Superfluid dynamics, equilibrium conditions, and centripetal forces

Mario Liu

Theoretische Physik, Universität Tübingen, 72076 Tübingen, Germany
E-mail: Mliu@uni-tuebingen.de

Received 22 August 2022
Accepted for publication 12 February 2023
Published 16 March 2023

Abstract. Thermodynamics of superfluids is revisited, clarifying two points. First, the density and pressure distribution for given normal and superfluid velocities is obtained, with the finding that counter heat currents give rise to a pressure depression and a centripetal force. Second, it is shown that the ideal two-fluid hydrodynamics is simply an assembly of equilibrium conditions—expressions of entropy being maximal.

Keywords: superfluid thermodynamics, two-fluid hydrodynamics, equilibrium conditions

Contents

1. Introduction ................................................................. 2
2. Superfluid thermodynamics .............................................. 3
   2.1. The chemical potential and pressure ................................ 3
   2.2. Treatment by Landau and others ................................... 5
   2.3. Equilibrium conditions ............................................. 6
   2.4. Equilibrium two-fluid dynamics .................................... 7
3. Dissipative fluxes ......................................................... 9

References ................................................................. 10
1. Introduction

Thermodynamics of superfluids was first studied by Khalatnikov and Landau [1, 2], later by Putterman and Uhlenbeck [3, 4]. It is revisited, because by Galilean transforming the energy and momentum between two inertial systems, they did not obtain the complete kinetic contribution to the chemical potential and pressure. More specifically, they found that $\mu - \mu_s$ (where $\mu$ is the general chemical potential, and $\mu_s$ the one in the frame of vanishing superfluid velocity, $v_s^i = 0$) is given by $(v_s^i)^2 - v_n^i v_s^i$. This is a pioneering result, useful for deriving the celebrated two-fluid hydrodynamics, but it lacks the information of the kinetic contribution in $\mu_s$. Only if this given, do we know the total kinetic contribution in $\mu$. The former is obtained below. With $\mu_0$ the rest frame chemical potential, it is

$$\mu_s = \mu_0(\rho,T) - \frac{1}{2} \frac{\partial \rho_n}{\partial \rho} w_i^2, \quad w_i \equiv (v_s^i - v_n^i).$$

To the best of my knowledge, this expression is new—in spite of the vast literature in superfluid $^4$He, $^3$He, solids, and smectics, see [5], and references therein. (Note that $w_i$ is $-v_n^i$ in the frame of $v_s^i = 0$.)

The complete motional contribution to $\mu$ enables one to calculate the density and pressure distribution in equilibrium for given velocities, especially under rotations and heat flows: first, there is a pressure decrease where heat flows, which should also be observable as a depression of the gas–liquid interface. A related effect exists under rotations, though not as pronounced: while the normal velocity $v_n^i$ gives rise to the usual centrifugal force, squeezing the density outward, any $w_i$ (whether positive or negative) diminishes this squeeze, partially drawing the density back. It acts as a centripetal force. At higher rotation rates with many superfluid vortices present, $w_i$ averaged over a number of vortices is small. But on smaller scales, at lower rotation rates, $w_i$ does vary more strongly.

Maximizing the entropy, one obtains one Euler–Lagrange condition for each state variable, which may be referred to as equilibrium conditions. The superfluid set of the conditions is given in section 2.3. Two of these are

$$\nabla_i T = 0, \quad \frac{\partial}{\partial t} v_n^i + \nabla_i \mu = 0,$$

stating that, in equilibrium, the temperature $T$ is always uniform, but the chemical potential $\mu$ is not. For two reasons, this seems troubling. First, $\frac{\partial}{\partial t} v_s^i \neq 0$ in equilibrium; and second, the Josephson equation $\frac{\partial}{\partial t} v_s^i + \nabla_i \mu = 0$ apparently relies on $\nabla_i \mu = 0$ to have no superfluid acceleration in equilibrium. If $\nabla_i \mu \neq 0$ sometimes, one starts to doubt the frame-independence of this important equation1. What I eventually found is reassuring: The two expressions $\frac{\partial}{\partial t} v_s^i + \nabla_i \mu = 0$ and $\frac{\partial}{\partial t} v_n^i + \nabla_i \mu = 0$ are not only compatible, they depend on each other, and may be employed with other equilibrium conditions to derive the ideal two-fluid hydrodynamics.

