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STABILITY OF THE SOLUTION TO INVERSE
OBSTACLE SCATTERING PROBLEM

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Abstract. It is proved that if the scattering amplitudes for two obstacles (from a large class of obstacles) differ a little, then the obstacles differ a little, and the rate of convergence is given. An analytical formula for calculating the characteristic function of the obstacle is obtained, given the scattering amplitude at a fixed frequency.

Introduction.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\Gamma$,

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \quad D' := \mathbb{R}^3 \setminus D, \quad k = \text{const} > 0; \quad u = 0 \quad \text{on} \quad \Gamma$$

$$u = \exp(ika \cdot x) + A(\alpha', \alpha, k)r^{-1} \exp(ikr) + o(r^{-1}), \quad r := |x| \to \infty, \quad \alpha' := x^{-1}.$$  \hspace{1cm} (2)

Here $\alpha$ is a given unit vector, $S^2$ is the unit sphere in $\mathbb{R}^3$, the function $A(\alpha', \alpha, k)$ is called the scattering amplitude (the radiation pattern). It is well known [1] that problem (1)-(2) has a unique solution, the scattering solution, so that the map $\Gamma \to A(\alpha', \alpha, k)$ is well defined. We consider the inverse obstacle scattering problem (IOSP): given $A(\alpha', \alpha) := A(\alpha', \alpha, k = 1)$ for all $\alpha', \alpha \in S^2$ and a fixed $k$ (for example, take $k = 1$ without loss of generality), find $\Gamma$.

Let us assume that $\Gamma \subset \gamma_\lambda$, where $\gamma_\lambda$ is the set of star-shaped (with respect to a common point $O$) surfaces, which are located in the annulus $0 < a_0 \leq |x| \leq a_1$, and whose equations $x_3 = \phi(x_1, x_2)$ in the local coordinates (in which $x_3$ is directed along the normal to $\Gamma$ at a point $s \in \Gamma$), have the property

$$\|\phi\|_{C^2, \lambda} \leq c_0, \hspace{1cm} (3)$$

$C^2, \lambda$ is the space of twice differentiable functions, whose second derivatives satisfy the Hölder condition of order $0 < \lambda \leq 1$, and $c_0$ is independent of $\phi$ and $\Gamma$.

Uniqueness of the solution to IOSP with fixed frequency data is first proved in [1, p. 85]. We are interested here in the stability problem: suppose that $\Gamma_j \in \gamma_\lambda$ generate $A_j(\alpha', \alpha)$, $j = 1, 2$, and

$$\max_{\alpha', \alpha \in S^2} |A_1(\alpha', \alpha) - A_2(\alpha', \alpha)| < \delta. \hspace{1cm} (4)$$

What can one say about the Hausdorff distance between $D_1$ and $D_2$: $\rho := \sup_{x \in \Gamma_1} \inf_{y \in \Gamma_2} |x - y|$. Let $\bar{D}_1$ denote a connected component of $D_1 \setminus D_2$, $D_{12} := D_1 \cup D_2$, $\Gamma_{12} := \partial D_{12}$, $D'_{12} := \mathbb{R}^3 \setminus D_{12}$, $\bar{\Gamma}_1 := \partial \bar{D}_1 := \Gamma'_1 \cup \bar{\Gamma}_2$, $\bar{\Gamma}_2 \subset \Gamma_2 := \partial D_2$, $\Gamma'_1 \subset \Gamma_1 := \partial D_1$. Let us assume, without loss of generality, that $\rho = |x_0 - y_0|$, $x_0 \in \Gamma'_1$, $y_0 \in \bar{\Gamma}_2$. Can one obtain a formula for calculating $\Gamma$, given $A(\alpha', \alpha)$ for all $\alpha', \alpha \in S^2$, $k = 1$ is fixed? No such formula is known for IOSP. For inverse potential scattering problem with fixed-energy data such a formula and stability estimates are obtained in [2], [3]. These results are based on the works [7],[8], [10]-[17], [19]-[21].

