Optimal Constructions of Hybrid Algorithms

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Abstract

We study on-line strategies for solving problems with hybrid algorithms. There is a problem $Q$ and $w$ basic algorithms for solving $Q$. For some $\lambda \leq w$, we have a computer with $\lambda$ disjoint memory areas, each of which can be used to run a basic algorithm and store its intermediate results. In the worst case, only one basic algorithm can solve $Q$ in finite time, and all the other basic algorithms run forever without solving $Q$. To solve $Q$ with a hybrid algorithm constructed from the basic algorithms, we run a basic algorithm for some time, then switch to another, and continue this process until $Q$ is solved. The goal is to solve $Q$ in the least amount of time. Using competitive ratios to measure the efficiency of a hybrid algorithm, we construct an optimal deterministic hybrid algorithm and an efficient randomized hybrid algorithm. This resolves an open question on searching with multiple robots posed by Baeza-Yates, Culberson and Rawlins. We also prove that our randomized algorithm is optimal for $\lambda = 1$, settling a conjecture of Kao, Reif and Tate.

1 Introduction

We study on-line strategies for solving problems with hybrid algorithms. There is a problem $Q$ and $w$ basic algorithms for solving $Q$. For some $\lambda \leq w$, we have a computer with $\lambda$ disjoint memory areas, each of which can be used to run a basic algorithm and store its intermediate results. In the worst case, only one basic algorithm can solve $Q$ in finite time, and all the other basic algorithms run forever without solving $Q$. To solve $Q$ with a hybrid algorithm constructed from the basic algorithms, we run a basic algorithm for some time, then switch to another, and continue this process until $Q$ is solved. The goal is to solve $Q$ in the least amount of time.

This optimization problem can be conveniently formulated as one of exploring an unknown environment with multiple robots [1, 2, 3, 4, 5, 7, 11]. At an origin, there are $w$ paths leading off into

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unknown territories. On one of the paths, there is a goal at an unknown distance \( n \) from the origin, and none of the other paths has a goal. Initially, there are \( \lambda \) robots standing at the origin. The robots can move back and forth on the paths to search for the goal. The objective is to minimize the total distance traveled by all the robots before the goal is found.

We use the notion of a competitive ratio, introduced by Sleator and Tarjan [13], to measure the efficiency of an exploration algorithm \( \mathcal{A} \). Let \( \text{cost}(\mathcal{A}) \) be the (worst-case or expected) total distance traveled by all the robots using \( \mathcal{A} \). Given a constant \( c \), we say that \( \mathcal{A} \) has a competitive ratio \( c \) if \( \text{cost}(\mathcal{A}) \leq c \cdot n + o(n) \).

An extreme case where there is only one robot, i.e., \( \lambda = 1 \), was studied by many researchers. In particular, Baeza-Yates, Culberson and Rawlins [3] presented an optimal deterministic algorithm. Kao, Reif and Tate [8] reported a randomized algorithm, proved its optimality for \( w = 2 \), and conjectured its optimality for all \( w > 2 \). In these two algorithms, the single robot searches the \( w \) paths in a cyclic fashion, and the returning positions on the paths form a geometric sequence. (A returning position on a path refers to a point where the robot starts moving towards the origin after advancing away from it.)

Another extreme case where the number of robots equals the number of paths, i.e., \( \lambda = w \), was studied by Azar, Broder and Manasse [1], who showed that the smallest competitive ratios are \( w \) for both deterministic and randomized algorithms. This ratio \( w \) can be achieved by the simple algorithm in which each robot explores a single path and keeps moving forward until the goal is found.

The general case \( 1 < \lambda < w \) has not been well understood, and its difficulty is twofold. Since \( \lambda > 1 \), an exploration algorithm must coordinate its robots in order to achieve optimality. Moreover, since \( \lambda < w \), some robots must move back and forth on two or more paths, and their returning positions on those paths are crucial for optimality. Our main results are that for all values of \( \lambda \) and \( w \), we construct

- a deterministic algorithm with the smallest possible competitive ratio, and
- an efficient randomized algorithm that is provably optimal for \( \lambda = 1 \).

Our deterministic algorithm resolves an open question of Baeza-Yates et al. [2] on exploration using multiple robots. The optimality proof for our randomized algorithm with \( \lambda = 1 \) settles the conjecture of Kao et al. [8] in the affirmative. Our results also imply that randomization can help reduce the competitive ratios if and only if \( \lambda < w \).

We discuss our deterministic exploration algorithm in \( \S \) and the randomized algorithm in \( \S \). The paper concludes with some directions for future research in \( \S \).

Throughout the paper, we label the \( w \) paths by \( 0, 1, \ldots, w-1 \) and the \( \lambda \) robots by \( 1, 2, \ldots \lambda \).

## 2 An optimal deterministic exploration algorithm

Let \( D(w, \lambda) \) denote the smallest competitive ratio for all deterministic exploration algorithms. The main result of this section is the next theorem.

**Theorem 2.1** \( D(w, \lambda) = \lambda + 2^{(w-\lambda+1)w-\lambda+1} \frac{(w-\lambda+1)^w-\lambda+1}{(w-\lambda)^w-\lambda}. \)
Baeza-Yates et al. [2] studied the case $\lambda = 1$, and their results can be restated as

$$D(w, 1) = 1 + 2 \frac{w^w}{(w - 1)^{w - 1}}. \quad (1)$$

Theorem 2.1 generalizes (1) and answers the open question in [2] on optimal exploration using multiple robots. To prove Theorem 2.1, we describe our deterministic algorithm in §2.1 and give a lower bound proof in §2.2.

