In this article we study the bordism groups of normally nonsingular maps \( f : X \to Y \) defined on pseudomanifolds \( X \) and \( Y \). To characterize the bordism of such maps, inspired by the formula given by Stong, we give a general definition of Stiefel-Whitney numbers defined on \( X \) and \( Y \) using the Wu classes defined by Goresky and Pardon in [9] and we show that in several cases the cobordism class of a normally nonsingular map \( f : X \to Y \) guarantees that these numbers are zero.

1. Introduction

The ambiental bordism of manifolds was presented by Thom in [14]. Conner and Floyd [4] extended this theory to bordism of maps between closed manifolds and there is a classical work of Stong [13], where the bordism class of maps between manifolds \( f : X \to Y \) is characterized in terms of so-called Stiefel-Whitney numbers of \( (f, X, Y) \).

Concerning the bordism on singular varieties, Siegel in [12] computed the bordism groups of \( \mathbb{Q} \)-Witt spaces, showing that in non trivial cases they are equal to the Witt groups. Pardon [11] computed the bordism groups of the “Poincaré duality spaces” defined by Goresky and Siegel in [10].

In this article we extend this notion to bordism groups of normally nonsingular maps \( f : X \to Y \) between pseudomanifolds. For closed smooth manifolds \( X \) and \( Y \) this definition becomes the Stong’s definition of cobordism of maps \( (f, X, Y) \) given in [13].

To characterize the bordism of such maps, inspired by the formula given by Stong, we give a general definition of Stiefel-Whitney numbers defined on \( X \) and \( Y \) using the Wu classes defined by Goresky and Pardon in [9] and we show that, in several cases the cobordism class of the map \( f \) guarantees that these numbers are zero. More precisely, we show how to extend the result of Stong in the case of normally nonsingular maps \( f : X \to Y \) in the following situations: Firstly we consider the case \( X \) is a locally orientable \( \mathbb{Z}_2 \)-Witt space of pure dimension \( a \) and \( Y \) an \( b \)-dimensional smooth manifold. Then we consider the case \( X \) is an \( a \)-dimensional smooth manifold and \( Y \) a locally orientable \( \mathbb{Z}_2 \)-Witt space of pure dimension \( b \). To conclude, we consider the general case where \( X \) and \( Y \) are locally orientable \( \mathbb{Z}_2 \)-Witt spaces.
2. Intersection homology

2.1. Pseudomanifolds.

Definition 2.1. A pseudomanifold (without boundary) of dimension $a$ is a compact space $X$ which is the closure of the union of $(a-1)$-dimensional simplices in any triangulation of $X$, and each $(a-1)$ simplex is a face of exactly two $a$-simplices.

A pseudomanifold (with boundary) of dimension $a$ is a compact space $X$ which is the closure of the union of $(a-1)$-dimensional simplices in any triangulation of $X$, and each $(a-1)$ simplex is a face of either one or two $a$-simplices. The boundary consists of simplices which are faces of only one $a$-simplex.

Every pseudomanifold admits a piecewise linear (P.L. for short) stratification, which is a filtration by closed subspaces $\emptyset \subset X_0 \subset X_1 \subset \ldots \subset X_{a-2} \subset X_a = X$, such that for each point $x \in X_i - X_{i-1}$ there is a neighborhood $U$ and a P.L. stratum preserving homeomorphism between $U$ and $\mathbb{R}^{a-i} \times C(L)$, where $L$ is the link of the stratum $X_i - X_{i-1}$ and $C(L)$ denotes the cone on $L$. Thus, if $X_i - X_{i-1}$ is non empty, it is a (non necessarily connected) manifold of dimension $i$, and is called the $i$-dimensional stratum of the stratification.

The singular part, denoted by $\Sigma X$, is contained in the element $X_{a-2}$ of the filtration.

