QUASILINEAR ELLIPTIC PROBLEMS WITH CYLINDRICAL SINGULARITIES AND MULTIPLE CRITICAL NONLINEARITIES:
EXISTENCE, REGULARITY, NONEXISTENCE

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Abstract. This work deals with existence of solutions for the class of quasilinear elliptic problems with cylindrical singularities and multiple critical nonlinearities that can be written in the form

\[-\text{div} \left( \frac{\nabla u |^{p-2}}{|y|^{ap}} \nabla u \right) - \mu \frac{u^{p-1}}{|y|^{p(a+1)}} = \frac{u^{p^*(a,b)-1}}{|y|^{p^*(a,b)}} + \frac{u^{p^*(a,c)-1}}{|y|^{p^*(a,c)}}, \quad (x, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k.

We consider $N \geq 3$, $1 \leq k \leq N$, $1 < p < N$, $\mu < \bar{\mu} \equiv \{[k - p(a + 1)]/p\}^p$, $a < (k - p)/p$, $a < b < c < a + 1$, $p^*(a,b) = Np/[N - p(a + 1 - b)]$, and $p^*(a,c) = Np/[N - p(a + 1 - c)]$; in particular, if $\mu = 0$ we can include the cases $(k - p)/p \leq a < k(N - p)/Np$ and $a < b < c < k(N - p(a + 1))/p(N - k) < a + 1$. The existence of a positive, weak solution $u \in D^1_a(\mathbb{R}^N \setminus \{y = 0\})$ is proved with the help of the mountain pass theorem. We also prove a regularity result, that is, using Moser’s iteration scheme we show that if $u \in D^1_an(\mathbb{R}^N \setminus \{y = 0\})$ is a weak solution to the related problem

\[-\text{div} \left( \frac{\nabla u |^{p-2}}{|y|^{ap}} \nabla u \right) - \mu \frac{u^{p-2}}{|y|^{p(a+1)}} = \frac{|u|^{p-2}u}{|y|^{p^{*}(a,b)}} + \frac{|u|^{p^{*}(a,c)-2}u}{|y|^{p^{*}(a,c)}}, \quad (x, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k,

then $u \equiv 0$ when either $1 < q < p^*(a,b)$, or $q > p^*(a,b)$ and $u \in L^q_{bp^*(a,b)/q,loc}(\mathbb{R}^N \setminus \{y = 0\}) \cap L^{\infty}_{bp^*(a,b)/q,loc}(\mathbb{R}^N \setminus \{y = 0\})$. This nonexistence of nontrivial solution is proved by using a Pohozaev-type identity.

1. Introduction and main results

The main goal of this work is to prove existence results for a class of quasilinear elliptic problems with cylindrical singularities and multiple critical nonlinearities that can be written in the form

\[-\text{div} \left( \frac{\nabla u |^{p-2}}{|y|^{ap}} \nabla u \right) - \mu \frac{u^{p-1}}{|y|^{p(a+1)}} = \frac{u^{p^*(a,b)-1}}{|y|^{p^*(a,b)}} + \frac{u^{p^*(a,c)-1}}{|y|^{p^*(a,c)}}, \quad (x, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k. \tag{1}

We consider $N \geq 3$, $1 \leq k \leq N$, $1 < p < N$, $0 \leq \mu < \bar{\mu} \equiv \{[k - p(a + 1)]/p\}^p$, $0 \leq a < (k - p)/p$, $a \leq b < c < a + 1$, $p^*(a,b) = Np/[N - p(a + 1 - b)]$, and $p^*(a,c) = Np/[N - p(a + 1 - c)]$; in particular, if $\mu = 0$ we can include the cases $(k - p)/p \leq a < k(N - p)/Np$ and $a < b < c < k(N - p(a + 1))/p(N - k) < a + 1$.

This class of problems arises in the study of standing waves in the anisotropic Schrödinger equation. It also appears in models of physical phenomena related to the equilibrium of the temperature in an anisotropic media that can be a ‘perfect insulator’ at some points, represented by the degenerate case $\inf_{|y| \to \infty} 1/|y|^{ap} = 0$, and can be a ‘perfect conductor’ at other points, represented by the singular case $\sup_{|y| \to 0} 1/|y|^{ap} = \infty$. Problem (1) also has some interest in astrophysics, where the dynamics of some galaxies is modeled with the use of cylindrical
weights due to their axial symmetry. For more details, see Dautray and Lions [14], Wang and Willem [37], Catrina and Wang [11], Badiale and Tarantello [5], Drábek [15], Ghergu and Rădulescu [21], and references therein. The pure mathematical interest in this class of problems is due to the fact that problem (1) can be regarded as a model for a general class of quasilinear elliptic problems with a cylindrical weight in the \( p \)-Laplacian differential operator and also involving multiple nonlinearities with cylindrical weights and critical Maz’ya’s exponents.

The choice for the intervals for the several parameters already specified is motivated by the following Maz’ya’s inequality, which plays a crucial role in our work since it allows the variational formulation of problem (1). Let \( N \geq 3 \), \( 1 \leq k \leq N \), \( \bar{z} = (x, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k \), \( 1 < p < N \), and either \( a < (k-p)/p \) and \( a \leq b \leq a+1 \), or \( (k-p)/p \leq a < k(N-p)/Np \) and \( a \leq b < k(N-p(a+1))/p(N-k) < a+1 \). Then there exists a positive constant \( C > 0 \) such that

\[
\left( \int_{\mathbb{R}^N} \frac{|u(z)|^{p^*(a,b)}}{|y|^{bp^*(a,b)}} dz \right)^{p/p^*(a,b)} \leq C \int_{\mathbb{R}^N} \frac{|\nabla u(z)|^p}{|y|^{ap}} dz
\]

for every function \( u \in C^\infty(\mathbb{R}^N \setminus \{|y|=0\}) \), where \( p^*(a,b) = Np/[N-p(a+1-b)] \) is the critical Maz’ya’s exponent.

The proof of inequality (2) can be found in the book by Maz’ya [26, Section 2.1.6]; the particular case \( k = N \) of inequality (2) was proved by Caffarelli, Kohn and Nirenberg [10]; see also Lin [25] for an inequality involving higher order derivatives in the case \( k = N \).

In what follows we present a very brief historical sketch for these types of problems, mainly concerning existence results. To avoid unnecessary repetitions, we write the class of problems in the form

\[
- \text{div} \left[ \frac{\nabla u}{|y|^{ap}} \nabla u \right] - \mu \frac{u^{p-1}}{|y|^{p(a+1)}} = \frac{u^{p^*(a,b)-1}}{|y|^{bp^*(a,b)}} + \lambda \frac{u^{p^*(a,c)-1}}{|y|^{cp^*(a,c)}}, \quad (x, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k.
\]

We also define the infimum

\[
\frac{1}{K(N, p, \mu, a, b)} \equiv \inf_{u \in C^\infty(\mathbb{R}^N \setminus \{|y|=0\})} \frac{\int_{\mathbb{R}^N} |\nabla u(z)|^p dz - \mu \int_{\mathbb{R}^N} \frac{|u(z)|^p}{|y|^{p(a+1)}} dz}{\left( \int_{\mathbb{R}^N} \frac{|u(z)|^{p^*(a,b)}}{|y|^{bp^*(a,b)}} dz \right)^{p/p^*(a,b)}},
\]

which is positive for \( \mu < \bar{\mu} \equiv 1/K(N, p, 0, a, a+1) \) and has an independent interest. For the determination of the optimal constant \( \bar{\mu} = [(k-p(a+1))/p]^p \), see the paper by Secchi, Smets and Willem [30].

First we consider the case \( \lambda = 0 \). For \( k = N \), which represents spherical weights, \( 1 < p < N \), \( \mu = 0 \), \( a = 0 \), \( b = 0 \), and \( p^*(a,a) = p^* = Np/(N-p) \), problem (3) was treated in the well known papers by Aubin [4] and Talenti [34]; they computed the value of the best constant \( K(N, p, 0, 0, 0) \) for the Sobolev inequality and presented the class of the functions that assume this infimum. For more information on best constants, see also the papers by Chou and Chu [12], Horiuchi [24] and references therein. Wang and Willem [37] showed the existence of solution to problem (3) in the case \( p = 2 \), which is the Laplacian operator, with critical spherical weights represented by \( k = N, 0 \leq a < (N-2)/2 \), \( a \leq b < a+1 \), with a homogeneous linear term, represented by \( 0 \leq \mu < \bar{\mu} \), and \( 2^*(a,b) < 2^* \equiv 2N/(N-2) \). Catrina and Wang [11] also considered these cases but with \( a < (N-2)/2 \) and obtained several existence, nonexistence, as well as symmetry breaking of solutions to problem (3). Problem (3) in the case of the \( p \)-Laplacian operator represented by \( 1 < p < N \), with critical spherical weights represented by \( k = N, a \leq b < a+1 \), and a homogeneous nonlinearity, represented by \( \mu < \bar{\mu} \), was also studied by Assunção, Carrião and Miyagaki [3], who extended the results by Wang and Willem [37]. For problems with cylindrical weights, represented by \( 1 \leq k \leq N \), we cite the paper by Musina [28], who studied the case \( N \geq 3, 1 \leq k \leq N, p = 2, a = 0, 0 < b \leq 1, \mu < \bar{\mu} \) and proved that problem (3) has a ground state solution; in particular, if \( k = 1 \) then the support of this solution
is the half-space $y \geq 0$. Additionally, if $b = 0$ and $0 < \mu < \bar{\mu}$, problem (3) has a ground state solution when either $2 < k \leq N$, or $k = 1$ and $N \geq 4$; in this last case the ground state also has support in a half-space. Gazzini and Musina [20] studied problem (3) in the case $N \geq 3$, $1 \leq k \leq N$, $1 < p < N$, $\mu = 0$, and either $0 < a < (k-p)/p$, and $a \leq b < a+1$, or $a \leq 0$ and $a < b < a+1$, and obtained an existence result; they also proved an existence result to problem (3) when $(k-p)/p \leq a < k(N-p)/(Np) + a + 1$ and $a < b < k(N-p)/(Np) < a + 1$ and $a < b < k(N-p)/(Np) + a + 1$ and $a < b < a+1$, and obtained an existence result; they also proved an existence result to problem (3) when $(k-p)/p \leq a < k(N-p)/(Np) + a + 1$ and $a < b < k(N-p)/(Np) < a + 1$ and $a < b < a+1$, and obtained an existence result; they also proved an existence result to problem (3) when $(k-p)/p \leq a < k(N-p)/(Np) + a + 1$ and $a < b < k(N-p)/(Np) < a + 1$ and $a < b < a+1$, and obtained an existence result; they also proved an existence result to problem (3) when $(k-p)/p \leq a < k(N-p)/(Np) + a + 1$ and $a < b < k(N-p)/(Np) < a + 1$ and $a < b < a+1$. In these cases, the infimum $1/K(N,p,0,a,b)$ is attained when either $p^*(a,b) < p^*$, or $p^*(a,a) = p^*$ and $1/K(N,p,0,a,b) < 1/K(N,p,0,0,0)$. Bhakta [8] generalized these results by considering problem (3) with a homogenous nonlinearly, represented by $\mu \neq 0$, with $N \geq 3$, $1 \leq k \leq N$, $1 < p < N$, and either $0 = a = b$ and $0 \leq \mu < \bar{\mu}$, or $a < (k-p)/p$, $a < b < a+1$ and $\mu < \bar{\mu}$, or still $0 < a = b$ and $\mu^* < \mu < \bar{\mu}$, where $\mu^* < \bar{\mu} [(N-1)/(Np)-ap^2/(N-\alpha)] < 0$. In each one of these cases, there exists solution to problem (3). For the case of a nonlinearity with a cylindrical weight that is not a pure power we cite the papers by Badiale and Tarantello [5] and by Sintzoff [31].

On the other hand, in the case $\lambda \neq 0$ we cite the paper by Filippucci, Pucci and Robert [19], where they proved some existence results for the problem (3) with $k = N$, $1 < p < N$, without singularities in the differential operator, represented by $a = 0$, but with a homogeneous nonlinearity, that is, $0 \leq \mu < \bar{\mu}$, and multiple critical nonlinearities, represented by $b = 0$ and $0 < c < 1$, that is, only one of them with spherical weight. For a generalization of this result, see Xuan and Wang [40], where the case $N \geq 3$, $k = N$, $1 < p < N$, $0 < \mu < \bar{\mu}$, $0 < a < (N-p)/p$, $a \leq b < a+1$, and $a \leq c < a+1$ is studied; see also Sun [33], who studied the case $N \geq 3$, $2 < k < N, 1 < p < k, 0 < \mu < \bar{\mu}$, $a = 0$, $b = 0$, and $0 < c < 1$. See also Ambrosetti, Brézis and Cerami [2] for a related problem with sublinear and superlinear nonlinearities in the case of the Laplacian operator without singular weights.

Inspired by Gazzini and Musina [20] and by Bhakta [8] regarding the nature of the cylindrical singularities, and by Filippucci, Pucci and Robert [19], by Xuan and Wang [40], and by Sun [33] with respect to the presence of multiple critical nonlinearities, our first result deals with existence of a positive, weak solution to problem (1). In its statement, we mention the Sobolev space $D_a^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})$, whose definition appears at the beginning of section 2.

Our first result reads as follows.

**Theorem 1.1.** Let $2 \leq k \leq N$, $1 < p < N$ and $a < (k-p)/p$; let $\bar{\mu} \equiv [(k-p(a+1))/p]^p$. Suppose that the parameters $b$ and $c$ verify one of the following cases.

1. $0 = a = b < c < 1$ and $\mu < \bar{\mu}$;
2. $a < b < c < a+1$, $\mu < \bar{\mu}$; in particular, if $\mu = 0$ we can include the cases $(k-p)/p \leq a < k(N-p)/Np$ and $a < b < c < k(N-p)/(Np) < a+1$.
3. $0 < a = b < c < a+1$, $\mu^* < \mu < \bar{\mu}$, where $\mu^* < \bar{\mu} [-ap^2/(N-\alpha)](N-1)/(N-p) < 0$; in particular, if $\mu = 0$, we can include the same special cases of item (2).

Then there exists a function $u \in D_a^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})$ such that $u > 0$ in $\mathbb{R}^N \setminus \{|y| = 0\}$ and $u$ is a weak solution to problem (1) in $\mathbb{R}^N \setminus \{|y| = 0\}$.

There are several difficulties to prove this existence result. In our case we consider $1 < p < N$, and the Sobolev space $D_a^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})$ does not have the structure of Hilbert spaces except in the particular case $p = 2$. Moreover, we have to deal with cylindrical critical singularities both on the differential operator and on the homogeneous nonlinearity, as well as on the multiple critical nonlinearities; hence, the operator is not uniformly elliptic. We also have to overcome with the lack of compactness because we consider critical exponents. Indeed, let $(u_n)_{n \in \mathbb{N}} \subset D_a^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})$ be a minimizing sequence to $1/K(N,p,\mu,a,b)$. Then, for arbitrary sequences $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ and $(\eta_n)_{n \in \mathbb{N}} \subset \mathbb{R}^{N-k}$, the sequence of functions $(\tilde{u}_n)_{n \in \mathbb{N}} \subset D_a^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})$ defined by

$$
\tilde{u}_n(x,y) \equiv t_n(N-p(a+1))/p \ u_n(t_n x + t_n y, t_n y)
$$

(5)
is also a minimizing sequence for $1/K(N, p, \mu, a, b)$ because all the integrals involved in the
definition of the infimum are invariant under the action of this group of transformations. Hence,
there exist non-compact minimizing sequences for $1/K(N, p, \mu, a, b)$; this means, for example,
that the mountain pass theorem yields Palais-Smale sequences, but not necessarily critical
points. To overcome this difficulty, we have to establish sufficient conditions under which the
Palais-Smale sequences have strongly convergent subsequences. Another difficulty is to prove
the almost everywhere convergence of the sequences involving integrals of the gradients of the
functions, since this result does not follow directly from the already established ones in the
literature. Due to these several aspects of problem (1), the classical methods of critical point
theory of the calculus of variations cannot be applied directly.

Additionally, the combination of multiple nonlinearities with critical exponents and cylin-
drical weights yields a more subtle and difficult problem because we have to perform a very
detailed analysis of the terms of the energy functional associated to them. The main difficulty
in this step is to understand the behavior of the Palais-Smale sequences. Indeed, in our problem
there is a phenomenon that Filippucci, Pucci and Robert [19] named ‘asymptotic competition’
between the energies carried by the two critical nonlinearities. And when one of them domi-
nates the other, there is the vanishing of the weakest one and we obtain solutions to problems
with only one critical nonlinearity; in other words, we do not obtain nontrivial solutions to
problem (1). Therefore, we have to avoid the dominance of one term over the other. To ac-
complished this goal, we will maintain the correspondence between each nonlinearity and its
singularity with the exponents given by the Maz’ya’s inequality, and we will choose a suitable
level for the mountain pass theorem which involves the best Maz’ya’s constant defined in (4)
and the functions that assume this value. This constitutes the major contribution of our work.

To prove Theorem 1.1, in section 2 we show that the energy functional verifies the hypothe-
ses of the mountain pass theorem; consequently, there exist Palais-Smale sequences for this
functional. Then, under appropriate hypotheses, we show that the energy level of these Palais-
Smale sequences are such that we can recover their strong convergence, up to passage to a
subsequence. In section 3 we study the structure of the Palais-Smale sequences that are weakly
convergent to zero; in this way we can identify an appropriate level to avoid the dominance
of the energy carried by one of the critical nonlinearities over the other. Finally, in section 4 we
show that the limit of this sequence is a nontrivial solution to problem (1).

To complement our existence theorem, we also study a regularity result of weak, positive
solution to a problem related to problem (1). As is usual in the theory of nonlinear elliptic
equations, to show the class of differentiability of the solution we use the iteration scheme intro-
duced by Moser [27]. This technique is also described in the books by Gilbarg and Trudinger [22, Seção 8.6] and by Struwe [32, Appendix B]; see also the paper by Brézis and Kato [9]. Applications
of this method can be found in the papers by Egnell [17], Chou and Chu [12], Chou and Geng [13], Xuan [39], Alves and Souto [1], Vassilev [36], and Bastos, Miyagaki and Vieira [6].

More precisely, consider the class of quasilinear elliptic equations with cylindrical singularities
and multiple nonlinearities

$$- \text{div} \left[ \frac{|\nabla u|^{p-2} \nabla u}{|y|^{\mu(p+1)}} \right] - \mu \frac{|u|^{p-2}u}{|y|^{\mu(p+1)}} = \frac{(u_+)^{p(a,b)-1}}{|y|^{\mu(p(a,b)}} + \frac{(u_+)^{p(a,c)-1}}{|y|^{\mu(p(a,c)}}$$

(6)

where the domain $\Omega \subset \mathbb{R}^{N-k} \times \mathbb{R}^k \setminus \{|y|=0\}$ is not necessarily bounded.

Our regularity result can be stated in the following way.

**Theorem 1.2.** Suppose that $1 \leq k \leq N$, $1 < p < N$, $a < (k-p)/p$, and $a \leq b < c < a+1$
and consider the domain $\Omega \subset \mathbb{R}^{N-k} \times \mathbb{R}^k \setminus \{|y|=0\}$, not necessarily bounded. If $u \in D^1_p(\Omega)$
is a weak solution to problem (6), then $u \in L^\infty_{\text{loc}}(\Omega)$.

