Vacuum Cherenkov radiation in spacelike Maxwell–Chern–Simons theory

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Abstract

A detailed analysis of vacuum Cherenkov radiation in spacelike Maxwell–Chern–Simons (MCS) theory is presented. A semiclassical treatment reproduces the leading terms of the tree-level result from quantum field theory. Moreover, certain quantum corrections turn out to be suppressed for large energies of the charged particle, for example, the quantum corrections to the classical MCS Cherenkov angle. It is argued that MCS-theory Cherenkov radiation may, in principle, lead to anisotropy effects for ultra-high-energy cosmic rays (UHECRs). In addition, a qualitative discussion of vacuum Cherenkov radiation from a modified-Maxwell term in the action is given, together with UHECR bounds on some of its dimensionless “coupling constants.”

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I. INTRODUCTION

A charged particle moving with constant velocity in a macroscopic medium is known to emit Cherenkov radiation if its velocity $v$ exceeds the phase velocity $v_{\text{ph}}$ of light in the medium \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. For a nondispersive isotropic medium with refractive index $n > 1$, the phase velocity of light $v_{\text{ph}} \equiv \omega/|k| = c/n$ is less than $c$, the velocity of light in vacuo. As $c$ corresponds to the maximum attainable velocity of a charged particle according to the theory of special relativity \cite{11}, it is then possible, for a sufficiently fast particle, to satisfy the Cherenkov condition $v > v_{\text{ph}}$ and radiate.

In certain Lorentz-violating theories of photons, the photon four-momentum $p_\mu = \hbar k_\mu \equiv \hbar (\omega/c, k)$ can, even in the vacuum, be spacelike ($\omega^2/c^2 - |k|^2 < 0$), so that $v_{\text{ph}} < c$. This last upper bound on the phase velocity allows for so-called “vacuum Cherenkov radiation,” that is, photon emission by a charged particle moving sufficiently fast in such a vacuum. The possibility of vacuum Cherenkov radiation in generic Lorentz-noninvariant theories has been discussed in, e.g., Refs. \cite{12, 13, 14}.

In this article, we study vacuum Cherenkov radiation in spacelike Maxwell–Chern–Simons (MCS) theory \cite{15, 16, 17}, continuing our previous work \cite{18}. Vacuum Cherenkov radiation in this theory has been studied earlier in Refs. \cite{19, 20}. Here, we are interested in the comparison with standard Cherenkov radiation in a macroscopic medium characterized by a refractive index. In addition, we will pay attention to quantum and spin effects, which will turn out to be relevant at large energies. The main focus of this article is theoretical, but applications to cosmic-ray physics will be briefly considered.

The article is organized as follows. In Sec. II, we present old and new results on vacuum Cherenkov radiation in MCS theory, with the fixed Chern–Simons vector taken to be purely spacelike. (The rather long expressions for the decay widths and radiation rates are relegated to Appendices A and B.) Possible physics applications of MCS-theory Cherenkov radiation include anisotropy effects for ultra-high-energy cosmic rays (UHECRs), as will be discussed in Sec. II B. In Sec. III we review certain well-known results on standard Cherenkov radiation in macroscopic media. These results are, in Sec. IV, applied to spacelike MCS theory in order to obtain a heuristic understanding of the expressions found in Sec. II. In Sec. V we give a qualitative discussion of vacuum Cherenkov radiation in another Lorentz-noninvariant theory with a modified-Maxwell term in the action. It is shown that UHECRs have the potential to set tight bounds on the “coupling constants” of the modified-Maxwell term (details are given in Appendix C together with a new bound based on the already available data). In Sec. VI we summarize our findings and discuss possible implications.

As to notation and conventions, we employ the Cartesian coordinates $(x^\mu) = (x^0, \mathbf{x}) = (ct, x^1, x^2, x^3)$, the Minkowski metric $(\eta_{\mu\nu}) = \text{diag}(+1, -1, -1, -1)$, and the totally antisymmetric Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$ with normalization $\epsilon_{0123} = 1$. Indices are lowered with the
Minkowski metric $\eta_{\mu\nu}$ and raised with the inverse metric $\eta^{\mu\nu}$. In most equations, we use natural units with $c = \hbar = 1$ but not always.

II. MCS-THEORY CHERENKOV RADIATION

A. Spacelike MCS theory

The electromagnetic MCS theory \cite{15, 16, 17} has the following action:

$$S_{\text{MCS}} = \int_{\mathbb{R}^4} d^4x \left( -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{4} m \epsilon_{\mu\nu\rho\sigma} \zeta^\mu A^\nu(x) F^{\rho\sigma}(x) \right),$$

with gauge field $A_\mu(x)$, Maxwell field strength $F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, Chern–Simons (CS) mass scale $m$, and fixed normalized CS vector $\zeta^\mu$. The background CS vector $\zeta^\mu$ can be timelike ($\zeta^\mu \zeta_\mu \equiv \zeta^\mu \eta_{\mu\nu} \zeta^\nu = 1$), null ($\zeta^\mu \zeta_\mu = 0$ with $\zeta^0 = 1$), or spacelike ($\zeta^\mu \zeta_\mu = -1$).

The timelike MCS theory appears to be inconsistent, that is, the theory violates unitarity or causality, or both \cite{15, 16}. In the present article, we specialize to the purely spacelike case,

$$\zeta^\mu \equiv (0, \zeta) \equiv (0, 0, 0, 1),$$

and assume the CS mass scale $m$ to be strictly nonzero and positive,

$$m > 0.$$ (2.2b)

The condition (2.2a) makes the propagation of light anisotropic and defines a class of preferred inertial frames, contradicting, thereby, the two axioms of the theory of special relativity \cite{11}.

The spacelike MCS theory has two photon modes with dispersion relations

$$\omega_{\pm}(k)^2 = |k|^2 \pm m \sqrt{|k|^2 \cos^2 \theta + m^2/4 + m^2/2},$$

where $\theta$ is the angle between wave vector $k$ and CS vector $\zeta$, or, more specifically, $\cos \theta = (k \cdot \zeta)/|k|$. The two polarization modes are denoted $\oplus$ and $\ominus$, corresponding to the different signs in (2.3). Further details on these polarization modes can be found in Sec. 2 of Ref. \cite{17} and Appendix A of Ref. \cite{18}. Note, finally, that the photon theory (2.1)–(2.2) is translation invariant but not rotation invariant, so that physical processes involving these photons (assuming Lorentz-invariant interactions) necessarily conserve energy-momentum but need not conserve angular momentum.
B. Vacuum Cherenkov radiation

We now add particles with electric charge $e$ and mass $M$ to the theory, taking the usual minimal coupling to the gauge field (i.e., replacing $\partial_\mu$ by $\partial_\mu + i e A_\mu$). The action of these charged particles is assumed to be Lorentz invariant, so that the Lorentz violation of the combined theory resides solely in the second term of (2.1). The $\ominus$ photon has, in fact, a spacelike four-momentum, which allows for Cherenkov radiation from any type of charged particle with mass $M$ and three-momentum $\mathbf{q}$, provided $\mathbf{q} \cdot \zeta \neq 0$ \[18\]. Hence, MCS-theory Cherenkov radiation has no threshold, in contrast to the situation for standard Cherenkov radiation in a nondispersive macroscopic medium (cf. Sec. IIIA). For later use, we introduce the following notations:

$$
\hat{\mathbf{q}} \equiv \frac{\mathbf{q}}{|\mathbf{q}|}, \quad q_\parallel \equiv \mathbf{q} \cdot \zeta, \quad q_\perp \equiv q_\perp / |q_\perp| \equiv \mathbf{q} - q_\parallel \zeta,
$$

(2.4)

for the unit vector in the $\mathbf{q}$ direction and momentum components parallel and orthogonal to the normalized CS vector $\zeta$. Remark that $q_\perp$ is defined to be nonnegative, whereas $q_\parallel$ can have an arbitrary sign.

