Orbit Fitting Based on Helmert Transformation

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Abstract  Orbit fitting is used in many GPS applications. For example, in Precise Point Positioning (PPP), GPS orbits (SP3 orbits) are normally retrieved either from IGS or from one of its Analysis Centers (ACs) with 15 minutes’ sampling, which is much bigger than the normal observation sampling. Therefore, algorithms should be derived to fit GPS orbits to the observation time. Many methods based on interpolation were developed. Using these methods the orbits fit well at the sampling points. However, these methods ignore the physical motion model of GPS satellites. Therefore, the trajectories may not fit the true orbits at the periods in between 2 sampling epochs. To solve this problem, we develop a dynamic approach, in which a model based on Helmert transformation is developed in GPS orbit fitting. In this orbit fitting approach, GPS orbits at sampling points are treated as pseudo-observations. Thereafter, Helmert transformation is built up between the pseudo-observations and dynamically integrated orbits at each epoch. A set of Helmert parameters together with corrections of GPS initial orbits are then modeled as unknown parameters. Results show that the final fit orbits have the same precision as the IGS final orbits.

Keywords  precise point positioning (PPP); IGS orbits; orbit fitting; Helmert transformation

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Introduction

Currently, GPS orbit products of IGS achieves a precision better than 5 cm (IGS: http://igsobs.jpl.nasa.gov/components/prods.html), which provides and enhances a lot of applications, e.g., PPP. In PPP, orbits and clocks of GPS satellites are normally fixed to a certain value. They may be retrieved either from IGS or from one of its ACs. Currently, the sampling rate of GPS orbits provided by IGS is 15 minutes. For high frequency data processing, algorithms should be derived to get the orbits at the observation time.

Many methods based on interpolation, such as Chebyshev polynomial [1-2], Lagrange polynomial [3], Newton polynomial [4] etc., were developed. These methods use polynomials to fit the trajectories of the GPS orbit. The orbits normally fit well at the sampling point. However, there is one problem of these methods that ignore the dynamical physical motion model of GPS satellites, i.e., Newton’s second law. Consequently, the orbits may not fit well at the epochs in between 2 sampling epochs.

Starting from this point, a model based on Helmert transformation is presented in GPS orbit fitting. Based on this model an orbit fitting approach is developed. First, we get the satellites’ initial orbits from the IGS final orbits. An orbit integration is then performed based on the derived initial orbits. Afterwards, Helmert transformation between IGS orbits and integrated orbits is set up at sampling epochs of the IGS orbits. Accumulating all the equations at each
epoch, the parameters of the model, including Hel- 
mert transformation parameters and corrections of 
initial orbits, are then estimated. By using the updated 
initial orbits, orbit integration can be performed again 
to get the new integrated orbits. This procedure can 
be iterated. Results show that the final integrated or-
bits have the same precision of the IGS final orbits.

1 Orbit integration

According to Newton’s second law, satellites’ mo-
tion equation and satellites’ initial orbits at epoch 
t_0 can be written as[5, 6],

\[ \begin{aligned}
\dot{x} &= F(x, t) \\
\dot{X} &= 0
\end{aligned} \]

(1)

Here, \( x_0 = (x_0, \dot{x}_0, p_0) \) are initial orbits includ-
ing positions, velocities and dynamic parameters 
(e.g., the solar radiation pressure parameters (SRP)) 
of the satellite, \( F(x, t) \) is the modeling equation of 
the complete set of forces acting on an orbiting satel-
lite[7-9]. With a proper integration method such as 
Adams- Cowell numerical integration, dynamic inte-
grated orbits \( x^* \) can be computed based on \( x_0 \).

