On Effective Constraints for the Riemann-Lanczos System of Equations.

S. Brian Edgar

Department of Mathematics, Linköpings universitet, Linköping, Sweden S-581 83.
E-mail: bredg@mai.liu.se

Abstract
There have been conflicting points of view concerning the Riemann–Lanczos problem in 3 and 4 dimensions. Using direct differentiation on the defining partial differential equations, Massa and Pagani (in 4 dimensions) and Edgar (in dimensions $n \geq 3$) have argued that there are effective constraints so that not all Riemann tensors can have Lanczos potentials; using Cartan’s criteria of integrability of ideals of differential forms Bampi and Caviglia have argued that there are no such constraints in dimensions $n \leq 4$, and that, in these dimensions, all Riemann tensors can have Lanczos potentials. In this paper we give a simple direct derivation of a constraint equation, confirm explicitly that known exact solutions of the Riemann-Lanczos problem satisfy it, and argue that the Bampi and Caviglia conclusion must therefore be flawed. In support of this, we refer to the recent work of Dolan and Gerber on the three dimensional problem; by a method closely related to that of Bampi and Caviglia, they have found an ‘internal identity’ which we demonstrate is precisely the three dimensional version of the effective constraint originally found by Massa and Pagani, and Edgar.
1. Introduction.
In two recent papers Dolan and Gerber [1,2] have revisited the Riemann–Lanczos problem [3], i.e., whether a Riemann tensor \( R_{abcd} \) can be generated from a 3-index tensor potential \( H_{abc} \),

\[
R_{abcd} = 2H_{ab[c,d]} + 2H_{cd[a,b]} \tag{1}
\]

where the potential \( H_{abc} \) satisfies

\[
H_{abc} = H_{[ab]c}, \quad H_{[abc]} = 0. \tag{2}
\]

In the literature†, there are two apparently conflicting answers to this problem in four dimensions: Massa and Pagani [5] have argued that:

... for the class of spacetime metrics satisfying \( R_{ab} = \lambda g_{ab} \) one of the integrability conditions of the system (1) takes the form \( R^2 - 2R_{abcd}R^{abcd} = 0 \), i.e., it imposes a restriction on the geometry itself

whereas Bampi and Caviglia [8] have argued that:

..., in the four-dimensional case no integrability condition is required. In other words, looking at the class of singular solutions allows a potential to exist without any restriction .... on the geometric structure of the underlying Riemann manifold

These two papers were written at the same time, and neither refers to the other. Of course this uncertainty would be dispelled if there were explicit examples of Lanczos potentials in reasonably general spaces; however, the only known examples are a few in flat [3,9] and conformally flat spaces [9] and a few very special spacetimes in [1,2].

Dolan and Gerber [1,2] rely on the results and to some extent on the method from [8], while stating that the paper of Massa and Pagani in [5], and the subsequent work by Edgar in [6,7],

... uses a totally different approach ... and is not applicable here.

We agree with Massa and Pagani [5] that if the representation (1) exists for arbitrary Riemann tensors then it would have very significant implications for both the mathematical viewpoint and physical applications of general relativity. Therefore it is important to determine whether the representation (1) exists for all spaces (as argued in [8,1,2]), or whether it only exists for a restricted number of spaces (as argued in [5,6,7]). If the latter case, it would be interesting to know whether there exist other effective constraints. Moreover, we believe that the work in [5,6,7] is very applicable to the work in [8,9] and [1,2], and a fuller understanding of the links between these two lines of investigation should clarify the apparent contradiction.

So, in this paper, we shall first demonstrate in a very simple unambiguous manner that (1) implies an integrability condition which is an effective constraint for dimensions \( n > 2 \) and therefore limits the class of spaces which can permit a Lanczos potential with properties (2) via (1). Moreover, we will show explicitly, that the Lanczos potentials of those very special spaces found in [1,2,3,9] satisfy this restriction. Furthermore, we demonstrate that the nontrivial 'internal identity' found recently by Dolan and Gerber in the three dimensional problem [1] is precisely the effective constraint found in [5,6,7], and we argue that the Bampi-Caviglia analysis [8] is therefore flawed. We propose that the Janet-Riquier approach used in [1] should be applied to other dimensions.

