Structure Entropy and Resistor Graphs *

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Abstract

The authors [20] defined the notion of structure entropy of a graph $G$ to measure the information embedded in $G$ that determines and decodes the essential structure of $G$. Here, we propose the notion of resistance of a graph as an accompanying notion of the structure entropy to measure the force of the graph to resist cascading failure of strategic virus attacks. We show that for any connected network $G$, the resistance of $G$ is $R(G) = H^1(G) - H^2(G)$, where $H^1(G)$ and $H^2(G)$ are the one- and two-dimensional structure entropy of $G$, respectively. According to this, we define the notion of security index of a graph to be the normalized resistance, that is, $\theta(G) = \frac{R(G)}{H^1(G)}$. We say that a connected graph is an $(n, \theta)$-resistor graph, if $G$ has $n$ vertices and has security index $\theta(G) \geq \theta$. We show that trees and grid graphs are $(n, \theta)$-resistor graphs for large constant $\theta$, that the graphs with bounded degree $d$ and $n$ vertices, are $(n, \frac{d}{2} - o(1))$-resistor graphs, and that for a graph $G$ generated by the security model [18, 19], with high probability, $G$ is an $(n, \theta)$-resistor graph, for a constant $\theta$ arbitrarily close to $1$, provided that $n$ is sufficiently large. To the opposite side, we show that expander graphs are not good resistor graphs, in the sense that, there is a global constant $\theta_0 < 1$ such that expander graphs cannot be $(n, \theta)$-resistor graph for any $\theta \geq \theta_0$. In particular, for the complete graph $G$, the resistance of $G$ is a constant $O(1)$, and hence the security index of $G$ is $\theta(G) = o(1)$. This shows that, for arbitrarily small constant $\epsilon > 0$, there is an $N$ such that for any $n \geq N$, the complete graph $G$ of $n$ vertices cannot be an $(n, \epsilon)$-resistor graph. Finally, we show that for any simple and connected graph $G$, if $G$ is an $(n, 1 - o(1))$-resistor graph, then there is a large $k$ such that the $k$-th largest eigenvalue of the Laplacian of $G$ is $o(1)$, giving rise to an algebraic characterization for the graphs that are secure against intentional virus attack.

1 Introduction

Shannon’s [28] metric measures the uncertainty of a probabilistic distribution as

$$H(p_1, \ldots, p_n) = - \sum_{i=1}^{n} p_i \log_2 p_i. \tag{1}$$

This metric and the associated concept of noise, have provided rich sources for both information science and technology. However, as pointed out by Brooks in [5], it had been a longstanding challenge to define the information that is embedded in a physical system, which determines and decodes the essential structure of the (observed and noisy) physical system. Such a metric, if well defined, may provide an approach to understand the folded three-dimensional structures of proteins. Shannon [29] himself realized that his metric of information fails to support the analysis of communication networks to answer the question such as characterization of the optimal communication networks. The answer for this question depends on a well-defined definition of the structure entropy, that is, the information embedded in a communication network.

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The challenge of the quantification of structural information becomes more and more important in the current information age, in which noisy big data with or without structures are assumed to support the world and our societies. Structural information may provide the principles for structuring the unstructured data and for discovering the knowledge from noisy data by removing the noises. To this end, the authors [20] introduced the notion of coding tree of a graph, and defined the structure entropy of a graph to be the minimum amount of information required to determine the codeword of the vertex in a coding tree for the vertex that is accessible from random walk with stationary distribution in the graph. The structure entropy of a graph is hence the information embedded in the graph. The structural information of a graph defined in this way allows us to decode the essential structure of the graph simultaneously at the same time when we measure the structure entropy of the graph.

In the present paper, we analyze communication networks based on the structural information theory [20]. Specifically, we investigate the security of networks against cascading failure of virus attacks. We introduce the notion of resistance of a graph and resistor graphs as accompanying notions of structure entropy of graphs to analyze the security of networks.

Network security has become a grand challenge in modern information science and computer science. An interesting discovery in network theory in the last few years is that network topology is universal in nature, society, and industry [3]. In fact, the current highly connected world is assumed to be supported by numerous networking systems. Real networks are not only too important to fail, but also too complicated to understand.

Erdős-Rényi proposed the first model [10, 11] (The ER model in short) to capture complex systems based on the assumption that real systems are evolved randomly. The ER model explores that if a graph is generated randomly, then the diameter of the graph is exponentially smaller than the size of the graph, referred to as the small world phenomenon. It has been shown that most real world networks do satisfy the small world phenomenon, giving rise to the first universal property of networks. However, can we really assume that real networks are purely random? Barabási and Albert [4] proposed a graph generator by introducing preferential attachment as an explicit mechanism, the model is thus called the preferential attachment (PA) model. Consequently, networks generated by the PA model naturally follow a power law. It has been shown that most real networks follow a power law. Consequently, power law has become the second universal property of networks [3].

As a matter of fact, real world networks are highly connected and naturally evolving, in which information spread easily and quickly. This is of course one of the main advantages of networks in both theory and applications. However, at the same time, this could be one of the main disadvantages of networks. Because, virus also quickly spreads all over the networks. It is due to this reason that network security has become a grand new challenge in the 21st century.

Networks may fail under attack due to different mechanisms [2, 22, 13, 14, 27, 1]. The first type is the physical attack of removal of some vertices or edges. It has been shown that in scale-free networks of the preferential attachment (PA) model [4], the overall network connectivity measured by the sizes of the giant connected components and the diameters does not change significantly under random removal of a small fraction of vertices, but vulnerable to removal of a small fraction of the high-degree vertices [1, 9, 24]. The second type is the cascading failure of attacks, which naturally appeared in rumor spreading, disease spreading, voting, and advertising [34, 2, 22]. One of the main features of networks in the current highly connected world is that failure of a few vertices of a network may generate a cascading failure of the whole network. It has been shown that in scale-free networks of the preferential attachment model even a weakly virulent virus can spread [25]. This explains a fundamental characteristic of security of networks [27].

For the physical attacks or random errors of removal of vertices, it was shown that the optimal networks resisting both physical attacks and random errors have at most three values of degrees for all the vertices of the networks [13], that networks having the optimal robustness resisting both high-degree vertices attacks and random errors, has a bimodal degree distribution [32]. To enhance the robustness of networks against biological virus spreading, it was proposed in [8] the acquaintance immunization strategy, which calls for the immunization of random acquaintances of randomly chosen vertices, and more recently, a security enhancing algorithm was proposed in [26] by randomly swapping two edges for a number of pairs of edges.

Li et al [18] proposed the security model of networks by using the idea of the Art of War [31]. It has been shown that with appropriate choices of parameters, the networks generated by the security model are secure against attacks of small scales [19]. Li and Pan [20] proposed the notion of structure entropy of networks to quantitatively measure the dynamical complexity of interactions and communications of the network, for each natural number $K$. However, it is an important open question to define a measure of security of a graph against cascading failure of intentional virus attacks. The authors of this paper and his coauthors [16] proposed the notions of resistance and security index of a graph by using the one- and two-dimensional structure entropy, and verified that both the resistance and security index measure the force of the graph to resist cascading failure of virus attacks. In [21], it was shown that resistance maximization is in fact the principle for defending networks against the super virus that infect all the neighbor vertices immediately. In the present paper, we establish the basic theory of resistance and
security index of graphs.

We organise the paper as follows. In Section 2 we introduce and analyze the notions of coding tree and structure entropy proposed by the authors in [20], we also introduce a variation of the structure entropy to study the relationship between the Shannon entropy and the structure entropy. In Section 3 we define the notions of resistance and security index of networks, and establish both the local and global resistance laws of networks. In Section 4 we introduce some basic results of the resistance and security indices of networks. In Section 5 we establish the theory of the resistance and security indices of the networks generated by the security model. In Section 6 we establish both the combinatorial and algebraic characterization theorems for the graphs with the optimal two-dimensional structure entropy, i.e., the security indices of the resistor graphs. In Section 7 we establish a lower bound of the resistance of bounded degree graphs. In Section 8 we show that the resistance of a complete graph is actually a universal constant $O(1)$. In Section 9 we establish both the combinatorial and algebraic characterization theorems for the resistor graphs. In Section 10 we summarise the results of the paper.

## 2 Structure Entropy of Graphs

The authors of this paper [20] proposed the notion of structure entropy of a graph to measure the information embedded in a physical system that decodes the essential structure of the system. In this section, we introduce the notion of structure entropy.

Before introducing the structure entropy, we recall the Huffman codes [12].

### 2.1 Huffman codes

Suppose that $\Sigma = \{1, 2, \cdots, n\}$ such that the probability $i$ occurs is $p_i$, for each $i$. Let $\sum_{i=1}^{n} p_i = 1$. We will encode the elements of $\Sigma$ by 0, 1-strings such that there is no codeword of an element is an initial segment of the codeword of another element of $\Sigma$.

Suppose that $T$ is a binary tree whose leaves are the codewords of the elements $1, 2, \cdots, n$. Suppose that element $i$ has probability $p_i$ with codeword $\alpha_i$ in $T$, then the average length of the codewords is

$$L^T(p_1, p_2, \cdots, p_n) = \sum_{i=1}^{n} p_i \cdot |\alpha_i|,$$

where $|\alpha_i|$ is the length of 0, 1-string $\alpha_i$.

The Huffman codes are to find the binary tree $T$ such that $L^T(p_1, p_2, \cdots, p_n)$ in Equation (2) is minimized. We define

$$L(p_1, p_2, \cdots, p_n) = \min_T L^T(p_1, p_2, \cdots, p_n),$$

where $T$ ranges over all the binary trees of $n$ leaves.

By definition, $L(p_1, p_2, \cdots, p_n)$ is the minimum average length of the binary representation of the alphabet $\Sigma$. Huffman codes achieve the minimum solution $L(p_1, p_2, \cdots, p_n)$.

It is known that

$$L(p_1, p_2, \cdots, p_n) \geq H(p_1, p_2, \cdots, p_n),$$

with equality holds when $p_i = 2^{-k}$ for some $k$, and for all $i$, where $H(p_1, p_2, \cdots, p_n)$ is the Shannon entropy of $p = (p_1, p_2, \cdots, p_n)$.

This means that the minimum average length of the binary representation of an element picked from a probability distribution is lower bounded by the Shannon entropy of the distribution, and the Shannon entropy is the tight lower bound of the minimum average length of the binary representations.

Before developing our theory, we recall a basic interpretation of the $\log_2$ function:

Let $p$ be a number with $0 < p < 1$. Suppose that $k$ is a natural number such that

$$\frac{1}{2^{k+1}} \leq p < \frac{1}{2^k},$$

which implies that

$$k + 1 \geq - \log_2 p > k.$$  

Equation (6) indicates that
(i) \(- \log_2 p\) is the information (or uncertainty) embedded in an item that occurs with probability \(p\).

(ii) \(\lceil - \log_2 p \rceil\) many bits are sufficient to express the item that occurs with probability \(p\).

(iii) The minimum length of the binary codeword of the item that occurs with probability \(p\) is exactly \(\lceil - \log_2 p \rceil\), which is greater than or equal to the information embedded in the item occurring with probability \(p\), that is, \(- \log_2 p\).

To define the structure entropy of a graph, we first need to encode a graph. Similarly to the Huffman codes, we encode a graph by a tree. However, it is a priority tree below, instead of a binary tree in the Huffman codes.

2.2 Priority tree

Definition 2.1. (Priority tree) A priority tree is a rooted tree \(T\) with the following properties:

(i) The root node is the empty string, written \(\lambda\).

A node in \(T\) is expressed by the string of the labels of the edges from the root to the node. We also use \(T\) to denote the set of the strings of the nodes in \(T\).

(ii) For every node \(\alpha\) in \(T\), there is a natural number \(k\) such that there are \(k\) edges linking \(\alpha\) to its \(k\) children.

The edges are labelled by

\[0 < 1 < \cdots < k - 1.\]

(Remark: (i) Unlike Huffman codes, we use an alphabet of the form \(\Sigma = \{0, 1, \cdots, k\}\) for each tree node \(\alpha\). In the Huffman codes, we always use the alphabet \(\Sigma = \{0, 1\}\).

(ii) Different nodes in \(T\) may have different numbers of children, i.e., different \(k\)'s.)

(iii) Every tree node \(\alpha\) is a string of numbers from 0 to some natural number.

For two tree nodes \(\alpha, \beta\), if \(\alpha\) is an initial segment of \(\beta\) as strings, then we write \(\alpha \subseteq \beta\). If \(\alpha \subseteq \beta\) and \(\alpha \neq \beta\), we write \(\alpha \subset \beta\).

2.3 Coding tree of a graph

Definition 2.2. (Coding tree of a graph) Let \(G = (V, E)\) be a graph. A coding tree of \(G\) is a priority tree \(T\) such that every tree node \(\alpha \in T\), there is a subset \(T_\alpha\) of the vertices \(V\), and such that the following properties hold:

(i) The root node \(\lambda\) is associated with the whole set \(V\) of vertices of \(G\), that is, \(T_\lambda = V\).

(ii) For every node \(\alpha \in T\), if \(\beta_1, \beta_2, \cdots, \beta_k\) are all the children of \(\alpha\), then \(\{T_\beta_1, \cdots, T_\beta_k\}\) is a partition of \(T_\alpha\).

(iii) For every leaf node \(\gamma \in T\), \(T_\gamma\) is a singleton.

Definition 2.3. (Codeword) Let \(G = (V, E)\) be a graph, and \(T\) be a coding tree of \(G\).

(i) For every node \(\alpha \in T\), we call \(\alpha\) the codeword of set \(T_\alpha\), and \(T_\alpha\) the marker of \(\alpha\).

(ii) For a leaf node \(\gamma \in T\), if \(T_\gamma = \{v\}\), then we say that \(\gamma\) is the codeword of \(v\), and \(v\) is the marker of \(\gamma\).

A coding tree \(T\) of a graph \(G\) satisfies the following

Definition 2.4. (Coding tree properties) Given a graph \(G\) and a coding tree \(T\) of \(G\), we assume that the following properties hold:

(i) For every node \(\alpha \in T\), the marker \(T_\alpha\) of \(\alpha\) is explicitly determined. This means that if we know \(\alpha\), then we have already known the marker \(T_\alpha\). This means that there is no uncertainty in \(T_\alpha\) once we know the codeword \(\alpha \in T\).

(ii) For every node \(\alpha \in T\), if we know the codeword \(\alpha\), then we simultaneously know \(\beta\) for all the codewords \(\beta\)’s in the branch between the root node \(\lambda\) and \(\alpha\) in \(T\), i.e., the \(\beta\) with \(\beta \subseteq \alpha\).
The advantage of the coding tree is the coding tree properties in Definition 2.4. The key to our definition of structure entropy is to use the coding tree properties above to reduce the uncertainty of a graph by a coding tree of the graph.

**Lemma 2.1.** Let \( G = (V, E) \) be a graph and \( T \) be a coding tree of \( G \). Then:

1. For every leaf node \( \gamma \in T \), there is a unique vertex \( v \) such that \( \gamma \) is the codeword of \( v \).
2. For every vertex \( v \in V \), there is a unique leaf node \( \gamma \in T \) such that \( v \) is the marker of \( \gamma \).

**Proof.** By the definition of coding tree.

By Lemma 2.1, the set of all the leaves in \( T \) is the set of codewords of the vertices \( V \). This property is the same as the Huffman codes, that is, all the leaf nodes of the tree are the codewords desired.

### 2.4 Structure entropy of a graph given by a coding tree

The authors [20] introduced the notion of structure entropy of a graph.

**Definition 2.5.** (Structure entropy of a graph by a coding tree, Li and Pan [20]) Let \( G = (V, E) \) be a graph, and \( T \) be a coding tree of \( G \). We define the structure entropy of \( G \) by coding tree \( T \) as follows:

\[
\mathcal{H}^T(G) = - \sum_{\alpha \neq \lambda, \alpha \in T} \frac{g_\alpha}{\text{vol}(G)} \cdot \log_2 \frac{\text{vol}(\alpha)}{\text{vol}(\alpha^-)},
\]

where \( g_\alpha = |E(T_\alpha, T_\lambda)| \), that is, the number of edges from the complement of \( T_\alpha \), i.e., \( T_\lambda \), to \( T_\alpha \), \( \text{vol}(G) \) is the volume of \( G \), that is, the total degree of vertices in \( G \), \( \text{vol}(\beta) \) is the volume of the vertices set \( T_\beta \), and \( \alpha^- \) is the parent node of \( \alpha \) in \( T \).

To understand Equation (7), we conclude the following items for the metric \( \mathcal{H}^T(G) \):

1. For every node \( \alpha \in T \), \( T_\alpha \) is the set of vertices associated with \( \alpha \). Suppose that, once we known \( \alpha \), we have already known the set \( T_\alpha \).
2. For each node \( \alpha \in T \) with \( \alpha \neq \lambda \), since \( \alpha^- \) is the parent node of \( \alpha \) in \( T \), the probability that the vertex \( v \in V \) from random walk with stationary distribution in \( G \) is in \( T_\alpha \) under the condition that \( v \in T_\alpha^- \) is \( \frac{\text{vol}(\alpha)}{\text{vol}(\alpha^-)} \).

Therefore the entropy (or uncertainty) of \( v \in T_\alpha \) under the condition that \( v \in T_\alpha^- \) is \( - \log_2 \frac{\text{vol}(\alpha)}{\text{vol}(\alpha^-)} \).
3. For every node \( \alpha \in T \), \( g_\alpha \) is the number of edges that random walk with stationary distribution arrives at \( T_\alpha \) from vertices \( T_\alpha \), the vertices outside \( T_\alpha \). Therefore, the probability that a random walk with stationary distribution is from outside \( T_\alpha \) to vertex in \( T_\alpha \) is \( \frac{g_\alpha}{\text{vol}(G)} \).

