Inviscid limit for Stochastic Navier-Stokes Equations under general initial conditions

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Abstract

We consider in a smooth and bounded two dimensional domain the convergence in the $L^2$ norm, uniformly in time, of the solution of the stochastic Navier-Stokes equations with additive noise and no-slip boundary conditions to the solution of the corresponding Euler equations. We prove, under general regularity on the initial conditions of the Euler equations, that assuming the dissipation of the energy of the solution of the Navier-Stokes equations in a Kato type boundary layer, then the inviscid limit holds.

Keywords: Inviscid limit; turbulence; additive noise; no-slip boundary conditions; boundary layer; energy dissipation.

1 Introduction

The study of the inviscid limit of the solutions of the Navier-Stokes equations is a classical topic in fluid mechanics. The Euler equations have very large classes of weak solutions, including non-dissipative ones [2], but the inviscid limit can in some cases furnish a selection principle [3]. In the case of domains without boundary several results are available in the deterministic case, see, for instance [5], [7], [9], [11], [17]. In the case of domains with boundary the difficulty of the problem changes drastically considering different boundary conditions also in the two dimensional case. Indeed, if we consider the so called Navier boundary conditions, some results are available both in the deterministic and in the stochastic case (see for example [4], [13]). The no-slip boundary conditions are more challenging. This is due to the appearance of the boundary layer. So far, only few results are available in this framework. They can be splitted in two macro-categories:

1. Conditioned results, namely proving that if the solution of the Navier-Stokes equations has a particular behavior in the boundary layer, then the inviscid limit holds true. These are the most common kind of results available for what concern the inviscid limit with no-slip boundary conditions. See for instance [6], [12], [20], [21].

2. Unconditioned results. They are based on strong assumptions about the symmetry of the domain and of the data [14], [15], or real analytic data [18], or the vanishing of the Eulerian initial vorticity in a neighborhood of the boundary [16].

The results of this paper go in the first direction. In particular our goal is to generalize the results of [12] to the stochastic framework and to not classical solutions of the Euler equations. We consider the stochastic Navier-Stokes equations with additive noise and no-slip boundary conditions in a smooth, bounded, two dimensional domain, proving that, under suitable assumptions on the behavior of their solutions in the boundary layer and of the additive noise, we have strong convergence to the solution of the deterministic Euler equations for vanishing viscosity. In section 2 we introduce the mathematical problem, giving some well known results about the well posedness of the stochastic Navier-Stokes equations with additive noise and the Euler equations, and stating our main theorems. In sections 3 and 4 we prove our main theorems. Lastly in section 5 we add some deterministic results related to Theorem 6 and Theorem 9.

Theorem 8 and Theorem 9 can be seen as introductory results for the analysis of the zero noise-zero viscosity limit following the idea of [1]. These kind of results are relevant for the analysis of a selection principle for the solutions of the Euler equations.
2 Main Results

Let $D \subseteq \mathbb{R}^2$ be smooth and bounded, $T > 0$ fixed and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a filtered probability space. Let $Z$ be a separable Hilbert space, denote by $L^2(\mathcal{F}_0, Z)$ the space of square integrable random variables with values in $Z$, measurable with respect to $\mathcal{F}_0$. Moreover, denote by $C_T([0, T]; Z)$ the space of continuous adapted processes $(X_t)_{t \in [0, T]}$ with values in $Z$ such that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|X_t\|^2_Z \right] < \infty$$

and by $L^2_Z(0, T; Z)$ the space of progressively measurable processes $(X_t)_{t \in [0, T]}$ with values in $Z$ such that

$$\mathbb{E} \left[ \int_0^T \|X_t\|^2_Z \, dt \right] < \infty.$$

Denote by $L^2(D; \mathbb{R}^2)$ and $H^k(D; \mathbb{R}^2)$ the usual Lebesgue and Sobolev spaces and by $H^k_0(D; \mathbb{R}^2)$ the closure in $W^k(D)$ of smooth compact support vector valued functions. Set

$$H = \{ f \in L^2(D; \mathbb{R}^2), \ \text{div} f = 0, \ f \cdot n|_{\partial D} = 0 \}, \ V = H^1_0(D; \mathbb{R}^2) \cap H, \ D(A) = H^2 \cap V.$$

We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the norm in $H$ respectively. Denote by $P$ the projection of $L^2(D; \mathbb{R}^2)$ on $H$ and define the unbounded linear operator $A : D(A) \subseteq H \to H$ by the identity

$$\langle Au, w \rangle = \langle \Delta u, w \rangle$$

for all $v \in D(A)$, $w \in H$. $A$ will be called the Stokes operator. It is well known (see for example [19]) that $A$ generates an analytic semigroup of negative type on $H$ and moreover $V = D \left( (-A)^{1/2} \right)$.

Let us consider a sequence of real Brownian motions $(W^k)_{k=1}^N$ adapted to $\mathcal{F}_t$ and a sequence of functions $(\sigma_k)_{k=1}^N \subseteq D(A)$. Let us, moreover, assume that $u^\nu_0 \in L^2(\mathcal{F}_0, H)$.

Let us consider the stochastic Navier-Stokes equations below. Some physical motivations for the introduction of this model can be found in [5].

$$\begin{cases}
du^\nu = -(-\nu \Delta u^\nu + \nabla u^\nu \cdot \nabla v^\nu + \nabla p^\nu) dt + \nu^{1/2} \sum_{k=1}^N \sigma_k^\nu dW^k_t, & t \in [0, T] \\
u^\nu(0) = u^\nu_0.
\end{cases} \quad (1)$$

**Definition 1** Given $u^\nu_0 \in L^2(\mathcal{F}_0, H)$, we say that a stochastic process $u^\nu$ is a weak solution of equation (1) if

$$u^\nu \in C_T([0, T]; H) \cap L^2_Z(0, T; V)$$

and for every $\phi \in D(A)$, we have

$$\langle u^\nu(t), \phi \rangle - \int_0^t \langle b(u^\nu(s), \phi, u^\nu(s)), \phi \rangle \, ds = \langle u^\nu_0, \phi \rangle + \nu \int_0^t \langle u^\nu(s), A\phi \rangle \, ds + \nu^{1/2} \sum_{k=1}^N \langle \sigma_k, \phi \rangle W^k_t,$$

for every $t \in [0, T]$, $\mathbb{P}$ a.s.

Under previous assumptions on the coefficient $\sigma_k$, equation (1) is well posed. Indeed the following theorem holds, see [8].

**Theorem 2** If $u^\nu_0 \in L^2(\mathcal{F}_0, H)$, $(\sigma_k)_{k=1}^N \subseteq D(A)$, there exists a unique weak solution of equation (1). Moreover the following relations hold:

$$\mathbb{E} \left[ \|u^\nu(t)\|^2 \right] + 2\nu \int_0^t \mathbb{E} \left[ \|u^\nu(s)\|^2 \right] \, ds = \mathbb{E} \left[ \|u^\nu_0\|^2 \right] + t\nu \sum_{k=1}^N \|\sigma_k\|^2 \quad (2)$$

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|u^\nu(t)\|^2 \right] \leq \mathbb{E} \left[ \|u^\nu_0\|^2_{L^2(D)} \right] + T\nu \sum_{k=1}^N \|\sigma_k\|^2 + K\nu^{1/2} \sum_{k=1}^N \mathbb{E} \left[ \int_0^T \langle u^\nu(s), \sigma_k \rangle^2 \, ds \right]^{1/2}, \quad (3)$$

$$\|u^\nu(t)\|^2 + 2\nu \int_0^t \|\nabla u^\nu(s)\|^2_{L^2(D)} \, ds = \|u^\nu_0\|^2 + t\nu \sum_{k=1}^N \|\sigma_k\|^2 + 2\nu^2 \sum_{k=1}^N \int_0^t \langle u^\nu(s), \sigma_k \rangle dW^k_s. \quad (4)$$

where $K$ is independent from $\nu$. 

2
Equation (2) will be called energy equality in the following, instead equation (4) will be called Itô formula.
For our purposes we will need a different relation satisfied by \( u^\nu \) that will be clarified by the following lemma.

**Lemma 3** Under the same assumptions of Theorem 2, if \( u^\nu \) is a weak solution of equation (7), then for each \( \phi \in C([0,T];V) \cap C^1([0,T];H) \)

\[
\langle u^\nu(t), \phi(t) \rangle = \langle u^\nu(0), \phi(0) \rangle + \int_0^t \langle u^\nu(s), \partial_s \phi(s) \rangle \, ds \\
- \nu \int_0^t \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} \phi(s) \rangle \, ds + \int_0^t b(u^\nu(s), \phi(s), u^\nu(s)) \, ds \\
+ \nu^2 \sum_{k=1}^N \langle \sigma_k, \phi(t) \rangle W^k_t - \nu^2 \sum_{k=1}^N \int_0^t \langle \sigma_k, \phi(s) \rangle W^k_s \, ds
\]

for every \( t \in [0,T], \, \mathbb{P} \) a.s.

**Proof.** Thanks to the regularity of the weak solution \( u^\nu \), by density we have that for each \( \phi \in V \)

\[
\langle u^\nu(t), \phi \rangle - \int_0^t b(u^\nu(s), \phi, u^\nu(s)) \, ds = \langle u^\nu(0), \phi \rangle + \nu^2 \sum_{k=1}^N \langle \sigma_k, \phi \rangle W^k_t \\
- \nu \int_0^t \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} \phi(s) \rangle \, ds,
\]

for every \( t \in [0,T], \, \mathbb{P} \) a.s. Let now \( \phi(t) \in C^1([0,T];H) \cap C([0,T];V) \). Let, moreover, \( \pi = \{0 = t_0 < t_1 < \cdots < T_n = T \} \) be a partition of \([0,T]\). Thus, using the identities

\[
\langle u^\nu(t_{i+1}), \phi(t_{i+1}) \rangle - \langle u^\nu(t_i), \phi(t_i) \rangle = \int_{t_{i}}^{t_{i+1}} \langle u^\nu(t), \partial_s \phi(s) \rangle \, ds,
\]

we get

\[
\langle \sigma_k W^k_{t_{i+1}}, \phi(t_{i+1}) \rangle - \langle \sigma_k W^k_{t_i}, \phi(t_i) \rangle = \int_{t_{i}}^{t_{i+1}} \langle \sigma_k W^k(t), \partial_s \phi(s) \rangle \, ds,
\]

for each \( t \in [0,T], \, \mathbb{P} \) a.s. Let now \( \phi(t) \in C^1([0,T];H) \cap C([0,T];V) \). Let, moreover, \( \pi = \{0 = t_0 < t_1 < \cdots < T_n = T \} \) be a partition of \([0,T]\). Thus, using the identities

\[
\langle u^\nu(t_{i+1}), \phi(t_{i+1}) \rangle - \langle u^\nu(t_i), \phi(t_i) \rangle = \int_{t_{i}}^{t_{i+1}} \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} \phi(t_i) \rangle \, ds \\
+ \int_{t_{i}}^{t_{i+1}} b(u^\nu(s), \phi(t_i), u^\nu(s)) \, ds \\
+ \int_{t_{i}}^{t_{i+1}} \langle u^\nu(t_{i+1}), \partial_s \phi(s) \rangle \, ds \\
- \nu^2 \sum_{k=1}^N \int_{t_{i}}^{t_{i+1}} \langle \sigma_k W^k(t_{i+1}), \partial_s \phi(s) \rangle \, ds \\
+ \nu^2 \sum_{k=1}^N \left( \langle \sigma_k W^k_{t_{i+1}}, \phi(t_{i+1}) \rangle - \langle \sigma_k W^k_{t_i}, \phi(t_i) \rangle \right)
\]

