Hypoellipticity and loss of derivatives

By J. J. Kohn*

(with an Appendix by Makhlouf Derridj and David S. Tartakoff)

Dedicated to Yum-Tong Siu for his 60th birthday.

Abstract

Let \{X_1, \ldots, X_p\} be complex-valued vector fields in \(\mathbb{R}^n\) and assume that they satisfy the bracket condition (i.e. that their Lie algebra spans all vector fields). Our object is to study the operator \(E = \sum X_i^* X_i\), where \(X_i^*\) is the \(L_2\) adjoint of \(X_i\). A result of Hörmander is that when the \(X_i\) are real then \(E\) is hypoelliptic and furthermore it is subelliptic (the restriction of a distribution \(u\) to an open set \(U\) is “smoother” than the restriction of \(Eu\) to \(U\)). When the \(X_i\) are complex-valued if the bracket condition of order one is satisfied (i.e. if the \(\{X_i, [X_i, X_j]\}\) span), then we prove that the operator \(E\) is still subelliptic. This is no longer true if brackets of higher order are needed to span. For each \(k \geq 1\) we give an example of two complex-valued vector fields, \(X_1\) and \(X_2\), such that the bracket condition of order \(k + 1\) is satisfied and we prove that the operator \(E = X_1^* X_1 + X_2^* X_2\) is hypoelliptic but that it is not subelliptic. In fact it “loses” \(k\) derivatives in the sense that, for each \(m\), there exists a distribution \(u\) whose restriction to an open set \(U\) has the property that the \(D^\alpha Eu\) are bounded on \(U\) whenever \(|\alpha| \leq m\) and for some \(\beta\), with \(|\beta| = m - k + 1\), the restriction of \(D^\beta u\) to \(U\) is not locally bounded.

1. Introduction

We will be concerned with local \(C^\infty\) hypoellipticity in the following sense. A linear differential operator operator \(E\) on \(\mathbb{R}^n\) is hypoelliptic if, whenever \(u\) is a distribution such that the restriction of \(Eu\) to an open set \(U \subset \mathbb{R}^n\) is in \(C^\infty(U)\), then the restriction of \(u\) to \(U\) is also in \(C^\infty(U)\). If \(E\) is hypoelliptic then it satisfies the following a priori estimates.

*Research was partially supported by NSF Grant DMS-9801626.
(1) Given open sets \( U, U' \) in \( \mathbb{R}^n \) such that \( U \subset \bar{U} \subset U' \subset \mathbb{R}^n \), a nonnegative integer \( p \), and a real number \( s_0 \), there exist an integer \( q \) and a constant \( C = C(U, p, q, s_0) \) such that
\[
\sum_{|\alpha| \leq p} \sup_{x \in U} |D^\alpha u(x)| \leq C \left( \sum_{|\beta| \leq q} \sup_{x \in U'} |D^\beta E u(x)| + \|u\|_{-s_0} \right),
\]
for all \( u \in C_0^\infty(\mathbb{R}^n) \).

(2) Given \( \varrho, \varrho' \in C_0^\infty(\mathbb{R}^n) \) such that \( \varrho' = 1 \) in a neighborhood of \( \text{supp}(\varrho) \), and \( s_0, s_1 \in \mathbb{R} \), there exist \( s_2 \in \mathbb{R} \) and a constant \( C = C(\varrho, \varrho', s_1, s_2, s_0) \) such that
\[
\|\varrho u\|_{s_1} \leq C(\|\varrho' E u\|_{s_2} + \|u\|_{-s_0}),
\]
for all \( u \in C_0^\infty(\mathbb{R}^n) \).

Assuming that \( E \) is hypoelliptic and that \( q \) is the smallest integer so that the first inequality above holds (for large \( s_0 \)) then, if \( q \leq p \), we say that \( E \) gains \( p - q \) derivatives in the sup norms and if \( q \geq p \), we say that \( E \) loses \( q - p \) derivatives in the sup norms. Similarly, assuming that \( s_2 \) is the smallest real number so that the second inequality holds (for large \( s_0 \)) then, if \( s_2 \leq s_1 \), we say that \( E \) gains \( s_1 - s_2 \) derivatives in the Sobolev norms and if \( s_2 \geq s_1 \), we say that \( E \) loses \( s_2 - s_1 \) derivatives in the Sobolev norms. In particular if \( E \) is of order \( m \) and if \( E \) is elliptic then \( E \) gains exactly \( m \) derivatives in the Sobolev norms and gains exactly \( m - 1 \) derivatives in the sup norms. Here we will present hypoelliptic operators \( E_k \) of order 2 which lose exactly \( k - 1 \) derivatives in the Sobolev norms and lose at least \( k \) derivatives in the sup norms.

Loss of derivatives presents a very major difficulty: namely, how to derive the \textit{a priori} estimates? Such estimates depend on localizing the right-hand side and (because of the loss of derivatives) the errors that arise are apparently always larger then the terms one wishes to estimate. This difficulty is overcome here by the use of subelliptic multipliers in a microlocal setting. In this introduction I would like to indicate the ideas behind these methods, which were originally devised to study hypoellipticity with gain of derivatives. It should be remarked that that for global hypoellipticity the situation is entirely different; in that case loss of derivatives can occur and is well understood but, of course, the localization problems do not arise.

We will restrict ourselves to operators \( E \) of second order of the form
\[
E u = -\sum_{i,j} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j},
\]
where \( (a_{ij}) \) is a hermitian form with \( C^\infty \) complex-valued components. If at some point \( P \in \mathbb{R}^n \) the form \( (a_{ij}(P)) \) has two nonzero eigenvalues of different
signs then $E$ is not hypoelliptic so that, without loss of generality, we will assume that $(a_{ij}) \geq 0$.

**Definition 1.** The operator $E$ is subelliptic at $P \in \mathbb{R}^n$ if there exists a neighborhood $U$ of $P$, a real number $\varepsilon > 0$, and a constant $C = (U, \varepsilon)$, such that

$$\|u\|_\varepsilon^2 \leq C(|(Eu, u)| + \|u\|^2),$$

for all $u \in C^\infty_0(U)$.

Here the Sobolev norm $\|u\|_s$ is defined by

$$\|u\|_s = \|\Lambda^s u\|,$$

and $\Lambda^s u$ is defined by its Fourier transform, which is

$$\hat{\Lambda^s u}(\xi) = (1 + |\xi|^2)^s \hat{u}(\xi).$$

We will denote by $H^s(\mathbb{R}^n)$ the completion of $C^\infty_0(\mathbb{R}^n)$ in the norm $\| \|_s$. If $U \subset \mathbb{R}^n$ is open, we denote by $H^s_{\text{loc}}(U)$ the set of all distributions on $U$ such that $\zeta u \in H^s(\mathbb{R}^n)$ for all $\zeta \in C^\infty_0(U)$. The following result, which shows that subellipticity implies hypoellipticity with a gain of $2\varepsilon$ derivatives in Sobolev norms, is proved in [KN].

**Theorem.** Suppose that $E$ is subelliptic at each $P \in U \subset \mathbb{R}^n$. Then $E$ is hypoelliptic on $U$. More precisely, if $u \in H^{-s_0} \cap H^s_{\text{loc}}(U)$ and if $Eu \in H^s_{\text{loc}}(U)$, then $u \in H^{s+2\varepsilon}_{\text{loc}}(U)$.

In [K1] and [K2] I introduced subelliptic multipliers in order to establish subelliptic estimates for the $\overline{\partial}$-Neumann problem. In the case of $E$, subelliptic multipliers are defined as follows.

**Definition 2.** A subelliptic multiplier for $E$ at $P \in \mathbb{R}^n$ is a pseudodifferential operator $A$ of order zero, defined on $C^\infty_0(U)$, where $U$ is a neighborhood of $P$, such that there exist $\varepsilon > 0$, and a constant $C = C(\varepsilon, P, A)$, such that

$$\|Au\|_\varepsilon^2 \leq C(|(Eu, u)| + \|u\|^2),$$

for all $u \in C^\infty_0(U)$.

If $A$ is a subelliptic multiplier and if $A'$ is a pseudodifferential operator whose principal symbol equals the principal symbol of $A$ then $A'$ is also a subelliptic multiplier. The existence of subelliptic estimates can be deduced from the properties of the set symbols of subelliptic multipliers. In the case of the $\overline{\partial}$-Neumann problem this leads to the analysis of the condition of "D’Angelo finite type." Catlin and D’Angelo, in [C] and [D’A], showed that D’Angelo finite type is a necessary and sufficient condition for the subellipticity of the $\overline{\partial}$-Neumann problem. To illustrate some of these ideas, in the case of an
operator $E$, we will recall Hörmander’s theorem on the sum of squares of vector fields.

Let $\{X_1, \ldots, X_m\}$ be vector fields on a neighborhood of the origin in $\mathbb{R}^n$.

**Definition 3.** The vectorfields $\{X_1, \ldots, X_m\}$ satisfy the *bracket condition* at the origin if the Lie algebra generated by these vector fields evaluated at the origin is the tangent space.

In [Ho], Hörmander proved the following

**Theorem.** If the vectorfields $\{X_1, \ldots, X_m\}$ are real and if they satisfy the bracket condition at the origin then the operator $E = \sum X_j^2$ is hypoelliptic in a neighborhood of the origin.

The key point of the proof is to establish that for some neighborhoods of the origin $U$ there exist $\varepsilon > 0$ and $C = C(\varepsilon, U)$ such that

$$
\|u\|_2^2 \leq C(\sum \|X_j u\|^2 + \|u\|^2),
$$

for all $u \in C_0^\infty(U)$. Here is a brief outline of the proof of estimate (1) using subelliptic multipliers. Note that

1. The operators $A_j = \Lambda^{-1} X_j$ are subelliptic multipliers with $\varepsilon = 1$, that is

$$
\|A_j u\|_2^2 \leq C(\sum \|X_j u\|^2 + \|u\|^2),
$$

for all $u \in C_0^\infty(U)$.

2. If $A$ is a subelliptic multiplier then $[X_j, A]$ is a subelliptic multiplier. (This is easily seen: we have $X_j^* = -X_j + a_j$ since $X_j$ is real and

$$
\|\|[X_j, A]u\|_2^2 \leq |(X_j Au, R^\varepsilon u)| + |(AX_j u, R^\varepsilon u)|
\leq |(Au, R^\varepsilon u)| + O(\|u\|^2) + |(Au, R^\varepsilon X_j u)| + |(AX_j u, R^\varepsilon u)|
\leq C \left(\|Au\|_2^2 + \sum \|X_j u\|^2 + \|u\|^2\right),
$$

where $R^\varepsilon = \Lambda^\varepsilon[X_j, A]$ and $\tilde{R}^\varepsilon = [X_j^*, R^\varepsilon]$ are pseudodifferential operators of order $\varepsilon$.)

Now using the bracket condition and the above we see that 1 is a subelliptic multiplier and hence the estimate (1) holds.

The more general case, where the $a_{ij}$ are real but $E$ cannot be expressed as a sum of squares (modulo $L_2$) has been analyzed by Oleinik and Radkevic (see [OR]). Their result can also be obtained by use of subelliptic multipliers and can then be connected to the geometric interpretation given by Fefferman and Phong in [FP]. The next question, which has been studied fairly extensively,
is what happens when subellipticity fails and yet there is no loss. A striking example is the operator on $\mathbb{R}^2$ given by

$$E = -\frac{\partial^2}{\partial x^2} - a^2(x) \frac{\partial^2}{\partial y^2},$$

where $a(x) \geq 0$ when $x \neq 0$. This operator was studied by Fedii in [F], who showed that $E$ is always hypoelliptic, no matter how fast $a(x)$ goes to zero as $x \to 0$. Kusuoka and Stroock (see [KS]) have shown that the operator on $\mathbb{R}^3$

$$E = -\frac{\partial^2}{\partial x^2} - a^2(x) \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2},$$

where $a(x) \geq 0$ when $x \neq 0$, is hypoelliptic if and only if $\lim_{x \to 0} \log a(x) = 0$. Hypoellipticity when there is no loss but when the gain is smaller than in the subelliptic case has also been studied by Bell and Mohamed [BM], Christ [Ch1], and Morimoto [M]. Using subelliptic multipliers has provided new insights into these results (see [K4]); for example Fedii’s result is proved when $a^2$ is replaced by $a$ with the requirement that $a(x) > 0$ when $x \neq 0$. In the case of the $\bar{\partial}$-Neumann problem and of the operator $\Box_b$ on CR manifolds, subelliptic multipliers are used to established hypoellipticity in certain situations where there is no loss of derivatives in Sobolev norms but in which the gain is weaker than in the subelliptic case (see [K5]). Stein in [St] shows that the operator $\Box_b + \mu$ on the Heisenberg group $\mathcal{H} \subset \mathbb{C}^2$, with $\mu \neq 0$, is analytic hypoelliptic but does not gain or lose any derivatives. In his thesis Heller (see [He]), using the methods developed by Stein in [St], shows that the fourth order operator $\Box_b^2 + X$ is analytic hypoelliptic and that it loses derivatives (here $X$ denotes a “good” direction). In a recent work, C. Parenti and A. Parmeggiani studied classes of pseudodifferential operators with large losses of derivatives (see [PP1]).

The study of subelliptic multipliers has led to the concept of multiplier ideal sheaves (see [K2]). These have had many applications notably Nadel’s work on Kähler-Einstein metrics (see [N]) and numerous applications to algebraic geometry. In algebraic geometry there are three areas in which multiplier ideals have made a decisive contribution: the Fujita conjecture, the effective Matsusaka big theorem, and invariance of plurigenera; see, for example, Siu’s article [S]. Up to now the use of subelliptic multipliers to study the $\bar{\partial}$-Neumann problem and the laplacian $\Box_b$ has been limited to dealing with Sobolev norms, Siu has developed a program to use multipliers for the $\bar{\partial}$-Neumann problem to study Hölder estimates and to give an explicit construction of the critical varieties that control the D’Angelo type. His program leads to the study of the operator

$$E = \sum_{1}^{m} X_j^* X_j,$$
where the \( \{X_1, \ldots, X_m\} \) are complex vector fields satisfying the bracket condition. Thus Siu’s program gives rise to the question of whether the above operator \( E \) is hypoelliptic and whether it satisfies the subelliptic estimate (1). These problems raised by Siu have motivated my work on this paper. At first I found that if the bracket condition involves only one bracket then (1) holds with \( \varepsilon = \frac{1}{4} \) (if the \( X_j \) span without taking brackets then \( E \) is elliptic). Then I found a series of examples for which the bracket condition is satisfied with \( k \) brackets, \( k > 1 \), for which (1) does not hold. Surprisingly I found that the operators in these examples are hypoelliptic with a loss of \( k - 1 \) derivatives in the Sobolev norms. The method of proof involves calculations with subelliptic multipliers and it seems very likely that it will be possible to treat the more general cases, that is when \( E \) given by complex vector fields and, more generally, when \( (a_{ij}) \) is nonnegative hermitian, along the same lines.

The main results proved here are the following:

**Theorem A.** If \( \{X_i, [X_i, X_j]\} \) span the complex tangent space at the origin then a subelliptic estimate is satisfied, with \( \varepsilon = \frac{1}{2} \).

**Theorem B.** For \( k \geq 0 \) there exist complex vector fields \( X_{1k} \) and \( X_2 \) on a neighborhood of the origin in \( \mathbb{R}^3 \) such that the two vector fields \( \{X_{1k}, X_2\} \) and their commutators of order \( k + 1 \) span the complexified tangent space at the origin, and when \( k > 0 \) the subelliptic estimate (1) does not hold. Moreover, when \( k > 1 \), the operator \( E_k = X_{1k}^* X_{1k} + X_2^* X_2 \) loses \( k \) derivatives in the sup norms and \( k - 1 \) derivatives in the Sobolev norms.

Recently Christ (see [Ch2]) has shown that the operators \( -\frac{\partial^2}{\partial s^2} + E_k \) on \( \mathbb{R}^4 \) are not hypoelliptic when \( k > 0 \).

**Theorem C.** If \( X_{1k} \) and \( X_2 \) are the vector fields given in Theorem B then the operator \( E_k = X_{1k}^* X_{1k} + X_2^* X_2 \) is hypoelliptic. More precisely, if \( u \) is a distribution solution of \( Eu = f \) with \( u \in H^{-s_2}(\mathbb{R}^3) \) and if \( U \subset \mathbb{R}^3 \) is an open set such that \( f \in H^{s_2}_{\text{loc}}(U) \), then \( u \in H^{s_2-k+1}_{\text{loc}}(U) \).

This paper originated with a problem posed by Yum-Tong Siu. The author wishes to thank Yum-Tong Siu and Michael Christ for fruitful discussions of the material presented here.

**Remarks.** In March 2005, after this paper had been accepted for publication, I circulated a preprint. Then M. Derridj and D. Tartakoff proved analytic hypoellipticity for the operators constructed here (see [DT]). The work of Derridj and Tartakoff used “balanced” cutoff functions to estimate the size of derivatives starting with the \( C^\infty \) local hypoellipticity proved here; then Bove, Derridj, Tartakoff, and I (see [BDKT]) proved \( C^\infty \) local hypoellipticity using the balanced cutoff functions, starting from the estimates for functions with
compact support proved here. Also at this time, in [PP2], Parenti and Parmegian, following their work in [PP1], gave a different proof of hypoellipticity of the operators discussed here and in [Ch2].

2. Proof of Theorem A

The proof of Theorem A proceeds in the same way as given above in the outline of Hörmander’s theorem. It works only when one bracket is involved because (unlike the real case) $\bar{X}_j$ is not in the span of the $\{X_1, \ldots, X_m\}$. The constant $\varepsilon = \frac{1}{2}$ is the largest possible, since (as proved in [Ho]) this is already so when the $X_i$ are real.

First note that $\|X_i^* u\|_2^2 \leq \|X_i u\|^2 + C\|u\|^2$, since

$$\|X_i^* u\|_2^2 = (X_i^* u, \Lambda^{-1} X_i^* u) = (X_i^* u, P^0 u) = (u, X_i P^0 u) = -(u, P^0 X_i u) + O(\|u\|^2);$$

hence,

$$\|X_i^* u\|_2^2 \leq C \left( \sum \|X_k u\|^2 + \|u\|^2 \right),$$

where $P^0 = \Lambda^{-1} \bar{X}_i$ is a pseudodifferential operator of order zero. Then we have

$$\|X_i^* u\|^2 = (u, X_i X_i^* u) = \|X_i u\|^2 + (u, [X_i, X_i^*] u)$$

$$\leq \|X_i u\|^2 + \|\Lambda^{\frac{1}{2}} u, \Lambda^{-\frac{1}{2}} [X_i, X_i^*] u\|$$

$$\leq \|X_i u\|^2 + C\|u\|_2^2.$$

To estimate $\|u\|_2^2$ by $C(\sum \|X_k u\|^2 + \|u\|^2)$ we will estimate $\|Du\|_2^2$ by $C(\sum \|X_k u\|^2 + \|u\|^2)$ for all first order operators $D$. Thus it suffices to estimate $Du$ when $D = X_i$ and when $D = [X_i, X_j]$. The estimate is clearly satisfied if $D = X_i$, if $D = [X_i, X_j]$ we have

$$\|X_i X_j u\|_2^2 = (X_i X_j u, \Lambda^{-1} [X_i, X_j] u) = (X_j X_i u, \Lambda^{-1} [X_i, X_j] u)$$

$$= (X_i X_j u, P^0 u) - (X_j X_i u, P^0 u);$$

the first term on the right is estimated by

$$(X_j X_i u, P^0 u) = (X_j u, X_i^* P^0 u) = -(X_j u, P^0 X_i^* u) + O(\|u\|^2 + \|X_j u\|^2)$$

$$\leq C(\|X_j u\| \|X_i^* u\| + \|u\|^2 + \|X_j u\|^2)$$

$$\leq 1.c. \sum (\|X_k u\|^2 + s.c. \|X_i^* u\|^2 + C\|u\|^2)$$

and the second term on the right is estimated similarly. Combining these we have

$$\|u\|_2^2 \leq C(\sum \|\frac{\partial u}{\partial x_i}\|_2^2 + \|u\|^2) \leq C(\sum \|X_k u\|^2 + \|u\|^2) + s.c. \|u\|_2^2;$$
hence
\[ \|u\|_2^2 \leq C \left( \sum \|X_k u\|^2 + \|u\|^2 \right) \]
which concludes the proof of theorem A.

