Electromagnetic fields in the exterior of an oscillating relativistic star – I. General expressions and application to a rotating magnetic dipole

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ABSTRACT

Relativistic stars are endowed with intense electromagnetic fields but are also subject to oscillations of various types. Here we investigate the impact that oscillations have on the electric and magnetic fields external to a relativistic star in vacuum. In particular, modelling the star as a relativistic polytrope with infinite conductivity, we consider the solution of the general relativistic Maxwell equations both in the vicinity of the stellar surface and far from it, once a perturbative velocity field is specified. In this first paper, we present general analytic expressions that are not specialized to a particular magnetic field topology or velocity field. However, as a validating example and an astrophysically important application, we consider a dipolar magnetic field and the velocity field corresponding to the rotation of the misaligned dipole. Besides providing analytic expressions for the electromagnetic fields produced by this configuration, we calculate, for the first time, the general relativistic energy loss through dipolar electromagnetic radiation. We find that the widely used Newtonian expression underestimates this loss by a factor of 2–6 depending on the stellar compactness. This correction could have important consequences in the study of the spin evolution of pulsars.

Key words: MHD – relativity – waves – stars: neutron – stars: oscillations – pulsars: general.

1 INTRODUCTION

The study of the electrodynamics of relativistic stars when these are undergoing normal-mode oscillations has the prospect of providing several pieces of information that are relevant not only for the astrophysics but also for the physics of these objects. Astronomical observations indicate, in fact, that compact relativistic stars are endowed with extremely intense magnetic fields, which reach surface strengths of the order of $\lesssim 10^{10}$ G in older neutron stars associated with recycled pulsars and low-mass X-ray binaries. Observations of young neutron stars, on the other hand, show surface magnetic fields that are much stronger and usually of the order of $10^{11}$–$10^{13}$ G. In addition to this, the phenomenology associated with soft gamma repeaters and anomalous X-ray pulsars suggests that the surface magnetic fields can become even stronger and up to $10^{14}$–$10^{15}$ G (Mereghetti & Stella 1995; Kouveliotou et al. 1998, 1999), where they could be produced through efficient dynamo processes (Thompson & Duncan 1995; Bonanno, Rezzolla & Urpin 2003).

However, intense electromagnetic fields are not the only important feature of compact objects. The same astronomical observations that indicate the presence of such fields, in fact, also make it manifest that these are associated with objects that are very compact and hence with strong gravitational fields. As a result, an accurate description of the electrodynamics of compact objects can only be made within a framework in which the relativistic corrections are properly accounted for. The literature investigating electromagnetic ‘test fields’ in curved space–times (i.e. fields whose energy density is small when compared with the average rest-mass energy density) is rather extensive and has considered both spherically symmetric space–times (Ginzburg & Ozernoy 1964; Anderson & Cohen 1970) and also slowly rotating space–times (Muslimov & Tsygan 1992; Muslimov & Harding 1997; Konno & Kojima 2000). More recently, Rezzolla, Ahmedov & Miller (2001a,b) have studied the interior and vacuum exterior electromagnetic fields produced by a rotating dipole moment comoving with the star and started a systematic approach to provide analytic expressions for the regions of space–time near the stellar surface (i.e. the so-called near zone). These results have also been extended beyond the low-frequency approximation by Kojima, Matsunaga & Okita (2003), who have resorted to a numerical approach to find solutions to the Maxwell equations that are approximate, but valid also at large distances (i.e. the so-called wave zone).

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If it is natural to expect strong electromagnetic fields in the vicinity of relativistic stars, it is equally natural to expect that these stars too, just like ordinary stars, will be subject to a variety of oscillation modes that become manifest through the emission of both gravitational waves and electromagnetic waves. Because both of these observational windows may soon provide data that could reach us simultaneously, they offer unique opportunities to test the physics and internal structure of these objects as well as the properties of matter at nuclear densities. The investigation of the electromagnetic fields produced by an oscillating magnetized star is, however, a rather complex problem, and this is testified by the scarce literature produced so far in this context. Exceptions in this sense are, to the best of our knowledge, the works of McDermott et al. (1984) and Muslimov & Tsygan (1986). While McDermott et al. have modelled the electric fields produced in the wave zone by a neutron star subject to both toroidal and spheroidal modes to calculate the electromagnetic damping of these oscillations, Muslimov & Tsygan have instead computed the exact analytical solutions for the electromagnetic fields produced, both in the near zone and in the wave zone, by an oscillating neutron star with a dipolar surface magnetic field. In both cases the neutron star was considered in vacuum and, for simplicity, the stellar magnetic field was considered unperturbed. More recently, Timokhin, Bisnovatyi-Kogan & Spruit (2000) have dropped the assumption of a vacuum region outside the stellar surface and performed an extensive investigation of the electrodynamics of the magnetosphere when this is perturbed as a result of toroidal oscillations.

An aspect that these three works share is the use of a Newtonian description of gravity and, as we have argued above, this is expected to be a poor approximation in the vicinity of the stellar surface. The aim of the present work is to lay out a general formalism for the calculation of electromagnetic ‘test fields’ in the curved space–time of a spherical relativistic star subject to perturbations. We do this by extending the approach developed for the electrodynamics of rotating relativistic stars (Rezzolla et al. 2001a,b) to include the possibility that the conducting crust may have a velocity field as a result of the stellar oscillations. As in our previous investigations (Rezzolla et al. 2001a,b), our main aim here is to provide analytic expressions in a form that is as simple as possible and can therefore be used also in astrophysical applications. In this first paper we will concentrate on the development of the mathematical formalism and in particular on the solution in the two distinct regions of the space for which exact solutions can be found, i.e. near the stellar surface where the relativistic corrections are most important (near zone), and at very large distances from it where the space–time approaches a flat one (wave zone). In both cases we find that two main corrections emerge in the expressions for the relativistic electromagnetic fields, and these are associated (i) to the amplification of the electromagnetic fields on the stellar surface produced by the local non-zero space–time curvature, and (ii) to the gravitational redshift which will affect the electromagnetic waves as these propagate in the curved space–time.

The expressions for the components of the electromagnetic fields presented here for the two regimes are completely general and do not refer to a specific magnetic field topology or to a precise velocity field. The application of these solutions to the case of those spheroidal and toroidal velocity fields most frequently encountered in the study of stellar oscillations will be presented in a companion paper (Rezzolla & Ahmedov, in preparation; hereafter Paper II). Here, however, to validate the expressions derived and to provide a comparison with known results, we consider a background magnetic dipole and the simplest of the perturbation velocity field topology or to a precise velocity field. The application of these solutions to the case of those spheroidal and toroidal velocity fields most frequently encountered in the study of stellar oscillations will be presented in a companion paper (Rezzolla et al. 2001a) to include the possibility that the conducting crust may have a velocity field as a result of the stellar oscillations. As in our previous investigations (Rezzolla et al. 2001a,b), our main aim here is to provide analytic expressions in a form that is as simple as possible and can therefore be used also in astrophysical applications. In this first paper we will concentrate on the development of the mathematical formalism and in particular on the solution in the two distinct regions of the space for which exact solutions can be found, i.e. near the stellar surface where the relativistic corrections are most important (near zone), and at very large distances from it where the space–time approaches a flat one (wave zone). In both cases we find that two main corrections emerge in the expressions for the relativistic electromagnetic fields, and these are associated (i) to the amplification of the electromagnetic fields on the stellar surface produced by the local non-zero space–time curvature, and (ii) to the gravitational redshift which will affect the electromagnetic waves as these propagate in the curved space–time.

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2 MAXWELL EQUATIONS IN A SCHWARZSCHILD SPACE–TIME

The investigation of the response of a relativistic and magnetized star upon a generic perturbation would require, in principle, the use of the coupled system of the Maxwell and Einstein equations. Such a formulation, however, is overly complicated for most astrophysical applications, and a few reasonable assumptions can be made to simplify the treatment to a more tractable form. The first of these assumptions concerns the corrections to the Einstein field equations introduced through the mass–energy contribution of the electromagnetic fields inside and outside the relativistic star. It is not difficult to show that these electromagnetic corrections are proportional to the electromagnetic energy density and are rather small in most compact and magnetized stars. Indeed, if \( \rho_\text{el} \) is the average rest-mass density of a star of mass \( M \) and radius \( R \) as
measured at infinity, these corrections are at most
\[
\frac{B^2}{8\pi G m(r)} \lesssim 2.2 \times 10^{-7} \left( \frac{B}{10^{15} \text{ G}} \right)^2 \left( \frac{1.4 M_\odot}{M} \right) \left( \frac{R}{15 \text{ km}} \right)^3,
\]
when compared with the total rest-mass energy density. Hereafter we will assume these corrections to be negligible for the stellar objects in which we are interested and thus treat the electromagnetic fields as ‘test fields’ in a given curved background. Other authors, interested also in the structural modifications produced in very highly magnetized stars (Bocquet et al. 1995; Konno, Obata & Kojima 1999, 2000; Oron 2002; Ioka & Sasaki 2004), have preferred not to make this approximation and have instead included the magnetic contribution to the space–time curvature. The second simplifying assumption is that we will also consider negligible the corrections produced by a global rotation of the space–time and induced by the rotation of the compact star (these corrections were discussed by Rezzolla et al. 2001a,b). The numerical analyses carried out by Geppert, Page & Zannias (2000) and Zanotti & Rezzolla (2002) show that this is a rather good approximation in most cases of astrophysical relevance.

As a result of these two assumptions, we can work in the background space–time of a non-rotating star, whose line element in a spherical coordinate system \((t, r, \theta, \phi)\) is given by
\[
ds^2 = g_{00}(r) \, dt^2 + g_{11}(r) \, dr^2 + r^2 \sin^2 \theta \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2.
\]
The portion of the space–time external to the star (i.e. for \(r \geq R\)) is simply given by the Schwarzschild solution with \(-g_{00} = N^2 = (1 - 2M/r)\) and \(g_{11} = 1/g_{00}\). For the portion of the space–time interior to the star, on the other hand, the metric functions can be specified in terms of two potentials \(A(r)\) and \(\Phi(r)\) so that
\[
g_{00} = -e^{2\Phi(r)}, \quad g_{11} = e^{2A(r)} = \left[ 1 - \frac{2m(r)}{r} \right]^{-1},
\]
where \(\rho(r)\) is the total energy density and \(m(r) = 4\pi \int_0^r r^2 \rho(r) \, dr\) its coordinate volume integral. The precise form of these potentials is obtained through the familiar solution of the Einstein equations for relativistic spherical stars, i.e. the TOV equations (Shapiro & Teukolsky 1983), and the metric functions are then matched continuously to the external Schwarzschild space–time so that
\[
-g_{00}(r=R) = N_R^2 = 1 - \frac{2M}{R} \quad \text{and} \quad g_{11}(r=R) = \frac{1}{N_R^2}.
\]
Within the external portion of this background space–time we select a family of static observers with four velocity components
\[
(u^\alpha)_{\text{obs}} = N^{-1}(1, 0, 0, 0), \quad (u^\alpha)_{\text{obs}} = N(-1, 0, 0, 0),
\]
and associate to them orthonormal frames with tetrad four vectors \(e_\beta = (e_0, e_r, e_\theta, e_\phi)\) and 1-forms \(\omega^\beta = (\omega^0, \omega^r, \omega^\theta, \omega^\phi)\), whose components are
\[
e_0^\beta = \frac{1}{N}(1, 0, 0, 0), \quad \omega_\theta^0 = N(1, 0, 0, 0),
\]
\[
e_r^\beta = N(0, 1, 0, 0), \quad \omega_\phi^r = \frac{1}{N}(0, 1, 0, 0),
\]
\[
e_\theta^\beta = \frac{1}{r}(0, 0, 1, 0), \quad \omega_\phi^\theta = r(0, 0, 1, 0),
\]
\[
e_\phi^\beta = \frac{1}{r \sin \theta}(0, 0, 0, 1), \quad \omega_\theta^\phi = r \sin \theta(0, 0, 0, 1),
\]
and which will become useful when determining the ‘physical’ components of the electromagnetic fields.