For the purpose of calculating the decay width, only the $q_\parallel$ dependence is nontrivial. We assume $|q_\parallel| \gg m$ and $M \gg m$, where $m$ is the positive CS mass scale and $M$ the mass of the charged particle. For MCS-theory Cherenkov radiation with $|q_\parallel| \sim M$, the allowed photon momentum component is very small, $|k_\parallel|$ running from zero to $O(m)$, while, for $|q_\parallel| \gg M$, the $|k_\parallel|$ maximum is roughly equal to $|q_\parallel|$. The tree-level amplitude square follows from the usual Feynman amplitude (Fig. 1). The result for a charged scalar is found to be given by

$$
|A|^2_{\text{scalar}} = e^2 \left( 4q^\mu q^\nu - 2q^\mu k^\nu - 2q^\nu k^\mu + k^\mu k^\nu \right) \epsilon_\mu(k) \epsilon_\nu(k),
$$

(2.5)

where $q^\mu$ is the initial four-momentum of the charged particle and $k^\mu$ the four-momentum of the emitted photon. The result for a charged Dirac spinor, averaged over initial spin components and summed over final spin components, has already been calculated in Ref. \[18\].
and reads
\[ |A|_{\text{spinor}}^2 = e^2 \left( 4q^\mu q^{\nu} - 2q^\mu k^{\nu} - 2q^{\nu} k^\mu + k^2 \eta^{\mu\nu} \right) \bar{\epsilon}_\mu(k) \epsilon_\nu(k). \] (2.6)

For both amplitude squares, the polarization rule \[18\] gives
\[ \bar{\epsilon}_\mu(k) \epsilon_\nu(k) \mapsto \frac{1}{2k^2 + m^2 \zeta^2} \left( -k^2 \eta_{\mu\nu} - m^2 \zeta_{\mu} \zeta_{\nu} + i m \epsilon_{\mu\nu\rho\sigma} \zeta^\rho k^\sigma \right). \] (2.7)

The decay widths of a particular momentum state of the scalar and spinor particles are then defined as the phase-space integral of the resulting amplitude squares,
\[ \Gamma_{\text{scalar/spinor}} = \int \mathcal{D}k \ |A|_{\text{scalar/spinor}}^2, \] (2.8)
where \( \mathcal{D}k \) is a shorthand notation for the phase-space measure; see Ref. \[18\] for details.

The decay widths for large parallel momentum components, \( |q_\parallel| \gg M \), are found to be given by
\[ \Gamma_{\text{scalar}} = \frac{1}{2} \alpha m |\hat{q} \cdot \zeta| \left( \ln(|q_\parallel|/m) + 2 \ln 2 - 1 \right) + \cdots \] (2.9)
and
\[ \Gamma_{\text{spinor}} = \frac{1}{2} \alpha m |\hat{q} \cdot \zeta| \left( \ln(|q_\parallel|/m) + 2 \ln 2 - 3/4 \right) + \cdots, \] (2.10)
with fine-structure constant \( \alpha \equiv e^2/4\pi \) (standard quantum electrodynamics of photons and electrons has \( \alpha \approx 1/137 \) and \( M \approx 511 \text{ keV} \)). The exact tree-level results for the decay widths with general three-momentum \( q \) are given in Appendix A.

Spacelike MCS theory, if physically relevant, thus predicts a direction-dependent lifetime of high-energy charged particles due to vacuum Cherenkov radiation. In principle, this nonstandard energy-loss mechanism may affect the propagation of UHECRs. Very possibly, even the numbers may work out, as the following example demonstrates.

Assume the CS mass scale \( m \) to be of cosmological origin (perhaps due to new low-energy/large-distance physics) and its numerical value to be given by the inverse of the size of the visible universe, \( m \sim 1/L_0 \approx 1/(10^{10} \text{lyr}) \approx 2 \times 10^{-33} \text{eV} \), which is consistent with astrophysical bounds \[15, 21\]. Also assume primary cosmic-ray protons with energy \( E_p = 10^{19} \text{eV} \) (i.e., just below the Greisen–Zatsepin–Kuzmin cutoff; cf. Refs. \[22, 23, 24\]) to have traveled over cosmological distances of the order of \( L_0 \sim 1/m \). With \( q_\parallel \equiv q \cdot \zeta \sim E_p \cos \theta_p \) and \( \alpha \ln(E_p/m) = O(1) \), the decay width (2.10) for these \( 10^{19} \text{eV} \) protons becomes \( \Gamma_p \sim (m/2)|\cos \theta_p| \) and there results a modest anisotropy for \( 10^{19} \text{eV} \) protons observed on Earth, with somewhat less protons from directions \( \hat{q} = \pm \zeta \) (having \( |\cos \theta_p| = 1 \)) than from orthogonal directions \( \hat{q} \perp \zeta \) (having \( |\cos \theta_p| = 0 \)). Alternatively, the lack of large-scale anisotropy (as suggested by the current experimental data \[22, 23, 24\] and assuming the
absence of strong extragalactic magnetic fields) would place a further upper bound on the mass scale $m$ of spacelike MCS theory at approximately $10^{-33}$ eV and perhaps a factor 10 better with the complete Auger data set. In this brief discussion, the expansion of the universe has not been taken into account, but this can, in principle, be done to leading order in $m$ \cite{23, 26}.

Needless to say, the example of the previous paragraph has been given for illustrative purposes only. But the fact remains that Lorentz-violating processes such as the one studied in the present article could turn out to be relevant to UHECR physics. Indeed, for cosmological applications of vacuum Cherenkov radiation, another important quantity to calculate is the radiation rate, which will be done in the next subsection.

C. Radiated energy rate

The energy-momentum loss of the charged particle (scalar or spinor) per unit of time is equal to the photon four-momentum weighted by the amplitude square and integrated over phase-space,

$$
\frac{dP^\mu}{dt} \equiv \int Dk \left|A\right|^2 k^\mu. \tag{2.11}
$$

Making the Ansatz

$$
\frac{dP^\mu}{dt} = \alpha m \left(K q^\mu + L m \zeta^\mu\right), \tag{2.12}
$$

one can calculate the dimensionless coefficients $K(q_\perp, q_\parallel)$ and $L(q_\perp, q_\parallel)$ for the spacelike MCS theory considered; see Appendix.\[3] The time component of this last expression then gives the rate of total radiated energy,

$$
\frac{dW}{dt} = \frac{dP^0}{dt} = \alpha m K q^0, \tag{2.13}
$$

for $\zeta^0 = 0$ from \[2.2a\]. As $t$ corresponds to the laboratory time, the path length of the charged particle is given by $l = \beta ct$, at least for uniform motion of the charged particle (i.e., neglecting radiation backreaction).

The radiated energy rate (Fig. 2) has three qualitatively different domains, again assuming $|q_\parallel| \gg m$ and $M \gg m$, while keeping $q_\perp$ arbitrary. For low momentum components compared to the particle mass, $m \ll |q_\parallel| \ll M$, the radiation rate is essentially the same for a scalar or spinor particle and given by

$$
\frac{dW}{dt} = \frac{\alpha}{4} m^2 \frac{|q_\parallel|^5}{M^5} + O(\alpha m^3 |q_\parallel|^5 / M^6). \tag{2.14}
$$
FIG. 2: Radiated energy rate $dW/dt$ from MCS-theory Cherenkov radiation, in units of $\alpha M^2$ and as a function of $|q||/M$, for a charged scalar particle with mass $M$, electric charge $e \equiv \sqrt{4\pi\alpha}$, and parallel momentum component $q_\parallel$ as defined by (2.4). The radiated energy rate is calculated from (2.13) and (B1a), for a particular choice of the CS mass scale, $m = 10^{-10}M$. The curve for a charged Dirac particle hardly differs from the curve for a charged scalar particle shown here.

For intermediate momentum components, $mM \ll m|q|| \ll M^2$, the radiation rate is again spin independent to leading order:

$$\frac{dW}{dt} = \frac{\alpha}{4} \frac{m^2 |q||^2}{M^2} + O(\alpha m^2). \quad (2.15)$$

Our results (2.14) and (2.15) agree with the expression (21) obtained classically by Lehnert and Potting [20] for a particular charge distribution and in the limit $M/m \to \infty$.

