Exact solution (by algebraic methods) of the lattice Schwinger model in the strong-coupling regime

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Abstract

Using the monomer–dimer representation of the lattice Schwinger model, with $N_f = 1$ Wilson fermions in the strong–coupling regime $(\beta = 0)$, we evaluate its partition function, $Z$, exactly on finite lattices. By studying the zeroes of $Z(k)$ in the complex plane $(\text{Re}(k), \text{Im}(k))$ for a large number of small lattices, we find the zeroes closest to the real axis for infinite stripes in temporal direction and spatial extent $S = 2$ and 3. We find evidence for the existence of a critical value for the hopping parameter in the thermodynamic limit $S \to \infty$ on the real axis at about $k_c \simeq 0.39$. By looking at the behaviour of quantities, such as the chiral condensate, the chiral susceptibility and the third derivative of $Z$ with respect to $1/2k$, close to the critical point $k_c$, we find some indications for a continuous phase transition.
1 Introduction

The Schwinger model [1], that is 2-dimensional QED (QED$_2$) with massless electrons, has always attracted the interest of theoreticians not only because it is a rather simple model, which can be solved analytically, but also (and mainly!) because many of its properties are quite similar to those of 4-dimensional QCD. Already in the basic version with $N_f = 1$ fermion flavours, one recovers a lot of QCD-like properties such as confinement for fermions, chiral symmetry breaking, due to the anomaly in the $U(1)$ axial current, and charge screening [1, 2, 3].

By virtue of this similarity, one is also tempted to consider the lattice version of the Schwinger model as a test-model for lattice 4-dimensional QCD (QCD$_4$). Also here one faces the problem of the choice of a lattice scheme for fermions: the two most common choices are the well-known Wilson fermions [4] and the staggered (or Kogut–Susskind [KS]) fermions [5, 6]. Most of the lattice calculations done up to now for the Schwinger model used the staggered fermion formulation [7, 8, 9], in which case the chiral limit is obtained by simply setting the bare fermion mass parameter $m$ appearing in the lattice Lagrangian to zero: $m \rightarrow 0$. All these lattice calculations seem to reproduce well the expected properties of the continuum massless Schwinger model, known from analytical results.

On the contrary, very little is known about the lattice Schwinger model with Wilson fermions. Our interest in the Wilson formulation of QED$_2$ comes from the recently made observation [10] that the critical point in the hopping parameter, at which the chiral limit is reached, may not agree with the naïve expectation (see below, for a more detailed discussion of this point). This may also be of relevance for an understanding of the complicated phase diagram found in the Wilson formulation of lattice QCD$_4$ with large number of flavours [11].

In the lattice action with Wilson fermions the bare fermion mass $m$ does not appear explicitly as in the lattice action with KS fermions. It contains as parameters the coupling constant $\beta$ and the hopping parameter $k$, which is related to the bare fermion mass. Therefore in the Wilson fermion formulation there is the problem of defining a chiral limit, corresponding to $m \rightarrow 0$. In lattice QCD$_4$ with Wilson fermions the chiral limit, at a given value of $\beta$, is reached when the hopping parameter $k$ approaches a certain critical value $k_c(\beta)$, often defined as the value of $k$ for which the pion mass $M_\pi$ vanishes. In fact, in the chiral limit of QCD$_4$ the pion becomes the Goldstone boson of the spontaneously-broken chiral symmetry and its mass is expected to vanish as $\sqrt{m}$ when $m \rightarrow 0$. It tacitly is assumed that this defines a critical point which coincides with a critical point (zero) of the partition function.

It is expected that the situation is similar for the lattice Schwinger model, in the sense that there will be a critical point for each value of the gauge coupling $k = k_c(\beta)$, which defines the chiral limit. The continuum chiral limit will be reached following this line up to $k_c(\beta = \infty) = 1/2d = 1/4$. However, we cannot determine $k_c(\beta)$ in the same way as in QCD$_4$, since in the Schwinger model with $N_f = 1$ we have no Goldstone boson in the chiral limit $m \rightarrow 0$: the $U(1)$ chiral symmetry is broken by the anomaly and one is left with a massive pseudoscalar, similar to the $\eta'$ in QCD$_4$. A determination of $k_c(\beta)$ thus has to proceed through the direct investigation of the singularities of $Z(k)$.

A quite common attitude is to assume that $k_c(\beta)$ coincides with the convergence radius
$\bar{k}(\beta)$ for the joint expansion in the hopping parameter $k$ and the inverse gauge coupling $\beta$. However, there is no proof that this is correct: for example, in the lattice Schwinger model with KS fermions we have that $\bar{k} \leq 1/2$ \cite{10}, while $k_c = \infty$. For Wilson fermions, one does not know the precise values of $\bar{k}(\beta)$ and $k_c(\beta)$. Our aim is to compare these two values at least in the strong coupling regime $\beta = 0$. It was already found in ref. \cite{10} that $\bar{k}(0) \leq 1/2$ and some indications from Monte Carlo simulations were reported, indicating that $k_c(0) \neq \bar{k}(0)$.

