of them are even not Lie subalgebras. Benkart showed that the structure of the Lie inner ideals are similar to the structure of the ad-nilpotent elements of Lie algebras [13]. Therefore, inner ideals are important in classifying Lie algebras because by using certain restriction on the ad-nilpotent elements one can distinguish the simple Lie algebras of classical type and of the non-classical ones in the case when \( p > 2 \). In several papers (see for example [14], [15] [18] and [19]) Fernández López et al. generalized Benkart’s theory over inner ideals.

In this paper, we discuss the structure of the infinite dimensional simple locally finite algebras. We start Section 2 with some preliminaries. Section 3 states some facts about the plain, diagonal and non-diagonal modules of finite dimensional Lie algebra and Section 4 consists of the infinite dimensional case where the some types of local systems of locally finite algebras (associative or Lie) are considered. Section 5 is the completion of Section 3 where the infinite dimensional cases of plain diagonal and non-diagonal Lie algebras are highlighted. In Section 6 we investigate the structure of (involution) simple and associative algebras. The main results of this paper are found in Sections 7 and 8, where the simple locally finite Lie algebras of simple and involution simple associative algebras are considered.

2. Preliminaries

A perfect Lie algebra is a Lie algebra \( L \) with the property \( L^{(1)} = L \) and a perfect associative algebra is an associative algebra \( A \) such that \( A^2 = A \) [4].

Definition 2.1. [1] A locally finite (associative, Lie,…etc) algebra is an algebra (associative, Lie,…etc) \( A \) over a field \( F \) in which for every finite set of elements in \( A \) we can find a finite dimensional subalgebra of \( A \) that contained it.

Recall that a set \( I \) is said to be a directed partially ordered set if there is an ordering relation \( \leq \) defined on \( I \) such that for each \( \alpha, \beta \in I \), there is \( \gamma \in I \) such that \( \alpha, \beta \leq \gamma \) [2].

Remark 2.2. Suppose that for each \( \alpha, \beta \in I \) with \( \alpha \leq \beta \) we set \( \alpha \leq \beta \). Then for each \( \alpha, \beta \in I \), there is \( \gamma \in I \) such that \( \alpha, \beta \leq \gamma \), so \( I \) is a directed partially ordered set. Thus, \( \lim_{\alpha \to \gamma} \mathscr{A} \) is the direct limits of an infinite chain of algebras \( (\mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots \subset \mathscr{A}_i \subset \cdots) \). Therefore, \( \mathscr{A} \) is the inductive limit \( \mathscr{A} = \lim_{\alpha \to \gamma} \mathscr{A} \) of the algebras \( \mathscr{A}_\alpha \).

We denote by \( M_n(F) \) the vector space of all \( n \times n \)-matrices together with the matrix multiplication defined on it.

Remark 2.3. Every \( M_n(F) \) can be generalized to be an \( (n+1) \times (n+1) \)-matrix \( M_{n+1}(F) \) by putting \( M_n(F) \) in the left upper hand corner and bordering the last column and row by 0's.

Example 2.4. As an example of locally finite associative algebra is the algebra \( M(q)(F) \) of infinite matrices with finite numbers of non-zero entries, that is,
\[ \mathcal{M}_\infty(\mathbb{F}) = \bigcup_{n=1}^{\infty} \mathcal{M}_n(\mathbb{F}). \tag{2.1} \]

By using the Lie multiplication in (1.1) \( \mathcal{M}_1(\mathbb{F}) \), we obtain a Lie algebra called the general linear Lie algebra \( \mathfrak{gl}_n(\mathbb{F}) = \mathcal{M}_n(\mathbb{F}) \). There are three simple Lie subalgebras of \( \mathfrak{gl}_n(\mathbb{F}) \). These are the special linear \( \mathfrak{sl}_n(\mathbb{F}) \), the Orthogonal \( \mathfrak{so}_n(\mathbb{F}) \) and the Symplectic \( \mathfrak{sp}_{2n}(\mathbb{F}) \) Lie algebras are subalgebras of \( \mathfrak{gl}_n(\mathbb{F}) \) which are defined, respectively, by

\[ \begin{align*}
\mathfrak{sl}_n(\mathbb{F}) &= \{ X \in \mathfrak{gl}_n(\mathbb{F}) : \text{tr}(X) = 0 \}; \\
\mathfrak{so}_n(\mathbb{F}) &= \{ X \in \mathfrak{gl}_n(\mathbb{F}) : X^T = -X \}; \\
\mathfrak{sp}_{2n}(\mathbb{F}) &= \{ X \in \mathfrak{gl}_n(\mathbb{F}) : X^T = -X \},
\end{align*} \tag{2.2, 2.3, 2.4} \]

where \( \text{tr}(X) \) is the trace of the matrix \( X \), \( X^T \) is the matrix transpose of \( X \) and \( X^T \) is the symplectic transpose of a matrix \( X \) defined by \( X^T = -JX' \) with \( J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \) (\( I_n \) is the identity \( n \times n \)-matrix).