### 2.1 A deterministic exploration algorithm

We first review the exploration algorithm for $\lambda = 1$ given by Baeza-Yates et al. [2]. This algorithm, referred to as $D(w, 1)$, is used as a subroutine in our algorithm for general $\lambda$. Let

$$f(w, i) = \begin{cases} 
(w/w - 1)^i & \text{for } i \geq 0, \\
0 & \text{for } i < 0.
\end{cases}$$

In algorithm $D(w, 1)$, the single robot searches the $w$ paths in a fixed cyclic order. The search proceeds in stages, starting from stage 0. In stage $i$, the robot searches path $i \mod w$ until position $f(w, i)$ and moves back to the origin if the goal is not found by then. If the goal is at position $f(w, i) + 1$, then the robot finds it in stage $i + w$, traveling a total distance of $f(w, i) + 1 + 2 \sum_{j=0}^{i+w-1} f(w, j)$.

Baeza-Yates et al. showed that the smallest competitive ratio of $D(w, 1)$ is

$$\lim_{i \to \infty} \frac{f(w, i) + 1 + 2 \sum_{j=0}^{i+w-1} f(w, j)}{f(w, i) + 1} \leq 1 + 2 \frac{w^w}{(w - 1)^{w - 1}}. \quad (2)$$

In our algorithm for general $\lambda$, for each $k < \lambda$, the $k$-th robot only searches path $k$. These $\lambda - 1$ robots simply advance on their own paths and never move towards the origin. Let $w' = w - \lambda + 1$. The $\lambda$-th robot explores the remaining $w'$ paths using $D(w', 1)$. The algorithm proceeds in rounds until the goal is found. In the $i$-th round, the robots move as follows:

- The $\lambda$-th robot chooses some path $p$ according to $D(w', 1)$ and searches it from the origin to position $f(w', i - w')$.
- All the robots then move in parallel from position $f(w', i - w')$ to position $f(w', i + 1 - w')$ on the paths where they stand.
- The $\lambda$-th robot continues to search path $p$ from position $f(w', i + 1 - w')$ to position $f(w', i)$ and then moves back to the origin.

We next analyze the above algorithm. If the goal is at position $f(w', i - w') + 1$ on some path, it is found in round $i$. By the time the goal is found, the first $\lambda - 1$ robots have each traveled a distance of $f(w', i - w') + 1$, and the $\lambda$-th robot a distance of $f(w', i - w') + 1 + 2 \sum_{j=0}^{i-1} f(w', j)$. Hence, the smallest competitive ratio of the above exploration algorithm is

$$\lim_{i \to \infty} \frac{\lambda \cdot (f(w', i - w') + 1) + 2 \sum_{j=0}^{i-1} f(w', j)}{f(w', i - w') + 1}.$$
By (2), the above formula is upper bounded by
\[ \lambda + 2 \frac{w' w'}{(w' - 1)^{w' - 1}} = \lambda + 2 \frac{(w - \lambda + 1)^{w - \lambda + 1}}{(w - \lambda)^{w - \lambda}}, \]
which equals the competitive ratio stated in Theorem 2.1.

2.2 A matching lower bound

Here, we prove that our deterministic algorithm in §2.1 is optimal by deriving a matching lower bound on the smallest competitive ratio \( r_A \) of any arbitrary deterministic exploration algorithm \( A(w, \lambda) \) with \( w \) paths and \( \lambda \) robots. Let \( t_0 \) to be the time when \( A(w, \lambda) \) commences. For \( i > 0 \), \( t_i \) denotes the \( i \)-th time when a robot of \( A(w, \lambda) \) starts moving towards the origin on some path. \( A(w, \lambda) \) can be partitioned into phases where phase \( i \) starts at \( t_{i-1} \) and ends at \( t_i \). The notion of a phase differs from that of a round in §2.1.

**Lemma 2.2** Given \( A(w, \lambda) \), there is a deterministic exploration algorithm \( A'(w, \lambda) \) such that \( r_{A'} \leq r_A \) and \( A'(w, \lambda) \) satisfies the following three properties:

- No two robots search the same path in the same phase.
- No robot moves towards the origin if some robot stays at the origin.
- As soon as a robot starts moving towards the origin on some path, all the other robots stop moving until that robot moves back to the origin and then advances on another path to a previously unsearched location.

**Proof.** Straightforward. □

If \( \lambda = 1 \), i.e., there is only one robot, then \( A(w, 1) \) can be characterized by a sequence \( \{(h_i, a_i), i \geq 1\} \) where \( a_i \) is the index of the path where the robot starts moving towards the origin in phase \( i \), and \( h_i \) is the distance that the robot searches on path \( a_i \) in phase \( i \). By simple calculation, the ratio \( r_A \) of \( A(w, 1) \) equals
\[ 1 + 2 \lim_{i \to \infty} \left( \frac{h_1 + \cdots + h_{i'-1}}{h_i} \right), \] (3)
where \( i' \) is the smallest index such that \( i' > i \) and \( a_{i'} = a_i \). Motivated by this finding, for any given \( \{(h_i, a_i), i \geq 1\} \), we define the corresponding ratio sequence \( \{H_i, i \geq 1\} \) by
\[ H_i = \frac{h_1 + \cdots + h_{i'-1}}{h_i} \] (4)
where \( i' \) is the smallest index with \( i' > i \) and \( a_{i'} = a_i \). Using \( H_i \), (3) can be written as
\[ 1 + 2 \lim_{i \to \infty} H_i. \]

