Minimal Information in Velocity Space

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Abstract

Jaynes’ transformation group principle can allow the computation of the prior density functions describing minimal knowledge for a velocity. The improper prior is uniform in the unbounded velocity space of classical mechanics. In the relativistic case, however, it reads

$$\mu(\beta_x, \beta_y, \beta_z) = \alpha \times \left[1 - (\beta_x^2 + \beta_y^2 + \beta_z^2)\right]^{-2},$$

but it can be rewritten as a uniform volumetric distribution, when the velocity space is given a non-trivial metric.

1 Introduction

Physicists seldom escape the need to express a state of incomplete knowledge of parameters describing a physical system. When certainty is not at hand, the appropriate language to describe information is probability. The Bayesian approach to probability has proved to be free of the inconsistencies induced by the classical “frequentist” theory, and to provide an elegant solution of many controversial problems, particularly in statistical mechanics\[1\]. However, the necessary assignment of a prior distribution describing minimal knowledge can be, when not neglected, a major difficulty. The search for the prior density function can lead to appropriate non-trivial results, even in the most elementary problems, as pointed out in the following, with the example of the velocity of a particle, in classical and special relativistic mechanics.

2 Minimal information and measure

I hereafter take the Bayesian viewpoint on probability theory\[1,2\], according to which the probability is a real-numbered measure of one’s belief in the validity of a logical proposition $A$, given incomplete knowledge $C$. By incomplete, I mean that $C$ does not allow one to establish the truth or falsehood of $A$ with certainty. In this scheme, a probability is always conditional, in the sense that it can be assigned to a proposition $A$, only assuming some previous knowledge $C$. This is clearly recognized by denoting the probability of proposition $A$ given information $C$ as $p(A|C)$.

Let $x$ be a parameter taking its value in a given range, and characterizing the physical system under study (from a chosen point of view). The knowledge $C$ obtained by some observation or experiment

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is then usually represented by a probability density that will be noted, following Tarantola [3] by an
overlined symbol, e.g.:

$$
\mathcal{T}_x : x \rightarrow \mathcal{T}_x(x).
$$

(1)

Proposition \(A\) can be a statement about the value of \(x\) (e.g.: \(A = (x \in [x_1, x_2])\)). The density \(\mathcal{T}_x\)
allows then the computation of the probability \(p(A|C)\) by the usual relation:

$$
p(A|C) = \int_{x_1}^{x_2} \mathcal{T}_x(x) \, dx.
$$

(2)

If a different parametrization \(y = T(x)\) is used to describe the system, then the information \(C\) induces
a probability density \(\mathcal{T}_y\) related to \(\mathcal{T}_x\) by

$$
\mathcal{T}_x(x) = \mathcal{T}_y(y) |J_T|.
$$

(3)

where \(|J_T|\) stands for the Jacobian of the change of parametrization \(T\). Equation (3) ensures that
the probability of proposition \(A = (x \in [x_1, x_2]) = [y \in T([x_1, x_2])]\) given information \(C\) is attributed
the same value whether the \(x\) or the \(y\) parametrization is used. This is hereafter referred to as the
relation of conservation of probabilities.

Now, prior to any observation, complete ignorance itself is also a state of knowledge, and it should
be described by a density function (d.f.) too. Indeed, the raw definition of the parameter \(x\) and
the properties it possesses are inevitably associated with some information \(I\): the state of minimal
information. The corresponding d.f. will be noted \(\mu_x\) and is usually termed the least informative
d.f., or the prior d.f. It must be emphasized that the form of \(\mu_x\) is generally not trivial, in particular,
\(\mu_x\) is not necessarily a constant. Evidence of this is given by the application of Eq.(3) to two
parametrizations \(x\) and \(y\), where minimal knowledge is described by

$$
\mu_x(x) = \mu_y(y) |J_T|.
$$

(4)

Hence the prior \(\mu\) cannot be constant in, for example, two parametrizations \(x\) and \(y\) related by a
nonlinear transformation \(T\). As a consequence, a general method of inference of the form of \(\mu\) must
be sought.

Jaynes [4] has proposed a “transformation group” method relying on the basic principle that “the
state of minimal information is described by the same d.f. in two different parametrizations in
which the problem is equivalently defined”, or, more widely stated by Tarantola and Valette [5], “the
least informative d.f. is form-invariant under the transformations that leave invariant the equations of
physics”. Therefore, in two parametrizations \(x\) and \(y\) related by such a transformation (i.e. \(y = T(x)\)),
Jaynes’ principle reads:

$$
\mu_x = \mu_y \quad \text{(i.e. } \forall y, \mu_y(y) = \mu_x(y) \text{ )}.
$$

(5)

Thus, the form of this function is constrained by the relation of conservation of probabilities (3),
which imposes the constraint

$$
\int \mu_x(x) \, dx = \int \mu_x[T(x)] \, d[T(x)].
$$

(6)

This relation must be valid for all \(x\) and for every allowed transformation \(T\).

