ONE DIMENSIONAL TOPOLOGICAL GALOIS THEORY

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ABSTRACT. In the preprint we present an outline of the one dimensional version of
topological Galois theory. The theory studies topological obstruction to solvability
of equations “in finite terms” (i.e. to their solvability by radicals, by elementary
functions, by quadratures and so on). The preprint is based on the author’s book on
topological Galois theory. It contains definitions, statements of results and comments
to them. Basically no proofs are presented.

The preprint was written as a part of the comments to a new edition (in prepara-
tion) of the classical book “Integration in finite terms” by J.F. Ritt.

1. Introduction. As was discovered by Camille Jordan the monodromy group of
an algebraic function is isomorphic to the Galois group of the associated extension
of the field of rational functions. Therefore the monodromy group is responsible for
the representability of an algebraic function by radicals (see [1]).

However, not only algebraic functions have the monodromy group. It is defined
for any solution of a linear differential equation whose coefficients are rational func-
tions and for many more functions, for which the Galois group does not make sense.
It is thus natural to try using the monodromy group for these functions instead of
the Galois group to prove that they do not belong to a certain Liouvillian class. This
particular approach is implemented in topological Galois theory (see [2]), which has
a one-dimensional version and a multidimensional version.

In the one-dimensional version we consider functions from Liouvillian classes as
multi-valued analytic functions of one complex variable. It turns out that there
exist topological restrictions on the way the Riemann surface of a function from a
certain Liouvillian class can be positioned over the complex plane. If a function does
not satisfy these restrictions, then it cannot belong to the corresponding Liouvillian
class.

Besides a geometric appeal, this approach has the following advantage. Topologi-
cal obstructions relate to branching. It turns out that if a function does not belong
to a certain Liouvillian class by topological reasons then it automatically does not
belong to a much wider class of functions. This wider class can be obtained if we
add to the Liouvillian class all single valued functions having at most countable set
of singularities and allow them to enter all formulas.

Key words and phrases. solvability by radicals, by elementary functions, by quadratures, by
generalized quadratures.

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The composition of functions is not an algebraic operation. In differential algebra, this operation is replaced with a differential equation describing it. However, for example, the Euler $\Gamma$-function does not satisfy any algebraic differential equation. Hence it is pointless to look for an equation satisfied by, say, the function $\Gamma(\exp x)$ and one can not describe it algebraically (but the function $y = \exp(\Gamma(x))$ satisfies the equation $y' = \Gamma'y$ over a differential field containing $\Gamma$ and it makes sense in the differential algebra). The only known results on non-representability of functions by quadratures and, say, the Euler $\Gamma$-functions are obtained by our method.

On the other hand, our method cannot be used to prove that a particular single valued meromorphic function does not belong to a certain Liouvillian class.

There are the following topological obstructions to representability of functions by generalized quadratures, $k$-quadratures and quadratures.

Firstly, the functions representable by generalized quadratures and, in particular, the functions representable by $k$-quadratures and quadratures may have no more than countably many singular points in the complex plane (see section 6).

Secondly, the monodromy group of a function representable by quadratures is necessarily solvable (see section 8). There are similar restrictions for for a function representable by generalized quadratures and $k$-quadratures. However, these restrictions are more involved. To state them, the monodromy group should be regarded not as an abstract group but rather as a transitive subgroup in the permutation group. In other terms, these restrictions make use not only of the monodromy group but rather of the monodromy pair of the function consisting of the monodromy group and the stabilizer of some germ of the function (see section 9).

One can prove that the only reasons for unsolvability in finite terms of Fuchsian linear differential equations are topological (see section 14). In other words, if there are no topological obstructions to solvability of a Fuchsian equation by generalized quadratures (by $k$-quadratures, by quadratures), then this equation is solvable by generalized quadratures (by $k$-quadratures or by quadratures respectively). The proof is based on a linear-algebraic part of differential Galois theory (dealing with linear algebraic groups and their differential invariants).

2. Solvability of equations in finite terms. An equation is solvable “in finite terms” (or is solvable “explicitly”) if its solutions belong to a certain class of functions. Different classes of functions correspond to different notions of solvability in finite terms.

A class of functions can be introduced by specifying a list of basic functions and a list of admissible operations. Given the two lists, the class of functions is defined as the set of all functions that can be obtained from the basic functions by repeated application of admissible operations. Below, we define Liouvillian classes of functions in exactly this way.

Classes of functions, which appear in the problems of integrability in finite terms, contain multivalued functions. Thus the basic terminology should be made clear. We work with multivalued functions “globally”, which leads to a more general understanding of classes of functions defined by lists of basic functions and of admissible operations. A multivalued function is regarded as a single entity. Operations on multivalued functions can be defined. The result of such an operation is a set of multivalued functions; every element of this set is called a function obtained from the given functions by the given operation. A class of functions is defined as the
set of all (multivalued) functions that can be obtained from the basic functions by repeated application of admissible operations.

### 3. Operations on multivalued functions.

Let us define, for example, the sum of two multivalued functions on a connected Riemann surface $U$.

**Definition 7.** Take an arbitrary point $a$ in $U$, any germ $f_a$ of an analytic function $f$ at the point $a$ and any germ $g_a$ of an analytic function $g$ at the same point $a$. We say that the multivalued function $\varphi$ on $U$ generated by the germ $\varphi_a = f_a + g_a$ is representable as the sum of the functions $f$ and $g$.

For example, it is easy to see that exactly two functions of one variable are representable in the form $\sqrt{x} + \sqrt{x}$, namely, $f_1 = 2\sqrt{x}$ and $f_2 \equiv 0$. Other operations on multivalued functions are defined in exactly the same way. *For a class of multivalued functions, being stable under addition means that, together with any pair of its functions, this class contains all functions representable as their sum.* The same applies to all other operations on multivalued functions understood in the same sense as above.

