LOWER BOUNDS ON THE COEFFICIENTS OF EHRHART POLYNOMIALS

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Abstract. We present lower bounds for the coefficients of Ehrhart polynomials of convex lattice polytopes in terms of their volume. Concerning the coefficients of the Ehrhart series of a lattice polytope we show that Hibi’s lower bound is not true for lattice polytopes without interior lattice points. The counterexample is based on a formula of the Ehrhart series of the join of two lattice polytope. We also present a formula for calculating the Ehrhart series of integral dilates of a polytope.

1. Introduction

Let $\mathcal{P}^d$ be the set of all convex $d$-dimensional lattice polytopes in the $d$-dimensional Euclidean space $\mathbb{R}^d$, i.e., all vertices of $P \in \mathcal{P}^d$ have integral coordinates and $\dim(P) = d$. The lattice point enumerator of a set $S \subset \mathbb{R}^d$, denoted by $G(S)$, counts the number of lattice (integral) points in $S$, i.e., $G(S) = \#(S \cap \mathbb{Z}^d)$. In 1962, Eugène Ehrhart (see e.g. [3, Chapter 3], [8]) showed that for $k \in \mathbb{N}$ the lattice point enumerator $G(kP)$, $P \in \mathcal{P}^d$, is a polynomial of degree $d$ in $k$ where the coefficients $g_i(P)$, $0 \leq i \leq d$, depend only on $P$:

$$G(kP) = \sum_{i=0}^{d} g_i(P) k^i. \tag{1.1}$$

The polynomial on the right hand side is called the Ehrhart polynomial, and regarded as a formal polynomial in a complex variable $z \in \mathbb{C}$ it is denoted by $G_P(z)$. Two of the $d + 1$ coefficients $g_i(P)$ are almost obvious, namely, $g_0(P) = 1$, the Euler characteristic of $P$, and $g_d(P) = \text{vol}(P)$, where $\text{vol}(\cdot)$ denotes the volume, i.e., the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$. It was shown by Ehrhart (see e.g. [3, Theorem 5.6], [8]) that also the second leading coefficient admits a simple geometric interpretation as lattice surface area of $P$:

$$g_{d-1}(P) = \frac{1}{2} \sum_{F \text{ facet of } P} \frac{\text{vol}_{d-1}(F)}{\det(\text{aff } F \cap \mathbb{Z}^d)}. \tag{1.2}$$

Here $\text{vol}_{d-1}(\cdot)$ denotes the $(d-1)$-dimensional volume and $\det(\text{aff } F \cap \mathbb{Z}^d)$ denotes the determinant of the $(d-1)$-dimensional sublattice contained in the affine hull of $F$. All other coefficients $g_i(P)$, $1 \leq i \leq d - 2$, have no such known explicit...
geometric meaning, except for special classes of polytopes. For this and as a general reference on the theory of lattice polytopes we refer to the recent book of Matthias Beck and Sinai Robins [3] and the references within. For more information regarding lattices and the role of the lattice point enumerator in convexity see [9].

In [4, Theorem 6] Ulrich Betke and Peter McMullen proved the following upper bounds on the coefficients $g_i(P)$ in terms of the volume:

$$g_i(P) \leq (-1)^{d-i} \text{stirl}(d,i) \text{vol}(P) + (-1)^{d-i-1} \frac{\text{stir}(d,i+1)}{(d-1)!}, \quad i = 1, \ldots, d-1.$$ 

Here $\text{stir}(d,i)$ denote the Stirling numbers of the first kind which can be defined via the identity $\prod_{i=0}^{d-1} (z-i) = \sum_{i=1}^{d} \text{stir}(d,i) z^i$.

In order to present our lower bounds on $g_i(P)$ in terms of the volume we need some notation. For an integer $i$ and a variable $z$ we consider the polynomial

$$(z+i)(z+i-1) \cdots (z+i-(d-1)) = d! \binom{z+i}{d},$$

and we denote its $r$-th coefficient by $C_{r,i}^d$, $0 \leq r \leq d$. For instance, it is $C_{0,i}^d = 1$, and for $0 \leq i \leq d-1$ we have $C_{0,i}^d = 0$. For $d \geq 3$ we are interested in

$$(1.3) \quad M_{r,d} = \min \{ C_{r,i}^d : 1 \leq i \leq d-2 \}.$$

Obviously, we have $M_{0,d} = 0$, $M_{d,d} = 1$ and it is also easy to see that (cf. Proposition 2.1 iii))

$$(1.4) \quad M_{d-1,d} = C_{d-1,1}^d = -\frac{d(d-3)}{2}.$$ 

With the help of these numbers $M_{r,d}$ we obtain the following lower bounds.