For large momentum components, $m|q|| \gg M^2$, the radiated energy rate depends on the spin of the charged particle:

$$\frac{dW_{\text{scalar}}}{dt} = \frac{\alpha}{4} m |q|| + O(\alpha M^2), \quad (2.16)$$

and

$$\frac{dW_{\text{spinor}}}{dt} = \frac{\alpha}{3} m |q|| + O(\alpha M^2), \quad (2.17)$$

1 We disagree, however, with the polarization pattern shown in Fig. 2 of Ref. [20]. The emitted radiation consists solely of $\oplus$ photons ($\ominus$ photons are kinematically not allowed), so that the radiation is essentially left polarized for wave vectors $\mathbf{k}$ in the approximate hemisphere around $\zeta$ and essentially right polarized for wave vectors $\mathbf{k}$ in the approximate hemisphere around $-\zeta$, with elliptical polarizations in a narrow band given by $|\mathbf{k} \cdot \zeta| \lesssim m$. (For details on the $\ominus$ polarization mode, see, e.g., the paragraphs below Eq. (2.13) of Ref. [17] and have, for generic $\mathbf{q}$ (that is, $|\mathbf{q} \cdot \zeta| \gg m$), the same circular polarization, left or right depending on the sign of $\mathbf{q} \cdot \zeta$. With the conserved helicity of the relativistic charged particle, the MCS Cherenkov process for generic $\mathbf{q}$ and large photon momentum component $|k|| \sim |q|| \gg M$ manifestly violates angular-momentum conservation [18].
with only a linear dependence on the initial parallel momentum component, compared to the quadratic dependence (2.15) for intermediate momentum components. The crossover from quadratic to linear behavior occurs at a momentum component $|q_\parallel|$ of the order of $M^2/m$. For the case of an electron, this crossover momentum takes the following numerical value:

$$|q_\parallel|_{\text{crossover}} \sim M^2 c/m \approx 1.3 \times 10^{44} \text{eV}/c \left( \frac{M}{511 \text{keV}/c^2} \right)^2 \left( \frac{2 \times 10^{-33} \text{eV}/c^2}{m} \right), \quad (2.18)$$

for an $m$ value corresponding to the inverse of the size of the visible universe (cf. the discussion at the end of Sec. II B). The tremendous energy appearing in (2.18) is, of course, experimentally out of reach. But the general considerations of this article may still be relevant to MCS theory in other applications (e.g., ultracold atomic systems [27] and photonic crystals [28]) or to other Lorentz-violating theories such as the one discussed in Sec. V.

At first sight, the difference between (2.16) and (2.17) is surprising since the leading-order terms of the decay width are the same for scalars and spinors, as shown by the logarithmic terms in (2.9) and (2.10). These last terms result, in fact, from the contribution of low-energy photons in the phase-space integral (2.8), where the difference between the two amplitude squares is small. But the radiated energy rate from (2.11) for $\mu = 0$ has an additional factor $k^0$ in the integrand and the integral gives, for sufficiently high energies of the charged particles, different rates for scalars and spinors. This spin dependence at ultrahigh energies is a genuine quantum effect and cannot be seen in the classical analysis of Refs. [19, 20].

For completeness, we give results for two further cases. First, consider charged massless spinors ($M = 0$), still with relatively large parallel momentum components, $|q_\parallel| \gg m$. It may then be of interest to calculate the radiated energy rate of left-handed and right-handed spinors independently. The result is

$$\frac{dW_{\text{spinor}, M = 0}}{dt} = \frac{\alpha}{3} m |q_\parallel| \left( 1 + \chi \text{sgn}(q_\parallel) \right) + O(\alpha m^2), \quad (2.19)$$

for spinor helicity $\chi = \pm 1/2$. In the corresponding decay width (2.10), only the subleading term $-3/4$ inside the outer parentheses is replaced by $-3 \left( 1 - 2 \chi \text{sgn}(q_\parallel) \right)/4$.

Second, consider charged massless scalar or spinor particles ($M = 0$), now with ultralow parallel momentum components, $|q_\parallel| \ll m$. The radiated energy rate is then given by

$$\frac{dW_{M = 0}}{dt} = \frac{\alpha}{5} |q_\parallel|^2 + O(\alpha |q_\parallel|^3/m). \quad (2.20)$$

The radiated energy rate (2.20) is not directly proportional to $m$ or $m^2$, but its numerical value is still very much less than $\alpha m^2$ because of the stated validity domain, $|q_\parallel| \ll m$.

The rest of the article is mainly concerned with a heuristic understanding of the results obtained in this section and to apply that understanding to another Lorentz-violating theory.
III. ST ANDARD CHERENKOV RADIA TION

A. Classical process

In this section, we recall some well-known results on standard Cherenkov radiation \[1, 2, 3, 4\] and refer to three monographs \[8, 9, 10\] for further details and references (useful discussions can also be found in the textbooks \[29, 30, 31\]). Specifically, we consider the propagation of an electrically charged particle in an isotropic dielectric \((\epsilon(\omega) \neq 1, \mu(\omega) = 1)\) with refractive index \(n(\omega) = \frac{c}{\omega} \sqrt{\epsilon(\omega)}\), for wave vector \(k\) and angular frequency \(\omega\) of the electromagnetic field. The particle has classical charge \(Q \neq 0\) (Coulomb potential \(V = \frac{Q}{4\pi r}\) in Heaviside–Lorentz units), mass \(M \geq 0\), velocity \(\beta \equiv v/c \leq 1\), three-momentum \(q\) and energy \(E = \sqrt{c^2 |q|^2 + M^2 c^4}\). In this section, we prefer to keep \(c\) and \(\hbar\) explicit.

Cherenkov radiation of a particular frequency \(\omega\) is emitted classically over a cone which makes an angle \(\theta_C(\omega)\) with the direction of motion of the charged particle. The numerical value of this polar angle can be determined by a Huygens-principle construction:

\[
\cos \theta_C(\omega) = \frac{1}{\beta n(\omega)},
\]

as long as \(\beta n(\omega) \geq 1\). The emitted radiation is linearly polarized with the magnetic field lying along the surface of the cone and the electric field orthogonal to it (angular momentum is manifestly conserved for a relativistic charged particle; compare with the last sentence of Footnote \[1\]). The energy radiated per unit of time and per unit of frequency is determined by the Frank–Tamm formula,

\[
\frac{d^2 w}{dt d\omega} = \beta Q^2 \frac{\sin^2 \theta_C(\omega)}{4\pi c} \omega, \tag{3.2}
\]

where, according to (3.1), the factor \(\sin^2 \theta_C(\omega)\) can be replaced by \(1 - (\beta n(\omega))^2\).

After integration of (3.2) over the allowed frequency range, the total radiated energy rate \(dW/dt\) for \(\beta = 1\) is infinite classically, unless the refractive index \(n(\omega)\) approaches unity fast enough for large \(\omega\). Indeed, if \(n(\omega) = 1\) above a cutoff frequency \(\omega_c\), the total radiated energy rate is of the order of

\[
\frac{dW}{dt} \sim \frac{Q^2}{4\pi c} \omega_c^2, \tag{3.3}
\]

purely by dimensional reasons.

B. Quantum effects

The expression (3.1) does not take energy-momentum conservation into account if the photon has energy \(\hbar\omega\) and effective momentum \(\hbar |k| = \hbar n(\omega) \omega/c\). The correct expression
for the Cherenkov angle is [5, 6]

\[ \cos \theta_C(\omega) = \frac{1}{\beta n(\omega)} \left( 1 + \frac{\hbar \omega}{2E} (n(\omega)^2 - 1) \right), \]  \hspace{1cm} (3.4)

as long as there is a real solution for \( \theta_C(\omega) \).

For constant refractive index \( n \), (3.4) gives a maximum photon energy

\[ \hbar \omega_{\text{max}} = 2E \frac{\beta n - 1}{n^2 - 1}. \]  \hspace{1cm} (3.5)

The quantum modification (3.4) makes the Cherenkov angle smaller than the classical value (as long as \( \beta n > 1 \)) and the Cherenkov cone shrinks to the forward direction as \( \omega \to \omega_{\text{max}} \).