2. Effective Constraints on Riemann–Lanczos system.

† We follow the conventions of [4]. Since there will be expressions which involve sums of terms with respectively the Ricci tensor and the square of Ricci tensor, a consistent convention is essential. In particular it should be noted that in [5,6,7] a different convention was used for the Ricci tensor and so there are some sign differences between some equations in this paper and their counterparts in [5,6,7].
Riemann-candidate–Lanczos problem in n-dimensional spaces

We consider first the more general problem in n-dimensional spaces of whether any Riemann-candidate tensor, \( \hat{R}_{abcd} \) — a tensor having the algebraic index symmetries of the Riemann tensor — can be generated from a potential \( H_{abc} \) by

\[
\hat{R}_{abcd} = 2H_{[ab}^{[c;d]} + 2H_{cd[a;b]}^{[a;\cdot]]}
\]  

where the potential \( H_{abc} \) satisfies (2).

Flat space

It is trivial to show that in flat space, although at the first derivative level we can only eliminate one of the potential terms,

\[
\hat{R}_{ab[cd;e]} = 2H_{[cd}^{[ab]} ; e]
\]  

by taking another derivative we can obtain

\[
\hat{R}^{[ab} _{[de;f]} c] = 0.
\]

It is important to check whether (5) is an effective constraint for arbitrary \( \hat{R}_{abcd} \) — in the sense that not all Riemann-candidates \( \hat{R}_{abcd} \) satisfy (5) — or whether there are situations when the lefthand side is identically zero; it is obvious that the left hand side is identically zero for dimensions \( n = 2 \), but in all other dimensions it is an effective constraint.†

Therefore the representation (3), in flat space \((n > 2)\), is valid only for the subset of Riemann-candidates which satisfy (5). This is an important example which illustrates two principles: it shows that a particular constraint on the geometry of the space, i.e., putting the curvature tensor \( R_{abcd} = 0 \), can imply a specific constraint on the Riemann-candidate \( \hat{R}_{abcd} \); on the other hand, although the constraint is quite restrictive, yet it does permit a significant class of exceptions. For instance, a set of Riemann candidates which satisfy the ‘flat space Bianchi equations’, \( \hat{R}_{ab[cd;e]} = 0 \); in this case \( H_{abc} = (h_{bc}^{[a} - h_{ac}^{[b}) \) and \( \hat{R}_{abcd} \) has the form of the Riemann tensor in the linearised theory \([3,9]\). Of course, this does not say anything directly about the Riemann–Lanczos problem, since in flat space the constraint is trivially satisfied. However we believe that the curved space analogue of this constraint (5) is the crucial equation in the Riemann-candidate–Lanczos problem and the Riemann–Lanczos problem.

Curved space

In general curved spaces we can carry out on (3) the same differentiation steps as led to (5), but this time the right hand side of equation (5) becomes complicated. However, we can easily find a much simpler subset, with significant properties. Noting that

\[
\hat{R}_{ab} \equiv \hat{R}^{i}_{aib} = 2H^i_{a[ib]} + 2H^i_{b[i;a]} \quad \text{and} \quad \hat{R} \equiv \hat{R}^i_i = 4H^i_{ij;j}
\]

we find that, after some rearranging of (3)†,

† This is an important point, and the effectiveness of the constraint should not be taken for granted. (In the Weyl-candidate–Lanczos problem \([9,10,11,12,13]\) a similar calculation gives an analogous equation which turns out to be trivially satisfied in 4 and 5 dimensions \([14,15]\); this was because any trace-free tensor \( A^{[abc]}_{[de;f]} \) is identically zero in dimensions \( n \leq 5 \) \([16,17]\).) To prove the effectiveness of this constraint we can simply choose a local Cartesian coordinate system \( x^i, i = 1, \ldots, n \) for \( n > 2 \) and construct a simple counterexample, e.g., \( R_{1312} = R_{1213} = \sin x^3 \sin x^2 \), all other Riemann tensor components zero.