It implies

\[
\langle u^\nu(T), \phi(T) \rangle = \langle u^\nu(0), \phi(0) \rangle - \int_0^T \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} \phi(s) \rangle \, ds \\
+ \int_0^T b(u^\nu(s), \phi(s), u^\nu(s)) \, ds \\
+ \int_0^T \langle u^\nu(s), \partial_s \phi(s) \rangle \, ds - \nu^2 \sum_{k=1}^N \int_0^T \langle \sigma_k W^k_{s_{2k}}, \partial_s \phi(s) \rangle \, ds \\
+ \nu^2 \sum_{k=1}^N \left( \langle \sigma_k W^k_T, \phi(T) \rangle - \langle \sigma_k W^k_0, \phi(0) \rangle \right).
\]
where \( s_+^u(s) = t_s \) if \( s \in [t_s, t_{s+1}] \) and \( s_-^u(s) = t_{s-1} \) if \( s \in [t_s, t_{s+1}] \). Taking the limit over a sequence of partitions \( \pi_N \) with size going to zero, we get

\[
\langle u^r(T), \phi(T) \rangle = \langle u^r(0), \phi(0) \rangle - \int_0^T \langle (-A)^{\frac{r}{2}} u^r(s), (-A)^{\frac{r}{2}} \phi(s) \rangle \, ds
+ \int_0^T b(u^r(s), \phi(s), u^r(s)) \, ds
+ \int_0^T \langle u^r(s), \partial_s \phi(s) \rangle \, ds - N \sum_{k=1}^N \int_0^T \langle \sigma_k W^k_T, \partial_s \phi(s) \rangle \, ds
+ \nu^r N \sum_{k=1}^N \left( \langle \sigma_k W^k_T, \phi(T) \rangle - \langle \sigma_k W^k_T, \phi(0) \rangle \right).
\]

(thanks to the regularity of \( u, \phi \), dominated convergence theorem and Itô isometry). The argument applies to a generic \( t \in [0, T] \), hence we have the thesis. ■

Let us now consider the Euler equations

\[
\begin{aligned}
\partial_t \bar{u} + \nabla \bar{u} \cdot \bar{u} + \nabla p &= \bar{f} \ (x, t) \in D \times (0, T), \\
\mathrm{div} \ ar{u} &= 0 \\
\bar{u} \cdot n |_{\partial D} &= 0 \\
\bar{u}(0) &= \bar{u}_0
\end{aligned}
\]

(5)

**Definition 4** Given \( \bar{u}_0 \in H, \bar{f} \in L^2(0, T; H) \) we say that \( \bar{u} \in C(0, T; H) \) is a weak solution of equation (5) if for every \( \phi \in C([0, T]; V) \cap C^1([0, T]; H) \)

\[
\langle \bar{u}(t), \phi(t) \rangle = \langle \bar{u}_0, \phi(0) \rangle + \int_0^t \langle \bar{u}(s), \partial_s \phi(s) \rangle \, ds + \int_0^t b(\bar{u}(s), \phi(s), \bar{u}(s)) \, ds + \int_0^t \langle \bar{f}(s), \phi(s) \rangle \, ds
\]

for every \( t \in [0, T] \) and the energy inequality

\[
\| \bar{u}(t) \|^2 \leq \| \bar{u}_0 \|^2 + 2 \int_0^t \langle f(s), \bar{u}(s) \rangle \, ds
\]

holds.

For what concern the well posedness of the Euler equations the following results hold true, see [11], [13].

**Theorem 5** If \( \bar{u}_0 \in C^{1+\varepsilon}(\bar{D}) \cap H \) and \( \bar{f} \in C^{1+\varepsilon}([0, T] \times \bar{D}) \cap L^2(0, T; H) \), then there exist \( \bar{u}, \bar{p} \) classical solutions of equation (5). Moreover, \( \bar{u}, \nabla \bar{u}, p, \nabla p \in C([0, T] \times \bar{D}) \), \( \bar{u} \) is unique and \( p \) is unique up to an arbitrary function of \( t \) which can be added to \( p \).

**Theorem 6** If \( \bar{f} = 0 \) a is a weak solution of the Euler equations with initial condition \( u_0 \in H \) and \( \bar{u} \) is the unique weak solution of the Euler equations with initial condition \( \bar{u}_0 \in H \cap C^{1+\varepsilon}(\bar{D}) \), then

\[
\| (u - \bar{u})(t) \|^2 \leq e^{2\| \nabla \bar{u} \|_{L^\infty([0, T] \times \bar{D})}} \| u_0 - \bar{u}_0 \|^2,
\]

Calling

\[
O_n = \left\{ u_0 \in H : \exists \bar{u}_0 \in H \cap C^{1+\varepsilon}(\bar{D}), \| u_0 - \bar{u}_0 \| < \frac{1}{n} e^{-3T\| \nabla \bar{u} \|_{L^\infty([0, T] \times \bar{D})}} \right\}
\]

where \( \bar{u} \) is the solution of the Euler equations with initial condition \( \bar{u}_0 \), then for each \( u_0 \in \bigcap_{n \geq 1} O_n =: \tilde{O} \) there exists a unique \( u \in C([0, T], H) \) weak solution of the Euler equations with initial condition \( u_0 \). Moreover the energy equality

\[
\| u(t) \|^2 = \| u_0 \|^2
\]

holds.

For each \( u_0 \in \tilde{O} \) we will say that \( \{ \bar{u}_m \} \) approximates \( u_0 \) in the sense of Theorem 5 if \( \bar{u}_m \in H \cap C^{1+\varepsilon}(\bar{D}) \) and

\[
\| u_0 - \bar{u}_m \| < \frac{1}{m} e^{-3T\| \nabla \bar{u} \|_{L^\infty([0, T] \times \bar{D})}}
\]

where \( \bar{u}_m \) is the solution of the Euler equations with initial condition \( \bar{u}_m \).

Lastly we introduce some results related to the boundary layer corrector of the solution of the Euler equations, see [12].
Proposition 7 Under the assumptions of Theorem 6
- it exists a smooth skew-symmetric matrix a such that $\bar{u} = \text{div} \ a$ on $\partial D$ and $a = 0$ on $\partial D$;
- let $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ a smooth function such that $\xi(0) = 1$, $\xi(r) = 0$ if $r \geq 1$ and
  $$z : D \to \mathbb{R}^+, \ z(x) = \xi(\rho/\delta) \text{ with } \rho = \text{dist}(x, \partial D)$$
  and $\delta$ a parameter which goes to 0 when $\nu$ goes to 0. Let, moreover, $v = \text{div}(za)$. Then,
  $$\bar{u} - v \in C([0,T];V) \cap C^1([0,T];H).$$

the following are equivalent:

Proposition 7

Now we can state our main theorems. Since the stochastic term in equation (1) goes to 0, we assume that the external
force in the Euler equations is identically 0. Theorem 8 is a generalization of the results in [12] to this stochastic framework
and also the idea of the proof is similar. In Theorem 9 we consider a wider set of initial conditions.

Theorem 8 If $\bar{u}_0 \in C^{1,*}(\bar{D})$ and under previous assumptions on $u_0^\nu$ and $\sigma_k$, if

$$\lim_{\nu \to 0} E \left[\|u_0^\nu - \bar{u}_0\|^2\right] = 0,$$

then the following are equivalent:

1. $\lim_{\nu \to 0} E \left[\sup_{t \in [0,T]} \|u^\nu - \bar{u}\|^2\right] = 0$.
2. $u^\nu(t) \to \bar{u}(t)$ in $L^2(\Omega \times D)$ for each $t \in [0,T]$.
3. $\lim_{\nu \to 0} \int_0^T E \left[\|\nabla u^\nu(t)\|_{L^2(D)}^2\right] dt = 0$.
4. $\lim_{\nu \to 0} \int_0^T E \left[\|\nabla u(t)\|_{L^2(D)}^2\right] dt = 0$.

Theorem 9 If $u_0 \in \bar{D}, u_0^\nu \in L^2(\mathcal{F}_0, H)$, $\lim_{n \to +\infty} E \left[\|u_0^\nu - u_0\|^2\right] = 0$. Let $u$ be the solution of the Euler equations with initial condition $u_0$, $u^\nu$ be the solution of the stochastic Navier-Stokes equations with viscosity $\nu_n$ and initial condition $u_0^\nu$. If

$$\lim_{n \to +\infty} \nu_n = 0, \quad \lim_{n \to +\infty} \nu_n \int_0^T E \left[\|\nabla u^\nu(t)\|_{L^2(\Gamma_{\nu_n})}^2\right] dt = 0,$$

then

$$\lim_{n \to +\infty} E \left[\sup_{t \in [0,T]} \|u^\nu - u\|^2\right] = 0.$$

Remark 10 Theorem 8 means that if convergence does not take place, the energy dissipation within the boundary layer
of width $\nu$ must remain finite as $\nu \to 0$. This suggests that something violent must have happened.

Remark 11 Theorem 7 is new also in the deterministic framework, namely taking $\sigma_k = 0 \forall \ k = 1, \ldots, n$. In section 5 we will prove this result in the deterministic framework for a non-zero external force.

Remark 12 $K$ will denote several constants dependent only from the solution of the Euler equations and its data, $\{\sigma_k\}_{k=1}^N$ and $T$ in the following.

5
3 Proof of Theorem 8

The proof of theorem 8 follows from a preliminary weaker result, namely under the same assumptions
\[
\lim_{\nu \to 0} \sup_{t \in [0,T]} \mathbb{E} \|u^\nu - u\|_{2t}^2 = 0.
\]
This is the analogous of the Kato’s result in this stochastic framework.

