TRANSFER OF CHARACTERS FOR DISCRETE SERIES REPRESENTATIONS OF THE UNITARY GROUPS IN THE EQUAL RANK CASE VIA THE CAUCHY-HARISH-CHANDRA INTEGRAL

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Abstract. As conjectured by T. Przebinda, the transfer of characters in the Howe’s correspondence should be obtained via the Cauchy-Harish-Chandra integral. In this paper, we prove that the conjecture holds for the dual pair \((G = U(p, q), G' = U(r, s))\), \(p + q = r + s\), starting with a discrete series representation \(\Pi\) of \(\tilde{U}(p, q)\).

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1. Introduction

Let \(W\) be a finite dimensional vector space over \(\mathbb{R}\) endowed with a non-degenerate, skew-symmetric, bilinear form \(\langle \cdot, \cdot \rangle\), \(\text{Sp}(W)\) be the corresponding group of isometries, \(\tilde{\text{Sp}}(W)\) be the metaplectic cover of \(\text{Sp}(W)\) (see [1, Definition 4.18]) and \((\omega, \mathcal{H})\) be the corresponding Weil representation (see [1, Section 4.8]). For every irreducible reductive dual pair \((G, G')\) in \(\text{Sp}(W)\), R. Howe proved (see [14, Theorem 1]) that there is a bijection between \(\mathcal{R}(\tilde{G}, \omega)\) and \(\mathcal{R}(\tilde{G'}, \omega)\) whose graph is \(\mathcal{R}(\tilde{G} \cdot \tilde{G'}, \omega)\) (where \(\mathcal{R}(\tilde{G}, \omega)\) is defined in Section 2). More precisely, to every \(\Pi \in \mathcal{R}(\tilde{G}, \omega)\), we associate a representation finitely generated admissible representation \(\Pi'\) of \(\tilde{G'}\) which has a unique irreducible quotient \(\Pi'\) such that \(\Pi \otimes \Pi' \in \mathcal{R}(\tilde{G} \cdot \tilde{G'}, \omega)\). We denote by \(\theta : \mathcal{R}(\tilde{G}, \omega) \ni \Pi \to \Pi' = \theta(\Pi) \in \mathcal{R}(\tilde{G'}, \omega)\) the corresponding bijection.

As proved by Harish-Chandra (see [7, Section 5] or Section 3), all the representations \(\Pi\) of \(\tilde{G}\) (resp. \(\Pi'\) of \(\tilde{G'}\)) appearing in the correspondence admit a character, i.e. a \(\tilde{G}\)-invariant distribution \(\Theta_{\Pi}\) on \(\tilde{G}\) (in the sense of Laurent Schwartz) given by a locally integrable function \(\Theta_{\Pi}\) on \(\tilde{G}\) which is analytic on \(\tilde{G}_{\mathbb{R}}\) (the set of regular elements of \(G_{\mathbb{R}}\)).

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The character $\Theta_{\Pi}$ determines the representation $\Pi$. In particular, one way to understand the Howe correspondence, i.e. to make the map $\theta$ explicit, is to understand the transfer of characters.

In his paper [23], T. Przebinda conjectured that the correspondence of characters should be obtained via the so-called Cauchy-Harish-Chandra integral that he introduced in [23]. We recall briefly the construction of this integral. Let $T : \tilde{Sp}(W) \to S'(W)$ be the embedding of the metaplectic group inside the space of tempered distributions on $W$ as in [1] Definition 4.23] (see also Remark 2.2) and $H_1, \ldots, H_n$ be a maximal set of non-conjugate Cartan subgroups of $G$ which are $i$-invariant, where $i$ is a Cartan involution on $G$. Every Cartan subgroup $H_i$ can be decomposed as $H_i = T_iA_i$, with $T_i$ maximal compact in $H_i$. Let $A'_1$ and $A''_1$ be the subgroups of $Sp(W)$ defined by $A'_1 = C_{Sp(W)}(A_1)$ and $A''_1 = C_{Sp(W)}(A'_1)$. One can easily check that $(A'_1, A''_1)$ form a dual pair in $Sp(W)$, which is not irreducible in general. For every function $\varphi \in \mathcal{C}^\infty_c(\hat{A}'_1)$, we define $\text{Chc}(\varphi)$ by

$$\text{Chc}(\varphi) = \int_{A'_1 \backslash W_{A''_1}} T(\varphi)(w)dw,$$

where $dw$ is a measure on the manifold $A'_1 \backslash W_{A''_1}$ defined in [23] Section 1. As mentioned in [23] Section 2] (see also Section 3), $\text{Chc}(\varphi)$ are well-defined and the corresponding map $\text{Chc} : \mathcal{C}^\infty_c(\hat{A}'_1) \to C$ is a distribution on $\hat{A}'_1$. For every regular element $h_i \in \hat{A}'_1$, we denote by $\text{Chc}_h$ the pull-back of $\text{Chc}$ through the map $\hat{G}' \ni \tilde{h} \mapsto \tilde{h} \in \hat{A}'_1$. Assume now that $\text{rk}(G) \leq \text{rk}(G')$. In [3] (see also Section 3, F. Bernon and T. Przebinda defined a map:

$$\text{Chc}^* : \mathcal{D}'(\hat{G}) \to \mathcal{D}'(\hat{G}')$$

where $\mathcal{D}'(\hat{G})$ is the set of $\hat{G}$-invariant distributions on $\hat{G}$. More precisely, if $\Theta$ is a $\hat{G}$-invariant distribution given by a locally integrable function $\Theta$ on $\hat{G}$, then, for every $\varphi \in \mathcal{C}^\infty_c(\hat{G}')$, we get:

$$\text{Chc}^*(\Theta)(\varphi) = \sum_{i=1}^n \frac{1}{|W(H_i)|} \int_{H_i}^{reg} \Theta(h_i) |\det(1 - \text{Ad}(h_i^{-1}))|_{\tilde{h}_i/\text{Chc}_{\tilde{h}_i}}(\varphi) d\tilde{h}_i.$$

The conjecture can be stated as follows:

**Conjecture 1.1.** Let $G_1$ and $G'_1$ be the Zariski identity components of $G$ and $G'$ respectively. Let $\Pi_1 \in \mathcal{A}(\hat{G}, \omega)$ satisfying $\Theta_{\Pi_1} = 0$ if $G = O(V)$, where $V$ is an even dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Then, up to a constant, $\text{Chc}^*(\Theta_{\Pi_1}) = \Theta_{\Pi'}$ on $\hat{G}'$. This result is well-known if the group $G$ is compact and had been proved recently in [24] in the stable range. In this paper, we investigate the case $(G, G') = (U(p, q), U(r, s))$, $p + q = r + s$ (p will always be assumed to be smaller or equal than q), in particular, the number of non-conjugate Cartan subgroups of $G$ is $p + 1$ (see Remark 2) and $\Pi_1 \in \mathcal{A}(\hat{G}, \omega)$ a discrete series representation of $G$. Let $\lambda$ be the Harish-Chandra parameter of $\Pi_1$. In this case, using Li’s result (see [17] Proposition 2.4] or Section 7, we get that $\Pi'_1 = \Pi'$ and using [20], we know that $\Pi'$ is a discrete series representations of $\hat{G}'$ (with Harish-Chandra parameter $\lambda'$), and the correspondence $\lambda \to \lambda'$ is known and explicit (see [20] Theorem 2.7). In order to prove that, up to a constant, $\text{Chc}^*(\Theta_{\Pi_1}) = \Theta_{\Pi'_1} = \Theta_{\Pi}$, we use a parametrisation of discrete series characters provided by Harish-Chandra (see [8] Lemma 44). More precisely, it follows from [3] and a result of Harish-Chandra (see [9] Theorem 2]) that the distribution $\text{Chc}^*(\Theta_{\Pi_1})$ is given by locally integrable function $\Theta_{\Pi'_1}$ analytic on $\hat{G}'^{reg}$. Using [3] Theorem 2.2], we proved in Proposition 4.1 that the value of $\Theta_{\Pi'_1}$ on $\hat{H}'^{reg}$, where $H'$ is the compact Cartan subgroup of $G'$, is of the form:

$$\Delta(h')\Theta_{\Pi'_1}(\tilde{h}') = C \sum_{c \in \mathcal{PC}(\gamma)} \sigma(\alpha)(c \hat{h}')^\gamma, \quad (\tilde{h}' \in \hat{H}'^{reg}).$$


where $C \in \mathbb{R}$, $\tilde{H}'$ is a double cover of $H'$ (see Section 5), chosen such that $\rho' = \frac{1}{2} \sum_{\alpha > 0} \alpha$ is analytic integral, $\hat{\rho}$ is a map from $\tilde{H}'$ into $H'$ (which is not an isomorphism of double covers in general), and $\lambda_{I_1}$ is a linear form on $\tilde{h}'$ depending on $\Pi$ which is conjugated to $\lambda$ under $\mathcal{H}_{R+$. Moreover, using results of \cite{5}, we proved in Proposition 5.7 that

$$\sup_{\tilde{g}'} \left| D(\tilde{g}')^{\chi}(\tilde{g}') \right| < \infty,$$

where $D$ is the Weyl denominator defined in Notations 5.3. Finally, applying \cite{5} Theorem 1.3 to our particular dual pair, it follows that $\varepsilon \text{Chc}^*(\Theta_{I_1}) = \chi_{\lambda_{I_1}}(\varepsilon) \text{Chc}^*(\Theta_{I_1})$ for every $\varepsilon \in \mathbb{Z}(\mathbb{H}(\mathfrak{g}'_{C_{I_1}}))$, where $\chi_{\lambda_{I_1}}$ is the character of $Z(\mathbb{H}(\mathfrak{g}'_{C_{I_1}}))$ obtained via the linear form $\lambda_{I_1}$ as in Remark 5.3 and then, using a result of Harish-Chandra (see \cite{5} Lemma 44) we get that $\text{Chc}^*(\Theta_{I_1})$ is the character of a discrete series representations of $\tilde{G}$ with Harish-Chandra parameter $\lambda_{I_1}$.

In Section 7 we prove, by using results of \cite{22} (see also \cite{17}), that $T(\Theta_{I_1})$ is a well-defined $\tilde{G} \cdot \tilde{G}'$-invariant distribution on $S^*(W)$ and we get in Corollary 7.4 the following equality:

$$T(\tilde{\Theta}_{I_1}) = C_{I_1 \tilde{H}'_I} T(\text{Chc}^*(\Theta_{I_1})),$$

where $C_{I_1 \tilde{H}'_I}$ is a constant depending on $\Pi$ and $\Pi'$. In particular, we can hope that the following diagram often commutes (up to a constant):

$$\xymatrix{ \mathcal{H}(G) \ar[d]_{T} \ar[r]^{\text{Chc}^*} & \mathcal{H}(G') \ar[d]_{T} \ar@<1ex>[l]^G \ar@<-1ex>[l]^{G'} \ar[d]_{\Theta} \ar[l]_{\Theta} \ar@<1ex>[l]^S \ar@<-1ex>[l]^{S'} \ar[r]_{\text{Chc}^*} & \mathcal{H}(G') \ar[d]_{T} \ar@<1ex>[l]^G \ar@<-1ex>[l]^{G'} \ar@<1ex>[l]^S \ar@<-1ex>[l]^{S'} \ar[r]_{\Theta} & \mathcal{H}(G') \ar[l]^G \ar[l]^{G'} \ar[l]^S \ar[l]^{S'} \ar@<1ex>[l]^G \ar@<-1ex>[l]^{G'} \ar@<1ex>[l]^S \ar@<-1ex>[l]^{S'} }$$

Moreover, according to Li’s result (see \cite{17} or Section 7), $\Pi$ can be embedded in $\omega$ as a subrepresentation, and by projecting onto the $\nu \otimes \Pi'$-isotypic component (where $\nu$ is the lowest $K$-type of $\Pi$ as in Theorem 5.4), we get (see Equation 12) the following equality:

$$\text{Chc}^*(\Theta_{I_1})(\varphi) = d_{I_1 \Theta} \int_{\tilde{K}} \int_{\tilde{G}} \Theta_{I_1}(k) \Theta_{I_1}(\tilde{g}) \varphi(\tilde{g}') \omega(k \tilde{g} \tilde{g}') d\tilde{g}' d\tilde{g} dk,$$

where $\varphi \in C_c^\infty(\tilde{G}')$ and $d_{I_1}$ is the formal degree of $\Pi$ (see Remark 5.2).

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2. **Howe correspondence and Cauchy-Harish-Chandra integral**

Let $W$ be a finite dimensional vector space over $\mathbb{R}$ endowed with a non-degenerate, skew-symmetric, bilinear form $(\cdot, \cdot)$. We denote by $\text{Sp}(W)$ the corresponding group of isometries, i.e.

$$\text{Sp}(W) = \{ g \in \text{GL}(W), (g(w), g(w')) = (w, w'), (\forall w, w' \in W) \},$$

and by $\tilde{\text{Sp}}(W)$ the metaplectic group as in [11] Definition 4.18: it’s a connected two-fold cover of $\text{Sp}(W)$. We will denote by $\pi : \tilde{\text{Sp}}(W) \to \text{Sp}(W)$ the corresponding covering map.
We say that a pair of subgroup \((G, G')\) of \(\text{Sp}(W)\) is a dual pair if \(G\) is the centralizer of \(G'\) in \(\text{Sp}(W)\) and vice-versa. The dual pair is said to be reductive if both \(G\) and \(G'\) act reductively on \(W\) and irreducible if we cannot find an orthogonal decomposition of \(W = W_1 \oplus W_2\) where both \(W_1\) and \(W_2\) are \(G \cdot G'\)-invariant. One can easily prove that the preimages \(G = \pi^{-1}(G)\) and \(G' = \pi^{-1}(G')\) in \(\widetilde{\text{Sp}}(W)\) form a dual pair in \(\widetilde{\text{Sp}}(W)\).

Let \((\omega, \mathcal{H})\) be the Weil representation of \(\widetilde{\text{Sp}}(W)\) corresponding to a fixed unitary character of \(\mathbb{R}\) and \((\omega^\infty, \mathcal{H}^\infty)\) be the corresponding smooth representation (see \([1, \text{Section 4.8}]\)). For a subgroup \(\tilde{\Pi}\), we denote by \(\text{N}(\tilde{\Pi})\), for every \(\tilde{\Pi} \leq \Pi\), the preimages \(\tilde{H}\) the set of conjugacy classes of irreducible admissible representations \((\tilde{\Pi}, \mathcal{H}_{\tilde{\Pi}})\) of \(\tilde{H}\) which can be realized as a quotient of \(\mathcal{H}^\infty\) by a closed \(\omega^\infty(\tilde{H})\)-invariant subspace.

As proved by R. Howe (see \([14, \text{Theorem 1}]\)), for every reductive dual pair \((G, G')\) of \(\text{Sp}(W)\), we have a one-to-one correspondence between \(\mathcal{H}(G, \omega)\) and \(\mathcal{H}(G', \omega)\) whose graph is \(\mathcal{H}(G \cdot G', \omega)\). More precisely, if \(\Pi \in \mathcal{H}(G, \omega)\), we denote by \(\text{N}(\Pi)\) the intersection of all the closed \(G\)-invariant subspaces \(\mathcal{N}\) such that \(\Pi \approx \mathcal{H}^\infty / \mathcal{N}\). Then, the space \(\mathcal{H}(\Pi) = \mathcal{H}^\infty / \text{N}(\Pi)\) is a \(G \cdot G'\)-module; more precisely, \(\mathcal{H}(\Pi) = \Pi \otimes \Pi'\), where \(\Pi'\) is a \(G'\)-module, not irreducible in general, but Howe’s duality theorem says that there exists a unique irreducible quotient \(\Pi'\) of \(\Pi'\) with \(\Pi' \in \mathcal{H}(G', \omega)\) and \(\Pi \otimes \Pi' \in \mathcal{H}(G \cdot G', \omega)\).

We will denote by \(\theta : \mathcal{H}(G, \omega) \rightarrow \mathcal{H}(G', \omega)\) the corresponding bijection.

**Notation 2.1.** We use here the notations of \([1]\). We denote by \(S'(W)\) the space of tempered distributions on \(W\) and by

\[
T : \widetilde{\text{Sp}}(W) \rightarrow S'(W)
\]

the injection of \(\widetilde{\text{Sp}}(W)\) into \(S'(W)\) (see \([1, \text{Definition 4.23}]\)). We denote by \(S'(W)\) the subset of \(\text{Sp}(W)\) given by

\[
\{g \in \text{Sp}(W), \det(g - 1) \neq 0\}
\]

and by \(\widetilde{S}'(W)\) its preimage in \(\widetilde{\text{Sp}}(W)\).

**Remark 2.2.** As explained in \([1]\), for every \(\tilde{\varphi} \in \widetilde{S}'(W)\), the distribution \(T(\tilde{\varphi})\) is defined by \(T(\tilde{\varphi}) = \Theta(\tilde{\varphi}) \chi_{c_{\omega}(\mu_{\tilde{\varphi}})}\), where \(\Theta\) is the character of the Weil representation \((\omega, \mathcal{H})\) defined in \([1, \text{Definition 4.23}]\), \(\chi_{c_{\omega}} : W \rightarrow \mathbb{C}\) is the function on \(W\) given by \(\chi_{c_{\omega}}(w) = \chi \left(\frac{1}{2}((g + 1)(g - 1)^{-1}w, w)\right)\) with \(g = \pi(\tilde{\varphi})\) and \(\mu_{\tilde{\varphi}}\) is the appropriately normalized Lebesgue measure on \(W\).