**Remark 2.5.**
1. It follows from [7] that \( t \) in (2.3) and \( r \) in (2.4) are involutions on \( \mathcal{M}_n(\mathbb{F}) \).
2. \( \mathfrak{gl}_n(\mathbb{F}), \mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_n(\mathbb{F}) \) and \( \mathfrak{sp}_{2n} \) are the Lie algebras of classical type.
3. \( \mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_n(\mathbb{F}) \) and \( \mathfrak{sp}_{2n} \) are called the simple Lie algebras of classical type.

The simple Lie algebras of classical type in Remark 2.5(2) can be constructed from a vector space \( V \) as follows: Consider the vector subspace \( \mathfrak{gl}(V) \) of \( \text{End}(V) \) together with the Lie multiplication defined in (1.1). Then we get the general \( \mathfrak{gl}(V) \) and the special \( \mathfrak{sl}(V) \) linear Lie algebras, where \( \mathfrak{sl}(V) \) is a subalgebra of \( \mathfrak{gl}(V) \) defined by \( \mathfrak{sl}(V) = \mathfrak{gl}(V), \mathfrak{gl}(V) [7] \).

If there is (skew)symmetric \( b \)-linear form \( \theta \) on \( V \), then we get the Orthogonal \( \mathfrak{so}(V, \theta) \) or the Symplectic \( \mathfrak{Sp}(V, \theta) \) Lie algebras, respectively. To simplify notations, we denote by \( \mathfrak{so}(V) \) and \( \mathfrak{sp}(V) \) to be the Orthogonal and the Symplectic Lie algebras, respectively.

**Lemma 2.6.** [7] Let \( V, V_1 \) and \( V_2 \) be vector spaces over \( \mathbb{F} \). Suppose that each of them is of dimension \( n \) and \( p = 0 \).

1. If \( * \) is an involution on the algebra \( \text{End}(V) = \mathcal{M}_n(\mathbb{F}) \), then there is a basis of \( V \) such that \( * \) is expressed as \( X \mapsto X^T \) or \( X \mapsto -X^T \) for each \( X \in \text{End}(V) \). In particular, \( \text{skew(End}(V)) = \mathfrak{so}_n(\mathbb{F}) \) or \( \mathfrak{sp}_{2n}(\mathbb{F}) \).

2. Let \( * \) be an involution defined on the algebra \( \text{End}(V_1) \bigoplus \text{End}(V_2) \) such that \( \text{End}(V_1)^* = \text{End}(V_2) \). Then there are bases of \( V_1 \) and \( V_2 \) such that \( * \) is expressed as \( (X_1, X_2) \mapsto (X_2^T, X_1) \) for each \( X_1 \in \text{End}(V_1) \bigoplus \text{End}(V_2) \). In particular,

\[ \text{skew(End}(V_1) \bigoplus \text{End}(V_2)) = \{ (X, X^T) : X \in \mathcal{M}_n(\mathbb{F}) \} = \mathfrak{gl}_n(\mathbb{F}) \]

**Example 2.8.** [3] Consider the locally finite associative algebra \( \mathcal{M}_\infty(\mathbb{F}) \) in Example 2.4. We construct three locally finite Lie subalgebras of \( \mathcal{M}_\infty(\mathbb{F}) \). Those are the stable special linear \( \mathfrak{sl}_\infty(\mathbb{F}) \), stable Symplectic \( \mathfrak{sp}_\infty(\mathbb{F}) \) and stable Orthogonal \( \mathfrak{so}_\infty(\mathbb{F}) \) Lie subalgebras of \( \mathcal{M}_\infty(\mathbb{F}) \) that defined to be the union (or the direct limit) of the natural embeddings, respectively,

\[ \begin{align*}
\mathfrak{s}_2(\mathbb{F}) &\hookrightarrow \mathfrak{s}_3(\mathbb{F}) \hookrightarrow \cdots \hookrightarrow \mathfrak{s}_n(\mathbb{F}) \hookrightarrow \cdots; \\
\mathfrak{sp}_2(\mathbb{F}) &\hookrightarrow \mathfrak{sp}_4(\mathbb{F}) \hookrightarrow \cdots \hookrightarrow \mathfrak{sp}_{2n}(\mathbb{F}) \hookrightarrow \cdots; \\
\mathfrak{s}_2(\mathbb{F}) &\hookrightarrow \mathfrak{s}_3(\mathbb{F}) \hookrightarrow \cdots \hookrightarrow \mathfrak{s}_n(\mathbb{F}) \hookrightarrow \cdots.
\end{align*} \]

**Definition 2.9.** [2] A locally finite (associative or Lie) algebra \( \mathcal{A} \) over a field \( \mathbb{F} \) is said to be locally semi(simple) in the case when for every finite set \( S \) of elements \( S \) of \( \mathcal{A} \) we can find a finite dimensional (semi)simple subalgebra of \( \mathcal{A} \) which contains \( S \).
Example 2.10. Let $A$ be a simple locally finite associative algebra over $F$. Then for every finite set of elements $S$ of $A$, there is a finite dimensional simple subalgebra $A_\alpha$ (for $\alpha = 1, 2, \ldots$) of $A$ that contains $S$, so there is a chain

$$A_1 \subset A_2 \subset A_3 \subset \ldots$$

of simple subalgebras of $A$ such that $A = \bigcup_{\alpha=1}^n A_\alpha$. Moreover, we can identify each $A_\alpha$ with $\mathcal{M}_{n_\alpha}(F)$ (for all $\alpha = 1, 2, \ldots$), where $n_\alpha$ is an integer number (because $F$ is algebraically closed). Note that each embedding $A_\alpha \subseteq A_{\alpha+1}$ is written as follows:

$$X \mapsto \text{diag}(X, \ldots, X, 0, \ldots, 0), \quad X \in \mathcal{M}_{n_\alpha}(F).$$