**Proposition 13** Under the same assumptions of Theorem 8, if
\[
\lim_{\nu \to 0} \mathbb{E} \|u^\nu_0 - \bar{u}_0\|^2 = 0,
\]
then the following are equivalent:
1. \(\lim_{\nu \to 0} \sup_{t \in [0,T]} \mathbb{E} \|u^\nu - \bar{u}\|^2 = 0\).
2. \(u^\nu(t) \to \bar{u}(t)\) in \(L^2(\Omega \times D)\) for each \(t \in [0,T]\).
3. \(\lim_{\nu \to 0} \int_0^T \mathbb{E} \|\nabla u^\nu(t)\|_{L^2(D)}^2\) \(dt = 0\).
4. \(\lim_{\nu \to 0} \int_0^T \mathbb{E} \|\nabla u^\nu(t)\|_{L^2(D)}^2\) \(dt = 0\).

**Proof.** 1. \(\Rightarrow\) 2. and 3. \(\Rightarrow\) 4. are obvious. We need only prove that 2. \(\Rightarrow\) 3. and 4. \(\Rightarrow\) 1.

2. \(\Rightarrow\) 3. By energy equality for each \(t = T\)
\[
\nu \mathbb{E} \left( \int_0^T \|\nabla u^\nu(s)\|_{L^2(D)}^2 \right) = \frac{1}{2} \mathbb{E} [\|u^\nu_0\|^2] - \frac{1}{2} \mathbb{E} \|u^\nu(T)\|^2 + T\nu \sum_{k=1}^N \|\sigma_k\|^2.
\]
Taking the limsup of this expression and exploiting the fact that under the assumptions
\[
\mathbb{E} [\|u^\nu_0 - \bar{u}_0\|^2] \to 0
\]
we get the thesis.

4. \(\Rightarrow\) 1. For each time \(t\) we have
\[
\mathbb{E} [\|u^\nu - \bar{u}\|^2] \leq \mathbb{E} [\|u^\nu\|^2] + \|\bar{u}\|^2 - 2\mathbb{E} [\langle u^\nu, \bar{u} \rangle]
\]
\[
\leq \mathbb{E} [\|u^\nu_0\|^2] + T\nu \sum_k \|\sigma_k\|^2 + \|\bar{u}_0\|^2 - 2\mathbb{E} [\langle u^\nu, \bar{u} \rangle]
\]
\[
\leq o(1) + 2\|\bar{u}_0\|^2 + T\nu \sum_k \|\sigma_k\|^2 - 2\mathbb{E} [\langle u^\nu, \bar{u} - v \rangle] - 2\mathbb{E} [\langle u^\nu, v \rangle]
\]
Then
\[
\mathbb{E} [\|u^\nu - \bar{u}\|^2] \leq o(1) + 2\|\bar{u}_0\|^2 + T\nu \sum_k \|\sigma_k\|^2 - 2\mathbb{E} [\langle u^\nu, \bar{u} - v \rangle] - 2\mathbb{E} [\langle u^\nu, v \rangle]
\]

(6)

To analyze the second-last term we use the weak formulation of \(u^\nu\) for the test function \(\bar{u} - v\).
\[
-2(E_{\nu}u^\nu(t), (\bar{u} - v)(t)) = -2(E_{\nu}(0), (\bar{u} - v)(0)) - 2\int_0^t (E_{\nu}(s), \partial_t (\bar{u} - v)(s)) ds +
\]
\[
2\nu \int_0^t \langle (A)^T u^\nu(s), (A)^T (\bar{u} - v)(s) \rangle ds - \int_0^t 2b(u^\nu(s), (\bar{u} - v)(s), u^\nu(s)) ds -
\]
\[
2\nu \sum_k \langle \sigma_k, (\bar{u} - v)(t) \rangle W_k + 2\nu \sum_{k=1}^N \int_0^t \langle \sigma_k, (\bar{u} - v)(s) \rangle W_k ds.
\]
Taking the expected value of the last expression we obtain

\[
-2\mathbb{E}[(u^\nu(t), (\bar{u} - v)(t))] + 2\mathbb{E}[\|u^\nu_0\|^2] = o(1) - 2\mathbb{E}\left[\int_0^t (u^\nu(s), \partial_s (\bar{u} - v)(s)) \, ds\right] + 2\mathbb{E}\left[\int_0^t ((-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} (\bar{u} - v)(s)) \, ds\right] - \mathbb{E}\left[\int_0^t 2b(u^\nu(s), (\bar{u} - v)(s), u^\nu(s)) \, ds\right].
\]

Moreover

\[
-\mathbb{E}[\langle u^\nu(s), \partial_s (\bar{u} - v)(s) \rangle] \leq \|v(t)\| \mathbb{E}[\|u^\nu(t)\|^2]^{\frac{1}{2}} \leq K\|v(t)\| \mathbb{E}[\|u^\nu(t)\|^2]^{\frac{1}{2}} \leq K\delta^{\frac{1}{2}} \to 0,
\]

\[
\mathbb{E}\left[\int_0^t b(u^\nu(s) - \bar{u}(s), \bar{u}(s), u^\nu(s) - \bar{u}(s)) \, ds\right] \leq \|\nabla \bar{u}\|_{L^\infty(0,T;D)} \mathbb{E}\left[\int_0^t \|u^\nu - \bar{u}(s)\|^2 \, ds\right].
\]

Using all these relations in equation (7), we get

\[
\mathbb{E}[\|u^\nu - \bar{u}\|^2] \leq o(1) + 2\|\bar{u}_0\|^2 - 2\mathbb{E}[\|u^\nu_0\|^2] + tr \sum_{k=1}^N \|\sigma_k\|^2 + 2\mathbb{E}\left[\int_0^t (\bar{u} - v)(s) \, ds\right] + 2\mathbb{E}\left[\int_0^t b(u^\nu(s), v(s), u^\nu(s)) \, ds\right] + 2\mathbb{E}\left[\int_0^t b(u^\nu(s), \bar{u}(s), \bar{u}(s), u^\nu(s)) \, ds\right] + 2\mathbb{E}\left[\int_0^t ((-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} (\bar{u} - v)(s)) \, ds\right] + 2\mathbb{E}\left[\int_0^t b(u^\nu(s), v(s), u^\nu(s)) \, ds\right] - 2\mathbb{E}\left[\int_0^t b(u^\nu(s) - \bar{u}(s), \bar{u}(s), u^\nu(s) - \bar{u}(s)) \, ds\right].
\]

thus, calling

\[
R(s) = tr \sum_{k=1}^N \|\sigma_k\|^2 + 2\mathbb{E}\left[\int_0^t b(u^\nu(s), v(s), u^\nu(s)) \, ds\right] + \mathbb{E}[b(u^\nu(s), v(s), u^\nu(s))]
\]

we have

\[
\mathbb{E}[\|u^\nu - \bar{u}\|^2] \leq o(1) + 2\mathbb{E}\left[\int_0^t (K\|u^\nu - \bar{u}(s)\|^2) + R(s) \, ds\right].
\]

If we are able to prove that \(\lim_{\nu \to 0} \int_0^t R(s) \, ds = 0\), then via Gronwall’s inequality we’ll get the thesis. The term related to \(\sigma_k\) is obvious. For what concerns the others:

\[
\|\mathbb{E}[b(u^\nu(s), v(s), u^\nu(s))]| \leq \mathbb{E}\left[\int_{\Gamma_0} |\nabla v(s)|^2 \frac{|u^\nu(s)|^2}{\rho^2} \, dx\right] \leq K\delta E \left[\frac{|u^\nu|^2}{\rho^2} \right]_{L^2(\Gamma_0)} \leq K\delta E \left[\|\nabla u^\nu\|_{L^2(\Gamma_0)}^2\right],
\]

\[
\text{Hardy–Littlewood ineq.} \leq K\delta E \left[\|\nabla u^\nu\|_{L^2(\Gamma_0)}^2\right].
\]
Moreover, thanks to previous relation and Corollary 14

Under the same assumptions of Proposition 13, if

Now the proof is similar to the previous one. For each time $t$ previous estimates and energy equality, via Holder inequality we get

Taking $\delta = cv$, we have

Exploiting the assumption

previous estimates and energy equality, via Holder inequality we get

**Corollary 14** Under the same assumptions of Proposition 13 if

then

**Proof.** Preliminarily, note that, starting from equation (3), we have

Now the proof is similar to the previous one. For each time $t$ we have

Let us rewrite $\langle u', \bar{u} - v \rangle$ thanks to the weak formulation of $u'$

Moreover, thanks to previous relation and

$$b(u', \bar{u}, u') = b(u', \bar{u}, u') = b(u' - \bar{u}, \bar{u}, u' - \bar{u}),$$

$$b(u', \bar{u}, u') = b(u', \bar{u}, u') = b(u' - \bar{u}, \bar{u}, u' - \bar{u}),$$
we have at time $t$

$$\|u^\nu - \bar{u}\|^2 = (\|u_0^\nu\|^2 + \|\bar{u}\|^2 - 2\langle u_0^\nu, (\bar{u} - v)(0) \rangle) + (\nu \sum_{k=1}^{N} \|\sigma_k\|^2 + 2\nu \sum_{k=1}^{N} \int_0^t \langle u^\nu(s), \sigma_k \rangle dW_k^s$$

$$- 2\nu \sum_{k=1}^{N} \langle \sigma_k, (\bar{u} - v)(t) \rangle W_k^t + 2\nu \sum_{k=1}^{N} \int_0^t \langle \sigma_k, (\bar{u} - v)(s) \rangle W_k^s ds$$

$$+ \left( 2 \int_0^t b(u^\nu, v, u^\nu)(s) ds - 2 \int_0^t b(u^\nu - \bar{u}, \bar{u}, u^\nu - \bar{u})(s) ds \right) +$$

$$(-2\langle u^\nu, v \rangle + 2\nu \int_0^t (-\bar{A} \frac{\partial}{\partial t} u^\nu(s), (-\bar{A} \frac{\partial}{\partial t} (\bar{u} - v))(s)) ds + 2 \int_0^t (u^\nu, \sigma_k v) ds)$$

$$= I_1(t) + I_2(t) + I_3(t) + I_4(t).$$

Thus

$$E \left[ \sup_{t \in [0,T]} \|u^\nu - u\|^2 \right] \leq E[\sup_{t \in [0,T]} I_1] + E[\sup_{t \in [0,T]} I_2] + E[\sup_{t \in [0,T]} I_3] + E[\sup_{t \in [0,T]} I_4]$$

$$E \left[ \sup_{t \in [0,T]} I_1 \right] = E \left[ \|u_0^\nu\|^2 + \|\bar{u}\|^2 - 2\langle u_0^\nu, (\bar{u} - v)(0) \rangle \right]$$

$$\leq - E \left[ \|u_0^\nu\|^2 + \|\bar{u}\|^2 + 2E \|u_0^\nu\|\|\bar{u}\| - \bar{u}(0) \right] + 2E \|u^\nu(0)\|$$

$$\|v(t)\|_{L^2(D)} \leq K \delta^{\frac{1}{2}}, \quad E[\|u_0^\nu - \bar{u}\|^2] + o(1) + K \delta^{\frac{1}{2}}.$$
omitted. For each \( m \)

