Sampling Theorems for Learning from Incomplete Measurements

Julián Tachella¹, Dongdong Chen² and Mike Davies²

¹Physics laboratory, CNRS & ENS de Lyon, France
²School of Engineering, University of Edinburgh, UK

January, 2022

Abstract

In many real-world settings, only incomplete measurement data are available which can pose a problem for learning. Unsupervised learning of the signal model using a fixed incomplete measurement process is impossible in general, as there is no information in the nullspace of the measurement operator. This limitation can be overcome by using measurements from multiple operators. While this idea has been successfully applied in various applications, a precise characterization of the conditions for learning is still lacking. In this paper, we fill this gap by presenting necessary and sufficient conditions for learning the signal model which indicate the interplay between the number of distinct measurement operators \( G \), the number of measurements per operator \( m \), the dimension of the model \( k \) and the dimension of the signals \( n \).

In particular, we show that generically unsupervised learning is possible if each operator obtains at least \( m > k + n/G \) measurements. Our results are agnostic of the learning algorithm and have implications in a wide range of practical algorithms, from low-rank matrix recovery to deep neural networks.

1 Introduction

In multiple sensing applications, we observe measurements \( y \in \mathbb{R}^m \) associated with a signal \( x \in \mathcal{X} \subseteq \mathbb{R}^n \), through the forward process

\[
y = Ax + \epsilon
\]  

(1)

where \( A \in \mathbb{R}^{n \times n} \) is a linear measurement operator and \( \epsilon \) denotes the noise affecting the measurements. This is the case of computed tomography [1], depth ranging [2] and non-line-of-sight imaging [3] to name a few. Estimating \( x \) from \( y \) is generally an ill-posed inverse problem due to the incomplete operator \( A \) with \( m < n \) and the presence of noise. Knowledge of the signal model is required to make this problem well-posed.

In many cases, obtaining ground-truth reconstructions \( x \) to learn the signal model might be very expensive or even impossible. For example, in medical imaging, it is not always possible to obtain fully sampled images of patients as they require long acquisition times. In astronomical imaging, it is impossible to obtain ground-truth references due to physical limitations. In electron-microscopy imaging [4], we can only measure 2D projections of a molecule. In these settings, we can only access measurements \( y \) for learning. Moreover, if the measurement process \( A \) is incomplete, it is fundamentally impossible to learn the model with only measurements \( y \), as there is no information
about the model in the nullspace of $A$. Thus, we end up with a chicken-and-egg problem: in order to reconstruct $x$ we need the signal model, but to learn this model we require some reconstructed samples $x$.

This fundamental limitation can be overcome by using information from multiple incomplete sensing operators $A_1, \ldots, A_G$, the general principle being that each operator can provide additional information about the model if it has a different nullspace. For example, in the image inpainting problem, Studer and Baraniuk [3] used the fact that the set of missing pixels may vary between observed images to learn a sparse dictionary model. Yang et al. [6] used multiple operators to learn a Gaussian mixture model in the context of hyperspectral imaging and high-speed video. Bora et al. [7] exploited this idea for learning a generative model in various imaging problems such as deblurring and compressed sensing. Matrix completion methods [8] exploit a similar principle, as the missing entries of each column (i.e., signal) are generally different. Ideally we would like to learn the signal model from only a small number of different measurement operators. We are thus motivated to determine typically how many such operators are required.

The problem can be formalized as follows. We first focus on the noiseless case to study the intrinsic identifiability problems associated to having only incomplete measurement data. The effect of noise will be discussed in Section 4. We assume that we observe a set of $N$ training samples $y_i$, where the $i$th signal is observed via $A_{g_i} \in \mathbb{R}^{m \times n}$, one of $G$ linear operators, i.e.,

$$y_i = A_{g_i}x_i$$

where $g_i \in \{1, \ldots, G\}$ and $i = 1, \ldots, N$. While we assume that the measurement operator $A_{g_i}$ is known for all observed signals, it is important to note that we do not know a priori if two observations $(y_i, A_{g_i})$ and $(y_i, A_{g_{i'}})$ are related to the same signal $x_i$. There are two natural questions regarding this learning problem:

**Q1. Signal Recovery** Is there a unique signal $x \in \mathcal{X}$ which verifies the measurements $y = A_g x$?

**Q2. Model Recovery** Can we uniquely identify the data distribution $p(x)$ and its support $\mathcal{X}$ from only measurements obtained via incomplete operators?

In general, there can be a unique solution for neither problem, just one or both. There might be a unique solution for signal recovery if the model is known, but it might be impossible to learn the model in the first place (e.g., blind compressed sensing [9]). The converse is also possible, that is, uniquely identifying a model without having enough measurements per sample to uniquely identify the associated signal (e.g., subspace learning from rank-1 projections [10]).

The answer to Q1 is well-known from generalized compressed sensing theory, see for example [11]. Unique signal recovery is possible if the signal model is low-dimensional, i.e., if $A_g$ has $m > 2k$ measurements, where $k$ is the model dimension. On the other hand, Q2 has been mostly studied in the context of matrix completion, where the set of signals is assumed to lie in a low-dimensional subspace of $\mathbb{R}^n$. Bora et al. [7] presented some results on the general setting, but only for the case where $G = \infty$ which is quite restrictive. In this paper, we provide sharp necessary and sufficient conditions which hold for any low-dimensional distribution (beyond linear subspaces) and only require a finite number of operators $G$. The main contributions of this paper are as follows:

- We show that unsupervised learning from a finite number of incomplete measurement operators is only possible if the model is low-dimensional. If the dimension of the model is large, i.e., $k \approx n$, the signal model cannot be uniquely identified if $m < n$.

