ON THE FUNDAMENTAL GROUPS OF NORMAL VARIETIES

DONU ARAPURA\textsuperscript{1}, ALEXANDRU DIMCA\textsuperscript{2}, AND RICHARD HAIN\textsuperscript{3}

Abstract. We show that the fundamental groups of normal complex algebraic varieties share many properties of the fundamental groups of smooth varieties. The jump loci of rank one local systems on a normal variety are related to the jump loci of a resolution and of a smoothing of this variety.

1. Introduction

By recent work of M. Kapovich, J. Kollár and C. Simpson it is known that any finitely presented group is the fundamental group of an irreducible projective variety; moreover, the singularities can be chosen to be of a specific type, see \cite{27, 28, 33}. On the other hand, a fundamental group $G$ of a smooth projective or quasi-projective variety satisfies a number of special properties. For instance, if $G = \pi_1(X, x)$, with $X$ quasi-projective smooth and $x \in X$, then it was shown by J. Morgan that $\mathfrak{g}_C$, the Malcev Lie algebra of $G$ over $\mathbb{C}$ has a natural mixed Hodge structure such that the associated graded Lie algebra $\text{Gr}_{W}^{*} \mathfrak{g}_C$ with respect to the weight filtration $W$, is isomorphic to $\mathbb{L}/J$, where $\mathbb{L}$ is a free Lie algebra with generators in degrees $-1$ and $-2$, and $J$ is a Lie ideal, generated in degrees $-2$, $-3$ and $-4$, see \cite[Corollary 10.3]{30} and \cite[Theorem 5.8]{26}. For smooth quasi-projective varieties, the list of known restrictions is much longer, see for instance \cite{1, 16, 17} and the references therein.

In this note, we consider the case of normal varieties. In a nutshell, we want to give some evidence that the fundamental groups of normal varieties behave like those of smooth varieties; see also the remarks at the end of section 2.3 in the first chapter of \cite{1}. In his 1974 ICM talk, P. Deligne stated in \cite[section 10]{13}, that Morgan’s theorem holds for $X$ a normal variety. We include a proof of Deligne’s result. When combined with \cite{3}, it puts strong restrictions on the solvable groups in the above class. We also bring additional support to the above feeling, by showing that all the specific properties of the fundamental groups of smooth varieties reflected by their resonance and characteristic varieties as explained in \cite{16} extends to the larger class

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of normal varieties, see Corollary 1.5. Using this result, we show in Corollary 1.6 that a right-angled Artin group is the fundamental group of a normal variety $X$ if and only if it is so for a smooth variety $X$.

Here is a detailed description of our results. For any finite connected CW complex $M$ we can define the first resonance varieties $R_k^1(M) \subset H^1(M, \mathbb{C})$ as the set of $u \in H^1(M, \mathbb{C})$ for which the corresponding Aomoto complex $H^*(M, \mathbb{C}), u \wedge)$ satisfies the condition

$$\dim H^1(H^*(M, \mathbb{C}), u \wedge) \geq k.$$ 

Similarly, the first characteristic variety $\Sigma_k^1(M) = V_k^1(M) \subset H^1(M, \mathbb{C}^*)$ is the set of rank one local systems $L$ (given by a representation in $\text{Hom}(\pi_1(M), \mathbb{C}^*) = H^1(M, \mathbb{C}^*)$) such that

$$\dim H^1(M, L) \geq k.$$ 

The restricted characteristic variety of $M$, denoted by $V_k^1(M)_1$, is the union of all irreducible components of the algebraic variety $V_k^1(M)$ which have strictly positive dimension and pass through the unit element 1. Recall that $R_k^1(M)$, $V_k^1(M)$ and $V_k^1(M)_1$ depend in fact only on the fundamental group $G = \pi_1(M)$ and hence may also be denoted by $R_k^1(G)$, $V_k^1(G)$ and $V_k^1(G)_1$. For simplicity, we set $R^1(M) = R^1_1(M)$, $V^1(M) = V^1_1(M)$ and $V^1(M)_1 = V^1_1(M)_1$.

If we apply these constructions to an irreducible normal variety $X$ of dimension $n \geq 2$ and to a resolution of singularities $p : \tilde{X} \to X$, we get pairs of resonance varieties $R_k^1(X)$ and $R_k^1(\tilde{X})$, and pairs of characteristic varieties $V_k^1(X)$ and $V_k^1(\tilde{X})$.

The first main result of this note is the following generalization of a main theorem in [2], and can be used to understand the relation among these varieties associated to $X$ and $\tilde{X}$.

**Theorem 1.1.** For any irreducible component $V$ of $V_k^1(X)$, with $k > 0$, having strictly positive dimension, there is a surjective morphism $f : X \to C$ with connected generic fibre to a smooth curve $C$ with negative Euler characteristic $E(C) \leq 0$ and a torsion character $\chi \in H^1(X, \mathbb{C}^*)$, such that $V$ is the translated affine torus $\chi f^* H^1(C, \mathbb{C}^*)$. Moreover, $p^*(V)$ is an irreducible component of $V_k^1(\tilde{X})$. Conversely, for an irreducible component $\hat{\tilde{V}}$ of $V_k^1(\tilde{X})$ satisfying the condition

$$(T) \text{ for any } L \in \hat{\tilde{V}} \text{ and } x \in X \text{ the restriction } L|_{F_x} \text{ is trivial, where } F_x = p^{-1}(x),$$

there is an irreducible component $V$ of $V_k^1(X)$ such that $p^*(V) = \hat{\tilde{V}}$. Finally, the zero dimensional components of $V_k^1(X)$ consist of torsion characters.

The second main result of this note is the following property, informally called Morgan’s obstruction and announced at the beginning of this Introduction.

**Theorem 1.2.** The graded Lie algebra $\text{Gr}^W_1 \mathfrak{g}$ of the (complex) Malcev Lie algebra $\mathfrak{g} = \mathfrak{g}(X, x)$ associated to $\pi_1(X, x)$ for a normal irreducible variety $X$ admits a presentation with generators of degree $-1$ and $-2$ and relations of degree $-2, -3$.
and $-4$. If $X$ is in addition projective, then the generators can be chosen only of degree $-1$ and the relations only of degree $-2$.

