Geometric angle structures on triangulated surfaces

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Abstract In this paper we characterize a function defined on the set of edges of a triangulated surface such that there is a spherical angle structure having the function as the edge invariant (or Delaunay invariant). We also characterize a function such that there is a hyperbolic angle structure having the function as the edge invariant.

§1. Introduction

Suppose $S$ is a closed surface and $T$ is a triangulation of $S$. Here by a triangulation we mean the following: take a finite collection of triangles and identify their edges in pairs by homeomorphism. Let $V, E, F$ be the sets of all vertices, edges and triangles in $T$ respectively. If $a, b$ are two simplices in triangulation $T$, we use $a < b$ to denote that $a$ is a face of $b$. Let $C(S, T) = \{(e, f) | e \in E, f \in F, such that e < f\}$ be set of all corners of the triangulation. An angle structure on a triangulated surface $(S, T)$ assigns each corner of $(S, T)$ a number in $(0, \pi)$. A Euclidean (or hyperbolic, or spherical) angles structure is an angle structure so that each triangle with the angle assignment is Euclidean (or hyperbolic, or spherical). More precisely, a Euclidean angle structure is a map $x : C(S, T) \rightarrow (0, \pi)$ assigning every corner $i$ (for simplicity of notation, we use one letter to denote a corner) a positive number $x_i$ such that $x_i + x_j + x_k = \pi$ whenever $i, j, k$ are three corners of a triangle. A hyperbolic angle structure is a map $x : C(S, T) \rightarrow (0, \pi)$ such that $x_i + x_j + x_k < \pi$. A spherical angle structure is a map $x : C(S, T) \rightarrow (0, \pi)$ such that

$$\begin{cases} x_i + x_j + x_k > \pi \\ x_j + x_k - x_i < \pi. \end{cases}$$

(1)

Actually it is proved in [B] that positive numbers $x_i, x_j, x_k$ are three inner angles of a spherical triangle if and only if they satisfy conditions (1).

Given an angle structure $x : C(S, T) \rightarrow (0, \pi)$, we define its edge invariant which is a function $D_x : E \rightarrow (0, 2\pi)$ such that $D_x(e) = x_i + x_{i'}$ where $i = (e, f), i' = (e, f')$ are two opposite corners facing the edge $e$. And we define its Delaunay invariant which is a function $\mathcal{D}_x : E \rightarrow (-2\pi, 2\pi)$ such that $\mathcal{D}_x(e) = x_j + x_k + x_{j'} + x_{k'} - x_i - x_{i'}$ where $i = (e, f), i' = (e, f')$ are two
opposite corners facing the edge $e$ and $j, k$ (or $j', k'$) are the other two corners of the triangle $f$ (or $f'$).

For the simplicity of notation, we use $G$ to denote a fixed geometry, where $G = E, H$ or $S$ means the Euclidean, hyperbolic or spherical geometry respectively. Now given a function $D : E \to (0, 2\pi)$ (or $D : E \to (-2\pi, 2\pi)$), we use $AG(S, T; D)$ (or $AG(S, T; D)$) to denote the set of all $G$ angle structures having $D$ (or $D$) as the edge (or Delaunay) invariant.

The motivation of considering these sets is the study of geometric cone metrics with prescribed edge invariant or Delaunay invariant on triangulated surfaces from the variational point of view. A Euclidean (or hyperbolic, or spherical) cone metric assigns each edge in $T$ a positive number such that the numbers on any three edges of a triangle in $T$ form three edge length of a Euclidean (or hyperbolic, or spherical) triangle. The variational method contains a variational problem and a linear programming problem. The variational problem is to show that the unique maximal point of a convex "capacity" defined on the set $AG(S, T; D)$ (or $AG(S, T; D)$) gives the unique geometric cone metric. The linear programming problem is to characterize the function $D$ (or $D$) such that the set $AG(S, T; D)$ (or $AG(S, T; D)$) is nonempty.

For Euclidean angle structures, the Delaunay invariant and the edge invariant are related by $2D_x(e) + D_x(e) = 2\pi$ for any $e$. Thus given two functions $D$ and $D$ satisfying $2D(e) + D(e) = 2\pi$ for any $e$, we have $AE(S, T; D) = AE(S, T; D)$. Therefore the problem of Euclidean cone metric with given edge invariant is equivalent to the problem of Euclidean cone metric with given Delaunay invariant. Rivin [Ri1] [Ri2] worked out the variational problem and the linear programming problem about $AE(S, T; D)$. Leibon [Le] worked out the variational problem and the linear programming problem about $AH(S, T; D)$. Luo [Lu] worked out the variational problem about $AS(S, T; D)$ the linear programming problem about which will be solved in this paper (theorem 1). Although the variational problems about $AH(S, T; D)$ and $AS(S, T; D)$ are still open, we will solve the linear programming problem about them in this paper (theorem 2 and 3).