3. The operators \( E_k \)

In this section we define the operators: \( L, \bar{L}, X_{1k}, X_2, \) and \( E_k \).

Let \( \mathcal{H} \) be the hypersurface in \( \mathbb{C}^2 \) given by:
\[ \Re(z_2) = -|z_1|^2. \]

We identify \( \mathbb{R}^3 \) with the Heisenberg group represented by \( \mathcal{H} \) using the mapping \( \mathcal{H} \to \mathbb{R}^3 \) given by \( x = \Re z_1, y = \Im z_1, t = \Im z_2 \). Let \( z = x + \sqrt{-1} y \). Let
\[ L = \frac{\partial}{\partial z_1} - 2\bar{z}_1 \frac{\partial}{\partial z_2} = \frac{\partial}{\partial z} + \sqrt{-1} \frac{\partial}{\partial t} \]
and
\[ \bar{L} = \frac{\partial}{\partial \bar{z}_1} - 2z_1 \frac{\partial}{\partial \bar{z}_2} = \frac{\partial}{\partial \bar{z}} - \sqrt{-1} \frac{\partial}{\partial t}. \]

Let \( X_{1k} \) and \( X_2 \) be the restrictions to \( \mathcal{H} \) of the operators
\[ X_{1k} = \bar{z}_1^k L = \bar{z}_1^k \frac{\partial}{\partial z} + \sqrt{-1} \bar{z}_1^{k+1} \frac{\partial}{\partial t}. \]

We set
\[ X_2 = \bar{L} = \frac{\partial}{\partial \bar{z}} - \sqrt{-1} \frac{\partial}{\partial t} \]
and
\[ E_k = X_{1k}^* X_{1k} + X_2^2 X_2 = -\bar{L}|z|^{2k} L - \bar{L}L. \]

By induction on \( j \) we define the commutators \( A^j_k \) setting \( A^1_k = [X_{1k}, X_2] \) and \( A^2_k = [A^{2-1}_k, X_2] \). Note that \( X_2, A^k_k \) and \( A^{k+1}_k \) span the tangent space of \( \mathbb{R}^3 \).

4. Loss of derivatives (part I)

In this section we prove that the subelliptic estimate (1) does not hold when \( k \geq 1 \). We also prove a proposition which gives the loss of derivatives in the sup norms which is part of Theorem B. To complete the proof of Theorem B, by establishing loss in the Sobolev norms, we will use additional microlocal analysis of \( E_k \), the proof of Theorem B is completed in Section 6.

**Definition 4.** If \( U \) is a neighborhood of the origin then \( \varphi \in C_0^\infty(U) \) is real-valued and is defined as follows \( \varphi(z, t) = \eta(z) \tau(t) \), where \( \eta \in C_0^\infty(\{ z \in \mathbb{C} \ | \ |z| < 2 \}) \) with \( \eta(z) = 1 \) when \( |z| \leq 1 \) and \( \tau \in C_0^\infty(\{ t \in \mathbb{R} \ | \ |t| < 2a \}) \) with \( \tau(t) = 1 \) when \( |t| \leq a \).
The following proposition shows that the subelliptic estimate (1) does not hold when \( k > 0 \).

**Proposition 1.** If \( k \geq 1 \) and if there exist a neighborhood \( U \) of the origin and constants \( s \) and \( C \) such that

\[
\|u\|_s^2 \leq C(\|z^j Lu\|^2 + \|\tilde{L} u\|^2),
\]

for all \( u \in C_0^\infty(U) \), then \( s \leq 0 \).

**Proof.** Let \( \lambda_0 \) and \( a \) be sufficiently large so that the support of \( g(\lambda z, t) \) lies in \( U \) when \( \lambda \geq \lambda_0 \). We define \( g_\lambda \) by

\[
g_\lambda(z, t) = g(\lambda z, t) \exp(-\lambda^\frac{1}{5}|z|^2 - it).
\]

Note that \( L\eta(z) = \tilde{L}\eta(z) = 0 \) when \( |z| \leq 1 \), that \( L(\tau) = iz\tau' \), and that \( \tilde{L}(\tau) = -iz\tau' \). Setting \( R^\lambda v(z, t) = v(\lambda z, t) \), we have:

\[
\begin{align*}
\tilde{z}^k L(g_\lambda) &= (\lambda\tilde{z}^k(R^\lambda L\eta)\tau + i\tilde{z}(R^\lambda \eta)\tau' + \lambda^\frac{1}{5}\tilde{z} R^\lambda g) \exp(-\lambda^\frac{1}{5}|z|^2 + it) \\
&= (\lambda(R^\lambda\tilde{L}\eta)\tau - iz(R^\lambda \eta)\tau') \exp(-\lambda^\frac{1}{5}|z|^2 + it).
\end{align*}
\]

Note that the restriction of \( |g_\lambda| \) to \( \mathcal{H} \) is

\[
|g_\lambda(z, t)| = g(\lambda z, t) \exp(-\lambda^\frac{1}{5}|z|^2).
\]

Now we have, using the changes of variables: first \((z, t) \mapsto (\lambda^{-1}z, t)\) and then \( z \mapsto \lambda^{-\frac{1}{5}}z\)

\[
\begin{align*}
\|g_\lambda\|^2 &= \frac{C}{\lambda^2} \int_{\mathbb{R}^2} \eta(z)^2 \exp(-2\lambda^\frac{1}{5}|z|^2) dxdy \\
&\geq \frac{C}{\lambda^2} \int_{\mathbb{R}^2} \exp(-2\lambda^\frac{1}{5}|z|^2) dxdy - \frac{C}{\lambda^2} \int_{|z| \geq 1} \exp(-2\lambda^\frac{1}{5}|z|^2) dxdy \\
&\geq \frac{C}{\lambda^\frac{1}{2}} - \frac{C}{\lambda^\frac{1}{2}} \exp(-\lambda^\frac{1}{5}) \int_{|z| \geq 1} \exp(-\lambda^\frac{1}{5}|z|^2) dxdy \\
&\geq \frac{C}{\lambda^\frac{1}{2}} - \frac{C}{\lambda^\frac{1}{2}} \exp(-\lambda^\frac{1}{5}).
\end{align*}
\]

Then we have

\[
\|g_\lambda\|^2 \geq \frac{\text{const.}}{\lambda^\frac{1}{5}}
\]

for sufficiently large \( \lambda \). Further, using the above coordinate changes to estimate the individual terms in (2) and in (3), we have

\[
\begin{align*}
\|\tilde{z}^k \lambda(R^\lambda L\eta)\tau \exp(-\lambda^\frac{1}{5}|z|^2 - it)\|^2 + \|\lambda(R^\lambda\tilde{L}\eta)\tau \exp(-\lambda^\frac{1}{5}|z|^2 - it)\|^2 &\leq C \exp(-\lambda^\frac{1}{5}) \int_{|z| \geq 1} \exp(-\lambda^\frac{1}{5}|z|^2) dxdy \leq \frac{C}{\lambda^\frac{1}{2}} \exp(-\lambda^\frac{1}{5}),
\end{align*}
\]
\[ \| |z|(R^\lambda \eta') \exp(-\lambda \hat{\xi}|z|)\|_2 \leq \frac{C}{\lambda^2} \int |z|^2 \exp(-2\lambda \hat{\xi}|z|) dxdy \leq \frac{C}{\lambda}, \]

and
\[ \| \lambda \hat{\xi} z^{k+1} R^\lambda g \exp(-\lambda \hat{\xi}|z|)\|_2 \leq C \lambda^{1-k} \int |z|^{2k+2} \exp(-2\lambda \hat{\xi}|z|) dxdy \leq \frac{C}{\lambda^{1-k}}. \]

Hence, if \( k \geq 1 \), we have
\[ \| \bar{z} L g \|_2 + \| \bar{L} g \|_2 \leq \frac{C}{\lambda^2}. \]

Since \( |x| \leq \frac{2}{\lambda} \) on the support of \( g^\lambda \) then we conclude, from the lemma proved below, that given \( \varepsilon \) there exists \( C \) such that
\[ \lambda^\varepsilon \| g^\lambda \| \leq C \| g \|_\varepsilon, \]
for sufficiently large \( \lambda \). It follows that, if \( k \geq 1 \) then the subelliptic estimate
\[ \| g^\lambda \|_2^\varepsilon \leq C(\| \bar{z} L g^\lambda \|_2 + \| \bar{L} g^\lambda \|_2) \]
implies that \( \lambda^{2\varepsilon - \frac{s}{2}} \leq C \lambda^{-\frac{s}{2}} \) which is a contradiction and thus the proposition follows. The following lemma then completes the proof. For completeness we include a proof which is along the lines given in [ChK].

**Lemma 1.** Let \( Q_\delta \) denote a bounded open set contained in the “slab” \( \{ x \in \mathbb{R}^n \mid |x_1| \leq \delta \} \). Then, for each \( \varepsilon > 0 \), there exists \( C = C(\varepsilon) > 0 \) such that
\[ \| u \| \leq C \delta^\varepsilon \| u \|_\varepsilon, \]
for all \( u \in C_0^\infty(Q_\delta) \) and \( \delta > 0 \).

**Proof.** Note that the general case follows from the case of \( n = 1 \). Since, writing \( x = (x_1, x') \), if for each fixed \( x' \) we have \( \| u(\cdot, x') \| \leq C \delta^\varepsilon \| u(\cdot, x') \|_\varepsilon \) then, after integrating with respect to \( x' \) and noting that \( (1 + |\xi|^2)^\varepsilon \leq (1 + |\xi|^2)^\varepsilon' \), we obtain the desired estimate. So we will assume that \( n = 1 \) and set \( x = x_1 \) and \( \xi = \xi_1 \). We define \( \| u \|_s \) by
\[ \| u \|_s^2 = \int |\xi|^{2s} \hat{u}(\xi)^2 d\xi. \]
We will show that, if \( s \geq 0 \), there exists a constant \( C \) such that
\[ \| u \|_s \leq C \| u \|_s, \]
for all \( u \in C_0^\infty((-1, 1)) \). First we have
\[ \| u(\xi) \| = \int e^{-ix\xi} u(x) dx \leq \sqrt{2} \| u \|. \]
Next, if $|\xi| \leq a \leq 1$,

$$(1 + \xi^2)^s |\hat{u}(\xi)|^2 \leq 2^{s+1} \|u\|^2$$

and

$$\int_{-\infty}^{\infty} (1 + \xi^2)^s |\hat{u}(\xi)|^2 d\xi$$

$$= \int_{|\xi| \leq a} \cdots + \int_{|\xi| > a} \cdots \leq 2^{s+2} a \|u\|^2 + \left(\frac{1}{a^2} + 1\right)^s \|u\|^2.$$ 

Hence if $a$ is small we obtain $\|u\|_s \leq C \|u\|_s$, as required. If $\text{supp}(u) \subset (-\delta, \delta)$ then set $u_\delta(x) = u(\delta x)$ so that $\text{supp}(u_\delta) \subset (-1, 1)$. Now

$$\hat{u}_\delta(\xi) = \frac{1}{\delta} \hat{u}\left(\frac{\xi}{\delta}\right)$$

so that $\|u_\delta\|^2 = \frac{1}{\delta^2} \|u\|^2$ and $\|\|u_\delta\|_s^2 = \delta^{2s-1} \|u\|_s^2$ which concludes the proof.

Next we prove that $E_k$ loses at least $k$ derivatives in the sup norms.

**Proposition 2.** If for some open sets $U$ and $U'$, with $\bar{U} \subset U'$, and for each $s_0$ there exists a constant $C = C(s_0)$ such that

$$\sum_{|\alpha| \leq p} \sup_{x \in U} |D^\alpha u(x)| \leq C \left( \sum_{|\beta| \leq q} \sup_{x \in U'} |D^\beta E_k u(x)| + \|u\|_{-s_0} \right),$$ 

for all $u \in C_0^\infty(\mathbb{R}^3)$, then $q \geq p + k$.

**Proof.** If $\delta > 0$ define $u_\delta$ by

$$u_\delta = (|z|^2 - \sqrt{-1}t)^p \log(|z|^2 + \delta - \sqrt{-1}t),$$

where $\log$ denotes the branch of the logarithm that takes reals into reals. Since $u_\delta$ is the restriction of $(-z_2)^p \log(-z_2 + \delta)$ to $\mathcal{F}$ we have $Lu_\delta = 0$. Then we have

$$\lim_{\delta \to 0} |D^p_\delta u_\delta(0)| = \infty.$$ 

Further

$$E_k u_\delta = -\bar{L}|z|^{2k} Lu_\delta$$

$$= 2k|z|^2 \left( -p(-z_2)^p-1 \log(-z_2 + \delta) + (-z_2)^p \log(-z_2 + \delta) + \frac{(-z_2)^p}{(-z_2 + \delta)} \right)$$

$$= 2k|z|^{2k} \left( p(|z|^2 - \sqrt{-1}t)^{p-1} \log(|z|^2 + \delta - \sqrt{-1}t) + \frac{(|z|^2 - \sqrt{-1}t)^p}{|z|^2 + \delta - \sqrt{-1}t} \right).$$
Note that $\|u_\delta\|_{-s_0}$ is bounded independently of $\delta$ when $s_0 \geq 3$. Thus, when $q \leq p + k - 1$, we have

$$\sum_{|\beta| \leq q} \sup_{x \in U'} |D^\beta E_k u_\delta(x)| \leq \text{const.}$$

with the constant independent of $\delta$. Hence, applying (5) to $u_\delta$ we obtain $q \geq p + k$. This concludes the proof of the proposition.

5. Notation

In this section we set down some notation which will be used throughout the rest of the paper.

1. Associated to the cutoff function $\varrho$ defined in Definition 1, is a $C^\infty$ function $\mu$ such that $L\varrho = \bar{z}\mu$ and $\bar{L}\varrho = \bar{z}\bar{\mu}$ (Such a $\mu$ exists since $L\varrho(z,t) = D_z\eta(z)\tau(t) + i\bar{z}\eta(z)D_t\tau(t)$.) Since $D_z\eta(z) = 0$ in a neighborhood of $z = 0$ we can set $\mu(z,t) = D_z\eta(z)\bar{\mu}(t) + i\eta(z)D_t\tau(t)$.)

2. Given cutoff functions $\varrho, \varrho'$, as in Definition 1, with $\varrho' = 1$ in a neighborhood of the support of $\varrho$, then we denote by $\{\varrho_i\}$ a special sequence of cutoff functions, each of which satisfies Definition 1 and such that: $\varrho_1 = \varrho$, $\varrho' = 1$ in a neighborhood of $\bigcup \varrho_i$, and $\varrho_{i+1} = 1$ in a neighborhood of the support of $\varrho_i$.

3. The abbreviations “s.c.” and “l.c.” will be used for “small constant” and “large constant”, respectively in the following sense. $A_u \leq \text{s.c.}B_u + \text{l.c.}C_u$ means that given any constant s.c. there exists a constant l.c. such that the inequality holds for all $u$ in some specified class.

4. We will use $\|u\|_{-\infty}$ to denote the following. Given $A_u$, the expression $A_u \leq \|u\|_{-\infty}$ means that: if for any $s_0$ there exist a constant $C = C(s_0)$ such that $A_u \leq C\|u\|_{-s_0}$ holds for all $u$ in some specified class.

6. Microlocalization on the Heisenberg group

Denote by $T$ the vector field defined by

$$T = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial t}.$$ 

Then

$$[L, \bar{L}] = \left[ \frac{\partial}{\partial z} + \sqrt{-1} \bar{z} \frac{\partial}{\partial \bar{t}}, \frac{\partial}{\partial \bar{z}} - \sqrt{-1} z \frac{\partial}{\partial t} \right] = 2T.$$
The following simple formula, which is obtained by integration by parts, is the starting point of all the estimates connected with the operators $E_k$.

**Lemma 2.** For $u \in C_0^\infty(\mathbb{R}^3)$ we have

\[ \|Lu\|^2 = 2(Tu, u) + \|Lu\|^2. \]

**Proof.** Since $L^* = -\bar{L}$ and $\bar{L}^* = -L$, we have

\[ \|Lu\|^2 = (Lu, Lu) = -(\bar{L}Lu, u) = -(\bar{L}Lu, u) = 2(Tu, u) + \|Lu\|^2. \]

We set $x_1 = x_2 = 3z$, and $x_3 = t$ and denote the dual coordinates by $\xi_1, \xi_2$, and $\xi_3$. For $(\alpha, t_0) \in \mathbb{C} \times \mathbb{R}$ we define

\[ z^\alpha = z - \alpha \text{ and } x^\alpha_3 = -2\alpha_2 x_1 + 2\alpha_1 x_2 + x_3 - t_0, \]

where $\alpha_1 = \Re \alpha$ and $\alpha_2 = \Im \alpha$. Then

\[ L = \frac{\partial}{\partial z^\alpha} + iz^\alpha \frac{\partial}{\partial x^\alpha_3} \]

and

\[ \bar{L} = \frac{\partial}{\partial z^\alpha} - iz^\alpha \frac{\partial}{\partial x^\alpha_3}. \]

We set $x^\alpha_1 = x_1 - \alpha_1$, $x^\alpha_2 = x_2 - \alpha_2$, and $x^\alpha = (x^\alpha_1, x^\alpha_2, x^\alpha_3)$. Let $\mathcal{F}_\alpha$ denote the the Fourier transform in the $x^\alpha$ coordinates; that is

\[ \mathcal{F}_\alpha u(\xi) = \int e^{-ix^\alpha \cdot \xi} u(x^\alpha) dx_1^\alpha x_2^\alpha x^\alpha_3. \]

**Definition 5.** Let $S^2 = \{\xi \in \mathbb{R}^3 | |\xi| = 1\}$ be the unit sphere. Suppose that $\mathcal{U}, \mathcal{U}_1$ are open subsets of $S^2$ with $\mathcal{U}_1 \subset \mathcal{U}$. For each such pair of open sets we define a set of $\gamma \in C^\infty(\mathbb{R}^3)$, with $\gamma \geq 0$, such that

1. $\gamma(\frac{\xi}{|\xi|}) = \gamma(\xi)$ when $|\xi| \geq 1$.
2. $\gamma(\xi) = 1$ when $\xi \in \mathcal{U}_1$.
3. $\gamma(\xi) = 0$ when $\xi \in S^2 - \mathcal{U}$.

To such a $\gamma$ and $\alpha \in \mathbb{C}$ we associate the operator $\Gamma_\alpha$ defined by

\[ \mathcal{F}_\alpha \Gamma_\alpha u(\xi) = \gamma(\xi) \mathcal{F}_\alpha u(\xi). \]

Let $\mathcal{U}^+, \mathcal{U}_1^+, \mathcal{U}_0, \mathcal{U}_1^0, \mathcal{U}^-$, and $\mathcal{U}_1^-$ be open subsets of $S^2$ defined as follows.

\[ \mathcal{U}^+ = \{\xi \in S^2 | \xi_3 > \frac{5}{9}\}, \quad \mathcal{U}_1^+ = \{\xi \in S^2 | \xi_3 > \frac{4}{9}\}, \]

\[ \mathcal{U}_0 = \{\xi \in S^2 | |\xi_3| < \frac{5}{6}\}, \quad \mathcal{U}_1^0 = \{\xi \in S^2 | |\xi_3| < \frac{2}{3}\}, \]

\[ \mathcal{U}^- = \{\xi \in S^2 | -\xi \in \mathcal{U}^+\}, \text{ and } \mathcal{U}_1^- = \{\xi \in S^2 | -\xi \in \mathcal{U}_1^+\}. \]
We denote by $\gamma^+$, $\gamma^0$, and $\gamma^-$ the corresponding functions and require further that $\gamma^+(\xi) = \gamma^-(\xi) = 0$ when $|\xi| \leq \frac{1}{2}$ and $\gamma^0(\xi) = 1$ when $\frac{1}{2} < |\xi| \leq 1$. The sets of these functions will be denoted by $\mathcal{G}^+$, $\mathcal{G}^0$, and $\mathcal{G}^-$, respectively. The corresponding operators are denoted by $\Gamma^+_\alpha$, $\Gamma^0_\alpha$, and $\Gamma^-_\alpha$. The sets of these operators will be denoted by $\mathfrak{S}^+_\alpha$, $\mathfrak{S}^0_\alpha$, and $\mathfrak{S}^-_{\bar{\alpha}}$, respectively. Given $(\alpha, t_0) \in \mathbb{C} \times \mathbb{R}$ the functions $\Gamma^+_\alpha u$, $\Gamma^0_\alpha u$, and $\Gamma^-_\alpha u$ will be referred to as microlocalizations of $u$ at $(\alpha, t_0)$ in the regions $+$, $0$, and $-$, respectively.