In the definition given above, not only the operation of addition plays a key role but also the operation of analytic continuation hidden in the notion of multivalued function. Indeed, consider the following example. Let $f_1$ be an analytic function defined on an open subset $V$ of the complex line $\mathbb{C}$ and admitting no analytic continuation outside of $V$, and let $f_2$ be an analytic function on $V$ given by the formula $f_2 = -f_1$. According to our definition, the zero function is representable in the form $f_1 + f_2$ on the entire complex line. By the commonly accepted viewpoint, the equality $f_1 + f_2 = 0$ holds inside the region $V$ but not outside.

Working with multivalued functions globally, we do not insist on the existence of a common region, were all necessary operations would be performed on single-valued branches of multivalued functions. A first operation can be performed in a first region, then a second operation can be performed in a second, different region on analytic continuations of functions obtained on the first step. In essence, this more general understanding of operations is equivalent to including analytic continuation to the list of admissible operations on the analytic germs.

### 4. Liouvillian classes of single variable functions.

In this section, we define Liouvillian classes of single variable functions (for many variables, the corresponding definitions can be found in [2]). We will describe these classes by lists of basic functions and admissible operations.

We will need the list of *basic elementary functions*. In essence, this list contains functions that are studied in high-school and which are frequently used in pocket calculators.

**List of basic elementary functions.**

1. All complex constants and an independent variable $x$.
2. The exponential, the logarithm and the power $x^\alpha$, where $\alpha$ is any complex constant.
3. Trigonometric functions: sine, cosine, tangent, cotangent.
4. Inverse trigonometric functions: arcsine, arccosine, arctangent, arccotangent.
Lemma 1. Basic elementary functions can be expressed through the exponentials and the logarithms with the help of complex constants, arithmetic operations and compositions.

Lemma 1 can be considered as a simple exercise. Its proof can be found in [3].

List of classical operations.
1. The operation of composition takes functions \( f, g \) to the function \( f \circ g \).
2. The arithmetic operations take functions \( f, g \) to the functions \( f + g, f - g, fg, \) and \( f/g \).
3. The operation of differentiation takes function \( f \) to the function \( f' \).
4. The meromorphic operation takes functions \( f_1, \ldots, f_n \) to the function \( F(f_1, \ldots, f_n) \) where \( F \) is a fixed meromorphic function of \( n \) complex variables.
5. The operation of integration takes function \( f \) to a solution of equation \( y' = f \) (the function \( y \) is defined up to an additive constant).
6. The operation of solving algebraic equations takes functions \( f_1, \ldots, f_n \) to the function \( y \) such that \( y^n + f_1 y^{n-1} + \cdots + f_n = 0 \) (the function \( y \) is not quite uniquely determined by functions \( f_1, \ldots, f_n \) since an algebraic equation of degree \( n \) can have \( n \) solutions).
7. The operation of solving linear differential equations takes functions \( f_1, \ldots, f_n \) to the function \( y \) such that \( y^{(n)} + f_1 y^{(n-1)} + \cdots + f_n = 0 \) (the function \( y \) is not uniquely determined by functions \( f_1, \ldots, f_n \) since a differential equation of order \( n \) has an \( n \) dimensional space of solutions).

We can now return to the definition of Liouvillian classes of single variable functions.

Functions representable by radicals. List of basic functions: all complex constants, an independent variable \( x \). List of admissible operations: arithmetic operations and the operation of taking the \( n \)-th root \( f^{1/n}, n = 2, 3, \ldots \), of a given function \( f \).

Functions representable by \( k \)-radicals. List of basic functions: all complex constants, an independent variable \( x \). List of admissible operations: arithmetic operations and the operation of taking the \( n \)-th root \( f^{1/n}, n = 2, 3, \ldots \), of a given function \( f \), the operation of solving algebraic equations of degree \( \leq k \).

Elementary functions. List of basic functions: basic elementary functions. List of admissible operations: compositions, arithmetic operations, differentiation.

Generalized elementary functions. This class of functions is defined in the same way as the class of elementary functions. We only need to add the operation of solving algebraic equations to the list of admissible operations.

Functions representable by quadratures. List of basic functions: basic elementary functions. List of admissible operations: compositions, arithmetic operations, differentiation, integration.

Functions representable by \( k \)-quadratures. This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations of degree at most \( k \) to the list of admissible operations.
Functions representable by generalized quadratures. This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations to the list of admissible operations.

5. Simple formulas with complicated topology. Developing topological Galois theory I followed the following plan:

I. To find a wide class of multivalued functions such that:
   a) it is closed under all classical operations;
   b) it contains all entire functions and all functions from each Liouvillian class;
   c) for functions from the class the monodromy group is well defined.

II. To use the monodromy group instead of the Galois group inside the class.

Let us discuss some difficulties that one need to overcome on this way.

Example. Consider an elementary function $f$ defined by the following formula:

$$f(z) = \ln \sum_{j=1}^{n} \lambda_j \ln(z - a_j)$$

where $a_j$ are different points in the complex line, and $\lambda_j \in \mathbb{C}$ are constants.

Let $\Lambda$ denote the additive subgroup of complex numbers generated by the constants $\lambda_1, \ldots, \lambda_n$. It is clear that if $n > 2$, then for almost every collection of constants $\lambda_1, \ldots, \lambda_n$, the group $\Lambda$ is everywhere dense in the complex line.

**Lemma 2.** If the group $\Lambda$ is dense in the complex line, then the elementary function $f$ has a dense set of logarithmic ramification points.

