A priori estimate for axially symmetric solutions to the
Navier-Stokes equations near the axis of symmetry

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Abstract. We consider the axially symmetric solutions with large swirl to
the Navier-Stokes equations. Let \( v_r, v_\varphi, v_z \) be the cylindrical coordinates
of velocity and \( \chi = v_{r,z} - v_{z,r} \) be a component of vorticity. Let \( V_2^k(\Omega^T) \)
be a space with the finite norm

\[
\|w\|_{V_2^k(\Omega^T)} = \|w\|_{L_\infty(0,T;H^k(\Omega))} + \|\nabla w\|_{L_2(0,T;H^k(\Omega))},
\]

\( k = 0, 1 \). We proved the a priori estimate for \( \|\chi\|_{V_2^0(\Omega_\zeta^T)} \), where \( \Omega_\zeta = \{x \in \Omega : r \leq 2r_0\} \) and \( r_0 \) is so small that the swirl \( u = rv_\varphi \) satisfies

\[
\|u\|_{C(0,T;C^{1/2}(\Omega_\zeta))} \leq \sqrt{\frac{5}{4}} \nu,
\]

where \( \nu \) is the viscosity and \( u \) vanishes on the axis of symmetry. (1) is
proved under the a priori estimate that \( v \in W^{2,1}_2(\Omega^T) \) (see [Z5]).
Next, having that \( \chi \in V_2^0(\Omega^T) \) (proof of the estimate for \( \|\chi\|_{V_2^0((\Omega\setminus\Omega_\zeta)^T)} \) is
not difficult) the elliptic problem \( v_{r,z} - v_{z,r} = \chi, v_{r,r} + v_{z,z} + \frac{\varphi}{r} = 0 \) implies
that \( v' = (v_r, v_z) \in V_2^1(\Omega^T) \), so \( v' \in L_{10}(\Omega^T) \). The above construction
implies the following step by step procedure. Let $v(0) \in H^1(\Omega)$ then $v \in W^{2,1}_2(\Omega^T)$. Next $\chi \in V^0_2(\Omega^T)$ implies that $v(T) \in H^1(\Omega)$, where $u_\varphi(T) \in H^1(\Omega)$ is shown separately. This will be a topic of the next paper (see [Z4]).

1. Introduction

The uniqueness and regularity of weak solutions to the Navier-Stokes equations are important open problems in the incompressible hydrodynamics. Since the problems are very difficult in a general case there are examined special solutions to the Navier-Stokes equations such as: two-dimensional, axially symmetric, helicoidal. The problems were solved for two-dimensional solutions in [L3] and in the case of helicoidal solutions in [MTL]. Moreover, the existence of global regular solutions which are close to the two-dimensional solutions was proved in [RZ, NZ, Z6, Z7]. So far the existence of regular axially symmetric solutions (see Definition 1.1) is still an open problem. However, the existence of regular axially symmetric solutions with vanishing swirl was proved long time ago in [L2, UY] the case with nonvanishing swirl is still open. Moreover, the existence of solutions which remain close to solutions from [L2, UY] was shown in [Z1, Z2, Z3, Z8]. In this paper and in [Z4] the existence of global regular axially symmetric solutions with nonvanishing swirl is proved in a periodic cylinder and with the slip boundary conditions on the lateral part of its boundary. The aim of this paper is to establish an a priori estimate for regular solutions in a neighbourhood of the axis of symmetry. In [Z4] a similar estimate is proved in a neighbourhood located at a positive distance from the axis of symmetry. Then in [Z4] by an appropriate partition of unity the estimate is derived in a whole cylinder. Finally in [Z4] by the Leray-Schauder fixed point theorem the existence of regular global axially symmetric solutions is established in a whole cylinder.

In this paper we consider the axially symmetric solutions (see Definition 1.1) to the following problem

$$
\begin{align*}
v_t + v \cdot \nabla v - \nu \Delta v + \nabla p &= 0 &\text{in } \Omega^T = \Omega \times (0, T), \\
\text{div } v &= 0 &\text{in } \Omega^T, \\
v \cdot \bar{n} &= 0 &\text{on } S^T, \\
\bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha &= 0, &\alpha = 1, 2, \text{ on } S^T_1 = S_1 \times (0, T), \\
\text{periodic boundary conditions} &\text{ on } S^T_2 = S_2 \times (0, T), \\
v|_{t=0} &= v_0 &\text{in } \Omega,
\end{align*}
$$

(1.1)
where $\Omega$ is an axially symmetric cylinder with boundary $S = S_1 \cup S_2$, $x = (x_1, x_2, x_3)$ is the Cartesian system of coordinates in $\mathbb{R}^3$ such that $x_3$-axis is the axis of the cylinder $\Omega$. By $S_1$ we denote the part of the boundary of the cylinder parallel to the $x_3$-axis and $S_2$ is perpendicular to it. Next, $v = v(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$ is the velocity of the considered fluid, $p = p(x, t) \in \mathbb{R}$ the pressure, $\nu > 0$ is the viscosity coefficient, $D(v) = \nabla v + \nabla v^T$ is the double symmetric part of $\nabla v$, $\bar{n}$ is the unit outward normal vector to $S_1$ and $\bar{\tau}_\alpha$, $\alpha = 1, 2$, is a tangent one.

To examine axially symmetric solutions to (1.1) we introduce the cylindrical coordinates $r, \varphi, z$ by the relations

$$x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = z.$$ 

Moreover, we introduce the vectors

$$\bar{e}_r = (\cos \varphi, \sin \varphi, 0), \quad \bar{e}_\varphi = (-\sin \varphi, \cos \varphi, 0), \quad \bar{e}_z = (0, 0, 1)$$

connected with the cylindrical coordinates.

Then, the cylindrical coordinates of $v$ are defined by the relations

$$v_r = v \cdot \bar{e}_r, \quad v_\varphi = v \cdot \bar{e}_\varphi, \quad v_z = v \cdot \bar{e}_z,$$

where the dot denotes the scalar product in $\mathbb{R}^3$. Finally, by $u = rv_\varphi$ we denote a swirl.

To describe the domain $\Omega$ in greater details we introduce the notation

$$\Omega = \{x \in \mathbb{R}^3 : r < R, \ |z| < a\},$$

$$S_1 = \{x \in \mathbb{R}^3 : r = R, \ |z| < a\},$$

$$S_2 = \{x \in \mathbb{R}^3, \ r < R, \ z \in \{-a, a\}\},$$

where $R$ and $a$ are given positive numbers.

**Definition 1.1.** By the axially symmetric solution we mean such solutions to problem (1.1) that

$$(1.2) \quad v_{r,\varphi} = v_{\varphi,\varphi} = v_{z,\varphi} = p_{,\varphi} = 0.$$ 

In the cylindrical coordinates equations (1.1) for the axially symmetric solutions can be expressed in the form (see [LL, Ko])

$$(1.3) \quad v_{r,t} + v \cdot \nabla v_r - \frac{v_r^2}{r} - \nu \Delta v_r + \nu \frac{v_r}{r^2} = -p_{,r},$$

$$(1.4) \quad v_{\varphi,t} + v \cdot \nabla v_\varphi + \frac{v_r}{r}v_\varphi - \nu \Delta v_\varphi + \nu \frac{v_\varphi}{r^2} = 0,$$

$$(1.5) \quad v_{z,t} + v \cdot \nabla v_z - \nu \Delta v_z = -p_{,z},$$

$$(1.6) \quad v_{r,r} + v_{z,z} = \frac{-v_r}{r},$$

where $v \cdot \nabla = v_r \partial_r + v_z \partial_z$, $\Delta u = \frac{1}{r}(ru_{,r}),_r + u_{,zz}$. 

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Let us introduce the $\varphi$-component of vorticity by

\[(1.7) \quad \chi = v_{r,z} - v_{z,r}.\]

Then $\chi$ is a solution to the problem (see [Z1])

\[(1.8) \quad \chi_t + v \cdot \nabla \chi - \frac{v_r}{r} \chi - \nu \left[ r \left( \frac{\chi}{r} \right)_{,r} + \chi_{,zz} + 2 \left( \frac{\chi}{r} \right)_{,r} \right] = \frac{2v_{\varphi}v_{\varphi,z}}{r}
\]

$\chi|_{t=0} = \chi_0$  
$\chi|_{S_1} = 0$,  $\chi|_{S_2}$ — periodic boundary condition.

The existence of global regular axially symmetric solutions with vanishing swirl was proved by Ladyzhenskaya and Ukhovski-Yudovich, independently, long time ago (see [L2, UY]). Special cases as axially symmetric domains without the axis of symmetry or the case with sufficiently small swirl near the axis of symmetry at time $t = 0$ were treated in [Z2, Z3], respectively. Moreover, global existence of large regular solutions with sufficiently small $v_\varphi$, $v_{r,\varphi}$, $v_{\varphi,\varphi}$, $v_{z,\varphi}$ at time $t = 0$ were proved in [Z1, Z8] in cylindrical and axially symmetric domains, respectively. We have to add that the slip boundary conditions were necessary.

Although a global $L_\infty$-estimate for swirl in the case of axially symmetric solutions was proved in [CL] (but under the technical assumption that $u$ vanishes at infinity and on the axis of symmetry) it was still not clear how to increase regularity of such weak solutions. Having the existence of weak solutions to problem (1.1) such that $v \in V^0_2(\Omega^T)$ (see the definition of this space in Section 2) we obtain that $v \cdot \nabla v \in L_{5/4}(\Omega^T)$. Then by [S, ZZ] we obtain that solutions to (1.1) are such that $v \in W^{2,1}_{5/4}(\Omega^T)$, $\nabla p \in L_{5/4}(\Omega^T)$. Then $v \in L_{5/2}(\Omega^T)$, $\nabla p \in L_{5/3}(\Omega^T)$, so $v \cdot \nabla v \in L_1(\Omega^T)$ and the technique from [S, ZZ] can not be again applied.

Passing to the cylindrical coordinates in estimates for the axially symmetric solution we obtain norms of weighted spaces (jacobian of transformation is $r$) so all imbeddings and interpolations are the same as in 3d case (see [Z3]).

Examining regularity of the axially symmetric solutions in a neighborhood $\Omega_*$ located at a positive distance from the axis of symmetry we showed in [Z2] that $v' = (v_r, v_z) \in V^1_2(\Omega^T_*)$ (see Definition 2.10 in Section 2). Then by imbedding we have

\[(1.9) \quad \|v'\|_{L_{10}(\Omega^T_*)} + \|\nabla v'\|_{L_{10/3}(\Omega^T_*)} \leq c\|v'\|_{V^1_2(\Omega^T_*)}
\]

(see Lemma 2.11 in Section 2).
Then from (1.9) we have that $v' \cdot \nabla v \in L^2(\Omega^T_*)$, $\nabla v \in L^2(\Omega^T_*)$, so by [S, ZZ] we obtain the existence of regular solutions such that $v \in W^{2,1}_{5/3}(\Omega^T_*)$, $\nabla p \in L^{5/2}(\Omega^T_*)$. Hence $\nabla v \in L^{5/2}(\Omega^T_*)$. Then $v' \cdot \nabla v \in L^2(\Omega^T_*)$, so [S, ZZ] implies that $v \in W^{2,1}_2(\Omega^T_*)$, $\nabla p \in L^2_2(\Omega^T_*)$.

In this paper we prove (1.9) in a neighborhood of the axis of symmetry (see Lemma 6.4). Combining the estimates near the axis and far from the axis we obtain (1.9) in the whole domain $\Omega$.

Then in [Z4] we prove the existence of regular solutions to problem (1.1) such that $v \in W^{2,1}_{5/3}(\Omega^T_*)$, $\nabla p \in L^{5/3}_2(\Omega^T_*)$.

To formulate the main results of this paper we need the following assumptions

**Assumptions**

1. Assume that there exist positive finite constants $d_1$ and $d_2$ such that
   \[ \|v_0\|_{L^2(\Omega)} \leq d_1, \]
   and
   \[ \|u_0\|_{L^\infty(\Omega)} \equiv d_2, \]
   where $u = rv_\varphi$, $u_0 = u|_{t=0}$.

2. Assume that $u_0 \in C^\alpha(\Omega)$, $\alpha \in (0, 1)$.

3. Assume that $\zeta = \zeta_1(r)$ is a cutoff smooth function such that $\zeta_1(r) = 1$ for $r \leq r_0$ and $\zeta_1(r) = 0$ for $2r_0 \leq r < R$.
   Assume that $\tilde{v}_\varphi = v_\varphi \zeta_1$, $\tilde{\chi} = \chi \zeta_1^2$ and
   \[ \left\| \frac{\tilde{v}_\varphi^2(0)}{r} \right\|_{L^2(\Omega)} < \infty, \quad \left\| \frac{\tilde{\chi}(0)}{r} \right\|_{L^2(\Omega)} < \infty. \]

4. Assume that $r_0$ is so small that
   \[ (1.10) \quad \|u\|_{L^\infty(\Omega^T_*)} \leq \frac{\sqrt{5}}{4} \nu, \]
   where $\Omega_{\zeta_1} = \Omega \cap \text{supp} \zeta_1$ (see assumptions of Lemma 5.3).

Now we describe restriction (1.10). We explain that it is not a smallness condition but restriction on $r_0$ only. In view of Assumption 2 and Lemma 2.7 we have that $u \in C^{\alpha,\alpha/2}(\Omega^T_*)$, where $\alpha$ is not larger that $1/2$.

But Assumption 1 and Lemma 2.3 imply
\[ \left\| \frac{u}{r^2} \right\|_{L^2(\Omega^T_*)} \leq \alpha d_1. \]

Hence $u$ vanishes on the axis of symmetry.

Therefore, since $u \in C(0, T; C^\alpha(\Omega))$ it follows that there exists $r_0$ so small that (1.10) can be satisfied.
**Theorem A.** (see Lemmas 5.3, 5.5) Let the Assumptions 1–4 hold. Then for solutions to problem (1.1) the following a priori estimate is satisfied

\[
\begin{align*}
&\left\| \frac{\tilde{v}_r}{r} \right\|_{L_4(\Omega^t)}^4 + \left\| \frac{\tilde{\chi}}{r} \right\|_{V_0^2(\Omega^t)}^2 + \left\| \nabla \left( \frac{\tilde{v}_r}{r} \right) \right\|_{L_2(\Omega^t)}^2 \\
&+ \left\| \frac{1}{r} \left( \frac{\tilde{v}_r}{r} \right) \right\|_{L_2(\Omega^t)}^2 \leq \varphi \left( \frac{1}{r_0}, d_1, d_2, \nu \right) [1 \\
&+ \left\| \tilde{v}_r^2(0) \right\|_{L_2(\Omega)} + \left\| \tilde{\chi}(0) \right\|_{L_2(\Omega)}^2 ] \equiv A_0^2, \quad t \leq T,
\end{align*}
\]

(1.11)

where \( \varphi \) is an increasing positive function and \( \tilde{v}_r = v_r \zeta \) and \( \zeta \) is introduced in Assumption 3.

**Theorem B.** (see Lemma 6.4) Let the assumptions of Theorem A be satisfied. Then (see Definition 2.9)

\[
\left\| \tilde{v}' \right\|_{V_1^2(\Omega^t)} \leq \varphi(A_0), \quad t \leq T,
\]

(1.12)

where \( v' = (v_r, v_z) \) and \( \varphi \) is also an increasing positive function.

To prove Theorem B we need Assumption 4. The assumption holds in view of Lemma 3.3. However, Lemma 3.3 is proved under the existence of local solution to (1.1) such that \( v \in W^{2,1}_r(\Omega^{T_*}) \), \( \nabla p \in L_2(\Omega^{T_*}) \) with \( T_* \) sufficiently small, the norm \( \| u \|_{C^{\alpha, \alpha/2}} \) does not depend on the local solution. This is an important factor letting (1.12) for any \( T \). In [Z4] there is shown how the local solution can be extended step by step without its norm increasing. This way guarantees the existence of global regular solutions.

There are many results concerning the sufficient conditions of regularity of axially symmetric solutions. In [CL] such condition is

\[
\int_0^T dt \left( \int_{\mathbb{R}^3} |v|^{\gamma} drd\varphi dz \right)^{\alpha/\gamma} < \infty, \quad \text{where } 1/\alpha + 1/\gamma \leq 1/2, \ 2 < \gamma < +\infty, \ 2 < \alpha \leq +\infty.
\]

In [SZ] the following condition is assumed

\[
\text{esssup}_{-1 \leq t \leq 0} \int_{\Omega} \frac{1}{r} |v(x,t)|^2 dx < \infty.
\]

Finally in [SS] there are considered the following conditions

\[
\text{esssup}_{Q(z_0,R)} r |\tilde{v}(x,t)| < \infty
\]

and

\[
\text{esssup}_{Q(z_0,R)} \sqrt{t_0 - t} |\tilde{v}(x,t)| < \infty
\]
where \( z_0 = (x_0, t_0), Q(z_0, T) = B(x_0, R) \times (t_0 - R^2, t_0), \) \( B(x_0, R) = \{ x \in \mathbb{R}^3 : |x - x_0| < R \}, \) \( \bar{v}(x, t) = v_r \bar{e}_r + v_z \bar{e}_z. \)

Now we outline the proofs of Theorems A and B. We have to emphasize that estimates (1.11) and (1.12) are a priori type estimates.

