Morphisms and automorphisms of skew-symmetric Lotka–Volterra systems

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Abstract

We study the basic relation between skew-symmetric Lotka–Volterra (LV) systems and graphs, both at the level of objects and morphisms, and derive a classification from it of skew-symmetric LV systems in terms of graphs as well as in terms of irreducible weighted graphs. We also obtain a description of their automorphism groups and of the relations which exist between these groups. The central notion introduced and used is that of decloning of graphs and of LV systems. We also give a functorial interpretation of the results which we obtain.

Keywords: Lotka–Volterra systems, graphs, integrability

(Some figures may appear in colour only in the online journal)
1. Introduction

In its most general form, a Lotka–Volterra (LV) system in dimension \( n \) is a dynamical system, described by the following system of differential equations:

\[
\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{i,j} x_i x_j, \quad i = 1, 2, \ldots, n. \tag{1.1}
\]

The coefficients \( \varepsilon_i \) and \( a_{i,j} \) are real or complex numbers, depending on whether one uses \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{C} \) as the base field. These equations first appeared in the study of population dynamics [16, 19, 20]. The LV systems which we study in this paper are skew-symmetric, which means that \( \varepsilon_i = 0 \) and \( a_{i,j} = -a_{j,i} \) for \( 1 \leq i, j \leq n \). These conditions do not only mean that we can interpret the coefficients \( a_{i,j} \) as the entries of a skew-symmetric matrix \( A \), but they also imply that (1.1) is a Hamiltonian system: as Hamiltonian structure we can take the quadratic Poisson structure \( \pi_A \) on \( \mathbb{F}^n \), defined in terms of the natural coordinates \( x_1, \ldots, x_n \) by the Poisson brackets \( \{ x_i, x_j \}_A := a_{i,j} x_i x_j \) for \( 1 \leq i, j \leq n \), and as Hamiltonian one can take the sum of all coordinates, \( H := x_1 + x_2 + \cdots + x_n \). It is well-known that \( \pi_A \) is indeed a Poisson structure.
(see for example [2, 15]), and it is clear that the Hamiltonian vector field \( X_H := \{ \cdot, H \}_A \) is given by (1.1) (with all \( \varepsilon_i \) equal to zero). In what follows we refer simply to skew-symmetric Lotka–Volterra systems as LV systems and we always take the Poisson structure \( \pi_A \) as its Hamiltonian structure.

It is natural to think of the skew-symmetric matrix \( A = (a_{ij}) \) as being the adjacency matrix of a graph, having an arc from the vertex \( i \) to the vertex \( j \) with value \( a_{ij} \) if \( a_{ij} \neq 0 \) and \( i < j \), and having no arc from \( i \) to \( j \) otherwise (see definition 2.1 below for a more intrinsic description of these graphs, which we will call skew-symmetric graphs). This well-known relation between graphs and LV systems has already been successfully used in the study of the dynamics of LV systems (see for example [5, 7]). In this paper, we use it for studying morphisms and automorphisms of LV systems, and we show how the classification of LV systems and of graphs are related.

A first result (proposition 3.2) states that morphisms between skew-symmetric graphs induce morphisms between the corresponding LV systems. By the latter, we mean a smooth map which preserves both the Poisson structure and the Hamiltonian. Moreover, the link between skew-symmetric graphs and LV systems is functorial, hence isomorphic graphs lead to isomorphic LV systems and graph automorphisms lead to automorphisms of LV systems. By construction, every such induced morphism is linear and a natural question is whether every linear morphism of LV systems is induced by a graph morphism. The answer is negative in general, even for automorphisms, but there is an easy characterization of the graphs for which every automorphism of the corresponding LV system is induced by a graph automorphism. We call these graphs, and the corresponding LV systems, irreducible: an irreducible graph is characterized by the fact that no two of its vertices have identical neighborhoods; said differently, its adjacency matrix has no equal rows (or columns, what amounts to the same).

For LV systems which are not irreducible, we show that the automorphism group is infinite, which proves that it is strictly larger than the automorphism group of the underlying graph. In order to have a precise description of the former automorphism group, we introduce the notion of decloning, which associates to a graph (and hence to its associated LV system) an irreducible graph (and an irreducible LV system). At the level of the graph this is done by identifying all vertices which have the same neighborhood; at the level of the LV system, the irreducible LV system is obtained by a Poisson reduction, which is a morphism of Hamiltonian systems. It leads to the following description of the automorphism group of the LV system, associated to any skew-symmetric graph \( \Gamma \):

\[
\text{Aut}(\text{LV}(\Gamma)) \simeq \prod_{\mathcal{L}(\Sigma)} \text{GL}^+(m, \mathbb{F}) \rtimes \text{Aut}(\Gamma, \mathcal{L}(\Sigma)).
\]

In this formula, \( \mathcal{L}(\Sigma) \) is the decloning of \( \Gamma \) and the integers \( \mathcal{L}(\Sigma) \) count for every vertex of \( \Gamma \) how many vertices of \( \Gamma \) have been identified by the decloning, in order to obtain \( \mathcal{L} \); finally, \( \text{GL}^+(m, \mathbb{F}) \) stands for the subgroup of \( \text{GL}(m, \mathbb{F}) \), fixing some particular non-zero vector. We also give a formula for the automorphism group of the graph \( \Gamma \),

\[
\text{Aut}(\Gamma) \simeq \prod_{\mathcal{L}(\Sigma)} \text{GL}(m, \mathbb{F}) \rtimes \text{Aut}(\Gamma, \mathcal{L}(\Sigma)).
\]

As we will show, all these automorphism groups fit naturally in a commutative diagram, whose top line contains the graphs and whose bottom line contains the corresponding LV systems.

We establish various properties of decloning. First, there is functoriality: we show that any surjective graph morphism induces a morphism between the decloned graphs, and similarly
at the level of their LV systems, leading to the following commutative diagram of categories and functors:

\[
\begin{array}{ccc}
\text{Gr} & \xrightarrow{\rho} & \text{Gr}_0 \\
\text{LV} & \xrightarrow{j_0} & \text{LV}_0 \\
\text{LV} & \xrightarrow{j_0} & \text{LV}_0
\end{array}
\]

In this diagram, \( \text{Gr} \) and \( \text{Gr}_0 \) stand for the categories of graphs, respectively irreducible graphs, with surjective morphisms, and \( j_0 \) stands for the natural embedding of \( \text{Gr}_0 \) in \( \text{Gr} \). Similarly, \( \text{LV} \) and \( \text{LV}_0 \) stand for the categories of LV systems, respectively irreducible LV systems, with surjective morphisms, and \( j_0 \) stands for the natural embedding of \( \text{LV}_0 \) in \( \text{LV} \). The decloning functors are denoted by \( \rho \) and \( \sigma \) and can be described as adjoint functors to \( j_0 \) respectively.

Decloning also plays an important role in the classification of LV systems. With some extra work, it follows from what precedes that if two irreducible LV systems \( \text{LV}(\Gamma) \) and \( \text{LV}(\Gamma') \) are linearly isomorphic, then the graphs \( \Gamma \) and \( \Gamma' \) are isomorphic; by decloning and thanks to a normal form that we give for linear morphisms onto irreducible LV systems, we show that the statement remains true for arbitrary LV systems, not necessarily irreducible ones. Moreover, we may replace in the statement ‘linearly isomorphic’ by ‘smoothly isomorphic’, since we show that when two LV systems are smoothly isomorphic, then they are linearly isomorphic. In conclusion we get one of the main results of this paper: two LV systems are smoothly isomorphic if and only if the underlying graphs are isomorphic. In that sense, the correspondence between graphs and LV systems is a bijective one. This result can be used to classify the LV systems having a specific property by translating the property, and working, on graphs. We have successfully done this by classifying the LV systems having the Kahan–Poisson property (see [9]).

Many LV systems have been shown to be integrable (i.e., Liouville integrable or superintegrable), often by using Lax equations [3, 4, 6, 8, 17, 18]. We show that an LV system \( \text{LV}(\Gamma) \) is integrable if and only if the declonedsystem \( \text{LV}(\Gamma') \) is integrable. Moreover, we give a construction which provides a (regular) Lax equation for \( \text{LV}(\Gamma) \) from a (regular) Lax equation for \( \text{LV}(\Gamma') \), if any. This leads in the particular case of the much studied Bogoyavlenskij systems \( B(n,k) \), for which a particularly elegant Lax pair is known [4], to a Lax pair for any cloned Bogoyavlenskij system; for the latter systems, we will provide also an alternative Lax pair.

We also give a population dynamics interpretation of cloning and decloning, showing that these operations are very natural.

The structure of the paper is as follows. In section 2 we introduce the basic graph terminology which we will use, we define graph decloning and establish its functorial properties. We use it to determine the group of graph automorphisms of a graph in terms of the graph automorphisms of its decloned graph. After recalling the basic definitions of LV systems, associated to skew-symmetric graphs, we introduce and study in section 3 the decloning of LV systems. Decloning of LV systems and the correspondence between graphs and LV systems are studied from the functorial point of view, leading to a description of the automorphism groups of LV systems and the classification of LV systems in terms of graphs. At the end of the section we give a population dynamics interpretation of cloning and decloning. Section 4 is devoted to integrability and to the construction of Lax equations for cloned LV systems, with special emphasis to the case of cloned Bogoyavlenskij systems.

Throughout the paper, \( F = \mathbb{R} \) or \( F = \mathbb{C} \).
2. Skew-symmetric graphs and their (auto-)morphisms

In this section, we introduce in the case of graphs the basic construction of decloning which we will use to study LV systems and establish its properties. We also show how the automorphism groups of a graph and of its decloned graph are related.

2.1. Skew-symmetric and weighted graphs

We consider two types of graphs, which we introduce first.

**Definition 2.1.** A skew-symmetric graph $\Gamma = (S, A)$ is a pair consisting of a finite set $S$, and a skew-symmetric map $A: S \times S \to \mathbb{F}$. A weighted graph $\Gamma = (S, \varpi)$ is a skew-symmetric graph $\Gamma = (S, A)$, equipped with a function $\varpi: S \to \mathbb{N}^*: \mathbb{N} \setminus \{0\}$.

The finite set $S$ in the definition of a graph $\Gamma$ or weighted graph $\Gamma = (S, \varpi)$ is called its vertex set. The cardinality of $S$ is denoted by $|S|$ and is called the order of $\Gamma$. When $S = \{1, 2, \ldots, n\}$, the map $A$ is naturally thought of as a (skew-symmetric $n \times n$) matrix; in general, we may also view $A$ as a matrix whose rows and columns are indexed by $S$. We therefore write $a_{s,t}$ for $A(s,t)$, where $s,t \in S$ and we call $A$ the adjacency matrix of $\Gamma$. One may think of $a_{s,t}$ as being the value of an arc from the vertex $s$ to the vertex $t$. We call $\varpi$ the weight vector of $\Gamma$ and we denote by $|\varpi| = \sum_{s \in S} \varpi(s)$ the total weight of $\varpi$ and denote by $\varpi_0$ the weight vector for which $\varpi_0(s) = 1$ for all $s \in S$.

The notion of a skew-symmetric graph, introduced above, is the skew-symmetric analog of the notion of a non-oriented (valued) graph, since the latter can be defined by replacing in definition 2.1 skew-symmetric by symmetric; in particular, one naturally associates with any graph a skew-symmetric graph by skew-symmetrizing its adjacency matrix. All results in this section are also valid for non-oriented graphs, but in sections 3 and 4 only skew-symmetric graphs and weighted graphs, as defined above, will be relevant, hence we only consider skew-symmetric graphs.

It is often useful to represent a (possibly weighted) skew-symmetric graph by a picture, where the following conventions turn out to be convenient: each vertex $s \in S$ is represented by a small circle containing $s$ as a label, and for each pair of vertices $s, t$, a single arrow is drawn between them when $a_{s,t} \neq 0$; we do not differentiate between the two ways of doing this, as indicated in figure 1.

When $a_{s,t} = 1$, which is often the case in our examples, we simply put an arrow from $s$ to $t$, omitting the value 1. For a weighted graph, we indicate the weight of a vertex as a label close to the vertex.

The above notions of graphs lead to the following natural definitions of their morphisms.

**Definition 2.2.** Let $\Gamma = (S, A)$ and $\Gamma' = (S', A')$ be two skew-symmetric graphs. A graph morphism $\Phi: \Gamma \to \Gamma'$ is a map $\Phi: S \to S'$ satisfying $a'_{\Phi(s), \Phi(t)} = a_{s,t}$ for all $s, t \in S$. Let $\varpi$ and $\varpi'$ be weight vectors for $\Gamma$, respectively for $\Gamma'$. A weighted graph morphism $\Psi: (\Gamma, \varpi) \to (\Gamma', \varpi')$ is a graph morphism $\Psi: \Gamma \to \Gamma'$, satisfying $\varpi' s(\Psi(s)) \leq \varpi(s)$ for all $s \in S$.

The identity map of $\Gamma'$ or of $(\Gamma, \varpi)$ (i.e., of $S$) is denoted $\text{Id}_{\Gamma}$. It is of course a (weighted) graph morphism, just like the composition of two (weighted) graph morphisms.
Figure 2. Every edge of the non-oriented graph $\Delta$ is replaced by a vertex and two arrows. A skew-symmetric graph is obtained, having the same automorphism group as $\Delta$.

Figure 3. On the left, a weighted graph $(\Gamma, \varpi)$ with vertex set $\{s, t, u, v\}$ whose adjacency matrix takes values in $\{-1, 0, 1\}$. On the right, the associated skew-symmetric graph $\Gamma^{\varpi}$.

Definition 2.2 leads at once to the definitions of (weighted) graph isomorphisms and (weighted) graph automorphisms; notice that when a graph morphism is bijective, its inverse is also a graph morphism, making it into a graph isomorphism; for a weighted graph morphism $\Psi$, which is bijective, one needs to ask in addition that it preserves the weight vectors, $\varpi(\Psi(s)) = \varpi(s)$ for all $s \in S$, for it to be an isomorphism. When two skew-symmetric graphs $\Gamma'$ and $\Gamma''$ are isomorphic, we write $\Gamma' \simeq \Gamma''$ or $\Phi : \Gamma' \simeq \Gamma''$, where $\Phi$ is an isomorphism, and the group of graph automorphisms of $\Gamma$ is denoted by Aut($\Gamma$); the same notations are used for weighted graphs $(\Gamma, \varpi)$. It is clear that $\text{Aut}(\Gamma)$ is a finite group; it is a subgroup of the symmetric group $S_{|\Gamma|}$. For a weighted graph $(\Gamma, \varpi)$ it is clear that $\text{Aut}(\Gamma, \varpi)$ is a subgroup of $\text{Aut}(\Gamma)$.

Remark 2.3. It was shown in [11] that any finite group $G$ is the group of automorphisms of a (finite) simple\footnote{$\Delta$ being simple means that it is non-oriented, without loops, and that all entries of its adjacency matrix are 0 or 1.} graph $\Delta = (S, E)$. It follows from this result that $G$ is also the group of automorphisms of a skew-symmetric graph $\Gamma$; to construct $\Gamma$ it suffices to replace in $\Delta$ every edge by a vertex and two incident arrows, as indicated in figure 2. Then $\Delta$ and $\Gamma$ have the same automorphism groups, and the result follows.

