Condition for emergence of complex eigenvalues
in the Bogoliubov-de Gennes equations

Y. Nakamura, M. Mine, M. Okumura, and Y. Yamanaka

I. INTRODUCTION

Experiments of the Bose-Einstein condensates (BECs) of neutral atomic gases, first realized in 1995 [1, 2, 3], have been offering challenging subjects in theoretical foundations of quantum many-body problem. Among others, some unstable phenomena of the BECs are very interesting, since to formulate unstable quantum many-body systems is an open problem. Observed examples of such phenomena are the split of a doubly quantized vortex into two singly quantized vortices [4], the decay of a condensate flowing in an optical lattice [5], and the decay of the initial configuration in a quenched ferromagnetic spinor BEC [6]. These phenomena are observed at very low temperatures where the dissipative mechanism brought by the thermal cloud is negligibly small. Such instability is called “dynamical instability,” and is distinguished from the “Landau instability,” in which the thermal cloud plays a dissipative role [7].

For the theoretical investigations of the instability, the Bogoliubov-de Gennes (BdG) equations [8, 9, 10] are employed. The BdG equations follow from linearization of the time-dependent Gross-Pitaevskii (TDGP) equation [11] and determine the excitation spectrum of the condensate. The BdG equations have complex eigenvalues for some cases (see below), and the presence of the complex eigenvalues is interpreted as the sign of dynamical instability. This instability is associated with the decay of the initial configuration of the condensate and can occur even at zero temperature. On the other hand, the Landau instability, which is characterized by the negative eigenvalues and in which the thermal cloud drives the system towards a lower energy state in a dissipative way, is impossible at very low temperature. It is reported that the BdG equations have complex eigenvalues in the cases where the condensate has a highly quantized vortex [12, 13, 14], where the condensate flows in an optical lattice [15], and where the condensate has gap solitons [16]. It is suggested that some degeneracy between a positive mode and a negative one in the BdG equations is necessary for the emergence of complex eigenvalues [12–18]. The analytical investigation on the problem is found in Ref. [15], where the instability of vortices in a binary mixture of BEC is studied in the weak interaction limit.

As for the dynamical instability, another treatment is known, which we refer to as the RK method given by Rossignoli and Kowalski [20]. In this method the quantum Hamiltonian of the quadratic form of creation and annihilation operators is considered, and the complex modes appear as a result of diagonalizing it with unusual operators which are neither bosonic nor fermionic ones. In the previous work [21], our group has analytically derived the condition for the existence of complex modes for the case where the condensate has a highly quantized vortex, by using the RK method. There a small coupling expansion is adopted, and the two-mode approximation on the Hamiltonian is assumed and is essential. The three-mode analysis is also performed, and it is confirmed that the condition for the existence of complex modes is not modified. However, it is not clarified why the two-mode approximation is crucial for the appearance of complex modes.

In this paper, we study analytically the BdG equations whose two-component eigenfunctions are not only of positive-norm but also of negative-norm or of zero-norm. It is shown that the degeneracy between a positive-norm mode and a negative-norm one is necessary for the emergence of the complex eigenvalues. To do this, we ex-
and the BdG equations in powers of a shift about a
value of the coupling constant at which all the eigenvalues
are real. The analytic study in Ref. [19] has already
indicated the importance of the degeneracy, which was
referred to as the frequency resonance of positive- and
negative-energy excitations there. It is emphasized that
the present analysis is quite simple and general, not
restricted to particular systems. Furthermore, our study
is based on the expansion of the BdG equations around a
finite (non-zero in general) coupling constant, while the
one in Ref. [19] is limited to the small coupling constant.
We apply the considerations developed here to the case
where the condensate has a highly quantized vortex, and
show that the degeneracy between the two modes
with different norms causes the complex eigenvalues. In
Section III, we expand both the GP and BdG
equations in powers of a shift about a
restricted to particular systems. Furthermore, our study is
based on the expansion of the BdG equations around a
finite (non-zero in general) coupling constant, while the
one in Ref. [19] is limited to the small coupling constant.

