Heterogeneous Facility Location Games

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Abstract

We study heterogeneous $k$-facility location games. In this model there are $k$ facilities where each facility serves a different purpose. Thus, the preferences of the agents over the facilities can vary arbitrarily. Our goal is to design strategy proof mechanisms that place the facilities in a way to maximize the minimum utility among the agents. For $k = 1$, if the agents’ locations are known, we prove that the mechanism that places the facility on an optimal location is strategy proof. For $k \geq 2$, we prove that there is no optimal strategy proof mechanism, deterministic or randomized, even when $k = 2$ there are only two agents with known locations, and the facilities have to be placed on a line segment. We derive inapproximability bounds for deterministic and randomized strategy proof mechanisms. Finally, we focus on the line segment and provide strategy proof mechanisms that achieve constant approximation. All of our mechanisms are simple and communication efficient. As a byproduct we show that some of our mechanisms can be used to achieve constant factor approximations for other objectives as the social welfare and the happiness.

1 Introduction

Facility location games lie in the intersection of AI, game theory, and social choice theory and have been studied extensively over the past years. The basic version of the problem was firstly studied by Procaccia and Tennenholtz [29]. In this setting, a central planner has to locate a facility on a real line based on the reported locations of selfish agents who want to be as close as possible to the facility. The goal of the planner is to locate the facility in a way that the sum of the utilities of the agents is maximized. However, the agents can misreport their locations in order to manipulate the planner and increase their utility. One main objective of the planner is to design procedures to locate the facility, called mechanisms, that incentivize the agents to report their true locations, i.e., the mechanisms are strategy proof.

When monetary payments are not allowed, that is the planner cannot pay the agents or demand payments from them, it is not always possible to design mechanisms that implement an optimal solution and remain strategy-proof. Thus, the goal is to design mechanisms that approximately maximize an objective function under the constraint that they are strategy proof. The term approximate mechanism design without money, introduced by Procaccia and Tennenholtz, is usually deployed for problems like the one described above. Procaccia and Tennenholtz studied homogeneous facility location games, where one, or two, identical facilities had to be placed on a real line and every agent wanted to be as close as possible to any of them. In this setting, the agents were reporting to the planner a point on the line and the objectives studied were the maximization of the social welfare or the minimum utility among the agents.

In many real-life scenarios though, both facilities and the preferences of the agents are heterogeneous; every facility serves a different need and every agent has potentially different needs from the others. Consider for example the case where the government is planning to build a school and a factory. Citizens’ preferences for these facilities might significantly differentiate. Those who work at the factory and also have children that go to school wish both facilities to be built close to their homes. Citizens without children might want the school to be built far because of the noise. Finally, those who do not work at the factory prefer its location to be far from their home to avoid emitted pollution.

The example above shows that an agent might want to be close to a facility, be away from a facility, or be indifferent about its presence. Feigenbaum and Sethuraman [10] studied 1-facility heterogeneous

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In [29] the objective was to minimize the social cost.
games where each agent reported his preferred location on the line, while it was known to the planner, whether he wanted to be close to, or away from, the facility. Zou and Li [36] extended the model of [10] for heterogeneous 2-facility games and studied the social utility objective for several different scenarios of the information the planner knows. Serafini and Ventre [30] studied heterogeneous 2-facility games on discrete networks. In their setting, each agent is located on a node of a graph and either is indifferent or wants to be close to each facility and the planner knows the location of every agent but not their preferences for the facilities.

In this paper, we extend the aforementioned models and study heterogeneous \( k \)-facility location games; simply \( k \)-facility games. Our main focus is to maximize the minimum utility among all the agents, termed Egalitarian. As a byproduct, we derive results for the social welfare, termed Utilitarian, and the recently proposed minimum happiness objective, termed Happiness. Happiness, which is reminiscent of the proportionality notion in resource allocation problems, is a fairness criterion for facility location problems introduced in [26]. The happiness of an agent is the ratio between the utility he gets under the locations of the facilities over the maximum utility the agent could get under any location. To the best of our knowledge, there is no prior work on this model. We note that while our model is a natural extension of the aforementioned models almost none of those results apply in our case.

1.1 Our contributions

We study several questions regarding heterogeneous \( k \)-facility games. Firstly, we focus on the case where there is only one facility to be located. Feigenbaum and Sethuraman [10] have proven that there is no deterministic strategy-proof mechanism with bounded approximation for Egalitarian for this case where the preferences of the agents are known and their locations are unknown. We study the complementary case where the locations are known and the preferences are not known to the planner. We prove that in this case the mechanism that places the facility on an optimal location for the reported preferences of the agents is strategy-proof. In fact, our result is much stronger since it holds for any combination of the following relaxations.

- The utility function of every agent can be any function that it is monotone with respect the distance between the location of the agent and the location of the facility. Thus, if an agent wants to be close to the facility, his utility decreases with the distance, and if he wants to be away from the facility, his utility increases.
- The domain \( D_i \) of every agent’s possible locations and the domain \( F \) of allowed locations for the facility can be any subset of \( \mathbb{R}^d \). In addition, it can be the case that \( F \cap D_i = \emptyset \) for every agent \( i \).

Next, we focus on the Egalitarian objective. We prove that there is no optimal deterministic strategy proof or strategy proof in expectation mechanism for \( k \)-facility games even for instances with \( k = 2 \), two agents, and known locations for the agents. We complement these results by deriving inapproximability bounds for deterministic and randomized strategy-proof mechanisms. The techniques we use are fundamentally different from [30], since in our model, the facilities can be located anywhere on the segment without any constraint, making the analysis more complex.

Then, we focus on 2-facility games and we propose strategy-proof mechanisms that achieve constant approximation ratio for the Egalitarian objective. All of our mechanisms are simple (i.e. if it requires minimal information from the agents (bit-wise)) and require limited communication. To the best of our knowledge, this is the first paper to study the communication complexity on facility location problems and how communication affects approximation. We propose two deterministic and two randomized mechanisms. The first deterministic mechanism, called Fixed, requires zero communication between the planner and the agents. On any instance, Fixed locates the facilities symmetrically away from the middle of the segment without requiring any information from the agents. Although this mechanism might seem naive, it achieves constant approximation. Furthermore, we prove that Fixed is optimal when no communication is allowed. No communication means that the agents do not transmit any bits to the planner before the locations for the facilities are decided, or equivalently that the facilities have to be located without getting any information from the agents. The second mechanism, termed Fixed*, utilizes the intuition gained from Fixed and chooses between five different location-combinations for the facilities and locates the facilities in one of them by using the information it got from the agents. Furthermore, every agent has to communicate only 5 bits of information to the agent. Our first
randomized mechanism, termed Random, places with half probability both facilities on the beginning of the segment and with half probability both facilities on the end of the segment. Random seems naive, but it achieves \( \frac{1}{3} \)-approximation, it is universally strategy-proof and requires zero communication. The second randomized mechanism, Random*, combines the ideas of Random and Fixed*, it is strategy-proof in expectation and improves upon Random by requiring again only 5 bits of information per agent.

For the special case where agents’ locations are known to the mechanism and all the agents are indifferent or want to be close to the facilities, we show how we can utilize the optimal mechanism for the 1-facility game and get a \( \frac{3}{4} \)-approximate strategy-proof mechanism for Egalitarian when \( k = 2 \).

As a byproduct, we show that Fixed and Random achieve the same approximation guarantee for Happiness and Utilitarian. Thus, we establish lower bounds that were not known before and complement the results of [36].

1.2 Further related work

There is a long line of work on homogeneous facility location games [1, 8, 14, 17, 22, 25, 24, 27, 35]. Different objectives and different utility functions have been studied as well. In [11] the objective was the sum of \( L_p \) norms of agent’s utilities, while in [12] it was the sum of least squares. In [15] introduced double-peaked utility functions. The obnoxious facility game on the line, where every agent wants to be away from the facilities, was introduced in [6] and later the model was extended for trees and cycles in [7]. In [23], the objective of least squares for obnoxious agents, was studied. The maximum envy was recently introduced as an objective for facility location games in [3]. In that paper as well as in [18] the authors studied the approximation of mechanisms according to additive errors. False-name proof mechanisms for the location of two identical facilities were studied in [31] while [32] gave a characterization of strategy-proof and group strategy-proof mechanisms in metric networks for 1-facility games with private locations of the agents. Since the conference version of this paper [2], three other papers on heterogeneous facility location games have appeared. In [29], the authors studied heterogeneous 2-facility games on a line segment, under the extra constraint where the locations between the two facilities have to be at least a certain distance. In [21], the authors studied heterogeneous facility location games where the agents were located on a line but the facility could be placed in a region on the plane.

Simple mechanisms received a lot of attention lately; see [10] for example and the references therein for simple auctions. Informally, a simple mechanism is easy to implement and allows the agents to “easily” deduce the strategy proofness of the mechanism. One way to capture simplicity is to use verifiably truthful mechanisms [3], where agents can check whether a mechanism is strategy-proof by using some, possibly exponential, algorithm. Simple mechanisms were formalized in [23] by introducing obviously strategy-proof mechanisms. [14] analysed this type of mechanisms for homogeneous 1-facility games.

After a long history in theoretical computer science [20], communication complexity problems have been studied in auction settings [3] and in facility location games [13] but with ordinal preferences of the agents as input to the mechanisms. Communication complexity has also been studied in other more general mechanism design problems [28, 34]. To the best of our knowledge, no one studied the communication complexity of facility location games on the line with cardinal utilities.

2 Model

In a \( k \)-facility game, there is a set \( N = \{1, \ldots, n\} \) of agents located in \( \mathbb{R}^d \) and a set of \( k \) distinct facilities \( F = \{1, \ldots, k\} \) that need to be placed in \( S \subseteq \mathbb{R}^d \). Each agent \( i \) is associated with a location \( x_i \in \mathbb{R}^d \) and a vector \( t_i \in \{-1, 0, 1\}^k \) that represents his preferences for the facilities.

If agent \( i \) wants to be far from facility \( j \), then \( t_{ij} = -1 \); if he is indifferent, then \( t_{ij} = 0 \); if he wants to be close to \( j \), then \( t_{ij} = 1 \). We will use \( y = (y_1, \ldots, y_k) \) to denote the locations of the facilities and \( s = (s_1, \ldots, s_n) \) to denote the profile of the agents, i.e. their declared tuples \( s_i = (x_i, t_i) , \forall i \in N \). A vector \( s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \) is the vector of tuples excluding \( s_i \), thus we can denote a profile as \( (s_i, s_{-i}) \).