2.2. Cloning of weighted graphs

In order to motivate the definition of graph decloning, given in the next subsection, we first introduce the operation of cloning. It associates with a weighted graph $(\Gamma, \varpi)$, with $\Gamma = (S, A)$, a skew-symmetric graph $\Gamma^{\varpi} = (S^{\varpi}, A^{\varpi})$. First, every vertex $s \in S$ gives rise to $\varpi(s)$ vertices in $S^{\varpi}$, which we denote by $s_1, s_2, \ldots, s_{\varpi(s)}$ and which we call the clones of the vertex $s$. So the constructed graph $\Gamma^{\varpi}$ has $|S^{\varpi}| = |\varpi|$ vertices. Second, the entries $a^{\varpi}_{s, t}$ of the (skew-symmetric) adjacency matrix $A^{\varpi}$ of $\Gamma^{\varpi}$ are defined by $a^{\varpi}_{s, t} := a_{s, t}$, for $s, t \in S$ and $1 \leq i \leq \varpi(s)$, $1 \leq j \leq \varpi(t)$. An example is given in figure 3.
Suppose that \( S = \{1, \ldots, n\} \), so that we can view \( A \) as a matrix in the usual way. Let us denote, for \( p, q \in \mathbb{N}^n \) by \( 1_{pq} \) the \( p \times q \) matrix all of whose entries are equal to 1. Then \( A^\pi \) is obtained from \( A \) by replacing each entry \( a_{ij} \) of \( A \) by the matrix \( a_{ij}1_{s(i), s(j)} \); the labeling of the rows and columns of \( A^\pi \) is given by \( 1, 1_2, \ldots, 1_{s(1)}, 2_1, \ldots, n_{s(n)} \), in that order. It is clear that for any \( s \in S \), the consecutive lines with labels \( s_1, s_2, \ldots, s_{s(i)} \) of \( A^\pi \) are identical, and similarly for the corresponding columns. It follows that \( A^\pi \) and \( A \) have the same rank. It also leads to the following definition:

**Definition 2.4.** A skew-symmetric graph \( \Gamma = (S, A) \) is said to be irreducible if \( A(s, t) \neq A(t, s) \) whenever \( s \neq t \). Otherwise it is said to be reducible. The same definition applies to weighted graphs.

Said differently, \((S, A)\) is reducible when the adjacency matrix of \( A \) has identical rows (or, equivalently, identical columns). It is clear that this property is invariant under (weighted) graph isomorphisms. By the above, if \((\Gamma, \varpi)\) is any weighted graph with \( \varpi \neq \varpi_0 \), then \( \Gamma^\varpi \) is reducible.

**Remark 2.5.** To the cloned graph \( \Gamma^\varpi = (S^\varpi, A^\varpi) \) of a weighted graph \((\Gamma, \varpi)\), with \( \Gamma = (S, A) \), one can add a weight vector \( \varpi' : S^\varpi \rightarrow \mathbb{N}^+ \) and consider the cloned graph \((\Gamma^\varpi)^{\varpi'}\). It is easy to see that the resulting graph is isomorphic to the cloned graph of \((\Gamma, \varpi')\), where the weight vector \( \varpi'' : S \rightarrow \mathbb{N}^+ \) is defined by \( \varpi''(s) := \sum_{i=1}^{\varpi(i)} \varpi(i, s) \) for all \( s \in S \). It follows that any repeated cloning of a weighted graph amounts to a single graph cloning. We will not elaborate on this fact, because we are mainly interested in decloning, which is an idempotent operation, as we will see shortly.

### 2.3. Decloning of skew-symmetric graphs

We now describe the inverse procedure, which we call decloning. Let \( \Gamma = (S, A) \) be a skew-symmetric graph; we construct its decloned graph \( \Gamma = (S, \Delta) \) and its weighted decloned graph \((\Gamma, \Delta')\). In a cloned graph the clones of a same vertex correspond to identical lines of the adjacency matrix \( A \), a property which can be used to partition \( a \) posteriori the vertex set \( S \) into parts containing the clones of each vertex. We therefore define an equivalence relation \( \sim \) on \( S \) by setting \( s \sim t \) if \( a_{s,t} = a_{t,s} \) for all \( u \in S \); said differently, if the lines (or, equivalently, the columns) of \( A \) with labels \( s \) and \( t \) are identical. Let \( \overline{S} := S/\sim \) and denote by \( p : S \rightarrow \overline{S} \) the canonical projection map; for \( s \in S \), we use the convenient notation \( \xi \) for \( p(s) \). We get a well-defined skew-symmetric map \( \Delta : \overline{S} \times \overline{S} \rightarrow \mathbb{F} \) by setting \( \Delta_{s,t} := a_{s,t} \), for all \( s, t \in S \). Notice that the latter definition is equivalent to saying that \( \Delta \) is the unique map making \( p : (S, A) \rightarrow (\overline{S}, \Delta) \) into a graph morphism. We call \( p \) the **decloning map** (of \( \Gamma \)). Finally, the weight vector \( \varpi_\Gamma \) is defined for \( \xi \in \overline{S} \) by \( \varpi_\Gamma(\xi) := \# \{ t \in S | s \sim t \} \); the decloning map can then also be viewed as a map (graph morphism) \( p : \Gamma \rightarrow (\overline{\Gamma}, \varpi_\Gamma) \). By construction, all lines of \( \Delta \) are different, so the decloned graph \( \Gamma = (\overline{S}, \Delta) \) is irreducible. When \( \Gamma \) is irreducible, \( \sim \) is trivial, so that \( \Gamma \sim \overline{\Gamma} \); as a consequence, \( \overline{\Gamma} \simeq \Gamma \) for any skew-symmetric graph \( \Gamma \). Cloning is the inverse procedure of decloning in the sense that \( \Gamma^\varpi \simeq \Gamma \) for any graph \( \Gamma \); in particular, every graph can be obtained by cloning a (unique) irreducible graph. Notice that if \( \varpi \) is any weight vector for \( \Gamma \), then \( \Gamma^\varpi \simeq \Gamma \) if and only if \( \Gamma \) is irreducible.

Figure 3, read from right to left, yields an example of decloning.

### 2.4. Decloning of graph morphisms

We show in this subsection that any surjective graph morphism can be decloned, i.e., it induces a unique (weighted) graph morphism between the (weighted) decloned graphs of the...
Lemma 2.6. Let $\Gamma = (S, A)$ and $\Gamma' = (S', A')$ be skew-symmetric graphs, with decloning maps $p : \Gamma \to (\Gamma, \varpi_\Gamma)$ and $p' : \Gamma' \to (\Gamma', \varpi_{\Gamma'})$. Suppose that $\Phi : \Gamma \to \Gamma'$ and $\Psi : \Gamma \to \Gamma'$ are two maps, making the following diagram commutative:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\Phi} & \Gamma' \\
p \downarrow & & \downarrow p' \\
(\Gamma, \varpi_\Gamma) & \xrightarrow{\Psi} & (\Gamma', \varpi_{\Gamma'})
\end{array}
$$

(2.1)

Then,

(a) $\Phi$ is a graph morphism if and only if $\Psi$ is a graph morphism.

(b) If $\Phi$ is surjective, then $\Phi$ is a graph morphism if and only if $\Psi$ is a weighted graph morphism. In this case, if $s, t \in S$, then $s \sim t$ if and only if $\Phi(s) \sim \Phi(t)$.

Proof. First, notice that the commutativity of the diagram can be written as $\Psi(s) = \Phi(s)$, for all $s \in S$. The conditions that $\Phi$, respectively $\Psi$, is a graph morphism mean that, for all $s, t \in S$,

$$
da'_{\Phi(s), \Phi(t)} = a_{s, t}, \quad \text{resp.} \quad d'_{\Phi(s), \Phi(t)} = a_{s, t}.'\]

By the definition of decloning, $a_{s, t} = a_{s, t}$ and $d'_{\Phi(s), \Phi(t)} = a_{s, t}$. So the equivalence in item (a) (and the inverse implication of the first equivalence in item (b)) follows from the commutativity of the diagram. In order to prove the remaining statements in item (b), suppose that $\Phi$ is a surjective graph morphism. For $s, t \in S$, by definition, $s \sim t$ if $a_{s, t} = a_{s, t}$ for all $u \in S$. Since $\Phi$ is a graph morphism, this is equivalent to $a'_{\Phi(s), \Phi(t)} = a'_{\Phi(s), \Phi(t)}$ for all $u \in S$. Since $\Phi$ is surjective, this means that $a'_{\Phi(s), \Phi(t)} = a'_{\Phi(s), \Phi(t)}$ for all $u \in S$, i.e., $\Phi(s) \sim \Phi(t)$. This shows the second statement in item (b); notice that the existence of $\Psi$ was not used to prove this equivalence. It follows, using the commutativity of the diagram and the surjectivity of $\Phi$, that

$$
\varpi_{\Gamma'}(\Psi(s)) = \varpi_{\Gamma'}(\Phi(s)) = \# \{ v \in S' | \Phi(s) \sim v \}
$$

$$
\leq \# \{ t \in S | \Phi(s) \sim \Phi(t) \} = \# \{ t \in S | s \sim t \} = \varpi_{\Gamma}(s),
$$

for all $s \in S$, so that $\Psi$ is a weighted graph morphism, which completes the proof of item (b). \hfill \Box

Notice that in the lemma, given $\Phi$, a map $\Psi$ making the diagram commutative is unique, since $p$ is surjective.

Proposition 2.7. Suppose that $\Gamma$ and $\Gamma'$ are two skew-symmetric graphs. As above, denote by $(\Gamma, \varpi_\Gamma)$ and $(\Gamma', \varpi_{\Gamma'})$ the weighted decloned graphs of $\Gamma$ and $\Gamma'$ and by $p$ and $p'$ the decloning maps.

(a) If $\Phi : \Gamma \to \Gamma'$ is a surjective graph morphism, then $\Phi$ induces a unique graph morphism $\Phi : \Gamma \to \Gamma'$ such that the following diagram of (surjective) graph morphisms is commutative:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\Phi} & \Gamma' \\
p \downarrow & & \downarrow p' \\
(\Gamma, \varpi_\Gamma) & \xrightarrow{\Phi} & (\Gamma', \varpi_{\Gamma'})
\end{array}
$$

(2.2)
Moreover, $\Phi : (\Gamma, \varpi_\Gamma) \to (\Gamma', \varpi_{\Gamma'})$ is a weighted graph morphism.

(b) If $\Gamma'$ is another skew-symmetric graph and $\Phi' : \Gamma' \to \Gamma''$ is another surjective graph morphism, then $\Phi' \circ \Phi = \Phi'' \circ \Phi$; also, $\text{Id}_{\Gamma} = \text{Id}_{(\Gamma, \varpi_\Gamma)}$.

(c) Suppose that $\Psi : (\Gamma, \varpi_\Gamma) \to (\Gamma', \varpi_{\Gamma'})$ is a weighted graph morphism, which is surjective. Then there exists a surjective graph morphism $\Phi : \Gamma \to \Gamma'$ such that $\Phi \circ \Phi = \Psi$.

**Proof.** We write $\Gamma = (S, A)$, as before. We first prove item (a). As we pointed out in the proof of lemma 2.6, the fact that $\Phi$ is a surjective graph morphism implies that for any $s, t \in S$, $s \sim t$ iff $\Phi(s) \sim \Phi(t)$. It follows that $\Phi$ is well-defined by $\Phi(s) = \Phi(a)$ and $\Phi$ is the unique map making (2.2) commutative. In view lemma 2.6, this shows item (a). The uniqueness of $\Phi$ in item (a) implies at once item (b). Suppose now that $\Psi : (\Gamma, \varpi_\Gamma) \to (\Gamma', \varpi_{\Gamma'})$ is a surjective weighted graph morphism. Since $\varpi_{\Gamma'}(\Psi(z)) \subseteq \varpi_\Gamma(z)$ for all $z \in S$, one can map surjectively the set of clones of any $z \in S$ to the clones of $\Psi(z)$; doing this for all $z \in S$ one gets a surjective map $\Phi$ such that $\Psi' \circ \Phi = \Psi \circ \rho$. By item (a) in lemma 2.6, $\Phi$ is a (surjective) graph morphism; by uniqueness, $\Phi = \Psi$. \hfill $\square$

The unique (weighted) graph morphism $\Phi$, induced by $\Phi$, is called its (weighted) decloned graph morphism. Property (b) in the proposition says that decloning is a functor. We will come back to this in section 2.6. In general, proposition 2.7 is false for a graph morphism $\Phi$ which is not surjective. A counterexample is given in figure 4.

2.5. Graph isomorphisms and automorphisms

We use proposition 2.7 to give a description of the graph automorphism group $\text{Aut}(\Gamma)$ of a skew-symmetric graph in terms of the automorphism group $\text{Aut}(\Gamma, \varpi_\Gamma)$ of its weighted decloned graph $(\Gamma, \varpi_\Gamma)$. Recall that $\Gamma$ is always supposed to be finite, so these groups are finite groups. We first consider graph isomorphisms.

**Proposition 2.8.** Let $\Gamma$ and $\Gamma'$ be two skew-symmetric graphs.

(a) Suppose that $\Phi : \Gamma \to \Gamma'$ is a surjective graph morphism, with weighted decloned graph morphism $\Phi : (\Gamma, \varpi_\Gamma) \to (\Gamma', \varpi_{\Gamma'})$. Then $\Phi$ is a weighted graph isomorphism if and only if $\Phi$ is a graph isomorphism.

(b) $\Gamma \simeq \Gamma'$ if and only if $(\Gamma, \varpi_\Gamma) \simeq (\Gamma', \varpi_{\Gamma'})$.

**Proof.** It follows from item (b) in proposition 2.7 that if $\Phi$ is a graph isomorphism, then $\Phi : \Gamma \to \Gamma'$ is a graph isomorphism and $\Phi : (\Gamma, \varpi_\Gamma) \to (\Gamma', \varpi_{\Gamma'})$ is a weighted graph isomorphism. Conversely, suppose that the induced map $\Phi : (\Gamma, \varpi_\Gamma) \to (\Gamma', \varpi_{\Gamma'})$ is a weighted graph isomorphism.
isomorphism. Then, for all $s \in S$, $\varpi_T(\Phi(s)) = \varpi_T(g)$, so that $|\Gamma| = |\pi_T| = |\varpi_T| = |\Gamma'|$. Since our graphs are finite and since $\Phi$ is assumed surjective, this shows that $\Phi$ is bijective, hence is a graph isomorphism. It proves item (a) and also the direct implication of item (b).

In order to prove the inverse implication of item (b), suppose that $\Psi : (\Gamma, \varpi_T) \simeq (\Gamma', \varpi_T)$ is a weighted graph isomorphism. Let $\Phi : \Gamma \to \Gamma'$ be a surjective graph morphism such that $\Phi\Psi = \Psi$, as provided by item (c) in proposition 2.7. Since $\Psi$ is bijective and weight-preserving,

$$|\Gamma'| = \sum_{\mathbf{w} \in \mathbf{S}'} \varpi_T'(\mathbf{w}) = \sum_{\mathbf{w} \in \mathbf{S}'} \varpi_T'(\Psi(\mathbf{w})) = \sum_{\mathbf{w} \in \mathbf{S}'} \varpi_T(\mathbf{w}) = |\Gamma|,$$

where $S$ and $S'$ stand for the vertex sets of $\Gamma$ and $\Gamma'$, as before. It follores 5 that the surjective graph morphism $\Phi$ is a bijection, hence a graph isomorphism, and $\Gamma \simeq \Gamma'$.

