Recursion Relation of Hyperelliptic Psi-Functions of Genus Two

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Abstract

A recursion relation of hyperelliptic $\psi$ functions of genus two, which was derived by D.G. Cantor (J. reine angew. Math. 447 (1994) 91-145), is studied. As Cantor’s approach is algebraic, another derivation is presented as a natural extension of the analytic derivation of the recursion relation of the elliptic $\psi$ function.

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§1. Introduction

For the case of elliptic curves, the elliptic $\psi$-function is defined by \[13, 14\],

$$\psi_n(u) = \frac{\sigma(nu)}{\sigma(u)^n}. \quad (1-1)$$

Using the addition formula of elliptic $\varphi$ functions,

$$-(\varphi(z) - \varphi(u)) = \frac{\sigma(z + u)\sigma(z - u)}{[\sigma(z)\sigma(u)]^2}, \quad (1-2)$$

we have a recursion relation \[13, 14\],

$$\psi_{n+m}\psi_{m-n} = \begin{vmatrix} \psi_{m-1}\psi_n & \psi_m\psi_{n+1} \\ \psi_m\psi_{n-1} & \psi_{m+1}\psi_n \end{vmatrix}, \quad (1-3)$$

Eq.(1-3) is proved as follows. The addition formula (1-2) becomes,

$$-(\varphi(nu) - \varphi(u)) = \frac{\psi_{n+1}(u)\psi_{n-1}(u)}{\psi_n(u)^2}, \quad (1-4)$$

and

$$-(\varphi(mu) - \varphi(nu)) = \frac{\psi_{n+m}(u)\psi_{m-n}(u)}{\psi_m(u)^2\psi_n(u)^2}. \quad (1-5)$$

Noting

$$[-\varphi(mu) + \varphi(nu)] + [\varphi(mu) - \varphi(u)] - [\varphi(nu) - \varphi(u)] = 0, \quad (1-6)$$

we have the relation (1-3). The formula (1-3) is a recursion relation of the elliptic $\psi$-function.

In this article we will consider an extension of these properties Eqs.(1-1)-(1-6) to those of a hyperelliptic curve $C$ of genus two given by an affine equation $y^2 = x^5 + \lambda_4x^4 + \lambda_3x^3 + \lambda_2x^2 + \lambda_1x + \lambda_0$ for certain complex numbers $\lambda$’s.

Recently Cantor found a generalization of the relation (1-3) of hyperelliptic $\psi$ function of genus two \[5\],

$$\psi_2^2\psi_m\psi_n\psi_{n+m}\psi_{m-n} = \begin{vmatrix} \psi_{m-2}\psi_n & \psi_{m-1}\psi_{n+1} & \psi_m\psi_{n+2} \\ \psi_{m-1}\psi_{n-1} & \psi_m\psi_n & \psi_{m+1}\psi_{n+1} \\ \psi_m\psi_{n-2} & \psi_{m+1}\psi_{n-1} & \psi_{m+2}\psi_n \end{vmatrix}, \quad (1-7)$$
by using Hankel determinant and Padé approximations. Here the hyperelliptic \( \psi \) function of genus two is defined as a polynomial \( \psi_m \) of \( x \) and \( y \), whose zeros \( P \) in the curve \( C \) are necessary and sufficient condition for the element \( m \cdot P \) in the related Jacobian (Jacobi variety) \( J \) to stay in the curve \( C \) embedded in \( J \) again. Due to the definition, \( \psi \) is an object of concern in number theory [5, 9-11]. (By primitive consideration, for any hyperelliptic curve we find that twice of the finite ramified points, \( i.e. \), zeros of \( y \), in the Jacobian exist in the curve again. In other words, \( \psi_2 \) must be \( y \) up to constant factor [9, 10]. However as the fact that \( \psi_2 \) is proportional to \( y \) is obtained by a trivial consideration, Cantor removed it after scaling \( \psi_m \) by \( \psi_2 \) for the sake of convenience from the viewpoint of study of cryptography. In fact in original formula of Eq.(1-7) in [5], the factor \( \psi_2^2 \) in the left hand side in Eq.(1-7) does not exist. In this article, we will faithfully follow the definition and employ convention of Ōnishi [9, 10] which requires some corrections to ones of [5].) We employ an analytic expression of \( \psi \) function [9, 10],

\[
\psi_n(u) = \frac{\sigma(nu)}{\sigma_2(u)n^2},
\]

where \( \sigma \) is the hyperelliptic \( \sigma \) function defined in §2, and \( u \) is a coordinate of the embedded curve \( C \) in the related Jacobian \( J \).