The main results are the following. For a triangulated surface $(S, T)$, a subset $X \subseteq F$, we use $|X|$ to denote the number of triangles in $X$ and we use $E(X)$ to denote the set of all edges of triangles in $X$.

**Theorem 1.** Given a triangulated surface $(S, T)$ and a function $D : E \to$
(0, π), the set $A(S, T; D)$ is nonempty if and only if for any subset $X \subseteq F$,

$$\pi |X| < \sum_{e \in E(X)} D(e).$$

**Theorem 2.** Given a triangulated surface $(S, T)$ and a function $D : E \to (0, 2\pi)$, the set $AH(S, T; D)$ is nonempty if and only if for any subset $X \subset F$,

$$\pi(|F| - |X|) > \sum_{e \notin E(X)} D(e).$$

**Theorem 3.** Given a triangulated surface $(S, T)$ and a function $D : E \to (-2\pi, 2\pi)$, the set $AS(S, T; D)$ is nonempty if and only if for any subset $X \subseteq F$,

$$\pi(|F| - |X|) > \sum_{e \notin E(X)} (\pi - \frac{1}{2}D(e)).$$

The paper is organized as follows. In section 2, we prove theorem 1 by using Leibon’s result. In section 3, we recall the duality theorem in linear programming. In section 4, following Rivin’s method, we prove theorem 2 and 3 by using the duality theorem.

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§2. Prove of theorem 1

First let us recall the Leibon’s result of characterization of the function $D$ such that the set $AH(S, T; D)$ is nonempty.

**Theorem 4.** (Leibon)[Le] Given a triangulated surface $(S, T)$ and a function $D : E \to (0, 2\pi)$, the set $AH(S, T; D)$ is nonempty if and only if for any subset $X \subseteq F$,

$$\pi |X| < \sum_{e \in E(X)} (\pi - \frac{1}{2}D(e)).$$

**Proof of theorem 1.** To show the conditions are necessary, for any $X \subseteq F$, we have $\sum_{e \in E(X)} D(e) = \sum_{e \in E(X)} (x_i + x_{i'})$, where $i, i'$ are two opposite corners facing the edge $e$. It turns out that the right hand side of the equation is equal
to $\sum_{f \in X} (x_i + x_j + x_k) + x_h$, where the corner $h = (e, f^*)$ with $e \in E(X)$ and $f^* \notin X$. Hence $\sum_{e \in E(X)} D(e) \geq \sum_{f \in X} (x_i + x_j + x_k) > \sum_{f \in X} \pi = \pi |X|.$

To show the conditions are sufficient, let us define a function $D : E \rightarrow (0, 2\pi)$ by setting $D(e) = 2\pi - 2D(e)$. Thus the conditions $\pi |X| < \sum_{e \in E(X)} (\pi - \frac{1}{2}D(e))$ which guarantee $AH(S, T; D)$ is nonempty by theorem 4. It follows that there is a solution for the inequalities

$$
\left\{ \begin{array}{l}
  x_i + x_j + x_k < \pi \\
  x_j + x_k + x_{j'} + x_{k'} - x_i - x_{j'} = D(e) \\
  x_i > 0
\end{array} \right.
$$

$i, j, k$ are three corners of a triangle

Let us define new variables $y_i$ for all $i \in C(S, T)$ by setting

$$y_i = \frac{\pi + x_i - x_j - x_k}{2}$$

provided $i, j, k$ are three corners of a triangle. And since $D(e) = 2\pi - 2D(e)$, the inequalities above are equivalent to

$$
\left\{ \begin{array}{l}
  y_i + y_j + y_k > \pi \\
  y_i + y_{i'} = D(e) \\
  y_j + y_k < \pi \\
  y_{j'} + y_{k'} - y_i - y_{i'} = D(e)
\end{array} \right.
$$

$i, j, k$ are three corners of a triangle

This solution obviously satisfies

$$
\left\{ \begin{array}{l}
  y_i + y_j + y_k > \pi \\
  y_i + y_{i'} = D(e) \\
  y_j + y_k - y_i < \pi \\
  y_i > 0
\end{array} \right.
$$

$i, j, k$ are three corners of a triangle

Thus we obtain an angle structure in $AS(S, T; D)$. QED

§3. Duality Theorem

We fix the notations as follows: $x = (x_1, \ldots, x_n)^t$ is a column vector in $\mathbb{R}^n$. The standard inner product in $\mathbb{R}^n$ is denoted by $a'x$. If $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, we denote its transpose by $A^t : \mathbb{R}^m \rightarrow \mathbb{R}^n$. Given two vectors $x, a$ in $\mathbb{R}^n$, we say $x \geq a$ if $x_i \geq a_i$ for all indices $i$. Also $x > a$ means $x_i > a_i$ for all indices $i$. 

