TITLE:
Fluctuation scaling limits for positive recurrent jumping-in diffusions with small jumps

AUTHOR(S):
Yamato, Kosuke; Yano, Kouji

CITATION:
Yamato, Kosuke ...[et al]. Fluctuation scaling limits for positive recurrent jumping-in diffusions with small jumps. Journal of Functional Analysis 2020, 279(7): 108655.

ISSUE DATE:
2020-10

URL:
http://hdl.handle.net/2433/268003

RIGHT:
© 2020. This manuscript version is made available under the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International license.; The full-text file will be made open to the public on 15 October 2022 in accordance with publisher’s ‘Terms and Conditions for Self-Archiving’; This is not the published version. Please cite only the published version. この論文は出版版ではありません。引用の際には出版版をご確認ご利用ください。
Fluctuation scaling limits for positive recurrent jumping-in diffusions with small jumps

Kosuke Yamato and Kouji Yano

Abstract

For positive recurrent jumping-in diffusions with small jumps, we establish distributional limits of the fluctuations of inverse local times and occupation times on the half line. For this purpose, we introduce and utilize eigenfunctions with modified Neumann boundary condition and apply the Krein-Kotani correspondence.

1 Introduction

We study a strong Markov process $X$ on the half line $[0, \infty)$ (resp. the real line $\mathbb{R}$) which has continuous paths up to the first hitting time of 0 and, as soon as $X$ hits 0, $X$ jumps into the interior $(0, \infty)$ (resp. $\mathbb{R} \setminus \{0\}$) and starts afresh. We call such a process $X$ a unilateral (resp. bilateral) jumping-in diffusion.

Let us consider the inverse local time $\eta$ at 0 of a jumping-in diffusion $X$, especially in the positive recurrent case where the inverse local time $\eta$ has a degenerate scaling limit:

$$\frac{1}{t} \eta(t) \xrightarrow{t \to \infty} b \in [0, \infty).$$  \hfill (1.1)

One of our two main aims is to establish its fluctuation scaling limit of the inverse local time $\eta$ of the form:

$$f(\gamma) \left( \frac{\eta(\gamma t)}{\gamma} - bt \right) \xrightarrow[\gamma \to \infty]{d} S(t) \quad \text{on } \mathbb{D}$$  \hfill (1.2)

for some function $f(\gamma)$ which diverges to $\infty$ as $\gamma \to \infty$ and some stable process $S(t)$ without negative jumps. Here $\mathbb{D}$ denotes the space of càdlàg paths from $[0, \infty)$ to $\mathbb{R}$ equipped with Skorokhod’s $J_1$-topology.

The other one is the fluctuation scaling limit of the occupation time on the half line of a bilateral jumping-in diffusion $X$:

$$A(t) = \int_0^t 1_{(0, \infty)}(X_s) ds.$$  \hfill (1.3)

In the positive recurrent case where $A(t)$ has a degenerate mean:

$$\frac{1}{t} A(t) \xrightarrow{t \to \infty} p \in (0, 1),$$  \hfill (1.4)
we establish the fluctuation scaling limits of the form:

\[ f(\gamma) \left( \frac{A(\gamma t)}{\gamma} - pt \right) \xrightarrow{\gamma \to \infty} Z(t) \]  

for a function \( f(\gamma) \) in (1.2) and some limit process \( Z(t) \). Here \( \xrightarrow{f.d.} \) denotes the convergence of finite-dimensional distributions.

**1.1 Main results**

By Feller [3] and Itô [6], it has shown that under the natural scale, a unilateral jumping-in diffusion can be characterized by a pair \((m,j)\) of a speed measure \(m\) and a jumping-in measure \(j\), both of which are Radon measures on \((0,\infty)\). The jumping-in diffusion is a strong Markov process on \([0,\infty)\) which behaves as a \(d^+dx\)-diffusion during staying in \((0,\infty)\) and jumps from the origin to \((0,\infty)\) according to \(j\), where \(d^+dx\) denotes the right-differentiation operator. We denote the jumping-in diffusion by \(X_{m,j}\) and its inverse local time at 0 by \(\eta_{m,j}\). For the precise description, see Section 5.

The following theorem gives the fluctuation scaling limit of the inverse local times of unilateral jumping-in diffusions with small jumps.

**Theorem 1.1.** Let \(\alpha \in (1,2)\). Assume \(X = X_{m,j}\) exists. Suppose the following hold:

(i) \(\int_0^\infty m(x,\infty)^2dx < \infty\),

(ii) \(m(x,\infty) \sim (\alpha - 1)^{-1}x^{1/\alpha - 1}K(x) \quad (x \to \infty)\) for a slowly varying function \(K\) at \(\infty\),

(iii) (Small jump condition) \(\kappa := \int_0^\infty xj(dx) < \infty\).

Then we have

\[ f(\gamma) \left( \frac{\eta_{m,j}(\gamma t)}{\gamma} - bt \right) \xrightarrow{\gamma \to \infty} S^{(\alpha)}(\kappa t) \quad \text{on} \ \mathbb{D} \ \text{with} \ f(\gamma) = \frac{1}{\gamma^{1/\alpha - 1}K(\gamma)}, \]  

where \(S^{(\alpha)}\) is a spectrally positive strictly \(\alpha\)-stable process whose Laplace exponent is

\[ \mathbb{E}[e^{-\lambda S^{(\alpha)}(t)}] = e^{-t\chi(\lambda)}, \quad \chi(\lambda) = -\frac{\Gamma(2 - \alpha)}{\Gamma(\alpha)} \frac{\alpha^{\alpha - 1}}{\alpha - 1} \lambda^{\alpha} \quad (\lambda > 0), \]  

and

\[ b = \int_0^\infty j(dx) \int_x^\infty m(y,\infty)dy. \]  

Theorem 1.1 will be proven in Section 6.

Note that a necessary and sufficient condition for the existence of \(X_{m,j}\) will be recalled at the beginning of Section 5.
The following theorem gives the fluctuation scaling limit of the occupation times on the half line of bilateral jumping-in diffusions with small jumps. As we will see in Section 7, a bilateral jumping-in diffusion can be characterized by two pairs \((m_+, j_+)\) and \((m_-, j_-)\), and thus we denote the corresponding process by \(X_{m_+, j_+; m_-, j_-}\).

**Theorem 1.2.** Let \(\alpha \in (1, 2)\). Assume \(X = X_{m_+, j_+; m_-, j_-}\) exists. Suppose the following hold:

(i) \(\int_{0^+} m_\pm(x, \infty)^2 dx < \infty\),

(ii) \(m_\pm(x, \infty) \sim w_\pm(\alpha - 1)^{-1} x^{1/\alpha - 1} K(x)\) \(x \to \infty\) for constants \(\alpha \in (1, 2)\) and \(w_\pm > 0\) and a slowly varying function \(K\) at \(\infty\),

(iii) \(\kappa_\pm := \int_0^{\infty} x j_\pm(dx) < \infty\).

Then we have

\[
f(\gamma) \left( \frac{A(\gamma t)}{\gamma} - pt \right) \xrightarrow{f.d.} (1-p)w_+ S^{(\alpha)}(\kappa_+ t) - pw_- \tilde{S}^{(\alpha)}(\kappa_- t)
\]

with \(f(\gamma) = \frac{1}{\gamma^{1/\alpha - 1} K(\gamma)}\), where

\[
b_\pm = \int_0^{\infty} j_\pm(dx) \int_0^x m_\pm(y, \infty) dy, \quad p = \frac{b_+}{b_+ + b_-}, \quad \tilde{\kappa}_\pm = \frac{\kappa_+}{b_+ + b_-},
\]

and, \(S^{(\alpha)}\) and \(\tilde{S}^{(\alpha)}\) are i.i.d. \(\alpha\)-stable processes characterized by (1.7).

Theorem 1.2 will be proven in Section 7.

**Example 1.3.** Let \(1 < \alpha < 2\) and \(0 < \beta < 1/\alpha\). Set

\[
m_\pm(x, \infty) = w_\pm(\alpha - 1)^{-1} (c + x)^{1/\alpha - 1}
\]

and

\[
j_\pm(dx) = \kappa_\pm(1 - \beta)x^{-\beta-1}1_{(0,1)}(x) dx
\]

for \(c \geq 0, w_\pm > 0\) and \(\kappa_\pm > 0\). Then we have the fluctuation limit (1.9) with \(f(\gamma) = \frac{1}{\gamma^{1/\alpha - 1}}\) and \(\int_0^{\infty} x j_\pm(dx) = \kappa_\pm\), where \(I_c = \int_0^{1} x^{-\beta-1} \{ (x + c)^{1/\alpha} - c^{1/\alpha} \} dx\) and

\[
b_\pm = I_c \alpha (1 - \beta) \kappa_\pm w_\pm, \quad p = \frac{\kappa_+ w_+}{\kappa_+ w_+ + \kappa_- w_-}, \quad \tilde{\kappa}_\pm = \frac{1}{I_c \alpha (1 - \beta)} \frac{\kappa_+}{\kappa_+ w_+ + \kappa_- w_-}.
\]
1.2 Background of our study

Here we explain the background of our study, briefly reviewing preceding works.

First, we recall the studies on the scaling limit of inverse local times for diffusions and jumping-in diffusions. For $\delta \in (0, 2)$, let us consider the reflecting Bessel diffusion $\tilde{X}^\delta$ of dimension $\delta$ starting from 0, which is a one-dimensional diffusion on $[0, \infty)$ whose local generator $L$ on $(0, \infty)$ is given by

$$\tilde{L}^\delta = \frac{1}{2} \frac{d^2}{dx^2} + \frac{\delta - 1}{2x} \frac{d}{dx}. \quad (1.14)$$

The scaled process $X^{(\alpha)} := \frac{1}{(2^\alpha)^2} (\tilde{X}^\delta)^{2\alpha}$ with $\alpha = 1 - \delta/2 \in (0, 1)$ has its local generator

$$L^{(\alpha)} = \frac{d}{dm^{(\alpha)}} \frac{d}{dx} \quad \text{with} \quad m^{(\alpha)}(0, x] = (1 - \alpha)^{-1} x^{1/\alpha - 1}. \quad (1.15)$$

It is well-known that the inverse local time at 0 of $X^{(\alpha)}$ is an $\alpha$-stable subordinator $S^{(\alpha)}$ characterized by (1.7):

$$E[e^{-\lambda S^{(\alpha)}(t)}] = e^{-t \chi(\lambda)}, \quad \chi(\lambda) = \frac{\Gamma(2 - \alpha)}{\Gamma(\alpha)} \frac{\alpha^{\alpha - 1}}{1 - \alpha} \lambda^\alpha \quad (\lambda > 0), \quad (1.16)$$

Stone [15] showed that, if a natural scale reflecting diffusion $X$ on $[0, \infty)$ has its speed measure $m$ satisfying

$$m(0, x] \sim (1 - \alpha)^{-1} x^{1/\alpha - 1} K(x) \quad (x \to \infty) \quad (1.17)$$

for $\alpha \in (0, 1)$ and a slowly varying function $K$ at $\infty$, then the following scaling limits hold:

$$\frac{1}{\gamma} X^{(\alpha)}(\gamma^{1/\alpha} K(\gamma)t) \xrightarrow{d_{\gamma \to \infty}} X^{(\alpha)}(t) \quad \text{on } \mathbb{D}, \quad (1.18)$$

$$\frac{1}{\gamma^{1/\alpha} K(\gamma)} \eta^{(\gamma)}(\gamma t) \xrightarrow{d_{\gamma \to \infty}} S^{(\alpha)}(t) \quad \text{on } \mathbb{D}, \quad (1.19)$$

where $\eta$ denotes the inverse local time at 0 of $X$. His method of the proof is to reduce the scaling limits to the continuity of $X$ and $\eta$ with respect to $m$. Yano [18, Thorem 2.6] has generalized this result to jumping-in diffusions with small jumps.

