COVERINGS OF ABELIAN GROUPS AND VECTOR SPACES

BALÁZS SZEGEDY

Abstract. We study the question how many subgroups, cosets or subspaces are needed to cover a finite Abelian group or a vector space if we have some natural restrictions on the structure of the covering system. For example we determine, how many cosets we need, if we want to cover all but one element of an Abelian group. This result is a group theoretical extension of the theorem of Brouwer, Jamison and Schrijver about the blocking number of an affine space. We show that these covering problems are closely related to combinatorial problems, including the so called additive basis conjecture, the three-flow conjecture, and a conjecture of Alon, Jaeger and Tarsi about nowhere zero vectors.

1. Introduction

A subgroup covering (coset covering) of the group $G$ is a collection of its subgroups (cosets of its subgroups) whose union is the whole group. A covering is called irredundant or minimal if none of its members can be omitted. B.H. Neumann observed that if a group $G$ has a finite irredundant (right) coset covering $H_1 \times_1, H_2 \times_2, \ldots, H_n \times_n$ then the index of the intersection of the subgroups $H_i$ is bounded above by some function of $n$. Let $f(n)$ (resp. $g(n)$) be the maximal possible value of $|G : \bigcap H_i|$ where $G$ is a group with a coset covering $\{H_i x_i | i = 1, \ldots, n\}$ (resp. subgroup covering $\{H_i | i = 1, \ldots, n\}$). Obviously we have $f(n) \geq g(n)$. M.J. Tomkinson proved that $f(n) = n!$ and that $g(n) \geq \frac{1}{2} \cdot 3^{2(n-1)/3}$. Since no super exponential lower bound has been found for $g(n)$, its order of magnitude is conjectured to be exponential. Let the functions $f_1(n)$ and $g_1(n)$ be similarly defined as $f(n)$ and $g(n)$ with the additional restriction that the group $G$ is always assumed to be Abelian. (Note that $f_1(n) \leq f(n)$ and $g_1(n) \leq g(n)$) L. Pyber pointed out that the order of magnitude of $g_1(n)$ is itself interesting.

We need the following definition.

Definition 1. Let $G$ be a fixed finite group. Let $f(G)$ (resp. $g(G)$) denote the minimal $k$ such that there exists an irredundant covering by $k$ cosets $\{H_i x_i | i = 1, \ldots, k\}$ (resp. subgroups $\{H_i | i = 1, \ldots, k\}$) of $G$ where $\bigcap H_i$ is trivial. (Note that the set of such subgroup coverings may be empty, and in this case we define $g(G)$ to be infinite)

Now we have that $g(G) \geq f(G)$. Pyber’s problem transforms to find a logarithmic lower bound for $g(A)$ in terms of $|A|$ if $A$ is an Abelian group.

Conjecture 2 (Pyber). There exists a fixed constant $c > 1$ such that $g(A) > \log_c |A|$ for all finite Abelian groups $A$.

Actually we believe that (in contrast with $f(n) = n!$) the growth of $f_1(n)$ is bounded above by some exponential function and thus
Conjecture 3. There exists a fixed constant $c_2 > 1$ such that $f(A) > \log_{c_2}|A|$ for all finite Abelian groups $A$.

We note that the worst known case (even for the function $f(A)$ and thus for $f_1(n)$) is the elementary Abelian 2-group $A = C_2^n$ ($n > 1$) where $f(G) = g(G) = n + 1$ (See Corollary 12). It suggests that perhaps 2 could be the true value for the constant $c$.

We have two results related to Conjecture 2 and Conjecture 3.

Theorem 4. Let $A$ be an Abelian group of order $p_1^{a_1}p_2^{a_2} \ldots p_n^{a_n}$. Then $g(A) \geq f(A) \geq 1 + \sum_{i=1}^{n} a_i$.

It means in particular that the inequality of Conjecture 2 holds with $c = p_n$ where $p_n$ is the last prime divisor of the order of $A$.

Alon and Füredi in [1] prove the surprising result that if want to cover all but one vertices of an $n$-dimensional cube, then we need at least $n$ hyperplanes. Actually they prove a more general result.

Theorem (Alon, Füredi). Let $h_1, h_2, \ldots, h_n$ be positive integers and let $V$ be the set of all lattice points $(y_1, y_2, \ldots, y_n)$ with $0 \leq y_i \leq h_i$. If we want to cover all but one of the points of $V$, then we need at least $h_1 + h_2 + \cdots + h_n$ hyperplanes.

Our next result is an analogy of the previous one. We determine, how many cosets we need, if we want to cover all but one element of an Abelian group. This result yields good lower bound for the size of an irredundant coset covering system if it contains a small coset.

Theorem 5. Let $A$ be an Abelian group of order $p_1^{a_1}p_2^{a_2} \ldots p_n^{a_n}$. Let $\phi(A)$ denote the minimal number $k$ for which there exists a system of subgroups $H_1, H_2, \ldots, H_k$ and elements $x_1, x_2, \ldots, x_k$ such that $G\{1\} = \bigcup H_i x_i$. Then $\phi(A) = \sum_{i=1}^{n} a_i(p_i-1)$.

Corollary 6. Let $A$ be an Abelian group and let $\{H_i x_i | i = 1 \ldots k\}$ be an irredundant coset covering of $A$. Then for all $i$

$$k \geq 1 + \log_2|G : H_i|$$

Note that Theorem 5 solves the special case of conjecture 3 when one of the cosets has size 1. In this case both conjectures hold with constant 2. Corollary 6 shows that if the covering system contains a ”small” subgroup of size less than $|A|^p$ for some $p < 1$ then both conjectures hold with constant $c = c_2 = 2/(1 - p)$.