‡ The left hand side of (7) is equivalent to \( 3\hat{R}^{[ab}_{abc] c} \) for \( n > 2 \); we begin instead with \( \hat{R}^c_a - 2\hat{R}^{ab}_{:ab} \) which means that our analysis also includes the case \( n = 2 \).
\[
\hat{R}^a_a - 2\hat{R}^{ab}_{;ab} = 4H^{ij}_{;ij}a^a - 4(H^i_{a[ib]} + H^i_{b[a]})^{ab} = 2R_a^aH^a_j + 4R^{ab}H_a^j_{;j:b} - 4R^{abc}H_{cab} + 2R^{abcd}H_{abcd}.
\]

(7)

In flat space this equation is the triple trace of (5).

The next step is the important one; by substituting \(\hat{R}_{abcd}\) in the last term on the right hand side we eliminate some of the remaining awkward potential terms as well as introducing the Riemann-candidate explicitly also on the righthand side,

\[
\hat{R}^a_a - 2\hat{R}^{ab}_{;ab} = 2R_a^aH^a_j + 4R^{ab}H_a^j_{;j:b} - 4R^{abc}H_{cab} + \frac{1}{2}R^{aijk}\hat{R}_{aijk}
\]

(8)

By decomposing \(R_{abcd}\) and \(\hat{R}_{abcd}\) into their trace-free parts with \(R_{ab} = S_{ab} + Rg_{ab}/n\), \(\hat{R}_{ab} = \hat{S}_{ab} + \hat{R}g_{ab}/n\), we obtain the alternative form,

\[
2\hat{S}^{ab}_{;ab} + \frac{2-n}{n}R^a_a = \left(\frac{4}{n} - 2\right)R_a^aH^a_j - 4S^{ab}H_a^j_{;j:b} + 4S_{abc}H_{cab} = \frac{1}{2}C^{abcd}\hat{C}_{abcd} + \frac{2}{n - 2}S^{ab}\hat{S}_{ab} + \frac{n - 2}{n}R\hat{R}.
\]

(9)

\(n = 2\)

Equation (9) is not valid for \(n = 2\), but the previous equation (8) is. However, when we substitute \(n = 2\) with \(R_{abcd} = Rg_{[a}g_{d]b}\) and \(\hat{R}_{abcd} = \hat{R}g_{[a}g_{d]b}\) into (8) the constraint collapses to a trivial identity, and so in two dimensions this particular constraint is not effective.

\(n > 2\)

We cannot conclude that (9) is an effective constraint on all geometries, and on all Riemann candidates, because of the existence of the potential \(H_{abc}\) and its derivatives alongside the Riemann candidate \(\hat{R}_{abcd}\) and the Riemann tensor \(R_{abcd}\). However for Einstein spaces, \(S_{ab} = R_{,a}\) we obtain an expression with no explicit terms in the potential \(H_{abc}\),

\[
2\hat{S}^{ab}_{;ab} + \frac{2-n}{n}R^a_a = -\frac{1}{2}C^{abcd}\hat{C}_{abcd} + \frac{n - 2}{n}R\hat{R}
\]

(10)

and in particular, for spaces of constant curvature, we obtain

\[
2\hat{S}^{ab}_{;ab} + \frac{2-n}{n}R^a_a = \frac{n - 2}{n}R\hat{R}.
\]

(11)

Therefore, for Einstein spaces, we find that the existence of a potential \(H_{abc}\) in (3) leads to an effective constraint because the terms involving the potential explicitly all disappear, and we get a condition (10) directly linking the background space geometry via the Riemann tensor \(R_{abcd}\) with the Riemann candidate \(\hat{R}_{abcd}\).

There are some very special situations where this restriction (10) is satisfied trivially; e.g., if the Riemann candidate \(\hat{R}_{abcd}\) satisfied a Bianchi like equation, \(\hat{R}_{ab[cd;e]} = 0\) and also has its Weyl and Ricci scalar part zero, \(\hat{C}_{abcd} = 0 = \hat{R}\). Of course we cannot conclude that in such situations a Lanczos potential will exist; we must also remember that there could be additional constraints at this order, or at higher orders of differentiation.

Turning to spaces other than Einstein spaces; important questions, with respect to a particular non-Einstein space, are if there exist more constraints, and whether all Riemann-candidates or some Riemann candidates or no Riemann candidates can be generated by a potential from (3). It is easy to see that there must always be some Riemann candidates, since in a particular space, if we choose a particular tensor \(H_{abc}\) with the symmetries (2), we can then define a Riemann-candidate via (3) which will automatically satisfy the constraint (9). However, our analysis is unable to tell us whether, in all non-Einstein spaces, for arbitrary
Riemann-candidates there exists a potential $H_{abc}$ with the symmetries (2) such that both the system (3) and the constraint (9) are satisfied.

