Abstract

Let $\Omega \subset \mathbb{R}^n$ be open and let $\mathcal{R}$ be a partial frame on $\Omega$, that is a set of $m$ linearly independent vector fields prescribed on $\Omega$ ($m \leq n$). We consider the issue of describing the set of all maps $F : \Omega \to \mathbb{R}^n$ with the property that each of the given vector fields is an eigenvector of the Jacobian matrix of $F$. By introducing a coordinate independent definition of the Jacobian, we obtain an intrinsic formulation of the problem, which leads to an overdetermined PDE system, whose compatibility conditions can be expressed in an intrinsic, coordinate independent manner. To analyze this system we formulate and prove a generalization of the classical Frobenius integrability theorems. The size and structure of the solution set of this system depends on the properties of the partial frame, in particular, whether or not it is in involution. A particularly nice subclass of involutive partial frames, called rich, can be completely analyzed. Involutive, but non-rich case is somewhat harder to handle. We provide a complete answer in the case of $m = 3$ and arbitrary $n$, as well as some general results for arbitrary $m$. The non-involutive case is far more challenging, and we only obtain a comprehensive analysis in the case $n = 3, m = 2$. Finally, we provide explicit examples illustrating the various possibilities. Our initial motivation for considering this problem comes from the geometric study of hyperbolic conservative systems in one spatial dimension.

Keywords: Jacobian matrix and map; affine connections; prescribed eigenvectors; integrability theorems; conservative systems; hyperbolic fluxes.

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1 Introduction

The present work deals with the construction of maps \( F: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) whose Jacobian matrix has a partially prescribed set of eigenvector fields on \( \Omega \). We consider this problem locally, i.e. in a sufficiently small neighborhood of a given point in \( \Omega \). The case when the full frame of \( n \) independent eigenvectors is prescribed has been considered in [7]. The generalization to a partially prescribed set of eigenvector fields allows a greater degree of flexibility in constructing such maps \( F \) and, in particular, permits maps \( F \) whose Jacobian matrix is not diagonalizable.

Another difference from the previous work is that all the overdetermined system of PDEs arising in the current paper are analyzed using smooth \( C^1 \) integrability theorems and, in particular, a recently proved generalization of the Frobenius theorem (see Section 3.3). This theorem allows us to remain in the smooth category, while in [7] we appealed in some cases to the Cartan-Kähler theorem, which requires analyticity assumptions.

Our motivation stems from the study of initial value problems for one dimensional conservative systems of the form

\[
    u_t + F(u)_x = 0, \quad u(0, x) = u_0(x),
\]

where \( t \in \mathbb{R} \) and \( x \in \mathbb{R} \) are the independent variables, \( u = u(t, x) \in \mathbb{R}^n \) is a vector of unknowns, and the flux function \( F \) is defined on some open set in \( \mathbb{R}^n \) and takes values in \( \mathbb{R}^n \). It is an outstanding open problem to provide an existence theory for the Cauchy problem for (1) which is general enough to cover nonlinear systems of physical interest and of initial data \( u_0(x) \) of “large” total variation. Such a theory is in place for near-equilibrium solutions (Glimm’s theorem [5]): global-in-time existence of a weak solution is guaranteed, provided the initial data \( u_0(x) \) have sufficiently small total variation. (For detailed accounts of this theory see [14, 2, 3].)

A key ingredient in the proof is the use of Riemann problems, i.e. initial value problems for (1) where the data \( u_0(x) \) consists of two constant states \( u^\pm \), separated by a jump discontinuity,

\[
    u_0(x) = \begin{cases} 
        u^- & x < 0 \\
        u^+ & x > 0 
    \end{cases}
\]

By knowing how to solve Riemann problems one can, via an approximation scheme, solve general (small variation) Cauchy problems.

The solution of (1)-(2) is a self-similar (function of \( x/t \)) fan of \( n \) waves emanating from the origin. These waves are determined from the eigenvalues and eigenvectors of the Jacobian matrix \( [D_u F] \). It is, therefore, of interest to gain an understanding of how the eigenstructure of \( F \) induce properties of the solutions \( u(t, x) \). It is then a basic question to what extent one can prescribe some or all of the eigenvectors of the flux \( F \).

\[\text{We employ } C^1 \text{ integrability theorems, but to avoid some technicalities } C^\infty\text{-smoothness is assumed throughout the paper.}\]
The present work is concerned with this last, purely geometric problem. A precise formulation of the problem is provided in Section 2. This first formulation, “Problem 1”, makes use of a chosen coordinate system. Section 3 provides the geometric framework required to obtain a coordinate-free formulation. We also state and prove a generalization of the Frobenius integrability theorem, which we use in this paper. In Section 4, we give an intrinsic (coordinate independent) definition of the Jacobian, and use it to reformulate Problem 1 in an intrinsic manner (see Problem 4). Exploiting the coordinate independent formulation we treat, in Section 5, the case when the prescribed, partial frame of eigenvectors-to-be, is in involution. In this case, the integrability condition of the $F(\mathcal{R})$ system lead to a closed algebro-differential system on eigenvalues-to-be $\lambda$’s. Section 6 analyzes the simplest non-involutive case of two prescribed vector fields in $\mathbb{R}^3$. Finally, Section 7 provides a list of examples that illustrate the results from the earlier sections.

2 Problem formulation

In this paper, $[D_\Psi \Psi]_u$ denotes the Jacobian matrix of a map $\Psi$ from an open subset $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$, relative to coordinates $u$. That is,

$$[D_\Psi \Psi] = \left[ \frac{\partial \Psi^i}{\partial u^j} \right]_{i,j=1,\ldots,n}. $$

We use the notation $[D_\Psi \Psi]_{u=\bar{u}}$, or simply $[D_\Psi \Psi]_{\bar{u}}$, when the matrix is evaluated at a point $\bar{u}$.

We consider the following problem:

**Problem 1.** Given an open subset $\Omega \subset \mathbb{R}^n$ on which we fix a coordinate system $u=(u^1,\ldots,u^n)$ and a point $\bar{u} \in \Omega$. Let $\mathcal{R} = \{R_1,\ldots,R_m\}$ be a set of $m \leq n$ smooth vector valued functions $R_i : \Omega \to \mathbb{R}^n$ which are linearly independent at $\bar{u}$. Then: describe the set $\mathcal{F}(\mathcal{R})$ of all smooth vector-valued functions

$$F(u) = [F^1(u),\ldots,F^n(u)]^T$$

defined near $\bar{u}$ and with the property that $R_1(u),\ldots,R_m(u)$ are right eigenvectors of the Jacobian matrix $[D_u F]_{\bar{u}}$ throughout a neighborhood of $\bar{u}$. In other words, we ask that there exist smooth, scalar functions $\lambda^i$ such that

$$[D_u F] R_i(u) = \lambda^i(u) R_i(u), \quad i = 1,\ldots,m, \tag{3}$$

holds on a neighborhood of $\bar{u}$.

As outlined in the Introduction, we are motivated by the construction of flux functions $F$ in systems of conservation laws of the form (1). The system (1) is called hyperbolic on $\Omega$ provided the Jacobian matrix $[D_u F]$ has a basis of real eigenvectors at each $u \in \Omega$, and it is called strictly hyperbolic if, in addition, all its eigenvalues are distinct at each $u \in \Omega$. We adopt the term flux for a vector-function satisfying (3), with adjectives hyperbolic, strictly hyperbolic or non-hyperbolic depending on the structure of eigenvectors and eigenvalues of $[D_u F]$, as described above.

In the list below, we clarify what we mean by “describe” in Problem 1 and make some preliminary observations about Problem 1.

1. *(PDE system)* Equations (3) comprise a system of $mn$ first order PDEs on $n+m$ unknown functions $\lambda^i$ and $F^j$:

$$\sum_{k} R_{ki} \frac{\partial F^j}{\partial u^k} = \lambda^i R_j^i, \quad \text{for } i = 1,\ldots,m, \quad j = 1,\ldots,n, \tag{4}$$

where $R_i(u) = [R_1^i(u),\ldots,R_m^i(u)]^T$, $i = 1,\ldots,m$. This system is overdetermined for all $n \geq m$, such that $n > 2$ and $m \geq 2$. Although derivatives of $\lambda$’s do not appear in the equations, these functions are not arbitrary parameters, but must, in turn, satisfy certain differential equations arising as differential consequence of (4).
2. (Vector space structure) Let $F_1, F_2 \in \mathcal{F}(\mathcal{R})$, have the domains of definitions $\Omega_1$ and $\Omega_2$, respectively. Since $\bar{u}$ belongs to both $\Omega_1$ and $\Omega_2$, the sum $F_1 + F_2$ is defined on the non-empty open neighborhood $\Omega_1 \cap \Omega_2$ of $\bar{u}$. It is easy to check that $F_1 + F_2$ belongs $\mathcal{F}(\mathcal{R})$ and that $\bar{a} F_1 \in \mathcal{F}$, where $\bar{a}$ is any real number, belongs to $\mathcal{F}(\mathcal{R})$. Thus $\mathcal{F}(\mathcal{R})$ is a vector space over $\mathbb{R}$. We will see below, that in some instances this is a finite dimensional vector space, while in others it is an infinite dimensional space. In the latter case, we describe the “size” of $\mathcal{F}(\mathcal{R})$ in terms of the number of arbitrary functions of a certain number of variables appearing in the general solution of $\mathcal{H}$. These arbitrary functions prescribe the values of $F$ and $\lambda$’s along certain submanifolds of $\Omega$. To obtain these results we use the integrability theorem stated in Section 3.3.

3. (Scaling invariance) Since eigenvectors are defined up to scaling, it is clear that

$$\mathcal{F}(R_1, \ldots, R_m) = \mathcal{F}(\alpha^1 R_1, \ldots, \alpha^m R_m)$$

for any nowhere zero smooth functions $\alpha^i$ on $\Omega$.

4. (Trivial solutions) For any $n + 1$ constants $\lambda, \bar{a}^1, \ldots, \bar{a}^n \in \mathbb{R}$ the “trivial” flux

$$F(u) = \lambda \begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix} + \begin{bmatrix} \bar{a}^1 \\ \vdots \\ \bar{a}^n \end{bmatrix}$$

satisfies $\mathcal{H}$. The set of such trivial solutions, denoted by $\mathcal{F}^{\text{triv}}$, is an $(n+1)$-dimensional vector subspace of $\mathcal{F}(\mathcal{R})$.

5. (Triviality is generic) It is worthwhile emphasizing that when $n > 2$ and $m \geq 2$, the compatibility conditions for $\mathcal{F}(\mathcal{R})$-system are closed, and thus almost all frames admit only trivial fluxes. One of the goals of the paper is to determine the properties of the frames that allow them to possess non-trivial fluxes, and in particular strictly hyperbolic fluxes.

The vector space $\mathcal{F}(\mathcal{R})$ will be called the flux space. We are, of course, only interested in non-trivial fluxes, and particularly in strictly hyperbolic fluxes due to their central role in the theory of conservation laws.

The next remark addresses the coordinate dependence of our formulation of Problem 4. In Section 4.2 we formulate a coordinate independent version (Problem 2), which, when expressed in an affine system of coordinates (see Definition 7) coincides with Problem 1. This intrinsic definition allows us to apply a geometric approach to analyze the solution set of PDE system $\mathcal{H}$.

Remark 2.1 (Coordinate dependence of the problem formulation). Assume $F(u) \in \mathcal{F}(\mathcal{R})$ for $\mathcal{R} = \{R_1, \ldots, R_m\}$, i.e. there exist $\lambda^1(u), \ldots, \lambda^m(u)$, such that system $\mathcal{H}$ is satisfied. Let a change of variables be described by a local diffeomorphism

$$u = \Phi(w).$$

It is not true, in general, that $\tilde{F}(w) = F(\Phi(w))$ belongs to $\mathcal{F}(\bar{\mathcal{R}})$, where $\bar{\mathcal{R}} = \{\tilde{R}_1(w), \ldots, \tilde{R}_m(w)\}$, with $\tilde{R}_i(w) = R_i(\Phi(w))$. Indeed:

$$[D_w (F \circ \Phi)] \tilde{R}_i = [D_u F]_{u=\Phi(w)} [D_w \Phi] R_i(\Phi(w)).$$

In general, $R_i(\Phi(w))$ is not an eigenvector of $[D_u F]_{u=\Phi(w)} [D_w \Phi]$.

Even if we transform $R_i(w)$’s, by treating them, more appropriately, as vector-fields:

$$R_i^*(w) = [D_w \Phi]^{-1} R_i(\Phi(w)),$$
then
\[
[D_w(F \circ \Phi)]R^i_w = [D_u F]_{u = \Phi(w)}[D_w \Phi][D_w \Phi]^{-1} R_i(\Phi(w))
\]
\[
= [D_u F]_{u = \Phi(w)} R_i(\Phi(w))
\]
\[
= \lambda_i(\Phi(w)) R_i(\Phi(w)) = \lambda_i(\Phi(w)) [D_w \Phi] R^i_w(w),
\]
and we see that \( R^i_w(w) \) is not an eigenvector of \([D_w(F \circ \Phi)]\), unless it is an eigenvector of \([D_w \Phi]\).

We also recall that the property of a matrix being the Jacobian matrix of some map is also coordinate dependent:

**Remark 2.2** (Coordinate dependence of the property of a matrix being a Jacobian matrix). Assume \( A(u) = [D_u F] \) for some smooth map \( F : \Omega \to \mathbb{R}^n \), and let a change of coordinates be given by a diffeomorphism \( u = \Phi(w) \). Then, it is not necessarily the case that matrix \( A(\Phi(w)) \) is a Jacobian matrix of any map in \( w \)-coordinates.

On the other hand, it is still possible to give a coordinate independent definition of the Jacobian linear map, as we do in Section 3.1. This is used to obtain a coordinate-independent formulation of Problem 1. We exploit this by working in frames that are adapted to the problem at hand, and we use the following geometric preliminaries.

### 3 Geometric preliminaries

Most of the notions and results, reviewed in this section, can be found in a standard differential geometry textbook. We included them to set up notation, as well as to make a paper self-contained. The notable exception is Section 3.3 where we state and prove a generalization of the Frobenius integrability theorem.

#### 3.1 Vector fields, flows, partial frames, involutivity, richness

It will be useful for us to give an intrinsic, coordinate free definition of a vector field as a linear first order differential operator on the set of functions

**Definition 1.** A smooth vector field \( \mathbf{r} \) on \( \Omega \) is an \( \mathbb{R} \)-linear map from the set of smooth functions \( C^\infty(\Omega) \) to itself that satisfying the product rule.

The set of all smooth vector fields will be denoted as \( \mathcal{X}(\Omega) \), and it is an infinite dimensional vector space over \( \mathbb{R} \) and a free \( n \)-dimensional module over \( C^\infty(\Omega) \). Relative to any coordinate system, a vector field \( \mathbf{r} \) evaluated at a point \( \bar{u} \in \Omega \) becomes a vector in \( \mathbb{R}^n \). We say that vector fields \( \mathbf{r}_1, \ldots, \mathbf{r}_m \) are independent at \( \bar{u} \in \Omega \) if vectors \( \mathbf{r}_1(\bar{u}), \ldots, \mathbf{r}_m(\bar{u}) \in \mathbb{R}^n \) are independent over \( \mathbb{R} \) relative to one and, therefore, to all coordinate systems.

**Definition 2.** A set of smooth vector fields \( \mathbf{r}_1, \ldots, \mathbf{r}_m \) on \( \Omega \) is called a partial frame on \( \Omega \) if they are independent for all \( \bar{u} \in \Omega \). If \( m = n \), then this set is called a frame.

It is easy to show that a frame comprises a basis of the module \( \mathcal{X}(\Omega) \) over \( C^\infty(\Omega) \), i.e., for any smooth vector field \( \mathbf{r} \in \mathcal{X}(\Omega) \) there are smooth functions \( R^1, \ldots, R^n \in C^\infty(\Omega) \), called the components of \( \mathbf{r} \) relative to frame \( \mathbf{r}_1, \ldots, \mathbf{r}_n \), such that \( \mathbf{r} = R^1 \mathbf{r}_1 + \cdots + R^n \mathbf{r}_n \). For a fixed coordinate system \( u^1, \ldots, u^n \), the frame \( \left\{ \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n} \right\} \) of partial derivatives is called a coordinate frame, but as we see below using non-coordinate frames can simplify a problem. The Lie bracket of two vector-fields can be defined as the commutator operator on functions.

**Definition 3.** Given two smooth vector field, their Lie bracket is the map \( C^\infty(\Omega) \to C^\infty(\Omega) \) defined by

\[
[\mathbf{r}_1, \mathbf{r}_2] \phi = \mathbf{r}_1(\mathbf{r}_2(\phi)) - \mathbf{r}_2(\mathbf{r}_1(\phi)).
\]
A standard calculation shows that \([\mathbf{r}_1, \mathbf{r}_2]\) is a vector field, i.e. a first order linear differential operator. Skew symmetry of the Lie bracket is obvious and Jacobi identity can be checked by an explicit calculation. Therefore, \(\mathfrak{X}(\Omega)\) has a structure of an infinite-dimensional real Lie algebra.

Given a frame \(\mathbf{r}_1, \ldots, \mathbf{r}_n\), we can write the following structure equations:

\[
[\mathbf{r}_i, \mathbf{r}_j] = \sum_{k=1}^{n} c_{ij}^k \mathbf{r}_k,
\]

where \(c_{ij}^k\), such that \(c_{ij}^k = -c_{ji}^k\), are smooth functions on \(\Omega\), called structure coefficients, or structure functions. In the conservation laws literature, these functions are called interaction coefficients because of their role in wave interaction formulas [5]. The Jacobi identity imply the following relationship on the structure coefficients:

\[
\begin{align*}
\mathbf{r}_l (c_{jk}^i) + \mathbf{r}_k (c_{ij}^l) + \mathbf{r}_j (c_{il}^k) &= 0, \quad 1 \leq i, j, k, l \leq n. \\

\end{align*}
\]

(7)

We will define two classes of partial frames with especially nice properties:

**Definition 4 (Involutive frame).** We say that a partial frame \(\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}\) is in involution if \([\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega)} \mathfrak{R}\) for all \(\mathbf{r}_i, \mathbf{r}_j \in \mathfrak{R}\).

The proof of the following proposition can be found in the proof of Theorem 6.5 of Spivak [15].

**Proposition 3.1.** Let \(\mathbf{r}_1, \ldots, \mathbf{r}_m\) be a partial frame in involution on \(\Omega\), then there is a commutative partial frame such that \(\tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_m\) on some open \(\Omega' \subset \Omega\), such that

\[
\text{span}_{\mathfrak{R}} \{\mathbf{r}_1|_u, \ldots, \mathbf{r}_m|_u\} = \text{span}_{\mathfrak{R}} \{\tilde{\mathbf{r}}_1|_u, \ldots, \tilde{\mathbf{r}}_m|_u\} \quad \text{for all } u \in \Omega'.
\]

**Proposition 3.2.** (Theorem 5.14 in [15]) If \(\mathbf{r}_1, \ldots, \mathbf{r}_m\) is a commutative partial frame on \(\Omega\), then in a neighborhood of each point \(\bar{u} \in \Omega\) there exist coordinate functions \(v_1, \ldots, v_n\), such that

\[
\mathbf{r}_i = \frac{\partial}{\partial v^i}, \quad i = 1, \ldots, m.
\]

**Definition 5 (Rich frame).** We say that a partial frame \(\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}\) is rich if every pair of its vector fields is in involution, i.e. \([\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega)} \{\mathbf{r}_i, \mathbf{r}_j\}\) for all \(i, j = 1, \ldots, m\).

In Lemma 5.6 we show that every rich partial frame \(\mathbf{r}_1, \ldots, \mathbf{r}_m\) can be scaled to become a commutative frame and so around each point one can find coordinates \(w^1, \ldots, w^n\), and non-zero functions \(\alpha^1, \ldots, \alpha^n\) such that

\[
\alpha^i \mathbf{r}_i = \frac{\partial}{\partial w^i}, \quad i = 1, \ldots, m.
\]

Classically, a conservative system is called rich if there are coordinate functions, called Riemann invariants, in which the system is diagonalizable. For definitions, and the fact that richness of a conservative system is equivalent to the richness of its eigenframe in the sense of our definition, we refer to [12], and Section 7.3 in [3]. Riemann invariants are exactly the coordinates appearing in Lemma 5.6 in the case of full frame: \(n = m\). The term rich refers to a large family of extensions (companion conservation laws) that strictly hyperbolic diagonalizable systems possess [3] [12].
3.2 Connection, symmetry, flatness, affine coordinates

We defined vector fields as directional derivatives of smooth functions. More generally, one can define directional derivatives of vector fields themselves by introducing the notion of a covariant derivative. We will use this notion to give a coordinate free definition of Jacobians and to express Problem 1 in a non-coordinate frame, which make it easier to find its solution.

Definition 6. A connection \( \nabla \) on \( \Omega \) is an \( \mathbb{R} \)-bilinear map

\[
\nabla : \mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \to \mathcal{X}(\Omega) \quad (r, s) \mapsto \nabla_r s
\]

such that for any smooth function \( \phi \) on \( \Omega \)

\[
\nabla_{\phi r}s = \phi \nabla_rs, \quad \nabla_r(\phi s) = r(\phi)s + \phi \nabla_rs.
\]

(8)

The vector field \( \nabla_r s \) is called the covariant derivative of \( s \) in the direction of \( r \).

Given a connection \( \nabla \) and a frame, \( \{r_1, \ldots, r_n\} \), for all \( i, j \in \{1, \ldots, n\} \) we can write

\[
\nabla_{r_i} r_j = \sum_{k=1}^{n} \Gamma^k_{ij} r_k,
\]

(9)

for some smooth functions \( \Gamma^k_{ij} \), called connection components, or Christoffel symbols. Conversely, due to \( \mathbb{R} \)-bilinearity and (8), for any choice of a frame and \( n^3 \) functions \( \Gamma^k_{ij} \), \( i, j, k = 1, \ldots, n \), formula (9) uniquely defines a connection on \( \Omega \).

Definition 7 (affine coordinates). Given a connection \( \nabla \), coordinate systems, such that relative to the corresponding coordinate frame all Christoffel symbols for \( \nabla \) are zero, are called affine.

Definition 8 (symmetry and flatness). A connection \( \nabla \) is symmetric if for all \( r, s \in \mathcal{X}(\Omega) \):

\[
\nabla_r s - \nabla_s r = [r, s].
\]

(10)

A connection \( \nabla \) is flat if for all \( r, s, t \in \mathcal{X}(\Omega) \):

\[
\nabla_r \nabla_s t - \nabla_s \nabla_r t = \nabla_{[r,s]} t.
\]

(11)

The above conditions are equivalent to the following relationships among the structure functions and Christoffel symbols relative to an arbitrary frame: for all \( i, j, k, s = 1, \ldots, n \),

\[
\Gamma^k_{ij} - \Gamma^k_{ji} = \delta^k_{ij} \quad \text{Symmetry} \quad (12)
\]

\[
r_s(\Gamma^j_{ki}) - r_k(\Gamma^j_{si}) = \sum_{l=1}^{n} (\Gamma^l_{ki} \Gamma^j_{sl} - \Gamma^l_{si} \Gamma^j_{kl} - c^l_{ks} \Gamma^j_{li}) \quad \text{Flatness}. \quad (13)
\]

A well known result, stated, for instance, in Proposition 1.1 in [13], implies that any flat and symmetric connection on \( \Omega \) admits an affine system of coordinates, and that any two affine coordinate systems are related by an affine transformation:

Proposition 3.3. A connection \( \nabla \) on an \( n \)-dimensional manifold \( M \) is symmetric and flat (has properties (12) and (13)) if and only if, it can be covered with an atlas of affine coordinate systems.