The map \(T\) can be extended to \(\widetilde{\text{Sp}}(W)\) and to \(\mathcal{E}_c^\infty(\widetilde{\text{Sp}}(W))\) by

\[
T(\varphi) = \int_{\widetilde{\text{Sp}}(W)} \varphi(\tilde{g}) T(\tilde{g}) d\tilde{g}, \quad (\varphi \in \mathcal{E}_c^\infty(\widetilde{\text{Sp}}(W))),
\]

where \(d\tilde{g}\) is the Haar measure on \(\widetilde{\text{Sp}}(W)\). As proved in \([1, \text{Section 4.8}]\), for every \(\varphi \in \mathcal{E}_c^\infty(\widetilde{\text{Sp}}(W))\), the distribution \(T(\varphi)\) on \(W\) is given by a Schwartz function on \(W\) still denoted by \(T(\varphi)\), i.e.

\[
T(\varphi)(\phi) = \int_{W} T(\varphi)(w) \phi(w) d\mu_{\tilde{\varphi}}(w), \quad (\phi \in S(W)).
\]
subset $W_{\Lambda'}$, which is $\Lambda'_i$-invariant and such that $\Lambda'_i \setminus W_{\Lambda'}$ is a manifold, endowed with a measure $d\omega$ such that for every $\phi \in C_c^\infty(W)$ such that $\text{supp}(\phi) \subseteq W_{\Lambda'}$,

$$\int_{W_{\Lambda'}} \phi(w)d\mu_{\Lambda'}(w) = \int_{\Lambda'_i \setminus W_{\Lambda'}} \int_{\Lambda'_i} \phi(aw)dad\omega.$$  

For every $\varphi \in C_c^\infty(\Lambda'_i)$, we denote by $\text{Chc}(\varphi)$ the following integral:

$$\text{Chc}(\varphi) = \int_{\Lambda'_i \setminus W_{\Lambda'}} T(\varphi)(d\omega).$$

According to Remark 2.2, the previous integral is well-defined and as proved in [23] Lemma 2.9], the corresponding map $\text{Chc} : C_c^\infty(\Lambda'_i) \to \mathbb{C}$ defines a distribution on $\Lambda'_i$.

For every $\tilde{h}_i \in \hat{H}_i$, we denote by $\tau_{\tilde{h}_i}$ the map:

$$\tau_{\tilde{h}_i} : \tilde{G}' \ni \tilde{g}' \mapsto \tilde{h}\tilde{g}' \in \tilde{\Lambda}'_i$$

and, for $\tilde{h}_i$ regular, by $\text{Chc}_{\tilde{h}_i} = \tau_{\tilde{h}_i}^* (\text{Chc})$, where $\tau_{\tilde{h}_i}^*$ is the pull-back of $\tau_{\tilde{h}_i}$ as defined in [13] Theorem 8.2.4]. In particular, for every $\tilde{h}_i \in \hat{H}_i^{\text{reg}}$, $\text{Chc}_{\tilde{h}_i}$ is a well-defined distribution on $\tilde{G}'$.

3. Explicit formulas of $\text{Chc}$ for unitary groups

Let $V = \mathbb{C}^{n+q}$ and $V' = \mathbb{C}^{r+s}$ be two complex vector spaces endowed with non-degenerate bilinear forms $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ respectively, with $(\cdot, \cdot)$ hermitian and $(\cdot, \cdot)'$ skew-hermitian, and let $(p, q)$ (resp. $(r, s)$) be the signature of $(\cdot, \cdot)$ (resp. $(\cdot, \cdot)'$). We assume that $p + q \leq r + s$. Let $\mathcal{B}_V = \{e_1, \ldots, e_n\}$, $n = p + q$ (resp. $\mathcal{B}_V' = \{e'_1, \ldots, e'_n\}$, $n' = r + s$) be a basis of $V$ (resp. $V'$) such that $\text{Mat}(\cdot, \cdot, \mathcal{B}_V) = \text{Id}_{p,q}$ (resp. $\text{Mat}(\cdot, \cdot, \mathcal{B}_V') = \text{Id}_{r,s}$). Let $G$ and $G'$ be the corresponding group of isometries, i.e.

$$G = G(V, (\cdot, \cdot)) \approx \left\{ g \in \text{GL}(n, \mathbb{C}), \mathcal{B}_V \right\} \right\}, \quad G' = G(V', (\cdot, \cdot)') \approx \left\{ g \in \text{GL}(n', \mathbb{C}), \mathcal{B}_V' \right\} \right\},$$

where $\approx$ is a Lie group isomorphism.

Let $H$ and $H'$ be the diagonal compact Cartan subgroups of $G$ and $G'$ respectively. By looking at the action of $H$ on the space $V$, we get a decomposition of $V$ of the form:

$$V = V_1 \oplus \ldots \oplus V_n,$$

where the spaces $V_a$ given by $V_a = \mathbb{C}e_a$ are irreducible $H$-modules. We denote by $J$ the element of $\mathfrak{h}$ such that $J = i\text{Id}_V$ and let $J_j = iE_{j,j}$. Similarly, we write

$$V' = V'_1 \oplus \ldots \oplus V'_{n'}$$

with $V'_a = \mathbb{C}e'_a$, $J'$ the element of $\mathfrak{h}'$ given by $J' = i\text{Id}_{V'}$ and $J'_j = iE_{j,j}$. Let $W = \text{Hom}_\mathbb{C}(V', V)$ endowed with the symplectic form $(\cdot, \cdot)$ given by:

$$\langle w_1, w_2 \rangle = \text{tr}_{\mathcal{B}_V}(w^*_2 w_1), \quad (w_1, w_2 \in W),$$

where $w^*_2$ is the element of $\text{Hom}(V, V')$ satisfying:

$$\langle w^*_2(v'), v \rangle = (v', w_2(v))' \quad (v \in V, v' \in V').$$

The space $W$ can be seen as a complex vector space by

$$(1) \quad iw = J \circ w \quad (w \in W).$$
We define a double cover \( \tilde{\text{GL}}_C(W) \) of the complex group \( \text{GL}_C(W) \) by:

\[
\tilde{\text{GL}}_C(W) = \left\{ \tilde{g} = (g, \xi) \in \text{GL}_C(W) \times C^\times, \xi^2 = \det(g) \right\}.
\]

Because \( p + q \leq r + s \), we get a natural embedding of \( \mathfrak{b}_C \) into \( \mathfrak{b}_C' \) and we denote by \( Z' = G' \) the centralizer of \( \mathfrak{h} \) in \( G' \). Let \( \tilde{H}' \) be the complexification of \( H' \) in \( \tilde{\text{GL}}_C(W) \). In particular, \( \tilde{H}' \) is isomorphic to

\[
\mathfrak{h}_C'/\left\{ \sum_{j=1}^q 2\pi i x_j, x_j \in \mathbb{Z} \right\}.
\]

We denote by \( \tilde{H}' \) the connected two-fold cover of \( H_C \) isomorphic to

\[
\mathfrak{h}_C'/\left\{ \sum_{j=1}^q 2\pi i x_j, \sum_{j=1}^q x_j \in 2\mathbb{Z}, x_j \in \mathbb{Z} \right\}.
\]

Let \( p : \tilde{H}' \rightarrow H_C \) the corresponding covering map. If \( \tilde{H}'_C \) is isomorphic to \( H_C \), we may choose an isomorphism \( \tilde{p} : \tilde{H}'_C \rightarrow H_C \) so that \( p = \tilde{p} \circ \tilde{p} \). Otherwise, \( \tilde{H}'_C \) coincides with the direct product \( H_C \times \{ \pm 1 \} \). In this case, we can define \( \tilde{p} : \tilde{H}'_C \rightarrow H_C \) to be the composition of \( p \) with the inclusion \( H_C \rightarrow H_C \times \{ \pm 1 \} \). Then, again \( p = \tilde{p} \circ \tilde{p} \).

**Remark 3.1.** (1) Let \( \Psi' := \Psi'((g_C', b_C'), \Psi'(t')) \) be a set of positive roots corresponding to \((1_C', b_C'), \Psi'(t') \) be a set of compact roots in \( \Psi' \), where \( t \) is the Lie algebra of \( K = U(r) \times U(s) \), and \( \Psi'_n \) be the set of non-compact roots of \( \Psi' \), i.e. \( \Psi'_n = \Psi' \setminus \Psi' \). The reason why we are considering the double cover \( \tilde{H}'_C \) of \( H'_C \) is to make the form \( \rho' = \frac{1}{4} \sum_{\alpha \in \Psi'_{\text{aff}}} \alpha \) analytic integral. For every analytic integral form \( \gamma \) on \( b'_C \), we will denote by \( h' \rightarrow h' \gamma \) the corresponding character on \( H'_C \).

(2) We know that, up to conjugation, the number of Cartan subgroups in \( U(r, s) \) is \( \min(r, s) + 1 \). Those Cartan subgroups can be parametrized by some particular subsets of \( \Psi'_n \). Let \( \Psi'_{\text{aff}} \) be the set of strongly orthogonal roots in \( \Psi' \) (see \([25, \text{Section 2}] \)).

For every \( \alpha \in \Psi'_{\text{aff}} \), we denote by \( c(\alpha) \) the element of \( G'_C \) given by:

\[
c(\alpha) = \exp \left( \frac{\pi}{4} (X_\alpha - X_{-\alpha}) \right),
\]

where \( X_\alpha \) (resp. \( X_{-\alpha} \)) is in \( g'_{C,\alpha} \) (resp. \( g'_{C,-\alpha} \)) and normalized as in \([25, \text{Equation 2.7}] \). For every subset \( S \) of \( \Psi'_{\text{aff}} \), we denote by \( c(S) \) the following element of \( G'_C \), defined by

\[
c(S) = \prod_{\alpha \in S} c(\alpha),
\]

and let

\[
b'(S) = g' \cap \text{Ad}(c(S))(b'_C).
\]

We denote by \( H'(S) \) the analytic subgroup of \( G' \) whose Lie algebra is \( b'(S) \). Then, \( H'(S) \) is a Cartan subgroup of \( G' \) and one can prove that all the Cartan subgroups are of this form (up to conjugation).

For every \( S \subseteq \Psi'_{\text{aff}} \), we will denote by \( H'_S \) the subgroup of \( H'_C \) given by:

\[
H'_S = c(S)^{-1} H'(S)c(S).
\]

Assume that \( r \leq s \). Then, we define \( \Psi'' = \left\{ e_i - e_j, 1 \leq i < j \leq r + s \right\} \), where \( e_i \) is the linear form on \( b'_C = C^{r+s} \) given by \( e_i(\lambda_1, \ldots, \lambda_{r+s}) = \lambda_i \). In this case, the set \( \Psi'' \) is equal to \( \left\{ e_i - e_{i+1}, 1 \leq t \leq r \right\} \). In particular, \( H''(0) = H' \) and if \( S_t = \{ e_1 - e_{r+1}, \ldots, e_t - e_{r+t} \} \), we get:

(2) \( H''_S = \left\{ h = \text{diag}(e^{X_1-Z_{r+1}}, \ldots, e^{X_1-Z_{r+s}}, e^{X_{r+1}}, \ldots, e^{X_r+X_{r+1}}, \ldots, e^{X_r+X_{r+s}}, e^{X_{r+s}}, \ldots, e^{X_{r+s}}), X_j \in \mathbb{R} \right\} \).
Fix a subset $S \in \Psi_n^{\text{reg}}$. We denote by $\tilde{H}_S$ the preimage of $H_S$ in $\tilde{H}_\mathbb{C}$. For every $\varphi \in \mathcal{C}_c^\infty(\tilde{G}')$, we denote by $\mathcal{H}_S \varphi$ the function of $\tilde{H}_S$ defined by:

$$\mathcal{H}_S \varphi(\tilde{h}') = \psi_{\Psi, \varphi}(\tilde{h}') \tilde{h}' \frac{\sum_{\alpha \in \Psi} (1 - \tilde{h}'^{-\alpha})}{\int_{G'/H(S)} \varphi(g' \cdot \tilde{h}') \tilde{h}' \cdot c(S)^{-1} g^{-1}) dg'}(\tilde{h}') \quad (\tilde{h}' \in \tilde{H}_S),$$

where $\Psi_{S, \varphi}$ is the subset of $\Psi'$ consisting of real roots for $H_S$ and $\psi_{\Psi, \varphi}$ is the function defined on $\tilde{H}_S^{\text{reg}}$ by

$$\psi_{\Psi, \varphi}(\tilde{h}') = \left( \prod_{\alpha \in \Psi} (1 - \tilde{h}'^{-\alpha}) \right).$$

To simplify the notations, we denote by $\Delta_{\varphi}(\tilde{h}')$ the quantity

$$\Delta_{\varphi}(\tilde{h}') = \frac{\tilde{h}'^{-1} \sum_{\alpha \in \Psi} (1 - \tilde{h}'^{-\alpha})}{(\tilde{h}')^{1 - \alpha}} \quad (\tilde{h}' \in \tilde{H}_S).$$

We define $\Delta_{\varphi}$ similarly, where $\Phi' = -\Psi'$.

**Remark 3.2.**
1. For every $\tilde{h}' \in \tilde{H}_S^{\text{reg}}$,

$$\Delta_{\varphi}(\tilde{h}') \Delta_{\varphi}(\tilde{h}') = \prod_{\alpha \in \Psi} (1 - \tilde{h}'^{-\alpha})(1 - \tilde{h}'^{-\alpha}).$$

Note that if $S = \emptyset$, we get for every $\alpha \in \Psi'$ and $\tilde{h}' \in \tilde{H}$ that $\tilde{h}'^{-\alpha} = \tilde{h}'^{\alpha}$. In particular, $\Delta_{\varphi}(\tilde{h}') \Delta_{\varphi}(\tilde{h}') = \prod_{\alpha \in \Psi} (1 - \tilde{h}'^{-\alpha})(1 - \tilde{h}'^{\alpha}) = \prod_{\alpha \in \Psi} [1 - \tilde{h}'^{-\alpha}]^2 = |\det(\text{Id} - \text{Ad}(\tilde{h}'))_g|$. Similarly, if $S \neq \emptyset$, we get for every $\alpha \in \Psi'$ and $\tilde{h}' \in \tilde{H}'$, there exists $\beta \in \Phi'$, independant on $\tilde{h}'$, such that $\tilde{h}'^{\alpha} = \tilde{h}'^{\beta}$. In particular, we get $\Delta_{\varphi}(\tilde{h}') \Delta_{\varphi}(\tilde{h}') = \prod_{\alpha \in \Psi} [1 - \tilde{h}'^{-\alpha}]^2$.

For every $\tilde{h}' \in \tilde{H}_S$, we denote by $|\Delta_{\varphi}(\tilde{h}')|^2 = \Delta_{\varphi}(\tilde{h}') \Delta_{\varphi}(\tilde{h}')$.

2. One can easily check that two Cartan subalgebras $\mathfrak{h}'(S_1)$ and $\mathfrak{h}'(S_2)$, with $S_1, S_2 \subseteq \Psi_n^{\text{reg}}$, are conjugate if and only if there exists an element of $\sigma \in \mathcal{W}$ sending $S_1 \cup (-S_1)$ onto $S_2 \cup (-S_2)$ (see [25 Proposition 2.16]).

The Weyl’s integration formula can be written with the previous notations as follows:

**Proposition 3.3 (Weyl’s Integration Formula).** For every $\varphi \in \mathcal{C}_c^\infty(\tilde{G}')$, we get:

$$\int_{\tilde{G}'} \varphi(\tilde{g}') d\tilde{g}' = \sum_{S \subseteq \Psi_n^{\text{reg}}} m_S \int_{\tilde{H}_S} \psi_{\Psi, \varphi}(\tilde{h}') \Delta_{\varphi}(\tilde{h}') \mathcal{H}_S \varphi(\tilde{h}') d\tilde{h}' .$$

where $m_S$ are complex numbers. Here, the subsets $S$ of $\Psi_n^{\text{reg}}$ are defined up to equivalence (see Remark 3.2).

**Proof.** See [3] Section 2, Page 3830].

**Remark 3.4.** In particular, if we fix $S \subseteq \Psi_n^{\text{reg}}$ and $\varphi \in \mathcal{C}_c^\infty(\tilde{G}')$ such that $\text{supp}(\varphi) \subseteq \tilde{G}' \cdot H(S)^{\text{reg}}$, the previous formula can be written as follow:

$$\int_{\tilde{G}'} \varphi(\tilde{g}') d\tilde{g}' = m_S \int_{\tilde{H}_S} \psi_{\Psi, \varphi}(\tilde{h}') \Delta_{\varphi}(\tilde{h}') \mathcal{H}_S \varphi(\tilde{h}') d\tilde{h}' .$$
Let $H_C, G_C \subseteq GL_C(W)$ the complexifications of $H$ and $G'$. We denote by $G_C^0$ the subgroup of $G_C$ consisting of elements commuting with the element $i$ introduced in Equation (1).

As proved in [3, Section 2], the character $\Theta$ defined in [11, Definition 4.23] extends to a rational function on $\tilde{H}_C \cdot \tilde{G}_C^0$ given by

$$\Theta(\tilde{h}\tilde{g'}) = (-1)^{2\epsilon} \frac{\det^\ast(\tilde{h}\tilde{g'})}{\det(1 - \tilde{h}\tilde{g'})}, \quad (\tilde{h} \in \tilde{H}_C, \tilde{g'} \in \tilde{G}_C^0),$$

where $2\epsilon$ is the maximal dimension of a real subspace of $W$ on which the symmetric form $(J, \cdot)$ is negative definite. More precisely, according to [3, Proposition 2.1], we get:

**Proposition 3.5.** For every $\tilde{h} \in \tilde{H}_C$ and $\tilde{h'} \in \tilde{H}_C'$, we get:

$$\det^\ast(\tilde{h})\omega_0 \Delta_\varphi(\tilde{h})\Theta(\tilde{h}\tilde{g'})\Delta_\Phi(\tilde{h'}) = \sum_{\sigma \in \Psi(\tilde{H}_C)} (-1)^{\nu_0} \frac{\sign(\sigma) \det^\ast(\sigma^{-1}(\tilde{h'}))_{\Psi(\tilde{Z})} \Delta_{\Phi}(\sigma^{-1}(\tilde{h'}))}{\det(1 - p(h)p(h'))_{\epsilon\omega_0}}.$$

where $\alpha \in (0, -1)$ depends only on the choice of the positive roots $\Psi$ and $\Phi'$, $k \in \{0, 1\}$ is defined by

$$k = \begin{cases} -1 & \text{if } n' - n \in 2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and $\omega^b$ is the set of elements of $W$ commuting with $b$.

