Manifestly covariant classical correlation dynamics I. General theory

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(Dated: May 29, 2009)

In this series of papers we substantially extend investigations of Israel and Kandrup on nonequilibrium statistical mechanics in the framework of special relativity. This is the first one devoted to the general mathematical structure. Basing on the action-at-a-distance formalism we obtain a single-time Liouville equation. This equation describes the manifestly covariant evolution of the distribution function of full classical many-body systems. For such global evolution the Bogoliubov functional assumption is justified. In particular, using the Balescu-Wallenborn projection operator approach we find that the distribution function of full many-body systems is completely determined by the reduced one-body distribution function. A manifestly covariant closed nonlinear equation satisfied by the reduced one-body distribution function is rigorously derived. We also discuss extensively the generalization to the general relativity especially an application to self-gravitating systems.

PACS numbers: 03.30.+p, 52.25.Dg

I. INTRODUCTION AND OVERVIEW

A. Some motivations

The fundamental problem of physics of relativistic classical many-body systems is one century old (for a review of early investigations see, e.g., Ref. [1]) and, still, remains one of the most important subjects in studies of relativity [2, 3, 4, 5]. At the microscopic level the basis of deterministic classical dynamics of relativistic many-body systems has been widely explored [6, 7, 8, 9]. At the macroscopic level relativistic hydrodynamical phenomena have been well documented [10, 11]. In particular, in recent years relativistic effects on macroscopic equilibrium and non-equilibrium phenomena have attracted considerable theoretical interests, and a variety of related subjects are under intensified investigations ranging from the internal evolution (or aging) of relativistic systems [12, 13], relativistic generalization of the Maxwell-Boltzmann distribution [8, 14] to relativistic Brownian motion [4]. A natural question, thus, is how various macroscopic relativistic phenomena emerge from deterministic relativistic classical many-body dynamics. Long time ago it was realized by the Brussels-Austin school that, as nonrelativistic many-body systems, the macroscopic evolution generally roots in the creation or annihilation of correlations between microscopic particles as they move in spacetime [5, 15, 16]. Thus, to investigate these issues one has to invent a theory of relativistic classical correlation dynamics aiming at understandings of the dynamics of distribution function of full many-body systems. Unfortunately, this is intellectually challenging and remains largely undeveloped.

From practical viewpoints a theory of relativistic classical correlation dynamics may be proven to be powerful in studies of various relativistic transport processes in plasma physics [17] and astrophysics [18, 19]. There substantial progresses have been made in past years by making use of various relativistic kinetic equations, which provide an adequate microscopic description for macroscopic relativistic hydrodynamics. In a pioneering paper by Heinz [20] it was realized that the relativistic classical kinetic equation may also become a powerful tool in studies of quark-gluon plasmas, which has nowadays spanned one of the most important branches in high-energy physics [21]. To derive relativistic (classical) kinetic equations there are a number of theoretical proposals. Among of them are two representative approaches: The first approach is largely phenomenological, where with the help of Stosszahlansatz kinetic equations are formulated as manifestly covariant versions of their Newtonian analogues so as to suit relativity principles [11]. The second approach [22, 23, 24, 25] starts from some relativistic BBGKY hierarchy and the cluster expansion is further applied. Then, in the crucial step one invokes the Bogoliubov functional assumption [26] to truncate the infinite hierarchy. As such, one obtains some relativistic kinetic equation. Yet, there does exist the notable exception in plasma physics, namely Klimontovich’s technique. With the help of this technique Klimontovich managed to justify the relativistic Landau equation [27] at the full microscopic level [28, 29]. Since then, it has become common to test the validity of various theories by justifying this kinetic equation [1, 15, 18, 19, 22, 24].

It is fair to say that for some important applications such as ultrarelativistic electromagnetic plasmas the particle interaction must be described by the quantum field theory. Therefore, a complete treatise of relativistic statistical effects is required to be built on quantum field theories. Then, a general prescription may be to generalize—a manifestly covariant manner—the Green’s function theory of nonequilibrium quantum statistical mechanics, pioneered by Schwinger, Kadanoff and Baym, Keldysh and Korenman [30, 31] so as to suit principles of quantum field theories. Such a task was undertaken for electron-positron plasmas some time ago [32]. The general formalism has been substantially renovated by making use of the functional-integral approach, emerging as
relativistic nonequilibrium quantum field theory \[33, 34\]. These approaches, in combination with the Wigner function technique, allow one to obtain a manifestly covariant quantum kinetic equation at the level of the weak coupling (Born) approximation which, importantly, recovers the relativistic Landau equation \[27, 28\] at the classical limit.

Nevertheless there are considerable practical and theoretical reasons which justify to intensify studies of the relativistic classical correlation dynamics. The most prominent one comes the very recent debate on the special relativistic generalization of the Maxwell-Boltzmann distribution \[3, 14\], triggered by the surprising finding of Horwitz and coworkers \[23\]. In Ref. \[23\] it was noticed that considerable conceptual and technical differences between the relativistic and Newtonian deterministic classical dynamics might completely disable simple generalizations obtained by the phenomenological approach. These authors proposed a manifestly covariant kinetic equation possessing a remarkable mathematical structure different from the one obtained either by other microscopic approaches \[15, 18, 24, 28, 32, 33, 34\] or by phenomenological arguments \[10, 11\]. The one century old belief—that the Jüttner equilibrium \[3, 14\] is established in relativistic dilute systems—is thereby challenged. At this stage it is completely unclear whether such a peculiar property is specific or universal to relativistic classical many-body systems. Thus, it is fundamentally important to derive as rigorously as possible the relativistic kinetic equation from generic classical many-body dynamics. Then, one may hope to prove or disprove kinetic equations proposed basing on phenomenological arguments and, furthermore, to obtain insight into the numerical experiment on the relativistic equilibrium of classical many-body systems \[3\]. Experiences in the Newtonian statistical mechanics suggest that the technique developed by the Brussels-Austin school \[35, 36\] may be perfectly suitable to fulfill such a task as it manages to derive a general kinetic equation \[37\] without resorting to either Stosszahlansatz or the truncation approximation. To employ the latter is well-known to be inevitable and, in fact, a crucial step in various microscopic approaches.

There are several important adjacent problems that may be explored within the scope of the relativistic classical correlation dynamics. One is the relativistic Brownian motion, which has received much attention in recent years (for example, see \[13\]). Another is to formulate the relativistic many-body equilibrium. For this Hakim conjectured long time ago that there exists a Lorentz invariant equilibrium hierarchy which is invariant under the spacetime translation and is merely determined by the Jüttner distribution \[1\]. However, to our best knowledge there have been no progresses reported so far.

Finally, new developments of the relativistic classical correlation dynamics are urged by the stellar dynamics. There, it is generally believed that a satisfactory framework is provided by the classical (nonequilibrium) statistical mechanics in the general relativity \[18\]. In particular, the kinetic equation describing the Brownian motion of self-gravitating systems suffers from a serious difficulty of the infrared divergence \[18, 38, 39\]. Such a divergence has been explored at the level of the Newtonian physics and been known to deeply root in the nonMarkovian and collective effect \[40, 41\], the latter of which, interestingly, finds the collective dielectric effect in classical plasmas \[42\] as the analogue. In the Newtonian physics the correlation dynamics encompasses a standard route towards the complete treatise of these effects. It is, therefore, a natural hope that the (general) relativistic classical correlation dynamics may help to understand this issue at deeper level.

### B. Nonequilibrium statistical mechanics: manifestly and nonmanifestly covariant formalism

In this series of papers we focus on the (special relativistic) classical statistical mechanics. In contrast to the relativity in the Newtonian physics the emergence of macroscopic irreversibility from deterministic classical dynamics nowadays is a well established subject. Indeed, for simple chaotic systems especially deterministic diffusive systems rigorous mathematical investigations, as well as various analytical theories \[43\] have shown that an initial phase space distribution (function) is decomposed into independent components each of which relaxes in an irreversible manner (Ruelle-Pollicott resonances), and after transient processes the evolution is dominated by diffusion at the macroscopic scale. For more complicated many-body systems (as we switch to in this paper) a milestone was put by Bogoliubov \[26\] who realized that, there, the evolution is a two-step process: In the first step (after transient decays) correlation functions relax into functionals of one-body distribution functions. In the second step the one-body distribution function evolves following a closed nonlinear kinetic equation, and is fully responsible for macroscopic hydrodynamics. Later on the concept of the Bogoliubov functional assumption was justified and substantially extended, especially in the profound work by the Brussels-Austin school using delicate diagrammatical \[35\] and projection operator \[36, 44\] approaches. In particular, the decomposition of many-body phase space distribution functions may be carried out in the way such that in the absence of interactions each component corresponds to some degree of correlations between particles. These correlations are preserved (mixed) in the absence (presence) of interactions justifying the terminology of “(classical) correlation dynamics”. As such, irreversible processes of approaching equilibrium, as well as the final equilibrium states are found to be completely determined by the so-called vacuum of correlations namely the component containing no correlations in the absence of interactions. This way the kinetic equation is no longer an approximation, rather, an exact theory to which the Liouville equation of full many-body systems converges in long times.
Then, a natural hope is to extend these concepts so as to suit relativity principles. At the early stages the Brussels-Austin school \[15, 45\] undertook systematic attempts (see Ref. \[13\] for a review) to generalize the Newtonian classical correlation dynamics to special relativity, which is built on the canonical formulation of relativistic dynamics of classical many-body systems \[8\] with the Hamiltonian, if necessary, including field degrees of freedom. Unfortunately, as pointed out by many authors especially in the notable critical analysis by Hakim \[1\], Israel and Kandrup \[18, 19\], and Kandrup \[19\], to proceed along this theoretical line one may have to overcome a number of conceptual and technical difficulties. The most serious problems are regarding covariance of the theory and particularly covariance of the evolution of many-body systems. Both deeply root in the absence of absolute time in many-body systems. For the former problem it should be stressed that by relativity principles it is perfectly legitimate to build any theories in either manifestly or non-manifestly covariant manner. It is exactly the latter that the Brussels-Austin school follows. Indeed, in the development of old relativistic classical correlation dynamics a preferred time coordinate is chosen and then a Liouville equation is formulated building on the Hamiltonian formalism. Furthermore, in an insightful work Balescu and Kotera realized that in this framework the Lorentz invariance must be understood in terms of the Lorentz group action representation the Liouvillian generates the distribution functions of many-particle phase space (if necessary, enlarged to the Lorentz group action on the distribution functions of many-particle phase space). Then, the underlying relativistic many-body dynamics is formulated in terms of the so-called action-at-a-distance formalism (see, e.g., Ref. \[9\] for a review) rather than the Hamiltonian formalism. Remarkably, in such a formalism fields are not considered as independent degrees of freedom. Instead, they are carried by complicated retarded (advanced) potentials. As a result, particles interact in a nonlocal manner. And the force acting on given particles must be viewed not only as a function of particle’s 4-position and 4-momentum vector, but also as a functional of the world line of all the other particles. Finally, particles are treated on different footing in the way that the N-particle system is divided into 1- and (\(N-1\))-particle subsystem coupling to each other. As such they managed to employ Willis-Picard projection operator approach \[40\] to obtain a closed nonlinear kinetic equation of one-body distribution function. The latter gives the relativistic Landau equation \[27, 28, 29\] from which, at the full microscopic level, Jüttner equilibrium distribution immediately follows. The success of the Israel-Kandrup formalism justified a number of far-reaching concepts. Among of them are: the global covariant evolution as conjectured by Hakim \[1\] and the legitimation of building relativistic classical non-equilibrium statistical mechanics on the action-at-a-distance formalism of deterministic many-body dynamics. The latter nowadays is widely accepted \[24, 24\]. Importantly, it was suggested that the one-body distribution function may, at least asymptotically, determine the covariant evolution of the system—the very nature of the Bogoliubov functional assumption.