First we describe a proof of estimate (1.12). Lemma 2.1 yields the energy type estimate for weak solutions to (1.1). Next Lemma 2.2 yields the existence of local solution to problem (1.1) such that \( v \in W^{2,1}_2(\Omega^{T_*}), \) \( \nabla p \in L_2(\Omega^{T_*}), \) where \( T_* \) is described by the assumptions of the lemma. Since we examine problem (1.1) in a bounded cylinder the \( L_\infty \) bound for \( u \) formulated in Lemma 2.5 is proved in Lemma 2.1 in [Z5]. Applying the DeGiorgi method developed by Ladyzhenskaya, Solonnikov, Uratseva (see [LSU, Ch. 2]) and assuming that \( v \in W^{2,1}_2(\Omega^T) \) we show that \( u \in C^{\alpha,\alpha/2}(\bar{\Omega}^{T_*}) \), where \( \alpha = \frac{1}{2} \) (see Lemmas 2.7, 3.1 and [Z5]).

Having the Hölder continuity of \( u \) Lemma 2.3 yields that \( u \) vanishes on the axis of symmetry. The two properties are crucial to satisfy (1.10) for sufficiently small \( r_0 \). Then in a series of lemmas (see below) estimate (1.11) is proved for \( t \leq T_* \). Then elliptic problem (4.1) implies (1.12). The estimate gives regularity near the axis of symmetry because of properties of the cut-off function \( \zeta = \zeta_1(r) \). From the proof of Lemma 5.3 in [Z4] we have that

\[
(1.13) \quad \tilde{v}' \in V^1_2(\Omega^{T_*}),
\]

where \( \tilde{v}' = v' \zeta_2(r) \) and \( \zeta_2(r) = 0 \) for \( r \leq \frac{r_0}{2} \) and \( \zeta_2(r) = 1 \) for \( r \geq \frac{3}{2} r_0 \).

Assuming that \( \{ \zeta_1(r), \zeta_2(r) \} \) compose a partition of unity in \( \Omega \) imbeddings (1.12) and (1.13) imply that \( v' \in V^1_2(\Omega^{T_*}) \). This property yields that \( v'(T_*) \in H^1(\Omega) \). To apply Lemma 2.2 for interval \([T_*, 2T_*]\) we need also that \( v_\varphi(T_*) \in H^1(\Omega) \), which is proved in Lemma 4.1 in [Z4].

Moreover, we show in [Z4] that there exists a constant \( \alpha_0 \) and a corresponding to it \( T_* > 0 \) such that if \( \| v(0) \|_{H^1(\Omega)} \leq \alpha_0 \) then \( \| v(T_*) \|_{H^1(\Omega)} \leq \alpha_0 \). This property implies that the local solution can be extended step by step.

Now we present shortly the steps of the proof of estimate (1.11). First we obtain (see Lemma 4.2)

\[
(1.14) \quad \left\| \nabla \left( \frac{\tilde{v}_r}{r} \right),_r \right\|^2_{L_2(\Omega)} + 6 \left\| \frac{1}{r} \left( \frac{\tilde{v}_r}{r} \right),_r \right\|^2_{L_2(\Omega)} \leq \left\| \left( \frac{\tilde{v}}{r} \right),_r \right\|^2_{L_2(\Omega)} + \varphi \) \text{ (norms of data),}
\]

where \( \varphi \) is an increasing positive function.
Next Lemma 5.1 yields

\[ \left\| \frac{\chi}{r} \right\|_{V^0_2(\Omega^t)}^2 \leq \frac{1}{\nu} \int_{\Omega^t} \frac{\tilde{v}_r^4}{r^4} \, dx \, dt + \varphi \text{ (norms of data)}. \] (1.15)

Finally, Lemma 5.2 implies the inequality

\[ \frac{1}{4} \int_{\Omega} \frac{\tilde{v}_r^4}{r^2} \, dx + \frac{3}{4} \nu \int_{\Omega^t} \left| \nabla \frac{\tilde{v}_r^2}{r} \right|^2 \, dx \, dt + \frac{1}{4} \nu \int_{\Omega^t} \frac{\tilde{v}_r^4}{r^4} \, dx \, dt \]

\[ \leq \int_{\Omega^t} \left| \frac{v_r}{r} \right| \frac{\tilde{v}_r^4}{r^4} \, dx \, dt + \varphi \text{ (norms of data)}. \] (1.16)

From (1.14)–(1.16) and under the assumption (see (5.10))

\[ \|u\|_{L^\infty(\Omega^t)} \leq \sqrt{\frac{5}{4}} \nu \] (1.17)

we obtain by Lemmas 5.3, 5.5 and Conclusion 5.6 the estimate (1.11).

The restriction (1.17) is satisfied in view of the Hölder continuity of \(u\) and the property that \(u\) vanishes on the axis of symmetry. Since \(u \in C^{1/2,1/4}\) we can calculate precisely the number \(r_0\) appearing in the definition of function \(\zeta_1(r)\). This implies that the r.h.s. of (1.11) depends on \(r_0\).

2. Auxiliary results

By \(c\) we denote a generic constant which changes its value from line to line. A constant \(c_k\) with index \(k\) is defined by the first formula, where it appears. By \(\varphi\) we denote the generic functions which changes its form from formula to formula and is always a positive and an increasing function. By \(c(\sigma)\) we denote a generic constant which increases with \(\sigma\).

Let us introduce the space

\[ V^0_2(\Omega^T) = \{ u : \|u\|_{V^0_2(\Omega^T)}^2 = \|u\|_{L^\infty(0,T;L^2(\Omega))} + \nu \|\nabla u\|_{L^2(\Omega^T)}^2 < \infty \} . \]

\textbf{Lemma 2.1.} Assume that \(v_0 \in L^2(\Omega)\). Then there exists a weak solution to problem (1.1) such that \(v \in V^0_2(\Omega^T)\) and

\[ \|v\|_{V^0_2(\Omega^T)} \leq c_0 \|v_0\|_{L^2(\Omega)} \equiv d_1 . \] (2.1)
From properties of $V_2^0(\Omega^T)$ we have (see [LSU, Ch. 2, Sect. 3])

\begin{equation}
\|v\|_{L^q(0,T;L^p(\Omega))} \leq c_2 \|v\|_{V_2^0(\Omega^T)},
\end{equation}

where

\[ \frac{3}{p} + \frac{2}{q} \geq \frac{3}{2}. \]

**Lemma 2.2.** (see Theorem 3.1 in [Z4]). Assume that $v_0 \in H^1(\Omega)$. Then for $T$ so small that

\[ c_\ast T^{1/2} \|v(0)\|_{H^1(\Omega)} \leq 1 \]

there exists a local solution to problem (1.1) such that $v \in W_2^{2,1}(\Omega^T)$, $\nabla p \in L_2(\Omega^T)$ and there exists a constant $c_0$ independent of $v$ and $p$ such that

\begin{equation}
\|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} \leq c_0 \|v(0)\|_{H^1(\Omega)}.
\end{equation}

**Lemma 2.3.** Let $v_0 \in L_2(\Omega)$. Then the weak axially symmetric solutions to problem (1.1) satisfy the estimate

\begin{equation}
\|v\|_{V_2^0(\Omega^T)} + \frac{\|v_r\|_{L^2(\Omega^T)}}{r} + \frac{\|v_\varphi\|_{L^2(\Omega^T)}}{r} \leq c_0^2 \|v_0\|_{L_2(\Omega)} \equiv d_1^2.
\end{equation}

The result is proved in Lemma 2.4 in [Z4].

Let us introduce the quantity (swirl)

\begin{equation}
\nabla \times v = r v_\varphi.
\end{equation}

We see that $u$ is a solution to the problem

\begin{align*}
u_t + v \cdot \nabla u - \nu \Delta u + 2\nu \frac{u_r}{r} = 0 & \quad \text{in } \Omega^T, \\
u|_{t=0} = u_0, & \quad \text{in } \Omega, \\
u,_{r} = \frac{2}{r}u & \quad \text{on } S_1^T, \\
\text{periodic boundary conditions} & \quad \text{on } S_2^T, \\
\text{div } v = 0 & \quad \text{in } \Omega,
\end{align*}

where the boundary condition (2.6)3 is derived in [Z1, Ch. 4, Lemma 2.1].
Lemma 2.4. (see Lemma 2.1 in [Z5]) Let $u$ be a solution to (2.6). Let $u_0 \in L_\infty(\Omega)$. Then there exists $c$ independent of $u$ such that

$$\|u\|_{L_\infty(\Omega^T)} \leq c\|u_0\|_{L_\infty(\Omega)} \equiv d_2. \tag{2.7}$$

From (2.4) and (2.5) we derive the estimate

$$\left\| \frac{u}{r^2} \right\|_{L_2(\Omega^T)} \leq d_1. \tag{2.8}$$

Lemma 2.5. Let the assumptions of Lemmas 2.3 and 2.4 hold. Then

$$\|v_\varphi\|_{L_4(\Omega^T)}^4 \leq \|rv_\varphi\|_{L_\infty(\Omega^T)}^2 \left\| \frac{v_\varphi}{r} \right\|_{L_2(\Omega^T)}^2 \leq d_1^2 d_2^2. \tag{2.9}$$

In view of (2.7) we can prove

Lemma 2.6. Assume that $u_0 \in C^\alpha(\Omega_\zeta)$, where $\Omega_\zeta = \Omega \cap \text{supp} \zeta$, $\zeta = \zeta(r)$ is smooth function such that $\zeta(r) = 1$ for $r \leq r_0$ and $\zeta(r) = 0$ for $r \geq 2r_0$, $2r_0 < R$. Then any solution to (2.6) satisfies

$$u \in C^{\alpha,\alpha/2}(\Omega_\zeta^T) \tag{2.10}$$

where $\alpha$ equals $1/2$ (see [Z5]).

We recall from [Z1, Ch. 2] the Hardy inequality

Lemma 2.7. Assume that $f \in L_{p,-\mu}(\Omega)$ and $f_r \in L_{p,1-\mu}(\Omega)$, where

$$L_{p,\nu}(\Omega) = \{u : \int_{\Omega} |u|^p r^{p\nu} \, dx < \infty\}, \quad p \in (1, \infty), \quad \nu \in \mathbb{R}. \tag{2.11}$$

Then

$$\|f\|_{L_{p,-\mu}(\Omega)} \leq \frac{1}{|\mu - \frac{2}{p}|} \|f_r\|_{L_{p,1-\mu}(\Omega)},$$

where $\mu \neq \frac{2}{p}$.

Remark 2.8. (see [Z10]) If $\mu > \frac{2}{p}$ then $f \notin L_{p,-\mu}(\Omega)$ because $f(0) = f|_{r=0} \neq 0$. In this case inequality (2.11) must be replaced by

$$\|f - f(0)\|_{L_{p,-\mu}(\Omega)} \leq \frac{1}{|\mu - \frac{2}{p}|} \|f_r\|_{L_{p,1-\mu}(\Omega)}. \tag{2.12}$$

Definition 2.9. We need also the space

$$V_k^0(\Omega^T) = \{u : \|u\|_{V_k^0(\Omega^T)} = \|u\|_{L_\infty(0,T;H^k(\Omega))} \leq \nu \|\nabla u\|_{L_2(0,T;H^k(\Omega))} < \infty\}, \quad k \in \mathbb{N} \cup \{0\}. \tag{2.13}$$

For $k = 0$ we have the space $V_2^0(\Omega^T)$.

Moreover

$$H^k(\Omega) = \left\{u : \|u\|_{H^k(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D_\alpha^0 u|^2 \, dx \right)^{1/2} < \infty \right\}, \quad k \in \mathbb{N} \cup \{0\}. \tag{2.14}$$

In [Z4] we prove
Lemma 2.10. Let $\zeta' = \zeta_2(r)$ be a smooth positive function such that $\zeta'(r) = 0$ for $r < r_0/2$ and $\zeta'(r) = 1$ for $r \geq r_0$. Let $\Omega' = \Omega \cap \text{supp } \zeta'$ and $\Omega_\zeta = \Omega \cap \{x \in \text{supp } \zeta' : \zeta'(r) = 1\}$. Assume that $v_0 \in L_2(\Omega)$ with $c_0\|v_0\|_{L_2(\Omega)} = d_1$. Assume also that $v_\varphi(0) \in L_{49/18}(\Omega', \chi(0) \in L_2(\Omega')$. Then

$$\|v'\|_{L_2(\Omega') \cap \{x \in \text{supp } \zeta' : \zeta'(r) = 1\}} \leq c(1/r_0, d_1)[d_1 + \|v_\varphi(0)\|_{L_{49/18}(\Omega')}] + \|\chi(0)\|_{L_2(\Omega')}$$

where $v' = (v_r, v_z)$.

Repeating the considerations from [BIN, Ch. 2, Sect. 2.16] we have

Lemma 2.11. Let $F(x) = \int_0^x f(y)dy$, $\beta > \frac{1}{p}$, $\|x^{-\beta}f(x)\|_{L_p(\mathbb{R}_\varepsilon)} < \infty$, $p \in (1, \infty)$, $\mathbb{R}_\varepsilon = \{x \in \mathbb{R} : 0 < \varepsilon < x\}$. Let $f(x) = 0$ for $x \in (0, \varepsilon]$. Then the following inequality holds

$$\left(\int_{\mathbb{R}_\varepsilon} |x^{-\beta} \int_0^x f(y)dy|^p dx \right)^{1/p} \leq \frac{1}{\beta - 1/p} \left(\int_{\mathbb{R}_\varepsilon} |x^{-\beta+1}f(x)|^p dx \right)^{1/p}. \tag{2.16}$$

Proof. Making the change of variable and applying the generalized Minkowski inequality (see [BIN, Ch. 2, Sect. 2.10]) we obtain

$$\left(\int_{\mathbb{R}_\varepsilon} |x^{-\beta} \int_0^x f(y)dy|^p dx \right)^{1/p} = \left(\int_{\mathbb{R}_\varepsilon} |x^{-\beta+1} \int_0^1 f(xt)dt|^p dx \right)^{1/p}$$

$$\leq \int_0^1 \left(\int_{\mathbb{R}_\varepsilon} |x^{-\beta+1}f(xt)|^p dx \right)^{1/p} dt$$

$$= \int_0^1 \left(\int_{\mathbb{R}_\varepsilon} |y^{-\beta+1}f(y)|^p dy \right)^{1/p} t^{\beta-1-1/p} dt$$

$$= \int_0^1 t^{\beta-1-1/p} \left(\int_{\mathbb{R}_\varepsilon} |y^{-\beta+1}f(y)|^p dy \right)^{1/p} dt \equiv I,$$

where we used that $f(y) = 0$ for $y \leq \varepsilon$. Integrating with respect to $t$ yields (2.16). This concludes the proof.

Taking $u(x) = \int_0^x f(y)dy$ and raising to the power $p$ the both sides of (2.16) we obtain

$$\int_{\mathbb{R}_\varepsilon} |r^{-\beta}u(r)|^p dr \leq \frac{1}{(\beta - 1/p)^p} \int_{\mathbb{R}_\varepsilon} |r^{-\beta+1}u, r|^p dr,$$

where $u, r = 0$ for $r \leq \varepsilon$. 

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Continuing,
\[
(2.17) \quad \int_{\mathbb{R}_\varepsilon} |r^{-\beta-1/p}u(r)|^p r dr \leq \frac{1}{(\beta - 1/p)^p} \int_{\mathbb{R}_\varepsilon} |r^{-\beta-1/p+1}u_r|^p r dr.
\]
Assuming that \( u = u(r, z) \) and integrating (2.17) with respect to \( z \) yields
\[
\int_{\mathbb{R} \times \mathbb{R}_\varepsilon} |r^{-\beta-1/p}u(r, z)|^p dx \leq \frac{1}{(\beta - 1/p)^p} \int_{\mathbb{R} \times \mathbb{R}_\varepsilon} |r^{-\beta-1/p+1}u_r(r, z)|^p dx,
\]
where \( dx = r dr dz \). Introducing \( \alpha = \beta + 1/p \) we obtain the following Hardy inequality
\[
(2.18) \quad \left( \int_{\mathbb{R} \times \mathbb{R}_\varepsilon} |r^{-\alpha}u_r(r, z)|^p dx \right)^{1/p} \leq \frac{1}{\alpha - 2/p} \left( \int_{\mathbb{R} \times \mathbb{R}_\varepsilon} |r^{-\alpha+1}u_{rr}(r, z)|^p dx \right)^{1/p},
\]
where \( \alpha > \frac{2}{p} \). \( u_r(r, z) = 0 \) for \( r \leq \varepsilon \).