3. Plain, diagonal and non-diagonal modules of finite dimensional Lie algebras.

Suppose that $L$ is perfect. Then there is a Levi (maximal semisimple) subalgebra $Q$ of $L$ such that $L = Q \oplus \mathcal{R}$, where $\mathcal{R}$ is a solvable radical of $L$ (Levi-Malcev Theorem). As $\mathcal{R}$ is an ideal of $L$, we have $L/\mathcal{R} = Q$. Let $V$ be a simple $L$-module. Since $A$ is perfect, $\text{Rad}(L)$ annihilate $V$, so $QV = V$ (because $V$ is simple). Let $Q_1, \ldots, Q_k$ be the simple ideals of $Q$ such that $Q = Q_1 \oplus \cdots \oplus Q_k$. Then $V$ is a completely reducible $Q$-module and $V = V_1 \oplus \cdots \oplus V_k$, where $V_i$ is a simple $Q_i$-module.

Remark 3.1. 1) If $Q \cong \mathfrak{sl}(V_i), \mathfrak{so}(V_i), \mathfrak{sp}(V_i)$ for each $1 \leq i \leq k$, then every natural $Q_i$-module $V_i$ is an $L$-module.

2) Suppose that $L \subseteq L'$ is a perfect Lie algebra. If $W$ is an $L'$-module, then $W_{\downarrow L}$ denotes the restriction of $W$ to $L$.

Definition 3.2. Suppose that $L$ is perfect and finite dimensional. Let $V$ be an $L$-module.

1. Suppose that $Q_i \cong \mathfrak{sl}(V_i)$ for each $1 \leq i < k$. Then $V$ is said to be a plain $L$-module if each $V_i$ is a natural $L_i$-module.

2. Suppose that $L'$ is a perfect Lie algebra such that $L'$ is finite dimensional. Let $V'_1, \ldots, V'_k$ be natural $L'$-modules. An embedding $L \subseteq L'$ is called a plain embedding if $(V'_1 \oplus \cdots \oplus V'_k)_{\downarrow L}$ is a plain $L$-module.

Example 3.3. Suppose that $L = \mathfrak{sl}_n(F)$ and $l' = \mathfrak{sl}_m(F)$ for some positive integers $n$ and $m$ with $n < m$. Let $V$ and $T$ be the natural and the trivial 1-dimensional $L$-modules, respectively. Then

1. The embedding $\mathfrak{sl}_n(F) \subseteq \mathfrak{sl}_m(F)$ is called a natural embedding if for every $L'$-module $V'$ we have, $$(V \oplus T) \oplus \cdots \oplus T.$$

2. The embedding $\mathfrak{sl}_n(F) \subseteq \mathfrak{sl}_m(F)$ is called a plain embedding if the $L'$-module $V'$ is plain, that is, $V' = V \oplus T \oplus \cdots \oplus T$ for some positive integers $\ell$ and $r$.

Example 3.4. The embedding $\mathfrak{sl}(V) \subseteq \mathfrak{sl}(W)$ is called a plain embedding if we can find a basis of $W$ such that $X \mapsto \text{diag}(1, \ldots, 1, X, \ldots, 0)$, (for all $X \in \mathfrak{sl}(V)$) where the integers $\ell$ and $z$ do not depend on $X$ and $z + \ell \text{dim}V = \text{dim}W$.

Definition 3.5. Suppose that $L$ is perfect and finite dimensional. Let $V$ be an $L$-module.

1. Suppose that $Q_i \cong \mathfrak{sl}(V_i), \mathfrak{so}(V_i), \mathfrak{sp}(V_i)$ for each $1 \leq i \leq k$. Then $V$ is said to be a diagonal $L$-module in the case when each $V_i$ is either a natural or a dual to natural $L$-module. Otherwise, $V$ is said to be a non-diagonal $L$-module.

2. Suppose that $L'$ is a perfect Lie algebra such that $L'$ is finite dimensional. Suppose that $V'_1, \ldots, V'_k$ are natural $L'$-modules. An embedding $L \subseteq L'$ is called a diagonal embedding if $(V'_1 \oplus \cdots \oplus V'_k)_{\downarrow L}$ is diagonal.
Example 3.6. Let and be classical simple Lie algebras (See Remark 2.5(3)) over . Suppose that , , be a natural, a dual and a trivial -dimensional -modules, respectively. Let be an -module. The embedding is diagonal if

for some positive integers and .

Example 3.7. The embedding is called a diagonal embedding if we can find a basis of such that

where , ( ) ∈ , … , ( ) ∈ , → , … , → , 0, … ,0 , ( ∈ ( ) ) where .

