Research article

Dynamics of an epidemic model with advection and free boundaries

Meng Zhao\(^1\), Wan-Tong Li\(^1,*\) and Yang Zhang\(^2\)

\(^1\) School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, P.R. China
\(^2\) Department of Mathematics, Harbin Engineering University, Harbin, 150001, P.R. China

* Correspondence: Email: wtli@lzu.edu.cn; Tel: +8609318912491.

Abstract: This paper deals with the propagation dynamics of an epidemic model, which is modeled by a partially degenerate reaction-diffusion-advection system with free boundaries and sigmoidal function. We focus on the effect of small advection on the propagation dynamics of the epidemic disease. At first, the global existence and uniqueness of solution are obtained. And then, the spreading-vanishing dichotomy and the criteria for spreading and vanishing are given. Our results imply that the small advection make the disease spread more difficult.

Keywords: epidemic model; partially degenerate; advection; free boundary; spreading and vanishing

1. Introduction

In order to describe the evolution of fecal-oral transmitted diseases in the Mediterranean regions, Capasso and Paveri-Fontana [1] proposed the following model

\[
\begin{cases}
    u'(t) = -au + cv, \\
    v'(t) = -bv + G(u),
\end{cases}
\]

where \(a, b, c\) are all positive constants, \(u(t)\) and \(v(t)\) denote the concentration of the infectious agent in the environment and the infective human population respectively. The coefficients \(a\) and \(b\) are the intrinsic decay rates of the infectious agent and the infective human population respectively, \(c\) represents the multiplication rate of the infectious agent due to the human infected population. The function \(G(u)\) stands for the force of infection of the human population due to the concentration of infectious agent. We assume that \(G(u)\) satisfies the two specific cases: (i) a monotone increasing function with constant concavity; (ii) a sigmoidal function of bacterial concentration tending to some finite limit, and with zero gradient at zero. These two cases contain most of the features of forces of infection in real epidemics. For some epidemic, if the density of infectious agent is small, the force of
infection of the humans will be weak and may tend to zero, and the function $G$ will satisfy case (ii). In this paper, we focus on such case, and assume that the function $G : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

(G1) $G \in C^2(\mathbb{R}^+)$, $G(0) = 0$, $G'(z) > 0$ for any $z > 0$ and $\lim_{z \to \infty} G(z) = 1$;

(G2) there exists $\xi > 0$ such that $G''(z) > 0$ for $z \in (0, \xi)$ and $G''(z) < 0$ for $z \in (\xi, \infty)$.

Denote

$$\theta = \frac{cG'(0)}{ab}.$$ 

Under two specific cases stated above, the global dynamics of the cooperative system (1.1) has been described in detail in [2]. It follows from [2, Theorem 4.3] that the global dynamics of (1.1) under conditions (G1) and (G2) can be described as follows:

(i) If $\theta < 1$ and $\frac{G(c)}{c} < \frac{ab}{c}$ for any $z > 0$, then the trivial solution is the only equilibrium for problem (1.1) and it is globally asymptotically stable in $\mathbb{R}^+ \times \mathbb{R}^+$.

(ii) If $\theta > 1$, then problem (1.1) has only one nontrivial equilibrium point $(u^*, v^*)$ in addition to $(0, 0)$ and it is globally asymptotically stable in $\mathbb{R}^+ \times \mathbb{R}^+$.

(iii) If $\theta < 1$ and $\frac{G(c)}{c} > \frac{ab}{c}$ for some $z_1 > 0$, then problem (1.1) has three equilibrium points:

$$E_0 = (0, 0), \quad E_1 = \left( K_1, \frac{aK_1}{c} \right) \quad \text{and} \quad E_2 = \left( K_2, \frac{aK_2}{c} \right),$$

where $0 < K_1 < K_2$ are the positive roots of $G(z) = \frac{ab}{c}z = 0$. In this case, $E_1$ is a saddle point, $E_0$ and $E_2$ are stable nodes.

In 1997, Capasso and Wilson [3] further considered spatial variation and studied the problem

$$\begin{align*}
  u_t &= d\Delta u - au + cv, \quad (t, x) \in (0, +\infty) \times \Omega, \\
  v_t &= -bv + G(u), \quad (t, x) \in (0, +\infty) \times \Omega, \\
  u(t, x) &= 0, \quad (t, x) \in (0, +\infty) \times \partial \Omega, \\
  u(0, x) &= u_0(x), \quad v(0, x) = v_0(x), \quad x \in \Omega,
\end{align*}$$

where $\Omega$ is bounded. By some numerical simulation, they speculated that the dynamical behavior of system (1.2) is similar to the ODE case. To understand the dispersal process of epidemic from outbreak to an endemic, Xu and Zhao [4] studied the bistable traveling waves of (1.2) in $x \in \mathbb{R}$.

The epidemic always spreads gradually, but the works mentioned above are hard to explain this gradual expanding process. To describe such a gradual spreading process, Du and Lin [5] introduced the free boundary condition to study the invasion of a single species. They considered the problem

$$\begin{align*}
  u_t - du_{xx} &= u(a - bu), \quad t > 0, \quad 0 < x < h(t), \\
  u_x(t, 0) &= 0, \quad u(t, h(t)) = 0, \quad t > 0, \\
  h'(t) &= -\mu u_x(t, h(t)), \quad t > 0, \\
  h(0) &= h_0, \quad u(0, x) = u_0(x), \quad 0 \leq x \leq h_0,
\end{align*}$$

and showed that (1.3) admits a unique solution which is well-defined for all $t \geq 0$ and spreading and vanishing dichotomy holds. Moreover, the criteria for spreading and vanishing are obtained: (i) for
h_0 \geq \frac{\pi}{2} \sqrt{\frac{a}{\mu}}$, the species will spread; (ii) for $h_0 < \frac{\pi}{2} \sqrt{\frac{a}{\mu}}$ and given $u_0(x)$, there exists $\mu^*$ such that the species will spread for $\mu > \mu^*$, and the species will vanish for $0 < \mu \leq \mu^*$. Finally, they gave the spreading speed of the spreading front when spreading occurs. Since then, many problems with free boundaries and related problems have been investigated, see e.g. [6–22] and their references.

In 2016, Ahn et al. [23] considered (1.2) with monostable nonlinearity and free boundaries. They obtained the global existence and uniqueness of the solution and spreading and vanishing dichotomy. Furthermore, by introducing the so-called spatial-temporal risk index

$$R_0^c(t) = \frac{G'(0) \xi}{a + d \left(\frac{\pi}{h(t) - g(t)}\right)^2},$$

they proved that: (i) if $R_0 = \frac{cG'(0)}{ab} \leq 1$, the epidemic will vanish; (ii) if $R_0^c(0) \geq 1$, the epidemic will spread; (iii) if $R_0^c(0) < 1$, epidemic will vanish for the small initial densities; (iv) if $R_0^c(0) < 1 < R_0$, epidemic will spread for the large initial densities. Recently, Zhao et al. [24] determined the spreading speed of the spreading front of problem described in [23].

Inspired by the work [23], we want to study (1.2) with bistable nonlinearity and free boundaries. Meanwhile, we also want to consider the effect of the advection. In 2009, Maitana and Yang [25] studied the propagation of West Nile Virus from New York City to California. In the summer of 1999, West Nile Virus began to appear in New York City. But it was observed that the wave front traveled 187 km to the north and 1100 km to the south in the second year. Therefore, taking account of the advection movement has the greater realistic significance. Recently, there are some works considering the advection. In 2014, Gu et al. [26] was the first time to consider the long-time behavior of problem (1.3) with small advection. Then, the asymptotic spreading speeds of the free boundaries was given in [27]. For more general reaction term, Gu et al. [10] studied the long time behavior of solutions of Fisher-KPP equation with advection $\beta > 0$ and free boundaries. For single equation with advection, there are many other works. For example, [28–34] and their references. Besides, there are also several works devoted to the system with small advection, such as, [35–40] and their references.

Taking account of the effect of advection, we consider

$$\begin{align*}
  u_t &= du_{xx} - \beta u_x - au + cv, \quad t > 0, \quad g(t) < x < h(t), \\
  v_t &= -bv + G(u), \quad t > 0, \quad g(t) < x < h(t), \\
  u(t, x) &= v(t, x) = 0, \quad t \geq 0, \quad x = g(t) \text{ or } x = h(t), \\
  g(0) &= -h_0, \quad g'(t) = -\mu u_x(t, g(t)), \quad t > 0, \\
  h(0) &= h_0, \quad h'(t) = -\mu u_x(t, h(t)), \quad t > 0, \\
  u(0, x) &= u_0(x), \quad v(0, x) = v_0(x), \quad -h_0 < x < h_0,
\end{align*}$$

(1.4)

where we use the changing region $(g(t), h(t))$ to denote the infective environment of disease, where the free boundaries $x = g(t)$ and $x = h(t)$ represent the spreading fronts of epidemic. Since the diffusion coefficient of $v$ is much smaller than that of $u$, we assume that the diffusion coefficient of $v$ is zero. When $u$ spreads into a new environment, some humans in the new environment may be infected. Hence, we can use $(g(t), h(t))$ to represent the habit of infective humans. We use $I_0 = (-h_0, h_0)$ to
denote the initial infective environment of epidemic. The initial functions \( u_0(x) \) and \( v_0(x) \) satisfy
\[
\begin{align*}
\{ u_0(x) \} & \in \mathcal{D} \quad (\text{for } x \in I_0, \ u_0(x) = 0 \text{ for } x \in \mathbb{R} \setminus I_0), \\
\{ v_0(x) \} & \in \mathcal{D}_2(\mathbb{R}) \quad (\text{for } x \in I_0, \ v_0(x) = 0 \text{ for } x \in \mathbb{R} \setminus I_0),
\end{align*}
\]
where \( p > 3 \). The derivation of the Stefan conditions \( h'(t) = -\mu u(t, h(t)) \) and \( g'(t) = -\mu v(t, g(t)) \) can be found in [41, 42]. In this paper, we always assume that \( G \) satisfies (G1)-(G2) and
\[
\text{(G3) } G(z) \text{ is locally Lipschitz in } z \in \mathbb{R}^+, \text{ i.e., for any } L > 0, \text{ there exists a constant } \rho(L) > 0 \text{ such that }
\end{align*}
\]
Furthermore, we assume that \( 0 < \beta < \beta^* \) with
\[
\beta^* = \begin{cases} 
\infty, & \theta < 1, \\
2 \sqrt{a(\frac{G(0)}{b} - a)}, & \theta > 1.
\end{cases}
\]
The rest of this paper is organized as follows. In Section 2, the global existence and uniqueness of solution, comparison principle and some results about the principal eigenvalue are given. Section 3 is devoted to the long time behavior of \((u, v)\). We get a spreading and vanishing dichotomy and give the criteria for spreading and vanishing. Finally, we give some discussions in Section 4.