---

1 This was my initial reason for the revisit of an earlier fixation, see [6–12].
Ideal dynamics is the full one without the dissipative terms; equilibrium dynamics, on the other hand, consists of conservation laws, the fluxes of which are assemblies of equilibrium conditions alone. Surprisingly, they are the same. The former does not contain any more information than the latter. We can use the equilibrium conditions to set up the ideal dynamics, then add in dissipative terms to arrive at the full dynamics. If generally true, this would systematize the task of setting up the macro-dynamics for any system, simplifying it considerably. Here, we show that it holds true for the two-fluid dynamics. Solids, nematic liquid crystals, granular media and polymers will be considered in a forthcoming paper. Any full dynamics—typically a set of nonlinear, partial differential equations—is in its essence an expression of the relaxation toward equilibrium.

2. Superfluid thermodynamics

2.1. The chemical potential and pressure

In superfluids, the energy $\varepsilon$ depends on the densities of mass $\rho$, entropy $s$, momentum $g_i$, and the superfluid velocity $v_i^s = (h/m)\nabla_i \varphi$. Writing

$$d\varepsilon = \mu d\rho + T ds + v_i^s dg_i + j_i^s dv_i^s,$$

defines the conjugate variables $\mu, T, v_i^s, j_i^s$. The same holds for the free energy, $f = \varepsilon - Ts$, with $df = \mu d\rho - s dT + v_i^s dg_i + j_i^s dv_i^s$. The infinitesimal Galilean transformation of the velocity $du_i$ is given by

$$dv_i^s = du_i, \quad dg_i = \rho du_i, \quad d\varepsilon = d(g_i^2/2\rho)|_\rho = g_i du_i,$$

holding $s, \rho = \text{const.}$ Inserting equation (4) into (3) or $df$, we obtain $g_i du_i = \rho v_i^s du_i + j_i^s du_i$, implying

$$g_i = \rho v_i^s + j_i^s,$$

and

$$\frac{\partial g_i}{\partial v_i^s}|_{\rho, v_i^s} = \frac{\partial j_i^s}{\partial v_i^s}|_{\rho, v_i^s}. \tag{6}$$

A Maxwell relation of equation (3),

$$\frac{\partial g_i}{\partial v_i^s}|_{v_i^s, \rho} = - \frac{\partial j_i^s}{\partial v_i^s}|_{v_i^s, \rho},$$

yields, in conjunction with equation (6),

$$j_i^s = \rho_s (v_i^s - v_i^s) = \rho_s w_i. \tag{7}$$

The superfluid density $\rho_s$ introduced here is, to lowest order in $w_i$, independent of $w_i$. This we shall assume from here on.
Superfluid dynamics, equilibrium conditions, and centripetal forces

Rewriting equations (5) and (7) as \( v^n_i = (g_i - \rho_s v_i^s) / \rho_n \), \( j^s_i = (\rho v_i^s - g_i) \rho_s / \rho_n \), where \( \rho_n = \rho - \rho_s \), we infer that the free energy is

\[
f - f_0(\rho, T) = \frac{g_i^2}{2\rho_n} - \frac{\rho_s}{\rho_n} g_i v_i^s + \frac{\rho \rho_s}{2 \rho_n} (v_i^s)^2,
\]

because \( v_i^n = \partial f / \partial g_i |_{\rho, T, v_i^n} \) and \( j_i^s = \partial f / \partial v_i^s |_{\rho, T, g_i} \). (This holds also for \( \varepsilon - \varepsilon_0 \), and the expression may be written as \( \frac{1}{2} [(\rho_s (v_i^n)^2 + \rho_n (v_i^s)^2)] \).

From equation (8), we can calculate the motional contributions to the chemical potential and pressure. Denoting \( \mu_0 \equiv \partial f_0 / \partial \rho |_{T, s} \), \( \mu_s' \equiv \partial \rho_s / \partial \rho |_{T, s} \), with \( \rho_s + \rho_s' = 1 \), we find

\[
\mu = \mu_0(\rho, T) + \frac{1}{2} [\rho_s' (v_i^n)^2 - (v_i^s)^2].
\]

The pressure is \( P = \mu \mu + v_i^n g_i - f = \mu \mu + T s + v_i^n g_i - \varepsilon \). Employing equations (8), (9) and denoting \( P_0(\rho, T) \equiv \mu \mu_0 - f_0 \), we may write the pressure as

\[
P = P_0(\rho, T) - w_i^n (\rho_s - \rho_s') / 2.
\]

Next, with \( \nabla_i \varepsilon = \mu \nabla_i \rho + T \nabla_i s + v_i^n \nabla_i g_i + j_i^s \nabla_i v_i^s \) and \( g_i = \rho v_i^s + \rho_s w_i \), see equations (5) and (7), the gradient of the pressure is

\[
\nabla_i P = \rho \nabla_i \mu + s \nabla_i T + g_i \nabla_i v_i^s - j_i^s \nabla_i v_i^s,
\]

\[
= \rho \nabla_i \mu + s \nabla_i T + \frac{1}{2} \rho \nabla_i (v_i^n)^2 - \frac{1}{2} \rho s \nabla_i (w_i^n)^2.
\]