The corresponding regularity theorem by Filippucci, Pucci and Robert [19] is a direct application
of the results by Pucci and Servadei [29], by Druet [16], and by Guedda and Verón [23].
In our case we cannot apply these theorems due to the presence of the singularity on the
differential operator and we have to prove the result independently. Inspired by Pucci and Servadei [29], in section 5 we show, through an inductive step, that \( u \in L^m_{\gamma, \text{loc}}(\Omega) \) for every \( m \in [1, \infty) \) and for some appropriate weight \( \gamma = \gamma(m) \in \mathbb{R}^+ \). The main difficulties involved in this part of the proof are related to the required estimates not only for one but for multiple critical nonlinearities with cylindrical weights; we also have to make estimates for the term of the energy functional involving the gradient which also has a cylindrical weight. In section 6 we show, using the Moser’s iteration scheme, that \( \lim_{m \to \infty} \| u \|_{L^m_{\gamma, \text{loc}}(\Omega)} \) is finite; finally, we conclude that \( u \in L^\infty_{\text{loc}}(\Omega) \).

We note that in Theorem 1.1 both critical exponents are the ones that make problem (1) invariant under the group of transformations defined by (5). A natural question is what happens when one of the nonlinearities has a different exponent. In other words, we consider the class of problems where the exponent in one of the nonlinearities is not critical as determined by Maz’ya’s inequality. In this case, we also observe the ‘asymptotic competition’ phenomenon, and there exists only the trivial solution to the problem.

More precisely, consider the problem

\[
- \text{div} \left( \frac{|\nabla u|^{p-2} \nabla u}{|y|^a} \right) - \mu \frac{|u|^{p-2} u}{|y|^{b(a+1)}} = \frac{|u|^{q-2} u}{|y|^{b(p^*(a,c)-2)}} + \frac{|u|^{p^*(a,c)-2} u}{|y|^{c(bp^*(a,c)-2)}}.
\]

Our nonexistence result of nontrivial solution reads as follows.

**Theorem 1.3.** Let \( 1 \leq k \leq N, 1 < p < N, a < (k-p)/p, a \leq b < c < a+1, 0 \leq \mu < \bar{\mu} \). If \( u \in D^1_{\text{a}, p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) is a weak solution to problem (7), then \( u \equiv 0 \) when either \( 1 < q < p^*(a, b) \), or \( q > p^*(a, b) \) and \( u \in L^q_{bp^*(a, b)/q, \text{loc}}(\mathbb{R}^N \setminus \{|y| = 0\}) \cap L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{|y| = 0\}) \).

As is usual in the proofs of nonexistence results, in section 7 we prove a Pohozaev-type identity. The main difficulty is to identify an expression involving at least two terms for the gradient and its corresponding cylindrical singularity due to the fact that we work in \( \mathbb{R}^{N-k} \times \mathbb{R}^k \). Then, in section 8 we show that applying the Pohozaev-type identity to a solution of a problem related to problem (7) leads to the vanishing of a somewhat involved integral. Finally, we show that if the dimensional balance involving one of the nonlinearities and its corresponding singular term does not verify the Maz’ya’s relation, then the norm of the solution is zero and we conclude that problem (7) only has the trivial solution.

2. Existence of Palais–Smale sequences

To use the direct method of the calculus of variations, we look for solutions to problem (1) in the Sobolev space \( D^1_{\text{a}, p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) defined as the completion of the space \( C^\infty_{\text{c}}(\mathbb{R}^N) \) of smooth functions with compact support with respect to the norm defined by

\[
\| \nabla u \|_{L^p_{\text{a}}(\mathbb{R}^N)} \equiv \left( \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|y|^{ap}} \, dz \right)^{1/p}.
\]

It is a well known fact that \( D^1_{\text{a}, p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) is a reflexive Banach space and that its elements can be identified with measurable functions up to subsets of measure zero. Additionally, from inequality (2) we can deduce that the embedding \( D^1_{a, p}(\mathbb{R}^N \setminus \{|y| = 0\}) \hookrightarrow L^{p^*(a, b)}(\mathbb{R}^N) \) is continuous, where \( L^{p^*(a, b)}(\mathbb{R}^N) \) denotes the Lebesgue space \( L^{p^*(a, b)}(\mathbb{R}^N) \) with weight \( |y|^{-bp^*(a, b)} \) and norm defined by

\[
\| u \|_{L^{p^*(a, b)}(\mathbb{R}^N)} \equiv \left( \int_{\mathbb{R}^N} \frac{|u(z)|^{p^*(a, b)}}{|y|^{bp^*(a, b)}} \, dz \right)^{1/p^*(a, b)}.
\]
Using Maz’ya’s inequality (2) we can show that the embedding $\mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\}) \hookrightarrow L^p_{a+1}(\mathbb{R}^N)$ is continuous for the parameters in the specified intervals. More precisely, the inequality
\[
\bar{\mu} \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^{p(a+1)}} \, dz \leq \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|y|^{ap}} \, dz
\]
is valid for every function $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\})$. From this inequality, for $\mu < \bar{\mu}$ we can define a norm $\| \cdot \| : \mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\}) \to \mathbb{R}$ by
\[
\|u\| \equiv \left( \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|y|^{ap}} \, dz - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^{p(a+1)}} \, dz \right)^{\frac{1}{p}}
\]
which is well defined in the Sobolev space $\mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\})$. We note that the inequalities
\[
\left(1 - \frac{\mu_+}{\bar{\mu}}\right) \|\nabla u\|_{L^p_a(\mathbb{R}^N)} \leq \|u\|^p \leq \left(1 + \frac{\mu_-}{\bar{\mu}}\right) \|\nabla u\|_{L^p_a(\mathbb{R}^N)}^p
\]
are valid for every function $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\})$, where $\mu_+ = \max\{\mu, 0\}$ and $\mu_- = \max\{-\mu, 0\}$. For that reason, the norms defined by (8) and (9) are equivalent.

A weak solution to problem (1) is a function $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\})$ such that the relation
\[
\int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|y|^{ap}} \nabla u \cdot \nabla v \, dz \leq \mu \int_{\mathbb{R}^N} \frac{|u|^{p-1}}{|y|^{p(a+1)}} v \, dz = \int_{\mathbb{R}^N} \frac{(u_p^{(a,b)})^{p-1}}{|y|^{bp(a,b)}} v \, dz + \int_{\mathbb{R}^N} \frac{(u_p^{(a,c)})^{p-1}}{|y|^{bp(a,c)}} v \, dz
\]
is valid for every function $v \in \mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\})$. Now we define the energy functional $\varphi : \mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\}) \to \mathbb{R}$ by
\[
\varphi(u) = \frac{1}{p} \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|y|^{ap}} \, dz - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^{p(a+1)}} \, dz - \frac{1}{p^*(a,b)} \int_{\mathbb{R}^N} \frac{(u_p^{(a,b)})^{p^*(a,b)}}{|y|^{bp^*(a,b)}} \, dz - \frac{1}{p^*(a,c)} \int_{\mathbb{R}^N} \frac{(u_p^{(a,c)})^{p^*(a,c)}}{|y|^{bp^*(a,c)}} \, dz,
\]
where we use the notation $u_+(x,y) = \max\{u(x,y), 0\}$. It is standard to verify that its Gâteaux derivative is given by
\[
\langle \varphi'(u), v \rangle = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|y|^{2p}} \langle \nabla u, \nabla v \rangle \, dz - \mu \int_{\mathbb{R}^N} \frac{|u|^2}{|y|^{2p(a+1)}} uv \, dz - \lambda \int_{\mathbb{R}^N} \frac{(u_p^{(a,b)})^{2-2}}{|y|^{bp^*(a,b)}} uv \, dz - \lambda \int_{\mathbb{R}^N} \frac{(u_p^{(a,c)})^{2-2}}{|y|^{bp^*(a,c)}} uv \, dz
\]
for every $u, v \in \mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\})$. Therefore, critical points of this functional are weak solutions to problem (1).

To prove Theorem 1.1, in the first place we show the existence of Palais-Smale sequences for suitable levels that will allow us to recover the compactness.

**Proposition 2.1.** Suppose that the hypotheses of Theorem 1.1 are valid and let $\varphi : \mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\}) \to \mathbb{R}$ be the energy functional defined in (11). Then there exist a Palais-Smale sequence for $\varphi$ at a level
\[
0 < d < d_* \equiv \min_{m \in \{a,c\}} \left\{ \left(1 - \frac{1}{p} \frac{1}{p^*(a,m)} \right) \mathcal{K}(N, p, \mu, a, m) \right\}.
\]

More specifically, there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\})$ such that
\[
0 < \lim_{n \to \infty} \varphi(u_n) = d < d_* \quad \text{and} \quad \lim_{n \to \infty} \varphi(u_n) = 0 \quad \text{strongly in } (\mathcal{D}^{1,p}_a(\mathbb{R}^N \setminus \{|y|=0\}))^*.
\]

To prove Proposition 2.1, we begin by showing that we can apply Ambrosetti and Rabinowitz’s mountain pass theorem. See Willem [38, Theorem 2.10].
Lemma 2.2. For the parameters in the specified intervals, the energy functional (11) verifies the hypotheses of the mountain pass theorem for every \( u \in D_1^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) such that \( u_+ \neq 0 \), that is,

1. \( \varphi(0) = 0 \) and there exist \( R, \lambda > 0 \) such that \( \varphi|_{\partial B(0)} \geq \lambda > 0 \).
2. For any \( u \in D_1^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \), there exists \( t_u > 0 \), such that \( \varphi(t_u u) \leq 0 \), and \( \|t_u u\| \geq R \).

Proof. Clearly we have \( \varphi(0) = 0 \); moreover, we can prove that \( \varphi \in C^1(D_1^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})) \) by using standard arguments. Using the definition (4) of the optimal constant of the Sobolev embedding, we obtain

\[
\varphi(u) \geq \frac{1}{p} \|u\|^p - \frac{\left(\frac{p^*(a,b)}{p}\right)}{p^*(a,b)} \|u\|^p - \frac{\left(\frac{p^*(a,c)}{p}\right)}{p^*(a,c)} \|u\|^p + \left(\frac{\left(\frac{p^*(a,b)}{p}\right)}{p^*(a,c)}\right) \|u\|^p.
\]

Since \( p < p^*(a,c) < p^*(a,b) \), there exist \( R, \lambda > 0 \) such that \( \varphi(u) \geq \lambda \) for every \( u \in D_1^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) that verifies the condition \( \|u\| = R \); moreover, \( \lim_{t \to +\infty} \varphi(t_u u) = -\infty \).

Now let \( t_u > 0 \) be a number such that \( \varphi(t_u u) < 0 \) for every \( t \geq t_u \), and \( \|t_u u\| > R \). To determine the maximum level, we consider the class of paths connecting the zero function to \( t_u u \), that is,

\[
\Gamma_u \equiv \{ \gamma \in C^0([0,1]; D_1^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})) \mid \gamma(0) = 0 \text{ and } \gamma(1) = t_u u \};
\]

finally, the energy level is given by

\[
d_u \equiv \inf_{\gamma \in \Gamma_u} \sup_{t \in [0,1]} \varphi(\gamma(t)) > 0.
\]

Thus, all the hypotheses of the mountain pass theorem are verified by the functional \( \varphi \). \( \square \)

Using the mountain pass theorem, there exists a sequence \( (u_n)_{n \in \mathbb{N}} \in D_1^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) such that

\[
\lim_{n \to \infty} \varphi(u_n) = d_u > 0 \quad \text{and} \quad \lim_{n \to \infty} \varphi'(u_n) = 0 \text{ strongly in } (D_1^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})).
\]

In the next two lemmas we show that the number \( d_u \) is below an appropriate level, for which we can recover the compactness of the Palais-Smale.

Lemma 2.3. Suppose that the hypotheses on one of the items of Theorems 1.1 are valid. Then there exists a function \( u \in D_1^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) such that \( u \geq 0 \) and

\[
d_u < \left( \frac{1}{p} - \frac{1}{p^*(a,b)} \right) K(N, p, \mu, a, b) \frac{p^*(a,b)}{p^*(a,b) - p}.
\]

Proof. Let \( u \in D_1^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) be a nonnegative function that attains the infimum \( 1/K(N, p, \mu, a, b) \) defined by (4); the proof of the existence of such a function can be seen in the paper by Bhakta [8, Theorems 1.1 and 1.2]. Using the definition of \( d_u \) given in the proof of Lemma 2.2, we obtain

\[
d_u \leq \sup_{t \geq 0} \varphi(t_u u).
\]

Now let the function \( f_1 : \mathbb{R}^+ \to \mathbb{R} \) be defined by

\[
f_1(t) \equiv \frac{p}{p} \left( \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|y|^{p(a+1)}} dz - \frac{\mu}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^{p(a+1)}} dz \right) - \frac{p^*(a,b)}{p^*(a,b)} \left( \int_{\mathbb{R}^N} \frac{|u|^{p^*(a,b)}}{|y|^{p^*(a,b) - p}} dz \right).
\]

Denoting by \( t_{\max} \) the point of maximum for \( f_1 \), we obtain

\[
d_u \leq \sup_{t \geq 0} \varphi(t_u u) \leq \sup_{t \geq 0} f_1(t) = f_1(t_{\max}) = \left( \frac{1}{p} - \frac{1}{p^*(a,b)} \right) K(N, p, \mu, a, b) \frac{p^*(a,b)}{p^*(a,b) - p}.
\]
To show that this inequality is strict, we argue by contradiction and we suppose that the equality is valid. Denoting by $t_0 > 0$ the factor of the extremal $u$ where the supremum of the energy functional is attained, we obtain
\[
d_u = \varphi(t_0 u) = f_1(t_0) - \frac{t_0^{p^*(a,c)}}{p^*(a,c)} \int_{\mathbb{R}^N} |u|^{p^*(a,c)} dz = f_1(t_{\text{max}}).
\]
This means that $f_1(t_{\text{max}}) < f_1(t_0)$, which is a contradiction. The result follows. \qed

**Lemma 2.4.** Suppose that the hypotheses on one of the items of Theorems 1.1 are valid. Then there exists a function $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N\{\|y\| = 0\})$ such that $u \geq 0$ and $0 < d_u < d_*$. Proof. If $d_u = [1/p - 1/p^*(a,b)]K(N,p,\mu,a,b)^{-p^*(a,b)/(p^*(a,b) - p)}$ then it suffices to consider the function $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N\{\|y\| = 0\})\{0\}$ given in Lemma 2.3 and we get $d_u < d^*$. Otherwise, let $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N\{\|y\| = 0\})$ be a nonnegative function such that the infimum $1/K(N,p,\mu,a,c)$ defined by (4) is attained and let the function $f_2: \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined by
\[
f_2(t) \equiv \frac{p}{p} \left( \int_{\mathbb{R}^N} |\nabla u|^p |y|^{ap} dz - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p |y|^{p(a+1)} dz \right) - \frac{p^{p^*(a,c)}}{p^*(a,c)} \int_{\mathbb{R}^N} |u|^{p^*(a,c)} dz.
\]
Arguing as in the proof of Lemma 2.3 we obtain the inequality $d_u < d^*$. Finally, the mountain pass theorem guarantees that $d_u > 0$. This concludes the proof of the lemma. \qed

**Proof of Proposition 2.1.** By Lemma 2.2 the energy functional $\varphi$ verifies the hypotheses of the mountain pass theorem. Hence, there exists a Palais-Smale sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N\{\|y\| = 0\})$ for the functional $\varphi$ at the level $d$; and by Lemma 2.4, we conclude that $d < d_*$. The proposition is proved. \qed

3. SEQUENCES WEAKLY CONVERGENT TO ZERO

Now we are going to study the behavior of the Palais-Smale sequences.

**Proposition 3.1.** Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N\{\|y\| = 0\})$ be a Palais-Smale sequence for the functional $\varphi$ at a level $d$ such that $0 < d < d_*$ as in Proposition 2.1 and suppose that the hypotheses on one of the items of Theorem 1.1 are valid. If $u_n \rightharpoonup 0$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N\{\|y\| = 0\})$ as $n \rightarrow \infty$, then for every $\delta > 0$ one of the following claims is valid.

1. $\lim_{n \rightarrow \infty} \int_{B_\delta(0)} \frac{(u_n)^{p^*(a,b)}}{|y|^{bp(a,b)}} dz = 0$ and $\lim_{n \rightarrow \infty} \int_{B_\delta(0)} \frac{(u_n)^{p^*(a,c)}}{|y|^{cp(a,c)}} dz = 0$.
2. $\limsup_{n \rightarrow \infty} \int_{B_\delta(0)} \frac{(u_n)^{p^*(a,b)}}{|y|^{bp(a,b)}} dz \geq \epsilon_0$ and $\limsup_{n \rightarrow \infty} \int_{B_\delta(0)} \frac{(u_n)^{p^*(a,c)}}{|y|^{cp(a,c)}} dz \geq \epsilon_0$ for some number $\epsilon_0 = \epsilon_0(N, \mu, a, c, d) > 0$.

To prove Proposition 3.1 we establish some lemmas.

**Lemma 3.2.** Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N\{\|y\| = 0\})$ be a Palais-Smale sequence as in Proposition 3.1. If $u_n \rightharpoonup 0$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N\{\|y\| = 0\})$ as $n \rightarrow \infty$, then for every compact subset $\omega \subset \mathbb{R}^N\{\|y\| = 0\}$, up to passage to a subsequence we have
\[
\lim_{n \rightarrow \infty} \int_{\omega} \frac{|u_n|^p}{|y|^{p(a+1)}} dz = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\omega} \frac{|u_n|^{p^*(a,c)}}{|y|^{cp(a,c)}} dz = 0,
\]
\[
\lim_{n \rightarrow \infty} \int_{\omega} \frac{|\nabla u_n|^p}{|y|^{ap}} dz = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\omega} \frac{\nabla u_n}{|y|^{ap}} dz = 0.
\]

**Proof.** Let $\omega \subset \mathbb{R}^N\{\|y\| = 0\}$ be a fixed compact subset. Thus, the expression $|y| + |y|^{-1}$ is bounded for every $z = (x,y) \in \omega$. Since $p^*(a,a+1) = p$, by Maz’ya’s inequality (2) and by the compact embedding $\mathcal{D}_a^{1,p}(\mathbb{R}^N\{\|y\| = 0\}) \hookrightarrow L_{a+1}^p(\mathbb{R}^N\{\|y\| = 0\})$, it follows that the first limit in (14) is valid. Likewise, using the fact that $a \leq c < a + 1$ we can show that the second limit in (14) is valid also.
To show that the limits (15) are valid we make some estimates. Let \( \eta \in C^\infty_c(\mathbb{R}^N \setminus \{|y| = 0\}) \) be a cut of function such that \( 0 \leq \eta \leq 1 \), with \( \text{supp} \nabla \eta \equiv \omega \).