In [8], I showed that the relations (1-3) and (1-7) can be regarded as discrete nonlinear difference equations themselves and are closely related to the discrete Painlevé equation I [12],

\[
\beta_{n+1} \beta_{n-1} = \frac{z}{\beta_n} + \frac{a}{\beta_n^2},
\]

and its higher rank analog. By setting \( \beta_n = \psi_{n+1}\psi_{n-1}/\psi_n^2 \), Eq.(1-3) is reduced to Eq.(1-9) [8]. Similarly we obtain several similar equations to Eq.(1-9) using the recursion relation (1-7). Thus Eq.(1-7) is a very interesting formula even from the viewpoint of the study of the nonlinear difference equation. Our motivation of this study is to find the extension of Eq.(1-9) from the viewpoint. In order to do, we believe that the investigation of the mathematical structure of Eq.(1-7) is very important.
Though Eq.(1-7) was obtained by Cantor, his approach is not out of the line of the derivation of Eqs.(1-1)-(1-6). His approach to Eq.(1-7) [5] is algebraic (and applicable to case of hyperelliptic curves of any genus).

In this article, we will give an analytic derivation of Eq.(1-7) as a simple extension of elliptic case shown in Eqs.(1-1)-(1-6) in the meaning of study of special functions. (In this article, we have used only the term “analysis” as a classical meanings or study of special functions or algebraic functions.) For the case of genus one, the left hand side of Eq.(1-3), $\psi_{m+n}\psi_{m-n}$ comes from the factor $\sigma(u + v)\sigma(u - v)$ in the right hand side of Eq.(1-2) due to the definition of $\psi$ in Eq.(1-1). The linear terms with respect to $\wp$ in the addition formula (1-2) play essential roles and the relation (1-6) is reduced to the recursion relation (1-3).

Our plan to prove the formula (1-7) is an extension of the idea given in Eq.(1-6). For the case of the genus two, using the hyperelliptic $\sigma$ function, hyperelliptic $\psi$ function is given by Eq.(1-8). On the other hand, the addition formula is given as a relation in the Jacobian [2, 4],

\[
\frac{\sigma(u + v)\sigma(u - v)}{\sigma(u)^2\sigma(v)^2} = -(\wp_{11}(u) - \wp_{11}(v) + \wp_{12}(u)\wp_{22}(v) - \wp_{12}(v)\wp_{22}(u)),
\]

for the hyperelliptic $\sigma$ and $\wp$ functions. As in the case of genus one, we will use the role of the linear terms of $\wp$ in Eq.(1-10) like in Eq.(1-6).

However there are two different points from the case of genus one. First is that the addition formula (1-10) contains the second order terms $\wp_{12}(u)\wp_{22}(v) - \wp_{12}(v)\wp_{22}(u))$ of $\wp$. Second is that the hyperelliptic $\psi$ function is defined over the embedded curve $C$ in the Jacobian $J$ whereas the the hyperelliptic $\sigma$ and $\wp$ functions and their relations are defined over the Jacobian $J$.

In order to overcome the difference, we introduce a function,

\[
\Psi_n := \frac{\sigma(nu)}{\sigma_2(u)^2},
\]
whose domain is $J$ and consider lifted quantities of both sides of the recursion relation (1-7). We evaluate the parts in terms of the relations of the $\varphi$-functions in the Jacobian $J$. After then, we constrain the quantities to the embedded curve $C$ in the Jacobian $J$. Thus §2 devotes the review of the hyperelliptic functions over the Jacobian $J$ and related analytic relations as a sense of study of special functions following the studies of hyperelliptic functions \cite{1, 2, 3, 4, 6, 9, 10}. Using the analytic properties, in §3, we will give the proof of the formula (1-7) or Theorem 3-5. As the notations becomes so complicated, we will not mention the idea of our derivation in detail here but in Remark 3-6. So we recommend that the reader should see the part in Remark 3-7 (1) before he starts to read §3.

Even though the notations becomes so complicated, the idea of our approach is very simple as an analytic process.

Before we finish the Introduction, we will comment on another variant of a generalization of the elliptic $\psi$ function and its recursion relation. Recently Kanayama found the another recursion relation on the function as a generalization of Eq.(1-1) to that of genus two,

$$
\Phi_n := \frac{\sigma(nu)}{\sigma(u)^n},
$$

(1-12)
defined over the Jacobian \cite{7}. $\Phi_n$ formally obeys the same as Eq.(1-3) \cite{7}. His approach is very resemble to ours. In order to compare Kanayama’s generalization \cite{7} of Eq.(1-3) to genus two case (defined over Jacobian $J$) with Cantor’s one \cite{5} (defined over the curve $C$ itself), I believe that our analytic approach is mathematically important.

I’m deeply indebted to Prof. Y. Ônishi for leading me this beautiful theory of $\psi$-function, sending me the work of Cantor and helpful discussions. I thank Prof. V. Z. Enolskii for sending me his interesting works and helpful comments. I am grateful to Prof. D. G. Cantor for critical reading of this manuscript and several helpful suggestions.
§2 Abelian Functions of Genus Two

This section reviews theory of hyperelliptic $\sigma$ and $\wp$ functions [1, 2, 3]. Recently Buchstaber, Enolskii, and Leykin wrote a paper [4] which contains very nice introduction to the theory in terms of modern language; it should be a guide to read this section.

In this article, we will deal with a hyperelliptic curve $C$ of genus two, given by an affine equation,

\[ y^2 = f(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5, \]

(2-1)

where $\lambda_5 \equiv 1$ and $\lambda_j$’s are complex numbers.