**Proof.** Let $g$ be the multivalued function defined by the formula

$$g(z) = \sum_{j=1}^{n} \lambda_j \ln(z - a_j).$$

Take a point $a \neq a_j$, $j = 1, \ldots, n$ and let $g_a$ be one of the germs of $g$ at $a$. A loop around the points $a_1, \ldots, a_n$ adds the number $2\pi i \lambda$ to the germ $g_a$, where $\lambda$ is an element of the group $\Lambda$. Conversely, every germ $g_a + 2\pi i \lambda$, where $\lambda \in \Lambda$, can be obtained from the germ $g_a$ by the analytic continuation along some loop. Let $U$ be a small neighborhood of the point $a$, such that the germ $g_a$ has a single-valued analytic continuation $G$ on $U$. The image $V$ of the domain $U$ under the map $G : U \to \mathbb{C}$ is open. Therefore, in the domain $V$, there is a point of the form $2\pi i \lambda$, where $\lambda \in \Lambda$. The function $G - 2\pi i \lambda$ is one of the branches of the function $g$ over the domain $U$, and the zero set of this branch in the domain $U$ is nonempty. Hence, one of the branches of the function $f = \ln g$ has a logarithmic ramification point in $U$.

The set $\Sigma$ of singular points of the function $f$ is a countable set (see section 6). Under assumptions of Lemma 2 the set $\Sigma$ is everywhere dense.

It is not hard to verify that the monodromy group (see section 7) of the function $f$ has the cardinality of the continuum. This is not surprising: the fundamental
group \( \pi_1(\mathbb{C} \setminus \Sigma) \) has obviously the cardinality of the continuum provided that \( \Sigma \) is a countable dense set.

One can also prove that the image of the fundamental group \( \pi_1(\mathbb{C} \setminus \{ \Sigma \cup b \}) \) of the complement of the set \( \Sigma \cup b \), where \( b \not\in \Sigma \), in the permutation group is a proper subgroup of the monodromy group of \( f \).

The fact that the removal of one extra point can change the monodromy group, makes all proofs more complicated.

Thus even simplest elementary functions can have dense singular sets and monodromy groups of cardinality of the continuum. In addition the removal of an extra point can change their monodromy groups.

6. Class of \( S \)-functions. In this section, we define a broad class of functions of one complex variable needed in the construction of topological Galois theory.

**Definition.** A multivalued analytic function of one complex variable is called a \( S \)-function, if the set of its singular points is at most countable.

Let us make this definition more precise. Two regular germs \( f_a \) and \( g_b \) defined at points \( a \) and \( b \) of the Riemann sphere \( S^2 \) are called equivalent if the germ \( g_b \) is obtained from the germ \( f_a \) by the analytic continuation along some path. Each germ \( g_b \) equivalent to the germ \( f_a \) is also called a regular germ of the multivalued analytic function \( f \) generated by the germ \( f_a \).

A point \( b \in S^2 \) is said to be a singular point for the germ \( f_a \) if there exists a path \( \gamma : [0, 1] \to S^2 \), \( \gamma(0) = a \), \( \gamma(1) = b \) such that the germ has no analytic continuation along this path, but for any \( \tau \), \( 0 \leq \tau < 1 \), it admits an analytic continuation along the truncated path \( \gamma : [0, \tau] \to S^2 \).

It is easy to see that equivalent germs have the same set of singular points. A regular germ is called a \( S \)-germ, if the set of its singular points is at most countable.

A multivalued analytic function is called a \( S \)-function if each its regular germ is a \( S \)-germ.

**Theorem 3 (on stability of the class of \( S \)-functions).** The class \( S \) of all \( S \)-functions is stable under the following operations:

1) differentiation, i.e. if \( f \in S \), then \( f' \in S \);
2) integration, i.e. if \( f \in S \) and \( g' = f \), then \( g \in S \);
3) composition, i.e. if \( g, f \in S \), then \( g \circ f \in S \);
4) meromorphic operations, i.e. if \( f_i \in S \), \( i = 1, \ldots, n \), the function \( F(x_1, \ldots, x_n) \) is a meromorphic function of \( n \) variables, and \( f = F(f_1, \ldots, f_n) \), then \( f \in S \);
5) solving algebraic equations, i.e. if \( f_i \in S \), \( i = 1, \ldots, n \), and \( f^n + f_1 f^{n-1} + \cdots + f_n = 0 \), then \( f \in S \);
6) solving linear differential equations, i.e. if \( f_i \in S \), \( i = 1, \ldots, n \), and \( f^{(n)} + f_1 f^{(n-1)} + \cdots + f_n f = 0 \), then \( f \in S \).

**Remark.** Arithmetic operations and the exponentiation are examples of meromorphic operations, hence the class of \( S \)-functions is stable under the arithmetic operations and the exponentiation.

**Corollary 4 (see [2]).** If a multivalued function \( f \) can be obtained from single valued \( S \)-functions by integration, differentiation, meromorphic operations, compositions, solutions of algebraic equations and linear differential equations, then the function \( f \) has at most countable number of singular points.
Corollary 5. A function having uncountably many singular points cannot be represented by generalized quadratures. In particular it cannot be a generalized elementary function and it cannot be represented by $k$-quadratures or by quadratures.

Example. Consider a discrete group $\Gamma$ of fractional linear transformations of the open unit ball $U$ having a compact fundamental domain. Let $f$ be a nonconstant meromorphic function on $U$ invariant under the action of $\Gamma$. Each point on the boundary $\partial U$ belongs to the closure of the set of poles of $f$, thus the set $\Sigma$ of singular points of $f$ contains $\partial U$. So $\Sigma$ has the cardinality of the continuum and $f$ cannot be expressed by generalized quadratures.

7. Monodromy group of a $S$-function. The monodromy group of a $S$-function $f$ is the group of all permutations of the sheets of the Riemann surface of $f$ which are induced by motions around the singular set $\Sigma$ of the function $f$. Below we discuss this definition more precisely.