**Claim 1.** It is valid the relation

\[
\int_{\mathbb{R}^N} \frac{|\nabla (\eta u_n)|^p}{|y|^{ap}} \, dz = \int_{\mathbb{R}^N} \frac{\eta |\nabla u_n|^p}{|y|^{ap}} \, dz + o(1).
\]

**Proof.** Applying the inequality \( ||X + Y|^p - |X|^p|| \leq C_p (||X||^{p-1} + ||Y||^{p-1}) ||Y|| \) to the values \( X = |\eta \nabla u_n|/|y|^a \) and \( Y = u_n \nabla \eta /|y|^a \), we obtain

\[
\left| \frac{|\nabla (\eta u_n)|^p}{|y|^a} - \frac{|\eta \nabla u_n|^p}{|y|^a} \right| \leq C_p \frac{|\eta \nabla u_n|^{p-1} |u_n \nabla \eta|}{|y|^a} + C_p \frac{|u_n \nabla \eta|^p}{|y|^{ap}}.
\]

(16)

To proceed, we apply Hölder’s inequality to the integral over \( \mathbb{R}^N \) of the first term on the right-hand side of inequality (16) to obtain

\[
\int_{\mathbb{R}^N} \frac{|u_n \nabla \eta|^p}{|y|^{ap}} \, dz \leq C_p \left( \int_{\mathbb{R}^N} \frac{|\nabla u_n|^p}{|y|^{ap}} \, dz \right)^{\frac{p-1}{p}} \left( \int_{\omega} \frac{|u_n|^p}{|y|^{p(a+1)}} \, dz \right)^{\frac{1}{p}} = o(1).
\]

On the other hand, using the first limit in (14), it follows that the integral over \( \mathbb{R}^N \) of the second term on the right-hand side of inequality (16) is such that

\[
\int_{\mathbb{R}^N} \frac{|u_n \nabla \eta|^p}{|y|^{ap}} \, dz \leq C \int_{\omega} \frac{|u_n|^p}{|y|^{p(a+1)}} \, dz = o(1).
\]

for some positive constant \( C > 0 \). Combining these results the claim follows. \( \square \)

Recall that \( (u_n)_{n \in \mathbb{N}} \) is a Palais-Smale sequence and that \( \eta^p u_n \in D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\}) \); for that reason, as \( n \to \infty \) we have

\[
\langle \varphi'(u_n), \eta^p u_n \rangle = o(1).
\]

(17)

Now we note that

\[
\lim_{n \to +\infty} \left\| \frac{u_n}{|y|^{a+1}} \right\|_{L^p(\omega)} = \lim_{n \to +\infty} \int_{\omega} \frac{|u_n|^p}{|y|^{p(a+1)}} \, dz = 0
\]

by the first limit in (14), where we used \( \omega = \text{supp} \nabla \eta \subset \mathbb{R}^N \setminus \{|y| = 0\} \). Furthermore, the sequence of the norms of the gradients \( (||\nabla u_n||_{L^p(\mathbb{R}^N \setminus \{|y| = 0\})})_{n \in \mathbb{N}} \subset \mathbb{R} \) is bounded due to the weak convergence \( u_n \rightharpoonup 0 \) in \( D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\}) \). Hence, using Hölder’s inequality and the fact that \( |y| \) is bounded in \( \text{supp} \nabla \eta = \omega \), we obtain

\[
\int_{\mathbb{R}^N} \frac{|\nabla u_n|^{p-1}}{|y|^{ap}} |\nabla u_n| \, dz \leq C \left( \int_{\mathbb{R}^N} \frac{|\nabla u_n|^p}{|y|^{ap}} \, dz \right)^{\frac{p-1}{p}} \left( \int_{\omega} \frac{|u_n|^p}{|y|^{p(a+1)}} \, dz \right)^{\frac{1}{p}} = o(1),
\]

as \( n \to +\infty \). Then, by the limits (14) and (17), by the previous inequality together with Claim 1 and Hölder’s inequality, it follows that

\[
\int_{\mathbb{R}^N} \frac{|\nabla (\eta u_n)|^p}{|y|^{ap}} \, dz = \int_{\mathbb{R}^N} \frac{(u_n)^{p^*(a,b)} \eta^p}{|y|^{bp^*(a,b)}} \, dz + o(1)
\]

\[
\leq \left( \int_{\mathbb{R}^N} \frac{(u_n)^{p^*(a,b)}}{|y|^{bp^*(a,b)}} \, dz \right)^{\frac{p^*(a,b)-p}{p^*(a,b)}} \left( \int_{\mathbb{R}^N} \frac{|\eta u_n|^{p^*(a,b)}}{|y|^{bp^*(a,b)}} \, dz \right)^{\frac{p}{p^*(a,b)}} + o(1)
\]

\[
\leq \left( \int_{\mathbb{R}^N} \frac{(u_n)^{p^*(a,b)}}{|y|^{bp^*(a,b)}} \, dz \right)^{\frac{p^*(a,b)-p}{p^*(a,b)}} K(N,p,\mu,a,b) \int_{\mathbb{R}^N} \frac{|\nabla (\eta u_n)|^p}{|y|^{ap}} \, dz + o(1).
\]

(18)

Consequently,

\[
\left( 1 - \left( \int_{\mathbb{R}^N} \frac{(u_n)^{p^*(a,b)}}{|y|^{bp^*(a,b)}} \, dz \right)^{\frac{p^*(a,b)-p}{p^*(a,b)}} K(N,p,\mu,a,b) \right) \int_{\mathbb{R}^N} \frac{|\nabla (\eta u_n)|^p}{|y|^{ap}} \, dz \leq o(1).
\]

(19)
On the other hand, since \((u_n)_{n \in \mathbb{N}} \subset D^1_a(\mathbb{R}^N \setminus \{|y| = 0\})\) is a Palais-Smale sequence at a level \(d\), direct computations show that

\[
d + o(1) = \varphi(u_n) - \frac{1}{p} \langle \varphi'(u_n), u_n \rangle
\]

\[
= \left( \frac{1}{p} - \frac{1}{p^a(a,b)} \right) \int_{\mathbb{R}^N} \left( \frac{u_n}{|y|^{bp^*(a,b)}} \right)^{\frac{p^*(a,b)}{p^a(a,b)}} \cdot \frac{1}{|y|^{p^a(a,b)}} \cdot dz + \left( \frac{1}{p} - \frac{1}{p^a(a,c)} \right) \int_{\mathbb{R}^N} \left( \frac{u_n}{|y|^{cp^*(a,c)}} \right)^{\frac{p^*(a,c)}{p^a(a,c)}} \cdot dz + o(1) \tag{20}
\]

This implies that

\[
\int_{\mathbb{R}^N} \left( \frac{u_n}{|y|^{bp^*(a,b)}} \right)^{\frac{p^*(a,b)}{p^a(a,b)}} dz \leq d \left( \frac{1}{p} - \frac{1}{p^a(a,b)} \right)^{-1} + o(1). \tag{21}
\]

Replacing inequality (21) in the relation (19), it follows that

\[
\left( 1 - \left( \frac{1}{p} - \frac{1}{p^a(a,b)} \right)^{-1} \right) K(N, p, \mu, a, b) \int_{\mathbb{R}^N} \frac{\nabla (\eta u_n)^p}{|y|^{ap}} dz \leq o(1).
\]

Finally, to show that the factor that multiplies the previous integral is positive, we use inequality (12). Hence the second limit in (15) is valid and this concludes the proof of the lemma. \(\square\)

For a given \(\delta > 0\), we define

\[
\alpha \equiv \limsup_{n \to +\infty} \int_{B_\delta(0)} \left( \frac{u_n}{|y|^{bp^*(a,b)}} \right)^{\frac{p^*(a,b)}{p^a(a,b)}} dz, \quad \beta \equiv \limsup_{n \to +\infty} \int_{B_\delta(0)} \left( \frac{u_n}{|y|^{cp^*(a,c)}} \right)^{\frac{p^*(a,c)}{p^a(a,c)}} dz, \quad \gamma \equiv \limsup_{n \to +\infty} \int_{B_\delta(0)} \left( \frac{\nabla u_n}{|y|^{ap}} \right) dz - \mu \int_{B_\delta(0)} \left( \frac{|u_n|^p}{|y|^{p(a+1)}} \right) dz. \tag{22}
\]

It follows from Lemma 3.2 that these values are well defined and do not depend on the choice of \(\delta > 0\).

**Lemma 3.3.** Let \((u_n)_{n \in \mathbb{N}} \subset D^1_a(\mathbb{R}^N \setminus \{|y| = 0\})\) be Palais-Smale sequence as in Proposition 3.1 and let \(\alpha, \beta, \) and \(\gamma\) be defined as in (22). If \(u_n \rightharpoonup 0\) weakly in \(D^1_a(\mathbb{R}^N \setminus \{|y| = 0\})\) as \(n \to +\infty\), then

\[
\alpha^{\frac{p}{p^a(a,b)}} \leq K(N, p, \mu, a, b) \gamma \quad \text{and} \quad \beta^{\frac{p}{p^a(a,c)}} \leq K(N, p, \mu, a, c) \gamma. \tag{23}
\]

**Proof.** Let \(R > \delta > 0\) and let \(\eta \in C^\infty(\mathbb{R}^N)\) be a cut off function such that \(\eta |_{B_\delta(0)} \equiv 1\) and \(\eta |_{\mathbb{R}^N \setminus B_R(0)} \equiv 0\). By the definition of the infimum (4) we have

\[
\left( \int_{\mathbb{R}^N} \left( \frac{\eta u_n}{|y|^{bp^*(a,b)}} \right)^{\frac{p^*(a,b)}{p^a(a,b)}} dz \right)^{\frac{p^a(a,b)}{p^*(a,b)}} \leq K(N, p, \mu, a, b) \left( \int_{\mathbb{R}^N} \frac{\nabla (\eta u_n)^p}{|y|^{ap}} dz - \mu \int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^{p(a+1)}} dz \right).
\]

This inequality, together with Claim 1 and Lemma 3.2 imply that

\[
\left( \int_{B_\delta(0)} \left( \frac{u_n}{|y|^{bp^*(a,b)}} \right)^{\frac{p^*(a,b)}{p^a(a,b)}} dz \right)^{\frac{p^a(a,b)}{p^*(a,b)}} \leq K(N, p, \mu, a, b) \left( \int_{B_\delta(0)} \frac{\nabla u_n}{|y|^{ap}} dz - \mu \int_{B_\delta(0)} \frac{|u_n|^p}{|y|^{p(a+1)}} dz \right) + o(1). \tag{24}
\]

Using this inequality, we conclude that

\[
\alpha^{\frac{p}{p^a(a,b)}} = \limsup_{n \to +\infty} \left( \int_{B_\delta(0)} \left( \frac{u_n}{|y|^{bp^*(a,b)}} \right)^{\frac{p^*(a,b)}{p^a(a,b)}} dz \right)^{\frac{p^a(a,b)}{p^*(a,b)}} \leq \limsup_{n \to +\infty} K(N, p, \mu, a, b) \left( \int_{B_\delta(0)} \frac{\nabla u_n}{|y|^{ap}} dz - \mu \int_{B_\delta(0)} \frac{|u_n|^p}{|y|^{p(a+1)}} dz \right).
\]
which is the first inequality in (23). The proof of the other inequality in (23) is similar.

**Lemma 3.4.** Let \((u_n)_{n \in \mathbb{N}} \subset D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\})\) be a Palais-Smale for the functional \(\varphi\) at the level \(d \in (0,d_*)\). If \(u_n \rightharpoonup 0\) weakly in \(D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\})\) as \(n \to +\infty\), then \(\gamma \leq \alpha + \beta\).

**Proof.** By the hypothesis on the sequence \((u_n)_{n \in \mathbb{N}} \subset D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\})\), we have

\[
0 = \limsup_{n \to \infty} \langle \varphi'(u_n), \eta u_n \rangle = \limsup_{n \to \infty} \left\{ \int_{B_\delta(0)} \frac{|\nabla u_n|^{p-2}}{|y|^{ap}} \nabla u_n \nabla (\eta u_n) \, dz - \mu \int_{B_\delta(0)} |u_n|^{p-2} |\eta u_n| \, dz - \int_{B_\delta(0)} \frac{(u_n)^{p^*(a,b)-2} - (u_n)^{p^*(a,c)-2}}{|y|^{bp^*(a,b)}} (u_n)_+ (\eta u_n) \, dz \right\}.
\]

Hence, as \(\eta \equiv 1\) in \(B_\delta(0)\), we obtain \(\gamma \leq \alpha + \beta\), which concludes the proof of the lemma.

**Proof of Proposition 3.1.** Let \((u_n)_{n \in \mathbb{N}} \subset D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\})\) be a Palais-Smale sequence for the functional \(\varphi\) at the level \(d \in (0,d_*)\). Lemmas 3.3 and 3.4 imply that

\[
\frac{\alpha}{p^*(a,b)} (1 - K(N, p, \mu, a, b) \alpha^{p^*(a,b)-p}) \leq K(N, p, \mu, a, b) \beta.
\]

Moreover, passing to the limit superior in both sides of inequality (21), we obtain

\[
\alpha \leq d \left( \frac{1}{p} - \frac{1}{p^*(a,b)} \right)^{-1}.
\]

Combining these two inequalities, we get

\[
\frac{\alpha}{p^*(a,b)} (1 - K(N, p, \mu, a, b) \left( d \left( \frac{1}{p} - \frac{1}{p^*(a,b)} \right)^{-1} \right)^{p^*(a,b)-p}) \leq K(N, p, \mu, a, b) \beta.
\]

Finally, to show that the factor that multiplies the integral is positive, again we use inequality (12). Hence, there exists a positive constant \(\delta_1 = \delta_1(N, p, \mu, a, b, d) > 0\) such that \(\alpha^{p^*(a,b)-p} \leq \delta_1 \beta\). Repeating the computations for \(\beta\), we conclude that there exists a positive constant \(\delta_2 = \delta_2(N, p, \mu, a, b, d) > 0\) such that \(\beta \frac{1}{p^*(a,b)} \leq \delta_2 \alpha\). In particular, from these inequalities it follows that there exists a positive constant \(\epsilon_0 = \epsilon_0(N, p, \mu, b, c, d) > 0\) such that either \(\alpha = 0\) and \(\beta = 0\), or \(\alpha \geq \epsilon_0\) and \(\beta \geq \epsilon_0\). This concludes the proof of the proposition.

4. **CONCLUSION OF THE PROOF OF THEOREM 1.1**

To prove Theorem 1.1 we have to estimate both the limits superior in Proposition 3.1.

**Lemma 4.1.** Let \((u_n)_{n \in \mathbb{N}} \subset D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\})\) be a Palais-Smale sequence for the functional \(\varphi\) at the level \(d \in (0,d_*)\). Then

\[
\min_{m \in \{b,c\}} \left\{ \limsup_{n \to +\infty} \int_{\mathbb{R}^N} \frac{(u_n)^{p^*(a,m)}}{|y|^{bp^*(a,m)}} \, dz \right\} > 0.
\]

**Proof.** Without loss of generality and arguing by contradiction, we can suppose that

\[
0 = \limsup_{n \to +\infty} \int_{\mathbb{R}^N} \frac{(u_n)^{p^*(a,b)}}{|y|^{bp^*(a,b)}} \, dz \leq \limsup_{n \to +\infty} \int_{\mathbb{R}^N} \frac{(u_n)^{p^*(a,c)}}{|y|^{bp^*(a,c)}} \, dz.
\]
Therefore, from the fact that $\langle \varphi'(u_n), u_n \rangle \to 0$ as $n \to \infty$, we obtain
\[
\int_{\mathbb{R}^N} \frac{\nabla u_n}{|y|^{p}} \, dz - \mu \int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^{p(a+1)}} \, dz = \int_{\mathbb{R}^N} \frac{(u_n)^{p^*(a,c)}}{|y|^{p^*(a,c)}} \, dz + o(1).
\]
Using this inequality and the definition of the infimum (4), it follows that
\[
\left( \int_{\mathbb{R}^N} \frac{(u_n)_{+}^{p^*(a,c)}}{|y|^{p^*(a,c)}} \, dz \right)^{\frac{p}{p^*(a,c)}} \leq K(N, p, \mu, a, c) \int_{\mathbb{R}^N} \frac{(u_n)_{+}^{p^*(a,c)}}{|y|^{p^*(a,c)}} \, dz + o(1).
\]
Consequently,
\[
\left( \int_{\mathbb{R}^N} \frac{(u_n)_{+}^{p^*(a,c)}}{|y|^{p^*(a,c)}} \, dz \right)^{\frac{p}{p^*(a,c)} - \frac{p^*(a,c) - p}{p^*(a,c)}} \leq o(1). \tag{26}
\]
On the other hand, using equality (20) and inequality (25), for $n \in \mathbb{N}$ big enough we have
\[
\int_{\mathbb{R}^N} \frac{(u_n)_{+}^{p^*(a,c)}}{|y|^{p^*(a,c)}} \, dz = d \left( \frac{1}{p} - \frac{1}{p^*(a,c)} \right)^{-1} + o(1).
\]
Replacing this equality in (26), it follows that
\[
\left( \int_{\mathbb{R}^N} \frac{(u_n)_{+}^{p^*(a,c)}}{|y|^{p^*(a,c)}} \, dz \right)^{\frac{p}{p^*(a,c)}} \times \left( 1 - K(N, p, \mu, a, c) \left( d \left( \frac{1}{p} - \frac{1}{p^*(a,c)} \right)^{-1} + o(1) \right) \right)^{\frac{p^*(a,c) - p}{p^*(a,c)}} \leq o(1). \tag{27}
\]
Using once again the definition (12) of $d_*$, we conclude that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} \frac{(u_n)_{+}^{p^*(a,c)}}{|y|^{p^*(a,c)}} \, dz = 0.
\]
And this equality, together with (20) imply that $d = 0$, which is a contradiction with the hypothesis that $d > 0$. This concludes the proof of the lemma. \hfill \Box

Using the value of $\epsilon_0 > 0$ identified in Proposition 3.1 and also Lemma 4.1, we can establish one more result that guarantees that the action of the group of transformations defined in (5) preserves the Palais-Smale sequences at the level $d$.

**Lemma 4.2.** Let $(u_n)_{n \in \mathbb{N}} \subset D^1_a p(\mathbb{R}^N \setminus \{|y| = 0\})$ be a Palais-Smale sequence for the functional $\varphi$ at a level $d \in (0, d_*)$ verifying the hypothesis of Proposition 3.1. Then there exists $\epsilon_1 \in (0, \epsilon_0/2)$ and there exist sequences $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $(\eta_n)_{n \in \mathbb{N}} \subset \mathbb{R}^{N-k}$ such that for every $\epsilon \in (0, \epsilon_1)$, there exists a sequence $(\tilde{u}_n)_{n \in \mathbb{N}} \subset D^1_a p(\mathbb{R}^N \setminus \{|y| = 0\})$, defined by (5) that is also a Palais-Smale sequence for the functional $\varphi$ at the level $d$. Moreover, this sequence verifies the equality
\[
\int_{B_1(0)} \frac{(\tilde{u}_n)_{+}^{p^*(a,b)}}{|y|^{p^*(a,b)}} \, dz = \epsilon \tag{28}
\]
for every $n \in \mathbb{N}$.