The total radiated energy rate is now given by

\[ \frac{dW}{dt} \sim \frac{Q^2}{4\pi c} \frac{E^2}{\hbar^2}, \]  \hspace{1cm} (3.6)

up to factors of order unity. Expression (3.6), compared to (3.3), makes clear that quantum theory \((\hbar \neq 0)\) renders the radiated energy rate finite by providing a cutoff on the frequency of the emitted radiation, even for the case of a frequency-independent refractive index.

C. Model

Consider, next, a refractive index which is assumed to behave as follows:

\[ n(\omega) \big|_{\text{model}} = 1 + \frac{\omega_0}{2\omega}, \]  \hspace{1cm} (3.7)

for \( \omega > \omega_0 \), with fixed angular frequency \( \omega_0 \). The assumed behavior of (3.7) certainly corresponds to “anomalous dispersion,” but the functional dependence on \( \omega \) is very different from that of standard macroscopic media with \( n(\omega) \sim 1 - \omega_p^2/\omega^2 \) for \( \omega \to \infty \); cf. Sec. 7.5 of Ref. [31]. The precise form of (3.7) is, in fact, chosen for comparison to MCS theory, as will become clear in the next section.

For the special behavior (3.7) of the refractive index, the quantum correction term in (3.4) turns out to be essentially frequency independent for \( \omega \gg \omega_0 \),

\[ \cos \theta_C(\omega) \big|_{\text{model}} = \frac{1}{\beta n(\omega)} \left( 1 + \frac{\hbar \omega_0}{2E} (1 + O(\omega_0/\omega)) \right), \]  \hspace{1cm} (3.8)

making the classical Cherenkov angle a good approximation for large particle energy \( E \gg \hbar \omega_0 \). For the model considered, the cutoff frequency is given by \( \omega_{\text{max}} = E/\hbar \) and the total radiated energy rate is finite.\(^2\) Making the replacements \( \beta = c|q|/E \) and

\(^2\) Classically, the radiation output in a medium with refractive index (3.7) would be proportional to \( \omega_0^2 \). This output would, however, be finite only for \( \beta < 1 \), because \( n(\omega) \) does not approach unity fast enough for \( \omega \to \infty \).
\( E = \sqrt{c^2 |q|^2 + M^2 c^4} \), one obtains the following high-energy behavior:

\[
\frac{dW}{dt} \mid_{\text{model}}^{\text{MCS}} = \frac{1}{2} \frac{Q^2}{4\pihc} \omega_0 E + \cdots ,
\]

(3.9)

where the ellipsis contains terms (involving logarithms) which are, at ultrahigh energies, small compared to the term shown. Note that the radiated energy rate (3.9) only grows linearly with \( E \), compared to the quadratic behavior (3.6) for the case of constant refractive index.

**IV. HEURISTICS OF MCS-THEORY CHERENKOV RADIATION**

**A. Refractive index**

High-energy \( \odot \) photons of MCS theory moving in a generic direction \( \hat{k} \) have, according to (2.3), a refractive index given by

\[
n_{\odot}(k) \equiv \frac{|k|}{\omega_-(k)} = 1 + \frac{m|\cos \theta|}{2|k|} + O\left(\frac{m^2}{|k|^2}\right),
\]

(4.1)

for \( |\cos \theta| \equiv |\hat{k} \cdot \zeta| \gg m/|k| \) and \( c = \hbar = 1 \). This is, in fact, the refractive index relevant to MCS-theory Cherenkov radiation from a highly energetic charged particle moving in a generic direction, where most radiation energy is carried away by photons with large parallel momentum components, \( |k \parallel| \equiv |\hat{k} \cdot \zeta| \gg m \). Dropping the suffix \( \odot \) on \( n \), the refractive index (4.1) can be written as

\[
n(\omega, \hat{k}) \mid_{\text{model}}^{\text{MCS}} = 1 + \frac{m|\cos \theta|}{2\omega} + O\left(\frac{m^2}{\omega^2}\right),
\]

(4.2)

in order to connect to the particular model discussed in Sec. III C.

**B. Cherenkov angle and radiated energy rate**

Setting \( \hbar \omega_0 \sim mc^2 \) in the model result (3.8) shows that, for high energies \( E \) of the charged particle, the quantum correction to the Cherenkov angle goes to zero as \( mc^2/E \). This behavior is quite different from that of standard Cherenkov radiation in a nondispersive medium, as given by (3.4) for \( \hbar \omega \sim E/2 \) and constant \( n \). The “good” high-energy behavior of MCS theory is perhaps not entirely surprising as the nonstandard term in the action (2.1) is super-renormalizable.

Specifically, for large energy \( E \gg M \gg m \) and generic direction \( \hat{q} \) of the charged particle \( (\hat{q} \cdot \zeta \neq 0) \), the MCS-theory Cherenkov radiation is emitted in a pencil beam around the \( \hat{q} \) direction with an angular dimension of the order of

\[
2 \theta_C(\omega, \hat{q}) \mid_{\text{model}}^{\text{MCS}} \sim 2 \sqrt{|\hat{q} \cdot \zeta| mc^2/(\hbar \omega)} \left(1 + O\left(Mc^2/E, mc^2/(\hbar \omega)\right)\right),
\]

(4.3)
for frequencies $\omega$ up to $\omega_{\text{max}} = E/\hbar$ and with $\hbar$ and $c$ temporarily reinstated.\(^3\) The factor $|\mathbf{\hat{q}} \cdot \zeta|$ under the square root of (4.3) corresponds to the absolute value of the cosine of the angle between the charged particle direction $\mathbf{\hat{q}}$ and the fixed CS direction $\zeta$, which, in turn, traces back to the cosine factor in the refractive index (4.2) of the individual photons.

We have also calculated the radiated energy rate $dW/dt$ from the model result (3.9), replacing $Q^2/(4\pi \hbar c)$ by $\alpha$ as defined below (2.10) and inserting an overall factor of $1/2$. This extra factor $1/2$ for the radiated energy rate is due to the fact that only one photon polarization ($\oplus$) contributes in MCS theory, the $\ominus$ photon having a timelike four-momentum. For intermediate momentum components, $mM < |q|| < M^2$, the adapted model result (3.9), taking into account the terms not shown explicitly, is numerically in good agreement with the original expression (2.15). For ultrahigh momentum components, $m|q|| > M^2$, the adapted model result (3.9), with $\omega_0$ replaced by $m|\cos \theta|$, gives immediately

$$
\left. \frac{dW}{dt} \right|_{\text{MCS}}^{\text{model}} = \frac{\alpha}{4} m |q|| + \cdots ,
$$

which agrees with the scalar radiation rate (2.16) calculated directly. The explanation for the slightly different spinor radiation rate (2.17) will be given in the next subsection.

### C. Spin effects

A charged particle with spin has a different interaction with the photon as a charged particle without spin. The Cherenkov radiation of a charged Dirac particle receives, therefore, an extra contribution (e.g., from the magnetic dipole moment) compared to the case of a charged scalar particle. From Eq. (2.39) of Ref. [8] or Eq. (280) of Ref. [9], we obtain the following extra contribution for a particle of spin $1/2$:

$$
\Delta \left( \frac{dW_{\text{spinor}}}{dt} \right) = \frac{\alpha}{\beta} \int_0^{\omega_{\text{max}}} d\omega \, \hbar \omega \left( \frac{\hbar^2 \omega^2}{4 E^2} \left( n(\omega)^2 - 1 \right) \right).
$$

Using the refractive index (4.2) and $\omega_{\text{max}} = E > m$ (again setting $c = \hbar = 1$), this expression gives a contribution with a linear momentum dependence,

$$
\Delta \left( \frac{dW_{\text{spinor}}}{dt} \right) \bigg|_{\text{MCS}}^{\text{model}} = \frac{\alpha}{12} m |q|| + \cdots ,
$$

which explains the difference between (2.16) and (2.17).

The crossover between the spin-independent behavior of (2.15) and the spin-dependent behavior of (2.16)-(2.17) occurs at a momentum component $|q||$ of order $M^2/m$, which has already been discussed in Sec. II C.