In a later version [3] of his probabilistic approach to inverse problems, Tarantola suggests the use
of volumetric probabilities \(\mu\) instead of probability densities \(\mathcal{T}\). Let \(x = (x^1, x^2, \ldots, x^n)\) be a given
parametrization of the physical system under study. Probability densities \(\mathcal{T}\) have to be multiplied by
the differential element \(dx = dx^1 \, dx^2 \cdots dx^n\) of the coordinates to get an infinitesimal probability,
whereas volumetric probabilities $f$ have to be multiplied by a volume element $dV(x^1, x^2, \ldots, x^n)$ to get the same equality

$$f(x^1, x^2, \ldots, x^n) \, dx^1 \, dx^2 \cdots dx^n = f(x^1, x^2, \ldots, x^n) \, dV(x^1, x^2, \ldots, x^n).$$

(7)

Tarantola chooses then to describe ignorance by a constant volumetric probability $\mu = \text{const}$. This arbitrary choice corresponds to the “natural” feeling that in the case of minimal knowledge, the probability is proportional to the volume element $dV$. From this choice, the search for the correct probability density $\mu_x$ turns into the search of the right volume element $dV(x)$. Indeed, Eq. (7) becomes

$$\mu_x(x^1, x^2, \ldots, x^n) \, dx^1 \, dx^2 \cdots dx^n = dV(x^1, x^2, \ldots, x^n) \times \text{const.}$$

(8)

From this point of view, Jaynes’ principle can be stated as: “In two parametrizations where the laws of physics take the same form, the volume element has the same form”. In other words, the volume element is unchanged under a transformation $T$ that “leave invariant the laws of physics”:

$$dV(x^1, x^2, \ldots, x^n) = dV[T(x^1, x^2, \ldots, x^n)].$$

(9)

For a sufficiently well-defined parameter, these conditions should permit the identification of an “objective” volume element. Furthermore, should the parameter space, where $x$ is defined, be given a metric, its form can be constrained. Indeed, if the line element is

$$dl^2 = g_{ij} \, dx^i \, dx^j,$$

(10)

the volume element is given by the square root of the determinant of the metric $g_{ij}$ multiplied by the differential of the parameters, that is:

$$dV(x^1, x^2, \ldots, x^n) = \sqrt{|g_{ij}|} \, dx^1 \, dx^2 \cdots dx^n.$$

(11)

We will hereafter use this principle to infer the appropriate least informative d.f. for the velocity of a particle with non-zero rest-mass, using both classical and relativistic mechanics, and the corresponding norms.

### 3 Velocity in classical mechanics

Let $K$ and $K'$ be two inertial reference frames where observers $O$ and $O'$ respectively, are at rest, and such that for each of them, the other is moving along the $x$ (or $x'$) axis in his reference frame. Let $P$ be a point-like material particle, moving freely along this common axis. The velocity of $P$ can be indicated by its value $v_x$ for $O$ and $v'_x$ for $O'$. Then, $v_x$ and $v'_x$ are related by the classical law of addition of velocities

$$v'_x = v_x - v_r,$$

(12)

where $v_r$ is the relative velocity of $O'$ as seen by $O$.

Let $\mu_x$ be the d.f. which represents minimal knowledge of this velocity for observer $O$, and $\mu'_x$ the corresponding least informative d.f. for the velocity of $P$ as seen by $O'$. The principle of Galilean

\footnote{This is perhaps where the automatic use of a constant probability density to describe ignorance originates, by misusing the differential of the coordinates as the volume element.}

\footnote{In the sense that it takes the same form whether one uses the $x$ or the $y = T(x)$ parametrization.}
relativity states that neither of the two observers is in a special situation with respect to the laws of mechanics. Therefore, by Jaynes’ principle, both observers $O$ and $O'$ must describe minimal knowledge of the particle’s velocity by the same d.f., that is

$$\mu_x = \mu'_x.$$  \hspace{1cm} (13)

Now, $v_x$ and $v'_x$ can be seen as two different parametrizations for the same physical system. The rule of conservation of probabilities (3) holds, and therefore the d.f. $\mu_x$ must obey the equivalent of Eq. (3), i.e.:

$$\mu_x(v_x) \, dv_x = \mu_x(v_x - v_r) \, dv_x$$  \hspace{1cm} (14)

This relation must be valid for all $v_r$ and all $v_x$. This implies

$$\mu_x(v_x) = \text{const.}$$  \hspace{1cm} (15)

Note that, as $-\infty < v_x < +\infty$, this d.f. is an improper prior (this is a common feature of a least informative d.f.).