Therefore what we are able to conclude for the Riemann-candidate–Lanczos problem is that:

- for $n = 2$ the constraint (9) linking the Riemann-candidate tensor and the geometry is trivially satisfied,
- for $n > 2$ the constraint (9) linking the Riemann-candidate tensor and the geometry is effective in some spaces, e.g., (10), and so not all Riemann-candidates can admit Lanczos potentials in all spaces via the representation (3);
- for $n > 2$, we know that there are some spaces and Riemann-candidate tensors which can admit Lanczos potentials via the representation (3) with (9) also satisfied identically; we do not know if there are more situations where Riemann-candidate tensors can admit Lanczos potentials via the representation (3), but if there are, then (9) will be satisfied identically; we also do not know if there are more constraints.

**Riemann–Lanczos problem in $n$-dimensional spaces**

When we specialise to the Riemann–Lanczos problem, i.e., $\hat{R}_{abcd} \equiv R_{abcd}$, we find that, as a consequence of (1), the left hand side of (9) is identically zero via the contracted Bianchi identity giving,

$$0 = \left(\frac{4}{n} - 2\right) R_{a}^{\ j} H^{aj} - 4S_{ab}^{\ j} H_{aj} + 4S_{abc}^{\ j} H_{aj} - \frac{1}{2} C^{abcd}C_{abcd} + \frac{2}{n - 2} S_{ab} S_{ab} + \frac{n - 2}{n} R^2$$

(12)

We can then conclude that for $n > 2$ we cannot have a Lanczos potential $H_{abc}$ for a space of (nonzero) constant curvature, and for $n \geq 4$ the only Einstein spaces which can have a Lanczos potential are those subjected to the restriction,

$$C^{aijk}C_{aijk} = \frac{2}{n} \frac{n - 2}{n} R^2.$$ 

(13)

This restriction is effective since there are no explicit terms involving the potential, and what we have is a direct condition on the geometry, which clearly not all Einstein spaces satisfy; in fact (13) is an additional invariant condition linking two Riemann scalar invariants in Einstein spaces.

This restriction (12) is only one scalar equation and so in general it, in itself, would not appear to be a very strong restriction on the class of Riemann tensors. For instance, in four dimensions we know that there exists fourteen Riemann scalar invariants — in general; however, when we specialise to vacuum 4-dimensional spacetimes we note that this constraint (13) excludes all Petrov types of the Weyl tensor except the very specialised Petrov type N. So, in vacuum in 4-dimensional spacetimes, (13) is a very strong restriction. Although Petrov type N spaces are not restricted by (13), we cannot conclude that they admit Lanczos potentials via (1); we must again remember that there could be additional constraints at this order, or at higher orders of differentiation.

As regards spaces other than Einstein spaces we know that there are a few explicit special examples of Riemann tensors with Lanczos potentials, e.g., some conformally flat spaces given in [9], and Debever, Gödel, Kasner 4-dimensional spacetimes and a Gödel 3-dimensional space given in [1,2].

To confirm the significance of the integrability condition (12) for the Riemann–Lanczos problem we have shown that all of these special examples (with the exception of the Kasner spacetime where the calculations were too complicated) are non-Einstein spaces and we have demonstrated explicitly that they satisfy (12). Whether potentials can be found for the Riemann tensors of all non-Einstein spaces cannot be decided from the above analysis.

‡ In [8], three examples of conformally flat 4-dimensional spacetimes are given and it is simple to confirm that the respective Lanczos potentials satisfy (12). In [2], Lanczos potentials are given for an example of a Debever and Gödel 4-dimensional spacetimes, and in [1] a Lanczos potentials is given for an example of a Gödel 3-dimensional space; in these spaces it is straightforward, with the help of GRTensorII [18], to confirm that (12) is satisfied.
Therefore what we are able to conclude for the Riemann-Lanczos problem is that:
• for \( n = 2 \) the constraint (12) on the Riemann tensor is trivially satisfied.
• for \( n > 2 \) the constraint (12) on the Riemann tensor is effective in some spaces, e.g. (13), and so not all Riemann tensors can admit Lanczos potentials via the representation (1);
• for \( n > 2 \), we do know that there are some special examples of Riemann tensors which can admit Lanczos potentials via the representation (1), and they also satisfy (12); we do not know if there are any others, but if there are, then the constraint (12) must be satisfied; we do not know if there are any more constraints.