The utility that agent \( i \) gets from facility \( j \), denoted as \( u_{ij}(x_i, t_i, y) \), depends on the distance \( \text{dist}(x_i, y_j) \) between the location of the agent and the location of the facility \( j \), and on the agent’s preference \( t_{ij} \) for that facility. We assume that \( u_{ij} \) follows the rules below:

- If \( t_{ij} = -1 \), then \( u_{ij}(x_i, t_i, y) \) is strictly increasing with \( \text{dist}(x_i, y_j) \).
- If \( t_{ij} = 0 \), then \( u_{ij}(x_i, t_i, y) \) is a constant independent of \( \text{dist}(x_i, y_j) \).
- If \( t_{ij} = 1 \), then \( u_{ij}(x_i, t_i, y) \) is strictly decreasing with \( \text{dist}(x_i, y_j) \).

The total utility agent \( i \) gets under \( y \) is defined as the sum of the utilities he gets for each of the facilities, i.e. \( u_i(x_i, t_i, y) = \sum_{j \in [k]} u_{ij}(x_i, t_i, y_j) \). We consider three different objective functions: \text{Égalitarian}, defined as \( \max_y \min_{s \in S_i} u_i(x_i, t_i, y) \); \text{Utilitarian} defined as \( \max_y \sum_i u_i(x_i, t_i, y) \); and \text{Happiness} defined as \( \max_y \min_{s \in S_i} \frac{u_i(x_i, t_i, y)}{u_i(x_i, t_i, y_j)} \) where \( u_i^*(x_i, t_i) = \max_y u_i(x_i, t_i, y) \).

A mechanism \( M \) is an algorithm that takes as input a profile \( s \) and outputs the locations of the facilities, \( y \). A mechanism is \text{deterministic} if it chooses \( y \) deterministically and it is \text{randomized} if \( y \) is chosen according to a probability distribution. Let \( \text{OPT}(s) \) and \( M(s) \) denote the optimal value and the value of mechanism \( M \) for an objective function under the profile \( s \) respectively. A mechanism \( M \) achieves an approximation ratio \( \alpha \leq 1 \), or it is \( \alpha \)-approximate, if for any type profile \( s_i \), it holds that \( M(s) \geq \alpha \cdot \text{OPT}(s) \). A mechanism is called strategy-proof if no agent can benefit by misreporting his preferences. Formally, a mechanism \( M \) is strategy-proof if for any true profile \( (s_i, s_{-i}) \) it returns locations \( y \) and any misreported profile \( (s_i', s_{-i}) \) it returns \( y' \), we have that \( u_i(x_i, t_i, y) \geq u_i(x_i, t_i, y') \).

A randomized mechanism is universally strategy proof if it is a probability distribution over deterministic strategy proof mechanisms and strategy proof in expectation if no agent can increase his expected utility by misreporting his type. Furthermore, a mechanism is called \text{false-name proof} if no agent can benefit by using multiple and different identities in the game.

### 2.1 Facility location on a line segment

A special case of \( k \)-facility games is when all the agents are located on the line segment \([0, \ell]\), where \( \ell > 0 \). This case is extensively studied in the literature \cite{26,40} since the definitions above are greatly simplified. For normalization purposes we assume that the maximum utility an agent can get from any facility is \( \ell \) and we define the utility function of agent \( i \) as follows.

\[
 u_{ij}(x_i, t_i, y_j) = \begin{cases} 
 |x_i - y_j|, & \text{if } t_{ij} = -1 \\
 \ell, & \text{if } t_{ij} = 0 \\
 \ell - |x_i - y_j|, & \text{if } t_{ij} = 1. 
\end{cases}
\]  

### 3 1-facility games with known locations

We first study the case where the locations of the agents are publicly known and only one facility has to be placed. We will show that the mechanism which places the facility on an optimal location using any \text{declaration-independent} tie-breaking rule is strategy-proof for \text{Égalitarian}, \text{Utilitarian}, \text{Happiness} objectives.

**Definition 1.** A mechanism \( M \) has a \text{declaration-independent} tie-breaking rule if it outputs the same \( y \) for any two profiles \( s \neq s' \) with \( M(s) = M(s') \).

Hence, a mechanism has a \text{declaration-independent} tie-breaking rule if it outputs the same location for the facility for all profiles that yield the same value for the objective we are trying to optimise. An example of such a rule is the lexicographic minimum.

### Mechanism 1 1-facility-\( \mathcal{O} \)-opt

**In:** For every agent \( i \): public location \( x_i \in \mathbb{R}^d \), private preference \( t_i \in \{-1, 0, 1\} \); region \( S \subseteq \mathbb{R}^d \); objective \( \mathcal{O} \); declaration-independent tie-breaking rule \( T \).

**Out:** Location \( y^* \in S \) for the facility.

1. Let \( Y \subseteq S \) such that every \( y \in Y \) optimizes \( \mathcal{O} \) for the given locations and preferences, excluding the agents with preference \( 0 \).
2. Choose \( y^* \in Y \) according to the tie-breaking rule \( T \).
Mechanism does not make any assumptions about the dimensions of the agents locations and the region \( S \). So, the actual locations of the agents can be in \( \mathbb{R}^d_i \) and the region \( S \subseteq \mathbb{R}_i^d \), where \( d_1 \neq d_2 \). In addition, \( S \) can be of an arbitrary form, i.e. it can be the union of several disjoint regions of \( \mathbb{R}^d \).

### 3.1 Analysis for Egalitarian objective

In this section we focus on the Egalitarian objective, i.e. \( O = \max_y \min_i u_i(x_i, t_i, y) \). In order to prove that Mechanism is strategy-proof for Egalitarian, we partition the agents into two sets \( T_1 \) and \( T_2 \). \( T_1 \) contains the agents with the minimum utility when the facility is placed on \( y^* \) and \( T_2 = N \setminus T_1 \). Since agents with preference type 0 have constant utility independently of \( \text{Egalitarian} \).

**Lemma 1.** No agent from \( T_2 \) can increase his utility by lying.

**Proof.** For the sake of contradiction suppose that an agent \( i \in T_2 \) with preference \( t_i \) declares preference \( t_i' \) and increases his utility. Let \( y' \) be the optimal location of the facility in this case. Since we have assumed that agent \( i \) increases his payoff, we have that \( u_i(x_i, t_i, y^*) < u_i(x_i, t_i, y') \). We will consider two cases depending on the declaration \( t_i' \).

- \( t_i' = 0 \). Recall, in this case, Mechanism excludes agent \( i \) from the computation of \( y' \). Since \( u_i(x_i, t_i, y^*) < u_i(x_i, t_i, y') \), we get that \( y^* \neq y' \). In addition, we get that \( \min_{j \neq i} u_j(x_j, t_j, y') > \min_{j \neq i} u_j(x_j, t_j, y^*) \); if this was not the case, the mechanism could return \( y^* \) and increase the value of the objective. Hence, we get that \( \min_{j \neq i} u_j(x_j, t_j, y') > \min_{j \neq i} u_j(x_j, t_j, y^*) \). This means that \( y' \) is a better solution than \( y^* \) for the Egalitarian objective, which contradicts the assumption that \( y^* \) is an optimal solution.

- \( t_i' \neq 0 \). The utility of agent \( i \) will change only if the location of the facility changes; this is due to the declaration-independent tie-breaking rule \( T \). This will happen only if \( u_i(x_i, t_i', y^*) < \min_{j \neq i} u_j(x_j, t_j, y^*) \). This means that \( u_i(x_i, t_i', y^*) < u_i(x_i, t_i, y^*) \). Without loss of generality let \( t_i' = 1 \). In this case \( \text{dist}(x_i, y^*) > \text{dist}(x_i, y') \), i.e. the new optimal location is closer to \( x_i \). But this means that \( u_i(x_i, t_i, y^*) < u_i(x_i, t_i, y') \) and since \( t_i = -1 \) this contradicts the assumption that agent \( i \) can increase his utility by misreporting his preference.

Next we prove that no agent from \( T_1 \) has an incentive to lie about his preferences.

**Lemma 2.** No agent from \( T_1 \) can increase his utility by lying.

**Proof.** We will prove the claim by contradiction. Suppose that an agent \( i \in T_1 \) with preference \( t_i \) can increase his utility by declaring \( t_i' \). Using exactly the same arguments as in Lemma we can see that \( t_i' \neq 0 \). Let \( y' \neq y^* \) be the optimal location for the facility when agent \( i \) declares \( t_i' \). Clearly, if \( y' = y^* \) agent \( i \) has no reason to lie. We now consider the following two cases:

- \( u_i(x_i, t_i', y^*) \geq u_i(x_i, t_i, y^*) \). Since we have assumed that agent \( i \) increases his utility by declaring \( t_i' \), we have that \( u_i(x_i, t_i', y^*) > u_i(x_i, t_i, y^*) \). Hence, \( \min_{j \neq i} u_j(x_j, t_j, y^*) > \min_{j \neq i} u_j(x_j, t_j, y^*) \) since an agent \( j \neq i \) who now has the minimum utility is the one who determines the new outcome \( y' \). We note that \( \min_{j \in N} u_j(x_j, t_j, y^*) \) should be strictly larger than \( \min_{j \in N} u_j(x_j, t_j, y^*) \) since Mechanism uses a declaration-independent tie-breaking rule. So the location should not change if the value of the objective remains the same. But then we have that \( \min_{j \in N} u_j(x_j, t_j, y^*) > \min_{j \in N} u_j(x_j, t_j, y^*) \) which contradicts the fact that \( y^* \) is an optimal location for the facility.

- \( u_i(x_i, t_i', y^*) < u_i(x_i, t_i, y^*) \). This means that agent \( i \) under the declaration \( t_i' \) has the smallest utility over all the agents. Hence, one of the following cases must be true since we assumed that \( t_i' \neq 0 \). The first one is when \( t_i = -1 \) and \( t_i' = 1 \). Since the utility of agent \( i \) under the declaration \( t_i' \) increased, it means that \( \text{dist}(x_i, y^*) < \text{dist}(x_i, y^*) \), i.e. the facility must be placed closer to his location \( x_i \). But this means that his utility under the true preference \( t_i \) decreased because the agent wants to be away from the facility. Similarly when \( t_i = 1 \) and \( t_i' = -1 \) the facility must be placed further away from the position of the agent, while the agent wants to be close to the facility. Hence, in both cases the utility of agent \( i \) decreases.
Hence, we have shown that for any declaration \( t \) theorem follows.

Theorem 1 complements in a sense the result of [10], where it was proven that there is no deterministic strategy-proof mechanism with bounded approximation for the Egalitarian objective for 1-facility games even on a line segments with known preferences but unknown locations.