2. Preliminaries

Firstly, we prove the existence and uniqueness of the solution.

**Lemma 2.1.** For any given \((u_0, v_0) \in \mathcal{D}(h_0) \times \mathcal{D}_2(h_0)\) and any \( \alpha \in (0, 1) \), there exists a \( T > 0 \) such that problem (1.4) admits a unique solution
\[
(u, v, g, h) \in \left( W_{1,2}^1(\Omega_T) \cap C^{1+\alpha, 1+\alpha}_{\text{loc}}(\Omega_T) \right) \times C^1([0, T]; L^\infty([g(t), h(t)])) \times \left[ C^{1+\alpha}_{\text{loc}}([0, T]) \right]^2,
\]
moreover,
\[
\|u\|_{W_{1,2}^1(\Omega_T)} + \|u\|_{C^{1+\alpha, 1+\alpha}(\Omega_T)} + \|g\|_{C^{1+\alpha}_{\text{loc}}([0, T])} + \|h\|_{C^{1+\alpha}_{\text{loc}}([0, T])} \leq C,
\]
where \( \Omega_T = \{(t, x) \in \mathbb{R}^2 : 0 \leq t \leq T, \ g(t) \leq x \leq h(t)\} \). \( C \) and \( T \) depend only on \( h_0, \alpha, \|u_0\|_{W_{1,2}^1([-h_0, h_0])} \) and \( \|v_0\|_{\infty} \).

**Proof.** This proof can be done by the similar arguments in [43]. But there are some differences. Hence, we give the details. Let
\[
y(t, x) = \frac{2x - g(t) - h(t)}{h(t) - g(t)}, \quad w(t, y) = u\left( t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2} \right),
\]
and
\[
z(t, y) = v\left( t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2} \right).
\]
Then problem (1.4) becomes
\[
\begin{cases}
w_t - dA^2w_{yy} + (\beta A - B)w_y = -aw + cz, & 0 < t < T, \quad -1 < y < 1, \\
w(t, -1) = w(t, 1) = 0, & 0 \leq t < T, \\
w(0, y) = u_0(h_0y) \pm w_0(y), & -1 < y < 1, 
\end{cases}
\]
(2.3)
\[
\begin{cases}
v_t = -bv + G(u), & 0 < t < T, \quad g(t) < x < h(t), \\
v(t, g(t)) = v(t, h(t)) = 0, & 0 \leq t < T, \\
v(0, x) = v_0(x), & -h_0 < x < h_0, 
\end{cases}
\]
(2.4)
and
\[
\begin{cases}
g'(t) = -\mu Aw_y(t, -1), & 0 < t < T, \\
h'(t) = -\mu Aw_y(t, 1), & 0 < t < T, \\
g(0) = -h_0, \quad h(0) = h_0, 
\end{cases}
\]
(2.5)
where
\[
A = A(g(t), h(t)) = \frac{2}{h(t) - g(t)} \quad \text{and} \quad B = B(g(t), h(t), y) = \frac{h'(t) + g'(t)}{h(t) - g(t)} + \frac{h'(t) - g'(t)}{h(t) - g(t)}.
\]
Denote \( g^* = -\frac{\mu}{h_0}u_0'(-h_0) \) and \( h^* = -\frac{\mu}{h_0}u_0'(h_0) \). For \( 0 < T \leq \frac{h_0}{2g^* + h^*} \), define
\[
\Delta_T = [0, T] \times [-1, 1].
\]
\[
\mathcal{D}_1T = \{ w \in C(\Delta_T) : w(0, y) = w_0(y), w(t, \pm 1) = 0, \|w - w_0\|_{C(\Delta_T)} \leq 1 \},
\]
\[
\mathcal{D}_2T = \{ g \in C^1([0, T]) : g(0) = -h_0, \quad g'(0) = g^*, \quad \|g' - g^*\|_{C([0, T])} \leq 1 \},
\]
\[
\mathcal{D}_3T = \{ h \in C^1([0, T]) : h(0) = h_0, \quad h'(0) = h^*, \quad \|h' - h^*\|_{C([0, T])} \leq 1 \}.
\]
It is easy to see that \( \mathcal{D}_T = \mathcal{D}_1T \times \mathcal{D}_2T \times \mathcal{D}_3T \) is a complete metric space with the metric
\[
d((w_1, g_1, h_1), (w_2, g_2, h_2)) = \|w_1 - w_2\|_{C(\Delta_T)} + \|g_1 - g_2\|_{C^1([0, T])} + \|h_1 - h_2\|_{C([0, T])}.
\]
For any given \( (w, g, h) \in \mathcal{D}_T \), there exist some \( \xi_1, \xi_2 \in (0, t) \) such that
\[
|g(t) + h_0| + |h(t) - h_0| = |g'(\xi_1)|t + |h'(\xi_2)|t \leq T(2 + g^* + h^*) \leq \frac{h_0}{2},
\]
which implies that
\[
2h_0 \leq h(t) - g(t) \leq 3h_0, \quad \forall \ t \in [0, T].
\]
Thus, \( A(g(t), h(t)) \) and \( B(g(t), h(t), y) \) are well-defined. By the definition of \( w \), we have
\[
u(t, x) = w\left(t, \frac{2x - g(t) - h(t)}{h(t) - g(t)}\right).
\]
(2.6)
Since \( |w(t, y)| \leq \|w_0\|_{L^\infty} + 1 \) for \( (t, y) \in \Delta_T \), we have
\[
|u(t, x)| \leq \|w_0\|_{L^\infty} + 1 \leq M_1, \quad \forall \ (t, x) \in [0, T] \times [g(t), h(t)].
\]
Define

\[
\tilde{v}_0(x) = \begin{cases} v_0(x), & x \in (-h_0, h_0), \\ 0, & x \in \mathbb{R} \setminus (-h_0, h_0) \end{cases}
\] and \( t_s := \begin{cases} t_s^e, & x \in [g(T), -h_0) \text{ and } x = g(t_s^e), \\ 0, & x \in [-h_0, h_0), \\ t_s^p, & x \in (h_0, h(T)] \text{ and } x = h(t_s^p). \end{cases} \)

For \( u \) defined as (2.6) and any given \( x \in [g(T), h(T)] \), we consider the following ODE problem

\[
\begin{align*}
\frac{dv}{dt} &= -bv + G(u(t, x)), & & t_s < t < T, \\
v(t_s, x) &= \tilde{v}_0(x).
\end{align*}
\] (2.7)

By the similar arguments as the step 1 in the proof of [44, Lemma 2.3], it is easy to show that (2.7) admits a unique solution \( v(t, x) \) for \( t \in [t_s, T_1] \), where \( T_1 \in \left(0, \frac{h_0}{2(\alpha + \beta)}\right] \). Hence, problem (2.4) has a unique solution \( v(t, x) \in C^1([0, T_1]; L^\infty([g(t), h(t)]) \). By the continuous dependence of the solution on parameters, we can have

\[\|v_x\|_{L^\infty(\Omega_{T_1})} \leq C_1.\]

Then

\[\|v_x\|_{L^\infty(\Omega_T)} \leq \|v_x\|_{L^\infty(\Omega_{T_1})} \leq C_1, \forall \, T \leq T_1.\]

For this \( v \), we can get

\[z(t, y) = v \left(t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2}\right).\]

For \((w, g, h)\) and \(z\) obtained above, we consider the following problem

\[
\begin{align*}
\tilde{w}_t - dA^2 \tilde{w}_{yy} + (\beta A - B)\tilde{w}_y &= -aw + cz, & & 0 < t < T, \quad -1 < y < 1, \\
\tilde{w}(t, -1) &= \tilde{w}(t, 1) = 0, & & 0 \leq t < T, \\
\tilde{w}(0, y) &= u_0(h_0 y), & & -1 < y < 1.
\end{align*}
\] (2.8)

Applying standard \( L^p \) theory and the Sobolev imbedding theorem, we can have there exists \( T_2 \in (0, T_1] \) such that (2.8) admits a unique solution \( \tilde{w}(t, y) \) and

\[\|\tilde{w}\|_{W^{1,2}_p(\Delta_{T_2})} + \|\tilde{w}\|_{C^{\frac{1}{2}, 1+\alpha}(\Delta_{T_2})} \leq C_2,\]

where \( C_2 \) is a constant depending only on \( h_0, \alpha \) and \( \|u_0\|_{W^{2,\alpha}_p([-h_0, h_0])} \). Then

\[\|\tilde{w}\|_{W^{1,2}_p(\Delta_T)} + \|\tilde{w}\|_{C^{\frac{1}{2}, 1+\alpha}(\Delta_T)} \leq \|\tilde{w}\|_{W^{1,2}_p(\Delta_{T_2})} + \|\tilde{w}\|_{C^{\frac{1}{2}, 1+\alpha}(\Delta_{T_2})} \leq C_2, \forall \, T \leq T_2.\] (2.9)

Define

\[
\tilde{g}(t) = -h_0 - \int_0^t \mu a(g(\tau), h(\tau))\tilde{w}_y(\tau, -1)d\tau, \\
\tilde{h}(t) = h_0 - \int_0^t \mu a(g(\tau), h(\tau))\tilde{w}_y(\tau, 1)d\tau,
\]
we can have \( \bar{g}(0) = -h_0, \bar{h}(0) = h_0, \)
\[
\bar{g}'(t) = -\mu A(g(t), h(t))\bar{w}_y(t, -1), \quad \bar{h}'(t) = -\mu A(g(t), h(t))\bar{w}_y(t, 1),
\]
and hence
\[
\|\bar{g}'\|_{C^2([0,T])}, \quad \|\bar{h}'\|_{C^2([0,T])} \leq \mu h_0 C_2 \leq C_1.
\] (2.10)

Now, we can define the mapping \( \mathcal{F} : D_T \to C(\Delta_T) \times C^1([0, T]) \times C([0, T]) \) by
\[
\mathcal{F}(w, g, h) = (\bar{w}, \bar{g}, \bar{h}).
\]