**Corollary 1.3.** The fundamental group $G = \pi_1(X)$ of a normal projective variety is 1-formal.

If $\Gamma$ is a finitely nilpotent group, we will say that it is quadratically presented if its Malcev Lie algebra has generators and relations in degree $-1$ and $-2$ respectively. The class of such groups is strongly restricted, see [11]. By the theorem [1.2] together with [3, thm 3.3], we deduce.

**Corollary 1.4.** If $X$ is normal and projective such that $\pi_1(X)$ is a solvable subgroup of $GL_n(\mathbb{Q})$, then $\pi_1(X)$ contains a quadratically presented nilpotent group of finite index.

The next result follows by using Theorems [1.1] and [1.2] and applying Theorem C in [10] to $\hat{X}$. We recall that a linear subspace $E \subset H^1(X, \mathbb{C})$ is said to be $p$-isotropic if the cup-product restriction morphism $E \otimes E \to H^2(X, \mathbb{C})$ has a $p$-dimensional image and, for $p = 1$, induces a pairing on $E$, see [10].

**Corollary 1.5.** Let $X$ be an irreducible normal variety. Set $G = \pi_1(X)$ and let $V^\alpha$ be the irreducible components in $V^1_k(X) = V^1_k(G)$ for some $k > 0$. Denote by $T^\alpha$ the tangent space at 1 to $V^\alpha$. Then the following hold.

1. Any tangent space $T^\alpha$ is a $p$-isotropic linear subspace of $H^1(X, \mathbb{C}) = H^1(G, \mathbb{C})$, defined over $\mathbb{Q}$ and of dimension at least $2p + 2$, for some $p = p(\alpha) \in \{0, 1\}$.
2. If $\alpha \neq \beta$, then $T^\alpha \cap T^\beta = 0$.
3. Suppose $G$ is a 1-formal group (e.g. $X$ is in addition projective) and let $R^\alpha$ be the irreducible components of the resonance variety $R^1_k(X) = R^1_k(G)$. Then the collection $T^\alpha$ coincides with the collection $R^\alpha$. In other words, in this case $R^1_k(X)$ is the tangent cone to $V^1_k(X)$ at the unit element 1.

As an application, we get the following analog of Theorem 11.7 in [10] (with exactly the same proof), telling us which right-angled Artin groups are fundamental groups of normal varieties. Recall that all right-angled Artin groups are 1-formal and hence they all pass Morgan’s obstruction recalled above.

**Corollary 1.6.** Let $\Gamma = (V, E)$ be a finite simplicial graph, with associated right-angled Artin group $G_\Gamma$. The following are equivalent.

1. There is a connected smooth algebraic variety $X$ such that $\pi_1(X) = G_\Gamma$.
2. There is a normal (or unibranch) algebraic variety $X$ such that $\pi_1(X) = G_\Gamma$.
3. The graph $\Gamma$ is a complete multipartite graph.
4. The group $G_\Gamma$ is a finite product of finitely generated free groups.

If $X$ is normal and projective, then $\pi_1(X) = G_\Gamma$ is a free abelian group of even rank. In particular, there are infinitely many right-angled Artin groups which are not isomorphic to fundamental groups of normal (or unibranch) varieties.
Finally, we explain how to compute characteristic varieties of normal projective varieties, in some cases, by smoothing them. Let $X$ be an analytic space equipped with a projective holomorphic map $f : X \to \Delta$ to the open disk which is smooth over $\Delta^* = \Delta - \{0\}$ and such that $X_0 = f^{-1}(0)$ is normal. Then $X$ is normal \cite[6.8.1]{[21]}.

Since it is convenient to allow a boundary, we replace $\Delta$, and accordingly $X$, by a smaller closed disk of radius, say $\epsilon$. Let us fix a holomorphic section $\sigma : \Delta \to X$ (which is not a restriction). We keep this notation throughout this section. It follows from our assumptions that $X^* = X - X_0 \to \Delta^*$ is a fibre bundle. We can also assume, after shrinking $\Delta$ if necessary, that we have a deformation retraction of $X$ to $X_0$ which collapses $\sigma(\Delta)$ to $\sigma(0)$. It follows that we have a specialization map

$$\pi_1(X_t, \sigma(t)) \to \pi_1(X_0, \sigma(0))$$

for any $t \in \partial \Delta$. The main results in this setting, comparing the invariants associated to the smooth general fiber $X_t$ to those of the normal special fiber $X_0$, are the following.

**Proposition 1.7.** The map (1.1) is surjective.

**Theorem 1.8.** With the same assumptions as above, with $t \in \partial \Delta$, the following holds.

1. The specialization map induces an isomorphism of vector spaces

$$H^1(X_0, \mathbb{Z}) \cong H^1(X_t, \mathbb{Z}).$$

2. The monodromy acts trivially on $H^1(X_t, \mathbb{Z})$.

3. The specialization map induces an injection of affine algebraic groups

$$H^1(X_0, \mathbb{C}^*) \hookrightarrow H^1(X_t, \mathbb{C}^*)$$

which is an isomorphism on the identity components, or equivalently the cokernel is finite.

4. Viewing (3) as an inclusion, we have

$$V^1(X_0) = V^1(X_t) \cap H^1(X_0, \mathbb{C}^*).$$

It follows from (4) above that each strictly positive dimensional irreducible component $V_i$ in $V^1(X_0)$ gives rise to a component $V_{i,t}$ in $V^1(X_t)$, which in turn, by \cite{[4]}, corresponds to a map $X_t \to C_{i,t}$ onto a smooth projective curve $C_{i,t}$ for any $t \in \Delta$. The next result shows that this family of maps $X_t \to C_{i,t}$ can be chosen to depend algebraically on $t \in \Delta$. 
Theorem 1.9. With the same assumptions as above, there exists a finite collection torsion characters $\chi_i \in H^1(X_0, \mathbb{C}^\times)$, $i = 1, \ldots, n, \ldots N$, and commutative diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{\pi_i} & C_i \\
\downarrow f & & \downarrow p_i \\
\Delta & \xrightarrow{} & \end{array}
$$

where $C_i \to \Delta, i = 1, \ldots n$ are relative smooth projective curves, such that

$$
V^1(X_t) \cap H^1(X_0, \mathbb{C}^\times) = \bigcup_{i=1}^{n} \chi_i \pi_{i,t}^* H^1(C_{i,t}, \mathbb{C}^\times) \cup \bigcup_{i=n+1}^{N} \{\chi_i\}
$$

for all $t \in \Delta$.

