Axisymmetric stability criterion for two gravitationally coupled singular isothermal discs

Yue Shen and Yu-Qing Lou

1 Physics Department, The Tsinghua Center for Astrophysics, Tsinghua University, Beijing 100084, China
2 Department of Astronomy and Astrophysics, The University of Chicago, 5640 S. Ellis Ave., Chicago, IL 60637 USA
3 National Astronomical Observatories, Chinese Academy of Sciences, A20, Datun Road, Beijing, 100012 China.

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ABSTRACT
Using the two-fluid formalism with the polytropic approximation, we examine the axisymmetric stability criterion for a composite system of gravitationally coupled stellar and gaseous singular isothermal discs (SIDs). Both SIDs are taken to be infinitely thin and are in a self-consistent steady background rotational equilibrium with power-law surface mass densities ($\propto r^{-1}$) and flat rotation curves. Recently, Lou & Shen (2003) derived exact solutions for both axisymmetric and nonaxisymmetric stationary perturbations in such a composite SID system and proposed the $D_s$-criterion of stability for axisymmetric perturbations. Here for axisymmetric perturbations, we derive and analyze the time-dependent WKBJ dispersion relation to study stability properties. By introducing a dimensionless stellar SID rotation parameter $D_s$, defined as the ratio of the constant stellar rotation speed $V_s$ to the constant stellar velocity dispersion $a_s$, one can readily determine the axisymmetric stability $D_s$-criterion numerically by identifying a stable range of $D_s$. Those systems which rotate too slow (collapse) or too fast (ring fragmentation) are unstable. We found that the stable range of $D_s^2$ depends on the mass ratio $\delta$ of the gaseous SID to the stellar SID and on the square of the ratio $\beta$ of the stellar velocity dispersion (which mimics the sound speed) to the gaseous isothermal sound speed. Increment of either $\delta$ or $\beta$ or both will diminish the stable range of $D_s^2$. The WKBJ results of instabilities provide physical explanations for the stationary configurations derived by Lou & Shen. It is feasible to introduce an effective $Q$ parameter for a composite SID system. The closely relevant theoretical studies of Elmegreen (1995), Jog (1996) and Shu et al. (2000) are discussed. A study of composite partial SID system reveals that an axisymmetric dark matter halo will promote stability of composite SID system against axisymmetric disturbances. Potential applications to disc galaxies, circumnucleus discs around nuclei of galaxies, and protostellar discs are briefly discussed.

Key words: stars: formation—ISM: general—galaxies: kinematics and dynamics—galaxies: spiral—galaxies: structure.

1 INTRODUCTION
In the context of galactic disc dynamics, there have been numerous investigations on criteria of axisymmetric and nonaxisymmetric disc instabilities (e.g., Binney & Tremaine 1987). For a single disc of either gaseous or stellar content, Safronov (1960) and Toomre (1964) originally derived a dimensionless $Q$ parameter to determine the local stability (i.e., $Q > 1$) against axisymmetric ring-like disturbances. Besides a massive dark matter halo, a more realistic disc galaxy involves both gas and stars. It is thus sensible to consider a composite system of one gas disc and one stellar disc under the gravitational control of the dark matter halo. Theoretical studies on this type of two-component disc systems were extensive in the past (Lin & Shu 1966, 1968; Kato 1972; Jog & Solomon 1984a, b; Bertin & Romeo 1988; Romeo 1992; Elmegreen 1995; Jog 1996; Lou & Fan 1998b, 2000a, b). In particular, it has been attempted to introduce proper definitions of an effective $Q$ parameter relevant to a composite disc system for the criterion of local axisymmetric instabilities (Elmegreen 1995; Jog 1996; Lou & Fan 1998b). The results of such a stability analysis may provide a basis for understanding composite disc dynamics (e.g., Lou & Fan 2000a, b; Lou & Shen 2003) and for estimating global star
formation rate in a disc galaxy (e.g., Kennicutt 1989; Silk 1997).

There are several reasons that lead us to once again look into this axisymmetric stability problem. A few years ago, Shu, Laughlin, Lizano & Galli (2000; see also Galli et al. 2001) studied stationary coplanar perturbation structures in an isopedically magnetized singular isothermal disc (SID) without using the usual WKBJ approximation. They found exact analytical solutions for stationary (i.e., zero pattern speed) configurations of axisymmetric and non-axisymmetric logarithmic spiral perturbations. According to their analysis, for axisymmetric perturbations with radial propagation, a SID with sufficiently slow rotation speed will collapse for sufficiently large radial perturbation scales, while a SID with sufficiently fast rotation speed will be unstable to ring fragmentations for relatively small radial perturbation scales (see Fig. 2 of Shu et al. 2000). For axisymmetric perturbations, the stable regime of SID rotation speed may be characterized by a dimensionless rotation parameter \( D \), which is the ratio of the constant SID rotation speed \( V \) to the isothermal sound speed \( a \). The critical values of the lowest and the highest \( D \) for axisymmetric stability can be derived from their marginal stability curves. Shu et al. (2000) supported their interpretations by invoking the local dispersion relation in the WKBJ approximation (Lin & Shu 2003). In a proper parameter regime, we demonstrate the correctness of interpretations by Shu et al. (2000), and in general, we clearly show the validity of using \( D \) parameter to demark stable and unstable regimes. To place our analysis in relevant contexts, we also discuss how the two effective \( Q \) parameters of Elmegreen (1995) and of Jog (1996) are related to our \( D \) parameter when other pertinent parameters are specified.

In order to make convincing arguments for the above analogy and our physical interpretations, we perform here a time-dependent local stability analysis using the WKBJ approximation for the composite SID system (Lou & Shen 2003). In a proper parameter regime, we demonstrate the correctness of interpretations by Shu et al. (2000), and in general, we clearly show the validity of using \( D \) parameter to demark stable and unstable regimes. To place our analysis in relevant contexts, we also discuss how the two effective \( Q \) parameters of Elmegreen (1995) and of Jog (1996) are related to our \( D \) parameter when other pertinent parameters are specified.

In Section 2, we derive the time-dependent WKBJ dispersion relation for the composite SID system and define several useful dimensionless parameters. In Section 3, we present the criterion for the SID system being stable against all axisymmetric disturbances. Finally, our results are discussed and summarized in Section 4.