**Theorem 1.1.** Let $P \in \mathcal{P}^d$, $d \geq 3$. Then for $i = 1, \ldots, d-1$ we have

$$g_i(P) \geq \frac{1}{d!} \left\{ (-1)^{d-i} \text{stir}(d+1,i+1) + (d! \text{vol}(P) - 1) M_{i,d} \right\}.$$ 

We remark that the coefficients $g_i(P)$, $1 \leq i \leq d-2$, might be negative and thus also the lower bounds given above. In general, the bounds of Theorem 1.1 are not best possible. For instance, in the case $i = d-1$ we get together with (1.4) the bound

$$g_{d-1}(P) \geq \frac{1}{(d-1)!} \left\{ d-1 - \frac{d-3}{2} d! \text{vol}(P) \right\}.$$ 

On the other hand, since the lattice surface area of any facet is at least $1/(d-1)!$ we have the trivial inequality (cf. (1.2))

$$(1.5) \quad g_{d-1}(P) \geq \frac{1}{2} \frac{d+1}{(d-1)!}.$$ 

Hence the lower bound on $g_{d-1}(P)$ given in Theorem 1.1 is only best possible if $\text{vol}(P) = 1/d!$. In the cases $i \in \{1,2, d-2\}$, however, Theorem 1.1 gives best possible bounds for any volume.
Corollary 1.2. Let $P \in P^d$, $d \geq 3$. Then

i) $g_1(P) \geq 1 + \frac{1}{2} + \cdots + \frac{1}{d-2} + \frac{2}{d-1} - (d-2)! \text{vol}(P)$,

ii) $g_2(P) \geq \frac{(-1)^d}{d!} \times \left\{ \text{stirl}(d+1,3) + \left( (-1)^d(d-2)! + \text{stirl}(d-1,2) \right) (d! \text{vol}(P) - 1) \right\}$,

iii) $g_{d-2}(P) \geq \begin{cases} \frac{1}{d!} \frac{(d-1)d(d+1)}{24} \{3(d+1) - d! \text{vol}(P)\} : & \text{if } d \text{ odd}, \\ \frac{1}{d!} \frac{(d-1)d}{24} \{3d(d+2) - (d-2) d! \text{vol}(P)\} : & \text{if } d \text{ even}. \end{cases}$

And the bounds are best possible for any volume.

For some recent inequalities involving more coefficients of Ehrhart polynomials we refer to [2]. Next we come to another family of coefficients of a polynomial associated to lattice polytopes.

The generating function of the lattice point enumerator, i.e., the formal power series

$$\text{Ehr}_P(z) = \sum_{k \geq 0} G_P(k) z^k,$$

is called the Ehrhart series of $P$. It is well known that it can be expressed as a rational function of the form

$$\text{Ehr}_P(z) = a_0(P) + a_1(P) z + \cdots + a_d(P) z^d \frac{1}{(1-z)^{d+1}}.$$

The polynomial in the numerator is called the $h^*$-polynomial. Its degree is also called the degree of the polytope [1] and it is denoted by $\text{deg}(P)$. Concerning the coefficients $a_i(P)$ it is known that they are integral and that

$$a_0(P) = 1, \quad a_1(P) = G(P) - (d+1), \quad a_d(P) = G(\text{int}(P)),$$

where $\text{int}(\cdot)$ denotes the interior. Moreover, due to Stanley’s famous non-negativity theorem (see e.g. [3, Theorem 3.12], [17]) we also know that $a_i(P)$ is non-negative, i.e., for these coefficients we have the lower bounds $a_i(P) \geq 0$. In the case $G(\text{int}(P)) > 0$, i.e., $\text{deg}(P) = d$, these bounds were improved by Takayuki Hibi [13] to

$$a_i(P) \geq a_i(P), 1 \leq i \leq \text{deg}(P) - 1. \tag{1.6}$$

In this context it was a quite natural question whether the assumption $\text{deg}(P) = d$ can be weaken (see e.g. [15]), i.e., whether these lower bounds (1.6) are also valid for polytopes of degree less than $d$. As we show in Example 1.4 the answer is already negative for polytopes having degree 3. The problem in order to study such a question is that only very few geometric constructions of polytopes are known for which we can explicitly calculate the Ehrhart series. In [3, Theorem 2.4, Theorem 2.6] the Ehrhart series of special pyramids and double pyramids over a basis $Q$ are determined in terms of the Ehrhart series of $Q$. In a recent paper Braun [6] gave a very nice product formula for the Ehrhart series of the free sum of two lattice polytopes, where one of the polytopes has to be reflexive. Here we consider a related construction, known as the join of two
polytopes [11]. As we learned by Matthias Beck the Ehrhart series of such a join is already described as Exercise 3.32 in the book [3] and it was personally communicated to the authors of the book by Kevin Woods. For completeness’ sake we present its short proof in Section 3.

**Lemma 1.3.** For $P \in \mathcal{P}^p$ and $Q \in \mathcal{P}^q$ let $P \ast Q$ be the join of $P$ and $Q$, i.e.,

$$P \ast Q = \text{conv}\{(x,0,0)^\top, (0_p, y, 1)^\top : x \in P, y \in Q\} \in \mathcal{P}^{p+q+1},$$

where $0_p$ and $0_q$ denote the $p$- and $q$-dimensional 0-vector, respectively. Then

$$\text{Ehr}_{P \ast Q}(z) = \text{Ehr}_P(z) \cdot \text{Ehr}_Q(z).$$

In order to apply this lemma we consider two families of lattice simplices. For an integer $m \in \mathbb{N}$ let

$$T_d^{(m)} = \text{conv}\{0, e_1, e_1 + e_2, e_2 + e_3, \ldots, e_{d-2} + e_{d-1}, e_{d-1} + m e_d\},$$

$$S_d^{(m)} = \text{conv}\{0, e_1, e_2, e_3, \ldots, e_{d-1}, m e_d\},$$

where $e_i$ denotes the $i$-th unit vector. It was shown in [4] that

$$\text{Ehr}_{T_d^{(m)}}(z) = 1 + (m - 1) z \left\lceil \frac{d}{2} \right\rceil \frac{1}{(1 - z)^{d+1}}$$

and

$$\text{Ehr}_{S_d^{(m)}}(z) = 1 + (m - 1) z \left(1 - z\right)^{d+1}.$$

Actually, in [4] the formula for $T_d^{(m)}$ was only proved for odd dimensions, but the even case can be treated completely analogously.

