HARDY-LITTLEWOOD INEQUALITIES AND FOURIER MULTIPLIERS ON SU(2)

RAUAN AKYLZHANOV, ERLAN NURSULTANOV, AND MICHAEL RUZHANSKY

Abstract. In this paper we prove a noncommutative version of Hardy–Littlewood inequalities relating a function and its Fourier coefficients on the group SU(2). As a consequence, we use it to obtain lower bounds for the $L^p$–$L^q$ norms of Fourier multipliers on the group SU(2), for $1 < p \leq 2 \leq q < \infty$. In addition, we give upper bounds of a similar form, analogous to the known results on the torus, but now in the noncommutative setting of SU(2).

1. Introduction

Let $\mathbb{T}^n$ be the $n$-dimensional torus and let $1 < p \leq q < \infty$. A sequence $\lambda = \{\lambda_k\}_{k \in \mathbb{Z}^n}$ of complex numbers is said to be a multiplier of trigonometric Fourier series from $L^p(\mathbb{T}^n)$ to $L^q(\mathbb{T}^n)$ if the operator

$$T_\lambda f(x) = \sum_{k \in \mathbb{Z}^n} \lambda_k \hat{f}(k) e^{ikx}$$

is bounded from $L^p(\mathbb{T}^n)$ to $L^q(\mathbb{T}^n)$. We denote by $m^q_p$ the set of such multipliers.

Many problems in harmonic analysis and partial differential equations can be reduced to the boundedness of multiplier transformations. There arises a natural question of finding sufficient conditions for $\lambda \in m^q_p$. The topic of $m^q_p$ multipliers has been extensively researched. Using methods such as the Littlewood-Paley decomposition and Calderon-Zygmund theory, it is possible to prove Hörmander-Mihlin type theorems, see e.g. Mihlin [Mih57, Mih56], Hörmander [Hor60], and later works.

However, these results imply that $\lambda \in m^q_p$ for all $1 < p < \infty$. In [Ste70], Stein raised the question of finding more subtle sufficient conditions for a multiplier to belong to some $m^q_p$, $p \neq 2$, without implying also that it belongs to all $m^q_p$, $1 < p < \infty$. In [NT00], Nursultanov and Tleukhanova provided conditions on $\lambda = \{\lambda_k\}_{k \in \mathbb{Z}^n}$ to belong to $m^q_p$ for the range $1 < p \leq 2 \leq q < \infty$. In particular, they established lower and upper bounds for the norms of multiplier $\lambda \in m^q_p$ which depend on parameters $p$ and $q$. Thus, this provided a partial answer to Stein’s question. Let us recall this result in the case $n = 1$:

Date: March 10, 2014.

2010 Mathematics Subject Classification. Primary 35G10; 35L30; Secondary 46F05;

Key words and phrases. Fourier multipliers, Hardy–Littlewood inequalities, noncommutative harmonic analysis.

The third author was supported by the EPSRC Grant EP/K039407/1.
Theorem 1.1. Let $1 < p \leq 2 \leq q < \infty$ and let $M_0$ denote the set of all finite arithmetic sequences in $\mathbb{Z}$. Then the following inequalities hold:

$$\sup_{Q \in M_0} \frac{1}{|Q|^{1 + \frac{1}{q} - \frac{1}{p}}} \left| \sum_{m \in Q} \lambda_m \right| \lesssim \|T_\lambda\|_{L^p \to L^q} \lesssim \sup_{k \in \mathbb{N}} \frac{1}{k^{1 + \frac{1}{q} - \frac{1}{p}}} \sum_{m=1}^{k} \lambda_m^*,$$

where $\lambda_m^*$ is a non-increasing rearrangement of $\lambda_m$, and $|Q|$ is the number of elements in the arithmetic progression $Q$.

In this paper we study the noncommutative versions of this and other related results. As a model case, we concentrate on analysing Fourier multipliers on the group $SU(2)$ of $2 \times 2$ unitary matrices with determinant one. Sufficient conditions for Fourier multipliers on $SU(2)$ to be bounded on $L^p$-spaces have been analysed by Coifman-Weiss [CW71b] and Coifman-de Guzman [CdG71], see also Chapter 5 in Coifman and Weiss’ book [CW71a], and are given in terms of the Clebsch-Gordan coefficients of representations on the group $SU(2)$. A more general perspective was provided in [RW13] where conditions on Fourier multipliers to be bounded on $L^p$ were obtained for general compact Lie groups.

Results about spectral multipliers are more known, for functions of the Laplacian (N. Weiss [Wei72] or Coifman and Weiss [CW74]), or of the sub-Laplacian on $SU(2)$, see Cowling and Sikora [CS01]. However, following [CW71b, CW71a, RW13], here we are rather interested in Fourier multipliers.

In this paper we obtain lower and upper estimates for the norms of Fourier multipliers acting between $L^p$ and $L^q$ spaces on $SU(2)$. These estimates explicitly depend on parameters $p$ and $q$. Thus, this paper can be regarded as a contribution to Stein’s question in the noncommutative setting of $SU(2)$. At the same time we provide a noncommutative analogue of Theorem 1.1. Briefly, let $A$ be the Fourier multiplier on $SU(2)$ given by

$$\hat{A} f(l) = \sigma_A(l) \hat{f}(l), \text{ for } \sigma_A(l) \in \mathbb{C}^{(2l+1) \times (2l+1)}, \ l \in \frac{1}{2} \mathbb{N}_0,$$

where we refer to Section 2 for definitions and notation related to the Fourier analysis on $SU(2)$. For such operators, in Theorem 3.1 for $1 < p \leq 2 \leq q < \infty$, we give two lower bounds, one of which is of the form

$$\sup_{l \in \frac{1}{2} \mathbb{N}_0} \frac{1}{(2l+1)^{1 + \frac{1}{p} - \frac{1}{q}}} \left( \frac{1}{2l+1} |\text{Tr } \sigma_A(l)| \right) \lesssim \|A\|_{L^p(SU(2)) \to L^q(SU(2))}.$$

A related upper bound will be given in Theorem 4.1 which will also contain two possible formulations, in terms of the density function and in terms of the non-increasing rearrangement. The latter, in analogy to Theorem 1.1, takes the form

$$\|A\|_{L^p(SU(2)) \to L^q(SU(2))} \lesssim \sup_{k \in \mathbb{N}} \frac{1}{k^{1 + \frac{1}{q} - \frac{1}{p}}} \sum_{m=1}^{k} m \lambda_m^*,$$

where $\lambda_m^*$ is the non-increasing rearrangement of the sequence

$$\lambda_m := \begin{cases} \|\sigma_A(l)\|_{op}, & m = (2l+1)^4, \ l \in \frac{1}{2} \mathbb{N}_0, \\ 0, & \text{otherwise,} \end{cases}$$
HARDY-LITTLEWOOD INEQUALITIES AND FOURIER MULTIPLIERS ON SU(2)

3

with respect to a suitably chosen measure.

The proof of the lower bound is based on the new inequalities describing the relationship between the “size” of a function and the “size” of its Fourier transform. These inequalities can be viewed as a noncommutative SU(2)-version of the Hardy-Littlewood inequalities obtained by Hardy and Littlewood in [HL27]. To explain this briefly, we recall that in [HL27], Hardy and Littlewood have shown that for $1 < p \leq 2$ and $f \in L^p(\mathbb{T})$, the following inequality holds true:

$$(1.4) \quad \sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\hat{f}(m)|^p \leq K \|f\|_{L^p(\mathbb{T})}^p,$$

arguing this to be a suitable extension of the Plancherel identity to $L^p$-spaces. While we refer to Section 2 and to Theorem 2.1 for more details on this, our analogue for this is the inequality

$$(1.5) \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l + 1)^{p-2} |\hat{f}(l)|^p_{\text{HS}} \leq c \|f\|_{L^p(SU(2))}^p, \quad 1 < p \leq 2,$$

which for $p = 2$ gives the ordinary Plancherel identity on SU(2), see (2.1). We refer to Theorem 2.2 for this and to Corollary 2.3 for the dual statement. For $p \geq 2$, the necessary conditions for a function to belong to $L^p$ are usually harder to obtain. In Theorem 2.5 we give such a result for $2 \leq p < \infty$ which takes the form

$$(1.6) \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l + 1)^{p-2} \left( \sup_{k \geq l} \frac{1}{2k + 1} \left| \text{Tr} \hat{f}(k) \right| \right)^p \leq c \|f\|_{L^p(SU(2))}^p, \quad 2 \leq p < \infty.$$

In turn, this gives a noncommutative analogue to the known similar result on the circle (which we recall in Theorem 2.4). Similar to (1.1), the averaged trace appears also in (1.6) – it is the usual trace divided by the number of diagonal elements in the matrix.

The results on the group SU(2) are usually quite important since, in view of the recently resolved Poincaré conjecture, they provide information about corresponding transformations on general closed simply-connected three-dimensional manifolds (see [RT10] for a more detailed outline of such relations). In our context, they give explicit versions of known results on the circle $\mathbb{T}$ or on the torus $\mathbb{T}^n$, in the simplest noncommutative setting of SU(2).

At the same time, we note that some results of this paper can be extended to Fourier multipliers on general compact Lie groups. However, such analysis requires a more abstract approach, and will appear elsewhere.

The paper is organised as follows. In Section 2 we fix the notation for the representation theory of SU(2) and formulate estimates relating functions with its Fourier coefficients: the SU(2)-version of the Hardy–Littlewood inequalities and further extensions. In Section 3 we formulate and prove the lower bounds for operator norms of Fourier multipliers, and in Section 4 the upper bounds. Our proofs are based on inequalities from Section 2. In Section 5 and Section 6 we complete the proofs of the results presented in previous sections.
We shall use the symbol $C$ to denote various positive constants, and $C_{p,q}$ for constants which may depend only on indices $p$ and $q$. We shall write $x \lesssim y$ for the relation $|x| \leq C|y|$, and write $x \asymp y$ if $x \lesssim y$ and $y \lesssim x$.

2. HARDY–LITTLEWOOD INEQUALITIES ON $SU(2)$

The aim of this section is to discuss necessary conditions and sufficient conditions for the $L^p(SU(2))$-integrability of a function by means of its Fourier coefficients. The main results of this section are Theorems 2.2 and 2.5. These results will provide a noncommutative version of known results of this type on the circle $T$. The proofs are given in Section 5 and in Section 6.

