The Winnability of Klondike Solitaire and Many Other Patience Games

Charlie Blake  thecharlieblake@gmail.com
Work undertaken while at School of Computer Science, University of St Andrews, St Andrews, UK
Ian P. Gent  ian.gent@st-andrews.ac.uk
School of Computer Science, University of St Andrews, St Andrews, UK (Corresponding Author)

Abstract

Our ignorance of the winnability percentage of the solitaire card game ‘Klondike’ has been described as “one of the embarrassments of applied mathematics” by Yan, Diaconis, Rusmevichientong, and Roy. Klondike, the game in the Windows Solitaire program, is just one of many single-player card games, generically called ‘patience’ or ‘solitaire’ games, for which players have long wanted to know how likely a particular game is to be winnable. A number of different games have been studied empirically in the academic literature and by non-academic enthusiasts. Here we show that a single general purpose Artificial Intelligence program named ‘Solvitaire’ can be used to determine the winnability percentage of 73 variants of 35 different single-player card games with a 95% confidence interval of ± 0.1% or better. For example, we report the winnability of Klondike as 81.945% ± 0.084% (in the ‘thoughtful’ variant where the player knows the rank and suit of all cards), a 30-fold reduction in confidence interval over the best previous result. The vast majority of our results are either entirely new or represent significant improvements on previous knowledge.

Authors’ Note: An earlier version of this paper was put on arXiv in June 2019. This version is significantly extended with new research and much greater detail given in several areas. All statements are correct as of January 2023, to the best of our belief.

1. Introduction

Patience games - single-player card games also known as ‘solitaire’ games1 - have been a popular pastime for more than 200 years (Ross & Healey, 1963). This popularity continues, with Microsoft Windows Solitaire – just one implementation of one patience game – being played 100 million times per day in 2020 (Jensen, 2020). We compute winnability percentages on random instances of many single-deck patience games using a general solver named ‘Solvitaire’. Almost all our results are either entirely new or significant improvements on previous knowledge. Where results were previously known, they were obtained using solvers specific to a particular game or small family of games. In contrast, Solvitaire solves a wide variety of patience games expressible in our flexible rule-description language. Based on depth-first backtracking search, it exploits a number of techniques to improve efficiency: transposition tables (Greenblatt, Eastlake, & Crocker, 1967; Smith, 2005), symmetry (Gent, Petrie, & Puget, 2006), dominances (Chu & Stuckey, 2015), and streamliners (Gomes & Sellmann, 2004; Wetter, Akgün, & Miguel, 2015).

Klondike,2 the game in Windows Solitaire, is just one example of hundreds of patience games that exist (Parlett, 1980). Understanding the range of games available requires understanding some key terminology: we give a very concise introduction in Section 2.

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1. Herein we use the word ‘patience’ as the traditional word in UK English while ‘solitaire’ is the US usage (Ross & Healey, 1963).
2. In the main text of this paper, we distinguish names of games by writing them in italics, e.g. Klondike.
The probability of winning has always been of interest to players, with advice published as to how likely a given game is to be winnable at least as long ago as 1890 (Cavendish, 1890). In this paper, we study 81 variants of about 40 different patience games. Not knowing the winnability of just one of these games, Klondike, has been called “one of the embarrassments of applied mathematics” (Yan et al., 2005). Only for a very small number of games, e.g. FreeCell (Fish, 2018), has this probability previously been known to a high degree of accuracy. For games with hidden cards, we follow standard practice in the literature of considering the ‘thoughtful’ variant (Yan et al., 2005), in which the ranks and suits of hidden cards are known to the player at the start of the game.

We are now able to report the winnability percentage of thoughtful Klondike and dozens of other games with a 95\% confidence interval within ±0.1\%. Remarkably, we achieve this with a solver which can be used for a very wide variety of games and is not highly optimised for any particular one. Our rule-sets and solver are flexible enough to include famous games such as Klondike, Canfield, FreeCell, Spider, Golf, Accordion, Black Hole and King Albert, all of which are very different from each other. We are not aware of any previous solver which can be used unchanged on any two of these games.

2. Terminology of Patience Games

Because terminology of patience games is not always the same in different sources, we describe very briefly the key terms we use in describing patience games in this paper. Much longer introductions to patience games can easily be found (Parlett, 1980). We use the word ‘game’ to refer to a particular set of rules for playing patience. A game is played with a number of complete ‘decks’ of cards, normally the standard deck with 13 cards of each of 4 suits. The rules of a game specify how the cards are placed before play starts: in the initial position some cards may be ‘hidden’ from the player, for example by being placed ‘face-down’. In this paper we follow previous work in studying ‘thoughtful’ variants of a game where the ranks and suits of hidden cards are known. With physical cards, the thoughtful variation is like the player peeking at each hidden card to see what it is. Electronic implementations with unlimited undos also become thoughtful, because the player can always go back to the start after finding any information they need in the game. We use the word ‘instance’ of a game to refer a particular arrangement of cards for that game, usually after random shuffling. Most games are won by rearranging cards so as to place them in order on a set of ‘foundations’, typically from A to K within each suit: in some cases the player is given some cards already placed on the foundation as a starter. In some games the goal instead is to move cards into a single ‘hole’, with consecutive cards required to be adjacent in rank but with no regard to suit: games vary whether one is allowed to loop round from K to A and vice versa. In some games, like Spider, cards are not built to individually but eliminated from the tableau when a complete sequence from A to K in a single suit has been constructed. An instance of a game is ‘winnable’ if there is any legal sequence of moves that leads to the goal predetermined by the rules of the game. In most games the main area of play is called the ‘tableau’. Cards can often be moved within the tableau: this is called ‘building’ one card onto another pile. In the scope of this paper, the card must be one lower in rank than the card it is placed on (with K considered one lower than A if appropriate). The ‘build policy’ determines additional rules: to be built on a card may need to be the same suit as the higher card, or of a suit of the opposite colour, or it may be allowed to be any suit. A sequence of consecutive built cards may be allowed to be moved together as one ‘group’: where allowed this may be with same restriction as the build policy, or sometimes a stricter
restriction that the group must all be the same suit. We refer to the number of tableau piles and their sizes at the start of the game as the ‘layout’. Typically a face-down card in the tableau is turned ‘face-up’ only when the card immediately covering it is moved. When a tableau pile becomes empty it is called a ‘space’: some games allow cards to be placed in spaces; sometimes the card placed in the space must be a K and in other games any card is allowed. In some games cards may be ‘worried back’: this means cards can be moved from a foundation to the tableau. Some games contain an ordered ‘stock’ of cards: often the player is allowed to ‘draw’ a given number of cards at a time. Most often stock cards are moved to a ‘waste’ pile, from which the top card can then be played to the tableau, while in others one card is dealt onto each tableau pile. Some games allow ‘redeals’, where the waste pile may be reused to form the stock again. Some games have a ‘reserve’ of cards which can be played onto the tableau or foundations but otherwise are static. ‘Free cells’ function like a reserve but cards may be moved from the tableau into free cells as well as in the other direction.

As an example we can now describe the game Klondike - the game in the Windows Solitaire program. A single standard deck is used and the goal is to build all cards on foundations in suit from A to K. The game begins with a tableau of 28 cards in a triangular form with piles from 1 to 7 cards, with all but the top card face-down. Face-up cards on the tableau may be built in alternating colour, and built groups may be moved. Face-down cards may not be moved. Spaces may be filled only by a K. Cards may be worried back from foundations to tableau. A stock of 24 cards may be drawn in groups of three, and redeals are allowed without limit. This description may be compared with Table 7. These rules are given to Solvitaire in a JSON format shown in Table 5. As another example, the full set of rules for Streets and Alleys is shown in Table 6: these also serve as the default value for any field not otherwise specified in a game’s JSON specification. For fuller description of our rules language, see Section 6.

Names of particular patience games are even more confused than the terminology for rules, with different names used for the same game and the same name used for different games. For example, the game we call Klondike in this paper is often just called ‘Patience’ (Parlett, 1980) or ‘Solitaire’ which are also names for the general family of single-player card games. Worse than that, Klondike can also be called ‘Canfield’ which is the name we use here for a completely different game. Both games have many other names, for example both being sometimes called ‘Demon’. Unfortunately this means we sometimes do not know what game is being referred to in historical documents: for example Stanislaw Ulam may have been referring to either game when he wrote that ‘Canfield Solitaire’ motivated his invention of Monte Carlo methods (Eckhardt, 1987). It is therefore particularly important for us to be clear on the name and rules we use for each game. We provide a concise summary of rules we used of most games studied in this paper in Table 7. Almost all games we studied can be described in this way, including all games for which we give the first reported results. The exceptional games that cannot be described in this framework are Accordion (BVS Development Corporation, 2003) and its variant with 18 cards (called ‘Late-Binding Solitaire’ by its originator) (Ross & Knuth, 1989), the two variants of Gaps (Helmstetter & Cazenave, 2004): their rules can be found in the papers just cited and are shown in our JSON format in Table 8.

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3. This prohibition means that thoughtful Klondike is subtly different from a variant in which all cards start face-up.
4. ‘Demon’ was the name used by Ian Gent’s mother for Canfield and by his father for Klondike.
3. History of Solving Patience Games

The winnability of patience games has interested people for many years, with many books on the topic providing estimates of how often each game can be won. In some cases, an expert’s views were astonishingly accurate: in the nineteenth century Cavendish (1890) said that the game Fan “with careful play, is slightly against the player”, while we show that it is \( 48.776\% \pm 0.099\% \) winnable.\(^5\) Other stated claims have been very inaccurate: (British) Canister was described by Parlett (1980) as “odds in favour”, while Table 2 shows that only slightly more than one in a million games are winnable. Distinguished scientists who have taken an interest in the question include Stanislaw Ulam, the inventor of computer-based Monte Carlo Methods, (Eckhardt, 1987), Donald Knuth, a Turing Award winner (Ross & Knuth, 1989), and Irving Kaplansky, a President of the American Mathematical Society (Mackenzie & Graham, 2019).

There are some patience games where there is no player choice required and the pleasure of the game is the purely mechanical playing out of the game. We do not pay attention to such games in this paper, but some solvability percentages have been calculated. For example, Clock Patience is provably won exactly \( \frac{1}{13} \) of the time (Jenkyns & Muller, 1981). Monte Carlo methods have shown Perpetual Motion to have a winnability of \( 8.6692 \pm 0.0017\% \) while superficially minor changes to the rules can increase this to \( 54.8033 \pm 0.0031\% \) winnability (Masten, 2022b; Clarke, 2009).

When Microsoft released one of the early versions of FreeCell, it included 32,000 different instances. It was conjectured that all were winnable, leading to an early example of internet crowdsourcing, the ‘Internet FreeCell Project’ led by Dave Ring in 1994-5 (Plante, 2012). People shared their solutions online for all instances except one, deal number 11982, which nobody could solve. This is now known to be unwinnable (Keller, 2015). At a similar time, Don Woods obtained an estimate of 99.999\% winnability for FreeCell from a computer study of a million random instances (Keller, 2015).\(^6\)

Since then, more computational experiments have given winnability estimates for a variety of games. These have been done both inside and outside the academic community. Some games have attracted academic attention, including Klondike (Bjarnason, Fern, & Tadepalli, 2009; Bjarnason, Tadepalli, & Fern, 2007; Yan et al., 2005), FreeCell (Paul & Helmert, 2016; Elyasaf, Hauptman, & Sipper, 2012; Dunphy & Heywood, 2003), Gaps (Helmstetter & Cazenave, 2004), King Albert (Roscoe, 2016) and Black Hole (Gent, Jefferson, Kelsey, Lynce, Miguel, Nightingale, Smith, & Tarim, 2007; Smith, 2005). For most patience games, however, the best known winnability estimates have been obtained by enthusiasts rather than academic or industrial researchers. Of games just mentioned, this includes FreeCell (Fish, 2018), and included Klondike (Birrell, 2017) and Black Hole (Fish, 2010) until the current paper. We are not aware of any academic predecessor to our work which studied a diverse range of patience games, but there have been substantial efforts across a range of games by enthusiasts including Shlomi Fish, Mark Masten (2022c), and Jan Wolter (2013b), among others. In summary, the world has owed far more to non-academic than academic research in knowing the winnability of patience games. We have used ideas from both academic and non-academic researchers. For example, for Klondike and Canfield we made essential use of both the \( K^+ \) representation of stock from academic research (Bjarnason et al., 2007) and the dominance described in Section 7.3 from non-academic research (Wolter, 2014d; Birrell, 2017). Note, however, that this pa-

\(^5\) We studied a very minor variant of Cavendish’s game, with sixteen piles of three and two piles of two instead of seventeen piles of three and one of one.

\(^6\) This is a pleasing example of a case where ‘99.999\%’ is not hyperbole for ‘almost always’ but is the scientifically established value to 5 significant figures.
per is not intended to give a complete survey of either academic or non-academic work on patience games.