Proposition 3.8. Let be a simple Lie algebra of rank greater than 10. Suppose that ⊆ ⊆ , where are all perfect and finite dimensional Lie algebras. Suppose that = 0 and is a non-diagonal embedding. If is non-trivial for every -module , then there is a natural -module such that is a non-diagonal -module.

4. Local Systems of Locally Finite Algebras

Definition 4.1. Suppose that is a locally finite algebra.

1. A local system of is a set ∈ of finite dimensional subalgebras of satisfying the following conditions:
   i. \( \cup \in \in \in \).
   ii. There exist \( \in \in \) for each pair \( \in \in \) such that \( \subseteq \subseteq \).

2. A local system of is called perfect in the case when are perfect algebras.

3. A local system of is called conical if it is perfect and if has a minimal element satisfying the following conditions:
   i. \( \subseteq \) for all \( \in \);
   ii. is simple;
   iii. If is a natural -module, then the restriction to contains a proper composition factor.

Remark 4.2. Definition 4.1(3.iii.) implies that the rank of every simple ideal of any Levi (maximal semisimple) subalgebra of \( \in \) (for every \( \in \) ) is greater than or equal to the rank of .

Lemma 4.3. Suppose that is a simple locally finite associative (or Lie) algebra. Suppose that \( \cap = 0 \).

The following holds:

1. If \( \cap \in \in \) is local system, then there exists \( \in \) for each \( \in \) such that \( \cap \cap \cap \cap = 0 \).

2. possesses a perfect local system.

3. If \( \cap \in \in \) is a local system of perfect algebras of \( \), then there exists \( \in \in \) for each \( \in \) such that \( \cap \cap = 0 \) for all \( \cap \in \).

Theorem 4.4. Suppose that \( \cap = 0 \) and is locally finite. Then

1. If \( \cap \) is simple with involution \( * \), then \( \cap \) contains a local system which is conical of arbitrary large rank;

2. If \( \cap \) is simple, then \( \cap \) contains a local system which is conical of arbitrary large rank.

Proposition 4.5. Suppose that \( \cap \) is simple and \( \in \in \) is a local system of \( \cap \). Let \( \cap \in \cap \in \) be a system of ideals such that \( \cap \) is an ideal of \( \cap \) for each \( \in \in \). Then either \( \cap \cap = 0 \) or for each \( \in \in \) there is \( \in \in \) with \( \cap \subseteq \cap \) and \( \cap \subseteq \cap \).
Proof. Put \( I = \bigcap_{a \in \Gamma} I_a \). Suppose that \( I \neq 0 \). Let
\[ I' = \bigcap_{a \in \Gamma} \left( X_a \mid X_a \text{ is an ideal of } \mathcal{A}_a \text{ with } X_a \not\supset I \right). \]
Then \( I' \) is an ideal of \( \mathcal{A}_\eta \) with \( Y_a \not\subseteq I' \) for each \( a \in I' \). Let \( X_a \) be a member of \( I' \). Then \( X_a \) is an ideal of \( \mathcal{A}_a \) with \( I \subseteq X_a \). Note that for any \( a \in \Gamma \), we have \( X_a \supset \mathcal{A}_a \). Hence, \( I' = \bigcup_{a \in \Gamma} I_a \) is an ideal of \( \mathcal{A} \) with \( I \subseteq I' \). Therefore, \( I \subseteq I' \). Since \( I \subseteq I' \), there is \( \alpha_k \in \Gamma \) such that \( \mathcal{A}_k \subseteq I_{k+1} \), as required.

5. Plain, diagonal and non-diagonal locally finite Lie algebras

Let \( L \subseteq L' \) be perfect Lie algebras. If \( L \) and \( L' \) are finite dimensional and \( V', \ldots, V_k \) are natural \( L' \)-modules, then an embedding \( L \subseteq L' \) is called a plain (resp. diagonal) embedding if \( \bigoplus_{a \in \Gamma} V_a \) is a plain (resp. diagonal) \( L \)-module.

Definition 5.1. [5] Suppose that \( L \) is simple. Then a plain (resp. diagonal) local system of \( L \) is a perfect local system \( \{ l_a \} \subseteq L' \) such that the embedding \( l_a \subseteq l_b \) is plain (resp. diagonal) for all \( \alpha \leq \beta \).

Example 5.2. If \( L \) is simple and \( p = 0 \), then by Lemma 4.2(2), \( L \) has a perfect local system, say \( \{ l_a \} \subseteq L' \). For each \( a \in \Gamma \), we denote by \( Q_a \) a Levi subalgebra of \( L \) with \( \{ Q_1, Q_2, \ldots, Q_a \} \) is the set of the simple ideals of \( Q_a \), where
\[ Q_a = Q_1^a \oplus \cdots \oplus Q_n^a. \]
Let \( V_k^a \) be the standard \( Q_a \)-module. As \( l_a \) is perfect, for each \( k \) there is a unique indecomposable \( l_k \)-module \( V_k^a \) in which the restriction \( V_k^a \mid_{l_a} \) is isomorphic to \( V_k^a \).