Let $F_{x_0}$ be the set of all germs of the $S$-function $f$ at point $x_0 \notin \Sigma$. Consider a closed curve $\gamma$ in $S^2 \setminus \Sigma$ beginning and ending at the point $x_0$. Given a germ $y \in F_{x_0}$ we can continue it along the loop $\gamma$ to obtain another germ $y_\gamma \in Y_{x_0}$. Thus each such loop $\gamma$ corresponds to a permutation $S_\gamma : F_{x_0} \to F_{x_0}$ of the set $F_{x_0}$ that maps a germ $y \in F_{x_0}$ to the germ $y_\gamma \in F_{x_0}$.

It is easy to see that the map $\gamma \to S_\gamma$ defines a homomorphism from the fundamental group $\pi_1(S^2 \setminus \Sigma, x_0)$ of the domain $S^2 \setminus \Sigma$ with the base point $x_0$ to the group $S(F_{x_0})$ of permutations. The monodromy group of the $S$-function $f$ is the image of the fundamental group in the group $S(F_{x_0})$ under this homomorphism.

Remark. Instead of the point $x_0$ one can choose any other point $x_1 \in S^2 \setminus \Sigma$. Such a choice will not change the monodromy group up to an isomorphism. To fix this isomorphism one can choose any curve $\gamma : I \to \mathbb{C}^N \setminus \Sigma$ where $I$ is the segment $0 \leq t \leq 1$ and $\gamma(0) = x_0$, $\gamma(1) = x_1$ and identify each germ $f_{x_0}$ of $f$ with its continuation $f_{x_1}$ along $\gamma$.

8. Strong non representability by quadratures. One can prove the following theorem.

Theorem 6 (see [2]). The class of all $S$-functions, having a solvable monodromy group, is stable under composition, meromorphic operations, integration and differentiation.

Definition. A function $f$ is strongly non representable by quadratures if it does not belong to a class of functions defined by the following data. List of basic functions: basic elementary functions and all single valued $S$-function. List of admissible operations: compositions, meromorphic operations, differentiation and integration.

Theorem 6 implies the following corollary.

Result on quadratures. If the monodromy group of an $S$-function $f$ is not solvable, then $f$ is strongly non representable by quadratures.

Example. The monodromy group of an algebraic function $y(x)$ defined by an equation $y^5 + y - x = 0$ is the unsolvable group $S_5$. Thus $y(x)$ provides an example of a function with finite set of singular points, which is strongly non representable by quadratures.

The following corollary 7 contains a stronger result on non representability of algebraic functions by quadratures.
**Corollary 7.** If an algebraic function of one complex variable has unsolvable monodromy group then it is strongly non representable by quadratures.

For algebraic functions of several complex variables there is a result similar to Corollary 7.

**9. The monodromy pair.** The monodromy group of a function $f$ is not only an abstract group but is also a transitive group of permutations of germs of $f$ at a non singular point $x_0$.

**Definition.** The monodromy pair of an $S$-function $f$ is a pair of groups, consisting of the monodromy group of $f$ at $x_0$ and the stationary subgroup of a certain germ of $f$ at $x_0$.

The monodromy pair is well defined, i.e. this pair of groups, up to isomorphisms, does not depend on the choice of the non singular point and on the choice of the germ of $f$ at this point.

**Definition.** A pair of groups $[\Gamma, \Gamma_0]$ is an almost normal pair if there is a normal subgroup $H$ of $\Gamma$ such that $H \subset \Gamma_0$ and the coset $\Gamma_0/H$ is finite.

**Definition.** The pair of groups $[\Gamma, \Gamma_0]$ is called an almost solvable pair of groups if there exists a sequence of subgroups

$$\Gamma = \Gamma_1 \supseteq \cdots \supseteq \Gamma_m, \quad \Gamma_m \subset \Gamma_0,$$

such that for every $i, 1 \leq i \leq m-1$ group $\Gamma_{i+1}$ is a normal divisor of group $\Gamma_i$ and the factor group $\Gamma_i/\Gamma_{i+1}$ is either a commutative group, or a finite group.

**Definition.** The pair of groups $[\Gamma, \Gamma_0]$ is called a $k$-solvable pair of groups if there exists a sequence of subgroups

$$\Gamma = \Gamma_1 \supseteq \cdots \supseteq \Gamma_m, \quad \Gamma_m \subset \Gamma_0,$$

such that for every $i, 1 \leq i \leq m-1$ group $\Gamma_{i+1}$ is a normal divisor of group $\Gamma_i$ and the factor group $\Gamma_i/\Gamma_{i+1}$ is either a commutative group, or a subgroup of the group $S_k$ of permutations of $k$ elements.

We say that group $\Gamma$ is almost solvable or $k$-solvable if pair $[\Gamma, e]$, where $e$ is the group containing only the unit element, is almost solvable or $k$-solvable respectively.

It is easy to see that an almost normal pair of groups $[\Gamma, \Gamma_0]$ is almost solvable or $k$-solvable if and only if the group $\Gamma$ is almost solvable or $k$-solvable respectively.

**10. Strong non representability by $k$-quadratures.** One can prove the following theorem.

**Theorem 8 (see [2]).** The class of all $S$-functions, having a $k$-solvable monodromy pair, is stable under composition, meromorphic operations, integration, differentiation and solutions of algebraic equations of degree $\leq k$.

**Definition.** A function $f$ is strongly non representable by $k$-quadratures if it does not belong to a class of functions defined by the following data. List of basic functions: basic elementary functions and all single valued $S$-function. List of admissible operations: compositions, meromorphic operations, differentiation, integration and solutions of algebraic equations of degree $\leq k$.

Theorem 8 implies the following corollary.
Result on \( k \)-quadratures. If the monodromy pair of an \( S \)-function \( f \) is not \( k \)-solvable, then \( f \) is strongly non representable by \( k \)-quadratures.

