Wave localization is a ubiquitous phenomenon. It refers to situations that transmitted waves in scattering media are trapped in space and remain confined in the vicinity of the initial site until dissipated. Based on a scaling analysis, the localization behavior in two and three dimensions is studied. It is shown that the localization transition is possible in two dimensional systems, supporting the recent numerical results.

Under proper conditions, multiple scattering leads to the unusual phenomenon of wave localization, a concept introduced by Anderson [5] to explain the metal-insulator transition induced by disorders in electronic systems and recently reviewed by Imada et al. [6]. That is, the electronic movement can be completely stopped due to multiple scattering by a sufficient amount of impurities in solids. It is believed that once the electronic movement is stopped, the electrons are trapped in space. The fact that this effect due to the wave nature of electrons has led to the conjecture that similar phenomena may also exist in the propagation of classical waves in randomly scattering media.

Considerable efforts have been devoted to the investigation of classical wave localization in random media. In most previous experimental studies, the apparatus is set up in such a way that waves are transmitted at one end of the scattering sample, then the scattered waves are recorded either on the other end to measure the transmission or are received at the transmitting site to measure the reflection from the sample. In either case, the measurement was done when both the transmitter and receiver are located outside the sample. The results are subsequently compared with the theory developed for classical wave localization [3] to infer the possible localization effect. In this way, observations of wave localization effects have been reported for water wave localization by random underwater topography [7], for acoustic waves [8], microwaves [9,10], and electrical resistivity [11]. An excellent account of multiple scattering was given in [12].

When propagating through a medium with many scatterers, waves will be repeatedly scattered to establish a process of multiple scattering of waves. It is now well known that multiple scattering gives rise to many fascinating phenomena, including the photonic or sonic band gaps in periodic structures [3], random lasers [13], and electrical resistivity [14]. An excellent account of multiple scattering was given in [15].

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In these measurements, two phenomena are thought as the indicator of localization effects. The first is the enhanced backscattering. As much discussed in the literature (e.g. Refs. [3,4,13]), the wave received at any spatial point is contributed by wave propagated along various paths. For the bistatic case, the random scattering leads to a destructive interference of scattered waves, thus reducing the transmission. For the backscattering situation, however, any random scattering path that returns to the transmitting source can always be followed by two opposite directions. The waves which propagate in the two opposite directions along a loop will acquire the same phase and therefore interfere constructively at the transmitting site, yielding the phenomenon of the enhanced backscattering. The second indicator is associated with the relation between the wave transmission and the sample size. The theory [13] predicts that once localization occurs, the wave transmission is expected to undergo a transition from a linear to quadratic decreasing, and eventually to follow an exponential decay. In line with the theory [13], it has been the prevailing view that all waves are localized in two dimensional (2D) systems with any amount of disorders.

The recent numerical simulation, however, shows that waves are not always localized in 2D random systems [17,18]. The work is done with acoustic propagation in water containing many randomly placed air-filled cylinders, by an exact method for multiple scattering. Unlike most previous cases, the numerical simulation has been done by placing an acoustic source inside the random array. The acoustic transmission for various frequencies is recorded by a receiver located outside the scattering array. It is found that while in a range of frequencies and for a sufficient amount of the air cylinders, the transmitted acoustic waves are indeed trapped or localized inside the random medium, the waves remain extended outside the localized regime.

An immediate criticism on this observation may be that the apparent state of wave propagation is an artifact of the finite size effect. In other words, it may be argued that the localization length exceeds the size of the scattering medium, thus the transmitted waves appear to be non-localized. Although it is true that it is impossible to simulate an infinite scattering medium, the observed phase transition between localized and extended wave transmission is not caused by the finite size effect [19]. Two main reasons support this viewpoint.
One, according to the analytic results [19], the localization length scales as $L \exp(L/\lambda)$, where $L$ is the mean free path of the scattering and $\lambda$ is the wavelength. If waves were localized for all frequencies, the theoretical calculation would lead to a smooth variation in the localization length. This is not observed. Second, as pointed out in [19], the wave localization would necessarily lead to a phase ordering. This can be understood as follows. In terms of wave field, $u$, the energy flow may be calculated from $J(\vec{r}) \sim i\text{Re}[u(\vec{r})\hat{\nabla}u(\vec{r})]$. Writing $u = |u|e^{i\theta}$, the flow becomes $\vec{J} \sim |u|^2\hat{\nabla}\theta$. Obviously, the energy will be localized when phase $\theta$ is constant (coherence) and $|u| \neq 0$. Vice versa, the vanishing energy flow should lead to the phase ordering. In the simulation in [17], as the sample size is enlarged, there is no tendency in the phase coherence for frequencies outside the localization regime. In spite of these, the fact that the results are numerical is discomforting.