The relations among the electric \(E^\alpha\) and magnetic \(B^\alpha\) four vector fields measured by an observer with four velocity \(u^\alpha\) can be expressed through the electromagnetic field tensor \(F_{\alpha\beta}\),
\[
F_{\alpha\beta} = 2u_\alpha E_\beta + \eta_{\alpha\beta\gamma\delta} u^\gamma B^\delta,
\]
where \(T_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})\) and \(\eta_{\alpha\beta\gamma\delta}\) is the pseudo-tensorial expression for the Levi–Civita symbol \(\epsilon_{\alpha\beta\gamma\delta}\).

The first pair of general relativistic Maxwell equations
\[
3F_{[\alpha\beta\gamma]} = 2(F_{\alpha\beta\gamma} + F_{\alpha\gamma\beta} + F_{\gamma\beta\alpha}) = 0,
\]
can then be projected along the tetrad four vectors carried by the proper observers and expressed in terms of the ‘physical’ electric and magnetic three vectors \(\vec{E}\) and \(\vec{B}\) (Thorne, Price & Macdonald 1986) as
\[
\vec{\nabla} \times \vec{B} = 0, \quad \partial_t \vec{B} = -\vec{\nabla} \times (N \vec{E}),
\]
where \(\vec{\nabla}\) represents the covariant derivative with respect to the spatial part of the metric (2). Note that equations (12) are not valid in a space–time admitting a rotational Killing vector (e.g. such as the one produced by a rotating relativistic star), and the general form they assume in that case is given in appendix D.

Once expressed in their ‘hatted’ component form, equations (12) simply become
\[
\sin \theta \partial_\theta(r^2 B^\theta) + \frac{r}{N} \partial_\phi \sin \theta B^\phi + \frac{r}{N} \partial_\phi B^\phi = 0,
\]
\[
\partial_r B^r = 0,
\]
\[
\partial_\theta B^\theta = 0,
\]
\[
\partial_\phi B^\phi = 0.
\]

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\[
\left( \frac{r \sin \theta}{N} \right) \partial_t B^\theta = \partial_\phi E^\phi - \partial_\phi (\sin \theta E^\phi),
\]
\[
\left( \frac{r \sin \theta}{N} \right) \partial_t B^\phi = -\partial_\phi E^\theta + \sin \theta \partial_\phi (rNE^\phi),
\]
\[
\left( \frac{r}{N} \right) \partial_t B^\theta = -\partial_\phi (rNE^\phi) + \partial_\phi E^\theta.
\]

Indicating now with \( \mathbf{u} \) the conductor four velocity and with \( \rho \) the proper charge density, \( \rho \mathbf{u}^\alpha \) will represent the convection current, and the second pair of Maxwell equations is then written as
\[
F^\alpha\beta = 4\pi J^\alpha = 4\pi (\rho \mathbf{u}^\alpha + j^\alpha),
\]
with the semicolon indicating the covariant derivative with respect to the metric (2) and \( J \) being the total electric charge current. Also these equations can be written in terms of the physical electric and magnetic vectors as
\[
\vec{\nabla} \cdot \vec{E} = 4\pi \rho, \quad \partial_t \vec{E} + 4\pi \vec{J} = -\vec{\nabla} \times (N \vec{B}),
\]
or, in component form, as
\[
N \sin \theta \partial_\phi (r^2 E^\phi) + r \partial_\phi (\sin \theta E^\phi) + r \partial_\phi E^\phi = 4\pi \rho \sin \theta,
\]
\[
\partial_\phi (\sin \theta B^\phi) - \partial_\phi B^\phi = \left( \frac{r \sin \theta}{N} \right) \partial_t E^\theta + 4\pi r \sin \theta J^\theta,
\]
\[
\partial_\phi B^\phi - \sin \theta \partial_\phi (rNB^\phi) = \left( \frac{r \sin \theta}{N} \right) \partial_t E^\phi + 4\pi r \sin \theta J^\phi,
\]
\[
\partial_\phi (rNB^\phi) - \partial_\phi B^\phi = \left( \frac{r}{N} \right) \partial_t E^\phi + 4\pi J^\phi.
\]
Note that the total electric charge current in equations (10) includes the conduction current \( j^\alpha \), which is associated with electrons having electrical conductivity \( \sigma \), and that Ohm’s law can be written as
\[
j_\alpha = \sigma F_{\alpha\beta} \mathbf{u}^\beta.
\]

So far, we have considered the star to be static and in equilibrium, but we can now introduce a fluid perturbation in terms of a four velocity \( \delta \mathbf{u} \), so that the fluid velocity is in general \( \mathbf{u} \equiv \delta \mathbf{u} \), where the \( \delta \) indicates that this is an Eulerian perturbation. The components of the velocity perturbation are then
\[
\delta \mathbf{u}^\alpha = \Gamma (1, \delta v^\alpha) = \Gamma \left( 1, e^{-\lambda} \frac{\delta v^\phi}{r}, \frac{\delta v^\theta}{r \sin \theta} \right)
\]
and
\[
\delta \mathbf{u}_d = \Gamma \left( -e^{2\phi}, e^{\lambda} \partial_\phi v^\theta, r \partial_\phi v^\phi, r \sin \theta \partial_\phi \delta v^\phi \right),
\]
where \( \delta v^\alpha \equiv \partial \mathbf{x}^\alpha / \partial t \) is the oscillation three velocity of the conducting stellar medium and \( \delta v^\phi, \delta v^\theta \) are the components of the oscillation three velocity in the orthonormal frame carried by the static observers in the stellar interior. Because we are interested in small velocity perturbations for which \( \delta v^\phi \ll 1 \), we can neglect terms \( \mathcal{O}(\delta v^3) \) and use the normalization for the four velocity \( w_\alpha w^\alpha = -1 \) to obtain
\[
\Gamma = \left[ -g_{00} \left( 1 + \frac{\delta v^\phi x^\phi}{g_{00}} \right) \right]^{-1/2} \geq e^{-\Phi}.
\]

Note that \( \delta \mathbf{u} \) is the only perturbation that we need to consider here. While, in fact, other perturbations (e.g. in pressure and in energy density) may be present and influence the stellar structure, these will not affect the induction equation, which will depend linearly on \( \delta \mathbf{u} \) only (cf. equations 10). Furthermore, for simplicity we will assume that \( \delta \mathbf{u} \) is a given, generic function and is not necessarily the solution of an eigenvalue problem (this was considered by Messios, Papadopoulas & Stergioulas 2001). Finally, because we treat the electromagnetic fields as ‘test fields’, we will not consider any perturbation to the space–time metric (2), and therefore work within the so-called Cowling approximation.

Note that the introduction of perturbations in the velocity will produce important modifications in the electric and magnetic fields in three different spatial regions: (i) the stellar interior, \( r < R \); (ii) the near zone, i.e. a vacuum region outside the stellar surface with radial extension \( r \gtrsim R \); and (iii) the wave zone, i.e. far away from the star, at \( r \gg R \), where the perturbations propagate as electromagnetic waves. Each of these modifications has observable consequences and the following sections are dedicated to discussing two of these regions for the general case of arbitrary velocity perturbations and generic magnetic field topology.

\[1\] We indicate with a tilde the 1-form basis of the static observers inside the star to distinguish this from the corresponding basis given in (6).
3 ELECTROMAGNETIC FIELDS IN THE STELLAR INTERIOR

We here concentrate on the modifications in the internal electromagnetic fields produced by velocity perturbations of different types. While not directly observable, these electromagnetic fields, and in particular the values they assume on the stellar surface, leave an important imprint on the electromagnetic waves produced by the oscillations and that will reach infinity.

We here start by using Ohm’s law (23) and the four velocity (24), to obtain the explicit components of the total current $J^\alpha$ as

$$J^1 = \rho_0 c^2 \delta v^1 + \sigma [E^1 + c^{-2} (\delta v^\phi B^\phi - \delta v^\phi B^\phi)],$$

(27)

$$J^\phi = \rho_0 c^2 \delta v^\phi + \sigma [E^\phi + c^{-2} (\delta v^r B^r - \delta v^r B^r)],$$

(28)

$$J^\theta = \rho_0 c^2 \delta v^\theta + \sigma [E^\theta + c^{-2} (\delta v^r B^r - \delta v^r B^r)].$$

(29)

A convenient way of simplifying the problem is that of considering the fluid as perfectly conducting, i.e. with $\sigma \to \infty$ (this is the limit of ideal magnetohydrodynamics). While idealized, this approximation is a rather good one in the present case, since the Ohmic diffusion time-scale is several orders of magnitude larger than the typical time-scale for the stellar oscillations. Once this assumption is made, the electric field in the interior of the star can be easily derived from (27)–(29) to be

$$E^\text{in}_1 = -c^{-2} (\delta v^\phi B^\phi - \delta v^\phi B^\phi),$$

(30)

$$E^\text{in}_\phi = -c^{-2} (\delta v^r B^r - \delta v^r B^r),$$

(31)

$$E^\text{in}_\theta = -c^{-2} (\delta v^r B^r - \delta v^r B^r),$$

(32)

where we have indicated with an index ‘int’ the field components for $r \leq R$. The Maxwell equations (13)–(16) combined with the expressions (30)–(32) will lead to the following set of magnetic induction equations:

$$(r \sin \theta) \partial_r B^\text{in}_1 = \sigma_0 (\delta v^r B^r_{\text{int}} - \delta v^r B^r_{\text{int}}) + \partial_\phi \left[ \sin \theta \left( \delta v^\phi B^\phi_{\text{int}} - \delta v^\phi B^\phi_{\text{int}} \right) \right],$$

(33)

$$(r \sin \theta) \partial_r B^\text{in}_\phi = \sigma_0 (\delta v^\phi B^\phi_{\text{int}} - \delta v^\phi B^\phi_{\text{int}}) - \sin \theta c^{-2} \partial_\phi \left[ \delta v^r B^r_{\text{int}} - \delta v^r B^r_{\text{int}} \right],$$

(34)

$$r \partial_\phi B^\text{in}_1 = \sigma_0 \left[ \left( \delta v^\phi B^\phi_{\text{int}} - \delta v^\phi B^\phi_{\text{int}} \right) \right] - \partial_\phi \left( \delta v^r B^r_{\text{int}} - \delta v^r B^r_{\text{int}} \right).$$

(35)

The boundary conditions for the magnetic field across the stellar surface $r = R$ can then be obtained after requiring continuity for the normal (i.e. $r$) component,

$$B^\text{ext}_{\text{int}}|_{r=R} = B^\text{in}_r,$$

(36)

while leaving the tangential (i.e. $\theta$ and $\phi$) components free to be discontinuous through surface currents,

$$B^\text{ext}_{\text{int}}|_{r=R} = B^\text{in}_\theta + 4\pi i^\phi,$$

(37)

$$B^\text{ext}_{\text{int}}|_{r=R} = B^\text{in}_\phi - 4\pi i^\phi,$$

(38)

where $i^\phi$ are the components of the surface current and $B^\phi_{\text{ext}} \equiv B^\text{in}_{\text{int}}|_{r=R}$ (the index ‘ext’ has been here used to refer to fields at $r \geq R$). In a similar way, the boundary conditions for the electric field across the stellar surface can be derived from imposing the continuity of the tangential components, while leaving the normal one free to be discontinuous through a surface charge distribution $\Sigma_s$. Simple algebra then gives

$$E^\text{ext}_{\text{int}}|_{r=R} = E^\text{in}_r|_{r=R} + 4\pi \Sigma_s = \frac{1}{N_R} \left( \delta v^\phi B^\phi_{R} - \delta v^\phi B^\phi_{R} \right) + 4\pi \Sigma_s,$$

(39)

$$E^\phi_{\text{ext}}|_{r=R} = E^\text{in}_\phi|_{r=R} = \frac{1}{N_R} (\delta v^\phi B^\phi_{R} - \delta v^\phi B^\phi_{R}),$$

(40)

$$E^\phi_{\text{ext}}|_{r=R} = E^\text{in}_\phi|_{r=R} = \frac{1}{N_R} (\delta v^\phi B^\phi_{R} - \delta v^\phi B^\phi_{R}),$$

(41)

where $\delta v^\phi_{R} \equiv \delta v^\phi|_{r=R}$ are the oscillation velocities at the stellar surface, the only relevant ones in this paper.

4 ELECTROMAGNETIC FIELDS IN THE NEAR ZONE

We now consider the form of the electromagnetic fields generated by stellar oscillations in a region of vacuum space exterior and close to the stellar surface. Our treatment in this section will be as general as possible and we will not restrict it to any specific magnetic field topology or velocity field. A more detailed discussion of the electromagnetic fields produced by those spheroidal and toroidal velocity fields most often discussed in the physics of compact stars will be presented in a companion paper (Paper II).

