Bose–Einstein condensation of nonrelativistic charged particles in a constant magnetic field

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The statistical mechanics of a system of non–relativistic charged particles in a constant magnetic field is discussed. The spatial dimension \( D \) is arbitrary with \( D \geq 3 \) assumed. Calculations are presented from first principles using the effective action method. For \( D \geq 5 \) the system has a phase transition with a Bose condensate. We show how the effective action method deals with in a very natural way with the condensate and study its rôle in the magnetization of the gas. For large values of the magnetic field we show how the magnetized gas in \( D \) spatial dimensions behaves like the free Bose gas in \((D - 2)\) spatial dimensions. Even though for \( D = 3 \) the magnetized gas does not have a phase transition for any non–zero value of the magnetic field, we show how the specific heat starts to resemble the result for the free gas as the magnetic field is reduced. A number of analytical approximations for the magnetization and specific heat are given and compared with numerical results. In this way we are able to study in precise detail how the \( B \to 0 \) limit of the magnetized gas is achieved.

I. INTRODUCTION

Bose–Einstein condensation (BEC) \( [1,2] \) has been an integral part of physics for many years. From a theoretical standpoint this phenomenon has been intensively studied as a possible means of explaining the superfluid transition in He\(^4\) \( [3] \), and also as an experimentally realisable possibility in dilute atomic gases. Recently BEC has been achieved in atomic systems of dilute alkali metal atoms \( [4–7] \), and for this reason a great deal of interest is now being focussed on these systems.

Another system worthy of study is the condensation (or lack thereof) of charged bosons in a homogeneous magnetic field. Initial work on the nonrelativistic case \( [8] \) was substantially improved on by Scharfroth \( [9] \), and was then generalized to dimensions other than three by May \( [10,11] \). The primary motivation for these efforts was to formulate a plausible theory of superconductivity. Although this attempt failed as far as standard low \( T_c \) superconductors are concerned, interest in the subject has revived as explanations of high \( T_c \) materials are now required \( [12] \).

More recently, several detailed studies have been made of the ideal system of nonrelativistic charged bosons, both numerically \( [13,14] \), and from a theoretical point of view in both two \( [15,16] \), and three \( [17,18] \) spatial dimensions. Also a detailed study of the boson gas in three dimensions with trapping harmonic potentials and the presence of a magnetic field has recently been made using the path–integral formalism \( [19] \). These authors find comparable behavior to that found using Mellin transforms, and a comparison between the two methods is very instructive. However due to constraints of space, the contrast between the two methods will not be discussed here in depth.

The behavior of the relativistic case has also been discussed \( [20–24] \). In addition more realistic systems which may occur in high \( T_c \) superconductors \( [25–27] \) have also been examined.

The outline of our paper is as follows. In section II we give a brief description of the free Bose gas in \( D \) spatial dimensions. We will later compare the analogous results for the magnetized gas to these free–field ones. Section III presents the effective action method and applies it to the magnetized gas. We concentrate initially on the specific heat and show how the presence of the magnetic field alters the behavior from that found for the free gas. We study numerically what happens for large and small magnetic fields. Results are also obtained for the magnetization. In Section IV we describe how it is possible to obtain analytical results for the critical temperature (when \( D \geq 5 \)), the magnetization, and the specific heat when the magnetic field is weak. We are able to analytically confirm the numerical results concerning the limit \( B \to 0 \). For the \( D = 3 \) gas we give approximations valid at the critical temperature for the free Bose gas. (Previous approximations were only valid for temperatures larger than this.) A number of appendices contain details of the expansions used for the analytical results.

II. BOSE–EINSTEIN CONDENSATION WITH NO EXTERNAL FIELD

In this section we wish to review very briefly some of the basic properties of the free Bose gas. The spatial dimension \( D \) will be arbitrary. For \( D = 3 \) the analysis is standard textbook material \( [28,29] \). The absence of Bose–Einstein condensation when \( D = 2 \) is also widely known \( [30,10,11] \). Spatial dimensions \( D > 3 \) have also been studied \( [31,32] \).

We will consider an ideal gas of \( N \) spinless bosons confined in a large box of volume \( V \) in \( D \) spatial dimensions. The infinite volume limit will be taken with \( N = \frac{V}{\lambda^D} \) fixed as is conventionally done. The energy levels of the system are...
\[ E_{n_i} = \frac{1}{2m} \sum_{i=1}^{D} \left( \frac{2\pi n_i}{L_i} \right)^2 \] (2.1)

before the infinite volume limit is taken if we impose periodic boundary conditions on the field. We set \( h = 1 \) throughout the paper. In (2.1), \( n_i = 0, \pm 1, \pm 2, \ldots \) and \( L_i \) is the length of the box in the \( i \)th direction. In section III we will show how the effective action formalism can be used to study this problem (see references [16, 33, 34] for reviews). For now we stick with the conventional thermodynamic expressions.

The particle number \( N \) is the chemical potential and \( \beta = T^{-1} \) in units with the Boltzmann constant \( k_B = 1 \). The grand canonical ensemble is used here.

By taking the infinite volume limit, we can replace the sums in (2.2) over \( n_i \) with integrals. After expanding the denominator of (2.2) in a geometric series, and using (2.3) for the energy levels, the integrals over \( n_i \) may be performed with the result

\[ U = \sum_{n_i} \frac{E_{n_i}}{e^{\beta(E_{n_i} - \mu)} - 1} \] (2.2)

where \( \mu \) is the chemical potential and \( \beta = T^{-1} \) in units with the Boltzmann constant \( k_B = 1 \). The grand canonical ensemble is used here.

The internal energy is given by

\[ U = \frac{D}{2} \beta V \left( \frac{m}{2\pi^2 \beta} \right)^2 \text{Li}_{D+2} \left[ e^{\beta \mu} \right]. \] (2.3)

We have defined the polylogarithm function \( \text{Li}_p [z] \) by

\[ \text{Li}_p [z] = \sum_{k=1}^{\infty} \frac{z^k}{k^p}. \] (2.4)

We note the property (assuming \( p > 1 \))

\[ \text{Li}_p [1] = \zeta (p), \] (2.5)

where \( \zeta (p) \) denotes the Riemann \( \zeta \)-function.

The essential feature of Bose–Einstein condensation as a phase transition is exhibited in the behavior of the specific heat. The specific heat at constant volume \( C_v \) is defined by

\[ C_v = \left( \frac{\partial U}{\partial T} \right)_{V,N} = -\beta^2 \left( \frac{\partial U}{\partial \beta} \right)_{V,N}. \] (2.6)

The particle number \( N \), given by

\[ N = V \left( \frac{m}{2\pi^2 \beta} \right)^2 \frac{\text{Li}_{D+2} \left[ e^{\beta \mu} \right]}{(2.7)} \]

is held fixed when computing the derivative in (2.6). The chemical potential is not fixed. From (2.3), (2.6) we find

\[ C_v = V \left( \frac{m}{2\pi^2 \beta} \right)^2 \frac{D(D+2)}{4} \text{Li}_{D+2} \left[ e^{\beta \mu} \right] \]

\[ -\frac{D}{2} \beta \zeta (D+2) \text{Li}_{D+2} \left[ e^{\beta \mu} \right]. \] (2.8)

Here \( \zeta \) denotes the derivative with respect to \( \beta \) holding \( V \) and \( N \) fixed. To calculate \( (\beta \mu) \zeta \) we differentiate both sides of (2.7) to give

\[ (\beta \mu) \zeta = \frac{D}{2} \frac{\text{Li}_{D+2} \left[ e^{\beta \mu} \right]}{\text{Li}_{D+2} \left[ e^{\beta \mu} \right]}. \] (2.9)

Substitution of (2.9) into (2.8) and use of (2.7) shows that

\[ C_v = \frac{D(D+2)}{4} \frac{\text{Li}_{D+2} \left[ e^{\beta \mu} \right]}{\text{Li}_{D+2} \left[ e^{\beta \mu} \right]}. \] (2.10)

If we confine ourselves to \( D \geq 3 \), as we do for the rest of the paper, then a critical temperature \( T_0 \) can exist at which the chemical potential \( \mu = 0 \). From (2.7) we find

\[ T_0 = \left( \frac{2\pi}{m} \right) \left[ \frac{N}{V \zeta(D+2)} \right]^{\frac{D}{D+2}}. \] (2.11)

For \( T < T_0 \) the chemical potential remains fixed at \( \mu = 0 \). It is easy to see that (2.10) only holds for \( T > T_0 \). When \( T < T_0 \), the specific heat may be evaluated from (2.8) by setting the term with \( (\beta \mu) \zeta \) to zero and also \( \mu \) to zero. This results in

\[ C_v = \frac{\beta}{\beta^2} \frac{D(D+2)}{4} \frac{\zeta (D+2)}{\zeta (D+2)}. \] (2.12)

for \( T < T_0 \).

The two expressions (2.10), (2.12) may be used to compute the specific heat for all temperatures, and also to study the behavior at \( T = T_0 \). It is easy to see from the polylogarithm function (2.4) that if \( p < 1 \), \( \text{Li}_p [z] \to \infty \) as \( z \to 1 \). This means that the second term of (2.10) vanishes as \( \mu \to 0 \) for \( D = 3, 4 \), but is finite for \( D \geq 5 \). Thus the specific heat is continuous at \( T = T_0 \) for \( D = 3, 4 \), but discontinuous at \( T = T_0 \) for \( D \geq 5 \). The discontinuity for \( D \geq 5 \) is easily computed in terms of Riemann \( \zeta \)-functions. The continuity of \( C_v \) for \( D = 3 \) is well known [20, 21]. The behavior for \( D > 3 \) can be found in [31, 32].

III. BOSE–EINSTEIN CONDENSATION IN A CONSTANT EXTERNAL MAGNETIC FIELD

When a magnetic field is applied to a gas of charged bosons in three spatial dimensions the energy spectrum (in the infinite volume limit) contains a discrete harmonic oscillator–like part as well as a continuous part. The discrete part is just the Landau level quantization [22]. For \( D > 3 \) there may be a number of discrete components because the magnetic field is not described by a vector, but by an antisymmetric tensor with more than one independent component [34]. For simplicity we will restrict our attention to the case of only a single nonzero component in the present paper. We wish to provide a treatment similar to that for the free Bose gas when a nonzero
magnetic field is present. In particular we will study the specific heat and see how the presence of a magnetic field alters the behavior from that found for the free Bose gas in section B. Also we will compute the magnetization and study the Meissner–Ochsenfeld effect in detail. The formalism used is the effective action method as reviewed in references 33, 34. This formalism allows the nonzero condensate (if there is one) to be treated in a very natural manner.

A. Thermodynamic potential and phase transitions: General formalism

The thermodynamic potential is usually defined by

$$\Omega_T = \frac{1}{\beta} \sum_n \ln \left[ 1 - e^{\beta(E_n - \mu)} \right]. \quad (3.1)$$

However, in the effective action method there is another term present if there is a nonzero condensate described by a background field $\bar{\Psi}$. This is

$$\Omega^{(0)} = \int d^D x \left\{ \frac{1}{2m} |D\bar{\Psi}|^2 - e\mu|\bar{\Psi}|^2 \right\}. \quad (3.2)$$

Here $D\bar{\Psi} = \nabla \bar{\Psi} - ieA\bar{\Psi}$ is the usual gauge–covariant derivative. The complete thermodynamic potential is

$$\Omega = \Omega^{(0)} + \Omega_{T \neq 0}. \quad (3.3)$$

(Actually, if we are interested in the dynamics of the magnetic field there will be an additional term involving a Maxwell action. We will consider this in section III D below). Given the thermodynamic potential, all quantities of interest can be calculated.

The presence of a condensate is signalled by a nonzero value for $\bar{\Psi}$. This is associated with symmetry breaking as discussed in the relativistic case 37, 38. In our case $\Psi$ must satisfy

$$\frac{\delta \Omega}{\delta \bar{\Psi}} = 0 = \frac{1}{2m} D^2 \bar{\Psi} + e\mu \bar{\Psi}. \quad (3.4)$$

We can solve this by expanding $\bar{\Psi}(x)$ in terms of the stationary state solutions to the Schrödinger equation:

$$\frac{-1}{2m} D^2 f_n(x) = E_n f_n(x). \quad (3.5)$$

If we write

$$\bar{\Psi}(x) = \sum_n C_n f_n(x) \quad (3.6)$$

for some coefficients $C_n$, and assume that the set of solutions $f_n(x)$ forms a complete set, then (3.4) results in

$$0 = (E_n - \mu)C_n. \quad (3.7)$$

We will define a critical value of $\mu$, say $\mu_c$, by

$$e\mu_c = E_0, \quad (3.8)$$

where $E_0$ is the lowest energy level. If $\mu < \mu_c$, then the only solution to (3.7) is for $C_n = 0$, which corresponds to $\Psi = 0$. There is no condensate in this case associated with symmetry breaking and a phase transition. However if $\mu$ can reach the value $\mu_c$ defined in (3.8) for some temperature $T_c$, then $C_0$ in (3.7) is undetermined and we can have a nonzero condensate described by

$$\bar{\Psi}(x) = C_0 f_0(x). \quad (3.9)$$

The temperature $T_c$ at which $\mu = \mu_c$ is called the critical temperature.

For the case of the free gas considered in section III we have $E_0 = 0$, so that $\mu_c = 0$. The critical temperature $T_c$ is then the value of the temperature at which the chemical potential vanishes as stated earlier. If the spatial dimension $D > 3$ a critical temperature exists and signals a phase transition with a nonzero value for $\bar{\Psi}$. ($\bar{\Psi}$ is constant for the free Bose gas). Associated with this phase transition is a growth in the number of particles in the ground state.