Two coordinate system \( u = (u^1, \ldots, u^n) \) and \( w = (w^1, \ldots, w^n) \), on an open subset \( \Omega \subset M \), are affine if and only if \( [w^1, \ldots, w^n]^T = C [u^1, \ldots, u^n]^T + \tilde{b} \), where \( u \) and \( w \) are treated as column vectors, \( C \in \mathbb{R}^{n \times n} \) is an \( n \times n \) invertible matrix and \( \tilde{b} \in \mathbb{R}^n \) is a constant vector.
Throughout the paper, we will use a particular connection, denoted \( \nabla \), defined by setting all Christoffel symbols to be zero, relative to the coordinate frame corresponding to the coordinate system \( u^1, \ldots, u^n \) fixed in Problem 1:

\[
\nabla \frac{\partial}{\partial u^j} = 0, \text{ for all } i, j = 1, \ldots, n. \tag{14}
\]

If the column vectors \( R \) and \( S \) are components of vector fields \( r \) and \( s \), respectively, in an affine coordinate system, then

the components of \( \nabla_r s \) are given by the column vector \( r(S) \), \( \tag{15} \)

where \( r \) is applied to each component of \( S \).

### 3.3 Integrability theorems

To analyze the “size” of the flux-space \( \mathcal{F}(\mathfrak{R}) \) in Problem 1, we use two integrability theorems: generalized Frobenius Theorem and Darboux Theorem.

The classical Frobenius theorem has three equivalent formulations: PDE formulations, vector-field formulation, and differential form formulation (see Spivak [15] Theorems 6.1, 6.5 and 7.14 Warner [16] Theorem 1.60, Remark 1.61, Theorem 2.32). For our generalization, we start with a vector-field formulation, Theorem 3.4, and then, as a consequence, prove its PDE formulation, Theorem 3.5, which we use further in the paper. When \( n = m \), both theorems are equivalent to the corresponding local versions of classical Frobenius theorem. A formulation of an appropriate global foliation version of the Theorem 3.4, as well as its differential form formulation are of interest, but fall outside of the scope of this paper.

In his thesis [1], the first author proved Theorem 3.5 directly, using contractive maps and Picard type argument. In the current proof, Picard type argument is hidden in the existence and uniqueness result for a flow of vectors field. A weaker version of Theorem 3.5 (with right hand-sides of (19) independent of \( \phi \)'s) appears in Lee [9], Theorem 19.27.

The vector field generalization of the Frobenius Theorem, which we rigorously formulate below, states that, given a local partial frame in involution \( s_1, \ldots, s_m \) on an open subset \( \mathcal{O} \subset \mathbb{R}^{n+p} \), where integers \( 1 \leq m \leq n \) and \( p \geq 1 \), and an \( (n-m) \)-dimensional embedded submanifold \( \Lambda \subset \mathcal{O} \), not tangent to any of the given vector fields \( s \)'s, one can locally extend \( \Lambda \) to an \( n \)-dimensional submanifold \( \Gamma \), tangent to each of the \( s \)'s at every point. Moreover, such an extension is locally unique. If \( n = m \), then \( \Lambda \) is a single point and we get a statement which is equivalent to a local vector-field version of the classical Frobenius theorem.

**Theorem 3.4.** Let \( s_1, \ldots, s_m \) be a partial frame in involution defined on an open subset \( \mathcal{O} \subset \mathbb{R}^{n+p} \), where \( 1 \leq m \leq n \) and \( p \geq 1 \). Let \( \Lambda \subset \mathcal{O} \) be an \( (n-m) \)-dimensional embedded submanifold, such that

\[
\operatorname{span}_\mathbb{R}\{s_1|z, \ldots, s_m|z\} \oplus T_z\Lambda \cong \mathbb{R}^n \hspace{1cm} \tag{16}
\]

for every point \( z \in \Lambda \). Then for every point \( \bar{z} \in \Lambda \), there exists an open neighborhood \( \mathcal{O}_{\bar{z}} \subset \mathcal{O} \) and an \( n \)-dimensional embedded submanifold \( \Gamma_{\bar{z}} \) of \( \mathbb{R}^{n+p} \), such that

1) \( \Lambda \cap \mathcal{O}_{\bar{z}} = \Lambda \cap \Gamma_{\bar{z}} \); 
2) \( s_i|z \in T_z\Gamma_{\bar{z}} \), for all \( i = 1, \ldots, m \) and for every point \( z \in \Gamma_{\bar{z}} \).

Manifold \( \Gamma_{\bar{z}} \) is locally unique, i.e. if there is another \( n \)-dimensional manifold \( \Gamma'_{\bar{z}} \), satisfying the two conditions stated above, then \( \Gamma_{\bar{z}} \cap \Gamma'_{\bar{z}} \) is also an \( n \)-dimensional manifold satisfying these conditions.

**Proof.** Since \( s_1, \ldots, s_m \) are in involution then, on an open neighborhood \( \mathcal{O}'_{\bar{z}} \subset \mathbb{R}^{n+p} \) of \( \bar{z} \in \Lambda \), by Proposition 3.1, there exists a partial commutative frame \( \tilde{s}_1, \ldots, \tilde{s}_m \), such that \( \operatorname{span}_\mathbb{R}\{\tilde{s}_1|z, \ldots, \tilde{s}_m|z\} = \)
span\{s_1, \ldots, s_m\} for all z ∈ O'_z. Since Λ is an embedded submanifold, by shrinking O'_z we may assume that \Lambda'_z = Λ ∩ O'_z is a coordinate neighborhood of \bar{z} in Λ, i.e. there exists an open set \mathcal{W}' ⊂ \mathbb{R}^{n-m} and a diffeomorphism

$$\psi: \mathcal{W}' \to \Lambda'_z,$$

such that \psi(0) = \bar{z}. Let \mathcal{B}_z = (-\varepsilon, \varepsilon)^m denote an open box in \mathbb{R}^m with sides 2\varepsilon centered at the origin. Let \exp^s(z) denote the flow of a vector field s on \mathcal{O}, i.e.

$$\frac{d}{de} \exp^s(z) = \mathbf{r}\big|_{\exp^s(z)}, \quad \exp^0\mathbf{s}(z) = z. \quad (17)$$

Then there exists an \varepsilon > 0, such that the map \Psi: \mathcal{W}' × \mathcal{B}_z → \mathbb{R}^{n+p}:

$$\Psi(w, \varepsilon) = \left(\exp^{s_1}(\bar{z}) \circ \cdots \circ \exp^{s_m}(\bar{z})\right)(\psi(w)). \quad (18)$$

is defined for all w ∈ \mathcal{W}' and ε ∈ \mathcal{B}_z. The map \Psi is smooth (see [8], pp. 371–379). Let \text{D}\Psi|_0: T_{(0)}\mathbb{R}^n → T_{\bar{z}}\mathbb{R}^{n+p} denote the differential of \Psi at the origin in \mathbb{R}^n and let \text{D}\psi|_0: T_{(0)}\mathbb{R}^{n-m} \to T_{\bar{z}}\mathbb{R}^{n+p} denote the differential of \psi at the origin in \mathbb{R}^{n-m}. Then vectors

$$\text{D}\Psi|_0 \left(\frac{\partial}{\partial w^i}\right) = \frac{\partial}{\partial w^i} \bigg|_{(w, \varepsilon) = 0} \Psi = \frac{\partial}{\partial w^i} \bigg|_{w = 0} \psi = \text{D}\psi|_0 \left(\frac{\partial}{\partial w^i}\right), \quad \text{where } i = 1, \ldots, n-m,$$

span the tangent space \text{T}_{\bar{z}}\Lambda. On the other hand,

$$\text{D}\Psi|_0 \left(\frac{\partial}{\partial \varepsilon^j}\right) = \frac{\partial}{\partial \varepsilon^j} \bigg|_{\varepsilon^j = 0} \exp^{s_j}(\bar{z}) = \bar{s}_j|_{\bar{z}}, \quad \text{for } j = 1, \ldots, m.$$ 

Therefore, due to (16), \text{D}\Psi|_0 has maximal rank n at 0 ∈ \mathbb{R}^n. Then, there exists an open subset of \mathcal{U} ⊂ \mathcal{W}' × \mathcal{B}_z ⊂ \mathbb{R}^n containing the origin, such that the restriction \Psi|_{\mathcal{U}}: \mathcal{U} → O_\bar{z} is an injective immersion. Define \Gamma_{\bar{z}} = \Psi(\mathcal{U}). By construction, \Gamma_{\bar{z}} is an n-dimensional submanifold of \mathbb{R}^{n+p}.

We next show that \Gamma_{\bar{z}} satisfies the tangency property 2) of the theorem, i.e. \text{s}_j|_{\bar{z}} ∈ T_{\bar{z}}\Gamma_{\bar{z}}, for all j = 1, ..., m and for every point \bar{z} ∈ Γ_{\bar{z}}. Since \Psi is an injective immersion, for any \bar{z} ∈ Γ_{\bar{z}}, there exists unique (w, \varepsilon) ∈ \mathcal{U}, such that \bar{z} = \Psi(w, \varepsilon). Since commutativity of the vector fields implies commutativity of the flows, we can pull exp^j \bar{s}_j to the most left in (18). Then, by (17) of an integral curve:

$$\text{D}\Psi|_{(w, \varepsilon)} \left(\frac{\partial}{\partial \varepsilon^j}\right) = \frac{\partial}{\partial \varepsilon^j} \bigg|_{(w, \varepsilon)} \exp^{s_j}(\bar{z}) \circ \cdots \circ \exp^{s_{j-1}}(\bar{z}) \circ \exp^{s_j}(\bar{z}) \circ \cdots \circ \exp^{s_1}(\bar{z}) \circ \psi(w) = s_j|_{\bar{z}}.$$ 

It remains to construct \mathcal{O}_{\bar{z}}, such that the intersection property 1) of the theorem is satisfied. Let \mathcal{W} be the projection of \mathcal{U} = \mathcal{U} ∩ (\mathcal{W}' × \{0\}) on \mathbb{R}^{n-m}. Then \mathcal{W} ⊂ \mathcal{W}' ⊂ \mathbb{R}^{n-m} is an open subset containing the origin. Define \Lambda_{\bar{z}} := \psi(\mathcal{W}), then \Lambda_{\bar{z}} is coordinate a chart on Λ centered at \bar{z}. By construction, \Lambda_{\bar{z}} = \Psi(\mathcal{U}) = \Lambda ∩ Γ_{\bar{z}}. Since Λ is embedded, there exists open subset \mathcal{O}_{\bar{z}} ⊂ \mathcal{O}'_{\bar{z}} ⊂ \mathbb{R}^{n+p} such that \Lambda_{\bar{z}} = \mathcal{O}_{\bar{z}} ∩ Λ. Then, by construction, \Lambda_{\bar{z}} = Λ ∩ Γ_{\bar{z}} = \Lambda ∩ \mathcal{O}_{\bar{z}}.

Local uniqueness of \Gamma_{\bar{z}} follows from the uniqueness of the integral curve of a given vector field originating at a given point, combined with the fact that any submanifold tangent to s_j must contain an open interval of the integral curve of s_j originating at each point of the submanifold.

We now formulate and prove a PDE version of the generalized Frobenius Theorem. A PDE system on p functions of n variables, considered in this theorem, prescribes derivative of each unknown function in the directions of m ≤ n vector fields comprising an involutive partial
frame. We will call such systems to be of \textit{generalized Frobenius type}. The theorem claims that under some natural integrability conditions, there is a unique solution of this system with an initial data prescribed along an \( m \)-dimensional manifold transversal to the given partial frame. For \( n = m \), this theorem is equivalent to the classical PDE version of the Frobenius Theorem (Theorem 6.1 in \cite{15}).

**Theorem 3.5** (Generalized Frobenius). Let \( \mathcal{R} = \{r_1, \ldots, r_m\} \) be a partial frame in involution on an open subset \( \Omega \subset \mathbb{R}^n \) with coordinates \((u^1, \ldots, u^n)\). Let \( \Theta \subset \mathbb{R}^p \) be an open subset with coordinates \((\phi^1, \ldots, \phi^p)\). Let \( h^i_j, i = 1, \ldots, p, \ j = 1, \ldots, m \), be given smooth functions on \( \Omega \times \Theta \). Consider a system of differential equations:

\[
 r_j(\phi^i(u)) = h^i_j(u, \phi(u)), \quad i = 1, \ldots, p; \ j = 1, \ldots, m. \tag{19}
\]

Assume the following integrability conditions

\[
 r_j (r_k (\phi^i)) - r_k (r_j (\phi^i)) = \sum_{l=1}^{m} c^l_{jk} r_l (\phi^i) \quad i = 1, \ldots, p; \ j, k = 1, \ldots, m, \tag{20}
\]

where the functions \( c^l_{jk} \)'s are defined by

\[
 [r_j, r_k] = \sum_{l=1}^{m} c^l_{jk} r_l, \tag{21}
\]

are identically satisfied on \( \Omega \times \Theta \) after substitution of \( h^i_j(u, \phi) \) for \( r_j(\phi^i(u)) \) as prescribed by the system \( \ref{19} \).

Then for any point \( \bar{u} \in \Omega \) and for any smooth initial data prescribed along any embedded submanifold \( \Xi \subset \Omega \) of codimension \( m \) containing \( \bar{u} \) and transversal to \( \mathcal{R} \), there is a unique smooth local solution of \( \ref{19} \). In other words, given arbitrary functions \( (g^1, \ldots, g^p) : \Xi \to \Theta \), there is an open subset \( \Omega' \subset \Omega \), containing \( \bar{u} \) and smooth functions \( (\alpha^1, \ldots, \alpha^p) : \Omega' \to \Theta \) satisfying \( \ref{19} \), such that \( \alpha^i|_{\Xi \cap \Omega} = g^i, i = 1, \ldots, p \).

**Proof.** Before staring a proof, we expand conditions \( \ref{20} \). After the first substitution of the derivatives of \( \phi^i \)'s as prescribed by \( \ref{19} \) into \( \ref{20} \), we get for \( i = 1, \ldots, p; \ j, k = 1, \ldots, m \):

\[
 r_j (h^i_k(u, \phi(u))) - r_k (h^i_j(u, \phi(u))) = \sum_{l=1}^{m} c^l_{jk} h^i_l(u, \phi(u)) \tag{22}
\]

Using the chain rule and again making a substitution prescribed by \( \ref{19} \) for the derivatives of \( \phi^i \)'s we get:

\[
 \sum_{l=1}^{n} \left( \frac{\partial h^i_k(u, \phi)}{\partial u^l} r_j(u^l) - \frac{\partial h^i_j(u, \phi)}{\partial u^l} r_k(u^l) \right) + \sum_{s=1}^{p} \left( \frac{\partial h^i_s(u, \phi)}{\partial \phi^s} h^i_j(u, \phi) - \frac{\partial h^i_j(u, \phi)}{\partial \phi^s} h^i_k(u, \phi) \right) = \sum_{l=1}^{m} c^l_{jk}(u) h^i_l(u, \phi). \tag{23}
\]

In order to use Theorem 3.4 for \( j = 1, \ldots, m \), we define vector-fields

\[
 s_j = r_j + \sum_{i=1}^{p} h^i \frac{\partial}{\partial \phi^i}.
\]

\(^2\) The resulting equations, explicitly written down as \( \ref{23} \), involve no derivatives of \( \phi \).

\(^3\) Here transversality means that \( \text{span}_{\mathbb{R}} \{r_1|_{\bar{u}}, \ldots, r_m|_{\bar{u}}\} \oplus T_{\bar{u}} \Xi = \mathbb{R}^n \) at very point \( \bar{u} \in \Xi \), where \( T_{\bar{u}} \Xi \) denotes the tangent space to \( \Xi \) at \( \bar{u} \).
on the open subset $\Omega \times \Theta \subset \mathbb{R}^{n+p}$. Independence of $s_1, \ldots, s_m$ follows from independence of $r_1, \ldots, r_m, \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}$. Using (23) and (24), we can show that $s_1, \ldots, s_m$ are in involution, and in fact satisfy the same structure equations as $r_1, \ldots, r_m$. Indeed, for all $j, k = 1, \ldots, m$:

$$[s_j, s_k] = [r_j, r_k] + \sum_{i=1}^p \left( r_j (h^i_k) - r_k (h^i_j) \right) \frac{\partial}{\partial u^i} = \sum_{l=1}^m c^l_{jk} \left( r_l + \sum_{i=1}^p h^i_l \frac{\partial}{\partial u^i} \right) = \sum_{l=1}^m c^l_{jk} s_l.$$

Define

$$\Lambda = \{(u, g(u)) \mid u \in \Xi\},$$

to be the graph of the map $g = (g^1, \ldots, g^p) : \Xi \to \mathbb{R}^p$. Then $\Lambda$ is an $(n-m)$-dimensional embedded submanifold of $\mathbb{R}^{n+p}$, such that (19) is satisfied. Thus by Theorem 5.9 there exist an open neighborhood $\mathcal{O} \subset \Omega \times \Theta$ of $\bar{z} = g(\bar{u})$ an $n$-dimensional manifold $\Gamma$, through the point $\bar{z} = g(\bar{u})$, such that $\Gamma \cap \Lambda = \Omega \cap \Lambda$ and vector-fields $s_1, \ldots, s_m$ are tangent to $\Gamma$ at every point of $\Gamma$. By possibly shrinking $\mathcal{O}$ and $\Gamma$ around $\bar{z}$, we may assume that

$$\Gamma = \{(u, \alpha(u)) \mid u \in \Omega'\},$$

is a graph of a map $\alpha = (\alpha^1, \ldots, \alpha^p) : \Omega' \to \Theta$, where $\Omega'$ is an open subset of $\Omega$, equal to the projection of $\mathcal{O}$ to $\mathbb{R}^n$. Since $\Gamma \cap \Lambda = \Omega \cap \Lambda$, we have $\alpha_{|\Xi \cap \Omega'} = g$. Then since $s_1, \ldots, s_m$ are tangent to $\Gamma$, we have

$$s_j \left( \phi^i - \alpha^i(u) \right) = 0 \text{ for all } u \in \Omega' \text{ and } i = 1, \ldots, p; \ j = 1, \ldots, m,$$

which is equivalent to

$$r_j (\alpha^i) = h^i_j (\alpha^i(u), u), \text{ for all } u \in \Omega' \text{ and } i = 1, \ldots, p; \ j = 1, \ldots, m,$$

and therefore $\alpha : \Omega' \to \Theta$ is a solution of the PDE system (19). Its uniqueness follows from the local uniqueness of $\Gamma$. \[ \square \]

We conclude this section by stating another integrability theorem, appeared as Theorem III in Book III, Chapter I of [4]. The PDE system on $p$ functions of $n$-variables, considered in this theorem, prescribes some subset of partial derivatives of each unknown function. A subset of derivatives prescribed for one of the unknown functions, may differ from a subset prescribed for the other. We will call such systems to be of the Darboux type. The Darboux theorem claims that provided the natural integrability conditions are satisfied, there is a unique solution for an appropriately prescribed initial data.

**Theorem 3.6.** (Darboux [3]) Let $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^p$ be open subsets, let $\bar{u} = (\bar{u}^1, \ldots, \bar{u}^p) \in \Omega$ be a fixed point, and let $h^i_j, \ i = 1, \ldots, p; \ j = 1, \ldots, m,$ be some given smooth functions on $\Omega \times \Theta$. Consider a system of differential equations on unknown functions $(\phi^1, \ldots, \phi^p) : \Omega \to \Theta$ of independent variables $u^1, \ldots, u^n$:

$$\frac{\partial \phi^i}{\partial u^j} = h^i_j (u, \phi), \quad j \in S_i, \ i = 1, \ldots, p. \quad (24)$$

Assume that the system prescribes compatible second order mixed derivatives in the following sense:

1. **C** Whenever two distinct derivatives $\frac{\partial \phi^i}{\partial u^j}$ and $\frac{\partial \phi^j}{\partial u^i}$ of the same unknown $\phi^i$ are present on the left hand side of (24), then the equation

$$\frac{\partial}{\partial u^k} \left[ h^i_j (u, \phi(u)) \right] = \frac{\partial}{\partial u^j} \left[ h^i_k (u, \phi(u)) \right]$$

contains (after expanding each side using the chain rule) only first order derivatives which appear in (24), and substitution from (24) for these first derivatives results in an identity in $u$ and $\phi$. \[ \square \]
Next, to describe the data, suppose a dependent variable $\phi^i$ appears differentiated in (24) with respect to $u^1, \ldots, u^n$. Then, letting $\bar{u}$ denote the remaining independent variables, we prescribe a smooth function $g^i(\bar{u})$ and require that

$$
\phi^i(u^1, \ldots, u^n) \bigg|_{u^1=\bar{u}^1, \ldots, u^n=\bar{u}^n} = g^i(\bar{u}).
$$

We make such an assignment of data for each $\phi^i$ that appears differentiated in (24). Then, under the compatibility condition (C), the problem (24) - (25) has a unique, local smooth solution for $u$ near $\bar{u}$.

**Remark 3.7.** If partial derivatives of all unknown functions are prescribed for the same set coordinates directions (i.e. $S_1 = \cdots = S_p$), the Darboux type system is of the generalized Frobenius type. Conversely, using Propositions 3.1 and 3.2, one can show that for any system of the generalized Frobenius type there is an equivalent Darboux type system, with all partial derivatives of all unknown functions prescribed for the same set coordinates directions. In this case, integrability conditions (C) of Theorem 3.6 are equivalent to integrability conditions in Theorem 3.5. However, the manifold $\Xi$ along which the initial data is allowed to be prescribed in Theorem 3.5 is more general than the coordinate subspace for which the data is prescribed in Theorem 3.6.

### 4 Coordinate-free formulation of the problem

In this section, we give an intrinsic (coordinate independent) formulation of Problem 1 which leads to a system of differential equations written in terms of the frame adapted to the problem. We derive some differential consequences of this system, which, in particular, lead to a set of necessary conditions for the existence of strictly hyperbolic fluxes.

#### 4.1 Intrinsic definition of the Jacobian and the $\mathcal{F}(\mathfrak{H})$-system

We start with an intrinsic definition of the Jacobian map $\mathcal{X}(\Omega) \to \mathcal{X}(\Omega)$ adapted from Remark 2.15 in [6].

**Definition 9.** Given a connection $\nabla$ on a smooth manifold $M$, the $\nabla$-Jacobian of a vector field $f \in \mathcal{X}(M)$ is the $C^\infty(M)$-linear map $J_f : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega)$ defined by:

$$
J_f(s) = \nabla_s f, \quad \forall s \in \mathcal{X}(\Omega).
$$

(26)

If $\{r_1, \ldots, r_n\}$ is a frame with Christoffel symbols $\Gamma^k_{ij}$ and $f = \sum_{i=1}^n F^i r_i$ then (26) implies:

$$
J_f(r_j) = \sum_{i=1}^n \left( r_j (F^i) + \sum_{k=1}^n \Gamma^i_{jk} F^k \right) r_i.
$$

(27)

Let $(u^1, \ldots, u^n)$ is an affine system of coordinates (see [11]) relative to a flat symmetric connection $\tilde{\nabla}$, let $f = \sum_{i=1}^n \tilde{F}^i(u) \frac{\partial}{\partial u^i}$, and let $\tilde{J}_f$ denote the $\tilde{\nabla}$-Jacobian of $f$. Then a direct computation shows that

$$
\tilde{J}_f \left( \frac{\partial}{\partial u^j} \right) = \sum_{i=1}^n \frac{\partial \tilde{F}^i}{\partial u^j} \frac{\partial}{\partial u^i},
$$

which corresponds exactly to the $j$-th column vector of the usual Jacobian matrix $[D_u \tilde{F}]$ of the vector valued function $\tilde{F}(u) = [\tilde{F}^1(u), \ldots, \tilde{F}^n(u)]^T$.