**Remark 3.6.** One can easily check that the space $\omega^b$ is given by

$$\omega^b = \sum_{i=1}^{n} \text{Hom}(V'_i, V_i).$$

For every $S \subseteq \Psi'_n$, we denote by $S$ the subset of $\{1, \ldots, r + s\}$ given by $S = \{j : \exists \alpha \in S \text{ such that } \alpha(J'_j) \neq 0\}$. Let $\sigma \in \mathcal{J}_n$ and $S \subseteq \Psi'_n$, we denote by $\Gamma_{\sigma, S}$ the subset of $b'$ defined as

$$\Gamma_{\sigma, S} = \left\{ Y \in b', (\langle \cdot, \cdot \rangle_{\epsilon\omega_0})_{\Sigma} \text{Hom}(V'_i, V_i) > 0 \right\},$$

and let $E_{\sigma, S} = \exp(\Gamma_{\sigma, S})$ the corresponding subset of $\tilde{H}_C'$, where $\exp$ is a choice of exponential map $\exp : \tilde{h}_C' \to \tilde{H}_C'$ obtained by choosing an element $1$ in $\pi^{-1}(1)$.

**Theorem 3.7.** For every $\tilde{h} \in \tilde{H} = \tilde{H}_0$ and $\varphi \in \mathcal{C}(\tilde{G})$, we get:

$$\det^\ast(\tilde{h})\omega_0 \Delta_\varphi(\tilde{h})\int_{\tilde{G}} \Theta(\tilde{h}\tilde{g'})\varphi(\tilde{g'})d\tilde{g'} = \sum_{\sigma \in \Psi'_{n, S}} \sum_{\mathcal{M}_S(\sigma)} \lim_{r \to 1} \int_{\tilde{H}_C'} \frac{\det^\ast(\sigma^{-1}(\tilde{h}'))_{\omega_0} \Delta_\Phi(\sigma^{-1}(\tilde{h}'))}{\det(1 - p(h)p(h'))_{\epsilon\omega_0}} e_{\Phi_S}(\tilde{h}')(\varphi(\tilde{h}'))d\tilde{h'},$$

where $\mathcal{M}_S(\sigma) = \frac{(-1)^{\nu_0} \varphi(\sigma)m_S}{\omega'(\tilde{Z}^r_C', H_C')}$.

The theorem 3.7 tells us how to compute $\text{Ch}_C$ for an element $\tilde{h}$ in the compact Cartan $H = H(0)$. Using [2], it follows that the value of $\text{Ch}$ on the other Cartan subgroups can be computed explicitly by knowing how to do it for the compact Cartan (we will assume, without loss of generality, that $p \leq q$, in particular, the number of Cartan subgroups of $G$, up to conjugation, is $p + 1$).
To simplify, we assume that \( r \in [26, \text{Section 2.3.6}] \). In particular, \( H(S_k) \) such that \( T(S_k) \) respectively and let \( H(S_i) \) denote by \( T \).

For every \( \alpha \) where \( C \) is a constant defined in \([2, \text{Theorem 0.9}]\), \( \epsilon \) and \( d \)

\[ S_i = \begin{cases} \{e_1 - e_{a+1}, \ldots, e_i - e_{a+1}\} & \text{if } r \leq s, \\ \{e_1 - e_{\beta+1}, \ldots, e_i - e_{\beta+1}\} & \text{otherwise} \end{cases} \]

where \( \alpha = \begin{cases} p & \text{if } r \leq p \\ r & \text{otherwise} \end{cases} \) and \( \beta = \begin{cases} p & \text{if } s \leq p \\ s & \text{otherwise} \end{cases} \).

For every \( i \in [0, p] \) and \( j \in [0, \min(r, s)] \), we denote by \( H(S_i) \) and \( H'(S_j) \) the Cartan subgroups of \( G \) and \( G' \) respectively and let \( H(S_i) = T(S_i)A(S_i) \) and \( H'(S_j) = T'(S_j)A'(S_j) \) be the decompositions of \( H(S_i) \) and \( H'(S_j) \) as in \([26, \text{Section 2.3.6}]\). In particular, \( H(S_k) = H'(S_k) \) for every \( k \in [0, \min(p, \min(r, s))] \).

To simplify, we assume that \( r \leq s \). We denote by \( V_{U,j} \) the subspace of \( V \) on which \( A(S_i) \) acts trivially and by \( V_{U,j} \) the orthogonal complement of \( V_{U,j} \) in \( V \). Let \( V_{U,j} = X_i \oplus Y_i \) be a complete polarization of \( V_{U,j} \). We assume that we have a natural embedding of \( V_{U,j} \) into \( V' \) such that \( X_i \oplus Y_i \) is a complete polarization with respect to \( (\cdot, \cdot) \) (i.e., \( i \leq r \)). Let \( U_j \) be the orthogonal complement of \( V_{U,j} \) in \( V' \); in particular, we get a natural embedding:

\[ \text{GL}(X_i) \times \text{G}(U_j) \subseteq G' = U(r, s). \]

We denote by \( T_1(S_i) \) the maximal subgroup of \( T(S_i) \) which acts trivially on \( V_{U,j} \) and let \( T_2(S_i) \) the subgroup of \( T(S_i) \) such that \( T(S_i) = T_1(S_i) \times T_2(S_i) \) with \( T_2(S_i) \subseteq G(V_{U,j}) \). In particular,

\[ H(S_i) = T_1(S_i) \times A(S_i) \times T_2(S_i). \]

Similarly, we get a decomposition of \( H'(S_i) \) as the form:

\[ H'(S_i) = T_1'(S_i) \times A'(S_i) \times T_2'(S_i). \]

Let \( \eta(S_i) \) and \( \eta'(S_i) \) be the nilpotent Lie subalgebras of \( u(p, q) \) and \( u(r, s) \) respectively given by

\[ \eta(S_i) = \text{Hom}(X_i, V_{U,j}) \oplus \text{Hom}(X_i, Y_i) \cap u(p, q), \quad \eta'(S_i) = \text{Hom}(U_j, X_i) \oplus \text{Hom}(X_i, Y_i) \cap u(r, s). \]

We will denote by \( W_{U,j} \) the subspace of \( W \) defined by \( \text{Hom}(U_i, V_{U,j}) \) and by \( P(S_i) \) and \( P'(S_i) \) the parabolic subgroups of \( G \) and \( G' \) respectively whose Levi factors \( L(S_i) \) and \( L'(S_i) \) are given by

\[ L(S_i) = \text{GL}(X_i) \times \text{G}(V_{U,j}), \quad L'(S_i) = \text{GL}(X_i) \times \text{G}(U_j), \]

and by \( N(S_i) := \exp(\eta(S_i)) \) and \( N'(S_i) := \exp(\eta'(S_i)) \) the unipotent radicals of \( P(S_i) \) and \( P'(S_i) \) respectively.

**Remark 3.9.** One can easily check that the forms on \( V_{U,j} \) and \( U_j \) have signature \( (p - i, q - i) \) and \( (r - i, s - i) \) respectively.

As proved in \([2, \text{Theorem 0.9}]\), for every \( \bar{h} = \tilde{t}_1 \tilde{a}_2 \in \bar{H}(S_i)^{\text{reg}} \) (using the decomposition of \( H(S_i) \) given in Equation (5)) and \( \varphi \in \mathfrak{g}'^\circ(G') \), we get:

\[ | \det(\text{Ad}(\bar{h}) - \text{Id})_{\eta(S_i)}|_{\text{Chc}(\varphi)} = \int_{\text{GL}(X_i)/T_1(S_i)A(S_i)} \int_{\text{G}(U_j)} \sigma(\tilde{t}_1 \tilde{a}_2 \tilde{y}) \text{Chc}_{w_0}(\tilde{t}_2 \tilde{y}) d'_{S_i}((\tilde{g}, \tilde{a}^{-1} \tilde{g}^{-1})) d'_{N(S_i)}((\tilde{g}, \tilde{a}^{-1} \tilde{g}^{-1})) d' \tilde{y} d' \tilde{g}, \]

where \( C \) is a constant defined in \([2, \text{Theorem 0.9}]\), \( \sigma \) is the character defined in \([2, \text{Lemma 6.3}]\), \( d_{S_i} : L(S_i) \to \mathbb{R} \) and \( d'_{S_i} : L'(S_i) \to \mathbb{R} \) are given by

\[ d_{S_i}(\bar{t}) = |\det(\text{Ad}(\tilde{t})_{\eta(S_i)})|^4, \quad d'_{S_i}(\bar{t}') = |\det(\text{Ad}(\tilde{t}')_{\eta'(S_i)})|^4, \quad (\tilde{t} \in \bar{L}(S_i), \tilde{t}' \in \bar{L}'(S_i)). \]
and \( \varphi_{N(S_i)}^{\widetilde{K}} \) is the Harish-Chandra transform of \( \varphi \), i.e. the function on \( \widetilde{L}(S_i) \) defined by:

\[
\varphi_{N(S_i)}^{\widetilde{K}}(P) = \int_{N(S_i)} \int_{\widetilde{K}} \varphi(k \tilde{n} h) dk dh, \quad (P \in \widetilde{L}(S_i)).
\]

One can easily check that \((G(V_{0,i}), G(U_i))\) is an irreducible dual pair in \( \text{Sp}(W_{0,i}) \) of the same type of \( G, G' \). Moreover, the element \( \tilde{t}_2 \) is contained in the compact Cartan of \( G(V_{0,i}) \). In particular, it follows from Theorem 3.17 that the integral

\[
\int_{G(U_i)} \varepsilon(\tilde{t}_2 \tilde{\gamma}) \text{Ch}_{\omega_g}(\tilde{t}_2 \tilde{\gamma}) d\tilde{\gamma} = \int_{N(S_i)} \int_{\widetilde{K}} \varphi(k \tilde{h} \tilde{\gamma}) g \tilde{h}^{-1} \tilde{\gamma} d\tilde{h} \tilde{\gamma}, \quad (P \in \widetilde{L}(S_i)).
\]

can be seen as a finite sum of integrals, where the test function \( \varphi \) is replaced by \( \varepsilon(\tilde{\gamma}) g \tilde{h}^{-1} \tilde{\gamma} \).

\[\text{Remark 3.11.} \quad \text{Assume that } j \in \{1, \ldots, n \} \text{ and } k \in \{0, \ldots, p - j \} \text{, we denote by } S^k_j = [e_{jr} - e_{pr} - e_{rj}] \text{ the subset of } \Psi^e_0(G(V_{0,i}), \tau_2(S_i)_{C}) \text{ and by } H(S^k_j) \text{ the corresponding Cartan subgroup of } G(V_{0,i}). \text{ By convention, } H(S^0_j) = T_2(S_i) \text{ is the compact Cartan subgroup of } G(U_i). \]

Assume that \( r \leq s \). For \( j \in \{1, \ldots, r \} \text{ and } k \in \{0, \ldots, r - j \} \text{, we denote by } S^k_j = [e_{jr} - e_{pr} - e_{rj}] \text{ the subset of } \Psi^e_0(g(U_{i}), \tau_2(S_i)_{C}) \text{ and by } H(S^k_j) \text{ the corresponding Cartan subgroup of } G(U_i). \text{ By convention, } H(S^k_j) = T_2(S_i) \text{ is the compact Cartan subgroup of } G(U_i).

\[\text{Notation 3.10.} \quad \text{For every } j \in \{1, \ldots, p \} \text{ and } k \in \{0, \ldots, p - j \}, \text{ we denote by } S^k_j = [e_{jr} - e_{pr} - e_{rj}] \text{ the subset of } \Psi^e_0(g(V_{0,i}), \tau_2(S_i)_{C}) \text{ and by } H(S^k_j) \text{ the corresponding Cartan subgroup of } G(V_{0,i}). \text{ By convention, } H(S^0_j) = T_2(S_i) \text{ is the compact Cartan subgroup of } G(U_i).
\]

We finish this section with a technical lemma which will be useful in Section 6.

Lemma 3.12. For every \( f \in \mathcal{E}_c(G'') \), we get:

\[
\int_{GL(X_j)/T(S_i) \times A'(S_j)} f_{N(S_j)}^{\widetilde{K}}(g_1 g_2 h g_2^{-1} g_1^{-1}) dg_2 dg_1 = \frac{D_{L(S_j)}(h)}{D_{L(S_j)}(\tilde{h})} \int_{GL(X_j)/T(S_i) \times A'(S_j)} \int_{G(U_i)/T(S_i)} f_{N(S_j)}^{\widetilde{K}}(g_1 g_2 h g_2^{-1} g_1^{-1}) dg_2 dg_1,
\]

\[\text{and } \varphi_{N(S_j)}^{\widetilde{K}} \text{ is the Harish-Chandra transform of } \varphi \text{, i.e. the function on } \widetilde{L}(S_j) \text{ defined by:}
\]

\[
\varphi_{N(S_j)}^{\widetilde{K}}(P) = \int_{N(S_j)} \int_{\widetilde{K}} \varphi(k \tilde{n} h) dk dh, \quad (P \in \widetilde{L}(S_j)).
\]
where $D_{L(S)}$ and $D_{L'(S)}$ are given by:

$$D_{L(S)}(\tilde{h'}) = |\det(\text{Id} - \text{Ad}(\tilde{h'})^{-1})|_{H'/V(S)}|^{\frac{1}{2}}, \quad D_{L'(S)}(\tilde{h'}) = |\det(\text{Id} - \text{Ad}(\tilde{h'})^{-1})|_{V(S)/V(S')}|^{\frac{1}{2}}.$$ 

**Proof.** As explained in [2, Appendix A], we have:

$$\int_{G/H(S)} f(\tilde{g}\tilde{h}^{-1})dg = \frac{D_{L(S)}(\tilde{h})}{D_{L'(S)}(\tilde{h})} \int_{H'/V(S)} f^{K}_{N(S)}(f\tilde{h}^{-1})df'$$

$$= \frac{D_{L(S)}(\tilde{h})}{D_{L'(S)}(\tilde{h})} \int_{\text{GL}(X)/T_{1}(S)\times \text{A}(S)} \int_{G(U)/H(S)} f^{K}_{N(S)}(g_{1}g_{2}\tilde{g}\tilde{h}^{-1}g_{1}^{-1})dg_{2}dg_{1},$$

where $D_{L(S)}(\tilde{h}) = D_{G}(\tilde{h'}) = |\det(\text{Ad}(\tilde{h'})^{-1} - \text{Id})|_{H'/V(S)}|^{\frac{1}{2}}$. Similarly, using that $H'(S_{i}) \subseteq P'(S_{j})$, we get:

$$\int_{G/H'(S_{j})} f(\tilde{g}\tilde{h}^{-1})dg = \frac{D_{L'(S)}(\tilde{h})}{D_{L'(S)}(\tilde{h})} \int_{H'/V(S)} f^{K}_{N(S)}(f\tilde{h}^{-1})df'$$

$$= \frac{D_{L'(S)}(\tilde{h})}{D_{L'(S)}(\tilde{h})} \int_{\text{GL}(X)/T_{1}(S)\times \text{A}(S)} \int_{G(U)/H(S)} f^{K}_{N(S)}(g_{1}g_{2}\tilde{g}\tilde{h}^{-1}g_{1}^{-1})dg_{2}dg_{1},$$

and the lemma follows. 

\[ \square \]

### 4. Transfer of invariant eigendistributions

We start this section by recalling the notion of invariant eigendistributions. We keep the notations of Appendix A. Let $G$ be a connected real reductive Lie group, $\mathcal{D}'(G)$ be the space of distributions of $G$, i.e., the continuous linear forms on $\mathcal{C}^\infty_c(G)$ and $D^0_G(G)$ the space of bi-invariant differential operators on $G$ as in Notations A.3. For every $f \in \mathcal{C}^\infty_c(G)$ and $g \in G$, we denote by $f^g$ the function of $\mathcal{C}^\infty_c(G)$ defined by $f^g(x) = f(g^{-1}x), x \in G$. We say that $T \in \mathcal{D}'(G)$ is a $G$-invariant distribution if $T(f^g) = T(f)$ for every $f \in \mathcal{C}^\infty_c(G)$ and $g \in G$.

**Definition 4.1.** A distribution $T$ on $G$ is an eigendistribution if there exists an algebra homomorphism $\chi_T : D^0_G(G) \rightarrow \mathbb{C}$ such that $D(T) = \chi_T(D)T$ for every $D \in D^0_G(G)$.

As proved by Harish-Chandra (see [9, Theorem 2]), for every invariant eigendistribution $T$ on $G$, there exists a locally integrable function $f_T$ on $G$ which is analytic on $G^\text{reg}$ such that $T = f_T$, i.e., for every $\varphi \in \mathcal{C}^\infty_c(G)$,

$$T(\varphi) = \int_G f_T(g)\varphi(g)dg.$$ 

**Remark 4.2.** (1) Using the isomorphism defined in Appendix A, Theorem A.6, an eigendistribution $T$ is an invariant distribution such that there exists a character $\chi_T$ of $Z(\mathfrak{g}(\mathfrak{g}))$ such that $T = \chi_T(z)T$ for every $z \in Z(\mathfrak{g}(\mathfrak{g}))$.

(2) Let $(\Pi, \mathcal{H})$ be a representation of $G$. Following [6], we say that the representation $\Pi$ is permissible if $\Pi(z)$ is a scalar multiple of the unit operator for every $z \in Z(G) \cap D$, where $D$ is the analytic subgroup of $G$ corresponding to $Z(l)$ (l being the Lie algebra of a maximal compact subgroup $\mathbf{K}$ of $G$). A permissible representation is said quasi-simple if there exists an homomorphism $\chi$ of $Z(\mathfrak{g}(\mathfrak{g}))$ into $\mathbb{C}$ such that $d\Pi(z)(\eta) = \chi(z)\eta$ for every $z \in Z(\mathfrak{g}(\mathfrak{g}))$ and $\eta$ in the Garding space $\text{Gar}(\Pi, \mathcal{H})$ (for the definition of $\text{Gar}(\Pi, \mathcal{H})$, see [6, Part II]). In particular, for such representations, Harish-Chandra proved that for every $\varphi \in \mathcal{C}^\infty_c(G)$, the operator $\Pi(\varphi)$ is a trace class operator (see [7, Section 5]) and the corresponding map
For every reductive group G, we denote by $C_{\infty}^0(H(G))$. This space is endowed with a natural topology (see [4, Section 3.3]). We denote by $J_{\infty}$. As proved in [4, Theorem 3.2.1], the map:

$$J_{\infty}: C_{\infty}^0(G) \to \mathcal{I}(G)$$

is well-defined and surjective. We denote by $\mathcal{I}^t(G)$ the transpose of $J_{\infty}$ defined by

$$J_{\infty}^t(T)(\varphi) = T(J_{\infty}(\varphi)), \quad (T \in \mathcal{I}^t(G), \varphi \in C_{\infty}^0(G)).$$

As proved in [4, Theorem 3.2.1], $J_{\infty}^t(T)$ is a G-invariant distribution on G and the corresponding map:

$$J_{\infty}^t: \mathcal{I}^t(G) \to \mathcal{I}^t(G)^G$$

is bijective.