In this work we will determine $k_c(0)$ by deriving analytically (using algebraic methods) the Lee–Yang zeroes \cite{12} of the partition function $Z(k)$ for the lattice Schwinger model, with $N_f = 1$ Wilson fermions, in the strong–coupling regime ($\beta = 0$). For a finite lattice $S \times T$ these zeroes have a non–vanishing imaginary part, in the complex plane $(Re(k), Im(k))$, indicating that there is no real critical point for a finite lattice. This remains true for $T \to \infty$ with finite $S$. Yet, enlarging the lattice, they show a tendency to move towards the real $k$–axis. By studying the zeroes in the complex plane of the partition function $Z(k)$ for a large number of small lattices, and then extrapolating to the thermodynamic limit $\infty \times \infty$, we will find evidence for the existence of a real critical value for the hopping parameter, at about $k_c \simeq 0.39$. We will also study some relevant quantities, such as the chiral condensate $<\bar{\psi}\psi>$, the chiral susceptibility and the third derivative, with respect to $1/2k$, of the partition function, in order to get some information about the question of the order of the phase transition.

2 The method

The action for the lattice Schwinger model is written as the sum of a gauge–action $S_G[U]$ and of a fermion–action $S_F[\psi, \bar{\psi}, U]$:

$$S = S_G[U] + S_F[\psi, \bar{\psi}, U].$$

(1)

The gauge part $S_G[U]$ is given by:

$$S_G[U] = \beta \sum_P \{1 - \frac{1}{2}(U_P + U_P^\dagger)\},$$

(2)

where $\beta \equiv 1/e^2$, $e$ being the usual electro–magnetic coupling constant. Since the gauge group is $U(1)$, the basic lattice gauge variable $U_\mu(n)$, corresponding to the link connecting the sites $n$ and $n + \hat{\mu}$, can be written in the form of a phase:

$$U_\mu(n) = \exp[i\phi_\mu(n)].$$

(3)

In eq.(2) $U_P$ stands for the usual $1 \times 1$ Wilson plaquette, constructed using the link variables $U_\mu(n)$.

In the strong coupling limit ($e^2 \to \infty$), the coefficient $\beta$ in front of $S_G[U]$ goes to zero and the total action $S$ reduces simply to the fermion action $S_F[\psi, \bar{\psi}, U]$. The action $S_F$ for
Wilson fermions (and only one flavour) can be written in the form:

\[
S_F = \frac{1}{2k} \sum_{n,m} \bar{\psi}(n) K_{nm}[U] \psi(m),
\]

where \( k \) is the so-called hopping parameter, and each \( K_{nm} \), for a given couple of lattice sites \( n \) and \( m \), is a matrix in Dirac space:

\[
K_{nm}[U] = \delta_{nm} \cdot \mathbf{I} - k \sum_{\mu} [(r - \gamma_{\mu}) U_{\mu}(n) \delta_{n+\hat{\mu},m} + (r + \gamma_{\mu}) U^\dagger_{\mu}(n - \hat{\mu}) \delta_{n-\hat{\mu},m}].
\]

Therefore \( K_{nm} \) is of the form:

\[
K_{nm} = \delta_{nm} \cdot \mathbf{I} - k \cdot M_{nm}[U],
\]

where the only non-vanishing matrices \( M_{nm} \) are those connecting neighbouring lattice sites:

\[
M_{n,n+\hat{\mu}}[U] = (r - \gamma_{\mu}) U_{\mu}(n), \\
M_{n,n-\hat{\mu}}[U] = (r + \gamma_{\mu}) U^\dagger_{\mu}(n - \hat{\mu}).
\]

The parameter \( r \), which satisfies \(|r| \leq 1\), is called the Wilson parameter. In the following we will consider only the case \( r = 1 \). The matrices \( \gamma_{\mu} \), with \( \mu = 1, 2 \) are the 2\( \times \)2 Euclidean Dirac matrices, corresponding to 1+1 space–time dimensions (in particular we will consider the index \( \mu = 1 \) as corresponding to the time dimension and the index \( \mu = 2 \) as corresponding to the space dimension): they satisfy the anticommutation relation \( \{ \gamma_i, \gamma_k \} = 2 \delta_{ik} \mathbf{I}_2 \), with \( \mathbf{I}_2 \) being the 2\( \times \)2 identity matrix. For our algebraic manipulations, we have chosen the following representation for the \( \gamma \)-matrices:

\[
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Finally, the partition function, in the strong coupling regime \( \beta = 0 \), is given by the following expression:

\[
Z(k) = \int [D\bar{\psi}D\psi] \int [DU] e^{-S_F[\psi,\bar{\psi},U]} \\
= \int [D\bar{\psi}\psi] \int [DU] e^{-\frac{1}{2k} \sum_{n,m} \bar{\psi}(n) K_{nm}[U] \psi(m)}.
\]