Equation (9) relates the distributions of density and velocities. For \( v_n = v_s = v \), the normal fluid behavior is restored: in a rotating equilibrium, if the center is stationary, we have \( \mu = \mu_0 - \frac{1}{2} v_s^2 = \text{const} \), see the discussion below equation (24) for detail. Since \( v_i \) increases toward the outer rim, \( \rho \) and \( \mu_0(\rho) \) increase to compensate. This is usually interpreted as a result of the universal centrifugal force pushing the density outwards: denoting vectors with bold letters, \( \mathbf{v} = \mathbf{r} \times \mathbf{r} \), we have \( \frac{1}{2} \nabla_i v_i^2 = \mathbf{v} \cdot \nabla_i \mathbf{r} = (\mathbf{v} \times \mathbf{r})_i \).

In a superfluid, equation (9), the same holds for \( \frac{1}{2} (v_i^n)^2 \). Yet any deviation of the superfluid, \( w \equiv v_s - v_n \), positive or negative, from a solid body rotation \( \mathbf{v} = \mathbf{r} \times \mathbf{r} \), even if it also rotates, operates as a centripetal force, diminishing this universality.

Including gravitation, the chemical potential remains constant, if we add in the gravitational potential \( \phi \) (see equations (9) and (22) below). Hence

\[
\nabla_i P_0 / \rho = \nabla_i \mu_0(\rho) = \nabla_i \left[ \frac{1}{2} (v_i^n)^2 - \frac{1}{2} \rho_s' w_i^n - \phi \right].
\]

\(^2\) The Landau–Lifshitz expression [13], \( P = -\varepsilon_s + T s + \mu \rho + \rho_s w_i^n \) (see equation (16) for the definition of \( \varepsilon_s \) and \( \rho_s \)) may be shown to be identical by simple algebra.

https://doi.org/10.1088/1742-5468/acbe98
In a quiescent super fluid, knowing the pressure and density implies the knowledge of the function \( P_0 = P_0(\rho) \). Hence, measuring the density and velocity distribution under rotation is an experimental check of equation (13).

The pressure \( P \) may be measured directly in the motional fluid. To obtain the relation between \( P \) and the two velocities, we need to replace \( P_0 \) in equation (10) using equation (13). But one can more easily go to equation (12), taking \( \nabla_i T = 0 \), \( \nabla_i \mu = -\nabla_i \phi \). Assuming incompressibility, \( \rho \) and \( \rho_s \) are constants and may be moved behind the gradient, implying

\[
P = \rho \left[ \frac{1}{2} (\nu_i^n)^2 - \phi \right] - \frac{1}{2} \rho_s w_i^2 + \text{const.} \tag{14}
\]

The first two terms show the classic behavior under rotation, with \( P \) growing linearly along the axis of gravitation \( -\hat{z} \), and quadratically with the radius. The second terms shows the superfluid modification.

At a stationary gas–liquid interface, the form is determined by \( P = \text{const.} \). If a heat flux \( w_i \) is applied, with no mass flux, the interface is depressed there, since we may then rewrite equation (14) using \( \rho v_i^n + \dot{j}_i^n = 0 \) as

\[
P = -\frac{1}{2} (\rho_s \rho_0 / \rho) w_i^2 - \rho \phi + \text{const.} \tag{15}
\]

### 2.2. Treatment by Landau and others

Before presenting their results, we draw a few auxiliary conclusions. In the frame \( v_i^n = 0 \), setting \( v_i^n \to 0 \), \( v_i^n \to (v_i^n - v_i^s) \) in equations (8) and (9), we have

\[
\varepsilon_s = \varepsilon_0(\rho, s) + \frac{1}{2} \rho_n w_i^2, \quad \mu_s = \mu_0 - \frac{1}{2} \rho_n w_i^2, \tag{16}
\]

implying that the general chemical potential, equation (9), may be written as

\[
\mu = \mu_s - v_i^n v_i^s + \frac{1}{2} (v_i^s)^2. \tag{17}
\]

The Galilean transformation by the velocity \( u_i \), taking a system with \( \hat{g}_i, \hat{\varepsilon} \) to one with \( g_i, \varepsilon \), is given by integrating equation (4) for constant \( \rho \),

\[
dg_i = \rho du_i \quad g_i = \hat{g}_i + \rho u_i, \tag{18}
\]

\[
d\varepsilon = g_i du_i = (\hat{g}_i + \rho u_i) du_i \quad \varepsilon = \hat{\varepsilon} + u_i \hat{g}_i + \frac{1}{2} \rho (u_i)^2. \tag{19}
\]

Landau and others started from the energy in the frame \( v_i^s = 0 \),

\[
d\varepsilon_s = \mu_s d\rho + T ds + (v_i^n - v_i^s) dj_i^n, \quad j_i^n = \rho_n (v_i^n - v_i^s), \tag{20}
\]

and employed equation (19) (with \( \varepsilon_s = \hat{\varepsilon}, j_i^n = \hat{g}_i \)) to arrive at equation (3), obtaining the relation between \( \mu \) and \( \mu_s \), equation (17).