**Definition 2-1** [1], [2], [4 Chapter 2], [9 p.384-386].

(1) Let us denote the homology of the hyperelliptic curve $C$ by

\[ H_1(X, \mathbb{Z}) = \mathbb{Z} \alpha_1 \oplus \mathbb{Z} \beta_1 \oplus \mathbb{Z} \alpha_2 \oplus \mathbb{Z} \beta_2, \]

(2-2)

where these intersections are given as $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = \delta_{i,j}$.

(2) The unnormalized differentials of first kind are defined by,

\[ du_1 := \frac{dx}{2y}, \quad du_2 := \frac{xdx}{2y}. \]

(2-3)

(3) The unnormalized differentials of second kind are defined by,

\[ dr_1 := \frac{1}{2y} (\lambda_3 x + 2\lambda_4 x^2 + 3\lambda_5 x^3) dx, \quad dr_2 := \frac{1}{2y} \lambda_5 x^2 dx. \]

(2-4)

(4) The unnormalized period matrices are defined by,

\[ 2\omega' := \left[ \int_{\alpha_1}^{\beta_1} du_1 \int_{\alpha_2}^{\beta_2} du_1 \right], 2\omega'' := \left[ \int_{\beta_1}^{\alpha_1} du_1 \int_{\beta_2}^{\alpha_2} du_1 \right], \quad \omega := \left[ \omega' \omega'' \right]. \]

(2-5)
(5) The normalized period matrices are defined by,
\[ t \left[ \frac{d\hat{u}_1}{d\hat{u}_2} \right] := \omega^{-1} t \left[ \frac{du_1}{du_2} \right], \quad \tau := \omega^{-1} \omega'', \quad \hat{\omega} := \begin{bmatrix} 1 \\ \tau \end{bmatrix}. \quad (2-6) \]

(6) We define the Abel map for symmetric product of the curve $C$ i.e., for points \{Q_1, Q_2\} in the curve $C$ by,
\[ \hat{w} : \text{Sym}^2(C) \longrightarrow \mathbb{C}^2, \quad \left( \hat{w}_k(Q_i) := \int_{\infty}^{Q_1} \hat{d}u_k + \int_{\infty}^{Q_2} \hat{d}u_k \right), \]
\[ w : \text{Sym}^2(C) \longrightarrow \mathbb{C}^2, \quad \left( w_k(Q_i) := \int_{\infty}^{Q_1} \hat{d}u_k + \int_{\infty}^{Q_2} \hat{d}u_k \right). \quad (2-7) \]

The Jacobian (Jacobi varieties) $\hat{J}$ and $J$ are defined as complex torus,
\[ \hat{J} := \mathbb{C}^2/\hat{\Lambda}, \quad J := \mathbb{C}^2/\Lambda. \quad (2-8) \]

Here $\hat{\Lambda}$ ($\Lambda$) is a lattice generated by $\hat{\omega}$ ($\omega$).

(7) The complete hyperelliptic integrals of the second kinds are defined by
\[ 2\eta' := \begin{bmatrix} \int_{\alpha_1} dr_1 & \int_{\alpha_2} dr_1 \\ \int_{\alpha_1} dr_2 & \int_{\alpha_2} dr_2 \end{bmatrix}, \quad 2\eta'' := \begin{bmatrix} \int_{\beta_1} dr_1 & \int_{\beta_2} dr_1 \\ \int_{\beta_1} dr_2 & \int_{\beta_2} dr_2 \end{bmatrix}. \quad (2-9) \]

(8) We will define the Riemann theta function over $\mathbb{C}^2$ characterized by $\hat{\Lambda}$,
\[ \theta \begin{bmatrix} a \\ b \end{bmatrix} (z) := \theta \begin{bmatrix} a \\ b \end{bmatrix} (z; \tau) := \sum_{n \in \mathbb{Z}^2} \exp \left[ 2\pi \sqrt{-1} \left\{ \frac{1}{2} t(n + a) \tau(n + a) + t(n + a)(z + b) \right\} \right], \quad (2-10) \]
for 2-dimensional vectors $a$ and $b$.

We will note that these contours in the integral are, for example, given in [4 p.19]. Thus above values can be explicitly computed for a given $y^2 = f(x)$. 

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Definition 2-2 [1 p.286], [B2 p.336, p.353, p.370], [4 p.32, p.35], [9 p.386-7].

We will introduce the coordinate \((u_1, u_2)\) in \(\mathbb{C}^2\) for points \((x, y)\) and \((x_2, y_2)\) of the curve \(y^2 = f(x)\) by \((u_1, u_2) := (w_1((x, y), (x_2, y_2)), w_2((x, y), (x_2, y_2)))\), i.e.,

\[
\begin{align*}
u_j & := \int_{\infty}^{(x,y_j)} du_j + \int_{\infty}^{(x_2,y_2)} du_j. \tag{2-11}
\end{align*}
\]

(1) The sigma function, which is a homomorphic function over \(\mathbb{C}^2\), is defined by,

\[
\sigma(u) := \sigma(u; \omega) := \gamma \exp(-\frac{1}{2} t \ w\omega' \omega'^{-1} u) \vartheta \left[ \frac{\delta''}{\delta'} \right] \left( (2\omega')^{-1} u; \tau \right), \tag{2-12}
\]

where \(\gamma\) is a certain constant, which is determined such that Proposition 2-4 holds, and,

\[
\begin{align*}
\delta' & := \left[ \begin{array}{c} 1 \\
\frac{1}{2} \end{array} \right], \quad \delta'' & := \left[ \begin{array}{c} 1 \\
\frac{1}{2} \end{array} \right]. \tag{2-13}
\end{align*}
\]

Let its derivative denote \(\sigma_\mu := \partial \sigma / \partial u_\mu\).