C. Overview of this series of papers

Despite of the significant achievements by Israel and Kandrup there are many important problems remained unsolved. First of all, it is not clear how to go beyond the weak coupling approximation within the Israel-Kandrup formalism. This issue is of great practical importance especially for transport processes in plasmas with electromagnetic interactions \[24, 29\]. Also, because of such a drawback one fails to predict—at the full microscopic level—kinetic equations and (local) equilibrium for dilute systems with moderate interaction strength. The solution to this problem may solve the puzzle of the relativistic generalization of the Maxwell-Boltzmann distribution which is currently undergoing intensified debate \[3, 14, 23\]. Then, in Ref. \[19\] the (local) equilibrium was studied under the kinetic approximation. There, the principal problem of formulating many-body equilibrium \[1\] has not been attacked. Finally, how to extend the Bogoliubov functional assumption rigorously remains largely unexplored.

In this series of papers we substantially extend the investigations of Israel and Kandrup \[18, 19\] and widely explore these issues in the framework of special relativity. We present a new, manifestly covariant classical correlation dynamics. As the basic viewpoint we proceed along the line of Refs. \[18, 19\] and formulate the under-
lying deterministic classical many-body dynamics in the action-at-a-distance formalism with $N$ motion equations as fundamental objects \cite{6, 7, 9}. Such formalism allows us to naturally work in physical coordinate and momentum. The apparent advantage is to admit a theory which is manifestly covariant at each step of the manipulations. Nevertheless this is by no means merely based on the aesthetic viewpoint. Indeed, we have been compelled to do so by significant progress achieved recently by both physicists and mathematicians. (i) In a 1-dimensional numerical simulation it is found that point-like collisions tend to drive a relativistic system into equilibrium described by Jüttner distribution \cite{3}. At the microscopic level the deterministic classical many-body dynamics of the underlying system may be described by the action-at-a-distance formalism in a rather straightforward manner. A statistical theory based on this formalism, in turn, is expected to explain the experimental discovery. (ii) The Kirpichev-Polyakov theorem \cite{5} partly justifies the long-standing conjecture of Hakim on mathematical foundations of relativistic statistical mechanics \cite{1}. It is shown that for 1-dimensional relativistic dynamics of classical charged systems, which is formulated in terms of the Wheeler-Feynmann formalism \cite{2}, an ordinary Cauchy problem as in the Newtonian mechanics can be stated. This is conceptually important because it suggests that the global evolution of relativistic many-body systems may be formulated in the way analogous to the Newtonian physics, despite that particles interact in a dramatically different manner.

The $N$ motion equations define a natural solution space namely the $8N$-dimensional $\Gamma$ phase space. (Throughout this work the mass-shell constraint is absorbed into the distribution functions.) For this phase space we may define a distribution function which, remarkably, depends on $N$ proper times. A significant difference from the Newtonian physics is that a bundle of $N$ world lines, rather than a representation point in the $\Gamma$ phase space underlies the subsequent analysis of the dynamics of distribution functions. Then, we formulate $N$ conservation equations. Moreover, for a large class of physical forces such as Lorentz forces the phase space volume element is invariant along world lines. As a result, despite of the absence of Hamiltonian these conservation equations collapse into the manifestly covariant Liouville equations. With the $N$ proper times identified we obtain a single-time Liouville equation which describes the manifestly covariant global evolution of distribution function. On this global evolution we may build a manifestly covariant theory of classical correlation dynamics by using the Balescu-Wallenborn projection operator approach \cite{30, 47}. It then follows that the reduced one-body distribution function fully determines the $N$-body distribution function, in particular, the entire correlation functions. The evolution of the reduced one-body distribution function obeys an exact closed kinetic equation. As such, we achieve the relativistic Bogoliubov functional assumption which, in contrast to earlier theories \cite{22, 24, 25}, is manifestly covariant. From the exact closed kinetic equation we recover various (Vlasov, Landau, Boltzmann) relativistic kinetic equations systematically. The solutions to these kinetic equations allow us to pass to macroscopic physical observables by carrying out appropriate average with respect to them.

It must be stressed that the present theory cannot serve as an alternative to the difficult problem of deterministic relativistic classical many-body dynamics. Neither are they equivalent. As the Newtonian physics in order to develop a theory of manifestly covariant classical correlation dynamics one actually requires only few assumptions regarding dynamical properties of deterministic relativistic many-body systems. These assumptions are exactly formulated in this paper. Although to prove them may be hard mathematical problems, they have apparent physical implications. Thus, these assumptions and thereby the present relativistic classical correlation dynamics are expected to be applicable for a large class of realistic relativistic many-body systems. It is also worth pointing out that compared to its Newtonian counterpart the classical correlation dynamics presented here suffers from additional technical complications. That is, the force acting on given particle is determined by the world lines of all the other particles and, therefore, as one passes from the exact closed kinetic equation to special kinetic equations appropriate approximations regarding the world line must be implemented. Indeed, in the present work we use the well known relativistic impulse approximation \cite{18, 19}. There, to fully determine interactions between two particles at given moment their trajectories are considered to be linear. To overcome this technical complication one, in principle, needs to either expand phase space so as to accommodate particle acceleration as well as its higher-order derivatives \cite{1}, or treat fields as independent degrees of freedom which is a tractable task. (However, the 1-dimensional relativistic dynamics of classical charged systems may be an exception, because according to the Kirpichev-Polyakov theorem the field degrees of freedom are redundant.) Since the present work aims at the principle problem of building a theory of manifestly covariant classical correlation dynamics on the action-at-a-distance formalism, we may leave this technical issue at this stage of conceptual development, in particular, if we ignore problems such as radiation reaction (for electromagnetic interactions).

We plan to explore various topics discussed above in this and the following \cite{48} paper. This first one is devoted to the general mathematical structure and is organized as follows: In Sec. II we first introduce preliminary concepts required for developing a theory of manifestly covariant global evolution. Then we derive a manifestly covariant single-time Liouville equation. In Sec. III the correlation pattern representation is established in a manifestly covariant manner. On this basis we apply the Balescu-Wallenborn projection operator approach to the single-time Liouville equation. In Sec. IV we present the relativistic version of the Bogoliubov functional assumption.
In particular, we prove the factorization theorem and derive the exact closed kinetic equation of reduced one-body distribution function. We close this paper by outlining further applications. In particular, we present an extensive discussion of generalizing the manifestly covariant classical correlation dynamics to the general relativity so as to treat self-gravitating systems, which is the main subject of Sec. V. Some technical details are presented in Appendix A and B.

The second paper (denoted as Paper II), written in a self-contained manner, is devoted to applications of the general principles to relativistic plasmas with electromagnetic interactions. The readers interested only in applications of the present theory may skip this first one and read the second one directly. There, the relativistic Vlasov, Landau and Boltzmann equation follow systematically from the exact manifestly covariant nonlinear equation which is closed, and satisfied by the reduced one-body distribution function. The collision integrals of the latter two justify the Jüttner distribution as the special relativistic generalization of the Maxwell-Boltzmann distribution. The collective effects, such as the issue of correlation at the relativistic many-body equilibrium are explored. We summarize this series of work in Paper II.

Finally we list some of the notations and conventions. We choose the unit system with the speed of light \( c = 1 \). To distinguish from the Minkowski 4-vector we use the bold font to denote the vector in the Euclidean space. To distinguish from the Minkowski 4-vector we use the bold font to denote the vector in the Euclidean space.

\[ \partial_\mu \] covariant derivative: \( \partial_\mu = \frac{\partial}{\partial x^\mu} \); \[ d^4z \] volume element in 4-dimensional Minkowski space: \( d^4z = dz^0dz^1dz^2dz^3 \); \[ \delta^{(d)}(f) \] d-dimensional Dirac function: \( d\Sigma_\mu = \frac{1}{d!} \epsilon_{\mu\nu\rho\lambda} dx^\nu \wedge dx^\rho \wedge dx^\lambda \) with \( \epsilon_{\mu\nu\rho\lambda} \) being \( \pm 1 \) when \( (\mu\nu\rho\lambda) \) is an even (odd) permutation of \((0123)\) and being 0 otherwise; \( x_i(\zeta) \) world line of particle \( i \); \( x_i(\zeta) \) 4-position of particle \( i \) at proper time \( \zeta \); \[ d\alpha \] volume element in the \( \mu \) phase space of particle \( i \): \( d\alpha = d^4x_i d^4p_i \).

II. MANIFESTLY COVARIANT LIOUVILE EQUATIONS

Let us consider a system consisting of \( N \) classical point particles, each of which has the rest mass \( m \). The particles have the action at a distance on other particles. The interaction propagates at the speed of light. (To simplify discussions throughout this series of papers we shall not consider the self-action.) The interactions carry energy and momentum to and from particles without the support of independent fields. This is the so-called action-at-a-distance formalism of relativistic classical many-body dynamics pioneered by Fokker [8] and Wheeler and Feynman [9]. The subject has been widely explored in the last half century [10]. In this section based on this formalism we derive a manifestly covariant single-time Liouville equation. Analogous to the Newtonian physics it then serves as the exact starting point of relativistic classical correlation dynamics.

A. Action-at-a-distance formalism

The history of the microscopic system is described by a bundle of \( N \) particle world lines which solve the following relativistic motion equations:

\[
\frac{dx_i^\mu}{d\tau_i} = \frac{p_i^\mu}{m} = u_i^\mu, \tag{1}
\]

\[
\frac{dp_i}{d\tau_i} = F_{\mu i}^\mu(x_i, p_i) + \sum_{j\neq i}^{N} F_{ij}^\mu(x_i, p_i). \tag{2}
\]

Here \( x_i^\mu(\tau_i), u_i^\mu(\tau_i), p_i^\mu(\tau_i) \) are the 4-position, 4-velocity and 4-momentum vector of particle \( i \) depending on the proper time \( \tau_i \), respectively, \( F_{\mu i}^\mu \) is the external force, and \( F_{ij}^\mu \) is the force acting on particle \( i \) by particle \( j \). Importantly, we assume that both forces do not depend on the acceleration of the acted particle, and consider the interacting force \( F_{ij}^\mu \) with the general form as follows [3]:

\[
F_{ij}^\mu(x_i, p_i) = \int_{-\infty}^{\infty} d\tau_j \ s(\rho_{ij}) \ F^{\mu\nu}(\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \zeta_{ij}) p_{\nu i}, \tag{3}
\]

where \( F^{\mu\nu} \) is an antisymmetric tensor, and the role of function \( s(\rho_{ij}) \) is to invariantly connect \( x_i \) with one (or several) points at the world line \( x_j(\tau_j) \). The arguments in Eq. (3) are defined as follows:

\[
\rho_{ij} = (x_i^\mu - x_j^\mu)(x_i^\mu - x_j^\mu),
\]

\[
\alpha_{ij} = (x_i^\mu + x_j^\mu)p_j^\mu - p_i^\mu(x_i^\mu - x_j^\mu),
\]

\[
\beta_{ij} = \frac{dp_{i}^\mu}{d\tau_j}(x_i^\mu - x_j^\mu), \quad \gamma_{ij} = p_i^\mu(x_j^\mu - x_i^\mu),
\]

\[
\zeta_{ij} = (x_i^\mu - x_j^\mu)\frac{dp_{j}^\mu}{d\tau_j} - \frac{dp_{i}^\mu}{d\tau_j}(x_i^\mu - x_j^\mu). \tag{4}
\]

The relation: \( p_i \cdot F_{ij}(x_i, p_i) = 0 \) is obvious because of the skew property of \( F^{\mu\nu} \). Such a relation is further enforced to external forces:

\[
p_i \cdot F_{\mu i}(x_i, p_i) = 0. \tag{5}
\]
Therefore, the mass-shell constraint:

\[ p^2 = m^2 \]  

(6)
is guaranteed. To fully determine the interacting force \( F_{ij} \) relies on the world line of particle \( j \) namely \( x_j(\tau_j) \). In Paper II we exemplify the interacting force in the case of electromagnetic interactions.