It would be interesting to find an example of a group $G$ which is the fundamental group of a normal variety, but not the fundamental group of a smooth variety.

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2. A basic fact on fundamental groups of normal analytic varieties

Let $X$ be an $n$-dimensional irreducible complex normal analytic variety, with $n \geq 2$, $X_{\text{reg}}$ the open subset of regular points of $X$ and $p : \hat{X} \to X$ a resolution of singularities for $X$ such that $p : p^{-1}(X_{\text{reg}}) \to X_{\text{reg}}$ is an isomorphism. We use this isomorphism to identify $X_{\text{reg}}$ to an open subset of $\hat{X}$ and denote by $i : X_{\text{reg}} \to \hat{X}$ the corresponding inclusions, such that $i = p \circ \hat{i}$. The first key result is the following, see for a proof (0.7) (B) in [18] or Proposition 2.10 in [29].

Theorem 2.1. The morphisms $p_* : \pi_1(\hat{X}) \to \pi_1(X)$ and $i_* : \pi_1(X_{\text{reg}}) \to \pi_1(X)$ are surjective. Moreover, for any proper closed analytic subset $A \subset X$, there is a surjection $\pi_1(X \setminus A) \to \pi_1(X)$ induced by the inclusion.

As remarked in (0.7) (B) in [18], the condition on $X$ being normal in the above result can be relaxed by asking that $X$ is unibranch, i.e. for any point $x \in X$, the analytic germ $(X, x)$ is irreducible. This class strictly contains the class of normal varieties, e.g. consider the product between a cuspidal curve and a smooth variety, but leads to the same family of topological spaces. Indeed, $X$ is a unibranch variety if and only if the normalization morphism $\nu : X' \to X$ is a homeomorphism (in both classical and Zariski topologies). However, in practice, it might be easier to check that a given variety is unibranch rather than normal. We leave the reader to reformulate the following results for unibranch varieties, and assume from now on that $X$ is normal unless stated otherwise.
Remark 2.2. (i) When $n = 2$, an alternative proof of Theorem 2.1 can be obtained using the following result, which is a special case of Proposition 3.2.B.2 in [6].

Let $X$ and $Y$ be two irreducible complex normal surfaces and $f : Y \to X$ a surjective proper morphism. If all the fibers of $f$ are connected (resp. simply-connected), then the morphism $f_* : \pi_1(Y) \to \pi_1(X)$ is surjective (resp. an isomorphism). When $f$ is a resolution of singularities for $X$, the fibers of $f$ are always connected by Zariski Main Theorem, and they are simply-connected if and only if for each singularity $(X, x)$ of $X$ the corresponding exceptional divisor is a tree of rational curves. This happens exactly when $X$ is a $\mathbb{Q}$-homology manifold, and hence in such a situation one has

\[(2.1) \quad \pi_1(X) = \pi_1(\hat{X}).\]

(ii) The morphism $\hat{\iota}_* : \pi_1(X_0) \to \pi_1(\hat{X})$ is clearly a surjection for any $n$ (by an obvious transversality argument) and moreover

\[(2.2) \quad \hat{\iota}^* : H^1(\hat{X}, \mathbb{C}) \to H^1(X_{\text{reg}}, \mathbb{C})\]

is an isomorphism when $n = 2$, see Lemma 2 in [17]. We do not know whether the isomorphisms (2.1) and (2.2) hold for $n > 2$ as well.

(iii) Easy examples (e.g. $X = S_6$, the degree 6 cyclic covering of $\mathbb{P}^2$ ramified along a Zariski sextic curve with 6 cusps situated on a conic) show that the other morphisms induced at $H^1$-level $\iota^*$ and $p^*$ are not isomorphisms in general, even when $n = 2$.

(iv) The monomorphism $p^* : H^1(X, \mathbb{Q}) \to H^1(\hat{X}, \mathbb{Q})$ implied by Theorem 2.1 shows that for a projective normal variety $X$, the first cohomology $H^1(X, \mathbb{Q})$ is a pure Hodge structure of weight 1, in particular the first Betti number $b_1(X)$ is even, see also Proposition 5.1 below for a precise formula. For an arbitrary normal variety $X$, it follows that $H^1(X, \mathbb{Q})$ has weights 1 and 2.

Remark 2.3. There are however differences between certain properties of fundamental groups of normal varieties and fundamental groups of smooth varieties. Perhaps the most basic is the following. If $X$ is a smooth complex (analytic) variety and $A$ is a closed subvariety of codimension $\text{codim} A \geq 2$, then the inclusion induces an isomorphism $\pi_1(X \setminus A) \to \pi_1(X)$. To see this, one can use the usual Thom transversality theorems for smooth real manifolds.

When $X$ is normal, this result is no longer true. For instance, when $(X, x)$ is the germ of a normal surface singularity, the fundamental group of (some representative of) $X$ is trivial, due to the conic local structure of analytic sets, see [10]. But the complement $X \setminus \{x\}$, which is homotopy equivalent to the link of the singularity, has a trivial fundamental group if and only if $(X, x)$ is a smooth germ, by the celebrated result of Mumford [31].
3. Proof of Theorem 1.1

Let $p : \hat{X} \to X$ be a resolution of singularities as above and note that by Chow’s Lemma we can suppose that $\hat{X}$ is a quasi-projective variety.

**Lemma 3.1.** If $L$ is a locally constant sheaf on $X$, then $p^* : H^1(X, L) \to H^1(\hat{X}, \hat{L})$ is injective, with $\hat{L} = p^*L$.

**Proof.** Note that we have by Leray Theorem

$$H^1(\hat{X}, \hat{L}) = \mathbb{H}^1(X, \mathbb{R}p_*\hat{L}).$$

The hypercohomology group in the right hand side can be computed using the usual $E_2$-spectral sequence

$$E_2^{p,q} = H^p(X, R^q p_* \hat{L}).$$

We have $E_{\infty}^{0,0} = E_2^{0,0} = H^1(X, R^0 p_* \hat{L})$ and $R^0 p_* \hat{L} = L$ since the fibers of $p$ are connected. From this we deduce an inclusion

(3.1) $$H^1(X, L) = E_{\infty}^{1,0} \to H^1(\hat{X}, \hat{L}).$$

which corresponds exactly to $p^*$.