First, let us fix the notation concerning the representations of the compact Lie group $SU(2)$. There are different types of notation in the literature for the appearing objects - we will follow the notation of Vilenkin [Vil68], as well as that in [RT10, RT13]. Let us identify $z = (z_1, z_2) \in \mathbb{C}^1 \times \mathbb{C}^1$, and let $\mathbb{C}[z_1, z_2]$ be the space of two-variable polynomials $f: \mathbb{C}^2 \to \mathbb{C}$. Consider mappings

$$T^l: SU(2) \to GL(V_l), \quad (t^l(u)f)(z) = f(zu),$$

where $l \in \frac{1}{2}\mathbb{N}_0$ is called the quantum number, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and where $V_l$ is the $(2l+1)$-dimensional subspace of $\mathbb{C}[z_1, z_2]$ containing the homogeneous polynomials of order $2l \in \mathbb{N}_0$, i.e.

$$V_l = \{f \in \mathbb{C}[z_1, z_2]: f(z_1, z_2) = \sum_{k=0}^{2l} a_k z_1^k z_2^{2l-k}, \quad \{a_k\}_{k=0}^{2l} \subset \mathbb{C}\}.$$ 

The unitary dual of $SU(2)$ is

$$\widehat{SU(2)} \cong \{t^l \in \text{Hom}(SU(2), U(2l + 1)): l \in \frac{1}{2}\mathbb{N}_0\},$$

where $U(d) \subset \mathbb{C}^{d \times d}$ is the unitary matrix group, and matrix components $t^l_{mn} \in C^\infty(SU(2))$ can be written as products of exponentials and Legendre-Jacobi functions, see Vilenkin [Vil68]. It is also customary to let the indices $m, n$ to range from $-l$ to $l$, equi-spaced with step one. We define the Fourier transform on $SU(2)$ by

$$\hat{f}(l) := \int_{SU(2)} f(u) t^l(u)^* \, du,$$

with the inverse Fourier transform (Fourier series) given by

$$f(u) = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l + 1) \text{Tr} \left( \hat{f}(l) t^l(u) \right).$$

The Peter-Weyl theorem on $SU(2)$ implies, in particular, that this pair of transforms are inverse to each other and that the Plancherel identity

$$(2.1) \quad \|f\|_{L^2(SU(2))}^2 = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l + 1) \|\hat{f}(l)\|_{\text{HS}}^2 =: \|\hat{f}\|_{\ell_2(\widehat{SU(2)})}^2$$

holds true for all $f \in L^2(SU(2))$. Here $\|\hat{f}(l)\|_{\text{HS}}^2 = \text{Tr} \left( \hat{f}(l) \hat{f}(l)^* \right)$ denotes the Hilbert-Schmidt norm of matrices. For more details on the Fourier transform on $SU(2)$ and
on arbitrary compact Lie groups, and for subsequent Fourier and operator analysis we can refer to [RT10].

There are different ways to compare the “sizes” of \( f \) and \( \widehat{f} \). Apart from the Plancherel’s identity (2.1), there are other important relations, such as the Hausdorff-Young or the Riesz-Fischer theorems. However, such estimates usually require the change of the exponent \( p \) in \( L^p \)-measurements of \( f \) and \( \widehat{f} \). Our first results deal with comparing \( f \) and \( \widehat{f} \) in the same scale of \( L^p \)-measurements. Let us remark on the background of this problem. In [HL27, Theorems 10 and 11], Hardy and Littlewood proved the following generalisation of the Plancherel’s identity.

**Theorem 2.1** (Hardy–Littlewood [HL27]). The following holds.

1. Let \( 1 < p \leq 2 \) and \( f \in L^p(\mathbb{T}) \), then

\[
\sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\widehat{f}(m)|^p \leq K_p \| f \|_{L^p(\mathbb{T})}^p,
\]

where \( K_p \) is a constant which depends only on \( p \).

2. Let \( 2 \leq p < \infty \). If \( \{\widehat{f}(m)\}_{m \in \mathbb{Z}} \) is a sequence of complex numbers such that

\[
\sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\widehat{f}(m)|^p < \infty,
\]

then there is a function \( f \in L^p(\mathbb{T}) \) with Fourier coefficients given by \( \widehat{f}(m) \), and

\[
\| f \|_{L^p(\mathbb{T})}^p \leq K'_p \sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\widehat{f}(m)|^p.
\]

Hewitt and Ross [HR74] generalised this theorem to the setting of compact abelian groups. Now, we give an analogue of the Hardy–Littlewood Theorem 2.1 in the noncommutative setting of the compact group SU(2).

**Theorem 2.2.** If \( 1 < p \leq 2 \) and \( f \in L^p(SU(2)) \), then we have

\[
\sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)\frac{5}{2} (2l + 1)^{p-4} \| \widehat{f}(l) \|_{\text{HS}}^p \leq c_p \| f \|_{L^p(SU(2))}^p.
\]

We can write this in the form more resembling the Plancherel identity, namely, as

\[
\sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)(2l + 1)^{\frac{5}{2}} (2l + 1)^{p-2} \| \widehat{f}(l) \|_{\text{HS}}^p \leq c_p \| f \|_{L^p(SU(2))}^p,
\]

providing a resemblance to both (2.2) and (2.1). By duality, we obtain

**Corollary 2.3.** If \( 2 \leq p < \infty \) and \( \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)\frac{5}{2} (2l + 1)^{p-4} \| \widehat{f}(l) \|_{\text{HS}}^p < \infty \), then \( f \in L^p(SU(2)) \) and we have

\[
\| f \|_{L^p(SU(2))}^p \leq c_p \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)^{\frac{5}{2}} (2l + 1)^{p-2} \| \widehat{f}(l) \|_{\text{HS}}^p.
\]

For \( p = 2 \), both of these statements reduce to the Plancherel identity (2.1).

Coming back to the Hardy–Littlewood Theorem 2.1, we see that the convergence of the series (2.3) is a sufficient condition for \( f \) to belong to \( L^p(\mathbb{T}) \), for \( p \geq 2 \). However,
this condition is not necessary. Hence, there arises the question of finding necessary conditions for \( f \) to belong to \( L^p \). In other words, there is the problem of finding lower estimates for \( \|f\|_{L^p} \) in terms of the series of the form (2.3). Such result on \( L^p(\mathbb{T}) \) was obtained by Nursultanov and can be stated as follows.

**Theorem 2.4 (Nur98).** If \( p \geq 2, f \in L^p(\mathbb{T}) \) and \( f \sim \sum_{m \in \mathbb{Z}} \hat{f}(m)e^{imx} \), then we have

\[
(2.7) \quad \sum_{k=1}^{\infty} k^{p-2} \left( \sup_{|e| \geq k} \frac{1}{|e|} \left| \sum_{m \in e} \hat{f}(m) \right| \right)^p \leq C \|f\|_{L^p(\mathbb{T})}^p,
\]

where \( M \) is the set of all finite arithmetic progressions in \( \mathbb{Z} \).

We now present a (noncommutative) version of this result on the group \( SU(2) \).

**Theorem 2.5.** If \( 2 \leq p < \infty \) and \( f \in L^p(SU(2)) \), then we have

\[
(2.8) \quad \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l+1)^{p-2} \left( \sup_{k \in \frac{1}{2} \mathbb{N}_0 \setminus k \geq l} \frac{1}{2k+1} \left| \text{Tr} \hat{f}(k) \right| \right)^p \leq c \|f\|_{L^p(SU(2))}^p.
\]

We note that for \( f \in L^2(SU(2)) \), this inequality is “weaker” than Plancherel’s identity (2.1), see Remark 2.6. However, its advantage is that it holds also for \( p \neq 2 \). This property will be important in further applications to Fourier multipliers in Theorem 3.1.

**Remark 2.6.** We note that for \( f \in L^2(SU(2)) \), this inequality is “weaker” than Plancherel’s identity (2.1), i.e., it follows from the Hilbert-Schmidt version of the Cauchy-Schwartz inequality, that

\[
(2.9) \quad \sum_{l \in \frac{1}{2} \mathbb{N}_0} \left( \sup_{k \in \frac{1}{2} \mathbb{N}_0 \setminus k \geq l} \frac{1}{2k+1} \left| \text{Tr} \hat{f}(k) \right| \right)^2 \lesssim \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l+1) \|\hat{f}(l)\|_{\text{HS}}^2 = \|f\|_{L^2(SU(2))}^2.
\]

**Proof.** It is enough to prove that

\[
(2.10) \quad \sup_{k \in \frac{1}{2} \mathbb{N}_0 \setminus k \geq l} \frac{1}{2k+1} \left| \text{Tr} \hat{f}(k) \right| \leq \|\hat{f}(l)\|_{\text{HS}}.
\]

It can easily be checked that \( \left| \text{Tr} \hat{f}(k) \right| \leq \|\hat{f}(k)\|_{\text{HS}} \sqrt{2k+1} \). Therefore, we have

\[
\frac{1}{2k+1} \left| \text{Tr} \hat{f}(k) \right| \leq \frac{1}{\sqrt{2k+1}} \|\hat{f}(k)\|_{\text{HS}}.
\]

Since \( f \in L^2(SU(2)) \), we have \( \|\hat{f}(l)\|_{\text{HS}} \to_{l \to \infty} 0 \). Hence

\[
\sup_{k \in \frac{1}{2} \mathbb{N} \setminus k \geq l} \|\hat{f}(k)\| \lesssim \|\hat{f}(l)\|_{\text{HS}}.
\]
Combining these inequalities, we get
\[
\sup_{k \geq l \in \mathbb{N}_0} \frac{1}{2k + 1} |\operatorname{Tr} \hat{f}(k)| \leq \sup_{k \geq l \in \mathbb{N}_0} \frac{1}{2k + 1} \|\hat{f}(k)\|_{\mathcal{HS}} \leq \frac{1}{\sqrt{2l + 1}} \|\hat{f}(l)\|_{\mathcal{HS}} \leq \|\hat{f}(l)\|_{\mathcal{HS}}.
\]
This proves inequality (2.10) and Remark 2.6.

For completeness, we give a simple argument for Corollary 2.3.

**Proof of Corollary 2.3.** The application of the duality of $L^p$ spaces and Plancherel’s identity (2.1) yields
\[
\|f\|_{L^p(SU(2))} = \sup_{g \in L^{p'}} \left( \frac{(f, g)_{L^p(SU(2))}}{\|g\|_{L^{p'}}^2} \right) = \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1) \operatorname{Tr} \left( \hat{f}(l) \hat{g}(l)^* \right),
\]
It is easy to see that
\[
(2l + 1) = (2l + 1)^{\frac{2}{p} - \frac{2}{p'} + \frac{2}{p'}},
\]
\[
\left| \operatorname{Tr} \left( \hat{f}(l) \hat{g}(l)^* \right) \right| \leq \|\hat{f}(l)\|_{\mathcal{HS}} \|\hat{g}(l)\|_{\mathcal{HS}}.
\]
Using these inequalities, applying Hölder inequality, for any $g \in L^{p'}$ with $\|g\|_{L^{p'}} = 1$, we have
\[
\|f\|_{L^p(SU(2))} \leq \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)^{\frac{2}{p} - \frac{2}{p'}} \|\hat{f}(l)\|_{\mathcal{HS}} (2l + 1)^{\frac{2}{p'} - \frac{2}{p}} \|\hat{g}(l)\|_{\mathcal{HS}}
\]
\[
\leq \left( \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)^{\frac{2}{p} - \frac{2}{p'}} \|\hat{f}(l)\|_{\mathcal{HS}}^p \right)^{\frac{1}{p'}} \left( \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)^{\frac{2}{p'} - \frac{2}{p}} \|\hat{g}(l)\|_{\mathcal{HS}}^{p'} \right)^{\frac{1}{p}}
\]
\[
\leq \left( \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)^{\frac{2}{p} - \frac{2}{p'}} \|\hat{f}(l)\|_{\mathcal{HS}}^p \right)^{\frac{1}{p'}} \|g\|_{L^{p'}} \leq \left( \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)^{\frac{2}{p} - \frac{2}{p'}} \|\hat{f}(l)\|_{\mathcal{HS}}^p \right)^{\frac{1}{p}},
\]
where we used Theorem 2.2 in the last line. This proves (2.6).