**B. Manifestly covariant single-time Liouville equation**

The motion equations (11) and (22) suggest that we may introduce the concept of \( 8N \)-dimensional \( \Gamma \) phase space. It is defined as the direct product of \( N \) flat \( \mu \) phase spaces, each of which corresponds to individual particle. That is,

\[ \Gamma = \bigotimes_{i=1}^{N} \mu_i, \quad \mu_i = M^4_{\mu} \otimes U^4_{\mu}, \]

(7)

where \( M^4_{\mu} \) and \( U^4_{\mu} \) are the 4-dimensional Minkowski spacetime and momentum space carried by particle \( i \), respectively. The volume element of the \( \Gamma \) phase space is \( d\Gamma = \prod_{i=1}^{N} d\tau_i d\mu_i \equiv d^4x_i d^4p_i \). There exists a fundamental difference between the \( \Gamma \) phase space defined above and its Newtonian counterpart: In contrast to the Newtonian physics \( X_N \equiv (x_1(\tau_1), p_1(\tau_1), \ldots, x_N(\tau_N), p_N(\tau_N)) \in \Gamma \) generally does not represent a microscopic state because, in general, it is not sufficient to uniquely determine the history of the full system namely the \( N \) particle world lines. Instead, in special relativity microscopic states have been conjectured to be \( N \) segments of particle world lines each of which is carried by individual particle \( \tilde{\omega} \). Furthermore, all the microscopic states constitute the space \( \Omega_N \) which has the dimensionality \( \geq 8N \). (Ref. [3] presents a non-trivial example where the equality holds.)

Given a bundle of \( N \) particle world lines satisfying Eq. (22), or more precisely, a microscopic state \( \tilde{\omega} \in \Omega^N \) the representation point \( X_N \) has the singular distribution as \( \prod_{i=1}^{N} \delta^{(4)}(x_i - x_i(\tau_i)) \delta^{(4)}(p_i - p_i(\tau_i)) \), which depends on \( \tilde{\omega} \) through the phase trajectory \( x_i(\tau_i), p_i(\tau_i), i = 1, \ldots, N \). By preparing a cloud of microscopic states centering at \( \tilde{\omega} \) and carrying out the average over this ensemble, we expect to smear the singular distribution at least along some directions (of the \( \Gamma \) phase space). More precisely, we wish to assert the existence of the following

**Assumption 2.1** There exists a probability measure on \( \Omega^N \) such that the average below

\[ D(x_1, p_1, \tau_1, \ldots, x_N, p_N; \tau_N; x_1[s], \ldots, x_N[s]) \]

\[ \equiv \left( \prod_{i=1}^{N} \delta^{(4)}(x_i - x_i(\tau_i)) \delta^{(4)}(p_i - p_i(\tau_i)) \right) \]

(8)

strongly peaks at \( (x_1(\tau_1), p_1(\tau_1), \ldots, x_N(\tau_N), p_N(\tau_N)) \).

**Remark.** Although the details of the average are not required for subsequent discussions, it is important to notice that \( D \) possesses \( N \) proper times and functionally depends on the \( N \) particle world lines, i.e., \( x_1[s], \ldots, x_N[s] \). Physically, it may be interpreted as a probability distribution function \([49]\) in the \( \Gamma \) phase space. Indeed, it is obvious that \( D \) has the following properties: (i) \( D \geq 0 \) and (ii) \( \int d\Gamma D = 1 \). Given an elementary Lebesgue-measurable subset \( \omega = \otimes_{i=1}^{N} \omega_i \subset \Gamma, \int_{\omega_i} d\Gamma D \) stands for the probability for particle \( i \) to be in \( \omega_i \) at \( \tau_i \) \( (i = 1, \ldots, N) \).

Similar to relativistic nonequilibrium mechanics built on the \( \mu \) phase space \([3]\), from the causality follows

**Lemma 2.2.** The distribution function \( D \) satisfies

\[ \lim_{\tau_i \to \pm \infty} D = 0, \quad \forall i \]

(9)
in the sense of Lebesgue measure.

**Proof.** Consider the probability density at the proper times \((\tau_1, \ldots, \tau_N)\) defined as

\[ \rho(x_1, \tau_1, \ldots, x_N, \tau_N; x_1[s], \ldots, x_N[s]) = \prod_{i=1}^{N} d^4p_i D(10) \]

Then, the probability for particle \( i \) to stay inside a finite Lebesgue-measurable volume \( \tilde{\omega}_i \equiv M^4_{\mu} \) at the proper time \( \tau_i \) \( (i = 1, \ldots, N) \) is

\[ n(\tau_1, \ldots, \tau_N) = \int_{\tilde{\omega}_1} d^4x_1 \cdots \int_{\tilde{\omega}_N} d^4x_N \rho . \]

(11)

Notice that for given \( i \) the world line \( x_i[s] \), having crossed \( \tilde{\omega}_i \), could not return. Thus,

\[ \lim_{\tau_i \to \pm \infty} n(\tau_1, \ldots, \tau_N) = 0 \]

(12)

for arbitrary \( \tilde{\omega}_i \). From this the lemma follows. Q.E.D.

However, the distribution function \( D \) has a remarkable property. That is, with the momenta integrated out it is reduced into a distribution function defined on the entire Minkowski spacetime. Thus, the distribution function \( D \) is not physical. Instead, to match macroscopic observations we have to further introduce the physical distribution function \( N \) such that with momenta integrated out it is reduced into a distribution function defined on a spacelike 3-surface–a subspace of the Minkowski spacetime. Mathematically, it is defined as

\[ N(x_1, p_1, \ldots, x_N, p_N; x_1[s], \ldots, x_N[s]) = \prod_{i=1}^{N} d\tau_i D(13) \]

which is the straightforward generalization of its \( \mu \) phase space counterpart \([3]\). The normalization of \( N \) follows from
Lemma 2.3. The physical distribution function $\mathcal{N}$ satisfies
\[
\int_{\Sigma_1 \otimes U_i^0} d\Sigma_{\mu_1} d^3 p_1 u_{\mu_1} \cdots \int_{\Sigma_N \otimes U_N^0} d\Sigma_{\mu_N} d^3 p_N u_{\mu_N} \mathcal{N} = 1.
\]
(14)

Here $\Sigma_i$ is an arbitrary spacelike 3-surface in $\mathcal{M}_i^4$ with the vectorial value of the differential form as $d\Sigma_{\mu_i}$.

Proof. For the volume element $d^4 x_i$ we have
\[
d^4 x_i = d\Sigma_{\mu_i} d x_{\mu_i}^0 = d\tau_i d\Sigma_{\mu_i} \frac{d x_{\mu_i}^0}{d\tau_i},
\]
where $d\Sigma_{\mu_i}$ is the differential form of an arbitrary spacelike 3-surface $\Sigma_i$:
\[
d\Sigma_{\mu_i} = \frac{1}{3!} \varepsilon_{\mu\nu\rho\lambda} d x_{\nu}^0 \land d x_{\rho}^0 \land d x_{\lambda}^0.
\]
(16)
Here $\varepsilon_{\mu\nu\rho\lambda}$ is $\pm 1$ when $(\mu\nu\rho\lambda)$ is an even (odd) permutation of $(0123)$ and is 0 otherwise.

Notice that $\int d\tau D = 1$. We then substitute Eq. (17) into it and integrate out $\tau_i$'s. With Eqs. (8) and (13) taken into account eventually we obtain Eq. (14). Q.E.D.

Remark. This lemma shows that the physical distribution function $\mathcal{N}$ is the analogue of the $\Gamma$ phase space distribution function in the Newtonian physics. The 7N-dimensional manifold on which $\mathcal{N}$ is normalized may be considered as the effective phase space. For the convenience below here we also introduce the space of physical one-body distribution $\mathcal{H}$ defined as:
\[
\mathcal{H} = \left\{ h | h : \Sigma \otimes U \rightarrow \mathbb{R}^+ \cup \{0\}, \lim_{N \rightarrow +\infty} N^{-1} \int_{\Sigma \otimes U} d\Sigma_{\mu} d^4 p u_{\mu} h = 1 \right\}.
\]
(18)

To further proceed we introduce the Liouvillian $\hat{L}$ which is decomposed into the free and interacting part, i.e., $\hat{L} = \hat{L}^0 + \lambda \hat{L}'$ with $\hat{L}^0$ and $\lambda \hat{L}'$ the free and interacting Liouvillian, respectively. They are defined as
\[
\hat{L}^0 = - \sum_{i=1}^N \left[ u_{\mu_i} \partial_{\mu_i} + F_{\text{ext}}^\mu (x_i, p_i) \frac{\partial}{\partial p_{\mu_i}} \right],
\]
\[
\lambda \hat{L}' = \sum_{i<j} \lambda \hat{L}'_{ij},
\]
\[
\lambda \hat{L}'_{ij} = - \left\{ F_{ij}^\mu (x_i, p_i) \frac{\partial}{\partial p_{\mu_i}} + F_{ji}^\mu (x_j, p_j) \frac{\partial}{\partial p_{\mu_j}} \right\},
\]
(19)
where $\hat{L}'_{ij}$ is the two-body interacting Liouvillian. Notice that here the dimensionless parameter $\lambda$ characterizes the interaction strength. Now we are ready to prove the following Liouville theorem which justifies the covariant notion of evolution in the $\Gamma$ phase space, and constitutes the exact starting point of the succeeding sections:

Theorem 2.4. If both the external and interaction force are conservative, i.e.,
\[
\frac{\partial}{\partial p_i^\mu} F_{\text{ext}} (x_i, p_i) = 0, \quad \frac{\partial}{\partial p_i^\mu} F_{ij} (x_i, p_i) = 0,
\]
then the distribution function $D(x_i, p_1, \tau_1 + \tau, \cdots, x_N, \tau_N) satisﬁes the following Liouville equation:
\[
\left( \frac{\partial}{\partial \tau} - \hat{L} \right) D = 0.
\]
(21)

Proof. Suppose that at the proper times $(\varsigma_1, \cdots, \varsigma_N)$ particles are in the volume $\omega = \prod_{i=1}^N d^4 x_i(\varsigma_i) d^3 p_i(\varsigma_i)$ with a probability density $D(x_i, p_1, \tau_1 + \tau, \cdots, x_N, \tau_N)$ satisﬁes the following Liouville equation:
\[
d^4 x_i(\varsigma_i) d^4 p_i(\varsigma_i) \prod_{i<j} d^4 x_j d^4 p_j D(\cdots x_i(\varsigma_i), p_i(\varsigma_i), \varsigma_i \cdots) = d^4 x_i(\varsigma'_i) d^4 p_i(\varsigma'_i) \prod_{i<j} d^4 x_j d^4 p_j D(\cdots x_i(\varsigma'_i), p_i(\varsigma'_i), \varsigma'_i \cdots)
\]
(22)
with all the irrelevant arguments in $D$ suppressed. Since the forces are conservative, the Lebesgue measure is conserved, i.e., $d^4 x_i(\varsigma_i) d^3 p_i(\varsigma_i) = d^4 x_i(\varsigma'_i) d^3 p_i(\varsigma'_i)$. Eq. (22) then gives
\[
D(\cdots x_i(\varsigma_i), p_i(\varsigma_i), \varsigma_i \cdots) = D(\cdots x_i(\varsigma'_i), p_i(\varsigma'_i), \varsigma'_i \cdots).
\]
(23)

Letting $\varsigma'_i \rightarrow \varsigma_i$ we obtain
\[
\left\{ \frac{\partial}{\partial \varsigma_i} + u_{\mu_i} \partial_{\mu_i} + \left[ F_{\text{ext}}^\mu (x_i, p_i) + \sum_{j \neq i}^N F_{ij}^\mu (x_i, p_i) \right] \frac{\partial}{\partial p_{\mu_i}} \right\} D(x_i, p_1, \varsigma_i, \cdots, x_N, \varsigma_N) = 0.
\]
(24)

Totally we have $N$ such Liouville equations with different evolution time $\varsigma_i$.