□

**Remark 3.2.** A continuation of this argument, shows that we have an exact sequence

$$0 \to H^1(X, L) \to H^1(\hat{X}, \hat{L}) \to H^0(X, R^1 p_* \hat{L}) \cong H^1(E, \hat{L}),$$

where $E$ is the exceptional divisor.

This Lemma combined with the fact that $p_\sharp$ is surjective yields the following.

**Corollary 3.3.** We have inclusions $H^1(X, \mathbb{C}) \subseteq H^1(\hat{X}, \mathbb{C})$, $H^1(X, \mathbb{C}^*) \subseteq H^1(\hat{X}, \mathbb{C}^*)$ and $V_1(X) \subseteq V_1(\hat{X})$ induced by $p^*$.

We discuss first the case of a non-translated component of positive dimension, i.e. we take $V \subseteq V_1(X)$. Since $\hat{X}$ is quasiprojective, we can apply [2] and find a map $g : \hat{X} \to C$ with generic connected fiber onto a smooth quasiprojective curve such that $g^*H^1(C, \mathbb{C}^*) \supseteq p^*V$. If we can show that $g$ factors through $X$, the theorem will follow. The main step for this is the following.

**Proposition 3.4.** For any point $x \in X$, the fibre $F_x = p^{-1}(y)$ maps to a point in $C$ under $g$.

**Proof.** Since the fiber $F_x$ is connected, it is enough to show that the restriction of $g$ to any irreducible component $F$ of $F_x$ is constant. Suppose that there is a component $F$ of $F_x$ such that the restriction $g : F \to C$ is dominant. By taking a generic linear section of the quasiprojective variety $F$ and taking the smooth part of an irreducible component of this intersection, we find a smooth curve $C' \subset F$ such that the restriction $g : C' \to C$ is still dominant. To continue we need the following.
Lemma 3.5. Let $\rho \in V$ be a nontorsion element. Then $p^*\rho|_F \neq 1$.

Proof of lemma. We know that $p^*\rho = g^*\rho'$ for some character $\rho' \in H^1(C, \mathbb{C}^*)$. The map $C' \to C$ being dominant, it follows that $H^1(C, \mathbb{C}) \to H^1(C', \mathbb{C})$ is injective (see for instance Lemma 6.10 in [10]), and thus $H^1(C, \mathbb{C}) \to H^1(F, \mathbb{C})$ is also injective. It follows that the map $H^1(C, \mathbb{C}^*)_1 \to H^1(F, \mathbb{C}^*)_1$ of connected components of the identity (where we view these as algebraic groups and the cohomology groups $H^1(C, \mathbb{C})$ and $H^1(F, \mathbb{C})$ as their Lie algebras) has finite kernel. Thus the pull-back of $\rho'$ to $F$ is nontrivial. \hfill \Box

On the other hand, we see that $p^*\rho|_F$ is the pullback of $\rho|_g$ which is necessarily trivial. This leads to a contradiction. Therefore Proposition 3.3 is proved. \hfill \Box

We thus get a mapping $h : X \to C$ such that $g = h \circ p$. Since $p$ is proper, if $K \subset C$ is closed then $h^{-1}(K) = p(g^{-1}(K))$ is closed. So that $h$ is continuous and therefore locally bounded. Moreover, $h$ is regular on the open set $X_0$ as $p$ is an isomorphism above $X_0$. Normality of $X$ shows then that $h$ is regular everywhere. If $L_C$ is a generic rank one local system on $C$, it follows from [2] that

\[\dim H^1(\hat{X}, g^*L_C) = -E(C),\]

in particular for such components the curve $C$ should have strictly negative Euler characteristic $E(C) = -k$. This result was reproved in [15], using only the existence of the map $g$ and the topology of constructible sheaves, hence Corollary 4.7 in [15] can be applied to the map $h$ and gives

\[\dim H^1(X, h^*L_C) = -E(C) = k.\]

This proves Theorem 1.1 in the case of a non-translated component.

Assume now that $V$ is translated and let $V'$ be the unique the irreducible component of $V^1(\hat{X})$ containing $p^*V$. Then by [2] we infer the existence of a map $g : \hat{X} \to C$ as above and of a torsion character $\chi' \in H^1(\hat{X}, \mathbb{C}^*)$ such that $V' = \chi'g^*H^1(C, \mathbb{C}^*)$. It follows from Theorem 5.3 in [15] that this character comes from a unique homomorphism $T(g) \to \mathbb{C}^*$, where

\[T(g) = \ker\{g_* : H_1(\hat{X}, \mathbb{Z}) \to H_1(C, \mathbb{Z})\} / \im\{i_* : H_1(F, \mathbb{Z}) \to H_1(\hat{X}, \mathbb{Z})\}.\]

Here $i : F \to \hat{X}$ is the inclusion of a generic fiber $F$ of $g$ and $T(g)$ is a finite Abelian group determined by the multiple fibers of $g$. Proceeding as above, we get a factorization $g = f \circ p$. It is clear that for any $c \in C$, $f^{-1}(c)$ is a multiple fiber of multiplicity $m_c > 0$ if and only if $g^{-1}(c)$ is a multiple fiber of multiplicity $m_c > 0$. It follows that the two finite groups $T(f)$ (defined exactly as above using $f$ instead of $g$) and $T(g)$ are isomorphic under $p^*$. Hence there is a torsion character $\chi \in H^1(X, \mathbb{C}^*)$ such that $\chi' = p^*\chi$. The injectivity of $p^*$ then implies that $V \subset \chi f^*H^1(C, \mathbb{C}^*)$, and hence we have equality, since $V$ is an irreducible component of $V^1(X)$ and $\chi f^*H^1(C, \mathbb{C}^*) \subset V^1(X)$. 