We apply the considerations developed here to the case
where the condensate has a highly quantized vortex, and
show that the degeneracy between the two modes
with different norms causes the complex eigenvalues. In
Section III, we expand both the GP and BdG
equations in powers of a shift about a
finite (non-zero in general) coupling constant, while the
one in Ref. [19] is limited to the small coupling constant.

This paper is organized as follows: In Section II, the
BdG equations are introduced and their properties are
shown. In Section III, we expand both the GP and BdG
equations in powers of a shift of the coupling constant,
and show that the degeneracy between the two modes
with different norms causes the complex eigenvalues. In
Section IV, we apply the analysis to the case where the
condensate has a highly quantized vortex. Section V is
devoted to the summary.

II. BOGOLIUBOV-DE GENNES EQUATIONS
AND THEIR PROPERTIES

We start with the following TDGP equation to describe
the dynamics of the condensate,

\[ i \frac{\partial}{\partial t} \Psi(x, t) = [K + V(x) + g|\Psi(x, t)|^2] \Psi(x, t), \] (1)

where \( K = -\frac{1}{2M} \nabla^2 \) with the mass of a neutral atom
\( M \), and \( V(x) \) and \( g \) are the trap potential for the atoms
and the coupling constant of self-interaction, respectively.
Throughout this paper \( \hbar \) is set to be unity. The
quantity \( |\Psi(x, t)|^2 \) corresponds to the density of condensate
atoms, and the total condensate number \( N \) is given by
\( N = \int dx^3 |\Psi(x, t)|^2 \). The stationary state is given as
\( \Psi(x, t) = \xi(x)e^{-i\mu t} \) with the chemical potential \( \mu \),
where the function \( \xi(x) \) satisfies the stationary Gross-Pitaevskii
(GP) equation,

\[ [K + V(x) - \mu + 2g|\xi(x)|^2] \xi(x) = 0. \] (2)

To study the excitation spectrum of the condensate,
let us consider a small fluctuation around the stationary
state as \( \Psi(x, t) = [\xi(x) + \delta \Psi(x, t)]e^{-i\mu t} \).
Substituting it into Eq. (1) and linearizing it with respect to
\( \delta \Psi(x, t) = u_n(x)e^{-iE_n t} + v_n(x)e^{iE_n t} \), we obtain the following
BdG equations,

\[ Ty_n(x) = E_n y_n(x). \] (3)

Here the doublet notation is introduced as

\[ y_n(x) = \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix}, \]

\[ T = \begin{pmatrix} \mathcal{L} & \mathcal{M} \\ -\mathcal{M}^* & -\mathcal{L} \end{pmatrix}, \] (5)

where

\[ \mathcal{L} = K + V(x) - \mu + 2g|\xi(x)|^2, \]

\[ \mathcal{M} = gE_n(x). \] (7)

The eigenvalue \( E_n \) is not necessarily real, since \( T \) is not
a Hermitian operator. The similar contents given in this
section below are presented in Ref. [22] for a general case,
and in Refs. [13, 19, 23] for the particular systems.

It is easily shown that the operator \( T \) has the following
algebraic properties [22, 23]:

\[ \sigma_3 T \sigma_3 = T^\dagger, \]

\[ \sigma_1 T \sigma_1 = -T^*, \] (9)

with the \( i \)-th Pauli matrix \( \sigma_i \). These relations regulate
the properties of the eigenfunctions in the following.

Let us introduce the “inner product” for arbitrary
doublets \( s(x) \) and \( t(x) \) as

\[ \langle s(x), t(x) \rangle = \int dx^3 s^\dagger(x)\sigma_3 t(x), \] (10)

and the squared “norm” \( \|s\|^2 \) as

\[ \|s\|^2 = \langle s(x), s(x) \rangle. \] (11)

Note that the squared “norm” may be zero or negative
due to the indefinite metric \( \sigma_3 \).