Furthermore, we let the $N$ proper times change uniformly, i.e.,
\[
d\varsigma_1 = \cdots = d\varsigma_N \equiv d\tau
\]
(25)
or equivalently, \( \zeta_i = \tau_i + \tau \) with \( \tau_i \)'s the initial proper times. Then, adding the \( N \) Liouville equations together Eq. (21) follows. Q.E.D.

Remarks. (i) In the absence of interactions the above single-time Liouville equation was derived by Hakim [1]. The manifestly covariant Liouville equation originates at the probability conservation law, and is irrespective of the Hamiltonian character of relativistic classical many-body dynamics. Of course, in the Hamiltonian formalism the (single-time) Liouville equation trivially exists albeit nonmanifestly covariant [13, 15]. In the action-at-a-distance formalism the classical many-body dynamics has non-Hamiltonian character. To justify the Liouville equation the constraint of conservative force must be imposed. Provided that such constraint is released one may enlarge the \( \Gamma \) phase space and subsequently arrive at a generalized single-time Liouville equation [1].

(ii) The single-time Liouville equation shows that the distribution function \( D \) depends on a proper time \( \tau \) parametrizing the global evolution in the \( \Gamma \) phase space. Yet, it still depends on the other \( N - 1 \) proper times associated with the initial condition of the global evolution. Upon passing from \( N \) Liouville equations (24) to the single-time Liouville equation (21) some details of deterministic classical many-body dynamics are lost. Thus, the single-time Liouville equation is not equivalent to relativistic motion equations. The peculiar feature of multiple proper times roots in the noninstantaneous feature of relativistic force and causes a conceptual difference from the Newtonian physics. In the latter case the instantaneous nature of forces allows one to parametrize the global evolution by the usual coordinate time. Consequently, in the sense of dynamics of distribution functions (single-time) Liouville equation is equivalent to motion equations.

(iii) For Eq. (21) let us integrate out \( \tau_i \)'s. Taking into account Eq. (18) we find

\[
\mathbb{L}N(x_1, p_1, \ldots, x_N, p_N; x_1[s], \ldots, x_N[s]) = 0. \quad (26)
\]

Provided that the functional dependence of \( N \) on \( x_1[s], \ldots, x_N[s] \) is released it then recovers the exact starting point of Refs. [18, 19] (in the framework of general relativity) and Ref. [24]. The very nature of such functional independence on the \( N \) particle world lines is that at given global proper time \( \tau \) the representation point is sufficient to uniquely determine the particle world lines. This does happen in the asymptotic sense. It turns out that, as the Newtonian physics, for sufficiently large proper time \( \tau \) the relativistic many-body systems may also lose the memory on “initial” world line segments—the profound change of deterministic classical dynamics. Consequently, the instantaneous evolution of \( N \) is determined by specified phase trajectories associated with \( N \) particles each of which, importantly, merely depends on the phase coordinate of given particle. In Sec. [IV] we will show that the single-time Liouville equation (21), indeed, admits such kind of solution in the thermodynamic limit: \( N \to +\infty \). However, we are not able to justify this picture for finite but sufficiently large \( N \).

III. CORRELATION DYNAMICS ANALYSIS OF GLOBAL EVOLUTION

In this section we wish to apply the Balescu-Wallenborn projection operator approach [36, 47] to the single-time Liouville equation (21). In doing so we achieve a theory of relativistic classical correlation dynamics in the way that at each step the manipulations are manifestly covariant. With this approach we manage to split the proper-time dependent distribution function into the kinetic and non-kinetic component. Each of them independently evolves, and the latter decays for sufficiently large global proper time.

It should be emphasized that given appropriate assumptions regarding the deterministic classical many-body dynamics, as to be specified below, (In other words, we will deal with some axiom dynamical system,) all the results presented in Sec. [III] and [IV] are mathematically rigorous.

A. Reduced distribution function representation

Let us first introduce the concept of reduced distribution function. From now on we denote the arguments: \( (x_i, p_i, \tau_i + \tau) \) of distribution functions as \( i \). Then, the reduced \( s \)-body distribution function is defined as

\[
D_s(i_1, \ldots, i_s; x_{i_1}, \ldots, x_{i_s}) \equiv \int \prod_{j=s+1}^N di_j D(1, \ldots, N; x_{i_1}, \ldots, x_{i_s}, x_j), \quad \forall 1 \leq i_1 < \cdots < i_s \leq N. \quad (27)
\]

Notice that here the particle groups \((i_1, \ldots, i_s)\) and \((i_{s+1}, \ldots, i_N)\) constitute a partition of the full system \((1, \ldots, N)\). Moreover, the reduced \( s \)-body distribution function \( D_s \) stands for the probability density for particle \( i_j \) to be at \((x_{i_j}, p_{i_j})\) at the proper time \( \tau_{i_j} + \tau \) \((1 \leq j \leq s)\). The normalization can be shown to be

\[
\prod_{j=1}^s \int di_j D_s(i_1, \ldots, i_s; x_{i_1}, \ldots, x_{i_s}) = 1. \quad (28)
\]

Then, for Eq. (21) we integrate out the phase coordinates of particle \( i_{s+1}, \ldots, i_N \). As a result we obtain

\[
\left\{ \frac{\partial}{\partial \tau} + \sum_{j=1}^s u_{ij}^\tau \partial_{i_j} - \sum_{j < j'} \lambda \hat{E}_{i_j, i_j'} \right\} D_s = \sum_{i_{s+1}=1}^N \int di_{s+1} \sum_{j=1}^s \lambda \hat{E}_{i_j, i_{s+1}} D_{s+1}, \quad (29)
\]
which is a manifestly covariant relativistic BBGKY hierarchy. Here in order to make the formula compact we have omitted all the arguments of $\mathcal{D}_s$ and $\mathcal{D}_{s+1}$. Notice that Eq. (29) differs from that studied in Refs. 22, 24, 25, where nonmanifestly covariant relativistic BBGKY hierarchies were derived for physical distribution functions.

Let us further define the distribution vector:

$$\mathbf{D} \equiv (\{D_1\}, \{D_2\}, \ldots, \{D_N\} \equiv D).$$ (30)

Notice that, for each $\{D_s\}$ there are $N!/[(N-s)!s!]$ components each of which corresponds to a $s$-particle group $(i_1, \ldots, i_s)$ with $1 \leq i_1 < \cdots < i_s \leq N$. With the help of this definition the BBGKY hierarchy is rewritten in a compact form:

$$\left( \frac{\partial}{\partial \tau} - \hat{\mathbf{L}} \right) \mathbf{D} = 0.$$ (31)

The projection to general $s$-particle states, denoted as $(i_1, \ldots, i_s)$ is identical to Eq. (29) provided that the matrix elements are set to be

$$(i_1, \ldots, i_j| \hat{\mathbf{L}}^0| i_1, \ldots, i_j) = -\sum_{k=1}^{j} u_{i_k} \partial_{i_k},$$

$$(i_1, \ldots, i_j| \lambda \hat{\mathbf{L}}' | i_1, \ldots, i_{j}, i_{j+1}) = -\int d_{i_{j+1}} \sum_{k=1}^{j} \lambda \hat{\mathbf{L}}_{ikj_{j+1}}.$$ (32)

and to be zero otherwise. Notice that the two-body Liouvillian $\lambda \hat{\mathbf{L}}_{ikj_{j+1}}$ is determined by the world lines of particles $i_k$ and $i_{j+1}$. Eqs. (30)-(32) may be considered as the reduced distribution function representation of the single-time Liouville equation (21).

Likewise, we may carry out the same program for the physical distribution function $\mathcal{N}$ and cast Liouville equation (26) into another manifestly covariant BBGKY hierarchy. The details are presented in Appendix A.

### B. Correlation pattern representation

To proceed further we introduce the so-called correlation pattern representation. Consider a general $s$-particle object $F(1, \ldots, s; x_{k_1}[s], \ldots, x_{k_s}[s])$ possessing the permutation symmetry with respect to $12\cdots s$, where $1 \leq k_i \leq N$, and $k_i \neq k_{i'}$, $\forall i \neq i'$. A unique correlation pattern, denoted as $|\Gamma_s\rangle$ (or $|\Gamma_s\rangle$) may be assigned to it in the following way: If $F$ is factorized into $s$ components each of which merely depends on $i \equiv (x_i, p_i, \tau_i + \tau)$ and $x_{k_i}[s]$ ($1 \leq i \leq s$), then $|\Gamma_s\rangle = |0_s\rangle = |1|2|\cdots|s\rangle$ (or $|\Gamma_s\rangle = |1|2|\cdots|s\rangle$), which is called vacuum state. In general, $F$ is at most partially factorized and called correlation state. And the correlation pattern describes the factorization structure. More precisely, suppose that $F$ possesses the factorization as follows:

$$F(1, \ldots, s; x_{k_1}[s], \ldots, x_{k_s}[s])$$

$$= \prod_{i=1}^{j} F_i(x_{k_i}[s], x_{k_i+1}[s], \ldots, x_{k_i+j_i}[s]),$$ (33)

where $P_i \equiv (s_i, s_i+1, \ldots, s_i+j_i) \neq \emptyset$, $i = 1, 2, \ldots, j < N$ constitute a partition of $(1, 2, \ldots, s)$:

$$P_1 \cup \cdots \cup P_j = (1, \ldots, s), P_i \cap P_{i'} = \emptyset, \forall i \neq i'.$$ (34)

Then, the correlation pattern is $|\Gamma_s\rangle = |P_1|P_2|\cdots|P_j\rangle$ (or $|\Gamma_s\rangle = |P_1|P_2|\cdots|P_j\rangle$). Notice that the correlation pattern possesses the permutation symmetry with respect to $P_1P_2\cdots P_j$. Therefore, two correlation patterns are considered to be identical if they differ only in the order of $P_i$.

With this definition the distribution vector may be decomposed in a more delicate manner:

$$\mathcal{D}_s(i_1, \ldots, i_s; x_{i_1}[s], \ldots, x_{i_s}[s])$$

$$= \sum_{\Gamma_s}|\Gamma_s\rangle\langle\Gamma_s|\mathcal{D}_s(i_1, \ldots, i_s; x_{i_1}[s], \ldots, x_{i_s}[s])$$

$$= \sum_{j=1}^{s} \sum_{P_1\cdots P_j} |P_1|P_2|\cdots|P_j\rangle\langle P_1|P_2|\cdots|P_j|\mathcal{D}_s(i_1, \ldots, i_s; x_{i_1}[s], \ldots, x_{i_s}[s])$$ (35)

for general reduced $s$-body distribution function, which is the reformulation of the cluster expansion. Here $P_1, \ldots, P_j$ is a partition of $(i_1 \ldots i_s)$. Notice that in the third line any two terms do not possess identical correlation pattern. Furthermore, in Appendix B we give the matrix elements of the preliminary operators $\hat{\mathbf{L}}^0$ and $\lambda \hat{\mathbf{L}}'$ in this representation. It should be stressed that such a cluster expansion differs from traditional one
that the distribution functions depend on particle world lines. Such a concept was first introduced by Klimontovich in the Newtonian context and was generalized to special relativity—in a manifestly covariant manner—by Hakim.