---

\(^3\) The mass scale $m$ enters the action (2.1) through the combination $mc/\hbar \equiv 1/\ell$, in terms of a fundamental length scale $\ell$. In this way, the leading term of (4.3) can be written as $2 \sqrt{|\mathbf{q} \cdot \zeta|} c/(\ell \omega)$, without explicit Planck constant $\hbar$. 
V. MODIFIED-MAXWELL-THEORY CHERENKOV RADIATION

The main focus of this article has been on explicit calculations of vacuum Cherenkov radiation in spacelike MCS theory coupled to Lorentz-invariant charged particles. In this section, we give a qualitative discussion of vacuum Cherenkov radiation in the only other possible gauge-invariant renormalizable theory for photons with Lorentz violation, namely, the so-called modified-Maxwell theory.

The action for modified-Maxwell theory can be written as follows \[32, 33\]:

\[
S_{\text{modM}} = \int_{\mathbb{R}^4} d^4 x \left( -\frac{1}{4} \left( \eta^{\mu\nu} \eta^{\rho\sigma} + \kappa^{\mu\nu\rho\sigma} \right) F_{\mu\nu}(x) F_{\rho\sigma}(x) \right),
\]

(5.1)

for a real dimensionless background tensor \(\kappa^{\mu\nu\rho\sigma}\) having the same symmetries as the Riemann curvature tensor and a double trace condition \(\kappa^{\mu\nu\rho\sigma} = 0\) (so that there are \(20 - 1 = 19\) independent components). All components of the \(\kappa\)–tensor in (5.1) are assumed to be very small, \(|\kappa^{\mu\nu\rho\sigma}| \ll 1\). Remark that the \(\kappa F F\) term in (5.1) is CPT even, whereas the \(m AF\) term in (2.1) is CPT odd.

Vacuum Cherenkov radiation for standard electrodynamics with the modified photonic action (5.1) has been studied classically by Altschul \[34, 35\]. Here, we can already make some general remarks on quantum effects, keeping \(\hbar\) and \(c\) explicit for the remainder of this section.

The nonstandard term in the action (5.1) is scale invariant, just as the standard term, and the modified photon dispersion relation is given by

\[
\omega(k)^2 = c^2 |k|^2 \left( 1 - \Theta(\hat{k}) \right),
\]

(5.2)

with \(\hat{k}\) the unit vector in the direction of \(k\) and \(\Theta\) a particular function of \(\hat{k}\), the components of the \(\kappa\)–tensor being considered fixed. Hence, the refractive index \(n = 1/\sqrt{1 - \Theta}\) depends on direction, not frequency.

However, according to (3.4) with \(n(\omega)\) replaced by \(n(\hat{k})\), quantum effects make the Cherenkov angle frequency dependent by an additional term proportional to the ratio of photon energy \(\hbar \omega\) and particle energy \(E\). Assuming a massless charged particle \((M = 0, \beta \equiv v/c = 1)\) and refractive index \(n = 1 + \delta n\) with \(\delta n = \delta n(\hat{k}) \geq 0\), we have for the Cherenkov relation (3.4):

\[
\cos \theta_C = 1 - n(1 - \hbar \omega/E) + O(\delta n^2),
\]

(5.3)

and for the corresponding factor entering the differential radiated energy rate (3.2):

\[
\sin^2 \theta_C = 2 \delta n (1 - \hbar \omega/E) + O(\delta n^2),
\]

(5.4)

so that anisotropy effects from \(\delta n\) occur already at zeroth order in \(\hbar \omega/E\) and explicit quantum effects at first order. Spin effects are of higher order in \(\omega\) but of the same order in \(\delta n\), according to (4.5).
Under the conditions stated, one expects for the generic radiated energy rate of a particle with electric charge \( e \equiv \sqrt{4\pi \alpha} \), mass \( M \geq 0 \), momentum \( \mathbf{q} \), and energy \( E \sim c|\mathbf{q}| \):

\[
\frac{dW_{\text{modM}}}{dt} = \alpha (\kappa qq) c^2/\hbar \sim \alpha \left( \xi_0 + \xi_1(\hat{\mathbf{q}}) \right) E^2/\hbar \bigg|_{E \gg E_{\text{thresh}}},
\]

in terms of the highly symbolic notation \((\kappa qq)\) for the appropriate contractions of the \(\kappa\)-tensor with two \(\mathbf{q}\)-vectors and nonnegative coefficients \(\xi_0\) and \(\xi_1\), the latter coefficient having a nontrivial direction dependence. The asymptotic behavior shown in (5.5) holds only for particle energies \(E\) well above the (direction-dependent) Cherenkov threshold, which has an order of magnitude given by

\[
E_{\text{thresh}} \sim Mc^2/\sqrt{\kappa},
\]

for an appropriate scale \(\kappa\) obtained from the \(\kappa^{\mu\nu\rho\sigma}\) components (\(\kappa\) is effectively set to zero if Cherenkov radiation is not allowed).

Our estimate for the threshold energy agrees with the result obtained by Altschul [34], as given by his Eq. (4) for a subset of \(\kappa\)-components (see below). His treatment of the radiation rate, however, is purely classical, as it neglects quantum effects on the Cherenkov angle and the differential radiation rate. He, then, introduces an energy cutoff \(\Lambda\) (possibly related to “new physics” which may or may not be required by causality) to make the total radiated energy rate finite, \(dW/dt \sim \alpha \kappa \Lambda^2/\hbar\), as given by his Eq. (7). But, as discussed in our Sec. III B, such a cutoff is already provided in the quantum theory by the energy \(E\) of the particle. With this cutoff \(E\), the radiation rate above threshold is really given by \(dW/dt \sim \alpha \kappa E^2/\hbar\), in agreement with (5.5) above. The difference between the asymptotic energy behaviors of the radiation rates (2.17) and (5.5) traces back to the fact that the CS term in (2.1) has a single derivative and the \(\kappa FF\) term in (5.1) has two. Once more, the total radiated energy rate for both Lorentz-violating theories is finite because of the frequency cutoff from standard quantum mechanics.

Possible signatures of the Lorentz-violating action (5.1) and the corresponding radiation rate (5.5) include nonstandard propagation effects for UHECRs, similar to the MCS effects discussed in the last three paragraphs of Sec. II B. In order to be specific, let us follow Altschul [34] by keeping only nine of the nineteen independent “coupling constants” from \(\kappa^{\mu\nu\rho\sigma}\) in (5.1), namely, those which do not lead to birefringence. Precisely these coupling constants, for flat spacetime denoted \(\tilde{\kappa}^{\mu\nu} \equiv \kappa^{\rho\mu\sigma\nu} \eta_{\rho\sigma}\) (symmetric and traceless in \(\mu, \nu\)), are only constrained at the \(10^{-16}\) level or worse [36, 37, 38]. Now, taking the most energetic UHECR event known today [39, 40] to correspond to a primary proton with energy \(E_p \approx 3 \times 10^{11}\) GeV and restmass \(M_p \approx 0.938\) GeV/c\(^2\), the Cherenkov threshold condition, \(M_p c^2/\sqrt{\kappa} \gtrsim E_p\), gives the following upper bound on the magnitude of generic \(\tilde{\kappa}^{\mu\nu}\) components: \(\tilde{\kappa} \lesssim 10^{-23}\). (A similar bound at the \(10^{-23}\) level has been derived kinematically by Coleman and Glashow [14].) Further details on our bound for generic \(\tilde{\kappa}^{\mu\nu}\) and similar bounds for special (nongeneric) \(\tilde{\kappa}^{\mu\nu}\) are given in Appendix C.
VI. SUMMARY

In Secs. II–IV of this article, we have arrived at a detailed understanding of vacuum Cherenkov radiation in spacelike MCS theory (2.1)–(2.2) coupled to Lorentz-invariant charged particles with spin 0 or 1/2.

Remarkably, quantum corrections to the Cherenkov angle $\theta_C$ and the amplitude square are suppressed because the refractive index $n$, for large photon momenta $|k|$, behaves as $1 + O(m/|k|)$, with $m$ the mass scale of the MCS theory. Quantum effects enter mainly by the condition on the maximum radiated photon energy and the resulting total radiated energy rate is, for ultrahigh particle energy $E$, proportional to $mE$.