Using the cylindrical coordinates to examine problem (1.1) we derive equations with coefficients which are singular on the axis of symmetry (see (3.5), (3.6), (4.1) , (4.11), (5.2)). To cancel the singularities we consider problem (1.1) in \( \Omega_\varepsilon = \{ x \in \Omega: 0 < \varepsilon < r < R \} \). Therefore, we have to add some boundary conditions on the surface \( r = \varepsilon \). We assume
\[
(2.19) \quad v \cdot \bar{n}|_{r=\varepsilon} = 0, \quad \bar{n} \cdot \mathcal{T}(v, p) \cdot \bar{n}_{\alpha}|_{r=\varepsilon} = 0, \quad \alpha = 1, 2,
\]
where \( \bar{n} \) is the unit outward normal vector to the surface \( r = \varepsilon \) so it is directed to the axis of symmetry.
In view of [Z1, Ch. 4, Lemma 2.4] conditions (2.19) imply
\[
(2.20) \quad v_r|_{r=\varepsilon} = 0, \quad v_{z,r}|_{r=\varepsilon} = 0, \quad \left( v_{\varphi,r} + \frac{1}{r} v_{\varphi} \right)|_{r=\varepsilon} = 0,
\]
where the last equation assumes the form
\[
(2.21) \quad u_{rr}|_{r=\varepsilon} = 0.
\]
Moreover, (2.20) implies
\[
(2.22) \quad \chi|_{r=\varepsilon} = 0.
\]
Considering problem (1.1) in \( \Omega_\varepsilon \) we should work with approximate functions \( v_\varepsilon, u_\varepsilon, \chi_\varepsilon \) but we omit the index \( \varepsilon \) for simplicity.
Repeating the proof of Lemma 2.3 in the case \( \Omega_\varepsilon \) we have

**Lemma 2.12.** Let \( v_0 \in L_2(\Omega) \). Then
\[
(2.23) \quad \|v_0\|_{V_0^2(\Omega_T^\varepsilon)} + \left\| \frac{v_r}{r} \right\|_{L_2(\Omega_T^\varepsilon)} + \left\| \frac{v_\varphi}{r} \right\|_{L_2(\Omega_T^\varepsilon)} \leq d_1.
\]
3. Existence in $B_2(Q_T, M, \gamma, r, \delta, \varkappa)$. Idea of the proof of Lemma 2.6

The Hölder continuity of $u$ in a neighborhood of the axis of symmetry is crucial in the proof of main estimate (5.28) (see also restriction (5.10) in Lemma 5.3).

In this Section we follow the ideas, definitions and results from [LSU, Ch. 2, Sects. 6, 7] which need some changes described in [Z5]. Since the result is very important we describe very roughly how the continuity follows. In this Section we base on the existence of weak solutions to problem (1.1).

**Definition 3.1.** We say that $u \in V_0^0(Q_T)$ such that $\sup_{Q_T}|u| \leq M$ belongs to $B_2(Q_T \cap \bar{Q}(\varrho, \tau), M, \gamma, r, \delta, \kappa)$ if the function $w(x, t) = \pm u(x, t)$ satisfies the inequalities

$$
\max_{t_0 \leq t \leq t_0 + \tau} \|w^{(k)}(x, t)\|_{L_2(B_{\varrho - \sigma_1 \varrho})}^2 \leq \|w^{(k)}(x, t_0)\|_{L_2(B_\varrho)}^2 + \gamma[(\sigma_1 \varrho)^{-2}\|w^{(k)}\|_{L_2(Q(\varrho, \tau))}^2 + \mu^\varrho(1+\varkappa)(k, \varrho, \tau)]
$$

(3.1)

and

$$
\|w^{(k)}\|_{V_0^0(Q(\varrho - \sigma_1 \varrho, \tau - \sigma_2 \tau))}^2 \leq \gamma\{[(\sigma_1 \varrho)^{-2} + (\sigma_2 \tau)^{-1}]\|w^{(k)}\|_{L_2(Q(\varrho, \tau))}^2 + \mu^\varrho(1+\varkappa)(k, \varrho, \tau)],
$$

(3.2)

where

$$
w^{(k)}(x, t) = \max\{w(x, t) - k; 0\},$$

$$Q(\varrho, \tau) = B_\varrho(x_0) \times (t_0, t_0 + \tau)$$

$$= \{(x, t) \in Q_T : |x - x_0| < \varrho, t_0 < t < t_0 + \tau\},$$

$Q_T = \Omega \times (0, T)$, $\varrho, \tau$ are arbitrary positive numbers, $\sigma_1, \sigma_2$ - arbitrary numbers from $(0, 1)$,

$$\mu(k, \varrho, \tau) = \int_{t_0}^{t_0 + \tau} \text{meas}^\varrho A_{k, \varrho}(t)dt,$$

$$A_{k, \varrho}(t) = \{x \in B_\varrho(x_0) : w(x, t) > k\}.$$  

The numbers $M, \gamma, \delta, \varkappa$ are arbitrary positive and numbers $r, q$ satisfy the relation

$$
\frac{2}{r} + \frac{3}{q} = \frac{3}{2}.
$$

(3.3)
Finally, $k$ is a positive number satisfying the condition

$$\text{esssup}_{Q(\varrho,\tau)} w(x,t) - k \leq \delta.$$  

**Lemma 3.2.** Assume that $u$ satisfies (2.6) and $|u| \leq M$. Assume that \( \text{div} v = 0, v \in L_{r'}(0,T;L_{q'}(\Omega)) \), \( \frac{3}{q'} + \frac{2}{r'} = 1 - \frac{3}{2}\kappa, \kappa > 0 \), $u$ is axially symmetric and periodic with respect to $x_3$.

Then $u^{(k)} \in B_2(Q_T \cap Q(\varrho,\tau), M, \gamma, r, \delta, \kappa)$, where $M, k, \delta$ satisfy (3.4) in the form

$$M - k \leq \delta \quad \text{and} \quad \kappa = \frac{1}{6}, \quad r = q = \frac{10}{3}.$$  

**Proof.** (see the proof of Lemma 3.2 in [Z5]).

**Lemma 3.3.** Let $u$ be a solution to (2.6).

Let \( \text{div} v' = 0, v' \in L_{r'}(0,T;L_{q'}(B_0)) \), \( \frac{3}{q'} + \frac{2}{r'} = 1 - \frac{3}{2}\kappa \). Then $u \in C^{\alpha,\alpha/2}(Q_T \cap Q(\varrho,\tau))$, where $\alpha = \frac{3\kappa}{2}$ and $\|u\|_{C^{\alpha,\alpha/2}} \leq c \max\{2M, 16\varrho_0^3\}$ with $c$ equal to some number.

For $v \in W^{2,1}_2(B_0^T)$ we have that $r' = q' = 10$ so $\kappa = \frac{1}{3}$ and $\alpha = \frac{1}{2}$.

The Lemma is proved in Theorem 5.4 in [Z5].

### 4. A priori estimates for $v_r$ and $v_z$ in a neighborhood of the $x_3$-axis

In this section we examine the elliptic problem for $v_r = v_r(r,z,t)$, $v_z = v_z(r,z,t)$,

$$v_{r,z} - v_{z,r} = \chi \quad \text{in} \ \Omega,$$

$$v_r + v_z + \frac{v_r}{r} = 0 \quad \text{in} \ \Omega,$$

$$v_r|_{S_1} = 0, \text{ periodicity with respect to } z.$$  

By localizing the problem to a neighborhood of the axis of symmetry we will be able to consider the result in $\mathbb{R}^3 \cap \{z : |z| < a\}$ with periodicity with respect to $z$.

Expressing (4.1) in the form

$$(rv_r)_r + (rv_z)_z = 0$$

we see that there exists a stream function $\psi$ such that

$$v_r = \frac{\psi_z}{r}, \quad v_z = -\frac{\psi_r}{r}.$$
Using (4.3) in (4.1) we obtain the equation

\[(4.4) \quad \left( \frac{\psi_z}{r} \right)_z + \left( \frac{\psi_r}{r} \right)_r = \chi. \]

To estimate solutions of (4.4) near the axis of symmetry we introduce a cut-off smooth function \( \zeta_1 = \zeta_1(r) \) such that \( \zeta_1(r) = 1 \) for \( r \leq r_0 \) and \( \zeta_1(r) = 0 \) for \( r \geq 2r_0 \), where \( 2r_0 < R \).

Therefore, we introduce the notation

\[(4.5) \quad \tilde{\psi} = \psi \zeta_1^2, \quad \tilde{\chi} = \chi \zeta_1^2. \]

In virtue of (4.5) we express (4.4) in the form

\[(4.6) \quad \left( \frac{\tilde{\psi}_z}{r} \right)_z + \left( \frac{\tilde{\psi}_r}{r} \right)_r = \tilde{\chi} + \frac{\psi_r}{r} \left( \zeta_1^2 \right)_r + \left( \psi \left( \zeta_1^2 \right)_r \right)_r \equiv \tilde{\chi}_*. \]

For (4.6) we have the following boundary conditions

\[(4.7) \quad \tilde{\psi} \text{ vanishes with all derivatives for } r \geq 2r_0,
\]

\[\tilde{\psi} \text{ is periodic with respect to } z \in [-a,a], \text{ so } \tilde{\psi}(r, (2k + 1)a, t) = \tilde{\psi}(r, (2k - 1)a, t), \quad k \in \mathbb{Z}.\]

From [Z1–Z3] we know that a natural energy type estimate for solutions to (1.8) is an estimate for \( \| \tilde{\psi} \|_{V_2^0(\Omega_T)} \), because in this case the nonlinearity of the second and the third terms on the l.h.s. of (1.8) is eliminated. Hence we will be looking for estimates for \( \tilde{\psi} \) in the case where the r.h.s. of (4.6) divided by \( r \) belongs to \( V_2^0(\Omega_T) \). This needs a very fast vanishing of \( \tilde{\psi} \) and \( \tilde{\chi}_* \) in a neighborhood of the axis of symmetry. To guarantee such vanishing we prove the existence of solutions to problem (4.6), (4.7) with such property. For this purpose we introduce a cylinder of radius \( r = \varepsilon < r_0 \). Then \( \tilde{\chi}_*|_{r \leq \varepsilon} = \chi \) and we assume that \( \chi|_{r=\varepsilon} = 0 \). We take the idea from [L2]. The assumption is motivated by estimations presented in Section 5, where we have to work with functions \( \chi, \frac{\psi}{r}, \frac{\psi}{r} \). Then considering the problems for \( \chi \) and \( v_\varphi \) we have to add additional artificial boundary conditions \( \chi|_{r=\varepsilon} = 0 \) and \( v_\varphi|_{r=\varepsilon} = 0 \) (see problems (5.2) and (5.7)). In this way we get approximate functions \( \chi_\varepsilon \) and \( v_\varphi \) but we shall omit the index \( \varepsilon \) for simplicity. Instead of the above approach we can treat the estimates in Section 4, 5, 6 as a priori estimates performed on smooth functions vanishing sufficiently fast near the axis of symmetry. However, we shall follow the first approach. To simplify problem (4.6), (4.7) we introduce the quantities \( \eta \) and \( \theta \) by the relations

\[(4.8) \quad \tilde{\psi} = \eta r^2, \quad \tilde{\chi}_* = \theta r, \]

\(15\)
where the restriction $\chi_{|r|\leq \varepsilon}$ implies that $\vartheta_{|r|\leq \varepsilon} = 0$. Using (4.8) in (4.6) yields

\begin{equation}
\eta_{zz} + \frac{3\eta_r}{r} = \vartheta
\end{equation}

By the definition of $\Delta$ in the cylindrical coordinates we obtain

\begin{equation}
\Delta \eta + \frac{2\eta_r}{r} = \vartheta.
\end{equation}

However, (4.10) contains a coefficient singular on the axis of symmetry we assume to consider (4.10) in domain $\Omega$, because the proof of Lemma 4.1 basis on weak formulation of (4.10) which does not contain any singular term on the axis of symmetry.

Then we examine the following boundary value problem

\begin{equation}
\Delta \eta + \frac{2\eta_r}{r} = \vartheta \quad \text{in } \Omega,
\end{equation}

\begin{equation}
\eta_{|r=2r_0} = 0, \quad \text{periodic boundary conditions}
\end{equation}

for $z \in \{(2k-1)a, (2k+1)a\}, \ k \in \mathbb{Z}$,

where $\vartheta_{|r\leq \varepsilon} = 0$.

We restrict our considerations to the case $k = 0$.

**Lemma 4.1.** Assume that $\vartheta, \vartheta_r \in L_2(\Omega), \vartheta, \varphi = 0, \vartheta_{|r=2r_0} = 0$, $\eta, \eta_r_{|r=2r_0} = 0$. Then there exists a solution to problem (4.10), (4.11) such that $\eta, \varphi = 0, \eta \in H^1(\Omega), \eta_r \in H_0^1(\Omega), \nabla \eta, z \in L_2(\Omega), \eta_{zr} \in H^1_0(\Omega)$, where

\begin{equation}
\|u\|_{H^1_0(\Omega)} = \|\nabla u\|_{L_2(\Omega)} + \left\| \frac{u}{r} \right\|_{L_2(\Omega)}
\end{equation}

and the estimate

\begin{equation}
\int_\Omega (|\eta|^2 + |\nabla \eta|^2 + |\nabla \eta_r|^2 + |\nabla \eta_z|^2 + |\nabla \eta_{zr}|^2) dx \\
+ \int_\Omega \left( \frac{\eta_r^2}{r^2} + \frac{\eta_{zr}^2}{r^2} \right) dx + \int_{-a}^{a} (\eta^2 + \eta_r^2 + \eta_z^2) r=0 dz \\
\leq c \int_{\Omega_{\varepsilon}} (\vartheta^2 + \vartheta_r^2) dx.
\end{equation}

holds.

**Proof.** Multiplying (4.11)$_1$ by $\eta$, integrating over $\Omega$ and using boundary conditions (4.11)$_{2,3}$ we obtain

\begin{equation}
\int_\Omega (\eta_r^2 + \eta_{zr}^2) dx - 2 \int_\Omega \eta_r \eta dr dz = \int_\Omega \vartheta \eta dx.
\end{equation}
The second term on the l.h.s. equals
\[-\int_{\Omega}(\eta^2)_rdrdz = \int_{-a}^{a}\eta^2|_{r=0}dz.\]

Hence, by the Poincare inequality we obtain from (4.13) the estimate

(4.14) \[\|\eta\|_{H^1(\Omega)} + \left(\int_{-a}^{a}\eta^2|_{r=0}dz\right)^{1/2} \leq c_1\|\vartheta\|_{L^2(\Omega)},\]

where \(c_1\) is an increasing function of \(r_0\).

We prove existence of solutions to problem (4.11) by the Galerkin method. Looking for weak solutions in the form

\[\eta^{(m)} = \sum_{i=1}^{m} a_{im}(t)\varphi_i(x),\]

where \(\{\varphi_i\}_{i=1}^{\infty}\) is a basis in \(H^1(\Omega)\) and \(a_{im}, i = 1, \ldots, m,\) satisfy

(4.15) \[\sum_{i=1}^{m} \int_{\Omega} a_{im}(t)\nabla'\varphi_i\nabla'\varphi_j dx - 2\sum_{i=1}^{m} \int_{\Omega} a_{im}(t)\varphi_{i,r}\varphi_{j} drdz \]
\[= \int_{\Omega} \vartheta \cdot \varphi_j dx, \quad j = 1, \ldots, m,\]

where \(\nabla' = (\partial_r, \partial_z)\). In view of (4.14) operator (4.11) is invertible so there exist solutions to (4.15) (see [LU, Ch. 3, Sect. 16]) and then by the Galerkin method we have the existence of solutions to problem (4.11) in \(H^1(\Omega)\).

We increase regularity of the weak solution by getting estimates for higher derivatives. These estimates can be derived precisely by applying differences and passing to the appropriate limits. For simplicity we shall only restrict our considerations to show appropriate a priori estimates for the weak solutions.