In this way we have assigned all the preliminary quantities to be used below, i.e., \( \mathcal{D} , \mathcal{L}^0 \) and \( \lambda \mathcal{L}' \) a unique decomposition in the correlation pattern representation. Thus, we may define the following vacuum and correlation operator, denoted as \( V \) and \( C \), respectively:

\[
V|\Gamma_\tau\rangle = \delta_{0,\Gamma_\tau} |\Gamma_\tau\rangle , \quad C|\Gamma_\tau\rangle = (1 - \delta_{0,\Gamma_\tau}) |\Gamma_\tau\rangle \tag{36}
\]

which are diagonal in the correlation pattern representation \( |\Gamma_\tau\rangle \) (and also in \( \langle \Gamma_\tau | ) \). It is easy to show that \( V \) and \( C \) constitute an orthogonal decomposition, i.e.,

\[
V + C = 1 , \quad V^2 = V , \quad C^2 = C , \quad VC = CV = 0 . \tag{37}
\]

With the correlation pattern representations of \( \mathcal{D} , \mathcal{L}^0 , \lambda \mathcal{L}' , \mathcal{V} \) and \( \mathcal{C} \) as building blocks, one may proceed to establish representations of more complicated operators with the help of following properties:

1. Completeness. \( \sum_r \sum_{\Gamma_r} |\Gamma_r\rangle \langle \Gamma_r| = 1 ; \)
2. Orthogonality. \( \langle \Gamma_r | \Gamma_{r'}\rangle = \delta_{rr'} \delta_{\Gamma_r \Gamma_{r'}} ; \)
3. Linearity. For arbitrary operators \( A \) and \( B \) : \( (A + B)|\Gamma_\tau\rangle = A|\Gamma_\tau\rangle + B|\Gamma_\tau\rangle . \)

C. Propagating operator and irreducible evolution operator

From now on we consider closed systems namely \( F^\mu_{\text{ext}} = 0 \) . In the absence of interactions, Eq. (21) is reduced into

\[
\left( \frac{\partial}{\partial \tau} - \mathcal{L}^0 \right) D = 0 . \tag{38}
\]

In this case, the evolution is determined by the propagating operator \( \hat{U}^0(\tau) \), or equivalently, its resolvent \( \mathcal{R}^0(z) \) which are defined as

\[
\hat{U}^0(\tau) = e^{\tau \mathcal{L}^0} , \quad \mathcal{R}^0(z) = \frac{1}{-iz - \mathcal{L}^0} . \tag{39}
\]

They are related through

\[
\hat{U}^0(\tau) = (2\pi)^{-1} \int_C dz e^{-iz\tau} \mathcal{R}^0(z) , \tag{40}
\]

where the contour \( C \) lies above all the singularities of the Laplace transform of \( \mathcal{D} \).

We further introduce the irreducible evolution operator \( \hat{E}(\tau) \) and its Laplace transform \( \hat{E}(z) \) :

\[
\hat{E}(z) = \sum_{n=0}^{\infty} \lambda^{n+1} \mathcal{L}^{n} \left[ \mathcal{C} \mathcal{R}^0(\tau) \mathcal{L}' \right]^n , \tag{41}
\]

\[
\hat{E}(\tau) = (2\pi)^{-1} \int_C dz e^{-iz\tau} \hat{E}(z) . \tag{42}
\]

Then, the axiom dynamical system that we will consider throughout this work is defined as such systems that satisfy the following

Assumption 3.1. The operators: \( V \hat{E}(z) V , V \mathcal{E}(z) \mathcal{R}^0(z) \mathcal{C} , C \mathcal{R}^0(z) \mathcal{E}(z) \mathcal{V} \) and \( C \mathcal{R}^0(z) + C \mathcal{R}^0(z) \mathcal{E}(z) \mathcal{R}^0(z) \mathcal{C} \) are regular and nonvanishing at \( z = 0 \).

We stress that this assumption merely introduces restrictions on microscopic interactions. A heuristic example where this assumption is applicable is a system composed of identical particles which interact with each other through short-ranged interactions.

D. Evolution of the kinetic component of distribution vector

Under Assumption 3.1, the following two theorems are straightforward generalizations of their Newtonian counterparts, and we shall not present the proof here.

Theorem 3.2. The distribution vector \( \vec{\mathcal{D}} \) may be decomposed into the kinetic component \( \hat{\Pi}_k \vec{\mathcal{D}} \) and the nonkinetic component \( \hat{\Pi}_{n.k.} \vec{\mathcal{D}} \) , i.e.,

\[
\vec{\mathcal{D}} = \hat{\Pi}_k \vec{\mathcal{D}} + \hat{\Pi}_{n.k.} \vec{\mathcal{D}} . \tag{42}
\]

The vacuum part of the former satisfies a closed hierarchy:

\[
\left( \frac{\partial}{\partial \tau} - \mathcal{V} \mathcal{F} \right) \mathcal{V} \hat{\Pi}_k \vec{\mathcal{D}} = 0 , \tag{43}
\]

where the operator \( \mathcal{V} \mathcal{F} \) is given by the following functional equation:

\[
\mathcal{V} \mathcal{F} = \mathcal{V} \hat{\mathcal{L}} + \int_0^\infty ds \mathcal{V} \mathcal{g}(s) \mathcal{F} \exp(-s \mathcal{V} \mathcal{F}) , \quad \mathcal{V} \mathcal{g}(s) \mathcal{F} = (2\pi)^{-1} \int_C dz e^{-isz} \mathcal{V} \hat{E}(z) \mathcal{R}^0(z) \mathcal{C} \lambda \mathcal{L}' \mathcal{F} . \tag{44}
\]

Remark. The operators \( \hat{\Pi}_k \) and \( \hat{\Pi}_{n.k.} \) are \( \tau \)-independent nonlinear functionals of the interacting Liouvillian \( \lambda \mathcal{L}' \). However, their explicit expressions are not needed for subsequent analysis and, therefore, we shall not present them here. Then, carrying out the Laplace transform [denoting the Laplace transform of \( \mathcal{g}(s) \) as \( \mathcal{g}_L \)] and substituting Eq. (41) into Eq. (44), we find

\[
\mathcal{V} \hat{g}_L \mathcal{F} = \sum_{n=1}^{\infty} \lambda^{n+1} \mathcal{V} \hat{\mathcal{L}}^{n} \left[ \mathcal{C} \mathcal{R}^0(\tau) \mathcal{L}' \right]^n \mathcal{F} . \tag{45}
\]
Notice that the operator $\mathbf{VFV}$ depends on the given particle world lines through the interacting Liouvillian $\lambda L'$. For the correlation part, as shown in the following theorem, it is fully determined by the vacuum part:

**Theorem 3.3.** The kinetic component of the distribution vector $\Pi_k \overrightarrow{D}$ satisfies

$$C \Pi_k \overrightarrow{D} = \int_0^\infty ds \int_0^s ds' \mathcal{U}(s - s') \mathcal{E}(s') \times \exp(-s \mathbf{VFV}) \Pi_k \overrightarrow{D}.$$  \hspace{1cm} (46)

Eqs. (13) and (19) constitute the main equations of the manifestly covariant classical correlation dynamics. Crucially, they differ from traditional correlation dynamics [15, 18, 30] in that both the operators $\Gamma$, $\mathcal{E}$, $\Pi_k$ and the distribution vector $\overrightarrow{D}$ are functionals of particle world lines. Such significant difference causes additional conceptual and technical complications.

**IV. BOGOLIUBOV FUNCTIONAL ASSUMPTION**

In the thermodynamic limit $N \to +\infty$ the kinetic component obeys an infinite hierarchy. In this section we show that such an infinite hierarchy may be reduced into a closed highly nonlinear equation which, remarkably, admits a solution uniquely determining the physical distribution function of full many-body systems. In doing so we achieve a manifestly covariant version of the Bogoliubov functional assumption.

**A. Hakim-Israel-Kandrup trial solution**

For the convenience below we introduce the mapping $\mathcal{X}$ which uniquely converts a representation point in $\mu$ space into a world line. It is defined as follows:

$$\mathcal{X} : (x, p) \in \mu \mapsto \mathcal{X}(x, p) \equiv x[s]$$  \hspace{1cm} (47)

with the world line $x[s]$ satisfying

(i) $x|_{s=0} = x$, $\frac{dx}{ds}|_{s=0} = p/m$;

(ii) $\forall s$, $\sqrt{\frac{d^2x}{ds^2} \frac{dx}{ds}} \leq 1$;

(iii) $d^2x ds^2$ is invariant along the phase trajectory: $(x[s], m \frac{dx}{ds}|_{x[s]})$.

The property (i) implies that $x[s]$ is a 1-dimensional submanifold of $M^4$ which passes through $x$ with the 4-momentum vector $p$, (ii) implies that the world line $\mathcal{X}(x, p)$ preserves the causality, and (iii) implies that particles move in the way as exposed to fictitious external field which introduces “conservative forces”. Notice that the details of $\mathcal{X}$, although complicated generally, are unimportant at this stage.

Eq. (13) is a closed hierarchy of $\Pi_k \overrightarrow{D}$. Thus, the latter as a whole may be regarded as a new distribution vector, denoted as $\overrightarrow{D}^0 \equiv \Pi_k \overrightarrow{D}$, which possesses the similar structure as $\overrightarrow{D}$, i.e., $\overrightarrow{D}^0 = (\{D^0_1\}, \{D^0_2\}, \cdots)$. In this part we come to study a particular solution of $\overrightarrow{D}^0$ with the general component as

$$D^0_0(i_1, \ldots, i_s; x_i[s], \ldots, x_i[s]) = D^0_s(i_1, \ldots, i_s; \mathcal{X}(x_1, p_1), \ldots, \mathcal{X}(x_s, p_s))$$

$$\equiv D^0_0(i_1, \ldots, i_s; \mathcal{X}_{i_1}, \ldots, \mathcal{X}_{i_s}).$$  \hspace{1cm} (48)

Also, we define the distribution vector $\overrightarrow{D}_\infty$ as follows:

$$\mathcal{N}(\Gamma_j(i_1, \ldots, i_s)|\overrightarrow{D}_\infty \equiv (\Gamma_j(i_1, \ldots, i_s)|\overrightarrow{D}_\infty, \forall (\Gamma_j(i_1, \ldots, i_s)), j \geq 1. \hspace{1cm} (49)$$

In Ref. [1]–in an implicit manner–Hakim noticed that in order to derive kinetic equations the distribution functions in the 8-dimensional $\mu$ phase space have to functionally depend on the particle world line, and the latter merely relies on the phase coordinates in the single particle $\mu$ phase space. In the work of Israel and Kandrup [18, 19] this concept was reinforced and explicitly formulated. Such an idea lies at the heart of Eq. (18). For this reason we may call $\overrightarrow{D}^0$ with the components given by Eq. (15) the Hakim-Israel-Kandrup (HIK) trial solution. From the HIK trial solution an important fact follows: In general, to determine expression of Eq. (13) explicitly requires calculations of the general matrix element $\langle i_1 | \cdots | i_j \exp\{\tau \mathbf{VFV}\}|i_1 | \cdots | i_{j'}\rangle$ with $j \leq j'$, where the sequence $i_1 \cdots i_{j+1}$ satisfies $i_s \neq i_s', \forall s \neq s'$. On one hand, $\exp\{\tau \mathbf{VFV}\}$ explicitly depends on the particle world lines given by the acted distribution vector. On the other hand, the particles, though having identical phase coordinates, may still be “distinguished” through their world lines which generally are coupled to each other. Consequently, the quantity $\langle i_1 | \cdots | i_j \exp\{\tau \mathbf{VFV}\}|i_1 | \cdots | i_{j'}\rangle |\overrightarrow{D}_\infty$ is sensitive to particles $i_j, i_{j+1} \cdots i_s$ in the intermediate states. The substantial simplification introduced by the HIK trial solution is just to remove this sensitivity by decoupling the particle world lines. That is,

$$\langle i_1 | \cdots | i_{j'} \langle i_{j+1} | \cdots | i_{j'} \rangle |\overrightarrow{D}_\infty = \langle i_1 | \cdots | i_{j'} \langle i_{j+1} | \cdots | i_{j'} \rangle \overrightarrow{D}_\infty,$$  \hspace{1cm} (50)

where the particle groups $(i_{j+1}, \cdots, i_{j'})$ and $(i_{j+1}', \cdots, i_{j'}')$ are not identical.

