Discrimination between pure states and mixed states

Chi Zhang,1‡ Guoming Wang,1† and Mingsheng Ying1‡

1State Key Laboratory of Intelligent Technology and Systems, Department of Computer Science and Technology Tsinghua University, Beijing, China, 100084

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In this paper, we discuss the problem of determining whether a quantum system is in a pure state, or in a mixed state. We apply two strategies to settle this problem: the unambiguous discrimination and the maximum confidence discrimination. We also proved that the optimal versions of both strategies are equivalent. The efficiency of the discrimination is also analyzed. This scheme also provides a method to estimate purity of quantum states, and Schmidt numbers of composed systems.

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I. INTRODUCTION

In many applications of quantum information, one of the important elements which affect the result of quantum process, is the purity of the quantum states produced or utilized. Hence, an interesting and important problem in quantum information is to estimate the purity of a quantum system [1, 2, 3, 4, 5]. This problem is also strongly related to the estimation of the entanglement of multiparty systems [6, 7, 8].

However, all above references considered this problem only in the simplest case of qubits, where both the definition and computation about purity are clear. Estimating the purity of a general quantum system is still open. In this paper, we first consider an extreme situation: given some copies of a quantum state, the task of us is to determine whether the state is pure or mixed. The process is called discrimination between pure states and mixed states. Then, by counting different results obtained in the above discriminations, we offer an effective method to estimate the purity of quantum states. The idea of discrimination between pure states and mixed states, was first mentioned in Ref.[8]. However, they did not study the problem formally and systematically, which is our aim in this paper. There are two different strategies to design the discrimination: the unambiguous discrimination [6, 7, 8], and the maximum confidence discrimination [10].

In the strategy of unambiguous discrimination, one can tell whether the quantum system is in a pure state or a mixed state without error, but a non-zero probability of inconclusive answer is allowed. In this paper, in order to simplify the presentation, we use the term “unambiguous” in a more general sense: it allows the success probability to be zero in some situation. On the other hand, in the maximum confidence discrimination, an inconclusive answer is not allowed, and after each discrimination, we must give a statement whether the quantum state is pure, or mixed. The discrimination is so named, because if an answer is given, the probability of obtaining a correct conclusion is maximized.

It is convenient to introduce some notations here. In the Hilbert space \( H^{\otimes n} \), we use \( H_{\text{sym}}^{\otimes n} \) to denote its symmetric subspace [11]. The orthogonal complement of \( H_{\text{sym}}^{\otimes n} \) is called the asymmetric subspace of \( H^{\otimes n} \), and denoted as \( H_{\text{asym}}^{\otimes n} \). We use \( \Phi(H_{\text{sym}}^{\otimes n}) \) and \( \Phi(H_{\text{asym}}^{\otimes n}) \) to represent the projectors of these two subspaces respectively.

In this paper, we prove that, given \( n \) copies of a quantum state \( \rho \) in Hilbert space \( H \), the optimal unambiguous discrimination and the maximum confidence discrimination can be carried out by the same measurement \( \{ \Pi_0 = \Phi(H_{\text{sym}}^{\otimes n}), \Pi_1 = \Phi(H_{\text{asym}}^{\otimes n}) \} \). The difference between these two discriminations comes only from the different explanations of the outcomes. In the unambiguous discrimination, the outcome ‘0’ is an inconclusive answer, and the outcome ‘1’ indicates that the system is in a mixed state. The drawback of the unambiguous discrimination is that, if the quantum system is in a pure state, people always fail to give a confirm answer. However, in the maximum confidence discrimination, the outcome ‘0’ indicates that the quantum system is considered to be in a pure state, and the outcome ‘1’ indicates a mixed state.

There are two natural assumptions in this paper. First, the purity of quantum states is invariant under any unitary operation. Suppose the purity of a quantum state \( \rho \) is represented by \( \mu(\rho) \), it must satisfy that \( \mu(\rho) = \mu(U\rho U^\dagger) \), for any unitary operator \( U \). We also assume that, when \( \rho \) is a pure state, \( \mu(\rho) = 1 \), otherwise \( 0 \leq \mu(\rho) < 1 \). For instance, the usually used purity of quantum states, \( \mu(\rho) = \text{Tr}^2(\rho) \), clearly satisfies these conditions. Second, the priori probability distributions of quantum states are also assumed invariant under unitary operations. Let us denote the priori probability density function as \( \eta(\rho) \), then \( \eta(\rho) = \eta(U\rho U^\dagger) \), for any unitary operator \( U \).