Differentiating (4.11) with respect \(r\), multiplying the result by \(\eta_{,r}\) and integrating over \(\Omega\) yields

\[\int_{\Omega} \Delta \eta_{,r} \eta_{,r} dx - 3\int_{\Omega} \frac{\eta^2_{,r}}{r^2} dx + 2\int_{\Omega} \frac{\eta_{,rr}\eta_{,r}}{r} dx = \int_{\Omega} \vartheta_{,r} \eta_{,r} dx,\]

where we used that \((\Delta \eta)_{,r} = \Delta \eta_{,r} - \frac{\eta_{,r}}{r^2} \).
Integrating by parts and using that $\eta_r|_{r=2r_0} = 0$ we obtain

$$\int_{\Omega} |\nabla \eta_r|^2 dx + 3 \int_{\Omega} \frac{\eta^2_r}{r^2} dx - \int_{\Omega} (\eta^2_r)_r dr dz = \int_{\Omega} \vartheta (\eta_{rr} r + \eta)_r dr dz.$$  

The last term on the l.h.s. of (4.16) equals $\int_{-a}^a \eta^2_r |_{r=0} dz$ and the r.h.s. we estimate by

$$\int_{\Omega} \left( \frac{\varepsilon_1}{2} \eta^2_{rr} + \frac{\varepsilon_2}{2} \frac{\eta^2_r}{r^2} \right) dx + \left( \frac{1}{2\varepsilon_1} + \frac{1}{2\varepsilon_2} \right) \int_{\Omega} \vartheta^2 dx.$$

Setting $\varepsilon_1 = 1$, $\varepsilon_2 = 5$, then (4.16) takes the form

$$\int_{\Omega} |\nabla \eta_r|^2 dx + \int_{-a}^a \frac{\eta^2_r}{r^2} dx + \int_{-a}^a \eta^2_r |_{r=0} dz \leq \frac{6}{5} \int_{\Omega} \vartheta^2 dx.$$

Differentiating (4.10) with respect to $z$, multiplying by $\eta_z$ and integrating over $\Omega$ yields

$$\int_{\Omega} \Delta \eta_z \eta_z dx + 2 \int_{\Omega} \eta_{zr} \eta_z dr dz = \int_{\Omega} \vartheta_{,z} \eta_z dx.$$

Integrating by parts and using the periodicity condition gives

$$\int_{\Omega} |\nabla \eta_z|^2 dx + 2 \int_{-a}^a \eta^2_z |_{r=0} dx \leq \int_{\Omega} \vartheta^2 dx.$$

Differentiating (4.10) with respect to $z$ yields

$$\Delta \eta_z + \frac{2\eta_{zr}}{r} = \vartheta_{,z}.$$

Differentiating (4.19) with respect to $r$ gives

$$\Delta \eta_{zr} - \frac{3\eta_{zr}}{r^2} + 2 \frac{\eta_{zrr}}{r} = \vartheta_{,zr}.$$

Multiplying (4.20) by $\eta_{zr}$, integrating over $\Omega$ and using that $\eta_{zr}|_{r=2r_0} = 0$ and (4.11)$_{2,3}$ we obtain

$$\int_{\Omega} |\nabla \eta_{zr}|^2 dx + 3 \int_{\Omega} \frac{\eta^2_{zr}}{r^2} dx - 2 \int_{\Omega} \frac{\eta_{zrr} \eta_{zr}}{r} dx$$

$$= - \int_{\Omega} \vartheta_{,zr} \eta_{zr} dx = \int_{\Omega} \vartheta_{,rr} \eta_{zr} dx,$$

$$\int_{\Omega} \vartheta_{,zr} \eta_{zr} dx.$$
where in view of the periodicity condition the integration by parts is performed in the r.h.s. of (4.21).

The last term on the l.h.s. of (4.21) equals

\[ -2 \int_{\Omega} \eta_{,zrr} \eta_{,zr} \, dr \, dz = - \int_{\Omega} (\eta_{,zr}^2)_{,r} \, dr \, dz = \int_{-a}^{a} \eta_{,zr}^2 |_{r=0} \, dz. \]

Summarizing, we obtain from (4.21) the estimate

(4.22) \[ \int_{\Omega} |\nabla \eta_{,zr}|^2 dx + 6 \int_{\Omega} \eta_{,zr}^2 \frac{1}{r^2} dx + 2 \int_{-a}^{a} \eta_{,zr}^2 |_{r=0} \, dz \leq \int_{\Omega} \varphi_r^2 \, dx. \]

From (4.14), (4.17), (4.18) and (4.22) we obtain (4.12). This concludes the proof.

**Lemma 4.2.** Let \( \zeta = \zeta_1(r) \) be a smooth function such that \( \zeta_1(r) = 1 \) for \( r \leq r_0 \) and \( \zeta_1(r) = 0 \) for \( r \geq 2r_0 \). Let \( \tilde{v}_r = v_r \zeta_1^2, \tilde{\chi} = \chi \zeta_1^2 \). Let \( \left( \frac{\tilde{\chi}}{r}, r \right) \in L_2(\Omega_\varepsilon), v_z, v_{z,r} \in L_2(\Omega_\varepsilon) \). Then for sufficiently smooth solutions to problem (1.1) the following inequality is valid

(4.23) \[ \left\| \nabla \left( \frac{\tilde{v}_r}{r} \right), r \right\|_{L_2(\Omega_\varepsilon)}^2 + 6 \right\| \frac{\tilde{v}_r}{r} \right\|_{L_2(\Omega_\varepsilon)}^2 \leq \left\| \left( \frac{\tilde{\chi}}{r} \right), r \right\|_{L_2(\Omega_\varepsilon)}^2 + c(1/r_0)(\|v_z\|_{L_2(\Omega_\varepsilon)}^2 + \|v_{z,r}\|_{L_2(\Omega_\varepsilon)}^2). \]

The inequality suggests fast decreasing of \( \tilde{v}_r \) and \( \tilde{\chi} \) approaching the axis of symmetry. This will be implied by the mentioned approximation and appropriate passing to the limit \( \varepsilon = 0 \).

**Proof.** Using the definitions of \( \tilde{\chi}_* \) by (4.6) and \( \vartheta \) by (4.8) we obtain

\[ \vartheta = \frac{\tilde{\chi}}{r} + \frac{1}{r} \left( \frac{\psi_{,zr}}{r} (\zeta_1^2), r \right) + \left( \frac{\psi (\zeta_1^2)_{,r}}{r} \right) \equiv \frac{\tilde{\chi}}{r} + \vartheta_*. \]
Hence we have

\[
\begin{align*}
\int_{\Omega} \vartheta_{z,r}^2 \, dx &= \int_{\Omega} \left[ \frac{1}{r} \left( \frac{\psi_{r,r}}{r} \zeta_1^2, r \right) \right] \, dx + \int_{\Omega} \left[ \frac{1}{r^2} \left( 2 \frac{\psi_{r,r}}{r} \zeta_1^2, r + \psi \left( \frac{\zeta_1^2, r}{r} \right) \right) \right] \, dx \\
&\leq c \int_{\Omega} \left[ \frac{1}{r^2} \left( 2 \frac{\psi_{r,r}}{r} \zeta_1^2, r + \psi \left( \frac{\zeta_1^2, r}{r} \right) \right) \right]^2 \, dx \\
&\quad + c \int_{\Omega} \left[ \left( \frac{\psi_{r,r}}{r} \right) \zeta_1^2, r + \frac{\psi_{r,r}}{r} \zeta_1^2, rr \\
&\quad + \left( \frac{\psi}{r} \right) \zeta_1^2, r + 2 \left( \frac{\psi}{r} \right) \zeta_1^2, rr + \frac{\psi}{r} \zeta_1^2, rrr \right]^2 \, dx \\
&\leq c \left( \frac{1}{r_0} \right) \int_{\Omega \cap \text{supp} \, \zeta_1, r} \left( v_z^2 + \left( \frac{\psi}{r} \right)^2 + v_{z,r}^2 \right) \, dx \\
&\leq c \left( \frac{1}{r_0} \right) \int_{\Omega} (v_z^2 + v_{z,r}^2) \, dx,
\end{align*}
\]

(4.24)

where we used that \( v_z = \frac{\psi}{r} \) and \( \text{supp} \, \zeta_1, r = \{ r : r_0 \leq r \leq 2r_0 \} \).

We employed also the Hardy inequality \( \int_{\Omega} \frac{\psi^2}{r} \, dx \leq \int_{\Omega} \frac{\psi^2}{r} \, dx \leq \int_{\Omega} v_z^2 \, dx \), where in view of (4.8) we have that \( \psi_{r,r}|_{r=0} = 0 \) and \( \psi|_{r=0} = 0 \).

In view of (4.24) and the expression for \( \vartheta \) we obtain from (4.22) the inequality

\[
\begin{align*}
\int_{\Omega} |\nabla \eta_{,z r}| \, dx + 6 \int_{\Omega} \eta_{,z r}^2 \, dx \\
&\leq \int_{\Omega} \left( \frac{\bar{\chi}}{r} \right)^2 \, dx + c \left( \frac{1}{r_0} \right) \int_{\Omega} (v_z^2 + v_{z,r}^2) \, dx.
\end{align*}
\]

(4.25)

Since \( \eta = \frac{\bar{\vartheta}}{r} \) we have that \( \eta, z = \frac{\bar{\vartheta}_{,z}}{r^2} = \frac{\bar{v}_z}{r} \). Then the second term on the l.h.s. of (4.25) takes the form

\[
\int_{\Omega} \left( \frac{\bar{v}_z}{r} \right)^2 \, dx.
\]

(4.26)

The first term on the l.h.s. of (4.25) equals

\[
\int_{\Omega} \left| \nabla \left( \frac{\bar{v}_z}{r} \right) \right|^2 \, dx.
\]

(4.27)
The term cannot be estimated from below by (4.26) because the Hardy inequality (2.13) does not hold in this case. Therefore, (4.25) takes the form

\[
\int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{v}_r}{r} \right) \right|^2 \, dx + 6 \int_{\Omega_\varepsilon} \frac{1}{r^2} \left( \frac{\tilde{v}_r}{r} \right)^2 \, dx \leq \int_{\Omega_\varepsilon} \left( \frac{\tilde{x}}{r} \right)^2 \, dx \\
+ c(1/r_0) \int_{\Omega_\varepsilon} (v_z^2 + v_{z,r}^2) \, dx.
\]

(4.28)

From (4.28) we obtain (4.23). This concludes the proof.

5. Estimate

Let us consider problem (1.8). Let \( \zeta = \zeta_1(r) \) be the cut-off function introduced in Section 4. Let

\[
\tilde{x} = \chi \zeta^2
\]

(5.1)

Then we introduce an approximate solution \( \tilde{x} \) to problem (1.8) as a solution to the problem

\[
\tilde{x}_t + v \cdot \nabla \tilde{x} - \frac{v_r}{r} \tilde{x} - \nu \left[ \left( r \left( \frac{\tilde{x}}{r} \right), _r, r \right) + \tilde{x}_{zz} + 2 \left( \frac{\tilde{x}}{r} \right), _r \right] \\
= v \cdot \nabla \zeta^2 \chi - \nu (\zeta_1^2, _r, r - \nu r \left( \frac{\chi}{r} \right), _r \zeta^2 \\
- 2\nu \left( \frac{\chi \zeta^2}{r}, _r \right) + 2 \tilde{v}_r \tilde{v}_r \tilde{z} \right) \quad \text{in} \quad \Omega_\varepsilon,
\]

(5.2)

\[
\tilde{x}|_{t=0} = \tilde{x}_0, \quad \tilde{x}|_{r=2r_0} = 0, \quad \tilde{x}|_{r=\varepsilon} = 0 \quad \text{and} \quad \tilde{x}|_{z=-a} = \tilde{x}|_{z=a}
\]

is periodic with respect to \( z \), where \( \tilde{v}_r = v_\varphi \zeta \).

Lemma 5.1. Let \( \Omega_\varepsilon = \{ x \in \mathbb{R}^3 : 0 < \varepsilon < r < R, |z| < a \} \). Assume that there exists a weak solution described by Lemma 2.3. Let \( \tilde{v}_r \in L_4(\Omega_\varepsilon T) \).
Let \( \chi \in L_2(\Omega \cap \text{supp} \zeta, r \times (0, T)) \), where \( \text{supp} \zeta, r = \{ r : r_0 \leq r \leq 2r_0 \} \).
Then for sufficiently smooth solutions to problem (1.1) we have

\[
\left\| \frac{\tilde{x}}{r} \right\|_{L_2(\Omega_\varepsilon \times (0, t))}^2 \leq c(1/r_0)d_1^2 + c(1/r_0)d_1\|\chi\|_{L_2(\Omega \cap \text{supp} \zeta, r \times (0, T))}^2 \\
+ \left\| \frac{\tilde{x}(0)}{r} \right\|_{L_2(\Omega_\varepsilon)}^2 \quad + \frac{1}{\nu} \int_0^t \int_{\Omega_\varepsilon} \tilde{v}_r^4 \, dx \, dt, \quad t \leq T,
\]

(5.3)
where $d_1$ is introduced in (2.1) and $\varepsilon < r_0$.

**Proof.** Multiplying (5.2) by $\tilde{\chi}/r$ and integrating over $\Omega_\varepsilon$ we obtain

$$\frac{1}{2} \frac{d}{dt} \left\| \frac{\tilde{\chi}}{r} \right\|_{L^2(\Omega_\varepsilon)}^2 + \nu \left\| \nabla \left( \frac{\tilde{\chi}}{r} \right) \right\|_{L^2(\Omega_\varepsilon)}^2 = \int_{\Omega_\varepsilon} \left[ v \cdot \nabla \chi^2 \frac{\tilde{\chi}}{r^2} - \nu (\chi \zeta^2)_r \frac{\tilde{\chi}}{r^2} - \nu \frac{r}{r^2} \chi \zeta^2 \frac{\tilde{\chi}}{r^2} \right] \, dx + 2 \int_{\Omega_\varepsilon} \frac{\tilde{v}_\varphi \tilde{v}_\varphi,z}{r} \, dx. \tag{5.4}$$

Now we estimate the particular terms in the r.h.s. of (5.4). The last term equals

$$\int_{\Omega_\varepsilon} \frac{(\tilde{v}_\varphi)^2}{r^2} \frac{\tilde{\chi}}{r} \, dx = - \frac{1}{2} \int_{\Omega_\varepsilon} \frac{\tilde{v}_\varphi^2}{r^2} \left( \frac{\tilde{\chi}}{r} \right) \, dx, \quad \leq \frac{\varepsilon_1}{2} \int_{\Omega_\varepsilon} \frac{(\tilde{\chi})^2}{r} \, dx \leq \frac{\varepsilon_1}{2} \int_{\Omega_\varepsilon} \left( \frac{\tilde{\chi}}{r} \right)^2 \, dx$$

To estimate the terms in the first integral on the r.h.s. of (5.4) we use properties of the cut-off function $\zeta = \zeta(r)$. The first expression under the square bracket is bounded by

$$c(1/r_0) \int_{\Omega \cap \text{supp } \zeta} |v_r| \chi^2 \, dx \leq \frac{c(1/r_0)}{r_0^3} \int_{\Omega \cap \text{supp } \zeta} \chi^2 \, dx.$$

The fourth term under the square bracket is estimated by

$$c(1/r_0) \int_{\Omega \cap \text{supp } \zeta} \chi^2 \, dx.$$

The second term under the square bracket equals to

$$- \nu \int_{\Omega} \chi_r \zeta^2 \frac{\tilde{\chi}}{r^2} \, dx - \nu \int_{\Omega} \chi \zeta^2 \frac{\tilde{\chi}}{r^2} \, dx \equiv I_1.$$  

Integrating by parts in the first term in $I_1$ it takes the form

$$- \frac{\nu}{2} \int_{\Omega} (\chi^2)_r \frac{\zeta^2 \zeta}{r^2} \, dx = \frac{\nu}{2} \int_{\Omega} \chi^2 \left( \frac{\zeta^2 \zeta}{r^2} \right)_r \, dx.$$
Hence
\[
|I_1| \leq c(1/r_0) \int_{\Omega \cap \text{supp } \zeta_r} \chi^2 dx.
\]
Similarly, the third term under the square bracket yields
\[
-\frac{\nu}{2} \int_{\Omega} \left( \left( \frac{\chi}{r} \right)^2 \right)_r \zeta^2 \chi^2 dx = \frac{\nu}{2} \int_{\Omega} \frac{\chi^2}{r^2} \left( \zeta^2 \right)_r \chi^2 dx \equiv I_2,
\]
so
\[
|I_2| \leq c(1/r_0) \int_{\Omega \cap \text{supp } \zeta_r} \chi^2 dx.
\]
Using the above estimates in (5.4) and assuming that \( \varepsilon_1 = \nu \) implies
\[
\frac{1}{2} \frac{d}{dt} \left\| \frac{\bar{\chi}}{r} \right\|_{L^2(\Omega_\varepsilon)}^2 + \frac{\nu}{2} \left\| \nabla \frac{\bar{\chi}}{r} \right\|_{L^2(\Omega_\varepsilon)}^2 \leq \frac{1}{2}\nu \int_{\Omega_\varepsilon} \frac{\bar{\varphi}^4}{r^4} dx
\]
\[
+ c(1/r_0) \int_{\Omega \cap \text{supp } \zeta_r} \chi^2 dx + c(1/r_0) \int_{\Omega \cap \text{supp } \zeta_r} |v_r| \chi^2 dx.
\]
Integrating (5.5) with respect to time and using Lemma 2.3 we obtain
\[
\left\| \frac{\bar{\chi}}{r} \right\|_{V^0_2(\Omega_\varepsilon \times (0,t))}^2 \leq \frac{1}{\nu} \left\| \frac{\bar{\varphi}}{r} \right\|_{L^4(\Omega_\varepsilon \times (0,t))}^4
\]
\[
+ c(1/r_0) d_1^2 + c(1/r_0) \int_0^t \int_{\Omega \cap \text{supp } \zeta_r} |v| \chi^2 dx dt + \left\| \frac{\bar{\chi}(0)}{r} \right\|_{L^2(\Omega_\varepsilon)}^2,
\]
where we used the definition of space \( V^0_2(\Omega^T) \) from the beginning of Section 2.
The third integral on the r.h.s. of (5.6) is estimated by
\[
c(1/r_0) \| v \|_{L^1_{\text{supp } \zeta_r \times (0,t)}} \| \chi \|_{L^2_{\text{supp } \zeta_r \times (0,t)}}^2 \leq c(1/r_0) d_1 \| \chi \|_{L^2_{\text{supp } \zeta_r \times (0,t)}}^2,
\]
where Lemma 2.3 was again used. In view of above estimates we obtain from (5.6) inequality (5.3). This concludes the proof. □

Let us consider (1.4). Multiplying (1.4) by \( \zeta = \zeta_1(r) \) and introducing the notation \( \tilde{\varphi} = v_\varphi \zeta \) we assume an approximate \( \tilde{\varphi} \) as a solution to the
problem

\begin{align}
\ddot{v}_\varphi, t + v \cdot \nabla \ddot{v}_\varphi + \frac{v_r}{r} \ddot{v}_\varphi - \nu \Delta \ddot{v}_\varphi + \nu \dddot{v}_\varphi = v \cdot \nabla \zeta v_\varphi - 2\nu v_\varphi \nabla \zeta \\
- \nu v_\varphi \Delta \zeta & \quad \text{in} \ \Omega_\varepsilon, \\
\ddot{v}_\varphi |_{t=0} = \ddot{v}_\varphi (0), \quad \ddot{v}_\varphi |_{r=2r_0} = 0, \quad \ddot{v}_\varphi |_{r=\varepsilon} = 0, \quad \ddot{v}_\varphi |_{z=-a} = \ddot{v}_\varphi |_{z=a},
\end{align}

so \( \ddot{v}_\varphi \) is periodic with respect to \( z \)

Lemma 5.2. Assume that \( \Omega_\varepsilon = \{ x \in \mathbb{R}^3 : 0 < \varepsilon < r < R, |z| < a \} \), \( \Omega_{\zeta,r} = \Omega \cap \text{supp} \zeta \). Assume that \( v \) is a weak solution to problem (1.1) satisfying (2.4), \( rv_\varphi (0) \in L_\infty (\Omega) \), \( \ddot{v}_\varphi (0) \in L_2 (\Omega_\varepsilon) \). Assume that \( v \) is sufficiently regular. Then the following a priori inequality holds

\begin{align}
\frac{1}{4} \int_{\Omega_\varepsilon} \ddot{v}_\varphi^4 dx + \frac{3}{4} \nu \int_{\Omega_\varepsilon} \left| \frac{\nabla \ddot{v}_\varphi}{r} \right|^2 dx dt + \frac{3}{4} \nu \int_{\Omega_\varepsilon} \ddot{v}_\varphi^4 r^4 dx dt \\
\leq \frac{3}{2} \int_0^t \int_{\Omega_\varepsilon} \frac{v_r}{r} \left| \frac{\ddot{v}_\varphi}{r^2} \right|^2 dx dt + c \left( \frac{1}{r_0} \right)^2 (1 + d_2) d_1^2 + \frac{1}{4} \left\| \frac{\ddot{v}_\varphi}{r} \right\|_{L_2 (\Omega_\varepsilon)}^2,
\end{align}

where \( c(1/r_0) \) is an increasing function.