The following lemma shows that the 0 microlocalization is elliptic for the operators $L$ and $\bar{L}$. In our estimates we will often encounter error terms which can be bounded by $C_{s_0} \| u \|_{-s_0}$ for every $s_0$; abusing notation we will bound such terms by “$\| u \|_{-\infty}$”.

**Lemma 3.** If $U$ is a neighborhood of $(\alpha, t_0)$ and if $\gamma^0, \tilde{\gamma}^0 \in \mathcal{G}^0$ with $\gamma^0 = 1$ in a neighborhood of the support of $\gamma^0$ then there exist constants $a > 0$ and $C > 0$ such that, if $|z - \alpha| < a$ on $U$, then

$$\| \Gamma^0_\alpha u \|_1 \leq C(\| \Gamma^0_\alpha L u \| + \| \tilde{\Gamma}^0_\alpha u \| + \| u \|_{-\infty})$$

and

$$\| \Gamma^0_\alpha u \|_1 \leq C(\| \Gamma^0_\alpha \bar{L} u \| + \| \tilde{\Gamma}^0_\alpha u \| + \| u \|_{-\infty}),$$

for all $u \in C^\infty_0(U)$.

**Proof.** If $\xi \in \mathcal{U}^0$ and if $|\xi| \geq 1$ then $|\xi_3| \leq \frac{5}{6}|\xi|$. Thus, if $\xi \in \mathcal{U}^0$, then $|\xi| \leq 6(|\xi_1| + |\xi_2|) + 1$. Now,

$$\| \Gamma^0_\alpha u \|_1^2 \leq C\left( \sum_{1}^{2} \| \frac{\partial}{\partial x_j} \Gamma^0_\alpha u \|^2 + \| \tilde{\Gamma}^0_\alpha u \|^2 + \| u \|^2_{-\infty} \right).$$

Let $U' \supset U$ be an open set such that $|z - \alpha| > 2a$ on $U'$ and let $\varphi \in C^\infty_0(U')$ satisfying $\varphi = 1$ in a neighborhood of $U$. Then

$$\| \Gamma^0_\alpha u \|_1^2 \leq C\left( \sum_{1}^{2} \| \frac{\partial}{\partial x_j} \Gamma^0_\alpha \varphi u \|^2 + \| \tilde{\Gamma}^0_\alpha u \|^2 + \| u \|^2_{-\infty} \right)$$

$$\leq C'\left( \sum_{1}^{2} \| \frac{\partial}{\partial x_j} \varphi \Gamma^0_\alpha u \|^2 + \| \tilde{\Gamma}^0_\alpha u \|^2 + \| u \|^2_{-\infty} \right)$$

$$\leq C''(\| \varphi \Gamma^0_\alpha u \|^2 + \| \tilde{\varphi} \Gamma^0_\alpha u \|^2$$

$$+ \max_{U'} |z - \alpha|^2 \| \frac{\partial}{\partial x_3} \Gamma^0_\alpha u \|^2 + \| \tilde{\Gamma}^0_\alpha u \|^2 + \| u \|^2_{-\infty})$$

$$\leq C''(\| \Gamma^0_\alpha L u \|^2 + \| \Gamma^0_\alpha \bar{L} u \|^2 + 4a^2 \| \Gamma^0_\alpha u \|^2 + \| \tilde{\Gamma}^0_\alpha u \|^2 + \| u \|^2_{-\infty}).$$

Hence, taking $a$ suitably small we obtain

$$\| \Gamma^0_\alpha u \|_1 \leq C(\| \Gamma^0_\alpha L u \|^2 + \| \Gamma^0_\alpha \bar{L} u \|^2 + \| \tilde{\Gamma}^0_\alpha u \|^2 + \| u \|^2_{-\infty}).$$
Furthermore, substituting \( \varphi \Gamma_0^a u \) for \( u \) in (6), we have
\[
\| L \varphi \Gamma_0^a u \|^2 = 2(T \varphi \Gamma_0^a u, \varphi \Gamma_0^a u) + \| L \varphi \Gamma_0^a u \|^2
\]
\[
\leq \text{s.e.} \| \frac{\partial}{\partial x_3} \Gamma_0^a u \|^2 + 1.c. \| \tilde{\Gamma}_0^a u \|^2 + \| u \|^2_{-\infty} + C \| \Gamma_0^a \tilde{L} u \|^2
\]
\[
\leq \text{s.e.} \| \Gamma_0^a u \|^2 + 1.c. \| \tilde{\Gamma}_0^a u \|^2 + \| u \|^2_{-\infty} + C \| \Gamma_0^a \tilde{L} u \|^2,
\]
and since
\[
\| L \varphi \Gamma_0^a u \|^2 \leq C \| \Gamma_0^a Lu \|^2 + \| \tilde{\Gamma}_0^a u \|^2 + \| u \|^2_{-\infty}
\]
we get
\[
\| \Gamma_0^a u \|_1 \leq C(\| \Gamma_0^a \tilde{L} u \| + \| \tilde{\Gamma}_0^a u \|^2 + \| u \|^2_{-\infty}).
\]
Similarly we obtain
\[
\| \Gamma_0^a u \|_1 \leq C(\| \tilde{\Gamma}_0^a \tilde{L} u \| + \| \Gamma_0^a u \|^2 + \| u \|^2_{-\infty}).
\]
This completes the proof of the lemma.

**Lemma 4.** If \( R^s \) is a pseudodifferential operator of order \( s \) then there exists \( C \) such that
\[
\| [R^s, \Gamma_0^+] u \| \leq C(\| \Gamma_0^a u \|_{s-1} + \| u \|_{-\infty})
\]
and
\[
\| [R^s, \Gamma_0^-] u \| \leq C(\| \Gamma_0^a u \|_{s-1} + \| u \|_{-\infty}).
\]

**Proof.** Since \( \gamma^0 = 1 \) on a neighborhood of the support of the derivatives of \( \gamma^+ \) it also equals one on a neighborhood of the support of the symbol of \( [R^s, \Gamma_0^+] \). Hence \( [R^s, \Gamma_0^+] = [R^s, \Gamma_0^+] \Gamma_0^a + R^{-\infty} \), where \( R^{-\infty} \) is a pseudodifferential operator whose symbol is identically zero. The same argument works for the term \( [R^s, \Gamma_0^-] \) and the lemma follows.

**Definition 6.** For each \( s \in \mathbb{R} \) we define the operator \( \Psi^a \) as follows. Let \( U^* \) and \( U_1^* \) be open sets in \( S^2 \) such that \( U^* = \{ \xi \in S^2 \mid |\xi_3| > \frac{1}{2} \} \) and \( U_1^* = \{ \xi \in S^2 \mid |\xi_3| > \frac{1}{3} \} \). Let \( \gamma^* \) be the function on \( \mathbb{R}^3 \) associated with \( U^* \), \( U^* \) such that \( \gamma^*(\xi) = 0 \) when \( |\xi| \leq \frac{1}{3} \) and \( \gamma^*(\xi) = 1 \) in the region \( \{ \xi \in \mathbb{R}^3 \mid \frac{1}{3} \leq |\xi_3| \leq \frac{1}{2} \} \). Then we set \( \psi^a(\xi) = (1 + |\xi_3|^2)^{\gamma^*}(\xi) \) and define \( \Psi^a \) by
\[
\mathcal{F}_a \Psi^a u(\xi) = \psi^a(\xi) \mathcal{F}_a u(\xi).
\]
Note that there exist positive constants \( c \) and \( C \) such that
\[
c(1 + |\xi|^2)^{\gamma^*}(\xi) \leq \psi^a(\xi) \leq C(1 + |\xi|^2)^{\gamma^*}(\xi).
\]
Hence \( \| \Psi^a \Gamma_0^au \| \sim \| \gamma^+ u \|_s \) and \( \| \Psi^a \Gamma_0^-u \| \sim \| \gamma^- u \|_s \); by \( \sim \) we mean that they differ by an operator of order \( -\infty \). Also, since \( \gamma^* = 1 \) on the supports of \( \gamma^+ \) and \( \gamma^- \), we have
\[
\Psi^s \Psi^s \gamma^+ \sim \Psi^{s+s^*} \gamma^+ \quad \text{and} \quad \Psi^s \Psi^s \gamma^- \sim \Psi^{s+s^*} \gamma^-.
\]
Lemma 5. There exists $C$ such that
\[ \|\Gamma^+_\alpha Lu\|^2 + \|\Gamma^+_\alpha u\|^2 \leq C(\|\Gamma^+_\alpha Lu\|^2 + \|\Gamma^+_\alpha u\|^2), \]
and
\[ \|\Gamma^-_\alpha Lu\|^2 + \|\Gamma^-_\alpha u\|^2 \leq C(\|\Gamma^-_\alpha Lu\|^2 + \|\Gamma^-_\alpha u\|^2), \]
for all $u \in C_0^\infty(U)$.

Proof. Taking $\varphi \in C_0^\infty$ with $\varphi = 1$ in a neighborhood of $\bar{U}$ we substitute $\varphi\Gamma^+_\alpha u$ for $u$ in (6) and obtain
\[ \|L\varphi\Gamma^+_\alpha u\|^2 = 2(T\varphi\Gamma^+_\alpha u, \varphi\Gamma^+_\alpha u) + \|\bar{L}\varphi\Gamma^+_\alpha u\|^2. \]
Now, we have
\[ (T\varphi\Gamma^+_\alpha u, \varphi\Gamma^+_\alpha u) = (TT\varphi_u, \varphi\Gamma^+_\alpha u) + O(\Gamma^+_\alpha u\|^2 + \|u\|^{-\infty}). \]
Since $F_\alpha(Tu) = \xi_3F_\alpha(u)$ we have $TT\varphi_u \sim \Psi^1\Gamma^+_\alpha \sim \Psi^\frac{1}{2}\Psi^\frac{1}{2}\Gamma^+_\alpha$ and
\[ (T\varphi\Gamma^+_\alpha u, \varphi\Gamma^+_\alpha u) = \|\Psi^\frac{1}{2}\Gamma^+_\alpha u\|^2 + O(\Gamma^+_\alpha u\|^2 + \|u\|^{-\infty}). \]
This proves the first part of the lemma, the second follows from the fact that $|\xi_3|\gamma^-(-\xi) = -\xi_3\gamma^+(-\xi)$. Then $\Psi^1\Gamma^-_\alpha \sim \Psi^\frac{1}{2}\Psi^\frac{1}{2}\Gamma^-_\alpha$, thus concluding the proof.

7. Loss of derivatives (part II)
Conclusion of the proof of Theorem B

In this section we conclude the proof of Theorem B by showing that if $k \geq 2$ then $E_k$ loses at least $k - 1$ derivatives in the Sobolev norms.

Proposition 3. Suppose that there exist two neighborhoods of the origin $U$ and $U'$, with $\bar{U} \subset U'$, and real numbers $s_1$ and $s_2$ such that if $g, g' \in C_0^\infty(U')$ with $g = 1$ on $U$ and $g' = 1$ in a neighborhood of the support of $g$, and if for any real number $s_0$ there exists a constant $C = C(g, g', s_0)$ such that
\[ \|g\varphi\|_{s_1} \leq C(\|g'\varphi\|_{s_2} + \|u\|_{-s_0}), \]
for all $u \in S$, then $s_2 \geq s_1 + k - 1$. Here $S$ denotes the Schwartz space of rapidly decreasing functions.

Proof. Let $\{\varphi_i\}$ and $\{\varphi'_i\}$ be sequences of cutoff functions in $C_0^\infty(U)$ and $C_0^\infty(U')$, respectively. We assume that $g(z, t) = \eta(|z|)\tau_i(t)$ and $g'_i(z, t) = \eta'_i(|z|)\tau'_i(t)$ as in Definition 1. We further assume that $\theta_0 = \varphi, \theta'_0 = g', \theta_{i+1} = 1$ in a neighborhood of the support of $\varphi_i$, and $g'_{i+1} = 1$ in a neighborhood of the support of $g'_i$ and that the $\eta_i(|z|)$ are monotone decreasing in $|z|$. We also choose $\gamma^+_i$ and $\gamma^0_i$ such that $\gamma^+_i \in G^+, \gamma^+_i = 1$, and $\gamma^0_i \in G^0$ and $\gamma^0_i = 1$ in neighborhoods of the supports of $\gamma^+_i$ and $\gamma^0_i$, respectively. Further we require
that $\gamma_i^0 = 1$ in a neighborhood of the support of derivatives of $\gamma_i^+$. Substituting $\Psi^{-s_1} \Gamma_0^+ u$ for $u$ in (7), replacing $s_0 + s_1$ by $s_0$, we have
\[
\| \varrho \Psi^{-s_1} \Gamma_0^+ u \|_{s_1} \leq C(\| \varrho' E_k \Psi^{-s_1} \Gamma_0^+ u \|_{s_2} + \| u \|_{-s_0}).
\]
Since $\gamma_1^+ \varrho_0^+ = \varrho_0^+$,
\[
\| \varrho \Psi^{-s_1} \Gamma_0^+ u \|_{s_1} = \| \Psi^{s_1} \Gamma_1^+ \varrho \Psi^{-s_1} \Gamma_0^+ u \| + O(\| u \|_{-1})
= \| \varrho \Gamma_0^+ u \| + \| \Psi^{s_1} \Gamma_1^+ [\varrho, \Psi^{s_1}] \Gamma_0^+ u \| + O(\| u \|_{-s_0}).
\]
Furthermore, $\Psi^{s_1} \Gamma_1^+ [\varrho, \Psi^{s_1}] \Gamma_0^+$ is an operator of order $-1$; hence we get
\[
\| \Psi^{s_1} \Gamma_1^+ [\varrho, \Gamma_0^+] \Psi^{-s_1} u \| \leq C(\| u \|_{-s_0})
\]
and
\[
\| \varrho \Gamma_0^+ u \| \leq C(\| \varrho' E_k \Psi^{-s_1} \Gamma_0^+ u \|_{s_2} + \| u \|_{-s_0}).
\]
Next we have
\[
\| \varrho' E_k \Psi^{-s_1} \Gamma_0^+ u \|_{s_2} \leq \| \Psi^{s_2-s_1} \Gamma_0^+ \varrho' E_k u \| + \| \varrho' E_k, \Psi^{-s_1} \Gamma_0^+ \| u \|_{s_2}.
\]
Since the symbol of $\gamma_1^0 \gamma_1^+ \varrho_1^+ = 1$ in a neighborhood of the symbol of $[\varrho' E_k, \Psi^{-s_1} \Gamma_0^+]$ and since the order of $[\varrho' E_k, \Psi^{-s_1} \Gamma_0^+]$ is $-s_1 + 1$, we have
\[
\| \| \varrho' E_k, \Psi^{-s_1} \Gamma_0^+ \| \| u \|_{-s_0} \leq C(\| \varrho_1^0 \Gamma_1^+ u \|_{s_2-s_1+1} + \| u \|_{-s_0}).
\]
Applying Proposition 3, we have
\[
\| \varrho_1^0 \Gamma_1^+ u \|_{s_2-s_1+1} \leq C(\| \varrho_1^0 E_k \Gamma_1^+ u \|_{s_2-s_1-1} + \| \varrho_1^0 \Gamma_1^+ u \|_{s_2-s_1} + \| u \|_{-\infty})
\]
so that
\[
\| \varrho' E_k \Psi^{-s_1} \Gamma_0^+ u \|_{s_2} \leq C(\| \Psi^{s_2-s_1} \Gamma_0^+ \varrho' E_k u \| + \| \varrho_1^0 \Gamma_1^+ u \|_{s_2-s_1-1} + \| \varrho_1^0 E_k \Gamma_1^+ u \|_{s_2-s_1-1} + \| u \|_{-s_0}).
\]
Therefore
\[
\| \varrho \Gamma_0^+ u \| \leq C(\| \Psi^{s_2-s_1} \Gamma_0^+ \varrho' E_k u \|
+ \| \varrho_1^0 E_k \Gamma_1^+ u \|_{s_2-s_1-1} + \| \varrho_1^0 E_k \Gamma_1^+ u \|_{s_2-s_1-1} + \| u \|_{-s_0}).
\]
Now we have
\[
\| \varrho_1^0 E_k \Gamma_1^+ u \|_{s_2-s_1-1} \leq \| \Psi^{s_2-s_1-1} \Gamma_1^+ \varrho_1^0 E_k u \| + \| \varrho_1^0 E_k, \Gamma_1^+ \| u \|_{s_2-s_1-1},
\]
again since $[\varrho_1^0 E_k, \Gamma_1^+]$ is an operator of order one and since $\varrho_2^0 \gamma_2^0 = 1$ in a neighborhood of its symbol, we get
\[
\| \| \varrho_2^0 E_k, \Gamma_1^+ \| \| u \|_{s_2-s_1-1} \leq C(\| \varrho_2^0 \Gamma_2^+ u \|_{s_2-s_1} + \| u \|_{-s_0}).
\]
Then, again applying Proposition 3, we have
\[
\| \varrho_2^0 \Gamma_2^+ u \|_{s_2-s_1} \leq C(\| \varrho_3^0 \Gamma_3^+ E_k \Gamma_2^+ u \|_{s_2-s_1-2} + \| u \|_{-\infty})
\]
so that
\[ \| \phi^+_2 E_k \Gamma^+_1 u \|_{s_2 - s_1 - 1} \leq C (\| \Psi^{s_2 - s_1 - 1} \Gamma^+_1 \phi^+_2 E_k u \| + \| \phi^+_3 \Gamma^+_0 E_k \Gamma^+_2 u \|_{s_2 - s_1 - 2} + \| u \|_{-s_0}). \]

Hence
\[ \| \phi^+_1 u \| \leq C (\| \Psi^{s_2 - s_1} \Gamma^+_0 \phi^+_1 E_k u \|
+ \| \Psi^{s_2 - s_1 - 1} \Gamma^+_1 \phi^+_2 E_k u \|
+ \| \phi^+_3 \Gamma^+_0 E_k \Gamma^+_2 u \|_{s_2 - s_1 - 2} + \| u \|_{-s_0}). \]

Proceeding inductively we obtain
\[ \| \phi^+_i u \| \leq C \left( \sum_{i=0}^{N} \| \Psi^{s_2 - s_1 - i} \Gamma^+_i \phi^+_i E_k u \|
+ \| \phi^+_N \Gamma^+_N E_k u \|_{s_2 - s_1 - N - 1} + \| u \|_{-s_0}. \right) \]

Since \( \| \Psi^{s_2 - s_1 - i} \Gamma^+_i \phi^+_i E_k u \| \) can be incorporated in the successive terms, we get, by choosing \( N \geq s_2 - s_1 + 1 - s_0 \)
\[ \| \phi^+_i u \| \leq C \left( \sum_{i=0}^{N} \| \Psi^{s_2 - s_1 - i} \Gamma^+_i \phi^+_i E_k u \| + \| u \|_{-s_0}. \right) \]