**Proposition 2.9.** Let $\Gamma = (S, A)$ be a skew-symmetric graph, whose weighted decloned graph is denoted by $(\Gamma, \varpi_T)$, where $\Gamma = (S, A)$. Then

$$0 \to \prod_{\mathbf{w} \in \mathbf{S}'} S_{\varpi_T'(\mathbf{w})} \to \text{Aut}(\Gamma) \to \text{Aut}(\Gamma, \varpi_T) \to 0$$

is a split short exact sequence of groups. As a consequence, $\text{Aut}(\Gamma)$ is a semi-direct product,

$$\text{Aut}(\Gamma) \simeq \prod_{\mathbf{w} \in \mathbf{S}'} S_{\varpi_T'(\mathbf{w})} \rtimes \text{Aut}(\Gamma, \varpi_T).$$

**Proof.** Let $s \in S$ and consider the equivalence class $\mathbf{s}$ of $s$, which we view here as a subset of $S$, of cardinality $\varpi_T(\mathbf{s})$. Let $\varsigma$ be any permutation of $\mathbf{S}/\mathbf{s} \simeq S_{\varpi_T(\mathbf{s})}$ and let $\Phi$ be the extension of $\varsigma$ to a permutation of $\mathbf{s}$ which fixes all vertices of $S \setminus \mathbf{s}$. According to lemma 2.6, with $\Gamma = \Gamma'$ and $\Psi = \text{Id}_{\mathbf{S}}$, $\Phi$ is a graph morphism, hence a graph automorphism. Since this can be done for any $s \in S$, and since for $s \neq t \in S$, the permutations of $S$ corresponding to $s$ and $t$ have disjoint support, these permutations generate a group of graph automorphisms of $\Gamma$, isomorphic to $\prod_{s \in S} S_{\varpi_T(s)}$; this accounts for its inclusion in $\text{Aut}(\Gamma)$. The surjectivity of the group homomorphism $\text{Aut}(\Gamma) \to \text{Aut}(\Gamma, \varpi_T) : \Phi \mapsto \Phi'$ is proven in the same way as item (b) in proposition 2.8. Since the neutral element of $\text{Aut}(\Gamma, \varpi_T)$ is $\text{Id}_G$, every graph automorphism in the kernel of this morphism can only permute equivalent vertices of $\Gamma$, hence is the graph automorphism corresponding to an element of $\prod_{s \in S} S_{\varpi_T(s)}$; conversely, the graph automorphism corresponding to any element of $\prod_{s \in S} S_{\varpi_T(s)}$ is in the kernel of the group homomorphism $\text{Aut}(\Gamma) \to \text{Aut}(\Gamma, \varpi_T)$. This shows that the short sequence is exact. To show that it is split, we need to construct a section of the surjection $\text{Aut}(\Gamma) \to \text{Aut}(\Gamma, \varpi_T)$. To do this, we fix a numbering of the elements of $\mathbf{s}$, for every $s \in S$, writing them as $s_1, s_2, \ldots, s_{\varpi_T(\mathbf{s})}$. Given $\Psi \in \text{Aut}(\Gamma, \varpi_T)$ we get a bijection $\overline{\Psi}$ of $\Gamma$ by setting $\overline{\Psi}(s_i) := \Psi(s_i)$, for $s \in S$ and $1 \leq i \leq \varpi_T(s)$. According to item (a) in lemma 2.6, $\overline{\Psi}$ is a graph (auto-)morphosis. Clearly, when $\Psi_1$ and $\Psi_2$ are two elements of $\text{Aut}(\Gamma, \varpi_T)$, then $\Psi_1 \circ \Psi_2 = \Psi_1 \circ \Psi_2$, so that the map $\Psi \mapsto \overline{\Psi}$ is a group homomorphism; it is a section because $\overline{\Psi} = \Psi$ for any $\Psi \in \text{Aut}(\Gamma, \varpi_T)$, as follows from the definitions of $\Psi$ and of decloning of surjective graph morphisms.\]

**2.6. Functorial interpretation**

We finish this section by giving a functorial interpretation\(^\text{6}\) of the above results, in particular of propositions 2.7 and 2.8. Let us denote by $\text{Gr}$ the category of skew-symmetric graphs (with

\(^{6}\) We only use the basics of category theory; see for example the book [1] which is freely available online.

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values in $\mathbb{F}$) with surjective graph morphisms and by $\text{Gr}_0$ the subcategory of irreducible graphs, whose embedding functor is denoted by $\iota_0$. According to item (b) in proposition 2.7, a functor $\rho : \text{Gr} \to \text{Gr}_0$ is defined for objects $\Gamma$ of $\text{Gr}$ by $\rho(\Gamma) := \sum \Gamma$ and for morphisms $\Phi : \Gamma \to \Gamma'$ by $\rho(\Phi) := \Phi : \sum \Gamma \to \sum \Gamma'$; we call $\rho$ the graph decloning functor. In categorical language, item (c) in proposition 2.7 implies, upon taking $\omega_\Gamma = \omega_{\Gamma'} = \omega_0$ that $\rho$ is a full functor. Also item (a) in proposition 2.7 says that $\text{Gr}_0$ is a reflective subcategory of $\text{Gr}$, with reflection functor the graph decloning functor $\rho$. Said differently, $\rho$ is an adjoint functor for $\iota_0$. Among other properties, it shows that decloning is a natural operation on graphs; in the next section we will see that decloning of LV systems has similar functorial properties.

3. LV systems

3.1. Basic definitions and properties

We first recall the basic definitions and properties of skew-symmetric LV systems. Let $A = (a_{ij})$ be any skew-symmetric $n \times n$ matrix with entries in $\mathbb{F}$, where $n$ is any positive integer. The LV system defined by $A$ is the Hamiltonian system on $\mathbb{F}^n$, defined by the following quadratic vector field:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_i x_j, \quad \text{for } i = 1, \ldots, n. \quad (3.1)$$

We have denoted the standard coordinates on $\mathbb{F}^n$ by $x_1, \ldots, x_n$. The matrix $A$ also defines a Poisson structure on $\mathbb{F}^n$,

$$\pi_A := \sum_{i<j} a_{ij} x_i \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_j} \quad (3.2)$$

The corresponding Poisson bracket is given by $\{x_i, x_j\}_A = a_{ij} x_i x_j$. It is a quadratic Poisson bracket, known as a diagonal Poisson bracket because of its particular form; the Poisson structure $\pi_A$ is also said to be log-canonical because it satisfies $\{\log x_i, \log x_j\}_A = a_{ij}$. If we denote by $H$ the sum of the coordinates, $H := x_1 + x_2 + \cdots + x_n$, then it is clear from (3.2) that (3.1) is the Hamiltonian vector field $\mathcal{X}_H = \{\cdot, H\}_A$. We call the Hamiltonian system $(\mathbb{F}^n, \pi_A, H)$ a (skew-symmetric) Lotka–Volterra system, or simply an LV system; notice that it is entirely determined by $A$.

Several basic properties of the LV system associated with $A$ can be read off from the matrix $A$. The rank of $\pi_A$ (which is by definition the maximal rank of $\pi_A$ at the points of $\mathbb{F}^n$) is equal to the rank of $A$. Moreover, if $(a_1, \ldots, a_n) \in \mathbb{N}^n$ is a null-vector of $A$, then $C := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is a Casimir function of $\pi_A$, i.e., $\mathcal{X}_C = 0$. This is also true if $(a_1, \ldots, a_n) \in \mathbb{N}^n$ upon properly interpreting $C$ as a function on an open subset of $\mathbb{F}^n$. When all the entries of $A$ are integers or rational numbers, as will be the case in the examples which we will discuss in section 4, a basis for the null-vectors of $A$ can be chosen in $\mathbb{N}^n$, yielding $n - \text{Rk} A$ independent rational Casimir functions of $\pi_A$.

The matrix $A$ is also useful for describing the reductions which are obtained by setting one or several of the coordinates $x_i$ equal to zero. Notice that such a reduction is a special type of Poisson reduction since for any collection $I \subset \{1, 2, \ldots, n\}$ the subspace of $\mathbb{F}^n$, defined by $x_i = 0$ for all $i \in I$, is a Poisson submanifold of $(\mathbb{F}^n, \pi_A)$. The reduced Poisson structure is therefore just the restriction of $\pi_A$ to the subspace. In particular, the reduced system is also an LV system and its matrix is obtained by removing from $A$ the $i$th row and $i$th column, for all $i \in I$, with as Hamiltonian the sum of all remaining coordinate functions. In general, the
3.2. LV systems associated with graphs

A smooth morphism of LV systems \( \phi : (\mathbb{R}^n, \pi_A, H) \rightarrow (\mathbb{R}^n, \pi_A, H') \) is a smooth\(^7\) map \( \phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \), which preserves the Poisson structure and the Hamiltonian; in terms of formulas this means that \( \{\phi^* F, \phi^* G\}_A = \phi^* \{F, G\}_A \) for any smooth functions \( F \) and \( G \) on \( \mathbb{R}^n \), and that \( \phi^* H = H \). It leads to the notion of smooth isomorphism of LV systems.

We will in what follows only consider linear morphisms and isomorphisms of LV systems. The reason for this is given by the following proposition which says that if two LV systems are smoothly isomorphic, they are linearly isomorphic.

**Proposition 3.1.** Suppose that \( \phi : (\mathbb{R}^n, \pi_A, H) \rightarrow (\mathbb{R}^n, \pi_A, H) \) is a smooth isomorphism of LV systems. Then \( \phi_0 \), the differential of \( \phi \) at the origin, is a (linear) isomorphism of LV systems.

**Proof.** We denote the standard coordinates on \( \mathbb{R}^n \) by \( x_1, x_2, \ldots, x_n \). We use the fact that any smooth function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) can be written as \( F = Q + L + c \), where \( c := F(0) \in \mathbb{R}^n \) and \( L := \sum_{k=1}^n x_k \frac{\partial F}{\partial x_k}(0) \), so that \( Q \) is a smooth function with \( Q(0) = 0 \) and \( \frac{\partial Q}{\partial x_k}(0) = 0 \) for \( k = 1, \ldots, n \). Applied to the functions \( \phi^* x_i : \mathbb{R}^n \rightarrow \mathbb{R}^n \) we write \( \phi^* x_i = Q_i + L_i + c_i \), for \( i = 1, \ldots, n \). Then the linear map \( \phi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is given by \( (L_1, \ldots, L_n) \). It is an isomorphism because \( \phi \) is a diffeomorphism. We need to show that

\[
\{L_i, L_j\}_A = a_{ij} L_i L_j, \quad \text{for } i, j = 1, \ldots, n.
\]  

(3.3)

When one of \( x_i \) and \( x_j \), say \( x_i \), is a Casimir function of \( \pi_A \), then \( \phi^* x_i \) is a Casimir function of \( \pi_A \), hence also its linear part \( L_i \) (for degree reasons); also \( a_{ij} = 0 \) for all \( j \), so both sides vanish. It remains to prove (3.3) when \( x_i \) and \( x_j \) are not Casimir functions of \( \pi_A \). We first show that if \( x_i \) is not a Casimir function, then \( c_i = 0 \). The proof goes by contradiction. Suppose that \( x_i \) is not a Casimir function of \( \pi_A \) and that \( c_i \neq 0 \). Then there exists a \( k \) with \( 1 \leq k \leq n \) such that \( a_{ik} \neq 0 \). On the one hand,

\[
\phi^* \{x_i, x_k\}_A = a_{ik} (\phi^* x_i)(\phi^* x_k) = a_{ik} (Q_i + L_i + c_i)(Q_k + L_k + c_k),
\]

while on the other hand

\[
\{\phi^* x_i, \phi^* x_k\}_A = \{Q_i, Q_k\}_A + \{Q_i, L_k\}_A + \{L_i, Q_k\}_A + \{L_i, L_k\}_A.
\]

Since \( \phi \) is a Poisson map, the above right hand sides are equal. Since the latter has no constant or linear terms, \( a_{ik} c_i c_k = 0 \) and \( a_{ik} c_i L_k + c_i L_k = 0 \), so that \( c_k = 0 \) and \( L_k = 0 \). The latter is impossible, since \( \phi_0 = (L_1, \ldots, L_n) \) is an isomorphism. We have arrived at a contradiction and may conclude that if \( x_i \) is not a Casimir function, then \( c_i = 0 \). Let us suppose now that \( x_i \) and \( x_j \) are not Casimir functions, so that \( c_i = c_j = 0 \). Then the quadratic parts of \( \phi^* \{x_i, x_j\}_A \) and \( \{\phi^* x_i, \phi^* x_j\}_A \) are respectively given by \( a_{ij} L_i L_j \) and \( \{L_i, L_j\}_A \); they are equal since \( \phi \) is a Poisson map, so that \( \{L_i, L_j\}_A = a_{ij} L_i L_j \). This proves (3.3), hence that \( \phi_0 \) is a (linear) Poisson diffeomorphism.

\[\square\]

3.2. LV systems associated with graphs

Since skew-symmetric graphs and LV systems both bijectively correspond to skew-symmetric matrices, it is natural to think of LV systems as being associated with graphs. Since the rows

\(^7\)By a slight abuse of terminology, we use the terms smooth and diffeomorphic both in the case of \( \mathbb{F} = \mathbb{R} \) and \( \mathbb{F} = \mathbb{C} \); strictly speaking we should say in the latter case holomorphic and biholomorphic.
and columns of the adjacency matrix of a skew-symmetric graph $\Gamma = (S, A)$ are indexed by the vertex set $S$ of $\Gamma$ we take a more intrinsic point of view and define the LV system associated with $\Gamma$, to be the Hamiltonian system $(\mathbb{F}[S], \pi_A, H_S)$; it is denoted by $\text{LV}(\Gamma')$. Here, $\mathbb{F}[S]$ stands for the vector space generated by $S$, with (unordered) basis $S$:

$$ \mathbb{F}[S] = \left\{ \sum_{s \in S} \alpha_s s \mid \alpha_s \in \mathbb{F} \text{ for all } s \in S \right\}. $$

Elements of $\mathbb{F}[S]$ are also denoted in (unordered) vector form $(\alpha_s)_{s \in S}$. The linear coordinates on $\mathbb{F}[S]$ associated with $S$ are denoted $x_s$, $s \in S$. Also, $\pi_A$ is the Poisson structure on $\mathbb{F}[S]$ whose associated Poisson bracket is given by

$$ \{x_s, x_t\}_A = a_{st} x_s x_t, \quad \text{for } s, t \in S. \quad (3.4) $$

The Hamiltonian $H_S$ is the sum of the coordinates, $H_S := \sum_{s \in S} x_s$. In this notation, the LV vector field $\mathcal{X}_{H_S}$ takes the form

$$ \dot{x}_s = x_s \sum_{t \in S} a_{st} x_t, \quad \text{for } s \in S. \quad (3.5) $$

When $\Gamma$ is equipped with a weight vector $\varpi$, we will say that $(\text{LV}(\Gamma'), \varpi)$ is a weighted LV system. Notice that replacing $S$ by $\{1, 2, \ldots, n\}$ amounts to totally ordering the elements of $S$, which can be done in $\#S!$ ways, making the link between skew-symmetric graphs and LV systems less intrinsic.

A morphism of LV systems, or simply an LV morphism, $\phi: (\mathbb{F}[S], \pi_A, H_S) \to (\mathbb{F}'[S], \pi_{A'}, H_{S'})$ is a linear Poisson map $\phi: (\mathbb{F}[S], \pi_A) \to (\mathbb{F}'[S'], \pi_{A'})$ for which $\phi^* H_{S'} = H_S$. Notice that $\phi$ does not entail a map from $S$ to $S'$, so there is no natural notion of a morphism of weighted LV systems; we will come back to this in section 3.5.

Several properties of a given LV system are more easily seen or explained from the underlying graph $\Gamma$ than from its adjacency matrix $A$. A simple example of this is that if $\Gamma$ is not connected, $\text{LV}(\Gamma)$ is a direct product (as a Hamiltonian system) of the LV systems which correspond to the connected components of the graph; also, reduction to the subspace defined by $x_s = 0$ for $s \in S_0 \subset S$ corresponds to removing from $\Gamma$ all vertices labeled by the entries of $S_0$ (one removes of course also all arrows which are incident with these vertices). Another example spelled out in proposition 3.2 below is that graph morphisms lead to morphisms of LV systems. Also, the cloning and decloning constructions for LV systems which are introduced in the next subsection derive naturally from the corresponding constructions for skew-symmetric graphs.

A few examples of graphs corresponding to some well-known examples are given in the figures 5–7 (see section 4.1 for more information on these examples).

We show in the following proposition that graph morphisms lead to LV morphisms between the corresponding LV systems. Any map $\Phi: S \to S'$ leads to a unique linear map $\phi: \mathbb{F}[S] \to \mathbb{F}'[S']$, extending $\Phi$, i.e., $\phi(s) = \Phi(s)$ for all $s \in S$. It is called the linear extension of $\Phi$. When $\Phi: \Gamma \to \Gamma'$ is a graph morphism, we denote its linear extension by $\text{LV}(\Phi)$.