Proof. Multiplying (5.7) by \( \ddot{v}_\varphi \frac{v_\varphi}{r^2} \) and integrating over \( \Omega_\varepsilon \times (0,t) \) we obtain

\begin{align}
\frac{1}{4} \int_{\Omega_\varepsilon} \ddot{v}_\varphi^4 dx + \frac{3}{4} \nu \int_{\Omega_\varepsilon} \left| \frac{\nabla \ddot{v}_\varphi}{r} \right|^2 dx dt + \frac{3}{4} \nu \int_{\Omega_\varepsilon} \ddot{v}_\varphi^4 r^4 dx dt \\
+ \int_{\Omega_\varepsilon} v \cdot \nabla \ddot{v}_\varphi \frac{v_\varphi}{r^2} dx dt \\
\leq -\int_0^t \int_{\Omega_\varepsilon} \frac{v_r}{r} \ddot{v}_\varphi^4 r^2 dx dt - \int_0^t \int_{\Omega_\varepsilon} v \cdot \nabla \zeta v_\varphi \ddot{v}_\varphi^2 r^2 dx dt \\
- 2\nu \int_0^t \int_{\Omega_\varepsilon} \nabla v_\varphi \nabla \zeta \ddot{v}_\varphi^2 r^2 dx dt - \nu \int_0^t \int_{\Omega_\varepsilon} v_\varphi \Delta \zeta \ddot{v}_\varphi^2 r^2 dx dt \\
+ \frac{1}{4} \left\| \frac{\ddot{v}_\varphi (0)}{r} \right\|_{L_2 (\Omega_\varepsilon)}^2,
\end{align}

where the expressions on the l.h.s. of the above inequality follow from the following calculations. The first, the fourth and the fifth terms from the
l.h.s. of (5.7) imply

\[
I \equiv \int_{\Omega_\varepsilon} \partial_t \tilde{v}_\varphi \frac{\tilde{v}_\varphi |\tilde{v}_\varphi|^2}{r^2} \, dx - \nu \int_{\Omega_\varepsilon} \Delta \tilde{v}_\varphi \frac{\tilde{v}_\varphi |\tilde{v}_\varphi|^2}{r^2} \, dx + \nu \int_{\Omega_\varepsilon} \tilde{v}_\varphi^4 \, dx \\
\equiv I_1 + I_2 + I_3,
\]

where

\[
I_1 = \frac{1}{4} \frac{d}{dt} \int_{\Omega_\varepsilon} \tilde{v}_\varphi^4 \frac{1}{r^2} \, dx.
\]

Integrating by parts in \(I_2\), using that \(\tilde{v}_\varphi|_{r=2r_0} = 0, \tilde{v}_\varphi|_{r=\varepsilon} = 0\) and that \(\tilde{v}_\varphi\) is periodic with respect to \(z\) we have

\[
I_2 = \nu \int_{\Omega_\varepsilon} \nabla \tilde{v}_\varphi \cdot \nabla \left( \frac{\tilde{v}_\varphi^3}{r^2} \right) \, dx = 3\nu \int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varphi|^2 \frac{\tilde{v}_\varphi^2}{r^2} \, dx - 2\nu \int_{\Omega_\varepsilon} \tilde{v}_\varphi \frac{\tilde{v}_\varphi^3}{r^2} \, drdz
\]

\[
= 3\nu \int_{\Omega_\varepsilon} \left| \nabla \tilde{v}_\varphi \frac{\tilde{v}_\varphi}{r} \right|^2 \, dx - \frac{\nu}{2} \int_{\Omega_\varepsilon} \left( \frac{\tilde{v}_\varphi^4}{r^2} \right)_r \, drdz
\]

\[
= \frac{3}{4} \nu \int_{\Omega_\varepsilon} \left| \frac{\nabla \tilde{v}_\varphi}{r} + \frac{\tilde{v}_\varphi^2}{r^2} \nabla r \right|^2 \, dx + \frac{\nu}{2} \int_{-a}^{a} \frac{\tilde{v}_\varphi^4}{r^2} \, drdz - \nu \int_{\Omega_\varepsilon} \frac{\tilde{v}_\varphi^4}{r^4} \, dx
\]

\[
= \frac{3}{4} \nu \int_{\Omega_\varepsilon} \left| \nabla \tilde{v}_\varphi \frac{\tilde{v}_\varphi}{r} \right|^2 \, dx - \frac{\nu}{2} \int_{\Omega_\varepsilon} \left| \frac{\tilde{v}_\varphi^4}{r^4} \right| \, dx + \frac{\nu}{2} \int_{-a}^{a} \frac{\tilde{v}_\varphi^4}{r^2} \, drdz
\]

The last term in \(I_2\) equals

\[
\frac{3}{8} \nu \int_{\Omega_\varepsilon} \partial_r \left( \frac{\tilde{v}_\varphi^4}{r^2} \right) \, drdz = -\frac{3}{8} \nu \int_{-a}^{a} \frac{\tilde{v}_\varphi^4}{r^2} \, drdz
\]
Hence
\[ I_2 = \frac{3}{4} \nu \int_{\Omega_\varepsilon} \left( \frac{\nabla \tilde{v}_\varphi}{r} \right)^2 dx - \frac{\nu}{4} \int_{\Omega_\varepsilon} \frac{\tilde{v}_\varphi^4}{r^4} dx + \frac{\nu}{8} \int_{-a}^a \frac{\tilde{v}_\varphi^4}{r^2} \bigg|_{r=\varepsilon} dz. \]

Therefore \( I \) takes the form
\[ I = \frac{1}{4} \frac{d}{dt} \int_{\Omega_\varepsilon} \frac{\tilde{v}_\varphi^4}{r^2} dx + \frac{3}{4} \nu \int_{\Omega_\varepsilon} \left| \frac{\nabla \tilde{v}_\varphi}{r} \right|^2 dx + \frac{3}{4} \nu \int_{\Omega_\varepsilon} \frac{\tilde{v}_\varphi^4}{r^4} dx + \frac{\nu}{8} \int_{-a}^a \frac{\tilde{v}_\varphi^4}{r^2} \bigg|_{r=\varepsilon} dz. \]

The last integral in \( I \) vanishes. Integrating the result with respect to time implies the three integrals on the l.h.s. of (5.9).

Finally, the last term on the l.h.s. of (5.9) equals
\[ \frac{1}{2} \int_{\Omega_t} \frac{v_r \tilde{v}_\varphi^4}{r^2} dx dt. \]

The second integral on the r.h.s. of (5.9) we estimate by
\[ \int_0^t \int_{\Omega_{\varepsilon,r}} |v|^2 \frac{|rv_\varphi|^3}{r^5} dx dt \leq c(1/r_0) \| rv_\varphi \|_{L^\infty(\Omega \times (0,t))}^3 \int_0^t \int_{\Omega} |v|^2 dx dt \]
\[ \leq c \left( \frac{1}{r_0} \right) d_2^3 d_1^2. \]

The third term on the r.h.s. of (5.9) takes the form
\[ -2 \nu \int_0^t \int_{\Omega} \nabla v_\varphi v_\varphi \frac{|v|^2}{r^2} \cdot \nabla \zeta \zeta |\zeta|^2 dx dt = -\nu \int_0^t \int_{\Omega} \nabla (v_\varphi^4) \cdot \nabla \zeta \zeta |\zeta|^2 dx dt \]
\[ = \nu \int_0^t \int_{\Omega} v_\varphi^4 \nabla \cdot \left( \frac{(\nabla \zeta \zeta |\zeta|^2}{r^2} \right) dx dt \equiv I_1. \]

Hence,
\[ |I_1| \leq c(1/r_0) \| rv_\varphi \|_{L^\infty(\Omega \times (0,t))}^2 \int_{\Omega} v_\varphi^2 dx dt \leq c(1/r_0) d_2^2 d_1^2. \]

Similarly, the last but one term on the r.h.s. of (5.9) we estimate by
\[ c \left( \frac{1}{r_0} \right) d_2^2 d_1^2. \]

Using the above estimates in (5.9) implies (5.8). This concludes the proof. \( \square \)
Lemma 5.3. Assume that \( \tilde{\chi}(0) / r \in L_2(\Omega), \) \( \tilde{\varphi}(0) / r \in L_4(\Omega), \) \( \chi \in L_{20}(\Omega_T). \) Let the assumptions of Lemmas 2.3, 2.4 hold. Assume that

\[
\|rv \varphi\|_{L_\infty(\Omega_t)} \leq \sqrt{\frac{5}{4}} \nu,
\]

Then the following a priori inequality holds

\[
L^2(\Omega^t_\varepsilon) \equiv \int_{\Omega^t_\varepsilon} \frac{\tilde{v}^4}{r^4} dx + \int_{\Omega^t_\varepsilon} \left| \tilde{\chi} \right|^2 + \int_{\Omega^t_\varepsilon} \left| \nabla \left( \frac{\tilde{v}}{r} \right) \right|^2 dx dt
\]

\[
+ \int_{\Omega^t_\varepsilon} \frac{1}{r^2} \left( \frac{\tilde{v}}{r} \right)^2 dx dt \leq \varphi(d_1, d_2) \| \chi \|^2_{L_{20}(\Omega \cap \text{supp } \zeta \times (0, t))}
\]

\[
+ \varphi(1/r_0, \nu, d_1, d_2) \left( 1 + \left\| \frac{v^2(0)}{r} \right\|^{2}_{L_2(\Omega)} + \left\| \frac{\tilde{\chi}}{r} \right\|^2_{L_2(\Omega)} \right),
\]

\( t \leq T, \)

where \( \tilde{v} = v \varphi \zeta, \tilde{\chi} = \chi \zeta^2, \tilde{v} = v \zeta^2, \zeta = \zeta_1(r), \zeta_1, r \neq 0 \) for \( r \in (r_0, 2r_0), \)

\( 2r_0 < R, \Omega_\varepsilon = \{ x \in \mathbb{R}^3 : 0 < \varepsilon < r < R, |z| < a \} \) and \( \varphi \) is an increasing positive function. Moreover, \( \Omega_\zeta = \Omega \cap \text{supp } \zeta. \)

Proof. From (4.28) we have

\[
\int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{v}}{r} \right) \right|^2 dx + 6 \int_{\Omega_\varepsilon} \frac{1}{r^2} \left( \frac{\tilde{v}}{r} \right)^2 dx \leq \int_{\Omega_\varepsilon} \left( \frac{\tilde{\chi}}{r} \right)^2 dx
\]

\[
+ c \int_{\Omega_\zeta} (v^2_z + v^2_{z,r}) dx,
\]

where \( \zeta = \zeta_1(r). \)

Integrating the above inequality with respect to time, using estimate (5.3) and Lemma 2.3 in the r.h.s. we obtain

\[
\int_{\Omega^t_\varepsilon} \left| \nabla \left( \frac{\tilde{v}}{r} \right) \right|^2 dx dt + 6 \int_{\Omega^t_\varepsilon} \frac{1}{r^2} \left( \frac{\tilde{v}}{r} \right)^2 dx dt \leq \frac{1}{\nu^2} \int_{0}^{t} \int_{\Omega^t_\varepsilon} \tilde{v}^4 dx dt
\]

\[
+ \frac{1}{\nu} c(1/r_0)d_1 \left\| \chi \right\|^2_{L_{20}(\Omega \cap \text{supp } \zeta, (0, t))} + \frac{1}{\nu} c(1/r_0)d_1^2
\]

\[
+ \frac{1}{\nu} \left\| \frac{\tilde{\chi}}{r} \right\|^2_{L_2(\Omega)}.
\]
To estimate the first term on the r.h.s. of (5.12) we employ (5.8)

\[
\int_{\Omega_{\varepsilon} \times (0,t)} \frac{\tilde{v}_r^4}{r^2} dx dt \leq \frac{2}{\nu} \int_{0}^{t} \int_{\Omega_{\varepsilon}} \frac{v_r^4}{r^2} dx dt
\]

\[
+ c \left( \frac{1}{r_0}, \nu \right) d_2^2 (1 + d_2) d_1^2 + \frac{2}{\nu} \left\| \frac{\tilde{v}_\varphi^2(0)}{r} \right\|_{L_2(\Omega)}^2.
\]

Let us recall that $\tilde{v}_r = v_\varphi \zeta^2$, $\tilde{\chi} = \chi \zeta^2$, $\tilde{v}_\varphi = v_\varphi \zeta$ and $\Omega_\zeta = \Omega \cap \text{supp } \zeta$.

The first term on the r.h.s. of (5.13) can be expressed in the form

\[
\frac{2}{\nu} \int_{0}^{t} \int_{\Omega_{\varepsilon}} \left| \frac{v_r \zeta^2}{r} \right| \frac{v_r \tilde{v}_\varphi^2}{r^2} dx dt = \frac{2}{\nu} \int_{0}^{t} \int_{\Omega_{\varepsilon}} \frac{\tilde{v}_r}{r^3} \frac{r^2 v_\varphi}{r^2} \tilde{v}_\varphi^2 dx dt
\]

\[
\leq \frac{2}{\nu} \left[ \varepsilon \int_{0}^{t} \int_{\Omega_{\varepsilon}} \frac{\tilde{v}_r^4}{r^4} dx dt + \frac{1}{2\varepsilon} \left\| \frac{r v_\varphi}{r^2} \right\|_{L_\infty(\Omega_\zeta^\varepsilon)}^4 \int_{0}^{t} \int_{\Omega_{\varepsilon}} \frac{\tilde{v}_\varphi^2}{r^6} dx dt \right].
\]

Setting $\varepsilon = \nu/2$ and using the estimate in the r.h.s. of (5.13) yields

\[
\int_{\Omega_\zeta^\varepsilon} \frac{\tilde{v}_\varphi^4}{r^4} dx dt \leq \frac{4}{\nu^2} \left\| r v_\varphi \right\|_{L_\infty(\Omega_\zeta^\varepsilon)}^4 \int_{\Omega_\zeta^\varepsilon} \frac{\tilde{v}_r^2}{r^6} dx dt
\]

\[
+ c \left( \frac{1}{r_0}, \nu \right) d_2^2 (1 + d_2) d_1^2 + \frac{4}{\nu} \left\| \frac{\tilde{v}_\varphi^2(0)}{r} \right\|_{L_2(\Omega)}^2.
\]

Employing (5.14) in the r.h.s. of (5.12) we obtain

\[
6 \int_{\Omega_\zeta^\varepsilon} \frac{1}{r^2} \left( \frac{\tilde{v}_r}{r} \right)^2 dx dt \leq \frac{4}{\nu^4} \left\| r v_\varphi \right\|_{L_\infty(\Omega_\zeta^\varepsilon)}^4 \int_{\Omega_\zeta^\varepsilon} \frac{\tilde{v}_r^2}{r^6} dx dt
\]

\[
+ c \left( \frac{1}{r_0}, \nu \right) d_2^2 (1 + d_2) d_1^2 + \frac{1}{\nu} c(1/r_0) d_1^2
\]

\[
+ c(1/r_0) d_1 \left\| \chi \right\|_{L_\infty(\Omega_\zeta^\varepsilon, \times (0,t))}^2
\]

\[
+ \frac{4}{\nu^2} \left\| \frac{\tilde{v}_\varphi^2(0)}{r} \right\|_{L_2(\Omega)}^2 \equiv \frac{4}{\nu^4} \left\| r v_\varphi \right\|_{L_\infty(\Omega_\zeta^\varepsilon)}^4 \int_{\Omega_\zeta^\varepsilon} \left| \frac{\tilde{v}_r}{r^3} \right|^2 dx dt + A_1^2,
\]

where $c(1/r_0)$ is an increasing function.