## 2 TWO-FLUID COMPOSITE SID SYSTEM

Here, we go through the basic fluid equations and derive the local dispersion relation in the WKBJ approximation (Lin & Shu 1964). For physical variables, we use either superscript or subscript \( s \) for the stellar SID and \( g \) for the gaseous SID. As both SIDs are assumed to have flat rotation curves with different speeds \( V_s \) and \( V_g \), we express the mean angular rotation speeds of the two SIDs in the forms of

\[
\Omega_s = V_s/r = a_s D_s/r, \quad \Omega_g = V_g/r = a_g D_g/r, \tag{1}
\]

where \( a_s \) and \( a_g \) are the velocity dispersion of the stellar SID and the isothermal sound speed in the gaseous SID, respectively. Dimensionless rotation parameters \( D_s \) and \( D_g \) are defined as the ratios of stellar SID rotation speed to velocity dispersion and gaseous SID rotation speed to sound speed.

* Subscript \( s \) indicates the relevance to the stellar SID.
speed, respectively. The two epicyclic frequencies are
\[ \kappa_s \equiv (2\Omega_s/r)[d(r^2\Omega_s)/dr]^{1/2} = \sqrt{2\Omega_s}, \]
\[ \kappa_g \equiv (2\Omega_g/r)[d(r^2\Omega_g)/dr]^{1/2} = \sqrt{2\Omega_g}. \]
Note that \( \kappa_s \) and \( \kappa_g \) are different in general.

By the polytropic assumption for the surface mass density and the two-dimensional pressure (e.g., Binney & Tremaine 1987), the fluid equations for the stellar disc, in the cylindrical coordinates \((r, \phi, z)\), \(z = 0\) plane are
\[ \frac{\partial u^s_r}{\partial t} + \frac{\partial u^s_\phi}{\partial r}(r\Sigma^s_u u^s_r) + \frac{\partial u^s_\phi}{\partial \phi}(\Sigma^s u^s_\phi) = 0, \tag{3} \]
\[ \frac{\partial u^s_\phi}{\partial t} + u^s_\phi \frac{\partial u^s_r}{\partial r} + \frac{\partial u^s_\phi}{\partial \phi}(r\Sigma^s u^s_\phi) - \frac{\partial \phi}{\partial r} \left( \frac{\Sigma^s u^s_\phi}{\Sigma^s} \right) = 0, \tag{4} \]
\[ \frac{\partial j^s_r}{\partial t} + u^s_\phi \frac{\partial j^s_\phi}{\partial r} + j^s_\phi \frac{\partial u^s_\phi}{\partial \phi} + \frac{\partial j^s_\phi}{\partial \phi}(r\Sigma^s u^s_\phi) - \frac{\partial \phi}{\partial r} \left( \frac{\Sigma^s j^s_\phi}{\Sigma^s} \right) = 0, \tag{5} \]
where \( \Sigma^s \) is the stellar surface mass density, \( \kappa_s \) is the radial component of the specific angular momentum, \( \alpha_s \) is the stellar velocity dispersion (or an effective “isothermal sound speed”), \( a^2 \Sigma^s \) stands for an effective two-dimensional polytropic approximation, and \( \phi \) is the total gravitational potential perturbation. For the corresponding fluid equations in the gaseous disc, we simply replace the relevant superscript or subscript \( s \) by \( g \). The two sets of fluid equations are coupled by the total gravitational potential \( \phi \) through the Poisson integral
\[ \phi(r, \phi, t) = \frac{\partial}{\partial t} \int_0^\infty \int_0^{\infty} \frac{\partial G \Sigma^s \phi(r, \psi, t) d\psi d\phi}{\sqrt{r^2 + \rho^2}} - 2\pi G \Sigma^s \phi(r, \psi, t) \]}
\[ \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) - 2\pi G \Sigma^s \phi(r, \psi, t) = 0, \tag{6} \]
where \( \Sigma = \Sigma^s + \Sigma^g \) is the total surface mass density. Here the stellar and gaseous SIDs interact mainly through the mutual gravity on large scales (Jog & Solomon 1984a,b; Bertin & Romeo 1988; Romeo 1992; Elmegreen 1995; Jog 1996; Lou & Fan 1998b, 2000a,b; Lou & Shen 2003).

Using the above equations for the steady background rotational equilibrium (indicated by an explicit subscript 0) with \( u_0^s = u_0^g = 0 \), \( \Omega_s = \delta_0^s/r^2 \), and \( \Omega_g = \delta_0^g/r^2 \), we obtain
\[ \Sigma_0^s = a_s^2(1 + D_0^s)/[2\pi G \rho(1 + \delta)], \tag{7} \]
\[ \Sigma_0^g = a_g^2(1 + D_0^g)/[2\pi G \rho(1 + \delta)], \tag{7} \]
and an important relation
\[ a_s^2(D_0^s + 1) = a_g^2(D_0^g + 1), \tag{8} \]
where \( \delta \equiv \Sigma_0^s/\Sigma_0^g \) is the SID surface mass density ratio, \( \beta \equiv a_s^2/a_g^2 \) stands for the square of the ratio of the stellar velocity dispersion to the gas sound speed. In our analysis, we specify values \( \delta \) and \( \beta \) to characterize different composite SID systems. For late-type spiral galaxies, gas materials are less than the stellar mass with \( \delta < 1 \). For young proto disc galaxies, gas materials may exceed stellar mass in general with \( \delta > 1 \). Thus, in our analysis and computations, both cases of \( \delta < 1 \) and \( \delta > 1 \) are considered. As the stellar velocity dispersion is usually larger than the gas sound speed \( a_s^2 > a_g^2 \), we then have \( \beta > 1 \). When \( \beta = 1 \), we have \( D_s = D_g \) by condition (8), which means the two SIDs may be treated as a single SID (Lou & Shen 2003).