**Example 1.4.** For $q \in \mathbb{N}$ odd and $l, m \in \mathbb{N}$ we have

$$\text{Ehr}_{T_q^{(l+1)} \ast S_p^{(m+1)}}(z) = \frac{1 + m z + l z^{\frac{q+1}{2}} + m l z^{\frac{q+3}{2}}}{(1 - z)^{p+q+2}}.$$

In particular, for $q \geq 3$ and $l < m$ this shows that (1.6) is, in general, false for lattice polytopes without interior lattice points.

Another formula for calculating the Ehrhart Series from a given one concerns dilates. Here we will show

**Lemma 1.5.** Let $P \in \mathcal{P}^d$, $k \in \mathbb{N}$ and let $\zeta$ be a primitive $k$-th root of unity. Then

$$\text{Ehr}_{kP}(z) = \frac{1}{k} \sum_{i=0}^{k-1} \text{Ehr}_P(\zeta^i z^\frac{1}{k}).$$

The lemma can be used, for instance, to calculate the Ehrhart series of the cube $C_d = \{x \in \mathbb{R}^d : |x_i| \leq 1, 1 \leq i \leq d\}$.

**Example 1.6.** For two integers $j, d$, $0 \leq j \leq d$, let

$$A(d,j) = \sum_{k=0}^{j} (-1)^k \binom{d+1}{k} (j-k)^d.$$

Proposition 1.10. Let $f(z) = a_2 z^2 + a_1 z + 1$, $a_i \in \mathbb{N}$, satisfying the inequalities in (1.8). Then $f$ is the $h^*$-polynomial of a lattice polytope.
Concerning lower bounds on the coefficients \(g_i(P)\) for 0-symmetric polytopes \(P\) we only know, except the trivial case \(i = d\), a lower bound on \(g_{d-1}(P)\) (cf. (1.5)). Namely

\[
g_{d-1}(P) \geq g_{d-1}(C_d^*) = \frac{2^{d-1}}{(d-1)!},
\]

where \(C_d^* = \text{conv}\{\pm e_i : 1 \leq i \leq d\}\) denotes the regular cross-polytope. This follows immediately from a result of Richard P. Stanley [18, Theorem 3.1] on the \(h\)-vector of "symmetric" Cohen-Macaulay simplicial complex.

Motivated by a problem in [12] we study in the last section also the related question to bound the surface area \(F(P)\) of a lattice polytope \(P\). In contrast to the \(g_i(P)\)'s the surface area is not invariant under unimodular transformations. In order to describe our result we denote by \(T_d\) the standard simplex \(T_d = \text{conv}\{0, e_1, \ldots, e_d\}\).

**Proposition 1.11.** Let \(P \in \mathcal{P}^d\). Then

\[
F(P) \geq \begin{cases} 
F(C_d^*) = \frac{2^d}{d!} d^{2 \frac{d}{d-1}}, & \text{if } P = -P, \\
F(T_d) = \frac{d+\sqrt{d}}{(d-1)!}, & \text{otherwise}. 
\end{cases}
\]

The paper is organized as follows. In the next section we give the proof of our main Theorem [1.11]. Then, in Section 3, we prove the Lemmas [1.3] and [1.5] and show how the Ehrhart series in the Examples [1.4] and [1.6] can be deduced. Moreover, we will give the proof of Proposition [1.10]. Finally, in the last section we provide a proof of Proposition [1.11] which in the symmetric cases is based on a isoperimetric inequality for cross-polytopes (cf. Lemma [1.11]).

## 2. Lower bounds on \(g_i(P)\)

In the following we denote for an integer \(r\) and a polynomial \(f(x)\) the \(r\)-th coefficient of \(f(x)\), i.e. the coefficient of \(x^r\), by \(f(x)|_r\). Before proving Theorem [1.11] we need some basic properties of the numbers \(C_{r,i}^d\) and \(M_{r,d}\) defined in the introduction (see [1.3]). We begin with some special cases.