Obviously, \( D_T \) is a bounded and closed convex set of \( C(\Delta_T) \times C^1([0, T]) \times C([0, T]) \), \( \mathcal{F} \) is continuous
in \( D_T \), and \( (w, g, h) \) is a fixed point of \( \mathcal{F} \) if and only if \( (w, v, g, h) \) solve (2.3), (2.4) and (2.5). By (2.9)
and (2.10), we have \( \mathcal{F} \) is compact and
\[
\|\bar{w} - w_0\|_{C(\Delta_T)} \leq C_2 T^\frac{1}{1+u}, \quad \|\bar{g}' - g\|_{C([0,T])} \leq C_3 T^{\frac{2}{1+u}}, \quad \|\bar{h}' - h'\|_{C([0,T])} \leq C_4 T^{\frac{2}{1+u}}.
\]

Therefore if we take \( T \leq \min\{T_2, C_2^{-1}, C_3^{-1}, C_4^{-1}\} \equiv T_3 \), then \( \mathcal{F} \) maps \( D_T \) into itself. It now follows
from the Schauder fixed point theorem that \( \mathcal{F} \) has a fixed point \( (w, g, h) \) in \( D_T \). Moreover, we have \( (w, v, g, h) \) solve (2.3), (2.4) and (2.5),
\[
\|w\|_{C^1([\Delta_T])} + \|w\|_{C^2([\Delta_T])} \leq C_2, \quad \|v_i\|_{L^\infty(\Omega_T)} \leq C_1, \quad \forall T \leq T_3.
\]

Define as before,
\[
\begin{align*}
    u(t, x) &= w\left(t, \frac{2x - g(t) - h(t)}{h(t) - g(t)}\right).
\end{align*}
\]

Then \( (u, v, g, h) \) solve (1.4), and satisfies (2.1) and (2.2).

In the following, we prove the uniqueness of \( (u, v, g, h) \). Let \( (u_i, v_i, g_i, h_i) \) \((i = 1, 2)\) be the two
solutions of problem (1.4) for \( T \in (0, T_3) \) sufficiently small. Let
\[
w_i(t, y) = u_i\left(t, \frac{(h_i(t) - g_i(t))y + h_i(t) + g_i(t)}{2}\right).
\]

Then it is easy to see that \( (w_i, v_i, g_i, h_i) \) solve (2.3), (2.4) and (2.5). Denoting
\[
A_i = A(g_i(t), h_i(t)), \quad B_i = B(g_i(t), h_i(t), y), \quad W = w_1 - w_2, \quad Z = z_1 - z_2, \quad G = g_1 - g_2, \quad H = h_1 - h_2,
\]
we can have
\[
\begin{align*}
    \begin{cases}
    W_t - dA^2_1 W_{yy} + (\beta A_1 - B_1)W_y = -aW + cZ \\
    \hspace{1cm}+(dA^2_1 - dA^2_2)w_{2yy} + [-(\beta A_1 - B_1) + (\beta A_2 - B_2)]w_{2y}, & 0 < t < T, \quad -1 < y < 1, \\
    W(t, -1) = W(t, 1) = 0, & 0 \leq t < T, \\
    W(0, y) = 0, & -1 < y < 1,
    \end{cases}
\end{align*}
\]
and
\[
\begin{align*}
    \begin{cases}
    G' = -\mu A_1 W_y(t, -1) + \mu (A_2 - A_1)w_2(t, -1), & 0 < t < T, \\
    H' = -\mu A_1 W_y(t, 1) + \mu (A_2 - A_1)w_2(t, 1), & 0 < t < T, \\
    G(0) = 0, \quad H(0) = 0.
    \end{cases}
\end{align*}
\] (2.11)
Using the $L^p$ estimates for parabolic equations and Sobolev imbedding theorem, we obtain
\[
\|W\|_{W^{1,2}_{0}(\Omega_T)} \leq C_4 \left( \|Z\|_{C(\Omega_T)} + \|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} \right),
\]
where $C_4$ depends on $C_2$, $C_3$ and the functions $A$ and $B$. Next we should estimate $\|z_1 - z_2\|_{C(\Omega_T)}$. For convenience, we define
\[
H_m(t) = \min\{h_1(t), h_2(t)\}, \quad H_M(t) = \max\{h_1(t), h_2(t)\},
\]
\[
G_m(t) = \min\{g_1(t), g_2(t)\}, \quad G_M(t) = \max\{g_1(t), g_2(t)\},
\]
\[
\Omega^G_{t_m H_M} = [0, T] \times \{G_m(t), H_M(t)\}.
\]
By direct calculations, we have
\[
\|z_1(t,y) - z_2(t,y)\|_{C(\Omega_T)} \leq \left| v_1 \left( t, \frac{(h_1(t) - g_1(t))y + h_1(t) + g_1(t)}{2} \right) - v_2 \left( t, \frac{(h_2(t) - g_2(t))y + h_2(t) + g_2(t)}{2} \right) \right|_{C(\Omega_T)}
\]
\[
+ \left| v_2 \left( t, \frac{(h_1(t) - g_1(t))y + h_1(t) + g_1(t)}{2} \right) - v_2 \left( t, \frac{(h_2(t) - g_2(t))y + h_2(t) + g_2(t)}{2} \right) \right|_{C(\Omega_T)}
\]
\[
\leq \|v_1(t,x) - v_2(t,x)\|_{C^{G_m H_M}(\Omega_T)} + \|v_2\|_{L^{\infty}(\Omega_T^{G_m H_M})} \left( \|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} \right).
\]
Now we estimate $|v_1 - v_2(t^*, x^*)|$ for any fixed $(t^*, x^*) \in \Omega^{G_m H_M}_T$. It will be divided into the following three cases.

**Case 1.** $x^* \in [-h_0, h_0]$.

Since (2.4) is equivalent to the following integral equation:
\[
v(t,x) = e^{-bt} \left[ v_0(x) + \int_0^t e^{bs} G(u)(s, x) ds \right],
\]
we have
\[
v_1(t,x) - v_2(t,x) = e^{-bt} \left[ \int_0^t e^{bs} (G(u_1) - G(u_2))(s, x) ds \right].
\]
Then,
\[
|v_1(t^*, x^*) - v_2(t^*, x^*)| \leq \frac{\rho(M_1)}{b} \|u_1 - u_2\|_{C^{G_m H_M}_T},
\]
(2.14)

**Case 2.** $x^* \in (h_0, H_m(t^*))$.

In this case, there exist $t_1^*, t_2^* \in (0, t^*)$ such that $h_1(t_1^*) = h_2(t_2^*) = x^*$. Without loss of generality, we may assume that $0 \leq t_1^* \leq t_2^*$. Then,
\[
v_1(t^*, x^*) - v_2(t^*, x^*) = e^{-bt^*} \left[ v_1(t_2^*, x^*) e^{bt_2^*} + \int_{t_1^*}^{t_2^*} e^{bs} (G(u_1) - G(u_2))(s, x^*) ds \right].
\]
Thus,
\[ |v_1(t^*, x^*) - v_2(t^*, x^*)| \leq |v_1(t_2^*, x^*)| + \frac{\rho(M_1)}{b}\|u_1 - u_2\|_{C(L_s^{\alpha_0 s}(\Omega))} \]

By (G1) and (G2), we can have that there exists \( \gamma \) such that \( G(z) \leq \gamma z \) for \( z \geq 0 \). Now we estimate \( v_1(t_2^*, x^*) \). Direct calculations give that
\[ v_1(t_2^*, x^*) = e^{-bt^*} \int_{t_1^*}^{t_2^*} e^{bt} G(u_1(s, x^*)) ds \leq \frac{\gamma}{b} \max_{s \in [t_1^*, t_2^*]} |u_1(t, x^*)| = \frac{\gamma}{b} \max_{s \in [t_1^*, t_2^*]} |(u_1 - u_2)(t, x^*)| \].
Hence,
\[ |v_1(t^*, x^*) - v_2(t^*, x^*)| \leq \frac{\gamma + \rho(M_1)}{b}\|u_1 - u_2\|_{C(L_s^{\alpha_0 s}(\Omega))} \].
(2.15)

**Case 3.** \( x^* \in [H_m(t^*), H_M(t^*)] \).
Without loss of generality, we assume that \( h_2(t^*) < h_1(t^*) \). In this case, there exists \( t_1^* \) such that \( h_1(t_1^*) = x^* \). Then \( v_1(t_1^*, x^*) = 0, u_2(t, x^*) = v_2(t, x^*) = 0 \) for \( t \in [t_1^*, t^*] \). Hence, \( V(t^*, x^*) = v_1(t^*, x^*) \) and
\[ v_1(t^*, x^*) = e^{-bt^*} \int_{t_1^*}^{t^*} e^{bt} G(u_1(s, x^*)) ds \leq \frac{\gamma}{b} \max_{s \in [t_1^*, t^*]} |u_1(t, x^*)| = \frac{\gamma}{b} \max_{s \in [t_1^*, t^*]} |(u_1 - u_2)(t, x^*)| \].
Hence,
\[ |v_1(t^*, x^*) - v_2(t^*, x^*)| \leq \frac{\gamma + \rho(M_1)}{b}\|u_1 - u_2\|_{C(L_s^{\alpha_0 s}(\Omega))} \].
(2.16)

By (2.14), (2.15) and (2.16), we have
\[ \|v_1 - v_2\|_{C(L_s^{\alpha_0 s}(\Omega))} \leq C_3\|u_1 - u_2\|_{C(L^{\alpha_0 s}(\Omega))} \],
(2.17)
where \( C_3 \) depends on \( b, \rho, M_1 \) and \( \gamma \). Now we estimate \( \|u_1(t, x) - u_2(t, x)\|_{C(L^{\alpha_0 s}(\Omega))} \),
\[ \|u_1(t, x) - u_2(t, x)\|_{C(L^{\alpha_0 s}(\Omega))} = \left\| w_1 \left( t, \frac{2x - g_1(t) - h_1(t)}{h_1(t) - g_1(t)} \right) \right\| - \left\| w_2 \left( t, \frac{2x - g_2(t) - h_2(t)}{h_2(t) - g_2(t)} \right) \right\|_{C(L^{\alpha_0 s}(\Omega))} \leq \|w_1(t_1, y) - w_2(t_1, y)\|_{C(\Delta_T)} + C_6 \left( \|G\|_{C_c([0, T])} + \|H\|_{C([0, T])} \right) \],
(2.18)
where \( C_6 \) only depends on \( h_0 \) and \( \|w_2\|_{C(\Delta_T)} \). By \( \bar{W}(0, y) = 0 \) and Sobolev imbedding theorem, we have
\[ \|W(t, y)\|_{C(\Delta_T)} \leq [W]_{C_2^{2,0}(\Delta_T)} T^{2\alpha} \leq C_7 T^{2\alpha} [W]_{C_2^{2,0}(\Delta_T)} \leq C_8 T^{2\alpha} [W]_{W^{1,2}_p(\Delta_T)}, \]
(2.19)
where \( C_7 \) and \( C_8 \) do not depend on \( T \). By (2.12), (2.13), (2.17), (2.18) and (2.19), we can get
\[ \|W\|_{W^{1,2}_p(\Delta_T)} \leq C_9 T^{2\alpha} [W]_{W^{1,2}_p(\Delta_T)} + C_{10} \left( \|G\|_{C_c([0, T])} + \|H\|_{C([0, T])} \right) \].