Given a characteristic frequency for the stellar oscillations $f_0$, the spatial extension of the near zone can then be estimated to be $\lambda \sim c/f_0$, which is the distance travelled by an electromagnetic wave in a time-scale $1/f_0$. For a typical neutron star of mass $M$ and radius $R$, the
fundamental frequency of oscillation is \( f_0 \sim \sqrt{GM/R^3} \sim 2-3 \) kHz, so that the near zone extends from \( R \) to \( \sim 100R \). Since we are neglecting terms \( O(\delta^2) \) in the Maxwell equations, we need not account for the displacement currents in equations (20)-(22) and set
\[
\partial_t E^i = \partial_j E^\phi = \partial_i E^\phi = 0, \tag{42}
\]
so that the second pair of Maxwell equations can be written as
\[
N \sin \theta \, \partial_\phi (r^2 E^\phi) + r \, \partial_\phi (\sin \theta \, E^\phi) + r \, \partial_\phi E^\phi = 0, \tag{43}
\]
\[
\partial_\phi (\sin \theta \, B^\phi) - \partial_\phi B^\phi = 0, \tag{44}
\]
\[
\partial_\phi B^\phi - \sin \theta \, \partial_\phi (r N B^\phi) = 0, \tag{45}
\]
\[
\partial_\phi (N r B^\phi) - \partial_\phi B^\phi = 0. \tag{46}
\]

The solutions to the Maxwell equations with sources represented by a bounded distribution of currents that vary harmonically with time is mostly easily found if the electromagnetic fields are expanded in their multipolar components. An efficient method to obtain these multipolar expansions has been proposed by Bouwkamp & Casimir (1954) in Newtonian electrodynamics and uses the general properties of solenoidal vector fields. Here we follow the same approach but extend it to relativistic electrodynamics. Consider therefore four scalar functions \( S, T, X \) and \( Z \), through which the general solutions to the vacuum Maxwell equations (13)-(16) and (43)-(46) can be written in the form
\[
B^\phi = -\frac{1}{r^2 \sin \theta} \left[ \sin \theta \, \partial_\phi (\sin \theta \, \partial_\phi S) + \partial_\phi^2 S \right], \tag{47}
\]
\[
B^\phi = \frac{N}{r} \partial_\phi S + \frac{1}{N r \sin \theta} \partial_\phi Z, \tag{48}
\]
\[
B^\phi = \frac{N}{r \sin \theta} \partial_\phi S - \frac{1}{N r \sin \theta} \partial_\phi Z \tag{49}
\]
and
\[
E^\phi = -\frac{1}{r^2 \sin \theta} \left[ \sin \theta \, \partial_\phi (\sin \theta \, \partial_\phi T) + \partial_\phi^2 T \right], \tag{50}
\]
\[
E^\phi = \frac{N}{r} \partial_\phi T + \frac{1}{N r \sin \theta} \partial_\phi X, \tag{51}
\]
\[
E^\phi = \frac{N}{r \sin \theta} \partial_\phi T - \frac{1}{N r} \partial_\phi X. \tag{52}
\]

We will refer to the functions \( S, T, X \) and \( Z \) as the ‘magnetic’ and ‘electric’ functions, respectively. Furthermore, to handle the angular derivatives analytically, we will assume that they can be separated in variables and expand the angular dependence in terms of spherical harmonics \( Y_{\ell m}(\theta, \phi) \),
\[
S(t, r, \theta, \phi) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell S_{\ell m}(t, r) Y_{\ell m}(\theta, \phi), \quad X(t, r, \theta, \phi) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell X_{\ell m}(t, r) Y_{\ell m}(\theta, \phi), \tag{53}
\]
\[
T(t, r, \theta, \phi) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell T_{\ell m}(t, r) Y_{\ell m}(\theta, \phi), \quad Z(t, r, \theta, \phi) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell Z_{\ell m}(t, r) Y_{\ell m}(\theta, \phi). \tag{54}
\]

Using this Ansatz and suppressing the summation symbols over the \( \ell, m \) indices, the general solutions (47)-(52) then take the form
\[
B^\phi = \frac{\ell (\ell + 1)}{r^2} \, S_{\ell m} Y_{\ell m}, \tag{55}
\]
\[
B^\phi = \frac{N}{r} \partial_\phi S_{\ell m} \, \partial_\phi Y_{\ell m} + \frac{1}{N r \sin \theta} \, Z_{\ell m} \partial_\phi Y_{\ell m}, \tag{56}
\]
\[
B^\phi = \frac{N}{r \sin \theta} \partial_\phi S_{\ell m} \, \partial_\phi Y_{\ell m} - \frac{1}{N r} \, Z_{\ell m} \partial_\phi Y_{\ell m}, \tag{57}
\]
for the magnetic field components, and
\[
E^\phi = \frac{\ell (\ell + 1)}{r^2} \, T_{\ell m} Y_{\ell m}, \tag{58}
\]
\[
E^\phi = \frac{N}{r} \partial_\phi T_{\ell m} \, \partial_\phi Y_{\ell m} + \frac{1}{N r \sin \theta} \, X_{\ell m} \partial_\phi Y_{\ell m}, \tag{59}
\]
\[
E^\phi = \frac{N}{r \sin \theta} \partial_\phi T_{\ell m} \, \partial_\phi Y_{\ell m} - \frac{1}{N r} \, X_{\ell m} \partial_\phi Y_{\ell m}, \tag{60}
\]
for the electric field components. We next concentrate on how to use the Maxwell equations to recast expressions (55)-(60) in a form more useful for astrophysical applications.
4.1 Magnetic fields

Substituting the general solutions (55)–(57) for the magnetic field into the Maxwell equations (44)–(46) gives two equations for the unknown functions \( Z_{\ell m} \) and \( S_{\ell m} \):

\[
\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta} Y_{\ell m}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} Y_{\ell m} \right] Z_{\ell m} = 0,
\]

\[
dr \left[ \left( 1 - \frac{2M}{r} \right) \frac{d}{dr} S_{\ell m} \right] - \frac{\ell (\ell + 1)}{r^2} S_{\ell m} = 0,
\]

which have the important property of depending on \( Z \) or \( S \) only, so that they can be solved separately. The first equation has indeed the trivial solution \( Z_{\ell m} = 0 \) since the content of the square brackets in (61) is a function of \( \theta \) and \( \phi \) only. The solution of (62), on the other hand, is less straightforward. It is convenient to introduce the new variable

\[ x \equiv 1 - \frac{r}{M} , \]

and factor out an explicit quadratic radial dependence in the functions \( S_{\ell m} \), which are then redefined as

\[ S_{\ell m} \equiv r^2 h(t, r) . \]

Using now (63) and (64), equation (62) takes the form

\[
dx \left\{ \left[ \frac{1 + x}{1 - x} \right] \frac{d}{dx} \left[ (1 - x)^2 h \right] \right\} + \frac{\ell (\ell + 1) h}{x} = 0 ,
\]

which can be solved in terms of Legendre functions of the second kind \( Q_\ell(x) \) (see Rezzolla et al. 2001a, for details), so that

\[ S_{\ell m}(t, r) = \left( 1 - \frac{x^2}{M} \right) \frac{d}{dx} \left[ (1 + x) \frac{dQ_\ell}{dx} \right] s_{\ell m}(t) . \]

Note that all of the time dependence in (66) is contained in the integration constants \( s_{\ell m}(t) \) and these are determined after suitable boundary conditions across the stellar surface are imposed.

We next consider how these expressions vary when a perturbation, caused for instance by a non-zero velocity field (24), is introduced. Hereafter, we will assume that a background magnetic field is present, which varies with time only on a time-scale that is much longer than that of the stellar oscillations and can therefore be considered static. Hereafter we will indicate with \( B_0^c \) the components of this zeroth-order (background) stellar magnetic field and with \( \delta B^c(t) \) the first-order perturbations, which clearly are the ones possessing a time dependence.

Within a linear regime in the perturbations, a convenient way to isolate the perturbed part of the magnetic fields is suggested by the structure of the solutions (55)–(57), which confines all of the time dependence in the \( s_{\ell m} \) terms. As a result, it is possible to express the magnetic field perturbation in terms of new, time-dependent integration constants \( \delta s_{\ell m}(t) \) that simply add to the background ones \( s_{\ell m} \). The new components of the magnetic field generated by a velocity perturbation will therefore have the generic form

\[ B^c = B_0^c + \delta B^c(t) = \frac{\ell (\ell + 1)}{M^3} \frac{d}{dx} \left[ (1 + x) \frac{dQ_\ell}{dx} \right] \left[ s_{\ell m} + \delta s_{\ell m}(t) \right] Y_{\ell m} . \]

\[ B^\theta = B_0^\theta + \delta B^\theta(t) = - \frac{1 + x}{(1 - x)^3} \frac{1}{M^3} \frac{d}{dx} \left[ (1 - x)^2 \frac{d}{dx} \left[ (1 + x) \frac{dQ_\ell}{dx} \right] \left[ s_{\ell m} + \delta s_{\ell m}(t) \right] \right] \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} Y_{\ell m} . \]

\[ B^\phi = B_0^\phi + \delta B^\phi(t) = - \frac{1 + x}{(1 - x)^3} \frac{1}{M^3} \frac{d}{dx} \left[ (1 - x)^2 \frac{d}{dx} \left[ (1 + x) \frac{dQ_\ell}{dx} \right] \left[ s_{\ell m} + \delta s_{\ell m}(t) \right] \right] \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_{\ell m} . \]

The values of the integration constants \( \delta s_{\ell m}(t) \) can be calculated rather straightforwardly if the oscillation modes are assumed to have a simple harmonic time dependence of the type \( \exp(-i\omega t) \), where \( \omega \) is the mode frequency. In this case, in fact, the radial component for the induction equation (33) at the surface of the star becomes

\[ \partial_t \left( \delta B^r_{\ell} \right) = -i \omega \partial_r \delta B^r_{\ell} = \frac{1}{R \sin \theta} \left\{ \partial_\phi \left( \delta v^r \partial_\phi B^\phi_{\ell} - \delta v^\phi \partial_\phi B^r_{\ell} \right) + \partial_\theta \left[ \sin \theta \left( \delta v^r B^\phi_{\ell} - \delta v^\phi B^r_{\ell} \right) \right] \right\} , \]

and the boundary condition for the continuity of the radial magnetic field (36) can then be used to determine the coefficients \( \delta s_{\ell m}(t) \) (see appendix B for details). Note that this procedure is different from the one used by Timokhin et al. (2000), in which the integration constants are instead obtained through the boundary conditions applied to the continuity of the tangential components of the electric field. Of course, the two procedures are equivalent and yield identical results.

A few remarks are worth making at this point. The first one is about the assumption of infinite conductivity for the stellar material, which implies that the magnetic field is advected with the fluid when this possesses a non-zero velocity. This is expressed by the general relativistic “frozen-flux” condition

\[ \partial_t \delta B = \vec{\nabla} \times (\delta \vec{v} \times \vec{B}_0) , \]

Note that the lapse function does not appear multiplying the magnetic field in expression (71). This is because of a cancellation when Ohm’s law is used to replace the contribution to the Maxwell equations coming from the electric field (see Appendix D for the expression of Ohm’s law in terms of physical three vectors).
and is, indeed, verified by expressions (67)–(69). The second one is that these expressions, although interesting for their generality, are not particularly useful if we do not restrict our attention to a specific background magnetic field configuration; this will be done in Section 6.1 where we concentrate on the specific form that equations (67)–(69) assume for a background static dipolar magnetic field. Finally, note that no reference is made at this point on the type of velocity perturbations introduced, as these are effectively incorporated in the expressions for the functions $\delta x_{\ell m}(t)$; in Sections 6.2 and 6.3 we will discuss the form of these functions for the case of a magnetic dipole in uniform rotation.

### 4.2 Electric fields

We next concentrate on the form of the electric field components and consider for this the solution to the Maxwell equations (14)–(16) and (43). Hereafter we will assume that the star has a zero net electric charge (i.e. $\int_{\Omega} \sqrt{\gamma} d^3 x = 0$, where $\gamma$ is the determinant of the three metric and $d^3 x$ is the coordinate volume element) and that the background magnetic field is stationary (i.e. $\partial_\nu B_\nu^B = 0$). In this case, the background electric fields will be identically zero (i.e. $E_\nu^B = 0$), but time-dependent, first-order electric fields $\delta E^\ell(t)$ may be present and induced by the perturbations in the background magnetic field.