Using the intrinsic definition of the Jacobian, we give an equivalent intrinsic formulation of Problem 1 which allows us to analyze it relative to a frame that is adapted to the problem.
Problem 2. Given a partial frame $\mathcal{F} = \{r_1, \ldots, r_m\}$ on an open subset $\Omega \subset \mathbb{R}^n$ ($n \geq m$), with a flat symmetric connection $\nabla$, and a fixed point $\bar{u} \in \Omega$; describe the set $\mathcal{F}(\mathcal{F})$ of smooth vector fields $f$ for which there exist an open neighborhood $\Omega' \subset \Omega$ of $\bar{u}$ and smooth functions $\lambda^i : \Omega' \to \mathbb{R}$, such that
\[
\nabla_{r_i} f = \lambda^i r_i, \quad \text{on } \Omega' \quad \text{for } i = 1, \ldots, m.
\] (28)

Remark 4.1. Problem 2 makes sense if we replace $\mathbb{R}^n$ with an arbitrary manifold $M$, and replace $\nabla$ with an arbitrary connection on the tangent bundle of $M$. In particular, it would be of interest to consider this problem on a Riemannian manifold with the Riemannian connection. These generalizations, however, fall outside of the scope of the current paper.

From Proposition 3.3 we know that any flat and symmetric connection admits an affine system of coordinates. If $F^1, \ldots, F^n$ are the components of $f$, and $R^1_i, \ldots, R^n_i$ are the components of $r_i$ in an affine system of coordinates, then (28) turns in a system of $mn$ first order PDE’s on $n + m$ unknown functions $F$’s and $\lambda$’s:
\[
\begin{align*}
  r_i(F^j) &= \lambda^i R^j_i, & &\text{for } i = 1, \ldots, m, \ j = 1, \ldots, n, \\
\end{align*}
\] (29)
which is equivalent to (23), because $r_i(F^j) = \sum R^j_i \frac{\partial F^j}{\partial u^i}$. Therefore, Problem 2 is, indeed, a coordinate-free formulation of Problem 1 and we can call system (28) the $\mathcal{F}(\mathcal{F})$-system. The set of vector fields satisfying (28) will be denoted $\mathcal{F}(\mathcal{F})$, and the elements of this set will be called fluxes for $\mathcal{F}$, the set fluxes always contains the set of identity fluxes $\mathcal{F}^{id}$, which we define by the property:
\[
\nabla_r f = r \quad \text{for all vector fields } r \in \mathcal{X}(\Omega). \quad \text{(30)}
\]

One can easily show that $\hat{f} \in \mathcal{F}^{id}$ if and only if relatively to any affine coordinates system $(u^1, \ldots, u^n)$:
\[
\hat{f} = [u^1, \ldots, u^n]^T + \bar{b}, \quad \text{for some } \bar{b} \in \mathbb{R}^n.
\]
The previously defined vector space of trivial fluxes (6), in this more abstract setting, corresponds to the vector space
\[
\mathcal{F}^{triv} = \{ f \in \mathcal{X}(\Omega) \mid \forall r \in \mathcal{X}(\Omega) \exists \lambda \in \mathbb{R}, \text{ such that } \nabla_r f = \lambda r \}. \quad \text{(31)}
\]
Equivalently, one can say that $\mathcal{F}^{triv} = \{ \lambda \hat{f} \mid \lambda \in \mathbb{R}, \hat{f} \in \mathcal{F}^{id} \}$ and, clearly, $\mathcal{F}^{id} \subset \mathcal{F}^{triv} \subset \mathcal{F}(\mathcal{F})$ for any partial frame $\mathcal{F}$.

4.2 Differential consequences of the $\mathcal{F}(\mathcal{F})$-system

We now derive the differential consequences of (28) implied by the flatness of the connection.

Proposition 4.2. Given a partial frame $\mathcal{F} = \{r_1, \ldots, r_m\}$, assume that $f \in \mathcal{F}(\mathcal{F})$ is a flux, and $s_1, \ldots, s_{n-m}$ is any completion of $\mathcal{F}$ to the full frame. Let the functions $a^i_k$ be defined by
\[
\nabla_{s_l} f = \sum_{k=1}^m a^i_k r_k + \sum_{k=m+1}^n a^i_k s_l, \quad l = 1, \ldots, n - m. \quad \text{(32)}
\]
Then the functions $\lambda^i, i = 1, \ldots, m$, prescribed by (28), and the functions $a^i_k, l = 1, \ldots, n - m, \ k = 1, \ldots, n$, satisfy the system of differential and algebraic equations:
\[ r_i(\lambda^j) = \Gamma^j_{ji}(\lambda^i - \lambda^j) + \sum_{l=m+1}^{n} a^l_i c^l_{ij} \text{ for all } 1 \leq i \neq j \leq m \] (33)

\[ \lambda^j \Gamma^k_{ij} - \lambda^i \Gamma^k_{ji} - c^k_{ij} \lambda^k = \sum_{l=m+1}^{n} a^k_i c^l_{ij} \text{ for all distinct triples } i, j, k \in \{1, \ldots, m\} \] (34)

\[ \lambda^j \Gamma^l_{ij} - \lambda^i \Gamma^l_{ji} = \sum_{l=m+1}^{n} a^l_i c^l_{ij} \text{ for all } 1 \leq i \neq j \leq m \text{ and } l = m + 1, \ldots, n. \] (35)

In the above equations, c’s and \( \Gamma \)'s are the structure functions and the Christoffel symbols for the frame:

\[
[r_i, r_j] = \sum_{k=1}^{m} c_{ij}^k r_k + \sum_{l=m+1}^{n} c_{ij}^l s_l,
\]

(36)

\[
\nabla_{r_i} r_j = \sum_{k=1}^{m} \Gamma^k_{ij} r_k + \sum_{l=m+1}^{n} \Gamma^l_{ij} s_l.
\]

(37)

Proof. Flatness condition (11) implies that

\[ \nabla_{r_i} \nabla_{r_j} f - \nabla_{r_j} \nabla_{r_i} f = \nabla_{[r_i, r_j]} f \text{ for all } i, j = 1, \ldots, m \] (38)

must hold on the solutions of (28), and, therefore,

\[ r_i(\lambda^j) r_j + \lambda^j \nabla_{r_i} r_j - r_j(\lambda^i) r_i - \lambda^i \nabla_{r_j} r_i = \nabla_{[r_i, r_j]} f \] (39)

Using (36) and (37), we obtain that (39) is equivalent to:

\[
\begin{align*}
& r_i(\lambda^j) r_j + \sum_{k=1}^{m} \lambda^j \Gamma^k_{ij} r_k + \sum_{l=m+1}^{n} \lambda^j \Gamma^l_{ij} s_l - r_j(\lambda^i) r_i - \sum_{k=1}^{m} \lambda^i \Gamma^k_{ji} r_k - \sum_{l=m+1}^{n} \lambda^i \Gamma^l_{ji} s_l \\
& = \sum_{k=1}^{m} c_{ij}^k \nabla_{r_k} f + \sum_{l=m+1}^{n} c_{ij}^l \nabla_{s_l} f
\end{align*}
\] (40)

It remains to rewrite the right-hand side of (40) in terms of the frame using (28) for the first sum and (32) for the second sum:

\[
\begin{align*}
& r_i(\lambda^j) r_j + \sum_{k=1}^{m} \lambda^j \Gamma^k_{ij} r_k + \sum_{l=m+1}^{n} \lambda^j \Gamma^l_{ij} s_l - r_j(\lambda^i) r_i - \sum_{k=1}^{m} \lambda^i \Gamma^k_{ji} r_k - \sum_{l=m+1}^{n} \lambda^i \Gamma^l_{ji} s_l \\
& = \sum_{k=1}^{m} c_{ij}^k \lambda^j r_k + \sum_{k=1}^{m} \sum_{l=m+1}^{n} a^k_i c^l_{ij} r_k + \sum_{l=m+1}^{n} a^l_i c^l_{ij} s_l
\end{align*}
\]

Collecting coefficients of the frame vector fields, we get a system of differential and algebraic equations (33) - (35).

It is worthwhile emphasizing that, in general, the structure functions c’s and the Christoffel symbols \( \Gamma \)’s appearing in (33) - (35), depend on the completion of \( \mathfrak{X} \) to a full frame.

Remark 4.3. We note that equations (33) - (35) do not provide a complete set of integrability conditions for the Frobenius-type system (28), (32), because it does not include conditions derived from \( \nabla_{r_i} \nabla_{s_j} f - \nabla_{s_j} \nabla_{r_i} f = \nabla_{[r_i, s_j]} f \) and \( \nabla_{s_i} \nabla_{s_j} f - \nabla_{s_j} \nabla_{s_i} f = \nabla_{[s_i, s_j]} f \). We will derive these
additional conditions, in Section 4 for \( m = 2, n = 3 \) case only, and we will observe how technical they become even in low dimensions.

However, we will see in Section 4 that if \( \mathcal{R} \) is an involutive partial frame, then \( (33) - (35) \) simplify to a system which involves unknown functions \( \lambda s \) only, and this system does provide a complete set of integrability conditions for \( (23) \). In the case of the full frame \( (m = n) \), equations \( (33) - (35) \) reduce to the \( \lambda \)-system introduced in \([7]\).

We can use equations \( (33) - (35) \) to obtain necessary conditions for \( \mathcal{F}(\mathcal{R}) \) to contain a strictly hyperbolic flux. As we will see below, these conditions are not sufficient except for the case of rich partial frames.

**Proposition 4.4** (necessary condition for strict hyperbolicity). Let \( \mathcal{R} = \{r_1, \ldots, r_m\} \) be a partial frame on \( \Omega \subset \mathbb{R}^n \) containing \( \bar{u} \). If there is a strictly hyperbolic flux \( f \in \mathcal{F}(\mathcal{R}) \) on some open neighborhood \( \Omega' \) of \( \bar{u} \) then for each pair of indices \( i \neq j \in \{1, \ldots, m\} \) the following equivalence condition holds:

\[
\nabla r_i r_j \in \text{span}_{C^\infty(\Omega')} \{r_i, r_j\} \quad \text{if and only if} \quad [r_i, r_j] \in \text{span}_{C^\infty(\Omega')} \{r_i, r_j\}
\]

(41)

**Proof.** If \( f \) is strictly hyperbolic on \( \Omega' \), then \( \mathcal{R} \) can be completed to a frame of eigenvectors \( r_1, \ldots, r_m, r_{m+1}, \ldots, r_n \), such that there exist functions \( \lambda^1, \ldots, \lambda^n : \Omega' \to \mathbb{R} \), with all distinct values at each point of \( \Omega' \), and

\[
\nabla r_i f = \lambda^i r_i, \quad i = 1, \ldots, n.
\]

In the statement of Proposition 4.2 let \( s_l = r_l \) for \( l = m + 1, \ldots, n \). Then \( a^i_l = \delta^i_l \lambda^l \), where \( \delta^i_l \) is the Kronecker delta function, and the algebraic conditions (34), (35) become

\[
\Gamma^k_{ij} \lambda^i - \Gamma^k_{ji} \lambda^j - c^k_{ij} \lambda^k = 0 \quad \text{for all} \quad 1 \leq i \neq j \leq m \quad \text{and} \quad 1 \leq k \leq n, \quad \text{such that} \quad k \neq i \quad \text{and} \quad k \neq j.
\]

(42)

Let us first assume that for some \( i, j \), such that \( 1 \leq i \neq j \leq m \), we have \( \nabla r_i r_j \in \text{span}_{C^\infty(\Omega')} \{r_i, r_j\} \) and \([r_i, r_j] \notin \text{span}_{C^\infty(\Omega')} \{r_i, r_j\}\). Then, from the latter condition, there exists \( k \in \{1, \ldots, n\} \), such that \( k \neq i \) and \( k \neq j \) and \( c^k_{ij} \neq 0 \), while the former condition implies that \( \Gamma^k_{ij} \equiv 0 \). Symmetry of \( \nabla \) implies that \( c^k_{ij} = -\Gamma^k_{ji} \neq 0 \), and then from (42) we have

\[
c^k_{ij} (\lambda^i - \lambda^j) \equiv 0.
\]

We then have \( \lambda^i = \lambda^j \) at least somewhere in \( \Omega' \), which contradicts our strict hyperbolicity assumption.

Let us now assume that for some \( i, j \) such that \( 1 \leq i \neq j \leq m \), we have \( \nabla r_i r_j \notin \text{span}_{C^\infty(\Omega')} \{r_i, r_j\} \) and \([r_i, r_j] \in \text{span}_{C^\infty(\Omega')} \{r_i, r_j\}\). Then, from the former condition, there exists \( k \in \{1, \ldots, n\} \), such that \( k \neq i \) and \( k \neq j \) and \( \Gamma^k_{ij} \neq 0 \), but from the latter condition we have \( c^k_{ij} \equiv 0 \). Symmetry of \( \nabla \) implies that \( \Gamma^k_{ij} = -\Gamma^k_{ji} \), and then from (42) we have

\[
\Gamma^k_{ij} (\lambda^i - \lambda^j) \equiv 0.
\]

We then have \( \lambda^i = \lambda^j \) at least somewhere in \( \Omega' \), which contradicts our strict hyperbolicity assumption. \( \square \)

## 5 Involutive partial frame

As it is discussed in Remark 4.3 above, the analysis of the \( \mathcal{F}(\mathcal{R}) \) system is much simpler when the partial frames \( \mathcal{R} \) is in involution. Partial frames of two “extreme” sizes: \( m = 1 \), that is all partial frames consisting of a single vector field, and \( m = n \), that is all full frames, fall into this
5.1 Arbitrary involutive partial frames

If a given partial frame \( \mathcal{R} \) is in involution, then for any completion of \( \mathcal{R} \) to a full frame \( \{r_1, \ldots, r_m, s_{m+1}, \ldots, s_n\} \), we have \( c_{ij}^k = 0 \) for all \( i, j = 1, \ldots, m, l = m+1, \ldots, n \) and, therefore, \( \Gamma^l_{ij} = \Gamma^l_{ji} \) due to the symmetry of the connection \((10)\). Then \((43) - (45)\) simplify to

\[
\begin{align*}
r_i(\lambda^j) &= \Gamma^j_{ji} (\lambda^i - \lambda^j) \quad \text{for all } 1 \leq i \neq j \leq m \\
\lambda^i \Gamma^k_{ij} - \lambda^j \Gamma^k_{ji} - c^k_{ij} \lambda^k &= 0 \quad \text{for all distinct triples } i, j, k \in \{1, \ldots, m\} \\
(\lambda^i - \lambda^j) \Gamma^l_{ji} &= 0 \quad \text{for all } 1 \leq i \neq j \leq m \text{ and } l = m+1, \ldots, n.
\end{align*}
\]

where \( c^i_\cdot \)'s and \( \Gamma^i_\cdot \)'s are defined by \((36)\) and \((37)\), respectively. Note that, due to involutivity of \( \mathcal{R} \), functions \( c^i_\cdot \), \( i, j, k \in \{1, \ldots, m\} \) do not depend on a choice of completion of \( \mathcal{R} \) to a frame, while \( \Gamma^i_\cdot \)'s, in general, do depend on a choice of such completion. We will call \((43) - (45)\) the \( \lambda \)-system, generalizing the terminology of \([7]\) to partial involutive frames.

The following proposition allows us, in the involutive case, to solve Problems 2 (and 1) in two steps: first find (or describe the set of) all solutions \( \lambda \) of system \((43) - (45)\), and then find (or describe the set of) all solutions \( f \) of \((28)\) for a given set of functions \( \lambda \)'s, satisfying \((43) - (45)\). This is possible because in the involutive case, equations \((43) - (45)\) provide a complete set of the integrability conditions for the \( \mathcal{F}(\mathcal{R}) \)-system \((28)\), as we show in the proof of the following proposition.

**Proposition 5.1 (\( \lambda(\mathcal{R}) \)-system).** If a partial frame \( \mathcal{R} = \{r_1, \ldots, r_m\} \) is in involution, then

1) For every \( f \in \mathcal{F}(\mathcal{R}) \), functions \( \lambda^1, \ldots, \lambda^m \) prescribed by \((28)\) satisfy \((43) - (45)\).

2) For every solution \( \lambda^1, \ldots, \lambda^m \) of \((43) - (45)\), and any smooth initial data for \( f \) prescribed along any embedded submanifold \( \Xi \subset \Omega \) of codimension \( m \) transverse to \( \mathcal{R} \), there is a unique smooth local solution of the \( \mathcal{F}(\mathcal{R}) \)-system \((28)\). In other words, given arbitrary smooth vector field \( \tilde{f} \) on \( \Xi \), there is an open subset \( \Omega' \subset \Omega \), containing \( \Xi \) and unique smooth extension \( f \) of \( \tilde{f} \) to \( \Omega' \) satisfying \((28)\).

**Proof.**

1) Equations \((43) - (45)\) are differential consequences of \((28)\), and, therefore, for every \( f \in \mathcal{F}(\mathcal{R}) \), functions \( \lambda^1, \ldots, \lambda^m \) prescribed by \((28)\) satisfy \((43) - (45)\).

2) Assume \( \lambda^1, \ldots, \lambda^m \) are solutions of \((43) - (45)\). In an affine system of coordinates \( u = (u^1, \ldots, u^n) \), equations \((28)\) turn into \((29)\). To simplify the notation we rewrite them in a vector-form:

\[
r_i(F)|_u = \lambda^i(u) R_i(u), \quad \text{for } i = 1, \ldots, m,
\]

where we assume that \( F \) and \( R_i \) are column vectors of the components of the vector-fields \( f \) and \( r_i \), respectively relative to the coordinate frame \( \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n} \).

The above system is of the form \((110)\) described in generalized Frobenius Theorem \((35)\). The integrability conditions become:

\[
r_i(\lambda^j) (R_j) + \lambda^j r_i(R_j) - r_j(\lambda^i) (R_i) - \lambda^i r_j(R_i) = \sum_{k=1}^m c^k_{ij} \lambda^k R_k
\]

(47)
Recalling that in the affine coordinates we have formula (45) for covariant derivatives, we see that (47) is equivalent to

\[
\mathbf{r}_i(\mathbf{r}^j) \mathbf{r}_j + \lambda^j \overline{\nabla}_{\mathbf{r}_i} \mathbf{r}_j - \mathbf{r}_j(\lambda^j) \mathbf{r}_i - \lambda^i \overline{\nabla}_{\mathbf{r}_j} \mathbf{r}_i = \sum_{k=1}^{m} c^k_{ij} \lambda^k \mathbf{r}_k, \tag{48}
\]

which, when written out in components relative to a completion of \( \mathfrak{R} \) to a frame \( \mathbf{r}_1, \ldots, \mathbf{r}_m \), \( s_1, \ldots, s_n \), is equivalent to (43) – (45). Components of the vector field \( \mathbf{f} \) provide the data for \( F \) of the type described in Theorem 5.5 and this theorem guarantees the existence of a locally unique solution of (40) with this data.

\[ \square \]

System (43) – (45) always has solution \( \lambda^1 = \cdots = \lambda^m \), but existence of other solutions of (43) – (45) is a subtle question. Moreover, even for non-trivial solutions of (43) – (45), existence of hyperbolic and strictly hyperbolic fluxes is a subtle question.

We note that conditions (44) and (45) immediately provide us with necessary conditions to existence of strictly hyperbolic solutions for Problem 1, in the case of involutive partial frames.

**Proposition 5.2** (necessary condition for strict hyperbolicity in the involutive case). *If a partial frame \( \mathfrak{R} = \{ \mathbf{r}_1, \ldots, \mathbf{r}_m \} \) is in involution, then the following conditions must be satisfied for all \( 1 \leq i \neq j \leq m \) on some open neighborhood \( \Omega' \subset \Omega \) of \( \bar{u} \), in order for the \( \mathcal{F}(\mathfrak{R}) \) set to contain a strictly hyperbolic flux:

\[
\overline{\nabla}_{\mathbf{r}_i} \mathbf{r}_j \in \text{span}_{C^\infty(\Omega')} \{ \mathbf{r}_i, \mathbf{r}_j \} \iff [\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega')} \{ \mathbf{r}_i, \mathbf{r}_j \} \tag{49}
\]

and

\[
\mathbf{r}_i \mathbf{r}_j \in \text{span}_{C^\infty(\Omega')} \mathfrak{R}, \tag{50}
\]

As a side remark, we observe that involutivity implies that \([\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega')} \mathfrak{R}\), and hence, due to the symmetry condition (10), we can replace condition \( 1 \leq i \neq j \leq m \) in (50) with \( 1 \leq i < j \leq m \).

**Proof.** Condition (49) is the same as (41) proved earlier. If for all open subsets \( \Omega' \subset \Omega \), there are \( 1 \leq i \neq j \leq m \), such that \( \overline{\nabla}_{\mathbf{r}_i} \mathbf{r}_j \notin \text{span}_{C^\infty(\Omega')} \mathfrak{R} \), then there exists \( m+1 \leq l \leq n \), such that \( \Gamma^i_{ij} \neq 0 \) on \( \Omega' \). From (45), it then follows that \( \lambda^i = \lambda^j \) at least somewhere on \( \Omega' \) and therefore \( \mathcal{F}(\mathfrak{R}) \) contains no strictly hyperbolic fluxes. \[ \square \]

The above conditions are not sufficient as will be illustrated by Example 5.3 in [7]. However, we can prove the following condition is sufficient.