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where \( C_9 \) depends on \( C_4, C_5 \) and \( C_8; C_{10} \) depends on \( C_1, C_5 \) and \( C_6 \). If \( T \in \min \{ T_3, (2C_9)^{-\frac{1}{2}} \} \equiv T_4, \)
\[
\|W\|_{W^{\frac{1}{2}}_p(\Delta_T)} \leq 2C_{10} \left( \|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} \right).
\] (2.20)

In the following, we estimate \( \|G\|_{C^1([0,T])} \) and \( \|H\|_{C^1([0,T])} \), Since \( G(0) = G'(0) = 0 \), we have
\[
\|G\|_{C^1([0,T])} = \max_{t \in [0,T]} G(t) + \max_{\xi \in [0,T]} G'(\xi) T + \max_{t \in [0,T]} G'(t)
\]
\[
\leq (1 + T) \max_{t \in [0,T]} \frac{G'(t) - G'(0)}{(t - 0)^{\frac{1}{2}}} T^{\frac{1}{2}} = T^{\frac{1}{2}} (1 + T) |G'|_{C^{\frac{1}{2}}([0,T])}.
\]

By (2.11), we have
\[
[G']_{C^{\frac{1}{2}}([0,T])} = C_{11} \left[ \|W_y(t, -1)\|_{C^{\frac{1}{2}}([0,T])} + (\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}) \right]_{C^{\frac{1}{2}}([0,T])},
\]
where \( C_{11} \) depends on \( \mu, A \) and \( h_0 \). It follows from the proof of [45, Theorem 1.1] that we have
\[
[W_y(t, y)]_{C^{\frac{1}{2}}([0,T])} \leq C_{12}[W_y(t, y)]_{C^{\frac{1}{2}}([0,T])} \leq C_{13}\|W\|_{W^{1,2}(\Delta_T)},
\]
where \( C_{12} \) and \( C_{13} \) do not depend on \( T \). Therefore, we have
\[
\|G\|_{C^1([0,T])} \leq C_{14} T^{\frac{1}{2}} (1 + T) (\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}),
\] (2.21)

where \( C_{14} \) depends on \( C_2, C_{10}, C_{11} \) and \( C_{13} \). Similarly, there exists \( C_{15} \) such that
\[
\|H\|_{C^1([0,T])} \leq C_{15} T^{\frac{1}{2}} (1 + T) (\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}).
\] (2.22)

It follows from (2.21) and (2.22) that
\[
\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} = C_{16} T^{\frac{1}{2}} (1 + T) (\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}) \leq \frac{1}{2} (\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])})
\]
if \( T \leq \min \{ T_4, 1, (4C_{10})^{-\frac{1}{2}} \} \equiv T_5 \), where \( C_{16} = C_{14} + C_{15} \). Hence, \( G = H = 0 \) for \( T \leq T_5 \). It follows from (2.20) that \( W = 0 \). This implies that \( u_1 \equiv u_2 \). By (2.17), we have \( v_1 \equiv v_2 \). The uniqueness is obtained.

Then it follows from the arguments in [23] that we can get the following estimates.

**Lemma 2.2.** Let \((u, v, g, h)\) be a solution of problem (1.4) defined for \( t \in (0, T_0] \), where \( T_0 \in (0, +\infty) \). Then there exist \( M_1, M_2 \) and \( M_3 \) independent of \( T_0 \) such that
(i) \( 0 < u(t, x) \leq M_1, 0 < v(t, x) \leq M_2 \) for \( t \in (0, T_0] \) and \( x \in [g(t), h(t)] \).
(ii) \( 0 < -g'(t), h'(t) \leq M_3 \) for \( t \in (0, T_0] \).

Just like the proof of [37, Theorem 3.2], we can obtain the global existence and uniqueness.

**Theorem 2.3.** The solution exists and is unique for all \( t > 0 \).

Then, we exhibit the following comparison principle, which can be proven by the similar argument in [23, Lemma 2.5].
**Theorem 2.4.** Assume that
\[ \bar{g}, \bar{h} \in C^1([0, +\infty)), \bar{u}(t, x), \bar{v}(t, x) \in C(\overline{D}) \cap C^{1,2}(D), \]
\[ \bar{u}(0, x) \in \mathcal{X}_1(h_0), \bar{v}(0, x) \in \mathcal{X}_2(h_0) \]
with
\[ D := \{ (t, x) \in \mathbb{R}^2 : 0 < t < \infty, \bar{g}(t) < x < \bar{h}(t) \}, \]
and \((\bar{u}, \bar{v}, \bar{g}, \bar{h})\) satisfies
\[
\begin{align*}
\bar{u}_t & \geq d\bar{u}_{xx} - \beta \bar{u}_x - a\bar{u} + c\bar{v}, & t > 0, \bar{g}(t) < x < \bar{h}(t), \\
\bar{v}_t & \geq -b\bar{v} + G(\bar{u}), & t > 0, \bar{g}(t) < x < \bar{h}(t), \\
\bar{u}(t, \bar{g}(t)) = \bar{u}(t, \bar{h}(t)) = 0, & t \geq 0, \\
\bar{v}(t, \bar{g}(t)) = \bar{v}(t, \bar{h}(t)) = 0, & t \geq 0, \\
\bar{g}(0) & \leq h_0, \bar{g}(t) \leq -\mu \bar{u}_x(t, \bar{g}(t)), & t > 0, \\
\bar{h}(0) & \geq h_0, \bar{h}(t) \geq -\mu \bar{u}_x(t, \bar{h}(t)), & t > 0, \\
\bar{u}(0, x) & \geq u_0(x), \bar{v}(0, x) \geq v_0(x), & -h_0 < x < h_0.
\end{align*}
\]

Then the solution \((u, v, g, h)\) of the free boundary problem (1.4) satisfies
\[ h(t) \leq \bar{h}(t), \ g(t) \geq \bar{g}(t), \ \forall \ t \geq 0, \]
\[ u(t, x) \leq \bar{u}(t, x), \ v(t, x) \leq \bar{v}(t, x), \ \forall \ t \geq 0, \ g(t) \leq x \leq \bar{h}(t). \]

**Remark 2.5.** The pair \((\bar{u}, \bar{v}, \bar{g}, \bar{h})\) in Theorem 2.4 is usually called an upper solution of problem (1.4). Similarly, we can define a lower solution by reversing all the inequalities in the suitable places.

In the following part, we consider the following eigenvalue problem
\[
\begin{align*}
-\lambda \phi &= d \phi_{xx} - \beta \phi_x - a\phi + \frac{cG'(0)}{b}\phi, & -l < x < l, \\
\phi(-l) &= \phi(l) = 0.
\end{align*}
\]  
(2.23)

Denote by \(\lambda_0(l)\) the principal eigenvalue of problem (2.23) with some fixed \(l\).

**Lemma 2.6.** \(\lambda_0(l)\) has the following form:
\[ \lambda_0(l) = \frac{\beta^2}{4d} + \frac{d \pi^2}{4l^2} - \left( \frac{cG'(0)}{b} - a \right). \]

**Proof.** We choose \(\beta\) to be small and determine it later. By a simple calculation, we can achieve the characteristic equation
\[ d\mu^2 - \beta \mu + \lambda - a + \frac{cG'(0)}{b} = 0, \]  
(2.24)
and let \(\mu_i (i = 1, 2)\) be the roots of (2.24). Then the solution of (2.23) is
\[ \phi(x) = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x}, \]
\[ c_1, c_2 \in \mathbb{R}. \]
where \( c_1 \) and \( c_2 \) will be determined later. Since \( \phi(-l) = \phi(l) = 0 \), we can derive that

\[
\Delta = \beta^2 - 4d \left( \lambda - a + \frac{cG'(0)}{b} \right) < 0.
\]

In fact, if \( \Delta = \beta^2 - 4d \left( \lambda - a + \frac{cG'(0)}{b} \right) \geq 0 \), we have \( \phi \equiv 0 \), which is a contradiction. Hence, (2.24) has two complex roots:

\[
\mu_1 = \frac{\beta + i \sqrt{4d \left( \lambda - a + \frac{cG'(0)}{b} \right) - \beta^2}}{2d}, \quad \mu_2 = \frac{\beta - i \sqrt{4d \left( \lambda - a + \frac{cG'(0)}{b} \right) - \beta^2}}{2d}.
\]

Then

\[
\phi(x) = c_1 e^{\frac{\mu_1 x}{2}} \left[ \cos \frac{\sqrt{4d \left( \lambda - a + \frac{cG'(0)}{b} \right) - \beta^2}}{2d} x + i \sin \frac{\sqrt{4d \left( \lambda - a + \frac{cG'(0)}{b} \right) - \beta^2}}{2d} x \right] + c_2 e^{\frac{\mu_2 x}{2}} \left[ \cos \frac{\sqrt{4d \left( \lambda - a + \frac{cG'(0)}{b} \right) - \beta^2}}{2d} x - i \sin \frac{\sqrt{4d \left( \lambda - a + \frac{cG'(0)}{b} \right) - \beta^2}}{2d} x \right].
\]

By \( \phi(-l) = \phi(l) = 0 \), we have \( c_1 = c_2 \) and

\[
\frac{\sqrt{4d \left( \lambda - a + \frac{cG'(0)}{b} \right) - \beta^2}}{2d} l = \frac{\pi}{2} + k\pi, \quad \forall \; k \in \mathbb{N}.
\]

When \( k = 0 \), \( \lambda \) attain its minimum, we have

\[
\lambda_0(l) = \frac{\beta^2}{4d} + \frac{d\pi^2}{4L^2} - \left( \frac{cG'(0)}{b} - a \right),
\]

and the corresponding eigenfunction \( \phi(x) = e^{\frac{\mu_1 x}{2}} \cos \left( \frac{\pi}{2} x \right) \). \( \square \)

Then we have the following properties about \( \lambda_0(l) \).