Let now \(L\) and \(L'\) be the rank one local systems on \(X\) and \(\hat{X}\) associated to the torsion character \(\chi\), and respectively \(\chi'\). Then, for a generic rank one local system \(L_C\) on \(C\), Corollary 4.7 in \cite{laza} yields the following equality
\[
\dim H^1(X, L \otimes h^*L_C) = -E(C) + |\Sigma(R^0h_*(L))|,
\]
and a similar formula of \(\dim H^1(\hat{X}, L' \otimes g^*L_C)\), where \(\Sigma(\mathcal{F})\) denotes the singular support of the constructible sheaf \(\mathcal{F}\). Hence in order to prove the the equality
\[
\dim H^1(X, L \otimes h^*L_C) = \dim H^1(\hat{X}, L' \otimes g^*L_C),
\]
it is enough to show that
\[
\Sigma(R^0h_*(L)) = \Sigma(R^0g_*(L')).
\]
Let \(c \in C\) be any point, and denote by \(D_c\) a small disc in \(C\) centered at \(c\), and set \(T_c = h^{-1}(D_c)\) and \(T'_c = g^{-1}(D_c)\). Then \(c \in \Sigma(R^0h_*(L))\) if and only if the restriction \(L|T_c\) is non trivial, and the same for \(c \in \Sigma(R^0g_*(L'))\), see for instance Lemma 4.2 in \cite{laza}. Since \(T'_c = p^{-1}(T_c)\) and \(L' = p^*L\) it follows that \(L|T_c\) is trivial implies that \(L'|T'_c\) is trivial.

Conversely, suppose that \(L'|T'_c\) is trivial and that \(E_c\) is the exceptional divisor of \(p : T'_c \to T_c\). Then \(L'|T'_c\) is trivial, and hence \(L|T_{c,reg}\) is trivial. The surjectivity \(\pi_1(T_{c,reg}) \to \pi_1(T_c)\) implied by Theorem \cite{simpson} yields the triviality of \(L|T_c\).

It remains to prove the following.

**Lemma 3.6.** The isolated components of \(V^1(X)\) consist of torsion points.

**Proof.** We employ the method of Simpson \cite{simpson}, along with its extension by Budur and Wang \cite{budur}. We start with the case where \(X\) is projective. By a specialization argument (\cite{simpson} p. 373) we can assume that \(p : \hat{X} \to X\) is defined over \(\mathbb{Q}\). Following Simpson, we introduce two spaces, the “Betti” space \(M_B(\hat{X}) = \text{Spec} \mathbb{Z}[H_1(\hat{X}, \mathbb{Z})]\), and the “de Rham” space \(M_{DR}(\hat{X})\) given as the moduli space of line bundles \(V\) on \(\hat{X}\) equipped with flat connections \(\nabla\). Both of these spaces are defined over \(\mathbb{Q}\). Over \(\mathbb{C}\), the map
\[
\mu : M_{DR}(\hat{X})(\mathbb{C}) \to M_B(\hat{X})(\mathbb{C}) = H^1(\hat{X}, \mathbb{C}^*)
\]
which sends a pair \((V, \nabla)\) to its monodromy, is an analytic isomorphism. We can identify \(V^1(X)\) with a subset of the right of \(\text{3.6}\), which we temporarily rename \(V_B^1(X)\). Let \(E \subset \hat{X}\) denote the exceptional divisor, which we assume to have normal crossings. Then for any local system \(L\) on \(X\), we have an isomorphism
\[
H^1(X, L) \cong \ker[H^1(\hat{X}, L) \to H^1(E, L)]
\]
obtained from Remark \cite{simpson} 3.2. This suggests the definition for the corresponding set \(V_{DR}^1(X) \subset M_{DR}(\hat{X})\). To \((V, \nabla) \in M_{DR}(\hat{X})\), we can associate its de Rham cohomology
\[
H^i(\hat{X}, (V, \nabla)) = \mathbb{H}^i(\hat{X}, \Omega^{\bullet}_{\hat{X}} \otimes V, \nabla) = H^i(\mathbb{R}\Gamma(\Omega^{\bullet}_{\hat{X}} \otimes V, \nabla))
\]
\[ H^i(E, (V, \nabla)) = H^i(Tot^* \mathrm{R}\Gamma(\Omega^*_E \otimes V, \nabla)) \]

where \( E_* \) is the standard simplicial resolution of \( E \) (as given for example in [20 \S4]). Let \( V^1_{DR}(X) \subset M_{DR}(\hat{X}) \) consist of those pairs \((V, \nabla)\) such that \((V, \nabla)|_E = (\mathcal{O}_E, d)\) and

\[ (3.7) \quad \ker[H^1((\hat{X}, (V, \nabla)) \to H^1(E, (V, \nabla))] \neq 0 \]

Both sets \( V^1_B(X) \) and \( V^1_{DR}(X) \) are defined over \( \bar{\mathbb{Q}} \) and correspond under \((3.6)\). Therefore we may apply [32 Cor 3.5] to conclude that an isolated point \( S_B \subset V^1_B(X) \) is torsion.

When \( X \) is only quasiprojective, we choose a normal compactification \( Y \supset X \) and a smooth projective compactification \( \hat{Y} \supset \hat{X} \) such that \( D = D_1 + \ldots + D_n = \hat{Y} - \hat{X} \) plus \( E \) is a divisor with normal crossings. We can proceed as above, but now \((3.6)\) is replaced by the diagram

\[
\begin{array}{ccc}
M_{DR}(\hat{Y}) & \longrightarrow & M_{DR}(\hat{Y}/D) \\
\downarrow \mu & & \downarrow \mu \\
M_B(\hat{Y}) & \longrightarrow & M_B(\hat{X})
\end{array}
\]

of \([8]\), where \( M_{DR}(\hat{Y}/D) \) is the moduli space of line bundles on \( \hat{Y} \) with connections with logarithmic singularities along \( D \). The map \( res \) assigns the vector of residues to a connection, and \( ev \) evaluates a local system on loops around the \( D_i \). The sets \( V^1_B(X) \subset M_B(\hat{X}) \) and \( V^1_{DR}(X) \subset M_{DR}(\hat{Y}/D) \) can be defined as above, where \((V, \nabla)|_E = (\mathcal{O}_E, d)\) as before, but the second condition \((3.7)\) is replaced by

\[ \ker[H^1(\hat{Y}, \Omega^*_C(\log D) \otimes V, \nabla)) \to H^1(Tot^*(\mathbb{R}\Gamma(\Omega^*_E(\log D) \otimes V, \nabla)))] \neq 0 \]

Both \( V^1_B(X) \) and \( V^1_{DR}(X) \) are defined over \( \bar{\mathbb{Q}} \), in a way compatible with the evident \( \bar{\mathbb{Q}} \) structures on \( \mathbb{C}^n \) and \((\mathbb{C}^*)^n \). Deligne’s comparison theorem [12 II, Thm 6.10] shows that \( \mu(V^1_{DR}(X) - res^{-1}[N^n]) = V^1_B(X) \). Let \( \tau \in V^1_B(X) \) be an isolated point. Then it follows that it is defined over \( \bar{\mathbb{Q}} \) and is the image of a \( \bar{\mathbb{Q}} \) point of \( V^1_{DR}(X) \). Therefore the Gelfond-Schneider theorem implies that \( \tau \) maps to an \( N \)-torsion point under \( ev \) for some \( N \geq 1 \). Therefore \( N\tau \in V^1_B(Y) = V^1_B(X) \cap M_B(\hat{Y}) \). This is torsion by the previous case of the lemma.

\[ \square \]

**Remark 3.7.** If \( L \) is an isolated point in some characteristic variety \( V^1_k(X) \), the following questions seem to be open.