They also derived the equation for \( v_i^s \) as \( \frac{\partial}{\partial t} v_i^s + \nabla_i [\frac{1}{2} (v_i^s)^2 + \mu_s] = 0 \), though \( \mu_s \) was denoted as \( \mu \). Because of equation (17), it is the same as the one obtained directly from the Josephson equation, see e.g. [8, 14].
and divergence-free superflow. Minimizing the total energy motion allowed in equilibrium. In essence, these are solid-body rotation, translation, is found in \[ \text{The first three conditions hold also in normal fluids, except for the fact that here, } \]

\begin{equation}
\nabla \cdot \mathbf{v} + (m/h)\mu = 0,
\end{equation}

or \( \frac{\partial}{\partial t} \mathbf{v}^s_i + \nabla_i (\mathbf{v} v^s_i + \mu) = 0. \) \tag{21}

### 2.3. Equilibrium conditions

First, we derive the equilibrium conditions for a closed, quiescent system, of given volume \( \int d^3r, \) energy \( \int \varepsilon_0 d^3r, \) and mass \( \int \rho d^3r, \) by maximizing the entropy \( \int s d^3r. \) This is equivalent to minimizing the energy for given entropy. Taking \( T_L, \mu_0 \) as constant Lagrange parameters, we vary \( \int \varepsilon_0 d^3r, \) with \( d\varepsilon_0 = \mu_0 d\rho + T_0 ds, \) as

\[
0 = \delta \int (\varepsilon_0 - T_L s - \mu_L \rho) \, d^3r = \int [(T_0 - T_L) \delta s + (\mu_0 - \mu_L) \delta \rho] \, d^3r.
\]

Since \( \delta s, \delta \rho \) vary independently, both brackets vanish, implying \( T_0, \mu_0 = \text{const}. \) Expressing these conditions locally, we have \( \nabla_i T_0 = 0, \) \( \nabla_i \mu_0 = 0. \)

Including gravitation, the energy is \( \tilde{\varepsilon}_0 = \varepsilon_0 + \rho \phi, \) with the potential \( \phi \) a fixed spatial function \([\phi = \rho(z - z_0) \text{ on the earth’s surface}],\) implying \( \mu_0(\rho) \equiv \partial \tilde{\varepsilon}_0 / \partial \rho = \mu_0 + \phi. \) Varying \( \int \tilde{\varepsilon}_0 d^3r \) under the same constraints, we obtain \( \nabla_i T_0 = 0 \) and

\[
\nabla_i \tilde{\mu}_0 = 0, \quad \text{or} \quad \nabla_i \mu_0 = -\nabla_i \phi.
\] \tag{22}

The pressure remains unchanged, \( \tilde{P}_0 = -\tilde{\varepsilon}_0 + \tilde{\mu}_0 \rho + T_0 s = P_0 \) for earth’s potential, as does its gradient. With \( \nabla_i \tilde{\varepsilon}_0 = \nabla_i \varepsilon_0 + \phi \nabla_i \rho + \rho \nabla_i \phi, \) we have

\[
\nabla_i \tilde{P}_0 = \rho \nabla_i \tilde{\mu}_0 + s \nabla_i T_0 - \rho \nabla_i \phi = -\rho \nabla_i \phi.
\] \tag{23}

Allowing for macroscopic motion, \( v_i^n, v_i^s \neq 0, \) the same procedure yields the type of motion allowed in equilibrium. In essence, these are solid-body rotation, translation, and divergence-free superflow. Minimizing the total energy \( \int (\varepsilon + \rho \phi) d^3r, \) equation (3), for given entropy \( S = \int s d^3r, \) mass \( M = \int \rho d^3r, \) momentum \( \mathbf{G} = \int \mathbf{g} d^3r, \) angular momentum \( \mathbf{L} = \int (\mathbf{r} \times \mathbf{g}) d^3r, \) and booster \( \mathbf{B} = \int (\rho \mathbf{r} - \mathbf{g} t) d^3r \) (see the explanation at the end of this section), yields:

\[
\nabla_i T = 0, \quad A_{ij} \equiv \frac{1}{2}(\nabla_i v^s_j + \nabla_j v^n_i) = 0, \quad \nabla_i \tilde{\mu} + \frac{\partial}{\partial t} v^n_i = 0, \quad \nabla_i j^s_i = 0.
\] \tag{24}