Definition 2.2. A map $f : X \to Y$ between pseudomanifolds is normally nonsingular if there exists a diagram

\[
\begin{array}{c}
N \xrightarrow{i} Y \times \mathbb{R}^k, \\
\pi \downarrow \downarrow p \\
X \xrightarrow{f} Y
\end{array}
\]

where $\pi : N \to X$ is a vector bundle with zero-section $s$, $i$ is an open embedding, $p$ is the first projection and one has $f = p \circ i \circ s$. The bundle $N = N_f$ is called the normal bundle.

2.2. Intersection Homology and Cohomology. All homology and cohomology groups will be considered with $\mathbb{Z}_2$ coefficients. Reference for this section is Goresky-MacPherson original paper [7].

The notion of perversity is fundamental for the definition of intersection homology and cohomology. A perversity $\underline{p}$ is a multi-index sequence of integers $(p(2), p(3), \ldots)$ such that $p(2) = 0$ and $p(c) \leq p(c + 1) \leq p(c) + 1$, for $c \geq 2$. Any perversity $\underline{p}$ lies between the zero perversity $\underline{0} = (0, 0, 0, \ldots)$ and the total perversity $\underline{t} = (0, 1, 2, 3, \ldots)$. In particular, we will use the lower middle perversity, denoted $\underline{m}$ and the upper middle perversity, denoted $\underline{n}$, such that

\[
\underline{m}(c) = \left\lfloor \frac{c-2}{2} \right\rfloor \quad \text{and} \quad \underline{n}(c) = \left\lfloor \frac{c-1}{2} \right\rfloor, \quad \text{for} \ c \geq 2.
\]

Let $X$ be an $a$-dimensional pseudomanifold and $\underline{p}$ a perversity. The intersection homology groups with $\mathbb{Z}_2$ coefficients, denoted $IH_{\underline{p}}^i(X)$, are the homology groups of the chain complex

\[
IC_{\underline{p}}^i(X) = \left\{ \xi \in C_i(X) \mid \begin{array}{c}
\dim(|\xi| \cap X_{a-c}) \leq i - c + p(c) \text{ and} \\
\dim(|\partial \xi| \cap X_{a-c}) \leq i - 1 - c + p(c)
\end{array} \right\},
\]

where
where $C_i(X)$ denotes the group of compact $i$-dimensional P.L. chains $\xi$ of $X$ with $\mathbb{Z}_2$ coefficients and $|\xi|$ denotes the support of $\xi$.

In fact $C_*(X)$ is the direct limit $\lim \rightarrow C^T_*(X)$, where $C^T_*(X)$ is the simplicial chain complex with respect to a triangulation $T$ and the direct limit is taken with respect to subdivision within the family of triangulations of $X$ compatible with the filtration of $X$.

The intersection cohomology groups with $\mathbb{Z}_2$ coefficients, denoted $IH^{a-i}_\bar{p}(X)$, are defined as the groups of the cochain complex

$$IC^{a-i}_\bar{p}(X) = \left\{ \gamma \in C^{a-i}(X) \mid \dim(|\gamma| \cap X_{a-c}) \leq i - c + p(c) \text{ and } \dim(|\partial \gamma| \cap X_{a-c}) \leq i - 1 - c + p(c) \right\},$$

where $C^{a-i}(X)$ denotes the abelian group, with $\mathbb{Z}_2$ coefficients, of all $(a-i)$-dimensional P.L. cochains of $X$ with closed supports in $X$.

The main properties of intersection homology that we will use are the following:

For any perversity $\bar{p}$, the Poincaré map $PD$, cap-product by the fundamental class of $X$ naturally factorizes in the following way $[7]$:

$$H^{a-i}(X) \xrightarrow{PD} H_i(X) \xrightarrow{\omega_X} IH^{\bar{p}}_i(X).$$

where $\alpha_X$ is induced by the cap-product by the fundamental class $[X]$ and $\omega_X$ is induced by the inclusion $IC^{\bar{p}}_i(X) \hookrightarrow C_i(X)$.

