INTEGRAL REPRESENTATION OF SOLUTIONS TO FUCHSIAN SYSTEM AND HEUN’S EQUATION

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Dedicated to Professor Masaki Kashiwara on his sixtieth birthday

ABSTRACT. We obtain integral representations of solutions to special cases of the Fuchsian system of differential equations and Heun’s differential equation. In particular, we calculate the monodromy of solutions to the Fuchsian equation that corresponds to Picard’s solution of the sixth Painlevé equation, and to Heun’s equation.

1. INTRODUCTION

The Fuchsian differential equation is a linear differential equation whose singularities are all regular. It frequently appears in a range of problems in mathematics and physics. For example, the famous Gauss hypergeometric differential equation is a canonical form of the second-order Fuchsian differential equation with three singularities on the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \). Global properties of solutions, i.e., the monodromy, often play decisive roles in the applications of these equations in physics and other areas of mathematics.

Heun’s differential equation is a canonical form of a second-order Fuchsian equation with four singularities, which is given by

\[
\frac{d^2y}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t} \right) \frac{dy}{dz} + \frac{\alpha \beta z - q}{z(z-1)(z-t)} y = 0,
\]

with the condition

\[
\gamma + \delta + \epsilon = \alpha + \beta + 1.
\]

Several approaches for analyzing Heun’s equation are known: including the Heun polynomial ([10]), Heun function ([10]), perturbation from the hypergeometric equation ([13]) and finite-gap integration ([21, 17, 11, 15]). Finite-gap integration is applicable for the case \( \gamma, \delta, \epsilon, \alpha - \beta \in \mathbb{Z} + 1/2, t \in \mathbb{C} \setminus \{0, 1\} \) and all \( q \), and results on the integral representation of solutions ([13]), the Bethe Ansatz ([13]), the Hermite-Krichever Ansatz ([11, 16]), the monodromy formulae by hyperelliptic integrals ([15]), the hyperelliptic-to-elliptic reduction formulae ([16]) and relationships with the Darboux transformation ([17]) have been obtained. In this paper, we obtain integral formulae of solutions for the case \( \gamma, \delta, \epsilon, \alpha + 1/2, \beta + 1/2 \in \mathbb{Z}, t \in \mathbb{C} \setminus \{0, 1\} \) and all \( q \), which then facilitates a calculation of the monodromy.

To obtain these formulae, we need to consider a Fuchsian system of differential equations with four singularities \( 0, 1, t, \infty \),

\[
\frac{dY}{dz} = \left( \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) Y,
\]

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where $A_0$, $A_1$, $A_t$ are $2 \times 2$ matrices with constant elements. We consider the case that $\det A_0 = \det A_1 = \det A_t = 0$, and $A_0 + A_1 + A_t = -\text{diag}(\kappa_1, \kappa_2)$ is a diagonal matrix. Let $\theta_i$ ($i = 0, 1, t$) denote the eigenvalues of $A_i$ other than 0, and $\theta_\infty = \kappa_1 - \kappa_2$. Under some assumptions the sixth Painlevé system is obtained by the monodromy preserving deformation. Here the sixth Painlevé system is defined by

$$
\frac{d\lambda}{dt} = \frac{\partial H_{VI}}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H_{VI}}{\partial \lambda},
$$

with the Hamiltonian

$$
H_{VI} = \frac{1}{t(t-1)} \left\{ \lambda(\lambda-1)(\lambda-t)\mu^2 \right. \\
- \left. \{ \theta_0(\lambda-1)(\lambda-t) + \theta_1 \lambda(\lambda-t) + (\theta_t - 1)\lambda(\lambda-1) \} \mu + \kappa_1(\kappa_2 + 1)(\lambda-t) \right\}.
$$

By eliminating $\mu$ in Eq. (1.4), we obtain the sixth Painlevé equation for $\lambda$,

$$
\frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left\{ \frac{1}{\lambda + 1} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t} \right\} \left( \frac{d\lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t} \right) \frac{d\lambda}{dt} \\
+ \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2} \left\{ \frac{(1-\theta_\infty)^2}{2} - \frac{\theta_0^2}{\lambda^2} + \frac{\theta_1^2}{(t-1)^2} + \frac{\theta_t^2}{(\lambda-t)^2} \right\},
$$

which is a non-linear ordinary differential equation of order two whose solutions do not have movable singularities other than poles. It is known that the sixth Painlevé systems have symmetry, and the action of the symmetry is called the Okamoto-Bäcklund transformation. The sixth Painlevé system has two-parameter solutions for the case $\theta_0 = \theta_1 = \theta_t = 1 - \theta_\infty = 0$, which are called Picard’s solution. By the Okamoto-Bäcklund transformation of the sixth Painlevé system, Picard’s solutions are transformed to the solutions for the case $(\theta_0, \theta_1, \theta_t, 1-\theta_\infty) \in O_1 \cup O_2$, where

$$
O_1 = \left\{ (\theta_0, \theta_1, \theta_t, 1-\theta_\infty) | \theta_0, \theta_1, \theta_t, 1-\theta_\infty \in \mathbb{Z}, \frac{1}{2} \right\},
$$

$$
O_2 = \left\{ (\theta_0, \theta_1, \theta_t, 1-\theta_\infty) | \theta_0, \theta_1, 1-\theta_\infty \in \mathbb{Z}, \theta_0 + \theta_1 + \theta_t + 1 - \theta_\infty \in 2\mathbb{Z} \right\}.
$$

For the case $(\theta_0, \theta_1, \theta_t, 1-\theta_\infty) \in O_1$, solutions of the Fuchsian system (Eq. (1.3)) are expressed in the form of the Hermite-Krichever Ansatz, which is a consequence of results presented in [18]. In the present study, we investigate solutions of the Fuchsian system for the case $(\theta_0, \theta_1, \theta_t, 1-\theta_\infty) \in O_2$.

These solutions will be shown to have integral representations whose integrands are functions in the form of the Hermite-Krichever Ansatz, and the Okamoto-Bäcklund transformation of the sixth Painlevé system, Picard’s solutions are transformed to the solutions for the case $(\theta_0, \theta_1, \theta_t, 1-\theta_\infty) = (0, 0, 0, 0)$, and we can calculate the monodromy explicitly. By considering the monodromy preserving deformation directly, we recover Picard’s solution of the sixth Painlevé equation.

The integral representations of solutions to the Fuchsian system follow from the results by Dettweiler-Reiter [15] and Filipuk [6] on the middle convolution (see section 3). By considering special cases, we obtain integral formulae of solutions to Heun’s equation for the case $\gamma, \delta, \epsilon, \alpha + 1/2, \beta + 1/2 \in \mathbb{Z}$, $t \in \mathbb{C} \setminus \{0, 1\}$ and all $q$, which are then available for calculating the monodromy. For the case $\gamma = \delta = \epsilon = \ldots$
$1, \alpha = 3/2, \beta = 1/2$, we have explicit representations of the integral, and so we obtain explicit representations of the monodromy.

This paper is organized as follows: In section 2, we introduce notation for the Fuchsian system with four singularities. In section 3, we review results on the middle convolution due to Dettweiler-Reiter and Filipuk, and combine their results. In section 4, we recall the Hermite-Krichever Ansatz. In section 5, we obtain explicit representations of the monodromy. Then Eq. (2.1) is Fuchsian, i.e., any singularities on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ are regular, and it may have regular singularities at $z = 0, 1, t, \infty$ on this sphere. Set

\begin{equation}
A_0 = \begin{pmatrix}
  u_0 + \theta_0 & -w_0 \\
  u_0(u_0 + \theta_0)/w_0 & -u_0
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
  u_1 + \theta_1 & -w_1 \\
  u_1(u_1 + \theta_1)/w_1 & -u_1
\end{pmatrix},
\end{equation}

where $Y = t(y_1(z), y_2(z)), t \neq 0, 1$, $A_0, A_1, A_t$ are $2 \times 2$ matrices with constant elements. Then Eq. (2.1) is Fuchsian, i.e., any singularities on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ are regular, and it may have regular singularities at $z = 0, 1, t, \infty$ on this sphere. Set

\begin{equation}
A_t = \begin{pmatrix}
  u_t + \theta_t & -w_t \\
  u_t(u_t + \theta_t)/w_t & -u_t
\end{pmatrix},
\end{equation}

where $u_0, w_0, u_1, w_1, u_t, w_t$ are defined by

\begin{equation}
w_0 = \frac{k\lambda}{t}, \quad w_1 = -\frac{k(\lambda - 1)}{t-1}, \quad w_t = \frac{k(\lambda - t)}{t(t-1)},
\end{equation}

\begin{align*}
u_0 &= -\theta_0 + \frac{\lambda}{t\theta_\infty} [\lambda(\lambda - 1)(\lambda - t)\mu^2 + \{2\kappa_1(\lambda - 1)(\lambda - t) - \theta_1(\lambda - t) - t\theta_t(\lambda - t) - \theta_1 - t\theta_t\}], \\
u_1 &= -\theta_1 - \frac{\lambda - 1}{(t - 1)\theta_\infty} [\lambda(\lambda - 1)(\lambda - t)\mu^2 + \{2\kappa_1(\lambda - 1)(\lambda - t) - \theta_1(\lambda - t) - t\theta_t(\lambda - t) - \theta_1 - (t - 1)\theta_t\}], \\
u_t &= -\theta_t + \frac{\lambda - t}{t(t - 1)\theta_\infty} [\lambda(\lambda - 1)(\lambda - t)\mu^2 + \{2\kappa_1(\lambda - 1)(\lambda - t) - \theta_1(\lambda - t) + t(\theta_\infty - \theta_t)(\lambda - t) + \theta_0 - (t - 1)(\theta_\infty - \theta_t)\}],
\end{align*}

2. FUCHSIAN SYSTEM WITH FOUR SINGULARITIES

We consider a system of ordinary differential equations,

\begin{equation}
\frac{dY}{dz} = A(z)Y, \quad A(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} = \begin{pmatrix}
  a_{11}(z) & a_{12}(z) \\
  a_{21}(z) & a_{22}(z)
\end{pmatrix},
\end{equation}

where $Y = t(y_1(z), y_2(z)), t \neq 0, 1$, $A_0, A_1, A_t$ are $2 \times 2$ matrices with constant elements. Then Eq. (2.1) is Fuchsian, i.e., any singularities on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ are regular, and it may have regular singularities at $z = 0, 1, t, \infty$ on this sphere. Set
and $\kappa_1 = (\theta_\infty - \theta_0 - \theta_1 - \theta_t)/2$, $\kappa_2 = -(\theta_\infty + \theta_0 + \theta_1 + \theta_t)/2$. Note that the eigenvalues of $A_i$ ($i = 0, 1, t$) are 0 and $\theta_i$. Set $A_\infty = -(A_0 + A_1 + A_t)$. Then

$$A_\infty = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$  

We denote the Fuchsian system (Eq. (2.1)) with Eqs. (2.2, 2.3) by $D_Y^t(\theta_0, \theta_1, \theta_\infty; \lambda, \mu; k)$. By eliminating $y_2(z)$ in Eq. (2.1), we have a second-order linear differential equation,

$$\frac{d^2y_1(z)}{dz^2} + \left( \frac{1 - \theta_0}{z} + \frac{1 - \theta_1}{z - 1} + \frac{1 - \theta_t}{z - t} - \frac{1}{z - \lambda} \right) \frac{dy_1(z)}{dz} + \left( \frac{\kappa_1(\kappa_2 + 1)}{z(z - 1)} + \frac{\lambda(\lambda - 1)\mu}{z(z - 1)(z - \lambda)} - \frac{t(t - 1)H}{z(z - 1)(z - t)} \right) y_1(z) = 0,$$

$$H = \frac{1}{t(t - 1)}[\lambda(\lambda - 1)(\lambda - t)\mu^2 - \{\theta_0(\lambda - 1)(\lambda - t) + \theta_1\lambda(\lambda - t) + (\theta_t - 1)\lambda(\lambda - 1)\} + \kappa_1(\kappa_2 + 1)(\lambda - t)],$$

which we denote by $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$. This equation has regular singularities at $z = 0, 1, t, \lambda, \infty$. The exponents of the singularity $z = \lambda$ are 0, 2, and this singularity is apparent (i.e. non-logarithmic). Note that the sixth Painlevé system

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda}$$

describes the condition for the monodromy preserving deformation of Eq. (2.1) with respect to the variable $t$.