**3. LOWER BOUNDS FOR FOURIER MULTIPLIERS ON SU(2)**

Let $A: C^\infty(SU(2)) \to \mathcal{D}'(SU(2))$ be a continuous linear operator. Here we are concerned with left-invariant operators which means that $A \circ \tau_g = \tau_g \circ A$ for the left-translation $\tau_g f(x) = f(g^{-1}x)$. Using the Schwartz kernel theorem and the Fourier inversion formula one can prove that the left-invariant continuous operator $A$ can be written as a Fourier multiplier, namely, as
\[
\hat{A}f(l) = \sigma_A(l) \hat{f}(l),
\]
for the symbol $\sigma_A(l) \in C^{(2l + 1) \times (2l + 1)}$. It follows from the Fourier inversion formula that we can write this also as
\[
Af(u) = \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1) \operatorname{Tr} \left( t^l(u) \sigma_A(l) \hat{f}(l) \right),
\]
(3.1)
where the symbol $\sigma_A(l)$ is given by
$$\sigma_A(l) = t^l(e)^* A t^l(e),$$
where $e$ is an identity matrix in $SU(2)$, and $(A t^l)_{m,k} = A(t^l_{m,k})$ is defined component-wise, for $-l \leq m, n \leq l$. We refer to operators in these equivalent forms as (non-commutative) Fourier multipliers. The class of these operators on $SU(2)$ and their $L^p$-boundedness was investigated in [CW71b, CW71a], and on general compact Lie groups in [RW13]. In particular, these authors proved Hörmander–Mikhlin type multiplier theorems in those settings, giving sufficient condition for the $L^p$-boundedness in terms of symbols. These conditions guarantee that the operator is of weak $(1,1)$-type which, combined with a simple $L^2$-boundedness statement, implies the boundedness on $L^p$ for all $1 < p < \infty$.

For a general (non-invariant) operator $A$, its matrix symbol $\sigma_A(u, l)$ will also depend on $u$. Such quantization (3.1) has been consistently developed in [RT10] and [RT13]. We note that the $L^p$-boundedness results in [RW13] also cover such non-invariant operators.

For a noncommutative Fourier multiplier $A$ we will write $A \in M^q_p(SU(2))$ if $A$ is a bounded operator from $L^p(SU(2))$ to $L^q(SU(2))$. We introduce a norm $\| \cdot \|$ on $M^q_p(SU(2))$ by setting
$$\|A\|_{M^q_p} = \|A\|_{L^p \rightarrow L^q}.$$ 
Thus, we are concerned with the question of what assumptions on the symbol $\sigma_A$ guarantee that $A \in M^q_p(SU(2))$. The sufficient conditions on $\sigma_A$ for $A \in M^q_p$ were investigated in [RW13]. The aim of this section is to give a necessary condition on $\sigma_A$ for $A \in M^q_p$, for $1 < p \leq 2 \leq q < \infty$.

Suppose that $1 < p \leq 2 \leq q < \infty$ and that $A: L^p(SU(2)) \rightarrow L^q(SU(2))$ is a Fourier multiplier. The Plancherel identity (2.1) implies that the operator $A$ is bounded from $L^2(SU(2))$ to $L^2(SU(2))$ if and only if $\sup_l \|\sigma_A(l)\|_{op} < \infty$. Different other function spaces on the unitary dual have been discussed in [RT10]. Following Stein, we search for more subtle conditions on the symbols of noncommutative Fourier multipliers ensuring their $L^p - L^q$ boundedness, and we now prove a lower estimate which depends explicitly on $p$ and $q$.

**Theorem 3.1.** Let $1 < p \leq 2 \leq q < \infty$ and let $A$ be a left-invariant operator on $SU(2)$ such that $A \in M^q_p(SU(2))$. Then we have

$$\sup_{l \in \frac{1}{2} \mathbb{N}_0} \min_{n \in \{-l, \ldots, +l\}} |\sigma_A(l)_{mn}| \lesssim \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))},$$

(3.2)

$$\sup_{l \in \frac{1}{2} \mathbb{N}_0} \frac{|\text{Tr} \, \sigma_A(l)|}{(2l + 1)^{1 + \frac{1}{p} + \frac{1}{q}}} \lesssim \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))},$$

(3.3)

One can see a similarity between (3.2), (3.3) and (1.1) as

$$\sup_{l \in \frac{1}{2} \mathbb{N}_0} \frac{1}{(2l + 1)^{\frac{1}{p} + \frac{1}{q}}} \left( \frac{1}{2l + 1} |\text{Tr} \, \sigma_A(l)| \right) \lesssim \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))}.$$
Fourier inversion formula we get
\[ \| t_{nn}^l \|_{L^p(SU(2))} \cong \frac{1}{(2l + 1)^{\frac{1}{p}}}. \]

Now, we use this result to establish a lower bound for the norm of \( A \in M^q_p(SU(2)) \). Let us fix an arbitrary \( l_0 \in \frac{1}{2} \mathbb{N}_0 \) and the corresponding diagonal element \( t_{nn}^l \). We consider \( f_{l_0}(g) \) such that its matrix-valued Fourier coefficient
\[ \widehat{f}_{l_0}(l) = \text{diag}(0, \ldots, 1, 0, \ldots) \delta^l_{l_0} \]
has only one non-zero diagonal coefficient 1 at the \( n^{th} \) diagonal entry. Then by the Fourier inversion formula we get \( f_{l_0}(g) = (2l_0 + 1) t_{nn}^l (g) \). By definition, we get
\[
\| A \|_{L^p \rightarrow L^q} = \sup_{f \neq 0} \frac{\| \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1) \text{Tr} \left( t^l(u) \sigma_A(l) \widehat{f}(l) \right) \|_{L^q(SU(2))}}{\| f \|_{L^p(SU(2))}}
\geq \frac{\| \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1) \text{Tr} \left( t^l(u) \sigma_A(l) \widehat{f}_{l_0}(l) \right) \|_{L^q(SU(2))}}{\| f_{l_0} \|_{L^p(SU(2))}}.
\]
Recalling (3.6), we get
\[
\| A \|_{L^p \rightarrow L^q} \geq \frac{\| (2l_0 + 1) \text{Tr} \left( t^l(u) \sigma_A(l) \widehat{f}_{l_0}(l) \right) \|_{L^q(SU(2))}}{\| f_{l_0} \|_{L^p(SU(2))}}.
\]
Setting \( h(g) := (2l_0 + 1) \text{Tr} \left( t^l(u) \sigma_A(l) \widehat{f}_{l_0}(l) \right) \), we have \( \widehat{h}(l) = 0 \) for \( l \neq l_0 \), and \( \widehat{h}(l_0) = \sigma_A(l_0) \widehat{f}_{l_0}(l_0) \). Consequently, we get
\[
\sup_{k \in \frac{1}{2} \mathbb{N}_0} \frac{1}{2k + 1} \text{Tr} \widehat{h}(k) = \begin{cases} 0, & l > l_0, \\ \frac{1}{2l_0 + 1} | \sigma_A(l_0)_{nn} |, & 1 \leq l \leq l_0. \end{cases}
\]
Using this, Theorem 2.5 and (3.5), we have
\[
\| A \|_{L^p \rightarrow L^q} \geq \frac{\left( \sum_{l=1}^{l_0} (2l + 1)^{q-2} \left( \frac{1}{2l_0 + 1} | \sigma_A(l_0)_{nn} | \right)^q \right)^{\frac{1}{q}}}{(2l_0 + 1)^{1 - \frac{1}{p}}}.
\]
where \( l_0 \) is an arbitrary fixed half-integer. Direct calculation now shows that
\[
\left( \sum_{l=1}^{l_0} (2l + 1)^{q-2} \left( \frac{1}{2l_0 + 1} | \sigma_A(l_0)_{nn} | \right)^q \right)^{\frac{1}{q}} = \frac{1}{2l_0 + 1} | \sigma_A(l_0)_{nn} | \left( \sum_{l=1}^{l_0} (2l + 1)^{q-2} \right)^{\frac{1}{q}} \approx \frac{| \sigma_A(l_0)_{nn} |}{(2l_0 + 1)^{\frac{1}{p} + \frac{1}{q}}}.
\]
Taking infimum over all \( n \in \{-l_0, -l_0 + 1, \ldots, l_0 - 1, l_0\} \) and then supremum over all half-integers, we have

\[
\|A\|_{L^p \to L^q} \gtrsim \sup_{l \in \frac{1}{2} \mathbb{N}_0} \min_{n \in \{-l, \ldots, l\}} |\sigma_A(l)_{nn}| (2l + 1)^{\frac{1}{p} - \frac{1}{q}}.
\]

This proves estimate (3.2). Now, we will prove estimate (3.3). Let us fix some \( l_0 \in \frac{1}{2} \mathbb{N}_0 \) and consider now \( f_{l_0}(u) := (2l_0 + 1)\chi_{l_0}(u) \), where \( \chi_{l_0}(u) = \text{Tr} t^{l_0}(u) \) is the character of the representation \( t^{l_0} \). Then, in particular, we have

\[
(3.7) \quad \widehat{f}_{l_0}(l) = \begin{cases} 1, & l = l_0, \\ 0, & l \neq l_0, \end{cases}
\]

where \( I_{2l+1} \in \mathbb{C}^{(2l+1) \times (2l+1)} \) is the identity matrix. Using the Weyl character formula, we can write

\[
\chi_{l_0}(u) = \sum_{k=-l_0}^{l_0} e^{ikt},
\]

where \( u = v^{-1} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} v \). The value of \( \chi_{l_0}(u) \) does not depend on \( v \) since characters are central. Further, the application of the Weyl integral formula yields

\[
\|f_{l_0}\|_{L^p(SU(2))} = (2l_0 + 1)\|\chi_{l_0}\|_{L^p(SU(2))} = (2l_0 + 1) \left( \int_0^{2\pi} \left| \sum_{k=-l_0}^{l_0} e^{ikt} \right|^p 2\sin^2 t \frac{dt}{2\pi} \right)^{\frac{1}{p}}.
\]

It is clear that \( \left| e^{(-l_0+1)t} \sum_{k=-l_0}^{l_0} e^{i(k+l_0+1)t} \right| = \left| \sum_{k=1}^{2l_0+1} e^{ikt} \right| \). We call \( D_{2l_0+1}(t) := \sum_{k=1}^{2l_0+1} e^{ikt} \) the Dirichlet kernel. Then, we apply inequality (B.3) from Appendix to the Dirichlet kernel \( D_{2l_0+1}(t) \), to get

\[
(3.8) \quad \|\chi_{l_0}\|_{L^p(SU(2))} \lesssim \|D_{2l_0+1}\|_{L^p(0,2\pi)} \approx (2l_0 + 1)^{1 - \frac{1}{p}}.
\]