We adapt the computations of the proof of Proposition 13 to analyze the second term, hence some explanation will be

In conclusion, if we take \( \delta = cv \), then

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u^v - u\|^2 \right] \leq (1) + K \nu \mathbb{E} \left[ \int_0^T \|\nabla u^v(s)\|_{L^2(\Gamma_{cv})}^2 \, ds \right] + K \nu \mathbb{E} \left[ \int_0^T \|\nabla u^v(s) - \bar{u}(s)\|_{L^2(\Gamma_{cv})}^2 \, ds \right]
\]

It is clear the almost all the terms goes to 0 thanks to the assumptions and Proposition 13. We need just to check that \( \nu \mathbb{E} \left[ \int_0^T \|\nabla u^v(s)\|_{L^2(D)}^2 \, ds \right] \) and \( \nu \mathbb{E} \left[ \int_0^T \|\nabla u^v\|_{L^2(\Gamma_{cv})}^2 \, ds \right] \) behave properly, but this is elementary, in fact:

\[
\nu \mathbb{E} \left[ \int_0^T \|\nabla u^v(s)\|_{L^2(D)}^2 \, ds \right] \leq \nu T \mathbb{E} \left[ \nu \int_0^T \|\nabla u^v(s)\|_{L^2(D)}^2 \, ds \right] \leq K \nu \mathbb{E} \left[ \nu \int_0^T \|\nabla u^v\|_{L^2(\Gamma_{cv})}^2 \, ds \right]^2 \nu \rightarrow 0.
\]

This completes the proof.

Theorem 8 follows immediately by Proposition 13 and Corollary 13.

4 Proof of Theorem 9

As in the previous section, we start with a weaker result with the supremum in time outside the expected value to obtain the stronger one with the supremum in time inside the expected value. The idea behind both the proofs is simply to introduce an approximation of \( u_0 \) in the sense of Theorem 9 then

\[
\|u^0 - u\|^2 \leq 2\|u^0 - \bar{u}^m\|^2 + 2\|\bar{u}^m - u\|^2,
\]

where \( \bar{u}^m \) is the solution of the Euler Equations with initial condition \( \bar{u}^m_0 \in H \cap C^{1,\alpha}(\bar{D}) \). Thus, the second term can be estimate via Theorem 4, the first one is analyzed exploiting techniques similar to the ones of the previous section.

Remark 15 If \( u_0 \in \bar{O} \) and \( \{\bar{u}^m_0\}_{m \in \mathbb{N}} \) approximates \( u_0 \) in the sense of Theorem 9 then

\[
\|u_0 - \bar{u}^m_0\|_{C^2(\bar{O})} \leq \frac{1}{m} \|\nabla \bar{u}^m\|_{L^\infty(\bar{O} \times \bar{D})}
\]

\[
\|u_0 - \bar{u}^m\|_{C^2(\bar{O})} \leq \frac{1}{m} \|\nabla \bar{u}^m\|_{L^\infty(\bar{O} \times \bar{D})}
\]

Lemma 16 Under the same assumptions of Theorem 4

\[
\lim_{m \to +\infty} \sup_{t \in [0,T]} \mathbb{E} \left[ \|u^0 - u\|^2 \right] = 0.
\]

Proof. Let \( \{\bar{u}^m_0\}_{m \in \mathbb{N}} \) approximating \( u_0 \) in the sense of Theorem 6 and \( \{\bar{u}_m\}_{m \in \mathbb{N}} \) the corresponding solutions of the Euler equations, then for each \( t, \, n, \, m \) we have

\[
\mathbb{E} \left[ \|u^n(t) - u(t)\|^2 \right] \leq 2\mathbb{E} \left[ \|u^n(t) - \bar{u}^m(t)\|^2 + 2\|\bar{u}^m(t) - u(t)\|^2 \right] \leq \frac{2}{m^2} + 2\mathbb{E} \left[ \|u^n(t) - \bar{u}^m(t)\|^2 \right].
\]

We adapt the computations of the proof of Proposition 13 to analyze the second term, hence some explanation will be omitted. For each \( m \) and \( \delta > 0 \) fixed, let us introduce the corretor of the boundary layer \( \bar{v}_m \). \( \bar{v}_m \) satisfies previous estimates with respect to a constant dependent from \( m \) and independent from \( t \), namely

\[
\|\bar{v}_m(t)\|_{L^\infty(D)} \leq K_m, \, \|\bar{v}_m(t)\|_{L^2(D)} \leq K_m^{\frac{1}{2}}, \, \|\partial_t \bar{v}_m(t)\|_{L^2(D)} \leq K_m^{\frac{1}{2}},
\]

\[
\|\bar{v}_m(t)\|_{L^\infty(D)} \leq K_m, \, \|\bar{v}_m(t)\|_{L^2(D)} \leq K_m^{\frac{1}{2}}, \, \|\partial_t \bar{v}_m(t)\|_{L^2(D)} \leq K_m^{\frac{1}{2}}.
\]

10
\[ \| \nabla v_m(t) \|_{L^\infty(D)} \leq K_m \delta^{-1}, \quad \| \nabla v_m(t) \|_{L^2(D)} \leq K_m \delta^{-1/2}, \quad \| \rho(t) \nabla v_m(t) \|_{L^\infty(D)} \leq K_m, \]
\[ \| \rho(t)^2 \nabla v_m(t) \|_{L^\infty(D)} \leq K_m \delta, \quad \| \rho(t) \nabla v_m(t) \|_{L^2(D)} \leq K_m \delta^{1/2}. \]

We have at time \( t \)
\[ E \left[ \| u^n - \bar{u}^m \|_2^2 \right] = E \left[ \| u^n \|_2^2 + \| \bar{u}^m \|_2^2 - 2E \left[ \langle u^n, \bar{u}^m \rangle \right] \right] \]
\[ \leq E \left[ \| u^n_0 \|_2^2 \right] + \nu t E \left[ \sum_k \| \sigma_k \|_2^2 + \| \bar{u}_0^m \|_2^2 - 2E \left[ \langle u^n, \bar{u}^m - v_m \rangle \right] \right] + 2E \left[ \langle u^n, v_m \rangle \right] \]
\[ = E \left[ \| u^n_0 - u_0 \|_2^2 \right] + \| u_0 \|_2^2 + 2E \left[ \langle u^n_0 - u_0, u_0 \rangle \right] + \nu t E \left[ \sum_k \| \sigma_k \|_2^2 + \| \bar{u}_0^m - u_0 \|_2^2 + \| u_0 \|_2^2 \right] \]
\[ + 2 \| u^n_0 - u_0, u_0 \| - 2E \left[ \langle u^n, \bar{u}^m - v_m \rangle \right] + 2E \left[ \langle u^n, v_m \rangle \right] \]
\[ \leq E \left[ \| u^n_0 - u_0 \|_2^2 \right] + 2 \| u_0 \|_2^2 + 2E \left[ \| u^n_0 - u_0 \|_2^2 \right] + \left[ \| u_0 \|_2^2 + 2E \left[ \| u^n_0 - u_0 \|_2^2 \right] + 2 \| \bar{u}_0^m - u_0 \| \right] \]
\[ + \nu t E \left[ \sum_k \| \sigma_k \|_2^2 - 2E \left[ \langle u^n, \bar{u}^m - v_m \rangle \right] + 2E \left[ \langle u^n, v_m \rangle \right] \right] \]
\[ \| v_m \|_{L_2(D)} \leq K_m \delta^{1/2} \]
\[ E \left[ \| u^n_0 - u_0 \|_2^2 \right] + 2 \| u_0 \|_2^2 + K E \left[ \| u^n_0 - u_0 \|_2^2 \right] \]
\[ + K \| u^n_0 - u_0 \| + \| \bar{u}_0^m - u_0 \| \]
\[ + K \| \bar{u}_0^m - u_0 \| \]
\[ + K \nu t \left[ \| u^n_0 - u_0 \|_2^2 \right] + 2 \| u_0 \|_2^2 + K E \left[ \| u^n_0 - u_0 \|_2^2 \right] + K \| u^n_0 - u_0 \| \]
\[ + K \nu t \left[ \| u^n_0 - u_0 \|_2^2 \right] + 2 \| u_0 \|_2^2 + K E \left[ \| u^n_0 - u_0 \|_2^2 \right] \]
\[ \| v_m \|_{L_2(D)} \leq K_m \delta^{1/2} \]

To analyze the second-last term we use the weak formulation of \( u^n \), taking \( \bar{u}^m - v_m \) as test function. We take directly the expected value of the weak formulation. Exploiting the relation
\[ E \left[ \langle u^n_0, \bar{u}_0^m \rangle \right] = \| u_0 \|_2^2 + E \left[ \langle u^n_0 - u_0, \bar{u}_0^m - u_0 \rangle \right] + E \left[ \langle u^n_0 - u_0, u_0 \rangle \right] + \langle u_0, \bar{u}_0^m - u_0 \rangle , \]
we get
\[ -2E \left[ \langle u^n(t), (u^m - v_m)(t) \rangle \right] + 2E \left[ \| u^n_0 \|_2^2 \right] = -2E \left[ \langle u^n_0 - u_0, \bar{u}_0^m - u_0 \rangle \right] - 2E \left[ \langle u^n_0 - u_0, u_0 \rangle \right] \]
\[ -2 \langle u^n_0, \bar{u}_0^m - u_0 \rangle + 2E \left[ \langle u^n_0, v_m(0) \rangle \right] \]
\[ -2E \left[ \int_0^t \langle u^n(s), \partial_s (\bar{u}^m - v_m)(s) \rangle \right] \]
\[ + 2 \nu E \left[ \int_0^t \langle (-A)^{\frac{1}{2}} u^n(s), (-A)^{\frac{1}{2}} (\bar{u}^m - v_m)(s) \rangle \right] \]
\[ - \nu E \left[ \int_0^t 2b(u^n(s), (\bar{u}^m - v_m)(s), u^n(s)) \right] \]
\[ \| v_m \|_{L_2(D)} \leq K_m \delta^{1/2} \]
\[ 2 \| u^n_0 - u_0 \| \left[ \| u^n_0 - u_0 \|_2^2 \right] \]
\[ + 2 \| u_0 \| \left[ \| u^n_0 - u_0 \|_2^2 \right] + 2 \| u_0 \|_2 \left[ \| u^n_0 \|_2 \right] + 2E \left[ \| u^n_0 \|_2^2 \right] + K \| u^n_0 - u_0 \| \]
\[ - 2E \left[ \langle u^n(s), \partial_s (\bar{u}^m - v_m)(s) \rangle \right] \]
\[ + 2 \nu E \left[ \int_0^t \langle (-A)^{\frac{1}{2}} u^n(s), (-A)^{\frac{1}{2}} (\bar{u}^m - v_m)(s) \rangle \right] \]
\[ - \nu E \left[ \int_0^t 2b(u^n(s), (\bar{u}^m - v_m)(s), u^n(s)) \right] \].