It is known that the sixth Painlevé system has symmetry of the extended affine Weyl group of type $F_4^{(1)}$ ([9], which is called the Okamoto-Bäcklund transformation. In particular the sixth Painlevé system is invariant under Okamoto’s transformation $s_2$ defined by

$$s_2 : \theta_0 \rightarrow \kappa_1 + \theta_0, \quad \theta_1 \rightarrow \kappa_1 + \theta_1, \quad \theta_t \rightarrow \kappa_1 + \theta_t, \quad \theta_\infty \rightarrow -\kappa_2,$$

$$\lambda \rightarrow \lambda + \kappa_1/\mu, \quad \mu \rightarrow \mu, \quad t \rightarrow t.$$  

Note that $s_2$ is involutive, i.e., $(s_2)^2 = 1$.

3. Middle convolution

Dettweiler and Reiter [4, 5] gave an algebraic analogue of Katz’ middle convolution functor, and Filipuk [8] applied them for the Fuchsian system with four singularities. We review and combine these authors’ results for the present setting. Note that the results of Dettweiler and Reiter are valid for Fuchsian equations of an arbitrary size and an arbitrary number of singular points. Let $A_0$, $A_1$, $A_t$ be matrices in $\mathbb{C}^{2 \times 2}$. For $\nu \in \mathbb{C}$, we define the convolution matrices $B_0, B_1, B_t \in \mathbb{C}^{6 \times 6}$ as follows:

$$(3.1) \quad B_0 = \begin{pmatrix} A_0 + \nu & A_1 & A_t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ A_0 & A_1 + \nu & A_t \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_0 & A_1 & A_t + \nu \end{pmatrix}.$$
We consider the following differential equation:

\[
dU \over dz = \left( \frac{B_0}{z} + \frac{B_1}{z-1} + \frac{B_t}{z-t} \right) U, \quad U \in \mathbb{C}^6
\]

We fix a base point \( o \in \mathbb{C} \setminus \{0, 1, t\} \). Let \( \alpha_i (i = 0, 1, t, \infty) \) be a cycle turning the point \( w = i \) anti-clockwise whose base point is \( o \). Let \( z \in \mathbb{C} \setminus \{0, 1, t\} \) and \( \alpha_z \) be a cycle turning the point \( w = z \) anti-clockwise. Let \( [\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta \) be the Pochhammer contour.

**Proposition 3.1.** ([5]) Assume that \( Y = {}^t(y_1(z), y_2(z)) \) is a solution to the differential equation

\[
dY \over dz = \left( \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) Y.
\]

For \( i \in \{0, 1, t, \infty\} \), the function

\[
U = \begin{pmatrix}
\int_{[\alpha, \alpha_i]} w^{-1}y_1(w)(z-w)^\nu dw \\
\int_{[\alpha, \alpha_i]} w^{-1}y_2(w)(z-w)^\nu dw \\
\int_{[\alpha, \alpha_i]} (w-1)^{-1}y_1(w)(z-w)^\nu dw \\
\int_{[\alpha, \alpha_i]} (w-1)^{-1}y_2(w)(z-w)^\nu dw \\
\int_{[\alpha, \alpha_i]} (w-t)^{-1}y_1(w)(z-w)^\nu dw \\
\int_{[\alpha, \alpha_i]} (w-t)^{-1}y_2(w)(z-w)^\nu dw
\end{pmatrix},
\]

satisfies differential equation (3.2).

**Proof.** It follows from a straightforward calculation that the function

\[
U = \begin{pmatrix}
z^{-1}y_1(z) \\
z^{-1}y_2(z) \\
(z-1)^{-1}y_1(z) \\
(z-1)^{-1}y_2(z) \\
(z-t)^{-1}y_1(z) \\
(z-t)^{-1}y_2(z)
\end{pmatrix}
\]

is a solution of Eq. (3.2) for the case \( \nu = -1 \) (see [5 Lemma 6.4]).

It is shown in [5 Lemma 6.2] that if \( U = {}^t(u_1(z), u_2(z), \ldots, u_6(z)) \) is a solution of Eq. (3.2) for the case \( \nu = \nu_1 \), then the function

\[
U = \begin{pmatrix}
\int_{[\alpha, \alpha_i]} u_1(w)(w-z)^{\nu_2-1} dw \\
\int_{[\alpha, \alpha_i]} u_2(w)(w-z)^{\nu_2-1} dw \\
\vdots \\
\int_{[\alpha, \alpha_i]} u_6(w)(w-z)^{\nu_2-1} dw
\end{pmatrix},
\]

is a solution of Eq. (3.2) for the case \( \nu = \nu_1 + \nu_2 \). By applying this result for the case \( \nu_1 = -1, \nu_2 = \nu + 1 \), we obtain the proposition. \( \square \)

We set

\[
\mathcal{L}_0 = \begin{pmatrix}
\text{Ker}(A_0) \\
0 \\
0
\end{pmatrix}, \quad \mathcal{L}_1 = \begin{pmatrix}
0 \\
\text{Ker}(A_1) \\
0
\end{pmatrix}, \quad \mathcal{L}_t = \begin{pmatrix}
0 \\
0 \\
\text{Ker}(A_t)
\end{pmatrix},
\]

\( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_t, \quad \mathcal{K} = \text{Ker}(B_0) \cap \text{Ker}(B_1) \cap \text{Ker}(B_t). \)
We fix an isomorphism between $\mathbb{C}^6/(\mathcal{K} + \mathcal{L})$ and $\mathbb{C}^m$ for some $m$. A tuple of matrices $m\nu(A) = (\bar{B}_0, \bar{B}_1, \bar{B}_t)$, where $\bar{B}_k$ ($k = 0, 1, t$) is induced by the action of $B_k$ on $\mathbb{C}^m \simeq \mathbb{C}^6/(\mathcal{K} + \mathcal{L})$, is called an additive version of the middle convolution of $(A_0, A_1, A_t)$ with the parameter $\nu$. Filipuk [6] established that, if $\nu = \kappa_1$, then $\mathbb{C}^6/(\mathcal{K} + \mathcal{L})$ is isomorphic to $\mathbb{C}^2$ and the isomonodromic deformation of the middle convolution system

\begin{equation}
\frac{d\tilde{Y}}{dz} = \left( \frac{\bar{B}_0 + \bar{B}_1}{z} + \frac{\bar{B}_t}{z-t} \right) \tilde{Y},
\end{equation}

gives the sixth Painlevé equation for the parameters transformed by Okamoto’s transformation $s_2$. Note that Boalch [2] obtained a geometric result on Okamoto’s transformation earlier by finding an isomorphism between a $2 \times 2$ Fuchsian equation and a $3 \times 3$ Fuchsian equation, which would be related with Filipuk’s result.

We now calculate explicitly the Fuchsian differential equation determined by the middle convolution that is required for our purpose, and which reproduces the result by Filipuk [6]. Let $A_0, A_1, A_t$ be the matrices defined by Eq.(2.2). If $\nu = \kappa_1$, then the spaces $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_t, \mathcal{K}$ are written as

\begin{equation}
\mathcal{L}_0 = \mathbb{C} \begin{pmatrix} w_0 \\ u_0 + \theta_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{L}_1 = \mathbb{C} \begin{pmatrix} 0 \\ 0 \\ w_1 \\ u_1 + \theta_1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{L}_t = \mathbb{C} \begin{pmatrix} 0 \\ 0 \\ 0 \\ w_t \\ u_t + \theta_t \end{pmatrix}, \quad \mathcal{K} = \mathbb{C} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\end{equation}

Set

\begin{equation}
S = \begin{pmatrix}
0 & 0 & 1 & w_0 & 0 & 0 \\
0 & 0 & 0 & u_0 + \theta_0 & 0 & 0 \\
P_{s1} & 0 & 0 & w_1 & 0 & 0 \\
P_{s2} & 0 & 0 & u_1 + \theta_1 & 0 & 0 \\
P_{s51} & P_{s52} & 1 & 0 & 0 & w_t \\
P_{s51} & P_{s52} & 0 & 0 & 0 & u_t + \theta_t \\
\end{pmatrix}, \quad S_{31} = \frac{t(1-t)}{\kappa_2 \kappa_4 (t_0 + \theta_0)} (u_1 + \theta_1), \quad S_{32} = \frac{1}{\theta_\infty} \left( \frac{w_0 - u_0 + \theta_0}{w_1} - \frac{w_0}{u_1 + \theta_1} \right), \quad S_{32} = \frac{1}{\theta_\infty} \left( \frac{w_0}{u_0 + \theta_0} - \frac{w_0}{u_0 + \theta_t} \right),
\end{equation}

and $\bar{U} = S^{-1}U$, where $U$ is a solution to Eq.(3.2). Then $\text{det} \bar{U} = k \lambda (\lambda - 1)(\lambda - t)\mu/(t(1-t)\theta_\infty)$ and $\bar{U}$ satisfies

\begin{equation}
\frac{d\bar{U}}{dz} = \begin{pmatrix} b_{11}(z) & b_{12}(z) & 0 & 0 & 0 & 0 \\
0 & b_{22}(z) & 0 & 0 & 0 & 0 \\
0 & b_{33}(z) & 0 & 0 & 0 & 0 \\
0 & 0 & b_{34}(z) & 0 & \frac{1}{\kappa_1} & 0 \\
0 & 0 & b_{54}(z) & 0 & \frac{\kappa_1}{z-1} & 0 \\
0 & 0 & b_{52}(z) & 0 & 0 & \frac{1}{\kappa_1} \end{pmatrix} \bar{U},
\end{equation}

where $b_{ij}(z)$ ($i = 1, 2, 3$) and $b_{ij}(z)$ ($i = 1, \ldots, 6$) are rational functions. Write $\bar{U} = ^t(\bar{u}_1(z), \bar{u}_2(z), \ldots, \bar{u}_6(z))$ and set $\tilde{g}_1(z) = \bar{u}_1(z)$, $\tilde{g}_2(z) = \bar{u}_2(z)$ and $\tilde{Y} = ^t(\tilde{g}_1(z), \tilde{g}_2(z))$. 