The proof of Theorem 5 uses character theory and some Galois theory. It is also worth mentioning that Theorem 5 implies that the blocking number of an affine space (i.e. the size of the smallest subset which intersect all hyperplanes) over the prime field $GF(p)$ is $1 + n(p - 1)$ which was proved (for arbitrary finite fields) by Brouwer, Schrijver and Jamison [4], [3] using polynomial method.

From the combinatorial point of view, the most important special case of the previously described covering problems is when the group $A$ is an elementary Abelian group $(C_p)^n$, and the covering system consists of hyperplanes (or affine hyperplanes). More generally, we can speak about hyperplane coverings of vector spaces over arbitrary finite fields. Many questions about graph colorings, nowhere zero flows or nowhere zero vectors can be translated to questions about special hyperplane coverings. However not much is known about such coverings. In Chapter 5, we present a character theoretic approach to hyperplane coverings in vector spaces.
The space of n-dimensional row vectors admits a natural scalar product. We prove the following:

**Theorem 7.** Let \( p \) be an odd prime and let \( A = \text{GF}(p)^n \). The space \( A \) is covered by the hyperplanes \( x_1 \perp, x_2 \perp, \ldots, x_k \perp \) if and only if for all vectors \( v \in A \) the number of 0-1 combinations of the vectors \( x_1, x_2, \ldots, x_k \) resulting \( v \) is even.

**Conjecture (Alon, Jaeger, Tarsi).** Let \( F \) be a finite field with \( q > 3 \) elements and let \( M \) be a nonsingular \( n \times n \) matrix over \( F \). Then there exists a nowhere zero (column) vector \( x \) (i.e. each component of \( x \) is non-zero) such that the vector \( Mx \) is also a nowhere zero vector.

With an elegant application of the polynomial method of Alon, Nathanson and Ruzsa (see: [2]) Alon and Tarsi prove [3] that the latter conjecture holds if \( F \) is a proper extension of some prime field \( \text{GF}(p) \). Actually they prove more. For example, from their results follows that if \( v \) is an arbitrary (column) vector and \( M \) is a nonsingular matrix over \( F \) then there exists a nowhere zero vector \( x \) such that \( Mx - v \) is a nowhere zero vector. It is reasonable to believe that the same statement holds over \( \text{GF}(p) \) where \( p \) is prime number and is bigger than 3. This conjecture will be called the choosability version of the Alon-Jaeger-Tarsi conjecture.

**Proposition 8.** A positive answer of conjecture 7 implies the Alon-Jaeger-Tarsi conjecture for \( F = \text{GF}(p) \) where \( p \geq c^2 \).

In Chapter 7, we discuss minimal hyperplane coverings.

**Definition 9.** Let \( V \) be an \( n \)-dimensional vectorspace over the finite field \( \text{GF}(q) \). Let \( h_q(n) \) denote the minimal number \( k \) such that there is collections of \( k \) hyperplanes (affine hyperplanes) of \( V \) which forms a minimal covering system and the intersection of these hyperplanes (the hyperplanes corresponding to these affine hyperplanes) is trivial.

Obviously we have \( h_q(n) \geq l_q(n) > n \). Let

\[
1 + \varepsilon_q = \inf_n l_q(n)/n.
\]

We conjecture that \( \varepsilon_q > 0 \) if \( q \) is an arbitrary prime power bigger than 2. This conjecture can be formulated in the following nice, self contained way.

**Conjecture 10.** Assume that for some prime power \( q > 2 \) the \( \text{GF}(q) \) vectorspace \( V \) is covered irredundantly by \( k \) affine hyperplanes \( H_1 + v_1, H_2 + v_2, \ldots, H_k + v_k \). Then the codimension of the intersection \( \bigcap_i H_i \) is at most \( k/(1 + \varepsilon_q) \) for some fixed positive constant \( \varepsilon_q \) which depends only on \( q \).

Using a result of Alon and Tarsi about nowhere zero points [3], we prove the following.

**Theorem 11.** If \( q \) is not a prime number then \( \varepsilon_q \geq \frac{1}{2} \).

The \( p = 3 \) case of Conjecture 10 is especially interesting because it is strongly related to the next two conjectures.

**Weak three flow conjecture** There exits a fixed natural number \( k \) such that if a graph \( G \) is at least \( k \)-connected then it admits a nowhere zero 3-flow.

It is well known that the next conjecture (for \( p = 3 \)) would imply the weak 3-flow conjecture.
Additive basis conjecture (Jaeger, Linial, Payan, Tarsi). For every prime \( p \) there is a constant \( c(p) \) depending only on \( p \) such that if \( B_1, B_2, \ldots, B_{c(p)} \) are bases of the \( GF(p) \) vectorspace \( V \) then all elements of \( V \) can be written as a zero-one linear combination of the elements of the union (as multisets) of the previous bases.

We show that \( \varepsilon_3 > 0 \) is equivalent with the additive basis conjecture for \( p = 3 \). For a prime number \( p > 3 \) we show that \( \varepsilon_p > 1 \) implies the choosability version of the Alon-Jaeger-Tarsi conjecture and that the latter one implies \( \varepsilon_p \geq 0.5 \). Note that Conjecture 5 implies that \( \varepsilon_p \geq \log_2 p - 1 \).

2. Notation and basics

Let \( A \) be a finite Abelian group. A linear character of \( A \) is a homomorphism from \( A \) to \( \mathbb{C}^* \). The linear characters of \( A \) are forming a group under the point wise multiplication (which is isomorphic to \( A \)) and they are forming a basis in the vector space of all function \( f : A \to \mathbb{C} \). The trivial character (which maps all elements of \( A \) to 1) will be denoted by \( 1_A \). The kernel of a linear character \( \chi \) is the set of those group elements \( g \in A \) for which \( \chi(g) = 1 \). We denote the kernel of \( \chi \) by \( \ker(\chi) \). It is easy to see that the subgroup \( H \leq A \) is the kernel of some linear character \( \chi \) if and only if \( A/H \) is cyclic.