For some systems it is possible to have a sudden growth in the occupancy of the ground state without a phase transition. In this case $\mu$ never reaches the critical value of $\mu_c$ but instead approaches it asymptotically. The speed at which the ground state particle number builds up depends on how fast $\mu$ approaches $\mu_c$. Because $\mu$ never reaches $\mu_c$ we have $\bar{\Psi} = 0$, and no symmetry breaking. As we will see below, for a charged Bose gas in a constant magnetic field the spatial dimension $D$ determines whether or not a phase transition occurs.

We can use our expression (3.3) for $\Omega$ to find the total charge $Q$, since $Q = -\frac{\partial \Omega}{\partial \mu}$ with $V, B, \beta, \text{ and } \bar{\Psi}$ held fixed. It is convenient to write

$$Q = Q_0 + Q_1 \quad (3.10)$$

where

$$Q_0 = -\frac{\partial \Omega^{(0)}}{\partial \mu} = e \int d^D |\bar{\Psi}|^2 = e |C_0|^2, \quad (3.11)$$

if we use (3.2), (3.9), and

$$Q_1 = -\frac{\partial \Omega_{T \neq 0}}{\partial \mu}. \quad (3.12)$$

From (3.1) we find

$$Q_1 = e \sum_n \frac{1}{e^{\beta(E_n - \mu)} - 1}. \quad (3.13)$$

If we can always solve $Q = Q_1$ for $\mu$ for all temperatures, then $\bar{\Psi} = 0$. There is no condensate, symmetry breaking, or phase transition in this case. If it is not possible to solve $Q = Q_1$ for $\mu$, then we must have $Q_0 \neq 0$ and find a nonzero value for $\bar{\Psi}$.
B. Thermodynamic potential: Constant magnetic field

The formalism outlined in section III A will now be applied to the $D$-dimensional charged Bose gas in a constant one-component magnetic field. We will assume $D \geq 3$ here and pick the magnetic field in the $z$–direction. It is possible to solve (3.3) for the energy levels and corresponding eigenfunctions $[35]$. We have (choosing $\hbar = c = 1$)

$$E_{n,k_i} = \left(n + \frac{1}{2}\right) \omega + \frac{1}{2m} \sum_{i=3}^{D} \left(\frac{2\pi k_i}{L_i}\right)^2$$  \hspace{1cm} (3.14)

where $n = 0, 1, \ldots$ labels the Landau level, and $k_i = 0, \pm 1, \ldots$ if we impose periodic boundary conditions on a box as in section II. We have defined

$$\omega = \frac{eB}{m}.$$  \hspace{1cm} (3.15)

The energy levels (3.14) are degenerate with degeneracy

$$g = \frac{eBL_1L_2}{2\pi}.$$  \hspace{1cm} (3.16)

As in section II we will be interested in the large box limit with $L_i \to \infty$. In this limit we can replace the sums over $k_i$ resulting when (3.14) is used in (3.1) with integrals. A change of variables gives

$$\Omega_{T \neq 0} = \frac{eBV}{2\pi \beta} \sum_{n=0}^{\infty} \int \frac{d^{D-2}k}{(2\pi)^{D-2}} \ln \left\{ 1 - e^{-\beta \left(\left(n + \frac{1}{2}\right)\omega + \frac{k^2}{2m} - \mu\right)} \right\}.$$  \hspace{1cm} (3.17)

This may be evaluated by expanding the logarithm in its Taylor series and then performing the integral over $k$. We find

$$\Omega_{T \neq 0} = -\omega V \left(\frac{m}{2\pi \beta}\right)^{\frac{D}{2}} \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} \frac{\omega^{-l}\beta^{-l/2} (n + \frac{1}{2}) \omega^{-\mu}}{(1 - e^{-\beta \omega})^l}$$  \hspace{1cm} (3.18)

where the second line has followed by performing the sum over $n$. At this stage it is useful to define a dimensionless temperature. We define

$$x = \beta \omega.$$  \hspace{1cm} (3.19)

$x$ is seen to be the ratio between the energy gap between successive energy levels $\omega$ and the thermal energy $k_B T$. The lowest energy level from (3.14) is $E_{0,0} = \frac{\omega}{2}$. From (3.8) the critical value for $\mu$ is

$$\epsilon \mu_c = \frac{\omega}{2}.$$  \hspace{1cm} (3.20)

We will define a dimensionless chemical potential $\epsilon$ by

$$\epsilon \mu = \omega \left(\frac{1}{2} - \epsilon\right).$$  \hspace{1cm} (3.21)

A phase transition is characterized by a critical temperature $T_c$ at which $\epsilon = 0$. The expression (3.18) may be written in terms of the dimensionless variables $x$ and $\epsilon$:

$$\Omega_{T \neq 0} = \omega V \left(\frac{m}{2\pi \beta}\right)^{\frac{D}{2}} \sum_{l=1}^{\infty} \frac{\omega^{-l}\beta^{-l/2} e^{-\epsilon \omega}}{(1 - e^{-\omega})^l}.$$  \hspace{1cm} (3.22)

It is convenient to introduce some compact notation for the class of sums we will encounter in order to simplify formulae. Let

$$\Sigma_{\epsilon}[\alpha, \delta] = \sum_{l=1}^{\infty} \frac{\omega^{-l}\beta^{-l/2} e^{-\epsilon \omega}}{(1 - e^{-\omega})^l}.$$  \hspace{1cm} (3.23)

With this notation we may write (3.22) as

$$\Omega_{T \neq 0} = \omega V \left(\frac{m}{2\pi \beta}\right)^{\frac{D}{2}} \epsilon \Sigma_{\epsilon}[-D, 0].$$  \hspace{1cm} (3.24)

Various thermodynamic quantities involve derivatives of the thermodynamic potential. We will initially consider the charge.

From (3.12), using (3.24) we have

$$Q_1 = \omega V \left(\frac{m}{2\pi \beta}\right)^{\frac{D}{2}} \frac{\partial}{\partial \mu} \Sigma_{\epsilon}[-D, 0]$$

$$= eV \left(\frac{m}{2\pi \beta}\right)^{\frac{D}{2}} x \Sigma_{\epsilon}[2 - D, 0].$$  \hspace{1cm} (3.25)

Whether or not a phase transition occurs is determined by the convergence or divergence of $\Sigma_{\epsilon}[2 - D, 0]$ as $\epsilon \to 0$. If this sum diverges as $\epsilon \to 0$ then we will always be able to solve $Q = Q_1$ for $\mu$. As discussed in section III A this means that there is no phase transition. From (3.23) it is easy to see that $\Sigma_{\epsilon}[\alpha, 0]$ diverges as $\epsilon \to 0$ for $\alpha \geq 2$. With $\alpha = 2 - D$, this means that there is no phase transition for $D = 3, 4$ dimensions. For $D \geq 5$, $\Sigma_{\epsilon}[2-D, 0]$ converges as $\epsilon \to 0$ and there is a phase transition. In this case (i.e. for $D \geq 5$) there is a nonzero condensate characterized by $\Psi \neq 0$.

For $D \geq 5$ we may use (3.9) to find $\tilde{\Psi}$. If we choose the gauge

$$A_1 = -By, \quad A_2 = \ldots = A_D = 0,$$  \hspace{1cm} (3.26)

for the vector potential, then

$$f_0 = \alpha e^{-\frac{1}{4} e \beta y^2}$$  \hspace{1cm} (3.27)

is the eigenfunction corresponding to the lowest energy $E_0 = \frac{\omega}{2}$. $\alpha$ is a constant chosen to normalize $f_0$. $\Omega^{(0)}$ is found by using (3.9) with (3.27) in the general expression (3.3). We will return to this when we discuss the magnetization in section III D.
C. The specific heat

The internal energy is given by \( U = \frac{\partial}{\partial \beta} (\beta \Omega) \) with \( V, \omega, \) and \( \beta \mu \) held fixed. Using (3.1) and (3.3) we find

\[
U = \int d^D x \left\{ \frac{1}{2m} |D \Psi|^2 \right\} + \frac{\partial}{\partial \beta} (\beta \Omega_{T \neq 0}). 
\]  
(3.28)

The first term accounts for any nonzero condensate, and the second term is easily seen to be

\[
\frac{\partial (\beta \Omega_{T \neq 0})}{\partial \beta} = \sum_n \frac{E_n}{e^{\beta (E_n - \epsilon_n)} - 1} \tag{3.29}
\]

which is the usual expression for the internal energy (see equation (2.3) with a different definition for \( \mu \)). With (3.3) and (3.9) we find

\[
U = E_0 |C_0|^2 + \frac{\partial}{\partial \beta} (\beta \Omega_{T \neq 0}). 
\]  
(3.30)

If \( E_0 = 0 \), then the contribution from \( \Omega^{(0)} \) to the internal energy vanishes, and the internal energy is given by the standard expression (3.29). This is the situation for the free Bose gas discussed in section II. For the constant magnetic field, \( E_0 = \frac{\omega}{2} \), so that if \( \Psi \neq 0 \) we must include the condensate contribution to obtain the correct expression for the energy.

If we use (3.11), then (3.30) may be written as

\[
U = \frac{\omega Q_0}{2e} + \frac{\partial}{\partial \beta} (\beta \Omega_{T \neq 0}). 
\]  
(3.31)

With (3.24) for \( \Omega_{T \neq 0} \), and differentiating with respect to \( \beta \) keeping \( \beta \mu, \omega, V \) fixed results in

\[
\frac{\partial (\beta \Omega_{T \neq 0})}{\partial \beta} = \omega V \left( \frac{m}{2\pi \beta} \right)^{\frac{D}{2}} \left\{ x \Sigma_2 [2-D, 1] + \frac{x \Sigma_1 [2-D, 0]}{2} \right\} + \frac{(D-2) \Sigma_1 [-D, 0]}{2}. 
\]  
(3.32)

Noting (3.25) allows us to write this as

\[
\frac{\partial}{\partial \beta} (\beta \Omega_{T \neq 0}) = \frac{\omega Q_0}{2e} + \omega V \left( \frac{m}{2\pi \beta} \right)^{\frac{D}{2}} \left\{ x \Sigma_2 [2-D, 1] + \frac{(D-2) \Sigma_1 [-D, 0]}{2} \right\}. 
\]  
(3.33)

Substitution of (3.33) back into (3.31) and noting from (3.10) that \( Q = Q_0 + Q_1 \), where \( Q \) is the total charge, results in

\[
U = \frac{\omega Q}{2e} + \omega V \left( \frac{m}{2\pi \beta} \right)^{\frac{D}{2}} \left\{ x \Sigma_2 [2-D, 1] + \frac{(D-2) \Sigma_1 [-D, 0]}{2} \right\}. 
\]  
(3.34)

The expression for the internal energy we have just obtained (3.34) holds whether there is a nonzero condensate (\( \Psi \neq 0 \)) or not. In cases where a phase transition does occur, the rôle of the condensate is crucial for obtaining the correct expression for the internal energy. If we had neglected the contribution coming from \( \Omega^{(0)} \) then (3.34) would have \( Q_1 \) in place of \( Q \). This would then lead to an erroneous expression for the specific heat, since \( Q \) is fixed whereas \( Q_1 \) is not.

The specific heat at constant volume was defined in (2.6). The quantities held fixed are \( V, Q, \Psi, \) and \( \omega = \frac{\mu}{m} \) when the differentiation is performed. This means that the first term in (3.34) makes no contribution to \( C_v \). Just like the free Bose gas discussed in section II, we must distinguish between the expressions above and below the critical temperature if there is a phase transition. For \( D \geq 5 \), the critical temperature \( T_c \) is defined by

\[
Q = eV \left( \frac{m}{2\pi \beta C} \right)^{\frac{D}{2}} x \Sigma_1 [2-D, 0] \bigg|_{\varepsilon = 0, x = x_c}. 
\]  
(3.35)

Here \( x_c = \beta_c \omega \) with \( \beta_c = T_c^{-1} \), and the sum \( \Sigma_1 [2-D, 0] \) evaluated with \( \varepsilon = 0 \) and \( x = x_c \). Unlike for the free Bose gas, it is not possible to solve for \( T_c \) analytically. We will return to an approximate evaluation of \( T_c \) in section IV. We have solved (3.35) numerically to find \( T_c \). The results for small values of \( \omega \) are shown in figure II.

As expected, when \( \omega \to 0 \) we have \( T_c \to T_0 \).