Let $(G, G')$ be an irreducible dual pair in $\text{Sp}(W)$ such that $\text{rk}(G) \leq \text{rk}(G')$ and $(\mathcal{I}(\tilde{G}), J_{\infty}(G))$, $(\mathcal{I}(\tilde{G}'), J_{\infty}(G'))$ be the corresponding space of orbital integrals on $\tilde{G}$ and $\tilde{G}'$ respectively. To simplify, we assume that both G and $G'$ are connected. For every function $\varphi \in C_{\infty}^0(\tilde{G}')$, we denote by $\text{Chc}(\varphi)$ the $\tilde{G}$-invariant function on $\tilde{G}'$ given by:

$$\text{Chc}(\varphi)(\tilde{h}) = \text{Chc}_{\tilde{h}}(\varphi), \quad (\tilde{h} \in H_{\tilde{h}}^{\text{reg}}).$$

In [3], the authors proved the following results:

**Theorem 4.4.** For every $\varphi \in C_{\infty}^0(\tilde{G}')$, $\text{Chc}(\varphi) \in \mathcal{I}(\tilde{G})$ and the corresponding map

$$\text{Chc}: C_{\infty}^0(\tilde{G}') \to \mathcal{I}(\tilde{G})$$

is continuous. Moreover, if $J_{\infty}(\varphi) = 0$, we get that $\text{Chc}(\varphi) = 0$, i.e. the map $\text{Chc}: C_{\infty}^0(\tilde{G}') \to \mathcal{I}(\tilde{G})$ factors through $\mathcal{I}(\tilde{G}')$ and we get a transfer of orbital integrals:

$$\text{Chc}: \mathcal{I}(\tilde{G}') \to \mathcal{I}(\tilde{G}).$$
By dualizing the previous map, we get \( \text{Chc}' : \mathcal{S}'(\widetilde{G}) \rightarrow \mathcal{S}'(\widetilde{G}) \) given by
\[
\text{Chc}'(\tau)(\phi) = \tau(\text{Chc}(\phi)) \quad (\tau \in \mathcal{S}'(\widetilde{G}), \phi \in \mathcal{S}(\widetilde{G})).
\]

By using the isomorphisms \( J'_G \) and \( J''_G \), we get a map \( \text{Chc}^* : \mathcal{S}'(\widetilde{G})^{\circ} \rightarrow \mathcal{S}'(\widetilde{G})^{\circ} \) given by \( \text{Chc}^* = J'_G \circ \text{Chc}' \circ (J''_G)^{-1} \).

We denote by \( \text{Eigen}(\widetilde{G}) \) (resp. \( \text{Eigen}(\widetilde{G}') \)) the set of invariant eigendistributions on \( \widetilde{G} \) (resp. \( \widetilde{G}' \)).

**Theorem 4.5.** The map \( \text{Chc}^* : \mathcal{S}'(\widetilde{G})^{\circ} \rightarrow \mathcal{S}'(\widetilde{G}')^{\circ} \) sends \( \text{Eigen}(\widetilde{G})^{\circ} \) into \( \text{Eigen}(\widetilde{G}')^{\circ} \).

**Remark 4.6.** If \( \Theta \) is a distribution on \( \widetilde{G} \) given by a locally integrable function \( \Theta \) on \( \widetilde{G} \), we get for every \( \varphi \in \mathcal{C}_c^{\infty}(\widetilde{G}) \) the following equality:
\[
\text{Chc}^*(\Theta)(\varphi) = \sum_{i=1}^n \frac{1}{|W(H_i)|} \int_{H_i} \Theta(h) \det(1 - \text{Ad}(h^{-1}))_{h|_{h_i}} \text{Chc}(\varphi)(h) \, dh,
\]
where \( H_1, \ldots, H_n \) is a maximal set of non-conjugate Cartan subgroups of \( G \).

We recall the following conjecture.

**Conjecture 4.7.** Let \( G_1 \) and \( G'_1 \) be the Zariski identity components of \( G \) and \( G' \) respectively. Let \( \Pi \in \mathcal{A}(\widetilde{G}, \omega) \) satisfying \( \Theta_{\Pi, G} = 0 \) if \( G = O(V) \), where \( V \) is an even dimensional vector space over \( \mathbb{R} \) or \( \mathbb{C} \). Then, up to a constant, \( \text{Chc}^*(\Theta_{\Pi}) = \Theta_{\Pi, G} \) on \( \widetilde{G}' \).

In few cases, the conjecture is well-known: if \( G \) is compact (see [23]) and if \( (G, G') \) is in the stable range (see [24]). In this paper, we are investigating the case \( \text{rk}(G) = \text{rk}(G') \), with \( \Pi \) a discrete series representation of \( G \). We will focus our attention on the dual pair of unitary groups satisfying \( \text{rk}(G) = \text{rk}(G') \), using some results of A. Paul that we recall in the next section.

## 5. Discrete series representations and a result of A. Paul

Let \( G \) be a connected real reductive Lie group.

**Definition 5.1.** We say that an irreducible representation \( (\Pi, (\mathcal{H}, <\cdot, \cdot>)) \) is a discrete series representation if all the functions \( \tau_{u,v} : g \mapsto <g(u), v> \in \mathbb{C} \), are in \( L^2(G) \), where
\[
\tau_{u,v} : G \ni g \rightarrow <g(u), v> \in \mathbb{C}.
\]

**Remark 5.2.** One can prove that the condition given in the previous definition is equivalent to say that the representation \( (\Pi, \mathcal{H}) \) is equivalent with a direct summand of the right regular representation of \( G \) on \( L^2(G) \).

Moreover, as recalled in [13] Section 9.3, for such a representation \( (\Pi, \mathcal{H}) \), there exists a positive number \( d_{\Pi} \) (depending on the Haar measure \( d_{G} \) on \( G \)), called the formal degree of \( \Pi \), such that for every \( u_1, u_2, v_1, v_2 \in \mathcal{H} \),
\[
\int_{G} \langle \Pi(g)u_1, v_1 \rangle \overline{\langle \Pi(g)u_2, v_2 \rangle} \, d_{G} = \frac{\langle u_1, u_2 \rangle \langle v_1, v_2 \rangle}{d_{\Pi}}.
\]

In his papers [8] and [10], Harish-Chandra gave a classification of the discrete series representations of \( G \). First of all, he proved that \( G \) has discrete series if and only if \( G \) has a compact Cartan subgroup (see [10] Theorem 13]). Let \( K \) be a maximal compact subgroup of \( G \) and \( H \) a Cartan subgroup of \( K \). He also proved that the set of discrete series is indexed by a lattice of \( ib' \). We say few words about this now. Let \( \Psi = \Psi(g_c, b_c) \) be the set of roots of \( g \), \( \Psi(t) = \Psi(t_c, b_c) \) be the set of compact roots of \( g, \rho = \frac{1}{4} \sum_{\alpha \in \Phi^r} \alpha \) and \( \rho(t) = \frac{1}{4} \sum_{\alpha \in \Phi^r(t)} \alpha \).
Theorem 5.6. Let $G$ be a dual pair of unitary groups in $\text{Sp}(2(p,q),\mathbb{R})$. More precisely, as proved in [8, Lemma 44], we have the following result.

Notation 5.3. For every $g \in G$, we denote by $D_g$ the function on $\mathbb{R}$ given by

$$D_g(t) = \det((t + 1)\text{Id}_g - \text{Ad}(g)) \quad (t \in \mathbb{R}).$$

In particular, $D_g(t) = \sum_{i=0}^n t^i D_i(g)$, with $n = \dim(G)$. The $D_i$'s are analytic on $G$ and let $l$ be the least integer such that $D_l \neq 0$. The integer $l$ is the rank of $g$. We denote by $D(g)$ the coefficient of $t^l$ in the previous polynomial and by $G^{\text{reg}}$ the set of $g \in G$ such that $D(g) \neq 0$.

Theorem 5.4. Let $\lambda$ be an element of $\mathfrak{h}^\ast$ such that $\lambda + \rho$ is analytic integral. Then, there exists a discrete series representation $(\Pi_\lambda, \mathcal{H}_\lambda)$ of $G$ such that:

1. The representation $\Pi_\lambda$ has infinitesimal character $\chi_\lambda$ as in Remark [4,9].
2. The linear form $\nu = \lambda + \rho - 2\rho(\mathfrak{t})$ is the highest weight of the lowest $\mathfrak{k}$-type for $\Pi_{\lambda_0}$ and the multiplicity of the corresponding representation $\Pi_\nu$ in $\Pi_{\lambda_0}$ is one.

The parameter $\lambda$ is called the Harish-Chandra parameter of $\Pi_\lambda$. Moreover, if we denote by $\Theta_\lambda$ the distribution character of $\Pi$ and by $\Theta_\lambda$ the corresponding locally integrable function on $G^{\text{reg}}$, we get that the restriction of $\Theta_\lambda$ of $\Pi$ to $H^{\text{reg}}$ is given by the following formula

$$\Theta_\lambda(\exp(X)) = (-1)^{\frac{\dim(G) - \dim(K)}{2}} \sum_{w \in W(\mathfrak{l})} s(w) \frac{e^{iW(X)\theta}}{\prod_{\alpha > 0} (e^{\alpha(X)} - e^{-\alpha(X)})}, \quad (X \in \mathfrak{h}^{\text{reg}}).$$

Remark 5.5. As proved in [8], for every discrete series $\Pi$ of $G$ with Harish-Chandra parameter $\lambda$, we get:

$$\sup_{g \in G^{\text{reg}}} |D(g)|^\frac{1}{2} |\Theta_\lambda(g)| < \infty.$$

The previous properties of $\Theta_\lambda$ characterize the discrete series characters inside the space of invariant distributions of $G$. More precisely, as proved in [8 Lemma 44], we have the following result.

Theorem 5.6. Let $\Theta_\lambda$ be $G$-invariant distribution on $G$ such that:

1. $\gamma(\lambda)(z)\Theta_\lambda, z \in Z(\mathfrak{h}(g_c))$.
2. $\sup_{g \in G^{\text{reg}}} |D(g)|^\frac{1}{2} |\Theta_\lambda(g)| < \infty$.
3. $\Theta_\lambda = 0$ pointwise on $H^{\text{reg}}$.

Then, $\Theta_\lambda = 0$.

The previous theorem will be central for us in Section 6 to prove the conjecture [4,7] for discrete series representations in the equal rank case. We now recall a key result of A. Paul for unitary groups. Let $(G, G') = (\text{U}(p, q), \text{U}(r, s))$ be a dual pair of unitary groups in $\text{Sp}(2(p + q)(r + s), \mathbb{R})$. As explained in [19, Section 1.2], the double cover of $\bar{U}(p, q)$ is isomorphic to

$$\bar{U}(p, q) \approx \{(g, \xi) \in \text{U}(p, q) \times \mathbb{C}^\ast, \xi^2 = \det(g)^{r-s}\}.$$

In particular, all the genuine admissible representations of $\bar{U}(p, q)$ are the form $\Pi \otimes \det^{\frac{r-s}{2}}$, where $\det^{\frac{r-s}{2}}$ is the genuine character of $\bar{U}(p, q)$ given by $\det^{\frac{r-s}{2}}(g, \xi) = \xi$ and $\Pi$ is an admissible representation of $\text{U}(p, q)$. From now on, we fix $p$ and $q$ and let $r$ and $s$ vary under the condition that $p + q = r + s$. In particular, under this condition, it follows from Equation (9) that the double cover of $\text{U}(p, q)$ stays the same when $r$ and $s$ vary.

In [19, Section 6], A. Paul proved the following theorem:
Theorem 5.7. For every genuine irreducible admissible representation \((\mathcal{H}_0, \pi)\) of \(\tilde{\mathbb{U}}(p, q)\), there exists a unique pair of integers \((r, s) = (r_1, s_1)\) such that \(p + q = r + s\) with \(\theta_{reg}(\pi) \neq 0\).

She also obtained more precise results for discrete series representations (see [19, Theorem 6.1] or [20, Theorem 2.7]).

Notation 5.8. We fix a basis \(\{e_1, \ldots, e_n\}\) of \(\mathfrak{h}^*\). In particular, every linear form \(\lambda\) on \(\mathfrak{h}\) can be written as \(\lambda = \sum_{i=1}^{n} \lambda_i e_i\) or also as \(\lambda = (\lambda_1, \ldots, \lambda_n)\).

Theorem 5.9. Let \(\Pi\) be a discrete series representation of \(\tilde{\mathbb{U}}(p, q)\), the corresponding representation \(\theta_{reg}(\Pi)\) is a discrete series representation of \(\tilde{\mathbb{U}}(r_1, s_1)\).

More precisely, if the Harish-Chandra parameter of \(\Pi\) is of the form

\[
\lambda = \lambda_{a,b} = (a_1, \ldots, a_d, b_1, \ldots, b_{p-a}, \gamma_1, \ldots, \gamma_b, \delta_1, \ldots, \delta_{q-b}),
\]

with \(a_i, b_j, \gamma_k, \delta_l \in \mathbb{Z} + \frac{1}{2}\) such that \(a_1 > \ldots > a_d > 0 > b_1 > \ldots > b_{p-a} > \gamma_1 > \ldots > \gamma_b > 0 > \delta_1 > \ldots > \delta_{q-b}\),

then \((r_1, s_1) = (a + q - b, b + p - a)\) and the corresponding Harish-Chandra parameter \(\lambda' = \lambda_{a,b}'\) of \(\theta_{reg}(\Pi)\) is of the form:

\[
\lambda'_{a,b} = (a_1, \ldots, a_d, \delta_1, \ldots, \delta_{q-b}, \gamma_1, \ldots, \gamma_b, b_1, \ldots, b_{p-a}).
\]

6. Proof of Conjecture 4.7 for discrete series representations in the equal rank case

In this section, we are interested in the dual pair \((G, G') = (U(p, q), U(r, s))\) such that \(p + q = r + s\). Without loss of generality, we assume that \(p \leq q\). In particular, the number of Cartan subgroups of \(G\), up to conjugation, is \(p + 1\). We denote by \(n = p + q\). Let \((V = \mathcal{C}^{p+q}, \langle \cdot, \cdot \rangle)\) and \((V' = \mathcal{C}^{rs'}, \langle \cdot, \cdot \rangle')\) be the hermitian and skew-hermitian spaces corresponding to \(G\) and \(G'\) respectively. In this case, the space \(W = \text{Hom}(V', V) = M((r + s) \times (p + q), \mathbb{C})\) and for every \(v \in W\), there exists a unique element \(w^* \in \text{Hom}(V, V') = M((p + q) \times (r + s), \mathbb{C})\) such that:

\[
(w(v'), v) = (v', w^*(v)), \quad (v \in V, v' \in V').
\]

One can prove that \(w^* = \text{Id}_{p+q} \mathcal{C} \text{Id}_{r+s}\) and the symplectic form \(\langle \cdot, \cdot \rangle\) on \(W\)

\[
\langle w, w' \rangle = \text{Re}(\text{tr}(w^* w)) = -\text{Im}(\text{tr}(\text{Id}_{p+q} \mathcal{C} \text{Id}_{r+s} w)) \quad (w, w' \in W).
\]

Let \(V_i = V'_i = \mathcal{C} e_i\). The subspaces \(\mathfrak{b}\) and \(\mathfrak{b}'\) of \(g\) and \(g'\) respectively given by:

\[
\mathfrak{b} = \mathfrak{b}' = \{y = (iX_1, \ldots, iX_n), X_i \in \mathbb{R}\}
\]

are Cartan subalgebras. Moreover, we get:

\[
W^\mathfrak{b} = \bigoplus_{i=1}^{n} \text{Hom}(V_i, V'_i) = \bigoplus_{i=1}^{n} \mathcal{C} \mathbb{R} E_{n,i}.
\]

Let \(\Pi\) be a discrete series of \(\tilde{\mathbb{U}}(p, q)\), \(\Theta_\Pi\) be the corresponding element of \(\mathcal{D}'(\mathcal{G})\), \(\Theta_\Pi\) the corresponding locally integrable function on \(\mathcal{G}\) such that \(\Theta_\Pi = T_{\theta_\Pi}\) and \(\chi_\Pi\) the infinitesimal character of \(\Pi\).

As recalled in Theorem 5.5, \(\text{Chc}'(\Theta_\Pi)\) is an element of \(\text{Eigen}(\mathcal{G})\). According to [9, Theorem 2], the distribution \(\Theta_\Pi' = \text{Chc}'(\Theta_\Pi)\) is given by a locally integrable function \(\Theta_\Pi'\) on \(\mathcal{G}\), analytic on \(\mathcal{G}^{\text{reg}}\).

Notation 6.1. From now on, we fix an element \(\tilde{\Pi}\) be an element of \(\pi^{-1}([-1])\). Let \(c : g^r \to G'\) the Cayley transform, where \(g^r\) and \(G'\) are defined in Section 2. As explained in [21, Lemma 3.5], there exists a unique smooth map \(\tilde{c} : g^r \to G'\) such that \(\pi \circ \tilde{c} = c\) and \(\tilde{c}(0) = -\tilde{1}\).
Theorem 6.2. The value of $\Theta_1^H$ on the compact Cartan $\tilde{H} = H(\emptyset)$ is given by the following formula:

$$\Delta_{\psi}(\tilde{h}')\Theta_1^H(\tilde{p}(\tilde{h}')) = \frac{C}{|\mathcal{W}(H)|} \sum_{\sigma \in \mathcal{P}_H} \varepsilon(\sigma)\det^* (\sigma(\tilde{h}')) \mathcal{W} \lim_{r \to \infty} \int_{\tilde{H}/\tilde{W}} \frac{\Theta_1^H(\tilde{p}(\tilde{h}))\Delta_{\psi}(\tilde{h}')}{\det(1 - p(\tilde{h})r\tilde{h}')} d\tilde{h}' \quad (\tilde{h}' \in \tilde{H})$$

where $H = H(\emptyset)$ is the compact Cartan of $G$ and $C = \chi_H(-1)\Theta(\tilde{h})C(1)\Phi$.  