---

1If the signal distribution is continuous, observing such a pair has probability 0.
We show that \( m \geq n/G \) measurements per operator are necessary for learning, and that for almost every set of \( G \) operators, \( m > k + n/G \) measurements per operator are sufficient.

A series of experiments demonstrate that our bounds accurately characterize the performance of popular learning algorithms on synthetic and real datasets.

1.1 Related Work

**Blind Compressed Sensing** The fundamental limitation of failing to learn a signal model from compressed data goes back to blind compressed sensing [9] for the specific case of models exploiting sparsity on an orthogonal dictionary. In order to learn the dictionary from compressed samples, [9] imposed additional constraints on the dictionary, while some subsequent papers [12, 13] removed these assumptions by proposing to use multiple operators \( A_g \) as studied here. This paper can be seen as a generalization of such results to more general signal models.

**Matrix Completion** Matrix completion consists of inferring missing entries of a data matrix \( Y = [y_1, \ldots, y_N] \), whose columns are generally inpainted samples from a low-dimensional distribution, i.e., \( y_i = A_g x_i \) where the operators \( A_g \) randomly select a subset of \( m \) entries of the signal \( x_i \). This problem can be viewed as the combination of model recovery, i.e., identifying the low-rank subspace that the columns of \( X = [x_1, \ldots, x_N] \) belong to, and signal recovery, i.e., reconstructing the individual columns. Assuming that the samples belong to a \( k \)-dimensional subspace can be imposed by recovering a rank-\( k \) signal matrix \( X \) from \( Y \). If the columns are sampled via \( G \) sufficiently different patterns \( A_g \) with the same number of entries \( m \), a sufficient condition [14] for uniquely recovering almost every subspace model is \( 2m \geq \left( 1 - 1/G \right) k + n/G \).

A similar necessary condition was shown in [15] for the case of high-rank matrix completion [16], which arises when the samples \( x_i \) belong to a union of \( k \)-dimensional subspaces. We show that unique recovery is possible for almost every set of \( G \) operators with \( m > k + n/G \) measurements, however the theory presented here goes beyond linear subspaces, being also valid for general low-dimensional models.

**Deep Nets for Inverse Problems** Despite providing very competitive results, most deep learning based solvers require measurements and signal pairs \( (x_i, y_i) \) in order to learn the reconstruction function \( y \mapsto x \) from incomplete measurements. A first step to overcome this limitation is due to Noise2Noise [17], where the authors show that it is possible to learn from only noisy samples. However, their ideas only apply to denoising settings where there is a trivial nullspace, as the operator \( A \) is the identity matrix. In Artifact2Artifact [18], it was empirically shown that it is possible to exploit different measurement operators to learn the reconstruction function in the context of magnetic resonance imaging. [7] proposed to learn a signal distribution from only incomplete measurements using multiple forward operators, however they only provide reconstruction guarantees for the case where an infinite number of operators \( A_g \) is available \(^3\), a condition that is not met in practice.

## 2 Signal Recovery Preliminaries

We denote the nullspace of \( A \) as \( \mathcal{N}_A \). Its complement, the range space of the pseudo-inverse \( A^\dagger \), is denoted as \( \mathcal{R}_A \), where \( \mathcal{R}_A \oplus \mathcal{N}_A = \mathbb{R}^n \) and \( \oplus \) denotes the direct sum. Throughout the paper, we

---

\(^2\) A larger number of measurements \( m = \mathcal{O}(k \log n) \) is required to guarantee a stable recovery when the number of patterns \( G \) is large [8].

\(^3\) Their result relies on the Cramér-Wold theorem, which is discussed in Section 5.
assume that the signals are sampled from a distribution \( p(x) \) supported on the signal set \( \mathcal{X} \subset \mathbb{R}^n \). Signal recovery has a unique solution if and only if the forward operator \( x \mapsto y \) is one-to-one, i.e., if for every pair of signals \( x_1, x_2 \in \mathcal{X} \) where \( x_1 \neq x_2 \) we have that

\[
Ax_1 \neq Ax_2 \quad (3)
\]

\[
A(x_1 - x_2) \neq 0 \quad (4)
\]

In other words, there is no vector \( x_1 - x_2 \neq 0 \) in the nullspace of \( A \). It is well-known that this is only possible if the signal set \( \mathcal{X} \) is low-dimensional. There are multiple ways to define the notion of dimensionality of a set in \( \mathbb{R}^n \). In this paper, we focus on the upper box-counting dimension which is defined for a compact subset \( S \subset \mathbb{R}^n \) as

\[
\text{boxdim}(S) = \limsup_{\epsilon \to 0} \frac{\log N(S, \epsilon)}{-\log \epsilon} \quad (5)
\]

where \( N(S, \epsilon) \) is the number of closed balls of radius \( \epsilon \) with respect to the norm \( \| \cdot \| \) that are required to cover \( S \). This definition of dimension covers both well-behaved models such as compact manifolds and more general low-dimensional sets. The mapping \( x \mapsto y \) is one-to-one for almost every forward operator \( A \in \mathbb{R}^{m \times n} \) if

\[
m > \text{boxdim}(\Delta \mathcal{X}) \quad (6)
\]

where \( \Delta \mathcal{X} \) denotes the normalized secant set which is defined as

\[
\Delta \mathcal{X} = \{ \Delta x \in \mathbb{R}^n | \Delta x = \frac{x_2 - x_1}{\|x_2 - x_1\|}, x_1, x_2 \in \mathcal{X}, x_2 \neq x_1 \}. \quad (7)
\]