An embedding \( l_a \subseteq l_b \) for \( \alpha < \beta \) is a diagonal embedding if
\[ V_k^a \mid_{l_a} \cong \{ v_1^a, \ldots, v_{n_a}^a, v_1^b, \ldots, v_{n_b}^b, \Gamma_a \}, \quad 1 \leq k \leq n_b, \]
where \( \Gamma_a \) is atrivial and one dimensional \( l_a \)-module and \( V_k^b \mid_{l_b} \) is the dual to \( V_k^b \).

Remark 5.3. [4] Suppose that \( \{ l_a \} \subseteq L' \) is a conical system of \( L \). Then all simple components of \( L_a \) (for each \( a \in \Gamma \)) are of classical type if the rank of \( L_a \) is greater than or equal to 9.

Definition 5.4. [5] Suppose that \( L \) is simple, then \( L \) is said to be plain (resp. diagonal) if \( L \) has a plain (resp. diagonal) local system.

Example 5.5. Consider the zero trace \( n \times n \)-matrices \( \mathfrak{x} \in M_n(F) \), then
1. \( \mathfrak{sl}_n(F) \) and \( \mathfrak{sl}_n(F) \) can be defined to be the limit of the sequence of the natural embeddings:
\[ \phi_1: \mathfrak{sl}_n(F) \Rightarrow \mathfrak{sl}_{n+1}(F) \]
and
\[ \phi_2: \mathfrak{sl}_{2n}(F) \Rightarrow \mathfrak{sl}_{2n+1}(F), \]
where \( \phi_1 \) and \( \phi_2 \) are defined as follows: \( \phi_1(X) = \text{diag}(X, 0) \) and \( \phi_2(X) = \text{diag}(X, X) \), respectively. Then \( \mathfrak{sl}_n(F) \) and \( \mathfrak{sl}_{2n}(F) \) are both simple of diagonal type.

2. A generalization of \( \mathfrak{sl}_{2n}(F) \) can be done as follows: Consider the sequence \( \mathfrak{g} = (\ell_1, \ell_2, \ell_3, \ldots) \) of the positive integers \( \ell_i \). Let \( \eta_n = \ell_1 \ell_2 \ell_3 \ldots \). Then \( \mathfrak{sl}_{\eta_n}(F) \) is defined to be the limit of the sequence of diagonal matrix embedding:
\[ \phi_2: \mathfrak{sl}_{\eta_n}(F) \Rightarrow \mathfrak{sl}_{\eta_{n+1}}, \]
where \( \phi_2 \) is defined as \( \phi_2(X) = \text{diag}(X, X, \ldots, X) \).
Definition 5.6. A subspace $B$ of $L$ is called an inner ideal of $L$ if $[B,[B,L]] \subseteq B$.

Theorem 5.7. Suppose that $p = 0$. The following holds:
1. [4] There exists a simple of diagonal type locally finite algebra that is not locally semisimple.
2. Suppose that $L$ is simple over $\mathbb{F}$, then
   i. [4] $L$ is semisimple as Lie algebra and locally perfect as well.
   ii. [9] $L$ contains a non-trivial inner ideal if it is of diagonal type and vice versa.

Definition 5.8. [5] Suppose that $L$ is simple. Then $L$ is called non-diagonal if there is no diagonal local system of $L$.

Recall that the map $ad: L \rightarrow gl(L)$, $ad(x) = [x,y]$ for all $x \in L$, is linear. The adjoint homomorphism $ad: L \rightarrow gl(L)$ is a linear map defined by $x \rightarrow ad_x$ for all $x \in L$ [16].

Example 5.9. Consider the Lie algebra $s_{ad}(F)$ which is defined to be the limit of the sequence of embeddings

$$s_{2}(F) \rightarrow s(s_{2}(F)) \equiv s_{3}(F) \rightarrow s(s_{3}(F)) \equiv s_{4}(F) \rightarrow \cdots$$

where all embeddings are induced by the adjoint map $x \rightarrow ad_x$ for all $x \in L$.

Theorem 5.10. [9] Suppose that $L$ is simple of non-diagonal type and $p = 0$. The following hold
1. If $\{L_\alpha\}_{\alpha \in \Gamma}$ is a conical system of $L$ of rank $\geq 10$, then for every $\alpha \in \Gamma$, there is $\beta \geq \alpha$ such that $L_\alpha \subseteq L_\beta$ is non-diagonal embedding for all $\gamma \geq \beta$.
2. $L$ has no non-zero proper inner ideals.

6. Simple and simple with Involution associative algebras.

Recall Wedderburn theorem (see [6, Theorem 1]) that if $\mathcal{A}$ is finite dimensional, then $\mathcal{A}$ can be written as $\mathcal{A} = \mathcal{S} \oplus \text{Rad}(\mathcal{A})$, where $\mathcal{S}$ is semisimple subalgebra of $\mathcal{A}$ and $\text{Rad}(\mathcal{A})$ is a nilpotent ideal (the radical) of $\mathcal{A}$.

Lemma 6.1. Suppose that $\mathcal{A}$ is semisimple and finite dimensional. If $p = 0$, then $[\mathcal{A}, \mathcal{A}]$ is a semisimple finite dimensional Lie algebra $\mathcal{A} \oplus \mathcal{F}$.