We close this brief history with a remarkable echo in our work of the origin of computer-based Monte Carlo methods, which are the method we use to compute winnability estimates throughout this paper. Monte Carlo methods were actually invented by Stanislaw Ulam with the idea of calculating the winnability of solitaire games, as he recalled:

*The first thoughts and attempts I made to practice [the Monte Carlo method] were suggested by a question which occurred to me in 1946 as I was convalescing from an illness and playing solitaires. The question was what are the chances that a Canfield laid out with 52 cards will come out successfully? After spending a lot of time trying to estimate them by pure combinatorial calculations, I wondered whether a more practical method than “abstract thinking” might not be to lay it out say one hundred times and simply observe and count the number of successful plays. This was already possible to envisage with the beginning of the new era of fast computers, and I immediately thought of problems of neutron diffusion and other questions of mathematical physics, and more generally how to change processes described by certain differential equations into an equivalent form interpretable as a succession of random operations.*

– Stanislaw Ulam, unpublished remarks 1983, quoted by Eckhardt (1987).

In this paper, we therefore achieve the idea of Ulam, giving a very precise estimate of the winnability of the solitaire he was playing using precisely the Monte Carlo methods he invented to achieve this. As discussed above, we do not know whether Ulam’s ‘Canfield’ was the game we call Klondike or Canfield. Whichever it may be, in this paper we have reduced the uncertainty of its winnability by a factor of more than 30 over the previous best estimate and to within ±0.1%.

4. Results Summary

We have experimented on numerous patience games. Our results fall into three categories: those for games already studied, those for main games which have not been studied before, and finally an extensive investigation into how varying the rules of Klondike affects winnability. We provide summary of our winnability estimates for each game in the following three subsections. Details of how these results were obtained occupy the bulk of the rest of this paper. Our focus in this paper has been on winnability, rather than accurate measures of time used for benchmarking purposes. However, we provide summary data of time used and nodes searched in the experiments reported in this paper in Table 11, page 40. Overall, the experiments reported in this paper used about 30 years of CPU-time.

4.1 Results for Previously Studied Games

Solvitaire is able to solve a wide range of previously-researched games, although we have not extended it to be able to search every game that has already been studied. Results are shown in Table 1, page 6. In most cases we improve on previous results, and in some very famous games the improvements are dramatic. For example, we have improved the 95% confidence interval for both Klondike and Canfield by a factor of 30 over the previous best known results. We have also used Solvitaire to identify bugs in previous solvers for those two games: see Section 7.2. All results except for Gaps (One Deal), Spider, and two variants of Klondike, have a 95% confidence interval within ±0.1%, and this is the first
Table 1: Comparison with previous work using a consistent methodology for calculating 95% confidence intervals (CI) described in Section 9.2. For numbers used for calculation of 95% CI values, see Table 11 (for Solitaire) and Table 9 (for data from the literature). State-of-the-art results for each game are in bold. Italics indicate results from the literature open to doubt, see accompanying note. [Th.] Thoughtful variant where position of all cards known at start.

| Game                | Variant | Solitaire 95% CI | Best Other 95% CI | Citation                          |
|---------------------|---------|------------------|-------------------|----------------------------------|
| Accordion           | [Th.]   | 99.9948 ± 0.0052% | 99.9999 ± 0.0064% | Masten (2022c)                   |
| Baker’s Game        |         | 95.053 ± 0.028%  | 75.011 ± 0.028%   | Pringle (2017, 2018)             |
| Black Hole          |         | 86.944 ± 0.022%  | 86.986 ± 0.053%   | Masten (2022c)                   |
| Canfield            | [Th.]   | 71.245 ± 0.031%  | 71.872 ± 1.059%   | Wolter (2013b)                   |
| Eight Off           |         | 99.880 ± 0.002%  | 99.8801 ± 0.0010% | Masten (2022c)                   |
| Fore Cell           |         | 85.617 ± 0.024%  | 85.605 ± 0.385%   | Keller (2015)                    |
| Gaps                | One Deal| 85.815 ± 3.717%  | 89.310 ± 9.124%   | Helmstetter and Cazenave (2004)  |
| Golf                |         | 45.109 ± 0.032%  | 45.077 ± 0.309%   | Wolter (2013b)                   |
| Klondike            | [Th.]   | 68.542 ± 0.092%  | 71.189 ± 8.678%   | Roscoe (2016, 2019)              |
| Late-Binding Solitaire |       | 47.021 ± 0.032%  | 45.418 ± 3.081%   | Ross & Knuth (1989)              |
| Simple Simon        |         | 89.322 ± 0.020%  | 89.319 ± 0.016%   | Masten (2022c)                   |
| Spider              | [Th.]   | 97.450 ± 0.034%  | 94.910 ± 5.090%   | Fish (2009)                      |
| Thirty Six          | [Th.]   | 94.674 ± 0.100%  | 94.488 ± 0.307%   | Wolter (2013b)                   |
| Trigon              |         | 15.996 ± 0.023%  | 16.008 ± 0.073%   | Wolter (2013b)                   |
| Worm Hole           |         | 99.886 ± 0.0074% | 99.8906 ± 0.0065% | Masten (2022c)                   |

Note: For discussion of Wolter’s code, see Section 7.2
Note: Using a solver by Shlomi Fish
Note: Michael Keller reports results obtained by Danny A. Jones
Note: For Basic Variant, raw results unstated in cited paper.

Note: For discussion of Birrell’s code, see Section 7.2
Note: Using a solver created by Don Woods
Note: Literature results mainly computer-solved but some human-solved

Time this has been achieved for ten of these games. There are games where Solitaire is not at good as existing solvers, as we discuss further in Section 10.
4.2 Results Only Obtained Using Solvitaire

The second class of results is those on which Solvitaire is responsible for the only good estimate of winnability that we know of. For new results, we have limited our presentation of results to those for which we can give a very small confidence interval. In Table 2, we give results for 20 games we experimented on ourselves, including some variants that we invented for the purposes of this paper to illustrate the flexibility of our rule language. Most games we give new results for are single-deck games, but we do report a good estimate for the two-deck game Mrs Mop. Additionally, Table 2 shows results for two variants of Carpet, for which the JSON rules were constructed and experiments performed by Masten (2022a).

All but one of the results shown in Table 2 have a 95% confidence interval within \( \pm 0.1\% \): the exception is Streets and Alleys, for which the number of unknown results limited us to \( \pm 0.2\% \).

One interesting game not included in Table 2 is one we invented based on Parlett’s (1980) game Black Hole with the addition of one freecell: we call the game “Worm Hole”. Using Solvitaire, we gave the first good estimate of winnability in an earlier version of our paper, but these results have now been improved on by (Masten, 2022d), as shown in Table 1. Interestingly, that improvements comes from the Masten’s use of a game-specific dominance we were not aware of.

Among the games we study is a stricter variant of thoughtful Canfield (invented for this paper) in which moves of partial piles are not allowed: our results show that about 3.7% of games are winnable with the weaker rules but cannot be won with the stronger.

4.3 Results on Variants of Klondike and Freecell

As well as their general comment on the embarrassment of not knowing the winnability of Klondike, Yan et al. (2005) also commented that “simple questions such as … How does this chance depend on the version I play? remain beyond mathematical analysis”. Solvitaire’s excellent performance and flexible rule-language gives an ideal framework to study this question. We studied a number of variants of the rules of Klondike to investigate how winnability of the thoughtful game was affected. As with our general results, we undertook both replications/improvements and new studies.

As a replication, we also experimented on a number of variants of FreeCell that have previously been experimented on, with results in Table 1, page 6. All results are consistent with previous work, with overlapping estimates of confidence interval. Several are significant improvements on knowledge. Table 1 also compares our own results with Birrell’s (2017) reported results for Klondike with varying draw sizes, and our results represent significant improvements.

For new studies we performed an extensive study of variants of Klondike. An important aspect of this study was to reuse results for one game on related games, as described below in Section 8: this greatly reduces the time needed to conduct such large sets of experiments.

Table 3 and Figure 1 show the results on varying numbers of draw size combined with whether or not ‘worrying back’ is allowed. As well as seeing the decline of solvability with increasing draw size, we also see the increasing importance of worrying back. By the largest draw size more than 9% of winnable games require it at least once. This is to be expected, as the reduced percentage winnability correlates with fewer routes to win, meaning that there are fewer ways to avoid worrying back.

We also experimented on varying some of the core rules of Klondike, specifically what is allowed to be placed in spaces and which suits are allowed for building piles. Table 4
| Game                                      | Confidence Interval                                                      |
|------------------------------------------|-------------------------------------------------------------------------|
|                                          | Percentage Range                                                       |
| Alpha Star                               | 47.794% ± 0.032%                                                        |
| American Canister                        | 5.606% ± 0.015%                                                         |
| Beleaguered Castle                       | 68.170% ± 0.099%                                                        |
| British Canister                         | 0.000129% ± 0.000008%                                                   |
| Canfield (Whole Pile Moves) [Th.]        | 67.562% ± 0.034%                                                        |
| Carpet [Th.][†]                          | 87.558% ± 0.021%                                                        |
| – “” – (Pre-founded Aces) †[Th.]         | 95.186% ± 0.014%                                                        |
| Delta Star                               | 34.413% ± 0.030%                                                        |
| East Haven [Th.]                         | 82.844% ± 0.100%                                                        |
| Fan                                      | 48.776% ± 0.099%                                                        |
| Fortune’s Favor [Th.]                    | 99.9999879% ± 0.0000022%                                                |
| Mrs Mop                                   | 97.992% ± 0.079%                                                        |
| Northwest Territory [Th.]                | 68.369% ± 0.094%                                                        |
| Raglan                                   | 81.226% ± 0.085%                                                        |
| Siegecraft                               | 99.136% ± 0.020%                                                        |
| Somerset                                 | 53.725% ± 0.097%                                                        |
| Spanish Patience                         | 99.863% ± 0.003%                                                        |
| Spiderette [Th.]                         | 99.620% ± 0.018%                                                        |
| Streets and Alleys                       | 51.187% ± 0.186%                                                        |
| Stronghold                               | 97.379% ± 0.042%                                                        |
| Thirty                                   | 67.454% ± 0.030%                                                        |
| Will O’ The Wisp [Th.]                   | 99.9240% ± 0.0027%                                                      |

Table 2: Solvability percentage: estimates of 95% confidence interval for patiences which were obtained for the first time using Solvitaire. † Carpet experiments were performed by Masten (2022a) with results shown in Table 9. Other experiments performed by us have results shown in Table 11. [Th.] Thoughtful variant where position of all cards known at start.
shows the results on \textit{Klondike} with draw size 3 and nine combinations of rules. We see that rules can be significantly more effect when combined than individually. For example, from the most liberal rules (top-left), restricting spaces to kings reduces winnability by only 0.068\% and changing the build policy to red-black reduces winnability by 4.964\%. However, combining the two restrictions reduces winnability by 17.978\%.

| Klondike Draw Size | Worrying Back Allowed | Worrying Back Not Allowed | % Critical |
|-------------------|-----------------------|---------------------------|------------|
| 1                 | 90.480±0.116\%        | 90.204±0.093\%            | 0.31\%     |
| 2                 | 88.620±0.135\%        | 88.289±0.112\%            | 0.37\%     |
| 3                 | 81.945±0.084\%        | 81.524±0.089\%            | 0.51\%     |
| 4                 | 69.337±0.098\%        | 68.723±0.095\%            | 0.89\%     |
| 5                 | 53.434±0.099\%        | 52.638±0.099\%            | 1.49\%     |
| 6                 | 35.854±0.095\%        | 34.982±0.094\%            | 2.43\%     |
| 7                 | 23.779±0.084\%        | 22.952±0.083\%            | 3.48\%     |
| 8                 | 12.276±0.065\%        | 11.703±0.064\%            | 4.67\%     |
| 9                 | 7.670±0.053\%         | 7.214±0.051\%             | 5.95\%     |
| 10                | 4.237±0.040\%         | 3.939±0.039\%             | 7.03\%     |
| 11                | 2.066±0.029\%         | 1.904±0.027\%             | 7.84\%     |
| 12                | 0.849±0.019\%         | 0.779±0.018\%             | 8.24\%     |
| 13                | 0.600±0.016\%         | 0.545±0.015\%             | 9.17\%     |

Table 3: Results for \textit{Klondike} with varying draw size and whether or not worrying back is allowed. The final column shows the percentage of winnable games with worrying back that cannot be won without using it least once.
Table 4: Our winnability estimates on variants of standard Klondike with draw size 3 and worrying back allowed. Rules vary on how cards can be built on in the tableau, and what cards if any may be placed into a space. Standard Klondike is in the central cell. The entry in italics for Not Allowed/Any Suit is as computed by our protocol but is not a useful confidence interval.

5. Exhaustive Search using AI Methods in Solvitaire

Solvitaire is a depth-first backtracking search solver over the state space of legal card configurations. We have prioritised the ability to obtain definite answers, i.e. to determine with certainty whether a given instance of a game is winnable or unwinnable. That is, that a key design decision (which has been very successful) was optimising for efficient exhaustive search for unwinnable instances, with less effort devoted to finding solutions quickly. From the initial position, all possible legal moves are constructed and then one chosen for exploration. This is repeated at each new position. If a position is reached where the game has been won, then search is finished. Alternatively, if no legal moves to a new position are possible, then search backtracks to the last parent of this position and tries an alternative move at that parent. If this process eventually exhausts the possibilities for the starting position, then the instance is proven to be unwinnable. In Solvitaire only the moves made are recorded, as opposed to full state representations, and backtracking is achieved by reversing the last move made.