**Proposition 2.1.** Let \(d \geq 3\). Then \(M_{0,d} = 0, M_{d,d} = 1\) and

i) \(M_{1,d} = C_{1,d-2}^d = -(d-2)!\),

ii) \(M_{2,d} = C_{2,d-2}^d = (d-2)! + (-1)^d \text{stirl}(d-1,2)\),

iii) \(M_{d-1,d} = C_{d-1,1}^d = -\frac{d(d-3)}{2}\),

iv) \(M_{d-2,d} = \begin{cases} 
C_{d-2,d-\frac{1}{3}}^d = -\frac{1}{4} \left(\frac{d+1}{3}\right), & \text{if } d \text{ odd}, \\
C_{d-2,d-\frac{2}{3}}^d = -\frac{1}{4} \left(\frac{d}{3}\right), & \text{if } d \text{ even}. 
\end{cases}\)

**Proof.** The cases \(M_{0,d}\) and \(M_{d,d}\) are trivial. Since \(C_{r,i}^d\) is the \((d-r)\)-th elementary symmetric function of \(\{l, l-1, \ldots, l-(d-1)\}\) we have \(C_{1,i}^d = (-1)^{d-i-1} i!\ (d-
for $i=1, \ldots, d-1$. Then

$$M_{1,d} = \min\{C_{1,i}^d : 1 \leq i \leq d-2\} = C_{1,d-2}^d = -(d-2)!$$

In the case $r = 2$ we obtain by elementary calculations that

$$C_{2,i}^d = i! \text{stir}(d-i,2) + (-1)^d (d-i-1)! \text{stir}(i+1,2)$$

$$= i! (d-i-1)! (-1)^{d-i} \left( \sum_{k=1}^{d-i-1} \frac{1}{k} - \sum_{k=1}^{i} \frac{1}{k} \right),$$

from which we conclude $M_{2,d} = C_{2,d-2}^d = (d-2)! + (-1)^d \text{stir}(d-1,2)$.

For iii) we note that

$$C_{d-1,i}^d = \sum_{j=i-(d-1)}^{i} j = \frac{d}{2}(d-1-2i),$$

and so $M_{d-1,d} = C_{d-1,1}^d$. Finally, for the value of $M_{d-2,d}$ we first observe that

$$C_{d-2,i}^d - C_{d-2,i-1}^d = (z+i)(z+i-1) \ldots (z+(d-1)) \left|_{d-2} \right. - (z+i-1) \ldots (z+i-(d-1))(z+i-d) \left|_{d-2} \right.$$  

$$= \sum_{j=-d+i+1}^{i-1} j (i-(-d+i)) = d \sum_{j=-d+i+1}^{i-1} j$$

$$= d(d-1)(-d+2i).$$

Thus the function $C_{d-2,i}^d$ is decreasing in $0 \leq i \leq \lfloor d/2 \rfloor$ and increasing in $\lfloor d/2 \rfloor < i \leq d$. So it takes its minimum at $i = \lfloor d/2 \rfloor$. First let us assume that $d$ is odd. Then

$$M_{d-2,d} = C_{d-2,d-1}^d = d! \left( \frac{z + (d-1)/2}{d} \right)_{d-2}$$

$$= z(z^2 - 1)(z^2 - 4) \ldots (z^2 - ((d-1)/2)^2) \left|_{d-2} \right. = -\sum_{i=0}^{(d-1)/2} i^2$$

$$= -\frac{1}{4} \left( d+1 \right) \left( d+1 \right).$$

The even case can be treated similarly. \hfill \square

In addition to the previous proposition we also need

**Lemma 2.2.**

i) $C_{r,i}^d = (-1)^{d-r}C_{r,d-1-i}^d$ for $0 \leq i \leq d-1$.

ii) Let $d \geq 3$. Then $M_{r,d} \leq 0$ for $1 \leq r \leq d-1$, and $M_{r,d} = 0$ only in the case $d = 3$ and $r = 2$.

**Proof.** The first statement is just a consequence of the fact that $C_{r,i}^d$ is the $(d-r)$-th elementary symmetric function of $\{l, l-1, \ldots, l-(d-1)\}$. For ii) we first observe that the case $d = 3$ follows directly from Proposition 2.1. Hence it
remains to show that \( M_{r,d} < 0 \) for \( d \geq 4 \) and \( 1 \leq r \leq d - 1 \). On account of i) it suffices to prove this when \( d - r \) is even and we will proceed by induction on \( d \).

The case \( d = 4 \) is covered by Proposition [2.1]. So let \( d \geq 5 \). By Proposition [2.1]i) we also may assume \( r \geq 2 \). It is easy to see that

\[
(2.1) \quad C^d_{r,i} = (i - d + 1) C^{d-1}_{r,i} + C^{d-1}_{r-1,i},
\]

and by induction we may assume that there exists a \( j \in \{1, \ldots, d - 3\} \) with \( C^d_{r-1,j} < 0 \). Observe that \( d - 1 - (r - 1) \) is even. If \( C^d_{r,j} \geq 0 \) we obtain by \((2.1)\) that \( C^d_{r,j} < 0 \) and we are done. So let \( C^d_{r,j} < 0 \). By part i) we know that

\[
C^d_{r,j} = (-1)^d (-1 )^{r - j} C_{r,j}^{d - 1} \quad \text{and} \quad C^d_{r-1,j} = (-1)^{d-r} C_{r-1,j}^{d-1}.
\]

Since \( d - r \) is even we conclude \( C_{r,d-2-j}^{d-1} > 0 \) and \( C_{r-1,d-2-j}^{d-1} < 0 \). Hence, on account of \((2.1)\) we get \( C_{r,d-2-j}^{d} < 0 \) and so \( M_{r,d} < 0 \).