The first three conditions hold also in normal fluids, except for the fact that here, \( v^n_i \) takes the place of \( v_i, \) since both are given as \( \partial \varepsilon / \partial \rho. \) The details of the calculation is found in [15] which, however, does not include gravitation\(^3\), see also [16]. To obtain the fourth condition, \( \nabla_i j^s_i = 0, \) we vary the last term in equation (3),

\[ \text{Since the kinetic energy in a normal fluid, } \frac{1}{2} m v^n_i^2 / \rho, \text{ depends only on } \rho, \text{ not on } s, \text{ there is no difference between } T_0 \text{ and the general } T. \text{ In superfluids, with } \rho, \text{ a function of both, one needs to distinguish between } T_0 \text{ and } T. \]

\[ \text{A cautionary remark: Including the gravitation, though the total energy } \tilde{w} \text{ is conserved, the momentum, angular momentum and booster no longer are. Nevertheless, on the earth surface } (\phi = Gz, \text{ or } \nabla_i \phi = G \hat{z}), \text{ } g_\perp = g \perp \hat{z}, \text{ } \ell_i = \ell \parallel \hat{z}, \text{ and the booster with } g_\perp, \ell_i \text{ remain conserved; hence } \nabla_\parallel \tilde{\mu} + \frac{\partial}{\partial t} v_\perp = 0 \text{ holds. And } \frac{\partial}{\partial t} v_i = 0 = \nabla_i \tilde{\mu} \text{ with } \text{a (co-moving) bottom}.} \]

\[ \text{https://doi.org/10.1088/1742-5468/acbe98} \]
Superfluid dynamics, equilibrium conditions, and centripetal forces

\[ \delta \int j_v^i d\Omega \phi d^3r = \oint j_v^i \delta \phi d^3r = \delta \nabla_i j_v^i \delta \phi d^3r = 0. \]

Taking \( \delta \phi = 0 \) at the surface of the system (properly isolated system, no external work), and \( \delta \phi \) arbitrary within the volume, we conclude \( \nabla_i j_v^i = 0 \).

Clearly, the temperature \( T \) is always uniform, even when the system is in motion, but the chemical potential \( \mu \) is not; shear flow is not permitted, \( A_{ij} = 0 \), rotation is, \( \Omega \equiv \frac{1}{2} \nabla \times v^n \neq 0 \). To understand why \( \partial_t v^n \neq 0 \), consider a system rotating with \( \Omega \) around its center of mass, \( R = R_0 + R_t \), that moves with a constant \( \dot{R} \). We have \( v^n = \dot{R} + \Omega \times (r - R) \), or \( \frac{\partial}{\partial t} v^n = \Omega \times \dot{R} = \nabla \mu \neq 0 \). These results are crucial for the considerations in section 2.4 to derive the equilibrium dynamics.

In the classic book by Landau and Lifshitz on statistical mechanics [13], a similar consideration is given (in the sections Macroscopic Motion and Rotating Bodies), see also [4]. Ignoring the booster, they obtained as condition \( \nabla \mu = 0 \). Yet both the angular momentum, \( \ell \equiv r \times g \), and the booster, \( b \equiv \rho r - gt \) are locally conserved quantities—none follows from rotational invariance, the other from Galilean invariance. Relativistically, both are closely related: \( b \) is the zeroth component of the four-angular momentum, the conservation of which is a result of the Lorentz invariance: \( \ell^{\alpha, \beta} = x^\alpha g^\beta - x^\beta g^\alpha \), \( x^\alpha = (ct, r) \), \( g^\alpha = (\varepsilon/c \approx \rho c, g) \). Since angular momentum conservation holds in all inertial systems, the zeroth component (that mixes with the other three under a Lorentz transformation) has to be conserved too. Ignoring the booster is similarly wrong as ignoring one of the three components of the angular momentum.