Example. The monodromy group of an algebraic function \( y(x) \) defined by an equation \( y^n + y - x = 0 \) is the permutation group group \( S_n \). For \( n \geq 5 \) the group \( S_n \) is not an \((n-1)\)-solvable group. Thus \( y(x) \) provides an example of a function with finite set of singular points which is strongly non representable by \((n-1)\)-quadratures.

This example can be generalized.

Corollary 9 (see [2]). If an algebraic function of one complex variable has non \( k \)-solvable monodromy group then it is strongly non representable by \( k \)-quadratures.

Theorem 10 (see [2]). An algebraic function of one variable whose monodromy group is \( k \)-solvable, can be represented by \( k \)-radicals.

Results similar to Corollary 9 and Theorem 10 hold also for algebraic functions of several complex variables.

11. Strong non representability by generalized quadratures. One can prove the following theorem.

Theorem 11 (see [2]). The class of all \( S \)-functions, having an almost solvable monodromy pair, is stable under composition, meromorphic operations, integration, differentiation and solutions of algebraic equations.

Definition. A function \( f \) is strongly non representable by generalized quadratures if it does not belong to a class of functions defined by the following data. List of basic functions: basic elementary functions and all single valued \( S \)-function. List of admissible operations: compositions, meromorphic operations, differentiation, integration and solutions of algebraic equations.

Theorem 11 implies the following corollary.

Result on generalized quadratures. If the monodromy pair of an \( S \)-function \( f \) is not almost solvable, then \( f \) is strongly non representable by generalized quadratures.

Suppose that the Riemann surface of a function \( f \) is a universal covering space over the Riemann sphere with \( n \) punched points. If \( n \geq 3 \) then the function \( f \) is strongly non representable by generalized quadratures. Indeed, the monodromy pair of \( f \) consists of the free group with \( n - 1 \) generators, and its unit subgroup. It is easy to see that such a pair of groups is not almost solvable.

Example. Consider the function \( z(x) \), which maps the upper half-plane onto a triangle with vanishing angles, bounded by three circular arcs. The Riemann surface of \( z(x) \) is a universal covering space over the sphere with three punched points.\(^1\) Thus \( z(x) \) is strongly non representable by generalized quadratures.

\(^1\) It is easy to see that the function \( z(x) \) maps its Riemann surface to the open ball whose boundary contains the vertices of the triangle. These properties of the function \( z(x) \) play the crucial role in Picard’s beautiful proof of his Little Picard Theorem.
Example. Let $K_1$ and $K_2$ be the following elliptic integrals, considered as the functions of the parameter $x$:

\[
K_1(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-t^2x^2)}} \quad \text{and} \quad K_2(x) = \int_0^{\frac{1}{x}} \frac{dx}{\sqrt{(1-t^2)(1-t^2x^2)}}.
\]

The functions $z(x)$ can be obtained from $K_1(x)$ and from $K_2(x)$ by quadratures. Thus both functions $K_1(x)$ and $K_2(x)$ are strongly non representable by generalized quadratures.

In the next section we will list all polygons $G$ bounded by circular arcs for which the Riemann map of the upper half-plan onto $G$ is representable by generalized quadratures.

12. Maps of the upper half-plane onto a curved polygon. Consider a polygons $G$ on the complex plane bounded by circle arcs, and the function $f_G$ establishing the Riemann mapping of the upper half-plane onto the polygon $G$. The Riemann–Schwarz reflection principle allows to describe the monodromy group $L_G$ of the function $f_G$ and to show that all singularities of $f_G$ are simple enough. This information together with Theorem 11 provide a complete classification of all polygons $G$ for which the function $f_G$ is representable in explicit form (see [2]).

If a polygon $\tilde{G}$ is obtained from a polygon $G$ by a linear transformation $w: \mathbb{C} \to \mathbb{C}$ then $f_{\tilde{G}} = w(f_G)$. Thus it is enough to classify $G$ up to a linear transformation.

1) The first case of integrability: the continuations of all sides of the polygon $G$ intersect at one point.

Mapping this point to infinity by a fractional linear transformation, we obtain a polygon $G$ bounded by straight line segments. All transformations in the group $L(G)$ have the form $z \to az + b$. All germs of the function $f = f_G$ at a non-singular point $c$ are obtained from a fixed germ $f_c$ by the action of the group $L(G)$ consisting of the affine transformations $z \to az + b$. The germ $R_c = (f''/f)_c$ is invariant under the action of the group $L(G)$. Therefore, the germ $R_c$ is a germ of a single valued function $R$. The singular points of $R$ can only be poles (see ). Hence the function $R$ is rational. The equation $f''/f = R$ is integrable by quadratures. This integrability case is well known. The function $f$ in this case is called the Christoffel–Schwarz integral.

2) The second case of integrability: there is a pair of points such that, for every side of the polygon $G$, these points are either symmetric with respect to this side or belong to the continuation of the side.

We can map these two points to zero and infinity by a fractional linear transformation. We obtain a polygon $G$ bounded by circle arcs centered at point 0 and intervals of straight rays emanating from 0 (see Figure 2). All transformations in the group $L(G)$ have the form $z \to az, z \to b/z$. All germs of the function $f = f_G$ at a non-singular point $c$ are obtained from a fixed germ $f_c$ by the action of the group $L(G)$:

\[
f_c \to af_c, f_c \to b/f_c.
\]

The germ $R_c = (f''/f_c)^2$ is invariant under the action of the group $L(G)$. Therefore, the germ $R_c$ is a germ of a single valued function $R$. The singular points of $R$ can only be poles (see ). Hence the function $R$ is rational. The equation $R = (f'/f)^2$ is integrable by quadratures
3) The finite nets of circles. To describe the third case of integrability we need to define first the finite net of circles on the complex plane. The classification of finite groups, generated by reflections in the Euclidian space $\mathbb{R}^3$ is well known. Each such group is the symmetry group of the following bodies:

1. a regular n-gonal pyramid;
2. a regular n-gonal diheron, or the body formed by two equal regular n-gonal pyramids sharing the base;
3. a regular tetrahedron;
4. a regular cube or icosahedron;
5. a regular dodecahedron or icosahedron.