Consider the stochastic process of random walks with stationary distribution in \( G \). It is the stochastic process as follows:

\[
X_0, X_1, X_2, \cdots,
\]

with the following properties:

1. Let \( x_0 \) be the vertex chosen in \( V \) with probability proportional to vertex degree, and \( X_0 \) be the codeword of vertex \( x_0 \) in \( T \). Suppose that \( X_i \) and \( x_i \) are defined.
2. Let \( x_{i+1} \) be the neighbor of \( x_i \) chosen uniformly and randomly among all the neighbors of \( x_i \) in \( G \). Then \( X_{i+1} \) is defined as the codeword of \( x_{i+1} \) in \( T \).

In the stochastic process in Equation (8), we are interested in the quantification of the entropy of \( X_{i+1} \) under the condition that we have already known \( X_i \), denoted by

\[
\tilde{H}(X_{i+1}|X_i).
\]

To compute \( \tilde{H}(X_{i+1}|X_i) \), suppose that \( x_i \) and \( x_{i+1} \) are as above, and that \( X_i = \alpha \in T \) is the codeword of \( x_i \) in \( T \). Notice that the codeword \( \alpha \) is a leaf node in \( T \). By Definition 2.4, we know \( T_\delta \) for all the nodes \( \delta \subseteq \alpha \), i.e., the initial segments of \( \alpha \) as strings.
Let $\gamma$ be the longest node $\delta \in T$ with $\delta \subseteq \alpha$ such that $x_{i+1} \in T_\delta$ holds. Then we know that $\gamma$ is an initial segment of the codeword of $x_{i+1}$ in $T$. To determine the codeword of $x_{i+1}$ in $T$, we only need to find the branch from $\gamma$ to a leaf node $\beta \in T$ such that $x_{i+1} \in T_\beta$. According to the analysis above, the information of $X_{i+1}$ under the condition of $X_i$ is:

$$\tilde{H}(X_{i+1} = \beta | X_i = \alpha) = - \sum_{\delta \in T, \gamma \subseteq \delta} \frac{g_\delta}{\text{vol}(\gamma)} \log_2 \frac{\text{vol}(\delta)}{\text{vol}(\gamma)},$$

where $\gamma = \alpha \cap \beta$ is the node in $T$ at which $\alpha$ and $\beta$ branch in $T$, $g_\delta$ is the number of edges in the cut $(T_\alpha, \bar{T}_\alpha)$.

We notice that, only if both $x_{i+1} \in T_\beta$ and $x_i \not\in T_\beta$ occur, we need to determine the codeword of $T_\beta$ in $\bar{T}_\beta$, for which the amount of information required is $- \log_2 \frac{\text{vol}(\beta)}{\text{vol}(\bar{T}_\beta)}$. So, intuitively, $\tilde{H}(X_{i+1} = \beta | X_i = \alpha)$ is the amount of information, in terms of the codeword of $T_\beta$ in $\bar{T}_\beta$, required to determine the codeword of $x_{i+1}$ under the condition that the codeword of $x_i$ is known. Note that we use the codewords of nodes in the coding tree to measure amount of information. That is why we use the notation $\tilde{H}(\cdot)$ to distinguish from the classic conditional entropy notation $H(\cdot)$.

Our definition of $H^T(G)$ in Definition 2.5 is

$$H^T(G) = \sum_{e=(x_i, x_{i+1}) \atop x_i, x_{i+1} \in V} \tilde{H}(X_{i+1} = \beta | X_i = \alpha),$$

where $X_i$ is the codeword of $x_i$, and $X_{i+1}$ is the codeword of $x_{i+1}$.

This measures the information required to determine the codeword of the vertex in $V$ that is accessible from random walk with stationary distribution in $G$, under the condition that the codeword of the starting vertex of the random walk is known.

### 2.5 Structure entropy

**Definition 2.6.** (Structure entropy of a graph, Li and Pan [20]) Let $G = (V, E)$ be a graph.

1. The structure entropy of $G$ is defined as

$$H(G) = \min_T \{H^T(G)\},$$

where $T$ ranges over all the coding trees of $G$.

[Remark: (i) We notice that the Huffman codes require to find a binary tree $T$ such that the $L^T(p_1, \cdots, p_n)$ in Equation 2 is minimized. In this case, Huffman codes have already been the optimum solution.

(ii) Our structure entropy of a graph requires to find a coding tree $T$ such that the $H^T(G)$ in Equation 7 is minimized. However, there is no algorithm achieving the optimal structure entropy so far. Although there are nearly linear time greedy algorithms for approximating the optimum coding tree, with remarkable applications. [20]]

2. For natural number $K$, the $K$-dimensional structure entropy of $G$ is defined as

$$H^K(G) = \min_T \{H^T(G)\},$$

where $T$ ranges over all the coding trees of $G$ of heights at most $K$.

[Remark: This allows us to study the structure entropy in different dimensions.]

The metric $H(G)$ has the following intuitions:

- The structure entropy $H(G)$ of $G$ is the least amount of information required to determine the codeword of the vertex in a coding tree that is accessible from random walk with stationary distribution in $G$.

- The structure entropy $H(G)$ is the information that determines and decodes the coding tree $T$ of $G$ that minimizes the uncertainty in positioning the vertex that is accessible from random walk in graph $G$.

Therefore, $H(G)$ is not only the measure of information, but decodes the structure of $G$ that minimises the uncertainty in the communications in the graph, which can be regarded as the ”essential" structure of the graph.
The $K$-dimensional structure entropy $\mathcal{H}_g^K(G)$ of $G$ has the similar intuitions as above.

We remark that the structure entropy has a rich theory with remarkable applications, more details are referred to [20]. Here we develop the theory of security by using the structure entropy with the cut module function as defined in [20]. It is interesting to notice that a theory of network security can be established by using only the one- and two-dimensional structure entropy developed in [20].

### 2.6 A variation of structure entropy

The structure entropy of a graph in Definitions 2.5 and 2.6 is often misunderstood as the average length of the codeword of the vertex that is accessible from random walk with stationary distribution in the graph. We argue that this is not the case.

To better understand the question, we introduce a variation of the structure entropy. It depends on a module function of a graph.

**Definition 2.7.** (Module function) Let $G = (V, E)$ be a connected graph. Let $\text{vol}(G)$ be the volume of $G$. A module function of $G$ is a function $g$ of the form:

\[
g : 2^V \rightarrow \{0, 1, \cdots, \text{vol}(G)\}.
\]

We define the structure entropy of a graph with a module function as follows.

**Definition 2.8.** (Structure entropy of a graph with a module function by a coding tree) Let $G = (V, E)$ be a graph, $g$ be a module function of $G$, and $T$ be a coding tree of $G$. We define the structure entropy of $G$ with module function $g$ by coding tree $T$ as follows:

\[
\mathcal{H}_g^T(G) = - \sum_{\alpha \in T, \alpha \neq \lambda} \frac{g(T_\alpha)}{\text{vol}(G)} \cdot \log_2 \frac{\text{vol}(\alpha)}{\text{vol}(\alpha^-)},
\]

where $\text{vol}(G)$ is the volume of $G$, $\text{vol}(\beta)$ is the volume of the vertices set $T_\beta$, and $\alpha^-$ is the parent node of $\alpha$ in $T$.

**Definition 2.9.** (Structure entropy of a graph with a module function) Let $G = (V, E)$ be a graph, and $g$ be a module function of $G$.

1. The structure entropy of $G$ with module function $g$ is defined as

\[
\mathcal{H}_g(G) = \min_T \{\mathcal{H}_g^T(G)\},
\]

where $T$ ranges over all the coding trees of $G$.

2. For natural number $K$, the $K$-dimensional structure entropy of $G$ with module function $g$ is defined as

\[
\mathcal{H}_g^K(G) = \min_T \{\mathcal{H}_g^T(G)\},
\]

where $T$ ranges over all the coding trees of $G$ of heights at most $K$.

[Remark: This allows us to study the structure entropy in different dimensions.]

The formula $\mathcal{H}_g^T(G) = - \sum_{\alpha \in T, \alpha \neq \lambda} \frac{g(T_\alpha)}{\text{vol}(\alpha)} \cdot \log_2 \frac{\text{vol}(\alpha)}{\text{vol}(\alpha^-)}$ is a generalization of $\mathcal{H}_g^T(G)$ in Definition 2.5 with the function $g$ here being an arbitrarily given module function, while the function $g$ in Definition 2.3 is the cut module function, that is, the number of edges in the cut.

In Definitions 2.8 and 2.9 the structure entropy of graph $G$ depends on a choice of a module function $g$. It is possible that there are many interesting choices for the module function $g$. We list a few of these as example:

1. For a subset $X$ of vertices $V$, $g(X)$ is the volume of $X$. In this case, $g$ is called the volume module function.
2. For each subset $X$ of $V$, $g(X)$ is the weights in the cut $(X, \bar{X})$ in $G$. In this case, we say that $g$ is the cut module function.
3. For a directed graph $G$ and for each subset $X$ of $V$, $g(X)$ is the weights of the flow from $\bar{X}$ to $X$. In this case, we call $g$ the flow module function.

For directed graphs, the flow module function would be essential to the structure entropy of the graphs.
In particular, there are module functions with additivity, with which the structure entropy collapses to the case of Shannon entropy.

**Definition 2.10.** (Additive module function) Let $G = (V, E)$ be a connected, simple graph with $n$ vertices and $m$ edges, and $g$ be a module function of $G$. We say that $g$ is an additive module function if for any disjoint sets $X$ and $Y$ of $V$,

$$g(X \cup Y) = g(X) + g(Y).$$  \hspace{1cm} (16)

**Theorem 2.1.** (Structural entropy of a graph with an additive function) Let $G = (V, E)$ be a connected, simple graph with $n$ vertices and $m$ edges, and let $g$ be an additive module function of $G$. For any coding tree $T$ of $g$, if $g$ satisfies the boundary condition

$$g(T_\alpha) = \begin{cases} d_\alpha & \text{if } \alpha \text{ is a leaf} \\ 2m & \text{if } \alpha = \lambda \end{cases},$$

then

$$\mathcal{H}_g^T(G) = - \sum_{\alpha \neq \lambda, \alpha \in T} \frac{g_\alpha}{2m} \log_2 \frac{V_\alpha}{V_{\alpha^-}},$$

\hspace{1cm} (17)

**Proof.** By Definition 2.10 noting that for every $\alpha \in T$, let $g_\alpha = g(T_\alpha)$ and $V_\alpha = \text{vol}(\alpha)$, we have:

$$\mathcal{H}_g^T(G) = - \sum_{\alpha \neq \lambda, \alpha \in T} \frac{g_\alpha}{2m} \log_2 \frac{V_\alpha}{V_{\alpha^-}} = - \sum_{\alpha \neq \lambda, \alpha \in T} \frac{g_\alpha}{2m} \log_2 V_\alpha + \sum_{\alpha \neq \lambda, \alpha \in T} \frac{g_\alpha}{2m} \log_2 V_{\alpha^-} = - \sum_{\alpha \neq \lambda, \alpha \in T} \frac{g_\alpha}{2m} \log_2 V_\alpha + \left( \sum_{\alpha \in T, \text{ non-leaf}} \frac{g_\alpha}{2m} \log_2 V_\alpha + \log_2(2m) \right),$$

by the additivity of $g$

$$= - \sum_{i=1}^n \frac{d_i}{2m} \log_2 \frac{d_i}{2m}. \hspace{1cm} (18)$$

The theorem follows. $\square$

**Definition 2.11.** (The length of the vertex accessible from random walk with stationary distribution) For a connected, simple graph $G = (V, E)$ of $n$ vertices and $m$ edges. Let $g$ be the volume module function of $G$ defined as: for any set $X$ of vertices $\bar{V}$, $g(X)$ is the volume of $X$. Suppose that $T$ is a coding tree of $G$. Then:

$$H^T_g(G) = - \sum_{\alpha \in T, \alpha \neq \lambda} \frac{V_\alpha}{2m} \log_2 \frac{V_\alpha}{V_{\alpha^-}},$$

where $V_\beta$ is the volume of $T_\beta$, $\alpha^-$ is the parent node of $\alpha$ in $T$.

Remarks: In this case, $H^T_g(G)$ is the amount of information required to describe the codeword in $T$ of the vertex that is accessible from random walk with stationary distribution in $G$, and is a lower bound of the average length of the codeword (in $T$) of the vertex that is accessible from random walk with stationary distribution in $G$.

**Theorem 2.2.** For any connected and simple graph $G = (V, E)$ with $n$ vertices and $m$ edges. For the module function $g(X) = \sum_{x \in X} d_x$, where $d_x$ is the degree of $x$ in $G$, and for any coding tree $T$ of $G$,

$$H^T_g(G) = - \sum_{i=1}^n \frac{d_i}{2m} \log_2 \frac{d_i}{2m} = \mathcal{H}^1(G),$$

where $d_i$ is the degree of vertex $i$ in $G$, $\mathcal{H}^1(G)$ is the one-dimensional structural entropy of $G$ \[20\].

**Proof.** Note that for any non-leaf node $\alpha \in T$, $V_\alpha = \sum_{\beta \in T, \beta^- = \alpha} V_\beta$, that is, $V_\alpha$ is an additive module function. The theorem follows from Theorem 2.1. $\square$
Theorem 2.2 shows that

- The information to describe the codeword of a tree of the vertex that is accessible from random walk with stationary distribution in G is independent of any coding tree T of G, and

- The minimum average length, written \( L(G) \), of the codeword in a coding tree of the vertex that is accessible from random walk with stationary distribution is greater than or equal to (or lower bounded by) the one-dimensional structure entropy \( H^1(G) \) \(^{[20]} \), or the Shannon entropy of the degree distribution of the graph. This means that

\[
L(G) = \Omega(\log_2 n),
\]

where \( n \) is the number of vertices in \( G \).

This property is in sharp contrast to the structure entropy. In fact, there are many graphs \( G \) such that the two-dimensional structure entropy \( H^2(G) = O(\log_2 \log_2 n) \), referred to \(^{[20]} \).

The proof of Theorem 2.1 also shows the reason why the structure entropy in Definitions 2.5 and 2.6 depend on the coding trees of a graph. The reason is that, the cut module function \( g \) in Definition 2.5 fails to have the additivity, since for any two disjoint vertex sets \( X \) and \( Y \), if there are edges between \( X \) and \( Y \), then \( g(X \cup Y) < g(X) + g(Y) \). This ensures that the structure entropy \( H^T(G) \) in Definition 2.5 depends on the coding tree \( T \) of \( G \). For this reason, the structure entropy provides the foundation for a new direction of information theory with rich theory and remarkable applications in many areas.

### 2.7 The relationship between Shannon entropy and the structure entropy

Given a connected graph \( G = (V, E) \), Theorem 2.2 implies that the information to describe the vertex that is accessible from random walk with stationary distribution cannot be reduced by any coding tree of the graph, so that the average length of the codewords of the vertex that is accessible from random walk with stationary distribution must be lower bounded by the entropy of the degree distribution of the graph.

However, the structure entropy of a graph defined in Definition 2.5 is determined by the coding tree of the graph, and hence the structure entropy is given in Definition 2.6. We have seen in \(^{[20]} \) that there are graphs \( G \) whose two-dimensional structure entropy is \( H^2(G) = O(\log_2 \log_2 n) \). This means that coding trees play an essential role in controlling a network by reducing the uncertainty of the interactions in the network.

Theorems 2.1 and 2.2, together with the theory in \(^{[20]} \), imply that structure entropy of a graph involves one more measure of the graph, which is the module function of graphs. This means that the variant of the structure entropy given in Definitions 2.5 and 2.6 is well-defined. In particular, using this variant of structure entropy, we are able to establish a new theory of structure entropy for directed graphs, in which case, the flow module function plays a role (project in progress).

The comparison between Shannon entropy and our structure entropy also suggests some interesting open questions. For example, Shannon entropy can be understood as the tight lower bound of the length of the binary expression of the item picked by the probabilistic distribution, and the tight lower bound for the number of bits required to guess the item chosen by the probabilistic distribution, and so on. Some of these measures such as the length of the binary expression is intuitive and concrete, and has geometric and physical meaning. However, the structure entropy has no such intuition. It is an open question to find a geometric or physical interpretation for the structure entropy in Definitions 2.5 and 2.6. Of course, the fundamental feature of structure entropy is that structure plays a role in information theory. This new feature leads to both information theoretical approach to graph theory and graphical approach to information theory, accompanying with the new notions of coding trees and module functions etc.

### 3 Resistance of Networks

In this section, we propose the notion of resistance and security index of a graph. The notions are built by using the one- and two-dimensional structure entropy introduced in \(^{[20]} \). We recall the one- and two-dimensional structure entropy \(^{[20]} \).
3.1 One- and two-dimensional structure entropy

According to Definition 2.5, the one-dimensional structure entropy of a graph $G = (V, E)$ is the following: Let $G = (V, E)$ be a connected graph with $n$ vertices and $m$ edges. For each vertex $i \in \{1, 2, \ldots, n\}$, let $d_i$ be the degree of $i$ in $G$, and let $p_i = d_i/2m$. Then the vector $p = (p_1, p_2, \ldots, p_n)$ is the stationary distribution of a random walk in $G$. By using this, we define the one-dimensional structure entropy or positioning entropy of $G$ by:

$$\mathcal{H}^1(G) = H(p) = H\left(\frac{d_1}{2m}, \ldots, \frac{d_n}{2m}\right) = -\sum_{i=1}^{n} \frac{d_i}{2m} \cdot \log_2 \frac{d_i}{2m}. \tag{22}$$

By Definition 2.5 we have:

**Definition 3.1.** (Structure entropy of $G$ by a partition, [20]) Given a connected graph $G = (V, E)$, suppose that $P = \{X_1, X_2, \ldots, X_L\}$ is a partition of $V$. We define the structure entropy of $G$ by $P$ as follows:

$$\mathcal{H}^P(G) := \sum_{j=1}^{L} \frac{V_j}{2m} H\left(\frac{d_{1(j)}}{V_j}, \ldots, \frac{d_{n(j)}}{V_j}\right) - \sum_{j=1}^{L} \frac{g_j}{2m} \log_2 \frac{V_j}{2m} = \sum_{j=1}^{L} \frac{V_j}{2m} \log_2 \frac{d_{1(j)}}{V_j} - \sum_{j=1}^{L} \frac{g_j}{2m} \log_2 \frac{d_{n(j)}}{V_j}, \tag{23}$$

where $L$ is the number of modules in partition $P$, $n_j$ is the number of vertices in module $X_j$, $d_{i(j)}$ is the degree of the $i$-th vertex in $X_j$, $V_j$ is the volume of module $X_j$, and $g_j$ is the number of edges with exactly one endpoint in module $j$.

