ON THE QUANTUM SYMMETRY OF THE CHIRAL ISING MODEL

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Abstract

We introduce the notion of rational Hopf algebras that we think are able to describe the superselection symmetries of two dimensional rational quantum field theories. As an example we show that a six dimensional rational Hopf algebra $H$ can reproduce the fusion rules, the conformal weights, the quantum dimensions and the representation of the modular group of the chiral Ising model. $H$ plays the role of the global symmetry algebra of the chiral Ising model in the following sense: 1) a simple field algebra $\mathcal{F}$ and a representation $\pi$ on $\mathcal{H}_\pi$ of it is given, which contains the $c = 1/2$ unitary representations of the Virasoro algebra as subrepresentations; 2) the embedding $U: H \to B(\mathcal{H}_\pi)$ is such that the observable algebra $\pi(A)^-$ is the invariant subalgebra of $B(\mathcal{H}_\pi)$ with respect to the left adjoint action of $H$ and $U(H)$ is the commutant of $\pi(A)$; 3) there exist $H$-covariant primary fields in $B(\mathcal{H}_\pi)$, which obey generalized Cuntz algebra properties and intertwine between the inequivalent sectors of the observables.

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1. Introduction

The Doplicher–Haag–Roberts program [1] for exploring the symmetries and the statistics of a field theoretical model merely from ‘observable’ data was carried out for localized charges in Minkowski space-times of dimensions $D > 2$ [2]. The set of superselection sectors $\{[\rho]\}$, which consists of the equivalence classes $[\rho]$ of certain endomorphisms $\rho$ of the observable algebra, can be characterized by representations of compact groups and their statistics is restricted to Bose or Fermi statistics. In two dimensions we expect a more rich structure in statistics, namely, braid group statistics will play the major role. But the possible symmetry structure, which is dual to the superselection sectors of a model is unknown yet. Nevertheless for a distinguished class of two dimensional field theories, for rational quantum field theories (RQFT), there is a hope to find the corresponding symmetry algebra structure. In a RQFT by definition [3] there are only finitely many superselection sectors and none of them obey permutation statistics except the vacuum sector. In this case the superselection sectors carry a unitary representation of the modular group $\Gamma = SL(2,\mathbb{Z})$ even in lack of conformal symmetry and the representations of the generators $S$ and $T$ of $\Gamma$ are given [3] in terms of the monodromy matrix and the statistics phases of the given model, respectively.

Since the nontrivial superselection sectors are thought to describe elementary particles whose inner degrees of freedom are finite the corresponding symmetry structure should have only finite dimensional irreducible representations. Since in a RQFT there can be only a finite number of different particles that give rise to only a finite number of superselection sectors the corresponding symmetry algebra should be finite dimensional as well. In the final stage of the reconstruction of a particular model we would like to have a simple field algebra $\mathcal{F}$ on which the symmetry algebra $H$ can act and the observable algebra $\mathcal{A}$ arises as the $H$-invariant subalgebra of $\mathcal{F}$. The inequivalent representations $\{\pi_r\}$ of the observables (with certain multiplicities) arise as subrepresentations of the representation $\pi$ of $\mathcal{F}$ on $\mathcal{H}_\pi$ and they are in one to one correspondence with the equivalence classes of endomorphisms $\{[\rho_r]\}$ through the vacuum representation $\pi_0$ of $\mathcal{A}$: $\pi_r \simeq \pi_0 \circ \rho_r$. Moreover, we would like to have a faithful realization $U$ of $H$ in $\mathcal{B}(\mathcal{H}_\pi)$, where $U(H)$ is the commutant of $\pi(\mathcal{A})$. But this means that $U(H)$ is a von Neumann algebra therefore $H$ should be a finite dimensional semisimple algebra, that is a finite direct sum of full matrix algebras. Thus the minimal central projectors of $H$ (that lead to the inequivalent representations of $H$) are in one to one correspondence with the inequivalent representations of the observables, because $U(H)$ and $\pi(\mathcal{A})^-$ have a common center.

These considerations and the requirements for the representations of $H$ below have lead us to the notion of rational Hopf algebras (RHA) that we think are able to describe the superselection symmetries of RQFTs.

1. Complete reducibility, finite dimensional irreducible representations.
2. Existence of a unit representation.
3. Existence of product of representations (unique up to unitary equivalence).
4. No loss of information taking products with the trivial representation.
5. Notion of contragredient representation (unique up to unitary equivalence): the product of a representation with its contragredient should contain the trivial representation — with multiplicity one in case of irreducible representations.
6. Commutativity of the product of representations up to unitary equivalence.
7. Associativity of the product of representations up to unitary equivalence.
8. Representations of the symmetry algebra do form a braided monoidal $C^*$-category.
9. $\hat{H}$, the set of equivalence classes of unitary irreducible representations of the symmetry algebra $H$ should give rise to a $|\hat{H}|$-dimensional unitary representation of the modular group $\Gamma$.

Since we want a RHA $H$ to reproduce not only the fusion rules of the superselection sectors $\{[\rho]\}$ of the corresponding observable algebra $\mathcal{A}$, but also their braid group representations, statistics parameters and the monodromy matrix arising from the statistics operators [1] we are to study algebra embeddings of $\nu: H \to M_n(H)$ type, that is amplifying monomorphisms $\nu$ of $H$, where $M_n(H)$ is an $n \times n$ matrix with entries in $H$, in order to mimic the endomorphisms $\rho: \mathcal{A} \to \mathcal{A}$ of the observables. The composition of amplimorphisms $\{\nu\}$ leads to an associative product $\times$. In order to describe the braiding properties of $\times$ one can introduce the notion of ‘statistics operator’ in $H$, which has properties similar to the ones of the statistics operator in algebraic field theory [4]. Therefore the statistics operators of $H$ naturally lead to statistics parameters, coloured braid group representations and the monodromy matrix of $H$. If we want to associate a RHA to an observable algebra as its global symmetry algebra we think that the analogous quantities should match.

Amplifying monomorphisms of a RHA $H$ are naturally provided by the coproduct $\Delta: H \to H \otimes H$

$$H \ni a \mapsto (1 \otimes e_r) \cdot \Delta(a) \in M_{n_r}(H), \quad (1.1)$$

where 1 is the unit element of $H$ and $e_r \in H$ is a central projector corresponding to a simple summand of dimension $n_r$ of $H$. If an embedding of type (1.1) is unit preserving it gives index $n_r^2$ [5], the ‘minimal’ embedding $H \mapsto H \otimes e^{11}$, where $e^{11}$ is a matrix unit, gives index one. If we want to reproduce a noninteger statistical dimension $d_r$ of a sector $[\rho_r]$, which is equal to the square root of the corresponding index $[\mathcal{A}: \rho_r(\mathcal{A})]$, we are forced to use a unit non-preserving coproduct in $H$ that can lead to an intermediate statistical dimension $1 < d_r < n_r$. Heuristically the origin of a possible unit non-preserving property of the coproduct can also be understood as the contradiction of the ‘symmetrization’ procedure one has to perform on the tensor product basis to decompose a product of two representations of $H$ and the braiding properties of fields in the product carrying these representation spaces. In the reconstruction of field theories in space-time dimensions $D > 2$ [2] the integer statistical dimension $d_r$ plays also the role of the dimension $n_r$ of the representation of the symmetry group and the multiplicity of the corresponding sector of the observables in the representation of the field algebra, that is $d_r = n_r$. In case of two dimensional field theories the latter roles can be played by the (integer) square root of the cardinality $n_r^2$ of a quasibasis [6] corresponding to a conditional expectation $E_r: \mathcal{A} \to \rho_r(\mathcal{A})$, while the noninteger statistical dimension $d_r$ arises as the square root of the index $d_r^2$ provided by this quasibasis. If $d_r$ is an integer we think that these two notions should coincide, i.e. $d_r = n_r$, as in the case $D > 2$.

In lack of a proof or a counterexample whether RHAs are dual to the superselection sectors of RQFTs or not we can try to test the ability of RHAs on particular RQFT models. Due to the lack of available examples we think that the chiral half of unitary rational
conformal models can be used since they also have only finite number of inequivalent representations of the chiral algebra (Virasoro, Kac–Moody, . . ., etc.) and this finite set of representations gives rise to a finite dimensional unitary representation of the modular group \( \Gamma \).

Since chiral conformal field theories live on the non-contractible ‘space-time’ \( S^1 \) an extra difficulty arises from the point of view of algebraic field theory: the set \( \mathcal{I} \) of open nondense intervals are not directed with respect to the partial ordering given by the inclusion of sets therefore the global observable algebra cannot be defined as the inductive limit of the local ones. One can only speak about a consistent family of von Neumann algebras of local observables \( \{ \mathcal{A}(I), I \in \mathcal{I} \} \) acting on a Hilbert space \( \mathcal{H}_0 \) and obeying isotony, locality and covariance with respect to the Möbius group \([7\). In this case a representation \( \pi \), by definition, is a consistent family of representations \( \{ \pi_I, I \in \mathcal{I} \} \) of the local observable algebras on some Hilbert space \( \mathcal{H}_\pi \), that is \( \pi_I \) is a representation of \( \mathcal{A}(I) \) on \( \mathcal{H}_\pi \) and \( J \subset I \) implies \( \pi_J|_{\mathcal{A}(I)} = \pi_I \). One can look for the universal algebra \( \mathcal{A} \) \([6\), which plays the role of the global observable algebra in the sense that there exists a unique representation \( \pi \) of \( \mathcal{A} \) on \( \mathcal{H}_\pi \) such that \( \pi|_{\mathcal{A}(I)} = \pi_I \).