Our motivation is to establish similar scaling limits for $1 < \alpha < 2$, in which case (1.18) and (1.19) fail. In Theorem 1.1, the scaling limit of speed measure is

$$\lim_{\gamma \to \infty} \frac{m^{(\alpha)}(\gamma x, \infty)}{m(\gamma, \infty)} = m^{(\alpha)}(x, \infty) := (\alpha - 1)^{-1} x^{1/\alpha - 1}. \quad (1.20)$$

Note that the boundary 0 for $L^{(\alpha)}$ ($1 < \alpha < 2$) is exit in the sense of Feller. The corresponding diffusion $X^{(\alpha)}$ is absorbed at 0 in finite time. Nevertheless we have the
compensated inverse local time for $X^{(\alpha)}$ which has been introduced by Kasahara and Watanabe [10]. In this paper, we study a fluctuation scaling limit for jumping-in diffusions with small jumps. We will show a continuity of inverse local times as Theorem 5.1 with respect to the pair of their speed measures and jumping-in measures when jumping-in measures concentrate to the origin. We will apply the continuity theorem to show Theorem 1.1 and 1.2, which reveal how the asymptotic behavior of speed measure and the moment of the jumping-in measure affect the limit process quantitatively.

As a preceding study to consider the scaling limits for $\alpha \geq 1$, there is a study by Kasahara and Watanabe [10]. They extended the continuity shown by Stone [15] for a more general class of speed measures. In the unilateral case, the speed measure $m$ comes from a string $m$, i.e. $m : (0, \infty) \to \mathbb{R}$ is a non-decreasing, right-continuous function: $m(a, b] = m(b) - m(a)$. For a string $m$ with

$$\int_{0^+} m(x)^2 dx < \infty, \quad (1.21)$$

they constructed a compensated inverse local time $T_m$ at 0 of a $d \frac{d^+}{dm} d\gamma$-diffusion. Although in Kasahara and Watanabe [10], the process $T_m$ is defined by a stochastic integral, it is also represented by the following limit:

$$T_m(t) = \lim_{n \to \infty} \int_{\ell_n}^{\infty} \ell(\ell^{-1}(t, 0), x) dm(x) + m(\epsilon_n) t, \quad (1.22)$$

where $\{\epsilon_n\}_{n}$ is a sequence of positive numbers converging to 0, $\ell$ is a local time of a standard Brownian motion and $\ell^{-1}(t, 0)$ is the right-continuous inverse of $\ell(t, 0)$. The limit in RHS is taken in the sense of convergence in distribution. Note that for a strictly increasing string $m$ with a regular boundary at 0, the inverse local time $\eta$ at 0 of a reflecting $d \frac{d^+}{dm} d\gamma$-diffusion is given by

$$\eta(t) = \int_{0}^{\infty} \ell(\ell^{-1}(t, 0), x) dm(x). \quad (1.23)$$

The process $T_m$ appears as a limit process of our continuity result Theorem 5.1. They showed the continuity of $T_m$ with respect to $m$ with (1.21) and apply it to the scaling limit of $T_m$:

**Theorem 1.4.** (Kasahara and Watanabe [10, Theorem 3.3]) Let $m$ be a string and $\alpha \in (1, 2)$. Assume $m$ satisfies the assumptions (i) and (ii) of Theorem 1.1 for a slowly varying function $K$ at $\infty$. Then we have

$$f(\gamma) \left( \frac{T_m(\gamma t)}{\gamma} - m(\infty) t \right) \xrightarrow{\gamma \to \infty} S^{(\alpha)}(t) \quad \text{on } \mathbb{D} \quad (1.24)$$

with $f(\gamma) = \frac{1}{\gamma^{1/\alpha - 1} K(\gamma)}$, where $S^{(\alpha)}$ is given in (1.7).

Now we recall the previous studies on the occupation time on the half line. Barlow, Pitman and Yor [1] have considered a skew Bessel diffusion $X^{(\alpha, p)}$ of dimension $2 - 2\alpha$ ($\alpha \in$
with the skewness parameter $p \in (0, 1)$. The process $X = X^{(\alpha, p)}$ is a bilateral $\frac{d}{dm} \frac{d}{dx}$ diffusion whose speed measure $m = m^{(\alpha, p)}$ is given by

$$m^{(\alpha, p)}(dx) = \frac{1}{\alpha} (1 - p)|x|^{1/\alpha - 2} \mathbb{1}_{\{x < 0\}} dx + \frac{1}{\alpha} px^{1/\alpha - 2} \mathbb{1}_{\{x > 0\}} dx. \quad (1.25)$$

They showed that $\frac{1}{t} A(t) \frac{d}{t \to \infty} Y_{\alpha, p}$ follows Lamperti's generalized arcsine law $\mu_{\alpha, p}$, which is characterized by its Stieltjes transform:

$$\int_0^1 \frac{\mu_{\alpha, p}(dx)}{\lambda + x} = \frac{p(\lambda + 1)^{\alpha - 1} + (1 - p)\lambda^{\alpha - 1}}{p(\lambda + 1)^{\alpha} + (1 - p)\lambda^{\alpha}} \quad (\alpha \in [0, 1], \ p \in [0, 1], \ \lambda > 0). \quad (1.26)$$

Watanabe [16] has given a necessary and sufficient conditions for general bilateral $\frac{d}{dm} \frac{d}{dx}$-diffusions to satisfy the following:

$$\frac{1}{t} A(t) \frac{d}{t \to \infty} Y_{\alpha, p}. \quad (1.27)$$

where $Y_{\alpha, p}$ is a random variable whose law is $\mu_{\alpha, p}$. The condition is given by some regularly varying conditions on the tail of the speed measure When $\alpha = 1$, the limit degenerates, indeed, $Y_{1, p} = p$. In this case, Kasahara and Watanabe [10] studied the fluctuation of the convergence (1.27):

**Theorem 1.5.** (Kasahara and Watanabe [10, Theorem 4.1])

Let $\alpha \in (1, 2)$. Let $m_\pm$ be Radon measures on $(0, \infty)$ and assume the assumption (ii) of Theorem 1.2 holds for constants $w_\pm > 0$ and a slowly varying function $K$ at $\infty$. Assume, moreover, that $m_+(0, \infty) + m_-(0, \infty) < \infty$. Let $m$ be a Radon measure on $\mathbb{R}$ such that

$$m[0, x] = m_+(0, x) \text{ and } m[-x, 0) = m_-(0, x) \text{ for } x > 0. \quad (1.28)$$

Then we have

$$f(\gamma) \left( \frac{A(\gamma t)}{\gamma} - pt \right) \frac{f.d.}{\gamma \to \infty} \left( 1 - p \right) w_+ S^{(\alpha)} \left( \frac{t}{m(\mathbb{R})} \right) - pw_- \tilde{S}^{(\alpha)} \left( \frac{t}{m(\mathbb{R})} \right) \quad (1.29)$$

with $f(\gamma) = \frac{1}{\gamma^{1/\alpha - 1}} K(\gamma)$, where $p = \frac{m(0, \infty)}{m(\mathbb{R})}$ and $S^{(\alpha)}$ and $\tilde{S}^{(\alpha)}$ are i.i.d. processes characterized by (1.7).

**Example 1.6.** Let $1 < \alpha < 2$. Consider $m_\pm$ given in (1.11) with $c > 0$. Then we have the fluctuation limit (1.29) with $f(\gamma) = \frac{1}{\gamma^{1/\alpha - 1}} K(\gamma)$, where

$$m(\mathbb{R}) = (w_+ + w_-)(\alpha - 1)^{-1} c^{1/\alpha - 1} \text{ and } p = \frac{w_+}{w_+ + w_-}. \quad (1.30)$$

Note that, in contrast with Example 1.3, we cannot take $c = 0$ since $m_+(0, \infty) = \infty$ and a bilateral $\frac{d}{dm} \frac{d}{dx}$-diffusion does not exist.
1.3 Our strategy for the proofs

Our main tool is the Laplace exponent \( \chi_{m,j} \) of \( \eta_{m,j} \) with \( E[e^{-\lambda \eta_{m,j}(1)}] = e^{-\chi_{m,j}(\lambda)} \), which can be represented, as we will see in Section 5, as

\[
\chi_{m,j}(\lambda) = \int_0^\infty (1 - g_m(\lambda; x)) j(dx) \quad (\lambda > 0).
\] (1.31)

Here \( u = g_m(\lambda; \cdot) \) \((\lambda > 0)\) is a unique solution to the ODE

\[
\frac{d}{dm} \frac{d^+}{dx} u = \lambda u, \quad u(0) = 1, \quad \lim_{x \to \infty} u^+(x) = 0, \quad u: \text{non-increasing},
\] (1.32)

where \( u^+ = \frac{d^+}{dx} u \). To show the distributional convergence (1.2), it is enough to prove the pointwise convergence of Laplace exponents of Lévy processes

\[
\tilde{\eta}_{m,j,\gamma}(t) := f(\gamma) \left( \frac{\eta_{m,j}(\gamma t)}{\gamma} - bt \right) \quad (\gamma > 0)
\] (1.33)

as \( \gamma \to \infty \). By changing variables, the Laplace exponent \( \tilde{\chi}_{m,j,\gamma} \) of \( \tilde{\eta}_{m,j,\gamma} \) can be represented by

\[
\tilde{\chi}_{m,j,\gamma}(\lambda) = \gamma \int_0^\infty \left( 1 - g_m \left( \frac{f(\gamma)}{\gamma} \lambda; x \right) \right) j(dx) - bf(\gamma)\lambda
\] (1.34)

\[
= \gamma \int_0^\infty \left( 1 - g_m(\lambda; x) \right) j_\gamma(dx) - b_\gamma \lambda
\] (1.35)

\[
= \chi_{m,\gamma,j}(\lambda) - b_\gamma \lambda
\] (1.36)

for appropriate Radon measures \( m_\gamma \) and \( j_\gamma \) and a constant \( b_\gamma \). Therefore our problem is reduced to the continuity of the Laplace exponent \( \chi_{m,j} \) with respect to \( m \) and \( j \), which we will prove in Theorem 5.1. For the proof, we need to analyze the behavior of \( g_m \) around 0. To this end, we introduce the eigenfunction of an initial value problem at 0.

When the boundary 0 for \( dm \) is regular, we have a unique solution \( u = \psi_m(\lambda; \cdot) \) to

\[
\frac{d}{dm} \frac{d^+}{dx} u = \lambda u, \quad u(0) = 0, \quad u^+(0) = 1 \quad (\lambda \in \mathbb{R})
\] (1.37)

and a unique solution \( u = \varphi_m(\lambda; \cdot) \) to

\[
\frac{d}{dm} \frac{d^+}{dx} u = \lambda u, \quad u(0) = 1, \quad u^+(0) = 0 \quad (\lambda \in \mathbb{R}).
\] (1.38)

When the boundary 0 for \( dm \) is exit, we still have \( \psi_m \) but do not have \( \varphi_m \): the initial value problem (1.38) has no solution. We would like to introduce a counterpart for \( \varphi_m \). For a string \( m \) with

\[
\int_{0^+} m(x)^2 dx < \infty,
\] (1.39)
we will show in Section 3 that there exists an eigenfunction \( u = \varphi^1_m(\lambda; \cdot) \) \((\lambda \in \mathbb{R})\) of the differential equation \( \frac{d}{dm} \frac{d}{dx} u = \lambda u \) with the modified Neumann boundary condition at 0:

\[
    u(0) = 1, \quad \lim_{x \to 0^+} (u^+(x) - \lambda m(x, 1)) = 0. \tag{1.40}
\]

Then for a suitable constant \( c^1_m(\lambda) \), we will have the following representation:

\[
    g_m(\lambda; x) = \varphi^1_m(\lambda; x) - c^1_m(\lambda) \psi_m(\lambda; x) \quad (\lambda > 0) \tag{1.41}
\]

and, utilizing this, we will show the continuity.