Many aspects of the search procedure need to be improved beyond the basic description above, and we use a number of techniques from Artificial Intelligence (AI) to do so. We describe the key improvements very briefly. We do not claim novelty for these improvements, as many have been applied before to patience solvers, singly and in various combinations (Wolter, 2014d; Birrell, 2017). Their use in combination in such a general solver does seem to be novel. Where appropriate, we adapt the search process as described above to incorporate the additional techniques efficiently.

We use transposition tables (Greenblatt et al., 1967; Smith, 2005) to avoid trying the same position twice. To do this we record every attempted position in a cache. Any position we might consider which is already in the cache can be ignored: its existence in the cache means that it would be potentially explored twice. Some care is needed to ensure that a cache hit correctly links to a previously explored position, so it is important to ensure that a complete game state is stored in the cache. For example, if the cache does not record whether cards in the layout are face-down or not, then obscure bugs can result. We never to need to retrieve any data from the cache except the existence of the state, so to save space we store a compressed representation of the state: this is not a highly optimised representation but is much smaller than the representation used for the active state. A secondary use of transposition tables is to avoid loops, i.e. a sequence of moves which arrives in a state previously visited as a parent of the current node. This actually reduces to the same case as the general one. If the transposition table becomes full, we
discard elements on a least-recently-used basis. The exception is that we never discard any ancestor of the current state, as otherwise loops can occur. If the transposition table is entirely full and all states in it are ancestors, then we give up on search and report that a memory-out has occurred. In extreme cases very large amounts of RAM are necessary, up to hundreds of gigabytes of RAM in some of the hardest problems we solved.

Symmetry in search problems has often been pointed out as an issue which can lead to much redundant search (Gent et al., 2006). That is the case in patience games where we can have equivalent but non-identical positions. A common example in patience games is that all spaces in the tableau are equivalent. We should not waste time trying a card in a second space if it did not work in the first. More subtly, if a sequence of moves precisely swaps two complete piles from an original position, then we should stop search as we have just returned to an equivalent position. We take a simple but effective approach to avoid this problem. Before storing states in a cache we reduce them to a canonical form, maintaining each group of indistinguishable locations such as tableau piles and freecells in a sorted order. For efficiency this order is maintained incrementally during search. Additionally, where a game does not use suits in any way (for example Black Hole) the canonical form can discard suit information for greater reduction.

As well as symmetries, there can be moves that are ‘safe’ to make without any need to consider the other alternatives on backtracking. This happens if we can prove that: if any move at all leads to a win, then there is also a win starting with the safe move. In other words, making such a move is guaranteed not to turn a winnable position into an unwinnable one. A simple example is that if the foundations are built up by suit and tableau moves build down in suit, it is safe to move any card from tableau to foundation whenever it is possible. We call such moves ‘dominances’ (Chu & Stuckey, 2015) as the safe move dominates the others which can be discarded. More complex analyses lead to more subtle definitions of dominance moves for games where cards build down in red-black order. As well as reduction in search space, states containing dominances don’t need to be entered into the transposition table, saving space. As well as committing to safe moves, dominances can also be used to forbid potential moves where we can show that it can be deferred if it needs to be made at all. In Section 7 we discuss dominances in depth - in particular the most widely used dominance, as well as extraordinarily subtle bugs we have found in proposed dominances. We also give a proof of correctness of a dominance that is particularly important in solving Klondike and Canfield effectively.

The final AI technique that we use is ‘streamliners’ (Gomes & Sellmann, 2004; Wetter et al., 2015). A streamliner imposes an additional property which does not necessarily hold in all solutions. A good streamliner is a property that greatly reduces the search space while also having a good chance of leaving at least one solution. If there is a solution found with a streamliner then we have proved the instance is winnable, but if not then we have to start search again without that property holding. We use two streamliners. First, in a game in which cards are moved to foundations, always make such a move when it is possible to do so. This is a very common technique of human players and massively reduces the search space while typically allowing most (but not all) winnable games to be won. Second, we pretend that cards have more symmetry than they do to increase the chance of cache hits. This is very relevant to games which build down in red-black order on the tableau, but up in suits on foundation. If we have a position that differs from a previously visited state only in suits (but not in colours) in the tableau, it is very

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7. Because our focus is on winnability of games, we do not insist that the safe sequences be the same length or shorter, so the dominances we use might not be appropriate in searches for the shortest winning sequence.
Table 5: Rules of Klondike in our JSON format. Note the use of a dominance in moving built groups to limit the available moves, as discussed in Section 7.3. If an unrestricted partial policy was given here, the same instances would be winnable.

unlikely to succeed if the first one does not. Exceptions do occur because of the differences between suits, but again the tradeoff is good for this streamliner. Because we prioritise giving proofs of unwinnability, we allocate 10% of the original time-limit for a streamlined search and if that fails to prove the game winnable, we allocate the original time-limit for a full-search. This is a run-time option since there are games where streamliners cannot possibly help. For many games where it does help, the streamlined search very commonly finds a solution very much faster than the full search would do, leading to greatly improved performance over a large set of instances. While past researchers have certainly used the idea of running a solver which might produce false negatives, thereby speeding up cases where a solution can be found (Fish, 2009), we do not think they have previously been used in the general way we do here.

Although not a general AI technique, we use the K+$^+$ representation of stock (Bjarnason et al., 2007) for patience games with stocks in which there are infinite redeals. This technique makes instantly available any card that could be played through a sequence of individual stock moves, and is an important optimisation in games such as Klondike and Canfield.

6. Configurable Rule-Sets

An important feature of our solver is that games are not hard-wired into the solver. That is, the input to the solver is a description of the rules of the game in a textual format in JSON, (Crockford, 2006) specifying values for different aspects of the game. The rules of Klondike in this format are shown in Table 5. See Section 2 for a description of patience terminology. The complete set of possible fields with their default values (which are the rules for Streets and Alleys) are shown in Table 6. For detailed rules of all games experimented on in this paper, see Appendix A. As well Configurable rule set also enables us to alter the rules of existing games to test how they affect the winnability of the game, as we showed in Section 4.3. However, our rules language does not cater to every possible patience: we were concerned that a much richer language might have made search less efficient. Our chosen tradeoff between expressiveness and complexity enabled us to obtain many new and improved results.
The use of a flexible rule description language gives us two huge advantages over all previous work in the area, which has allowed at most a limited flexibility of game definition within a relatively small family. The most obvious advantage is the wide range of games that can be experimented on without any adaptation at all of the underlying search engine. This can be seen throughout this paper, where we experimented on dozens of very different games, as well as many minor variants of some. Games that we had not considered at all can be tested just by constructing appropriate JSON input: Masten (2022a) did this to use Solvitaire to find the winnability of two variants of (thoughtful) Carpet. The second advantage is that, when we fixed bugs or introduced optimisations for a particular rule, all games using that rule gained the advantage of improved results. For example, the dominance we prove in Section 7.3 has previously been used only in special-purpose solvers for Canfield and Klondike but could be applied without change to Northwest Territory, where it massively improved our ability to solve this game. As well as greater efficiency this enhances robustness of our results, since any remaining bugs for a given rule will have had to escape detection in any game they applied to.

7. Correct and Incorrect Dominances for Patience Games

The use of ‘dominances’ has proven to be important in AI search (Chu & Stuckey, 2015): as an example from 1962, the ‘pure literal’ rule in the classic DPLL algorithm is a dominance (Davis, Logemann, & Loveland, 1962). Although not under the name, the importance of dominances has been recognised by previous researchers, since they can greatly reduce the search space to explore (Keller, 2012; Masten, 2022d; Wolter, 2013a; Birrell, 2017). There are two key cases of dominances in searching patience games. First, a move which we can commit to making in a given situation and therefore avoid backtracking from the choice, because we know that if any solution exists, there is a solution where this move is made next. Second, a possible move which we can decide not to attempt at all, because we know that if that move leads to a solution, there is another way of winning the game without making that move next.

In Section 7.1, we discuss the most commonly used dominance in playing patience games, allowing moves to foundations to be committed to. In Section 7.2, we discuss examples of some very subtle bugs in apparent dominances in both Solvitaire and other solvers we studied. These plausibly look like dominances, but turned out to be incorrect and can lead to the wrong answer being given for winnability of some instances. In Section 7.3, we discuss an important dominance which applies to key games like Klondike and Canfield, and which can greatly improve Solvitaire’s performance. Given its importance and the errors we identified in other dominances, we give a proof of correctness for this dominance.

7.1 Safe Moves To Foundations

In many patience games, the goal is to move cards to the foundations. Beginners often make such moves whenever possible, but this is not always safe. However, an important family of dominances make these moves when it is genuinely safe to do so, and can thus be used to reduce search.

The most typical games build up by suit on the foundations but build down in alternating colour on the tableau. In such games we can automatically move a card to the foundation if it is at most two more than the current card on foundations of the opposite colour and at most three more than the current card on foundation of the other suit of the same colour (Keller, 2015, 2012). For example, if the foundations have been built to 8♠, 7♦, 9♥, 8♠, it is safe to build the 10♥ from tableau to foundation unconditionally.
Table 6: Rules of Streets and Alleys in our JSON format. These are also default values used for any game where that value is unspecified. The fields ‘accordion’ and ‘sequences’ are used for Accordion-like and Gaps-like games respectively.
The only use we could have for the $10♥$ is to put a black 9 on it, which in turn can only
be used to put the $8♦$ on. But all these cards could instead - and preferably - go to the
foundation immediately, so there is no need for them on the tableau and therefore not for
the $10♥$. Following this, it would not be safe to put up the $J♥$ to foundation, because we
might want to keep it to build down $10♠$ and $9♥$.\footnote{It is unclear where this dominance originated, perhaps being invented independently multiple times. Keller (2012) described it as ‘a clear and obvious rule’ and states it was implemented in some of the earliest FreeCell programs.}

If a game does not allow worrying back, then we can use a slightly stronger rule. The
rule as above applies but we can also move to foundation unconditionally if the card is no
more than one higher ranked than the foundations of the opposite colour (Keller, 2012).
The reason is that there are no cards of the opposite colour that can possibly by built in
the foundation onto this card. On the other hand, when a game does allow worrying back,
then we can add a related dominance. We can ban worrying back from foundations to
tableau if the card replaced on the tableau would be eligible for automatic movement to
the tableau under the first dominance: such a move would lead to a pointless loop. While
seemingly minor, this is important as it ensures that progress is not reversed unnecessarily.
This is a slightly stronger and generalised version of a dominance proposed by Bjarnason
et al. (2007) for Klondike.

Similar, but less complex, dominances are available with other building rules than the
standard red-black. If the build policy is by suit, then we can always require cards to be
moved from the tableau to foundations if they can be, since no other card can be built
onto them. If the build policy is that building is regardless of suit, then we can move
a card to foundation if it is no more than two higher than the lowest card yet built to
foundation.

These dominances apply to moving cards from the tableau, as well as from a freecell
or the reserve. However, it is not safe to enforce this dominance from the stock - as we
discuss in Section 7.2. The exception is when the stock draw size is 1 and infinite redeals
of stock are allowed: in this case the stock can be treated as if it were a reserve.

Solvitaire implements all the preceding dominances. We do not offer a proofs of cor-
rectness since they are widely known and believed to be correct. However, all of the
preceding discussion concerns single-deck games. While they are likely to generalise to
multi-deck games, we do not apply them out of an abundance of safety.

7.2 Incorrect Optimisations in Existing Solvers for Klondike and Canfield

We tested Solvitaire’s results on 50,000 instances each against the best existing solvers for
Klondike (Birrell, 2017) and Canfield (Wolter, 2014d). Where both the existing solver and
Solvitaire determine the answer, they should both agree that a given instance is winnable
or unwinnable. Where there was disagreement on a specific instance, we looked at the
solution produced by whichever solver claimed the game was winnable, which we could
check by hand for correctness. In some cases the inconsistency was due to a different
understanding of the rules, in which case we always revised our rules to match those of
the existing solver. Some bugs remained, and where the bug was in Solvitaire we corrected
it, but some bugs were found in existing solvers.

We discovered the same incorrect dominance in both an earlier version of Solvitaire
and in Birrell’s (2017) Klondike Solver. This concerned worrying-back, i.e. returning a
card from foundation to the tableau. It might seem that it would be unnecessary ever
to do this immediately after placing the same card from tableau to foundation, but we
can construct instances in which it is necessary. In one such example, we move the $3♣$. 
to foundation, revealing the previously hidden $4\heartsuit$: the only winning continuation is to reverse this immediately, then move the $2\heartsuit$ onto the $3\heartsuit$, uncovering the $5\clubsuit$ onto which we can now move the pile under $4\heartsuit$. We believe Klondike Solver incorrectly reports six of its first 50,000 random instances to be unwinnable, due to this or other bugs. Given the much smaller samples of 1,000, We do not know if the results reported by Birrell (2017) are affected.