Since we examine the chiral Ising model in terms of Neveu–Schwarz (NS) and Ramond (R) Majorana fermion modes we also use this description of the global observables: \( \mathcal{A}_\mathcal{M} \) is given as a direct sum of NS and R fermion mode bilinears. But this observable algebra \( \mathcal{A}_\mathcal{M} \) has two (related) drawbacks: first, it is not simple, therefore its irreducible representations are not faithful, second, it contains ‘nonlocal’ observables in a sense that for example the R bilinears somehow know that they can live only on the twofold covering of the underlying spacetime \( S^1 \). Of course the very choice of this observable algebra is unavoidable if one intends to describe all of the superselection sectors by endomorphisms of the observable algebra \( \mathcal{A}_\mathcal{M} \) \([7\). But we think that in a true local treatment the natural thing would be to restrict ourselves to the net of local — in both sense — observables \( \{ \mathcal{A}(I), I \in \mathcal{I} \} \) and to allow localized amplifying homomorphisms \( \nu: \mathcal{A} \to M_n(\mathcal{A}) \) of them. Then it is the localized amplimorphism \( \nu \) that should ‘cut’ and ‘sew together’ the \( n \) copies of \( S^1 \) on a ‘common’ interval and ‘mix’ their local observables in a way that the arising irreducible representation of \( \{ \mathcal{A}(I), I \in \mathcal{I} \} \) is globally meaningful only on the \( n \)-fold covering \( \tilde{S}^1 \) of \( S^1 \). The localized transportable endomorphisms may pick up a nontrivial monodromy by transportation with \( 2\pi \) thus one may obtain a nonequivalent endomorphism to the original one. In case of localized transportable amplimorphisms \( \rho: \mathcal{A} \to M_n(\mathcal{A}) \) the equivalence can be ensured by local observables in \( M_n(\mathcal{A}) \). In this description of the R sector we would expect that \( n = 2 \) because locally the twofold covering of \( S^1 \) ‘looks’ as doubled intervals of \( S^1 \). This treatment can be supported by the description of conformal field theories given in \([9\) and by the fact that even in the global description \( H \)-covariance requires the use of proper amplimorphisms in case of the R sector.

To show that the \( c = 1/2 \) unitary Virasoro representations can be obtained from the vacuum sector through amplifying homomorphisms of the observables we derive \( H \)-covariant multiplet matrix fields, or primary fields, vertex operators in conformal field theoretical language, that obey a slightly modified weak F-algebra (generalized Cuntz algebra \([10\]) relations \([11\). The modification is due to the unit non-preserving property of the coproduct of the global symmetry algebra \( H \) of the chiral Ising model.
The organization of the paper is as follows. In Chapter 2.1 we give the defining properties of a RHA $H$ and examine products of amplifying monomorphisms of $H$. For the description of their braiding properties we introduce the notion of the statistics operator in $H$ that leads to the notion of statistics parameters and the monodromy matrix of $H$. In Chapter 2.2 RHAs are constructed that obey Ising fusion rules. We recover from $H$ the conformal weights ($\text{mod } 1$), the central charge ($\text{mod } 8$) and the modular group representation of the chiral Ising model. In Chapter 3 we discuss the field theory of the chiral Ising model. We show that $H$ arises as the commutant of the observables in the representation $\pi$ of the field algebra $F$. After examining various $H$-actions on $B(H \pi) H$-covariant primary fields are constructed, which obey generalized Cuntz algebra properties and lead to amplifying homomorphisms of the observables. Finally, Chapter 4 contains a short discussion and an outlook.

2. Rational Hopf algebras

2.1. Statistics operators and parameters in a rational Hopf algebra

Here we give a short summary about the defining properties (points corresponding to requirements in the Introduction) and about the construction of the statistics operator, statistics parameters and the monodromy matrix of a rational Hopf algebra $H$. The detailed description will be given in [12].

The defining properties are as follows:

1. $H$ is an associative finite dimensional semisimple $*$-algebra with unit.
2. The counit $\epsilon: H \rightarrow \mathbb{C}$ is a unit preserving $*$-homomorphism.
3. The coproduct $\Delta: H \rightarrow H \otimes H$ is a $*$-monomorphism. $(a \otimes b)^* = a^* \otimes b^*$.
4. The counit obeys the property

$$\left(\epsilon \otimes \text{id}\right) \circ \Delta(a) = \rho a \rho^* \quad \left(\text{id} \otimes \epsilon\right) \circ \Delta(a) = \lambda a \lambda^*$$

with unitaries $\rho, \lambda \in H$.
5. The antipode $S$ is a linear $*$-antiautomorphism of $H$. There exist nonzero elements $l, r \in H$ such, that for all $a \in H$

$$a^{(1)} \cdot l \cdot S(a^{(2)}) = l \cdot \epsilon(a), \quad S(a^{(1)}) \cdot r \cdot a^{(2)} = \epsilon(a) \cdot r,$$

where $a^{(1)} \otimes a^{(2)} \equiv \Delta(a)$.
6. Quasi cocommutativity: there exists $R \in H \otimes H$ such that

$$\Delta'(a) \cdot R = R \cdot \Delta(a), \quad a \in H;$$

$$\Delta'(1) \cdot R = R = R \cdot \Delta(1),$$

$$R \cdot R^* = \Delta'(1), \quad R^* \cdot R = \Delta(1),$$

where $\Delta'$ denotes the coproduct with interchanged tensor product factors.
7. Quasi coassociativity: there exists $\varphi \in H \otimes H \otimes H$ such that

$$\left(\Delta \otimes \text{id}\right) \circ \Delta(a) \cdot \varphi = \varphi \cdot \left(\text{id} \otimes \Delta\right) \circ \Delta(a), \quad a \in H;$$

$$\left(\Delta \otimes \text{id}\right) \circ \Delta(1) \cdot \varphi = \varphi = \varphi \cdot \left(\text{id} \otimes \Delta\right) \circ \Delta(1),$$

$$\varphi \cdot \varphi^* = \left(\Delta \otimes \text{id}\right) \circ \Delta(1), \quad \varphi^* \cdot \varphi = \left(\text{id} \otimes \Delta\right) \circ \Delta(1).$$
8.a Triangle identity:

$$(id \otimes \epsilon \otimes id)\varphi = (\lambda^* \otimes 1)\Delta(1 \otimes \rho).$$

8.b Square identities:

$$S(\varphi_1) \cdot r \cdot \varphi_2 \cdot l \cdot S(\varphi_3) = 1 = \varphi_1^* \cdot l \cdot S(\varphi_2^*) \cdot r \cdot \varphi_3^*.$$ 

8.c Pentagon identity:

$$(\Delta \otimes id \otimes id)\varphi \cdot (id \otimes id \otimes \Delta)\varphi = (\varphi \otimes 1) \cdot (id \otimes \Delta \otimes id)\varphi \cdot (1 \otimes \varphi).$$

8.d Hexagon identities:

$$\varphi_{231} \cdot (\Delta \otimes id) R \cdot \varphi_{123} = R_{13} \cdot \varphi_{132} \cdot R_{23},$$

$$\varphi_{312}^* \cdot (id \otimes \Delta) R \cdot \varphi_{123}^* = R_{13} \cdot \varphi_{213}^* \cdot R_{12},$$

where if $\varphi \equiv \varphi_{123} = \sum \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ then $\varphi_{231} = \sum \varphi_2 \otimes \varphi_3 \otimes \varphi_1.$

9. The monodromy matrix $Y \in M_{|\hat{H}|}(H)$ of the symmetry algebra is invertible.

We note that rational Hopf algebras share a lot of properties of quasitriangular quasi Hopf algebras [13] and weak quasi Hopf algebras [14] in the sense that the coproduct is not necessarily coassociative and unit preserving, respectively. The main difference is in the $^\ast$-algebra properties and in the most restrictive property 9. of rational Hopf algebras, since the latter is the only one among the nine properties that excludes group algebras of finite non-Abelian groups.

The representations of $H$ are $D: H \to M_n(C)$ $^\ast$-algebra homomorphisms. Due to 1. they are completely reducible. The matrix units $\{e_{ij}^r \in H \mid r \in \hat{H}, i, j = 1, \ldots, n_r\}$ provide us a linear basis in $H,$ where $\hat{H}$ is the index set of minimal central projectors in $H.$ The defining unitary irreducible representations $D_r, r \in \hat{H}$ of $H$ are given as

$$D_{ij}^r(a) := a_{ij}^r, \quad a = \sum_{p \in H} \sum_{i, j = 1}^{n_p} a_{ij}^p e_{ij}^p, \quad a \in H, \quad a_{ij}^p \in C.$$

The counit $\epsilon$ in 2. is considered as the one dimensional trivial representation. The coproduct in 3. allows us to define product of representations:

$$(D_1 \times D_2)(a) := (D_1 \otimes D_2)(\Delta(a)) \equiv D_1(a^{(1)}) \otimes D_2(a^{(2)}).$$

Since we do not require the unit preserving property for the coproduct null-representation are allowed, but the one-dimensional null representation is not considered to be irreducible. 4. means that product representations with the trivial one lead to equivalent representations to the original ones. Using the antipode in 5. one can define the contragredient $\bar{D}$ of a representation $D,$ namely: $\bar{D}(a) := D^t(Sa), a \in H,$ where $t$ denotes the transposition of a matrix. Properties of $S$ ensure that $\bar{D}: H \to M_n(H)$ is a $^\ast$-homomorphism, where $n$ is the dimension of $D.$ Moreover, $l$ and $r$ serve as natural $(D \times \bar{D}|\epsilon)$ and $(\epsilon|\bar{D} \times D)$
intertwiners. Properties 6–7. ensure the commutativity and associativity of the product of representations up to unitary equivalence. The identities 8.a–d are responsible for the correct braided monoidal structure of the representations. The definition of the monodromy matrix will be given after the construction of the statistics operator.