**Proposition 5.3** (sufficient condition for strict hyperbolicity in the involutive case). *Assume, that functions \( \lambda^1, \ldots, \lambda^m \) satisfying (43) – (45) are such that for some \( \bar{u} \in \Omega \), all \( m \) numbers \( \lambda^1(\bar{u}), \ldots, \lambda^m(\bar{u}) \) are distinct. Then on an open neighborhood \( \bar{u} \) there exists a strictly hyperbolic flux \( \mathbf{f} \), such that

\[
\overline{\nabla}_{\mathbf{r}_i} \mathbf{f} = \lambda^i \mathbf{r}_i, \quad i = 1, \ldots, m.
\]

**Proof.** Let \( R_i \) be the column vector of components of \( \mathbf{r}_i \) in an affine system of coordinates \( u^1, \ldots, u^n \), and let \( R = [R_1 \ldots R_m] \) be an \( n \times m \) matrix comprised of these column vectors. Since \( \mathbf{r}_i, i = 1, \ldots, m \) are independent at \( \bar{u} \), there is a non-zero \( n \times m \) minor of \( R(\bar{u}) \). Due to continuity the same minor is non-zero on some open neighborhood of \( \bar{u} \). Let \( \{i_1, \ldots, i_m\} \) be the row indices of the submatrix corresponding to this minor. Up to permuting coordinate functions \( u^1, \ldots, u^n \) we may, in order to simplify the notation, assume that \( i_j = j \). Then the set of vector-fields \( \mathbf{r}_1, \ldots, \mathbf{r}_m, \frac{\partial}{\partial u^{i_1}}, \ldots, \frac{\partial}{\partial u^{i_m}} \) are independent and, therefore, a submanifold \( \Xi \) defined by \( u^i = \bar{u}^i \) for \( i = 1, \ldots, m \) is transversal to \( \mathfrak{R} \).
For \( l \in \{ m+1, \ldots, n \} \), choose arbitrary constants \( \bar{\lambda}^l \), such that all \( n \) real numbers \( \lambda^1(\bar{u}), \ldots, \lambda^m(\bar{u}), \bar{\lambda}^{m+1}, \ldots, \bar{\lambda}^n \) are distinct. Define

\[
\bar{F}(\bar{u}^1, \ldots, \bar{u}^m, u^{m+1}, \ldots, u^n) = [0, \ldots, 0, \bar{\lambda}^{m+1} u^{m+1}, \ldots, \bar{\lambda}^n u^n]^T
\]

and let \( F \) be an extension of \( \bar{F} \) such that \( [D_uF] R_i(u) = \lambda^i R_i(u) \), where \( i = 1, \ldots, m \) guaranteed by Proposition 5.1. Then

\[
[D_uF](\bar{u}) = \begin{bmatrix}
\frac{\partial F_1}{\partial u^1}(\bar{u}) & \cdots & \frac{\partial F_1}{\partial u^m}(\bar{u}) & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
\frac{\partial F_m}{\partial u^1}(\bar{u}) & \cdots & \frac{\partial F_m}{\partial u^m}(\bar{u}) & \bar{\lambda}^{m+1} \\
\vdots & \vdots & \vdots & \ddots \\
\frac{\partial F_n}{\partial u^1}(\bar{u}) & \cdots & \frac{\partial F_n}{\partial u^m}(\bar{u}) & \bar{\lambda}^n
\end{bmatrix}
\]

where empty spaces are filled with zero. At the point \( \bar{u} \), the matrix \( [D_uF] \) has \( n \) distinct real eigenvalues \( \lambda^1(\bar{u}), \ldots, \lambda^m(\bar{u}), \bar{\lambda}^{m+1}, \ldots, \bar{\lambda}^n \). Since the entries of \( [D_uF] \) are smooth real functions, a standard argument, involving the implicit function theorem, implies that there is an open neighborhood \( \Omega' \subset \Omega \) of \( \bar{u} \), such that at every point of \( \Omega' \) the matrix \( [D_uF] \) has \( n \) distinct real eigenvalues, and, therefore, \( F \) is strictly hyperbolic on \( \Omega' \).

\[
\square
\]

Remark 5.4 (Single vector field case). When \( \mathcal{R} = \{ r_1 \} \), all three conditions (43) - (45) trivially hold. We, therefore, can assign \( \lambda^1 \) to be any function on \( \Omega \). Then, by Proposition 5.1, for every assignment of the vector field \( \bar{f} \) on an \((n-1)\)-dimensional manifold \( \Xi \), transverse to \( r_1 \), there exists unique local vector field \( f \) such \( \bar{\nabla}_{r_1} f = \lambda^1 r_1 \) and \( f|_{\Xi} = \bar{f}|_{\Xi} \). Thus the general solution of the \( \mathcal{F}(\mathcal{R}) \)-system (28) depends on one arbitrary function of \( n \)-variables (this is the functions \( \lambda^1 \)) and \( n \) functions of \( n-1 \) variables, that locally describe the initial data for the vector field \( f \). Due to Proposition 5.3, the \( \mathcal{F}(\mathcal{R}) \)-set contains strictly hyperbolic fluxes.

Remark 5.5 (Full frame). If \( \mathcal{R} \) is a full frame, it, of course, in involution. In this case (43) - (45) trivially holds and the remaining equations, (43) and (44), comprise an algebro-differential system, called the \( \lambda \)-system and analyzed in details in [7]. According to Proposition 5.1, for every solution of the \( \lambda \)-system and for every assignment of the vector field \( f \) at a point \( \bar{u} \in \Omega \), there exists a locally unique solution \( f \) of (28) such that \( f|_{\bar{u}} \) is prescribed. This can be also seen directly as follows. Since a full frame of eigenvector fields are given, once eigenfunctions are found, the Jacobian matrix \( [D_uF] \) can be immediately obtained. The \( i \)-th row of \( [D_uF] \) is the gradient of \( F^i \), and \( F^i \) itself can be recovered in the standard manner by solving a sequence of ODEs. If the value of \( F^i(\bar{u}) \) is prescribed, then functions \( F^i \) is unique.

5.2 Rich partial frame

Rich frames comprise a particularly nice subclass of involutive frames. Recall that according to Definition 5.1, a partial frame \( \mathcal{R} = \{ r_1, \ldots, r_m \}, 1 \leq m \leq n \), is called rich, if it is pairwise in involution: \( \{ r_i, r_j \} \in \text{span}\{ r_1, r_1 \} \) for all \( i, j \in \{ 1, \ldots, m \} \). This case trivially includes all partial frames consisting of a single vector-field. Also this case includes all involutive partial frames consisting of two vector fields.

Let \( \{ r_1, \ldots, r_m, s_{m+1}, \ldots, s_n \} \) to be any completion of \( \mathcal{R} \) to a frame and let, as usual, use \( c \) and \( \Gamma \) to denote the corresponding structure functions and Christoffel symbols of the connection \( \nabla \), respectively. Since \( \mathcal{R} \) is rich and due to the symmetry of the connection, we have

\[
c^l_{ij} = 0 \quad \text{and} \quad \Gamma^l_{ij} = \Gamma^l_{ji} \quad \text{for all distinct triples} \quad i, j, l, \quad \text{such} \quad 1 \leq i, j, m, \quad 1 \leq l \leq n.
\]
Then the $\mathcal{F}(\mathcal{R})$-system \ref{eq:13} - \ref{eq:15} becomes
\[
\mathbf{r}_i(\lambda^j) = \Gamma_{ij}^l (\lambda^i - \lambda^j) \quad \text{for all } 1 \leq i \neq j \leq m \tag{52}
\]
\[
\Gamma_{ij}^l (\lambda^i - \lambda^j) = 0 \quad \text{for all } 1 \leq i < j \leq m, 1 \leq l \leq n \text{ such that } l \neq i \text{ and } l \neq j. \tag{53}
\]

In the rich case, the necessary conditions for the $\mathcal{F}(\mathcal{R})$-set to contain strictly hyperbolic fluxes, spelled out in the Proposition \ref{prop:52} become
\[
\tilde{\nabla}_i r_j \in \text{span}_{C^\infty(\Omega)} \{ \mathbf{r}_i, \mathbf{r}_j \} \quad \text{for all } 1 \leq i \neq j \leq m. \tag{54}
\]

Theorem \ref{thm:57} shows that, for a rich partial frame, this necessary conditions are also sufficient. Moreover, for the frames that satisfy \ref{eq:54}, the proposition describes the "size" of the set $\mathcal{F}(\mathcal{R})$. Theorem \ref{thm:58} describes the "size" of the set $\mathcal{F}(\mathcal{R})$ for partial frames that do not satisfy \ref{eq:54}, and therefore, do not admit strictly hyperbolic fluxes.

The following lemma allows us to introduce a coordinate system adapted to a given rich partial frame and subsequently to invoke Darboux theorem to describe the solution set of the $\mathcal{F}(\mathcal{R})$-system.

**Lemma 5.6.** Assume a partial frame $\mathcal{R} = \{ \mathbf{r}_1, \ldots, \mathbf{r}_m \}$ on $\Omega$ is rich, then in a neighborhood of every point $\bar{u} \in \Omega$ there exist

1) positive scalar functions $\alpha^1, \ldots, \alpha^m$, such that vector fields $\tilde{\mathbf{r}}_i = \alpha^i \mathbf{r}_i$, $i = 1, \ldots, n$ commute, i.e. $[\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j] = 0$ for all $i, j \in \{1, \ldots, m\}$;

2) local coordinate functions $(w^1, \ldots, w^n)$, such that $\tilde{\mathbf{r}}_i = \frac{\partial}{\partial w^i}$, $i = 1, \ldots, m$.

**Proof.** 1) For a rich partial frame $\mathcal{R}$ the following structure equations hold:
\[
[\mathbf{r}_i, \mathbf{r}_j] = c^i_{ij} \mathbf{r}_i + c^j_{ij} \mathbf{r}_j \quad i, j = 1, \ldots, m,
\]
where structure functions $c^k_{ij}$ are independent of completion of $\mathcal{R}$ to a frame. We will show that the condition $[\mathbf{r}_i, \mathbf{r}_j] = 0$ leads to a PDE system on $\alpha$’s of generalized Frobenius type. Indeed,
\[
[\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j] = [\alpha^i \mathbf{r}_i, \alpha^j \mathbf{r}_j] = \alpha^i \alpha^j [\mathbf{r}_i, \mathbf{r}_j] + \alpha^i \mathbf{r}_i (\alpha^j) \mathbf{r}_j - \alpha^j \mathbf{r}_j (\alpha^i) \mathbf{r}_i
\]
\[
= \alpha^i (\alpha^j c^i_{ij} - c^j_{ij} (\alpha^i)) \mathbf{r}_i - \alpha^j (\alpha^j c^j_{ij} - c^j_{ij} (\alpha^j)) \mathbf{r}_j. \tag{55}
\]
Then $[\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j] = 0$ if and only if $\beta^i = \ln(\alpha^i)$ satisfies the PDE system.
\[
\mathbf{r}_j (\beta^i) = c^i_{ij}(u) \quad \text{for all } 1 \leq i \neq j \leq m \tag{56}
\]
To this system we add equations:
\[
\mathbf{r}_j (\beta^i) = 0 \quad \text{for all } 1 \leq j \leq m, \tag{57}
\]
making an additional requirement that, for each $i = 1, \ldots, m$, $\beta^i$ is constant along the integral curve of $r^i$. Since $c^i_{ij} = 0$, we can combine \ref{eq:55} and \ref{eq:56} in one system of $m^2$ equations on $m$ unknown functions $\beta$ of $n$ variables of generalized Frobenius type:
\[
\mathbf{r}_j (\beta^i) = c^i_{ij}(u) \quad \text{for all } 1 \leq i, j \leq m \tag{58}
\]
We now write out the integrability conditions \ref{eq:20}, prescribed in Theorem \ref{thm:55},
\[
\mathbf{r}_j (c^i_{ik}) - \mathbf{r}_k (c^j_{ik}) = c^j_{ik} c^i_{ij} + c^j_{ik} c^i_{jk} \quad \text{for all } 1 \leq i, j, k \leq m \tag{59}
\]
and note that they are satisfied due to Jacobi identities \ref{eq:7}.

Due to Theorem \ref{thm:55} we can prescribe any initial value for $\beta$’s along a submanifold transversal to $\mathcal{R}$ and get a unique solution of \ref{eq:58} on an open neighborhood of $\bar{u}$ with this initial data. Then positive functions $\alpha^i = e^{\beta^i}$ satisfy requirements of the theorem.
2) This is a direct consequence of Proposition 3.2.

Due to Lemma 5.6 and thanks to the scaling invariance of Problems 1 and 2 we may assume that the given rich partial frame is commutative. We then can use a local coordinate system \( w^1, \ldots, w^n \), such that \( r_i = \frac{\partial}{\partial w^i}, \) for \( i = 1, \ldots, m \). We complete \( \mathfrak{R} \) to a frame \( \{ r_1, \ldots, r_m, s_{m+1}, \ldots, s_n \} \), where \( s_l = \frac{\partial}{\partial w^l} \), for \( l = m+1, \ldots, n \). The commutativity of the frame and the symmetry of the connection \( \widetilde{\nabla} \) imply the following conditions on the structure coefficients (36) and Christoffel symbols (37) for this frame:

\[
c^i_{rs} = 0 \text{ and } \Gamma^i_{rs} = \Gamma^i_{sr} \text{ for all } l, s, r \in \{1, \ldots, n\}.
\] (60)

Then equations (43) – (45) become:

\[
\begin{align*}
\frac{\partial}{\partial w_i} (\lambda^j) &= \Gamma^j_{ji} (\lambda^i - \lambda^l) \quad \text{for all } 1 \leq i \neq j \leq m \\
\Gamma^i_{ij} (\lambda^i - \lambda^j) &= 0 \quad \text{for all } 1 \leq i < j \leq m, 1 \leq l \leq n, \text{ such that } l \neq i \text{ and } l \neq j.
\end{align*}
\] (61) (62)

Assuming that the Christoffel symbols \( \Gamma \)'s and the unknown functions \( \lambda \)'s are expressed in \( w \)-coordinates, we can treat (61) – (62) as a system of PDE’s with simple linear constrains on the unknown functions \( \lambda \)'s:

**Theorem 5.7.** If a partial frame \( \mathfrak{R} = \{ r_1, \ldots, r_m \} \) is rich and it satisfies conditions (54), then the set \( \mathcal{F}(\mathfrak{R}) \) of all local solutions of (28) near \( \bar{u} \) depends on

- \( m \) arbitrary functions of \( n - m + 1 \) variables, prescribing, for \( j = 1, \ldots, m \), a function \( \lambda^j \) along an arbitrary \( (n - m + 1) \)-dimensional manifold \( \Xi_j \) containing \( \bar{u} \) and transverse to the set of vector-fields \( \{ r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_m \} \);

- \( n \) functions of \( n - m \) variable\(^\dagger\), prescribing components of a vector field \( f \) along an arbitrary \( (n - m) \)-dimensional manifold \( \Xi \) transverse to the partial frame \( \mathfrak{R} \).

The above data uniquely determines \( f \) in an open neighborhood of \( \bar{u} \). The \( \mathcal{F}(\mathfrak{R}) \)-set always contains strictly hyperbolic fluxes.

**Proof.**

1) As has been discussed above, after rescaling, we may assume that \( \mathfrak{R} \) is a commutative frame and we choose a coordinate system such that \( r_i = \frac{\partial}{\partial w^i}, i = 1, \ldots, m \). Conditions (54) are invariant under rescaling of \( \mathfrak{R} \) and imply that

\[
\Gamma^i_{ij} \equiv 0 \quad \text{for all } 1 \leq i \neq j \leq m, 1 \leq l \leq n, \text{ such that } l \neq i \text{ and } l \neq j,
\] (63)

and therefore equations (62) trivially hold. Equations (61) is of the Darboux type and we proceed by verifying the integrability conditions (C) stated in Theorem 3.6. For this purpose we substitute partial derivatives prescribed by (61), into equality of mixed partials conditions:

\[
\frac{\partial}{\partial w^k} \left( \frac{\partial \lambda^j}{\partial w^i} \right) = \frac{\partial}{\partial w^i} \left( \frac{\partial \lambda^j}{\partial w^k} \right), \text{ for all distinct triples } i, j, k \in \{1, \ldots, m\}.
\]

The first substitution leads to

\[
\frac{\partial}{\partial w^k} \left( \Gamma^j_{ij} (\lambda^i - \lambda^l) \right) = \frac{\partial}{\partial w^i} \left( \Gamma^j_{jk} (\lambda^k - \lambda^l) \right), \text{ for all distinct triples } i, j, k \in \{1, \ldots, m\},
\]

\(^\dagger\)Example 7.1 demonstrates that, when a general solution of an \( \mathcal{F}(\mathfrak{R}) \)-system is explicitly written out, some of the arbitrary functions of \( n - m \) variables may be absorbed into arbitrary functions of \( n - m + 1 \) variables (a larger number of variables). This is a standard phenomena arising in applications of integrability theorems.
and the subsequent substitution (using abbreviated notation $\partial_i = \frac{\partial}{\partial u_i}$) leads to the condition:

\[
\left( \partial_i \Gamma^j_{ik} - \partial_k \Gamma^j_{ij} \right) \lambda^j + \left( \Gamma^j_{ij} \Gamma^i_{ik} + \Gamma^j_{jk} \Gamma^k_{ji} - \Gamma^j_{ij} \Gamma^k_{jk} - \partial_i \Gamma^j_{jk} \right) \lambda^k = 0. \tag{64}
\]

which must hold for all triples of pairwise distinct indices $i, j, k \in \{1, \ldots, m\}$. We will use flatness condition (13) to show that all $\lambda^i$’s appear (64) with identically zero coefficients. We first substitute $s = j$ in (13) and we assume that $i, j, k \in \{1, \ldots, m\}$ are pairwise distinct indices. Then using (60) and (63), we obtain that for all triples of pairwise distinct indices $i, j, k \in \{1, \ldots, m\}$:

\[
- \partial_k \Gamma^j_{ij} = \Gamma^j_{jk} \Gamma^i_{ji} - \Gamma^j_{ij} \Gamma^k_{ji} - \Gamma^j_{jk} \Gamma^k_{ji}. \tag{65}
\]

This immediately implies that the coefficient, $\Gamma^j_{jk} \Gamma^i_{ji} - \Gamma^j_{ij} \Gamma^k_{ji}$, of $\lambda^i$ in (64) is identically zero. Interchanging $k$ and $i$ in (65), we obtain:

\[
- \partial_i \Gamma^j_{jk} = \Gamma^j_{ij} \Gamma^k_{jk} - \Gamma^j_{jk} \Gamma^k_{ij} - \Gamma^j_{ij} \Gamma^i_{ik}. \tag{66}
\]

and, therefore, the coefficient of $\lambda^k$ in (64) is identically zero. We note that the right-hand sides of the identities (65) and (66) are equal and, therefore, the coefficient, $\partial_i \Gamma^j_{jk} - \partial_k \Gamma^j_{ij}$, of $\lambda^j$ in (64) is identically zero.

Thus, we have verified the integrability conditions (C) stated in Theorem 3.6 do hold for the PDE system (61). We conclude that, for a fixed point $\tilde{u} \in \Omega$, whose $u$-coordinates are $(\tilde{w}^1, \ldots, \tilde{w}^m)$ and $w$-coordinates are $(\bar{w}^1, \ldots, \bar{w}^m)$ and any assignment of $m$ arbitrary functions of $n - m + 1$ variables:

\[
\tilde{\lambda}^i (\bar{w}^1, \ldots, \bar{w}^i, \bar{w}^{i+1}, \ldots, \bar{w}^m, w^1, \ldots, w^{m+1}, \ldots, w^n), i = 1, \ldots, m
\]

on the subsets $\Xi_i \subset \Omega$, where $w^j = \bar{w}^j$, for $j = 1, \ldots, i - 1, i + 1, \ldots, m$, there is a unique local solution $\lambda^1, \ldots, \lambda^m$ of (61), such that $\lambda^i|_{\Xi_i \cap \Omega} = \tilde{\lambda}^i|_{\Xi_i \cap \Omega}$ on some open subset $\Omega' \subset \Omega$ containing $\tilde{u}$. Thus the general solution $\lambda$ of (61) depends on $m$ arbitrary functions of $n - m + 1$ variables.

2) Recalling that for a rich frame, satisfying (54), the system (52) is equivalent to the $\lambda$-system (43) – (45), we use Proposition 5.1 to conclude that for any solution $\lambda$ of (52) and any smooth initial data for $\mathbf{f}$ prescribed along any embedded submanifold $\Xi \subset \Omega$ of codimension $m$ transversal to $\mathcal{R}$, there is a unique smooth local solution of $\mathcal{F}(\mathcal{R})$-system (28). In local coordinates, the initial data can be defined by $n$ functions (components of $\mathbf{f}$) of $n - m$ variables (local coordinates on $\Xi$). Therefore, for a given solution $\lambda$ of (52), the general solution $\mathbf{f}$ of $\mathcal{F}(\mathcal{R})$-system (28) depends on $n$ arbitrary functions of $n - m$ variables.

3) We can always choose $\lambda^1, \ldots, \lambda^m$ in Part 1) of the proof, such that all $m$ real numbers $\lambda^1(\tilde{u}), \ldots, \lambda^m(\tilde{u})$ are distinct. Let $\lambda_1, \ldots, \lambda_m$ be the corresponding solutions of (52). Then the existence of strictly hyperbolic fluxes in the $\mathcal{F}(\mathcal{R})$-set follows from Proposition 5.3.

We observe that in single vector field case ($m = 1$), the conclusion of Theorem 5.7 is consistent with the observation made in Remark 5.4. The first part of the proof of Theorem 5.7 is a rather straightforward generalization of the proof of Theorem 4.3 in [7], where the $\lambda$-system (52) was considered in the case of the full frame ($m = n$). In a similar way, we can generalize Theorem 4.4 in [7] to treat the case when necessary conditions (54) for strict hyperbolicity are not
satisfied. In this case, the algebraic relationship (63) implies that there exist \(i, j \in \{1, \ldots, m\}\), such that \(i \neq j\) and \(\lambda^i \equiv \lambda^j\), and therefore, there are no strictly hyperbolic fluxes in the \(\mathcal{F}(\mathcal{R})\)-set. A rather involved description of the \(\mathcal{F}(\mathcal{R})\)-set is given by the following theorem, whose proof can be easily spelled out by combining the arguments in the proofs of Theorem 4.4 in \[7\] and Theorem 5.7 above. The argument is rather technical and is not reproduced here.

**Theorem 5.8.** Let \(\mathcal{R} = \{r_1, \ldots, r_m\}\) be a rich partial frame on an open subset \(\Omega \subset \mathbb{R}^n\), that does not satisfy conditions (53). Then the system (52) – (55) imposes multiplicity condition (54) on \(\lambda^i\)’s in the following sense. There are disjoint subsets \(A_1, \ldots, A_{s_0} \subset \{1, \ldots, m\}\) (\(s_0 \geq 1\)) of cardinality two or more, and such that (54) – (55) impose the equality \(\lambda^i = \lambda^j\) if and only if \(i, j \in A_\alpha\) for some \(\alpha \in \{1, \ldots, s_0\}\). Let \(l = \sum_{\alpha=1}^{s_0} |A_\alpha| \leq m\) and \(s_1 = m - l\). By relabeling indices we may assume that \(\{1, \ldots, m\} \setminus \bigcup_{\alpha=1}^{s_0} A_\alpha = \{1, \ldots, s_1\}\).

The set \(\mathcal{F}(\mathcal{R})\) of all local solutions of (28) near \(\bar{u}\) depends on

- \(s_1\) arbitrary functions \(\bar{\lambda}^1, \ldots, \bar{\lambda}^{s_1}\) of \(n - m + 1\) variables, prescribing, for \(j = 1, \ldots, s_1\), the data for function \(\lambda^j\), so that \(\lambda^j|_{\bar{\Xi}_j} = \bar{\lambda}^j\), where \(\bar{\Xi}_j\) is an arbitrary \((n - m + 1)\)-dimensional manifold \(\bar{\Xi}_j\) containing \(\bar{u}\) and transverse to the set of vector-fields \(\{r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_m\}\);
- \(s_0\) arbitrary functions \(\kappa^1, \ldots, \kappa^{s_0}\) of \(m - n\) variables, prescribing, for \(j = s_1 + 1, \ldots, m\), the data for functions \(\lambda^j\), so that when \(j \in A_\alpha\) for some \(\alpha = 1, \ldots, s_0\) when \(j \in A_\alpha\) for some \(\alpha = 1, \ldots, s_0\), then \(\lambda^j|_{\bar{\Xi}_j} = \kappa^\alpha\), where \(\bar{\Xi}_j\) is an \((n - m)\)-dimensional manifold passing through \(\bar{u}\) and transverse to \(\mathcal{R}\);
- \(n\) functions of \(n - m\) variables prescribing components of a vector field \(\mathbf{f}\) along an arbitrary \((n - m)\)-dimensional manifold \(\Xi\) transverse to the partial frame \(\mathcal{R}\).