![FIG. 1. For \( D \geq 5 \) this shows the value of \( \beta_c \) compared to the free Bose gas critical value of \( \beta_0 \). For \( D = 3, 4 \) because there is no critical temperature when a magnetic field is present we have plotted the inverse temperature of the specific heat maximum in units of \( \beta_0 \). For small values of \( \omega \), \( \beta_c \) is close to \( \beta_0 \) (an approximate analytical expression will be given in section IV). As the strength of the magnetic field is increased the deviation of \( T_c \) from \( T_0 \) becomes more pronounced, but lessens with increasing dimension. In all cases where a phase transition occurs, the critical temperature is lower than the result for the free Bose gas as observed in III. Let \( C_v^{\omega} \) denote the specific heat for \( T > T_c \) and \( C_v^{\sigma} \) be the specific heat for \( T < T_c \). Using (3.34) we have]
\[
C_v^\omega = xV \left( \frac{m}{2\pi\beta} \right)^\frac{D}{2} \left\{ 2x^2\Sigma_3 [4 - D, 2] + x^2\Sigma_2 [4 - D, 1] + (D - 2)x\Sigma_3 [2 - D, 1] + \frac{D(D - 2)}{4}\Sigma_1 [-D, 0] + \beta x \left( \frac{\partial(\omega x)}{\partial\beta} \right)_Q \right. \\
\left. \times \left[ \Sigma_2 [4 - D, 1] + \left( \frac{D - 2}{2} \right) \Sigma_1 [2 - D, 0] \right] \right\}. \quad (3.36)
\]

The terms in \( \left( \frac{\partial(\omega x)}{\partial\beta} \right)_Q \) arise from the fact that the charge \( Q \) rather than the chemical potential \( \mu \) is held fixed. By setting \( Q = Q_1 \) in (3.25) (valid since \( T > T_c \)), and differentiating both sides with respect to \( \beta \) with \( Q \) held fixed gives
\[
\beta x \left( \frac{\partial(\omega x)}{\partial\beta} \right)_Q = -\frac{x}{\Sigma_1 [4 - D, 0]} \left[ x\Sigma_2 [4 - D, 1] + \left( \frac{D - 2}{2} \right) \Sigma_1 [2 - D, 0] \right]. \quad (3.37)
\]

Substitution of (3.37) into (3.36) leads to
\[
C_v^\omega = xV \left( \frac{m}{2\pi\beta} \right)^\frac{D}{2} \left\{ 2x^2\Sigma_3 [4 - D, 2] + x^2\Sigma_2 [4 - D, 1] + (D - 2)x\Sigma_3 [2 - D, 1] \right. \\
\left. \frac{D(D - 2)}{4}\Sigma_1 [-D, 0] \right. \\
\left. - \frac{x^2(\Sigma_3 [4 - D, 1])^2}{\Sigma_1 [4 - D, 0]} \right. \\
\left. - \left( \frac{D - 2}{2} \right)^2 \frac{(\Sigma_1 [2 - D, 0])^2}{\Sigma_1 [4 - D, 0]} \right\}. \quad (3.38)
\]

For \( T < T_c \) we have \( \omega = 0 \) fixed. (Equivalently, \( \mu = \frac{\omega}{2} \) is fixed). The result of this is that \( C_v^\omega \) is given by (3.38) but with the last two terms set to zero, and \( \omega = 0 \) in all other terms:
\[
C_v^\omega = xV \left( \frac{m}{2\pi\beta} \right)^\frac{D}{2} \left\{ 2x^2\Sigma_3 [4 - D, 2] + x^2\Sigma_2 [4 - D, 1] + (D - 2)x\Sigma_3 [2 - D, 1] \right. \\
\left. + \frac{D(D - 2)}{4}\Sigma_1 [-D, 0] \right\}_{\omega = 0}. \quad (3.39)
\]

By comparing \( C_v^\omega \) in (3.38) with \( C_v^\omega \) in (3.39) it can be seen that whether or not the specific heat is continuous at the critical temperature is determined by the behavior of \( \Sigma_1 [4 - D, 0] \) as \( \omega \rightarrow 0 \). The two expressions will only agree if \( \Sigma_1 [4 - D, 0] \) diverges in this limit. From the definition (3.23) this only happens for \( D = 5, 6 \) (Recall that we are assuming \( D \geq 5 \) here so that \( T_c \) exists). We conclude that the specific heat for the charged Bose gas in a constant magnetic field is continuous at the critical temperature for \( D = 5, 6 \) and discontinuous for \( D \geq 7 \).

In the cases \( D = 3, 4 \) where there is no phase transition, \( \omega \) never vanishes. The specific heat is given by (3.38) in these two cases for all temperatures. When \( D = 3, 4 \) the specific heat is a perfectly smooth function of temperature.

Graphs showing the specific heat for \( D = 3—7 \) are shown in figures 2—6.

FIG. 2. This shows the specific heat at constant volume in units of the total number of particles as a function of the inverse temperature in units of \( \beta_0 \) where \( \beta_0 = T_0^{-1} \) with \( T_0 \) given in (2.11) for \( D = 3 \). For comparison the free gas result is also shown (\( \omega = 0 \)).

FIG. 3. The specific heat at constant volume in units of the total number of particles for the magnetized gas with \( D = 4 \). The top curve (\( \omega = 0 \)) is the result for the free Bose gas.
FIG. 4. The specific heat at constant volume in units of the total number of particles for the magnetized gas with $D = 5$. The curves are given as a function of $\beta = T^{-1}$ in units of $\beta_0$ rather than $\beta_C$. The discontinuous free gas result is labeled $\omega = 0$.

For the $D = 3, 4$ gases, the presence of the magnetic field is seen in figures 2-4 to round off the familiar sharp behavior at the transition temperature for the free gas. In both cases as the magnetic field is reduced the curves for the specific heat tend towards the free-field results. The maximum in the specific heat becomes sharper, and the temperature at which the maximum occurs tends towards the value $T_0$ as the magnetic field is reduced. For $B \neq 0$, although there is no phase transition characterized by a critical temperature and a nonzero condensate, as $B$ is reduced the specific heat starts to look more and more like the free gas result. We will look at this analytically in section IV. In both cases, as $\beta \to 0$ (or $T \to \infty$) the specific heats for $D = 3, 4$ approach the classical Maxwell–Boltzmann results of $1.5$ (for $D = 3$) and $2$ (for $D = 4$).

The behavior of the specific heat as $B$ is increased is also of interest. Increasing the value of $B$ tends to reduce the specific heat maximum and to broaden the curves. In fact for large values of $B$ if we examine figure 2 it can be seen that the curves approach the value of $\frac{3}{2}$ as $B$ is reduced before rising sharply to the classical result of $\frac{5}{2}$. This demonstrates that the specific heat for the gas with $D = 3$ resembles the specific heat for the free gas with $D = 1$ in a strong magnetic field. The value of $\frac{3}{2}$ is the classical value for the one–dimensional gas. For the magnetized gas with $D = 4$, figure 2 shows that the specific heat approaches the value of $1$ in a strong magnetic field before the sharp rise to the classical value of $2$. Again this shows the reduction in the dimension by 2 since $\frac{C_V}{N} \to 1$ as $\beta \to 0$ for the free Bose gas in two spatial dimensions. This is totally consistent with the approach used in references [40,41] in which the leading behavior of thermodynamic quantities was studied in a general setting by using the lowest energy solutions. As $B$ is increased, the gap between the ground state and the excited states becomes larger; thus the leading contribution would be expected to come from the ground state.

The results for the specific heat of the 5–dimensional gas are shown in figure 4. In this case the specific heat for the free Bose gas is discontinuous. Nevertheless as the magnetic field is reduced the specific heat curves approach the free gas result. The peaks of the specific heat start to become sharper and the slope of the curve steeper. The classical Maxwell–Boltzmann result of $\frac{5}{2}$ is reached as $\beta \to 0$. Just as for the cases $D = 3, 4$, the gas exhibits a reduction in the effective dimension for large values of the magnetic field. This time we would expect to find the specific heat curves looking more and more like the familiar form for the specific heat of the free gas in three spatial dimensions. For clarity we have given the large magnetic field results on a separate graph in figure 7 over a large temperature range.

The reduction in dimension should be evident with the specific heat approaching the value $\frac{3}{2}$, the classical free gas result for $D = 3$, before rising to the value $\frac{5}{2}$ as $\beta \to 0$. 

FIG. 5. The specific heat at constant volume in units of the total number of particles for the magnetized gas with $D = 6$. The discontinuous free gas result is shown as $\omega = 0$.

FIG. 6. The specific heat at constant volume in units of the total number of particles for the magnetized gas with $D = 7$. This is the first spatial dimension for which the specific heat has a discontinuity at the critical temperature. The free gas result ($\omega = 0$) is also shown.
for weak fields the specific heat resembles the free 5–dimensional gas, and for strong magnetic fields the apparent reduction in dimension again takes place with ω = 0 in figure 6.

The specific heats for D = 6, 7 are shown in figures 2, 3 respectively. For D = 6, we see the peaks sharpen as the free gas result is approached. Like the 5–dimensional gas the curves are always continuous, but the slope becomes greater as B is reduced. For strong magnetic fields the specific heat in the presence of a magnetic field is discontinuous. For strong magnetic fields the specific heat resembles the free 5–dimensional gas, and for weak fields the free Bose gas limit is approached.

D. Magnetization

Even though for D = 3 there is no phase transition which can be associated with Bose–Einstein condensation, by studying the magnetization of the charged gas Schafroth showed that the gas exhibited the Meissner–Ochsenfeld effect. The generalization to other spatial dimensions was performed later. We will show how the formalism described in section III A can be used to obtain the magnetization. In particular, the role of the condensate ¯Ψ for D ≥ 5 will be examined carefully.

The simplest way to see the effects of magnetization is by studying how the field equations for electromagnetism are affected. To do this we must include a term in the thermodynamic potential (3.3) for the electromagnetic field. We will use Heaviside–Lorentz rationalized units as usual in quantum field theory. (A discussion of the various units and how this alters the expression for the magnetization is given in [41]. Of course the physics of the situation should be independent of this arbitrary choice). We add on

\[
\Omega_{\text{em}} = \int d^Dx \left( \frac{1}{4} F_{ij} F^{ij} - J_{\text{ext}}^i A_i \right) \tag{3.40}
\]
to (3.3), where \( F_{ij} = \partial_i A_j - \partial_j A_i \) is the field strength tensor describing the magnetic field, and \( J_{\text{ext}}^i \) is the externally applied current which is responsible for setting up the magnetic field. The complete thermodynamic potential is

\[
\Omega = \Omega_{\text{em}} + \Omega^{(0)} + \Omega_T \neq 0 \tag{3.41}
\]
where \( \Omega^{(0)} \) and \( \Omega_T \neq 0 \) are given in (3.1) and (3.2).

Variation of \( \Omega \) with respect to the magnetic field \( F_{ij} \) results in

\[
\partial_j H^{ij} = J_{\text{ext}}^i \tag{3.42}
\]
where

\[
H^{ij} = F^{ij} + 2 \frac{\delta}{\delta F_{ij}} \left( \Omega^{(0)} + \Omega_T \neq 0 \right). \tag{3.43}
\]

\( H^{ij} \) is the D–dimensional analogue of the usual vector \( H \) in 3-dimensional electromagnetism. As explained earlier it is necessary to treat the magnetic field as a tensor if \( D \neq 3 \). For more details of this analysis see reference [53].

We have treated the magnetic field generally in (3.42–3.43). Specializing now to a single component field of strength \( B \) \( (F_{12} = F_{21} = B) \), equation (3.43) can be written as

\[
H = B - M, \tag{3.44}
\]
where

\[
M = -\frac{\delta}{\delta B} \left( \Omega^{(0)} + \Omega_T \neq 0 \right), \tag{3.45}
\]
and \( H^12 = -H^{21} = H \). Equation (3.44) is the conventional \( B–H \) relation found in three spatial dimensions, but with the notation defined here it can be seen to hold for all \( D \). In (3.45) \( M \) is the magnetization. This approach is seen to avoid any ambiguity between what Schafroth called the acting and microscopic fields.

We can now split \( M \) in (3.45) into two pieces in an obvious way. The derivative in (3.45) is a functional derivative, and because \( B \) is a constant for our problem we can define

\[
M^{(0)} = -\frac{1}{V} \frac{\partial \Omega^{(0)}}{\partial B}, \tag{3.46}
\]
\[
M_T \neq 0 = -\frac{1}{V} \frac{\partial \Omega_T \neq 0}{\partial B}. \tag{3.47}
\]

\( M_T \neq 0 \) is easily computed using (3.24) to be

\[
M_T \neq 0 = \frac{e}{m} \left( \frac{m}{2\pi \hbar} \right) \frac{\varphi}{2} \left\{ \Sigma_1 [-D,0] - \frac{x}{2} \Sigma_1 [2-D,0] - x \Sigma_2 [2-D,1] \right\}. \tag{3.48}
\]
Recalling (3.25) we have
\[ M_{T \neq 0} = -\frac{1}{2mV} Q_1 + \frac{e}{m} \left( \frac{m}{2\pi\beta} \right) \frac{\Phi}{\beta} \{ \Sigma_1 [-D, 0] \\
- x \Sigma_2 [2 - D, 1] \}. \] (3.49)

In cases where no phase transition occurs, we have \( Q_1 = Q \) so that the first term of (3.49) is constant. Also if \( \Psi = 0 \) then \( M^{(0)} = 0 \), so that the total magnetization is given by (3.49) with \( Q_1 = Q \).

When a phase transition does occur we need \( M^{(0)} \). With our gauge choice (3.26), and using (3.2) for \( \Omega^{(0)} \), we find
\[ M^{(0)} = -\frac{e^2 B}{mV} \int d^d x y^2 |\bar{\Psi}|^2. \] (3.50)

\( \bar{\Psi} \) was given by (3.25) with (3.27). It is easy to show that
\[ \int d^d x y^2 |f_0|^2 = \frac{1}{2eB}, \] (3.51)

which results in
\[ M^{(0)} = -\frac{e}{2mV} |C_0|^2 = -\frac{Q_0}{2mV}, \] (3.52)

after using (3.11). We may now combine (3.49) and (3.52) to read
\[ M = -\frac{Q}{2mV} + \frac{e}{m} \left( \frac{m}{2\pi\beta} \right) \frac{\Phi}{\beta} \{ \Sigma_1 [-D, 0] \\
- x \Sigma_2 [2 - D, 1] \}. \] (3.53)

since \( Q = Q_0 + Q_1 \) is the total charge.

The result in (3.53) is the exact expression for the magnetization which holds even if there is a phase transition. Had the condensate \( \bar{\Psi} \) been ignored, we would have obtained (3.49) rather than (3.53). In the true expression (3.53) the first term is constant, whereas in (3.49) the first term is not constant if there is a phase transition. Thus, neglect of the condensate for \( D \geq 5 \) would lead to an erroneous result for the magnetization.

A selection of graphs for the dimensionless magnetization \( \mathcal{M} = \frac{mV M}{Q} \) is shown in figure 8. Although the expression for the magnetization will not be correct for \( D \geq 5 \) if the condensate is neglected it can be seen in that the case \( D = 5 \), the graph is smooth and continuous for all values of the field. For \( D = 7 \) a kink is apparent. This seeming incongruity can be explained by the fact that the magnetization is calculated from a first derivative of the thermodynamic potential whereas the heat capacity comes from a second derivative. Hence the magnetization curve is smooth in those cases \( (D = 5, 6) \) where the heat capacity is continuous.