To derive the expressions for these perturbed electric fields, we substitute the decomposed expressions (50)–(52) into the Maxwell equations (14)–(16) and (43) to obtain the following set of equations for the unknown functions $T_{\ell m}$ and $X_{\ell m}$:

$$
\frac{d}{dr} \left( 1 - \frac{2M}{r} \right) \frac{d}{dr} T_{\ell m} - \ell(\ell + 1) \frac{1}{r^2} T_{\ell m} = 0, \tag{72}
$$

$$
\ell(\ell + 1) Y_{\ell m} X_{\ell m} = -r^2 \partial_\ell B^\ell, \tag{73}
$$

$$
\partial_\ell X_{\ell m} \partial_\ell Y_{\ell m} = -\frac{r}{N} \partial_\ell B^\ell, \tag{74}
$$

$$
\partial_\ell X_{\ell m} \partial_\ell Y_{\ell m} = -\frac{r \sin \theta}{N} \partial_\ell B^\ell. \tag{75}
$$

It is apparent that equation (72) is the same as (62) but for the unknown functions $\delta T_{\ell m}(t)$ and the integration constants $\delta T_{\ell m}(t)$:

$$
T_{\ell m} = (1 - x)^2 M^2 \frac{d}{dx} \left[ (1 + x) \frac{dQ_t}{dx} \right] \delta T_{\ell m}(t). \tag{76}
$$

In general, that is for a multipolar magnetic field, the ‘electric’ functions $X_{\ell m}(t)$ are determined directly from the time derivatives of the ‘magnetic’ functions $\delta T_{\ell m}$ (cf. equations 55 and 73), i.e.

$$
X_{\ell m} = -\partial_\ell \delta T_{\ell m} = \frac{(1 - x)^2}{M} \frac{d}{dx} \left[ (1 + x) \frac{dQ_t}{dx} \right] \delta X_{\ell m}(t), \tag{77}
$$

where $\delta X_{\ell m}(t) = -\partial_\ell [\delta T_{\ell m}(t)]$. Expression (77) underlines that the functions $X_{\ell m}$ represent the contribution to the electric field coming from the time variation of the magnetic field, with the coefficients $\delta X_{\ell m}$ being determined, also in this case, through the use of suitable boundary conditions.

Using expressions (76) and (77) and standard recurrence formulae for the Legendre functions that we will recall in Appendix A, the solution for the vacuum electric field (58)–(60) can be written as

$$
\delta E^\ell = \frac{\ell^2(\ell + 1)}{(1 - x)^2} [\ell Q_{\ell - 1} - (\ell + 1 - \ell x) Q_\ell] \delta T_{\ell m}(t) Y_{\ell m}, \tag{78}
$$

$$
\delta E^\delta = \frac{\ell^2(\ell + 1)}{(1 - x)^2} N \frac{1}{1-x} [x Q_{\ell - 1} \delta T_{\ell m}(t) \partial_\ell Y_{\ell m} + \frac{M(1 - x)}{N \sin \theta} \frac{d}{dx} \left[ (1 + x) \frac{dQ_t}{dx} \right] \delta x_{\ell m}(t) \partial_\ell Y_{\ell m}], \tag{79}
$$

$$
\delta E^\beta = \frac{\ell^2(\ell + 1)}{(1 - x)^2} N \sin \theta \frac{1}{x} [x Q_{\ell - 1} \delta T_{\ell m}(t) \partial_\ell Y_{\ell m} - M \frac{1 - x}{N} \frac{d}{dx} \left[ (1 + x) \frac{dQ_t}{dx} \right] \delta x_{\ell m}(t) \partial_\ell Y_{\ell m}], \tag{80}
$$

where, again, $Q_\ell = Q_\ell(x)$. Note how equations (78)–(80) clearly indicate that, given a magnetic field with multipolar components up to the order $\ell$, the corresponding electric field will have multipolar components up to the order $\ell + 1$. This is another aspect of the interconnection between electric and magnetic fields contained in the Maxwell equations.

Finally, the integration constants $\delta t_{\ell m}$ and $\delta x_{\ell m}$ can be calculated from the jump conditions (40)–(41) of the tangential electric field (79)–(80) at the stellar surface and are given by

$$
\delta t_{\ell m}(t) = \frac{(1 - x)^2}{\ell^2(\ell + 1)^2} \frac{1}{x R_\ell Q_{\ell}(x_R) - Q_{\ell - 1}(x_R)}^{-1} \times \int d\Omega \left\{ \partial_\ell Y^*_{\ell m} \left[ \delta v^\ell_{\ell m}(t) B^\ell_R - \delta v^{\beta}_{\ell m}(t) B^\beta_R \right] - i \frac{m Y^*_{\ell m}}{\sin \theta} \left[ \delta v^\delta_{\ell m}(t) B^\delta_R - \delta v^{\beta}_{\ell m}(t) B^\beta_R \right] \right\}, \tag{81}
$$

$$
\delta x_{\ell m}(t) = -\frac{3(1 - x)^2}{8 M f_R} \int d\Omega \left\{ \partial_\ell Y^*_{\ell m} \left[ \delta v^\ell_{\ell m}(t) B^\ell_R - \delta v^{\beta}_{\ell m}(t) B^\beta_R \right] + \frac{m Y^*_{\ell m}}{\sin \theta} \left[ \delta v^\delta_{\ell m}(t) B^\delta_R - \delta v^{\beta}_{\ell m}(t) B^\beta_R \right] \right\}. \tag{82}
$$

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where \( Q_{\ell}(x_\alpha) \equiv Q_{\ell}(1 - R/M) \), \( d\Omega = \sin \theta \, d\theta \, d\phi \) and \( f_R \) is just shorthand notation for
\[
f_R \equiv -\frac{3}{8} \left( \frac{R}{M} \right)^3 \left[ \ln N_R^2 + \frac{2M}{R} \left( 1 + \frac{M}{R} \right) \right] = -\frac{3}{8} \left( \frac{R}{M} \right)^3 \left[ \ln \left( 1 - \frac{2M}{R} \right) + \frac{2M}{R} \left( 1 + \frac{M}{R} \right) \right]. \tag{83}
\]

### 5 ELECTROMAGNETIC FIELDS IN THE WAVE ZONE

The treatment of the electromagnetic fields in the wave zone can start from considering the Maxwell equations (13)–(16) and (19)–(22) in the case when electric charges and currents are absent. In particular, inserting equations (15) and (16) in the time derivative of equation (20) and using the expression for (19) in vacuum (i.e. for \( \rho_c = 0 \)), one obtains the following wave equation for the radial component of the electric field on a curved Schwarzschild background:
\[
\partial_\ell^2 E^\ell - \frac{N^2}{r^2} \partial_\ell \left[ N^2 \partial_\ell (r^2 E^\ell) \right] - \frac{N^2}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\ell E^\ell) - \frac{N^2}{r^2 \sin^2 \theta} \partial_\ell^2 E^\ell = 0. \tag{84}
\]

Substituting the decomposition (58) for the radial component of the electric field in terms of spherical harmonics into the wave equation (84) yields the well-known Regge–Wheeler equation describing the dynamics of vector perturbations in a Schwarzschild space–time (Chandrasekhar 1992):
\[
\left( \partial_\ell^2 - \partial_{\ell m}^2 \right) T_{\ell m} + \ell (\ell + 1) \frac{N^2}{r^2} T_{\ell m} = 0, \tag{85}
\]
where \( r_s \equiv r + 2M \ln(r/2M - 1) \) is the tortoise coordinate. (Similar equations can be obtained also for the other functions \( S_{\ell m}, X_{\ell m} \) and \( Z_{\ell m} \)).

Although equation (85) has been extensively studied in the past and contains information about general relativistic effects such as the scattering off the curved background space–time (see Malec, Murchadha & Chmaj 1998, for a recent investigation), it does not represent the most convenient way to evaluate the components of the magnetic and electric fields in the wave zone. There, in fact, the relativistic corrections \( \mathcal{O}(M/r) \) can be reasonably neglected and we can seek guidance in determining the components of the electromagnetic fields from simpler Newtonian expressions.

We note that assuming a flat background space–time for the form of the electromagnetic fields in the wave zone is clearly an approximation, but it is not in contrast with the general relativistic approach followed so far. As will become clearer in the following, in fact, the final expressions that we derive on a flat background still need to be completed through the specification of suitable boundary conditions to be imposed at the stellar surface and these will indeed contain the general relativistic character of the curved space–time. Furthermore, adopting a Newtonian framework as a guideline is not only a good approximation, but also has two important advantages. The first one is that it is possible to find an analytic expression for the electromagnetic fields which will depend on outgoing spherical Hankel functions and their derivatives (cf. equations 97–102). In a fully general relativistic treatment, instead, the numerical solution of an ordinary differential equation is necessary to obtain the form of the electromagnetic fields in the wave zone (this will be discussed in Paper II). The second advantage is that, in the specific but important application of the present formalism to a rotating magnetic dipole, the well-known Newtonian expressions can find general relativistic corrections that have simple physical interpretations (this will be discussed in Section 6).

We therefore express the electromagnetic fields in the wave zone through two scalar functions \( U \) and \( V \) (also sometimes referred to as the ‘Debye potentials’), whose angular dependence is expanded in a series of spherical harmonics and that have a harmonic time dependence
\[
U(t, r, \theta, \phi) \equiv r U_{\ell m}(r) Y_{\ell m}(\theta, \phi) e^{-i\omega t}, \quad V(t, r, \theta, \phi) \equiv r V_{\ell m}(r) Y_{\ell m}(\theta, \phi) e^{-i\omega t}. \tag{86}
\]

Note that in the curved space–time exterior to the star, the observed frequency for the stellar oscillations will be different for different observers and therefore is a function of position. In particular, if \( k^\alpha \) is the null wavevector associated with the electromagnetic fields in the wave zone and \( u^\alpha \) the four velocity of an observer, then \( \omega \equiv -k^\alpha u_\alpha \) will be the frequency measured by such an observer. Denoting with \( \omega_R \equiv \omega(r = R) \) the angular frequency of oscillation measured by an observer at the stellar surface, the corresponding electromagnetic wave frequency at a generic radial position \( r > R \) will be subject to the standard gravitational redshift and given by (see Appendix C for a somewhat different derivation)
\[
\omega(r) = \omega_R \frac{N_R}{N} = \omega_R \sqrt{\frac{R - 2M}{r - 2M}} \frac{r}{R}. \tag{87}
\]

In the asymptotically flat regions of the space–time where we are considering the wave zone solutions to the Maxwell equations, the electromagnetic waves will have reached their asymptotic form and their frequency can then be considered to be simply \( \omega \simeq \omega_R N_R = \text{constant} \).

Under these assumptions, and following the approach suggested by Bouwkamp & Casimir (1954) and Casimir (1960), the components of the electromagnetic fields in the wave zone assume the generic form
\[
B_{\ell}^t = \frac{1}{r} U_{\ell m} Y_{\ell m} e^{-i\omega t}, \tag{88}
\]
\[
B_{\ell}^i = \frac{1}{r (\ell + 1)} \left[ \partial_\ell (r U_{\ell m}) \partial_\ell Y_{\ell m} - \frac{i\omega}{\sin \theta} r V_{\ell m} \partial_\ell Y_{\ell m} \right] e^{-i\omega t}, \tag{89}
\]
\[ B^\phi = \frac{1}{r(\ell + 1)} \left[ \partial_r (r V_{\ell m}) \frac{1}{\sin \theta} \partial_\theta Y_{\ell m} + i \omega V_{\ell m} \partial_\theta Y_{\ell m} \right] e^{-i\omega t}, \]

and

\[ E^\phi = \frac{1}{r} V_{\ell m} Y_{\ell m} e^{-i\omega t}, \]

\[ E^\phi = \frac{1}{r(\ell + 1)} \left[ \partial_r (r V_{\ell m}) \frac{1}{\sin \theta} \partial_\theta Y_{\ell m} + i \omega r V_{\ell m} \partial_\theta Y_{\ell m} \right] e^{-i\omega t}, \]

\[ E^\phi = \frac{1}{r(\ell + 1)} \left[ \partial_r (r V_{\ell m}) \frac{1}{\sin \theta} \partial_\theta Y_{\ell m} - i \omega r V_{\ell m} \partial_\theta Y_{\ell m} \right] e^{-i\omega t}. \]

It is useful to note that, while no difference would be introduced in the expressions of the radial components of the magnetic and electric fields (i.e. equations 88 and 91), the general relativistic expressions for the remaining components would be corrected by coefficients of the type \( \sim N \) or \( \sim 1/N \), and effectively very small as soon as one considers regions of the space–time away from the stellar surface (Paper II).