According to the definition, $\mathcal{H}^P(G)$ is the average number of bits required to determine the code $(i, j)$ of the vertex of the graph that is accessible from random walk with stationary distribution in $G$, where $i$ is the code of the vertex in its own module, and $j$ is the code of the module of the accessible vertex in the whole network $G$.

Now we turn to define the two-dimensional structure entropy of $G$.

**Definition 3.2.** (Two-dimensional structure entropy, [20]) Given a connected graph $G$, define the structure entropy of $G$ by:

$$\mathcal{H}^2(G) = \min_{P} \{\mathcal{H}^P(G)\}, \tag{24}$$

where $P$ runs over all the partitions of $G$.

Clearly, the definition of $\mathcal{H}^2(G)$ in Definition 3.2 is the same as that in Definition 2.9 for $K = 2$.

For the one- and two-dimensional structure entropy, we will use some fundamental results from [20].

**Theorem 3.1.** (Lower bound of one-dimensional structure entropy of simple graphs) Let $G = (V, E)$ be an undirected, connected, and simple graph with $m$ edges, i.e., $|E| = m$. Then:

$$\mathcal{H}^1(G) \geq \frac{1}{2} \left(\log_2 m - 1\right).$$

**Theorem 3.2.** (Lower bound of one-dimensional structure entropy of graphs of balanced weights) Let $G = (V, E)$ be a connected graph with weight function $w$. Let $m = |E|$ be the number of edges. If the ratio of maximum weight and minimum weight is at most $m^\epsilon$, that is $\frac{\max_{e \in E} w(e)}{\min_{e \in E} w(e)} \leq m^\epsilon$, for some constant $\epsilon < 1$, then:

$$\mathcal{H}^1(G) \geq \frac{1}{2} \left(1 - \epsilon\right) \log_2 m - 1.$$ 

Given a graph $G = (V, E)$, and a subset $S$ of $V$, the conductance of $S$ in $G$ is given by

$$\Phi(S) = \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}, \tag{25}$$

where $E(S, \bar{S})$ is the set of edges with one endpoint in $S$ and the other in the complement of $S$, i.e. $\bar{S}$, and $\text{vol}(X)$ is the sum of degrees $d_x$ for all $x \in X$. The conductance of $G$ is defined to be the minimum of $\Phi(S)$ over all subsets $S$’s, that is:

$$\Phi(G) = \min_{S \subseteq V} \{\Phi(S)\}. \tag{26}$$
Definition 3.3. (Resistance of networks, Li and Pan [20]) Let $G = (V, E)$ be a connected graph. Suppose that there is partition $\mathcal{P}$ of $G$ such that random walk in $G$ with stationary distribution easily goes to a small module $X_j$ of $\mathcal{P}$ after which it is not easy to escape from the module $X_j$. In this case, the spreading of the virus is restrained by the partition $\mathcal{P}$ of $G$. To characterise the scenario, we define the resistance of $G$ given by partition $\mathcal{P}$.

Definition 3.4. (Resistance of a graph by a partition $\mathcal{P}$, Li and Pan [20]) Let $G = (V, E)$ be a connected graph and $\mathcal{P}$ be a partition of $G$. The resistance of $G$ given by $\mathcal{P}$ is defined as follows:

$$R^\mathcal{P}(G) = -\sum_{j=1}^{L} \frac{V_j - g_j}{2m} \log_2 \frac{V_j}{2m},$$

where $V_j$ is the volume of the $j$-th module $X_j$ of $\mathcal{P}$, and $g_j$ is the number of edges from $X_j$ to the vertices outside $X_j$.

We define the notion of resistance of a graph $G$.

Definition 3.5. (Resistance of networks, Li and Pan [20]) Let $G$ be a connected graph. We define the resistance of $G$ as follows:

$$R(G) = \max_{\mathcal{P}} \{R^\mathcal{P}(G)\},$$

where $\mathcal{P}$ runs over all the partitions of $G$.

Intuitively, the resistance $R(G)$ measures the force of $G$ to resist the cascading failure of virus attacks in $G$. As a matter of fact, the authors and their coauthors have shown experimentally that the resistance of a network does measure the force of the network to resist cascading failures of virus attacks [16], and that resistance maximization is the principle for defending the networks from virus attacks [21].

For the resistance of graph $G$ by $\mathcal{P}$, we have the following resistance principle of networks.

Theorem 3.5. (Resistance principle, Li and Pan, [20]) Let $G = (V, E)$ be a connected graph. Suppose that $\mathcal{P}$ is a partition of $V$ with the notations the same as that in the definitions of $\mathcal{H}^1(G)$ and $\mathcal{H}^\mathcal{P}(G)$. Then the positioning entropy of $G$, $\mathcal{H}^1(G)$, and the structure entropy of $G$ by given $\mathcal{P}$, i.e., $\mathcal{H}^\mathcal{P}(G)$, satisfy the following properties:

1. (Additivity law of one-dimensional structure entropy) The positioning entropy of $G$ satisfies:

$$\mathcal{H}^1(G) = -\sum_{j=1}^{L} \frac{V_j}{2m} \sum_{i=1}^{n_j} \frac{d^{(j)}_i}{V_j} \log_2 \frac{d^{(j)}_i}{V_j} - \sum_{j=1}^{L} \frac{V_j}{2m} \log_2 \frac{V_j}{2m}.$$  

2. (Local resistance law of networks)

$$R^\mathcal{P}(G) = -\sum_{j=1}^{L} \frac{V_j - g_j}{2m} \log_2 \frac{V_j}{2m} = \mathcal{H}^1(G) - \mathcal{H}^\mathcal{P}(G)$$

Theorem 3.4. (Lower bounds of two-dimensional structure entropy of simple graphs) Let $G = (V, E)$ be an undirected, connected and simple graph with number of edges $|E| = m$. Then the two-dimensional structure entropy of $G$ satisfies

$$\mathcal{H}^2(G) = \Omega(\log_2 \log_2 m).$$

[Remark: In [20], the authors first defined the one- and two-dimensional structure entropy and then extended to the high-dimensional cases. In that paper, we used the notion “partitioning tree” in the definition of high dimensional structure entropy. Here we use the notion of coding tree of a graph. We would hope that this new notion is better for people to understand the structural information theory.]

The notion of structure entropy may have fundamental accompanying notions, for instance, noises, coding and decoding etc. The authors introduced the notion of resistance as an accompanying notion of structure entropy in [20]. It is interesting that resistance is determined only by the one- and two-dimensional structure entropy of the graph.

Given a network $G = (V, E)$, consider the following scenario: Suppose that there is a virus which randomly spreads in $G$. Suppose that there is partition $\mathcal{P}$ of $G$ such that random walk in $G$ with stationary distribution easily goes to a small module $X_j$ of $\mathcal{P}$ after which it is not easy to escape from the module $X_j$. In this case, the spreading of the virus is restrained by the partition $\mathcal{P}$ of $G$. To characterise the scenario, we define the resistance of $G$ given by partition $\mathcal{P}$.
(3) Assume that for each \( j \), \( V_j \leq m \), for \( m = |E| \). Then

\[
\mathcal{R}^P(G) = -\sum_{j=1}^{L} (1 - \Phi(X_j)) \frac{V_j}{2m} \log_2 \frac{V_j}{2m} = \mathcal{H}^1(G) - \mathcal{H}^P(G)
\]  

(33)

where \( \Phi(X_j) \) is the conductance of \( X_j \) in \( G \).

By the local resistance law in Theorem 3.5 we have:

**Theorem 3.6.** (Global resistance law of networks, Li and Pan [20]) Let \( G \) be a connected graph. Then, we have

\[
\mathcal{R}(G) = \mathcal{H}^1(G) - \mathcal{H}^2(G).
\]

(34)

**Proof.** By Theorem 3.5 □

According to Theorem 3.6, we define the security index of a graph \( G \) to be the normalised resistance of \( G \). That is,

**Definition 3.5.** (Security index of a graph) Let \( G \) be a connected graph. We define the security index of graph \( G \) as follows:

\[
\theta(G) = \frac{\mathcal{R}(G)}{\mathcal{H}^1(G)}.
\]

(35)

By the definition in Equation (30) and the resistance principle of networks by partitions, we have that the resistance of a connected graph \( G \) satisfies:

According to Theorem 3.6 for a connected graph \( G \), we have:

\[
\mathcal{R}(G) = \mathcal{H}^1(G) - \mathcal{H}^2(G).
\]

Notice that \( \mathcal{H}^2(G) \leq \mathcal{H}^1(G) \).

By the definition of security index in Equation (35) and by the result in Equation (34), we have that the security index of a connected graph \( G \) satisfies the following:

\[
\theta(G) = 1 - \frac{\mathcal{H}^2(G)}{\mathcal{H}^1(G)}.
\]

(36)

Based on the security index, we introduce the following:

**Definition 3.6.** (Resistor graph) Let \( G = (V, E) \) be a connected graph of \( n \) vertices and \( m \) edges. Let \( \theta \) be a number in \((0, 1)\). We say that \( G \) is an \((n, \theta)\)-resistor graph, if:

\[
\theta(G) \geq \theta.
\]

(37)

By Theorem 3.3, for any expander graph \( G \), the conductance \( \Phi(G) \) is a large constant \( \alpha \), therefore, the security index of \( G \) is \( \theta(G) < 1 - \alpha \) for a large constant \( \alpha \). This means that expanders are not good resistor graphs.

### 4 Basic Theorems for Classic Structures

Trees and grid graphs perhaps are the most natural and most frequently used data structures. The authors [20] have established some lower and upper bounds of the \( K \)-dimensional structure entropy of the graphs. Here we prove the basic theorems of the resistances and security indices of the classical data structures.
4.1 Resistance and security index of trees

In this subsection, we consider the resistance and security indices of complete binary trees. Similar results can be generalized easily to any trees with constant bounded degrees. A complete binary tree is a tree whose non-leaf nodes has exactly two children and every leaf node has the same depth (In this section, for notational simplicity, we define the depth of a node to be the number of nodes on the unique path from this node to the root). So the complete binary tree of depth $H$ has exactly $2^H - 1$ nodes.

**Theorem 4.1.** Let $T$ be a complete binary tree of depth $H$ and thus of size $n = 2^H - 1$. Then:

1. The resistance of $T$ is $R(T) \geq \log_2 n - \log_2 \log_2 n - 5 - o(1) = (1 - o(1)) \cdot \log_2 n$, and

2. The security index of $T$ is $\theta(T) \geq 1 - \frac{\log_2 \log_2 n}{\log_2 n} - O\left(\frac{1}{\log_2 n}\right) = 1 - o(1)$.

**Proof.** We will prove that

(i) $H^1(T) \geq \log_2 n - 1$, and

(ii) $H^2(T) \leq \log_2 \log_2 n + 4 + o(1)$.

Then Theorem 4.1 follows immediately.

To calculate $H^1(T)$, note that in $T$, there are $2^H - 1$ nodes, and one of them is the root of degree 2, $2^{H-1}$ of them are leaves of degree 1, and $2^{H-1} - 2$ of them are intermediate nodes of degree 3. The total volume of $T$ is thus $2^{H+1} - 4$. So

\[
H^1(T) = \frac{2}{2^H} - \frac{2}{2^{H+1}} - 2 \left(\frac{2^{H-2}}{2^H} - 2\right) - \frac{3}{2^H} \left(\log_2 \frac{2^H - 2}{2^H} + \log_2 \frac{2^H - 3}{2^H - 2} + \log_2 \frac{2^H - 4}{2^H - 2}\right)\\
\geq H - 1\\
\geq \log_2 n - 1.
\]

To prove $H^2(T) \leq \log_2 \log_2 n + 4 + o(1)$, it suffices to define a partition $P$ of the nodes in $T$ such that $H^P(T) \leq \log_2 \log_2 n + 4 + o(1)$. We define $P$ as follows. Let $1 \leq k \leq H$ be an integer. We partition every subtree whose root is a node of depth $H - k + 1$ as a module and the remaining part consisting of all the nodes of depth at most $H - k$ as a module. Now we have $2^{H-k}$ complete binary subtrees, each of which, denoted by $T_j$, $j = 1, 2, \ldots, 2^{H-k}$, has a size $2^k - 1$ and another complete binary subtree, denoted by $T'$, which has a size $2^{H-k} - 1$. A simple calculation indicates that for each $T_j$, its volume $\text{vol}(T_j) = 2^{k+1} - 3$, and the volume of $T'$ is $\text{vol}(T') = 3 \cdot 2^{H-k} - 4$.

For each $T_j$, we have

\[
- \sum_{v \in T_j} \frac{d_v}{2m} \log_2 \frac{d_v}{\text{vol}(T_j)} = -\left(\frac{2^{k+1} - 3}{2m} \cdot \frac{3}{2^{k+1} - 3} - \frac{1}{2m} \cdot \frac{1}{2^{k+1} - 3}\right)\\
\leq \frac{1}{2m} \left([2^{k+1} - 1] \cdot 3(k + 1) + 2^{k+1}(k + 1)\right)\\
\leq \frac{2^{k+1}}{2m}(k + 1).
\]

So

\[
- \sum_{j=1}^{2^{H-k}} \frac{\text{vol}(T_j)}{2m} \sum_{v \in T_j} \frac{d_v}{\text{vol}(T_j)} \log_2 \frac{d_v}{\text{vol}(T_j)} = -\sum_{j=1}^{2^{H-k}} \sum_{v \in T_j} \frac{d_v}{\text{vol}(T_j)} \log_2 \frac{d_v}{\text{vol}(T_j)}\\
\leq \frac{2^{H-k}}{2m}(k + 1).
\]
Note that each $T_j$ has exactly one global edge connecting to $T'$. So the number of global edges for each $T_j$ is $g_j = 1$. We have

$$- \sum_{j=1}^{2^{n-k}} \frac{g_j}{2m} \log_2 \frac{\text{vol}(T_j)}{2m} = -2^{n-k} \frac{1}{2m} \log_2 \frac{2^{k+1} - 3}{2m}$$

$$= \frac{2^{n-k}}{2m} \left[ \log_2 2m - (k + 1) + O \left( \frac{1}{2^k} \right) \right].$$

Then consider the subtree $T'$. Note that all the nodes in $T'$ except for the root of $T$ which has degree 2, have degree 3. So

$$- \sum_{v \in T'} \frac{d_v}{2m} \log_2 \frac{d_v}{\text{vol}(T')} = -2^{n-k} \cdot \frac{3}{2m} \log_2 \frac{3}{2 \cdot 2^{H-k} - 4} - 2 \frac{2m}{2m} \log_2 \frac{2}{3 \cdot 2^{H-k} - 4}$$

$$\leq \frac{2^{H-k}}{2m} \cdot 3(\text{H} - k).$$

Note that $T'$ has $2^{H-k}$ global edges, each of which joins a subtree $T_j$. We have

$$- \frac{g_{T'}}{2m} \log_2 \frac{\text{vol}(T')}{2m} = -2^{n-k} \log_2 \frac{3 \cdot 2^{H-k} - 4}{2m}$$

$$= \frac{2^{H-k}}{2m} \cdot \left[ \log_2 2m - (H - k) + O \left( \frac{1}{2^{H-k}} \right) \right].$$

So in all, noting that $\log_2 2m = \log_2(2^{H+1} - 4) \leq H + 1$, the structure entropy of $T$ by partition $\mathcal{P}$ is

$$\mathcal{H}(T) = - \sum_{j=1}^{2^{n-k}} \frac{\text{vol}(T_j)}{2m} \sum_{v \in T_j} \frac{d_v}{\text{vol}(T_j)} \log_2 \frac{d_v}{\text{vol}(T_j)} - \frac{2^{n-k}}{2m} \log_2 \frac{\text{vol}(T_j)}{2m}$$

$$- \sum_{v \in T'} \frac{d_v}{2m} \log_2 \frac{d_v}{\text{vol}(T')} - \frac{g_{T'}}{2m} \log_2 \frac{\text{vol}(T')}{2m}$$

$$\leq \frac{2^{H-k}}{2m} \cdot 2^{k+1}(k + 1) + \frac{2^{H-k}}{2m} \cdot \left[ \log_2 2m - (k + 1) + O \left( \frac{1}{2^k} \right) \right]$$

$$+ \frac{2^{H-k}}{2m} \cdot 3(\text{H} - k) + \frac{2^{H-k}}{2m} \cdot \left[ \log_2 2m - (H - k) + O \left( \frac{1}{2^{H-k}} \right) \right]$$

$$\leq \frac{2^{H-k}}{2^{H+1} - 4} \cdot \left[ (2^{k+1} + 1)(k + 1) + 4(\text{H} - k) + O \left( \frac{1}{2^k} \right) \right] + O \left( \frac{1}{2^{H-k}} \right)$$

$$\leq \frac{2^k + 1}{2^{H-k}} + O \left( \frac{k + 1}{2^k} + \frac{k + 1}{2^H} + \frac{H - k}{2H} \right).$$

When we choose $k + 1 = [\log_2 H]$, the above value is at most $[\log_2 H] + 4 + o(1)$, which is $\log_2 \log_2 n + 4 + o(1)$. Theorem 4.1 follows.

### 4.2 Resistance and security index of grid graphs

In this subsection, we consider the resistance and the security indices of grid graphs. An $n \times n$ grid graph $G = (V, E)$ is a graph defined on the vertex set $V = \{v_{i,j} : i, j \in \mathbb{Z}^+, 1 \leq i, j \leq n\}$ and the edge set $E = \{(v_{i,j}, v_{i,j'}) : |j - j'| = 1\} \cup \{(v_{i,j}, v_{i',j}) : |i - i'| = 1\}$. 