Moreover
\[ -E \left[ \langle u^n(s), \partial_s (\bar{u}^m - v_m)(s) \rangle \right] \]
\[ \| \partial_t v_m(1) \|_{L_2(D)} \leq K_m \delta^{1/2} \]
\[ = E \left[ \langle u^n(s), \partial_s \bar{u}^m(s) \rangle \right] \]
\[ \leq E \left[ \langle u^n(s), \nabla (\bar{u}^m(s)) \rangle \right] + K_m \delta^{1/2} .
\]

Thanks to previous relations and noting that
\[ b(u^n, \bar{u}^m, u^n) - b(u^n, \bar{u}^m, \bar{u}^m) = b(u^n, \bar{u}^m, u^n - \bar{u}^m) \]
we can continue the estimate of \( \mathbb{E} \left[ \| u^n - \bar{u}^m \| \right]^2 \):

\[
\mathbb{E} \left[ \| u^n - \bar{u}^m \| \right]^2 \leq K_m \delta + 2K \mathbb{E} \left[ \| u^n_0 - u_0 \| \right]^2 + \| u^n_0 - u_0 \| K \| \bar{u}^m_0 - u_0 \| + K_n
\]

Arguing as in the proof of Proposition 13, we have

\[
\mathbb{E} \left[ \int_0^T b(u^n, v_m, u^n) \, ds \right] \| \nabla v_m(t) \|_{L^\infty} \leq K_m \delta \quad K_m \delta \mathbb{E} \left[ \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)} \, ds \right]
\]

\[
\| \nabla\bar{u}^m(t) \|_{L^2(D)} \leq K_m \delta^{-1/2} \mathbb{E} \left[ \int_0^T \| \nabla u^n \|_{L^2(D)} \, ds \right] + \mathbb{E} \left[ \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)} \, ds \right].
\]

Thanks to these relations, we can continue the estimate of \( \mathbb{E} \left[ \| u^n - \bar{u}^m \| \right]^2 \):

\[
\mathbb{E} \left[ \| u^n - \bar{u}^m \| \right]^2 \leq K_m \delta + 2K \mathbb{E} \left[ \| u^n_0 - u_0 \| \right]^2 + \| u^n_0 - u_0 \| K \| \bar{u}^m_0 - u_0 \| + K_n
\]

\[
+ 2\| \nabla u^n \|_{L^\infty(0, T; L^2(D))} \nu_n \mathbb{E} \left[ \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)} \, ds \right] + \mathbb{E} \left[ \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)} \, ds \right].
\]

Taking \( \delta = c\nu_n \), by Gronwall’s inequality and Holder’s inequality, we have

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \| u^n - \bar{u}^m \| \right]^2 \leq (K_m \nu_n^{1/2} + K \| u^n_0 - u_0 \| \right)^2 + \| u^n_0 - u_0 \| K \| \bar{u}^m_0 - u_0 \| + K_n
\]

\[
+ 2\| \nabla u^n \|_{L^\infty(0, T; L^2(D))} \nu_n \mathbb{E} \left[ \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)} \, ds \right] + \mathbb{E} \left[ \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)} \, ds \right]^{1/2}
\]

\[
+ K_m \mathbb{E} \left[ \nu_n \mathbb{E} \left[ \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)} \, ds \right] \right]^{1/2} + \mathbb{E} \left[ \int_0^T \| \nabla u^n \|_{L^2(D)} \, ds \right]^{1/2}. \]

Taking the limsup with respect to \( n \) of this expression for \( m \) fixed, we have

\[
\limsup_{n \to +\infty} \sup_{t \in [0, T]} \mathbb{E} \left[ \| u^n - \bar{u}^m \| \right]^2 \leq K \| u^n_0 - u_0 \| e^{2T \| \nabla u^n \|_{L^\infty(0, T; D)}} \quad \text{Remark 15} \quad K \leq \frac{K_m}{m} e^{-T \| \nabla u^n \|_{L^\infty(0, T; D)}}, \tag{7}
\]

12
Coming back to
\[ \mathbb{E} \left[ \| u^n(t) - u(t) \|^2 \right] \leq \frac{2}{m^2} + 2 \mathbb{E} \left[ \| u^n(t) - \bar{u}^m(t) \|^2 \right]. \]
If we fix \( \epsilon > 0 \) and \( m \) such that \( \frac{2 - \epsilon}{m^2} < \epsilon \), then taking the limsup with respect to \( n \) of previous expression for \( m = m \) we have
\[
\limsup_{n \to +\infty} \mathbb{E} \left[ \| u^n(t) - u(t) \|^2 \right] \leq \epsilon.
\]
We have the thesis from the arbitrariness of \( \epsilon \).

**Proof of Theorem 9** Let \( \{ \bar{u}^n_m \}_{m \in \mathbb{N}} \) approximating \( u_0 \) in the sense of Theorem 9 and \( \{ \bar{u}_m \}_{m \in \mathbb{N}} \) the corresponding solutions of the Euler equations, then for each \( t, n, m \) we have
\[
\| u^n(t) - u(t) \|^2 \leq 2\| u^n(t) - \bar{u}^m(t) \|^2 + 2\| \bar{u}^m(t) - u(t) \|^2 \leq \frac{2}{m^2} + 2\| u^n(t) - \bar{u}^m(t) \|^2.
\]
We adapt the computations of the proof of Corollary 13 and Lemma 10 to analyze the last term, hence some explanation will be omitted. For each \( m \) and \( \delta > 0 \) fixed, let us introduce the corrector of the boundary layer \( v_m \), it satisfies previous estimates. We have at time \( t \)
\[
\| u^n - \bar{u}^m \|^2 = \| u^n \|^2 + \| \bar{u}^m \|^2 - 2\langle u^n, \bar{u}^m \rangle \leq \| u^n \|^2 + \| v_m \|^2 + \frac{2}{m^2} + 2\| u^n \|^2 - 2\| u^n - \bar{u}^m \|^2 + 2\| v_m \|^2 - 2\| u^n - \bar{u}^m \|^2.
\]
Let us rewrite \( \langle u^n, v_m \rangle \) thanks to the weak formulation of \( u^n \)
\[
-2\langle u^n, v_m \rangle = -2\langle u^n, (\bar{u}^m - v_m)(0) \rangle - 2\int_0^t \langle u^n(s), \partial_t (\bar{u}^m - v_m)(s) \rangle \, ds
\]
\[
+ 2\sqrt{\nu_n} \sum_{k=1}^N \langle \sigma_k, (\bar{u}^m - v_m)(s) \rangle W_k^t + 2\sqrt{\nu_n} \sum_{k=1}^N \int_0^t \langle \sigma_k, (\bar{u}^m - v_m)(s) \rangle W_k^t \, ds.
\]
Moreover
\[
-2\langle u^n(s), \partial_t (\bar{u}^m - v_m)(s) \rangle = 2\langle u^n(s), \partial_t v_m(s) \rangle + 2\langle u^n(s), \nabla \bar{u}^m \cdot \bar{u}^m \rangle.
\]
Thanks to previous relations and noting that
\[
b(u^n, \bar{u}^m, u^n) - b(u^n, v_m, u^n) = b(u^n - \bar{u}^m, \bar{u}^m, u^n - \bar{u}^m)
\]
we have
\[
\| u^n - \bar{u}^m \|^2 = \| u^n \|^2 + \| \bar{u}^m \|^2 - 2\langle u^n, (\bar{u}^m - v_m)(0) \rangle + (\nu_n) \sum_{k=1}^N \| \sigma_k \|^2 + 2\sqrt{\nu_n} \sum_{k=1}^N \int_0^t \langle u^n(s), \sigma_k \rangle \, dW_k^t
\]
\[
- 2\sqrt{\nu_n} \sum_{k=1}^N \langle \sigma_k, (\bar{u}^m - v_m)(t) \rangle W_k^t + 2\sqrt{\nu_n} \sum_{k=1}^N \int_0^t \langle \sigma_k, (\bar{u}^m - v_m)(s) \rangle W_k^t \, ds + (2\int_0^t b(u^n, v_m, u^n)(s) \, ds - 2\int_0^t b(u^n - \bar{u}^m, \bar{u}^m, u^n - \bar{u}^m)(s) \, ds + (-2\langle u^n, v_m \rangle + 2\sqrt{\nu_n} \int_0^t ((-A)\frac{1}{2} u^n(s) - (A)\frac{1}{2} (\bar{u}^m - v_m)(s)) \, ds + 2\int_0^t \langle u^n, \partial_t v_m \rangle \, ds)
\]
\[
= I_1(t) + I_2(t) + I_3(t).
\]
Thus
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| u^n - \bar{u}^m \| \right] \leq \mathbb{E} \left[ \sup_{t \in [0,T]} I_1 \right] + \mathbb{E} \left[ \sup_{t \in [0,T]} I_2 \right] + \mathbb{E} \left[ \sup_{t \in [0,T]} I_3 \right] + \mathbb{E} \left[ \sup_{t \in [0,T]} I_4 \right]
\]
13
\[ E \left[ \sup_{t \in [0,T]} I_1 \right] = E \left[ \|u_0^m\|^2 + \|\bar{u}_0^m\|^2 - 2\langle u_0^m, (\bar{u}^m - v_m)(0) \rangle \right] \]
\[ = -E \|u_0^m\|^2 + 2E \|u_0^m - \bar{u}_0^m\|^2 + 2E \|u_0^m - v_m(0)\|^2 \]
\[ \leq -E \|u_0^m - u_0^m\|^2 - \|u_0^m\|^2 - 2E \left[ \|u_0^m - u_0\| + \|u_0 - \bar{u}_0^m\| \right] + 2E \left[ \|u_0^m\| \|\bar{u}_0^m - u_0^m\| \right] + 2E \|u_0^m\| \|v_m(0)\| \]
\[ \|v_m(t)\|_{L^2(D)} \leq K_m \theta^\frac{1}{2} \left[ E \|u_0^m - u_0\|^2 + K_E \|u_0^m - u_0\|^2 \right]^{1/2} + \|\bar{u}_0^m - u_0\|^2 + K \|\bar{u}_0^m - u_0\| + K_m \delta^\frac{1}{2} \left[ \|u_0^m\|^2 \right]^{1/2}. \]

