Torsion in extra-dimensions

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Abstract. We consider a variant of the 5 dimensional Kaluza-Klein theory within the framework of Einstein-Cartan formalism. By imposing a set of constraints on torsion and Ricci rotation coefficients, we show that the torsion components are completely expressed in terms of the metric. and the Ricci tensor in 5D corresponds exactly to what one would obtain from torsion-free general relativity on a 4D hypersurface. The contributions of the scalar and vector fields of the standard K-K theory to the Ricci tensor and the affine connections are completely nullified by the contributions from the torsion. As a consequence, geodesic motions do not distinguish the torsion free 4D space-time from a hypersurface of 5D space-time with torsion satisfying the constraints. Since torsion is not an independent dynamical variable in this formalism, the modified Einstein equations are different from those in the general Einstein-Cartan theory. This leads to important cosmological consequences such as the emergence of cosmic acceleration.

1. Introductory remarks
Torsion, a geometrical property of space-time, represents spin degrees of freedom. In analogy to Einstein equations where the energy-momentum of matter fields is coupled to the geometrical Riemannian metric, we can couple angular momentum and spin of matter fields to the geometrical torsion. The resulting theory of gravity, known as the Einstein-Cartan theory [1] treats the metric and torsion as two independent geometrical characteristics of space-time [2]. This geometrical property of space-time, torsion arises in the definition of covariant derivative operator of vector and tensor fields, forming an anti-symmetric part of the affine connection coefficients (ACCs). In the absence of torsion, metric compatibility leads the determination of the ACCs in terms of the metric components and their derivatives, known as Christoffel symbols, which are symmetric.

Historically, beginning with the Kaluza-Klein (KK) theory, there has been a great interest in introducing extra dimensions of space-time to unify gravity with elementary particle interactions. In the KK theory, the scalar and vector fields, which are the extra dimensional components of the metric tensor contribute to the affine connection and the Ricci tensor and hence modify their values from the corresponding values in 4D space-time[3]. The contribution of these fields to the Einstein tensor are normally interpreted as gravity induced matter.

In the present work, we incorporate torsion into 5D KK theory[4]. In general torsion introduces new parameters other than the metric. But with the choice of a set of minimal conditions on torsion, so as to restrict torsion to the extra dimension, we determine all its components in terms of the metric. Interestingly, the imposed conditions lead to a complete cancellation between the
modifications induced by the extra dimensional metric components and the contributions from the torsion. Thus the Ricci tensor in 5D space-time in the resulting formalism is exactly the same as the Ricci tensor in a torsion free 4D space-time. We conclude by briefly discussing some ensuing physical consequences.

2. Basic Formalism

We begin by defining and collecting together from reference [5], the relevant standard relations we need in both the coordinate and the inertial frames. Let \((i, j, k, \ldots)\) and \((A, B, \ldots)\) denote coordinate and inertial frame indices respectively and \(\hat{e}_i = \partial_i\) and \(\hat{\theta}^i = dx^i\) be the basis of the tangent and dual spaces at each point in space-time. We define the corresponding inertial basis to be \(\hat{e}^A = e^i_A \hat{e}_i\) and \(\hat{\theta}^A = e^i_j \hat{\theta}^i\), where the vielbeins \(e^i_A\) and \(e^j_B\) satisfy the orthonormality conditions,

\[
e^i_A e^j_B = \delta^A_B, \quad e^i_i e^j_B = \delta^B_j.
\]

By definition, the metric in the inertial frame is Minkowskian \(\eta_{AB}\), and the metric tensor in the coordinate system is \(g_{ij} = e^i_A e^j_B \eta_{AB}\) and \(g^{ij} = e^i_A e^j_B \eta^{AB}\). The covariant derivative operator \(\tilde{\nabla}\) can be defined in terms of the coordinate basis, or equivalently in terms of the inertial frame basis,

\[
\tilde{\nabla}_{\hat{e}_i} \hat{e}_j = \tilde{\Gamma}^k_{ij} \hat{e}_k, \quad \tilde{\nabla}_{\hat{e}_A} \hat{e}_B = \omega^C_{AB} \hat{e}_C,
\]

where \(\tilde{\Gamma}^i_{jk}\) and \(\omega^A_{BC}\) are the affine and the Ricci rotation coefficients respectively. The relationship between these two quantities follows from the transformation laws between the coordinate frame(\(\hat{e}_i\)) and inertial frame(\(\hat{e}^A\)),

\[
\omega^A_{BC} = e^i_B (\tilde{\nabla}_{\hat{e}_i} e^j_C) e^j_A.
\]