Following the standard normalization convention, we also eliminate the factor \( 1/2k \) appearing in the exponent in eq.(9) by re-scaling the fermion fields with \( \sqrt{2k} \):

\[
\bar{\psi} = \sqrt{2k} \bar{\psi}, \quad \tilde{\psi} = \sqrt{2k} \tilde{\psi}.
\]

When evaluating a matrix element of the form \( < \prod_{n=1}^N \tilde{\psi}_{A_n}(x_n) \bar{\psi}_{B_n}(y_n) > \), in terms of the re-scaled fields, we must however keep in mind that the original correlation function is obtained
by multiplying with \((1/2k)^N\). When considering a lattice with \(S\) lattice sites in the space direction and \(T\) lattice sites in the time direction, for a total of \(N = S \times T\) lattice sites, the partition function (9) becomes:

\[
Z(k) = (2k)^{-2N} \tilde{Z}(k),
\]

where we have defined:

\[
\tilde{Z}(k) \equiv \int [D\tilde{\psi}D\tilde{\psi}] \int [DU] e^{-\sum_{n,m} \tilde{\psi}(n)K_{nm}[U]\tilde{\psi}(m)}.
\]

It is exactly \(\tilde{Z}(k)\) that we have evaluated with algebraic methods for a large series of small lattices. First of all we have put \(\tilde{Z}(k)\) in a comfortable form for subsequent algebraic manipulations. Making use of the explicit expression (3) for the link variables \(U\) and remembering the Grassmann properties of the fermion fields, one finds the monomer–dimer representation for the partition function \(\tilde{Z}(k)\):

\[
\tilde{Z}(k) \equiv \int [D\tilde{\psi}D\tilde{\psi}] \prod_n F(n)L_1(n)L_2(n),
\]

where \(F(n)\) is the monomer term at lattice site \(n\), coming from the mass–term in the action (\textit{i.e.} the bilinear diagonal term not containing the gauge fields \(U\)):

\[
F(n) = 1 - \tilde{\psi}_1(n)\tilde{\psi}_1(n) - \tilde{\psi}_2(n)\tilde{\psi}_2(n) + \tilde{\psi}_1(n)\tilde{\psi}_2(n)\tilde{\psi}_1(n)\tilde{\psi}_2(n).
\]

(The indices 1 and 2 are the Dirac indices). The quantities \(L_\mu(n)\), with \(\mu = 1, 2\), are the dimer terms defined on links of the lattice. They result from the direct integration over the gauge field \(U_\mu(n) = \exp[i\phi_\mu(n)]\), corresponding to the link \(n \to n + \hat{\mu}\). Explicitly:

\[
L_\mu(n) = \int_{-\pi}^{\pi} \frac{d\phi_\mu(n)}{2\pi} e^{\{M_\mu(n)\exp[i\phi_\mu(n)] + N_\mu(n)\exp[-i\phi_\mu(n)]\}},
\]

where \(M_\mu(n)\) and \(N_\mu(n)\) are given by:

\[
\begin{align*}
M_\mu(n) &= k\tilde{\psi}(n)(1 - \gamma_\mu)\tilde{\psi}(n + \hat{\mu}), \\
N_\mu(n) &= k\tilde{\psi}(n + \hat{\mu})(1 + \gamma_\mu)\tilde{\psi}(n).
\end{align*}
\]

Thanks to the particularly simple form (3) of the gauge variables \(U\) in the case of a \(U(1)\) gauge group, the integration in (15) can be performed in an elementary way, after having expanded as a power–series the first exponential. Making use of the explicit representation (8) for the \(\gamma\)–matrices and of the Grassmann properties of the fermion fields, one finds the following rather simple expressions for the one–link integrals \(L_1(n)\) and \(L_2(n)\):

\[
\begin{align*}
L_1(n) &= 1 + M_1(n)N_1(n) \\
 &= 1 + 4k^2\tilde{\psi}_2(n)\tilde{\psi}_2(n + 1)\tilde{\psi}_1(n + 1)\tilde{\psi}_1(n), \\
L_2(n) &= 1 + M_2(n)N_2(n) \\
 &= 1 + k^2\tilde{\psi}(n)\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}\tilde{\psi}(n + \hat{2}) \cdot \tilde{\psi}(n + \hat{2})\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}\tilde{\psi}(n).
\end{align*}
\]
After inserting the expressions (14) and (17) in (13), one is left with an integral over the fermionic variables, which must be evaluated according to the integration rules for the Grassmann variables. Doing this “manually” turns out to be extremely boring and time-consuming, even for the very small $2 \times 2$ lattice. For this reason we have developed an algebraic method for evaluating the integral (13). We have used the algebraic computer language “Mathematica” and have implemented the basic rules for the Grassmann algebra:

$$\{\eta_i, \eta_j\} = 0, \quad \int d\eta_i = 0, \quad \int d\eta_i \eta_k = \delta_{ik}. \quad (18)$$