The group algebra \( \mathbb{C}A \) consists of the formal linear combinations of the group elements. The fact that some Abelian groups are imagined as additive structures can cause some confusion because the concept of group algebra suggests that the group operation is the multiplication. For example we will work in the group algebra of the additive group of a finite vector space \( V \). In this structure all vectors from \( V \) are linearly independent and the group algebra \( \mathbb{C}V^+ \) consists of the formal \( \mathbb{C} \)-linear combinations of the elements of \( V \). The product of two vectors \( v_1 \) and \( v_2 \) is the vector \( z = v_1 + v_2 \) with coefficient 1. If we add together \( v_1 \) and \( v_2 \) in the group algebra then it has nothing to do with the element \( z \). Another source of confusion is that the identity element of the group algebra is the zero vector with coefficient one. The identity element of the group algebra is always denoted by \( 1 \). For a good reference about characters and group algebras see [5].

Let \( V \) be an \( n \) dimensional vector space. A Hyperplane of \( V \) is a subspace of co-dimension 1. We say that the hyperplanes \( H_1, H_2, \ldots, H_k \) are independent (or the set \( \{H_1, H_2, \ldots, H_k\} \) is independent) if the co-dimension of their intersection is \( k \). If \( V \) is represented as the space of row vectors of length \( n \) then there is a natural scalar product on \( V \) defined by \( (x, y) = \sum_{i=1}^{n} x_i y_i \). The vectors \( x_1, x_2, \ldots, x_k \in V \) are linearly independent if and only if the hyperplanes \( x_1^+, x_2^+, \ldots, x_k^+ \) are linearly independent. If \( V \) is a row space, then the usual basis will be always denoted by \( b_1, b_2, \ldots, \) their orthogonal spaces will be denoted by \( B_1, B_2, \ldots \) and we call them basis hyperplanes. An affine hyperplane is the set \( H + v \) where \( H \) is a hyperplane and \( v \) is a vector. If \( A = H + v \) is an affine hyperplane, we say that \( H \) is the hyperplane corresponding to \( A \). A collection of affine hyperplanes \( A_1, A_2, \ldots, A_k \) is called independent if the corresponding hyperplanes are independent.
3. Proof of Theorem 4.

Let $\Omega = \{H_1x_1, H_2x_2, \ldots, H_kx_k\}$ be a coset system of the Abelian group $A$. We say that $\bigcap_{i=1}^{k} H_i$ is the subgroup intersection of $\Omega$. If $S$ is a subset of $A$, we say that $S$ is covered by $\Omega$ if it is contained in the union of the elements of $\Omega$. Let $M$ be a subgroup of $A$. We denote by $\Omega/M$ the coset system in $A/M$ consisting of the images of the cosets $H_1x_1, H_2x_2, \ldots, H_kx_k$ under the homomorphism $A \to A/M$. By abusing the notation, we denote by $\Omega \cap M$ the system consisting of the cosets $H_1x_1 \cap M$, $H_2x_2 \cap M$, $\ldots$, $H_kx_k \cap M$.

Proof of theorem 4. For a natural number $n$ with prime decomposition $p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ let $\lambda(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_m$. We prove Theorem 4 by induction on the order of the Abelian group $|A|$. During the proof we will frequently use the fact that the coset structure of $A$ is translation invariant. Let $\Omega = \{H_1x_1, H_2x_2, \ldots, H_kx_k\}$ be a minimal coset covering system of $A$ with trivial subgroup intersection. We have to show that $k \geq 1 + \lambda(|A|)$.

Let $K$ be a maximal subgroup of $A$ containing $H_1$. Note the $K$ has prime index in $A$ and so $\lambda(|A|) = \lambda(|K|) + 1$. Using the fact that any translation of the system $\Omega$ is again a minimal coset covering system with trivial subgroup intersection, we can assume that $x_1 \notin K$. This means that $H_1x_1$ is disjoint from $K$ and that $H_2x_2, \ldots, H_kx_k$ covers $K$. Let $\Omega_1 \subseteq \{H_2x_2, \ldots, H_kx_k\} \subset \Omega$ be minimal with the property that it covers $K$. There are two possibilities. The first one is that the subgroup intersection of $\Omega_1$ is trivial. In this case the subgroup intersection of $\Omega_1 \cap K$ is also trivial and then by induction we can deduce that

$$k \geq 1 + f(K) \geq 2 + \lambda(|K|) = 1 + \lambda(|A|)$$

which finishes the proof.

The second one is that the subgroup intersection of $\Omega_1$ is not trivial. Let $M_1$ denote the subgroup intersection of $\Omega_1$. Since the factor group $K/(K \cap M_1)$ is covered minimally by $\Omega_1/M_1$ with trivial subgroup intersection, we have by induction that $|\Omega_1/M_1| = |\Omega_1| \geq 1 + \lambda(K/(K \cap M_1))$. Let $y_1$ be an element of $A$ which is not covered by the cosets in $\Omega_1$. Clearly the whole coset $M_1y_1$ does not intersect any coset from $\Omega_1$. Let $\Omega_2 \subset \Omega$ be a minimal covering system for $M_1y_1$ and let $M_2$ be the subgroup intersection of $\Omega_1 \cup \Omega_2$. Using translation invariance we have that $\Omega_2 \geq 1 + \lambda(M_1/M_2)$.

Now we define a process. Assume that the $\Omega_i$, $M_i$ and $y_i$ is already constructed for $1 \leq i \leq t$ and the subgroup intersection $M_i$ is still not trivial. Let $\Omega_{t+1} \subset \Omega$ be a minimal covering for $M_ty_t$. Let $M_{t+1}$ denote the subgroup intersection of the system $\bigcup_{i=1}^{t+1} \Omega_i$. If $M_{t+1}$ is not trivial then let $y_{t+1}$ be an element which is not covered by the system $\bigcup_{i=1}^{t+1} \Omega_i$.