For perversities $\bar{p}$ and $\bar{r}$ such that $\bar{p} + \bar{r} \leq \bar{i}$, the intersection product

$$IH^{\bar{p}}_i(X) \times IH^{\bar{r}}_j(X) \rightarrow IH^{\bar{p}+\bar{r}}_{(i+j)-a}(X)$$

is well defined.

The natural homomorphism $IH^{\bar{p}}_{\bar{p}-i}(X) \rightarrow IH^{\bar{p}}_i(X)$, cap-product by the fundamental class $[X]$, is an isomorphism.

### 3. Witt spaces and Wu classes

In this section we use definitions and notations of M. Goresky [6] and M. Goresky and W. Pardon [9]. First of all, let us fix notations in the smooth case.

Let $X$ be an $a$-dimensional manifold. We will denote by $w^i(X) \in H^i(X)$ the Stiefel-Whitney cohomology classes (S-W cohomology classes) of the tangent bundle $TX$. The Stiefel-Whitney homology classes (S-W homology classes) of $TX$ denoted by $w_{a-i}(X) \in H_{a-i}(X)$ are their images by Poincaré duality. Let $i: X \hookrightarrow V$ be the inclusion of differentiable manifolds, then one has the naturality formula $i^*(w^i(V)) = w^i(X)$.

In the singular case, the **Steenrod square operations** are defined in intersection cohomology by M. Goresky [6] §3.4] as follows:
Definition 3.1. Let $X$ be an $a$-dimensional pseudomanifold. Suppose $\bar{c}$ and $\bar{d}$ are perversities such that $2\bar{c} \leq \bar{d}$. For any $i$ with $0 \leq i \leq [a/2]$ the “Steenrod square” operation

$$S^q: IH^i_d(X) \to IH^{i+j}_d(X) \to \mathbb{Z}_2$$

is given by multiplication with the intersection cohomology $i^{th}$-Wu class of $X$:

$$v^i(X) = v^i_d(X) \in IH^i_d(X).$$

One defines $v^i(X) = 0$, for $i > [a/2]$.

Definition 3.2. ([9], Definition 10.1) A stratified pseudomanifold $X$ is a $\mathbb{Z}_2$-Witt space if for each stratum of odd codimension $2k + 1$, $IH^i_k(L) = 0$, where $L$ is the link of the stratum.

For such spaces, the middle intersection homology group satisfies the Poincaré duality over $\mathbb{Z}_2$.

In the following, we will use the notion of locally orientable Witt-space that we recall:

Definition 3.3. ([9], Definition 10.2) A stratified pseudomanifold $X$ is a locally orientable Witt space if it is both locally orientable and a $\mathbb{Z}_2$-Witt space.

Let $X$ be a $\mathbb{Z}_2$-Witt space, then the Wu classes $v^i(X)$ lift canonically to $IH^i_d(X) = IH^i_0(X)$ (see [9] §10). We denote by $v_{a-i}(X) \in IH^i_{a-i}(X)$ the (homology) $(a - i)^{th}$-Wu class of $X$, in intersection homology, dual to the Wu class $v^i(X)$ (denoted by $IV^i \in IH^i(X)$ in [6]).

Definition 3.4. ([6] [12]) One defines the Whitney classes by

$$IW_{a-i}(X) = \sum_{\ell+j=i} S^q^\ell v^j(X) \in IH^i(X) = H_{a-i}(X).$$

The pullback of the intersection cohomology Whitney class under a normally nonsingular map is given by the following theorem ([6] 5.3]):

Theorem 3.5. Let $X$ and $Y$ be $\mathbb{Z}_2$-Witt spaces and $f : X \to Y$ a normally nonsingular map with normal bundle $N_f$. Then one has, in $IH^*(X)$:

$$f^*(IW(Y)) = W(N_f) \cup IW(X)$$

where $W(N_f)$ is the Whitney cohomology class (in $H^*(X)$) of the normal bundle $N_f$.