In the typical configuration that we intend to consider, which is inspired by astronomical observations, the star is endowed with a background magnetic field that is essentially stationary on the time-scale of the stellar oscillations. The background electric fields will then be either zero or induced by the time variations of the background magnetic field (such as those produced by the stellar oscillations or by its rotation). In this case, no background electromagnetic fields will be present in the wave zone, and expressions (88)–(93) will then effectively refer to the perturbations. As such, they should be denoted with a symbol ‘\( \delta \)’ but, in order to keep the expressions compact and since it is not necessary to distinguish them from the background expressions, the symbol ‘\( \delta \)’ will be omitted hereafter.

Substituting the above expressions (88)–(93) into the Maxwell equations evaluated in a flat space–time, these can be recast into ordinary wave-like equations for the Debye potentials \( U \) (or \( V \)), i.e.

\[ \partial^2 \tilde{U} - \partial^2 \tilde{U} - \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta U) - \frac{1}{r^2 \sin^2 \theta} \partial^2_\theta U = 0, \]

which would instead take the form

\[ \partial^2 \tilde{U} - N^2 (N^2 \partial_\phi \tilde{U}) - N^2 \left[ \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta \tilde{U}) + \frac{1}{r^2 \sin^2 \theta} \partial^2_\theta \tilde{U} \right] = 0 \]

on a Schwarzschild background. Note that, while equations (94) and (95) are not wave equations for the potentials \( U \) and \( V \) (or the corresponding relativistic ones \( \tilde{U} \) and \( \tilde{V} \)), they are so for the potentials \( U_{\ell m} \) and \( V_{\ell m} \) (or the corresponding relativistic ones \( \tilde{U}_{\ell m} \) and \( \tilde{V}_{\ell m} \)).

The main advantage of the use of a flat background space–time for the study of the electromagnetic fields in the wave zone region comes from the fact that, while no analytic solution can be found for equation (95), an analytic solution to equation (94) (regular everywhere except at \( r = 0 \)) can be expressed in terms of outgoing spherical Hankel functions \( H_\ell \omega r \) in the form

\[ U_{\ell m}(r) = [\ell (\ell + 1)]^{1/2} H_\ell (\omega r) u_{\ell m}, \quad V_{\ell m}(r) = -[\ell (\ell + 1)]^{1/2} H_\ell (\omega r) v_{\ell m}, \]

so that the components for the magnetic fields are given by the general expressions

\[ B^\phi = \frac{e^{-i\omega t}}{\sqrt{\ell (\ell + 1)}} H_\ell (\omega r) u_{\ell m} Y_{\ell m}, \]

\[ B^\phi = \frac{e^{-i\omega t}}{\sqrt{\ell (\ell + 1)}} \left[ D H_\ell (\omega r) u_{\ell m} \partial_\theta Y_{\ell m} - \omega H_\ell (\omega r) V_{\ell m} \frac{m Y_{\ell m}}{\sin \theta} \right], \]

\[ B^\phi = \frac{1}{\sqrt{\ell (\ell + 1)}} \left[ D H_\ell (\omega r) u_{\ell m} \frac{m Y_{\ell m}}{\sin \theta} - \omega H_\ell (\omega r) V_{\ell m} \partial_\theta Y_{\ell m} \right], \]

while the electric field components are expressed as

\[ E^\phi = -\frac{e^{-i\omega t}}{\sqrt{\ell (\ell + 1)}} H_\ell (\omega r) v_{\ell m} Y_{\ell m}, \]

\[ E^\phi = -\frac{e^{-i\omega t}}{\sqrt{\ell (\ell + 1)}} \left[ D H_\ell (\omega r) V_{\ell m} \partial_\theta Y_{\ell m} + \omega H_\ell (\omega r) u_{\ell m} \frac{m Y_{\ell m}}{\sin \theta} \right], \]

\[ E^\phi = -i \frac{e^{-i\omega t}}{\sqrt{\ell (\ell + 1)}} \left[ D H_\ell (\omega r) V_{\ell m} \frac{m Y_{\ell m}}{\sin \theta} + \omega H_\ell (\omega r) u_{\ell m} \partial_\theta Y_{\ell m} \right]. \]

Here, we have denoted the radial derivative as \( D H_\ell \omega r \equiv r^{-1} \partial_r [r H_\ell \omega r] \) and we recall that the spherical Hankel functions have a simple radial fall-off in the case of small arguments, i.e.

\[ H_\ell (\omega r) \approx -i(2\ell - 1)!!(\omega r)^{-\ell - 1}, \quad D H_\ell (\omega r) \approx i(2\ell - 1)!!(\omega r)^{-\ell - 2}\omega = -\frac{H_\ell}{r}, \] for \( \omega r \approx \omega_\Omega R \ll 1, \)
while they exhibit a typical oscillatory behaviour (in space) in the limit of large arguments (see, for instance, Arfken & Weber 2001), i.e.
\[ H_x(\omega r) \approx (-i)^{\ell+1} e^{i\omega r} r^{\ell+1}, \quad D H_x(\omega r) \approx (-i)^{\ell} e^{i\omega r} = i \omega \ell, \quad \text{for} \quad \omega r \to \infty. \] (104)

We also note that the Newtonian expressions (97)–(102) are not new and have indeed been discussed by a number of authors before (see, for instance, McDermott et al. 1984). However, new and general relativistic corrections are introduced in equations (97)–(102) once the integration coefficients \( u_{\ell m} \) and \( v_{\ell m} \) are specified through the matching of the electromagnetic fields (97)–(102) at the stellar surface with the help of the boundary conditions (40)–(41):
\[ v_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \mathcal{D} H_x(\omega R) N_R \int d\Omega \left\{ \partial_\theta Y_{\ell}^m \left[ \delta \nu^\theta B_\theta^\theta - \delta v^\theta B_\theta^\theta \right] + \frac{m}{\sin \theta} \frac{Y_{\ell}^m}{\sin \theta} \left[ \delta \nu^\phi B_\theta^\phi - \delta v^\phi B_\theta^\phi \right] \right\}, \] (105)
\[ u_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} H_x(\omega R) N_R \int d\Omega \left\{ i \partial_\phi Y_{\ell}^m \left[ \delta \nu^\phi B_\theta^\phi - \delta v^\phi B_\theta^\phi \right] + \frac{m}{\sin \theta} \frac{Y_{\ell}^m}{\sin \theta} \left[ \delta \nu^\phi B_\theta^\phi - \delta v^\phi B_\theta^\phi \right] \right\}, \] (106)
and of the relation
\[ \int d\Omega \left( \partial_\theta Y_{\ell}^m \partial_\theta Y_{\ell}^m - \frac{m Y_{\ell}^m}{\sin \theta} \frac{m Y_{\ell}^m}{\sin \theta} \right) = -\ell(\ell+1). \] (107)

Indeed, and as will become apparent in the discussion in the following section, all of the general relativistic effects are embodied in the proper specification of the integration constants (105) and (106). In the case of the electromagnetic radiation produced by a rotating dipole, for instance, they will be responsible for the most important quantitative corrections of general relativistic nature.

6 A USEFUL APPLICATION: A ROTATING MAGNETIC DIPOLE

In the following, we will discuss the application of the expressions derived in Sections 4 and 5 for a configuration frequently discussed in astrophysics: the electromagnetic fields and the corresponding energy losses produced when a relativistic and massive magnetic dipole moment rotates in vacuum. Although idealized, this simple model has been widely used to provide a first quantitative description of the phenomenon associated with pulsars. The expressions for the electromagnetic fields and for the energy losses through dipolar electromagnetic radiation have been derived long ago within a Newtonian description (Pacini 1968) and, to the best of our knowledge, have not been extended before to a fully general relativistic framework. This complex problem, however, can find a simple solution within the formalism discussed so far if it is recast into the problem of determining the electromagnetic fields in the near and wave zones of a relativistic and perfectly conducting stellar crust moving with a velocity field
\[ \delta v^\alpha = \frac{1}{N^2} \left( 1, 0, 0, \Omega \right), \] (108)
and whose corresponding components of the three velocity are
\[ \delta v^\ell = \left( 0, 0, \Omega r \sin \theta \right), \] (109)
where \( r \ll R \) and \( \Omega \) is the angular velocity of the star as measured by a distant observer.

Being axisymmetric, the velocity field (108) will not introduce first-order perturbations in the near and wave zones if the magnetic dipole is aligned with the stellar rotation axis. However, if a non-zero inclination angle \( \chi \) is present between the stellar magnetic dipole and the rotation axis, this velocity field will introduce a time modulation of the magnetic and electric field components in the near zone as well as the emission of electromagnetic waves in the wave zone. These variations can be compared with the expressions for the electromagnetic fields of a rotating relativistic magnetized star available in the literature (Konno & Kojima 2000; Rezzolla et al. 2001a,b), thus offering a useful test-bed for the formalism developed so far. Before doing that, however, it is necessary to calculate the expressions for the static background dipolar magnetic field using the general expressions (67)–(69).

6.1 Near zone background magnetic fields of a static dipole

Hereafter, we will calculate the dipolar properties of the background magnetic field by considering the general expressions (67)–(69) with \( \ell = 1 \), but will also allow for non-zero values of the index \( m \) to investigate the important departures from axisymmetry. Furthermore, we will assume the magnetic dipole to be varying on a time-scale much longer than that set by the stellar oscillations so that it can effectively be considered static. In this case, the scalar functions \( S_{1m} \) in (66) are time-independent and can be easily calculated to be
\[ S_{1m}(r) = \frac{r^2}{2M^2} \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \left| S_{1m} \right|, \] (110)
after using the expression for the first Legendre function of the second kind\(^3\)
\[ Q_1 = \frac{x}{2} \ln \left( \frac{x+1}{x-1} \right) - 1. \] (111)

\(^3\)Note that in Rezzolla et al. (2001a), the incorrect expression was given for \( Q_1 \). In particular, equation (89) of that paper shows the expression for \( Q_0 \); the correct expression for \( Q_1 \) was however used in the calculations reported there.
Substituting (110) into equations (67)–(69) yields the following components for the background magnetic field in the near zone:

\[ B^i = \frac{1}{M^2} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] s_{im} Y_{lm} = \sqrt{\frac{2}{3\pi M^7}} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] \left[ \sqrt{2} s_{10} \cos \theta - \text{Re} \left( s_{11} e^{i\phi} \right) \sin \theta \right], \]

\[ B^\theta = \frac{N}{M^2 r} \left[ \frac{r}{M} \ln N^2 + 1 \right] s_{im} \partial_\theta Y_{lm} = -\sqrt{\frac{3}{2\pi M^7 r}} \left[ \ln N^2 + 1 \right] \left[ \sqrt{2} s_{10} \sin \theta \cos \phi \right], \]

\[ B^\phi = \frac{N}{M^2 r \sin \theta} \left[ \frac{r}{M} \ln N^2 + 1 \right] s_{im} \partial_\phi Y_{lm} = -\sqrt{\frac{3}{2\pi M^7 r}} \left[ \ln N^2 + 1 \right] \left[ \text{Re} \left( s_{11} e^{i\phi} \right) \cos \theta \right], \]

where we have chosen \( s_{1m} = 0 \) for \( m < 0 \) and where \( \text{Re}(A) \) is the real part of \( A \).