1. Is \( p^*(L) \) an isolated point in some characteristic variety of \( \hat{X} \)?
2. What is the relation between \( \dim H^1(X, L) \) and \( \dim H^1(\hat{X}, p^*L) \)?
4. Proof of Theorem 1.2 and of Corollary 1.3

The proof of Theorem 1.2 follows closely the proof of Theorem 5.8 in [26], which itself gives a new proof of Corollary (10.3) in Morgan’s paper [30].

Though both of these results discuss only smooth varieties, the construction by Hain of a mixed Hodge structure (MHS for short) on the (rational) Malcev Lie algebra \( g = g(X, x) \) associated to \( \pi_1(X, x) \) for any complex algebraic variety in [24], [25] allows us to proceed as follows.

By Remark 2.2, (iv) we know that \( H_1(X) \) is a pure Hodge structure of weight 1 for \( X \) normal projective (resp. has weights 1 and 2 for \( X \) normal quasi projective), and hence by duality, the homology group \( H_1(X) \) is a pure structure of weight \(-1\) for \( X \) projective (resp. has weights \(-1\) and \(-2\) for \( X \) quasi projective).

There is a canonical homomorphism \( H^*(g) \to H^*(X) \) which is an isomorphism in degree 1 and injective in degree 2. One can prove that it is a morphism of MHS in all degrees. However, all we need to know here is that it is a morphism in degrees \( \leq 2 \). We prove the last statement because it is much simpler. That it is an isomorphism in degree 1 is a direct consequence of [24, Thm. 6.3.1(c)]. That it is a morphism in degree 2 follows by the argument given in [26, Prop. 7.1]. In the current case, the argument, though formally the same, is simpler as the coefficient module is the trivial local system \( \mathbb{Q}_X \).

Since \( g \) is topologically generated as a Lie algebra by the image of any section of the surjection \( g \to H_1(X, \mathbb{Q}) \), it follows that \( g \) is a Hodge Lie algebra all of whose weights are negative. Then Proposition 5.2 in [26] implies that there is a canonical Lie algebra isomorphism

\[
(4.1) \quad g_\mathbb{C} = \prod_{j \geq 1} \text{Gr}^W_{-j} g_\mathbb{C},
\]

and hence the first claim in Theorem 5.8 in [26] holds for \( X \) normal.

Using now Corollary 5.7 in [26], we get the following version of the second claim in Theorem 5.8 in [26].

**Lemma 4.1.** There is a morphism of graded vector spaces

\[
\delta : \text{Gr}^W_\ast H_2(X, \mathbb{C}) \to \mathbb{L}(\text{Gr}^W_\ast H_1(X, \mathbb{C}))
\]

such that

\[
\text{Gr}^W_\ast g_\mathbb{C} = \mathbb{L}(\text{Gr}^W_\ast H_1(X, \mathbb{C}))/\langle \delta(\text{Gr}^W_\ast H_2(X, \mathbb{C})) \rangle.
\]

*Here \( \mathbb{L}(E) \) denotes the free Lie algebra spanned by a finite dimensional \( \mathbb{C} \)-vector space \( E \), see Chapter 3 in [1] for details.*

Let us study the image of \( \delta \) in Lemma 4.1 for \( X \) normal. We know that \( H_2(X, \mathbb{C}) \) has weights \( 0, -1, -2, -3 \) and \(-4\). The degree 2 part of \( \mathbb{L}(\text{Gr}^W_\ast H_1(X, \mathbb{C})) \) has weights strictly less than \(-1\) because it is a quotient of \( \otimes^2 H_1(X, \mathbb{C}) \). Therefore the graded pieces \( \text{Gr}^W_0 H_2(X, \mathbb{C}) \) and \( \text{Gr}^W_{-1} H_2(X, \mathbb{C}) \) are mapped to zero under \( \delta \). Hence
the image of \( \delta \) coincides with the image of \( \bigoplus_{j=2,3,4} \text{Gr}_j W H_2(X, \mathbb{C}) \) which consists of elements of weight \(-2, -3\) and \(-4\) in \( L(\text{Gr}_j W H_2(X, \mathbb{C})) \).

When \( X \) is projective, it is enough to show that the image of \( \delta \) consists of quadratic elements, i.e. elements in \( \otimes^2 H_1(X, \mathbb{C}) \). Since \( X \) is projective, we know that \( H_2(X, \mathbb{C}) \) has weights 0, \(-1\) and \(-2\). The graded pieces \( \text{Gr}_j W H_2(X, \mathbb{C}) \) and \( \text{Gr}_j H_2(X, \mathbb{C}) \) are mapped to zero under \( \delta \) for the same reasons as above. Hence the image of \( \delta \) coincides with the image of \( \text{Gr}_j W H_2(X, \mathbb{C}) \) which consists of elements of weight \(-2\) in \( L(\text{Gr}_j W H_2(X, \mathbb{C})) = L(H_1(X, \mathbb{C})) \). Such elements are clearly contained in \( \otimes^2 H_1(X, \mathbb{C}) \). This completes our proof of Theorem 1.2.

To prove Corollary 1.3, recall that the (finitely presentable) fundamental group \( \pi_1(X, x) \) is 1-formal if and only if its Malcev Lie algebra \( g_{\mathbb{C}} \) is isomorphic to the degree completion of a quadratically presented Lie algebra, see for instance Proposition 3.20 in [1].