In section II we prove that $\rho \leq c_1 \left( \frac{|\ln|\ln|\delta||}{|\ln|\delta||} \right)^{c_2}$ as $\delta \to 0$. We also prove some inversion formula, but it is an open problem to make an algorithm out of this formula. In Remark 3, we comment on some recent papers [4-6] in which attempts are made to study the stability problem and point out a number of errors in these papers. Our result, formulated as Theorem 1 in section II, is stronger than the results announced in Theorem 1 in [4], Theorem 1 in [5] and Theorem 2.10 in [6].
II. Stability Result and a Reconstruction Formula.

**Theorem 1.** Under the assumptions of section I, one has $\rho(\delta) \leq c_1 \left( \frac{\ln |\ln \delta|}{\ln \delta} \right)^{c_2}$, where $c_1$ and $c_2$ are positive constants independent of $\delta$.

**Proposition 1.** There exists a function $\nu_\epsilon(\alpha, \theta) \in L^2(S^2)$ such that

$$-4\pi \lim_{\epsilon \to 0} \int_{S^2} A(\theta', \alpha) \nu_\epsilon(\alpha, \theta) d\alpha = \frac{\lambda^2}{2} \tilde{\chi}_D(\lambda).$$

Here $\lambda \in \mathbb{R}^3$ is an arbitrary fixed vector, $\chi_D(x) := \begin{cases} 1, & x \in D \\ 0, & x \notin D \end{cases}$, $\tilde{\chi}_D(\lambda) := \int_{\mathbb{R}^3} \exp(-i \lambda \cdot x) \chi_D(x) dx$, $\theta, \theta' \in M := \{ \theta : \theta \in \mathbb{C}^3, \theta \cdot \theta = 1 \}$, $\theta' - \theta = \lambda$, and $A(\theta', \alpha)$ is defined by the absolutely convergent series

$$A(\theta', \alpha) = \sum_{k=0}^{\infty} A_k(\alpha) Y_k(\theta'), \quad \theta' \in M, \quad A_k(\alpha) := \int_{S^2} A(\alpha', \alpha) Y_k(\alpha') d\alpha',$$

where $Y_k(\alpha)$ are the orthonormal in $L^2(S^2)$ spherical harmonics, $Y_k(\theta')$ is the natural analytic continuation of $Y_k(\alpha')$ from $S^2$ to $M$, and the series (6) converges absolutely and uniformly on compact subsets of $S^2 \times M$.

**Remark 1.** The stability result given in Theorem 1 is similar to the one in [3], p. 9, formula (2.42), for inverse potential scattering.

**Remark 2.** Proposition 1 claims the existence of the inversion formula (5). An open problem is to construct the function $\nu_\epsilon(\alpha, \theta)$ algorithmically, given the data $A(\alpha', \alpha)$ \quad \forall \alpha', \alpha \in S^2$.

**Proof of Theorem 1.** First, we prove that $\rho(\delta) \to 0$ as $\delta \to 0$. Then, we prove that $|u_2| \leq c \rho$ in $\tilde{D}_1$. Next, we prove that $|u_2(x)| \leq c e^{c'\rho'}$ (*) if $\text{dist}(x, \Gamma_1') = O(\rho)$, where $|\ln \epsilon| = cN(\delta)$, $N(\delta) := |\ln \delta|/|\ln |\ln \delta||$. From (*) Theorem 1 follows. By $c$, $c'$, $c$, $c_j$ various positive constants, independent of $\delta$ and on $\Gamma \in \gamma_\lambda$, are denoted.