In this way we were able to write computer programs (for “Mathematica”) for evaluating products of polynomials of arbitrary strings of Grassmann variables (like those appearing in eq.(13)) and for integrating them. In practice we have used the following strategy for calculating the partition function (13) for a given lattice of size $S \times T$. First of all we have computed the transfer matrix, i.e. the products of all one–link–terms, $L_1(n)$ and $L_2(n)$, and all mass–terms $F(n)$ belonging to a given space–like line $(x,t)$ with $x = 1, 2, \ldots, S$. We call this object a “line”. It is a function of all Grassmann fields belonging to the sites of the line under consideration:

$$\text{line}[t, 0] = \prod_{x=1}^{S} F(x,t) L_1(x,t) L_2(x,t). \quad (19)$$

In evaluating this product we have already taken into account the torus–like topology of the lattice, in form of periodic boundary conditions for the Grassmann fields along space–like lines. The partition function is represented in terms of $\text{line}[t, 0]$ as:

$$\tilde{Z}(k) = \int [D\bar{\psi} D\tilde{\psi}] \prod_{t=1}^{T} \text{line}[t, 0]. \quad (20)$$

Starting with the object $\text{line}[t, 0]$ one can now evaluate composite objects, such as the product of two adjacent lines: $\text{line}[t, 0] \cdot \text{line}[t+1, 0]$. By virtue of eqs.(19) and (17), all other lines, different from $\text{line}[t, 0]$ or $\text{line}[t+1, 0]$, do not depend on the Grassmann fields on the sites $(x, t+1), \ x = 1, 2, \ldots, S$. Therefore we can integrate the product $\text{line}[t, 0] \cdot \text{line}[t+1, 0]$ with respect to these Grassmann fields and obtain:

$$\text{line}[t, 1] = \int \prod_{x=1}^{S} d\bar{\psi}(x, t+1)d\tilde{\psi}(x, t+1)[\text{line}[t, 0] \cdot \text{line}[t + 1, 0]]. \quad (21)$$

We can then proceed in the same way and construct more extended objects. For example we can multiply $\text{line}[t, 1]$ with $\text{line}[t + 2, 0]$, or even with $\text{line}[t + 2, 1]$, and integrate over the Grassmann fields lying on the line $(x, t + 2), \ x = 1, 2, \ldots, S$: In general we obtain $\text{line}[t, l]$ by performing the integration over the $(l)$ lines of intermediate Grassmann fields at $x = t + 1, \ldots, t + l$. Finally, for $l = T − 1$ the resulting object covers just the entire lattice $S \times T$ and we must again take into account the torus–like topology of the lattice, i.e. impose anti–periodic boundary conditions for the Grassmann fields along time–like lines (In fact, it turns out to be irrelevant, for the final result, if we impose periodic or anti–periodic
boundary conditions for these Grassmann fields). In the last step, we can integrate over the remaining Grassmann fields and obtain the final result for the partition function (13).

It turns out that the CPU time required in this approach is entirely controlled by the initial spatial extent $S$ of the lattice as this determines the number of Grassmann fields one can combine in a given string of fields. Doubling the length in time direction does not lead to a drastic increase of the computer time as the resulting object, line$[t, 2l]$ contains exactly the same number and types of strings of Grassmann fields as line$[t, l]$: the only additional complication results from the more complex structure of the coefficients of these strings, which become higher order polynomials in $k$ for increasing $T$. As a curiosity, if we try to follow the strategy of “extending” in the space–direction, instead of extending in the time–direction, we need (for our programs) CPU times which are orders of magnitude larger than in the previous case. This is simply due to the space–time asymmetry in the representation (8): but of course the partition function for a lattice $S \times T$, namely $\tilde{Z}(k, S, T)$, is exactly equal to the partition function $\tilde{Z}(k, T, S)$ for a lattice $T \times S$ (One can always choose a representation for the $\gamma$ matrices in which $\tilde{\gamma}_1 = \gamma_2$ and $\tilde{\gamma}_2 = \gamma_1$).

3 Results

Following the computational strategy discussed in the previous section, we have evaluated the partition function $\tilde{Z}(k, S, T)$, given by (13), for a large number of lattices of the form $S \times T$ with $S = 2$ and 3 ranging from $2 \times 2$ up to $2 \times 32$ and from $3 \times 3$ up to $3 \times 16$. These calculations could be performed on a workstation. For $S \geq 4$ considerably more computer time and memory would be required.

From the Grassmann properties of the fermion fields, it immediately follows that the function $\tilde{Z}(k, S, T)$, for a given lattice having $N = S \times T$ sites, is a polynomial of order $2N$ in the hopping parameter $k$,

$$\tilde{Z}(k, S, T) = \sum_{n=0}^{N} a_{2n} k^{2n},$$

On a lattice with $N$ lattice sites there are $4N$ different fermion fields (four fields for each site), so that, by virtue of the properties of the Grassmann algebra, one can construct strings of fermion fields with at most $4N$ fields. And since each power $k^2$ is always accompanied by four fermion fields, one can at most construct a string with $4N$ fields with a coefficient (proportional to ) $k^{2N}$ in front of it.