The inclusion $j : X \hookrightarrow V$ provides an unique morphism $j^* : IH^n_{\bar{q}-i}(V) \to IH^{n-i}_q(X)$ (see [6] §(3.4)). The result comes from the commutative diagram

$$\begin{array}{ccc}
IH^n_{\bar{q}-i}(X) & \xrightarrow{j^*} & IH^n_{\bar{q}-i}(V) \\
\downarrow \cong & & \downarrow \cong \\
IH^0_i(X) & \xrightarrow{j_X^*} & IH^0_{i+1}(V)
\end{array}$$

where the bottom map $j_X^*$ is defined by the upper one. We have:

Corollary 3.6. Let us consider the inclusion $j : X \hookrightarrow V$ of the $\mathbb{Z}_2$-Witt space $X$ in a $\mathbb{Z}_2$-Witt space $V$ such that $\Sigma V \subset X$, so that the normal bundle $N_i$ is trivial. Then one has:

$$j_X^*(v_{i+1}(V)) = v_i(X).$$
4. Cobordism of maps

Definition 4.1. Let $f : X \to Y$ be a normally nonsingular map between pseudomanifolds of dimensions $a$ and $b$ respectively. The triple $(f, X, Y)$ bords if there exist:

1. Pseudomanifolds $V$ and $W$ with dimensions $a + 1$ and $b + 1$, respectively, such that $\partial V = X$ and $\partial W = Y$; $\Sigma V \subset X$ and $\Sigma W \subset Y$.

2. $F : V \to W$ normally nonsingular such that $F|_X = f$.

We will denote $(f, X, Y) = \partial(F, V, W)$.

The definition implies that $V \setminus X$ and $W \setminus Y$ are smooth manifolds. If $X$ (resp. $Y$) is a manifold, then $W$ (resp. $V$) is a manifold with smooth boundary.

If we consider $X$ and $Y$ closed smooth manifolds, this definition becomes the Strong’s definition to cobordism of maps $(f, X, Y)$ in [13]. In this case Strong defines the Stieffel-Whitney (S-W for short) numbers associated to the map $(f, X, Y)$; these numbers allow to characterize the bordism properties among such maps. We recover here results described by Stong which are necessary to better understand our main results.

Definition 4.2. [13] Let us consider a map $f : X \to Y$, where $X$ and $Y$ are manifolds of dimensions $a$ and $b$, respectively. Define $f^! : H^i(X) \to H^{i+b-a}(Y)$ in such a way that for any $\alpha \in H^i(X)$, we define $f^!(\alpha) : H^{i+b-a}_i(Y) \to \mathbb{Z}_2$ such that for each $\beta \in H^{i+b-a}_i(Y)$,

$$f^!(\alpha)(\beta) = \langle f^*(\tilde{\beta}) \cup \alpha, [X] \rangle \in \mathbb{Z}_2,$$

where $\tilde{\beta} \in H^{a-i}(Y)$ is the Poincaré dual of $\beta$.

Remark 4.3. According to Atiyah and Hirzebruch [1], the map $f^!$ can be described in the following way: let us consider $h : X \to S^s$ an imbedding of $X$ in some $s$-dimensional sphere $S^s$ and $T$ a tubular neighborhood of $(f \times h)(X)$ in $Y \times S^s$, then $f^!$ is the composition of the maps:

$$H^i(X) \xrightarrow{\varphi} H^{i+s+b-a}(T/\partial T) \xrightarrow{c^*} H^{i+s+b-a}(Y \times S^s) \simeq H^{i+b-a}(Y),$$

where $\varphi$ denotes the Thom isomorphism and $c : Y \times S^s \to T/\partial T$ is the contraction.

5. Main results

In this section we show how to extend the result in the case of singular spaces and normally nonsingular maps $f : X \to Y$. Firstly we consider the case $X$ is a locally orientable $\mathbb{Z}_2$-Witt space of pure dimension $a$ and $Y$ a $b$-dimensional smooth manifold. Then we consider the case $X$ is an $a$-dimensional smooth manifold and $Y$ is a locally orientable $\mathbb{Z}_2$-Witt space of pure dimension $b$. To conclude, we consider the general case where $X$ and $Y$ are locally orientable $\mathbb{Z}_2$-Witt spaces.

