Entropy-based Bounds on Dimension Reduction in $L_1$

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October 22, 2018

Abstract

We show that for every large enough integer $N$, there exists an $N$-point subset of $L_1$ such that for every $D > 1$, embedding it into $\ell_1^d$ with distortion $D$ requires dimension $d$ at least $N^{\Omega(1/D^2)}$, and that for every $\epsilon > 0$ and large enough integer $N$, there exists an $N$-point subset of $L_1$ such that embedding it into $\ell_1^d$ with distortion $1 + \epsilon$ requires dimension $d$ at least $N^{1-O(1/\log(1/\epsilon))}$. These results were previously proven by Brinkman and Charikar [JACM, 2005] and by Andoni, Charikar, Neiman, and Nguyen [FOCS 2011]. We provide an alternative and arguably more intuitive proof based on an entropy argument.

1 Introduction

We prove the following theorem.

**Theorem 1.1.** For every large enough integer $N$, there exists an $N$-point subset of $L_1$ such that for every $D > 1$, embedding it into $\ell_1^d$ with distortion $D$ requires dimension $d$ at least $N^{\Omega(1/D^2)}$. Moreover, for every $\epsilon > 0$ and large enough integer $N$, there exists an $N$-point subset of $L_1$ such that embedding it into $\ell_1^d$ with distortion $1 + \epsilon$ requires dimension $d$ at least $N^{1-O(1/\log(1/\epsilon))}$.

Both parts of Theorem 1.1 were previously known. The first part (embedding with large distortion) was first shown by Brinkman and Charikar [BC05], and later with a simpler proof by Lee and Naor [LN04]. The second part (embedding with low distortion) was recently shown by Andoni, Charikar, Neiman, and Nguyen [ACNN11]. Our proof is based on an entropy argument, and is arguably more intuitive.

The set of points we use is identical to the one used by Andoni et al. [ACNN11]. For completeness, we briefly describe it here (see also Figure 1 for an illustration). For integers $k \geq 2, n \geq 1$, we define the so-called “recursive cycle” graph $G_{k,n}$ and associate with each vertex a label in $\{0,1\}^k$. The set of all labels will be our point set $P_{k,n}$ in $\ell_1$. First, for $k \geq 2$, let $G_{k,1}$ be the cycle of length $2k$, with two distinguished antipodal vertices (i.e., of distance $k$), call them “left” and “right”. For $0 \leq i \leq k$, the $i$th vertex on the top path from the left to the right vertex is labeled with the vector

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Figure 1: $G_{3,2}$ with our labeling and orientation of the edges and the labels on vertices in $\{0, 1\}^9$.

$(0, \ldots, 0, 1, \ldots, 1)$ with $k - i$ zeros and $i$ ones, and the $i$th vertex on the bottom path is associated with the vector $(1, \ldots, 1, 0, \ldots, 0)$ with $i$ ones and $k - i$ zeros. Notice that the $\ell_1$ distance between the labels of any two adjacent vertices is 1, whereas that between the labels of any two antipodal vertices is $k$.

For $n \geq 2$, define $G_{k,n}$ as the graph obtained from $G_{k,n-1}$ by replacing each edge with a copy of $G_{k,1}$ and identifying the distinguished vertices with the original endpoints of the edge. The number of vertices in $G_{k,n}$ is easily seen to be $N_{k,n} := \frac{(2k - 2)(2k)^n + 2k}{2k - 1} \leq (2k)^n$.

For the labels, we first take the labels in $G_{k,n-1}$ and duplicate each coordinate $k$ times. This defines the labels for those vertices coming from $G_{k,n-1}$. For the newly added vertices on each cycle that replaced an edge of $G_{k,n-1}$, we replace the $k$ coordinates on which the two distinguished nodes of that cycle differ with the same labeling of $G_{k,1}$ described earlier. Notice the following two properties: the $\ell_1$ distance between the labels of any two adjacent vertices is 1, and for $1 \leq \ell \leq n$, the distance between any two antipodal vertices in level $\ell$ is $k^{n-\ell+1}$. We remark that these two properties are also satisfied by the shortest path metric on $G_{k,n}$, but since that metric is not in $\ell_1$, it is not good enough for the purpose of proving dimension reduction in $\ell_1$.

Finally, we label the edges of $G_{k,1}$ by elements of $[2k]$ starting from the left vertex and going along the cycle, and extend this to a labeling of $G_{k,n}$ by elements of $[2k]^n$ in a recursive way, with the coordinates labeling the location of the edge from the top layer to the bottom layer (see Figure 1).