We have computed the partition functions $\tilde{Z}(k, S, T)$ for ($S = 2, T = 2, \ldots, 32$) and for ($S = 3, T = 3, \ldots, 16$): each of them is a polynomial of order $2N = 2(S \times T)$ in $k$, with $a_0 = 1$ ($\tilde{Z}(k = 0, S, T) = 1$) and $a_{2n} \geq 0$ for all $n$. The magnitude of these coefficients generally increases with the order of $k$, apart from deviations in the very first coefficients due the torus–like topology of the lattice which allows for special “trajectories”, made up of chains of links wrapping around the lattice. In Table 1 and Table 2 we report the list of these coefficients for the two lattices $2 \times 32$ and $3 \times 16$ respectively. Note that for $S$ and $T$ even the partition function $\tilde{Z}(k, S, T)$ is a polynomial of the form $\sum_{n=0}^{N/2} a_{4n} k^{4n}$. This is due to the larger set of symmetries on such lattices, as will be discussed below. Let us first
discuss the distribution of zeroes of the partition function. In [Fig.1] and in [Fig.2] we show the distribution of the complex zeroes, \((Re(k), Im(k))\), of the partition function \(\tilde{Z}(k, S, T)\) for the \(2 \times 32\) and \(3 \times 16\) lattices, respectively.

By virtue of (22), \(\tilde{Z}(k)\) is a polynomial in \(k^2\) with real coefficients: so, if \(\tilde{k}\) is a complex zero of \(\tilde{Z}(k)\), also \(-\tilde{k}\) and \(\tilde{k}^*\) (the complex–conjugate of \(\tilde{k}\)) will be zeroes of \(\tilde{Z}(k)\). As a consequence of this, the distribution of zeroes \((x, y)\) in the complex plane \((Re(k), Im(k))\) is invariant under the \textit{parity} transformation \((x, y) \rightarrow (-x, -y)\) (\textit{P–symmetry}) and under the \textit{complex–conjugate} transformation \((x, y) \rightarrow (x, -y)\) (\textit{C–symmetry}). In other words, the distribution of zeroes turns out to be symmetric under reflections with respect to the real and/or the imaginary \(k\)–axis: this is evident from [Fig.1] and [Fig.2]. The distribution of zeroes in [Fig.1], for the lattice \(2 \times 32\), has an additional symmetry under reflections with respect to the axis \(Re(k) - Im(k) = 0\) and/or to the axis \(Re(k) + Im(k) = 0\). This additional symmetry is typical for lattices of size \(S \times T\), where both \(S\) and \(T\) are even numbers. In fact, it turns out that for this class of lattices the partition function \(\tilde{Z}(k)\) may be written in the following form:

\[
\tilde{Z}(k) = \int \prod_{l \, \text{odd}} d\bar{\psi}(l)d\bar{\psi}(l) \prod_{m \, \text{odd}} F(m) \prod_{n \, \text{even}} \text{cross}(n),
\]

where a given site \((s, t)\) (with \(s = 1, \ldots, S\) and \(t = 1, \ldots, T\)) is said to be \textit{even} or \textit{odd} if the integer number \(s + t\) is respectively even or odd. While \(F(m)\) is the usual monomer term which we have introduced before in eq.(14), \text{cross}(n)\) is a new object obtained by multiplying the monomer term in the site \(n\) with the four dimer terms starting from or ending at the site \(n\), and finally integrating with respect to the fermion fields in \(n\):

\[
\text{cross}(n) = \int d\bar{\psi}(n)d\psi(n)F(n)L_1(n)L_2(n)L_1(n-\hat{1})L_2(n-\hat{2}).
\]

By explicitly evaluating this expression, one finds that \(\text{cross}(n)\) may be written as \(1 + k^4\alpha(\bar{\psi}\psi)\), where \(\alpha(\bar{\psi}\psi)\) is a sum of products of four fermion fields defined in the neighbouring sites of \(n\) (i.e., \(n+1, n-1, n+\hat{2}\) and \(n-\hat{2}\)). Therefore the partition function \(\tilde{Z}(k)\) will be a polynomial in \(k^4\), with real (and positive) coefficients. As a consequence, if \(\tilde{k}\) is a solution of \(\tilde{Z}(\tilde{k}) = 0\), also \(i\tilde{k}\) will be a solution. In other words, the distribution of zeroes of \(\tilde{Z}(k)\) in the complex \(k\)–plane will be invariant under the transformation \((x, y) \rightarrow (-y, x)\): we will call this an \(I–symmetry\). After combining this \(I–symmetry\) with the other \(P–\) and \(C–symmetries\), one immediately recognizes that the distribution of zeroes is invariant under reflections not only with respect to the real or the imaginary \(k\)–axes, but also with respect to the axes \(Re(k) - Im(k) = 0\) and \(Re(k) + Im(k) = 0\): this is in fact what one observes in [Fig.1].