The above proof yields the following result. For stronger or related statements refer to Theorem 5 in [23] (for a proof see Theorem 12.8 in [22]) showing the formality of \( \mathbb{Q} \)-manifolds, and Theorem 11 in [7], showing the formality of \( \mathbb{V} \)-manifolds.

**Corollary 4.2.** The fundamental group \( G = \pi_1(X) \) of a projective variety \( X \) which is a \( \mathbb{Q} \)-manifold is 1-formal.

Indeed, if \( X \) is a proper variety which is a \( \mathbb{Q} \)-manifold, then the Deligne MHS on \( H^k(X, \mathbb{Q}) \) is in fact pure of weight \( k \) for any \( k \), see Theorem 8.2.4 in [14]. It follows that the above proof for Corollary 1.3 works in this new setting as well.

**Remark 4.3.** There are examples of smooth varieties \( X \) such that \( H^1(X, \mathbb{Q}) \) is a pure Hodge structure of weight 1 and \( \pi_1(X, x) \) is not 1-formal, see for instance section 10 in [16].

### 5. Proof of Proposition 1.7 and of Theorems 1.8 and 1.9

First we prove Proposition 1.7. The map \( \pi_1(X^*) \to \pi_1(X) \) is surjective by Theorem 2.1. The image of \( \pi_1(\partial \Delta) \to \pi_1(X) \) induced by \( \sigma \) is trivial. Since we have a split extension \( 1 \to \pi_1(X_i) \to \pi_1(X^*) \to \pi_1(\partial \Delta) \to 1 \), the proposition follows.

The following result is needed in the proof of Theorem 1.8 but we think it has an independent interest as well.

**Proposition 5.1.** If \( V \) is a normal projective variety, the first Betti number \( b_1(V) \) of \( V \) satisfies the equality

\[
b_1(V) = 2 \dim H^1(V, \mathcal{O}_V).
\]

**Proof.** Let \( \pi : \tilde{V} \to V \) be a resolution of singularities. Then if \( L \) is a line bundle on \( V \), we see by the projection formula that \( L = \pi_* \pi^* L = \pi_* \mathcal{O}_{\tilde{V}} \otimes L = L \) since \( \pi_* \mathcal{O}_{\tilde{V}} = \mathcal{O}_V \) by normality. Therefore \( \pi^* : \text{Pic}^0(\tilde{V}) \to \text{Pic}^0(V) \) is injective. Consequently,
$\text{Pic}^0(V)$ is an abelian variety. The exponential sequence $1 \to \mathbb{Z} \to \mathcal{O}_V \to \mathcal{O}_V^* \to 1$ \cite[p 142]{19}, implies

$$\text{Pic}^0(V) = \frac{H^1(V, \mathcal{O}_V)}{H^1(V, \mathbb{Z})}$$

It follows that $H^1(V, \mathbb{Z})$ is a lattice in $H^1(V, \mathcal{O}_V)$. This proves the proposition. \hfill \square

Now we turn to the proof of Theorem 1.8. Proposition 1.7 shows that the specialization map $H^1(X_t, \mathbb{Z}) \to H^1(X_0, \mathbb{Z})$ is surjective. Let $K$ denote the kernel. Dualizing gives an exact sequence

$$0 \to H^1(X_0, \mathbb{Z}) \to H^1(X_t, \mathbb{Z}) \to \text{Hom}(K, \mathbb{Z})$$

Thus the Betti numbers satisfy $b_1(X_0) \leq b_1(X_t)$. Since all the above groups in (5.1) are torsion free, to show (1) it is enough to prove that the $b_1(X_0) = b_1(X_t)$. To see this, note that $\dim H^1(X_t, \mathcal{O}_{X_t})$ is upper semicontinuous as function of $t$ \cite{?}. When combined with Proposition 5.1, we get an inequality in the opposite direction $b_1(X_0) \geq b_1(X_t)$. Therefore assertion (1) follows. The image of the specialization map clearly lies in the invariant part $H^1(X_t, \mathbb{Z})^T$, where $T$ denotes local monodromy. This implies (2). (3) is consequence of (1) and proposition 1.7.

We now turn to the last assertion (4). Given a rank one local system $L$ on $X_0$, we can view it as a local system on $X$, and therefore on $X_t$ by restriction. Proposition 1.7 implies that the map

$$H^1(X_0, L) \to H^1(X_t, L)$$

is injective because it can be identified with an edge map for Hochschild-Serre spectral sequence, see for instance \cite{8}, p. 171 (where the homology case is treated and hence an epimorphism is obtained). One has to remark that the character $\rho_L$ associated to $L$ can be regarded as being defined on $H_1(X_0, \mathbb{Z})$, has a lift to $H_1(X_t, \mathbb{Z})$ and hence has a trivial restriction to the kernel $K$.

This immediately gives an inclusion $V^1(X_0) \subseteq V^1(X_t)$. We need the reverse inclusion. By Beauville’s structure theorem \cite{1}, the set $V^1(X_t)$ is Zariski closed, and torsion points are Zariski dense in each component. Thus we may suppose that $L \in V^1(X_t) \cap H^1(X_0, \mathbb{C}^*)$ is $n$-torsion and then show that $L \in V^1(X_0)$. By Kummer theory, we may choose an étale cover $\tilde{X} = \text{Specan}(\bigoplus_{i=0}^{n-1} L^\otimes i) \xrightarrow{p} X$ such that $p^*L$ is trivial \cite[§17]{5}. We have

$$H^1(\tilde{X}, \mathbb{C}) = \bigoplus H^1(X, L^\otimes i)$$

We can pick out the individual summands on the right by decomposing the left side into irreducibles under the action of the Galois group $\mathbb{Z}/n\mathbb{Z}$ of the covering. By applying (1) to $\tilde{X} \to \Delta$, we find that $H^1(X_0, \mathbb{C}) \cong H^1(\tilde{X}_t, \mathbb{C})$. Since restriction is clearly $\mathbb{Z}/n\mathbb{Z}$ equivariant, we obtain $H^1(X_0, L) \cong H^1(X_t, L)$. Therefore $L \in V^1(X_0)$.\hfill \square
Now we pass to the proof of Theorem 1.9. By Theorem 1.1 we can find torsion characters $\chi_i$ and maps to curves $\pi_{i,0}: X_0 \to C_{i,0}$ such that

$$V^1(X_0) = \bigcup_{i=1}^n \chi_i \pi_{i,0}^* H^1(C_{i,0}, \mathbb{C}^*) \cup \bigcup_{i=n+1}^N \{\chi_i\}$$