The zero-field spontaneous magnetization plotted on the graphs is of the Schafroth \([9]\) form
\[ M = -\frac{Q}{2mV} \left[ 1 - \left( \frac{T}{T_0} \right)^{\frac{1}{3}} \right] \] (3.54)

and it can be seen that as the field \( B \to 0 \), this limit is recovered.

**FIG. 8.** The dimensionless magnetization \( \mathcal{M} = \frac{mV M}{Q} \) for the charged Bose gas in 3, 5, and 7 spatial dimensions. As the number of dimensions increases, the diamagnetism due to macroscopic occupation of the lowest Landau level in the low temperature (high \( \frac{1}{\beta} \)) region becomes more pronounced. As can be seen from the graphs, the general behavior of the magnetized gas is similar in cases when a phase transition is absent \( (D = 3) \) or present \( (D = 5, 7) \). The even dimensional cases \( (D = 4, 6) \) have been omitted for brevity as they merely interpolate between the odd ones shown above.
IV. ANALYTICAL EXPANSIONS

So far we have presented mainly the results obtained from a numerical evaluations of the sums (3,23), since it is not possible to obtain exact results. However due to the form of the exponential involving $e^{-ix}$, in order to obtain reliable numerical results directly from (3,23) it is necessary to include an increasing number of terms as $x$ decreases. In this section we will discuss a reliable method for obtaining approximate analytical expressions for various situations when $\varepsilon$ and $x$ are small by finding asymptotic expansions for $\Sigma_\varepsilon[\alpha, \delta]$.

The basic method we will use here involves the Mellin–Barnes contour integral representation for the exponential function:

$$e^{-v} = \int_{c-i\infty}^{c+i\infty} \frac{d\theta}{2\pi i} \Gamma(\theta) v^{-\theta}. \quad (4.1)$$

Here $c$ is a constant with $\Re\, c > 0$ so that the contour lies to the right of the poles of the Gamma function. This is basically equivalent to the method used by Robinson [42] to obtain asymptotic expansions for the Bose–Einstein functions. It was used for the 3-dimensional magnetized gas by Daicic and Frankel [17], and has been used to discuss Bose–Einstein condensation in a harmonic oscillator potential in reference [43]. There are various ways in which (4.1) can be used to obtain expansions over a range of $x, \varepsilon$ as discussed in references [43, 44]. We will content ourselves with the simplest presentation here.

The details of how (4.1) may be used for the class of sums (3,23) are given in Appendix A for the situations of interest to us. The basic method is to use (4.1) to convert (3,23) into a contour integral. The contour may be closed in the left hand side of the complex plane and the result evaluated by the residue theorem. The even and odd spatial dimensions differ somewhat in the pole structure of the integrand. The net result is an asymptotic series for $\Sigma_\varepsilon[\alpha, \delta]$ which can be used to approximate the specific heat, magnetization and other thermodynamic quantities.

A. Critical temperature

For $D \geq 5$ the magnetized Bose gas is characterized by a well defined critical temperature $T_c$ which satisfies (3.33). It is not possible to evaluate $T_c$ in closed form. However we can use our asymptotic expansion of $\Sigma_\varepsilon[2-D,0]$ to obtain an approximate result for weak magnetic fields. We have from Appendix A

$$Q \simeq eV \left( \frac{m}{2\pi\beta_C} \right)^{\frac{D}{2}} \left\{ \zeta_\varepsilon\left(\frac{D}{2}\right) + \frac{\varepsilon}{2} \zeta_\varepsilon\left(\frac{D-2}{2}\right) + \ldots \right\}. \quad (4.2)$$

if only the two leading terms are included. This assumes $x_C \ll 1$. With the free Bose gas critical temperature defined by

$$Q = eV \left( \frac{m}{2\pi\beta_0} \right)^{\frac{D}{2}} \zeta_\varepsilon\left(\frac{D}{2}\right) \quad (4.3)$$

we find

$$0 \simeq 1 - \left( \frac{\beta_C}{\beta_0} \right)^{\frac{D}{2}} \zeta_\varepsilon\left(\frac{D}{2}\right) + \frac{x_C}{2} \zeta_\varepsilon\left(\frac{D-2}{2}\right). \quad (4.4)$$

Since $x_C$ is assumed small we see that $\beta_C \simeq \beta_0$. It is easy to show from (4.4) that

$$T_C \simeq T_0 - \frac{1}{D} \frac{\zeta_\varepsilon\left(\frac{D-2}{2}\right) eB}{\zeta_\varepsilon\left(\frac{D}{2}\right) m} \quad (4.5)$$

to leading order in $\frac{eB}{m}$. This shows that $T_C \to T_0$ as $B \to 0$. Furthermore, for a fixed charge density, the critical temperature is lower when a non zero magnetic field is present. This is consistent with our earlier numerical results.

A cruder estimate of $T_c$ was given in reference [38] which had the same linear behavior as in (4.5) but with a different numerical factor in front of $B$. Our result (4.5) is a special case of the multi–component magnetic field presented in reference [44].

It is possible to improve on the linear approximation of (4.5) by working consistently to higher order in the expansions. It is necessary to deal with $D = 5, 6$ separately from $D > 6$ because of the range of the terms retained.

For $D = 5$ we find

$$T_C \simeq T_0 - \frac{1}{5} \frac{\zeta_\varepsilon\left(\frac{5}{2}\right)}{5 \zeta_\varepsilon\left(\frac{5}{2}\right)} x_0 - \frac{1}{5 \pi^2} \frac{\zeta_\varepsilon\left(\frac{3}{2}\right)}{\zeta_\varepsilon\left(\frac{5}{2}\right)} x_0^3 + \left[ \frac{\zeta_\varepsilon\left(\frac{3}{2}\right)}{100 \zeta_\varepsilon\left(\frac{5}{2}\right)} - \frac{\zeta_\varepsilon\left(\frac{3}{2}\right)}{30 \zeta_\varepsilon\left(\frac{5}{2}\right)} \right] x_0^2 + \mathcal{O}(x_0^3). \quad (4.6)$$

For $D = 6$ we find

$$T_C \simeq T_0 - \frac{\pi^2}{36 \zeta_\varepsilon\left(\frac{5}{2}\right)} x_0 + \frac{x_0^2}{6 \zeta_\varepsilon\left(\frac{5}{2}\right)} \times \left[ \frac{\zeta_\varepsilon\left(\frac{3}{2}\right)}{\pi^2} + \frac{\pi^4}{216 \zeta_\varepsilon\left(\frac{5}{2}\right)} + \frac{1}{6} \left( \ln \frac{x_0}{2\pi} - 6 \right) + \ldots \right] \quad (4.7)$$

For $D \geq 7$ we have

$$T_C \simeq T_0 - \frac{1}{D} \frac{\zeta_\varepsilon\left(\frac{D-2}{2}\right)}{\zeta_\varepsilon\left(\frac{D}{2}\right)} x_0 + \frac{1}{6D} \frac{\zeta_\varepsilon\left(\frac{D-2}{2}\right)}{\zeta_\varepsilon\left(\frac{D}{2}\right)} \times \left[ \frac{3(D-2)}{2D} \right] \frac{\zeta_\varepsilon\left(\frac{D-2}{2}\right)}{\zeta_\varepsilon\left(\frac{D}{2}\right)} - 1 \right] x_0^2 + \ldots \quad (4.8)$$

It is worth remarking that the result given by May [11] is only correct if it is taken to linear order in $B$.

We can also obtain an approximate expression for the charge in the condensate when $T \leq T_c$ for $D \geq 5$. When $T \leq T_c$ we have $\varepsilon = 0$, so (3,23) gives us

$$Q_1 = eV \left( \frac{m}{2\pi\beta} \right)^{\frac{D}{2}} x\Sigma_\varepsilon[2-D,0]|_{\varepsilon=0}. \quad (4.9)$$
By using (3.35), which defines the critical temperature \( T_c \), we find (noting \( x = \beta \omega, x_C = \beta_C \omega \))

\[
Q_1 = Q \left( \frac{x_C}{x} \right)^{D-2} \frac{\zeta_{\alpha}(\frac{D-2}{2})}{\zeta_{\alpha}(\frac{D}{2})} \frac{\sum_{l=1}^{\infty} [2 - D, 0]_{l=0}}{\sum_{l=0}^{\infty} [2 - D, 0]_{l=0}}. \tag{4.10}
\]

From (3.10) we find that the charge in the condensate is

\[
Q_0 = Q \left\{ 1 - \left( \frac{x_C}{x} \right)^{D-2} \frac{\zeta_{\alpha}(\frac{D-2}{2})}{\zeta_{\alpha}(\frac{D}{2})} \right\} \left\{ \frac{\sum_{l=1}^{\infty} [2 - D, 0]_{l=0}}{\sum_{l=0}^{\infty} [2 - D, 0]_{l=0}} \right\}. \tag{4.11}
\]

This result is exact. If we now use the approximate analytical expressions for \( \Sigma \) we obtain

\[
Q_0 = Q \left\{ 1 - \left( \frac{x_C}{x} \right)^{D-2} \frac{\zeta_{\alpha}(\frac{D-2}{2})}{\zeta_{\alpha}(\frac{D}{2})} \right\} \left\{ \frac{\sum_{l=1}^{\infty} [2 - D, 0]_{l=0}}{\sum_{l=0}^{\infty} [2 - D, 0]_{l=0}} \right\}. \tag{4.12}
\]

It is worth remarking that the accuracy of any of our approximate expressions can be increased by simply including more terms.

From (4.12) we see that as \( \omega \to 0 \) because \( \frac{x_C}{x} = \frac{T}{T_0} \), and we know that \( T_c \to T_0 \), we recover the free Bose gas result

\[
Q_0 = Q \left[ 1 - \left( \frac{T}{T_0} \right)^{D-2} \right] \tag{4.13}
\]

The term in (4.12) which involves \( (x - x_c) \) represents the lowest order correction to the free field result in a weak magnetic field.

In figure 3 we have plotted the free gas result (4.13), the exact numerical result, and our approximation (4.12) for the case \( D = 5 \). It can be seen that (4.12) is very close to the true result.

![Figure 3](image)

FIG. 3. The ratio of the charge contained in the ground state over the total charge for i) the gas in (4.14), ii) the exact result (4.14), and iii) the approximation (4.12) (for \( D = 5 \), \( \omega = 1 \)).

**B. Magnetization for \( D \geq 5 \)**

The magnetization may be evaluated using (3.53). It is necessary to distinguish the cases \( D = 3, 4 \) for which no phase transition occurs from \( D \geq 5 \). We will deal with \( D \geq 5 \) in this section, leaving \( D = 3, 4 \) until later. With \( D \geq 5 \) we must deal with \( T \geq T_c \). When \( T \geq T_c \) we have \( \varepsilon \equiv 0 \) and the results are more complicated. In addition, the expansions break down once \( \varepsilon \) grows too large.

For \( T \leq T_c \) we use (3.53) with \( \varepsilon = 0 \):

\[
M = -\frac{Q}{2mV} \left\{ 1 - 2 \left( \frac{T}{T_c} \right)^{D-2} \right\} \left[ \frac{(\sum_{l=1}^{\infty} [2 - D, 0]_{l=0})}{\sum_{l=0}^{\infty} [2 - D, 0]_{l=0}} \right]_{T=0}. \tag{4.14}
\]

This can be rewritten if we eliminate \( \left( \frac{m}{2\pi} \right)^{D-2} \) using (3.35). We find

\[
M = -\frac{Q}{2mV} \left\{ 1 - 2 \left( \frac{T}{T_c} \right)^{D-2} \right\} \left[ \frac{(\sum_{l=1}^{\infty} [2 - D, 0] - x \Sigma_{l=0} [2 - D, 1])}{\sum_{l=0}^{\infty} [2 - D, 0]_{l=0}} \right]_{T=0}. \tag{4.15}
\]

It is now a straightforward matter to use the results of Appendix A to expand the sums.