Proof. Let $\varphi$ be a function in $\mathcal{C}_c^{\infty}(G')$. According to Remark 2, we get that

$$\Theta_1^H(\varphi) = \frac{1}{|\mathcal{W}(H)|} \int_{\tilde{H}/\tilde{W}} \Theta_1^H(\tilde{h})\det(1 - Ad(\tilde{h}^{-1}))_{\tilde{h}}\varphi(\tilde{h})d\tilde{h},$$

where $H(S_t)$ is a set of Cartan subgroups as in Remark 2 and let $H = H(\emptyset)$ the compact Cartan of $G$. Now, if we assume that supp($\varphi$) $\subseteq G' \cdot \tilde{H}$, then

$$\Theta_1^H(\varphi) = \frac{1}{|\mathcal{W}(H)|} \int_{\tilde{H}/\tilde{W}} \Theta_1^H(\tilde{h})\det(1 - Ad(\tilde{h}^{-1}))_{\tilde{h}}\varphi(\tilde{h})d\tilde{h}$$

According to Equation 8 and Theorem 3.7, we get:

$$\Theta_1^H(\varphi) = \frac{(-1)^{\mu}C}{|\mathcal{W}(H)|} \int_{\tilde{H}/\tilde{W}} \Theta_1^H(\tilde{h})\Delta_{\psi}(\tilde{h}) \left( \int_{G'} \Theta(\tilde{h}\tilde{g}')\varphi(\tilde{g}')d\tilde{g}' \right) d\tilde{h}$$

$$= \frac{(-1)^{\mu+1}C}{|\mathcal{W}(H)|} \int_{\tilde{H}/\tilde{W}} \Theta_1^H(\tilde{h})\Delta_{\psi}(\tilde{h})\det^* (\tilde{h}) \left( \int_{G'} \Theta(\tilde{h}\tilde{g}')\varphi(\tilde{g}')d\tilde{g}' \right) d\tilde{h}$$

$$= -\frac{Cm_0}{|\mathcal{W}(H)|} \sum_{\sigma \in \mathcal{P}_H} \varepsilon(\sigma) \lim_{r \to \infty} \int_{\tilde{H}/\tilde{W}} \Theta_1^H(\tilde{h})\Delta_{\psi}(\tilde{h})\det^* (\tilde{h}) \int_{G'} \frac{\det^* (\sigma^{-1}(\tilde{h}'))_{\tilde{w}}}{\det(1 - p(\tilde{h})r(\tilde{h}'))_{\tilde{w}}\mathcal{W}} \mathcal{W}(\varphi)(\tilde{h}')d\tilde{h}'$$

With such assumptions on the support of $\varphi$, we get using Equation 3:

$$\Theta_1^H(\varphi) = \int_{G'} \Theta_1^H(\tilde{g}')\varphi(\tilde{g}')d\tilde{g}' = m_0 \int_{G'} \Delta_{\psi}(\tilde{h}').\mathcal{W}(\Theta_1^H(\varphi)(\tilde{h}'))d\tilde{h}'$$

$$= -m_0 \int_{G'} \Theta_1^H(\tilde{h}').\Delta_{\psi}(\tilde{h}').\mathcal{W}(\varphi)(\tilde{h}')d\tilde{h}'$$

By identifications, we get, up to a constant, that:

$$\Delta_{\psi}(\tilde{h}')\Theta_1^H(\tilde{p}(\tilde{h}')) = \frac{C}{|\mathcal{W}(H)|} \sum_{\sigma \in \mathcal{P}_H} \varepsilon(\sigma)\det^* (\sigma^{-1}(\tilde{h}')) \mathcal{W} \lim_{r \to \infty} \int_{\tilde{H}/\tilde{W}} \frac{\Theta_1^H(\tilde{p}(\tilde{h}))\Delta_{\psi}(\tilde{h})\det^* (\tilde{h})}{\det(1 - p(\tilde{h})r(\tilde{h}'))_{\tilde{w}} \mathcal{W}} d\tilde{h}$$

and the theorem follows. □

We know that the set of roots for $(\varphi, b)$ is given by

$$\{\pm(e_i - e_j), 1 \leq i < j \leq n\}.$$ 

Let $K = U(p) \times U(q)$ be a maximal compact subgroup of $G$. Let $\Psi(t) = \Psi(t_c, b_c)$ be a set of compact positive roots given by:

$$\Psi(t) = \{e_i - e_j, 1 \leq i < j \leq p\} \cup \{e_i - e_j, p + 1 \leq i < j \leq n\}.$$
The compact Weyl group \( W' = W(K, H) \) is \( \mathcal{P} \times \mathcal{Q} \). Let \( \lambda = \sum_{i=1}^{p+q} \lambda_i e_i \) be the Harish-Chandra parameter of \( \Pi \).

Using Theorem 5.4 the value of \( \Theta_{\Pi} \) on \( \mathcal{H}^{\text{reg}} \) is given by:

\[
\Theta_{\Pi}(\hat{\rho}(\hat{h})) = (-1)^{p+q} \sum_{\beta \in \mathcal{P} \times \mathcal{Q}} \epsilon(\beta) \prod_{h>0} \left( \frac{(\beta \hat{h})^4}{(h^2 - h^{-2})} \right) \quad (\hat{h} \in \mathcal{H}^{\text{reg}}),
\]

with \( \alpha_{p,q} = \frac{\dim(G) - \dim(K)}{2} = pq \). Using that \( W = \bigoplus_{i=1}^{n} \text{Hom}(V_i, V_i) \), we get:

\[
\det(1 - p(h) \rho(h'), \pi) = \prod_{i=1}^{n} \left(1 - h_i (r h')^{-1}_{\sigma(i)}\right) = (-1)^n \prod_{i=1}^{n} \left(r h' \right)^{-1}_{\sigma(i)} \prod_{i=1}^{n} \left(h_i - (r h')_{\sigma(i)}\right),
\]

and

\[
\det^{\frac{1}{2}}(\sigma^{-1}(\hat{h}'))_{W} = \prod_{i=1}^{n} h_i^{\frac{1}{2}} \quad \det^{\frac{1}{2}}(\hat{h})_{W} = \prod_{i=1}^{n} h_i^{\frac{1}{2}}.
\]

To simplify the notations, we will denote by \( \xi \) the element of \( \mathfrak{h}_C^{\circ} \) given by \( \xi = \sum_{i=1}^{n} \frac{1}{2} e_i \). We recall a basic Cauchy integral formula.

**Lemma 6.3.** Let \( k \in \mathbb{Z} \) and \( a \in \mathbb{C} \setminus S^1 \). Then,

\[
\frac{1}{2i\pi} \int_{S^1} \frac{e^{az}}{z-a} \, dz = \begin{cases} a^k & \text{if } k \geq 0 \text{ and } |a| < 1 \\ -a^k & \text{if } k < 0 \text{ and } |a| > 1 \\ 0 & \text{otherwise} \end{cases}
\]

For every \( \hat{h} \in \mathcal{H}^{\text{reg}} \), we get from Theorem 6.2:

\[
\Delta_{\mathcal{W}}(\hat{h}') \Theta_{\Pi}'(\hat{\rho}(\hat{h}')) = C \sum_{\sigma \in \mathcal{P} \times \mathcal{Q}} \sum_{n, \rho(\hat{h})} \epsilon(\sigma) \epsilon(\beta) \prod_{i=1}^{n} h_i^{\frac{1}{2}} \lim_{\rho(\hat{h}) \to \infty} \int_{\mathcal{H}} \frac{(\beta \hat{h})^4}{\prod_{i=1}^{n} (h_i - (r h')_{\sigma(i)})} \, dh
\]

\[
= C \sum_{\sigma \in \mathcal{P} \times \mathcal{Q}} \sum_{n, \rho(\hat{h})} \epsilon(\sigma) \epsilon(\beta) \prod_{i=1}^{n} h_i^{\frac{1}{2}} \lim_{\rho(\hat{h}) \to \infty} \int_{\mathcal{H}} \frac{(\beta \hat{h})^{4 + \xi}}{\prod_{i=1}^{n} (h_i - (r h')_{\sigma(i)})} \, dh
\]

\[
= 2C \sum_{\sigma \in \mathcal{P} \times \mathcal{Q}} \sum_{n, \rho(\hat{h})} \epsilon(\sigma) \epsilon(\beta) \prod_{i=1}^{n} h_i^{\frac{1}{2}} \lim_{\rho(\hat{h}) \to \infty} \int_{\mathcal{H}} \frac{h_i^{\beta + \frac{1}{2}}}{\prod_{i=1}^{n} (h_i - (r h')_{\sigma(i)})} \, dh
\]

\[
= \frac{2C \prod_{i=1}^{n} h_i^{\frac{1}{2}}}{(2i\pi)^n} \sum_{\sigma \in \mathcal{P} \times \mathcal{Q}} \sum_{n, \rho(\hat{h})} \epsilon(\sigma) \epsilon(\beta) \lim_{\rho(\hat{h}) \to \infty} \int_{S^1} \frac{z^{\beta - \frac{1}{2}}}{z - (r h')_{\sigma \beta^{-1}(i)}} \, dz,
\]

where \( C = \frac{(-1)^{pq} C}{\mathcal{W}(H)} \).
Lemma 6.4. For every \( \sigma \in \mathcal{S}_{r+1} \), the space \( E_{\sigma,0} \) is given by

\[
E_{\sigma,0} = \left\{ h' = (e^{-X_1}, \ldots, e^{-X_r}) \in H^r_{\mathbb{C}} \mid \begin{cases} 
X_{\sigma(i)} > 0 & \text{if } i \in \{1, \ldots, p\} \text{ and } \sigma(i) \in \{1, \ldots, r\} \\
X_{\sigma(i)} < 0 & \text{if } i \in \{1, \ldots, p\} \text{ and } \sigma(i) \in \{r + 1, \ldots, r + s\} \\
X_{\sigma(i)} < 0 & \text{if } i \in \{p + 1, \ldots, p + q\} \text{ and } \sigma(i) \in \{1, \ldots, r\} \\
X_{\sigma(i)} > 0 & \text{if } i \in \{p + 1, \ldots, p + q\} \text{ and } \sigma(i) \in \{r + 1, \ldots, r + s\} 
\end{cases} \right\}
\]

Proof. Let \( w = \sum_{i=1}^{n} w_i E_{i,\sigma(i)} \in H^r_{\mathbb{C}}, \sigma \in \mathcal{S}_{r+1} \) and \( y = (iX_1, \ldots, iX_n) \in b' \), with \( X_j \in \mathbb{R} \). Then,

\[
\langle y(\sigma(w)), \sigma(w) \rangle = \langle y\left(\sum_{i=1}^{n} w_i E_{i,\sigma(i)}\right), \sum_{i=1}^{n} w_i E_{i,\sigma(i)}\rangle = -\langle \sum_{i=1}^{n} w_i y_{\sigma(i)} E_{i,\sigma(i)}, \sum_{i=1}^{n} w_i E_{i,\sigma(i)}\rangle
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Im}(\text{tr}(\overline{w_i} E_{i,\sigma(i)} E_j E_{i,\sigma(i)}))
\]

\[
= \sum_{i=1}^{n} \text{Im}(\text{tr}(\overline{w_i} E_{i,\sigma(i)} E_{i,\sigma(i)}))
\]

\[
= \sum_{i=1}^{n} |w_i|^2 X_{\sigma(i)} - \sum_{i=p+1}^{n} |w_i|^2 X_{\sigma(i)} = \sum_{i=p+1}^{n} |w_i|^2 X_{\sigma(i)} + \sum_{i=p+1}^{n} |w_i|^2 X_{\sigma(i)}
\]

In particular, using Equation (4), we get:

\[
\Gamma_{\sigma,0} = \left\{ y = (iX_1, \ldots, iX_n) \in b' \mid \begin{cases} 
X_{\sigma(i)} > 0 & \text{if } i \in \{1, \ldots, p\} \text{ and } \sigma(i) \in \{1, \ldots, r\} \\
X_{\sigma(i)} < 0 & \text{if } i \in \{1, \ldots, p\} \text{ and } \sigma(i) \in \{r + 1, \ldots, n\} \\
X_{\sigma(i)} < 0 & \text{if } i \in \{p + 1, \ldots, n\} \text{ and } \sigma(i) \in \{1, \ldots, r\} \\
X_{\sigma(i)} > 0 & \text{if } i \in \{p + 1, \ldots, n\} \text{ and } \sigma(i) \in \{r + 1, \ldots, n\} 
\end{cases} \right\}
\]

The result follows using that \( E_{\sigma,0} = \exp(\Gamma_{\sigma,0}) \).

\[\square\]

Proposition 6.5. Let \( \Pi \in \mathcal{R}(\mathbb{U}(p, q), \omega) \) be a discrete series representation of Harish-Chandra parameter \( \lambda_{a,b} \), as in Theorem 5.3 and let \( (r, s) = (\tau_{\Pi}, s_{\Pi}) \) the unique integers such that \( \theta_{a,b}(\Pi) \neq 0 \). The value of \( \theta_{0}^\Pi \) on \( H^r_{\mathbb{C}} \) is given by

\[
\Delta_{\psi}(\tilde{h}^r) \Theta_{0}^\Pi(\tilde{h}^r) = 2(-1)^{p-r-b} \xi(\tau_{a,b}) C \sum_{\sigma \in \mathcal{S}_{r+1}} \mathcal{E}(\sigma, \tau_{a,b}^r) C \xi(\sigma(\tilde{h}^r)^{\tau_{a,b}^r})
\]

where \( \tau_{a,b} \in \mathcal{S}_{r+1} \) is defined by:

- If \( r \leq p \), \( \tau_{a,b} = (a + 1, p + b + 1)(a + 2, p + b + 2) \ldots (r, p + q) \),
- If \( p + 1 \leq r \leq p + b \), \( \tau_{a,b} \in \text{Stab}_{\mathcal{S}_{r+1}} \{(a, 1, \ldots, a) \cup \{r + 1, \ldots, p + b\}\} \) and satisfies:

\[
\tau_{a,b}(a + 1) = p + b + 1, \ldots, \tau_{a,b}(r) = r + s, \tau_{a,b}(p + b + 1) = a + 1, \ldots, \tau_{a,b}(p + q) = r.
\]
If $r \geq p + b + 1$, $\tau_{a,b} \in \text{Stab}_{\mathcal{F}_r}((1, \ldots, a) \cup \{p + b + 1, \ldots, r\})$ and satisfies

$$\tau_{a,b}(a + 1) = r + 1, \quad \tau_{a,b}(p + b) = r + s, \quad \tau_{a,b}(r + 1) = a + 1, \quad \tau_{a,b}(r + s) = p + b + 1.$$ 

**Notation 6.6.** For every subset $[i_1, \ldots, i_k]$ of $\{1, \ldots, p\}$ (resp. $[p + 1, \ldots, p + q]$, $[1, \ldots, r]$ or $[r + 1, \ldots, r + s]$), we denote by $[i_1, \ldots, i_k]^c$ the set $\{1, \ldots, p\} \setminus [i_1, \ldots, i_k]$ (resp. $[p + 1, \ldots, p + q] \setminus [i_1, \ldots, i_k]$, $[1, \ldots, r] \setminus [i_1, \ldots, i_k]$ or $[r + 1, \ldots, r + s] \setminus [i_1, \ldots, i_k]$).

For two subsets $[a_1, \ldots, a_m]$ and $[b_1, \ldots, b_n]$ of $\{1, \ldots, p + q\}$, we denote by $\mathcal{S}^{[b_1, \ldots, b_n]}_{[a_1, \ldots, a_m]}$ the groups of bijections between $[a_1, \ldots, a_m]$ and $[b_1, \ldots, b_n]$.

Similarly, for every $\beta \in \mathcal{F}_p \times \mathcal{F}_q$, we denote by $\mathcal{S}^{[\beta]}_{[\beta]}$ the groups of bijections between $\{\beta(a_1), \ldots, \beta(a_m)\}$ and $[b_1, \ldots, b_n]$.

Obviously,

$$\mathcal{F}_p \times \mathcal{F}_q = \bigcup_{[i_1, \ldots, i_k] \subseteq \{1, \ldots, p\}} \bigcup_{[j_1, \ldots, j_l] \subseteq \{1, \ldots, p + q\}} \mathcal{S}^{[j_1, \ldots, j_l]}_{[i_1, \ldots, i_k]} \times \mathcal{S}^{[\beta]}_{[\beta]} \times \mathcal{S}^{[\beta]}_{[\beta]} \times \mathcal{S}^{[j_1, \ldots, j_l]}_{[p + 1, \ldots, p + q] \cup \{p + 1, \ldots, p + q\} \cup \{p + 1, \ldots, p + q\} \cup \{p + 1, \ldots, p + q\}}$$

for every $1 \leq t \leq p$.