Before moving on, we point out a few interesting properties associated with the acoustic scattering by parallel air-cylinders in water. (1) The air-cylinders in water are strong acoustic scatter due to the large contrast in the acoustic impedance. At low frequencies, there appears a resonant scattering. (2) In a wide range of frequencies above the resonance, the scattering is nearly isotropic, in the way that the backward scattering strength is not negligible compared to the forward scattered strength. In most previous cases, the backscattered wave is neglected. This approximation is only valid for weak tenuous scattering [21]. (3) Though complicated, the scattering by an array of many air-cylinders allows for an exact formulation and can be evaluated to desired degrees of accuracy [17,21]. And it is fair to mention that the formulation has been applied successfully to inspect the recent experiments [22]. (4) Experimentally, the air-filled cylinders can be any gas enclosure with a thin insignificant elastic shell. In short, these properties make air-cylinders in water an ideal 2D system for theoretical and experimental localization studies.

Now we analyze the acoustic localization in water with many air-cylinders. Upon inspection, we believe that the evident contradiction with the previous claim is due to the difference in the ways that the wave localization is inferred or interpreted. It has been thought that enhanced backscattering is a precursor to localization. Our numerical results shows, however, that there is no direct link between backscattering enhancement and localization [24]. Furthermore, as mentioned above, most previous measurements are performed when both transmitter and receiver are located outside the scattering medium. The study of whether the waves are localized or extended is obscured by boundary effects such the reflection and deflection effects. These effects attenuate waves, resulting in possibly an exponential decay in transmission and thus making the data interpretation ambiguous. It is highly plausible that the inhibition in the wave transmission does not necessarily guarantee that the wave can be actually trapped in the medium once the transmitting source is moved into the randomly scattering medium. In other words, it is necessary to differentiate the situation that the wave is blocked from transmission from the situation that wave can be actually localized in the medium; we believe that the latter case is in fact what the concept of wave localization is meant to be [24]. We stress that whether waves are localized or extended is an intrinsically property of the system that is supposed to be infinite. This property does not depend on the source, and should not depend on the boundary either: thus a genuine analysis should not be plagued by boundary effects not only
in the localization region but also in the non-localization region. We believe that while the source is placed inside the medium with increasing sizes, the infinite system can be mimicked and the localization property can be probed without ambiguity.

To discern the observation in [17], we adopt a scaling analysis by analogy with that presented in [1]. Consider that a transmitting source is inside a homogeneous random medium. To account for the fact that the source is inside the medium, we take the geometry as shown in Fig. 1. We consider the cylindrical and spherical scaling for 2D and 3D respectively. In line with the discussion in [7], for small resistance \( R \) the medium is assumed to follow the ohmic behavior. This leads to

\[
R \sim \begin{cases} 
\frac{L}{L_0}, & \text{for } 1D \\
\ln(L/L_0), & \text{for } 2D, \\
\frac{1}{L_0} - \frac{1}{L}, & \text{for } 3D.
\end{cases}
\]

where \( L_0 \) refers to the microscopic size [7]. This is valid when \( R \) is small. In the other limit that the resistance is large, exponential wave localization is expected. By taking into the geometric factors, the resistance thus grows as

\[
R \sim L^{d-1} e^{L/L_1},
\]

where \( d \) denotes the dimension and \( L_1 \) is the localization length.

The scaling function is defined as

\[
\beta(R(L)) = \frac{d \ln R}{d \ln L}.
\]

Taking into account Eqs. (1) and (3), the asymptotic behavior for \( \beta \) in one dimension is

\[
\beta \sim \begin{cases} 
1, & \text{for } \ln(R) \to -\infty \\
\ln(R), & \text{for } \ln(R) \to \infty
\end{cases}
\]

It is clear from this equation, the localization behavior is the same as that predicted in [3]. We can also obtain the asymptotic behavior for the scaling function in both 2D or 3D as

\[
\beta \sim \begin{cases} 
e^{-\ln(R)}, & \text{for } \ln(R) \to -\infty \\
\ln(R), & \text{for } \ln(R) \to \infty
\end{cases}
\]

This equation indicates that the wave localization behavior in 2D and 3D should be similar. What is expected for 3D may also appear in 2D.