1. \( D = 5 \)

From (A11) with \( \alpha = -5 \) and \( \varepsilon = 0 \) we have

\[
\Sigma_1 [-5, 0] \approx \frac{\zeta_{\alpha}(\frac{5}{2})}{x^2} + \frac{\zeta_{\alpha}(\frac{3}{2})}{2} + \frac{x \zeta_{\alpha}(\frac{1}{2})}{12}
+ \frac{x^2 \zeta_{\alpha}(\frac{1}{2})}{4\pi^2} - \frac{x^3 \zeta_{\alpha}(\frac{3}{2})}{720} + O(x^5). \tag{4.16}
\]

From (A29) with \( \alpha = -3, \varepsilon = 0 \) and \( \delta = 1 \) we have

\[
\Sigma_2 [-3, 1] \approx \frac{\zeta_{\alpha}(\frac{7}{2})}{x^2} - \frac{\zeta_{\alpha}(\frac{3}{2})}{12} - \frac{3x^2 \zeta_{\alpha}(\frac{1}{2})}{8\pi^2}
- \frac{x^3 \zeta_{\alpha}(\frac{1}{2})}{960\pi} + O(x^4). \tag{4.17}
\]

From (A11) with \( \alpha = -3 \) and \( \varepsilon = 0 \) we have

\[
\Sigma_1 [-3, 0]_{x=x_c} \approx \frac{\zeta_{\alpha}(\frac{5}{2})}{x_c^2} + \frac{\zeta_{\alpha}(\frac{3}{2})}{2} + \frac{x^2 \zeta_{\alpha}(\frac{1}{2})}{2\pi^2}
+ \frac{\zeta_{\alpha}(\frac{1}{2})}{12} x_c + \frac{x^3 \zeta_{\alpha}(\frac{1}{2})}{2880\pi} + O(x_c^5). \tag{4.18}
\]
It is now easy to show that the asymptotic expansion of the magnetization is given by

\[ M = -\frac{Q}{2mV} \left\{ 1 - \left( \frac{T}{T_c} \right)^{\frac{3}{2}} \right. \right. - \left. \left. \left( \frac{T}{T_c} \right)^{\frac{3}{2}} f_5(x, x_c) \right\} \]  

(4.19)

where

\[ f_5(x, x_c) \approx \left( x - x_c \right) \zeta_n \left( \frac{3}{2} \right) + \frac{5x^2}{4\pi^2} \zeta_n \left( \frac{5}{2} \right) - \frac{x^2}{2\pi^2} \zeta_n \left( \frac{1}{2} \right)
\]
\[ + \left( \frac{x^2}{3} - \frac{2x_c}{6} \right) \zeta_n \left( 2 \right) - \frac{x^3}{12} \zeta_n \left( 3 \right)
\]
\[ + \left( x_c - \frac{x}{3} \right) \frac{x^2}{2\pi^2} \zeta_n \left( \frac{1}{2} \right) - \frac{5x_c x^3}{8\pi^2} \zeta_n \left( \frac{5}{2} \right) + \ldots \]  

(4.20)

By taking \( B \to 0 \) we are left with the spontaneous magnetization

\[ M(B \to 0) \simeq -\frac{Q}{2mV} \left[ 1 - \left( \frac{T}{T_0} \right)^{\frac{3}{2}} \right] \]  

(4.21)

since \( T_c \to T_0 \) and \( f_5(x, x_c) \to 0 \) in this limit. This is the 5-dimensional version of Schafroth’s result, and agrees with May [1]. When \( B \neq 0 \) there are corrections to the Schafroth form as shown by the terms in \( f_5(x, x_c) \).

We can also compare our result with that given by May [1]. To do this we must replace \( T_c \) in (4.19) with \( T_0 \). We will only work to first order in the magnetic field and use (4.19). It is easily shown that

\[ M \simeq -\frac{Q}{2mV} \left\{ 1 - \left( \frac{T}{T_0} \right)^{\frac{3}{2}} \right. \right. - \left. \left. \left( \frac{T}{T_0} \right)^{\frac{3}{2}} f_6(x, x_c) \right\} \]  

(4.22)

This agrees with May’s result where a different method was used. There are two comments to make here. The first is that had we taken the expansion beyond linear order in \( x \) the results obtained would not agree with that of May’s method for reasons already mentioned. Secondly, care must be exercised in using (4.22) because we have assumed \( T \leq T_c \) in its derivation. This means that (4.22) does not hold at \( T = T_0 \), as can be seen from (4.24). If we set \( T = T_0 \) in (4.22) we would conclude that \( M \) was positive, leading to paramagnetic rather than diamagnetic behavior. This is clearly wrong. We will return to results for \( T = T_0 \) later. There is of course nothing wrong with taking \( T = T_c \) in (4.19).

2. \( D = 6 \)

From (A8), (A9) with \( \varepsilon = 0, k = 2 \) we have

\[ \Sigma_1[-4, 0]_{\varepsilon=0} \simeq \frac{\zeta_n(3)}{x_c} + \frac{\zeta_n(2)}{2} - x_c \left( \frac{\zeta_n(2)}{2\pi^2} + \frac{\ln \frac{x_c}{2\pi} - \gamma}{12} \right)
\]
\[ + x_c^3 \frac{\pi^2}{8640} + \mathcal{O}(x_c^5) \]  

(4.23)

From (A8) with \( \varepsilon = 0, k = 3 \) we find

\[ \Sigma_1[-6, 0] \simeq \frac{\zeta_n(4)}{x_c} + \frac{\zeta_n(3)}{2} + \frac{x_c \zeta_n(2)}{12} - \frac{x^2 \zeta_n(3)}{8\pi^2} + \frac{x^3}{1440} \]  

(4.24)

In this case all higher order terms in the asymptotic expansion vanish. Finally from (A22) with \( \varepsilon = 0, \delta = 1 \) and \( k = 2 \) we find

\[ \Sigma_2[-4, 1] \simeq \frac{\zeta_n(4)}{x^2} - \frac{x_c \zeta_n(3)}{12} + \frac{x_c \zeta_n(2)}{4\pi^2} - \frac{x^2}{480} \]  

(4.25)

Again higher order terms in the asymptotic expansion vanish. In obtaining (4.25) various relations of the Riemann \( \zeta \)-function were used. The expression for the magnetization becomes

\[ M = -\frac{Q}{2mV} \left\{ 1 - \left( \frac{T}{T_c} \right)^{\frac{3}{2}} \right. \right. - \left. \left. \left( \frac{T}{T_c} \right)^{\frac{3}{2}} f_6(x, x_c) \right\} \]  

(4.26)

where

\[ f_6(x, x_c) \approx -\left( x - x_c \right) \frac{\pi^2}{6\zeta_n(3)} + 3x^2 + \frac{\pi^4 x_c}{216\zeta_n^2(3)} - \frac{\pi^4 x_c^2}{144\zeta_n^3(3)}
\]
\[ - \frac{x_c}{\zeta_n(3)} \left[ \frac{\zeta_n(2)}{2\pi^2} + \frac{\ln \frac{x_c}{2\pi} - \gamma}{12} \right]
\]
\[ + x_c^3 \left[ \frac{\pi^6}{1728\zeta_n^6(3)} + \frac{\pi^2}{6\zeta_n^2(3)} \left( \frac{\zeta_n(2)}{2\pi^2} + \frac{\ln \frac{x_c}{2\pi} - \gamma}{12} \right) \right]
\]
\[ - x_c^2 \left[ \frac{\pi^6}{2592\zeta_n^6(3)} + \frac{\pi^2}{18\zeta_n^2(3)} \left( \frac{\zeta_n(2)}{2\pi^2} + \frac{\ln \frac{x_c}{2\pi} - \gamma}{12} \right) \right]
\]
\[ - \frac{x^2}{4\zeta_n(3)} \left( \frac{x}{4} + \frac{x_c}{4} \right) + \ldots \]  

(4.27)

Again we see the Meissner–Ochsenfeld effect as \( B \to 0 \), with the magnetization of the Schafroth form.
3. \( D \geq 7 \)

Using the results given in Appendix \( \sqrt{2} \), and working only to quadratic order in \( x, x_c \) (to avoid treating even and odd dimensions separately) we find

\[
M = - \frac{Q}{2mV} \left\{ 1 - \left( \frac{T}{T_c} \right)^{D-2} \right\} f_D(x, x_c)
\]

where

\[
f_D(x, x_c) = \left( x - x_c \right) \frac{\zeta_R (D-2)}{\zeta_R (\frac{D}{2})} + \frac{x_c^2}{4} \left[ \frac{\zeta_R^2 (D-2)}{\zeta_R (\frac{D}{2})} - \zeta_R \frac{(D-2)}{3 \zeta_R (\frac{D}{2})} \right] - x x_c \frac{\zeta_R^2 (D-2)}{6 \zeta_R (\frac{D}{2})} + \ldots .
\]

Since \( f_D(x, x_c) \to 0 \) as \( B \to 0 \) we recover the Schafroth magnetization.

C. Specific Heat

In a similar manner to the previous section analytical expansions may be obtained for the specific heat capacities in the dimensions of interest to us in this paper. It is more convenient to express these results in the form they were given previously, \( i.e. \) as expansions of the ratio \( M_N \). These expansions are constructed from the expression for the heat capacity (either equation \( 3.38 \) if above the transition temperature, or \( 3.39 \) if below it) as well as that for the number density (obtained from \( 2.25 \) as \( N = \frac{Q}{\pi} \)). The various sums present in these equations are treated as before by substituting the appropriate analytic approximations given in appendix \( \sqrt{2} \). Brief comments on the form of the expansions are made in each case.

1. \( D = 3 \)

From figure \( \sqrt{2} \) it can be seen that the charged Bose gas in three spatial dimensions does not exhibit a phase transition with finite field, but approaches the zero field result in the \( B \to 0 \) limit. The maximum of the heat capacity is always lower than the zero field limit, a fact which can clearly be seen from the analytic expansion

\[
\frac{C_v}{N} \simeq \frac{15 \zeta_R (\frac{3}{2})}{4 \zeta_R (\frac{3}{2})} - \frac{3 x^{\frac{3}{2}}}{4 \sqrt{\pi} \zeta_R (\frac{3}{2}) \zeta_R (\frac{3}{2}, \varepsilon)} \left[ 6 \zeta_R^3 (\frac{3}{2}) + 5 \pi \zeta_R (\frac{3}{2}) \zeta_R (\frac{3}{2}, \varepsilon) \zeta_R (\frac{3}{2}, \varepsilon) \right]
\]

\[
+ 5 \pi (2 \varepsilon - 1) \zeta_R (\frac{3}{2}) \zeta_R (\frac{3}{2}) \zeta_R (\frac{3}{2}) - 24 \zeta_R^3 (\frac{3}{2}) \zeta_R (\frac{3}{2})
\]

\[
- 4 \pi \zeta_R^2 (\frac{3}{2}) \zeta_R (\frac{3}{2}, \varepsilon) \zeta_R (\frac{3}{2}, \varepsilon) + O(x^\frac{3}{2}),
\]

\[
\]

2. \( D = 4 \)

In even dimensions, as noted in the appendix, the analysis is complicated by the presence of double poles. These give rise to terms in \( \ln x \) which dominate over the higher power terms which are the only ones present in the odd dimensional cases. In addition, the situation is complicated by terms involving \( \ln \varepsilon \) and \( \psi(\varepsilon) \). The expansion is given by

\[
\frac{C_v}{N} \simeq \frac{35 \zeta_R (\frac{3}{2})}{4 \zeta_R (\frac{3}{2})} - \frac{25 \zeta_R^3 (\frac{3}{2})}{4 \zeta_R^3 (\frac{3}{2})} + \frac{25 \pi x^{\frac{1}{2}} \zeta_R (\frac{3}{2})}{4 \zeta_R^3 (\frac{3}{2})}
\]

\[
+ 5 \pi (2 \varepsilon - 1) \zeta_R (\frac{3}{2}) \zeta_R (\frac{3}{2}) - 5 \zeta_R^3 (\frac{3}{2}) \zeta_R (\frac{3}{2})
\]

\[
- 2 \zeta_R^2 (\frac{3}{2}) \zeta_R^2 (\frac{3}{2}) + O(x^{\frac{1}{2}}),
\]

while below \( T_c \) equation \( 3.39 \) must be used, and hence

\[
\frac{C_v}{N} \simeq \frac{35 \zeta_R (\frac{3}{2})}{4 \zeta_R (\frac{3}{2})} + \frac{15 \zeta_R^2 (\frac{3}{2})}{8 \zeta_R^2 (\frac{3}{2})} + \frac{35 \pi x^{\frac{1}{2}} \zeta_R (\frac{3}{2})}{4 \zeta_R^2 (\frac{3}{2})}
\]

\[
+ 7 x^{\frac{3}{2}} (6 \zeta_R (\frac{3}{2}) \zeta_R (\frac{3}{2}) + 5 \zeta_R (\frac{3}{2}) \zeta_R (\frac{3}{2})
\]

\[
+ O(x^3).
\]
4. $D = 6$

The $D = 6$ case is similar to the $D = 4$ case except that here too a phase transition occurs below a critical temperature. Again the presence of logarithmic terms complicates the expression. For $T > T_c$

$$
\frac{C_v}{N} \simeq \frac{2}{15\pi^2 \zeta_R(3)} \times \frac{x}{90\pi^2 \zeta_R^2(3)} \times [8^{1/2} (2 \varepsilon - 1) (2^{1/2} - 15\pi^2 \zeta_R^2(3))
+ 29160 \zeta_R^3(3) \{ \zeta_R(0, 1 + \varepsilon) - \ln \varepsilon \}]
+ 162(2\varepsilon - 1)x \ln x
+ O \left( x^2 \right),
$$

(4.34)

whilst for $T \leq T_c$

$$
\frac{C_v}{N} \simeq \frac{2\pi^4}{15\zeta_R(3)} + \frac{x}{90\zeta_R(3)} + \frac{x^2 \pi^2}{180\zeta_R^2(3)} \times [24\pi^2 \zeta_R^2(-1, 1) - 2\pi^2 - 125\zeta_R(3)]
+ O \left( x^2 \ln x \right). 
$$

(4.35)

5. $D = 7$

$D = 7$ is the first case in which the heat capacity is wholly discontinuous. Above the critical temperature the expression

$$
\frac{C_v}{N} \simeq \frac{7}{4\zeta_R(\frac{3}{2})} \zeta_R(\frac{5}{2}) - 7\zeta_R^2(\frac{3}{2})
+ \frac{x(2^{14} - 7 \ln 2^{14})}{8\zeta_R(\frac{3}{2}) \zeta_R(\frac{5}{2})} \left[ 5\zeta_R^2(\frac{3}{2}) \zeta_R(\frac{5}{2}) 
+ 9\zeta_R(\frac{5}{2}) \zeta_R(\frac{3}{2}) - 7\zeta_R^2(\frac{3}{2}) \{ \zeta_R(\frac{5}{2}) + \zeta_R(\frac{3}{2}) \} \right]
- \frac{49\pi^2 x^2 \zeta_R(\frac{3}{2}, 0) x \ln x}{2\zeta_R^2(\frac{3}{2})} + O(x^2),
$$

(4.36)

holds whereas below it

$$
\frac{C_v}{N} \simeq \frac{63\zeta_R(\frac{3}{2})}{4\zeta_R(\frac{5}{2})} + \frac{7x(5\zeta_R^2(\frac{3}{2}) - 9\zeta_R(\frac{4}{2}) \zeta_R(\frac{5}{2}))}{8\zeta_R^2(\frac{3}{2})}
+ \frac{x^2}{16\zeta_R^2(\frac{3}{2})} \left[ 5\zeta_R(\frac{3}{2}) \zeta_R(\frac{5}{2}) - 35\zeta_R(\frac{4}{2}) \zeta_R(\frac{5}{2}) 
- 21\zeta_R(\frac{1}{2}) \zeta_R(\frac{3}{2}) \right] + O(x^3).
$$

(4.37)

must be used.