In our argument, the results in Kotani [12] play a crucial role. He showed that the class of strings \( m \) satisfying \( \int_{0^+} m(x)^2 dx < \infty \) have one-to-one correspondence to a class of Herglotz functions, in Kotani [12] and Kasawara and Watanabe [10] it was shown that the correspondence is bi-continuous in a certain sense. See Section 2 for the detail. These results are an extension of the Krein correspondence which has been used in the studies of one-dimensional diffusions (see e.g. Kotani and Watanabe [13] or Kasahara [7]). Applying his result, we can obtain the explicit form of the coefficient \( c^1_m(\lambda) \) in (1.41) and its continuous dependence on \( m \).

We show Theorem 1.2 as an application of Theorem 1.1. The key to the proof is to establish the tail behavior of the Lévy measure of the inverse local time, which is obtained by applying Tauberian theorems to Theorem 1.1. We then appeal to the Itô excursion theory which connects the occupation time on the half line with the inverse local time.

Let us remark on the non-degenerate case, that is, \( \frac{1}{t} A(t) \) converges in law to a non-degenerate distribution. This case falls down to Lamperti’s generalized arcsine law, which has been thoroughly studied in Watanabe [16] in the context of one-dimensional diffusions. He gave a necessary and sufficient condition for the convergence by the methods of double Laplace transforms and Williams formula via the excursion theory. His methods are still valid in our situation. See Appendix B.

### 1.4 Outline of the paper

The remainder of the present paper is organized as follows: In Section 2, we briefly review some results on Feller’s classification of boundary and the Krein-Kotani correspondence. In Section 3, we construct the function \( \varphi^1_m \) when the boundary 0 is exit and establish some elementary estimates for them. In Section 4, we represent \( g_m \) as a linear combination of \( \varphi^1_m \) and \( \psi_m \) and, determine the coefficient. In Section 5, we show a continuity of inverse local times with respect to their speed measures and jumping-in measures. In Section 6, we study the fluctuation scaling limit of inverse local times of jumping-in diffusions. In Section 7, we discuss the fluctuation of the occupation time on the half line of bilateral jumping-in diffusions. In appendix A, we show a continuity theorem for Laplace transforms of random variables which may have negative values. In appendix B, we treat the case \( \frac{1}{t} A(t) \) converges to non-degenerate distribution.
Acknowledgements We would like to thank Shinichi Kotani, who read an early draft of this paper and gave us valuable comments. We also would like to thank Yuji Kasahara. His advice improved Appendix A. The research of Kouji Yano was supported by JSPS KAKENHI Grant Number JP19H01791, JP19K21834 and JP18K03441.

2 The Krein-Kotani correspondence

We consider the state interval \((a, \infty)\) for \(a = 0\) or \(-\infty\). Let \(dm\) be a Radon measure on \((a, \infty)\) with full support and let \(s\) be a strictly increasing continuous function \((a, \infty)\). We define

\[
I = \int_a^1 ds(y) \int_y^\infty dm(z), \quad J = \int_a^1 dm(y) \int_y^\infty ds(z). \tag{2.1}
\]

Feller’s classification of the boundary \(a\) is as follows:

\[
\begin{array}{c|cc}
J < \infty & I < \infty & I = \infty \\
J = \infty & \text{regular} & \text{exit} \\
\end{array} \tag{2.2}
\]

It is well-known that for such \(m\) and \(s\), there is a diffusion \(X\) on \((a, \infty)\) or \([-\infty, \infty)\) whose local generator on \((a, \infty)\) is given as \(\frac{d}{dm} \frac{d}{ds}\). We call \(m\) the speed measure of \(X\) and \(s\) the scale function of \(X\). It is also well-known that if the boundary \(a\) is regular or exit, we have such a diffusion \(X\) and \(X\) hits or is killed at \(a\) in finite time. When the boundary \(a\) is regular, the boundary \(a\) can be reflecting, absorbing or elastic and, when \(a\) is exit, the boundary \(a\) for \(X\) is necessarily absorbing. When the boundary \(a\) is natural or entrance, such a diffusion \(X\) also exists and \(X\) does not approach \(a\) in finite time. For a diffusion \(X\) with the generator \(\frac{d}{dm} \frac{d}{ds}\), the process \(s(X)\) is also a diffusion and its generator is represented as \(\frac{d}{dm} \frac{d}{ds} \tilde m(x) = m(s^{-1}(x))\). We say that \(s(X)\) is the diffusion \(X\) under the natural scale. We may always assume natural scale without loss of generality.

Let us consider the state interval \((-\infty, \infty)\). We briefly summarize some results on the Krein-Kotani correspondence. See Kotani [12] for the details. A function \(w : (-\infty, \infty) \to [0, \infty]\) is called a string on \((-\infty, \infty)\) when \(w\) is non-decreasing and right-continuous. We denote as \(M_{\text{circ}}\) the set of strings \(w : (-\infty, \infty) \to [0, \infty]\) satisfying

\[
\int_{-\infty}^b x^2 dw(x) < \infty \tag{2.3}
\]

for some \(b \in \mathbb{R}\). For an element \(w \in M_{\text{circ}}\) and \(\lambda > 0\), we consider the solution \(u = f_w(\lambda; \cdot)\) to the following ODE:

\[
\frac{d}{dw} \frac{d^+}{dx} u = \lambda u, \quad u(-\infty) = 1, \quad u^+(-\infty) = 0 \quad (x < \ell). \tag{2.4}
\]

Here \(\ell = \inf \{x \in \mathbb{R} \mid w(x) = \infty\}\). Then define

\[
h_w(\lambda) = b + \int_{-\infty}^b \left( \frac{1}{f_w(\lambda; x)} \right)^2 dx + \int_{\ell}^b \frac{dx}{f_w(\lambda; x)} \quad (\lambda > 0). \tag{2.5}
\]
for some $b \in \mathbb{R}$. Note that $h_w(\lambda)$ is finite for every $\lambda > 0$ and the function $h_w$ does not depend on the choice of $b$. Moreover, if we define $\tilde{h}_w(-\lambda) = h_w(\lambda)$ ($\lambda > 0$), then the function $\tilde{h}_w$ is a Herglotz function, that is, $\tilde{h}_w$ can be extended to the holomorphic function on the upper half plane and, it maps the upper half plane to itself. Hence from the general theory of Herglotz functions, for a constant $\alpha \in \mathbb{R}$ and a Radon measure $\sigma$ on $[0, \infty)$ with $\int_0^\infty \frac{\sigma(d\xi)}{\xi^2+1} < \infty$, we have the following representation:

$$h_w(\lambda) = \alpha + \int_{0-}^\infty \left( \frac{1}{\xi + \lambda} - \frac{\xi}{\xi^2 + 1} \right) \sigma(d\xi). \tag{2.6}$$

We note that the measure $\sigma$ in RHS of (2.6) is the spectral measure of the differential operator $-\frac{d}{dw} \frac{d}{dx}$. Hence we call $h(w; \cdot)$ the spectral characteristic function of $w$.

Let $\mathcal{H}$ be the set of functions which are expressed in the form of RHS of (2.6) for a constant $\alpha \in \mathbb{R}$ and a Radon measure $\sigma$ on $[0, \infty)$ with $\int_0^\infty \frac{\sigma(d\xi)}{\xi^2+1} < \infty$. It was proved in Kotani [12] that the map $\mathcal{M}_{\text{circ}} \ni w \mapsto h_w \in \mathcal{H}$ is bijective. We call this correspondence the Krein-Kotani correspondence.

Let us consider the state interval $(0, \infty)$. A function $m: (0, \infty) \to (-\infty, \infty)$ is called a string on $(0, \infty)$ when $m$ is non-decreasing and right-continuous. We introduce the set of strings $\mathcal{M}_1$ which we mainly treat in the present paper. A string $m$ is an element of $\mathcal{M}_1$ when it is strictly increasing, right-continuous and satisfies $\int_{0+} m(x)^2 dx < \infty$. Note that the condition $\int_{0+} m(x)^2 dx < \infty$ implies the boundary 0 for a $\frac{d}{dm} \frac{d}{dx}$-diffusion is exit or regular. For a string $m$ on $(0, \infty)$, we define

$$m^*(x) = \inf\{y > 0 \mid m(y) > x\} \quad (x \in \mathbb{R}). \tag{2.7}$$

Then $w(x) = m^*(x)$ is a string on $(-\infty, \infty)$. We call $m^*$ the dual string of $m$.

We have the following fact:

For $m \in \mathcal{M}_1$, its dual string $m^*$ is an element of $\mathcal{M}_{\text{circ}}$. \tag{2.8}

In fact, from an elementary computation, it is easily checked that

$$\int_{0+} m(x)^2 dx < \infty \iff \int_{-\infty} x^2 dm^*(x) < \infty. \tag{2.9}$$

For $m \in \mathcal{M}_1$, we define

$$H_m(\lambda) = h_{m^*}(\lambda) \quad (\lambda > 0). \tag{2.10}$$

Kasahara and Watanabe [11] have shown that the Laplace exponent of $T_m$ is represented by $\lambda H_m(\lambda)$:

**Theorem 2.1.** (Kasahara and Watanabe [11, Theorem 1]) For $m \in \mathcal{M}_1$, it holds that

$$\mathbb{E}[e^{-\lambda T_m(t)}] = e^{-t \lambda H_m(\lambda)} \quad (\lambda > 0). \tag{2.11}$$
Example 2.2. For $0 < \alpha < 1$ or $1 < \alpha < 2$, we define $m^{(\alpha)}$ by

$$m^{(\alpha)}(x) = (1 - \alpha)^{-1} x^{1/\alpha - 1},$$

which is compatible with (1.15) and (1.20). Then it holds that

$$H_{m^{(\alpha)}}(\lambda) = \frac{\Gamma(2 - \alpha)}{\Gamma(\alpha)} \frac{\alpha^{\alpha - 1}}{1 - \alpha} \lambda^{\alpha - 1}. \quad (2.13)$$

See [12, Example 4] for the details.

An important consequence of the Krein-Kotani correspondence is the following theorem shown in Kasahara and Watanabe [10] which asserts a kind of continuity of the Krein-Kotani correspondence.

**Theorem 2.3.** (Kasahara and Watanabe [10, Theorem 2.9]) Let $m_n, m \in \mathcal{M}_1$ and $\sigma \geq 0$. Assume the following hold:

(i) $\lim_{n \to \infty} m_n(x) = m(x)$ for every continuity point $x$ of $m$,

(ii) $\lim_{x \to +0} \limsup_{n \to \infty} |\int_0^x m_n(y)^2 dy - \sigma^2| = 0$.

Then we have for every $\lambda > 0$

$$\lim_{n \to \infty} H_{m_n}(\lambda) = H_m(\lambda) - \sigma^2 \lambda. \quad (2.14)$$

### 3 Construction of eigenfunctions of the generator

In this section, we introduce the function $\varphi_1^m$ which is the $\lambda$-eigenfunction of the generator $\frac{d}{dm} + \frac{d}{dx}$ and satisfies the condition which we call modified Neumann boundary condition.

We prepare some notation.