### 3.2 Analysis for the Utilitarian objective

In this section we focus on the Utilitarian objective, i.e. \( \mathcal{O} = \max_y \sum_i u_i(x_i, t_i, y) \). Again, since agents with preference type 0 have constant utility independently of \( y \), we will assume that there is no agent \( i \) with \( t_i = 0 \).

**Theorem 2.** Mechanism \( \mathcal{M} \) is an optimal strategy-proof for the Utilitarian objective.

Proof. We will prove the theorem by contradiction. So, assume that there exists an agent \( i \) who can increase his utility by declaring \( t'_i \neq t_i \). Let \( y^* \) be the optimal location of the facility when \( i \) declares \( t_i \) and \( y' \neq y^* \) be the location of the facility when he declares \( t'_i \). So, by assumption, we have that \( u_i(x_i, t_i, y^*) < u_i(x_i, t_i, y') \).

Firstly, assume that \( t'_i = 0 \). Then, Mechanism \( \mathcal{M} \) excludes agent \( i \) from the computation of \( y' \). In addition, since the mechanism uses a declaration-independent tie-breaking rule and \( y' \neq y^* \), it must be true that

\[
\sum_{j \neq i} u_j(x_j, t_j, y^*) < \sum_{j \neq i} u_j(x_j, t_j, y').
\]

If this was not the case, we would increase the value of the objective by choosing \( y^* \) instead. Thus, since we assumed that \( u_i(x_i, t_i, y^*) < u_i(x_i, t_i, y') \), we get that \( \sum_{j \neq i} u_j(x_j, t_j, y^*) < \sum_{j \neq i} u_j(x_j, t_j, y') \) which contradicts the assumption that \( y^* \) maximizes the social welfare.

Having established that \( t'_i \neq 0 \), we consider the following two cases depending on the utilities of the rest of the agents under \( y^* \) and \( y' \). In what follows we will assume that \( t_i = 1 \) and \( t'_i = -1 \); the arguments for \( t_i = -1 \) and \( t'_i = 1 \) are similar.

- \( \sum_{j \neq i} u_j(x_j, t_j, y^*) \geq \sum_{j \neq i} u_j(x_j, t_j, y') \). Then, as above, we get that \( y^* \) does not maximize the welfare objective since we have assumed that \( u_i(x_i, t_i, y^*) < u_i(x_i, t_i, y') \).

- \( \sum_{j \neq i} u_j(x_j, t_j, y^*) \neq \sum_{j \neq i} u_j(x_j, t_j, y') \). Since \( y^* \neq y' \) and since Mechanism \( \mathcal{M} \) has a declaration-independent tie-breaking rule, it should be true that

\[
\sum_{j \neq i} u_j(x_j, t_j, y^*) + u_i(x_i, t'_i, y^*) < \sum_{j \neq i} u_j(x_j, t_j, y') + u_i(x_i, t'_i, y').
\]

So, we get that \( u_i(x_i, t'_i, y^*) < u_i(x_i, t'_i, y') \) and since we have assumed that \( t'_i = -1 \) we get that \( \text{dist}(x_i, y^*) < \text{dist}(x_i, y') \). This, in turn means that \( u_i(x_i, t_i, y^*) > u_i(x_i, t_i, y') \) which is a contradiction. Hence, we have shown that for any declaration \( t'_i \) the utility of the agent cannot increase. Thus, the theorem follows.

### 3.3 Analysis for the other objectives

Observe that in the analysis in Sections 3.1 and 3.2 the only assumptions about the utility functions of the agents is that they are monotone with respect to the distance between the location of the agent and the location of the facility. Thus, every agent can have his own type of utility function, completely different than the types of the other agents.

Recall, for Happiness we have that \( \mathcal{O} = \max_y \min_i \frac{u_i(x_i, t_i, y)}{u'_i(x_i, t_i)} \) where \( u'_i(x_i, t_i) = \max_y u_i(x_i, t_i, y) \). Observe though, \( u'_i(x_i, t_i) \) is a constant, hence the analysis of Section 3.1 applies here as well. So, can set get the following as a corollary of Theorem 1.
Corollary 1. Mechanism $\mathcal{M}$ is an optimal strategy-proof for the Happiness objective.

4 Inapproximability results

In the remainder of the paper, unless specified otherwise, we study the Egalitarian objective. In this section, we provide inapproximability results for strategy-proof mechanisms for 2-facility games. We show that the second facility changes dramatically the landscape of strategy-proofness. We prove that the extension of the optimal mechanism for two facilities, i.e. placing the facilities on the locations that maximize the objective under the declared preferences of the agents, is not strategy proof even in the setting of a line segment with two agents and known locations. Furthermore, we provide inapproximability results for strategy-proof mechanisms.

We first prove that there is no 0.851-approximate deterministic strategy proof and then extend it to strategy-proof in expectation mechanisms.

Theorem 3. There is no $\alpha$-approximate deterministic strategy proof mechanism for the 2-facility game with $\alpha \geq 0.851$.

Proof. Let us consider the instances $I$ and $I'$ depicted in Figure 1. Each white circle corresponds to an agent. Agent $a_1$ is located on 0 and agent $a_2$ on $x > \frac{x}{2}$, where the exact value of $x$ will be specified later in the proof. Without loss of generality we assume that $\ell = 1$. Firstly, we will prove that the mechanism that places the facilities on their optimal locations is not strategy-proof even when the locations of the agents are known. Then, we will use these instances to derive our inapproximability result.

On instance $I$ agents $a_1$ and $a_2$ have preferences $t_1 = (-1, 1)$ and $t_2 = (0, 1)$ respectively. It is not hard to see that the optimal locations for the facilities are $y_1 = 1$ and $y_2 = \frac{x}{2}$ where each agent gets utility $2 - \frac{x}{2}$. The optimal locations of the facilities are depicted by black circles in the figure.

On instance $I'$ agent $a_1$ has the same preferences as on instance $I$ while the preferences of agent $a_2$ are $t'_2 = (-1, 1)$. The optimal locations for the facilities in this instance are $y_1 = 1$ and $y_2 = x$ where each agent gets utility $2 - x$.

Instances $I$ and $I'$ show that the mechanism which places the facilities on the optimal locations is not strategy-proof. On instance $I$ agent $a_2$ can declare $t'_2 = (-1, 1)$ and increase his utility from $2 - \frac{x}{2}$ to 2.

Next, we focus the on the inapproximability result for any deterministic mechanism. The high level idea of the proof is as follows. We assume that we know a strategy-proof mechanism $M$ that achieves the best possible approximation ratio for the problem. Firstly, we focus on instance $I$ where we show that $M$ always places the first facility on 1 and we derive the approximation guarantee of $M$ on $I$ as a function of the location $y_2$ of the second facility. Then, we turn our attention to instance $I'$ and we observe that for the location $y'_2$ of the second facility in this case, it should be true that $y'_2 \leq y_2$. Using this, we consider the two possible cases for the location $y_1$ of facility $f_1$ with respect to $x$ and we derive bounds on the approximation ratio of $M$, in each case as a function of $x$. Then, we optimize on the value of $x$ and derive the claimed bound.

So, let $M$ be a strategy-proof mechanism that achieves the best possible approximation for the Egalitarian objective. We first argue that on instance $I$ mechanism $M$ should place facility $f_1$ on $z = 1$. If this was not the case, the utility of $a_1$ would strictly increase by the movement of $f_1$ to 1, while the utility of $a_2$ would remain the same. Hence, the approximation ratio of $M$ would strictly improve by placing $f_1$ on 1, contradicting the assumption that $M$ achieves the best approximation guarantee.

Figure 1: Example for preferences in $\{-1, 0, 1\}^2$. 
Next, suppose that $M$ places facility $f_2$ on $y_2 \leq x$ on instance $I$. Since $M$ is strategy proof, $f_2$ cannot be placed on any $y'_2 > y_2$ on instance $I'$. If $y'_2 > y_2$, then agent $a_2$ from $I$ could declare preferences $\ell'_2 = (-1, 1)$ and increase his utility (assuming that $x > \frac{36}{41}$). We consider the following two cases regarding the location $y'_1$ in which $M$ places $f_1$ on $I'$:

- **$y'_1 \geq x$.** Then, obviously $y_1 = 1$ since otherwise the utility of both agents in $I'$ is decreasing and thus $M$ does not achieve the maximum approximation. So, under $M$ agent $a_2$ on instance $I'$ gets utility at most $u'_2 = 2 - 2x + y_2$ while $a_2$ gets utility $u_1' = 2 - 2y_2 \geq u_2'$ (since $x \geq y_2$). Thus $M$ achieves an approximation of $\frac{2-2x+y_2}{2-2y_2}$. Furthermore, on instance $I$ agent $a_1$ gets utility $u_1 = 2 - y_2$, since as explained earlier, $M$ places $f_1$ on 1, while $a_2$ gets utility $u_2 = 2 - x + y_2 \geq u_1$ when $y_2 \geq \frac{x}{2}$. Clearly if $y_2 < \frac{x}{2}$ the utility of $a_1$ gets worse. Thus, the approximation of $M$ on instance $I$ is $\frac{2-2y_2}{2-2x}$. Observe that the approximation guarantee of $M$ on $I$ is decreasing with $y_2$ while on $I'$ it is increasing with $y_2$. So, if we optimize the approximation guarantee and solve for $y_2$ we get that $y_2 = \frac{6x-2x^2}{8-4x}$. Thus, if $y'_1 > x$, the approximation of $M$ is at most

$$\frac{4 - 2 \cdot \frac{6x-2x^2}{8-4x}}{4-x} = \frac{4x^2 - 24x + 32}{3x^2 - 20x + 32}$$

(2)

- If $M$ on instance $I'$ places $f_1$ on $y'_1 < x$, then observe that there is no location $y'_2$ for $f_2$ such that both agents get utility strictly larger than 1. Thus, in this case $M$ achieves approximation at most

$$\frac{1}{2-x}$$

(3)

Observe that the approximation guarantee in (2) increases with $x$ while in (3) it decreases with $x$. So if we optimize on the approximation guarantee of $M$, we have to solve for $x$ the equation $-4x^3 + 29x^2 - 60x + 32 = 0$. The unique solution in $[0, 1]$ is $x = \frac{13 - \sqrt{34}}{8}$. Using this value in (2) and (3) we get that any deterministic strategy proof mechanism on instances $I$ and $I'$ achieves approximation less than 0.851.

The inapproximability bound can be extended to strategy proof in expectation mechanisms.

**Theorem 4.** There is no $\alpha$-approximate strategy proof in expectation mechanism for the 2-facility game with $\alpha \geq 0.851$.