Then we have
\[
(3.12) \quad \frac{d\tilde{Y}}{dz} = \begin{pmatrix}
 b_{11}(z) & b_{12}(z) \\
 b_{21}(z) & b_{22}(z)
\end{pmatrix} \tilde{Y}.
\]

The elements \(b_{11}(z), b_{12}(z), b_{21}(z), b_{22}(z)\) are calculated explicitly and Eq. (3.12) coincides with the Fuchsian differential equation \(D_Y(\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_\infty; \bar{\lambda}, \bar{\mu}; \bar{k})\) (see Eq. (2.11)), where
\[
\begin{align*}
\bar{\theta}_0 &= \frac{\theta_0 - \theta_1 - \theta_t + \theta_\infty}{2}, \\
\bar{\theta}_1 &= -\frac{\theta_0 + \theta_1 - \theta_t + \theta_\infty}{2}, \\
\bar{\theta}_\infty &= \frac{\theta_0 + \theta_1 + \theta_t + \theta_\infty}{2},
\end{align*}
\]
\[
\bar{\lambda} = \lambda + \kappa_1/\mu, \quad \bar{\mu} = \mu, \quad \bar{k} = k.
\]

The functions \(\tilde{y}_1(z)\) and \(\tilde{y}_2(z)\) are expressed as
\[
(3.14) \quad \tilde{y}_1(z) = \frac{\bar{\lambda}(u_0 + \theta_0)}{\lambda} u_1(z) - \frac{k\bar{\lambda}}{t-1} u_2(z) + \frac{(\bar{\lambda} - 1)(u_1 + \theta_t)}{\lambda - 1} u_3(z)
\]
\[
+ \frac{k(\bar{\lambda} - 1)}{t-1} u_4(z) + \frac{(\bar{\lambda} - t)(u_t + \theta_t)\mu}{\lambda - t} u_5(z) + \frac{k(\bar{\lambda} - t)}{t(1-t)} u_6(z),
\]
\[
\tilde{y}_2(z) = \frac{\theta_0}{\kappa_2} \left( \frac{-tu_0(u_0 + \theta_0)}{k\lambda} u_1(z) + u_0 u_2(z) + \frac{(t - 1)u_1(u_1 + \theta_t)}{k(\lambda - 1)} u_3(z) \right)
\]
\[
+ \frac{u_1 u_4(z) + \frac{(t - 1)u_1(u_1 + \theta_t)}{k(\lambda - t)} u_5(z) + u_1 u_6(z)}{u_1(z)}.
\]

It follows from Proposition 3.1 that the function \(U = ^i(u_1(z), u_2(z), \ldots, u_6(z))\) given by Eq. (3.1) is a solution to Eq. (3.2). Combining with the relations \(y_2(w) = (dy_1(w)/dw - a_{11}(w)y_1(w))/a_{12}(w),\) \(y_1(w) = (dy_2(w)/dw - a_{22}(w)y_2(w))/a_{21}(w)\) and Eq. (3.14), the functions \(\tilde{y}_1(z)\) and \(\tilde{y}_2(z)\) are expressed as the integral in the following proposition by means of a straightforward calculation:

**Proposition 3.2.** Set \(\kappa_1 = (\theta_\infty - \theta_0 - \theta_1 - \theta_t)/2\) and \(\kappa_2 = -\theta_\infty + \theta_0 + \theta_1 + \theta_t)/2\). If \(Y = ^i(y_1(z), y_2(z))\) is a solution to the Fuchsian differential equation \(D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)\) (see Eq. (2.11)), then the function \(\tilde{Y} = ^i(\tilde{y}_1(z), \tilde{y}_2(z))\) defined by
\[
(3.15) \quad \tilde{y}_1(z) = \int_{[\alpha, \alpha]} \left\{ \kappa_1 y_1(w) + (w - \bar{\lambda}) dy_1(w)/dw \right\} \frac{(z - w)^{\kappa_1}}{w - \lambda} dw,
\]
\[
\tilde{y}_2(z) = -\frac{\theta_0}{\kappa_2} \int_{[\alpha, \alpha]} \frac{dy_2(w)}{dw} (z - w)^{\kappa_1} dw,
\]
satisfies the Fuchsian differential equation \(D_Y(\kappa_1 + \theta_0, \kappa_1 + \theta_1, \kappa_1 + \theta_t, -\kappa_2; \lambda + \kappa_1/\mu, \mu; k)\) for \(i \in \{0, 1, t, \infty\}\).

Therefore, if we know a solution to the differential equation \(D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)\), then we have integral representations of solutions to the Fuchsian differential equation \(D_Y(\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_t, \bar{\theta}_\infty; \bar{\lambda}, \bar{\mu}; \bar{k})\) obtained by Okamoto’s transformation \(s_2\). It can be shown that, if \(\kappa_2 \neq 0, \kappa_1 \not\in \mathbb{Z}\) and \(\theta_i \not\in \mathbb{Z}\) for some \(i \in \{0, 1, t, \infty\}\), then the function \(\tilde{y}_1(z)\) is non-zero for generic \(\lambda\) and \(\mu\) (see \(\Delta\) Lemma 6.6)). On the other hand, a solution to Eq. (2.11) for the case \(\theta_0, \theta_1, \theta_t, \theta_\infty \in \mathbb{Z} + 1/2\) can be expressed in the form of the Hermite-Krichever Ansats. In the next section, we recall the Hermite-Krichever Ansats.
4. HERMITE-KRICHEVER ANSatz

We rewrite Eq.(2.5) in elliptical form. Recall that Eq.(2.5) is written as

\[
\frac{d^2 y_1(z)}{dz^2} + \left( \frac{1 - \theta_0}{z} + \frac{1 - \theta_1}{z - 1} + \frac{1 - \theta_t}{z - \lambda} - \frac{1}{z - \mu} \right) \frac{dy_1(z)}{dz} + \left( \kappa_1(\kappa_2 + 1) \right) \frac{\lambda(\lambda - 1)\mu}{z(z - 1)(z - \lambda)} - \frac{t(t - 1)H}{z(z - 1)(z - t)} \right) y_1(z) = 0,
\]

and \( H \) is determined by

\[
H = \frac{1}{t(t - 1)} \left[ \lambda(\lambda - 1)(\lambda - t)\mu^2 - \{\theta_0(\lambda - 1)(\lambda - t) + \theta_1\lambda(\lambda - t) + (\theta_t - 1)\lambda(\lambda - 1)\} \mu + \kappa_1(\kappa_2 + 1)(\lambda - t) \right].
\]

Let \( \wp(x) \) be the Weierstrass \( \wp \)-function with periods \((2\omega_1, 2\omega_3)\), \( \omega_0(= 0), \omega_1, \omega_2(= -\omega_1 - \omega_3), \omega_3 \) be half-periods and \( e_i = \wp(\omega_i) \) \((i = 1, 2, 3)\). Set

\[
z = \frac{\wp(x) - e_1}{e_2 - e_1}, \quad t = \frac{e_3 - e_2}{e_1 - e_2}, \quad \lambda = \frac{\wp(\delta) - e_1}{e_2 - e_1}
\]

For \( t \in \mathbb{C} \setminus \{0, 1\} \), there exists a pair of periods \((2\omega_1, 2\omega_3)\) such that \( t = (\wp(\omega_3) - \wp(\omega_2))/\wp(\omega_1) - \wp(\omega_2)) \). The value \( \delta \) is determined up to the sign \( \pm \) and the periods \( 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \). Set

\[
\theta_0 = l_1 + 1/2, \quad \theta_1 = l_2 + 1/2, \quad \theta_t = l_3 + 1/2, \quad \theta_\infty = -l_0 + 1/2,
\]

\[
f(x) = y_1(z)z^{-l_1/2}(z - 1)^{-l_2/2}(z - t)^{-l_3/2}.
\]

Then Eq.(4.1) is transformed to

\[
- \frac{d^2 f(x)}{dx^2} + \frac{\wp'(x)}{\wp(x) - \wp(\delta)} \frac{d}{dx} + \frac{\bar{s}}{\wp(x) - \wp(\delta)} + \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i) + C \right) f(x) = 0,
\]

\[
\bar{s} = -4(e_2 - e_1)^2\lambda(\lambda - 1)(\lambda - t) \left( \mu - \frac{l_1}{2\lambda} - \frac{l_2}{2(\lambda - 1)} - \frac{l_3}{2(\lambda - t)} \right),
\]

\[
C = 4(e_2 - e_1) \{ \lambda(1 - \lambda)\mu - t(1 - t)H \} + (l_1 + l_2 + l_3 + l_0 + 1)(l_1 + l_2 + l_3 - l_0)e_3
\]

\[
- 2(l_1l_2e_3 + l_2l_3e_1 + l_3l_1e_2) + 2(l_1 + l_2 + l_3)((e_2 - e_1)\lambda + e_1) + \sum_{i=1}^{3} l_i(l_i + 2)e_i,
\]

and Eq.(4.2) is equivalent to the equality

\[
C = 4(e_2 - e_1)\lambda(\lambda - 1)(\lambda - t)\mu \left\{ \mu - \frac{l_1 + \frac{1}{\lambda}}{\lambda} - \frac{l_2 + \frac{1}{\lambda - 1}}{\lambda - 1} - \frac{l_3 + \frac{1}{\lambda - t}}{\lambda - t} \right\} + \sum_{i=1}^{3} l_i(l_i + 2)e_i
\]

\[
+ ((e_2 - e_1)\lambda + e_1)\{(l_1 + l_2 + l_3 + l_0 + 2)(l_1 + l_2 + l_3 - l_0 + 1) - 2\}
\]

\[
- 2(l_1l_2e_3 + l_2l_3e_1 + l_3l_1e_2),
\]
which shows that the regular singularities \( x = \pm \delta \) are apparent. The sixth Painlevé equation (Eq. 4.6) for \( \lambda = (\varphi(\delta) - e_1)/(e_2 - e_1) \) also has an elliptical representation

\[
\frac{d^2 \delta}{dt^2} = -\frac{1}{4\pi^2} \left\{ \frac{(1 - \theta^2)}{2} \varphi'(\delta) + \frac{\theta_0^2}{2} \varphi'(\delta + 1) + \frac{\theta_1^2}{2} \varphi'(\delta + \tau + 1) + \frac{\theta_2^2}{2} \varphi'(\delta + \tau) \right\},
\]

where \( \omega_1 = 1/2, \omega_3 = \tau/2 \) and \( \varphi(z) = (\partial/\partial z)\varphi(z) \) (see [8, 12, 19]), and it is related to the monodromy preserving deformation of Eq. 4.5 by the variable \( \tau = \omega_3/\omega_1 \).

We recall that a solution to Eq. 4.5 can be expressed in the form of the Hermite-Krichever Ansatz if \( l_0, l_1, l_2, l_3 \in \mathbb{Z} \). Note that the condition \( \theta_\infty, \theta_0, \theta_1, \theta_\tau \in \mathbb{Z} + 1/2 \) corresponds to the condition \( l_0, l_1, l_2, l_3 \in \mathbb{Z} \). Set

\[
\Phi_i(x, \alpha) = \frac{\sigma(x + \omega_i - \alpha)}{\sigma(x + \omega_i)} \exp(\zeta(\alpha)x), \quad (i = 0, 1, 2, 3).
\]

**Proposition 4.1.** (18) Set \( \tilde{l}_0 = |l_0 + 1/2| + 1/2 \) and \( \tilde{l}_i = |l_i + 1/2| - 1/2 \) \((i = 1, 2, 3)\). For \( l_0, l_1, l_2, l_3 \in \mathbb{Z} \), we have polynomials \( Q(\lambda, \mu), P_1(\lambda, \mu), \ldots, P_6(\lambda, \mu) \) such that if \( P_2(\lambda, \mu) \neq 0 \) then there exists a solution \( f_{HK}(x; l_0, l_1, l_2, l_3; \lambda, \mu) \) to Eq. 4.5 of the form

\[
f_{HK}(x; l_0, l_1, l_2, l_3; \lambda, \mu) = \exp(\kappa x) \left( \sum_{i=0}^{3} \sum_{j=0}^{\tilde{l}_i-1} b^{(i)}_j \frac{d^j}{dx^j} \Phi_i(x, \alpha) \right)
\]

for some values \( \alpha, \kappa \) and \( b^{(i)}_j \) \((i = 0, 1, 2, 3, j = 0, \ldots, \tilde{l}_i - 1)\), and the values \( \alpha \) and \( \kappa \) are expressed as

\[
\varphi(\alpha) = \frac{P_1(\lambda, \mu)}{P_2(\lambda, \mu)}, \quad \varphi'(\alpha) = \frac{P_3(\lambda, \mu)}{P_4(\lambda, \mu)} \sqrt{-Q(\lambda, \mu)},
\]

\[
\kappa = \frac{P_5(\lambda, \mu)}{P_6(\lambda, \mu)} \sqrt{-Q(\lambda, \mu)}.
\]

Regarding the periodicity of the function \( f_{HK}(x) = f_{HK}(x; l_0, l_1, l_2, l_3; \lambda, \mu) \), we have

\[
f_{HK}(x + 2\omega_j) = \exp(-2\eta_j \alpha + 2\omega_j \zeta(\alpha) + 2\kappa \omega_j) f_{HK}(x),
\]

for \( j = 1, 3 \), where \( \eta_j = \zeta(\omega_j) \).