The analysis of the others is similar to what we have done before:

\[ E \left[ \sup_{t \in [0,T]} I_2 \right] \leq 2E \left[ \sup_{t \in [0,T]} \int_0^t b(u^n, v^n, u^n)(s) \, ds \right] + 2E \left[ \sup_{t \in [0,T]} \int_0^t b(u^n - \bar{u}^m, \bar{u}^m, u^n - \bar{u}^m)(s) \, ds \right] \]
\[ \leq 2E \left[ \int_0^T |b(u^n, v^n, u^n)(s)| \, ds \right] + 2E \left[ \int_0^T |b(u^n - \bar{u}^m, \bar{u}^m, u^n - \bar{u}^m)(s)| \, ds \right] \]
\[ \leq 2E \left[ \int_0^T \|\rho^2 \nabla v_m(s)\|_{L^2(D)} \|u^n\|_{L^2(D)} \|u^n - \bar{u}^m\|^2(s) \, ds \right] + 2E \left[ \int_0^T \|\bar{u}^m(s)\|_{L^2(D)} \|\nabla \bar{u}^m(s)\|_{L^2(D)} \, ds \right] \]
\[ \leq 2K_m \delta^\frac{1}{2} \left[ \int_0^T \|\nabla u^n(s)\|^2_{L^2(D)} \, ds \right] + 2\|\bar{u}^m\|_{L^2(D)} E \left[ \int_0^T \|u^n - \bar{u}^m\|^2(s) \, ds \right] \]

Let us analyze all the elements of \( I_2 \) exploiting previous energy equalities and properties of Brownian motion:

\[ E \left[ \sup_{t \in [0,T]} TV_m \sum_{k=1}^N \|\sigma_k\|^2 \right] < K_V \]

\[ E \left[ \sup_{t \in [0,T]} 2 \sqrt{\nu_m} \sum_{k=1}^N \langle \sigma_k, \bar{u}^m - v_m(t) \rangle W^h_k \right] \]
\[ \leq K \sqrt{\nu_m} \left\| \bar{u}^m - v_m \right\|_{L^2(D)} \left[ E \left[ \sup_{t \in [0,T]} \left\| W^h_k(t) \right\|^2 \right] \right] \]
\[ \leq K \sqrt{\nu_m} \left\| \bar{u}^m - v_m \right\|_{L^2(D)} \]

\[ E \left[ \sup_{t \in [0,T]} 2 \sqrt{\nu_m} \sum_{k=1}^N \int_0^t \langle \sigma_k, u - v(s) \rangle W^h_k(s) \, ds \right] \leq K \sqrt{\nu_m} \left\| \bar{u}^m - v_m \right\|_{L^2(D)} \]

It remains only to analyze \( I_4 \). Some of the estimates below use tricks already presented, hence some details have been omitted.

\[ 2\nu_m E \left[ \sup_{t \in [0,T]} \int_0^t \langle (A)^\frac{1}{2} u^n, (A)^{-\frac{1}{2}} (\bar{u}^m - v_m) \rangle \, ds \right] \leq 2\nu_m E \left[ \int_0^T \|\nabla \bar{u}^m\|_{L^2(D)} \|\nabla u^n(s)\| \, ds \right] \]
\[ + 2\nu_m E \left[ \int_0^T K_m \delta^\frac{1}{2} \|\nabla u^n(s)\|^2_{L^2(D)} \, ds \right] \]

\[ E \left[ \sup_{t \in [0,T]} \langle u^n, v_m \rangle \right] \leq E \left[ \sup_{t \in [0,T]} \left\| u^n \right\| \sup_{t \in [0,T]} \left\| v_m \right\| \right] \leq K_m \delta^\frac{1}{2} E \left[ \sup_{t \in [0,T]} \left\| u^n \right\|^2 \right]^{1/2} \]
\[ \leq K_m \delta^\frac{1}{2}. \]
In conclusion, if we take $\Delta = \epsilon v$, then by Holder’s inequality

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} \|u^n - \bar{u}^m\|^2 \right] \leq \mathbb{E}\left[ \|u^n_0 - u_0\|^2 \right] + K\mathbb{E}\left[ \|u^n_0 - u_0\|^2 \right]^{1/2} + \left\|\bar{u}^m - u_0\right\| + K\left\|\bar{u}^m - u_0\right\|
$$

$$
+ K\sqrt{n} + K\sqrt{\nu_n} + K\sqrt{\nu_n}\|\bar{u}^m - u_0\|_{L^\infty([0,T], L^2(D))}
$$

$$
+ 2K\nu_n \mathbb{E}\left[ \int_0^T \|\nabla u^n(s)\|_{L^2(D \setminus \bar{D}_n)}^2 \, ds \right] + 2\|\nabla v^n\|_{L^\infty([0,T] \times D)} \mathbb{E}\left[ \int_0^T \|u^n - \bar{u}^m(s)\|^2 \, ds \right]^{1/2}
$$

$$
+ \|\nabla \bar{u}^m\|_{L^\infty([0,T], L^2(D))} \nu_n \int_0^T \|\nabla u^n(s)\|_{L^2(D)}^2 \, ds
$$

$$
+ 2K\left[ \int_0^T \nu_n \|\nabla u^n\|_{L^2(D \setminus \bar{D}_n)}^2 \, ds \right]^{1/2} + K\nu_n.
$$

Taking the limsup with respect to $n$ of this expression for $m$ fixed we have

$$
\limsup_{n \to +\infty} \mathbb{E}\left[ \sup_{t \in [0,T]} \|u^n - \bar{u}^m\|^2 \right] \leq \left\|\bar{u}^m_0 - u_0\right\|^2 + K\left\|\bar{u}^m_0 - u_0\right\|
$$

$$
+ 2T\|\nabla \bar{u}^m\|_{L^\infty([0,T] \times D)} \limsup_{n \to +\infty} \left( \sup_{t \in [0,T]} \mathbb{E}\left[ \|u^n - \bar{u}^m\|^2 \right] \right)
$$

$$
\leq \left|\nu \bar{u}^m_0 - u_0\right|^2 + K\left|\bar{u}^m_0 - u_0\right| + K\left\|\bar{u}^m_0 - u_0\right\| + \frac{K}{m}.
$$

Coming back to

$$
\|u^n(t) - u(t)\|^2 \leq \frac{2}{m^2} + 2\|u^n(t) - \bar{u}^m(t)\|^2.
$$

If we fix $\epsilon > 0$ and $\bar{m}$ such that

$$
2\|\bar{u}^m_0 - u_0\|^2 + 2K\|\bar{u}^m_0 - u_0\| + 2\frac{K\bar{m} + 1}{m^2} < \epsilon,
$$

then taking the expected value of the supremum in time of the previous expression for $m = \bar{m}$ we have

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} \|u^n(t) - u(t)\|^2 \right] \leq \frac{2}{m^2} + 2\mathbb{E}\left[ \sup_{t \in [0,T]} \|u^n(t) - \bar{u}^m(t)\|^2 \right].
$$

Taking the limsup with respect to $n$ of the last inequality we have

$$
\limsup_{n \to +\infty} \mathbb{E}\left[ \sup_{t \in [0,T]} \|u^n(t) - u(t)\|^2 \right] < \epsilon.
$$

We have the thesis from the arbitrariness of $\epsilon$. ■

5 A Deterministic Remark

As anticipated in Remark 11 in this section we prove an inviscid limit result in the deterministic framework, analogous to Theorem 9 for a particular class of external forces. This result extends the setting considered by Kato in 12 and it is the object of Theorem 19.

Lemma 17. If $u$ is a weak solution of the Euler equations with initial condition $u_0 \in H$ and external force $f \in L^2(0, T; H)$ and $\bar{u}$ is the unique weak solution of the Euler equations with initial condition $\bar{u}_0 \in H \cap C^{1,s}(\bar{D})$ and external force $f \in L^2(0, T; H) \cap C^{1,s}((0, T) \times \bar{D})$, then

$$
\|u - \bar{u}\|_{L^2(0, T, H)}^2 \leq \epsilon^2 \|\nabla u\|_{L^\infty((0, T) \times \bar{D})}^2 \|u_0 - \bar{u}_0\|^2
$$

$$
+ 2\sqrt{T} \|f\|_{L^2(0, T, H)} \left( \sqrt{2}\|u_0\|^2 + 4T\|f\|_{L^2(0, T, H)}^2 + \sqrt{2}\|\bar{u}_0\|^2 + 4T\|\bar{f}\|_{L^2(0, T, H)}^2 \right).
$$

15
For each $K \geq 1$, calling

$$O^K_n = \{ u_0 \in H, f \in L^2(0, T; H): \exists \bar{u}_0 \in H \cap C^{1, \alpha}(D), f \in L^2(0, T; H) \cap C^{1, \alpha}([0, T] \times D),$$

$$\|u_0 - \bar{u}_0\| < \frac{1}{n} e^{-KT\|\nabla u\|_{L^{\infty}([0, T] \times D)}}, \|f - \bar{f}\| < \frac{1}{n} e^{-2KT\|\nabla u\|_{L^{\infty}([0, T] \times D)}} \}$$

where $\bar{u}$ is the solution of the Euler equations with initial condition $\bar{u}_0$ and external force $\bar{f}$, then for each $(u_0, f) \in \cap_{n \geq 1} O^K_n =: O^K$ there exists a unique $u \in C([0, T], H)$ weak solution of the Euler equations with initial condition $u_0$ and external force $f$. Moreover the energy equality

$$\|u(t)\|^2 = \|u_0\|^2 + 2 \int_0^t \langle f, u \rangle \, ds$$

holds.