Equation (4) is the basis for our discussion. From the asymptotic behavior in Eqs. (4) and (3), we may sketch the universal curves in \( d = 1, 2, 3 \) dimensions. The central assumption here is continuity [7]: once wave is localized, the increasing sample size would always mean more localization. This assumption has been discussed in some detail in [7]. It is obvious that the 1D situation is a replicate of that shown in [7]. The result is that all waves are localized in one dimension for any given amount of disorders.

The situations in two or three dimensions are more subtle. Two possibilities are shown in Fig. 2. In the first instance shown by Fig. 2(a), as \( \ln(R) \) increases, the scaling function \( \beta \) may decrease, then crosses the horizontal axis and reaches a minimum before increasing to follow the linear relation for large \( \ln(R) \). The crossing of the horizontal axis produces two fixed points: A and B. At both points, \( \beta \) vanishes. It is clear that A and B are respectively the stable and unstable fixed points. Point B separates the localization state and the extended state. When \( \ln(R) \) is greater than B, the increasing sample size leads to an infinite resistance, thus the waves become localized inside the medium. When \( \ln(R) \) is below point B, increasing sample size leads the system to the fixed point A, at which the increasing \( L \) will no longer affect the resistance. On the first sight, this feature seems awkward. After inspection, it becomes clear that it is actually a clear indicator of a wave propagating state, i.e., the extended state. This can be understood as follows. As the transmitted wave propagates, the wave coherence starts to decrease, yielding the way to incoherence. The total wave is the addition of the coherence and incoherence waves [4]. When there is no absorption, by energy conservation the total wave transmission, an appropriation of the inverse of the resistance, will not change along the propagation path. The transmission will thus not vary as the sample size changes. Therefore the feature at point A actually reflects the law of energy conservation. This picture has indeed been supported by the previous simulations [17,23].

FIG. 2. Plots of \( \beta \) vs \( \ln(R) \) for 1, 2, and 3 dimensions.

FIG. 1. We consider the cylindrical and spherical scaling for 2D or 3D respectively.
The second possibility is shown in Fig. 2(b). The scaling function $\beta$ will not drop below the horizontal axis. In this case, all waves in two and three dimensions are localized like in the 1D situation. For example, this is expected to occur when the amount of disorders is exceedingly large [25,26]. Previous results affirming that all waves are localized in 2D random media may fit in this possibility.

Based upon the above scaling analysis, we argue that the observation of wave localization in 2D reported in [17] follows the behavior illustrated by Fig. 2(a). There are critical points separating the localized state from the extended state. When waves are localized in the medium, the waves follow the exponential localization, as clearly shown by Fig. 3 in [17]. Outside the localized regime, the waves remain extended in space. The averaged transmission consequently is nearly constant along the traveling path in the radial direction. Therefore the observation of [17] finds the explanation.

In summary, we have shown a scaling analysis of wave localization in randomly scattering media. We pointed out that the differentiation should be made with respect to whether the transmitting source is placed inside or outside the random media. The problem with the latter is that the effects from other sources such as boundary reflection, scattering into other directions cannot be excluded. These effects may result in phenomena which could have been attributed incorrectly to the localization effect. In the analysis when the source is outside the medium [1], the asymptotic behavior in the Ohmic region was derived under the assumption that the current flows uniformly in one direction. This is only possible with properly scaled sources and the presence of confining boundaries, obviously in conflict with the proclamation that whether it is the localization or extended state is the intrinsic property of the medium and should not rely on a boundary nor the source. The vagueness is avoided when the source is inside the medium. The present analysis shows that wave localizations in 2D and 3D random systems are similar. The transition from the propagating state to the localized state is possible in both two and three dimensions. Finally, we note that recent experiments on electronic systems also suggest a metal-insulator transition in two dimensions in contrast to the previous assertion; the mechanism in these systems is still unclear.

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