5.1. Case of a map $f : X \to Y$, with $Y$ a smooth manifold.

Let $f : X \to Y$ be a normally nonsingular map, with $X$ a locally orientable $\mathbb{Z}_2$-Witt space of pure dimension $a$ and $Y$ a $b$-dimensional smooth manifold.
Definition 5.1. Let us define the map \( f_B : IH^p_i(X) \to IH^p_i(Y) \) in such a way that the following diagram commutes

\[
\begin{array}{ccc}
H_i(X) & \xrightarrow{f_*} & H_i(Y) \\
\downarrow{\omega_X} & & \downarrow{\omega_Y} \\
IH^p_i(X) & \xrightarrow{f_B} & IH^p_i(Y)
\end{array}
\]

i.e. \( f_B = (\omega_Y)^{-1} \circ f_* \circ \omega_X \), where the map \( \omega_Y \) is an isomorphism since \( Y \) is smooth.

We denote by \( \tilde{f}_B \) the map obtained by composition

\[
IH^p_i(X) \xrightarrow{\omega_X} H_i(X) \xrightarrow{f_*} H_i(Y) \xrightarrow{PD} H^{b-i}(Y)
\]

with Poincaré duality \( PD \).

Definition 5.2. For any partition \( \iota = (\iota_1, \ldots, \iota_s) \) and \( r \) numbers \( u_1, \ldots, u_r \) satisfying

\[
(\iota_1 + \cdots + \iota_s) + u_1 + \cdots + u_r + r(b - a) = b,
\]

let us denote \( w^\iota(Y) = w^{\iota_1}(Y) \cdots w^{\iota_r}(Y) \). The S-W numbers of any triple \((f, X, Y)\) are defined by

\[
\langle w^\iota(Y).\tilde{f}_B(v_{a-u_1}(X)).\cdots.\tilde{f}_B(v_{a-u_r}(X)), [Y] \rangle.
\]

Theorem 5.4. Let \( f : X \to Y \) be a normally nonsingular map, with \( X \) a locally orientable \( \mathbb{Z}_2 \)-Witt space of pure dimension \( a \) and \( Y \) a \( b \)-dimensional smooth manifold. If \((f, X, Y)\) bords, then for any partition \( \iota \) and \( r \) numbers \( u_1, \ldots, u_r \) satisfying \((5.3)\), the S-W numbers

\[
\langle w^\iota(Y).\tilde{f}_B(v_{a-u_1}(X)).\cdots.\tilde{f}_B(v_{a-u_r}(X)), [Y] \rangle
\]

are zero.

Proof. As \((f, X, Y)\) bords, one has \((f, X, Y) = \partial(F, V, W)\). We may define a map

\[
\tilde{F}_B : IH^p_i(V) \to IH^p_i(W) = H_i(W) \to H^{b+1-i}(W)
\]

in the same way that we defined \( f_B \).

One has:

\[
\langle w^\iota(V).\tilde{f}_B(v_{a-u_1}(V)).\cdots.\tilde{f}_B(v_{a-u_r}(V)), \partial[W] \rangle =
\]

\[
\langle j^*w^\iota(W).j^*\tilde{F}_B(v_{a-u_1}(V)).\cdots.\tilde{f}_B(v_{a-u_r}(V)), \partial[W] \rangle,
\]

by corollary 3.6 and commutativity of the following diagram:
COBORDISM OF MAPS ON $\mathbb{Z}_2$-WITT SPACES

$$H_{i+1}(V) \xrightarrow{F_*} H_{i+1}(W)$$

$$H_i(X) \xrightarrow{j_*} H_i(Y)$$

$\omega_Y$

$\omega_X$

$H^p_i(X) \xrightarrow{\tilde{F}_B} H^{b-i}(Y)$

$\tilde{f}^*$

$\delta$

$IH^{p_i}(X) \xrightarrow{j_*} IH^{p_i}(Y)$

$PD \simeq$

$PD \simeq$

$\bar{H}_{i+1}(V) \xrightarrow{\tilde{F}_B} H^{b-i}(W)$. 

