Comparative analysis of dynamic state recognition between permutation entropy and weighted permutation entropy

Wenxiang Luo\textsuperscript{1,2}, Li Wan\textsuperscript{2,*} and Hui Liu\textsuperscript{2}

\textsuperscript{1}Software Engineering Institute of Guangzhou, Guangzhou City, Guangdong Province, 510990, China
\textsuperscript{2}Guangzhou University, Guangzhou City, Guangdong Province, 510006, China
*Corresponding author’s e-mail: wanli@gzhu.edu.cn

Abstract. Permutation entropy (PE) and weighted permutation entropy (WPE) are indexes of the complex system. In this paper, the nonlinear sequence of two chaotic states is simulated by Logistic mapping, and the effectiveness of the PE and WPE algorithms in chaotic state recognition is compared and analyzed with different sliding step sizes. The results show that the value of WPE changes more obviously than PE in different chaotic states, and the recognition effect of WPE is better than PE in dynamic states, which can provide theoretical support for further analyzing the complexity characteristics of actual data.

1. Introduction
Entropy is a measure of system complexity, which can be used for time series dynamic state recognition. After the establishment of information theory, the theory and application of entropy have been developed rapidly. In recent years, new entropies, such as approximate entropy [1], sample entropy [2], and permutation entropy [3], have been proposed based on information entropy. Weighted permutation entropy [4] is an improved method proposed by Fadlallah in 2013 for the calculation of permutation entropy without considering the amplitude information in time series, which has good anti-noise and anti-interference ability. The WPE algorithm is simple and operable, and can effectively magnify the small changes of time series. It is widely used in signal mutation detection and random signal analysis, which has achieved good application effects in mechanical fault diagnosis [5], medicine [6], and other fields [7-10].

This paper uses Logistic mapping to simulate chaotic sequences to test the effectiveness of the permutation entropy and weighted permutation entropy algorithms in identifying different dynamics, providing theoretical support for further complexity analysis of actual data.

2. Methods
2.1 permutation entropy
Given a time series $X = \{x(n), n = 1, 2, \cdots, N\}$, reconstructed the phase space of $X$ and obtained the matrix $X_k$ is in the following:
(1) (1 ) (1 ( 1) )

\[
\left[ \begin{array}{ccc}
  x(1) & x(1 + \tau) & \cdots & x(1 + (m - 1)\tau) \\
  \vdots & \vdots & \ddots & \vdots \\
  x(i) & x(i + \tau) & \cdots & x(i + (m - 1)\tau) \\
  \vdots & \vdots & \ddots & \vdots \\
  x(K) & x(K + \tau) & \cdots & x(K + (m - 1)\tau)
\end{array} \right], \quad i = 1, 2, \ldots, K,
\]

Where \( K = N - (m - 1)\tau \), \( m \) is the embedding dimension, and \( \tau \) is the delay time. Rearranges \( X_i = (x(i), x(i + \tau), \ldots, x(i + (m - 1)\tau)) \) in ascending order can get

\[
x(i + (j_1(i) - 1)\tau) \leq x(i + (j_2(i) - 1)\tau) \leq \cdots \leq x(i + (j_m(i) - 1)\tau),
\]

where \( i = 1, 2, \ldots, K \), \( \pi_i = (j_1(i), j_2(i), \ldots, j_m(i)) \) is some permutation of \( (1, 2, \ldots, m) \) which represents the index of the column of the elements in \( X_i \). If the elements are equal, they are sorted by the size of the element index. Obviously, it has at most \( m! \) different arrangement of species in \( m \) elements. For instance, given a time series \( \{7, 6, 5, 8, 6\} \), when \( m = 3 \) and \( \tau = 1 \), the first reconstructed component is \( (7, 6, 5) \).

The index of \( (7, 6, 5) \) is \( (2, 1, 0) \) since \( x_{t+2} < x_{t+1} < x_t \), Similarly, the index of the second reconstructed component \( (6, 5, 8) \) is \( (2, 1, 0) \), and the index of the third reconstructed component \( (5, 8, 6) \) is \( (0, 2, 1) \) [11].