By definition, we get

\[
\|A\|_{L^p \to L^q} = \sup_{f \neq 0} \frac{\|\sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1) \text{Tr} t^{l}(u)\sigma_A(l) \widehat{f}(l)\|_{L^q(SU(2))}}{\|f\|_{L^p(SU(2))}} \geq \frac{\|\sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1) \text{Tr} t^{l_0}(u)\sigma_A(l) \widehat{f}_{l_0}(l)\|_{L^q(SU(2))}}{\|\widehat{f}_{l_0}\|_{L^p(SU(2))}}.
\]

Recalling (3.7), we obtain

\[
\|A\|_{L^p \to L^q} \gtrsim \frac{(2l_0 + 1) \text{Tr} (t^{l_0}(g)\sigma_A(l_0))}{\|\widehat{f}_{l_0}\|_{L^p(SU(2))}}.
\]
Setting \( h(g) := (2l_0 + 1) \text{Tr} \left( t^{l_0}(g)\sigma_A(l_0) \right) \), we have \( \hat{h}(l) = 0 \) for \( l \neq l_0 \), and \( \hat{h}(l_0) = \sigma_A(l_0) \). Consequently, we get

\[
\sup_{k \geq l_0} \frac{1}{2k + 1} \left| \text{Tr} \hat{h}(k) \right| = \begin{cases} 0, & l > l_0, \\ \frac{1}{2l_0 + 1} \left| \text{Tr} \sigma_A(l_0) \right|, & 1 \leq l \leq l_0. \end{cases}
\]

Using this and Theorem 2.5, we have

\[
\|A\|_{L^p \rightarrow L^q} \gtrsim \left( \sum_{l=1}^{l_0} (2l + 1)^{q-2} \left| \frac{1}{2l_0 + 1} \left| \text{Tr} \sigma_A(l_0) \right| \right|^q \right)^{\frac{1}{q}} \left( \sum_{l=1}^{l_0} (2l + 1)^{-\frac{1}{p}} \right)^{\frac{1}{q}}
\]

where \( l_0 \) is an arbitrary fixed half-integer. Direct calculation shows that

\[
\left( \sum_{l=1}^{l_0} (2l + 1)^{q-2} \left| \frac{1}{2l_0 + 1} \left| \text{Tr} \sigma_A(l_0) \right| \right|^q \right)^{\frac{1}{q}} = \frac{1}{2l_0 + 1} \left| \text{Tr} \sigma_A(l_0) \right| \left( \sum_{l=1}^{l_0} (2l + 1)^{-\frac{1}{p}} \right)^{\frac{1}{q}}
\]

Taking supremum over all half-integers, we have

\[
\|A\|_{L^p \rightarrow L^q} \gtrsim \sup_{l \in \frac{1}{2} \mathbb{N}_0} \frac{\left| \text{Tr} \sigma_A(l) \right|}{(2l + 1)^{1+\frac{1}{p}+\frac{1}{q}}}
\]

This proves the estimate (3.3) \( \square \)

### 4. Upper bounds for Fourier multipliers on SU(2)

In this section we give a noncommutative SU(2) analogue of the upper bound for Fourier multipliers, analogous to the one on the circle \( \mathbb{T} \) in Theorem 1.1.

In the proof, we will use the Lorentz sequence spaces. For the basic background on the properties of Lorentz spaces that we use we refer to Appendix A. For more details see e.g. Stein and Weiss [SW71, Chapter 5].

We will give two formulations of the upper bound in Theorem 4.1. For the Lorentz type formulation, we consider the measure space \((\mathbb{N}, \Sigma_{\mathbb{N}}, \nu)\) that we define by

\[
\Sigma_{\mathbb{N}} := \{Q \subset \mathbb{N}: Q \text{ is finite}\}
\]

and

\[
\nu(Q) := \sum_{n \in Q} n.
\]

We will consider the non-increasing rearrangements with respect to the measure \( \nu \). For examples of this, we refer to Appendix A.
Theorem 4.1. If $1 < p \leq 2 \leq q < \infty$ and $A$ is a left-invariant operator on $SU(2)$, then we have

\begin{equation}
\|A\|_{L^p(SU(2)) \to L^q(SU(2))} \lesssim \sup_{s > 0} \left( \sum_{l \in \frac{1}{2}N_0} (2l + 1)^4 \right)^{\frac{1}{p} + \frac{1}{q}}.
\end{equation}

Moreover, if we define the sequence

\begin{equation}
\lambda_n := \begin{cases} \|\sigma_A(l)\|_{op}, & n = (2l + 1)^4, \ l \in \frac{1}{2}N_0, \\ 0, & \text{otherwise}, \end{cases}
\end{equation}

and let $\lambda^*_n$ be the non-increasing rearrangement of $\lambda_n$ with respect to $\nu$, then we have

\begin{equation}
\|A\|_{L^p(SU(2)) \to L^q(SU(2))} \lesssim \sup_{k \in \mathbb{N}} \frac{1}{k^{\frac{1}{p} + \frac{1}{q}}} \sum_{n=1}^{k} n \lambda^*_n.
\end{equation}

Proof. Let $1 < p \leq 2 \leq q < \infty$ and let $f \in L^p(SU(2))$. By (3.1), we have

\begin{equation}
\|Af\|_{L^q(SU(2))} = \left\| \sum_{l \in \frac{1}{2}N_0} (2l + 1) \text{Tr} \left( t^l(u)\sigma_A(l)\hat{f}(l) \right) \right\|_{L^q(SU(2))}.
\end{equation}

For technical convenience we introduce an embedding of matrix Fourier coefficients $\hat{f}(l)$ into a sequence space as follows:

\begin{equation}
\{\hat{f}(l)\}_{l \in \frac{1}{2}N_0} \ni \hat{f}(l) \mapsto a_n := \begin{cases} (2l + 1)^{-\frac{3}{2}} \|\hat{f}(l)\|_{HS}, & n = (2l + 1)^4, \ l \in \frac{1}{2}N_0, \\ 0, & \text{otherwise}. \end{cases}
\end{equation}

Then, we also embed the symbol $\sigma_A(l)$ into a Lorentz sequence space by using formula (4.2).

Let us denote by $a^*_n$, $\lambda^*_n$ the non-increasing rearrangements of $a_n$ and $\lambda_n$, respectively, with respect to measure $\nu$. Consequently, there is a permutation $p: \mathbb{N} \to \mathbb{N}$ such that $a^*_n = a_p(n)$. Let us denote by $\zeta$ the corresponding permutation on $\frac{1}{2}N_0$ given by $\zeta_l = \frac{p(2l + 1) - 1}{2}$.

Using this notation, we can reformulate Theorem 2.2 and Corollary 2.3 as follows: if $1 < p \leq 2$ and $f \in L^p(SU(2))$, then

\begin{equation}
\sum_{l \in \frac{1}{2}N_0} (2\zeta_l + 1)^{\frac{3p}{2} - 4} \|\hat{f}(\zeta_l)\|_{HS}^p \lesssim \|f\|_{L^p(SU(2))}^p.
\end{equation}

Consequently, as the counterpart of Corollary 2.3, for $2 \leq q < \infty$, we have

\begin{equation}
\|f\|_{L^q(SU(2))}^q \lesssim \sum_{l \in \frac{1}{2}N_0} (2\zeta_l + 1)^{\frac{3q}{2} - 4} \|\hat{f}(\zeta_l)\|_{HS}^q.
\end{equation}

We will first prove the estimate

\begin{equation}
\|A\|_{M^p} \lesssim \sup_{k \in \mathbb{N}} \frac{1}{k^{\frac{p}{2} - 1}} \frac{1}{k} \sum_{n=1}^{k} \lambda^*_n \nu_n,
\end{equation}

where $\lambda_n := \frac{1}{k^{\frac{p}{2} - 1}} \frac{1}{k} \sum_{n=1}^{k} \lambda^*_n \nu_n$. 


which is equivalent to (4.3), with \( \nu_n = \nu\{n\} = n \) denoting the measure we are using. Using (4.6), we get

\[
\|Af\|_q^{\nu(SU(2))} \lesssim \sum_{l \in \mathbb{N}_0} (2 \zeta_l + 1)^{q-4} \|\sigma_A(\zeta_l) \hat{f}(\zeta_l)\|_{HS}^q
\]

\[
\leq \sum_{l \in \mathbb{N}_0} (2 \zeta_l + 1)^{q-4} (\|\sigma_A(\zeta_l)\|_{op} \|\hat{f}(\zeta_l)\|_{HS})^q
\]

\[
= \sum_{l \in \mathbb{N}_0} (2 \zeta_l + 1)^{4(q-2)} (\|\sigma_A(\zeta_l)\|_{op} \|\hat{f}(\zeta_l)\|_{HS}^q (2 \zeta_l + 1)^{4}.
\]

Now if we recall notation (4.2) and (4.4), we obtain

\[
\|Af\|_q^{\nu(SU(2))} \lesssim \sum_{n \in \mathbb{N}} n^{q-2} (\lambda_n^* a_n^*)^q \nu_n.
\]

We now claim that we have the estimate

\[
\sum_{n=1}^{\infty} n^{q-2} (\lambda_n^* a_n^*)^q \nu_n \lesssim \sum_{m=0}^{2m+1-1} \sum_{n=2m} (\lambda_n^* a_n^*)^q \nu_n.
\]

We will often use the following simple auxiliary lemma

**Lemma 4.2.** Suppose \( \gamma \in \mathbb{R}_+ \). Then we have

\[
\sum_{n=2^m}^{2^{m+1}-1} n^\gamma \approx 2^{m(\gamma+1)}.
\]

To prove (4.9), we split the sum \( \sum_{n=1}^{\infty} n^{q-2} (\lambda_n^* a_n^*)^q \nu_n \) into dyadic blocks:

\[
\sum_{n=1}^{\infty} n^{q-2} (\lambda_n^* a_n^*)^q \nu_n = \sum_{m=0}^{2m+1-1} \sum_{n=2^m}^{2^{m+1}-1} n^{q-2} (\lambda_n^* a_n^*)^q \nu_n.
\]

Since \( \lambda_n^* a_n^* \) is a non-increasing sequence, we can estimate

\[
\sum_{n=2^m}^{2^{m+1}-1} n^{q-2} (\lambda_n^* a_n^*)^q \nu_n \leq (\lambda_{2m}^* a_{2m}^*)^q \sum_{n=2^m}^{2^{m+1}-1} n^{q-2} \nu_n.
\]

The application of Lemma 4.2 yields

\[
\sum_{n=2^m}^{2^{m+1}-1} n^{q-2} \nu_n = \sum_{n=2^m}^{2^{m+1}-1} n^{q-1} \approx 2^{qm}.
\]