**Proof.** Let $\lambda \equiv \lim\sup_{n \to \infty} \int_{\mathbb{R}^N} \frac{(u_n)_{+}^{p^*(a,b)}}{|y|^{p^*(a,b)}} \, dz$. It follows from Lemma 4.1 that $\lambda > 0$. Let $\epsilon_1 \equiv \min\{\epsilon_0/2, \lambda\}$; for the rest of the proof we fix $\epsilon \in (0, \epsilon_1)$. Passing to a subsequence, still denoted by $(u_n)_{n \in \mathbb{N}} \subset D^1_a p(\mathbb{R}^N \setminus \{|y| = 0\})$, for every $n \in \mathbb{N}$ there exists $(r_n)_{n \in \mathbb{N}} > 0$ such that
\[
\int_{B_{r_n}(0)} \frac{(u_n)_{+}^{p^*(a,b)}}{|y|^{p^*(a,b)}} \, dz = \epsilon.
\]
Due to the invariance of this integral under the action of the group of homoteties and translations, it is standard to prove that the sequence \( (\tilde{u}_n)_{n \in \mathbb{N}} \subset D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) verifies the equality (28) and also the conclusions of Proposition 3.1. This concludes the proof of the lemma. \( \square \)

**Proposition 4.3.** Let \( (u_n)_{n \in \mathbb{N}} \subset D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) be a Palais-Smale for the functional \( \varphi \) at the level \( d \) and let \( (\tilde{u}_n)_{n \in \mathbb{N}} \subset D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) be the sequence defined in (5). Then there exists a function \( u_\infty \in D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) such that \( \tilde{u}_n \rightharpoonup u_\infty \) weakly in \( D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) as \( n \to \infty \), possibly after passage to a subsequence. Furthermore, \( \tilde{u}_\infty > 0 \) in \( \mathbb{R}^N \setminus \{|y| = 0\} \) and \( \tilde{u}_\infty \) is a weak solution to problem (1).

**Proof.** Let \( (u_n)_{n \in \mathbb{N}} \subset D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) be a Palais-Smale for the functional \( \varphi \) at the level \( d \in (0, d_*) \) and let \( (\tilde{u}_n)_{n \in \mathbb{N}} \subset D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) be the sequence defined in (5).

The sequence \( (\tilde{u}_n)_{n \in \mathbb{N}} \) is bounded in \( D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \). Indeed, since \( p < p^*(a, c) < p^*(a, b) \), we have

\[
c_1 + c_2 \| \tilde{u}_n \| \geq \varphi'(\tilde{u}_n) - \frac{1}{p^*(a, c)} \left( \frac{1}{p} - \frac{1}{p^*(a, c)} \right) \| \tilde{u}_n \|^p - \frac{1}{p^*(a, b)} \| \tilde{u}_n \|^p - \frac{1}{p^*(a, c)} \| \tilde{u}_n \|^p,
\]

and our claim follows. Thus, there exists a function \( \tilde{u}_\infty \in D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) such that, up to passage to a subsequence still denoted in the same way, \( \tilde{u}_n \rightharpoonup \tilde{u}_\infty \) weakly in \( D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \) as \( n \to \infty \). Additionally, we have that \( (|\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n)_{n \in \mathbb{N}} \subset (L_a^p(\mathbb{R}^N))^N \), \( (\tilde{u}_n)_{n \in \mathbb{N}} \subset L_b^{p^*(a, b)}(\mathbb{R}^N) \) and \( (\tilde{u}_n)_{n \in \mathbb{N}} \subset L_b^{p^*(a, c)}(\mathbb{R}^N) \) are bounded sequences in these spaces. Hence, we have the following convergences:

1. \( (|\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n)_{n \in \mathbb{N}} \rightharpoonup T \) weakly in \( (L_a^p(\mathbb{R}^N))^N \), for some \( T \in (L_a^p(\mathbb{R}^N))^N \);
2. \( (|\tilde{u}_n|^{p^*(a, b)} \tilde{u}_n)_{n \in \mathbb{N}} \rightharpoonup |\tilde{u}_\infty|^{p^*(a, b)} \tilde{u}_\infty \) weakly in \( L_b^{p^*(a, b)}(\mathbb{R}^N) \);
3. \( (|\tilde{u}_n|^{p^*(a, c)} \tilde{u}_n)_{n \in \mathbb{N}} \rightharpoonup |\tilde{u}_\infty|^{p^*(a, c)} \tilde{u}_\infty \) weakly in \( L_b^{p^*(a, c)}(\mathbb{R}^N) \);
4. \( (|\tilde{u}_n|^{p^*(a, b)} \tilde{u}_n)_{n \in \mathbb{N}} \rightharpoonup |\tilde{u}_\infty|^{p^*(a, c)} \tilde{u}_\infty \) weakly in \( L_b^{p^*(a, c)}(\mathbb{R}^N) \).

Now we show that this weak limit \( \tilde{u}_\infty \) is not identically zero. We argue by contradiction and we suppose that \( \tilde{u}_\infty \equiv 0 \). Applying Proposition 3.1, by equality (28) the first case is excluded; and since \( 0 < \epsilon < \epsilon_0/2 \), again by equality (28) we have

\[
\epsilon_0 \leq \int_{B_{\epsilon_0}(0)} \frac{(u_n)_{p^*(a,b)}}{|y|^{p^*(a,b)}} \, dz = \epsilon < \frac{\epsilon_0}{2},
\]

which is a contradiction. Hence, we necessarily have \( \tilde{u}_\infty \neq 0 \).

In what follows we show that \( \tilde{u}_\infty \) is a weak solution to problem (6). From the convergence \( \varphi'(\tilde{u}_n) \to 0 \) in \( (D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}))^* \) as \( n \to +\infty \), we obtain

\[
o(1) = \langle \varphi'(\tilde{u}_n), v \rangle = \int_{\mathbb{R}^N} \frac{|\nabla \tilde{u}_n|^{p-2}}{|y|^{p-1}} \langle \nabla \tilde{u}_n, \nabla v \rangle \, dz - \mu \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{p^*(a+1)} \tilde{u}_nv}{|y|^{p^*(a+1)}} \, dz
- \int_{\mathbb{R}^N} \frac{(\tilde{u}_n)_{p^*(a,b)} \tilde{u}_nv}{|y|^{p^*(a,b)}} \, dz - \int_{\mathbb{R}^N} \frac{(\tilde{u}_n)_{p^*(a,c)} \tilde{u}_nv}{|y|^{p^*(a,c)}} \, dz,
\]

for every \( v \in D_{\alpha}^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\}) \). And from the weak convergences previously determined, it follows that

\[
\mu \int_{\mathbb{R}^N} \frac{|\tilde{u}_\infty|^{p^*(a,b)} \tilde{u}_\infty v}{|y|^{p^*(a,b)}} \, dz \to \mu \int_{\mathbb{R}^N} \frac{|\tilde{u}_\infty|^{p^*(a,c)} \tilde{u}_\infty v}{|y|^{p^*(a,c)}} \, dz
\]

and

\[
\int_{\mathbb{R}^N} \frac{(\tilde{u}_\infty)_{p^*(a,b)} \tilde{u}_\infty v}{|y|^{p^*(a,b)}} \, dz \to \int_{\mathbb{R}^N} \frac{(\tilde{u}_\infty)_{p^*(a,c)} \tilde{u}_\infty v}{|y|^{p^*(a,c)}} \, dz.
\]
This follows from Lemmas the almost everywhere convergence of the sequence of the gradients of our original sequence.

The next two lemmas are crucial to prove the almost everywhere convergence of the sequence to denote positive constants, which are independent from and from However, these constants can change from one passage to the other in the computations.

We remark that the sequence \((\tilde{u}_n)_{n, m, \epsilon}\) is crescent. Let us also fix \(j \in \mathbb{N}\) and consider the cut off function \(\eta \in C_{\infty}^{0}(B_{R_{j}+1})\) such that \(0 \leq \eta \leq 1\) and also \(\eta(z) = 1\) for every \(z \in B_{R_{j}}\).

We recall that for every \(X, Y \in \mathbb{R}^{N}\) it is valid the inequality \(|X|^{p-2}X - |Y|^{p-2}Y, X - Y) \geq 0\), which follows from the monotonicity of the function \(t \mapsto |t|^{p-2}t\). Moreover, the equality is valid if, and only if, \(X = Y\). Hence,

\[
\langle |\nabla \tilde{u}_n|^{p-2}\nabla \tilde{u}_n - |\nabla \tilde{u}_\infty|^{p-2}\nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle \geq 0.
\]

Given \(0 < \epsilon < 1\), \(n \in \mathbb{N}\) and \(m \in \mathbb{N}\) with \(m \geq 1\), we define the subsets

\[
E_m = \{ z \in \mathbb{R}^{N} : |\tilde{u}_\infty(z)| > m \},
\]

\[
A_{n, m, \epsilon}^{+} = \{ z \in \mathbb{R}^{N} : |\tilde{u}_\infty(z)| \leq m \text{ and } |\tilde{u}_n(z) - \tilde{u}_\infty(z)| \geq \epsilon \},
\]

\[
A_{n, m, \epsilon}^{-} = \{ z \in \mathbb{R}^{N} : |\tilde{u}_\infty(z)| \leq m \text{ and } |\tilde{u}_n(z) - \tilde{u}_\infty(z)| < \epsilon \}.
\]

In this way, we have

\[
\int_{\mathbb{R}^{N}} \left| \frac{\langle |\nabla \tilde{u}_n|^{p-2}\nabla \tilde{u}_n - |\nabla \tilde{u}_\infty|^{p-2}\nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \right|^\frac{1}{p} \, dz
\]

\[
= \int_{E_m} \left| \frac{\langle |\nabla \tilde{u}_n|^{p-2}\nabla \tilde{u}_n - |\nabla \tilde{u}_\infty|^{p-2}\nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \right|^\frac{1}{p} \, dz
\]

\[
+ \int_{A_{n, m, \epsilon}^{+}} \left| \frac{\langle |\nabla \tilde{u}_n|^{p-2}\nabla \tilde{u}_n - |\nabla \tilde{u}_\infty|^{p-2}\nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \right|^\frac{1}{p} \, dz
\]

\[
+ \int_{A_{n, m, \epsilon}^{-}} \left| \frac{\langle |\nabla \tilde{u}_n|^{p-2}\nabla \tilde{u}_n - |\nabla \tilde{u}_\infty|^{p-2}\nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \right|^\frac{1}{p} \, dz.
\]
is bounded. Thus, by Hölder’s inequality we obtain
\[
\int_{E_m} \left| \frac{\langle \nabla \bar{u}_n \rangle}{|y|^{ap}} \right|^\frac{1}{p} dz \leq C |E_m|^{\frac{p-1}{p}}. \tag{33}
\]
Moreover, denoting by \( \chi_A : \mathbb{R}^N \to \mathbb{R} \) the characteristic function on the subset \( A \subset \mathbb{R}^N \), Hölder’s and Maz’ya’s inequalities imply that
\[
|E_m| \leq \frac{1}{m} \int_{E_m} \chi_{E_m}(z) dz \leq \frac{1}{m} C |E_m|^{\frac{p-1}{p}},
\]
that is, \( |E_m| \leq C/m^p \). Combining this result with inequality (33) we obtain an estimate for the first term on the right-hand side of equality (32), namely,
\[
\int_{E_m} \left| \frac{\langle \nabla \bar{u}_n \rangle}{|y|^{ap}} \right|^\frac{1}{p} dz \leq C \left\{ \{ z \in B_{R_{j+1}} \mid \bar{u}_n(z) - \bar{u}_\infty \mid \geq \epsilon \} \right\}^{\frac{p-1}{p}}. \tag{34}
\]
Applying Hölder’s inequality to the second term of equality (32), we obtain
\[
\int_{E_m} \left| \frac{\langle \nabla \bar{u}_n \rangle}{|y|^{ap}} \right|^\frac{1}{p} dz \leq C \left\{ \{ z \in B_{R_{j+1}} \mid \bar{u}_n(z) - \bar{u}_\infty \mid \geq \epsilon \} \right\}^{\frac{p-1}{p}}.
\]
Moreover, we have \( \chi_{\{ z \in B_{R_{j+1}} \mid \bar{u}_n(z) - \bar{u}_\infty \mid \geq \epsilon \} \} \leq \chi_{B_{R_{j+1}}} \) and, \( \chi_{\{ z \in B_{R_{j+1}} \mid \bar{u}_n(z) - \bar{u}_\infty \mid \geq \epsilon \} \to 0 \) a.e. \( \mathbb{R}^N \) as \( n \to \infty \). Since \( B_{R_{j+1}}(0) \subset \mathbb{R}^N \) is bounded for every \( n \in \mathbb{N} \), by the Lebesgue dominated convergence theorem it follows that \( \{ z \in B_{R_{j+1}} \mid \bar{u}_n(z) - \bar{u}_\infty \mid \geq \epsilon \} \to 0 \) as \( n \to \infty \). Hence, there exists \( n(\epsilon) \in \mathbb{N} \) independent from \( m \), such that \( \{ z \in B_{R_{j+1}} \mid \bar{u}_n(z) - \bar{u}_\infty \mid \geq \epsilon \} < \epsilon \) for every \( n \in \mathbb{N} \) such that \( n > n(\epsilon) \). Inequality (35) together with previous one imply that the second term on the right-hand side of equality (32) is such that
\[
\int_{E_m} \left| \frac{\langle \nabla \bar{u}_n \rangle}{|y|^{ap}} \right|^\frac{1}{p} dz \leq C \left\{ \{ z \in B_{R_{j+1}} \mid \bar{u}_n(z) - \bar{u}_\infty \mid \geq \epsilon \} \right\}^{\frac{p-1}{p}}. \tag{36}
\]
for every \( n \in \mathbb{N} \) such that \( n \geq n(\epsilon) \).
Finally, the third term on the right-hand side of the equality (32) can be estimated using Hölder’s inequality. Indeed,
\[
\int_{E_m} \left| \frac{\langle \nabla \bar{u}_n \rangle}{|y|^{ap}} \right|^\frac{1}{p} dz = \int_{E_m} \left| \frac{\langle \nabla \bar{u}_n \rangle}{|y|^{ap}} \right|^\frac{1}{p} dz = \int_{E_m} \left| \frac{\langle \nabla \bar{u}_n \rangle}{|y|^{ap}} \right|^\frac{1}{p} dz \leq C \left\{ \{ z \in B_{R_{j+1}} \mid \bar{u}_n(z) - \bar{u}_\infty \mid \geq \epsilon \} \right\}^{\frac{p-1}{p}}. \tag{37}
\]
Now we define the function $\psi_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$
\psi_\epsilon(\sigma) \equiv \begin{cases} 
\sigma, & \text{if } |\sigma| \leq \epsilon; \\
\epsilon \text{sign}(\sigma), & \text{if } |\sigma| > \epsilon.
\end{cases}
$$

To estimate the integral on the right-hand side of inequality (37) we note that the set $A_{n,m,\epsilon}^-$ can be written as

$$
A_{n,m,\epsilon}^- = \{ z \in B_{R_j+1} : |\tilde{u}_\infty(z)| \leq m \text{ and } |\tilde{u}_n(z) - \psi_m(\tilde{u}_\infty(z))| < \epsilon \}.
$$

In this way, we obtain

$$
\int_{B_{R_j+1}} \chi_{A_{n,m,\epsilon}^-} \frac{\eta(\nabla \tilde{u}_n)^{p-2} \nabla \tilde{u}_n \cdot \nabla (\tilde{u}_n - \tilde{u}_\infty)}{|y|^{ap}} \, dz \\
= \int_{B_{R_j+1}} \chi_{A_{n,m,\epsilon}^-} \eta(\nabla \tilde{u}_n)^p \, dz - \int_{B_{R_j+1}} \chi_{A_{n,m,\epsilon}^-} \eta(\nabla \tilde{u}_n)^{p-2} \nabla \tilde{u}_n \cdot \nabla \tilde{u}_\infty \, dz \\
\leq \int_{B_{R_j+1}} \chi_{\{z \in B_{R_j+1} : |\tilde{u}_n(z) - \psi_m(\tilde{u}_\infty(z))| < \epsilon\}} \frac{\eta(\nabla \tilde{u}_n)^p}{|y|^{ap}} \, dz \\
- \int_{B_{R_j+1}} \chi_{\{z \in B_{R_j+1} : |\tilde{u}_n(z) - \psi_m(\tilde{u}_\infty(z))| < \epsilon\}} \frac{\eta(\nabla \tilde{u}_n)^{p-2} \nabla \tilde{u}_n \cdot \nabla \tilde{u}_\infty}{|y|^{ap}} \, dz \\
= \int_{B_{R_j+1}} \eta(\nabla \tilde{u}_n)^{p-2} \nabla \tilde{u}_n \cdot \nabla (\psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty))) \, dz \\
\leq \int_{B_{R_j+1}} \frac{(\nabla \tilde{u}_n)^{p-2} \nabla \tilde{u}_n \cdot (\eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty)))}{|y|^{ap}} \, dz \\
+ \int_{B_{R_j+1}} (\nabla \tilde{u}_n)^{p-2} \nabla \tilde{u}_n \cdot (\eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty))) \, dz \\
\leq \int_{B_{R_j+1}} (\nabla \tilde{u}_n)^{p-2} \nabla \tilde{u}_n \cdot (\eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty))) \, dz + C\epsilon.
$$

(38)

Now we use the function $\eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty)) \in D_\sigma^1 R^N \backslash \{ |y| = 0 \}$ as a test function in the definition of weak solution to the problem (6) to estimate the integral in (38). Thus, we obtain

$$
\int_{\mathbb{R}^N} \frac{(\nabla \tilde{u}_n)^{p-2} \nabla \tilde{u}_n \cdot (\eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty)))}{|y|^{ap}} \, dz \\
= \langle \varphi'(\tilde{u}_n), \eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty)) \rangle + \mu \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{p-2} \tilde{u}_n (\eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty)))}{|y|^{p(a+1)}} \, dz \\
+ \int_{\mathbb{R}^N} \frac{(\tilde{u}_n)^{p_+(a,b)-2}(\tilde{u}_n)_+(\eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty)))}{|y|^{p_+(a,b)}} \, dz \\
+ \int_{\mathbb{R}^N} \frac{(\tilde{u}_n)^{p_+(a,c)-2}(\tilde{u}_n)_+(\eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty)))}{|y|^{p_+(a,c)}} \, dz.
$$

(39)

Since the sequences $(||\tilde{u}_n||_{L^p_{\kappa+1}(\cdot)})_{n \in \mathbb{N}} \subset \mathbb{R}$, $(||\tilde{u}_n||_{L^{p_+(a,b)}(\cdot)})_{n \in \mathbb{N}} \subset \mathbb{R}$ and $(||\tilde{u}_n||_{L^{p_+(a,c)}(\cdot)})_{n \in \mathbb{N}} \subset \mathbb{R}$ are bounded, regardless of $n \in \mathbb{N}$, it follows from inequality (38) and equality (39) that

$$
\int_{B_{R_j+1}} \chi_{A_{n,m,\epsilon}^-} \frac{\eta(\nabla \tilde{u}_n)^{p-2} \nabla \tilde{u}_n \cdot \nabla (\tilde{u}_n - \tilde{u}_\infty)}{|y|^{ap}} \, dz \leq \langle \varphi'(\tilde{u}_n), \eta \psi_\epsilon(\tilde{u}_n - \psi_m(\tilde{u}_\infty)) \rangle + C\epsilon.
$$