The above data uniquely determines \(\mathbf{f}\) in an open neighborhood of \(\bar{u}\). The \(\mathcal{F}(\mathcal{R})\)-set never contains strictly hyperbolic fluxes.

### 5.3 Non-rich involutive frames consisting of three vector-fields

The lowest cardinality of a partial frame, for which involutive, non-rich scenario may appear, is \(m = 3\) case. In \[7\], we treated the case when \(m = n = 3\), i.e. the full frame case. We now generalize these results to \(n \geq 3\). Generalization to \(m > 3\) would require a consideration of a large number of cases and was not performed here.

We first treat the case when \(\mathcal{R}\) satisfies the necessary conditions of Proposition 5.2 for the existence of strictly hyperbolic fluxes. We choose an arbitrary completion of \(\mathcal{R}\) to a frame and write out the \(\lambda\)-system (43) – (46). The differential part (43) becomes:

\[
\begin{align*}
\mathbf{r}_2(\lambda^1) &= \Gamma_{12}(\lambda^2 - \lambda^1) \\
\mathbf{r}_3(\lambda^1) &= \Gamma_{13}(\lambda^3 - \lambda^1) \\
\mathbf{r}_1(\lambda^2) &= \Gamma_{21}(\lambda^1 - \lambda^2) \\
\mathbf{r}_3(\lambda^2) &= \Gamma_{23}(\lambda^3 - \lambda^2) \\
\mathbf{r}_1(\lambda^3) &= \Gamma_{31}(\lambda^1 - \lambda^3) \\
\mathbf{r}_2(\lambda^3) &= \Gamma_{32}(\lambda^2 - \lambda^3).
\end{align*}
\]

Algebraic equations (44) can be written as:

\[
A_\lambda \begin{bmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{bmatrix} = 0,
\quad \text{where} \quad A_\lambda = \begin{bmatrix} c_{12} & r_{12} & r_{13} \\ r_{21} & c_{21} & c_{23} \\ r_{31} & c_{31} & c_{32} \end{bmatrix}.
\]

5 It is clear that for all \(i \neq j\), such that \(\nabla_{\mathbf{r}_i} \mathbf{r}_j \notin \text{span}\{\mathbf{r}_1, \mathbf{r}_2\}\), equations (53) imply a multiplicity condition \(\lambda^i = \lambda^j\). Less obviously, (52) may impose additional multiplicity conditions on \(\lambda^i\)’s. See the proof of Lemma 4.5 in \[7\] for more details.
Condition (50) in Proposition 5.2 implies that (14) is trivial. We also note that, since \( R \) is involutive and satisfies conditions in Proposition 5.2 for all \( i, j, k \in \{1, 2, 3\} \), the structure coefficients \( c^k_{ij} \) and Christoffel symbols \( \Gamma^k_{ij} \) are independent of the completion of \( R \) to a frame, and therefore the system (67) – (68) can be written out without specifying a completion to a full frame. Our goal is to describe the solution set of (67) – (68).

Looking more closely at matrix \( A_\lambda \) we make the following observations

- From the symmetry of the connection it follows that the last column of \( A_\lambda \) is the sum of the first two columns and therefore \( \text{rank} \ A_\lambda \leq 2 \).
- Non-richness of \( R \) implies that at least one of \( c \)'s appearing in \( A_\lambda \) is non zero and therefore \( \text{rank} \ A_\lambda \geq 1 \).
- Condition (49) in Proposition 5.2 implies that, for each row in \( A_\lambda \), either all three entries are zero, or all three entries are non-zero.

Following the same argument as in Section 3 of [7], one can show that if \( \text{rank} \ A_\lambda = 2 \) at \( \bar{u} \), then the three eigenfunctions must coincide in a neighborhood of \( \bar{u} \), i.e. \( \lambda^1 = \lambda^2 = \lambda^3 = \lambda \) for some functions \( \lambda \) and, therefore, \( \mathcal{F}(R) \) does not contain strictly hyperbolic fluxes. Moreover, (67) implies that \( \lambda \) is constant along the integral manifolds of the involutive frame \( R \), and we can prescribe an arbitrary value of \( \lambda \) along a manifold \( \Xi \) transverse to \( R \). Otherwise, \( \text{rank} \ A_\lambda = 1 \), and we may assume without loss of generality, that \( c^1_{23} \neq 0 \). The first equation in (68) cannot be solved for \( \lambda^1 \) and this solution can be substituted in (67). After simplifications we get a system that specifies the derivatives of the two unknown functions \( \lambda^2 \) and \( \lambda^3 \) on \( \mathcal{R}^n \) along a partial involutive frame \( r_1 \), \( r_2 \) and \( r_3 \):

\[
\begin{align*}
    r_1(\lambda^2) &= \frac{\Gamma^2_{23} \Gamma^1_{23}}{c^1_{32}} (\lambda^2 - \lambda^3), \\
    r_2(\lambda^3) &= \frac{\Gamma^1_{23} (\Gamma^3_{32} - \Gamma^1_{12}) - c^1_{32} r_2 (\Gamma^1_{32} c^1_{32})}{\Gamma^1_{32}} (\lambda^2 - \lambda^3), \\
    r_3(\lambda^2) &= -\frac{\Gamma^2_{23} (\lambda^2 - \lambda^3)}{\Gamma^1_{32}}, \\
    r_1(\lambda^3) &= \frac{\Gamma^2_{23} (\lambda^2 - \lambda^3)}{c^1_{32}} , \\
    r_2(\lambda^3) &= \frac{\Gamma^3_{23} (\lambda^2 - \lambda^3)}{\Gamma^1_{32}}, \\
    r_3(\lambda^2) &= \frac{\Gamma^1_{23} (\Gamma^3_{32} - \Gamma^2_{23}) + c^1_{32} r_3 (\Gamma^1_{32} c^1_{32})}{\Gamma^2_{23}} (\lambda^2 - \lambda^3),
\end{align*}
\]

(69)

This system looks identical to the system (3.22) in [7], however, in [7], we had \( n = 3 \), while here \( n \geq 3 \) and, therefore, the classical Frobenius theorem, used in [7], is not sufficient in this case, and, therefore, we appeal to a more general Theorem 3.5. To verify the integrability conditions we rewrite (69)

\[
    r_i(\lambda^s) = \phi_i^s(u)(\lambda^2 - \lambda^3) \quad \text{for} \ i = 1, 2, 3 \text{ and } s = 2, 3,
\]

(70)

where \( \phi_i^s \) are known functions of \( \Gamma \)'s, given by the right-hand sides in (69). Then the integrability conditions amount to:

\[
\left[ r_i(\phi^s_j) - r_j(\phi^s_i) + \phi^s_j (\phi^3_i - \phi^3_j) - \phi^s_i (\phi^2_i - \phi^2_j) \right] (\lambda^2 - \lambda^3) = \sum_{k=1}^{3} c^s_{ij} \phi^s_k (\lambda^2 - \lambda^3),
\]

(71)

where \( 1 \leq i < j \leq 3, s = 2, 3 \) and \( c^s_{ij} = \Gamma^s_{ij} - \Gamma^s_{ji} \).

These conditions are satisfied if \( \lambda^2 = \lambda^3 \) in a neighborhood of \( \bar{u} \), in which case, the first equation in (68) implies \( \lambda^1 = \lambda^2 = \lambda^3 = \lambda \) and, as above, the functions \( \lambda \) must be constant along the integral manifolds of the involutive frame \( R \), and we can prescribe an arbitrary value...
of \( \lambda \) along a manifold \( \Xi \) transverse to \( \mathcal{R} \). For a strictly hyperbolic flux to exist the following six conditions must hold:

\[
\begin{align*}
\mathbf{r}_i(\phi_j^2) - \mathbf{r}_j(\phi_i^2) &= 0, \\
\mathbf{r}_i(\phi_j^3) - \mathbf{r}_j(\phi_i^3) &= -\phi_j^2\phi_i^3 - \phi_i^2\phi_j^3 + \sum_{k=1}^{3} c_{ij}^k \phi_k^2, \\
\mathbf{r}_i(\phi_j^3) - \mathbf{r}_j(\phi_i^3) &= \phi_j^2\phi_i^3 - \phi_i^2\phi_j^3 + \sum_{k=1}^{3} c_{ij}^k \phi_k^3
\end{align*}
\]

(72)

(73)

Conditions (72) – (73), in the case of full frames in \( \mathbb{R}^3 \), were derived in [7], and Examples 5.1 and 5.3 in [7] show that these compatibility conditions may or may not be satisfied: they must be checked for each case individually. If these integrability conditions are met then, according to Theorem 3.5, the general solution to the \( \lambda \)-system depends on two functions of \( n - 3 \) variables prescribing the values of \( \lambda^2 \) and \( \lambda^3 \) along any two \( n - 3 \) dimensional manifold passing through \( \bar{u} \) and transverse to \( \mathcal{R} \). Function \( \lambda^1 \) is then determined by the first equation in (68). Combining the above argument with Propositions 5.1 and Propositions 5.2 we arrive to the following theorem:

**Theorem 5.9.** Assume \( \mathcal{R} = \{ \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \} \) is a non-rich partial frame in involution, on a neighborhood \( \Omega \) of \( \bar{u} \), satisfying conditions (49) and (50) in Proposition 5.2. For \( i, j, k \in \{ 1, 2, 3 \} \), let \( c_{ij}^k \) and \( \Gamma_{ij}^k \) be defined by

\[
[r_i, r_j] = \sum_{k=1}^{3} c_{ij}^k r_k \quad \nabla r_i r_j = \sum_{k=1}^{3} \Gamma_{ij}^k r_k.
\]

Up to permutation of indices and by shrinking \( \Omega \) we may assume \( c_{23}^1 \) is nowhere zero on \( \Omega \).

- If the matrix \( A_\lambda \) defined in (68) has rank 1 and that (72) - (73) are satisfied in a neighborhood of \( \bar{u} \), then the solution set \( \mathcal{F}(\mathcal{R}) \) of system (28) depends on \( n - 2 \) arbitrary functions of \( n - 3 \) variables (2 of those determine the values \( \lambda^2 \) and \( \lambda^3 \), while \( n \) of those determine the values \( \mathbf{f} \) along an \( (n - 3) \)-dimensional manifold passing through \( \bar{u} \) and transverse to \( \mathcal{R} \). The set \( \mathcal{F}(\mathcal{R}) \) contains strictly hyperbolic fluxes.

- If the matrix \( A_\lambda \) defined in (68) has rank 2 at \( \bar{u} \) or (72) - (73) are not satisfied at \( \bar{u} \), then then the three eigenfunctions must coincide in a neighborhood of \( \bar{u} \), i.e. \( \lambda^1 = \lambda^2 = \lambda^3 = \lambda \) for some functions \( \lambda \), such that \( \lambda \) is constant along the integral manifolds of the involutive frame \( \mathcal{R} \), and can take arbitrary values along a manifold \( \Xi \) transverse to \( \mathcal{R} \). The solution set \( \mathcal{F}(\mathcal{R}) \) of system (28) depends on \( n + 1 \) arbitrary functions of \( n - 3 \) variables (1 of those determine the values \( \lambda \) and \( n \) of those determine the values \( \mathbf{f} \) along an \( (n - 3) \)-dimensional manifold passing through \( \bar{u} \) and transverse to \( \mathcal{R} \). The set \( \mathcal{F}(\mathcal{R}) \) does not contain strictly hyperbolic fluxes.

When the partial frame \( \mathcal{R} \) does not satisfy the necessary conditions of Proposition 5.2 for the existence of strictly hyperbolic fluxes, then the algebraic conditions (44) and (45) force two or more of eigenfunctions to be equal to each other, and we can prove the following result:

**Theorem 5.10.** Assume \( \mathcal{R} = \{ \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \} \) is a non-rich partial frame in involution, on a neighborhood \( \Omega \) of \( \bar{u} \), such that \( \mathcal{R} \) does not satisfy condition (44) or condition (51) in Proposition 5.2. Then there are exactly two possibilities:

either the \( \lambda \)-system (43) – (45) implies that \( \lambda^1 = \lambda^2 = \lambda^3 = \lambda \), in a neighborhood of \( \bar{u} \), where a function \( \lambda \) is constant along the integral manifolds of the involutive frame \( \mathcal{R} \) and may take arbitrary values on an \( (n - 3) \)-dimensional manifold \( \Xi_0 \) passing through \( \bar{u} \) and transverse to \( \mathcal{R} \).
or, up to permutation of indices, the λ-system (43) – (45) implies that λ₁ = λ² = λ, but allows the possibility that λ ≠ λ³ in a neighborhood of ̅u. In this case, the function λ³ is uniquely determined by its values on an (n − 2)-dimensional manifold Ξ₁ passing through ̅u and transverse to {r₁, r₂} and the function λ is uniquely determined by its values on an (n − 3)-dimensional manifold Ξ₂ passing through ̅u and transverse to ℜ, and.

In both cases, the λ-system (43) – (45) has a locally unique solution with the data, described above, and for each such solution, the F(ℜ)-system (28) has a locally unique solution determined by the values of f on an (n − 3)-dimensional manifold Ξ passing through ̅u and transverse to ℜ. The set F(ℜ) contains no strictly hyperbolic fluxes.

Proof.

1) If condition (49) is not satisfied, then equations (44) imply that at least two functions among λ₁, λ² and λ³ are identically equal to each other in neighborhood of ̅u. Similarly, if condition (50) is not satisfied than equations (45) imply that at least two functions among λ₁, λ² and λ³ coincide in neighborhood of ̅u. In either case the set F(ℜ) does not contain strictly hyperbolic fluxes.

2) If (44) and (45) imply that all three are equal, i.e. λ₁ = λ² = λ³ = λ, then, the differential part (43) of the λ-system, implies that the function λ is constant along the integral manifolds of the involutive frame ℜ. In this case, the system (43) trivially satisfies the assumptions of Theorem 3.5, which implies that for any assignment of λ along an (n − 3)-dimensional manifold Ξ₀ passing through ̅u and transverse to ℜ, there is unique such function in a neighborhood of ̅u.

3) If (44) and (45), imply that only two of λ's coincide, e.g. λ₁ = λ² = λ, but they don’t imply that they must be equal to λ³, then one can argue that c’s and Γ satisfy the following conditions

\[ \Gamma_{13}^{23} \neq 0, \Gamma_{13}^{23} = 0 \quad \text{and} \quad \Gamma_{13}^{23} = 0, \]  

and the λ-system (43) – (45) becomes:

\[
\begin{align*}
\mathbf{r}_2(\lambda) &= 0 \\
\mathbf{r}_3(\lambda) &= \Gamma_{13}^{23}(\lambda^3 - \lambda) \\
\mathbf{r}_1(\lambda) &= 0 \\
\mathbf{r}_3(\lambda) &= \Gamma_{13}^{23}(\lambda^3 - \lambda) \\
\mathbf{r}_1(\lambda^3) &= \Gamma_{13}^{23}(\lambda - \lambda^3) \\
\mathbf{r}_2(\lambda^3) &= \Gamma_{13}^{23}(\lambda - \lambda^3).
\end{align*}
\]  

If \( \Gamma_{13}^{23} \neq 0, \) the the second and the fourth equations in the above system imply that \( \lambda = \lambda^3, \) and therefore again \( \lambda^1 = \lambda^2 = \lambda^3 = \lambda, \) and we arrive to the situation considered in part 2) of the proof. If

\[
\Gamma_{23}^{13} = \Gamma_{13}^{23},
\]

we end up with the system

\[
\begin{align*}
\mathbf{r}_1(\lambda) &= 0 \\
\mathbf{r}_2(\lambda) &= 0 \\
\mathbf{r}_3(\lambda) &= \Gamma_{13}^{23}(\lambda^2 - \lambda) \\
\mathbf{r}_1(\lambda^3) &= \Gamma_{13}^{23}(\lambda - \lambda^3) \\
\mathbf{r}_2(\lambda^3) &= \Gamma_{13}^{23}(\lambda - \lambda^3).
\end{align*}
\]
We subtract equations (82) from (85), and equation (83) from (86), and introduce a new unknown functions \( \mu = \lambda^3 - \lambda \). We obtain:

\[
\begin{align*}
  r_1(\lambda) &= 0 \\
  r_2(\lambda) &= 0 \\
  r_3(\lambda) &= \Gamma^1_{13}\mu \\
  r_1(\mu) &= -\Gamma^3_{31}\mu \\
  r_2(\mu) &= -\Gamma^3_{32}\mu.
\end{align*}
\]

By assumption \( \{r_1, r_2, r_3\} \) are in involution, the first condition in (74) implies that the vector fields \( r_1 \) and \( r_2 \) are in involution. Thus we can first apply Theorem 3.5 to the sub-system (85) - (86), whose integrability condition,

\[
r_2(\Gamma^3_{31}) - r_1(\Gamma^3_{32}) = c_{21}^2 \Gamma^3_{32} + c_{21}^1 \Gamma^3_{31}
\]

is satisfied as shown in Lemma 3.6 of [7], due to the flatness and symmetry property of the connection, combined with conditions (74) and (76). Thus there is unique solution \( \mu \) for the subsystem (85) - (86) with any data prescribed along an \((n - 2)\)-dimensional manifold \( \Xi_1 \) passing through \( \bar{u} \) and transversal to \( r_1, r_2 \). Any solution \( \mu \) can be substituted into (84), and then we apply Theorem 3.5 to the sub-system (82) - (84), whose integrability condition

\[
\begin{align*}
  r_2(\Gamma^1_{13}) &= \Gamma^3_{23}\Gamma^1_{13} \\
  r_1(\Gamma^1_{13}) &= \Gamma^3_{13}\Gamma^1_{13}
\end{align*}
\]

As it is shown in Lemma 3.6 of [7], conditions (85) hold identically on \( \Omega \) due to the flatness and symmetry property of the connection, combined with conditions (74) and (76). Then Theorem 3.5 guarantees that there exists a locally unique solution of system sub-system (82) - (84), with the values of function \( \lambda \) prescribed along an \((n - 3)\)-dimensional manifold \( \Xi_2 \) passing through \( \bar{u} \) and transversal to \( \mathcal{R} \). Recalling that \( \mu = \lambda^3 - \lambda \), we conclude that \( \lambda \) is uniquely determined by its values on an \((n - 3)\)-dimensional manifold \( \Xi_1 \) passing through \( \bar{u} \) and transversal to \( \mathcal{R} \), and function \( \lambda^3 \) is uniquely determined by its values on an \((n - 2)\)-dimensional manifold \( \Xi_2 \) passing through \( \bar{u} \) and transversal to \( \{r_1, r_2\} \).

4) It follows from Proposition 5.1 that for each solution of the \( \lambda \) system, \( \mathcal{F}(\mathcal{R}) \)-system (28) has a locally unique solution determined by the values of \( f \) on an \((n - 3)\)-dimensional manifold \( \Xi \) passing through \( \bar{u} \) and transversal to \( \mathcal{R} \).

\[ \square \]

6 Non-involutive partial frames of two vector fields in \( \mathbb{R}^3 \).

In the non-involutive case, the differential consequences (33) – (35) of the \( \mathcal{F}(\mathcal{R}) \)-system (28) involve additional functions \( a \)'s, so instead of the “\( \lambda \)-system”, we get the “\( \lambda \)-\( a \)-system”, and, moreover, (33) – (35) do not provide a complete set of the integrability conditions for the \( \mathcal{F}(\mathcal{R}) \)-system. This makes the non-involutive case to be much harder to analyze than the involutive case, and we are able to treat only the lowest dimension where such scenario can arise: \( \mathcal{R} = \{r_1, r_2\} \) is a partial frame in \( \mathbb{R}^3 \), such that at a fixed point \( \bar{u} \in \Omega \):

\[
[r_1, r_2]|_\bar{u} \notin \text{span}_\mathbb{R}\{r_1|_\bar{u}, r_2|_\bar{u}\}.
\]

The \( \mathcal{F}(\mathcal{R}) \)-system then consists of two equations:

\[
\nabla_{r_1} f = \lambda^1 r_1 \text{ and } \nabla_{r_2} f = \lambda^2 r_2.
\]

26
and the necessary conditions (41) for strict hyperbolicity become
\[ \nabla_{r_1} r_2 |_{\tilde{u}} \notin \text{span}_R \{ r_1 |_{\tilde{u}}, r_2 |_{\tilde{u}} \} \quad \text{and} \quad \nabla_{r_2} r_1 |_{\tilde{u}} \notin \text{span}_R \{ r_1 |_{\tilde{u}}, r_2 |_{\tilde{u}} \}. \] (91)

Below we state two theorems that describe the size and the structure of the flux space \( \mathcal{F}(R) \) for partial frames \( R \) satisfying (91). The proofs of the Theorems 6.1 and 6.2 rely on the sequences of lemmas listed below. We remind the reader that \( \mathcal{F}^{\text{triv}} \) denotes the 4-dimensional space of trivial fluxes.

**Theorem 6.1.** Let \( R = \{ r_1, r_2 \} \) a non-involutive partial frame on an open neighborhood of \( \tilde{u} \in \mathbb{R}^3 \) satisfying conditions (91). Then
1) A non-zero flux \( f \in \mathcal{F}(R)/\mathcal{F}^{\text{triv}} \) is either strictly hyperbolic or non-hyperbolic.
2) If \( \dim \mathcal{F}(R)/\mathcal{F}^{\text{triv}} > 1 \), then \( \mathcal{F}(R) \) contains strictly hyperbolic fluxes.
3) If \( \mathcal{F}(R) \) contains a non-hyperbolic flux, then for any vector field \( s \) completing \( R \) to a local frame, the following identity holds on an open neighborhood of \( \tilde{u} \):
\[ \Gamma^3_{12} \Gamma^3_{21} - 2 (c^3_{12})^2 = \Gamma^3_{11} \Gamma^3_{22}, \] (92)
where \( c \)'s and \( \Gamma \)'s are structure components and Christoffel symbols for connection \( \nabla \) relative to the frame \( r_1, r_2, s \).

Although identity (92) is a closed condition, and, therefore, is restrictive, Examples 7.7 and 7.9 demonstrate that there are partial frames whose set of fluxes contains non-hyperbolic fluxes. On the other hand, Examples 7.5, 7.6, 7.8 and 7.11 show that there are partial frames for which all non-trivial fluxes are strictly hyperbolic.

**Theorem 6.2.** Let \( R = \{ r_1, r_2 \} \) a non-involutive partial frame on an open neighborhood of \( \tilde{u} \in \mathbb{R}^3 \) satisfying conditions (91). Let \( s \) be any completion of \( r_1 \) and \( r_2 \) to a local frame near \( \tilde{u} \) and let \( \Gamma \)'s be Christoffel symbols for connection \( \nabla \) relative to this frame. Assume further that the following condition is satisfied:
\[ \Gamma^3_{22}(\tilde{u}) \Gamma^3_{11}(\tilde{u}) - 9 \Gamma^3_{12}(\tilde{u}) \Gamma^3_{21}(\tilde{u}) \neq 0. \] (93)

Then
1) \( 0 \leq \dim \mathcal{F}(R)/\mathcal{F}^{\text{triv}} \leq 4 \).
2) For each \( k = 0, \ldots, 4 \) there exists \( R \), satisfying assumptions of the theorem, such that \( \dim \mathcal{F}(R)/\mathcal{F}^{\text{triv}} = k \).