**Proof.**

Estimate: For what concern the solution with smooth initial condition and external force, thanks to Theorem 5, $\bar{u}$ is a classical solution and the energy equality holds, namely

$$\|\bar{u}(t)\|^2 = \|\bar{u}(0)\|^2 + 2 \int_0^t \langle \bar{f}, \bar{u} \rangle \, ds \leq \|\bar{u}(0)\|^2 + 2 \int_0^t \|f\|\|\bar{u}\| \, ds \leq \|\bar{u}(0)\|^2 + 2T \|f\|^2 \|\bar{u}\| L_{x=0,T,H} + \frac{\|\bar{u}\|^2_{L^2(0,T,H)}}{2}.$$  

Thus

$$\sup_{t \in [0, T]} \|\bar{u}\|^2 \leq 2\|\bar{u}_0\|^2 + 4T\|f\|^2 L^2(0,T,H).$$  

(8)

The same estimate holds for any weak solution of the Euler equations (if it exists) with initial condition $u_0$ and external force $f$ thanks to energy inequality and the same computations. Let us consider the weak formulation satisfied by $\bar{u}$ using as test function $\tilde{u}$. Then at time $t$

$$\langle u, \tilde{u} \rangle = \langle u_0, \tilde{u}_0 \rangle + \int_0^t \langle u, \partial_t \tilde{u} \rangle \, ds + \int_0^t \langle b(u, \tilde{u}, u) \rangle \, ds + \int_0^t \langle f, \tilde{u} \rangle \, ds$$

$$= \langle u_0, \tilde{u}_0 \rangle + \int_0^t \langle b(u - \bar{u}, \bar{u}, u - \bar{u}) \rangle \, ds + \int_0^t \langle f, \tilde{u} \rangle \, ds + \int_0^t \langle \tilde{f}, u \rangle \, ds$$

$$\leq \langle u_0, \tilde{u}_0 \rangle + \|\tilde{u}\|_{L^{\infty}(0,T;L^{\infty}(D))} \int_0^t \|u - \tilde{u}\|^2 \, ds + \int_0^t \langle \tilde{f}, u \rangle \, ds + \int_0^t \langle \tilde{f}, \tilde{u} \rangle \, ds.$$

Thus

$$\|u - \bar{u}\|^2 = \|u\|^2 + \|\bar{u}\|^2 - 2\langle u, \bar{u} \rangle$$

$$\leq \|u_0 - \tilde{u}_0\|^2 + 2 \int_0^t \langle f, \tilde{u} \rangle \, ds + \|u_0\|^2 + 2 \int_0^t \langle f, u \rangle \, ds - 2\langle u, \bar{u} \rangle$$

$$\leq \|u_0 - \tilde{u}_0\|^2 + 2 \|\tilde{u}\| L^{\infty}(0,T;L^{\infty}(D)) \int_0^t \|u - \tilde{u}\|^2 \, ds + 2 \int_0^t \|u - \bar{u}\| f - \bar{f} \, ds$$

$$+ 2\sqrt{T} \|f - \bar{f}\|_{L^2(0,T,H)} \left( \|u\|_{L^\infty(0,T,H)} + \|\bar{u}\|_{L^\infty(0,T,H)} \right)$$

$$\leq \|u_0 - \tilde{u}_0\|^2 + 2 \|\tilde{u}\| L^{\infty}(0,T;L^{\infty}(D)) \int_0^t \|u - \tilde{u}\|^2 \, ds$$

$$+ 2\sqrt{T} \|f - \bar{f}\|_{L^2(0,T,H)} \left( \sqrt{2\|u_0\|^2 + 4T\|f\|^2 L^2(0,T,H)} + \sqrt{2\|\tilde{u}_0\|^2 + 4T\|f\|^2 L^2(0,T,H)} \right).$$

Thus

$$\|u - \bar{u}(t)\|^2 \leq e^{2\|\tilde{u}\|_{L^{\infty}(0,T;L^{\infty}(D))} \|u_0 - \tilde{u}_0\|^2} + 2\sqrt{T} \|f - \bar{f}\|_{L^2(0,T,H)} \left( \sqrt{2\|u_0\|^2 + 4T\|f\|^2 L^2(0,T,H)} + \sqrt{2\|\tilde{u}_0\|^2 + 4T\|f\|^2 L^2(0,T,H)} \right).$$
Existence: Let \( (u_0, f) \in \mathcal{O}_K \) and \( \{(\tilde{u}_n^m, \tilde{f}^m)\}_{n \in \mathbb{N}} \) a sequence which approximates \((u, f)\) in the sense of the theorem, namely \( \tilde{u}_0^m \in H \cap C^{1,\alpha}(D), \tilde{f}^m \in L^2(0,T; H) \cap C^{1,\alpha}([0,T] \times D) \) and

\[
\|u_0 - \tilde{u}_n^m\| < \frac{1}{n} e^{-K'T\|\nabla \tilde{u}^m\|_{L^\infty([0,T] \times D)}},
\]

\[
\|f - \tilde{f}^m\|_{L^2(0,T;H)} < \frac{1}{m} e^{-4T\|\nabla \tilde{u}^m\|_{L^\infty([0,T] \times D)}},
\]

where \( \tilde{u}^m \) is the solution of the Euler equations with initial condition \( \tilde{u}_0^m \) and external force \( \tilde{f}^m \). We will prove that \( \{\tilde{u}^m\} \) is a Cauchy sequence in \( C(0,T; H) \) and the solution of the Euler equations with initial condition \( u_0 \) and external force \( f \) is unique.

Preliminarily, note that if \( \|a - b\|^2 \leq \alpha \), then \( \|a\|^2 \leq 4\|b\|^2 + \frac{4}{\alpha} \).

For what concern uniqueness, if \( u^1 \) and \( u^2 \) are two solutions of the Euler equations with initial condition \( u_0 \) and external force \( f \), then at time \( t \)

\[
\|u^1(t) - u^2(t)\|^2 \leq 2\|u^1(0) - \tilde{u}_0^m\|^2 + 2\|u^2(0) - \tilde{u}_0^m\|^2
\]

\[
\leq 4e^{2\|\nabla \tilde{u}^m\|_{L^\infty([0,T] \times D)}(\|u_0 - \tilde{u}_0^m\|^2 + \|f\|_{L^2(0,T;H)})} + 2\sqrt{T}\|f - \tilde{f}^m\|_{L^2(0,T;H)} \left(\sqrt{2\|u_0\|^2 + 4T\|f\|^2_{L^2(0,T;H)}} + \sqrt{2\|u_0^m\|^2 + 4T\|f^m\|^2_{L^2(0,T;H)}}\right)
\]

\[
\leq \frac{1}{n^2} \left(1 + 2\sqrt{T} \left(\sqrt{2\|u_0\|^2 + 4T\|f\|^2_{L^2(0,T;H)}} + \sqrt{8\|u_0\|^2 + 16T\|f\|^2_{L^2(0,T;H)}} + \frac{8 + 16T}{\alpha}\right)\right).
\]

From the last inequality the uniqueness of the solution is evident. Lastly let us consider \( \|\tilde{u}^n - \tilde{u}^m\|^2 \) for \( n \geq m \). We have

\[
\|\tilde{u}^n - \tilde{u}^m\|^2 \leq e^{2T\|\nabla \tilde{u}^m\|_{L^\infty([0,T] \times D)}}(\|\tilde{u}_0^n - \tilde{u}_0^m\|^2 + 2\|u^m(0) - u^n(0)\|^2)
\]

\[
+ 2\sqrt{T}\|f^n - f^m\|_{L^2(0,T;H)} \left(\sqrt{2\|u_0^n\|^2 + 4T\|f^n\|^2_{L^2(0,T;H)}} + \sqrt{2\|u_0^m\|^2 + 4T\|f^m\|^2_{L^2(0,T;H)}}\right)
\]

\[
\leq e^{2T\|\nabla \tilde{u}^m\|_{L^\infty([0,T] \times D)}}(2\|\tilde{u}_0^n - u^n(0)\|^2 + 2\|u^n(0) - u^n(0)\|^2)
\]

\[
+ 2\sqrt{T}\|f^n - f^m\|_{L^2(0,T;H)} \left(\sqrt{2\|u_0^n\|^2 + 4T\|f^n\|^2_{L^2(0,T;H)}} + \sqrt{2\|u_0^m\|^2 + 4T\|f^m\|^2_{L^2(0,T;H)}}\right)
\]

\[
+ 2\sqrt{T}\|f^n - f^m\|_{L^2(0,T;H)} \left(\sqrt{2\|u_0^n\|^2 + 4T\|f^n\|^2_{L^2(0,T;H)}} + \sqrt{2\|u_0^m\|^2 + 4T\|f^m\|^2_{L^2(0,T;H)}}\right)
\]

\[
\leq C(T,\|u_0\|,\|f\|_{L^2(0,T;H)}) \left(\frac{1}{n^2} + \frac{1}{m^2}\right).
\]

The last inequality implies existence.

Energy: Let \( (u_0, f) \in \mathcal{O}_K \) and \( \{(\tilde{u}_n^m, \tilde{f}^m)\}_{n \in \mathbb{N}} \) a sequence which approximates \((u, f)\) in the sense of the theorem like in the previous step. Then for each \( n \in \mathbb{N} \)

\[
\|\tilde{u}^n(t)\|^2 = \|\tilde{u}_0^n\|^2 + \int_0^t \langle \tilde{u}^n(s), \tilde{f}^m(s) \rangle \, ds.
\]

Exploiting the fact that \( \tilde{u}^n \in C([0,T]; H), \tilde{f}^m \in L^2(0,T; H) \cap C^{1,\alpha}([0,T] \times D) \)

\[
\|u_0 - \tilde{u}_0^m\| < \frac{1}{m} e^{-4T\|\nabla \tilde{u}^m\|_{L^\infty([0,T] \times D)}},
\]

\[
\|f - \tilde{f}^m\|_{L^2(0,T;H)} < \frac{1}{m} e^{-4T\|\nabla \tilde{u}^m\|_{L^\infty([0,T] \times D)}},
\]

where \( \tilde{u}^m \) is the solution of the Euler equations with initial condition \( \tilde{u}_0^m \) and external force \( \tilde{f}^m \).
Remark 18 If \((u_0, f) \in \bar{O}\) and \(\{(\tilde{u}_0^m, \tilde{f}^m)\}_{m \in \mathbb{N}}\) approximates \((u_0, f)\) in the sense of Theorem [17] then
\[
\|\tilde{u}_0^m - u_0\| \leq e^{2T\|\nabla \tilde{u}_0^m\|_{L^\infty(\Omega \times I)}} \left( \|u_0 - \tilde{u}_0^m\|^2 + 2\sqrt{T}\|f - \tilde{f}^m\|_{L^2(0, T; H)} \left( \sqrt{2\|u_0\|^2 + 4T\|f\|_{L^2(0, T; H)}^2} + \sqrt{2\|\tilde{u}_0^m\|^2 + 4T\|\tilde{f}^m\|_{L^2(0, T; H)}^2} \right) \right) \\
\leq \frac{1}{m^2} \left( 1 + 2\sqrt{T} \left( \sqrt{2\|u_0\|^2 + 4T\|f\|_{L^2(0, T; H)}^2} + \sqrt{8\|u_0\|^2 + 16T\|f\|_{L^2(0, T; H)}^2 + \frac{8 + 16T}{3}} \right) \right) \\
\leq \frac{K(T, \|u_0\|, \|f\|_{L^2(0, T; H)})}{m^2}.
\]
Thanks to Lemma [17] we are able to prove a Kato type inviscid limit result also in the deterministic framework.