**Step 1.** Proof of the relation $\rho(\delta) \to 0$ as $\delta \to 0$. Assume the contrary:

$$\rho_n := \rho(\delta_n) \geq c > 0 \quad \text{for some sequence} \quad \delta_n \to 0.$$  \hspace{1cm} (7)

Let $\Gamma_{j_n}$, $j = 1, 2$, be the corresponding sequences of the boundaries, $\Gamma_{j_n} \in \gamma_\lambda$. Due to assumption (3), one can select a convergent in $C^2(\mu)$, $0 < \mu < \lambda$, subsequence, which we denote $\Gamma_{j_n}$ again. Thus $\Gamma_{j_n} \to \Gamma_j$ as $n \to \infty$. From (7) it follows that (1) $\rho(D_1, D_2) \geq c > 0$, where $D_j$ is the obstacle with the boundary $\Gamma_j$. By the known continuity of the map $\Gamma_j \to A_j$, $\Gamma_j \in \gamma_\mu$, it follows that $A_1(\alpha', \alpha) - A_2(\alpha', \alpha) = 0$.

By the uniqueness theorem [1, p. 85] it follows that $\Gamma_1 = \Gamma_2$. Thus, $\rho(D_1, D_2) = 0$ which is a contradiction to (1). This contradiction proves that $\rho(\delta) \to 0$ as $\delta \to 0$.

**Step 2.** Proof of the estimate $|u_2(x)| \leq c \rho$ for $x \in \tilde{D}_1$. It is known that $\|u_2\|_{C^2(D_2)} \leq c$, where $u_2 = u_2(x, \alpha)$ is the scattering solution corresponding to the obstacle $D_2$. Since $u_2 = 0$ on $\tilde{\Gamma}_2$, one has $|u_2(x)| \leq (\max_{x \in D_1} |\nabla u_2|) \rho \leq c \rho$.

**Step 3.** Proof of the estimate $|v(x)| \leq c e^{d'}$, where $v := u_2 - u_1$ and $d := \text{dist}(x, \Gamma_1')$. 

From [3, p. 26, formulas (4.12), (4.17), (2.28)], one has

\[ |v(x)| \leq c \exp\{-\gamma N(\delta)\}, \quad |x| > a_2, \quad N(\delta) := \frac{|\ln \delta|}{\ln |\ln \delta|}, \quad \gamma := \ln \frac{a_2}{a_1} > 0, \quad (8) \]

\( a_2 > a_1 \) is an arbitrary fixed number, \( a_2 \leq |x| \leq a_2 + 1 \) (in [3] it is assumed \( a_2 > a_1 \sqrt{2} \), but \( a_2 > a_1 \) is sufficient). Let us derive from (8), from equation (1) for \( v(x) \), from the radiation condition for \( v(x) \), and from the estimate \( \|v\|_{C^2(D_{12}')} \leq c \), the estimate:

\[ |v(x)| \leq ce^{\epsilon x'}, \quad x \in D_{12}, \quad c_3 \rho \leq d \leq c_4 \rho, \quad c_3 > 0, \quad d = \text{dist}(x, \Gamma_1'), \quad (9) \]

If (9) is proved, then Theorem 1 follows. Indeed, \( |v(x)| = |v(s) + \nabla v \cdot (x - s)| = O(\rho) \leq ce^{\epsilon x'} \) if \( d \) satisfies (9). Here we use: 1) \( v = u_2 - u_1 = u_2 \) on \( \Gamma_1' \), \( |u_2| = O(\rho) \) on \( \Gamma_1' \), since \( u_2 = 0 \) on \( \Gamma_2 \), and \( |\nabla u_2| \leq c \), 2) \( |x - s| = O(\rho) \) if \( \text{dist}(x, \Gamma_1') = O(\rho) \), and 3) \( 0 < c \leq |\nabla v| \leq \epsilon \) if \( d \) satisfies (9). The last claim follows from the continuity of \( \nabla v(x) \), smallness of \( \rho, \rho(\delta) \to 0 \) as \( \delta \to 0 \), and the fact that \( |\nabla u_j|_{\Gamma_1'} \neq 0 \) almost everywhere (otherwise, by the uniqueness of the solution to the Cauchy problem for (1), one concludes that \( u_j = 0 \) in \( D_{12}' \), which contradicts (2), since, by (2), \( |u_j| \to 1 \) as \( |x| \to \infty \)). Thus \( \ln \rho \leq c \epsilon \ln \epsilon \), or \( (s) \frac{|\ln \rho|}{\ln(\rho - 1)} \leq c(\ln(\epsilon^{-1})) \), where \( \rho \) and \( \epsilon \) are small numbers, \( 0 < \rho, \epsilon < 1, c, c' > 0, \) and \( c \) stands for different constants. It follows from \( (s) \) that \( \rho \leq \{c(\ln(\epsilon^{-1}))\}^{-1} \), \( \omega \to 0 \) as \( \epsilon \to 0 \). From the definition (8) of \( \epsilon \), one gets the estimate of Theorem 1. Thus, the proof of Theorem 1 is completed as soon as (9) is proved.