So, we obtain:

$$\left\langle j^* \left( w^i(W).\tilde{F}_B(v_{a-u_1}(V)) \cdot \ldots \cdot \tilde{F}_B(v_{a-u_r}(V)) \right), \partial[W] \right\rangle =$$

$$\left\langle \delta j^* \left( w^i(W).\tilde{F}_B(v_{a-u_1}(V)) \cdot \ldots \cdot \tilde{F}_B(v_{a-u_r}(V)) \right), [W, \partial W] \right\rangle = 0,$$

where

$$H^k(W) \xrightarrow{\delta} H^{k}(Y) \xrightarrow{\tilde{f}} H^{k+1}(W, \partial W)$$

is part of a long exact sequence, so that $\delta j^* = 0$. □

5.2. Case of a map $f : X \to Y$, with $X$ a smooth manifold.

Let $f : X \to Y$ be a normally nonsingular map, with $X$ an $a$-dimensional smooth manifold and $Y$ a locally orientable $\mathbb{Z}_2$-Witt space of pure dimension $b$.

Since $f$ is a normally nonsingular map one may consider the normal bundle $N_f$ over $X$, and $i : N_f \to Y \times \mathbb{R}^{s+1}$ an open imbedding. Let $\tilde{T}$ be a tubular neighborhood of $(f \times h)(X)$ in $Y \times \mathbb{R}^{s+1}$, where $h : X \to \mathbb{R}^{s+1}$ is defined in such a way that the following diagram commutes.

$$\begin{array}{ccc}
N_f & \xrightarrow{i} & Y \times \mathbb{R}^{s+1} \\
\sigma \downarrow & & \downarrow f \times h \\
X & \xrightarrow{f \times h} & 
\end{array}$$

We denote by $S^s$ the $s$-dimensional sphere in $\mathbb{R}^{s+1}$ and by $T$ the intersection $T = \tilde{T} \cap (Y \times S^s)$. Following the remark, there exists a map $\phi$ which is the composition of the maps:

$$H^i(X) \xrightarrow{\varphi} H^{i+s+b-a}(T/\partial T) \xrightarrow{c'} H^{i+s+b-a}(Y \times S^s) \simeq H^{i+b-a}(Y),$$

here $\varphi$ denotes the Thom homomorphism and $c : Y \times S^s \to T/\partial T$ is the contraction. The last isomorphism is given by the K"unneth formula for a product of a smooth manifold with a $\mathbb{Z}_2$-Witt space [3].
Since $X$ is a smooth manifold, $\alpha_X : H^i(X) \to IH^p_{a-i}(X)$ is an isomorphism, then one defines the map $f_B$ by commutativity of the following diagram, i.e. as being $f_B = \alpha_Y \circ \phi \circ \alpha_X^{-1}$

$$\begin{array}{ccc} H^i(X) & \xrightarrow{\phi} & H^{b-(a-i)}(Y) \\ \alpha_X \downarrow \cong & & \downarrow \alpha_Y \\ IH^p_{a-i}(X) & \xrightarrow{f_B} & IH^p_{a-i}(Y). \end{array}$$

For any $u$ with $0 \leq u \leq b$, let $v_u(Y) \in IH^p_{u}(Y)$ the Wu class of $Y$, dual of $v^{b-u}(Y) \in IH^p_{b-u}(Y)$ and $w_{b-u}(X)$ the homology Whitney class of $X$, so that $f_B(w_{b-u}(X)) \in IH^p_{b-u}(Y)$. For any $u$ with $0 \leq u \leq b$ the S-W intersection numbers

$$v_u(Y) \cdot f_B(w_{b-u}(X))$$

are well defined.