**Definition 3.1.** For $m \in \mathcal{M}_1$, we define as follows:

$$G_m(x) = \int_0^x m(y) dy \quad (x \geq 0), \quad (3.1)$$

$$\tilde{m}(x) = m(x) - m(1) \quad (x > 0), \quad (3.2)$$

$$G_1^m(x) = \int_0^x \tilde{m}(y) dy \quad (x \geq 0). \quad (3.3)$$

**Definition 3.2.** Let $U$ be a function of bounded variation on $(0, \infty)$ and $f$ be a function such that $\int_0^x |f||dU| < \infty$ for every $x > 0$. Here $|dU|$ is the total variation measure of the Stieltjes measure $dU$. We define

$$U \cdot f(x) = \int_0^x f(y) dU(y). \quad (3.4)$$
**Definition 3.3.** For \( m \in \mathcal{M}_1 \), we define
\[
s(x) = x \ (x \geq 0),
\]
(3.5)
\[
\psi_m(\lambda; x) = \sum_{k=0}^{\infty} \lambda^k ((s \bullet m \bullet)^k s)(x) \quad (\lambda \geq 0, \ x \geq 0),
\]
(3.6)
\[
g_m(\lambda; x) = \psi_m(\lambda; x) \int_x^{\infty} \frac{dy}{\psi_m(\lambda; y)^2} \quad (\lambda \geq 0, \ x \geq 0)
\]
(3.7)
where \((s \bullet m \bullet)^1 s = s \bullet m \bullet s, (s \bullet m \bullet)^2 s = s \bullet m \bullet s \bullet m \bullet s, \) etc.

The convergence in the RHS of (3.6) and (3.7) follows from the following proposition.

**Proposition 3.4.** For \( m \in \mathcal{M}_1 \), the following hold for every \( x \geq 0, \ \lambda \geq 0 \) and \( d \geq 0 \):
\[
(s \bullet m \bullet)^d s(x) \leq x E_m^d(0; x),
\]
(3.8)
\[
m \bullet (s \bullet m \bullet)^{d-1} s(x) \leq E_m^{d-1}(0; x),
\]
(3.9)
\[
0 \leq \psi_m(\lambda; x) - \sum_{k=0}^{d-1} \lambda^k (s \bullet m \bullet)^k s(x) \leq x \lambda^d E_m^d(\lambda; x),
\]
(3.10)
\[
0 \leq \psi_m(\lambda; x) - 1 - \sum_{k=0}^{d-1} \lambda^{k+1} m \bullet (s \bullet m \bullet)^k s(x) \leq \lambda^{d+1} E_m^d(\lambda; x)
\]
(3.11)
where \((m \bullet s)^d(x) = \left( \int_0^{x} ydm(y) \right)^d \) and \( E_m^d(\lambda; x) = (1/d!)(m \bullet s)^d(x)e^{\lambda(m \bullet s)(x)} \).

**Proof.** First we show (3.9) by induction. The case \( d = 0 \) is obvious. Assume (3.9) holds for \( k \geq 0 \). Then it holds that
\[
m \bullet (s \bullet m \bullet)^k s(x) = m \bullet s \bullet (m \bullet (s \bullet m \bullet)^{k-1} s)(x)
\]
(3.12)
\[
\leq m \bullet s \bullet E_m^k(0; x)
\]
(3.13)
\[
\leq \int_0^{x} E_m^k(0; y)gdm(y)
\]
(3.14)
\[
= E_m^{k+1}(0; x),
\]
(3.15)
which shows (3.9) for \( d = k + 1 \). By induction we obtain (3.9). From (3.9), we have
\[
(s \bullet m \bullet)^d s(x) = s \bullet m \bullet (s \bullet m \bullet)^{d-1} s(x)
\]
(3.16)
\[
\leq \int_0^{x} E_m^d(0; y)dy
\]
(3.17)
\[
\leq x E_m^d(0; x)
\]
(3.18)
and, we obtain (3.8). Next we show (3.10). Since it holds that

\[
\psi_m(\lambda; x) - \sum_{k=0}^{d-1} \lambda^k (s \bullet m \bullet)^k s(x) = \sum_{k=d}^{\infty} \lambda^k (s \bullet m \bullet)^k s(x) = \sum_{k=0}^{\infty} \lambda^{k+d} s \bullet (m \bullet s \bullet)^k (m \bullet (s \bullet m \bullet)^{d-1}) (x)
\]

\[
\leq \lambda^d m \bullet (s \bullet m \bullet)^{d-1} s(x) \sum_{k=0}^{\infty} \lambda^k (s \bullet m \bullet)^k s(x),
\]

we have from (3.8) and (3.9),

\[
(3.21) \leq \lambda^d E^d_m(0; x) \sum_{k=0}^{\infty} \frac{x^k (m \bullet s)^k(x)}{k!} = x \lambda^d E^d_m(\lambda; x).
\]

The proof of (3.11) is similar and so we omit it. □

**Remark 3.5.** (i) The function \(u = \psi_m(\lambda; \cdot)\) is the unique solution of the integral equation:

\[
u(x) = x + \lambda \int_0^x (x - y) u(y) \, dm(y) \quad \text{for } \lambda \in \mathbb{R} \text{ and } x \in [0, \infty).
\]

In other words, the function \(u = \psi_m(\lambda; \cdot)\) is the unique solution of the ODE \(\frac{d}{dm} \frac{d^+}{dx} u = \lambda u\) satisfying the boundary condition \(u(0) = 0\) (Dirichlet) and \(u^+(0) = 1\).

(ii) The function \(u = g_m(\lambda; \cdot)\) is the unique, non-negative and non-increasing solution of the equation \(\frac{d}{dm} \frac{d^+}{dx} u = \lambda u\) \((\lambda \geq 0)\) satisfying the boundary condition \(u(0^+) = 1\) and \(\lim_{x \to \infty} \frac{d^+}{dx} u(x) = 0\). In fact, since we have

\[
g_m \psi^+_m - g_m^+ \psi_m = g_m \psi^+_m - \psi^+_m \left( \int_x^\infty \frac{dy}{\psi_m(y)^2} \right) \psi_m + 1 = 1,
\]

it follows that

\[
0 = d(g_m \psi^+_m - g_m^+ \psi_m) = g_m d\psi^+_m - \psi_m dg_m^+ = \lambda g_m \psi_m dm - \psi_m dg_m^+.
\]

Hence we obtain

\[
\frac{d}{dm} \frac{d^+}{dx} g_m = \lambda g_m
\]

(this argument is due to Itô [5]).

Here we introduce the \(\lambda\)-eigenfunction announced in the beginning of this section.

**Definition 3.6.** For \(m \in \mathcal{M}_1, x \geq 0\) and \(\lambda \geq 0\), we define

\[
\varphi^1_m(\lambda; x) = 1 + \sum_{k=0}^{\infty} \lambda^{k+1} (s \bullet m \bullet)^k G^1_m(x).
\]
The convergence of the summation in RHS of (3.27) follows from the following proposition.

**Proposition 3.7.** Let $m \in \mathcal{M}_1$. Then for any $d \geq 0$, the following hold for every $x \geq 0$ and $\lambda \geq 0$:

\[
\left| \varphi^1_m(\lambda; x) - 1 - \sum_{k=0}^{d} \lambda^{k+1}(s \bullet m \bullet)^k G^1_m(x) \right| \leq \lambda^{d+2} x \cdot S_m(x) \cdot E^d_m(\lambda; x), \tag{3.28}
\]

\[
\left| (\varphi^1_m)^+(\lambda; x) - \lambda \tilde{m}(x) - \sum_{k=1}^{d-1} \lambda^{k+1} m \bullet (s \bullet m \bullet)^{k-1} G^1_m(x) \right| \leq \lambda^{d+1} \cdot S_m(x) \cdot E^d_m(\lambda; x) \tag{3.29}
\]

where $S_m(x) = \sup_{y \in [0, x]} |m \bullet G^1_m(y)|$.

**Proof.** We only show (3.28); the proof of (3.29) is similar. From the definition of $\varphi^1_m$ and Proposition 3.4, for $x \geq 0$, we have

\[
\left| \varphi^1_m(\lambda; x) - 1 - \sum_{k=0}^{d} \lambda^{k+1}(s \bullet m \bullet)^k G^1_m(x) \right| = \left| \sum_{k=d+1}^{\infty} \lambda^{k+1}(s \bullet m \bullet)^k G^1_m(x) \right| \tag{3.30}
\]

\[
\leq S_m(x) \lambda^{2} \sum_{k=d}^{\infty} \lambda^k (s \bullet m \bullet)^k s(x) \tag{3.31}
\]

\[
\leq \lambda^{d+2} x \cdot S_m(x) \cdot E^d_m(\lambda; x). \tag{3.32}
\]

\]

The following theorem gives a characterization of the function $\varphi^1_m$ in terms of the integral and differential equations.

**Theorem 3.8.** For $m \in \mathcal{M}_1$ and $\lambda \geq 0$, the following hold:

(i) The summation in (3.27) converges uniformly on every compact subset of $[0, \infty)$.

(ii) The function $u = \varphi^1_m(\lambda; \cdot) - 1$ is the unique solution of the integral equation

\[
u(x) = \lambda G^1_m(x) + \lambda \int_0^x (x - y) u(y) dm(y). \tag{3.33}
\]

Equivalently, the function $u = \varphi^1_m(\lambda; \cdot)$ is the unique solution of the equation \( \frac{d}{dm} \frac{d}{dx} u = \lambda u \) with the boundary condition:

\[
u(0) = 1, \quad \lim_{x \to +0} \left( u^+(x) - \lambda \tilde{m}(x) \right) = 0. \tag{3.34}
\]

We call this boundary condition the **modified Neumann boundary condition**.
Proof. The assertion (i) is obvious from Proposition 3.7 and for the assertion (ii), it is easily checked that $u = \varphi_m^1(\lambda; \cdot) - 1$ is the solution of the integral equation (3.33). We prove the uniqueness. Let $u$ and $v$ be the solution of the integral equation (3.33). Define $w(x) = u(x) - v(x)$. Then it holds that

$$w(x) = \lambda \int_0^x dy \int_0^y w(z) dm(z) \quad (x \in [0, \infty)). \quad (3.35)$$

Hence we have $w(x) \leq x^{\lambda m(0+)} \int_0^x w(y) dm(y)$ for every $x \geq 0$ and $k \geq 1$. Then it follows that $w(x) = 0 \ (x \geq 0)$.

We introduce a subset of $\mathcal{M}_1$.

Definition 3.9. Define

$$\mathcal{M}_0 = \{ m \in \mathcal{M}_1 \mid \lim_{x \to +0} m(x) > -\infty \}. \quad (3.36)$$

A string $m \in \mathcal{M}_0$ with $m(0+) \geq 0$ is called a Krein’s string and extensively used in the studies of one-dimensional diffusions (see e.g. [7], [12]). In these studies, the unique solution $u = \varphi_m$ of the differential equation $\frac{d}{dm} \frac{d}{dx} u = \lambda u \ (\lambda \geq 0)$ with $u(0) = 1$ and $u^+(0) = \lambda m(0+)$ plays an important role. The following proposition gives a relation between $\varphi_m$ and $\varphi_m^1$.