In Eq.(1), we can define \( \delta = x - x^* \). Based on the 
Taylor expansion, we get

\[ \delta = \frac{\partial F(x, t)}{\partial x} \delta \]

(2)

The solution of Eq.(2) can be expressed as

\[ \delta = \Psi(t, t_0) \delta_0 \]

(3)

where \( \delta_0 = x_0 - x^* \) is the orbit corrections at initial 
epoch \( t_0 \). Substituting Eq.(3) into Eq.(2), we have the following equation.

\[ \begin{aligned}
\Psi(t, t_0) &= \frac{\partial F(x, t)}{\partial x} \Psi(t, t_0) \\
\Psi(t_0, t_0) &= I
\end{aligned} \]

(4)

where \( I \) is the unit matrix, and \( \Psi(t, t_0) \) is called transi-

tion matrix. It can be expressed in detail as

\[ \Psi(t, t_0) = \begin{pmatrix}
\frac{\partial r}{\partial t} & \frac{\partial r}{\partial t_0} & \frac{\partial r}{\partial p} \\
\frac{\partial \dot{r}}{\partial t} & \frac{\partial \dot{r}}{\partial t_0} & \frac{\partial \dot{r}}{\partial p} \\
\frac{\partial \ddot{r}}{\partial t} & \frac{\partial \ddot{r}}{\partial t_0} & \frac{\partial \ddot{r}}{\partial p} \\
\frac{\partial p}{\partial t} & \frac{\partial p}{\partial t_0} & \frac{\partial p}{\partial \dot{p}}
\end{pmatrix} \]

(5)

From the numeric integration we can get transition 
matrix as well as integrated orbits \( x^* \).

2 Orbit fitting based on Helmert transformation

Helmert transformation is mostly used to express 
differences between reference frames[10]. It is used by 
the International GNSS Service (IGS) community to 
analyze the systematic differences between ACs and 
to combine products from different ACs to get the fi-
al IGS products[11]. As we know, each software may 
differ in dynamic models and processing approaches, 
consequently the software difference results in the 
difference of the reference frame defined. Considering 
the systematic differences between our integrated or-
bits and the IGS final orbits, we can build up Hel-
mert transformation between the dynamic integrated 
orbits and the IGS final orbits. At epoch \( t \), it can be 
expressed as

\[ \begin{pmatrix}
X_K^i \\
Y_K^i \\
Z_K^i
\end{pmatrix} = \begin{pmatrix}
\Delta X \\
\Delta Y \\
\Delta Z
\end{pmatrix} + (1 + K')R_i (\alpha') R_z (\beta') R_i (\gamma') \begin{pmatrix}
X_T^i \\
Y_T^i \\
Z_T^i
\end{pmatrix} \]

(6)

where \( (X_K^i, Y_K^i, Z_K^i) \) are the IGS final orbits; \( (X_T^i, Y_T^i, Z_T^i) \) are the dynamically integrated orbits 
expressed in Earth-fixed reference frame, which can be obtained 
by using the following transformation[9]:

\[ \begin{pmatrix}
X_T^i \\
Y_T^i \\
Z_T^i
\end{pmatrix} = Q(t_i) R(t_i) \begin{pmatrix}
X_I^i \\
Y_I^i \\
Z_I^i
\end{pmatrix} \]

(7)

where \( (X_I^i, Y_I^i, Z_I^i) \) are integrated orbits expressed in 
inertial reference frame, and \( Q(t_i), R(t_i), W(t_i) \) are the 
matrices for precession-nutation, Earth rotation and 
pole wobble, respectively.

We can rewrite the model as

\[ r_k' = T + (1 + K)R_i \cdot R_z \cdot r_i' \]

(8)

where \( r_k', r_i' \) are the denotation of the IGS final orbit 
(in Earth-fixed frame), the dynamically integrated or-
bit (in inertial frame); \( T, K \) represent the translation 
and the scale parameters of the Helmert transforma-
tion, and \( R_i, R_z \) are the rotation matrices in Eq.(6)
and Eq.(7), respectively. In $R$, parameters are: Earth pole $x_p, y_p$ and the rates $\dot{x}_p, \dot{y}_p$, time parameter $UT1-UTC$ ($dUT1$) and the rate $dUT1$.