(40)
Regarding the second integral on the right-hand side of inequality (37), we note that it can be written in the form
\[
\int_{B_{R_{j+1}}} \chi_{A_{n,m,s}} \frac{\eta \langle |\nabla \tilde{u}_\infty|^p - 2 \nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \, dz
= \int_{B_{R_{j+1}}} \chi_{\{z \in \mathbb{R}^N : |\tilde{u}_\infty(z)| < m\}} \frac{\eta \langle |\nabla \tilde{u}_\infty|^p - 2 \nabla \tilde{u}_\infty, \nabla (\psi \tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \, dz.
\]

(41)

Now we combine inequalities (34), (36), (40), and (41) with equality (32) and we deduce that, for every \( n > n(\epsilon) \) and for every \( m \in \mathbb{N}^* \), it is valid the inequality
\[
\int_{\mathbb{R}^N} \frac{\eta \langle |\nabla \tilde{u}_n|^p - 2 \nabla \tilde{u}_n - |\nabla \tilde{u}_\infty|^p - 2 \nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \, dz
\leq \frac{C}{m^{\frac{p-1}{p}}} + C \epsilon^{-\frac{1}{p}} + C \left\{ - \int_{B_{R_{j+1}}} \chi_{\{\tilde{u}_\infty \leq m\}} \frac{\eta \langle |\nabla \tilde{u}_\infty|^p - 2 \nabla \tilde{u}_\infty, \nabla (\psi \tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \, dz + \langle \varphi'(\tilde{u}_n), \eta \psi \tilde{u}_n - \tilde{u}_\infty \rangle \right\}.
\]

(42)

For \( n \geq n(\epsilon) \) fixed, the sequence \( (\eta \psi \tilde{u}_n - \psi_m(\tilde{u}_\infty))_{n \in \mathbb{N}} \subset D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\}) \) is bounded. By the reflexivity of the Sobolev space \( D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\}) \), up to passage to a subsequence always denoted in the same way, there exists a function \( \eta \psi \tilde{u}_n(\tilde{u}_n - \tilde{u}_\infty) \in D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\}) \) such that \( \eta \psi \tilde{u}_n(\tilde{u}_n - \psi_m(\tilde{u}_\infty)) \rightharpoonup \eta \psi \tilde{u}_n(\tilde{u}_n - \tilde{u}_\infty) \) weakly in \( D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\}) \) as \( m \to \infty \). Passing to the limit as \( m \to \infty \) in inequality (42) it follows that
\[
\int_{\mathbb{R}^N} \frac{\eta \langle |\nabla \tilde{u}_n|^p - 2 \nabla \tilde{u}_n - |\nabla \tilde{u}_\infty|^p - 2 \nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \, dz
\leq C \epsilon^{-\frac{1}{p}} + C \left\{ - \int_{\mathbb{R}^N} \frac{\eta \langle |\nabla \tilde{u}_\infty|^p - 2 \nabla \tilde{u}_\infty, \nabla (\psi \tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \, dz + \langle \varphi'(\tilde{u}_n), \eta \psi \tilde{u}_n - \tilde{u}_\infty \rangle \right\}.
\]

(43)

On the other hand, by the definition of weak solution and by the fact that the space \( C^0(\overline{B_{R_{j+1}}}) \) is dense in \( L^{p+1}_{a+1}(B_{R_{j+1}}) \), we deduce that
\[
\nabla \psi \tilde{u}_n(\tilde{u}_n - \tilde{u}_\infty) \rightharpoonup 0 \text{ weakly in } L^{p+1}_{a+1}(B_{R_{j+1}})
\]
as \( n \to +\infty \). Since the sequence \( (\psi \tilde{u}_n(\tilde{u}_n - \tilde{u}_\infty))_{n \in \mathbb{N}} \subset D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\}) \) is also bounded, up to the passage to a subsequence we have
\[
\nabla \psi \tilde{u}_n(\tilde{u}_n - \tilde{u}_\infty) \rightharpoonup 0 \text{ weakly in } D^{1,p}_a(\mathbb{R}^N \setminus \{|y| = 0\})
\]
as \( n \to \infty \). Passing to the limit superior as \( n \to \infty \) in inequality (43), it follows that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \frac{\eta \langle |\nabla \tilde{u}_n|^p - 2 \nabla \tilde{u}_n - |\nabla \tilde{u}_\infty|^p - 2 \nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \, dz \leq C(\epsilon^{-\frac{1}{p}} + \epsilon^{\frac{1}{2p}}).
\]

And since \( 0 < \epsilon < 1 \) is arbitrary, using the Cantor’s diagonal argument we conclude that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \frac{\eta \langle |\nabla \tilde{u}_n|^p - 2 \nabla \tilde{u}_n - |\nabla \tilde{u}_\infty|^p - 2 \nabla \tilde{u}_\infty, \nabla (\tilde{u}_n - \tilde{u}_\infty) \rangle}{|y|^{ap}} \, dz = 0.
\]

The lemma is proved. \( \square \)

**Lemma 4.5.** Suppose that the hypotheses of Proposition 4.3 are valid. Then, as \( n \to \infty \) we have \( \nabla \tilde{u}_n \to \nabla \tilde{u}_\infty \) a. e. in \( \mathbb{R}^N \).
Proof. As in the proof of Lemma 4.4, let us cover the space $\mathbb{R}^N$ by a sequence of balls $(B_{R_j})_{j \in \mathbb{N}} \subset \mathbb{R}^N$ centered at the origin, where the sequence of radii $(R_j)_{j \in \mathbb{N}} \subset \mathbb{R}$ is crescent. Let us also fix $j \in \mathbb{N}$ and consider the cut off function $\eta \in C_0^\infty(B_{R_{j+1}})$ such that $0 \leq \eta \leq 1$ and also $\eta(z) = 1$ for every $z \in B_{R_j}$.

We recall that there exists a constant $C > 0$ such that $\langle |X|^{s-2}X - |Y|^{s-2}Y, X - Y \rangle \geq C|X - Y|^s$ for every $X, Y \in \mathbb{R}^N$ with $s \geq 2$.

Using this inequality and Lemma 4.4 it follows that
\[
\limsup_{n \to \infty} \int_{B_{R_j}} \frac{|\nabla \tilde{u}_n - \nabla \tilde{u}_\infty|^p}{|y|^p} \, dz = 0,
\]
that is, $|\nabla \tilde{u}_n|^p \to |\nabla \tilde{u}_\infty|^p$ strongly in $L^1(B_{R_j})$ as $n \to \infty$ and up to a subsequence, still denoted in the same way, we deduce that $\nabla \tilde{u}_n \to \nabla \tilde{u}_\infty$ a.e. in $B_{R_j}$ as $n \to \infty$. Since $j \in \mathbb{N}$ is arbitrary, using Cantor’s diagonal argument we conclude that $\nabla \tilde{u}_n \to \nabla \tilde{u}_\infty$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. The lemma is proved. □

Proof of Theorem 1.1. The conclusion follows directly from Propositions 2.1 and 4.3. □

In the proof of Theorem 1.1 we used, among other ideas, the existence results by Bhakta [8, Theorems 1 and 2], which do not apply in the case $a = b$ and $\mu < 0$; however, problem (1) still has solution in this situation. As can be expected, to prove this existence result we have to use ideas different from the ones already used. We adapt some arguments by Filippucci, Pucci and Robert [19] by making a convenient translation in the variable of the cylindrical singularity and, through an analysis of the asymptotic behavior of this function, we obtain the existence result that can be stated as follows.

**Theorem 4.6.** Let $1 \leq k \leq N$, $1 < p < N$, $0 = a = b < c < 1$, and $\mu < 0$. Then there exists $u \in D_a^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})$ such that $u > 0$ in $\mathbb{R}^N \setminus \{|y| = 0\}$ and $u$ is a weak solution to problem (1).

Proof. Let $v \in D_a^{1,p}(\mathbb{R}^N \setminus \{|y| = 0\})$ be a nonnegative function that assumes the optimal constant $1/K(N, p, 0, 0, 0)$. Let $e_2 \in \mathbb{R}^k$ be a unitary vector and set $v_\alpha(z) \equiv v(x, y - \alpha e_2)$. Moreover, let the functional $\varphi$ be defined as in (11).

**Claim 2.** It is valid the inequality
\[
\max_{t \geq 0} \varphi(tv_\alpha) < \left(\frac{1}{p} - \frac{1}{p^*(0, 0)}\right) K(N, p, \mu, 0, 0) - \frac{p^{*}(0,0)}{p^*(0,0)-p}.
\]

Proof. Let the function $f_3 : \mathbb{R}^+ \to \mathbb{R}$ be defined by
\[
f_3(t) \equiv \frac{1}{p} \left\|v_\alpha\right\|^{p^*} - \frac{1}{p^*(0,0)} \left(\int_{\mathbb{R}^N} (v_{\alpha+})^{p^*(0,0)} \, dz\right) \varphi^{*(0,0)}.
\]

Denoting by $t_{\max}$ the point of maximum for $f_3$ and using the invariance of the three integrals involved in the definition of the infimum under the translations, we obtain
\[
\lim_{\alpha \to +\infty} f_3(t_{\max}) = \lim_{\alpha \to +\infty} \left(\frac{1}{p} - \frac{1}{p^*(0,0)}\right) \left\{ \int_{\mathbb{R}^N} |\nabla v_\alpha|^p \, dz - \mu \int_{\mathbb{R}^N} \frac{|v_\alpha|^p}{|y + \alpha e_2|^p} \, dz \right\}^{p^*(0,0)} \frac{p^{*(0,0)}}{p^*(0,0)-p}
\]
\[
= \left(\frac{1}{p} - \frac{1}{p^*(0,0)}\right) K(N, p, 0, 0, 0) - \frac{p^{*}(0,0)}{p^*(0,0)-p}.
\]

And since $\mu < 0$ implies $1/K(N, p, 0, 0, 0) < 1/K(N, p, \mu, 0, 0)$, we deduce that
\[
\max_{t \geq 0} \varphi(tv_\alpha) \leq \sup_{t \geq 0} f_3(t) = f_3(t_{\max}) < \left(\frac{1}{p} - \frac{1}{p^*(0,0)}\right) K(N, p, \mu, 0, 0) - \frac{p^{*}(0,0)}{p^*(0,0)-p},
\]
for $\alpha \in \mathbb{R}^+$ big enough. This concludes the proof of the claim. 

Combining the definition (13) of $d_a$ and Claim 2, it follows that

$$0 < d_v \leq \max_{t \geq 0} \varphi (tv_a) < \left(\frac{1}{p} - \frac{1}{p^* (0, 0)}\right) K(N, p, \mu, 0, 0)^{-\frac{p^*(0, 0)}{p}},$$

that is, the conclusion of Lemma 2.3 still holds in the case $a = b = 0$ and $\mu < 0$.

Our next goal is to prove that the conclusion of Lemma 2.4 also holds in this case. Using the definition (12) of $d_{*}$, we obtain the inequality $0 < d_v < d_{*}$.

The conclusions of Proposition 3.1 are still valid in the case $a = b = 0$ and $\mu < 0$, with the same proof. Finally, combining Propositions 2.1 and 4.3 we obtain the conclusion of the theorem.

We can unify the proofs of the existence of solution to problem (1) in the cases $0 \leq a = b < (k - p)/p$ and $\mu < 0$ by considering the subspace of $\mathcal{D}_{a,p} (\mathbb{R}^N \setminus \{|y| = 0\})$ of the functions that are cylindrically symmetric in the variable of the singularity, that is, in the subspace of the functions such that $u(z) = u(x, y) = u(x, |y|)$ for every $y \in \mathbb{R}^k$. However, in this case the solution obtained is not necessarily a ground state solution, due to some possible breaking of symmetry. Our result can be stated as follows.

**Theorem 4.7.** Let $k \geq 2$, $1 < p < N$, $0 \leq a < \frac{k - p}{p}$, $a = b < c < a + 1$ and $\mu < \bar{\mu}$. Then problem (1) has a nonnegative solution.

**Proof.** Let us define the subspace

$$\mathcal{D}_{a,cyl} (\mathbb{R}^N \setminus \{|y| = 0\}) = \{ u \in \mathcal{D}_{a,p} (\mathbb{R}^N \setminus \{|y| = 0\}) : u(z) = u(x, |y|) \}$$

as the subset of $\mathcal{D}_{a,p} (\mathbb{R}^N \setminus \{|y| = 0\})$ consisting of functions cylindrically symmetric in the variable of the singularity. Similarly, we also define the infimum

$$\frac{1}{K_{cyl} (N, p, \mu, a, b)} \equiv \inf_{u \in \mathcal{D}_{a,cyl} (\mathbb{R}^N \setminus \{|y| = 0\})} \frac{\int_{\mathbb{R}^N} |\nabla u|^p |y|^{-ap} dz - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^{p(a+1)}} dz}{\left( \int_{\mathbb{R}^N} (u_+)^{p^*(a,b)} |y|^{bp^*(a,b)} dz \right)^{\frac{1}{p^*(a,b)}}}.$$  

(46)

Using the existence result by Bhakta [8, Teorema 1.3], we deduce that there exists a nonnegative, cylindrically symmetric function $u(z) = u(x, y) = u(x, |y|) \in \mathcal{D}_{a,cyl} (\mathbb{R}^N \setminus \{|y| = 0\})$ that assumes the infimum $1/K_{cyl} (N, p, \mu, a, a)$. Using a convenient multiple of this function, we obtain a nonnegative, cylindrically symmetric solution to problem

$$- \operatorname{div} \left( \frac{|\nabla u|^{p-2}}{|y|^{-ap}} \nabla u \right) - \mu \frac{|u|^{p-2} u}{|y|^{p(a+1)}} = \frac{|u|^{p^*(a,a)} - 2 - 2}{|y|^{bp^*(a,a)}}, \quad (x, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k.$$

Finally, defining

$$0 < d < d_{*,cyl} \equiv \min_{m \in \{a, c\}} \left\{ \left( \frac{1}{p} - \frac{1}{p^*(a, m)} \right) K_{cyl} (N, p, \mu, a, m)^{-\frac{p^*(a,m)}{p^*(a,m)-p}} \right\},$$

we can repeat the proofs of the several results in sections 2 and 3; thus, we can extend Theorem 1.1 to the cases $0 \leq a = b < (k - p)/p$ and $\mu < \bar{\mu}$. This concludes the proof of the theorem. 

□
5. The inductive step of the Moser’s iteration scheme

In this section we state some auxiliary results that are important to prove our regularity theorem. It is worth mentioning that our result does not follow directly from the corresponding regularity result by Filippucci, Pucci, and Robert [19], since they applied the conclusions of the theorems by Druet [16], and by Guedda and Veron [23]; in our case we have to deal, among other things, with the singularity on the differential operator. In our proof we apply Moser’s iteration scheme and we follow closely the arguments by Pucci and Servadei [20, Theorem 2.2].

Our first result can be stated as follows.

**Lemma 5.1.** Let \( \Omega \subset \mathbb{R}^N \setminus \{|y|=0\} \) be a not necessarily bounded domain and let \( \eta: \Omega \to \mathbb{R} \) be a differentiable function with compact support. Given the constants \( d > 0 \) and \( l > 1 \), let the functions \( \rho: \mathbb{R} \to \mathbb{R} \) and \( \zeta: \mathbb{R} \to \mathbb{R} \) be defined by

\[
\rho(t) \equiv t \min \{|t|^{dp}, p^r\} \quad \text{and} \quad \zeta(t) \equiv \rho(t)|\eta|^p.
\]

Then \( \zeta(u) \in \mathcal{D}_{a,0}^{1,p}(\Omega) \).

**Proof.** We begin by defining the subsets

\[
\Omega_0 \equiv \{z \in \text{supp } \eta : |u(z)|^d < l\}, \quad \Omega_1 \equiv \{z \in \text{supp } \eta : |u(z)|^d = l\},
\]

\[
\Omega_2 \equiv \{z \in \text{supp } \eta : |u(z)|^d > l\}.
\]

Using a result by Gilbarg and Trudinger [22, Lemma 7.7], it follows that \( \rho \in \mathcal{D}_{a}^{1,p}(\Omega_1) \) and \( \rho \in \mathcal{D}_{a}^{1,p}(\Omega_2) \), because \( u \in \mathcal{D}_{a,\text{loc}}^{1,p}(\Omega) \). It remains to show that \( \rho \in \mathcal{D}_{a}^{1,p}(\Omega_0) \).

To do this we consider the function \( G: \mathbb{R} \to \mathbb{R} \) defined by

\[
G(t) \equiv \begin{cases} 
|t|^{pd}, & \text{if } |t|^d \leq l; \\
(pd + 1)|l|^d t - pd|l|^{(pd+1)/d}, & \text{if } |t|^d > l, \\
(pd + 1)|l|^d t + pd|l|^{(pd+1)/d}, & \text{if } t < -|l|^d.
\end{cases}
\]

It is clear that \( G(0) = 0 \) and \( G \in C^1(\mathbb{R}) \); moreover, \( G' \in L^\infty(\mathbb{R}) \). Since \( \rho(u) = G \circ u|_{\Omega_0} \), using another result by Gilbarg and Trudinger [22, Lemma 7.5] we deduce that \( \rho \in \mathcal{D}_{a}^{1,p}(\Omega_0) \). In \( \Omega_0 \) we have

\[
\nabla \rho(u) = \nabla(G(u)) = \nabla(u|u|^{pd}) = (1 + pd)|u|^{pd} \nabla u.
\]

In \( \Omega_2 \) we have \( |u(z)|^d > l \); hence, \( \rho(u) = u \min \{|u|^{pd}, p^r\} = u p^r \) and \( \nabla \rho(u) = \nabla(p^r u) = p^r \nabla u \). Finally, in \( \Omega_1 \) we have \( \nabla \rho(u) = 0 \) because \( |u(z)|^d = l^{1/d} \). Thus, we obtain

\[
\nabla \rho(u) = \begin{cases} 
(pd + 1)|l|^d \nabla u, & \text{if } z \in \Omega_0; \\
0, & \text{if } z \in \Omega_1; \\
|l|^d \nabla u, & \text{if } z \in \Omega_2.
\end{cases}
\]

And since \( \eta \in C^1(\Omega) \), it follows that \( \zeta \in \mathcal{D}_{a,0}^{1,p}(\Omega) \). This completes the proof of the lemma. \(\Box\)

Now we prove a crucial result that has an independent value.