It is shown from Eq. (8) that the operator \( T \) is pseudo
Hermitian, namely,

\[ \langle s(x), T t(x) \rangle = \langle T s(x), t(x) \rangle. \] (12)

Because of the conjugate symmetry of the “inner product,”

\[ \langle s, t \rangle = \langle t, s \rangle^*, \] (13)

Eq. (12) implies that the matrix element \( \langle s, Ts \rangle \) is
always real. Furthermore, Eq. (12) leads to

\[ (E_n - E_m^*) \langle y_m, y_n \rangle = 0, \] (14)

and

\[ \text{Im}(E_n)\|y_n\|^2 = 0, \] (15)

where \( y_n \) and \( y_m \) are the eigenfunctions of the BdG equations
which belong to the eigenvalues \( E_n \) and \( E_m \), respectively. Two eigenvectors \( y_n \) and \( y_m \) are orthogonal to
each other for $E_n^* \neq E_m$. If $E_n$ is a complex eigenvalue \[[\text{Im}(E_n)] \neq 0\], then we have $\|y_n\|^2 = 0$. Equivalently, if $\|y_n\|^2 \neq 0$, then $E_n$ is real.

Next, if $y$ is an eigenfunction belonging to a non-zero real eigenvalue $E$, one can show from Eq. (9) that $z \equiv \sigma_1 y^*$ is an eigenfunction belonging to $-E$. One also finds $\|y\|^2 = -\|z\|^2$, as is seen in Refs. [13, 14, 19, 22, 23].

Hereafter, we denote positive- and negative-norm eigenfunctions by $y_n$ and $z_n$, respectively, which make a pair and are related to each other as $z_n = \sigma_1 y_n$. Without losing the generality, the absolute values of their norms can be set to unity,

\[\begin{align*}
(y_n, y_m) &= \delta_{nm}, \\
(z_n, z_m) &= -\delta_{nm}, \\
(y_n, z_m) &= 0.
\end{align*}\]  

(16) (17) (18)

III. GENERAL CONDITION FOR EMERGENCE OF COMPLEX EIGENVALUES

In this section, in order to derive the condition for the emergence of complex eigenvalues, we employ the perturbation theory, dividing the coupling constant $g$ as

\[g = g_0 + \varepsilon g',\]  

(19)

with a sufficiently small parameter $\varepsilon$. We consider the situation that all the eigenvalues are real for the unperturbed coupling constant $g_0$ and that the complex mode arises due to the perturbation $\varepsilon g'$ in Eq. (19). This approach is quite general and is applicable to most dynamical instabilities of the BECs such as the splitting of a vortex, the decay of the flowing condensate in an optical lattice. Similarly to the following analysis, it is also possible to develop the perturbation theory not of the coupling constant but of the strength of the optical lattice potential for the flowing condensate in an optical lattice, although we do not present it in this paper.

The quantities appearing in the GP and BdG equations, (11) and (23), are expanded in terms of $\varepsilon$ as

\[\begin{align*}
\xi(x) &= \xi^{(0)}(x) + \varepsilon \xi^{(1)}(x) + O(\varepsilon^2), \\
y(x) &= y^{(0)}(x) + \varepsilon y^{(1)}(x) + O(\varepsilon^2), \\
\mu &= \mu^{(0)} + \varepsilon \mu^{(1)} + O(\varepsilon^2), \\
E &= E^{(0)} + \varepsilon E^{(1)} + O(\varepsilon^2).
\end{align*}\]  

(20) (21) (22) (23)

We obtain at the orders of $\varepsilon^0$ and $\varepsilon^1$,

\[\begin{align*}
0 &= \mathcal{L}_0 \xi^{(0)} - \mathcal{M}_0 \xi^{(0)*}, \\
0 &= \mathcal{L}_0 \xi^{(1)} - \mathcal{M}_0 \xi^{(1)*} + \mathcal{L}' \xi^{(0)} - \mathcal{M}' \xi^{(0)*}, \\
0 &= (T_0 - E^{(0)}) y^{(0)}(x) = 0, \\
(T_0 - E^{(0)}) y^{(1)}(x) + (T' - E^{(1)}) y^{(0)}(x) &= 0.
\end{align*}\]  

(24) (25) (26) (27)

where

\[\begin{align*}
T_0 &= \begin{pmatrix} \mathcal{L}_0 & \mathcal{M}_0 \\ -\mathcal{M}_0^* & -\mathcal{L}_0 \end{pmatrix}, \\
T' &= \begin{pmatrix} \mathcal{L}' & \mathcal{M}' \\ -\mathcal{M}'^* & -\mathcal{L}' \end{pmatrix},
\end{align*}\]  