4
A linear programming problem \((P)\) is to minimize an objective function 
\[ z = a^t x \]
subject to the restrain conditions
\[
\begin{align*}
Ax &= b \\
x &\geq 0
\end{align*}
\]
where \(x \in \mathbb{R}^n\), \(b \in \mathbb{R}^m\) and \(A : \mathbb{R}^n \to \mathbb{R}^m\) is a linear transformation. We call a point \(x\) satisfying the restrain conditions a feasible solution and denote the set of all the feasible solutions by \(D(P) = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}\). An optimal solution \(x\) for \((P)\) is a feasible solution so that the objective function \(z\) realizes the minimal value. The dual problem \((P^*)\) of \((P)\) is to maximize
\[ z = b^t y \]
subject to
\[ A^t y \leq a, y \in \mathbb{R}^m. \]
Let us recall the duality theorem in linear programming. The proof of the theorem can be found in the book [KB].

**Theorem 5.** The following statements are equivalent.
(a) Problem \((P)\) has an optimal solution.
(b) \(D(P) \neq \emptyset\) and \(D(P^*) \neq \emptyset\).
(c) Both problem \((P)\) and problem \((P^*)\) have optimal solutions so that the minimal value of \((P)\) is equal to the maximal value of \((P^*)\).

In applications that we are interested, there is a special case that the objective function \(z = 0\) for \((P)\). Thus the optimal solution exists if and only if \(D(P) \neq \emptyset\). Thus we obtain the following corollary.

**Corollary 6.** For \(A : \mathbb{R}^n \to \mathbb{R}^m\) and \(b \in \mathbb{R}^m\), the set \(\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} \neq \emptyset\) if and only if the maximal value of \(z = b^t y\) on \(\{y \in \mathbb{R}^m | A^t y \leq 0\}\) is non-positive.

§4. **Proof of theorem 2 and 3**

By following Rivin’s method in [Ri2], we will prove a lemma about the closure of \(AH(S,T;D)\) in \(\mathbb{R}^{3|E|} = \{(x_i)^t, i \in C(S,T)\}\). The closure of \(AH(S,T;D)\) consists of all the points satisfying
\[
\begin{align*}
x_i + x_j + x_k &\leq \pi \quad i, j, k \text{ are three corners of a triangle} \\
x_i + x_{i'} = D(e) \quad i, i' \text{ are two opposite corners facing an edge } e \\
x_i &\geq 0
\end{align*}
\]

**Lemma 7.** Given a triangulated surface \((S,T)\) and a function \(D : E \to [0,2\pi]\), the closure of \(AH(S,T;D)\) is nonempty if and only if for any subset
\( X \subseteq F, \)
\[
\pi(|F| - |X|) \geq \sum_{e \notin E(X)} D(e).
\]

**Proof.** The linear programming problem \((P)\) with variables \(x = (..., x_i, ..., t_f, ...)\) indexed by \(C(S, T) \cup F\) is to minimize the objective function \(z = 0\) subject to the restrain conditions
\[
\begin{align*}
&x_i + x_j + x_k + t_f = \pi \quad i, j, k \text{ are three corners of a triangle} f \\
&x_i + x_{i'} = D(e) \quad i, i' \text{ are two opposite corners facing an edge } e \\
&x_i \geq 0 \\
&t_f \geq 0
\end{align*}
\]

The dual problem \((P^*)\) with variable \(y = (..., y_f, ..., y_e, ...)\) indexed by \(E \cup F\) is to maximize the objective function \(z = \sum_{f \in F} \pi y_f + \sum_{e \in E} D(e) y_e\) subject to the restrain conditions
\[
\begin{align*}
y_f &\leq 0 \\
y_f + y_e &\leq 0 \quad \text{whenever } e < f.
\end{align*}
\]

Since the closure of \(AH(S, T; D)\) is nonempty is equivalent to that the set \(D(P)\) is nonempty, by corollary 6, the latter one is equivalent to that the maximal value of the objective function of \((P^*)\) is non-positive.