**Lemma 2.7.** The following assertions hold:

(i) \( \lambda_0(l) \) is continuous and strictly decreasing in \( l \),

\[
\lim_{l \to 0} \lambda_0(l) = \infty, \quad \lim_{l \to \infty} \lambda_0(l) = \frac{\beta^2}{4d} - \left( \frac{cG'(0)}{b} - a \right).
\]

(ii) If \( \frac{cG'(0)}{ab} > 1 \) and \( 0 < \beta < 2 \sqrt{d \left( \frac{cG'(0)}{b} - a \right)} \), then there exists

\[
l' = 2d\pi \sqrt{4d \left( \frac{cG'(0)}{b} - a \right) - \beta^2}
\]

such that \( \lambda_0(l') = 0 \). Furthermore, \( \lambda_0(l) > 0 \) for \( 0 < l < l' \), and \( \lambda_0(l) < 0 \) for \( l > l' \).

(iii) If \( \frac{cG'(0)}{ab} \leq 1 \), then \( \lambda_0(l) > \frac{\beta^2}{4d} - \left( \frac{cG'(0)}{b} - a \right) > 0 \).

**Proof.** By the expression of \( \lambda_0(l) \) in Lemma 2.6, the proof of lemma is obvious. We omit it here. \( \square \)
3. Spreading and vanishing

Firstly, we give the definitions of spreading and vanishing of the disease:

**Definition 3.1.** We say that vanishing happens if

\[ h_\infty - g_\infty < \infty \text{ and } \lim_{t \to \infty} (\|u(t, \cdot)\|_{C([g(t), h(t)])} + \|v(t, \cdot)\|_{C([g(t), h(t)])}) = 0, \]

and spreading happens if

\[ h_\infty - g_\infty = \infty \text{ and } \limsup_{t \to \infty} (\|u(t, \cdot)\|_{C([g(t), h(t)])} + \|v(t, \cdot)\|_{C([g(t), h(t)])}) > 0. \]

Then, we give the following lemmas.

**Lemma 3.2.** Let \((u, v, g, h)\) be the solution of (1.4). If \(h_\infty - g_\infty < \infty\), then there exists a constant \(C > 0\) such that

\[ \|u(t, \cdot)\|_{C^1([g(t), h(t)])} \leq C, \quad \forall t > 1. \]  

Moreover,

\[ \lim_{t \to \infty} g'(t) = \lim_{t \to \infty} h'(t) = 0. \]

**Proof.** We can use the method in [46, Theorem 2.1] to get (3.1). Then the proof of (3.2) can be done as [16, Theorem 4.1].

**Lemma 3.3.** Let \(d, \mu, h_0\) be positive constants, \(w \in C^{1+\alpha}_{\text{loc}}([0, \infty) \times [g(t), h(t)])\) and \(g, h \in C^{1+\frac{\alpha}{2}}([0, \infty))\) for some \(\alpha > 0\). We further assume that \(w_0(x) \in \mathcal{L}^1(h_0)\). If \((w, g, h)\) satisfies

\[
\begin{aligned}
 w_t &\geq dw_{xx} - \beta w_x - aw, & t > 0, & g(t) < x < h(t), \\
 w(t, x) &= 0, & t \geq 0, & x \leq g(t), \\
 w(t, x) &= 0, & t \geq 0, & x \geq h(t), \\
 g(0) &= -h_0, & g'(t) &\leq -\mu w_x(t, g(t)), & t > 0, \\
 h(0) &= h_0, & h'(t) &\geq -\mu w_x(t, h(t)), & t > 0, \\
 w(0, x) &= w_0(x) \geq 0, & -h_0 < x < h_0,
\end{aligned}
\]

and

\[ \lim_{t \to \infty} g(t) = g_\infty > -\infty, \quad \lim_{t \to \infty} g'(t) = 0, \quad \lim_{t \to \infty} h(t) = h_\infty < \infty, \quad \lim_{t \to \infty} h'(t) = 0, \]

\[ \|w(t, \cdot)\|_{C^1([g(t), h(t)])} \leq M, \quad \forall t > 1 \]

for some constant \(M > 0\). Then

\[ \lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} w(t, x) = 0. \]

**Proof.** It can be proved by the similar arguments in [16, Theorem 4.2].

By above Lemmas 3.2 and 3.3, we can derive the following result.
Theorem 3.4. If \( h_\infty - g_\infty < \infty \), then
\[
\lim_{t \to \infty} (\|u(t, \cdot)\|_{C([g(t), h(t)])} + \|v(t, \cdot)\|_{C([g(t), h(t)])}) = 0.
\]

Proof. Firstly, we can use the method in the proof of [46, Theorem 2.1] to get
\[
\|u\|_{C^1([0, \infty) \times [g(t), h(t)])} + \|g\|_{C^1([0, \infty) \times [0, \infty)})} + \|h\|_{C^1([0, \infty)})} \leq C.
\]
Recall that \( u \) satisfies (3.3). By Lemmas 3.2 and 3.3, we can get \( \lim_{t \to \infty} \|u(t, \cdot)\|_{C([g(t), h(t)])} = 0 \). Noting that \( v(t, x) \) satisfies
\[
v_t = -bv + G(u), \quad t > 0, \quad g(t) < x < h(t)
\]
and \( G(u) \to 0 \) uniformly for \( x \in [g(t), h(t)] \) as \( t \to \infty \), we have \( \lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0 \). \( \Box \)

Lemma 3.5. If \( G(z) < \frac{ab}{c} \) for any \( z > 0 \), then \( h_\infty - g_\infty < \infty \).

Proof. Direct calculations yield
\[
\frac{d}{dt} \int_{g(t)}^{h(t)} (u(t, x) + \frac{c}{b}v(t, x)) \, dx
\]
\[
= \int_{g(t)}^{h(t)} (u_t + \frac{c}{b}v_t) \, dx
\]
\[
= \int_{g(t)}^{h(t)} (du_{xx} - \beta u_x - au + \frac{c}{b}G(u)) \, dx
\]
\[
= -\frac{d}{\mu} (h'(t) - g'(t)) + \int_{g(t)}^{h(t)} \left( -au + \frac{c}{b}G(u) \right) \, dx.
\]
Integrating from 0 to \( t \) gives
\[
\int_{g(t)}^{h(t)} (u(t, x) + \frac{c}{b}v(t, x)) \, dx
\]
\[
= \int_{g(h_0)}^{h(h_0)} (u_0(x) + \frac{c}{b}v_0(x)) \, dx - \frac{d}{\mu} (h(t) - g(t))
\]
\[
+ \frac{d}{\mu} 2h_0 + \int_0^{h(h_0)} \int_{g(s)}^{h(s)} \left( -au + \frac{c}{b}G(u) \right) \, dx \, ds.
\]
Since \( u \geq 0, \nu \geq 0 \) and \( G(u) \leq \frac{ab}{c}u \) for \( u \geq 0 \), we have
\[
h(t) - g(t) \leq \frac{\mu}{d} \int_{g(h_0)}^{h(h_0)} \left( u_0(x) + \frac{c}{b}v_0(x) \right) \, dx + 2h_0 < \infty.
\]
Letting \( t \to \infty \), we have \( h_\infty - g_\infty < \infty \). \( \Box \)

Lemma 3.6. Assume that \( G(z_1) > \frac{ab}{c} \) for some \( z_1 > 0 \). If \( \lambda_0(h_0) > 0 \) holds, then vanishing will happen provided that \( u_0 \) and \( v_0 \) are sufficiently small.
Proof. We prove this result by constructing the appropriate upper solution. Let \( \phi \) be the corresponding eigenfunction of \( \lambda_0(h_0) \). Since \( \lambda_0(h_0) > 0 \), we can choose some small \( \delta \) such that

\[
-\delta - \frac{\beta h_0 \delta^2}{2d(2 + \delta)} + \frac{3}{4} \lambda_0 \frac{1}{(1 + \delta)^2} > 0.
\]

Set

\[
\sigma(t) = h_0(1 + \delta - \frac{\delta}{2} e^{-\delta t}), \quad t \geq 0,
\]

\[
\bar{u}(t, x) = c e^{-\delta t} \phi \left( \frac{x h_0}{\sigma(t)} \right) e^{\frac{b}{2} (1 - \frac{h_0}{\sigma(t)})}, \quad t \geq 0, \quad -\sigma(t) \leq x \leq \sigma(t),
\]

\[
\bar{v}(t, x) = \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \frac{h_0^2}{\sigma^2} \bar{u}, \quad t \geq 0, \quad -\sigma(t) \leq x \leq \sigma(t).
\]