With increasing lattice size the zero with the smallest imaginary part comes closer to the real axis. This zero of the partition function has been plotted in [Fig.3] for various lattices of size \(S \times T\). As can be seen, for increasing \(T\) and fixed \(S\) the imaginary part of the zeroes approaches a finite, non–zero value, indicating that there is no phase transition on infinite stripes of extent \(S < \infty\). We have then used a polynomial fit to derive, from the two previous series of points, the asymptotic values of the roots nearest to the real \(k\)–axis corresponding to the lattices \(2 \times \infty\) and \(3 \times \infty\). In practice, we have performed separate
fits of real and imaginarily parts of the sequence of roots for given $S$, using the polynomial ansatz, $P(T) = a_0 + a_1/T + a_2/T^2 + \ldots$. The number of coefficients in the polynomials was chosen so as to minimize the chi-squared value. It turns out that for the lattices with $S = 2$ good fits are obtained if one uses a polynomial with six coefficients for the real part of the roots and a polynomial with five coefficients for the imaginary part. Instead, for the lattices with $S = 3$ good fits are obtained using polynomials with only three coefficients for both the real and the imaginary part of the roots.

In this way we have found that the root nearest to the real $k$–axis is approximately $(0.464 \pm 0.003, 0.172 \pm 0.004)$ for the $2 \times \infty$ lattice and approximately $(0.43 \pm 0.01, 0.09 \pm 0.01)$ for the $3 \times \infty$ lattice.

If one tries to linearly extrapolate these two points, assuming the existence of a real critical value $k_c$ in the thermodynamic limit $S \to \infty$ and $T \to \infty$ (see also below), one finds approximately (within the accuracy of our linear extrapolation):

$$k_c \simeq 0.39 .$$

(25)

Although the spatial extent of our lattices is still quite small, it is interesting to analyze the complex zeroes of infinite stripes under the assumption of a second order phase transition in the thermodynamic limit $S \to \infty$. The real and the imaginary parts of the zero closest to the real axis are then expected to scale like:

$$Re(k(S)) = k_c + \frac{a}{S^{1/\nu}},$$

$$Im(k(S)) = \frac{b}{S^{1/\nu}}.$$  

(26)

Using this ansatz for our $S = 2$ and $S = 3$ results we find from the $S$–dependence of $Im(k)$ the critical exponent $\nu \simeq 0.63$ and using this to extract $k_c$ from $Re(k)$ we again find $k_c \simeq 0.39$. Of course, in order to get a realistic estimate of the errors on these numbers, one needs to consider also lattices with much larger $S$.

This is surely an extremely interesting result: the value we have found for $k_c$ is considerably smaller than the value $\bar{k} = 1/2$ we would have expected from naïve argumentations. Since for $\beta = \infty$ (theory with free fermions, since $e^2 = 1/\beta \to 0$) it is well known that $k_c = 1/4$ (in general it will be $k_c = 1/2d$, with $d$ space–time dimensions), we are led by the result (25) to believe in the existence of a line of phase transition from $(\beta = 0, k_c \simeq 0.39)$ to $(\beta = \infty, k_c = 1/4)$. (See also ref. [14], where this same result was found making use of the so–called eight–vertex model [15] for the analytical computations.)

In order to get some further information about the question of the order of the phase transition, we have studied the behaviour of the (re–scaled) chiral condensate $<\bar{\psi}\psi>$, the (re–scaled) chiral susceptibility $\chi$ and the (re–scaled) third derivative, with respect to $1/2k$, of the logarithm of the partition function $Z$ in the vicinity of the critical point $k_c$. The re–scaled chiral condensate $<\bar{\psi}\psi>$ is given by the following expression in terms of the partition function $Z(k)$ (see eqs.(5), (9) and (11)):

$$<\bar{\psi}\psi> = -\frac{1}{(2k)N} \frac{d}{d(1/2k)} ln(Z) = -2 + \frac{k}{N} \left( \frac{1}{Z(k)} \frac{dZ(k)}{dk} \right),$$

(27)
where $N = S \times T$ is, as usual, the number of sites of the lattice. In [Fig.4] we report the behaviour of $\bar{Q} \equiv -< \bar{\psi} \psi>$ as a function of $k$, for the two lattices $2 \times 32$ and $3 \times 16$. The re–scaled chiral susceptibility $\tilde{\chi}$ is defined as:

$$
\tilde{\chi} \equiv \frac{1}{(2k)^2 N d(1/2k)^2} \ln(Z) \equiv \sum_n <\bar{\psi}(n)\psi(0)> - N <\bar{\psi} \psi>^2
$$

$$
= -2 + \frac{k}{N} \left[ k \left( \frac{1}{Z} \frac{dZ}{dk} \right) - k \left( \frac{1}{Z} \frac{d^2Z}{dk^2} \right)^2 + 2 \left( \frac{1}{Z} \frac{d^3Z}{dk^3} \right) \right].
$$

(28)