Now we are able to give the proof of our main Theorem.

**Proof of Theorem [1.1].** We follow the approach of Betke and McMullen used in [4, Theorem 6]. By expanding the Ehrhart series at \( z = 0 \) one gets (see e.g. [3, Lemma 3.14])

\[
(2.2) \quad G_P(z) = \sum_{i=0}^{d} a_i(P) \left( z + \frac{d-i}{d} \right).
\]

In particular, we have

\[
(2.3) \quad \frac{1}{d!} \sum_{i=0}^{d} a_i(P) = g_d(P) = \text{vol}(P).
\]

For short, we will write \( a_i \) instead of \( a_i(P) \) and \( g_i \) instead of \( g_i(P) \). With this notation we have

\[
(2.4) \quad d! g_r = d! G_P(z)|_r = d! \sum_{i=0}^{d} a_i \left( z + \frac{d-i}{d} \right)|_r = C^d_{r,d} + (a_1 C^d_{r,d-1} + a_d C^d_{r,0}) + \sum_{i=2}^{d-1} a_i C^d_{r,d-i}.
\]

Since \( C^d_{r,d-1} \geq 0 \) we get with Lemma [2.2]i) that \( C^d_{r,d-1} = |C^d_{r,0}| \). Together with \( a_1 = G(P) - (d+1) \geq G(\text{int}(P)) = a_d \) and \( C^d_{r,d} = (-1)^{d-r} \text{stirl}(d+1, r+1) \) we find

\[
(2.5) \quad d! g_r \geq (-1)^{d-r} \text{stirl}(d+1, r+1) + \sum_{i=2}^{d-1} a_i C^d_{r,d-i} = (-1)^{d-r} \text{stirl}(d+1, r+1) + \sum_{i=2}^{d-1} a_i \left( C^d_{r,d-i} - M_{r,d} \right) + \sum_{i=1}^{d} a_i M_{r,d} - (a_1 + a_d) M_{r,d} \\
\geq (-1)^{d-r} \text{stirl}(d+1, r+1) + (d! \text{vol}(P) - 1) M_{r,d},
\]

where \( \text{stirl}(n, k) \) denotes the Stirling number of the first kind.
where the last inequality follows from the definition of $M_{r,d}$ and the non-positivity of $M_{r,d}$ (cf. Proposition 2.1 and Lemma 2.2 ii)).

We remark that for $d \geq 3$, $r \in \{1, \ldots, d-1\}$ and $(r,d) \neq (2,3)$ we can slightly improve the inequalities in Theorem 1.1 because in these cases we have $M_{r,d} < 0$ (cf. Lemma 2.2 ii)), and since $C_{r,d}^{d-1}$ is the $(d-r)$-th elementary symmetric function of $\{0, \ldots, d-1\}$ we also know $C_{r,d}^{d-1} > 0$ for $1 \leq r \leq d-1$. Hence we get (cf. (2.4) and (2.5))

$$d! g_{s} = C_{r,d}^{d} + \sum_{i=1}^{d} a_{i} C_{r,d-i}^{d}$$

$$= C_{r,d}^{d} + a_{1} \left( C_{r,d-1}^{d} - M_{r,d} \right) + \sum_{i=2}^{d} \left( C_{r,d-i}^{d} - M_{r,d} \right) + \sum_{i=1}^{d} a_{i} M_{r,d}$$

$$\geq (-1)^{d-r} \text{stirl}(d+1, r+1) + 2a_{1}(P) + (d! \text{vol}(P) - 1)M_{r,d}$$

$$= (-1)^{d-r} \text{stirl}(d+1, r+1) - 2(d+1) + 2G(P) + (d! \text{vol}(P) - 1)M_{r,d}.$$

Corollary 1.2 is an immediate consequence of Theorem 1.1 and Proposition 2.1.

Proof of Corollary 1.2. The inequalities just follow by inserting the value of $M_{r,d}$ given in Proposition 2.1 in the general inequality of Theorem 1.1. Here we also have used the identities

$$\text{stirl}(d+1, 2) = (-1)^{d+1} d! \sum_{i=1}^{d} \frac{1}{i} \quad \text{and} \quad \text{stirl}(d+1, d-1) = \frac{3d+2}{4} \left( \begin{array}{c} d+1 \\ 3 \end{array} \right).$$

It remains to show that the inequalities are best possible for any volume. For $r = d-2$ we consider the simplex $T_{d}^{(m)}$ (cf. (1.7)) with $a_{0}(T_{d}^{(m)}) = 1$, $a_{[d/2]}(T_{d}^{(m)}) = (m-1)$ and $a_{i}(T_{d}^{(m)}) = 0$ for $i \notin \{0, [d/2]\}$. Then $\text{vol}(T_{d}^{(m)}) = m/d!$ and on account of Proposition 2.1 we have equality in (2.4) and (2.5).