**Theorem 4.2.** Let $G = (V, E)$ be an $n \times n$ grid graph. Then the resistance and the security index of $G$ satisfies:

1. The resistance of $G$ is $R(G) \geq \log_2[n(n-1)] - 2 \log_2 \log_2 n - O(1)$, and
2. The security index of $G$ is $\rho(G) \geq 1 - \frac{2 \log_2 \log_2 n}{\log_2[n(n-1)]} - O \left( \frac{1}{\log_2 n} \right)$.

**Proof.** We will prove that

(i) $\mathcal{H}^1(G) \geq \log_2[n(n-1)]$, and
(ii) $H^2(G) \leq 2 \log_2 \log_2 n + O(1)$.

Then Theorem 4.2 follows.

To calculate $H^1(G)$, note that in a $n \times n$ grid, there are four vertices (corners) of degree 2, $4(n-2)$ vertices (sides) of degree 3 and $(n-2)^2$ vertices of degree 4. The total volume of $G$ is thus $4n(n-1)$. So

$$H^1(G) = -4 \cdot \frac{2}{4n(n-1)} \log_2 \frac{2}{4n(n-1)} - 4(n-2) \cdot \frac{3}{4n(n-1)} \log_2 \frac{3}{4n(n-1)}$$

$$= \left[ \frac{2}{n(n-1)} + \frac{3(n-2)}{n(n-1)} \right] \cdot \log_2 [n(n-1)] + \frac{3(n-2)(2 - \log_2 3) + 2}{n(n-1)}$$

$$\geq \log_2 [n(n-1)] + \frac{3(n-2)(2 - \log_2 3) + 2}{n(n-1)}$$

To prove $H^2(G) \leq 2 \log_2 \log_2 n + O(1)$, similarly to the proof of Theorem 4.1, we find a partition $P$ for the vertices in $G$ to witness the upper bound. We divide $G$ into sub-grids of size $k \times k$. For notational simplicity, assume that $n$ can be divided by $k$. So we have exactly $\left(\frac{n}{k}\right)^2$ such sub-grids. For each sub-grid, denoted by $G_j$, let $d_{i,j}^*(j)$ denote the degree of the $i$-th vertex, which is 4 for most vertices, 3 for border vertices and 2 for corner vertices of $G$. By the extremum property of the entropy function $H(\cdot)$, the positioning entropy within $G_j$ satisfies

$$H \left( \frac{d_{1,j}^*(j)}{\text{vol}(G_j)}, \ldots, \frac{d_{k,j}^*(j)}{\text{vol}(G_j)} \right) \leq \log_2 k^2 = 2 \log_2 k.$$  

So

$$\sum_j \frac{\text{vol}(G_j)}{2m} \cdot H \left( \frac{d_{1,j}^*(j)}{\text{vol}(G_j)}, \ldots, \frac{d_{k,j}^*(j)}{\text{vol}(G_j)} \right) \leq 2 \log_2 k.$$  

Since the total number of global edges is

$$\sum_j g_j = 2n \left( \frac{n}{k} - 1 \right),$$

and noting that $m = 2n(n-1)$, we have

$$- \sum_j \frac{g_j}{2m} \log_2 \frac{\text{vol}(G_j)}{2m} \leq \left( \sum_j g_j \right) \cdot \frac{1}{2m} \cdot \log_2 2m \leq \frac{n - k}{2k(n-1)} \cdot (2 \log_2 n + 2) \leq \frac{\log_2 n + 1}{k}.$$  

So in all, we have that the structure entropy of $G$ by partition $P$ is

$$H^P(G) = \sum_j \frac{\text{vol}(G_j)}{2m} \cdot H \left( \frac{d_{1,j}^*(j)}{\text{vol}(G_j)}, \ldots, \frac{d_{k,j}^*(j)}{\text{vol}(G_j)} \right) - \sum_j \frac{g_j}{2m} \log_2 \frac{\text{vol}(G_j)}{2m}$$

$$\leq 2 \log_2 k + \frac{\log_2 n + 1}{k}.$$  

Let $k = \Theta(\log_2 n)$, then $H^P(G) \leq 2 \log_2 \log_2 n + O(1)$. Theorem 4.2 follows.  

Theorems 4.1 and 4.2 show that the classical graphs such as trees and grid graphs can be used as the basic module of secure networks. This is a surprising result, since it means that secure networks may be constructed by using the basic structures.
5 Resistance of Bounded Degree Graphs

In this section, we give a lower bound for the resistance of regular graphs.

Theorem 5.1. Let $G = (V, E)$ be a simple, connected graph of bounded degree $d$ for some constant $d$. Then

$$R(G) \geq \left(\frac{2}{d} - o(1)\right) \cdot \log_2 n. \quad (38)$$

Proof. We only have to show that there is a partition $P = V_1 \cup V_2 \cup \cdots \cup V_L$, such that

$$R(G) \geq \mathcal{H}^L(G) - \mathcal{H}^P(G) = - \sum_{j=1}^{L} (1 - \Phi(V_j)) \cdot \frac{\vol(V_j)}{\vol(G)} \log_2 \frac{\vol(V_j)}{\vol(G)} \geq \left(\frac{2}{d} - o(1)\right) \cdot \log_2 n. \quad (39)$$

Since $G$ is connected, consider a spanning tree, denoted by $T$, of $G$. Since $G$ has bounded degree, so is $T$. Pick an arbitrary vertex $v$ of $T$ as the root. Then the depth of every other vertex $v$ is the length of the unique path from $r$ to $v$, and every vertex on this path other than $v$ is called an ancestor of $v$. We say that the $k$-th ancestor of $v$ is the one which has distance $k$ from $v$.

We define a partition of vertices in $T$ recursively in the following way. Let $l = \lceil \log_2 n \rceil - 1$ and denote $T_0 = T$. For $i = 0, 1, 2, \ldots$, find the deepest vertex, denoted by $v_i$, of $T_i$ (break ties arbitrarily), and denote by $v_{i+1}$ the ancestor of $v_i$ which has distance $l$ from $v_i$. Take the subtree rooted by $v_{i+1}$, denoted by $V_{i+1}$, as a module, and then delete $T_{i+1}$ from $T$. This procedure will not end until $T_1$ is empty. Suppose that the last module is $V_L$. Then $P = V_1 \cup V_2 \cup \cdots \cup V_L$ is a partition of $V$. Next, we show that for this partition $P$, Inequality (39) is satisfied.

Note that each $T_i$, has bounded degree $d$, and so the size of each $T_i$ is at most $d^{d+1} \leq \log_2 n$, and $L \geq n/\log_2 n$. Except for the last module $V_L$, the size of each $V_i$ is certainly at least $l$. For each $V_i$, since it is connected in the spanning tree $T$ of $G$, it is also connected in $G$. So there are at least $|V_i| - 1$ edges in $G$ whose two endpoints are both in $V_i$, and $\vol(V_i) \geq 2(|V_i| - 1) + 1$. Thus, for each $V_i$, we have

$$\Phi(V_i) \leq \frac{\vol(V_i) - 2(|V_i| - 1)}{\vol(V_i)} \leq \frac{d|V_i| - 2(|V_i| - 1)}{d|V_i|} = 1 - \frac{2}{d} + \frac{2}{d|V_i|}.$$ 

Therefore, for sufficiently large $n$.

$$R(G) \geq \mathcal{H}^L(G) - \mathcal{H}^P(G)$$

$$\geq - \sum_{j=1}^{L} (1 - \Phi(V_j)) \frac{\vol(V_j)}{\vol(G)} \log_2 \frac{\vol(V_j)}{\vol(G)}$$

$$\geq - \sum_{j=1}^{L-1} \left(\frac{2}{d} - \frac{2}{d|V_j|}\right) \frac{\vol(V_j)}{\vol(G)} \log_2 \frac{\vol(V_j)}{\vol(G)}$$

$$\geq - \sum_{j=1}^{L-1} \left(\frac{2}{d} - \frac{2}{dL}\right) \frac{\vol(V_j)}{\vol(G)} \log_2 \frac{\vol(V_j)}{\vol(G)}$$

$$\geq \frac{2}{d} \cdot \left(1 - \frac{1}{L}\right) \cdot \frac{\vol(G) - \vol(V_L)}{\vol(G)} \cdot \log_2 \frac{\vol(V_j)}{\vol(G)}$$

$$\geq \frac{2}{d} \cdot \left(1 - \frac{1}{L}\right) \cdot \frac{\vol(G) - \vol(V_L)}{\vol(G)} \cdot \log_2 \frac{\vol(G) - \vol(V_L)}{d \log_2 n}$$

$$\geq \frac{2}{d} \cdot \left(1 - \frac{1}{L}\right) \cdot \frac{\vol(G) - d \log_2 n}{\vol(G)} \cdot \log_2 \frac{\vol(G) - d \log_2 n}{d \log_2 n}$$

$$= \left(\frac{2}{d} - o(1)\right) \cdot \log_2 n.$$ 

This completes the proof of the theorem.

\[\square\]

1In this paper, whenever we say a proposition holds for “sufficiently large” values, we mean that there is some large enough value such that the proposition holds for all values larger than this one.
6 Resistance of Complete Graphs

As mentioned before, Theorem 3.3 indicates that expander graphs are not good resistor graphs.

In this section, we analyze the resistance of the "most expanding" graphs, i.e., the complete graphs. We show that the resistance of a complete graph is as low as a constant $O(1)$. First, we answer the following question: When a partition $P$ is given on a graph, to achieve a larger resistance (or equivalently, a smaller two-dimensional structural information) from $P$, how to split or merge the modules in $P$. For two subsets of vertices $X$ and $Y$, denote by $e(X,Y)$ the number of edges whose one endpoint is in $X$ and the other in $Y$. The following lemma gives the criteria.

Lemma 6.1. (Merging-Splitting Lemma) Let $G = (V,E)$ be a regular graph. Let $P_1 = X_1 \cup X_2 \cup \cdots \cup X_L$ and $P_2 = Y_1 \cup Y_2 \cup \cdots \cup X_L$ be two partitions of $V$ whose only difference is the module $X_1$ in $P_1$ being split into two modules $Y_1 \cup Y_2$ in $P_2$. Then $\mathcal{H}^{P_2}(G) \geq \mathcal{H}^{P_1}(G)$ if and only if

$$e(Y_1,Y_2) \cdot \log_2 \frac{n}{|X_1|} \geq e(Y_1,Y_1) \cdot \log_2 \frac{|X_1|}{|Y_1|} + e(Y_2,Y_2) \cdot \log_2 \frac{|X_1|}{|Y_2|},$$

and $\mathcal{H}^{P_2}(G) \leq \mathcal{H}^{P_1}(G)$ if and only if

$$e(Y_1,Y_2) \cdot \log_2 \frac{n}{|X_1|} \leq e(Y_1,Y_1) \cdot \log_2 \frac{|X_1|}{|Y_1|} + e(Y_2,Y_2) \cdot \log_2 \frac{|X_1|}{|Y_2|}.$$

Proof. The proof is straightforward. For any non-empty subset of vertices $X$, let $H(X)$ denote the entropy of the degree distribution of vertices in $X$. By Definition 3.1,

$$\mathcal{H}^{P_2}(G) - \mathcal{H}^{P_1}(G) = \left( \frac{\text{vol}(Y_1)}{\text{vol}(G)} \cdot H(Y_1) + \frac{\text{vol}(Y_2)}{\text{vol}(G)} \cdot H(Y_2) - \frac{\text{vol}(X_1)}{\text{vol}(G)} \cdot H(X_1) \right) - \left( \Phi(Y_1) \cdot \frac{\text{vol}(Y_1)}{\text{vol}(G)} \cdot \log_2 \frac{|Y_1|}{|X_1|} + \Phi(Y_2) \cdot \frac{\text{vol}(Y_2)}{\text{vol}(G)} \cdot \log_2 \frac{|Y_2|}{|X_1|} \right)$$

$$- \Phi(X_1) \cdot \frac{\text{vol}(X_1)}{\text{vol}(G)} \cdot \log_2 \frac{|X_1|}{|Y_1|} \cdot \frac{|X_1|}{|Y_2|}.$$

(Note that $e(X_1, X_1) = e(Y_1, Y_1) + e(Y_2, Y_2) - 2e(Y_1,Y_2)$)

$$= \left( \frac{\text{vol}(Y_1)}{\text{vol}(G)} \cdot \log_2 \frac{|Y_1|}{|X_1|} + \frac{\text{vol}(Y_2)}{\text{vol}(G)} \cdot \log_2 \frac{|Y_2|}{|X_1|} \right)$$

$$- \frac{\text{vol}(X_1)}{\text{vol}(G)} \cdot \log_2 \frac{|X_1|}{|Y_1|} \cdot \frac{|X_1|}{|Y_2|}.$$

(Note that $G$ is regular)

$$= \frac{\text{vol}(Y_1)}{\text{vol}(G)} \cdot \log_2 \frac{|Y_1|}{|X_1|} + \frac{\text{vol}(Y_2)}{\text{vol}(G)} \cdot \log_2 \frac{|Y_2|}{|X_1|}$$

$$- \frac{2e(Y_1,Y_2)}{\text{vol}(G)} \cdot \log_2 \frac{|X_1|}{|Y_1|} \cdot \frac{|X_1|}{|Y_2|}.$$

So $\mathcal{H}^{P_2}(G) \geq \mathcal{H}^{P_1}(G)$ if and only if

$$e(Y_1,Y_2) \cdot \log_2 \frac{n}{|X_1|} \geq e(Y_1,Y_1) \cdot \log_2 \frac{|X_1|}{|Y_1|} + e(Y_2,Y_2) \cdot \log_2 \frac{|X_1|}{|Y_2|},$$

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and \( \mathcal{H}^P(G) \leq \mathcal{H}^P(G) \) if and only if

\[
\epsilon(Y_1, Y_2) \cdot \log_2 \left( \frac{n}{|X_1|} \right) \leq \epsilon(Y_1, Y_1) \cdot \log_2 \left( \frac{|X_1|}{|Y_1|} \right) + \epsilon(Y_2, Y_2) \cdot \log_2 \left( \frac{|X_1|}{|Y_2|} \right).
\]

The lemma follows.

By Lemma \([6.1]\), we know that to reduce the structure information, a large module (large \(|X_1|\)) tends to split into pieces (in the case that \( \mathcal{H}^P(G) < \mathcal{H}^P(G) \)), while small modules (small \(|Y_1|\) and \(|Y_2|\)) tend to merge into big ones (in the case that \( \mathcal{H}^P(G) > \mathcal{H}^P(G) \)).

For complete graphs, we have the following theorem.

**Theorem 6.1.** Let \( G \) be a complete graph with \( n \) vertices. Then

\[
\mathcal{R}(G) = O(1).
\]

**Proof.** In the complete graph \( G \) of size \( n \), since each vertex has degree \( n - 1 \), a subset of vertices of size \( x \) has volume \((n - x)/(n - 1)\), vol\((G) = n(n - 1)\).

Suppose that \( P = V_1 \cup V_2 \cup \cdots \cup V_L \) be the partition of \( V \) such that \( \mathcal{H}^2(G) = \mathcal{H}^P(G) \). Let \( n_j = |V_j| \) for each \( j \in [L] \). Then

\[
\mathcal{R}(G) = -\sum_{j=1}^{L} \left( 1 - \Phi(V_j) \right) \cdot \frac{\text{vol}(V_j)}{\text{vol}(G)} \log_2 \frac{\text{vol}(V_j)}{\text{vol}(G)}
\]

\[
= -\sum_{j=1}^{L} \left( 1 - \frac{n - n_j}{n - 1} \right) \cdot \frac{n_j}{n} \log_2 \frac{n_j}{n}
\]

\[
= -\sum_{j=1}^{L} \left( \frac{n_j - 1}{n - 1} n_j \right) \log_2 \frac{n_j}{n}.
\]

Next, we prove that when \( P \) makes \( \mathcal{H}^P(G) \) minimized, the size of modules in \( P \) should be the same \(^2\). Suppose that \( x \) and \( y \) are the sizes of two modules and \( x + y = a \) for some fixed \( a \). We will show that when other modules are fixed, averaging \( x \) and \( y \), that is \( x = y = a/2 \), or \( x = 0 \) while \( y = a \), or \( x = a \) while \( y = 0 \) makes the structural information (under this partition) minimized. This means that for two modules in a partition, to reduce the structure information, they tend to be the same size, or merge into a single module. This holds for every pair of modules, which implies that the module sizes in the optimal partition are averaged.

Note that

\[
(n - 1)n \cdot \mathcal{R}(G) = -\sum_{j=1}^{L} (n_j - 1)n_j \cdot \log_2 \frac{n_j}{n}.
\]

We only have to show that the function

\[
f(x) \overset{\Delta}{=} -x(x - 1) \cdot \log_2 \frac{x}{n} - (a - x)(a - x - 1) \cdot \log_2 \frac{a - x}{n}
\]

achieves maximum at \( x = a/2 \), or 0, or \( a \), when \( 0 \leq x \leq a \). The first derivative and the second derivative of \( f(x) \) satisfies

\[
\ln 2 \cdot f'(x) = -(2x - 1) \cdot \ln x - [2(x - a) + 1] \cdot \ln(a - x) + 2(2x - a) \cdot \ln n + (a - 2x),
\]

\[
\ln 2 \cdot f''(x) = -2 \cdot \ln [x(a - x)] + \frac{a}{x(a - x)} + 4 \cdot \ln n - 6.
\]

Note that the function

\[
g(x) = -2 \cdot \ln x + \frac{a}{x} + 4 \cdot \ln n - 6
\]

strictly decreases monotonically for \( x > 0 \). There is at most one root for \( g(x) \), and consequently, there are at most two roots for \( f''(x) \), and thus, there are at most two inflection points for \( f'(x) \). Observing that \( f'(a/2) = 0 \), \( \lim_{x \to 0^+} f'(x) = -\infty \) and \( \lim_{x \to a^-} f'(x) = +\infty \), we know that there are at most three maximal values and two

\(^2\)Since the size of a module should be an integer, here we say that two modules has the same size if the deficit is at most one. But for the notational simplicity, we assume that \( n \) can always be divided by the parameters we suppose, and the error will be absorbed in the notation \( O(\cdot) \).
minimal values for \( f(x) \) in the interval \([0, a]\) because of at most two inflection points in this interval, and these three (possible) maximal points take values at \( x = 0 \) or \( a/2 \) or \( a \). This means that \( x = y = a/2 \), or \( x = 0 \) while \( y = a \), or \( x = a \) while \( y = 0 \) makes the structural information minimized when other modules are fixed.