Proposition 3.10. If $m \in \mathcal{M}_0$, then for $\lambda \geq 0$ the function $u = \varphi_m^1(\lambda; \cdot) + \lambda m(1) \psi_m(\lambda; \cdot)$ is the unique solution of the integral equation:

$$u(x) = 1 + \lambda \int_0^x (x - y) u(y) dm(y), \quad (3.37)$$

where we regard $dm\{0\} = m(0+)$. If, in addition, $m(0+) \geq 0$, then the function $u(x) = \varphi_m^1(\lambda; \cdot) + \lambda m(1) \psi_m(\lambda; \cdot)$ is the unique non-decreasing solution of the differential equation:

$$\frac{d}{dm} \frac{d}{dx} u = \lambda u, \quad u(0) = 1, \quad u^+(0) = \lambda m(0+). \quad (3.38)$$

Proof. Setting $v = \varphi_m^1 - 1$, we have from (3.33),

$$\lambda \int_0^x dy \int_0^y \left( \varphi_m^1 + \lambda m(1) \psi_m \right) dm(z)$$

$$= \lambda \int_0^x dy \int_0^y (v + 1 + \lambda m(1) \psi_m) dm(z) + \lambda m(0+)x$$

$$= v - \lambda G_m^1 + \lambda \int_0^x (m(y) - m(0+)) dy + \lambda m(1)(\psi_m - x) + \lambda m(0+)x \quad (3.41)$$

$$= v + \lambda m(1) \psi_m \quad (3.42)$$
Since \( \varphi_m^1(\lambda; \cdot) \) and \( \psi_m(\lambda; \cdot) \) are linearly independent solutions of the equation \( \frac{d}{dm} \frac{d^+}{dx} u = \lambda u \), for \( \lambda > 0 \) the function \( g_m(\lambda; \cdot) \) can be represented as
\[
g_m(\lambda; \cdot) = \varphi_m^1(\lambda; \cdot) - c_m^1(\lambda) \psi_m(\lambda; \cdot) \tag{3.43}
\]
by a constant \( c_m^1(\lambda) \).

4 Representation of spectral characteristic functions

In this section, we show for \( m \in \mathcal{M}_1 \) it holds that
\[
c_m^1(\lambda) = \lambda H_m(\lambda) - \lambda m(1) \quad (\lambda > 0).
\]
This result is well-known in the case the boundary 0 is regular. Therefore this is an extension of the result to a class of exit boundaries. The essential tool is the Krein-Kotani correspondence.

We note the following well-known result on one-dimensional diffusion theory (see e.g. [13] for the proof).

**Proposition 4.1.** Let \( m \in \mathcal{M}_1 \) with \( m(0+) \geq 0 \) and \( u = \varphi_m(\lambda; \cdot) \) be the unique solution of the following differential equation:
\[
\frac{d}{dm} \frac{d^+}{dx} u = \lambda u, \quad u(0) = 1, \quad u^+(0) = \lambda m(0+) \quad (\lambda \geq 0).
\]
Then the following holds:
\[
g_m(\lambda; \cdot) = \varphi_m(\lambda; \cdot) - \lambda H_m(\lambda) \psi_m(\lambda; \cdot) \quad (\lambda > 0). \tag{4.2}
\]

The following proposition is obvious from Proposition 3.10 and Proposition 4.1.

**Proposition 4.2.** Let \( m \in \mathcal{M}_0 \) and \( \lambda > 0 \). It holds that
\[
c_m^1(\lambda) = \lambda H_m(\lambda) - \lambda m(1). \tag{4.3}
\]

We generalize Proposition 4.2 by approximation argument.

**Theorem 4.3.** Let \( m \in \mathcal{M}_1 \) and \( \lambda > 0 \). It holds that
\[
c_m^1(\lambda) = \lambda H_m(\lambda) - \lambda m(1). \tag{4.4}
\]

**Proof.** This proof is essentially due to that of Theorem 1 of [7]. We take a decreasing sequence \( \{a_n\}_n \) which diverges to \(-\infty\). Fix \( \lambda > 0 \). Define strings \( \{m_n\}_n \) as \( m_n(x) = \max\{m(x), a_n\} \). We denote
\[
\varphi_n^1(x) = \varphi_m^1(x), \quad \varphi^1(x) = \varphi_m^1(x), \quad etc. \tag{4.5}
\]
Since \( m_n(0+) > -\infty \), it holds that from Proposition 4.2
\[
c_n^1 = \lambda H_n - \lambda m_n(1). \tag{4.6}
\]
In order to prove
\[ g_m = (\varphi_n^1 + \lambda m(1)\psi_n) - \lambda H_n^*\psi_n, \] (4.7)
 it is enough to show that as \( n \to \infty \),
\[ H_n \to H, \quad \varphi_n^1(x) \to \varphi^1(x), \quad \psi_n(x) \to \psi(x), \quad g_n(x) \to g(x) \] (4.8)
hold for some \( x > 0 \). The first one directly follows from Theorem 2.3. At first, we prove \( \lim_{n \to \infty} \psi_n(x) = \psi(x) \) for every \( x > 0 \). Since it holds that \( (m_n \bullet s)(x) \leq (m \bullet s)(x) \), from Proposition 3.4, we have
\[ x \leq \psi_n(x) \leq xe^{\lambda(m\bullet s)(x)}, \quad 1 \leq \psi^+_n(x) \leq e^{\lambda(m\bullet s)(x)}. \] (4.9)
Then from Ascoli-Arzela theorem and the diagonal argument, we can take a subsequence \( \{n_k\}_k \) such that \( \{\psi_{n_k}\}_k \) converges to some \( \tilde{\psi} \) uniformly on every compact subset of \([0, \infty)\).

Then from Remark 3.5, the following holds:
\[ \psi_{n_k}(x) = x + \lambda \int_0^x (x - y)\psi_{n_k}(y)dm_n(y). \] (4.10)

Then by \( n \to \infty \), we obtain
\[ \tilde{\psi}(x) = x + \lambda \int_0^x (x - y)\tilde{\psi}(y)dm(y). \] (4.11)
Hence by the uniqueness of the solution of the integral equation, we obtain \( \tilde{\psi} = \psi \). This argument also holds if we start from any subsequence of \( \{\psi_n\}_n \). Hence we have
\[ \lim_{n \to \infty} \psi_n(x) = \psi(x) \] (4.12)
for every \( x \in [0, \infty) \) and \( \lambda > 0 \). Since we have (3.7) and (4.9), by the dominated convergence theorem, we obtain
\[ \lim_{n \to \infty} g_n(x) = g(x) \] (4.13)
for every \( x > 0 \) and \( \lambda > 0 \). Finally, we prove that \( \lim_{n \to \infty} \varphi_n^1(x) = \varphi^1(x) \). From the definition of \( \varphi_n^1 \), we can easily check the following holds for \( x \in (0, \infty) \):
\[ |\varphi_n^1(x)| \leq 1 + \lambda \int_0^x |m(y)|dy + \lambda^2 e^{\lambda(m\bullet s)(x)} \cdot \sup_{y \in [0,x]} |m \bullet G^1_m(y)|, \] (4.14)
\[ |(\varphi_n^1)^+(x)| \leq \lambda |m(x)| + \lambda^2 e^{\lambda(m\bullet s)(x)} \cdot \sup_{y \in [0,x]} |m \bullet G^1_m(y)|. \] (4.15)
Then we can take a subsequence \( \{n_k\}_k \) such that \( \varphi_{n_k}^1 \) converges to some function \( \tilde{\varphi} \) uniformly on every compact subset of \((0, \infty)\). Since the function \( u = \tilde{\varphi} - 1 \) is a solution of the integral equation
\[ u(x) = \lambda G^1_m(x) + \lambda \int_0^x (x - y)u(y)dm(y). \] (4.16)
From the uniqueness of the solution, we obtain
\[ \tilde{\varphi}(x) = \varphi^1(x). \] (4.17)
Hence it holds that \( \lim_{n \to \infty} \varphi_n^1(x) = \varphi^1(x) \). □
The following proposition shows that the second order at 0 of $g_m$ is $x$.

**Proposition 4.4.** Let $m \in M_1$. Then the following holds:

$$
\lim_{x \to 0} \frac{1 - g_m(\lambda; x) + \lambda G_m(x)}{x} = \lambda H_m(\lambda) \quad \text{for every } \lambda > 0.
$$

**Proof.** From Theorem 4.3 and Theorem 3.8 (ii), it holds that

$$
1 - g_m + \lambda G_m = \lambda (H_m - \lambda m(1)) \psi_m + \lambda m(1)x - \int_0^x dy \int_0^y (\varphi_m^1 - 1) dm(y).
$$

Since $\psi_m^+(0+) = 1$ holds, we obtain the desired result.

\[\square\]

5 **Convergence of inverse local times**

We consider a strong Markov process $X$ on $[0, \infty)$ which has continuous paths and natural scale on $(0, \infty)$ and as soon as $X$ hits the origin, it jumps into $(0, \infty)$. The process $X$ has the local generator $L$ of the form:

$$
Lu(x) = \frac{d^-}{dm} dx + \frac{d^+}{dx} u(x) \quad \text{for } x \in (0, \infty)
$$

subject to Feller’s boundary condition [3]

$$
\int_0^\infty (u(x) - u(0)) j(dx) = 0,
$$

where $m$ and $j$ are Radon measures on $(0, \infty)$. A necessary and sufficient condition for the existence of $X = X_{m,j}$ for a given pair $(m, j)$ is the following (see Itô [6] and Rogers [14]):

\[\begin{array}{l}
\{(i) \ j(1, \infty) + \int_0^1 x j(dx) + \int_0^1 |G_m(x)| j(dx) < \infty, \\
\quad (ii) \ j(0, 1) = \infty. \}
\end{array}\]

We briefly summarize the construction of a sample path of the process $X_{m,j}$ via Itô’s excursion theory. Let $m$ and $j$ be Radon measures on $(0, \infty)$. Assume $(m, j)$ satisfies the condition (C). Define $T_0(e) = \inf\{s > 0 \mid e(s) = 0\} \ (e \in \mathbb{D})$ and $E$ as the set of all elements $e$ in $\mathbb{D}$ which satisfy that $e(u) = 0$ for every $u \geq T_0(e)$ if $T_0(e) < \infty$. Let $P_x^m \ (x \in (0, \infty))$ be the law of the diffusion process with speed measure $dm$ starting from $x$ and killed at 0. Then the following measure $n_{m,j}$ is the excursion measure of the process $X_{m,j}$:

$$
n_{m,j}(A) = \int_0^\infty P_x^m(A) j(dx) \quad (A \in \mathcal{B}(E)).
$$
We construct the sample paths of $X_{\sigma,\tau}$. We define $N(dsde)$ as a Poisson random measure on $(0,\infty) \times \mathbb{E}$ having intensity measure $dx \otimes n_{\sigma,\tau}$ being defined on a probability space $(\Omega, \mathcal{F}, P)$. Here $dx$ is the Lebesgue measure. We define $D(p) = \{ s \in (0, \infty) \mid N(\{ s \} \times \mathbb{E}) = 1 \}$ and the map $p : D(p) \to \mathbb{E}$ such that $p(s)$ ($s \in D(p)$) is the only one element of the support of the measure $N(\{ s \} \times de)$. We define the process $\eta_{\sigma,\tau}$ as follows:

$$\eta_{\sigma,\tau}(u) = \int_{(0,u] \times \mathbb{E}} T_0(e) N(dsde). \quad (5.5)$$

Then we construct $X_{\sigma,\tau}$ as follows:

$$X_{\sigma,\tau}(t) = \begin{cases} p[u](t - \eta_{\sigma,\tau}(u^-)) & \text{if } u \in D(p) \text{ and } \eta_{\sigma,\tau}(u^-) \leq t < \eta_{\sigma,\tau}(u), \\ 0 & \text{otherwise.} \end{cases} \quad (5.6)$$

Then $\eta_{\sigma,\tau}$ plays the role of the inverse local time at 0 of $X_{\sigma,\tau}$. This process $X_{\sigma,\tau}$ thus constructed is the jumping-in diffusion associated to $\sigma$ and $\tau$ and started from 0. We have for $\lambda > 0$,

$$\chi_{\sigma,\tau}(\lambda) = -\log P[e^{-\lambda \eta_{\sigma,\tau}(1)}] = \int_0^\infty (1 - e^{-\lambda s}) n_{\sigma,\tau}(T_0 \in du) \quad (5.7)$$

$$= \int_0^\infty P^m_x[1 - e^{-\lambda T_0}] j(dx) \quad (5.8)$$

It is well-known that the following holds (see e.g. [5]):

$$g_{\sigma}(\lambda;x) = P^m_x[e^{-\lambda T_0}]. \quad (5.10)$$

Hence we obtain the following representation:

$$\chi_{\sigma,\tau}(\lambda) = \int_0^\infty (1 - g_{\sigma}(\lambda;x)) j(dx) \quad (\lambda > 0). \quad (5.11)$$

Here we establish a continuity theorems for $\eta_{\sigma,\tau}$.