In addition, the effects of the charged particle’s spin are not negligible and change the coefficient of the leading term of the radiation rate. Since the radiated MCS photons are typically circularly polarized, effects of angular momentum nonconservation play a role. But, like quantum effects for the amplitude square, further spin effects are suppressed at ultrahigh energies.

Vacuum Cherenkov radiation is quite different in modified-Maxwell theory (5.1), as shown in Sec. V. The corresponding radiation rate has not been calculated exactly but its generic asymptotic behavior has been established and was given in (5.5). Contrary to the case of MCS-theory Cherenkov radiation, modified-Maxwell-theory Cherenkov radiation does have an energy threshold, which allows for UHECR bounds on certain combinations of the nineteen “coupling constants” from $\kappa^{\mu\nu\rho\sigma}$ in the modified-Maxwell action (5.1).

As discussed in Appendix C, future UHECR bounds on the nine nonbirefringent coupling constants from $\kappa^{\mu\nu\rho\sigma}$ can be expected at the $10^{-23}$ level, at least, for primary protons. At this moment, we only have a bound at the $10^{-23}$ level for a special choice of coupling constants (the spatially isotropic case 2 in Appendix C). Together with existing bounds on the ten remaining birefringent coupling constants from $\kappa^{\mu\nu\rho\sigma}$ at the $10^{-32}$ level [36], these expected UHECR bounds at the $10^{-23}$ level may suggest that the CPT–even Lorentz-violating $\kappa_{FF}$ term in (5.1) is effectively absent. If true, this constitutes one more hint (see, e.g., Refs. [44, 45] for other hints) in support of the fundamental role of Lorentz invariance at the high-energy/small-distance frontier of physics.

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4 The same can perhaps not be said of the CPT–odd $mAF$ term in (2.1), for which there exists at least one physical mechanism [41, 42, 43] that naturally gives small values for the mass scale $m$, namely, proportional to the inverse of the size of the universe.
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APPENDIX A: MCS DECAY WIDTHS

In this Appendix, analytic tree-level results for the decay widths from MCS Cherenkov radiation are presented. Considered are charged scalar and spinor particles with electric charge $e$, mass $M$, and generic three-momentum $\mathbf{q} \equiv \mathbf{q}_\parallel + \mathbf{q}_\perp$. The mass-shell condition of the charged particle is given by the standard Lorentz-invariant expression, $E^2 = q_\perp^2 + q_\parallel^2 + M^2$. The theory considered is standard quantum electrodynamics with an additional spacelike Chern–Simons (CS) term in the photonic action, where $m$ denotes the Lorentz-violating CS mass scale and $\zeta$ the normalized CS vector. See Secs. II A and II B for details. All calculations of this article have been performed with MATHEMATICA 5.0 [46].

The different terms of the decay widths will be ordered as follows: first, an inverse-hyperbolic term and, then, terms with descending powers of $m$. With this ordering and natural units ($c = \hbar = 1$), the scalar decay width is found to be given by

$$\Gamma_{\text{scalar}}(q_\perp, q_\parallel) = \frac{\alpha m}{16 E (q_\perp^2 + M^2)^{1/2}} \left( \left( 4(M^2 + 2q_\parallel^2) - m^2 \right) \arcsinh(2k_{\text{max}}/m) ight. $$

$$+ 2(2|q_\parallel| + k_{\text{max}}) m - 4|q_\parallel| \sqrt{m^2 + 4k_{\text{max}}^2} - 8 M^2 k_{\text{max}}/m \Bigg), \quad (A1)$$

and the spinor decay width by

$$\Gamma_{\text{spinor}}(q_\perp, q_\parallel) = \frac{\alpha m}{16 E (q_\perp^2 + M^2)^{1/2}} \left( \left( 4(M^2 + 2q_\parallel^2) + m^2/2 \right) \arcsinh(2k_{\text{max}}/m) \right.$$ 

$$+ 2(2|q_\parallel| - k_{\text{max}}) m - (4|q_\parallel| - k_{\text{max}}) \sqrt{m^2 + 4k_{\text{max}}^2} - 8 M^2 k_{\text{max}}/m \Bigg), \quad (A2)$$

in terms of the fine-structure constant $\alpha \equiv e^2/(4\pi)$ and the maximum photon momentum component $k_{\text{max}}$ defined by

$$k_{\text{max}}(q_\parallel) \equiv \frac{2m|q_\parallel| \left( m + 2 \sqrt{q_\parallel^2 + M^2} \right)}{m^2 + 4M^2 + 4m \sqrt{q_\parallel^2 + M^2}} \geq 0. \quad (A3)$$

The spinor decay width has already been given in Ref. [18] but has been included here for comparison with the scalar result.
APPENDIX B: MCS RADIATION RATE COEFFICIENTS

In this Appendix, analytic tree-level results for the coefficients $K$ and $L$ of the MCS Cherenkov energy-momentum-loss rate (2.12) are presented, the first of which completely determines the radiated energy rate (2.13) of the charged particle considered (scalar or spinor). See Appendix A for further details.

The radiation rate coefficients for a charged scalar particle are given by

\[ K_{\text{scalar}}(q_\perp, q_\parallel) = \frac{1}{128E(q_\parallel^2 + M^2)^{3/2}} \left( 3 \left( 4(M^2 + 2q_\parallel^2) - m^2 \right) m^2 \text{arcsinh}(2k_{\text{max}}/m) ight. \]

\[ + (12|q_\parallel| + 8k_{\text{max}}) m^3 \]

\[ - 2\sqrt{m^2 + 4k_{\text{max}}^2} (6|q_\parallel| + k_{\text{max}}) m^2 \]

\[ - 8 \left( 2M^2(|q_\parallel| + 2k_{\text{max}}) + |q_\parallel|(4q_\parallel^2 + 4|q_\parallel|k_{\text{max}} - 3k_{\text{max}}^2) \right) m \]

\[ + 8\sqrt{m^2 + 4k_{\text{max}}^2} \left( M^2(2|q_\parallel| + k_{\text{max}}) + 2q_\parallel^2 (2|q_\parallel| - k_{\text{max}}) \right) \]

\[ - 32M^2|q_\parallel| k_{\text{max}}^2/m^2 \right), \quad (B1a) \]

\[ L_{\text{scalar}}(q_\perp, q_\parallel) = \frac{\text{sgn}(q_\parallel)}{128E(q_\parallel^2 + M^2)^{3/2}} \left( -|q_\parallel|(4(M^2 + 4q_\parallel^2) - 3m^2) m \text{arcsinh}(2k_{\text{max}}/m) \right. \]

\[ - 4 \left( -M^2 + 2|q_\parallel|(3|q_\parallel| + k_{\text{max}}) \right) m^2 \]

\[ + 2\sqrt{m^2 + 4k_{\text{max}}^2} \left( -2M^2 + |q_\parallel|(4|q_\parallel| + k_{\text{max}}) \right) m \]

\[ - 8 \left( 2M^4 + M^2(4q_\parallel^2 - 4|q_\parallel|k_{\text{max}} - k_{\text{max}}^2) + 2q_\parallel^2 k_{\text{max}}(3|q_\parallel| + k_{\text{max}}) \right) \]

\[ + 8M^2 \sqrt{m^2 + 4k_{\text{max}}^2} \left( 2M^2 + |q_\parallel|(|q_\parallel| - 3k_{\text{max}}) \right) / m \]

\[ - 32M^4k_{\text{max}}^2/m^2 \right), \quad (B1b) \]

and those for a charged spin 1/2 particle by

\[ K_{\text{spinor}}(q_\perp, q_\parallel) = \frac{1}{192E(q_\parallel^2 + M^2)^{3/2}} \left( 3 \left( 6(M^2 + 2q_\parallel^2) + m^2 \right) m^2 \text{arcsinh}(2k_{\text{max}}/m) \right. \]

\[ + (10|q_\parallel| - 12k_{\text{max}}) m^3 \]

\[ - 2\sqrt{m^2 + 4k_{\text{max}}^2} (5|q_\parallel| - 3k_{\text{max}}) m^2 \]