Using the induction hypothesis and translation invariance we get that $|\Omega_{t+1}| \geq 1 + \lambda(M_t/M_{t+1})$. Assume that $M_t$ is trivial, and thus $r$ is the length of the previous process. Now we have that

$$|\Omega| \geq \sum_{i=1}^{r} |\Omega_i| \geq r + \lambda(|K|) \geq 1 + \lambda(|A|).$$

□

Using Theorem 1. we obtain precise result for $(C_2)^n$. 
Corollary 12. $f((C_2)^n) = n + 1$

Proof. Theorem 1. implies that $f((C_2)^n) \geq n + 1$. Let $H_i$ $(1 \leq i \leq n)$ be the subgroup consisting of all elements with 0 at the i-th place. The group $(C_2)^n$ is the union of the groups $H_i$ and the element $(1, 1, \ldots, 1)$. \hfill $\Box$

4. PROOF OF THEOREM 5.

For a natural number $n$ with prime decomposition $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, let $\tau(n) = \sum_{i=1}^{s} \alpha_i(p_i - 1)$. Let $\phi(A)$ denote the smallest number $k$ for which there is a collection of cosets $H_1x_1, H_2x_2, \ldots, H_kx_k$ in the Abelian group $A$ such that

$$A \setminus \{1\} = \bigcup_{i=1}^{k} H_ix_i.$$ 

Lemma 13. $\phi(A) \leq \tau(|A|)$

Proof. We go by induction on $|A|$. Let $B < A$ be a subgroup of index $p_1$. The group $A$ is a disjoint union of $p_1$ cosets of $B$. Using the statement for $B$ we obtain the result for $A$. \hfill $\Box$

Lemma 14. Let $B$ and $C$ be two Abelian groups with $|B|, |C| = 1$. Then $\phi(B \times C) \geq \phi(B) + \phi(C)$.

Proof. If two groups have coprime order then a subgroup of their direct product is a direct product of their subgroups. It follows that for $H \leq B \times C$ and $g \in B \times C$ there are subgroups $H_1 \leq B$, $H_2 \leq C$ and elements $g_1 \in B$, $g_2 \in C$ such that $Hg = \{(h_1g_1, h_2g_2) | h_1 \in H_1, h_2 \in H_2\}$. Assume that $(B \times C) \setminus \{1\} = \bigcup_{i=1}^{k} K_ig_i$ where $K_i < B \times C$, $g_i \not\in K_i$ and $k = \phi(B \times C)$. If $K_ig_i$ intersects $(B, 1) \leq B \times C$ then it does not intersect $(1, C)$ otherwise the 1 would be an element of $K_ig_i$. (The analogous statement holds if $K_ig_i$ intersects $(1, C)$.) This implies that $k \geq \phi(B) + \phi(C)$.

Lemma 15. If $H < G$ and $g \not\in H$ then there exists a subgroup $K$ of $G$ such that $H \leq K$, $g \not\in K$, $G/K$ is cyclic.

Proof. Let $K$ be maximal with the property $H \leq K < G$, $g \not\in K$. In the factor group $G/K$ every nontrivial subgroup $K_2$ contains $Kg$ otherwise the preimage of $K_2$ under the homomorphism $G \to G/K$ would be bigger than $K$ and would not contain $g$. It follows that $G/K$ can’t be a direct product of two proper subgroups because one of them would not contain $Kg$. Using the structure theorem of finite Abelian groups we obtain that $G/K$ must be cyclic of prime power order. \hfill $\Box$

Lemma 16. If $P$ is an Abelian group of order $p^\alpha$ for some prime $p$ and integer $\alpha$, then $\phi(P) \geq \alpha(p-1)$.

Proof. Let $k = \phi(P)$ and $P \setminus \{1\} = \bigcup_{i=1}^{k} H_ig_i$ (where $g_i \not\in H_i$). Using Lemma 15 we obtain that there are subgroups $K_i$ $(1 \leq i \leq k)$ with $H_i \leq K_i$, $g_i \not\in K_i$ and $P/K_i$ is cyclic for all $1 \leq i \leq k$. Now we have $P \setminus \{1\} = \bigcup_{i=1}^{k} K_ig_i$ and for each $K_i$ there exists a linear character $\chi_i$ of $P$ such that $\ker(\chi_i) = K_i$. Clearly the product $\prod_{i=1}^{k} (\chi_i - \chi_i(g_i))1_P)$ takes the value zero on every element $1 \neq g \in P$ but it is nonzero on the element 1. From this we obtain the following equality

$$\prod_{i=1}^{k} (\chi_i - \chi_i(g_i))1_P) = \prod_{i=1}^{k} (1 - \chi(g_i))/|P|)(\sum_{\chi \in \text{Irr}(P)} \chi)$$
The linear characters of $P$ are forming a basis of the vector space of $P \to \mathbb{C}$ functions, and thus after expanding both sides of the above equation, the coefficients of the characters must coincide. On the left hand side each coefficient is a sum of roots of unities thus they are algebraic integers. On the right hand side every character has coefficient $\prod_{i=1}^{k} (1 - \chi(g_i))/|P|$, and thus this number is an algebraic integer. The $|P|$-th cyclotomic field $F$ is a normal extension of $\mathbb{Q}$, and the degree of the field extension $F/\mathbb{Q}$ is $p^{n-1}(p - 1)$. Using the fact that the Galois norm of an algebraic integer is an integer we deduce that $n(|P|) = p^{\alpha_{p-1}(p-1)}$ divides $\prod_{i=1}^{k} n(1 - \chi(g_i))$ where $n(x)$ denotes the Galois norm of $x$ in the field extension $F/\mathbb{Q}$. An easy calculation shows that $n(1 - \chi(g_i)) = p^{\alpha_{p-\operatorname{logp}(\alpha(\chi(g_i)))}} \leq p^{\alpha_{n-1}}$ where $\alpha(\chi(g_i))$ denotes the multiplicative order of $\chi(g_i)$. The last inequality completes the proof.