To show the conditions \(\pi(|F| - |X|) \geq \sum_{e \notin E(X)} D(e)\) for any \(X \subseteq F\) are necessary, for any \(X \subseteq F\), let
\[
y_f = \begin{cases} 0 & \text{if } f \in X \\ -1 & \text{if } f \notin X \end{cases} \quad \text{and } y_e = \begin{cases} 0 & \text{if } e \in E(X) \\ 1 & \text{if } e \notin E(X) \end{cases}
\]

We claim that \((y_f, y_e)\) is a feasible solution. In fact, given a pair \(e < f\), if \(f \in X\), we must have \(e \in E(X)\), then \(y_f + y_e = 0\). If \(f \notin X\), then \(y_f + y_e = -1 + y_e \leq 0\).

By the assumption that the maximal value of the objective function of \((P^*)\) is non-positive, since \((y_f, y_e)\) is feasible, we have \(0 \geq z(y_f, y_e) = \sum_{f \notin X} \pi y_f + \sum_{e \notin E(X)} D(e) y_e = \pi(|X| - |F|) + \sum_{e \notin E(X)} D(e)\).

To show the conditions are sufficient, take an arbitrary feasible solution \((y_f, y_e)\). If \(y_f = 0\) for all \(f\), from \(y_f + y_e \leq 0\), we know \(y_e \leq 0\). Hence...
\[ z(y_f, y_e) = \sum_{e \in E} D(e)y_e \leq 0, \text{ since } D(e) \in [0, 2\pi]. \] Otherwise, define \( X = \{ f \in F| y_f = 0 \} \subset F, \) and let \( a = \max\{y_f, f \notin X\} \). We have \( a < 0 \). Define

\[
y_f^{(1)} = \begin{cases}
y_f = 0 & \text{if } f \in X \\
y_f - a & \text{if } f \notin X
\end{cases}
\quad \text{and } y_e^{(1)} = \begin{cases}
y_e & \text{if } e \in E(X) \\
y_e + a & \text{if } e \notin E(X)
\end{cases}
\]

We claim that \((y_f^{(1)}, y_e^{(1)})\) is a feasible solution. In fact, \( y_f^{(1)} \leq 0 \). Given a pair \( e < f \), if \( f \in X \), we must have \( e \in E(X) \), then \( y_f^{(1)} + y_e^{(1)} = y_f + y_e \leq 0 \). If \( f \notin X \) and \( e \notin E(X) \), then \( y_f^{(1)} + y_e^{(1)} = y_f - a + y_e + a \leq 0 \). If \( f \notin X \) but \( e \in E(X) \), there exists another triangle \( f' \in X \) so that \( e < f' \), then \( y_e = y_e + y_e' \leq 0 \). Therefore \( y_f^{(1)} + y_e^{(1)} = y_f - a + y_e \leq y_f - a \leq 0 \), since \( a \) is the maximum.

Now the value of the objective function is \( z(y_f^{(1)}, y_e^{(1)}) = z(y_f, y_e) + a(\pi(|X| - |F|) + \sum_{e \in E(X)} D(e)) \geq z(y_f, y_e) \), according to the conditions. Note the number of 0’s in \( \{y_f^{(1)}\} \) is more than that in \( \{y_f\} \). By the same procedure, after finite steps, it ends at a feasible solution \((y_f^{(n)}, y_e^{(n)})\). We have \( z(y_f^{(n)}, y_e^{(n)}) \leq 0 \). Since the value of the objective function does not increase, therefore \( 0 \geq z(y_f^{(n)}, y_e^{(n)}) \geq \ldots \geq z(y_f^{(1)}, y_e^{(1)}) \geq z(y_f, y_e) \). QED

**Proof of theorem 2.** Let \( x_i = a_i + \varepsilon \) for any \( i \in C(S, T) \), where \( a_i \geq 0 \) and \( \varepsilon \geq 0 \). The linear programming problem \((P)\) with variables \( \{..., a_i, ..., \varepsilon\} \) is to minimize the objective function \( z = -\varepsilon \) subject to the restrain conditions

\[
\begin{align*}
a_i + a_j + a_k + 3\varepsilon &\leq \pi & \text{if \( i, j, k \) are three corners of a triangle} \\
a_i + a_j + 2\varepsilon &\leq D(e) & \text{if \( i, j \) are two opposite corners facing an edge } e \\
a_i &\geq 0 \\
\varepsilon &\geq 0
\end{align*}
\]

The dual problem \((P^*)\) with variable \( y = (..., y_f, ..., y_e, ...) \) indexed by \( E \cup F \) is to maximize the objective function \( z = \sum_{f \in F} \pi y_f + \sum_{e \in E} D(e)y_e \) subject to the restrain conditions

\[
\begin{align*}
y_f &\leq 0 \\
y_f + y_e &\leq 0 & \text{whenever } f < e \\
3 \sum_{f \in F} y_f + 2 \sum_{e \in E} y_e &\leq -1
\end{align*}
\]

