BALANCED NORMAL CONES AND FULTON-MACPHERSON’S INTERSECTION THEORY

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Dedicated to Bob MacPherson on the occasion of his 60th birthday

ABSTRACT. Let X be a subscheme of a reduced scheme Y. Then Y has a flat degeneration to the normal cone C_XY of X, and this degeneration plays a key step in Fulton and MacPherson’s “basic construction” in intersection theory. The intersection product has a canonical refinement as a sum over the components of C_XY, for X and Y depending on the given intersection problem. The cone C_XY is usually not reduced, which leads to the appearance of multiplicities in intersection formulae.

We describe a variant of this degeneration, due essentially to Samuel, Rees, and Nagata, in which Y flatly degenerates to the “balanced” normal cone \( \overline{C}_{XY} \). This space is reduced, and has a natural map onto the reduction \( (C_XY)_{\text{red}} \) of \( C_XY \). The multiplicity of a component now appears as the degree of this map. Hence intersection theory can be studied using only reduced schemes. Moreover, since the map \( \overline{C}_XY \rightarrow (C_XY)_{\text{red}} \) may wrap multiple components of \( \overline{C}_XY \) around one component of \( C_XY \), writing the intersection product as a sum over the components of \( \overline{C}_XY \) gives a further canonical refinement.

In the case that X is a Cartier divisor in a projective scheme Y, we describe the balanced normal cone in homotopy-theoretic terms, and prove a useful upper bound on the Hilbert function of \( \overline{C}_XY \).

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1. INTRODUCTION

1.1. Normal cones and balanced normal cones. Let $R$ be a commutative Noetherian ring with unit – indeed, all rings encountered this paper will have these properties – and let $I$ be an ideal. For $r \in R$, define $q(r)$ as the largest $n$ such that $r \in I^n$, or $\infty$ if there is no largest $n$ (e.g. if $r = 0$). Then the associated graded ring is defined as

$$\text{gr } R := \bigoplus_{n \in \mathbb{N}} \{r : q(r) \geq n\} / \{r : q(r) \geq n + 1\}.$$ 

One of its virtues is that it has a map not only to, but from, $R/I$. (Whereas $R$ doesn’t naturally have a map from $R/I$.) Moreover, the map $R/I \to \text{gr } R$ is an inclusion (as the $n = 0$ summand).

Following Samuel (our reference is Rees’ book [Re]), define $\overline{q}(r) := \lim_{n \to \infty} q(r^n)/n$, the homogenization of the filtration $q$. Samuel proved that this limit exists. Nagata and Rees showed that it is rational-valued with bounded denominator. Rees gave a formula for $\overline{q}$, using Rees algebras, which we recall in section 2. Nagata proved that $\overline{q} - q$ is bounded [Re theorem 4.21], which implies that

$$\overline{\text{gr }} R := \bigoplus_{n \in \mathbb{Q}} \{r : \overline{q}(r) \geq n\} / \{r : \overline{q}(r) > n\}$$

is again Noetherian [Re] lemma 2.46]. Note that this grading is by $\mathbb{Q}$, not (usually) by $\mathbb{N}$.

Example 1. Let $R = \mathbb{F}[x]$, $I = \langle x^2 \rangle$. Then $q(x^n) = \lceil n/2 \rceil$, and $\text{gr } R = \mathbb{F}[\bar{x}, \bar{y}] / \langle \bar{x}^2, \bar{y}^2 \rangle$, where the subscripts indicate the degrees. Whereas $\overline{q}(x^n) = n/2$, and $\overline{\text{gr }} R = \mathbb{F}[\bar{x}, \bar{y}] / \langle \bar{x}^2 - \bar{y} \rangle$ written for comparison.

We urge the reader to check the details of this example, as examples 2 and 4 build upon this one.

The following intuition seems to be useful. In $\text{gr } R$ in the example above, $x$ is “rushed” into the degree 0 piece (rather than waiting until degree 1/2 where it “belongs”), and by degree 0 standards its square (which has $q = 1$) vanishes. It is this premature appearance of $x$ that leads to its nilpotency in $\text{gr } R$.

Proposition 1. Assume that the ideal satisfies $\cap_{j=1}^{\infty} \overline{I}^j = \{0\}$.

- There is a flat degeneration of $R$ to $\text{gr } R$.
- $\overline{\text{gr }} R$ has no nilpotents.
- If $R$ has no nilpotents, there is a flat degeneration of $R$ to $\overline{\text{gr }} R$.
- There are natural maps $\text{gr } R \leftrightarrow \text{gr } R_0 = R/I$.
- There are natural maps $\overline{\text{gr }} R \leftrightarrow \overline{\text{gr }} R_0 = R/\sqrt{I}$.

Proof. We start with the second claim. Let $0 \neq \bar{r} \in \overline{\text{gr }} R$ be nilpotent, so $\bar{r}^M = 0$. Then $\overline{q}(\bar{r}^M) > Mn$. Hence $\overline{q}(\bar{r}) > n$, a contradiction.

The intersection $\cap_{j=1}^{\infty} \{r : \overline{q}(r) \geq j\}$ is the ideal $\sqrt{\cap_{j=1}^{\infty} \overline{I}^j}$, which under the assumption $\cap_{j=1}^{\infty} \overline{I}^j = \{0\}$ is just the nilpotents. Then the first and third claims use the Rees algebra to provide the flat family [Ei sec. 6.5].

The fourth claim is obvious. For the fifth, we need to compute $\overline{\text{gr }} R_0 = R/(r : \overline{q}(r) > 0)$. Then

$$\{r : \overline{q}(r) > 0\} = \{r : \exists M, q(r^M) > 0\} = \sqrt{I}. \quad \square$$
Much of this paper is concerned with the natural map $\text{gr} R \to \overline{\text{gr}} R$, which we take up in the next section.

Young algebraic geometers are strictly indoctrinated to regard killing nilpotents as a bad habit; information is being thrown away. To allay their suspicions, we emphasize that $\overline{\text{gr}} R$ is not just $\text{gr} R$ mod its nilpotents (though as we shall see, it contains that as a subring). If $R$ and $I$ are graded so that one can speak of Hilbert functions, then $R, \text{gr} R, \overline{\text{gr}} R$ all have the same Hilbert function, whereas $\overline{\text{gr}} R$ modulo its nilpotents will have a smaller Hilbert function (unless it has no nilpotents). In this sense, the information usually recorded in nilpotents is just showing up in a different way.

Since the denominators in $\overline{q}$ are bounded, one may be tempted to clear them by rescaling the grading. This seems to carry no benefit, and only serves to make the map $\text{gr} R \to \overline{\text{gr}} R$ no longer graded.

Given a subscheme $W$ of a scheme $V$, hence an ideal sheaf $I$ inside the structure sheaf $R$, we can define the normal cone $C_W V$ to $W$ and the balanced normal cone $\overline{C}_W V$, using $\text{gr}$ and $\overline{\text{gr}}$ respectively. The term “balanced” is chosen to evoke the idea that the grading is carefully weighted to avoid creating nilpotents.

1.2. The maps $\text{gr} R \to \overline{\text{gr}} R$ and $\overline{C}_W V \to C_W V$.

**Lemma 1.** Let $I \leq R$ be an ideal in a ring $R$. Then there is a natural map $\beta : \text{gr} R \to \overline{\text{gr}} R$. Moreover, if $\phi : R \to S$ takes $\phi(1) \leq J$ for $J$ an ideal in $S$, then there is a natural commuting square

\[
\begin{array}{ccc}
\text{gr} R & \xrightarrow{\beta_R} & \overline{\text{gr}} R \\
\downarrow \quad & & \downarrow \\
\text{gr} S & \xrightarrow{\beta_S} & \overline{\text{gr}} S
\end{array}
\]

where the associated gradeds are the obvious ones.

**Proof.** The main fact used is that $\overline{q} \geq q$. The statements then follow more or less directly from the definitions. \qed

Since $\overline{\text{gr}} R$ has no nilpotents, the kernel of $\beta : \text{gr} R \to \overline{\text{gr}} R$ is plainly at least the nilpotents.

**Proposition 2.** The kernel of $\beta : \text{gr} R \to \overline{\text{gr}} R$ is exactly the nilpotent elements in $\text{gr} R$. If $\text{gr} R$ has no nilpotents, then $\beta$ is an isomorphism (and otherwise not).

**Proof.** We need the calculation

\[
\overline{q}(r) > n \iff \exists m, \overline{q}(r) > n + \frac{1}{m} \iff \exists M > 0, q(r^M) > Mn.
\]

Let $\overline{r}$ denote the image of $r$ in $\text{gr} R_{\overline{q}(r)}$. If $\beta(\overline{r}) = 0$, then $\overline{q}(r) > q(r)$, so $\exists M > 0, q(r^M) > Mn$. Hence $(\overline{r})^M = 0$. So the kernel is exactly the nilpotents.

If $\text{gr} R$ has no nilpotents, then there does not exist $r \in \text{gr} R_n \setminus \{0\}$ with $r^M = 0$. So $q(r^M)$ is not more than $Mn$; indeed $q(r^M) = Mn$ for all $M$. Hence $\overline{q}(r) = n$, and $\overline{q} = q$. Thus $\overline{\text{gr}} = \text{gr}$ naturally.

Since $\overline{\text{gr}} R$ has no nilpotents, $\text{gr} R$ can only be isomorphic to it if it too has no nilpotents. \qed
We now switch over to the geometric point of view, in which we map from the balanced normal cone to the ordinary one. The above proposition tells us that the map $\overline{C_WV} \to C_WV$ factors as $\overline{C_WV} \to (C_WV)_{\text{red}} \hookrightarrow C_WV$, where $(C_WV)_{\text{red}}$ denotes the reduction of $C_WV$. So $(C_WV)_{\text{red}}$ serves as an intermediary when trying to compare the spaces $C_WV$ and $\overline{C_WV}$. This motivates our looking at Chow groups, since $A_\ast(C_WV) = A_\ast((C_WV)_{\text{red}})$.

**Theorem 1.** Let $W$ be a closed subscheme of $V$, where $V$ is reduced. The induced map $\beta : \overline{C_WV} \to C_WV$ is proper, with finite fibers. Assume now that $V$ is quasiprojective. Then the two maps $\overline{C_WV} \to (C_WV)_{\text{red}}, (C_WV)_{\text{red}} \hookrightarrow C_WV$ induce the same Chow class in $A_\ast((C_WV)_{\text{red}})$.

(We expect that the hypothesis on $V$ is largely unnecessary.)

Note that these two Chow classes are induced on $(C_WV)_{\text{red}}$ in very different ways, as we go over in section 5. The inclusion $(C_WV)_{\text{red}} \hookrightarrow C_WV$ defines a class by taking the sum of the top-dimensional components weighted by the lengths of the local rings on the target. Whereas the surjection $\overline{C_WV} \to (C_WV)_{\text{red}}$ defines a class by taking the sum of the top-dimensional components weighted by the degree of the map over those components.