Proof of Theorem 5. According to Lemma 13 it is enough to prove that
\[ \phi(A) \geq \tau(|A|). \]
We go by induction on $|A|$. If $|A|$ is a prime power then Lemma 10 yields the result. If $A$ is not a prime power then $A = B \times C$ where $|B|, |C| = 1$ and using the statement for $B$ and $C$, Lemma 14 completes the proof.

proof of Corollary 6. Let $g$ be an element of $H_i x_i$ which is not covered by $H_j x_j$ for all $j \neq i$. Lemma 13 shows that there is a coset system $\Omega$ consisting of $\tau(|H_i|)$ cosets whose union is $H_i x_i \setminus \{g\}$. The union of the system $\Omega \cup \{H_j x_j \mid j \neq i, 1 \leq j \leq k\}$ is $A \setminus \{g\}$, so translating it with $g^{-1}$ we can apply Theorem 5. We obtain that $k - 1 + \tau(|H_i|) \geq \tau(|A|)$ and thus $k \geq 1 + \tau(|A:H_i|)$. It means in particular that $k \geq 1 + \log_2 |G:H_i|$.

5. Hyperplane coverings and characters

Now we describe our character theoretical approach to hyperplane covering problems. Let $p$ be a fixed prime number, let $\omega = e^{2\pi i/p}$ and let $A = (C_p)^n$. We regard $A$ as the $n$-dimensional row vector space over $GF(p)$.

Lemma 17. The space $A$ is covered by the hyperplanes $x_1 \perp, x_2 \perp, \ldots, x_k \perp$ if and only if the equation
\[ (x_1 - 1)(x_2 - 1) \ldots (x_k - 1) = 0. \]
is satisfied in the group algebra $\mathbb{C}[A]$ where $1$ denote the identity element of $A$ (which is actually the zero vector, if we think of $A$ as a vector space).

Note that substraction in the previous lemma is the group algebra substraction and not the vector substraction.

Proof. The function
\[ f : (x_1, x_2, \ldots, x_n) \to ((y_1, y_2, \ldots, y_n) \to \omega^{x_1 y_1 + x_2 y_2 + \ldots + x_n y_n}) \]
gives an isomorphism between $A$ and its character group $A^*$. Moreover $f$ can be uniquely extended to an algebra isomorphism between the group algebra $\mathbb{C}[A]$ and the character algebra $\mathbb{C}[A^*]$. Note that the character algebra is just the algebra of all functions $A \to \mathbb{C}$ with the point wise multiplication. Clearly we have that
\[ x^\perp = \ker(f(x)) \text{ for all row vectors } x \in A. \] It follows that the space \( A \) is covered by the hyperplanes \( x_1^\perp, x_2^\perp, \ldots, x_k^\perp \) if and only if
\[ (f(x_1) - 1_A)(f(x_2) - 1_A) \cdots (f(x_k) - 1_A) = 0. \]
Applying \( f^{-1} \) to both side of the previous equation we obtain the statement of the lemma.

The previous lemma gives a characterization of covering systems in terms of orthogonal vectors. Our following theorem gives a group algebra free characterization of coverings in terms of orthogonal vectors if \( p \) is an odd prime.

**Theorem 18.** Let \( p \) be an odd prime and let \( A = (C_p)^n \). The space \( A \) is covered by the hyperplanes \( x_1^\perp, x_2^\perp, \ldots, x_k^\perp \) if and only if for all vectors \( v \in A \) the number of 0-1 combinations of the vectors \( x_1, x_2, \ldots, x_k \) resulting \( v \) is even.

**Proof.** Let \( F \) be the algebraic closure of the field with two elements. Since \( p \) is odd, \( F \) contains a \( p \)-th root of unity \( \omega \) and thus we can repeat everything what we did over \( C \). We obtain that the space \( A \) is covered by the hyperplanes \( x_1^\perp, x_2^\perp, \ldots, x_k^\perp \) if and only if the equation
\[ (x_1 - 1)(x_2 - 1) \cdots (x_k - 1) = 0. \]
holds in the group algebra \( F[A] \). Since \( F \) has characteristic 2 we don’t have to care about the signs in the previous formula. The rest of the proof is straightforward by expanding the formula.

6. **on the Alon-Jaeger-Tarsi conjecture**

The following lemma shows the relationship between hyperplane coverings and the Alon-Jaeger-Tarsi conjecture.

**Lemma 19.** Let \( p \) be a fixed prime number and let \( n \) be a fixed natural number. The following statements are equivalent.

1. The \( n \)-dimensional vector space over \( GF(p) \) can’t be covered by the union of two independent sets of hyperspaces.
2. If \( M \) is a non singular \( n \times n \) matrix over \( GF(p) \) then there exists a nowhere zero vector \( x \) such that \( Mx \) is also a nowhere zero vector.

**Proof.** (1)\( \Rightarrow \) (2) Let \( x_1, x_2, \ldots, x_n \) denote the rows of \( M \), and let \( H_i = x_i^\perp \) for \( 1 \leq i \leq n \). Since \( M \) is non singular we have that the subspaces \( H_1, H_2, \ldots, H_n \) are independent. Let \( S_i \) be the hyperspace consisting of the row vectors with a zero at the \( i \)-th component. It follows from (1) that there exists a vector \( y \) which is not contained in the union of the spaces \( H_i, S_i \) (\( 1 \leq i \leq n \)). Clearly \( y \) is a nowhere zero vector such that \( My^T \) is also a nowhere zero vector.