A sequence \( \{(h_i, a_i), i \geq 1\} \) is a \( w \)-sequence if \( h_i > 0 \) and \( a_i \) is an integer for all \( i \) as well as \(|\{i \mid a_i = j\}| = \infty \) for at least \( w \) integers \( j \). A \( w \)-sequence \( \{(h_i, a_i), i \geq 1\} \) is a cyclic sequence if
$a_i = i \mod w$. Since the integers $a_i$ are uniquely specified in a cyclic sequence, we represent a cyclic sequence by $\{s_i, i \geq 1\}$. The corresponding ratio sequence, denoted by $\{S_i, i \geq 1\}$, is thus defined by

$$S_i = \frac{s_1 + \cdots + s_{i+w-1}}{s_i}. \quad (5)$$

**Fact 1** (see [4]) For every cyclic $w$-sequence, $\lim_{i \to \infty} S_i \geq \frac{w^w}{(w-1)^{w-1}}$.

**Lemma 2.3** For each $w$-sequence $\{(h_i, a_i), i \geq 1\}$, there exists a cyclic $w$-sequence $\{s_i, i \geq 1\}$ such that $\lim_{i \to \infty} H_i \geq \lim_{i \to \infty} S_i$.

**Proof.** See Appendix A. \qed

Intuitively, Lemma 2.3 shows that we can modify a deterministic exploration algorithm to search the paths in a cyclic order without increasing its competitive ratio.

**Lemma 2.4** If $\mathcal{A}(w, \lambda)$ has a finite competitive ratio and satisfies the properties of Lemma 2.2, then there exists a $(w - \lambda + 1)$-sequence $\{(h_i, a_i), i \geq 1\}$ such that

$$r_{\mathcal{A}} \geq \lambda + 2 \lim_{i \to \infty} H_i,$$

where $\{H_i, i \geq 1\}$ is as defined in [4].

**Proof.** First, we inductively define the sequence $\{(h_i, a_i), i \geq 1\}$ and a sequence of $w$-dimensional vectors $\pi_i = (\pi_i(0), \ldots, \pi_i(w-1))$. Then, we prove the inequality claimed in the lemma.

We first define $(h_1, a_1)$ and $\pi_1$ by looking at the first phase of $\mathcal{A}(w, \lambda)$. Assume that at time $t_1$, a robot starts moving towards the origin on path $j$. Let $h_1$ be the distance that the robot has searched on path $j$. Define

$$\pi_1 = (0, 1, 2, \ldots, w-1) \text{ and } a_1 = \pi_1(j).$$

Once $\pi_{i-1}$ is defined, we define $(h_i, a_i)$ and $\pi_i$ by looking at phase $i$ of $\mathcal{A}(w, \lambda)$. Recall that phase $i$ starts at time $t_{i-1}$ and ends at time $t_i$. Assume that path $l$ is the unique path that is not searched in phase $i-1$ but is searched in phase $i$. Also assume that at time $t_i$, a robot starts moving towards the origin on path $k$. Let $h_i$ be the distance that the robot has searched on path $k$. For $j = 0, \ldots, w-1$, let

$$\pi_i(j) = \begin{cases} 
\pi_{i-1}(l) & \text{if } j = k, \\
\pi_{i-1}(k) & \text{if } j = l, \\
\pi_{i-1}(j) & \text{if } j \neq k \text{ and } j \neq l; 
\end{cases}$$

i.e., we switch the $k$-th entry and the $l$-th entry of $\pi_{i-1}$ to obtain $\pi_i$. Let $a_i = \pi_i(k)$. Since $\mathcal{A}(w, \lambda)$ has a finite competitive ratio, $\{(h_i, a_i), i \geq 1\}$ is an infinite sequence. For every $i \geq 1$, $t_i$ is the time when a robot starts moving towards the origin on some path $p$. Let $i' > i$ be the index such that phase $i'$ is the first phase that path $p$ is searched again after $t_i$. Such $i'$ exists and is finite.
Claim 2.1 For all \(i \geq 1\), \(a_i' = a_i\) and \(a_j \neq a_i\) for \(j = i + 1, \ldots, i' - 1\).

To prove this claim, assume that \(a_i = \pi_i(k)\), i.e., a robot starts moving towards the origin on path \(k\) immediately after \(t_i\). In phase \(i + 1\), that robot moves back on path \(k\) to the origin and then searches another path \(l\) with \(l \neq k\). Then, some robot starts moving towards the origin on path \(k_1\) with \(k_1 \neq k\) right after \(t_{i+1}\). Since \(k \neq l\) and \(k \neq k_1\), by the definition of \(\pi_{i+1}\),

\[
\pi_{i+1}(k) = \pi_i(k).
\]

By the choice of \(i'\), path \(k\) must be idle from \(t_{i+1}\) to \(t_{i' - 1}\). Hence,

\[
\pi_j(k) = \pi_{j-1}(k) = \cdots = \pi_{i+1}(k) = \pi_i(k) = a_i. \tag{6}
\]

Moreover, \(a_j \neq \pi_j(k)\). Thus, from (6),

\[
a_j \neq a_i \text{ for } j = i + 1, \ldots, i' - 1. \tag{7}
\]

By the choice of \(i'\), path \(k\) is reused in phase \(i'\). Assume that a robot searches path \(k_2\) immediately after \(t_{i'}\). By the inductive procedure for defining \(\pi_{i'}\) and \(a_{i'}\),

\[
\pi_{i'}(k_2) = \pi_{i'-1}(k) \text{ and } a_{i'} = \pi_{i'}(k_2). \tag{8}
\]

Combining (6) and (8), we have \(a_i' = a_i\). This and (7) conclude the proof of Claim 2.1.

Claim 2.2 \(\{(h_i, a_i), i \geq 1\}\) is a \((w - \lambda + 1)\)-sequence.