In another language one can say that the coproduct \( \Delta \) provides us a possibly unit non-preserving embedding of \( H \) into \( H \otimes H \). Of course, we are interested in such embeddings only up to inner unitary automorphisms of \( H \otimes H \). This means that the algebra \( H \) with the coproduct

\[
\Delta_U(a) = U \Delta(a) U^*, \quad U \in U_2 \equiv \{ V \in H \otimes H \mid VV^* = 1 \otimes 1 \} \tag{2.3}
\]

is considered to be equivalent to the original RHA if their representations are equivalent from a category theoretical point of view. One can show that this equivalence always holds with elements in \( U = \sum_k U_{1k} \otimes U_{2k} \equiv U_1 \otimes U_2 \in U_2 \) since defining

\[
\rho_U = \epsilon(U_1)U_2 \rho, \quad \lambda_U = U_1 \epsilon(U_2) \lambda, \tag{2.4a}
\]

\[
l_U = U_1 l S(U_2), \quad r_U = S(U_1^*) r U_2^*, \tag{2.4b}
\]

\[
R_U = U_{21} R U_{12}^*, \quad \varphi_U = U_{12} [ (\Delta \otimes id) U ] \varphi [ (id \otimes \Delta) U^* ] U_{23}^*, \tag{2.4c}
\]

\( H_U \equiv (H, \epsilon, \Delta_U, \rho_U, \lambda_U, R_U, \varphi_U; S, l_U, r_U) \) also satisfies 1–9. We can use this ‘gauge freedom’ \( U_2 \) to reach a canonical form of a rational Hopf algebra. With suitable choice of \( U \in U_2 \) one achieves the more familiar properties

4.

\[
(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta
\]

8.a

\[
(id \otimes \epsilon \otimes id) \varphi = \Delta(1)
\]

of the counit and \( \varphi \) instead of 4. and 8.a. Then one proves that

\[
(\epsilon \otimes id \otimes id) \varphi = \Delta(1) = (id \otimes id \otimes \epsilon) \varphi, \tag{2.5a}
\]

\[
\epsilon(R_1) \cdot R_2 = 1 = R_1 \cdot \epsilon(R_2) \tag{2.5b}
\]

and \( \epsilon \circ S = \epsilon \) fulfil as well. Using the remaining gauge freedom \( l \) and \( r \) can be transformed into invertible central elements of \( H \) and one of them can be even positive. Using an other gauge freedom

\[
S_U(a) = U S(a) U^*, \quad l_U = l U^*, \quad r_U = U r, \quad U \in U_1 \equiv \{ V \in H \mid VV^* = 1 \} \tag{2.6}
\]

one proves that \( S \) can be chosen as \( S(e_r^{ij}) = e_{\tilde{r}}^{ij}, r \in H, i, j = 1, \ldots, n_r, \) where \( r \mapsto \tilde{r} \) is the involution describing the ‘charge conjugation’ among isomorphic direct summands \( e_r H \) and \( e_{\tilde{r}} H \) in \( H \).
To obtain information about the braiding properties of the coproduct we will use amplifying monomorphisms, or amplimorphisms, for short, of \( H \) instead of representations. The benefit of this choice stems from the existence of a left inverse of an amplimorphism, which can lead to the notion of conditional expectations, statistics parameters and index. To stress the similarity of the following construction described in details in \([12]\) to that of in algebraic field theory \([4]\) we use the same name for the corresponding quantities.

An amplimorphism of \( H \) is a \( ^* \)-algebra monomorphism \( \nu: H \rightarrow M_n(H) \). A left inverse \( \Phi_\nu: M_n(H) \rightarrow H \) of \( \nu \) is a positive linear map having the property

\[
\Phi_\nu(1_n) = 1, \quad \Phi_\nu(\nu(a) \cdot B \cdot \nu(c)) = a \cdot \Phi_\nu(B) \cdot c, \quad a, c \in H, \ B \in M_n(H).
\]

The linear space of intertwiners between the amplimorphisms \( \mu: H \rightarrow M_m(H) \) and \( \nu: H \rightarrow M_n(H) \) is

\[
(\mu|\nu) = \{ T \in \text{Mat}(m \times n, H) | \mu(a)T = T\nu(a), a \in H, \mu(1)T = T = T\nu(1) \}.
\]

Amplimorphisms \( \nu_1 \) and \( \nu_2 \) are called equivalent, \( \nu_1 \sim \nu_2 \) if there is an equivalence \( T \) in the intertwiner space \( (\nu_1|\nu_2) \), that is

\[
TT^* = \nu_1(1), \quad T^*T = \nu_2(1).
\]

One can define subobjects, direct sums and product of amplimorphisms. The latter is given by

\[
(\mu \times \nu)^{i_1j_1,i_2j_2}(a) := \mu^{i_1j_2}(\nu^{i_2j_1}(a)), \quad a \in H,
\]

and it is associative. The product \( T_1 \times T_2 \) of intertwiners \( T_i \in (\mu_i|\nu_i), i = 1, 2 \) and is defined as

\[
T_1 \times T_2 := \mu_1(T_2) \cdot (T_1 \otimes I_{n_2}) = (T_1 \otimes I_{m_2}) \cdot \nu_1(T_2) \in (\mu_1 \times \mu_2|\nu_1 \times \nu_2).
\]

An amplimorphism \( \mu: H \rightarrow M_m(H) \) always leads to a representation: we have only to compose it with the ‘vacuum representation’ that is with the trivial representation, the counit \( \epsilon \):

\[
D_\mu := \epsilon \circ \mu: H \rightarrow M_m(C), \quad D^{ij}_\mu(a) := \epsilon(\mu^{ij}(a)), \quad a \in H; \ i, j = 1, \ldots, m.
\]

On the other hand every nonzero representation \( D \) of \( H \) defines a special amplimorphism \( \mu_D: H \rightarrow M_m(H) \) by the help of the coproduct:

\[
\mu_D(a) := a^{(1)} \otimes D(a^{(2)}), \quad a \in H,
\]

where \( m \) is the dimension of the representation \( D \).

We call an amplimorphism \( \nu: H \rightarrow M_n(H) \) natural if \( \nu \sim \mu_D \), i.e. if there is a representation \( D: H \rightarrow M_m(C) \) and an equivalence \( T \in (\mu_D|\nu) \subset \text{Mat}(m \times n, H) \). The equivalences \( \mu_{D_1} \sim \nu \sim \mu_{D_2} \) imply that \( D_1 \) and \( D_2 \) are unitary equivalent representations.
A natural amplimorphism \( \nu \sim \mu_D \) is called irreducible if the representation \( D \) is irreducible. The identity amplimorphism \( id \) is the special amplimorphism corresponding to the trivial representation, that is to the counit: \( id(a) \equiv \mu_\epsilon(a) = a^{(1)} \otimes \epsilon(a^{(2)}) = a \).

The product of two special amplimorphisms is a natural amplimorphism since it is
\[
\mu_{D_1} \times \mu_{D_2} = \text{Ad}[(id \otimes D_1 \otimes D_2)\varphi] \circ \mu_{D_1 \times D_2},
\]
that is
\[
\mu_{D_1} \times \mu_{D_2}(a) = [\varphi_1 \otimes D_1(\varphi_2) \otimes D_2(\varphi_3)] \cdot \mu_{D_1 \times D_2}(a) \cdot [\varphi_1^* \otimes D_1(\varphi_2^*) \otimes D_2(\varphi_3^*)] = a^{(11)} \otimes D_1(a^{(12)}) \otimes D_2(a^{(2)}).
\]

If \( \nu_1, \nu_2 \) are natural amplimorphisms with equivalences \( T_i \in (\nu_i|\mu_{D_i}), i = 1, 2 \) then their product is natural because
\[
(T_1 \times T_2) \cdot \varphi_1 \otimes D_1(\varphi_2) \otimes D_2(\varphi_3) \in (\nu_1 \times \nu_2|\mu_{D_1 \times D_2})
\]
is an equivalence. Therefore natural amplimorphisms are closed with respect to the product \( \times \). We stress that this product is associative by its definition (2.8a) even if the coproduct is only quasi coassociative. In case of special amplimorphisms the equality
\[
[(\mu_1 \times \mu_2) \times \mu_3](a) = a^{(111)} \otimes D_1(a^{(112)}) \otimes D_2(a^{(12)}) \otimes D_3(a^{(2)}) = [\mu_1 \times (\mu_2 \times \mu_3)](a)
\]
can be seen using the pentagon identity.

The braiding of amplimorphisms is described by the statistics operator \( \varepsilon \). For special amplimorphisms \( \mu_1, \mu_2 \) corresponding to representations \( D_1 \) and \( D_2 \) the statistics operator \( \varepsilon(\mu_1; \mu_2) \) is a unitary intertwiner
\[
\mu_2 \times \mu_1(a) \cdot \varepsilon(\mu_1; \mu_2) = \varepsilon(\mu_1; \mu_2) \cdot \mu_1 \times \mu_2(a), \quad a \in H,
\]
\[
\varepsilon(\mu_1; \mu_2) \cdot \varepsilon(\mu_1; \mu_2)^* = \mu_2 \times \mu_1(1),
\]
\[
\varepsilon(\mu_1; \mu_2)^* \cdot \varepsilon(\mu_1; \mu_2) = \mu_1 \times \mu_2(1),
\]
which is defined as
\[
\varepsilon(\mu_1; \mu_2) = [(id \otimes D_2 \otimes D_1)\varphi] \cdot [id \otimes P_{12}] \cdot [id \otimes (D_1 \otimes D_2)(R)] \cdot [(id \otimes D_1 \otimes D_2)\varphi^*],
\]
where \( P_{12} : \mathbf{C}^{m_1} \otimes \mathbf{C}^{m_2} \to \mathbf{C}^{m_2} \otimes \mathbf{C}^{m_1} \) interchanges the tensor product factors. In case of natural amplimorphisms \( \nu_1, \nu_2 \) the statistics operator \( \varepsilon(\nu_1, \mu_1; \nu_2, \mu_2) \) is given by
\[
\varepsilon(\nu_1, \mu_1; \nu_2, \mu_2) := (T_2 \times T_1)^* \cdot \varepsilon(\mu_1; \mu_2) \cdot (T_1 \times T_2),
\]
where \( T_i \in (\mu_i|\nu_i), i = 1, 2 \) are equivalences to the corresponding special amplimorphisms \( \mu_i, i = 1, 2 \).

The statistics operator obeys the properties similar to that of in algebraic field theory:

i) \( \varepsilon(\nu_1, \mu_1; \nu_2, \mu_2) \) is an equivalence from \( \nu_1 \times \nu_2 \) to \( \nu_2 \times \nu_1 \),
ii) \( \varepsilon(\nu_1, \mu_1; \nu_2, \mu_2) \) is independent of the choice of the special amplimorphisms \( \mu_1, \mu_2 \), therefore we can write
\[
\varepsilon(\nu_1; \nu_2) := \varepsilon(\nu_1; \mu_1; \nu_2, \mu_2),
\]

iii) initial conditions
\[
\varepsilon(\nu; id) = \nu(1) = \varepsilon(id; \nu),
\]

iv) let \( \nu_i \sim \tilde{\nu}_i \) and \( T_i \in (\tilde{\nu}_i|\nu_i) \) equivalences for \( i = 1, 2 \), then
\[
\varepsilon(\tilde{\nu}_1; \tilde{\nu}_2) = (T_2 \times T_1) \cdot \varepsilon(\nu_1; \nu_2) \cdot (T_1 \times T_2)^*,
\]
v) for composition of natural morphisms one has the hexagonal identities
\[
\varepsilon(\nu_1 \times \nu_2; \nu_3) = (\varepsilon(\nu_1; \nu_3) \otimes I_2) \cdot \nu_1(\varepsilon(\nu_2; \nu_3)),
\]
\[
\varepsilon(\nu_1; \nu_2 \times \nu_3) = \nu_2(\varepsilon(\nu_1; \nu_3)) \cdot (\varepsilon(\nu_1; \nu_2) \otimes I_3),
\]
vii) \( \varepsilon_{ab} \equiv \varepsilon(\nu_a; \nu_b) \) obeys the coloured braid relation
\[
\varepsilon_3(\varepsilon_{12}) \cdot (\varepsilon_{13} \otimes I_2) \cdot \nu_1(\varepsilon_{23}) = (\varepsilon_{23} \otimes I_1) \cdot \nu_2(\varepsilon_{13}) \cdot (\varepsilon_{12} \otimes I_3).
\]
The statistics operator \( \varepsilon_\nu \) of a natural amplimorphism \( \nu \) is defined as \( \varepsilon_\nu := \varepsilon(\nu; \nu) \).