**Proposition 3.2.** Let $\Gamma$ and $\Gamma'$ be two skew-symmetric graphs and suppose that $\Phi: \Gamma \to \Gamma'$ is a graph morphism.

(a) $\text{LV}(\Phi): \text{LV}(\Gamma) \to \text{LV}(\Gamma')$ is an LV morphism.

(b) If $\Phi': \Gamma' \to \Gamma''$ is another graph morphism, then $\text{LV}(\Phi' \circ \Phi) = \text{LV}(\Phi') \circ \text{LV}(\Phi)$; also, $\text{LV}(\text{Id}_\Gamma) = \text{Id}_{\text{LV}(\Gamma)}$.  

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Figure 5. The LV system corresponding to the left graph is the Kac–van Moerbeke system KM(6); the one corresponding to the right graph is B(6, 2).

Figure 6. The LV system corresponding to the left graph is the direct sum of two copies of KM(3); the one on the right corresponds to LV(6, 0).

Figure 7. The two graphs are obtained by removing the vertex 6 from the right graph in figures 5 and 6, respectively. The corresponding LV systems are reductions of the original ones. Notice that the right one is LV(5, 0).

(c) $\Phi$ is a graph isomorphism if and only if $LV(\Phi)$ is an LV isomorphism.

Proof. Let us write $\Gamma = (S, A)$ and $\Gamma' = (S', A')$, and let us denote the linear map $LV(\Phi)$ by $\phi$. The linear coordinates on $F[S]$ and on $F[S']$ are respectively denoted by $x_s$, with $s \in S$ and $y_u$, with $u \in S'$. By definition, $\phi$ is given by $\phi(\sum_{s \in S} \alpha_s) = \sum_{u \in S'} \delta_{\Phi(s)} \Phi(u)$, where $\alpha_s \in F$ for all $s \in S$; in vector form, this is written $\phi((\alpha_s)_{s \in S}) = (\sum_{u \in S'} \alpha_s \delta_{\Phi(s)} u)_{u \in S'}$. We show that the corresponding algebra homomorphism $\phi^*$ is given, for $u \in S'$, by

$$\phi^* y_u = \sum_{\Phi(u) = u} x_s.$$  \tag{3.6}
Figure 8. There exists no graph morphism between the left and right graphs. However, a morphism \( \phi \) between their LV systems is defined by \( \phi^* y_v = x_s + x_t \) and \( \phi^* y_u = x_a \).

To prove this formula, which is an equality of linear functions on \( \mathbb{F}[S] \), it suffices to check that both sides of (3.6) take the same value when evaluated on any element of \( S \). For \( t \in S \), we have

\[
(\phi^* y_u)(t) = y_u(\Phi(t)) = \delta_{u,\Phi(t)},
\]

\[
\left( \sum_{\Phi(j)=u} x_j \right)(t) = \sum_{\Phi(j)=u} x_j(t) = \sum_{\Phi(j)=u} \delta_{u,j} = \delta_{u,\Phi(t)},
\]

which proves (3.6). To show that \( \phi \) is a Poisson map, we need to verify that

\[
\{ \phi^* y_u, \phi^* y_v \}_A = \phi^* \{ y_u, y_v \}_{A'},
\]

for all \( u, v \in S' \). First, notice that since \( \{ y_u, y_v \}_A' \) is a multiple of \( y_u y_v \), both sides of (3.7) are zero when \( u \) or \( v \) do not belong to \( \text{Im}(\Phi) \). We may therefore suppose that \( u, v \in \text{Im}(\Phi) \). Since \( \Phi \) is a graph morphism, if \( \Phi(s) = u \) and \( \Phi(t) = v \), then \( a_{u,v}^t = a_{s,t} \). Therefore,

\[
\{ \phi^* y_u, \phi^* y_v \}_A = \sum_{\Phi(j)=u} \sum_{\Phi(j)=v} \{ x_s, x_t \}_A = \sum_{\Phi(j)=u} \sum_{\Phi(j)=v} a_{s,t}x_s x_t
\]

\[
= a_{u,v}^t \left( \sum_{\Phi(j)=u} x_s \right) \left( \sum_{\Phi(j)=v} x_t \right) = a_{u,v}^t \phi^* (y_u) \phi^* (y_v)
\]

\[
= \phi^* (a_{u,v}^t y_u y_v) = \phi^* \{ y_u, y_v \}_{A'}.\]

Moreover,

\[
\phi^* H_S = \sum_{u \in S'} \phi^* y_u = \sum_{u \in S'} \sum_{\Phi(j)=u} x_j = \sum_{j \in S} x_j = H_S.
\]

It follows that \( \phi \) is an LV morphism, which is item (a) (recall that \( \phi = \text{LV}(\Phi) \)). For the proof of item (b), it suffices to verify that \( \text{LV}(\Phi' \circ \Phi) \) and \( \text{LV}(\Phi'^r \circ \Phi) \) take the same value at every \( s \in S \subset \mathbb{F}[S] \), just like \( \text{LV}(\text{Id}_{\Gamma}) \) and \( \text{Id}_{\text{LV}(\Gamma)} \), but that is trivial. The proof of the direct implication in item (c) follows from items (a) and (b); the inverse implication in item (c) follows from the fact that if \( \text{LV}(\Phi) \) is an isomorphism, then \( \Phi \) is bijective since \( \text{LV}(\Phi) \) is the linear extension of \( \Phi \).

\[
\Box
\]

Items (a) and (b) of the proposition imply that \( \text{LV} \) is a functor. We will come back to this in section 3.7.

Not all morphisms between LV systems are induced by graph morphisms, even when the underlying graphs are irreducible. A simple counterexample is given in figure 8.
3.3. Cloning and decloning of LV systems

We define in this paragraph the cloning and decloning of LV systems, leaving the more delicate issue of the decloning of LV morphisms to section 3.4. We do this by using the corresponding constructions for the underlying graphs. Let \( \Gamma \) be a skew-symmetric graph and let \( \varpi \) be a weight vector for \( \Gamma \). Recall that we denote by \( \Gamma^{\varpi} \) the cloning of \( (\Gamma, \varpi) \) and by \( \Gamma \) and \( \varpi \Gamma \) respectively the decloning and the weighted decloning of \( \Gamma \). Their LV systems are by definition the cloning of the LV system \( \text{LV}(\Gamma) \) with weight vector \( \varpi \), respectively the (weighted) decloning of \( \text{LV}(\Gamma) \). Thus, the cloning of the weighted LV system \( (\text{LV}(\Gamma), \varpi) \) is \( \text{LV}(\Gamma^{\varpi}) \) and the decloning, respectively weighted decloning, of \( \text{LV}(\Gamma) \) is \( \text{LV}(\Gamma^{\varpi}) \), respectively \( \text{LV}(\varpi \Gamma) \). When \( \Gamma \) is irreducible, \( \Gamma \simeq \bigoplus_i \) we will say that \( \text{LV}(\Gamma) \) is irreducible, otherwise that it is reducible.

We give a more explicit description of these systems, which we do first in the case of cloning. Let \( (\Gamma, \varpi) \) be a weighted graph, with \( \Gamma = (S, A) \). Recall that \( \Gamma^{\varpi} \) is the skew-symmetric graph \( (S^{\varpi}, A^{\varpi}) \), where the elements of \( S^{\varpi} \) are all \( s_i \) with \( s \in S \) and \( 1 \leq i \leq \varpi(s) \). Thus, the phase space \( \mathbb{F}[S^{\varpi}] \) of \( \text{LV}(\Gamma^{\varpi}) \) has dimension \( |\varpi| \) and is equipped with coordinate functions \( x_{s_i} \), with \( s \) and \( i \) as above. Since the entries of \( A^{\varpi} \) are given by \( a_{x_{s_i}, x_{s_j}} = a_{i, j} \), for \( s, t \in S \) and \( 1 \leq i \leq \varpi(s), 1 \leq j \leq \varpi(t) \), the Poisson structure \( \pi_{\varpi} \) is given, as in (3.4), by the following diagonal Poisson brackets:

\[
\{x_{s_i}, x_{s_j}\}_{A^{\varpi}} = a_{i, j} x_{s_i} x_{s_j}.
\] (3.8)

In this formula, \( s, t \in S \) and \( 1 \leq i \leq \varpi(s) \) and \( 1 \leq j \leq \varpi(t) \). Since the matrices \( A \) and \( A^{\varpi} \) have the same rank, the corresponding Poisson structures \( \pi_A \) and \( \pi_{\varpi} \) have the same rank (equal to the rank of \( A \)). Since for any \( s \in S \), the lines with labels \( s_1, s_2, \ldots, s_{\varpi(s)} \) are the same, the functions \( x_{s_i}/x_{s_j} \) are (independent) Casimir functions of \( \pi_{\varpi} \), for \( s \in S \) and \( i = 2, 3, \ldots, \varpi(s) \); this follows also easily from (3.8). Since the Hamiltonian of \( \text{LV}(\Gamma^{\varpi}) \) is the sum of the coordinate functions, it is given by

\[
H_{S^{\varpi}} = \sum_{s \in S} \sum_{i=1}^{\varpi(s)} x_{s_i},
\] (3.9)

and the Hamiltonian vector field of \( H_{S^{\varpi}} \) takes the simple form

\[
\dot{x}_{s_i} = x_{s_i} \sum_{t \in S} \sum_{j=1}^{\varpi(t)} a_{i, j} x_{s_j}, \quad \text{for } s \in S \text{ and } i = 1, \ldots, \varpi(s).
\] (3.10)

We now give a more explicit description of decloning of LV systems, which we do in two different ways. We first show that decloning amounts to a special type of Poisson reduction [15, section 5.2.2], given by a Poisson submersion, as stated in the following proposition:

**Proposition 3.3.** Let \( \Gamma \) be any graph and let \( p : \Gamma \to \bigoplus_i \) denote the decloning map of \( \Gamma \). Then \( \text{LV}(p) : \text{LV}(\Gamma) \to \text{LV}(\Gamma_{\varpi}) \) is an LV morphism, which is a surjective submersion, so that \( \text{LV}(\Gamma_{\varpi}) \) is obtained from \( \text{LV}(\Gamma) \) by Poisson reduction. Explicitly, \( \text{LV}(p) \) is given by

\[
\text{LV}(p) : \mathbb{F}[S] \to \mathbb{F}[S] \quad (\alpha_s)_{s \in S} \mapsto \left( \sum_{s \in S} \alpha_s \right)_{s \in S},
\] (3.11)

and is called the decloning map of \( \text{LV}(\Gamma) \).
Proof. Since \( p : \Gamma \rightarrow \hat{\Sigma} \) is a graph morphism, \( \text{LV}(p) \) is an LV morphism (proposition 3.2). Since \( p \) is surjective, \( \text{LV}(p) \) is surjective on \( \hat{\Sigma} \) so it is a surjective linear map, hence a (surjective) submersion. The explicit formula for \( \text{LV}(p) \) is an easy transcription of (3.6).

Alternatively, one can describe decloning of LV systems as a different type of Poisson reduction, obtained by putting several of the coordinates \( \chi_{x_i} \) equal to zero; recall from section 3.1 that this amounts to restricting the LV system to an LV system on a subspace, and recall from section 3.2 that this amounts to removing from the graph \( \Gamma \) the corresponding vertices \( s_i \). Applied to the current case, in which \( \hat{\Sigma} \) is obviously isomorphic to the subgraph \( \hat{\Gamma}_0 \) of \( \Gamma \), induced by any set \( S_0 \) of representatives of \( S \) modulo \( \sim \), we get that \( \text{LV}(\hat{\Sigma}) \cong \text{LV}(\hat{\Gamma}_0) \) and that \( \text{LV}(\hat{\Gamma}_0) \) is obtained by Poisson reduction from \( \text{LV}(\Gamma) \). The above definitions of cloning, decloning and irreducibility of LV systems are \textit{a priori} unsatisfactory because we have not shown yet that if \( \text{LV}(\Gamma) \cong \text{LV}(\Gamma') \) then \( \Gamma \cong \Gamma' \), and hence that \( \text{LV}(\hat{\Sigma}) \cong \text{LV}(\hat{\Sigma}') \). To do this, one needs to study LV morphisms in more detail, which will be done in the next subsection.

3.4. Decloning of morphisms of LV systems

We show in this subsection that surjective morphisms of LV systems can be decloned. To do this, and in order to prove some related results, we will use the following key lemma:

Lemma 3.4. Let \( \phi : (\mathbb{F}[S], \pi_S) \rightarrow (\mathbb{F}[S'], \pi_{S'}) \) be a linear Poisson morphism. It is assumed that \( \text{Im} \phi \) is not contained in a coordinate hyperplane \( y_u = 0, u \in S' \). We denote, as before, the coordinates on \( \mathbb{F}[S] \) and on \( \mathbb{F}[S'] \) by \( x_s, w \), with \( s \in S \), respectively by \( y_u, \) with \( u \in S' \). Let \( B = (\beta_{s, t}) \) be the matrix defined by \( \phi^* y_u = \sum_{s \in S} \beta_{u, s} x_s \). Suppose that \( \beta_{u, s} \beta_{t, s} \neq 0 \) for some \( u \neq v \in S' \) and some \( s \in S \). Then \( u \sim v \).

Proof. We first express the fact that \( \phi \) is a Poisson map. For \( u, v \in S' \) we have that

\[
\phi^* \{ y_u, y_v \}_{\mathbb{F}[S]} = d^{'}_{u,v}(\phi^* y_u)(\phi^* y_v) = \sum_{s \in S} d^{'}_{u,v} \beta_{u, s} \beta_{v, s} x_s x_t,
\]

\[
\{ \phi^* y_u, \phi^* y_v \}_{\mathbb{F}[S]} = \sum_{s \in S} a_{s, t} \beta_{u, s} \beta_{v, s} x_s x_t.
\]

Taking the coefficient of \( x_s x_t \) in these expressions we find that \( \phi \) is a Poisson map if and only if

\[
(a_{u,v} - a_{s,t}) \beta_{u,s} \beta_{v,t} + (d^{'}_{u,v} + a_{s,t}) \beta_{u,s} \beta_{v,t} = 0, \quad \text{for all } s, t \in S, u, v \in S'.
\]

(3.12)

Suppose now that \( \phi \) is a Poisson map and that \( \beta_{u,s} \beta_{v,t} \neq 0 \) for some \( u \neq v \in S' \) and some \( s \in S \). We show that \( u \sim v \), i.e., that \( a_{u,v} = a_{s,t} \) for all \( w \in S' \). Replacing \( b \) by \( s \) in (3.12) we find, since \( a_{s,t} = 0 \), that \( 2a_{u,v} \beta_{u,s} \beta_{v,t} = 0 \), so that \( a_{u,v} = 0 = a_{v,s} \), which shows that \( a_{u,w} = a_{v,w} \) for \( w = v \), and also for \( w = u \). Let \( w \in S' \), different from \( u \) and \( v \). We distinguish two cases. Suppose first that \( \beta_{u,s} = 0 \) and let \( t \) be such that \( \beta_{w,t} \neq 0 \); such a \( t \) exists because \( \text{Im} \phi \) is not contained in the coordinate hyperplane \( y_u = 0 \). If we replace \( v \) by \( w \) in (3.12) we get \( (d^{'}_{u,w} - a_{s,t}) \beta_{u,s} \beta_{v,t} = 0 \), so that \( d^{'}_{u,w} = a_{s,t} \). Similarly, if we replace \( u \) by \( w \) in (3.12) we get \( d^{'}_{v,w} = a_{s,t} \). It follows that \( d^{'}_{u,w} = d^{'}_{v,w} \). When \( \beta_{u,s} \neq 0 \) we take \( t = s \) and proceed as in the first case, with \( v = w \) (resp. \( u = w \)), to find \( a_{u,w} = 0 \) (resp. \( a_{v,w} = 0 \)). This leads again to \( a_{u,w} = a_{v,w} \) and completes the proof that \( u \sim v \). 