Let \( \tilde{\tau} \in C^\infty \) with \( \tilde{\tau} = 1 \) on the support of \( \tau^+_N \); then \( \tau^+_i E_k u = \tau^+_i E_k \tilde{\tau} u \) when \( i \leq N \) so that replacing \( u \) by \( \tilde{\tau} u \) we obtain
\[ \| \phi^+_i \tilde{\tau} u \| \leq C \left( \sum_{i=0}^{N} \| \Psi^{s_2 - s_1 - i} \Gamma^+_i \phi^+_i \tilde{\tau} u \| + \| \tilde{\tau} u \|_{-s_0}. \right) \]

Hence, since \( \gamma^0 = 1 \) in a neighborhood of the support of the symbol of \( [\Gamma^+, \tilde{\tau}] \) and thus can be incorporated in the estimate as above, we have
\[ \| \phi^+_i u \|^2 \leq C \left( \sum_{i=0}^{N} \| \Psi^{s_2 - s_1 - i} \Gamma^+_i \phi^+_i E_k u \|^2 + \| \tilde{\tau} u \|_{-s_0}^2. \right) \]

Choosing \( \tilde{\gamma}^+ \) so that \( \tilde{\gamma}^+ = 1 \) in a neighborhood of the supports of the \( \gamma^+_i \), we have \( \tilde{\gamma}^+ = 1 \) in a neighborhood of the support of the symbol of \( \Psi^{s_2 - s_1 - i} \Gamma^+_i \phi^+_i E_k. \)

Then we obtain
\[ \| \phi^+_i u \|^2 \leq C \left( \sum_{i=0}^{N} \| \Psi^{s_2 - s_1 - i} \Gamma^+_i \tilde{\gamma}^+ E_k u \|^2 + \| \tilde{\tau} u \|_{-s_0}^2. \right) \]

We define \( h_\lambda \) by
\[ h_\lambda(z, t) = \exp \left( -\lambda^2 (|z|^2 - it) \right), \]

since \( \tilde{\tau} h_\lambda \in S \) and obtain
\[ \| \phi^+_i \tilde{\tau} h_\lambda \| \leq C \left( \| \Psi^{s_2 - s_1} \Gamma^+_i \tilde{\tau} E_k h_\lambda \| + \| \tilde{\tau} h_\lambda \|_{-s_0}. \right) \].
Assuming that \( \eta(|z|) \) is monotone decreasing we have \( \eta(|z|) \geq \eta(\lambda|z|) \); hence, setting \( \eta_\lambda(z) = \eta(\lambda|z|) \), we obtain

\[
\|g \Gamma^+ h_\lambda\| \geq \|\eta_\lambda \tau \Gamma^+ h_\lambda\|.
\]

Then, setting \( x' = (x_1, x_2) \), \( y' = (y_1, y_2) \), and \( \xi' = (\xi_1, \xi_2) \) and changing variables \( \lambda y' \rightarrow y' \), \( \xi' \rightarrow \lambda \xi' \), and \( \xi_3' \rightarrow \xi_3' + \lambda^2 \), we get

\[
\eta_\lambda \tau \Gamma^+ h_\lambda(x) = \int \exp(i(x - y) \cdot \xi) \tau(y_3) \gamma^+(\xi) \exp(-\lambda^2(|y'|^2 - iy_3)) dyd\xi
\]
\[
= \int \exp(i(x' - y') \cdot \xi' + x_3 \xi_3 - y_3(\xi_3 - \lambda^2)) \tau(y_3) \gamma^+(\xi) \exp(-\lambda^2|y'|^2) dyd\xi
\]
\[
= \lambda^{-2} \int \exp(i(\lambda x' - y') \cdot \xi')
\]
\[
+ (x_3 - y_3) \tau(y_3) \gamma^+(\lambda \xi', \lambda^2 + \xi_3) \exp(-|y'|^2) dyd\xi.
\]

Making the change of variables \( \lambda x' \rightarrow x' \) we have

\[
\|\eta_\lambda \tau \Gamma^+ h_{\lambda, \delta}\|^2
\]
\[
= \frac{1}{\lambda^6} \int \left| \int \exp(i(x - y) \cdot \xi) \tau(y_3) \gamma^+(\lambda \xi', \lambda^2 + \xi_3) \exp(-|y'|^2) dyd\xi \right|^2 dx.
\]

Given \( (\xi', \xi_3) \in \text{supp}(\gamma^+) \) we have

\[
\lim_{\lambda \rightarrow \infty} \left| \frac{\lambda \xi'}{\xi_3 + \lambda^2} \right| = 0,
\]

and there exists \( \tilde{\lambda} \) such that \( \gamma^+(\lambda \xi', \lambda^2 + \xi_3) = 1 \) when \( \lambda \geq \tilde{\lambda} \). Hence we have \( \lim_{\lambda \rightarrow \infty} \gamma^+(\lambda \xi', \lambda^2 + \xi_3) = 1 \); thus there exist \( \lambda_0 \) such that

\[
\|\eta_\lambda \tau \Gamma^+ h_\lambda\|^2 \geq \frac{1}{2\lambda^6} \int \left| \int \exp(i(x - y) \cdot \xi) \tau(y_3) \exp(-|y'|^2) dyd\xi \right|^2 dx,
\]

when \( \lambda \geq \lambda_0 \), therefore there exists \( C \) independent of \( \lambda \) such that

\[
\|g \Gamma^+ h_\lambda\| \geq \frac{C}{\lambda^3},
\]

when \( \lambda \geq \lambda_0 \).

Next, we will estimate the term \( \|\tilde{\tau} h_\lambda\|_{-s_0} \). We will use the facts that \( \frac{1}{m!}\tilde{L}^m(\tilde{z}^m) = 1 \) and that \( \tilde{L}(h_\lambda) = 0 \). Taking \( m \leq s_0 \), we have

\[
\mathcal{F}(\Lambda^{-s_0} \tilde{h}_\lambda)(\xi) = \int (1 + |\xi|^2)^{\frac{m}{2}} \tilde{\tau}(x_3) \exp(-i(x \cdot \xi - \lambda^2 x_3) - \lambda^2 |z|^2) dx
\]
\[
= \frac{1}{m!} \int \tilde{L}^m(\tilde{z}^m)(1 + |\xi|^2)^{\frac{m}{2}} \tilde{\tau}(x_3) \exp(-i(x \cdot \xi - \lambda^2 x_3) - \lambda^2 |z|^2) dx
\]
\[
= \frac{1}{m!} \int \tilde{z}^m(1 + |\xi|^2)^{\frac{m}{2}} \tilde{L}^m(\tilde{\tau}(x_3) \exp(-i(x \cdot \xi - \lambda^2 x_3) - \lambda^2 |z|^2) dx
\]
\[
= -\frac{1}{m!} \int z^m (1 + |\xi|^2)^{\frac{m}{2}} \exp(-\lambda^2 |z|^2 - ix_3) \bar{L}^m (\tau(x_3) \exp(-i x \cdot \xi)) dx
= -\frac{1}{m!} \int z^m (1 + |\xi|^2)^{\frac{m}{2}} \exp(-\lambda^2 |z|^2) \bar{L}^m (\tau(x_3) \\
\cdot \exp(-ix' \cdot \xi' - ix_3 (\xi_3 - \lambda^2)) dx
\]
and
\[
\bar{L}^m (\tau(x_3) \exp(-ix' \cdot \xi' + i - ix_3 (\xi_3 - \lambda^2))
= \sum_{j=0}^{m} a_j (x_3) z^j (i \xi_1 + 2 \lambda \xi_3) \exp(-ix' \cdot \xi' - ix_3 (\xi_3 - \lambda^2)).
\]
Thus, setting \(w^{(m)}(x, \xi) = \sum_{j=0}^{m} a_j (x_3) z^j (i \xi_1 + 2 \lambda \xi_3) \exp(-ix' \cdot \xi' - ix_3 (\xi_3 - \lambda^2))\) and denoting the corresponding pseudodifferential operator by \(W^{(m)}\), we have
\[
||\tau h_\lambda||_{-s_a} = C||W^{(m)} h_\lambda||_{-s_a} \leq C||z^m \bar{\tau}' h_\lambda||_{m-s_a} \leq C||z^m \bar{\tau}' h_\lambda||,
\]
where \(\bar{\tau}' \in C^\infty_0 (\mathbb{R})\) and \(\bar{\tau}' = 1\) in a neighborhood of the support of \(\bar{\tau}\). Now, changing coordinates \(\lambda z \rightarrow z\), we get
\[
||z^m \bar{\tau}' h_\lambda||^2 = \int |z|^{2m} \bar{\tau}' (x_3)^2 \exp(-2\lambda^2 |z|^2) dx \leq \frac{C}{\lambda^{2m+2}}.
\]
To estimate the remaining terms we have
\[
E_k h_\lambda (z, t) = -2(k + 1)^2 |z|^{2k} h_\lambda (z, t).
\]
Therefore, with the coordinate change \(\lambda x' \rightarrow x'\), we get
\[
\mathcal{F} (\Psi^s \Gamma^+ \tau E_k h_\lambda) (\xi)
= C \mathcal{F} (\Psi^s \Gamma^+ \lambda^2 |z|^{2k} h_\lambda) (\xi)
= \lambda^{-2} (1 + \xi_3^2) \bar{\gamma}^+ (\xi) \mathcal{F} \left( (\tau(x_3) |z|^{2k} \exp(-\lambda^2 |z|^2)) (\xi') (\xi_3 - \lambda^2)\right)
= \lambda^{-2} \mathcal{F} \left( (1 + \xi_3^2) \bar{\gamma}^+ (\xi) \bar{\tau}(\xi_3 - \lambda^2) \right)(\xi') (\lambda^{-1} \xi').
\]
Then, integrating and making the changes of coordinates \(\xi' \rightarrow \lambda \xi', \xi_3 \rightarrow \xi_3 + \lambda^2\), we get
\[
||\Psi^s \Gamma^+ \tau E_k h_\lambda||^2
\leq C \lambda^{-4k-4} \int (1 + \xi_3^2) \bar{\gamma}^+ (\xi) \bar{\tau}(\xi_3 - \lambda^2) |z|^{2k} \exp(-|z|^2) \mathcal{F} \mathcal{F} (\xi') ^2 d\xi
\leq C \lambda^{-4k-2} \int (\lambda^2 + (\xi_3 + \lambda^2)^2) \bar{\gamma}^+ (\lambda \xi', \xi_3 + \lambda^2) \bar{\tau}(\xi_3) |z|^{2k} \exp(-|z|^2) \mathcal{F} (\xi') ^2 d\xi.
\]
Then if \(s \geq 0\) and if \(\lambda\) is sufficiently large we have
\[
||\Psi^s \Gamma^+ \tau E_k h_\lambda||^2 \leq C \lambda^{4s-4k-2}.
\]
We assume $k \geq 1$; if $s_2 - s_1 < 0$ then

$$\|\Psi^{s_1-s_2} \mathcal{G}^+ h_\lambda\|^2 \leq C \left( \|\tilde{\mathcal{G}}^+ \mathcal{E}_k h_\lambda\|^2 + \|\tilde{\mathcal{G}} h_\lambda\|^2_{s_0} \right) \leq C \lambda^{-4k-2}$$

and, by Lemma 1,

$$\|\Psi^{s_1-s_2} \mathcal{G}^+ h_\lambda\|^2 = \|\Psi^{s_1-s_2} \eta \mathcal{G}^+ h_\lambda\|^2 + O(\|\mathcal{G}^+ h_\lambda\|^2_{s_0})$$

$$\geq C \lambda^{2s_2-2s_1} \|\eta \mathcal{G}^+ h_\lambda\|^2 - C' (\|\mathcal{G}^+ h_\lambda\|^2_{s_0})$$

$$\geq C (\lambda^{2s_2-2s_1} - \lambda^{-2m-2}).$$

This implies that for large $\lambda$ we have $\lambda^{2s_2-2s_1-2} \leq C (\lambda^{-4k-2} + \lambda^{-2m-2})$, which is a contradiction, so that $s_2 - s_1 \geq 0$ and

$$C_1 \lambda^{-6} \leq C_2 \|\mathcal{G}^+ h_\lambda\|^2 \leq C \left( \|\Psi^{s_2-s_1} \tilde{\mathcal{G}}^+ \mathcal{E}_k h_\lambda\|^2 + \|\tilde{\mathcal{G}} h_\lambda\|^2_{s_0} \right)$$

$$\leq C_3 (\lambda^{4s_2-4s_1-4k-2} + \lambda^{-2m-2}).$$

Therefore, if $m$ large we get $C_1 \leq 2C_3 \lambda^{4(s_2-s_1-k+1)}$ for large $\lambda$. Hence $s_2 - s_1 - k + 1 \geq 0$, which concludes the proof of the proposition and also of Theorem B.

8. Elliptic and subelliptic microlocalizations

In this section we will show that the a priori estimates for the operator $E_k$ gain two derivatives in the 0 microlocalization and gains one derivative in the $-1$ microlocalization, these gains are in the Sobolev norms. Without loss of generality we will deal only with microlocalizations near the origin, taking $\alpha = 0$ and setting $\mathcal{G}^0 = \mathcal{G}^0_0$ and $\mathcal{G}^- = \mathcal{G}^-_0$. The subscript $\alpha$ will be dropped from the corresponding operators.

**Proposition 4.** Let $U$ and $U'$ be neighborhoods of the origin with $U \subset U'$, and $|z| \leq a$ on $U'$, where $a$ is sufficiently small as in Lemma 3. Suppose that $\varrho \in C_0^\infty(U)$ and $\varrho' \in C_0^\infty(U')$ with $\varrho' = 1$ on a neighborhood of $\bar{U}$. Further suppose that $\gamma^0, \tilde{\gamma}^0 \in \mathcal{G}^0$ with $\tilde{\gamma}^0 = 1$ on a neighborhood of the support of $\gamma^0$. Then, given $s, s_0 \in \mathbb{R}$, there exists $C = C(\varrho, \varrho', \gamma^0, \tilde{\gamma}^0, s, s_0)$ such that

$$\|\varrho \mathcal{G}^0 u\|^2_{s+2} + \|\varrho \mathcal{G}^0 z^k L u\|^2_{s_1+1} + \|\varrho \mathcal{G}^0 \tilde{L} u\|^2_{s_1+1} \leq C(\|\varrho' \mathcal{G}^0 E_k u\|^2 + \|u\|^2_{s_0}),$$

for all $u \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz class of rapidly decreasing functions.

**Proof.** Let $\{\varrho_i\}$ be a sequence of functions such that $\varrho_i \in C_0^\infty(U)$, $\varrho_0 = \varrho$, $\varrho_{i+1} = 1$ in a neighborhood of the support of $\varrho_i$, and such that $\varrho' = 1$ in a neighborhood of the supports of all the $\varrho_i$. Let $\{\gamma_i^0\}$ be a sequence in $\mathcal{G}^0$ such that $\gamma_i^0 = \gamma^0$, $\gamma_{i+1}^0 = 1$ in a neighborhood of the support of $\gamma_i$, and $\tilde{\gamma}_i^0 = 1$ in a neighborhood of the supports of all the $\gamma_i^0$. Then substituting $\varrho \Lambda^{s+1} \Gamma^0_1 u$ for $u$ in Lemma 3 we have

$$\|\Gamma^0 \varrho \Lambda^{s+1} \Gamma^0_1 u\|^2_1 \leq C (\|\Gamma^0 \mathcal{L} \varrho \Lambda^{s+1} \Gamma^0_1 u\|^2 + \|\varrho \Lambda^{s+1} \Gamma^0_1 u\|^2).$$
Hence
\[ \| \Gamma^0 \varrho \Lambda^{s+1} \Gamma_1^0 u \|_1^2 \leq C(\| \Gamma^0 z^k L \varrho \Lambda^{s+1} \Gamma_1^0 u \|_1^2 + \| \Gamma^0 \bar{L} \varrho \Lambda^{s+1} \Gamma_1^0 u \|_1^2 + \| \varrho \Lambda^{s+1} \Gamma_1^0 u \|_1^2). \]
Then
\[ \| \Gamma^0 \varrho \Lambda^{s+1} \Gamma_1^0 u \|_1^2 = \| \varrho \Gamma^0 u \|_{s+2}^2 + O(\| \Lambda^1 [\Gamma^0 \varrho, \Lambda^{s+1}] \Gamma_1^0 u \|_1^2 + \| \Lambda^{s+2}[\Gamma^0, \varrho] \Gamma_1^0 u \|_1^2 + \| \Lambda^{s+2} \varrho(\Gamma^0 \Gamma_1^0 - \Gamma^0) u \|_1^2). \]
Since \([\Gamma^0 \varrho, \Lambda^{s+1}]\) is a pseudodifferential operator of order \(s+1\) and since \(\varrho_1 = 1\) on the support of its symbol, we have
\[ \| \Lambda^1 [\Gamma^0 \varrho, \Lambda^{s+1}] \Gamma_1^0 u \|_1^2 \leq C(\| \varrho_1 \Gamma_1^0 u \|_{s+1}^2 + \| \Gamma_1^0 u \|_{-\infty}^2). \]
The operator \([\Gamma^0, \varrho]\) is of order \(-1\) and \(\varrho_1 = 1\) on the support of its symbol, so that
\[ \| \Lambda^{s+2}[\Gamma^0, \varrho] \Gamma_1^0 u \|_1^2 \leq C(\| \varrho_1 \Gamma_1^0 u \|_{s+1}^2 + \| \Gamma_1^0 u \|_{-\infty}^2). \]
The symbol of the operator \(\Lambda^{s+2} \varrho(\Gamma^0 \Gamma_1^0 - \Gamma^0)\) is zero so that
\[ \| \Lambda^{s+2} \varrho(\Gamma^0 \Gamma_1^0 - \Gamma^0) u \|_1^2 \leq C \| u \|_{-\infty}^2. \]
Then we obtain
\[ \| \varrho \Gamma^0 u \|_{s+2}^2 \leq C(\| \Gamma^0 \varrho \Lambda^{s+1} \Gamma_1^0 u \|_{s+1}^2 + \| \varrho_1 \Gamma_1^0 u \|_{s+1}^2 + \| u \|_{-\infty}^2), \]
so that
\[ \| \varrho \Gamma^0 u \|_{s+2}^2 \leq C(\| \Gamma^0 z^k L \varrho \Lambda^{s+1} \Gamma_1^0 u \|_{s+1}^2 + \| \Gamma^0 \bar{L} \varrho \Lambda^{s+1} \Gamma_1^0 u \|_{s+1}^2 + \| \varrho \Lambda^{s+1} \Gamma_1^0 u \|_{s+1}^2 + \| u \|_{-\infty}^2). \]
The following lemma which involves a vector field \(X\) will be applied with \(X = z^k L\) and \(X = \bar{L}\).

**Lemma 6.** If \(X\) is a complex vector field on \(\mathbb{R}^3\) then
\[ \| \Gamma^0 X \varrho \Lambda^{s+1} \Gamma_1^0 u \|_1^2 \leq C(\| \varrho \Gamma^0 X u \|_{s+1}^2 + \| \varrho_1 \Gamma_1^0 u \|_{s+1}^2 + \| u \|_{-\infty}^2) \]
and
\[ \| \varrho \Gamma^0 X u \|_{s+1}^2 = (\Lambda^1 \varrho \Gamma^0 X \Lambda^s \varrho \Gamma^0 u) \]
\[ + O(\| \varrho_1 \Gamma_1^0 u \|_{s+1}^2 + \| \varrho \Gamma^0 u \|_{s+2}^2 \| \varrho_1 \Gamma_1^0 u \|_{s+1}^2 + \| \varrho_1 \Gamma_1^0 X u \|_{s+2}^2 + \| \varrho_1 \Gamma_1^0 u \|_{s+1}^2 \| \Gamma^0 X u \|_{s+1}^2 + \| u \|_{-\infty}^2), \]
for all \(u \in S\).