In [Fig.5] we report the behaviour of $\tilde{\chi}$ as a function of $k$, for the same two lattices. It is easy to verify that $\tilde{\chi} = -k(d\bar{Q}/dk) - \bar{Q}$, so that $\tilde{\chi}$ behaves as $\tilde{\chi} \simeq -\bar{Q} \rightarrow -2$ for $k \rightarrow 0$. Finally, we also consider the third derivative, with respect to $1/2k$, of the logarithm of the partition function $Z$:

$$
\tilde{\varphi} \equiv -\frac{1}{(2k)^3 N d(1/2k)^3} \ln(Z) = \sum_{m,n} <\bar{\psi}(m)\psi(m)\bar{\psi}(n)\psi(n)> - 3N <\bar{\psi} \psi> \tilde{\chi} - N^2 <\bar{\psi} \psi>^3
$$

$$
= -4 + \frac{k}{N} \left[ k^2 \left( \frac{1}{Z} \frac{d^2Z}{dk^2} \right) + 6k \left( \frac{1}{Z} \frac{d^3Z}{dk^3} \right) - 3k^2 \left( \frac{1}{Z} \frac{d^2Z}{dk^2} \right) \left( \frac{1}{Z} \frac{d^2Z}{dk^2} \right) \right]
$$

$$
+ \frac{k}{N} \left[ 2k^2 \left( \frac{1}{Z} \frac{d^3Z}{dk^3} \right)^2 - 6k \left( \frac{1}{Z} \frac{d^3Z}{dk^3} \right)^2 + 6 \left( \frac{1}{Z} \frac{d^2Z}{dk^2} \right) \right].
$$

(29)

The behaviour of $\tilde{\varphi}$ as a function of $k$ is shown in [Fig.6], for the two lattices $2 \times 32$ and $3 \times 16$. It is easy to demonstrate that $\tilde{\varphi} = k(d\tilde{\chi}/dk) + 2\tilde{\chi}$: from this one can immediately derive that $\tilde{\varphi}$ behaves as $\tilde{\varphi} \simeq 2\tilde{\chi} \rightarrow -4$ for $k \rightarrow 0$.

Clearly the three quantities are closely related. The sharpening of the crossover in $<\bar{\psi} \psi>$ is reflected in the rise of the peak in $\tilde{\chi}$ with increasing $S$, and the narrowing of the peak in $\tilde{\chi}$ is expressed in terms of the rapidly rising peaks with opposite signature in $\tilde{\varphi}$. Certainly the behaviour of these three quantities is consistent with that expected for a continous phase transition, i.e. a second– or third–order phase transition.

4 Conclusions

We have evaluated analytically, using algebraic methods, the partition function $Z$ for the lattice Schwinger model, with $N_f = 1$ Wilson fermions, in the strong–coupling regime ($\beta = 0$). For a given lattice $S \times T$, the partition function is of the form $Z(k, S, T) = (2k)^{-2N} \bar{Z}(k, S, T)$, where $N = S \times T$ is the total number of lattice sites and $\bar{Z}(k, S, T)$ is a polynomial in $k$ of order $O(2N)$. By studying the zeroes in the complex plane $(Re(k), Im(k))$ of the partition
function $\tilde{Z}(k, S, T)$ for a large series of small lattices $S \times T$, we have found evidence for the existence of a critical value for the hopping parameter, which in the thermodynamic limit, $S, T \to \infty$ lies on the real axis at about $k_c \simeq 0.39$. We are led by this result to believe in the existence of a line of phase transition from $(\beta = 0, k_c \simeq 0.39)$ to $(\beta = \infty, k_c = 1/4)$. In order to determine the order of the transition, it is clearly important to study in more detail the density of the zeroes near $k_c$. This requires larger values of $S$. We have analyzed the chiral condensate $\langle \bar{\psi}\psi \rangle$, the chiral susceptibility and the third derivative, with respect to $1/2k$, of the partition function, in order to get some insights into the question of the order of the phase transition. Even though the present analysis does not yet allow to draw a definite conclusion on the order of the transition we have found some indications that the phase transition might be third order or even second order.

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FIGURE CAPTIONS

Fig. 1. The distribution of zeroes of the partition function  \( \tilde{Z}(k, 2, 32) \) in the complex plane \((\text{Re}(k), \text{Im}(k))\).

Fig. 2. The distribution of zeroes of the partition function  \( \tilde{Z}(k, 3, 16) \) in the complex plane \((\text{Re}(k), \text{Im}(k))\).

Fig. 3. The zeroes of  \( \tilde{Z}(k, S, T) \) closest to the real \( k \)–axis for various lattices of size \( S \times T \): 
\textit{crosses} refer to lattices with \( S = 2 \) and \( T = 2, 3, 4, 6, 8, 10, 16, 32 \), while \textit{triangles} refer to lattices with \( S = 3 \) and \( T = 3, 4, 5, 6, 8, 9, 10, 12, 16 \).