It follows from Proposition 4.1 that a solution to the Fuchsian differential system \( D_{y_1}(l_1 + 1/2, l_2 + 1/2, l_3 + 1/2, -l_0 + 1/2; \lambda, \mu; k) \) is expressed in the form of the Hermite-Krichever Ansatz for the case \( l_0, l_1, l_2, l_3 \in \mathbb{Z} \) by setting \( z = (\varphi(x) - e_1)/(e_2 - e_1), \quad t = (e_3 - e_2)/(e_1 - e_2), \quad y_1(z) = z^{l_1/2}(z-1)^{l_2/2}(z-t)^{l_3/2}f_{HK}(x; l_0, l_1, l_2, l_3; \lambda, \mu), \quad y_2(z) = (dy_1(z)/dz - a_{11}(z)y_1(z))/a_{12}(z). \)

We now consider the Hermite-Krichever Ansatz for the case \( l_0 = l_1 = l_2 = l_3 = 0 \) in detail, which was demonstrated in [18]. The differential equation 4.5 is written as

\[
\left\{ -\frac{d^2}{dx^2} + \frac{\varphi'(x)}{\varphi(x) - \varphi(\delta)} \frac{d}{dx} - \frac{4\mu\lambda(\lambda - 1)(\lambda - t)(e_2 - e_1)^2}{\varphi(x) - \varphi(\delta)} + C \right\} f(x) = 0,
\]

We assume that \( \delta \not\equiv 0 \mod \omega_1\mathbb{Z} \oplus \omega_3\mathbb{Z} \). The condition that the regular singularities \( x = \pm \delta \) are apparent is written as

\[
C = 2(2\lambda(\lambda - 1)(\lambda - t)\mu^2 - (3\lambda^2 - 2(1+t)\lambda + t)\mu)(e_2 - e_1),
\]
(see Eq. (4.6)). We consider Eq. (4.12) with the condition in Eq. (4.13). The polynomial $Q(\lambda, \mu)$ in Eq. (4.10) is calculated as
\begin{equation}
Q(\lambda, \mu) = -2\mu(2\lambda - 1)(2\lambda - 1)(2\lambda - t)/\sqrt{e_2 - e_1}.
\end{equation}
There exists a solution $f_{HK}(x)(= f_{HK}(x; 0, 0, 0; \lambda, \mu))$ to Eq. (4.12) that can be expressed in the form of the Hermite-Krichever Ansatz as
\begin{equation}
f_{HK}(x) = \tilde{b}_0(0) \exp(\kappa x) \Phi_0(x, \alpha),
\end{equation}
if $\mu \neq 0$. The values $\alpha$ and $\kappa$ are determined as
\begin{equation}
\varphi(\alpha) = e_1 + (e_2 - e_1) \left( \lambda - \frac{1}{2\mu} \right), \quad \varphi'(\alpha) = -\frac{(e_2 - e_1)^2 \sqrt{-Q(\lambda, \mu)}}{2\mu^2},
\end{equation}
and we have
\begin{equation}
\lambda = -\frac{1}{e_2 - e_1} \left\{ \varphi(\alpha) - e_1 - \frac{\varphi'(\alpha)}{2\kappa} \right\}, \quad \mu = -\frac{(e_2 - e_1)\kappa}{\varphi'(\alpha)}.
\end{equation}

5. Integral representation of solutions to Fuchsian system

We show that solutions to the Fuchsian system (Eq. (2.1)) have integral representations for the case $\theta_0, \theta_1, \theta_t, \theta_\infty \in \mathbb{Z}$, $\theta_0 + \theta_1 + \theta_t + \theta_\infty \in 1 + 2\mathbb{Z}$ by use of the function in the form of Hermite-Krichever Ansatz.

**Theorem 5.1.** Assume that $l_0, l_1, l_2, l_3 \in \mathbb{Z}$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}$. Let $f_{HK}(x)(= f_{HK}(x; l_0, l_1, l_2, l_3; \lambda, \mu))$ be the solution expressed in the form of the Hermite-Krichever Ansatz in Proposition 4.1. Set
\begin{equation}
\tilde{\varphi}(x) = \frac{\varphi(x) - e_1}{e_2 - e_1}, \quad \kappa_1 = -\frac{l_0 + l_1 + l_2 + l_3 + 1}{2}.
\end{equation}
Then the function $\tilde{y}_1^{(i)}(z)$ $(i = 0, 1, 2, 3)$ defined by
\begin{equation}
\tilde{y}_1^{(i)}(z) = \int_{-\tilde{\varphi}^{-1}(z) + 2\omega_i} \tilde{\varphi}(\xi) \left\{ \sum_{j=1}^3 \frac{l_j}{\varphi(\xi) - e_j} \right\} \frac{df_{HK}(\xi)}{d\xi} d\xi
+ \left( \tilde{\varphi}(\xi) - \lambda - \frac{\kappa_1}{\mu} \right) \left( \prod_{j=1}^3 (\varphi(\xi) - e_j)^{1/2} \right) \left( \frac{\tilde{\varphi}(\xi) - e_1}{\tilde{\varphi}(\xi) - \lambda} \right)^{l_1} d\xi,
\end{equation}
is a solution to the Fuchsian differential equation $D_{\tilde{y}_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty ; \tilde{\lambda}, \tilde{\mu})$ (see Eq. (2.3)), where
\begin{equation}
\tilde{\theta}_0 = -\frac{l_0 + l_1 - l_2 - l_3}{2}, \quad \tilde{\theta}_1 = \frac{l_0 - l_1 + l_2 - l_3}{2}, \quad \tilde{\theta}_t = \frac{l_0 - l_1 - l_2 + l_3}{2},
\end{equation}
\begin{equation}
\tilde{\theta}_\infty = \frac{l_0 + l_1 + l_2 + l_3}{2} + 1, \quad \tilde{\lambda} = \lambda + \frac{\kappa_1}{\mu}, \quad \tilde{\mu} = \mu.
\end{equation}
For the case $\kappa_1 \leq -1$, we interpret the integral as a half of the value integrated over the cycle from a point sufficiently close to $\xi = -\tilde{\varphi}^{-1}(z) + 2\omega_i$, turning around the point $\xi = -\tilde{\varphi}^{-1}(z) + 2\omega_i$ clockwise, moving to the point sufficiently close to $\xi = \tilde{\varphi}^{-1}(z)$, turning around the point $\xi = \tilde{\varphi}^{-1}(z)$ anticlockwise and returning to the initial point.
Proof. By changing the variable \( w = (\varphi(\xi) - e_1)/(e_2 - e_1) \), substituting \( y_1(w) = w^{l_2/2}(w-1)^{l_2/2}(w-t)^{l_2/2}f_{HK}(\xi) \) and multiplying Eq. (3.15) by \( tk_2(u_0 + \theta_0)/(-\lambda(\lambda-t)\mu)\tilde{y}_1(z) \), we obtain the integrand. We consider the integral contour \([\alpha_z, \alpha_t]\). Let \( \alpha \in \mathbb{C} \setminus \{0,1,t\} \) be the initial point of the contour in the \( w \)-plane, and \( \pm x_0 \) (resp. \( \pm x \)) be the point such that \( \varphi(\pm x_0) = o \) (resp. \( \varphi(\pm x) = z \)). We choose \( x_0 \) sufficiently close to \( x \). The contour \( \alpha_z \) in the \( z \)-plane corresponds to the contour whose initial point is \( x_0 \) and turning \( x \) anticlockwise and returning either to \( x_0 \) or to the contour whose initial point is \( -x_0 \) and turning \(-x \) anticlockwise and returning to \( -x_0 \), depending on the choice of branching. The contour \( \alpha_\infty \) (resp. \( \alpha_0, \alpha_1, \alpha_t \)) in the \( w \)-plane corresponds either to the contour whose initial point is \( x_0 \) and ends at \( -x_0 \) (resp. \( -x_0 + 2\omega_1, -x_0 + 2\omega_2, -x_0 + 2\omega_3 \)) or to the reverse contour. By analytic continuation along the cycle \( \alpha_z \), the integrand is multiplied by \(-1 \) because of the factor \((z - \varphi(\xi))^\kappa_1 \) \((\kappa_1 \in \mathbb{Z} + 1/2) \), and the integral tends to zero in the limit as \( x_0 \to x \) for the case \( \kappa > -1 \). Hence the contour \([\alpha_z, \alpha_\infty]\) (resp. \([\alpha_z, \alpha_0]\), \([\alpha_z, \alpha_1]\), \([\alpha_z, \alpha_t]\)) corresponds to a contour that runs twice from \( x \) to \(-x \) (resp. \(-x + 2\omega_1, -x + 2\omega_2, -x + 2\omega_3 \)). We therefore obtain the theorem. \( \square \)

Note that if \( l_0, l_1, l_2, l_3 \in \mathbb{Z} \) and \( l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1 \), then the integral is equal to zero. It follows from the assumption \( l_0, l_1, l_2, l_3 \in \mathbb{Z} \) and \( l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} \) that \( \theta_0, \theta_1, \theta_\infty \in \mathbb{Z} \) and \( \theta_0 + \theta_1 + \theta_\infty \in 1 + 2\mathbb{Z} \).