Direct computations yield

\[
\bar{u}_t - d \bar{u}_{xx} + \beta \bar{u}_x + \alpha \bar{u} - c \bar{v} \\
= \bar{u} \left( -\delta - \frac{\phi' x h_0 \sigma'}{\phi \sigma^2} + \frac{\beta h_0 x \sigma'}{2d \sigma^2} \right) \\
- d e^{-\delta t} e^{\frac{b}{2} (1 - \frac{h_0}{\sigma(t)})} \left[ \frac{\phi''}{\sigma(t)} \right] + 2 \phi \frac{h_0 \beta}{2d} \left( 1 - \frac{h_0}{\sigma(t)} \right) + \phi \left( \frac{\beta}{2d} \right)^2 \left( 1 - \frac{h_0}{\sigma(t)} \right)^2 \\
+ \beta e^{-\delta t} e^{\frac{b}{2} (1 - \frac{h_0}{\sigma(t)})} \left[ \phi' \frac{h_0}{\sigma(t)} + \phi' \frac{h_0}{\sigma(t)} b \left( 1 - \frac{h_0}{\sigma(t)} \right) \right] + \alpha \bar{u} - c \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \frac{h_0^2}{\sigma^2} \bar{u} \\
= \bar{u} \left( -\delta - \frac{\phi' x h_0 \sigma'}{\phi \sigma^2} + \frac{\beta h_0 x \sigma'}{2d \sigma^2} \right) \\
+ e^{-\delta t} e^{\frac{b}{2} (1 - \frac{h_0}{\sigma(t)})} \left[ \frac{h_0^2}{\sigma^2} \right] + \phi \left( \frac{\beta}{4d} \right)^2 \left( 1 - \frac{h_0^2}{\sigma^2} \right) + \alpha \bar{u} - c \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \frac{h_0^2}{\sigma^2} \bar{u} \\
\geq \bar{u} \left( -\delta - \frac{\beta h_0 x \sigma'}{2d \sigma} + \frac{3}{4} \lambda_0 \frac{h_0^2}{\sigma^2} \right) + \left( 1 - \frac{h_0^2}{\sigma^2} \right) \left( \beta \frac{\bar{u}}{4d} + \alpha \bar{u} \right) \\
> \bar{u} \left( -\delta - \frac{\beta h_0 \delta^2}{2d(2 + \delta)} + \frac{3}{4} \lambda_0 \frac{1}{(1 + \delta)^2} \right) > 0,
\]

and

\[
\bar{v}_t + b \bar{v} - G(\bar{u}) \\
= - \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \frac{h_0^2 \sigma'}{\sigma^3} \bar{u} + \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \frac{h_0^2}{\sigma^2} (\bar{u}_x + b \bar{u}) - G'(\xi) \bar{u} \\
\geq - \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \frac{h_0^2 \sigma'}{\sigma^3} \bar{u} + \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \frac{h_0^2}{\sigma^2} \left[ -\delta - \frac{\beta h_0 \delta^2}{2d(2 + \delta)} + b \right] \bar{u} - G'(\xi) \bar{u} \\
= \bar{u} \left( \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \frac{h_0^2}{\sigma^2} \left[ -\delta - \frac{\beta h_0 \delta^2}{2d(2 + \delta)} + b \right] + G'(0) \frac{h_0^2}{\sigma^2} \left[ 1 - \frac{2 \delta^2}{b(2 + \delta)} \right] \right) \leq B
\]
for all \( t > 0 \) and \(-\sigma(t) < x < \sigma(t)\), where \( \xi \in (0, \overline{u}) \). Let

\[
e = \frac{\delta^2 h_0 (1 + \frac{\delta}{2})}{2\mu} \min \left\{ -\frac{1}{\phi'(h_0)} e^{-\frac{\delta}{\sigma} \phi(h_0)}, \frac{1}{\phi'(-h_0)} e^{-\frac{\delta}{\sigma} \phi(-h_0)} \right\}.
\]

Since \( \overline{u} \leq \epsilon e^{\frac{\delta}{\sigma} \phi(h_0)} \), we can choose \( \delta \) to be sufficiently small such that \( B > 0 \). Noting that

\[
\sigma'(t) = h_0 \frac{\delta^2}{2} e^{-\delta t}, \quad \overline{u}_s(t, \sigma(t)) = e^{\delta t} \phi'(h_0) \frac{h_0}{\sigma} e^{\frac{\delta}{\sigma} (\sigma(t) - h_0)},
\]

then we have

\[
\begin{align*}
\overline{u}_t & \geq d\overline{u} - \beta \overline{u} - a\overline{u} + c\overline{v}, & t > 0, & -\sigma(t) < x < \sigma(t), \\
\overline{v}_t & \geq -b \overline{v} + G(\overline{u}), & t > 0, & -\sigma(t) < x < \sigma(t), \\
\overline{u}(t, -\sigma(t)) = \overline{u}(t, \sigma(t)) = 0, & t \geq 0, \\
\overline{v}(t, -\sigma(t)) = \overline{v}(t, \sigma(t)) = 0, & t \geq 0, \\
\sigma(0) & \leq \overline{u}, & t > 0, & -\mu \overline{u}_s(t, -\sigma(t)), \\
\sigma(0) & \geq h_0, & \sigma'(t) & \geq \overline{u}_s(t, \sigma(t)),
\end{align*}
\]

If \( u_0 \) and \( v_0 \) are sufficiently small such that

\[
u_0(x) \leq \epsilon \phi \left( \frac{x}{1 + \frac{\delta}{2}} \right) e^{\frac{\delta x}{\sigma(0)}}, \quad \forall \ x \in [-h_0(1 + \frac{\delta}{2}), h_0(1 + \frac{\delta}{2})]
\]

and

\[
v_0(x) \leq \left( \frac{\sigma'(0)}{b} + \frac{\lambda_0}{4c} \right) \frac{1}{(1 + \frac{\delta}{2})^2} \epsilon \phi \left( \frac{x}{1 + \frac{\delta}{2}} \right) e^{\frac{\delta x}{\sigma(0)}}, \quad \forall \ x \in [-h_0(1 + \frac{\delta}{2}), h_0(1 + \frac{\delta}{2})],
\]

then

\[
u_0(x) \leq \overline{u}(0, x), \quad \forall \ x \in (-h_0, h_0).
\]

Applying Theorem 2.4 gives that \( h(t) \leq \sigma(t) \) and \( g(t) \geq -\sigma(t) \). Hence, \( h_\infty - g_\infty \leq 2h_0(1 + \delta) < \infty \). By Theorem 3.4, we have \( \lim_{t \to \infty} \|u(t, \cdot)\|_{C([\sigma(t), h(t)])} + \|v(t, \cdot)\|_{C([\sigma(t), h(t)])} = 0 \).

By Lemma 3.6, we can derive the following corollary directly.

**Corollary 3.7.** Assume that \( \frac{G(\xi)}{z_1} > \frac{ab}{c} \) for some \( z_1 > 0 \), then the following statements holds:

(i) If \( \frac{G(0)}{z_1} < 1 \), then vanishing will happen for \( u_0 \) and \( v_0 \) sufficiently small.

(ii) If \( \frac{G(0)}{ab} > 1 \) and \( h_0 < l' \), then vanishing will happen for \( u_0 \) and \( v_0 \) sufficiently small.

**Lemma 3.8.** Assume that \( \frac{G(\xi)}{z_1} > \frac{ab}{c} \) for some \( z_1 > 0 \) and \( \frac{G(0)}{ab} > 1 \). If \( h_0 > l' \), then spreading will happen.
We first note that there exists $\lambda_0(h_0)$. Since $\frac{cG'(0)}{ab} > 1$ and $h_0 > l^*$, we have $\lambda_0(h_0) < 0$. Then we construct a suitable lower solution. Since

$$\frac{cG'(0)}{b} + \frac{\lambda_0}{4} = \frac{\beta^2}{4d} + \frac{d\pi^2}{4\ell^2} + a - \frac{3\lambda_0}{4} > 0,$$

we can define

$$u(t, x) = \epsilon \phi(x), \ t \geq 0, \ -h_0 \leq x \leq h_0,$$

$$v(t, x) = \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right) \epsilon \phi(x), \ t \geq 0, \ -h_0 \leq x \leq h_0.$$

Direct computations yield

$$u_t - \frac{\partial^2 u}{\partial x^2} + \beta u_x + au - cv = \epsilon \left(-d \phi_{xx} + \beta \phi_x + \alpha \phi - \frac{cG'(0)}{b} \phi - \frac{\lambda_0}{4}\phi\right) = \frac{3}{4} \lambda_0 \epsilon \phi < 0,$$

and

$$v_t + b v - G(u) = \epsilon \phi \left(G'(0) - G'(\xi) + \frac{b\lambda_0}{4c}\right)$$

for all $t > 0$ and $-h_0 < x < h_0$, where $\xi \in (0, u)$. We can choose $\epsilon$ small enough such that

$$G'(0) - G'(\xi) + \frac{b\lambda_0}{4c} \leq 0, \ \epsilon \phi(x) \leq u_0(x), \ \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right) \epsilon \phi(x) \leq v_0(x).$$

Then

$$\begin{cases}
  u_t - \frac{\partial^2 u}{\partial x^2} - \beta u_x - au + cv, & t > 0, \ -h_0 < x < h_0, \\
  v_t - bv + G(u), & t > 0, \ -h_0 < x < h_0, \\
  u(t, -h_0) = u(t, h_0) = 0, & t \geq 0, \\
  v(t, -h_0) = v(t, h_0) = 0, & t \geq 0, \\
  0 \leq -\mu u_x(t, h_0), \ 0 \leq -\mu u_x(t, h_0), & t > 0, \\
  u(0, x) \leq u_0(x), \ v(0, x) \leq v_0(x), & -h_0 < x < h_0.
\end{cases}$$

It follows from Remark 2.5 that $u(t, x) \geq u(t, x)$ in $[0, \infty) \times [-h_0, h_0]$. Hence,

$$\lim_{t \to \infty} ||u(t, \cdot)||_{C([l, \ell])} \geq \epsilon \phi(x) > 0.$$

By Theorem 3.4, we have $h_\infty - g_\infty = \infty$. □

**Lemma 3.9.** Assume that $\frac{G(z_1)}{z_1} > \frac{ab}{c}$ for some $z_1 > 0$ and $\frac{cG'(0)}{ab} > 1$. If $h_0 < l^*$, then $h_\infty - g_\infty = \infty$ provided that $u_0$ and $v_0$ are sufficiently large.