For small perturbations denoted by subscripts 1, we obtain the following linearized equations
\[ \frac{\partial \Sigma_1^s}{\partial t} + \frac{\partial}{\partial \phi} \left( r \Sigma_0^s u_1^s \right) + \frac{\partial}{\partial \phi} \left( \frac{\partial \Sigma_1^s}{\partial \phi} \right) = 0, \tag{9} \]
\[ \frac{\partial u_1^s}{\partial t} + \frac{\partial}{\partial \phi} \left( r \Sigma_0^s u_1^s \right) - 2\pi G \Sigma_0^s - \frac{\partial}{\partial \phi} \left( \frac{\Sigma_0^s \partial j_1^s}{\Sigma_0^s} \right) = 0, \tag{10} \]
\[ \frac{\partial \Sigma_1^g}{\partial t} + \left( \frac{\partial \Sigma_1^g}{\partial \phi} \right) - 2\pi G \Sigma_0^g = 0, \tag{11} \]
for the stellar SID, and the corresponding equations for the gaseous SID with subscript or superscript \( s \) replaced by \( g \) in equations (9)–(11). The total gravitational potential perturbation is
\[ \phi_1(r, \phi, t) = \int_0^{\infty} \int_0^{\infty} \frac{G \Sigma_0^s + \Sigma_0^g}{r^2 + \rho^2 - 2\pi G \rho} d\psi d\phi, \tag{12} \]
In the usual WKBJ approximation, we write the coplanar axisymmetric perturbations with the periodic dependence of \( \exp(ikr + i\omega t) \) where \( k \) is the radial wavenumber and \( \omega \) is the angular frequency. With this dependence in the perturbation equations, we obtain the local WKBJ dispersion relation for the composite SID system in the form of
\[ (\omega^2 - \kappa_s^2 - k^2 a_s^2 + 2\pi G \kappa_s^0) \times \omega^2 - k^2 a_g^2 + 2\pi G \kappa_g^0 + 2\pi G \kappa_s^0 \]
\[ - (2\pi G \kappa_s^0)(2\pi G \kappa_g^0) = 0. \tag{13} \]
Jog & Solomon (1984a) have previously derived the similar dispersion relation with \( V_s = V_g \) and thus \( \kappa_s = \kappa_g \). The stability analyses of Elmegreen (1995) and Jog (1996) start with the same dispersion relation of Jog & Solomon (1984a). The WKBJ approach of Jog & Solomon (1984a) is generally applicable to any locally prescribed properties of a two-fluid disc system. In earlier theoretical studies including that of Jog & Solomon (1984a), it is usually taken that the rotation speeds of the two fluid discs are equal. Our model of two coupled SIDs here is self consistent globally with a central singularity and with different SID rotation speeds in general. When one applies the local WKBJ analysis to our composite SID system, the procedure is identical to that of Jog & Solomon (1984a) but with different local SID speeds \( V_s \) and \( V_g \) and hence different epicyclic frequencies \( \kappa_s \) and \( \kappa_g \). Therefore, our dispersion (13) appears strikingly similar to that of Jog & Solomon (1984a) yet with different \( \kappa_s \) and \( \kappa_g \). In short, if one prescribes two different disc speeds and thus two different epicyclic frequencies in the analysis of Jog
& Solomon (1984a), the resulting dispersion relation should be our equation (13).

In our analysis, the rotation speeds of the stellar and the gaseous SIDs differ in general. In particular, since $a_s^2$ is usually higher than $a_g^2$, it follows from equations (1) and (8) that $Ω_s < Ω_g$. That is, the stellar disc rotates somewhat slower than the gaseous disc does. Such phenomena have been observed in the solar neighborhood as well as in external galaxies (K. C. Freeman, 2003, private communications). This is related to the so-called asymmetric drift phenomena (e.g., Mihalas & Binney 1981) and can be understood in terms of the Jeans equations (Jeans 1919) originally derived by Maxwell from the collisionless Boltzmann equation. In essence, random stellar velocity dispersions produce a pressure-like effect such that mean circular motions become slower (Binney & Tremaine 1987). In our treatment, this effect of stellar velocity dispersions is modelled in the polytropic approximation (see eqns (2) and (8)). Biermann & Davis (1960) discussed the possibility that the mean rotation speeds of the stars and gas in the Galaxy may be different and reached the opposite conclusion that a gas disc should rotate slower than a stellar disc does. However, their analysis based on a virial theorem that excludes the effect of stellar velocity dispersions. In other words, the stress tensor due to stellar velocity dispersions in the Jeans equation has been ignored in their considerations.

3 AXISSYMMETRIC STABILITY ANALYSIS

We study the axisymmetric stability problem based on the WKBJ dispersion relation. Similar to previous analysis (Toomre 1964; Elmegreen 1995; Jog 1996; Lou & Fan 1998b), we also derive an effective $Q$ parameter. This $Q_{\text{eff}}$ is expected to depend on $D_s^2$ once $δ$ and $β$ are known.

For the convenience of analysis, we define notations

\[
\begin{align*}
H_1 &\equiv κ^2 + k^2 a_s^2 - 2πG|k|Σ_0^s, \\
H_2 &\equiv κ_g^2 + k^2 a_g^2 - 2πG|k|Σ_0^g, \\
G_1 &\equiv 2πG|k|Σ_0^s, \\
G_2 &\equiv 2πG|k|Σ_0^g,
\end{align*}
\]

(14)

to express dispersion relation as an explicit quadratic equation in terms of $ω^2$, namely

\[
\omega^4 - (H_1 + H_2)ω^2 + (H_1H_2 - G_1G_2) = 0.
\]

(15)
The two roots $\omega^2_+ + ω^2_-$ of equation (15) are

\[
\omega^2_\pm(k) = \frac{1}{2} \{ (H_1 + H_2) \\
\pm [(H_1 + H_2)^2 - 4(H_1H_2 - G_1G_2)]^{1/2} \}.
\]

(16)
The $ω^2_+$ root is always positive, which can be readily proven.\footnote{The determinant $Δ ≡ (H_1 - H_2)^2 + 4G_1G_2 ≥ 0$.}

\footnote{This conclusion holds true if $H_1 + H_2 ≥ 0$. If $H_1 + H_2 < 0$, then $H_1H_2 - G_1G_2 < 0$ and the conclusion still holds.}

We naturally focus on the $ω^2_+$ root, namely

\[
ω^2_+(k) = \frac{1}{2} \{ (H_1 + H_2) \\
- [(H_1 + H_2)^2 - 4(H_1H_2 - G_1G_2)]^{1/2} \},
\]

(17)
to search for stable conditions that make the minimum of $ω^2_+(k) ≥ 0$ for all $k$.