All these groups of isometries, except for the group of dodecahedron or icosahedron, are solvable.

The intersections of the unit sphere, whose center coincides with the barycenter of the body, with the mirrors, in which the body is symmetric, is a certain net of great circles. Stereographic projections of each of them is a net net of circles on complex plane defined up to a fractional linear transformation. The nets corresponding to the bodies listed above will be called the finite nets of circles.

4) The third case of integrability: every side side of a polygon $G$ belongs to some finite net of circles. In this case the function $f_G$ has finitely many branches. Since all singularities of the function $f_G$ are algebraic (see [2]), the function $f_G$ is an algebraic function. For all finite nets but the net of dodecahedron or icosahedron, the algebraic function $f_G$ is representable by radicals. For the net of dodecahedron or icosahedron the function $f_G$ is representable by radicals and solutions of degree five algebraic equations (in other words $f_G$ is representable by $k$-radicals).

5) The strong non representability. Our results imply the following:

Theorem 12 (see [2]). If a polygon $G$ bounded by circles arcs does not belong to one of the three cases described above, then the function $f_G$ is strongly non representable by generalized quadratures.

13. Non solvability of linear differential equations. Consider a homogeneous linear differential equation

$$y^{(n)} + r_1 y^{(n-1)} + \cdots + r_n y = 0,$$

whose coefficients $r_i$’s are rational functions of the complex variable $x$. The set $\Sigma \subset \mathbb{C}$ of poles of $r_i$’s is called the set of singular points of the equation (3). At a point $x_0 \in \mathbb{C} \setminus \Sigma$ the germs of solutions of (3) form a $\mathbb{C}$-linear space $V_{x_0}$ of dimension $n$. The monodromy group $M$ of the equation (3) is the group of all linear transformations of the space $V_{x_0}$ which are induced by motions around the set $\Sigma$. Below we discuss this definition more precisely.

Consider a closed curve $\gamma$ in $\mathbb{C} \setminus \Sigma$ beginning and ending at the point $x_0$. Given a germ $y \in V_{x_0}$ we can continue it along the loop $\gamma$ to obtain another germ $y_\gamma \in V_{x_0}$. Thus each such loop $\gamma$ corresponds to a map $M_\gamma : V_{x_0} \to V_{x_0}$ of the space $V_{x_0}$ to itself that maps a germ $y \in V_{x_0}$ to the germ $y_\gamma \in V_{x_0}$. The map $M_\gamma$ is linear since an analytic continuation respects the arithmetic operations. It is easy to see that the map $\gamma \to M_\gamma$ defines a homomorphism of the fundamental group $\pi_1(\mathbb{C} \setminus \Sigma, x_0)$ of the domain $\mathbb{C} \setminus \Sigma$ with the base point $x_0$ to the group $GL(n)$ of invertible linear transformations of the space $V_{x_0}$. 
The monodromy group $M$ of the equation (3) is the image of the fundamental group in the group $GL(n)$ under this homomorphism.

Remark. Instead of the point $x_0$ one can choose any other point $x_1 \in \mathbb{C} \setminus \Sigma$. Such a choice will not change the monodromy group up to an isomorphism. To fix this isomorphism one can choose any curve $\gamma : I \to \mathbb{C}^N \setminus \Sigma$ where $I$ is the segment $0 \leq t \leq 1$ and $\gamma(0) = x_0$, $\gamma(1) = x_1$ and identify each germ $y_{x_0}$ of solution of (*) with its continuation $y_{x_1}$ along $\gamma$.

Lemma 13. The stationary subgroup in the monodromy group $M$ of the germ $y \in V_{x_0}$ of almost every solution of the equation (3) is trivial (i.e. contains only the unite element $e \in M$).

Proof. The monodromy group $M$ contains countable many linear transformations $M_i$. The space $L_i \subset V_{x_0}$ of fixed points of a non identity transformation $M_i$, is a proper subspace of $V_{x_0}$. The union $L$ of all subspaces $L_i$ is a measure zero subset of $V_{x_0}$. The stationary subgroup in $M$ of $y \in V_{x_0} \setminus L$ is trivial.

Theorem 14 (see [2]). If the monodromy group of the equation (3) is not almost solvable (is not $k$-solvable, or is not solvable) then its almost every solution is strongly non representable by generalized quadratures (correspondingly, is strongly non representable by $k$-quadratures, or is strongly non representable by quadratures).

Consider a homogeneous system of linear differential equations

\begin{equation}
(4) \quad y' = Ay
\end{equation}

where $y = (y_1, \ldots, y_n)$ is the unknown vector valued function and $A = \{a_{i,j}(x)\}$ is a $n \times n$ matrix, whose entries are rational functions of the complex variable $x$. One can define the monodromy group of the equation (4) exactly in the same way as it was defined for the equation (3).

We will say that a vector valued function $y = (y_1, \ldots, y_n)$ belongs to a certain class of functions if all its components $y_i$ belong to this class. For example the statement "a vector valued function $y = (y_1, \ldots, y_n)$ is strongly non representable by generalized quadratures" means that at least one component $y_i$ of $y$ is strongly non representable by generalized quadratures.

Theorem 15. If the monodromy group of the system (4) is not almost solvable (is not $k$-solvable, or is not solvable) then its almost every solution is strongly non representable by generalized quadratures (correspondingly, is strongly non representable by $k$-quadratures, or is strongly non representable by quadratures).