**Theorem 5.1.** Let $m_n, m \in \mathcal{M}_1$ and $j_n$ be a Radon measure on $(0, \infty)$ and assume $(m_n, j_n)$ satisfies (C). Suppose the following hold:

(i) $\lim_{n \to \infty} m_n(x) = m(x)$ for every continuity point $x$ of $m$,

(ii) $\lim_{x \to +0} \limsup_{n \to \infty} \int_0^x m_n(y)^2 dy = 0$,

(iii) $j_n(dx) \overset{w}{\to} 0$ on $[1, \infty]$,

(iv) $x j_n(dx) \overset{w}{\to} \kappa \delta_0(dx)$ on $[0,1]$ for a constant $\kappa > 0$. 

19
Then if we take \( b_n = - \int_0^1 G_{m_n}(x) j_n(dx) \), we have

\[
\eta_{m_n,j_n}(t) - b_n t \xrightarrow{n \to \infty} T_m(\kappa t) \quad \text{on } \mathbb{D}.
\] (5.12)

Here \( T_m(t) \) is the Lévy process without negative jumps whose Laplace exponent is \( \lambda H_m(\lambda) \).

**Proof.** From (5.11) and Proposition A.1, it is enough to show the following for every \( \lambda > 0 \):

\[
\lim_{n \to \infty} \left( \int_0^\infty (1 - g_{m_n}(\lambda; x)) j_n(dx) + \lambda \int_0^1 G_{m_n}(x) j_n(dx) \right) = \kappa \lambda H_m(\lambda). \quad (5.13)
\]

Fix \( \lambda > 0 \). Since the function \( 1 - g_{m_n} \) is bounded continuous, we see from the assumption (iii) that (5.13) is equivalent to the following (with \( \lambda \) being omitted):

\[
\lim_{n \to \infty} \int_0^1 (1 - g_{m_n} + \lambda G_{m_n}) dj_n = \kappa \lambda H_m. \quad (5.14)
\]

From (3.43), we have

\[
g_{m_n} = \varphi_{m_n}^1 - c_{m_n}^1 \psi_{m_n} = (1 + \lambda G_{m_n}^1 + \Phi_{m_n}^1) - c_{m_n}^1 (x + \Psi_{m_n}) \quad (5.15)
\]

where \( \Phi_{m_n}^1 = \varphi_{m_n}^1 - 1 - \lambda G_{m_n}^1, \Psi_{m_n} = \psi_{m_n} - x \). Note that from Schwarz’s inequality, we have for \( x \in (0,1] \),

\[
(m_n \cdot s)(x) \leq x \tilde{m}_n(x) - \int_0^x \tilde{m}_n(y)dy \leq \int_0^x (-\tilde{m}_n(y))dy \leq \sqrt{x} \left( \int_0^x \tilde{m}_n(y)^2dy \right)^{1/2}, \quad (5.16)
\]

and hence for a fixed constant \( \delta \in (0,1) \), from Proposition 3.7, we have for a constant \( C > 0 \),

\[
\int_0^1 |\Phi_{m_n}^1| dj_n \leq \lambda^2 \int_0^\delta x S_{m_n}(x)e^{\lambda(m_n \cdot s)} j_n(dx) + \lambda^2 \int_0^1 x S_{m_n}(x)e^{\lambda(m_n \cdot s)} j_n(dx) \leq C \lambda^2 \left( \int_0^\delta \tilde{m}_n(y)^2dy \right) \int_0^\delta x j_n(dx) + C \lambda^2 \int_0^1 x j_n(dx). \quad (5.17)
\]

Then from the assumptions (ii), (iii) and (iv), it follows that

\[
\limsup_{n \to \infty} \int_0^1 |\Phi_{m_n}^1| dj_n \leq \kappa C \lambda^2 \limsup_{n \to \infty} \left( \int_0^\delta \tilde{m}_n(y)^2dy \right) \xrightarrow{\delta \to 0} 0 \quad \text{by (ii).} \quad (5.19)
\]

Similarly, by Proposition 3.4, we can show

\[
\lim_{n \to \infty} \int_0^1 \Psi_{m_n} dj_n = 0. \quad (5.20)
\]
Since we have
\[
1 - g_{m_n} + \lambda G_{m_n} = 1 - ((1 + \lambda G^1_{m_n} + \Phi^1_{m_n}) - c^1_{m_n}(x + \Psi_{m_n})) + \lambda G_{m_n}
\]
(5.21)
\[
= -\Phi^1_{m_n} + \lambda H_{m_n} x + c^1_{m_n} \Psi_{m_n}.
\]
(5.22)

Note that from Theorems 2.3 and 4.3, it holds that
\[
\lim_{n \to \infty} H_{m_n} = H_m, \quad \sup_n c^1_{m_n} < \infty.
\]
(5.23)

Hence from (5.19) and (5.20), we obtain (5.14).

6 Scaling limit of inverse local times

Applying the continuity theorem shown in the previous section, we establish the scaling limit of \( \eta_{m,j} \).

We reduce the scaling limit of \( \eta_{m,j} \) to the continuity of it with respect to \( m \) and \( j \) by the change of variables. For every \( m \in \mathcal{M}_1, a, b > 0 \) and \( x > 0 \), it holds that

\[
g_{am^b}^b(\lambda; x) = g_m(\frac{a}{b}, \lambda; bx),
\]
(6.1)
\[
\psi_{am^b}^b(\lambda; x) = \frac{1}{b} \psi_m(\frac{a}{b}, \lambda; bx),
\]
(6.2)
\[
\varphi_{am^b}^1(\lambda; x) = \varphi_m^1(\frac{a}{b}, \lambda; bx),
\]
(6.3)

where \( m^b(x) = m(bx) \).

Now we proceed to prove Theorem 1.1.

**Proof of Theorem 1.1.** We may assume \( m(\infty) = 0 \) without loss of generality. Define

\[
m_{\gamma}(x) = \frac{m(\gamma x)}{\gamma^{1/\alpha - 1}K(\gamma)}; \quad j_{\gamma}(dx) = \gamma j(d(\gamma x)); \quad \bar{b}_{\gamma} = -\int_0^\infty G_m dj_{\gamma}.
\]
(6.4)

Then

\[
\frac{1}{\gamma^{1/\alpha}K(\gamma)}(\eta_{m,j}(\gamma t) - b_{\gamma}t) \overset{d}{=} \eta_{m_{\gamma},j_{\gamma}}(t) - \bar{b}_{\gamma}t \quad \text{on } \mathbb{D}.
\]
(6.5)

Then it is enough to show that \( \{m_{\gamma}\}_\gamma \) and \( \{j_{\gamma}\}_\gamma \) satisfy the assumptions of Theorem 5.1 with \( m = m^{(\alpha)} \) and

\[
\lim_{\gamma \to \infty} (\bar{b}_{\gamma} - b_{\gamma}) = 0,
\]
(6.6)
where $b_γ = - \int_0^1 G_{m,j} dγ$. It is easily checked that the assumption (i) of Theorem 5.1 holds. Then we show that the following hold:

$$\lim \limsup_{\delta \to 0} \lim_{\gamma \to \infty} \int_0^\delta m_\gamma(x)^2 dx = 0, \quad (6.7)$$

$$j_\gamma(dx) \xrightarrow{\gamma \to \infty} 0 \quad \text{on } [1, \infty], \quad (6.8)$$

$$x j_\gamma(dx) \xrightarrow{\gamma \to \infty} \kappa \delta_0(dx) \quad \text{on } [0, 1]. \quad (6.9)$$

From Karamata’s theorem [2, Proposition 1.5.8], we have

$$\lim_{\gamma \to \infty} \int_0^\delta m_\gamma(x)^2 dx = (\alpha - 1)^{-2} \int_0^\delta x^{2/\alpha - 2} dx = \frac{1}{(2/\alpha - 1)(\alpha - 1)^2} \delta^{2/\alpha - 1}. \quad (6.10)$$

Hence we obtain (6.7). Next we prove (6.6). By changing variables, we have

$$\int_1^\infty |G_{m,j}\gamma(dx) = \frac{1}{\gamma^{1/\alpha - 1} K(\gamma)} \int_\gamma^\infty |G_{m,j}(dx). \quad (6.11)$$

Again by Karamata’s theorem, it holds that $\lim_{x \to \infty} \frac{G_m(x)}{axm(x)} = 1$. Then for any $\epsilon > 0$, there exists some $R > 0$ such that for every $x \geq R$ it holds that $\left| \frac{G_m(x)}{axm(x)} \right| < 1 + \epsilon$. Then for $x \geq R$, it follows that

$$\frac{1}{\gamma^{1/\alpha - 1} K(\gamma)} \int_\gamma^\infty |G_{m,j}(dx) \leq \frac{- (1 + \epsilon) m(\gamma)}{\gamma^{1/\alpha - 1} K(\gamma)} \int_\gamma^\infty x j(dx). \quad (6.12)$$

Then from assumption (i) and (ii), we have (6.6). Next we show (6.8). By changing variables, for every bounded continuous function $f : [0, 1] \to \mathbb{R}$, it holds that

$$\lim_{\gamma \to \infty} \int_\gamma^\infty f(x) x j_\gamma(dx) = \lim_{\gamma \to \infty} \int_0^\gamma f(\gamma^{-1} x) x j(dx) = f(0). \quad (6.14)$$

Hence (6.9) holds. The proof is complete. \hfill \Box

We determine the tail behavior of excursion length as a corollary of Theorem 1.1.