2.4. Equilibrium two-fluid dynamics

Now we show the equivalence between equilibrium and ideal dynamics. The latter is given as

\[
\frac{\partial}{\partial t} s + \nabla_i f_i = 0, \quad f_i^{\text{eq}} = s v_i^n, \tag{25}
\]

\[
\frac{\partial}{\partial t} \rho + \nabla_i j_i = 0, \quad j_i = \rho v_i^n + j_i^s, \tag{26}
\]

\[
\frac{\partial}{\partial t} v_i^s + \nabla_i (v_i^n v_k^s + \hat{\mu}) = 0, \quad \hat{\mu}^{\text{eq}} = \mu, \tag{27}
\]

\[
\frac{\partial}{\partial t} g_i + \nabla_k \sigma_{ik} = 0, \quad \sigma_{ik}^{\text{eq}} = P \delta_{ik} + g_i v_k^n + v_i^n j_k^s, \tag{28}
\]

\[
\frac{\partial}{\partial t} \varepsilon + \nabla_i Q_i = 0, \quad Q_i^{\text{eq}} = \mu g_i + v_k^n v_k^s j_i^s + v_i^n (s T + v_k^n g_k). \tag{29}
\]

These are five (true or quasi) conservation laws, with the fluxes specified. They have been derived [1, 4] by inserting equations (25)–(28) into equations (3) and (29),

\[
\frac{\partial}{\partial t} \varepsilon = \mu \frac{\partial}{\partial t} \rho + T \frac{\partial}{\partial t} s + v_i^n \frac{\partial}{\partial t} g_i + j_i^s \frac{\partial}{\partial t} v_i^s = -\nabla_i Q_i, \tag{30}
\]

requiring that the four fluxes are such that they can be combined to form \( \nabla_i Q_i \). This is referred to as the hydrodynamic procedure. Uniqueness is certainly an issue, though more in principle than in practice. The above fluxes do satisfy the requirement, and

https://doi.org/10.1088/1742-5468/acbe98
one realizes quickly that alternative fluxes are hard to construct. The lack of rigor is compensated, and any remaining doubts eliminated, by experiments.

This paper aims to prove that given the equilibrium conditions (24), and the fact that in equilibrium, scalar variables and parameters are stationary in the frame $\mathbf{v}^n = 0$, one also arrives at the above fluxes. Employing the notation $D_t A \equiv (\frac{\partial}{\partial t} + v^n_k \nabla_k) A$ for any scalar $A$, we have, in equilibrium,

$$D_t \rho = 0, \quad D_t s = 0, \quad D_t \rho_s = 0. \quad (31)$$

Starting with $D_t s = 0$, we add $s \nabla_i v^m_i = 0$ to yield equation (25). Next, to derive equation (26), we consider angular momentum and booster conservation,

$$\frac{\partial}{\partial t} \epsilon_m = (r \times \frac{\partial}{\partial t} g)_m = \epsilon_{mki} r_k \frac{\partial}{\partial t} g_i = -\epsilon_{mki} r_k \nabla_j \sigma_{ij}$$

$$= -\nabla_j [\epsilon_{mki} r_k \sigma_{ij}] + \epsilon_{mki} \sigma_{ik},$$

$$\frac{\partial}{\partial t} b_i = r_i \frac{\partial}{\partial t} \rho - t \frac{\partial}{\partial t} g_i - g_i = -r_i \nabla_j j_j + t \nabla_j \sigma_{ij} - g_i$$

$$= \nabla_j (t \sigma_{ij} - r_i j_j) + j_i - g_i,$$

concluding that (both in and off equilibrium)

$$\sigma_{ij} = \sigma_{ji}, \quad j_i = g_i. \quad (32)$$

Given $g_i$, equation (5), we know $j_i$, equation (26).

For any vector $\mathbf{B}$, we denote

$$D_t \mathbf{B} \equiv (\frac{\partial}{\partial t} + v^n_k \nabla_k - \Omega \times) \mathbf{B}, \quad (\mathbf{B} \times \Omega)_i = B_k \nabla_i v^m_k. \quad (33)$$

The second equation holds because $(\mathbf{B} \times \Omega)_i = \frac{1}{2}(\nabla_i v^m_i - \nabla_k v^m_k) B_k$ and because $A_{ij} = 0$. Next, we show the validity of

$$D_t \mathbf{v}^n = -\nabla \mu, \quad D_t \mathbf{v}^s = -\nabla \bar{\mu}. \quad (34)$$

We have $\frac{\partial}{\partial t} \mathbf{v}^n = D_t \mathbf{v}^n$ because $v^n_k \nabla_k v^m_i + (\mathbf{v}^n \times \Omega)_i = 0$, and may rewrite the equilibrium condition as $D_t \mathbf{v}^n = -\nabla \mu$. The second of equation (34) is the same as the first of equation (27), because $\nabla_i (v^n_k v^m_k) = v^n_k \nabla_i v^m_k + v^m_k \nabla_i v^n_k = v^n_k \nabla_k v^m_i + (\mathbf{v}^n \times \Omega)_i$. (Remember $\nabla_i v^m_k = \nabla_k v^m_i \sim \nabla_i \nabla_k \varphi$.)