For given \( \theta_0, \theta_1, \theta_\infty \) such that \( \theta_0, \theta_1, \theta_\infty \in \mathbb{Z} \) and \( \theta_0 + \theta_1 + \theta_\infty \in 1 + 2\mathbb{Z} \), we have integral representations of solutions by choosing \( l_0, l_1, l_2, l_3 \) appropriately. More precisely, we have the following corollary:

**Corollary 5.2.** Assume that \( \theta_0, \theta_1, \theta_\infty \in \mathbb{Z} \) and \( \theta_0 + \theta_1 + \theta_\infty \in 1 + 2\mathbb{Z} \). Set

\[
\begin{align*}
    l_0 &= \frac{-\theta_0 - \theta_1 - \theta_\infty + 1}{2}, \\
    l_1 &= \frac{\theta_0 - \theta_1 - \theta_\infty - 1}{2}, \\
    l_2 &= \frac{-\theta_0 + \theta_1 - \theta_\infty - 1}{2}, \\
    l_3 &= \frac{-\theta_0 - \theta_1 + \theta_\infty - 1}{2},
\end{align*}
\]

\[\tilde{\varphi}(z) = \frac{\varphi(x) - e_1}{e_2 - e_1}, \quad \tilde{\kappa}_1 = \frac{\theta_0 + \theta_1 + \theta_\infty - 1}{2}.
\]

Let \( f_{HK}(x) = (f_{HK}(x; l_0, l_1, l_2, l_3; \lambda - \tilde{\kappa}_1/\mu, \mu)) \) be the solution expressed in the form of the Hermite-Krichever Ansatz in Proposition 4.1. Then the function \( \tilde{y}_1^{(i)}(z) \) \((i = 0, 1, 2, 3)\) defined by

\[
\tilde{y}_1^{(i)}(z) = \int_{\tilde{\varphi}^{-1}(z) + 2\omega_i}^{\tilde{\varphi}^{-1}(z)} \left\{ \frac{\tilde{\kappa}_1}{e_2 - e_1} + (\tilde{\varphi}(\xi) - \lambda) \sum_{j=1}^{3} \frac{l_j}{2(e_2 - e_1) e_3} \right\} \tilde{\varphi}'(\xi) f_{HK}(\xi) d\xi
\]

\[+ (\tilde{\varphi}(\xi) - \lambda) \frac{df_{HK}(\xi)}{d\xi} \left[ \prod_{j=1}^{3} (\varphi(\xi) - e_j)^{l_j/2} \frac{(z - \tilde{\varphi}(\xi)\tilde{\kappa}_1)}{\tilde{\varphi}(\xi) - \lambda + \kappa_1/\mu} \right] d\xi,
\]

is a solution to the Fuchsian differential equation \( D_{\tilde{y}_1}(\theta_0, \theta_1, \theta_\infty; \lambda, \mu) \). For the case \( \kappa_1 \leq -1 \), we interpret the integral as the one in Theorem 5.1.

Let \( a_{11}(z), a_{12}(z) \) be the functions defined in Eq. (2.1) and set \( \tilde{y}_2^{(i)}(z) = (d\tilde{y}_1^{(i)}(z)/dz - a_{11}(z)\tilde{y}_1^{(i)}(z))/a_{12}(z) \) \((i = 0, 1, 2, 3)\). Then the function \( Y = \tilde{t}(\tilde{y}_1^{(i)}(z), \tilde{y}_2^{(i)}(z)) \) is a solution to the Fuchsian differential system \(DY(\theta_0, \theta_1, \theta_\infty; \lambda, \mu; k) \) (see Eq. (2.1)). Note that the function \( \tilde{y}_2^{(i)}(z) \) is also expressed as the form like Eq. (5.5) by combining the
expression in Eq. (3.15), the relation \( y_1(w) = w^{1/2}(w - 1)^{1/2}(w - t)^{1/2} f_{HK}(\tilde{\varphi}^{-1}(w)) \) and the relation among \( dy_2(w)/dw, dy_1(w)/dw \) and \( y_1(w) \).

We can calculate the monodromy of the Fuchsian system \( D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k) \) and the Fuchsian equation \( D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu) \) for the case \( \theta_0, \theta_1, \theta_t, \theta_\infty \in \mathbb{Z} \) and \( \theta_0 + \theta_1 + \theta_t + \theta_\infty \in 1 + 2\mathbb{Z} \) in principal by considering the integral representations of solutions and their asymptotics around the singularities. We will do this for the case \( (\theta_0, \theta_1, \theta_t, 1 - \theta_\infty) = (0, 0, 0, 0) \) in the next section.

6. Integral representation of solutions to the Fuchsian equation for the case \( (\theta_0, \theta_1, \theta_t, 1 - \theta_\infty) = (0, 0, 0, 0) \)

We consider the integral representation of solutions to the Fuchsian equation for the case \( (\theta_0, \theta_1, \theta_t, 1 - \theta_\infty) = (0, 0, 0, 0) \). For this case, the function \( f_{HK}(\xi) \) in the integrand of Eq. (5.2) is written in the form of the Hermite-Krichever Ansatz for the case \( l_0 = l_1 = l_2 = l_3 = 0 \), and it is described by Eq. (4.17). The values \( \lambda, \mu \) for the case \( l_0 = l_1 = l_2 = l_3 = 0 \) and the values \( \alpha, \kappa \) are related by Eq. (4.17). By substituting Eq. (4.17) into Eq. (5.5) and the integral representation of the function \( \tilde{y}_2(i)(z) \) like Eq. (5.5), multiplying by appropriate constants and applying the formula \( \varphi(x) - \varphi(\xi) = -\sigma(x + \xi)\sigma(x - \xi)/(\sigma(x)^2\sigma(\xi)^2) \), we have the following proposition:

**Proposition 6.1.** Set

\[
(6.1) \quad f_i(x) = \int_{-\infty + 2\omega_i}^x \frac{e^{(\kappa + \zeta(\alpha))\xi} \sigma(x)\sigma(x - \alpha)}{\sqrt{\sigma(x - \xi)\sigma(x + \xi)}} d\xi,
\]

\[
g_i(x) = \frac{1}{4k(e_2 - e_1)} \int_{-\infty + 2\omega_i}^x \left( \frac{\varphi'(\xi) + \varphi(\alpha)}{\varphi(\xi - \alpha)} \right) \frac{e^{(\kappa + \zeta(\alpha))\xi} \sigma(x)\sigma(x - \alpha)}{\sqrt{\sigma(x - \xi)\sigma(x + \xi)}} d\xi.
\]

The function \( f_i(x), g_i(x) \) \( (i = 0, 1, 2, 3, z = (\varphi(x) - e_1)/(e_2 - e_1)) \) is a solution to the Fuchsian differential system \( D_Y(0, 0, 0, 1; \lambda, \mu; k) \), where

\[
(6.2) \quad \lambda = \frac{\varphi(\alpha) - e_1}{e_2 - e_1}, \quad \mu = -\frac{(e_2 - e_1)\kappa}{\varphi'(\alpha)}.
\]

In particular, the function \( f_i(x) \) \( (i = 0, 1, 2, 3, z = (\varphi(x) - e_1)/(e_2 - e_1)) \) is a solution to the Fuchsian differential equation \( D_{y_1}(0, 0, 0, 1; \lambda, \mu) \) and the differential equation can also be written as

\[
(6.3) \quad \frac{d^2 y}{dx^2} + \left\{ \sum_{j=1}^3 \frac{1}{2(\varphi(x) - e_j)} - \frac{1}{\varphi(x) - \varphi(\alpha)} \right\} \varphi'(x) \frac{dy}{dx} + \left\{ -\kappa^2 - \frac{\varphi'(\alpha)}{\varphi(x) - \varphi(\alpha)} \frac{\kappa + \varphi(x) - \varphi(\alpha)}{\varphi'(\alpha)} \right\} y = 0.
\]

The monodromy matrix for the Fuchsian differential system \( D_Y(0, 0, 0, 1; \lambda, \mu; k) \) with respect to a basis of solutions \( \left\{ \begin{pmatrix} y_1^{[1]}(z) \\ y_2^{[1]}(z) \end{pmatrix}, \begin{pmatrix} y_1^{[2]}(z) \\ y_2^{[2]}(z) \end{pmatrix} \right\} \) along a cycle \( \gamma \) coincides with the monodromy matrix for the Fuchsian differential equation \( D_{y_1}(0, 0, 0, 1; \lambda, \mu) \)
with respect to a basis of solutions \( \{ y_1^{(1)}(z), y_2^{(2)}(z) \} \) along the cycle \( \gamma \). Hence we investigate the monodromy matrices for the Fuchsian differential equation \( D_{y_1}(0, 0, 0, 1; \lambda, \mu) \) by applying integral representations of solutions \( f_i(x) \) \((i = 0, 1, 2, 3)\).

Assume that \( \alpha \not\equiv 0 \mod \omega_1Z \oplus \omega_3Z \). By considering the exponents of the singularities, we have a basis of local solutions to the Fuchsian differential equation \( D_{y_i}(0, 0, 0, 1; \lambda, \mu) \) about \( x = 0 \) and \( \omega_i \) \((i = 1, 2, 3)\) of the form

\[
(6.4) \quad s_1^{(0)}(x) = x + c_2^{(0)}x^2 + \ldots, \quad s_2^{(0)}(x) = s_1^{(0)}(x) \log x + c_2^{(0)}x^2 + \ldots,
\]

\[
s_1^{(i)}(x) = 1 + c_1^{(i)}(x - \omega_i) + \ldots, \quad s_2^{(i)}(x) = s_1^{(i)}(x) \log(x - \omega_i) + c_0^{(i)} + c_1^{(i)}(x - \omega_i) + \ldots.
\]

Let \( \gamma_i \) \((i = 0, 1, 2, 3)\) be the cycle turning anti-clockwise around \( x = \omega_i \), and \( f_i^\gamma \) be the function which is continued analytically along the cycle \( \gamma \). Then we have

\[
(6.5) \quad (s_1^{(i)}(x), s_2^{(i)}(x)) = \left( s_1^{(i)}(x), s_2^{(i)}(x) \right) \begin{pmatrix} 1 & 2\pi\sqrt{-1} \\ 0 & 1 \end{pmatrix} \quad (i = 0, 1, 2, 3).
\]

We now relate the local solutions \((s_1^{(i)}(x), s_2^{(i)}(x))\) \((i = 0, 1, 2, 3)\) to the solutions of integral representations \( f_0(x), \ldots, f_3(x) \). Since \( f_0(x) \) is a solution to the Fuchsian differential equation \( D_{y_1}(0, 0, 0, 1, \lambda, \mu) \), it is expressed as a linear combination of \( s_1^{(0)}(x) \) and \( s_2^{(0)}(x) \). We set \( \xi = x\nu \). Since \( \lim_{x \to 0} \sigma(x)/x = 1 \), we have the following asymptotic limit as \( x \to 0 \):

\[
(6.6) \quad f_0(x) = \int_{-1}^1 \frac{e^{x(\kappa(\alpha))\nu}\sigma(x)\sigma(x\nu - \alpha)}{\sqrt{x(x(1 + \nu))\sigma(x(1 - \nu))}}xd\nu \sim \int_{-1}^1 \frac{x^2\sigma(\alpha)}{x\sqrt{(1 + \nu)(1 - \nu)}}d\nu = \sigma(-\alpha)\pi x.
\]

Hence we have

\[
(6.7) \quad f_0(x) = \sigma(-\alpha)\pi s_1^{(0)}(x).
\]