The magnitude of $x_p, y_p$ is less than 1" and the magnitude of $dUT1$ is less than 1 second (15" in angle). The magnitude of $\dot{x}_p, \dot{y}_p$ and $dUT1$ are even smaller. Ignoring the effects of $\dot{x}_p, \dot{y}_p$ and $dUT1$, Eq.(8) can be rewritten as

$$\begin{bmatrix}
X'_k \\
Y'_k \\
Z'_k
\end{bmatrix} = \begin{bmatrix}
\Delta X \\
\Delta Y \\
\Delta Z
\end{bmatrix} + (1+K)R_i(\alpha)R_i(\beta)R_i(\gamma)R_{nc} \begin{bmatrix}
X'_i \\
Y'_i \\
Z'_i
\end{bmatrix}$$

where, $R_{nc}$ is the residual matrix. Comparing Eq.(9) to Eq. (8),

$$\alpha = \alpha' + y_p, \beta = \beta' + x_p, \gamma = \gamma' + dUT1$$

Defining $R_i(R) = R_i(\alpha)R_i(\beta)R_i(\gamma)$, we can rewrite Eq.(9) as

$$r'_i = T + (1 + K)R_i(R) \cdot R_{nc} \cdot r'_i$$

where $T, K, R$ represent Helmert parameters($H$) ($\Delta X, \Delta Y, \Delta Z, K, \alpha, \beta, \gamma$).

The linearization of Eq.(11) reads as

$$r'_i = r'_{i,0} + \frac{\partial r'_i}{\partial \psi} \psi \label{eq:12}$$

The parameters to be estimated are expressed in Eq.(13), where $dH$ are the corrections of Helmert transformation parameters, $dr'_i$ are the corrections of integrated orbits at the current epoch.

$$dv = (dH, dr'_i) \label{eq:13}$$

Considering Eq.(3) and Eq.(4), we can transform the parameter $dr'_i$ to initial orbit corrections $d\psi$ using Eq.(14):

$$dr'_i = \Psi(t, t_0) d\psi$$

Therefore, the final parameters can be expressed as:

$$dx = (dH, d\psi) \label{eq:15}$$

The design matrix is:

$$A_i = \frac{\partial r'_i}{\partial \psi} = \begin{bmatrix} \frac{\partial r'_i}{\partial H} & \frac{\partial r'_i}{\partial \psi} \cdot \Psi(t, t_0) \end{bmatrix}\label{eq:16}$$

Eq.(12) can be formed at each epoch $t_i$. Accumulating all the epochs (normally 96 epochs) in the IGS final orbits, we get the observation equations,

$$A_i dx = L \label{eq:17}$$

Here,

$$A_i = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}, L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix} \Rightarrow \begin{bmatrix} r^1_k - r^1_{k,0} \\ \vdots \\ r^n_k - r^n_{k,0} \end{bmatrix} \label{eq:18}$$

The normal equation can be written as

$$Nd\psi = b \label{eq:19}$$

According to the Least Square Estimation (LSE) theory, the final normal equation equals to the accumulation of the normal equation at each epoch, i.e.,

$$N = \sum_{i=1}^{\hat{N}} (A_i^T A_i), \quad b = \sum_{i=1}^{\hat{N}} A_i^T L_i \label{eq:20}$$

The solution of Eq.(19) can be iterated. By using the new initial orbits from Eq.(19), the new integrated orbits can be generated. Based on the final estimated initial orbits, the final integrated orbits can be generated, which are final fit orbits.

### 3 Data processing

As a validation of the model and the processing approach, we fit the GPS orbits on the day 062 of year 2005 to the IGS final orbits. The initial satellites’ positions and velocities of orbit integration are derived from the IGS orbits by interpolation. The initial SRP parameters are set to zero. The dynamic models are listed in Table 1.

| Table 1 Dynamic models |
|------------------------|
| Gravity model          | EGM96 (8×8)          |
| Tide                   | Solid earth tide     |
| N-body                 | Sun & Moon           |
| Solar radiation pressure| BERNE[12-13]         |

According to Chen[14], the satellites’ orbits achieve a similar precision under different parameter sets of our model. Therefore, we estimate only Helmert translation parameters. To sum up, the estimated parameters include Helmert translation parameters, corrections of initial orbits (positions, velocities and SRP parameters).