(2)\( \Rightarrow \) (1)

Assume that \( V \) is an \( n \)-dimensional vector space covered by the independent hyperspace sets \( \Omega_1 \) and \( \Omega_2 \). We can assume that both \( \Omega_1 \) and \( \Omega_2 \) are maximal independent sets. It is easy to see that we can represent \( V \) as a row space such that the hyperspaces in \( \Omega_1 \) are exactly the spaces formed by all vectors with a zero at a fixed component. Let \( x_1, x_2, \ldots, x_n \) be a system of non zero vectors whose orthogonal spaces are exactly the hyperspaces in \( \Omega_2 \). It is clear that the vectors \( x_i \) (\( 1 \leq i \leq n \)) are linearly independent. Let \( M \) be a matrix such that its row vectors are \( x_i \) \( 1 \leq i \leq n \). Now \( M \) contradicts the assumption of (2).
Lemma 21. Let $M$ be an $n$ by $n$ matrix over the field $GF(p)$ and let $\{x_1, x_2, \ldots, x_n\}$ be the rows of $M$. Moreover let $b_i$ be the $i$-th row of the $n$ by $n$ identity matrix. Then $M$ is an AJT if and only if
\[(b_1 - 1)(b_2 - 1) \cdots (b_n - 1)(x_1 - 1)(x_2 - 1) \cdots (x_m - 1) \neq 0.\]
in the group algebra $C[V^+]$ where $V$ denote the space of $n$ dimensional row vectors. If $p$ is odd and $F$ is the algebraic closure of the field with two elements then the same statements holds if we replace $C$ by $F$.

Proof. The proof is straightforward from Lemma 12 and Theorem 18.\[\square\]

Definition 22. Let $B$ be a subset of the $n$ dimensional $GF(p)$ space $V$. Let $C(B)$ denote the set of all vectors $v$ for which the number of zero-one combinations of the elements from $B$ resulting $v$ is odd. In particular if $B$ is a linearly independent set then $C(B)$ is the set of all zero-one combinations of the elements from $B$. We say that $C(B)$ is the cube determined by the set $B$. Let $A_1, A_2, \ldots, A_n$ be two element subsets of $GF(p)$. We say that the vector set $\{(a_1, a_2, \ldots, a_n) | a_i \in A_i\}$ is a combinatorial cube in the $n$-dimensional row space.

Using our character theoretic approach we obtain the following characterization of AJT-s.

Theorem 23. Let $M$ be an $n$ by $n$ matrix over the field $GF(p)$ where $p > 2$. Let $X$ be the set formed by the rows of $M$ and let $B$ be the ordinary basis of the $n$ dimensional row-space. $M$ is an AJT if and only if the set $C(X) \cap (C(B) + v)$ has odd number of points for some vector $v$.

Proof. Let $F$ be the algebraic closure of the field with two elements. Recall that the elements of the group algebra $F[V^+]$ are formal $F$-linear combinations of the group elements.

Using Lemma 21 and that $1 = -1$ in characteristic 2 we get that $M$ is an AJT if and only if
\[
(b_1 + 1)(b_2 + 1) \cdots (b_n + 1)(x_1 + 1)(x_2 + 1) \cdots (x_m + 1) = \sum_{S_1 \subseteq \{1,2,\ldots,n\}} \sum_{S_2 \subseteq \{1,2,\ldots,m\}} \prod_{i \in S_1} b_i \prod_{i \in S_2} x_i
\]
is not zero in the group algebra $F[V^+]$. Let $y$ be a fixed vector in $V$. To determine the coefficient of $y$ in the previous product we have to compute the number of the solutions of the following equation in $F[V^+]$ where $S_1 \subseteq \{1, 2, \ldots, n\}$, $S_2 \subseteq \{1, 2, \ldots, m\}$.

$$
\prod_{i \in S_1} b_i \prod_{i \in S_2} x_i = y
$$

Since it does not contain any addition, it can be translated into the following equation in $V$.

$$
\sum_{i \in S_1} b_i + \sum_{i \in S_2} x_i = y
$$

The number of the solutions of the previous equation is clearly

$$
|C(X) \cap (-C(B) + y)| = |C(X) \cap (C(B) - (1, 1, \ldots, 1) + y|
$$

and the parity of this number gives the coefficient of $y$. It follows that $M$ is an AJT if and only if there is a vector $v$ for which $|C(X) \cap C(B) + v|$ is an odd number. □

As a consequence of the previous lemma we obtain the following.

**Corollary 24.** Let $M$ be an $n$ by $n$ matrix, and let $X$ be the set formed by the rows of $M$. Then $M$ is an AJT if and only if there is a combinatorial cube which has odd intersection with $C(X)$.

**Proof.** It is clear that if $M$ is an AJT and $N$ is obtained from $M$ by multiplying the rows by non zero scalars then $N$ is an AJT too. Applying the previous theorem to all possible such $N$ the proof is straightforward. □

7. MINIMAL HYPERPLANE COVERINGS

**Lemma 25.** Let $q$ be a prime power which is not a prime. Let $V$ be an $n$ dimensional vector space over $GF(q)$ and let $B_1$ and $B_2$ be two bases of $V$. Then each vector $v \in V$ can be written as a nowhere zero linear combination (i.e. neither coefficient is zero) of $B_1 \cup B_2$.

**Proof.** We write each vector as a row vector in the basis $B_1$. Let $M$ be a matrix whose rows are the vectors from $B_2$. According to the results of Alon and Tars in [3] there is a nowhere zero (row) vector $x$ such that $v - xM$ is a nowhere zero vector $y$. It means that $v = xM + y$ which yields the required linear combination. □

**Lemma 26.** Let $M$ be a matroid on the set $E$. If $|E| \geq r(E)k$ for a natural number $k$ then there is a subset $X \subseteq E$ such that $X$ as a matroid has $k$ disjoint bases.

