Contact characteristics of a sphere with a layered elastic half-space

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Abstract. On the basis of a simplified rigid model of a layered elastic body, an engineering technique for determining the parameters of a contact is proposed for the introduction of a spherical indenter into it. The model is based on the dependence of the displacement of the points of the half-space along the axis of symmetry on the magnitude of the applied distributed load. The reduced elasticity modulus and the Poisson's ratio are determined depending on the elastic properties of the base and coating materials, the thickness of the coating and the radius of the contact area. Expressions are given for determining the parameters of a contact when a spherical indenter is introduced into a layered body. The obtained results are compared with the exact solution of the spatial axisymmetric problem for describing the stress-strain state in an elastic layer when a spherical indenter is introduced into it, using the Fourier-Bessel integral transformation method. An analysis of the comparison of the results obtained with the results of exact solutions makes it possible to recommend the proposed engineering method for practical use.

1. Introduction

At present, the possibilities of increasing the service life of the connections of machine parts by changing the design or improving the materials by optimizing their microstructure are practically exhausted. In this regard, one of the promising ways to improve the operational performance of the joints of machine parts, including sealing joints and friction units, is to apply coatings to their working surfaces, or to form modified layers based on metals, ceramics, polymers [1]. Experience in the operation of friction units and seals with such coatings shows that their antifriction properties and sealing ability are determined not only by the properties of the coating material, but also by its thickness [2]. Known recommendations on the choice of coating thickness are based on experimental data, often contradictory. The lack of a theory of contact interaction of rough surfaces through the coating layer does not allow developing reliable methods for predicting the friction characteristics of tri-shears and tightness of seals at the design stage, which requires costly and time-consuming experimental tests.

In the framework of the theory of elasticity, the presence of a coating means that an elastic body should be considered with varying values of the elastic modulus and Poisson’s ratio depending on the distance to the surface [3].

2. Solution methods

Contact problems for bodies with varying in depth mechanical characteristics were considered in [3, 4, 5, 6, 7 and others]. However, it is not possible to apply the obtained results for solving practical
problems of tribology and hermetology. In [8], an approximate solution of the axisymmetric contact problem for an elastic layer of finite thickness lying on a rigid base is presented, which may be of interest when using polymer coatings.

A separate group should include engineering methods for solving contact problems based on simplifying hypotheses, for example, the representation of a layered body as a topcoat-composite – structures with special mechanical properties depending on the coating thickness and the mechanical properties of the base and coating materials [9, 10].

As the method was developed, the authors of [11, 12, 13, 14, 15] determined the reduced modulus of elasticity and the Poisson’s ratio for any values of coating thickness under axisymmetric loading of a layered half-space based on the rigid model of a layered body.

In general, the axisymmetric load is

$$p(r) = p_0 \left(1 - r^2/a^2\right)^\beta, \quad 0 \leq r \leq a,$$  \hspace{1cm} (1)

where $0 \leq \beta \leq 0.5$, $p_0 = p_m(1 + \beta)$, $p_m$ is the mean pressure, $P = p_m \pi a^2$.

Following the classical approach, based on the application of potential Boussinesq functions [16], to move any point along the axis of symmetry inside a homogeneous half-space when loaded with a distributed load (1) are determined by the expression:

$$u_z = \frac{1 + \nu}{2\pi E^*} \left[2(1 - \nu) \psi - \frac{d\psi}{dz}\right], \quad \psi = \int_{r}^{R} \frac{p(r)}{R} r dr d\phi, \quad R = \sqrt{r^2 + z^2}.$$  \hspace{1cm} (2)

Taking into account the expression (1) and the fact that $\rho = r/a$ and $\zeta = z/a$, after integration, we have

$$\psi = \frac{\pi p_0 a}{1.5} \frac{1}{\sqrt{1 + \zeta^2}} F_1 \left(\frac{1}{2}, 1.5; 2.5; \frac{1}{1 + \zeta^2}\right),$$  \hspace{1cm} (3)

where $F_1(a, b; c; x)$ is the Gauss hypergeometric function, $E^* = E/(1 - \nu^2)$, $\zeta = z/a$.