(2) The hyperelliptic \(\wp\)-function over the Jacobian \(\mathcal{J}\) is defined by,

\[
\wp_{\mu\nu}(u) := -\frac{\partial^2}{\partial u_\mu \partial u_\nu} \log \sigma(u). \tag{2-14}
\]

(3) When \((x_2, y_2)\) is the infinity point, \((u_1, u_2)\) is a function only of \((x, y) \in C\) and we refer this \(u\) by \(u \in \iota(C) \subset \mathcal{J}\); the operation makes the curve \(C\) embedded into the Jacobian \(\mathcal{J}\), \(\iota : C \hookrightarrow \mathcal{J}\).

**Proposition 2-3.**

Let us express \(\wp_{\mu\nu} := \partial \wp_{\mu\nu}(u) / \partial u_\rho\) and \(\wp_{\mu\nu\rho\lambda} := \partial^2 \wp_{\mu\nu}(u) / \partial u_\mu \partial u_\nu\). Then hyperelliptic \(\wp\)-functions obey the relations,

\[
\begin{align*}
(H - 1) & \quad \wp_{2222} - 6\wp_{22}^2 = 2\lambda_3 \lambda_5 + 4\lambda_4 \wp_{22} + 4\lambda_5 \wp_{21} - 12\lambda_6 \wp_{11}, \\
(H - 2) & \quad \wp_{2221} - 6\wp_{22}\wp_{21} = 4\lambda_4 \wp_{21} - 2\lambda_5 \wp_{11}, \\
(H - 3) & \quad \wp_{2211} - 4\wp_{21}^2 - 2\wp_{22}\wp_{11} = 4\lambda_2 \wp_{31} + 2\lambda_3 \wp_{21}.
\end{align*}
\]
(H − 4) \( \varphi_{2111} - 6\varphi_{21}\varphi_{11} = -2\lambda_0\lambda_5 - 2\lambda_1(\varphi_{22}) + 4\lambda_2\varphi_{21}, \)

(H − 5) \( \varphi_{1111} - 6\varphi_{11}^2 = -4\lambda_0\lambda_4 + 2\lambda_1\lambda_3 - 12\lambda_0(\varphi_{22}) + 4\lambda_1\varphi_{21} + 4\lambda_2\varphi_{11}. \)  

(I − 0) \( \varphi_{112} = \varphi_{222}\varphi_{12} + \varphi_{122}\varphi_{22}, \)

(I − 1) \( \varphi_{222}^2 = 4(\varphi_{22}^3 + \varphi_{12}\varphi_{22} + \lambda_4\varphi_{22}^2 + \varphi_{11} + \lambda_3\varphi_{22} + \lambda_2), \)

(I − 2) \( \varphi_{222}\varphi_{221} = 4(\varphi_{12}\varphi_{22}^2 - \varphi_{12}^2 + \lambda_3\varphi_{12} - \lambda_1) + \lambda_3\varphi_{12} + \lambda_4\varphi_{12}\varphi_{22}), \)

(I − 3) \( \varphi_{221}^2 = 4(\varphi_{11}\varphi_{22} - (\varphi_{11}\varphi_{22} - \varphi_{12}^2 + \lambda_3\varphi_{12} - \lambda_1)\varphi_{22} - \varphi_{11}\varphi_{12} + \lambda_4\varphi_{11}\varphi_{22}) \)

\[ + \lambda_3\varphi_{12}\varphi_{22} - \lambda_4(\varphi_{11}\varphi_{22} - \varphi_{12}^2 + \lambda_3\varphi_{12} - \lambda_1) \]

\[ + \lambda_4\lambda_3\varphi_{12} - \lambda_1\varphi_{22} - \lambda_1\lambda_4 + \lambda_0. \)  

**Proof.** See p.155-6 in [3], p.86-88 in [4], or [6].

We also have the additional formula for them.

**Proposition 2-4.**

(1) \[ \frac{\sigma(u + v)\sigma(u - v)}{\sigma(u)^2\sigma(v)^2} = Q(u, v), \]

where

\[ Q(u, v) := -(\varphi_{11}(u) - \varphi_{11}(v) + \varphi_{12}(u)\varphi_{22}(v) - \varphi_{12}(v)\varphi_{22}(u)). \]

(2) \[ \varphi_{ij}(u + v) + \varphi_{ij}(u - v) = 2\varphi_{ij}(u) - \frac{Q_{ij}Q - Q_iQ_j}{Q^2}, \]

where \( Q_i(u, v) := \partial_u Q \) and \( Q_{ij}(u, v) := \partial_{u_j} Q_j. \)

**Proof.** See p.372, p.381 in [2] or p.109-114 in [4].
Proposition 2-5.

In terms of $(x,y)$ and $(x_2,y_2)$ in Eq. (2-11), the $\wp$ functions are expressed by

$$\wp_{12} = xx_2, \quad \wp_{22} = x + x_2.$$  \hspace{1cm} (2-20)

Proof. See p.377 in [2] or p.38, p.84 in [4].

