A framework of crossover of scaling law: dynamical impact of viscoelastic surface

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Abstract

In this paper, it was succeeded that a crossover of scaling laws is described as continuous process and it emerges as a result of the interference from self-similar variable of the higher class of the self-similarity on the dynamical impact of solid sphere onto a viscoelastic surface. All the physical factors including the size of spheres, the impact of velocity are successfully summarized to the primal dimensionless numbers which construct a self-similar solution of the second kind, which represents the balance between dynamics involved in the problem. The self-similar solution gives two different scaling laws by the perturbation method describing the crossover. These theoretical predictions are compared with experimental results to show good agreement. It was suggested that a hierarchical structure of similarity plays a fundamental role on crossover, which offers a fundamental insight to self-similarity in general.

1 Introduction

"Scaling never appears by accident"[1]. Scaling law is the representation of physical law, which is expressed by a power-functional relation between physical parameters (e.g., Boyle’s law is the inverse-proportional relation between pressure and volume $P \sim V^{-1}$)

\begin{equation}
y = At^\alpha
\end{equation}

in which $y$, $t$ are physical parameters, $A$ is prefactor, $\alpha$ is power exponent. It is quite general and basic concept in physics. It enable us to connect theory with experiment as theoretical verification is generally performed through the reference of the scaling relation obtained by the experimental observation[2]. On the other hand, one observes the case in which a scaling law transforms to another in different scale of physical variables, $y = At^\alpha \rightarrow Bt^\gamma$, which we call crossover of scaling law, in a wide variety of fields: the mechanics of continua[3, 4], soft matter[5], quantum physics[6], critical phenomena[7, 8] and so on. Such phenomena are very interesting for application and biology as it is expected that they can be associated with the invention of functional materials[9, 10], and may play an important role on the biological functions[11–13]. Crossover of scaling law can be formalized as the process of transition of scaling law by the continuous change of a scale parameter. However, the studies of crossovers generally focus on each scaling law in the extreme limit independently. As a result, they failed to formalize it as the continuous process. Besides the mechanism to realize the crossover is not focused on.
The appearance of stable scaling law can be understood as an intermediate asymptotic\cite{14–24}, which is defined as an asymptotic representation of a function valid in a certain scale range. Barenblatt has formalized the idealization of physical theory in terms of dimensional analysis. Dimensional analysis gives a self-similar solution of which variables are dimensionless numbers consisting of the physical quantities involved in the phenomena. Considering the dimensions of parameters, the scaling law of Eq. (1) can be transformed to $\Pi = y/At^\alpha$. Later the dependence on other dimensionless numbers, say $\theta = t^{1/\beta}$, is investigated to obtain $\Pi = \Phi(\theta)$. If the dimensionless function $\Phi$ converges to a finite limit, $\Phi \rightarrow \text{const}$ as $\theta \rightarrow 0$, which corresponds to complete similarity\cite{25}, and that a single dimensionless number is remained, an intermediate asymptotic is obtained, which results in the scaling law corresponding to Eq. (1), while it is locally valid in the range in which its asymptotic is maintained in this case; it is $y = At^\alpha$ ($\theta \ll 1, 0 < t \ll x^{1/\beta}$).\cite{14}. This formalization facilitates us to understand the idealization of physics and the dimensionless numbers though his theory is limited to the case of a single scaling law and has not extended to how crossovers occur.

In this paper, I develop Barenblatt’s idea for crossover of scaling law. In terms of the concept of the intermediate asymptotic, it is expected that crossover must correspond to the case in which its idealization is broken. As previously mentioned, a single scaling law is obtained in the condition in which its dimensionless function converges to a finite limit, $\Phi = \text{const}$. Conversely speaking, the incomplete convergence of dimensionless function, $\Phi \neq \text{const}$, namely the interference of another dimensionless number may generate crossover of scaling law. Thus, $\Phi$ is a mechanism to change the intermediate asymptotics to another. I will demonstrate that we can understand crossover by such a framework. If we find $\Phi$, we can describe the crossover as continuous process.

In this work, on the dynamical impact of solid sphere onto the viscoelastic surface, I will show how Maxwell model becomes mechanism to generate crossover of scaling law and how it constitutes a self-similar solution in which two dimensionless variables are related, which corresponds to the above-mentioned framework. Maxwell model is frequently applied to the various phenomena in which the behaviors change on different time-scale, such as earthquake\cite{26, 27}, and fracture\cite{28, 29}. In the context of contact mechanics\cite{30–32}, in which features of contact are drastically changed\cite{33, 34} depending on the form of contact, the viscoelastic materials have been interesting materials\cite{35–37}. It is recognized that the viscoelasticity plays an important roll on the adhesion of interface and closing or opening crack between surface\cite{28}. In this work, I focus on the viscoelastic behavior derived from bulk property. I will show that a new scaling law appears in viscoelastic regime and that there exist a self-similar solution which governs this crossover and two scaling law, which corresponds to a self-similar solution of the second kind.

2 Experiment

The experiments have been performed using a viscoelastic surface made of polydimethylsiloxane (PDMS) (Fig. 1). The PDMS (SILPOT\textsuperscript{TM} 184 W/C, DOW) surface was prepared by mixing curing agent and base by the proportion of 1:40 and pour into the mold. After leaving for 3 hr 30 min at 60°C, the surface was solidified, of which thickness $h = 7.5$ mm, the fraction of contact $\phi = 1$, elastic modulus $E \simeq 0.78$ MPa and viscous coefficient $\mu = 141$ Pa·s. The elastic modulus and viscous coefficient were estimated by fitting the experimental results. The prepared PDMS surface was so viscous and smooth that the ball did not rebound by simply dropping the ball due to the adhesion effect. In order to eliminate such an adhesion effect, the PDMS surface was coated with grease (WD-40). The metallic ball (Tsubaki Nakashima co., ltd., SUJ2) was suspended by the electromagnet (ESCO Co.,Ltd., EA984CM-1) of which magnet force is controlled. Once the ball is released from the electromagnet, it starts to free fall and collides with the PDMS surface (Fig. 2)\cite{38}. After the contact of surface, it reaches to the maximum deformation $\delta_m$ then rebounds to release from surface again.