The existence of this constraint for the Riemann–Lanczos problem was originally demonstrated in four dimensions by Massa and Pagani [5] who set up the problem in ordinary tensor notation, but carried out the actual derivation of the crucial constraint equation in tensor-valued differential forms; this calculation was quite involved, and strictly 4-dimensional\(^\dagger\). The tensor-valued differential form part of the derivation of the integrability condition was rederived by Edgar in [6] in ordinary tensor notation, but the argument was still strictly 4-dimensional. Subsequently, a more direct and complete derivation of the constraint equation — with no explicit dimension imposed — was given in [7], and an even simpler variation by Höglund [19]. The derivation given above for the Riemann-candidate–Lanczos problem is based on the version in [19].

3. Effective Constraints for the Parallel Problem.
In their investigations Bampi and Caviglia [8,9] did not in fact deal with \( \hat{R}_{abcd} \) and \( H_{abc} \) directly but rather with their respective counterparts \( N_{abcd} \) and \( T_{abc} \) which satisfied

\[
N_{abcd} = 2T_{ab[c;d]} + 2T_{cd[a;b]} \tag{14}
\]

where \( N_{abcd} \) and \( T_{abc} \) have only the respective symmetries,

\[
T_{abc} = T_{[ab]c} \quad \text{and} \quad N_{abcd} = N_{[ab][cd]} = N_{[cdab]} \tag{15}
\]

Their motivation for studying this parallel problem was that they were able to show that this problem and the Riemann-candidate–Lanczos problem were mathematically equivalent — in four dimensions. For other dimensions, any positive results for the existence of potentials for all \( N_{abcd} \) would also apply to the narrower Riemann-candidate–Lanczos problem; but negative results for the parallel problem would, in general, be irrelevant to the narrower Riemann-candidate–Lanczos problem.

\( n > 4 \).
We can immediately find the integrability condition

\[
N_{[abcd;c]} = 0 \tag{16}
\]

and confirm that this is always an effective constraint. So we can conclude that in this parallel problem not all tensors \( N_{abcd} \) can be written in terms of a potential; this result does not permit us to draw any conclusion about the associated Riemann-candidate–Lanczos problem.

\( n > 2 \).
If we carry out again the antisymmetrisation over 5 indices as in (16) we just obtain the trivial identity in dimensions 3 and 4. So instead, to find effective constraints for \( n > 2 \), we have to carry out the same

\(^\dagger\) In [5] a different sign convention was used for the Ricci tensor from that used in this paper, and this convention was also used in [6,7]; so there are some sign differences between this version of the equation and that in [5,6,7].
procedure as in Section 2 involving two differentiations; but since we already know that there are restrictive
integrability conditions for the Riemann-candidate–Lanczos problem, there is no purpose in investigating
further the parallel problem as a means of investigating the narrower Riemann-candidate–Lanczos problem.
However, for completeness we add that in the calculations leading to the constraint (8), the only index
symmetries used were those of the type (15), and so we can deduce that the parallel problem is subject to
the constraint
\[ N^a_{\;a} - 2N^{ab}_{\;ab} = 2R_a T^{a j} + 4R^{ab} T^{a j}_{\;j b} - 4R_{abc} T^{c m b} + \frac{1}{2} R^{a i j k} N_{a i j k}. \]  
which in flat space simplifies to
\[ N^a_{\;a} - 2N^{ab}_{\;ab} = 0. \]  
As argued in the last section, these constraints are effective.

In summary, we can conclude that, since our investigation found the existence of effective constraints in the
parallel problem, we cannot draw any conclusions about the original Riemann-candidate–Lanczos problem,
since in all cases the constraints in the parallel problem may or may not be present in the narrower Riemann-
candidate–Lanczos problem. However, the occurrence of the integrability conditions (16) and (17,18), and
the role played by dimension will be of interest in the next section.