**Proof.** Let $X$ be a minimal subset of $E$ with the property $|X| \geq r(X)k$. According to Edmond’s matroid packing theorem, the maximal number of pairwise disjoint bases in $X$ equals

$$
\min \left\{ \left| \frac{|X| - |Y|}{r(X) - r(Y)} \right| : Y \subseteq X, r(Y) < r(X) \right\}.
$$

The minimality of $X$ implies that for an arbitrary subset $Y \subseteq X$ with $r(Y) < r(X)$ we have that $|Y| < r(Y)k$. It follows that $|Y| - r(Y)k < |X| - r(X)k$ and so $(|X| - |Y|)/(r(X) - r(Y)) > k$. □
Theorem 27. Let \( q \) be a prime power which is not a prime. Let \( V \) be a vector space over \( GF(q) \) which is covered irredundantly by \( k \) affine hyperplanes \( H_i + v_i \) \((1 \leq i \leq k)\). Then the co-dimension of the intersection of the hyperplanes \( H_i \) \((1 \leq i \leq k)\) is less than \( \frac{2p}{3k} \).

Proof. It is easy to see that the space \( V/\bigcap_{1\leq i \leq n} H_i \) is covered irredundantly by the images of \( H_i + v_i \) so we can assume that \( \bigcap H_i \) is trivial. We go by contradiction. Assume that \( \dim(V) = n \geq \frac{2p}{3k} \). Without loss of generality we can assume that \( H_1, H_2, \ldots, H_n \) are independent hyperplanes, and \( v_i \) is the zero vector for \( 1 \leq i \leq n \). We can choose a basis \( B = \{ b_i | 1 \leq i \leq n \} \) such that the previous hyperplanes are exactly the orthogonal spaces of the basis elements. Let \( W = \bigcap_{i> n} H_i \). From our assumption it follows that \( \dim(V/W) \leq k - n \leq \frac{1}{3}n \). Let \( p_i \) be the image of \( b_i \) \((i = 1 \ldots n)\) under the homomorphism \( V \to V/W \). From Lemma 26 it follows that there are two disjoint index sets \( I_1, I_2 \subseteq \{1 \ldots n\} \) such that \( \{p_i | i \in I_1\} \) and \( \{p_i | i \in I_2\} \) are bases of the same subspace \( T \leq W \). Let \( j \) be an element in \( I_1 \), and let \( x = \sum_{i=1}^{n} \lambda_ib_i \) be an element in \( H_j \leq V \) which is not covered by \( H_i + v_i \) for \( i \neq j, 1 \leq i \leq k \). Since the hyperplanes \( H_i \) \((1 \leq l \leq n)\) does not cover \( x \) for \( l \neq j \) it follows that \( \lambda_i \neq 0 \) for all \( l \neq j \). Let \( y = \sum_{i=1}^{n} \lambda_ip_i \) and \( y_1 = \sum_{i \in I_1 \cup I_2} \lambda_ip_i \). Lemma 28 implies that \( y_1 \) can be written as a nowhere zero linear combination of the vectors \( p_i \) \((i \in I_1 \cup I_2)\) and thus \( y \) can be written in the form \( \sum_{i=1}^{n} \mu_ip_i \) where \( \mu_i \neq 0 \) for \( 1 \leq i \leq n \). Let \( z = \sum_{i=1}^{n} \mu_ib_i \). The vector \( z \) is a preimage of \( y \) under the homomorphism \( V \to W \) and so \( z - v \in W \). Since \( z \) is a nowhere zero vector in the basis \( B \) we have that it is not contained in \( H_1, H_2, \ldots, H_n \). Let \( t > n \) be a number for which \( H_t + v_t \) contains \( z \). By definition of \( W \), \( H_t + v_t \) contains the set \( z + W \). This contradicts the assumption that \( v \) is covered only by \( H_j \).

Note that the condition on \( q \) was hidden in Lemma 26 when we used a result of [3]. This means that the choosability version of the Alon-Jaeger-Tarsi conjecture would imply the analogue statement for an arbitrary prime number bigger than 3. It can also be seen (from the previous proof) that the following weak conjecture implies \( \varepsilon_p > 0 \) if \( p > 2 \).

Weak conjecture For every prime \( p > 2 \) there is a constant \( c_2(p) \) depending only on \( p \) such that if \( B_1, B_2, \ldots, B_{c_2(p)} \) are bases of the \( GF(p) \) vectorspace \( V \) then all elements of \( V \) can be written as a nowhere zero linear combination of the elements of the union (as multisets) of the previous bases.

The next result shows that the weak conjecture is equivalent with \( \varepsilon_p > 0 \).

Lemma 28. If \( \varepsilon_p > 0 \) then the weak conjecture holds for \( p \) with any \( c_2(p) = k > \frac{1+\varepsilon_p}{\varepsilon_p} \).