§3. Recursion Relation of Hyperelliptic $\psi$-function of Genus Two

Let us define a hyperelliptic $\psi$ function of genus two as

$$\psi_n(u) = \frac{\sigma(nu)}{\sigma_2(u)^{n^2}},$$  \hspace{1cm} (3-1)

for $u \in \iota(C) \subset J$. In general, for $u \in \iota(C) \subset J$, $\sigma$ and $\sigma_2(\equiv \partial\sigma/\partial u_2)$ vanish but by cancellation of zeros, the formula (3-1) is well-defined and has a finite value.

We wish to derive the relation (1-7) over the embedded curve $\iota(C)$ but the quantities and relations in the previous section are defined over $J$. Thus let us introduce a quantity,

$$\Psi_n(u) = \frac{\sigma(nu)}{\sigma_2(u)^{n^2}},$$  \hspace{1cm} (3-2)

for $u \in J$, which is limit as $\psi_n(u) = \lim_{u \rightarrow \iota(C)} \Psi_n(u)$. After considering relations over $J$, we will restrict some quantities into $\iota(C)$. We also note that even if the local parameter $u$ is in $\iota(C)$, the point $nu$ $(n > 1)$ is in $J$ but not in $\iota(C)$ in general.

Using Eq.(3-2), the addition formula (2-17) is expressed by

$$\frac{\Psi_{m+n}(u)\Psi_{m-n}(u)}{\Psi_n(u)^2\Psi_m(u)^2} = Q(mu, nu).$$  \hspace{1cm} (3-3)

For simplicity, we assume that $n$ and $m$ are natural numbers greater than 1 and $u$ is a generic point in $\iota(C)$ hereafter,

Lemma 3-1.
We introduce a quantity $\Xi_3(mu, u)$ as
\begin{equation}
\Xi_3(mu, u) := \frac{1}{\Psi_m(u)^3} \left( \Psi_{m-1}(u)^2 \Psi_{m+1}(u) + \Psi_{m+1}(u)^2 \Psi_{m-2}(u) \right), \quad (3-4)
\end{equation}
for $u \in J$. Then $\Xi_3(mu, u)$ becomes,
\begin{align*}
\Xi_3(mu, u) &= \Psi_1(u)^6 \left( Q(mu, u) [2Q(mu, u)^2 + Q_{11}(mu, u) \\
&- \varphi_{22}(mu)Q_{12}(mu, u) + \varphi_{12}(u)Q_{22}(mu, u)] \\
&- Q_1(mu, u)^2 - \varphi_{22}(u)Q_1(mu, u)Q_2(mu, u) \\
+ \varphi_{12}(u)Q_2(mu, u)^2 \right). \quad (3-5)
\end{align*}

**Proof.** First we can rewrite $\Xi_3$ as,
\begin{equation}
\Xi_3(mu, u) = \Psi_1(u)^6 \frac{\Psi_{m-1}(u)^2 \Psi_{m+1}(u)^2}{\Psi_m(u)^4 \Psi_1(u)^4} \left( \frac{\Psi_{m+1}(u)^2 \Psi_m(u)}{\Psi_{m-1}(u)^2 \Psi_1(u)^2} + \frac{\Psi_{m-2}(u) \Psi_m(u)}{\Psi_{m-1}(u)^2 \Psi_1(u)^2} \right) \quad (3-6)
\end{equation}
Using Proposition 2-4 (1) and (2) sequentially, it becomes
\begin{align*}
\Psi_1(u)^6 Q(mu, u)^2 (Q((m+1)u, u) + Q((m-1)u, u)) \\
= \Psi_1(u)^6 Q(mu, u)^2 \left( 2\varphi_{11}(u) - 2\varphi_{11}(mu) \\
+ \frac{Q_{11}(mu, u)Q(mu, u) - Q_1(mu, u)^2}{Q(mu, u)^2} \\
- \varphi_{22}(u) \left( 2\varphi_{12}(mu) - \frac{Q_{12}(mu, u)Q(mu, u) - Q_1(mu, u)Q_2(mu, u)}{Q(mu, u)^2} \right) \\
+ \varphi_{12}(u) \left( 2\varphi_{22}(mu) - \frac{Q_{22}(mu, u)Q(mu, u) - Q_2(mu, u)^2}{Q(mu, u)^2} \right) \right) \\
= \Psi_1(u)^6 \left( 2Q(mu, u)^2 (\varphi_{11}(u) - \varphi_{11}(mu) + \varphi_{22}(u)\varphi_{12}(mu) - \varphi_{12}(u)\varphi_{22}(mu)) \\
+ Q_{11}(mu, u)Q(mu, u) - Q_1(mu, u)^2 \\
- \varphi_{22}(u)(Q_{12}(mu, u)Q(mu, u) - Q_1(mu, u)Q_2(mu, u)) \\
+ \varphi_{12}(u)(Q_{22}(mu, u)Q(mu, u) - Q_2(mu, u)^2) \right). \quad (3-7)
\end{align*}
By arranging these terms, we obtain Eq.(3-5). ♦

Hereafter, let us consider the restriction of these quantities to the embedded curve $\iota(C) \subset J$. 

\begin{equation}
\text{Proof.} \quad \text{— Continued —}
\end{equation}
Lemma 3-2.