By expanding the $ω^2_+(k)$ root in terms of definition (14), we obtain an expression involving parameters $κ_s$, $κ_g$, $a_s^2$, $a_g^2$, $Σ_0^s$, $Σ_0^g$, and $k$. For a composite SID system with power-law surface mass densities, flat rotation curves, and the equilibrium properties of $κ_s$, $κ_g$, $Σ_0^s$ and $Σ_0^g$, equation (17) can also be expressed in terms of four parameters $a_s^2$, $D_s^2$, $δ$ and $β$, namely

\[
ω^2(K) = \frac{a_s^2}{2k^2} [A_2K^2 + A_1K + A_0 - ϕ^{1/2}],
\]

(18)
where $K ≡ |k|r$ is the dimensionless local radial wavenumber and the relevant coefficients are defined by

\[
\begin{align*}
A_2 &≡ 1 + 1/β, \\
A_1 &≡ -(1 + D_s^2), \\
A_0 &≡ 4D_s^2 + 2(1 - 1/β), \\
ϕ &≡ B_4K^4 + B_3K^3 + B_2K^2 + B_1K + B_0,
\end{align*}
\]

(19)
with

\[
\begin{align*}
B_4 &≡ (1 - 1/β)^2, \\
B_3 &≡ 2(1 + D_s^2)(1 - 1/β)(δ - 1)/(1 + δ), \\
B_2 &≡ (D_s^2 - 1 + 2/β)(D_s^2 + 3 - 2/β), \\
B_1 &≡ 4(1 + D_s^2)(1 - 1/β)(1 - δ)/(1 + δ), \\
B_0 &≡ 4(1 - 1/β^2).
\end{align*}
\]

As $a_s^2$ has been taken out as a common factor of equation (18), the most relevant part is simply

\[
A_2K^2 + A_1K + A_0 - ϕ^{1/2}
\]

(20)
that involves the stellar rotation parameter $D_s^2$ and the two ratios $δ$ and $β$. We need to determine the value of $K_{\text{min}}$ at which $ω^2(K)$ takes the minimum value, in terms of $D_s^2$. We then substitute this $K_{\text{min}}$ into expression (20) and derive the condition for $D_s^2$ that makes this minimal $ω^2_{\text{min}} ≥ 0$.

3.1 The $D_s$–criterion in the WKBJ regime

To demonstrate the stability properties unambiguously, and to confirm our previous interpretations for the marginal stability curves in a composite SID system (Lou & Shen 2003), we first present some numerical results that contain the same physics and are simple enough to be understood.

According to equation (18), $ω^2_+$ is a function of rotation parameter $D_s^2$ and wavenumber $K$. We then use $K$ as the horizontal axis and $D_s^2$ as the vertical axis to plot contours of $ω^2_+$ numerically. A specific example of $δ = 0.2$ and $β = 10$ is shown in Fig. 1. The shaded regions are where $ω^2_+$ takes on negative values and the blank region is where $ω^2_+$ takes on positive values. In this manner, we clearly obtain marginal stability curves (solid lines) along which $ω^2_+ = 0$. 

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Physically, the shaded regions of negative $\omega^2$ values are unstable, while the blank region of positive $\omega^2$ values is stable. From such contour plots of $\omega^2$, one can readily determine the specific range of stellar rotation parameter $D^2$ such that the composite SID system is stable against axisymmetric disturbances. We therefore confirm our previous interpretations for the marginal stability curves for a composite SID system (Lou & Shen 2003). For a comparison, Fig. 2 shows the corresponding marginal stability curves with $\delta = 0.2$ and $\beta = 10$ that we derived previously (Lou & Shen 2003) as the stationary axisymmetric perturbation configuration where $\alpha$ is a dimensionless effective radial wavenumber (see also Shu et al. 2000). The apparent difference in Figs. 1 and 2, mainly in the collapse regions, arises because the results of Fig. 2 are exact perturbation solutions without using the WKBJ approximation.

Now in the WKBJ approximation, the stable range of $D^2$ can be read off from Fig. 1. For the case of $\delta = 0.2$ and $\beta = 10$, the composite SID system is stable from $D^2 = 0.1193$ at $K = 0.5121$ to $D^2 = 4.1500$ at $K = 3.9237$. In comparison with the single SID case (see Fig. 2 of Shu et al. 2000), this stable range of $D^2$ diminishes. With other sets of $\delta$ and $\beta$ parameters, we can derive similar diagrams as Fig. 2 as have been done in our recent work, for examples, Figs. 7 – 10 of Lou & Shen (2003). In the WKBJ approximation, we show variations with $\delta$ and $\beta$ in Figs. 3 and 4. When $\delta$ is larger, the system tends more likely to be unstable for small $\beta$. In reference to Fig. 10 of Lou & Shen (2003) with $\delta = 1$ and $\beta = 30$, we plot contours of $\omega^2$ in Fig. 5 in the WKBJ approximation. In this case, the stable range of $D^2$ disappears, that is, the composite SID system cannot be stable against axisymmetric disturbances. We note again that the collapse region with small wavenumber deviates from the exact result of Fig. 10 of Lou & Shen (2003), due to the obvious limitation of the WKBJ approximation. For the unstable region of large wavenumber, which we labelled as ring fragmentation previously (Lou & Shen 2003),
of angular momentum, SID rotation plays a stabilizing role to modify the onset of Jeans collapse (e.g., Chandrasekhar 1961). Hence, there appears a critical $D_z^2$ below which exists the collapse regime. Of course, too large a $D_z^2$ causes the other type of ring fragmentation instabilities to set in. For most disc galaxies currently observed, the rotation parameter $D_z^2$ is typically large enough to avoid such collapses.

Elsewhere, the collapse regime studied here may have relevant astrophysical applications, e.g., in circumnuclear discs around nuclei of galaxies and protostellar discs around protostellar cores etc., where $D_z^2$ may be small enough (either a high velocity dispersion $a_s$ or a low rotation speed $V_s$) to initiate collapses during certain phases of disc system evolution. In fact, for a low rotation speed $V_s$, we note from the perspective of evolution, that those proto disc galaxies of low rotation speeds in earlier epochs, will Jeans collapse and lead to other morphologies, while those proto disc galaxies of sufficiently fast rotation speeds in earlier epochs will gradually evolve into disc galaxies we observe today. Meanwhile, ring fragmentation instabilities are thought to be relevant to global star formation in a disc galaxy – another important aspect of galactic evolution.