**Proof.** We will use again the instances from Figure 1 to prove the claim setting $x = \frac{13 - \sqrt{34}}{8}$ and on $I'$ it is $2 - x$.

So, let $M$ be a strategy-proof in expectation mechanism. Observe that on instance $I$ the mechanism should place the facility $f_1$ on 1 for the same reason as the one mentioned in the proof of Theorem 3. Every other location for $f_1$ decreases the approximation guarantee of $M$. Suppose now that $M$ places $f_2$ on $y \in [0, 1]$ according to the probability distribution $p(y)$. Without loss of generality we can assume that $p(y) = 0$ for every $y > x$. This is because the approximation guarantee of $M$ can increase if we place the facility on $x$ instead of some $y > x$. Hence, on instance $I$ under $M$ agent $a_1$ gets utility 1 from $f_1$ and utility $\int_0^x p(y)dy$ from facility $f_2$, so $u_1 = 2 - \int_0^x p(y)dy$ in total. Similarly, agent $a_2$ gets utility 1 from $f_1$ and utility $1 - x + \int_0^x p(y)dy$ from facility $f_2$, so $u_2 = 2 - x + \int_0^x p(y)dy$ in total. Then, since $u_2 < u_1$, the approximation guarantee of $M$ on $I$ is at most

$$\frac{2}{4-x} \cdot \left( 2 - \int_0^x p(y)dy \right)$$

(4)

We now consider two cases according to the location in which $M$ places facility $f_1$ on instance $I'$. If $M$ places $f_1$ on $y'_1 \geq x$, then without loss of generality we can assume that $f_1$ is placed on 1 since every other location decreases the utility of both agents. So suppose that $M$ places $f_1$ on 1 with some probability.

Furthermore, suppose that $M$ places $f_2$ on $y$ according to the probability distribution $\pi(y)$ when $f_1$ is placed on 1. Observe that we can assume that $M$ does not place $f_2$ on $y > x$, since the utility of both agents could increase by placing it on $x$ instead. Thus, on instance $I'$, agent $a_2$ gets utility $1 - x$.
from facility \( f_1 \) and utility \( 1 - x + \int_0^x \pi(y) y dy \) from facility \( f_2 \), so \( u_2^0 = 2 - 2x + \int_0^x \pi(y) y dy \) in total. Similarly, agent \( a_1 \) gets total utility \( u_1^1 = 2 - \int_0^x \pi(y) y dy > u_2^0 \). Since \( M \) is strategy proof it must hold that \( \int_0^x \pi(y) y dy \leq \int_0^x p(y) y dy \). If this was not the case, then agent \( a_2 \) from instance \( I \) could declare preferences \((-1,1)\) and increase its utility. As a result the approximation guarantee of \( M \) on \( I' \) is at most

\[
\frac{1}{2 - x} \cdot \left( 2 - 2x + \int_0^x p(y) y dy \right)
\]

(5)

\( M \) achieves the best approximation on both instances when the quantities from (4) and (5) are equal. Hence, if we equalize them and solve for the integral we get that \( \int_0^x \pi(y) y dy \leq \int_0^x p(y) y dy \). If this was not the case, then agent \( a_2 \) from instance \( I \) could declare preferences \((-1,1)\) and increase its utility. As a result the approximation guarantee of \( M \) on \( I' \) is at most

\[
\frac{1}{2 - x} \cdot \left( 2 - 2x + \int_0^x p(y) y dy \right)
\]

(5)

5 Deterministic Mechanisms

In this section, we propose deterministic strategy-proof mechanisms. An initial approach would be to consider each facility independently and place it to its optimal location. As we have already proved this mechanism is strategy-proof by placing each facility independently to its optimal position. However, it achieves poor approximation if the agents want to be away from the facilities. Consider the case with \( n \) agents located on \( 0, \frac{\ell}{n}, \frac{2\ell}{n}, \ldots, \frac{(n-1)\ell}{n}, \ell \) and each having preferences \((-1, -1)\). Observe that the optimal location for one facility is to be placed on \( \frac{\ell}{n} \) since this location maximizes the minimum distance between any agent and the facility. Thus, both facilities will be placed on the same location \( \frac{\ell}{n} \). Then the agent located in 0 has utility \( \frac{2\ell}{n} \), the minimum over all the agents. It is not hard to see that an optimal solution is to place facility \( f_1 \) on 0 and facility \( f_2 \) on \( \ell \) resulting in a utility of \( \ell \) for each agent. Hence, the mechanism that places the facilities independently to their optimal locations is \( \frac{n}{\ell} \)-approximate.

The example above provides evidence that a mechanism with good approximation ratio should not put both facilities on the same location if there are agents who have preference -1 for both facilities; in the worst case there is an instance with an agent located in the exact same location where the facilities are placed and with preferences \((-1, -1)\) resulting in a zero approximation. On the other hand, the facilities should not be placed far away from each other. This is because, in the worst case again, an agent might have preference -1 for the facility that is close to his location and preference 1 for the facility that is far from him.

Using the intuition gained from the discussion above we propose a mechanism for the 2-facility game that comprises these ideas and places the facilities symmetrically away from the endpoints of the segment. Mechanism \textbf{Fixed} depicts our approach. It does not use any information from the agents, thus it is de facto strategy-proof.

Definition 2 (Fixed Mechanism). Let \( z_f = 1 - \frac{\ell}{\sqrt{2}} \). \textbf{Fixed} mechanism sets \( y_1 = z_f \cdot \ell \) and \( y_2 = (1 - z_f) \cdot \ell \).

Theorem 5. \textbf{Fixed} is \( z_f \simeq 0.292\)-approximate.

Proof. Tables 1 and 2 show the utility the agent located on \( x_i \) gets under \( y = (z \cdot \ell, (1 - z) \cdot \ell) \) and the corresponding ratio. Our goal is to find a \( z \in [0, \ell] \) that maximizes the minimum ratio. Thus, the optimal guarantee for \textbf{Fixed} is achieved when \( \frac{\ell}{z} = \frac{z \cdot \ell - 2\ell z^2}{\ell - 2\ell z^2} \). If we solve for \( z \), the feasible solution is \( z_f = (1 - \frac{\ell}{\sqrt{2}}) \ell \) and the approximation guarantee follows.

Finally, observe that if the number of facilities to be placed is at least two, then \( \max_{x \in [0, \ell]} \min_{y} u_i(x, t, y) \geq \max_{t} \min_{x} u^*_i(x, t) \geq \ell \). Thus, \textbf{Fixed} can be used for both \textsc{Egalitarian} and \textsc{Happiness} objectives and since it does not use any information from the agents, it possesses all the desirable properties like group strategy proofness and false name proofness.

Theorem 5 shows the sharp contrast between 1-facility and 2-facility games where both locations and preferences are private. Recall that [10] proved that for 1-facility games there is no deterministic
strategy-proof mechanism with bounded approximation guarantee. Observe furthermore that \textit{Fixed} does not require any information from the agents. Next, we prove that it is optimal when no communication is allowed.

\textbf{Theorem 6.} \textit{Fixed} is the optimal deterministic mechanism when no communication is allowed.

\textit{Proof.} Let \( M \) be any deterministic mechanism that places the facilities with no communication. Since \( M \) is deterministic, it places them on the same locations for any instance. So, let \( y_1 \cdot \ell \) and \( y_2 \cdot \ell \) be the locations of the first and the second facility respectively. Without loss of generality assume that \( 0 \leq y_1 \leq y_2 \leq 1 \). We will prove our claim by contradiction. So, for the sake of contradiction assume that the approximation ratio of \( M \) is strictly better than \( z = (1 - \frac{2z}{\sqrt{2}}) \). Without loss of generality we assume that \( y_1 \leq \frac{1}{2} \). Consider the following two instances. On the first instance there is only one agent on \( y_1 \cdot \ell \) with preferences \((-1, 1)\). The utility of the agent under \( M \) is \((y_2 - y_1) \cdot \ell \). The optimal solution places both facilities on \( \ell \) and the agent gets utility \((2 - 2y_1) \cdot \ell \). So, the approximation ratio of \( M \) is \( \frac{2z - 2y_1}{2z} \). Since the approximation of \( M \) is strictly greater than \( z \), we get that

\[ y_1 < \frac{y_2 - 2z}{1 - 2z} \]  

(6)

Now, consider the instance where there is only one agent on 0 with preferences \((-1, 1)\). Under \( M \), the agent gets utility \((1 + y_1 - y_2) \cdot \ell \). The optimal solution for this instance places the first facility on \( \ell \), the second one on 0, and the agent gets utility \( 2\ell \). Hence, the approximation guarantee of \( M \) on this instance is \( \frac{1 + y_1 - y_2}{2} \). Again, since we assume that the approximation is strictly greater than \( z \), we get that

\[ y_1 > 2z + y_2 - 1 \]  

(7)

The combination of Equations (6) and (7) dictates that \( y_2 > 3 - \frac{1}{2z} - 2z > 1 - z \). Similarly using another two instances we can prove that \( y_1 < z \). More specifically, we use the instance where there is only one agent on \( y_2 \cdot \ell \) with preferences \((-1, -1)\) and the instance where there is only one agent on \( \ell \) with preferences \((1, -1)\). Finally, consider again the instance where there is only one agent on 0 with preferences \((-1, 1)\). Recall that the approximation guarantee of the mechanism on this instance is \( \frac{1 - y_1 - y_2}{2} \). So, since \( y_1 < z \) and \( y_2 > 1 - z \), we get that the approximation guarantee is strictly smaller than \( z \) which is a contradiction. Our claim follows. \( \square \)

\textbf{5.1 \textit{Fixed}\textsuperscript{+} mechanism}

In order to describe \textit{Fixed}\textsuperscript{+}, we need to introduce the following events:

- \( L_j \): Every agent wants facility \( j \) below \( \ell/2 \). Formally, for every agent \( i \) with \( x_i \leq \frac{\ell}{2} \) it holds that \( t_{ij} \in \{0, 1\} \) and for every agent \( i \) with \( x_i > \frac{\ell}{2} \) it holds that \( t_{ij} \in \{0, -1\} \).