Expressions (112)–(114) are the general relativistic vacuum magnetic field components in the near zone of a relativistic spherical star with a dipolar magnetic field. To be fully determined, they further need the specification of the coefficients \( s_{im} \), which can be calculated through the use of suitable boundary conditions across the stellar surface. As mentioned in the previous section, these integration constants represent the way in which the information about the properties of the star, and their general relativistic corrections, are imposed upon the electromagnetic fields both in the near zone and in the wave zone. One way of determining these boundary conditions could be by ensuring the continuity of the radial component of the magnetic field across the stellar surface, but this would require knowledge of the value of the magnetic field components at the stellar surface, and these are not necessarily easy to measure. Alternatively, the coefficients can be computed after multiplying both sides of (112) by \( Y_{l mL}^* \), and exploiting the orthogonality of the spherical harmonics: i.e. \( \int d\Omega Y_{lm} Y_{l mL}^* = \delta_{lL} \delta_{mM} \). Doing so yields the following expressions for the unknown coefficients:

\[ s_{lm} = \frac{M^3 r^2}{r^2 \ln N^2 + 2M(r + M)} \int B^l Y_{lm}^* d\Omega, \]

which must hold at any radius \( r \) and, in particular, in the asymptotically flat portion of the space–time, where \( M/r \ll 1 \) and the magnetic field components assume their Newtonian expressions,

\[ (B^i)_{\text{Newton}} = \frac{B_0 R^3}{2\pi} \left( 2(\cos \chi \cos \theta + \sin \chi \sin \theta \cos \phi), \cos \chi \sin \theta - \sin \chi \cos \theta \cos \phi, \sin \chi \sin \phi \right), \]

where \( B_0 \equiv 2\mu /R^3 \) is the (Newtonian) value of the magnetic field at the polar axis. Exploiting this property, the integrals on the right-hand side of equation (115) can be computed analytically, giving

\[ s_{10} = -\frac{\sqrt{3\pi}}{4} B_0 R^3 \cos \chi, \quad s_{11} = \frac{\sqrt{3\pi} B_0 R^3}{2} \sin \chi. \]

In the limit of flat space–time, i.e. for \( M/r \to 0 \) and \( M/r \to 0 \), expressions (112)–(114) give

\[ \lim_{M/r \to 0, M/r \to 0} B^i = -\frac{8}{3\pi^4} s_{im} Y_{lm} = \left( \frac{R}{r} \right)^3 B^i_R, \]

\[ \lim_{M/r \to 0, M/r \to 0} B^\theta = \frac{4}{3\pi^3} s_{im} \partial_\theta Y_{lm} = -\frac{1}{2} \left( \frac{R}{r} \right)^3 \partial_\theta B^\theta_R, \]

\[ \lim_{M/r \to 0, M/r \to 0} B^\phi = \frac{4}{3\pi^3} s_{im} \partial_\phi Y_{lm} = -\frac{1}{2} \left( \frac{R}{r} \right)^3 \partial_\phi B^\phi_R, \]

where \( B^i_R = B_0 (\cos \chi \cos \theta + \sin \chi \sin \theta \cos \phi) \) is the radial component of the magnetic field at the stellar surface. As expected, expressions (118)–(120) coincide with the near zone solutions of Muslimov & Tsygan (1986) for the dipolar magnetic field of a Newtonian magnetized star (cf. equations 18 of Muslimov & Tsygan 1986) and that a toroidal magnetic field can be produced by a poloidal one when this is not axisymmetric (cf. equations 114 and 120).

6.2 Near zone electromagnetic fields from a rotating magnetic dipole

Given the background dipolar magnetic fields discussed in the previous section, we now examine the electromagnetic fields that are produced when the stellar dipole is inclined at an angle \( \chi \) with respect to the stellar polar axis and rotates uniformly with an angular velocity \( \Omega \) as measured by a distant observer. In this case, the integration constants \( s_{im} \) for the background magnetic field are the same as in (117), while the values for the integration constants \( \delta s_{im} \) relative to the perturbed magnetic field are computed to be (see Appendix B for details)

\[ \delta s_{10} = 0, \quad \delta s_{11} = \frac{\sqrt{3\pi} B_0 R^3}{2} \left( e^{i2\phi} - 1 \right) \sin \chi, \]

where, again, we have chosen \( \delta s_{1m} = 0 \) for \( m < 0 \). Using now (117) and (121) in expressions (67)–(69), the complete magnetic field components produced in the near zone by the relativistic rotating magnetic dipole are then given by the general expressions

\[ B^i = \frac{3R^3}{8M^7} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] B_0 (\cos \chi \cos \theta + \sin \chi \sin \theta \cos \lambda), \]

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\[ B^\ell = \frac{3R^3 N}{8M^2 r} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right] B_0 (\cos \chi \sin \theta - \sin \chi \cos \theta \cos \lambda), \]  
(123)

\[ B^0 = \frac{3R^3 N}{8M^2 r} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right] B_0 (\sin \chi \sin \lambda), \]
(124)

where \( \lambda = \phi - \Omega t \). Although derived in a different way, expressions (122)–(124) coincide with the magnetic field components of a magnetized relativistic star in the slow-rotation approximation (cf. equations 97–99 of Rezzolla et al. 2001a). Furthermore, in the case in which \( \chi = 0 \), expressions (122)–(124) simply reduce to the magnetic field components of a relativistic dipole (Ginzburg & Ozerney 1964; Anderson & Cohen 1970).

We next consider the expressions for the electric field using equations (78)–(80) with \( \ell = 2 \) for a quadrupolar electric field and exploiting the relation between the \( X_{\ell m} \) and \( S_{\ell m} \) functions (cf. equation 77), which leaves as the only non-zero component

\[ X_{2m} = \frac{r^2}{2} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] \delta_{2m}(t). \]

With some straightforward algebra, we finally obtain that the components of the electric field produced in the near zone by the rotating magnetic dipole assume the form

\[ \delta E^0 = \frac{1}{3} \frac{E_0}{g_R} \frac{1}{N_R} \left[ \left( 3 - \frac{2r}{M} \right) \ln N^2 + \frac{2M^2}{3r^2} \right] \left[ \cos \chi (3 \cos^2 \theta - 1) + 3 \sin \chi \sin \theta \cos \theta \cos \lambda \right], \]
(126)

\[ \delta E^\ell = -\frac{1}{2} \frac{E_0}{g_R} \frac{N}{N_R} \left[ \left( \frac{1}{r} - \frac{r}{M} \right) \ln N^2 - 2 - \frac{2M^2}{3r^2 N^2} \right] \left[ 2 \cos \chi \sin \theta \cos \theta - (\cos^2 \theta - \sin^2 \theta) \sin \chi \cos \lambda \right], \]
(127)

\[ \delta E^\ell = -\frac{1}{2} \frac{E_0}{g_R} \frac{N}{N_R} \left[ \left( \frac{1}{r} - \frac{r}{M} \right) \ln N^2 - 2 - \frac{2M^2}{3r^2 N^2} \right] \sin \chi \cos \theta \sin \lambda, \]
(128)

where we express with \( g_R \) the constant coefficient

\[ g_R = \left( 1 - \frac{R}{M} \right) \ln N_R - \frac{2M^2}{3R^2 N_R^2} - 2 = 1 - \frac{R}{M} \ln \left( 1 - \frac{2M}{R} \right) - \frac{2}{3} \left( \frac{M}{R} \right)^2 \left( \frac{R}{R - 2M} \right)^2 - 2, \]
(129)

and where, in analogy with the corresponding Newtonian expression, we define the electric field \( E_0 \) as

\[ E_0 = (f_R B_0) \frac{\Omega}{N_R} R = (f_R B_0) \Omega_R R. \]
(130)

It is worth noting that the coefficient \( f_R \) introduced in expression (83) has also an important physical meaning since it gives a simple measure of the relativistic corrections that lead to an effective amplification of the magnetic fields in the vicinity of the stellar surface. In particular, if we indicate with \( \hat{B}_0 \) the radial component of the magnetic field measured at the surface of the relativistic star for an aligned non-rotating magnetic dipole, i.e. \( B_0 \equiv B^r (r = R, \theta = 0, \chi = 0, \Omega = 0) \), then the ratio

\[ \frac{\hat{B}_0}{B_0} = \frac{B_0 R^3}{2 \mu_0} = f_R \]
(131)

represents the general relativistic amplification of the magnetic field strength (cf. equation 122). This amplification factor \( f_R \) is a direct consequence of the space–time curvature produced by the star and is shown in the small inset of Fig. 1, where it takes values between 1.1 and 1.5 for the values of the stellar compactness \( M/R \) usually associated with compact stars. With simple algebra and proper order accounting, it is not difficult to show that \( f_R = 1 \) in the Newtonian limit (cf. equation 93 of Rezzolla et al. 2001a), so that \( E_0 = \Omega R B_0 \) will then represent the well-known electric field induced by the rotation of a Newtonian magnetized star.

Also for the electric fields, it is possible to show that expressions (126)–(128) coincide with the solutions found for the exterior electric field of a rotating misaligned dipole in a Schwarzschild space–time (cf. equations 124–126 of Rezzolla et al. 2001a). Furthermore, in the limiting case of a flat space–time, i.e. \( M/r, M/R \to 0 \), the expressions coincide with the near zone solutions of Deutsch (1955). Overall, the results presented in this section confirm the consistency of the procedure followed and show the flexibility of the formalism presented here.

### 6.3 Wave zone electromagnetic fields from a rotating magnetic dipole

The expressions for the electromagnetic fields produced in the wave zone by a rotating magnetic dipole were calculated long ago in Newtonian gravity, dating back to the work of Deutsch (1955), who presented them in terms of a complex but complete multipolar expansion. The extension of Deutsch’s results to a general relativistic framework using the procedure discussed in Section 5 is rather straightforward, but we will consider it here only for the lowest-order multipoles. In particular, we will discuss the dipolar (i.e. those with \( \ell = 1 \)) and quadrupolar (i.e. those with \( \ell = 2 \)) parts of the electric and magnetic fields in the wave zone since these are the ones with the slowest radial fall-off and are therefore the ones mainly responsible for the energy losses.

After some lengthy but straightforward algebra in which conditions (105) and (106) are written explicitly in terms of the magnetic field components (122)–(124) and of the velocity field (108), it is possible to determine the different integration constants \( u_{\ell m} \) and \( v_{\ell m} \) to be used...
in expressions (97)–(102). More specifically, when \( \ell = 1 \) one finds that \( v_{im} = 0 \) and the only non-zero coefficient is given by \( u_{11} \) and has an explicit expression

\[
u_{11} = -i \sqrt{\frac{4\pi}{3}} \Omega^2 f_B B_0 \sin \chi.
\]

(132)

Using this result, the dipolar parts of the electromagnetic fields (97)–(102) induced in the wave zone by the rotation of the dipole are then expressed as the real parts of the following solutions:

\[
B^\ell = -i \frac{\Omega^2 R^3}{N r^2} f_B B_0 \sin \theta \sin \chi e^{i(\Omega r - \ell + \phi)},
\]

(133)

\[
B^\phi = \frac{1}{2} \frac{\Omega^2 R^3}{r} f_B B_0 \cos \theta \sin \chi e^{i(\Omega r - \ell + \phi)},
\]

(134)

\[
B^\theta = \frac{i}{2} \frac{\Omega^2 R^3}{r} f_B B_0 \sin \chi e^{i(\Omega r - \ell + \phi)},
\]

(135)

\[
E^\ell = 0,
\]

(136)

\[
E^\phi = \frac{i}{2} \frac{\Omega^2 R^3}{r} E_0 \sin \chi e^{i(\Omega r - \ell + \phi)} = B^\phi,
\]

(137)

\[
E^\theta = \frac{1}{2} \frac{\Omega^2 R^3}{r} E_0 \cos \theta \sin \chi e^{i(\Omega r - \ell + \phi)} = -B^\phi.
\]

(138)

Since the wave zone is located well outside the light cylinder, i.e. at \( r \gg r_c \equiv 1/\Omega \), expressions (133)–(138) show that in this region the electromagnetic fields behave essentially as radially outgoing waves for which \( |B^\ell|/B^\phi| \sim |B^\ell|/B^\phi| \sim 1/\Omega r \ll 1 \).