The addition of a gaseous SID to a stellar SID will decrease the overall stability against axisymmetric disturbances at any wavelengths in general, that is, the stable range of $D_z^2$ shrinks. However, for the two types of instabilities, namely, the collapse instability at small wavenumbers and the ring-fragmentation instability at relatively large wavenumbers, the gravitational coupling of the two SIDs play different roles. For instance, a composite SID system tends more likely to fall into ring fragmentations but less likely to collapse. For either larger $\delta$ or larger $\beta$, this tendency becomes more apparent.

This trend can be understood physically. For the ring-fragmentation regime, suppose the stellar SID by itself alone is initially stable and a gaseous SID component is added to the stellar SID system in a dynamically consistent manner. The gaseous SID may be either stable or unstable by itself alone. By itself, the gaseous SID tends to be more unstable if the sound speed $a_s$ becomes smaller or $\Sigma_0$ becomes larger according to the definition of the $Q$ parameter (valid for the ring fragmentation instability), which means larger $\beta$ or larger $\delta$, respectively. Thus, with either $\beta$ or $\delta$ or both being larger, a composite SID system tends to be more vulnerable.
to ring fragmentations. The interpretation for the collapse regime is associated with the dynamical coupling of surface mass density $\Sigma$ and epicyclic frequency $\kappa$ that leads to the following conclusion, namely, a lower velocity dispersion of the gas component seems to prevent collapse. This should be understood from condition (8): for larger $\beta$, the interpretation for the collapse region appears to decrease when either $\delta$ or $\beta$ increases. The stable regime of $D^2_s$ may disappear for a case like $\delta = 1$ and $\beta = 30$. When $\beta = 1$, the problem is independent of $\delta$, that is, $\delta$ can be arbitrary (Lou & Shen 2003).

Table 1. The overall stable ranges of $D^2_s$ against axisymmetric disturbances at any wavelengths for different sets of $\delta$ and $\beta$. The values in parentheses are derived by Lou & Shen (2003) without the WKBJ approximation and are preferred to describe the collapse region. The overall stable range of $D^2_s$ appears to decrease when either $\delta$ or $\beta$ increases. The stable range of $D^2_s$ may disappear for a case like $\delta = 1$ and $\beta = 30$. When $\beta = 1$, the problem is independent of $\delta$, that is, $\delta$ can be arbitrary (Lou & Shen 2003).

| $\delta$ | $\beta$ | lower limit of $D^2_s$ | higher limit of $D^2_s$ |
|----------|---------|----------------------|----------------------|
| -        | 1       | 0.1716(0.9261)       | 5.8284(5.6259)       |
| 0.2      | 1.5     | 0.1350(0.8406)       | 5.3974(5.2117)       |
| 5        | 0.120(0.7705) | 4.4782(4.3558) |  |
| 10       | 0.1193(0.7692) | 4.1500(4.0611) |  |
| 30       | 0.1185(0.7539) | 3.8062(3.7670) |  |
| 1        | 1.5     | 0.0596(0.6556)       | 4.5945(4.3532)       |
| 5        | 0.0411(0.4478) | 2.1275(2.0853) |  |
| 10       | 0.0401(0.4221) | 1.0947(1.0834) |  |
| 30       | (-)     | (-)                  | (-)                  |
| 5        | 3       | 0.0039(0.1418)       | 1.7397(1.6718)       |
| 10       | 5       | 0.0011(-)            | 0.5696(0.5317)       |

3.2 Effective $Q$ parameters

It is natural to extend the concept of the $Q$ parameter for a single disc to an effective $Q$ parameter for a composite disc system for the purpose of understanding axisymmetric stability properties. In the following, we discuss effective $Q$ parameters for a composite SID system in reference to the works of Elmegreen (1995) and Jog (1996). There are several points worth noting in our formulation. First, $\kappa_s$ and $\kappa_g$ are related to each other but are allowed to be different in general. Secondly, the background surface mass densities are related to SID rotation speeds through the polytropic approximation and the steady background rotational equilibrium. For the convenience of notations, the effective $Q$ parameters introduced by Elmegreen (1995) and by Jog (1996) are denoted by $Q_E$ and $Q_J$, respectively.

3.2.1 The $Q_E$ parameter of Elmegreen

Starting from equation (18) or (22), we define an effective $Q_E$ parameter in a composite SID system following the procedure of Elmegreen (1995) but with two different epicyclic frequencies $\kappa_s$ and $\kappa_g$. The minimum of $\omega^2$ is given by

$$\omega^2_{\min} = \frac{a^2}{2\pi^2}[A_2K_{\min}^2 + A_1K_{\min} + A_0 - \varphi^{1/2}]$$

$$= \frac{a^2}{2\pi^2}(\varphi^{1/2} - A_2K_{\min}^2 - A_1K_{\min})(Q_E^2 - 1)$$

where

$$Q_E^2 = \frac{A_0}{\varphi^{1/2} - A_2K_{\min}^2 - A_1K_{\min}}$$

and $\varphi$ takes on the value at $K = K_{\min}$. As $K_{\min}$ depends on $D^2_s$, $\delta$ and $\beta$, so does $Q_E^2$. For specified parameters $\delta$ and $\beta$ of a composite SID system, the parameter $Q_E^2$, giving the criterion of axisymmetric stability (i.e. stable when $Q_E^2 > 1$), corresponds to the rotation parameter $D^2_s$ of the stellar disc. One purpose of the present analysis is to establish the correspondence between the axisymmetric stability condition and the marginal stability curves of axisymmetric stationary perturbation configuration derived recently by Lou & Shen (2003) for a composite SID system. The pertinent results for a single SID can be found in Shu et al. (2000).