Our argument remains valid if \( |v| = O(\rho^m) \) with some \( m, 0 < m < \infty \). Such an inequality is always true for a solution \( v \) to elliptic equation (1) unless \( \rho = 0 \) (see [26, p.14]).

Proof of (9). Since \( \|v\|_{C^2(D_{12})} \leq c, v(x) \) vanishes at infinity, and \( v \) solves (1), one can represent \( v(x) \) in \( D_{12}' \) by the volume potential: \( v(x) = \int_{D_{12}} g(x - y) f(y) dy, f \in C^\mu(D_{12}), g(x) := \frac{\exp(|iy|)}{4\pi|x|}. \) The function \( |x - y| = \sqrt{r^2 + 2r|y|\cos \theta + |y|^2}^{1/2} := R \) admits analytic continuation on the complex plane \( z = r \exp(iv) \) to the sector \( S_\phi := \{ \arg z < \phi, z^2 - 2z|y|\cos \theta + |y|^2 \neq 0 \} \) for \( z \) in this sector. We use the branch of \( R \) for which \( \text{Im} R \geq 0, \) and \( \text{Re} R \) at \( m = 0 \geq 0. \) The argument of \( R^2 := z^2 - 2z|y|\cos \theta + |y|^2 \) is defined so that it belongs to the interval \( [0, 2\pi) \), so that the analytic continuation of \( g(x - y) \) to the sector \( S_\phi \) is bounded there. It is crucial to have at least boundedness of the norm \( \|v\|_{C^1(D_{12})} \). Indeed, \( (1) \) implies that one can extend \( v \) from \( D_{12}' \) to \( D_{12} \) as \( C^1(\mathbb{R}^3) \) functions. This is true although the boundary \( \partial D_{12} \) may be nonsmooth to the degree which prevents using the known extension theorems (Stein’s theorem, for example). The way to go around this difficulty is to extend \( u_1 \) and \( u_2 \) separately to \( D_1 \) and \( D_2 \) respectively, and then take \( v = u_2 - u_1 \) as the extension. If \( v \in C^2(\mathbb{R}^3) \) satisfies the radiation condition and the Helmholtz equation, and is \( C^2 \) in the interior and in the exterior of \( D_{12} \), then it is representable as a sum of the volume and single-layer potentials, and our argument, which uses analytic continuation, goes through. Without this assumption the argument is not valid and the conclusion fails, as the following example shows.