Wolter (2013a), provided the best previous analysis of Canfield giving statistics over 50,000 tests of 35,606 solved, 13,730 proved unsolvable, and 664 indeterminate. The code for his solver is available (Wolter, 2014d). As published, the code gives different results because it implements a rule that only entire columns or the bottom card alone can be moved. This is different from the game rules that Wolter (2014a) gives himself, where partial built piles may be moved instead of just whole columns. A minor change to the published code restores the game to Wolter’s (2014a) rules and after doing this we obtained identical results to those Wolter (2013a) reported. After making this change, we compared results between Wolter’s solver and ours for Canfield. There remained discrepancies which revealed Solvitaire to have both an unintended rule and a separate bug. When these were corrected we still found a small number of different results, which led to the discovery of two obscure bugs in Wolter’s code, arising from an incorrect dominance rule. This was a dominance which forced moves to the foundation be made when an appropriate card was in the last two cards in the stock, because playing these cards could (apparently) never prevent another card being played. Unfortunately, if the number of cards in waste is not a multiple of the number of cards played from stock (typically 3), then immediately playing the last card in stock prevents access to the card at the top of the waste pile, and possibly others. For much more subtle reasons, it is not safe to allow the penultimate card in the stock to be played. While rare, we did see examples of random games where Wolter’s code incorrectly reported winnable games as impossible. For example, in one game in which the base card was $5\diamond$, the stock started $3\spadesuit \ 6\spadesuit \ 6\diamond$ and ended $K\spadesuit \ 7\diamond \ 5\diamond \ Q\clubsuit$. There was no solution if the $5\diamond$ (the second last card in the stock) was played immediately. To win, the player has to wait until the $6\diamond$ and $6\clubsuit$ are both played consecutively. Having delayed the play of $5\diamond$ allows it to be played now, uncovering the $7\diamond$ which can be put on the $6\diamond$. The situation is the curious one that if we have already played $5\diamond$ earlier, then after $6\diamond$ we are able to play either the $7\diamond$ or the $6\clubsuit$ but not both. We believe a very weak version of Wolter’s dominance is correct: when the last card of stock (not the last two) meets the conditions of Section 7.1 and the stock is currently at a multiple of the draw size, the last card can be moved to foundation. We did not implement this in our code.

To correct these two bugs we rewrote Wolter’s code to allow dominance moves only for the last card in stock and only when the number of cards in the waste pile is a multiple of the number of cards played from stock. With these corrections our code does not disagree on any of 50,000 instances we tested. Using this corrected code with the parameters Wolter (2013a) previously used, we obtained 35,605 solved, 13,671 proved unsolvable, and 724 indeterminate instances: if those results had been reported, Table 1 would have a confidence interval of $71.929\% \pm 1.118\%$ for Wolter’s results. This confidence interval did not, at that time, overlap with our results, leading us to investigate closely the pseudorandom generators for both programs. We found flaws in both generators,

9. Curiously, the implemented rule is precisely that given by Parlett (1980), but we do not know whether this was intentional. We cannot check this since Jan Wolter died on 1 January, 2015. We are happy to have this opportunity to pay tribute to him both for his excellent work on solitaire solving programs, and for his openness in making his code publicly available, allowing us to build on his work.

10. Perhaps Wolter corrected the code but never pushed to Google code, or alternatively computed the results before some later code change.
with Wolter’s code producing identical instances on repeated seeds, e.g. the same results for seeds 12 and 1212, and ours a slightly biased sample. We corrected our generator appropriately and our results are now consistent with Wolter’s, as shown in Table 1.

The general point we make in this section is not a criticism of other programmers, but to emphasise the ease with which apparently correct optimisations can in fact be wrong, and to show the difficulty that can arise in locating the errors. Additionally, it shows the power of Solvitaire in being able to run such extensive comparisons with other solvers that it is able to find very rare inconsistencies, and the benefit to other games of fixing bugs found while investigating one game.

7.3 Proof of Correctness of an Existing Dominance

In studying code by Wolter (2014d) for Canfield and Birrell (2017) for Klondike, we noticed an interesting dominance in both. This is that moves of built piles on the tableau are only allowed if either the entire pile is being moved or only a part of a pile is being moved and it is possible to build to the foundation the card above11 the top card in the built pile being moved. Our experiments failed to show any case where this optimisation changed results. Wolter has died and Birrell (2018) did not have a correctness proof. We have not found this optimisation documented in the literature, and its correctness is not obvious, so we give a proof here, we believe for the first time.

We wish to generalise the theorem, to allow its application in cases which don’t use a standard four-suit deck or the common red-black building policy. To do so we assume that the build policy has the property that, given any two cards, the two sets of places those cards can move to by the build policy is either identical or disjoint. We call such a build policy an indistinguishable building policy, since it allows no distinction between two cards which can move to the same place. For example, in the classic games using red-black building by rank, any two cards of different colours or ranks have disjoint cards they can be built on, while two cards of the same colour and rank can be built on exactly the same set of cards. Another indistinguishable policy would be in a game using five identical decks with three suits in which cards must be built in the same suit only: here there would be five copies of e.g. 9♣, each of which could be built on 10♣ but not on 10♦ or 10♥. Despite its flexibility this generalisation still excludes some build policies. An example is ‘different-suit’. This does not meet the condition because both spades and diamonds can move to clubs, while spades but not diamonds can move to diamonds. Counterexamples to the theorem would occur if we allowed this build policy. For example, we might need a group headed by 9♠ to move from 10♣ to 10♦ to allow the 9♦ to move under the 10♣, as it cannot be moved to the 10♦.

Similarly, we also assume that the same build policy controls moving groups and individual cards. Some games, such as Spider, use a policy where individual cards can be moved in any suit but built groups can only be moved if they are all the same suit. This means that moves of groups can be necessary to establish sequences of the same suit, even if the card above is not buildable.

**Theorem 1.** We consider any patience or solitaire game which: has a tableau which builds down according to an ‘indistinguishable’ build policy (as defined above); allows moves of complete or incomplete built piles as a single move according to same policy as for individual cards; the only place a card can move from the tableau is to another tableau pile or to a foundation; is won by moving all cards to the foundations; and contains no rules

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11. For clarity, in the built pile 10♣9♥8♠ we say the 10♣ is above the 9♥ while the 8♠ is below the 9♥. The possible confusion is that 10♣ is placed physically underneath the 9♥ when played on a table.
invalidating moves by constraints on their order in the move sequence. For any instance of such a game, if the instance is winnable with the original rules, then it is also winnable with the restriction that an incomplete built pile may only be moved if the card above the moved partial pile can be built immediately to foundation.

**Proof** The proof will proceed by permuting the move order, e.g. swapping the order of consecutive moves. This is the reason we require that moves are not made invalid by their position in the move sequence: so the proof would not apply to a game where moves to foundation could only be made if the position in the move sequence were divisible by 3.

Consider any winning sequence of moves in the original rules, \( m_1, m_2, \ldots, m_n \), containing at least one invalid move under the restriction. Suppose that the move \( m_i \) is the last move in the sequence which is not valid under the additional restriction. Note that \( i < n \) since the last move in a winning game must be moving a card to the foundation pile. By adapting the moves \( m_i \) and \( m_{i+1} \), we will create a new sequence of moves which is legal and a winning sequence. The new sequence will either have fewer invalid moves than the original, or have the same number but with the last invalid move closer to the end of the sequence than before. This means that by repeated application of the process we will eventually obtain a sequence of moves that wins the game and has zero invalid moves. We define a move \( m \) as being a pair \([c, t]\) where \( c \) is the card being moved and \( t \) is the card or location it is moved to. (We assume some unambiguous notation where cards \( T \) card being moved by a move, and \( m \) moves. We define a move \( m \) as being a pair \([c, t]\) where \( c \) is the card being moved and \( t \) is the card or location it is moved to. (We assume some unambiguous notation where cards are moved to locations such as spaces which are currently empty.) We write \( C(m) \) for the card being moved by a move, and \( T(m) \) for the card or location moved to, i.e. if \( m = [c, t] \) then \( C(m) = c \) and \( T(m) = t \). We remark that, in games with multiple decks, there will be distinct cards with the same suit/rank, but these are still considered separate cards.

Since it is the last invalid move, \( m_i \) must be the move of a partial pile where the card above is in sequence but cannot be built to foundation. For \( m_{i+1} \) we do know: it exists since \( i < n \); it is a legal move; and if it is a partial pile move the card above it is buildable to foundation. We now show by case analysis how to replace moves \( m_i, m_{i+1} \) in the sequence. In most cases the adjustment is straightforward. Before describing the straightforward cases, we consider the most critical, difficult, case.

**Case 1.** The critical case is where move \( m_{i+1} \) is moving a card or pile onto the pile just vacated by the original move \( m_i \) (which was by hypothesis the last invalid move). We can illustrate by example in the case of a red-black build policy: this might be a move of a three card pile \( 10\heartsuit 9\spadesuit 8\heartsuit \) from the \( J\spadesuit \) to \( J\heartsuit \) at a time when the \( J\heartsuit \) could not be moved to foundation, followed immediately by a move of the \( 10\heartsuit \) to the \( J\spadesuit \), i.e. \( m_i = [10\heartsuit, J\heartsuit] \) and \( m_{i+1} = [10\heartsuit, J\spadesuit] \). We deal with this case first by omitting move \( m_i \), thus reducing the number of illegitimate moves by one, and then replacing \( m_{i+1} \) by a move \( m'_{i+1} = [C(m_{i+1}), T(m_i)] \). In the example, we delete the move of \( 10\heartsuit \) and change the move of the \( 10\heartsuit \) to be to the \( J\spadesuit \) instead of the \( J\heartsuit \), i.e. \( m'_{i+1} = [10\heartsuit, J\spadesuit] \). The move \( m'_{i+1} \) must be a legal move because the build policy is indistinguishable so cannot allow \( m_{i+1} \) and disallow \( m'_{i+1} \). Move \( m'_{i+1} \) must also be valid under the partial pile restriction since move \( m_{i+1} \) was.

We now have to consider the remaining moves in \( m_{i+2}, \ldots, m_n \). We create moves \( m'_{i+2}, \ldots, m'_j \) until we have identical layouts again in the original and new sequence of moves, after which we retain moves \( m_{j+1}, \ldots, m_n \). Until then, we will maintain an invariant property, that the cards \( T(m_i) \) and \( T(m_{i+1}) \) (\( J\spadesuit \) and \( J\heartsuit \)) remain in the tableau, that at least one of them has a card built below it, that the piles under those cards are swapped in the new sequence compared to the original, and that all other cards in the layout are identical. This invariant certainly holds after the deletion of \( m_i \) and the replacement of
$m_{i+1}$ by $m'_{i+1}$. Now we assume the invariant is true up to move $m'_{k-1}$ and consider move $m_k$. If this move does not involve either of the affected piles, then and necessarily retains the invariant, so we simply set $m'_k = m_k$. However, when the move does involve at least one of the affected piles, it must fall into one of the following five subcases. In the first three we have to adapt the sequence of moves to a new one to retain the invariant, with the last two being simpler.

(a) If move $m_k$ is of $C(m_i)$ to $T(m_{i+1})$ (e.g. of 10♠ to J♦ in our example) then we simply delete the move completely. Because of the invariant the cards below $C(m_i)$ are already at the intended target location, so we need do nothing. Included in this sub-case is where move $m_{i+1}$ is an exact reverse of $m_i$ and both are deleted. In general, the final move sequence is

$$m_1, \ldots, m_{i-1}, m'_{i+1}, \ldots, m'_{k-1}, m_{k+1}, \ldots, m_n$$

(b) If move $m_k$ is of $T(m_{i+1})$ to foundation [respectively $T(m_i)$], e.g. moves J♦ to foundation in our example [respectively J♡], then it now has a card built below it (by the invariant) so the move is not currently possible. By the invariant, the card $T(m_i)$ must itself be clear, since the card $T(m_{i+1})$ was clear in the original [respectively $T(m_{i+1})$ must be clear]. This means that we can now insert the move $m'_k = [C(m_i), T(m_i)]$: this is legal by indistinguishability and it is now a valid move under the restriction because $T(m_i)$ is now buildable [respectively set $m'_k = [C(m_{i+1}), T(m_{i+1})]$]. The move $m_k$ is now legal, positions are identical, and the new sequence contains one less invalid move. The final move sequence is

$$m_1, \ldots, m_{i-1}, m'_{i+1}, \ldots, m'_{k-1}, m'_k, m_{k+1}, \ldots, m_n$$

(c) If move $m_k = [C_k, T_k]$ removes the last card below $T(m_i)$ and $T(m_{i+1})$, e.g. moves either 10♣ or 10♠ in our example, leaving both J♡ or J♦ uncovered, then we set $m'_k$ to the same move $[C_k, T_k]$. The move $m'_k$ must be legal, by indistinguishability. If the move is to foundation or the card covering $C_k$ is now buildable to foundation, the move is valid under the restriction and the sequence has one less invalid move. If the move $m'_k$ is invalid, then the sequence has the same number of invalid moves as before (since we deleted $m_i$), but the last is nearer the end of the sequence (since $n - k > n - i$). By the invariant the layouts are now identical so the final move sequence is

$$m_1, \ldots, m_{i-1}, m'_{i+1}, \ldots, m'_{k-1}, m'_k, m_{k+1}, \ldots, m_n$$

(d) If move $m_k$ is from (or to) a card on a pile below either $T(m_i)$ or $T(m_{i+1})$, but is not covered by one of the above sub-cases (e.g. of 8♠ from the 9♡ to the 9♢ in our example) then we can make the unchanged move $m_k$ now. That is, we move from $C(m_k)$ to $T(m_k)$: by the invariant the identical card that $m_k$ was originally moved from (or to) is below the other one in the revised sequence. The invariant is thus retained.