(28) (29)

with

\[\begin{align*}
\mathcal{L}_0 &= K + V - \mu^{(0)} + 2g_0 |\xi^{(0)}|^2, \\
\mathcal{L}' &= 2g' |\xi^{(0)}|^2 + 4g_0 \text{Re}\left(\xi^{(0)} \xi^{(1)*}\right) - \mu^{(1)}, \\
\mathcal{M}_0 &= g_0 \xi^{(0)*2}, \\
\mathcal{M}' &= g' \xi^{(0)*2} + 2g_0 \xi^{(0)} \xi^{(1)}.
\end{align*}\]  

(30) (31) (32) (33)

Note that the operator $T'$ is pseudo-Hermitian. In the case where no degeneracy is present, the first-order eigenvalue is

\[E^{(1)} = \begin{pmatrix} y^{(0)}, T'y^{(0)} \end{pmatrix}.\]  

(34)

Since $T'$ is pseudo-Hermitian, $E^{(1)}$ is always real [see the discussion below Eq. (12)]. Hence, if all the eigenvalues are real and there is no degeneracy for the unperturbed coupling constant $g_0$, then no complex eigenvalue exists for $g = g_0 + \varepsilon g'$.

Next, let us turn to degenerate cases. Here we assume for simplicity that only a single pair of modes is degenerate.

First, we consider the case in which two positive-norm modes $y_1^{(0)}$ and $y_2^{(0)}$ share the same eigenvalue $E^{(0)}$ which is real. Using the degenerate perturbation theory, we obtain the following secular equation which determines $E^{(1)}$,

\[\begin{vmatrix}
(y_1^{(0)}, T'y_1^{(0)}) - E^{(1)} & (y_1^{(0)}, T'y_2^{(0)}) \\
(y_2^{(0)}, T'y_1^{(0)}) & (y_2^{(0)}, T'y_2^{(0)}) - E^{(1)}
\end{vmatrix} = 0.\]  

(35)

This implies that $E^{(1)}$ is the eigenvalue of the Hermitian matrix,

\[\begin{vmatrix}
(y_1^{(0)}, T'y_1^{(0)}) & (y_1^{(0)}, T'y_2^{(0)}) \\
(y_2^{(0)}, T'y_1^{(0)}) & (y_2^{(0)}, T'y_2^{(0)})
\end{vmatrix},\]  

(36)

hence $E^{(1)}$ is real. Likewise, it is also shown that no complex eigenvalue arises from the degeneracy between two negative-norm modes, $z_1^{(0)}$ and $z_2^{(0)}$.

Suppose that the two modes of one positive- and one negative-norms, $y_1^{(0)}$ and $z_2^{(0)}$, are degenerate for a real eigenvalue $E^{(0)}$. We then obtain the secular equation,

\[\begin{vmatrix}
(y_1^{(0)}, T'y_1^{(0)}) - E^{(1)} & (y_1^{(0)}, T'z_2^{(0)}) \\
(z_2^{(0)}, T'y_1^{(0)}) & (z_2^{(0)}, T'z_2^{(0)}) + E^{(1)}
\end{vmatrix} = 0.\]  

(37)
which implies that $E^{(1)}$ is the eigenvalue of the non-Hermitian matrix
\[
\begin{pmatrix}
(y_1^{(0)}, T' y_1^{(0)}) & (y_2^{(0)}, T' z_2^{(0)}) \\
(z_2^{(0)}, T' y_1^{(0)}) & (z_2^{(0)}, T' z_2^{(0)})
\end{pmatrix}.
\] (38)

In this case $E^{(1)}$ can be complex. Actually, $E^{(1)}$ becomes complex if the following condition is satisfied,
\[
\left|\begin{pmatrix}
y_1^{(0)}, T' y_1^{(0)} \\
z_2^{(0)}, T' z_2^{(0)}
\end{pmatrix} + \begin{pmatrix}
z_2^{(0)}, T' y_1^{(0)} \\
y_1^{(0)}, T' z_2^{(0)}
\end{pmatrix}\right| < 2 \left|\begin{pmatrix}
y_1^{(0)}, T' z_2^{(0)} \\
z_2^{(0)}, T' y_1^{(0)}
\end{pmatrix}\right|,
\] (39)

where Eqs. (12) and (13) are employed.