Our task now is to find the solution $K_{\min}$. As $\kappa_s \neq \kappa_g$ in general, our computation turns out to be somewhat more complicated or involved than that of Elmegreen (1995) who took $\kappa_s = \kappa_g$. It becomes more difficult to solve this algebraic problem of order higher than the third analytically. However, if one multiplies both sides of equation (17) by the positive root $\omega^2_+ + A_0$, a simple expression appears

$$W \equiv \omega^2_+ \omega^2 = H_1H_2 - G_1G_2$$

Instead of minimizing $\omega^2$, we solve for $K_c$ in terms of $D^2_s$ at which $W \equiv \omega^2_+ \omega^2_-$ attains the minimum value. This $K_c$ must satisfy the following equation

$$\frac{dW}{dK} = \frac{d(H_1H_2 - G_1G_2)}{dK} = 0$$

By substituting $H_1$, $H_2$, $G_1$ and $G_2$ into equation (26) and regarding $\delta$ and $\beta$ as specified parameters, equation (26) can be reduced to a cubic equation of $K$ involving $D^2_s$, namely,

$$K^3 + aK^2 + bK + c = 0$$

where

$$a = -3(D^2_s + 1)(1 + \beta \delta)/[4(1 + \delta)]$$
In most cases \( \parallel \), equation (27) has only one real solution in the form of

\[
K_c = (x - q/2)^{1/3} + (-x - q/2)^{1/3} - a/3 ,
\]

where \( x = (q^2/4 + p^3/27)^{1/2} \), \( p = b - a^2/3 \) and \( q = 2a^3/27 - ab/3 + c \). For a single real root, it is required that \( q^2/4 + p^3/27 > 0 \). Through numerical computations, we find that this condition is met for most pairs of \( \delta \) and \( \beta \).

We then regard this \( K_c \) as an estimator** for \( K_{min} \) to determine \( Q^2_k \). As already known in our analysis (Lou & Shen 2003), when the ratio \( \beta \) is equal to 1, the properties of a composite SID system have something in common with those of a single SID. Moreover, the stationary axisymmetric configuration becomes the same as a single SID. We expect that the present problem should reduce to a single SID case when \( \beta = 1 \). We consider below the special case of \( \beta = 1 \).

With \( \beta = 1 \) in equation (27), \( K_c \) can be determined in terms of \( D^2_2 \) without involving \( \delta \) at all. We then use \( K_c \) as \( K_{min} \) in the definition of \( Q^2_k \), namely, condition (24). One can readily verify that \( Q^2_k > 1 \) for \( 3 - 2\sqrt{2} < D^2_2 < 3 + 2\sqrt{2} \); \( Q^2_k < 1 \) for \( D^2_2 > 3 + 2\sqrt{2} \) or \( D^2_2 < 3 - 2\sqrt{2} \); and \( Q^2_k = 1 \) for \( D^2_2 = 3 \pm 2\sqrt{2} \). The range of \( D^2_2 \) that makes \( Q^2_k > 1 \) is \( 0.1716 < D^2_2 < 5.8284 \), which is exactly the same as the \( Q^2_k \) estimator for the single SID of Shu et al. (2000). Based on estimates of \( Q \) parameter alone, Shu et al. (2000) made physical interpretations for their axisymmetric stationary perturbation solution curves for a single SID. According to their interpretation, the maximum \( D^2_2 \) is \( 0.9320 \) of the collapse branch and the minimum \( D^2_2 = 5.410 \) of the ring-fragmentation branch (see Fig. 2 of Shu et al. 2000) encompass the required range of \( D^2_2 \) for stability against axisymmetric disturbances. As the formulation of Shu et al. (2000) is exact in contrast to the WKBJ approximation, the stable range of \( D^2_2 \) slightly deviates from that determined by the effective \( Q \) parameter. And for the upper bound of \( D^2_2 \), with a larger wavenumber (or a smaller wavelength), the two approaches lead to consistent results as expected.

In our recent analysis (Lou & Shen 2003), we obtained curves for stationary axisymmetric perturbations given various combinations of \( \delta \) and \( \beta \) for a composite SID system. These curves differ from those for a single SID and the stable range of \( D^2_2 \) varies for different combinations of \( \delta \) and \( \beta \). It is found that this stable range of \( D^2_2 \) may be significantly reduced for certain values of \( \delta \) and \( \beta \), as shown in Fig. 7–9 of Lou & Shen (2003) and in the analysis of Section 3.1. Moreover, when \( \delta \) and \( \beta \) are sufficiently large, the stable range of \( D^2_2 \) may no longer exist, as shown in Fig. 10 of Lou & Shen (2003) and Fig. 5 here. For these cases, a composite SID system becomes vulnerable to axisymmetric instabilities.

For \( \delta = 0.2 \) and \( \beta = 10 \), as shown in Fig. 6 of Lou & Shen (2003) and also Fig. 2 of this paper, we search for the approximate stable range of \( D^2_2 \) with the WKBJ dispersion relation (15) for a composite SID system. By using the \( Q^2_k \) defined by equation (24), one can directly plot the curve of \( Q^2_k \) versus \( D^2_2 \) as shown in Fig. 6, from which one can identify the approximate stable range of \( D^2_2 \) from \( \sim 0.1193 \) to \( \sim 4.1500 \) where \( Q^2_k = 1 \). Several other curves with different \( \delta \) and \( \beta \) are also shown in Fig. 6.

One thing that should be emphasized is that the \( Q^2_k \) parameter as defined by equation (24) must be positive, otherwise it may occasionally lead to incorrect conclusions regarding the sign of \( \omega^2 \). This procedure can be cumbersome. Alternatively, Jog (1996) used a seminumerical way to define another effective \( Q \) parameter, referred to as \( Q_j \) here, that is different from \( Q^2_k \) defined by equation (24).

3.2.2 The \( Q_j \) parameter of Jog

The \( Q_j \) parameter can be derived by solving cubic equation (27) for \( K \). It may happen that equation (27) gives three real solutions. In order to obtain an analytical form of \( K_{min} \), one must identify the absolute minimum of \( \omega^2 \). This procedure can be cumbersome. Alternatively, Jog (1996) used a seminumerical way to define another effective \( Q \) parameter, referred to as \( Q_j \) here, that is different from \( Q^2_k \) defined by equation (24).