**Example 2.** Let $V$ be the line with coordinate $x$, and $W$ the doubled origin (defined by $x^2 = 0$). Then $C_WV$ is the doubled line, whereas $\overline{C_WV}$ is just the ordinary line; see example for these calculations. The map $\overline{C_WV} \to (C_WV)_{\text{red}} \hookrightarrow C_WV$ is the squaring map from the line to the (reduction of the doubled) line.

It can happen that $\overline{C_WV}$ has more components than $C_WV$, not because a component collapses (since we know there are finite fibers), but because multiple components of $\overline{C_WV}$ cover the same component of $C_WV$. When this happens, we get a refinement of the multiplicities in the fundamental class of $C_WV$; the multiplicity of a component $F \subseteq C_WV$ is the sum over those components $F \subseteq \overline{C_WV}$ whose image is $F$, of the degree of the map $F \to F$.

**Example 3.** Let $R = \mathbb{F}[a, b]/\langle a^2 - b^2 \rangle$, so $V := \text{Spec } R$ is the union of two lines. Let $I = \langle b \rangle$, so $W$ is a double point at the origin. $C_WV$ is a trivial line bundle over $W$, $q(a) = 0$, $\overline{q}(a) = 1$, and $\overline{gr} R \cong R$. The map $\overline{C_WV} \to C_WV$ maps the two lines onto the reduction of $C_WV$.

In this way, the fundamental class of $C_WV$ is a sum of the two (equal) Chow classes induced by the lines in $\overline{C_WV}$.

To prove theorem we need a number of basic results about balanced normal cones, which will come in section 2. The proof itself will come in section 5. It is a bit involved, which seems to be inherent in the fact that the two classes on $(C_WV)_{\text{red}}$ are induced in different ways.

A simpler proof will appear in [AK], where we show that the $gr$ $R$-modules $gr R$, $\overline{gr} R$ are $K$-equivalent. This implies theorem at least in characteristic 0 and in the rational Chow group.

1.3. **The “basic construction” in intersection theory.** We recall the basic construction from [FM].
Let \( i : X \hookrightarrow Y \) be an inclusion (soon, a regular embedding), and \( f : V \to Y \) a morphism. Let \( W \) be the pullback, so we have a square

\[
\begin{array}{ccc}
W & \hookrightarrow & V \\
\downarrow & & \downarrow \\
X & \hookrightarrow & Y
\end{array}
\]

Now replace each of the big schemes (\( Y \) and \( V \)) by the normal cones to the subschemes. This allows us to reverse the horizontal arrows, replacing inclusions by epimorphisms.

\[
\begin{array}{ccc}
W & \hookrightarrow & C_W V \\
\downarrow & & \downarrow \\
X & \hookrightarrow & C_X Y
\end{array}
\]

This is no longer a pullback diagram; we only have a map from \( C_W V \) to the actual pullback \( N \). Hence \( C_W V \) defines a Chow class on \( N \). (While it is not hard to check that the map \( C_W V \to N \) is an inclusion, this property doesn’t seem to play any role in the construction.)

For purposes of intersection theory, it turns out to be useful to require that \( i \) be a regular embedding, i.e. that \( C_X Y \) be a vector bundle. This is because Fulton and MacPherson’s goal is to define a Chow class down on \( W \) (not up on \( N \)), which they call the “refined intersection product” of \( X \) and \( V \). (It can be thought of as a cap product, where the regular embedding \( X \hookrightarrow Y \) plays the role of the cobordism class and the map \( V \to Y \) that of the bordism class.) This is done using a Thom-Gysin isomorphism \( A_*(N) \cong A_{*-d}(W) \), which holds if \( N \) is a vector bundle of some dimension \( d \). This is guaranteed if \( C_X Y \) is a vector bundle, motivating that condition. This completes the basic construction.

How do things change in what we will call the **balanced basic construction**, where we instead use balanced normal cones?

First, we will require that \( V \) and \( Y \) are reduced, in order that their degenerations to balanced normal cones be flat degenerations. (These are not particularly stringent assumptions from the point of view of intersection theory, where the most important case is \( X \) smooth and \( Y = X \times X \). However one should note that if \( X \) is regularly embedded in \( Y \), but not reduced, it does not follow that \( \overline{C_X Y} \) is a vector bundle over \( X_{\text{red}} \).)

As before, by passing to the cones we can reverse the horizontal arrows. However, these reversed maps are no longer epimorphisms – they only hit the reductions (thanks to the last part of proposition 1).

\[
\begin{array}{ccc}
W & \hookrightarrow & \overline{W_{\text{red}}} \leftarrow \overline{C_W V} \\
\downarrow & & \downarrow \\
X & \hookrightarrow & \overline{X_{\text{red}}} \leftarrow \overline{C_X Y}
\end{array}
\]

If we assume that \( X \) is smooth and \( f \) is a regular embedding, then \( C_X Y \) is reduced (as it is a vector bundle over something reduced). Hence \( \overline{C_X Y} = C_X Y \), and the pullback to \( W \) is again \( N \).

However, even in this case, \( W \) and \( C_W V \) are typically not reduced. So \( \overline{C_W V} \), which is reduced, is something new. It too maps (though usually not injectively) to the pullback bundle \( N \), and this map factors as \( \overline{C_W V} \to (C_W V)_{\text{red}} \hookrightarrow N_{\text{red}} \hookrightarrow N \).

**Theorem 2.** Assume that \( X, Y, V \) are reduced, with \( X \hookrightarrow Y \) regularly embedded and \( V \to Y \) a morphism.
Fulton and MacPherson’s refined intersection product $X \cdot V \in A^*(W)$, usually calculated with $C_W V$, can be calculated equally well with $\overline{C}_W V$.

**Proof.** Since the map $\overline{C}_W V \to N$ factors through $\beta : \overline{C}_W V \to C_W V$, theorem 1 implies that $\overline{C}_W V$ and $C_W V$ induce the same Chow class on $N$. □

**Example 4.** Let $Y, V$ be affine lines with coordinates $y, v$, let $X$ be the origin in $Y$, and let $V \to Y$ be the squaring map $y = v^2$. Then $W$ is the doubled origin in $V$, defined by $v^2 = 0$.

In ordinary intersection theory, the normal cones $C_W V, C_X Y$ and the pullback $N$ are all trivial line bundles, over $W, X, W$ respectively. The map $C_W V \to N$ is an isomorphism, inducing the fundamental class on $W$, which is the twice the class of the reduced point $W_{\text{red}}$.

In the balanced basic construction, the balanced normal cone $\overline{C}_W V$ is the trivial line bundle over $W_{\text{red}}$ and the map $\overline{C}_W V \to N$ is the squaring map, rather than an isomorphism. We calculate this on the algebra side, where the diagram above is

\[
\begin{array}{c}
\mathbb{F}[v]/(v^2) \to \mathbb{F} \\
\uparrow \quad \uparrow \\
\mathbb{F} \quad \mathbb{F} \leftrightarrow \mathbb{F}[v^{1/2}] \\
\end{array}
\]

Here the parenthesized subscripts indicate the degree in these graded rings. In the graded map on the right, $y \mapsto v^2$. The pushout $\text{Fun}(N_{\text{red}})$ of that right square is obviously $\mathbb{P}[y]$, so this squaring map is the one $\overline{C}_W V \to N_{\text{red}}$ claimed above, inducing twice the fundamental Chow class of $N_{\text{red}}$. The Gysin map then takes that to twice the fundamental class of the reduced point $W_{\text{red}}$ as predicted by theorem 2.

Because the space $\overline{C}_W V$ can have more components than $C_W V$, as in example 3, we can refine Fulton and MacPherson’s “refined intersection products” further as a sum over the components of $\overline{C}_W V$.

**Example 5.** The further refinement in example 3 only reflected the fact that $V$ itself was reducible. This example, revisited in section 3, shows the refinement can be nontrivial even when $V$ is irreducible.

Let $Y = \text{Spec} \mathbb{F}[a, b]$, $X$ the $a$-axis, and $V$ the nodal cubic curve $b^2 = a^2(a + 1)$. Their intersection $W = X \cap V$ is a double point at the origin (the node of the cubic) and a reduced point at $(-1, 0)$. The map from $C_W V$ to the pullback $W \times \mathbb{A}^1$ of the (trivial) normal bundle $C_X Y$ is an isomorphism, inducing the fundamental Chow class on $W \times \mathbb{A}^1$ and thereby on $W$.

In this case, the Chow ring calculation (on the projective plane) gives 3 times the class of a point. The refined intersection product just calculated splits this as $3 = 2 + 1$, from the double point and single point.

To compute the balanced normal cone $\overline{C}_W V$, present $V = \text{Spec} R$ using $R = \mathbb{F}[a, b, c]/(c - a(a + 1), b^2 - ac, b^2(a + 1) - c^2)$. Then $c^2 \in \langle b \rangle^2$, so $\overline{\mathbb{F}}(c)$ is plainly at
least 1. As we will be able to compute later (using proposition 4), in fact \( q(a) = 0 \), \( q(c) = 1 \), and

\[
\mathcal{R} = \mathbb{F}[a, b, c]/(a(a+1), ac, b^2(a+1)-c^2) = \mathbb{F}[a, b, c]/((a+1) \cap (a, b-c) \cap (a, b+c)).
\]

So \( \mathcal{C}_YX \) is an isolated line union a pair of intersecting lines, and the sum of these components further refines the intersection calculation as \( 3 = 1 + (1+1) \).

Unlike \( R \) and \( \text{gr} \ R \), this ring \( \mathcal{R} \) is not generated over \( \mathbb{F} \) by two elements.

It would be interesting to find the branch locus of the map \( \mathcal{C}_WV \rightarrow (C_WV)_{\text{red}} \) in genuine intersection theory examples, and see what that and more refined degeneracy loci mean for enumerative questions. It would also be interesting to see how the monodromy group of the branched cover relates to the “Galois group” of the enumerative problem [Ha].

We now outline the rest of the paper. In section 2 we describe Rees’ formula for \( q \) and give the basic results about \( \text{gr} \ R \). When \( \text{gr} \ R \) is reduced, then \( \mathcal{R} = \text{gr} \ R \); we present a number of examples to show some possible reasons that \( \mathcal{R} \neq \text{gr} \ R \). In section 3 we study the intersection of a variety in affine space with a hyperplane, and geometrically describe the normal cone (and under certain conditions, the balanced normal cone) as flat limits. In section 4 we introduce the ring \( \mathcal{R} \) with which to further study \( \mathcal{R} \) in the case that \( I \) is principal, and we compute several examples. Finally, in section 5 we prove theorem 1.

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2. Properties of \( \mathcal{R} \)

2.1. Rees’ formula for \( q \). In this section \( R \) is a ring without nilpotents. (And commutative, Noetherian, and with unit, as per our standing assumptions.)

Assume to begin with that \( R \) is an integrally closed domain, and \( I \) is a principal ideal \( \langle b \rangle \). Let \( D_1, \ldots, D_n \) be the components of \( I \)'s vanishing set, and \( v_i \) the corresponding valuations.