As is well known, the affine connection by itself is not a tensor, but its antisymmetric part, the torsion, is a tensor.

\[
T^i_{jk} = \tilde{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{kj}.
\]

Again, using the transformation laws between the coordinate and the inertial frames, we have

\[
T^i_{jk} = e^A_j e^B_k e^i_A T^A_{BC}.
\]

Furthermore, with the standard assumption of metric compatibility, namely \(\nabla_{\hat{e}_i} g_{jk} = 0\), we obtain

\[
\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + K^i_{jk},
\]

where \(\tilde{\Gamma}^i_{jk}\) is the Christoffel connection,

\[
\tilde{\Gamma}^i_{jk} = \left\{ \begin{array}{l} \Gamma^i_{jk} \\ K^i_{jk} \end{array} \right\} = \frac{1}{2} g^{im} [\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}],
\]

and \(K^i_{jk}\) is the contorsion tensor.

\[
K^i_{jk} = \frac{1}{2} [T^i_{jk} + T^i_{j,k} + T^i_{k,j}].
\]
3. 5D space-time Metric

Consider a foliation of the 5D space-time in terms of a family of 4D hypersurfaces, which are parametrized by the coordinate system \( \{x^\mu\} \), where \( (\mu, \nu, ...) \) denote the coordinate indices on these hypersurfaces. Let \( x^5 \) denote the parametrization of the family, and 5 denote the corresponding coordinate index. The hypersurface coordinates \( \{x^a\} \) together with \( x^5 \) will then span the entire 5D space-time. Let the metric and its inverse on each of the hypersurfaces be \( g_{\mu \nu} \) and \( g^{\mu \nu} \) respectively, which can in principle depend on the \( x^5 \) coordinate. Let \( \{e^a_\mu, e^\mu_5\} \) denote the tetrad system on these hypersurfaces satisfying the orthonormality relations \( e^a_\mu e^\mu_\beta = \delta^a_\beta \) and \( e^\mu_5 e^\nu_5 = \delta^\mu_\nu \). Here \( (a, b, ...) \) denote the tetrad indices on these 4D hypersurfaces. The metric on the hypersurface is then given by \( g_{\mu \nu} \) the tetrad system on these hypersurfaces satisfies the orthonormality relations \( e^a_\mu e^\mu_\beta = \delta^a_\beta \) and \( e^\mu_5 e^\nu_5 = \delta^\mu_\nu \). Here \( (a, b, ...) \) denote the tetrad indices on these 4D hypersurfaces. The metric on the hypersurface is then given by \( g_{\mu \nu} = e^a_\mu e^\nu_\beta \eta_{ab} \) and its inverse \( g^{\mu \nu} = e^a_\mu e^\nu_\beta \eta^{ab} \).

We will now construct the vielbeins in the 5D space-time by extending the tetrad system on the 4D hypersurfaces. We take the components of the 5D vielbeins to be \( e^a_\mu, e^\mu_5 \) and \( e^\mu_5 = (e^\mu_a, e^\mu_5, e^\mu_i, e^\mu_j) \). The index 5 corresponds to the fifth dimension of the inertial frame. The orthonormality relations in 5D (eq. 1) immediately leads to

\[
e^\mu_5 = 0, \quad e^5_a = -e^\mu_a A_\mu, \quad e^5_5 = \Phi^{-1}, \quad e^5_5 = \Phi.
\]

The metric in the 5D is then given by \( g_{ij} = e_i^A e_j^B \eta_{AB} \) and \( g^{ij} = e_i^A e_j^B \eta^{AB} \).

\[
g_{\mu \nu} = g_{\mu \nu} + \epsilon A_\mu A_\nu \Phi^2, \quad g_{55} = \epsilon A_\mu \Phi^2, \quad g^{55} = \epsilon \Phi^2, \\
g^{\mu \nu} = g^{\mu \nu} + g^{55} = -A_\mu, \quad g^{55} = A_\mu A^\lambda + \epsilon \Phi^{-2}.
\]

The raising and lowering of indices on the vector field is done w.r.t the 4D metric \( g_{\mu \nu} \). The parameter \( \epsilon = \pm 1 \) denotes whether the extra dimension is space-like or time-like. Note that the induced metric on the hypersurfaces (induced by the 5D geometry), \( g_{\mu \nu} + \epsilon A_\mu A_\nu \Phi^2 \) is different from the 4D metric \( g_{\mu \nu} \) on them, but quite evidently related by a gauge transformation.

