ON THE ERGODIC BEHAVIOUR OF AFFINE VOLterra PROCESSES

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ABSTRACT. We show the existence of a stationary measure for a class of multidimensional stochastic Volterra systems of affine type. These processes are in general not Markovian, a shortcoming which hinders their large-time analysis. We circumvent this issue by lifting the system to a measure-valued stochastic PDE introduced by Cuchiero and Teichmann [CT20], whence we retrieve the Markov property. Leveraging on the associated generalised Feller property, we extend the Krylov-Bogoliubov theorem to this infinite-dimensional setting and thus establish an approach to the existence of invariant measures. We present concrete examples, including the rough Heston model from Mathematical Finance.

1. INTRODUCTION

We are interested in the large-time behaviour of multidimensional Stochastic Volterra Equations (SVEs) of affine type which take the form

\[ V_t = V_0 + \int_0^t K(t-s)b(V_s) \, ds + \int_0^t K(t-s)\sigma(V_s) \, dW_s, \quad t \geq 0, \]  

(1.1)

where \( \sigma^T : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) and \( b : \mathbb{R}^d \to \mathbb{R}^d \) are affine. Define \( \mathbb{R}_+ := [0, \infty) \); the kernel \( K \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^{d \times d}) \) is a diagonal matrix with each component being the Laplace transform of some signed measure on \( \mathbb{R}_+ \), such as the Riemann-Liouville kernel \( t \mapsto t^{\alpha-1}, \alpha \in (1/2, 1] \).

The main result of this paper is the existence of an invariant measure for the Markovian lift of (1.1) introduced by Cuchiero and Teichmann [CT20]; it entails the existence of a stationary measure for \( V \). The renewed interest in SVEs stems from the emergence of a new class of stochastic volatility models which surrender the comfort of Markovianity for the sake of consistency with the data. Two examples in this direction are the rough Heston model, which arises as the scaling limit of a high-frequency model governed by Hawkes processes [EEFR18], and the rough Bergomi model which appeared in [BFG16]. The former is affine in the sense of (1.1) while the latter is not. The term affine Volterra process was coined in [ALP17] where general
existence and uniqueness are derived. We note here however that despite the large number of asymptotic results, very little is known about the ergodic behaviour of rough volatility models.

In fact, to the best of our knowledge, only three related results exist in this direction. The first one is a large deviation principle for the rescaled log-price under the rough Heston model [FGS20]; it is derived from computing the limit of its characteristic function, which is known in semi-closed form [EER19]. The second one is [GR20], where the authors proved the existence of an invariant measure for the asset price under the assumption that the non-Markovian variance process has a stationary distribution. Upon completion of this work, we became aware of the very recent (and the third known related result to us) [FJ22]. There, using completely different methods than ours, the authors prove the existence and characterise the properties of the invariant measure associated to a special case (1.1), where the diffusion matrix $\sigma$ is taken to be diagonal (so it does not include the rough Heston model for example). While our results currently only provide conditions for existence of an invariant measure (see however the discussion in Section 6), our method is generally applicable to SVE models as long as one can verify that the lift is a generalised Feller process and that certain bounds hold and importantly we note that it parallels the Markovian theory. Other notable results in related directions are [EBDK22, CT19].

When the process is Markovian, one gains access to transition semigroups, which in turn allow to study the ergodic behaviour. A multitude of tools have been developed to prove existence and uniqueness of invariant measures, as well as asymptotic stability and rates of convergence. The most prominent in the literature are based either on showing that the transition semigroup has the (strong) Feller property or on dissipativity methods, which are related to Lyapunov function techniques. We refer to [DPZ92, DPZ96] for an overview of these approaches.

A different point of view is given by the theory of Random Dynamical Systems where Markovianity is replaced by a cocycle property for the driving noise, e.g. (fractional) Brownian motion [Arn98, GAKN09]. Furthermore, in this context one analyses the asymptotic behaviour by taking the starting time to $-\infty$ and looking at the system at $t = 0$, instead of starting at $t = 0$ and looking at $t$ goes to $+\infty$.

This inspired Hairer’s theory of Stochastic Dynamical Systems, which aimed at studying the large-time behaviour of an SDE driven by additive fractional noise [Hai02]. To reconcile the Markovian ergodic theory with non-Markovian processes, his main idea was to augment the state space with all the past fractional noise, and to define a Feller semigroup on this augmented space. This led to further advances with multiplicative noise in the case $H > 1/3$ [HO07, HP11], and for the discretised version of the SDE [Var19].

The more recent literature on large-time behaviour of fractional processes has flourished under the umbrella of rough paths theory, in particular regarding multiscale systems [CKMZ16, GL20,
GLS20, LS20, PIX21, PIX20]. Their asymptotic properties were also investigated thanks to Malliavin calculus [BGS21] and the stochastic sewing lemma [HL20]. As usual with fractional stochastic integrals however, these frameworks do not accommodate for the highly irregular paths (Hurst exponent in $(0, 1/4)$) found in rough volatility models.

The idea of [Hai05] with augmenting the state space to recover Markovianity made its way to rough volatility modelling, albeit with a mild adaptation. Motivated by hedging applications in the rough Heston model, El Euch and Rosenbaum [EER18] showed that since the variance process is non-Markovian one needs to include the whole forward variance curve; this suggests to consider the system $(S_t, (\mathbb{E}[V_{s+t}|\mathcal{F}_s])_{s \geq 0})$. The forward variance even satisfies a Stochastic Partial Differential Equation (SPDE) [AE19], which allowed to further characterise the Markovian structure of the model. A similar SPDE lift for another singular SVE was in fact derived in an earlier work [MS15]. The analysis of the rough Heston model is facilitated by its affine structure, unlike the rough Bergomi model, its main competitor. Yet, the forward variance curve of the latter also satisfies an SPDE as it belongs to the realm of polynomial processes [CSF21].

The most conducive SPDE lift for our purposes, though, was introduced by Cuchiero and Teichmann in [CT20]. Unlike the previous approaches, they argue it is more sensible to start from the Markovian lift before solving the SVE. More precisely, they consider the one-dimensional measure-valued SPDE

$$
\frac{d\lambda_t}{dx} = -x\lambda_t(dx) dt + \nu(dx) dX_t,
$$

where $\lambda_0$ and $\nu$ are signed measures and $X$ is an Itô semimartingale of the form

$$
X_t = -\beta \int_0^t \langle 1, \lambda_s \rangle \, ds + \sigma \int_0^t \sqrt{\langle 1, \lambda_s \rangle} \, dW_s,
$$

where $\langle y, \lambda \rangle := \int_0^\infty y(x) \lambda(dx)$ for all $y \in C_0(\mathbb{R}_+)$. In fact, one should interpret the SPDE (1.2) in the mild sense such that

$$
\langle y, \lambda_t \rangle = \int_0^\infty e^{-tx} y(x) \lambda_0(dx) + \int_0^\infty \left( \int_0^t e^{-(t-s)x} y(x) \, dX_s \right) \nu(dx).
$$

Applying the stochastic Fubini theorem, the total mass $\langle 1, \lambda \rangle$ thus solves the affine SVE (1.1) in dimension one, with $K(t) = \int_0^\infty e^{-tx} \nu(dx)$ and $\lambda_0$ being the Dirac mass at zero multiplied by $V_0$.

This setting is not limited to the univariate case and, recalling our initial motivation, we observe that the rough Heston model implies the lift

$$
\begin{aligned}
\frac{d\lambda_1^1}{dx} &= -x\lambda_1^1(dx) dt + \delta_0(dx) \left( \frac{1}{2} (1, \lambda_1^2) dt + \sqrt{(1, \lambda_1^2)^2} dW_1^1 \right), \\
\frac{d\lambda_2^1}{dx} &= -x\lambda_2^1(dx) dt + \nu(dx) \left( \beta(\theta - \langle 1, \lambda_1^2 \rangle) dt + \sigma \sqrt{(1, \lambda_1^2)^2} dW_2^1 \right), \\
\frac{d\lambda_2^2}{dx} &= -x\lambda_2^2(dx) dt + \nu(dx) \left( \beta(\theta - \langle 1, \lambda_2^2 \rangle) dt + \sigma \sqrt{(1, \lambda_2^2)^2} dW_2^2 \right),
\end{aligned}
$$
where $W^1$ and $W^2$ are correlated Brownian motions. In this framework, $\langle 1, \lambda^1_t \rangle$ represents the log asset price and $\langle 1, \lambda^2_t \rangle$ the instantaneous variance.

The added value of this approach lies in the generalised Feller property, a notion introduced in [DT10], which is satisfied by the solution to (1.2) (albeit with $\theta = 0$) and opens the gates to many parallels with Markovian ergodic theory. Indeed, it is an extension of the standard Feller property to spaces that are not locally compact. Interestingly enough, in [DT10], the authors also prove weak existence and uniqueness of the mild solution of (1.2) by computing its Laplace transform

$$\mathbb{E}[\exp((y_0, \lambda_t))] = \exp((y_t, \lambda_0)),$$

where $y$ is the unique solution of a non-linear PDE.

The ergodic theory of space-time SPDEs is classical by now, with the monographs [DPZ92, DPZ96], and in particular the vast literature on the 2D stochastic Navier-Stokes equation which was the stage of several profound advances such as the asymptotic coupling technique and the asymptotic Feller property [FM95, Hai05, HM06, EMS01, EM01]. More recent results include [BKS20, CKNP21, KKMS20].