The idea of the proof is the following. Given a low-distortion embedding of $P_{k,n}$ into $\ell_1^d$, we naturally obtain a mapping that maps each edge of the graph $G_{k,n}$ to a $d$-dimensional vector (namely, the difference between the two embedded endpoints) whose $\ell_1$ norm is close to 1. Assume for sim-
plicity that this norm is exactly 1; assume moreover that the vector has non-negative coordinates. (In the proof we will show how to reduce the general case to this case.) So we can equivalently view this mapping as an encoding from $[2k]^n$ to probability distributions over $[d]$. Using the second property mentioned above, one can obtain the following crucial property of the encoding: For any $\ell \in [n]$ and any $x_1, \ldots, x_{\ell-1} \in [2k]$, if we are given $x_1, \ldots, x_{\ell-1}$ together with the encoding of $(x_1, \ldots, x_n) \in [2k]^n$, where $x_1, \ldots, x_n$ are chosen uniformly, then we have a good probability to guess $x_\ell \mod k$ (perfect probability in case of no distortion). A basic information theoretic argument now provides a lower bound on $d$ of any such encoding. For instance, in the case there is no distortion, the encoding allows us to predict $x_\ell \mod k$ as above with certainty, and the information theoretic argument gives the tight bound $d \geq k^n$. We note that this simple yet powerful information theoretic argument appears in various different contexts, such as that of quantum random access codes [Nay99].

2 Preliminaries

All logarithms are base 2. We use $[k]$ to denote the set $\{1, \ldots, k\}$. We now list a few basic definitions and facts from information theory. Although not really needed for our proof, the interested reader can find an introduction to the area in [CT06]. We let $H(\delta) := -\delta \log \delta - (1 - \delta) \log(1 - \delta)$ denote the binary entropy function. For a random variables $X$ on a domain $[d]$ obtaining each value $i \in [d]$ with probability $p_i$, the entropy of $X$ is given by $H(X) := -\sum p_i \log p_i$, and is always at most $\log d$. For two random variables $X, Y$, the conditional entropy $H(X \mid Y)$ is the expectation of $H(X \mid Y = y)$ over $y$ chosen according to $Y$; this can be seen to equal $H(XY) - H(Y)$. Finally, the mutual information $I(X : Y)$ is defined as $H(X) + H(Y) - H(XY) = H(X) - H(X \mid Y)$, and the conditional mutual information $I(X : Y \mid Z)$ is the expectation of $I(X : Y \mid Z = z)$ over $z$ chosen according to $Z$, or equivalently, $H(X \mid Z) + H(Y \mid Z) - H(XY \mid Z)$. The data processing inequality says that applying a function cannot increase mutual information, $I(f(X) : Y) \leq I(X : Y)$.

The following claim (which is essentially what is known as Fano’s inequality) shows that if one random variable can be used to predict another random variable, then their mutual information cannot be too small.

Claim 2.1. Assume $X$ is a random variable uniformly distributed over $[k]$. Let $Y$ be another random variable, and assume that there exists some function $f$ with range $[k]$ such that $f(Y) = X$ with probability at least $p \geq 1/2$. Then $I(X : Y) \geq \log k - (1 - p) \log(k - 1) - H(p)$.

Proof. By the data processing inequality,

$I(X : Y) \geq I(X : f(Y)) = H(X) - H(X \mid f(Y)) = \log k - H(X \mid f(Y))$, 

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so it suffices to bound $H(X \mid f(Y))$ from above. Since conditioning cannot increase entropy,

\[
H(X \mid f(Y)) = H(1_{X=f(Y)}, X \mid f(Y))
\]

\[
= H(1_{X=f(Y)} \mid f(Y)) + H(X \mid 1_{X=f(Y)}, f(Y))
\]

\[
\leq H(1_{X=f(Y)}) + H(X \mid 1_{X=f(Y)}, f(Y))
\]

\[
\leq H(p) + (1 - p) \log (k - 1).
\]

\[
H(1_{X=f(Y)}) = H(X) = H(p) + (1 - p) \log (k - 1).
\]

3 Proof

Our main technical theorem is the following.