Therefore, we have

\[
\sum_{n=2^m}^{2^{m+1}-1} n^{q-2} (\lambda_n^* a_n^*)^q \nu_n \lesssim (2 \lambda_{2m}^* a_{2m}^*)^{2m}.
\]
Since $\lambda_n^* \cdot a_n^*$ is non-increasing, we can also estimate
\[
2^{\frac{m}{q'}} \lambda_n^* a_n^* \lesssim \left( \sum_{n=2^{m-1}}^{2^m-1} (\lambda_n^* a_n^*)^{q'} \right)^{\frac{1}{q'}}.
\]
Combining these inequalities and using that $2^m \leq 2^{\frac{m}{q'}}$, we get
\[
\sum_{n=2^m}^{2^{m+1}-1} n^{q-2} (\lambda_n^* a_n^*)^{q'} \nu_n \lesssim \left( \sum_{n=2^{m-1}}^{2^m-1} (\lambda_n^* a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}}.
\]
Finally, we obtain
\[
\sum_{n=1}^{\infty} n^{q-2} (\lambda_n^* a_n^*)^{q'} \nu_n = \sum_{m=0}^{\infty} \sum_{n=2^m}^{2^{m+1}-1} n^{q-2} (\lambda_n^* a_n^*)^{q'} \nu_n
\]
\[
= (\lambda^*_1 a^*_1)^q + \sum_{m=1}^{\infty} \sum_{n=2^m}^{2^{m+1}-1} n^{q-2} (\lambda_n^* a_n^*)^{q'} \nu_n \lesssim (\lambda^*_1 a^*_1)^q + \sum_{m=1}^{\infty} \left( \sum_{n=2^{m-1}}^{2^m-1} (\lambda_n^* a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}}
\]
\[
\lesssim (\lambda^*_1 a^*_1)^q + \sum_{m=0}^{\infty} \left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^* a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}} \lesssim \sum_{m=0}^{\infty} \left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^* a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}}.
\]
This proves the claimed inequality (4.9). We now write $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Using Hölder inequality
\[
\left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^* a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}} \leq \left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^* a_n^*)^{r} \nu_n \right)^{\frac{q}{q'}} \left( \sum_{n=2^m}^{2^{m+1}-1} (a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}},
\]
we get
\[
\sum_{n=1}^{\infty} n^{q-2} (\lambda_n^* a_n^*)^{q'} \nu_n \lesssim \sum_{m=0}^{\infty} \left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^* a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}}
\]
\[
\lesssim \sum_{m=0}^{\infty} \left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^* a_n^*)^{r} \nu_n \right)^{\frac{q}{q'}} \left( \sum_{n=2^m}^{2^{m+1}-1} (a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}}
\]
\[
\lesssim \sup_{m \in \mathbb{N}_0} \left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^* a_n^*)^{r} \nu_n \right)^{\frac{q}{q'}} \sum_{m=0}^{\infty} \left( \sum_{n=2^m}^{2^{m+1}-1} (a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}}.
\]
We now claim that we can estimate
\[
\sum_{m=0}^{\infty} \left( \sum_{n=2^m}^{2^{m+1}-1} (a_n^*)^{q'} \nu_n \right)^{\frac{q}{q'}} \lesssim \sum_{n=1}^{\infty} (n^{r'} a_n^*)^{q'} \nu_n = \|a\|_{r',q'}^q.
\]
Indeed, arguing as above and applying Lemma 4.2 to the sums
\[
\sum_{n=2^m}^{2^{m+1}-1} \nu_n, \quad \sum_{n=2^{m-1}}^{2^m-1} \nu_n, \quad \text{with } \nu_n = n,
\]
we see that
\[
\sum_{m=0}^{\infty} \left( \sum_{n=2^m}^{2^{m+1}-1} (a_n^*)^p \nu_n \right)^{\frac{2}{p'}} = (a_1^*)^q + \sum_{m=1}^{\infty} \left( \sum_{n=2^m}^{2^{m+1}-1} (a_n^*)^p \nu_n \right)^{\frac{2}{p'}}
\]

(since \(a_n^*\) is decreasing)
\[
\lesssim (a_1^*)^q + \sum_{m=1}^{\infty} \left( \sum_{n=2^m}^{2^{m+1}-1} \nu_n \right)^{\frac{2}{p'}} \lesssim (a_1^*)^q + \sum_{m=1}^{\infty} \left( a_{2^m}^* 2^{m\frac{2}{p'}} \right)^q (2^m)^{\frac{2}{p'p}}.
\]

Since \(M_p^q = M_{q'}^{p'}\) we may assume that \(q < p'\), for otherwise we have \(p' < (q')' = q\).
Therefore \((2^m)^{\frac{2}{p'}} \leq 2^m\) and we get
\[
(a_1^*)^q + \sum_{m=1}^{\infty} \left( a_{2^m}^* 2^{m\frac{2}{p'}} \right)^q (2^m)^{\frac{2}{p'}} \leq (a_1^*)^q + \sum_{m=1}^{\infty} \left( a_{2^m}^* 2^{m\frac{2}{p'}} \right)^q 2^m
\]
\[
\lesssim (a_1^*)^q + \sum_{m=1}^{\infty} \left( a_{2^m}^* 2^{m\frac{2}{p'}} \right)^q \sum_{n=2^{m-1}}^{2^m-1} 1 = (a_1^*)^q + \sum_{m=1}^{\infty} \left( a_{2^m}^* 2^{m\frac{2}{p'}} \right)^q \sum_{n=2^{m-1}}^{2^m-1} \frac{\nu_n}{n}
\]
(since \(2^{m-1} a_{2^m}^* \leq n^{\frac{1}{p'}} a_n^*\), 
\(2^{m-1} \leq n \leq 2^m - 1\))
\[
\lesssim (a_1^*)^q + \sum_{m=1}^{\infty} \sum_{n=2^{m-1}}^{2^m-1} \left( n^{\frac{1}{p'}} a_n^* \right)^q \frac{\nu_n}{n} \lesssim \sum_{m=0}^{\infty} \sum_{n=2^m}^{2^{m+1}-1} \left( n^{\frac{1}{p'}} a_n^* \right)^q \frac{\nu_n}{n}.
\]

This proves (4.11). We claim that we also have
\[
(4.12) \quad \left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^*)^q \nu_n \right)^{\frac{1}{q}} \lesssim \frac{1}{k^{1/q}} \sum_{n=1}^{k} \lambda_n^* \nu_n, \quad 2^m \leq k \leq 2^{2m}.
\]

Indeed, recalling that \(\lambda_n^*\) is a non-increasing sequence, we have
\[
\left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^*)^q \nu_n \right)^{\frac{1}{q}} \leq \lambda_{2^m}^* \left( \sum_{n=2^m}^{2^{m+1}-1} \nu_n \right)^{\frac{1}{q}}.
\]

Now, we apply Lemma 4.2 to get
\[
\sum_{n=2^m}^{2^{m+1}-1} \nu_n = \sum_{n=2^m}^{2^{m+1}-1} n \cong 2^{2m}.
\]

Hence
\[
\left( \sum_{n=2^m}^{2^{m+1}-1} (\lambda_n^*)^q \nu_n \right)^{\frac{1}{q}} \lesssim \lambda_{2^m}^* (2^{2m})^{\frac{1}{q}}.
\]

Further, we split the last expression into two multiples
\[
\lambda_{2^m}^* (2^{2m})^{\frac{1}{q}} = 2^m \lambda_{2^m}^* \cdot (2^{2m})^{\frac{1}{q} - 1}.
\]
We estimate the first term as
\[
2^{2m} \lambda_{2^m}^* \leq \sum_{n=2^{m-1}}^{2^m-1} \lambda_n^* \nu_n.
\]

Combining these inequalities, we have
\[
2^{2m} \lambda_{2^m}^* \cdot (2^{2m})^{\frac{1}{q} - 1} \leq \frac{1}{(2^{2m})^{1 - \frac{1}{q}}} \sum_{n=2^{m-1}}^{2^m-1} \lambda_n^* \nu_n.
\]

Since \(2^m \leq k \leq 2^{2m}\) and \(\lambda_n^*\) is non-negative for any \(n\), we get
\[
\left( \sum_{n=2^{m-1}}^{2^m-1} (\lambda_n^*)^q \nu_n \right)^{\frac{1}{q}} \leq \frac{1}{k^{1 - \frac{1}{q}}} \sum_{n=1}^{k} \lambda_n^* \nu_n = \frac{1}{k^{\frac{1}{q} + \frac{1}{p}}} \sum_{n=1}^{k} \lambda_n^* \nu_n,
\]
where we used the relation \(\frac{1}{q} = \frac{1}{p} - \frac{1}{q}\). This proves (4.12).

Using inequalities (4.11), (4.12) and embedding \(l_{p', q'} \hookrightarrow l_{p', q}\), summarising the above steps, we have
\[
\|Af\|_{L^p(SU(2))}^q \leq \sum_{n=1}^{\infty} \left( \frac{n}{q}\right)^{q/2} (\lambda_n^*)^q \nu_n \leq \sum_{n=1}^{\infty} \left( \frac{\sum_{m=0}^{2m-1} (\lambda_n^*)^{q/2} \nu_n}{\sum_{m=2^m}^{2^{m+1}-1} (\lambda_n^*)^{q/2} \nu_n} \right)^{q/2} \nu_n
\]
\[
\leq \left( \sup_{k \in \mathbb{N}} \frac{1}{k^{\frac{1}{q} + \frac{1}{p}}} \sum_{n=1}^{k} \lambda_n^* \nu_n \right)^{q/2} \sum_{n=1}^{\infty} \left( \frac{n}{q} \right)^{q/2} \nu_n \leq \left( \Psi_{p,q} \|a\|_{l_{p', q}} \right)^q.
\]

Now, we rewrite inequality (4.5) in terms of the Lorentz sequence space (see Appendix A) norm as
\[
\|a\|_{l_{p', q}} \leq \|f\|_{L^p(SU(2))}, \quad 1 < p \leq 2.
\]

Thus, we have
\[
\|Af\|_{L^p(SU(2))} \leq \Psi_{p,q} \|a\|_{l_{p', q}} \leq \Psi_{p,q} \|f\|_{L^p(SU(2))}
\]

This proves (4.7). Now, we have
\[
\|A\|_{M_p^q} \leq \sup_{k \in \mathbb{N}} \frac{1}{k^{\frac{1}{q}}} \sum_{n=1}^{k} \lambda_n^* \nu_n,
\]
where \(\frac{1}{q} = \frac{1}{p} - \frac{1}{q}\) and \(1 < p \leq 2 \leq q < \infty\).