As one can expect, the literature on measure-valued SPDEs is less developed. However, there exist connections with measure-valued branching processes (aka superprocesses) which are also Markovian processes characterised by their Laplace transform, only the PDE satisfied by $y$ takes a different form [Li11]. We note that the ergodic behaviour of superprocesses has been studied extensively thanks to their Laplace transform [CRY17, Eth93, Fri19, Isc86, KLM19], which is a hopeful message for us. Furthermore, the analogy with our SPDE lift does not stop there: the density field of some of these superprocesses also satisfies an SPDE with an affine structure [KS88, LX04], and in the continuous-state case branching processes share properties with affine SDEs [KRM12]. However, superprocesses take values in spaces of non-negative measures, which can be locally compact, in which case it makes sense to use the standard Feller property.

The rest of the paper is organised as follows. In Section 2 we discuss the mathematical framework, the generalised Feller property, and present our main results, Proposition 2.4 and Theorem 2.7. In Section 3 we introduce a condition for existence of an invariant measure in an abstract setting that then leads to the proof of Proposition 2.4. In Section 4, we consider the multidimensional SPDE, prove that there is a unique solution to the multidimensional lift that is a generalised Feller process and further discuss its properties in Proposition 4.3. Section 5 concludes the proof of Theorem 2.7 by showing that under Assumption 2.6, the bound in Proposition 2.4 holds. Section 6 discusses future directions.
2. Framework and Main Results

We recall some of the notations introduced in [CT20]. Let $X$ be a completely regular Hausdorff topological space. We say that $\varrho : X \rightarrow (0, \infty)$ is an admissible weight function if the sets $K_R := \{x \in X : \varrho(x) \leq R\}$ are compact for all $R > 0$ [CT20, Definition 2.1]. The supremum norm on $\mathbb{R}$ is denoted $\|\cdot\|_\infty$. The vector space
\[
B^\varrho(X) := \left\{ f : X \rightarrow \mathbb{R} : \sup_{x \in X} \varrho(x)^{-1} \|f(x)\|_\infty < \infty \right\},
\]
equipped with the norm
\[
\|f\|_\varrho := \sup_{x \in X} \varrho(x)^{-1} \|f(x)\|_\infty , \tag{2.1}
\]
is a Banach space, and $C_b(X) \subset B^\varrho(X)$. Moreover, the space $B^\varrho(X)$ is defined as the closure of $C_b(X)$ in $B^\varrho(X)$ [CT20, Definition 2.3], and is also a Banach space when equipped with the norm (2.1). Generalised Feller semigroups are the analogue of standard Feller semigroups on the space of continuous functions vanishing at infinity on locally compact spaces. They are bounded, positive, linear, strongly continuous operators.

**Definition 2.1** (Definition 2.5 of [CT20]). A family of bounded linear operator $P_t : B^\varrho(X) \rightarrow B^\varrho(X)$ for $t \geq 0$ is called generalised Feller semigroup if

(i) $P_0 = I$, the identity on $B^\varrho(X)$,
(ii) $P_{t+s} = P_t P_s$, for all $t, s \geq 0$,
(iii) For all $f \in B^\varrho(X)$ and $x \in X$, $\lim_{t \downarrow 0} P_t f(x) = f(x)$,
(iv) There exist $C > 0$ and $\varepsilon > 0$ such that for all $t \in [0, \varepsilon]$, $\|P_t f\|_{L^\varrho(X)} \leq C$,
(v) $P_t$ is positive for all $t \geq 0$, that is, for any $f \in B^\varrho(X)$ such that $f \geq 0$, then $P_t f \geq 0$.

Let $\mathcal{P}(X)$ be the space of probability measures on $X$.

**Definition 2.2.** For all $t \geq 0$, we define $P_t^*$ as the adjoint of $P_t$, that is, for all $\varphi \in B^\varrho(X)$ and $\mu \in \mathcal{P}(X)$, they satisfy
\[
\int_X P_t \varphi(x) \mu(dx) = \langle P_t \varphi, \mu \rangle = \langle \varphi, P_t^* \mu \rangle.
\]

We will call $(\lambda_t)_{t \geq 0}$ the process associated to the semigroup $(P_t)_{t \geq 0}$ (and reciprocally) if, for all $t \geq 0$, $P_t^* \gamma$ is the law of $\lambda_t$ whenever $\gamma$ is the law of $\lambda_0$. From the display above, one deduces that in that case $P_t \varphi(\lambda_0) = \mathbb{E}_\gamma [\varphi(\lambda_t)]$.

**Definition 2.3.** We say that $\mu$ is an invariant measure if $P_t^* \mu = \mu$ for all $t \geq 0$.

2.1. Main results. We start with a condition for existence in a general state-space $Y^*$, defined as the dual of a Banach space $Y$ and equipped with its weak-$*$-topology. We also define the strong norm $\|\lambda\|_{Y^*} := \sup_{y \in Y, \|y\|_Y \leq 1} \langle y, \lambda \rangle$. 

Proposition 2.4. If $(\lambda_t)_{t \geq 0}$ is a generalised Feller process taking values in $Y^*$ and
\[ \sup_{t \geq 0} \mathbb{E}[\|\lambda_t\|_{Y^*}] < \infty, \] (2.2)
then it has an invariant measure.

Remark 2.5. Equation (2.2) corresponds to $\sup_{t \geq 0} \mathbb{E}[g(\lambda_t)] < \infty$ for the choice of weight function $g(\lambda) := 1 + \|\lambda\|_{Y^*}$.

Our goal is to apply this to $Y^*$-valued SPDEs which we introduce now. Let $d \geq 1$, we consider the Banach space $Y := C_b(\mathbb{R}_+, \mathbb{R}^d)$ and its dual $Y^* = \mathcal{M}(\mathbb{R}_+, \mathbb{R}^d)$. We then have $\langle y, \lambda \rangle := \sum_{i=1}^d \langle y^i, \lambda^i \rangle := \sum_{i=1}^d \int_0^\infty y^i(x) \lambda^i(dx)$ for all $y \in Y$, $\lambda \in Y^*$. Moreover, the weight function is $g(\lambda) := 1 + \|\lambda\|_{Y^*}$, the set $X$ introduced in section 1.3 is a subset of $Y^*$ and the space $B^d(X)$ is defined in the same way.

We consider the $Y^*$-valued multidimensional SPDE
\[ d\lambda_t(dx) = -x\lambda_t(dx)\,dt + \nu(dx)\,dX_t, \] (2.3)
where $X$ is an $\mathbb{R}^d$-semimartingale:
\[ X_t := \int_0^t b(\Xi_s)\,ds + \int_0^t \sigma(\Xi_s)dW_s, \] (2.4)
with $\Xi := (\Xi^1, \ldots, \Xi^d)$, $\lambda_t(dx) \in \mathbb{R}^d$, $\nu(dx) \in \mathbb{R}^{d \times d}$, $W$ is an $m$-dimensional Brownian motion, and $b \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d \times m}$ are such that
\[ b_i(x) = -\sum_{k=1}^d \beta_{ik} x_k; \quad \sigma_{ij}(x) = \sum_{k=1}^d \sigma_{ijk} \sqrt{x_k} + c_{ijk}, \quad \text{for all } x = (x^1, \ldots, x^d) \in \mathbb{R}^d. \] (2.5)

Let us define $\tilde{d} \in [0, d]$ and the state space $D := \mathbb{R}_+^{\tilde{d}} \times \mathbb{R}^{d-\tilde{d}}$.

Assumption 2.6. The following conditions hold:

a) $\lambda_0(dx) = V_0\delta_0(dx)$ for some $V_0 \in D$;
b) for all $i \in [1, \tilde{d}]$, $\nu^i(dx) = x^{\alpha(i)-1}\,dx$, where $\alpha(i) \in (\frac{1}{2}, 1]$;
c) $c_{ijk} = 0$ for all $i, j, k$;
d) if $i \leq \tilde{d}$ and $i \neq k$, then $\sigma_{ijk} = \beta_{ik} = 0$ for all $j$;
e) if $i > \tilde{d}$ and $k > \tilde{d}$, then $\sigma_{ijk} = \beta_{ik} = 0$ for all $j$.

Our main result is the following.

Theorem 2.7. If Assumption 2.6 holds, then the solution to (2.3) has an invariant measure.

Remark 2.8. Assumption 2.6 has specific structural restrictions on the coefficients $c, \beta, \sigma$ under which Theorem 2.7 holds. We note here though that these are only sufficient conditions and in fact, intermediate results of independent interest, such as Proposition 4.3 and Lemma 5.2, do not require these conditions.
Condition d) states that for \( i \leq \bar{d} \) the total mass \( \overline{\lambda}^i \) of the \( i \)th component is an autonomous square-root process. The last condition entails that for \( i > \bar{d} \) the \( i \)th component is explicitly given in terms of the autonomous square-root processes. Heuristically, in the mild sense, (2.3) reads

\[
\langle y, \lambda_i \rangle = \langle e^{-t} y, \lambda_0 \rangle + \int_0^t y(x) \left( \int_0^x e^{-x(t-s)} \, dX_s \right) \nu(dx),
\]

for any \( y \in Y \) and \( \lambda_0 \in Y^* \), and for all \( i \in [1, d] \),

\[
\langle y^i, \lambda^i_0 \rangle = \langle e^{-t} y^i, \lambda^0_0 \rangle + \int_0^t y^i(x) \left( \int_0^x e^{-x(t-s)} \, dX^i_s \right) \nu(dx).
\]

Assumption 2.6 then implies

\[
dX^i_t = -\beta_{iM} \overline{\lambda}^i_t \, dt + \sqrt{\overline{\lambda}^i_t} \sum_{j=1}^m \sigma_{ij} \, dW^j_t,
\]

\[
dX^i_t = -\sum_{k=1}^{\bar{d}} \beta_{ik} \overline{\lambda}^i_t \, dt + \sum_{j=1}^m \sum_{k=1}^{\bar{d}} \sigma_{ijk} \sqrt{\overline{\lambda}^i_t} \, dW^j_t,
\]

if \( i \in [1, \bar{d}] \),

if \( i \in [\bar{d} + 1, d] \).