Example 1: Let \( D := \{ x : |x| \leq 1, x \in \mathbb{R}^3 \}, v = v_\ell := \frac{h^{(1)}(\ell)}{h^{(1)}(1)} Y_\ell(x^0), \) where \( h^{(1)}(\ell) \) is the spherical Hankel function, \( Y_\ell(x^0) \) is the normalized in \( L^2(S^2) \) spherical harmonic. It is well known that \( h^{(1)}(\ell) \sim i \frac{(\ell + 1)!}{(\ell + 2)!b} \) as \( \ell \to \infty \) uniformly in \( 1 \leq \ell \leq b, b < \infty \) is arbitrary. Therefore \( v_\ell \sim \sim \ell^{-\ell+1} \) as \( \ell \to \infty \). In any annulus \( A := \{ x : 1 < a_2 \leq y \leq b \}, \) one has \( \|v_\ell\|_{L^2(A)} \leq c_\ell^{-\ell+1} \) as \( \ell \to \infty \). On the other hand \( \|v_\ell\|_{L^2(S^2)} = 1 \) for all \( \ell \). Thus, for sufficiently large \( \ell \) the solution \( v_\ell \) to Helmholtz equation is as small as one wishes in the annulus \( A \), but it is not small at the boundary \( \partial D \) for any \( \ell \) or its \( L^2(\partial D) \) norm is one. The reason for the solution to fail to be small on \( \partial D \) is that the \( C^1 \) norm of \( v_\ell \) is unbounded, as \( \ell \to \infty \), on \( \partial D \).
Let us continue the proof of (9). The function $v(r, r^0, \alpha)$, where $\alpha$ is the same as in (2), $r^0 := x/r$, and $r = |x|$, admits an analytic continuation to the sector $S$ on the complex plane $z$, $S := \{z : |\arg(z - r(x^0))| < \phi\}$, $\phi > 0$, $r = r(x^0)$ is the equation of the surface $\Gamma^0$ in the spherical coordinates with the origin at the point $O$, and $v(z, x^0, \alpha)$ is bounded in $S$. The angle $\phi$ is chosen so that the cone $K$ with the vertex at $r(x^0)$, axis along the normal to $\Gamma^0$ at the point $r(x^0)$, and the opening angle $2\phi$, belongs to $D_1/2$. Such a cone does exist because of the assumed smoothness of $\Gamma_j$. The analytic continuation of this type was used in [18]. It follows from (8) that $\sup_{r \geq r_0} |v(r)| \leq \epsilon$, and $\sup_{z \in S} |v(z)| \leq \epsilon$, since $\Im(z^2 - 2z|y| \cos \theta + |y|^2|/2 \geq 0$ in $S$. From this and the classical theorem about two constants [22, p. 296], one gets $|v(z)| \leq ce^{h(z)}$, where $h(z) = h(z, L, Q)$ is the harmonic measure of the set $\partial S \setminus L$ with respect to the domain $Q := S \setminus L$ at the point $z \in Q$. Here $L$ is the ray $[a_2, +\infty)$, $\partial S$ is the union of two rays, which form the boundary of the sector $S$, and of the ray $L$. The proof is completed as soon as we demonstrate that $h(z) \sim kd^2$ as $z \to r(x^0)$ along the real axis, $d := |z - r(x^0)|$, $k = \text{const} > 0$, $c = \text{const} > 0$. This, however, is clear: let $r(x^0)$ be the origin, and denote $z - r(x^0)$ by $z$. If one maps conformally the sector $S$ onto the half-plane $Rez \geq 0$ using the map $w = z^c$, $c = \frac{i}{2\alpha}$, then the ray $L$ is mapped onto the ray $L := [a_2', +\infty)$, and (see [22, p. 293]) $h(z, L, Q) = h(z^c, L', Q')$, where $Q'$ is the image of $Q$ under the mapping $z \mapsto z^c = w$. By the Hopf lemma [23, p. 34], $\frac{\partial h(0, L', Q')}{\partial w} > 0$, $h(0, L', Q') = 0$, so $h(w, L', Q') \sim kw = k\bar{z}^c$ as $z \to 0$, and (9) is proved. Theorem 1 is proved. □