Condition (93) arises in the proof of Lemma 6.5. Example 7.11 illustrates that there are partial frames with non-trivial fluxes, for which (93) does not hold. However, from the proof of Lemma 6.5, one can see that analyzing the size \( \mathcal{F}(R) \) in this case becomes rather technical and we left this non-generic case for the future work.

**Lemma 6.3.** Conditions (92) and (93) are independent of the choice of a vector-field \( s \) that completes \( R \) to a frame.

**Proof.** Consider two completions of \( R \) to a local frame in a neighborhood \( \Omega \) of \( \tilde{u} \). The first one is given by a vector field \( s \), while the second one is given by a vector field \( s' \). Then, we can express \( s' \) as linear combination of \( \{ r_1, r_2, s \} \):
\[ s' = \alpha r_1 + \beta r_2 + \gamma s, \]
for some smooth functions \( \alpha, \beta \) and \( \gamma \), such that \( \gamma \) is nowhere zero on \( \Omega \). Let \( c \)'s and \( \Gamma \)'s be the structure components and Christoffel symbols for connection \( \nabla \) relative to the frame \( r_1, r_2, s \).
and let $c'$s and $\Gamma'$s be the structure components and Christoffel symbols for connection $\tilde{\nabla}$ relative to the frame $r_1, r_2, s'$. Then for $i, j = 1, 2$:

$$\tilde{\nabla}_r r_j = \Gamma^1_{ij} r_1 + \Gamma^2_{ij} r_2 + \Gamma^3_{ij} s = \Gamma^1_{ij} r_1 + \Gamma^2_{ij} r_2 + \Gamma^3_{ij} \left( s' - \alpha r_1 - \beta r_2 \right)$$

$$= \left( \Gamma^1_{ij} - \frac{\alpha}{\gamma} \Gamma^3_{ij} \right) r_1 + \left( \Gamma^2_{ij} - \frac{\beta}{\gamma} \Gamma^3_{ij} \right) r_2 + \frac{1}{\gamma} \Gamma^3_{ij} s'.$$

Therefore, $\Gamma^3_{ij} \tilde{\nabla}_r r_j$ and so $c^3_{ij} \tilde{\nabla}_r r_j$ for $i, j = 1, 2$. Then (93) and (92) hold for $c'$s and $\Gamma'$s if and only if they hold for $c'$s and $\Gamma'$s.

The fact that conditions (92) and (93) are independent of the completion of $\mathfrak{R}$ to a frame suggests that they can be written as some relations among vector fields $r_1, r_2, [r_1, r_2]$, $\tilde{\nabla}_r r_2, r_1$ and $\tilde{\nabla}_r r_2, r_1$. However, we have not discovered such expressions.

**Lemma 6.4.** Let $\mathfrak{R} = \{r_1, r_2\}$ be a non-involutive partial frame satisfying (91). Let $s = [r_1, r_2]$ and for $1 \leq i, j, k \leq 3$, let $c_{ij}^2$ denote the structure functions and let $\Gamma^0_{ij}$ denote the Christoffel symbols for connection $\tilde{\nabla}$ relative to the local frame $\{r_1, r_2, s\}$. For functions $\lambda^1$ and $\lambda^2$ defined on an open neighborhood $\Omega$ of $\tilde{u}$ the following two conditions are equivalent:

1) There is a solution $f$ of $\mathcal{F}(\mathfrak{R})$ system (91), for the prescribed functions $\lambda^1$ and $\lambda^2$.

2) Functions $\lambda^1$ and $\lambda^2$, together with functions $a^1$ and $a^2$, defined by

$$a^1 = - r_2 (\lambda^1) - \Gamma^{12}_{12} (\lambda^1 - \lambda^2)$$

$$a^2 = r_1 (\lambda^2) - \Gamma^{21}_{21} (\lambda^1 - \lambda^2)$$

satisfy the following system of 6 equations:

$$r_1 (\lambda^1) = \frac{1}{\Gamma^{13}_{21}} \left( \Psi_1 (\lambda^1 - \lambda^2) + \Gamma^{31}_{11} a^1 + 2 \Gamma^{32}_{12} a^2 \right),$$

$$r_2 (\lambda^2) = \frac{1}{\Gamma^{13}_{21}} \left( \Psi_2 (\lambda^1 - \lambda^2) - 2 \Gamma^{31}_{21} a^1 - \Gamma^{32}_{22} a^2 \right),$$

$$r_2 (a^1) = (\Gamma^{21}_{13} \Psi^{23}_{12} - \Gamma^{12}_{21}) (\lambda^1 - \lambda^2) + (c_2^{13} - \Gamma^{12}_{21}) a^1 - \Gamma^{22}_{22} a^2,$$

$$r_1 (a^2) = (\Gamma^{21}_{13} \Psi^{23}_{12} + \Gamma^{22}_{21}) (\lambda^1 - \lambda^2) - \Gamma^{11}_{11} a^1 + (c_1^{23} - \Gamma^{12}_{21}) a^2,$$

$$r_1 (a^1) - s (\lambda^1) = \Gamma^{13}_{12} \Psi^{13}_{12} (\lambda^1 - \lambda^2) - (\Gamma^{11}_{11} - c_1^{13}) a^1 - \Gamma^{12}_{12} a^2,$$

$$r_2 (a^2) - s (\lambda^2) = \Gamma^{23}_{21} \Psi^{23}_{21} (\lambda^1 - \lambda^2) - \Gamma^{21}_{21} a^1 + (c_2^{23} - \Gamma^{22}_{22}) a^2,$$

where

$$\Psi_1 = \Gamma^{12}_{21} (\Psi^{23}_{21} - \Gamma^{31}_{11}) - r_1 (\Gamma^{31}_{12}) \quad \text{and} \quad \Psi_2 = \Gamma^{32}_{21} (\Psi^{23}_{12} - \Gamma^{31}_{11}) + r_2 (\Gamma^{31}_{12}).$$

Moreover, for every $\lambda^1$ and $\lambda^2$ satisfying condition 2), there exists unique, up to adding a constant vector, flux $f$ satisfying (91).

**Proof.** We note that due to the symmetry of $\tilde{\nabla}$ and our definition of $s$ we have

$$\Gamma^1_{12} - \Gamma^3_{12} = c_{12}^1 = 0, \quad \Gamma^2_{12} - \Gamma^3_{21} = c_{12}^2 = 0, \quad \Gamma^3_{12} - \Gamma^3_{21} = c_{12}^3 = 1.$$
(1 \Rightarrow 2) Assume for \( \lambda^1 \) and \( \lambda^2 \), there exists \( f \) such that (90) holds. Then flatness condition (111) implies that

\[
\nabla_{[r_1,r_2]} f = \nabla_{r_1} \nabla_{r_2} f - \nabla_{r_2} \nabla_{r_1} f.
\]

We recall that \( s = [r_1, r_2] \), expand the right-hand side, substitute (90), and use (103), to derive that

\[
\nabla_s f = a^1 r_1 + a^2 r_2 + a^3 s,
\]

with \( a^1 \) and \( a^2 \) given by (111) and (115), and

\[
a^3 = \Gamma_{12}^3 \lambda^2 - \Gamma_{21}^3 \lambda^1.
\]

We record following simple consequences of (106) and the last equation in (103) that is repeatedly used below.

\[
\lambda^1 - a^3 = \Gamma_{12}^3 \lambda^1 - \lambda^2 \text{ and } \lambda^2 - a^3 = \Gamma_{21}^3 \lambda^1 - \lambda^2.
\]

By expanding the flatness identity

\[
\nabla_{[r_1,s]} f = \nabla_{r_1} \nabla_s f - \nabla_s \nabla_{r_1} f
\]

we obtain

\[
\begin{align*}
\begin{aligned}
r_1(a^1) &= s(\lambda^1) + \Gamma_{13}^1 \lambda^1 - (\Gamma_{11}^1 - c_{13}^3) a^1 - \Gamma_{12}^1 a^2 - \Gamma_{13}^1 a^3, \\
&= s(\lambda^1) + \Gamma_{13}^1 \lambda^1 - (\Gamma_{11}^1 - c_{13}^3) a^1 - \Gamma_{12}^1 a^2,
\end{aligned}
\end{align*}
\]

(coefficient of \( r_1 \))

\[
\begin{align*}
\begin{aligned}
r_1(a^2) &= \Gamma_{31}^2 \lambda^1 + c_{13}^2 \lambda^2 - \Gamma_{21}^2 a^1 + (c_{13}^2 - \Gamma_{12}^3) a^2 - \Gamma_{13}^3 a^3, \\
&= (\Gamma_{31}^2 \Gamma_{21}^3 + \Gamma_{31}^3) \lambda^1 - \Gamma_{21}^2 a^1 + (c_{13}^2 - \Gamma_{12}^3) a^2,
\end{aligned}
\end{align*}
\]

(coefficient of \( r_2 \))

\[
\begin{align*}
\begin{aligned}
r_1(a^3) &= \Gamma_{31}^3 \lambda^1 - \Gamma_{11}^3 a^1 - \Gamma_{12}^3 a^2 - \Gamma_{13}^3 a^3, \\
&= \Gamma_{31}^3 \Gamma_{12}^3 \lambda^1 - \Gamma_{11}^3 a^1 - \Gamma_{12}^3 a^2,
\end{aligned}
\end{align*}
\]

(coefficient of \( s \))

where \( a^3 \) was eliminated from the right-hand sides of the above equations using (107).

Similarly, identity

\[
\nabla_{[r_2,s]} f = \nabla_{r_2} \nabla_s f - \nabla_s \nabla_{r_2} f
\]

leads to

\[
\begin{align*}
\begin{aligned}
r_2(a^1) &= c_{23}^1 \lambda^1 + \Gamma_{32}^1 \lambda^2 + (c_{23}^1 - \Gamma_{12}^1) a^1 - \Gamma_{12}^3 a^2 - \Gamma_{13}^3 a^3, \\
&= (\Gamma_{23}^1 \Gamma_{12}^3 - \Gamma_{32}^1) \lambda^1 - \Gamma_{12}^3 a^2 + (c_{23}^1 - \Gamma_{12}^3) a^1 - \Gamma_{12}^3 a^2,
\end{aligned}
\end{align*}
\]

(coefficient of \( r_1 \))

\[
\begin{align*}
\begin{aligned}
r_2(a^2) &= s(\lambda^2) + \Gamma_{23}^2 \lambda^2 - \Gamma_{21}^2 a^1 + (c_{23}^2 - \Gamma_{22}^3) a^2 - \Gamma_{23}^3 a^3, \\
&= s(\lambda^2) + \Gamma_{23}^2 \Gamma_{21}^3 \lambda^1 - \Gamma_{21}^2 a^1 + (c_{23}^2 - \Gamma_{22}^3) a^2,
\end{aligned}
\end{align*}
\]

(coefficient of \( r_2 \))

\[
\begin{align*}
\begin{aligned}
r_2(a^3) &= \Gamma_{32}^3 \lambda^2 - \Gamma_{21}^3 a^1 - \Gamma_{22}^3 a^2 - \Gamma_{32}^3 a^3, \\
&= \Gamma_{32}^3 \Gamma_{21}^3 \lambda^1 - \Gamma_{32}^3 a^1 - \Gamma_{32}^3 a^2.
\end{aligned}
\end{align*}
\]

(coefficient of \( s \))

We note that (113), (110), (109), (114) coincide with (98), (99), (100), and (101), respectively. To show the remaining two equations, (96) and (97), we note that equations (111)
Lemma 6.5. Let $\lambda$ be a partial frame satisfying assumptions of the Theorem 6.2. Then the set of pairs of functions $\lambda(\mathcal{R}) = \{(\lambda^1, \lambda^2)\}$ satisfying condition 2) of Lemma 6.4 is a real vector space of dimension at most 5.

Proof. It is straightforward to check that $\lambda(\mathcal{R})$ is a vector space. To prove the bound on its dimension, we prolong the system of equations (104) – (107), listed in condition 2) of Lemma 6.4, to a system of the Frobenius-type on 5 unknown functions $\lambda^1$, $\lambda^2$, $a^1$, $a^2$, and $\tau$, where we define

$$\tau = s(\lambda^2) \text{ for } s = [r_1, r_2].$$

(120)

This is done in the following steps.

1) By expanding the right-hand side of the commutator relationship

$$[r_1, r_2](\lambda^1)$$

and (115) express the derivatives of $a_3$ in the $r_1$ and $r_2$ directions, respectively. However, these derivatives can be also obtained by differentiating (106) and substituting (104) and (95):

$$r_1(a_3) = \Gamma^3_{12} r_1(\lambda^2) - \Gamma^3_{21} r_1(\lambda^1) + r_1(\Gamma^3_{12}) (\lambda^2 - \lambda^1) = (\Gamma^3_{12} \Gamma^2_{12} - r_1(\Gamma^3_{12})) (\lambda^1 - \lambda^2) + \Gamma^3_{12} a^2 - \Gamma^3_{21} r_1(\lambda^1),$$

$$r_2(a_3) = \Gamma^3_{12} r_2(\lambda^2) - \Gamma^3_{21} r_2(\lambda^1) + r_2(\Gamma^3_{12}) (\lambda^2 - \lambda^1) = \Gamma^3_{12} r_2(\lambda^2) + (r_2(\Gamma^3_{12}) - \Gamma^3_{21} \Gamma^1_{12}) (\lambda^2 - \lambda^1) + \Gamma^3_{21} a^1,$$

where we used that, due to the last equation in (103), derivatives of $\Gamma$ are equal. From (111) and (116) we obtain:

$$\Gamma^3_{21} r_1(\lambda^1) = (\Gamma^3_{12} (\Gamma^2_{21} - \Gamma^3_{31}) - r_1(\Gamma^3_{12})) (\lambda^1 - \lambda^2) + \Gamma^3_{11} a^1 + 2 \Gamma^3_{12} a^2$$

Similarly, from (115) and (117) we obtain:

$$\Gamma^3_{12} r_2(\lambda^2) = (r_2(\Gamma^3_{12}) + \Gamma^3_{21} (\Gamma^3_{32} - \Gamma^1_{12}) (\lambda^1 - \lambda^2) - 2 \Gamma^3_{21} a^1 - \Gamma^3_{22} a^2.$$
and substitution of the expressions for \( \mathbf{r}_1(\lambda^1), \mathbf{r}_2(\lambda^1), \mathbf{r}_1(\lambda^2), \mathbf{r}_2(\lambda^2), \mathbf{r}_1(a^1), \mathbf{r}_2(a^1), \mathbf{r}_1(a^2), \mathbf{r}_2(a^2) \) from (101) – (104), we obtain

\[
2 \Gamma_{21}^3 s(\lambda^1) + 2 \Gamma_{12}^3 s(\lambda^2) = \Gamma_{21}^3 \left( A_1 (\lambda^1 - \lambda^2) + B_1 a^1 + C_1 a^2 \right),
\]

where

\[
A_1 = -r_2 \left( \frac{\bar{Y}_1}{\Gamma_{21}^1} \right) - r_1 (\Gamma_{12}^1) - \frac{Y_1 Y_2}{\Gamma_{21}^1 \Gamma_{12}^1} - \Gamma_{12}^3 (\Gamma_{13}^1 + 2 \Gamma_{23}^2) - \frac{\Gamma_{12}^3}{\Gamma_{21}^1} (\Gamma_{23}^1 \Gamma_{12}^1 - \Gamma_{13}^1) + \Gamma_{12}^1 \Gamma_{21}^1
\]

\[
B_1 = -r_2 \left( \frac{\Gamma_{12}^1}{\Gamma_{12}^3} \right) - \frac{Y_1 (\Gamma_{21}^1 - 1)}{\Gamma_{21}^1 \Gamma_{12}^1} - \frac{\Gamma_{12}^3 c_{21}}{\Gamma_{21}^1 \Gamma_{12}^1} + 2 \frac{\Gamma_{12}^3 \Gamma_{21}^1}{\Gamma_{21}^1} + \Gamma_{11}^1 - c_{13}^1
\]

\[
C_1 = 2 \frac{r_2 (\Gamma_{21}^1)}{(\Gamma_{12}^3)^2} - \frac{Y_1 \Gamma_{12}^3}{\Gamma_{21}^1 \Gamma_{12}^1} - 2 \frac{\Gamma_{12}^1}{\Gamma_{21}^1} + \frac{\Gamma_{12}^3 \Gamma_{12}^1}{\Gamma_{12}^3} + 2 \frac{\Gamma_{12}^3 \Gamma_{21}^1}{\Gamma_{21}^1} (\Gamma_{22}^2 - c_{23}^1)
\]

2) By expanding the right-hand side of the commutator relationship

\[
\mathbf{s}(\lambda^2) = [\mathbf{r}_1, \mathbf{r}_2](\lambda^2)
\]

and substitution of the expressions for \( \mathbf{r}_1(\lambda^1), \mathbf{r}_2(\lambda^1), \mathbf{r}_1(\lambda^2), \mathbf{r}_2(\lambda^2), \mathbf{r}_1(a^1), \mathbf{r}_2(a^1), \mathbf{r}_1(a^2), \mathbf{r}_2(a^2) \) from (101) – (104), we obtain

\[
2 \Gamma_{21}^3 s(\lambda^1) + 2 \Gamma_{12}^3 s(\lambda^2) = \Gamma_{12}^3 \left( A_2 (\lambda^1 - \lambda^2) + B_2 a^1 + C_2 a^2 \right).
\]

\[
A_2 = r_1 \left( \frac{\bar{Y}_2}{\Gamma_{12}^3} \right) - r_2 \left( \frac{\Gamma_{21}^3}{\Gamma_{21}^3} \right) - \frac{Y_1 Y_2}{\Gamma_{12}^3 \Gamma_{12}^1} - \Gamma_{21}^3 (\Gamma_{23}^1 + 2 \Gamma_{13}^2) - \frac{\Gamma_{21}^3}{\Gamma_{21}^3} (\Gamma_{13}^3 \Gamma_{21}^1 + \Gamma_{31}^3) + \Gamma_{21}^3 \Gamma_{21}^1
\]

\[
B_2 = -2 \frac{r_1 (\Gamma_{21}^3)}{(\Gamma_{12}^3)^2} + \frac{Y_2 \Gamma_{12}^3}{\Gamma_{12}^3 \Gamma_{12}^1} + 2 \frac{\Gamma_{21}^3}{\Gamma_{21}^3} + \frac{\Gamma_{21}^3}{\Gamma_{12}^3} + \frac{\Gamma_{21}^3}{\Gamma_{12}^3} (\Gamma_{11}^3 - c_{13}^1)
\]

\[
C_2 = -r_1 \left( \frac{\Gamma_{21}^3}{\Gamma_{12}^3} \right) + \frac{Y_2 (\Gamma_{12}^3 + 1)}{\Gamma_{12}^3 \Gamma_{12}^1} - \frac{\Gamma_{21}^3 c_{13}^1}{\Gamma_{12}^3} + 2 \frac{\Gamma_{21}^3 \Gamma_{12}^1}{\Gamma_{12}^3} + \Gamma_{22}^3 - c_{23}^1
\]

3) Observe that (121) and (125) have identical left-hand sides, and so their right-hand side must be equal. It turns out that this indeed the case. In fact,

\[
\Gamma_{21}^3 A_1 \equiv \Gamma_{12}^3 A_2, \quad \Gamma_{21}^3 B_1 \equiv \Gamma_{12}^3 B_2 \quad \text{and} \quad \Gamma_{21}^3 C_1 \equiv \Gamma_{12}^3 C_2
\]

due to flatness condition (111). To show the A-identity in (128), we first compute \( \Gamma_{21}^3 A_1 - \Gamma_{12}^3 A_2 \) by substituting \( Y_1 \) and \( Y_2 \) into (122) and (126) and making various simplifications. We obtain

\[
\Gamma_{21}^3 A_1 - \Gamma_{12}^3 A_2 = -s(\Gamma_{12}^3) + \Gamma_{12}^3 r_2 (\Gamma_{12}^3) - \Gamma_{21}^3 r_1 (\Gamma_{32}^3) - \Gamma_{21}^3 r_2 (\Gamma_{32}^3) + \Gamma_{31}^3 r_2 (\Gamma_{32}^3) - \Gamma_{31}^3 r_2 (\Gamma_{32}^3)
\]
\[
+ \Gamma_{21}^3 r_2 (\Gamma_{12}^3) - \Gamma_{12}^3 (\Gamma_{23}^3) - \Gamma_{11}^3 (\Gamma_{23}^3) - \Gamma_{13}^3 (\Gamma_{23}^3) + \Gamma_{22}^3 (\Gamma_{21}^3) + \Gamma_{31}^3 (\Gamma_{23}^3).
\]

We then expand the identity

\[
\Gamma_{12}^3 \left( \nabla_{r_1} \nabla_s r_1 - \nabla_s \nabla_{r_1} \nabla_{r_1} - \nabla_{r_1} r_1 \right) - \Gamma_{12}^3 \left( \nabla_{r_1} \nabla_s r_2 - \nabla_s \nabla_{r_1} r_2 - \nabla_{r_1} r_2 \right) \equiv 0.
\]

and observe that the coefficient of \( s \) in (131) equals to the left hand side of (130). Similarly, we use the \( s \) coefficient of the expanded identity \( \nabla_{r_1} \nabla_{r_2} r_1 - \nabla_{r_2} \nabla_{r_1} r_1 \equiv \nabla_{r_2} r_2 \) to show the B-identity of (129), and the \( s \) coefficient of the expanded identity \( \nabla_{r_1} \nabla_{r_2} r_2 - \nabla_{r_2} \nabla_{r_1} r_2 \equiv \nabla_{r_2} r_2 \) to show the C-identity of (129).

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4) Introducing a new unknown function \( \tau \), defined by (120), we solve (121) for \( s(\lambda^1) \):

\[
s(\lambda^1) = -\frac{\Gamma^3_{21}}{\Gamma^3_{23}} \tau + \frac{1}{2} A_1 (\lambda^1 - \lambda^2) + \frac{1}{2} B_1 a^1 + \frac{1}{2} c_1 a^2
\]

and rewrite (100) and (101) as

\[
\begin{align*}
r_1(a^1) &= \left( \frac{1}{2} A_1 + \Gamma^1_{13} \Gamma^3_{12} \right) (\lambda^1 - \lambda^2) + \left( \frac{1}{2} B_1 - \Gamma^1_{11} + c_1^3 \right) a^1 + \left( \frac{1}{2} C_1 - \Gamma^1_{12} \right) a^2 - \frac{\Gamma^3_{21}}{\Gamma^3_{23}} \tau, \\
r_2(a^2) &= \Gamma^3_{22} \Gamma^3_{21} (\lambda^1 - \lambda^2) - \Gamma^3_{21} a^1 + (c_3^3 - \Gamma^3_{22}) a^2 + \tau,
\end{align*}
\]

(133)

5) To complete the system (94) - (99), (120), (132), (133) and (134) to a system of the Frobenius type we need to express the remaining derivatives \( s(a^1) \), \( s(a^2) \), \( r_1(\tau) \), \( r_2(\tau) \) and \( s(\tau) \) as functions of \( \lambda^1 \), \( \lambda^2 \), \( a^1 \), \( a^2 \) and \( \tau \). For this purpose, we continue to consider various consequences of commutator relationships.