The representation (2.6) allows to derive the equation satisfied by \( \overline{\lambda} \) in certain cases of interest.

Example 2.9.

- The one-dimensional Volterra square-root process, appearing as the variance in the rough Heston model [EEFR18] satisfies these conditions with \( \bar{d} = d = 1 \) and \( \alpha(i) = \alpha \in (\frac{1}{2}, 1) \):

\[
\overline{\lambda}_t = V_0 + \int_0^t (t-s)^{\alpha-1} \left( -\beta \overline{\lambda}_s \right) \, ds + \int_0^t (t-s)^{\alpha-1} \sigma \sqrt{\overline{\lambda}_s} \, dW_s.
\]

Its lift is the one-dimensional SPDE introduced in [CT20].

- The two-dimensional rough Heston model is also covered where \( \bar{d} = 1, \ d = m = 2, \alpha(1) = \alpha \in (\frac{1}{2}, 1) \) and \( \alpha(2) = 1 \):

\[
\begin{align*}
\overline{\lambda}^1_t &= V_0^1 + \int_0^t (t-s)^{\alpha-1} \left( -\beta \overline{\lambda}^1_s \right) \, ds + \int_0^t (t-s)^{\alpha-1} \sigma \sqrt{\overline{\lambda}^1_s} \, dW^1_s \\
\overline{\lambda}^2_t &= -\frac{1}{2} \int_0^t \overline{\lambda}^2_s \, ds + \int_0^t \rho \sqrt{\overline{\lambda}^1_s} \, dW^1_s + \int_0^t \overline{\rho} \sqrt{\overline{\lambda}^2_s} \, dW^2_s.
\end{align*}
\]

More precisely, the coefficients are \( \sigma_{ij1} = 0 \) for all \( i, j = 1, 2 \) and

\[
\beta = \begin{pmatrix} 0 & 1/2 \\ 0 & \beta \end{pmatrix}; \quad \sigma_{..2} = \begin{pmatrix} \overline{\rho} & \rho \\ 0 & \sigma \end{pmatrix}.
\]

- One can also consider extensions of the previous example with \( \alpha(2) < 1 \), and higher dimensional systems where \( \bar{d} > 1 \) and the square-root processes feed back into the dynamics of \( \overline{\lambda}^i_t \), i.e. \( \beta_{ik} > 0 \) for all \( k \leq \bar{d} \).
Proof of Theorem 2.7. This is a combination of Propositions 2.4, 4.3 and 5.1. More precisely, Proposition 2.4 states that, for a generalised Feller process \((\lambda_t)_{t \geq 0}\), the bound (2.2) is a sufficient condition for the existence of an invariant measure. Proposition 4.3 shows that there exists a unique solution \((\lambda_t)_{t \geq 0}\) to (1.2) and that it is indeed a generalised Feller process. Finally, Proposition 5.1 ensures the condition holds.

Theorem 2.7 delivers a stationary distribution for \((\lambda_t)_{t \geq 0}\) which satisfies the SVE

\[
\lambda_t = V_0 + \int_0^t K(t-s) \, dX_s,
\]

where \(K(t) = \int_0^\infty e^{-xt} \nu(dx)\) and \(X\) solves (2.4). The choice of \(\nu\) in Assumption 2.6 yields the Riemann-Liouville kernel \(K(t) = t^{\alpha(i) - 1} / \Gamma(\alpha(i) \Gamma(1 - \alpha))\), and weak existence and uniqueness were derived in [ALP17].

Corollary 2.10. Under the conditions of Theorem 2.7, the unique weak solution of (1.1) has a stationary distribution.

Proof. Let \(\mu^*\) denote the invariant measure of \(\lambda\). If the law of \(\lambda_t\) is \(\mu^*\) for some \(t \geq 0\), then \(\lambda_{t+s} = (1, \lambda_{t+s})\) is equal in distribution to \((1, \lambda_t) = \lambda_t\).

Remark 2.11. The authors in [CT20] provide two types of lifts for the SVE (1.1). The forward curve lift (Section 5.2 of [CT20]) lies in \(C(\mathbb{R}^2_+)\) where a tightness criterion depends on Arzelá-Ascoli theorem, as for the original SVE. However, this Kolmogorov-type criterion seems unsuitable for large time since the bound depends on the time horizon ([ALP17, Lemma 2.4] or [JP22, Lemma 3.12]). The other type of lift, introduced in Section 5.1 of [CT20] and displayed in (1.2), takes value in the space of signed measures on \(\mathbb{R}_+\) for which there exists a nice tightness criterion.

3. A CONDITION FOR EXISTENCE

This section aims at proving Proposition 2.4. We start by stating and proving an extension of Krylov-Bogoliubov theorem [DPZ92, Theorem 11.7] to the setting of generalised Feller processes on \(B^\varrho(X)\) for any completely regular Hausdorff topological space \(X\).

Lemma 3.1. Let \((P_t)_{t \geq 0}\) be a generalised Feller semigroup. Suppose that there exists \(\gamma \in \mathcal{P}(X)\) and a strictly positive sequence \(T_n\) going to \(+\infty\) as \(n\) goes to \(+\infty\) such that

(i) the sequence of measures \(Q_{T_n}^\gamma := \frac{1}{T_n} \int_0^{T_n} P_t^\gamma \, dt\) converges weakly to some \(\mu \in \mathcal{P}(X)\);
(ii) for any \(X\)-valued random variable \(\lambda\) with distribution \(\gamma\), \(\sup_{t \geq 0} P_t \varrho(\lambda) < \infty\).

Then \(\mu\) is an invariant measure for \((P_t)_{t \geq 0}\).
Remark 3.2. By Prokhorov’s theorem [DPZ92, Theorem 2.3], (i) is equivalent to tightness of \((P_t^* \gamma)_{t \geq 0}\). The original Krylov-Bogoliubov theorem for standard Feller semigroups does not require (ii).

Proof. Fix \(t > 0\) and \(\varphi \in \mathcal{B}_c(X)\), then by definition \(P_t \varphi \in \mathcal{B}_c(X)\). Let \(R > 0\), since \(f \in \mathcal{B}_c(X)\) if and only if \(f|_{K_R} \in \mathcal{C}(K_R)\) (see [CT20, Equation (2.3)]), we need to work on \(K_R\) to apply the weak convergence of (i):

\[
\langle \varphi, P_t^* \mu \rangle = \langle P_t \varphi, \mu \rangle = \int_{X \setminus K_R} P_t \varphi \, d\mu + \int_{K_R} P_t \varphi \, d\mu = \int_{X \setminus K_R} P_t \varphi \, d\mu + \lim_{n \uparrow \infty} \int_{K_R} P_t \varphi \, dQ_t^* \gamma.
\]

Then we go back to \(X\) to apply the adjoint property

\[
\int_{K_R} P_t \varphi \, dQ_t^* \gamma = \frac{1}{T_n} \int_0^{T_n} \langle P_t \varphi, P_s^* \gamma \rangle \, ds - \int_{X \setminus K_R} P_t \varphi \, dQ_t^* \gamma
\]

\[
= \frac{1}{T_n} \int_t^{t+T_n} \langle \varphi, P_s^* \gamma \rangle \, ds - \int_{X \setminus K_R} P_t \varphi \, dQ_t^* \gamma.
\]

We observe that

\[
\frac{1}{T_n} \int_t^{t+T_n} \langle \varphi, P_s^* \gamma \rangle \, ds = \frac{1}{T_n} \int_0^{T_n} \langle \varphi, P_s^* \gamma \rangle \, ds + \frac{1}{T_n} \int_{T_n}^{t+T_n} \langle \varphi, P_s^* \gamma \rangle \, ds - \frac{1}{T_n} \int_0^t \langle \varphi, P_s^* \gamma \rangle \, ds,
\]

where the second and third terms tend to zero as \(n\) goes to infinity, while the first one is \(\langle \varphi, Q_t^* \gamma \rangle\).