Proof of Proposition 1. It is proved in [2, p. 183] that the set $\{u_N(s, \alpha)\}_{\nu\alpha \in S^2}$ is complete in $L^2(\Gamma)$. This implies existence of a function $\nu_c(\alpha, \theta)$ such that

$$\left\| \int_{S^2} u_N(s, \alpha)\nu_c(\alpha, \theta)ds - \frac{\partial \exp(i\theta \cdot s)}{\partial N_s}\right\|_{L^2(\Gamma)} < \epsilon,$$

where $\epsilon > 0$ is arbitrarily small fixed number, $N_s$ is the exterior normal to $\Gamma$ at the point $s$, and $\theta \in M$ is an arbitrary fixed vector. It is well known [1, p. 52], that

$$-4\pi A(\theta', \alpha) = \int_{\Gamma} \exp(-i\theta' \cdot s)u_N(s, \alpha)ds.$$  

Multiply (11) by $\nu_c(\alpha, \theta)$, integrate over $S^2$ and use (10), to get

$$-4\pi \lim_{\epsilon \to 0} \int_{S^2} A(\theta', \alpha)\nu_c(\alpha, \theta)ds = \int_{\Gamma} \exp(-i\theta' \cdot s)\frac{\partial \exp(i\theta \cdot s)}{\partial N_s}ds.$$  

Note that

$$\int_{\Gamma} \exp(-i\theta' \cdot s)\frac{\partial \exp(i\theta \cdot s)}{\partial N_s}ds = \frac{1}{2} \int_{\Gamma} \frac{\partial \exp[-i(\theta' - \theta) \cdot s]}{\partial N_s}ds$$

$$= \frac{1}{2} \int_D \nabla^2 \exp(-i\lambda \cdot x)dx = -\frac{\lambda^2}{2} \chi_D(\lambda)$$

where the first equation is obtained with the help of Green’s formula. From (12) and (13) one obtains (5). Proposition 1 is proved. □

Remark 3. In [4]-[5] attempt is made to obtain stability results for IOSP, but several errors invalidate the proofs in [4], [5] and [6] related to stability for IOSP. Let us point out some of the errors. Lemma 5, as stated in [4, p. 83], repeated as Lemma 4 in [5], claims that if a solution to a homogeneous Helmholtz equation in the exterior of a bounded domain $D$ is small in the annulus $R \leq |x| \leq R + 1$, $|v| \leq \epsilon$ in the annulus, then $|v|_{\partial D} \leq c \log \epsilon^{-c_1}$. This is incorrect as Example 1 shows. Lemma 3 in [4] is wrong (factor
\( \rho^{2m} \) is forgotten in the argument). In fact, stronger results have been published earlier [17], [2], [3]. In [5] Lemma 2 is intended as a correction of Lemma 3 in [4] (without even mentioning [4]), but its proof is also wrong: the factor \( \rho^{2m} \) is not estimated. There are other mistakes in [5] (e.g., the known asymptotics of Hankel functions in [5, p. 538] is given incorrectly). In [6] these mistakes are repeated (p. 600). There are claims in [6] that: a) there is a gap in the Schiffer’s proof of the uniqueness theorem for IOSP with the data \( A(a',\alpha_0,k) \) \( \forall a' \in S^2, \forall k > 0, \alpha_0 \in S^2 \) is fixed [6, p. 605], b) that Theorem 6 in [8] is incorrect, and the proof of Lemma 5 in [8] contains a flaw [6, p. 588]. These claims are wrong, and no justifications of the claims are given. The remark concerning Shiffer’s proof in [6, p. 605, line 1] is irrelevant (see [1, pp. 85-86]).

It should be noted that the arguments in [4]-[5] are based on the well known estimates of Landis [9] for the stability of the solution to the Cauchy problem, but no references to the work of Landis are given. In [6] it is not mentioned that the concept of completeness of the set of products of solutions to PDE (which is discussed in [6]) has been introduced and widely used for the proof of the uniqueness theorems in inverse problems in the works [2], [13], [19]-[21] (see also references in [2], [13]). In [24] and [25] two theorems are announced which contradict each other (Theorem 1 in [25] and Theorem 2 in [24]).

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