The processes are observed by high speed camera (FASTCAM SA1.1, 768×768 pixel, 10000 fps). The size of sphere $R$ is differed as 3.0, 4.0, 5.0, 6.0, 7.0 and 8.0 mm, of which density $\rho = 7800$ kg·m$^{-3}$. The collision experiments were performed for 6 times in each conditions. The information of velocity, maximum deformation, contact time and so on were extracted from the movies by...
image analysis which is programmed by Python using Open CV. These numerical estimations were used to calculate the relevant dimensionless numbers.

3 Maxwell viscoelastic foundation model

Here we think about the problem in which a rigid sphere is free fall onto the viscoelastic surface (See Fig. 3). In our experiment condition, I assume that the adhesion effect is eliminated by the coating with grease, thus only the viscoelastic bulk property contributes. In this case, the surface is modeled by the viscoelastic-foundation model in which the stress deformation is described by foundations which are arranged in parallel[39]. The foundation model is simplified model to describe the stress but widely applied to the viscoelastic materials. In our model, which we call Maxwel viscoelastic model (MVF model), each foundation consists of a dashpot (viscous coefficient $\mu$) and a spring (elastic modulus $E$), which are serially connected. In this case, the stress $\sigma$ and the deformation $\epsilon$ can be related by the following differential equation with time $t$, $\mu \frac{d\delta}{dt} + \sigma = \mu \frac{d\delta}{dt}$, which corresponds to Maxwell model. By assuming the deformation by the impact of sphere of which deformation is $\delta$, thickness of surface is $h$, the deformation can be described by $\epsilon = \frac{\delta}{h}$. Thus, the rate of deformation is described by $\frac{d\epsilon}{dt} = \frac{1}{h} \frac{d\delta}{dt}$. In this model, it is assumed that the main contribution of deformation is due to $\frac{d\delta}{dt} \simeq \text{const}$ for the foundation, which is supported by the observation of the experiments[40]. In this case, the differential equation is solved as $\sigma (t) = \mu \frac{d\delta}{dt} \left[ 1 - e^{-\frac{Et}{\mu}} \right]$. As the rate of deformation is independent of the position of contact within the contact area $\pi a^2$ where $a$ is contact radius, thus the energy of deformation is described by

$$E_{MVF} = \frac{\pi \mu \phi R \delta_0 \delta}{h} \frac{d\delta}{dt} \left[ 1 - \exp \left( -\frac{Et_c}{\mu} \right) \right]$$

(2)

where $R$ is radius of sphere, $\phi$ is the fraction of contact and $t_c$ is contact time. $\phi = 1$ in the plane surface.

$E_{MVF}$ is quite characteristic as it transforms depending on the contact time $\mu / Et_c = De$, Deborah number[41]. Supposing $De \gg 1$ which can be realized by the fast-time impact due to the following scaling $t_c \sim \delta_m / v_i$ where $\delta_m$ is maximum deformation, $v_i$ is the impact-velocity,
Fig. 3 (Color online) The geometrical parameters involved in the collision between elastic surface with its thickness \(h\) and solid sphere with its radius \(R\) in the impact-velocity \(v_i\). The deformation \(\delta\) and diameter of contact \(a\) are generated by the collision onto the viscoelastic surface.

and the relation of \(\frac{d\delta}{dt} = v_i\), Taylor expansion is applied to \(E_{MVF}\) as follows; \(E_{MVF} = \frac{\pi\mu\phi R\delta_m^2}{h}v_i\left(\frac{E\delta_m}{\mu v_i} - \cdots\right) \approx \frac{\pi\mu\phi R\delta_m^2}{h} = E_{el}\), which corresponds to the elastic energy[24, 42]. This results shows that \(E_{MVF}\) experiences a transition to fully elastic energy or the energy mixed with viscous component depending on the contact time.

Suppose that the kinetic energy of the solid ball with the density \(\rho\) is converted to \(E_{MVF}\), we have

\[
\frac{2}{3} \pi R^3 \rho v_i^2 = \frac{\pi\mu\phi R\delta_m^2}{h}v_i\left[1 - \exp\left(-\frac{E\delta_m}{\mu v_i}\right)\right];
\]

it is energy exchange at which the deformation is maximized.

4 Dimensional analysis

In this section, I intend to demonstrate how MVF model gives rise to crossover of scaling law through dimensional analysis. Eq. (3) suggests the possibility to change the form of energy depending on the contact time. Now here I intend to visualize the dynamics which involve in the problem by exploring self-similar structure.

In order to see the self-similar structure, here I perform the dimensional analysis[43]. The physical parameters which are involved are summarized to the following function \(\delta_m = f (R, h, \phi, \rho, \mu, E, v_i)\). The dimensions of the function are described as follows; \([\delta_m] = L, [R] = L, [h] = L, [\phi] = 1, [\rho] = M/L^3, [\mu] = M/LT, [E] = M/LT^2, [v_i] = L/T\) by LMT unit. By selecting \(R, \rho, E\) as the governing parameters with independent dimensions, which are defined as the parameters which cannot be represented as a product of the remaining parameters, following self-similar variables are defined:

\[
\Pi = \frac{\delta_m}{R}, \kappa = \frac{h}{R}, \eta = \frac{\rho v_i^2}{E}, \theta = \frac{\mu}{E^{1/2} \rho^{1/2} R} (4)
\]

then we have \(\Pi = \Phi(\phi, \kappa, \theta, \eta)\) where \(\eta\) corresponds to Cauchy number, dimensionless velocity-component.