To prove this claim, the only nontrivial property of the sequence that we need to verify is that there are at least \((w - \lambda + 1)\) integers \(j\) such that \(|\{i \mid a_i = j\}| = +\infty\). By Claim 2.1, it suffices to prove that there exist \((w - \lambda + 1)\) integers \(j\) such that

\[
a_i = j \text{ for some } i. \tag{9}
\]

Without loss of generality, we label the \(w\) paths in such a way that

- the label of the path where a robot starts moving towards the origin immediately after \(t_1\) is 0;
- paths 1, 2, \ldots, \(w - \lambda\) are not searched before \(t_1\);
- \(i_1 < i_2 < \cdots < i_{w-\lambda}\) where \(i_j\) is the first phase in which path \(j\) is searched.

By the assumption on \(a_0\) and the definition of \(a_1\),

\[
a_1 = 0. \tag{10}
\]

For \(j = 1, \ldots, w - \lambda\), let \(j^*\) be the label of the path where a robot starts moving towards the origin immediately after \(t_{i_j}\). By the definitions of \(\{\pi_i, i \geq 1\}\) and \(\{(h_i, a_i), i \geq 1\}\), \(\pi_{i_j}(j^*) = \pi_{i_j-1}(j) = \cdots = \pi_1(j) = j\). Therefore,

\[
a_{i_j} = \pi_{i_j}(j^*) = j \text{ for } 1 \leq j \leq w - \lambda.
\]

By this and (10), at least \(w - \lambda + 1\) integers \(j\) satisfy (9), finishing the proof of Claim 2.2.
Continuing the proof of Lemma 2.4, for each $i \geq 1$, let $p$ be the path where a robot starts moving towards the origin right after $t_i$. Let $T$ be the first time when path $p$ is searched for exactly distance $h_i$ in phase $i'$. By the properties stated in Lemma 2.2, the $\lambda$ robots stand at different paths at time $T$. Let $d_1, d_2, \ldots, d_{\lambda-1}$ be the distances that the robots except the one on path $p$ are from the origin at time $T$. Let $d_j = \min\{d_1, d_2, \ldots, d_{\lambda-1}\}$. There are two cases.

**Case 1:** $d_j \geq h_i$. If the goal is on path $p$ at distance $h_i + 1$, then when $A(w, \lambda)$ finds the goal, its performance ratio is at least

$$\frac{d_1 + d_2 + \cdots + d_{\lambda-1} + 2(h_1 + \cdots + h_{i'-1}) + h_i + 1}{h_i + 1} \geq \frac{h_i}{h_i + 1} \left(\lambda + \frac{2(h_1 + \cdots + h_{i'-1})}{h_i}\right). \quad (11)$$

**Case 2:** $d_j < h_i$. Let $p'$ be a path that has been searched up to distance $d_j$ at time $T$. If the goal is on path $p'$ at distance $d_j + 1$, then when $A(w, \lambda)$ finds the goal, its performance ratio is at least

$$\frac{d_1 + d_2 + \cdots + d_j + 1 + \cdots + d_{\lambda-1} + 2(h_1 + \cdots + h_{i'-1}) + h_i}{d_j + 1} \geq \frac{d_j + 1}{d_j} \left(\lambda + \frac{2(h_1 + \cdots + h_{i'-1})}{h_i}\right). \quad (12)$$

Since $A(w, \lambda)$ has a finite competitive ratio, $\lim_{i \to \infty} h_i = \lim_{j \to \infty} d_j = +\infty$. Therefore, by (11) and (12),

$$r_A \geq \lim_{i \to \infty} \left(\lambda + \frac{2(h_1 + \cdots + h_{i'-1})}{h_i}\right) = \lambda + 2 \lim_{i \to \infty} H_i.$$

This and Claims 2.1 and 2.2 conclude the proof of Lemma 2.4.

Fact 1 and Lemmas 2.2, 2.3 and 2.4 give a lower bound proof for Theorem 2.1.

### 3 A randomized exploration algorithm

We give our randomized exploration algorithm for general $\lambda$ in §3.1 and prove its optimality for $\lambda = 1$ in §3.2.

#### 3.1 A randomized exploration algorithm for general $\lambda$

We first review the randomized search algorithm of Kao et al. [8] for $\lambda = 1$, which we refer to as $\mathcal{R}(w, 1)$. Choose $r_w > 1$ such that

$$\frac{r_w - 1}{(r_w - 1) \ln r_w} = \min_{r > 1} \frac{r - 1}{(r - 1) \ln r}.$$ 

Such $r_w$ exists and is unique [8]. $\mathcal{R}(w, 1)$ proceeds as follows:

1. $\sigma \leftarrow$ a random permutation of $\{0, \ldots, w-1\}$;
2. $\epsilon \leftarrow$ a random real number uniformly chosen from $[0, 1)$;
3. \( d \leftarrow r^\epsilon_w \);

4. \( i \leftarrow 1; \)

5. repeat
   
   explore path \( \sigma(i) \) up to distance \( d \);
   if goal not found then return to origin;
   
   \( d \leftarrow d \cdot r_w; \) 
   \( i \leftarrow (i + 1) \mod w; \)

   until the goal is found.

Let \( R(w, \lambda) \) denote the smallest competitive ratio for randomized exploration algorithms for \( w \) paths and \( \lambda \) robots. Let

\[
\bar{R}(w) = 1 + \frac{2}{w} \frac{r^w_w - 1}{(r^w_w - 1) \ln r^w_w}.
\]

**Fact 2** (see [8]) \( R(w, 1) \leq \bar{R}(w) \).

We now construct our randomized exploration algorithm for general \( \lambda \) using \( R(w, 1) \) as a subroutine. First, we pick a random permutation \( \sigma \) of \( \{0, 1, \ldots, w - 1\} \). For the first \( \lambda - 1 \) robots, robot \( i \) searches only path \( \sigma(i) \). These robots search their own paths at the same constant speed. The \( \lambda \)-th robot searches the remaining \( w - \lambda + 1 \) paths using \( R(w - \lambda + 1, 1) \) also at a constant speed. The speeds are coordinated by a parameter \( v \) such that at any given time, the total distance traveled by the \( \lambda \)-th robot is \( v \) times the distance traveled by each of the first \( \lambda - 1 \) robots. By choosing an appropriate \( v \), we can prove the next theorem.