- \( H_j \): Every agent wants facility \( j \) above \( \ell/2 \). Formally, for every agent \( i \) with \( x_i \leq \frac{\ell}{2} \) it holds that \( t_{ij} \in \{0, -1\} \) and for every agent \( i \) with \( x_i > \frac{\ell}{2} \) it holds that \( t_{ij} \in \{0, 1\} \).

| \( t_i \) \( u_i(x_i, t_i, y) \) \( u_i^*(x_i, t_i) \) \ Ratio | \( t_i \) \( u_i(x_i, t_i, y) \) \( u_i^*(x_i, t_i) \) \ Ratio |
|---|---|---|---|---|---|
| 1, 1 | \( \ell + 2x_i \) | \( 2\ell \) | \( \geq \frac{\ell}{2} \) | 1, 1 | \( (1 + 2z) \cdot \ell \) | \( 2\ell \) | \( \geq \frac{\ell}{2} \) |
| -1, 1 | \( 2z \cdot \ell \) | \( 2\ell - x_i \) | \( \geq z \) | -1, 1 | \( 2x_i \) | \( 2\ell - x_i \) | \( \geq \frac{\ell}{2} \) |
| 1, -1 | \( (2 - 2z) \cdot \ell \) | \( 2\ell - x_i \) | \( \geq \frac{\ell}{z} \) | 1, -1 | \( 2\ell - 2x_i \) | \( 2\ell - x_i \) | \( \geq \frac{\ell}{2\sqrt{2}} \) |
| -1, -1 | \( \ell - 2x_i \) | \( 2\ell - 2x_i \) | \( \geq \frac{\ell}{z} \) | -1, -1 | \( (1 - 2z) \cdot \ell \) | \( 2\ell - 2x_i \) | \( \geq \frac{\ell}{2\sqrt{2}} \) |

Table 1: Case analysis when \( x_i \leq z \cdot \ell \) or \( x_i \geq (1 - z) \cdot \ell \).

Table 2: Case analysis when \( z \cdot \ell < x_i < (1 - z) \cdot \ell \).
**Fixed** superscript mechanism

**Input:** Locations $x_1, \ldots, x_n$ and preferences $p_1, \ldots, p_n$.

**Output:** Locations $y_1$ and $y_2$.

Set $z_d = \frac{7}{22} \approx 0.31$.

1. If events $L_1$ and $L_2$ occur, then set $y_1 = y_2 = z_d \cdot \ell$.
2. Else if events $L_1$ and $H_2$ occur, then set $y_1 = z_d \cdot \ell$ and $y_2 = (1 - z_d) \cdot \ell$.
3. Else if events $H_1$ and $H_2$ occur, then set $y_1 = y_2 = (1 - z_d) \cdot \ell$.
4. Else if events $H_1$ and $L_2$ occur, then set $y_1 = (1 - z_d) \cdot \ell$ and $y_2 = z_d \cdot \ell$.
5. Else set $y_1 = z_d \cdot \ell$ and $y_2 = (1 - z_d) \cdot \ell$.

**Lemma 3.** Fixed superscript is strategy-proof.

**Proof.** We will prove that there is no deviation that can yield strictly higher utility for any agent $i \in N$. Fix an arbitrary declaration for all the agents except agent $i$. For every $j \in \{1, 2\}$, let $y_j$, respectively $y_j'$, denote the location where Fixed superscript places facility $j$ when agent $i$ declares truthfully, respectively non-truthfully, his preference for facility $j$. Let us define $w_{ij} := |y_j' - x_i| - |y_j - x_i|$. Then, observe that the difference $\Delta$ between the utility that agent $i$ gets by reporting truthfully and misreporting, can be written as $\Delta = t_{i1} \cdot w_{i1} + t_{i2} \cdot w_{i2}$. Hence, there exists a profitable deviation for agent $i$ if and only if there is a declaration such that $\Delta < 0$. To prove that such declaration does not exist, we will use Tables 3 and 4 assuming first that agent $i$ is located on $x_i \leq \frac{\ell}{2}$. This is a concise representation of all cases that bypasses the repetitive case analysis. Table 3 presents possible preferences of agent $i$ when the mechanism places the facilities through Step 2 and it is interpreted as follows. If $t_{ij}$ at Step $k$ can be 0 or 1, then we write “+$” on the corresponding cell of the table; if $t_{ij}$ at Step $k$ can be either 0 or $-1$, then we write “$-$” on the corresponding cell. For example, if the mechanism places the facilities through Step 3, then for agent $i$ it must hold that $t_{i1} \in \{-1, 0\}$ and $t_{i2} \in \{-1, 0\}$. In Table 3 the $(k, l)$th cell shows the signs of $w_{i1}$ and $w_{i2}$ when the outcome of the mechanism changes from Step $k$ to Step $l$, where Step $k$ corresponds to the outcome when agent $i$ truthfully declares his preferences and Step $l$ corresponds to the outcome of the mechanism when agent $i$ lies. So, the $(2, 3)$th cell of Table 3 corresponds to the case where Fixed superscript under the true declaration would place the facilities through Step 2, but under the misreport of agent $i$ would place them through Step 3. In addition, the signs $(+, 0)$ mean that $w_{i1} > 0$ and $w_{i2} = 0$. So, using this information alongside the information from the agent’s preferences from the third row of Table 3, we can deduce that $\Delta \geq 0$ under this change. If we apply the same reasoning, we will see that $\Delta \geq 0$ for all possible cases when $x_i \leq \frac{\ell}{2}$. Hence, there is no profitable deviation for agent $i$, when $x_i \leq \frac{\ell}{2}$.

| Step | $t_{i1}$ | $t_{i2}$ |
|------|---------|---------|
| Step 1 | + | + |
| Step 2 | + | - |
| Step 3 | - | - |
| Step 4 | - | + |
| Step 5 | + | + |

Table 3: Preferences of agent $i$ for every step of Fixed superscript, when $x_i \leq \frac{\ell}{2}$.

| True | Lie |
|------|-----|
|      |     |
| Step 1 | (0,0) | (0,0) | (+,+) | (0,+) | (0,+) |
| Step 2 | (0,-) | (0,0) | (+,0) | (+,-) | (0,0) |
| Step 3 | (-,0) | (0,0) | (0,0) | (0,-) | (0,0) |
| Step 4 | (-,0) | (0,0) | (+,0) | (0,0) | (+,-) |
| Step 5 | (0,-) | (0,0) | (+,0) | (+,-) | (0,0) |

Table 4: Signs for $(w_{i1}, w_{i2})$ when $x_i \leq \frac{\ell}{2}$.

Similarly when $x_i > \frac{\ell}{2}$, we can use Tables 5 and 3 and see again that $\Delta \geq 0$ in every case, thus the lemma follows.

**Theorem 7.** Fixed superscript is $1 - 2z_d \approx 0.366$-approximate.
Table 5: Preferences of agent $i$ for every step of Fixed$^\dagger$, when $x_i > \frac{d}{2}$.

| Step | $t_{i1}$ | $t_{i2}$ |
|------|----------|----------|
| Step 1 | -        | -        |
| Step 2 | -        | +        |
| Step 3 | +        | +        |
| Step 4 | +        | -        |
| Step 5 | -        | +        |

Proof. In order to prove our claim, we will focus on the agent that gets the minimum utility under Fixed$^\dagger$. We will prove that for every possible combination of his preferences and his location the agent gets at least $\frac{2z_d}{y_d}$ fraction of the utility he would get under an optimal solution. So, let $i$ be an agent that gets minimum utility under Fixed$^\dagger$. Without loss of generality, we will assume that he is located below $\frac{\ell}{2}$. Observe that for the preference combinations $(0,1),(1,0),(0,-1),(-1,0)$ the agent gets utility at least $\ell$, while the maximum utility he can get is trivially bounded by $2\ell$. Hence, if the agent’s preferences are any of these combinations, then under any location for the facilities the agent gets at least half of his maximum utility and the mechanism is at least $\frac{1}{2}$-approximate.

- $p_i = (1,1)$. Observe that if there exists an agent with preferences $(1,1)$, then Fixed$^\dagger$ will locate the facilities either through Step 1 or through Step 2. Observe that under any of these steps, agent $i$ gets utility at least $\ell$, while the maximum utility he can get is bounded by $2\ell$. So, the mechanism is $\frac{1}{2}$-approximate in any of these steps.

- $p_i = (1,-1)$. When there exists an agent below $\frac{\ell}{2}$ with preferences $(1,-1)$, Fixed$^\dagger$ will place the facilities either through Step 3 or through Step 4. Observe that both steps place the facilities in the same way. If we check Tables 1 and 2 we can see that in any case the ratio of the mechanism is greater than $\frac{1}{2}$.

- $p_i = (-1,1)$. When there exists an agent below $\frac{\ell}{2}$ with preferences $(-1,1)$, then Fixed$^\dagger$ will place the facilities either through Step 5 or through Step 4. In the worst case scenario when Step 5 is used, agent $i$ is located at $x_i = \frac{\ell}{2} - \epsilon$, for some $\epsilon > 0$. In this case his utility is $u_i = \ell + 2\epsilon$, so the ratio of the mechanism is greater than $\frac{1}{2}$.

The utility of $i$ from Fixed$^\dagger$ in this step is $u_i = 2z_d\ell$, when $x_i \leq z_d$ and $u_i = 2x_i$, when $z_d < x_i \leq \frac{\ell}{2}$. If Step 5 is chosen, there should exist an agent $i_0$ with $x_{i0} \geq \frac{\ell}{2}$ and $t_{i02} = 1$ or with $x_{i0} < \frac{\ell}{2}$ and $t_{i02} = -1$. The utility of the optimal solution for agent $i$, gets maximized when agent $i_0$ is located in $x_{i0} = \frac{\ell}{4}$ with preferences $t_{i0} = (0,1)$. In this case the optimal places $f_1$ on $y_1 = \ell$ and the $f_2$ on $y_2 = \frac{\ell}{2}$. The utility of $i$ under opt is then $u_i = \frac{2\ell}{\ell}$, and the approximation in this step is

$$\frac{8z_d}{7\ell}$$  \hspace{1cm} (8)

- $p_i = (-1,-1)$. When there exists an agent below $\frac{\ell}{2}$ with preferences $(-1,-1)$, then Fixed$^\dagger$ will place the facilities either through Step 4 or through Step 5. When Step 4 is used by the mechanism, agent $i$ gets utility $(1 - z_d - x_i) \cdot 2\ell$ while the optimal value is trivially bounded by $(1 - x_i) \cdot 2\ell$. Hence the approximation guarantee from this step for $x_i \leq \frac{\ell}{2}$ is

$$\frac{1 - z_d - x_i}{1 - x_i} \geq 1 - 2z_d$$  \hspace{1cm} (9)

When Step 5 is used, the worst case instance for the mechanism is when agent $i$ is located on $z_d$ and there is another agent on $\frac{\ell}{2} - \epsilon$ with preferences $(1,0)$. The optimal mechanism will place $f_1$ on $y_1 = \frac{3\ell + 2z_d - 2\epsilon}{4}$ and $f_2$ on $y_2 = z_d$. Then, the utility of agent $i$ in the optimal solution is $\frac{7\ell + 2z_d - 2\epsilon}{3}$ while the utility he gets under Fixed$^\dagger$ is $(1 - 2z_d) \cdot \ell$. Hence, the approximation ratio of Fixed$^\dagger$ is

$$\frac{4\ell - 8z_d}{7\ell + 2z_d}$$  \hspace{1cm} (10)
We first observe from (9) and (10) that \(1 - 2z_d \leq \frac{1 - 8z_d}{1 + 2z_d}\). Hence, the value of \(z_d\) for which the approximation guarantee is maximized can be found if we equalize (8) and (9): \(\frac{8z_d}{1 + 2z_d} = 1 - 2z_d \Rightarrow z_d = \frac{1}{2} \approx 0.31\). Then the approximation ratio of the mechanism is \(\frac{1}{12} \approx 0.36\).