To go further to consider the self-similarity structure, the following solution is quite helpful,

\[
\Pi = \text{const} \left(\frac{\kappa}{\phi}\right)^{\frac{1}{3}}\eta^{\frac{1}{3}} (5)
\]

which corresponds to Chastel-Gondret-Mongrue (CGM) solution[42, 44, 45]. CGM solution is obtained in the case in which the kinetic energy is fully transformed to elastic energy on foundation model, which corresponds to the solution obtained from Eq. (3) in case of \(De \gg 1\). I have shown previously that \(E_{MVF}\) turns to be the elastic energy in high-velocity impact. It is expected that the scaling solution Eq. (5) experiences a kind of operation and transforms to another in the low-velocity impact in which \(De < 1\). This transformation is expected to give rise to a crossover. If we here define newly a dimensionless number \(\Psi = \frac{\Pi}{\kappa \eta^2}\) from Eq. (5), \(\Psi\) is no more constant in case of \(De < 1\) but a parameter which depends on another dimensionless number, De. This dependence can be identified from Eq. (3) by defining a new dimensionless number \(Z = \frac{\Pi}{\delta_m^{1/2} \mu v_i}\), as follows,

\[
\Psi = \frac{2}{3} \frac{Z}{1 - \exp(-Z)} (6)
\]

though \(Z\) equals to \(1/\text{De}\). Supposing \(\Psi = \Phi(Z)\), \(\Phi\) converges to a finite limite as \(Z\) goes to zero[46]. Therefore, equation (6) belongs to a self-similar solution of the second kind[47], which is defined as the power-exponents of self-similar variables cannot be determined by dimensional analysis and mathematically corresponds to a fractal[48]. Here \(\Phi(Z)\) is the function that transforms CGM solution.

Note that there is a hierarchical structure on the self-similar solution in Eq. (6) depending on the convergence of dimensionless function (See Fig. 4). \(\Pi, \kappa, \theta, \phi\) and \(\eta\) belong to a similarity-class which forms a following similarity structure: \(\Pi = \Phi(\phi, \kappa, \theta, \eta)\). Here I call this class similarity of the
first class as it is generated through dimensional analysis. In the first class, each parameter belong to dimensionless physical quantities. On the other hand, $\Psi$ and $Z$ belong to an another similarity-class to form the following similarity structure: $\Psi = \Phi (Z)$ where $\Psi = \frac{\eta^3}{\kappa \eta^2}$ and $Z = \frac{\Pi}{\eta^{1/2}}$. I call this class similarity of the second class [49]. The variables of the second class normalizes the difference of the variables of the first class to integrate the single lines, which corresponds to the data collapse [50, 51].

In the second class, self-similar variables represents the dynamics of energy involved in the process. $\Psi$ represents the proportion of kinetic energy and elastic energy while $Z$ represents the proportion of viscous energy and elastic energy, which corresponds to Deborah number. One can find that $\Phi (Z)$ represents the interference of viscous components. If $Z$ goes to 0, which can be achieved by the high-velocity impact or short-time contact, $\Phi \to \text{const}$, then we have the CGM solution, which is realized in the case in which the kinetic energy fully transforms to elastic energy. Here the convergence of $\Phi (Z)$ means the inactiveness of $Z$. Thus in case of $Z \ll 1$, the impact is elastically dominant, which we call elastic impact and it gives $1/3$ power-law on $\eta$. However, when this idealized condition is not satisfied ($Z > 1$), which can be realized by the low-velocity impact, the viscous component $\Phi (Z)$ interferes $\Psi$. In this region, the viscosity contributes in the impact, then it changes scaling law. This impact corresponds to the viscoelastic impact.

We cannot see the actual scaling behavior of $\Pi$ and $\eta$ in viscoelastic regime from the second class. They belong to the first class and their behaviors are not simply consistent with $\Psi$ and $Z$ as one see $\frac{\Pi^3}{\kappa \eta^3} = \Phi \left( \frac{\Pi}{\eta^{1/2}} \right)$, which shows that $\Pi$ and $\eta$ are included in both variables of the higher class, $\Psi$ and $Z$. In order to know the scaling behavior of $\Pi$ and $\eta$, here I apply the third term perturbation method [52], then we have

$$\Pi = \frac{\kappa}{54 \phi \theta^2} + \left( \frac{\kappa^2}{486 \phi^2 \theta^3} \right)^{1/3} \eta^{5/3} + \left( \frac{2 \kappa}{3 \phi} \right)^{1/3} \eta^{3/2} \quad (7)$$

as $\varepsilon = \frac{1}{\eta^{1/2}} \to 0$ [53].

As we can see, Eq. (7) includes two different power exponents as $\eta^{1/3}$ and $\eta^{1/6}$, which suggests that intermediate asymptotics appear depending on $\theta$, $\eta$ or $Z$. In case of the impact of high velocity and/or smaller sphere, which corresponds to $\eta \gg 1$ and/or $\theta \gg 1$ and $Z \ll 1$, $\eta^{1/3}$ is dominant. Conversely in the region of viscoelastic impact in which $Z > 1$, realized by low-velocity $\eta \ll 1$ and/or the impact of larger sphere $\theta \ll 1$, $\eta^{1/6}$ is dominant while the intermediate behavior may be realized in $Z \sim 1$.