(e) Any move $m_k$ of a pile of cards starting from $T(m_i)$ or $T(m_{i+1})$ or any card above them can be retained unchanged with $m'_k = m_k$ and the invariant is retained.

All remaining cases are essentially straightforward because we can simply swap consecutive moves $m_i$ and $m_{i+1}$, sometimes with minor changes. In each case the position after the second move is identical in each sequence, and we have either removed an invalid move or moved it one move closer to the end of the sequence, as required.
Case 2. If the moves $m_i$ and $m_{i+1}$ are entirely unrelated then we can simply swap the order of the moves as they do not affect each other. I.e. we create a new move sequence

$$m_1, \ldots m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots m_n$$

Note that the swap cannot affect the validity of $m_{i+1}$ under the restriction: by hypothesis it was a valid move and it remains so. However, it is possible that, in its new position, the move $m_i$ is now a valid move under the restriction. If that happens we have reduced the number of invalid moves, but if not we have moved the last valid move one closer to the end of the move sequence.

Case 3. We can make consecutive moves from the same pile, i.e. have $C(m_{i+1})$ be either the card above $C(m_i)$ or a card in sequence above it. Because move $m_i$ disobeyed the restriction, the move $m_{i+1}$ cannot be of the card above $C(m_i)$ to foundation, so the only remaining possibility is a second consecutive move between tableau piles. Again we can swap the order of the moves. The new move sequence is

$$m_1, \ldots m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots m_n$$

Notice that in this case the card $C(m_i)$ (and any partial pile below it) is moved twice instead of just once, but ends in an identical position. The move $m_{i+1}$ must still be valid under the restriction in the earlier position, while $m'_i$ remains invalid, but appears one move nearer the end of the sequence.

Case 4. We can make consecutive moves to the same pile. In this case again we simply swap moves $m_i$ and $m_{i+1}$. The analysis is the same as in the previous case, except that this time it is the card $C(m_{i+1})$ (and possibly partial pile below it) that is moved twice instead of once. Again, $m_i$ remains invalid, but appears one move closer to the end of the sequence.

Case 5. The final possibility is that the second move is from the pile the first move went to. This gives a number of possibilities depending on the card moved the second time: the second card moved can be the same as the first card, a card above it in the new pile, or a card below it.

- If the same card is moved twice, then the moves cannot be an immediate reversal of moves since that was covered as case (a) in the Critical Case above. Thus, we can replace the two moves with a single move bypassing the intermediate position, $m'_i = [C(m_i), T(m_i + 1)]$. This move $m'_i$ may still be invalid under the restriction but is one move closer to the end. In this case the final sequence of moves is

$$m_1, \ldots m_{i-1}, m'_i, m_{i+2}, \ldots m_n$$

- If the second card is either above or below the first moved card, then again we can just swap the order of the two moves, giving the sequence

$$m_1, \ldots m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots m_n$$

The result is the same, with the invalid move being one later in the sequence. If the card $C(m_{i+1})$ was above the first moved card in the second pile, then the card $C_{m_i}$ and any pile below it is only are now only moved once instead of twice. If the card $C(m_{i+1})$ was below $C(m_i)$ then the card $C(m_i)$ and any cards below it are now moved only once.
In all of the cases analysed above, we are able to do one of two things. We either produce a new sequence with one less invalid move under the restriction, or produce a sequence with the same number of invalid moves but the last one nearer the end of the sequence. Iterating this procedure must inevitably lead to a solution with zero invalid moves under the restriction. Therefore we can impose the restriction without making any instance unsolvable.

In fact a slightly adapted proof would also justify a slightly stronger restriction: as well as insisting that a foundation move is available next, we could also insist that the card is actually moved to foundation, slightly reducing the search space in some cases. However, the weaker restriction is what we implemented in our code and experiments. An analogue of the stronger restriction for the game of Worm Hole has proved to be important in effective search (Masten, 2022d). The important of similar techniques in different circumstances illustrates the need to be able to reason more effectively about dominances in general, to allow them to be used when it is correct to do so, without the need for the detailed kind of proof we presented here.

7.4 Future Work in Dominances

The importance of dominances for patience solving is very high, but a number of significant problems remain with their application, which should be addressed in future work. First, one has to discover dominances in the first place. They can be difficult to notice and are often not well publicised in the literature. There is also the overhead of implementation as they can be quite specialised: for example we have not implemented possible dominances to forbid making pointless moves of a card on the tableau which clears a space that cannot be usefully used. Apart from discovering and implementing dominances in the first place, ensuring their correctness can be very difficult. This is difficult within a single game, as discussed in Section 7.2, but even more difficult when using the general rule language such as provided by Solvitaire. Unusual combinations of rules may invalidate a dominance which is valid in very similar games. To reduce this problem, in Solvitaire we do automate the use of the dominance of Section 7.1 but only under strict conditions which we hope make it correct. For the dominance of Section 7.3, we require it to be specified in the JSON statement of the rules of the game, avoiding automatically applying it incorrectly. There is a very close link between streamliners and dominances, since a streamliner is just an incorrect dominance, so unifying their treatment would be interesting. Ideally we would like to be able to apply dominances and streamliners automatically, correctly, and generally. Achieving this remains a key challenge for future work in patience solving, automatically, correctly, and generally. Achieving this remains a key challenge for future work in patience solving.

8. Relationships between Games

In some cases one ruleset is stronger than another, in that any legal move in the stronger game would also be legal in the weaker one. For example, Worm Hole is identical to Black Hole but with the addition of a free cell. Any instance that can be won as the stronger game of Black Hole must automatically be winnable in Worm Hole via the same sequence of moves: Some examples from the rules of Klondike further illustrate the concept.

- Allowing worrying back makes a game strictly weaker. If we can win the game without worrying back then we can make the same sequence of moves in the game that allows worrying back.
• Allowing nothing to be put into empty spaces is strictly stronger than allowing only kings to put in spaces, which in turn is strictly stronger than allowing any card to be put in spaces.

• Allowing building down in any suit is strictly weaker than building down in red/black ordering. It is also strictly weaker than building down in same-suit ordering. However, red/black and same-suit orderings are incomparable.

• Any draw size from the stock is strictly weaker than any multiple of it. For example, draw size 2 is strictly weaker than draw size 4 and 6. While draw sizes 4 and 6 are incomparable, they are each strictly weaker than draw size 12.

Where one game is stronger than another, winnable instances of the stronger game must be winnable in the weaker, and unwinnable instances of the weaker game must be unwinnable in the stronger. We can use this to reduce greatly the set of instances that must be tested to obtain results between games.

Despite the extreme amounts of time we spent on the hardest Klondike instances, we still found some that were proved unwinnable in weaker games or winnable in stronger ones. Of 396 instances that were not solved directly, 7 were found winnable in games where the stock is drawn in units of 6 instead of 3, and one more where the stock is drawn in units of 9. While valid playthroughs for the original game, the reduced search space in the stronger game allowed the winning moves to be found faster. A further 231 instances were shown to be unwinnable when cards are drawn from the stock in ones. It might seem surprising that it is easier to prove the instance unwinnable in a weaker game. The reason is that drawing cards by one allows for a dominance that is not valid when drawing by 3. Drawing cards by one with unlimited redeals makes all cards in the stock available at any time, so we can apply the dominance described in Section 7.1 to the stock as well as to the tableau: this is invalid with other draw sizes. When the dominance is applied it reduces the size of the search space and thus allows all possibilities to be exhausted. This is an example of relaxation in a search problem (Hooker, 2002). Taken together, we were able to use these approaches to resolve 239 of the 396 unknown instances, leaving only 157.

When computing results on two games which were strictly stronger/weaker than each other, we used an identical set of instances in each case. This gives us two significant advantages. First, it acts to reduce the statistical variance in computing the difference in winnability between the games. Second, we were able to exploit the linkage between games to avoid recomputing winnability results we already knew. For example, since draw size 5 is strictly weaker than draw size 10, we did not need to test draw size 10 on any of the 465,656 instances proven unwinnable at size 5. Having found the 42,372 winnable instances with worrying back at draw size 10, these were the only ones we needed to test for winnability without worrying back. This greatly reduces the time taken to compute accurate winnability percentages across a range of related games. This could be used as an additional form of streamliner when searching for solutions for individual patiences - e.g. when trying to win a game with worrying back one could first try it without, which greatly reduces the search space while often not greatly reducing the chances of winning. This is an interesting area for future research that we have not yet investigated.

Table 4 shows up a weakness in Solvitaire’s ability to solve patience games. Of a million instances, more than 97% could not be determined when combining building in any suit with spaces not being fillable. We discuss this weakness further in Section 10.
9. Methods

9.1 Implementation, Testing and Debugging

Solvitaire is implemented in the C++ programming language. During development, code was profiled to identify hotspots in code which needed optimisation. Some areas which did not turn out to be critical were surprising; for example the code to find available moves is barely optimised despite being used at each node in search.

We used a number of strategies to test our code and reduce bugs to a minimum. First, we used unit, integration and performance tests to guard against regressions in the code. As we introduced new game features we created bespoke, simplified games to target the added functionality. Our tests were build upon these game types, using hand-crafted deals with known solutions. We also had a performance benchmark script, which measured the performance of the solver on a number of benchmark instances to let us know if our latest code changes had slowed it down.

Second, we ran strict and loose versions of particular games over identical instances. Where Solvitaire reported a looser versions of a game as but the stricter version as winnable, a bug was indicated which we then fixed.

Third, we could test our work on the macroscopic scale, by comparing overall results obtained using Solvitaire on games also estimated by previous researchers. For example, we discuss in Section 7.2 that this allowed us to identify and fix a bug in our pseudorandom instance generator. Table 1 shows that, where we were able to compute confidence intervals for related work, all our 95% confidence intervals now overlap with the best existing estimate. Given the complete independence of our implementations with those of many different past researchers, this strongly suggests that bugs that significantly affect winnability percentages are unlikely.

Finally, at the microscopic level, for the games FreeCell, Canfield, and Klondike, we tested individual instances to make sure our solver gave consistent results with existing solvers. For FreeCell, we ran Solvitaire on each of the 102,075 unsolvable instances of FreeCell found by Fish (2018): all were correctly identified as unsolvable except for two that could not be determined. We tested Solvitaire against the best existing solvers for each of Canfield (Wolter, 2014d) and Klondike (Birrell, 2017) on 50,000 individual instances each. Detailed study of individual inconsistent results allowed us to determine which solver was correct. If the bug was in Solvitaire, we then corrected it. As discussed in Section 7.2, we also we found problems in both existing solvers. Although this happened in very rare cases, it indicates the detailed work that allowed us to discover such rare bugs in existing solvers.

We cannot rule out that bugs remain in our code that might affect winnability of some games, especially using unusual combinations of rules we have not tested exhaustively. Availability of our codebase will enable future researchers to identify any remaining bugs in our code (Blake & Gent, 2019). The code includes random generation of instances which is portable across different machines, so other researchers should be able to recreate the same test instances to check our results against theirs.

9.2 Statistics

Each random instance is necessarily either solvable or unsolvable, and therefore the true picture for any given game is it behaves as a binomial with probability \( p \) of success. As discussed in Section 8, in some cases we used winnability facts from stronger or weaker games where this was guaranteed correct, saving considerable time. We used the following consistent protocol for measuring a confidence interval on the estimate of winnability
percentage. From a sample, if we know the number of winnable and unwinnable games, we calculate a 95% confidence interval for the true value of \( p \) using Wilson’s method (Agresti & Coull, 1998). When some instances’ winnability are unknown, e.g. due to timeouts, we form the most conservative possible interval by calculating the interval both on the assumption that every unknown instance is unwinnable and on the assumption that every unknown instance is winnable. We then report the range from the lower bound of the first interval to the upper bound of the second. While it would be nice to be less conservative and get a smaller interval, no other totally general approach seems valid: for example, in Spider it is very likely that almost all unresolved instances are winnable, while in Klondike most long-running instances turn out to be unwinnable. We normally report percentage winnability to 3 decimal places, but give more places where winnability is very close to either 0 or 100%. We use the most conservative possible rounding: given the number of digits we are reporting, we round the lower bound down and the upper bound up. Given the range calculated, we report it from the centre, plus or minus half the range (with the centre chosen arbitrarily from the two choices where the range is odd in the last digit). For calculating equivalent intervals for comparison with previous work, in most cases we could deduce the raw numbers of solved, unsolvable and indeterminate cases from past publications, and calculate the confidence interval that would result from the same protocol. While a confidence interval we compare against may not be the same as that reported in a previous paper, our comparisons with previous results are on a like-for-like basis without being dependent on varying methodologies for estimating the range of solvability used by different authors. Statistics were calculated using R (R Core Team, 2016).