Counting all the permutations of \( \pi_i \), the probability of each permutation is

\[
p(\pi_i) = \frac{f(\pi_i)}{K}, \quad K = N - (m - 1)\tau,
\]

where \( 0 < r \leq m! \), \( 0 < f(\pi_r) \leq N - (m - 1)\tau \). The permutation entropy (PE) is defined as:

\[
PE(m) = -\sum_{r=1}^{m!} p(\pi_r) \log(p(\pi_r)),
\]

where \( \log \) is with base 2. \( PE(m) \) gets the maximum value \( \log(m!) \) when \( p(\pi_r) = 1/(m!) \). In practical applications, \( \log(m!) \) is used to normalize \( PE(m) \), that is:

\[
0 \leq PE(m) = PE(m)/\log(m!) \leq 1.
\]

The value of \( PE(m) \) is consistent with the randomness of the time series. The larger the value of \( PE(m) \) is, the higher the randomness of the time series is.

### 2.2 weighted permutation entropy

The permutation entropy algorithm does not consider the specific value of the time series but shows its complexity by examining the permutation pattern of the time series. This algorithm is simple and easy to understand. In the calculation of PE, because the amplitude information in the time series is not considered, the amplitude difference between the sequences with the same permutation is ignored, which will inevitably lead to the reconstructed components with different amplitude to obtain the same \( PE(m) \) value. To solve this problem, Fadlallah (2013) proposed an improved algorithm of permutation entropy --weighted permutation entropy (WPE). WPE considers the amplitude information of time series when calculating the probability of a permutation. In order to clarify the difference with PE, formula (3) in the PE algorithm is rewritten as follows:

\[
p(\pi_r) = \frac{\sum_{i=1}^{K} 1_{\text{type}(u) = \pi_r}(X_i)}{\sum_{i=1}^{K} 1_{\text{type}(u) \in \Pi}(X_i)},
\]

where \( 1_u(v) = \begin{cases} 1, & v \in u \\ 0, & v \notin u \end{cases} \), \( \Pi = \{\pi_i\}_{i=1}^{m!} \).

The probability of each permutation pattern in the WPE algorithm is defined as:
\[ p(w_{r}) = \frac{\sum_{i=1}^{K} l_{\text{type}(u)-\sigma_{i}}(X_{r})w_{r}}{\sum_{i=1}^{K} l_{\text{type}(u)-\sigma_{i}}(X_{r})w_{r}}, \tag{7} \]

where \( w_{r} \) is the weight of the reconstructed component \( X_{r} \), which is represented by the variance of \( X_{i} \):

\[ w_{r} = \frac{1}{m} \sum_{q=1}^{m} \left[ x(i + (q - 1)\tau) - \bar{X}_{i} \right]^2, \tag{8} \]

\( \bar{X}_{i} \) is the mean of the reconstructed component \( X_{i} \), which is defined by

\[ \bar{X}_{i} = \frac{1}{m} \sum_{q=1}^{m} x(i + (q - 1)\tau). \tag{9} \]

WPE is an improved algorithm of PE, which is defined as:

\[ \text{WPE}(m) = -\sum_{r=1}^{m} p(w_{r}) \log(p(w_{r})), \tag{10} \]

where \( \log \) is with base 2. It can be obtained the normalized form by

\[ 0 \leq \text{WPE}(m) = \text{WPE}(m)/\log(m!) \leq 1. \tag{11} \]

Weighted permutation entropy considers the amplitude information of reconstructed components when calculating the probability of a certain permutation, which can fully reflect the detailed characteristics of time series, and has a good effect on the applications that need to describe the detailed characteristics of time series.

### 3. Dynamic state recognition comparison

A chaotic ideal time series (IS) with length 1000 is simulated by Logistic mapping, and its iterative equation is as follows:

\[ x_{n+1} = \lambda x_{n} (1 - x_{n}), \quad x_{n} \in [0, 1]. \tag{12} \]

\( \lambda \) represents the growth rate of the Pest Model where \( \lambda \in [0, 4] \). For \( x_{0} = 0.7 \), IS consists of two parts, and the parameters of the 500 data before and after are \( \lambda = 3.6 \) and \( \lambda = 3.7 \) respectively, both of which are in a chaotic state. The evolution curve of IS is shown in figure 1.

![Figure 1. Chaotic ideal time series IS](image)

According to the structure of IS, the evolution of the system changes from a low growth rate of 3.6 to a high growth rate of 3.7 at \( n = 501 \), and the complexity of the first 500 points of IS sequence is lower than that of the last 500 points.