5 Result and discussion

In previous section, it was expected that scaling law of CGM solution experiences the interference from Deborah number $Z$. This interference is described by Eq. (6) as self-similar solution of the second kind, $\Psi = \Phi (Z)$. The self-similar solution directly describes the dynamics between kinetic component, elasticity and viscosity. It consists the similarity of the second class though actual scaling behavior is understood by Eq. (7) through the perturbation method, which predicts the existence of the crossover of scaling law on $\Pi$ and $\eta$. Each equation describes the different class of self-similarity.

Figure 5 shows the self-similar variables in different self-similarity class. Figure 5 (a - f) demonstrates the self-similarity of the first class, which is the power-law relation between $\Pi$ and $\eta$ in different size of spheres while Figure 5 (g) demonstrates the self-similarity of the second class, which is...
their Ψ and Z. As we can see, the plots of Π and η reveal gradually different scaling law from \( R = 3.0 \text{ mm to 8.0 mm} \). Equation (7) predicts that Π and η have different power law depending on \( θ \) and \( η \), finally summarized to \( Φ(\eta) \). As the prediction mentioned in previous section suggests, the impacts of the smaller spheres \( (R = 3.0, 4.0 \text{ mm}) \) which tends to have smaller \( Z \) follow \( 1/3 \) power-law, which corresponds to elastic impact (Fig. 5 (a, b)). \( Φ(Z) \) belonging to the plots of \( 1/3 \) power-law shows the smaller \( Z \) and tendency to convergence to a finite limit, which signifies that \( Φ(Z) \), viscous component hardly contribute. On the other hand, larger spheres \( (R = 7.0, 8.0 \text{ mm}) \) reveal different power-law, which is closer to \( 1/6 \) power-law in low-velocity (Fig. 5 (e, f)). The plots revealing \( 1/6 \) power-laws belong to the larger \( Z \) and Ψ increases in Fig. 5 (g), which means that viscous component \( Φ(Z) \) contributes and it is viscoelastic impacts. The dash-dots lines which are described by Eq. (7) are consistent with the power-law behavior of the plots. The plots of intermediate size of spheres \( (R = 5.0, 6.0 \text{ mm}) \) reveals slightly intermediate scaling law between \( 1/3 \) and \( 1/6 \) though this behavior is well described by the Eq. (7). All these plots roughly follow the line described by the self-similar solution of Eq. (6) in Fig. 5 (g).

We see that some plots that belong to high-velocity impacts in larger sphere \( (R > 5.0 \text{ mm}) \) having smallest \( Z \), reveals larger deviation in Fig. 5 (g). This may be because the indentation of high-velocity impact of larger sphere tend to be so intense that the assumption of foundation model could be violated. In the contact mechanics, the deformation is assumed to be smaller, compared with radius of sphere. As the deviation is larger in the case in which the indentation is large compared to the thickness of surface, the contact may be no more elastic. Another reason can be the interaction of the surface, as adhesion is not considered. To realize this assumption, the adhesion effect was decreased by coating grease though it may not be enough to remove these interaction completely, which may result in the dependence on the size of spheres. However, the plots belonging to intermediate velocities tend to follow the single line well. We can say that overall behaviors well correspond to the theory. The scaling behaviors were well described by theoretical lines in Fig. 5 (a-f), which demonstrates the good consistency.

The model assumes the main contribution is due to \( \frac{dv_i}{dt} = v_i \), which corresponds to the square deformation. This assumption is well justified by the observation (see Fig. S2)[40]. The observation shows that the rate of deformation corresponds to the impact velocity \( v_i \), then the domain in which the rate of deformation is constant is maintained, then it steeply decreases in the end. The attractors of deformation are quite similar in different impact-velocity (Fig. S2 (b)). Thus, the different impact-velocity makes the different contact time (Fig. S2 (c)). The lower the impact-velocity is, the longer the contact time is, which results in the different feature of impact and different scaling laws.

The adhesion effect plays the important roll on the viscoelastic impact though in this work we focused on the roll of viscoelasticity from the bulk property. As the adhesion effect is largely eliminated by coating the surface with the grease and the viscoelastic effect is limited in the crack closing[28], it is expected that the adhesion effect is quite limited in this work. The consideration of the adhesion effect did not improve the data collapse. Falcon et al. reported that the gravity effect on the repeated bouncing ball[54]. In this work, as the maximal deformation and the impact velocity, which is determined by the energy exchange, are focused, the roll of gravity is not apparent. However, it is indirectly related with the impact velocity while the gravity fields is constant in this experiment. This effect does not make difference on the results, which is the limitation of this work. 

\( E_{MVF} \) reveals an interesting feature as it transforms its form qualitatively depending on \( Z \). It should be noted that such a behavior does not appear from Kelvin-Voigt model, which is another model for viscoelastic materials and consists of the spring and dash pot are parallely connected. Therefore, Kelvin-Voigt model does not make crossover and it is clear that Kelvin-Voigt model is not appropriate for this problem. On the other hand, it is found that Zener model in which Kelvin-Voigt model and Maxwell model are combined, also reveals the similar crossover between \( η^{1/3} \) and \( η^{1/6} \) with different coefficient on Eq. (7) by the perturbation method. Thus the Maxwell element is essential for the crossover.

In my hypothesis of crossover of scaling law, it was expected that the interference of another dimensionless number generates the crossover.
Fig. 5 (Color online) The different hierarchical structure of self-similarity. (a) - (f) Self-similarity of the first class: the power law relations $\Pi$ and $\eta$ in different size of sphere. The dashed lines indicates the slope of 1/6, the solid line indicates the slope of 1/3 and colored dot-dashed line indicates Eq. (7) in each size of spheres. (g) Self-similarity of the second class: the plots between $\Psi$ and $Z$. $R = 3.0$ mm ($\bullet$), 4.0 mm ($\triangle$), 5.0 mm ($\times$), 6.0 mm ($\phi$), 7.0 mm ($\blacksquare$) and 8.0 mm ($\downarrow$) where $\Pi = \delta_m/R$, $\eta = \rho v_i^2/E$, $\Psi = \frac{\Pi^3}{\eta \kappa}$, and $Z = \frac{\Pi}{\eta^{1/2}} = \frac{E \delta_m}{\mu v_i^3}$. The red line in Figure (g) is Eq. (6). The dashed line roughly indicates the line separating the region.