**Lemma 5.2.** Let \( \Omega \subset \mathbb{R}^N \setminus \{|y|=0\} \) be a not necessarily bounded domain. Consider the parameters in the already specified intervals and let \( u \in \mathcal{D}_{a}^{1,p}(\Omega) \) be a weak solution to problem (6). For every \( d > 0 \) it is valid the proposition

\[
\text{if } u \in \mathcal{D}_{a,\text{loc}}^{1,p}(\Omega) \cap L^{p+1}_{a,\text{loc}}(\Omega), \text{ then } u \in L^{p^a(a,b)(d+1)}_{a/(d+1),\text{loc}}(\Omega) \cap L^{p^a(a,c)(d+1)}_{c/(d+1),\text{loc}}(\Omega). \tag{49}
\]

**Proof.** Let the function \( u \in \mathcal{D}_{a,\text{loc}}^{1,p}(\Omega) \) be a weak solution to problem (6). We define the function \( f: \Omega \times \mathcal{D}_{a,\text{loc}}^{1,p}(\Omega) \to \mathbb{R} \) by

\[
f(z, u) = \mu \frac{|u|^{p-2} u}{|y|^{p(a+1)}} + \frac{(u_+)^{p^a(a,b)-1}}{|y|^{p^a(a,b)}} + \frac{(u_+)^{p^a(a,c)-1}}{|y|^{p^a(a,c)}},
\]

where \( u_+ = \max\{0, u\} \). Since \( u \in \mathcal{D}_{a,\text{loc}}^{1,p}(\Omega) \), we have

\[
\|u\|_{\mathcal{D}_{a,\text{loc}}^{1,p}(\Omega)} < \infty.
\]

This implies that \( f \) is locally bounded on \( \Omega \times \mathcal{D}_{a,\text{loc}}^{1,p}(\Omega) \), and hence, by the previous lemmas, we can apply the Moser’s iteration scheme. In particular, for every \( \delta > 0 \), there exists a \( \delta > 0 \) such that

\[
\text{if } u \in \mathcal{D}_{a,\text{loc}}^{1,p}(\Omega) \cap L^{p+1}_{a,\text{loc}}(\Omega), \text{ then } u \in L^{p^a(a,b)(d+1)}_{a/(d+1),\text{loc}}(\Omega) \cap L^{p^a(a,c)(d+1)}_{c/(d+1),\text{loc}}(\Omega). \tag{49}
\]

We denote by \( \mathcal{D}_{a}^{1,p}(\Omega) \) the completion of \( \mathcal{D}_{a,\text{loc}}^{1,p}(\Omega) \) with respect to the norm \( \|u\|_{\mathcal{D}_{a}^{1,p}(\Omega)} = \|u\|_{\mathcal{D}_{a,\text{loc}}^{1,p}(\Omega)} \). Then, for every \( \delta > 0 \), there exists a \( \delta > 0 \) such that

\[
\text{if } u \in \mathcal{D}_{a}^{1,p}(\Omega) \cap L^{p+1}_{a,\text{loc}}(\Omega), \text{ then } u \in L^{p^a(a,b)(d+1)}_{a/(d+1),\text{loc}}(\Omega) \cap L^{p^a(a,c)(d+1)}_{c/(d+1),\text{loc}}(\Omega). \tag{49}
\]
where we denote $z = (x, y) \in \Omega$. And for $l > 1$ we define the function $\tilde{\rho}: \mathbb{R} \to \mathbb{R}$ by $\tilde{\rho}(t) \equiv t \min\{|t|^d, l\}$. Arguing as we have already done in the proof of Lemma 5.1, we obtain

$$\nabla \tilde{\rho}(u) = \begin{cases} (d+1)|u|^d \nabla u, & \text{if } z \in \Omega_0; \\ 0, & \text{if } z \in \Omega_1; \\ l \nabla u, & \text{if } z \in \Omega_2. \end{cases} \tag{50}$$

Moreover, it is valid the inequality

$$|\nabla u|^{p-1}|\rho| \leq |\nabla \tilde{\rho}|^{p-1}|\tilde{\rho}| \quad \text{a. e. in } \Omega. \tag{51}$$

Using the function $\zeta = \rho |\eta|^p$ as a test function in equation (10), as well the definition of $f$, we obtain

$$\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \zeta \rangle dz \leq \int_{\Omega} |\tilde{\rho} \eta|^p \left( |\mu| \frac{1}{|y|^{p(a+1)}} + |u|^{p^* (a,b) - p} \frac{1}{|y|^{bp^* (a,b)}} + |u|^{p^* (a,c) - p} \frac{1}{|y|^{cp^* (a,c)}} \right) dz. \tag{52}$$

Using the definition of the function $\zeta$, together with Bernoulli’s inequality, that is, $0 < pd + 1 < (d + 1)^p$, for $d \geq -1$ and $p > 1$, we deduce that

$$\int_{\Omega} \frac{|\nabla u|^{p-2}}{|y|^{ap}} \langle \nabla u, \nabla \zeta \rangle dz \leq (pd + 1) \int_{\Omega_0} \frac{|\nabla u|^{p-2}}{|y|^{ap}} |\eta|^p |u|^{pd} \langle \nabla u, \nabla u \rangle dz + \int_{\Omega_2} \frac{|\nabla u|^{p-2}}{|y|^{ap}} |\eta|^p |u|^{p^*} \langle \nabla u, \nabla u \rangle dz$$

$$\quad + \int_{\Omega} \frac{|\nabla u|^{p-2}}{|y|^{ap}} \rho \rho |\eta|^p \langle \nabla u, \nabla \eta \rangle dz > \frac{(pd + 1)}{(d + 1)^p} \int_{\Omega} \frac{|\eta | |\nabla \tilde{\rho}|^p}{|y|^{ap}} \eta^p dz - p \int_{\Omega} \frac{|\nabla u|^{p-1}}{|y|^{ap}} |u| \min\{|u|^{pd}, p\} |\eta|^p |\nabla \eta| dz. \tag{53}$$

Isolating the first term on the right-hand side of the previous inequality, and using inequalities (51) and (52), we deduce that

$$\frac{(pd + 1)}{(d + 1)^p} \int_{\Omega} \frac{|\eta | |\nabla \tilde{\rho}|^p}{|y|^{ap}} \eta^p dz < \int_{\Omega} |\tilde{\rho} \eta|^p \left( |\mu| \frac{1}{|y|^{p(a+1)}} + |u|^{p^* (a,b) - p} \frac{1}{|y|^{bp^* (a,b)}} + |u|^{p^* (a,c) - p} \frac{1}{|y|^{cp^* (a,c)}} \right) dz$$

$$\quad + p \int_{\Omega} \frac{|\eta | |\nabla \tilde{\rho}|^{p-1}}{|y|^{ap}} |\tilde{\rho} \nabla \eta| dz. \tag{54}$$

On the other hand, using Young’s inequality on the integrand of the last term but without the singularity, we find

$$|\eta | |\nabla \tilde{\rho}|^{p-1} |\tilde{\rho} \nabla \eta| \leq \frac{|\eta | |\nabla \tilde{\rho}|^p}{p' \epsilon^p} + \frac{e^p |\tilde{\rho} \nabla \eta|^p}{p}. \tag{55}$$

and since $1 < pd + 1$, it follows that

$$\frac{1}{(d + 1)^p} \int_{\Omega} \frac{|\eta | |\nabla \tilde{\rho}|^p}{|y|^{ap}} dz < \int_{\Omega} |\tilde{\rho} \eta|^p \left( |\mu| \frac{1}{|y|^{p(a+1)}} + |u|^{p^* (a,b) - p} \frac{1}{|y|^{bp^* (a,b)}} + |u|^{p^* (a,c) - p} \frac{1}{|y|^{cp^* (a,c)}} \right) dz$$

$$\quad + p - 1 \frac{e^p}{e^p} \int_{\Omega} \frac{|\eta | |\nabla \tilde{\rho}|^p}{|y|^{ap}} dz + e^p \int_{\Omega} \frac{|\tilde{\rho} \nabla \eta|^p}{|y|^{ap}} dz. \tag{56}$$

Consequently,

$$\left( \frac{1}{(d + 1)^p} - \frac{p - 1}{e^p} \right) \int_{\Omega} \frac{|\eta | |\nabla \tilde{\rho}|^p}{|y|^{ap}} dz < \int_{\Omega} |\tilde{\rho} \eta|^p \left( |\mu| \frac{1}{|y|^{p(a+1)}} + |u|^{p^* (a,b) - p} \frac{1}{|y|^{bp^* (a,b)}} + |u|^{p^* (a,c) - p} \frac{1}{|y|^{cp^* (a,c)}} \right) dz$$

$$\quad + e^p \int_{\Omega} \frac{|\tilde{\rho} \nabla \eta|^p}{|y|^{ap}} dz. \tag{57}$$
Now we choose \(\epsilon^p = 2(p - 1)(d + 1)^p\) and we denote \(c_1 = \max\{2, 2p(p - 1)^{p - 1}\};\) hence,
\[
\|\eta \nabla \tilde{\rho}\|^p_{L^p_\eta(\Omega)} < 2(d + 1)^p \int_\Omega |\tilde{\rho}|^p \left( \frac{\|\mu\|^p}{|y|^{p(a + 1)}} + \frac{|u|^{p^*(a,b) - p}}{|y|^{b p(a,b)}} + \frac{|u|^{p^*(a,c) - p}}{|y|^{c p(a,c)}} \right) \, dz \\
+ 2^p(p - 1)^{p - 1}(d + 1)^p \int_\Omega \frac{\|\nabla \eta\|^p}{|y|^{\alpha p}} \, dz \\
< c_1(d + 1)^p \int_\Omega |\tilde{\rho}|^p \left( \frac{|\eta|}{|y|^{p(a + 1)}} + \frac{|u|^{p^*(a,b) - p}}{|y|^{b p(a,b)}} + \frac{|u|^{p^*(a,c) - p}}{|y|^{c p(a,c)}} + \|\nabla \eta\|^p \right) \, dz.
\] (53)

Applying Hölder’s inequality to the second and third integrals on the right-hand side of inequality (53), we deduce that
\[
\int_\Omega |\tilde{\rho}|^p \frac{|u|^{p^*(a,b) - p}}{|y|^{b p(a,b)}} \, dz \leq \|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,b)}}(\Omega)} \|u\|^{p^*(a,b) - p}_{L^p_{b^{p^*(a,b)}}(\text{supp } \eta)} \tag{54}
\]
and
\[
\int_\Omega |\tilde{\rho}|^p \frac{|u|^{p^*(a,c) - p}}{|y|^{c p(a,c)}} \, dz \leq \|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,c)}}(\Omega)} \|u\|^{p^*(a,c) - p}_{L^p_{b^{p^*(a,c)}}(\text{supp } \eta)} \tag{55}
\]
Applying Maz’ya’s inequality, together with inequality \((X + Y)^p \leq 2^{p - 1}(X^p + Y^p)\) for \(X, Y \in \mathbb{R}^+\) and \(p > 1\), we obtain
\[
\|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,b)}}(\Omega)} \leq K(N, p, \mu, a, b) \|\nabla(\tilde{\rho} \eta)\|^p_{L^p_\eta(\Omega)} \\
\leq 2^{p - 1}K(N, p, \mu, a, b) \left( \|\eta \nabla \tilde{\rho}\|^p_{L^p_\eta(\Omega)} + \|\tilde{\rho} \eta\|^p_{L^p_\eta(\Omega)} \right). \tag{56}
\]
Combining inequalities (53), (54), (55) and (56), we deduce that
\[
\|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,b)}}(\Omega)} \leq 2^{p - 1}K(N, p, \mu, a, b)[c_1(d + 1)^p + 1] \int_\Omega |\tilde{\rho}|^p \left( \frac{\|\mu\|^p}{|y|^{p(a + 1)}} + \frac{|\nabla \eta|^p}{|y|^{\alpha p}} \right) \, dz \\
+ 2^pK(N, p, \mu, a, b)c_1(d + 1)^p \|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,b)}}(\Omega)} \|u\|^{p^*(a,b) - p}_{L^p_{b^{p^*(a,b)}}(\text{supp } \eta)} \\
+ 2^{p - 1}K(N, p, \mu, a, b)c_1(d + 1)^p \|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,c)}}(\Omega)} \|u\|^{p^*(a,c) - p}_{L^p_{b^{p^*(a,c)}}(\text{supp } \eta)}. \tag{57}
\]
Similarly, we obtain
\[
\|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,c)}}(\Omega)} \leq 2^{p - 1}K(N, p, \mu, a, c)[c_1(d + 1)^p + 1] \int_\Omega |\tilde{\rho}|^p \left( \frac{\|\mu\|^p}{|y|^{p(a + 1)}} + \frac{|\nabla \eta|^p}{|y|^{\alpha p}} \right) \, dz \\
+ 2^pK(N, p, \mu, a, c)c_1(d + 1)^p \|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,b)}}(\Omega)} \|u\|^{p^*(a,b) - p}_{L^p_{b^{p^*(a,b)}}(\text{supp } \eta)} \\
+ 2^{p - 1}K(N, p, \mu, a, c)c_1(d + 1)^p \|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,c)}}(\Omega)} \|u\|^{p^*(a,c) - p}_{L^p_{b^{p^*(a,c)}}(\text{supp } \eta)}. \tag{58}
\]
Since \(u \in L^{p^*(a,b)}(\Omega) \cap L^{p^*(a,c)}(\Omega)\), for every \(z_0 \in \text{supp } \eta\) there exists \(R = R(z_0) > 0\) such that \(B_R(z_0) \subseteq \text{supp } \eta\) and it is valid the inequality
\[
\max_{m \in \{b,c\}} \left( \int_{B_R(z_0)} \frac{|u|^{p^*(m)}}{|y|^{mp^*(m)}} \, dz \right)^{\frac{p^*(m) - p}{p^*(m)}} \leq \min_{m \in \{b,c\}} \left[ 2^{p - 1}K(N, p, \mu, a, m)c_1(d + 1)^p \right]^{-1}. \tag{59}
\]
Now we choose the cut off function \(\eta \in C_0^\infty(\Omega)\) with the following additional properties: the support of \(\eta\) is such that \(\text{supp } \eta \subseteq B_{2R}(z_0), 0 \leq \eta \leq 1, \eta \equiv 1\) in \(B_R(z_0)\), and \(|\nabla \eta| \leq 2/R\).

Using this cut off function in inequality (57), together inequality (59) with the appropriate values for \(m \in \{b, c\}\) on both sides, as well as the fact that \(|\mu||y|^{-1}|\eta|^p \leq c_2\) in \(\text{supp } \eta\) for some positive constant \(c_2 > 0\), we deduce that
\[
\|\tilde{\rho} \eta\|^p_{L^p_{b^{p^*(a,b)}}(B_{2R}(z_0))}
\]
where the notation $c$ Since obtain Finally, since the cut off function Using an index notation, we rewrite part of proposition (49) in the form 

\[
\frac{2p+1}{3} K(N, p, \mu, a, b) \left[ c_3 \left( d + 1 \right)^{p^2} + 1 \right] \int_{B_{2R}(z_0)} |\tilde{\rho}|^p \left( \frac{|\mu| |\eta|^p}{|y|^{ap}} + \frac{\nabla \eta|^p}{|y|^{ap}} \right) dz \\
+ \frac{1}{3} \left\| \tilde{\rho} \eta \right\|_{L^{p,a,c}((u,c), B_{2R}(z_0))}^p
\]

\[
= \frac{2p+1}{3} K(N, p, \mu, a, b) \left[ c_1 \left( d + 1 \right)^{p^2} + 1 \right] \left( c_2 + \frac{2p}{R^p} \right) \int_{\text{supp } \eta} \left( \frac{|\eta|^{d+1}}{|y|^{a}} \right)^p dz \\
+ \frac{1}{3} \left\| \tilde{\rho} \eta \right\|_{L^{p,a,c}((u,c), B_{2R}(z_0))}^p
\]

\[
= c_3 \left\| u \right\|_{L^{p(a+1)}_{0,c}(\text{supp } \eta)}^{p(a+1)} + \frac{1}{3} \left\| \tilde{\rho} \eta \right\|_{L^{p,a,c}((u,c), B_{2R}(z_0))}^p
\]

where $c_3 = 2p K(N, p, \mu, a, b) \left[ c_1 \left( d + 1 \right)^{p^2} + 1 \right] \left( c_2 + \frac{2p}{R^p} \right)$.

Similarly, we obtain

\[
\left\| \tilde{\rho} \eta \right\|_{L^{p,a,c}((u,c), B_{2R}(z_0))}^p \leq c_4 \left\| u \right\|_{L^{p(a+1)}_{0,c}(\text{supp } \eta)}^{p(a+1)} + \frac{1}{3} \left\| \tilde{\rho} \eta \right\|_{L^{p,a,c}((u,c), B_{2R}(z_0))}^p
\]

where $c_4 = 2p K(N, p, \mu, a, b) \left[ c_1 \left( d + 1 \right)^{p^2} + 1 \right] \left( c_2 + \frac{2p}{R^p} \right)$.

Adding termwise both sides of inequalities (60) and (61), it follows that

\[
\left\| \tilde{\rho} \eta \right\|_{L^{p,a,b}((u,c), B_{2R}(z_0))}^p + \left\| \tilde{\rho} \eta \right\|_{L^{p,a,c}((u,c), B_{2R}(z_0))}^p \leq \frac{3}{2} \left( c_3 + c_4 \right) \left\| u \right\|_{L^{p(a+1)}_{0,c}(\text{supp } \eta)}^{p(a+1)}.
\]

Passing to the limit as $l \to \infty$ in the first term on the left-hand side of inequality (62), we obtain

\[
\left\| u \right\|_{L^{p(a+1)}_{a/(d+1)}(B_{2R}(z_0))}^{p(a+1)} \leq \frac{3}{2} \left( c_3 + c_4 \right) \left\| u \right\|_{L^{p(a+1)}_{a/(d+1)}(\text{supp } \eta)}^{p(a+1)}.
\]

Since $\text{supp } \eta$ can be covered by a finite number of balls with these properties, say $M$ balls, using the notation $c_5 = M \left( (3/2)(c_3 + c_4) \right)^{1/(pd+p)}$ we infer that

\[
\left\| u \right\|_{L^{p(a,b)}((d+1)}_{b/(d+1)}(B_{2R}(z_0))}^{p(a,b)} \leq c_5 \left\| u \right\|_{L^{p(a+1)}_{a/(d+1)}(\text{supp } \eta)}^{p(a+1)}.
\]

Similarly, we have

\[
\left\| u \right\|_{L^{p(a,c)}((d+1)}_{a/(d+1)}(\text{supp } \eta)}^{p(a,c)} \leq c_5 \left\| u \right\|_{L^{p(a+1)}_{a/(d+1)}(\text{supp } \eta)}^{p(a+1)}.
\]

Finally, since the cut off function $\eta \in C^1_0(\Omega)$ is arbitrary, we conclude the proof of the lemma. \hfill \square

Now we use the iteration scheme to conclude that $u \in L^m_{\gamma,loc}$ for every $m \in [1, +\infty)$ and some appropriate weight $\gamma$.

**Lemma 5.3.** Let $\Omega \subset \mathbb{R}^N \setminus \{|y| = 0\}$ be a not necessarily bounded domain. Consider the parameters in the already specified intervals and let $u \in D^{1,p}_a(\Omega)$ be a weak solution to problem (6). Then $u \in L^m_{\gamma,loc}$, for every $m \in [1, +\infty)$ and some appropriate weight $\gamma = \gamma(m)$.

**Proof.** Using an index notation, we rewrite part of proposition (49) in the form

If $u \in D^{1,p}_a(\Omega) \cap L^{p(d,a,c)}_{a/(d+1),loc}(\Omega)$, then $u \in L^{p(d+1)}_{a/(d+1),loc}(\Omega)$. 


Now we choose these indexes so that  \( d_0 = 0, e_0 = 0, d_i + 1 = (d_{i-1} + 1)p^* (a, b)/p \), and 
\( e_i + 1 = (d_{i-1} + 1)a/b \) for \( i = 1, 2, 3, \ldots \); this implies that 
\( d_i + 1 = (p^*(a, b)/p)^i \) and \( e_i + 1 = (p^*(a, b)/p)^i (a/b) \) for \( i = 1, 2, 3, \ldots \); therefore,

If \( u \in D_{a, \text{loc}}^{1,p} (\Omega) \cap L_{b \text{loc}}^{p(p^*(a, b)/p)^{i-1}} (\Omega) \), then \( u \in L_{b \text{loc}}^{p(p^*(a, b)/p)^i} (\Omega) \), \( (i = 2, 3, 4, \ldots) \).