We relocate to \(K_R\) for the weak convergence:

\[
\lim_{n \uparrow \infty} \frac{1}{T_n} \int_t^{t+T_n} \langle \varphi, P_s^* \gamma \rangle \, ds = \lim_{n \uparrow \infty} \left( \int_{K_R} \varphi \, dQ_t^* \gamma + \int_{X \setminus K_R} \varphi \, dQ_t^* \gamma \right)
\]

\[
= \int_{K_R} \varphi \, d\mu + \lim_{n \uparrow \infty} \int_{X \setminus K_R} \varphi \, dQ_t^* \gamma.
\]

Overall, this yields our objective plus a remainder

\[
\langle \varphi, P_t^* \mu \rangle = \langle \varphi, \mu \rangle - \int_{X \setminus K_R} \varphi \, d\mu + \int_{X \setminus K_R} P_t \varphi \, d\mu + \lim_{n \uparrow \infty} \left( \int_{X \setminus K_R} \varphi \, dQ_t^* \gamma - \int_{X \setminus K_R} P_t \varphi \, dQ_t^* \gamma \right)
\]

\[
=: \langle \varphi, \mu \rangle + \varepsilon(R).
\]

To deal with \(\varepsilon(R)\), we apply a Fatou-type lemma for measures [DE97, Theorem A.3.12]. For any \(f \in \mathcal{B}_c(X)\),

\[
\left| \int_{X \setminus K_R} f \, d\mu \right| \leq \sup_{x \in X \setminus K_R} \frac{|f(x)|}{\varrho(x)} \int_X \varrho \, d\mu \leq \sup_{x \in X \setminus K_R} \frac{|f(x)|}{\varrho(x)} \lim_{n \uparrow \infty} \int_X \varrho \, dQ_t^* \gamma
\]

\[
= \sup_{x \in X \setminus K_R} \frac{|f(x)|}{\varrho(x)} \lim_{n \uparrow \infty} \frac{1}{T_n} \int_0^{T_n} P_s \varrho(\lambda) \, ds,
\]
where $\lambda$ is a $X$-valued random variable with distribution $\gamma$. Therefore,

$$
\varepsilon(R) \leq 2 \left( \sup_{x \in X \setminus K_R} \left| \varphi(x) \right| + \left| P_t \varphi(x) \right| \right) \left( \sup_{t \geq 0} P_t \varrho(\lambda) \right).
$$

By Equation (2.3) in [CT20] we have for any $f \in B^\varrho(X)$,

$$
\lim_{R \to \infty} \sup_{x \in X \setminus K_R} \left| f(x) \right| \varrho(x) = 0.
$$

The uniform bound $\sup_{t \geq 0} P_t \varrho(\lambda) < \infty$ thus ensures that $\varepsilon(R)$ vanishes as $R$ goes to $+\infty$. □

**Proof of Proposition 2.4.** Let $(P_t)_{t \geq 0}$ be the generalised Feller semigroup associated to $(\lambda_t)_{t \geq 0}$. Hence, for any $X$-valued random variable $\lambda_0$, $P_t \varrho(\lambda_0) = E[\varrho(\lambda_t)]$.

In [CT20], the analysis is performed with the weight function $\varrho(\lambda) = 1 + \|\lambda\|_{Y^*}^2$. The only necessary estimate is $E[\varrho(\lambda_t)] \leq C \varrho(\lambda_0)$, for some finite constant $C > 0$, and we note that the inequality $E[1 + \|\lambda_t\|_{Y^*}^2] \leq C(1 + \|\lambda_0\|_{Y^*}^2)$ implies $E[1 + \|\lambda_t\|_{Y^*}] \leq \sqrt{2C}(1 + \|\lambda_0\|_{Y^*})$. Therefore, the weight function $\varrho(\lambda) = 1 + \|\lambda\|_{Y^*}$ is also admissible, and the assumption (2.2) implies

$$
\sup_{t \geq 0} E[\varrho(\lambda_t)] < \infty.
$$

(3.1)

This yields condition (ii) of Lemma 3.1.

Furthermore, by [BD19, Lemma 2.9], for all $t \geq 0$, let $\Lambda_t$ be the $Y^*$-valued random variable with distribution $Q_t^* \gamma$. The family $(\Lambda_t)_{t \geq 0}$ is tight if and only if there exists a tightness function $G : Y^* \to \mathbb{R}_+$ such that $\sup_{t \geq 0} E[G(\Lambda_t)] < \infty$. We observe that an admissible weight function is also a tightness function by design, hence proving tightness of $(\Lambda_t)_{t \geq 0}$ boils down to showing

$$
\sup_{T \geq 0} \frac{1}{T} \int_0^T E[\varrho(\lambda_t)] \, dt < \infty,
$$

which is a consequence of (3.1). Therefore, both conditions of Lemma 3.1 are satisfied as soon as (2.2) holds. □

### 4. The multidimensional SPDE

For simplicity, we assume that $\nu$ is diagonal, which releases some indexation load and appoints the same kernel for drift and diffusion. We can then define the matrix $K(t) := \langle e^{-t}, \nu \rangle$ and since $\nu$ is diagonal this yields $K^i(t) = \langle e^{-t}, \nu^i \rangle_i$ where the superscript $i$ stands for the $i$th component of the diagonal.

We replicate the setting of Markovian affine processes characterised in [DFS03] and cast it in the appropriate multidimensional adaptation of [CT20]. In the setting of (2.3), we want the total mass $\overline{\lambda}$ of each component to be either a square root process or an Ornstein-Uhlenbeck-type process; this is achieved by the following set of assumptions:
Assumption 4.1. The following hold:

- if \( i < d \) then \( c_{ijk} = 0 \) for all \( j, k \); if moreover \( k \neq i \) then \( \sigma_{ijk} = \beta_{ik} = 0 \) for all \( j \);
- if \( i > d \) and \( k > d \), then \( \sigma_{ijk} = 0 \) for all \( j \).

Remark 4.2. This assumption is strictly weaker than Assumption 2.6, hence Examples 2.9 are still covered.

We can now display the equations satisfied by the total mass of each component. For each \( i \in [1, d] \), \( \overline{x}^i \) is then an autonomous square-root process living in \( \mathbb{R}_+ \), and satisfies

\[
\overline{x}^i = (e^{-t}, \lambda^i_0) + \int_0^t K^i(t - s)(-\beta_i \overline{x}^i_s) \, ds + \int_0^t K^i(t - s) \sqrt{\lambda^i_s} \sum_{j=1}^m \sigma_{ij} \, dW^j_s,
\]

(4.1)

with no feedback from the other components. For each \( i \in [d + 1, d] \), \( \overline{x}^i \) is an OU-type process living in \( \mathbb{R} \), which allows feedback from every component in the drift but only from the first \( \overline{d} \) square-root components in the diffusion:

\[
\overline{x}^i = (e^{-t}, \lambda^i_0) + \int_0^t K^i(t - s) \left(-\sum_{k=1}^d \beta_{ik} \overline{x}^k_s\right) \, ds + \sum_{j=1}^m \sum_{k=1}^d \int_0^t K^i(t - s) \left(\sigma_{ijk} \sqrt{\lambda^k_s} + c_{ijk}\right) \, dW^j_s.
\]

(4.2)

The reason is that a component needs to lie on the positive cone if its square root appears somewhere in the SPDE.

We now go through the one-dimensional results of [CT20, Section 4] and highlight the differences with the multidimensional case presented here. Instead of the spaces \( \mathcal{E}^w \) introduced in [CT20, (4.7)], we define for all \( n \in \mathbb{N} \),

\[
\mathcal{E}^n := \left\{ \lambda_0 \in Y^* : \overline{m}^n_i \geq 0, \text{ for all } i \in [1, \overline{d}], \overline{m}^n_i \in \mathcal{M}(\mathbb{R}_+, \overline{R}) \text{ for all } i \in [\overline{d} + 1, d] \text{ and } t \geq 0 \right\},
\]

where

\[
\begin{align*}
\overline{m}^n_i &= (e^{-t}, \lambda^i_0) - \left(\beta_i - n \sum_{j=1}^m \sigma^2_{ij}\right) \int_0^t K^i(t - s) \overline{m}^n_i \, ds, \\
\text{d}\overline{m}^n_i &= A^* \overline{m}^n_i \, dt + \nu \left(\sum_{k=1}^d \beta_{ik} \overline{m}^n_k - n \sum_{j=1}^m \sum_{k=1}^d \left(\sigma_{ijk} \sqrt{\overline{m}^n_k} + c_{ijk}\right)^2\right) \, dt, \quad \overline{m}^n_0 = \lambda^i_0, \quad i \in [\overline{d} + 1, d].
\end{align*}
\]

Note that \( \eta \) satisfies a deterministic equation and the condition for \( i \in [\overline{d} + 1, d] \) in the definition of \( \mathcal{E}^n \) is necessarily satisfied, while we want to restrict the state space of the square-root processes to \( \mathbb{R}_+ \). These definitions allow to define the invariant space \( \mathcal{E} := \cap_{n \in \mathbb{N}} \mathcal{E}^n \) and its polar cone \( \mathcal{E}^* := \left\{ y \in Y : \langle y, \lambda \rangle \leq 0 \text{ for all } \lambda \in \mathcal{E} \right\} \). For \( \mathcal{E} \) to be well defined we had to let the coefficients grow with \( n \) at a uniform speed across the components, which is a notable difference with the one-dimensional framework of [CT20] where a single speed \( w \) was needed.
In order to define the generator of the SPDE, we introduce the set $D := \{ y \in Y : \langle y, \nu \rangle \text{ is well-defined} \}$ and, for each $y \in D$, the set $F$ of Fourier basis elements of the form
\[
 f_y : E \to [0, 1]; \lambda \mapsto \exp(\langle y, \lambda \rangle).
\] (4.3)

We also recall that the resolvent of the second kind corresponding to $K$ is the kernel $R \in L^1(\mathbb{R}_+, \mathbb{R}^{d \times d})$ such that
\[
 K*K = R = R*K = K - R.
\]

The following is the analogue of [CT20, Theorem 4.17].