Observe that since Fixed\(^+\) asks for the exact location of every agent, it requires arbitrarily large communication; this happens for example when the location \(x_i\) of an agent \(i\) is irrational. However, a closer look shows that this is not necessary. Fixed\(^+\) only needs to know whether an agent is located below or above \(\frac{\ell}{2}\). One bit suffices for this piece of information; an agent transmits 0 if he is below \(\frac{\ell}{2}\) and 1 if he is above \(\frac{\ell}{2}\). Furthermore, his preference for each facility requires two bits, so Fixed\(^+\) requires only five bits per agent. An interesting question is whether there exists a deterministic mechanism that achieves better approximation when every agent communicates \(O(1)\) bits.

6 Randomized mechanisms

In this section, we propose two randomized mechanisms, Random and Random\(^+\) that achieve constant approximation ratio and are universally strategy-proof and strategy-proof in expectation respectively. Random requires zero communication and Random\(^+\) can be implemented using five bits per agent.

**Definition 3 (Random mechanism).** Random sets \(y_1 = y_2 = 0\) with probability \(\frac{1}{2}\) and \(y_1 = y_2 = \ell\) with probability \(\frac{1}{2}\).

**Theorem 8.** Random is universally strategy proof and achieves \(\frac{1}{2}\) approximation.

**Proof.** Firstly, it is easy to see that the mechanism is universally strategy proof since in each case, the mechanism chooses a fixed location, which is strategy-proof. We will prove that every agent gets utility at least \(\frac{1}{2}\) in expectation from every facility. Suppose that agent \(i \in N\) is located on \(x_i\) and has preferences \(t_i\). Let us study the expected utility that the agent gets from facility \(j\). If \(t_{ij} = 1\), then the agent’s utility is \(\ell - x_i\) when \(y_j = 0\) and \(x_i\) when \(y_j = \ell\). If \(t_{ij} = -1\), then the agent gets utility \(x_i\) if \(y_j = 0\) and \(\ell - x_i\) if \(y_j = \ell\). If \(t_{ij} = 0\), then the agent gets utility \(\ell\) irrespectively from \(y_j\). As a result, the agent gets utility at least \(\frac{\ell}{2}\) in expectation from each facility. So in total the agent in expectation gets utility at least \(\ell\). Since, the maximum utility is trivially bounded by \(2\ell\), the theorem follows.

Although Random seems naive, it achieves the best approximation so far, using zero communication as well.

**Theorem 9.** Random is the optimal mechanism when no communication is allowed.

**Proof.** For the purpose of contradiction suppose there is a mechanism \(M\) achieving an approximation strictly higher than \(\frac{1}{2}\). Let \(p(y_1, y_2)\) be the joint probability distribution of the facilities \(y_1\) and \(y_2\).

Consider now an instance \(I_1\) with one agent \(a_1\) located at 0 having preferences \((1, 1)\). His utility is then

\[u_1 = \int_0^1 \int_0^1 p(y_1, y_2) \cdot (1 - y_1) + p(y_1, y_2) \cdot (1 - y_2) dy_1 dy_2 = 2 \int_0^1 \int_0^1 p(y_1, y_2) dy_1 dy_2 = 2 - \omega\]

where \(\omega = \int_0^1 \int_0^1 p(y_1, y_2)(y_1 + y_2) dy_1 dy_2\). In \(I_1\) the optimal solution places both facilities in 0 resulting in a utility of \(u_1^* = 2\). The approximation ratio of \(M\) in \(I_1\) is then

\[1 - \frac{1}{2} \cdot \omega\] (11)

Similarly consider another instance \(I_2\) with one agent \(a_2\) located at 0 having preferences \((-1, -1)\). The utility of \(a_2\) is then \(u_2 = \int_0^1 \int_0^1 p(y_1, y_2)(y_1 + y_2) dy_1 dy_2 = \omega\). In \(I_2\) the optimal solution places both facilities in 1 resulting in a utility of \(u_2^* = 2\). Thus the approximation ratio of \(M\) in \(I_2\) is

\[\frac{1}{2} \cdot \omega\] (12)

Combining (11) and (12) we derive that \(\omega = 1\) and thus the approximation of \(M\) is \(\frac{1}{2}\), a contradiction.
We should note that Random can be extended for $k$-facility games, for any $k$, and achieve $\frac{1}{2}$ approximation. Furthermore, we use the intuition obtained from it to construct Random*. The first four steps of Random* are the same as in Fixed*, so again we will use the events $L_j$ and $H_j$ introduced in the previous section.

**Random** mechanism

**Input**: Locations $x_1, \ldots, x_n$ and preferences $p_1, \ldots, p_n$.

**Output**: Locations $y_1$ and $y_2$.

Set $z_r = \frac{13 - \sqrt{161}}{8}$

1. If events $L_1$ and $L_2$ occur, then set $y_1 = y_2 = z_r \cdot \ell$.
2. Else if events $L_1$ and $H_2$ occur, then set $y_1 = z_r \cdot \ell$ and $y_2 = (1 - z_r) \cdot \ell$.
3. Else if events $H_1$ and $H_2$ occur, then set $y_1 = y_2 = (1 - z_r) \cdot \ell$.
4. Else if events $H_1$ and $L_2$ occur, then set $y_1 = (1 - z_r) \cdot \ell$ and $y_2 = z_r \cdot \ell$.
5. Else with probability $\frac{1}{2}$ set $y_1 = y_2 = z_r \cdot \ell$ and with probability $\frac{1}{2}$ set $y_1 = y_2 = (1 - z_r) \cdot \ell$.

**Lemma 4.** Random* is strategy proof in expectation.

**Proof.** Steps 1-4 of Random* are similar to the steps of Fixed, so any deviation of agent $i$ that changes the outcome between Steps 1-4 will not result in a better outcome for the agent, as it is shown in Lemma 3. Therefore we only need to examine any deviation to, or from, Step 5.

**Deviation to Step 5:** Similarly to Lemma 3, let us denote by $\Delta = t_{i1} \cdot w_{i1} + t_{i2} \cdot w_{i2}$ the difference between the expected utility of agent $i$ when he reports truthfully and when misreporting and Random* implements Step 5; here $w_{i1} := \frac{1}{2} |z_r \cdot \ell - x_i| + \frac{1}{2} |(1 - z_r) \cdot \ell - x_i| - |y_j - x_i|$. Tables 7 and 9 present the signs of the preferences of agent $i$ when $x_i \leq \frac{\ell}{2}$ and $x_i > \frac{\ell}{2}$ respectively; recall “+” corresponds to preferences in $\{0, 1\}$ and “-” to preferences in $\{-1, 0\}$. Each cell of the tables [8] and [10] (for $x_i \leq \frac{\ell}{2}$ and $x_i > \frac{\ell}{2}$ respectively) presents the signs of $(w_{i1}, w_{i2})$ when agent $i$ misreports so that Step 5 is followed by Random*. It can be easily verified that for all possible combinations we have that $\Delta \geq 0$. Thus, any misrepresentation of the preferences of agent $i$ that changes the outcome of Random* from Step 1-4 to Step 5 does not increase the utility of the agent.

| True | Lie | Step 5 |
|------|-----|--------|
| Step 1 | (+,+)|        |
| Step 2 | (+,-)|        |
| Step 3 | (-,+)|        |
| Step 4 | (-,-)|        |

**Table 7:** Preferences of agent $i$ for Steps 1-4 of Random*, when $x_i \leq \frac{\ell}{2}$.

| True | Lie | Step 5 |
|------|-----|--------|
| Step 1 | (-,-)|        |
| Step 2 | (-,+)|        |
| Step 3 | (+,-)|        |
| Step 4 | (+,+)|        |

**Table 8:** Signs for $(w_{i1}, w_{i2})$ when $x_i \leq \frac{\ell}{2}$.

| True | Lie | Step 5 |
|------|-----|--------|
| Step 1 | (-,-)|        |
| Step 2 | (-,+)|        |
| Step 3 | (+,-)|        |
| Step 4 | (+,+)|        |

**Table 9:** Preferences of agent $i$ for Step 1-4 of Random*, when $x_i > \frac{\ell}{2}$.

**Table 10:** Signs for $(w_{i1}, w_{i2})$ when $x_i > \frac{\ell}{2}$.

**Deviation from Step 5:** In order for the outcome of the mechanism to change, the preference of $i$ must change sign from “+” to “-” or from “-” to “+”. Similarly as above, $\Delta = t_{i1} \cdot w_{i1} + t_{i2} \cdot w_{i2}$ denotes
the difference between the expected utility of agent $i$ when he reports truthfully and Step 5 is employed, and when misreporting. Now, we have that $w_{ij} := |y_j - x_i| - \frac{1}{2}|z_i - y_i - \frac{1}{2}(1 - z_i)|\ell - x_i|$. Each cell of column $c$ of the tables [11] and [12] presents the only possible signs of $(t_{1i}, t_{2i})$ of agent $i$ in Step 5 when he reports his true preference, that can result in the step of column $c$ if he misreports. As an example, consider the case where $t_{1i} \leq 0$ when $x_i \leq \frac{c}{2}$. If $i$ changes his declaration to $\ell_{1i}' \geq 0$ Step 1 cannot be followed. Again, it can be easily verified that $\mathbf{A} \geq 0$ in every case, hence there is no profitable deviation for agent $i$.

| True | Lie | Step 1 | Step 2 | Step 3 | Step 4 |
|------|-----|--------|--------|--------|--------|
|      |     | (-,-)  | (+,+)  | (+,+)  | (+,-)  |
| Step 5 |     |         |         |         |         |

Table 11: Signs of $(t_{1i}, t_{2i})$ when $x_i \leq \frac{c}{2}$.

| True | Lie | Step 1 | Step 2 | Step 3 | Step 4 |
|------|-----|--------|--------|--------|--------|
|      |     | (+,+)  | (-,-)  | (-,-)  | (-,-)  |
| Step 5 |     |         |         |         |         |

Table 13: Signs of $(t_{1i}, t_{2i})$ when $x_i > \frac{c}{2}$.

| True | Lie | Step 1 | Step 2 | Step 3 | Step 4 |
|------|-----|--------|--------|--------|--------|
|      |     | (-,-)  | (+,+)  | (+,+)  | (+,-)  |
| Step 5 |     |         |         |         |         |

Table 12: Signs of $(w_{1i}, w_{2i})$ when $x_i \leq \frac{c}{2}$.

| True | Lie | Step 1 | Step 2 | Step 3 | Step 4 |
|------|-----|--------|--------|--------|--------|
|      |     | (+,+)  | (+,-)  | (-,-)  | (-,-)  |
| Step 5 |     |         |         |         |         |

Table 14: Signs of $(w_{1i}, w_{2i})$ when $x_i > \frac{c}{2}$.