Theorem 19 If \((u_0, f) \in \bar{O}, u_0^m \in H, f^m \in L^2(0, T; H),\)
\[
\lim_{n \to +\infty} \|u_0^m - u_0\| = 0, \quad \lim_{n \to +\infty} \|f^m - f\|_{L^2(0, T; H)} = 0.
\]
Let \(u\) be the solution of the Euler equations with initial condition \(u_0\) and external force \(f\), \(w^m\) be the solution of the deterministic Navier-Stokes equations with
viscosity \(\nu_n\), initial condition \(u_0^m\) and external force \(f^m\). If
\[
\lim_{n \to +\infty} \nu_n = 0, \quad \lim_{n \to +\infty} \nu_n \int_0^T \|\nabla w^m(t)\|_{L^2(I, \nu_n)}^2\ dt = 0,
\]
then
\[
\lim_{n \to +\infty} \sup_{t \in [0, T]} \|u^n - u\|^2 = 0.
\]
Proof. The proof is an adaptation of previous stochastic arguments, the only novelty is the presence of deterministic external forces. Hence we just give details on the new elements.

Let \(\{(u_0^m, f^m)\}_{m \in \mathbb{N}}\) approximating \((u_0, f)\) in the sense of Theorem [17] and \(\{\tilde{u}_m\}_{m \in \mathbb{N}}\) the corresponding solutions of the Euler equations, then for each \(n, m\) we have
\[
\|u^n(t) - u(t)\|^2 \leq 2\|u^n(t) - \tilde{u}_m(t)\|^2 + 2\|\tilde{u}_m(t) - u(t)\|^2
\]
\[
\leq K \|u^n - \tilde{u}_m\|^2 + 2\|u^n - u\|^2 + 2\|\tilde{u}_m - u\|^2
\]

Remark 19 \[
\|u^n(t) - u(t)\|^2 \leq K \frac{n^2}{m^2} + 2\|u^n - u\|^2 + 2\|\tilde{u}_m - u\|^2.
\]

For each \(m\) and \(\delta > 0\) fixed, let us introduce the corrector of the boundary layer \(v_m\), it satisfies previous estimates. We have at time \(t\)
\[
\|u^n - \tilde{u}_m\|^2 = \|u^n\|^2 + \|\tilde{u}_m\|^2 - 2\langle u^n, \tilde{u}_m \rangle
\]
\[
\|u^n\|^2 \leq \|u^n - \tilde{u}_m\|^2 + 2\|u^n\|^2 + K\|u^n - u_0\|^2 + \|\tilde{u}_m - u_0\|^2 + K\|\tilde{u}_m - u_0\|^2
\]
\[
+ 2\int_0^t \langle f^n, u^n \rangle\ ds + 2\int_0^t \langle \tilde{f}^m, \tilde{u}_m \rangle\ ds - 2\langle u^n, \tilde{u}_m - v_m \rangle + K\|\tilde{u}_m\|^2.
\]
To analyze the second-last term we use the weak formulation of \(u^n\), taking \(\tilde{u}_m - v_m\) as test function. Exploiting the relation
\[
\langle u^n, \tilde{u}_m \rangle = \langle u^n, u_0 \rangle, \quad \langle u^n - u_0, \tilde{u}_m - u_0 \rangle + \langle u^n - u_0, u_0 \rangle + \langle u_0, \tilde{u}_m - u_0 \rangle,
\]
we get
\[
eg \langle u^n(t), (u^n - v_m)(t) \rangle + 2\|u^n\|^2 \leq \|v_m\|_{L^2(D)} \leq K m^{-\frac{1}{2}}\|u^n - u_0\|^2 + K\|\tilde{u}_m - u_0\|^2 + K m^{-\frac{1}{2}}\|v_m\|_{L^2(D)} \leq K m^{-\frac{1}{2}}\|u^n - u_0\|^2 + K\|\tilde{u}_m - u_0\|^2 + K m^{-\frac{1}{2}}\|v_m\|_{L^2(D)} \leq 2\int_0^t \langle u^n(s), \partial_s (u^n - v_m)(s) \rangle\ ds
\]
\[
+ 2\nu_n \int_0^t \langle (-A)^{\frac{1}{2}} u^n(s), (-A)^{\frac{1}{2}} (u^n - v_m)(s) \rangle\ ds
\]
\[
- \int_0^t 2b(u^n(s), (u^n - v_m)(s), u^n(s))\ ds
\]
\[
- 2\int_0^t \langle f^n, (u^n - v_m) \rangle\ ds.
\]
Moreover
\[ -\langle u^n(s), \partial_s(\bar{u}^m - v_m)(s) \rangle \leq K_m \delta^\frac{1}{2} + K \| u^n_0 - u_0 \|^2 + \| \bar{u}^m_0 - u_0 \|^2 + K \| \bar{u}^m - u_0 \| + K \| u^n_0 - u_0 \| \]
\[ = \nu_n \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)}^2 \] ds

Thanks to previous relations and noting that
\[ b(u^n, \bar{u}^m, u^n) - b(u^n, \bar{u}^m, \bar{u}^m) = b(u^n - \bar{u}^m, \bar{u}^m, u^n - \bar{u}^m) \]
we can continue the estimate of \( \| u^n - \bar{u}^m \|^2 \):
\[ \| u^n - \bar{u}^m \|^2 \leq K_m \delta^\frac{1}{2} + K \| u^n_0 - u_0 \|^2 + \| \bar{u}^m_0 - u_0 \|^2 + K \| \bar{u}^m - u_0 \| + K \| u^n_0 - u_0 \| \]
\[ + 2 \int_0^T b(u^n, v_m, u^n) \] ds + \[ 2 \nu_n \int_0^T \| \nabla \bar{u}^m \|_{L^\infty([0,T] \times D)} \int_0^T \| u^n - \bar{u}^m \|^2 \] ds
\[ + 2 \int_0^T (f^n - \bar{f}^m, u^n - \bar{u}^m) \] ds.

Arguing as in the stochastic case, we have
\[ \int_0^T b(u^n, v_m, u^n) \] ds \leq K_m \delta \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)}^2 \] ds
\[ + \nu_n \int_0^T \| \nabla \bar{u}^m \|_{L^\infty([0,T] \times D)} \int_0^T \| u^n - \bar{u}^m \|^2 \] ds.

For what concern the new term
\[ \int_0^T (f^n - \bar{f}^m, u^n - \bar{u}^m) \] ds \leq \int_0^T \| f^n - \bar{f}^m \| \| u^n - \bar{u}^m \| \] ds
\[ \leq \sqrt{T} \| f^n - \bar{f}^m \|_{L^2(0,T; H)} (\| u^n \|_{L^\infty(0,T; H)} + \| \bar{u}^m \|_{L^\infty(0,T; H)}) \]
\[ \leq K \star \nu_n \int_0^T \| \nabla \bar{u}^m \|_{L^\infty(0,T; L^2(\Gamma_s))} \int_0^T \| u^n - \bar{u}^m \|^2 \] ds.

Thanks to this relations we can continue the estimate of \( \| u^n - \bar{u}^m \|^2 \):
\[ \| u^n - \bar{u}^m \|^2 \leq K_m \delta^\frac{1}{2} + K \| u^n_0 - u_0 \|^2 + \| \bar{u}^m_0 - u_0 \|^2 + K \| \bar{u}^m - u_0 \| + K \| u^n_0 - u_0 \| \]
\[ + K_m \delta \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)}^2 \] ds + \[ K \star \nu_n \int_0^T \| \nabla \bar{u}^m \|_{L^\infty(0,T; L^2(\Gamma_s))} \int_0^T \| u^n - \bar{u}^m \|^2 \] ds
\[ + 2 \int_0^T \| \nabla \bar{u}^m \|_{L^\infty(0,T; L^2(\Gamma_s))} \int_0^T \| u^n - \bar{u}^m \|^2 \] ds.

Taking \( \delta = \epsilon \nu_n \), by Gronwall’s inequality and Holder’s inequality we have
\[ \sup_{t \in [0,T]} \| u^n - \bar{u}^m \|^2 \leq \left( K_m \epsilon \nu_n^2 \right)^{\frac{1}{2}} + K \| u^n_0 - u_0 \|^2 + \| \bar{u}^m_0 - u_0 \|^2 + K \| \bar{u}^m - u_0 \| + K \| u^n_0 - u_0 \| \]
\[ + K_m \epsilon \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)}^2 \] ds + \[ K \star \nu_n \int_0^T \| \nabla \bar{u}^m \|_{L^\infty(0,T; L^2(\Gamma_s))} \int_0^T \| u^n - \bar{u}^m \|^2 \] ds
\[ + K_m \left( \epsilon \nu_n \int_0^T \| \nabla u^n \|_{L^2(\Gamma_s)}^2 \right) \] ds. 
\[ e^{2T \epsilon \nu_n \| \nabla \bar{u}^m \|_{L^\infty(0,T; L^2(\Gamma_s))}}. \]
Taking the limsup with respect to $n$ of this expression for $m$ fixed we have
\[
\limsup_{n \to +\infty} \sup_{t \in [0,T]} \|u^n - \bar{u}^m\|^2 \leq \left( K \|u^n_0 - u_0\| + \|\bar{u}^m_0 - u_0\| \right)^2 + \|f^m - f\|_{L^2(0,T;H)} e^{2T\|\nabla u^m\|_{L^\infty([0,T \times D])}} \tag{9}
\]
\[
\leq \frac{K}{m} + \frac{K}{m^2}. \tag{10}
\]
Coming back to
\[
\|u^n(t) - u(t)\|^2 \leq \frac{K}{m^2} + 2\|u^n(t) - \bar{u}^m(t)\|^2.
\]
If we fix $\epsilon > 0$ and $\bar{m}$ such that $\frac{\sqrt{2K}}{\bar{m}} < \epsilon$, then taking the limsup with respect to $n$ of previous expression for $m = \bar{m}$ we have
\[
\limsup_{n \to +\infty} \|u^n(t) - u(t)\|^2 \leq \epsilon.
\]
We have the thesis from the arbitrariness of $\epsilon$. $\blacksquare$

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