**Proof.** To simplify the notations, we will denote by $R(\sigma, \lambda_{a,b}, \beta, \sigma) \in \mathcal{F}_p \times \mathcal{F}_q$, the following term:

$$R(\sigma, \lambda_{a,b}, \beta) = \lim_{r \to \infty} \prod_{i=1}^{n} \int_{S^1} z^{(r \sigma)^{(i)}} dz$$

According to Lemmas 6.3 and 6.4, we get that $R(\sigma, \lambda_{a,b}, \beta) \neq 0$ if and only if

$$\sigma \circ \beta^{-1} \in \bigcup_{[i_1, \ldots, i_k] \subseteq \{1, \ldots, p\}} \bigcup_{[j_1, \ldots, j_l] \subseteq \{1, \ldots, r\}} \mathcal{S}^{[j_1, \ldots, j_l]}_{[i_1, \ldots, i_k]} \times \mathcal{S}^{[\beta]}_{[\beta]} \times \mathcal{S}^{[\beta]}_{[\beta]} \times \mathcal{S}^{[j_1, \ldots, j_l]}_{[p + 1, \ldots, p + q] \cup \{p + 1, \ldots, p + q\} \cup \{p + 1, \ldots, p + q\} \cup \{p + 1, \ldots, p + q\}}$$

We first assume that $r \leq p$. In this case, using that $[a + 1, \ldots, p] = [a + 1, \ldots, r] \cup \{r + 1, \ldots, p\}$, we get

$$\mathcal{F}_p \times \mathcal{F}_q = \left( \bigcup_{[i_1, \ldots, i_k] \subseteq \{1, \ldots, p\}} \bigcup_{[j_1, \ldots, j_l] \subseteq \{1, \ldots, r\}} \mathcal{S}^{[j_1, \ldots, j_l]}_{[i_1, \ldots, i_k]} \times \mathcal{S}^{[\beta]}_{[\beta]} \times \mathcal{S}^{[\beta]}_{[\beta]} \times \mathcal{S}^{[j_1, \ldots, j_l]}_{[p + 1, \ldots, p + q] \cup \{p + 1, \ldots, p + q\} \cup \{p + 1, \ldots, p + q\} \cup \{p + 1, \ldots, p + q\}} \circ \sigma_1 \right)$$

where $\sigma_1 = (a + 1, p + b + 1)(a + 2, p + b + 2) \cdots (r, p + q)$. For every $\beta \in \mathcal{F}_p \times \mathcal{F}_q$, there exists exactly $r!$ elements in $\sigma \in \mathcal{F}_p \times \mathcal{F}_q$ such that $\sigma \circ \beta^{-1} \in \mathcal{F}_p \times \mathcal{F}_q \circ \sigma_1$. Then,

$$\sum_{\sigma \in \mathcal{F}_p \times \mathcal{F}_q} \sum_{\beta \in \mathcal{S}^{[\beta]}_{[\beta]}} \epsilon(\sigma \beta) \lim_{r \to \infty} \prod_{i=1}^{n} \int_{S^1} z^{(r \sigma)^{(i)}} dz = (-1)^{p + q - a - b} (2i\pi)^{p + q} \sum_{\tau \in \mathcal{F}_p \times \mathcal{F}_q} \epsilon(\tau) \prod_{i=1}^{n} \int_{S^1} h^{(i \tau)^{(i)}} dz = (-1)^{p + q - a - b} (2i\pi)^{p + q} \sum_{\tau \in \mathcal{F}_p \times \mathcal{F}_q} \epsilon(\tau (1 \theta)^{(1 \theta)} \tau^{-1}) \sum_{i=1}^{n} \epsilon(\tau (1 \theta)^{(1 \theta)} \tau^{-1})^{(i \theta)}$$

where $\xi = \frac{n}{2} \theta$, i.e.

$$\Delta (1 \theta)^{(1 \theta)} = 2(1 \theta)^{p + q - a - b} \epsilon(\sigma_1) \sum_{\tau \in \mathcal{F}_p \times \mathcal{F}_q} \epsilon(\tau (1 \theta)^{(1 \theta)}) \tau^{-1}(1 \theta).$$

Now assume that $r > p$. We distinguish two cases. If $p + 1 \leq r \leq p + b$, then,

$$\mathcal{F}_p \times \mathcal{F}_q = \left( \bigcup_{[i_1, \ldots, i_k] \subseteq \{1, \ldots, p\}} \bigcup_{[j_1, \ldots, j_l] \subseteq \{1, \ldots, r\}} \mathcal{S}^{[j_1, \ldots, j_l]}_{[i_1, \ldots, i_k]} \times \mathcal{S}^{[\beta]}_{[\beta]} \times \mathcal{S}^{[\beta]}_{[\beta]} \times \mathcal{S}^{[j_1, \ldots, j_l]}_{[p + 1, \ldots, p + q] \cup \{p + 1, \ldots, p + q\} \cup \{p + 1, \ldots, p + q\} \cup \{p + 1, \ldots, p + q\}} \circ \sigma_2 \right)$$

where $\sigma_2 = (a + 1, p + b + 1)(a + 2, p + b + 2) \cdots (r, p + q)$.
for every $\eta \in \mathcal{S}_{p+b}$ satisfying $\eta[1, \ldots, a] = [1, \ldots, a]$, $\eta[r+1, \ldots, p+b] = [r+1, \ldots, p+b]$, $\eta[a+1, \ldots, r] \subseteq [p+b+1, \ldots, r+s]$ and $\eta[p+b+1, \ldots, p+q] \subseteq [a+1, \ldots, r]$. Let $\sigma_2$ be the element of $\text{Stab}_{\mathcal{S}_{p+b}}([1, \ldots, a] \cup [r+1, \ldots, p+b])$ given by

$$\sigma_2(a+1) = p + b + 1, \ldots, \sigma_2(r) = r + s, \sigma_2(p + b + 1) = a + 1, \ldots, \sigma_2(p + q) = r.$$  

This element satisfy the previous conditions and we get:

$$\Delta_{\Psi}(\hat{h}')\Theta_1'(\hat{p}(\hat{h}')) = 2(-1)^{p+q-a-b}\epsilon(\sigma_2)\sum_{\tau \in \mathcal{S}_{p+b}} e(\tau)(\tau\hat{h}')^{\tau\sigma_2(a)}.$$  

Similarly, $r \geq p + b + 1,$

$$\mathcal{S}_{r} \times \mathcal{S}_{r} = \left( \bigcup_{(1 \ldots, a) \in [1 \ldots, r]} \bigcup_{(p + 1 \ldots, p + b) \in [r + 1 \ldots, r + s]} \mathcal{S}_{(1 \ldots, a)} \times \mathcal{S}_{(p + 1 \ldots, p + b)} \times \mathcal{S}_{(r + 1 \ldots, r + s)} \right) \circ \eta,$$

for every $\eta \in \mathcal{S}_{r+2}$ satisfying $\eta[1, \ldots, a] = [1, \ldots, a]$, $\eta[p+b+1, \ldots, r] = [p+b+1, \ldots, r]$, $\eta[a+1, \ldots, p+b] \subseteq [r+1, \ldots, r+s]$ and $\eta[r+1, \ldots, r+s] \subseteq [a+1, \ldots, p+b+1]$. Let $\sigma_3$ be the element of $\text{Stab}_{\mathcal{S}_{p+b}}([1, \ldots, a] \cup [p + b + 1, \ldots, r])$ given by

$$\sigma_3(a + 1) = r + 1, \ldots, \sigma_3(p + b) = r + s, \sigma_3(r + 1) = a + 1, \ldots, \sigma_3(s + r) = p + b + 1.$$  

This element satisfies the previous conditions and we get:

$$\Delta_{\Psi}(\hat{h}')\Theta_1'(\hat{p}(\hat{h}')) = 2(-1)^{p+q-a-b}\epsilon(\sigma_3)\sum_{\tau \in \mathcal{S}_{p+b}} e(\tau)(\tau\hat{h}')^{\tau\sigma_3(a)}.$$  

\[\square\]

**Proposition 6.7.** For every $\Pi \in \mathcal{R}(\tilde{U}(p, q), \omega)$, we get

$$\sup_{\tilde{\eta} \in \mathcal{G}_{\Psi}} |D(\tilde{\eta})|^2 |\Theta_1'(\tilde{\eta})| < \infty.$$  

We first need to introduce some notations.

**Notation 6.8.** Let $k \in \left[1, \min(r, s)\right]$, we denote by $\eta^+_{\mathcal{S}_{k}} = \text{Ad}(c_{\Pi}(S_k))^{-1}(\eta'(S_k)) \subseteq \eta^+_\Pi$, where $\eta^+_\Pi = \bigoplus_{\alpha \in \Psi} \eta^+_{\mathcal{S}_{\alpha}}$ where $\eta^+_{\mathcal{S}_{\alpha}}$ is the eigenspace corresponding to $\alpha \in \Psi$.

By keeping the notations of Section 3 we get that $\Psi'$ can be decomposed as follow:

\[\Psi' = \Psi'((g(U_k)) \times \Psi'((u(U_k))) \cup \Psi'((\eta'(S_k))), \quad (k \in \left[1, \min(p, \min(r, s))\right]).\]

Finally, we denote by $\Psi'((g(U_k)))$ the Weyl group corresponding to $(g(U_k)_{\mathcal{S}_{k}} \cup l_2(S_k))_{\mathcal{S}_{k}}$.

**Lemma 6.9.** For every $\hat{h}' \in \tilde{H}(S_k)^{\text{reg}}$, $\det(\text{Id} - \text{Ad}(\hat{h}'))_{\eta'(S_k)} \in \mathbb{R}_+^*.$

*Proof.* To make things easier, we will consider $S_k = \{e_1 - e_{r+s-1}, \ldots, e_k - e_{r+s}\}$. As in Equation (2), we have:

$$H_{S_k} = c(S_k)^{-1} H(S_k) c(S_k) = \left\{ h' = (e^{q_0 + x_1}, \ldots, e^{q_0 + x_t}, t_1, \ldots, t_{r+s-2k}, e^{q_0 + x_1}, \ldots, e^{q_0 + x_t}, t) \in U(1), q_0, x \in \mathbb{R} \right\}.$$

We denote by $h'_1 = c(S_k)^{-1} H(S_k)$. Obviously, we get:

$$\det(\text{Id} - \text{Ad}(\hat{h}'))_{\eta'(S_k)} = \det(\text{Id} - \text{Ad}(h'_1))_{\eta'(S_k)} = \det(\text{Id} - \text{Ad}(h'_1))_{\eta'(S_k)} = \det(\text{Id} - \text{Ad}(h'_1))_{\eta'(S_k)}.$$

We know that

$$\det(\text{Id} - \text{Ad}(h'_1))_{\eta'(S_k)} = \prod_{\alpha \in \Psi'(\eta'(S_k))} (1 - h''_{\eta'(S_k)}),$$
and that
\[\Psi'(\eta'(S_2)) = \{e_i - e_j, 1 \leq i \leq k, k + 1 \leq j \leq r + s - k\} \cup \{e_i - e_j, k + 1 \leq i \leq r + s - k, r + s - k + 1 \leq j \leq r + s\}.
\]
Let \(\alpha_1 = e_i - e_j\), with \(1 \leq i \leq k, k + 1 \leq j \leq r + s - k\) and let \(\alpha_2 = e_{j+s-r+1}\). Then,
\[
(1 - \hat{h}_1^m)(1 - \hat{h}_1^{m_2}) = (1 - e^{i \hat{h}_1 \hat{t}_{j-k}})(1 - \hat{t}_{j-k} e^{-i \hat{h}_1 \hat{t}_{j-k}}) = |1 - e^{i \hat{h}_1 \hat{t}_{j-k}}|^2,
\]
and the result follows.

\[
\square
\]

**Proof of Proposition 4.7** Without loss of generality, we can assume that \(r \leq s\). We distinguish two cases. We first start with \(p \leq r\). Note that in this case, \(H_\alpha = H_\alpha^\prime\) (resp. \(H(S_i) = H'(S_i)\)) for every \(0 \leq i \leq p\), with \(S_i = \{e_{i-1} - e_{i+1}, \ldots, e_i - e_{i+r}\}\) as in Notation [2.3.3.6.2].

In this case, we get, using [2.3.3.6.4], that for every \(\varphi \in \mathcal{C}_c(\hat{G}')\):
\[
\Theta_{\tilde{H}}(\varphi) = \int_{\hat{G}'} \Theta_{\tilde{H}}(\tilde{g}) \varphi(\tilde{g}) d\tilde{g} = \sum_{i=0}^\infty \int_{\mathcal{L}_0} \mathcal{E}_{\mathcal{L}_{0}}(\hat{h}) \mathcal{H}_0(\Theta_{\tilde{H}}(\hat{h})) d\hat{h}
\]
\[
= \sum_{i=0}^\infty \int_{\mathcal{L}_0} \mathcal{E}_{\mathcal{L}_{0}}(\hat{h}) \mathcal{H}_0(\Theta_{\tilde{H}}(\hat{h})) d\hat{h} = e^{-i \hat{h}_1 \hat{t}_0} \int_{\mathcal{L}_0} \mathcal{E}_{\mathcal{L}_{0}}(\hat{h}) \mathcal{H}_0(\Theta_{\tilde{H}}(\hat{h})) d\hat{h}
\]
\[
= e^{-i \hat{h}_1 \hat{t}_0} \int_{\mathcal{L}_0} \Theta_{\tilde{H}}(\hat{h}) d\hat{h} = e^{-i \hat{h}_1 \hat{t}_0} \int_{\mathcal{L}_0} \varphi(\hat{g}) \hat{g}^{-1} d\hat{g}
\]
\[
(11) + \sum_{i=1}^p m_i \int_{\mathcal{L}_0} \mathcal{E}_{\mathcal{L}_{0}}(\hat{h}) \mathcal{H}_0(\Theta_{\tilde{H}}(\hat{h})) d\hat{h} = e^{-i \hat{h}_1 \hat{t}_0} \int_{\mathcal{L}_0} \varphi(\hat{g}) \hat{g}^{-1} d\hat{g}
\]

where \(\mathcal{H}_0(\Theta_{\tilde{H}}(\hat{h})) = \mathcal{H}_0(\hat{h}) \mathcal{H}_0(\Theta_{\tilde{H}}(\hat{h}))\) as in Remark [2.3.3.6.2]. \(L'(S_i)\) is defined in Section [2.3.3.6.1] and \(\Lambda(c(S_i), \hat{h})\) is given by:
\[
\Lambda(c(S_i), \hat{h}) = \frac{D_{L'(S_i)}(c(S_i), \hat{h})}{D_{L'(S_i)}(c(S_i), \hat{h})} = \frac{\det(\text{Id} - Ad(c(S_i), \hat{h})(c(S_i))^{-1})}{\det(\text{Id} - Ad(c(S_i), \hat{h})(c(S_i))^{-1})}
\]

Using that \(\Theta_{\tilde{H}} = \text{Chc}^\ast(\Theta_{\tilde{H}})\), we get:
\[
\Theta_{\tilde{H}}(\varphi) = \sum_{i=0}^\infty \int_{\mathcal{L}_0} \Theta_{\tilde{H}}(\hat{h}) d(\text{Id} - Ad(\hat{h})^{-1})_{S_i} \varphi(\hat{h}) d\hat{h}
\]
Using Equation [7.1], we get that:
\[
\Theta_{\tilde{H}}(\varphi) = \sum_{i=0}^p \int_{\mathcal{L}_0} \int_{\mathcal{L}_0} \int_{\mathcal{L}_0} \Theta_{\tilde{H}}(\tilde{g}) d(\text{Id} - Ad(\tilde{g})^{-1})_{S_i} \varphi(\tilde{g}) d\tilde{g}
\]
\[
(12) = \sum_{i=0}^p \int_{\mathcal{L}_0} \int_{\mathcal{L}_0} \int_{\mathcal{L}_0} \Theta_{\tilde{H}}(\tilde{g}) d(\text{Id} - Ad(\tilde{g})^{-1})_{S_i} \varphi(\tilde{g}) d\tilde{g}
\]

\[
\int_{\mathcal{L}_0} \varphi(\tilde{g}) d\tilde{g} = \int_{\mathcal{L}_0} \varphi(\tilde{g}) d\tilde{g}
\]

\[
21
\]
Let \( i \in [1, r] \) and \( \varphi \in \mathcal{C}_c^\infty(\overline{G}) \) such that \( \text{supp}(\varphi) \subseteq \overline{G} \cdot \overline{\text{R}(S_j)} \). On one hand, using Equation (11), we get:

\[
\Theta_i(\varphi) = m \int_{\mathcal{T}_j(S_j)} \Theta_i(\varphi_{i^*}(\cdot, \cdot)^{-1}) \Delta G_i(\cdot)^{-1} \Delta G_i(\cdot)^{-1} \lambda(\cdot) \int_{U(S_j)/H(S_j)} \varphi_{N(S_j)}(\cdot) \left( \lambda(\cdot) \right)^{-1} \epsilon(\cdot, \cdot) d\lambda_{(\cdot)}
\]

\[
= m \int_{\mathcal{T}_j(S_j)} \int_{\mathcal{T}_j(S_j)} \epsilon(\cdot, \cdot) \varphi_{N(S_j)}(\cdot) \left( \lambda(\cdot) \right)^{-1} \int_{U(S_j)/H(S_j)} \varphi_{N(S_j)}(\cdot) \left( \lambda(\cdot) \right)^{-1} \epsilon(\cdot, \cdot) d\lambda_{(\cdot)}
\]

In particular, for every \( j < i \), we get from Equation (3):

\[
\Theta_j(\varphi) = m \int_{\mathcal{T}_j(S_j)} \int_{\mathcal{T}_j(S_j)} \epsilon(\cdot, \cdot) \varphi_{N(S_j)}(\cdot) \left( \lambda(\cdot) \right)^{-1} \int_{U(S_j)/H(S_j)} \varphi_{N(S_j)}(\cdot) \left( \lambda(\cdot) \right)^{-1} \epsilon(\cdot, \cdot) d\lambda_{(\cdot)}
\]

On the other hand, it follows from Equation (12) that

\[
\Theta_i(\varphi) = \sum_{j=0}^{\min(m, r)} C_j \int_{\mathcal{T}_j(S_j)} \gamma_{N(S_j)} \left( \lambda(\cdot) \right)^{-1} \epsilon(\cdot, \cdot) \varphi_{N(S_j)}(\cdot) \left( \lambda(\cdot) \right)^{-1} \int_{U(S_j)/H(S_j)} \varphi_{N(S_j)}(\cdot) \left( \lambda(\cdot) \right)^{-1} \epsilon(\cdot, \cdot) d\lambda_{(\cdot)}
\]