**Theorem 3.1** \( R(w, \lambda) \leq \frac{1}{w} \left( (\lambda - 1) + \sqrt{(w - \lambda + 1) \bar{R}(w - \lambda + 1)} \right)^2. \)

**Proof.** The smallest competitive ratio of our exploration algorithm is at most

\[
\frac{\lambda - 1}{w} ((\lambda - 1) + v) + \frac{w - \lambda + 1}{w} \left( \frac{\lambda - 1}{v} + 1 \right) \bar{R}(w - \lambda + 1).
\]

At \( v = \sqrt{(w - \lambda + 1) \bar{R}(w - \lambda + 1)} \), this expression assumes its minimum

\[
\frac{1}{w} \left( (\lambda - 1) + \sqrt{(w - \lambda + 1) \bar{R}(w - \lambda + 1)} \right)^2.
\]

\( \square \)

Combining Theorem 2.1 and 3.1, we can prove the following corollary.

**Corollary 3.2** If \( \lambda < w \), then the smallest competitive ratio of randomized exploration algorithms is always smaller than that of deterministic ones.
Proof. By Theorems 2.1 and 3.1, we only need to prove

\[
\frac{1}{w} \left( (\lambda - 1) + \sqrt{(w - \lambda + 1) \bar{R}(w - \lambda + 1)} \right)^2 < \lambda + 2 \frac{(w - \lambda + 1)^{w-\lambda+1}}{(w - \lambda)^{w-\lambda}}. \tag{13}
\]

Let \(a = \bar{R}(w - \lambda + 1)\), by Fact 2, which is a competitive ratio of a randomized exploration algorithm with one robot and \(w - \lambda + 1\) paths. Let \(b = 1 + 2 \frac{(w-\lambda+1)^{w-\lambda+1}}{(w-\lambda)^{w-\lambda+1}}\), by Theorem 2.1, which is the smallest competitive ratio of deterministic algorithms also for one robot and \(w - \lambda + 1\) paths. Since \(a < b\), to prove (13), it is suffice to show that

\[
\frac{1}{w} \left( (\lambda - 1) + \sqrt{(w - \lambda + 1) a} \right)^2 \leq \lambda - 1 + a,
\]

which is equivalent to \( (a - \sqrt{w - \lambda + 1})^2 \geq 0 \). \(\square\)

### 3.2 A matching lower bound for \(\lambda = 1\)

The next theorem shows that our randomized algorithm is optimal for \(\lambda = 1\).

**Theorem 3.3** \(R(w, 1) \geq \bar{R}(w)\).

The proof of this theorem uses a strengthened version of Fact 3 below. Let \(\bar{s} = \{s_i, i \geq 0\}\) denote an infinite sequence of positive numbers. Let

\[
S_w = \{\{s_i, i \geq 0\} | \lim_{i \to \infty} s_i = \infty, s_0 = 1, \text{ and for all } i \geq 0, s_{i+w} > s_i\}.
\]

For any \(\epsilon > 0\) and \(\bar{s} = \{s_i, i \geq 0\} \in S_w\), let

\[
G_w(\epsilon, \bar{s}) = \epsilon \sum_{i=0}^{\infty} \frac{s_i + \cdots + s_{i+w-1}}{s_{i+w}}.
\]

An exploration algorithm with one robot is called cyclic if it searches the paths one after another in a cyclic order.

**Fact 3** (see [8]) The smallest competitive ratio of any cyclic randomized exploration algorithm is at least

\[
\sup_{\epsilon > 0} \inf_{\bar{s} \in S_w} \{1 + \frac{2}{w} G_w(\epsilon, \bar{s})\}.
\]

Note that if \(w = 2\), i.e., there are only two paths, then every exploration algorithm is cyclic. This is not true for \(w \geq 3\). Ma and Yin [9] strengthened Fact 3 by removing the cyclic assumption about exploration algorithms.

**Fact 4** (see [9]) The lower bound stated in Fact 3 also holds for any arbitrary randomized exploration algorithms that may or may not be cyclic.
Proof. By Yao’s formulation of von Neumann’s minimax principle \[14\], it suffice to lower bound the competitive ratios of all deterministic algorithms against a chosen probability distribution. The proof idea is to show that an optimal algorithm against the distribution used by Kao et al. \[8\] for Fact 3 can be modified to be cyclic. □

By Fact 4, to prove Theorem 3.3, we only need to prove the next theorem. Let

\[ C_w = \frac{r_w - 1}{(r_w - 1) \ln r_w}. \]

Theorem 3.4 \[\sup_{\epsilon > 0} \inf_{\bar{s} \in S_w} G_w(\epsilon, \bar{s}) \geq C_w.\]

The proof of Theorem 3.4 is divided into three parts. In \[\S 3.2.1\], we lower bound the infinite sum \[G_w(\epsilon, \bar{s})\] by the finite sum

\[ H(k, \bar{s}(\epsilon)) = \frac{-\epsilon + \sum_{i=0}^{k-1} \frac{s_i(\epsilon) + \cdots + s_{i+w-1}(\epsilon)}{(s_i(\epsilon))^{1+\epsilon}}}{\ln s_k(\epsilon)}. \]

In \[\S 3.2.2\], we lower bound this finite sum by \[C_w\]. Finally in \[\S 3.2.3\], we complete the proof of Theorem 3.4.