\( \square \)
Using the lemma, we find a simple description of surjective LV morphisms, when the target system is irreducible.

**Proposition 3.5.** Let \( \phi : (\mathbb{F}[S], \pi_A, H_S) \to (\mathbb{F}[S'], \pi_{A'}, H_{S'}) \) be a surjective LV morphism, where the target system is supposed irreducible.

(a) There exists for any \( u \in S' \) a (unique) non-empty subset \( S_u \) of \( S \), such that \( \phi : \mathbb{F}[S] \to \mathbb{F}[S'] \) is given by

\[
\phi(\alpha_t)_{t \in S} = \left( \sum_{s \in S_u} \alpha_s \right)_{u \in S'} \tag{3.13}
\]

the subsets \( (S_u)_{u \in S'} \) form a partition of \( S \); in particular, if \( u \neq v \in S' \) then \( S_u \cap S_v = \emptyset \).

(b) For any \( u \neq v \in S' \), one has \( a_{u,v} = a'_{u,v} \) for all \( s \in S_u \) and \( t \in S_v \); in particular, \( a_{s,t} \) is independent of \( s \in S_u \) and of \( t \in S_v \).

**Proof.** In order to prove item (a), we apply lemma 3.4 in case the target system is irreducible, i.e., the graph \((S', A')\) is irreducible. Then \( u \sim v \) for \( u, v \in S' \) implies that \( u = v \), so that by the lemma, \( \beta_{u,v} = 0 \) for all \( s \in S \) and all \( u \neq v \in S' \). If we define \( S_u := \{ s \in S | \beta_{u,t} \neq 0 \} \) for \( u \in S' \) then this implies that if \( u \neq v \in S' \) then \( S_u \cap S_v = \emptyset \). It follows from the definition of \( S_u \) that \( \phi^* y_u = \sum_{s \in S_u} \beta_{u,v} x_s \), where the scalars \( \beta_{u,v} \), with \( s \in S_u \), are non-zero. Since every \( s \in S \) appears in at most one \( S_u \), we can deduce from

\[
\sum_{s \in S} x_s = H_S = \phi^* H_{S'} = \sum_{u \in S'} \phi^* y_u = \sum_{u \in S'} \sum_{s \in S_u} \beta_{u,v} x_s
\]

that all \( \beta_{u,v} \) with \( s \in S_u \) are equal to 1, and that every element \( s \in S \) belongs to one of the \( S_u \).

It follows that

\[
\phi^* y_u = \sum_{s \in S_u} x_s, \tag{3.14}
\]

which is equivalent to (3.13). Moreover, the surjectivity of \( \phi \) implies that each subset \( S_u \) is non-empty, so that \( (S_u)_{u \in S'} \) is a partition of \( S \). This proves item (a).

In order to prove item (b), let \( u \neq v \in S' \). Then, in view of (3.14), and since \( \phi \) is a Poisson map,

\[
\sum_{s \in S_u} \sum_{t \in S_v} a_{u,t} x_s x_t = \{ \phi^* y_u, \phi^* y_v \}_A = \phi^* \{ y_u, y_v \}_{A'} = \sum_{s \in S_u} \sum_{t \in S_v} a'_{u,v} x_s x_t.
\]

Since \( S_u \cap S_v = \emptyset \), this implies item (b). \( \square \)

Before proving that surjective LV morphisms can be decloned, we prove a lemma which is an analog of the first item of lemma 2.6.

**Lemma 3.6.** Let \( \Gamma \) and \( \Gamma' \) be skew-symmetric graphs, with decloning maps \( p : \Gamma \to \Gamma' \) and \( p' : \Gamma' \to \Gamma'' \). Suppose that \( \phi \) and \( \psi \) are two linear maps, making the following diagram commutative:
If $\phi$ is an LV morphism, then $\psi$ is an LV morphism.

**Proof.** Suppose that $\phi$ is an LV morphism and consider the map $\varphi := \text{LV}(p') \circ \phi = \psi \circ \text{LV}(p)$, which is an LV morphism. For any functions $F, G$ on $\text{LV}(\Gamma')$ (i.e., on $\mathbb{F}[\Sigma']$, with $\Gamma' = (\Sigma', A')$),

\[
\begin{align*}
\text{LV}(p') & \circ \varphi = \text{LV}(p) \circ \psi = \{ F, G \}_A \equiv \{ \phi^* F, \phi^* G \}_A; \\
\text{LV}(p') & \circ \psi = \text{LV}(p) \circ \varphi = \{ F, G \}_A = \{ \phi^* F, \phi^* G \}_A,
\end{align*}
\]

where we have used that $\varphi$ and $\text{LV}(p)$ are Poisson morphisms. Since $\text{LV}(p)$ is surjective, $\text{LV}(p')^*$ is injective and we can conclude that $\psi^* \{ F, G \}_A = \{ \varphi^* F, \varphi^* G \}_A$. This shows that $\psi$ is a Poisson map. Similarly, since $\varphi$ and $\text{LV}(p)$ preserve the respective Hamiltonians,

\[
\text{LV}(p') \circ \psi \mathbf{H}_A = \varphi^* \mathbf{H}_A = \mathbf{H} = \text{LV}(p)^* \mathbf{H}_A,
\]

we may conclude from the injectivity of $\text{LV}(p)^*$ that $\psi^* \mathbf{H}_A = \mathbf{H}_A$. This shows that $\psi$ is an LV morphism.

It is easy to construct a counterexample for the inverse implication of the previous lemma, though the analogous property for graphs is an equivalence (see lemma 2.6). We now show that surjective LV morphisms can be declined, just like surjective graph morphisms (proposition 2.7). Notice that, again, weighted LV systems are not considered here, because we do not have a notion of morphism between such systems.

**Proposition 3.7.** Suppose that $\phi : \text{LV}(\Gamma) \to \text{LV}(\Gamma')$ is a surjective LV morphism. Denote by $\Gamma'$ and $\Gamma''$ the decloned graphs of $\Gamma$ and $\Gamma'$.

(a) The LV morphism $\phi$ induces a unique LV morphism $\phi : \text{LV}(\Gamma) \to \text{LV}(\Gamma')$ such that the following diagram of (surjective) LV morphisms is commutative:

\[
\begin{array}{ccc}
\text{LV}(\Gamma) & \xrightarrow{\phi} & \text{LV}(\Gamma') \\
\text{LV}(p) \downarrow & & \downarrow \text{LV}(p') \\
\text{LV}(\Gamma) & \xrightarrow{\phi} & \text{LV}(\Gamma')
\end{array}
\]

(b) If $\phi' : \text{LV}(\Gamma') \to \text{LV}(\Gamma'')$ is another surjective LV morphism, then $\phi' \circ \phi = \phi' \circ \phi$; also,

\[
\text{Id}_{\text{LV}(\Gamma')} = \text{Id}_{\text{LV}(\Gamma')},
\]

**Proof.** Consider again the map $\varphi := \text{LV}(p') \circ \phi$, which is a surjective LV morphism with irreducible target. By item (a) in proposition 3.5, $\varphi(\alpha_s)_{s \in S} = \left( \sum_{q \in S_q} \alpha_s \right)_{s \in S_q}$, for all $(\alpha_s)_{s \in S} \in \mathbb{F}[S]$ and for some partition $(S_q)_{q \in \mathbb{Q}}$ of $S$. Suppose that $s \sim t$ with $s \in S_q$ and $t \in S_r$. We show by contradiction that $s = t$, which shows that $s \sim t$. Assume therefore that $s \neq t$, still assuming that $s \sim t$. In view of item (b) of proposition 3.5, $a_{qs} = a_{qs}^*$, independently of $q \in S_q$ and $r \in S_r$; since $a_{qs} = 0$
(because $s \sim t$), $a'_w^u = 0$. This shows that $a'_w^u = a'_w^u$ when $w = u$ or $w = v$. For $w \in S'$ different from $u$ and $v$ and $r \in S'_w$, again by the cited item, and since $s \sim t$,
\[
a'_s^w = a_{s,r} = a_{r,s} = a'_r^w,
\]
so that $a'_w^u = a'_w^u$ for all $w \in S'$, which shows that $u \sim v$. Since $\Gamma'$ is irreducible, this implies that $u = v$, which contradicts the assumption that $u \neq v$. Therefore $u = v$. The partition of $S$, defined by $\sim$, is therefore a refinement of the partition $(S'_w)_{w \in S'}$. This allows us to define a map
\[
\phi : \mathbb{F}[S] \to \mathbb{F}[S']
\]

\[
(\gamma_a)_{a \in S} \mapsto \left( \sum_{u \in S'} \gamma_a^u \right)_{u \in S'}. \tag{3.17}
\]

We show that $\phi$ makes $(3.16)$ into a commutative diagram. Let $(\alpha_a)_{a \in S} \in \mathbb{F}[S]$. Then, by $(3.11)$, the definition $(3.17)$ of $\phi$ and $(3.13)$, in that order, we get

\[
(\phi \circ \text{LV}(p))(\alpha_a)_{a \in S} = \phi \left( \sum_{a \in S} \alpha_a \right) = \left( \sum_{a \in S} \sum_{u \in S} \alpha_a \right)_{u \in S'} = \left( \sum_{a \in S} \alpha_a \right)_{u \in S'} = \phi(\alpha_a)_{a \in S} = (\text{LV}(p') \circ \phi)(\alpha_a)_{a \in S}.
\]

It follows that $\phi \circ \text{LV}(p) = \text{LV}(p') \circ \phi$, as was to be shown. In view of lemma 3.6, $\phi$ is an LV morphism. This proves item (a). Item (b) follows at once from it by the uniqueness of $\phi$.

The linear map $\phi_*$, induced by $\phi$, is called its decloned LV morphism. It is clear that the above proposition says that decloning of LV systems is a functor, just like decloning of graphs. We will come back to this in section 3.7.

3.5. Isomorphisms of LV systems

We now consider LV isomorphisms. We have already seen in figure 8 that not all morphisms of LV systems are induced by graph morphisms, even when the underlying graphs are irreducible. However, isomorphisms (and in particular automorphisms) of irreducible LV systems are induced by graph morphisms, as we show in the following proposition:

**Proposition 3.8.** Let $\Gamma$ and $\Gamma'$ be two skew-symmetric graphs, with $\Gamma'$ assumed irreducible. If $\phi : \text{LV}(\Gamma) \to \text{LV}(\Gamma')$ is an LV isomorphism, then $\phi = \text{LV}(\Phi)$ for a unique graph isomorphism $\Phi : \Gamma \to \Gamma'$.

**Proof.** Suppose that $\phi : \text{LV}(\Gamma) \to \text{LV}(\Gamma')$ is an LV isomorphism. According to proposition 3.5, $\phi$ is of the form $\phi(\alpha_a)_{a \in S} = (\sum_{u \in S_u} \alpha_u)_{u \in S'}$, where the subsets $S_u$ form a partition of $S$, indexed by $S'$. Since $\# S = \# S'$ every part $S_u$ of $S'$ must be a singleton, and we can define a bijection $\Phi : \Gamma \to \Gamma'$ by letting $S_u = \{\Phi(u)\}$. Then $\phi$ takes the simple form $\phi(\alpha_a)_{a \in S} = (\alpha_{\Phi(u)})_{u \in S'}$, i.e., $\phi$ simply permutes the coordinates, as dictated by $\Phi$ and $\phi^* y_u = x_{\Phi(u)}$ for all $u \in S$. Let $u, v \in S'$ and denote $s := \Phi(u)$ and $t := \Phi(v)$. Then
\[
a_{uv, xv} = \phi^* y_u, \phi^* y_v = \phi^* \{y_u, y_v\}_{st} = a'_{u,v,x,v},
\]
which shows that $\Phi$ is a graph morphism; $\phi$ is the linear extension of $\Phi$, hence $\phi = \text{LV}(\Phi)$. Since the partition in subsets $S_u$ is uniquely determined by $\phi$, the map $\Phi$ is unique.
Thanks to proposition 3.8, we can define the notion of a weighted LV isomorphism between irreducible LV systems, and show that the graph underlying an LV system is unique, up to isomorphism.

**Definition 3.9.** Let \((\Gamma, \varpi)\) and \((\Gamma', \varpi')\) be weighted irreducible graphs. Let \(\psi : \text{LV}(\Gamma) \to \text{LV}(\Gamma')\) be an LV isomorphism and let \(\Psi : \Gamma \to \Gamma'\) denote the graph isomorphism for which \(\psi = \text{LV}(\Psi)\). Then \(\psi : (\text{LV}(\Gamma), \varpi) \to ((\text{LV}(\Gamma'), \varpi')\) is said to be an isomorphism of weighted LV systems, or simply a weighted LV isomorphism if \(\Psi^* \varpi' = \varpi\), i.e., when \(\psi : (\Gamma, \varpi) \to (\Gamma', \varpi')\) is a weighted graph isomorphism.

**Proposition 3.10.** Let \(\Gamma\) and \(\Gamma'\) be skew-symmetric graphs. Then \(\Gamma \simeq \Gamma'\) if and only if \(\text{LV}(\Gamma) \simeq \text{LV}(\Gamma')\).

**Proof.** According to proposition 3.2, we only need to show the inverse implication. As before, we write \(\Gamma = (\Sigma, A)\) and \(\Gamma' = (\Sigma', A')\), and we denote their weighteddecloned graphs by \((\Sigma, \varpi_{\Gamma})\) and \((\Sigma', \varpi_{\Gamma'})\). The coordinates on \(F[\Sigma]\) and on \(F[\Sigma']\) are respectively denoted by \(x_s\) with \(s \in \Sigma\) and \(y_s\) with \(s \in \Sigma'\). Suppose that \(\phi : \text{LV}(\Gamma) \to \text{LV}(\Gamma')\) is an LV isomorphism. We know from lemma 3.4 that if \(u, v \in \Sigma\), with \(u \sim v\), then the functions \(x_s\) of which \(\phi^* x_u\) and \(\phi^* x_v\) are linear combinations are all different. Let \(y \in \Sigma'\) and denote its clones by \(u_1, u_2, \ldots, u_{|\Sigma'|}\). Then \(\phi^* y_{u_1}, \phi^* y_{u_2}, \ldots, \phi^* y_{u_{|\Sigma'|}}\) depend only on a certain set of coordinate functions \(x_s\); since \(\phi\) is an isomorphism, their number is also \(|\Sigma'|\), so they are \(x_{s_1}, x_{s_2}, \ldots, x_{s_{|\Sigma'|}}\). The notation suggests that the vertices \(s_i\) are all equivalent, so that \(\varpi_{\Gamma'}(\underline{y}) \leq \varpi_{\Gamma'}(\underline{\phi})\) and this is the case. Indeed, since the above functions \(y_u\) are in involution, the same is true for the functions \(x_s\), and the claim follows from item (a) in proposition 3.5. We therefore get a bijection \(\Psi : \Sigma \to \Sigma'\), such that for any \(s \in \Sigma\), only the functions \(y_s\), with \(u\) being a clone of \(\Psi(s)\), depend on the functions \(x_s\), with \(s\) a clone of \(\Psi(s)\); and \(\Psi\) satisfies \(\varpi_{\Gamma'}(\Psi(s)) \leq \varpi_{\Gamma'}(\underline{\phi})\) for all \(s \in \Sigma\); since \(|\Sigma'| = |\Sigma|\) (because \(\phi\) is an isomorphism), we must have equality, \(\varpi_{\Gamma'}(\Psi(s)) = \varpi_{\Gamma'}(\underline{\phi})\) for all \(s \in \Sigma\). The isomorphism \(\phi\), induced by \(\phi\), as given by proposition 3.7, is just the linear extension of \(\Psi\). Therefore, \(\phi : (\text{LV}(\Gamma), \varpi_{\Gamma}) \to (\text{LV}(\Gamma'), \varpi_{\Gamma'})\) is a weighted LV isomorphism, induced by a weighted graph isomorphism \(\Psi : (\Sigma, \varpi_{\Gamma}) \to (\Sigma', \varpi_{\Gamma'})\). By item (b) in proposition 2.8, \(\Gamma\) and \(\Gamma'\) are isomorphic, as was to be shown.