The following relations hold over $\iota(C)$:

1. For $u \in \iota(C)$,
   \[ \varphi_{12}(2u) = -x^2, \quad \varphi_{22}(2u) = 2x. \]  
   \[ (3-8) \]

2. For $u \in \iota(C)$,
   \[ \psi_2(u) = 2y. \]  
   \[ (3-9) \]

3. \[ \lim_{u \to \iota(C)} (\Psi_1(u)^2 \psi_{12}(u)) = x^2, \]
   \[ \lim_{u \to \iota(C)} (\Psi_1(u)^2 \psi_{12}(u)) = -x, \]
   \[ \lim_{u \to \iota(C)} (\Psi_1(u)^2 \psi_{22}(u)) = 1. \]  
   \[ (3-10) \]

Proof. Proposition 2-5 leads (1) directly. On (2), $\psi_2(u) = \frac{\sigma(2u)}{\sigma_2(u)}$ was given in [9].

From Proposition 2-5, we have

\[ x = \lim_{u \to \iota(C)} \frac{\varphi_{12}(u)}{\varphi_{22}(u)} = \lim_{u \to \iota(C)} \frac{\sigma_1(u)}{\sigma_2(u)}. \]  
   \[ (3-11) \]

Noting the definition of $\varphi$ and using the relation,

\[ \lim_{u \to \iota(C)} (\Psi_1(u)^2 \psi_{12}(u)) = \lim_{u \to \iota(C)} \left( \frac{\sigma(u)^2}{\sigma_2(u)^2} - \frac{\sigma_1(u)\sigma_j(u) - \sigma_{ij}(u)\sigma(u)}{\sigma(u)^2} \right). \]  
   \[ (3-12) \]

(3) is obtained.

Lemma 3-3.

1. \[ \lim_{u \to \iota(C)} Q(mu, 2u) = (\varphi_{11}(mu) - \varphi_{11}(2u) - 2\varphi_{12}(mu)x - \varphi_{22}(mu) x^2). \]  
   \[ (3-13) \]

2. The quantities
   \[ q(mu, u) := \lim_{u \to \iota(C)} \Psi_1(u)^2 Q(mu, u), \quad q_i(mu, u) := \lim_{u \to \iota(C)} \Psi_1(u)^2 Q_i(mu, u), \]
\[ q_{ij}(\mu, u) := \lim_{\| \rightarrow \|} \Psi_1(u)^2 Q_{ij}(\mu, u), \quad (3-14) \]

become
\[ q(\mu, u) = -x^2 + \varphi_{12}(\mu) + \varphi_{22}x, \]
\[ q_{ij}(\mu, u) = \varphi_{12ij}(\mu) + \varphi_{22ij}(\mu)x, \]
\[ q_i(\mu, u) = \varphi_{12i}(\mu) + \varphi_{22i}(\mu)x. \quad (3-15) \]

(3) By letting \( \xi_3(\mu, u) := \lim_{u \in \| (C)} \Xi_3(\mu, u), \)
\[ \xi_3(\mu, u) = q(\mu, u) \left( 2q(\mu, u)^2 - q_{12}(\mu, u) - xq_{22}(\mu, u) \right) \]
\[ - q_2(\mu, u) \left( q_1(\mu, u) + xq_2(\mu, u) \right). \quad (3-16) \]

**Proof.** Addition formula (2-17) of \( Q(\mu, 2u) \) is restricted to \( \| (C) \) noting Lemma 3-2 (3), and then (1) is obtained. Similarly (2) and (3) are proved. In (3), we note that
\[ \lim_{u \rightarrow \| (C)} \Psi_1(u)^2 Q_{11}(\mu, u) = 0, \quad \lim_{u \rightarrow \| (C)} \Psi_1(u)^2 Q(\mu, u)Q_{11}(\mu, u) = 0, \quad (3-17) \]
due to the order of the zero at \( \| (C) \).