We will now impose a set of constraints on torsion and the Ricci rotation coefficients consistent with Cartan’s first structure equation [5] that relates torsion and connection coefficients. With a minimal modification of the standard general relativity in mind, we chose the set of constraints [7] so that the torsion components on the 4D hypersurface vanish (see also [8]), yet leaving non-vanishing torsion components completely determined in terms of the metric components.

**Condition 1**: \( T^a_{\ BC} = 0 \)

Using eq. 5 and noting that \( e^\mu_5 = 0 \), we find \( T^\mu_{\ 5k} = 0 \). This implies that the only nonzero components of torsion are \( T^5_{\ 5k} \). To determine these, we impose the following condition on the Ricci rotation coefficients,

**Condition 2**: \( \omega^5_{\ BC} = 0 \)

Condition 2 along with metric compatibility implies \( \omega^A_{\ B5} = 0 \). We can now use eq. 3 to write,

\[
\omega^A_{\ B5} = e^i_B (\nabla_i e^j_5) e^A_j = e^i_B (\partial_i e^j_5 + \tilde{\Gamma}^j_{\ ik} e^k_5) e^A_j = 0
\]

Since \( e^\mu_5 = e^5_5 = 0 \), the above equation implies,

\[
\tilde{\Gamma}^\mu_{\ 55} = 0, \quad \tilde{\Gamma}^5_{\ 55} = -e^5_5 \partial_5 e^5_5
\]

\[\text{There exists another class of vielbeins that satisfy the orthonormality relations. But they are essentially related to eq. 9 by gauge. Here, we choose to work with the vielbeins given by eq. 9 to make the formalism readily comparable to the existing Kaluza-Klein literature.}\]
Using the above equations along with eq.6, we can express the contorsion in terms of the Christoffel symbols $\Gamma$ and the vielbeins,

\[
\begin{align*}
K^\mu_{i5} &= -\tilde{\Gamma}^\mu_{i5}, \\
K^5_{i5} &= -\left(\tilde{\Gamma}^5_{i5} + e_5^i \partial_5 e_5^5\right) = -\tilde{\Gamma}^5_{i5} + J_i,
\end{align*}
\]

where $J_i \equiv \Phi^{-1} \partial_i \Phi$.

The above equations are sufficient to determine all the components of torsion in terms of the metric components

\[
\begin{align*}
T_{ij}^\mu &= T_{55}^5 = 0, \\
T_{\mu\nu}^5 &= 2\partial_{[\mu} A_{\nu]} + 2J_{[\mu} A_{\nu]}, \\
T_{\mu5}^5 &= J_\mu - \partial_5 A_\mu - A_\mu J_5,
\end{align*}
\]

A remarkable result that follows is that in addition to yielding the non-vanishing components of torsion in terms of the metric components, the solution to eq. 13 also yields the following condition on the metric on the 4D hypersurfaces.

\[
\partial_5 g_{\mu\nu} = 0.
\]

This implies all the hypersurfaces in the foliating family have the same 4D metric. To place things in perspective, we observe that in the standard Kaluza-Klein theory, the assumption of cylindrical condition makes all the quantities, namely $g_{5\mu\nu}$, $A_\mu$ and $\Phi$ independent of $x^5$. Whereas in our formulation, the constraints automatically imply $g_{\mu\nu}$ is independent of $x^5$, while $A_\mu$ and $\Phi$ could still depend on $x^5$.