So from now on, we can suppose that \( x \) is the size of each modules in \( P \), and so \( L = n/x \) (suppose that \( n \) can be divided by \( x \)). We have

\[
(n - 1)n \cdot R(G) = -\frac{n}{x}(x - 1)x \cdot \log_2 \frac{x}{n} = -n(x - 1) \cdot \log_2 \frac{x}{n}.
\]

Define function

\[
h(x) \triangleq \ln 2 \cdot (n - 1) \cdot R(G) = -(x - 1) \cdot \ln \frac{x}{n},
\]

To compute the maximum value of \( h(x) \), note that

\[
h'(x) = \ln \frac{n}{x} + \frac{1}{x} - 1, \quad h''(x) = -\frac{x + 1}{x^2} < 0.
\]

Thus, \( h(x) \) takes the maximum value at \( x = x_0 \) where \( x_0 \) is the unique root of \( h'(x) \). That is,

\[
\ln \frac{x_0}{n} = \frac{1}{x_0} - 1.
\]

Therefore,

\[
h(x_0) = -(x_0 - 1) \cdot \ln \frac{x_0}{n} = x_0 + \frac{1}{x_0} - 2 \leq n + \frac{1}{n} - 2.
\]

So

\[
R(G) \leq \frac{n + \frac{1}{n} - 2}{\ln 2 \cdot (n - 1)} = \frac{1}{\ln 2} \cdot \frac{n - 1}{n} < \log_2 e.
\]

Adding the error caused by the deficit of module sizes, we have \( R(G) = O(1) \). This completes the proof of the theorem. \( \square \)

In the above proof, note that \( h'(n/e) > 0 \), and when \( n \geq 7 \), \( h'(n/2) < 0 \), which means that \( n/e < x_0 < n/2 \), and in the optimal partition \( P \), the number of modules \( L = 2 \) or \( 3 \) while each module has size \( n/2 \) or \( n/3 \) for \( n \geq 7 \). Theorem \( 6.1 \) means that any partition \( P \) of the complete graph saves only a constant bits of information.

Theorem \( 6.1 \) indicates that the resistance of the complete graphs is \( O(1) \), and the security index of the complete graphs is \( O \left( \log_{\log_2 e} n \right) = o(1) \), where \( n \) is the number of vertices of the graph, so that the complete graphs are far from resistor graphs.

The arguments above clearly indicate that

**Theorem 6.2.** For arbitrarily small constant \( \epsilon > 0 \), there is an \( N \) such that for any \( n \geq N \), the complete graph of \( n \) vertices cannot be an \((n, \epsilon)\)-resistor graph.

**Proof.** By the proof of Theorem \( 6.1 \) \( \square \)

### 7 Resistance and Security Index of the Networks of the Security Model

Li, Li, Pan and Zhang \([18]\) proposed the security model of networks. Here we establish the theory of resistance and security index of the networks generated by the security model.

The model proceeds as follows.

**Definition 7.1.** (Security model, \([18]\)) Given a homophyly (or affinity) exponent \( a \geq 0 \) and a natural number \( d \),

1) Let \( G_{n_0} \) be an initial graph of size \( n_0 \) such that each vertex has a distinct color and called seed, where \( n_0 \) is an arbitrary positive integer.

   For each step \( i > n_0 \), let \( G_{i-1} \) be the graph constructed at the end of step \( i - 1 \), and \( p_i = 1/(\log i)^a \).

2) At step \( i \), we create a new vertex, \( v \).

3) With probability \( p_i \), \( v \) chooses a new color, in which case,
i) we call $v$ a seed,

ii) (preferential attachment) create an edge $(v, u)$ where $u$ is chosen with probability proportional to the degrees of vertices in $G_{i-1}$, and

iii) (randomness) create $d - 1$ edges $(v, u_j)$, where each $u_j$ is chosen randomly and uniformly among all seed vertices in $G_{i-1}$.

4) Otherwise, then $v$ chooses an old color, in which case,

i) (randomness) $v$ chooses uniformly and randomly an old color as its own color and

ii) (homophily and preferential attachment) create $d$ edges $(v, u_j)$, where $u_j$ is chosen with probability proportional to the degrees of all vertices of the same color as that of $v$ in $G_{i-1}$.

We use $S(n, a, d)$ to denote the model with affinity exponent $a$, average number of edges $d$ and number of vertices $n$.

The authors [19] have shown that

1. (Uniform threshold security theorem) Let $G$ be a graph constructed from $S(n, a, d)$ with $p_1 = \log^{-a} i$ for homophily exponent $a > 4$ and for $d \geq 4$. Let the uniform threshold $\phi = O\left(\frac{1}{\log^b n}\right)$ for $b = \frac{a}{2} - 2 - \epsilon$ for arbitrarily small $\epsilon > 0$.

Then with probability $1 - o(1)$ (over the construction of $G$), there is no initial set of poly-logarithmic size which causes a cascading failure set of non-negligible size. Precisely, we have that for any constant $c > 0$,

$$\Pr_{G \in nS(n, a, d), \, G = (V, E)} \left[ \forall S \subseteq V, |S| = \lceil \log^c n \rceil, \left| \inf_{G}^{\phi}(S) \right| = o(n) \right] = 1 - o(1),$$

where $\inf_{G}^{\phi}(S)$ is the infection set of $S$ in $G$ with uniform threshold $\phi$.

2. (Random threshold security theorem) Let $a > 6$ be the homophily exponent, and $d \geq 4$. Suppose that $G$ is a graph generated from $S(n, a, d)$.

Then with probability $1 - o(1)$ (over the construction of $G$), there is no initial set of poly-logarithmic size which causes a cascading failure set of non-negligible size. Formally, we have that for any constant $c > 0$,

$$\Pr_{G \in nS(n, a, d), \, G = (V, E)} \left[ \forall S \subseteq V, |S| = \lceil \log^c n \rceil, \left| \inf_{G}^{R}(S) \right| = o(n) \right] = 1 - o(1).$$

In the present paper, we establish the resistance and security indices of the network of the security model.

**Theorem 7.1.** (Resistance theorem of the networks of the security model) Given affinity exponent $a > 0$ and natural number $d > 1$, let $G = (V, E)$ be a network of the security model with $n$ vertices, affinity exponent $a$ and average number of edges $d$. Then, with probability $1 - o(1)$,

1. the resistance of $G$ is $\mathcal{R}(G) = \Omega(\log n)$, and

2. the security index of $G$ is $\theta(G) = 1 - o(1)$.

Theorem 7.1 ensures that for every affinity exponent $a > 0$ and for every edge parameter $d$, for the networks of the security model with affinity $a$ and edge parameter $d$ and with sufficiently large number of vertices $n$, with probability $\approx 1$, the resistances of the networks are as large as $\Omega(\log n)$, and the security indices of the networks are close to 1. Therefore the networks are secure against cascading failures of arbitrarily strategic virus attacks.

The proof of Theorem 7.1 consists of two parts, the first part is a lower bound of one-dimensional structure entropy of the networks, and the second part is the two-dimensional structure entropy of the networks. For the first part, we use Theorems 3.1 and 3.2.

Therefore, for simple or balanced graphs $G$, we have:

$$\mathcal{H}^1(G) = \Omega(\log n).$$

For the second part, we investigate the structure entropy of the networks given by the natural partition classified by colors.

Let $G = (V, E) \in S(n, a, d)$ be a network of $n$ vertices generated from our Security model. Every vertex is associated with a color, we say that the classification of the vertices by colors is the natural community structure of
In so doing, a natural community of $G$ is a maximal homochromatic set. We note that each natural community contains a seed, which is the first vertex specified in the community. We use $N$ to denote the natural community structure of $G$.

We approximate the two-dimensional structure entropy of $G$ by $\mathcal{H}^N(G)$.

First, we introduce some notations and basic probabilistic tools.

For every $t$, we use $G_t$ to denote the graph obtained at the end of time step $t$ of the construction of $G$, and $C_t$ to denote the set of seed vertices of $G_t$.

For an edge $e = (u, v)$, we call $e$ a local edge, if the two endpoints $u$, $v$ share the same color, and a global edge, otherwise.

The probabilistic bounds used are referred to the appendix.

For estimating $\mathcal{H}^N(G)$, we establish the following fundamental properties of the networks generated by the security model.

**Theorem 7.2.** (Fundamental theorem of the networks of the security model) Given $a \geq 0$ and $d \geq 2$, let $G = (V, E)$ be a graph of $n$ vertices generated from $S(n, a, d)$. Let $T_1 = \log^{a+1} n$ and $T_2 = \frac{n}{\log^2 n}$ for some positive constant $b$. Then the following properties hold.

1. With probability $1 - o(1)$, for all $t \geq T_1$, $\frac{1}{2 \log^{a+1} n} \leq |C_t| \leq \frac{2}{\log^{a+1} n}$.

2. When $\frac{a}{d} > 0$, for each homochromatic set $S$, if $t > T_S \geq T_1$, then the expectation of its size at time step $t$ is $\Theta(\log^{a+1} n - \log^{a+1} t_S)$, where $t_S$ is the time step at which the seed vertex of $S$ is created.

3. With probability $1 - o(1)$, every homochromatic set in $G$ has a size upper bounded by $4 \log^{a+1} n$.

4. For each homochromatic set $S$, if $t_S \geq T_2$, then for sufficiently large $n$ the number of global edges in $G$ associated to $S$, denoted by $g_S$, satisfies that

   (i) if $a > 1$, then $E(g_S) \leq \frac{2}{a}(a + 1)b^2(\log \log n)^2$;

   (ii) if $a = 1$, then $E(g_S) \leq 8b^2(\log \log n)^2$;

   (iii) if $0 < a < 1$, then $E(g_S) \leq 5b^2(\log \log n)^2$.

The proof of Theorem 7.2 is referred to the appendix.

Then we turn to prove Theorem 7.1.

**Proof.** (Proof of Theorem 7.1) According to Equation (11), it suffices to show that, when $a > 0$, with probability $1 - o(1)$, the two-dimensional structure entropy of $G$ is $\mathcal{H}^2(G) = o(\log n)$, which is negligible compared to its one-dimensional structure entropy $\mathcal{H}^1(G)$. Thus, the resistance of $G$ is approximately $\mathcal{H}^1(G)$, which is $\Omega(\log n)$, and the security index of $G$ is $1 - o(1)$.

To establish an upper bound for $\mathcal{H}^2(G)$, it suffices to give a partition for the vertices in $G$ with $\mathcal{H}^P(G) = o(\log n)$. Let $N$ be the natural partition given by the homochromatic sets.

Recall Equation (21)

$$\mathcal{H}^N(G) = -\sum_{j=1}^{L} \frac{V_j}{2m} \sum_{i=1}^{n_j} d^{(j)}_i \log_2 \frac{d^{(j)}_i}{V_j} - \sum_{j=1}^{L} \frac{g_j}{2m} \log_2 \frac{V_j}{2m}.$$ 

Set the first term of $\mathcal{H}^N(G)$ by $H_1 = -\sum_{j=1}^{L} \frac{V_j}{2m} \sum_{i=1}^{n_j} d^{(j)}_i \log_2 \frac{d^{(j)}_i}{V_j}$, and for each homochromatic set $X_j$, set $H_j = -\sum_{i=1}^{n_j} d^{(j)}_i \cdot \log_2 \frac{d^{(j)}_i}{V_j}$. By Theorem 7.2, with probability $1 - o(1)$, for each $j$, $n_j \leq 4 \log^{a+1} n$. Since the uniform distribution gives rise to the maximum entropy, we have that with probability $1 - o(1)$,

$$H_j \leq \log_2 n_j = O(\log \log n),$$

and by averaging, we have

$$H_1 = \sum_{j=1}^{L} \frac{V_j}{2m} H_j = O(\log \log n).$$

Moreover,

$$-\sum_{j=1}^{L} \frac{g_j}{2m} \log_2 \frac{V_j}{2m} \leq \sum_{j=1}^{L} \frac{g_j}{2m} \log_2 \frac{n_j}{2m} = \frac{\log_2 2m}{2m} \cdot \sum_{j=1}^{L} g_j.$$
Let $m_g$ be the number of global edges in $G$. Then $\sum_{j=1}^{L} g_j = 2m_g$. By the construction of $G$, $m_g = d|C_n|$, where $|C_n|$ is the number of colors in $G$ (and also the number of homochromatic sets in $G$ and the number of modules in $\mathcal{P}$). By Theorem 7.2 with probability $1 - o(1)$, the size $|C_n|$ of $C_n$ is at most $2n/\log^a n$. Therefore the second term of $\mathcal{H}^N(G)$ is

$$- \sum_{j=1}^{L} \frac{g_j}{2m} \log_2 \frac{V_j}{2m} \leq \frac{\log_2 2m}{2m} \cdot \frac{4dn}{\log^a n} = O(\log^{1-a} n).$$

Putting all together, we have that, with probability $1 - o(1)$,

$$\mathcal{H}^N(G) = H_1 - \sum_{j=1}^{L} \frac{g_j}{2m} \log_2 \frac{V_j}{2m} = O(\log \log n + \log^{1-a} n).$$

So, if $0 < a < 1$, then with probability $1 - o(1)$,

$$\mathcal{H}^N(G) = O(\log^{1-a} n).$$

If $a \geq 1$, then with probability $1 - o(1)$,

$$\mathcal{H}^N(G) = O(\log \log n).$$

In both cases, $\mathcal{H}^N(G) = o(\log n)$. Theorem 7.1 follows.

8 Eigenvalues of the Laplacian of Resistor Graphs

In this section, we study the eigenvalues of the Laplacian of the resistor graphs. At first, we introduce some results on high order Cheeger’s inequality which we will use.

Let $G = (V, E)$ be an undirected graph with $|V| = n$ and $|E| = m$. The Laplacian of $G$ is defined to be the $n \times n$ matrix $L = I - D^{-1/2}AD^{-1/2}$, where $A$ is the adjacency matrix of $G$ and $D$ is the diagonal matrix whose $(v, v)$-th entry is the degree of vertex $v$. So the spectrum of $L$ satisfies $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$, where $\lambda_k$ is the $k$-th eigenvalue of $L$.

For a subset of vertices $S \subseteq V$, define the conductance of $S$ to be

$$\Phi(S) = \frac{|E(S, \overline{S})|}{\min\{\text{vol}(S), \text{vol}(\overline{S})\}},$$

where $|E(S, \overline{S})|$ is the number of edges between $S$ and its complement $\overline{S}$.

Lee, Gharan and Trevisan [15] defined the $k$-way conductance of a graph $G$ as follows:

$$\phi(k) = \min_{S_1, S_2, \ldots, S_k \subseteq V} \max\{\Phi(S_i)\},$$

where the minimum runs over all collections of disjoint non-empty subsets $S_1, S_2, \ldots, S_k \subseteq V$.

The high-order Cheeger’s inequalities [15] state that the $k$-way conductance of $G$ is bounded by the $k$-th eigenvalue of $L$ in the following forms:

$$\frac{\lambda_k}{2} \leq \phi(k) \leq O(k^2)\sqrt{\lambda_k},$$

$$\frac{\lambda_k}{2} \leq \phi(k) \leq O(\sqrt{\lambda_{2k}} \log k).$$

In this section, we investigate the eigenvalues of the Laplacian of the graphs with optimum two-dimensional structure entropy, i.e., $\mathcal{H}^N(G) = O(\log_2 \log_2 n)$. 

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8.1 Characterization of the graphs with small two-dimensional structure entropy

**Theorem 8.1.** (Combinatorial property theorem of the graphs with two-dimensional structure entropy $O(\log_2 \log_2 n)$) Let $G = (V, E)$ be a graph with number of edges $m = |E|$, volume $\text{vol}(G)$ and no isolated vertices. Let $w : E \to \mathbb{R}^+$ be the weight function satisfying $\frac{\max_{e \in G} \{w(e)\}}{\min_{e \in G} \{w(e)\}} \leq W$, for some constant $W \geq 1$. If $\mathcal{H}^2(G) \leq c \log_2 \log_2 m$ for some constant $c > 0$ and any sufficiently large $m$, then for any $\varepsilon > 0$, and sufficiently large $m$, there is a set of modules of vertices, denoted by $A$, satisfying

1. $\text{vol}(A) \geq (1 - 2\varepsilon) \cdot \text{vol}(G)$;
2. For each module $X \in A$, $\Phi(X) \leq 1/\log_2^{1-\varepsilon} m$;
3. For each module $X \in A$, $|X| \leq \log^{3e/\varepsilon} m$.

**Theorem 8.1** implies that if $\mathcal{H}^2(G) = O(\log \log m)$, then almost all vertices belong to a module of conductance $\varepsilon$ and size $\log^{O(1/\varepsilon)} m$.

**Proof.** Since the one- and two-dimensional structural entropies depend on the relative weights on edges instead of the absolute values of the weights, for notational simplicity in our proof, we assume that the least weight on edge is 1 while the largest one is $W$. We also assume that there is no isolated vertices in $G$ since isolated vertices do not change any parameters in the theorem.