The term almost every means that the complement has Lebesgue measure 0 in the space of linear measurement operators \( \mathbb{R}^{m \times n} \). The normalized secant set of models of dimension \( k \) generally has dimension \( 2k \), requiring \( m > 2k \) measurements to ensure signal recovery. For example, the union of \( k \)-dimensional subspaces requires at least \( 2k \) measurements\(^4\) to guarantee one-to-one-ness \([20]\). This includes well-known models such as \( k \)-sparse models (e.g., convolutional sparse coding \([21]\)) and co-sparse models (e.g., total variation \([22]\)). In the regime \( k < m \leq 2k \), the subset of signals where one-to-oneness fails is at most \((2k - m)\)-dimensional \([19]\).

### 3 Uniqueness of Any Model?

A natural first question when considering uniqueness of the model is: can we recover any distribution \( p(x) \) observed via forward operators \( A_1, \ldots, A_G \), even in the case where \( \mathcal{X} \) is the full \( \mathbb{R}^n \)? We show that, in general, the answer is no.

Uniqueness can be analysed from the point of view of the characteristic function of \( p(x) \), defined as \( \varphi(w) = \mathbb{E}\{e^{iw^\top x}\} \) where the expectation is taken with respect to \( p(x) \) and \( i = \sqrt{-1} \) is the imaginary unit. If two distributions have the same characteristic function, then they are necessarily the same almost everywhere. Each forward operator provides information about a subspace of the characteristic function as

\[
\mathbb{E}\{e^{iw^\top A_y x}\} = \mathbb{E}\{e^{iw^\top A_y A_y^\top x}\} = \mathbb{E}\{e^{i(A_y^\top A_y)^\top x}\} = \varphi(w) \mathbf{1}_{w \in \mathbb{R}^n} \quad (8)
\]

\[
\mathbb{E}\{e^{iA_y^\top A_y w}\} = \mathbb{E}\{e^{iA_y^\top A_y A_y w}\} = \mathbb{E}\{e^{i(A_y^\top A_y)^\top w}\} = \varphi(w) \mathbf{1}_{w \in \mathbb{R}^n} \quad (9)
\]

\[
\mathbb{E}\{e^{iA_y^\top A_y w}\} = \mathbb{E}\{e^{iA_y^\top A_y A_y w}\} = \mathbb{E}\{e^{i(A_y^\top A_y)^\top w}\} = \varphi(w) \mathbf{1}_{w \in \mathbb{R}^n} \quad (10)
\]

\(^4\)While the bound in \([6]\) guarantees unique signal recovery, more measurements (e.g., an additional factor of \( \mathcal{O}(\log n) \) measurements) are typically necessary in order to have a stable inverse \( f : y \mapsto x \), i.e., possessing a certain Lipschitz constant. A detailed discussion can be found for example in \([11]\).
where $1_S$ denotes the indicator function on the set $S$. Given that $m < n$, the characteristic function is only observed in the subspaces $R_{A_g}$ for all $g \in \{1, \ldots, G\}$. For any finite number of operators, the union of these subspaces does not cover the whole $\mathbb{R}^n$, and hence there is loss of information, i.e., the signal model cannot be uniquely identified.

In the case of an infinite number of operators $G = \infty$, the Cramér-Wold theorem guarantees uniqueness of the signal distribution if all possible one dimensional projections ($m = 1$) are available \cite{23,7}. However, in most practical settings we can only access a finite number of operators and many distributions will be non-identifiable.

4 Uniqueness of Low-Dimensional Models

Most models appearing in signal processing and machine learning are assumed to be approximately low-dimensional, with a dimension $k$ which is much lower than the ambient dimension $n$. As discussed in Section 2, the low-dimensional property is the key to obtain stable reconstructions, e.g., in compressed sensing. In the rest of paper, we impose the following assumptions on the model:

**A1** The signal set $\mathcal{X}$ is either

(a) A bounded set with box-counting dimension $k$.

(b) An unbounded conic set whose intersection with the unit sphere has box-counting dimension $k - 1$.

This assumption has been widely adopted in the inverse problems literature, as it is a necessary assumption to guarantee signal recovery. Our definition of dimension covers most models used in practice, such as union of subspaces (simple subspace models, convolutional sparse coding models, $k$-sparse models), low-rank matrices and compact manifolds. In many applications, the main goal is to learn the reconstruction function $f : y \rightarrow x$, which only requires knowledge of the low-dimensional support $\mathcal{X}$ and not the full distribution $p(x)$. Moreover, if there is a one-to-one reconstruction function, uniqueness of the support necessarily implies uniqueness of $p(x)$:

**Proposition 4.1.** Assume there is a measurable one-to-one mapping between measurements and signals. Then it is possible to uniquely recovery the signal distribution $p(x)$ if its support $\mathcal{X}$ is known.