Substituting expression (3) into (2) and taking into account that $\frac{d\psi}{dz} = \frac{d\psi}{d\zeta}$, we get

$$u_z = \frac{p_m a}{E^*} \cdot K(\zeta, \beta, \nu)$$  \hspace{1cm} (4)

$$K(\zeta, \beta, \nu) = \frac{1}{\sqrt{1 + \zeta^2}} \left[F_1 \left(\frac{1}{2}, 1 + \beta; 2 + \beta; \frac{1}{1 + \zeta^2}\right) - \frac{\zeta}{2(1 - \nu)} \cdot \frac{d}{d\zeta} \left[\frac{1}{\sqrt{1 + \zeta^2}} F_1 \left(\frac{1}{2}, 1 + \beta; 2 + \beta; \frac{1}{1 + \zeta^2}\right)\right]\right].$$  \hspace{1cm} (5)

For Hertz load distribution with $\beta = 0.5$ the expression for $K(\zeta, \beta, \nu)$ can be represented as:

$$K(\zeta, 0.5, \nu) = \arctg \zeta + \frac{\nu}{1 - \nu} \zeta(1 - \zeta \arctg \zeta).$$  \hspace{1cm} (6)

Simplify the notation by accepting

$$K(0.0.5, \nu) = K(0), \quad K(\delta_1, 0.5, \nu) = K(\delta_1),$$  \hspace{1cm} (7)

As the analysis showed, the function $K(\zeta)$ to a small extent depends on the values of the Poisson coefficient. Moreover, $K(0) = \pi/2$, with $\zeta \to 0$, $K(\zeta) \to 0$.

For a layered elastic body, the reduced elastic modulus and the Poisson’s ratio is determined by the expressions:

$$E_{01}^* = E_{11}^* F_1, \quad F_1 = \frac{K_{01}(0)}{K_{01}(0) - K_{11}(\delta_1) + K_{01}(\delta_1)} I_e;$$  \hspace{1cm} (8)

$$\nu_{01} = \nu_1 + \left(\nu_0 - \nu_1\right) \frac{1 - F_1^{-1}}{1 - I_e}$$  \hspace{1cm} (9)
Since the rough surface for solving problems of tribology and hermetology is represented as a set of spherical asperities distributed over height, it is of practical interest to determine the contact characteristics when the sphere is introduced into a layered elastic half-space.

3. Description of the proposed method

Consider the contact of a rigid sphere of radius $R$ with a layered half-space (Figure 1).

![Figure 1](image1.png)

**Figure 1.** The design scheme of contacting (a) and the scheme of contact of a sphere with homogeneous half-spaces, having characteristics $\nu_1$, $E_1$ (b) and $\nu_0$, $E_0$ (c).

The displacement of the point O (Figure 1a) can be represented as the sum of the displacements of the coating and the base under the load $P$

$$w_0 = w_5 + w_4.$$

Scheme 1a can be represented as a stiffness model (Figure 2a).

![Figure 2](image2.png)

**Figure 2.** Modeling a layered half-space: a) – the original scheme of a layered half-space (Figure 1); b), c) – stiffness schemes of homogeneous half-spaces.

Then the displacement $w_0 = P(s_1 + s_0)$, where $s_1$, $s_0$ – stiffness of the layer and the base material, $P = \pi a^2 p_m$.

Consider the insertions of spheres of radius $R$ into two homogeneous half-spaces with elastic characteristics $\nu_1$, $E_1$ and $\nu_0$, $E_0$, under loads $P_1$ and $P_0$ (Figure 2b and 2c) According to [16], the radius of the contact area of a sphere with a homogeneous half-space

$$a = \left(\frac{3PR}{4E^2}\right)^{\frac{1}{3}}. ~ (10)$$

Taking into account (6) and (10), the compression of a homogeneous layer with a thickness of $\delta$ (Figure 2b) and the corresponding rigidity are determined by the equation

$$w_1 = w_0 = \frac{1.5P_1}{\pi} \left(\frac{4}{3E_1^2 P_1 R}\right)^{\frac{1}{3}} [K_i(0) - K_i(\delta)], \quad s_1 = \frac{w_1}{P_1} = \frac{1.5}{\pi} \left(\frac{4}{3E_1^2 P_1 R}\right)^{\frac{1}{3}} [K_i(0) - K_i(\delta)]. ~ (11)$$
The displacement of the point $A_0$ and the corresponding stiffness $s_0$ are equal to

$$w_{A_0} = w_A = \frac{1.5P}{\pi} \left( \frac{4}{3E_0^2PR} \right) \frac{1}{2} K_0(\delta), \quad s_0 = \frac{w_{A_0}}{P_0} = \frac{1.5}{\pi} \left( \frac{4}{3E_0^2PR} \right) \frac{1}{2} K_0(\delta)$$  \hfill (12)