Fig.1 shows the orbit differences between our final fit orbits and the IGS orbits of GPS PRN01, where
we see smoothing periodical variations. The period fits well with GPS’ revolution period. The range of the variations is within (~4,4) cm in each direction.

Fig.2 to Fig.4 summarize the statistics of the orbit differences of each satellite. Fig.2 shows the absolute value of the mean orbit differences, where we see that most of them are smaller than 2 mm. This shows that the systematic errors between our fit orbits and the IGS orbits are absorbed quite well by using our model.

Fig.3 shows the mean RMS of the orbit differences. The RMS is less than 3 cm in each direction and 3D RMS is less than 4 cm for each satellite, which is similar with the current precision of the IGS final orbits. Fig.4 shows the mean differences of the distance from satellite to the Earth’s center (3D range), where we see that all of them are smaller than 4 cm.

4 Conclusion

As the orbit fitting performed for GPS satellites shows, the proposed model achieves the orbits with similar precision as IGS final orbits.

The orbit fitting with other Helmert parameter settings was carried out, and we obtained similar results.
as what was shown already. In our approach, the fit orbits retain the dynamic properties of the satellite and the sampling of the fit orbits depends on the integration interval (we set it to 9.375 seconds). With the smoothing variation of satellites’ orbits within this short period, the misfit problem of the interpolation methods is solved.

The fit orbits derived from our research have obvious periodicities, which follow, in general, the revolution periods of the GPS satellites. This may be due to some dynamic models’ deficiency, e.g., SRP model. In order to better understand the reason, further investigations are needed.

References

[1] Yu Peng, Sun Xuejin, Zhao Shijun (2004) Chebyshev polynomial fitting model for GPS orbit calculation [J]. Meteorological Science and Technology, 32(3):198-200
[2] Kong Qiaoli (2006) Using Chebyshev polynomial to fit the precise satellite ephemeris [J]. Bulletin of Surveying and Mapping, 8:1-3
[3] Cai Yanhui, Cheng Pengfei, Li Xiyin (2003) On calculating coordinates of satellites by lagrange interpolation and chebyshev polynomial simulation [J]. GNSS World of China, 28(3):10-13
[4] Hong Ying, Ou Jikun, Peng Bibo (2006) Three interpolation methods for precise ephemeris and clock offset of GPS satellite [J]. Geomatics and Information Science of Wuhan University, 31(6):516-518
[5] Wang Jiexian (1997) Precise orbit determination and positioning of GPS [M]. Shanghai: Tongji University Press
[6] Chen Junping (2007) On precise orbit determination of low earth orbiters[D]. Shanghai: Shanghai Tongji University
[7] Beutler G (2004) Methods of celestial mechanics volume i: physical, mathematical, and numerical principles[M]. Berlin, Heidelberg: Springer-Verlag
[8] Beutler G (2004) Methods of celestial mechanics volume ii: application to planetary system, geodynamics and satellite geodesy[M]. Berlin, Heidelberg: Springer-Verlag
[9] McCarthy D D, Petit G (2004) IERS conventions, (IERS Technical Note No. 32)[C]. Frankfurt am Main: Verlag des Bundesamts für Kartographie und Geodäsie
[10] Boucher C, Altamimi Z, Sillard P, Feissel-Vernier M (2004) The ITRF2000 (IERS Technical Note No. 31)[C]. Frankfurt am Main: Verlag des Bundesamts für Kartographie und Geodäsie
[11] Beutler G, Kouba J, Springer T (1995) Combining the orbits of IGS processing centers[J]. Bulletin Géodésique, 69(4): 200-222
[12] Chen Junping, Wang Jiexian(2006)Solar radiation pressure models for the GPS satellites[J]. Acta Astronomica Sinica, 47(3):310-319
[13] Springer T A, Beutler G, Rothacher G(1998)A new solar radiation pressure model for the GPS Satellites[J]. GPS Solutions, 2(3): 50-62
[14] Chen Junping, Wang Jiexian(2008)Reduced-dynamic precise orbit determination for low earth orbiters based on Helmert transformation [J]. Artificial Satellite, 42(3): 155-165