**Proof.** We have
\[ \| \Gamma^0 X \varrho \Lambda^{s+1} \Gamma_1^0 u \| \leq || \Gamma^0 \varrho \Lambda^{s+1} \Gamma_1^0 X u \| + || \Gamma^0 [X, \varrho \Lambda^{s+1} \Gamma_1^0] u \|. \]
The operator \(P = \Gamma^0 \varrho \Lambda^{s+1} \Gamma_1^0 X - \Lambda^{s+1} \Gamma_1^0 \varrho \Gamma^0 X\) is of order \(s+1\) and \(\varrho_1 \gamma_1^0 = 1\) in a neighborhood of the symbol of \(P\); hence
\[ \| Pu \| \leq C(|| P \varrho_1 \Gamma_1^0 u \| + || u \|_{-\infty}) \leq C(|| \varrho_1 \Gamma_1^0 u \|_{s+1} + || u \|_{-\infty}). \]
Replacing $\rho$ Proceeding inductively, we obtain

$$\|\Gamma^0 \rho A^{s+1} \Gamma^0_1 X u\| \leq C(\|\rho \Gamma^0 X u\|_{s+1} + \|u\|_{-\infty}).$$

Furthermore, $\Gamma^0 [X, \rho A^{s+1} \Gamma^0]$ is of order $s + 1$ and $\rho_1 \gamma_1^0 = 1$ in a neighborhood of the support of its symbol so that

$$\|\Gamma^0 [X, \rho A^{s+1} \Gamma^0] u\| \leq C(\|\rho_1 \Gamma^0_1 u\|_{s+1} + \|u\|_{-\infty}),$$

which proves the first part of the lemma.

For the second part of the lemma we write

$$\|\rho \Gamma^0 X u\|_{s+1}^2 = (\Lambda^{s+1} \rho \Gamma^0 X u, \Lambda^{s+1} \rho \Gamma^0 X u)
= (\Lambda^{s+1} \rho \Gamma^0 X u, [\Lambda^{s+1} \rho \Gamma^0, X] u) + ([X^*, \Lambda^{s+1} \rho \Gamma^0] X u, \Lambda^{s+1} \rho \Gamma^0 u)
+ (\Lambda^s \rho \Gamma^0 X^* X u, \Lambda^{s+1} \rho \Gamma^0 u).$$

Then, since $[\Lambda^{s+1} \rho \Gamma^0, X]$ is of order $s + 1$ and $\rho_1 \gamma_1^0 = 1$ in a neighborhood of its symbol,

$$\|[\Lambda^{s+1} \rho \Gamma^0, X] u\|^2 \leq C(\|\rho_1 \Gamma^0_1 u\|_{s+1}^2 + \|u\|_{-\infty}^2).$$

Then

$$( [X^*, \Lambda^{s+1} \rho \Gamma^0] X u, \Lambda^{s+1} \rho \Gamma^0 u) = ( (\Lambda^{s+1} \rho \Gamma^0)^* X u, [X^*, \Lambda^{s+1} \rho \Gamma^0] u) + ((\Lambda^{s+1} \rho \Gamma^0)^* X u, [X^*, \Lambda^{s+1} \rho \Gamma^0]^* u).$$

Let $Q = (\Lambda^{s+1} \rho \Gamma^0)^*, [X^*, \Lambda^{s+1} \rho \Gamma^0]$; then $Q$ has order $2s + 1$ and $\rho_1 \gamma_1^0 = 1$ in a neighborhood of its symbol. Thus

$$\| (Q X u, u) \| \leq C(\|Q \rho_1 \Gamma^0_1 X u, \rho_1 \Gamma^0_1 u\| + \|u\|_{-\infty}^2)
\leq C(\|\rho_1 \Gamma^0_1 X u\|_{s+1}^2 + \|\rho_1 \Gamma^0_1 u\|_{s+1}^2 + \|u\|_{-\infty}^2).$$

The symbol of the operator $(\Lambda^{s+1} \rho \Gamma^0)^* - \Lambda^{s+1} \rho \Gamma^0$ is zero, the order of $[X^*, \Lambda^{s+1} \rho \Gamma^0]^*$ is $s + 1$ and $\rho_1 \gamma_1^0 = 1$ on a neighborhood of its support. Hence

$$\| (\Lambda^{s+1} \rho \Gamma^0)^* X u, [X^*, \Lambda^{s+1} \rho \Gamma^0]^* u) \| \leq C(\|\rho \Gamma^0 X u\|_{s+2} \|\rho_1 \Gamma^0_1 u\|_{s+1} + \|u\|_{-\infty}^2).$$

Combining these we conclude the proof of the lemma.

Returning to the proof of the proposition, by using the above lemma, when $X = \zeta^k L$ and when $X = L$, we obtain

$$\|\rho \Gamma^0 \rho_1^0 u\|_{s+2}^2 + \|\rho \Gamma^0 \zeta^k L u\|_{s+1}^2 + \|\rho \Gamma^0 \bar{L} u\|_{s+1}^2
\leq C(\|\rho \Gamma^0 E_k u\|_{s+2}^2 + \|\rho \Gamma_1^0 \zeta^k L u\|_{s+1}^2 + \|\rho \Gamma_1^0 \bar{L} u\|_{s+1}^2 + \|u\|_{-\infty}^2).$$

Replacing $\rho$ by $\phi_1$, $\rho_1$ by $\phi_{i+1}$, $\Gamma^0$ by $\Gamma^0_i$, $\Gamma^0_i$ by $\Gamma^0_{i+1}$, and $s$ by $s - i$ we obtain

$$\|\phi_{i+1} \Gamma^0_1 u\|_{s+2-i}^2 + \|\phi_{i+1} \zeta^k L u\|_{s+1-i}^2 + \|\phi_{i+1} \bar{L} u\|_{s+1-i}^2
\leq C(\|\phi_{i+1} \Gamma_{E_k} u\|_{s-i}^2 + \|\phi_{i+1} \Gamma^0_{i+1} \zeta^k L u\|_{s-i}^2 + \|\phi_{i+1} \Gamma^0_{i+1} \bar{L} u\|_{s-i}^2 + \|u\|_{-\infty}^2).$$

Proceeding inductively, we obtain
\[
\|q\Gamma^0 u\|_{s+2}^2 + \|q\Gamma^0 z^k Lu\|_{s+1}^2 + \|q\Gamma^0 \bar{L} u\|_{s+1}^2
\]
\[
\leq C \left( \sum_{i=0}^{N} \|q\Gamma_1^0 E_k u\|_{s-i}^2 + \|q \Gamma_{N+1}^0 \Gamma_{N+1}^0 z^k Lu\|_{s-N}^2 + \|q \Gamma_{i+N} \Gamma_1^0 \bar{L} u\|_{s-N}^2 + \|u\|_{2\infty}^2 \right).
\]

Setting \(N \geq s_0 + s + 1\) we conclude the proof of the proposition since
\[
\|q\Gamma_1^0 E_k u\|_{s-i}^2 \leq C(\|q\Gamma^0 E_k u\|_s^2 + \|u\|_{2\infty}^2).
\]

**Proposition 5.** Given neighborhoods of the origin \(U\) and \(U'\) with \(\bar{U} \subset U'\); suppose that \(q \in C_0^\infty(U)\) and \(q' \in C_0^\infty(U')\) with \(q' = 1\) on a neighborhood of \(\bar{U}\).

Further suppose that \(\gamma, \gamma^- \in \mathcal{G}^-\) with \(\gamma^- = 1\) on a neighborhood of the support of \(\gamma^-\). Then, given \(s, s_0 \in \mathbb{R}\), there exists \(C = C(q, q', \gamma, \gamma^-, s, s_0)\) such that
\[
\|q\Gamma^- u\|_{s+1}^2 + \|q\Gamma^- z^k Lu\|_{s+\frac{1}{2}}^2 + \|q\Gamma^- \bar{L} u\|_{s+\frac{1}{2}}^2 \leq C(\|q\Gamma^- E_k u\|_s^2 + \|u\|_{s-N}^2),
\]
for all \(u \in \mathcal{S}\).

*Proof.* The proof is entirely analogous to that of the above proposition. We use Lemma 5 in place of Lemma 3 and substitute \(q\Lambda^{s+\frac{1}{2}} \Gamma^- u\) for \(u\) we obtain
\[
\|\Gamma^- q\Lambda^{s+\frac{1}{2}} \Gamma^- u\|_{\frac{1}{2}}^2 \leq C(\|\Gamma^0 \bar{L} q\Lambda^{s+\frac{1}{2}} \Gamma^- u\|_s^2 + \|q\Lambda^{s+\frac{1}{2}} \Gamma^- u\|_s^2).
\]
Then one proceeds exactly as above to obtain the proof.

In the case \(k = 0\) the vector fields \(L\) and \(\bar{L}\) play exactly the same role and so we obtain the following.

**Proposition 6.** Given neighborhoods of the origin \(U\) and \(U'\) with \(\bar{U} \subset U'\).

Suppose that \(q \in C_0^\infty(U)\) and \(q' \in C_0^\infty(U')\) with \(q' = 1\) on a neighborhood of \(\bar{U}\).

Further suppose that \(\gamma^+, \gamma^+ \in \mathcal{G}^+\) with \(\gamma^+ = 1\) on a neighborhood of the support of \(\gamma^+\). Then, given \(s, s_0 \in \mathbb{R}\), there exists \(C = C(q, q', \gamma, \gamma^+, s, s_0)\) such that
\[
\|q\Gamma^+ u\|_{s+1}^2 + \|q\Gamma^+ Lu\|_{s+\frac{1}{2}}^2 + \|q\Gamma^+ \bar{L} u\|_{s+\frac{1}{2}}^2 \leq C(\|q\Gamma^+ E_0 u\|_s^2 + \|u\|_{s-N}^2),
\]
for all \(u \in \mathcal{S}\).

9. **The operator \(E_0\) and gain of derivatives**

Since \(E_0\) is a real operator, it can be written as \(E_0 = -X^2 - Y^2\), where \(X = \frac{1}{\sqrt{2}}RL\) and \(Y = \frac{1}{\sqrt{2}}\bar{Z}L\). Thus it is one of the simplest operators that satisfy Hörmander’s condition and it is well understood. Nevertheless, it is instructive to write it in terms of \(L\) and \(\bar{L}\) and analyze it microlocally in the framework of the previous section. The operator \(E_0\) gains one derivative. As we have seen the operators \(E_k\) do not gain derivatives when \(k > 0\) and \(z = 0\); in a neighborhood on which \(z \neq 0\) they do gain derivatives and they also gain in the 0 and \(-\) microlocalizations.
In the analysis of $E_0$ we can assume, without loss of generality, that $\alpha = 0$ and we set $\gamma = \gamma_0$, and $\Gamma = \Gamma_0$. The basic observation is that the gain of derivatives in the $+$ and $-$ microlocalizations is controlled by the operators $\bar{L}L$ and $\bar{L}\bar{L}$, respectively. In the 0 microlocalization the gain of derivatives is controlled by both $\bar{L}L$ and $\bar{L}\bar{L}$ independently. Propositions 4 and 5 give a priori estimates for $E_0$ in the 0 and $-$ microlocalizations, respectively. Proposition 6 gives these estimates for the $+$ microlocalization. Here we show how to go from the a priori estimates to hypoellipticity. In particular we prove that $E_0$ is hypoelliptic and that $E_k$ is hypoelliptic on open sets on which $z \neq 0$ and that the 0 and $-$ microlocalizations of the operators $E_k$ are hypoelliptic.

**Proposition 7.** If $u$ is a distribution such that for some open set $V \subset \mathbb{R}^3$ the restriction of $E_0u$ to $V$ is in $C^\infty(V)$ then the restriction of $u$ to $V$ is also in $C^\infty(V)$. More precisely, if $E_0u \in H^s_{\text{loc}}(V)$ then $u \in H^{s+1}_{\text{loc}}(V)$.

**Proof.** Assuming that $E_0u \in H^s_{\text{loc}}(V)$, it suffices to show that any $P \in V$ has a neighborhood $U \subset V$ such that for any $\varrho \in C^\infty_0(U)$ we have $\varrho u \in H^{s+1}(\mathbb{R}^3)$. Without loss of generality we may assume that $P = 0$. Now choose neighborhoods $U$ and $U'$ of $P$ such that $\bar{U} \subset U'$ and $|z| \leq a$ on $U'$, as in Proposition 4. Let $\varrho \in C^\infty_0(U)$, let $\varrho' \in C^\infty_0(U')$ with $\varrho' = 1$ in a neighborhood of the support of $\varrho$, and let $\theta \in C^\infty_0(\mathbb{R}^3)$ such that $\theta = 1$ on a neighborhood of $\bar{U}'$. Since $u$ is a distribution there exists an $s_0 \in \mathbb{R}$ such that $\theta u \in H^{-s_0}(\mathbb{R}^3)$. Then, choosing $\gamma^+, \gamma^0$, and $\gamma^-$ such that $\gamma^+ + \gamma^0 + \gamma^- \geq \text{const.} > 0$ and combining Propositions 4, 5, and 6 we obtain the a priori estimate

$$
\|\varrho u\|_{s+1}^2 + \|\varrho Lu\|_{s+\frac{1}{2}}^2 + \|\varrho\bar{L}u\|_{s+\frac{1}{2}}^2 \leq C(\|\varrho' E_0u\|_s^2 + \|u\|_{s-s_0}^2),
$$

for all $u \in C^\infty(\mathbb{R}^3)$. Let $\chi \in C^\infty_0(\mathbb{R}^3)$ with $\chi(0) = 1$. For $\delta > 0$ we define the smoothing operator $S_\delta$ by $F(S_\delta u)(\xi) = \chi(\delta \xi)\hat{u}(\xi)$. The important facts are that:

1. If $\delta > 0$ then for any distribution $v$ the function $S_\delta v \in C^\infty(\mathbb{R})$.

2. If $v$ is a distribution and if $\|S_\delta v\|_s$ is bounded independently of $\delta$ then $v \in H^s(\mathbb{R}^3)$.

3. If $v \in H^s(\mathbb{R}^3)$ then $\lim_{\delta \to 0} \|S_\delta v - v\|_s = 0$.

4. For $\delta \geq 0$ the operator $S_\delta$ is a pseudodifferential operator which is uniformly of order zero.

Replacing $u$ by $S_\delta \varrho u$ in Lemma 6 and in the proofs of Propositions 4, 5, and 6 and using item 4 above we obtain

$$
\|S_\delta \varrho u\|_{s+1}^2 \leq C(\|S_\delta \varrho' E_0u\|_s^2 + \|S_\delta \varrho' u\|_{s+\frac{1}{2}}^2 + \|\bar{S}_\delta \varrho u\|_{2-s_0}^2),
$$
where $\tilde{S}_\delta$ has the symbol $\tilde{\chi}(\delta \xi)$ with $\tilde{\chi} = 1$ in a neighborhood of the support of $\chi$. Choose $m$ so that $-s_0 \geq s+1-m$, then substituting $s+1-m+j$ for $s$ above we obtain, by induction on $j$, that $\|S_\delta u\|_{s+1}^2$ is bounded independently of $\delta$. Hence $gu \in H^{s+1}(\mathbb{R}^3)$ thus concluding the proof.

Next we will show that in any region in which $z \neq 0$ the operator $E_k$ is hypoelliptic with a gain of one derivative.

**Proposition 8.** If $V \subset \mathbb{R}^3$ is an open set, with the property that $z \neq 0$ on $V$, and if $u$ is a distribution such that the restriction of $E_k u$ to $V$ is in $C^\infty(V)$, then the restriction of $u$ to $V$ is also in $C^\infty(U)$. More precisely, if $E_k u \in H^s_{loc}(V)$ then $u \in H^{s+1}_{loc}(V)$.

**Proof.** Let $P \in V$ then $P = (\alpha, t_0)$ with $\alpha \neq 0$. Let $U$ be a neighborhood of $P$ such that on $U$ we have $|z - \alpha| < a$, where $a$ is chosen as in Lemma 3, and also such that on $U$ we have $|z| \geq b > 0$. Then

$$\|Lu\|^2 \leq b^{-2k}\|z^k Lu\|^2,$$

for all $u \in C^\infty_0(U)$. Hence Propositions 4, 5, and 6 hold with $\gamma$ replaced by $\gamma_\alpha$. The proof is then concluded using the same argument as above, replacing $S_\delta$ with $S_{\alpha, \delta}$, which is defined by $F_\alpha(S_{\alpha, \delta}) = \chi(\delta \xi)F_\alpha u(\xi)$.

Now we prove microlocal hypoellipticity in the 0 and $-$ microlocalizations.

**Proposition 9.** Given neighborhoods of the origin $U$ and $U'$ with $\hat{U} \subset U'$ and $|z| \leq a$ on $\hat{U}'$, where $a$ is sufficiently small as in Lemma 3, suppose that $g \in C^\infty_0(U)$ and $g' \in C^\infty_0(U')$ with $g' = 1$ on a neighborhood of $\hat{U}$. Further suppose that $\gamma_0 \in G^0$. Then, given $s \in \mathbb{R}$, if $u$ is a distribution such that $g' E_k u \in H^s(\mathbb{R}^3)$ then $g\Gamma^0 u \in H^{s+2}(\mathbb{R}^3)$.

**Proof.** The proof consists of proving the following estimate

$$\|S_\delta g\Gamma^0 u\|_{s+2}^2 \leq C(g' E_k u\|_{s}^2 + \|u\|_{-s_0}^2).$$

Its proof is exactly analogous to the proof of Proposition 4. Replacing $u$ by $S_\delta u$ the same proof as of Lemma 6 using $X S_\delta$ instead of $X$ gives

$$\|S_\delta g\Gamma^0 X u\|_{s+1}^2 = (\Lambda^s S_\delta g\Gamma^0 X u, \Lambda^{s+2} S_\delta g\Gamma^0 u)$$

$$+ O(||S_\delta g\Gamma^0 u\|_{s+2}^2 ||S_\delta g\Gamma^0 u\|_{s+1}^2)$$

$$+ ||S_\delta g\Gamma^0 X u\|_{s}^2 + ||S_\delta g\Gamma^0 u\|_{s+1} ||g\Gamma^0 X u\|_{s+1} + ||u\|_{-\infty}^2).$$

The argument then proceeds exactly as in Proposition 4 and shows that $\|S_\delta g\Gamma^0 u\|_{s+2}^2$ is bounded independently of $\delta$ completing the proof.

For the $-$ microlocalization we the following result follows from an argument entirely analogous to the above proposition.
Proposition 10. Given neighborhoods of the origin $U$ and $U'$ with $\bar{U} \subset U'$ and $|z| \leq a$ on $U'$, where $a$ is sufficiently small as in Lemma 3. Suppose that $\varrho \in C_0^\infty(U)$ and $\varrho' \in C_0^\infty(U')$ with $\varrho' = 1$ on a neighborhood of $\bar{U}$. Further suppose that $\gamma^- \in G^0$. Then, given $s \in \mathbb{R}$, if $u$ is a distribution such that $\varrho' E_k u \in H^s(\mathbb{R}^3)$ then $\varrho \Gamma^- u \in H^{s+1}(\mathbb{R}^3)$.

10. The operator $E_1$: no loss, no gain

As was shown in Section 5 the operator $E_1$ does not gain any derivatives. Here we will give a proof of an a priori estimate which shows that it does not lose any derivatives. More precisely, the estimate will show that $E_1$ does not lose any derivatives after it is proved that $E_1$ is hypoelliptic. This will be done using the same estimate with an appropriate smoothing operator in Section 14. As we have seen all the operators $E_k$ gain a derivative in regions where $z \neq 0$ and in the 0 and $-$ microlocalizations. Thus the remaining case is the $+$ microlocalization when $z = 0$. Since the operators $E_k$ are invariant under translation in the $t$ direction it will suffice to consider neighborhoods of the origin. In this section we will present a direct proof of the a priori estimates for $E_1$ which will rely on the following lemma. This proof however cannot be adopted to prove the corresponding a priori estimate for the operator $F_1 = E_1 + c$ unless $c \geq 0$. In fact the same estimates will be proved when we treat the general case of $E_k$ with $k \geq 1$. However that treatment is much more complicated so it might be worthwhile to note this simpler proof.