Using Lemma [5,12] and the equality \( c(S_j)c_i^j = c(S_j) \), we get:

\[
\Theta_i(\varphi) = \sum_{j=0}^{\min(m, r)} C_j \int_{\mathcal{T}_j(S_j)} \gamma_{N(S_j)} \left( \lambda(\cdot) \right)^{-1} \epsilon(\cdot, \cdot) \varphi_{N(S_j)}(\cdot) \left( \lambda(\cdot) \right)^{-1} \int_{U(S_j)/H(S_j)} \varphi_{N(S_j)}(\cdot) \left( \lambda(\cdot) \right)^{-1} \epsilon(\cdot, \cdot) d\lambda_{(\cdot)}
\]
As explained in Remark 3.11, for every $0 \leq j \leq i$, we have the following decomposition $H'(S_i) = T'_1(S_i) \times \pi'(S_j) \times T'_1(S_{i+j}) \times \pi'(S_{i+j})$. In particular, every element $h \in H'(S_i)$ can be written as $h = t_{ij, \alpha} h_j t_{i, \alpha}$. We get similar results for $H'_S$. In particular, we get that the value of $\Theta'_{ij}$ on $H'(S_i)$ is given by:

$$\Theta'_{ij}(c(S_i)\hat{p}(\hat{h})c(S_i)^{-1})|\Delta_G(\hat{h})|^{2} \Lambda(c(S_i)\hat{p}(\hat{h})c(S_i)^{-1}) = \sum_{j=0}^{\min(i,p)} C_j \sum_{\sigma \in W(U_{ij})} \varepsilon(\sigma)$$

$$\Delta_{\Psi'_{ij}(U_{ij})}(\hat{h}t_{ij}) \det^{-2}(\sigma^{-1}(\hat{h}t_{ij})) W_{ij}^{2, \sigma} \Delta_{\Psi'_{ij}(U_{ij})}(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1})$$

$$\lim_{\sigma \to 1} \int_{T_{x,y}} \frac{\Delta_{\Psi'_{ij}(U_{ij})}(\hat{h}t_{ij}) \det(\sigma^{-1}(\hat{h}t_{ij})) W_{ij}^{2, \sigma} \Delta_{\Psi'_{ij}(U_{ij})}(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1}) \det(Id - \Ad(c(S_j)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_j)^{-1}))_{\pi(S_j)} \det(1 - p(\hat{h}t_{ij}))_{\pi(S_j)} d\hat{h}}{\Delta_{\Psi'_{ij}(U_{ij})}(\hat{h}) \det(Id - \Ad(c(S_j)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_j)^{-1}))_{\pi(S_j)} \det(1 - (\hat{h}t_{ij}))_{\pi(S_j)}}$$

Using Equation 10, we get

$$\frac{\Delta_{\Psi'_{ij}(U_{ij})}(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1})}{\Delta_{\Psi'_{ij}(U_{ij})}(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1})}|\Delta_G(\hat{h}t_{ij})|^{2} \Lambda(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1}) = \prod_{\sigma \in \Psi'_{ij}(U_{ij})} |1 - (\hat{h}t_{ij})_{\pi(S_j)}(\hat{h}t_{ij})_{\pi(S_j)}|$$

and

$$\frac{\Delta_{\Psi'_{ij}(U_{ij})}(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1})}{\Delta_{\Psi'_{ij}(U_{ij})}(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1})}|\Delta_G(\hat{h}t_{ij})|^{2} \Lambda(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1}) = \prod_{\sigma \in \Psi'_{ij}(U_{ij})} |1 - (\hat{h}t_{ij})_{\pi(S_j)}(\hat{h}t_{ij})_{\pi(S_j)}|$$

so in particular

$$\frac{\Delta_{\Psi'_{ij}(U_{ij})}(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1})}{\Delta_{\Psi'_{ij}(U_{ij})}(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1})}|\Delta_G(\hat{h}t_{ij})|^{2} \Lambda(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1}) = 1$$

Finally,

$$\Delta_{\Psi'(U_{ij})}(c(S_i)\hat{p}(\hat{h})c(S_i)^{-1}) = \sum_{j=0}^{\min(i,p)} C_j \sum_{\sigma \in W(U_{ij})} \varepsilon(\sigma) \prod_{\sigma \in \Psi'(U_{ij})} |1 - (\hat{h})_{\pi(S_j)}(\hat{h})_{\pi(S_j)}|$$

$$\lim_{\sigma \to 1} \int_{T_{x,y}} \frac{\Theta_{ij}(c(S_j)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_j)^{-1}) \Delta_{\Psi'(U_{ij})}(c(S_i)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_i)^{-1}) \det(Id - \Ad(c(S_j)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_j)^{-1}))_{\pi(S_j)} \det(1 - p(\hat{h}t_{ij}))_{\pi(S_j)} d\hat{h}}{\Delta_{\Psi'(U_{ij})}(\hat{h}) \det(Id - \Ad(c(S_j)\hat{p}(t_{ij}\hat{h}t_{ij})c(S_j)^{-1}))_{\pi(S_j)} \det(1 - (\hat{h}t_{ij}))_{\pi(S_j)}}$$

Again, we get from Equation 10 that

$$\Delta_{\Psi'}(t_{ij} \hat{h}) = \prod_{\sigma \in \Psi'(U_{ij})} \Delta_{\Psi'(U_{ij})}(\hat{h}) \Delta_{\Psi'(U_{ij})}(t_{ij} \hat{h})$$

$$= \prod_{\sigma \in \Psi'(U_{ij})} \Delta_{\Psi'(U_{ij})}(\hat{h}) \prod_{\sigma \in \Psi'(U_{ij})} (1 - (t_{ij} \hat{h}))^{\alpha}$$

then,

$$\Delta_{\Psi'}(\hat{h}) \Theta'_{ij}(c(S_i)\hat{p}(\hat{h})c(S_i)^{-1}) = \sum_{j=0}^{\min(i,p)} C_j \sum_{\sigma \in W(U_{ij})} \varepsilon(\sigma) \det^{-2}(\sigma^{-1}(\hat{h}t_{ij})) W_{ij}^{2, \sigma} \varepsilon(c(S_i)\hat{p}(\hat{h}t_{ij})c(S_i)^{-1})$$
with Harish-Chandra parameter $\lambda_{a,b}$ as in Theorem 5.9. For every $z \in \mathbb{Z}(\mathbb{W}(q(r + s, \mathbb{C})))$, we get:

$$z\Theta^r_{\Pi} = \chi'_{\lambda_{a,b}}(z)\Theta^r_{\Pi},$$

where $\chi'_{\lambda_{a,b}} = \lambda'_{a,b}(\gamma(z))$ as in Appendix A Remark A.9.

**Proof.** Obviously, in the equal rank case, $\mathbb{W}(\mathbb{G}_{\Pi}) = \mathbb{W}(\mathbb{G}_{\Pi}^*)$. It follows from [13] Theorem 1.4] that $\text{Chc}^*(z\Theta_{\Pi}) = z\text{Chc}(\Theta_{\Pi})$. Because $\lambda_{a,b}$ and $\lambda'_{a,b}$ are conjugated under $\mathcal{F}_{r+s}$, the result follows from Theorem 5.10.

**Corollary 6.11.** For every discrete series representation $\Pi$ of $\mathbb{U}(p, q)$, we get

$$\text{Chc}^*(\Theta_{\Pi}) = C\Theta_{\Pi,c}(\Pi),$$

with $C \in \mathbb{C}$.

**Proof.** Using Theorem 5.6 it follows from Propositions 6.3 and 6.7 and Lemma 6.10 that, up to a scalar, $\Theta_{\Pi}^r = \text{Chc}^*(\Theta_{\Pi})$ is either the character of a discrete series representations of $\mathbb{U}(r, s)$ with Harish-Chandra parameter $\tau_{a,b}(\lambda_{a,b})$ if $(r, s) = (r_1, s_1)$ or 0 if $(r, s) \neq (r_1, s_1)$. The result follows from Theorem 5.4 because $\tau_{a,b}(\lambda_{a,b})$ and $\lambda'_{a,b}$ (as in Theorem 5.9) are conjugated under $\mathcal{F}_r \times \mathcal{F}_s$.
Corollary 6.12. If \((G, G') = (U(p, q), U(r, s)), p + q = r + s\) and \(\Pi \in \hat{\mathcal{R}}(\tilde{G}, \omega)\) a discrete series representation of \(\tilde{G}\). Then, the conjecture \(\ref{conjecture2.7}\) holds.

Proof. It follows from Theorem \(\ref{theorem2.2}\) because \(\Pi' = \Pi'_1\).

\(\square\)

7. A commutative diagram and a remark on the distribution \(\Theta_{\Pi'}\)

We start this section by recalling a result of T. Przebinda (see \(\ref{przebinda22}\)). Let \((G, G') = (G(V, (\cdot, \cdot), G(V', (\cdot, \cdot))))\) be an irreducible reductive dual pair in \(\text{Sp}(W)\). As proved in \(\ref{przebinda14}\) (see also Section \(\ref{section2}\)), the representations appearing in the correspondence are realized as quotients of \(\mathcal{H}^\infty\), the set of smooths vectors of the metaplectic representation \((\omega, \mathcal{H})\). Let \(\Pi \in \hat{\mathcal{R}}(\tilde{G}, \omega), \Pi'\) the corresponding element of \(\hat{\mathcal{R}}(G, \omega)\) and \(N(\Pi \otimes \Pi')\) the \(\omega^\infty(\tilde{G} \cdot \tilde{G}')\)-equivariant subspace of \(\mathcal{H}^\infty\) such that \(\Pi \otimes \Pi' \approx \mathcal{H}^\infty/N(\Pi \otimes \Pi')\) as in Section \(\ref{section2}\).

In particular,

\[(\Pi \otimes \Pi')^* \approx (\mathcal{H}^\infty/N(\Pi \otimes \Pi'))^* \approx \text{Ann}(N(\Pi \otimes \Pi')) = \{\alpha \in \mathcal{H}^\infty^*, \alpha(X) = 0, (\forall X \in N(\Pi \otimes \Pi'))\} \subseteq \mathcal{H}^\infty^*\],

i.e. there exists a unique element, up to a constant, \(\Gamma_{\Pi \otimes \Pi'} \in \text{Hom}(\mathcal{H}^\infty^*, \mathcal{H}^\infty^*)\) such that \(\text{Hom}_{G, G'}(\omega, \Pi \otimes \Pi') = \mathbb{C} \cdot \Gamma_{\Pi \otimes \Pi'}\).

Remark 7.1. Let \(W = X \otimes Y\) be a complete polarization of \(W\). It is well-known that we can realize the representation \(\omega\) on \(\mathcal{H} = L^2(X)\): this is the Schrödinger model. Moreover, the space of smooth vectors of \(\omega\) is the Schwartz space \(S(X)\) of \(X\).

Using the isomorphisms \(\mathcal{H} : S^*(W) \rightarrow S^*(X \times X)\) and \(\text{Op} : S^*(X \times X) \rightarrow \text{Hom}(S(X), S^*(X))\) (see \(\ref{halb12}\) Equations \(\ref{equation143}\) and \(\ref{equation146}\)), there exists a unique distribution \(f_{\Pi \otimes \Pi'} \in S^*(W)\) such that \(\Gamma_{\Pi \otimes \Pi'} = \text{Op} \circ \mathcal{H}(f_{\Pi \otimes \Pi'})\). The distribution \(f_{\Pi \otimes \Pi'}\) is called the intertwining distribution corresponding to \(\Pi \otimes \Pi'\).

As explained in \(\ref{halb17}\) Section \(\ref{section2}\), the situation turns out to be slightly easier when \(\text{dim}(V) \leq \text{dim}(V')\) and \((\Pi, \mathcal{H}_{\Pi})\) a discrete series representation of \(\tilde{G}\). Under those hypothesis, the space \(\mathcal{H}_{\Pi}^\infty \otimes \mathcal{H}_{\Pi'}^\infty\) has a natural structure of \(\tilde{G}\)-modules. Using the scalar products on \(\mathcal{H}\) and \(\mathcal{H}_{\Pi}\), we get a natural inner product \(\langle \cdot, \cdot \rangle\) on \(\mathcal{H}_{\Pi}^\infty \otimes \mathcal{H}_{\Pi'}^\infty\). We denote by \(\langle \cdot, \cdot \rangle_{\Pi}\) the following form on \(\mathcal{H}_{\Pi}^\infty \otimes \mathcal{H}_{\Pi'}^\infty\):

\[\langle \Phi, \Phi' \rangle_{\Pi} = \int_{\tilde{G}} \langle \Phi, (\omega \otimes \Pi)(g) \Phi' \rangle dg, \quad (\Phi, \Phi' \in \mathcal{H}_{\Pi}^\infty \otimes \mathcal{H}_{\Pi'}^\infty).\]

One can easily prove that in this context, the previous integral converges absolutely. We denote by \(R(\Pi)\) the radical of the form \(\langle \cdot, \cdot \rangle_{\Pi}\) and we still denote by \(\langle \cdot, \cdot \rangle_{\Pi}\) the non-degenerate form we got on \(H(\Pi) = \mathcal{H}_{\Pi}^\infty \otimes \mathcal{H}_{\Pi'}^\infty/R(\Pi)\). The group \(\tilde{G}\) acts naturally on \(\Pi(\Pi)\) and we denote by \(\theta_0(\Pi)\) the corresponding \(\tilde{G}\)-module.

Theorem 7.2 (\(\ref{halb17}\) Section \(\ref{section2}\)).

1. There exists \(u_0, v_0 \in \mathcal{H}_{\Pi}^\infty\) and \(x, y \in \mathcal{H}_{\Pi'}^\infty\) such that

\[\int_{\tilde{G}} \langle u_0 \otimes x, (\omega \otimes \Pi)(g)(v_0 \otimes y) \rangle_{\Pi} \neq 0.\]

Moreover, we get

\[\int_{\tilde{G}} |(\omega(g)u, v)|^2 dg < +\infty, \quad (u, v \in \mathcal{H}_{\Pi}^\infty).\]

2. The representation \(\Pi\) can be embedded in \(\omega\) as an irreducible subrepresentation and \(\theta_0(\Pi)\) defines an irreducible unitary representation on the completion of \(H(\Pi)\) (completion with respect to \(\langle \cdot, \cdot \rangle_{\Pi}\)).

3. The map \(\Pi \rightarrow \theta_0(\Pi')\) coincide with Howe’s duality correspondence.
We get the following proposition.

**Proposition 7.3.** Let \((G, G') = (U(V), U(V'))\) with \(\dim(V) \leq \dim(V')\) and \(Π\) be a discrete series representation of \(G\). The intertwining distribution is given by \(T(\Theta_{Π}) = \int_{G} \Theta_{Π}(g) T(\tilde{g}) dg\).

**Proof.** As explained in [22, Theorem 3.1], the previous Lemma follows if the following condition

\[
\int_{G} |\Omega(\tilde{g})||\Theta_{Π}(\tilde{g})| |dg| < \infty
\]

is satisfied, where \(Ω\) is defined in Appendix B. Using Lemma B.1, it follows that there exists \(C_Ω > 0\) such that

\[
\int_{G} |\Omega(\tilde{g})||\Theta_{Π}(\tilde{g})| |dg| \leq C_Ω \int_{G} \Xi(\tilde{g}) |\Theta_{Π}(\tilde{g})| |dg|.
\]

Using the fact that every discrete series satisfies the strong inequality (see [26, Section 5.1]), it follows from [26, Lemma 5.1.3] that

\[
\int_{G} \Xi(\tilde{g}) |\Theta_{Π}(\tilde{g})| |dg| < \infty,
\]

and the proposition follows.

**Corollary 7.4.** Assume that \((G, G') = (U(p,q), U(r,s))\), with \(p + q = r + s\), and let \(Π\) be a discrete series representation of \(G\). Then, there exists a constant \(C_{Π(Π)} \in C\) such that \(T(\Theta_{Π}) = C_{Π(Π)} T(\chi C(\Theta_{Π}))\).

**Proof.** The proof of this corollary follows from Corollary 6.11 and Proposition 7.3.

We finish this section with a remark concerning the global character \(Θ_{Π'}\). \(Π' = θ(Π)\) and \(Π \in \mathcal{H}(G, Ω)\) a discrete series representation. We proved in Corollary 6.11 that \(\chi C(Θ_{Π}) = Θ_{Π'}\) if \(rk(G) = rk(G')\). But the global character \(Θ_{Π'}\) can be obtained via \(Θ_{Π}\) in a different way.

As before, we assume that \(rk(G) \leq rk(G')\). In particular, every discrete series representation \(Π \in \mathcal{H}(G, Ω)\) is a sub-representation of \(ω\) and let \(\mathcal{H}(Π)\) be the \(Π\)-isotypic component of \(\mathcal{H}\). As explained in Proposition 7.3, \(T(\Theta_{Π})\) is well-defined. Moreover, using [11, Section 4.8], the operator \(ω(\Theta_{Π})\) is a well-defined operator of \(\mathcal{H}\) and one can check that \(\mathcal{H}(Π) := ω(\Theta_{Π})\) is a projection operator onto \(\mathcal{H}(Π)\). As a \(G \times G'\)-modules, we get \(\mathcal{H}(Π) = Π \otimes Π'\).

Let \(l\) be the Harish-Chandra parameter of \(Π\) and \(ν\) the lowest \(K\)-type of \(Π\). In particular, according to Theorem 5.4 as a \(K \times G'\), we get:

\[
\mathcal{H}(Π) = \bigoplus_{ξ \in K_Π} m_ξ Π_ξ \otimes Π' = Π_ν \otimes Π' \oplus \bigoplus_{ξ \neq ν \in K_Π} m_ξ Π_ξ \otimes Π',
\]

where \(Π_ξ\) is a \(K\)-module of highest weight \(ξ\) and \(K_{Π}\) is the set of irreducible representations of \(K\) such that \(Hom_K(\mathcal{H}_ξ, \mathcal{H}) \neq \{0\}\). We denote by \(\mathcal{H}(Π)(ν)\) the \(ν\)-isotypic component of \(\mathcal{H}(Π)\). We denote by \(\mathcal{P}_ν : \mathcal{H}(Π) \rightarrow \mathcal{H}(Π)(ν)\) the corresponding projection operator and let \(\mathcal{P}_{Π,ν} = \mathcal{P}_ν \circ Π_ν\). Clearly, \(\mathcal{P}_{Π,ν} = Π(\mathcal{P}_{Π,ν}) \circ ω(\Theta_{Π})\). In particular, for every \(φ \in \mathcal{H}(G)\), we get:

\[
Θ_{Π}(φ) = \frac{1}{d_ν} \text{tr}(\mathcal{P}_{Π,ν} \circ ω(φ)) = \frac{1}{d_ν} \text{tr}(\mathcal{P}_{Π,ν} \circ ω(φ)).
\]
In particular, if \( \text{rk}(G) = \text{rk}(G') \), we get using Corollary [6,11] that:

\[
\sum_{r=0}^{p} \frac{1}{|W(H_i)|} \int_{H_i} \Theta_{1}(h_i) |\det(\text{Ad}(h_i)^{-1})| C_{\hbar/|h_i|} \psi(h_i) dh_i = d_{1,\tr} \left( \int_{K} \int_{G} \int_{G'} \Theta_{1}(k) \Theta_{1}(g) \psi(g') \omega(kgg') dg' dg dk \right).
\]

**Appendix A. Some standard isomorphisms**

A.0.1. Universal envelopping algebra of \( \mathfrak{g} \) as differential operators on \( G \). Let \( M \) be a real connected manifold of dimension \( n \). We denote by \( \mathcal{C}^\infty(M) \) the space of smooth functions and \( \mathcal{C}^\infty_c(M) \) the space of compactly supported function on \( \mathcal{C}^\infty(M) \).