We conclude that the degeneracy between modes of positive- and negative-norms, $y$ and $z$, is essential for the emergence of the complex eigenvalue. This can be extended to the cases where more than two modes are degenerate, because the degeneracy between $y$’s and $z$’s gives rise to a non-Hermitian matrix such as (38). Our conclusion is consistent with the suggestions in Refs. 13-14. Note that for the emergence of the complex eigenvalues the degeneracy is necessary but not sufficient and that the inequality in Eq. (39) must be satisfied in addition.

IV. APPLICATION TO THE CASE OF A HIGHLY QUANTIZED VORTEX AND VALIDITY OF TWO-MODE APPROXIMATION

In this section, as an application of the formulation developed in the preceding section, we derive the expression of the condition for the emergence of the complex eigenvalues in a trapped BEC with a highly quantized vortex. The application elucidates why the two-mode approximation is valid in our previous work where the RK method is used [21].

We consider a harmonic trap potential with a cylindrical symmetry:
\[
V(r, z) = \frac{1}{2} M (\omega_r^2 r^2 + \omega_z^2 z^2),
\] (40)

where $r = \sqrt{x^2 + y^2}$. Let us assume that the vortex is created along the $z$-axis. Then the order parameter $\xi(x)$ can be written as
\[
\xi(x) = \sqrt{\frac{N}{2\pi}} e^{i\delta(x)} f(r, z),
\] (41)

with the winding number $\kappa$ which is an integer. Without losing the generality, $\kappa$ is assumed to be a positive integer. We consider the case of $g_0 = 0$, which corresponds to the weak coupling limit. The quantities $L_0$, $M_0$, $L'$, and $M'$ are written as
\[
\begin{align*}
L_0 &= K + V - \mu^{(0)}, \\
L' &= -\mu^{(1)} + 2g' |\xi^{(0)}|^2, \\
M_0 &= 0, \\
M' &= g' \xi^{(0)} \xi^{(0)*}.
\end{align*}
\] (42-45)

The eigenfunctions of the unperturbed BdG equations are written as
\[
\begin{align*}
y_{n,l,m}^{(0)}(r, \theta, z) &= \left(U_{n,l,m}(r, \theta, z) \right), \\
z_{n,l,m}^{(0)}(r, \theta, z) &= \left(U_{n,l,m}^*(r, \theta, z) \right),
\end{align*}
\] (46-47)

which belong to the eigenvalues $\epsilon_{n,l,m}$ and $-\epsilon_{n,l,m}$, respectively. Here, $n = 0, 1, 2, \ldots$ is the principal quantum number, $l = 0, \pm 1, \pm 2, \ldots$ is the magnetic quantum number, and $m = 0, 1, 2, \ldots$ is the quantum number along the $z$ axis. The function $U_{n,l,m}(r, \theta, z)$ is expressed as
\[
U_{n,l,m}(r, \theta, z) = \sqrt{\frac{1}{2\pi}} e^{i(l+\kappa)\theta} u_{n,l,m}(r, z),
\] (48)

with the solution $u_{n,l,m}(r, z)$ of the following eigen-equation:
\[
-\frac{1}{2M} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{(l+\kappa)^2}{r^2} + \frac{\partial^2}{\partial z^2}\right) + V(r, z) - \mu^{(0)} \right) u_{n,l,m}(r, z) = \epsilon_{n,l,m} u_{n,l,m}(r, z).
\] (49)