In terms of volumetric probabilities, a comparison of the result (15) and Eq. (8) gives immediately the correct volume element:

$$dV(v_x) \propto dv_x.$$  \hspace{1cm} (16)

The former case can be extended to 2 and 3 dimensional cases, in which the velocities measured by two equivalent observers are related by

$$v'_x = R(v - v_r),$$  \hspace{1cm} (17)

where $v'$, $v$ and $v_r$ are 2 or 3-vectors respectively standing for the velocity of: $P$ seen by $O'$, $P$ seen by $O$, and $O'$ seen by $O$. $R$ is, depending on the case, a 2-D or 3-D rotation matrix.

Since $O$ and $O'$ are equivalent observers, they must describe minimal knowledge of the velocity by the same d.f. However, as previously, $v$ and $v'$ can be seen as two different parametrizations of the same system and hence, the rule of conservation of probabilities implies:

$$\mu_x(v) \, dv = \mu_x[R(v - v_r)] \, dv'.$$  \hspace{1cm} (18)

The Jacobian of the transformation (17) being equal to 1, this also leads to

$$\mu_x(v) = \text{const.}$$  \hspace{1cm} (19)

The deduction of the volume element is, as previously, straightforward. For the 2-D case, we get

$$dV(v_x, v_y) = \text{const.} \times dv_x \, dv_y$$

$$dV(v_x, v_z) = \text{const.} \times dv_x \, dv_z,$$

$$dV(v_y, v_z) = \text{const.} \times dv_y \, dv_z$$  \hspace{1cm} (20)

and for the 3-D case, we get, equivalently,

$$dV(v_x, v_y, v_z) = \text{const.} \times dv_x \, dv_y \, dv_z.$$  \hspace{1cm} (21)

This allows the computation of an “objective” metric in velocity space. For simplicity’s sake, it is convenient to use the polar coordinate system $(v, \theta, \varphi)$ related to the cartesian coordinates $(v_x, v_y, v_z)$ by the relations:

$$\begin{align*}
  v_x &= v \cos \theta \\
  v_y &= v \sin \theta \sin \varphi \\
  v_z &= v \sin \theta \cos \varphi
\end{align*}$$  \hspace{1cm} (22)
For reasons of isotropy, the length element must take the form:
\[ dl^2 = f_1(v) \, dv^2 + f_2(v) [d\theta^2 + \sin^2 \theta \, d\phi^2] . \] (23)

Equation (16) corresponds to a fixed direction of \( v \), i.e. \( \theta = 0 \). This leads to
\[ dl^2 = g_{11}^2 = \text{const.}, \]
\[ g_{11}^2 = a^2 . \] (24)

In polar coordinates, equations (20) and (21) read
\[ dV(v, \theta) = \text{const.} \times v \, dv \, d\theta \]
\[ dV(v, \theta, \phi) = \text{const.} \times v^2 \sin \theta \, dv \, d\theta \, d\phi . \] (25)

This imposes
\[ f_2(v) = \text{const.} \times v^2, \]
and again, symmetry considerations impose \( a = b \). Thus the invariant metric of the velocity space in classical mechanics takes the Euclidean form
\[ dl^2 = a^2 [dv^2 + v^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)] = a^2 (dv_x^2 + dv_y^2 + dv_z^2) . \] (27)

This length element, together with the choice of a constant volumetric probability to describe minimal information, ensures that the prior for the velocity expressed in cartesian coordinates is constant, in one, two and three dimensional cases, as required by Eqs. (15) and (19).

4 Velocity in special relativity

We now express the velocities in terms of their ratio to the speed of light (i.e. \( \beta = v/c \)). The difference from the former section is that, instead of the Galilean laws for the addition of velocities, the relativistic laws have now to be applied.