Note that CGM solution is normalized to a dimensionless number $\Psi$ then $\Phi(Z)$ describes how $\Psi$ is interfered and changed by $Z$. This process is experimentally described in Fig. 5 (g). In the end, Maxwell model constitutes such an interference as a self-similar solution Eq. (6). When $\Psi$ behaves as constant $\Phi = \text{const}$, CGM solution is obtained as an intermediate asymptotics then $\Pi \sim \eta^{1/3}$. However, in the region of $\Phi \neq \text{const}$, which means the interference of another dimensionless number $Z$, another scaling law appears, $\Pi \sim \eta^{1/6}$. In this region, the viscoelastic effect starts to interfere. This is the fundamental mechanism of the crossover of scaling law in this problem. The self-similar solution, Eq. (6) changes the CGM solution qualitatively. It is expected that such a class of self-similar solutions may exist on other crossovers of scaling law. Thus we expect; there
exist a self-similar solution on crossover of scaling law. However, it should be noted that such a solution belongs to the higher hierarchy of self-similarity as dynamics involved in the problem is described on higher hierarchy, which suggests that the hierarchy of self-similarity is quite important on the crossover.

We see that Eq. (6) describes the process of crossover continuously. Generally, in the studies of crossover, the scaling behaviors in each domain are investigated independently. However, in this work, we started from the idealized region in which a scaling law had already obtained as CGM solution, then we define a dimensionless number $\Psi$ from CGM solution and identified the self-similar solution $\Phi$ from MVF model. As we see previously, $\Phi$ describes the degree of transformation of scaling law which is normalized in $\Psi$. This strategy is quite unique and may effective to other problem. We generally find idealized regions in which the problem is simpler and certain scaling law appears in most of problem even though the non-idealized problem is difficult to attack. By starting from idealized region and introducing the scaling law itself as a dimensionless number, one may find how this idealized region is changed, which may leads to the way to extend the problem into non-idealized region. One notes that $\Phi$ does not only qualitatively decide the crossover but also it numerically expresses the balance between the dimensionless numbers. This numerical balance accurately decides the balance of coefficients of Eq. (7), which enables us to describe the exact behaviors of crossover more accurately.

As the framework and mechanism of crossover is proposed, it is expected that such a framework may exist in other problems. All the stable scaling laws should be intermediate asymptotics in which dimensionless functions converge. Thus the transition of scaling laws must be given by the violation of this idealization. Yasuda et al. also reported the primal dimensional number to change the scaling laws\[55\]. Barenblatt reported the dependence of power exponents by another dimensionless numbers\[56, 57\] though in these cases the exact form of dimensionless function $\Phi$ were unclear and the hierarchy was not focused on. In the present work, I succeeded to identify the exact form of dimensionless functions. The insights in the present work suggested that the investigation of hierarchy of the higher class can be the clue to describe crossover.

Finally, the combination of dimensionless number listed on Eq. (4) is not the only possible selection. However, this combination is plausible in terms of the perturbation. It suggests that the selection of dimensionless numbers is not arbitrary as it is related with the strategy of perturbation. $\theta = \frac{\mu}{E^{1/2} \rho^{1/2} R}$, appearing naturally in this problem and playing an essential role, is also an interesting dimensionless number as it can be expressed as $\theta = \frac{Re}{Ca^{1/2}}$ in which $Re$ is Reynolds number and $Ca$ is Cauchy number. $\theta$ is here indicates the proportion of viscosity, elasticity and inertia.

6 Conclusion

The above discussion with experimental results confirms the validity of Eq. (6) with Eq. (7) as the fundamental equation of this problems. In this paper, I have succeeded to obtain the self-similar solutions governing the exact behavior of crossover of scaling law theoretically, which well corresponds to experimental results. Then I have succeeded to demonstrate the framework and mechanism of crossover of scaling law; crossover of scaling is generated by the interference of another self-similar variables of higher class, which corresponds to the framework mentioned in the introduction. It suggests that there always exists a self-similar solution on crossover.

The method exercised in this work is unique in terms of methodology. This work succeeded to describe the crossover of scaling law as continuous process. The degree of this interference is quantitatively and qualitatively estimated and it enable us to describe crossover more accurately. This accuracy was guaranteed by the coefficients of the different scaling laws which was given by the self-similar solution of the second class, which suggests that the coefficients are essential on the accurate description of crossover.

Finally, intermediate asymptotics is the locally valid asymptotic expression while we have found that this locality is governed by the self-similar solution of the higher class in this paper. This framework is simple and expected to be quite general in physics. Besides, it is similar to critical phenomena in which the transition of phase
is generated by the continuous parameter variation. Therefore, the present work supplies interesting insights for the concept of self-similarity, nonequilibrium theory and critical phenomena, for a wide variety of fields in physics in general.

**Supplementary information.**

**Data availability.** The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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The case in which dimensionless function obtained by dimensional analysis $\Phi(\xi, \eta)$ converges to a finite limit as $\eta \to 0$ or $\infty$ corresponds to complete similarity or similarity of the first kind in $\eta$ while the case in which the complete similarity is not satisfied but the convergence of dimensionless function is recovered by constructing the new self-similar variables as $\Pi/\eta^\xi$ and $\xi/\eta^\zeta$, corresponds to incomplete similarity or similarity of the second kind.