**Proposition 4.3.** Let Assumption 4.1 hold. Assume moreover that $\lambda_0 \in E$, $K \in L_{loc}^2(\mathbb{R}_+, \mathbb{R}^{d \times d})$ and, for all $i \in \llbracket 1, d \rrbracket$ and $w > 0$, $K^i$ and $R^i_w$ are non-negative, where $R^i_w$ is the resolvent of the second kind of $wK^i$.

(i) The SPDE (2.3) admits a unique Markovian solution with values in $E$ given by a generalised Feller semigroup $(P_t)_{t \geq 0}$ on $B(\mathbb{R})$. The generator $A : F \to B(\mathbb{R})$ associated to the semigroup $(P_t)_{t \geq 0}$ reads
\[
 Af_y(\lambda) = f_y(\lambda) \left( \langle Ay, \lambda \rangle + \langle yb(\lambda), \nu \rangle + \frac{1}{2} \langle y\sigma^T(\lambda), \nu \rangle \langle y\sigma(\lambda), \nu \rangle \right) =: f_y(\lambda)R(y, \lambda),
\] (4.4)
where the coefficients are given in (2.5).

(ii) This generalised Feller process allows to construct a probabilistically weak and analytically mild solution, i.e. for $y \in Y$,
\[
 \langle y, \lambda_t \rangle = \langle ye^{-t \cdot}, \lambda_0 \rangle + \int_0^t \langle ye^{-(t-s) \cdot}, \nu \rangle dX_s.
\]

(iii) The affine transform formula is satisfied, i.e.
\[
 \mathbb{E}_{\lambda_0}[\exp(\langle y_0, \lambda_t \rangle)] = \exp(\langle y_t, \lambda_0 \rangle),
\]
where $y_t$ solves
\[
 \langle \partial_t y_t, \lambda \rangle = R(y_t, \lambda),
\]
for all $\lambda \in E$, $y_0 \in E_*$, $t \geq 0$, where $R$ is defined in (4.4). Furthermore, $y_t \in E_*$ for all $t \geq 0$.

(iv) For any $\lambda_0 \in E$, the corresponding stochastic Volterra equation given by
\[
 \mathcal{X}_t = \langle 1, \lambda_t \rangle = \langle e^{-t \cdot}, \lambda_0 \rangle + \int_0^t K(t-s) dX_s
\]
admits a probabilistically weak solution.

(v) For all $u \in \mathbb{R}$, the Laplace transform of the Volterra equation $\mathcal{X}_t$ is
\[
 \mathbb{E}_{\lambda_0}[e^{u\mathcal{X}_t}] = \exp \left( u\langle e^{-t \cdot}, \lambda_0 \rangle + \int_0^t \langle e^{(t-s) \cdot}, \lambda_0 \rangle R(\langle y_s, \nu \rangle) ds \right).
\]

\[\text{This is the multidimensional version of the generator given in [CT20, (4.21)]}, \text{ where there seems to be a mild typo.}\]
Proof. The entirety of [CT20, Section 4] aims at proving [CT20, Theorem 4.17]. Our result follows from the same lines with a few modifications and remarks that we highlight here. The Lemmas, Assumptions, Remarks, Propositions and Theorems we refer to in this proof are all from [CT20], unless stated otherwise.

Modifications. Since $e^{-t}\nu \in Y^*$ for all $t > 0$ and
\[
\int_0^t \|e^{-s}\nu\|_{Y^*}^2 \, ds = \int_0^t K(s)^2 \, ds < \infty,
\]
Assumption 4.5 holds. Moreover, Assumption 4.9 is satisfied thanks to our assumption on $K^i$ and $R^i_w$ and Remark 4.10.

The space of signed measures is a vector space, hence a convex cone, and so are $E^n$ for all $n \in \mathbb{N}$. The weak-$\ast$-continuity of the solution map and the bound of $\varrho$ derived in Proposition 4.6 hold without modification. Essentially, every bound remains by first getting the bound for the autonomous one-dimensional square-root for $i \in [1, \tilde{d}]$ and then plugging it in the other rows.

The invariant spaces $E^n$ then satisfy all the necessary properties of Proposition 4.8 and Lemma 4.11 because they stay in their convex cone. This yields the generalised Feller property on $B(E)$ of Theorem 4.13.

The variation of constant method continues to apply in multiple dimensions and for OU type processes, hence Proposition 4.14 still holds.

We denote the operator $A^\ast \lambda (dx) = -x \lambda (dx)$ for all $\lambda \in \mathcal{M}(\mathbb{R}_+, \mathbb{R}^d)$. In the proof of Theorem 4.17, for $i \in [1, \tilde{d}]$, we replace Equation (4.26) of [CT20] by
\[
d\lambda_{i,n}^t = A^\ast \lambda_{i,n}^t \, dt + \nu \left( \beta_{ii} - n \sum_{j=1}^m \sigma_{ij}^2 \lambda_{i,n}^t \right) \lambda_{i,n}^t \, dt + \nu \sum_{j=1}^m \sum_{k=1}^d \left( \sigma_{ijk} \sqrt{\lambda_{i,n}^t} + c_{ijk} \right) \lambda_{i,n}^t \, dt + dN_{ijk,n}^t,
\]
where $N_{ijk,n}^t$ is a jump process that jumps by $1/n$ and with intensity $n^2 \sigma_{ijk}^2 \lambda_{i,n}^t$. Similarly, for all $i \in [\tilde{d} + 1, d]$,
\[
d\lambda_{i,n}^t = A^\ast \lambda_{i,n}^t \, dt + \nu \left( \sum_{k=1}^d \beta_{ik} \lambda_{i,n}^{k,n} \, dt + \sum_{j=1}^m \sum_{k=1}^d \left( -n \left( \sigma_{ijk} \sqrt{\lambda_{i,n}^{k,n}} + c_{ijk} \right) \lambda_{i,n}^{k,n} \, dt + dN_{ijk,n}^{k,n} \right) \right),
\]
where $N_{ijk,n}^{k,n}$ is a jump process that jumps by $1/n$ and with intensity $n^2 \left( \sigma_{ijk} \sqrt{\lambda_{i,n}^{k,n}} + c_{ijk} \right)^2$.

The same arguments as in the proof of Theorem 4.17 allow to conclude that the limit of $\lambda^n$ as $n$ goes to infinity is a generalised Feller process which generator coincides with that of (2.3) that we compute now.
The generator. Let us define, for any \( y \in \mathcal{E}, \lambda_0 \in \mathcal{E} \) and \( i \in \llbracket 1, d \rrbracket \),
\[
S_t^i := \langle y^i, \lambda_0^i \rangle = \langle y^i e^{-t}, \lambda_0^i \rangle + \int_0^t \langle y^i e^{-(t-s)}, \nu^i \rangle dX_s^i.
\]
This is not a semimartingale, therefore we introduce this approximation for any \( \varepsilon > 0 \):
\[
S_t^{i, \varepsilon} := \langle y^i e^{-t}, \lambda_0^i \rangle + \int_0^t \langle y^i e^{-(t+s)}, \nu^i \rangle dX_s^i.
\]
We define \( F(s) := e^s \) such that \( F(S_t) = e^{\langle y, \lambda_t \rangle} = e^{\sum_i \langle y^i, \lambda_t^i \rangle} = f_y(\lambda_t) \) and by convergence in law \( P_t f_y(\lambda_0) = \mathbb{E}[F(S_t)] = \lim_{\varepsilon \to 0} \mathbb{E}[F(S_t^{\varepsilon})] \). By Itô’s formula,
\[
F(S_t^{\varepsilon}) = F(S_0^{\varepsilon}) + \sum_{i=1}^d \int_0^t \partial_i F(S_r^{\varepsilon}) dS_r^{i, \varepsilon} \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij} F(S_r^{\varepsilon}) d\lbrack S^{i, \varepsilon}, S^{j, \varepsilon} \rbrack_r,
\]
where, letting \( A_y(x) = -xy(x) \), we obtain
\[
dS_r^{i, \varepsilon} = \left( \langle A_y e^{-r}, \lambda_0^i \rangle + \langle y e^{-r}, \nu^i \rangle b_i(\lambda_r) + \int_0^t \langle A_y e^{(r+s-u)}, \nu^i \rangle dX_u^i \right) dr + \langle y e^{-r}, \nu^i \rangle \sigma_i(\lambda_r) dB_r.
\]
Hence, for all \( \lambda_0 \in \mathcal{E} \),
\[
\mathbb{E}[F(S_t^{\varepsilon})] = e^{\langle y, \lambda_0 \rangle} + \int_0^t \mathbb{E} \left[ F(S_r^{\varepsilon}) \sum_{i=1}^d \left( \langle A_y e^{-r}, \lambda_0^i \rangle + \langle y e^{-r}, \nu^i \rangle b_i(\lambda_r) + \int_0^t \langle A_y e^{(r+s-u)}, \nu^i \rangle b_i(\lambda_u) du \right) \right. \\
+ \frac{F(S_r^{\varepsilon})}{2} \sum_{i,j=1}^d \langle y e^{-r}, \nu^i \rangle \langle y e^{-r}, \nu^j \rangle \sigma_i \sigma_j(\lambda_r) \biggr] dr.
\]
Therefore, we obtain
\[
\lim_{\varepsilon \to 0} \frac{P_t f_y(\lambda) - f_y(\lambda)}{t} = \lim_{\varepsilon \to 0} \frac{\mathbb{E}[F(S_t^{\varepsilon})] - f_y(\lambda)}{t} \\
= f_y(\lambda) \sum_{i=1}^d \left( \langle A_y^i, \lambda^i \rangle + \langle y^i, \nu^i \rangle b_i(\lambda) \right) + \frac{f_y(\lambda)}{2} \sum_{i,j=1}^d \left( \langle y^i, \nu^i \rangle \langle y^j, \nu^j \rangle \sigma_i \sigma_j(\lambda) \right) \\
= f_y(\lambda) \left( \langle A_y, \lambda \rangle + \langle y b(\lambda), \nu \rangle + \frac{1}{2} \langle \sigma(\lambda), \nu \rangle \langle \sigma(\lambda), \nu \rangle \right),
\]
where the coefficients were given in (2.5). This yields item (i), while item (ii) follows along the same lines as in Theorem 4.17.