The conjugate \( \tilde{\nu} \) of an amplimorphism \( \nu: H \rightarrow M_n(H) \) is defined as
\[
\tilde{\nu}(a) := S[\nu(S(a))]^t, \tag{2.17}
\]
where \( S \) is the antipode and \( ^t \) refers to the transposed matrix. The conjugate \( \tilde{\mu}_D \) of a special amplimorphism \( \mu_D \) is natural since one proves that \( \tilde{\mu}_D \sim \mu_{\tilde{D}} \), where \( \tilde{D} \) is the contragredient representation of \( D \). Conjugation of amplimorphisms is involutive up to equivalence. The conjugate \( \tilde{T} \) of an intertwiner \( T \in (\nu_1|\nu_2) \) is defined as \( \tilde{T} := S[T]^t \). It is also involutive up to equivalences. One easily proves that \( \tilde{T} \in (\tilde{\nu}_2|\tilde{\nu}_1) \). Thus using product of equivalences one proves that the conjugate \( \tilde{\nu} \) of a natural amplimorphism \( \nu \) is natural.

A partial isometry \( P_{\mu} \in (\mu_{\tilde{D}} \times \mu_D)|id \) for special amplimorphisms (with nonzero \( D \)) can be given as
\[
P_{ij}^{\mu} := \frac{1}{\sqrt{\text{tr} D(rr^*)}} \varphi_1 \cdot D^{ji} (\varphi_3 r^* S(\varphi_2)), \quad i, j = 1, \ldots \dim D. \tag{2.18}
\]
Thus a partial isometry \( P_\nu \in (\tilde{\nu} \times \nu)|id \) for a natural amplimorphism \( \nu \sim \mu_D \) can also be given. A standard left inverse \( \Phi_\nu: M_n(H) \rightarrow H \) of a natural amplimorphism \( \nu: H \rightarrow M_n(H) \) (\( \nu \sim \mu_D \) for a nonzero representation \( D \)) is defined as
\[
\Phi_\nu(A) := P_\nu^* \cdot \tilde{\nu}(A) \cdot P_\nu, \quad A \in M_n(H). \tag{2.19}
\]
Indeed, $\Phi_\nu$ is a positive linear map having the properties (2.7).

The statistics parameter matrix $\Lambda_\nu \in M_n(H)$ and the statistical parameter $\lambda_\nu \in H$ of a natural amplimorphism $\nu : H \to M_n(H)$ is defined as

\[
\Lambda_\nu := \Phi_\nu(\varepsilon_\nu), \quad \lambda_\nu := \Phi_\nu(\Lambda_\nu).
\]  

(2.20)

One proves that the statistics parameter depends only on the equivalence class of the corresponding amplimorphism and it is in the center of $H$. For an irreducible amplimorphisms $\nu_r, r \in \hat{H}$ it has the form

\[
\lambda_r = \frac{\omega_r}{d_r} \cdot 1,
\]

(2.21)

where the pure phase $\omega_r$ is the statistics phase and the positive real number $d_r$ is the statistical dimension of the irreducible representation $r$. Now we can give the definition of the monodromy matrix $Y \in M_{|\hat{H}|}(H)$. It is defined as

\[
Y_{rs} := d_r d_s \cdot \Phi_r \Phi_s (\varepsilon(\nu_r; \nu_s) \cdot \varepsilon(\nu_s; \nu_r)), \quad r, s \in \hat{H}.
\]

(2.22)

If has the form $Y_{rs} = y_{rs} \cdot 1, y_{rs} \in \mathbb{C}$ and if it is invertible then similarly to [3] one proves that

\[
V(S)_{rs} = \frac{1}{|\sigma|} \cdot y_{rs}, \quad V(T)_{rs} = \left( \frac{\sigma}{|\sigma|} \right)^{\frac{1}{2}} \cdot \delta_{rs} \omega_r, \quad \sigma = \sum_{r \in \hat{H}} d_r^2 \omega_r^{-1}
\]

(2.23)

provides us a unitary representation $V$ of the modular group $\Gamma$.

It is easy to see that the statistics parameters and the monodromy matrix are independent of the gauge choice in $U_1, U_2$, while the statistics operators are invariant up to unitary equivalence.

2.2. Rational Hopf algebras with Ising fusion rules

In this Section we construct RHAs, whose irreducible unitary representations obey the same fusion rules as the primary fields of the chiral Ising model.

Since the Virasoro algebra has three inequivalent unitary representations at $c = 1/2$ the RHA $H$ we are looking for should be a direct sum of three full matrix algebras. The statistical dimensions, or quantum dimensions in conformal field theoretical language [15], in the chiral Ising model corresponding to the sectors of conformal weights 0, 1/2 and 1/16 are $d_0 = 1, d_1 = 1$ and $d_2 = \sqrt{2}$, respectively. Our ansatz is to choose the dimensions $n_r$ of the corresponding simple direct summands $M_{n_r}, r = 0, 1, 2$ of $H$ as the smallest integer that obey $d_r \leq n_r$. This choice gives $n_r = d_r$ if $d_r$ is an integer and associates one dimensional direct summands in $H$ to Abelian sectors, or simple currents in conformal field theoretical language. Thus $H = M_1 \oplus M_1 \oplus M_2$ as a *-algebra* and its unit is the

* After this work was completed K. Szlachányi called into our attention the preprint of V. Schomerus [16] where in terms of Clebsch–Gordan coefficients a six dimensional weak quasitriangular quasi Hopf algebra is given as a possible example that describes the fusion rules of the chiral Ising model.
sum of the primitive central idempotents: \(1 = e_0 + e_1 + e_2\). The central projector \(e_0\) of the first \(M_1\) is chosen to be an integral in \(H\), that is \(ae_0 = e(a) \cdot e_0, a \in H\) [17], thus using the matrix units as a linear basis of \(H\) the count is given as

\[
\epsilon(f) = \begin{cases} 
1, & f = e_0; \\
0, & f = e_1, e_2^{11}, e_2^{12}, e_2^{21}, e_2^{22}. 
\end{cases}
\] (2.24)

In order to reproduce the fusion rules of the chiral Ising model the nonzero fusion coefficients in the decomposition of products of irreducible representations of \(H\) should read as

\[
N^0_{00} = N^0_{11} = N^0_{22} = N^1_{01} = N^1_{10} = N^1_{22} = N^2_{02} = N^2_{20} = N^2_{12} = N^2_{21} = 1,
\] (2.25)

where 0, 1, 2 refer to the primitive central idempotents \(e_0, e_1, e_2\). The coproduct gives an embedding of \(H\) into the semisimple matrix algebra

\[
H \otimes H = M_1^{(00)} \oplus M_1^{(01)} \oplus M_1^{(10)} \oplus M_1^{(11)} \oplus M_2^{(02)} \oplus M_2^{(02)} \oplus M_2^{(12)} \oplus M_2^{(21)} \oplus M_4^{(22)}
\]

and the fusion coefficient \(N^s_{pr}\) is just the number of lines in the corresponding Bratteli diagram between the simple summands \(e_s H \leftrightarrow e_p H \otimes e_r H\). The most general coproduct, which is consistent with the fusion rules (2.25) and with axioms 3. and 4., is given on the basis elements as

\[
\Delta(e_0) = e_0 \otimes e_0 + e_1 \otimes e_1 + U_{22}(e_2^{11} \otimes e_2^{11}) U_{22}^*,
\]

\[
\Delta(e_1) = e_0 \otimes e_1 + e_1 \otimes e_0 + U_{22}(e_2^{22} \otimes e_2^{22}) U_{22}^*,
\]

\[
\Delta(e_2^{i}) = e_0 \otimes e_2^{ij} + e_2^{ij} \otimes e_0 + U_{12}(e_1 \otimes e_2^{ij}) U_{12}^* + U_{21}(e_2^{ij} \otimes e_1) U_{21}^*, \quad i, j = 1, 2;
\]

with unitaries \(U_{12}, U_{21}\) and \(U_{22}\) in the simple direct summands \(M_2^{(12)}, M_2^{(21)}\) and \(M_4^{(22)}\), respectively. Using the gauge freedom \(U_2\) we can transform the ~\(U\)-s into unit matrices reaching a cocommutative coproduct. We note that this coproduct is nothing else than the coproduct on the group algebra \(CS_3\) truncated by the projection

\[
\Delta(1) = 1_{00} + 1_{01} + 1_{10} + 1_{11} + 1_{02} + 1_{20} + 1_{12} + 1_{21} + 1_{21} + 1_{22} + e_2^{11} \otimes e_2^{11} + e_2^{22} \otimes e_2^{22},
\] (2.27)

where we used the notation \(1_{rs} = e_r \otimes e_s\). The reached coproduct is not coassociative in the direct summands \(M_4^{(122)}, M_4^{(221)}\) and \(M_8^{(222)}\). Therefore we have to introduce a nontrivial associator \(\varphi\). The most general \(\varphi\) that satisfies axioms 7., 8.a’ and 8.c can be written in the form

\[
\varphi = 1_{000} + 1_{001} + 1_{002} + 1_{012} + 1_{011} + 1_{012} + 1_{020} + 1_{021} + 1_{021} + 1_{100} + 1_{101} + 1_{102}
\]

\[
+ 1_{110} + 1_{111} + 1_{120} + 1_{200} + 1_{201} + 1_{210} + \omega_1 \cdot 1_{112} - 1_{121} + \omega_1^* \cdot 1_{211}
\]