For $r = 1, 2$ and $d \geq 4$ we consider the $(d-4)$-fold pyramid $\tilde{T}_{d}^{(m)}$ over $T_{4}^{(m)}$ given by $\tilde{T}_{d}^{(m)} = \text{conv}\{ T_{4}^{(m)}, e_{5}, \ldots, e_{d} \}$. Then $\text{vol}(\tilde{T}_{d}^{(m)}) = m/d!$ and in view of (1.7) and [3, Theorem 2.4] we obtain

$$a_{0}(\tilde{T}_{d}^{(m)}) = 1, \quad a_{2}(\tilde{T}_{d}^{(m)}) = m-1 \quad \text{and} \quad a_{i}(\tilde{T}_{d}^{(m)}) = 0, \ i \notin \{0, 2\}.$$ 

Again, by Proposition 2.1 we have equality in (2.4) and (2.5). 

3. Ehrhart series of some special polytopes

We start with the short proof of Lemma 1.3.

Proof of Lemma 1.3. Since

$$\text{Ehr}_{P}(z) \text{Ehr}_{Q}(z) = \sum_{k \geq 0} \left( \sum_{m+l=k} G_{P}(m)G_{Q}(l) \right) z^{k},$$
it suffices to prove that the Ehrhart polynomial $G_{P*Q}(k)$ of the lattice polytope $P*Q \in \mathbb{P}^{p+q+1}$ is given by

$$G_{P*Q}(k) = \sum_{m+l=k} G_P(m)G_Q(l).$$

This, however, follows immediately from the definition since

$$k(P*Q) = \{\lambda (x, o_q, 0)^\top + (k - \lambda)(o_p, y, 1)^\top : x \in P, y \in Q, 0 \leq \lambda \leq k\}. \quad \Box$$

Example [1.4] in the introduction shows an application of this construction. For Example [1.6] we need Lemma [1.5].

**Proof of Lemma [1.5]** With $w = z^\frac{1}{k}$ we may write

$$\frac{1}{k} \sum_{i=0}^{k-1} \text{Ehr}_P(\zeta^i w) = \frac{1}{k} \sum_{i=0}^{k-1} \sum_{m \geq 0} G_P(m)(\zeta^i w)^m = \frac{1}{k} \sum_{m \geq 0} G_P(m)w^m \sum_{i=0}^{k-1} \zeta^i m.$$ 

Since $\zeta$ is a $k$-th root of unity the sum $\sum_{i=0}^{k-1} \zeta^i m$ is equal to $k$ if $m$ is a multiple of $k$ and otherwise it is $0$. Thus we obtain

$$\frac{1}{k} \sum_{i=0}^{k-1} \text{Ehr}_P(\zeta^i w) = \sum_{m \geq 0} G_P(mk)w^{mk} = \sum_{m \geq 0} G_{kp}(m)z^m = \text{Ehr}_{kp}(z). \quad \Box$$

As an application of Lemma [1.5] we calculate the Ehrhart series of the cube $C_d$ (cf. Example [1.6]). Instead of $C_d$ we consider the translated cube $2\tilde{C}_d$, where $\tilde{C}_d = \{x \in \mathbb{R}^d : 0 \leq x_i \leq 1, 1 \leq i \leq d\}$. In [3, Theorem 2.1] it was shown that $a_i(\tilde{C}_d) = A(d, i + 1)$ where $A(d, i)$ denotes the Eulerian numbers. Setting $w = \sqrt{z}$ Lemma [1.5] leads to

$$\text{Ehr}_{C_d}(z) = \frac{1}{2} \left( \text{Ehr}_{\tilde{C}_d}(w) + \text{Ehr}_{\tilde{C}_d}(-w) \right)$$

$$= \frac{1}{2} \left( \sum_{i=1}^{d} A(d, i) \frac{w^{i-1}}{(1 - w)^{d+1}} + \sum_{i=1}^{d} A(d, i) \frac{(-w)^{i-1}}{(1 + w)^{d+1}} \right)$$

$$= \frac{1}{2} \frac{1}{(1 - z)^{d+1}} \left( \sum_{i=1}^{d} A(d, i) w^{i-1} (1 + w)^d + \sum_{i=1}^{d} A(d, i) (-w)^{i-1} (1 - w)^d \right)$$

$$= \frac{1}{(1 - z)^{d+1}} \left( \sum_{i=1}^{d} A(d, i) \sum_{j=0, i + j - 1 \text{ even}}^{d+1} \binom{d+1}{j} w^{i+j-1} \right)$$
Substituting \(2l = i + j - 1\) gives

\[
\text{Ehr}_{C_d}(z) = \frac{1}{(1-z)^{d+1}} \left( \sum_{l=0}^{d} \sum_{i=2l-d}^{2l+1} \binom{d+1}{2l+1-i} A(d, i) w^{2l} \right)
\]

\[
= \frac{1}{(1-z)^{d+1}} \left( \sum_{l=0}^{d} \sum_{j=0}^{d+1} \binom{d+1}{j} A(d, 2l+1-j) \right),
\]

which explains the formula in Example 1.6.