**Proof.** If $\mathcal{X}$ is known and there is a measurable one-to-one mapping from each observed signal $y_i$ to $\mathcal{X}$, then it is possible to obtain $p(x)$ as the push-forward of the measurement distribution. \qed

Before delving into the main theorems, we present a simple example which provides intuition of how a low-dimensional model can be learned via multiple projections $A_g$:

**Learning a one-dimensional subspace** Consider a toy signal model with support $\mathcal{X} \subset \mathbb{R}^3$ which consists of a one-dimensional linear subspace spanned by $\phi = [1, 1, 1]^	op$, and $G = 3$ measurement operators $A_1, A_2, A_3 \in \mathbb{R}^{2 \times 3}$ which project the signals into the $x(3) = 0$, $x(2) = 0$ and $x(1) = 0$ planes respectively, where $x(i)$ denotes the $i$th entry of the vector $x$. The example is illustrated in Figure 1. The first operator $A_1$ imposes a constraint on $\mathcal{X}$, that is, every $x \in \mathcal{X}$ should verify $x(1) - x(2) = 0$. Without more operators providing additional information about $\mathcal{X}$, this constraint yields a plane containing $\mathcal{X}$, and there are infinitely many one-dimensional models that would fit the training data perfectly. However, the additional operator $A_2$ adds the constraint $x(2) - x(3) = 0$, which is sufficient to uniquely identify $\mathcal{X}$ as

$$
\hat{\mathcal{X}} = \mathcal{X} = \{v \in \mathbb{R}^3 \mid v(1) - v(2) = v(2) - v(3) = 0\}
$$
Figure 1: Toy example of a 1-dimensional subspace embedded in $\mathbb{R}^3$. If we only observe the projection of the signal set into the plane $x(3) = 0$, then there are infinite possible lines that are consistent with the measurements. Adding the projection into the $x(1) = 0$ plane, allows us to uniquely identify the signal model.

is the desired 1-dimensional subspace. Finally, note that in this case the operator $A_3$ does not restrict the signal set further, as the constraint $x(1) - x(3) = 0$ is verified by the other two constraints.

The ideas from the one-dimensional subspace example can be generalized and formalized as follows: for each projection $A_g$, we can constrain the model support $\mathcal{X}$ by considering the set

$$\hat{\mathcal{X}}_g = \{ v \in \mathbb{R}^n \mid v = \hat{x}_g + u, \hat{x}_g \in \mathcal{X}, u \in \mathcal{N}_{A_g} \}$$

(11)

which has dimension at most $n - (m - k)$. Note that the true signal model is a subset of $\hat{\mathcal{X}}_g$. The inferred signal set belongs to the intersection of these sets

$$\hat{\mathcal{X}} = \bigcap_{g \in \mathcal{G}} \hat{\mathcal{X}}_g$$

(12)

which can be expressed concisely as

$$\hat{\mathcal{X}} = \{ v \in \mathbb{R}^n \mid A_g(x_g - v) = 0, g = 1, \ldots, G, x_1, \ldots, x_G \in \mathcal{X} \}$$

(13)

Even though we have derived the set $\hat{\mathcal{X}}$ from a purely geometrical argument, the constraints in (13) also offer a simple algebraic intuition: the inferred signal set consists of the points $v \in \mathbb{R}^n$ which verify the following system of equations

$$\begin{bmatrix} A_1 & \vdots & A_G \end{bmatrix} v = \begin{bmatrix} A_{1}x_1 \\ \vdots \\ A_{G}x_G \end{bmatrix}.$$ 

(14)

for all possible choices of $G$ points $x_1, \ldots, x_G$ in $\mathcal{X}$. In other words, given a dataset of $N$ incomplete measurements $\{A_gx_i\}_{i=1}^N$, it is possible to build $\hat{\mathcal{X}}$ by trying all the possible combinations of $G$ samples and keeping only the points $v$ which are the solutions of (14).

It is trivial to see that $\mathcal{X} \subseteq \hat{\mathcal{X}}$, but when can we guarantee $\hat{\mathcal{X}} = \mathcal{X}$? As in the previous toy example, if there are not enough constraints, e.g., if we have a single $A$ and no additional transformations, the inferred set will have a dimension larger than $k$, containing undesired aliases. In particular, we have the following lower bound on the minimum number of measurements:

$^5$Despite providing a good intuition, this procedure for estimating $\mathcal{X}$ is far from being practical as it would require an infinite number of observed samples if the dimension of the signal set is not trivial $k > 0$. 

6
Proposition 4.2 (Theorem 1 in [24]). A necessary condition for model uniqueness is that 
\[ m \geq n/G \].

*Proof.* In order to have model uniqueness, the system in (14) should only admit a solution if 
\[ v = x_1 = \cdots = x_G \]. If \( m < n/G \) there is more than one solution for any choice of \( x_1, \ldots, x_G \in X \) as 
the matrix on the left hand side of (14) has rank smaller than \( n \).

Note that this necessary condition does not take into account the dimension of the model. As 
discussed in Section 3, a sufficient condition for model uniqueness must depend on the dimension of 
the signal set \( k \). Our main theorem shows that \( k \) additional measurements per operator are sufficient 
for model recovery:

**Theorem 4.3.** For almost every set of \( G \) mappings \( A_1, \ldots, A_G \in \mathbb{R}^{m \times n} \), the signal model \( X \) can be 
uniquely recovered if the number of measurements verify \( m > k + n/G \).

The proof of Theorem 4.3 is included in Section 5. If we have a large number of independent 
operators \( G \geq n \), Theorem 4.3 states that only \( m > k + 1 \) measurements are sufficient for unique 
model recovery, which is slightly smaller (if the model is not trivial, i.e., \( k > 1 \)) than the number of 
measurements typically needed for signal recovery \( m > 2k \). In this case, it is possible to uniquely 
identify the model, without necessarily having a unique reconstruction of each observed signal. 
However, as discussed in Section 2 for \( k < m \leq 2k \), the subset of signals which cannot be uniquely 
recovered is at most \((2k - m)\)-dimensional.