**Proof.** We first note that there exists $\sqrt{T^*} > l^*$ such that $\lambda_0(\sqrt{T^*}) < 0$.

Inspired by the argument of [8, proposition 5.3], we consider

$$\begin{cases}
  -d \varphi'' - \left(\frac{1}{2} + \sqrt{T^*} + 1\right) \varphi' = \tilde{\lambda}_0 \varphi, \ 0 < x < 1, \\
  \varphi'(0) = \varphi(1) = 0.
\end{cases} \tag{3.4}$$
It is well-known that the first eigenvalue $\tilde{\lambda}_0$ of (3.4) is simple and the corresponding eigenfunction $\varphi$ can be chosen positive in $[0, 1)$ and $||\varphi||_{L^\infty(0,1)} = 1$. Moreover, one can easily see that $\tilde{\lambda}_0 > 0$ and $\varphi'(x) < 0$ in $(0, 1)$. We extend $\varphi$ to $[-1, 1]$ as an even function. Then clearly

$$
\begin{cases}
-d\varphi'' - \left(\frac{1}{2} + \sqrt{T^* + 1}\right) \frac{1}{\sqrt{T + \varrho}} \varphi' \varphi = \tilde{\lambda}_0 \varphi, & -1 < x < 1, \\
\varphi(-1) = \varphi(1) = 0.
\end{cases}
$$

Now we construct a suitable lower solution to (1.4). Define

$$
\eta(t) = \sqrt{t + \varrho}, \quad 0 \leq t \leq T^*,
$$

$$
u(t, x) = \begin{cases} 
\frac{m}{(t + \varrho)^{k+1}} \varphi\left(\frac{x}{\sqrt{t + \varrho}}\right), & 0 \leq t \leq T^*, -\eta(t) < x < \eta(t), \\
0, & 0 \leq t \leq T^*, |x| \geq \eta(t),
\end{cases}
$$

where the constants $\varrho$, $m$, $k$ are chosen as follows:

$$0 < \varrho \leq \min \left\{1, h_0^2\right\}, \quad k \geq \tilde{\lambda}_0 + a(T^* + 1), \quad m \geq \frac{(T^* + 1)^k}{2\mu \min\{\varphi(-1), -\varphi'(1)\}}.
$$

Let

$$
t_s := \begin{cases} 
t_s^1, & x \in [-\eta(T^*), -\sqrt{\varrho}) \text{ and } x = -\eta(t_s^1), \\
0, & x \in [-\sqrt{\varrho}, \sqrt{\varrho}], \\
t_s^2, & x \in (\sqrt{\varrho}, \eta(T^*)) \text{ and } x = \eta(t_s^2)
\end{cases}
$$

and

$$
v_0(x) = \begin{cases} 
\frac{\pi}{2} + \frac{\pi}{2} \cos\left(\frac{x}{\sqrt{\varrho}}\right), & -\sqrt{\varrho} \leq x \leq \sqrt{\varrho}, \\
0, & |x| > \sqrt{\varrho},
\end{cases}
$$

where we choose $\varepsilon$ small enough such that

$$v_0(x) \leq v_0(x), \quad \forall \ x \in (-\sqrt{\varrho}, \sqrt{\varrho}).
$$

Then we define

$$
u(t, x) = e^{-bt} \left(\int_{t_s}^t e^{br} G(u(\tau, x))d\tau + v_0(x)\right), \quad t_s \leq t \leq T^*, -\eta(t) \leq x \leq \eta(t).
$$

Direct computations yield

$$
u_t - d\nu_{xx} + \beta u_x + au - cv
\leq - \frac{m}{(t + \varrho)^{k+1}} \left[ k \varphi + \frac{x}{2 \sqrt{t + \varrho}} \varphi' - \sqrt{t + \varrho} \varphi' - a(t + \varrho) \varphi \right]
\leq - \frac{m}{(t + \varrho)^{k+1}} \left[ k \varphi + \left(\frac{1}{2} + \sqrt{T^* + 1}\right) \frac{1}{\sqrt{T + \varrho}} \varphi' + a(T^* + 1) \varphi \right]
\leq - \frac{m}{(t + \varrho)^{k+1}} \left[ d \varphi'' + \left(\frac{1}{2} + \sqrt{T^* + 1}\right) \frac{1}{\sqrt{T + \varrho}} \varphi' + \tilde{\lambda}_0 \varphi \right] = 0.
$$
and
\[ \nu + bv - G(u) = 0, \quad 0 < t \leq T^*, \quad -\eta(t) < x < \eta(t). \]

For \( x \in [-\sqrt{\rho}, \sqrt{\rho}] \), we have \( t_x = 0 \). Then
\[ \varphi(x) = \varphi_0(x), \quad \forall \, x \in [-\sqrt{\rho}, \sqrt{\rho}]. \]

Moreover,
\[
\begin{align*}
\eta'(t) + \mu u_x(t) &= \frac{1}{2} \frac{\nu}{\sqrt{1 + \rho}} + \frac{\mu m}{(t + \rho)^{k+\frac{1}{2}}} \varphi'(1) \leq 0, \quad \forall \, t \in (0, T^*), \\
\eta'(t) - \mu u_x(t) &= \frac{1}{2} \frac{\nu}{\sqrt{1 + \rho}} - \frac{\mu m}{(t + \rho)^{k+\frac{1}{2}}} \varphi'(-1) \leq 0, \quad \forall \, t \in (0, T^*).
\end{align*}
\]

If \( u_0 \) is sufficiently large such that \( u(0, x) = \frac{m}{\sqrt{\rho}} \varphi\left(\frac{x}{\sqrt{\rho}}\right) \leq u_0(x) \) for \( x \in [-\sqrt{\rho}, \sqrt{\rho}] \), then we have
\[
\begin{align*}
\eta(x) &\leq \frac{1}{\sqrt{1 + \rho}} - \beta u_x - au + cv, \\
\nu &\leq -bv + G(u), \\
u(t, x) &= \gamma(t, x) = 0, \quad 0 \leq t \leq T^*, \quad x \leq -\eta(t), \\
u(t, x) &= \gamma(t, x) = 0, \quad 0 \leq t \leq T^*, \quad x \geq \eta(t), \\
-\eta'(t) &\geq -\mu u_x(t, -\eta(t)), \quad 0 < t \leq T^*, \\
\eta'(t) &\leq -\mu u_x(t, \eta(t)), \quad 0 < t \leq T^*, \\
\eta(0, x) &\leq u_0(x), \quad \nu(0, x) \leq \nu_0(x), \quad -\eta(0) < x < \eta(0).
\end{align*}
\]

Noting that \( \eta(0) = \sqrt{\rho} \leq h_0 \), we can use Remark 2.5 to conclude that \( h(t) \geq \eta(t) \) and \( g(t) \leq -\eta(t) \) in \([0, T^*]\). Specially, we obtain \( h(T^*) \geq \eta(T^*) = \sqrt{T^* + \rho} > \sqrt{T^*} \) and \( g(T^*) < -\sqrt{T^*} \). Then
\[
(-\Gamma', \Gamma') \subseteq (-\sqrt{T^*}, \sqrt{T^*}) \subseteq (g(t), h(t)), \quad \forall \, t \geq T^*.
\]

Hence, we have \( h_\infty - g_\infty = +\infty \) by Lemma 3.8.

Next, we present the sharp criteria on initial value, which separates spreading and vanishing.

**Theorem 3.10.** For some \( \gamma > 0 \) and \( \omega_1 \) and \( \omega_2 \) in \( (h_0, h_\infty) \), let \((u, v, g, h)\) be a solution of (1.4) with \((u_0, v_0) = \gamma(\omega_1, \omega_2)\), then the following statements holds:

(i) Assume that \( \frac{G'(0)}{ab} < 1 \). If \( \frac{G(z)}{z} < \frac{ab}{c} \) for any \( z > 0 \), then vanishing will happen. If \( \frac{G(z)}{z} > \frac{ab}{c} \) for some \( z_1 > 0 \), then vanishing will happen for \( u_0 \) and \( v_0 \) sufficiently small.

(ii) Assume that \( \frac{G'(0)}{ab} > 1 \) and \( 0 < \beta < 2 \sqrt{d \left( \frac{G'(0)}{b} - a \right)} \). If \( \frac{G(z)}{z} < \frac{ab}{c} \) for any \( z > 0 \), then vanishing will happen. If \( \frac{G(z)}{z} > \frac{ab}{c} \) for some \( z_1 > 0 \), then the following will hold:

(a) If \( h_0 > \Gamma' \), then spreading will happen; (b) If \( h_0 < \Gamma' \), then there exists \( \gamma' \in (0, \infty) \) such that spreading occurs for \( \gamma > \gamma' \), and vanishing happens for \( 0 < \gamma \leq \gamma' \).

**Proof.** This theorem follows from Lemma 3.5, Corollary 3.7, Lemmas 3.8 and 3.9. The conclusion (b) can be proven by the same arguments in [23, Theorem 4.3].
Finally, we give the asymptotic behavior of (1.4) when spreading happens.

**Theorem 3.11.** Assume that \( \frac{cG(0)}{ab} > 1 \), \( 0 < \beta < 2 \sqrt{d\left(\frac{cG(0)}{b} - a\right)} \) and \( \frac{\delta_1}{z_1} > \frac{ab}{c} \) for some \( z_1 > 0 \). If \( h_{\infty} - g_{\infty} = \infty \), then

\[
(u^*, v^*(x)) = \lim \inf_{t \to \infty} (u(t, x), v(t, x)) \leq \lim \sup_{t \to \infty} (u(t, x), v(t, x)) \leq (u^*, v^*)
\]

for \( x \in \mathbb{R} \), where \((u^*(x), v^*(x))\) will be given in the proof.

**Proof.** We denote by \((u(t), v(t))\) the solution of (1.1) with

\[
u(0) = \|u_0\|_{L^\infty([-h_0, h_0])} \quad \text{and} \quad v(0) = \|v_0\|_{L^\infty([-h_0, h_0])}.
\]

Applying the comparison principle gives

\[
(u(t, x), v(t, x)) \leq (u(t), v(t)) \quad \text{for} \quad t > 0 \quad \text{and} \quad g(t) \leq x \leq h(t).
\]

Since \( \frac{cG(0)}{ab} > 1 \), \( \lim_{t \to \infty} (u(t), v(t)) = (u^*, v^*) \). Hence,

\[
\lim \sup_{t \to \infty} (u(t, x), v(t, x)) \leq (u^*, v^*) \quad \text{uniformly for} \quad x \in \mathbb{R}.
\]

By Lemma 2.7, we can find some \( L > l^* \) such that \( \lambda_0(L) < 0 \), where \( \lambda_0(L) \) is the principal eigenvalue of problem (2.23) with \( l = L \) and \( \phi(x) \) is the corresponding eigenfunction. For such \( L \), it follows from \( h_{\infty} - g_{\infty} = \infty \) that there exists \( T_L \) such that

\[
[-L, L] \subset [g(t), h(t)], \quad \forall \ t \geq T_L.
\]

Let \((u(t, x), v(t, x)) = \delta \left( \phi(x), \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \phi(x) \right)\), then we can choose small \( \delta \) such that

\[
\begin{align*}
    u_t - du_{xx} + \beta u_x + au - cv &\leq 0, & t > T_L, \ -L < x < L, \\
    v_t + bv - G(u) &\leq 0, & t > T_L, \ -L < x < L, \\
    u(t, x) = v(t, x) = 0, & t \geq T_L, \ x = -L \text{ or } x = L, \\
    u(T_L, x) \leq u(T_L, x), \ v(T_L, x) \leq v(T_L, x), & -L < x < -L.
\end{align*}
\]

Applying the comparison principle gives that

\[
(u(t, x), v(t, x)) \geq \delta \left( \phi(x), \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \phi(x) \right), \ t \geq T_L, \ -L \leq x \leq L.
\]

We extend \( \delta \left( \phi(x), \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \phi(x) \right) \) to \((u^*(x), v^*(x))\) by defining

\[
(u^*(x), v^*(x)) = \begin{cases} 
\delta \left( \phi(x), \left( \frac{G'(0)}{b} + \frac{\lambda_0}{4c} \right) \phi(x) \right), & -L \leq x \leq L, \\
0, & x < -L \text{ or } x > L.
\end{cases}
\]

Then we have \( \lim \inf_{t \to \infty} (u(t, x), v(t, x)) \geq (u^*(x), v^*(x)) \) for \( x \in \mathbb{R} \). \( \square \)
4. Discussion

In this paper, we have dealt with a partially degenerate epidemic model with free boundaries and small advection. At first, we obtain the global existence and uniqueness of the solution. Then the effect of small advection is considered. We have proved that the results is similar to that in [20, 23] under the condition $0 < \beta < \beta^*$. But we should explain that, for the case that $\frac{G'(0)}{ab} > 1$ and $\beta \geq 2 \sqrt{d \left( \frac{G'(0)}{b} - a \right)}$, the criteria for spreading and vanishing is hard to get by using the results of eigenvalue problem to construct the suitable upper and lower solution. We will study it in the future. When spreading occurs, the precise long-time behavior also needs a further consideration.