To prove the third item, one only needs to notice that
\[
f_{y_t}(\lambda) \partial_t y_t(\lambda) = \partial_t f_{y_t}(\lambda) = Af_{y_t}(\lambda) = f_{y_t}(\lambda) R(y_t, \lambda),
\]
where the second equality corresponds to the backward Kolmogorov equation and the third follows from the definition of the generator in (4.4).

The fourth point is a consequence of the second with \( y \equiv 1 \).
Finally, let \( y_0 \equiv u \in \mathbb{R} \), such that from (ii),
\[
\mathbb{E} \left[ e^{uY_t} \right] = \mathbb{E} \left[ e^{\langle y_0, \lambda_t \rangle} \right] = e^{\langle y_0, \lambda_0 \rangle},
\]
where
\[
\langle y_t, \lambda_0 \rangle = u \langle e^{-t}, \lambda_0 \rangle + \int_0^t \langle e^{(t-s)} \cdot \lambda_0 \rangle \mathcal{R} \left( \langle y_s, \nu \rangle \right) ds,
\]
which clearly indicates that \( \psi_t = \langle y_t, \nu \rangle \) solves
\[
\psi_t = uK(t) + \int_0^t K(t-s)\mathcal{R}(\psi_s) ds.
\]
This concludes the proof of the proposition.

5. Verifying the condition for existence

We proved in the previous section that the solution of (2.3) is indeed a generalised Feller process, hence all that is left to apply Proposition 2.4 is to show that the bound
\[
\sup_{t \geq 0} \mathbb{E} \left[ ||\lambda_t||_{Y_*} \right] < \infty
\]
holds. In this section, we specify the conditions on the parameters of the SPDE that ensure (5.1).

We start with some comments on the possible choices of initial condition \( \lambda_0 \); recall that \( \lambda_0 \) is admissible if it belongs to \( \mathcal{E} \). By Assumption 5.2 and Remark 5.3 in [CT20], \( e^{-r \nu} \in \mathcal{E} \) for all \( r > 0 \). We can identify different conditions for initial conditions of OU rows or square-root rows. Clearly, for \( i \in [\tilde{d} + 1, d] \), \( \lambda_0 \) can be any signed measure on \( \mathbb{R} \). For \( i \in [1, \tilde{d}] \), by Remark 5.4 in [CT20] and Example 2.2 in [AE18], are allowed all \( \lambda_0 \in Y^* \) such that \( t \mapsto \int_0^\infty e^{-tx} \lambda_0(dx) \in \mathcal{G}_K \), where \( \mathcal{G}_K \) includes in particular Hölder continuous, non-decreasing functions \( g \) with \( g(0) \geq 0 \), and functions of the form \( g = V_0 + K \ast \theta \) such that \( \theta(s) ds + V_0 L(ds) \) is a non-negative measure, where \( V_0 > 0 \) and \( L \) is the resolvent of first kind of \( K \). This yields at least two options:

- \( \lambda_0(dx) = V_0 \delta_0(dx) \), with \( V_0 > 0 \);
- \( \lambda_0(dx) = V_0 \delta_0(dx) + x^{-\alpha-\mu} dx \) where \( 0 < \mu < 1 - \alpha \), such that
\[
\int_0^\infty e^{-xt} \lambda_0(dx) = V_0 + \Gamma(1-\alpha-\mu)t^{\alpha+\mu-1} = V_0 + \frac{\Gamma(1-\alpha-\mu)\Gamma(\alpha+\mu)}{\Gamma(\alpha)\Gamma(\mu)} \int_0^t (t-s)^{\alpha-1}s^{\mu-1} ds,
\]
which is equal to \( V_0 + (K \ast \theta)(t) \) with \( \theta \) and \( L \) non-negative.

We will be particularly interested in the case of constant initial condition because it allows us to compute \( \mathbb{E}[Y_t] \) explicitly, see (5.4). This is the main result of this section.

**Proposition 5.1.** Under Assumption 2.6, the bound (5.1) holds.

We need to state two intermediate lemmas, the proofs of which are postponed to the end of the section, before being in a position to prove Proposition 5.1.
Lemma 5.2. The bound (5.1) holds if \( \|\lambda_0\|_{Y^*} < \infty \) and, for all \( i \in [1, d] \),

\[
\sup_{t \geq 0} E \left[ \int_0^\infty \left( \int_0^t e^{-\gamma(t-s)} dX^i_s \right)^2 \nu^i(dx) \right] < \infty. \tag{5.2}
\]

Remark 5.3. In the case where \( \nu = \delta_0 \) there is no kernel, hence this expectation boils down to \( E[|X^i_t|] \) where \( X^i \) is a Markovian process. For example, Condition (5.2) holds if \( X \) is a multidimensional OU process, which is a Gaussian process with bounded variance, and if \( X \) is a one-dimensional square-root process, which is positive with bounded expectation.

Lemma 5.4. For all \( \alpha \in (\frac{1}{2}, 1] \), \( \mu \in (0, 1] \) and \( \beta > 0 \) (\( E_{\alpha, \beta} \) is the Mittag-Leffler function [Erd55]),

\[
\sup_{t \geq 0} \int_0^t (t-s)^{\mu-1} \left( 1 - \beta s^\alpha E_{\alpha, \alpha+1}(-\beta s^\alpha) \right) ds < \infty.
\]

Proof of Proposition 5.1. First notice that \( \|\lambda_0\|_{Y^*} = V_0 < \infty \). For all \( i > \tilde{d} \), since \( c_{ijk} = 0 \) and \( \beta_{ik} = 0 \) for \( k > \tilde{d} \),

\[
\mathbb{E} \left[ \int_0^t \int_0^t e^{-\gamma(t-s)} dX^i_t \right] \nu^i(dx) \leq \mathbb{E} \left[ \sum_{k=1}^d \int_0^t \left( \int_0^t e^{-\gamma(t-s)} \beta_{ik} \lambda^k_s ds \right) \nu^i(dx) \right] + \int_0^\infty \mathbb{E} \left[ \int_0^t e^{-\gamma(t-s)} \sum_{j=1}^m \sum_{k=1}^d \sigma_{ijk} \sqrt{\lambda^k_s} dB^j_t \right] \nu^i(dx)
\]

\[
\leq \sum_{k=1}^d \mathbb{E} \left[ \int_0^t (t-s)^{\alpha(i)-1} \beta_{ik} \lambda^k_s ds \right] + \int_0^\infty \left( \mathbb{E} \left[ \int_0^t e^{-2\gamma(t-s)} dm \sum_{j=1}^m \sum_{k=1}^d \sigma_{ij}^2 \lambda^k_s ds \right] + \frac{1}{2} \right) \nu^i(dx),
\tag{5.3}
\]

where we used Itô’s isometry. By Lemma 5.2 it is sufficient to prove that the latter is uniformly bounded in \( t \geq 0 \). Recall that if \( k \leq \tilde{d} \) then \( \lambda^k \) is one-dimensional and, by [ALP17, Lemma 4.2],

\[
\mathbb{E}[\lambda^k_t] = V_0^k - V_0^k \beta_{kk} t^\alpha E_{\alpha, \alpha+1}(-\beta_{kk} t^\alpha),
\tag{5.4}
\]

hence the first term of the previous expression is uniformly bounded by Lemma 5.4.