With all the $K^i_{jk}$ determined from eqns. 14 and 8, we now use eq. 6 to calculate all the affine connection coefficients.

\[
\begin{align*}
\tilde{\Gamma}^\lambda_{55} &= \tilde{\Gamma}^\lambda_{55} = \tilde{\Gamma}^\lambda_{55} = 0, \\
\tilde{\Gamma}^5_{\mu\nu} &= \nabla_\mu A_\nu + J_\mu A_\nu, \\
\tilde{\Gamma}^5_{5\mu} &= \partial_5 A_\mu + J_5 A_\mu, \\
\tilde{\Gamma}^5_{\mu5} &= J_\mu, \quad \tilde{\Gamma}^\lambda_{55} = J_5, \quad \tilde{\Gamma}^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu}.
\end{align*}
\]

Here $\Gamma^\lambda_{\mu\nu}$ corresponds to the Christoffel symbols (analogous to eq. 7) obtained from torsion free 4D space-time with metric $g_{\mu\nu}$. Note that the components of 5D Christoffel symbols along the hypersurface coordinates is not equal to the Christoffel symbols calculated on 4D spacetime, that is, $\Gamma^\lambda_{\mu\nu} \neq \Gamma^\lambda_{\mu\nu}$. The symbol $\nabla_\mu$ corresponds to the covariant derivative operator on the torsion-free 4D space-time, where the Christoffel symbols are exactly the affine connection coefficients.

Substituting the above connection coefficients in the Ricci tensor defined by

\[
R_{ik} = \partial_k \tilde{\Gamma}^j_{ji} - \partial_j \tilde{\Gamma}^j_{ki} + \tilde{\Gamma}^j_{km} \tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^j_{jm} \tilde{\Gamma}^m_{kj},
\]

we find

\[
\tilde{R}_{\mu\nu} = R_{\mu\nu}, \quad \tilde{R}_{\mu5} = \tilde{R}_{5\mu} = \tilde{R}_{55} = 0.
\]

Here $R_{\mu\nu}$ represents the Ricci tensor on the torsion-free 4D space-time. It also follows that the 5D Ricci scalar is exactly the same as the Ricci scalar in the torsion free 4D space time, that is $\tilde{R} = R$. We also note, in general, the presence of torsion makes the Ricci tensor non-symmetric, but the constraints we have imposed on the torsion leaves the Ricci tensor symmetric.

It is straightforward to see that the formalism and the results obtained thus far are not specific to 4 and 5 dimensions, they can be generalized to any arbitrary D and D+1 dimensions. We shall now consider some implications of the formalism with respect to geodesic equations and solutions to Einstein equations.
4. Geodesic Equations
The 5D geodesic equations split into
\[
\begin{align*}
\dot{x}^5 &+ \tilde{\Gamma}^5_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \left( \tilde{\Gamma}^5_{\mu 5} + \tilde{\Gamma}^5_{5\mu} \right) \dot{x}^\mu \dot{x}^5 + \Gamma^5_{55} (\dot{x}^5)^2 = 0, \\
\dot{x}^\lambda &+ \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 
\end{align*}
\]
We note that the components of the geodesic equations along the hypersurface are exactly the same as the geodesic equations in the torsion free 4D space-time. This is in contrast with the conventional Kaluza Klein theory where the 4D geodesic equations are modified by the presence of the fields \( A_\mu \) and \( \Phi \). In this formalism, the presence of torsion completely nullifies the effect of these fields in the 4D geodesic equations. Furthermore, it is worth noting that the geodesic of a particle can be confined to a 4D hypersurface by requiring \( \dot{x}^5 = 0 \). From eq. 19, we see that this requires the additional condition,
\[
\tilde{\Gamma}^5_{\mu\nu} = \nabla_\mu A_\nu + J_\mu A_\nu = 0.
\]
If this condition can be satisfied, then there will be no observable difference, as far as a test particle is concerned, whether we live in a torsion free 4D space-time or on a hypersurface within the 5D space-time with torsion. However, it is apparent that this condition is a strong constraint requiring the vector and scalar fields satisfy the above equation for a given 4D metric. It is conceivable that for some 4D metrics, no choice of vector and scalar fields would satisfy the above constraint. In such cases, the particle would be free to move in the \( x^5 \) direction unless constrained by an external force or if the fifth co-ordinate is compact and small as usually assumed in most adaptations of the Kaluza-Klein theory.