Our present article is organized as follows. In section [11] we provide the optimal unambiguous discrimination between pure states and mixed states. And, in section [14] we provide the maximum confidence discrimination between pure states and mixed states. We also generalize the unambiguous discrimination between pure states and mixed states to a “semi-unambiguous” estimation for ranks of quantum states in section [13] which can also be seen as an estimation for the Schmidt num-
II. OPTIMAL UNAMBIGUOUS DISCRIMINATION

In this section, we consider the unambiguous discrimination between pure states and mixed states. Suppose we are given $n$ copies of a quantum state, which is in the Hilbert space $H$. The unambiguous discrimination is described by a POVM measurement on the Hilbert space $H^\otimes n$. The measurement is comprised by three positive operators, $\Pi_p$, $\Pi_m$, and $\Pi_?$, satisfying that

$$\text{Tr}(\Pi_p \rho^\otimes n) = 0,$$

for any mixed state $\rho$, \hspace{1cm} (1)

$$\langle \psi \rangle^\otimes n \Pi_m \langle \psi \rangle^\otimes n = 0,$$

for any pure state $\langle \psi \rangle$, and \hspace{1cm} (2)

$$\Pi_? = I - \Pi_m - \Pi_p.$$ \hspace{1cm} (3)

Therefore, if the outcome is ‘$p$’, the system is assured to be in a pure state; if the outcome is ‘$m$’, the system is in a mixed state; and outcome ‘?’ denotes an inconclusive answer.

The efficiency of the discrimination is

$$p = \int_{0 \leq \mu(\rho) < 1} \text{Tr}(\rho^\otimes n \Pi_m) \eta(\rho) d\rho$$

$$+ \int_{\mu(\rho) = 1} \text{Tr}(\rho^\otimes n \Pi_p) \eta(\rho) d\rho.$$ \hspace{1cm} (4)

The optimal unambiguous discrimination is the one with the maximum efficiency, and we have the following theorem.

**Theorem 1** The optimal unambiguous discrimination between pure states and mixed states is a POVM measurement $\{\Pi_p, \Pi_m, \Pi_?\}$, such that

$$\Pi_p = 0,$$

$$\Pi_m = \Phi(H^\otimes n_{\text{sym}}),$$

$$\Pi_? = \Phi(H^\otimes n_{\text{asym}}),$$ \hspace{1cm} (5)

where $\Phi(H^\otimes n_{\text{sym}})$ and $\Phi(H^\otimes n_{\text{asym}})$ are the projectors of symmetric subspace and asymmetric subspace of $H^\otimes n$ respectively.

**Proof.** For a mixed state $\rho$, whose spectrum decomposition is $\rho = \sum_{i=1}^m \lambda_i |\phi_i\rangle \langle \phi_i|$, and for any $n$-tuple chosen from $\{1, \cdots, m\}$, $\pi = (\pi_1, \cdots, \pi_n)$, where repetition is allowed, let us introduce the following two definitions,

$$\lambda_\pi = \prod_{j=1}^n \lambda_{\pi_j},$$ \hspace{1cm} (6)

and

$$|\phi_\pi\rangle = \otimes_{j=1}^n |\phi_{\pi_j}\rangle.$$

Then,

$$\rho^\otimes n = \sum_\pi \lambda_\pi |\phi_\pi\rangle \langle \phi_\pi|,$$

where $\pi$ ranges over all $n$-tuples chosen from $\{1, \cdots, m\}$.

Because $\Pi_p$ is a positive operator, from Eq.(4) and Eq.(1),

$$\langle \phi_\pi | \Pi_p | \phi_\pi \rangle = 0,$$ \hspace{1cm} (9)

for any product state $|\phi_\pi\rangle$. Therefore, when the system is in a pure state $|\psi\rangle$, it satisfies that

$$\langle \psi | \Pi_p | \psi \rangle^\otimes n = 0,$$ \hspace{1cm} (10)

i.e., for any situation, the probability of getting the ‘$p$’ result is always zero, which means that without loss of generality, we can simply let $\Pi_p = 0$.