In the previous section we showed that the elliptic microlocalization $\Gamma^0 u$ is smooth whenever $E_k u$ is smooth. Thus we do not have to keep track of just which microlocalizing operator in $\mathfrak{g}^0$ is used; in order to simplify the calculations we will write $u^0$ instead of $\Gamma^0 u$. Similarly, since all the commutators with $\Gamma^+$ that arise are dominated as follows $\|[\Gamma^+, R^+] u\| \leq C(\|\Gamma^0 u\|_{s-1} + \|u\|_{-\infty})$, we will write $u^+$ instead of $\Gamma^+$.

Lemma 7. Given a bounded open set $U \subset \mathbb{R}^3$ there exists $C > 0$ such that

$$\|u\|^2 \leq C(\|\bar{z} Lu\|^2 + \|\bar{L} u\|^2),$$

for all $u \in C_0^\infty(U)$.

Proof. If $u \in C_0^\infty(U)$ we have

$$\|u\|^2 = (L(z)u, u) = -(z Lu, u) - (zu, \bar{L} u) \leq \|\bar{z} Lu\| \|u\| + \|zu\| \|\bar{L} u\| \leq \text{s.c.} \|u\|^2 + \text{l.c.}(\|\bar{z} Lu\|^2 + \|\bar{L} u\|^2).$$

Absorbing the first term on the right into the left-hand side completes the proof.
The other estimate we will use here is given in Lemma 5 with $\alpha = 0$, namely

$$(8) \quad \|Lu^+\|^2 + \|u^+\|^2 \leq C(\|Lu^+\|^2 + \|u^+\|^2 + \|u\|^2_{-\infty}),$$

for all $u \in C_0^\infty(U)$.

**Proposition 11.** Let $U$ be a bounded neighborhood of the origin such that $|z| \leq a$ on $U$, let $\varrho, \varrho' \in C_0^\infty(U)$ with $\varrho' = 1$ in a neighborhood of the support of $\varrho$. Then, given $s, s_0 \in \mathbb{R}$ there exists $C = C(\varrho, \varrho', s, s_0)$ such that

$$\|\Psi^s z\varrho u^+\| \leq C(\|\Psi^s \varrho u\| + \|\Psi^s \varrho' u\| + \|u\|_{-s_0}),$$

for all $u \in C_0^\infty(\mathbb{R}^3)$.

**Proof.** We assume that $u \in C_0^\infty(\mathbb{R}^3)$ and replace $u$ in (8) by $\varrho' z\Psi^s u$. Then, following the method of Proposition 4, we get

$$\|\varrho' z\Psi^s u\| \leq C(\|zL\varrho' \Psi^s u\|^2 + \|L\varrho' \Psi^s u\|^2 + \|\varrho'' \Psi^s u\|^2 + \|u\|^2_{-\infty})$$

$$\leq C(\|\varrho' \Psi^s(E_k u)^+\| + \|\varrho'' \Psi^s u\| + \|\varrho'' \Psi^s u\|^2 + \|u\|^2_{-\infty})$$

Next, we replace $u$ by $\varrho \Psi^s \varrho^1 u^+$ in Lemma 8 and, with the use of Lemma 1 and the fact that

$$\|\varrho' z\Psi^s u\|^2 = \|z\varrho' \Psi^s \varrho^1 u\|^2 + O(\|u\|^2_{-\infty}),$$

we obtain

$$\|\varrho \Psi^s \varrho^1 u^+\|^2 \leq C(\|\varrho E_k u\|^2_{s+\frac{1}{2}} + \|L\varrho \Psi^s \varrho^1 u\|^2 + \|u\|^2_{-\infty})$$

$$\leq C(\|\varrho E_k u\|^2_{s+\frac{1}{2}} + \|L\varrho \Psi^s \varrho^1 u\|^2 + \|\varrho' \varrho^0 u\|^2_{s+\frac{1}{2}} + \|u\|^2_{-\infty})$$

$$\leq C(\|\varrho E_k u\|^2_{s+\frac{1}{2}} + \|z\varrho' \Psi^s \varrho^1 u\|^2 + \|\varrho' \varrho^0 u\|^2_{s+\frac{1}{2}} + \|u\|^2_{-\infty})$$

Then, redefining $\varrho'$ and $\varrho''$, we conclude the proof.

**11. Estimates of $\varrho Lu^+$ and of $\varrho L\varrho u^+$**

In this section we begin to prove the *a priori* estimates for the operators $E_k$ with $k \geq 1$. These will be derived from the estimate (8) and the estimates in the 0 microlocalization. The main difficulty is the localization in space; one cannot have a term with the cutoff function $\varrho$ between $u$ and $L$, or $\varrho L\varrho u$.
the term also contains suitable powers of $z$ and $\bar{z}$. Substituting $\varrho \Psi^s \bar{L}u$ for $u$ in (8) we have
\[ \| \bar{L} \varrho \Psi^s \bar{L}u^+ \|^2 + \| \varrho \Psi^s \bar{L}u^+ \|^2 \lesssim C (\| L \varrho \Psi^s \bar{L}u^+ \|^2 + \| \varrho \Psi^s \bar{L}u^+ \|^2 + \| u \|^2_{-\infty}), \]
so that,
\[
\| \varrho \Psi^s \bar{L}u^+ \|^2 + \| \varrho \Psi^s \bar{L}^2u^+ \|^2 + \| \varrho \Psi^s \bar{L}u^+ \|^2 \\
\lesssim C (\| \varrho \Psi^s \bar{L}u^+ \|^2 + \| \varrho' \Psi^s \bar{L}u^+ \|^2 + \| \varrho' u^0 \|^2_{s+1} + \| u \|^2_{-\infty}) \\
\lesssim C (\| \varrho \Psi^s \bar{L}u^+ \|^2 + \| \varrho' \Psi^s \bar{L}u^+ \|^2 + \| \varrho' \bar{E}_k u \|^2_{s-1} + \| u \|^2_{-\infty}).
\]
Since $\bar{L}\bar{L} = -\bar{L}E_k - \bar{L}^2 |z|^{2k} \bar{L}$, we have
\[
\| (\varrho \Psi^s \bar{L} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \| \
\lesssim C (\| (\varrho \Psi^s \bar{L}E_k u^+ , \varrho \Psi^s \bar{L}u^+) \| + \| (\varrho \Psi^s \bar{L}^2 |z|^{2k} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \|) \
\lesssim 1.c. \| \varrho \bar{E}_k u \|^2 + s.c. \| \varrho \bar{L} \bar{L}u^+ \|^2 + C (\| \varrho \Psi^s \bar{L}^2 |z|^{2k} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \|.
\]
Then, to estimate $(\| \varrho \Psi^s \bar{L}^2 |z|^{2k} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \|$, we have
\[
\bar{L}^2 |z|^{2k} \bar{L} = -k \bar{L} \bar{z} \bar{z}^{k-1} \bar{L} + \bar{L} |z|^{2k} \bar{L} \\
= -k \bar{L} \bar{z} |z|^{2(k-1)} + \bar{L} \bar{L} \bar{z} \bar{z}^{k-1} - 2k \bar{z} \bar{z}^{k-1} \bar{L} + \bar{L} |z|^{2k} \bar{L} - 2 |z|^{2k} T \bar{L} \\
= -k \bar{L} \bar{z} |z|^{2(k-1)} - 4k \bar{z} \bar{z}^{k-1} \bar{L} + k(k-1) \bar{L} \bar{z} \bar{z}^{k-2} + k \bar{z} \bar{z}^{k-1} \bar{L} \\
+ \bar{L} |z|^{2k} \bar{L} - 2 |z|^{2k} T \bar{L},
\]
and, using integration by parts, we get
\[
\| (\varrho \Psi^s \bar{L} |z|^{2(k-1)} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \| \leq 1.c. \| z^{2(k-1)} \varrho \Psi^s u^+ \|^2 + \mathcal{E}_1, \\
\| (\varrho \Psi^s \bar{L} \bar{z} \bar{z}^{k-1} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \| \leq 1.c. \| z^{2k-1} \varrho \Psi^s \bar{L} u^+ \|^2 + \mathcal{E}_2, \\
(k-1) \| (\varrho \Psi^s \bar{L} \bar{z} \bar{z}^{k-2} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \| \leq (k-1) (1.c. \| z^{2(k-1)} \varrho \Psi^s u^+ \|^2 + \mathcal{E}_3), \\
\| (\varrho \Psi^s \bar{L} \bar{z} \bar{z}^{k-1} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \| \leq 1.c. \| z^{2k-1} \varrho \Psi^s u^+ \|^2 + \mathcal{E}_4, \\
\| (\varrho \Psi^s \bar{L} |z|^{2k} \bar{L} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \| \leq \mathcal{E}_4,
\]
and
\[
\| (\varrho \Psi^s |z|^{2k} T \bar{L} \bar{L}u^+ , \varrho \Psi^s \bar{L}u^+) \| \leq \mathcal{E}_2,
\]
where
\[
\mathcal{E}_1 \sim \| \varrho' u^0 \|^2 + \| \varrho' \Psi^s \bar{L}u^+ \|^2 + \| u \|^2_{-\infty}, \\
\mathcal{E}_2 \sim s.c. \| \varrho \Psi^s \bar{L} \bar{L}u^+ \|^2 + \mathcal{E}_1, \\
\mathcal{E}_3 \sim s.c. \| \varrho \Psi^s \bar{L}^2u^+ \|^2 + \mathcal{E}_1, \\
\mathcal{E}_4 \sim s.c. \| \varrho \Psi^s \bar{L} \bar{L}u^+ \|^2 + \mathcal{E}_1.
\]
Again, let \( \{ \varrho_i \} \) be a sequence of cutoff functions as defined in Section 2. Then substituting \( \varrho_i \) for \( \varrho \), \( s - \frac{k-1}{2} \) for \( s \), and \( \varrho_{i+1} \) for \( \varrho' \), we get

\[
\| \varrho_i \Psi^s \overline{\varrho^k L u^+} \| \leq C \| \varrho_{i+1} \Psi^{s+1} \overline{\varrho^{k-1} L u^+} \|
\]

\[
\leq C(\| \varrho' E_k u \| + \| \varrho_{i+1} \Psi^{s+1} \overline{\varrho^{k-1} L u^+} \| + \| \varrho_i \Psi^{s+1} \overline{\varrho^{k-1} L u^+} \| + \| u \|_{-\infty} ).
\]

Then we obtain the following, by substituting these inequalities into each other for successive \( i \)

\[
\| \varrho \Psi^s \overline{\varrho E_k u^+} \| \leq C \left( \sum_{i=1}^{N} \left( \| \varrho_i \Psi^{s+1} \overline{\varrho E_k u^+} \| + \| \varrho_{i+1} \Psi^{s+1} \overline{\varrho E_k u^+} \| + \| \varrho_i \Psi^{s+1} \overline{\varrho E_k u^+} \| + \| u \|_{-\infty} \right) \right).
\]

Given \( s_o \) we choose \( N > 2(s - s_o) + 1 \) then we obtain the following estimate which will be repeatedly used in establishing the \textit{a priori} estimates for \( E_k \)

\[
\| \varrho \Psi^s \overline{\varrho E_k u^+} \| \leq C(\| \varrho E_k u^+ \| + \| \varrho' E_k u^+ \| + \| \varrho' \Psi^{s+1} \overline{\varrho E_k u^+} \| + \| u \|_{-\infty} )
\]

\[(9) \quad \| \varrho \Psi^s \overline{\varrho E_k u^+} \| \leq C(\| \varrho E_k u^+ \| + \| \varrho' E_k u^+ \| + \| \varrho' \Psi^{s+1} \overline{\varrho E_k u^+} \| + \| u \|_{-\infty} ).
\]
12. Estimates of $\|z^j \psi^{s+j} \varrho u^+\|

**Lemma 8.** If $a > 0$ then for $m \in \mathbb{Z}^+$ and a small constant s.c. there exists a constant l.c. such that

$$\sum_{j=1}^{m-1} \|z^j \psi^{s+j} \varrho u^+\|^2 \leq \text{l.c.} \|z^m \psi^{s+ma} \varrho u^+\|^2 + \text{s.c.} \|\psi^s \varrho u\|^2 + C(\|\varrho' u^0\|^2_{s+(m-1)a-1} + \|u\|^2_{-\infty}),$$

for all $u \in C^\infty(U)$.

**Proof.** For $m = 2$ we have

$$\|z \psi^{s+a} \varrho u^+\|^2 = (\|z^2 \psi^{s+2a} \varrho u^+\|, \psi^s \varrho u^+) + O(\|\varrho' u^0\|^2_{s+a-1} + \|u\|^2_{-\infty}) \leq \text{l.c.} \|z^2 \psi^{s+2a} \varrho u^+\|^2 + \text{s.c.} \|\psi^s \varrho u^+\|^2 + C(\|\varrho' u^0\|^2_{s+a-1} + \|u\|^2_{-\infty}).$$

For $m > 2$ we assume

$$\sum_{j=1}^{m-2} \|z^j \psi^{s+j} \varrho u^+\|^2 \leq \text{l.c.} \|z^{m-1} \psi^{s+(m-1)a} \varrho u^+\|^2 + \text{s.c.} \|\psi^s \varrho u\|^2 + C(\|\varrho' u^0\|^2_{s+(m-2)a-1} + \|u\|^2_{-\infty}),$$

and we have

$$\|z^{m-1} \psi^{s+(m-1)a} \varrho u^+\|^2 = (\|z^m z \psi^{s+ma} \varrho u^+\|, \psi^s \varrho u^+) + O(\|\varrho' u^0\|^2_{s+(m-1)a-1} + \|u\|^2_{-\infty}) \leq \text{l.c.} \|z^m \psi^{s+ma} \varrho u^+\|^2 + \text{s.c.} \|z^{m-2} \psi^{s+(m-2)a} \varrho u^+\|^2 + C(\|\varrho' u^0\|^2_{s+(m-1)a-1} + \|u\|^2_{-\infty}).$$

Adding this to the above and absorbing the term multiplied by s.c. in the right-hand side we conclude the proof.

**Lemma 9.** If $0 < j < m$ and if $\frac{ma}{j} < B$ then for any s.c. and any $N$ there exists $C_N$ such that

$$\|z^j \psi^{s+A} \varrho u^+\|^2 \leq \text{s.c.} \|z^m \psi^s B \varrho u^+\|^2 + \|\psi^s \varrho u^+\|^2 + C(\|\varrho' u^0\|^2_{s+B-1} + C_N \|u^+\|^2_{-N} + C(\|\varrho' u^0\|^2_{s+B-1} + \|u\|^2_{-\infty}),$$

for all $u \in C^\infty_0(U)$.

**Proof.** With $a = \frac{A}{j}$ we have

$$\|z^j \psi^{s+A} \varrho u^+\|^2 \leq \text{l.c.} \|z^m \psi^{s+ma} \varrho u^+\|^2 + \text{s.c.} \|\psi^s \varrho u^+\|^2 + C(\|\varrho' u^0\|^2_{s+ma-1} + \|u\|^2_{-\infty}).$$

Since $ma = \frac{ma}{j} < B$,

$$\psi^{s+ma}(\xi) \leq \text{s.c.} \psi^{s+B}(\xi) + \text{l.c.} (1 + |\xi|^2)^{-\frac{N}{2}}.$$
Then
\[ \|z^m \Psi^{s+ma} qu^+\|^2 = \|\Psi^{s+ma}z^m qu^+\|^2 + O(\|u_0\|^2_{s+ma-1} + \|u\|^2_{-\infty}) \]
\[ \leq \text{s.c.} \|\Psi^{s+j} qu^+\|^2 + C_N \|u^+\|^2_{-\infty} \]
\[ + O(\|u_0\|^2_{s+ma-1} + \|u\|^2_{-\infty}) \]
\[ \leq \text{s.c.} \|z^m \Psi^{s+j} qu^+\|^2 + C_N \|u^+\|^2_{-\infty} \]
\[ + O(\|u_0\|^2_{j+ma-1} + \|u\|^2_{-\infty}). \]

Combining with the above we conclude the proof of the lemma.

**Lemma 10.** If \( \sigma = \frac{1}{2\pi} \) and if \( 1 \leq j \leq k \) then
\[ \|z^j \Psi^{s+j} qu^+\|^2 \leq C(\|\phi' E_k u\|_s^2 + \|\phi' u_0\|_s^2 + \|u\|^2_{-\infty}), \]
for all \( u \in C_0^\infty(U) \).

**Proof.** First note that
\[ \|\phi^k \Psi^s u^+\|^2 \leq C(\|\phi' E_k u\|_{\frac{s}{s-k}}^2 + \|\phi' \Psi^{s-k} u^+\|^2 + \|u\|^2_{-\infty}). \]

Then, replacing \( s \) by \( s + k \sigma \), since \( k \sigma - \frac{1}{2} = 0 \), we have
\[ \|z^j \Psi^{s+j} qu^+\|^2 \leq C(\|z^k \Psi^{s+k} qu^+\|^2 + \|\Psi^s qu\|^2 + \|\phi' u_0(\frac{s}{s-k})_{a-1} + \|u\|^2_{-\infty}) \]
\[ \leq C(\|z^k L \Psi^s qu^+\|^2 + \|z^k L \Psi^s qu^+\|^2 + \|\phi' u^+\|_s^2 + \|u\|^2_{-\infty}) \]
\[ \leq C(\|\phi' E_k u\|_s^2 + \|\phi' u_0\|_s^2 + \|u\|^2_{-\infty}). \]

**Lemma 11.** There exists a \( C > 0 \) such that
\[ \|\phi \Psi^{s+\sigma} u^+\| \leq C(\|\phi' E_k u\|_{s+\sigma+k-1}^2 + \|\phi' u_0\|_s^2 + \|u\|^2_{-\infty}), \]
for all \( u \in C_0^\infty(U) \).