Proof. Consider the set of the simple ideals $\{S_1, \ldots, S_k\}$ of $\mathcal{A}$, so $\mathcal{A} = S_1 \oplus \cdots \oplus S_k$. Then for each $1 \leq i \leq k$, we have $S_i \cong \mathcal{F}$ for some integer $n_i$, so $[S_i, S_j] \cong \mathcal{F}$ (see (2.2)). Thus,

$$[\mathcal{A}, \mathcal{A}] = [S_1, S_1] \oplus \cdots \oplus [S_k, S_k] \cong \mathcal{F} \oplus \cdots \oplus \mathcal{F}.$$ 

Therefore, $[\mathcal{A}, \mathcal{A}]$ is a semisimple and finite dimensional, as required. $\blacksquare$

Definition 6.2. [1] An associative algebra $\mathcal{A}$ is said to be an involution simple associative algebra if the only $*$-invariant ideals of $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$.

We have the following result. See [1, Proposition 2.3] for the proof.

Proposition 6.3. Every involution simple algebra $\mathcal{A}$ over $\mathbb{F}$ is either simple with involution $*$ or the $\mathcal{A} = \mathcal{Q} \oplus \mathcal{Q}^*$, where $\mathcal{Q}$ is simple ideal.

We will need the following well-known result. See for example [7].
Lemma 6.4. Let $\mathcal{A}$ be semisimple and finite dimensional with involution $\ast$. Suppose that $p = 0$. Then $\{\text{skew}(\mathcal{A}), \text{skew}(\mathcal{A})\}$ is semisimple Lie algebra.

Proof. Let $S_1, \ldots, S_k$ be the simple ideals of $\mathcal{A}$, so $\mathcal{A} = S_1 \oplus \ldots \oplus S_k$. Then by Proposition 6.3, for each $1 \leq i \leq k$, we have $S_i$ is either simple with involution $\ast$ or $S_i = U_i \oplus U_i^\ast$ for some simple ideals $U_i$ and $U_i^\ast$ of $S_i$. Thus, by using Lemma 2.6, we get that

$$\text{skew}(S_i) \cong \{a_{i1}(F), \ldots, a_{in}(F)\}$$

if $S_i$ is simple with involution

Thus, $[\text{skew}(S_i), \text{skew}(S_j)] \cong a_{i1}(F), \ldots, a_{in}(F)$, $a_{i1}(F)$ for each $i = 1, \ldots, k$. Therefore,

$$[\text{skew}(\mathcal{A}), \text{skew}(\mathcal{A})] = [\text{skew}(S_1), \text{skew}(S_1)] \oplus \ldots \oplus [\text{skew}(S_k), \text{skew}(S_k)],$$

is semisimple and finite dimensional.

Definition 6.5. A system $\{\mathcal{A}_\alpha\}_{\alpha \in \Gamma}$ is called an $*$-invariant system if for each $\mathcal{A}_\alpha \in \{\mathcal{A}_\alpha\}_{\alpha \in \Gamma}$ we have $\alpha_\alpha \in \mathcal{A}_\alpha$ for all $\alpha_\alpha \in \mathcal{A}_\alpha$.

We have the following lemma (See [1]).

Lemma 6.6. Let $\mathcal{A}$ be locally finite with involution $\ast$. Then $\mathcal{A}$ contains an $*$-invariant system.

Proof. Consider the local system $(A_\alpha)_{\alpha \in \Gamma}$ of $\mathcal{A}$. Then for each $\alpha \in \Gamma$, consider the subalgebra $A_\alpha$ of $\mathcal{A}$ that generated by $\mathcal{A}_\alpha + A_\alpha^\ast$. Since $\mathcal{A}_\alpha^\ast \in \mathcal{A}$, for all $\alpha \in \mathcal{A}_\alpha$, we get that $A_\alpha$ is an $*$-invariant subalgebra of $\mathcal{A}$. Thus, $\{A_\alpha\}_{\alpha \in \Gamma}$ is a $*$-invariant local system of $\mathcal{A}$.

Proposition 6.7. If $\mathcal{A}$ is simple with involution and $p = 0$.

1. [1] $\mathcal{A}$ has an $*$-invariant conical system of large rank.

2. [9] If $\{A_\alpha\}_{\alpha \in \Gamma}$ is a $*$-invariant conical system of $\mathcal{A}$, then for every $\alpha \in \Gamma$ there exists $\alpha' \in \Gamma$ satisfying that for all $*$-invariant maximal ideals $I$ of $A_{\beta} \geq \alpha'$ we have $A_\alpha \cap I = 0$.

7. Locally finite Lie algebras of simple associative algebras

Definition 7.1. [1] 1) An associative algebra $\mathcal{A}$ is said to be an envelope of a Lie algebra $L$ if:

i. $L$ is a subalgebra of $\mathcal{A}$.

ii. $L$ generates $\mathcal{A}$.

2) An envelope $\mathcal{A}$ of $L$ is said to be a $B$-envelope of $L$ if $L = [\mathcal{A}, \mathcal{A}]$.