Then, from Eq.(2), we know that $\Pi_m$ is orthogonal to any $|\psi\rangle^\otimes n$, where $|\psi\rangle \in H$. It is known that the span space of all $|\psi\rangle^\otimes n$ is just the symmetric subspace of $H^\otimes n$, which has been denoted as $H^\otimes n_{\text{sym}}$ [12]. Thus, the support space of $\Pi_m$ must be in the asymmetric subspace $H^\otimes n_{\text{asym}}$, i.e., $\Pi_m = \Phi(H^\otimes n_{\text{asym}})$. The probability of determining a mixed state $\rho$ is

$$p(m|\rho) = \text{Tr}(\rho^\otimes n \Pi_m) \leq \text{Tr}(\rho^\otimes n \Phi(H^\otimes n_{\text{asym}})).$$ \hspace{1cm} (11)

Hence, the optimal unambiguous discrimination is the measurement $\{\Pi_p, \Pi_m, \Pi_?\}$ given in Eq.(5).

Under the optimal unambiguous discrimination, when the quantum system is in a pure state, the result is sure to be inconclusive. If the quantum system is in a mixed state $\rho$, the probability of receiving an inconclusive answer is

$$p(\pi|\rho) = \text{Tr}(\rho^\otimes n \Pi_\pi)$$

$$= \sum_\pi \lambda_\pi \langle \phi_\pi | \Phi(H^\otimes n_{\text{sym}}) | \phi_\pi \rangle$$

$$= \sum_\pi \lambda_\pi \text{per}(\Gamma_\pi),$$ \hspace{1cm} (12)

where $\Gamma_\pi$ is the Gram matrix derived from $\{|\phi_{\pi_1}\rangle, \cdots, |\phi_{\pi_n}\rangle\}$, and $\text{per}(A)$ denotes the permanent of the matrix $A$, i.e.,

$$\text{per}(A) = \sum_\sigma \prod_i A(i, \sigma(i)),$$ \hspace{1cm} (13)

where $\sigma$ ranges over all permutation on $n$ symbols [11].

Let $\pi$ is an $n$-tuple valued in $\{1, \cdots, m\}$. We use $n^\pi_i$ to denote the number of occurrences of $i$ in $\pi$, where $i = 1, \cdots, m$. Because for any two eigenvectors of $\rho$ with non-zero eigenvalues, $\langle \phi_i | \phi_j \rangle = \delta_{i,j}$,

$$\Gamma_\pi = \bigoplus_{i=1}^m I_{n^\pi_i}.$$ \hspace{1cm} (14)
where $I_n^m$ is the $n^m$-dimensional identity matrix. Consequently, from Eq.(12)

\[
p(\pi|\rho) = \sum_p \frac{\lambda_p}{n!} \prod_{i=1}^{m} n_i^{n_i!} \\
= \sum_{\sum_i n_i = n} \prod_{i=1}^{m} \frac{n_i!}{n_i^{n_i!}} \prod_{i=1}^{m} \lambda_i^n \frac{\prod_{i=1}^{m} n_i!}{n!} \\
= \sum_{\sum_i n_i = n} \prod_{i=1}^{m} \lambda_i^n.
\]

(15)

III. SEMI-UNAMBIGUOUS ESTIMATION OF SCHMIDT NUMBER

As we know, the entanglement of a bipartite quantum system is closely related to the purity of one of its subsystems. Whether a subsystem is in a pure state is equivalent to whether the total quantum system is in a product state. Hence, the measurement given in the above section also provides an unambiguous estimation for entanglement of bipartite quantum systems. Moreover, in this section, we will provide a natural generalization, which can be called semi-unambiguous estimation of the Schmidt number of bipartite systems.

The Schmidt number of a bipartite system equals to the rank of the quantum state in each of its subsystems. Hence, estimating the Schmidt number is equivalent to estimating the rank of quantum states. First, let us reconsider the discrimination between pure states and mixed states. In the discrimination, the ‘m’ result means that the rank of the state is no less than 2, while the inconclusive answer can also be considered as a trivial conclusion that the rank of the state is no less than 1. Although the discrimination does not offer the exact value of the rank of the quantum state, it offers a lower bound for the rank. Moreover, the lower bound is assured to be correct. In this way, we can call the discrimination between pure states and mixed states also a “semi-unambiguous” estimation for the rank of quantum states. A more general “semi-unambiguous” estimation of the rank of quantum states can be defined as a POVM measurement on $H^\otimes n$ with operators \{\Pi_1, \Pi_2, \cdots, \Pi_m\}, where $m$ is the dimension of $H$. The measurement satisfies that for any quantum state $\rho$ whose rank is $k$, Tr($\Pi_i \rho^{\otimes n}$) = 0, for any $i > k$. Thus, whenever the outcome $k$ is observed, we can make sure that the rank of $\rho$ is no less than $k$.