In view of the Hardy inequality (2.18) for $p = 2$ inequality (5.15) takes the form

\[
6 \int_{\Omega_\zeta^\varepsilon} \frac{1}{r^2} \left( \frac{\tilde{v}_r}{r} \right)^2 dx dt \leq \frac{4}{\nu^4} \left\| r v_\varphi \right\|_{L_\infty(\Omega_\zeta^\varepsilon)}^4 \int_{\Omega_\zeta^\varepsilon} \frac{1}{r^2} \left( \frac{\tilde{v}_r}{r} \right)^2 dx dt + A_1^2.
\]
Assuming that
\[6 - \frac{4}{\nu^4} \left\| rv_\varphi \right\|_{L^\infty(\Omega)}^4 \geq 1\]
we obtain from (5.16) the inequality
\[
\int_{\Omega_t} \frac{1}{r^2} \left( \frac{\bar{v}_r}{r} \right)^2 dx dt \leq A_1^2.
\]
Employing (5.18) in the r.h.s. of (5.14) yields
\[
\int_{\Omega_t} \frac{\tilde{v}_r^4}{r} dx dt \leq \frac{2}{\nu^2} d_2^4 A_1^2 + c A_1^2
\]
Exploiting (5.19) in (5.3) and (5.12) implies
\[
\left\| \frac{\chi}{r} \right\|_{V^a(\Omega_t)}^2 + \int_{\Omega_t} \left| \nabla \left( \frac{\bar{v}_r}{r} \right) \right| dx dt + \int_{\Omega_t} \frac{1}{r^2} \left( \frac{\bar{v}_r}{r} \right)^2 dx dt \leq c(1/r_0, \nu, d_1, d_2) A_1^2.
\]
This concludes the proof of the lemma.

**Remark 5.4.** Condition (5.10) looks like a smallness condition, however it can be justified by the following explanations.

By Lemma 3.3 we have that \( u = rv_\varphi \in C^{\alpha, \alpha/2}(\Omega^T) \), \( \alpha = 1/2 \). Then (2.10) implies that \( u |_{r=0} = 0 \).

Hence for a sufficiently small neighborhood of the \( x_3 \)-axis, so for sufficiently small \( r_0 \), restriction (5.10) can be satisfied. Since \( \alpha = 1/2 \) the number \( r_0 \) must be small but can be calculated precisely. Then a bound in the a priori estimate, which we are looking for, becomes an increasing function of \( 1/r_0 \).

The local existence is proved in interval \((0, T_*)\) and \( v \in W^{2,1}_2(\Omega^T) \). Next by Theorem B, Lemma 2.11 and Lemma 6.1 we are able to obtain a priori estimate for \( v \in W^{2,1}_2(\Omega^T) \), \( T > T_* \). Then \( v(T_*) \in H^1(\Omega) \) and Lemma 2.2 yields existence in \( W^{2,1}_2(\Omega \times (T_*, 2T_*)) \).

Continuing, we have existence in \( W^{2,1}_2(\Omega^T) \) for any \( T > 0 \). This implies that in each time interval \((kT_*, (k+1)T_*)\), \( k \in \mathbb{N} \), restriction (5.10) is satisfied.

To derive any estimate from (5.11) we have to estimate \( \left\| \chi \right\|_{L^4(\Omega \times \text{supp } \zeta, r \times (0,T))} \). For this purpose we need
Lemma 5.5. Let
\[ A_0 = \varphi(1/r_0, \nu, d_1, d_2) \left( 1 + \frac{\tilde{v}_\varphi^2(0)}{r} \right)^2 + \frac{\tilde{\chi}(0)}{r} \right)^2 \]
be finite, where \( \tilde{v}_\varphi = v_\varphi \zeta, \tilde{\chi} = \chi \zeta^2 \) and the smooth cut-off function \( \zeta \) is described in Lemma 5.3. Then there exists a positive constant \( c \) such that
\[ \| \chi \|_{L^2(\Omega, \cap \text{supp } \zeta_r \times (0, T))} \leq c A_0. \]

Proof. From (5.11) we have
\[ \left\| \frac{\tilde{\chi}}{r} \right\|_{L^{10/3}(\Omega \times (0, t))} \leq c_1 \left\| \frac{\chi}{r} \right\|_{L^{10/3}(\Omega, \cap \text{supp } \zeta, r \times (0, T))} + A_0. \]

Let us introduce the sets
\[ \Omega^{(\lambda)}_\varepsilon = \{(r, z) : 0 < \varepsilon < r \leq r_0 - \lambda, |z| < a\} \]
and connect with them a set of cut-off smooth functions such that
\[ \zeta^{(\lambda)} = \begin{cases} 1 & \text{for } (r, z) \in \Omega^{(\lambda)}_\varepsilon \\ 0 & \text{for } (r, z) \in \Omega_\varepsilon \setminus \Omega^{(\lambda/2)}_\varepsilon. \end{cases} \]
To simplify notation we denote \( \eta = \frac{\tilde{\chi}}{r} \). Then (5.22) can be expressed in the form
\[ \| \eta \|_{L^{10/3}(\Omega^{(\lambda)}_\varepsilon \times (0, t))} \leq c_1 \| \eta \|_{L^{10/3}(\Omega^{(\lambda/2)}_\varepsilon \setminus \Omega^{(\lambda)}_\varepsilon \times (0, t))} + A_0. \]
From (5.23) we have
\[ \int_{\Omega^{(\lambda)}_\varepsilon \times (0, t)} |\eta|^{10/3} dx dt \leq c_2 c_1^{10/3} \int_{\Omega^{(\lambda/2)}_\varepsilon \setminus \Omega^{(\lambda)}_\varepsilon \times (0, t)} |\eta|^{10/3} dx dt + c_2 A_0^{10/3}, \]
where \( c_2 = 2^{10/3} - 1 \).
Adding
\[ c_2 c_1^{10/3} \int_{\Omega^{(\lambda)}_\varepsilon \times (0, t)} |\eta|^{10/3} dx dt \]

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to both sides of (5.24) we obtain
(5.25)\[
\int_{\Omega_{\epsilon}^{(\lambda)} \times (0, t)} |\eta|^{10} dx dt \leq \frac{c_2 c_1^{10}}{1 + c_2 c_1^{10/3}} \int_{\Omega_{\epsilon}^{(\lambda/2)} \times (0, t)} |\eta|^{10} dx dt + \frac{c_2}{1 + c_2 c_1^{10/3}} A_0^{10/3}.
\]
Introducing the notation
\[
f(\lambda) = \int_{\Omega_{\epsilon}^{(\lambda)} \times (0, t)} |\eta|^{10} dx dt, \quad \mu = \frac{c_2 c_1^{10/3}}{1 + c_2 c_1^{10/3}} < 1,
\]
\[
K = \frac{c_2}{1 + c_2 c_1^{10/3}} A_0^{10/3}
\]
we obtain from (5.25) the inequality
(5.26)\[
f(\lambda) \leq \mu f(\lambda/2) + K
\]
Hence we derive the estimate
(5.27)\[
f(\lambda) \leq \sum_{j=0}^{\infty} \mu^j K = \frac{1}{1 - \mu} K
\]
Therefore, Lemma 5.5 is proved.

**Conclusion 5.6.** In view of (5.21) inequality (5.11) implies the a priori estimate
(5.28)\[
L^2(\Omega_{\epsilon}^t) \leq c A_0, \quad t \leq T.
\]
where \(L\) is defined by (5.11).

**Remark 5.7.** All estimates in this section are made in \(\Omega_{\epsilon} = \{(r, z) \in \Omega : 0 < \epsilon < r < R, |z| < a\}, \epsilon > 0\). Then we introduced approximate functions \(\chi_{\epsilon}, v_{\phi\epsilon}, v_{r\epsilon}\) which are such that
(5.29)\[
v_{\phi\epsilon}|_{r=\epsilon} = 0, \quad v_{r\epsilon}|_{r=\epsilon} = 0, \quad \chi_{\epsilon}|_{r=\epsilon} = 0.
\]
They are defined by problems (5.2), (5.7) and (4.1), where the appropriate conditions from (5.29) are added. Then estimate (5.28) takes the form
(5.30)\[
L_{\epsilon}^2(\Omega_{\epsilon}^t) \leq c A_0,
\]
where \(L_{\epsilon}\) means that it depends on \(\chi_{\epsilon}, v_{\phi\epsilon}, v_{r\epsilon}\). Then by appropriate passing with \(\epsilon\) to 0 we obtain (5.30) for \(\epsilon = 0\). The passing will be performed in [Z4].
6. Decay estimates

In this section we derive estimates guaranteeing global existence.

The calculations in Lemma 6.1 are formal but they can be performed in the more rigorous way presented in Sections 4 and 5.

**Lemma 6.1.** Let the assumptions of Lemmas 2.3 and 2.4 hold. Let \( \bar{v}_\varphi(0) \in L^2(\Omega), \bar{\chi}(0) \in L^2(\Omega) \), where \( \bar{v}_\varphi = v_\varphi \zeta, \bar{\chi} = \chi \zeta^2 \) and \( \zeta = \zeta(r) \) is introduced in Lemma 5.1. Let

\[
\|rv_\varphi\|_{L^\infty(\Omega^T)} \leq \sqrt{\frac{3}{4} \nu}
\]

Then

\[
\frac{1}{\nu^2} \left\| \frac{\bar{v}_\varphi^2(t)}{r} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\bar{\chi}(t)}{r} \right\|_{L^2(\Omega)}^2 \leq \frac{1}{1 - \varepsilon} \left[ \varphi_1(1/r_0, d_1, d_2) + \varphi_2(1/r_0, d_1) \varepsilon^{-3} \right] d_1^2
\]

\[
+ \frac{1}{1 - \varepsilon} \left( \frac{1}{\nu^2} \left\| \frac{\bar{v}_\varphi^2(0)}{r} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\bar{\chi}(0)}{r} \right\|_{L^2(\Omega)}^2 \right) e^{-\nu_* t},
\]

where \( \varphi_1, \varphi_2 \) are increasing positive functions of their arguments, \( \nu_* > 0 \) and \( \varepsilon \in (0, 1) \).

**Proof.** Let us introduce the notation

\[ \Omega_\zeta = \Omega \cap \text{supp } \zeta, \quad \Omega_{\zeta,r} = \Omega \cap \text{supp } \zeta, \]

From (5.5) we have

\[
\frac{d}{dt} \left\| \frac{\bar{\chi}}{r} \right\|_{L^2(\Omega)}^2 + \nu \left\| \nabla \frac{\bar{\chi}}{r} \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\nu} \int_{\Omega} \frac{\bar{v}_\varphi^4}{r^4} dx + c_1(1/r_0) \int_{\Omega_{\zeta,r}} \chi^2 dx + c_2(1/r_0) \int_{\Omega_{\zeta,r}} |v| \chi^2 dx,
\]

where \( c_1(r_0) \sim r_0^{-4}, c_2(r_0) \sim r_0^{-3} \).
Multiplying \( \frac{\ddot{v}_\varphi}{r} \frac{|\ddot{v}_\varphi|^2}{r^2} \) and integrating over \( \Omega \) we obtain

\[
\frac{1}{4} \frac{d}{dt} \left\| \frac{\ddot{v}_\varphi}{r} \right\|_{L_2(\Omega)}^2 + \frac{3 \nu}{4} \left\| \frac{\nabla \ddot{v}_\varphi}{r} \right\|_{L_2(\Omega)}^2 + \frac{3 \nu}{4} \int \frac{\dddot{v}_\varphi^4}{r^4} dx 
\]

(6.3)

\[
\leq \|rv_\varphi\|_{L_\infty(\Omega)}^2 \int_\Omega \frac{|\ddot{v}_\varphi|}{r^3} \frac{\dddot{v}_\varphi^2}{r^2} dx - \int_\Omega v \cdot \nabla \zeta v \ddot{v}_\varphi \dddot{v}_\varphi^2 dx 
- 2 \nu \int_\Omega \nabla v_\varphi \nabla \zeta v_\varphi \ddot{v}_\varphi \dddot{v}_\varphi^2 dx - \nu \int_\Omega v_\varphi \Delta \zeta \ddot{v}_\varphi \dddot{v}_\varphi^2 dx.
\]

Now we examine the particular terms on the r.h.s. of (6.3). The second integral is bounded by

\[
\frac{\varepsilon}{2} \int_\Omega \frac{\ddot{v}_\varphi^4}{r^4} dx + \frac{c_3(r_0)}{2 \varepsilon} \int_\Omega v^2 v_\varphi^4 dx,
\]

where \( c_3(r_0) \sim r_0^{-2} \) and the second integral is estimated by

\[
\frac{c_4(r_0)}{2 \varepsilon} \|rv_\varphi\|_{L_\infty(\Omega)}^4 \int_\Omega v^2 dx,
\]

where \( c_4(r_0) \sim r_0^{-6} \).

Summarizing the second integral on the r.h.s. of (6.3) is bounded by

(6.4)

\[
\frac{\varepsilon}{2} \int_\Omega \frac{\ddot{v}_\varphi^4}{r^4} dx + \frac{c_3(r_0)}{2 \varepsilon} \|rv_\varphi\|_{L_\infty(\Omega)}^4 \int_\Omega v^2 dx.
\]

The third integral on the r.h.s. of (6.3) we express in the form

\[
-2 \nu \int_\Omega \nabla v_\varphi \nabla \zeta \frac{v_\varphi v_\varphi^2 \zeta^3}{r^2} dx = -\nu \int_\Omega \nabla v_\varphi \nabla \zeta \frac{\zeta^3}{r^2} dx 
= \nu \int_\Omega v_\varphi^4 \nabla \left( \frac{\zeta^3}{r^2} \right) dx = \nu \int_\Omega v_\varphi^4 \Delta \zeta \frac{\zeta^3}{r^2} dx + \frac{3}{2} \nu \int_\Omega v_\varphi^4 |\nabla \zeta|^2 \frac{\zeta^2}{r^2} dx 
- \nu \int_\Omega v_\varphi^4 \nabla \zeta \cdot \nabla r \frac{\zeta^3}{r^2} dx.
\]

Therefore the sum of the third and the fourth terms from the r.h.s. of (6.3) equals

\[
-\nu \int_\Omega v_\varphi^4 \Delta \zeta \frac{\zeta^3}{r^2} dx + \frac{3}{2} \nu \int_\Omega v_\varphi^4 |\nabla \zeta|^2 \frac{\zeta^2}{r^2} dx - \nu \int_\Omega v_\varphi^4 \nabla \zeta \cdot \nabla r \frac{\zeta^3}{r^2} dx \equiv I_1.
\]
Hence

\[ |I_1| \leq c_5(r_0) \int_{\Omega_{\xi,r}} v_\phi^4 \, dx, \]

where \( c_5(r_0) \sim r_0^{-4} \).