5. Einstein’s equations
To obtain the field equations, we need to vary the action with respect to the independent dynamical variables of the theory. In Einstein-Cartan theory, the independent variables are the metric and the torsion. By imposing the geometrical constraints using Cartan structure equations, we have expressed torsion in terms of the 5D metric. Hence, taking the Lagrangian density to be the Ricci scalar and varying the action with respect to the 5D metric, we obtain the following modified Einstein equations.
\[
\begin{align*}
\mathcal{R}^\nu_{\mu} - \frac{1}{2} \mathcal{R}^\nu_\delta \delta^\nu_\mu + H^\nu_\mu &= \Sigma^\nu_\mu \\
- A^\alpha \mathcal{R}_\mu\alpha - A^\alpha H_\mu\alpha &= \Sigma^5_\mu \\
- \frac{1}{2} \mathcal{R} &= \Sigma^5_5
\end{align*}
\]
where
\[
H^\mu_\nu = \nabla_{(\mu} B_{\nu)} - (\nabla \cdot B) g^\mu_\nu + J_{(\mu} B_{\nu)} - (J \cdot B) g^\mu_\nu
\]
and \( B_\mu = T^5_{\mu5} \) is a vector in 4D torsion free space time.

In the above equations \( \Sigma \) is the stress tensor that one would obtain when matter fields are included in the Lagrangian prior to variation of action. Its 4D components will be the observed stress energy tensor and can be identified with the stress energy tensor of the usual 4D torsion free general relativity. Its conservation requires
\[
\nabla_\nu H^\nu_\mu = 0.
\]
Note that when \( H^\mu_\nu = 0 \), eq. 21 reduces to the torsion free 4D Einstein equations. One could set \( \Sigma^5_\mu \) to zero and use eqns. 22, 25 to solve for the fields \( A_\mu \) and \( \Phi \).
6. Robertson-Walker cosmology:
To illustrate an application of the formalism, let us consider spatially flat homogenous and isotropic universe with metric
\[ ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right). \]  

(26)

The assumption of homogeneity and isotropy of the 4D geometry requires that \( A_\mu = (A_t, 0, 0, 0) \) and \( A_t \) and \( \Phi \) are functions of \( t \) and not the spatial coordinates. To further simplify, we shall also assume that the fields do not depend on \( x^5 \). From eq. 14, these constraints imply that \( B_\mu = J_\mu \) and the only non vanishing component of \( J_\mu \) is \( J_t \), and of \( A_\mu = A_t \). Applying the conservation equation (eq. 25) yields

\((i) \ \dot{\Phi} = 0, J_t = 0, \) or \( (ii) \ \dot{\Phi} = \dot{a}(t), J_t = \ddot{a}/\dot{a}. \)  

(27)

Case \((i)\) yields \( H_{\mu\nu} = 0 \), and eq. 21 yields the usual Friedman equation along with matter conservation.
\[ (\dot{a}/a)^2 = \frac{8\pi}{3} \rho. \]

Case \((ii)\) yields non vanishing \( H_{\mu\nu} \), which when applied to eq. 21 gives the modified Friedman equation
\[ (\dot{a}/a)^2 + (\ddot{a}/a) = \frac{8\pi}{3} \rho, \]

(28)

along with matter conservation which implies \( \rho a^3 = \rho_o \) is a constant in a matter dominated universe.

To solve the equation, we specify initial conditions at the current instant of time, \( a = 1, \dot{a} = H_o, \) the Hubble constant and \( \ddot{a} = -q_o H_o^2 \), where \( q_o \) is the current deceleration parameter. These conditions can be used to calculate \( \rho_o \), the current matter density (including dark matter). Taking \( q_o = -0.5 \), which is consistent with the current observations \cite{6}, the solution to eq. 28 is plotted in figure 1. From the dashed-dot curve (scale factor), note that the universe started expanding at \( t = -0.518 H_o^{-1} \), from a size of \( a = 2/3 \), prior to which it was in a contracting phase. This is in contrast to the solution of the usual Friedmann equations which yields a big bang
\( a = 0 \) at \( t = -0.667 H_0^{-1} \). From the solid curve, note that the acceleration is currently positive but decreasing and would become negative beyond \( t = 0.319 H_0^{-1} \). This is qualitatively consistent with the analysis of observed data in [9], but is in sharp contrast with the standard \( \Lambda \)CDM model which predicts that the acceleration would continue to increase for ever. Clearly, more detailed studies are needed to understand the full implications of this formalism on cosmology.

7. Concluding remarks
Torsion is an integral, geometrical property of space time. The model we have explored has a very general mathematical result pertaining to affine connection and Ricci tensor in Einstein-Cartan theory in higher dimensions. In higher dimensional theories, it provides an alternative way to confine gravity in a torsion free lower dimension, yet modifying it in a significant way to be relevant to astrophysics and cosmology. We have explored some of the interesting consequences of the model and in the process of further study.

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