3.2.1 Lower bounding \[G_w(\epsilon, \bar{s})\] by \[H(k, \bar{s}(\epsilon))\]

Lemma 3.5 For all \(\epsilon > 0\), there exists \(\bar{s}(\epsilon) = \{s_i(\epsilon), i \geq 0\} \in S_w\) such that for all \(k\) with \(s_k(\epsilon) > 1,\)

\[ \inf_{\bar{s} \in S_w} G_w(\epsilon, \bar{s}) \geq H(k, \bar{s}(\epsilon)). \]

Proof. By the definition of infimum, for all \(\epsilon > 0\), there exists \(\bar{s}(\epsilon) = \{s_i(\epsilon), i \geq 0\} \in S_w\) such that

\[ \inf_{\bar{s} \in S_w} G_w(\epsilon, \bar{s}) + \epsilon^2 \geq G_w(\epsilon, \bar{s}(\epsilon)) \]

\[ = \epsilon \sum_{i=0}^{k-1} \frac{s_i(\epsilon) + \cdots + s_{i+w-1}(\epsilon)}{(s_i(\epsilon))^{1+\epsilon}} + (s_k(\epsilon))^{-\epsilon} \left( \epsilon \sum_{i=k}^{\infty} \frac{s_i(\epsilon) + \cdots + s_{i+w-1}(\epsilon)}{(s_i(\epsilon))^{1+\epsilon}} \right). \]

Since \(\bar{s}'(\epsilon) = \{s_i'(\epsilon) = \frac{s_{i+1}(\epsilon)}{s_k(\epsilon)}, i \geq 0\} \in S_w\),

\[ \inf_{\bar{s} \in S_w} G_w(\epsilon, \bar{s}) + \epsilon^2 = \left( \epsilon \sum_{i=0}^{k-1} \frac{s_i(\epsilon) + \cdots + s_{i+w-1}(\epsilon)}{(s_i(\epsilon))^{1+\epsilon}} \right) + (s_k(\epsilon))^{-\epsilon} G_w(\epsilon, \bar{s}') \]

\[ \geq \left( \epsilon \sum_{i=0}^{k-1} \frac{s_i(\epsilon) + \cdots + s_{i+w-1}(\epsilon)}{(s_i(\epsilon))^{1+\epsilon}} \right) + (s_k(\epsilon))^{-\epsilon} \inf_{\bar{s} \in S_w} G_w(\epsilon, \bar{s}). \]
Since \( s_k(\epsilon) > 1 \) and \( \frac{\epsilon}{1-x^{-\epsilon}} \geq \frac{1}{\ln x} \) for all \( x > 1 \),
\[
\inf_{\vec{s} \in S_w} G_w(\epsilon, \vec{s}) \geq \frac{\epsilon}{1 - (s_k(\epsilon))^{1+\epsilon}} \left( -\epsilon + \sum_{i=0}^{k-1} \frac{s_i(\epsilon) + \cdots + s_{i+w-1}(\epsilon)}{(s_k(\epsilon))^{1+\epsilon}} \right) \\
\geq \frac{1}{\ln s_k(\epsilon)} \left( -\epsilon + \sum_{i=0}^{k-1} \frac{s_i(\epsilon) + \cdots + s_{i+w-1}(\epsilon)}{(s_k(\epsilon))^{1+\epsilon}} \right).
\]

\( \square \)

**Lemma 3.6** There exist a strictly increasing integer sequence \( \{p_i, i \geq 0\} \) and a sequence \( \vec{s}(\epsilon_n) \) such that
\[
\lim_{n \to \infty} H(k_n, \vec{s}(\epsilon_n)) \text{ exists and is finite}
\]
and
\[
\sup_{\epsilon > 0} \inf_{\vec{s} \in S_w} G_w(\epsilon, \vec{s}) \geq \lim_{n \to \infty} H(k_n, \vec{s}(\epsilon_n)),
\]
where \( \epsilon_n = \frac{1}{p_n} \), \( 0 < p_n \leq k_n \leq p_n + w - 1 \), and \( s_{kn}(\epsilon_n) = \max\{s_{p_n}(\epsilon_n), \ldots, s_{p_n+w-1}(\epsilon_n)\} \).

**Proof.** This lemma follows from the fact that by Fact 4 and Lemma 3.5, \( H(k, \vec{s} (\epsilon)) \) is bounded. \( \square \)

### 3.2.2 Lower Bounding \( H(k, \vec{s} (\epsilon)) \) by \( C_w \)

This is the most difficult part of the proof of Theorem 3.4 and requires six technical lemmas, namely, Lemmas 3.7 through 3.12. The proofs of these lemmas are given in Appendix B.

We first rewrite
\[
H(k_n, \vec{s}(\epsilon_n)) = \frac{1}{\ln s_k(\epsilon_n)} \left( -\epsilon_n + \sum_{i=0}^{k_n-1} \frac{s_i(\epsilon_n) + \cdots + s_{i+w-1}(\epsilon_n)}{(s_i(\epsilon_n))^{1+\epsilon_n}} \right) \\
= \frac{1}{\ln s_k(\epsilon_n)} \left( -\epsilon_n + \sum_{j=0}^{w-1} \sum_{i=0}^{k_n-j} \frac{s_{i+j}(\epsilon_n)}{(s_i(\epsilon_n))^{1+\epsilon_n}} \right).
\]

Hence, by Lemma 3.6, there is a constant \( C \) such that
\[
C \geq H(k_n, \vec{s}(\epsilon_n)) \geq \frac{-\epsilon_n + \sum_{j=0}^{w-1} L_n(j)}{\ln s_k(\epsilon_n)} \text{,}
\]
(14)

where
\[
L_n(0) = \sum_{i=0}^{k_n-1} \frac{1}{(s_i(\epsilon_n))^{\epsilon_n}}
\]
and
\[
L_n(j) = \sum_{i=0}^{k_n-j} \frac{s_{i+j}(\epsilon_n)}{(s_i(\epsilon_n))^{1+\epsilon_n}}
\]
for \( j = 1, \ldots, w - 1 \). Note that \( s_{kn} > 1 \) implies \( \ln s_{kn} > 0 \).