In what follows, $U(L)$ is denoted to be the universal enveloping algebra of $L$ and $A(L)$ the augmented ideal of $U(L)$ which defined to be the ideal of $U(L)$ of codimension $1$. Recall that universal enveloping algebra $U(L)$ of $L$ is an infinite-dimensional associative algebra [16]. If $A$ is a $B$-envelope of $L$, then $A$ can be considered as the augmented ideal $A(L)$ of $U(L)$ [1]. Therefore, and there is a $1-1$ correspondence between $A$ and $H_A$ with the following property $H_{A} \cap L = 0$, $A(L)/H_{A} \cong \mathcal{A}$.

Remark 7.2. We say that $\mathcal{A} \leq \mathcal{C}$ if and only if $H_{\mathcal{A}} \geq H_{\mathcal{C}}$.

Theorem 7.3. [1] If $L$ is simple plain and $p = 0$, then $L$ generates two $B$-envelopes associative algebras $\mathcal{A}_+$ and $\mathcal{A}_-$ such that:

1. The radical $\text{Rad}(\mathcal{A}_+)$ annihilates $\mathcal{A}_-$.

2. $\mathcal{A}_+/\text{Rad}(\mathcal{A}_+)$ is a simple $B$-envelope of $L$.

3. If $\mathcal{A}_+$ is a $B$-envelope of $L$, then $\mathcal{A}_+/\text{Rad}(\mathcal{A}_+) \leq \mathcal{A} \leq \mathcal{A}_+$ or $\mathcal{A}_-/\text{Rad}(\mathcal{A}_-) \leq \mathcal{A} \leq \mathcal{A}_-$.

4. The inverse of the mapping in (v) is defined by $\mathcal{A} \rightarrow |\mathcal{A}, \mathcal{A}|$. 

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Recall that a subspace \( B \) of \( L \) is called an inner ideal of \( L \) if \( [B, [B, L]] \subseteq B \) (see Definition 5.6). \( B \) is called abelian in the case when \( [B, B] = \{0\} \). An inner ideal of the Lie algebra \( \mathfrak{A}(-) \) is called Jordan-Lie in the case when \( B^2 = \{0\} \) [10].

**Theorem 7.5.** Let \( \mathfrak{A} \) be simple and \( p = 0 \).

1. \([\mathfrak{A}, \mathfrak{A}]\) is a simple and plein. Moreover, \( \mathfrak{A} \) is \( \mathfrak{B} \)-envelope of \( [\mathfrak{A}, \mathfrak{A}] \).
2. \([\mathfrak{A}, \mathfrak{A}]\) contains a proper inner ideal.
3. \([\mathfrak{A}, \mathfrak{A}]\) is regular if and only if there is a right \( \mathfrak{B} \)-ideal of \( \mathfrak{A} \) which is \( \mathfrak{A} \)-regular.

**Proof.** Part (1) is proved in [1]. For the proof see [1, Theorem 2.12].

2. By (1), \( \mathfrak{A}^{(1)} \) is a simple and diagonal, so by Theorem 7.2.1, \( L \) contains a non-trivial inner ideal, as required.

3. This is proved in [9]. For the proof see [9, Corollary 4, i.4].

4. Let \( B \) be an inner ideal of \( [\mathfrak{A}, \mathfrak{A}] \). By using (3), we get that \( [B, B] \subseteq B^2 = \{0\} \). □

**Definition 7.6.** [9] An inner ideal \( B \) of \( \mathfrak{A}(-) \) (or \([\mathfrak{A}, \mathfrak{A}]\)) is called regular if \( B \) is Jordan-Lie and \( B \mathfrak{A} B \subseteq B \).

**Lemma 7.7.** [10] If \( p \neq 2,3 \), then an inner ideal \( B \) of \([\mathfrak{A}, \mathfrak{A}]\) is regular if and only if there is right \( \mathfrak{B} \) and left \( \mathfrak{L} \) ideals of \( \mathfrak{A} \) with \( \mathfrak{B} \mathfrak{L} = 0 \) such that

\[
\mathfrak{B} \mathfrak{L} \subseteq B \subseteq \mathfrak{B} \cap \mathfrak{L} \mathfrak{F} [\mathfrak{A}, \mathfrak{A}].
\]

We will need the following proposition. It represents a special case of [10, Proposition 6.20].

**Proposition 7.8.** [10] If \( p \neq 2,3 \), then Jordan-Lie inner ideals of \( \mathfrak{A}(-) \) and of \([\mathfrak{A}, \mathfrak{A}]\) are regular.

**Theorem 7.9.** Let \( \mathfrak{A} \) be simple and locally semisimple. If \( p = 0 \), then

1. \([\mathfrak{A}, \mathfrak{A}]\) is locally semisimple as Lie algebra.
2. Consider the system \( \{\mathfrak{A}_\alpha\}_{\alpha \in \Gamma} \) of \( \mathfrak{A} \). If \( \mathfrak{B} \) is inner ideal of \( \mathfrak{A} \), \( \mathfrak{B} \cap [\mathfrak{A}, \mathfrak{A}] \) is a inner ideal of \( \mathfrak{A} \).
3. Every inner ideal \( \mathfrak{A}^{(1)} \) is regular.
4. Every proper inner ideal of \( \mathfrak{A}^{(1)} \) can be written as \( \mathfrak{B} \mathfrak{L} \) for some right \( \mathfrak{B} \) and left \( \mathfrak{L} \) ideals of \( \mathfrak{A} \) with \( \mathfrak{B} \mathfrak{L} = 0 \).