The value of $P_1$ is determined from the condition of equality of displacements ($z = \delta$) of a layered elastic body under load $P$ and a homogeneous material under load $P_1$:

$$P_1 = \frac{F_1^*}{E_0^*} \left( \frac{K_{01}(0) - K_{01}(\delta)}{K_1(0) - K_1(\delta)} \right) \frac{1}{2} P.$$  \hfill (13)

The value of $P_0$ is determined from the condition of equality of displacements ($z = \delta$) of a layered elastic body under load $P$ and a homogeneous material under load $P_0$. As a result, we get

$$P_0 = \frac{E_0^*}{E_0^*} \left( \frac{K_{01}(\delta)}{K_0(\delta)} \right) \frac{1}{2} P.$$  \hfill (14)

From equality of displacements $w_1 = w_0$ и $w_{A_0} = w_A$ it is followed that:

$$P = P_1 \frac{s_1}{s_1 + s_0} + P_0 \frac{s_0}{s_1 + s_0}. \hfill (15)$$

Substituting the expressions (11), (12), (13), (14) into (15) we have

$$E_{01}^* = E_1^* \cdot F_{1R}, \quad F_{1R} = K_{01}(0) \left( \frac{(K_{1}(0) - K_{01}(\delta))^{1/2}}{K_{1}(0) - K_{01}(\delta)} \frac{1}{2} I_1 \right)^{-1} \hfill (16)$$

In the case of the approximation of the values of $K_{01}(\delta)$ to $K_{01}(\delta)$ we obtain the equation (8).

The maximum discrepancies in the dependences of $F_{in}(\delta)$ and $F_{in}(\delta)$ in the range $\delta = 0...10$ with $I_1 = 0.1$ do not exceed 5% and the average discrepancies do not exceed 1%. Assuming the applicability of the Hertz theory for a layered elastic body with a reduced modulus of elasticity $E_{01}^* = E_1^* \cdot F_{1R}$, we determine the relative contact characteristics when introducing a sphere - radius of the contact area $\bar{a} = a/R$, insertion value $\bar{w}_0 = w_0/R$ maximum pressure at the contact area $\bar{p}_0 = p_0 / E_1^*$—depending on the dimensionless load $\bar{P} = P / (E_1^* R^2)$. For $a/R \leq 0.4$ when defining contact parameters, it should to use the expression for a paraboloid of rotation [16]:

$$\bar{P} = \frac{P}{E_1^* R^2} = \frac{4}{3} \bar{a}^3, \quad \bar{w}_0 = w_0 / R = \bar{a}^2, \quad \bar{p}_0 = \frac{p_0}{E_1^*} = \frac{1}{\pi} \left( \frac{6}{E_1^*} \right)^{1/3} \bar{P}^{1/3} R^{2/3}. \hfill (17)$$

Equations for the spherical indenter are given in [17].

4. Results and discussion

The exact solution of the spatial axisymmetric problem for describing the stress-strain state in an elastic layer with the indentation of a spherical indenter into it is given in [2, p. 56-62], which used the method of the integral Fourier-Bessel transform. This problem was also considered in [18, 19, 20]. Below, in table 1, some data of the exact solution from [2, p. 68-69, table 2.1] for comparison with the results obtained by the proposed engineering methodology. The elastic moduli and Poisson’s ratios of the coating and base
materials are respectively: $E_0 = 201$ GPa, $\nu_0 = 0.3$; $E_1 = 2.39$ GPa, $\nu_1 = 0.38$; Spherical indenter radius $R = 2.5 \times 10^{-3}$ m, the thickness of the polymer coating is rigidly bonded to the base, $\delta = 10^{-4}$ m.