Let $\mathcal{P}$ be a partition of vertices in $G$ such that $\mathcal{H}^p(G) \leq c \log_2 \log_2 m$. Define

$$J = \{ j : V_j \in \mathcal{P}, H_j \leq \varepsilon^{-1} \cdot c \log_2 \log_2 m \},$$

and $\overline{J} = |\mathcal{P}| \setminus J$, where $V_j$ is the $j$-th module of $\mathcal{P}$ and

$$H_j = - \sum_{v \in V_j} \frac{d_v}{\text{vol}(V_j)} \log_2 \frac{d_v}{\text{vol}(V_j)}$$

is the one-dimensional structure entropy of $V_j$. Since

$$\sum_{j \in [\mathcal{P}]} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot H_j \leq \mathcal{H}^p(G) \leq c \log_2 \log_2 m,$$

we have

$$\sum_{j \in \mathcal{P}} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot \varepsilon^{-1} c \log_2 \log_2 m \leq \sum_{j \in \mathcal{P}} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot H_j \leq c \log_2 \log_2 m.$$  

So

$$\sum_{j \in \mathcal{P}} \frac{\text{vol}(V_j)}{\text{vol}(G)} \leq \varepsilon,$$

and

$$\sum_{j \in J} \frac{\text{vol}(V_j)}{\text{vol}(G)} \geq 1 - \varepsilon,$$

which means that the total volume of the modules in $V_j$ for $j \in \overline{J}$, denoted by $\text{vol}(\overline{J})$, is negligible.

Define

$$J' = \{ j : V_j \in \mathcal{P}, \Phi_j \leq \frac{1}{\log_2^{1-\varepsilon} m} \},$$

and $\overline{J'} = |\mathcal{P}| \setminus J'$. Then we will show that the total volume of the modules in $V_j$ for $j \in \overline{J'}$, denoted by $\text{vol}(\overline{J'})$, is also negligible. To this end, the following lemma will be useful.

**Lemma 8.1.** Let $X$ be a subset of vertices in $G$ with one-dimensional structure entropy $\mathcal{H}^1(X)$ and conductance $\Phi(X)$. Let $m_X$ be the number of edges whose two end-points are both in $X$. If $W \leq m_X$ for some $\varepsilon \geq 0$, then we have

$$\mathcal{H}^1(X) \geq \frac{1 - \Phi(X)}{2} \cdot [(1 - \varepsilon) \log_2 m_X - 3].$$

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Proof. We call the edges whose two end-points are both in \( X \) to be \textit{local edges}, and the edges in the cut \((X, \overline{X})\), each of which has exactly one end-point in \( X \), to be \textit{global edges}. Let

\[
g_X = \sum_{e \in (X, \overline{X})} w(e)
\]

be the total weight of global edges. So \( g_X = \Phi(X) \cdot \text{vol}(X) \). Let \( G_X \) be the subgraph induced by the vertices in \( X \). So \( \text{vol}(G_X) = \text{vol}(X) - g_X \) and the number of edges in \( G_X \) is \( m_X \). Since \( W \leq m_X \), by Theorem 21 in [19], the one-dimensional structure entropy of \( G_X \) satisfies

\[
\mathcal{H}^1(G_X) \geq \frac{1}{2}[(1 - \epsilon) \log_2 m_X - 1].
\]

For each vertex \( v \in X \), let \( g_v \) be the total weight of global edges associated with \( v \). Let \( v_i (1 \leq i \leq |X|) \) be the \( i \)-th vertex in \( X \). Suppose that \( v_1 \) is the vertex with the largest weighted degree in \( X \) (break ties arbitrarily). Define the weighted degree distribution

\[
X' = \left\{ \frac{d_{v_1} + \sum_{i=2}^{|X|} g_{v_i}}{\text{vol}(X)}, \frac{d_{v_2} - g_{v_2}}{\text{vol}(X)}, \ldots, \frac{d_{v_{|X|}} - g_{v_{|X|}}}{\text{vol}(X)} \right\},
\]

which is the degree distribution obtained by associating all the global edges to the vertex \( v_1 \). To establish the relationship among \( \mathcal{H}^1(X), \mathcal{H}(X') \) and \( \mathcal{H}^1(G_X) \), we introduce the following property for Shannon entropy.

Lemma 8.2. Let \( l \geq 2 \) be an integer and \( p = \{p_1, p_2, \ldots, p_l\} \) be a probability distribution satisfying \( \sum_{i=1}^l p_i = 1 \). Suppose that \( p_1 \geq p_2 \). Then for any \( 0 \leq \alpha \leq p_2 \), let \( p' = \{p_1 + \alpha, p_2 - \alpha, p_3, \ldots, p_l\} \), and we have \( H(p) \geq H(p') \).

Proof. Let function

\[
\begin{align*}
f(\alpha) &= H(p) - H(p') \\
&= (p_1 + \alpha) \log_2 (p_1 + \alpha) + (p_2 - \alpha) \log_2 (p_2 - \alpha) - p_1 \log_2 p_1 - p_2 \log_2 p_2.
\end{align*}
\]

So its first derivative

\[
f'(\alpha) = \log_2 \frac{p_1 + \alpha}{p_2 - \alpha} \geq 0,
\]

for \( 0 \leq \alpha \leq p_2 \). Since \( f(0) = 0 \), \( f(\alpha) \geq 0 \) for \( 0 \leq \alpha \leq p_2 \). Lemma 8.2 follows. \( \square \)

Because of the symmetry of function \( H(\cdot) \), Lemma 8.2 holds not only for \( p_1 \) and \( p_2 \), but for any \( p_i \) and \( p_j \). By the construction of the distribution \( X' \), which can be viewed as associating one by one the global edges on \( v_i \), for \( 2 \leq i \leq |X| \) to \( v_1 \), in which at each step, by Lemma 8.2, the entropy of the intermediate degree distribution is decreasing. Thus, \( \mathcal{H}(X') \geq \mathcal{H}(X) \).

On the other hand, letting \( d'_{v_1} = d_{v_1} + \sum_{i=2}^{|X|} g_{v_i} \) and \( d'_{v_i} = d_{v_i} - g_{v_i} \) for \( 2 \leq i \leq |X| \), by the additivity of \( H(\cdot) \), we know that

\[
\begin{align*}
H(X') &= H \left( \frac{d'_{v_1}}{\text{vol}(X)}, \frac{d'_{v_2}}{\text{vol}(X)}, \ldots, \frac{d'_{v_{|X|}}}{\text{vol}(X)} \right) \\
&= H \left( \frac{d'_{v_1}}{\text{vol}(X)}, \frac{\text{vol}(X) - d'_{v_1}}{\text{vol}(X)} \right) \\
&\quad + \frac{\text{vol}(X) - d'_{v_1}}{\text{vol}(X)} \sum_{i=2}^{|X|} \left( \frac{d'_{v_i}}{\text{vol}(X)} \log_2 \frac{\text{vol}(X) - d'_{v_i}}{\text{vol}(X) - d'_{v_1}} \right),
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{H}^1(G_X) &= H \left( \frac{d'_{v_1} - g_X}{\text{vol}(X) - g_X}, \frac{d'_{v_2}}{\text{vol}(X) - g_X}, \ldots, \frac{d'_{v_{|X|}}}{\text{vol}(X) - g_X} \right) \\
&= H \left( \frac{d'_{v_1} - g_X}{\text{vol}(X) - g_X}, \frac{\text{vol}(X) - d'_{v_1}}{\text{vol}(X) - g_X} \right) \\
&\quad + \frac{\text{vol}(X) - d'_{v_1}}{\text{vol}(X) - g_X} \sum_{i=2}^{|X|} \left( \frac{d'_{v_i}}{\text{vol}(X) - d'_{v_i}} \log_2 \frac{\text{vol}(X) - d'_{v_i}}{\text{vol}(X) - d'_{v_1}} \right).
\end{align*}
\]
Comparing this two equations, we have that
\[ \text{vol}(X) \cdot \left[ H(X') - H \left( \frac{d_{XJ}}{\text{vol}(X)} \cdot \frac{\text{vol}(X) - d_{XJ}}{\text{vol}(X)} \right) \right] = (\text{vol}(X) - gX) \cdot \left[ H^1(G_X) - H \left( \frac{d_{XJ} - gX}{\text{vol}(X) - gX} \cdot \frac{\text{vol}(X) - d_{XJ}}{\text{vol}(X) - gX} \right) \right], \]

Recall that
\[ H^1(G_X) \geq \frac{1}{2} \log \frac{\text{vol}(X)}{\text{vol}(X) - gX} \geq \frac{1}{2} \log m_X - 1, \]
and \( \Phi(X) = gX / \text{vol}(X) \), we have that
\[ H(X') \geq \frac{1 - \Phi(X)}{2} \cdot (1 - \epsilon) \log m_X - 3. \]
Recall that \( H^1(X) \geq H(X') \). Lemma 8.1 follows. \( \square \)

Then we show that \( \text{vol}(J') \) is negligible. Assume that there is a constant \( \epsilon_0 > 0 \) such that \( \text{vol}(J') \geq \epsilon_0 \cdot \text{vol}(G) \). Then the total volume of the modules in \( V_j \) for \( j \in J' \), denoted by \( \text{vol}(J') \), is at most \( (1 - \epsilon_0) \cdot \text{vol}(G) \). Since all the modules \( V_j \) for \( j \in J' \) have a conductance at most \( 1 / \log^{1-\epsilon} m \), then the conductance of the union of all the modules \( V_j \) for \( j \in J' \) is also at most \( 1 / \log^{1-\epsilon} m \) because some global edges for the modules are possible to be local edges in the union. So the total weight of edges in the cut \( (\bigcup_{j \in J'} V_j, \bigcup_{j \in \overline{J'}} V_j) \), denoted by \( g_{J'} \), is at most \( \text{vol}(J') / \log^{1-\epsilon} m \), which means that the conductance of \( \bigcup_{j \in J'} V_j \), denoted by \( \Phi(J') \), is at most \( \text{vol}(J') / (\text{vol}(J') \cdot \log^{1-\epsilon} m) \leq (1 - \epsilon_0) / \epsilon_0 \cdot \log^{1-\epsilon} m \). Let \( m_{J'} \) be the number of edges whose two end-points are both in \( \bigcup_{j \in J'} V_j \). Recall that for each edge \( e, 1 \leq w(e) \leq W \). Then
\[ m_{J'} \geq \frac{\text{vol}(J') - g_{J'}}{W} = \frac{(1 - \Phi(J')) \cdot \text{vol}(J')}{W} \geq \frac{\epsilon_0 \cdot m}{W} \cdot \frac{1 - \epsilon_0 - \epsilon_0 \cdot \log^{1-\epsilon} m}{W}. \]

Since \( W \leq m_{J'} \) for sufficiently large \( m \), by Lemma 8.1 the one-dimensional structure entropy of \( \bigcup_{j \in J'} V_j \) satisfies
\[ H^1(J') \geq \frac{1 - \Phi(J')}{2} \cdot (1 - \epsilon) \log m_{J'} - 3 = \Omega(\log m). \]
By the additivity of entropy function, we have that
\[ H^1(J') = \sum_{j \in J'} \frac{\text{vol}(V_j)}{\text{vol}(J')} \cdot H_j + \sum_{j \in J'} \left( \frac{\text{vol}(V_j)}{\text{vol}(J')} \cdot \log \frac{\text{vol}(V_j)}{\text{vol}(J')} \right) = \Omega(\log m). \]
Since \( \text{vol}(J') \geq \epsilon_0 \cdot \text{vol}(G) \), we know that
\[ \sum_{j \in J'} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot H_j + \sum_{j \in J'} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot \log \frac{\text{vol}(V_j)}{\text{vol}(G)} = \Omega(\log m). \]
Note that \( \Phi_j > 1 / \log^{1-\epsilon} m \). Thus,
\[ H^P(G) = \sum_{j \in J'} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot H_j + \sum_{j \in J'} \Phi_j \cdot \log \frac{\text{vol}(V_j)}{\text{vol}(G)} = \Omega(\log^5 m), \]
which contradicts the fact that \( H^P(G) \leq c \log_2 \log_2 m \) for some constant \( c > 0 \) and sufficiently large \( m \). Therefore, for any \( \epsilon > 0 \), \( \text{vol}(J') \leq \epsilon \cdot \text{vol}(J) \) is negligible.

Now, define \( A = J \cap J' \) to indicate the set of modules both in \( J \) and \( J' \). So for any \( j \in A \), \( H_j \leq \epsilon^{-1} \cdot c \log_2 \log_2 m \) and \( \Phi_j \leq 1 / \log^{1-\epsilon} m \). The total volume of modules in \( A \) is at least \( \text{vol}(G) - \text{vol}(J) - \text{vol}(J') \geq (1 - 2\epsilon) \cdot \text{vol}(G) \). So we only have to show that the size of each module \( V_j \) in \( A \) has a size at most \( \log^{3/\epsilon} m \), and then Theorem 8.1 follows.
By Lemma\[8.1\] for each \( j \in A \),
\[
H_j \geq \frac{1 - \Phi_j}{2} \cdot [(1 - \epsilon) \log_2 m_j - 3],
\]
if \( W \leq m_j' \), where \( m_j \) is the number of edges whose two end-points are both in \( V_j \). This condition holds for any \( m_j = \omega(1) \) and in this case, \( \epsilon \) can be arbitrarily close to 0. Otherwise, \( m_j = O(1) \) and we have already shown that the size of \( V_j \) is at most \( \log^{3c/\epsilon} n \) for sufficiently large \( m \) since there is no isolated vertex in \( G \).

Since \( H_j \leq \epsilon^{-1} \cdot c \log_2 m, \) we have that
\[
\epsilon^{-1} \cdot c \log_2 m \geq \frac{1 - \Phi_j}{2} \cdot [(1 - \epsilon) \log_2 m_j - 3].
\]

Since \( \Phi_j \leq 1/\log_2^{1-\epsilon} m, \) we have that
\[
(1 - \epsilon) \log_2 m_j \leq \frac{2c \log_2 m}{\epsilon \cdot (1 - 1/\log_2^{1-\epsilon} m)} + 3.
\]

Thus, for sufficiently large \( m \), both the number of edges and the number of vertices in \( V_j \) is at most \( \log^{3c/\epsilon} m \).

This completes the proof of Theorem\[8.1\].

By appropriately choosing the parameters in the proof of Theorem\[8.1\] we have the following:

**Theorem 8.2.** Let \( G = (V, E) \) be a graph with number of edges \( m = |E| \) and volume \( \text{vol}(G) \). Let \( w : E \to \mathbb{R}^+ \) be the weight function satisfying \( \max_{e \in E} \{w(e)\} \leq W, \) for some constant \( W \geq 1 \). If for any \( c > 0 \), \( \mathcal{H}^2(G) \leq c \log_2 m \) for any sufficiently large \( m \), then for any \( \epsilon, \phi > 0 \), and sufficiently large \( m \), there is a set of modules of vertices, denoted by \( A \), satisfying

1. \( \text{vol}(A) \geq (1 - 2\epsilon) \cdot \text{vol}(G) \);
2. For each module \( X \in A \), \( \Phi(X) \leq \phi \);
3. For each module \( X \in A \), \( |X| \leq m^\phi \).

Theorem\[8.1\] shows that \( \mathcal{H}^2(G) = O(\log \log n) \) guarantees a nice combinatorial property of the graph. On the other hand, combinatorial properties ensure that \( \mathcal{H}^2(G) = O(\log \log n) \).

**Theorem 8.3.** (Combinatorial properties guarantee \( \mathcal{H}^2(G) = O(\log \log n) \)) Let \( G = (V, E) \) be a connected and balanced graph of size \( n = |V| \). Then both (1) and (2) below hold:

1. If there is a set of modules \( A \) satisfying
   
   - (i) \( \text{vol}(A) = (1 - o(1)) \cdot \text{vol}(G) \), where \( \text{vol}(A) \) is the sum of the weighted degrees of all the nodes in the modules in \( A \);
   - (ii) For each module \( X \in A \), its size \( |X| = n^{o(1)} \);
   - (iii) For each module \( X \in A \), its conductance \( \Phi(X) = o(1) \),

   then the two-dimensional structural information of \( G \) is \( \mathcal{H}^2(G) = o(\log n) \).

2. If there is a set of modules \( A \) satisfying
   
   - (i) \( \text{vol}(A) = \left(1 - O\left(\frac{\log \log n}{\log n}\right)\right) \cdot \text{vol}(G) \);
   - (ii) For each module \( X \in A \), \( |X| = \log^{O(1)} n \);
   - (iii) For each module \( X \in A \), \( \Phi(X) = O\left(\frac{\log \log n}{\log n}\right) \),

   then \( \mathcal{H}^2(G) = O(\log \log n) \).
Proof. Consider the partition of nodes of $G$ that consists of the modules in $A$ and the module, denoted by $S$, composed by the nodes not in $A$. Note that

$$H^2(G) \leq \sum_{X \in A} \frac{\text{vol}(X)}{\text{vol}(G)} H^1(X) + \sum_{X \in A} \Phi(X) \frac{\text{vol}(X)}{\text{vol}(G)} \log_2 \frac{\text{vol}(X)}{\text{vol}(G)} \text{vol}(S) \frac{\text{vol}(S)}{\text{vol}(G)}.$$ 

For (1), for each $X \in A$, we have

(i) $H^1(X) \leq \log_2 |X| = o(\log n)$,

(ii) $\Phi(X) = o(1)$,

(iii) $\sum_{X \in A} (\text{vol}(X)/\text{vol}(G)) \log_2 (\text{vol}(X)/\text{vol}(G)) \leq \log_2 n$,

(iv) $\text{vol}(S) = o(1) \cdot \text{vol}(G)$, $H^1(S) \leq \log_2 n$, and

(v) $\Phi(S) \cdot (\text{vol}(S)/\text{vol}(G)) \log_2 (\text{vol}(S)/\text{vol}(G)) \leq 1$.

So in all, $H^2(G) = o(\log n)$.

For (2), for each $X \in A$, we have:

(i) $H^1(X) \leq \log_2 |X| = O(\log \log n)$,

(ii) $\Phi(X) = O(\log \log n / \log n)$,

(iii) $\sum_{X \in A} (\text{vol}(X)/\text{vol}(G)) \log_2 (\text{vol}(X)/\text{vol}(G)) \leq \log_2 n$,

(iv) $\text{vol}(S) = O(\log \log n / \log n) \cdot \text{vol}(G)$,

(v) $H^1(S) \leq \log_2 n$, and

(vi) $\Phi(S) \cdot (\text{vol}(S)/\text{vol}(G)) \log_2 (\text{vol}(S)/\text{vol}(G)) \leq 1$.

So we have $H^2(G) = O(\log \log n)$.