\[ - 4 \left( 6M^2(|q_\parallel| + 2k_{\text{max}}) + 12|q_\parallel|^3 + 12q_\parallel^2 k_{\text{max}} - 3|q_\parallel|k_{\text{max}}^2 + 2k_{\text{max}}^3 \right) m \]

\[ + 4\sqrt{m^2 + 4k_{\text{max}}^2} \left( 3M^2(2|q_\parallel| + k_{\text{max}}) + 2|q_\parallel|(6q_\parallel^2 - 3|q_\parallel|k_{\text{max}} + k_{\text{max}}^3) \right) \]

\[ - 48M^2|q_\parallel| k_{\text{max}}^2/m^2 \right), \quad (B2a) \]
\[
L_{\text{spinor}}(q_\perp, q_\parallel) = \frac{\text{sgn}(q_\parallel)}{192 E (q_\parallel^2 + M^2)^{3/2}} \left( -3|q_\parallel| \left( 2(M^2 + 4q_\parallel^2) + m^2 \right) m \arcsinh(2k_{\text{max}}/m) \\
- 2(M^2 + 6q_\parallel^2 - 6|q_\parallel|k_{\text{max}}) m^2 \\
+ 2\sqrt{m^2 + 4k_{\text{max}}^2} \left( M^2 - 3|q_\parallel| (2|q_\parallel| - k_{\text{max}}) \right) m \\
- 4 \left( 6M^4 + 3M^2 (4|q_\parallel|^2 - 4k_{\text{max}}|q_\parallel| + k_{\text{max}}^2) \\
- 2|q_\parallel|k_{\text{max}} (6q_\parallel^2 - 3|q_\parallel|k_{\text{max}} + k_{\text{max}}^2) \right) \\
+ 4M^2 \sqrt{m^2 + 4k_{\text{max}}^2} \left( 6M^2 + 12q_\parallel^2 - 9k_{\text{max}}|q_\parallel| + 2k_{\text{max}}^2 \right) / m \\
- 48M^4 k_{\text{max}}^2 / m^2 \right),
\]

with \( E \equiv q^0 = \sqrt{q_\perp^2 + q_\parallel^2 + M^2} \) the energy of the charged particle (from the initial state factor) and \( k_{\text{max}} \) the maximum photon momentum component defined by (A3).

In closing, we remark that the factor \( E \) in the denominator of (B1a) or (B2a) cancels out in the corresponding radiated energy rate (2.13), so that this rate only depends on the magnitude of the parallel momentum component, \( |q_\parallel| \).

**APPENDIX C: UHECR BOUNDS ON NONBIREFRINGENT MODIFIED-MAXWELL THEORY**

In this Appendix, vacuum Cherenkov bounds are discussed for certain “coupling constants” of modified-Maxwell theory (5.1) over flat Minkowski spacetime with a metric \( \eta_{\mu\nu} \) as defined in Sec. 1. Specifically, the following Ansatz for \( \kappa_{\mu\nu\rho\sigma} \) is considered [34]:

\[
\kappa_{\mu\nu\rho\sigma} = \frac{1}{2} \left( \eta^{\mu\rho} \tilde{\kappa}^{\nu\sigma} - \eta^{\mu\sigma} \tilde{\kappa}^{\nu\rho} - \eta^{\nu\rho} \tilde{\kappa}^{\mu\sigma} + \eta^{\nu\sigma} \tilde{\kappa}^{\mu\rho} \right),
\]

(C1)

in terms of the nine components of a symmetric and traceless matrix \( \tilde{\kappa}^{\alpha\beta} \). The corresponding modified-Maxwell theory has no birefringence.

It is possible to derive an upper bound on a combination of the coupling constants \( \tilde{\kappa}^{\mu\nu} \) from the observation of a UHECR event with a primary proton moving in the direction \( \hat{q} \) and having an ultrarelativistic energy \( E_p \gg M_p c^2 \). Very briefly, the argument runs as follows [13, 14]: an ultra-high-energy cosmic proton can arrive on Earth only if it does not lose energy by vacuum Cherenkov radiation, which requires the proton energy \( E_p \) to be close to or below threshold, \( E_p \lesssim E_{\text{thresh}} \). Using the explicit result for the threshold energy [34], this condition can be written as the following upper bound:

\[
R(\tilde{\kappa}_{ij} \hat{q}_i \hat{q}_j + 2 \tilde{\kappa}_{0j} \hat{q}_j + \tilde{\kappa}_{00}) \lesssim (M_p c^2 / E_p)^2 = 10^{-20} \left( \frac{10^{10} \text{GeV}}{E_p} \right)^2 \left( \frac{M_p}{\text{GeV} / c^2} \right)^2,
\]

(C2)
with the ramp function \( R(x) \equiv x \theta(x) \), defined in terms of the step function \( \theta(x) = 1 \) for \( x \geq 0 \) and \( \theta(x) = 0 \) for \( x < 0 \). For a negative (spacelike) argument of the ramp function on the left-hand-side of (C2), the phase velocity of light in the specified direction is larger than \( c \) and vacuum Cherenkov radiation is impossible (the maximum attainable velocity of the charged particle being equal to \( c \)); see below for further comments.

In order to simplify the discussion, it will be assumed that many UHECRs with energy of \( 10 \text{ EeV} = 10^{10} \text{ GeV} = 10^{19} \text{ eV} \) or more will be available in the future and that they come from all directions in space [22, 23, 24]. Throughout this article (except in the last equation of this Appendix), we take the primary particle of the UHECR considered to be a proton, but for a primary nucleus we can simply replace the proton mass \( M_p \) in (C2) by the relevant mass \( M_{\text{nucleus}} \). (The primary-particle type is expected to correlate with, e.g., the atmospheric depth of the air-shower maximum; cf. Ref. [40].) With these assumptions, only three simple cases will be discussed in detail, leaving a complete discussion to the moment when the radiation rate (5.5) has been calculated exactly. By the way, the implicit assumption behind the bounds of this Appendix is that no extremely small numerical factors appear in the final version of the radiation rate (5.5) and that, therefore, the Cherenkov threshold condition (C2) is relevant; cf. Ref. [14].

The first case considered has all space-space components of \( \tilde{\kappa}_{\mu \nu} \) vanishing,

\[
\text{case 1 : } \tilde{\kappa}_{ij} \equiv 0, \text{ for } i, j \in \{1, 2, 3\}, \quad (C3)
\]

with \( \tilde{\kappa}_{00} \equiv \tilde{\kappa}_{jj} \) also vanishing. Then, only the linear \( \hat{\mathbf{q}} \) term on the left-hand-side of (C2) remains. Assuming to have observed 10 EeV protons with directions \( \hat{\mathbf{q}} = (\pm 1, 0, 0), (0, \pm 1, 0), \) and \( (0, 0, \pm 1) \), the following bounds are obtained:

\[
\text{case 1 : } |\tilde{\kappa}_{0j}| \lesssim (1/2) \times 10^{-20}, \quad (C4)
\]

for \( j = 1, 2, 3 \).

The second case has spatial isotropy for the terms on the diagonal of the \( \tilde{\kappa}_{\mu \nu} \) matrix and vanishing off-diagonal terms:

\[
\text{case 2 : } \tilde{\kappa}_{11} \equiv \tilde{\kappa}_{22} \equiv \tilde{\kappa}_{33} \equiv \tilde{\kappa}_{00}/3 \text{ and } \tilde{\kappa}_{\mu \nu} \equiv 0, \text{ for } \mu \neq \nu. \quad (C5)
\]

The largest component of \( \tilde{\kappa}_{\mu \nu} \) is \( \tilde{\kappa}_{00} \). Assuming to have observed at least one 300 EeV proton [30, 40], its particular direction \( \hat{\mathbf{q}} \) being irrelevant, the following bound is obtained for nonnegative \( \tilde{\kappa}_{00} \):

\[
\text{case 2 : } 0 \leq \tilde{\kappa}_{00} \lesssim (3/4) \times 10^{-23}, \quad (C6)
\]

having used a numerical value \( E_p = 300 \text{ EeV} \) for the right-hand-side of (C2). There is no Cherenkov bound for negative \( \tilde{\kappa}_{00} \) and the previous electron-anomalous-magnetic-moment bound (at the two \( \sigma \) level) is:

\[
\text{case 2 : } 0 < -\tilde{\kappa}_{00} \lesssim (3/2) \times 3 \times 10^{-8} \approx 5 \times 10^{-8}, \quad (C7)
\]
which follows from Eq. (2.7) of Ref. [37] with an absolute value on its left-hand-side.