Assuming \( \varepsilon = \frac{\nu}{2} \) in (6.4) and employing the above estimates in (6.3) we obtain

\[
\frac{1}{4} \frac{d}{dt} \left| \frac{\tilde{v}_\phi^2}{r} \right|^2_{L^2(\Omega)} + \frac{3\nu}{4} \left| \frac{\nabla \tilde{v}_\phi}{r} \right|^2_{L^2(\Omega)} + \frac{\nu}{2} \int_{\Omega} \frac{\tilde{v}_\phi^4}{r^4} \, dx \leq \frac{r_0^2}{r} \int_{\Omega} \tilde{v}_\phi^2 \, dx + \frac{c_4(r_0)}{\nu} \int_{\Omega_{\xi,r}} v_\phi^4 \, dx + \frac{c_5(r_0)}{\Omega_{\xi,r}} \int v_\phi^4 \, dx.
\]

Applying the Hölder and the Young inequalities to the first term on the r.h.s. of (6.5) we have

\[
\frac{d}{dt} \left| \frac{\tilde{v}_\phi^2}{r} \right|^2_{L^2(\Omega)} + \frac{3\nu}{4} \left| \frac{\nabla \tilde{v}_\phi}{r} \right|^2_{L^2(\Omega)} + \frac{\nu}{4} \int_{\Omega} \frac{\tilde{v}_\phi^4}{r^4} \, dx \leq \frac{r_0^2}{r} \int_{\Omega} \tilde{v}_\phi^2 \, dx + \frac{c_4(r_0)}{\nu} \int_{\Omega_{\xi,r}} v_\phi^4 \, dx + \frac{c_5(r_0)}{\Omega_{\xi,r}} \int v_\phi^4 \, dx.
\]

Using estimate (4.28) in (6.2) yields

\[
\frac{d}{dt} \left| \frac{\tilde{\chi}}{r} \right|^2_{L^2(\Omega)} + \frac{\nu}{2} \left| \frac{\nabla \tilde{\chi}}{r} \right|^2_{L^2(\Omega)} + \frac{3\nu}{\Omega} \int \left( \frac{\tilde{v}_r}{r^2} \right)^2 \, dx \leq \frac{1}{\nu} \int_{\Omega} \tilde{v}_\phi^4 \, dx + c_1(r_0) \int_{\Omega_{\xi,r}} \tilde{\chi}^2 \, dx + c_2(r_0) \int_{\Omega_{\xi,r}} |v| \chi^2 \, dx + c_6(r_0) \int_{\Omega_{\xi,r}} (v_x^2 + v_{z,r}^2) \, dx,
\]

where \( c_6(r_0) \sim r_0^{-4} \).
Multiplying (6.6) by $\frac{4}{\nu^2}$, adding to (6.7) and applying the Hardy inequality to the first integral on the r.h.s. of (6.6) we obtain

\[
\frac{d}{dt} \left( \frac{1}{\nu^2} \left\| \frac{\partial \varphi}{\partial r} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\tilde{v}}{r} \right\|_{L^2(\Omega)}^2 \right) + \frac{3}{\nu} \left\| \nabla \frac{\partial \varphi}{\partial r} \right\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \left\| \nabla \frac{\tilde{v}}{r} \right\|_{L^2(\Omega)}^2 \\
+ 3\nu \int_\Omega \frac{1}{\nu^2} \left( \frac{v_r}{r} \right)^2 dx \leq \frac{4}{\nu^3} \|rv\varphi\|_{L^\infty(\Omega)}^4 \int_\Omega \frac{1}{\nu^2} \left( \frac{v_r}{r} \right)^2 dx \\
+ \frac{4c_4(r_0)}{\nu^3} \|rv\varphi\|_{L^\infty(\Omega)}^4 \int_{\Omega_{\varsigma,r}} v^2 dx + \frac{4c_5(r_0)}{\nu^2} \int_{\Omega_{\varsigma,r}} v^4 dx \\
+ c_1(r_0) \int_{\Omega_{\varsigma,r}} \chi^2 dx + c_2(r_0) \int_{\Omega_{\varsigma,r}} |v|\chi^2 dx \\
+ c_6(r_0) \int_{\Omega_{\varsigma,r}} (v_r^2 + v_{r,r}^2) dx.
\]

(6.8)

Assuming

\[
3\nu \geq \frac{4}{\nu^3} \|rv\varphi\|_{L^\infty(\Omega_{\varsigma,r})}^4
\]

we obtain from (6.8) the inequality

\[
\frac{d}{dt} \left( \frac{1}{\nu^2} \left\| \frac{\partial \varphi}{\partial r} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\tilde{v}}{r} \right\|_{L^2(\Omega)}^2 \right) \\
+ \nu \left( \frac{3}{\nu^2} \left\| \nabla \frac{\partial \varphi}{\partial r} \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \nabla \frac{\tilde{v}}{r} \right\|_{L^2(\Omega)}^2 \right) + \frac{\nu}{2} \left\| \nabla \frac{\tilde{v}}{r} \right\|_{L^2(\Omega)}^2 \\
\leq \frac{4c_4(r_0)}{\nu^3} \int_{\Omega_{\varsigma,r}} v^2 dx + \frac{4c_5(r_0)}{\nu^2} \int_{\Omega_{\varsigma,r}} v^4 dx \\
+ c_1(r_0) \int_{\Omega_{\varsigma,r}} \chi^2 dx + c_2(r_0) \int_{\Omega_{\varsigma,r}} |v|\chi^2 dx \\
+ c_6(r_0) \int_{\Omega_{\varsigma,r}} (v_r^2 + v_{r,r}^2) dx.
\]

(6.10)

Using the Poincare inequality in the second expression on the l.h.s. of
(6.10) we can express (6.10) in the form
\[
\frac{d}{dt} \left( \frac{1}{\nu^2} \left\| \frac{\nu^2}{r} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\chi'}{r} \right\|_{L^2(\Omega)}^2 \right) + \nu \left( \frac{1}{\nu^2} \left\| \frac{\nu^2}{r} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\chi}{r} \right\|_{L^2(\Omega)}^2 \right) + \frac{\nu}{2} \left\| \nabla \frac{\chi}{r} \right\|_{L^2(\Omega)}^2 
\]
\[
\leq 4c_4(r_0) \frac{d}{dt} \int_{\Omega_{\zeta,r}} v^2 dx + \frac{4c_5(r_0)}{\nu^2} \int_{\Omega_{\zeta,r}} v^4 dx 
\]
\[
+ c_1(r_0) \int_{\Omega_{\zeta,r}} \chi^2 dx + c_2(r_0) \int_{\Omega_{\zeta,r}} |v|^2 \chi^2 dx 
\]
\[
+ c_6(r_0) \int_{\Omega_{\zeta,r}} (v_z^2 + v_{z,r}^2) dx, 
\]
where \( \nu_0 = \min \left\{ \frac{\nu}{2} c_p, \frac{3\nu}{c_p} \right\} \) and \( c_p \) is the constant from the Poincare inequality.

Multiplying (6.11) by \( e^{\nu_0 t} \), integrating the result with respect to time we obtain
\[
\left( \frac{1}{\nu^2} \left\| \frac{\nu^2}{r} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\chi}{r} \right\|_{L^2(\Omega)}^2 \right) e^{\nu_0 t} + \frac{\nu}{2} \int_0^t \left\| \nabla \frac{\chi}{r} \right\|_{L^2(\Omega)}^2 e^{\nu_0 t'} dt' 
\]
\[
\leq c_7(r_0, d_1, d_2) d_1^2 e^{\nu_0 t} + c_2(r_0) \int_0^t \int_{\Omega_{\zeta,r}} |v|^2 |\chi|^2 dx e^{\nu_0 t'} dt' 
\]
\[
+ \left( \frac{1}{\nu^2} \left\| \frac{\nu^2}{r} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\chi}{r} \right\|_{L^2(\Omega)}^2 \right), 
\]
where \( c_7(r_0) \sim r_0^{-4} \).

The second term on the r.h.s. is estimated by
\[
c_2(r_0) \int_0^t \left\| v \right\|_{L^2(\Omega_{\zeta,r})} \left\| \chi \right\|_{L^4(\Omega_{\zeta,r})}^2 dt' \leq \varepsilon \int_0^t \left\| \nabla \frac{\chi}{r} \right\|_{L^2(\Omega_{\zeta,r})}^2 dt' 
\]
\[
+ c(1/r_0, d_1) \varepsilon^{-3} d_1^2, 
\]
where \( \varepsilon \in (0, 1) \). Let
\[
X(t) = \frac{1}{\nu^2} \left\| \frac{\nu^2}{r} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\chi}{r} \right\|_{L^2(\Omega)}^2 
\]
\[
(6.14) 
\]
Employing (6.13) and (6.14) in (6.12) and applying the local iteration argument (see [LSU, Ch. 4, Sect. 10]) we obtain

\[ X(t) \leq \frac{1}{1-\varepsilon} \left[ c \gamma(r_0, d_1, d_2) + c(1/r_0, d_1)\varepsilon^{-3}d_1^2 \right] \]

\[ + \frac{1}{1-\varepsilon} X(0)e^{-\nu_* t}, \]

where \( \varepsilon \in (0, 1) \). This concludes the proof.

Let us express inequality (6.15) in the form

\[ X(t) \leq A + \frac{1}{1-\varepsilon} X(0)e^{-\nu_* t}, \]

where \( X(t) \) is defined by (6.14). Then we have

**Lemma 6.2.** Let \( T > 0 \) be so large that

\[ \frac{1}{1-\varepsilon} e^{-\nu_* T} < 1, \quad \varepsilon \in (0, 1). \]

Then for any \( k \in \mathbb{N} \) the following estimate is valid

\[ X((k+1)T) \leq A \left( \frac{1}{1-\varepsilon} \right)^{k+1} X(0)e^{-\nu_* kT}. \]

Proof follows easily by iteration.

**Remark 6.3.** In reality (6.1) is a global estimate, however, sometimes, it is convenient to examine problem (1.1) step by step in time. This may decrease a number of restrictions imposed on data (see [Z6]). Since (6.1) holds for any \( T \) we see that (6.17) is not very restrictive. Moreover, \( A \) is large because it is proportional to \( r_0^{-4}\varepsilon^{-3} \), where \( \varepsilon \in (0, 1) \) and \( r_0 \) must be so small that the restriction holds

\[ \|u\|_{L_\infty(\Omega_T^\varepsilon)} \leq \sqrt{\frac{3}{4}} \nu. \]

The above condition can be satisfied either by applying Lemma 3.3 and (2.8) with the Hölder exponent calculated in [Z5] or by applying the local in time regularity result (see Lemmas 2.2, 2.3). However, in the both cases \( r_0 \) must be small. We emphasize that smallness of \( r_0 \) implies that bounds either in (6.1) or in (6.18) are large.

Now we shall show that the estimate

\[ \left\| \frac{\tilde{\chi}}{r} \right\|_{V_2^0(\Omega_T)} \leq cA_0 \equiv c\varphi(1/r_0, d_1, d_2) \left[ 1 + \left\| \frac{\tilde{\eta}_\varphi(0)}{r} \right\|_{L_2(\Omega)} + \left\| \frac{\tilde{\chi}(0)}{r} \right\|_{L_2(\Omega)} \right] \]
implies the following bound
\[(6.20) \quad \|\tilde{v}'\|_{V^2_\Omega} \leq \varphi(A_0),\]
where \(v' = (v_r, v_z)\).

For this purpose we come back to Section 4. Let us recall the notation
\[\eta = \frac{\tilde{\psi}}{r^2}, \quad \vartheta = \frac{\tilde{\chi}}{r},\]
and \(\eta\) is a solution to (4.10),
\[(6.21) \quad \Delta \eta + \frac{2\eta}{r} = \vartheta, \quad \Delta \eta = \frac{1}{r}(r\eta_r)_r + \eta_{zz}.\]

**Lemma 6.4.** Assume that (6.19) holds and \(A_0\) is finite, where \(\varphi\) is an increasing positive function, \(d_1\) is defined by (2.4), \(d_2\) by (2.7) and \(\tilde{\varphi}(0) = v(0)\), \(\tilde{\chi}(0) = \chi(0)\). Moreover, \(\zeta = \zeta(r)\) is a smooth cut off function such that \(\zeta(r) = 1\) for \(r \leq r_0\) and \(\zeta(r) = 0\) for \(r \geq 2r_0\). Finally, \(r_0\) is so small that
\[\|u\|_{L^\infty(\Omega_\zeta)} \leq \sqrt[4]{\frac{5}{4}}\nu\]
(see (5.10)), where \(\Omega_\zeta = \Omega \cap \text{supp} \zeta\). Then (6.20) holds.

**Proof.** Let us recall that \(\eta\) and \(\vartheta\) have compact supports with respect to \(r\) and they are periodic with respect to \(z\). From Section 4 formula (4.12) we have the estimate
\[(6.22) \quad \int_{\Omega_\epsilon} (\eta^2 + |\nabla \eta|^2 + |\nabla \eta_r|^2 + |\nabla \eta_z|^2 + |\nabla \eta_{zr}|^2) dx \leq c \int_{\Omega_\epsilon} (\vartheta^2 + \vartheta^2_r) dx \equiv I_1.\]

From (6.22) we have
\[\int_{\Omega_\epsilon} |\nabla \eta_{zr}|^2 dx = \int_{\Omega_\epsilon} \left| \nabla \left( \frac{\tilde{v}_r}{r} \right) \right|^2 \leq I_1,\]
so
\[(6.23) \quad \int_{\Omega_\epsilon} \left| \nabla \left( \frac{\tilde{v}_r}{r} \right) \right|^2 dx = \int_{\Omega_\epsilon} \left[ \left( \frac{\tilde{v}_r}{r} \right)_{rr} + \left( \frac{\tilde{v}_r}{r} \right)_{r2} \right] dx \leq I_1,\]
where we used that \( \eta = \frac{\bar{\psi}}{r^2}, \bar{\psi} = \psi \zeta_1^2, \bar{v}_r = v_r \zeta_1^2, \bar{v}_z = v_z \zeta_1^2 \),

\[
v_r = \frac{\psi_z}{r}, \quad v_z = -\frac{\psi_r}{r}, \quad \eta_z = \frac{\bar{\psi}_z}{r^2} = \frac{\bar{v}_r}{r},
\]

(6.24) \[
\eta_{,r} = \left( \frac{\bar{\psi}}{r^2} \right)_{,r} = \frac{\psi_r \zeta_1^2}{r^3} - \frac{2\psi}{r^3} \zeta_1^2 + \frac{\psi}{r^2} \zeta_1^2, r
\]

\[
= -\frac{v_z}{r} \zeta_1^2 - \frac{2\psi \zeta_1^2}{r^3} + \frac{\psi_r}{r^2} \zeta_1^2, r = -\frac{v_z}{r} \zeta_1^2 + \frac{2\bar{\psi}}{r^3} + \frac{\psi}{r^2} \zeta_1^2, r.
\]

From (6.23) we have

\[
\int_{\Omega} \left( \frac{\bar{\psi}_r}{r} \right)^2 dx \leq I_1,
\]

so

\[
\left( \frac{\bar{\psi}_r}{r} \right)^{,rr} = \frac{\bar{v}_{r,rr}}{r} - 2 \frac{\bar{v}_{r,r}}{r^2} + 2 \frac{\bar{v}_r}{r^3}.
\]

Hence

\[
\int_{\Omega} \frac{\bar{v}_{r,rr}^2}{r^2} dx \leq c \left( \int_{\Omega} \left( \frac{\bar{v}_r}{r} \right)^2 dx + \int_{\Omega} \frac{\bar{v}_{r,r}^2}{r^4} dx + \int_{\Omega} \frac{\bar{v}_{r,r}^2}{r^6} dx \right) \equiv J_1.
\]

To estimate \( J_1 \) we calculate

\[
\int_{\Omega} \frac{1}{r^2} \left| \frac{\bar{v}_{r,r}}{r} \right|^2 dx
\]

\[
= \int_{\Omega} \frac{1}{r^2} \left[ \left( \frac{\bar{v}_r}{r} \right)_{,r} - \frac{\bar{v}_r}{r^2} \right]^2 dx \leq c \left[ \int_{\Omega} \frac{1}{r^2} \left( \frac{\bar{v}_r}{r} \right)^2 dx + \int_{\Omega} \frac{\bar{v}_{r,r}^2}{r^6} dx \right].
\]

Therefore, in view of (6.23), we obtain

\[
J_1 \leq cI_1 + c \int_{\Omega} \frac{1}{r^2} \left( \frac{\bar{v}_r}{r} \right)^2 dx + c \int_{\Omega} \frac{\bar{v}_{r,r}^2}{r^6} dx \equiv J_2.
\]

To estimate \( J_2 \) we see that (6.22) implies

\[
\int_{\Omega} \frac{\eta_{,rr}^2}{r^2} dx \leq cI_1
\]

so

\[
\int_{\Omega} \frac{1}{r^2} \left( \frac{\bar{v}_r}{r} \right)^2 dx \leq cI_1.
\]

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Moreover, by the Hardy inequality (2.18) for $p = 2$, we have
\[
\int_{\Omega} \frac{\tilde{v}_r^2}{r^6} \, dx = \int_{\Omega} \frac{1}{r^4} \left| \frac{\tilde{v}_r}{r} \right|^2 \, dx \leq \int_{\Omega} \frac{1}{r^2} \left( \frac{\tilde{v}_r}{r} \right)^2 \, dx \leq c I_1.
\]
Using the above estimate in $J_2$ we obtain
\[
(6.25) \quad \int_{\Omega} \frac{\tilde{v}_{rr}^2}{r^2} \, dx \leq c I_1.
\]

Differentiating (6.21) twice with respect to $z$, multiplying the result by $\eta_{zz}$ and integrating over $\Omega$ we obtain
\[
\int_{\Omega} |\nabla \eta_{zz}|^2 \, dx - \int_{\Omega} (\eta_{zz}^2)_{,r} \, drrdz = \int_{\Omega} \partial_z \eta_{zz} \, dx,
\]
where the integration by parts was performed.
Continuing, we have
\[
\int_{\Omega} |\nabla \eta_{zz}|^2 \, dx + \int_{-a}^{a} \eta_{zz}^2 \, dz = \int_{\Omega} \eta_{zz}^2 \, dx \equiv c I_2.
\]
Since
\[
\eta_{zz} = \left( \frac{\tilde{v}_r}{r} \right)_{zz} = \frac{\tilde{v}_{rzz}}{r},
\]
we obtain
\[
(6.26) \quad \int_{\Omega} \frac{\tilde{v}_{rzz}^2}{r^2} \, dx \leq c I_2.
\]