The explicit form of $u_{n,l,m}(r, z)$ is given by the Laguerre polynomials $L_n^l(x)$ and the Hermite polynomials $H_m(x)$ as
\[
\begin{align*}
u_{n,l,m}(r, z) &= C_{n,l,m} e^{-\frac{1}{2} \omega_z^2 r^2 (\alpha_{\perp} r)^{l+|\kappa|}} L_n^{l+|\kappa|}(\alpha_{\perp} r^2) \\
&\quad \times e^{-\frac{1}{2} \omega_z^2 z^2} H_m(\alpha_z z),
\end{align*}
\] (50)

with
\[
C_{n,l,m} = \sqrt{\frac{2^{\frac{n}{2}} \omega_z}{\pi}} \sqrt{\frac{n!}{2^m \cdot m! \cdot (|l+\kappa|+n)!}},
\] (51)

while the eigenvalue is given as
\[
\epsilon_{n,l,m} = \omega_z (2n + |l+\kappa| + 1) + \omega_z \left( m + \frac{1}{2} \right) - \mu^{(0)}.
\] (52)

The normalization constant [41] is determined so that
\[
\int d^3 x U_{n,l,m}^*(r, \theta, z) U_{n', l', m'}(r, \theta, z) = \delta_{n,n'} \delta_{l,l'} \delta_{m,m'},
\] (53)

and $\|y_{n,l,m}^{(0)}\|^2 = -\|z_{n,l,m}^{(0)}\|^2 = 1$.

The unperturbed order parameter and the chemical potential are obtained as
\[
\begin{align*}
\xi^{(0)}(r, \theta, z) &= \sqrt{N} U_{0,0,0}(r, \theta, z), \\
\mu^{(0)} &= \omega_z (|\kappa|+1) + \frac{1}{2} \omega_z.
\end{align*}
\] (54-55)

Substituting Eqs. (12)-(15) into Eq. (25), we obtain
\[
\begin{align*}
\mu^{(1)} &= D \frac{(2\kappa)!}{(|\kappa|)!^2},
\end{align*}
\] (56)
Calculating the inner products, we finally obtain the expression for the matrix elements of $T'$.

\[
D = \frac{g'N}{\sqrt{8\pi}} \frac{\alpha_z^2 \alpha_x}{2\kappa + 2}.
\]  

(57)

\[
\begin{align*}
(y^{(0)}_{n, l, m}, T' y^{(0)}_{n', l', m'}) &= \delta_{ll'} \frac{g'N}{\pi} \int dr dz \, \mu u_{n,l,m} u_{n',l,m'} - \delta_{nn'} \delta_{ll'} \mu (1), \\
(z^{(0)}_{n, l, m}, T' z^{(0)}_{n', l', m'}) &= \delta_{ll'} \frac{g'N}{\pi} \int dr dz \, \mu u_{n,l,m} u_{n',l,m'} + \delta_{nn'} \delta_{ll'} \mu (1), \\
(y^{(0)}_{n, l, m}, T' z^{(0)}_{n', l', m'}) &= \delta_{l,-l'} \frac{g'N}{2\pi} \int dr dz \, \mu u_{n,l,m} u_{n',l',-m'}.
\end{align*}
\]

(58)  

(59)  

(60)

We notice that the matrix elements in Eqs. (58) and (59) vanish unless $l = l'$, while that in Eq. (60) vanishes unless $l = -l'$. According to the general formulation of the degenerate perturbation theory, the whole secular determinant is a product of secular determinants of submatrices corresponding to the subspaces spanned by the degenerate eigenfunctions. We therefore inquire the degeneracy in Eqs. (58), (59). Concerning Eqs. (58) and (59), the degeneracy condition $\epsilon_{n,l,m} = \epsilon_{n',l',m'}$ implies $l = l'$, and then $n = n'$ and $m = m'$. On the other hand, concerning Eq. (60), the condition is given by $\epsilon_{n,l,m} = -\epsilon_{n',l',m'}$, and is satisfied when $l = -l'$, and $n = n' = m = m' = 0$ and $|l| \leq \kappa$. Consequently, only the pairs of $y^{(0)}_{0,1,0}$ and $z^{(0)}_{0,-1,0}$ are degenerate and give the non-trivial secular determinants. Thus, the whole secular determinant is a product of the following three kinds of factors:

\[
\begin{align*}
(i) \quad &\left( y^{(0)}_{n, l, m}, T' y^{(0)}_{n, l, m} \right) - E^{(1)}, \\
(ii) \quad &\left( z^{(0)}_{n, l, m}, T' z^{(0)}_{n, l, m} \right) + E^{(1)}, \\
(iii) \quad &\begin{vmatrix}
(y^{(0)}_{0, l, 0}, T' y^{(0)}_{0, l, 0}) - E^{(1)} & (y^{(0)}_{0, l, 0}, T' z^{(0)}_{0, -l, 0}) \\
(z^{(0)}_{0, -l, 0}, T' y^{(0)}_{0, l, 0}) & (z^{(0)}_{0, -l, 0}, T' z^{(0)}_{0, -l, 0}) + E^{(1)}
\end{vmatrix}.
\end{align*}
\]

It is clear that the only term which can cause $E^{(1)}$ complex is (iii), which is the form of the determinant of a $2 \times 2$ matrix, as discussed in the preceding section. This analytic result justifies the two-mode analysis assumed in our previous work [21], and clarifies which two modes are relevant.

Now the condition for the emergence of the complex mode is given by

\[
\begin{align*}
\left| \left( y^{(0)}_{0, l, 0}, T' y^{(0)}_{0, l, 0} \right) + (z^{(0)}_{0, -l, 0}, T' z^{(0)}_{0, -l, 0}) \right| &< 2 \left| \left( z^{(0)}_{0, -l, 0}, T' y^{(0)}_{0, l, 0} \right) \right|.
\end{align*}
\]

(61)

Calculating the inner products, we finally obtain the expression of the condition as

\[
\begin{align*}
\frac{(2\kappa + l)!}{2^l \kappa!(\kappa + l)!} + \frac{(2\kappa - l)!}{2^{-l} \kappa!(\kappa - l)!} &< \frac{(2\kappa)!}{\kappa! \sqrt{(\kappa + l)!}(\kappa - l)!}.
\end{align*}
\]

(62)

We can see from the inequality that the complex eigenvalues appear when $|l| = 2$ with $\kappa = 2$, $|l| = 2, 3$ with $\kappa = 3$, or $|l| = 2, 3$ with $\kappa = 4$. In the case of $\kappa = 0, 1$, the inequality is never satisfied and no complex eigenvalue appears. This result is consistent with that of the numerical studies of the BdG equations [12, 13], and reproduces our previous one [21].

In our previous analysis [21], we did not treat the BdG equations but the RK Hamiltonian with the two-mode approximation, although its justification was not clear then. In this section, we have confirmed for the system...
of a highly quantized vortex system that the two-mode analysis in our previous treatment is justified, as the crucial factor of the whole secular determinant is the secular determinant of a $2 \times 2$ matrix. Furthermore, it is now clear which two modes are relevant.

V. SUMMARY

We have investigated the condition for the appearance of the dynamical instability of the Bose-condensed system, whose sign is interpreted to be the emergence of the complex eigenvalues in the BdG equations, using the perturbation theory for both the GP and BdG equations with respect to the coupling constant. We note in comparison with Ref. [10] that our method is quite simple and general without being confined to particular systems, and that the perturbative expansion is made around a finite (non-zero in general) coupling constant.

Our important conclusion is that the emergence of the complex eigenvalues in the BdG equations is attributed to the degeneracy between a positive-norm eigenmode and a negative-norm one. The expression of the condition for the existence of the complex eigenvalues follows from the secular equation of the $2 \times 2$ matrix. We have considered the situation that all the eigenvalues are real for the unperturbed coupling constant $g_0$, and that the perturbation in the coupling constant gives rise to the complex modes.

Inspired by the analysis on the flowing condensate in an optical lattice with sufficiently small lattice height [14], suggesting that the complex eigenvalues are caused by the degeneracy, Taylor and Zaremba solved the BdG equations in the weak potential limit and confirmed the suggestion [22]. Similarly as in the case of the flowing condensate in an optical lattice, the perturbation with respect to the potential strength can be considered. Following the similar discussion as we developed in this paper, we develop the perturbation theory not of the coupling constant but of the potential strength. Then it will turn out that the appropriate degeneracy at the 0-th order of the BdG equations is necessary for the emergence of the complex eigenvalues. In this way, our method presented in this paper includes the method of Taylor and Zaremba.