Then \( q(r) \geq n \iff r \in \langle b^n \rangle \implies v_i(r) \geq nv_i(b) \), or put another way,

\[
q(r) \leq \min_i \frac{v_i(r)}{v_i(b)}.
\]

The same bound follows for \( q \). Rees’ theorem, in this special case, says that \( q \) is actually given by this formula.

Example 6. Let \( P \) be a lattice polytope, and \( R \) the homogeneous coordinate ring of the projective toric variety \( X = X_P \), which has a basis given by lattice points in dilations of \( P \). Let \( b \) be the degree 1 element corresponding to some lattice point \( p \in P \). Then the valuations \( \{v_i\} \) in the formula for \( q \) correspond to the facets of \( P \) not containing \( p \). If \( o \) is any lattice point in \( P \) and \( r \) the corresponding ring element, then \( v_i(r) \) is the distance of \( o \) to the \( i \) facet, measured in lattice units.

Let \( f : P \rightarrow \mathbb{R} \) denote the continuous piecewise-linear function measuring the distance of \( q \) to a far wall of \( P \) along the straight line connecting \( p \) and \( q \); it takes
the value 1 at \( p \), 0 on all facets \( F \) not containing \( p \), and varies linearly on the cone from \( p \) to \( F \). Then if \( r \in R \) is a basis element corresponding to a lattice point \( p \in P \), we have \( \overline{\varphi}(r) = f(p) \). (More generally, if \( r \in R \) is a basis element corresponding to a lattice point \( o \) in the \( k \)-fold dilation \( kP \) of \( P \), we have \( \overline{\varphi}(r) = k f(o/k) \).) An example is in figure 1.

The ring \( \overline{\varphi} R \) is the homogeneous coordinate ring of a union of projective toric varieties, whose components are (weighted) cones on the facets of \( P \) not containing \( p \). This reducibility arises from the fact that in the associated graded, the product of two basis elements can be zero, which happens if and only if when projecting away from \( p \) the corresponding points in \( P \) do not project to a common facet.

![Figure 1](image)

**Figure 1.** A lattice polytope, the valuation \( \overline{\varphi} \) (and its level sets) evaluated on the generators, and the polyhedral complex arising from \( \overline{\varphi} R \).

More general reduced torus-equivariant degenerations of toric varieties were studied in [Al].

Several phenomena must be dealt with to generalize Rees’ formula to the case of a general ring \( R \) and nonprincipal \( I \). First, if \( I \) is not principal, we work with the blowup algebra. One interesting aspect of this is that if \( \text{Spec} R \) or \( \text{Spec} R/I \) are singular along \( \text{Spec} R/I \), then the exceptional locus in the blowup may have more components than \( \text{Spec} R/I \) does itself, and we need them for Rees’ formula.

Second, if \( R \) is not integrally closed, the valuations \( \{v_i\} \) may not be simply associated to divisors. For example, let \( R = \mathbb{F}[x, y]/(xy) \), and \( I = (x + y^2) \). Then \( I \) vanishes at the origin, to order 1 if one approaches along the \( x \)-axis and order 2 if one approaches along the \( y \)-axis. Hence “the order of vanishing at the origin” is not well-defined.

(For \( I = \langle b \rangle \), the natural condition is that \( R \) be integrally closed inside \( R[b^{-1}] \). This will show up in a different guise in section [4].)

Finally, if \( \text{Spec} R \) is reducible, \( I \) may vanish altogether on some components. If \( r \) doesn’t vanish on those, then plainly \( r^n \notin I \) for any \( r \), so \( \overline{\varphi}(r) = 0 \). If contrariwise \( r \) does vanish on them, we can remove those components by passing to \( R/\text{ann}(I) \) and compute \( \overline{\varphi} \) in that ring.

In the rest of this section we assume \( I = \langle b \rangle \).

**Lemma 2** (Samuel). Let \( R = \bigoplus_{n \in \mathbb{N}} R_n \) be a graded Noetherian ring. Then \( R \) is finitely generated as an algebra over \( R_0 \).

**Proof.** Let \( r_1, \ldots, r_k \) be homogeneous generators of the augmentation ideal \( R_+ \). Let \( r \) be a homogeneous element of positive degree. Then \( r = \sum c_i r_i \) for some homogeneous \( (c_i) \) with \( \deg c_i < \deg r \). By induction, these \( c_i \) are polynomials in the \( (r_j) \) with coefficients from \( R_0 \). Hence \( r \) is such a polynomial too.
By the direct sum assumption, every element of \( R \) is a sum of such \( rs \) and an element of \( R_0 \).

\[ \qed \]

**Proposition 3.** Let \( 1 = \langle b \rangle \) be a principal ideal in a ring \( R \).

- The multiplication map \( b \cdot : \text{gr } R_i \to \text{gr } R_{i+1} \) is always onto.
- The multiplication map \( b \cdot : \text{gr } R_i \to \text{gr } R_{i+1} \) is \( 1 : 1 \) for all large \( i \).
- The multiplication map \( b \cdot : \text{gr } R_i \to \text{gr } R_{i+1} \) is always 1:1 for \( i > 0 \). If \( b \) is not a zero divisor in \( R \), then \( b \cdot : \text{gr } R_0 \to \text{gr } R_1 \) is also 1:1.
- The multiplication map \( b \cdot : \text{gr } R_i \to \text{gr } R_{i+1} \) is onto for all large \( i \).

Note that the first two claims are only interesting for \( i \in \mathbb{N} \), whereas the second two are interesting for \( i \in \mathbb{Q} \).

**Proof.** The first claim is essentially tautological.

For the second, consider the ascending chain \( \text{ann}(b) \leq \text{ann}(b^2) \leq \text{ann}(b^3) \leq \ldots \) of annihilator ideals in \( \text{gr } R \). Let \( j \) be the stage at which it stabilizes. Then for \( c \in \text{gr } R \), \( b^j c \neq 0 \) implies \( b^k c \neq 0 \) for any \( k \geq j \). With this and the first claim, we see that if \( d \in \text{gr } R_k \setminus \{ 0 \} \), \( k \geq j \), then \( bd \neq 0 \).

The third and fourth don’t depend on \( R \) as much as \( R \) mod its nilpotents, so we assume now that \( R \) has none.

For the third, let \( c \in R \). If \( b \) vanishes on a component of \( \text{Spec } R \) on which \( c \) doesn’t, then \( 1 \) is a zero divisor and \( \mathfrak{g}(c) = 0 \). So now we assume that \( c \) vanishes on each component of \( \text{Spec } R \) on which \( b \) vanishes, and we can pass to \( R/\text{ann}(b) \).

Now, \( \mathfrak{g}(c) = \min_i \frac{v_i(c)}{v_i(b)} \). Then

\[
\mathfrak{g}(bc) = \min_i \frac{v_i(bc)}{v_i(b)} = \min_i \frac{v_i(b) + v_i(c)}{v_i(b)} = \min_i 1 + \frac{v_i(c)}{v_i(b)} = 1 + \mathfrak{g}(c).
\]

So \( c \neq 0 \) implies \( \text{gr } (bc) \neq 0 \). Contrapositively, the only way for \( \text{gr } (c \neq 0) \) to be annihilated by \( b \cdot \) is for \( b \) to be a zero divisor and \( \mathfrak{g}(c) = 0 \). This gives the third claim.

For the last claim, since \( \text{gr } R \) is Noetherian, let \( g_1, \ldots, g_G \) generate \( \text{gr } R \) as an algebra over \( \text{gr } R_0 \) (using lemma \( \mathbb{Q} \)). Then any monomial in the \( \{ g_i \} \) of high degree must involve some \( g_i \) to a high power. Since each \( g_i \in \sqrt{\langle b \rangle} \), having \( g_i \) to a high power means that a factor of \( b \) can be extracted. This establishes the fourth claim.

The following gives a characterization of \( \mathfrak{g} \) that is useful for verifying examples, and in section \( \mathbb{Q} \) will also be of use in interpreting balanced normal cones geometrically. It uses the concept of **homogeneous** filtrations \( p \), meaning \( p(r^n) = np(r) \forall r \in R, n \in \mathbb{N} \).

**Proposition 4.** The filtration \( \mathfrak{g} \) is the unique minimum homogeneous filtration \( p \) with \( p(b) = 1 \). In other words, let \( p \) be a homogeneous filtration on \( R \) such that \( p(b) = 1 \). Then \( p(r) \geq \mathfrak{g}(r) \forall r \in R \).

If \( R = \mathbb{F}[a_1, \ldots, a_n, b]/I \), and \( w_1, \ldots, w_n \geq 0 \) are lower bounds on \( \mathfrak{g}(a_1), \ldots, \mathfrak{g}(a_n) \), then let \( p \) be the (possibly inhomogeneous) filtration induced on \( R \) from the filtration \( p(b^B \prod_i a_i^{n_i}) = B + \sum_i n_i w_i \) on the polynomial ring. If the associated graded to \( p \) has no nilpotents, then \( p = \mathfrak{g} \).
Proof. By the existence of the limit $\overline{r}$, given $r \in \mathbb{R}$, $\varepsilon > 0$, for all large $n$ we have $q(r^n)/n \geq \overline{r}(r) - \varepsilon$. Hence $r^n = ab^{[n(\overline{r}(r) - \varepsilon)]}$ for some $a \in \mathbb{R}$. Then

$$p(r) = \frac{1}{n}p(r^n) = \frac{1}{n}p(ab^{[n(\overline{r}(r) - \varepsilon)]}) \geq \frac{1}{n}p(b^{[n(\overline{r}(r) - \varepsilon)]})$$

$$= \frac{1}{n}[n(\overline{r}(r) - \varepsilon)] \geq \frac{1}{n}(n(\overline{r}(r) - \varepsilon) - 1) = \overline{r}(r) - \varepsilon - \frac{1}{n}$$

hence $p(r) \geq \overline{r}(r)$.

For the second part, saying that the associated graded to $p$ has no nilpotents is the same as saying that $p$ is homogeneous. Plainly $p(b) = 1$. So by the first part, $p \geq \overline{r}$. Since $p$ is the smallest filtration with $p(a_i) = w_i$, and $w_i \leq \overline{r}(a_i)$ by assumption, we have $p \neq \overline{r}$. So $p = \overline{r}$. \qed

Note that not every homogeneous filtration on a polynomial ring mod an ideal is of the form in the second part of the proposition – for example, the $(x + y)$-adic filtration on $\mathbb{F}[x, y]$. We will only be able to apply the second part of proposition 4 when the generating set has been chosen felicitously.

In some of the examples to come, we will present $R$ as a polynomial ring modulo an ideal. We’ll determine some lower bounds $\{w_i\}$ on the $\overline{r}$s of the variables, including $\overline{r}(b) = 1$, and consider the induced (a priori inhomogeneous) filtration $p$. To compute the associated graded to $p$, we check that the generating set of the ideal is a Gröbner basis with respect to some term order respecting this weighting of the variables, and replace each relation by its lowest-weight component. To be sure we’re satisfying proposition 4 it remains to check that the associated graded has no nilpotents. When all goes well and that turns out to be true, we learn three things: $p = \overline{r}$, our lower bounds $\{w_i\}$ were correct, and each filtered piece of $R$ intersected with the linear span of the variables is spanned by a subset thereof.