Combined with proposition 3.1 and item (b) of proposition 2.8, we get the following result on the classification of LV systems:

**Theorem 3.11.** Let \(\Gamma\) and \(\Gamma'\) be two skew-symmetric graphs. The following are equivalent:

(a) The LV systems \(\text{LV}(\Gamma)\) and \(\text{LV}(\Gamma')\) are smoothly isomorphic;
(b) The LV systems \(\text{LV}(\Gamma)\) and \(\text{LV}(\Gamma')\) are linearly isomorphic;
(c) The graphs \(\Gamma\) and \(\Gamma'\) are isomorphic;
(d) The weighted (irreducible) graphs \((\Gamma, \varpi_{\Gamma})\) and \((\Gamma', \varpi_{\Gamma'})\) are isomorphic.

The classification of LV systems, modulo smooth isomorphisms, is therefore the same as the classification of (weighted irreducible) graphs, modulo (weighted) graph isomorphisms.

### 3.6. Automorphisms of LV systems

We now turn to LV automorphisms. We denote, for an irreducible weighted graph \((\Gamma, \varpi)\), by \(\text{Aut}(\text{LV}(\Gamma), \varpi)\) the group of automorphisms of the weighted LV system \((\text{LV}(\Gamma), \varpi)\). It is a subgroup of \(\text{Aut}(\text{LV}(\Gamma))\). Since \(\Phi^* \varpi_0 = \varpi_0\) for any \(\Phi \in \text{Aut}(\Gamma)\), the groups \(\text{Aut}(\text{LV}(\Gamma), \varpi_0)\) and
\textnormal{Aut}(LV(\Gamma))$ are isomorphic. Moreover, proposition 3.8 also leads to the following description of the automorphism group of an irreducible LV system.

\textbf{Proposition 3.12.} Let $(\Gamma, \varpi)$ be a weighted graph, where $\Gamma$ is assumed irreducible.

(a) If $\phi \in \textnormal{Aut}(LV(\Gamma))$, then $\phi = LV(\Phi)$ for a unique $\Phi \in \textnormal{Aut}(\Gamma)$.
(b) The map $LV: \textnormal{Aut}(\Gamma) \rightarrow \textnormal{Aut}(LV(\Gamma))$ is a group isomorphism, as well as its restriction $LV_0: \textnormal{Aut}(\Gamma, \varpi) \rightarrow \textnormal{Aut}(LV(\Gamma), \varpi)$.

\textbf{Proof.} Item (a) is a particular case of proposition 3.8. According to item (c) of proposition 3.2, the restriction of the functor $LV$ to $\textnormal{Aut}(\Gamma)$ (which we still denote by $LV$) takes values in $\textnormal{Aut}(LV(\Gamma))$. Item (b) of the same proposition says that this restriction is a group homomorphism. According to item (a) it is bijective, hence $LV: \textnormal{Aut}(\Gamma) \rightarrow \textnormal{Aut}(LV(\Gamma))$ is a group isomorphism. Since the condition of preserving the weight $\varpi$ is by definition the same for a weighted LV isomorphism as for a weighted graph morphism, the isomorphism $LV$ further restricts to a group isomorphism $LV_0: \textnormal{Aut}(\Gamma, \varpi) \rightarrow \textnormal{Aut}(LV(\Gamma), \varpi)$.

When $\Gamma$ is reducible, the group of automorphisms of $LV(\Gamma)$ is much larger than $\textnormal{Aut}(\Gamma)$. It is in fact infinite, as we show in the following proposition:

\textbf{Proposition 3.13.} Let $(\Gamma, \varpi)$ be a weighted graph, with $\Gamma = (S, A)$, and with cloned graph $\Gamma^\varpi = (S^\varpi, A^\varpi)$. Suppose that $\phi$ is a vector space automorphism of $F[S^\varpi]$, leaving for every $s \in S$ the subspace $\textnormal{Span}\{x_i|1 \leq i \leq \varpi(s)\}$ invariant; suppose also that $\phi$ leaves the sum of the associated coordinates $x_i$ invariant, i.e., $\phi^* \sum_{i=1}^{\varpi(s)} x_i = \sum_{i=1}^{\varpi(s)} x_i$. Then $\phi$ is an LV automorphism of $LV(\Gamma^\varpi)$.

\textbf{Proof.} Let $\phi$ be a vector space automorphism of $F[S^\varpi]$ and suppose that $\phi$ leaves for every $s \in S$ the subspace $\textnormal{Span}\{x_i|1 \leq i \leq \varpi(s)\}$ invariant. Then, for any $s \in S$ and $i = 1, 2, \ldots, \varpi(s)$,

$$\phi^* x_i = \sum_{j=1}^{\varpi(i)} \phi^* (\varepsilon_i^j x_j), \quad \text{where } \left(\frac{\phi^* (\varepsilon_i^j)}{\phi^* (\varepsilon_j^i)}\right) \in \textnormal{GL}(\varpi(s), F).$$

The condition that $\phi^* \sum_{i=1}^{\varpi(s)} x_i = \sum_{i=1}^{\varpi(s)} x_i$ amounts to $\sum_{i=1}^{\varpi(s)} \phi^* (\varepsilon_i^j) = 1$ for $j = 1, \ldots, \varpi(s)$ and therefore $\phi^* H_{\varpi} = H_{\varpi}$. Let us denote the linear coordinates on $F[S]$ by $x_j$, where $s \in S$. For $s, t \in S$ and $i \in \{1, 2, \ldots, \varpi(s)\}$, $j \in \{1, 2, \ldots, \varpi(t)\}$, because $\{x_i, x_j\}_{A_\varpi} = \alpha_{ij} x_i x_j$, we have

$$\left\{\phi^* x_i, \phi^* x_j\right\}_{A_\varpi} = \sum_{k=1}^{\varpi(i)} \sum_{l=1}^{\varpi(t)} \phi^* (\varepsilon_k^l) \phi^* (\varepsilon_j^i) \alpha_{ij} x_k x_l = \alpha_{ij} \phi^* x_i \phi^* x_j$$

$$= \phi^* \left\{x_i, x_j\right\}_{A_\varpi},$$

and therefore $\phi$ is indeed an LV automorphism of $LV(\Gamma^\varpi)$.

Propositions 3.12 and 3.13 lead at once to the following corollary. We denote the group of matrices $\{a_{ij}\} \in \textnormal{GL}(n, F)$ for which $\sum_{i=1}^{n} a_{ij} = 1$ for $j = 1, \ldots, n$ by $\textnormal{GL}^+(n, F)$. Since it is the group of invertible matrices fixing a non-zero vector, it is isomorphic to an affine group, $\textnormal{GL}^+(n, F) \simeq \textnormal{Aff}(n - 1, F) \simeq \mathbb{R}^{n-1} \rtimes \textnormal{GL}(n - 1, F)$.  

\begin{equation*}
\textnormal{GL}^+(n, F) \simeq \textnormal{Aff}(n - 1, F) \simeq \mathbb{R}^{n-1} \rtimes \textnormal{GL}(n - 1, F). 
\end{equation*}
In the notation of proposition 3.13, it follows from the proposition that the LV morphism associated to a permutation of the clones of a vertex \( s \in S \) is an element of \( \text{GL}^+(\varpi(s), F) \), viewed as an automorphism of \( \text{LV}(\Gamma) \).

**Corollary 3.14.** Let \( \Gamma \) be a skew-symmetric graph. The LV system \( \text{LV}(\Gamma) \) is irreducible if and only if its automorphism group \( \text{Aut}(\text{LV}(\Gamma)) \) is finite. \( \square \)

**Proposition 3.15.** Suppose that \( \Gamma = (S, A) \) is a skew-symmetric graph, and denote its weighted decloned graph by \( (\Gamma, \varpi_\Gamma) \), with \( \Gamma = (\bar{S}, \bar{A}) \). The following diagram is commutative and its lines are split short exact sequences.

\[
\begin{array}{cccccc}
0 & \rightarrow & \prod_{s \in \bar{S}} S_{\varpi_\Gamma(s)} & \rightarrow & \text{Aut}(\Gamma) & \rightarrow & \text{Aut}(\Gamma, \varpi_\Gamma) & \rightarrow & 0 \\
& & \downarrow & & \downarrow \cong \text{LV} & & \downarrow & & \\
0 & \rightarrow & \prod_{s \in \bar{S}} \text{GL}^+(\varpi_\Gamma(s), F) & \rightarrow & \text{Aut}(\text{LV}(\Gamma)) & \rightarrow & \text{Aut}(\text{LV}(\Gamma), \varpi_\Gamma) & \rightarrow & 0
\end{array}
\]

As a consequence, \( \text{Aut}(\text{LV}(\Gamma)) \) is a semi-direct product,

\[
\text{Aut}(\text{LV}(\Gamma)) \cong \prod_{s \in \bar{S}} \text{GL}^+(\varpi_\Gamma(s), F) \rtimes \text{Aut}(\Gamma, \varpi_\Gamma).
\]

**Proof.** Since \( \Gamma \simeq \Gamma \) in a canonical way, we can identify the vertex set \( S \) of \( \Gamma \) with the vertex set \( \varpi_\Gamma \) of \( \Gamma \) and so \( S = \{ s | s \in \bar{S}, i = 1, 2, \ldots, \varpi_\Gamma(s) \} \). The left square of the diagram is commutative because the leftmost vertical arrow in the diagram is the restriction of \( \text{LV} \) to a subgroup. The right square is commutative because of the definition of weighted decloning of LV automorphisms, in case the automorphism is induced by a graph automorphism. We have already shown in proposition 2.9 that the top line is a split short exact sequence. It implies, by commutativity of the diagram, that \( \text{Aut}(\text{LV}(\Gamma)) \rightarrow \text{Aut}(\text{LV}(\Gamma), \varpi_\Gamma) \) is surjective. The kernel of this map consists of the automorphisms of \( F[\varpi_\Gamma] \), leaving for every \( s \in \bar{S} \) the subspace \( \text{Span}\{ s | 1 \leq i \leq \varpi_\Gamma(s) \} \) invariant, as well as the sum of the associated coordinates \( x_s \), i.e., \( \phi^\ast \sum_{i=1}^{\varpi_\Gamma(s)} x_s = \sum_{i=1}^{\varpi_\Gamma(s)} \phi x_s \), so it is precisely \( \prod_{s \in \bar{S}} \text{GL}^+(\varpi_\Gamma(s), F) \). The commutativity of the diagram and the fact that \( \text{LV}_0 \) is an isomorphism imply that, since the upper line is split, the lower line is also split. \( \square \)

### 3.7 Functorial interpretation

We have already given a functorial interpretation of the results obtained in section 2 (see section 2.6). We now complete this with the results of section 3. Consider the following diagram:

\[
\begin{array}{ccc}
\text{Gr} & \overset{\phi}{\rightarrow} & \text{Gr}_0 \\
\downarrow \text{LV} & & \downarrow \text{LV}_0 \\
\text{LV} & \overset{\varphi}{\rightarrow} & \text{LV}_0
\end{array}
\]

Its top line has already been discussed in section 2.6: \( \text{Gr} \) is the category of skew-symmetric graphs with surjective morphisms and \( \text{Gr}_0 \) is the full subcategory of irreducible graphs, with
inclusion functor \( i_0 \). Also, \( \rho \) is the decloning functor. We define similarly \( LV \) to be the category whose objects are LV systems \( (F[S], A, H_S) \) and whose morphisms are surjective LV morphisms, i.e., surjective linear Poisson maps, which preserve the Hamiltonian. The full subcategory of irreducible LV systems is denoted \( LV_0 \). According to propositions 3.3 and 3.7, decloning of LV systems defines a functor \( \sigma \), which is according to item (a) in proposition 3.7 a reflection functor for the inclusion functor \( j_0 : LV_0 \to LV \). In particular, \( \sigma \) is an adjoint functor for \( j_0 \). Proposition 3.2 says that we have a functor \( LV : Gr \to LV \) which associates to a skew-symmetric graph \( \Gamma \) the corresponding LV system \( LV(\Gamma) \), and to a surjective graph morphism \( \Phi : \Gamma \to \Gamma' \) the LV morphism \( LV(\Phi) : LV(\Gamma) \to LV(\Gamma') \). It is an embedding functor and its restriction to \( Gr_0 \) is the functor \( LV_0 \), which takes by definition values in \( LV_0 \). According to proposition 3.10 the functor \( LV \), and therefore also its restriction \( LV_0 \), is a conservative functor (it reflects isomorphisms).

The commutativity of the diagram, to wit the fact that \( LV_0 \circ \rho = \sigma \circ LV \) and \( LV \circ i_0 = j_0 \circ LV_0 \), is clear on objects; on morphisms, it follows for the first equality from the fact that the decloning of a surjective LV morphism, induced by a graph morphism, is just the decloned graph morphism, viewed as an LV morphism; on morphisms, the second equality follows at once from the fact that \( LV_0 \) is a restriction of the functor \( LV \) to the subcategories \( Gr_0 \) and \( LV_0 \) of \( Gr \) and \( LV \).

### 3.8. Population dynamics interpretation of (de-)cloning

LV systems first appeared in the context of population dynamics \([16, 20]\). In this context, \( F = \mathbb{R} \), the set \( S \) is the set of species and for each \( s \in S \) the function \( x_s \) is the number of individuals belonging to the species \( s \). The differential equations

\[
\dot{x}_s = \sum_{t \in S} a_{s,t} x_s x_t, \quad \text{for all } s \in S,
\]

describe the evolution of the number of individuals of each species; the real parameter \( a_{s,t} \) governs the interaction between the species \( s \) and \( t \), with \( a_{s,t} \) being positive (respectively negative) meaning that the number of individuals of species \( s \) will grow (respectively shrink) proportionally to the number of individuals of species \( s \), to the number of individuals of species \( t \), and to \( a_{s,t} \). In this model, cloning is the natural procedure of subdividing each species in subspecies, the interaction between the subspecies of two different species being the same as the interaction between the original species, with no interaction between two subspecies of the same species. Also here, decloning is more important than cloning, since it amounts to simplifying the model by assembling similar species in a single one. From that point of view, the decloning map \( LV(p) \) of proposition 3.3 is crucial.

We show in the next proposition that under the reasonable assumption that the interaction between two species does not depend on the other species, proposition 3.3 only holds when these interactions are quadratic, with linear contribution by each of the two species, i.e., are LV models.

**Proposition 3.16.** Let \( S \) be a finite set and suppose that \( \pi \) is a Poisson structure on \( F[S] \) for which \( \{x_s, x_t\} = \pi_{s,t} = \pi_{t,s}(x_s, x_t) \), i.e., the function \( \pi_{s,t} \) depends on \( x_s \) and \( x_t \) only. Let \( \varpi : S \to \mathbb{N}^n \) be a weight vector and consider the bivector field \( \pi \) on \( F[S] \) defined by setting \( \{x_s, x_t\} := \pi_{s,t}(x_s, x_t) \). Suppose that \( \pi \) is a Poisson structure and that the decloning map \( LV(p) \) is a Poisson map. Then, for every \( s \in S \) for which \( \varpi(s) > 1 \), \( \pi_{s,t} \) is a linear function of \( x_t \). In particular, if \( \varpi(s) > 1 \) and \( \varpi(t) > 1 \) then \( \pi_{s,t} = a_{s,t} x_t x_t \) for some \( a_{s,t} \in \mathbb{F} \), and if all \( \varpi(s) > 1 \) then \( \pi \) is a diagonal Poisson structure and \( (F[S], \pi, H_S) \) is an LV system, with \( \pi = \pi_A \), where \( A = (a_{s,t})_s \).
The proof which we will give is based on the following lemma:

**Lemma 3.17.** Let \( F(\alpha, \beta) \) be a non-zero function in two variables. In case \( \mathbb{F} = \mathbb{R} \), we assume \( F \) to be smooth; if \( \mathbb{F} = \mathbb{C} \), then \( F \) is assumed to be holomorphic. Let \( m, n \in \mathbb{N} \) with \( m > 1 \) and \( n > 0 \). If

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} F(\alpha_i, \beta_j) = F\left(\sum_{i=1}^{m} \alpha_i, \sum_{j=1}^{n} \beta_j\right) \quad \text{for all } \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \mathbb{F},
\]

then

\[
F(\alpha + \alpha', \beta) = F(\alpha, \beta) + F(\alpha', \beta) \quad \text{for all } \alpha, \alpha', \beta \in \mathbb{F},
\]

(3.19)

so that \( F \) is \( \mathbb{F} \)-linear in its first argument, \( F(\alpha, \beta) = \alpha G(\beta) \) for some function \( G \) (in one variable).