Let us define \( \tilde{\nu}^i(dx) := x^{-\frac{\alpha}{2}} \nu^i(dx). \) For any \( f : \mathbb{R}^+ \to \mathbb{R}^+ \), we can split the following integral and apply Jensen’s inequality to both terms

\[
\int_0^\infty \sqrt{f(x)} \nu^i(dx) = \int_0^1 \sqrt{f(x)} \nu^i(dx) + \int_1^\infty \sqrt{f(x)} \tilde{\nu}^i(dx)
\]

\[
\leq \sqrt{\nu^i((0, 1])} \sqrt{\int_0^1 f(x) \nu(dx)} + \sqrt{\tilde{\nu}^i((1, \infty])} \sqrt{\int_1^\infty f(x) x \tilde{\nu}^i(dx)}
\]

\[
\leq \frac{\Gamma(1-\mu)}{1-\mu} \sqrt{\int_0^\infty f(x) \nu^i(dx)} + \frac{\Gamma(1-\mu)}{\mu - \frac{1}{2}} \sqrt{\int_0^\infty f(x) x \tilde{\nu}^i(dx)}.
\]
We then set \( f(x) = \mathbb{E} \left[ \int_0^t e^{-2x(t-s)} \sigma_{ijk}^x \lambda^k_s \, ds \right] \). Both terms lead to the same type of kernel, after an application of Fubini’s theorem:
\[
\int_0^\infty e^{-2x(t-s)} \nu^i(dx) = 2^{\nu-1} (t-s)^{\alpha(i)-1};
\]
\[
\int_0^\infty e^{-2x(t-s)} x \nu^i(dx) = \int_0^\infty e^{-2x(t-s)} x^{\frac{1}{2} - \alpha(i)} \frac{dx}{\Gamma(1 - \alpha)} = \frac{2^{\alpha(i)} - \frac{1}{2} \Gamma(\alpha(i) - 1)}{\Gamma(1 - \alpha)} (t-s)^{\alpha(i) - \frac{3}{2}}.
\]
We are left to consider integrals of the type \( \int_0^t (t-s)^{\mu-1} \mathbb{E} \left[ \lambda^k_s \right] \, ds \), for \( \mu \in (0,1] \), which are uniformly bounded by Lemma 5.4.

On the other hand, if \( i < \tilde{d} \),
\[
\mathbb{E} \left[ \int_0^\infty \left| \int_0^t e^{-2x(t-s)} \lambda^k_s \, ds \right| \nu^i(dx) \right] \leq \mathbb{E} \left[ \int_0^\infty \left( \int_0^t e^{-2x(t-s)} \beta_i \lambda^k_s \, ds \right) \nu^i(dx) \right]
\]
\[
+ \int_0^\infty \mathbb{E} \left[ \left| \int_0^t e^{-2x(t-s)} \sum_{j=1}^m \sigma_{ij} \sqrt{\lambda^k_s} \, dB_j^s \right| \nu^i(dx) \right],
\]
which boils down to (5.3).

**Remark 5.5.** The case \( c_{ijk} > 0 \) is unlikely to go through by splitting the integrals as we did here. Consider (5.2) where \( X \) is a one-dimensional OU process and split the Lebesgue and Itô integrals as in (5.3). The stochastic integral then reads
\[
\int_0^\infty \mathbb{E}[|B_t^x|] x^{-\alpha} \, dx,
\]
where \( B_t^x := \int_0^t e^{-x(t-s)} \, dW_s \). Since \( \mathbb{E}[|B_t^x|] = \sqrt{\frac{1 - e^{-2x^2}}{x \pi}} \), this entails that (5.5) grows as \( t^{\alpha - \frac{3}{2}} \), and thus is not uniformly bounded.

We conclude with the proof of Lemmas 5.2 and 5.4.

**Proof of Lemma 5.2.** Let \( \mathcal{B}(\mathbb{R}_+) \) denote the Borel subsets of \( \mathbb{R}_+ \). For a signed measure \( \lambda \) on \( \mathbb{R}_+ \), define the upper variation \( \mathcal{U} \) (resp. lower variation \( \mathcal{L} \)) as \( \mathcal{U}(\lambda) := \sup \{ \lambda(A) : A \in \mathcal{B}(\mathbb{R}_+) \} \) (resp. inf), and such that the total variation corresponds to \( ||\lambda||_{Y^*} = \mathcal{U}(\lambda) - \mathcal{L}(\lambda) \).

Let us fix \( t > 0 \), then there exist two increasing sequences of sets \( (U_n)_{n \geq 1} \) and \( (L_n)_{n \geq 1} \), both in \( \mathcal{B}(\mathbb{R}_+) \), such that \( \lambda_t(U_n) \) is non-negative for all \( n \in \mathbb{N} \) and increases towards \( \mathcal{U}(\lambda_t) \) as \( n \) goes to \( +\infty \), and analogously, \( \lambda_t(L_n) \) is non-positive and decreases to \( \mathcal{L}(\lambda_t) \). We will use the representation
\[
||\lambda_t||_{Y^*} = \lim_{n \uparrow \infty} (\lambda_t(U_n) - \lambda_t(L_n)) = \lim_{n \uparrow \infty} \left( \int_{U_n} \lambda_t(dx) - \int_{L_n} \lambda_t(dx) \right).
\]
Intuitively, one should think of the limit of $U_n$ (resp. $L_n$) as the subset of $\mathbb{R}^d_+$ on which $\lambda_t$ is positive (resp. negative). This way, the function maximising $\sup_{\|y\| \leq 1} \langle y, \lambda_t \rangle$ (if it exists) corresponds to the limit of $1_{U_n} - 1_{L_n}$.

We did not assume that $\|\lambda_t\|_Y$. should be finite. Yet, we can apply the monotone convergence theorem on the representation (5.6)

$$
\mathbb{E}[\|\lambda_t\|_Y.] = \lim_{n \uparrow + \infty} \mathbb{E} \left[ \int_{U_n} \lambda_t(dx) - \int_{L_n} \lambda_t(dx) \right] = \lim_{n \uparrow + \infty} \mathbb{E} \left[ \int_0^\infty (1_{U_n} - 1_{L_n}) \left( e^{-xt} \lambda_0(dx) + \int_0^t e^{-x(t-s)} dX_s \nu(dx) \right) \right]. \tag{5.7}
$$

Since $\|\lambda_0\|_Y. < \infty$ by assumption, the first term is finite. Starting from (5.7), we consider the worst possible sets $U_n$ and $L_n$, recall that $\nu$ is diagonal and the form of $X$:

$$
\mathbb{E} \left[ \int_0^\infty (1_{U_n} - 1_{L_n})(x) \left( \int_0^t e^{-x(t-s)} dX_s \right) \nu(dx) \right] = \mathbb{E} \left[ \sum_{i=1}^d \int_0^\infty (1_{U_n} - 1_{L_n})(x) \left( \int_0^t e^{-x(t-s)} dX^i_s \right) \nu^i(dx) \right] \leq \sum_{i=1}^d \mathbb{E} \left[ \int_0^\infty \left| \int_0^t e^{-x(t-s)} dX^i_s \right| \nu^i(dx) \right],
$$

which yields the claim. □

**Proof of Lemma 5.4.** First recall that, since $\alpha > 0$,

$$
E_{\alpha, \alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \alpha)}, \quad \text{for any } z \in \mathbb{R}.
$$

For $\mu \in (0, 1]$, we have

$$
\int_0^t (t-s)^{\mu-1} s^{\alpha(n+1)} \, ds = t^{\alpha(n+1)+\mu} B(\mu, \alpha(n+1) + 1) = t^{\alpha(n+1)+\mu} \frac{\Gamma(\mu) \Gamma(\alpha(n+1) + 1)}{\Gamma(\alpha(n+1) + 1 + \mu)}.
$$

Hence,

$$
\int_0^t (t-s)^{\mu-1} \left( 1 - \beta s^\alpha E_{\alpha, \alpha+1}(-\beta s^\alpha) \right) \, ds = \frac{t^\mu}{\mu} - \beta \Gamma(\mu) \sum_{n=0}^{\infty} \frac{(-\beta)^n t^{\alpha(n+1)+\mu}}{\Gamma(\alpha(n+1) + 1 + \mu)} = \frac{t^\mu}{\mu} - \beta \Gamma(\mu) e^{\alpha+\mu} E_{\alpha, \alpha+\mu+1}(-\beta t^\alpha). \tag{5.8}
$$

As $z$ tends to $-\infty$, Formula (21) in [Erd55] reads (since $|\arg(-z)| = |0| < (1 - \alpha/2)\pi$)

$$
E_{\alpha, \mu}(z) = -\sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k\alpha)} + O(|z|^{-m-1}),
$$
so that, with $z = -\beta t^\alpha$, as $t \uparrow \infty$, we obtain

$$E_{\alpha,\alpha+\mu+1}(-\beta t^\alpha) = \frac{1}{\beta t^\alpha \Gamma(\mu + 1)} + O(t^{-2\alpha}).$$

We conclude by (5.8) that

$$\int_0^t (t-s)^{\mu-1}(1 - \beta s^\alpha E_{\alpha,\alpha+1}(-\beta s^\alpha)) \, ds = O(1),$$

independently of $t$. \qed

6. Outlook

It is of interest to extend the results to more general Volterra processes. For instance, by shifting the generator of the lift, it is possible to add a drift in the coefficient, for instance in the one-dimensional case $b(x) = \beta(\theta - x)$, and prove that this still produces a generalised Feller process.

We have investigated the validity of the Condition (3.1) in this case, but it is not clear at this point how to complete the proof.

Building on the generalised Feller property, we also aim at showing the uniqueness of the invariant measure. The analogue of the strong Feller property is granted in our setting since $B^d(X)$ already includes the space of bounded functions; uniqueness shall follow if $(\lambda_t)_{t \geq 0}$ satisfies a sort of recurrence property. One can also hope to characterise the unique invariant measure using tools from Malliavin calculus or the form of the Laplace transform. Once ergodicity is established, it is interesting to identify the rate of convergence and how it varies with $\alpha$, thereby linking roughness of the volatility and convergence to the stationary measure.
References

[AE18] E. Abi Jaber and O. El Euch. Markovian structure of the Volterra Heston model. *Statistics and Probability Letters*, 149:63–72, 2018.