14. Solvability of Fuchsian equations. The differential field of rational functions of $x$ is isomorphic to the differential field $\mathfrak{R}$ of germs of rational functions at the point $x_0 \in \mathbb{C} \setminus \Sigma$. Consider the differential field extension $\mathfrak{R}\{y_1, \ldots, y_n\}$ of $\mathfrak{R}$ where the germs $y_1, \ldots, y_n$ form a basis in the space $V_{x_0}$ of solutions of the equation (3) at $x_0$.

Lemma 16. Every linear map $M_i$ from the monodromy group of equation (3), can be uniquely extended to a differential automorphism of the differential field $\mathfrak{R}\{y_1, \ldots, y_n\}$ over the field $\mathfrak{R}$.
Proof. Every element \( f \in \mathcal{R}\{y_1, \ldots, y_n\} \) is a rational function of the independent variable \( x \), the germs of solutions \( y_1, \ldots, y_n \) and their derivatives. It can be continued meromorphically along the curve \( \gamma \in \pi_1(\mathbb{C} \setminus \Sigma, x_0) \) together with \( y_1, \ldots, y_n \).

This continuation gives the required differential automorphism, since the continuation preserves the arithmetical operations and differentiation, and every rational function of \( x \) returns back to its original values (since it is a single-valued valued function). The differential automorphism is unique because the extension is generated by \( y_1, \ldots, y_n \).

The differential Galois group (see [2], [3]) of the equation (3) over \( \mathcal{R} \) is the group of all differential automorphisms of the differential field \( \mathcal{R}\{y_1, \ldots, y_n\} \) over the differential field \( \mathcal{R} \). According to Lemma 32 the monodromy group of the equation (3) can be considered as a subgroup of its differential Galois group over \( \mathcal{R} \).

The differential field of invariants of the monodromy group action is a subfield of \( \mathcal{R}\{y_1, \ldots, y_n\} \), consisting of the single-valued functions. Differently from the algebraic case, for differential equations the field of invariants under the action of the monodromy group can be bigger than the field of rational functions. The reason is that the solutions of differential equations may grow exponentially in approaching the singular points or infinity.

Example. All solutions of the simplest differential equation \( y' = y \) are single-valued exponential functions \( y = C \exp x \), which are not rational.

For a wide class of Fuchsian linear differential equations all the solutions, while approaching the singular points and the point infinity, grow polynomially.

The following Frobenius theorem is an analog for Fuchsian equations of C.Jordan theorem (see [4]) for algebraic equations.

**Theorem (Frobenius).** For Fuchsian differential equations the subfield of the differential field \( \mathcal{R}\{y_1, \ldots, y_n\} \), consisting of single-valued functions, coincides with the field of rational functions.

A system of linear differential equations (4) is called a Fuchsian system if the matrix \( A \) has the following form:

\[
A(x) = \sum_{i=1}^{k} \frac{A_i}{x - a_i},
\]

where the \( A_i \)'s are constant matrices. Linear Fuchsian system of differential equations in many ways are similar to linear Fuchsian differential equations.

In construction of explicit solutions of linear differential equations the following theorem is needed.

**Theorem (Lie–Kolchin).** Any connected solvable algebraic group acting by linear transformations on a finite-dimensional vector space over \( \mathbb{C} \) is triangularizable in a suitable basis.

Using Frobenius Theorem and Lie–Kolchin Theorem one can prove that the only reasons for unsolvability of Fuchsian linear differential equations and systems of linear differential equations are topological. In other words, if there are no topological obstructions to solvability then such equations and systems of equations are solvable. Indeed, the following theorems hold:
Theorem 17 (see [2]). If the monodromy group of the linear Fuchsian differential equation (3) is almost solvable (is $k$-solvable, or is solvable) then its every solution is representable by generalized quadratures (correspondingly, is representable by $k$-quadratures, or is representable by quadratures).

Theorem 18 (see [2]). If the monodromy group of the linear Fuchsian system differential equations (4) is almost solvable (is $k$-solvable, or is solvable) then its every solution is representable by generalized quadratures (correspondingly, is representable by $k$-quadratures, or is representable by quadratures).

15. Fuchsian systems with small coefficients. In general the monodromy group of a given Fuchsian equation is very hard to compute. It is known only for very special equations, including the famous hypergeometric equations. Thus Theorems 17 and 18 are not explicit. If the matrix $A(x)$ in the system (4) is triangular then one can easily solve the system by quadratures. It turns out that if the matrix $A(x)$ has the form (5), where the matrices $A_i$’s are sufficiently small, then the system (4) with a non triangular matrix $A(x)$ is unsolvable by generalized quadratures for a topological reason.

Theorem 19 (see [2]). If the matrices $A_i$’s are sufficiently small, $\|A_i\| < \varepsilon(a_1, \ldots, a_k, n)$, then the monodromy group of the system

$$y' = \left(\sum_{i=1}^{k} \frac{A_i}{x-a_i}\right)y$$

is almost solvable if and only if the matrices $A_i$’s are triangularizable in a suitable basis.

Corollary 20. If in the assumptions of Theorem 19 the matrices $A_i$’s are not triangularizable in a suitable basis then almost every solution of the system (6) is strongly non representable by generalized quadratures.

16. Polynomials invertible by radicals. In 1922 J.F.Ritt published (see [5]) the following beautiful theorem which fits nicely into topological Galois theory.

Theorem (J.F. Ritt). The inverse function of a polynomial with complex coefficients can be represented by radicals if and only if the polynomial is a composition of linear polynomials, the power polynomials $z \rightarrow z^n$, Chebyshev polynomials and polynomials of degree at most 4.

Outline of proof (following [6]).

1) Every polynomial is a composition of primitive ones: Every polynomial is a composition of polynomials that are not themselves compositions of polynomials of degree > 1. Such polynomials are called primitive. Recall that a permutation group $G$ acting on a non-empty set $X$ is called primitive if $G$ acts transitively on $X$ and $G$ preserves no nontrivial partition of $X$. A polynomial is primitive if and only if the monodromy group of inverse of the polynomial acts primitively on its branches.