In the one-dimensional case, the relativistic equivalent of Eq. (12) reads
\[ \beta' = \frac{\beta - \alpha}{1 - \alpha \beta} . \] (28)

where \( \beta' = v'/c \) and \( \alpha = v_r/c \). It is convenient to use the parametrization defined by \( b = \text{arctanh} \beta \), \( a = \text{arctanh} \alpha \) and \( b' = \text{arctanh} \beta' \), in which Eq. (28) takes exactly the same form as Eq. (12), that is :
\[ b' = b - a . \] (29)

Consequently, Jaynes’ principle leads to :
\[ p_b(b) = \text{const}. \] (30)

Back in the \( \beta \) parametrization (using Eq. (3)), this becomes
\[ p_\beta(\beta) = \frac{\text{const}.}{1 - \beta^2} . \] (31)

Note that as \(-1 < \beta < 1\), this d.f. is not normalizable, cf Eq. (19).
For the two and three dimensional cases, the relativistic law of addition of velocities reads:

\[
\begin{align*}
\beta'_x &= (\beta_x - \alpha)/(1 - \alpha \beta_x) \\
\beta'_y &= (\beta_y/(1 - \alpha^2)^{1/2})/ (1 - \alpha \beta_x) \\
\beta'_z &= (\beta_z/ (1 - \alpha^2)^{1/2})/ (1 - \alpha \beta_x)
\end{align*}
\]

(32)

It is more convenient to use the parametrization \((\beta, \cos \theta, \varphi)\) related to \((\beta_x, \beta_y, \beta_z)\) by

\[
\begin{align*}
\beta_x &= \beta \cos \theta \\
\beta_y &= \beta \sin \theta \sin \varphi \\
\beta_z &= \beta \sin \theta \cos \varphi
\end{align*}
\]

(33)

and similarly for the primed reference frame. These equations lead, after some calculation, to the following relations between \((\beta, \cos \theta, \varphi)\) and \((\beta', \cos \theta', \varphi')\)

\[
T_{\alpha} : \begin{pmatrix} \beta \\ \cos \theta \\ \varphi \end{pmatrix} \rightarrow \begin{pmatrix} \beta' \\ \cos \theta' \\ \varphi' \end{pmatrix} = \begin{pmatrix} [(1 - \alpha \beta \cos \theta)^2 - (1 - \beta'^2)(1 - \alpha^2)]^{1/2}(1 - \alpha \beta \cos \theta)^{-1} \\ (\beta \cos \theta - \alpha)[(1 - \alpha \beta \cos \theta)^2 - (1 - \beta'^2)(1 - \alpha^2)]^{-1/2} \end{pmatrix}
\]

(34)

One can check that the Jacobian of such a transformation allows the writing of a simple equality:

\[
\frac{\beta'^2}{(1 - \beta'^2)^2} \, d\beta' \, d(\cos \theta') \, d\varphi' = \frac{\beta^2}{(1 - \beta^2)^2} \, d\beta \, d(\cos \theta) \, d\varphi.
\]

(35)

The equivalent of Eq. (6), expressing both Jaynes' principle and the principle of special relativity reads

\[
\mu_{\beta \cos \theta \varphi}(\beta', \cos \theta', \varphi') \, d\beta' \, d(\cos \theta') \, d\varphi' = \mu_{\beta \cos \theta \varphi}(\beta, \cos \theta, \varphi) \, d\beta \, d(\cos \theta) \, d\varphi,
\]

(36)

which means that two observers \(O\) and \(O'\) at rest in two inertial frames \(K\) and \(K'\) moving with a relative velocity \(\alpha\), describe the minimal information about a particle’s velocity \((\beta, \cos \theta, \varphi)\) by the same d.f. \(\mu_{\beta \cos \theta \varphi}\). Comparison of Eqs. (35) and (36) yields an obvious solution for \(\mu_{\beta \cos \theta \varphi}(\beta, \cos \theta, \varphi)\), which reads:

\[
\mu_{\beta \cos \theta \varphi}(\beta, \cos \theta, \varphi) = \frac{\beta^2}{(1 - \beta^2)^2} \times a
\]

(37)

where \(a\) is a constant.

We can use isotropy arguments to assert that the corresponding metric of the parameter space must take the form

\[
dl^2 = g_1(\beta) \, d\beta^2 + g_2(\beta) \left[ \frac{d(\cos \theta)^2}{\sin^2 \theta} + \sin^2 \theta \, d\varphi^2 \right].
\]

(38)

So that we have from (38) and (31), for the fixed, known direction of the velocity \(\theta = 0, \, d\theta = 0\)

\[
[g_1(\beta)]^{1/2} = \frac{\alpha}{1 - \beta^2}.
\]

(39)

The more general three-dimensional case similarly leads from (38) and (39) to

\[
[g_1(\beta) \, g_2(\beta)^2]^{1/2} = b \times \frac{\beta^2}{(1 - \beta^2)^2}.
\]