**Proof.**
\[ \|\Psi^{s+\sigma} qu^+\|^2 = (L(z) \Psi^{s+\sigma} qu^+, \Psi^{s+\sigma} qu^+) \]
\[ = -(zL \Psi^{s+\sigma} qu^+, \Psi^{s+\sigma} qu^+) - (z \Psi^{s+\sigma} qu^+, L \Psi^{s+\sigma} qu^+) \]
\[ \leq 1.c. \|z \Psi^{s+\sigma} qu^+\|^2 + C \|\Psi^{s+\sigma} qu^+\|^2 + "\text{error}". \]

where,
\[ "\text{error} \leq \text{s.c.} \|\Psi^{s+\sigma} qu^+\|^2 + C(\|z \Psi^{s+\sigma} qu^+\|^2 + \|u_0\|^2_{s+\sigma} + \|u\|^2_{-\infty}). \]

In the estimate of the “error” the first term on the right gets absorbed and the other terms are estimated as follows.
\[ \|z \Psi^{s+\sigma} qu^+\|^2 \leq C(\|\phi' E u\|_s^2 + \|\phi' u^+\|_s^2 + \|u\|^2_{-\infty}). \]
The third term, which is microlocalized in the elliptic region, is estimated by
\[ \|\varrho u^0\|^2_{s+\sigma} \leq C(\|\varrho Eu\|^2_{s+\sigma-2} + \|\varrho'u\|^2). \]

Hence we get
\[ \|\Psi^{s+\sigma}\varrho u^+\|^2 \leq C\left(\|z\Psi^{s+\sigma}\varrho L u^+\|^2 + \|\Psi^{s+\sigma}\varrho L u^+\|^2 + \|\varrho E u\|^2_{s+\sigma-2} + \|\varrho'u\|^2_{s+\sigma-2} + \|u\|^2_{-\infty}\right). \]

From (9) we have
\[ \|\Psi^{s+\sigma}\varrho L u^+\|^2 \leq C(\|\varrho E u\|^2_{s+\sigma-2} + \|\varrho'u\|^2_{s+\sigma-2} + \|u\|^2_{-\infty}). \]

So the term that remains to be estimated is \(\|z\Psi^{s+\sigma}\varrho L u^+\|^2\), and we have
\[ \|z\Psi^{s+\sigma}\varrho L u^+\|^2 = (|z|^2\Psi^{s+\sigma+\frac{1}{2}}\varrho L u^+, \Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+) \]
\[ + O(\|u^0\|^2_{s+\sigma-2} + \|u\|^2_{-\infty}) \leq 1.c. |z|^2\Psi^{s+\sigma+\frac{1}{2}}\varrho L u^+|^2 + s.c. \|\Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+\|^2 \]
\[ + O(\|u^0\|^2_{s+\sigma-2} + \|u\|^2_{-\infty}) \]

and
\[ \|\Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+\|^2 = (\Psi^{s+\sigma}\varrho L u^+, \Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+) \]
\[ - (\Psi^{s+\sigma}\varrho u^+, \tilde{L}\Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+) + \mathcal{E}_1 \]
\[ = - (\Psi^{s+\sigma}\varrho u^+, [\tilde{L}, \Psi^{s+\sigma-1}\varrho L]u^+) \]
\[ - (\Psi^{s+\sigma}\varrho u^+, \Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+) + \mathcal{E}_1 \]
\[ \leq C(\|\Psi^{s+\sigma}\varrho u^+\|^2 + \|[\tilde{L}, \Psi^{s+\sigma-1}\varrho L]u^+\|^2) \]
\[ + \|\Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+\|^2 + \mathcal{E}_1. \]

The second term is estimated as follows
\[ [\tilde{L}, \Psi^{s+\sigma-1}\varrho L]u^+ = [\tilde{L}, \Psi^{s+\sigma-1}]\varrho L u^+ + \Psi^{s+\sigma-1}\tilde{L}(\varrho) L u^+ \]
\[ - 2\Psi^{s+\sigma-1}\varrho T u^+ + \Psi^{s+\sigma-1}\varrho L L u^+ \]
so that
\[ \|[\tilde{L}, \Psi^{s+\sigma-1}]\varrho L u^+\|^2 \leq C(\|\Psi^{s+\sigma-1}\varrho T u^+\|^2 + \|u\|^2_{-\infty}) \]
\[ \leq C(\|\varrho'u\|^2_{s+\sigma-2} + \|u\|^2_{-\infty}), \]

and
\[ \|\Psi^{s+\sigma-1}\tilde{L}(\varrho) L u^+\|^2 + \|\Psi^{s+\sigma-1}\varrho T u^+\|^2 \]
\[ \leq C(\|z\Psi^{s+\sigma}\varrho'u^+\|^2 + \|\Psi^{s+\sigma}\varrho u\|^2) + \mathcal{E}_2. \]

Furthermore we have
\[ \|\Psi^{s+\sigma-1}\varrho L L u^+\|^2 \leq C(\|\varrho'E_k u\|^2_{s+\sigma-\frac{1}{2}} + \mathcal{E}_3). \]
The terms $E$ are bounded as follows

$$E_1 \leq C(\|u^0\|_{s+\sigma}^2 + \|z\Psi^{s+\sigma}u^+\|^2 + \|u\|_{-\infty}^2).$$

By Lemma 10 we get

$$E_1 \leq C(\|\phi E_ku\|_{s_\sigma}^2 + \|\phi'u\|_{s_\sigma}^2 + \|u\|_{-\infty}^2),$$

$$E_2 \leq C(\phi'u\|_{s+\sigma-1}^2 + E_1) \leq C'E_1,$$

and

$$E_3 \leq C(\|z^{2k-1}\phi u^+\|_{s_\sigma}^2 + \|z^{2k-2}\phi u^+\|_{s-\frac{1}{2}}^2 + E_2) \leq C'E_2.$$

Hence we have

$$\|\Psi^{s+\sigma}\phi u^+\|^2 + \|z\Psi^{s+\sigma}\phi Lu^+\|^2$$

$$\leq C(\|z^{2s+\frac{1}{2}}\phi Lu^+\|^2 + \|\phi E_ku\|_{s_\sigma}^2 + \|\phi'u\|_{s_\sigma}^2 + \|u\|_{-\infty}^2).$$

To estimate the first term on the right we will use Lemma 8 as follows.

$$\|z^{2s+\frac{1}{2}}\phi Lu^+\|^2 \leq C(\|z\Psi^{s+\sigma}\phi Lu^+\|^2 + \|\phi u^0\|_{s+\sigma+\frac{1}{2}}^2 + \|u\|_{-\infty}^2).$$

We apply Lemma 8 with $a = \frac{1}{2}$, $m = k - 1$, $s$ replaced by $s + \sigma$, and $u$ replaced by $zLu$ to obtain

$$\|z^{2\Psi^{s+\sigma}+\frac{1}{2}}Lu^+\|^2 \leq 1.c.\|z^{k-1}\Psi^{s+\sigma}+\frac{k-1}{2}Lu^+\|^2 + \text{s.c.}\|z\Psi^{s+\sigma}Lu^+\|^2$$

$$+ \|\phi u^0\|_{s+\sigma+\frac{k-1}{2}}^2 + \|u\|_{-\infty}^2$$

$$\leq 1.c.\|z^{k}\Psi^{s+\sigma}+\frac{k-1}{2}Lu^+\|^2 + \text{s.c.}\|z\Psi^{s+\sigma}Lu^+\|^2$$

$$+ C(\|\phi u^0\|_{s+\sigma+\frac{k}{2}}^2 + \|u\|_{-\infty}^2).$$

Therefore we have

$$\|\Psi^{s+\sigma}\phi u^+\|^2$$

$$\leq C(\|z^k\Psi^{s+\sigma}+\frac{k-1}{2}Lu^+\|^2 + \|\phi E_ku\|_{s+\sigma+\frac{k-1}{2}-2}^2 + \|\phi'u\|_{s_\sigma}^2 + \|u\|_{-\infty}^2)$$

$$\leq C(\|\Psi^{s+\sigma}+\frac{k-1}{2}Lu^+\|^2 + \|\phi E_ku\|_{s+\sigma+\frac{k-1}{2}-2}^2 + \|\phi'u\|_{s_\sigma}^2 + \|u\|_{-\infty}^2).$$

Next, from Lemma 8 with $m = k$, $a = \frac{1}{2}$ and $s$ replaced by $s + \sigma$, we have

$$\|z\Psi^{s+\sigma}+\frac{1}{2}Lu^+\|^2 \leq 1.c.\|z^k\Psi^{s+\sigma}+\frac{k}{2}Lu^+\|^2$$

$$+ \text{s.c.}\|\Psi^{s+\sigma}Lu^+\|^2 + C(\|\phi u^0\|_{s+\sigma+\frac{k}{2}-1}^2 + \|u\|_{-\infty}^2)$$

$$\leq C(\|\Psi^{s+\sigma}+\frac{k-1}{2}Lu^+\|^2 + E_1.$$
and
\[ \| z^k \Psi^{s+\sigma+\frac{k}{2}} g'u^+ \|^2 \leq C \| \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi}^k L u^+ \|^2 + \mathcal{E}_1 \]
\[ = -C(\| \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} L |z|^{2k} L u^+, \| \Psi^{s+\sigma+\frac{k-1}{2}} g'u^+ \|) \]
\[ - 2C(\| \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi}^k L u^+, \| \Psi^{s+\sigma+\frac{k-1}{2}} z^{k+1} \mu u^+ \|) + \mathcal{E}_2 \]

since
\[ \| (\Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi}^k L u^+, \Psi^{s+\sigma+\frac{k-1}{2}} z^{k+1} \mu u^+) \| \leq s.c. \| z^k \Psi^{s+\sigma+\frac{k}{2}} g'u^+ \|^2 \]
\[ + 1.c. \| z^{k+1} \Psi^{s+\sigma+\frac{k-1}{2}} g'u^+ \|^2 + \mathcal{E}_2. \]

Hence we obtain
\[ \| z^k \Psi^{s+\sigma+\frac{k}{2}} g'u^+ \|^2 \leq C \| \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi}^k L u^+ \|^2 + \mathcal{E}_2 \]
\[ \leq C \| (\Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} E_k u^+, \Psi^{s+\sigma+\frac{k-1}{2}} g'u^+) \| + \mathcal{E}_3 \]
\[ \leq C \| (\Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} E_k u^+, \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} L \bar{\Phi} u^+) \| + \mathcal{E}_3 \]
\[ \leq C \| (\bar{\Phi} E_k u^+, \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} L \bar{\Phi} u^+) \|^2 + \| \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} \bar{\Phi} L \bar{\Phi} u^+ \|^2 + \mathcal{E}_5 \]
\[ \leq C \| (\bar{\Phi} E_k u^+, \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} L \bar{\Phi} u^+) \|^2 + \| \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} \bar{\Phi} \bar{\Phi} L \bar{\Phi} \bar{\Phi} u^+ \|^2 + \mathcal{E}_5 \]
\[ \leq C \| (\Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} L \bar{\Phi} u^+, \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} \bar{\Phi} \bar{\Phi} L \bar{\Phi} \bar{\Phi} u^+) \| + \mathcal{E}_5 \]
\[ \leq C \| (\Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} L \bar{\Phi} u^+, \Psi^{s+\sigma+\frac{k-1}{2}} \bar{\Phi} \bar{\Phi} \bar{\Phi} L \bar{\Phi} \bar{\Phi} u^+) \|^2 + \mathcal{E}_5 \]

Thus, applying Lemma 9 with \( m = 2k - 1, j = 2k - 2, A = k - \frac{3}{2}, B = k - 1 \), and \( s \) replaced by \( s + \sigma \), we have
\[ \frac{mA}{j} = \frac{2k - 1}{2k - 2} (k - \frac{3}{2}) < k - 1 = B. \]

Now,
\[ \| z^{2k-2} \Psi^{s+\sigma+k-\frac{3}{2}} g'u^+ \|^2 \leq s.c. \| z^{2k-1} \Psi^{s+\sigma+k-1} g'u^+ \|^2 + \mathcal{E}_7. \]

Replacing \( g'u^+ \) by \( z^{k-1} g'u^+ \) and \( s \) by \( s + \frac{k-2}{2} \) we obtain
\[ \| z^{2k-1} \Psi^{s+\sigma+k-1} g'u^+ \|^2 \]
\[ \leq C \| (\Psi^{s+\sigma+k-\frac{3}{2}} \bar{\Phi}^2 \bar{\Phi} \bar{\Phi} \bar{\Phi} \bar{\Phi} z^{2k-1} L u^+) \|^2 + \mathcal{E}_8 \]
\[ \leq C \| (\Psi^{s+\sigma+k-\frac{3}{2}} \bar{\Phi}^2 \bar{\Phi} \bar{\Phi} \bar{\Phi} \bar{\Phi} z^{2k-1} L u^+) \|^2 + \mathcal{E}_8 \]
\[ \leq C \| (\Psi^{s+\sigma+k-\frac{3}{2}} \bar{\Phi}^2 \bar{\Phi} \bar{\Phi} \bar{\Phi} \bar{\Phi} z^{2k-1} L u^+) \|^2 + \mathcal{E}_8 \]
\[ \leq C \| (\Psi^{s+\sigma+k-\frac{3}{2}} \bar{\Phi}^2 \bar{\Phi} \bar{\Phi} \bar{\Phi} \bar{\Phi} z^{2k-1} L u^+) \|^2 + \mathcal{E}_8 \]
\[ \leq C \| (\Psi^{s+\sigma+k-\frac{3}{2}} \bar{\Phi}^2 \bar{\Phi} \bar{\Phi} \bar{\Phi} \bar{\Phi} z^{2k-1} L u^+) \|^2 + \mathcal{E}_8 \]
\[ \leq C \| (\Psi^{s+\sigma+k-\frac{3}{2}} \bar{\Phi}^2 \bar{\Phi} \bar{\Phi} \bar{\Phi} \bar{\Phi} z^{2k-1} L u^+) \|^2 + \mathcal{E}_8 \]
\[ \leq C \| (\Psi^{s+\sigma+k-\frac{3}{2}} \bar{\Phi}^2 \bar{\Phi} \bar{\Phi} \bar{\Phi} \bar{\Phi} z^{2k-1} L u^+) \|^2 + \mathcal{E}_8 \]
\[ \leq C \| (\Psi^{s+\sigma+k-\frac{3}{2}} \bar{\Phi}^2 \bar{\Phi} \bar{\Phi} \bar{\Phi} \bar{\Phi} z^{2k-1} L u^+) \|^2 + \mathcal{E}_8 \]
To complete the proof of the a priori estimate we will analyze the error terms:

\[ E_1 \sim \| z^k \psi^{s+\sigma+\frac{k-1}{2}} g u^+ \|^2 + \| g' u^0 \|_{s+\sigma+\frac{k-1}{2}} + \text{s.c.} \| \psi^{s+\sigma} g u^+ \|^2 + \| u \|_{-\infty}^2, \]

\[ E_2 \sim E_1 + \| z^{k+1} \psi^{s+\sigma+\frac{k-1}{2}} g' u^+ \|^2, \]

\[ E_3 \sim E_2 + \| g' u^0 \|_{s+\sigma+\frac{k}{2}} + \text{s.c.} \| \psi^{s+\sigma+\frac{k+1}{2}} g z^k u^+ \|^2, \]

\[ E_4 \sim E_3 + \| \psi^{s+\sigma+\frac{k+1}{2}} g \bar{L} u^+ \|^2, \]

\[ E_5 \sim E_4 + \| z^k \psi^{s+\sigma} g' u^+ \|^2, \]

\[ E_6 \sim E_5 + \| z^{2k-1} \psi^{s+\sigma+k-\frac{3}{2}} g' u^+ \|^2 + \| z^{2k-2} \psi^{s+\sigma+k-\frac{5}{2}} g' u^+ \|^2, \]

\[ E_7 \sim E_6 + \| g u^0 \|_{s+\sigma+k-2} + \| u^+ \|_{-N}, \]

\[ E_8 \sim E_7 + \| \psi^{s+\sigma+k-\frac{3}{2}} g' z^k \bar{L} u^+ \|^2 + \| \psi^{s+\sigma+k-\frac{5}{2}} g' \bar{L} u^+ \|^2, \]

\[ E_9 \sim E_8 + \text{s.c.} \| \psi^{s+\sigma+k-1} g z^{2k-1} \bar{L} u^+ \|^2 + \| z^{2k-1} \psi^{s+\sigma+k-2} g' u^+ \|^2, \]

and

\[ E_{10} \sim E_9 + \text{s.c.} \| \psi^{s+\sigma+k-1} g \bar{L} u^+ \|^2 + \| \psi^{s+\sigma+k-2} g' \bar{L} u^+ \|^2. \]

The “admissible” errors are \( \| g' u \|_s^2 + \| u \|_{-\infty}^2 \). The terms involving \( u^0 \) are all bounded by const.\( \| g' E_k u \|_{s+\sigma+k-2}^2 \) modulo admissible errors. The terms involving a small constant s.c. are absorbed in the left. The term \( \| z^k \psi^{s+\sigma} g' u^+ \|^2 \) is bounded by const.\( \| g' E_k u \|_s^2 \), and the remaining terms can be bounded by a constant times \( A(s, g') \), where \( A(s, g') \) is defined by

\[ A(s, g') = \| z^k \psi^{s+\sigma+\frac{k-1}{2}} g' u^+ \|^2 + \| z^{2k-1} \psi^{s+\sigma+k-\frac{3}{2}} g' u^+ \|^2 + \| \psi^{s+\sigma+k-\frac{7}{2}} g' \bar{L} u^+ \|^2. \]

Repeating the same estimates with \( s \) replaced by \( s - \frac{1}{2} \) we replace the error \( A(s, g') \) by \( A(s - \frac{1}{2}, g') \). Repeating this process \( 2k-2 \) times (and redefining \( g' \)) we obtain the desired a priori estimate, namely:

\[ \| \psi^{s+\sigma} g u^+ \|^2 \leq C(\| g' E_k u \|_{s+\sigma+k-1}^2 + \| g' u \|_s^2 + \| u \|_{-\infty}^2). \]
14. Smoothing

To conclude the proof of Theorem C we will apply the above estimate to the smoothing of a solution. Given a distribution solution $u$ of $E_k u = f$ with $f$ whose restriction to $U$ is in $C^\infty(U)$, we wish to show that the restriction of $u$ to $U$ is in $C^\infty$. Without loss of generality we assume that the distribution $u$ has compact support and lies in $H^{-s_0}(\mathbb{R}^3)$. For $\delta > 0$ we will define a smoothing operator $K_\delta$ such that $K_\delta u \in C^\infty$ and $\lim_{\delta \to 0} K_\delta (\check{u}^+) \sim \check{u}^+$.

**Definition 7.** Let $\omega \in C^\infty_0(\mathbb{R})$, with $\omega(0) = 1$ and let $\kappa_\delta(\xi) = \omega(\delta \xi_3)\gamma^+(\xi)$ and

$$
\hat{K}_\delta u(\xi) = \kappa_\delta(\xi)\hat{u}(\xi),
$$

where $\gamma^+(\xi) = 1$ in a neighborhood of the support of $\hat{u}^+$.

**Lemma 12.** If $\|K_\delta (\check{u}^+)\|_s \leq C$ and if $\check{u}'^0 \in H^s$ then $\check{u}^+ \in H^s$.

**Proof:** We have

$$
\|K_\delta (\check{u}^+) - \check{u}^+\|_s \leq \|K_\delta ((\check{u}^+) - (\check{u}^+))\|_s + C\|\check{u}'^0\|_s
$$

and

$$
\lim_{\delta \to 0} (1 + |\xi|^2)^{\frac{s}{2}} \omega(\delta \xi_3)(\check{u}(\xi)) = (1 + |\xi|^2)^{\frac{s}{2}}(\check{u}^+(\xi)).
$$

Then $(\check{u}^+) \in H^s$ and since $(\check{u}^+) - \check{u}^+$ is supported in the elliptic region $\Omega^0$ we have

$$
\|\check{u}^+\|_s \leq \|(\check{u})^+\|_s + C\|\check{u}'^0\|_s,
$$

thus concluding the proof.

**Lemma 13.** For $\delta > 0$, $K_\delta$ is a pseudodifferential operator of order $-\infty$ which is of order zero uniformly in $\delta$. $K_\delta$ has the following commutation properties.

1. $[E, K_\delta](I - \Gamma^0)$ is a pseudodifferential operator of order $-\infty$ uniformly in $\delta$.

2. If $R^s$ is a pseudodifferential operator of order $s$ then

$$
[R^s, K_\delta] = \Gamma^0 R^s_\delta - 1 + \Psi^{s-1} R^0_\delta + R^{-\infty}_\delta,
$$

where $R^{s-1}_\delta$, $R^0_\delta$, and $R^{-\infty}_\delta$ are pseudodifferential operators of orders $-\infty$ for $\delta > 0$ and of orders $s-1$ and $0$ uniformly in $\delta$.

**Proof.** Number 1 follows from the fact that when $|\xi| \geq 1$ then $\gamma^0(\xi) = 1$ on the support of these symbols. To deal with number 2 we write the principal symbol of $[R^s, K_\delta]$. Setting $x_1 = x$, $x_2 = y$ and $x_3 = t$, we have

$$
\sum_j \frac{\partial \kappa_\delta}{\partial \xi_j} \frac{\partial r^s}{\partial x_j} = \delta \omega'(\delta \xi_3)\gamma^+ \frac{\partial r^s}{\partial x_3} + \sum_j \omega(\delta \xi_3) \frac{\partial \gamma^+}{\partial \xi_j} \frac{\partial r^s}{\partial x_j}.
$$
The lemma then follows, since
\[ \delta \omega' (\delta \xi_3) \tilde{\gamma} + \frac{\partial \tau^s}{\partial x_3} = \xi_3^{-1} \gamma + \left\{ \tilde{\gamma} + \xi_3^{-s} \delta \xi_3 \omega'(\delta \xi_3) \frac{\partial \tau^s}{\partial x_3} \right\}, \]
where \( \tilde{\gamma} = 1 \) in a neighborhood of the support of \( \gamma \) and equals zero in a neighborhood of the origin. The expression in braces is the symbol of an operator of order zero uniformly in \( \delta \).