**Lemma 3-4.**

For \( u \in \| (C) \), we introduce the quantities,
\[ \xi_0(\mu, nu) := \psi_2(u)^2 \frac{\psi_{m-n}(u)\psi_{m+n}(u)}{\psi_m(u)^2\psi_n(u)^2}, \quad \xi_1(\mu) := \frac{\psi_{m-2}(u)\psi_{m+2}(u)}{\psi_m(u)^2}, \]
\[ \xi_2(\mu, nu) := \frac{\psi_{m-1}(u)\psi_{m+1}(u)}{\psi_m(u)^2\psi_n(u)^3} \left( \psi_{n-1}(u)^2\psi_{n+2}(u) + \psi_{n+1}(u)^2\psi_{n-2}(u) \right) \]
\[ - \frac{\psi_{n-1}(u)\psi_{n+1}(u)}{\psi_n(u)^2\psi_m(u)^3} \left( \psi_{m-1}(u)^2\psi_{m+2}(u) + \psi_{m+1}(u)^2\psi_{m-2}(u) \right). \quad (3-18) \]

They satisfy the following relations:

(1)
\[ \xi_0(\mu, nu) = -4y^2(\varphi_{11}(\mu) - \varphi_{11}(nu) - 2\varphi_{12}(\mu)\varphi_{12}(nu) - \varphi_{22}(\mu)\varphi_{22}(nu)). \quad (3-19) \]
Here we use Eq.(2-18), Eq.(3-3) and Eq.(3-15). Let us consider the parts of \( \xi_2 (mu, nu) \) consists of \( \xi_3 (mu, u) \) and \( \xi_3 (nu, u) \) due to the definition in Lemma 3-3,

\[
\xi_2 (mu, nu) = \frac{\psi_{m-1}(u)\psi_{m+1}(u)}{\psi_m(u)^2} \xi_3 (nu, u) - \frac{\psi_{n-1}(u)\psi_{n+1}(u)}{\psi_n(u)^2} \xi_3 (mu, u)
\]

\[= (x^2 - \psi_{22}(nu)x - \psi_{12}(nu))\xi_3 (mu, u) \]

\[\quad - (x^2 - \psi_{22}(mu)x - \psi_{12}(mu))\xi_3 (nu, u). \quad (3-22)\]

Here we use Eq.(2-18), Eq.(3-3) and Eq.(3-15). Let us consider the parts of \( \xi_3 (mu, u) \) in Eq.(3-16),

\[
q_{12}(mu, u) = 6\varphi_{12}(mu)\varphi_{22}(mu) - 4(\varphi_{11}(mu)\varphi_{22}(mu) - \varphi_{12}(mu))^2 + 2\lambda_2 \varphi_{12}(mu)
\]

\[\quad + x(6\varphi_{12}(mu)\varphi_{22}(mu) - 2\varphi_{11}(mu) + 4\lambda_1 \varphi_{12}(mu)), \]

\[
q_{22}(mu, u) = 6\varphi_{12}(mu)\varphi_{22}(mu) - 2\varphi_{11}(mu) + 4\lambda_1 \varphi_{12}(mu)
\]

\[\quad + x(6\varphi_{22}(mu)^2 + 4\varphi_{12}(mu) + 4\lambda_1 \varphi_{22}(mu) + 2\lambda_2), \]

\[
q_1(mu, u) = \varphi_{112}(mu) + x\varphi_{221}(mu), \quad q_2(mu, u) = \varphi_{122}(mu) + x\varphi_{222}(mu). \quad (3-23)\]

By using above parts, the constituent of the first term in \( \xi_3 (mu, u) \) is expressed by

\[
2q(mu, u)^2 - q_{12}(mu, u) - xq_{22}(mu, u)
\]

\[= 2x^4 - 4x^3 \varphi_{22}(mu)
\]

\[\quad - (4\varphi_{22}(mu)^2 + 8\varphi_{12}(mu) + 4\lambda_1 \varphi_{22}(mu) + 2\lambda_2)x^2
\]

\[\quad - (8\varphi_{12}(mu)\varphi_{22}(mu) - 4\varphi_{11}(mu) + 8\lambda_1 \varphi_{12}(mu))x
\]

\[\quad - (2\varphi_{11}(mu)\varphi_{22}(mu) + 2\varphi_{12}(mu)^2 + 2\lambda_2 \varphi_{12}(mu)), \quad (3-24)\]
and the second term in $\xi_3(\mu, u)$ becomes

$$q_2(\mu, u)\left(q_1(\mu, u) + xq_2(\mu, u)\right)$$

$$= (\varphi_{122}(\mu) + x\varphi_{222}(\mu))\varphi_{122}(\mu) + 2x\varphi_{221}(\mu) + x^2\varphi_{222}(\mu)$$

$$= \varphi_{222}(\mu)^2x^3 + 3\varphi_{122}(\mu)\varphi_{222}(\mu)x^2$$

$$+ (\varphi_{12}(\mu)\varphi_{222}(\mu)^2 - \varphi_{22}(\mu)\varphi_{122}(\mu)\varphi_{222}(\mu) + 2\varphi_{122}(\mu)^2)x$$

$$- \varphi_{22}(\mu)\varphi_{122}(\mu)^2 - \varphi_{12}(\mu)\varphi_{122}(\mu)\varphi_{222}(\mu).$$

(3-25)

Using the relation (I-0)-(I-3) in Proposition 2-3, $q_2(\mu, u)\left(q_1(\mu, u) + xq_2(\mu, u)\right)$ is reduced to the formula consisting only of $\varphi_{ij}(\mu)$. After tedious computations, we have the relation $\xi_2(\mu, nu)$.)