D. Expansions for $T$ near $T_0$

As noted in Schafroth’s original paper [9] the approximation (for the 3-dimensional gas)

$$
M \simeq -\frac{Q}{2mV} \left[ 1 - \left( \frac{T}{T_0} \right)^\frac{3}{2} \right]
$$

(4.38)

will break down when $T$ becomes too close to $T_0$. This can be substantiated by a direct numerical evaluation of $M$ and comparison with (4.35) as we showed in [14]. The Schafroth criterion for validity of (4.38) can be derived in a simple manner as described in [17]. An obvious question to ask is that if (4.38) does not hold as $T$ becomes close to $T_0$, is there another simple approximation which can be used? This was first studied by Daicic and Frankel [17], who showed by expanding about $\mu = 0$ that the magnetization $M \simeq -eB^\frac{1}{2}$ with $c > 0$ a constant. This approximation was valid for values of $T$ closer to $T_0$ than Schafroth’s approximation, but still broke down as $T \to T_0$. In reference [44] we showed how it was possible to evaluate the magnetization in a temperature range which included $T = T_0$. In this section we will present the details of this result and generalize the analysis to spatial dimensions $D > 3$.

For $D = 3$ the critical temperature $T_0$ for the free Bose gas is defined by

$$
Q = eV \left( \frac{m}{2\pi \beta_0} \right)^\frac{3}{2} \zeta_R \left( \frac{3}{2} \right)
$$

(4.39)

where $\beta_0 = T_0^{-1}$. For $B \neq 0$ we have

$$
Q = eV \left( \frac{m}{2\pi \beta} \right)^\frac{3}{2} x \Sigma_1 [-1, 0].
$$

(4.40)

Equating these two expressions gives

$$
x \Sigma_1 [-1, 0] = \left( \frac{x}{x_0} \right)^\frac{3}{2} \zeta_R \left( \frac{3}{2} \right)
$$

(4.41)

where $x_0 = \beta_0 \omega$. The aim now is to solve this for $\varepsilon$ when the magnetic field is weak and $T$ is close to $T_0$ (meaning $x$ is close to $x_0$). Because we assume a weak magnetic field, we may use the first few terms in the asymptotic expansion of $\Sigma_1 [-1, 0]$ obtained from Appendix A. We will approximate (4.41) as

$$
\left( \frac{x}{x_0} \right)^\frac{3}{2} \zeta_R \left( \frac{3}{2} \right) \simeq \zeta_R \left( \frac{3}{2} \right) + (\pi x)^\frac{1}{2} \zeta_R \left( \frac{1}{2}, \varepsilon \right)
+ \zeta_R \left( \frac{1}{2} \right) \left( \frac{1}{2} - \varepsilon \right) x + . . .
$$

(4.42)

Higher order terms could easily be included to improve the accuracy of the result.

Suppose that we concentrate first on $T = T_0$. Call the value of $\varepsilon$ at $T = T_0$, $\varepsilon_0$. Then (4.42) gives us

$$
0 = \zeta_R \left( \frac{1}{2}, \varepsilon_0 \right) + (\pi x)^\frac{1}{2} \zeta_R \left( \frac{1}{2}, \varepsilon_0 \right) \left( \frac{1}{2} - \varepsilon_0 \right) x_0 + . . .
$$

(4.43)

(The next term is of order $x_0^\frac{3}{2}$). If we let $B \to 0$ then $x_0 = \beta_0 \omega \to 0$. Thus as $B \to 0$ we must have $\varepsilon_0 \to a$ where $a$ is defined by

$\omega$
0 = \zeta_\nu \left( \frac{3}{2}, a \right). \quad (4.44)

The value of $a$ can be found numerically with the result $a = 0.302721829\ldots$. We have verified this by solving (4.41) numerically for decreasing values of $B$ and found that $\epsilon \to a$ as $B$ is reduced. Because $\mu = \omega \left( \frac{3}{2} - \epsilon \right)$ this result is still consistent with the expectation that $\mu \to 0$ as $B \to 0$. We have essentially determined how fast $\mu$ vanishes as $B \to 0$.

For small, but non-zero, values of $B$ we can try to solve (4.43). In order to obtain a consistent expansion from (4.43) it is fairly clear that we must have

$$\epsilon_0 \simeq a + a_1 x_0^{\frac{1}{2}} + a_2 x_0 + \ldots$$

(4.45)

for some coefficients $a_1, a_2, \ldots$ which can be found by substituting (4.43) into (4.43) and working to a consistent order in $x_0$. It is easily shown that

$$\zeta_\nu \left( \frac{3}{2}, \epsilon_0 \right) \simeq \frac{1}{2} a_1 x_0^{\frac{1}{2}} \zeta_\nu \left( \frac{3}{2}, a \right) + O (x_0). \quad (4.46)$$

Use of (4.43) and (4.45) in (4.43) fixes

$$a_1 = \frac{2 \zeta_\nu \left( \frac{3}{2} \right)}{\pi \zeta_\nu \left( \frac{5}{2}, a \right)} \left( \frac{1}{2} - a \right). \quad (4.47)$$

It should be clear how we can obtain an approximation for $\epsilon_0$ to any order in $x_0$ by extending the procedure we have just described to higher order.

So far we have just concentrated on the evaluation of $\epsilon$ at the single temperature $T_0$. Suppose that we now extend this to temperatures which are close to $T_0$. By a simple extension of the analysis just presented it is possible to show

$$\epsilon \simeq a + a_1 x_0^{\frac{1}{2}} + \frac{6 \zeta_\nu \left( \frac{3}{2} \right)}{\pi \zeta_\nu \left( \frac{5}{2}, a \right)} x_0^{\frac{1}{2}} \left[ 1 - \left( \frac{x}{x_0} \right)^{\frac{1}{2}} \right] + \ldots. \quad (4.48)$$

Consistency of this expansion requires

$$1 - \left( \frac{x}{x_0} \right)^{\frac{1}{2}} \ll x_0^{\frac{1}{2}}. \quad (4.49)$$

In particular, the approximation is good at $x = x_0$.

The approximation we have found for $\epsilon$ may now be used in the expressions for the specific heat and the magnetization. For the specific heat at $T = T_0$ we find

$$\frac{C_v}{N} \simeq \frac{15 \zeta_\nu \left( \frac{3}{2} \right)}{4 \zeta_\nu \left( \frac{3}{2} \right)} - x_0^{\frac{1}{2}} + \frac{9 \zeta_\nu \left( \frac{3}{2} \right)}{2 \pi \zeta_\nu \left( \frac{5}{2}, a \right)} x_0^{\frac{1}{2}} \left( 1 - \frac{a}{\frac{1}{2}} \right)$$

$$+ x_0 \left\{ \frac{3}{4} \left( \frac{1}{2} - a \right) + \frac{9 \zeta_\nu \left( \frac{3}{2} \right)}{\pi \zeta_\nu \left( \frac{5}{2}, a \right)} \right\}$$

$$- \frac{2 \zeta_\nu \left( \frac{3}{2}, \epsilon_0 \right) \zeta_\nu \left( \frac{5}{2}, a \right)}{2 \pi \zeta_\nu \left( \frac{5}{2}, a \right)} \left( \frac{1}{2} - a \right)$$

$$+ O \left( \frac{1}{x_0^2} \right). \quad (4.50)$$

This shows that as $B \to 0$ the specific heat approaches the free field result at $T = T_0$, confirming analytically the trend we found numerically. It also provides an analytic proof that for small $x_0$, the presence of the magnetic field lowers the value of the specific heat from the free field value.

For the magnetization we find (at $T = T_0$)

$$M \simeq -\frac{Q}{2mV} \left\{ \frac{6 \pi \zeta_\nu \left( \frac{3}{2} \right) \zeta_\nu \left( \frac{5}{2}, a \right)}{\zeta_\nu \left( \frac{3}{2} \right)} x_0^{\frac{1}{2}} \right\}$$

$$- 2 \zeta_\nu \left( \frac{3}{2} \right) \left( \frac{1}{6} - a + a^2 \right) x_0 + \ldots \quad (4.51)$$

This provides confirmation of the $M \simeq -c B^{\frac{3}{2}}$ magnetization law of Daicic and Frankel [17], but at a lower temperature (and therefore the coefficient of $B^{\frac{3}{2}}$ differs from that of [17]). In addition, we have computed the next order correction to the leading $B^{\frac{3}{2}}$ behavior.

Having presented the case $D = 3$ in some detail, we will now examine other dimensions in less detail. For $D = 4$, in place of (4.41) we have

$$x \Sigma_\nu \left[ -2, 0 \right] = \left( \frac{x}{x_0} \right)^{2} \zeta_\nu \left( 2 \right), \quad (4.52)$$

with $T_0$ now defined by

$$Q = eV \left( \frac{m}{2 \pi \beta_0} \right)^2 \zeta_\nu \left( 2 \right). \quad (4.53)$$

(Here $x_0 = \beta_0 \omega$ with $\beta_0 = T_0^{-1}$ as before, and we note $\zeta_\nu \left( 2 \right) = \frac{x_0}{\beta_0}$). Using the expansion for $\Sigma_\nu \left[ -2, 0 \right]$ of Appendix A valid for small $x$ and small $\epsilon x$ we obtain

$$0 = \left\{ 1 - \left( \frac{x}{x_0} \right)^{2} \right\} \zeta_\nu \left( 2 \right) + \left( \epsilon - \frac{1}{2} \right) x \ln \left( \epsilon x \right) - \epsilon \epsilon x + \ldots. \quad (4.54)$$

Concentrating on the temperature $T = T_0$ at which $\epsilon = \epsilon_0$ we find to lowest order, that $\epsilon_0$ must satisfy

$$0 = \left( \epsilon_0 - \frac{1}{2} \right) \ln \left( \epsilon_0 x_0 \right) - \epsilon_0. \quad (4.55)$$

Assuming $\epsilon_0$ is finite as $x_0 \to 0$, then in (4.55) we may use $\left| \ln x_0 \right| \gg \left| \ln \epsilon_0 \right|$ to find

$$\epsilon_0 \simeq 1 \left( 1 + \frac{1}{\ln x_0} \right) \quad (4.56)$$

if only the leading term is kept. This shows that $\epsilon_0 \to \frac{1}{2}$ from below as $x_0 \to 0$. Unlike the case $D = 3$, for which $\lim_{B \to 0} \frac{\epsilon}{2} = \frac{1}{2} - a$ was non-zero, this time we find $\lim_{B \to 0} \frac{\epsilon}{2} = 0$. Again by including more terms it is possible to render the approximation for $\epsilon_0$ more accurate. Because of the presence of the logarithm in (4.56), the convergence of $\epsilon_0$ towards the value of $\frac{1}{2}$ is very slow.
If we expand the magnetization for $T = T_0$ using (4.56) it is found that

$$M \simeq -\frac{Q}{2mV\pi^2} \ln\left(\frac{\eta}{x_0}\right) \quad (4.57)$$

for small $x_0$. (Only the leading term has been included here). For the specific heat we have

$$\frac{C_v}{N} \simeq 6\zeta_n(3) - \frac{4\zeta_n(2)}{\ln\left(\frac{4\pi}{\eta}\right)} + \ldots \quad (4.58)$$

if only the leading order term is kept. Again we see the $B$ as $D > 5$. For odd $D$, we must distinguish between $D = 3$ and $D = 5$. For odd $D$ we use (A16) with $\alpha = 2 - D$ and $\delta = 0$ to obtain

$$0 = \zeta_n\left(\frac{D-2}{2}\right)\left(\frac{1}{2} - \varepsilon_0\right) + \frac{1}{\ln(4\pi)}\left(\varepsilon_0^2 - \varepsilon_0 + \frac{1}{6}\right)x_0$$

$$+ \Gamma\left(\frac{4-D}{2}\right)\zeta_n\left(\frac{D-4}{2}, \varepsilon_0\right)x_0^{\frac{D-4}{2}} + \mathcal{O}\left(x_0^3\right). \quad (4.60)$$

By letting $x_0 \to 0$ (corresponding to $B \to 0$) we see that $\varepsilon_0 \to \frac{1}{2}$. In order to see how fast $\varepsilon_0 \to \frac{1}{2}$ as $x_0 \to 0$ we must distinguish between $D = 5$ and $D > 5$. For $D = 5$, the next to leading order term is of order $x_0^\frac{3}{2}$ and we find

$$\varepsilon_0 = \frac{1}{2} + \frac{1}{\sqrt{\pi}} - 1\left(x_0^\frac{3}{2}\right) + \mathcal{O}\left(x_0\right). \quad (4.61)$$

For $D > 5$ the next to leading order term is of order $x_0$ and we find

$$\varepsilon_0 = \frac{1}{2} - \frac{1}{24\zeta_n\left(\frac{D-4}{2}\right)} + \ldots \quad (4.62)$$

(The next term in (4.62) is of order $x_0^\frac{3}{2}$ if $D = 7$ and $x_0^2$ for $D > 7$). In the even dimensional case we may use (A16) with $k = \frac{D-2}{2}$ and $\delta = 0$ in (4.58). For $D = 6$ we find

$$\varepsilon_0 = \frac{1}{2} + \frac{x_0\ln x_0}{4\pi^2} + \mathcal{O}\left(x_0\right). \quad (4.63)$$

For $D > 6$ the result in (4.62) holds. Therefore for all $D \geq 5$ we find that $\varepsilon_0 \to \frac{1}{2}$ as $B \to 0$, proving that $\frac{\varepsilon_0}{n} \to 0$ as $B \to 0$. This contrasts for the result for $D = 3$.