From

\[
[r_1, s](\lambda^2) = r_1(s(\lambda^2)) - s(r_1(\lambda^2))
\]

expanding the left-hand side and substituting (120) and (95) into the right-hand side, we get:

\[
c_1^3 r_1(\lambda^2) + c_2^3 r_2(\lambda^2) + c_3^3 s(\lambda^2) = r_1(\tau) - s(\Gamma^2_{21} (\lambda^1 - \lambda^2) + a^2).
\]

(135)

By substituting the already known expressions of the directional derivatives, \( r_1(\lambda^2) \), \( r_2(\lambda^2) \), \( s(\lambda^2) \) and \( s(\lambda^1) \), given by (95), (97), (120) and (123) into (135), respectively, we obtain:

\[
r_1(\tau) - s(a^2) = \mathcal{L}_1(\lambda^1 - \lambda^2, a^1, a^2, \tau),
\]

(136)

where \( \mathcal{L}_1 \) is some known, linear in its arguments function with coefficients depending on \( c \)'s, \( \Gamma \)'s and their derivatives. The explicit expression of \( \mathcal{L}_1 \) is too long to be included.

From

\[
[r_2, s](\lambda^2) = r_2(s(\lambda^2)) - s(r_2(\lambda^2))
\]

expanding the left-hand side and substituting (120) and (97) into the right-hand side, we get:

\[
c_1^3 r_1(\lambda^2) + c_2^3 r_2(\lambda^2) + c_3^3 s(\lambda^2) = r_2(\tau) - s\left( \frac{1}{\Gamma^3_{12}} \left( \Gamma_{2} (\lambda^1 - \lambda^2) - 2 \Gamma^3_{21} a^1 - \Gamma^3_{22} a^2 \right) \right).
\]

(137)

By substituting the already known expressions of the directional derivatives, \( r_1(\lambda^2) \), \( r_2(\lambda^2) \), \( s(\lambda^2) \) and \( s(\lambda^1) \) into (137) we obtain

\[
\Gamma^3_{12} r_2(\tau) + 2 \Gamma^3_{21} s(a^1) + \Gamma^3_{22} s(a^2) = \mathcal{L}_2(\lambda^1 - \lambda^2, a^1, a^2, \tau),
\]

(138)

where function \( \mathcal{L}_2 \) is linear in its arguments with coefficients depending on \( c \)'s, \( \Gamma \)'s and their derivatives.

Similarly from the commutator relationships

\[
[r_2, s](\lambda^1) = r_2(s(\lambda^1)) - s(r_2(\lambda^1)) \quad \text{and} \quad [r_1, s](\lambda^1) = r_1(s(\lambda^1)) - s(r_1(\lambda^1))
\]

we obtain equations

\[
- \Gamma^3_{12} r_2(\tau) + \Gamma^3_{21} s(a^1) = \mathcal{L}_3(\lambda^1 - \lambda^2, a^1, a^2, \tau),
\]

(139)
Lemma 6.6. Let \( u \in \Omega \) and \( \lambda \) for complex eigenfunctions come in conjugate pairs, in the three real eigenfunctions. However, by (90), it possesses two real eigenfunctions \( \lambda^1, \lambda^2, a^1, a^2, \tau \), appearing in (90), a possibility of the third function being complex is excluded, and, therefore, \( \lambda^1 \) and \( \lambda^2 \) are non-hyperbolic means that the operator \( \nabla_{\ast} f \) does not possess three real eigenfunctions. However, by (90), it possesses two real eigenfunctions \( \lambda^1 \) and \( \lambda^2 \). Since complex eigenfunctions come in conjugate pairs, in the \( n = 3 \) case, a possibility of the third eigenfunction being complex is excluded, and, therefore, \( f \) must possess a generalized eigenvector field, which we denote \( s \). Let \( c \)'s and \( \Gamma \)'s and their derivatives. Therefore, we obtain a Frobenius-type system. If the integrability conditions for this system are not identically satisfied, they will impose additional relationships on \( \lambda, \lambda^2, a^1, a^2, \tau \), reducing the size of the solution set.

The fifteen equations (94) - (99), (120), (132), (133), (134), (141) and (142) can be used to solve for expressions \( s(a^1), s(a^2), r_1(\tau) \) and \( r_2(\tau) \) as linear functions of \( \lambda^1 - \lambda^2, a^1, a^2, \tau \), with coefficients depending on \( c \)'s, \( \Gamma \)'s and their derivatives.

Finally, \( s(\tau) = [r_1, r_2](\tau) = r_1(r_2(\tau)) - r_2(r_1(\tau)) \), (142)

after substitution of already known expressions of the derivatives, \( r_1(\tau), r_2(\tau), r_1(\lambda^1), r_2(\lambda^2), r_1(\lambda^2), r_2(\lambda^1), r_1(a^1), r_2(a^1), r_1(a^2), r_2(a^2), r_1(a^2), r_2(a^2), r_1(\tau), r_2(\tau), r_1(\lambda^1), r_2(\lambda^2), r_1(\lambda^2), r_2(\lambda^1), r_1(a^1), r_2(a^1), r_1(a^2), r_2(a^2), r_1(a^2), r_2(a^2), r_1(\tau), r_2(\tau) \), also becomes a linear function of \( \lambda^1 - \lambda^2, a^1, a^2, \tau \), with coefficients depending on \( c \)'s, \( \Gamma \)'s and their derivatives.

6) The fifteen equations (94) - (99), (120), (132), (133), (134), (141) and (142) can be used to express all the directional derivatives of functions \( \lambda^1, \lambda^2, a^1, a^2, \tau \) as linear combinations of \( \lambda^1 - \lambda^2, a^1, a^2, \tau \) with coefficients depending on \( c \)'s, \( \Gamma \)'s and their derivatives. Therefore, we obtain a Frobenius-type system. If the integrability conditions for this system are not identically satisfied, they will impose additional relationships on \( \lambda^1, \lambda^2, a^1, a^2, \tau \), reducing the size of the solution set.

7) The Frobenius-type system (94) - (99), (120), (132), (133), (134), (141) and (142) was obtained as a consequence of condition 2) of Lemma 6.4. Therefore, the vector space of pairs functions \( \lambda(\mathfrak{W}) = \{\lambda^1, \lambda^2\} \) satisfying this condition is at most 5.

\[ \lambda^1 = \lambda^2 = \lambda, \text{ where } \lambda \text{ is a non-constant function.} \]

Proof. We recall that \( f \) being non-hyperbolic means that the operator \( \nabla_{\ast} f \), does not posses three real eigenfunctions. However, by (90), it possesses two real eigenfunctions \( \lambda^1 \) and \( \lambda^2 \). Since complex eigenfunctions come in conjugate pairs, in the \( n = 3 \) case, a possibility of the third eigenfunction being complex is excluded, and, therefore, \( f \) must posses a generalized eigenvector field, which we denote \( s \). Let \( c \)'s and \( \Gamma \)'s and their derivatives and Christoffel symbols for \( \nabla \), relative to the frame \{\( r_1, r_2, s \)\}.

1. \( \lambda^1 = \lambda^2 \) To prove by contradiction, we assume that \( \lambda^1 \neq \lambda^2 \). Then either

\[ \nabla_s f = r_1 + \lambda^1 s \quad \text{or} \quad \nabla_s f = r_2 + \lambda^2 s. \]
Without loss of generality, we assume that the second equality holds (otherwise relabel \( r_1 \) and \( r_2 \)). Then (104) together with (90) and the second equation in (143) imply:
\[
c_{12}^1 \lambda^1 r_1 + c_{12}^2 \lambda^2 r_2 + c_{12}^3 (r_2 + \lambda^2 s) = \nabla_r (\lambda^2 r_2) - \nabla_{r_2} (\lambda^1 r_1) .
\] (144)

Using (90) again and collecting the coefficients with \( s \), we obtain
\[
c_{12}^3 \lambda^2 = \lambda^2 \Gamma^3_{12} - \lambda^1 \Gamma^3_{21} \text{ or, equivalently, } \Gamma^3_{21} (\lambda^2 - \lambda^1) = 0 .
\]

Conditions in (147) imply that \( \Gamma^3_{21} \neq 0 \) on an open neighborhood of \( \bar{u} \), and, therefore, \( \lambda^1 = \lambda^2 \) on this neighborhood.

2. (\( \lambda \neq \text{const} \)) Let \( \lambda^1 = \lambda^2 = \lambda \). Then, since \( s \) is a generalized eigenvector field, we must have
\[
\nabla_s f = \alpha r_1 + \beta r_2 + \lambda s ,
\] (145)
where \( \alpha \) and \( \beta \) are some functions, such that \( \alpha(\bar{u}) \) or \( \beta(\bar{u}) \) is non-zero. To prove by contradiction, we assume that \( \lambda \) is a constant function in a neighborhood of \( \bar{u} \). Then (104), together with (90), (146) imply that
\[
c_{12}^1 \lambda r_1 + c_{12}^2 \lambda r_2 + c_{12}^3 (\alpha r_1 + \beta r_2 + \lambda s) = \lambda \nabla_r r_2 - \lambda \nabla_{r_2} r_1 .
\] (146)

On the left-hand-side of (146), we notice that \( c_{12}^1 \lambda r_1 + c_{12}^2 \lambda r_2 + c_{12}^3 \lambda s = \lambda [r_1, r_2] \). At the same time, the right-hand side of (146) equals to \( \lambda [r_1, r_2] \) due to the symmetry condition (19). Then \( \alpha r_1 + \beta r_2 = 0 \), which contradicts our assumption that vectors \( r_1|_u \) and \( r_2|_u \) are independent and \( \alpha \) and \( \beta \) are some functions such that \( \alpha(\bar{u}) \) or \( \beta(\bar{u}) \) is non zero. Thus \( \lambda \) is a non-constant function.

Lemma 6.7. Let \( \mathcal{R} = \{r_1, r_2\} \) be a non-involutive partial frame satisfying conditions (91). Assume \( f \in \mathcal{F}(\mathcal{R}) \) is a non-hyperbolic flux. Then all other non-hyperbolic fluxes in \( \mathcal{F}(\mathcal{R}) \) are of the form \( cf + (a \text{ trivial flux}) \) where \( c \neq 0 \in \mathbb{R} \).

Proof. 1) Let \( f \in \mathcal{F}(\mathcal{R}) \) be a non-hyperbolic flux. From Lemma 6.6, it follows that there exists a non-constant function \( \lambda \) in a neighborhood of \( \bar{u} \), such that \( f \) and \( \lambda^1 = \lambda^2 = \lambda \) satisfy (90). It is straightforward to check that \( cf + \lambda \mathbf{f} \), where \( \lambda \in \mathbb{R} \) and \( \mathbf{f} \in \mathcal{F}^{id} \) (see (90) to recall the definition of identity fluxes) is a non-hyperbolic flux, which together with \( \lambda^1 = \lambda^2 = c \lambda + \lambda \) satisfy (90). Recalling (31), we conclude that \( cf + (a \text{ trivial flux}) \) belongs to \( \mathcal{F}(\mathcal{R}) \) and clearly those fluxes are non-hyperbolic. It remains to show that any non-hyperbolic flux in \( \mathcal{F}(\mathcal{R}) \) is of this form.

2) Lemma 6.4 implies that function \( \lambda \) together with functions \( a^1 \) and \( a^2 \), defined by
\[
a^1 = -r_2(\lambda) \quad \text{ and } \quad a^2 = r_1(\lambda)
\] (147) (148)

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satisfy the following equations (these are (90)–(101) in the case when \( \lambda^1 = \lambda^2 \)):

\[
\begin{align*}
\mathbf{r}_1(\lambda) &= \frac{1}{\Gamma_{21}^3} \left( \Gamma_{11}^3 a^1 + 2 \Gamma_{12}^3 a^2 \right), \\
\mathbf{r}_2(\lambda) &= -\frac{1}{\Gamma_{12}^3} \left( 2 \Gamma_{21}^3 a^1 + \Gamma_{22}^3 a^2 \right), \\
\mathbf{r}_2(a^1) &= (c_{23}^3 - \Gamma_{21}^1) a^1 - \Gamma_{22}^1 a^2, \\
\mathbf{r}_1(a^2) &= -\Gamma_{11}^1 a^1 + (c_{13}^3 - \Gamma_{12}^1) a^2, \\
\mathbf{r}_1(a^1) - s(\lambda) &= -(\Gamma_{11}^1 - c_{13}^3) a^1 - \Gamma_{12}^1 a^2, \\
\mathbf{r}_2(a^2) - s(\lambda) &= -\Gamma_{21}^1 a^1 + (c_{23}^3 - \Gamma_{22}^1) a^2,
\end{align*}
\]

where, \( \Gamma \)'s are Christoffel symbols for \( \bar{\nabla} \) relative to the frame \( \{ \mathbf{r}_1, \mathbf{r}_2, s = [\mathbf{r}_1, \mathbf{r}_2] \} \). Equations (147) and (148) immediately imply that:

\[
s(\lambda) = \mathbf{r}_1(\mathbf{r}_2(\lambda)) - \mathbf{r}_2(\mathbf{r}_1(\lambda)) = -\mathbf{r}_1(a^1) - \mathbf{r}_2(a^2).
\]

Then from (155), together with (153) and (154), we obtain:

\[
\begin{align*}
s(\lambda) &= \frac{1}{3} (\Gamma_{11}^1 + \Gamma_{21}^2 - c_{13}^3) a^1 + \frac{1}{3} (\Gamma_{22}^1 + \Gamma_{12}^1 - c_{23}^3) a^2, \\
\mathbf{r}_1(a^1) &= \frac{1}{3} (-2 \Gamma_{11}^1 + \Gamma_{21}^2 + 2 c_{13}^3) a^1 + \frac{1}{3} (\Gamma_{22}^1 - 2 \Gamma_{12}^1 - c_{23}^3) a^2, \\
\mathbf{r}_2(a^2) &= \frac{1}{3} (\Gamma_{11}^1 - 2 \Gamma_{21}^2 - c_{13}^3) a^1 + \frac{1}{3} (-2 \Gamma_{22}^1 + \Gamma_{12}^1 + 2 c_{23}^3) a^2,
\end{align*}
\]

From Lemma 6.6 we know that \( \lambda \) is a non-constant function, and, therefore, at least one of its derivatives in the frame directions must be non-zero. Examining (147), (148) and (156), we conclude that at least one of the functions \( a^1 \) or \( a^2 \) is non-zero. Without loss of generality, we assume that \( a^1 \neq 0 \) (otherwise, relabel \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \)).

3) Equations (147), (148), (149), (150) imply:

\[
\begin{bmatrix}
\Gamma_{11}^3 & \Gamma_{12}^3 + 1 \\
\Gamma_{21}^3 - 1 & \Gamma_{22}^3
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix} = 0
\]

Since \( [a^1, a^2]^T \) is a non-zero vector, matrix \( M = \begin{bmatrix}
\Gamma_{11}^3 & \Gamma_{12}^3 + 1 \\
\Gamma_{21}^3 - 1 & \Gamma_{22}^3
\end{bmatrix} \) must have rank less than 2 in order for (159) to have a solution, i.e.

\[
\Gamma_{11}^3 \Gamma_{22}^3 - (\Gamma_{12}^3 + 1)(\Gamma_{21}^3 - 1) = 0.
\]

Substituting \( c_{12}^3 = 1 \) in (160) and simplifying, we get condition

\[
\Gamma_{12}^3 \Gamma_{21}^3 - (c_{12}^3)^2 = \Gamma_{11}^3 \Gamma_{22}^3.
\]

4) At least one of the expressions \( \Gamma_{12}^3 + 1 \) or \( \Gamma_{21}^3 - 1 \) is non-zero (if both are zero, then \( c_{21}^3 = -2 \), which contradicts our assumption that \( c \)'s are the structure constants for the
frame \( \{ r_1, r_2, s = [r_1, r_2] \} \). Then (159) has a one-parametric family of solutions. In part 2) of the proof, we argued that we may assume that \( a^1 \neq 0 \). Then, from (159), we can express

\[
a^2 = \alpha a^1, \tag{162}
\]

where \( \alpha(u) \) is some known function expressible in terms of \( \Gamma \)'s. (Explicitly, if \( \Gamma^3_{12} \neq -1 \), then \( \alpha = \frac{\Gamma^3_{22} \Gamma^2_{11} - \Gamma^3_{12} \Gamma^2_{22}}{\Gamma^3_{12} + 1} \), otherwise, we can show that \( \Gamma^3_{22} \neq 0 \) and \( \alpha = \frac{3}{\Gamma^3_{22}} \).)

Substitution of (162) into (147), (148), (151), (156), and (157), leads to equations:

\[
\begin{align*}
\mathbf{r}_1(\lambda) &= \alpha a^1, \\
\mathbf{r}_2(\lambda) &= -a^1, \\
s(\lambda) &= \alpha_1 a^1, \\
r_1(a^1) &= \alpha_2 a^3, \\
r_2(a^1) &= \alpha_3 a^1, 
\end{align*}
\]

where \( \alpha, \alpha_1, \alpha_2, \alpha_3 \) are some known functions, expressible in terms of \( \Gamma \)'s and their directional derivatives. Substituting (166) and (167), in the commutator relationship, we conclude that

\[
s(a^1) = r_1(r_2(a^1)) - r_2(r_1(a^1)) = \alpha_4 a^1, \tag{168}
\]

where \( \alpha_4 \) is another known function, expressible in terms of \( \Gamma \)'s and their directional derivatives. System (163) – (168) is a Frobenius-type system on two unknown functions, \( \lambda \) and \( a^1 \), and so its solution depends on at most two arbitrary constants.

5) From parts 1) and 2) of the proof, it follows that, there exists a non-constant functions \( \lambda \) and \( a^1 = -r_2(\lambda) \), satisfying (163) – (168), and then we immediately have a two parametric family of solution \( \lambda_{c,\bar{\lambda}} = c \lambda + \bar{\lambda}, \ a^1_c = c a^1, \) where \( c, \bar{\lambda} \) are arbitrary constants. From part 4), we conclude that there is no other solution. On the other hand, each \( \lambda_{c,\bar{\lambda}} \), with \( c \neq 0 \), corresponds to a three-parametric family of non-hyperbolic fluxes \( cf + \bar{\lambda} \tilde{f} \), where \( f \in \mathcal{F}^{\text{id}} \).

We conclude that any non-hyperbolic flux in \( \mathcal{F}(\mathcal{R}) \) is of the form \( cf + (a \text{ trivial flux}) \). \( \square \)

Remark 6.8. From (147) and (148) it follows that if \( f \) is a nonhyperbolic flux for \( \mathcal{R} = \{ r_1, r_2 \} \), then \( s = [r_1, r_2] \) is a generalized eigenvector field of \( f \). Indeed,

\[
\tilde{\nabla}_{[r_1, r_2]} f = \tilde{\nabla}_{r_1} \tilde{\nabla}_{r_2} f - \tilde{\nabla}_{r_2} \tilde{\nabla}_{r_1} f = \tilde{\nabla}_{r_1}(\lambda r_2) - \tilde{\nabla}_{r_2}(\lambda r_1) = a^1 r_1 + a^2 r_2 + \lambda [r_1, r_2]. \tag{169}
\]

Proof of Theorem 6.1

1) We want to show that a non-zero flux \( f \in \mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}} \) is either strictly hyperbolic or non-hyperbolic. Assume that there exists a non-strictly hyperbolic flux \( f \in \mathcal{F}(\mathcal{R}) \). This means that \( f \) has the third eigenvector field \( r_3 \) and at least two of the corresponding eigenvalue functions \( \lambda^1, \lambda^2 \) and \( \lambda^3 \) coincide in a neighborhood of a fixed point \( \bar{u} \in \Omega \). Examining the \( r_3 \) component of the expended flatness condition (104), we conclude that

\[
\Gamma^3_{12} \lambda^2 - \Gamma^3_{21} \lambda^1 = c^3_{12} \lambda^3, \tag{170}
\]

where here \( c \)'s and \( \Gamma \)'s denote structure coefficients and Christofel symbols for \( \tilde{\nabla} \), relative to the frame \( \{ r_1, r_2, r_3 \} \). Equation (170) must hold as an identity in an neighborhood of \( \bar{u} \), and it can be rewritten as

\[
\Gamma^3_{12} (\lambda^2 - \lambda^3) - \Gamma^3_{21} (\lambda^1 - \lambda^3) \equiv 0, \tag{171}
\]
From the assumption of the theorem it follows that $\Gamma^3_{12} \neq 0$, $\Gamma^3_{21} \neq 0$, and $\Gamma^3_{12} \neq \Gamma^3_{21}$. Then, from (164), we conclude that if any two of the functions $\lambda^1, \lambda^2, \lambda^3$ are equal then all there of them must be equal: $\lambda^1(u) = \lambda^2(u) = \lambda^3(u) = \lambda(u)$. This implies that $\nabla_r f = \lambda r$ for any $r \in \mathcal{X}(\Omega)$. Therefore, from the flatness conditions
\[
\nabla_{[r_1, r_2]} f = \nabla_{r_1} \nabla_{r_2} f - \nabla_{r_2} \nabla_{r_1} f \quad \text{for} \quad i = 2, 3
\]
we can deduce that:
\[
\lambda [r_1, r_i] = r_i (\lambda r_i) - r_i (\lambda r_1) \quad \text{for} \quad i = 2, 3.
\]
Since the right-hand side of the above equality is $\lambda [r_1, r_i] + r_1(\lambda) r_i - r_i(\lambda) r_1$, and $r_1, r_2, r_3$ are independent we conclude $r_i(\lambda) = 0$ for $i = 1, 2, 3$ and, therefore, $\lambda \equiv \hat{\lambda} \in \mathbb{R}$ is a constant function. This implies that $f$ is a trivial flux, and the statement is proven.

2) From Lemma 6.7, if $\mathcal{F}(\mathfrak{R})$ contains strictly hyperbolic fluxes, then up to adding a trivial flux, it contains exactly one-parametric family of non-hyperbolic fluxes. Therefore, if $\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} > 1$, then $\mathcal{F}(\mathfrak{R})$ contains hyperbolic fluxes, and, from the first statements of the theorem, we know that all non-trivial hyperbolic fluxes in $\mathcal{F}(\mathfrak{R})$ are strictly hyperbolic.

3) In the proof of Lemma 6.7 (see (161)), we showed that if $\mathcal{F}(\mathfrak{R})$ contains non-hyperbolic fluxes, then (92) holds with $c$’s and $\Gamma$’s being structure components and Christoffel symbols for the connection $\nabla$ relative to the frame $\{r_1, r_2, [r_1, r_2]\}$. Then Lemma 6.3 asserts that (92) holds with $c$’s and $\Gamma$’s corresponding to any completion $\{r_1, r_2, s\}$ of $\mathfrak{R}$ to a frame.

**Proof of Theorem 6.2**

1. We want to show that $0 \leq \dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} \leq 4$. Lemma 6.5 asserts that under the assumptions of Theorem 6.2 the set of pairs of functions $\lambda(\mathfrak{R}) = \{\lambda^1, \lambda^2\}$ satisfying condition 2) of Lemma 6.4 is a real vector space of dimension at most 5. In addition, Lemma 6.4 implies for every $\lambda^1$ and $\lambda^2$ satisfying condition 2), there exists unique, up to adding a constant vector in $\mathbb{R}^3$, flux $f$ satisfying (38). Thus $\dim \mathcal{F}(\mathfrak{R}) \leq 8$. On the other hand, $\mathcal{F}(\mathfrak{R})$ contains a 4-dimensional subspace of trivial fluxes and, therefore, the stated inequalities hold.