Similarly,

If \( u \in D_{a, \text{loc}}^{1,p} (\Omega) \cap L_{c \text{loc}}^{p(p^*(a, c)/p)^{i-1}} (\Omega) \), then \( u \in L_{c \text{loc}}^{p(p^*(a, c)/p)^i} (\Omega) \), \( (i = 2, 3, 4, \ldots) \).

This concludes the proof of the lemma.

\[ \square \]

6. Conclusion of the Proof of Theorem 1.2

Now we finish the proof of Theorem 1.2, that is, we show that if the function \( u \in D_a^{1,p}(\Omega) \) is a weak solution to problem (6), then \( u \in L_{\text{loc}}^{\infty}(\Omega) \).

\textbf{Proof of Theorem 1.2.} Let \( E \subseteq \mathbb{R}^N \) be a bounded domain with boundary of class \( C^1 \) and let \( \tilde{\Omega} \) be a bounded set so that \( E \subseteq \tilde{\Omega} \subseteq \Omega \). Repeating the same arguments of section 5 and choosing the cut off function \( \eta: \Omega \rightarrow \mathbb{R} \) so that \( 0 \leq \eta \leq 1 \), \( \text{supp} \ \eta \subseteq \tilde{\Omega} \) and \( \eta \equiv 1 \) in \( E \), by the inequality similar to inequality (53), we deduce that

\[
\| \nabla \tilde{\rho} \|_{L^p(E)}^p \leq c_1 (d + 1)^2 \int_E |\tilde{\rho}|^p \left( \frac{|\mu|}{|y|^{p(a+1)}} + |u|^{p^*(a,b)-p} \right) \, dz.
\]

(65)

Using Maz’ya’s inequality, we have

If \( u \in D_{a, \text{loc}}^{1,p}(E) \cap L_{b, \text{loc}}^p (E) \), then \( u \in L_{b, \text{loc}}^{p^*(a,b)} (E) \).

Hence, by the inductive step proved in section 5, we obtain

If \( u \in D_{a, \text{loc}}^{1,p}(E) \cap L_{b, \text{loc}}^{p^*(a,b)} (E) \), then \( u \in L_{b/p, \text{loc}}^{p^*(a,b)/p} (E) \).

Applying Hölder’s inequality to the second term in inequality (65), we deduce that

\[
\int_E \frac{|\tilde{\rho}|^p |u|^{p^*(a,b)-p}}{|y|^{bp^*(a,b)}} \, dz \leq \| \tilde{\rho} \|_{L_{b \text{loc}}^{p^*(a,b)/r^*(a,b)}(E)} \| u \|_{L_{b \text{loc}}^{p^*(a,b)/p}(E)}^{p^*(a,b)-p} \| L_{b/p, \text{loc}}^{p^*(a,b)/p}(E),
\]

(66)

where \( r(a,b) = (p^*(a,b)/p)^2 - pp^*(a,b) + pp^*(a,b) > 1 \) because \( a \leq b < a + 1 \).

By the choice of the set \( E \), by the continuity of the embedding \( D_a^{1,p} \hookrightarrow L_{b}^{p^*(a,b)}(E) \) and by inequalities (65) and (66), it follows that

\[
\| \tilde{\rho} \|_{L_{b}^{p^*(a,b)}(E)}^p \leq K(N, p, \mu, a, b) \left( \int_E \left| \frac{\nabla \tilde{\rho}}{|y|^a} \right|^p \, dz - \mu \int_E \left| \frac{\tilde{\rho}}{|y|^{a+1}} \right|^p \, dz \right) 
\]

\[
\leq K(N, p, \mu, a, b) c_1 (d + 1)^2 \left\| \mu \| L_{a+1}^p(\Omega) \right\| + K(N, p, \mu, a, b) c_1 (d + 1)^2 \left\| \tilde{\rho} \|_{L_{b \text{loc}}^{p^*(a,b)/r(a,b)}(E)} \right\| u \|_{L_{b \text{loc}}^{p^*(a,b)/p}(E)}^{p^*(a,b)-p} 
\]

\[
- \mu K(N, p, \mu, a, b) \| \tilde{\rho} \|_{L_{a+1}^p(\Omega)}^p.
\]

(67)

Passing to the limit as \( k \to \infty \) and using the definition of \( \tilde{\rho} \), we obtain

\[
\| u \|_{L_{b/p, \text{loc}}^{p^*(a,b)/p}(E)}^{p^*(a,b)} \leq K(N, p, \mu, a, b) c_1 (d + 1)^2 \left\| \mu \right\|_{L_{b/p, \text{loc}}^{p^*(a,b)/p}(E)}^{p^*(a,b)-p} 
\]

\[
+ K(N, p, \mu, a, b) c_1 (d + 1)^2 \left\| u \right\|_{L_{b \text{loc}}^{p^*(a,b)/p}(E)}^{p^*(a,b)-p} 
\]

\[
- \mu K(N, p, \mu, a, b) \| u \|_{L_{b/p, \text{loc}}^{p^*(a,b)/p}(E)}^{p^*(a,b)}.
\]

(68)
Now we are going to estimate the first and the last terms on the right-hand side of inequality (68). Since \( u \in L^m_{loc}(\Omega) \) for every \( m \in [1, +\infty) \) and \( E \Subset \overline{\Omega} \subset \Omega \) is a bounded set, we have

\[
|y| \geq \frac{1}{|E|}
\]

for every \( z = (x, y) \in E \). Hence, applying Hölder’s inequality, we obtain

\[
\| u \|^{p(d+1)}_{L^{r(d+1)}(E)} \leq \left( \int_E \left( \frac{1}{|y|} \right)^{p^*(a,b) - pr(a,b)} \frac{u^{p^*(a,b) - 1}}{p^*(a,b)} \, dz \right)^{1/p} \left( \int_E \left( \frac{u^{pr(a,b)}}{|y|^{d+1}} \right)^{p^*(a,b)(d+1)} \, dz \right)^{1/p^*(a,b)}
\]

\[
= M_E^{s(a,b)} |E|^{t(a,b)} \| u \|^{p(d+1)}_{L^{p^*(a,b)(d+1)}(E)},
\]

where \( |E| \) denotes the measure of the subset \( E \), \( s(a,b) = p(a+1) - b(p^*(a,b) - p) - bp^2/p^*(a,b) \) and \( t(a,b) = p/p^*(a,b) - p^2/(p^*(a,b)^2) \).

Substituting inequality (69) in (68) and using the fact that \((d+1)p^2 > 1\), we obtain

\[
\| u \|^{p(d+1)}_{L^{p^*(a,b)(d+1)}(E)} \leq K(N, p, \mu, a, b)c_1 (d+1)p^2 |\mu| M_E^{s(a,b)} |E|^{t(a,b)} \max\{\| u \|^{p(d+1)}_{L^{p^*(a,b)(d+1)}(E)}, 1\}
\]

\[
+ K(N, p, \mu, a, b)c_1 (d+1)p^2 \| u \|^{p^*(a,b) - p}_{L^{p^*(a,b)^2/p}(E)} \max\{\| u \|^{p(d+1)}_{L^{p^*(a,b)(d+1)}(E)}, 1\}
\]

\[
+ (d+1)p^2 |\mu| K(N, p, \mu, a, b) M_E^{s(a,b)} |E|^{t(a,b)} \max\{\| u \|^{p(d+1)}_{L^{p^*(a,b)(d+1)}(E)}, 1\},
\]

that is,

\[
\| u \|^{p(d+1)}_{L^{p^*(a,b)(d+1)}(E)} \leq (A(d+1)p)^{\frac{d+1}{d+1}} \max\{\| u \|^{p(d+1)}_{L^{p^*(a,b)(d+1)}(E)}, 1\},
\]

(70)

where

\[
A^p = A(u)^p \equiv \max \left\{ 3K(N, p, \mu, a, b)c_1 |\mu| M_E^{s(a,b)} |E|^{t(a,b)}, 3K(N, p, \mu, a, b)c_1 \| u \|^{p^*(a,b) - p}_{L^{p^*(a,b)^2/p}(E)}, 3|\mu| K(N, p, \mu, a, b) M_E^{s(a,b)} |E|^{t(a,b)} \right\}.
\]

Now we use Moser’s iteration scheme. Choosing \( d + 1 = r(a, b) = r > 1 \) in inequality (70), we get

\[
\| u \|^{p^*(a,b)r}_{L^r_b(E)} \leq (A^r)^{\frac{r}{r+1}} \max\{\| u \|^{p^*(a,b)r}_{L^r_b(E)}, 1\}.
\]

(71)

Choosing \( d + 1 = r(a, b)^2 = r^2 \) in inequality (70) and using inequality (71), we get

\[
\| u \|^{p^*(a,b)r^2}_{L^{r^2}_b(E)} \leq (A^{r^2})^{\frac{r^2}{r^2+1}} \max\{\| u \|^{p^*(a,b)r^2}_{L^{r^2}_b(E)}, 1, (A^r)^{-\frac{r}{r+1}}\}.
\]

(72)

In general, choosing \( (d+1) = r(a, b)^j = r^j \) for \( j \in \mathbb{N} \), we get

\[
\| u \|^{p^*(a,b)r^j}_{L^{r^j}_b(E)} \leq (A^{\frac{r^j}{j+1} \cdots + \frac{r}{2} + \frac{1}{2}}) (r^p)^{\frac{1}{j+1} \cdots + \frac{1}{2}}
\]

\[
\times \max \left\{ \| u \|^{p^*(a,b)r^j}_{L^r_b(E)}, 1, \max \left\{ A^{-\frac{1}{j+1} (r^p)^{-\frac{1}{j+1}}} ; \ldots ; A^{-\frac{1}{j+1} \cdots + \frac{1}{j+1} - 1} (r^p)^{-\frac{1}{j+1} \cdots + \frac{1}{2}} \right\} \right\}
\]

\[
\leq A^{\alpha^*_r p^{\alpha^*_r}} \max \left\{ \| u \|^{p^*(a,b)r^j}_{L^r_b(E)}, 1, M_{j-1} \right\},
\]

(72)
where $M_1 \equiv 1$ and $M_{j-1} \equiv \max_{1 \leq i \leq j-1} \{A^{-\sigma_i - r^*_i} \}$ for $j > 1$, with $\sigma_i = \sum_{k=1}^i r^{-k}$ and $\tau_i = \sum_{k=1}^i kr^{-k}$. And since $r = r(a,b) > 1$, we have

$$M_{j-1} = \begin{cases} A^{-\sigma_j - r^{-i}} & \text{if } A \leq 1; \\ A^{-\frac{i}{r}r^{-i}} & \text{if } A > 1. \end{cases}$$

Passing to the limit as $i \to \infty$, it follows that

$$\lim_{i \to \infty} \sigma_i = \frac{1}{r - 1}, \quad \lim_{i \to \infty} \tau_i = \frac{r}{(r - 1)^2}, \quad \text{and} \quad \lim_{i \to \infty} M_{j-1} \equiv M = \begin{cases} A^{-\frac{i}{r}r^{-i}} & \text{if } A \leq 1; \\ A^{-\frac{i}{r}r^{-i}} & \text{if } A > 1. \end{cases}$$

Therefore,

$$\|u\|_{L^\infty(E)} \leq A^{rac{i}{r}r^{-i}} \max \left\{ \|u\|_{L^{p^*_i(a,b)}(E)}, 1, M \right\}.$$ Finally, since the subset $E \subseteq \Omega$ is arbitrary, the proof of the theorem is complete. \hfill \Box

7. A Pohozaev-type identity

In this section we prove Theorem 1.3 by using a Pohozaev-type identity, whose underlying principle is the comparison of two different variations of the energy functional at a critical point. For more details on these types of identities, see Ekeland and Ghoussoub [18, Remark 2.1].

Before we prove our first lemma, however, we note that in the case $1 < q < p^*(a,b)$, if $u \in D^1_{a,b}(\mathbb{R}^N \setminus \{|y| = 0\})$, then $u \in L^q_{\text{loc}}(\mathbb{R}^N \setminus \{|y| = 0\})$; consequently, the definition of weak solution makes sense. On the other hand, in the case $q > p^*(a,b)$ the same conclusion is valid if we suppose additionally that $u \in L^{p^*_i(a,b)/q,\text{loc}}(\mathbb{R}^N \setminus \{|y| = 0\}) \cap L^{p,\text{loc}}(\mathbb{R}^N \setminus \{|y| = 0\})$.

Lemma 7.1. Let $\eta, u \in C_c^\infty(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} \langle \nabla u, \nabla (\langle z, \nabla (\eta u) \rangle) \rangle \, dz + \frac{N - p(a + 1)}{p} \int_{\mathbb{R}^N} \eta \frac{|
abla u|^p}{|y|^p} \, dz = B(u, \eta) \quad (73)$$

where

$$B(u, \eta) = \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} u \langle \nabla u, \nabla \eta \rangle \, dz + \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} u \sum_{i,j} z_i \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_i} \eta \, dz$$

$$+ \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} \langle \nabla u, \nabla \eta \rangle \langle z, \nabla u \rangle \, dz + \frac{p - 1}{p} \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} \langle z, \nabla \eta \rangle \, dz. \quad (74)$$

Proof. To prove equality (73) we expand the expression $\langle \nabla u, \nabla (\langle z, \nabla (\eta u) \rangle) \rangle$. In this way, we obtain

$$\int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} \langle \nabla u, \nabla (\langle z, \nabla (\eta u) \rangle) \rangle \, dz$$

$$= \int_{\mathbb{R}^N} \eta \frac{|
abla u|^p}{|y|^p} \, dz + \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} u \sum_{i=1}^{N} \frac{\partial}{\partial z_i} \frac{\partial^2}{\partial z_i \partial z_j} \eta \, dz$$

$$+ \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} u \langle \nabla u, \nabla \eta \rangle \, dz + \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} u \sum_{i=1}^{N} \frac{\partial}{\partial z_i} \frac{\partial^2}{\partial z_j \partial z_i} \eta \, dz$$

$$+ \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} \langle \nabla u, \nabla \eta \rangle \langle z, \nabla u \rangle \, dz + \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} \langle z, \nabla \eta \rangle \, dz. \quad (74)$$

Now we expand the second term on the right-hand side of inequality (74). Using the divergence theorem and recalling that $\eta \in C_c^\infty$ we obtain

$$\int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} \sum_{i,j=1}^{N} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_i} \eta \, dz = \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^p} \sum_{i=1}^{N} \frac{\partial}{\partial z_i} \left( \frac{|
abla u|^p}{p} \right) \, dz$$
By a density argument, there exists a sequence \((\varphi_n)\) in \(C^\infty_0(\mathbb{R}^N \setminus \{y = 0\})\) such that \(\lim_{n \to \infty} \varphi_n = u\) in \(C^1_{\text{loc}}(\mathbb{R}^N \setminus \{y = 0\})\) and \(W^{2,1}_{\text{loc}}(\mathbb{R}^N \setminus \{y = 0\})\). Applying Lemma 7.1 to the functions \(\eta\) and \(\varphi_n\), and passing to the limit as \(n \to \infty\) in equality (73) we conclude the proof of the lemma.

8. Conclusion of the proof of Theorem 1.3

Now we can use the Pohozaev-type identity (73) to show that there exists no nontrivial solution to problem (7).

Lemma 8.1. Let \(f \in C^0((\mathbb{R}^N \setminus \{y = 0\}) \times \mathbb{R})\) and let \(u \in D^{1,p}_a(\mathbb{R}^N \setminus \{y = 0\}) \cap C^1(\mathbb{R}^N \setminus \{y = 0\}) \cap W^{2,1}_{\text{loc}}(\mathbb{R}^N \setminus \{y = 0\})\) be a solution to problem

\[
-\text{div} \left[ \frac{|\nabla \xi|^{p-2}}{|y|^{ap}} \nabla \xi \right] = f(z, \xi) \quad \text{in} \quad \mathbb{R}^N. \tag{76}
\]

Suppose that the function \(F(z, \xi) = \int_0^t f(z, \tau) d\tau\) is such that \(F \in C^1((\mathbb{R}^N \setminus \{y = 0\}) \times \mathbb{R})\); suppose also that \(\xi f(\cdot, \xi), F(\cdot, \xi), \sum_{i=1}^N \partial F/\partial z_i(\cdot, \xi) \in L^1_a(\mathbb{R}^N)\). Then

\[
\int_{\mathbb{R}^N} \left[ \frac{N - p(a + 1)}{p} \xi f(z, \xi) - NF(z, \xi) - \sum_{i=1}^N z_i \frac{\partial F}{\partial z_i}(z, \xi) \right] dz = 0. \tag{77}
\]

Proof. Let us consider the test function \(\eta \in C^\infty_0(\mathbb{R}^N \setminus \{y = 0\})\); let us also consider the sequence \((\varphi_n)\) in \(C^\infty_0(\mathbb{R}^N \setminus \{y = 0\})\) such that \(\lim_{n \to \infty} \varphi_n = \xi \in C^1_{\text{loc}}(\mathbb{R}^N \setminus \{y = 0\}) \cap W^{2,1}_{\text{loc}}(\mathbb{R}^N \setminus \{y = 0\})\).