**Theorem 5.5.** Let $f : X \to Y$ be a normally nonsingular map, with $X$ an $a$-dimensional smooth manifold and $Y$ a locally orientable $\mathbb{Z}_2$-Witt space of pure dimension $b$. If $(f, X, Y)$ bords, then for any $0 \leq u \leq b$ the S-W numbers

$$v_u(Y) \cdot f_B(w_{b-u}(X))$$

are zero.

**Proof.** If $(f, X, Y) = \partial(F, V, W)$, one has

$$H^i(V) \overset{\cong}{\to} H^{i+s+b-a}(T'/\partial T') \overset{\cong}{\to} H^{i+s+b-a}(W \times S^s) \cong H^{i+b-a}(W),$$

where $V$ is embedded in $S^s$ and $T'$ is a tubular neighborhood of $(F \times h)(V)$, which gives rise to the corresponding map $F_B$. Therefore we can consider the following diagram, where PD denotes the Poincaré duality

$$\begin{array}{ccc} H^i(X) & \xrightarrow{f_B} & H^i(Y) \\ \alpha_X \downarrow & & \uparrow \alpha_Y \\ IH^p_{a-i}(X) & \xrightarrow{f_B} & IH^p_{a-i}(Y) \\ \omega_X & & \omega_Y \\ PD & & PD \\ H^i(X) & \xrightarrow{c^*} & H^{i+s+b-a}(T/\partial T) \\ c & & c^* \\ H^{i+s+b-a}(Y \times S^s) \cong H^{i+b-a}(Y) \end{array}$$

since we had defined $f_B$ and $F_B$ the result follows in the same way of the proof of Theorem 5.4 \hfill \Box

### 5.3. The general case.

In the general case $X$ and $Y$ are locally orientable $\mathbb{Z}_2$-Witt spaces of dimensions $a$ and $b$ respectively. It is not always possible to define an unique map $f_B$ as done in the other cases, however we can show that for any map $f_B$ considered, the bordism condition of $(f, X, Y)$ implies that the corresponding S-W numbers are zero.

First we show the following lemma.
Lemma 5.6. Let $f : X \to Y$ be a normally nonsingular map and $(f, X, Y) = \partial(F, V, W)$. Given a map $f_B$ there exists a map $F_B$ such that the following diagram commutes.

\[
\begin{array}{ccc}
IH^m_u(X) & \xrightarrow{j_X} & IH^m_{u+1}(V) \\
f_B & & F_B \\
IH^m_u(Y) & \xrightarrow{j_Y} & IH^m_{u+1}(W).
\end{array}
\]

Proof. The diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j_X} & V \\
f \downarrow & & F_B \\
Y & \xrightarrow{j_Y} & W
\end{array}
\]

is a cartesian diagram. Then we can apply Proposition 10.7 in [2] (see also [8]).

One has equality of sheaves on $Y$:

\[
j_Y^* F_B \mathcal{A} = f^* j_X^* \mathcal{A}
\]

for any sheaf $\mathcal{A}$ on $V$. That provides a commutative diagram of complexes of sheaves on $Y$ (perverse intersection sheaves for the middle perversity $\bar{m}$).

\[
f^* \mathcal{IC}^\bullet_X \xrightarrow{j_X^*} f^* j_X^* (\mathcal{IC}^\bullet_V) = j_Y^* F_B (\mathcal{IC}^\bullet_V) \]

Let us remind that intersection homology is obtained by taking hypercohomology of the perverse intersection sheaf:

\[
IH^m_u(Y) = \mathbb{H}^{b-u}(Y; \mathcal{IC}^\bullet_Y)
\]