Theorems 8.1 and 8.3 together give a combinatorial characterization for the graphs $G$ with $H^2(G) = O(\log \log n)$. However, in this characterization, the combinatorial property is complicated. This is the disadvantage of the combinatorial characterization theorem. We then look for simpler characterizations of the graphs.

8.2 Algebraic properties of the graphs with $H^2(G) = O(\log \log n)$

In this part, we show that for any connected graph $G$, if the two-dimensional structure entropy $H^2(G) = O(\log \log n)$, then there are many eigenvalues of the Laplacian of $G$ that are close to 0. The proof of the result is actually an application of Theorem 8.1.

Theorem 8.4. (Algebraic characterization theorem of the graphs with two-dimensional structure entropy $O(\log_2 \log_2 n)$)

For every weighted graph $G = (V, E, w)$ with number of edges $m = |E|$ and weight function $w : E \to \mathbb{R}^+$ satisfying $\frac{\max_{e \in E} w(e)}{\min_{e \in E} w(e)} \leq W$ for some constant $W \geq 1$, if $H^2(G) \leq c \log_2 \log_2 m$ for some constant $c > 0$ and any sufficiently large $m$, then for any $\varepsilon > 0$ and sufficiently large $m$, there is an integer

$$k \geq \frac{(1 - 2\varepsilon) \cdot m}{W \cdot \log_2^{\frac{n_0}{2}} m}$$

such that $\lambda_k \leq 2 / \log_2^{1-\varepsilon} m$.

Theorem 8.4 implies that if $H^2(G) = O(\log \log m)$, then there is an integer $k = \Omega(n/\text{poly log } m)$ such that the $k$-th largest eigenvalue $\lambda_k$ of the Laplacian of $G$ satisfies $\lambda_k = o(1)$. 

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Proof. By Theorem 8.1, we know that there is a subset of vertices \( A \subseteq V \) satisfying the three properties stated in Theorem 8.1. Assume again that the least weight on edge is 1 while the largest one is \( W \). For each module \( X \in A \), denote by \( g_X \) the total weight of global edges of \( X \). Then we have

\[
\Phi(X) = \frac{g_X}{\text{vol}(X)} \leq \frac{1}{\log_2(1 - \epsilon) m}.
\]

Moreover, since \( |X| \leq \log_2^{3 \epsilon / 2} m \), we have

\[
\text{vol}(X) - g_X \leq W \cdot \log_2^{6 \epsilon / 2} m.
\]

Combining the two inequalities above, we have

\[
\text{vol}(X) \leq \frac{W \cdot \log_2^{6 \epsilon / 2} m}{1 - \log_2^{(1 - \epsilon) m}}.
\]

Since \( \text{vol}(A) \geq (1 - 2 \epsilon) \cdot \text{vol}(G) \) and \( \text{vol}(G) \geq 2m \), the total number of modules in \( A \) is at least

\[
\frac{(1 - 2 \epsilon) \cdot 2m}{W \log_2^{6 \epsilon / 2} m} \geq \frac{(1 - 2 \epsilon) \cdot m}{W \log_2^{6 \epsilon / 2} m} \triangleq k_0
\]

for sufficiently large \( m \). This means that we can find at least \( k_0 \) disjoint modules in \( G \), each of which has conductance at most \( 1 / \log_2^{1 - \epsilon} m \). By the high-order Cheeger’s inequalities, there is an integer \( k \geq k_0 \) such that the \( k \)-way conductance of \( G \) is at most \( 1 / \log_2^{1 - \epsilon} m \), and so \( \lambda_k \leq 2 / \log_2^{1 - \epsilon} m \). The theorem follows.

For any connected and balanced graph \( G \), if \( \mathcal{H}^{2}(G) = O(\log \log n) \), then the security index \( \theta(G) \) of \( G \) is \( 1 - o(1) \). By Theorem 8.4 in this case, there is a large \( k \) such that the \( k \)-th largest eigenvalue \( \lambda_k \) of the Laplacian of \( G \) is less than a small constant \( \epsilon > 0 \). In the next section, we will show that this result can be further strengthened.

It is interesting to notice that each eigenvalue is in \([0, 2]\), and that the summation of all the eigenvalues \( \lambda_i \) of \( G \) is less than a small constant \( \epsilon > 0 \). In the next section, we will show that this result can be further strengthened.

It is interesting to notice that each eigenvalue is in \([0, 2]\), and that the summation of all the eigenvalues \( \lambda_i \) of \( G \) is less than a small constant \( \epsilon > 0 \). In the next section, we will show that this result can be further strengthened.

9 Eigenvalues of the Laplacian of the Security Graphs

For a given resistor graph, we first establish the following combinatorial characterization theorem.

**Theorem 9.1.** *(Combinatorial property theorem of resistor graphs)* Let \( G = (V, E) \) be a graph with number of edges \( m = |E| \) and volume \( \text{vol}(G) \). Let \( w : E \rightarrow \mathbb{R}^+ \) be the weight function satisfying \( \frac{\max_{e \in G} w(e)}{\min_{e \in G} w(e)} \leq W \), for some constant \( W \geq 1 \). If the security index \( \theta(G) \geq 1 - \theta \) for some constant \( \theta \), then for any \( \epsilon > 0 \), \( \phi > \theta \), there is a constant \( \alpha < 1 \) (related to \( \theta \) and \( \phi \)), such that for any sufficiently large \( m \), there is a set of modules of vertices, denoted by \( A \), satisfying

1. \( \text{vol}(A) \geq (1 - \alpha - \epsilon) \cdot \text{vol}(G) \);
2. For each module \( X \in A \), \( \Phi(X) \leq \phi \);
3. For each module \( X \in A \), \( |X| \leq 2^{\mathcal{H}}(G) \frac{3\theta}{1 - \theta} \).

**Proof.** We also suppose that the edge weights range from 1 to \( W \). Since \( \theta(G) \geq 1 - \theta \), there is a partition \( \mathcal{P} \) on \( V \) such that \( \mathcal{H}^\mathcal{P}(G) \leq \theta \cdot \mathcal{H}^\mathcal{I}(G) \). Define

\[
J = \{ j : V_j \in \mathcal{P}, H_j \leq \epsilon^{-1} \cdot \theta \mathcal{H}^\mathcal{I}(G) \},
\]

where \( V_j \) is the \( j \)-th module of \( \mathcal{P} \) and

\[
H_j = -\sum_{e \in V_j} \frac{d_e}{\text{vol}(V_j)} \log_2 \frac{d_e}{\text{vol}(V_j)}.
\]
is the one-dimensional structure entropy of $V_j$. Thus the fraction of total volume of $\mathcal{J}$ satisfies

$$\sum_{j \in \mathcal{J}} \frac{\text{vol}(V_j)}{\text{vol}(G)} \leq \varepsilon.$$

Define

$$J' = \{ j : V_j \in \mathcal{P}, \Phi_j \leq \phi \},$$

and $\mathcal{J} = [\mathcal{P}] \setminus J'$. Then we will show that the fraction of the total volume of the modules in $V_j$ for $j \in \mathcal{J}$, denoted by $\text{vol}(\mathcal{J})$, is at most some constant $\alpha$. Let $\text{vol}(\mathcal{J}) = \varepsilon_0 \cdot \text{vol}(G)$. Then $\mathcal{H}^\mathcal{P}(G) \leq \theta \cdot \mathcal{H}^1(G)$ means that

$$(1 - \theta) \cdot \sum_{j \in [\mathcal{P}]} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot H_j \leq (\theta - \Phi_j) \cdot \hat{H}(J') + (\theta - \Phi_j) \cdot \hat{H}(\mathcal{J}),$$

where

$$\hat{H}(J') = \sum_{j \in J'} \frac{\text{vol}(V_j)}{\text{vol}(G)} \log_2 \frac{\text{vol}(V_j)}{\text{vol}(G)}$$

and

$$\hat{H}(\mathcal{J}) = \sum_{j \in \mathcal{J}} \frac{\text{vol}(V_j)}{\text{vol}(G)} \log_2 \frac{\text{vol}(V_j)}{\text{vol}(G)}.$$

By the definition of $J'$, we have

$$(1 - \theta) \cdot \sum_{j \in [\mathcal{P}]} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot H_j \leq \theta \cdot \hat{H}(J') + (\theta - \phi) \cdot \hat{H}(\mathcal{J}).$$

Replacing $\text{vol}(J')$ and $\text{vol}(\mathcal{J})$ with $\varepsilon_0 \cdot \text{vol}(G)$ and $(1 - \varepsilon_0) \cdot \text{vol}(G)$, respectively, we get

$$(1 - \theta) \cdot \sum_{j \in [\mathcal{P}]} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot H_j \leq \theta (1 - \varepsilon_0) \cdot H'(J') + (\theta - \phi) \varepsilon_0 \cdot H(\mathcal{J}) - \theta (1 - \varepsilon_0) \log_2 (1 - \varepsilon_0) - (\theta - \phi) \varepsilon_0 \log_2 \varepsilon_0,$$

where

$$H(J') = \sum_{j \in J'} \frac{\text{vol}(V_j)}{\text{vol}(J')} \log_2 \frac{\text{vol}(V_j)}{\text{vol}(J')}$$

and

$$H(\mathcal{J}) = \sum_{j \in \mathcal{J}} \frac{\text{vol}(V_j)}{\text{vol}(\mathcal{J})} \log_2 \frac{\text{vol}(V_j)}{\text{vol}(\mathcal{J})}.$$}

This implies that

$$(1 - \theta) \cdot \sum_{j \in [\mathcal{P}]} \frac{\text{vol}(V_j)}{\text{vol}(G)} \cdot H_j \leq \theta \cdot H(J') - [\theta \cdot H(J') + (\phi - \theta) \cdot H(\mathcal{J})] \cdot \varepsilon_0 + 2\theta.$$
Since \( \Phi_j \leq \phi \), we have that

\[
(1 - \epsilon) \log_2 m_j \leq \frac{2\theta \cdot H^1(G)}{\epsilon(1 - \phi)} + 3.
\]

Therefore, for sufficiently large \( m \), both the number of edges \( m_j \) and the number of vertices in \( V_j \) is upper bounded by \( 2^{\frac{H^1(G)}{m} \cdot \frac{2\theta}{\epsilon(1 - \phi)}} \). This completes the proof of Theorem 9.1.

By Theorem 9.1 we have the following algebraic property theorem of the general resistor graphs.

**Theorem 9.2.** (Algebraic property theorem of resistor graphs) For every weighted graph \( G = (V, E, w) \) with number of edges \( m = |E| \) and weight function \( w : E \to \mathbb{R}^+ \) satisfying \( \frac{\max_{e \in E} \{w(e)\}}{\min_{e \in E} \{w(e)\}} \leq W \) for some constant \( W \geq 1 \), if the security index \( \theta(G) \geq 1 - \theta \) for some constant \( \theta \), then for any \( \epsilon > 0 \), \( \phi > \theta \), there is a constant \( \alpha < 1 \) such that for any sufficiently large \( m \), there is an integer

\[
k \geq \frac{2(1 - \alpha - \epsilon)(1 - \phi) \cdot m}{W \cdot 2^{H^1(G)} \cdot \frac{2\theta}{\epsilon(1 - \phi)(1 - \epsilon)}}
\]

such that \( \lambda_k \leq 2\phi \).

**Proof.** By the proof of Theorem 8.4.

### 10 Conclusions and Discussion

We proposed the notion of resistance of a graph to measure the force of the graph to resist cascading failures of strategic virus attacks. The resistance of a graph \( G \) is the maximum number of bits required to determine the codeword of the module of the graph that is accessible from random walk from which random walk cannot escape. We found the resistance law of networks that the resistance of a graph is the difference of the one- and two-dimensional structure entropy of the graph. Here, for a graph \( G \) and a natural number \( K \), the \( K \)-dimensional structure entropy of \( G \) is the least number of bits required to determine the \( K \)-dimensional codeword of the vertex that is accessible from the random walk with stationary distribution in \( G \). We defined the security index of a graph \( G \) to be the normalised resistance of \( G \). We propose the notion of \( (n, \theta) \)-resistor graph. For a large constant \( \theta \) (that is, less than and close to 1), an \( (n, \theta) \)-resistor graph is a connected graph with \( n \) vertices, and with security index greater than or equal to \( \theta \). We showed that for a tree with bounded weights or grid graphs \( G \), the resistance of \( G \) is \( \Omega(\log n) \) and the security index of \( G \) is \( 1 - o(1) \). The results demonstrate that the natural structures such as trees and grid graphs have the important property of high resistance and high security against virus attacks. We showed that for the networks \( G \) of the security model with affinity exponent \( \alpha > 0 \) and edge parameter \( d \geq 2 \), the resistance of \( G \) is maximised as \( \Omega(\log n) \), and the security index of \( G \) is maximised as \( 1 - o(1) \), for sufficiently large \( n \). Therefore, the security model does generate the networks of high resistances and high security indices. We also establish both a combinatorial and an algebraic characterization theorems of the resistor graphs. In particular, we show that for a large constant \( \theta \), for an \( (n, \theta) \)-resistor graph, and for any small constant \( \epsilon > 0 \), there is a large \( k \) such that the \( k \)-th largest eigenvalue of the Laplacian of the graph is less than \( \epsilon \). Our results provide the fundamental theory for network security, with potential applications in the security engineering of networks.

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Fact 10.1. For any real $\lambda > 0$ and any $X$ variables.

Proof. Note that Lemma 10.1. We will use the following form of Chernoff bound.

Lemma 10.2. (Azuma’s inequality) Let $X_1, \ldots, X_n$ be a vector of positive entries. Let $e = (c_1, \ldots, c_n)$ be a vector of positive entries. Let a sequence of random variables $X_0, X_1, \ldots, X_n$ be a martingale. If it is $e$-Lipschitz, that is, $|X_i - X_{i-1}| \leq c_i$ for $i = 1, \ldots, n$, then for any $\lambda > 0$,

$$\Pr[X_n \leq X_0 - \lambda] \leq \exp \left( -\frac{\lambda^2}{2 \sum_{i=1}^{n} c_i^2} \right),$$

$$\Pr[X_n \geq X_0 + \lambda] \leq \exp \left( -\frac{\lambda^2}{2 \sum_{i=1}^{n} c_i^2} \right).$$

We will use the following form of Azuma’s inequality for supermartingales.

Lemma 10.3. (Supermartingale inequality, Theorem 2.40) For a filter $\{0, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$, suppose that a non-negative random variable $X_i$ is $\mathcal{F}_i$-measurable for $0 \leq i \leq n$. Let $B$ be the bad set associated with the following admissible conditions: (that is, the set of events that the conditions fail to hold.)

$$E(X_i|\mathcal{F}_{i-1}) \leq X_{i-1},$$

$$\text{Var}(X_i|\mathcal{F}_{i-1}) \leq \sigma_i^2 + \phi_i X_{i-1},$$

$$X_i - E(X_i|\mathcal{F}_{i-1}) \leq a_i + M,$$

where $\sigma_i, \phi_i, a_i$ and $M$ are non-negative constants. Then we have

$$\Pr(X_n \geq X_0 + \lambda) \leq \exp \left( -\frac{\lambda^2}{2(\sum_{i=1}^{n} \sigma_i^2 + a_0^2 + (X_0 + \lambda)(\sum_{i=1}^{n} \phi_i) + M \lambda / 3)} \right) + \Pr(B).$$

The following fact will also be very useful in our proofs.

Fact 10.1. For any real $x$,

$$\frac{1}{x} + 1 \leq \log \left( 1 + \frac{1}{x} \right) \leq \frac{1}{x}.$$

Proof. Note that $1 + y \leq e^y$ holds for all real $y$. The fact is obtained by replacing $y$ with $-\frac{1}{x+1}$ and $\frac{1}{x}$, respectively.

The following expansion of power series is folklore.

Fact 10.2. For any $u > 0$ and $|x| \leq 1$,

$$(1 \pm x)^u = 1 \pm ux + \frac{u(u-1)}{2!} x^2 \pm \frac{u(u-1)(u-2)}{3!} x^3 + \cdots + (-1)^m \frac{u(u-1) \cdots (u-m+1)}{m!} x^m + \cdots.$$
Appendix B: Proof of Theorem 7.2

Proof. (Proof of Theorem 7.2) For (1). By the construction of $G$, the expectation of $|C_t|$ is

$$E(|C_t|) = n_0 + \sum_{i=3}^{t} \frac{1}{\log^a i}.$$  

By indefinite integral

$$\int \left( \frac{1}{\log^a x} - \frac{a}{\log^{a+1} x} \right) dx = \frac{x}{\log^a x} + C,$$

we know that if $t \geq T_1$ is large enough (when $n$ is large enough), then

$$\sum_{i=3}^{t} \frac{1}{\log^a i} \leq 1 + \int_{2}^{t} \frac{1}{\log^a x} dx$$

$$\leq \int_{2}^{t} \frac{6}{5} \left( \frac{1}{\log^a x} - \frac{a}{\log^{a+1} x} \right) dx$$

$$\leq \frac{4t}{3\log^a t},$$

where $\frac{6}{5}$ and $\frac{4}{3}$ are chosen arbitrarily among the numbers larger than 1. Similarly,

$$\sum_{i=3}^{t} \frac{1}{\log^a i} \geq \int_{2}^{t} \frac{1}{\log^a x} dx$$

$$\geq \int_{2}^{t} \left( \frac{1}{\log^a x} - \frac{a}{\log^{a+1} x} \right) dx$$

$$\geq \frac{3t}{4\log^a t}.$$

By the Chernoff bound (Lemma 10.1), since $t \geq T_1$ and $n_0$ is a constant, with probability $1 - \exp(-\Omega(\frac{t}{\log a})) = 1 - o(n^{-1})$, we have $\frac{t}{2\log^a t} \leq |C_t| \leq \frac{2t}{\log^a t}$. By the union bound, such an inequality holds for all $t \geq T_1$ with probability $1 - o(1)$.

we define the following event:

**Definition 10.1.** Let $E$ be the event that, for all $i \geq T_1$, $\frac{t}{2\log^a t} \leq |C_t| \leq \frac{2t}{\log^a t}$. By the discussion above, $E$ happens with probability $1 - o(1)$. We will assume and use this event frequently throughout our proofs.