Proceeding in a similar manner when \( \ell = 2 \) and requiring the solutions to be regular at \( \theta = 0 \), one finds that \( u_{2m} = 0 \) and the only non-zero coefficient is then given by \( v_{21} \), whose explicit form is

\[
v_{21} = \frac{i}{3} \sqrt{\frac{4\pi}{5}} \Omega^3 f_B B_0 \sin \chi.
\]

(139)

As a result, the quadrupolar parts of the electromagnetic fields induced in the wave zone by the rotation of the magnetic dipole are expressed as the real parts of the following solutions:

\[
B^\ell = 0,
\]

(140)

\[
B^\phi = \frac{1}{12} \frac{\Omega^4 R^3}{r} f_B B_0 \cos \theta \sin \chi e^{i(\Omega r - \ell + \phi)},
\]

(141)

\[
B^\theta = \frac{i}{12} \frac{\Omega^4 R^3}{r} f_B B_0 (\sin^2 \theta - \cos^2 \theta) \sin \chi e^{i(\Omega r - \ell + \phi)},
\]

(142)

\[
E^\ell = \frac{1}{2} \frac{\Omega^4 R^3}{N r^2} E_0 \sin \theta \sin \chi e^{i(\Omega r - \ell + \phi)},
\]

(143)

\[
E^\phi = \frac{i}{12} \frac{\Omega^4 R^3}{r} E_0 (\sin^2 \theta - \cos^2 \theta) \sin \chi e^{i(\Omega r - \ell + \phi)} = B^\phi,
\]

(144)

\[
E^\theta = \frac{1}{12} \frac{\Omega^4 R^3}{r} E_0 \cos \theta \sin \chi e^{i(\Omega r - \ell + \phi)} = -B^\phi.
\]

(145)

Also in this case the electromagnetic fields behave essentially as radially outgoing waves for which \( |E^\ell/E^\phi| \sim |E^\ell/E^\phi| \sim 1/\Omega r \ll 1 \).

A few aspects of expressions (133)–(145) are worth noting. The first and most obvious one is that all of these components are identically zero when the magnetic dipole is aligned with the rotation axis (i.e. \( \chi = 0 \)), underlining that an axisymmetric configuration cannot emit electromagnetic waves. The second is that all components have a phase term \( \propto \exp[\{i\ell (\Omega r - \ell + \phi)] \) expressing the wave nature of the solutions, where the term \( \propto \exp(i\Omega r) \) is inherited from the spherical Hankel functions once \( \omega = \Omega \). In the limit of flat space–time, these solutions coincide with those in equations (87) and (88) of Michel & Li (1999) for the dipolar and quadrupolar fields, respectively. Finally, because expressions (133)–(138) and (140)–(145) represent the dominant electromagnetic fields in the wave zone, they provide the largest contributions to the electromagnetic losses that will be computed in the following section.

### 6.4 Electromagnetic radiation losses from a rotating magnetic dipole

Using the results of the previous section, we can estimate the luminosity carried away by the dipolar electromagnetic radiation \( \mathcal{L}_{\text{em}} \) in terms of the integral of the radial component of the Poynting vector \( \mathbf{P} \),

\[
\mathcal{L}_{\text{em}} \equiv \int_\Sigma \mathbf{P} \cdot d\mathbf{S} = \frac{1}{4\pi} \int_\Sigma (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S},
\]

(146)

where the integral is made over the two sphere \( \partial \Sigma \) of radius \( r \gg 1/\Omega > R \) and surface element \( d\mathbf{S} \).
Substituting in (146) the expressions (98), (99) and (101), (102) for the electric and magnetic fields in the wave zone, it is not difficult to find that the radial component of the Poynting vector is given by

\[ P' = \frac{1}{4\pi} [E^\phi B^\phi - E^\phi B^\phi] = \frac{1}{4\pi \ell (\ell + 1)} \left[ \phi \cdot \mathbf{H} \cdot (\mathbf{H} \cdot \mathbf{v}) \left( |\mathbf{u}|^2 + |\mathbf{v}|^2 \right) \left( \frac{m \mathbf{v}}{\sin^2 \theta} \mathbf{v} \right) \right], \tag{147} \]

and is basically determined by the square of the integration constants \( u \) and \( v \) (cf. equations 105 and 106). In the case of purely dipolar radiation, \( u_{11} \) is the only non-zero coefficient (cf. equations 132) and the power (146) radiated in the form of dipolar electromagnetic radiation is then

\[ L_{\text{em}} = \frac{1}{8\pi} \left( |u_{11}|^2 + |v_{11}|^2 \right) = \frac{|u_{11}|^2}{8\pi} = \frac{\Omega_1^2 R^6 B_0^2}{6c^3} \sin^2 \chi. \tag{148} \]

When compared with the equivalent Newtonian expression for the rate of electromagnetic energy loss through dipolar radiation (Pacini 1968; Landau & Lifshitz 1987),

\[ \left( L_{\text{em}} \right)_{\text{Newton}} = \frac{\Omega_1^4 R^6 B_0^2}{6c^3} \sin^2 \chi, \tag{149} \]

it is easy to realize that the general relativistic corrections emerging in expression (148) are due partly to the field amplification at the stellar surface (i.e. \( B_0 = f_R B_0 \); cf. equation 131) and partly to the increase in the effective rotational angular velocity produced by the gravitational redshift (i.e. \( \Omega = \Omega_0 N_0 \); cf. equation 87). Overall, therefore, the presence of a curved space–time has the effect of increasing the rate of energy loss through dipolar electromagnetic radiation by an amount that can be easily estimated to be

\[ \left( \frac{L_{\text{em}}}{L_{\text{em}} \text{Newton}} \right) = \left( \frac{f_R}{N_0} \right)^2, \tag{150} \]

and whose dependence is shown in Fig. 1 with a solid line. Although expression (148) has been obtained neglecting the curvature effects for the electromagnetic fields in the wave zone region (cf. Section 5), it is straightforward to realize that already in this simplified setup the Newtonian expression (149) underestimates the electromagnetic radiation losses by a factor that is between 2 and 6 in the typical range for the compactness of relativistic stars. Furthermore, when referred to a typical neutron star with magnetic field \( B_0 = 10^{12} \) G, \( R = 12 \) km and angular velocity \( \Omega = 500 \) rad s\(^{-1}\), expression (148) gives an energy-loss rate

\[ L_{\text{em}} \approx 1.1 \times 10^{39} \left( \frac{f_R}{N_0} \right)^2 \left( \frac{R}{1.2 \times 10^6 \text{ cm}} \right)^6 \left( \frac{\Omega}{500 \text{ rad s}^{-1}} \right)^4 \left( \frac{B_0}{10^{12} \text{ G}} \right)^2 \sin^2 \chi \text{ erg s}^{-1}, \tag{151} \]

where \( f_R/N_0 \approx 4 \) for \( M \approx 1.4 \) M\(_{\odot}\).

The expression for the energy loss (148) can also be used to determine the spin evolution of a pulsar that is converting its rotational energy into electromagnetic radiation. Following the simple arguments proposed more than 30 years ago (Pacini 1968; Gunn & Ostriker 1969a,b), it is possible to relate the electromagnetic energy loss \( L_{\text{em}} \) directly to the loss of rotational kinetic energy \( E_{\text{rot}} \) defined as

\[ E_{\text{rot}} = \frac{1}{2} \int d^3 \mathbf{x} \sqrt{\mathbf{v}} e^{-\Phi/\ell} \rho (\delta \mathbf{v}^2), \tag{152} \]

where \( \rho \) is the stellar energy density. Introducing now the general relativistic moment of inertia of the star,

\[ I = \int d^3 \mathbf{x} \sqrt{\mathbf{v}} e^{-\Phi/\ell} \rho r^2 \sin^2 \theta, \tag{153} \]

whose Newtonian limit gives the well-known expression for a spherical distribution of matter \( I \equiv (I)_{\text{Newton}} = \frac{5}{3} MR^2 \), the energy budget is then readily written as

\[ \dot{E}_{\text{rot}} = \frac{d}{dt} \left( \frac{1}{2} I \dot{\Omega}^2 \right) = -L_{\text{em}}. \tag{154} \]

Of course, in enforcing the balance (154), we are implicitly assuming all the other losses of energy (e.g. those due to gravitational waves) to be negligible. This can be a reasonable approximation except during the initial stages of the pulsar’s life, during which the energy losses due to emission of gravitational radiation will dominate because of the steeper dependence on the angular velocity (i.e. \( E_{\text{GW}} \propto \Omega^6 \)).

Expression (154) can also be written in a more useful form in terms of the pulsar’s most important observables: the period \( P \) and its time derivative \( \dot{P} = dP/dt \). In this case, in fact, using expressions (148) and (154), it is not difficult to show that

\[ PP = \left( \frac{2\pi^2}{3c^3} \right) \frac{R^6 B_0^2}{I} \left( \frac{f_R}{N_0} \right) \left( PP \right)_{\text{Newton}}, \tag{155} \]

where the Newtonian expression is given by (Gunn & Ostriker 1969a,b)

\[ (PP)_{\text{Newton}} = \left( \frac{2\pi^2}{3c^3} \right) \frac{R^6 B_0^2}{I}. \tag{156} \]

Also in this case it is not difficult to realize that general relativistic corrections will be introduced through the amplification of the magnetic field and of the stellar angular velocity, as well as of the stellar moment of inertia (i.e. \( I/I \)).
Expressions (148) and (155) could be used to investigate the rotational evolution of magnetized neutron stars with predominant dipolar magnetic field anchored in the crust, either when isolated (Page, Geppert & Zannias 2000) or when present in a binary system (Lavagetto et al. 2004). Indeed, a detailed and accurate investigation of this type has already been performed by Page et al. (2000), who have paid special attention to the general relativistic corrections that need to be included for a correct modelling of the thermal evolution but also of the magnetic and rotational evolution. It should be remarked, however, that in their treatment Page et al. (2000) have adopted an expression for the loss of rotational kinetic energy that is similar to expression (155), in that it accounts for the magnetic field amplification due to the curved background space–time, but does not include the corrections due to the gravitational redshift. As a result, the general relativistic electromagnetic luminosity estimated by Page et al. (2000) is smaller than that computed here, and it is shown with a dashed line in Fig. 1 for comparison (labelled as PGZ). As the authors themselves discuss, their choice was based on simple phenomenological considerations and was made in the absence of a more systematic treatment of the electromagnetic fields produced by a rotating and magnetized relativistic star. We hope that the results reported here will be of help in all of those investigations that consider the thermal, magnetic and rotational evolution of magnetized neutron stars.

7 CONCLUSIONS

There is undoubted observational evidence that strong electromagnetic fields are often present in the vicinity of relativistic stars. At the same time, it is natural to expect that these stars will be subject to perturbations of various types, which will induce oscillations responsible for the emission of both electromagnetic and gravitational waves. Clearly, the possibility of detecting simultaneously both types of signal will provide important information about the mass and radius of these objects and hence give the ability to investigate the properties of matter at nuclear densities. Building on the general relativistic treatment of electrodynamics of magnetized neutron stars developed by Rezzolla et al. (2001a,b), we have studied here the electromagnetic fields generated when the perfectly conducting crust of a magnetized relativistic star possesses a non-zero velocity field. The star has been modelled as a relativistic polytrope with infinite conductivity and in vacuum. The relativistic corrections induced by a rotation in the space–time as well as those produced by the electromagnetic energy densities have been neglected. We expect both of these approximations to be satisfactory for the large majority of rapidly rotating and magnetized neutron stars.

As in our previous investigations (Rezzolla et al. 2001a,b), we have paid special care to providing expressions for the electromagnetic fields that, besides being analytical, are written in a form that is as simple as possible, highlights the relation between the general relativistic and Newtonian expressions, and can therefore also be used in astrophysical applications. This first paper has been mostly devoted to the development of the mathematical formalism that distinguishes the solution near the stellar surface (where the electromagnetic fields are quasi-stationary) from the solution at very large distances (where the electromagnetic fields approach a wave-like behaviour). In both of these regions and with suitable approximations, in fact, exact solutions to the general relativistic Maxwell equations can be found and have been shown to contain two main corrections when compared with the corresponding Newtonian expressions. The first one is related to the...
amplification of the electromagnetic fields on the stellar surface caused by the background space–time curvature, while the second one is due to the gravitational redshift that will affect the electromagnetic waves as these propagate in the curved space–time. It should be noted, in fact, that, while the wave zone solutions to the Maxwell equations formally coincide with the ones derived in a flat space–time, the integration constants defined at the surface of the star imprint important general relativistic corrections, which, in turn, convey information about the mass and radius of the oscillating star.