**Table 1.** The results of calculations comparing the values of the relative insertion of a spherical indenter into the elastic layer, rigidly bonded to the base, carried out according to [2] and according to the proposed method.

| $a \cdot 10^4$ (m) | $P$ (N) | $p_0$ (GPa) | $w_0 \cdot 10^4$ (m) | $\log \bar{P}$ | $\bar{w}_0$ | $\bar{w}_{0z}$ | $(\bar{w}_0 - \bar{w}_{0z})/\bar{w}_0$ (%) |
|--------------------|-------|-----------|-------------------|-----------|-------|-------|----------------|
| 0.05               | 0.0002| 3.69      | 0.0106            | -7.941    | 4.24 $\times 10^{-6}$ | 3.999 $\times 10^{-6}$ | 5.67 |
| 0.1                | 0.0017| 7.39      | 0.0413            | -7.012    | 1.652 $\times 10^{-6}$ | 1.6 $\times 10^{-6}$ | 3.13 |
| 0.2                | 0.0135| 14.8      | 0.1570            | -6.112    | 6.28 $\times 10^{-5}$ | 6.4 $\times 10^{-5}$ | -1.92 |
| 0.3                | 0.0464| 23        | 0.322             | -5.575    | 1.288 $\times 10^{-4}$ | 1.21 $\times 10^{-4}$ | 6.05 |
| 0.5                | 0.2250| 40        | 0.804             | -4.89     | 3.216 $\times 10^{-4}$ | 3.24 $\times 10^{-4}$ | -0.76 |
| 0.8                | 1.04  | 73        | 1.842             | -4.225    | 7.368 $\times 10^{-4}$ | 7.842 $\times 10^{-4}$ | -6.43 |
| 1                  | 2.23  | 102       | 2.738             | -3.894    | 1.095 $\times 10^{-3}$ | 1.156 $\times 10^{-3}$ | -5.56 |
| 2                  | 28.3  | 359       | 9.781             | -2.79     | 3.912 $\times 10^{-3}$ | 4.225 $\times 10^{-3}$ | -7.98 |
| 3                  | 126   | 804       | 21.45             | -2.142    | 8.58 $\times 10^{-3}$ | 8.464 $\times 10^{-3}$ | 1.35 |
| 4                  | 427   | 1440      | 38.17             | -1.612    | 0.015            | 0.016            | -5.64 |
| 5                  | 1046  | 2267      | 60.33             | -1.222    | 0.024            | 0.025            | -4.76 |
| 8                  | 6962  | 5824      | 163.1             | -0.399    | 0.065            | 0.062            | 4.2  |
| 10                 | 16952 | 8990      | 260.4             | -0.013    | 0.104            | 0.1              | 4.13 |
| 15                 | 82556 | 19155     | 616.3             | 0.675     | 0.247            | 0.235            | 4.56 |
| 20                 | 247400| 31973     | 1137              | 1.151     | 0.455            | 0.416            | 8.51 |

For given values of $a$ (column 1), the indentation force $P$, the maximum pressure at the contact area $p_0$ and the insertion value $w_0$ (columns 2–4). To summarize the results, we compare the dimensionless quantities. For dimensionless load, the relative indentation of $\bar{w}_0$ is determined according to [2] and according to the proposed engineering technique $\bar{w}_{0z}$ (columns 6 and 7, respectively). The maximum error in absolute value did not exceed 10%, and the average error for the given values of the relative implementation was 4.7%, which indicates the acceptability of the proposed method for engineering calculations of the parameters of contact of a spherical indenter with a layered elastic body.

The corresponding graphical dependences of the relative indentation on the dimensionless load, calculated by different methods, are presented in Figure 3. Dots indicate dependencies according to the data given in Table 1. The solid line corresponds to the data calculated by equation (17) for the values of $a$ given in column 1 of the table 1. The dashed line corresponds to an approximate solution of the axisymmetric contact problem for an elastic layer of finite thickness [8]. As follows from Figure 3 and tabel 1, an approximate solution is acceptable only for $a \leq \delta$.

**Figure 3.** Dependence of the relative introduction $\bar{w}_0$ on the dimensionless load $\bar{P}$.
Figure 4 shows the dependences of the relative radius of the contact area \( \bar{a} \), and the relative maximum contact pressure \( \bar{p}_0 \) on the dimensionless load \( \bar{P} \) obtained from the Equation (17) under the assumption of the Hertz theory.