4. ‘Generic’, ‘Ordinary’ and ‘Singular’ solutions.

From the type of investigation in Sections 2 and 3 on the constraints due to integrability conditions, we are
only able to directly draw limited conclusions. For more complete conclusions we need a procedure which
will distinguish between the respective situations where there are indeed effective constraints (even if we
cannot find them explicitly), and where there are no effective constraints and the existence of a potential
is always guaranteed. This is clearly the role of Cartan’s local criteria of integrability of ideals of exterior
forms [20,21,22], as set out in [8,9]. Furthermore, if the system is not in involution and we do find some
effective constraints, we can prolong the original system to take account of these constraints, and then also
use Cauchy’s criteria to analyse the prolonged system; if the prolonged system is not in involution then the
process can be repeated.

As noted in the last section Bampi and Caviglia [8] considered the parallel problem (14) for the tensors
\( N_{abcd} \) and \( T_{abc} \) as a means of studying the Riemann-candidate–Lanczos problem (3). We shall now discuss their
results, and compare with our results in the previous two sections.

\( n > 4 \). In [8] it is stated that in higher dimensions there will be non-trivial restrictions on the data. In
fact we have obtained this result very easily in (16). This result has no direct relevance to the narrower
Riemann-candidate–Lanczos problem.

\( n = 4 \). In their first paper, Bampi and Caviglia [9] showed that the equation (14) does not always admit a solution
for a given \( N_{abcd} \). More precisely, they showed the non-existence of solutions under certain generic conditions
on \( N_{abcd} \), or as Massa and Pagani [5] pointed out, Bampi and Caviglia [9] showed that the representation
(3) does not exhaust the totality of the set of tensors \( N_{abcd} \) (which in four dimensions is equivalent to the
set of Riemann-candidates \( \hat{R}_{abcd} \)), and hence, since the Riemann tensors themselves are only a proper subset
of this larger class of Riemann-candidates, this result says nothing about the validity of (3) for Riemann
tensors.

\( \dagger \) Of course if the results for the parallel problem had been positive, then these results could have been
specialised to the narrower Riemann-candidate–Lanczos problem. This is the case in the related Weyl-
candidate–Lanczos problem also considered in [9].
This generic result for the Riemann-candidate–Lanczos problem given by Bampi and Caviglia [9] was strengthened in their second paper [8] where it was stated in Theorem 1 that there never exists ‘regular’ (‘ordinary’) solutions to (14) for any tensor $N_{abcd}$ in four dimensions; so this result now includes the Riemann–Lanczos problem, i.e., there never exists ‘regular’ (‘ordinary’) solutions to (1) for any Riemann tensor $R_{abcd}$ in four dimensions. If we interpret ‘regular’ solutions to mean the existence of the most general solutions with no constraints on the class of tensors $N_{abcd}$ and the underlying space, then the conclusion in Theorem 1 does not contradict the results in Section 3 of this paper.

The difficulty is with Theorem 2 in [8]. The original system (14) is not in involution and so is prolonged, and as a result of the analysis of the prolonged system, Bampi and Caviglia find no constraints, and conclude in Theorem 2 [8] that in four dimensions, although the representation (14) never permits ‘regular’ (‘ordinary’) solutions to (14), it always admits ‘singular’ (‘nonordinary’) solutions† for all tensors $N_{abcd}$ (equivalently for all Riemann-candidates $\hat{R}_{abcd}$). Therefore, in [8] the ‘singular’ solutions are argued to have what appears to be exactly the same general properties as ‘regular’ solutions, with no constraints on the data $N_{abcd}$ or on the background space.

Normally we expect that such ‘singular’ solutions would involve some restrictions on the set of tensors $N_{abcd}$ and/or on the underlying space. However, Bampi and Caviglia [8] claim that these ‘singular’ solutions are, to their own surprise, not subject to any integrability conditions, and therefore the class of ‘singular’ solutions allows a potential to exist without any restriction on $N_{abcd}$ and on the geometric structure of the underlying Riemann manifold.