For the HIK trial solution the factorization theorem below shows that Eq. (13) may be reduced into a single closed equation of reduced one-body distribution function, which is the generalization of Clavin’s theorem.
Proposition 3.4. In the limit $N \to +\infty$ the infinite hierarchy \((\ref{eq:17})\) may be reduced into a single closed equation of reduced one-body distribution function as follows:

$$
\left\{ \frac{\partial}{\partial \tau} + u^n_i \partial_{\mu_1} - \int d2 \lambda \hat{L}_{12} \tilde{D}(2; \chi_2) \right\} \tilde{D}(1; \chi_1) = \sum_{j \geq 2} \int d2 \cdots \int d j \langle 1 | V(\Gamma - \hat{\Sigma}) V | 1 \cdots | j \rangle \prod_{s=1}^j \tilde{D}(s; \chi_s),
$$

\hspace{1cm} (51)

Furthermore, the stationary solution with respect to the $\tau$-parametrized evolution above may be reduced into the following closed equation:

$$
\left\{ u^n_i \partial_{\mu_1} - \int d\Sigma_2 d^4 p_2 u^n_i \lambda \hat{L}_{12} f(2) \right\} f(1) = \mathbb{K}[f]
$$

\hspace{1cm} (52)

Proof. Let us consider the HIK trial solution. We may formally solve the hierarchy \((\ref{eq:17})\) and project the solution $\tilde{D}^0(\tau)$ to the one-body vacuum state say $|i\rangle$. Consequently, we obtain (In order to make the formula compact for the moment we suppress all the arguments of $\tilde{D}^0$ except the proper time $\tau$ parametrizing the global evolution.)

$$
\langle i | \tilde{D}^0(\tau) = \langle i | \exp\{\tau V \Gamma V\} | i \rangle \langle i | \tilde{D}^0(0) + \sum_{j=1}^N \sum_{i_1 \cdots i_j=1}^N \langle i | \exp\{\tau V \Gamma V\} | i | i_1 \cdots | i_j \rangle \langle i | i_1 \cdots | i_j | \tilde{D}^0(0),
$$

\hspace{1cm} (53)

where the sequence $i_1 \cdots i_j$ satisfies $i_s \neq i, \forall s$ and $i_s \neq i_s', \forall s \neq s'$. Without loss of generality the sequence is ordered in the way that it starts from $i$. Because of the particle symmetry namely Eq. \((\ref{eq:18})\) we may simplify Eq. \((\ref{eq:20})\) as (Without loss of generality we set $i = 1$.)

$$
\langle 1 | \tilde{D}^0(\tau) = \sum_{j=1}^N \frac{(N-1)!}{(N-j)!} \langle 1 | \exp\{\tau V \Gamma V\} | 1 \cdots | j \rangle \langle 1 \cdots | j | \tilde{D}^0(0),
$$

\hspace{1cm} (54)

where the combinatorial factors arises from the particle symmetry. Notice that the matrix element of the operator $\exp\{\tau V \Gamma V\}$ is well defined in the limit $N \to +\infty$ because the world lines are given by the mapping $\chi'$. Multiplying both sides by $N$ and taking Eq. \((\ref{eq:21})\) into account we find

$$
\langle 1 | \tilde{D}_\infty(\tau) = \sum_{j=1}^\infty \langle 1 \cdots | j | \tilde{D}^0(0) \rangle \langle 1 \cdots | j | \tilde{D}_\infty(0)
$$

\hspace{1cm} (55)

from Eq. \((\ref{eq:22})\).

In general, for finite $N$ projecting $\tilde{D}^0(\tau)$ to a vacuum state, say $\langle i_1 \cdots | i_j \rangle, 1 < j < N$ we obtain

$$
\langle i_1 \cdots | i_j | \tilde{D}^0(\tau) = \langle i_1 \cdots | i_j \exp\{\tau V \Gamma V\} V \tilde{D}^0(0)
$$

\hspace{1cm} (56)

where for fixed $n$ the sequence $i_1 \cdots i_n$ satisfies $i_s \neq i_{s'}, \forall s \neq s', 1 \leq s, s' \leq n$. In the second equality for the intermediate vacuum state the particle order again needs to be distinguished and, without loss of generality, we set the leading $j$ particles to be $i_1 i_2 \cdots i_j$. Because of the particle symmetry with the limit $N \to +\infty$ taken we obtain (setting $i_k = k, k = 1, \cdots, j$)

$$
\langle 1 \cdots | j | \tilde{D}_\infty(\tau) = \sum_{n=j}^\infty \langle 1 \cdots | j \exp\{\tau V \Gamma V\} | 1 \cdots | n \rangle \langle 1 \cdots | n | \tilde{D}_\infty(0) \rangle \equiv R
$$

\hspace{1cm} (57)

or equivalently,

$$
\tilde{D}_\infty(\tau) = \exp\{\tau V \Gamma V\} \tilde{D}_\infty(0)
$$

\hspace{1cm} (58)
by noticing \( \langle 1 | \cdots | j \exp\{\tau V\Gamma V\} | 1 \cdots | n \rangle = 0 \) for \( j > n \).

We then come to analyze \( R \). For this purpose we consider the diagrammatical presentation of general matrix element: \( \langle 1 | \cdots | j \exp\{\tau V\Gamma V\} | 1 \cdots | n \rangle, j \leq n \) describing the transition from the intermediate to final vacuum state. Fully parallel to the Newtonian physics \[36\], starting from Eq. (44) and employing the correlation pattern representation of operators \( \hat{\mathcal{U}}^0 \) and \( \lambda \hat{\mathcal{W}}' \) (see Appendix \[9\], one may show that diagrams representing the transition element are composed of \( j \) disconnected clusters. Each of them involves a particle group \( P_i \) \((i = 1, 2, \cdots, j)\) with \( 1 + j_i \) particles \((j_i \geq 0)\) which are labeled as \( i \) and \( s_i, s_i + 1, \cdots, s_i + j_i \). Upon propagating to the left particles \( s_i, s_i + 1, \cdots, s_i + j_i \) are annihilated and eventually at the left-most side only particle \( i \) is left giving the vacuum state \( \langle i \rangle \). Thus, we factorize the matrix element \( R \) into \( j \) components:

\[
R = \sum_{n=j}^{\infty} \sum_{P_i} \prod_{i=1}^{j} \langle i | \exp\{\tau V\Gamma V\} | 0_i(P_i) \rangle \langle 0_i(P_i) | \hat{\mathcal{D}}_\infty(0) \rangle 
\]

\[
= \sum_{s_1=1}^{\infty} \cdots \sum_{s_1=1}^{\infty} \prod_{i=1}^{j} \langle i | \exp\{\tau V\Gamma V\} | i_1 \cdots i_{s_i} \rangle \langle i_1 \cdots i_{s_i} | \hat{\mathcal{D}}_\infty(0) \rangle 
\]

\[
= \prod_{i=1}^{j} \sum_{s_i=1}^{\infty} \langle i | \exp\{\tau V\Gamma V\} | i_1 \cdots i_{s_i} \rangle \langle i_1 \cdots i_{s_i} | \hat{\mathcal{D}}_\infty(0) \rangle, \quad (60)
\]

where in the second equality we use the particle symmetry namely Eq. (60) to make the change of variables: \( P_i \to i_1, i_2, \cdots, i_{s_i} \) in such a way that \( i_1 \equiv i \) and \( i_s \not\equiv i_{s'}, s \not\equiv s' \).

With Eq. (60) substituted Eq. (57), together with Eq. (55) then gives

\[
\langle 1 | \cdots | j | \hat{\mathcal{D}}_\infty(\tau) \rangle = \prod_{i=1}^{j} \sum_{s_i=1}^{\infty} \langle i | \exp\{\tau V\Gamma V\} | i_1 \cdots i_{s_i} \rangle \langle i_1 \cdots i_{s_i} | \hat{\mathcal{D}}_\infty(0) \rangle, \quad \forall j \geq 1. \quad (61)
\]

Such an infinite hierarchy is solved by

\[
\langle 1 | \cdots | j | \hat{\mathcal{D}}_\infty(\tau) \rangle = \prod_{s=1}^{j} ND_1(\tau; \lambda_s) 
\]

\[
= \prod_{s=1}^{j} {\mathcal{D}}(x_s, p_s, \tau_s + \tau; \lambda(x_s, p_s)), \quad \forall j \geq 1, \quad (62)
\]

where in the last equality we have retrieved all the arguments of the distribution functions. Indeed, with such a solution inserted the infinite hierarchy (61) is reduced into a single closed equation as follows:

\[
{\mathcal{D}}(1; \lambda_1) = \sum_{j=1}^{\infty} \langle 1 | \exp\{\tau V\Gamma V\} | \cdots | j \rangle \prod_{s=1}^{j} {\mathcal{D}}(s; \lambda_s), \quad (63)
\]

Taking the derivative with respect to \( \tau \) we obtain

\[
\frac{\partial}{\partial \tau} {\mathcal{D}}(1; \lambda_1) = \sum_{j=1}^{\infty} \langle 1 | \exp\{\tau V\Gamma V\} | \cdots | j \rangle \prod_{s=1}^{j} \langle \tau_s(\lambda_s) | \cdots | \lambda_1 \rangle, \quad (64)
\]

which is the differential form of Eq. (63). We thus prove the first part of the proposition.

Let us now fix a spacelike 3-surface \( \Sigma_i \) for particle \( i \) and denote the coordinate as \( \sigma_i \). Then, the stationary \( {\mathcal{D}}(i; \lambda_i) \)--with respect to the \( \tau \)-parametrized evolution--assumes the following form:

\[
{\mathcal{D}}(i; \lambda_i) = \tilde{f}(\sigma_i, p_i; \lambda(x, p_i)(\tau - \xi_i)). \quad (65)
\]

Here \( \xi_i \) is the proper time when the world line \( \lambda(x, p_i) \) passes through the phase point \( (x, p_i) \), and
\[ f(\sigma, p; X(x, p)) = \text{distribution function peaking at the spacelike 3-surface } \Sigma. \]

Inserting Eq. (65) into Eq. (51) and integrating out \( \gamma_i, i \geq 2 \) and \( \gamma_1 \), we obtain a general kinetic equation (62) with the collision integral given by

\[ K[f] = \sum_{j \geq 2} \int_{\Sigma_2 \cup U_2^j} d\mu_{2j} d^4 p_2 u_2^\mu \cdots \int_{\Sigma_1 \cup U_1^j} d\mu_{1j} d^4 p_j u_j^\mu \]

\[ \times \langle 1|V(\Gamma - \hat{\Sigma})V|\cdots|j \rangle \prod_{i=1}^j f(i), \quad (66) \]

where

\[ f(x, p) = \int ds \hat{f}(\sigma, p; X(x, p))(s) \in \mathcal{H}. \quad (67) \]

Notice that in Eq. (66), the two-body interacting Liouvillian \( \hat{\Sigma} \) is a functional of the world lines \( X(x, p) \) and \( X'(x, p) \). The second part of the proposition then follows. Q.E.D.