By the theorem 5(c), the maximal value of the objective function of \((P^*)\) is negative is equivalent to that the minimal value of the objective function of
We claim that \((y_f, y_e)\) is a feasible solution. If fact, as in lemma 7, we can check \(y_f + y_e \leq 0\) for any pair \(e < f\). Furthermore
\[
3 \sum_{f \in F} y_f + 2 \sum_{e \in E} y_e = 3 \sum_{f \notin X} (-1) + 2 \sum_{e \notin E(X)} 1 = 3(|X| - |F|) + 2(|E| - |E(X)|) = 3|X| - 2|E(X)| + 2|E| - 3|F| = 3|X| - 2|E(X)| \leq -1
\]
since \(2|E| = 3|F|\). Now \((y_f, y_e)\) is feasible implies that \(z(y_f, y_e) < 0\) which is equivalent to \(\pi(|F| - |X|) < \sum_{e \notin E(X)} D(e)\).

To show the conditions are sufficient, by the proof of lemma 7 we know the maximal value of the objective function of \((P^*)\) is \(\leq 0\) under the conditions. We try to show it cannot be 0. Assume that \((y_f, y_e)\) is a feasible solution satisfying \(z(y_f, y_e) = 0\). We claim that \(y_f = 0\) for all \(f\). Otherwise, as in the proof of lemma 7, we can find another feasible solution \((y_f^{(1)}, y_e^{(1)})\) and we can check that \(z(y_f^{(1)}, y_e^{(1)}) = z(y_f, y_e) + a(\pi(|X| - |F|) + \sum_{e \notin E(X)} D(e)) > z(y_f, y_e) = 0\), according to the conditions. It is contradiction since the maximal value of the objective function of \((P^*)\) is \(\leq 0\).

Now from \(y_f = 0\) for all \(f\) we see \(y_e \leq 0\). Since \(0 = z(y_f, y_e) = \sum_{e \in E} D(e)y_e\) and \(D(e) > 0\), we get \(y_e = 0\) for all \(e\) and therefore \((y_f, y_e) = (0, 0)\). But \((y_f, y_e) = (0, 0)\) does not satisfy \(3 \sum_{f \in F} y_f + 2 \sum_{e \in E} y_e \leq -1\). It is a contradiction since we assume that \((y_f, y_e)\) is a feasible solution. This proves that the maximal value of the objective function of \((P^*)\) is negative.

QED

**Proof of theorem 3.** Given two functions \(D : E \rightarrow (0, 2\pi)\) and \(\mathcal{D} : E \rightarrow (-2\pi, 2\pi)\) satisfying \(2D(e) + \mathcal{D}(e) = 2\pi\) for any \(e\), we claim that
$AH(S, T; D) \neq \emptyset$ is equivalent to $AS(S, T; D) \neq \emptyset$. By this claim, theorem 3 is true as a corollary of theorem 2.

In fact, $AS(S, T; D)$ is the set of solutions for the inequalities

$$\begin{cases}
    x_i + x_j + x_k > \pi & \text{i, j, k are three corners of a triangle} \\
    x_j + x_k - x_i < \pi & \text{i, j, k are three corners of a triangle} \\
    x_j + x_k + x_j' + x_k' - x_i - x_i' = D(e) \\
    x_i > 0
\end{cases}$$

Let us define new variables $y_i$ for all $i \in C(S, T)$ by setting

$$y_i = \frac{\pi + x_i - x_j - x_k}{2}$$

provided $i, j, k$ are three corners of a triangle. Since $2D(e) + D(e) = 2\pi$, we see that the inequalities above are equivalent to

$$\begin{cases}
    y_i + y_j + y_k < \pi & \text{i, j, k are three corners of a triangle} \\
    y_i > 0 \\
    y_i + y_i' = D(e) & \text{i, i' are two opposite corners facing an edge e} \\
    y_j + y_k < \pi & \text{j, k are two corners of a triangle}
\end{cases}$$

Since $y_i + y_j + y_k < \pi$ implies $y_j + y_k < \pi$, we can omit the latter one. Equivalently, we get

$$\begin{cases}
    y_i + y_j + y_k < \pi & \text{i, j, k are three corners of a triangle} \\
    y_i > 0 \\
    y_i + y_i' = D(e) & \text{i, i' are two opposite corners facing an edge e}
\end{cases}$$

Now the set of solutions of the inequalities above is exactly $AH(S, T; D)$. Thus we see $AH(S, T; D) \neq \emptyset$ is equivalent to $AS(S, T; D) \neq \emptyset$. QED

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