In order to study the spreading of disease, the asymptotic spreading speed of the spreading fronts is one of the most important subjects. To estimate the precise asymptotic spreading speed, we need to study the corresponding semi-wave problem or some other new technique. This may be not an easy task and deserves further study. We will consider it in another paper.

Due to the advection term, we find that the spreading barrier $l^*$ becomes larger if we increase the size of $\beta$ for $\beta \in (0, \beta^*)$. This means that if $\beta \in (0, \beta^*)$, the more large the size of advection is, the more difficult the disease will spread. This result may provide us a suggestion in controlling and preventing the disease. It may be an effective measure to make the infectious agents move along a certain direction by artificial means.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. V. Capasso and S. L. Paveri-Fontana, A mathematical model for the 1973 cholera epidemic in the European Mediterranean region, *Rev. d’Epidemiol. Santé Publique*, 27 (1979), 32–121.
2. H. H. Wilson, *Ordinary Differential Equations*, Addison-Wesley Publ. Comp., London, 1971.
3. V. Capasso and R. E. Wilson, Analysis of a reaction-diffusion system modeling man-environment-man epidemics, *SIAM J. Appl. Math.*, 57 (1997), 327–346.
4. D. Xu and X. Q. Zhao, Erratum to: “Bistable waves in an epidemic model”, *J. Dyn. Differ. Eq.*, 17 (2005), 219–247.
5. Y. Du and Z. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.*, 42 (2010), 377–405.
6. Y. Du, Z. Guo and R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, *J. Funct. Anal.*, 265 (2013), 2089–2142.
7. Y. Du and Z. Lin, The diffusive competition model with a free boundary: invasion of a superior or inferior competitor, *Discrete Contin. Dyn. Syst. Ser. B*, **19** (2014), 3105–3132.
8. Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, *J. Eur. Math. Soc.*, **17** (2015), 2673–2724.
9. J. Ge, K. I. Kim, Z. Lin, et al., A SIS reaction-diffusion-advection model in a low-risk and high-risk domain, *J. Differ. Eq.*, **259** (2015), 5486–5509.
10. H. Gu, B. Lou and M. Zhou, Long time behavior of solutions of Fisher-KPP equation with advection and free boundaries, *J. Funct. Anal.*, **269** (2015), 1714–1768.
11. J. Guo and C. Wu, On a free boundary problem for a two-species weak competition system, *J. Dynam. Differ. Eq.*, **24** (2012), 873–895.
12. K. I. Kim, Z. Lin and Q. Zhang, An SIR epidemic model with free boundary, *Nonlinear Anal. Real. World Appl.*, **14** (2013), 1992–2001.
13. J. Wang and L. Zhang, Invasion by an inferior or superior competitor: a diffusive competition model with a free boundary in a heterogeneous environment, *J. Math. Anal. Appl.*, **423** (2015), 377–398.
14. M. Wang, On some free boundary problems of the prey-predator model, *J. Differ. Eq.*, **256** (2014), 3365–3394.
15. M. Wang and J. Zhao, Free boundary problems for a Lotka-Volterra competition system, *J. Dyn. Differ. Eq.*, **26** (2014), 655–672.
16. M. Wang and J. Zhao, A free boundary problem for the predator-prey model with double free boundaries, *J. Dyn. Differ. Eq.*, **29** (2017), 957–979.
17. M. Wang and Y. Zhang, Dynamics for a diffusive prey-predator model with different free boundaries, *J. Differ. Eq.*, **264** (2018), 3527–3558.
18. W. T. Li, M. Zhao and J. Wang, Spreading fronts in a partially degenerate integro-differential reaction-diffusion system, *Z. Angew. Math. Phys.*, **68** (2017), Art. 109, 28 pp.
19. A. K. Tarboush, Z. Lin and M. Zhang, Spreading and vanishing in a West Nile virus model with expanding fronts, *Sci. China Math.*, **60** (2017), 841–860.
20. J. Wang and J. F. Cao, The spreading frontiers in partially degenerate reaction-diffusion systems, *Nonlinear Anal.*, **122** (2015), 215–238.
21. X. Bao, W. Shen and Z. Shen, Spreading speeds and traveling waves for space-time periodic nonlocal dispersal cooperative systems, *Commun. Pure Appl. Anal.*, **18** (2019), 361–396.
22. B. S. Han and Y. Yang, An integro-PDE model with variable motility, *Nonlinear Anal. Real. World Appl.*, **45** (2019), 186–199.
23. I. Ahn, S. Beak and Z. Lin, The spreading fronts of an infective environment in a man-environment-man epidemic model, *Appl. Math. Model.*, **40** (2016), 7082–7101.
24. M. Zhao, W. T. Li and W. Ni, Spreading speed of a degenerate and cooperative epidemic model with free boundaries, *Discrete Contin. Dyn. Syst. Ser. B*, in press (2019).
25. N. A. Maidana and H. Yang, Spatial spreading of West Nile Virus described by traveling waves, *J. Theoret. Biol.*, **258** (2009), 403–417.
26. H. Gu, Z. Lin and B. Lou, Long time behavior of solutions of a diffusion-advection logistic model with free boundaries, *Appl. Math. Lett.*, **37** (2014), 49–53.

27. H. Gu, Z. Lin and B. Lou, Different asymptotic spreading speeds induced by advection in a diffusion problem with free boundaries, *Proc. Amer. Math. Soc.*, **143** (2015), 1109–1117.

28. J. Ge, C. Lei and Z. Lin, Reproduction numbers and the expanding fronts for a diffusion-advection SIS model in heterogeneous time-periodic environment, *Nonlinear Anal. Real World Appl.*, **33** (2017), 100–120.

29. H. Gu and B. Lou, Spreading in advective environment modeled by a reaction diffusion equation with free boundaries, *J. Differ. Eq.*, **260** (2016), 3991–4015.

30. Y. Kaneko and H. Matsuzawa, Spreading speed and sharp asymptotic profiles of solutions in free boundary problems for nonlinear advection-diffusion equations, *J. Math. Anal. Appl.*, **428** (2015), 43–76.

31. H. Monobe and C. H. Wu, On a free boundary problem for a reaction-diffusion-advection logistic model in heterogeneous environment, *J. Differ. Eq.*, **261** (2016), 6144–6177.

32. N. Sun, B. Lou and M. Zhou, Fisher-KPP equation with free boundaries and time-periodic advections, *Calc. Var. Partial Differ. Eq.*, **56** (2017), 61–96.

33. L. Wei, G. Zhang and M. Zhou, Long time behavior for solutions of the diffusive logistic equation with advection and free boundary, *Calc. Var. Partial Differ. Eq.*, **55** (2016), 95–128.

34. Y. Zhao and M. Wang, A reaction-diffusion-advection equation with mixed and free boundary conditions, *J. Dynam. Differ. Eq.*, **30** (2018), 743–777.

35. Q. Chen, F. Li and F. Wang, A reaction-diffusion-advection competition model with two free boundaries in heterogeneous time-periodic environment, *IMA J. Appl. Math.*, **82** (2017), 445–470.

36. M. Li and Z. Lin, The spreading fronts in a mutualistic model with advection, *Discrete Contin. Dyn. Syst. Ser. B.*, **20** (2015), 2089–2105.

37. C. Tian and S. Ruan, A free boundary problem for Aedes aegypti mosquito invasion, *Appl. Math. Model.*, **46** (2017), 203–217.

38. M. Zhang, J. Ge and Z. Lin. The invasive dynamics of Aedes aegypti mosquito in a heterogenous environment (in Chinese), *Sci. Sin. Math.*, **48** (2018), 999–1018.

39. L. Zhou, S. Zhang and Z. Liu, A free boundary problem of a predator-prey model with advection in heterogeneous environment, *Appl. Math. Comput.*, **289** (2016), 22–36.

40. M. Zhu, X. Guo and Z. Lin, The risk index for an SIR epidemic model and spatial spreading of the infectious disease, *Math. Biosci. Eng.*, **14** (2017), 1565–1583.

41. G. Bunting, Y. Du and K. Krakowski, Spreading speed revisited: analysis of a free boundary model, *Netw. Heterog. Media*, **7** (2012), 583–603.

42. Z. Lin, A free boundary problem for a predator-prey model, *Nonlinearity*, **20** (2007), 1883–1892.

43. M. Wang, H. Huang and S. Liu, A logistic SI epidemic model with degenerate diffusion and free boundary, preprint, (2019).

44. J. F. Cao, Y. Du, F. Li, et al., The dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries, *J. Funct. Anal.*, (2019), https://doi.org/10.1016/j.jfa.2019.02.013.
45. M. Wang, Existence and uniqueness of solutions of free boundary problems in heterogeneous environments, *Discrete Contin. Dyn. Syst. Ser. B*, 24 (2019), 415–421.

46. M. Wang, A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment, *J. Funct. Anal.*, 270 (2016), 483–508.

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