V. DISCUSSION AND CONCLUSIONS

This paper has studied the thermodynamic properties of the ideal charged Bose gas in some detail. Spatial dimensions with $D \geq 3$ have been examined since even for the free gas it is known that the properties of the gas are sensitive to the spatial dimension. The specific heat was calculated numerically as well as analytically, for small values of the magnetic field. We also performed calculations of the magnetization and showed how the effective action method could be used to account for the condensate when $D \geq 5$.

One motivation for our study was to understand in more detail the behavior of the magnetized gas for $D = 3$. When a magnetic field is present, no matter how small, there is no phase transition; however when there is no magnetic field the system does exhibit a phase transition with a non–zero condensate. On physical grounds we would expect that there should be some sense in which the system in a small magnetic field should behave in almost the same way as the free gas. By examining the specific heat it is possible to see that as the magnetic field is reduced, the curves start to resemble the specific heat for the free gas. So long as the magnetic field remains non–zero the specific heat is always smooth, with the specific heat maximum approaching the the free Bose gas transition temperature as the magnetic field is reduced. The results of section V may be used to study this analytically. Although we did not show it, it is straightforward to show that the derivative of the specific heat becomes discontinuous as $B \to 0$, exactly as in the case for the free Bose gas. Similar remarks apply to gases in other dimensions.

We also studied the behavior in large magnetic fields. Here we found that the specific heat for a gas in $D$ spatial dimensions looked like the specific heat for the free gas in $(D - 2)$ spatial dimensions over a range of temperatures. Once the temperature gets too large this effective reduction in dimension disappears.

By using the Mellin–Barnes integral transform we were able to obtain a number of analytical approximations. Although, for $D \geq 5$, it is not possible to solve for the critical temperature exactly, it is possible to obtain good estimates when the magnetic field is weak. These approximations can then be used to study the Meissner–Ochsenfeld effect. We were also able to obtain reliable approximations for the first time valid at the free gas condensation temperature. As a by–product our calculations showed exactly how the chemical potential vanishes as $B \to 0$.

The most obvious way that the calculations given in this paper should be extended is by the inclusion of Coulomb interactions among the charged particles. Clearly this is essential before it will be possible to study Bose–Einstein condensation of charged particles in a reliable way. Given the recent experimental advances in the cooling and trapping of atomic gases, it may become
increasingly important to study this problem for trapped ions. A less pressing extension of our work is to relativistic charged particles. Daicic and Frankel [15,17] have done a study of this already in various cases; however, it would be easily possible to extend the analysis of our paper to obtain the specific heat for the first time.

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APPENDIX A: DETAILS OF THE MELLIN–BARNES TRANSFORMATIONS

1. \( \kappa = 1 \) case

The sum \( \Sigma_1 [\alpha, \delta] \) defined in (1.23) is

\[
\Sigma_1 [\alpha, \delta] = \sum_{l=1}^{\infty} \frac{l^2 \varepsilon^{-lx(\varepsilon+\delta)}}{(1 - e^{-lx})}. \tag{A1}
\]

By using the binomial expansion on the denominator we find

\[
\Sigma_1 [\alpha, \delta] = \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} l^2 \varepsilon^{-lx(n+\varepsilon+\delta)}. \tag{A2}
\]

This is now in a form where (4,1) can be used for the exponential. There are a variety of ways to use (4.1) on
suppose that we concentrate on $\delta = 0$ initially. If we are interested in the $\varepsilon \to 0$ limit, it proves advantageous to separate off the $n = 0$ term in (A2) before using (4.1), which is useful for isolating terms which diverge as $\varepsilon \to 0$, although we do not have to do this. We find

$$
\Sigma_1 [\alpha, 0] = \text{Li}_{-\alpha} [e^{-x\varepsilon}] + \int_{c-i\infty}^{c+i\infty} \frac{d\theta}{2\pi i} \Gamma(\theta) x^{-\theta} \times \zeta_{\alpha} \left( \frac{\theta}{\varepsilon}, \varepsilon + 1 \right) \tag{A3}
$$

if it is noted that the sums over $l$ and $n$ can be done in terms of the Riemann and Hurwitz $\zeta$–functions respectively. If the $n = 0$ term is not removed, then in place of (A3) we find just the integral part with $\zeta_{\alpha} (\theta, \varepsilon)$ in place of $\zeta_{\alpha} (\theta, 1 + \varepsilon)$.

At this stage it is necessary to identify a number of special cases. If $\alpha = 0$, then the poles of the Riemann and Hurwitz $\zeta$–functions both occur at $\theta = 1$ and the integrand has a double pole. If $\alpha = -2k$ where $k = 1, 2, 3, \ldots$, then the pole of the Riemann $\zeta$–function coincides with one of the poles of the $\Gamma$–function and again the integrand has a double pole. For $\alpha \neq 0, -2, -4, \ldots$ the functions occurring in (A3) all have distinct poles, so that the integrand only has simple poles.

\begin{itemize}
\item[a.] $\alpha = 0, \delta = 0$

Setting $\alpha = 0$ in (A3) results in $\text{Li}_0 [e^{-x\varepsilon}]$. This is simply

$$
\text{Li}_0 [e^{-x\varepsilon}] = \frac{1}{(e^{x\varepsilon} - 1)} \tag{A4}
$$

which follows directly from the definition (2.4). Evaluation of the integral by residues results in

$$
\Sigma_1 [0, 0] \approx \frac{1}{(e^{x\varepsilon} - 1)} - \frac{\psi(1 + \varepsilon) + \ln x}{x} - \sum_{p=0}^{\infty} \frac{(x)^{2p+1}}{\Gamma(2p+2)} \zeta_{\alpha} (-1 - 2p) \times \zeta_{\alpha} (-1 - 2p, 1 + \varepsilon). \tag{A5}
$$

We use the symbol $\approx$ here to denote that the result is an asymptotic expansion. The Riemann and Hurwitz $\zeta$–functions in (A3) may be related to the Bernoulli numbers and polynomials [15]. The divergence as $\varepsilon \to 0$ is contained in the first term. When $\varepsilon x$ is small, we may use [15]

$$
\frac{1}{(e^{x\varepsilon} - 1)} = \frac{1}{\varepsilon x} - \frac{1}{2} \sum_{p=0}^{\infty} \frac{(\varepsilon x)^{2p+1}}{\Gamma(2p+2)} \zeta_{\alpha} (-1 - 2p) \tag{A6}
$$

to obtain

$$
\Sigma_1 [0, 0] \approx \frac{1}{\varepsilon x} - \frac{1}{2} \sum_{p=0}^{\infty} \frac{(\varepsilon x)^{2p+1}}{\Gamma(2p+2)} \zeta_{\alpha} (-1 - 2p) \times \{ (2p+1) \zeta_{\alpha} (-1 - 2p, 1 + \varepsilon) \}. \tag{A7}
$$

b. $\alpha = -2k, k = 1, 2, 3, \ldots; \delta = 0$

We set $\alpha = -2k$ in (A3) and note that the integrand has simple poles at $\theta = 1$ and $\theta = -p$ for $p \neq k - 1$, and a double pole at $\theta = 1 - k$. It is straightforward to expand about these poles and use the residue theorem to obtain

$$
\Sigma_1 [-2k, 0] \approx \text{Li}_0 [e^{-x\varepsilon}] + \frac{\zeta_{\alpha} (k+1)}{x} \frac{(-x)^{k-1}}{\Gamma(k)} \zeta_{\alpha} (1-k, 1+\varepsilon) + (\gamma + \psi(k) - \ln x) \zeta_{\alpha} (1-k, 1+\varepsilon) \tag{A8}
$$

$$
+ \sum_{p=0, p \neq k-1}^{\infty} \frac{(-x)^p}{\Gamma(p+1)} \zeta_{\alpha} (k-p) \times \zeta_{\alpha} (-p, 1+\varepsilon).
$$

Here $\zeta_{\alpha} (1-k, 1+\varepsilon)$ denotes the derivative of $\zeta_{\alpha} (s, 1+\varepsilon)$ with respect to $s$ evaluated at $s = 1 - k$. For $k = 1$, $\zeta_{\alpha} (0, 1+\varepsilon)$ may be found in [15]. It is possible to generalize the procedure described in [15] to obtain $\zeta_{\alpha} (1-k, 1+\varepsilon)$ for other values of $k$, although we have not found it possible to obtain as simple a result for $k = 1$.

The values we need are given in Appendix B.

The polylogarithm $\text{Li}_k [e^{-x\varepsilon}]$ may be expanded for small $\varepsilon x$, also using (4.1). This was originally described by Robinson [12]. We find

$$
\text{Li}_k [e^{-x\varepsilon}] \approx \left( \frac{(-x)^{k-1} (\psi(k) + \gamma - \ln (x\varepsilon))}{\Gamma(k)} \right) + \sum_{l=0; l \neq k-1}^{\infty} \frac{(-x)^l}{\Gamma(l+1)} \zeta_{\alpha} (k-l). \tag{A9}
$$

Finally we note that [15]

$$
\psi(k) + \gamma = 1 + \frac{1}{2} + \ldots + \frac{1}{k-1} \tag{A10}
$$

for $k > 1$, and vanishes for $k = 1$. 

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c. \( \alpha \neq 0, -2, -4, \ldots; \delta = 0 \)

In this case all of the poles of the integrand in (A3) are simple, and we obtain

\[
\Sigma_i [\alpha, 0] \simeq \text{Li}_\frac{\alpha}{2} [e^{-\varepsilon x}] + \frac{\zeta_\alpha(1 - \frac{\alpha}{2})}{x} + \Gamma \left( 1 + \frac{\alpha}{2} \right) x^{-\left(1 + \frac{\alpha}{2}\right)} \zeta_\alpha \left( 1 + \frac{\alpha}{2}, 1 + \varepsilon \right) + \sum_{p=0}^{\infty} \frac{(-x)^p}{\Gamma(p + 1)} \zeta_\alpha \left( - \left[ p + \frac{\alpha}{2} \right] \right) \times \zeta_\alpha \left( - p, 1 + \varepsilon \right). \tag{A11} \]

The expansion of the polylogarithm for small \( \varepsilon x \) is again described in reference [12] and is

\[
\text{Li}_\frac{\alpha}{2} [e^{-\varepsilon x}] \simeq \Gamma(1 + \frac{\alpha}{2}) (\varepsilon x)^{-1 - \frac{\alpha}{2}} + \sum_{l=0}^{\infty} \frac{(-\varepsilon x)^l}{\Gamma(l + 1)} \zeta_\alpha (l + 1). \tag{A12} \]

We also require the expansion of \( \Sigma_i [\alpha, \delta] \) for \( \delta > 0 \). The difference between \( \delta = 0 \) and \( \delta \neq 0 \) can be seen from (A12): when \( \delta = 0 \) the \( n = 0 \) term can lead to a divergent sum over \( l \) when \( \varepsilon \to 0 \). For \( \delta > 0 \) the sum over \( l \) is always convergent, even when \( \varepsilon = 0 \). Thus for \( \delta > 0 \) we do not have to separate off the \( n = 0 \) term. Using (A11) for (A2) we obtain

\[
\Sigma_i [\alpha, \delta] = \int_{e^{-i\infty}}^{e^{+i\infty}} \frac{d\theta}{2\pi i} \theta x^{-\theta} \zeta_\alpha \left( \theta - \frac{\alpha}{2} \right) \zeta_\alpha \left( \theta, \varepsilon + \delta \right). \tag{A13} \]

The same cases arise as for \( \delta = 0 \), and the analysis is similar. We will simply list the results.

d. \( \alpha = 0, \delta > 0 \)

\[
\Sigma_i [0, \delta] \simeq \frac{1}{x} (\psi(\varepsilon + \delta) + \ln x) - \sum_{p=0}^{\infty} \frac{x^{2p+1}}{\Gamma(2p+2)} \zeta_\alpha (-1 - 2p) \times \zeta_\alpha (-1 - 2p, \varepsilon + \delta). \tag{A14} \]

e. \( \alpha = -2k, k = 1, 2, 3, \ldots; \delta > 0 \)

\[
\Sigma_i [-2k, \delta] \simeq \frac{\zeta_\alpha (k + 1)}{x} \frac{(-x)^{k-1}}{\Gamma(k)} \zeta_\alpha' (1 - k, \varepsilon + \delta) + (\gamma + \psi(k) - \ln x) \zeta_\alpha (1 - k, \varepsilon + \delta) + \sum_{p=0}^{\infty} \frac{(-x)^p}{\Gamma(p+1)} \zeta_\alpha (k-p) \times \zeta_\alpha (- p, \varepsilon + \delta). \tag{A15} \]

f. \( \alpha \neq 0, -2, -4, \ldots; \delta > 0 \)

\[
\Sigma_i [\alpha, \delta] \simeq \frac{\zeta_\alpha (1 - \frac{\alpha}{2})}{x} + \Gamma \left( 1 + \frac{\alpha}{2} \right) x^{-\left(1 + \frac{\alpha}{2}\right)} \zeta_\alpha \left( 1 + \frac{\alpha}{2}, \varepsilon + \delta \right) + \sum_{p=0}^{\infty} \frac{(-x)^p}{\Gamma(p + 1)} \zeta_\alpha \left( - \left[ p + \frac{\alpha}{2} \right] \right) \times \zeta_\alpha \left( - p, \varepsilon + \delta \right). \tag{A16} \]

One final comment we wish to make is that (A14)–(A16) hold even for \( \delta = 0 \), although as we explained, in this case it may be more difficult to isolate the terms which diverge as \( \varepsilon \to 0 \). Nevertheless, these results can be used if desired.