![Graph showing log \( \bar{a} \) vs log \( \bar{P} \) and log \( \bar{p}_0 \) vs log \( \bar{P} \)](image)

**Figure 4.** Dependence of the relative radius of the contact area \( \bar{a} \) and the relative maximum contact pressure \( \bar{p}_0 \) on a dimensionless load

As follows from Figures 3 and 4, the best coincidence of the results of the proposed method with the exact solution [2] occurs when determining the indentation of the indenter (Figure 3). This is quite understandable, since the basis of the rigid model of a layered body is the dependence of the displacements along the axis of symmetry on the applied force. The systematic error in determining the radius of the contact area and the maximum contact pressure is related to the difference in the contact from the Hertz’s pressure distribution.

5. Conclusion

1. Using a simplified model of a layered body, an engineering technique has been developed for determining contact parameters when introducing a spherical indenter into it.

2. A comparison was made of the results obtained with the data of an exact solution of a similar problem [2], which presents certain difficulties for its practical application in engineering calculations, due to their complexity and laboriousness. The coincidence of the graphical dependencies of the contact parameters from the dimensionless force, which varies within ten orders of magnitude, with the results of accurate solutions [2] allows us to recommend the proposed engineering technique for practical use.

3. The best coincidence of the results according to the proposed method with the exact solution [2] takes place when determining the implementation value of a spherical indenter (Figure 3). The maximum error in this case did not exceed 10%, and the average error for the reduced values of the relative implementation was 4.7%. An approximate solution for the indentation of a sphere into an elastic layer of finite thickness [8] is acceptable only for \( a \leq \delta \).

4. The proposed method is recommended for use for solving problems of contacting a rough surface through a layer of polymer coating. In this case, the contact of a single asperity should be considered taking into account the influence of the remaining contacting asperities [21].

References

[1] Kovalev EP, Ignat’ev MB, Semenov AP et al 2004 *J. Friction & Wear* **25** 316

[2] Makushkin AP 1993 *Polymers in friction and seal assemblies at low temperatures* (Moscow: Mechanical Engineering) p 288

[3] Torskaya EV 2014 *Modeling of frictional interaction of bodies with coverings: diss* (Moscow) p 251
[4] Aleksandrov VM and Mkhitaryan SM 1983 Contact problems for bodies with elastic coatings and interlayers (Moscow: Nauka) p 488
[5] Giannakopoulos AT and Suresh T 1997 J Solids Struct. 34 2357
[6] Aizikovich SM, Aleksandrov VM, Vasiliev AS, Krenev LI and Trubchik IS 2011 Analytic solutions of mixed axisymmetric problems for functional gradient media (Moscow: Fizmatlit) p 192
[7] Potelezhko VP (2006) Mechanics and physics of processes on the surface and in the contact of solid bodies and machine parts 2 27
[8] Argatov II 2004 Problems of mechanical engineering and reliability of machines 6 35
[9] Voronin NA 1993 Friction and wear 14 250
[10] Voronin NA 2002 Friction & Wear 23 583
[11] Ogar PM, Tarasov VA and Fedorov IB 2013 Modern tech.. Syst. an.. Model. 1 2
[12] Ogar PM and Tarasov VA 2013 Adv. Mat. Res. 677 267
[13] Ogar PM, Gorokhov DB and Kozhevnikov AS 2016 Modern tech.. Syst. an. Model 4 37
[14] Kozhevnikov AS 2017 Mechanics of the XXI century 16 211
[15] Ogar P, Kozhevnikov A and Fedorov V 2018 MATEC Web of Conf. 224 02051
[16] Johnson K 1989 Contact mechanics (Moscow: Mir) p 510
[17] Argatov II and Dmitriev NN 2003 Fundamentals of the theory of elastic discrete contact (S. – Peterburg: Politechnica)
[18] O’Sullivan TC and King RB 1988 ASME J. Tribol. 110:235
[19] Peng W and Bhushan B 2001 Wear 249:741
[20] Peng W 2001 Contact mechanics of multilayered rough surfaces in tribology: diss (The Ohio State University) p 129
[21] Ogar P, Alpatov Yu, and Gorokhov D 2018 MATEC Web of Conf. 224 02051