9.3 Experimental Setup

Monte Carlo methods using pseudo-random generation were used to create instances of each game. We used the Mersenne twister generator (Matsumoto & Nishimura, 1998) mt19337 provided by the C++ standard library to generate a stream of pseudorandom numbers. The stream of numbers passed all tests for randomness in the Dieharder test suite, v3.31.1 (Brown, Eddelbuettel, & Bauer, 2019), simulating the way it was used in our code with the initial seed incremented after every 52 random numbers. We wrote our own code to create instances from the stream of numbers: this generator is portable so should produce identical results for the same seed, and is included in our code for Solvitaire. In running experiments, a critical point is that runs which were unresolved are included in our statistics. In many cases we re-ran failed seeds with larger computational resources, but where we could never resolve the instance, they are included in our data as unknown. It would be improper to ignore them and rerun with a new seed as hard instances can have a different likelihood of being winnable to a new random seed. Having decided on a sample size for an experiment we used a consecutive sequence of seeds for that experiment. Seeds for each instance are recorded in our data. As well as winnability, we recorded many other features of search such as run-time, memory usage, cache usage, and search depth. We do not report those statistics in detail but they are available in full in our data files: Table 11 gives an overview by game of run-time and nodes searched.

Experiments mostly used the Cirrus UK National Tier-2 HPC Service at EPCC (see acknowledgements). Additionally, a small number of our results presented here were obtained on local compute-servers at the University of St Andrews. In selecting experimental parameters such as sample size, number of cores used per machine, timeout limits, and cache sizes, we made choices intended to optimise the computing resources and time available. For example, for some games it was critical to run with very large amounts of RAM,
reducing the number that could be run in parallel on one machine. In some cases, we accepted a small number of timeouts in order to get a very large sample size (e.g. American Canister). In others where there were many timeouts, we focussed on a smaller sample size but very long runtimes to minimise the number of unknowns (e.g. Gaps One Deal). All results were obtained using Solvitaire, but to save CPU time we sometimes reused results for one game for a related game, as described in Section 8. During experiments, minor changes were made to Solvitaire: our data files indicate the version used for each experiment. As a consistency check, we compared results of the current version (0.10.1) against our reported results by testing 1000 instances (or 100 in two very hard games). In all cases where both versions completed, all features of search including precise numbers of nodes were identical.

10. Evaluation of Solvitaire

We have achieved considerable success using Solvitaire, as we have reported throughout this paper. In this section we reflect on the strengths and weaknesses of our program, Solvitaire, and how future work can build on our success.

A remarkable feature of some games is the extraordinary depths that search can reach while still being successful. For example, in Beleaguered Castle, one game was solved at a search depth of more than 190 million, while another was proved unwinnable with a maximum depth of more than 27 million. The latter case involved a total search of just over 1 billion nodes, so an average of less than 40 nodes per search depth. This indicates a very unusual search space, since normally the number of nodes grows exponentially with depth. It remains open whether understanding this unusual behaviour can help improve search.

A significant feature of our work is that most predecessors have written a special program for each main game, while our single program Solvitaire can solve games from a simple textual description. This gives two significant advantages over previous work. First, the uniform approach enables us to implement advanced AI techniques just once but apply them to many games. Second, we can gain improved confidence in correctness through bugfixes from in one game automatically applying to all others. Therefore our work has significant value even in games where previous studies have been done.

We regard it as remarkable that we have been able to obtain so many new and improved results using a general purpose patience playing program. Normally, we would expect a general purpose program to be significantly outperformed by specially written programs. In some cases we have obtained very much improved results over previous work, but this may be due to the availability of significant computer time on modern CPUs rather than an improved solver. We have not attempted a quantitative comparison of our solvers with competing ones in terms of, say, run times. Nevertheless, it remains clear that Solvitaire is an outstanding solver in most games we have evaluated it on, while not necessarily better than its competitors in the few games that those competitors exist.

While giving us many advantages, our configurable rule set has some limitations. For example, it does not allow for games with a fixed number of redeals of stock, and not being able to find the shortest possible winning sequence. We also do not allow for some key rules such as pairing, eliminating games like Doublets and many others. These limitations were conscious in the sense that in the expressivity/speed tradeoff, we prioritised getting good performance on games we could express rather than total generality.

There are some games where we could not improve on previous results, as seen in Table 1. Some examples of this are very near to 100% winnability, such as FreeCell, Spider,
and Accordion. We may have been less effective for these games due to our entirely general approach of prioritising proving whether a game was winnable or not, rather than fastest possible finding of a winning sequence when it existed. A particularly interesting example where another solver outperforms Solvitaire is Worm Hole. We invented this game to show the flexibility of our ruleset and got reasonable results, but since doing so Masten (2022d) has reported a dominance we were not aware of which allows for improved performance. This again shows the value of dominances in patience solving, as we discussed in Section 7, and how much more remains to be done in this area.

One important weakness we have identified is that in some games, many instances are unwinnable but Solvitaire is unable to prove them so. A particularly clear example of this is seen in Table 4. We believe the problem is the combination of a very liberal rule for moving cards (in this case any suit) with a very strict restriction (in this case unfillable spaces). The liberal rule makes the search space very large, while the restriction means that the location of a small number of cards can make the game unwinnable. In this game, imagine a King covering a Queen of the same suit in the tableau. The Queen can never be reached because the King cannot be moved to a space, while the King cannot be built to the foundation because the Queen is not available. Yet there can be literally billions of potential paths that Solvitaire might have to explore, leading to timing out. We have seen similar problems in other games where a game with many possible moves can be unwinnable for small local reasons, and thrashing occurs. While we did not report results here, we saw this with the game ‘Alina’ (Gent, 2022), where Solvitaire has never proved one to be unwinnable even though our human examination shows that many cannot be won. It should be possible to create solvers which combine the exploratory search strength of Solvitaire while also adding more reasoning ability to exclude possibilities. Indeed, it is remarkable that Solvitaire is so successful on so many games despite this issue.

11. Conclusions

We have shown that a single depth-first search based solver, Solvitaire, is able to produce state-of-the-art results across a very wide variety of patience games. We achieved this by combining a variety of general AI search techniques. In doing so, we have obtained many entirely new results across a wide variety of games. We have also greatly improved the state-of-the-art winnability estimates on many games including some of the most famous games such as Klondike (often just called ‘Solitaire’) and Canfield. In a pleasing callback to their origin, we have now used Monte Carlo methods to answer the question that caused Stanislaw Ulam to invent Monte Carlo methods.

Despite the level of interest we described in Section 3, we are surprised that this study is the first of its kind, i.e. an academic study of the winnability of many different patience games. Previous studies within academia have tended to focus one game, possibly with some variants. There have been more wide ranging studies done outside the academic literature, for example by Masten (2022c), Wolter (2013b), and others. Showing that general AI methods can be applied across patience games, we hope very much that other researchers will build on what we have done and no doubt greatly improve on it.

While we believe we have made a significant contribution to the study of games that have occupied humans for uncounted hours, much remains to be done. Without doubt, the most interesting question we leave open is one we have not attempted to tackle at all. In games with hidden cards like Klondike, what is the true probability of winning from a starting position? We have always solved the ‘thoughtful version’. When faced with a game of the classic Solitaire, Klondike, with no peeking on physical cards or electronic undo
button, what is the best attainable probability of winning, and how does one obtain this? For many games, this remains a very hard problem, with an answer that is not currently known for Klondike even within a factor of two. While the thoughtful winnability gives an upper bound, it gives us no direct information about a lower bound.

Data and Code Availability

Full experimental results reported in this paper are available at figshare.com with DOI doi:10.6084/m9.figshare.8311070.v4 (Gent & Blake, 2023). This dataset includes all runs used to report data in this paper, together with other material such as details of testing and analysis scripts used to compute winnability estimates reported here. The code for Solvitaire is open-source under the GNU GPL Version 2 licence. The code used for this version of the paper is available in Zenodo at identifier doi:10.5281/zenodo.3529524 (Blake & Gent, 2019). Development history of Solvitaire is also available on Github at URL https://github.com/thecharlesblake/Solvitaire.

Author Contributions

IPG proposed and supervised the project. CB and IPG jointly made high-level design decisions. CB made all low-level design decisions, implemented Solvitaire, and named it. CB and IPG debugged Solvitaire, and ran exploratory experiments. IPG ran the full experiments reported here and analysed them. IPG constructed the proof of Theorem 1. IPG drafted the paper, with CB and IPG revising it.

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Appendix A. Rules Of Patience Games

Table 7 shows the rule for most main games in this paper. Exceptional games not in this table are Accordion, Late-Binding Solitaire and Gaps variants, for which detailed rules in JSON are shown in Table 8. To present most games in a uniform format, Table 7 uses a very concise notation which is explained below. Where variants of a game are reported in this paper, the rules are as given here except for the stated change, e.g. Fore Cell (Same Suit) has the same rules as Fore Cell except with BP set to =.

**Game:** Name of game, plus citation which gives to name and rules we use (but is not necessarily the inventor of the game). **Symbols Used:** **Game invented for this paper**

**Decks:** Number of complete decks used in the game, of 52 cards by default. **Symbols Used:** ‡ Deck of 32 cards, we use A+2-8 of each suit.

**Foundations:** Rules for Foundations or Hole. Number of cards initially placed in foundations, • for hole being used, or S for Spider-type elimination of suits. **Symbols Used:** ✓ Worrying back from foundations to tableau is allowed. × Worrying back is not allowed. ¶ Random base of foundation. § A K not considered adjacent in rank.

**Tableau:** Number of tableau piles, plus shape of tableau. **Symbols Used:** □ piles all of same length except possibly for some piles of one extra length; △ piles in triangular form; solid shapes indicates that cards face-down except the top card, otherwise all cards face-up.

**BP:** Build Policy, rule by which one card may be placed on another in the tableau. Where allowed, cards must be one lower in rank (including from K on A if Foundations start on random base). **Symbols Used:** × building not allowed; * card of any suit allowed; rb card must be of opposite colour (red on black or black on red); = card of same suit.

**MG:** Move of Groups, whether or not a consecutive sequence of built cards may be moved as a unit in the tableau. **Symbols Used:** × not allowed; ✓ allowed with the same restriction as BP; = allowed for sequence of cards of the same suit. + Only entire piles may be moved.

**SP:** What card may be put in a free space in the tableau, or sequence if MG allows it. **Symbols Used:** × Spaces may not be filled; ✓ Spaces may be filled by any card; K Spaces may be filled by a K only (or card one rank below foundation base if random). † Space must be refilled immediately from stacked reserve until that is empty, then may be filled freely. †† Space must be refilled immediately from waste (or stock if empty).

**Stock:** The first number indicates the number of cards in the stock; the second symbol indicates the number of cards drawn at a time from a stock, with □ indicating that one card from stock is dealt to each tableau pile; the third number indicates whether no redeals are allowed when the stock is empty or an infinite number are.

**FC:** The number of Free Cells in the game, if any, followed by the number of free cells that are filled at the start of the game.

**Reserve:** The size of any Reserve in the game. S indicates that the reserve is ‘stacked’, i.e. only the top card of it is available for play. Otherwise all cards are available at any time.
## Table 7: Detailed rules of main games studied in this paper, excepting those in Table 8. See page 32 for key. **Original game.**