Let us estimate the right-hand side here as
\[
k^{\frac{1}{q} - 1} \sum_{n=1}^{k} \lambda_n^* \nu_n = k^{\frac{1}{q} - 1} \sum_{n=1}^{k} n^{\frac{1}{q} + 1} \lambda_n^* n^{-\frac{1}{q}} \leq \left( \sup_{n \in \mathbb{N}} n^{\frac{1}{q} + 1} \lambda_n^* \right) \left( k^{\frac{1}{q} - 1} \sum_{n=1}^{k} n^{-\frac{1}{q}} \right)
\]
\[
\leq \sup_{n \in \mathbb{N}} n^{\frac{1}{q} + 1} \lambda_n^* = \sup_{n \in \mathbb{N}} n^{\frac{1}{p} + \frac{1}{q}} \lambda_n^*.
\]
Now, we use the properties of decreasing rearrangements. It can be proven (see [Gra08]) that

$$\sup_{n \in \mathbb{N}} n^{\frac{1}{p} + \frac{1}{q'}} \lambda_n^* = \sup_{s > 0} s \left( \mu_\lambda(s) \right)^{\frac{1}{p} + \frac{1}{q'}},$$

where

$$\mu_\lambda(s) = \nu \{ n \in \mathbb{N} : |\lambda_n| > s \}.$$

Here, the measure $\nu$ is given by

$$\nu(Q) = \sum_{n \in Q} n, \quad Q \in \Sigma_\mathbb{N}.$$ 

Hence, we get

$$\sup_{n \in \mathbb{N}} n^{\frac{1}{p} + \frac{1}{q'}} \lambda_n^* = \sup_{s > 0} s \left( \sum_{n \in \mathbb{N} : |\lambda_n| > s} n \right)^{\frac{1}{p} + \frac{1}{q'}}.$$

Recall definition (4.2),

$$\lambda_n = \begin{cases} \| \sigma_A(l) \|_{op}, & n = (2l + 1)^4, l \in \frac{1}{2} \mathbb{N}_0, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, we have

$$\sum_{n \in \mathbb{N} : |\lambda_n| > s} n = \sum_{l \in \frac{1}{2} \mathbb{N}_0 : \| \sigma_A(l) \|_{op} > s} (2l + 1)^4.$$ 

We then get

$$\sup_{n \in \mathbb{N}} n^{\frac{1}{p} + \frac{1}{q'}} \lambda_n^* = \sup_{s > 0} s \left( \sum_{l \in \frac{1}{2} \mathbb{N}_0 : \| \sigma_A(l) \|_{op} > s} (2l + 1)^4 \right)^{\frac{1}{p} + \frac{1}{q'}},$$

completing the proof. $\square$

5. Proof of Theorem 2.2

In this section we prove the Hardy–Littlewood type inequality given in Theorem 2.2.

**Proof of Theorem 2.2** Let $\nu$ give measure $\frac{1}{(2l + 1)^4}$ to the set consisting of the single point $l, l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and measure zero to a set which does not contain any of these points. We will show that the sub-linear operator

$$Tf := \{(2l + 1)^\frac{5}{2} \| \hat{f}(l) \|_{HS} \}_{l \in \frac{1}{2} \mathbb{N}_0}$$

is well-defined and bounded from $L^1(SU(2))$ to $L^p(\frac{1}{2} \mathbb{N}_0, \nu)$ for $1 < p \leq 2$, with

$$\| Tf \|_{p, \nu} = \left( \sum_{l \in \frac{1}{2} \mathbb{N}_0} ((2l + 1)^\frac{3}{2} \| \hat{f}(l) \|_{HS})^p \cdot (2l + 1)^{-4} \right)^{\frac{1}{p}}.$$
This will prove Theorem 2.2. We first show that $T$ is of type $(2, 2)$ and weak type $(1, 1)$. Using Plancherel’s identity (2.1), we get

$$
\|Tf\|_{L^2(\frac{1}{2}N_0, \nu)}^2 = \sum_{l \in \frac{1}{2}N_0} (2l + 1)^{\frac{3}{2}} - 4 \|\hat{f}(l)\|_{\text{HS}}^2 = \sum_{l \in \frac{1}{2}N_0} (2l + 1) \|\hat{f}(l)\|_{\text{HS}}^2 = \|\hat{f}\|_{L^1(\text{SU}(2))}^2 = \|f\|_{L^2(\text{SU}(2))}^2.
$$

Thus, $T$ is of type $(2, 2)$.

Further, we show that $T$ is of weak type $(1, 1)$; more precisely we show that

$$
(5.1) \quad \nu \left\{ l \in \frac{1}{2}N_0 : (2l + 1)^{\frac{3}{2}} \|\hat{f}(l)\|_{\text{HS}} > y \right\} \leq 4 \frac{\|f\|_{L^1(\text{SU}(2))}}{y}.
$$

The left-hand side here is the sum $\sum 1/(2l + 1)^{\frac{3}{2}}$ taken over those $l \in \frac{1}{2}N_0$ for which $(2l + 1)^{\frac{3}{2}} \|\hat{f}(l)\|_{\text{HS}} > y$. From the definition of the Fourier transform it follows that

$$
\|\hat{f}(l)\|_{\text{HS}} \leq \sqrt{2l + 1} \|f\|_{L^1(\text{SU}(2))}.
$$

Therefore, we have

$$
y < (2l + 1)^{\frac{3}{2}} \|\hat{f}(l)\|_{\text{HS}} \leq (2l + 1)^{\frac{3}{2} + \frac{1}{2}} \|f\|_{L^1(\text{SU}(2))}.
$$

Using this, we get

$$
\left\{ l \in \frac{1}{2}N_0 : (2l + 1)^{\frac{3}{2}} \|\hat{f}(l)\|_{\text{HS}} > y \right\} \subset \left\{ l \in \frac{1}{2}N_0 : (2l + 1) > \left( \frac{y}{\|f\|_{L^1}} \right)^{\frac{3}{2}} \right\}
$$

for any $y > 0$. Consequently,

$$
\nu \left\{ l \in \frac{1}{2}N_0 : (2l + 1)^{\frac{3}{2}} \|\hat{f}(l)\|_{\text{HS}} > y \right\} \leq \nu \left\{ l \in \frac{1}{2}N_0 : (2l + 1) > \left( \frac{y}{\|f\|_{L^1}} \right)^{\frac{3}{2}} \right\}.
$$

We set $w := \left( \frac{y}{\|f\|_{L^1(\text{SU}(2))}} \right)^{\frac{1}{3}}$. Now, we estimate $\nu \left\{ l \in \frac{1}{2}N_0 : (2l + 1) > w \right\}$. By definition, we have

$$
\nu \left\{ l \in \frac{1}{2}N_0 : (2l + 1) > \left( \frac{y}{\|f\|_{L^1}} \right)^{\frac{3}{2}} \right\} = \sum_{n > w} \frac{1}{n^2}.
$$

In order to estimate this series, we introduce the following lemma.

**Lemma 5.1.** Suppose $\beta > 1$ and $w > 0$. Then we have

$$
(5.2) \quad \sum_{n > w} \frac{1}{n^\beta} \leq \begin{cases} \frac{\beta}{\beta - 1}, & w \leq 1, \\ \frac{1}{\beta - 1} \left( \frac{1}{w^{\beta - 1}} \right), & w > 1. \end{cases}
$$

The proof is rather straightforward. Now, suppose $w \leq 1$. Then applying this lemma with $\beta = 4$, we have

$$
\sum_{n > w} \frac{1}{n^4} \leq \frac{4}{3}.
$$
Since $1 \leq \frac{1}{w^3}$, we obtain
\[ \sum_{n>n_0}^{\infty} \frac{1}{n^4} \leq \frac{4}{3} \leq \frac{4}{3} \frac{1}{w^3}. \]

Recalling that $w = \left(\frac{y}{\|f\|_{L^1(SU(2))}}\right)^{\frac{1}{3}}$, we finally obtain
\[ \nu \left\{ l \in \frac{1}{2} \mathbb{N}_0 : (2l + 1) > \left(\frac{y}{\|f\|_{L^1}}\right)^{\frac{1}{3}} \right\} = \sum_{n>n_0}^{\infty} \frac{1}{n^4} \leq \frac{4}{3} \frac{\|f\|_{L^1(SU(2))}}{y}. \]

Now, if $w > 1$, then we have
\[ \sum_{n>n_0}^{\infty} \frac{1}{n^4} \leq \frac{1}{3} \frac{1}{w^3} = \frac{4}{3} \frac{\|f\|_{L^1}}{y}. \]

Finally, we get
\[ \nu \left\{ l \in \frac{1}{2} \mathbb{N}_0 : (2l + 1) > \left(\frac{y}{\|f\|_{L^1}}\right)^{\frac{1}{3}} \right\} \leq \frac{4}{3} \frac{\|f\|_{L^1(SU(2))}}{y}. \]

This proves (5.1). By Marcinkewicz interpolation theorem with $(\alpha_1, \beta_1) = (1, 1), (\alpha_2, \beta_2) = (2, 2)$, and $(\alpha, \beta) = (p, p)$, we have
\[ \left( \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l + 1)^{\frac{p}{2} - 4} \|\widehat{f}(l)\|_{L^p}^{p} \right)^{\frac{1}{p}} \leq \|Tf\|_{p,\nu} \leq c_p \|f\|_{L^p(SU(2))}. \]

This completes the proof of Theorem 2.2. \qed

6. PROOF OF THEOREM 2.5

In this section we prove Theorem 2.5.

Proof of Theorem 2.5 We first simplify the expression for Tr $\widehat{f}(k)$. By definition, we have
\[ \widehat{f}(k) = \int_{SU(2)} f(u)T_k(u)^* \, du, \quad k \in \frac{1}{2} \mathbb{N}_0, \]
where $T_k$ is a finite-dimensional representation of SU(2) as in Section 2. Using this, we get
\[ \text{Tr} \, \widehat{f}(k) = \int_{SU(2)} f(u) \chi_k(u) \, du, \quad (6.1) \]
where $\chi_k(u) = \text{Tr} \, T_k(u), k \in \frac{1}{2} \mathbb{N}_0$, where we changed the notation from $t_k$ to $T_k$ to avoid confusing with the notation that follows. The characters $\chi_k(u)$ are constant on the conjugacy classes of SU(2) and we follow [Vil68] to describe these classes explicitly.
It is well known from linear algebra that any unitary unimodular matrix \( u \) can be written in the form 
\[
\delta = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix},
\]
where \( \lambda = e^{\frac{it}{2}} \) and \( \frac{1}{\lambda} = e^{-\frac{it}{2}} \) are the eigenvalues of \( u \). Moreover, among the matrices equivalent to \( u \) there is only one other diagonal matrix, namely, the matrix \( \delta' \) obtained from \( \delta \) by interchanging the diagonal elements.

Hence, classes of conjugate elements in \( SU(2) \) are given by one parameter \( t \), varying in the limits \(-2\pi \leq t \leq 2\pi\), where the parameters \( t \) and \(-t \) give one and the same class. Therefore, we can regard the characters \( \chi_k(u) \) as functions of one variable \( t \), which ranges from 0 to \( 2\pi \).

The special unitary group \( SU(2) \) is isomorphic to the group of unit quaternions. Hence, the parameter \( t \) has a simple geometrical meaning - it is equal to angle of rotation which corresponds to the matrix \( u \).

Let us now derive an explicit expression for the \( \chi_k(u) \) as function of \( t \). It was shown e.g. in [RT10] that \( T^k(\delta) \) is a diagonal matrix with the numbers \( e^{-int}, -k \leq n \leq k \) on its principal diagonal.