We consider the asymptotics of \( f_0(x) \) in the limit as \( x \to \omega_i \) \((i = 1, 2, 3)\). By using the formula \( \sigma(x + 2\omega_i) = -\sigma(x)\exp(2\eta_i(x + \omega_i)) \), we have

\[
(6.8) \quad f_0(x) = \int_0^1 \frac{e^{x(\kappa(\alpha))\nu}\sigma(x)\sigma(x\nu - \alpha)}{\sqrt{-e^{2\eta_i(x(1 + \nu) - \omega_i)}\sigma(x(1 + \nu) - 2\omega_i)\sigma(x(1 - \nu))}}xd\nu
\]

\[
+ \int_{-1}^0 \frac{e^{x(\kappa(\alpha))\nu}\sigma(x\nu - \alpha)}{\sqrt{-e^{2\eta_i(x(1 - \nu) - \omega_i)}\sigma(x(1 - \nu) - 2\omega_i)\sigma(x(1 + \nu))}}xd\nu
\]

\[
\sim \int_0^1 \frac{e^{x(\kappa(\alpha))\nu - \eta_i(2x - \omega_i)}\sigma(x\nu - \alpha)}{x\sqrt{(1 + \nu - 2\omega_i/x)(1 - \nu)}}d\nu + \int_{-1}^0 \frac{e^{-x(\kappa(\alpha))\nu + \eta_i(2x - \omega_i)}\sigma(x\nu - \alpha)}{x\sqrt{(1 - \nu - 2\omega_i/x)(1 + \nu)}}d\nu
\]

\[
\sim -\log(\omega_i - x)e^{\omega_i(\kappa(\alpha) - \eta_i)}\sigma(\omega_i)\sigma(\omega_i - \alpha) - \log(\omega_i - x)e^{\omega_i(-\kappa(\alpha) - \eta_i)}\sigma(\omega_i)\sigma(-\omega_i - \alpha)
\]

\[
= -\log(\omega_i - x)e^{\omega_i(\kappa(\alpha) - \eta_i)}\sigma(\omega_i)\sigma(\omega_i - \alpha)(1 - e^{-2\omega_i(\kappa(\alpha)) + 2\eta_i\alpha}).
\]

Since \( f_0(x) \) is a solution to Eq. \((2.1)\), it can be expressed as a linear combination of \( s_1^{(i)}(x) \) and \( s_2^{(i)}(x) \), and we have

\[
(6.9) \quad f_0(x) = -e^{\omega_i(\kappa(\alpha) - \eta_i)}\sigma(\omega_i)\sigma(\omega_i - \alpha)(1 - e^{-2\omega_i(\kappa(\alpha)) + 2\eta_i\alpha})s_2^{(i)}(x) + e^{(0,i)}s_1^{(i)}(x),
\]
for some constant $c^{(0,i)}$. Next, we express the function $f_i(x)$ ($i = 1, 2, 3$) as a linear combination of $s_1^{(j)}(x)$ and $s_2^{(j)}(x)$ for $j \in \{0, 1, 2, 3\}$. We set $\xi = (\omega_i - x)\nu + \omega_i$, whereupon we have

$$f_i(x) = \int_{-1}^{1} \frac{e^{x(\kappa + \zeta(\alpha))(\omega_i - x)\nu + \omega_i - \alpha}}{\sqrt{\sigma((\omega_i - x)(1 + \nu))\sigma((\omega_i - x)(1 - \nu) - 2\omega_i)}} (x - \omega_i) d\nu. \tag{6.10}$$

Similarly, we have

$$f_i(x) \sim \sqrt{-1}\pi \sigma(\omega_i) \sigma(\omega_i - \alpha) e^{\omega_i(\kappa + \zeta(\alpha) - \eta_j)} (1 - e^{2(\omega_i - \omega_j)(\kappa + \zeta(\alpha) + 2(\eta_j - \eta_i)\alpha)} \log(\omega_j - x), \tag{6.11}$$

$$f_i(x) \sim -\sqrt{-1}\sigma(-\alpha) (1 - e^{2\omega_i(\kappa + \zeta(\alpha) - 2\eta_i\alpha)} x \log x, \tag{6.12}$$

for some constants $c^{(i,0)}$ and $c^{(i,j)}$.

We consider the monodromy matrices on the basis $(f_0(x), f_1(x))$. Set

$$e[i] = \exp(2\omega_i(\kappa + \zeta(\alpha)) - 2\eta_i\alpha), \quad (i = 1, 2, 3). \tag{6.14}$$

It follows from Eqs. $(6.5, 6.7, 6.13)$ that

$$(f_0^{\gamma_0}(x), f_1^{\gamma_0}(x)) = (\sigma(-\alpha) \pi s_1^{(0, \gamma_0)}(x), -\sqrt{-1}\sigma(-\alpha) (1 - e[1]) s_2^{(0, \gamma_0)}(x) + c^{(i,0)} s_1^{(0, \gamma_0)}(x))$$

$$(\sigma(-\alpha) \pi s_1^{(0)}(x), -\sqrt{-1}\sigma(-\alpha) (1 - e[1]) s_2^{(0)}(x) + 2\pi \sqrt{-1}s_1^{(0)}(x)) + c^{(i,0)} s_1^{(0)}(x))$$

$$(f_0(x), f_1(x)) \left( \begin{array}{cc} 1 & 2(1 - e[1]) \\ 0 & 1 \end{array} \right). \tag{6.15}$$

Similarly, it follows from Eqs. $(6.5, 6.9, 6.13)$ that

$$(f_0^{\gamma_1}(x), f_1^{\gamma_1}(x)) = (f_0(x), f_1(x)) \left( \begin{array}{cc} 1 & 0 \\ -2(1 - 1/e[1]) & 1 \end{array} \right). \tag{6.16}$$

If $e[1] \neq 0$, then it follows from the asymptotic limits as $x \to 0$ and $x \to \omega_1$ that the functions $f_0(x)$ and $f_1(x)$ form a basis of solutions to Eq. $(2.1)$, and the functions $f_j(x)$ ($j = 2, 3$) are written as linear combinations of $f_0(x)$ and $f_1(x)$. Write $f_j(x) = \tilde{c}_{0,j} f_0(x) + \tilde{c}_{1,j} f_1(x)$. Then the coefficients $\tilde{c}_{0,j}, \tilde{c}_{1,j}$ are determined by considering the asymptotic limits as $x \to \omega_1$ and $x \to 0$, and we have

$$\tilde{c}_{0,j} = \frac{e[1] - e[j]}{1 - e[1]}, \quad \tilde{c}_{1,j} = \frac{1 - e[j]}{1 - e[1]}. \tag{6.17}$$
Therefore
\begin{equation}
(f_0^{(j)}(x), f_1^{(j)}(x)) = (f_0^{(j)}(x), f_1^{(j)}(x)) \begin{pmatrix}
1 & -\tilde{c}_{0,j}/\tilde{c}_{1,j} \\
0 & 1/\tilde{c}_{1,j}
\end{pmatrix}
\end{equation}

\begin{equation}
= (f_0(x), f_j(x)) \begin{pmatrix}
1 & 0 \\
-2(1 - 1/e[j]) & 1
\end{pmatrix} \begin{pmatrix}
1 & -\tilde{c}_{0,j}/\tilde{c}_{1,j} \\
0 & 1/\tilde{c}_{1,j}
\end{pmatrix}
\end{equation}

\begin{equation}
= (f_0(x), f_1(x)) \begin{pmatrix}
1 & \tilde{c}_{0,j} \\
0 & \tilde{c}_{1,j}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-2(1 - 1/e[j]) & 1
\end{pmatrix} \begin{pmatrix}
1 & -\tilde{c}_{0,j}/\tilde{c}_{1,j} \\
0 & 1/\tilde{c}_{1,j}
\end{pmatrix}
\end{equation}

\begin{equation}
= (f_0(x), f_1(x)) \begin{pmatrix}
1 + 2(e[1] - e[j])(e[j] - 1) & 2(e[1] - e[j])^2 \\
-2(e[1] - e[j])^2 / (e[j] - 1) & 1 - 2(e[1] - e[j])(e[j] - 1) / (e[j] - 1)
\end{pmatrix}
\end{equation}

and we have obtained the monodromy matrices for the basis \((f_0(x), f_1(x))\) on the cycles \(\gamma_2, \gamma_3\).

We consider the monodromy preserving deformation with respect to the basis \((f_0(x), f_1(x))\). Assume that the values \(e[1], e[3]\) are preserved while varying the ratio \(\omega_3/\omega_1\). Then the monodromy is preserved by Eqs. (6.15, 6.16) and the equality \(e[1] + e[2] + e[3] = 0\). Since the values \(e[1], e[3]\) are preserved by monodromy preserving deformation, we have
\begin{equation}
-2\eta_1 \alpha + 2\omega_1 \zeta(\alpha) + 2\kappa \omega_1 = \pi \sqrt{-1} C_1,
\end{equation}

\begin{equation}
-2\eta_3 \alpha + 2\omega_3 \zeta(\alpha) + 2\kappa \omega_3 = \pi \sqrt{-1} C_3,
\end{equation}

for constants \(C_1\) and \(C_3\). By Legendre’s relation, \(\eta_3 \omega_3 - \eta_1 \omega_1 = \pi \sqrt{-1}/2\), we have
\begin{equation}
\alpha = C_3 \omega_1 - C_1 \omega_3,
\end{equation}

\begin{equation}
\kappa = \zeta(C_1 \omega_3 - C_3 \omega_1) + C_3 \eta_1 - C_1 \eta_3,
\end{equation}

Recall that the sixth Painlevé equation has an elliptical representation (see Eq. (4.17)), and it is a differential equation on \(\delta\) with respect to the variable \(\tau = \omega_3/\omega_1\). For the case \((\theta_0, \theta_1, \theta_1, 1 - \theta_\infty) = (0, 0, 0, 0)\), this equation is written as \(d^2 \delta / d\tau^2 = 0\). The variables \(\lambda\) and \(\delta\) are related by \(\lambda = (\varphi(\delta) - e_1)/(e_2 - e_1)\). With regards to the integral representations of solutions to \(Dy_1(0, 0, 0, 1; \lambda, \mu)\), we have the relations
\begin{equation}
\lambda = \frac{1}{e_2 - e_1} \{\varphi(\alpha) - e_1\}, \quad \mu = -\frac{(e_2 - e_1) \kappa}{\varphi'(\alpha)}.
\end{equation}

Hence \(\alpha\) plays the role of \(\delta\) mod \(2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}\), and Eq. (6.20) corresponds to Picard’s solution to the sixth Painlevé equation for the case \((\theta_0, \theta_1, \theta_1, 1 - \theta_\infty) = (0, 0, 0, 0)\) by setting \(\omega_1 = 1/2\) and \(\omega_3 = \tau/2\). We therefore reproduce Picard’s solution by determining the monodromy of the corresponding Fuchsian equation.

7. Integral representation of solutions to Heun’s equation

In section 5, we obtained that, if \(\theta_0, \theta_1, \theta_1, \theta_\infty \in \mathbb{Z}\) and \(\theta_0 + \theta_1 + \theta_i + \theta_\infty \in 1 + 2\mathbb{Z}\), then we have integral representations of solutions to the Fuchsian equation \(Dy_1(\theta_0, \theta_1, \theta_1, \theta_\infty; \lambda, \mu)\) (see Eq. (5.22)) and the Fuchsian system \(Dy(\theta_0, \theta_0, \theta_1, \theta_\infty; \lambda, \mu, k)\). In this section we obtain integral representations of solutions to Heun’s equation by a suitable choice of the parameters \(\lambda\) and \(\mu\).