Since $\xi_0(nu, \mu) + \xi_1(\mu) - \xi_1(nu) - \xi_2(\mu, nu) = 0$, we have a part of the recursion relation of $\psi$-function of genus two. From the definition of $\psi_n$, we have the relations, $\psi_0 \equiv \psi_1 \equiv 0$ for $u \in \iota(C)$ [9, 10], the recursion relation can be extended to all integers $n$ and $m$ ($m \geq n \geq 0$).

**Theorem 3-5.**

*For integers $m, n$, ($m \geq n \geq 0$),

$$\psi_2\psi_2\psi_m\psi_n\psi_{n+m}\psi_{m-n} = \begin{vmatrix} \psi_{m-2}\psi_n & \psi_{m-1}\psi_{n+1} & \psi_m\psi_{n+2} \\ \psi_{m-1}\psi_{n-1} & \psi_m\psi_n & \psi_{m+1}\psi_{n+1} \\ \psi_m\psi_{n-2} & \psi_{m+1}\psi_{n-1} & \psi_{m+2}\psi_n \end{vmatrix}. \quad (3-26)$$

**Remark 3-6.**

(1) As mentioned in Introduction, let us summarize this article here. In fact we derived the recursion relation (3-26) but the process we stepped is too confused and looks haphazard. Thus we will mention the idea of our derivation and recapitulate the process here.
In our proofs of the relations in Theorem 3-5 and Eq.(1-3), which are cases of genera one and two, addition formulae play important roles. Especially, linear terms of ϕ-function in the addition formulae are essential. For the case of genus one, the role of the linear term with respect to ϕ was mentioned in Eq.(1-6). Even for the case of genus two, the idea to derive the recursion relation (3-26) is the same.

As we have the addition formula of ϕ in Eq.(1-10) or Eqs.(2-17) and (2-18), the factor σ(u + v)σ(u − v) in the left hand side of Eq.(2-17) is an essential part of ψ_{m+n}ψ_{m−n} in the left hand side of Eq.(3-26) since ψ is defined as Eq.(3-1) in terms of the σ function. Combination of the linear terms with respect to ϕ’s in Eq.(2-18) can be canceled like the genus one case. Thus by introducing ξ_0 and ξ_1 in Eq.(3-18), we considered ξ_0(nu, mu) + ξ_1(mu) − ξ_1(nu) like Eq.(1-6); from Eqs.(3-19) and (3-20) in Lemma 3-4, these linear terms without x nor x^2 vanishes. Identification 2y with ψ_2 in Eq.(3-9) roughly gives a recursion relation. However in the addition formula Eq.(2-18), there appear the quadratic terms on ϕ’s and thus we need some technical manipulations. ξ_2 in Eq.(3-18) appears to remove the excess terms in ξ_0(nu, mu) + ξ_1(mu) − ξ_1(nu) due to the formula (3-21).

Further ψ function of genus two is defined over a curve itself ψ(C) rather than the Jacobian J but the ϕ’s and σ’s are defined over the Jacobian J. Hence in order to perform the process mentioned above, we need consider the restriction of the domain of the functions and the relations. We started with functions over the Jacobian J, i.e. Ψ in Eq.(3-2), Ξ’s in Eq.(3-4), Q in Eq.(2-17) and so on, and then investigated them. After restricting their domain to ψ(C), we obtained Theorem 3-5.

Accordingly we emphasize again that the idea of our derivations of the recursion relations of genera one and two is very simple and in the derivation, linear terms of ϕ-function in the addition formulae play essential roles.
(2) After seeing the success of the idea of the proofs of genera one and two, one might consider more general relation for higher genus. However he must encounter the difficulty. Because in the addition formula for the case of genus three,

\[
\frac{-\sigma(u + v)\sigma(u - v)}{\sigma(u)^2\sigma(v)^2} = \left(\wp_{31}(u) - \wp_{31}(v)\right)^2 - \left(\wp_{31}(u) - \wp_{31}(v)\right)\left(\wp_{22}(u) - \wp_{22}(v)\right)
\]

\[
- \left(\wp_{33}(u) - \wp_{33}(v)\right)\left(\wp_{11}(u) - \wp_{11}(v)\right)
\]

\[
+ \left(\wp_{12}(u) - \wp_{12}(v)\right)\left(\wp_{23}(u) - \wp_{23}(v)\right).
\]

(3-27)

there is no linear term with respect to \(\wp\) for the hyperelliptic \(\sigma\) and \(\wp\) functions of genus three. The \(\psi\) function of genus three is also given by

\[
\psi_n(u) = \frac{\sigma(nu)}{\sigma_2(u)^n},
\]

(3-28)

for a coordinate \(u\) of the embedded hyperelliptic curve in its related Jacobian of genus three [11]. In other words, the attempt of the extension of the derivation Eq.(1-3)-(1-6) might fail for the case of genus three. In order to reveal the difference between the properties of \(\psi\) functions of the cases \(g \leq 2\) and \(g = 3\), we believe that our approach is also mathematically important.

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