In order to calculate in general the Ehrhart series of the prism \(P = \{(x, x_d) \in \mathbb{R}^d : x \in Q, x_d \in [0, m]\}\) where \(Q \in \mathcal{P}^{d-1}\), \(m \in \mathbb{N}\) (cf. Example 1.7), we use the differential operator \(T\) defined by \(\frac{d}{dz}\). Considered as an operator on the ring of formal power series we have (cf. e.g. [3, p. 28])

\[
(3.1) \quad \sum_{k \geq 0} f(k) z^k = f(T) \frac{1}{1-z}
\]

for any polynomial \(f\). Since \(G_P(k) = (m k + 1) G_Q(k)\) we deduce from (3.1)

\[
\text{Ehr}_P(z) = (m T + 1) \text{Ehr}_Q(z) = m z \frac{d}{dz} \text{Ehr}_Q(z) + \text{Ehr}_Q(z).
\]

Thus

\[
\text{Ehr}_P(z) = m z \sum_{i=0}^{d} \binom{i}{d+i} z^i (1-z) + \sum_{i=0}^{d-1} m d a_i(Q) z^i \frac{(1-z)}{(1-z)^{d+1}}
\]

\[
= \sum_{i=0}^{d-1} \binom{m i + 1}{d+i} a_i(Q) z^i (1-z) + \sum_{i=0}^{d-1} m d a_i(Q) z^{i+1} \frac{(1-z)}{(1-z)^{d+1}}
\]

\[
= \frac{1}{(1-z)^{d+1}} \sum_{i=1}^{d} ((m i + 1) a_i(Q) + (m(d - i + 1) - 1) a_{i-1}(Q)) z^i,
\]

which is the formula in Example 1.7.

Finally, we come to the classification of \(h^*\)-polynomials of degree 2.

\textbf{Proof of Proposition 1.10.} We recall that \(a_1(P) = G(P) - (d+1)\) and \(a_d(P) = G(\text{int}(P))\) for \(P \in \mathcal{P}^d\). In the case \(a_2 = 1, a_1 = 7\) the triangle \(\text{conv}\{0, 3e_1, 3e_2\}\) has the desired \(h^*\)-polynomial. Next we distinguish two cases:

i) \(a_2 < a_1 \leq 3a_2 + 3\). For integers \(k, l, m \) with \(0 \leq l, k \leq m + 1\) let \(P \in \mathcal{P}^2\) given by \(P = \text{conv}\{0, l e_1, e_2 + (m + 1) e_1, 2 e_2, 2 e_2 + k e_1\}\). Then it is easy to see that \(a_2(P) = m\) and \(P\) has \(k + l + 4\) lattice points on the boundary. Thus \(a_1(P) = k + l + m + 1\).

ii) \(a_1 \leq a_2\). For integers \(l, m \) with \(0 \leq l \leq m\) let \(P \in \mathcal{P}^3\) given by \(P = \text{conv}\{0, e_1, e_2, -l e_3, e_1 + e_2 + (m + 1) e_3\}\). The only lattice points contained in \(P\) are the vertices and the lattice points on the edge \(\text{conv}\{0, -e_3\}\). Thus \(a_3(P) = 0\) and \(a_1(P) = l\). On the other hand, since \((l + m + 1)/6 = \text{vol}(P) = (\sum_{i=0}^{3} a_i(P))/6\) (cf. 2.3) it is \(a_2(P) = m\). \(\square\)
4. 0-SYMMETRIC LATTICE POLYTOPES

In order to study the surface area of 0-symmetric polytopes we first prove an isoperimetric inequality for the class of cross-polytopes.

**Lemma 4.1.** Let \( v_1, \ldots, v_d \in \mathbb{R}^d \) be linearly independent and let \( C = \text{conv}\{\pm v_i : 1 \leq i \leq d\} \). Then

\[
\frac{F(C)^d}{\text{vol}(C)^{d-1}} \geq \frac{2^d}{d!} \frac{d^2}{d^3},
\]

and equality holds if and only if \( C \) is a regular cross-polytope, i.e., \( v_1, \ldots, v_d \) form an orthogonal basis of equal length.

**Proof.** Without loss of generality let \( \text{vol}(C) = 2^d/d! \). Then we have to show

\[
(4.1) \quad F(C)^d \geq \frac{2^d}{d!} \frac{d^2}{d^3}.
\]

By standard arguments from convexity (see e.g. [10, Theorem 6.3]) the set of all 0-symmetric cross-polytopes with volume \( 2^d/d! \) contains a cross-polytope \( C^* = \text{conv}\{\pm w_1, \ldots, \pm w_d\} \), say, of minimal surface area. Suppose that some of the vectors are not pairwise orthogonal, for instance, \( w_1 \) and \( w_2 \). Then we apply to \( C^* \) a Steiner-Symmetrization (cf. e.g. [10, pp. 169]) with respect to the hyperplane \( H = \{x \in \mathbb{R}^d : w_i x = 0\} \). It is easy to check that the Steiner-symmetral of \( C^* \) is again a cross-polytope \( \tilde{C}^* \), say, with \( \text{vol}(\tilde{C}^*) = \text{vol}(C^*) \) (cf. [10, Proposition 9.1]). Since \( C^* \) was not symmetric with respect to the hyperplane \( H \) we also know that \( F(\tilde{C}^*) < F(C^*) \) which contradicts the minimality of \( C^* \) (cf. [10, p. 171]).