A sliding window with length \( M = 150 \) is used to select the data of IS to form a subsequence. The PE and WPE algorithms are used to calculate the entropy value of the subsequence for \( m = 3 \), \( \tau = 2 \) respectively. Selecting a new subsequence with the step size of \( S = 10 \) and \( S = 20 \), respectively, and calculating the corresponding entropy value of the new subsequence until the end of IS, which can be obtained a sequence of entropy values that changes gradually with the step size, as shown in figure 2.
As shown in figure 2, the entropy value calculated by the two algorithms presents two different stable states before and after $n = 500$, indicating that IS belongs to different chaotic states. According to the process of sliding data, the edge of the two kinds of chaotic data should be at the beginning of the entropy rising interval, about at $n = 501$, which is consistent with the structure of IS. The entropy increase interval obtained by the WPE algorithm is steeper than that obtained by the PE algorithm with better differentiation. On the other hand, according to the analysis of the sliding data process, when the two kinds of chaotic data occupy half of the sliding window size respectively, the entropy value reaches the maximum and then decreases to a similar level with the latter kind of data. The entropy obtained by WPE decreases after reaching the maximum value, which is consistent with the process of sliding data. However, the entropy obtained by PE does not reach the maximum value in the interval (500, 650), and then decreases to a certain extent. Therefore, WPE performs better than PE in dynamic recognition of chaotic time series.

4. Conclusion
The nonlinear sequences of different chaotic states are simulated by Logistic mapping, and the effectiveness of PE and WPE in chaotic state recognition is compared and analyzed with different sliding step sizes. The results show that the change of entropy value of WPE is more evident than that of PE in different chaotic states, indicating that WPE can effectively magnify the slight change of time series. Furthermore, WPE is better than PE in dynamic state identification, which can provide theoretical support for further analyzing the complexity characteristics of actual data.

Acknowledgments
This work was financially supported by the National Natural Science Foundation of China (41872246); Scientific Research Project of the Software Engineering Institute of Guangzhou (ky201909).
References

[1] Pincus, S. M. (1991) Approximate entropy as a measure of system complexity. Proceedings of the National Academy of Sciences, 88: 2297-2301.

[2] Richman, J. S., Moorman, J. R. (2000) Physiological time-series analysis using approximate entropy and sample entropy. American Journal of Physiology-Heart and Circulatory Physiology, 278: 2039-2049.

[3] Bandt, C., Pompe, B. (2002) Permutation entropy: A natural complexity measure for time series. Physical Review Letters, 88: 174102.

[4] Fadlallah, B., Chen, B. D., Keil, A., et al. (2013) Weighted-permutation entropy: A complexity measure for time series incorporating amplitude information. Physical Review E, 87: 022911.

[5] Ding, J. X., Wang, Z. Y., Yao, L. G., et al. (2021) Rolling bearing fault diagnosis based on GCMWPE and parameter optimization SVM. China Mechanical Engineering, 32: 147-155.

[6] Lee, C. H., Chen, S. H., Jiang, B. C., et al. (2020) Estimating postural stability using improved permutation entropy via TUG accelerometer data for community-dwelling elderly people. Entropy, 20, 1097.

[7] Chen S. J., Shang P. J., Wu Y. (2018) Multivariate multiscale fractional order weighted permutation entropy of nonlinear time series. Physica A: Statistical Mechanics and its Applications, 515: 217-231.

[8] Wu P., Guo L. L., Duan Y. Y., et al. (2019) Control loop performance monitoring based on weighted permutation entropy and control charts. The Canadian Journal of Chemical Engineering, 97: 1488-1495.

[9] He C., Wu T., Liu C. C., et al. (2020) A novel method of composite multiscale weighted permutation entropy and machine learning for fault complex system fault diagnosis. Measurement, 158, 107748.

[10] Li, Y. X., Geng, B., Jiao, S. B. (2021) Refined composite multi-scale reverse weighted permutation entropy and its applications in ship-radiated noise. Entropy, 23, 476.

[11] Luo, W. X., Wan, L., Lai, S. M. (2018) Comparison of sequence mutational detection of two kinds of moving permutation entropy. Journal of Xiamen University of Technology, 26: 91-96.