| Game                  | Rules Citation     | Decks | Foundations | Tableau | TC | BP | MG | SP | Stock | FC | Reserve |
|-----------------------|--------------------|-------|-------------|---------|----|----|----|----|-------|----|---------|
| Alpha Star            | (Gent, 2022)       | 1     | 4 ×         |         | 12 □ | 48 = | ✓  | ✓  |       |    |         |
| American Canister     | (Warfield, 2016a)  | 1     | 0 ×         |         | 8 □ | 52 rb | ✓  | ✓  |       |    |         |
| Baker’s Game          | (Parlett, 1980)    | 1     | 4 ×         |         | 8 □ | 52 = | ✓  | ✓  | 4 0   |    |         |
| Beleaguered Castle    | (Parlett, 1980)    | 1     | 4 ×         |         | 8 □ | 48 * | ✓  | ✓  |       |    |         |
| Black Hole            | (Parlett, 1980)    | 1     | • ×         |         | 17 □ | 51 × | ✓  | ✓  |       |    |         |
| (British) Canister    | (Parlett, 1980)    | 1     | 0 ×         |         | 8 □ | 52 rb × | K |    |       |    |         |
| Canfield              | (Wolter, 2014a)    | 1     | 1 × ¶       |         | 4 □ | 4 rb ✓ | †  | ✓  | 34 3 ♣ | 13 S|         |
| Delta Star            | (Gent, 2022)       | 1     | 4 ×         |         | 12 □ | 48 = | ✓  | ✓  |       |    |         |
| East Haven            | (Warfield, 2016b)  | 1     | 0 ✓         |         | 7 ■ | 21 rb | ✓  | ✓  | 31 □ | 0  |         |
| Eight Off             | (Parlett, 1980)    | 1     | 0 ×         |         | 8 □ | 48 = | ✓  | ✓  | K     | 8 4 |         |
| Fan                   | (Parlett, 1980)    | 1     | 0 ✓         |         | 18 □ | 52 = | x  | K  |       |    |         |
| Fore Cell             | (Keller, 2015)     | 1     | 0 ×         |         | 8 □ | 48 rb | ✓  | ✓  | K     | 4 4 |         |
| Fortune’s Favor       | (Parlett, 1980)    | 1     | 4 ×         |         | 12 □ | 12 = | x  | †† | 36 1 0|    |         |
| Freecell              | (Keller, 2015)     | 1     | 0 ×         |         | 8 □ | 52 rb | ✓  | ✓  |       |    | 4 0     |
| Golf                  | (Parlett, 1980)    | 1     | • × ¶ §     |         | 7 □ | 35 × | x  | x  | 16 1 0|    |         |
| King Albert           | (Parlett, 1980)    | 1     | 0 ✓         |         | 9 △ | 45 rb | ✓  | ✓  |       |    | 7       |
| Klondike              | (Bjarnason et al., 2007) | 2 S × | 13 □ 104 * = | ✓  | ✓  |       |    | 4 2   |       |
| Mrs Mop               | (Parlett, 1980)    | 2     | S ×         |         | 10 □ | 50 = | x  | K  |       |    |         |
| Northwest Territory   | (Warfield, 2017a)  | 1     | 0 ✓         |         | 8 △ | 36 rb | ✓  | ✓  | K     | 16 |         |
| Raglan                | (Parlett, 1980)    | 1     | 4 ✓         |         | 9 △ | 42 rb | ✓  | ✓  |       |    | 6       |
| Seahaven Towers       | (Warfield, 2018; Cabral, 2019) | 1     | 0 ×         |         | 10 □ | 50 = | x  | K  |       |    | 4 2     |
| Siegecraft            | (Wikipedia Contributors, 2017) | 1     | 4 ×         |         | 8 □ | 48 * | x  | ✓  | 1 0   |    |         |
| Simple Simon          | (Parlett, 1980)    | 1     | S ×         |         | 10 □ | 52 = | ✓  | ✓  |       |    |         |
| Somerset              | (Parlett, 1980)    | 1     | 0 ✓         |         | 10 □ | 52 rb | ✓  | ✓  |       |    |         |
| Spanish Patience      | (Warfield, 2017b)  | 1     | 0 ✓         |         | 13 □ | 52 * | x  | ✓  |       |    |         |
| Spiderette            | (Parlett, 1980)    | 1     | S ×         |         | 10 □ | 52 = | ✓  | ✓  | 24 □ | 0  |         |
| Spider                | (Parlett, 1980)    | 2     | S ×         |         | 10 □ | 54 * | =  | ☑  | 50 □ 0|    |         |
| Streets & Alleys      | (Parlett, 1980)    | 1     | 0 ×         |         | 8 □ | 52 * | x  | ✓  |       |    |         |
| Stronghold            | (Warfield, 2019)   | 1     | 0 ×         |         | 8 □ | 52 * | x  | ✓  |       |    | 1 0     |
| Thirty                | (Parlett, 1980)    | 1     | 0 ×         |         | 5 □ | 30 * | ✓  | ✓  |       |    | 2       |
| Thirty Six            | (Wolter, 2014b)    | 1     | 0 ×         |         | 6 □ | 36 * | ✓  | ✓  | 16 1 0|    |         |
| Trigon                | (Wolter, 2014c)    | 1     | 0 ×         |         | 7 △ | 28 = | ✓  | K  | 24 3 ∞|    |         |
| Will O’ The Wisp      | (Parlett, 1980)    | 1     | S ×         |         | 7 ■ | 21 * | =  | ✓  | 31 □ | 0  |         |
| Worm Hole             | **                 | 1     | • ×         |         | 17 □ | 51 × | x  | x  | 1 0   |    |         |
"foundations":,
  "present": false,
"tableau piles":
  "count": 0,
"accordion":
  "size": 52,
  "moves": ["L1", "L3"],
  "build policies": ["same-suit", "same-rank"]

Rules of Accordion. The rules of Late-Binding Solitaire are the same except with size 18.

"foundations":
  "present": false,
"tableau piles":
  "count": 0,
"sequences":
  "count": 4,
  "direction": "L",
  "build policy": "same-suit",
  "fixed suit": false

Rules of Gaps (One Deal). The rules for Gaps (Basic Variant) are the same except with fixed suit true.

Table 8: Rules in our JSON format of unusual games omitted from Table 7.

Appendix B. Summary of Data from the Literature

Results from previous researchers on winnability of patience games is widely spread, and presented in numerous different forms. Here we present the raw data used to generate confidence intervals for existing results throughout this paper. Table 9 shows the raw data for the best results we could find for each game, while Table 10 gives the archival URL for pages giving the reported data. Archival URLs are particularly important: for example, many results were originally discussed in Yahoo groups, which were deleted in 2020. This list only includes games we have compared with Solvitaire. For other games not included here, useful starting points are the summaries of Keller (2015), Masten (2022c) and Wolter (2013b).
| Game                  | Sample | Winnable | Unwinnable | Unknown |
|----------------------|--------|----------|------------|---------|
| Accordion            | 3 × 10^7 | 30,000,000 | 0          | 0       |
| Baker’s Game         | 10^6   | 7,501,119 | 2,498,881  | 0       |
| Black Hole           | 1.6 × 10^6 | 1,391,771 | 208,229    | 0       |
| Canfield [Th.]       | 50,000 | 35,606   | 13,730     | 664     |
| Carpet [Th.]         | 10^6   | 8,755,758 | 1,244,242  | 0       |
| – Pre-founded Aces   | 10^6   | 9,518,603 | 481,397    | 0       |
| Eight Off            | 5 × 10^7 | 49,940,034 | 59,966     | 0       |
| Fore Cell            | 32,000 | 27,395   | 4,605      | 0       |
| – Same Suit          | 10^6   | 105,560  | 894,440    | 0       |
| FreeCell             | 8,589,934,591 | 8,589,832,516 | 102,075    | 0       |
| – 0 Cells            | 8,589,934,591 | 18,577,014 | 8,571,181,674 | 175,903 |
| – 1 Cell             | 100,000 | 19,473   | 80438      | 89      |
| – 2 Cells            | 400,000 | 317,873  | 82126      | 1       |
| – 3 Cells            | 10^6   | 993,600  | 6,380      | 20      |
| – 4 Piles            | 32,000  | 5       | 31995      | 0       |
| – 5 Piles            | 32,000  | 1,266    | 30,713     | 0       |
| – 6 Piles            | 32,000  | 19,685   | 12,184     | 131     |
| – 7 Piles            | 32,000  | 31,641   | 357        | 2       |
| Gaps                 | One Deal | 100     | 88         | 4       | 8     |
| – Basic Variant      | 10,000 | 2,480   | 7,520      | 0       |
| Golf                 | 100,000 | 45,077  | 54,923     | 0       |
| King Albert          | 100    | 72       | 28         | 0       |
| – Draw 1             | 1,000  | 919      | 62         | 19      |
| – Draw 2             | 1,000  | 801      | 71         | 28      |
| – Draw 3             | 1,000  | 836      | 149        | 15      |
| Klondike [Th.]       | 1,000  | 709      | 285        | 6       |
| – Draw 5             | 1,000  | 526      | 473        | 1       |
| – Draw 6             | 1,000  | 345      | 655        | 0       |
| – Draw 7             | 1,000  | 233      | 767        | 0       |
| Late-Binding Solitaire | 1,000    | 454      | 546        | 0       |
| Seahaven Towers      | 1.5 × 10^7 | 13,397,816 | 1,602,184  | 0       |
| Simple Simon         | 5,000  | 4,533    | 0          | 467     |
| – Spider [Th.]       | 32,000 | 31,998   | 0          | 2       |
| – Thirty Six [Th.]   | 100,000 | 94,327   | 5,343      | 330     |
| – Trigon             | 10^6   | 160,076  | 839,924    | 0       |
| – Worm Hole          | 10^6   | 998,908  | 1,092      | 0       |

Table 9: Totals from the literature used in this paper. Numbers in italics indicate issue discussed in accompanying note. For archival URLs giving source of data in this table, see Table 10.
| Game       | Variant       | Archival URL                                                                 |
|------------|---------------|------------------------------------------------------------------------------|
| Accordion  |               | 20220425085012/https://masten.000webhostapp.com/Accordion.html               |
| Baker’s Game |            | 20220425085149/https://masten.000webhostapp.com/BakersGame.html              |
| Black Hole |               | 20220425085046/https://masten.000webhostapp.com/BlackHole.html               |
| Canfield   |               | 20180429220704/https://politaire.com/article/canfield.html                   |
| Carpet     |               | 2020728095714/https://masten.000webhostapp.com/Carpet.html                  |
| Eight Off  |               | 20220426164250/https://masten.000webhostapp.com/EightOff.html               |
| Fore Cell  | Same Suit     | 20181215222456/http://solitairelaboratory.com/fcfaq.html                     |
|            |               | 20220426164250/https://masten.000webhostapp.com/EightOff.html               |
| FreeCell   | 0 Cells       | 2020419155553/https://github.com/shlomif/freecell-pro-0fc-deals/blob/master/README.md |
|            | 2 Cells       | 20130719010443/https://fc-solve.blogspot.com/2012/09/two-freecell-solvability-report-for.html |
|            | Others        | 20221221122054/https://igp.host.cs.st-andrews.ac.uk/KellerMillion.htm       |
| Gaps       |               | 20180729133856/https://link.springer.com/content/pdf/10.1007/978-0-387-35706-5_22.pdf |
| Golf       |               | 20170625031422/https://politaire.com/article/golf.html                      |
| King Albert |              | 20220618044831/https://arxiv.org/pdf/1611.08418.pdf                         |
| Klondike   |               | 20160218015922/https://github.com/ShootMe/Klondike-Solver/blob/master/Statistics.txt |
| Late-Binding Solitaire | | 20180409232321/https://l.stanford.edu/pub/cstr/reports/cs/tr/89/1269/CS-TR-89-1269.pdf |
| Seahaven Towers | | 20220426164824/https://masten.000webhostapp.com/SeahavenTowers.html           |
| Simple Simon |            | 20220428151919/https://fc-solve.shlomifish.org/mail-lists/fc-solve-discuss/archive/0974.html |
| Spider     |               | 20210305230502/https://www.tranzoa.net/~alex/plspider.htm                    |
| Thirty Six |               | 201706250101624/https://politaire.com/article/thirtysix.html                 |
| Trigon     |               | 20170625011319/https://politaire.com/article/trigon.html                     |
| Worm Hole  |               | 20220426164749/https://masten.000webhostapp.com/WormHole.html               |

Table 10: Archival URLs for sources of data reported in Table 9. Note that URLs are not necessarily those of citations in Table 1. URLs are relative to https://web.archive.org/web/. For the original URL, delete the numerical prefix and first slash.
### Appendix C. Summary Statistics of Experiments Reported Here

Table 11 summarises the results of Solvitaire on all patiences experimented on in this paper.

The first four columns report on the winnability statistics from which confidence intervals were calculated. The total sample is given as well as the number proven winnable, proven unwinnable, and unknown. In some cases, results for particular instances were not run on this particular game but deduced from related games, as described in Section 8.

The final four columns give summary search statistics for all our data provided in our auxiliary data. The number of runs is the number of times Solvitaire was executed on that specific game in our experimental set, and so therefore can be either higher or lower than the sample size in the first column. Lower numbers than the sample arises if other games are used to deduce results while higher numbers result from rerunning instances that Solvitaire initially failed to resolve. The final three columns give, to 2 significant figures, the mean number of nodes searched, the mean CPU time per run in CPU-seconds, and the total CPU time over all runs in CPU-days. These statistics are intended to give a general idea of the ease or difficulty that Solvitaire had with any game, as well as the total resources we devoted to that game. However, they are not well-suited for benchmarking Solvitaire against alternative solvers, because the focus of our experiments was to obtain high-quality estimates of winnability percentage. In particular, experiments were run on a variety of machines with different specifications, often varying within a single experiment. CPU times are as recorded by our internal timing mechanism: while we did often record a slightly more accurate external timing mechanism, which was typically \( \approx 10\% \) higher, this statistic was not available for all instances.