According to solution (17), \( \omega^2 \) will be positive or negative if \( H_1 H_2 - G_1 G_2 > 0 \) or \( < 0 \). The critical condition of neutral stability is thus \( H_1 H_2 - G_1 G_2 = 0 \), which can be

\[
b = D^2_2 (\beta + 1) + \beta - 1 ,
\]
\[
c = - (D^2_2 + 1) [\beta D^2_2 (1 + \delta) + \beta - 1] / [2(1 + \delta)] .
\]

** When \( K_n \) of Bertin & Romeo (1988) give the proof that as long as \( 1/\beta > 0.0294 \) and \( \delta > 0.172 \), there exists only one minimum of \( \omega^2 \). When there are more than one local minima for \( \omega^2 \), one should choose \( K_n \) for the smallest minimum of \( \omega^2 \).

\[\parallel\text{Figure 6. Curves of } Q^2_k \text{ versus } D^2_2 \text{ which intersect the horizontal line } Q^2_k = 1. \text{ For each curve, the two points of intersection at } Q^2_k = 1 \text{ give the stable range of } D^2_2 \text{ for a composite SID against axisymmetric disturbances. The } \beta = 1 \text{ case gives the same result of a single SID.}\]
cast into the form of
\[ \frac{2\pi G\Sigma_0^a}{\kappa_2^2 + k^2a_2^2} + \frac{2\pi G\Sigma_0^g}{\kappa_3^2 + k^2a_3^2} = 1. \]

One can now define a function \( F \) such that
\[ F \equiv \frac{2\pi G\Sigma_0^a}{\kappa_2^2 + k^2a_2^2} + \frac{2\pi G\Sigma_0^g}{\kappa_3^2 + k^2a_3^2} = 1. \]

where expressions (2), (7), (8) and \( K \equiv \|k\|r \) are used. We then search for \( K_{\text{min}} \) at which a composite SID system becomes hardest to be stabilized, that is, when \( \omega_2 \) reaches the minimum. The effective \( Q \) parameter, \( Q_j \), can thus be

\[ Q_j \equiv \frac{2}{1 + Q_j^2} \equiv \frac{K_{\text{min}}(D_j^2 + 1)/(1 + \delta)}{2D_j^2 + K_{\text{min}}^2} \]

\[ + \frac{K_{\text{min}}\beta(D_j^2 + 1)\delta/(1 + \delta)}{2[\beta(D_j^2 + 1) - 1] + K_{\text{min}}^2} \]

(30)

(see eqns. (5) and (6) of Jog 1996). It follows that \( Q_j^2 < 1 \) or \( > 1 \) correspond to instability or stability, respectively. Therefore, for a given set of \( \delta, \beta, \) and \( D_j^2 \), one can numerically determine the value of \( K_{\text{min}} \). By inserting this \( K_{\text{min}} \) into equation (30), one obtains the value of \( Q_j^2 \) for any given set of \( \{\delta, \beta, D_j^2\} \). It is then possible to explore relevant parameter regimes of interest. Jog (1996) introduced three parameters, namely, the two Toomre \( Q \) parameters for each disc and the surface mass density ratio of the two discs, to determine \( Q_j^2 \). Equivalently, we use three dimensionless parameters \( \delta, \beta, \) and \( D_j^2 \) instead. Figs. 7 and 8 show variations of \( Q_j^2 \) with \( D_j^2 \). Similar to \( Q_E \), the value of \( Q_j^2 > 1 \) when \( D_j^2 \) falls in the same ranges. When \( \delta \) is fixed and \( \beta \) increases, the left and right bounds of the stable \( D_j^2 \) range move towards left together, while the width of the stable range appears to decrease. We have revealed the same trend of variations in our recent work (Lou & Shen 2003). It is possible for the left bound to disappear when \( \beta \) is sufficiently large, and it is easier to achieve this when \( \delta \) becomes larger. For a sufficiently large \( \delta \), the increase of \( \beta \) may completely suppress the stable range of \( D_j^2 \) as shown in Fig. 5 for \( \delta = 1 \) and \( \beta = 30 \).

In comparison to \( Q_E \), the \( Q_j \) definition is valid for all parameter regimes and avoids improper situations for unusual \( \delta \) and \( \beta \) values. However, to find the value of \( Q_j^2 \), one must perform numerical exploration for each given \( D_j^2 \), while for \( Q_E \), the procedure is analytical as long as there exists a stable range of \( D_j^2 \). Regardless, the critical values of \( D_j^2 \) found by \( Q_E \) and \( Q_j \) are equivalent.

### 3.3 Composite partial SID system

In disc galaxies, there are overwhelming evidence for the existence of massive dark matter halos as inferred from more or less flat rotation curves. To mimic the gravitational effect of a dark matter halo, we include gravity terms \( \partial \Phi/\partial r \) and \( \partial \Phi/\partial \varphi \) in the radial and azimuthal momentum equations (4) and (5), respectively, where \( \Phi \) is an axisymmetric gravitational potential due to the dark matter halo. It is convenient to introduce a dimensionless parameter \( F \equiv \phi/(\phi + \Phi) \) for the fraction of the disc potential relative to the total potential (e.g., Syer & Tremaine 1996; Shu et al. 2000; Lou 2002). The background rotational equilibrium is thus modified by this additional \( \Phi \) accordingly. As before, we write \( \Omega_\alpha = a_\alpha D_\alpha/r \), \( \Omega_\beta = a_\beta D_\beta/r \), and \( k_\delta = \sqrt{2}\Omega_\delta \), \( k_\eta = \sqrt{2}\Omega_\eta \).

The equilibrium surface mass densities now become
\[ \Sigma_0^a = \frac{a_\alpha^2(1 + D_j^2)F}{2\pi G r(1 + \delta)} \]

\[ \Sigma_0^g = \frac{a_\beta^2(1 + D_j^2)F\delta}{2\pi G r(1 + \delta)} \]

where \( 0 \leq F \leq 1 \). In our perturbation analysis, the dynamic response of this axisymmetric massive dark matter halo to coplanar perturbations in a composite SID system is
ignored (Syer & Tremaine 1996; Shu et al. 2000; Lou 2002). We thus derive similar perturbation equations as those of a full SID case \((F = 1)\) and consequently the similar dispersion relation as equation (13) yet with modified equilibrium properties. Now following the same procedure of WKBJ perturbation analysis described in Section 3.1, we can plot contours of \(\omega^2\) as a function of rotation parameter \(D_s^2\) and wavenumber \(K\) for a composite partial SID system. The case of \(F = 1\) corresponds to a composite full SID system that has been studied in Section 3. When \(F\) becomes less than 1 (i.e., a composite partial SID system), which means the fraction of the dark matter halo increases, the stable range of \(D_s^2\) becomes enlarged, as shown in Figs. 9 and 10 for \(\delta = 0.2\) and \(\beta = 10\). From these contour plots, it is clear that the introduction of a dark matter halo tends to stabilize a composite partial SID system. For late-type disc galaxies, one may take \(F = 0.1\) or smaller. Such composite partial SID systems are stable against axisymmetric disturbances in a wide range of \(D_s^2\). Moreover, those composite full SID systems that are unstable may be stabilized by the presence of a dark matter halo, as shown by the example of Fig. 11 for \(\delta = 1\), \(\beta = 30\) and \(F = 0.5\) (this case is unstable for a composite full SID system).