2.2. Examples. Here are some of the nonobvious possible behaviors of $\overline{r}$ and $\overline{gr}$.

2.2.1. The limit $\overline{r}$ need not be achieved. One way of thinking about the limit $\lim_{n \to \infty} \frac{q(r^n)}{n}$ is to take the limit through a subsequence $1 = n_1|n_2|n_3|\cdots$, which is easily seen to be increasing:

$$q(r) = \frac{q(r^{n_1})}{n_1} \leq \frac{q(r^{n_2})}{n_2} \leq \frac{q(r^{n_3})}{n_3} \leq \cdots \leq \overline{r}(r).$$

Many people’s first guess, upon learning the definition of $\overline{r}$, is that the limit $\overline{r}(r)$ is achieved for some finite $n$. This turns out to be true if $R$ is integrally closed.

**Proposition 5** (Rees). Let $R$ be an integrally closed domain, and $I = \langle b \rangle$. Then there exists $N > 0$ such that $\overline{r}(r) = \frac{1}{N}q(r^N)$.

Proof. Let $N$ be the least common multiple of the valuations $v_i(b)$, so $N\overline{r}$ is $\mathbb{N}$-valued. Then for any $r$, the rational function $r^N/b^{[N\overline{r}(r)]}$ satisfies the valuative criterion for integrality. (We asked that $R$ be a domain so that $b$ is not a zero divisor.) Since $R$ is integrally closed, $r^N/b^{[N\overline{r}(r)]} = s$ for some $s \in R$. Hence $q(r^N) \geq N\overline{r}(r)$, but we already knew the opposite inequality. \qed
Example 7. This is a variant of example 3 with the same geometry. Let $R = \mathbb{F}[a, b]/(a^2 - ab)$, and $I = \langle b \rangle$. Then $a^n = ab^{n-1}$, and in fact $q(a^n) = n - 1$. Taking the limit, $\overline{q}(a) = 1$. But for no $n$ is $q(a^n)/n = \overline{q}(a)$.

2.2.2. $\overline{gr}R \neq gr R$ despite being integer-graded. We’ve already shown that $q = \overline{q}$, if and only if $gr R$ has no nilpotents, and if and only if $\overline{gr}R \equiv gr R$. One obvious reason for $\overline{gr}R$ to be different from $gr R$ is if $v_i(b) > 1$ for some valuation $v_i$ in Rees’ formula, and $\overline{gr}R$ to have support in other than integer degrees. Geometrically, this corresponds to the divisor $b = 0$ not being generically reduced. (It is still possible for $\overline{gr}R$ to be integer-graded, as example 3 shows.)

This raises the question: if the divisor $b = 0$ is generically reduced, does that force $\overline{gr}R = gr R$? To construct a counterexample, it will suffice to make the divisor generically reduced but not reduced, hence not satisfying Serre’s criterion S1. So the ambient $\text{Spec} R$ shouldn’t satisfy Serre’s criterion S2, the canonical example being the union of two planes in 4-space.

Let $R = \mathbb{F}[b, c, d, e]/(d(b - d), dc, e(b - d), ec)$, the union of the $d = e = 0$ plane and $b - d = c = 0$ plane. Then the $b$-divisor is $\text{Spec} \mathbb{F}[b, c, d, e]/(b, d^2, dc, ed, ec)$, supported on the $b = d = e = 0$ line union the $b = d = c = 0$ line, with an extra point embedded at the origin.

Since $d^N = db^{N-1}$ for all $N > 1$, we see $\overline{q}(d) \geq \frac{N-1}{N}$. So $\overline{q}(d) \geq 1$, and the lower bounds 1 we can guess for the $\overline{q}$ of the variables are $\overline{q}(b), \overline{q}(d) \geq 1, \overline{q}(c), \overline{q}(e) \geq 0$. (Note that $\overline{q}(d) \neq q(d) = 0$.)

The relations are homogeneous with respect to this weighting, hence the associated graded $\overline{gr}R$ turns out to be isomorphic to $R$:

$$\overline{gr}R = \mathbb{F}[b(1), c, d(1), e]/(ec, d(b - d), dc, e(b - d)).$$

Since this has no nilpotents, we can use proposition 4 to know that we have correctly calculated $\overline{q}$. (Side note: the fact that $\overline{gr}R \equiv R$ doesn’t mean that $\overline{gr}R$ is boring – rather, it has served as a means of discovering a grading with which to better understand $R$ itself.)

Whereas $gr R = \mathbb{F}[b(1), c, d, e]/(ec, d^2, dc, ed)$, whose quotient by $\sqrt{0} = \langle d \rangle$ is $\mathbb{F}[b(1), c, e]/(ec)$.

Geometrically, the map $gr R \to gr R/\langle d \rangle \to \overline{gr}R$ corresponds (in reverse) to a pair of planes meeting at a point, mapping onto a pair of planes meeting along a line, mapping into a thickening of that scheme along the line.

This hints at a strengthening of theorem 11 which is implied by our result to appear in [AK]. Let $U \subseteq C_W V$ be the open locus over which $\beta$ is a local isomorphism. Then it seems the map $\overline{C_W V} \setminus \beta^{-1}(U) \to C_W V \setminus U$ takes the fundamental Chow class to the fundamental Chow class.)

2.2.3. $\overline{gr}R \neq gr R$ despite the divisor being reduced. It is curious that this can only happen if $b$ is a zero divisor, as we now prove.

Proposition 6. Let the ring $R$ contain the element $b$, and assume that $b$ is not a zero divisor, and $R/\langle b \rangle = gr R_0$ has no nilpotents. Then $gr R$ has no nilpotents, so $\overline{q} = q$ and $\overline{gr}R = gr R$.

Proof. Assume $c \in R$ is nonzero, and $q(c) = n > 0$, so $c$ has image $\hat{c} \in gr R_n$. Assume also that $c^m = 0$, so $q(c^m) > mn$. 


Then we can write \( c = ab^n \) and \( c^m = db^{mn+1} \), where \( q(a) = 0 \). So \( c^m = a^mb^{mn} = db^{mn+1} \), hence \( b^{mn}(a^m - bd) = 0 \). Since \( b \) is not a zero divisor, \( a^m - bd = 0 \), so \( q(a^m) \geq 1 \). Therefore, \( a^m \) is a nilpotent element of \( \text{gr} \ R_0 \), contradiction. \( \Box \)

To find an example in which \( \text{gr} \ R \) has nilpotents, but only after degree \( 0 \), we therefore need to allow \( b \) to be a zero divisor. The proof above suggests \( I = \langle c - ab, c^2 - db^4 \rangle \), which is almost good enough, we just need to take its radical (using [M2]):

\[
I = \sqrt{\langle c - ab, c^2 - db^4 \rangle} = \langle ab - c, ac - b^2d, c^2 - b^4d \rangle = \langle b, c \rangle \cap \langle c - ab, a^2 - b^2d \rangle
\]

Let \( R = \mathbb{F}[a, b, c, d]/I \). Then \( R \) has no nilpotents, and neither does \( R/\langle b \rangle = \mathbb{F}[a, b, c, d]/\langle b, c \rangle \).

But \( q(c) = 1 \), \( q(c^2) = 4 \), so \( c \) gives a nilpotent element of \( \text{gr} \ R_1 \).

In fact \( \overline{\mathbb{F}}(c) = 2 \), and

\[
\text{gr} R = \mathbb{F}[a_{(0)}, b_{(1)}, c_{(2)}, d_{(0)}]/\langle ab, ac, c^2 - b^4d \rangle
\]

where the parenthesized subscripts indicate the degrees. This, too, can be checked with proposition [4].

A more standard Gröbner basis calculation tells us

\[
\text{gr} \ R = \mathbb{F}[a_{(0)}, b_{(1)}, c_{(2)}, d_{(0)}]/\langle c, a^2b \rangle.
\]

Geometrically, the map \( \text{gr} \ R \to \overline{\text{gr}} \ R \) corresponds (in reverse) to a union of a plane and a surface along a line, mapping to a union of a plane and a double plane along a line, where the map is generically \( 1 : 1 \) on the first component and \( 2 : 1 \) on the second.

3. Some normal cones and balanced normal cones as flat limits

Let \( R = \mathbb{F}[a_1, \ldots, a_{n-1}, b] \) be a polynomial ring in \( n \) variables, and \( I \) a radical ideal. Let \( Y = \mathbb{A}^n = \text{Spec} \ R \), and let \( X = \mathbb{A}^{n-1} \) be the \( b = 0 \) hyperplane. Let \( V = \text{Spec} \ R/I \), and \( W = X \cap V \). We interpret the “basic construction” in this case in terms of a transparent geometric limit, and under the hypotheses of proposition [4] do the same for the balanced version. We include this description only for illustration, and in this section do not give full proofs (though they are quite straightforward from the theory of Gröbner degenerations).

The basic construction, in this case, goes from

\[
\begin{array}{ccc}
W & \hookrightarrow & V \\
\downarrow & & \downarrow \\
\mathbb{A}^{n-1} & \hookrightarrow & \mathbb{A}^n
\end{array}
\quad \quad \quad \quad \quad \quad
\begin{array}{ccc}
W & \hookrightarrow & \mathbb{C}_W V \\
\downarrow & & \downarrow \\
\mathbb{A}^{n-1} & \hookrightarrow & \mathbb{A}^n
\end{array}
\]

Hence \( \mathbb{C}_W V \) maps into the pullback \( W \times \mathbb{A}^1 \), inducing a Chow class on \( W \times \mathbb{A}^1 \) and thereby on the intersection \( W \).

There is a geometric picture of the passage to the normal cone \( C_{\mathbb{A}^{n-1}} \mathbb{A}^n \cong \mathbb{A}^n \). Let the circle \( \mathbb{G}_m \) act on \( \mathbb{A}^n \) by

\[
t \cdot (a_1, \ldots, a_{n-1}, b) := (a_1, \ldots, a_{n-1}, tb).
\]

Then \( \mathbb{C}_W V \) can be computed as the flat limit \( \lim_{t \to \infty} t \cdot V \), stretching \( V \) away from \( V \cap \mathbb{A}^{n-1} \).

\[\text{1} \text{Instead of the relation } c^2 - bd^4, \text{ we might equally well have used } c^2 - bd^3, \text{ in which case } \overline{\mathbb{F}}(c) = 1 + 2. \]

We preferred \( \overline{\mathbb{F}}(c) = 2 \) to emphasize that the advantages of \( \overline{\text{gr}} \ R \) over \( \text{gr} \ R \) are not merely due to the rational grading.
Two things can happen to any particular component \( K \subset V \) under this limit. If \( K \subset \mathbb{A}^{n-1} \), then \( t \cdot K = K \) for all \( t \) including \( t = \infty \). The map \( C_W V \to W \times \mathbb{A}^1 \) restricts to a map \( K \to K \times \mathbb{A}^1 \), inducing the zero Chow class.