**Conclusion of proof of Theorem C.** Substituting \( K_\delta u \) for \( u \) in (10) we obtain
\[ \| \Psi^{s+\sigma} \varrho K_\delta u^+ \|^2 \leq C(\| \varrho' E_k K_\delta u \|^2_\infty + \| \varrho' K_\delta u \|^2_s + \| K_\delta u \|^2_\infty). \]
Then we have
\[ \| K_\delta (\varrho u^+) \|^2_\infty \leq C(\| \Psi^{s+\sigma} \varrho K_\delta u^+ \|^2 + \| \varrho' u \|^2_\infty), \]
\[ \| [\varrho' E_k, K_\delta] u \|^2_\infty \leq C(\| \varrho'' u \|^2_\infty + \| \varrho' u \|^2_\infty), \]
and
\[ \| K_\delta u \|^2_\infty \leq C\| u \|^2_\infty. \]
Further
\[ \| \varrho'' u \|^2_\infty \leq C(\| \varrho''' E_k u \|^2 + \| u \|^2_\infty). \]
Therefore, changing notation for the cutoff functions, we get
\[ \| K_\delta (\varrho u^+) \|^2_\infty \leq C(\| \varrho' E_k u \|^2_\infty + \| \varrho' u \|^2_\infty + \| u \|^2_\infty). \]
Therefore, if \( u \in H^{-s_0} \), if \( u^+ \in H^{s_0}_{\text{loc}}(U) \), and if \( E_k u \in H^{s_0+k-1}_{\text{loc}}(U) \) then \( u^+ \in H^{s_0+k}_{\text{loc}}(U) \). It then follows that if \( u \in H^{-s_0} \) and if \( E_k u \in H^{s_1}_{\text{loc}}(U) \) then \( u^+ \in H^{s_1+k-1}_{\text{loc}}(U) \). Since, under the same assumptions, we have \( u^0 \in H^{s_1+2}_{\text{loc}}(U) \) and \( u^- \in H^{s_1+1}_{\text{loc}}(U) \) we conclude that \( u \in H^{s_1+k-1}_{\text{loc}}(U) \), thus proving Theorem C.

15. Local existence in \( L^2 \)

The a priori estimates for \( E_k \) imply the following local existence result.

**Theorem.** If \( P \in U \subset \mathbb{R}^3 \) with \( U \) an open set, then there exists a neighborhood \( U_1 \subset \bar{U}_1 \subset U \), with \( P \in U_1 \), such that if \( f \in H^{k-1}_{\text{loc}}(U) \) then there exists \( u \in L^2(U_1) \) and \( E_k u = f \) in \( U_1 \).

**Proof.** Let \( U_1 \) be a small neighborhood of \( P \). In Lemma 11 set \( \varrho = 1 \) in a neighborhood of \( \bar{U}_1 \) and set \( u = v \in C_c^\infty(U_1) \) so that \( \varrho u = v \) and \( [\Psi^{s+\sigma}, \Gamma^+] \) is an operator of order \( -\infty \) on \( C_c^\infty(U_1) \). Hence we obtain
\[ \| \Psi^{s+\sigma} u^+ \|^2 \leq C(\| E_k v \|^2_\infty + \| v \|^2_\infty), \]
for all \( v \in C_0^\infty(U_1) \). Setting \( s + \sigma + k = 0 \) and combining with the estimates for \( v^0 \) and \( v^- \), we obtain

\[
\|v\|_{-k+1}^2 \leq C(\|E_k v\|_2^2 + \|v\|_{-k+1-\sigma}^2).
\]

Then, if the diameter of \( U_1 \) is sufficiently small, we have

\[
\|v\|_{-k+1-\sigma}^2 \leq \text{small const.} \|v\|_{-k+1}^2.
\]

Hence

\[
\|v\|_{-k+1} \leq \text{const.} \|E_k v\|,
\]

for all \( v \in C_0^\infty(U_1) \).

Let \( \mathcal{W} = C_0^\infty(U_1) \) and let \( K : \mathcal{W} \to \mathbb{C} \) be the linear functional defined by \( Kw = (v, f) \) with \( w = E_k v \). Then

\[
|Kw| = |(v, f)| \leq \|v\|_{-k+1} \|f\|_{k-1} \leq C\|w\|.
\]

So \( K \) is bounded on \( \mathcal{W} \); hence it can be extended to a bounded linear functional on \( L^2(U_1) \). Therefore there exists \( u \in L^2(U_1) \) such that \( Kw = (w, u) \), that is \( (v, f) = (E_k v, u) = (v, E_k u) \). Thus \( E_k u = f \) in \( L^2(U_1) \), which completes the proof.

Princeton University, Princeton, NJ
E-mail address: kohn@math.princeton.edu

References

[BM] D. Bell and S. Mohammed, An extension of Hörmander’s theorem for infinitely degenerate second-order operators, Duke Math. J. 78 (1995), 453–475.

[BDKT] A. Bove, M. Derridj, J. J. Kohn, and D. S. Tartakoff, Hypoellipticity for a sum of squares of complex vector fields with large loss of derivatives, preprint.

[C] D. Catlin, Necessary conditions for the subellipticity of the \( \bar{\partial} \)-Neumann problem, Ann. of Math. 117 (1983), 147–171; Subelliptic estimates for the \( \bar{\partial} \)-Neumann problem on pseudoconvex domains, Ann. of Math. 126 (1987), 131–191.

[Ch1] M. Christ, Hypoellipticity in the infinitely degenerate regime, in Complex Analysis and Geometry (Columbus, OH, 1999), 59–84, Ohio State Univ. Math. Res. Inst. Publ. 9, de Gruyter, Berlin, 2001.

[Ch2] ———, A counterexample for sums of squares of complex vector fields, preprint, 2004.

[ChK] M. Christ and G. E. Karadjo, Local solvability for a class of partial differential operators with double characteristics, preprint.

[D’A] J. P. D’Angelo, Real hypersurfaces, orders of contact, and applications, Ann. of Math. 115 (1982), 615–637.

[DT] M. Derridj and D. Tartakoff, Local analytic hypoellipticity for a sum of squares of complex vector fields with large loss of derivatives, preprint.

[F] V. S. Fedii, A certain criterion for hypoellipticity, Mat. Sb. 14 (1971), 15–45.

[FP] C. Fefferman and D. H. Phong, The uncertainty principle and sharp Gårding inequalities, Comm. Pure Appl. Math. 34 (1981), 285–331.
Appendix:
Analyticity and loss of derivatives

By MAKHLOUF DERRIDJ and DAVID S. TARTAKOFF

Abstract

In [2], J. J. Kohn proves $C^\infty$ hypoellipticity for a sum of squares of complex vector fields which exhibit a large loss of derivatives. Here, we prove analytic hypoellipticity for this operator.
1. Introduction and outline

In [2], J. J. Kohn proves hypoellipticity for the operator

\[ P = LL^* + (\bar{z}^k L)^*(\bar{z}^k L), \quad L = \frac{\partial}{\partial z} + iz \frac{\partial}{\partial t}, \]

for which there is a large loss of derivatives — indeed in the \textit{a priori} estimate one bounds only the Sobolev norm of order \(-(k - 1)/2\), and thus there is a loss of \(k - 1\) derivatives: \(Pu \in H^s_{\text{loc}} \implies u \in H^{s-(k-1)}_{\text{loc}}\).

We show in this note that solutions of \(Pu = f\) with \(f\) real analytic are themselves real analytic in any open set where \(f\) is. In so doing we use an \textit{a priori} estimate which follows easily from that established by Kohn for this operator, namely for test functions \(v\) of small support near the origin:

\[ \|Lv\|^2_0 + \|\bar{z}^k Lv\|^2_0 + \|v\|^2_{k-1} \lesssim |(Pv, v)_{L^2}|. \]

In fact, in [5] (see also [1]), we give a rapid and direct derivation of (1.1) for this operator and similar estimates for more degenerate operators.

The first two terms on the left of this estimate exhibit maximal control in \(L\) and \(\bar{z}^k L\), but only these complex directions. Hence in obtaining recursive bounds for derivatives it is essential to keep one of these vector fields available for as long as possible. For this, we will construct a carefully balanced localization of high powers of \(T = -2i\partial/\partial t\) and use the estimate repeatedly, reducing the order of powers of \(T\) but accumulating derivatives on the localizing functions. These Ehrenpreis type localizing functions work ‘as if analytic’ up to a prescribed order, with all constants independent of that order, as in [3], [4], but eventually the good derivatives (\(\bar{L}\) or \(\bar{z}^k L\)) are lost and we must use the third term on the left of the estimate, absorb the loss of \(\frac{k-1}{2}\) derivatives, introduce a new localizing function of larger support and start the whole process again, but with only a (fixed) fraction of the original power of \(T\).

2. Observations and simplifications

Our first observation is that we know the analyticity of the solution for \(z\) different from 0 from the earlier work of the second author [3], [4] and Trèves [6]. Thus, modulo brackets with localizing functions whose derivatives are supported in the known analytic hypoelliptic region, we take all localizing functions independent of \(z\).

Our second observation is that it suffices to bound derivatives measured in terms of high powers of the vector fields \(L\) and \(\bar{T}\) in \(L^2\) norm, by standard arguments, and indeed estimating high powers of \(L\) can be reduced to bounding high powers of \(\bar{T}\) and powers of \(T\) of half the order, by repeated integration by parts. Thus our overall scheme will be to start with high powers (order \(2p\)) of
L or $\bar{T}$, use integration by parts and the a priori estimate repeatedly to reduce to treating $T^p u$ in a slightly larger set.

And to do this, we introduce a new special localization of $T^p$ adapted to this problem.

3. The localization of high powers of $T$

The new localization of $T^p$ may be written in the form:

$$ (T^{p_1,p_2})_\varphi = \sum_{\substack{a \leq p_1 \\
 b \leq p_2}} \frac{L^a \circ z^a \circ T^{p_1-a} \circ \varphi^{(a+b)} \circ T^{p_2-b} \circ \varphi \circ \bar{T}}{a! b!}. $$

Here by $\varphi^{(r)}$ we mean $(-i\partial/\partial t)^r \varphi(t)$ since near $z = 0$ we have seen that we may take the localizing function independent of $z$. Note that the leading term (with $a + b = 0$) is merely $T^{p_1} \varphi T^{p_2}$ which equals $T^{p_1+p_2}$ on the initial open set $\Omega_0$ where $\varphi \equiv 1$.

We have the commutation relations:

$$ [L, (T^{p_1,p_2})_\varphi] \equiv L \circ (T^{p_1-1,p_2})_\varphi', $$

$$ [\bar{T}, (T^{p_1,p_2})_\varphi] \equiv (T^{p_1,p_2-1})_{\varphi'} \circ \bar{T}, $$

$$ [(T^{p_1,p_2})_\varphi, z] \equiv (T^{p_1-1,p_2})_{\varphi'} \circ z, $$

and

$$ [(T^{p_1,p_2})_\varphi, \bar{T}] \equiv \bar{T} \circ (T^{p_1,p_2-1})_{\varphi'}, $$

where the $\equiv$ denotes modulo $C^{p_1-p_1'+p_2-p_2'}$ terms of the form

$$ \left[ L^{p_1-p_1'} \circ z^{p_1-p_1'} \circ T^{p_2} \circ \varphi^{(p_1-p_1'+p_2-p_2'+1)} \circ T^{p_2'} \circ T^{p_2-p_2'} \circ \bar{T}^{p_2-p_2'} \right] \left( \frac{1}{(p_1-p_1')!(p_2-p_2')!} \right) $$

with either $p_1' = 0$ or $p_2' = 0$, i.e., terms where all free $T$ derivatives have been eliminated on one side of $\varphi$ or the other. Thus if we start with $p_1 = p_2 = p/2$, and iteratively apply these commutation relations, the number of $T$ derivatives not necessarily applied to $\varphi$ is eventually at most $p/2$.

4. The recursion

We insert first $v = (T^{\tilde{p},\tilde{p}})\varphi u$ in the a priori inequality, then bring $(T^{\tilde{p},\tilde{p}})\varphi$ to the left of $P = -L\bar{T} - \bar{T}z^{k-\tilde{k}}L$ since $P u$ is known and analytic. We have, omitting for now the ‘subelliptic’ term,

$$ \|\bar{T}(T^{\tilde{p},\tilde{p}})\varphi u\|_0^2 + \|\bar{T}^k L(T^{\tilde{p},\tilde{p}})\varphi u\|_0^2 \lesssim \|(P(T^{\tilde{p},\tilde{p}})\varphi u, (T^{\tilde{p},\tilde{p}})\varphi u)_{L^2}\| $$

$$ \lesssim \|(T^{\tilde{p},\tilde{p}})\varphi Pu, (T^{\tilde{p},\tilde{p}})\varphi u)_{L^2}\| + \|(P, (T^{\tilde{p},\tilde{p}})\varphi u)_{L^2}\| $$
and, by the above bracket relations,

\[ ([P, (T^\frac{5}{2} \bar{z})] \varphi) u, (T^\frac{5}{2} \bar{z}) \varphi) u \]

\[ = - ([L \bar{\varphi}, (T^\frac{5}{2} \bar{z})] \varphi) u, (T^\frac{5}{2} \bar{z}) \varphi) u - ([L \bar{z} \bar{\varphi}, (T^\frac{5}{2} \bar{z})] \varphi) u, (T^\frac{5}{2} \bar{z}) \varphi) u \]

\[ = - (L(T^\frac{5}{2} \bar{z} - 1) \varphi) L u, (T^\frac{5}{2} \bar{z}) \varphi) u - (L(T^\frac{5}{2} \bar{z} - 1) \bar{\varphi}) \bar{L} u, (T^\frac{5}{2} \bar{z}) \varphi) u \]

\[ - \sum_{k'=1}^{k} (L z^k (T^\frac{5}{2} \bar{z} - 1) \varphi) z^{k-k'} \bar{z}^k L u, (T^\frac{5}{2} \bar{z}) \varphi) u \]

\[ - \sum_{k'=0}^{k-1} (L z^k \bar{z}^{k-k'} (T^\frac{5}{2} \bar{z} - 1) \varphi) \bar{z}^{k-k'} \bar{z}^k L u, (T^\frac{5}{2} \bar{z}) \varphi) u \]

\[ - (L z^k \bar{z}^{k} L (T^\frac{5}{2} \bar{z} - 1) \varphi) \bar{L} u, (T^\frac{5}{2} \bar{z}) \varphi) u, \]

with the same meaning for \( \equiv \) as above. In every term, no powers of \( z \) or \( \bar{z} \) have been lost, though some may need to be brought to the left of the \( (T^q_i \bar{z}) \bar{\varphi} \) with again no loss of powers of \( z \) or \( \bar{z} \) and a further reduction in order, every bracket reduces the order of the sum of the two indices \( p_1 \) and \( p_2 \) by one (here we started with \( p_1 = p_2 = p/2 \)), picks up one derivative on \( \varphi \), and leave the vector fields over which we have maximal control in the estimate intact and in the correct order. Thus we may bring either \( L \bar{z}^k \) or \( L \) to the right as \( \bar{z}^k L \) or \( L \), and use a weighted Schwarz inequality on the result to take maximal advantage of the a priori inequality. Iterations of all of this continue until there remain at most \( p/2 \) free \( T \) derivatives (i.e., the \( T \) derivatives on at least one side of \( \varphi \) are all 'corrected' by good vector fields) and perhaps as many as \( p/2 \) \( L \) or \( \bar{L} \) derivatives, and we may continue further until, at worst, these remaining \( L \) or \( \bar{L} \) derivatives bracket two at a time to produce more \( T \)'s, with corresponding combinatorial factors. After all of this, there will be at most \( T^{\frac{3p}{2}} \) remaining, and a factor of \( \frac{p!}{2} \sim \frac{p}{2} \)!

It is here that the final term on the left of the a priori inequality is used, in order to bring the localizing function out of the norm after creating another balanced localization of \( T^{3p/4} \) with a new localizing function of Ehrenpreis type with slightly larger support, geared, roughly, to \( 3p/4 \) instead of to \( p \).

Recall that such such localizing functions \( \psi \) may be constructed for any \( N \) and satisfy

\[ |\psi^{(r)}| \leq \left( \frac{C}{e} \right)^{r+1} N^r, \quad r \leq 2N \]

where \( C \) is independent of \( N \) and \( e = \text{dist}(\{\psi \equiv 1\},(\text{supp } \psi)^c) \).

5. Conclusion of the proof

Finally, this entire process, which reduced the order from \( p \) to at most \( 3p/4 \), (or more precisely to at most \( 3p/4 + (k - 1)/2 \)), is repeated, over and
over, each time essentially reducing the order by a factor of \(3/4\). After at most \(\log(3/4)p\) such iterations we are reduced to a bounded number of derivatives, and, as in \([3]\) and \([4]\), all of these nested open sets may be chosen to fit in the one open set \(\Omega'\) where \(Pu\) is known to be analytic, and all constants chosen independent of \(p\) (but depending on \(Pu\)). The fact that in those works one full iteration reduced the order by half played no essential role — a factor of \(3/4\) works just as well.

To be precise, the sequence of open sets, \(\{\Omega_j\}\), each compactly contained in the next, with \(\Omega_{\log(3/4)p} = \Omega'\), have separations \(d_j = \text{dist}(\Omega_j, \Omega_{j+1}')\), \(\sum d_j = \text{dist}(\Omega_0, \Omega'') = d\), which need to be picked carefully. The localizing functions \(\{\varphi_j\}\) with \(\varphi_j \in C_0^\infty(\Omega_j) \equiv 1 \text{ on } \Omega_j\) satisfy

\[
|\varphi_j^{(r)}| \leq (C/d_j)^{r+1}((3/4)^j)^r, \quad r \leq 2(3/4)^j p.
\]

We shall take the \(d_j = \frac{1}{(j+1)^2} \sum \frac{1}{(j+1)^2}\).

Now at most \((3/4)^j p\) derivatives will fall on \(\varphi_j\), and most of the effect of the derivatives will be balanced by corresponding factorials in the denominator, as in \((3.2)\), roughly the powers of \((3/4)^j p\) in \((5.3)\) in view of Stirling’s formula. In addition, as noted immediately before the last paragraph in Section 4, there will be factorials corresponding to the diminution of powers of \(T\). What will not be balanced are the powers of \(d_j^{-1}\), but the product of these factors will contribute

\[
\prod_{j=1}^{\log(3/4)p} (j^2)^{(3/4)^j p} = \left(\prod_{j=1}^{\log(3/4)p} j^{(3/4)^j p}\right)^{2p} = C^p,
\]

which, together with the factorials just mentioned, proves the analyticity of the solution in \(\Omega_0\).

---

5 rue de la Juviniere, 78350 Les Loges en Josas, France
E-mail address: derridj@club-internet.fr

University of Illinois at Chicago, Chicago IL
E-mail address: dst@uic.edu

References

[1] A. Bove, M. Derridj, J. J. Kohn, and D. S. Tartakoff, Hypoellipticity for a sum of squares of complex vector fields with large loss of derivatives, preprint.

[2] J. J. Kohn, Hypoellipticity and loss of derivatives, *Ann. of Math.* 162 (2005), 943–982.

[3] D. S. Tartakoff, Local analytic hypoellipticity for \(\square_b\) on nondegenerate Cauchy Riemann manifolds, *Proc. Nat. Acad. Sci. U.S.A.* 75 (1978), 3027–3028.

[4] ———, The local real analyticity of solutions to \(\square_b\) and the \(\bar{\partial}\)-Neumann problem, *Acta Math.* 145 (1980), 117–204.

[5] ———, Analyticity for singular sums of squares of degenerate vector fields, *Proc. Amer. Math. Soc.*, to appear.

[6] F. Trèves, Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications to the \(\bar{\partial}\)-Neumann problem, *Comm. Partial Differential Equations* 3 (1978), 475–642.

(Received April 13, 2005)