To further lower bound \( H(k_n, \vec{s}(\epsilon_n)) \), we first work on \( L_n(0) \) to show that \( s_{kn}(\epsilon_n) \to \infty \) as \( n \to \infty \). Then, we work on \( L_n(1) \) to show that \( s_n(\epsilon_n) \) grows somewhat smoothly at a rate exponential in \( n \). Finally, we lower bound all \( L_n(j) \).

The next lemma is frequently used in this section.
Lemma 3.7 For every positive integer \( m \) and for all \( \epsilon, x_0, \ldots, x_m > 0 \),
\[
\frac{x_1}{x_0^{1+\epsilon}} + \frac{x_2}{x_1^{1+\epsilon}} + \cdots + \frac{x_m}{x_{m-1}^{1+\epsilon}} \geq \frac{m}{(1+\epsilon)^m} \left( \frac{x_m}{x_0} \right)^{E_{\epsilon}(m)}
\]
where \( E_{\epsilon}(m) = \frac{\epsilon^{(1+\epsilon)^m}}{(1+\epsilon)^m-1} \).

The next two lemmas give some properties of the sequence \( \{s_{k_n}(\epsilon_n), k \geq 0\} \).

Lemma 3.8 \( \lim_{n \to \infty} s_{k_n}(\epsilon_n) = \infty \).

For all \( n \), pick \( h_n \in \{k_n - w + 1, \ldots, k_n - 1\} \) such that
\[
s_{h_n}(\epsilon_n) = \min\{s_{k_n-w+1}(\epsilon_n), \ldots, s_{k_n-1}(\epsilon_n)\}.
\]
Since \( p_n \to \infty \) as \( n \to \infty \), we assume without loss of generality that \( k_n - w + 1 \geq 0 \). Since \( s_{k_n}(\epsilon_n) > 1 \), there exists a unique \( \nu_n \) such that
\[
s_{h_n}(\epsilon_n) = (s_{k_n}(\epsilon_n))^{1-\nu_n}.
\]
By the choice of \( k_n \) and the monotonicity of \( S_w \), \( s_{k_n}(\epsilon_n) \geq s_{h_n}(\epsilon_n) \) and thus \( \nu_n \geq 0 \).

Lemma 3.9 \( \lim_{n \to \infty} \nu_n = 0 \) and for some finite \( \Delta > 1 \), \( \lim_{n \to \infty} (s_{k_n}(\epsilon_n))^{\frac{1}{\nu_n}} = \Delta \).

The next lemma estimates \( L_n(0) \) and \( L_n(1) \).

Lemma 3.10 \( \lim_{n \to \infty} \frac{L_n(0)}{\ln s_{k_n}(\epsilon_n)} \geq \frac{1}{\ln \Delta} \) and \( \lim_{n \to \infty} \frac{L_n(1)}{\ln s_{k_n}(\epsilon_n)} \geq \frac{\Delta}{\ln \Delta} \).

The next two lemmas estimate \( L_n(2), \ldots, L_n(w-1) \). For all integers \( n \geq 0 \), let \( b_n = \epsilon_n + C \ln s_{k_n}(\epsilon_n) \). In light of Lemma 3.8, we assume \( \ln s_{k_n}(\epsilon_n) \geq 1 \) and thus \( b_n \geq 1 \); otherwise we can replace \( \{p_n, n \geq 0\} \) with a subsequence for which these bounds hold.

Lemma 3.11 For all \( n \geq 0 \) and \( i \in \{0, 1, \ldots, w-1\} \), \( b_n^{(w-1)(1+\epsilon_n)^{w-1}} \geq s_i(\epsilon_n) \).

Lemma 3.12 For \( j = 2, \ldots, w-1 \) and \( u = 0, \ldots, j-1 \), \( \lim_{n \to \infty} L_n(j) \geq \frac{\Delta^u}{\ln \Delta} \).

3.2.3 Proof of Theorem 3.4

From (14) and Lemmas 3.10 and 3.12, we have \( \lim_{n \to \infty} H(k_n, s(\epsilon_n)) \geq \frac{1+\Delta+\cdots+\Delta^{w-1}}{\ln \Delta} \). By the definition of \( r_w \) and the fact \( \Delta > 1 \) from Lemma 3.9, \( \frac{1+\Delta+\cdots+\Delta^{w-1}}{(\Delta-1)\ln \Delta} = \frac{\Delta^{w-1}}{(\Delta-1)\ln \Delta} \geq C_w \). Combining this with Lemma 3.6, we complete the proof of Theorem 3.4 and thus that of Theorem 3.3.
4 Future research directions

Our deterministic exploration algorithm is optimal for all \( \lambda \), but our randomized algorithm is only shown to be optimal for \( \lambda = 1 \). For general \( \lambda \), it may be possible to obtain better or even optimal competitive ratios by holding \( \lambda - 1 \) robots still while one robot moves towards the origin, as in our deterministic algorithm. This technique is not essential in our deterministic algorithm, but there is evidence that it may be useful in the randomized case. We further conjecture that there exists an optimal randomized algorithm in which one robot searches \( w - \lambda + 1 \) paths and each of the other robots searches only one of the remaining paths.