With the exception of two games, full data for all instances we experimented on is provided in our online dataset (Gent & Blake, 2023). The exceptions are British Canister and Fortune’s Favor, which were so easy and had winnability so close to 0/1 that we used samples of size \( 10^9 \). We only retained instances which are one of: in the first \( 10^7 \) instances; or took more than 1 sec. to solve; or had the rare result (winnable for British Canister or unwinnable for Fortune’s Favor). While this means complete data is not available, it seemed a reasonable compromise between ability to check our work and excessive storage requirements. Note that, since all instances of both games were solved, the winnability of all \( 10^9 \) individual instances can be checked from our data.

| Game Variant | Winnability Statistics | Search Statistics |
|--------------|------------------------|------------------|
|              | Sample | Winnable | Unwinnable | Unknown | Runs | Mean Nodes | Mean CPU (seconds) | Σ CPU (days) |
| Accordion | \( 10^6 \) | 999,996 | 0 | 4 | 1,000,116 | 5.1 \times 10^6 | 9.8 | 110 |
| Alpha Star | \( 10^7 \) | 4,779,474 | 5,220,526 | 0 | 10,000,000 | 7.7 \times 10^2 | 0.0037 | 0.42 |
| American Canister | \( 10^7 \) | 560,567 | 9,439,428 | 5 | 10,000,179 | 9.1 \times 10^4 | 0.61 | 71 |
| Baker’s Game | \( 10^7 \) | 7,505,266 | 2,494,734 | 0 | 10,000,000 | 7.8 \times 10^4 | 0.42 | 49 |
| Beleaguered Castle | \( 2 \times 10^6 \) | 1,362,720 | 635,919 | 1,361 | 2,671,263 | 2.6 \times 10^6 | 4.9 | 150 |
| Black Hole | \( 10^7 \) | 8,694,457 | 1,305,543 | 0 | 10,000,000 | 4.3 \times 10^5 | 2.8 | 330 |
| Game                  | Variant | Winnability Statistics | Search Statistics |
|----------------------|---------|-----------------------|-------------------|
|                      |         | Sample Winnable | Unwinnable | Unknown | Runs Mean Nodes | Mean CPU (seconds) | Σ CPU (days) |
| British Canister     | †       | $10^9$          | 1,290       | 999,999,710 | 0       | $10,001,326$ | $74$ | 0.000095 | n/a     |
| Canfield             |         | $10^7$          | 7,124,239   | 2,875,241  | 520     | 10,000,000  | $1.6 \times 10^6$ | 4.8 | 560     |
| Canfield (Whole pile)|         | $10^7$          | 6,755,771   | 3,243,482  | 747     | 10,000,000  | $2.4 \times 10^6$ | 6.3 | 730     |
| Delta Star           |         | $10^7$          | 3,441,247   | 6,588,753  | 0       | 10,000,000  | $1.0 \times 10^3$ | 0.0035 | 0.4     |
| East Haven           |         | $2 \times 10^6$ | 1,655,944   | 342,169    | 1,887   | 2,075,274   | $1.7 \times 10^6$ | 5.4 | 130     |
| Eight Off            |         | $10^7$          | 9,988,054   | 11,946     | 0       | 10,000,000  | $1.4 \times 10^4$ | 0.046 | 5.3     |
| Fan                  |         | $10^6$          | 487,759     | 512,241    | 0       | 1,000,000   | $6.3 \times 10^5$ | 1   | 12      |
| Fore Cell            |         | $10^7$          | 8,561,569   | 1,438,082  | 349     | 10,000,000  | $3.6 \times 10^5$ | 0.71 | 82      |
| Fore Cell (BP =)     |         | $10^7$          | 1,056,397   | 8,943,603  | 0       | 10,000,000  | $4.8 \times 10^3$ | 0.015 | 1.7     |
| Fortunes Favor       | †       | $10^9$          | 999,999,881 | 119        | 0       | $10,294,763$ | $2.1 \times 10^4$ | 0.068 | n/a     |
| FreeCell             |         | $10^7$          | 9,999,890   | 110        | 0       | 10,000,016  | $7.6 \times 10^4$ | 0.37 | 43      |
| FreeCell (FC 0)      |         | $10^7$          | 21,354      | 9,978,617  | 29      | 10,000,111  | $2.8 \times 10^4$ | 0.057 | 6.7     |
| FreeCell (FC 1)      |         | $10^6$          | 193,335     | 806,370    | 295     | 1,000,749   | $1.4 \times 10^6$ | 3.3 | 39      |
| FreeCell (FC 2)      |         | $10^6$          | 795,341     | 204,449    | 210     | 1,000,440   | $8.3 \times 10^5$ | 2.3 | 26      |
| FreeCell (FC 3)      |         | $10^6$          | 993,580     | 6,410      | 10      | 1,000,021   | $1.4 \times 10^5$ | 0.32 | 3.7     |
| FreeCell (4 Piles)   |         | $10^7$          | 864         | 9,999,136  | 0       | 10,000,000  | $1.5 \times 10^3$ | 0.0021 | 0.25    |
| FreeCell (5 Piles)   |         | $10^6$          | 38,577      | 961,392    | 31      | 1,000,173   | $5.5 \times 10^5$ | 1.1 | 12      |
| FreeCell (6 Piles)   |         | $2 \times 10^6$ | 1,227,828   | 770,982    | 1,190   | 2,003,743   | $5.6 \times 10^6$ | 13 | 290     |
| FreeCell (7 Piles)   |         | $10^6$          | 988,556     | 11,417     | 27      | 1,000,061   | $2.7 \times 10^5$ | 0.62 | 7.2     |
| Gaps (Basic Variant) |         | $10^7$          | 2,490,171   | 7,509,829  | 0       | 10,000,000  | $7.2 \times 10^5$ | 3.4 | 400     |
| Gaps (One Deal)      |         | $10^4$          | 8,285       | 1,107      | 608     | 11,416      | $7.2 \times 10^8$ | 2,000 | 260     |
| Golf                 |         | $10^7$          | 4,510,859   | 5,489,141  | 0       | 10,000,000  | $6.8 \times 10^5$ | 1.6 | 180     |
| King Albert          |         | $2 \times 10^6$ | 1,370,321   | 628,618    | 1,061   | 2,011,590   | $4.8 \times 10^6$ | 15 | 360     |
| Klondike             |         | $10^6$          | 819,371     | 180,472    | 157     | 1,005,717   | $2.9 \times 10^7$ | 75 | 870     |
| Klondike (WB ×)      |         | $10^6$          | 815,114     | 184,637    | 249     | 819,759     | $3.5 \times 10^6$ | 10 | 95      |
| Klondike (SP ✓, BP *)|         | $10^6$          | 999,233     | 763        | 4       | 3,366       | $1.1 \times 10^7$ | 26 | 1.0     |
| Klondike (SP ✓)      |         | $10^6$          | 949,577     | 50,406     | 17      | 180,629     | $7.5 \times 10^5$ | 2.1 | 4.4     |
| Klondike (SP ✓, BP =)|         | $10^6$          | 407,620     | 592,380    | 0       | 931,055     | $1.4 \times 10^4$ | 0.032 | 0.34    |
| Game         | Variant                  | Winnability Statistics | Search Statistics |
|--------------|--------------------------|------------------------|-------------------|
|              |                          | Sample Winnable Unwinnable Unknown | Runs Mean Nodes Mean CPU (seconds) Σ CPU (days) |
| Klondike (BP *) | 10⁶ 998,155 1,033 812 | 180,629 4.2 × 10⁷ 75 160 |
| Klondike (BP =) | 10⁶ 68,945 931,055 0 | 1,000,000 4.0 × 10⁴ 0.063 0.73 |
| Klondike (SP ×, BP *) | 10⁶ 24,068 1,134 974,798 | 2,645 2.8 × 10⁸ 600 18 |
| Klondike (SP ×) | 10⁶ 20,757 977,411 1,832 | 819,528 3.9 × 10⁷ 99 940 |
| Klondike (SP ×,BP =) | 10⁶ 1,772 998,228 0 | 68,945 3.3 × 10⁴ 0.051 0.040 |
| Klondike (Draw 1) | 10⁶ 904,226 94,629 1,145 | 180,837 6.4 × 10⁷ 190 400 |
| Klondike (Draw 1, WB ×) | 10⁶ 901,702 97,622 676 | 90,168 4.9 × 10⁷ 150 150 |
| Klondike (Draw 2) | 10⁶ 885,476 113,084 1,440 | 511,234 2.3 × 10⁷ 60 360 |
| Klondike (Draw 2, WB ×) | 10⁶ 882,409 116,624 967 | 167,750 4.4 × 10⁷ 110 220 |
| Klondike (Draw 4) | 10⁶ 693,296 306,564 140 | 779,013 2.9 × 10⁶ 8.5 77 |
| Klondike (Draw 4, WB ×) | 10⁶ 687,198 312,729 73 | 572,605 1.6 × 10⁶ 4.2 28 |
| Klondike (Draw 5) | 10⁶ 534,329 465,656 15 | 999,774 1.0 × 10⁶ 3.3 39 |
| Klondike (Draw 5, WB ×) | 10⁶ 526,376 473,621 3 | 494,742 3.4 × 10⁵ 0.94 5.4 |
| Klondike (Draw 6) | 10⁶ 358,539 641,460 1 | 819,759 3.0 × 10⁵ 1 9.6 |
| Klondike (Draw 6, WB ×) | 10⁶ 349,817 650,183 0 | 350,494 1.7 × 10⁵ 0.45 1.8 |
| Klondike (Draw 7) | 10⁶ 237,786 762,214 0 | 1,000,000 1.6 × 10⁵ 0.45 5.2 |
| Klondike (Draw 7, WB ×) | 10⁶ 229,522 770,478 0 | 237,755 1.2 × 10⁵ 0.31 0.86 |
| Klondike (Draw 8) | 10⁶ 122,759 877,241 0 | 1,000,000 6.9 × 10⁴ 0.18 2.1 |
| Klondike (Draw 8, WB ×) | 10⁶ 117,024 882,976 0 | 122,753 9.0 × 10⁴ 0.24 0.34 |
| Klondike (Draw 9) | 10⁶ 76,699 923,301 0 | 819,759 5.1 × 10⁴ 0.13 1.3 |
| Klondike (Draw 9, WB ×) | 10⁶ 72,140 927,860 0 | 76,676 7.6 × 10⁴ 0.2 0.18 |
| Klondike (Draw 10) | 10⁶ 42,372 957,628 0 | 534,352 4.1 × 10⁴ 0.11 0.66 |
| Klondike (Draw 10, WB ×) | 10⁶ 39,392 960,608 0 | 42,371 6.1 × 10⁴ 0.16 0.08 |
| Klondike (Draw 11) | 10⁶ 20,655 979,345 0 | 905,431 1.3 × 10⁴ 0.034 0.36 |
| Klondike (Draw 11, WB ×) | 10⁶ 19,037 980,963 0 | 20,654 4.5 × 10⁴ 0.12 0.029 |
| Klondike (Draw 12) | 10⁶ 8,489 991,511 0 | 358,540 1.1 × 10⁴ 0.03 0.12 |
| Klondike (Draw 12, WB ×) | 10⁶ 7,788 992,212 0 | 8,488 3.3 × 10⁴ 0.088 0.0087 |
| Klondike (Draw 13) | 10⁶ 5,998 994,002 0 | 905,431 4.3 × 10³ 0.011 0.12 |
| Game                        | Variant | Winnability Statistics | Search Statistics |
|-----------------------------|---------|-----------------------|-------------------|
|                            | Sample | Winnable | Unwinnable | Unknown | Runs | Mean Nodes | Mean CPU (seconds) | Σ CPU (days) |
| Klondike (Draw 13,WB ×)     | 10⁶    | 5,444    | 994,556    | 0       | 5,997 | 3.4 × 10⁴ | 0.092              | 0.0064      |
| Late-Binding Solitaire     | 10⁷    | 4,702,154 | 5,297,846  | 0       | 10,000,000 | 6.7 × 10⁴ | 0.054              | 6.3         |
| Mrs Mop                    | 2 × 10⁶| 1,958,661 | 38,969     | 2,370   | 2,004,805 | 1.3 × 10⁷ | 38                 | 880         |
| Northwest Territory        | 10⁶    | 683,669  | 316,287    | 44      | 1,001,297 | 4.9 × 10⁶ | 21                 | 240         |
| Raglan                     | 10⁶    | 812,184  | 187,650    | 166     | 1,000,009 | 4.1 × 10⁵ | 0.98               | 11          |
| Seahaven Towers            | 10⁷    | 8,933,178 | 1,066,822  | 0       | 10,000,000 | 8.4 × 10⁴ | 0.12               | 14          |
| Siegecraft                 | 10⁶    | 991,378  | 8,595      | 27      | 1,000,054 | 1.8 × 10⁵ | 0.31               | 3.6         |
| Simple Simon               | 10⁴    | 97,257   | 2,847      | 10      | 1,000,000 | 6.0 × 10⁴ | 0.15               | 1.7         |
| Somerset                   | 2 × 10⁶| 1,073,962 | 924,968    | 1,070   | 2,004,561 | 5.5 × 10⁵ | 2.9                | 68          |
| Spanish Patience           | 10⁷    | 9,986,239 | 13,746     | 15      | 10,000,028 | 2.0 × 10⁴ | 0.090              | 10          |
| Spider                     | 10⁴    | 9,731    | 0          | 269     | 11,494  | 2.6 × 10⁸ | 810                | 110         |
| Spiderette                 | 10⁶    | 996,153  | 3,751      | 96      | 1,000,000 | 1.0 × 10⁶ | 1.8                | 21          |
| Streets and Alleys         | 2 × 10⁶| 1,021,425 | 973,933    | 4,642   | 2,012,134 | 1.6 × 10⁷ | 29                 | 670         |
| Stronghold                 | 10⁶    | 973,689  | 26,106     | 205     | 1,000,320 | 1.7 × 10⁶ | 3.4                | 40          |
| Thirty                     | 10⁷    | 6,745,425 | 3,254,508  | 67      | 10,000,000 | 1.4 × 10⁵ | 0.22               | 26          |
| Thirtysix                  | 10⁶    | 946,196  | 52,704     | 1,100   | 1,001,085 | 9.2 × 10⁶ | 18                 | 200         |
| Trigon                     | 10⁷    | 1,599,605 | 8,400,395  | 0       | 10,000,000 | 2.7 × 10³ | 0.015              | 1.7         |
| Will o’ the Wisp           | 10⁷    | 9,992,300 | 7,487      | 213     | 10,000,906 | 2.9 × 10⁵ | 1.3                | 150         |
| Worm Hole                  | 10⁶    | 998,881  | 1,104      | 15      | 1,000,662 | 2.3 × 10⁷ | 41                 | 470         |

Table 11: Summary statistics for winnability and search for experiments reported in this paper. For search statistics, number of runs is precise with other figures given to two significant figures. † Run times for the full set of 10⁹ instances were not recorded - see main text on page 37. WB : Worrying back. SP : spaces. BP : Build Policy, FC: number of free cells.