See Supplemental Material for Complete similarity and incomplete similarity.

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INTERMEDIATE ASYMPTOTICS

In this section, I briefly explain the concept of intermediate asymptotics which is formalized by Barenblatt[1–4] by using a simple example. Intermediate asymptotics is an asymptotic representation of a function valid in a certain range of independent variables, which corresponds to a kind of the formalization of the idealization which accompanies with the construction of physical model. To understand this concept, the following problem of dimensional analysis might be helpful. Imagine that the circle is pictured on the surface of the sphere (See Fig. S1). In this problem, the physical parameters that are involved are the surface area of circle \( S \), radius of the circle \( r \) and the radius of sphere \( R \). Here we would like to know the scaling behavior between \( S \) and \( r \). Therefore we assume the functional relation as follows: \( S = f(r, R) \).

![Fig. S1: (Color online) A circle of which radius is \( r \) and surface area is \( S \), described on a sphere of which radius is \( R \).](image)

In this case, we attempt to obtain the exact scaling behavior by dimensional consideration. According to dimensional analysis, as the dimension of physical parameters \([S] = L^2\), \([r] = L\) and \([R] = L\), selecting \( r \) as a governing parameter of independent dimension, we have the following dimensionless function,

\[
\Pi = \Phi (\theta)
\]  

where \( \Pi = \frac{S}{r^2} \) and \( \theta = \frac{r}{R} \). Eq. S1 suggests that we expect the following scaling relation, \( S \sim r^2 \), if \( \Pi \) is constant. However, we easily find that this guess depends on the behavior of \( \Phi \).

By the geometrical consideration, in this case we can calculate the exact form of \( \Phi \) as follows,

\[
\Phi (\theta) = 2\pi \frac{1 - \cos \theta}{\theta^2}.
\]

To know the behavior of \( \Phi \) in the case in which \( \theta \to 0 \), which corresponds to the increase of \( R \) or the decrease of \( r \), Taylor expansion is applied to Eq. (S2) then we have,

\[
\Pi = \Phi (\theta) \simeq \pi - \frac{\pi}{12} \theta^2 \cdots + \frac{1}{\theta \to 0} \pi.
\]
As Eq. (S3) shows, Φ converges to a finite limit π, then we have a following intermediate asymptotics as \( \Pi = S r_2 \),

\[
S = \pi r^2 \quad (0 < r \ll R) \quad (S4)
\]
as far as the asymptotic condition \( \theta \ll 1 \), corresponding to \( 0 < r \ll R \), is satisfied.

Note that the scaling law Eq. (S4) is valid in the scale range \((0 < r \ll R)\), in which the circle is significantly smaller than the sphere. Therefore, Eq. (S4) is an asymptotic expression which is valid in the certain range of variable \( r \). This scaling law formalized \textit{locally} is an intermediate asymptotic in this problem. Barenblatt insisted that this framework is applicable to the construction of physical model.

The important point of this concept is that every physical problem has dimension and can be applied dimensional analysis to obtain dimensionless function \( \Phi \). By considering the convergence of \( \Phi \), some self-similar variables can be selected to have the idealized solution effectively and practically as the convergence of \( \Phi \) can be verified by the experimental or simulational results even if the exact form of \( \Phi \) is not obtained. This procedure give rise to the strategy of Barenblatt as it is formalized in the recipe[5].

The second important point is that this process, in which one screens the self-similar variables of \( \Phi \) depending on their convergence, corresponds to idealization of the problems. More or less, all the physical models involve idealizations such as ignorance of friction force, ignorance of quantum or relativity effect. All these assumption corresponds to the idealizing process of dimensionless function. For example, ideal gas equation can be considered as an intermediate asymptotic valid in the range where the volume of molecules \( b \) and the molecular interaction \( a \) are negligible on van der Waals equation as follows,

\[
p = \left( \frac{nRT}{V} \right) - \left( \frac{an^2}{V^2} \right) \rightarrow \frac{nRT}{V} \left( \frac{an^2}{V^2} \ll p \ll \frac{RT}{b} \right). \quad (S5)
\]

This idealizing scale range is satisfied as far as \( \Pi_a = \frac{an^2}{RT^2} \ll 1 \) and \( \Pi_b = \frac{p}{RT} \ll 1 \). The interested readers may refer to Ref.[6] for further discussion related with phenomenology, renormalization and asymptotic analysis on physics.

This concept suggests that every physical theory is \textit{locally} valid. This localization is quantitatively and qualitatively formalized by the intermediate asymptotics. In the present work, the author focuses on this point and consider the case of the transition of this \textit{locality}.

**COMPLETE SIMILARITY AND INCOMPLETE SIMILARITY**

In this section, I briefly explain complete similarity and incomplete similarity, as well as the self-similarity of the first kind and the self-similarity of the second kind [7]. They are the category in terms of the convergence of dimensionless function. Zeldovich noted that there exists the type of self-similarity[8]. As the previous section showed, the self-similar solution is obtained by dimensional analysis. Supposing that a certain physical function,

\[
y = f(t, x, z) \quad (S6)
\]
in which \( y, t, x \) and \( z \) are certain physical quantities which have physical dimensions. Selecting \( t \) as a governing parameter with independent dimension, which is defined as physical parameters which cannot be represented as a product of the remaining parameters, then we apply dimensional analysis to have

\[
\Pi = \Phi(\eta, \xi) \quad (S7)
\]

where \( \Pi = y/t^\alpha, \eta = x/t^\beta \) and \( \xi = z/t^\gamma \). \( \alpha, \beta \) and \( \gamma \) can be fully determined by the consideration of dimension of parameters through dimensional analysis.