So we can assume that the vectors \( w_i \) are pairwise orthogonal. Next suppose that \( \|w_1\| > \|w_2\| \), where \( \| \cdot \| \) denotes the Euclidean norm. Then we apply Steiner-Symmetrization with respect to the hyperplane \( H \) which is orthogonal to \( w_1 - w_2 \) and bisecting the edge \( \text{conv}\{w_1, w_2\} \). As before we get a contradiction to the minimality of \( C^* \).

Thus we know that \( w_i \) are pairwise orthogonal and of same length. By our assumption on the volume we get \( \|w_i\| = 1, 1 \leq i \leq d \), and it is easy to calculate that \( F(C^*) = (2^d/d!)d^{3/2} \). So we have

\[
F(C) \geq F(C^*) = \frac{2^d}{d!} \frac{d^3}{d^2},
\]

and by the foregoing argumentation via Steiner-Symmetrizations we also see that equality holds if and only \( C \) is a regular cross-polytope generated by vectors of unit-length. \( \square \)

The determination of the minimal surface area of 0-symmetric lattice polytopes is an immediate consequence of the lemma above, whereas the non-symmetric case does not follow from the corresponding isoperimetric inequality for simplices.

**Proof of Proposition 1.11.** Let \( P \in \mathcal{P}^d \) with \( P = -P \). Then \( P \) contains a 0-symmetric lattice cross-polytope \( C = \text{conv}\{\pm v_i : 1 \leq i \leq d\} \), say, and by the
monotonicity of the surface area and Lemma 4.1 we get
\begin{equation}
F(P) \geq F(C) \geq \left( \frac{2^d}{d!} \right)^\frac{1}{d^2} \frac{\text{vol}(C)}{d^\frac{d-1}{d}}.
\end{equation}
Since \( v_i \in \mathbb{Z}^d \), \( 1 \leq i \leq d \), we have \( \text{vol}(C) = (2^d/d!) |\det(v_1, \ldots, v_d)| \geq 2^d/d! \), which shows by (4.2) the 0-symmetric case.

In the non-symmetric case we know that \( P \) contains a lattice simplex \( T = \{ x \in \mathbb{R}^d : a_i x \leq b_i, 1 \leq i \leq d + 1 \} \), say. Here we may assume that \( a_i \in \mathbb{Z}^n \) are primitive, i.e., \( \text{conv}\{0, a_i\} \cap \mathbb{Z}^n = \{0, a_i\} \), and that \( b_i \in \mathbb{Z} \). Furthermore, we denote the facet \( \{ x \in \mathbb{R}^d : a_i x = b_i \} \) by \( F_i \), \( 1 \leq i \leq d + 1 \). With these notations we have \( \det(\text{aff} F_i \cap \mathbb{Z}^n) = \|a_i\| \) (cf. [14, Proposition 1.2.9]). Hence there exist integers \( k_i \geq 1 \) with
\begin{equation}
\text{vol}_{d-1}(F_i) = k_i \frac{\|a_i\|}{(d-1)!},
\end{equation}
and so we may write
\begin{equation}
F(P) \geq F(T) = \sum_{i=1}^{d+1} \text{vol}_{d-1}(F_i) \geq \frac{1}{(d-1)!} \sum_{i=1}^{d+1} \|a_i\|.
\end{equation}
We also have \( \sum_{i=1}^{d+1} \text{vol}_{d-1}(F_i)a_i/\|a_i\| = 0 \) (cf. e.g. [10, Theorem 18.2]) and in view of (4.3) we obtain \( \sum_{i=1}^{d+1} k_i a_i = 0 \). Thus, since the \( d+1 \) lattice vectors \( a_i \) are affinely independent we can find for each index \( j \in \{1, \ldots, d\} \) at least two vectors \( a_{i_1} \) and \( a_{i_2} \) having a non-trivial \( j \)-th coordinate. Hence
\begin{equation}
\sum_{i=1}^{d+1} \|a_i\|^2 \geq 2d.
\end{equation}
Together with the restrictions \( \|a_i\| \geq 1, 1 \leq i \leq d + 1 \), it is easy to argue that \( \sum_{i=1}^{d+1} \|a_i\| \) is minimized if and only if \( d \) norms \( \|a_i\| \) are equal to 1 and one is equal to \( \sqrt{d} \). For instance, the intersection of the cone \( \{ x \in \mathbb{R}^{d+1} : x_i \geq 1, 1 \leq i \leq d + 1 \} \) with the hyperplane \( H_\alpha = \{ x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i = \alpha \}, \alpha \geq d + 1 \), is the \( d \)-simplex \( T(\alpha) \) with vertices given by the permutations of the vector \( (1, \ldots, 1, \alpha - d)^T \) of length \( \sqrt{d + (\alpha - d)^2} \). Therefore, a vertex of that simplex is contained in \( \{ x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i^2 \geq 2d \} \) if \( \alpha \geq d + \sqrt{d} \). In other words, we always have
\begin{equation}
\sum_{i=1}^{d+1} \|a_i\| \geq d + \sqrt{d},
\end{equation}
which gives the desired inequality in the non-symmetric case (cf. (1.3)).

We remark that the proof also shows that equality in Proposition 1.11 holds if and only if \( P \) is the \( o \)-symmetric cross-polytope \( C^*_d \) or the simplex \( T_d \) (up to lattice translations).

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