Multiplying both sides of equation (76) by \(\langle z, \nabla (\eta \varphi_n) \rangle\) and using the divergence theorem, we obtain

\[
\int_{\mathbb{R}^N} |\nabla \xi|^{p-2} |y|^{ap} \langle \nabla \xi, \nabla (z, \nabla (\eta \varphi_n)) \rangle dz = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla \xi|^{p-2} |y|^{ap} \langle \nabla \xi, \nabla (z, \nabla (\eta \varphi_n)) \rangle dz = \lim_{n \to \infty} \int_{\mathbb{R}^N} f(z, \xi) \langle z, \nabla (\eta \varphi_n) \rangle dz = \int_{\mathbb{R}^N} \xi f(z, \xi) \langle z, \nabla \eta \varphi_n \rangle dz = \int_{\mathbb{R}^N} \xi f(z, \xi) \langle z, \nabla (\eta \xi) \rangle dz + \int_{\mathbb{R}^N} \eta \sum_{i=1}^N z_i \frac{\partial F(z, \xi)}{\partial z_i} dz - \int_{\mathbb{R}^N} \eta \sum_{i=1}^N z_i \frac{\partial F}{\partial z_i}(z, \xi) dz = \int_{\mathbb{R}^N} \xi f(z, \xi) \langle z, \nabla \eta \rangle dz - \int_{\mathbb{R}^N} \sum_{i=1}^N \frac{\partial (\eta z_i)}{\partial z_i} F(z, \xi) dz - \int_{\mathbb{R}^N} \eta \sum_{i=1}^N z_i \frac{\partial F}{\partial z_i}(z, \xi) dz. \tag{78}
\]

On the other hand, multiplying both sides of equation (76) by the test function \(\eta \varphi_n\), we obtain

\[
\int_{\mathbb{R}^N} |\nabla \xi|^{p-2} |y|^{ap} \langle \nabla \xi, \nabla (\eta \xi) \rangle dz = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla \xi|^{p-2} |y|^{ap} \langle \nabla \xi, \nabla (\eta \varphi_n) \rangle dz \]
Expanding the left-hand side of the previous equality, it follows that

\[
\int_{\mathbb{R}^N} \eta \frac{|\nabla \xi|^p}{|y|^ap} \, dz = \int_{\mathbb{R}^N} f(z, \xi) \eta \xi \, dz - \int_{\mathbb{R}^N} \xi \frac{|\nabla \xi|^{p-2}}{|y|^ap} \langle \nabla \xi, \nabla \eta \rangle \, dz.
\]  

(79)

Substituting equalities (78) and (79) in equality (73), we find that

\[
\int_{\mathbb{R}^N} \xi f(z, \xi) \langle z, \nabla \eta \rangle \, dz - \int_{\mathbb{R}^N} \sum_{i=1}^N \frac{\partial(\eta z_i)}{\partial z_i} F(z, \xi) \, dz - \int_{\mathbb{R}^N} \eta \sum_{i=1}^N \frac{\partial F}{\partial z_i}(z, \xi) \, dz + \frac{N - p(a + 1)}{p} \int_{\mathbb{R}^N} f(z, \xi) \eta \xi \, dz - \frac{N - p(a + 1)}{p} \int_{\mathbb{R}^N} \xi \frac{|\nabla \xi|^{p-2}}{|y|^ap} \langle \nabla \xi, \nabla \eta \rangle \, dz
\]

\[
= \int_{\mathbb{R}^N} \frac{|\nabla \xi|^{p-2}}{|y|^ap} \xi \langle \nabla \xi, \nabla \eta \rangle \, dz + \int_{\mathbb{R}^N} \frac{|\nabla \xi|^{p-2}}{|y|^ap} \xi \sum_{i,j} z_i \frac{\partial \xi}{\partial z_j} \frac{\partial^2 \eta}{\partial z_j \partial z_i} \, dz
\]

\[
+ \int_{\mathbb{R}^N} \frac{|\nabla \xi|^{p-2}}{|y|^ap} \langle \nabla \xi, \nabla \eta \rangle \langle z, \nabla \xi \rangle \, dz + \frac{p - 1}{p} \int_{\mathbb{R}^N} \frac{|\nabla \xi|^p}{|y|^ap} \langle z, \nabla \eta \rangle \, dz.
\]  

(80)

Now we expand the second term on the left-hand side of the previous equality, and we obtain

\[
\int_{\mathbb{R}^N} \sum_{i=1}^N \frac{\partial(\eta z_i)}{\partial z_i} F(z, \xi) \, dz = \int_{\mathbb{R}^N} \langle \nabla \eta, z \rangle F(z, \xi) \, dz + \int_{\mathbb{R}^N} N \eta F(z, \xi) \, dz.
\]

Substituting the previous equality in equality (80) and reordering the terms, it follows that

\[
\frac{N - p(a + 1)}{p} \int_{\mathbb{R}^N} f(z, \xi) \eta \xi \, dz - \int_{\mathbb{R}^N} N \eta F(z, \xi) \, dz - \int_{\mathbb{R}^N} \eta \sum_{i=1}^N \frac{\partial F}{\partial z_i}(z, \xi) \, dz + \frac{N - p(a + 1)}{p} \int_{\mathbb{R}^N} \xi \frac{|\nabla \xi|^{p-2}}{|y|^ap} \langle \nabla \xi, \nabla \eta \rangle \, dz
\]

\[
= \frac{N - ap}{p} \int_{\mathbb{R}^N} \xi \frac{|\nabla \xi|^{p-2}}{|y|^ap} \langle \nabla \xi, \nabla \eta \rangle \, dz + \int_{\mathbb{R}^N} \frac{|\nabla \xi|^{p-2}}{|y|^ap} \xi \sum_{i,j} z_i \frac{\partial \xi}{\partial z_j} \frac{\partial^2 \eta}{\partial z_j \partial z_i} \, dz
\]

\[
+ \int_{\mathbb{R}^N} \frac{|\nabla \xi|^{p-2}}{|y|^ap} \langle \nabla \xi, \nabla \eta \rangle \langle z, \nabla \xi \rangle \, dz + \frac{p - 1}{p} \int_{\mathbb{R}^N} \frac{|\nabla \xi|^p}{|y|^ap} \langle z, \nabla \eta \rangle \, dz
\]

\[- \int_{\mathbb{R}^N} \xi f(z, \xi) \langle z, \nabla \eta \rangle \, dz + \int_{\mathbb{R}^N} \langle \nabla \eta, z \rangle F(z, \xi) \, dz.
\]  

(81)

Now we must estimate each one of the terms on the right-hand side of equality (81). For the first and second terms, using Hölder’s inequality we obtain

\[
\left| \frac{N - ap}{p} \int_{\mathbb{R}^N} \xi \frac{|\nabla \xi|^{p-2}}{|y|^ap} \langle \nabla \xi, \nabla \eta \rangle \, dz \right| \leq \frac{N - ap}{p} \| \nabla \xi \|^{p-1}_{L_p^p(\text{supp } \nabla \eta)} \| \xi \|_{L_p^p(\mathbb{R}^N)} \| \nabla \eta \|_{L^p(\mathbb{R}^N)}
\]

and

\[
\left| \int_{\mathbb{R}^N} \frac{|\nabla \xi|^{p-2}}{|y|^ap} \xi \sum_{i,j} z_i \frac{\partial \xi}{\partial z_j} \frac{\partial^2 \eta}{\partial z_j \partial z_i} \, dz \right| \leq c \| \nabla \xi \|^{p-1}_{L_p^p(\text{supp } \nabla \eta)} \| \xi \|_{L_p^p(\mathbb{R}^N)} \left\| |z| \sum_{i=1}^N \nabla \left( \frac{\partial \eta}{\partial z_i} \right) \right\|_{L^p(\mathbb{R}^N)}.
\]

For the third and fourth terms, we find

\[
\left| \int_{\mathbb{R}^N} \frac{|\nabla \xi|^{p-2}}{|y|^ap} \langle \nabla \xi, \nabla \eta \rangle \langle \nabla z, \nabla \xi \rangle \, dz \right| \leq \| \nabla \eta \|_{L_p^p(\text{supp } \nabla \eta)} \| \nabla \xi \|_{L_p^p(\text{supp } \nabla \eta)}
\]

and

\[
\left| \frac{p - 1}{p} \int_{\mathbb{R}^N} \frac{|\nabla \xi|^p}{|y|^ap} \langle z, \nabla \eta \rangle \, dz \right| \leq \frac{p - 1}{p} \| \nabla \eta \|_{L_p^p(\text{supp } \nabla \eta)} \| \nabla \xi \|_{L_p^p(\text{supp } \nabla \eta)}.
\]
Finally, for the fifth and sixth terms, we find
\[
\left| \int_{\mathbb{R}^N} (\nabla \eta, z) (F(z, \xi) - \xi f(z, \xi)) \, dz \right| \leq \|\nabla \eta\|_\infty \int_{\text{supp} |\nabla \eta|} |F(z, \xi) - \xi f(z, \xi)| \, dz.
\]

Our objective now is to study the asymptotic behavior of the bounds from above of these several inequalities. To do this we choose an appropriate cut off function \( \eta \). Consider the function \( h \in C^\infty(\mathbb{R}) \) such that \( h|_{ \{ t \leq 1 \} } \equiv 0 \), \( h|_{ \{ t \geq 2 \} } \equiv 1 \) and \( 0 \leq h \leq 1 \). Given \( \epsilon > 0 \) small enough, we define \( \eta_\epsilon(z) = h(|z|/\epsilon) \) if \( |z| \leq 3\epsilon \), \( \eta_\epsilon(z) = h(1/(\epsilon |z|)) \) if \( |z| \geq (2\epsilon)^{-1} \), and \( \eta_\epsilon(z) = 1 \) otherwise. We also choose the function \( h \) such that \|h'\| \leq 2 \; ; \text{with these choices, we have} \eta_\epsilon \in C^\infty(\mathbb{R} \setminus \{0\}) \). Using \( \eta = \eta_\epsilon \), we show that the bounds from above vanish as \( \epsilon \to 0 \).

In the first place, we estimate the term \( |\nabla \eta| \). Since the function \( \eta \) is radial, we use the notation \( |z| = r \). Hence, for \( r < 2\epsilon \) we obtain
\[
\eta_\epsilon(r) = h \left( \frac{r}{\epsilon} \right) \quad \text{and} \quad |\eta_\epsilon'(r)| \leq \left| h' \left( \frac{r}{\epsilon} \right) \right| \frac{1}{\epsilon} \leq \frac{2}{\epsilon}.
\]
In a similar way, if \( r > 1/(2\epsilon) \), then it follows that
\[
\eta_\epsilon(r) = h \left( \frac{1}{r \epsilon} \right) \quad \text{and} \quad |\eta_\epsilon'(r)| \leq \left| h' \left( \frac{1}{r \epsilon} \right) \right| \frac{1}{r \epsilon^2} \leq \frac{2}{r^2 \epsilon}.
\]

Now we estimate the term \( \|\nabla \eta\|_{L^N(\mathbb{R}^N)}^N \). Using the properties of the cut off function \( \eta \), we obtain
\[
\|\nabla \eta\|_{L^N(\mathbb{R}^N)}^N \leq \int_{r < r < 2r} |\nabla \eta|^N \, dv + \int_{1/2r < r < 1} |\nabla \eta|^N \, dv
\leq \frac{2N}{\epsilon^N} \int_{r < r < 2r} \frac{1}{r^{N-1}} \, dr + \frac{2N}{\epsilon^N} \int_{1/2r < r < 1} \frac{1}{r^{2N}} \, dr
= \frac{2^{2+N}}{N} (2^N - 1).
\]
Therefore, \( \|\nabla \eta\|_{L^N(\mathbb{R}^N)}^N \) is finite and does not depend on \( \epsilon \).

Moreover, we have
\[
\left\| \sum_{i=1}^N \left| \nabla \left( \frac{\partial \eta}{\partial z_i} \right) \right| \right\|_{L^N(\mathbb{R}^N)} \leq c \int_{\mathbb{R}^N} \sum_{i=1}^N \left| \nabla \left( \frac{\partial \eta}{\partial z_i} \right) \right| \, dz.
\]
By the properties of the cut off function \( \eta \) this term is finite, as well as the term \( \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} \).

Following up, we show that \( \|\nabla \eta\|^N_{L^1(\text{supp} |\nabla \eta|)} \to 0 \) as \( \epsilon \to 0 \). Since \( \|\nabla \xi\|_{L^1(\mathbb{R}^N)} < +\infty \), we consider balls centered at the origin with radii \( 0 < r_\epsilon < R_\epsilon \) such that \( \text{supp} |\nabla \eta| \subset B_{r_\epsilon}(0) \cup (\mathbb{R}^N \setminus B_{R_\epsilon}(0)) \); for example, we can take \( r_\epsilon = 2\epsilon \) and \( R_\epsilon = 1/2\epsilon \). As \( \epsilon \to 0 \) we obtain \( r_\epsilon \to 0 \) and \( R_\epsilon \to +\infty \); hence,
\[
\int_{\text{supp} |\nabla \eta|} |\nabla \xi| \, dz \leq \int_{B_{r_\epsilon}(0) \cup (\mathbb{R}^N \setminus B_{R_\epsilon}(0))} |\nabla \xi| \, dz \to 0.
\]
Using this same argument, we can show that
\[
\int_{\text{supp} |\nabla \eta|} |F(z, \xi) - \xi f(z, \xi)| \, dz \to 0
\]
as \( \epsilon \to 0 \). As a result, we obtain equality (77). The lemma is proved.

\begin{lemma}
If \( u \in D^{1,p}_a(\mathbb{R}^N \setminus \{ |y| = 0 \}) \cap C^1(\mathbb{R}^N \setminus \{ |y| = 0 \}) \cap W^{1,1}_{\text{loc}}(\mathbb{R}^N \setminus \{ |y| = 0 \}) \) is a weak solution to problem (7) for \( q > 1 \) and \( q \neq p^*(a, b) \), then \( u \equiv 0 \).
\end{lemma}
Proof. To use Lemma 8.1 we need to prove that $u \in L^q_{bp^*(a,b)/q}(\mathbb{R}^N)$. To do this, we use the test function $\eta_k u$ in the weak solution to problem (7), where $\eta_k \in C_c^\infty(\mathbb{R}^N \setminus \{|y| = 0\})$ is the same function defined in the proof of Lemma 8.1. Thus, we have

$$\int_{\mathbb{R}^N} \frac{|
abla u|^{p-2}}{|y|^{ap}} \langle \nabla u, \nabla (\eta_k u) \rangle \, dz - \mu \int_{\mathbb{R}^N} \frac{\eta_k |u|^p}{|y|^{p(a+1)}} \, dz = \int_{\mathbb{R}^N} \frac{\eta_k |u|^{p^*(a,c)}}{|y|^{cp^*(a,c)}} \, dz + \int_{\mathbb{R}^N} \frac{\eta_k |u|^q}{|y|^{bp^*(a,b)}} \, dz. \tag{82}$$

In what follows we estimate the first term on the right-hand side of equality (82), and we obtain

$$\left| \int_{\mathbb{R}^N} \frac{\eta_k |u|^{p^*(a,c)}}{|y|^{cp^*(a,c)}} \, dz \right| \leq \int_{\mathbb{R}^N} \left| \frac{u}{|y|^{p^*(a,c)}} \right|^p \, dz \leq K(N, p, \mu, a, c) \left( \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^{ap}} \, dz - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^{p(a+1)}} \, dz \right) \frac{p^*(a,c)}{p}.$$ We also have

$$\left| \int_{\mathbb{R}^N} \frac{|
abla u|^{p-2}}{|y|^{ap}} \langle \nabla u, \nabla (\eta_k u) \rangle \, dz \right| \leq c + \left( \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^{ap}} \, dz \right)^{\frac{p}{p-1}} \left( \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|y|^{ap^*}} \, dz \right)^{\frac{1}{p^*}} \left( \int_{\mathbb{R}^N} |\nabla \eta|^N \, dz \right)^{\frac{1}{N}},$$

where we have applied Maz’ya’s and Hölder’s inequalities to obtain three factors that are finite and independent from $\epsilon$.

To estimate the second term on the left-hand side of equality (82), we write

$$\left| \mu \int_{\mathbb{R}^N} \frac{\eta_k |u|^p}{|y|^{p(a+1)}} \, dz \right| \leq K(N, p, \mu, a, a + 1) \left( \int_{\mathbb{R}^N} \frac{|
abla u|^p}{|y|^{ap}} \, dz - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^{p(a+1)}} \, dz \right),$$

which is finite because $u \in D^{1,p}_{a}(\mathbb{R}^N \setminus \{|y| = 0\})$. Therefore, passing to the limit as $\epsilon \to 0$, we deduce that $u \in L^q_{bp^*(a,b)/q}(\mathbb{R}^N)$.

Now we can apply Lemma 8.1 to the functions

$$f(z, \xi) = \mu \frac{|\xi|^{p-2}}{|y|^{p(a+1)}} \xi + \frac{|\xi|^{p^*(a,c) - 2}}{|y|^{cp^*(a,c)}} \xi + \frac{|\xi|^{q-2}}{|y|^{bp^*(a,b)}} \xi$$

and

$$F(z, \xi) = \mu \frac{|\xi|^p}{p |y|^{p(a+1)}} + \frac{1}{p^*(a,c)} \frac{|\xi|^{p^*(a,c)}}{|y|^{cp^*(a,c)}} + \frac{1}{q} \frac{|\xi|^q}{|y|^{bp^*(a,b)}}.$$ In this way, from equality (77) we deduce that

$$0 = \left( \frac{N - p(a+1)}{p} - \frac{N}{p} + (a + 1) \right) \mu \int_{\mathbb{R}^N} \frac{|\xi|^p}{|y|^{p(a+1)}} \, dz + \left( \frac{N - p(a+1)}{p} - \frac{N}{p^*(a,c)} + c \right) \int_{\mathbb{R}^N} \frac{|\xi|^{p^*(a,c)}}{|y|^{cp^*(a,c)}} \, dz + \left( \frac{N - p(a+1)}{p} - \frac{N}{q} + \frac{bp^*(a,b)}{q} \right) \int_{\mathbb{R}^N} \frac{|\xi|^q}{|y|^{bp^*(a,b)}} \, dz \leq \left( \frac{1}{p^*(a,b)} - \frac{1}{q} \right) (N - bp^*(a,b)) \int_{\mathbb{R}^N} \frac{|\xi|^q}{|y|^{bp^*(a,b)}} \, dz.$$ By hypothesis we have $q \neq p^*(a,b)$; moreover, $a < (N - p)/p$ implies that $(N - bp^*(a,b)) \neq 0$. Hence, $\|\xi\|_{L^q_{bp^*(a,b)/q}(\mathbb{R}^N)} = 0$, that is, $\xi \equiv 0$. This concludes the proof of the lemma. □

Lemma 8.3. Let $u \in D^{1,p}_{a}(\mathbb{R}^N \setminus \{|y| = 0\})$ be a weak solution to problem (7), where $1 < q < p^*(a,b)$. Then $u \in D^{1,p}_{a}(\mathbb{R}^N \setminus \{|y| = 0\}) \cap C^1(\mathbb{R}^N \setminus \{|y| = 0\}) \cap W^{2,1}_{loc}(\mathbb{R}^N \setminus \{|y| = 0\})$. 
Proof. We begin by writing problem (7) in the form
\[ -\text{div} \left[ \frac{\left| \nabla u \right|^{p-2}}{|y|^{ap}} \nabla u \right] = f(x, u), \]
where
\[ f(x, u) = \mu \frac{|u|^{p-2}u}{|y|^{p(a+1)}} + \frac{|u|^{p^*(a,c)-2}u}{|y|^{p^{*}(a,c)}} + \frac{|u|^{q-2}u}{|y|^{bq(a,b)}}. \]
Hence, for every subset \( \omega \in \mathbb{R}^N \setminus \{|y|=0\} \) there exists a positive constant \( C(\omega) > 0 \) such that
\[ |f(x, u)| \leq C(\omega) \left( \mu \frac{|u|^{p-2}u}{|y|^{p(a+1)}} + \frac{|u|^{q-2}u}{|y|^{bq(a,b)}} \right). \]
In this way, from Theorem 1.2 it follows that \( u \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{|y|=0\}) \). Using a result by Tolksdorf [35], we deduce that \( u \in C^1(\mathbb{R}^N \setminus \{|y|=0\}) \cap W^{2,1}_{\text{loc}}(\mathbb{R}^N \setminus \{|y|=0\}) \). The lemma is proved.

Proof of Theorem 1.3. In the case \( 1 < q < p^*(a,b) \) the proof follows immediately from Lemmas 8.2 and 8.3. In the case \( q > p^*(a,b) \) we cannot apply directly Lemma 8.3 because the conclusion of Theorem 1.2 is not valid. To overcome this difficulty, we suppose additionally that \( u \in L^{q}_{bp^*(a,b)/q,\text{loc}}(\mathbb{R}^N \setminus \{|y|=0\}) \cap L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{|y|=0\}); \) the conclusion follows.

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