Let \( u = u_1\delta u_1^{-1} \). Since characters are constant on conjugacy classes of elements, we get
\[
\chi_k(u) = \chi_k(\delta) = \text{Tr} \left( T^k(\delta) \right) = \sum_{n=-k}^{k} e^{int}.
\]

It is natural to express the invariant integral over \( SU(2) \) in (6.1) in new parameters, one of which it \( t \).

Since special unitary group \( SU(2) \) is diffeomorphic to the unit sphere \( S^3 \) in \( \mathbb{R}^4 \) (see, e.g., [RT10]), with
\[
SU(2) \ni u = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \leftrightarrow \varphi(u) = x = (x_1, x_2, x_3, x_4) \in S^3,
\]
we have
\[
\int_{SU(2)} f(u)\chi_k(u) \, du = \int_{S^3} f(x)\chi_k(x) \, dS,
\]
where \( f(x) := f(\varphi^{-1}(x)) \), and \( \chi_k(x) := \chi_k(\varphi^{-1}(x)) \). In order to find an explicit formula for this integral over \( S^3 \), we consider the parametrisation
\[
x_1 = \cos \frac{t}{2}, \quad x_2 = v, \quad x_3 = \sqrt{\sin^2 \frac{t}{2} - v^2 \cdot \cos h}, \quad x_4 = \sqrt{\sin^2 \frac{t}{2} - v^2 \cdot \sin h}, \quad (t, v, h) \in D,
\]
where \( D = \{(t, v, h) \in \mathbb{R}^3: |v| \leq \sin \frac{t}{2}, 0 \leq t, h \leq 2\pi\} \).

The reader will have no difficulty in showing that 
\[
dS = \sin \frac{t}{2} dt dv dh.
\]
Therefore, we have
\[
\int_{S^3} f(x) \chi_k(t) dS = \int_D f(h, v, t) \chi_k(t) \sin \frac{t}{2} dh dv dt.
\]
Combining this and (6.4), we get
\[
\text{Tr} \hat{f}(k) = \int_D f(h, v, t) \chi_k(t) \sin \frac{t}{2} dh dv dt.
\]
Thus, we have expressed the invariant integral over SU(2) in the parameters \( t, v, h \).

The application of Fubini’s Theorem yields
\[
\int_D f(h, v, t) \chi_k(t) \sin \frac{t}{2} dh dv dt = \int_0^{2\pi} \chi_k(t) \sin \frac{t}{2} dt \int_{-\sin \frac{t}{2}}^{2\pi} dv \int_0^{2\pi} f(h, v, t) dh.
\]
Combining this and (6.3), we obtain
\[
\text{Tr} \hat{f}(k) = \int_0^{2\pi} dt \sum_{n=-k}^k e^{int} \sin \frac{t}{2} \int_{-\sin \frac{t}{2}}^{2\pi} dv \int_0^{2\pi} f(h, v, t) dh.
\]
Interchanging summation and integration, we get
\[
\text{Tr} \hat{f}(k) = \sum_{n=-k}^k \int_0^{2\pi} e^{int} \sin \frac{t}{2} dt \int_{-\sin \frac{t}{2}}^{2\pi} dv \int_0^{2\pi} f(h, v, t) dh.
\]
By making the change of variables \( t \to 2t \), we get
\[
(6.5) \quad \text{Tr} \hat{f}(k) = \sum_{n=-k}^k \int_0^{\pi} \int_{-\sin t}^{\sin t} e^{-i2nt} \cdot 2 \sin t dt \int_{-\sin t}^{\sin t} dv \int_0^{2\pi} f(h, v, 2t) dh.
\]
Let us now apply Theorem 2.4 in \( L^p(T) \). To do this we introduce some notation. Denote
\[
F(t) := 2 \sin t \int_{-\sin t}^{\sin t} \int_0^{2\pi} f(h, v, 2t) dh dv, \quad t \in (0, \pi).
\]
We extend \( F(t) \) periodically to \([0, 2\pi)\), that is \( F(x + \pi) = F(x) \). Since \( f(t, v, h) \) is integrable, the integrability of \( F(t) \) follows immediately from Fubini’s Theorem. Thus function \( F(t) \) has a Fourier series representation
\[
F(t) \sim \sum_{k \in \mathbb{Z}} \hat{F}(k)e^{ikt},
\]
where the Fourier coefficients are computed by

$$\hat{F}(k) = \int_{[0,2\pi]} F(t)e^{-ikt} dt.$$  

Let $A_k$ be a $2k+1$-element arithmetic sequence with common difference of 2 and its initial terms is $-2k$, i.e.,

$$A_k = \{-2k, -2k+2, \ldots, 2k\} = \{-2k + 2j\}_{j=0}^{2k}.$$  

Using this notation and (6.5), we have

(6.6)  \[ \text{Tr} \hat{f}(k) = \sum_{n \in A_k} \hat{F}(n). \]

Define

$$B = \bigcup_{k=1}^{\infty} A_k.$$  

Using $B \subset M$ and (6.6), we have

(6.7)  \[ \sup_{k \in \frac{1}{2} \mathbb{N}_0} \frac{1}{2k+1} \left| \text{Tr} \hat{f}(k) \right| \leq \sup_{e \in B, |e| \geq 2l+1} \frac{1}{|e|} \left| \sum_{i \in e} \hat{F}(i) \right| \leq \sup_{e \in M, |e| \geq 2k+1} \frac{1}{|e|} \left| \sum_{i \in e} \hat{F}(i) \right|. \]

Denote by $m = 2l+1$. If $l$ runs over $\frac{1}{2} \mathbb{N}_0$, then $m$ runs over $\mathbb{N}$. Using (6.7), we get

(6.8)  \[ \sum_{l \in \frac{1}{2} \mathbb{N}_0} (2l+1)^{p-2} \left( \sup_{k \in \frac{1}{2} \mathbb{N}_0, 2k+1 \geq 2l+1} \frac{1}{2k+1} \left| \text{Tr} \hat{f}(k) \right| \right)^p \leq \sum_{m \in \mathbb{N}} m^{p-2} \left( \sup_{e \in M, |e| \geq m} \frac{1}{|e|} \left| \sum_{i \in e} \hat{F}(i) \right| \right)^p. \]

Application of inequality (2.7) yields

(6.9)  \[ \sum_{m \in \mathbb{N}} m^{p-2} \left( \sup_{e \in M, |e| \geq m} \frac{1}{|e|} \left| \sum_{i \in e} \hat{F}(i) \right| \right)^p \leq c \| F \|_{L^p(0,2\pi)}^p. \]

Using Hölder inequality, we obtain

$$\int_0^\pi |F(t)|^p dt \leq 2^{3p-2} \pi^{p-1} \int_0^\pi \sin t dt \int_0^{2\pi} |f(h,v,2t)|^p dv.$$  

By making the change of variables $t \to \frac{1}{2}$ in the right hand side integral, we get

$$\int_0^\pi |F(t)|^p dt \leq (8\pi)^{\frac{p-1}{p}} \left( \int_0^{2\pi} \sin \frac{t}{2} dt \int_0^{2\pi} |f(h,v,t)|^p dv \right)^{\frac{1}{p}}.$$
Thus, we have proved that
\[(6.10) \quad \|F\|_{L^p(0,\pi)} \leq c_p \|f\|_{L^p(SU(2))},\]
where \(c_p = (8\pi)^{\frac{p-1}{p}}\). Combining (6.7), (6.9) and (6.10), we obtain
\[
\sum_{m \in \mathbb{N}} m^{p-2} \left( \sup_{k \in \mathbb{N}_0} \frac{1}{2k+1} \left| \text{Tr} \hat{f}(k) \right| \right)^p \leq c \|f\|_{L^p(SU(2))}.
\]
This completes the proof. \(\square\)

**Appendix A. Lorentz Spaces and Embeddings**

The study of Lorentz spaces goes back to the work of Lorentz [Lor50], see also [Hun66], [SW71], or [Gra08] for more recent results concerning functional properties of Lorentz spaces. They play an important role in the theory of Banach function spaces, in particular in the interpolation theory of sublinear operators.

Let \(f\) be a complex-valued measurable function defined on a \(\sigma\)-finite measure space \((X, A, \mu)\). For \(s \geq 0\), define the distribution function \(\mu_f\) of \(f\) as
\[(A.1) \quad \mu_f(s) = \mu\{x \in X: |f(x)| > s\}.
\]
By \(f^*\) we mean the non-increasing rearrangement of \(f\) given as
\[(A.2) \quad f^*(t) = \sup\{s > 0: \mu_f(s) \geq t\}, \quad t \geq 0.
\]

**Definition A.1.** Given \(f\) a measurable function on a measure space \((X, A, \mu)\) and \(0 < p, q \leq \infty\), define
\[(A.3) \quad \|f\|_{L^p,q} = \begin{cases} \left( \int_0^\infty \left( t^\frac{1}{p} f^*(t) \right)^q \frac{dt}{t} \right)^\frac{1}{q}, & \text{if } q < \infty, \\ \sup_{t>0} t^\frac{1}{p} f^*(t), & \text{if } q = \infty. \end{cases}
\]

The set of all \(f\) with \(\|f\|_{L^p,q}\) is denoted by \(L^p,q(X, \mu)\) and is called the Lorentz space with indices \(p\) and \(q\). The Lorentz spaces are a “logarithmic” refinement of the Lebesgue spaces that can help to discern more delicate integrability properties of a function.

The sequence Lorentz spaces \(l^p,q\) in Section 4 are the special cases of the Lorentz spaces \(L^p,q(X, A, \mu)\) for \(X = \mathbb{N}\) and \(\Sigma_X = \{Q \subset \mathbb{N}: Q \text{ is finite}\}\), and
\[
\mu(Q) = \sum_{n \in Q} n.
\]

Now, we consider an example.
Example A.2. Let $X$, $A$ and $\mu$ be as above. We consider a sequence $a = \{1, 2, 3, 0, \ldots, \}$ and compute its non-increasing rearrangement with respect to measure $\mu$. Using definition (A.1), we compute the distribution function

$$\mu_a(s) = \mu_a(s) = \mu\{n \in \mathbb{N} : |a(n)| > s\} = \sum_{n \in \mathbb{N} : |a(n)| > s} n = \begin{cases} 0, & s \geq 3, \\ 3, & 2 \leq s < 3, \\ 5, & 1 \leq s < 2, \\ 6, & 0 \leq s < 1. \end{cases}$$

Using this, one can easily calculate

$$a_n^* = \sup\{s > 0 : \mu_a(s) \geq n\} = \begin{cases} \sup\{s > 0 : \mu_a(s) \geq 1\} = 3, & n \in \{1, 2, 3\} \\ \sup\{s > 0 : \mu_a(s) \geq 4\} = 2, & n \in \{4, 5\} \\ \sup\{s > 0 : \mu_a(s) \geq 6\} = 1, & n = 6, \\ \sup\{s > 0 : \mu_a(s) \geq n\} = 0, & n \geq 7. \end{cases}$$

Thus, we have

$$a = \{1, 2, 3, 0, \ldots, \} \longrightarrow a^* = \{3, 3, 3, 2, 2, 1, 0, \ldots, \}$$