\[
+ e_0 \otimes e_2^{11} \otimes e_2^{11} + e_0 \otimes e_2^{22} \otimes e_2^{22} + e_2^{11} \otimes e_2^{11} \otimes e_0 + e_2^{22} \otimes e_2^{22} \otimes e_0
\]

\[
+ e_2^{11} \otimes (e_0 - e_2) \otimes e_2^{11} + e_2^{22} \otimes (e_0 + e_1) \otimes e_2^{22}
\]

\[
+ \omega_1 e_1 \otimes e_2^{12} \otimes e_2^{12} + \omega_1^* \omega_2^* \otimes e_1 \otimes e_2^{21} \otimes e_2^{21} - \omega_1 \omega_2 \cdot e_2^{12} \otimes e_2^{12} \otimes e_2^{12} - \omega_1 \omega_2 \cdot e_2^{21} \otimes e_2^{21} \otimes e_2^{21}
\]

\[
- \omega_1 \omega_2 \cdot e_2^{12} \otimes e_2^{12} \otimes e_1 - \omega_1^* \omega_2 \cdot e_2^{21} \otimes e_2^{21} \otimes e_1
\]

\[
+ \frac{\alpha}{\sqrt{2}} \cdot \left[ e_2^{11} \otimes e_2^{11} \otimes e_2^{11} + e_2^{12} \otimes e_2^{12} \otimes e_2^{21} - \omega_1 \omega_2 \cdot \left(e_2^{11} \otimes e_2^{12} \otimes e_2^{12} + e_2^{12} \otimes e_2^{12} \otimes e_2^{22}\right)
\]

\[
+ \omega_1^* \omega_2^* \cdot \left(e_2^{21} \otimes e_2^{21} \otimes e_2^{11} + e_2^{22} \otimes e_2^{21} \otimes e_2^{21}\right) + e_2^{11} \otimes e_2^{22} \otimes e_2^{12} + e_2^{22} \otimes e_2^{22} \otimes e_2^{22}\right] \right],
\]
where the parameters $\omega_1, \omega_2$ are pure phases and $\alpha = \pm 1$. The most general $R$-matrix that satisfies axiom 6., 8.d and the equalities (2.5b) is

$$ R = e_0 \otimes e_0 + e_0 \otimes e_1 + e_1 \otimes e_0 - e_1 \otimes e_1 + e_0 \otimes e_2 + e_2 \otimes e_0 \\
+ i \alpha_1 (e_1 \otimes e_2 + e_2 \otimes e_1) + \omega_1^{(22)} \cdot e_1^{11} \otimes e_2^{11} + \omega_1^{(22)} \cdot e_2^{22} \otimes e_2^{22}, $$

(2.29a)

where

$$ \omega_1^{(22)} = e^{-i \frac{\pi}{5} \left[ \alpha_1 - 2(1-\alpha) - 4(1-\alpha_2) \right]}, \quad \omega_2^{(22)} = e^{i \frac{\pi}{5} \left[ 3\alpha_1 + 2(1-\alpha) + 4(1-\alpha_2) \right]}, $$

(2.29b)

and $\alpha_1, \alpha_2$ can have the values $\pm 1$.

Since the sectors of the Ising model are selfconjugate on the basis of the gauge freedom $U_1$ in (2.6) we choose the transposition as the antipode. Then the general solution for the intertwiners $l, r \in H$ that obey 5. is given by

$$ l = l_0 \cdot e_0 + l_1 \cdot e_1 + l_2 \cdot e_1^{11}, \quad r = r_0 \cdot e_0 + r_1 \cdot e_1 + r_2 \cdot e_2^{11}, \quad l_p, r_p \in \mathbb{C}, \ p = 0, 1, 2. $$

(2.30)

The square identities imply the relations

$$ r_0 l_0 = 1, \quad r_1 l_1 = 1, \quad r_2 l_2 = \alpha \sqrt{2}. $$

(2.31)

We make the choice

$$ l_0 = r_0 = l_1 = r_1 = 1, \quad \alpha l_2 = r_2 = 2^{\frac{3}{2}}. $$

(2.32)

Using the remaining gauge freedom in $U_2$ that leaves the coproduct and the intertwiners $\lambda, \rho, l, r, R$ fix we can transform the phases $\omega_1, \omega_2$ in (2.28) into one. Thus we have found eight inequivalent $H = M_1 \oplus M_1 \oplus M_2$ algebras obeying Ising fusion rules and satisfying axioms 1–8. Using a $U_2$ gauge transformation, which is nontrivial only in the $e_2 H \otimes e_2 H$ summand

$$ U_{22} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \in M_4^{(22)}, $$

(2.33)

and applying the transformation properties (2.4) we reach the canonical form of these algebras, where $l$ and $r$ are invertible central elements:

$$ \epsilon(f) = \begin{cases} 1, & f = e_0; \\
0, & f = e_1, e_2^{11}, e_2^{12}, e_2^{21}, e_2^{22}; \end{cases} $$

(2.34a)

$$ f^* = \begin{cases} f, & f = e_0, e_1, e_2^{11}, e_2^{22}, \\
e_2^{21}, & f = e_2^{12}; \\
e_2^{12}, & f = e_2^{21}; \end{cases} \quad S(f) = \begin{cases} f, & f = e_0, e_1, e_2^{11}, e_2^{22}, \\
e_2^{21}, & f = e_2^{12}; \\
e_2^{12}, & f = e_2^{21}; \end{cases} $$

(2.34b)
\[ \Delta(e_0) = e_0 \otimes e_0 + e_1 \otimes e_1 + \frac{1}{2} \cdot [e^{11}_2 \otimes e^{11}_2 + e^{12}_2 \otimes e^{12}_2 + e^{21}_2 \otimes e^{21}_2 + e^{22}_2 \otimes e^{22}_2], \]
\[ \Delta(e_1) = e_0 \otimes e_1 + e_1 \otimes e_0 + \frac{1}{2} \cdot [e^{11}_2 \otimes e^{12}_2 - e^{12}_2 \otimes e^{11}_2 - e^{21}_2 \otimes e^{21}_2 + e^{22}_2 \otimes e^{22}_2], \]
\[ \Delta(e^{ij}_2) = e_0 \otimes e^{ij}_2 + e^{ij}_2 \otimes e_0 + e_1 \otimes e^{ij}_2 + e^{ij}_2 \otimes e_1; \quad i, j = 1, 2; \]
\[ R = e_0 \otimes e_0 + e_0 \otimes e_1 + e_1 \otimes e_0 - e_1 \otimes e_1 \\
+ e_0 \otimes e_2 + e_2 \otimes e_0 + i \alpha_1 \cdot (e_1 \otimes e_2 + e_2 \otimes e_1) \\
+ \frac{\omega}{\sqrt{2}} \cdot [e^{11}_2 \otimes e^{11}_2 + e^{22}_2 \otimes e^{22}_2 - i \alpha_1 \cdot (e^{12}_2 \otimes e^{12}_2 + e^{21}_2 \otimes e^{21}_2)]; \]
\[ \varphi = (e_0 + e_1) \otimes (e_0 + e_1) \\
+ (e_0 + e_1) \otimes (e_0 + e_1) \otimes e_2 + e_2 \otimes (e_0 + e_1) \otimes (e_0 + e_1) \\
+ e_0 \otimes e_2 \otimes e_0 + e_0 \otimes e_2 \otimes e_1 + e_1 \otimes e_2 \otimes e_0 - e_1 \otimes e_2 \otimes e_1 \\
+ e_0 \otimes e^{11}_2 \otimes e^{11}_2 + e_2 \otimes e^{22}_2 \otimes e^{22}_2 + e^{11}_2 \otimes e^{11}_2 \otimes e_0 + e^{22}_2 \otimes e^{22}_2 \otimes e_0 \\
- e_1 \otimes e^{11}_2 \otimes e^{11}_2 + e_2 \otimes e^{22}_2 \otimes e^{22}_2 + e^{11}_2 \otimes e^{11}_2 \otimes e_1 - e^{22}_2 \otimes e^{22}_2 \otimes e_1 \\
+ e^{11}_2 \otimes e^{11}_2 \otimes e_0 + e^{22}_2 \otimes e^{22}_2 \otimes e_0 - e^{12}_2 \otimes e_1 \otimes e^{12}_2 - e^{21}_2 \otimes e_1 \otimes e^{21}_2 \\
+ \frac{\alpha}{\sqrt{2}} \cdot [e^{11}_2 \otimes e^{11}_2 \otimes e^{11}_2 - e^{12}_2 \otimes e^{12}_2 \otimes e^{12}_2 + e^{21}_2 \otimes e^{21}_2 \otimes e^{21}_2 + e^{11}_2 \otimes e^{11}_2 \otimes e^{11}_2 \\
+ e^{22}_2 \otimes e^{22}_2 \otimes e^{22}_2 - e^{12}_2 \otimes e^{12}_2 \otimes e^{12}_2 + e^{22}_2 \otimes e^{22}_2 \otimes e^{22}_2 + e^{21}_2 \otimes e^{21}_2 \otimes e^{21}_2 + e^{21}_2 \otimes e^{21}_2 \otimes e^{21}_2]; \]
\[ r = e_0 + e_1 + 2^{-\frac{1}{4}} \cdot e_2, \quad l = e_0 + e_1 + 2^{-\frac{1}{4}} \alpha \cdot e_2, \]
where \( \alpha, \alpha_1, \alpha_2 = \pm 1 \) and \( \omega = \exp[(i \pi/8)(\alpha_1 + 2(1 - \alpha) + 4(1 - \alpha_2))] \).