Furthermore, we have applied the analysis to the case where the condensate has a highly quantized vortex, and have rederived the expression of the condition for emergence of the complex eigenvalues. We have confirmed why the two-mode approximation applied in our previous work [21] is valid, deriving the form of the secular equation, which is a product of $2 \times 2$ matrix or less.

Acknowledgments

Y.N. is supported partially by “Ambient SoC Global COE Program of Waseda University” of the Ministry of Education, Culture, Sports, Science and Technology, Japan. M.M. is supported partially by the Grant-in-Aid for The 21st Century COE Program (Physics of Self-organization Systems) at Waseda University, and Waseda University Grant for Special Research Projects. This work is partly supported by Grant-in-Aid for Scientific Research (C) (No. 17540364) from the Japan Society for the Promotion of Science and one for Young Scientists (B) (No. 17740258) from the Ministry of Education, Culture, Sports, Science and Technology, Japan. The authors thank the Yukawa Institute for Theoretical Physics at Kyoto University for offering us the opportunity to discuss this work during the YITP workshop YITP-W-07-07 on “Thermal Quantum Field Theories and Their Applications”.

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198 (1995).
[2] K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
[3] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. 75, 1687 (1995).
[4] Y. Shin, M. Saba, M. Vengalattore, T. A. PASQUINI, C. Sanner, A. E. Leanhardt, M. Premkis, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 93, 160406 (2004).
[5] L. Fallani, L. De Sarlo, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, Phys. Rev. Lett. 93, 140406 (2004).
[6] L. E. Sadler, J. M. Higbie, S. R. Leslie, M. Vengalattore, and D. M. Stamper-Kurn, Nature 443, 312 (2006).
[7] L. Pitaevskii, and S. Stringari, Bose-Einstein Condensation, (Oxford University Press, New York, 2003), p.255.
[8] N. N. Bogoliubov, J. Phys. (Moscow) 11, 23 (1947).
[9] P. G. de Gennes, Superconductivity of Metals and Alloys (Benjamin, New York, 1966).
[10] A. L. Fetter, Ann. of Phys. 70, 67 (1972).
[11] F. Dalfovo, S. Giorgini, L. P. Pitaevskii and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[12] H. Pu, C. K. Law, J. H. Eberly, and N. P. Bigelow, Phys. Rev. A 59, 1533 (1999).
[13] Y. Kawaguchi and T. Ohmi, Phys. Rev. A 70, 043610 (2004).
[14] B. Wu and Q. Niu, Phys. Rev. A 64, 061603(R) (2001).
[15] B. Wu and Q. Niu, New J. Phys. 5, 104 (2005).
[16] K. M. Hilligsoe, M. K. Oertbhaler, and K. -P. Marlzin, Phys. Rev. A 66, 063605 (2002).
[17] W. Zhang, D. L. Zhou, M. -S. Chang, M. S. Chapman, and L. You, Phys. Rev. Lett. 95, 180403 (2005).
[18] D. C. Roberts and M. Ueda, Phys. Rev. A. 73, 053611 (2006).
[19] D. V. Skryabin, Phys. Rev. A 63, 013602 (2000).
[20] R. Rossignoli and A. M. Kowalski, Phys. Rev. A 72,
[21] E. Fukuyama, M. Mine, M. Okumura, T. Sunaga, and Y. Yamanaka, Phys. Rev. A 76, 043608 (2007).
[22] M. Mine, M. Okumura, T. Sunaga, and Y. Yamanaka, Ann. Phys. (N.Y.) 322, 2327 (2007).
[23] K. Kobayashi, M. Mine, M. Okumura, and Y. Yamanaka, e-print arXiv:0706.0959, Ann. Phys. (N.Y.) (to be published).
[24] We use the following conventions: $L^k_n(x) = \frac{e^x x^{-k} d^m}{dx^m}(e^{-x} x^{n+k})$, $H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} e^{-x^2}$.
[25] E. Taylor and E. Zaremba, Phys. Rev. A 68, 053611 (2003).