It is more interesting when \( K \not\subset \mathbb{A}^{n-1} \). Then \( \lim_{t \to \infty} t \cdot K = (K \cap \mathbb{A}^{n-1}) \times \mathbb{A}^1 \), and the map \( C_W V \to W \times \mathbb{A}^1 \) restricts to an isomorphism \( (K \cap \mathbb{A}^{n-1}) \times \mathbb{A}^1 \cong (K \cap \mathbb{A}^{n-1}) \times \mathbb{A}^1 \), inducing the fundamental class. The Thom-Gysin isomorphism then takes that to the fundamental class of \( K \cap \mathbb{A}^{n-1} \) inside \( W \).

In all, the intersection class on \( W \) is given by the fundamental classes of the thickenings of the \((\dim V - 1)\)-dimensional components of \( W \), leaving out those components that were components of \( V \).

In the balanced basic construction,

\[
\begin{array}{ccc}
W & \hookrightarrow & V \\
\downarrow & & \downarrow \\
\mathbb{A}^{n-1} & \hookrightarrow & \mathbb{A}^n
\end{array}
\]

Assume now we are in the case of proposition 4 where \( \overline{\pi}(a_i) = w_i \) for \( i = 1, \ldots, n - 1 \), and \( \overline{\pi} \) is induced from the filtration \( \overline{\pi}(b^n \prod_i a_i^{n_i}) = B + \sum_i n_i w_i \). Fix a number \( N > 0 \) such that each \( N w_i \in \mathbb{N} \).

In this case \( \overline{C}_W V \) can also be computed as a limit. Let \( \mathbb{G}_m \) act on \( \mathbb{A}^n \) by

\[
t \cdot (a_1, \ldots, a_{n-1}, b) := (t^{N w_i} a_1, \ldots, t^{N w_{n-1}} a_{n-1}, t^N b).
\]

Then it is not hard to show that \( \overline{C}_W V = \lim_{t \to \infty} t \cdot V \).

**Example 8.** Let \( V \) be the parabola \( \{b = a^2\} \) in the \( ab \)-plane, so \( W \) is a double point at the origin. Then \( t \cdot V \) is the skinny parabola \( \{b/t = a^2\} \), whose limit as \( t \to \infty \) is a double line. The map \( C_W V \to W \times \mathbb{A}^1 \) is an isomorphism.

In the balanced construction, \( \overline{\pi}(a) = 1/2 \), and we need \( N \) even. So \( t \cdot V \) is the parabola \( \{b/t^N = (a/t^{N/2})^2\} \), which is to say, \( t \cdot V = V \cong \overline{C}_W V \). The map \( \overline{C}_W V \to W \times \mathbb{A}^1 \) is a double cover of the reduction of \( W \times \mathbb{A}^1 \).

**Example 9.** Recall the nodal cubic \( V = \text{Spec } \mathbb{R}, \mathbb{R} = \mathbb{F}[a, b]/(b^2 - a^2(a + 1)) \) from example 5. The limit picture of the usual normal cone stretches this nodal cubic vertically, resulting in a line at \( a = -1 \) and a double line at \( a = 0 \).

In this case \( \mathbb{G}_\mathbb{R} \) was not generated by two variables; we needed to introduce \( c = a(a + 1) \). Geometrically, \( V \) is stretched into the third dimension. In terms of the \( \mathbb{R} \)-picture, the points in \( W = V \cap \{b = 0\} \) are left alone, the points elsewhere in \( a < 0 \) are pushed behind the page, and the points in \( a > 0 \) are pulled out of the page. The local picture of an \( \times \) through the origin is rotated a bit about the \( b \) axis, leaving the \( ab \)-plane.

The limit picture of the balanced normal cone stretches not only the vertical dimension, but the new third dimension (since \( \overline{\pi}(c) = 1 \)). In the limit, one has a vertical line through the point \((-1, 0, 0) \in W \), and the local picture of an \( \times \) has been stretched to a union of two lines lying in the \( a = 0 \) plane.

It would be interesting to study the relation of balanced normal cones and dynamical intersection theory (which in the context of this section, defines the intersection as the flat limit of \( V \cap \{b = t\} \) as \( t \to 0 \)). Very preliminary investigation suggests that where
usual dynamical intersection theory studies how solutions collide as \( t \to 0 \), the balanced version keeps track also of how fast they collide.

4. The Cartier case: the ring \( \widehat{gr} \) and a homotopy interpretation

We saw in proposition 3 that the multiplication operator \( b \cdot \) on \( \widehat{gr} R \) is always 1:1 above degree 0, so that

\[
\widehat{gr} R \to R/\sqrt{I} \oplus \widehat{gr} R/\text{ann}(b)
\]

is an injection. This proposition also told us that on \( \overline{\text{gr}} R/\text{ann}(b) \), multiplying by \( b \) is 1:1 and in high degrees, onto. That suggests that we fill in the holes in small degrees, which we do now.

The map \( \overline{\text{gr}} R \) to the fraction ring \( \overline{\text{gr}} R[b^{-1}] \) has kernel \( \text{ann}(b) \). Define

\[
\overline{\text{gr}} R := \text{the integral closure of } \overline{\text{gr}} R/\text{ann}(b) \text{ in } \overline{\text{gr}} R[b^{-1}].
\]

**Lemma 3.** Let \( R \) be a \( \mathbb{Q}_{\geq 0} \)-graded ring with a homogeneous element \( b \) such that \( b \cdot : R_n \to R_{n+1} \) is 1:1 for all \( n \) and onto for large \( n \). Let \( r \in R_k \) be homogeneous. Then \( r/b^{|k|} \) is integral over \( R \), and even over \( R_0[b] \).

**Proof.** Pick \( d > 0 \) such that \( kd \in \mathbb{N} \). Then \( r^d \in R_{kd} \), and to show \( r \) is integral it is enough to show \( r^d \) is integral. In this way we can reduce to the case \( k \in \mathbb{N} \), which we assume hereafter.

Fix \( N \in \mathbb{N} \) such that \( b \cdot : R_n \to R_{n+1} \) is onto for all \( n \geq N \). Then \( R_N \) is a finite module over \( R_0 \) (proof: take a homogeneous generating set for the ideal \( \oplus_{n \geq N} R_n \); the elements in \( R_N \) generate \( R_N \) as an \( R_0 \)-module). Since we can use \( b \cdot \) to identify all these \( R_n \) for \( n \geq N, n \in \mathbb{N} \), we will denote this module by \( R_{N \gg 0} \). By multiplying by \( b^N \), any homogeneous element \( s \in R \) has an image \( s' \) in \( R_{N \gg 0} \).

Now consider the sequence \( 1', r', (r')^2', \ldots \) in \( R_{N \gg 0} \). They generate an \( R_0 \)-submodule of \( R_{N \gg 0} \), but only finitely many are needed to generate. Hence for some \( m > 0 \), we can write \( (r^m)' = \sum_{i<m} c_i (r^i)' \) with each \( c_i \in R_0 \). Lifting back to \( R \), this becomes \( r^m = \sum_{i<m} c_i b^{k(m-i)} r^i \). So \( r \) satisfies a monic polynomial with \( R_0[b] \)-coefficients. \( \square \)

**Theorem 3.** Let \( R \) be a ring and \( b \) an element, inducing \( \overline{\text{gr}}, \overline{\text{gr}} R, \overline{\text{gr}} R \) as above.

Then the natural map

\[
\overline{\text{gr}} R \to R/\sqrt{I} \oplus \overline{\text{gr}} R
\]

is a graded inclusion, and an isomorphism in high degrees. The multiplication map

\[
b \cdot : \overline{\text{gr}} R_n \to \overline{\text{gr}} R_{n+1}
\]

is an isomorphism for all rational \( n \geq 0 \).

(The common reflex is to conclude from this that \( \overline{\text{gr}} R \cong (\overline{\text{gr}} R_0)[b] \). But \( \overline{\text{gr}} R \) is rationally graded, not integrally, so this result is merely specifying a periodicity in the grading.)

If \( R^* \) is a graded ring with \( R^0 = \mathbb{F} \) an algebraically closed field, and \( b \in R^* \) is homogeneous for this grading, then \( \overline{\text{gr}} R \) splits naturally as a finite direct sum of doubly graded rings \( \{A_i\}_{i=1}^{\ldots m} \) with each \( (A_i)^0 = (A_i)_0^0 = \mathbb{F} \).
Proof. The map given is the composite
\[ \overline{gr} R \to R/\sqrt{I} \oplus \overline{gr} R/\text{ann}(b) \to R/\sqrt{I} \oplus \overline{gr} R \]
of two graded inclusions (taking \( R/\sqrt{I} \) to be degree 0), and hence is one also.

Using proposition 3 choose \( N > 0 \) such that \( b^n : \overline{gr} R_n \to \overline{gr} R_{n+1} \) is an isomorphism for all \( n \geq N \). Then if \( c/b^k \in \overline{gr} R_n \) for \( n \geq N \), we know \( c \in (\overline{gr} R/\text{ann}(b))_{n+k} \), which we can identify with \( \overline{gr} R_{n+k} \) since \( n+k \geq N+k \geq N > 0 \). Then by the assumption on \( N, c = b^kd \) for some \( d \), so \( c/b^k = d \in \overline{gr} R_n \). This shows that the inclusion \( \overline{gr} R_n \to \overline{gr} R_n \) is onto for \( n \geq N \).

To see that \( b^n \) is an isomorphism for all rational \( n \geq 0 \), we apply lemma 3 to \( \overline{gr} R \).

Since \( \overline{gr} R \) stands between \( \overline{gr} R/\text{ann}(b) \) and its full normalization, it is finite over \( \overline{gr} R/\text{ann}(b) \). In particular \( \overline{gr} R^0 \) is a finite-dimensional \( \mathbb{F} \)-algebra. Since \( \overline{gr} R^0 \) has no nilpotents and \( \mathbb{F} \) is algebraically closed, we find \( \overline{gr} R^0 \cong \mathbb{F} \). In more detail, \( \overline{gr} R^0 \) has a unique \( \mathbb{F} \)-basis \( (\pi_1, \ldots, \pi_m) \) up to reordering, with \( \pi_i^2 = \pi_j^2 = 0 \) for \( i \neq j \).

Again since \( \overline{gr} R^0 \) is finite-dimensional without nilpotents, all of \( \overline{gr} R^0 \) must be in \( \overline{gr} R_0 \).

Let \( A_i = \pi_i \overline{gr} R \) as an algebra with unit \( \pi_i \). Then \( \overline{gr} R = \oplus A_i \) as claimed. \( \square \)

It was to obtain a theorem like this that first led the author to the study of balanced normal cones, to study the Hilbert function of \( R^\bullet \) in terms of \( R/\sqrt{I} \) and \( \overline{gr} R \). In a future publication \( [Kn] \) we will use theorem 3 inductively to study standard bases of homogeneous coordinate rings.