Recall that Heun’s differential equation is defined by
\begin{equation}
\frac{d^2 y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right) \frac{dy}{dz} + \frac{\alpha \beta z - q}{z(z-1)(z-t)} y = 0,
\end{equation}
with the condition $\gamma + \delta + \epsilon = \alpha + \beta + 1$. This equation has an elliptical representation: Set

$$\tag{7.2} z = \frac{\varphi(x) - e_1}{e_2 - e_1}, \quad t = \frac{e_3 - e_1}{e_2 - e_1}, \quad f(x) = yz^{-1_1}(z - 1)^{-1_2}(z - t)^{-1_3},$$

then Heun’s equation (Eq. $\tag{7.1}$) is transformed to

$$\tag{7.3} \left( -\frac{d^2}{dx^2} + \sum_{i=0}^3 l_i(l_i + 1)\varphi(x + \omega_i) - E \right) f(x) = 0,$$

where

$$\tag{7.4}
l_0 = \alpha - \beta - 1/2, \quad l_1 = -\gamma + 1/2, \quad l_2 = -\delta + 1/2, \quad l_3 = -\epsilon + 1/2, \quad E = (e_2 - e_1)(-4q + (-(\alpha - \beta)^2 + 2\gamma^2 + 6\gamma\epsilon + 2\epsilon^2 - 4\gamma - 4\epsilon - \delta^2 + 2\delta + 1)/3

+ (-(\alpha - \beta)^2 + 2\gamma^2 + 6\gamma\delta + 2\delta^2 - 4\gamma - 4\delta - \epsilon^2 + 2\epsilon + 1)t/3).$$

We obtained in section 5.3 that, if $l_0, l_1, l_2, l_3 \in \mathbb{Z}$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}$, then the function $\tilde{y}_1^{(i)}(z)$ defined by

$$\tag{7.5}
\tilde{y}_1^{(i)}(z) = \int_{\tilde{\varphi}^{-1}(z) + 2\omega_i}^{\tilde{\varphi}^{-1}(z)} \left\{ \frac{\kappa_1}{e_2 - e_1} + \left( \tilde{\varphi}(\xi) - \lambda - \frac{\kappa_1}{\mu} \right) \sum_{i=1}^3 \frac{l_i}{2(\tilde{\varphi}(\xi) - e_i)} \right\} \varphi'(\xi) f_{HK}(\xi)

+ \left( \tilde{\varphi}(\xi) - \lambda - \frac{\kappa_1}{\mu} \right) \frac{df_{HK}(\xi)}{d\xi} \left( \prod_{i=1}^3 (\tilde{\varphi}(\xi) - e_i)^{l_i/2} \right) \frac{(z - \tilde{\varphi}(\xi))^{\kappa_1}}{(\tilde{\varphi}(\xi) - \lambda)\xi} d\xi,$$

$(i = 0, 1, 2, 3)$ is a solution to the Fuchsian differential equation $D_{y_1}(\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_t, \bar{\theta}_\infty; \lambda + \kappa_1/\mu, \mu)$, where

$$\tag{7.6}
\bar{\theta}_0 = \frac{-l_0 + l_1 - l_2 - l_3}{2}, \quad \bar{\theta}_1 = \frac{-l_0 - l_1 + l_2 - l_3}{2}, \quad \bar{\theta}_t = \frac{-l_0 - l_1 - l_2 + l_3}{2}, \quad \bar{\theta}_\infty = \frac{2}{2} + 1, \quad \tilde{\varphi}(x) = \frac{\varphi(x) - e_1}{e_2 - e_1}, \quad \kappa_1 = \frac{l_0 + l_1 + l_2 + l_3 + 1}{2},$$

and the function $f_{HK}(x)$ is defined in Theorem $\text{5.1}$. The Fuchsian equation $D_{y_1}(\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_t, \bar{\theta}_\infty; \lambda + \kappa_1/\mu, \mu)$ has an apparent singularity at $z = \lambda + \kappa_1/\mu$. We consider the confluence of the apparent singularity $z = \lambda + \kappa_1/\mu$ to the regular singularity $z = \infty$. Set $\mu = 0$. Then the Fuchsian equation is written as Heun’s equation

$$\tag{7.7}
\frac{d^2y}{dz^2} + \left( \frac{1 - \bar{\theta}_0}{z} + \frac{1 - \bar{\theta}_1}{z - 1} + \frac{1 - \bar{\theta}_t}{z - t} \right) \frac{dy}{dz}

+ \tilde{\kappa}_1(\tilde{\kappa}_2 + 2)z + \tilde{\kappa}_1(1 - \tilde{\theta}_\infty)\lambda - \tilde{\kappa}_1((\tilde{\kappa}_2 + \tilde{\theta}_t + 1)t + (\tilde{\kappa}_2 + \tilde{\theta}_t + 1)z) y = 0,$$

where $\tilde{\kappa}_1 = (\tilde{\theta}_\infty - \bar{\theta}_0 - \bar{\theta}_1 - \bar{\theta}_t)/2$ and $\tilde{\kappa}_2 = -(\tilde{\theta}_\infty + \bar{\theta}_0 + \bar{\theta}_1 + \bar{\theta}_t)/2$. We have $1 - \bar{\theta}_0, 1 - \bar{\theta}_1, 1 - \bar{\theta}_t, \tilde{\kappa}_1 + 1/2, \tilde{\kappa}_2 + 2 + 1/2 \in \mathbb{Z}$. For the case $\bar{\theta}_\infty = 1$, we set $\mu = bs^2$, \ldots
\( \lambda = c/s \) and consider the limit \( s \to 0 \). Then we have

\[
(7.8) \quad \frac{d^2 y}{dz^2} + \left( \frac{1 - \tilde{\theta}_0}{z} + \frac{1 - \tilde{\theta}_1}{z - 1} + \frac{1 - \tilde{\theta}_t}{z - t} \right) \frac{dy}{dz} + \frac{\kappa_1(\kappa_2 + 2)z + \kappa_1 b c^2 - \kappa_1((\kappa_2 + \tilde{\theta}_t + 1)t + (\kappa_2 + \tilde{\theta}_1 + 1))}{z(z - 1)(z - t)} y = 0,
\]

The following theorem follows from Eq. (7.7) by substituting the parameters as indicated:

**Theorem 7.1.** (i) Assume that \( \gamma + \delta + \epsilon = \alpha + \beta + 1 \), \( \gamma, \delta, \epsilon, \alpha + 1/2, \beta + 1/2 \in \mathbb{Z} \). Set

\[
(7.9) \quad \tilde{l}_0 = \alpha - 3/2, \quad \tilde{l}_1 = \delta + \epsilon - \alpha - 1/2, \quad \tilde{l}_2 = \gamma + \epsilon - \alpha - 1/2, \quad \tilde{l}_3 = \gamma + \delta - \alpha - 1/2.
\]

Let \( f_{HK}(x) = f_{HK}(x; \tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3; \lambda, \mu) \) be the function expressed in the form of the Hermite-Krichever Ansatz. Set \( \tilde{\varphi}(x) = (\varphi(x) - e_1)/(e_2 - e_1) \) and

\[
(7.10) \quad F(\xi; \lambda, \mu, m) = \mu^m \left[ \left\{ -\frac{\beta}{e_2 - e_1} + \left( \tilde{\varphi}(\xi) - \lambda + \frac{\beta}{\mu} \right) \sum_{i=1}^{3} \frac{\tilde{l}_i}{2(\varphi(\xi) - e_i)} \right\} \tilde{\varphi}'(\xi)f_{HK}(\xi) + \left( \tilde{\varphi}(\xi) - \lambda + \frac{\beta}{\mu} \right) \frac{df_{HK}(\xi)}{d\xi} \right] \left( \prod_{i=1}^{3}(\varphi(\xi) - e_i)^{\tilde{l}_i/2} \right).
\]

If \( \alpha - \beta \neq 1 \) (resp. \( \alpha - \beta = 1 \)) and the integrand in Eq. (7.11) (resp. Eq. (7.12)) has a non-zero finite limit as \( \mu \to 0 \) (resp. \( s \to 0 \)) for some \( m \), then the functions

\[
(7.11) \quad \tilde{y}_1^{(i)}(z) = \int_{\tilde{\varphi}^{-1}(z) + 2\omega_i}^{\tilde{\varphi}^{-1}(z)} \lim_{\mu \to 0} F(\xi; \lambda, \mu, m) \frac{(z - \tilde{\varphi}(\xi))^{-\beta}}{(\tilde{\varphi}(\xi) - \lambda)} d\xi,
\]

\[
(7.12) \quad \tilde{y}_1^{(i)}(z) = \int_{\tilde{\varphi}^{-1}(z) + 2s}^{\tilde{\varphi}^{-1}(z)} \lim_{s \to 0} F(\xi; c/s, b s^2, m) \frac{(z - \tilde{\varphi}(\xi))^{-\beta}}{(\tilde{\varphi}(\xi) - c/s)} d\xi,
\]

\( bc^2 = t(\alpha - \epsilon) + (\alpha - \delta) + q/(1 - \alpha) \), \( (\alpha - \beta = 1) \).
(i = 0, 1, 2, 3) are solutions to Heun's equation (Eq. (7.1)).
(ii) Assume that \(l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2\) and \(l_0 + l_1 + l_2 + l_3 \in 1 + 2\mathbb{Z}\). Set

\[
\begin{align*}
\tilde{l} &= \frac{l_0 + l_1 + l_2 + l_3}{2}, \quad \tilde{l}_0 = \frac{l_0 - l_1 - l_2 - l_3}{2} - 1, \\
\tilde{l}_1 &= \frac{-l_0 + l_1 - l_2 - l_3}{2}, \quad \tilde{l}_2 = \frac{-l_0 - l_1 + l_2 - l_3}{2}, \quad \tilde{l}_3 = \frac{-l_0 - l_1 - l_2 + l_3}{2},
\end{align*}
\]

\[
F(\xi; \lambda, \mu, m) = \mu^m \left[ \frac{\tilde{l}}{e_2 - e_1} + \left( \tilde{\varphi}(\xi) - \lambda - \frac{\tilde{l}}{\mu} \right) \left( \sum_{i=1}^{3} \frac{\tilde{l}_i}{2(\varphi(\xi) - e_i)} \right) \varphi'(\xi) f_{HK}(\xi) \\
+ \left( \tilde{\varphi}(\xi) - \lambda - \frac{\tilde{l}}{\mu} \right) \frac{df_{HK}(\xi)}{d\xi} \right] \left( \prod_{i=1}^{3} (\varphi(\xi) - e_i)^{\tilde{l}_i/2} \right).
\]

If \(l_0 \neq 1/2\) (resp. \(l_0 = 1/2\)) and the integrand in Eq. (7.14) (resp. Eq. (7.13)) has a non-zero finite limit as \(\mu \rightarrow 0\) (resp. \(s \rightarrow 0\)) for some \(m\), then the functions

\[
f^{(i)}(x) = \left( \prod_{j=1}^{3} (\varphi(x) - e_j)^{-\tilde{l}_j/2} \right) \int_{x-2s}^{x} \lim_{\mu \rightarrow 0} F(\xi; \lambda, \mu, m) \frac{(\varphi(x) - \varphi(\xi))^{\tilde{l}_i}}{(\tilde{\varphi}(\xi) - \lambda)} d\xi,
\]

\[
\lambda = \frac{E + (l_3 - l_1)(2l_0 + l_1 + l_3)e_1 + (l_3 - l_2)(2l_0 + l_2 + l_3)e_2}{(e_1 - e_2)(l_1 + l_2 + l_3 + l_0)(2l_0 - 1)} + \frac{e_1}{e_2 - e_1}, \quad (l_0 \neq 1/2),
\]

\[
f^{(i)}(x) = \left( \prod_{j=1}^{3} (\varphi(x) - e_j)^{-\tilde{l}_j/2} \right) \int_{x-2s}^{x} \lim_{s \rightarrow 0} F(\xi; c/s, b s^2, m) \frac{(\varphi(x) - \varphi(\xi))^{\tilde{l}_i}}{(\tilde{\varphi}(\xi) - c/s)} d\xi,
\]

\[
bc^2 = \frac{E + (l_3 - l_1)(l_1 + l_3 + 1)e_1 + (l_3 - l_2)(l_2 + l_3 + 1)e_2}{(e_1 - e_2)(2l_1 + 2l_2 + 2l_3 + 1)}, \quad (l_0 = 1/2),
\]

(i = 0, 1, 2, 3) are solutions to the elliptic representation of Heun’s equation (Eq. (7.3)).