To identify $\bar{\mu}$, we resort to the microscopic Josephson equation, $\frac{\hbar}{m} \frac{\partial}{\partial t} \varphi = -\mu$, which accounts for the evolution of $\varphi$ in a system at rest. If $v_i^m \neq 0$, it changes to $(\hbar/m)D_t \varphi = -\bar{\mu}$. Applying the gradient yields $\bar{\mu}^\text{eq} = \mu$.

The Josephson equation may seem an input, but is not. It can be deduced from the form of $j_i$, via energy conservation. Proceeding as prescribed by equation (30) we obtain (terms containing $\mu, \bar{\mu}^\text{eq}$ alone are displayed)

$$\nabla_i (Q_i) = \mu \nabla_i j_i^s + j_i^s \nabla_i \bar{\mu}^\text{eq} + \cdots = \nabla_i (\mu g_i) + j_i^s \nabla_i (\bar{\mu}^\text{eq} - \mu) + \cdots$$

Only for $\bar{\mu}^\text{eq} = \mu$, is the energy $\varepsilon$ conserved, $\nabla_i Q_i = \nabla_i (\mu g_i + \cdots)$.
Having introduced \( v_i^s \) as the new state variable, the one crucial assumption to arrive at the two-fluid dynamics is to take it transforming as a velocity. This yields \( g_i \), hence \( j_i \), and also the Josephson equation.

An additional important conclusion from equation (34) is that in a rotating equilibrium, if the center of mass is at rest, \( \dot{\mathbf{R}} = 0 \), with \( \nabla \mu = 0 \), both \( v^s \) in the \( v^n \)-frame and \( v^n \) itself are stationary: \( D_t v^n = 0 \), \( \frac{\partial}{\partial t} v^n = 0 \). But both change with time if \( \dot{\mathbf{R}} \neq 0 \), even in equilibrium.

Finally, because \( g = \rho v^n + j^s \), \( D_t (\rho v^n) = v^n D_t \rho + \rho D_t v^n \), and \( D_t j^s = (v^s - v^n) \)

\[
D_t g = D_t (\rho v^n) = -\rho \nabla \mu. \tag{35}
\]

Given equation (11), we note (28) is the same as (35), because \( \nabla_i T = 0 \), \( g_k \nabla_i v_i^k = (g \times \Omega)_i \), \( \nabla_i (g_i v_i^k) = v_i^k \nabla_i g_i \), \( \nabla_k (v_i^s j^k_i) - j^k_i \nabla_i v_i^k = 0 \). Also, the stress \( g_i v_i^k + v_i^j j^k_i = \rho (v_i^k + v_i^j j^k_i) \) is symmetric, equation (32).

If we did not know in advance what the momentum flux is, these steps may appear to lack uniqueness. Yet another form of the momentum flux is quite improbable: First set \( j_i^s = 0 \) to consider the stress tensor for normal fluids. Facing the need to rewrite \( \rho \nabla \mu \) of equation (35) as \( \nabla_i \sigma_{ij} \), there is little choice other than switching to the pressure, \( \nabla_i P \). Rewriting the remaining term \( v_i^k \nabla_i g_i \) as \( \nabla_k (g_i v_i^k) \) is a familiar step, with \( g_i v_i^k = \rho \nabla_i v_i^k \) symmetric. Going back to superfluid, only \( g_i v_i^k + v_i^j j^k_i \) is symmetric, and \( \nabla_k (v_i^s j^k_i) = j^k_i \nabla_i v_i^k \) cancels the same term in the pressure gradient.

Energy conservation \( \frac{\partial}{\partial t} \varepsilon + \nabla_i Q_i = 0 \) is not independent. To calculate \( Q_i \), one proceeds as in equation (30), now keeping every term. Since all fluxed are known, it is easily done. That all terms combine to form the divergence of \( Q_i \), is clearly an added argument against alternative fluxes.

To add gravitation, we write \( \frac{\partial}{\partial t} g_i + \nabla_k \sigma_{ik} = -\rho \nabla_i \phi \), equations (22) and (23), implying \( \frac{\partial}{\partial t} \varepsilon + \nabla_i Q_i = -\rho v_i^n \nabla_i \phi \), since \( \frac{\partial}{\partial t} \varepsilon = v_i^n \frac{\partial}{\partial t} g_i + \cdots = -\rho v_i^n \varepsilon \nabla_i \phi + \cdots \).