For (2). By the construction of $G$, the expectation of $|S|$ at time step $t$ is

$$E(|S|) = 1 + \sum_{i=t_{S}+1}^{t} \left( 1 - \frac{1}{\log^a i} \right) \cdot \frac{1}{|C_t|}.$$  

By (1), we know that $E$ holds with probability $1 - o(1)$. Thus, if $a > 0$, then at time step $t$,

$$E(|S|) = \Theta \left( \sum_{i=t_{S}+1}^{t} \left( 1 - \frac{1}{\log^a i} \right) \cdot \frac{\log^a t}{t} \right)$$

$$= \Theta \left( \int_{t_{S}}^{t} \frac{\log^a x}{x} dx \right)$$

$$= \Theta(\log^{a+1} t - \log^{a+1} t_{S}).$$

For (3). It suffices to show that with probability $1 - o(n^{-1})$, the homochromatic set of the first color $\kappa$ has size $4 \log^{a+1} n$. Then the result follows from the union bound.

Let $S_{\kappa}$ be the set of vertices sharing color $\kappa$. Conditioned on the event $E$, for large enough $n$,
Therefore, with probability \(1 - o(n^{-1})\), the size of \(S_t\) is at most \(4 \log^{a+1} n\).

For (4). We need to bound the number of global edges with one endpoint in \(S_t\).

For \(t \geq t_s\), define \(S[t]\) to be the snapshot of \(S\) at time step \(t\), and \(\partial(S)[t]\) to be the set of edges from \(S[t]\) to \(\overline{S[t]}\), the complement of \(S[t]\). So \(\partial(S)[t]\) is in fact the set of global edges of \(S\) at time step \(t\) and \(g_{S} = |\partial(S)[n]|\). Denote by \(D(S)[t]\) the total degree of vertices in (the volume of) \(S[t]\). In our proof, we first give a recurrence for the expected value of \(D(S)[t]\) at any time step \(t > t_s\), and then show that \(\partial(S)[n]\) is not expectedly too many.

By the construction of \(G_t\), when a new vertex is created, the volume it contributes to the network is \(2d\). By (1), we know that the volume of \(G_t\) is \(2d(1 + o(1))t\), where \(o(t)\) is contributed by \(G_{n_0}\). The recurrence of \(D(S)[t]\) satisfies

\[
E[D(S)[t] | D(S)[t - 1]] \leq D(S)[t - 1] + \frac{1}{\log^2 t} \left[ \frac{D(S)[t - 1]}{2d(t - 1)} + (d - 1) \cdot \frac{1}{|C_{t-1}|} \right] + \left( 1 - \frac{1}{\log^2 t} \right) \cdot \frac{2d}{|C_{t-1}|} \tag{44}
\]

We suppose the event \(\mathcal{E}\) that for all \(t \geq T_1 = \log^{a+1} n\), \(\frac{t}{2 \log^2 t} \leq |C_t| \leq \frac{2t}{\log^2 t}\), which almost surely holds by (1). It also holds for \(t \geq T_2\) for sufficiently large \(n\). On this condition, recalling that \(d \geq 2\), we have

\[
E[D(S)[t] | D(S)[t - 1]] \leq D(S)[t - 1] \left[ 1 + \frac{1}{2(t - 1) \log^a t} \right] + \frac{4d \log^a t}{t}. \tag{45}
\]

Taking expectation on both sides, we have

\[
E(D(S)[t]) \leq E(D(S)[t - 1]) \left[ 1 + \frac{1}{2(t - 1) \log^a t} \right] + \frac{4d \log^a t}{t}. \tag{46}
\]

Then we analyze this recurrence for the cases of \(a \geq 1\) and \(a < 1\), respectively.

When \(a \geq 1\), since for sufficiently large \(n\) and thus for sufficiently large \(t\) with \(t \geq t_s \geq T_2\), we have

\[
9d \log^a t \log \frac{t + 1}{t} \geq 9d \log^a t \log \frac{t + 1}{t} - \frac{9d \log t}{2(t - 1)}
\]

\[
\geq 9d \log^a t \log \frac{t + 1}{t} \geq \frac{9d \log^a t}{2(t - 1)}
\]

\[
\geq \frac{4d \log^a t}{t}. \tag{47}
\]

where the second inequality follows from Fact[10.1] Applying it to Inequality (46), we have

\[
E(D(S)[t]) - 9d \log^a(t + 1) \leq \left[ 1 + \frac{1}{2(t - 1) \log^a t} \right] \cdot (E(D(S)[t - 1]) - 9d \log^a t).
\]

Recursively, we have

\[
E(D(S)[t]) \leq \theta_t \cdot [E(D(S)[t_S]) - 9d \log^a(t_S + 1)] + 9d \log^a(t + 1)
\]

Relevant equations:

\[
E(|S_t|) = 1 + \sum_{i=3}^{n} \left( 1 - \frac{1}{\log^a i} \right) \cdot \frac{1}{|C_i|}
\]

\[
\leq T_1 + \sum_{i=T_1+1}^{n} \left( 1 - \frac{1}{\log^a i} \right) \cdot \frac{2 \log^a i}{i}
\]

\[
\leq 3 \log^{a+1} n.
\]

By the Chernoff bound,

\[
Pr[|S_t| > 4 \log^{a+1} n] = o(n^{-1}).
\]
holds for all \( t_S < t \leq n \), where
\[
\theta_t = \prod_{i=t_S+1}^t \left[ 1 + \frac{1}{2(i-1) \log^a i} \right].
\]

Note that \( E(D(S)[t_S]) = d \). So
\[
E(D(S)[t]) \leq 9d \log^{a+1}(t+1) - \theta_t \cdot [9d \log^{a+1}(t_S + 1) - d].
\] (48)

When \( 0 < a < 1 \), since for sufficiently large \( n \) and thus for sufficiently large \( t \),
\[
\frac{9d \log^a t}{2(t-1)} - 9d \cdot \frac{\log^2 a (t+1) - \log^2 a t}{2t}
\]
\[
\geq 4d \log^a t \frac{d \log^a t}{t},
\] (49)

where the first inequality follows from the fact that \( \log(t+1) - \log t = \log \left(1 + \frac{1}{t}\right) \leq \frac{1}{t} \) and so when \( a < 1 \),
\[
\lim_{t \to \infty} \frac{\log^2 a (t+1) - \log^2 a t}{\log^a t} = \lim_{t \to \infty} t \cdot \frac{\log a (t+1) - 1}{\log a t} \cdot \frac{\log a t}{\log a (t+1) + \log a t}
\]
\[
\leq \lim_{t \to \infty} t \cdot \frac{\log a (t+1) - 1}{\log a t} \cdot \frac{\log a t}{\log a (t+1) + \log a t}
\]
\[
\leq \lim_{t \to \infty} \frac{2 \log^a a (t+1)}{\log a t} = 0.
\]

Applying Inequality (49) to (46), we have
\[
E(D(S)[t]) + 9d \log^2 a (t+1) \leq \left[ 1 + \frac{1}{2(t-1) \log^a t} \right] \cdot (E(D(S)[t-1]) + 9d \log^a t).
\]

Recursively, we have
\[
E(D(S)[t]) \leq \theta_t \cdot [E(D(S)[t_S]) + 9d \log^2 a (t_S + 1)] - 9d \log^2 a (t+1)
\]
holds for all \( t_S < t \leq n \), and so
\[
E(D(S)[t]) \leq \theta_t \cdot [9d \log^2 a (t_S + 1) + d] - 9d \log^2 a (t+1).
\] (50)

Note that by the construction of \( G \),
\[
E(g_S) \leq \sum_{t=t_S}^n \frac{1}{\log^a t} \left[ \frac{E(D(S)[t])}{2d(t-1)} + E \left( \frac{d-1}{|C_{t-1}|} \right) \right].
\] (51)

Let
\[
U_1 = \sum_{t=t_S}^n \frac{E(D(S)[t])}{2d(t-1) \log^a t}
\]
and
\[
U_2 = \sum_{t=t_S}^n E \left( \frac{d-1}{|C_{t-1}| \cdot \log^a t} \right).
\]

So \( E(g_S) \leq U_1 + U_2 \). Recall that in the proof of (1), we have shown that for each time step \( t \geq T_1 (\geq T_2) \), with probability at least \( 1 - \exp(-\Omega(\log t - t)) \), we have \( \frac{1}{2 \log t} \leq |C_t| \leq \frac{2}{\log t} \). So for some constant \( c > 0 \),
\[
E \left( \frac{d-1}{|C_{t-1}| \cdot \log^a t} \right) \leq \frac{2(d-1)}{t} + t \cdot \exp \left( -\frac{ct}{\log^a t} \right),
\]
and so
\[
U_2 \leq \sum_{t=t_s}^n E \left( \frac{d - 1}{|C_{t-1}| \cdot \log^2 t} \right) = O(d \cdot (\log n - \log t_s)) = O(d \cdot \log \log n).
\]

Next, we will bound \(U_1\) by using Inequalities \ref{eq:s} and \ref{eq:o} for different values of \(a\).

When \(a \geq 1\), we have
\[
U_1 \leq \sum_{t=t_s}^n \frac{9d \log^{a+1}(t+1) - \theta_1 \cdot [9d \log^{a+1}(t_s + 1) - d]}{2d(t-1) \log^2 t}.
\]

Since \(\theta_1 > 1\), for sufficient large \(n\), we have
\[
U_1 \leq \sum_{t=t_s}^n \frac{9 \log^{a+1}(t+1) - [9 \log^{a+1}(t_s + 1) - 1]}{2(t-1) \log^2 t}
\]
\[
\leq \frac{9}{2} \left[ \sum_{t=t_s}^n \frac{\log t}{t-1} - \log^{a+1}(t_s + 1) \sum_{t=t_s}^n \frac{1}{(t-1) \log^2 t} \right]
\]
\[
\leq \frac{9}{2} \left( \int_{t_s}^n \frac{\log x}{x} dx - \log^{a+1} t_s \int_{t_s}^n \frac{1}{x \log x} dx \right).
\]

If \(a > 1\), then
\[
U_1 \leq \frac{9}{2} \log^2 n \cdot \left[ \frac{1}{2} \left( 1 + \frac{1}{a-1} \right) \left( \log t_s \log n \right)^2 + \frac{1}{a-1} \left( \log t_s \log n \right)^{a+1} \right]
\]
\[
= \frac{9}{2} \log^2 n \cdot \left[ \frac{1}{2} + \frac{a+1}{2(a-1)} \left( 1 - b \log \log n \right)^2 + \frac{1}{a-1} \left( 1 - b \log \log n \right)^{a+1} \right].
\]

By Fact \ref{fact:10.2},
\[
\left( 1 - b \log \log n \right)^{a+1} \leq 1 - \frac{(a+1)b \log \log n}{\log n} + \frac{(a+1)ab^2(\log \log n)^2}{2 \log^2 n}.
\]

Thus,
\[
U_1 \leq \frac{9}{2} \log^2 n \cdot \left[ \frac{1}{2} - \frac{a+1}{2(a-1)} \left( 1 - \frac{2b \log \log n}{\log n} + \frac{b^2(\log \log n)^2}{\log^2 n} \right) \right]
\]
\[
+ \frac{1}{a-1} \left( 1 - \frac{(a+1)b \log \log n}{\log n} + \frac{(a+1)ab^2(\log \log n)^2}{2 \log^2 n} \right) \right]
\]
\[
= \frac{9}{4} (a+1)b^2(\log \log n)^2.
\]

Note that \(E(q_s) = U_1 + U_2\) and \(U_2 = O(\log \log n)\). For sufficiently large \(n\), \(E(q_s) \leq \frac{9}{2} (a+1)b^2(\log \log n)^2\).

(4)(i) follows.

If \(a = 1\), then
\[
U_1 \leq \frac{9}{2} \left( \int_{t_s}^n \frac{\log x}{x} dx - \log^2 t_s \int_{t_s}^n \frac{1}{x \log x} dx \right)
\]
\[
= \frac{9}{2} \left[ \frac{1}{2} (\log^2 n - \log^2 t_s) - \log^2 t_s \cdot (\log \log n - \log \log t_s) \right]
\]
\[
= \frac{9}{2} \left[ \frac{1}{2} (\log^2 n - \log^2 t_s) - \log^2 t_s \cdot \log \left( 1 + \frac{b \log \log n}{\log n - b \log \log n} \right) \right]
\]
\[
\leq \frac{9}{2} \left[ \frac{1}{2} (\log^2 n - \log^2 t_s) - \log^2 t_s \cdot \frac{b \log \log n}{\log n} \right]
\]
\[
= \frac{9}{2} \left[ \frac{1}{2} \log^2 n - \frac{1}{2} (\log n - b \log \log n)^2 - (\log n - b \log \log n)^2 \cdot \frac{b \log \log n}{\log n} \right]
\]
\[
= \frac{9}{2} \left[ \frac{3}{2} b^2(\log \log n)^2 - \frac{(b \log \log n)^3}{\log n} \right]
\]
\[
\leq \frac{27}{4} b^2(\log \log n)^2.
\]
Since $E(g_S) = U_1 + U_2$ and $U_2 = O(\log \log n)$, when $n$ is large enough, $E(g_S) \leq 8b^2(\log \log n)^2$. (4)(ii) follows.

When $a < 1$, applying Inequality (50) to (51), we have

$$U_1 \leq \sum_{t=t_0}^n \frac{\theta_t \cdot (9d \log 2^a (t_S + 1) + d) - 9d \log 2^a (t + 1)}{2d(t - 1) \log^a t}$$

$$= \frac{9}{2} \left[ \sum_{t=t_0}^n \frac{\theta_n \log 2^a t_S}{(t - 1) \log^a t} - \sum_{t=t_0}^n \frac{\log 2^a (t + 1)}{(t - 1) \log^a t} \right]$$

$$= \frac{9}{2} \left( \theta_n \log 2^a t_S \cdot \int_{t_0}^n \frac{1}{x \log^a x} \, dx - \int_{t_0}^n \frac{\log^a x \, dx}{x} \right) + O \left( \frac{1}{n} \right)$$

$$= \frac{9}{2} \left( \theta_n \log 2^a t_S \cdot \log 1-a n - \log 1-a t_S - \frac{\log 1+a n - \log 1+a t_S}{1+a} \right) + O \left( \frac{1}{n} \right)$$

$$= \frac{9}{2(1-a)} \log 1+a n \log 2^a t_S - \frac{9}{2} \left( \frac{\theta_n}{1-a} - \frac{1}{1+a} \right) \log 1+a n$$

To deal with the factor $(\theta_n - 1)$, we need the following lemma.

**Lemma 10.4.** For sufficiently large $n$,

$$\theta_n - 1 \leq \frac{b \log \log n}{\log^a n}.$$  

Note that by the above lemma, for sufficiently large $n$

$$U_1 \leq \frac{9}{2} b \cdot \frac{b \log \log n}{\log^a n} \log 2^a n \log \log n + O \left[ \frac{(\log \log n)^2}{\log^{1-a} n} \right]$$

$$\leq \frac{9}{2} b^2 (\log \log n)^2.$$  

Note that $E(g_S) = U_1 + U_2$ and $U_2 = O(\log \log n)$. For sufficiently large $n$, $E(g_S) \leq 5b^2(\log \log n)^2$. (4)(iii) follows.

To complete the proof, we prove Lemma 10.4.

**Proof.** Recall that

$$\theta_n = \prod_{i=t_2+1}^n \left[ 1 + \frac{1}{2(i - 1) \log^a i} \right].$$

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Then
\[
\log \theta_n = \sum_{i=t_s+1}^{n} \log \left[ 1 + \frac{1}{2(i-1) \log^a i} \right]
\]
\[
\leq \sum_{i=t_s+1}^{n} \frac{1}{2(i-1) \log^a i}
\]
\[
\leq \frac{1}{2} \int_{t_s}^{n} \frac{1}{x \log^a x}
\]
\[
= \frac{1}{2(1-a)} \cdot \left( \log 1^{1-a} n - \log 1^{1-a} t_s \right)
\]
\[
= \frac{\log 1^{-a} n}{2(1-a)} \cdot \left[ 1 - \left( 1 - \frac{b \log \log n}{\log n} \right)^{1-a} \right]
\]
\[
= \frac{\log 1^{-a} n}{2(1-a)} \cdot \left[ (1-a) \cdot \frac{b \log \log n}{\log n} - \frac{(1-a)(-a)}{2} \cdot \left( \frac{b \log \log n}{\log n} \right)^2 \right]
\]
\[+ O \left( \frac{\log \log n}{\log n} \right)^3 \]
\[
= \frac{b \log \log n}{2 \log^a n} + O \left( \frac{\log \log n)^2}{\log 1^{1-a} n} \right).
\]

Thus, for sufficiently large \( n \), \( \log \theta_n \leq \frac{3b \log \log n}{4 \log^a n} \), which implies that
\[ \theta_n \leq (\log n)^{\frac{3b}{4 \log^a n}}. \]

A key observation is that, for any constant \( c \), by l’Hôpital’s rule,
\[
\lim_{n \to \infty} \frac{(\log n)^{\frac{1}{\log \log n} - 1}}{\log \log n} = \lim_{y \to \infty} \frac{y^{\frac{1}{\log y} - 1}}{\log y} = \lim_{y \to \infty} \frac{(y^{\frac{1}{\log y} - 1})'}{\log y}
\]
\[
= \lim_{y \to \infty} \frac{c(1-a \log y)}{y^{1+a} \log y} \cdot \frac{y^{1+a}}{1-a \log y}
\]
\[
= \lim_{y \to \infty} c \cdot \frac{y^{\frac{1}{\log y} - 1}}{\log y} = c.
\]

Thus, for any \( \epsilon > 0 \), if \( n \) is large enough, then
\[ \theta_n - 1 \leq \frac{3b}{4} \cdot \frac{1}{1+\epsilon} \cdot \frac{\log \log n}{\log^a n}. \]

Let \( \epsilon = \frac{1}{2} \), then the lemma follows.

This completes the proof of Theorem 7.3.