(40)
We can thus infer the form of the two functions $g_1$ and $g_2$. Imposing that when $\beta$ tends towards 0 (i.e. the velocity of the particle is low compared to $c$), the metric takes the classical form (27), it is straightforward to obtain

$$
dl^2 = a^2 \left[ \frac{1}{(1 - \beta^2)^2} \, d\beta^2 + \frac{\beta^2}{(1 - \beta^2)} (d\theta^2 + \sin^2 \theta \, d\varphi^2) \right]. \tag{41}
$$

It is then purely a technical matter to get the corresponding form of the metric tensor for the Cartesian $(\beta_x, \beta_y, \beta_z)$ coordinates. The result is

$$
|g_{ij}| = a^2 \left[ 1 - \left( \beta_x^2 + \beta_y^2 + \beta_z^2 \right) \right] \left[ \begin{array}{ccc}
1 - \beta_y^2 - \beta_z^2 & \beta_x \beta_y & \beta_x \beta_z \\
\beta_x \beta_y & 1 - \beta_x^2 - \beta_z^2 & \beta_y \beta_z \\
\beta_x \beta_z & \beta_y \beta_z & 1 - \beta_x^2 - \beta_y^2 \\
\end{array} \right]. \tag{42}
$$

This metric allows the direct computation of the prior for a velocity in one, two, and three dimensional cases. In particular, both Eqs. (37) and (42) yield :

$$
\mu_{\beta_x \beta_y \beta_z}(\beta_x, \beta_y, \beta_z) = a \left[ 1 - \left( \beta_x^2 + \beta_y^2 + \beta_z^2 \right) \right]. \tag{43}
$$

5 Comments

The volume element induced by Eq. (41) leads to an infinite volume for the whole velocity space $-1 < \beta < 1$, $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$. This is also related to the non-normalizability of the d.f. $\mu_{\beta \cos \theta \varphi}$ inferred in Eq. (37). As $\beta$ tends towards 1, the “distance” between two “close” points $l([\beta, \beta + d\beta])$ tends to infinity. This should have been expected, since the equivalent classical case is $v \to +\infty$. The length element (41) also implies that in velocity space, the invariant “distance” of a point to the sphere $\beta = 1$ is infinite, in any reference frame, therefore, the statement that a velocity $v$ is “close” to $c$ is meaningless. Another consequence is, that choosing 0.95 $c$ as an “average” value for $v$, when one has the information $0.9 \, c \leq v \leq c$, appears logically nonsense. In any case, in the energy parametrization $E = E_0 \left( 1 - \beta^2 \right)^{-1/2}$, the corresponding knowledge would only give a lower value for the energy of the particle, and therefore, it would not come to one’s mind to try to summarize this information in terms of one “average” value.

The metric (41) and (42) is “objective” in the sense that it defines a measure element in the velocity space which is invariant under any Lorentz velocity-transformation, accordingly to the special principle of relativity. Both the metric (42) and the densities (31) or (43) deduced from the special principle of relativity yield to the metric (27) and the densities (15) and (19) respectively, when the classical approximation $c \to +\infty$ is made.

It might appear strange that $\mu_{\beta x y z} \neq \mu_x \mu_y \mu_z$. Indeed, it is a counter-example of the intuitive general equality postulated by Tarantola $\mu$. Actually, in the Galilean case, the equality holds, from (13) and (11). Yet, for the relativistic case, it is clear that it is not valid anymore. Indeed, the appearance of transverse terms in the metric (42) makes minimal information on the velocity in one fixed and known direction depend on the orthogonal velocity. This is a direct consequence of the relativity of time, and should not therefore be surprising. It illustrates however, how important it is to clearly define the problem before any prior is to be inferred by Jaynes’ method. In other words, the parameters have to be precisely characterized, and the invariance transformation groups and subgroups found, previously to the application of the transformation-group method.

To summarize, the prior density functions for a velocity have been found in 1, 2 and 3 dimensional cases, in Galilean and special relativity, using Jaynes’ transformation group method. The inference
of the corresponding invariant “objective” measure in the velocity space illustrates the relations between measure, volume element and least-informative density function. Even in this very simple case, the result (i.e., Eqs. (42) and (43)) is non-trivial. As the assignment of prior probabilities is a necessary first step before any consistent (i.e., Bayesian) probability-based inference method is put into practice, this shows that the precise definition of the parameters must be given one’s full attention. In less obvious cases, this task is more difficult, but it can also lead to important conclusions. An application of this technique to derive the minimal information description of cosmological parameters is underway, and appears very promising.

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