Before providing the semi-unambiguous estimation of the rank of quantum states. We first introduce some fundamental knowledge about group representation theory needed here. For details, please see Ref. [13].

A Young diagram $[\lambda] = [\lambda_1, \cdots, \lambda_k]$, where $\sum \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k > 0$, is a graphical representation of a partition of a natural number $n$. It consists of $n$ cells, arranged in left-justified rows, where the number of cells in the $i$th row is $\lambda_i$.

A Young tableau is obtained by placing the numbers $1, \cdots, n$ in the $n$ cells of a Young diagram. If the numbers form an increasing sequence along each row and each column, the Young tableau is called standard Young tableau. For a given Young diagram $[\lambda]$, the number of standard Young tableau can be calculated with the hook length formula, and denoted by $f[\lambda]$. In this paper, we use $T_r^{[\lambda]}$ to denote the $r$th standard Young tableau, where $r = 1, \cdots, f[\lambda]$.

The Hilbert space $H^{\otimes n}$, where the dimension of $H$ is $m$, can be decomposed into a set of invariant subspaces under operation $U^{\otimes n}$, for any unitary operation $U$ on $H$. Each of the subspaces corresponds to a standard Young tableau $T_r^{[\lambda]}$, where the number of rows in $[\lambda]$ is no more than $m$. So, we can denote the subspaces as $H_r^{[\lambda]}$, and denote its projector as $\Phi(H_r^{[\lambda]})$. Then, we have

\[H^{\otimes n} = \bigoplus_{[\lambda],r} H_r^{[\lambda]},\]

(16)

For instance, the symmetric subspace $H_{sym}^{\otimes n}$ is just one of these subspaces, $H_{sym}^{\otimes n}$.

For a quantum state $\rho$ in $H$, whose rank is $k$, the support space of $\rho^{\otimes n}$ is in the sum of subspaces $H_r^{[\lambda]}$, where the number of rows in Young diagram $[\lambda]$ is no greater than $k$. Therefore, a semi-unambiguous estimation of the rank of quantum states can be designed as a POVM measurement \{\Pi_1, \cdots, \Pi_m\}, such that

\[\Pi_i = \sum_{h([\lambda]) = i} \sum_r \Phi(H_r^{[\lambda]}),\]

(17)

where $h([\lambda])$ is the number of rows in $[\lambda]$. As said above, if the rank of $\rho$ is $k$, Tr($\Pi_i \rho^{\otimes n}$) = 0, for any $i > k$. Thus, once an ‘i’ result is observed, we can assert that the rank of $\rho$ is no less than $i$. For $n$ copies of a bipartite quantum system, through measuring any of its subsystems with the measurement given in Eq. (17), we can semi-unambiguously estimate the Schmidt number of the whole system.

IV. MAXIMUM CONFIDENCE DISCRIMINATION

In this section, we consider a different strategy for determining whether the quantum system is in a pure state, which is called “maximum confidence discrimination” [10].

The discrimination is still a POVM measurement \{\Pi_p, \Pi_m\}. But when the outcome is ‘p’, the quantum system is believed in a mixed state; otherwise, the outcome is ‘m’, and the quantum state is considered to be pure. A maximum confidence strategy is to maximize the reliability of the conclusion, i.e., let the following two probabilities be maximized,

\[p(\text{pure}|p) = \frac{\int_{\mu(\rho)=1} \eta(\rho) \text{Tr}(\rho^{\otimes n}\Pi_p) d\rho}{\int_{\mu(\rho)\leq1} \eta(\rho) \text{Tr}(\rho^{\otimes n}\Pi_p) d\rho},\]

(18)
and
\[ p(\text{mixed}|m) = \frac{\int_{\mu(p) < 1} \eta(\rho) \text{Tr}(\rho \otimes^n \Pi_m) d\rho}{\int_{\mu(p) \leq 1} \eta(\rho) \text{Tr}(\rho \otimes^n \Pi_m) d\rho}. \] (19)

Clearly, Eq. (18) and Eq. (19) do not always get maximum values at the same time. However, on the assumptions about unitary invariance of \( \eta(\rho) \) and \( \mu(\rho) \), we can prove that there exists a measurement \( \{ \Pi_p, \Pi_m \} \) maximizing both Eq. (18) and Eq. (19), as the following theorem states.