For the rest of this section, we make the assumptions of the latter part of the theorem, namely that \( R^\bullet \) is a graded ring with \( R^0 = \mathbb{F} \) an algebraically closed field, and \( b \in R^\bullet \) is a homogeneous element. Let \( Y = \text{Proj} R^\bullet \) and \( X = \text{Proj} R/\langle b \rangle \) the divisor \( b = 0 \). Let \( \overline{C}_X Y = \text{Proj} \overline{gr} R^\bullet \). Write \( \overline{C}_X Y \) for \( \text{Proj} \overline{gr} R \). Then by the last part of the theorem above, \( \overline{C}_X Y \) is a disjoint union of weighted cones \( \{ \text{Proj} A \} \).

We can now interpret some of these ring maps geometrically:

\[
\begin{align*}
\overline{gr} R & \to \overline{gr} R[b^{-1}] & \iff & & \overline{C}_X Y \setminus X & \to \overline{C}_X Y \\
\overline{gr} R & \to R/\sqrt{I} \oplus \overline{gr} R/\text{ann}(b) & \iff & & X_{\text{red}} \cup \overline{C}_X Y \setminus X & \to \overline{C}_X Y \\
\overline{gr} R & \to R/\sqrt{I} \oplus \bigoplus A & \iff & & X_{\text{red}} \cup \bigcup A \text{Proj} A & \to \overline{C}_X Y
\end{align*}
\]

4.1. Examples.

**Example 10.** Recall \( R = \mathbb{F}[a, b]/(a^2 - ab) \) from example 3 with \( \overline{gr}(a) = \overline{gr}(b) = 1 \).

Now \( f = ab^{-1} \) is integral, since \( f(f - 1) = b^{-2}a(a - b) = 0 \), and \( \overline{gr} R = \mathbb{F}[b, f]/(f(f - 1)) \cong \mathbb{F}[b] \oplus \mathbb{F}[b] \).

Geometrically, the divisor \( b = 0 \) is a double point at the intersection of the two lines \( \text{Spec} R \). The normal cone \( \text{Spec} \overline{gr} R \) is the trivial line bundle over the double point. The balanced normal cone \( \text{Spec} \overline{gr} R \cong \text{Spec} R \) is just the two intersecting lines. Whereas \( \text{Spec} \overline{gr} R \) pulls apart the two lines; it is the full normalization.

**Example 11.** Let \( R = \mathbb{F}[b, c, d]/(c(c^2 - bd)) \), so \( X = \text{Proj} R \) is the union of a line and a conic in the plane. Using proposition 4 we find \( \overline{gr} R \cong R \), with \( b \in \overline{gr} R_1, c \in \overline{gr} R_{1/2}, d \in \overline{gr} R_0 \).
Then \( f = c^2/b \in \mathfrak{gr} R[b^{-1}] \) is integral, because \( f(f - d) = cb^{-2}c(c^2 - bd) = 0 \). In fact

\[
\mathfrak{gr} R = \mathbb{F}[b, c, d, f]/\langle c(f - d), bf - c^2, f(f - d) \rangle,
\]
so \( \text{Proj} \mathfrak{gr} R/\langle b \rangle \) is the disjoint union of the point \( \text{Proj} \mathfrak{R}/\langle b, c \rangle \) and the (doubly fat) point \( \text{Proj} \mathfrak{R}/\langle b, f - d, c^2 \rangle \). Whereas \( \text{Proj} \mathfrak{gr} R/\langle b \rangle \) is only one (triply fat) point.

Note that \( \text{Proj} \mathfrak{gr} R \) is not the full normalization of \( X \), which would pull the two components apart at both ends.

**Example 12.** Recall the ring \( R = \mathbb{F}[b, c, d, e]/\langle d(b - d), e(b - d), dc, ec \rangle \) from subsection 2.2.2, the union of the \( d = e = 0 \) plane and \( b - d = c = 0 \) plane. We found that the associated graded \( \mathfrak{gr} R \) turns out to be isomorphic to \( R \), with \( \mathfrak{g}(b) = \mathfrak{g}(d) = 1, \mathfrak{g}(c) = \mathfrak{g}(e) = 0 \).

The first few graded pieces of \( \mathfrak{gr} R \) are

\[
\begin{align*}
\mathfrak{gr} R_0 &= \mathbb{F}[c, e]/\langle ce \rangle \\
\mathfrak{gr} R_1 &= b(\mathfrak{gr} R_0) \oplus Fd \\
\mathfrak{gr} R_2 &= b^2(\mathfrak{gr} R_0) \oplus Fbd \\
\mathfrak{gr} R_3 &= b^3(\mathfrak{gr} R_0) \oplus Fb^2d \\
\vdots
\end{align*}
\]

so \( \mathfrak{g} : R_0 \to R_1 \) is 1:1, and is an isomorphism in all higher degrees. This suggests we look at the element \( f = db^{-1} \). It is indeed integral, satisfying \( f(f - 1) = 0 \).

We know that \( \mathfrak{gr} R \) should be the \( b \)-cone over \( \mathfrak{gr} R_0 \), so should have no relations involving \( b \); each will end up replaced by relations in degree 0. In fact

\[
\mathfrak{gr} R = \mathbb{F}[b, c, f, e]/\langle f(f - 1), fc, e(f - 1), ec \rangle,
\]
geometrically the cone over the disjoint union of the two lines \( c = f - 1 = 0, e = f = 0 \).

**Example 13.** From subsection 2.2.3 recall the ring

\[
\mathfrak{gr} R = \mathbb{F}[a_{(0)}, b_{(1)}, c_{(2)}, d_{(0)}]/\langle ab, ac, c^2 - b^4d \rangle
\]

where the parenthesized subscripts indicate the degrees. The first few graded pieces are

\[
\begin{align*}
\mathfrak{gr} R_0 &= \mathbb{F}[a, d] \\
\mathfrak{gr} R_1 &= b(\mathfrak{gr} R_0) \\
\mathfrak{gr} R_2 &= b^2(\mathfrak{gr} R_0) \oplus c(\mathfrak{gr} R_0) \\
\mathfrak{gr} R_3 &= b^3(\mathfrak{gr} R_0) \oplus bc(\mathfrak{gr} R_0) \\
\mathfrak{gr} R_4 &= b^4(\mathfrak{gr} R_0) \oplus b^2c(\mathfrak{gr} R_0) \\
\vdots
\end{align*}
\]

In this example \( b \) is a zero divisor, and \( \mathfrak{g} : \mathfrak{gr} R_0 \to \mathfrak{gr} R_1 \) is not 1:1. All later maps are 1:1, but only become onto at and after \( \mathfrak{gr} R_2 \to \mathfrak{gr} R_3 \).

The relation \( c^2 - b^4d = 0 \) says that \( e = cb^{-2} \) is integral, as \( e^2 = d \). In fact

\[
\mathfrak{gr} R = \mathbb{F}[a, b, e, d]/\langle a, e^2 - d \rangle
\]

where we’ve lost the component that lived in \( b = 0 \).
Let Proposition 7. (obvious formula the reuse of the name. With it, we can define
Proof. First, define a filtration on $R$. Our goal is to show that

Example 14. This is an irreducible example with $\widetilde{\text{gr}} R \neq \text{gr} R$.
Consider $R = \mathbb{F}[b, c, d, f]/(b^2 f + bcd + c^3)$, the homogeneous coordinate ring of a cubic surface. It has a $\mathbb{P}^1$ of singularities, along $b = c = 0$. It is easy to see that $c^{2n+1} \in \langle b^n \rangle$, and the lower bounds $\langle c \rangle \geq 1/2, \langle d \rangle, \langle f \rangle \geq 0$ suggest the degeneration $\mathbb{F}[b(1), c(1/2), d(0), f(0)]/(bcd + c^3)$. Applying proposition 4 we see that we have correctly computed $\widetilde{\mathbb{F}}$ and $\text{gr} R$.

The first few graded pieces of $\text{gr} R$ are
\[
\begin{align*}
\text{gr} R_0 & = \mathbb{F}[d, f] \\
\text{gr} R_2 & = \mathbb{F}[d, f]c \\
\text{gr} R_1 & = \mathbb{F}[d, f]b \oplus \mathbb{F}[d, f]c^2 \\
\text{gr} R_1 & = \mathbb{F}[d, f]bc \\
\text{gr} R_2 & = \mathbb{F}[d, f]bc^2 \\
\end{align*}
\]

Multiplication by $b$ should give a “1-fold periodicity” on $\text{gr} R$, suggesting we let $y = b^{-1}c^2$ fill in the hole seen in degree 0. Then $y^2 + dy = b^{-2}c(c^3 + bcd) = 0$, so $y$ is indeed integral over $R$. In fact
\[
\text{gr} R = \mathbb{F}[b, c, d, f, y]/(y^2 + dy, by - c^2) = (\mathbb{F}[d, f, y]/\langle y(y + d) \rangle) \langle b, c \rangle/\langle by - c^2 \rangle.
\]

Consider the map $\text{gr} R \hookrightarrow \text{gr} R$ after $b$ is killed:
\[
\mathbb{F}[c, d, f]/\langle c^3 \rangle \to \mathbb{F}[c, d, f, y]/(y^2 + dy, c^2)
\]

Taking Proj, this is a map from a bouquet of two $\mathbb{P}^1$s (one with multiplicity 2) onto a single $\mathbb{P}^1$ (with multiplicity 3), 1 : 1 at the north pole intersection but otherwise 2 : 1.

It will be useful later to know that the integral closure can be taken before or after $\text{gr} R$.

Proposition 7. Let $R$ be a ring and $b \in R$. Let $\widetilde{R}$ denote the integral closure of $R/\text{ann}(b)$ in $R[b^{-1}]$. Then $\text{gr} R \cong \text{gr} \widetilde{R}$.

Proof. First, define a filtration on $R[b^{-1}]$ again called $\mathcal{F}$ by $\mathcal{F}(p/b^k) = \mathcal{F}(pb) - (k + 1)$. By Rees’ formula for $\mathcal{F}$, this formula is well-defined. (If $b$ is not a zero divisor, the more obvious formula $\mathcal{F}(p) - k$ works as well.) Plainly it restricts to $\mathcal{F}$ on $R/\text{ann}(b)$, justifying the reuse of the name. With it, we can define $\text{gr} (R[b^{-1}])$, easily seen to be isomorphic to $(\text{gr} R)[b^{-1}]$. Now the $\text{gr}$ of
\[
R/\text{ann}(b) \hookrightarrow \widetilde{R} \hookrightarrow R[b^{-1}]
\]
gives $\text{gr}(R/\text{ann}(b)) \hookrightarrow \text{gr} \widetilde{R} \hookrightarrow \text{gr} R[b^{-1}] = (\text{gr} R)[b^{-1}]$.

Our goal is to show that $\text{gr} \widetilde{R}$ is the integral closure of $\text{gr}(R/\text{ann}(b))$ in $\text{gr} (R[b^{-1}])$.