Now we turn to the computation of the statistics parameters and the monodromy matrix. Using the defining irreducible representations for the special amplimorphisms \( \mu_r, r = 0, 1, 2 \) the corresponding statistics operators are given as:

\[ \bar{\varepsilon}_{00} = \bar{\varepsilon}_{01} = \bar{\varepsilon}_{10} = 1 = -\bar{\varepsilon}_{11}, \]
\[ \bar{\varepsilon}_{02} = \bar{\varepsilon}_{20} = \begin{pmatrix} e_0 + e_1 + e^{11}_2 & 0 \\ 0 & e_0 + e_1 + e^{22}_2 \end{pmatrix}, \]
\[ \bar{\varepsilon}_{12} = i \alpha_1 \begin{pmatrix} e_0 - e_1 \\ e^{21}_2 \\ e_0 - e_1 \end{pmatrix}, \quad \bar{\varepsilon}_{21} = i \alpha_1 \begin{pmatrix} e_0 - e_1 \\ -e^{21}_2 \\ e_0 - e_1 \end{pmatrix}, \]
\[ \bar{\varepsilon}_{22} = \frac{\varepsilon}{\sqrt{2}} \begin{pmatrix} e_0 + e_1 + (1 + i \alpha_1)e^{11}_2 & 0 & 0 & -i \alpha_1(e_0 - e_1) \\ 0 & (1 + i \alpha_1)e^{11}_2 & 0 & 0 \\ 0 & 0 & (1 - i \alpha_1)e^{22}_2 & 0 \\ -i \alpha_1(e_0 - e_1) & 0 & 0 & e_0 + e_1 + (1 - i \alpha_1)e^{22}_2 \end{pmatrix}. \]
Since the defining irreducible representations are exactly self-conjugate, \( \bar{D}_r \equiv D_r, r = 0, 1, 2; \) one can easily compute the corresponding left inverses:

\[
\Phi_0 = id_H; \quad \Phi_1(f) = \begin{cases} 
  e_1, & f = e_0; \\
  e_0, & f = e_1; \\
  f, & f = e^{ij}_2;
\end{cases} \quad \Phi_2([f]) = \begin{cases} 
  \frac{1}{2} \cdot e^{ij}_2, & [f] = [e_0]^{i,j}; \\
  \frac{1}{2} \cdot e^{ij}_2, & [f] = [e_1]^{i,j}; \\
  \frac{1}{2} \cdot (e_0 + e_1), & [f] = [e^{11}_2]^{1,1}; \\
  \frac{1}{2} \cdot (e_0 - e_1), & [f] = [e^{12}_2]^{1,2}; \\
  \frac{1}{2} \cdot (e_0 - e_1), & [f] = [e^{21}_2]^{2,1}; \\
  \frac{1}{2} \cdot (e_0 + e_1), & [f] = [e^{22}_2]^{2,2}; \\
  0, & \text{otherwise},
\end{cases}
\]

with \( i, j = 1, 2. \) Therefore the statistics parameter matrices read as

\[
\Lambda_0 = 1, \quad \Lambda_1 = -1, \quad \Lambda_2 = \frac{\omega}{\sqrt{2}}, \quad \begin{pmatrix}
  e_0 + e_1 + e^{11}_2 & 0 \\
  0 & e_0 + e_1 + e^{22}_2
\end{pmatrix}.
\]

Applying the left inverses again one obtains the statistics parameters \( \lambda_r, \) dimensions \( d_r \) and phases \( \omega_r \) of the irreducible amplimorphisms \( \mu_r, r = 0, 1, 2: \)

\[
\lambda_0 = 1, \quad \lambda_1 = -1, \quad \lambda_2 = \frac{\omega}{\sqrt{2}} \cdot 1, \\
\frac{d_0 = 1, \quad d_1 = 1, \quad d_2 = \sqrt{2},}{\omega_0 = 1, \quad \omega_1 = -1, \quad \omega_2 = \omega.}
\]

From the statistics operators \( \hat{\varepsilon}(\mu_r; \mu_s) \) one computes the monodromy matrix \( Y \in M_{|\hat{H}|}(H): \)

\[
Y = 1 \otimes \begin{pmatrix}
  1 & 1 & \sqrt{2} \\
  1 & 1 & -\sqrt{2} \\
  \sqrt{2} & -\sqrt{2} & 0
\end{pmatrix}.
\]

It is invertible, thus (2.23) leads to a unitary representation of the modular group \( \Gamma \) in \( M_3(\mathbb{C}) \) and we have obtained eight inequivalent rational Hopf algebras corresponding to the eight possible choice of \( \alpha, \alpha_1, \alpha_2 = \pm 1. \) All of these algebras have the same fusion rules, the same statistical dimensions and monodromy matrix but the statistics phase \( \omega_2 = \exp[(i\pi/8)(\alpha_1 + 2(1 - \alpha) + 4(1 - \alpha_2))] \) is different, namely, it can be any of the primitive 16th root of unity, that is \( \omega_2 = \exp[2\pi i(2n + 1)/16], n = 0, \ldots, 7. \) Analyzing the statistics weights \( w_r := (1/2\pi i) \cdot \log \omega_r = (2n + 1)/16 \in [0, 1) \) and the ‘central charge’ \( c := (-8/2\pi i) \cdot \log(\sigma/|\sigma|) = (2n + 1)/2 \in [0, 8) \) of these RHAs

| \( w_0 \) | \( w_1 \) | \( w_2 \) | \( c \) |
|---|---|---|---|
| \( H(\text{Ising}) \) | 0 | 1/2 | 1/16 | 1/2 |
| \( H(\tilde{A}_1(2)) \) | 0 | 1/2 | 3/16 | 3/2 |
| \( H(\tilde{E}_8(2)) \) | 0 | 1/2 | 15/16 | 15/2 |
| \( H(\tilde{B}_r(1)) \) | 0 | 1/2 | (2r + 1)/16 | (2r + 1)/2 |

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we can ‘identify’ the different rational Hopf algebras as the symmetry algebras of the chiral Ising model, the level two $SU(2), E_8$ and level one $SO(2r + 1)$ Kac–Moody algebras or chiral WZW theories. All of these chiral field theories contain three inequivalent irreducible unitary representations of the corresponding chiral observable algebra. They obey the same fusion rules, the same statistics dimensions as the chiral Ising model, the representation of the modular generator $S$ is the same, only the conformal weights and the Virasoro central charges are different and they correspond to the values in the table, mod $1$ and mod $8$, respectively.

3. The field algebra of the chiral Ising model

3.1. Realization of the symmetry algebra $H$

The field theory of the chiral Ising model is described in [8] by real NS and R Majorana fields on the twofold covering $\tilde{S}^1$ of the compactified light cone $S^1$. The NS and R fields are ‘periodic’ and ‘antiperiodic’ on $S^1$, respectively. In terms of fermion modes this universal Majorana algebra $\mathcal{M}$ is the unital $^\ast$-algebra given by the generators

$$1, Y, B_n, n \in \frac{1}{2}\mathbb{Z}, \quad Y^* = Y, \quad B_n^* = B_{-n}$$

(3.1a)

together with the relations

$$\{B_n, B_m\} = \frac{1}{2}\delta_{n+m,0}[1 + (-1)^{2n}Y], \quad B_n Y = (-1)^n B_n = Y B_n, \quad Y^2 = 1. \quad (3.1b)$$

$\mathcal{M}$ is a direct sum of the simple $^\ast$-algebras NS and R, the corresponding projections are $(1 \mp Y)/2$, respectively. Choosing a polarization, that is a prescription of creation and annihilation operators, the NS and R algebras have only one faithful unitary irreducible representations up to unitary equivalence [18]. These are the Fock representations $\pi_{NS}$ and $\pi_R$ on the Hilbert spaces $H_{NS}$ and $H_R$ characterized by the cyclic vacuum vectors $\Phi_{NS} \in H_{NS}$ and $\Phi_R \in H_R$:

$$\pi_{NS}(B_n)\Phi_{NS} \equiv b_n\Phi_{NS} = 0, \quad n > 0, \quad n \in \mathbb{Z} + \frac{1}{2}; \quad \pi_{NS}(Y)\Phi_{NS} = -\Phi_{NS},$$

$$\pi_R(B_n)\Phi_R \equiv b_n\Phi_R = 0, \quad n > 0, \quad n \in \mathbb{Z}; \quad \pi_R(Y)\Phi_R = \Phi_R.$$  

(3.2)

The observable algebra $\mathcal{A}_M$ is a direct sum generated by of NS and R bilinears. The Hilbert space $\mathcal{H} = H_{NS} \oplus H_R$ is decomposed into four irreducible representations of the observable algebra $\mathcal{A}_M$

$$\mathcal{H} = (H_0 \oplus H_{\frac{1}{16}}) \oplus (H_{\frac{1}{16}} \oplus H_{\frac{1}{16}}),$$

(3.3)

characterized by the subscripts which are the conformal weights, because one can build the $c = 1/2$ unitary representations of the Virasoro algebra in terms of infinite sums of normal ordered fermion bilinears. The irreducible subspaces in (3.3) carry the three inequivalent representations $\pi_0, \pi_{\frac{1}{16}}$ and $\pi_{\frac{1}{16}}$ of the observable algebra.

In order to show that the rational Hopf algebra $H$ constructed in the previous Chapter is the global symmetry algebra of the chiral Ising model let us first describe the field theory on the Hilbert space $\mathcal{H}$ in (3.3). For an easy treatment we give an explicit equivalent
realization of $\mathcal{B}(\mathcal{H})$ by embedding $\mathcal{M}$ into the simple *-algebra $\mathcal{F} = M_2(\text{NS})$, which consists of two by two matrices with entries in the NS algebra.

The *-monomorphisms $\iota: \mathcal{M} \to M_2(\text{NS})$ is given as

$$
\iota(B_n) = \begin{cases} 
(B_n B_+ B_- & 0 \\
0 & B_n B_- B_+ 
\end{cases}, \quad n \in \pm (N + \frac{1}{2}); \\
(0 & B_+ \\
0 & 0), \quad n = \frac{1}{2}; \\
(0 & 0 \\
B_- & 0), \quad n = -\frac{1}{2}; \\
(B_n \pm \frac{1}{2} B_- B_+ & 0 \\
0 & B_n \pm \frac{1}{2} B_+ B_-), \quad n \in \pm N; \\
\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & B_- \\
B_+ & 0 \end{pmatrix}, \quad n = 0;
\end{cases}
$$

(3.4)

where $B_{\pm} = B_{\pm \frac{1}{2}}$. Thus the universal Majorana algebra is embedded into $\mathcal{F} = M_2(\text{NS})$, since we have constructed a direct sum image of R and NS in $\mathcal{F}$. The orthogonal projections $Y_\mp = (\mathbf{1}_2 \mp \iota(Y))/2$ to the images of the NS and R algebras are given as

$$
Y_- = \begin{pmatrix} B_+ B_- & 0 \\
0 & B_- B_+ \end{pmatrix}, \quad Y_+ = \begin{pmatrix} B_- B_+ & 0 \\
0 & B_+ B_- \end{pmatrix},
$$

(3.5)

where $\mathbf{1}_2 = \mathbf{1}_{\text{NS}} \otimes I_2$ is the unit element of $M_2(\text{NS})$.