**Theorem 2** The maximum confidence discrimination between pure states and mixed states is a POVM measurement \( \{ \Pi_p, \Pi_m \} \), such that
\[ \Pi_p = \Phi(H_{\text{sym}}^n), \] \[ \Pi_m = \Phi(H_{\text{asym}}^n), \] (20)
where \( \Phi(H_{\text{sym}}^n) \) and \( \Phi(H_{\text{asym}}^n) \) are as in Theorem 1.

**Proof.** First, we consider the construction of \( \Pi_p \). From the assumptions that \( \eta(\rho) = \eta(U\rho U^\dagger) \) and \( \mu(\rho) = \mu(U\rho U^\dagger) \),
\[ p(\text{pure}|p) = \frac{\int_{\mu(p) = 1} \eta(\rho) \text{Tr}(\rho \otimes^n \Pi_p) d\rho}{\int_{\mu(p) \leq 1} \eta(\rho) \text{Tr}(\rho \otimes^n \Pi_p) d\rho} \]
\[ = \frac{\int_{\mu(p) = 1} \eta(\rho) \text{Tr}(\rho \otimes^n \Pi_p(U^\dagger)^\otimes^n) d\rho}{\int_{\mu(p) \leq 1} \eta(\rho) \text{Tr}(\rho \otimes^n \Pi_p(U^\dagger)^\otimes^n) d\rho}, \] (21)
for any unitary operation \( U \). Hence, if \( \Pi_p \) maximizes Eq. (18), so does \( \int U^\otimes \Pi_p(U^\dagger)^\otimes dU \) with respect to the normalized invariant measure \( dU \) of the unitary group \( U(m) \). Hence, we can choose the operator \( \Pi_p \) to satisfy that
\[ \Pi_p = \int U^\otimes \Pi_p(U^\dagger)^\otimes dU, \] (22)
which shows that \( \Pi_p \) commutes with any unitary operator of the form \( U^\otimes^n \). Thus, from the representation theory of classical groups in Ref. [14], \( \Pi_p \) can be expressed as a linear combination of permutation operators
\[ \Pi_p = \sum_{\sigma} \alpha_\sigma V_\sigma, \] (23)
where \( \alpha_\sigma \in C \), \( \sigma \) ranges over all permutations of \( n \) elements, and \( V_\sigma \) is the permutation operator derived from \( \sigma \), i.e.,
\[ V_\sigma |\psi_1\rangle |\psi_2\rangle \cdots |\psi_n\rangle = |\psi_{\sigma^1}\rangle |\psi_{\sigma^2}\rangle \cdots |\psi_{\sigma^n}\rangle. \] (24)

For any state \( |\varphi\rangle \) in the symmetric subspace \( H_{\text{sym}}^n \), \( V_\sigma |\varphi\rangle = |\varphi\rangle \), so \( \Pi_p |\varphi\rangle = (\sum \alpha_\sigma |\varphi\rangle \), which indicates that
\[ \Pi_p = \alpha \Phi(H_{\text{sym}}^n) + \Pi'_p, \] (25)
where \( \Pi'_p \) is a positive operator whose support space is in \( H_{\text{sym}}^n \), and \( \alpha = \sum \alpha_\sigma \). Because for any pure state \( \rho = |\psi\rangle \langle \psi| \), the support space of \( \rho \otimes^n \) is in the symmetric subspace \( H_{\text{sym}}^n \), \( \text{Tr}(\rho \otimes^n \Pi_p) = 0 \) for any \( \mu(p) = 1 \). Therefore, the numerator of Eq. (18) does not change if we substitute \( \Pi_p \) with \( \alpha \Phi(H_{\text{sym}}^n) \), and the denominator diminishes or remains the same. So, the optimal \( \Pi_p \) has the form of \( \alpha \Phi(H_{\text{sym}}^n) \) for any constant \( \alpha \).