2. \( \kappa = 2 \) case

The sum \( \Sigma_2 [\alpha, \delta] \) defined in (A23) is

\[
\Sigma_2 [\alpha, \delta] = \sum_{l=1}^{\infty} \frac{\theta^2 e^{-l\varepsilon(\delta + \theta)}}{1 - e^{-l\varepsilon}}. \tag{A17} \]

The analysis of this sum proceeds in the same way as for \( \Sigma_i [\alpha, \delta] \). Binomial expansion of the denominator of (A15) gives

\[
\Sigma_2 [\alpha, \delta] = \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \frac{\theta^2 (n+1) e^{-l\varepsilon(n+\varepsilon+\delta)}}{(n+1)!}. \tag{A18} \]

We only require the result for \( \delta = 1 \) in the main part of the paper, so that the sum over \( l \) is convergent even for \( \varepsilon = 0 \). We do not separate off any terms. Application of (A17) to (A18) results in

\[
\Sigma_2 [\alpha, \delta] = \int_{e^{-i\infty}}^{e^{+i\infty}} \frac{d\theta}{2\pi i} \Gamma(\theta) x^{-\theta} \zeta_\alpha \left( \theta - \frac{\alpha}{2} \right) \times \left\{ \zeta_\alpha \left( \theta - 1, \varepsilon + \delta \right) + (1 - \varepsilon - \delta) \zeta_\alpha \left( \theta, \varepsilon + \delta \right) \right\}. \tag{A19} \]

(The sum over \( l \) is a Riemann \( \zeta \)-function, and the sum over \( n \) can be done in terms of the Hurwitz \( \zeta \)-function as shown).
a. $\alpha = 0$

The integrand of $[A19]$ has a simple pole at $\theta = 2$, a double pole at $\theta = 1$, and simple poles at $\theta = 0, -1, -2, \ldots$. We find

$$\Sigma_2 [0, \delta] \simeq \frac{\zeta_n(2)}{x^2} - \frac{1}{4}(\varepsilon + \delta)^2 + \frac{1}{2}(\varepsilon + \delta) - \frac{5}{24}$$

$$- \frac{1}{x} \left[ \frac{1}{2} - \varepsilon - \delta - (1 - \varepsilon - \delta) \psi(\varepsilon + \delta) \right]$$

$$- (1 - \varepsilon - \delta) \ln x$$

$$+ \sum_{p=0}^{\infty} \frac{(x)^{2p+1}}{\Gamma(2p+2)} \zeta_n (-1 - 2p)$$

$$\times \{ \zeta_n (-2 - 2p, \varepsilon + \delta)$$

$$+ (1 - \varepsilon - \delta) \zeta_n (-1 - 2p, \varepsilon + \delta) \}. \quad (A20)$$

b. $\alpha = 2$

The integrand of $[A19]$ has a double pole at $\theta = 2$, and single poles at $\theta = 1, 0, -1, \ldots$. The residue theorem gives,

$$\Sigma_2 [2, \delta] \simeq - \frac{(1 - \varepsilon - \delta)}{2x}$$

$$+ \frac{1}{x^2} \left[ 1 - \ln x - \psi(\varepsilon + \delta) \right]$$

$$+ (1 - \varepsilon - \delta) \zeta_n (2, \varepsilon + \delta)$$

$$+ \sum_{p=0}^{\infty} \frac{(x)^{2p}}{\Gamma(2p+1)} \zeta_n (-1 - 2p)$$

$$\times \{ \zeta_n (-1 - 2p, \varepsilon + \delta)$$

$$+ (1 - \varepsilon - \delta) \zeta_n (-2, \varepsilon + \delta) \}. \quad (A21)$$

c. $\alpha = -2k, k = 1, 2, 3, \ldots$

The integrand of $[A19]$ has simple poles at $\theta = 2, 1$, a double pole at $\theta = 1-k$, and simple poles at $\theta = -p, p = 0, 1, 2, 3, \ldots$ with $p \neq k-1$. We find

$$\Sigma_2 [-2k, \delta] = \frac{\zeta_n(2+k)}{x^2} + \frac{(1 - \varepsilon - \delta) \zeta_n (1+k)}{x}$$

$$+ \frac{(-x)^{k-1}}{\Gamma(k)} (\gamma + \psi(k) - \ln x)$$

$$\times \{ \zeta_n (-k, \varepsilon + \delta)$$

$$+ (1 - \varepsilon - \delta) \zeta_n (-1 - k, \varepsilon + \delta) \}$$

$$+ \frac{(-x)^{k-1}}{\Gamma(k)} \{ \zeta_n' (-k, \varepsilon + \delta)$$

$$+ (1 - \varepsilon - \delta) \zeta_n' (1 - k, \varepsilon + \delta) \}$$

$$+ \sum_{p=0}^{\infty} \frac{(-x)^p}{\Gamma(p+1)} \zeta_n(k-p)$$

$$\times \{ \zeta_n (-p-1, \varepsilon + \delta)$$

$$+ (1 - \varepsilon - \delta) \zeta_n (-p, \varepsilon + \delta) \} \} \quad (A22)$$

d. $\alpha \neq 2, 0, -2, -4, \ldots$

The integrand only has simple poles in this case, and we find that

$$\Sigma_2 [\alpha, \delta] = \Gamma \left( 1 + \frac{\alpha}{2} \right) x^{-(1+\frac{\alpha}{2})} \left\{ \zeta_n \left( \frac{\alpha}{2}, \varepsilon + \delta \right) + (1 - \varepsilon - \delta) \zeta_n \left( 1 + \frac{\alpha}{2}, \varepsilon + \delta \right) \right\}$$

$$+ \frac{\zeta_n (2 - \frac{\alpha}{2})}{x^2}$$

$$+ \sum_{p=0}^{\infty} \frac{(-x)^p}{\Gamma(p+1)} \zeta_n (-p - \frac{\alpha}{2})$$

$$\times \{ \zeta_n (-p-1, \varepsilon + \delta)$$

$$+ (1 - \varepsilon - \delta) \zeta_n (-p, \varepsilon + \delta) \} \} \quad (A23)$$

e. $\kappa = 3$ case

From (3.23) we have

$$\Sigma_3 [\alpha, \delta] = \sum_{l=1}^{\infty} \frac{1}{(1 - e^{\lambda 2\pi i})^3}. \quad (A24)$$

Again we are only interested in $\delta > 0$. Following the now familiar steps of binomially expanding the denominator, using $[1.1]$ for the exponential, and performing the sums in terms of $\zeta$–functions results in

$$\Sigma_3 [\alpha, \delta] \simeq \int_{e^{-\infty i}}^{e^{+\infty i}} \frac{d\theta}{2\pi i} \Gamma(\theta) x^{\theta - \frac{\alpha}{2}}$$

$$\times \{ \zeta_n (\theta - 2, \varepsilon + \delta)$$

$$+ (3 - 2\varepsilon - 2\delta) \zeta_n (\theta - 1, \varepsilon + \delta)$$

$$+ (1 - \varepsilon - \delta) (2 - \varepsilon - \delta) \zeta_n (\theta, \varepsilon + \delta) \} \}. \quad (A25)$$

f. $\alpha = 0$

The integrand in $[A25]$ has simple poles at $\theta = 3, 2, 0, -1, -2, \ldots$ and a double pole at $\theta = 1$. We find

$$\Sigma_3 [0, \delta] = \frac{\zeta_n(3)}{x^3} + \frac{3 - 2\varepsilon - 2\delta}{2x^2}$$

$$+ \frac{1}{2x} \left( \frac{3}{2}(\varepsilon + \delta)^2 - \frac{7}{2}(\varepsilon + \delta) + \frac{17}{12} \right)$$

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\(\frac{1}{4} \{ \zeta_H (-2, \varepsilon + \delta) + (3 - 2 \varepsilon - 2 \delta) \zeta_H (-1, \varepsilon + \delta) + (1 - \varepsilon - \delta)(2 - \varepsilon - \delta) \left( \frac{1}{2} - \varepsilon - \delta \right) \} \)

\[ \approx \frac{(3 - 2 \varepsilon - 2 \delta) \zeta_H (2 + k)}{2x^2} + \frac{(1 - \varepsilon - \delta)(2 - \varepsilon - \delta) \zeta_H (1 + k)}{2x} + \frac{(-x)^{k-1}}{2 \Gamma (k)} \{ \gamma + \psi (k) \}
\]

\[ [ \zeta_H (-1 - k, \varepsilon + \delta) + (3 - 2 \varepsilon - 2 \delta) \zeta_H (-k, \varepsilon + \delta) + (1 - \varepsilon - \delta)(2 - \varepsilon - \delta) \zeta_H (1 - k, \varepsilon + \delta)] + \zeta_H' (-1 - k, \varepsilon + \delta) + (3 - 2 \varepsilon - 2 \delta) \zeta_H' (-k, \varepsilon + \delta) - \ln x + (1 - \varepsilon - \delta)(2 - \varepsilon - \delta) \zeta_H' (1 - k, \varepsilon + \delta) \]

\[ + \sum_{p=0}^{\infty} \frac{(-x)^{k-p}}{2 \Gamma (p + 1)} \zeta_H (k - p)
\]

\[ \times \{ \zeta_H (-p - 2, \varepsilon + \delta) + (3 - 2 \varepsilon - 2 \delta) \zeta_H (-p - 1, \varepsilon + \delta) + (1 - \varepsilon - \delta)(2 - \varepsilon - \delta) \zeta_H (-p, \varepsilon + \delta) \} \]. (A26)

\( \text{h. } \alpha = -2k, k = 1, 2, 3, \ldots \)

The integrand of (A25) has simple poles at \( \theta = 3, 1, 0, -1, -2, \ldots \) and a double pole at \( \theta = 2 \) for \( \alpha = 2 \). When \( \alpha = 4 \) we have a double pole at \( \theta = 3 \), and simple poles at \( \theta = 2, 1, 0, \ldots \). Because we do not require these cases in our paper we will not give the results here.

\( \text{i. } \alpha \neq 4 - 2k, k = 0, 1, 2, \ldots \)

In this case all the poles of (A25) are simple and we obtain

\[ \Sigma_s [\alpha, \delta] \approx \Gamma \left( 1 + \frac{\alpha}{2} \right) \frac{2^{(-1 + \frac{\alpha}{2})}}{2} \{ \zeta_H \left( \frac{\alpha}{2} - 1, \varepsilon + \delta \right) + (3 - 2 \varepsilon - 2 \delta) \zeta_H \left( \frac{\alpha}{2}, \varepsilon + \delta \right) + (1 - \varepsilon - \delta)(2 - \varepsilon - \delta) \zeta_H \left( 1 + \frac{\alpha}{2}, \varepsilon + \delta \right) \}
\]

\[ + \frac{\zeta_H (3 - \frac{\alpha}{2})}{x^3} + \frac{(3 - 2 \varepsilon - 2 \delta) \zeta_H \left( 2 - \frac{\alpha}{2} \right)}{2x^2} + \frac{(1 - \varepsilon - \delta)(2 - \varepsilon - \delta) \zeta_H \left( -p, \varepsilon + \delta \right)}{2x} \]

\[ + \sum_{p=0}^{\infty} \frac{(-x)^{p}}{2 \Gamma (p + 1)} \zeta_H \left( -p + \frac{\alpha}{2} \right)
\]

\[ \times \{ \zeta_H (-p - 2, \varepsilon + \delta) + (3 - 2 \varepsilon - 2 \delta) \zeta_H (-p - 1, \varepsilon + \delta) + (1 - \varepsilon - \delta)(2 - \varepsilon - \delta) \zeta_H (-p, \varepsilon + \delta) \}. \) (A28)

\section*{APPENDIX B: DERIVATIVES OF THE HURWITZ \( \zeta \)-FUNCTION}

The derivative of \( \zeta_H (s, a) \) with respect to \( s \) at \( s = 0 \) is given by (13)

\[ \zeta_H' (0, a) = \ln \left( \frac{\Gamma (a)}{\sqrt{2 \pi}} \right). \] (B1)

The derivative makes use of the Plana summation formula to derive an integral representation for \( \zeta_H (s, a) \) (the Hermite representation) which is then differentiated with respect to \( s \). In a similar manner the derivatives at other values of \( s \) may be calculated, although the results are less simple than (B1). For \( s = -1 \) we find

\[ \zeta_H' (-1, a) = \frac{(6a^2 (\gamma + 1) + 6a + (1 - \gamma - \ln 2 \pi))^{12}}{12} + \ln \left( \left( \frac{\Gamma (a)}{\sqrt{2 \pi}} \right)^{\alpha} \right) + \frac{\zeta_H' (2)}{2 \pi^2} - \sum_{t=1}^{\infty} \frac{(-1)^{t+1} \zeta_H (t + 1) a^{t+2}}{(t + 2)}. \] (B2)

Here \( \zeta_H' (2) \) is the derivative of the Riemann \( \zeta \)-function given by

\[ \zeta_H' (2) = - \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \] (B3)

we find...
\[ \zeta_H'(-2, a) = \frac{(a^3(8\gamma + 6) + 9a^2 + a(3 - 2\gamma - 2\ln 2\pi))}{12} \]
\[ + \ln \left( \left\{ \frac{\Gamma(a)}{\sqrt{2\pi}} \right\}^{a^2} \right) + \left( \frac{4a\zeta_H'(2) - \zeta_H(3)}{4\pi^2} \right) \]
\[ - \sum_{t=1}^{\infty} \frac{(-1)^{t+1}a^{(t+3)}(t + 4)\zeta_H(t + 1)}{(t + 2)(t + 3)}. \] (B4)

A useful check on the results is provided by the general relation
\[ \frac{\partial}{\partial a} \zeta_H'(s, a) = -\zeta_H(s + 1, a) - s\zeta_H'(s + 1, a). \] (B5)