**Theorem 3.1.** For any $k \geq 2$, $n \geq 1$ the following holds. Assume $f : [2k]^n \to \mathbb{R}^d$ satisfies that for all $x_1, \ldots, x_n \in [2k], \|f(x_1, \ldots, x_n)\|_1 \leq 1$ and, moreover, that for some $\varepsilon < 1/(k-1)$, and for all $\ell \in [n]$, $x_1, \ldots, x_{\ell-1} \in [2k]$, and $r \in [k-1],\[
\frac{1}{2k} \left\| \sum_{b=1}^r (f(x_1, \ldots, x_{\ell-1}, b) + f(x_1, \ldots, x_{\ell-1}, b+k)) \right\|_1 - \sum_{b=r+1}^k (f(x_1, \ldots, x_{\ell-1}, b) + f(x_1, \ldots, x_{\ell-1}, b+k)) \right\|_1 
\geq 1 - \varepsilon
\]

(1)

where $f(x_1, \ldots, x_\ell)$ denotes the average of $f(x_1, \ldots, x_n)$ over $x_{\ell+1}, \ldots, x_n$ chosen uniformly in $[2k]$. Then

\[
d \geq 2^{(\log k - \delta \log (k-1)) - H(\delta)n - 1 - \frac{1}{2}}
\]

(2)

where $\delta := (k-1)\varepsilon/2 < 1/2$.

Before proving the theorem, let us explain how it implies Theorem 1.1. Consider any embedding $F$ of $P_{k,n}$ into $\ell^1_1$ with distortion at most $1/(1 - \varepsilon)$ for some $\varepsilon < 1/(k-1)$. By scaling $F$, we can assume that it is 1-Lipschitz (i.e., it does not expand any distance) and that distances are not contracted by more than $1 - \varepsilon$. Let $f$ be the function that maps each $x \in [2k]^n$ to $F(u) - F(v)$, where $u$ is the label of the right endpoint of the edge labeled by $x$ and $v$ is the label of its left endpoint. Since $F$ is 1-Lipschitz, $\|f(x)\|_1 \leq 1$ for all $x \in [2k]^n$. Moreover, it is not difficult to see that $f$ satisfies Eq. (1) (see Figure 2). Hence, Theorem 3.1 implies that the bound in Eq. (2) holds.
For the first part of Theorem 1.1, we fix \( k = 2 \). We obtain that for any \( D \geq 1 \), any distortion-\( D \) embedding of \( G_{2,n} \) (so \( \epsilon = 1 - 1/D \) and \( \delta = 1/2 - 1/(2D) \)) must have dimension at least
\[
2^{(1-H(1/2-1/2D))n-1} - \frac{1}{2} = 2^{\Omega(n/D^2)} = N_{2,n}^{\Omega(1/D^2)}.
\]
For the second part of Theorem 1.1, choosing \( k \approx 1/(\epsilon \log(1/\epsilon)) \) and noting that \( \delta \log k = O(1) \), we obtain that the dimension must be at least
\[
(2k)^n 2^{(-\delta \log k - 2)n-1} - \frac{1}{2} = N_{k,n}^{1-O(1/\log(1/\epsilon))}.
\]

**Proof of Theorem 3.1.** We start by considering the case that for all \( x_1, \ldots, x_n \in [2k], f(x_1, \ldots, x_n) \) has non-negative coordinates and \( \ell_1 \)-norm 1. We will later see how this implies the general case.

Making this assumption allows us to think of \( f(x_1, \ldots, x_n) \) as a probability distribution over \([d]\). Let \( X = (X_1, \ldots, X_n) \) and \( M \) be two random variables where \( X \) is uniformly distributed over \([2k]^n\) and \( M \) is distributed over \([d]\) according to \( f(X) \). Using the chain rule for mutual information we obtain
\[
\log d \geq H(M) \geq I(X : M) = I(X_1 : M) + I(X_2 : M \mid X_1) + \cdots + I(X_n : M \mid X_1, \ldots, X_{n-1}).
\]

The following lemma implies that for any \( \ell \in [n] \),
\[
I(X_\ell : M \mid X_1, \ldots, X_{\ell-1}) \geq \log k - \delta \log(k-1) - H(\delta)
\]
(this is true even conditioned on any fixed value of \( X_1, \ldots, X_{\ell-1} \), and not just on average) and therefore
\[
d \geq 2^{(\log k - \delta \log(k-1) - H(\delta)n}.
\]

**Lemma 3.2.** Let \( A \) and \( B \) be two random variables such that \( A \) is uniformly distributed over \([2k]\) and for any \( a \in [2k] \), conditioned on \( A = a \), \( B \) is distributed according to some probability distribution \( P_a \) on \([d]\).

Assume that for all \( r \in [k-1] \),
\[
\frac{1}{2k} \left\| \sum_{a=1}^r (P_a + P_{a+k}) - \sum_{a=r+1}^k (P_a + P_{a+k}) \right\|_1 \geq 1 - \epsilon.
\]

Then \( I(A : B) \geq \log k - \delta \log(k-1) - H(\delta) \).