On the other hand, if we choose \( \Phi(H_{\text{asym}}^n) \) as \( \Pi_m \), then for any pure state \( \rho \), whose purity \( \mu(\rho) = 1 \), we have \( \text{Tr}(\rho \otimes^n \Pi_m) = 0 \), and Eq. (19) has the maximum value 1. To satisfy the condition \( \Pi_p + \Pi_m = I \), let \( \alpha = 1 \), \( \Pi_p = \Phi(H_{\text{sym}}^n) \). This completes the proof. \( \square \)

It is easy to see that the optimal unambiguous discrimination and the maximum confidence discrimination are the same measurement \( \{ \Pi_0 = \Phi(H_{\text{sym}}^n), \Pi_1 = \Phi(H_{\text{asym}}^n) \} \). The difference between the two discriminations is the meaning of the ‘0’ result. In the former discrimination, the ‘0’ result means an inconclusive answer; however, in the latter discrimination, if a ‘0’ result is obtained, the quantum system is considered in a pure state.

From Eq. (15), for \( n \) copies of a quantum state \( \rho \), under the measurement of \( \{ \Pi_0, \Pi_1 \} \) given above, the probability of receiving a ‘0’ result is
\[ p_0(n) = \sum_{\sum i : n_i = n} \prod_{i=1}^m \lambda_i^{n_i}, \] (26)
where \( \lambda_1, \ldots, \lambda_m \) are the eigenvalues of \( \rho \). As we know, the above quantity is the complete symmetric polynomial of degree \( n \) for \( \{ \lambda_1, \ldots, \lambda_m \} \), which is usually denoted by \( h_n(\lambda_1, \ldots, \lambda_m) \). From Ref. [15], the complete symmetric polynomials can be derived from a generating function
\[ H_m(t) = \sum_{k \geq 0} h_k(\lambda_1, \ldots, \lambda_m)t^k = \frac{1}{\prod_{i=1}^m (1 - t\lambda_i)}. \] (27)

Let \( \lambda^* \) stand for the maximum eigenvalue of \( \rho \), then, if we have \( n \) copies of the states, the probability of judging it to be pure can be evaluated as follows:
\[ p_0(n) = \sum_{\sum i : n_i = n} \prod_{i=1}^m \lambda_i^{n_i} \leq \left( \frac{n + m - 1}{n} \right) (\lambda^*)^n. \] (28)

Then, if the quantum system is in a pure state, \( p_0(n) \) will always be 1, otherwise \( \lambda^* < 1 \), and \( p_0(n) \) will converge to zero with exponential convergence rate.

In section III, we discuss the semi-unambiguous estimation of the rank of quantum states, which is given in Eq. (17). An open problem is whether this measurement also offers a maximum confidence estimation of ranks of quantum states, if we consider the result ‘i’ as a claim that the rank of the quantum state is \( i \).
V. ESTIMATING PURITY OF STATES

The maximum confidence discrimination between pure states and mixed states provides a natural intuition for the purity of a quantum system, i.e., the greater the probability of getting a ‘0’ result, the closer it is to a pure state. Hence, by repetitively performing the measurement, and counting the proportion of ‘0’ results, we can estimate the probability of judging the system being pure, which, in some sense, reflects some information about the purity of the system. However, a more interesting conclusion is that, no matter how people define the purity of quantum states, as long as it satisfies the condition of unitary invariant, it can be well estimated through a set of maximum confidence discriminations.

On the assumption of unitary invariant, the purity of a quantum state \( \rho, \mu(\rho) = \mu(\text{diag}(\lambda_1, \cdots, \lambda_m)) \), where \( \lambda_1, \cdots, \lambda_m \) are the eigenvalues of \( \rho \). Hence, \( \mu(\rho) \) is a function of its eigenvalues. Estimating the purity of a quantum state \( \rho \) can be reduced to estimating the eigenvalues of \( \rho \). The characteristic polynomial of \( \rho \) is a polynomial, whose roots are the eigenvalues, i.e.,

\[
\det(xI - \rho) = \prod_{i=1}^{m} (x - \lambda_i) = \sum_{j=0}^{m} a_j x^{m-j}.
\] (29)

If we can successfully estimate every coefficient \( a_j, j = 0, \cdots, m \), the eigenvalues can be estimated by solving the equation \( \sum_{j=0}^{m} a_j x^{m-j} = 0 \).