**Proof.** 1. Consider the semisimple system \( \{\mathfrak{A}_\alpha\}_{\alpha \in \Gamma} \). Then by Lemma 6.1, \( \{\mathfrak{A}_\alpha \cap \mathfrak{A}_\beta\} \subseteq [\mathfrak{A}_\alpha, \mathfrak{A}_\beta] \), so \( \mathfrak{A}^{(1)} \) is locally semisimple Lie algebra, as required.

2. By Theorem 7.5(3), that is, \( B^2 = \{0\} \), so we only need to prove that \( B \mathfrak{A} B \subseteq B \). Let \( \mathfrak{S} = \{S_\alpha\}_{\alpha \in \Gamma} \) be a semisimple local system of \( \{\mathfrak{A}_\alpha\} \), where \( S_\alpha \mathfrak{F} \mathfrak{F} \) is a semisimple local system of \( \mathfrak{A} \). By (1), \( B \mathfrak{A} B = \mathfrak{B}_\mathfrak{L} \subseteq \mathfrak{S}_\mathfrak{B} \mathfrak{L} \mathfrak{B} \). Since \( \mathfrak{S}_\mathfrak{B} \mathfrak{L} \mathfrak{B} \) is semisimple, By Proposition 7.8, \( \mathfrak{B}_\mathfrak{L} \) is a regular inner ideal of \( \mathfrak{S}_\mathfrak{B} \mathfrak{L} \mathfrak{B} \), so \( \mathfrak{B}_\mathfrak{L} \mathfrak{S}_\mathfrak{B} \mathfrak{B} \subseteq \mathfrak{B}_\mathfrak{B} \). Thus, \( x \in \mathfrak{B}_\mathfrak{B} \subseteq B \). Therefore, \( B \) is regular.

4. This follows from (3) and Lemma 7.7. □
8. Locally finite Lie algebras of involution simple associative algebras

Definition 8.1. An associative algebra \( \mathcal{A} \) with an involution is called \( B^\ast \)-envelope if \( \mathcal{A} \) is an envelope of \( L \) and \( L = K(1) \), where \( K = \text{skew}(\mathcal{A}) \).

Theorem 8.2. [1] If \( p = 0 \), then \( L \) generates a unique \( B^\ast \)-envelope associative algebras \( \mathcal{A} \) such that
1. The Jacobson radical \( \text{Rad}(\mathcal{A}) \) annihilates \( \mathcal{A} \).
2. \( \mathcal{A}/\text{Rad}(\mathcal{A}) \) is a simple \( B^\ast \)-envelope of \( L \).
3. If \( \mathcal{A} \) is a \( B^\ast \)-envelope of \( L \), then either \( \mathcal{A}/\text{Rad}(\mathcal{A}) \leq \mathcal{A} \leq \mathcal{A} \).
4. The mapping \( L \mapsto \mathcal{A}/\text{Rad}(\mathcal{A}) \) is a \( 1 \sim 1 \) correspondence between \( L \) and the set of all involution simple infinite dimensional locally finite associative algebras.
5. The inverse of the linear transformation in (iv) is defined to be \( \mathcal{A} \mapsto [\text{skew}(\mathcal{A}), \text{skew}(\mathcal{A})] \).

Theorem 8.3. [1] Let \( p = 0 \). Then \( L = [K, K] \) is simple and diagonal. Moreover, \( \mathcal{A} \) is \( B^\ast \)-envelope of \( [K, K] \).

An inner ideal of \( K = \text{skew}(\mathcal{A}) \) or \( [K, K] \) is said to be Jordon-Lie if \( B^2 = 0 \) [23].

Definition 8.4. [9] An inner ideal \( B \) of \( K = \text{skew}(\mathcal{A}) \) (or \( [K, K] \)) is said to be a \( * \)-regular if \( B \) is Jordan-Lie and \( \text{skew}(B^\ast B) \subseteq B \).

Lemma 8.5. [10] Suppose that \( K = \text{skew}(\mathcal{A}) \) and \( p = 0 \). An inner ideal \( B \) of \( K(1) \) is \( * \)-regular if and only if there exists left ideal \( \mathcal{U} \) of \( \mathcal{A} \) satisfying \( \mathcal{U}^\ast \mathcal{U} = 0 \) such that \( \mathcal{U} \mathcal{U} \subseteq \mathcal{U} \mathcal{U}^\ast \mathcal{U} \subseteq K(1) \).

Theorem 8.6. [9] If \( p = 0 \) and \( \mathcal{A} \) is locally \( * \)-semisimple, then the following hold.
1. \( [K, K] \) is locally semisimple.
2. Suppose that \( [K, K] \) is non-isomorphic to \( \mathfrak{so}_n(\mathbb{F}) \), then
   i. If \( B \) is inner ideal of \( K(1) \), then \( B \) is \( * \)-regular;
   ii. If \( B \) is inner ideal of \( K(1) \) can be written in form \( \mathcal{U} \mathcal{U}^\ast \) for some left \( \mathcal{U} \) ideal of \( \mathcal{A} \) with \( \mathcal{U}^\ast \mathcal{U} = 0 \).

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