We denote by \( \mathcal{D}(M) \) the set of derivations of \( \mathcal{C}^\infty(M) \), i.e.

\[ \mathcal{D}(M) = \{ X : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M), X(fg) = X(f)g + fX(g) \}. \]

The space \( \mathcal{D}(M) \) is the set of \( \mathcal{C}^\infty \)-vectors fields of \( M \).

**Definition A.1.** A continuous endomorphism \( D \) of \( \mathcal{C}^\infty_c(M) \) is called a differential operator if whenever \( U \) is an open set in \( M \) and \( f \) a function of \( \mathcal{C}^\infty_c(M) \), vanishing on \( U \), then \( Df \) vanishes on \( U \).

**Proposition A.2.** Let \( D \) be a differential operator on \( M \). For each \( p \in M \) and each open connected neighbourhood \( U \) of \( p \) on which the local coordinates system \( \Psi : x \rightarrow (x_1, \ldots, x_n) \) is valid, there exists a finite set of functions \( a_a \) of class \( \mathcal{C}^\infty \) such that for each \( f \in \mathcal{C}^\infty_c(M) \) with support contained in \( U \),

\[
Df(x) = \begin{cases} 
\sum_{a=a_1, \ldots, a_n} a_a(x) D^a f \circ \Psi^{-1}(x) & \text{if } x \in U \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** The proof of this result can be found in [11, Proposition 1].

**Notation A.3.** We denote by \( \text{D}(M) \) the set of differential operators.

From now on, we assume that \( G = M \) is a connected Lie group. We denote by \( (L, \mathcal{L}^2(G, dg)) \) the left regular representation. Obviously, the space \( \mathcal{C}^\infty_c(G) \) is \( G \)-invariant.

We define an action of \( G \) on \( \text{D}(G) \) by

\[
\tau(g) \text{D}(f) = L_g \circ \text{D}(f \circ L_{g^{-1}}) \quad (g \in G, f \in \mathcal{C}^\infty_c(G), \text{D} \in \text{D}(G)).
\]

**Definition A.4.** We say that \( \text{D} \in \text{D}(G) \) is left-invariant if \( \tau(g) \text{D} = \text{D} \) for all \( g \in G \), i.e. \( L_g \circ \text{D}(f) = \text{D}(f \circ L_{g}^{-1}) \).

We denote by \( \text{D}_L(G) \) the set of left-invariant differential operators of \( G \). Similarly, we say that \( \text{D} \) is right invariant if \( \tau_1(g)\text{D} = \text{D} \) for every \( g \in G \), where

\[
\tau_1(g) \text{D}(f) = R_{g^{-1}} \circ \text{D}(f \circ R_g) \quad (f \in \mathcal{C}^\infty_c(G)).
\]

The operator \( \text{D} \) is said to be bi-invariant if \( \tau(g) \tau_1(h) \text{D} = \text{D} \) for every \( g, h \in G \), i.e. \( R_h^{-1} \circ L_g \circ \text{D}(L_{g^{-1}} \circ f \circ R_h) = \text{D}(f) \) for every \( f \in \mathcal{C}^\infty_c(G) \).
Notation A.5. We denote by $D^G(G)$ the set of right-invariant differential operators and by $D^G_G(G)$ the set of bi-invariant differential operators.

We recall the following result.

Theorem A.6. The natural embedding $\mathfrak{g} \to D^G(G)$ extends to an algebra isomorphism $U(\mathfrak{g}) \to D^G_G(G)$.

Moreover, its restriction to $Z(U(\mathfrak{g}))$ is isomorphic $D^G_G(G)$.

Proof. The proof of this result can be found in [12]. □

A.0.2. Harish-Chandra isomorphism. Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by $W = W(\mathfrak{g}, \mathfrak{h})$ the corresponding Weyl group. We denote by $\eta^+$ the subalgebra of $\mathfrak{g}$ given by $\eta^+ = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \mathbb{C}X_\alpha$, where $\mathfrak{g}_\alpha = \mathbb{C}X_\alpha$. Similarly, we denote by $\mathcal{N}$ and $\mathcal{P}$ the following subspaces of $\mathcal{U}(\mathfrak{g})$ given by:

$$
\mathcal{N} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} Y_\alpha \mathcal{U}(\mathfrak{g}) \quad \mathcal{P} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} X_\alpha \mathcal{U}(\mathfrak{g}),
$$

where $Y_\alpha$ is a basis of $\mathfrak{g}_{-\alpha}$.

Lemma A.7. We get the following decomposition:

$$
\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathcal{P} + \mathcal{N}).
$$

We denote by $p_1 : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ the natural projection corresponding to Equation (14). We restrict this map to $Z(\mathcal{U}(\mathfrak{g}))$. We denote by $\xi_1$ the map:

$$
\xi_1 : \mathfrak{h} \ni h \to \xi_1(h) = h - \rho(h).1 \in S(\mathfrak{h}),
$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \alpha \in \mathfrak{h}^*$. Using the universal property, we can extend the map $\xi_1$ to $S(\mathfrak{h})$. We denote by $\gamma$ the map:

$$
\gamma = \xi_1 \circ p_1 : Z(\mathcal{U}(\mathfrak{g})) \to S(\mathfrak{h})^W.
$$

Theorem A.8. The map $\gamma$ is an algebra homomorphism which is injective. Moreover, $\text{Im}(\gamma) = S(\mathfrak{h})^W$ and then:

$$
\gamma : Z(\mathcal{U}(\mathfrak{g})) \to S(\mathfrak{h})^W.
$$

is a bijection.

Remark A.9. Harish-Chandra’s isomorphism classify all the possible infinitesimal character. Indeed, let $\lambda : \mathfrak{h} \to \mathbb{C}$ be a linear map. Using the universal property of the symmetric algebra, the linear form $\lambda$ can be extended to a linear map $\lambda : S(\mathfrak{h}) \to \mathbb{C}$ and by using the map $\lambda$, we get a map $\chi_\lambda : Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$ given by:

$$
\chi_\lambda(z) = \lambda(\gamma(z)) \quad (z \in Z(\mathcal{U}(\mathfrak{g}))).
$$
We recall the following theorem.

**Theorem A.10.** Let \( g \) be a complex reductive Lie algebra and \( \mathfrak{h} \) a Cartan subalgebra of \( g \). Then every homomorphism of \( Z(\mathbb{U}(g)) \) into \( \mathbb{C} \) sending 1 to 1 is of the form \( \chi_A, \lambda \in \mathfrak{h}^* \). If \( \lambda \) and \( \lambda' \) are in \( \mathfrak{h}^* \), then \( \chi_A = \chi_A' \) if and only if \( \lambda \) and \( \lambda' \) are in the same \( \mathfrak{u} \)-orbit.

In particular, \( \text{Spec}(Z(\mathbb{G}(g))) = \mathfrak{b}' \backslash \mathfrak{u}' \).

The proof of this result can be found in [16].

**Appendix B. A general lemma for unitary groups**

Let \( U \) be a maximal compact subgroup of \( \text{Sp}(W) \). It is well-known that the restriction of \( \omega \) to \( \widetilde{U} \) is a direct sum of irreducible representations whose multiplicity is one. Moreover, the lowest \( \widetilde{U} \)-type \( V_\omega \) is one-dimensional. Let \( x \) be a non-zero vector in \( V_\omega \) and let \( \Omega \) be the function on \( \text{Sp}(W) \) given by

\[
\Omega(\tilde{g}) = \langle \omega(\tilde{g})x, x \rangle, \quad (\tilde{g} \in \text{Sp}(W)).
\]

We denote by \( \xi_\omega \) the (unitary) character of \( \widetilde{K} \) such that \( \omega(\tilde{k})x = \xi_\omega(\tilde{k})x, \tilde{k} \in \widetilde{K} \). One can check that for every \( \tilde{k}_1, \tilde{k}_2 \in \widetilde{K} \) and \( \tilde{g} \in \widetilde{G} \),

\[
\Omega(\tilde{k}_1\tilde{k}_2) = \langle \omega(\tilde{k}_1\tilde{k}_2)x, x \rangle = \langle \omega(\tilde{k}_2)x, \omega(\tilde{k}_1^{-1})x \rangle = \xi_\omega(\tilde{k}_2k_1^{-1})(\langle \omega(\tilde{g})x, x \rangle) = \xi_\omega(\tilde{k}_2k_1^{-1})\Omega(\tilde{g}).
\]

In particular, the map \( \tilde{G} \ni \tilde{g} \rightarrow |\Omega(\tilde{g})| \in \mathbb{C} \) is \( \widetilde{K} \)-bi-invariant. In particular, using the decomposition \( \text{Sp}(W) = \widetilde{K}\widetilde{A}\widetilde{K} \) as in [26] Section 3.6.7, with \( A = \text{Cl}(A^\times), A^\times = \exp(a^\times) \), \( a \) the maximal split Cartan subalgebra of \( \text{sp}(W) \) and \( a^\times = \{ H \in a, a(H) > 0, a \in a^* \} \), we get for every \( \tilde{g} = \tilde{k}_1\tilde{a}\tilde{k}_2 \in \widetilde{K}\widetilde{A}\widetilde{K} \) that \( |\Omega(\tilde{g})| = |\Omega(\tilde{a})| \).

We denote by \( \Xi \) the \( \widetilde{K} \)-bi-invariant function defined in [26] Section 4.5.3.

**Lemma B.1.** Let \( (G, G') = (U(V), U(V')) \subseteq \text{Sp}(V \otimes V'_{\mathbb{H}}) \) be a dual pairs of unitary groups such that \( \dim_{\mathbb{C}}(V) \leq \dim_{\mathbb{C}}(V') \). Then, there exists a constant \( C_{\Omega} > 0 \) such that

\[
|\Omega(\tilde{g})| \leq C_{\Omega}\Xi(\tilde{g}), \quad (\tilde{g} \in \widetilde{G}).
\]

**Remark B.2.** As explained in [22] Section 6.4.1, for an irreducible reductive dual pair \( (G, G') \), there exist a constant \( C = C_{d,d'} > 0 \) such that

\[
|\Omega(\tilde{c}(X))| = C|\det(\tilde{c}(\text{Id} - X))|^{\frac{d}{2}}|\det(\text{Id} - JX)|^{-\frac{d'}{2}}, \quad (X \in \mathfrak{g}'),
\]

where \( d = \dim_{\mathbb{C}}(V) \) and \( d' = \dim_{\mathbb{C}}(V') \).

**Proof.** We start by determining the value of \( \Omega \) for the dual pair \( (G, G') = (U(1, 1), U(1)) \). Let \( G = \text{KAK} \) be the decomposition of \( G \) as in [26] Section 3.6.7. In this case,

\[
A = \left\{ \begin{pmatrix} \text{ch}(X) & \text{sh}(X) \\ \text{sh}(X) & \text{ch}(X) \end{pmatrix}, X \in \mathbb{R}^*_+ \right\}.
\]

Let \( a(X) \in \mathfrak{a}^\times \) and \( b(X) \in \mathfrak{a}^\times \) such that \( a(X) = c(b(X)) \). One can easily check that

\[
b(X) = \begin{pmatrix} 0 & a(X) \\ a(X) & 0 \end{pmatrix},
\]

where \( a(X) = \frac{\text{sh}(X)}{\text{ch}(X) - 1} \). Note that \( a(X) = \frac{1}{\text{th}(\frac{X}{2})} \).
Let $\mathcal{B} = \{e_1, e_2\}$ be a basis of $V$ such that $\text{Mat}_{\mathcal{B}}(\cdot, \cdot) = \text{Id}_{1,1}$. Then, using that $\mathcal{B}_R = \{e_1, e_2, ie_1, ie_2\}$ is a basis of the real vector space $V_R$, it follows that:

$$\det_{\mathcal{B}}(\text{Id} - b(X)) = \det \begin{pmatrix} 1 & -\alpha(X) & 0 & 0 \\ -\alpha(X) & 1 & 0 & 0 \\ 0 & 0 & 1 & -\alpha(X) \\ 0 & 0 & -\alpha(X) & 1 \end{pmatrix} = (1 - \alpha(X)^2)^2.$$  

Similarly, using that

$$J = \text{Mat}_{\mathcal{B}_R}(\cdot, \cdot)_R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

we get:

$$\det(i\text{Id} - Jb(X)) = \det \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - \det \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & -1 & -\alpha(X) \\ 0 & 0 & \alpha(X) & 0 \\ 0 & 0 & \alpha(X) & 0 \\ 1 & \alpha(X) & 0 & 0 \end{pmatrix} = (1 + \alpha(X)^2)^2.$$  

Using that

$$\frac{\text{th}(X)^2 - 1}{\text{th}(X)^2 - 1} = \frac{-1}{\text{ch}(X)},$$

it follows from Remark 4.2 that there exists $C > 0$ such that:

$$|\Omega(\tilde{c}(b(X)))| = C \frac{1 - \alpha(X)^2}{1 + \alpha(X)^2} = C \frac{\text{ch}(X)}{\text{ch}(X)^2}.$$  

In particular, for the dual pair $(G, G') = (U(1, 1), U(n))$, we get for every $a(X) = c(b(X)) \in \Lambda^c \subseteq G$ that:

$$|\Omega(\tilde{c}(b(X)))| = C \frac{\text{ch}(X)}{\text{ch}(X)^n}.$$  

From [26] Theorem 4.5.3, we know that $\tilde{a}(X)^{-p} \leq \Xi(\tilde{a}(X))$, with $\tilde{a}(X) = \tilde{c}(b(X))$. In our case, we get that $\tilde{a}(X)^{-p} = e^{-X}$ and using that for every $n \geq 1, \text{ch}(X)^n \geq \text{ch}(X) \geq e^X$, it follows that:

$$|\Omega(\tilde{c}(b(X)))| = \frac{C}{\text{ch}(X)^n} \leq \frac{C}{\text{ch}(X)} \leq C e^{-X} \leq C \Xi(\tilde{a}(X)).$$

In particular, using the $\tilde{K}$-bi-invariance of $\Omega$ and $\Xi$, we get that $\Omega(\tilde{g}) \leq C \Xi(\tilde{g})$ for every $\tilde{g} \in \tilde{G}$ for $(G, G') = (U(1, 1), U(n))$. One can easily check that $G'$ can be replaced by $U(r, s)$ and the computations are similar.
We can now extend it to \((G, G') = (U(p, p), U(n))\). In this case,
\[
A = \left\{ D = \begin{pmatrix} D_1(X) & D_2(X) \\ D_2(X) & D_1(X) \end{pmatrix}, X \in \mathbb{R}^{+}_{p} \right\}
\]
where for \(X = (X_1, \ldots, X_p)\), \(D_1(X) = \text{diag}(\text{ch}(X_1), \ldots, \text{ch}(X_p))\) and \(D_2(X) = \text{diag}(\text{sh}(X_1), \ldots, \text{sh}(X_p))\). One can easily check that there exists \(C > 0\) such that
\[
|\Omega(\tilde{c}(b(X)))| = \frac{C}{\prod_{i=1}^{p} \text{ch}(X_i)^p}.
\]
In this case, \(\rho = \sum_{i=1}^{2p} \frac{2p - 2i + 1}{2} e_i\), and from Equation 2, we get
\[
\tilde{a}(X)^{-\rho} = \text{diag}(e^{-X_1}, \ldots, e^{-X_p}, e^{X_1}, \ldots, e^{X_p})^{-\rho} = \prod_{k=1}^{p} e^{-2pX_k}
\]
If \(n \geq 2p\), it follows that \(\text{ch}(X_k)^{\rho} \geq \text{ch}(X_k)^{2p} \geq e^{2pX_k}\) for every \(k \in \{1, p\}\) and then,
\[
|\Omega(\tilde{c}(b(X)))| = \frac{C}{\prod_{i=1}^{p} \text{ch}(X_i)^{\rho}} \leq C \prod_{k=1}^{p} e^{-2pX_k} \leq \Xi(\tilde{a}(X)).
\]
Again, \(U(n)\) can be replaced by \(U(r, s)\) as long as \(r + s \geq 2p\). Finally, it \(G = U(p, q)\), with \(p \leq q\), we get that:
\[
A = \left\{ D = \begin{pmatrix} D_1(X) & D_2(X) \\ D_2(X) & D_1(X) \end{pmatrix}, X \in \mathbb{R}^{+}_{p} \right\}
\]
where \(D_1(X) = \text{diag}(\text{ch}(X_1), \ldots, \text{ch}(X_p))\), \(D_2(X) = \text{diag}(\text{sh}(X_1), \ldots, \text{sh}(X_p)), 0_{p,q}\), where \(0_{p,q}\) is the zero matrix of \(\text{Mat}(p, q - p)\), and
\[
D_3(X) = \begin{pmatrix} \text{diag}(\text{ch}(X_1), \ldots, \text{ch}(X_p)) & 0_{p,q-p} \\ 0_{q-p,p} & 0_{q-p,q-p} \end{pmatrix},
\]
and one can check that the computations done for \(U(p, p)\) extends easily to \(U(p, q)\). The lemma follows.

\[\square\]

**Remark B.3.** One can easily see that the condition \(\dim_{C}(V) \leq \dim_{C}(V')\) of Lemma B.1 does not mean that similar estimates cannot be obtained in some cases if \(\dim_{C}(V) > \dim_{C}(V')\). Indeed, it follows from the proof of Lemma B.1 that the inequality (15) can be obtained for \((G, G') = (U(1, 1), U(1))\).

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