Remark. (i) By definition the distribution function satisfies \( \dot{D}|_{\gamma_2, \gamma_3} = 0 \) and thereby is compatible with Lemma 2.2. \( f \) may take a particular form: \( f_K(x, p) \delta(\gamma_1 - \gamma_i) \) with \( f_K(x, p) \in \mathcal{H} \), which was given in an implicit manner in Ref. [19]. (ii) An equation similar to Eq. (51) has been obtained by Hakim [1] for dilute gases with scalar and electromagnetic interactions. There, the equation is derived under the weak coupling approximation (namely the second order interaction expansion). In contrast, provided that the force is conservative Eq. (51) presented here is exact including all the higher order (three-, four-body, etc.) correlations. It encompasses a route to go beyond the weak coupling approximation for dilute gases, and is essential to the justification of the relativistic Boltzmann equation (to be detailed in Paper II).

B. Physical correlation functions

The solution of Eq. (52) fully determines the entire physical correlation functions. Indeed, let us replace \( \Pi \rightarrow \Pi_0 \) in Eq. (49) with \( \Pi_0 \), the components of which are given by Eq. (55). Taking into account Eq. (55), by straightforward calculations we may find the hierarchy of physical correlations in the limit: \( N \rightarrow +\infty \), denoted as \( \mathcal{C} \), to be

\[ \mathcal{C} = \int_0^\infty ds \int_0^s ds' \mathcal{C}| \dot{\Sigma}^0(s - s')\hat{\Sigma}(s')V \]

\[ \exp(-sV|\Gamma|) = f(i) \quad (68) \]

In the derivation we notice that the vacuum part of the physical distribution vector is given by

\[ \langle i_1| \cdots |i_n \rangle = \prod_{j=1}^n f(i_j) \quad (69) \]

for arbitrary vacuum state \( \langle i_1| \cdots |i_n \rangle \).

Alternatively, we may project Eq. (52) to the general \( j \)-particle correlation pattern \( \Gamma_{ij}(i_1, \cdots, i_j) \neq 0 \) and arrive at

\[ \langle \Gamma_{ij}| \hat{\Gamma}_\infty = \sum_{n=0}^\infty \int_0^\infty ds \int_0^s ds' \prod_{j=1}^n f(i_j) \quad (70) \]

\[ \times \langle \langle \Gamma_{ij}| \mathcal{C}| \dot{\Sigma}^0(s - s')\hat{\Sigma}(s')V \exp(-sV|\Gamma|) |i_1| \cdots |i_n \rangle \]

This exact formula provides a principle to calculate arbitrary physical correlation functions, provided that the physical one-body distribution function namely \( f(x, p) \) is given. Eqs. (52) and (70) show that (after transient processes) physical correlation functions relax into functionals of physical one-body distribution function. The latter obeys closed kinetic equation (54) from which a relativistic hydrodynamic description stems. Thus, we justify the manifestly covariant Bogoliubov functional assumption.

V. SOME REMARKS ON FURTHER APPLICATIONS

It has been a long standing problem to reconcile statistical mechanics and relativity principles. In recent years this subject has become fundamentally important to studies in many fields as mentioned in the introductory section. In 1984 Israel and Kandrup made substantial progresses in this direction [18, 19]. There, the authors formulated manifestly covariant classical nonequilibrium statistical mechanics and successfully applied it to relativistic plasmas with electromagnetic interactions. In particular, the relativistic Landau equation (27) was justified at the full microscopic level. It differed remarkably from earlier attempts [15, 28] in that the theory is manifestly covariant at each step of the manipulation. As such, the formulated nonequilibrium statistical mechanics (particularly various approximations unavoidable on top of it) is guaranteed to suit relativity principles automatically. That impeding further development lies, as pointed out in the original paper [19], in that the Israel-Kandrup formalism fails to go beyond the weak coupling approximation and to capture the collective effects. (It is important that the latter heals the well-known logarithmic divergence of the Landau equation.) One of the main purposes of this series of papers, indeed, is to attack this difficult problem.

In this paper we start from the single-time Liouville equation (21) and derive rigorously the manifestly covariant closed nonlinear equation (42), satisfied by the reduced one-body distribution function. (The latter indicates that a necessary condition to establish the irreversibility is the introduction of the thermodynamic
limit $N \rightarrow +\infty$.) The solution to Eq. (52), in turn, determines the hierarchy of correlations namely Eq. (70). We remark that although Eq. (52) is exact, it differs from the usual relativistic kinetic equations [10, 27] in the non-Markovian feature rendering the failure of $H$-theorem [40]. This feature was known long time ago in the Newtonian physics (for example, the Prigogine-Resibois master equation [51]). It may, as pointed out in a pioneering study [40], play essential roles in understanding the infrared divergence of the collision integral of self-gravitating systems (which we will discuss below). It is only beyond the correlation scale (provided that the two-body correlation is short-ranged) that the general equation (52) converges to a usual kinetic equation. (The very latter leads to a macroscopic hydrodynamic description.) In Paper II we exemplify this scenario in the case of relativistic plasmas with electromagnetic interactions. There, it will become clearer that Eqs. (52) and (70) allow one to overcome the difficulty encountered in the Israel-Kandrup formalism. Furthermore, we manage to justify the manifestly covariant Boltzmann equation [10, 11], and are able to explore— at the quantitative level—the origin of the logarithmic divergence of the relativistic Landau equation. They both are extremely hard to be achieved within the Israel-Kandrup formalism.

It should be stressed that the general theory, under Assumption 2.1. and 3.1, is rigorous. These two assumptions essentially require that (classical) chaotic dynamics sufficiently develops for the underlying relativistic many-body system. Similar to their Newtonian analogues to justify them mathematically is an extremely hard task and is unnecessary at this stage. Rather, from the practical viewpoint the theory presented here may be applicable to a large class of systems where the interacting force preserves the mass-shell constraint and, moreover, does not depend on the acceleration of the acted particle. A heuristic example is a system composed of identical particles which interact with each other through a weak enough massive scalar field. In this case the interacting force is

$$F_{ij}^\mu(x_i, p_i) = \lambda_0 (\eta^\mu\nu - u_i^\mu u_j^\nu) \partial_\nu \Phi_{ij}(x)|_{x=x_i}, \quad (71)$$

$$(\partial_\mu \partial^\mu + \kappa^2) \Phi_{ij}(x) = -4\pi\lambda_0 \int d\tau_j \delta^{(4)}(x - x_j(\tau_j)),$$

where $|\lambda_0 \Phi_{ij}| \ll m$, the interaction strength is $\propto \lambda_0^2$, and $\kappa$ is the mass of the scalar field. Notice that the interacting force preserves the mass-shell constraint, i.e., $u_i \cdot F_{ij}(x_i, p_i) = 0$, but does not satisfy the conservative condition namely Eq. (20). Nevertheless it turns out that the general theory well applies in this case and, furthermore, the entire program of Paper II may be readily carried over to this system. In particular, because of the short-ranged nature of the interacting force a well defined Landau collision integral results under the weak coupling approximation.

### A. Relativistic self-gravitating systems

In this series of papers we limit ourselves to the special relativity. The theoretic scope presented here may be further extended to the general relativity and, thus, find important applications in astrophysics. In particular, there are no difficulties which impede generalizing the present manifestly covariant classical correlation dynamics to self-gravitating systems composed of relativistic star clusters or galaxies interacting merely gravitationally. Since this subject is far beyond the scope of this series of papers we here report briefly some preliminary observations and leave the thorough analysis for future studies.

In formulating the classical correlation dynamics of relativistic self-gravitating systems an additional difficulty arises, namely to treat the “gravitational force”. We here follow the prescription of Israel and Kandrup [18]. First of all, we fix a background geometry ($\mathcal{M}^4, g_{\mu\nu}(x)$). The 8-dimensional $\mu$ phase space [compared with Eq. (7)] associated with particle, say $i$ is now defined as $\mu_i : \{(x_i, p_i)|x_i \in \mathcal{M}^4, g_{\mu\nu}(x_i)p_i^\mu p_j^\nu = m^2\}$. The background metric $g_{\mu\nu}(x)$ is such chosen that it solves some field equation with smoothed-out matter distribution as the source. Crucially, given particle $i$ the realistic path $x_i[\cdot]$—dictated by underlying classical many-body dynamics—is the geodesic generated by another metric $g'_{\mu\nu}(x)$. It generally deviates from the geodesic in the background geometry ($\mathcal{M}^4, g_{\mu\nu}(x)$) (and, as such, do not preserve the phase volume of $\mu_i$.) This very deviation is driven by the difference of the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ and $\Gamma'^\lambda_{\mu\nu}$ (associated with the metrics $g_{\mu\nu}$ and $g'_{\mu\nu}$, respectively), i.e., $\delta \Gamma^\lambda_{\mu\nu}(x) = \Gamma'^\lambda_{\mu\nu}(x) - \Gamma^\lambda_{\mu\nu}(x)$, and describes the gravitational force completely. Provide that $\delta \Gamma^\lambda_{\mu\nu}(x)$ varies over a scale much smaller than the radius of the space-time curvature, it may be determined by some linear field equation.

To substantiate these observations we need to make the following replacement in Eq. (21) ($\Delta^\mu_{\nu\nu} = g_{\mu\nu} - u_{\mu i}u_{\nu i}$):

$$\mathcal{L}^0 = -\sum_{i=1}^N \left[ u_i^\mu \partial_{\mu i} + m\Gamma^\lambda_{\mu\nu}(x_i)u_{\lambda i}u_i^\nu \frac{\partial}{\partial p_{\mu i}} \right], \quad (72)$$

$$\lambda \mathcal{L}'_{ij} = m\delta \Gamma^\lambda_{\nu\rho}(i, j) \partial_{\rho i} \Delta^\nu_{\mu\nu} - \delta \Gamma^\lambda_{\nu\mu}(i, j), \quad (73)$$

where in Eq. (73) the second term is obtained by exchanging the particle labels of the first term. Here we notice that because of the linear field approximation mentioned above the perturbed Christoffel symbol of particle $i$ namely $\delta \Gamma^\lambda_{\mu\nu}(x_i)$ consists of $N - 1$ contributions from all the other particles, and $\delta \Gamma^\lambda_{\nu\mu}(i, j)$ is the one associated with particle $j$.

Then, one may further proceed to formally carry out the entire program presented here. In particular, we obtain, similar to Eq. (72), an exact closed nonlinear equa-
tion satisfied by the one-body distribution function:
\[
\left\{ \begin{aligned}
u_i^\mu \partial_{\mu 1} + m \Gamma_{\mu 3}^\lambda(x_1) u_{\lambda 1} u_\delta \frac{\partial}{\partial \rho_{\mu 1}} \\
- \int_{\Sigma_2} d\Sigma_{\mu 2} \int \frac{d^4 p_2}{\sqrt{g}} u_2^\lambda \lambda_{12} f(2) \right\} f(1) = \mathbb{K}_g[f] \tag{74}
\]
where \( f \) is normalized according to
\[
\lim_{N \to \infty} N^{-1} \int_{\Sigma_1} d\Sigma_{\mu 1} f^4 \mu^2 f(x, p) = 1 \quad \text{with } \Sigma \text{ a spacelike 3-surface, and } \mathbb{K}_g[f] \text{ is the collision integral (The explicit form may be readily found which is yet unnecessary for present discussions.)}. 
\]
Because the perturbed Christoffel symbol is weak enough to the leading order \( \lambda \)-expansion Eq. (74) then gives
\[
\left\{ \begin{aligned}
u_i^\mu \partial_{\mu 1} + m \Gamma_{\mu 3}^\lambda(x_1) u_{\lambda 1} u_\delta \frac{\partial}{\partial \rho_{\mu 1}} \\
- \int_{\Sigma_2} d\Sigma_{\mu 2} \int \frac{d^4 p_2}{\sqrt{g}} u_2^\lambda \lambda_{12} f(2) \right\} f(1) = 0 \tag{75}
\]
the solution to which self-consistently determines the background geometry, i.e., \((M^4, g_{\mu \nu}(x))\) through the Einstein field equation \((G_{\mu \nu} = \text{a function of the Christoffel symbol}):\)
\[
G_{\mu \nu} = -8\pi \int \frac{d^4 p}{\sqrt{g}} u^\mu u_\nu f(x, p). \tag{76}
\]
Eqs. (75) and (76) constitute the well-known Boltzmann-Einstein equations [18].