**Proof.** Let us first recall the standard argument that (3.19) implies that \( F \) is \( \mathbb{F} \)-linear in its first argument. The condition implies that \( F \) is \( \mathbb{Q} \)-linear in its first argument, hence \( \mathbb{R} \)-linear by continuity. When \( \mathbb{F} = \mathbb{C} \) we fix \( \beta \) and write \( K(\alpha) := F(\alpha, \beta) \). As before, \( K \) is \( \mathbb{R} \)-linear and therefore \( K(0) = 0 \), which implies that \( K(\alpha) = \alpha \alpha' K(\alpha) \) for \( \alpha, \beta \in \mathbb{C} \) and a holomorphic function \( K \) for which \( \lambda K(\lambda \alpha) = K(\alpha) \) for all \( \lambda \in \mathbb{C} \). It follows that \( K = 0 \), so that \( K(\alpha) = \alpha \alpha' \) is \( \mathbb{C} \)-linear.

We now prove that (3.18) implies (3.19). First notice that, taking all \( \alpha_i \) and \( \beta_j \) equal to zero in (3.18), we get \( F(0, 0) = 0 \), while taking \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) equal to zero we get \( F(0, 0) = 0 \) for all \( \beta \). It is therefore clear that, when \( n = 1 \), (3.18) implies (3.19). For \( n > 1 \), taking \( \alpha_2, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) equal to zero to find \( F(0, \beta) = 0 \) for all \( \alpha \) and therefore, to finish the proof, we take \( \alpha_1, \ldots, \alpha_m, \beta_2, \ldots, \beta_n \) equal to zero to find \( F(\alpha, \beta_1) = F(\alpha, \beta_2) = F(\alpha_1 + \alpha_2, \beta_1) \) for all \( \alpha_1, \alpha_2, \beta_1 \in \mathbb{F} \), which is (3.19).

**Proof of proposition 3.16.** Suppose that \( \varpi(s) > 1 \) for some \( s \in S \), and let \( t \in S \) with \( t \neq s \). Since \( \text{LV}(p) \) is assumed to be a Poisson map,

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} \pi_{ij}(x_i, x_j) = \{x_1 + x_2 + \cdots + x_{\mu(s)}, x_1 + x_2 + \cdots + x_{\mu(t)}\} = \\
\{\text{LV}(p)^* x_i, \text{LV}(p)^* x_j\} = \text{LV}(p)^* \{x_i, x_j\} = \\
\text{LV}(p)^* \pi_{ij} = \pi_{ij} \left( \sum_{i=1}^{\mu(s)} x_i, \sum_{j=1}^{\mu(t)} x_j \right).
\]

It now suffices to apply lemma 3.17 with \( F = \pi_{ij} \) and \( m = \varpi(s) \) and \( n = \varpi(t) \) to conclude that \( \pi_{ij} \) is a linear function of \( x_i \). When also \( \varpi(t) > 1 \), then \( \pi_{ij} \) is a linear function of its two arguments \( x_i \) and \( x_j \), so \( \pi_{ij} = \alpha_{ij} x_i x_j \) for some \( \alpha_{ij} \in \mathbb{F} \). This implies that \( \pi \) is a diagonal Poisson structure if all weights \( \varpi(s) \) are at least 2.

**4. Integrability and Lax equations**

We have seen in section 3 that cloning and decloning play a special role in the classification of LV systems and in the description of their automorphism groups. In this section, we
study cloning and decloning of LV systems from the point of view of integrability and of Lax equations. In section 4.1 we recall the main examples of integrable LV systems. We show in section 4.2 that cloning and decloning preserve integrability, and we construct in section 4.3 a Lax equation for the cloning of a large class of important examples, the Bogoyavlenskij systems (recalled below).

4.1. Integrable examples

For a generic skew-symmetric graph $\Gamma$, the Hamiltonian system $LV(\Gamma)$ is not integrable, though several infinite families of LV systems are known to be integrable. Before giving a few examples of such families, we recall two basic notions of integrability, adapted to the case of LV systems. Recall that for an $n$-dimensional LV system $LV(\Gamma)$, with $\Gamma = (S, A)$, the rank of the Poisson structure $\pi_A$ is the rank of the skew-symmetric matrix $A$. Therefore, one needs for Liouville integrability $n - 1$ independent first integrals, including the Hamiltonian, which are pairwise in involution (meaning that their Poisson bracket is zero). We also recall that for superintegrability one needs $n - 1$ independent first integrals (the Poisson structure does not intervene in the definition of superintegrability).

In the examples which follow, we give some families of LV systems which are Liouville or superintegrable (conjecturally, for one of them). In each of the examples, we will use $S_n := \{1, 2, \ldots, n\}$ as the vertex set of the underlying graph.

**Example 4.1.** The best known example of an integrable LV system is the $n$-particle periodic Kac–van Moerbeke system $KM(n)$, where $n \geq 3$; it is the LV system $LV(\Gamma)$ where $\Gamma$ is a circuit with $n$ vertices. See figure 5, where the graph underlying $KM(6)$ is given. If we label the consecutive vertices in the circuit as $1, 2, \ldots, n$, then the adjacency matrix $A$ of $\Gamma$ is given by

$$A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & -1 \\
-1 & 0 & 1 & \ldots & 0 & \\
0 & -1 & 0 & \vdots & \\
\vdots & \ddots & \ddots & 0 & \\
0 & 0 & 1 & \vdots & \\
1 & 0 & \ldots & \ldots & -1 & 0
\end{pmatrix},$$

(4.1)

and so the $KM(n)$ vector field is given by

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1}), \quad \text{for} \quad i = 1, \ldots, n,$$

(4.2)

with the understanding that the indices are periodic modulo $n$, so that $x_{n+1} = x_1$ and $x_0 = x_n$. The rank of $A$ is $n - 1$ when $n$ is odd and $n - 2$ otherwise. In the first case the product $x_1x_2\ldots x_n$ is a Casimir function; in the second case, both products $x_1x_3\ldots x_{n-1}$ and $x_2x_4\ldots x_n$ are Casimir functions. An additional $\left\lfloor \frac{n-1}{2} \right\rfloor$ independent polynomial first integrals (including the Hamiltonian), in involution, are constructed from a Lax equation (with spectral parameter) for $KM(n)$, see [10] and section 4.3 below. This accounts for the Liouville integrability of $KM(n)$.

It is clear that a circuit with $n$ vertices is irreducible with automorphism group the cyclic group $C_n$. In view of proposition 3.12, it follows that $\text{Aut}(KM(n)) = C_n$; the only automorphisms of $KM(n)$ are those permutations of the variables $x_i$ which respect the cyclic order $1, 2, \ldots, n$.

**Example 4.2.** An LV system $LV(\Gamma)$ is said to be of maximal interaction if its defining graph $\Gamma$ is a tournament graph, i.e. every vertex is adjacent to all other vertices; said differently, the
entries of the adjacency matrix $A$ of $\Gamma$ satisfy $a_{i,j} \neq 0$ for $i \neq j$. A prime example is when $a_{i,j} = 1$ for all $i < j$.

\[
A = \begin{pmatrix}
0 & 1 & 1 & \ldots & 1 & 1 \\
-1 & 0 & 1 & 1 & \ldots & 1 \\
-1 & -1 & 0 & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & -1 & 0 & \ldots & -1 & 0
\end{pmatrix}.
\]  

(4.3)

We call the corresponding LV system LV($n$, 0), as we did in [6]. See figures 6 and 8 which correspond to $n = 6$ and $n = 5$ respectively. For general $n$, the Hamiltonian vector field of LV($n$, 0) is given by

\[
\dot{x}_i = x_i \left( \sum_{j=1}^{i-1} x_j - \sum_{j=i+1}^{n} x_j \right), \quad \text{for } i = 1, \ldots, n.
\]  

(4.4)

The rank of $A$ is $n - 1$ when $n$ is odd; a rational Casimir function is then given by

\[
C := \frac{x_1 x_3 x_5 \ldots x_n}{x_2 x_4 \ldots x_{n-1}}.
\]

When $n$ is even, the rank of $A$ is $n$ and so the constant functions are the only Casimir functions. For any $n$, the system possesses $n - 1$ independent rational first integrals (including the Hamiltonian, which is polynomial), making it superintegrable (see [18]). Among these rational first integrals, which can be constructed using Darboux polynomials, $\left\lceil \frac{n+1}{2} \right\rceil$ integrals in involution can be chosen, making the system also Liouville integrable. See also [14] for an integrable generalization.

The vertices of the graph $\Gamma$, underlying LV($n$, 0) all have a different outdegree, so $\Gamma$ is irreducible and has trivial automorphism group. According to proposition 3.12, $\text{Aut}(\text{LV}(n, 0))$ is the trivial group.

**Example 4.3.** The KM($n$) system has been generalized by Bogoyavlenskij [4] to the case of interaction between neighbors at distance at most $k$, with $k < n/2$ (the KM($n$) system corresponds to $k = 1$). It is called the Bogoyavlenskij system, denoted $B(n, k)$. The defining graph $\Gamma_{n,k} = (S_n, A_{n,k})$ of $B(n, k)$ is the graph with vertex set $S_n$ and with an arrow from $i$ to $j$ whenever $0 < j - i \leq k$ or $0 < i - j \leq n - k$. Said differently, there is an arrow from every vertex to the next $k$ vertices with respect to the cyclic order on $S_n$. See for example figure 5 for the graph $\Gamma_{6,2}$. The adjacency matrix $A_{n,k}$ is the skew-symmetric Toeplitz matrix of order $n$, whose first line is given by

\[
(0, 1, 1, \ldots, 1, 0, 0, \ldots, 0, -1, -1, \ldots, -1).
\]

The Hamiltonian vector field $X_{H_{F_{n,k}}}$ takes the symmetric form

\[
\dot{x}_i = x_i \sum_{j=1}^{k} (x_{i+j} - x_{i-j}), \quad \text{for } i = 1, \ldots, n,
\]  

(4.5)

where the indices are again taken modulo $n$ and in $S_n$. Bogoyavlenskij gives in [4] a Lax pair $(L, M)$ with spectral parameter for $B(n, k)$, which we will recall and use in section 4.3. The
coefficients of the characteristic polynomial of $L$ are first integrals of $B(n, k)$, which have been shown to be in involution by using the theory of $r$-matrices (see [17]). For small values of $n$ (and all $k < n/2$), one shows by direct computation that this yields enough independent first integrals for Liouville integrability; conjecturally, this holds for all $n$ (and all $k < n/2$). For $k = 1$ one gets $B(n, 1) = \text{KM}(n)$, which is Liouville integrable (see example 4.1 above) and for $n = 2k + 1$, with $k$ arbitrary, one gets the so-called Bogoyavlenskij–Itoh system, which is also known to be Liouville integrable (see [3, 12, 13]). In both cases, the first integrals are obtained from the Lax operator $L$ in (4.8). The graph $G_{nk}$ is irreducible and its automorphism group is the cyclic group $C_{2m}$. It follows that the automorphism group of $B(n, k)$ is the cyclic group $C_n$.

**Example 4.4.** As we already pointed out in section 3.1, LV systems can be reduced by setting one or several of the variables $x_i$ equal to zero. In general, the reduced systems, which are still LV systems, may be integrable or not. The LV system $LV(n, 0)$ (see example 4.2) can be seen as such a reduction of the Bogoyavlenskij–Itoh system $B(2n - 1, n - 1)$, and it is Liouville integrable with rational first integrals; moreover it is superintegrable. Reductions of $LV(n, 0)$ are of the form $LV(m, 0)$ with $m < n$ (see figure 7), hence are also Liouville and superintegrable. For the more general systems $LV(n, k)$, which are also integrable reductions of the Bogoyavlenskij–Itoh system $B(2n - 2k - 1, n - k - 1)$, see [6]. The reductions of $\text{KM}(n)$ are also Liouville integrable with polynomial first integrals since they are products of open (non-periodic) KM systems as can again be easily seen from their underlying graphs: the underlying graph of the $n$-particle open KM system is the chain with vertices $\{1, 2, \ldots, n\}$ and an arrow from $i$ to $i + 1$ for $i = 1, 2, \ldots, n - 1$. The first integrals of the latter system are obtained by reduction from the first integrals of $\text{KM}(n + 1)$, in particular they are polynomial; they are sufficient in number to ensure Liouville integrability.

### 4.2. Integrability

We show in this subsection that if an LV system $LV(\Gamma)$ is Liouville integrable, or superintegrable, then also any other LV system $LV(\Gamma')$ having the same decloning, i.e., for which $\Gamma = \Gamma'$. In particular, $LV(\Gamma)$ is Liouville integrable, or superintegrable, if and only if the decloned system $LV(\Gamma)$ is Liouville integrable, or superintegrable. Recall from section 4.1 that if $\Gamma = (S, A)$ is a skew-symmetric graph of order $n$, then $LV(\Gamma)$ is $n$-dimensional and for the Liouville integrability of $LV(\Gamma)$, we need $n - \frac{1}{2} \text{Rk} A$ independent functions in involution, among which the Hamiltonian $H_S$; for the superintegrability of $LV(\Gamma)$, we need $n - 1$ independent first integrals. Let $\pi$ be a weight vector for $\Gamma$. Then $LV(\Gamma^\pi)$ has dimension $|\pi|$, so that in particular, for the

- Liouville integrability of $LV(\Gamma^\pi)$, we need $|\pi| - \frac{1}{2} \text{Rk} A$ independent functions in involution, among which the Hamiltonian $H_S$;
- Superintegrability of $LV(\Gamma^\pi)$, we need $|\pi| - 1$ independent first integrals.

**Proposition 4.5.** Let $\Gamma$ and $\Gamma'$ be two skew-symmetric graphs with the same decloned graph, i.e., $\Gamma = \Gamma'$. If the LV system $LV(\Gamma)$ is Liouville integrable, or superintegrable, then the same is true for $LV(\Gamma')$.

**Proof.** It is clear that is enough to show that if $\Gamma$ is irreducible then $LV(\Gamma)$ is Liouville integrable (or superintegrable) if and only if $LV(\Gamma^\pi)$ is Liouville integrable (resp. superintegrable), where $\pi$ is any weight vector for $\Gamma$.