This conclusion in Theorem 2 [8] contradicts the results in Section 3 of this paper; in particular the very simple obvious effective constraint (17) for flat space (flat space is not excluded in the analysis in [8]). We suspect that this claim of no constraints on these ‘singular’ solutions is not due simply to a misinterpretation of the properties of ‘singular’ solutions but, more fundamentally, to a fault in the calculations in [8]. We would point out that the only constraint — the ‘internal identity’ $A_{[abcd]} = 0$ in the notation of [8] — discovered in the calculations for Theorem 2 in [8] is precisely the very obvious integrability condition (18) which we found directly in Section 3, and which is of course trivially satisfied in four dimensions. What is most surprising about the method of application of Cauchy’s criteria in [8,9] is that the possibility of constraints existing after two differentiations — involving linear combination of components of $B_{abcdef}$ in the notation of [8,9] — does not arise; whereas, from our work in Section 3, and in particular the simple effective constraint equation (18), it is clear that this is precisely where we expect constraints. Unfortunately, it is not easy to check the accuracy of the argument in [8] at this level, since no explicit details were given leading to the conclusion that $s'_3 = s_3$, for the Cartan characters.

$n = 3$. For the parallel problem, it is stated in [8] that, under ‘generic’ conditions, there does not exist any ‘regular’ solution because of the existence of an internal identity; however in their second paper [8] Bampi and Caviglia state that, by the same argument as in four dimensions, there exist ‘singular’ solutions independently of the choice of $\hat{R}_{abcd}$ and the geometry of the space. So we have here the same situation as in four dimensions, involving a contradiction with the effective constraint found in Section 3. Of course, we should remember that the parallel problem is not equivalent to the Riemann-candidate–Lanczos problem in three dimensions, and a negative result in the former has no direct relevance to the latter.

Significantly, in the next section, we will find that Dolan and Gerber [1], using an alternative but related method to [8] for the direct Riemann–Lanczos problem, have found an explicit constraint which is effective;

† There is no precise explanation in [8] of the difference between ‘regular’ and ‘singular’ solutions, but rather an analogy is given. It would seem that Bampi and Caviglia [8] view the distinction as simply a technical matter: solutions involving the maximal Cartan characters are ‘regular’ solutions, while solutions from a prolonged system with less than the maximal Cartan characters are ‘singular’ solutions.
and this constraint is precisely the three dimensional version of the effective constraint which we discussed in Section 2.

\( n = 2 \). In [9] it is stated that there will always be ‘regular’ solutions, and there is no contradiction with the integrability condition (9) in Section 2 since with the substitution \( n = 2 \) the constraint is trivially satisfied.

Finally we turn to the results on the differential gauge. It is stated in [8] that even when an arbitrary differential gauge condition on \( H_{ab}^{c} {}_c \) (e.g. \( H_{ab}^{c} {}_c = 0 \)) is put alongside the condition (3), there will always be ‘singular’ solutions in four dimensions. As we have discussed above, we believe that the result that there are always ‘singular’ solutions is flawed. But the question arises, in those special cases where there are ‘singular’ solutions (with a restriction on the Riemann-candidates and/or the geometry) whether the differential gauge can be chosen arbitrarily. We believe that this question of gauge is still, in general, open.

5. The Riemann-Lanczos problem by the Janet-Riquier approach.

Dolan and Gerber [1,2] have considered the direct Riemann–Lanczos problem (1) with the symmetries (2) (not the parallel problem as in [8,9]) from the exterior derivative viewpoint along the same lines as in [8,9]; but also they have considered the problem directly as a system of partial differential equations in two and three dimensions [1] using the related Janet-Riquier [23,24] approach. In fact their analysis is valid also for the more general Riemann-candidate–Lanczos problem (3).

\( n = 2 \). This problem has been shown by the Janet-Riquier approach always to have solutions; in fact it is a very simple problem which has also been integrated directly in [1]. There is no contradiction with the integrability condition (9) in Section 2, since we have already noted for \( n = 2 \) that the constraint is not effective.