2. For $k = 0, \ldots, 4$, Examples 7.4–7.8 exhibit partial frames, satisfying the assumptions of the theorem, such that $\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} = k$.

**7 Examples**

The examples, provided in this section, illustrate the main results of the paper and also provide a proof for the existence statement in Theorem 6.2. The computations were performed in the computer algebra system MAPLE by setting up systems of differential equations for $f$ and $\lambda$’s and using a built in command pdsolve to solve them.

**7.1 Rich partial frames**

For a rich partial frame satisfying conditions (54), Theorem 5.7 describes the degree of freedom for prescribing $\lambda$’s and $f$’s satisfying the $\mathcal{F}(\mathfrak{R})$-system (28). The theorem also asserts that $\mathcal{F}(\mathfrak{R})$ contains strictly hyperbolic fluxes. The following three examples demonstrate these results. They also underscore the following interesting phenomenon: a hyperbolic flux corresponding to a rich partial frame may have a non-rich full frame. In fact, we found examples with three different scenarios: in Example 7.1 all strictly hyperbolic fluxes in $\mathcal{F}(\mathfrak{R})$ are rich, in Example 7.2...
all hyperbolic (strictly and non-strictly) fluxes in \( F(\mathcal{R}) \) are non-rich, and finally in Example 7.3 \( F(\mathcal{R}) \) contains both rich and non-rich strictly hyperbolic fluxes.

In the following examples, \( n = 3, m = 2 \). The standard affine coordinates in \( \mathbb{R}^3 \) for the connection \( \nabla \) are denoted by \((u, v, w)\). We start with a simple example, a partial frame given by the first two standard vectors in \( \mathbb{R}^3 \):

**Example 7.1.** Let \( r_1 = [1,0,0]^T \) and \( r_2 = [0,1,0]^T \) comprise a partial frame \( \mathcal{R} \) on \( \mathbb{R}^3 \). It is clear that \( \mathcal{R} \) satisfies the assumptions of Theorem 5.7, and as predicted by this theorem \( \lambda^1 \) and \( \lambda^2 \), satisfying (43) - (45) are parametrized by two functions of two variables:

\[
\lambda^1 = \phi(u, w) \quad \text{and} \quad \lambda^2 = \psi(v, w). \quad (172)
\]

For each such pair of \( \lambda^1 \) and \( \lambda^2 \) we get a family of fluxes in \( F(\mathcal{R}) \) parametrized by three arbitrary functions of one variable, \( g, h \) and \( k \):

\[
f = \begin{bmatrix}
\int_u^w \phi(s, w) \, ds + g(w) \\
\int_v^w \psi(s, w) \, ds + h(w) \\
k(w)
\end{bmatrix}. \quad (173)
\]

On the other hand, we could start by parametrizing the set \( \mathcal{F}(\mathcal{R}) \) by two arbitrary functions \( \Phi \) and \( \Psi \) of two variables and an arbitrary function \( k \) of one variable:

\[
f = [\Phi(u, w), \Psi(v, w), k(w)] \quad \text{with} \quad \lambda^1 = \frac{\partial \Phi}{\partial u}, \quad \lambda^2 = \frac{\partial \Psi}{\partial v}. \quad (174)
\]

Of course, (174) is equivalent to (172) - (173), but, in (174), arbitrary functions \( g, h \) are absorbed into \( \Phi \) and \( \Psi \). Although (174) is simpler, (172) - (173) more closely illustrate the argument in the proof of Theorem 5.7. Obviously, for almost all choices of \( \Phi, \Psi \) and \( k \), the resulting flux is strictly hyperbolic.

We finally argue that all strictly hyperbolic fluxes in \( F(\mathcal{R}) \) are rich. Let \( r_3 \) be the third eigenvector field of a hyperbolic flux \( f \in \mathcal{F}(\mathcal{R}) \). Since \( r_3 \) is linearly independent of \( r_1 \) and \( r_2 \), it can be, up to rescaling, written as \( r_3 = [a, b, 1]^T \), where \( a \) and \( b \) are some functions on \( \mathbb{R}^3 \). Then, since \( \nabla r_1, r_1 = \nabla r_3, r_2 = 0 \), we have, in particular, that

\[
\Gamma_{31}^2 = \Gamma_{32}^1 = 0 \quad \text{and, therefore,} \quad c_{13}^2 = \Gamma_{13}^2 \quad \text{and} \quad c_{23}^1 = \Gamma_{23}^2. \quad (175)
\]

We also have

\[
\Gamma_{12}^3 = \Gamma_{21}^3 = c_{12}^3 = 0. \quad (176)
\]

Substituting (175) and (176) into (44) produces two equations:

\[
\Gamma_{23}^1 (\lambda^3 - \lambda^1) = 0 \quad \text{and} \quad \Gamma_{13}^2 (\lambda^3 - \lambda^2) = 0. \quad (177)
\]

If \( \Gamma_{23}^1 \neq 0 \) or \( \Gamma_{13}^2 \neq 0 \), then (177) implies that \( \lambda^3 = \lambda^1 \) or \( \lambda^3 = \lambda^2 \), and, therefore, \( f \) is not strictly hyperbolic. If \( \Gamma_{23}^1 = 0 \) and \( \Gamma_{13}^2 = 0 \), then (176) implies that \( c_{23}^1 = 0 \) and \( c_{13}^2 = 0 \), and therefore \( f \) is rich. Thus \( \mathcal{F}(\mathcal{R}) \) does not contain non-rich strictly hyperbolic fluxes.

On the contrary, the following example presents a rich pair of vector fields, satisfying (53), which admits only non-rich hyperbolic fluxes.

**Example 7.2.** Consider a partial frame \( \mathcal{R} \) consisting of the vector fields \( r_1 = [1,0,0]^T \) and \( r_2 = [w,1,0]^T \) on \( \Omega \subset \mathbb{R}^3 \), such that \( w \neq 0 \). We have \( [r_1, r_2] = 0 \), \( \nabla r_1, r_2 = 0 \) and \( \nabla r_2, r_1 = 0 \), and, therefore, we are in the case considered in Theorem 5.7. As predicted by Theorem 5.7, the degree of freedom for prescribing \( \lambda^1 \) and \( \lambda^2 \) consists of two arbitrary functions of two variables:

\[
\lambda^1 = \phi(w, v - u) \quad \text{and} \quad \lambda^2 = \psi(w, v). \quad (178)
\]
We now present examples of non-involutive partial frames $R$.

### 7.2 Non-involutive partial frames of two vector fields in $\mathbb{R}^3$

By construction, with the eigenvalues $-\frac{1}{w}$, one can find functions $h, g$ and $k$, so that the resulting flux is strictly hyperbolic. For a concrete example, let $\phi(w, v - \frac{u}{w}) = -\frac{1}{w}$ and $\psi(v, w) = 0$, $g(w) = h(w) = 0$ and $k(w) = -\frac{1}{w} - \log w$. We observe that the flux

$$f = \begin{bmatrix} v - \frac{u}{w} \\ 0 \\ -\frac{1}{w} - \log w \end{bmatrix}$$

is strictly hyperbolic with the eigenvalues

$$\lambda^1 = -\frac{1}{w}; \quad \lambda^2 = 0; \quad \lambda^3 = \frac{1 - w}{w^2},$$

and with the third eigenvector given by $r_3 = [u, 0, 1]^T$.

We now show that, although the partial frame $R$ is rich, the corresponding set of fluxes $\mathcal{F}(R)$ does not contain any rich hyperbolic fluxes. Indeed, let $r_3$ be the third eigenvector of a strictly hyperbolic flux in $\mathcal{F}(R)$. Up to a scaling, any vector field, which is linearly independent from $r_1$ and $r_2$, is of the form $r_3 = [a, b, 1]^T$, where $a$ and $b$ are arbitrary functions on $\mathbb{R}^3$. Since $[r_3, r_2] = [1, 0, 0]^T$, we have $c_{32} = 1$, and, therefore, there is no rich hyperbolic fluxes in $\mathcal{F}(R)$.

Finally, we present an example of a rich partial frame $R$, which admits both rich and non-rich strictly hyperbolic fluxes.

**Example 7.3.** Consider a partial frame $R$, consisting of the vector fields $r_1 = [1, -\sqrt{u}, 0]^T$ and $r_2 = [1, -\sqrt{u}, 0]^T$ on $\Omega \subset \mathbb{R}^3$, such that $u \neq 0$. One can directly check that the assumption of Theorem 5.7 are satisfied.

Adjoining the third vector field $r_3 = [0, 0, 1]^T$, we obtain a full rich frame, which also satisfies hypothesis of Theorem 5.7, and therefore it admits strictly hyperbolic fluxes, all of which, by construction belong to $\mathcal{F}(R)$. We do not include the general explicit expression for these fluxes, which is rather long and involves special functions.

On the other hand, if we adjoin vector field $r_3 = [0, 0, -u]^T$, we obtain a non-rich full frame (with $c_{32} = -\frac{1}{u}$), such that modulo $\mathcal{F}^{uv}$, it has a one parametric family of strictly hyperbolic fluxes:

$$f = a [v, \frac{u^2}{2} + w, 0]^T,$$

with the eigenvalues

$$\lambda^1 = -\sqrt{u}; \quad \lambda^2 = \sqrt{u}; \quad \lambda^3 = 0.$$

By construction, $\mathcal{F}(R)$ contains fluxes (179), and, thus, it contains both rich and non-rich strictly hyperbolic fluxes.

### 7.2 Non-involutive partial frames of two vector fields in $\mathbb{R}^3$

We now present examples of non-involutive partial frames $R = \{r_1, r_2\}$ on some open subsets of $\mathbb{R}^3$, which illustrate Theorems 6.1 and 6.2. We continue with the examples, which satisfy
all the hypothesis of Theorem 6.2. These examples support the second claim of this theorem, assuring that for each \( k = 0, \ldots, 4 \), there exists \( R \), satisfying assumptions of the theorem, such that \( \dim F(R)/F^{\text{triv}} = k \).

**Example 7.4** \((\dim F(R)/F^{\text{triv}} = 0)\). For a partial frame \( R \) consisting of vector fields \( r_1 = [0, 1, u]^T \) and \( r_2 = [w, 0, 1]^T \) all fluxes are trivial.

**Example 7.5** \((\dim F(R)/F^{\text{triv}} = 1)\). For a partial frame \( R \) consisting of vector fields \( r_1 = [v, u, w]^T \) and \( r_2 = [u, w, v]^T \), on an open subset \( \Omega \subset \mathbb{R}^3 \), where these vectors are independent, the non-trivial fluxes form a one-parametric family:

\[
f = \frac{c_1}{(u + v + w)^2} \begin{bmatrix} -\frac{1}{2}u^2 - uv & -(u + v)(u + w) - \frac{1}{2}v^2 \\ v + \frac{1}{2}w^2 \\ u \end{bmatrix}.
\]

This frame does not satisfy condition (92) and, therefore, in agreement with Theorem 6.1 all non-trivial fluxes are strictly hyperbolic with eigenfunctions:

\[
\lambda_1 = c_1 \frac{u - v}{(u + v + w)^2}, \quad \lambda_2 = c_1 \frac{v - w}{(u + v + w)^2}, \quad \lambda_3 = 0.
\]

with the third eigenvector equal to \( r_3 = [u, v, w]^T \).

**Example 7.6** \((\dim F(R)/F^{\text{triv}} = 2)\). For a partial frame \( R \) consisting of vector fields \( r_1 = [-1, 0, v + 1]^T \) and \( r_2 = [\frac{w}{v-1}, -1, u]^T \), defined on an appropriate open subset of \( \mathbb{R}^3 \), the set of non-trivial fluxes forms a two-dimensional vector space:

\[
f = c_1 \begin{bmatrix} \frac{((v - 1)u + w)Ei(v - 1) - e^{1-v}u}{(v + 1)((1 - v)u - w)Ei(v - 1) + (2(v + 1)u + w)e^{1-v}} \\ \frac{1}{2} \left((v - 1)^2 Ei(v - 1) - (3v + 2)e^{1-v}\right) \\ uv + w \\ u (1 - v^2) - vw \end{bmatrix} + c_2 \begin{bmatrix} uv + w \\ u (1 - v^2) - vw \\ \frac{v^2}{4} \\ v \end{bmatrix},
\]

where \( Ei \) is the exponential integral:

\[
Ei(x) = \int_1^\infty \frac{e^{-tx}}{t} \, dt.
\]

This frame does not satisfy condition (92) and, therefore, in agreement with Theorem 6.1 all non-trivial fluxes are strictly hyperbolic with eigenfunctions:

\[
\lambda_1 = -c_1 (2Ei(v - 1) + e^{1-v}) - c_2;
\]

\[
\lambda_2 = c_1 ((v - 1)Ei(v - 1) + v e^{1-v}) + c_2 v;
\]

\[
\lambda_3 = c_1 e^{1-v} + c_2.
\]

The third eigenvector of \( [DF] \) is:

\[
r_3 = \begin{bmatrix} c_1 Ei(v - 1) + c_2 \\ 0 \\ c_1 (2 e^{1-v} + (v - 1)Ei(v - 1)) - c_2 (v - 1) \end{bmatrix}.
\]

\(^{6}\)technically, we should say “the set of non-trivial fluxes and the zero flux form a two-dimensional vector space.”
Example 7.7 (dim $F(\mathcal{R})/F^{\text{triv}} = 3$). For a partial frame $\mathcal{R}$ consisting of vector fields $r_1 = [1, \sqrt{w}, 0]^T$ and $r_2 = [u, 0, -w]^T$ the set of non-trivial fluxes forms a three-dimensional vector space:

$$f = c_1 \begin{bmatrix} 3uwv - u^2 - v^2w \\ uwv \\ vuv^3 - uw^2 \end{bmatrix} + c_2 \begin{bmatrix} v \\ uw \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} u\sqrt{w} - v \\ u \sqrt{w} \\ 0 \end{bmatrix}. $$

In this case, when $c_1 = 0$ and $c_2 = \frac{1}{2}c_3$, we obtain a one parametric family of non-hyperbolic fluxes:

$$f^{\text{nh}} = c \begin{bmatrix} u\sqrt{w} - \frac{1}{2}v \\ \frac{1}{2}u \sqrt{w} \\ \frac{1}{3}v \end{bmatrix}. $$

with the eigenfunctions:

$$\lambda^1 = \lambda^2 = \frac{1}{2}c\sqrt{w}. $$

For an extra reassurance, we can confirm that $\mathcal{R}$ satisfies the necessary condition (92) for admitting non-hyperbolic fluxes. To check this conditions we complete $\mathcal{R}$ to a frame, for instance, by adjoining the vector field $s = [r_1, r_2] = [1, \frac{1}{2}\sqrt{w}, 0]^T$. One can also confirm an observation made in Remark 6.8 that $s$ is a generalized eigenvector:

$$\vec{\nabla}_s f = \frac{1}{2}c\sqrt{w}s + \frac{1}{4}c\sqrt{w}r_1. $$

In agreement with Theorem 6.1 all other fluxes are strictly hyperbolic with the eigenfunctions:

$$\lambda^1 = c_1(v\sqrt{w} + uw) + c_2\sqrt{w}; $$
$$\lambda^2 = c_1 \left( \frac{3}{2}v\sqrt{w} - uw \right) + c_3 \frac{1}{2}\sqrt{w}; $$
$$\lambda^3 = c_1(2v\sqrt{w} - 3uw) - c_2\sqrt{w} + c_3\sqrt{w}. $$

The third eigenvector of $[DF]$ is:

$$r_3 = \begin{bmatrix} c_1 \left( 2u^2w - 2v^2 \right) + c_1 c_2 \left( 3\sqrt{w}u - v \right) - c_1 c_3 \left( \sqrt{w}u + v \right) + c_2^2 - 2c_3 \\ \frac{1}{2}(c_1 v + c_2) \left( 2c_1 \left( \sqrt{w}u + v \right) - c_3 \sqrt{w} + c_2 \right) \\ -2u w \left( c_1 \left( \sqrt{w}u + v \right) + c_2 \sqrt{w} \right) \end{bmatrix}. $$

We observe that, in the last example, the relationship of $r_3$ on $c_1$, $c_2$ and $c_3$ is non-linear.

Example 7.8 (dim $F(\mathcal{R})/F^{\text{triv}} = 4$). For a partial frame $\mathcal{R}$ consisting of vector fields $r_1 = [1, 0, v]^T$ and $r_2 = [0, 1, -u]^T$, the set of non-trivial fluxes forms a four-dimensional vector space:

$$f = c_1 \begin{bmatrix} 2u(w + uu) \\ 2v(w - uu) \\ w^2 + 3uwv \end{bmatrix} + c_2 \begin{bmatrix} 2u^2 \\ w - uu \\ 2u^2v \end{bmatrix} + c_3 \begin{bmatrix} uv + w^2 \\ -2v^2 \\ 2uw^2 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 2v \\ w - uu \end{bmatrix}. $$

This frame does not satisfy condition (92) and, therefore, in the agreement with Theorem 6.1 all non-trivial fluxes are strictly hyperbolic with eigenfunctions:

$$\lambda^1 = 2c_1(w + 3uv) + 4c_2u - 2c_3v; $$
$$\lambda^2 = 2c_1(w - 3uv) - 2c_2u - 4c_3v + 2c_4; $$
$$\lambda^3 = 2c_1w + c_2u - c_3v + c_4. $$
The third eigenvector of \( |Df| \) is:

\[
\mathbf{r}_3 = \begin{bmatrix}
-2c_1u - c_3 \\
2c_1v + c_2 \\
2c_1u v + 2c_2u + 2c_3v - c_4
\end{bmatrix}.
\]

For the contrast, we show another maximal-dimensional example, where \( \mathcal{F}(\mathbb{R}) \) contains non-hyperbolic fluxes.

**Example 7.9** (\( \dim \mathcal{F}(\mathbb{R})/\mathcal{F}^{\text{triv}} = 4 \)). For a partial frame \( \mathcal{R} \) consisting of vector fields \( r_1 = [1, 0, 2u]^T \) and \( r_2 = [0, 1, u]^T \), the set of non-trivial fluxes forms a four-dimensional vector space:

\[
f = c_1 \begin{bmatrix}
u (uv - w) \\
-2v (2uv - w) \\
-6uv (uv - w) - 2w^2
\end{bmatrix} + c_2 \begin{bmatrix}
u^2 \\
2(2uv - w) \\
2u^2v
\end{bmatrix} + c_3 \begin{bmatrix}
w - uv \\
2v^2 \\
2uv^2
\end{bmatrix} + c_4 \begin{bmatrix}
0 \\
v \\
2uv - w
\end{bmatrix}.
\]

In this case, when \( c_1 = c_3 = c_4 = 0 \) and \( c_2 = 1 \), we obtain a one-parametric family of non-hyperbolic fluxes:

\[
f^{\text{nh}} = c \begin{bmatrix}
u^2 \\
2(2uv - w)
\end{bmatrix}
\]

with the eigenfunctions:

\[\lambda^1 = \lambda^2 = 2cu.\]

In agreement with Theorem 6.10, all other fluxes are strictly hyperbolic with the eigenfunctions:

\[
\lambda^1 = -c_1w + 2c_2u + c_3v; \\
\lambda^2 = 2c_1(w - 3u v) + 2c_2u + 4c_4 v + c_4; \\
\lambda^3 = 2c_1(3u v - 2w) + 2c_2u - 2c_3v - c_4.
\]

**Remark 7.10.** The following interesting property can be observed in Examples 7.3, 7.9, where \( 1 \leq \dim \mathcal{F}(\mathbb{R})/\mathcal{F}^{\text{triv}} \leq 4 \). For basis fluxes \( f_1, \ldots, f_k \) presented in these examples (\( k = 2, \ldots, 4 \), depending on an example), the corresponding Jacobian matrices, \( DF_1, \ldots, DF_k \), have the additivity of eigenvalues property, called the \( \mathcal{L} \)-property in Motzkin’s and Taussky’s papers [10, 11]. By construction, \( r_1 \) and \( r_2 \) are eigenvectors of \( DF_1, \ldots, DF_k \), and, therefore, it is obvious, that if \( \lambda_1, \ldots, \lambda_k \) are the eigenvalues for \( r_1 \) of \( DF_1, \ldots, DF_k \), respectively, and \( \lambda_1^2, \ldots, \lambda_k^2 \) are the eigenvalues for \( r_2 \) of \( DF_1, \ldots, DF_k \), respectively, then for \( f = c_1f_1 + \cdots + c_kf_k \), the Jacobian matrix \( DF \) has the eigenvalue \( \lambda^1 = c_1\lambda_1 + \cdots + c_k\lambda_k^1 \) for the eigenvector \( r_1 \) and the eigenvalue \( \lambda^2 = c_1\lambda_1^2 + \cdots + c_k\lambda_k^2 \) for the eigenvectors \( r_2 \). However, it is surprising that the third eigenvalues also “add up”. Indeed, is still true in all of the examples that \( \lambda^3 = c_1\lambda_1^3 + \cdots + c_k\lambda_k^3 \) is the third eigenvalue of \( DF \), where \( \lambda_1, \ldots, \lambda_k \) are the third eigenvalues of \( DF_1, \ldots, DF_k \), despite the fact that these matrices have non-collinear third eigenvectors \( r_{3,1}, \ldots, r_{3,k} \).

We finish with an example demonstrating that even when the first assumption of the Theorem 6.2 i.e. the necessary conditions [11] for strict hyperbolicity, holds, the second assumption given by the condition [33] may not hold.

**Example 7.11.** Consider a partial frame \( \mathcal{R} \), defined on an open subset of \( \mathbb{R}^3 \), where \( w > 0 \), consisting of vector fields \( r_1 = [1, 0, w]^T \) and \( r_2 = [0, 1, -\frac{w}{r}\ln(w) + u]^T \). This partial frame
satisfies the necessary condition for strict hyperbolicity. A vector field \( s = [0, 0, 1]^T \) completes \( \mathcal{R} \) to a frame, and one can easily verify that relative to this frame:

\[
\Gamma_{22}^3(u) \Gamma_{11}^3(u) - 9 \Gamma_{12}^3(u) \Gamma_{21}^3(u) \equiv 0. \tag{180}
\]

In fact, this example was obtained by setting up a differential equation on the components of vector fields \( r_1 \) and \( r_2 \), induced by the identity (180) and finding its particular solution.

For this partial frame the vector space \( \mathcal{F}(\mathcal{R})/\mathcal{F}_{\text{triv}} \) is one-dimensional:

\[
f = c \begin{bmatrix}
\frac{1}{2} e^{-u} \\
\frac{1}{2} e^{-u} w \\
e^{-u} w (u - \frac{9}{8} \ln(w) + \frac{1}{8})
\end{bmatrix}.
\]

This frame does not satisfy condition (92) and, therefore, in the agreement with Theorem 6.1 all non-trivial fluxes are strictly hyperbolic with eigenfunctions:

\[
\lambda^1 = -\frac{1}{8} c e^{-u}; \\
\lambda^2 = c e^{-u} \left( u - \frac{9}{8} \ln(w) \right); \\
\lambda^3 = 0
\]

and the third eigenvector field is \( r_3 = [0, 1, 0]^T \).

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M. Benfield, SAN DIEGO, CA (mike.benfield@gmail.com).

H. K. Jenssen, DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY (jenssen@math.psu.edu).

I. A. Kogan, DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY (iakogan@ncsu.edu).