Let $X$ and $Y$ be two Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ respectively, and suppose that $X \subset Y$. As usual, we say that $X$ is embedded into $Y$, and write $X \hookrightarrow Y$, if there is a constant $C$ such that

$$\|x\|_Y \leq C\|x\|_X, \quad \text{for all } x \in X.$$

Proposition A.3. Suppose $1 < p < \infty$ and $1 < q < r \leq \infty$. Then there exists a constant $c_{p,q,r}$ (which depends on $p, q$ and $r$) such that

$$\|f\|_{L_{p,r}} \leq c_{p,q,r}\|f\|_{L_{p,q}}. \quad \text{(A.4)}$$

Proof. We have

$$\frac{1}{t^q} f^*(t) = \left\{ \frac{p}{q} \int_0^t \left[ s^q f^*(s) \right]^{\frac{1}{q}} \frac{ds}{s} \right\}^{\frac{1}{q}} \leq \left\{ \frac{p}{q} \int_0^t \left[ s^q f^*(s) \right]^{\frac{1}{q}} \frac{ds}{s} \right\} \leq \left( \frac{p}{q} \right)^{\frac{1}{q}} \|f\|_{L_{p,q}},$$

since $f^*$ is decreasing. Hence, taking the supremum over all $t > 0$, we obtain

$$\|f\|_{L_{p,\infty}} \leq \left( \frac{p}{q} \right)^{\frac{1}{q}} \|f\|_{L_{p,q}}. \quad \text{(A.5)}$$

This establishes (A.4) in the case $r = \infty$. Finally, when $r < \infty$, we have

$$\|f\|_{L_{p,r}} = \left\{ \int_0^t \left[ s^q f^*(s) \right]^{\frac{1}{q} r} \frac{dt}{t} \right\}^{\frac{1}{r}} \leq \|f\|_{L_{p,\infty}} \|f\|_{L_{p,q}}^{\frac{1}{r}}. \quad \text{(A.6)}$$

Inequality (A.5) combined with (A.6) gives (A.4) with $c_{p,q,r} = \left( \frac{p}{q} \right)^{\frac{r-q}{r}}$. □

For more details on Lorentz spaces one can refer to [SW71], [Gra08] and references therein.
Appendix B. Net spaces and Dirichlet kernel

Let \( \mu \) be the 1-dimensional Lebesgue measure in \( \mathbb{T}^1 \) and let \( M \) be a fixed family of measurable subsets of \( \mathbb{T}^1 \). We call \( M \) a net in what follows. For a function \( f \) defined and integrable on each \( e \) in \( M \) let

\[
\overline{f}(t, M) := \sup_{e \in M} \left| \int_{e} f(x) \, d\mu \right|,
\]

where the supremum is taken over all \( e \in M \), of measure \( |e| := \mu(e) > t \), \( t \in (0, \infty) \).

If \( t > 2\pi \), then we set \( \overline{f}(t, M) = 0 \). We call \( \overline{f}(t, M) \) the mean function of \( f \) with respect to the net \( M \).

Let \( N_{p,q}(M), 0 < p, q < \infty \), be the set of functions \( f \) such that

\[
\|f\|_{N_{p,q}(M)} := \left( \int_0^\infty (\overline{f}^p(t, M)q \frac{dt}{t}) \right)^{\frac{1}{p}} < \infty.
\]

Suppose \( Q \) is an interval in \( \mathbb{T} \), \( m \in \mathbb{N} \), \( d \in \mathbb{R}_+ \). The set of the form

\[
Q^d_m = \bigcup_{0 \leq k \leq m} (Q + kd)
\]

is called a harmonic interval in \( \mathbb{T} \). We denote by \( M_0 \) the set of all harmonic intervals in \( \mathbb{T} \).

Theorem B.1. Let \( 2 < p < \infty \). Let \( M_0 \) be a harmonic net in \( \mathbb{T}^1 \) and let \( M_1 \) be the collection of all compact sets in \( \mathbb{T}^1 \). If \( f \in N_{p',q}(M_1) \), then we have

\[
\|f\|_{N_{p,q}(M_0)} \leq \|\hat{f}\|_{l_{p,q}} \leq c_1 \|f\|_{N_{p',q}(M_1)},
\]

where \( p' = \frac{1}{p-1} \).

Using the duality of \( L^p \) spaces and Abel transformation one can easily deduce the following corollary.

Corollary B.2. Suppose \( 1 < p < 2 \) and \( f \in L^p(\mathbb{T}) \). Then we have

\[
\|f\|_{L^p(\mathbb{T})} \lesssim \left( \sum_{n \in \mathbb{Z}} m^{2p-2} |\hat{f}_n - \hat{f}_{n+1}|^p \right)^{\frac{1}{p}} + \sup_{n \in \mathbb{Z}} n^{\frac{1}{p'}} |\hat{f}_n|.
\]

We recall that

\[
D_N(t) = \sum_{k=1}^N e^{ikt}.
\]

Using this corollary, one can give the \( L^p \) norm characterisation of the Dirichlet kernel \( D_N \).

Proposition B.3. Suppose \( 1 < p < 2 \). We have

\[
\|D_N\|_{L^p(\mathbb{T})} \cong N^{\frac{1}{p'}}
\]
Proof. We apply Theorem B.1 to $D_N$ with $2 < p' < \infty$ and $q = p$,

$$c_0 \|D_N\|_{N,p,p(M_0)} \leq \|\hat{D}_N\|_{L_{p',p}} \leq \|D_N\|_{N,p,p(M_1)}.$$ 

Since $N_{p,p}(M_1) = L^p(T^n)$ (see [Nur98]), the lower estimate of $\|D_N\|_{L^p(T^n)}$ in (B.3) is a consequence of the Hardy-Littlewood inequality. Further, we use Corollary B.2 to get the upper estimate:

$$\|D_N\|_{L^p(T^n)} \lesssim \left( \sum_{n \in \mathbb{Z}} n^{2p-2} |\hat{D}_N n - \hat{D}_N n+1|^p \right)^{\frac{1}{p}} + \sup_{n \in \mathbb{Z}} n^{\frac{1}{p'}} |\hat{D}_N n|.$$ 

We note that

$$\hat{D}_N n = \begin{cases} 1, & n \in \{1, \ldots, N\}, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, we have

$$\|D_N\|_{L^p(T^n)} \lesssim N^{\frac{1}{p'}}.$$ 

This proves Proposition B.3. \qed

References

[CdG71] R. R. Coifman and M. de Guzmán. Singular integrals and multipliers on homogeneous spaces. Rev. Un. Mat. Argentina, 25:137–143, 1970/71. Collection of articles dedicated to Alberto González Domínguez on his sixty-fifth birthday.

[CS01] M. Cowling and A. Sikora. A spectral multiplier theorem for a sublaplacian on SU(2). Math. Z., 238(1):1–36, 2001.

[CW71a] R. R. Coifman and G. Weiss. Analyse harmonique non-commutative sur certains espaces homogènes. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin, 1971. Étude de certaines intégrales singulières.

[CW71b] R. R. Coifman and G. Weiss. Multiplier transformations of functions on SU(2) and $\sum_2$. Rev. Un. Mat. Argentina, 25:145–166, 1971. Collection of articles dedicated to Alberto González Domínguez on his sixty-fifth birthday.

[CW74] R. R. Coifman and G. Weiss. Central multiplier theorems for compact Lie groups. Bull. Amer. Math. Soc., 80:124–126, 1974.

[Gra08] L. Grafakos. Classical Fourier analysis, volume 249 of Graduate Texts in Mathematics. Springer, New York, second edition, 2008.

[GT80] S. Giulini and G. Travaglini. $L^p$-estimates for matrix coefficients of irreducible representations of compact groups. Proc. Amer. Math. Soc., 80(3):448–450, 1980.

[HL27] G. H. Hardy and J. E. Littlewood. Some new properties of Fourier constants. Math. Ann., 97(1):159–209, 1927.

[Hör60] L. Hörmander. Estimates for translation invariant operators in $L^p$ spaces. Acta Math., 104:93–140, 1960.

[HR74] E. Hewitt and K. A. Ross. Rearrangements of $L^r$ Fourier series on compact Abelian groups. Proc. Lond. Math. Soc. (3), 29:317–330, 1974.

[Hun66] R. A. Hunt. On $L(p, q)$ spaces. Enseignement Math. (2), 12:249–276, 1966.

[Lor50] G. G. Lorentz. Some new functional spaces. Ann. of Math. (2), 51:37–55, 1950.

[Mih56] S. G. Mihlin. On the theory of multidimensional singular integral equations. Vestnik Leningrad. Univer., 11(1):3–24, 1956.

[Mih57] S. G. Mihlin. Singular integrals in $L_p$ spaces. Dokl. Akad. Nauk SSSR (N.S.), 117:28–31, 1957.

[NT00] E. D. Nursultanov and N. T. Tleukhanova. Lower and upper bounds for the norm of multipliers of multiple trigonometric Fourier series in Lebesgue spaces. Funktsional. Anal. i Prilozhen., 34(2):86–88, 2000.
[Nur98] E. Nursultanov. Net spaces and inequalities of Hardy-Littlewood type. *Sb. Math.*, 189(3):399–419, 1998.

[RT10] M. Ruzhansky and V. Turunen. *Pseudo-differential operators and symmetries. Background analysis and advanced topics*, volume 2 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010.

[RT13] M. Ruzhansky and V. Turunen. Global quantization of pseudo-differential operators on compact Lie groups, SU(2), 3-sphere, and homogeneous spaces. *Int. Math. Res. Not. IMRN*, (11):2439–2496, 2013.

[RW13] M. Ruzhansky and J. Wirth. On multipliers on compact Lie groups. *Funct. Anal. Appl.*, 47(1):87–91, 2013.

[Ste70] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[SW71] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[Vil68] N. J. Vilenkin. *Special functions and the theory of group representations*. Translated from the Russian by V. N. Singh. Translations of Mathematical Monographs, Vol. 22. American Mathematical Society, Providence, R. I., 1968.

[Wei72] N. J. Weiss. *Lp estimates for bi-invariant operators on compact Lie groups*. *Amer. J. Math.*, 94:103–118, 1972.

RAUAN AKYLZHANOV:
DEPARTMENT OF MATHEMATICS
MOSCOW STATE UNIVERSITY, KAZAKH BRANCH
AND GUMILYOV EURASIAN NATIONAL UNIVERSITY,
ASTANA, KAZAKHSTAN
E-mail address akilzhanoff@yandex.ru

ERLAN NURLUSTANOV:
DEPARTMENT OF MATHEMATICS
MOSCOW STATE UNIVERSITY, KAZAKH BRANCH
AND GUMILYOV EURASIAN NATIONAL UNIVERSITY,
ASTANA, KAZAKHSTAN
E-mail address er-nurs@yandex.ru

MICHAEL RUZHANSKY:
DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
180 QUEEN'S GATE, LONDON SW7 2AZ
UNITED KINGDOM
E-mail address m.ruzhansky@imperial.ac.uk