Taking hypercohomology

\[
\mathbb{H}^{b-u}(Y; \bullet)
\]

of the previous diagram, one obtains:

\[
\begin{array}{ccc}
\mathbb{H}^{b-u}(X; \mathcal{IC}^\bullet_X) & \xrightarrow{j_X^*} & \mathbb{H}^{b-u}(V; \mathcal{IC}^\bullet_V) \\
f_B & & F_B \\
\mathbb{H}^{b-u}(Y; \mathcal{IC}^\bullet_Y) & \xrightarrow{j_Y^*} & \mathbb{H}^{b-u}(W; \mathcal{IC}^\bullet_W)
\end{array}
\]

and the Lemma follows. □

Theorem 5.7. Let $f : X \to Y$ be a normally nonsingular map, with $X$ and $Y$ locally orientable $\mathbb{Z}_2$-Witt spaces of pure dimension $a$ and $b$ respectively. Then for any $u$ with $0 \leq u \leq b$, the S-W numbers $\langle v_u(Y), f_B(v_{b-u}(X)), [Y] \rangle$ are zero.
Proof. The diagram of Lemma 5.6 can be written in the cohomology setting

\[
\begin{align*}
    IH^n_{\bar{m}}(X) \xrightarrow{f^B} IH^n_{\bar{m}}(Y) \\
    \downarrow j_X^* \quad \downarrow j_Y^* \\
    IH^n_{\bar{m}}(V) \xrightarrow{F^B} IH^n_{\bar{m}}(W),
\end{align*}
\]

where \( \bar{m} + \bar{n} = \bar{t} \) and we use the same notation for corresponding maps \( j_X^* \) and \( j_Y^* \).

Let us consider the homology class \( v_{b-u}(Y) \in IH^n_{b-u}(Y) \), that will be written \( v^u(Y) \in IH^n_{\bar{m}}(Y) \) in the cohomology setting.

Then \( v^u(Y) = j_Y^* v^u(W) \) where \( v^u(W) \in IH^n_{\bar{n}}(W) \) is the corresponding cohomology Wu class to the homology Wu class \( v_{b+1-u}(W) \in IH^n_{\bar{m}+1}(W) \) of \( W \).

Let us consider the cohomology Wu class \( v^{a-u}(X) \in IH^n_{\bar{m}}(X) \) corresponding for the cohomology Wu class \( v_u(X) \in IH^n_{\bar{m}}(X) \). Then \( v^{a-u}(X) = j_X^*(v^{a-u}(V)) \) where \( v^{a-u}(V) \in IH^n_{\bar{n}}(V) \) is the corresponding cohomology Wu class to the homology class \( v_{u+1}(V) \in IH^n_{\bar{n}+1}(V) \).

One has

\[ f^B(v^{a-u}(X)) = f^B j_X^*(v^{a-u}(X)) \in IH^n_{\bar{m}}(Y). \]

The intersection product

\[ v_{b-u}(Y) \cdot f_B(v_u(X)) \in IH^n_{b-u}(Y) \times IH^n_{a-u}(Y) \to IH^n_0(Y) \]

corresponds to the product

\[ v^u(Y) \cup f^B(v^{a-u}(X)) \in IH^n_{\bar{m}}(Y) \times IH^n_{\bar{n}}(Y) \to IH^n_0(Y). \]

One has

\[
\langle v^u(Y) \cup f^B(v^{a-u}(X)), [Y] \rangle = \]

\[
\langle j_Y^*v^u(W) \cup f^B j_X^*(v^{a-u}(V)), [Y] \rangle = \]

\[
\langle j_Y^*v^u(W) \cup j_Y^*F^B(v^{a-u}(V)), [Y] \rangle = \]

\[
\langle j_Y^* [v^u(W) \cup F^B(v^{a-u}(V))], \partial[W] \rangle = \]

\[
\langle \delta j_Y^* [v^u(W) \cup F^B(v^{a-u}(V))], [W, \partial W] \rangle = 0 \]

where the first equality is a consequence of the Theorem 5.3 of Goresky [6], the second one is from Lemma 5.6 and the fourth equality is obtained in an analogous way than the proof of Theorem 5.5. \( \square \)
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