**Corollary 6.1.** Under the same assumptions in Theorem 1.1, we have

$$n_{m,j} [T_0 > s] \sim \frac{\kappa a^{-1}}{\Gamma(\alpha)} s^{-\alpha} L^\sharp(s)^{-\alpha} \quad (s \to \infty), \quad (6.15)$$

where $L^\sharp(x)$ be a is Bruijn conjugate of $L(x) = K(x^\alpha)$.\hfill \Box
Proof. From Theorem 1.1, for every $\lambda > 0$, it holds that

$$\gamma^\alpha \left( \chi_{m,j} \left( \frac{\lambda}{\gamma K(\gamma)} \right) - \frac{b\lambda}{\gamma K(\gamma)} \right) \xrightarrow{\gamma \to \infty} \kappa \lambda H_m(\omega)(\lambda). \quad (6.16)$$

Here $b = -\int_0^\infty G_m(x) j(dx)$. Hence it holds that

$$u(\lambda) := \chi_{m,j}(\lambda) - b\lambda \sim C\lambda^\alpha L^\sharp(1/\lambda)^{-\alpha} \quad (\lambda \to +0) \quad (6.17)$$

where $C = \kappa H_m(\alpha)(1)$. Since $u(\lambda) = \int_0^\infty \mathbb{P}_x^{m}[1 - e^{-\lambda T_0}] j(dx) - b\lambda$, the function $-u'(\lambda)$ is completely monotone. From the monotone density theorem [2, Theorem 1.7.2b]), we obtain

$$u'(\lambda) \sim C\alpha \lambda^{\alpha-1} L^\sharp(1/\lambda)^{-\alpha} \quad (\lambda \to +0). \quad (6.18)$$

Let $\nu$ be a Radon measure on $(0, \infty)$ defined by

$$\nu(dx) = n_{m,j}[T_0 > x]dx. \quad (6.19)$$

We note that $\nu(0, \infty) = P[T_0] = -\int_0^\infty G_m(x) j(dx) = b$ and

$$\hat{\nu}(\lambda) := \int_0^\infty e^{-\lambda x} \nu(dx) = \frac{\chi_{m,j}(\lambda)}{\lambda} \quad (\lambda > 0). \quad (6.20)$$

Then it holds that

$$\hat{\nu}'(\lambda) = \frac{\lambda \chi'_{m,j}(\lambda) - \chi_{m,j}(\lambda)}{\lambda^2} = \frac{\lambda u'(\lambda) - u(\lambda)}{\lambda^2}. \quad (6.21)$$

From (6.17) and (6.18), we have

$$\hat{\nu}'(\lambda) \sim C(\alpha - 1)\lambda^{\alpha-2} L^\sharp(1/\lambda)^{-\alpha} \quad (\lambda \to +0) \quad (6.22)$$

Then from Kasahara [8, Theorem 2.1] with $\beta = 2 - \alpha$ and $n = 1 > 2 - \alpha$, it follows that

$$\nu[s, \infty) \sim \frac{\kappa \alpha^{\alpha-1}}{(\alpha - 1) \Gamma(\alpha)} s^{-\alpha+1} L^\sharp(s)^{-\alpha} \quad (s \to \infty). \quad (6.23)$$

Again from the monotone density theorem, we obtain (6.15). \qed

7 Limit theorems for the occupation times on the half line of bilateral jumping-in diffusions

Let $m_+, m_- \in \mathcal{M}_1$ and $j_+, j_-$ be Radon measures on $(0, \infty)$ and suppose $(m_+, j_+)$ and $(m_-, j_-)$ satisfy (C). In this section, we treat bilateral jumping-in diffusion processes i.e. Markov processes on $\mathbb{R}$ which behaves like $X_{m_+, j_+}$ while $X$ is positive and like $-X_{m_-, j_-}$ while $X$ is negative and as soon as the process hits the origin it is thrown into $\mathbb{R} \setminus \{0\}$ according to $j_+$ and $j_-$. The precise definition is as follows. Take two independent
Proof of Theorem 1.2. Define $N_{m_+,j_+}$ and $N_{m_-,j_-}$ whose intensity measures are $n_{m_+,j_+}$ and $n_{m_-,j_-}$ on a common probability space. Define $N_{m_-,j_-}(dsde) = \tilde{N}_{m_-,j_-}(dsd(-e))$. Then we define bilateral jumping-in diffusion process $X = X_{m_+,j_+,m_-j_-}$ from the excursion point process $N_{m_+,j_+} + N_{m_-,j_-}$.

Define
\[
A(t) = \int_0^t 1_{(0,\infty)}(X(s))ds \quad (7.1)
\]
for $t \geq 0$ and we study the fluctuation of the mean occupation time on the half line $(1/t)A(t)$ as $t \to \infty$, in the case the limit degenerates, that is,
\[
\lim_{t \to \infty} \frac{1}{t} A(t) \xrightarrow{P} p \in (0,1). \quad (7.2)
\]

Now we proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. Define $\eta = \eta_{m_+,j_+} + \eta_{m_-,j_-}$ and $\Delta \eta(t) = \eta(t) - \eta(t-)$. Define $\ell$ as the right-continuous inverse of $\eta$, which serves as the local time at 0 of $X$. It is not difficult to check that for every $t > 0$, it holds that
\[
\eta_{m_+,j_+}(\ell(t)) - \Delta \eta(\ell(t)) \leq A(t) \leq \eta_{m_+,j_+}(\ell(t)), \quad (7.3)
\]
\[
\eta_{m_-,j_-}(\ell(t)) - \Delta \eta(\ell(t)) \leq \eta_{m_-,j_-}(\ell(t)-) \leq t - A(t) \leq \eta_{m_-,j_-}(\ell(t)). \quad (7.4)
\]
Define $\tilde{\eta} = (1-p)\eta_{m_+,j_+} - p\eta_{m_-,j_-}$. Then from (7.3) and (7.4), the process $A(t) - pt = (1-p)A(t) - p(t - A(t))$ satisfies the following:
\[
|(A(t) - pt) - \tilde{\eta}(\ell(t))| \leq \Delta \eta(\ell(t)) \quad (t \geq 0). \quad (7.5)
\]

Step 1: Let us show the following:
\[
\frac{1}{\gamma^{1/\alpha}K(\gamma)} \Delta \eta(\ell(\gamma t)) \xrightarrow{d.} 0. \quad (7.6)
\]
For this, it is enough to show
\[
\frac{1}{\gamma^{1/\alpha}K(\gamma)} \Delta \eta(\ell(\gamma t)) \xrightarrow{P} 0 \quad \text{for every } t \geq 0. \quad (7.7)
\]
Take $\epsilon > 0$ and $\delta > 0$ and define $c_\pm(\delta) = \frac{1 \pm \delta}{b_+ + b_-}$. Then it holds that
\[
P \left[ \Delta \eta(\ell(\gamma t)) > \gamma^{1/\alpha}K(\gamma)\epsilon \right] \leq P(\ell(\gamma t) \notin [c_-(\delta)\gamma t,c_+(\delta)\gamma t]) \quad (7.8)
\]
\[
+ P \left[ \sup_{s \in [c_-(\delta)\gamma t,c_+(\delta)\gamma t]} \Delta \eta(s) > \gamma^{1/\alpha}K(\gamma)\epsilon \right]. \quad (7.9)
\]
\[
\frac{1}{\gamma^{1/\alpha}K(\gamma)} \Delta \eta(\ell(\gamma t)) \xrightarrow{P} 0 \quad (7.10)
\]
By the definition of $\ell(t)$, we have
\[
P[\ell(t) \notin [c_-(\delta)\gamma t, c_+(\delta)\gamma t]] \leq P[\eta(c_-(\delta)\gamma t) > \gamma t] + P[\eta(c_+(\delta)\gamma t) \leq \gamma t]. \tag{7.11}
\]
From Theorem 1.1, we have $\eta(\gamma)/\gamma \xrightarrow[\gamma \to \infty]{} b_+ + b_-$. Therefore we have $(7.11) \to 0 \ (\gamma \to \infty)$. Recall that $N := N_{m_+, j_+} + N_{m_-, j_-}$ is the excursion point process of $X$. Define $n = n_{m_+, j_+} + n_{m_-, j_-}$. Then we have
\[
P \left[ \sup_{s \in [c_-(\delta)\gamma t, c_+(\delta)\gamma t]} \Delta \eta(s) > \gamma^{1/\alpha} K(\gamma)\epsilon \right] \tag{7.12}
\]
\[
= 1 - P \left[ N \left\{(s, e) \in [0, \infty) \times \mathbb{E} \mid s \in [c_-(\delta)\gamma t, c_+(\delta)\gamma t], T_0(e) > \gamma^{1/\alpha} K(\gamma)\right\} = 0 \right] \tag{7.13}
\]
\[
= 1 - \exp\left(-\gamma t(c_+ - c_-)\right) n \left[T_0 > \gamma^{1/\alpha} K(\gamma)\epsilon\right]. \tag{7.14}
\]
Therefore from Corollary 6.1, we have
\[
\limsup_{\gamma \to \infty} P \left[ \Delta \eta(\gamma t) > \gamma^{1/\alpha} K(\gamma)\epsilon \right] \leq 1 - \exp\left(-\gamma t(c_+ - c_-)\right) \frac{\kappa \alpha^{-1}}{\Gamma(\alpha)} e^{-\alpha}. \tag{7.15}
\]
Since $c_+ - c_- \to 0$ as $\delta \to +0$, we obtain (7.7).

Step 2: Here we follow the argument in [9] and [10]. Since $P = \frac{b_+}{b_+ + b_-}$, it holds that
\[
f(\gamma)\tilde{\eta}(\gamma t) = (1 - p)(\eta_{m_+, j_+}(\gamma t) - b_+\gamma t) - p(\eta_{m_-, j_-}(\gamma t) - b_-\gamma t). \tag{7.16}
\]
Then from Theorem 1.1, we have the following convergence on $\mathbb{D} \times \mathbb{D}$ in $J_1$-topology:
\[
\left(f(\gamma)\tilde{\eta}(\gamma t), \frac{1}{\gamma} \eta(\gamma t)\right) \xrightarrow[\gamma \to \infty]{d} ((1 - p)w_+ T_{m_+}(\kappa_+ t) - pw_- T_{m_-}(\kappa_- t), (b_+ + b_-)t). \tag{7.17}
\]
Since the right-continuous inverse process of $\tilde{\eta}(\gamma t)$ is $\ell(\gamma t)$, the following convergence holds on $\mathbb{D}$ in $M_1$-topology (see e.g. [17, Theorem 13.2.3, 13.6.1]):
\[
f(\gamma)\tilde{\eta}(\ell(\gamma t)) \xrightarrow[\gamma \to \infty]{d} (1 - p)w_+ T_{m_+}(\kappa_+ t) - pw_- T_{m_-}(\kappa_- t). \tag{7.18}
\]
Since the limit process has no fixed jumps, $M_1$-convergence implies the finite-dimensional convergence. \hfill \Box

**Example 7.1.** Let $1 < \alpha < 2$ and $0 < \beta < 1/\alpha$. Define
\[
m_\pm(x) = w_\pm(\alpha - 1)^{-1}x^{1/\alpha - 1} \tag{7.19}
\]
and
\[
j_\pm(dx) = \kappa_\pm(1 - \beta)x^{-\beta - 1}1_{(0, 1)}(x)dx \tag{7.20}
\]
for $w_\pm > 0$ and $\kappa_\pm > 0$. Then we have for $\kappa_\pm = \int_0^\infty x j_\pm(dx)$,
\[
b_\pm = \frac{\alpha^2(1 - \beta)}{1 - \alpha \beta}w_\pm \kappa_\pm, \quad p = \frac{w_+ \kappa_+}{w_+ \kappa_+ + w_- \kappa_-}, \quad \tilde{\kappa}_\pm = \frac{1 - \alpha \beta}{\alpha^2(1 - \beta)} \frac{\kappa_\pm}{w_+ \kappa_+ + w_- \kappa_-}. \tag{7.21}
\]
Hence it holds from Theorem 1.2
\[
\frac{1}{\gamma^{1/\alpha - 1}} \left( A(\gamma t) - pt \right) \xrightarrow[\gamma \to \infty]{f.d.} (1 - p)w_+ S^{(\alpha)}(\kappa_+ t) - pw_- S^{(\alpha)}(\kappa_- t). \tag{7.22}
\]
A Appendix: Continuity theorem for Laplace transforms

Although the following result seems to be well-known, we cannot find its proof anywhere, and so we prove it here for convenience of the readers.