Let us denote, as before, the Poisson structures of $LV(\Gamma)$ and of $LV(\Gamma^\pi)$ respectively by $\pi_\Gamma$ and $\pi_{\pi_\Gamma}$; the standard coordinates on $\mathbb{F}[S]$ and on $\mathbb{F}[S^\pi]$ are denoted respectively by $x_i$ and $x_{i_s}$, with $s \in S$ and $i \in \{1, 2, \ldots, \pi(s)\}$; also, $n$ stands for the order of $\Gamma$. We recall from
section 3.3 that \( \pi_A \) and \( \pi_A^\infty \) have the same rank, equal to the rank of the adjacency matrix \( A \) of \( \Gamma \); let us denote this even integer by \( 2r \). Recall also from that section that we have a first set of \( |\varpi| - n \) independent Casimir functions for \( \pi_A^\infty \), which are of the form \( x_{i_j}/x_{i_j} \), for \( s \in S \) and \( i = 2, 3, \ldots, \varpi(s) \). We consider new coordinates \( y_{s_i} \), with \( s \in S \) and \( i \in \{1, 2, \ldots, \varpi(s)\} \), on an open dense subset of \( \mathbb{F}[S^\infty] \); they are defined by:

\[
y_{s_i} := LV(p)^* x_s = \sum_{i=1}^{\varpi(s)} x_{i_j}, \quad \text{for } s \in S,
\]

\[
y_{j_s} := x_{i_j}/x_{i_j}, \quad \text{for } s \in S, \text{ and } j \in \{2, 3, \ldots, \varpi(s)\},
\]

where \( LV(p) : \mathbb{F}[S^\infty] \to \mathbb{F}[S] \) is the decloning map, defined in proposition 3.3. It is clear from (3.10) that, in the new coordinates, \( LV(\Gamma^\infty) \) decouples into a subsystem, isomorphic to \( LV(\Gamma) \), and a trivial system, i.e., a system with no dynamics: explicitly, the decoupled system reads

\[
\dot{y}_{s_i} = \sum_{r \in S} a_{rs} y_{r_s}, \quad \dot{y}_{j_s} = 0, \quad s \in S, \quad j \in \{2, 3, \ldots, \varpi(s)\}.
\]

Suppose that \( LV(\Gamma) \) is Liouville integrable, respectively superintegrable, with independent first integrals \( F_1, F_2, \ldots, F_r \). On \( \mathbb{F}[S^\infty] \) we consider the functions \( LV(p)^* F_1, \ldots, LV(p)^* F_r \), as well as the Casimir functions \( y_{s_i} \) for \( s \in S \) and \( i \in \{2, 3, \ldots, \varpi(s)\} \). The former functions are first integrals for \( LV(\Gamma^\infty) \) because \( LV(p) \) is a Poisson map, with \( LV(p)^* H_1 = H_2 \); for the same reason, they are in involution when the functions \( F_1, \ldots, F_r \) are in involution. The functions \( LV(p)^* F_1, \ldots, LV(p)^* F_r \) are independent because \( LV(p) \) is a submersion; since they only depend on the variables \( y_{s_i} \), with \( s \in S \), they are also independent of the Casimirs \( y_{s_i} \), with \( i > 1 \). It follows that we have enough independent functions in involution for the Liouville integrability, respectively enough independent functions for the superintegrability of \( LV(\Gamma^\infty) \).

In order to prove the inverse implication, suppose that \( LV(\Gamma^\infty) \) is Liouville integrable, respectively superintegrable, with first integrals \( G_1, G_2, \ldots, G_r \) complemented with the Casimir functions \( \{y_{s_i} : s \in S, i \in \{2, 3, \ldots, \varpi(s)\}\} \). If we fix these Casimir functions to any specific values \( c_{i_s} \in \mathbb{F} \), then the restricted functions \( G_1, \ldots, G_r \) depend only on \( y_{s_i} \), with \( s \in S \), hence are pullbacks under the Poisson submersion \( LV(p) \) of some functions \( F_1, \ldots, F_r \), defined on an open subset of \( \mathbb{F}[S] \). Since these restricted functions are also first integrals, it follows as before that the functions \( F_1, \ldots, F_r \) are first integrals of \( LV(\Gamma) \), and that they are in involution when the functions \( G_1, \ldots, G_r \) are in involution. Moreover, for generic values of the constants \( c_{i_s} \), they are independent. It follows that \( LV(\Gamma) \) is Liouville integrable, respectively superintegrable.

Since the decloning map \( LV(p) \) is a Poisson map, the construction of the integrals in the above proof shows that if \( \Gamma \) is a skew-symmetric graph for which \( LV(\Gamma) \) is Liouville or superintegrable, then \( LV(p) : LV(\Gamma) \to LV(\Gamma^\infty) \) is a morphism of integrable systems.

4.3. Lax equations

In this section we show how a Lax equation for an LV system \( LV(\Gamma) \) leads to a Lax equation for any of the cloned systems \( LV(\Gamma^\infty) \). In the case of \( LV(\Gamma) = B(n, k) \), we will also provide an alternative Lax equation.

Suppose that \( L = [L, M] \) is a Lax equation for the LV system \( LV(\Gamma) \), where \( \Gamma = (S, A) \) is any skew-symmetric graph of order \( n \). In order to indicate explicitly how \( L \) and \( M \) depend on the \( x \)-variables, we write the matrices \( L \) and \( M \) also as \( L(x) \) and \( M(x) \). We construct a Lax
equation for \( LV(\Gamma_n) \), where \( \varpi \) is any weight vector for \( \Gamma \). To do this, we use the decloning map \( \chi := LV(p) : F[\mathbb{S}^n] \to F[\mathbb{S}] \), defined in proposition 3.3. We also let \( y_i := \chi x_i = \sum_{r=1}^{n} x_{\varpi(s)} \) for all \( s \in \mathbb{S} \). Then the vector field (3.10) of \( LV(\Gamma_n) \) can be written in the compact form

\[
\dot{x}_i = x_i \sum_{r=1}^{k} a_{i r} y_r, \quad \text{for } s \in \mathbb{S} \text{ and } i = 1, \ldots, \varpi(s). \tag{4.7}
\]

Summing up these equations for \( i = 1, \ldots, \varpi(s) \), we get

\[
\dot{y}_s = \sum_{i=1}^{\varpi(s)} \dot{x}_i = \sum_{i=1}^{\varpi(s)} x_i \sum_{r=1}^{n} a_{i r} y_r = y_i \sum_{r=1}^{n} a_{i r} y_r,
\]

which has exactly the same form as (3.1), for which we have the Lax equation \( \hat{L}(x) = [L(x), M(x)] \). It follows that \( \hat{L}(y) = [L(y), M(y)] \) is a Lax equation for \( LV(\Gamma_n) \). Suppose that the former Lax equation is regular, which means that the dynamics is completely determined by it. Then the latter Lax equation is also regular, because the Poisson manifold \( (F[\mathbb{S}^n], \pi_{\mathbb{S}^n}) \) has Casimirs functions \( x_{ sulph }, x_{ sulph } \), for \( 1 \leq s \leq n \) and \( i = 2, \ldots, \varpi(s) \). Also, when the former Lax equation depends on a parameter, then so does the latter Lax equation. Finally, if \( F \) is a first integral of \( LV(\Gamma) \) which appears as a coefficient of the characteristic polynomial of \( L(x) \), then \( \chi^* F \) is a first integral of \( LV(\Gamma_n) \) which appears as a coefficient of the characteristic polynomial of \( L(y) \); in particular, if the Lax operator \( L(x) \) provides enough first integrals to prove the Liouville integrability of \( LV(\Gamma) \) then so does \( L(y) \) for \( LV(\Gamma_n) \).

We now turn to the particular case of \( B(n, k) \). Let \( k \) and \( n \) be fixed, with \( k < n/2 \) and denote as in example 4.3 by \( \Gamma_{n,k} = (S_n, A_{n,k}) \) the graph underlying the LV system \( B(n, k) \). Bogoyavlenskij [4] constructed for \( B(n, k) \) the following Lax equation with spectral parameter \( \lambda \):

\[
\hat{L} = [L, M], \quad \text{where } \begin{cases} L := L_0 + \lambda \Delta, \\ M := M_0 - \lambda \Delta^{k+1}, \end{cases} \quad \begin{cases} L_0 := X \Delta^{-k}, \\ M_0 := \sum_{i=k+1}^{n-1} \Delta^i X \Delta^{-i}, \end{cases} \tag{4.8}
\]

and where \( \Delta \) is the circular shift matrix defined by \( \Delta_{i,j} := \delta_{i+j,1,j} \) and \( X \) is the diagonal matrix \( \text{diag}(x_1, x_2, \ldots, x_n) \); notice that \( M_0 \) is also a diagonal matrix. We fix a weight vector \( \varpi \) for \( \Gamma_{n,k} \). It follows from what precedes that \( \hat{L}(y) = [L(y), M(y)] \) is a Lax equation for the cloned system \( LV(\Gamma_{n,k}) \), where we recall that \( y_i = \chi^* x_i \) for \( i = 1, \ldots, n \); said differently, the latter Lax equation is obtained by replacing in (4.8) everywhere the diagonal matrix \( X \) by the diagonal matrix \( \chi^* X \).

In what follows we provide a different Lax equation \( \tilde{L} = [\mathcal{L}, \mathcal{M}] \) for the cloned system \( LV(\Gamma_{n,k}) \). The new Lax equation, which is also regular and also depends on a spectral parameter, has the advantage that all phase variables \( x_{ sulph } \) of the cloned system appear as entries of the new Lax operator \( \tilde{L} \). The matrices \( \mathcal{L} \) and \( \mathcal{M} \) will be block matrices with \( N \times N \) square blocks of order \( n \), where \( N \) is the maximum number of clones of a vertex, \( N := \max\{ \varpi(s) | s \in S_n \} \); for \( 1 \leq i, j \leq N \) the \((i, j)\)-block of such a block matrix \( N \) is denoted by \( N_{ij} \). We let \( x_{ sulph } = 0 \) for \( \varpi(s) < i \leq N \) and \( s \in S_n \), so that every vertex of \( \Gamma \) has now the same number of clones. With this notation, the cloned Bogoyavlenskij system \( LV(\Gamma_{n,k}) \) can be written in the simpler form

\[
\dot{x}_i = x_i \sum_{r=1}^{k} (y_{i+r} - y_{i-r}), \quad \text{for } 1 \leq s \leq n \text{ and } 1 \leq i \leq N. \tag{4.9}
\]
where the indices of the variables \( y_i \) are again taken modulo \( n \) and in \( S_n \). Let us denote, for \( 1 \leq i \leq N \), by \( X^{(i)} \) the diagonal matrix \( \text{diag}(x_1, x_2, \ldots, x_n) \), and in analogy with (4.8), let

\[
L^{(i)} := L^{(i)}_0 + \lambda \Delta, \quad M^{(i)} := M^{(i)}_0 - \lambda \Delta^{k+1},
\]

\[
L^{(i)}_0 := X^{(i)} \Delta^{-k}, \quad M^{(i)}_0 := \sum_{t=k+1}^{n-1} \Delta^t X^{(i)} \Delta^{-t}. \tag{4.10}
\]

We first show that, in terms of these matrices, (4.9) can be written as the following equation of Lax type:

\[
\dot{L}^{(i)}_0 = \left[ L^{(i)}_0, \chi^* M_0 \right] = \sum_{r=1}^N \left[ L^{(i)}_0, M^{(r)}_0 \right]. \tag{4.11}
\]

To do this, we first rewrite the first equality of (4.11), using (4.8) and (4.10); we also use that \( X^{(i)} \) and \( \Delta^t Y \Delta^{-t} \) are diagonal matrices, hence commute, where \( Y := \chi^* X \):

\[
\dot{X}^{(i)} = \sum_{t=k+1}^{n-1} X^{(i)} \Delta^{-t} Y \Delta^{k-t} - \sum_{t=k+1}^{n-1} \Delta^t Y \Delta^{-t} X^{(i)}
\]

\[
= X^{(i)} \left( \sum_{t=k+1}^{n-1} \Delta^{-t} Y \Delta^{k-t} - \sum_{t=k+1}^{n-1} \Delta^t Y \Delta^{-t} \right)
\]

\[
= X^{(i)} \sum_{t=1}^k \left( \Delta^t Y \Delta^{-t} - \Delta^{n-t} Y \Delta^{t-n} \right).
\]

The equivalence with (4.9) then follows by taking in both sides of the latter equality, which are diagonal matrices, the \((s,s)\)th entry.

**Proposition 4.6.** Let \( L \) and \( M \) be the square matrices of order \( nN \), whose blocks are defined for \( 1 \leq i, j \leq N \) by \( L^{(i,j)} := L^{(j)} \) and

\[
M^{(i,j)} := \delta_{ij} \sum_{r=1}^N \left( M^{(r)}_0 + \Delta^k X^{(r)} \Delta^{-k} \right) - \Delta^k X^{(j)} \Delta^{-k} - \lambda \Delta^{k+1}.
\]

Then \( \hat{L} = [L, M] \) is a Lax equation with spectral parameter for the cloned Bogoyavlenskij system \( LV(\Gamma_{nN}^{(k)}) \).

**Proof.** For \( 1 \leq i, j \leq N \) we show that the \((i,j)\)-blocks of both sides of \( \hat{L} = [L, M] \) agree.

On the one hand, we get by using (4.11),

\[
\hat{L}^{(i,j)} = \hat{L}^{(j)} = \hat{L}^{(j)}_0 = \sum_{r=1}^N \left[ L^{(r)}_0, M^{(r)}_0 \right]. \tag{4.12}
\]

The \((i,j)\)-block of the commutator \([L, M] = LM - ML\) is by block multiplication given by
\[ \sum_{r=1}^{N} \left( \Delta^k X(r) \Delta^{-k} + \lambda \Delta^{k+1} \right) L^{(r)} = \sum_{r=1}^{N} L^{(r)} \left( \Delta^k X(r) \Delta^{-k} + \lambda \Delta^{k+1} \right) \]
\[ + \sum_{r=1}^{N} \left[ L^{(r)} M_0^{(r)} + \Delta^k X(r) \Delta^{-k} \right] = \sum_{r=1}^{N} \left( A_0(r) + \lambda A_1(r) + \lambda^2 A_2(r) \right), \]

with
\[ A_0(r) = L^{(r)}_0 \Delta^k X(r) \Delta^{-k} - L^{(r)}_0 \Delta^k X^{(r)} \Delta^{-k} + L^{(r)}_0 M_0^{(r)} - M_0^{(r)} L^{(r)}_0, \]
\[ A_1(r) = \Delta^{k+1} L^{(r)}_0 + \Delta^{k+1} \left( X^{(r)} - X^{(r)} \right) \Delta^{-k} - L^{(r)}_0 \Delta^{k+1} + \left[ \Delta, M_0^{(r)} \right], \]
\[ A_2(r) = \Delta^{k+2} - \Delta^{k+2} = 0. \]

In order to compute \( A_1(r) \), first notice that
\[ \left[ \Delta, M_0^{(r)} \right] = \sum_{i=k+1}^{n-1} \left( \Delta^{i+1} X^{(r)} \Delta^{-i} - \Delta^{i} X^{(r)} \Delta^{1-i} \right) \]
\[ = X^{(r)} \Delta - \Delta^{k+1} X^{(r)} \Delta^{-k} = \left[ L^{(r)}_0, \Delta^{k+1} \right], \]
where we have used the definitions (4.10) and \( \Delta^a = \text{Id}_n \). It follows, using again the definition of \( L^{(r)}_0 \), that
\[ A_1(r) = \Delta^{k+1} X^{(r)} \Delta^{-k} - L^{(r)}_0 \Delta^{k+1} + \left[ \Delta, M_0^{(r)} \right] \]
\[ = \Delta^{k+1} X^{(r)} \Delta^{-k} - \Delta^{k+1} L^{(r)}_0 = 0. \]

Using again the definition of \( L^{(r)}_0 \), the first two terms of \( A_0(r) \) cancel, so that \( A_0(r) = \left[ L^{(r)}_0, M_0^{(r)} \right] \).

Summing up, we get
\[ [\mathcal{L}, \mathcal{M}]_{(i,j)} = \sum_{r=1}^{N} A_0(r) = \sum_{r=1}^{N} \left[ L^{(r)}_0, M_0^{(r)} \right], \]
so that, using (4.12), \( \dot{\mathcal{L}}_{(i,j)} = [\mathcal{L}, \mathcal{M}]_{(i,j)} \), as was to be shown.

\[ \square \]

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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