The faithful unitary irreducible representation $\pi$ of $\mathcal{F}$ is given by $\pi_{\text{NS}} \otimes id_2$ on $\mathcal{H}_\pi = \mathcal{H}_{\text{NS}} \otimes C^2$, where $\mathcal{H}_{\text{NS}}$ is the Fock representations of the NS algebra. Since the commutants $\iota(\text{NS}) \cap \iota(\text{NS})'$ and $\iota(\text{R}) \cap \iota(\text{R})'$ are both trivial the representations $\pi_{\text{NS}} := \pi \circ \iota_{\text{NS}}$ on the Hilbert space $\pi(Y_-) \mathcal{H}_\pi$ and $\pi_{\text{R}} := \pi \circ \iota_{\text{R}}$ on $\pi(Y_+) \mathcal{H}_\pi$ are both irreducible. Thus $\pi_{\text{NS}}$ and $\pi_{\text{R}}$ are unitary equivalent to the Fock representations $\pi_{\text{NS}}$ and $\pi_{\text{R}}$ in (3.2), respectively.

Restricting ourselves to the observable subalgebra $\mathcal{A} \equiv \iota(\mathcal{A}_\mathcal{M})$ the total Hilbert space $\mathcal{H}_\pi$ is decomposed into four irreducible representations as before

$$
\mathcal{H}_\pi = (\mathcal{H}_0 \oplus \mathcal{H}_\frac{1}{2}) \oplus (\mathcal{H}_{\frac{1}{2}+} \oplus \mathcal{H}_{\frac{1}{2}-}).
$$

(3.6)

The corresponding cyclic vacuum vectors with respect to the observables are given as

$$
\Omega_0 = \Omega_{\text{NS}} = |0\rangle \otimes f_1 \in \mathcal{H}_0, \quad \Omega_1 = \pi_{\text{NS}}(B_-) \Omega_{\text{NS}} = |\frac{1}{2}\rangle \otimes f_2 \in \mathcal{H}_{\frac{1}{2}+}, \\
\Omega_1^1 = \Omega_{\text{R}} = |0\rangle \otimes f_2 \in \mathcal{H}^1_{\frac{1}{2}+}, \quad \Omega_2^2 = \pi_{\text{R}}(\sqrt{2}B_0) \Omega_{\text{R}} = |\frac{1}{2}\rangle \otimes f_1 \in \mathcal{H}^2_{\frac{1}{2}+},
$$

(3.7)

where $|0\rangle$ is the Fock vacuum in $\mathcal{H}_{\text{NS}}, |\frac{1}{2}\rangle = b_- |0\rangle$ and $\{f_1, f_2\}$ is an orthonormal basis in $C^2$.

Now the commutant of $\pi(\mathcal{A})$ in $\mathcal{B}(\mathcal{H}_\pi)$, which is isomorphic to the semisimple matrix algebra $M_1 \oplus M_1 \oplus M_2$, can be given explicitly. If $P_{\pm}: \mathcal{H}_{\text{NS}} \to \mathcal{H}_{\text{NS}}$ denote the projections $(\mathbf{1}_{\text{NS}} \pm (-1)^F)/2 \in \mathcal{B}(\mathcal{H}_{\text{NS}})$, where $F$ is the fermion number operator, then a unit preserving
*-monomorphism \( U: H \to \mathcal{B}(\mathcal{H}_\pi) \) can be given by defining the the images of matrix units of the rational Hopf algebra \( H \) in (2.34) as

\[
U(e_0) = P_+ b_+ b_\pm \otimes f^{11} + P_+ b_- b_+ \otimes f^{22}, \quad U(e_1) = P_- b_+ b_- \otimes f^{11} + P_- b_- b_+ \otimes f^{22},
\]

\[
U(e_{11}^i) = P_+ b_+ b_- \otimes f^{11} + P_+ b_- b_+ \otimes f^{22}, \quad U(e_{12}^i) = P_+ b_+ \otimes f^{21} - P_+ b_- \otimes f^{11},
\]

\[
U(e_{21}^i) = P_- b_- \otimes f^{12} - P_- b_+ \otimes f^{21}, \quad U(e_{22}^i) = P_- b_- b_+ \otimes f^{11} + P_- b_- b_- \otimes f^{22},
\]

(3.8)

where \( f^{ij}, i, j = 1, 2 \), are the matrix units on the tensor factor \( \mathbb{C}^2 \) of \( \mathcal{H}_\pi \) corresponding to the basis \( \{ f_1, f_2 \} \). This identification is correct because the vacuum vector of \( \mathcal{H}_\pi \) is \( H \)-invariant

\[
U(a)\Omega_0 = \epsilon(a) \cdot \Omega_0, \quad a \in H,
\]

(3.9a)

and the other vacuum vectors of the direct sum observables \( \pi(A) \) give rise to the defining representations of \( H \):

\[
U(a)\Omega_r^i = \sum_{k=1}^{n_r} \Omega_r^k \cdot D_r^{ki}(a), \quad r = 1, 2, \quad i = 1, \ldots, n_r.
\]

(3.9b)

We note that \( U(e_{12}^1) \) and \( U(e_{21}^2) \) have fermionic character. The weak closure \( \pi(A)^- \) of \( \pi(A) \) contains \( U(e_r), r \in \hat{H} \), which shows that the fermion number has an observable meaning only on the NS sectors \( \mathcal{H}_0 \) and \( \mathcal{H}_\pm \) because \( U(e_{11}^1) \) and \( U(e_{22}^2) \) does not belong to \( \pi(A)^- \).

3.2. \( H \)-actions on the field algebra

Having a unitary realization \( U \) of the symmetry algebra \( H \) we would like to characterize the elements of \( \mathcal{B}(\mathcal{H}_\pi) \) by the representations of \( H \). The maps \( \lambda \) and \( \rho \) defined as

\[
H \otimes \mathcal{B}(\mathcal{H}_\pi) \ni a \otimes F \mapsto \lambda_a(F) := U(a)F,
\]

\[
H \otimes \mathcal{B}(\mathcal{H}_\pi) \ni a \otimes F \mapsto \rho_a(F) := FU(a)
\]

(3.10a)

are left and right \( H \)-actions, respectively, on \( \mathcal{B}(\mathcal{H}_\pi) \) therefore \( \mathcal{B}(\mathcal{H}_\pi) \) is an \( H \)-bimodule. Combining \( \rho \) with the antipode \( S \) we obtain a left action \( \Lambda: H \otimes H \otimes \mathcal{B}(\mathcal{H}_\pi) \to \mathcal{B}(\mathcal{H}_\pi): \)

\[
\Lambda_{a \otimes b}(F) = U(a)FU(S(b)).
\]

(3.10b)

One can define the left adjoint action* \( \alpha: H \otimes \mathcal{B}(\mathcal{H}_\pi) \to \mathcal{B}(\mathcal{H}_\pi) \)

\[
\alpha_a(F) := \Lambda_{\Delta(a)}(F) = U(a^{(1)})FU(S(a^{(2)}))
\]

(3.11)

in order to characterize the observables in an other way. If \( A \) is an observable, i.e. if it commutes with \( U(H) \) then

\[
U(a^{(1)})AU(S(a^{(2)})) = AU(a^{(1)})U(S(a^{(2)})) = \epsilon(a) \cdot A,
\]

(3.12)

* But let us note that the module algebra properties do not fulfil with this action since \( H \) is not coassociative.
Thus let \( B \) be the one dimensional orthogonal projections in the decomposition

\[
B = \sum_{i=1}^{n_0} p_i = p_1 + p_2 \equiv e_1^{11} \otimes e_2^{22} + e_2^{22} \otimes e_2^{11}, \quad p_i p_j = \delta_{ij} p_j, \quad p_i^* = p_i, \tag{3.16}
\]

then \( \mathcal{N} \) can be decomposed as

\[
\mathcal{N} = \bigoplus_{i=1}^{2} \mathcal{N}_i, \quad \mathcal{N}_i = \Lambda_{p_i}(\mathcal{N}). \tag{3.17}
\]

Writing a general element \( \hat{F} \in \pi(\mathcal{F}) \) in the form

\[
\hat{F} = \left( \begin{array}{cc} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{array} \right), \tag{3.18a}
\]

where \( \hat{A}, \ldots, \hat{D} \) are polynomials in NS generators with the ordering that the four possible \( b_+ b_-, b_- b_+, b_+, b_- \) terms stand at the end of each monom:

\[
\hat{A} = A^\pm b_+ b_- + A^\mp b_- b_+ + A^+ b_+ + A^- b_-, \tag{3.18b}
\]
i.e. $A^\pm, \ldots, D^\pm$ do not contain already the generators $b_+, b_-$, one checks that $B_0$ is just the observable algebra $\pi(A)^-$:

$$
\alpha_{e_0} (\hat{F}) = \left( \begin{array}{cc} A_0^+ b_+ b_- & B_0^+ b_+-
(\frac{1}{2}) \left( \begin{array}{cc}
(A_0^+ + D_0^+) b_- b_+ & (B_1^- + C_1^+) b_- (A_0^- + D_0^-) b_- b_+
\end{array} \right) \right). (3.19)
$$

The two terms on the right hand side correspond to NS and R observables, respectively, while the subscripts 0 and 1 denote even and odd parts of the polynomials: $A^+ = A_0^+ + A_1^+, \ldots$, etc. The linear subspaces $B_1, B_2, N$ of $B(\mathcal{H}_\pi)$ look as follows:

$$
\alpha_{e_1} (\hat{F}) = \left( \begin{array}{cc} A_1^+ b_+ b_- & B_0^+ b_+
(\frac{1}{2}) \left( \begin{array}{cc}
(A_0^+ - D_0^+) b_- b_+ & (B_1^- - C_1^+) b_- (D_0^- - A_0^+) b_- b_+
\end{array} \right) \right),
$$

$$
\alpha_{e_2} (\hat{F}) = \left( \begin{array}{cc} A^+ b_+ + A^- b_- & B^\pm b_+ b_- + B^\mp b_- b_+
C^\pm b_+ b_- + C^\mp b_- b_+
D^\pm b_+ + D^\mp b_-
\end{array} \right),
$$

$$
\Lambda_{p_1+p_2} (\hat{F}) = \left( \begin{array}{cc} A_1^+ b_+ b_- & B_0^+ b_-
C_0^+ b_+ b_- \end{array} \right).
$$