**Proposition A.1.** Let $X_n, X$ be real-valued random variables such that

$$E[e^{-\lambda X_n}] < \infty, \ E[e^{-\lambda X}] < \infty \quad (n \in \mathbb{N}, \lambda > 0). \quad (A.1)$$

Assume the following holds:

$$\lim_{n \to \infty} E[e^{-\lambda X_n}] = E[e^{-\lambda X}] \quad (\lambda > 0). \quad (A.2)$$

Then we have

$$X_n \xrightarrow[n \to \infty]{} X. \quad (A.3)$$

**Proof.** Define $Y_n := e^{-X_n}$. By (A.1) and (A.2), we have

$$\sup_n E[Y_n^\lambda] < \infty \quad (\lambda > 0). \quad (A.4)$$

In particular, the sequence $\{Y_n\}$ is tight. Take a subsequence $\{Y_{n_k}\}_k$ such that

$$Y_{n_k} \xrightarrow[k \to \infty]{} Y \quad (A.5)$$

holds for some non-negative random variable $Y$. Then from (A.4), it holds that

$$\lim_{k \to \infty} E[Y_{n_k}^\lambda] = E[Y^\lambda] \quad (\lambda > 0). \quad (A.6)$$

From (A.2), we obtain

$$E[Y^\lambda] = E[e^{-\lambda X}] \quad (\lambda > 0). \quad (A.7)$$

Since $Y^s \leq Y \lor 1$ and $e^{-sX} \leq e^{-X} \lor 1$ for $0 < s < 1$, we have from the dominated convergence theorem

$$P[Y > 0] = \lim_{s \to 0^+} E[Y^s] = \lim_{s \to 0} E[e^{-sX}] = 1. \quad (A.8)$$

Hence we can define $Z = - \log Y$ and it follows from (A.7) that $X_{n_k} \xrightarrow[k \to \infty]{} Z$ and

$$E[e^{-\lambda Z}] = E[e^{-\lambda X}] \quad (\lambda > 0). \quad (A.9)$$

For the desired result, it suffices to prove $Z \overset{d}{=} X$.

From (A.1), we can define

$$\varphi_Z(w) := E[e^{-wZ}] \quad (\text{Re} \ w \geq 0). \quad (A.10)$$
Then it is enough to show that \( \varphi_Z \) is analytic on \( \{ w : \text{Re}(w) > 0 \} \) and continuous on \( \{ \text{Re}(w) \geq 0 \} \). Since we have \( x \leq e^x \) and \( xe^{-x} \leq 1/e \) for every \( x \geq 0 \), it holds

\[
|Z e^{-wZ}| = |Z e^{-(\text{Re}(w)Z)}| \leq \frac{1}{e \text{Re}(w)} \vee e^{-(1+\text{Re}(w))Z} \quad (\text{Re}(w) > 0).
\] (A.11)

Hence \( \varphi_Z \) is analytic on \( \{ w : \text{Re}(w) > 0 \} \). Take \( \xi \in \mathbb{R} \) and a sequence \( \{s_n\}_n \) in \( \{ w : \text{Re}(w) > 0 \} \) converging to 0. Then we have \( |e^{-(s_n+i\xi)Z}| \leq e^{-Z} \vee 1 \) for large \( n \) and from the dominated convergence theorem we obtain

\[
\varphi_Z(i\xi) = \lim_{n \to \infty} \varphi_Z(s_n + i\xi),
\] (A.12)

which proves the continuity of \( \varphi_Z \) on \( \{ w : \text{Re}(w) \geq 0 \} \).

\[ \square \]

B Appendix: Convergence of occupation times on the half line to non-degenerate distributions

Here we treat the case \( \frac{1}{t}A(t) \) converges in law to non-degenerate distribution. As we mentioned in Section 1, we can apply the methods used in [16], which are double Laplace transforms and Williams formula.

The following proposition called Williams formula can be proved by almost the same way in [4]. Though it is shown only for the Brownian motion in [4], the proof is essentially due to the property of the process that it does not jump from the positive (negative) side to the negative (positive) side without visiting the origin, and therefore we can extend the result to our situation.

Proposition B.1 (Williams formula). Let \( m_+, m_- \in \mathcal{M}_1 \) and \( j_+, j_- \) be Radon measures on \( (0, \infty) \) and suppose \( (m_+, j_+) \) and \( (m_-, j_-) \) satisfy (C). Then the following hold:

\[
A^{-1}(t) = t + \eta_{m_-, j_-}(\eta_{m_+, j_+}^{-1}(t)),
\] (B.1)

Here \( A^{-1}(t) \) and \( \eta_{m_+, j_+}^{-1}(t) \) are the right-continuous inverse processes of \( A(t) \) and \( \eta_{m_+, j_+}(t) \), respectively.

We can also prove the following two propositions by the same argument in [16]. For the proof of the former, Williams formula is utilized.

Proposition B.2. Under the same assumption in Proposition B.1, we have for every \( \lambda > 0 \) and \( \mu > 0 \),

\[
\int_0^\infty e^{-\mu t} E[e^{-\lambda A(t)}] dt = \frac{\chi_{m_+, j_+}(\lambda + \mu)/(\lambda + \mu) + \chi_{m_-, j_-}(\mu)/\mu}{\chi_{m_+, j_+}(\lambda + \mu) + \chi_{m_-, j_-}(\mu)}.
\] (B.2)

Proposition B.3. Under the same assumption in Proposition B.1, let \( \zeta \) be a real-valued random variable. Then \( \frac{1}{t}A(t) \xrightarrow{d} \zeta \) is equivalent to the following holds:

\[
\lim_{\gamma \to \infty} \int_0^\infty e^{-\mu t} E[e^{-\frac{2}{\gamma} A(t)}] dt = \int_0^\infty e^{-\mu t} E[e^{-\lambda \zeta}] dt \quad \text{for every } \lambda, \mu > 0.
\] (B.3)
The following theorem can be proved by the same argument in Theorem 1.1.

**Theorem B.4.** Let \( m \in \mathcal{M}_1 \), \( j \) be a Radon measure on \((0, \infty)\) and \( K \) be a slowly varying function at \( \infty \) and, assume \((m, j)\) satisfies \((C)\). Suppose the following hold:

(i) \( m(x) \sim (1 - \alpha)^{-1} x^{1/\alpha - 1} K(x) \quad (x \to \infty) \) for a constant \( \alpha \in (0, 1) \),

(ii) \( \kappa := \int_0^\infty x j(dx) < \infty. \)

Then we have

\[
\frac{1}{\gamma^{1/\alpha} K(\gamma)} \eta_{m,j}(\gamma t) \xrightarrow{\gamma \to \infty} T_{m(\alpha)}(\kappa t) \quad \text{on } \mathbb{D},
\]

\[
n_{m,j}[T_0 > s] \sim \frac{\kappa s^{-\alpha} L^s(s)^{\alpha}}{\Gamma(\alpha)} (s \to \infty),
\]

where \( L^s(x) \) is a de Bruijn conjugate of \( L(x) = K(x^\alpha) \).

The following is the desired limit theorem. Note that the definition of \( \mu_{\alpha, p} \) is given in (1.26).

**Theorem B.5.** Let \( m_+, m_- \in \mathcal{M}_1 \) and \( j_+, j_- \) be Radon measures on \((0, \infty)\) and suppose \((m_+, j_+)\) and \((m_-, j_-)\) satisfy \((C)\). Assume the following hold:

(i) \( m_\pm(x) \sim w_\pm (1 - \alpha)^{-1} x^{1/\alpha - 1} K(x) \quad (x \to \infty) \) for constants \( \alpha \in (0, 1) \), \( w_\pm > 0 \) and a slowly varying function \( K \) at \( \infty \),

(ii) \( \kappa_\pm := \int_0^\infty x j_\pm(dx) < \infty. \)

Then we have

\[
\frac{1}{t} A(t) \xrightarrow{t \to \infty} Y_{\alpha, p},
\]

where \( p = \frac{\kappa_+ w_+^\alpha}{\kappa_+ w_+^\alpha + \kappa_- w_-^\alpha} \) and \( Y_{\alpha, p} \) is distributed as \( \mu_{\alpha, p} \).

**Example B.6.** Let \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). Set

\[
m_\pm(0, x) = w_\pm (1 - \alpha)^{-1} (c + x)^{1/\alpha - 1}
\]

and

\[
j_\pm(dx) = \kappa_\pm (1 - \beta) x^{-\beta - 1} 1_{(0,1)}(x) dx
\]

for \( c \geq 0 \), \( w_\pm > 0 \) and \( \kappa_\pm > 0 \). Then we have the limit (B.6) with \( \int_0^\infty x j_\pm(dx) = \kappa_\pm. \)

28
References

[1] M. Barlow, J. Pitman, and M. Yor. Une extension multidimensionnelle de la loi de l’arc sinus. In Séminaire de Probabilités, XXIII, volume 1372 of Lecture Notes in Math., pages 294–314. Springer, Berlin, 1989.

[2] N. H. Bingham, C. M. Goldie, and J. L. Teugels. Regular variation, volume 27 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1987.

[3] W. Feller. The parabolic differential equations and the associated semi-groups of transformations. Ann. of Math. (2), 55:468–519, 1952.

[4] N. Ikeda and S. Watanabe. Stochastic differential equations and diffusion processes, volume 24 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1981.

[5] K. Itô. Essentials of stochastic processes, volume 231 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 2006. Translated from the 1957 Japanese original by Yuji Ito.

[6] K. Itô. Poisson point processes and their application to Markov processes. Springer-Briefs in Probability and Mathematical Statistics. Springer, Singapore, 2015. Mimeographic original in 1969.

[7] Y. Kasahara. Spectral theory of generalized second order differential operators and its applications to Markov processes. Japan. J. Math. (N.S.), 1(1):67–84, 1975/76.

[8] Y. Kasahara. Tails of the first hitting times of linear diffusions. Tsukuba J. Math., 40(1):55–79, 2016.

[9] Y. Kasahara and S. Kotani. On limit processes for a class of additive functionals of recurrent diffusion processes. Z. Wahrsch. Verw. Gebiete, 49(2):133–153, 1979.

[10] Y. Kasahara and S. Watanabe. Brownian representation of a class of Lévy processes and its application to occupation times of diffusion processes. Illinois J. Math., 50(1-4):515–539, 2006.

[11] Y. Kasahara and S. Watanabe. Remarks on Krein-Kotani’s correspondence between strings and Herglotz functions. Proc. Japan Acad. Ser. A Math. Sci., 85(3):22–26, 2009.

[12] S. Kotani. Krein’s strings with singular left boundary. Rep. Math. Phys., 59(3):305–316, 2007.

[13] S. Kotani and S. Watanabe. Krein’s spectral theory of strings and generalized diffusion processes. In Functional analysis in Markov processes (Katata/Kyoto, 1981), volume 923 of Lecture Notes in Math., pages 235–259. Springer, Berlin-New York, 1982.
[14] L. C. G. Rogers. Itô excursion theory via resolvents. *Z. Wahrsch. Verw. Gebiete*, 63(2):237–255, 1983.

[15] C. Stone. Limit theorems for random walks, birth and death processes, and diffusion processes. *Illinois J. Math.*, 7:638–660, 1963.

[16] S. Watanabe. Generalized arc-sine laws for one-dimensional diffusion processes and random walks. In *Stochastic analysis (Ithaca, NY, 1993)*, volume 57 of *Proc. Sympos. Pure Math.*, pages 157–172. Amer. Math. Soc., Providence, RI, 1995.

[17] W. Whitt. *Stochastic-process limits*. Springer Series in Operations Research. Springer-Verlag, New York, 2002. An introduction to stochastic-process limits and their application to queues.

[18] K. Yano. Convergence of excursion point processes and its applications to functional limit theorems of Markov processes on a half-line. *Bernoulli*, 14(4):963–987, 2008.