[AE19] E. Abi Jaber and O. El Euch. Multi-factor approximation of rough volatility models. *SIAM Journal on Financial Mathematics*, 10(2):309–349, 2019.

[ALP17] E. Abi Jaber, M. Larsson, and S. Pulido. Affine Volterra processes. *Annals of Applied Probability*, 29(5):3155–3200, 2017.

[Arn98] L. Arnold. *Random Dynamical Systems*. Springer-Verlag, NY, 1998.

[BD19] A. Budhiraja and P. Dupuis. *Analysis and Approximation of Rare Events Representations and Weak Convergence Methods*. Springer, 2019.

[BFG16] C. Bayer, P. K. Friz, and J. Gatheral. Pricing under rough volatility. *Quantitative Finance*, 16(6):887–904, 2016.

[BGS21] S. Bourguin, S. Gailus, and K. Spiliopoulos. Typical dynamics and fluctuation analysis of slow-fast systems driven by fractional Brownian motion. *Stochastics and Dynamics*, 21(07):2150030, 2021.

[BKS20] O. Butkovsky, A. Kulik, and M. Scheutzow. Generalized couplings and ergodic rates for SPDEs and other Markov models. *The Annals of Applied Probability*, 30(1):1–39, 2020.

[CKMZ16] P. K. Chevyrev, I. Friz, A. Korepanov, I. Melbourne, and H. Zhang. Multiscale systems, homogenization, and rough paths. *In International Conference in Honor of the 75th Birthday of SRS Varadhan*. Springer, Cham, 2016.

[CKNP21] L. Chen, D. Khoshnevisan, D. Nualart, and F. Pu. Spatial ergodicity for SPDEs via Poincaré-type inequalities. *Electronic Journal of Probability*, 26:1–37, 2021.

[CRY17] Z.-Q. Chen, Y.-X. Ren, and T. Yang. Law of large numbers for branching symmetric hunt processes with measure-valued branching rates. *Journal of Theoretical Probability*, 30:898–931, 2017.

[CSF21] C. Cuchiero and S. Svaluto-Ferro. Infinite-dimensional polynomial processes. *Finance and Stochastics*, 25:383–426, 2021.

[CT19] J. Cuchiero and J. Teichmann. Markovian lifts of positive semidefinite affine Volterra-type processes. *Decisions in Economics and Finance*, 42:407–448, 2019.

[CT20] J. Cuchiero and J. Teichmann. Generalized Feller processes and Markovian lifts of stochastic Volterra processes: the affine case. *Journal of Evolution Equations*, 20:1301–1348, 2020.

[DE97] P. Dupuis and R. S. Ellis. *A Weak Convergence Approach to the Theory of Large Deviations*. Wiley, 1997.

[DFS03] D. Duffie, D. Filipović, and W. Schachermayer. Affine processes and applications in finance. *The Annals of Applied Probability*, 13(3):984–1053, 2003.

[DPZ92] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. in: Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992.

[DPZ96] G. Da Prato and J. Zabczyk. *Ergodicity for Infinite Dimensional Systems*. London Mathematical Society Lecture Note Series, Series Number 229, 1996.

[DT10] P. Dörsek and J. Teichmann. A semigroup point of view on splitting schemes for stochastic (partial) differential equations. Preprint available at: arXiv:1011.2651, 2010.

[EBDK22] F. Espen Benth, N. Detering, and P. Krühner. Stochastic volterra integral equations and a class of first-order stochastic partial differential equations. *Stochastics*, 2022.

[EEFR18] O. El Euch, M. Fukasawa, and M. Rosenbaum. The microstructural foundations of leverage effect and rough volatility. *Finance and Stochastics*, 22(2):241–280, 2018.

[EER18] O. El Euch and M. Rosenbaum. Perfect hedging in rough Heston models. *Annals of Applied Probability*, 28(6):3813–3856, 2018.
[EER19] O. El Euch and M. Rosenbaum. The characteristic function of rough Heston models. *Mathematical Finance*, 29(1):3–38, 2019.

[EM01] W. E. and J.-C. Mattingly. Ergodicity for the Navier-Stokes equation with degenerate random forcing: Finite-dimensional approximation. *Communications on Pure and Applied Mathematics*, 54(11):1386–1402, 2001.

[EMS01] W. E., J.-C. Mattingly, and Y. Sinai. Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation. *Communications in Mathematical Physics*, 224:83–106, 2001.

[Erd55] A. Erdélyi. *Higher transcendental functions*, volume 3. McGraw-Hill Book Company, 1955.

[Eth93] A. M. Etheridge. Asymptotic behaviour of measure-valued critical branching processes. *Proceedings of the American mathematical society*, 118(4), 1993.

[FGS20] M. Forde, S. Gerhold, and B. Smith. Small-time, large-time, and $H \to 0$ asymptotics for the rough Heston model. *Mathematical Finance*, 31(1), 2020.

[FJ22] M. Friesen and P. Jin. Volterra square-root process: Stationarity and regularity of the law. Preprint https://arxiv.org/abs/2203.08677v1, 2022.

[FM95] F. Flandoli and B. Maslowski. Ergodicity of the 2-D Navier-Stokes equation under random perturbations. *Communications in Mathematical Physics*, 171:119–141, 1995.

[Fri19] M. Friesen. Long-time behavior for subcritical measure-valued branching processes with immigration. Preprint available at: arXiv:1903.05546, 2019.

[GAKN09] M. J. Garrido-Atienza, P. E. Kloeden, and A. Neuenkirch. Discretization of stationary solutions of stochastic systems driven by fractional Brownian motion. *Applied Mathematics and Optimization*, 60:151–172, 2009.

[GL20] J. Gehringer and X.-M. Li. Functional limit theorems for the fractional Ornstein-Uhlenbeck process. *Journal of Theoretical Probability*, 2020.

[GLS20] J. Gehringer, X.-M. Li, and J. Sieber. Functional limit theorems for Volterra processes and applications to homogenization. Preprint available at: arXiv:2002.04832, 2020.

[GR20] B. Gerencsér and M. Rásonyi. Invariant measures for fractional stochastic volatility models. Preprint available at: arXiv:2002.04832, 2020.

[Hai02] M. Hairer. Exponential mixing properties of stochastic pdes through asymptotic coupling. *Probability Theory and Related Fields*, 124:345–380, 2002.

[Hai05] M. Hairer. Ergodicity of stochastic differential equations driven by fractional Brownian motion. *The Annals of Probability*, 33(2):703–758, 2005.

[HL20] M. Hairer and X.-M. Li. Averaging dynamics driven by fractional Brownian motion. *The Annals of Probability*, 48(4), 2020.

[HMO6] M. Hairer and J.-C. Mattingly. Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Annals of Mathematics*, 164:993–1032, 2006.

[HO07] M. Hairer and A. Ohashi. Ergodic theory for SDEs with extrinsic memory. *The Annals of Probability*, 35(5):1950–1977, 2007.

[HP11] M. Hairer and N. S. Pillai. Ergodicity of hypoelliptic SDEs driven by fractional Brownian motion. *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, 47(2):601–628, 2011.

[Isc86] I. Iscoe. Ergodic theory and a local occupation time for measure-valued critical branching Brownian motion. *Stochastics: An International Journal of Probability and Stochastic Processes*, 18:197–243, 1986.

[JP22] A. Jacquier and A. Pannier. Large and moderate deviations for stochastic Volterra systems. *Stochastic Processes and their Applications*, 2022.

[KKMS20] D. Khoshnevisan, K. Kim, C. Mueller, and S.-Y. Shiu. Phase analysis for a family of stochastic reaction-diffusion equations. Preprint available at: arXiv:2012.12512, 2020.

[KLM19] E. T. Kolkovska and J. A. López-Mimbela. Survival of some measure-valued markov branching processes. *Stochastic Models*, 35(2):221–233, 2019.
[KRM12] M. Keller-Ressel and A. Mijatović. On the limit distributions of continuous-state branching processes with immigration. *Stochastic Processes and Their Applications*, 122(6):2329–2345, 2012.

[KS88] N. Konno and T. Shiga. Stochastic partial differential equations for some measure-valued diffusions. *Probability Theory and Related Fields*, 79:201–225, 1988.

[Li11] Z. Li. *Measure-Valued Branching Markov Processes*. Springer, Probability and Its Applications, 2011.

[LS20] X.-M. Li and J. Sieber. Slow-fast systems with fractional environment and dynamics. Preprint available at: arXiv:2012.01910, 2020.

[WX04] Z. Li, H. Wang, and J. Xiong. A degenerate stochastic partial differential equation for superprocesses with singular interaction. *Probability Theory and Related Fields*, 130:1–17, 2004.

[MS15] L. Mytnik and T. S. Salisbury. Uniqueness for Volterra-type stochastic integral equations. Preprint arXiv:1502.05513, 2015.

[PIX20] B. Pei, Y. Inahama, and Y. Xu. Averaging principles for mixed fast-slow systems driven by fractional Brownian motion. Preprint available at: arXiv:2001.06945, 2020.

[PIX21] B. Pei, Y. Inahama, and Y. Xu. Averaging principle for fast-slow system driven by mixed fractional Brownian rough path. *Journal of Differential Equations*, 301:202–235, 2021.

[Var19] M. Varvenne. *Ergodicity of fractional stochastic differential equations and related problems*. PhD thesis, Université Paul Sabatier - Toulouse III, 2019.

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