**Operators Sharing the Same Nullspace** Theorem 4.3 applies to almost every choice of \( G \) 
measurement operators in \( \mathbb{R}^{m \times n} \). However, there is a particular choice of operators where it is 
impossible to uniquely identify the model. Despite having measure zero in the space of \( G \) linear 
operators, it is useful to characterize this worst case scenario:

**Proposition 4.4.** A necessary condition for the identifiability of the model \( X \) is that the range \( \mathcal{R}_{A_g} \) 
is not equal for all \( g \in \{1, \ldots, G\} \).

*Proof.* If \( \mathcal{R}_{A_g} = \mathcal{R}_{A_g}' \) for all \( g \neq g' \), then the system in (14) will have multiple solutions for any 
choice of \( x_1, \ldots, x_G \in X \) as the matrix on the left hand side of (14) has rank \( m < n \).

**Operators with Different Number of Measurements** The results of the previous subsections 
can be easily extended to the setting where each measurement operator has a different number 
of measurements, i.e., \( A_1 \in \mathbb{R}^{m_1 \times n} \), \( A_G \in \mathbb{R}^{m_G \times n} \). In this case, the necessary condition in 
Proposition 4.2 is \( \sum_{g=1}^{G} m_g \geq n \), and the sufficient condition in Theorem 4.3 is \( \frac{1}{G} \sum_{g=1}^{G} m_g > k + n/G \). 
As the proofs mirror the ones of Proposition 4.2 and Theorem 4.3, we leave the details to the reader.

**Noisy measurement data** Surprisingly, the results of this section are also theoretically valid if 
the measurements are corrupted by independent additive noise \( \epsilon \), i.e., \( y = A_g x + \epsilon \), as long as the 
noise distribution is known and has a nowhere zero characteristic function (e.g., Gaussian noise):

**Proposition 4.5.** For a fixed noise distribution, if its characteristic function is nowhere zero, 
then there is a one-to-one mapping between the space of clean measurement distributions and noise 
measurement distributions.

The proof is included in the Appendix. If the clean measurement distribution can be uniquely 
identified, we can then apply Theorem 4.3 and Propositions 4.2 and 4.4. Note that this only guarantees 
model identifiability and makes no claims on the sample complexity of any learning process.
5 Proof of Main Theorem

This section includes the proof of Theorem 4.3. The reader who is more interested in the implications of the theorem than in the proof mechanism can skip this section. The proof relies on lemmas 4.5 and 4.6 in [19] which are a generalization of Whitney’s embedding theorem (the proof of this lemma is discussed in the Appendix):

**Lemma 5.1** (Lemmas 4.5 and 4.6 in [19]). Let $S$ be a bounded subset of $\mathbb{R}^n$, and let $G_0, G_1, \ldots, G_t$ be Lipschitz maps from $S$ to $\mathbb{R}^m$. For each integer $r \geq 0$, let $S_r$ be the subset of $z \in S$ such that the rank of the $m \times t$ matrix

$$
\Phi_z = [G_1(z), \ldots, G_t(z)]
$$

is $r$, and let $\text{boxdim}(S_r) = k_r$. For each $\alpha \in \mathbb{R}^t$ define $G_\alpha(z) = G_0 + \Phi_z \alpha$. If for all integers $r \geq 0$ we have that $r > k_r$, then $G_\alpha^{-1}(0)$ is empty for almost every $\alpha \in \mathbb{R}^t$.

We can now present the proof of Theorem 4.3:

**Proof.** In order to have model uniqueness, we require that the inferred signal set $\hat{X}$ defined in (13) equals the true set $X$, or equivalently that their difference

$$
\hat{X} \setminus X \equiv \{v \in \mathbb{R}^n \setminus X \mid A_1(x_1 - v) = \cdots = A_G(x_G - v) = 0, x_1, \ldots, x_G \in X\}
$$

is empty, where $\setminus$ denotes set difference. Let $S \subset \mathbb{R}^{n(G+1)}$ be the set of all vectors $z = [v, x_1, \ldots, x_G]^\top$ with $v \in \mathbb{R}^n \setminus X$ and $x_1, \ldots, x_G \in X$. The difference set defined in (16) is empty if and only if for any $z \in S$ we have

$$
\begin{bmatrix}
-A_1 & 1 \\
\vdots & \vdots \\
-A_G & 1
\end{bmatrix}
\begin{bmatrix}
-1 & v \\
1 & x_1 \\
\vdots & \vdots \\
1 & x_G
\end{bmatrix}
\neq 0
$$

for all $z \in S$ we have

$$
G_\alpha(z) \neq 0
$$

where $G_\alpha$ maps $z \in S$ to $\mathbb{R}^{mG}$. Let $\alpha = [\text{vec}(A_1)^\top, \ldots, \text{vec}(A_G)^\top]^\top \in \mathbb{R}^{mnG}$, then as a function of $\alpha$ we can also write (17) as

$$
\begin{bmatrix}
(x_1 - v)^\top \otimes I_m \\
\vdots \\
(x_G - v)^\top \otimes I_m
\end{bmatrix}
\neq 0
$$

where $\otimes$ is the Kronecker product and we used the fact that $A(x_G - v) = (x_G - v)^\top \otimes I_m \text{vec}(A)$. As $v$ does not belong to the signal set, the matrix on the left hand side of (19) has rank $mG$ for all $z \in S$. We treat the cases of bounded and conic signal sets separately, showing in both cases that, for almost every $\alpha \in \mathbb{R}^{mnG}$, the condition in (19) holds for all $z \in S$ if $m > k + n/G$:

**Bounded signal set** Let $S_\rho$ be a subset of $S$ defined as

$$
S_\rho = \{z \in \mathbb{R}^{n(G+1)} \mid z = [v^\top, x_1^\top, \ldots, x_G^\top]^\top, x_1, \ldots, x_G \in X, \|v\|_2 \leq \rho\}.
$$

As $S_\rho$ is bounded, we have $\text{boxdim}(S_\rho) \leq kG + n$. Thus, if $mG > kG + n$, Lemma 5.1 states that for almost every $\alpha \in \mathbb{R}^{mn}$ holds for all $z \in S_\rho$. As $S$ can be decomposed as a countable union of $S_\rho$ of increasing radius, i.e., $S = \bigcup_{\rho \in \mathbb{N}} S_\rho$, and a countable union of events of measure zero has measure zero, then for almost every $\alpha$ all $z \in S$ verifies (19) if $m > k + n/G$.
Figure 2: Reconstruction probability of a $k$-dimensional subspace using incomplete measurements arising from $G$ independent operators for different $k$. The curve in red shows the bound of Theorem 4.3, $m > k + n/G$.

**Conic signal set** If the signal set is conic, then $S$ is also conic. Hence, due to the linearity of (17) with respect to $z$, there exists $z \in S$ which does not verify (17) if and only if for any bounded set $B$ containing an open neighbourhood of 0, there exists a $z \in S \cap B$ which does not verify (17). As $\text{boxdim}(S \cap B) \leq Gk + n$, Lemma 5.1 states that for almost every $\alpha$, all $z \in S$ verifies (19) as long as $m > k + nG$.

### 6 Experiments

#### 6.1 Subspace Learning

We consider the problem of learning a $k$-dimensional subspace model from partial observations, where the signals $x_i$ are generated from a standard Gaussian distribution on the low-dimensional subspace. The observations $y_i$ are obtained by randomly choosing one out of $G$ operators $A_1, \ldots, A_G \in \mathbb{R}^{m \times n}$, each composed of iid Gaussian entries of mean 0 and variance $n^{-1}$. In order to recover the signal matrix $X = [x_1, \ldots, x_N]$, we solve the following low-rank matrix recovery problem

$$\begin{align*}
\arg\min_X & \quad \|X\|_* \\
\text{s.t.} & \quad A_g x_i = y_i \quad \forall i = 1, \ldots, N
\end{align*}$$

(21)

where $\| \cdot \|_*$ denotes the nuclear norm. A recovery is considered successful if $\sum_i \|\hat{x}_i - x_i\|^2 < 10^{-1}$, where $\hat{x}_i$ is the estimated signal for the $i$th sample. We use a standard matrix completion algorithm [25] to solve (21). The ambient dimension is fixed at $n = 50$, and the experiment is repeated for $k = 1, 10, 40$. For each experiment we set $N = 150k$ in order to have enough samples to estimate the subspaces [14]. Figure 2 shows the probability of recovery over 25 Monte Carlo trials for different numbers of measurements $m$ and operators $G$. The reconstruction probability exhibits a sharp transition which follows the bound presented in Theorem 4.3 i.e., $m > k + n/G$.

#### 6.2 Deep Networks

We next consider the problem of directly learning the reconstruction function $f_\theta : y \mapsto x$ using a deep neural network with weights denoted by $\theta$. We train a network that aims to achieve data consistency
Table 1: Average PSNR in dB achieved by a residual network and a non-residual network for the MNIST reconstruction task with $G = 40$ operators, each with $m = 100$ measurements.

|                  | $\phi = \text{Id}$ | $\phi$ w. residual | $\phi$ w/o residual |
|------------------|----------------------|---------------------|---------------------|
| PSNR             | 10.02                | 10.32               | 21.36               |

for all the training data via the following unsupervised loss (note no ground truth data is used in training)

$$\arg\min_\theta \sum_{i=1}^N \|y_i - A_g f_\theta(y_i)\|^2. \quad (22)$$

We use the standard MNIST dataset which has an approximate box-counting dimension $k = 12$ [26]. The dataset contains $N = 60000$ training samples, and these are partitioned such that $N/G$ different samples are observed via each operator. The entries of the forward operators are sampled from a Gaussian distribution with zero mean and variance $n^{-1}$. The test set consists of 10000 samples, which are also randomly divided into $G$ parts, one per operator. Similarly to [1] for a sample associated with the $g$th operator, we define $f_\theta(x) = \phi_\theta \circ A_g^\dagger$ where $\phi_\theta : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a trainable network whose aim is to map $A_g^\dagger y$ to the signal set $X$. The networks are trained using the Adam optimizer.

Figure 3: Schematic of the fully connected autoencoder used in the MNIST experiments. Number of nodes are indicated at the bottom of each layer.

When evaluating the ability of neural networks to perform unsupervised learning from incomplete data it is necessary to ensure that we are not just observing the inductive bias of the network, which has been shown can provide a powerful image model without any training [27, 28]. In order to minimize the impact of the inductive bias of the networks’ architecture, we use fully connected layers which do not exploit any spatial image prior. It is easy to see that a valid solution to the optimization problem in (22) is just the identity $\phi_\theta(x) = x$. To avoid learning the identity, we require that $\phi_\theta(x)$ exploits the low-dimensionality of the signal set, so we use an autoencoder architecture with 3 hidden layers with 1000, 32 and 1000 neurons, as shown in Figure 3. We use relu non-linearities between layers, except at the output of the last layer.