If \( \Phi \) converges to a finite limit as \( \xi \) goes to zero or infinity, this case corresponds to complete similarity or similarity of the first kind in the similarity parameter \( \xi \). In this case, \( \xi \) can be excluded on the consideration and we have an intermediate asymptotics. Once \( \eta \) and \( \xi \) both satisfy the complete similarity then \( \Phi \rightarrow \text{const} \) as \( \eta \gg 1 \) and \( \xi \gg 1 \), then we have a following intermediate asymptotic, \( y = \text{const} \ t^\alpha \ (0 < t \ll x^{\gamma/\beta}, \ 0 < t \ll z^{\gamma/\gamma}) \). When the self-similar variables satisfies the condition of complete similarity, \( \Pi = \Phi(\xi, \eta) \) is corresponds to a self-similar solution of the first kind. In the previous section, as Eq. (S3) shows, the dimensionless function converges to a finite limit, therefore the problem belongs to complete similarity and Eq. (S1) is a self-similar solution of the first kind.

On the other hand, in the case in which the complete similarity is not satisfied, namely \( \Phi \) does not converge to a finite limit as \( \eta \) goes to zero or infinity, but the convergence is recovered by constructing new self-similar variables as
the power-law monomial using dimensionless variables, this case corresponds to incomplete similarity or similarity of the second kind. In this case, we may have the following self-similar solution, which is called self-similar solution of the second kind,

\[ \Psi = \Phi(Z) \]  

(S8)

where \( \Psi = \Pi/\eta^\zeta \) and \( Z = \xi/\eta^\epsilon \).

The first important point is that the power exponent \( \zeta \) and \( \epsilon \) cannot be determined by dimensional analysis in case of the second kind while it is possible in case of the first kind. We may occasionally determine \( \zeta \) and \( \epsilon \) by the method for nonlinear eigenvalue problems[9] or renormalization group theory[10, 11] though we may consider them as simply empirical numbers[12]. It was suggested that self-similarity of the second kind corresponds to fractal[13, 14], which was elaborated by Mandelbrot. The fractal is a geometrical object which is lacking in a characteristic length. If the objects possess a certain characteristic length, the scale of the object is apparent by scale transformation. On the other hand, the scale of fractal objects is not apparent but self-similar as the scale transformation. Such a geometrical property corresponds to the divergence of dimensionless function in dimensional analysis.

The second important point is that there exists a hierarchy of self-similarity. Note that we can find a parallelism between the first kind and the second kind. Dimensional analysis transforms \( y = f(t, x, z) \) to \( y/t^\alpha = \Phi(x/t^\beta, z/t^\gamma) \) while \( \Pi = \Phi(\eta, \xi) \) is transformed to \( \Pi/\eta^\zeta = \Phi(\xi/\eta^\epsilon) \) in case of the second kind. In the present study, I focused on this hierarchy though the first kind and the second kind refer to the property of the convergence of dimensionless function, not to the classes to which dimensionless parameters belong. Thus, I introduced a word, class to characterize the hierarchical structure.

By considering the convergence of the self-similar variables, the self-similar structure of the problem are explored, and intermediate asymptotics is finally obtained. Depending on the type of similarity, the asymptotics is called intermediate asymptotics of the first kind or intermediate asymptotics of the second kind.

THE TIME EVOLUTION OF DEFORMATION

In this work, the model assumes the main contribution of deformation is due to \( \frac{d\delta}{dt} = v_i \), which corresponds to the square deformation. This behavior is well supported by the observation of experiment. Fig. 2(a) shows the time evolution of deformation for \( R = 8.0 \) mm. After the contact, the rate of deformation corresponds to the impact velocity and the rate of deformation is maintained for a while then it steeply decreases in the end. Fig. 2(b) shows the comparison of each impacts for normalized deformation and time. The deformation is normalized by the maximum deformation \( \delta_m \) and the time is normalized by \( t_{\text{max}} \) at which \( \delta \) reaches to \( \delta_m \). One can find that all the attractors overlap almost completely, which signifies the attractors are similar. This means that the lower the impact-velocity is, the longer the contact time is. This relation is well observed in Fig. 2(c), which shows the linear relation between \( t_{\text{max}} \) and \( \delta_m/v_i \).

THE CONVERGENCE OF EQ. (6).

Eq. (6) is seemingly a indeterminate form as \( Z \to 0 \) though it converges to a finite limit as follows. Using L'Hôpital’s rule, then we have

\[ \lim_{Z \to 0} \frac{2}{3} \frac{Z}{1 - e^{-Z}} = \lim_{Z \to 0} \frac{2}{3} \frac{(Z)'}{(1 - e^{-Z})'} = \frac{2}{3}. \]  

(S9)

THE DERIVATION OF EQ. (7)

According to Eq. (6), in the self-similarity of the first class, we have

\[ \frac{\Pi^3 \phi}{\kappa \eta} = \frac{2}{3} \left( \frac{\Pi}{\eta^{3/2}} \right) \left[ 1 - \exp \left( -\frac{\Pi}{\eta^{3/2}} \right) \right]. \]  

(S10)
FIG. S2: (Color online) (a) The time evolution of deformation $\delta$ and (b) the normalized deformation $\delta/\delta_m$ for $R = 8.0$ mm, $v_i = 314$ mm/s, $Z = 4.13$ (●), $v_i = 615$ mm/s, $Z = 2.75$ (●), $v_i = 735$ mm/s, $Z = 2.48$ (●), $v_i = 1100$ mm/s, $Z = 2.07$ (●), $v_i = 1480$ mm/s, $Z = 1.82$ (●), $v_i = 2254$ mm/s, $Z = 1.47$ (●) and $v_i = 3210$ mm/s, $Z = 1.22$ (●). $\delta_m$ is maximum deformation and $t_{\text{max}}$ is the time at which $\delta$ reaches to $\delta_m$. The vertical dashed line indicates the moment of contact time and the horizontal dashed line indicates the configuration at $\delta = 0$. (c) The dependence between $t_{\text{max}}$ and $\delta_m/v_i$ for each size of sphere.