Recall the famous Viete’s theorem, it is easy to know

\[
a_0 = e_0(\lambda_1, \cdots, \lambda_m) = 1
\]

\[
a_1 = -e_1(\lambda_1, \cdots, \lambda_m) = -\sum_{i=1}^{m} \lambda_i
\]

\[
a_2 = e_2(\lambda_1, \cdots, \lambda_m) = \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1}\lambda_{i_2}
\]

\[\cdots\]

\[
a_k = (-1)^k e_k(\lambda_1, \cdots, \lambda_m) = (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}
\]

\[
\cdots
\]

\[
a_m = (-1)^m e_m(\lambda_1, \cdots, \lambda_m) = (-1)^m \lambda_1 \lambda_2 \cdots \lambda_m.
\] (30)

Here, the polynomial \( e_k(\lambda_1, \cdots, \lambda_m) \) is the \( m \)-th elementary symmetric polynomial of \( \{\lambda_1, \cdots, \lambda_m\} \) \[15\], whose generating function is,

\[
E_m(t) = \sum_{i=0}^{m} e_i(\lambda_1, \cdots, \lambda_m)t^i = \prod_{i=1}^{m}(1 + t\lambda_i).
\] (31)

Combined with Eq. (27), we have that \( H(t)E(-t) = 1 \), so

\[
\sum_{r=0}^{k} (-1)^r e_r h_{m-r} = 0,
\] (32)

for any \( k \geq 1 \), if we set \( e_r(\lambda_1, \cdots, \lambda_m) = 0 \), when \( r > m \). Here, for simplicity, we use \( e_k, h_l \) to denote \( e_k(\lambda_1, \cdots, \lambda_m), h_l(\lambda_1, \cdots, \lambda_m) \) respectively. Then, it is not hard to see that

\[
e_k = \begin{pmatrix}
h_1 & h_2 & h_3 & \cdots & h_{k-1} & h_k \\
1 & h_1 & h_2 & \cdots & h_{k-2} & h_{k-1} \\
0 & 1 & h_1 & \cdots & h_{k-3} & h_{k-2} \\
0 & 0 & 1 & \cdots & h_{k-4} & h_{k-3} \\
0 & 0 & 0 & \cdots & 1 & h_2 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix},
\] (33)

Clearly, \( h_1 = \sum_{i=1}^{m} \lambda_i = \text{Tr}(\rho) = 1 \). As stated in section \[IV\], for any \( k \geq 2 \), \( h_k \) is the probability of receiving ‘0’ result, when we measure \( \rho^k \) by the measurement \( \{\Pi_0 = \Phi(H^\otimes_k), \Pi_1 = \Phi(H^\otimes_k)\} \). Therefore, if we have \( N \) copies of quantum state \( \rho \), where \( N \) is much larger than \( m \), we can estimate the eigenvalues of \( \rho \) in the following strategy.

First, separate the \( N \) copies into \( m \) groups, the \( k \)th group has \( kN_k \) copies of the quantum state. Then, operate the measurement \( \{\Pi_0 = \Phi(H^\otimes_k), \Pi_1 = \Phi(H^\otimes_k)\} \) on \( \rho^k \) for \( N_k \) times in the \( k \)th group. Suppose among these results, the number of ‘0’ results is \( S_k \), then we can estimate \( p_0(k) \), i.e., \( h_k \) by \( \frac{S_k}{N_k} \). Then, through Eq. (33), we can estimate every \( e_k \), where \( 1 \leq k \leq m \). Hence, from Eq. (30), the characteristic polynomial of \( \rho \), whose roots are the eigenvalues we want to estimate is known. The task remained for us is to solve the equation given in Eq. (29).

VI. CONCLUSION

In this paper, we investigate the discrimination between pure states and mixed states, which may play an important role in further study for estimating the purity of quantum states. The discrimination is described by POVM measurements \( \{\Pi_0 = \Phi(H^\otimes_n), \Pi_1 = \Phi(H^\otimes_n)\} \) on \( n \) copies of the quantum state being discriminated. If the ‘0’ result is considered as an inconclusive answer, the measurement is the optimal unambiguous discrimination. On the other hand, if the ‘0’ result is considered as a hint that the quantum system is in a pure state, the discrimination is the maximum confidence discrimination. We also provide a semi-unambiguous estimation for the rank of quantum states, which also can be used to estimate the Schmidt number of bipartite quantum systems. Finally, we give a strategy to estimate the purity of quantum systems.
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