We first show $\text{gr} \widetilde{R}$ is integral over $\text{gr} R$. Let $r \in R$ lie over $\mathcal{F} \in \text{gr} R$. Then if $r/b^k$ satisfies a monic polynomial $p \in R[x]$, its image $\mathcal{F}/b^k$ satisfies $\mathcal{F} \in \text{gr} R[x]$. Hence we have maps
\[
\text{gr}(R/\text{ann}(b)) \hookrightarrow \text{gr} \widetilde{R} \hookrightarrow \text{gr} R.
\]

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Since the composite is an isomorphism in large degrees, so is the second inclusion.

Now it is enough to know that the map \( b : \overline{gr} \mathbb{R}_n \to \overline{gr} \mathbb{R}_{n+1} \) is onto for all \( n \geq 0 \). Let \( \overline{p} \in \overline{gr} \mathbb{R}_{n+1} \) be the image of \( p \in \mathbb{R}_{n+1} \), so \( \overline{q}(p) = n + 1 \geq 1 \). Then by Rees’ formula and the valuative criterion for integrality, \( p/b \) is integral over \( \mathbb{R}/\text{ann}(b) \), so \( p/b \in \mathbb{R}_n \), giving a preimage of \( \overline{p} \) in \( \overline{gr} \mathbb{R}_n \).

\[ \square \]

4.2. **A homotopical analogy.** Let \( X \) be a topological space, and \( D \) a closed subset. Then we can think of \( X \) as built from \( D \) with \( X \setminus D \) attached. A standard homotopical operation at this point is to study \((X, X \setminus D)\) by collapsing \( X \setminus D \) to a point, or at least something contractible.

That is slightly too brutal for us. First thicken \( D \) to an open neighborhood \( D_+ \) such that \( D_+ \) retracts to \( D \) and \( X \setminus D \) retracts to \( X \setminus D_+ \). Then separate \( X \setminus D \) into connected components \( W_i \), with each \( W_- := W \setminus D_+ \) a closed retract of \( W \). Now let \( X' \) be \( X \) with each \( W_- \) collapsed to its own point. In good cases, up to homotopy this means we replace \( W \) with the cone on \( W \cap D \).

**Example 15.** Let \( X \) be a circle and \( D \) a point on it. Then \( X \setminus D \) is connected, so there is only one connected component \( W \), and \( W \) is already contractible. But in the (trivial) passage from \( X \) to \( X' \) we don’t replace \( W \) by the cone on the point \( \overline{W} \cap W = X \setminus (X \setminus D) = D \); that would flatten the circle \( X \) to an interval. Rather, we replace \( W \) with the cone on \( \overline{W} \cap \overline{D} \), which is two points.

In bad cases like this example, we are still replacing \( W \) with a cone – it’s just not the cone on \( W \cap D \), but instead a sort of link of that inside \( W \). In the above example, the link was two points.

In the algebraic geometry, the passage from \( X \to X' \) parallels the flat degeneration \( R \to \overline{gr} R \). The decomposition of the open set \( X' \setminus D \) into connected components \( U \) corresponds to the decomposition of the fraction ring \( \overline{gr} R[b^{-1}] \) as a direct sum. “Being a cone” is replaced by having a periodic grading.

The most subtle point in the above topological picture is the fact that we don’t replace each \( W \) with a cone on \( W \cap D \), but on something that *maps* to \( W \cap D \). In the algebraic geometry, this reflects the fact that the inclusion \( \overline{gr} R/\text{ann}(b) \to \overline{gr} R \) may not be an isomorphism. In this way, perhaps one should think of the map \( \overline{gr} R/\text{ann}(b) \to \overline{gr} R \) as sort of an attaching map when building a complex. In example 14 above, the attaching map is the one from the bouquet of two \( \mathbb{P}^1 \)s to a single \( \mathbb{P}^1 \).

5. **Proof of theorem**

Since we have to deal seriously with Chow classes in this section, we list some simple properties we will need of them:

1. If \( \phi : W \to V \) is proper, there is an induced map \( \phi_* : A_*(W) \to A_*(V) \) of their Chow groups, and these maps are functorial.
2. Any scheme \( W \) has a “fundamental class” \( [W] \in A_{\dim W}(W) \). Consequently, a proper map \( \pi : W \to V \) induces a class \( [\pi_*] := \pi_*([W]) \in A_{\dim W}(V) \). (For some authors, the fundamental class of a nonequidimensional scheme is inhomogeneous, and ours is merely the component in top degree.)
(3) If \(W = \bigcup W_i\) where each \(\dim (W_i \cap W_j) < \dim W\), then \(\deg W = \sum_i \dim W_i = \dim W\).

(4) The inclusion \(W_\text{red} \hookrightarrow W\) if the reduction of \(W\) induces a map \(\mathbb{A}_n(W_\text{red}) \to \mathbb{A}_n(W)\). Consequently, we can pull back \(\deg W\) to a class on \(W_\text{red}\).

(5) If \(\phi_i : W_i \to V\) is a flat family of schemes each proper over \(V\), then \(\{\phi_i\}\) is constant in \(t\).

(6) If \(\nu : V' \to V\) is the normalization of a reduced scheme \(V\), then \(\nu = \nu\).

(7) If \(\nu : V \to V'\) is the blowup of \(V\) along a subscheme \(W\) containing no component of \(V\), then \(\nu = \nu\).

(8) Let a composite \(W \to V \to Z\) of proper maps take the fundamental class of \(W\) to that of \(Z\), i.e. \([\beta \circ \alpha] = [Z]\). Then \([\alpha] = [V]\) and \([\beta] = [Z]\).

One way these interrelate is the following. Consider the flat family of schemes over \(V\) in which \(V\) degenerates to \(C_W V\). If \(W = V_\text{red}\), then the map \(C_W V \to V\) is proper, hence (by property 5) induces the fundamental class on \(V\). But \(C_W V \to V\) factors as \(C_W V \to W \to V\), giving us another way to see the class on \(W = V_\text{red}\) induced from the thickening \(V\).

The proof of theorem \([\text{I}]\) involves three reductions (much the same as in [Re, chapter 4]):

(1) We excise any components of \(V\) contained completely within \(W\).
(2) We blow up \(V\) along \(W\).
(3) We normalize \(V\) along \(W\).

We’ll justify these reductions in propositions \([\text{II}, \text{III}],\) and \([\text{IV}].\) Then we’ll develop the tools to address what for us is the fundamental case, that \(W\) is defined in \(V\) by the vanishing of a nonzero divisor \(b\), and \(V\) is normal along \(W\).

We’ll call a map \(\nu : V \to Z\) **volumetric** if it takes the fundamental class to the fundamental class. (There does not seem to be a standard term for this, and we don’t seriously propose this as the right name for this concept beyond its frequent use in this section.)

Theorem \([\text{I}]\) is thus the statement that \(\beta : C_W V \to C_W V\) is volumetric.

**Proposition 8.** Let \(V\) be a reduced scheme and \(W\) a closed subscheme. Let \(V' = \overline{V \setminus W}\) and \(W' = V' \cap W\). Then theorem \([\text{I}]\) holds for the pair \((W \subseteq V)\) if it holds for the pair \((W' \subseteq V')\).

**Proof.** Note first that \(\dim W' < \dim V\). Applying lemma \([\text{I}]\) to the map of pairs

\[
\begin{align*}
\begin{array}{ccc}
W \cup W' & \hookrightarrow & W \cup V' \\
\downarrow & & \downarrow \\
W & \hookrightarrow & V
\end{array}
\end{align*}
\]

gives a commuting square

\[
\begin{align*}
\begin{array}{ccc}
W \cup C_W V' & \to & W \cup C_W V' \\
\downarrow & & \downarrow \\
C_W V & \to & C_W V.
\end{array}
\end{align*}
\]

Let \(R, R/I\) denote the coordinate rings of \(V\) and \(W\).

It is easy to check that the \(I\)-adic filtration on \(R\) is restricted from the \(I\)-adic filtration on \(R/\sqrt{I} \oplus R/\text{ann}(I)\) along the obvious map (which is an inclusion since \(R\) has no nilpotents). Consequently, the map \(\text{gr} R \to R/\sqrt{I} \oplus \text{gr} (R/\text{ann}(I))\) is an inclusion, and an isomorphism in positive degrees. Geometrically, the map \(W \cup C_W V' \to C_W V\) is onto, finite-to-one, and \(1 : 1\) away from \(W' \subseteq C_W V\). Consequently it is volumetric (which uses \(\dim W' < \dim V\)).

Exactly the same argument applies to the balanced normal cones.

Since the vertical maps in the square are volumetric, if the top map is volumetric, then so is the bottom one. \[\square\]
Proposition 9. Let $V$ be a reduced scheme with $W \subseteq V$ a closed subscheme containing no components. Let $\hat{V}$ denote the blowup of $V$ along $W$ and $\tilde{W}$ denote the exceptional divisor. Then theorem \( \text{I} \) holds for the pair $(W \subseteq V)$ if it holds for the pair $(W \subseteq \hat{V})$.

Proof. Consider the commuting diagram

\[
\begin{array}{ccc}
\hat{C}_W V & \rightarrow & C_W \hat{V} \\
\downarrow & & \downarrow \\
C_W V & \rightarrow & C_W V
\end{array}
\]

Here $C_W \hat{V}$ denotes the blowup along $W$; the isomorphism $C_W \hat{V} \cong \hat{C}_W V$ then follows directly from the definitions. The vertical map on the right is volumetric by property \( \text{I} \). Once we prove that the vertical map on the left is volumetric, we’re done.

To prove that $\hat{C}_W V \rightarrow C_W V$ is volumetric, we show that it sits intermediate to a blowup $\hat{C}_W V \rightarrow \tilde{W}$, and apply properties \( \text{I} \) and \( \text{II} \). If $V = \text{Spec } R$ and $W$ is defined by $I$, let $\mathbf{J}$ be the ideal in $R$ given by $\oplus_{n \geq 1} R_n$; a sort of $\mathbf{gr}$-analogue of the augmentation ideal. Then

\[
\begin{array}{c}
\hat{C}_W V = \text{Spec } \mathbf{gr} R \\
\tilde{W} \rightarrow \hat{C}_W V = \text{Proj } \mathbf{gr} R \oplus \mathbf{J} \oplus \mathbf{J}^2 \oplus \ldots \\
\tilde{W} \rightarrow \mathbf{J} = \text{Proj } R \oplus I \oplus I^2 \oplus \ldots \\
\tilde{W} = \text{Proj } R/I \oplus I^2/I^3 \oplus \ldots
\end{array}
\]

Let $S$ denote the graded ring $R \oplus I \oplus I^2 \oplus \ldots$, and $S_0$ the ideal generated by the I in the 0th graded piece. So $\hat{V} = \text{Proj } S$, $\tilde{W} = \text{Proj } S/S_0$. Our goal is to show that $\mathbf{gr}_{S_0} S$, the coordinate ring of $\hat{C}_W \hat{V}$, includes naturally into the coordinate ring of $\tilde{W}$. In degree 0 they are both $\mathbf{gr} R$, and using $q \leq \mathbf{r}$ we can see $\mathbf{gr} R \otimes_R I^k \leq \mathbf{J}^k$, completing the proof. \( \square \)

Proposition 10. Let $V = \text{Spec } R$ be a reduced scheme and $b \in R$ a nonzero divisor, with $W$ the subscheme defined by $b = 0$. Let $\hat{R}$ denote the integral closure of $R$ in $R[b^{-1}]$, and $\hat{V} = \text{Spec } \hat{R}$, and $\tilde{W}$ the subscheme defined by $b = 0$. Then theorem \( \text{I} \) holds for the pair $(W \subseteq V)$ if it holds for the pair $(W \subseteq \hat{V})$.