Table 1 shows the performance for the case of $G = 40$ operators with $m = 100$ measurements each for $\phi_\theta(x)$ being (i) the identity, (ii) the autoencoder with a residual connection which allows it to learn the identity mapping and (iii) the autoencoder with no residual connection which enforces a low-dimensional representation. The network with residual connection fails to learn, obtaining a similar performance to the simple pseudo-inverse, whereas the autoencoder without residual obtains a significant improvement of more than 10 dB. This is in line with our theory where we have seen that a requirement for unsupervised learning from incomplete measurements is that the model is low dimensional. We next explore the ability of the autoencoder without residual connection to perform unsupervised learning as a function of the number of operators $G$ and measurements $m$. 

10
Figure 4: Average test PSNR for the MNIST dataset, for different number of random Gaussian operators and measurements per operator. The curve in red shows the necessary condition of Theorem 4.3, \( m > k + n/G \).

Figure 5: Reconstructed images of the MNIST test set with \( G \) random Gaussian operators. The first column shows the reconstruction obtained via a pseudo-inverse. PSNR values are shown on the top right corner of the images.

Figure 4 shows the average test peak-signal-to-noise ratio (PSNR) achieved by the trained model for \( G = 1, 10, 20, 30, 40 \) and \( m = 1, 100, 200, 300, 400 \). The results follow closely the bound presented in Section 4 which is indicated by the red dashed line, as the network is only able to learn the reconstruction mapping when the sufficient condition \( m > k + n/G \) is verified. In sampling regimes below this condition, the performance is similar to simply applying the pseudo-inverse \( A_g^\dagger \). Figure 5 shows examples of reconstructed images for networks trained with different number of operators and measurements.

Finally, we replace Gaussian operators for \( G \) different random inpainting masks. The inpainting operators have a diagonal structure which has zero measure in \( \mathbb{R}^{m \times n} \), however our sufficient condition still provide a reasonable lower bound on predicting the performance, as shown in Figure 6. It is likely that due to the coherence between measurement operators and images (both operators and MNIST images are sparse), more measurements are required to obtain good reconstructions than in the case of Gaussian operators. Figure 7 shows examples of reconstructed images for different number of operators and measurements.
Figure 6: Average test PSNR for the MNIST dataset, for different number of random inpainting operators and measurements per operator. The curve in red shows the necessary condition of Theorem 4.3, $m > k + n/G$.

Figure 7: Reconstructed images of the MNIST test set with $G$ random inpainting operators. The first column shows the reconstruction obtained via a pseudo-inverse. PSNR values are shown on the top right corner of the images.

7 Conclusions and Future Work

We have presented sampling theorems for the unsupervised learning of signal models from incomplete measurements using multiple measurement operators. Our bounds characterize the interplay between the fundamental properties of the problem: the ambient dimension, the data dimension and the number of measurement operators. Moreover, the bounds are agnostic of the learning algorithms and provide useful necessary and sufficient conditions for designing principled sensing strategies. Our results shed light into the setting where access to ground truth data cannot be guaranteed which is of extreme importance in various applications.

Extending the present theory to the case where the operators $A_g$ present some problem specific constraints (e.g., they are inpainting matrices and/or the different operators are related by some transformation) is an interesting avenue of future research. We also leave the study of robustness to noise as well as the cases where the signal model is only approximately low dimensional and/or the ambient space is infinite dimensional (i.e., signals with a continuous representation) for future work.
Acknowledgements

This work is supported by the ERC C-SENSE project (ERCADG-2015-694888).

References

[1] Kyong Hwan Jin, Michael T McCann, Emmanuel Froustey, and Michael Unser. Deep convolutional neural network for inverse problems in imaging. *IEEE Transactions on Image Processing*, 26(9):4509–4522, 2017.

[2] Joshua Rapp, Julian Tachella, Yoann Altmann, Stephen McLaughlin, and Vivek K Goyal. Advances in single-photon lidar for autonomous vehicles: Working principles, challenges, and recent advances. *IEEE Signal Processing Magazine*, 37(4):62–71, 2020.

[3] Matthew O’Toole, David B Lindell, and Gordon Wetzstein. Confocal non-line-of-sight imaging based on the light-cone transform. *Nature*, 555(7696):338–341, 2018.

[4] Harshit Gupta, Thong H Phan, Jaejun Yoo, and Michael Unser. Multi-cryogan: Reconstruction of continuous conformations in cryo-em using generative adversarial networks. In *European Conference on Computer Vision*, pages 429–444. Springer, 2020.

[5] Christoph Studer and Richard G Baraniuk. Dictionary learning from sparsely corrupted or compressed signals. In *2012 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 3341–3344. IEEE, 2012.

[6] Jianbo Yang, Xuejun Liao, Xin Yuan, Patrick Llull, David J. Brady, Guillermo Sapiro, and Lawrence Carin. Compressive sensing by learning a gaussian mixture model from measurements. *IEEE Transactions on Image Processing*, 24(1):106–119, 2015.

[7] Ashish Bora, Eric Price, and Alexandros G. Dimakis. AmbientGAN: Generative models from lossy measurements. In *International Conference on Learning Representations*, 2018.

[8] Emmanuel J Candès and Benjamin Recht. Exact matrix completion via convex optimization. *Foundations of Computational mathematics*, 9(6):717, 2009.

[9] Sivan Gleichman and Yonina C Eldar. Blind compressed sensing. *IEEE Transactions on Information Theory*, 57(10):6958–6975, 2011.

[10] Y. Chen, Y. Chi, and A. J. Goldsmith. Exact and stable covariance estimation from quadratic sampling via convex programming. *IEEE Transactions on Information Theory*, 61(7):4034–4059, 2015.