By multiplying $\kappa/\Pi \phi$ and we have the following form from Eq. (S10)

$$\frac{2}{3} = \Pi^2 \theta^\phi \frac{1}{\kappa \eta^{1/2}} \left[ 1 - \exp \left( -\frac{\Pi}{\theta \eta^{1/2}} \right) \right].$$ (S11)

In order to see the actual behavior of $\Pi$, I applied the third term perturbation method. As it belongs to the problem of the singular perturbation[15], therefore here we assume

$$\Pi = \frac{1}{\varepsilon \eta} (\Pi_0 + \varepsilon^\alpha \Pi_1 + \varepsilon^{2\alpha} \Pi_2 + \cdots)$$ (S12)

where $\gamma$ and $\alpha$ are constant, $\varepsilon = 1/\theta \eta^{1/2}$.

By applying the Taylor expansion on the exponential part and substituting Eq. (S12) into Eq. (S11), we have

$$\theta^\phi \frac{\Pi^2}{\kappa} \left\{ \varepsilon \Pi - \frac{1}{2} \varepsilon^2 \Pi^2 + \frac{1}{6} \varepsilon^3 \Pi^3 + \cdots \right\} = \frac{2}{3}$$

$$\Rightarrow \theta^\phi \frac{\Pi^2}{\kappa} e^{1-2\gamma} \left( \Pi_0^2 + 2\varepsilon^\alpha \Pi_1 \Pi_0 + 2\varepsilon^{2\alpha} \Pi_2 \Pi_0 + \varepsilon^{2\alpha} \Pi_0^2 + \cdots \right) \left\{ \varepsilon^{1-\gamma} (\Pi_0 + \varepsilon^\alpha \Pi_1 + \varepsilon^{2\alpha} \Pi_2 + \cdots) \right. - \frac{1}{2} \varepsilon^{2-2\gamma} (\Pi_0^2 + 2\varepsilon^\alpha \Pi_1 \Pi_0 + 2\varepsilon^{2\alpha} \Pi_2 \Pi_0 + \varepsilon^{2\alpha} \Pi_1^2 + \cdots + \frac{1}{6} \varepsilon^{3-3\gamma} (\Pi_0^3 \cdots) \cdots \right\} = \frac{2}{3}$$ (S13)

as $\varepsilon \to 0$.

To balance each terms, we find that $\gamma = 2/3$ and $\alpha = 1/3$ then we obtain,

$$\theta^\phi \frac{\Pi^2}{\kappa} \left( \Pi_0^2 + 2\varepsilon^{1/3} \Pi_1 \Pi_0 + 2\varepsilon^{2/3} \Pi_2 \Pi_0 + \varepsilon^{2/3} \Pi_1^2 + \cdots \right) \left\{ \Pi_0 + \varepsilon^{1/3} \Pi_1 + \varepsilon^{2/3} \Pi_2 + \cdots \right. - \frac{1}{2} \varepsilon^{1/3} \left( \Pi_0^2 + 2\varepsilon^{1/3} \Pi_1 \Pi_0 + 2\varepsilon^{2/3} \Pi_2 \Pi_0 + \varepsilon^{2/3} \Pi_1^2 + \cdots + \frac{1}{6} \varepsilon^{2/3} (\Pi_0^3 \cdots) \cdots \right) = \frac{2}{3}.$$ (S14)

From this we have

$$O(1) \Rightarrow \theta^\phi \frac{\Pi^3}{\kappa} = \frac{2}{3}$$

$$\Pi_0 = \left( \frac{2}{3} \right)^{\frac{1}{3}} \frac{1}{\theta^{2/3}} \left( \frac{\kappa}{\phi} \right)^{\frac{1}{3}}.$$ (S15)
\[ O \left( \varepsilon^{1/3} \right) \leftrightarrow 3\Pi_0^2\Pi_1 - \frac{1}{2}\Pi_0^4 = 0 \]
\[ \Pi_1 = \frac{1}{6}\Pi_0^2 = \frac{1}{\theta^{4/3}} \left( \frac{\kappa}{\phi} \right)^{\frac{2}{3}} \left( \frac{1}{486} \right)^{\frac{1}{3}} \]  \hspace{1cm} (S16)

\[ O \left( \varepsilon^{2/3} \right) \leftrightarrow 3\Pi_0^2\Pi_2 - 2\Pi_0^3\Pi_1 + \frac{1}{6}\Pi_0^6 + 3\Pi_0\Pi_2^2 = 0 \]
\[ \Pi_2 = \frac{2}{3}\Pi_0\Pi_1 - \frac{1}{18}\Pi_0^3 - \frac{\Pi_2^2}{\Pi_0} = \frac{1}{54\theta^2} \frac{\kappa}{\phi} \]  \hspace{1cm} (S17)

Using results of Eq. (S15), Eq. (S16), Eq. (S17), \( \gamma = 2/3 \) and \( \alpha = 1/3 \) for Eq. (S12) then we have a following result,
\[ \Pi = \frac{\kappa}{54\theta^2} + \left( \frac{\kappa^2}{486\theta^2} \right)^{\frac{1}{3}} \eta^{\frac{1}{3}} + \left( \frac{2\kappa}{3\phi} \right)^{\frac{1}{3}} \zeta^{\frac{1}{3}} \]  \hspace{1cm} (S18)

which corresponds to the Eq. (7).

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