Fig. 4. The re–scaled chiral condensate,  \( \tilde{Q} = -<\tilde{\psi}\tilde{\psi}> \), as a function of \( k \), for the two lattices \( 2 \times 32 \) and \( 3 \times 16 \).

Fig. 5. The re–scaled chiral susceptibility \( \chi \) (defined in the text by eq.(28)) as a function of \( k \), for the two lattices \( 2 \times 32 \) and \( 3 \times 16 \).

Fig. 6. The re–scaled third derivative \( \tilde{\phi}(k) \) of \( \ln(Z) \) with respect to \( 1/2k \) (defined in the text by eq.(29)), for the two lattices \( 2 \times 32 \) and \( 3 \times 16 \).
Complex zeroes of $Z_{(2 \times 32)}(k)$
Complex zeroes of $Z_{(3x16)}(k)$
$Q = -\langle \psi \psi \rangle$ as a function of $k$
$\chi$ as a function of $k$
$\varphi$ as a function of $k$
Table 1: The coefficients of the polynomial \( \tilde{Z}(k, 2, 32) = \sum_{n=0}^{32} a_{4n} k^{4n} \).

| \( n \) | \( a_{4n} \) (for the 2 \times 32 lattice) |
|-------|-----------------------------------------|
| 0     | 1                                       |
| 1     | 512                                      |
| 2     | 129024                                   |
| 3     | 21331968                                 |
| 4     | 2602369024                               |
| 5     | 249774997504                             |
| 6     | 19638234644480                           |
| 7     | 1300255043747840                         |
| 8     | 73961277759684608                        |
| 9     | 3668969473236271104                      |
| 10    | 160569828529865228288                    |
| 11    | 625590573448198504448                    |
| 12    | 218535073687128684625920                 |
| 13    | 688374618372455350140928                 |
| 14    | 196354174723439285612478464             |
| 15    | 5089687819224732967794376704            |
| 16    | 120172957550769852891363540992          |
| 17    | 2588628553768726317536821379072         |
| 18    | 50910684319442948897317601673216        |
| 19    | 914109924603780980467032270045184        |
| 20    | 14970604113895231454059152342515712      |
| 21    | 223213979431827944404264819095502848     |
| 22    | 3021076338904294430787178751238078464   |
| 23    | 36958171640461419289379070931541950464   |
| 24    | 406259327271308991154552486860991496192 |
| 25    | 3980443123196403924685803149026276671488 |
| 26    | 3437785395966486622345748168690925155712 |
| 27    | 257704138028993570924127408162720968605696 |
| 28    | 163968939294763403316669897664996583474816 |
| 29    | 855974942640752278806067258779563498825184 |
| 30    | 34663799157267802700683573313175561579790336 |
| 31    | 97874256443064108849693716776331356231696384 |
| 32    | 146811384664566690398085268839091235998019584 |
Table 2: The (non-zero) coefficients of the polynomial $\tilde{Z}(k, 3, 16) = \sum_{n=0}^{48} a_{2n} k^{2n}$.

| n  | $a_{2n}$ (for the $3 \times 16$ lattice)    |
|-----|------------------------------------------|
| 0   | 1                                        |
| 3   | 1024                                     |
| 4   | 768                                      |
| 5   | 6144                                     |
| 6   | 516096                                   |
| 7   | 811008                                   |
| 8   | 6334464                                  |
| 9   | 174620672                                |
| 10  | 431947776                                |
| 11  | 3203923968                               |
| 12  | 45588021248                               |
| 13  | 153911033856                             |
| 14  | 106385698048                              |
| 15  | 9906042175488                             |
| 16  | 40917798223872                            |
| 17  | 261009935695872                           |
| 18  | 1857033179496448                          |
| 19  | 8585692506488832                          |
| 20  | 50608039715143680                         |
| 21  | 303834862874263552                        |
| 22  | 1470365700697620480                       |
| 23  | 8023810955173429248                       |
| 24  | 42904506735217606656                      |
| 25  | 205822945618110185472                    |
| 26  | 1036084831241631694848                    |
| 27  | 5045499989985857634304                    |
| 28  | 23116972970746438483968                   |
| 29  | 10690351205303436687360                   |
| 30  | 476889028736929390133248                  |
| 31  | 203848818744602390429696                  |
| 32  | 8604050273287356083601408                 |
| 33  | 34923088125822911234703360               |
| 34  | 136172615876993812147470336              |
| 35  | 515227117640733139176259584              |
| 36  | 1857978729092258641800593408             |
| 37  | 6360979177485248012381847552             |
| 38  | 20656762024719021784826904576            |
| 39  | 62433587250829914862254030848            |
| 40  | 174406412859094327618009300992           |
| 41  | 446095973409032980536257150976           |
| 42  | 1025256020010205168904800567296          |
| 43  | 208088022191744886698175102976           |
| 44  | 3738100459121645858840962689024          |
| 45  | 561014810982451164480363560960          |
| 46  | 7107243167143739985776879861760        |
| 47  | 6755399440834383071115485380608         |
| 48  | 5629499534329800464442257309696         |