[11] Anthony Bourrier, Mike E Davies, Tomer Peleg, Patrick Pérez, and Rémi Gribonval. Fundamental performance limits for ideal decoders in high-dimensional linear inverse problems. *IEEE Transactions on Information Theory*, 60(12):7928–7946, 2014.

[12] Jorge Silva, Minhua Chen, Yonina C Eldar, Guillermo Sapiro, and Lawrence Carin. Blind compressed sensing over a structured union of subspaces. *arXiv preprint arXiv:1103.2469*, 2011.

[13] Mohammad Aghagolzadeh and Hayder Radha. New guarantees for blind compressed sensing. In *2015 53rd Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1227–1234. IEEE, 2015.
[14] Daniel L Pimentel-Alarcón, Nigel Boston, and Robert D Nowak. A characterization of deterministic sampling patterns for low-rank matrix completion. IEEE Journal of Selected Topics in Signal Processing, 10(4):623–636, 2016.

[15] Daniel Pimentel-Alarcon and Robert Nowak. The information-theoretic requirements of subspace clustering with missing data. In International Conference on Machine Learning, pages 802–810. PMLR, 2016.

[16] Brian Eriksson, Laura Balzano, and Robert Nowak. High-rank matrix completion. In Artificial Intelligence and Statistics, pages 373–381. PMLR, 2012.

[17] Jaakko Lehtinen, Jacob Munkberg, Jon Hasselgren, Samuli Laine, Tero Karras, Miika Aittala, Timo Aila, et al. Noise2noise. In International Conference on Machine Learning. PMLR, 2018.

[18] Jiaming Liu, Yu Sun, Cihat Eldeniz, Weijie Gan, Hongyu An, and Ulugbek S Kamilov. Rare: Image reconstruction using deep priors learned without groundtruth. IEEE Journal of Selected Topics in Signal Processing, 14(6):1088–1099, 2020.

[19] Tim Sauer, James A Yorke, and Martin Casdagli. Embedology. Journal of statistical Physics, 65(3):579–616, 1991.

[20] T. Blumensath and M. E. Davies. Sampling theorems for signals from the union of finite-dimensional linear subspaces. IEEE Transactions on Information Theory, 55(4):1872–1882, 2009.

[21] Hilton Bristow, Anders Eriksson, and Simon Lucey. Fast convolutional sparse coding. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 391–398, 2013.

[22] Leonid I Rudin, Stanley Osher, and Emad Fatemi. Nonlinear total variation based noise removal algorithms. Physica D: nonlinear phenomena, 60(1-4):259–268, 1992.

[23] Harald Cramér and Herman Wold. Some theorems on distribution functions. Journal of the London Mathematical Society, 1(4):290–294, 1936.

[24] Dongdong Chen, Julián Tachella, and Mike E Davies. Equivariant imaging: Learning beyond the range space. In Proceedings of the IEEE/CVF International Conference on Computer Vision (ICCV), pages 4379–4388, October 2021.

[25] Jian-Feng Cai, Emmanuel J Candès, and Zuowei Shen. A singular value thresholding algorithm for matrix completion. SIAM Journal on optimization, 20(4):1956–1982, 2010.

[26] Matthias Hein and Jean-Yves Audibert. Intrinsic dimensionality estimation of submanifolds in rd. In Proceedings of the 22nd international conference on Machine learning, pages 289–296, 2005.

[27] Dmitry Ulyanov, Andrea Vedaldi, and Victor Lempitsky. Deep image prior. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 9446–9454, 2018.

[28] Julian Tachella, Junqi Tang, and Mike Davies. The neural tangent link between cnn denoisers and non-local filters. In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (CVPR), pages 8618–8627, June 2021.
A Proofs.

We first present the proof of Proposition 4.5:

**Proof.** Consider the noisy measurements associated to the $g$th operator $A_g$, as $z = y + \epsilon$, where $z$ are the observed noisy measurements, $y$ are the clean measurements and $\epsilon$ is additive noise (independent of $y$). The characteristic function of the sum of two independent random variables is given by the multiplication of their characteristic functions, i.e.,

$$\varphi_z(w) = \varphi_y(w)\varphi_\epsilon(w)$$

(23)

where $\varphi_z$, $\varphi_y$ and $\varphi_\epsilon$ are the characteristic functions the noisy measurement, clean measurements and noise distributions, respectively. If the characteristic function of the noise distribution is nowhere zero, we can uniquely identify the characteristic function of the clean measurement distribution as

$$\varphi_y(w) = \frac{\varphi_z(w)}{\varphi_\epsilon(w)}$$

(24)

The clean measurement distribution is fully characterized by its characteristic function $\varphi_y(w)$. We end the proof by noting that the same reasoning applies to the measurements of every operator $A_g$ with $g \in \{1, \ldots, G\}$.

The proof of Lemma 5.1 follows standard covering arguments and may be sketched as follows. From the dimensionality assumption, the set $S_r$ can be essentially covered by $O(\epsilon^{-kr})$ $\epsilon$-balls. Furthermore, for any $z \in S_r$, the probability (measured with respect to $\alpha \in \mathbb{R}^t$) that $G_\alpha(z)$ maps to the neighborhood of 0 scales as $\epsilon^r$. Hence the probability of this happening for any of the points in the cover scales as $\epsilon^{r-kr}$. If we take $r > kr$ then the probability of such an event tends to zero as we shrink $\epsilon$. Full details can be found in the proofs in [19].

15