\( n = 3 \). Using the Janet-Riquier approach, Dolan and Gerber [1] have found that the original Riemann–Lanczos problem is not in involution and so there are no ‘regular’ solutions. After one prolongation, obtained by adding one ‘internal identity’, they found that the prolonged system was involutive. They point out that their ‘internal identity’ is not trivial, and they give it in invariant form as

\[
\sum_{\alpha} f^{(R)}_{\alpha} a_i b_i c_i + f^{(R)} C_{ij} a_i b_j = 0
\]

This is precisely the constraint

\[
f^{(R)}_{\alpha} a_i b_i c_i = 0.
\]

since there is only one component of (21) in three dimensions; and since in three dimensions there is no trace-free part to \( f^{(R)}_{\alpha} a_i b_i c_i \), the constraint (21) can be rewritten more compactly as

\[
2 f^{(R)}_{\alpha} a_i b_i c_i + f^{(R)} C_{ij} a_i b_j = 0
\]

When the substitution (20) is made into this last equation, we obtain

\[
R^{a} a - 2 R^{\alpha}_{;ab} = 4 H^{ij} a_i b_j a - 4 \left( H^{i} a_i b_j + H^{i} b_i c_i \right) a
\]

which is precisely (the Riemann tensor version of) equation (7), and leads to an effective constraint (8) as shown. Of course, for a Riemann tensor, the left hand side will be identically zero from the Binchi identity, but we can see that the analysis also gives the constraint for the more general Riemann-candidate–Lanczos problem.
Therefore, the 'internal identity' above (19) is precisely the three dimensional version of the effective constraint (12) for the Riemann–Lanczos problem found in Section 2. Furthermore, since the prolonged system, created by adding this constraint, has been shown to be involutive in [1] this must be the only constraint. Also in [1] there is an explicit example in three dimensions of a Lanczos potential, which can be interpreted as a singular solution for the unprolonged problem, or as a regular solution for the prolonged problem. We have confirmed directly that it satisfies the constraint (9) in Section 2 (equivalently the 'internal identity' (19) which was found in [1]).

6. Conclusion.
We have confirmed that two successive differentiations of the defining equations (3) of the Riemann-candidate–Lanczos problem, leads to an effective constraint in \( n > 2 \) dimensions, and known solutions of the problem have been shown explicitly to satisfy this constraint. Furthermore, we have shown that the results in [5,6,7] are very relevant to the work of Dolan and Gerber [1,2]; in particular the 'internal identity' found in [1] for the three dimensional Riemann–Lanczos problem is precisely the effective constraint found in [5,6,7]. The existence of this effective constraint contradicts the results in [8] for three and four dimensions. It is significant that, using a method similar to the approach in [8,9], in three dimensions, Dolan and Gerber have found in [1] exactly the three dimensional version of the effective constraint originally found by [5,6,7]. This reinforces our suspicion that the approach in [8,9] is flawed, and an attempt should be made to reinvestigate the prolonged system at the level of the second derivatives of the Riemann tensor for all dimensions from the exterior differential system viewpoint. (Dolan and Gerber [2] have briefly discussed the problem in some other dimensions from the exterior differential system viewpoint as used in [8,9], but have not developed it further.) However, it seems that the Janet-Riquier method for partial differential equations is shorter and perhaps more transparent, and so it would be preferable to first apply this approach, as used in two and three dimensions in [1], to other dimensions; then this could be compared with the exterior differential system approach.

We also note the significant role that the dimension of the space has played at a number of crucial places in our arguments; whether a constraint is effective or not can depend on the dimension. The constraint (8) is effective for \( n > 2 \), but trivial for \( n = 2 \); in the parallel problem as well as the constraint (17) for \( n > 2 \), an additional constraint (16) occurs for \( n > 4 \). The role of dimension is even more subtle for the Weyl-Lanczos problem where complicated constraints involving the second derivatives of the Weyl tensor are valid only for dimensions \( n \geq 6 \), while constraints involving the third derivatives of the Weyl tensor are valid only for dimensions \( n \geq 5 \); there are no constraints for \( n = 4 \) [10 - 15].

In summary, we note that not all Riemann candidates and Riemann tensors have potentials in dimensions \( n > 2 \), because of the existence of an effective constraint; although there are special cases where such potentials do exist. From [1] it is known that prolongation with this one constraint gives an involutive system, in three dimensions; it is still an open question whether prolongation with this one constraint can lead to involution in higher dimensions; a preliminary investigation of the flat space case by the Janet-Riquier approach would suggest that all the equations corresponding to (5) may be needed.

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