Let us further replace \( \mathcal{X}(x, p) \) [cf. Eq. (17)] by the phase trajectory generated by the geodesic in the background geometry. Provided that the self-gravitating system is dilute, by taking into account the so-called locality assumption we are able to show that the collision integral is
\[
\mathbb{K}_g[f] = \int \frac{d^4 p}{\sqrt{g}} \left( \frac{\partial}{\partial p_1^\mu} - \frac{\partial}{\partial p_2^\mu} \right) e_g^{\mu \nu} \left( \frac{\partial}{\partial p_1^\nu} - \frac{\partial}{\partial p_2^\nu} \right) f(x_1, p_1) f(x_1, p_2) \tag{77}
\]
\[
e_g^{\mu \nu} = 2m^4 \left[ 1 - 2(u_1 \cdot u_2)^2 \right]^2 \times \int \frac{d^4 k}{\sqrt{g}} \delta(k \cdot u_1) \delta(k \cdot u_2) \frac{k \mu k \nu}{(k \cdot k)^2} \]
up to the \( \lambda^2 \) accuracy (namely the weak coupling approximation to be detailed in Paper II). This is the Israel-Kandrup collision integral [18] well-known to astrophysicists.

The collision integral (77) justifies that the classical correlation dynamics can be well applied to relativistic self-gravitating systems, which was questioned by Kandrup long time ago [19]. It should be stressed, however, that such a justification is at the level of formal manipulations. The serious difficulty is that this collision integral suffers from the infrared divergence, and may be amounted to the failure of the locality assumption. (The ultraviolet divergence is due to the failure of the weak coupling approximation which may be readily healed and, thus, of no special interests [18].) Such a divergence, like plasmas [38], finds its origin at the long-ranged nature of gravitational forces, which was noticed by Chandrasekhar long time ago [38]. However, in (Coulomb) plasmas the Debye screening renders the effective potential short-ranged. As a result, the two-body correlation is short-ranged justifying the locality assumption, which results in a well defined kinetic equation [12]. (One of the central issue of Paper II, indeed, is the detailed analysis of the analogue in the manifestly covariant classical correlation dynamics.) This scenario nevertheless breaks down in self-gravitating systems. Indeed, experiences in Newtonian self-gravitating systems suggest that the screening is dynamical [11]. The general formalism presented here may serve as a useful technique for exploring this fundamental issue, which we leave for future studies.

**Acknowledgments**

I am deeply grateful to Q. K. Lu for numerous fruitful discussions at the early stage of this work, and especially to S. L. Tian for invaluable help. I also would like to thank M. Courbage and M. Garst for useful conversations, and especially to C. Kiefer for his interests and encouragements. This work is supported by Transregio SFB 12 of the Deutsche Forschungsgemeinschaft and was partly done in Institute of Henri Poincare.

**APPENDIX A: BBGKY HIERARCHY OF PHYSICAL DISTRIBUTION FUNCTIONS**

Let us introduce the concept of the physical distribution vector as follows:
\[
\vec{\mathcal{N}} \equiv (\{N_1\}, \{N_2\}, \ldots, \{N_N\}) \equiv \mathcal{N}. \tag{A1}
\]
The general component above is defined as
\[
N_i(x_{i_1}, p_{i_1}, \ldots, x_{i_s}, p_{i_s}; \{x_{i_1}\}, \ldots, \{x_{i_s}\}) \equiv \prod_{j=s+1}^{N} \int_{\Sigma_{i_j} \otimes U_{i_j}^4} d\Sigma_{\mu i_j} d^4 p_{i_j} u^\mu_{i_j} N \tag{A2}
\]
for \( 1 \leq i_1 < \cdots < i_s \leq N, 1 \leq s \leq N. \) The normalization of the component is given by
\[
\int_{\Sigma_{i_1} \otimes U_{i_1}^4} d\Sigma_{\mu i_1} d^4 p_{i_1} u^\mu_{i_1} \cdots \int_{\Sigma_{i_s} \otimes U_{i_s}^4} d\Sigma_{\mu i_s} d^4 p_{i_s} u^\mu_{i_s} \times N_i(x_{i_1}, p_{i_1}, \ldots, x_{i_s}, p_{i_s}; \{x_{i_1}\}, \ldots, \{x_{i_s}\}) = 1 \tag{A3}
\]
for arbitrary spacelike 3-surface \( \Sigma_{i_1} \).

Similar to Sec. III A from the Liouville equation (26) we obtain
\[
\vec{\mathcal{L}} \vec{\mathcal{N}} = 0. \tag{A4}
\]
Upon projecting it to the general s-particle state \( (i_1, \cdots, i_s) \) we use the same matrix elements except that
the annihilation vertex (see Appendix B) is replaced by

\[ \langle i | \lambda \hat{L}_{ij} | \Gamma_2(i,j) \rangle = \lambda \hat{L}_{ij} , \]

\[ \langle i | \lambda \hat{L}_{ij} | \Gamma_2(i,j) \rangle = \int dj \lambda \hat{L}_{ij} , \]

\[ \langle j | \lambda \hat{L}_{ij} | \Gamma_2(i,j) \rangle = \int di \lambda \hat{L}_{ij} , \]

and all the other matrix elements vanish. In general, using the linearity we find that for matrix elements \( \langle \Gamma_s'(i_1, \cdots, i_{s'}) | \lambda \hat{L}_s | \Gamma_s(j_1, \cdots, j_s) \rangle \) to not vanish it is necessary that \((j_1, \cdots, j_s) = (i_1, \cdots, i_{s'})\) or \((i_1, \cdots, i_{s'}, j_{s'+1})\). Besides, provided that the correlation pattern \( \Gamma_s(j_1, \cdots, j_s) = |P_1| P_2 | \cdots | P_j \) with

\[ P_1 \cup \cdots \cup P_j = (j_1, j_2, \cdots, j_s) , \]

\[ P_i \cap P_{i'} = \emptyset , \forall i \neq i' , \]

then the matrix element is invariant under the particle permutation if both particles are in the same subset. This statement is also applicable to the correlation pattern \( \langle \Gamma_s \rangle \).

Table 1 presents the matrix elements with \( s' = 1, 2 \). For the matrix elements \( \langle \Gamma_s'(i_1, \cdots, i_{s'}) | \lambda \hat{L}_s | \Gamma_s(j_1, \cdots, j_s) \rangle \) with \( s' > 2 \), we set up the diagrammatical rules as follows:

1. The correlation pattern \( \Gamma_s(j_1, \cdots, j_s) \) is given by the right-most part of the diagram. If particles are correlated then we connect them.

2. For each particle draw a horizontal line which stands for the propagation from the right to the left. If particle is annihilated during the propagation it is drawn by a dotted line, otherwise by a solid line.

3. Draw all diagrams with an interacting vertex which may be either Fig. II (a) or (b). The obtained diagram, as a whole, gives a possible final correlation pattern which is read out according to the following rule: If particles are connected (irrespective of solid/dotted line) then they are correlated.

4. Compare each possible final correlation pattern with \( \langle \Gamma_s'(i_1, \cdots, i_{s'}) \rangle \). If they are not identical then we assign the value zero to the corresponding diagram.

5. Otherwise, depending on the type of the interaction vertex we assign the value \( \lambda \hat{L}_{ij} \) or \( \int dk \lambda \hat{L}_{ik} \) accordingly. Here \( i, j, k \) are the particles joining the vertex.

6. Summing up all the nonvanishing diagrams then gives the matrix element.

**APPENDIX B: LIOUVILLIAN IN CORRELATION PATTERN REPRESENTATION**

In this appendix we will give the matrix elements of the free and interacting Liouvillian in the correlation pattern representation. For this purpose it is sufficient to know the matrix elements of the operators \( -u^\mu \partial_{\mu i} \) and \( \lambda \hat{L}_{ij} \) because of the linearity. Let us project the BBGKY hierarchy (A) to the correlation pattern \( \langle \Gamma_s(i_1, \cdots, i_s) \rangle \) then substitute the cluster expansion (B) into it. Comparing with Eq. (A) (setting \( \lambda = 0 \)) we find that the operator \( -u^\mu \partial_{\mu i} \) preserves a given correlation pattern \( \Gamma_s(i_1, \cdots, i_s) \) (namely the free motion). That is,

\[ \langle \Gamma_s(i_1, \cdots, i_s) | (-u^\mu \partial_{\mu i}) | \Gamma_s(i_1, \cdots, i_s) \rangle = -u^\mu \partial_{\mu i} \]

(B1)

if \( i \in (i_1, \cdots, i_s) \). All the other matrix elements vanish.

Similarly, we find that, given a two-particle correlation pattern \( \Gamma_2(i, j) \), the two-body interacting Liouvillian

\[ \lambda \hat{L}_{ij} , i \neq j \] either preserves the particles (Fig. II (a)) or annihilates one particle via the manipulation of integration (Fig. II (b)). That is,
Table 1. Matrix element \( \langle \Gamma_s' | \lambda \hat{\Sigma}' | \Gamma_s \rangle \) with \( s' = 1, 2 \). All the other matrix elements not listed here are zero.

| \( \langle i | \) | \( \langle i | \hat{\Sigma}' | i \rangle = 0 \) | \( \langle i | \hat{\Sigma}' | i \rangle = 0 \) | \( \langle i | \hat{\Sigma}' | i \rangle = 0 \) |
| \( \langle i | \hat{\Sigma}' | j \rangle \) | \( \langle i | \hat{\Sigma}' | j \rangle = 0 \) | \( \langle i | \hat{\Sigma}' | j \rangle = 0 \) | \( \langle i | \hat{\Sigma}' | j \rangle = 0 \) |
| \( \langle i | \hat{\Sigma}' | k \rangle \) | \( \langle i | \hat{\Sigma}' | k \rangle = 0 \) | \( \langle i | \hat{\Sigma}' | k \rangle = 0 \) | \( \langle i | \hat{\Sigma}' | k \rangle = 0 \) |

For illustrations here we give some examples: 

\( \langle i j k | \lambda \hat{\Sigma}' | i j k \rangle = \lambda (\hat{L}'_{ij} + \hat{L}'_{ik}) \) (Fig. 1 (c) and (d)),

\( \langle i j k | \lambda \hat{\Sigma}' | i j k \rangle = \int dk \lambda (\hat{L}'_{ij} + \hat{L}'_{ik}) \) (Fig. 1 (e) and (f)),

and \( \langle i j k | \lambda \hat{\Sigma}' | i j k \rangle = 0 \).
that in many condensed matter and high-energy liter-
atures (for examples, Refs. [21 [31]) the “Boltzmann”
equation is in fact “Boltzmann-like” which, typically,
is the kinetic (transport) equation of Vlasov, Fokker-
Planck, Landau, or of Boltzmann but with the scattering
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