Proof. We go by contradiction. Assume that the weak conjecture is not true with \( c_2(p) = k \). Let \( n \) be the minimal dimension where the conjecture is false (with \( c(p) = k \)) and assume that the bases \( B_1, B_2, \ldots, B_k \) are forming a counter example in the \( n \) dimensional space \( V \). Let \( M \) be an \( n \times nk \) matrix whose columns are the vectors from the previous bases. According to our assumption there is a vector \( v \) such that there is no nowhere zero vector \( x \) with \( Mx = v \). Let say that an index set \( I \subseteq \{1\ldots nk\} \) is a blocking set if for all \( x \in GF(p)^{nk} \) with \( Mx = v \) there is a \( j \in I \) such that the \( j \)-th coordinate of \( x \) is zero. Let \( I \) be a minimal blocking set. First we prove by contradiction that \( I = \{1\ldots nk\} \). Assume that \( P = \{1\ldots nk\} \setminus I \) is not empty. Let \( j \) be an element of \( P \), let \( y \) be the \( j \)-th column of \( M \) and let \( W \) be the
factor space \( V/\langle y \rangle \). Let \( P_1, P_2, \ldots, P_k \) be the images of the bases \( B_1, B_2, \ldots, B_k \) under the homomorphism \( V \to W \). It is clear that each \( P_i \) contains a basis for \( W \) and by minimality of \( n \) it follows that each vector \( x \in W \) is a nowhere zero linear combination of the elements in \( P_1, P_2, \ldots, P_k \). In particular the image of \( v \) can be written as such a nowhere zero combination. It means that there is a vector \( x \in \text{GF}(p)^{nk} \) for which \( Mx = v \) and all but the \( j \)-th coordinate of \( x \) are not zero. It contradicts the assumption that \( I \) is a blocking set. Now we have that \( \{1..nk\} \) is a minimal blocking set and thus for each \( j \in \{1..nk\} \) there is a vector \( x_j \in \text{GF}(p)^{nk} \) such that all but the \( j \)-th coordinate of \( x_j \) are not zero and \( Mx_j = v \). Let \( U \) be the affine hyperplane consisting of all \( x \) for which \( Mx = v \). For all \( j \in \{1..nk\} \) let \( H_j \leq U \) be the affine hyperplane consisting of those elements \( x \) whose \( j \)-th coordinate is zero. Now the affine space \( U \) is covered irredundantly by the affine hyperplanes \( H_j \). Since \( \dim(U) = n(k - 1) \) it follows that \( \frac{k}{k-1} \geq 1 + \epsilon_p \). □

8. COLORINGS AND FLOWS

In this section we outline the relation between colorings, flows and hyperplane coverings. Let \( G \) be a finite, loopless graph with vertex set \( V(G) \) and edge set \( E(G) \). Let \( q \) be a prime power and let \( W \) be the vector space of all functions \( V(G) \to \text{GF}(q) \). For two functions \( f, g \in W \) we define their scalar product by
\[
(f, g) = \sum_{v \in V(G)} f(v)g(v)
\]
We associate a vector \( v_e \in W \) to each edge \( e \in E(G) \) such that \( v_e \) takes 1 and \(-1\) at the two different endpoints of \( e \), and it takes 0 everywhere else.

**Lemma 29.** \( G \) is colorable with \( q \) colors if and only if the orthogonal spaces of the vectors \( v_e \) do not cover the whole space \( W \).

**Proof.** We can think of \( W \) as the set of all possible (not necessary proper) colorings of \( G \). It is clear that a vector \( v \in W \) is orthogonal to \( v_e \) for some \( e \in E(G) \) if and only if \( v \) takes the same value at the endpoints of \( e \). It means that \( G \) has a proper coloring with \( q \) colors if and only if there is a vector \( v \in W \) which is not contained in any of the spaces \( v_e^\perp \).

Combining the previous lemma with Theorem 18 one gets the following peculiar characterization of colorability.

**Proposition 30.** If \( q \) is an odd prime then \( G \) can be colored by \( q \) colors if and only if there is a vector \( v \in W \) such that the number of zero-one combinations of the vectors \( v_e \) resulting \( v \) is odd.

Note that the space \( v_e^\perp \) depends only on the one dimensional space spanned by \( v_e \). It means that the vectors \( v_e \) can be replaced by any nonzero representative from their one dimensional spaces, which gives an even stronger version of the previous proposition. We also note that the ”if” direction remains true if we delete the condition that \( q \) is a prime number.
Let $G = (V, E)$ be a directed graph and let $A$ be an Abelian group. An $A$-flow on $G$ is a function $f : E \rightarrow A$ such that for all $v \in V$

$$\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e).$$

where $\delta^+(v)$ denote the set out going edges and $\delta^-(v)$ denote the set of in coming edges. If $f(e) \neq 0$ for all $e \in E$ then $f$ is called a nowhere zero flow. Clearly the existence of a nowhere zero flow on $G$ is independent of the orientation of $G$. If $G$ is undirected we will say that it admits a nowhere zero $A$-flow if some (an thus every) orientation of it admits a nowhere zero $A$-flow. Let $G$ be a fixed graph with a fixed direction and consider the set $B$ of all possible flows on $G$. It is clear that $B$ is a subgroup of the direct product $A^E$ and one can prove easily that $B \cong A^{ |E| - |V| + m}$ where $m$ denotes the number of connected components of $G$. For each edge $e$ there is a subgroup $B_e \leq B$ consisting of those flows which vanish on $e$. Clearly, $G$ has a nowhere zero flow if and only if the subgroups $B_e$ do not cover the group $B$. Moreover, it is also clear that the intersection of the subgroups $B_e$ is trivial. It means in particular that if $G$ is a graph which is "edge-minimal" respect to the property of having no nowhere zero flow (i.e. $G$ has no nowhere zero flow, but after deleting any edge, the resulting graph always has one) then the number of edges is less than $g_1(|B|)$ where $g_1$ is the function defined in the introduction. Note that if $A$ has a finite field structure, then the group $B$ can be regarded as a vectorspace over $A$ with hyperplane system $\{B_e | e \in E\}$.

9. Hierarchy of Conjectures

\[
\begin{array}{cccccccc}
\text{case } p > 3 & \text{AJT} & \text{C-AJT} & \varepsilon_p & \varepsilon_p \geq 0.5 & \varepsilon_p > 1 & \varepsilon_p \geq \log_2(p) - 1 & c_2 = 2 \\
\text{case } p = 3 & \text{AB} & \text{W} & \varepsilon_p > 0 & \text{WT} & \text{WT} & \text{AB} & \text{WT} & \text{C-AJT} & \text{choosability version of AJT} \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{AJT} & \text{Alon-Jaeger-Tarsi conjecture} & \\
\text{C-AJT} & \text{choosability version of AJT} & \\
\text{AB} & \text{additive basis conjecture} & \\
\text{W} & \text{weak conjecture} & \\
\text{WT} & \text{weak three flow conjecture} & \\
\end{array}
\]

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