Let $V_i = \pi_{i,0}^* H^1(C_{i,0}, \mathbb{C}^*)$ and let $V_{i,t} \subset H^1(X_t, \mathbb{C}^*)$ denote its image under specialization. By Theorem 1.8, we have

$$V^1(X_t) \cap H^1(X_0, \mathbb{C}^*) = \bigcup_{i=1}^n \chi_i V_{i,t} \cup \bigcup_{i=n+1}^N \{\chi_i\}$$

for $t \neq 0$. By applying Theorem 1.1 to $X_t$, we see that $V_{i,t}$ is necessarily the preimage of a map to a curve $\pi_{i,t}: X_t \to C_{i,t}$. The remaining issue is to show that these curves $C_{i,t}$ and the maps $\pi_{i,t}$ fit together into a family, and this extends to $t = 0$. It is convenient to first linearize these objects by setting

$$U_{i,t} = \pi_{i,t}^* H^1(C_{i,t}, \mathbb{Z})$$

This is a sub Hodge structure of $H^1(X_t, \mathbb{Z})$. This determines an abelian variety

$$A_{i,t} = \frac{U_{i,t}^* \otimes \mathbb{C}}{U_{i,t}^* + F^{-1}U_{i,t}^* \otimes \mathbb{C}}$$

which is a quotient of the Albanese $\text{Alb}(X_t)$. We have a map $a_t: X_t \to A_{i,t}$ given by composing the Albanese map $X_t \to \text{Alb}(X_t)$, determined by the base point $\sigma(t)$, with the quotient map. We recover the curve $C_{i,t}$ from $U_{i,t}$ by taking the $im(a_t)$.

We can characterize the complexification $U_{i,t,\mathbb{C}}$ as the connected component of $\exp^{-1} V_{i,t}$ containing $0$, where $\exp: H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$ is the exponential map. This also determines the lattice via $U_{i,t} = U_{i,t,\mathbb{C}} \cap H^1(X_t, \mathbb{Z})$. This new characterization shows that $U_{i,t}$ is the image of $U_{i,0}$ under specialization.

Viewing local systems as bundles, we can see that

$$H^1(X_0, \mathbb{Z}) \times \Delta^* \cong \bigcup_{t \in \Delta^*} H^1(X_t, \mathbb{Z}) = R^1 f_* \mathbb{Z}|_{\Delta^*}$$

is constant, where the isomorphism is given by specialization. The previous discussion shows that

$$U_i = \bigcup_{t \neq 0} U_{i,t} \subset \bigcup H^1(X_t, \mathbb{Z})$$

forms a sublocal system. Since the fibres are sub Hodge structures of $H^1(X_t, \mathbb{Z})$, it follows that $U_i$ is a sub-variation of Hodge structure of $= R^1 f_* \mathbb{Z}|_{\Delta^*}$, which we denote by $U_i$. We can now construct a bundle

$$A_i = \frac{U_{i,\mathbb{C}}^*}{U_i^* + F^{-1}U_{i,\mathbb{C}}^*}$$
of Abelian varieties over $\Delta$. When restricted to $\Delta^*$, this is a quotient of the relative Albanese $Alb(X^*/\Delta^*)$. Let $\alpha : X^* \to Alb(X^*/\Delta^*) \to A_i$ be the composite, where the first map is a relative version of the Albanese map. We can see that the fibre over $t$ is precisely $a_t : X_t \to A_{i,t}$ constructed above. Let $C_i^* = image(a) \subset A_i|_{\Delta^*}$. Then the fibres of $C_i^*$ are exactly the $C_{i,t}$, $t \in \Delta^*$.

To complete the proof, we have to show that the maps $\pi_i^* : X_i^* \to C_i^*$ constructed above extend. Since the monodromy of the family $C_i^*$ is trivial, and in particular unipotent, we can conclude first of all that this has an extension $C_i$ to a stable curve over $\Delta$, and secondly that $C_i$ is in fact smooth over 0. Now let $\Gamma \subset X \times C_i$ denote the closure of the graph of $\pi_i^*$. Let $p : \Gamma \to X$ and $q : \Gamma \to C_i$ denote the projections. Given $x_0 \in X_0$, we claim that $p^{-1}(x_0)$ maps to a point in $C_i$ under $q$. Suppose not, then arguing exactly as in the proof of proposition 3.4, we would see that the induced map $r : H^1(C_i, 0, C) \to H^1(p^{-1}(x_0), C)$ would have to be nonzero. We can choose an analytic arc passing through $x_0$ and not contained in $X_0$. After normalizing and shrinking the arc, we obtain a holomorphic function $g : \Delta \to X_0$ such that $g(0) = x_0$ and $g \circ f : \Delta \to \Delta$ is a finite cover ramified at 0. Let $t$ be nonzero and sufficiently small, then we have a diagram

$$
\begin{array}{ccc}
H^1(C_0, \mathbb{C}) & \xrightarrow{r} & H^1(p^{-1}(x_0), \mathbb{C}) \\
\cong & & \\
H^1(C_t, \mathbb{C}) & \to & H^1(p^{-1}((g \circ f)^{-1}(t)), \mathbb{C}) = 0
\end{array}
$$

This shows that the map $r = 0$. So the claim is proved, and it implies that $\Gamma$ is the graph of a map $\pi_i : X \to C_i$ extending $\pi_i^*$. By chasing the diagram

$$
\begin{array}{ccc}
H^1(C_{i,0}, \mathbb{C}^*) & \to & H^1(X_0, \mathbb{C}^*) \\
\cong & & \\
H^1(C_{i,t}, \mathbb{C}^*) & \to & H^1(X_t, \mathbb{C}^*)
\end{array}
$$

We can see that $\pi_{i,0}^* H^1(C_{i,0}, \mathbb{C}^*) = V_i$, and this completes the proof.

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Department of Mathematics, Purdue University, West Lafayette, IN 47907, U.S.A.
E-mail address: dvb@math.purdue.edu

Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06100 Nice, France.
E-mail address: dimca@unice.fr

Department of Mathematics, Duke University, Durham, NC 27708-0320, U.S.A.
E-mail address: hain@math.duke.edu