From the subspaces $\mathcal{B}_r^k = \alpha_{e_{r,k}} (B(\mathcal{H}_\pi))$ one can construct the fields

$$
\mathcal{B}_r^k \ni F_r^k \mapsto \tilde{F}_r^k \equiv U(\varphi_1^\star) F_r^k U(S(\varphi_2^\star) r \varphi_3^\star) \in \tilde{\mathcal{B}}_r^k
$$

obeying $H$-covariant intertwining properties

$$
U(a) \tilde{F}_r^k = \sum_{k'=1}^{n_r} \tilde{F}_r^{k'} D_{r}^{k,k'} (a^{(1)}) U(a^{(2)}) \equiv \tilde{F}_r^{k'} U(\nu_r^{k,k'} (a)), \quad a \in H, (3.22)
$$

that is they induce amplimorphisms of $\nu_r(a) = D_r(a^{(1)} \otimes a^{(2)}$ type of the symmetry algebra $H$. The elements of $\tilde{\mathcal{B}}_r^k, k = 1, \ldots, n_r, r \neq 0$ are the ‘true’ charged fields because they map $H$-covariant states into $H$-covariant ones:

$$
U(a) \cdot \tilde{F}_r^k \Psi_s^{j} = \sum_{k',j'} \tilde{F}_r^{k'j'} (D_r \times D_s)^{k'j',kj} (a), \quad \Psi_s^{j} \in U(e_s^{ij}) \mathcal{H}_\pi. (3.23)
$$

The map $F \mapsto \tilde{F}$ in (3.21) is the identity on the observables due to the square identity 8.b, while the fields in $N$ are mapped into zero due to (3.14).

### 3.3. $H$-covariant multiplet fields

From the linear subspaces $\tilde{\mathcal{B}}_r^k, k = 1, \ldots, n_r, \ r \in \tilde{H}$ one can construct $H$-covariant multiplet matrix fields $\tilde{F}_r = \{ \tilde{F}_r^{k,k} | \kappa, k = 1, \ldots, n_r \} \in M_{n_r} (B(\mathcal{H}_\pi))$ obeying slightly modified weak F-algebra (generalized Cuntz algebra [10]) properties [11]:

$$
U(a) \tilde{F}_r^{k,k} = \sum_{k'=1}^{n_r} \tilde{F}_r^{k,k'} U(\nu_r^{k,k'} (a)), \quad \kappa = 1, \ldots, n_r, \ a \in H, (3.24a)
$$

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where $\nu_r$ is the $H$-amplimorphism

$$
\nu_r(a) = D_r(a^{(1)}) \otimes a^{(2)}.
$$

Clearly, $U(\nu_r(1)) \in M_{n_r}(B(H_\pi))$ is a projection, that is $\tilde{F}_r \in M_{n_r}(B(H_\pi))$ is a partial isometry. The matrix elements $U(\nu_r^{kk'}(1))$ commute with the observables. In our case $\nu_r^{kk'}(1) = \delta_{kk'} \cdot E^k_r$, where the projections $E^k_r \in H$ differ from the unit 1 of $H$ only if the amplimorphism $\nu_r$ of $H$ is unit non-preserving:

$$
E_0 = E_1 = 1, \quad E_2 = e_0 + e_1 + e_2^{11}, \quad E_2 = e_0 + e_1 + e_2^{22}.
$$

Due to (3.24) $\tilde{F}^{kk}_r$ obeys the following intertwining property with an observable $A \in \pi(A)$ (summation is supressed):

$$
\tilde{F}^{kk}_r A = U(1) \tilde{F}^{kk}_r A = \tilde{F}^{kk'}_r U(\nu_r^{kk'}(1)) A = \tilde{F}^{kk'}_r A U(\nu_r^{kk'}(1)) = \tilde{F}^{kk'}_r A \tilde{F}^{kk'}_r = \rho_r^{kk'}(A) \tilde{F}^{kk'}_r.
$$

$\rho$ is an amplimorphism of the observables $\pi(A)$ into $M_{n_r}(\pi(A)^-)$ since $\rho_r^{kk'}(A)$ is observable

$$
U(a) \rho_r^{kk'}(A) = U(a) \tilde{F}^{kk}_r A \tilde{F}^{kk'}_r = \tilde{F}^{kk}_r U(\nu_r^{kk'}(a)) \tilde{F}^{kk'}_r = \tilde{F}^{kk}_r A [U(a^*) \tilde{F}^{kk'}_r] = \rho_r^{kk'}(A) U(a),
$$

moreover

$$
\rho_r^{kk'}(A) \rho_r^{kk''}(B) = \rho_r^{kk'''}(AB), \quad \rho_r^{kk'''}(A) = \rho_r^{kk'}(A^*)
$$

fulfil as well. An explicit choice of the covariant multiplet matrices is:

$$
\tilde{F}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{F}_1 = \begin{pmatrix} (P_+ - P_-)b_- b_+ & b_+ \\ b_- & (P_+ - P_-)b_+ b_- \end{pmatrix},
$$

$$
\tilde{F}_2 = \begin{pmatrix} xP_- b_+ + P_- b_+ & xP_+ b_- b_- + xP_- b_+ + P_- b_- + xP_+ b_+ b_- \& xP_- b_+ b_- & xP_+ b_- + P_- b_+ - xP_+ b_- \\ xP_+ b_- b_+ + P_+ b_+ b_- & -xP_- b_+ \& 0 & P_+ b_- b_+ \\ 0 & P_+ b_- b_+ & 0 & -P_- b_+ \end{pmatrix},
$$

where $x = 1/\sqrt{2}$ and the two by two blocks in $\tilde{F}_2$ correspond to the matrix elements $\tilde{F}^{kk}_r$, $\kappa, k = 1, 2$. Writing a general NS and R element in $\pi(A)$ in the form (see (3.19))

$$
O_{NS} = \begin{pmatrix} A_0 b_+ b_- & B_1 b_+ \\ C_1 b_- & D_0 b_- b_+ \end{pmatrix}, \quad O_R = \begin{pmatrix} E_0 b_- b_+ & E_1 b_- \\ E_1 b_+ & E_0 b_+ b_- \end{pmatrix},
$$

(3.30)
the induced amplimorphisms $\rho_1$ and $\rho_2$ of the observables read as:

\[
\begin{align*}
\rho_1(O_{NS}) &= \begin{pmatrix} D_0 b_+ b_- & -C_1 b_+ \\
-B_1 b_- & A_0 b_+ b_+ \end{pmatrix}, \\
\rho_1(O_R) &= \begin{pmatrix} E_0 b_- b_+ & -E_1 b_- \\
-E_1 b_+ & E_0 b_+ b_- \end{pmatrix}, \\
\rho_2(O_{NS}) &= \begin{pmatrix} A_0 b_+ b_- & 0 & 0 & B_1 b_- \\
0 & A_0 b_+ b_- & B_1 b_+ & 0 \\
0 & C_1 b_- & D_0 b_+ b_+ & 0 \\
C_1 b_+ & 0 & 0 & D_0 b_+ b_- \end{pmatrix}, \\
\rho_2(O_R) &= \begin{pmatrix} E_0 b_+ b_- & E_1 b_+ & 0 & 0 \\
E_1 b_- & E_0 b_- b_+ & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\tag{3.31a}
\]

Due to the choice of the multiplet matrices the images are in $\pi(A)$, that is they do not contain the elements $U(e_0), U(e_1)$. Therefore (3.31) defines amplimorphisms of the observable algebra $A$. The amplimorphism $\rho_2: A \to M_2(A)$ is not unit preserving, but of course a left inverse exists. The representations $\tilde{\pi}_r$ of $A$ on the (finite multiple) of the vacuum Hilbert space $H_0 \otimes \mathbb{C}^{n_r}$, which are equivalent to the representations $\pi_r(A) \equiv \pi_r|_{H_r(A)}$ are given as

\[
\begin{align*}
\tilde{\pi}_r(A)(\Psi \otimes z_\kappa) &= \left(\pi_0 \otimes id_{n_r}\right)\left(\sum_{\kappa',\kappa''}^{n_r} \rho_{2}^{\kappa',\kappa''}(A) \otimes z^{\kappa',\kappa''}\right)(\Psi \otimes z_\kappa) \\
&= \sum_{\kappa'=1}^{n_r} \pi_0(\rho_{2}^{\kappa',\kappa}(A))\Psi \otimes z_{\kappa'},
\end{align*}
\tag{3.32}
\]

where $A \in A, \Psi \in H_0, \{z_\kappa\}$ is an orthonormal basis in $\mathbb{C}^{n_r}$ and $z^{\kappa',\kappa''}$ are the corresponding matrix units.

4. Discussion and outlook

Apart from the previously discussed chiral Ising model there is a less trivial case [19] when a rational Hopf algebra may describe the superselection symmetry of a chiral field theory: a rational Hopf algebra $H = M_1 \oplus M_2 \oplus M_2 \oplus M_1$ can reproduce the fusion rules, the conformal weights, the quantum dimensions and the modular group representation of the level 3 integrable representations of the $\hat{A}_1$ Kac–Moody algebra. It is an example for a RHA that provides us the third smallest index $(3 + \sqrt{5})/2$ in the Jones classification [5]. Of course, there are lots of examples for RHAs with coassociative and unit preserving coproduct: every double $\mathcal{D}(G)$ of a finite group $G$ is a RHA. They describe the global symmetries of $G$-orbifold models [20] and $G$-spin chains [11].

Thus we conjecture that the global symmetry algebras of unitary chiral rational conformal field theories are provided by rational Hopf algebras. In that case the classification of rational Hopf algebras leads to a partial classification of unitary RCFTs, namely, the
possible fusion rules, the conformal weights \((mod 1)\), the Virasoro central charge \((mod 8)\) and the modular group representations can be classified.

Dropping axiom 9. one can study degenerate RHAs having degenerate monodromy matrices. They can correspond to two dimensional field theories with finite number of superselection sectors, where not only the vacuum sector obeys permutation group statistics \([3]\).

Finally, we would like to mention that the low temperature behaviour of three dimensional quantum impurity problems and the multi-channel Kondo effect can be described in terms of chiral conformal field theories \([21]\). Therefore rational Hopf algebras may emerge as symmetry algebras of an impurity atom coupled to a three dimensional electron gas.

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