Proof. Since $b$ is not a zero divisor, the map $\hat{C}_W V \rightarrow \tilde{W}$ is volumetric. (This uses properties \( \text{I} \) and \( \text{II} \) since $\hat{C}_W V$ is a partial normalization of $\tilde{W}$.) By proposition \( \text{I} \), $\hat{C}_W V \cong \tilde{W}$. Thus we have a commutative diagram

\[
\begin{array}{ccc}
\hat{C}_W V & \cong & C_W \hat{V} \\
\downarrow & & \downarrow \\
C_W V & \rightarrow & C_W V
\end{array}
\]

It remains to argue that $C_W \hat{V} \rightarrow C_W V$ is volumetric. Since $b$ is not a zero divisor, $W$ and $\tilde{W}$ contain no components of the normal cones. Hence we can test the lengths of all components by passing to the projectivization $\mathbb{P} C_W \hat{V} \rightarrow \mathbb{P} C_W V$. But since $\tilde{W}$ and $W$ are Cartier, this is just the map $\tilde{W} \rightarrow W$ which is volumetric by properties \( \text{I} \) and \( \text{II} \). \( \square \)

We are now ready to approach the basic case of theorem \( \text{I} \) We start with a couple of lemmas \( \text{I} \) and \( \text{II} \) giving equalities of Chow classes.
Lemma 4. Let \( \pi : W \to W_{\text{red}} \) be a map such that the composite \( W_{\text{red}} \hookrightarrow W \to W_{\text{red}} \) is the identity. Then the two Chow classes induced on \( W_{\text{red}} \) by these maps are the same.

Proof. Consider the induced morphism \( A(W_{\text{red}}) \to A(W) \to A(W_{\text{red}}) \). The inclusion \( \iota \) induces a class by pulling back \([W]\) along \((\iota_*)^{-1}\). The projection \( \pi \) induces a class by mapping \([W]\) forward using \( \pi_* \). By functoriality, these two maps \((\iota_*)^{-1}, \pi_* : A(W) \to A(W_{\text{red}})\) are the same. \(\square\)

The following lemma is stronger than we need, but aids the intuition.

Lemma 5. Let \( R \) be a finitely generated commutative algebra over a field \( \mathbb{F} \). Let \( \{g_1, \ldots, g_m\} \subseteq R \) be a list of elements whose images generate the quotient \( R/\sqrt{0} \) as an \( \mathbb{F} \)-algebra. Then there exists a list \( \{n_1, \ldots, n_k\} \) of nilpotents in \( R \) such that the concatenation \( \{g_1, \ldots, g_m, n_1, \ldots, n_k\} \) generate \( R \) as an \( \mathbb{F} \)-algebra.

Proof. Let \( I \) be the nilpotent radical of \( R \), and consider the associated graded \( \text{gr} R \). This is again Noetherian, so its augmentation ideal is finitely generated, and by homogeneous elements; pick generators \( \{n'_1, \ldots, n'_k\} \) the images of some elements \( \{n_1, \ldots, n_k\} \) from \( R \). The monomials in these generators have a maximum possible degree, so \( I^M = 0 \) for \( M \) large enough.

Now we claim that \( \{g'_1, \ldots, g'_m, n'_1, \ldots, n'_k\} \) generate \( \text{gr} R \) as an \( \mathbb{F} \)-algebra. The argument is exactly the same as in the proof of lemma 2.

Knowing \( \{g'_1, \ldots, g'_m, n'_1, \ldots, n'_k\} \) generate \( \text{gr} R \), we now claim that \( \{g_1, \ldots, g_m, n_1, \ldots, n_k\} \) generate \( R \). Let \( r \) be an element to generate. Then its image \( r' \in \text{gr} R_{q(r)} \) can be written as \( p(g', n') \) for some \( \mathbb{F} \)-polynomial \( p \), so \( q(r - p(g, n)) > q(r) \), where \( q \) as before measures the depth in the I-adic filtration. So to generate \( r \), it is enough to generate \( r - p(g, n) \), which is deeper in the I-adic filtration. Since \( I^M = 0 \) for \( M \) large enough, this algorithm terminates. \(\square\)

The next lemma is sort of a dual to property 5 in it the base ring is a quotient of each element of the family, rather than a subring. It seems unlikely that the quasiprojectivity asked of \( W \) is necessary, and we hope that someone more fluent with Chow classes can sidestep it.

Lemma 6. Let \( V \to \mathbb{A}^1 \) be a quasiprojective flat family over a field \( \mathbb{F} \), whose reduction \( V_{\text{red}} \) is a trivial family \( W \times \mathbb{A}^1 \). Then each thickening \( V_t \) of \( W \) induces the same Chow class on \( W \).

Proof. The fundamental Chow class of \( V_t \) is the linear combination of its top-dimensional components, each weighted by the length of the local ring at the generic point of the component.

These lengths don’t change if we replace \( V \) by the quasiaffine cone over it, so we can assume \( V \) quasiaffine. Nor do they change if we algebraically close the base field \( \mathbb{F} \). Doing so will allow us to test the lengths at general-enough points, rather than the generic points.

Now use lemma 5 to pick coordinates \( \{g_1 = t, g_2, \ldots, g_m, n_1, \ldots, n_k\} \) on the affine closure of \( V \), where \( g_2, \ldots, g_m \) are coordinates on \( W \), and \( n_1, \ldots, n_k \) vanish on \( V_{\text{red}} \).
By imposing \((m-1)-\dim W\) general affine-linear conditions on the variables \((g_2, \ldots, g_m)\), we can \(W\) down to a set \(W_0\) of reduced points, with at least one general point in each top-dimensional component \(F\) of \(W\). (We use \(\mathbb{P}\) infinite to guarantee the existence of general-enough linear conditions.) These same conditions cut \(V\) down to a new flat family \(V_0\).

For each top-dimensional component \(F\) of \(W\), pick a point \(w \in W_0\) in the smooth locus of \(F\). Shrink \(V_0\) to the subscheme supported on \([w] \times \mathbb{A}^1\). As an \(\mathbb{P}[t]\)-module, the ring of functions on \(V_0\) is finitely generated, and torsion-free, hence free. Therefore \(\dim_{\mathbb{P}} \text{Fun}(V_t)\), the coefficient of \([F]\) in \([V_t]\), is constant in \(t\).

**Proof of theorem**

As usual, we take \(V = \text{Spec } R\), with \(W\) defined by the ideal \(I\).

The first claims of the theorem – that \(\overline{C}_W V \to C_W V\) is proper with finite fibers – are implied by the claim that \(\overline{g} R\) is a finite module over the image \(R/\sqrt{I}[b]\) of \(\text{gr } R\) in \(\overline{g} R\). To see that this module is finitely generated, let \((g_1, \ldots, g_k)\) be homogeneous generators of \(\overline{g} R\) over \(\overline{g} R_0\), which we can take to all be in positive degree (using lemma 2). By proposition \(3\) after some degree \(N\) the multiplication map \(b : \overline{g} R_n \to \overline{g} R_{n+1}\) is onto. Only finitely many monomials in the \((g_i)\) have degree \(\leq N\), and these monomials generate \(\overline{g} R\) as an \(R/\sqrt{I}[b]\)-module.

Next, we use propositions \(8, 9, 10\) to reduce to the case that \(I = \langle b \rangle\) is a principal ideal, \(b\) is not a zero divisor, and \(R\) is integrally closed in \(R[b^{-1}]\). Three effects of the integrality are that

- \(\text{gr } R = R/I[b]\)
- \(q(m) \geq 1 \iff q(r) \geq 1\)
- each \(r\) can be written as \(b^m(a)\) with \(q(a) < 1\).

Now consider the following diagrams:

\[
\begin{array}{ccc}
\text{gr } R & \to & (\text{gr } R)/\langle b \rangle [b] \\
\text{gr } R & \to & \text{gr } (R/I) \\
\downarrow & & \downarrow \\
R/\sqrt{I}[b] & = & R/\sqrt{I}[b]
\end{array}
\]

We first define these rings, and the vertical maps. The dashed arrows mean “has a flat degeneration to”, and point from general fiber to special fiber.

We’ve already defined \(\overline{g} R\), and noted the inclusion \(R/\sqrt{I} \hookrightarrow \overline{g} R\) as the degree 0 part. Extend this map to \(t : R/\sqrt{I}[b] \to \overline{g} R\) by taking \(b\) to the evident element of \(\overline{g} R_1\). (In fact, this map is an inclusion, with image the integer-graded part of \(\overline{g} R\).)

Now filter \(\overline{g} R\) by the \(\langle b \rangle\)-adic filtration, and take the associated graded; call this \(\text{gr } \overline{g} R\). This operation is trivial on the image of \(R/\sqrt{I}[b]\), so there is still a natural map \(R/\sqrt{I}[b] \to \text{gr } \overline{g} R\).

We must understand this ring \(\text{gr } \overline{g} R\). Each homogeneous element of \(\overline{g} R\) is of the form \(b^m a\) with \(\deg a \in [0, 1]\). Then the product in \(\text{gr } \overline{g} R\) is \((b^m a) \cdot (b^m c) = b^{m+\deg c} a\) if \(\deg a + \deg c < 1\), and 0 otherwise. Put another way, \(\text{gr } \overline{g} R \cong (\overline{g} R)/\langle b \rangle [b]\).

Coming from the other end, the \(\overline{g}\)-filtration on \(R\) induces one on \(R/I\), whose associated graded we call \(\overline{g} (R/I)\). Since \(q(m) \geq 1 \iff q(r) \geq 1\), this ring can be identified with \((\overline{g} R)/\langle b \rangle [b]\). With this, we can further identify the rightmost vertical map in the left diagram with the leftmost in the right diagram, up to adjoining \(b\).
Each of these vertical maps induces a Chow class on $R/\sqrt{I}[b]$, the reduction of $gr\ R$. Applying property 5, lemma 4, and lemma 6 to squares 1, 2, and 4, we see that the four classes are the same. In particular $\iota_*([Spec\ gr\ R]) = (\pi^*)^{-1}([Spec\ gr\ R]) \in A(Spec\ R/\sqrt{I}[b])$.

Applying $\pi_*$ to both sides, we get $(\pi \circ \iota)_* ([Spec\ gr\ R]) = [Spec\ gr\ R] \in A(Spec\ gr\ R)$, as was to be shown.

It seems worth noting that the vertical maps $\iota$ and $\pi$ can’t naturally be reversed, helping to motivate the course of the above proof.

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