**Proof.** Let \( A' = ((A - 1) \mod k) + 1 \), and notice that \( A' \) is uniformly distributed on \([k]\). By the data processing inequality, \( I(A : B) \geq I(A' : B) \). For any \( a \in [k] \), let \( Q_a := (P_a + P_{a+k})/2 \) be the distribution of \( B \) conditioned on \( A' = a \). Our assumption says that for all \( r \in [k-1] \),
\[
\frac{1}{k} \left\| \sum_{a=1}^r Q_a - \sum_{a=r+1}^k Q_a \right\|_1 \geq 1 - \epsilon.
\]

We need the following easy claim.
Claim 3.3. For any \( p_1, \ldots, p_k \geq 0 \),

\[
\left( \sum_{i=1}^{k} p_i \right) - \max\{p_1, \ldots, p_k\} \leq \frac{1}{2} \left( \sum_{i=1}^{k-1} \left( \sum_{i=1}^{k} p_i \right) - \left| \sum_{i=1}^{r} p_i - \sum_{i=r+1}^{k} p_i \right| \right).
\]

Proof. Let \( r^* \in \{0, \ldots, k-1\} \) be the largest such that the expression inside the absolute value is negative. Then the sum of the absolute values at \( r = r^* \) and \( r = r^* + 1 \) is exactly \( 2p_{r^*+1} \). The claim follows.

By applying the inequality to each of the \( d \) coordinates of the probability distributions \( Q_a \), and summing the results, we obtain

\[
1 - \frac{1}{k} \| \max\{Q_1, \ldots, Q_k\} \|_1 \leq \frac{1}{2} \sum_{r=1}^{k-1} \left( 1 - \frac{1}{k} \| \sum_{a=1}^{r} Q_a - \sum_{a=r+1}^{k} Q_a \|_1 \right)
\]

and hence

\[
\frac{1}{k} \| \max\{Q_1, \ldots, Q_k\} \|_1 \geq 1 - \frac{1}{k} \| \max\{Q_1, \ldots, Q_k\} \|_1 - (k-1)\varepsilon/2 = 1 - \delta.
\]

Consider the function that maps each \( j \in [d] \) to the \( a \in [k] \) that maximizes \( \Pr[Q_a = j] \). This function correctly predicts \( A' \) from \( B \) with probability \( \frac{1}{k} \| \max\{Q_1, \ldots, Q_k\} \|_1 \). The lemma now follows from Claim 2.1.

We now show how to derive a similar bound for any \( f \) as in the statement of the theorem. Let \( f : [2k]^n \rightarrow \mathbb{R}^d \) be such that for all \( x \in [2k]^n \), \( f(x) \) has \( \ell_1 \) norm at most 1. Define \( g : [2k]^n \rightarrow \mathbb{R}^{2d+1} \) by the concatenation

\[
g(x) := \max\{f(x), 0\} \cdot \max\{-f(x), 0\} \cdot 1 - \|f(x)\|_1.
\]

Obviously, for all \( x \), \( g(x) \) is non-negative and has \( \ell_1 \) norm 1. Moreover, the linear operator that maps any \( y \in \mathbb{R}^{2d+1} \) to the vector \( (y_j - y_{j+d})_{j=1}^{d} \in \mathbb{R}^d \) cannot increase the \( \ell_1 \) norm and maps \( g(x) \) to \( f(x) \) for all \( x \). Therefore Eq. (1) holds for \( g \), and the theorem follows.

Acknowledgments

I thank the organizers of the workshop “Metric embeddings, algorithms and hardness of approximation” in the Institut Henri Poincaré, where this work started. I also thank Moses Charikar for the inspiring talk he gave there, and Assaf Naor and Ofer Neiman for useful discussions.

References

[ACNN11] A. Andoni, M. S. Charikar, O. Neiman, and H. L. Nguyen. Near linear lower bounds for dimension reduction in \( \ell_1 \). In Proc. of 52nd IEEE FOCS. 2011.

[BC05] B. Brinkman and M. Charikar. On the impossibility of dimension reduction in \( \ell_1 \). J. of the ACM, 52(5):766–788, 2005.
[CT06] T. M. Cover and J. A. Thomas. *Elements of information theory (2nd edition)*. Wiley, 2006.

[LN04] J. R. Lee and A. Naor. Embedding the diamond graph in $L_p$ and dimension reduction in $L_1$. *Geometric and Functional Analysis*, 14(4):745–747, 2004.

[Nay99] A. Nayak. Optimal lower bounds for quantum automata and random access codes. In *Proc. of 40th IEEE FOCS*, pages 369–376. 1999.