**Theorem 10.** Random$^+$ is $(\frac{1}{2} + z_r) \simeq 0.538$-approximate.

**Proof.** To prove our claim, we will focus on the agent that gets the minimum utility under Random$^+$. We will prove that for every possible combination of his preferences and his location the agent gets a fraction of $\frac{1}{2} + z_r$ of the utility he would get under an optimal solution. So, let $i$ be an agent that gets minimum utility under Random$^+$. Without loss of generality, we will assume that he is located below $\frac{c}{2}$.

- $p_i = (1, 1)$. If there exists an agent below $\frac{c}{2}$ with preferences $(1, 1)$, then Random$^+$ will place the facilities through Step [1] or through Step [3]. We will consider each case separately. If the facilities are placed due to Step [1] then the utility of the agent is at least $(1 + 2z_r)\ell$, while the maximum utility the agent can get is $2\ell$. Hence, the approximation guarantee of the mechanism in this case is $\frac{1}{2} + z_r$. For the case where the facilities are placed due to Step [3] we have to consider the following subcases. Firstly, if $x_i \geq z_r$, then the expected utility of the agent is $(1 + 2z_r)\ell$ while the optimum is bounded by $2\ell$, hence the mechanism achieves $\frac{1}{2} + z_r$ approximation. If $x_i < z_r$, we have to further consider two cases depending on the reason Step [3] was triggered. The first one is that there exists an agent $i'$ with $x_{i'} < \frac{c}{2}$ that has preference $-1$ for one of the two facilities. Then, the optimal value for the objective is upper bounded by $\frac{2\ell}{2}$. Hence, since we assumed that agent $i$ has the minimum utility under Random$^+$, we get that it achieves $\frac{1}{2}$ approximation in this case. The second subcase is when there exists an agent $i'$ with $x_{i'} > \frac{c}{2}$ that has preference 1 for one of the two facilities. Then, the optimum is again upper bounded by $\frac{2\ell}{2}$ and the mechanism achieves the claimed approximation ratio.

- $p_i = (1, 0)$. Observe that the analysis for the case $p_i = (0, 1)$ is symmetric, hence it will be omitted. When there exists an agent below $\frac{c}{2}$ with preferences $(0, 1)$, then Random$^+$ will place the facilities either through Step [1] or through Step [2] or through Step [3]. As in the previous case, it is not hard to see that under any of these steps the utility of agent $i$ is at least $(\frac{2\ell}{2} + z_r)\ell$, while the maximum utility he can get is bounded by $2\ell$. So, the approximation guarantee follows.

- $p_i = (1, -1)$. Observe that the analysis for the case $p_i = (-1, 1)$ is symmetric, hence it will be omitted. When there exists an agent below $\frac{c}{2}$ with preferences $(-1, 1)$, then Random$^+$ will locate the facilities either through Step [2] or through Step [3]. When the mechanism locates the facilities through Step [2], then the worst case instance occurs when there is only one agent $i$ on $\frac{c}{2}$. Then, the agent gets utility $\ell$, while an optimal solution locates the first facility on $\frac{c}{2}$ and the second facility on 0 yielding utility $\frac{3\ell}{2}$. Thus, the mechanism is $\frac{3}{2}$-approximate. So, for the chosen value of $z_r$,
the mechanism is \((\frac{1}{2} + z_r)\)-approximate. If, on the other hand, the mechanism locates the facilities through Step 3, then the expected utility of agent \(i\) is \(\ell\) irrespectively of his location \(x_i\). In order to construct a worst-case instance, it suffices to consider instances with only two agents, since more agents can only restrict more the set of optimal solutions, which implies that the optimal value can only decrease. The “loosest” constraint to the optimal that triggers Step 5 too, is when there is an agent on \(\frac{x}{2} + \epsilon\) for an arbitrarily small \(\epsilon\) with preferences \((1, 0)\). Then, the worst case instance for the mechanism, in terms of approximation guarantee, is when \(x_i = 0\) and the optimal utility the agent \(i\) gets is bounded by \(\frac{1}{2}:\) the first facility is located at \(\frac{x}{2}\) and the second one at \(\ell\). Hence, the mechanism is \(\frac{1}{2}\)-approximate. So, for the chosen value of \(z_r\), the mechanism is \((\frac{1}{2} + z_r)\)-approximate.

- \(p_i = (-1, 0)\). Observe that the analysis for the case \(p_i = (0, -1)\) is symmetric, hence it will be omitted. When there exists an agent below \(\frac{x}{2}\) with preferences \((-1, 0)\), then \(\text{Random}^+\) will place the facilities either through Step 3 or through Step 4 or through Step 5. When Step 3 or Step 4 is used, agent \(i\) gets utility \(2\ell - z, \ell - x_i\), while the optimal utility is bounded by \(2\ell - x_i\) by locating both facilities on \(\ell\). So, since \(x_i < \frac{x}{2}\), the approximation of \(\text{Random}^+\) is at least \(1 - \frac{z}{2} > \frac{1}{2} + z_r\), for the chosen value of \(z_r\). If Step 5 is used, then agent \(i\) gets utility at least \((\frac{1}{2} - z_r) \cdot \ell\). Furthermore, we can trivially bound the optimal utility by \(2\ell\), so, for the chosen value of \(z_r\) we get that \(\text{Random}^+\) is \((\frac{1}{2} + z_r)\)-approximate.

- \(p_i = (-1, -1)\). When there exists an agent below \(\frac{x}{2}\) with preferences \((-1, -1)\), then \(\text{Random}^+\) will locate the facilities either through Step 3 or through Step 5. When Step 3 is used by the mechanism, then agent \(i\) gets utility \((1 - z_r - x_i) \cdot 2\ell\) while the optimal value is trivially bounded by \((1 - x_i) \cdot 2\ell\). Hence, since \(x_i < \frac{x}{2}\), the approximation guarantee of the mechanism is bounded by \(1 - \frac{z}{2} > \frac{1}{2} + z_r\). If Step 5 is used, then using similar arguments as in the case where \(p_i = (1, -1)\), we can construct the worst case instance for the mechanism by locating an agent with preferences \((1, 0)\) on \(\frac{x}{2} - \epsilon\) and by setting \(x_i = z_r \ell\). Then, under Step 5 agent \(i\) gets utility \((1 - 2z_r) \cdot \ell\) and his utility under the optimal location for the facilities is bounded by \((\frac{z}{2} - z_r) \cdot \ell\); the first facility is located on \((\frac{z}{2} + z_r) \cdot \ell\) and the second one on \(\ell\). Hence, \(\text{Random}^+\) is \((1 - 2z_r)/(\frac{z}{2} - z_r)\)-approximate. It is not hard to verify that for the chosen value of \(z_r\) we get that \((1 - 2z_r)/(\frac{z}{2} - z_r) = \frac{1}{2} + z_r\).

Using the same technique as for \(\text{Fixed}^+\), \(\text{Random}^+\) can be implemented in a communication-efficient way where each agent sends only five bits to the planner.

7 Two-preference instances

In this section, we study \(k\)-facility games where all the agents have preferences in \(\{0, 1\}^k\), \(\{1, -1\}^k\), or in \(\{0, -1\}^k\), which we call two-preference instances. The non-existence of optimal deterministic strategy-proof mechanisms can be extended even on two-preference instances with three agents.

Theorem 11. For any \(k \geq 2\), there is no optimal deterministic strategy proof mechanism for \(k\)-facility games even on two-preference instances with three agents and known locations.

The proof of the theorem follows from the instances of Figures 2, 3, 4. As in Theorem 8, white circles correspond to agents and black circles to the optimal locations.

We now show how we can modify \(\text{Fixed}^+\) by changing the value of \(z_f\) and achieve better approximation guarantees. We denote the mechanisms as \(\text{Fixed}^{(0,1)}\), for preferences in \(\{0, 1\}^k\), and \(\text{Fixed}^{(0,-1)}\), for preferences in \(\{-1, 0\}^k\). Furthermore, for \(k = 2\) we derive a new deterministic mechanism termed \(OPT^2\), for the case where all agents have preferences in \(\{0, 1\}^2\) and their locations are known.

Definition 4. \(\text{Fixed}^{(0,1)}\) sets \(y_1 = \ldots = y_k = \frac{x}{2} \).

Theorem 12. \(\text{Fixed}^{(0,1)}\) is \(\frac{1}{2}\)-approximate.

Proof. Observe that for every agent \(i\) and any facility \(j\) it holds that \(u_{ij}(x_i, t, y_j) \geq \ell - |x_i - \frac{x}{2}| \geq \frac{\ell}{2}\). Hence, \(u_i(x_i, t, y) \geq \frac{\ell}{2}\). Observe, however, that \(\max_j u_i(x_i, t, y) \leq k \cdot \ell\). Hence, agent \(i\) under \(y\) gets at least half of his maximum utility.
(a) Instance $I$

| Preference | Utility |
|------------|---------|
| $(1, 1)$   | $0$     |

(b) Instance $I'$

| Preference | Utility |
|------------|---------|
| $(1, 1)$   | $0$     |

Figure 2: Example for preferences in $\{0, 1\}^2$. The agent located on $0$ in the instance $I$ can declare preferences $(1, 1)$ and increase his utility by moving the facility $f_2$ closer to $0$.

(1) Instance $I$

| Preference | Utility |
|------------|---------|
| $(-1, 1)$  | $0$     |

(b) Instance $I'$

| Preference | Utility |
|------------|---------|
| $(-1, 1)$  | $0$     |

Figure 3: Example for preferences in $\{-1, 1\}^2$. The agent located on $\ell - \epsilon$ in the instance $I$ can declare preferences $(-1, 1)$ and increase his utility by moving the facility $f_2$ closer to $\ell - \epsilon$.