The third case is based on a preferred background vector $\xi$, which, without loss of generality, can be chosen as $\xi \equiv (0, 0, 1)$. The components of $\kappa_{\mu\nu}$ are then given by

$$\begin{align*}
\text{case 3 : } & \tilde{\kappa}_{00} \equiv -\tilde{\kappa}_{11} \equiv -\tilde{\kappa}_{22} \equiv \tilde{\kappa}_{33}/3 \quad \text{and} \quad \tilde{\kappa}_{\mu\nu} \equiv 0, \quad \text{for } \mu \neq \nu. 
\end{align*}$$

(C8)

For this case, the largest component of $\tilde{\kappa}_{\mu\nu}$ is $\tilde{\kappa}_{33}$. Denoting the angle between the particle momentum $q$ and background vector $\xi$ by $\theta_3$, the Cherenkov threshold condition (C2) reads

$$\begin{align*}
\text{case 3 : } & 0 \leq (4/3) \tilde{\kappa}_{33} \cos^2 \theta_3 \lesssim 10^{-20},
\end{align*}$$

(C9)

for the numerical values of $E_p$ and $M_p$ used on the right-hand-side of (C2). Assuming the observation of a 10 EeV cosmic proton moving parallel or antiparallel to the background vector $\xi$ in the 3–direction (this proton having $\cos^2 \theta_3 = 1$), the following bound is obtained for nonnegative $\tilde{\kappa}_{33}$:

$$\begin{align*}
\text{case 3 : } & 0 \leq \tilde{\kappa}_{33} \lesssim (3/4) \times 10^{-20}.
\end{align*}$$

(C10)

Similar to case 2, there is no Cherenkov bound for negative $\tilde{\kappa}_{33}$ and the previous microwave-oscillator bound (at the two $\sigma$ level) is:

$$\begin{align*}
\text{case 3 : } & 0 < -\tilde{\kappa}_{33} \lesssim (9/4) \times (501/100) \times 10^{-14} \approx 1.1 \times 10^{-13},
\end{align*}$$

(C11)

which follows from Table II of Ref. [38(a)] with the identification $\tilde{\kappa}_e = (4/9) \text{ diag}(1, 1, -2) \tilde{\kappa}_{33}$ for the case considered with $\xi \equiv (0, 0, 1)$ in arbitrary coordinates.

In view of the above results, three general remarks are in order. First, the overall sign of the $\kappa$–tensor in (5.1) is physically relevant, as the Cherenkov threshold condition (C2) makes clear. Second, cases 2 and 3 arise from a common Ansatz: $\kappa_{\mu\nu} \propto \xi_{\mu}\xi_{\nu} - \eta_{\mu\nu} \xi_{\rho}\xi^{\rho}/4$, with $\xi^\mu = (1, 0, 0, 0)$ for case 2 and $\xi^\mu = (0, \xi) = (0, 0, 0, 1)$ for case 3. Third, case 2 with negative $\tilde{\kappa}_{00}$ and case 3 with negative $\tilde{\kappa}_{33}$ have phase and group velocities of light larger than the velocity $c$ encoded in the causal structure of Minkowski spacetime (recall $x^0 = ct$) and it is not clear if these theories are physically consistent; cf. Refs. [16, 33]. Admittedly, case 1 is not perfect either, with velocities of light larger than $c$ in certain directions. But case 2 with nonnegative $\tilde{\kappa}_{00}$ (or case 3 with nonnegative $\tilde{\kappa}_{33}$) does appear to be physical and precisely the isotropic case 2 with nonnegative $\tilde{\kappa}_{00}$ has the strongest bound, namely, Eq. (C6).

The three special cases discussed here give an idea of what a bound of the type (C2) may or may not imply for the coupling constants $\kappa_{\mu\nu}$. Now consider generic $\kappa_{\mu\nu}$ components with a positive left-hand-side in (C2) of order $\tilde{\kappa}$, which is, most likely, one of the conditions defining the “physical domain” of modified-Maxwell theory (5.1) coupled to Lorentz-invariant

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charged particles. Then, the observation of a particular 300 EeV proton \[39, 40\] gives the tentative bound discussed in Sec. V:

\[
0 \leq \tilde{\kappa}^{\text{generic}}_{\nu\mu} \mid_{\text{physical domain}} \lesssim 10^{-23} \left( \frac{3 \times 10^{11} \text{GeV}}{E_p} \right)^2 \left( \frac{M_p}{\text{GeV}/c^2} \right)^2,
\]

(C12)

with the above-mentioned \textit{caveat} on the nature of the primary [for an iron (Fe) nucleus, the bound is increased by a factor \((56)^2 \approx 3 \times 10^3\) to an approximate value of \(3 \times 10^{-20}\)]. Bound (C12) is called tentative because a \textit{precise} definition of “\textit{generic} \tilde{\kappa}_{\mu\nu}” has not been given.

However, with the reconstructed shower path available \[39\], the proton direction of this particular 300 EeV event \((n = 1)\) is known within certain errors, \(\hat{q} = \hat{q}^{(1)}\), and can simply be inserted on the left-hand-side of (C2), with the right-hand-side taking the numerical value \(10^{-23}\). If more 300 EeV protons become available in the future \((n = 1, \cdots, N)\), expression (C2) generates \(N\) bounds with \(\hat{q} = \hat{q}^{(n)}\) on the left-hand-side and \(10^{-23}\) on the right-hand-side, which can then be analyzed further (giving proper confidence limits, for example).

The following \textit{Ansatz} for the coupling constants \(\tilde{\kappa}^{\mu\nu}\) may turn out to be useful in the analysis:

\[
(\tilde{\kappa}^{\mu\nu}) \equiv \text{diag}(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \tilde{\kappa}^{00} + (\delta \tilde{\kappa}^{\mu\nu}), \quad \delta \tilde{\kappa}^{00} = 0,
\]

(C13)

with one independent variable \(\pi^{00}\) for the “spatially isotropic part” of \(\tilde{\kappa}^{\mu\nu}\) and eight independent variables in \(\delta \tilde{\kappa}^{\mu\nu}\) which need not be small. For a large number \(N\) of 300 EeV protons distributed isotropically and restricting to the physical domain of the coupling constants, the \textit{sum} of the \(N\) bounds mentioned in the previous paragraph will give approximately the same bound on the isotropy variable \(\pi^{00}\) as in (C6) for case 2 above, but now without assumptions on the other coupling constants \(\delta \tilde{\kappa}^{\mu\nu}\).

Returning to the present situation, there is already the following bound on nonnegative \(\pi^{00}\) from the current number of more or less isotropic 10 EeV cosmic-ray events (notably from AGASA and HiRes \[22, 23\] in the northern hemisphere and from preliminary data of Auger \[24(b)\] in the southern hemisphere):

\[
0 \leq \pi^{00} \mid_{\text{physical domain}} \lesssim 2 \times 10^{-17} \left( \frac{M_{\text{primary}}}{M_{\text{Fe}}} \right)^2,
\]

(C14)

where, most likely, light primaries can be selected by appropriate cuts on the shower-maximum depth and other characteristics. Here, a relatively narrow energy band around 10 EeV has been considered, but, more generally, bound (C14) scales as the inverse square of the average cosmic-ray energy. Incidentally, the reference frame in which bound (C14) holds is the one in which the cosmic-ray energies are measured. This new astrophysics bound on the spatially isotropic part of the coupling constants \(\tilde{\kappa}^{\mu\nu}\) at the \(10^{-17}\) level (or at the \(10^{-20}\) level for proton primaries) improves significantly upon the direct laboratory bound at the \(10^{-7}\) level \[38(b)\] or the indirect laboratory bound at the \(10^{-8}\) level \[37\].
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