The general expressions for the electromagnetic fields presented here do not refer to any specific magnetic field topology or velocity field and are, in this sense, completely general. However, to validate the expressions derived and to consider an astrophysically important test-bed, we have also examined the form that these electromagnetic fields assume when the background magnetic fields are of dipolar type and the crustal velocity field that is produced by the uniform rotation of the dipole. This configuration is often used as the simplest model for the electromagnetic emission observed from pulsars and has been extensively investigated in the past. While analytic expressions are available within a Newtonian framework both in the near zone and in the wave zone, the general relativistic expressions are limited to the near zone. A direct comparison with these expressions and the perfect agreement found have shown that the formalism developed here is flexible and robust. Furthermore, the use of analytic expressions for the components of the electromagnetic fields in the wave zone has allowed us to calculate, for the first time, the general relativistic expression for the electromagnetic energy loss through dipolar radiation. A simple comparison with the corresponding Newtonian expression shows that the latter underestimate this loss by a factor of 2–6, depending on the stellar compactness.

Work is now in progress to apply the formalism presented here to those spheroidal and toroidal velocity fields most frequently encountered in the study of stellar oscillations. The results will be presented in a forthcoming companion paper (Paper II).

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APPENDIX A: DERIVATIVES OF LEGENDRE FUNCTIONS

In this appendix we briefly sketch the lengthy but simple algebra needed for the calculation of the functions \( Q_{\ell m} \) and their radial derivatives, which are used in the calculation of the electric field in the near zone. Using the recurrence formulae (see, for instance, Jeffrey 1995)

\[
(x^2 - 1) \frac{dQ_{\ell}}{dx} = \ell(xQ_{\ell} - Q_{\ell-1}),
\]

(A1)

\[
(\ell + 1)Q_{\ell+1} - (2\ell + 1)xQ_{\ell} + \ell Q_{\ell-1} = 0,
\]

(A2)

where, we recall, \( x = 1 - r/M \), it is possible to show that the radial second-order derivative of Legendre functions of the second kind takes the form

\[
\frac{d}{dx} \left( (1 + x) \frac{dQ_{\ell}}{dx} \right) = \frac{\ell}{(x - 1)^2} \{x(1 - \ell)Q_{\ell} + Q_{\ell-1}\}.
\]

(A3)

As a result, the expressions contained in the electric field components (78)–(80) can be written explicitly as

\[
\frac{d}{dr} \left( (1 + x) \frac{dQ_{\ell}}{dx} \right) = -M \frac{d}{dx} \left( (x - 1)^2 \left( (1 + x) \frac{dQ_{\ell}}{dx} \right) \right) = -\frac{M \ell^2 (\ell + 1)}{(x + 1)}(xQ_{\ell} - Q_{\ell-1}).
\]

(A4)

APPENDIX B: CALCULATION OF THE INTEGRATION CONSTANTS \( \partial_\ell \delta S_{\ell m} \)

Inserting into (70) the value of the magnetic field (47)–(49) at the surface of the star

\[
B_\| = -\frac{1}{R^2} (\nabla^2 \delta S)_{\| R}, \quad B^\theta = \frac{N_R}{R} (\partial_\theta \partial_\theta S)_{\| R}, \quad B^\phi = \frac{N_R}{R \sin \theta} (\partial_\phi \partial_\phi S)_{\| R},
\]

and making use of expression (55), it is possible to derive that

\[
(\partial_\ell \delta S_{\ell m} Y_{\ell m})_{\| R} = \frac{1}{\ell(\ell + 1)} \left\{ (\nabla^2 \delta S) \left[ \frac{1}{R \sin \theta} [\partial_\theta (\sin \theta \partial_\theta \delta v^\theta) + \partial_\phi \delta v^\phi] \right.ight.
\]

\[
+ \left. \left[ N_R \partial_\ell, (\nabla^2 \delta S) \delta v^\ell + \frac{1}{R} \partial_\theta (\nabla^2 \delta S) \delta v^\theta + \frac{1}{R \sin \theta} \partial_\phi (\nabla^2 \delta S) \delta v^\phi \right] + N_R \left[ \partial_\theta \partial_\phi \partial_\theta \delta v^\ell + \frac{1}{\sin^2 \theta} \partial_\phi \partial_\theta \partial_\phi \delta v^\ell \right] \right\}_{\| R},
\]

(B2)

where \( \nabla^2 \) is the angular part of the Laplacian, i.e.

\[
\nabla^2 = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\phi^2.
\]

(B3)

It is interesting to note that in the limit of flat space–time our expression (B2) coincides with equation (B2) of Timokhin et al. (2000), which was instead derived after requiring the continuity of the tangential electric field at the stellar surface. Multiplying this equation by \( Y_{\ell m}^* \), integrating it over the solid angle and using equation (66), we can finally obtain an expression for the coefficients \( \partial_\ell (\delta S_{\ell m}) \) in the expansion of \( \delta S \) in spherical harmonics:

\[
(\partial_\ell \delta S_{\ell m})_{\| R} = \frac{\ell}{M} \left\{ -\frac{1}{1 + \ell} Q_{\ell} + Q_{\ell-1} \right\} \partial_\ell \delta S_{\ell m}(t)
\]

\[
= \frac{1}{\ell(\ell + 1)} \int d\Omega Y_{\ell m}^* \left\{ (\nabla^2 \delta S) \left[ \frac{1}{R \sin \theta} [\partial_\theta (\sin \theta \partial_\theta \delta v^\theta) + \partial_\phi \delta v^\phi] \right.ight.
\]

\[
+ \left. \left[ N_R \partial_\ell, (\nabla^2 \delta S) \delta v^\ell + \frac{1}{R} \partial_\theta (\nabla^2 \delta S) \delta v^\theta + \frac{1}{R \sin \theta} \partial_\phi (\nabla^2 \delta S) \delta v^\phi \right] \right\}_{\| R}.
\]

(B4)

APPENDIX C: A DIFFERENT APPROACH TO THE GRAVITATIONAL REDSHIFT

Here we provide a somewhat different derivation of the well-known result that photons in a gravitational potential undergo a gravitational redshift. For doing this we will exploit the existence in this space–time of a time-like Killing vector \( \xi^\mu \) such that

\[
\xi_{\mu,\nu} + \xi_{\mu,\nu} = 0.
\]

(C1)

Electromagnetic waves propagate along geodesics and the associated null wavevector \( k^\mu \) will be tangent to these trajectories and parallel transported along them, i.e.

\[
k_{\mu,\nu} k^\nu = 0.
\]

(C2)

We now introduce two important frequencies: the first one is the frequency measured by an observer with four velocity \( u^\alpha \) and defined as

\[
\omega \equiv -k^\alpha u_\alpha.
\]

(C3)
while the second one is the frequency associated with the time-like Killing vector $\xi^a$ and defined as

$$\omega_\xi = -k^a \xi_a.$$  \hfill (C4)

The two frequencies have marked differences: while (C3) depends on the observer chosen and is therefore a function of position, (C4) is a conserved quantity that remains constant along the trajectory followed by the electromagnetic wave. This is easily verified after using equations (C1) and (C2), which give

$$\left( k^a \xi_a \right)_\beta \beta^\beta = \left( k^a \xi_a \right)_\beta \beta^\beta = k^a \partial_\beta k^\beta \xi_a + \xi_{a,\beta} k^\beta = 0.$$  \hfill (C5)

We can now exploit this property to measure how the frequency changes with position and is therefore redshifted in the space–time (2).

Assume the Killing vector to have components

$$k^a \equiv (1, 0, 0, 0), \quad \xi_a \equiv N^2(-1, 0, 0, 0),$$ \hfill (C6)

so that $\omega_\xi = -k_0 = \text{constant}$. The frequency of an electromagnetic wave emitted at the surface of the star $r = R$ and measured by an observer with four velocity $u^a$ parallel to $\xi^a$ (i.e. a static observer) will be

$$\omega_R = -\left. (k_0 u^a) \right|_{r=R} = N_R^{-1} \omega_\xi,$$ \hfill (C7)

so that at a generic radial position $r$

$$N(r) \omega_0(r) = \text{constant} = N_0 \omega_R.$$ \hfill (C8)

This expression coincides with equation (87) in the text.

**APPENDIX D: MAXWELL EQUATIONS IN A SPACE–TIME WITH A ROTATION**

If the space–time admits a rotational Killing three vector $\vec{\beta}$, the full set of Maxwell equations can then be written as (Thorne et al. 1986; Rezzolla et al. 2001a, b)

$$\vec{\nabla} \cdot \vec{B} = 0,$$ \hfill (D1)

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_e,$$ \hfill (D2)

$$(\partial_t - \mathcal{L}_\beta) \vec{B} = -\vec{\nabla} \times (N \vec{E}),$$ \hfill (D3)

$$(\partial_t - \mathcal{L}_\beta) \vec{E} = \vec{\nabla} \times (N \vec{B}) - 4\pi \vec{j},$$ \hfill (D4)

where, again, $\vec{\nabla}$ represents the covariant derivative with respect to the spatial part of the metric, $\mathcal{L}_\beta$ is the Lie derivative along $\vec{\beta}$ and $N$ is the lapse. The system of equations (D1)–(D4) is completed with Ohm’s law, which takes the general form

$$\vec{j} = \sigma \left( \vec{E} + (\vec{v} + \vec{\beta}) \times \vec{B} \right).$$ \hfill (D5)

The Lie derivative of a three vector $\vec{A}$ along the vector field $\vec{\beta}$ is not particularly difficult to calculate. However, the form of its components in a space–time with line element (2) and in the frame (6)–(9) is not easy to find in the literature. For this reason we give them here as a useful reference:

$$(\mathcal{L}_\beta \vec{A})^\gamma = e^{-\Lambda} \beta^\beta \partial_\gamma \vec{A} + \frac{\beta_\gamma}{r} \partial_\beta \vec{A} + \frac{\partial_\beta \vec{A}}{r \sin \theta} \partial_\theta \vec{A} - e^{-\Lambda} \vec{A} \partial_\beta \beta^\beta - \frac{A^\beta}{r} \partial_\beta \vec{A} - \frac{A^\beta}{r \sin \theta} \partial_\theta \vec{A}.$$ \hfill (D6)

$$(\mathcal{L}_\beta \vec{A})^\beta = e^{-\Lambda} \beta^\beta \partial_\beta \vec{A} + \frac{\beta^\beta}{r} \partial_\beta \vec{A} + \frac{\partial_\beta \vec{A}}{r \sin \theta} \partial_\theta \vec{A} - e^{-\Lambda} \vec{A} \partial_\beta \beta^\beta - \frac{A^\beta}{r} \partial_\beta \vec{A} - \frac{A^\beta}{r \sin \theta} \partial_\theta \vec{A} + e^{-\Lambda} \vec{A} \partial_\beta \beta^\beta - \frac{A^\beta}{r \sin \theta} \partial_\theta \vec{A},$$ \hfill (D7)

$$(\mathcal{L}_\beta \vec{A})^\phi = e^{-\Lambda} \beta^\beta \partial_\phi \vec{A} + \frac{\beta_\phi}{r} \partial_\beta \vec{A} + \frac{\partial_\beta \vec{A}}{r \sin \theta} \partial_\theta \vec{A} - e^{-\Lambda} \vec{A} \partial_\beta \beta^\beta - \frac{A^\beta}{r} \partial_\beta \vec{A} - \frac{A^\beta}{r \sin \theta} \partial_\theta \vec{A} + e^{-\Lambda} \vec{A} \partial_\beta \beta^\beta - \frac{A^\beta}{r \sin \theta} \partial_\theta \vec{A} + e^{-\Lambda} \vec{A} \partial_\beta \beta^\beta - \frac{A^\beta}{r \sin \theta} \partial_\theta \vec{A},$$ \hfill (D8)

If the Lie derivative is now taken along the rotational Killing vector

$$\vec{\beta} = \frac{1}{2} \frac{\beta_\phi}{r \sin \theta} \vec{e}_\phi = -\omega r \sin \theta \vec{e}_\phi,$$ \hfill (D9)

as for the space–time of a rapidly rotating relativistic star, expressions (D6)–(D8) reduce to

$$(\mathcal{L}_\beta \vec{A})^\gamma = -\omega \partial_\phi \vec{A}^\gamma,$$ \hfill (D10)

$$(\mathcal{L}_\beta \vec{A})^\beta = -\omega \partial_\phi \vec{A}^\beta,$$ \hfill (D11)

$$(\mathcal{L}_\beta \vec{A})^\phi = \vec{A} \omega \partial_\phi \vec{e}_\phi.$$. \hfill (D12)

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