(1) Instance $I$

| Preference | Utility |
|------------|---------|
| $(-1, -1)$ | $0$     |

(b) Instance $I'$

| Preference | Utility |
|------------|---------|
| $(-1, -1)$ | $0$     |

Figure 4: Example for preferences in $\{-1, 0\}^2$. The agent located on $0$ in the instance $I$ can declare preferences $(-1, -1)$ and increase his utility by moving the facility $f_2$ away from $0$. Observe that for the Instance $I'$ there are two optimal solutions ($y_1 = 0, y_2 = \ell$ and $y_1 = \ell, y_2 = 0$). However, this does not affect the correctness of our example assuming that the mechanism chooses a solution deterministically.

Definition 5. $\text{Fixed}^{(0, -1)}$ sets $y_1 = \ldots = y_{\lfloor \frac{k}{2} \rfloor} = 0$ and $y_{\lceil \frac{k}{2} \rceil} = \ldots = y_k = \ell$.

Theorem 13. $\text{Fixed}^{(0, -1)}$ is $\frac{|\frac{k}{2}|}{k}$-approximate.

Proof. Observe that since $t_i \in \{0, -1\}^k$ it holds that $u_i(x_i, t_i, y) = \sum_j \left[ \frac{1}{2} \right] \cdot x_i + \left[ \frac{1}{2} \right] \cdot (\ell - x_i) \geq \left[ \frac{1}{2} \right] \cdot \ell$. Observe though that $\max_y u_i(x_i, t_i, y) \leq k \cdot \ell$. Hence, $\text{Fixed}^{(0, -1)}$ is at least $\frac{|\frac{k}{2}|}{k}$-approximate.

Definition 6. $\text{OPT}^2$ places each of the two facilities independently on its optimal location.

It is not hard to see that $\text{OPT}^2$ is strategy-proof. This is because we know that when agents’ locations are known, the mechanism that places one facility on the leftmost optimal location is strategy proof. Therefore, since the mechanism places each facility independently no agent can increase his utility by lying.

Theorem 14. $\text{OPT}^2$ is $\frac{3}{4}$-approximate.

Proof. Before we analyze the approximation guarantee of the mechanism, let us first study the locations in which the mechanism places the facilities. Since the preferences of each agent are in $\{0, 1\}^2$, it is not hard to see that the optimal location for each facility is the median point between the locations of the leftmost and the rightmost agents that want to be close to the facility.
Without loss of generality, we can assume that the agent with the minimum utility under \( OPT^2 \), denoted by \( a_1 \), has preferences \((1, 1)\). If \( t_i = (1, 0) \), then the agent would have utility at least \( \frac{3}{4} \ell \) since any other agent who wants to be close to the first facility is located in distance at most \( \ell \) from \( a_1 \)’s location. The maximum utility the agent can get is \( 2 \ell \), so the mechanism is then \( \frac{3}{4} \)-approximate.

Assume that \( a_1 \) is located on \( x \leq \frac{\ell}{2} \). Then, without loss of generality, we can assume that he is located on 0 since for any other location the agent would be closer to the facilities and thus his utility would increase. Then, observe that agent \( a_1 \), alongside with the rightmost agents, will define the locations of the facilities. Observe that if the rightmost agent has preferences \((1, 1)\), then \( OPT^2 \) is optimal. So, we can assume that the rightmost agent, denoted by \( a_{r-1} \), has preferences \((0, 1)\). In the worst case, \( a_{r-1} \) will be located on \( \ell \) since for every other location the utility of agent \( a_1 \) will be higher. We have to consider the two possible preferences for the second rightmost agent with preference 1 for the first facility and prove that \( OPT^2 \) achieves the desired approximation. We will use \( a_i \) to denote this agent and \( x_i \) to denote his location.

Firstly, we consider the case where agent \( a_i \) has preferences \((1, 1)\) and \( x_i \geq \frac{\ell}{2} \). The utilities of the agents for the facilities under the locations \((y_1, y_2)\), where \( y_2 \leq x_i \), are \( u_1 = 2\ell - y_1 - y_2 \), \( u_i = 2\ell - 2x_i + y_1 + y_2 \) and \( u_{r-1} = \ell + y_2 \). \( OPT^2 \) will place the facilities to \( y_1 = \frac{\ell}{2} \) and \( y_2 = \frac{\ell}{2} \) and the utility of agent \( a_i \) will be \( u_1 = \frac{3\ell - x_i}{2} \). Observe that the locations of the facilities that make the utilities of these three agents equal provide an upper bound on the utility that agent \( a_1 \) gets under the optimal solution, since any other solution would yield lower utility for at least one of these agents. If we find the locations of the facilities that equalize the utilities for the agents we get \( y_1 = 2x_i - \ell \) and \( y_2 = \ell - x_i \) and thus the optimal utility for agent \( a_i \) is bounded by \( 2\ell - x_i \). Hence, \( OPT^2 \) is \( \alpha = \frac{3\ell - x_i}{2\ell} \) approximate.

In the case where \( x_i < \frac{\ell}{2} \), it is not difficult to see that agent \( a_1 \) gets utility at least \( \frac{3}{4} \ell \) under \( OPT^2 \). Observe that under the optimal solution the utility of the agents is bounded by \( \frac{3}{4} \ell \), since there are no locations for the facilities where both \( a_1 \) and \( a_{r-1} \) get more than \( \frac{3}{4} \ell \). Thus, in this case the mechanism is \( \frac{3}{4} \)-approximate.

If the preferences of \( a_i \) are \((1, 0)\), then similar analysis can be applied.

### 8 Utilitarian and Happiness

In this section we show that \( \text{Fixed} \), \( \text{Fixed}^{(0,1)} \), \( \text{Fixed}^{(0, -1)} \), and \( \text{Random} \) achieve the same approximation guarantees for \text{Utilitarian} and \text{Happiness} objectives as \text{Egalitarian}. All mechanisms remain strategy proof since they do not require any information from the agents. Recall, \text{Utilitarian} is the sum of the utilities of the agents, formally \( \sum_i u_i(x_i, t_i, y) \) and \text{Happiness} is \( \min_i \frac{u_i(x_i, t_i, y)}{u_1(x_i, t_i)} \), where \( u_i^*(x_i, t_i) = \max_y u_i(x_i, t_i, y) \).

**Theorem 15.** For \text{Utilitarian} and \text{Happiness} objectives the following hold. \( \text{Fixed} \) is \( \frac{1}{2} \)-approximate. \( \text{Fixed}^{(0,1)} \) is \( \frac{1}{2} \)-approximate. \( \text{Fixed}^{(0, -1)} \) is \( \frac{1}{3} \)-approximate. \( \text{Random} \) is \( \frac{1}{5} \)-approximate.

**Proof.** In the proofs of Theorems 5, 12, 13 and 10 it is proven that for every agent \( i \) holds that \( \frac{u_i(x_i, t_i, y)}{u_i^*(x_i, t_i)} \geq \alpha \), where \( \alpha \) is the approximation ratio of the corresponding mechanism. Hence, the claim for \text{Happiness} already follows from those proofs since they capture the definition of \text{Happiness}. For \text{Utilitarian}, observe that \( OPT_w = \max_y \sum_i u_i(x_i, t_i, y) \leq \sum_i u_i^*(x_i, t_i) \). So, from the proofs of the aforementioned theorems we get that \( u_i(x_i, t_i, y) \geq u_i^*(x_i, t_i) \cdot \alpha \) for every \( i \). So, if we sum over \( i \) we get that \( \sum_i u_i(x_i, t_i, y) \geq \alpha \cdot \sum_i u_i^*(x_i, t_i) \geq \alpha \cdot OPT_w \) and the theorem follows.

The observing reader may wonder whether the approximation guarantee of \( \text{Fixed} \) for \text{Utilitarian} contradicts the result of 30. Recall, 30 proved that there is no deterministic strategy-proof mechanism for \text{Utilitarian} with approximation ratio better than \( \frac{2}{3} \). However, a closer look will reveal that in order to establish that result, the following assumptions must be made. Firstly, that every agent wants to be close to the first facility and away from the second facility. Furthermore, they defined the utility of an agent located on \( x_1 \) to be \( u_i(x_1, y) = |x_1 - y_1| - |x_1 - y_2| \). This different definition of utility is crucial for deriving those negative results and this is the reason why our results do not contradict theirs.
9 Discussion

In this paper, we studied heterogeneous facility locations on the line segment. To the best of our knowledge, this is the first systematic study of this model for the Egalitarian objective. We derived inapproximability results for strategy-proof mechanisms for Egalitarian even for instances with known locations and two agents. Furthermore, we derived strategy-proof mechanisms that achieve constant approximation for Egalitarian, some of which also achieve the same guarantee for Utilitarian and Happiness objectives.

All of our mechanisms are simple and can be implemented in a communication-efficient way. More specifically, every mechanism needs zero or five bits of information from every agent. Communication efficiency is crucial for real life scenarios. Consider the example of the factory and the school discussed in the introduction. If thousand of citizens live on this street, then our mechanisms require only their preferences and whether they live on the western part of the street or on the east one and not their full address saving a huge amount of time to the planner. To the best of our knowledge, this is the first time that communication complexity is studied for facility location problems. We strongly believe that there is much to be said about facility location mechanisms and communication complexity. Firstly, it would be really interesting to understand how limited communication affects the approximation guarantee of mechanisms. Is there a better randomized mechanism than Random when no communication is allowed? Are there better mechanism than Fixed$^+$ and Random$^+$ when every agent is allowed to communicate $O(1)$ bits? Can Fixed$^+$ and Random$^+$ be extended for $k \geq 3$ facilities?

Another intriguing avenue of research is to use communication complexity to define “simple” mechanisms. Recently [Li] defined the obviously strategy proof (OSP) mechanisms to capture the simplicity of mechanisms. Intuitively, a mechanism is obviously strategy-proof if it remains incentive compatible even when some of the agents are not fully rational. The formal definition of OSP is quite technical, and thus we decided not to include it in our paper since it would deviate from its main theme. However, we strongly believe that some of our mechanisms, if not all of them, should be obviously strategy proof [Li]. Fixed and Random do not use any information from the agents. In both Fixed$^+$ and Random$^+$, if an agent knows the declarations of the rest of the agents, then he can verify that he cannot increase his utility by misreporting his type using $O(1)$ space. We believe that this kind of mechanisms are de facto simple and deserve further studying.
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