We consider the limits \(\lambda \pm \kappa_1/\mu \rightarrow 0, 1, t\). The following equations are obtained by setting \(\lambda = -\kappa_1/\mu, \lambda = 1 - \kappa_1/\mu, \lambda = t - \kappa_1/\mu\) in the Fuchsian equation \(D_y y_{\theta_0, \theta_1, \theta_t, \theta_\infty, \lambda, \mu}\):

\[
\frac{d^2y}{dz^2} + \left( \frac{1}{z} + \frac{1 - \tilde{\theta}_1}{z - 1} + \frac{1 - \tilde{\theta}_t}{z - t} \right) \frac{dy}{dz} + \frac{\tilde{\kappa}_1(\tilde{\kappa}_2 + 1)z + t \tilde{\theta}_0 \mu}{z(z - 1)(z - t)} y = 0,
\]

\[
\frac{d^2y}{dz^2} + \left( \frac{1 - \tilde{\theta}_0}{z} + \frac{1 - \tilde{\theta}_1}{z - 1} + \frac{1 - \tilde{\theta}_t}{z - t} \right) \frac{dy}{dz} + \frac{\tilde{\kappa}_1(\tilde{\kappa}_2 + 1)(z - 1) + (1 - t) \tilde{\theta}_1 \mu}{z(z - 1)(z - t)} y = 0,
\]

\[
\frac{d^2y}{dz^2} + \left( \frac{1 - \tilde{\theta}_0}{z} + \frac{1 - \tilde{\theta}_1}{z - 1} + \frac{-\tilde{\theta}_t}{z - t} \right) \frac{dy}{dz} + \frac{\tilde{\kappa}_1(\tilde{\kappa}_2 + 1)(z - t) + t(t - 1) \tilde{\theta}_t \mu}{z(z - 1)(z - t)} y = 0,
\]
where the parameters are defined as for the case $\mu = 0$. For the case $\tilde{\theta}_i = 0$ ($i = 0, 1, t$), we set $\mu = c/s$, $\lambda = i - \kappa_1/\mu + bs^2$, and consider the the limit $s \to 0$. Then we have

\begin{align}
(7.19) \quad & \frac{d^2 y}{dz^2} + \left( \frac{1 - \tilde{\theta}_1}{z - 1} + \frac{1 - \tilde{\theta}_1}{z - t} \right) \frac{dy}{dz} + \frac{\tilde{\kappa}_1(\tilde{\kappa}_2 + 1)}{z(z - 1)(z - t)} y = 0, \\
(7.20) \quad & \frac{d^2 y}{dz^2} + \left( \frac{1 - \tilde{\theta}_0}{z} + \frac{1 - \tilde{\theta}_1}{z - t} \right) \frac{dy}{dz} + \frac{\tilde{\kappa}_1(\tilde{\kappa}_2 + 1)(z - 1) + (t - 1)bc^2}{z(z - 1)(z - t)} y = 0, \\
(7.21) \quad & \frac{d^2 y}{dz^2} + \left( \frac{1 - \tilde{\theta}_0}{z} + \frac{1 - \tilde{\theta}_1}{z - 1} \right) \frac{dy}{dz} + \frac{\tilde{\kappa}_1(\tilde{\kappa}_2 + 1)(z - t) + t(1 - t)bc^2}{z(z - 1)(z - t)} y = 0.
\end{align}

Note that we have similar propositions to Theorem 7.1

We consider the integral representations of solutions to Heun’s equation for the case $\gamma = \delta = \epsilon = 1$ and $\alpha = 3/2, \beta = 1/2$, i.e. the case $l_0 = 1/2, l_1 = l_2 = l_3 = -1/2$. Recall that the functions

\begin{equation}
(7.22) \quad f_i(x) = \int_{-x + 2\omega_i}^{x} \frac{e^{(\kappa + \zeta(\alpha))\xi}}{\sqrt{\sigma(x - \xi)\sigma(x + \xi)}} d\xi, \quad z = \frac{\varphi(x) - e_1}{e_2 - e_1},
\end{equation}

for $i = 0, 1, 2, 3$ are solutions to the Fuchsian differential equation $D_{\eta_i}(0, 0, 0, 1; (\varphi(\alpha) - e_1)/(e_2 - e_1), -(e_2 - e_1)\kappa/\varphi'(\alpha))$ (see Proposition 6.1). The condition $s \to 0$ in Theorem 7.1 implies the condition $\alpha \to 0$ while setting $\kappa = -\zeta(\alpha) + \tilde{\kappa}$. Therefore, it follows from Eq. (7.22) that the functions

\begin{equation}
(7.23) \quad f_i(x) = \int_{-x + 2\omega_i}^{x} \frac{e^{\tilde{\kappa}x}}{\sqrt{\sigma(x - \xi)\sigma(x + \xi)}} d\xi,
\end{equation}

for $i = 0, 1, 2, 3$ are solutions to Heun’s equation

\begin{equation}
(7.24) \quad \frac{d^2 y}{dz^2} + \left( \frac{1}{z} + \frac{1}{z - 1} + \frac{1}{z - t} \right) \frac{dy}{dz} + \frac{3z + (3e_1 - \tilde{\kappa}^2)/(e_2 - e_1)}{4z(z - 1)(z - t)} y = 0,
\end{equation}

by setting $z = (\varphi(x) - e_1)/(e_2 - e_1)$, and the functions

\begin{equation}
(7.25) \quad f^{(i)}(x) = \left( \prod_{i=1}^{3} (\varphi(x) - e_i) \right)^{1/4} \int_{-x + 2\omega_i}^{x} \frac{e^{\tilde{\kappa}\xi}}{\sqrt{\sigma(x - \xi)\sigma(x + \xi)}} d\xi
= \left( \frac{\sigma(x - \omega_1)\sigma(x - \omega_2)\sigma(x - \omega_3)}{\sigma(x)} \right)^{1/2} \int_{-x + 2\omega_i}^{x} \frac{e^{\tilde{\kappa}\xi}}{\sqrt{\sigma(x - \xi)\sigma(x + \xi)}} d\xi,
\end{equation}

for $i = 0, 1, 2, 3$ are solutions to Heun’s equation in elliptical form for the case $l_0 = 1/2, l_1 = l_2 = l_3 = -1/2$,

\begin{equation}
(7.26) \quad -\frac{d^2 y}{dx^2} + \frac{3}{4} \varphi(x) - \frac{1}{4} \sum_{i=1}^{3} \varphi(x + \omega_i) + \tilde{\kappa}^2 f(x) = 0,
\end{equation}

The monodromy matrix of solutions to Eq. (7.24) can be expressed in the form of those in section 6 by substituting $\kappa = -\zeta(\alpha) + \tilde{\kappa}$ and $\alpha = 0$. In fact, if $e^{2\omega_1\tilde{\kappa}} \neq 1$ then the functions $f_0(x)$ and $f_1(x)$ are linearly independent, and the monodromy matrices
are written as

\begin{align}
(f_0^{\gamma_0}(x), f_1^{\gamma_0}(x)) &= (f_0(x), f_1(x)) \left( \begin{array}{cc} 1 & 2(1 - e^{2\omega_1 \tilde{\kappa}}) \\ 0 & 1 \end{array} \right), \\
(f_0^{\gamma_1}(x), f_1^{\gamma_1}(x)) &= (f_0(x), f_1(x)) \left( \begin{array}{cc} 1 & 0 \\ -2(1 - e^{-2\omega_1 \tilde{\kappa}}) & 1 \end{array} \right), \\
(f_0^{\gamma_2}(x), f_1^{\gamma_2}(x)) &= (f_0(x), f_1(x)) \\
&\quad \left( \begin{array}{cc} 1 + 2(e^{2\omega_1 \tilde{\kappa}} - e^{2\omega_2 j \tilde{\kappa}}) & \frac{(e^{2\omega_1 \tilde{\kappa}} - e^{2\omega_2 j \tilde{\kappa}} - 1)}{e^{2\omega_1 \tilde{\kappa}}} \\ -2 & 1 - 2(e^{2\omega_1 \tilde{\kappa}} - e^{2\omega_2 j \tilde{\kappa}}) \frac{(e^{2\omega_1 \tilde{\kappa}} - e^{2\omega_2 j \tilde{\kappa}} - 1)}{e^{2\omega_1 \tilde{\kappa}}} \end{array} \right),
\end{align}

which are obtained by analytic continuation of Eqs. (6.15, 6.16, 6.18) on the limit \( \alpha \to 0 \). The monodromy matrices of solutions to Eq. (7.24) are written as products of the monodromy matrices in Eq. (7.27) and the scalar that is determined by the branching of \( (\sigma(x - \omega_1)\sigma(x - \omega_2)\sigma(x - \omega_3)/\sigma(x))^{1/2} \). If \( \tilde{\kappa} = 0 \), then the integrals in Eq. (7.23) are written as

\begin{align}
\int_{z}^{\infty} \frac{dw}{\sqrt{(w - z)(w - e_1)(w - e_2)(w - e_3)}}, \quad \int_{e_1}^{z} \frac{dw}{\sqrt{(w - z)(w - e_1)(w - e_2)(w - e_3)}},
\end{align}

for \( i = 1, 2, 3 \) by setting \( w = \nu(\xi) \) and \( z = \varphi(x) \). These integrals coincide with the formula for the density function on root asymptotics of spectral polynomials for the Lame operator discovered by Borcea and Shapiro [3] (see also [19]).

The limits \( \lambda + \kappa_1/\mu \to 0, t \) correspond respectively to the limits \( \alpha \to e_1, e_2, e_3 \). The functions

\begin{align}
f_{i'}(x) = \int_{-x + 2\omega_{i'}}^{x} \frac{e^{(\kappa + \eta)\xi} \sigma(x) \sigma(\xi - \omega_i)}{\sqrt{\sigma(x - \xi)\sigma(x + \xi)}} \, d\xi, \quad (i' = 0, 1, 2, 3)
\end{align}

are solutions to the following Heun’s equations;

\begin{align}
\frac{d^2 y}{dz^2} + \left( \frac{1}{z - 1} + \frac{1}{z - t} \right) \frac{dy}{dz} + \frac{z - \kappa^2/(e_2 - e_1)}{4z(z - 1)(z - t)} y = 0, \quad (i = 1), \\
\frac{d^2 y}{dz^2} + \left( \frac{1}{z} + \frac{1}{z - t} \right) \frac{dy}{dz} + \frac{z - 1 - \kappa^2/(e_2 - e_1)}{4z(z - 1)(z - t)} y = 0, \quad (i = 2), \\
\frac{d^2 y}{dz^2} + \left( \frac{1}{z} + \frac{1}{z - t} \right) \frac{dy}{dz} + \frac{z - t - \kappa^2/(e_2 - e_1)}{4z(z - 1)(z - t)} y = 0, \quad (i = 3),
\end{align}

by setting \( z = (\varphi(x) - e_1)/(e_2 - e_1) \), and we have similar results for Heun’s equations in elliptical form for the case \( l_0 = -l_1 = l_2 = l_3 = -1/2, \ l_0 = l_1 = -l_2 = l_3 = -1/2, \ l_0 = l_1 = l_2 = -l_3 = -1/2 \) respectively. The monodromy matrices are expressed in similar forms as Eq. (7.27).

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