We differentiate (6.21) twice with respect to $r$, multiply the result by $r^2 \eta_{,rr}$ and integrate over $\Omega$. Then we obtain
\[
(6.27) \quad \int_{\Omega} (\Delta \eta)_{,rr} r^2 \eta_{rr} \, dx + 2 \int_{\Omega} \left( \frac{\eta_{,r}}{r} \right)_{,rr} \, dx = \int_{\Omega} \partial_{,rr} r^2 \eta_{rr} \, dx.
\]
Using
\[
\Delta \eta = \eta_{rr} + \eta_{zz} + \frac{\eta_{,r}}{r}
\]
we have
\[
(\Delta \eta)_{,rr} = \eta_{rrrr} + \eta_{rrzz} + \frac{\eta_{rrrr}}{r} - 2 \frac{\eta_{rr}}{r^2} + 2 \frac{\eta_{,r}}{r^3}
\]
\[
= \Delta \eta_{,rr} - \frac{2 \eta_{rr}}{r} + 2 \frac{\eta_{,r}}{r^3}.
\]
Then (6.27) takes the form

\[
\int_{\Omega} r^2 \Delta \eta_{rrr} \eta_{rrr} dx - 6 \int_{\Omega} \eta_{rr}^2 dx + 6 \int_{\Omega} \frac{\eta_{r} \eta_{rr}}{r} dx + 2 \int_{\Omega} \eta_{rrr} \eta_{rrr} r dx = \int_{\Omega} \vartheta_{r} \eta_{rrr}^2 r dx.
\]

(6.28)

Integrating by parts in the first term on the l.h.s. we obtain

\[
\int_{\Omega} r^2 \Delta \eta_{rrr} \eta_{rrr} dx = - \int_{\Omega} \nabla \eta_{rrr} \cdot \nabla (r^2 \eta_{rrr}) dx
\]

\[= - \int_{\Omega} |\nabla \eta_{rrr}|^2 r^2 dx - 2 \int_{\Omega} \eta_{rrr} \eta_{rrr} r dx.
\]

Next,

\[
6 \int_{\Omega} \frac{\eta_{r} \eta_{rr}}{r} dx = 3 \int_{\Omega} (\eta_{r}^2)_r drdz = -3 \int_{-a}^{a} \eta_{r}^2 |_{r=0} dz,
\]

\[
\int_{\Omega} \vartheta_{r} r^2 \eta_{rrr} dx = \int_{\Omega} \vartheta_{r} \eta_{rrr} r^3 drdz = - \int_{\Omega} \vartheta_{r} (r^3 \eta_{rrr})_r drdz.
\]

In view of the above results equation (6.28) takes the form

\[
\int_{\Omega} |\nabla \eta_{rrr}|^2 r^2 dx + 6 \int_{\Omega} \eta_{rrr}^2 dx + 3 \int_{-a}^{a} \eta_{r}^2 |_{r=0} dz + \int_{-a}^{a} \eta_{r}^2 r^2 |_{r=0} dz
\]

(6.29)

\[= \int_{\Omega} \vartheta_{r} (r^3 \eta_{rrr} + 3r^2 \eta_{rrr}) drdz.
\]

Applying the Hölder and the Young inequalities in the r.h.s. of (6.29) and using that functions \( \vartheta \) and \( \eta \) have compact supports with respect to \( r \), so \( \vartheta \) and \( \eta \) vanish for \( r \geq 2r_0 \), we obtain

\[
\int_{\Omega} |\nabla \eta_{rrr}|^2 r^2 dx + \int_{\Omega} \eta_{rrr}^2 dx \leq c(r_0) \int_{\Omega} \vartheta_{r}^2 dx.
\]

(6.30)
Since $\eta = \frac{\tilde{\psi}}{r}$, (6.30) implies

$$\int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{\psi}}{r^2} \right) \right|^2 r^2 \, dx \leq c I_1.$$  

Continuing, we see that

$$\int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{\psi}}{r^2} \right) \right|^2 r^2 \, dx = \int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{\psi}_r}{r} - 2 \frac{\tilde{\psi}}{r^3} \right) \right|^2 r^2 \, dx$$

$$= \int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{v}_z}{r} + \frac{\psi \zeta_1}{r} - 2 \frac{\tilde{\psi}}{r^3} \right) \right|^2 r^2 \, dx \leq c I_1.$$  

Then

(6.31)

$$\int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{v}_z}{r} \right) \right|^2 r^2 \, dx \leq c \int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{\psi}}{r^3} \right) \right|^2 r^2 \, dx + c \int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\psi \zeta_1}{r^3} \right) \right|^2 r^2 \, dx + c I_1.$$  

To examine the first integral on the r.h.s. of (6.31) we recall that $\tilde{\psi} = \eta^2 r$.

Then

$$\int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{\psi}}{r^3} \right) \right|^2 r^2 \, dx = \int_{\Omega_\varepsilon} \left| \nabla \left( \eta r \right) \right|^2 r^2 \, dx = \int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\eta r - \eta^2}{r^2} \right) \right|^2 r^2 \, dx$$

$$\leq c \int_{\Omega_\varepsilon} \left( \left| \nabla \eta \right|^2 + \left| \nabla \left( \frac{\eta}{r^2} \right) \right|^2 \right) r^2 \, dx$$

$$\leq c \int_{\Omega_\varepsilon} \left| \nabla \eta \right|^2 dx + c \int_{\Omega_\varepsilon} \left( \frac{\eta^2}{r^2} + \frac{\left| \nabla \eta \right|^2}{r^2} + \frac{\eta^2}{r^4} \right) dx \equiv K_1.$$  

In view of (6.22) we obtain

$$K_1 \leq c I_1 + c \int_{\Omega_\varepsilon} \left( \frac{\eta^2}{r^2} + \frac{\eta^2}{r^4} \right) dx.$$  

since the support of $\eta$ with respect to $r$ is compact, because $\eta$ vanishes for $r \geq 2r_0$, we have

$$\int_{\Omega_\varepsilon} \frac{\eta^2}{r^2} dx \leq c(r_0) \int_{\Omega_\varepsilon} \frac{\eta^2}{r^2 + 2\delta} dx \leq c(r_0) \int_{\Omega_\varepsilon} \frac{\eta^2}{r^2} dx$$

$$\leq c(r_0) \int_{\Omega_\varepsilon} \frac{\eta^2}{r^2} dx \leq c(r_0) I_1.$$  

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where $\delta \in (0, 1)$.

By the Hardy inequality (2.18) for $p = 2$ with $\eta, \xi = 0$ for $r \leq \varepsilon$ we obtain

$$\int_{\Omega_{\varepsilon}} \frac{\eta^2}{r^4} dx \leq c \int_{\Omega_{\varepsilon}} \frac{\eta^2}{r^2} dx \leq cI_1.$$ 

Hence,

$$K_1 \leq cI_1$$

and (6.31) assumes the form

(6.32) \[ \int_{\Omega_{\varepsilon}} \left| \nabla \left( \frac{\tilde{v}_z}{r} \right) \right|^2 r^2 dx \leq cI_1 + c \int_{\Omega_{\varepsilon}} \left| \frac{\psi\zeta_1^2}{r^2} \right|^2 r^2 dx. \]

Since

$$\nabla \left( \frac{\tilde{v}_z}{r} \right) = \nabla \frac{\tilde{v}_z}{r} - \frac{\tilde{v}_z}{r^2} - \frac{\nabla \tilde{v}_z}{r^2} + 2 \frac{\tilde{v}_z}{r^3} \text{ in } \Omega_{\varepsilon}$$

we obtain from (6.32) the inequality

$$\int_{\Omega_{\varepsilon}} \left| \nabla \tilde{v}_{z,r} \right|^2 r^2 dx \leq cI_1 + c \int_{\Omega_{\varepsilon}} \left( \left| \frac{\tilde{v}_{z,r}}{r^2} \right|^2 + \left| \frac{\tilde{v}_{z,z}}{r^2} \right|^2 + \frac{\tilde{v}_z^2}{r^6} \right) r^2 dx$$

$$+ c \int_{\Omega_{\varepsilon}} \left| \nabla \frac{\psi\zeta_1^2}{r^2} \right|^2 r^2 dx$$

(6.33) \[ \leq cI_1 + c \int_{\Omega_{\varepsilon}} \left( \frac{\tilde{v}_z^2}{r^2} + \frac{\tilde{v}_{z,z}^2}{r^2} + \frac{\tilde{v}_z^2}{r^4} \right) dx + c \int_{\Omega_{\varepsilon}} \left| \nabla \left( \frac{\psi\zeta_1^2}{r^2} \right) \right|^2 r^2 dx \]

Now we shall estimate the middle integral on the r.h.s. of (6.33). Since $v_z = -\frac{\psi}{r}$ and $\tilde{\psi} = \eta r^2$ we have that

$$\tilde{v}_{z,z} = -\eta_{z,z} r - 2\eta_{z,r} + \frac{\psi\zeta_1^2}{r}$$

$$\tilde{v}_z = -\eta_{z,r} r - 2\eta + \frac{\psi r}{r} \zeta_1^2, \quad v_{z,r} = -\eta_{z,r} r - 3\eta_{z,r} + \left( \frac{\psi \zeta_1^2}{r} \right)_{,r}.$$
Then, in view of (6.22), we have
\[
\int_{\Omega_\varepsilon} \frac{\tilde{v}^2_{z,r}}{r^2} \, dx \leq c \int_{\Omega_\varepsilon} \left( \frac{\eta^2}{r^2} + \frac{\eta^2_{,rr}}{r^2} + \frac{1}{r^2} \left( \frac{\psi}{r \zeta_{1,r}} \right)^2 \right) \, dx
\]
\[
\leq c I_1 + c \left( \frac{1}{r_0} \right) \int_{\Omega_\varepsilon} \left( \frac{\psi^2}{r \zeta_{1,r}} \right)^2 \, dx,
\]
\[
\int_{\Omega_\varepsilon} \frac{\tilde{v}^2_{z,z}}{r^2} \, dx \leq c \int_{\Omega_\varepsilon} \left( \eta^2_{,rz} + \eta^2_{,r^2} + v^2_{r} \zeta^2_{1,r} \right) \, dx \leq c I_1 + c(1/r_0) \int_{\Omega_\varepsilon} v^2_r \, dx,
\]
\[
\int_{\Omega_\varepsilon} \frac{\tilde{v}^2_{z}}{r^4} \, dx \leq c \int_{\Omega_\varepsilon} \left( \frac{\eta^2_r}{r^2} + \frac{\eta^2_{,r^2}}{r^4} \right) \, dx + c(1/r_0) \int_{\Omega_\varepsilon} \psi^2 \, dx.
\]

In view of (6.22) and the Hardy inequality we have
\[
\int_{\Omega_\varepsilon} \left( \frac{\eta^2_r}{r^2} + \frac{\eta^2_{,r^2}}{r^4} \right) \, dx \leq c I_1.
\]

Since \( v_r|_{r=0} = 0 \) we have that \( \psi_z|_{r=0} = 0 \), so \( \psi|_{r=0} = 0 \) can be chosen. Then
\[
\psi = \int_0^r \psi_{,r} \, dr = \int_0^r \frac{\psi_r}{r} \, r \, dr
\]
and
\[
(6.34) \quad \int_{\Omega} \psi^2 \, dx \leq c \int_{\Omega} v^2_z \, dx.
\]

Moreover,
\[
\int_{\Omega_\varepsilon} \left( \frac{\psi}{r} \zeta_{1,r}^2 \right)^2 \, dx \leq c(1/r_0) \left( \int_{\Omega_\varepsilon} \frac{\psi^2}{r^2} \, dx + \int_{\Omega} \psi^2 \, dx \right) \leq c(1/r_0) \int_{\Omega} v^2_z \, dx.
\]

Finally, the last integral on the r.h.s. of (6.33) can be estimated by
\[
c \left[ \int_{\Omega_\varepsilon} \left| \frac{\psi r \zeta_{1,r}^2}{r} \right|^2 \right] \, dx + \int_{\Omega_\varepsilon} \left| \frac{\psi z \zeta_{1,r}^2}{r} \right|^2 \, dx + \int_{\Omega_\varepsilon} \left| \psi \left( \frac{\zeta_{1,r}^2}{r^2} \right) \right|^2 \, dx
\]
\[
\leq c(r_0) \left[ \int_{\Omega} (v^2_r + v^2_z) \, dx + \int_{\Omega} \psi^2 \, dx \right] \leq c(1/r_0) \int_{\Omega} (v^2_r + v^2_z) \, dx.
\]
Therefore, (6.33) implies

\[(6.35) \quad \int_{\Omega_\varepsilon} |\nabla \tilde{v}_{z,r}|^2 dx \leq c I_1 + c(1/r_0) \int_{\Omega} (v_r^2 + v_z^2) dx.\]

From (6.25), (6.26) and (6.35) after passing with \(\varepsilon\) to 0 we have that

\[(6.36) \quad \|\tilde{v}'\|_{L^2(0,T;H^2(\Omega))} \leq c A_0,\]

where

\[A_0 = \varphi \left( d_1, d_2, \frac{1}{r_0} \right) \left( 1 + \left\| \frac{\tilde{v}_r^2(0)}{r_0} \right\|_{L^2(\Omega)} + \left\| \frac{\tilde{x}(0)}{r_0} \right\|_{L^2(\Omega)} \right)\]

Similarly, we have

\[(6.37) \quad \|\tilde{v}'\|_{L^\infty(0,T;H^1(\Omega))} \leq c A_0.\]

This concludes the proof of Lemma 6.4.

Finally, we need

**Lemma 6.5.** Assume that \(\tilde{\chi}_r \in V_2^0(\Omega_\varepsilon^t), \tilde{v}' \in V_2^0(\Omega_\varepsilon^t), \tilde{v}' = (v_r, v_z).\) Then

\[(6.38) \quad \|\tilde{v}/r\|_{V_2^1(\Omega_\varepsilon^t)} \leq c (\|\tilde{x}/r\|_{V_2^0(\Omega_\varepsilon^t)} + (1/r_0)\|v'\|_{V_2^0(\Omega_\varepsilon^t)}),\]

where \(t \leq T.\)

**Proof.** From (6.23) we have

\[\int_{\Omega_\varepsilon} \left| \nabla \left( \frac{\tilde{v}_r}{r} \right) \right|^2 dx \leq c \int_{\Omega_\varepsilon} (\vartheta^2 + \vartheta_{r,r}^2) dx.\]

In view of definition of \(\vartheta\) (see (4.6) and (4.7)) we derive

\[\int_{\Omega_\varepsilon} (\vartheta^2 + \vartheta_{r,r}^2) dx \leq c(1/r_0) \int_{\Omega_\varepsilon} (v_z^2 + v_{z,r}^2) dx + c \int_{\Omega_\varepsilon} \left( \left| \frac{\tilde{x}}{r} \right|^2 + \left| \left( \frac{\tilde{x}}{r} \right)_{r,r} \right|^2 \right) dx.\]

From (6.26) we obtain

\[\int_{\Omega_\varepsilon} \frac{\tilde{v}_{r,z}^2}{r_0^2} dx \leq c \int_{\Omega_\varepsilon} \vartheta_{z,z}^2 dx.\]
where
\[ \int_{\Omega_e} \vartheta^2 \, dx \leq \int_{\Omega_e} \left( \frac{\tilde{x}}{r} \right)^2 \, dx + c(1/r_0) \int_{\Omega_e} (\tilde{v}_r^2 + v_{z,z}^2) \, dx. \]

In view of the above inequalities we obtain
\[
\int_{\Omega_e} \left| \nabla^2 \left( \frac{\tilde{v}_r}{r} \right) \right|^2 \, dx \leq c \int_{\Omega_e} \left( \left| \frac{\tilde{x}}{r} \right|^2 + \left| \nabla \frac{\tilde{x}}{r} \right|^2 \right) \, dx
\]
\[ + c(1/r_0) \int_{\Omega_e} (\tilde{v}_r^2 + v_{z}^2 + |\nabla v_z|^2) \, dx. \]

Inequality (4.18) yields
\[
\int_{\Omega_e} |\nabla \eta_{z,z}|^2 \, dx \leq \int_{\Omega_e} \vartheta^2 \, dx,
\]
where the l.h.s. of (6.40) equals
\[
\int_{\Omega_e} \left| \nabla \frac{\psi_{z,z}}{r^2} \right|^2 \, dx = \int_{\Omega_e} \left| \nabla \frac{\tilde{v}_r}{r} \right|^2 \, dx
\]
and the r.h.s. of (6.40) is estimated by
\[
\int_{\Omega_e} \vartheta^2 \, dx \leq \int_{\Omega_e} \left| \frac{\tilde{x}}{r} \right|^2 \, dx + c(1/r_0) \int_{\Omega_e} v_{z}^2 \, dx,
\]
where (6.34) was used.
Hence, (6.40) takes the form
\[
\int_{\Omega_e} \left| \nabla \frac{\tilde{v}_r}{r} \right|^2 \, dx \leq c \int_{\Omega_e} \left| \frac{\tilde{x}}{r} \right|^2 \, dx + c(1/r_0) \int_{\Omega_e} v_{z}^2 \, dx.
\]

Adding the integral with respect to time of (6.39) to \( L_\infty \)-norm with respect to time of (6.41) implies (6.38). This concludes the proof.
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