2) Reduction to the case of primitive polynomials: A composition of polynomials is invertible by radicals if and only if each polynomial in the composition is invertible by radicals. Indeed, if each of the polynomials in composition is invertible by radicals, then their composition also is. Conversely, if a polynomial $R$ appears in
the presentation of a polynomial $P$ as a composition $P = Q \circ R \circ S$ and $P^{-1}$ is representable by radicals, then $R^{-1} = Q \circ P^{-1} \circ S$ is also representable by radicals. Thus it is enough to classify only the primitive polynomials invertible by radicals.

3) A result on solvable primitive permutation groups containing a full cycle: A primitive polynomial is invertible by radicals if and only if the monodromy group of inverse of the polynomial is solvable. Since it acts primitively on its branches and contains a full cycle (corresponding to a loop around the point at infinity on the Riemann sphere), the following group-theoretical result of Ritt is useful for the classification of polynomials invertible by radicals:

**Theorem (on primitive solvable groups with a cycle).** Let $G$ be a primitive solvable group of permutations of a finite set $X$ which contains a full cycle. Then either $|X| = 4$, or $|X|$ is a prime number $p$ and $X$ can be identified with the elements of the field $F_p$, so that the action of $G$ gets identified with the action of the subgroup of the affine group $AGL_1(p) = \{ x \to ax + b | a \in (F_p)^*, b \in F_p \}$ that contains all the shifts $x \to x + b$.

4) Solvable monodromy groups of inverse of primitive polynomials: It can be shown by applying Riemann–Hurwitz formula that among the groups in Theorem on primitive solvable groups with a cycle, only the following groups can be realized as monodromy groups of inverse of primitive polynomials: 1. $G \subset S_4$, 2. Cyclic group $G = \{ x \to x + b \} \subset AGL_1(p)$, 3. Dihedral group $G = \{ x \to \pm x + b \} \subset AGL_1(p)$.

5) Description of primitive polynomials invertible by radicals: It can be easily shown (see for instance [see Ritt 22], [Khovanskii 07 Variations], [Burda Khovanskii11 Branching]) that the following result holds:

**Theorem 21.** If the monodromy group of inverse of a primitive polynomial is a subgroup of the group $\{ x \to \pm x + b \} \subset AGL_1(p)$, then up to a linear change of variables the polynomial is either a power polynomial or a Chebyshev polynomial.

Thus the polynomials whose inverse have monodromy groups 1-3 are respectively 1. Polynomials of degree four. 2. Power polynomials up to a linear change of variables. 3. Chebyshev polynomials up to a linear change of variables.

In each of these cases the fact that the polynomial is invertible by radicals follows from solvability of the corresponding monodromy group or from explicit formulas for its inverse (see for instance [BurdaKhovanskii11Branching]).

17. **Polynomials invertible by $k$-radicals.** In this section we discuss the following generalization of J.F.Ritt’s Theorem.

**Theorem 22 (see [6]).** A polynomial invertible by radicals and solutions of equations of degree at most $k$ is a composition of power polynomials, Chebyshev polynomials, polynomials of degree at most $k$ and, if $k \leq 14$, certain primitive polynomials whose inverse have exceptional monodromy groups. A description of these exceptional polynomials can be given explicitly.

The proofs rely on classification of monodromy groups of inverse of primitive polynomials obtained by Müller based on group-theoretical results of Feit and on previous work on primitive polynomials whose inverse have exceptional monodromy groups by many authors. Besides the references to these highly involved and technical results an outline of the proof of Theorem 22 is not complicated and it resembles the outline of the proof of Ritt’s Theorem.
Let us start with some background on representability by $k$-radicals.

**Definition.** Let $k$ be a natural number. A field extension $L/K$ is $k$-radical if there exists a tower of extensions $K = K_0 \subset K_1 \subset \ldots \subset K_n$ such that $L \subset K_n$ and for each $i$, $K_{i+1}$ is obtained from $K_i$ by adjoining an element $a_i$, which is either a solution of an algebraic equation of degree at most $k$ over $K_i$, or satisfies $a_i^m = b$ for some natural number $m$ and $b \in K_i$.

**Theorem 23 (see [2]).** A Galois extension $L/K$ of fields of characteristic zero is $k$-radical if and only if its Galois group is $k$-solvable.

An algebraic function $z = z(x)$ of one or several complex variables is said to be representable by $k$-radicals if the corresponding extension of the field of rational functions is a $k$-radical extension.

Theorem 23 and C. Jordan’s Theorem (see [1]) imply the following corollary.

**Corollary 24.** An algebraic function is representable by $k$-radicals if and only if its monodromy group is $k$-solvable.

(Note that Theorem 10 above coincides with a part of Corollary 23).

Let us outline briefly the main steps in the proof of Theorem 22:

**Outline of proof of Theorem 22.**

1. Exactly as in Ritt’s theorem one can show that a composition of polynomials is invertible by $k$-radicals if and only if each polynomial in the composition is invertible by $k$-radicals. Thus one can reduce Theorem 22 to the case of primitive polynomials.

2. Feit and Jones totally classified all primitive permutation groups of $n$ elements containing a full cycle.

3. Using the this classification and Riemann-Hurwitz formula, Müller listed all groups of permutations of $n$ elements which are monodromy groups of inverses of degree $n$ primitive polynomials.

4. For each group from Müller’s list of groups of permutations of $n$ elements one can determine the smallest $k$ for which it is $k$-solvable and choose the exceptional groups for which $k$ is smaller than $n$.

5. For each such exceptional group one can explicitly describe polynomials whose inverse has the exceptional monodromy group.

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