3. Dissipative fluxes

Off equilibrium, the entropy production does not vanish, \( \frac{\partial}{\partial t} s + \nabla_i f_i = R/T \neq 0 \), neither do the equilibrium conditions,

\[
\nabla_i T, A_{ij}, \nabla_i j^k_i, \frac{\partial}{\partial t} v_i^n + \nabla_i \mu \neq 0,
\]

and are now referred to as thermodynamic forces. As they completely characterize a macro state, both in and off equilibrium, we may expand \( R \) in them, \( R = \kappa (\nabla_i T)^2 + \cdots \), to second order. There are no constant or linear terms, as \( R \) vanishes and is minimal in equilibrium. At the same time, dissipative fluxes appear, \( f = f^{eq} + f^D \), \( \tilde{\mu} = \tilde{\mu}^{eq} + \tilde{\mu}^D \), \( \sigma_{ij} = \sigma_{ij}^{eq} + \sigma_{ij}^D \), where \( j_i^D = j_i - j_i^{eq} \) vanishes identically. Proceeding again as in equation (30), including now the dissipative fluxes, we find an expanded \( Q_i \), and

\[
R = j_i^D \nabla_i \mu + f^D_i \nabla_i T + \tilde{\mu}^D \nabla_i j_i^s + \sigma_{ij}^D A_{ij}, \tag{36}
\]

https://doi.org/10.1088/1742-5468/acbe98
enabling us to conclude that \( f^D, \hat{\mu}^D, \sigma^D_{ij}, A_{ij} \) are linear combinations of \( \nabla_i T, \nabla_{ij}s, A_{ij} \) as prescribed by the Onsager rules—with \( f^D_i \sim \nabla_i T \), we indeed have \( R = \kappa(\nabla_i T)^2 + \cdots \). Since \( j^D_i \equiv 0 \), neither \( \frac{\partial}{\partial t}v^D_i + \nabla_i \hat{\mu} \) nor \( \nabla_i \hat{\mu} \) serves as a fourth thermodynamic force.

The basic insight of the above consideration is the central role of equilibrium conditions, both in and off equilibrium. The two-fluid hydrodynamics is simply the addition of equilibrium dynamics plus thermodynamic forces pushing the system back to equilibrium. There are indications that this is true of many more systems, including solid, nematic/smectic liquid crystals, polymeric solutions and granular media.

References

[1] Khalatnikov I M 2018 *An Introduction to the Theory of Superfluidity* (Boca Raton, FL: CRC Press)
[2] Landau L D and Lifshitz E M 1987 *Fluid Mechanics* (Course of Theoretical Physics vol 6) (Amsterdam: Elsevier)
[3] Putterman S and Uhlenbeck G E 1969 Thermodynamic equilibrium of rotating superfluids *Phys. Fluids* **12** 2229
[4] Putterman S J 1974 *Superfluid Hydrodynamics* (Amsterdam: North Holland)
[5] Hofmann J and Zwerger W 2021 Hydrodynamics of a superfluid smectic *J. Stat. Mech.* 033104
[6] Liu M 1975 Hydrodynamics of \(^3\)He near the A transition *Phys. Rev. Lett.* **35** 1577
[7] Liu M 1978 Two possible types of superfluidity in crystals *Phys. Rev.* B **18** 1165
[8] Liu M and Cross M C 1978 Broken spin-orbit symmetry in superfluid \(^3\)He and B-phase dynamics *Phys. Rev. Lett.* **41** 250
[9] Liu M and Cross M C 1979 Gauge wheel of superfluid \(^3\)He *Phys. Rev. Lett.* **43** 296
[10] Liu M 1982 Broken relative symmetry and the hydrodynamics of superfluid \(^3\)He *Physica* B+C **109** 1615–28
[11] Liu M 1998 Rotating superconductors and the frame-independent London equation *Phys. Rev. Lett.* **81** 3223
[12] Jiang Y and Liu M 2001 Rotating superconductors and the London moment: thermodynamics versus microscopics *Phys. Rev. B* **63** 184506
[13] Landau L D, Lifshitz E M and Pitaevskii L P 1980 *Statistical Physics, Part I* (Oxford: Pergamon)
[14] Hu C-R and Saslow W M 1977 Hydrodynamics of \(^3\)He-A with arbitrary textures *Phys. Rev. Lett.* **38** 605–9
[15] Kostädt P and Liu M 1998 Three ignored densities, frame-independent thermodynamics and broken Galilean symmetry *Phys. Rev.* E **58** 5535
[16] Liu M 2018 Thermodynamics and constitutive modeling *Alert Doctoral School 2018: Energetical Methods in Geomechanics* ed I Einav and E Gerolymatou (France: The Alliance of Laboratories in Europe for Education, Research and Technology) pp 3–42

https://doi.org/10.1088/1742-5468/acbe98