4 DISCUSSIONS AND SUMMARY

Our model of a composite (full or partial) SID system is an idealization of any actual disc systems, such as disc galaxies, circumnuclear discs surrounding nuclei of galaxies, or protostellar discs. One key feature of the SID model is the flat rotation curve. One major deficiency in galactic application is the central singularity that sometimes even gives rise to conflicting theoretical interpretations (see Shu et al. 2000). The isothermality is yet another simplifying approximation. In galactic applications, the common wisdom is to introduce a bulge around the center to avoid the central singularity. By a local WKBJ stability analysis of axisymmetric perturbations, this model can be utilized as usual with qualifications. We now discuss the WKBJ results in the context of our own Galaxy, the Milky Way, for which the necessary observational data are available, such as the gas fraction, the stellar and the gaseous velocity dispersions and the epicyclic frequency, and so forth.

There are several inadequacies of our model. First, the isothermal assumption implying constant \(a_s\) and \(a_\varphi\) throughout the disc system is a gross simplification; the velocity dispersion \(a_s\) in the Milky Way decreases with increasing radius (e.g., Lewis & Freeman 1989). Secondly, by the polytropic approximation, the surface mass densities are characterized by power-law \(\propto r^{-1}\) profiles, which appear not to be the case of the Milky Way (e.g., Caldwell & Ostriker 1981). Nevertheless, to estimate local stability properties of a disc portion, we may take \(a_s = 50\,\text{km s}^{-1}\), \(V_s = 220\,\text{km s}^{-1}\), \(\delta = 0.1\) and flat rotation speed \(V_\varphi = 220\,\text{km s}^{-1}\) around 4 kpc from the center of our Galaxy. For a massive dark matter halo such that \(F = 0.1\), the relevant parameters to determine the
effective $Q$ parameter for a composite partial SID system are $\delta = 0.1$, $\beta = 50$, $D^2 = 20$ and $F = 0.1$. Following the definition of Jog (1996), the value of $Q_J \gg 1$. While for full SIDs with $F = 1$, we get $Q_J = 0.17$ which means local instability. Obviously, the dark matter halo plays the key role in stabilizing the system.

The composite partial SID model with power-law $\propto r^{-1}$ surface mass densities may not describe the Milky Way well. But for those disc galaxies with flat rotation curves and approximate power-law $\propto r^{-1}$ surface mass densities, our model should offer a sensible local criterion for axisymmetric instability. For example, given radial variations of velocity dispersions and gas mass fraction, we may deduce locally unstable zones.

For circumnuclear discs around nuclei of galaxies, the stellar velocity dispersion $a_s$ can be as high as several hundred kilometers per second along the line of sight (e.g., Whitmore et al. 1985; McElroy 1994); this may then lead to small $D^2$. Within such a parameter regime, the collapse instability may set in during a certain phase of system evolution. For instance, in the case of Fig. 2 with $\delta = 0.2$ and $\beta = 10$, we choose $V_s = 200$ km s$^{-1}$ and $a_s = 300$ km s$^{-1}$ in a circumnuclear disc region of a disc galaxy. The resulting $D^2 = 0.44$ would give rise to collapse and the composite SID system might eventually evolve into a bulge of high velocity dispersion.

Another possible collapse situation of interest corresponds to small $D^2$ for sufficiently low disc rotation speed $V$. Potential applications include but not limited to protostellar discs in the context of star formation. We here briefly discuss several qualitative aspects. For a protostellar disc system, the two disc components here may be identified with the relatively “hot” gas disc and the relatively “cool” dust disc. Interactions of radiation fields from the central protostar with cool dusts in the disc lead to infrared emissions. In our idealized treatment, the gas and dust discs are treated as two gravitationally coupled thin SIDs. This simple model treatment may need to be complemented by other processes of coupling such as collisions between gas and dust particles. When such a composite disc system rotates at a low speed (e.g., Shu et al. 1987), the rotation parameter $D$ could be low enough to initiate large-scale collapse. The subsequent dynamical processes will drive the composite SID system to more violent star formation activities.

So far we have investigated the axisymmetric stability problem for a composite system of gravitationally coupled stellar and gaseous SIDs using the WKBJ approximation and we now summarize the results of our analysis.

First, because the single SID case studied by Shu et al. (2000) may be regarded as the special case of a composite SID system, the results of our WKBJ analysis clearly support the physical interpretations of Shu et al. (2000) for the marginal stability curves of axisymmetric perturbations. The “ring fragmentation regime” is related to the familiar $Q$ parameter (Safronov 1960; Toomre 1964). The presence of the Jeans “collapse regime” can be traced to the SID model where self-gravity, effective pressure, and SID rotation compete with each other.

Secondly, the recent study of Lou & Shen (2003) generalizes that of Shu et al. (2000) and describes exact perturbation solutions in a composite SID system. We obtained marginal stability curves for axisymmetric perturbations that are qualitatively similar to those of Shu et al. (2000) and that depend on additional dimensionless parameters. It is natural to extend the interpretations of Shu et al. (2000). Through the WKBJ analysis here, we now firmly establish the presence of two regimes of “ring fragmentation” and “collapse” in a composite SID system. It is fairly straightforward to apply our exact $D$-criterion for a composite SID system, not only for disc galaxies, but also for other disc systems including circumnuclear discs and protostellar discs etc.

Thirdly, in the WKBJ approximation, it is also possible to relate our $D$-criterion to the $Q_E$ criterion similar to that of Elmegreen (1995). Because $\kappa_s$ and $\kappa_g$ are different in general, our WKBJ treatment is not the same as that of Elmegreen (1995).

Fourthly, also in the WKBJ approximation, we relate our $D$-criterion to the $Q_J$ criterion as defined by Jog (1996) but with $\kappa_s \neq \kappa_g$ in general. We find that the $Q_J$ criterion is fairly robust in the WKBJ regime.

Finally, we further consider the axisymmetric stability of a composite partial SID system to include the gravitational effect from an axisymmetric dark matter halo. The stabilizing effect of the dark matter halo is apparent.

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