Our exploration algorithms minimize the total distance traveled by all the robots. It would be of interest to minimize the total parallel exploration time. We conjecture that the optimal competitive ratios of parallel time are achieved when the paths are partitioned as even as possible.

In the layered graph traversal problem [6, 10, 12], one robot searches for a certain goal in a graph. The robot can shortcut between paths without going through the origin, and while exploring one path, it can obtain free information about the other paths. An analog of our work would be to study how to search a layered graph with multiple robots.

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A Proof of Lemma 2.3

The sequence \( \{(h_i, a_i), i \geq 1\} \) characterizes a deterministic exploration algorithm \( \mathcal{A}(w, 1) \), and the smallest competitive ratio of this algorithm is \( 1 + 2 \lim_{i \to \infty} H_i \). If \( \mathcal{A}(w, 1) \) does not have a finite competitive ratio, then Lemma 2.3 holds trivially. Thus, we assume that \( \mathcal{A}(w, 1) \) has a finite competitive ratio.

Claim A.1 \( \lim_{i \to \infty} h_i = \infty \).

Proof. Assume for contradiction that the claim does not hold. Then there exist a constant \( M < \infty \) and a subsequence \( \{h_{i_k}, k \geq 1\} \) of \( \{h_i, i \geq 1\} \) such that \( h_{i_k} \leq M \) for all \( k \). Therefore,

\[
\lim_{i \to \infty} H_i \geq \lim_{k \to \infty} h_{i_k} \geq \lim_{k \to \infty} \frac{h_1 + \cdots + h_{i_k} - 1}{h_{i_k}} \geq \frac{1}{M} \lim_{k \to \infty} (h_1 + \cdots + h_{i_k} - 1),
\]

which equals \( \infty \) from the assumption that \( \mathcal{A}(w, 1) \) has a finite competitive ratio. \( \square \)

By Claim A.1, for any \( M < \infty \), \( |\{h_i \mid h_i \leq M\}| \) is finite. Hence, we can sort \( \{h_i, i \geq 1\} \) into a sorted sequence \( \{s_i, i \geq 1\} \). Observe that

\[
s_1 + \cdots + s_i \leq h_1 + \cdots + h_i, \text{ for all } i \geq 1.
\] (15)
The sequence \(\{s_i, i \geq 1\}\) is our desired cyclic \(w\)-sequence, and its corresponding ratio sequence \(\{S_i, i \geq 1\}\) is uniquely defined as in (3). To prove \(\lim_{i \to \infty} H_i \geq \lim_{i \to \infty} S_i\), it suffices to show that for each \(j\) large enough, there is a \(j^*\) such that

\[
S_j \leq H_{j^*} \quad \text{and} \quad j^* \to \infty \quad \text{as} \quad j \to \infty. \tag{16}
\]

For any fixed \(j\) that is sufficiently large, we consider two cases.

*Case 1:* There exists some \(t \geq j + w - 1\) such that \(h_t \leq s_j\). Since \(t' > t\) is the smallest index with \(a_{t'} = a_t\), \(t' - 1 \geq t \geq j + w - 1\). Hence, by (15),

\[
S_j = \frac{s_1 + \cdots + s_{j+w-1}}{s_j} \leq \frac{h_1 + \cdots + h_{j+w-1}}{s_j} \leq \frac{h_1 + \cdots + h_{t'}}{h_t} = H_t. \tag{17}
\]

Let \(j^* = t\). Then, (16) follows from (17) and the fact \(j^* \geq j\).

*Case 2:* \(h_t > s_j\) for all \(t \geq j + w - 1\). Since \(\{h_1, \ldots, h_{j+w-2}\}\) contains all \(h_t\) with \(h_t \leq s_j\), \(\{h_1, \ldots, h_{j+w-2}\}\) contains \(\{s_1, \ldots, s_j\}\) as a subset. Thus,

\[
|\{h_t \mid h_t > s_j \text{ and } 1 \leq t \leq j + w - 2\}| \leq (j + w - 2) - j = w - 2. \tag{18}
\]

Since \(\{(h_t, a_t), i \geq 1\}\) is a \(w\)-sequence, there are \(w\) distinct integers \(v_1, v_2, \ldots, v_w\), each appearing infinitely many times in \(\{a_i, i \geq 1\}\). Since \(j\) is sufficiently large, we can assume without loss of generality that each \(v_k\) appears at least once in \(\{a_1, \ldots, a_j\}\). For \(k\) with \(1 \leq k \leq w\), let \(j(k)\) be the largest index with \(a_{j(k)} = v_k\). Among \(h_{j(1)}, \ldots, h_{j(w)}\), by (18), at least two, say, \(h_{j(k_1)}\) and \(h_{j(k_2)}\), are less than \(s_j\). By the choices of \(j(k_1)\) and \(j(k_2)\), both \(j'(k_1)\) and \(j'(k_2)\) are at least \(j + w - 1\).

Without loss of generality, we assume \(j'(k_1) > j + w - 1\). By (14),

\[
S_j = \frac{s_1 + \cdots + s_{j+w-1}}{s_j} \leq \frac{h_1 + \cdots + h_{j+w-1}}{s_j} \leq \frac{h_1 + \cdots + h_{j'(k_1)-1}}{h_{j(k_1)}} = H_{j(k_1)}. \tag{19}
\]

Now, let \(j^* = j(k_1)\). Since \(v_{k_1}\) appears infinitely many times in \(\{a_i, i \geq 1\}\), \(j(k_1) \to \infty\) as \(j \to \infty\). Now, (16) follows from (19).

Cases 1 and 2 together complete the proof of Lemma 2.3.

### B Proofs of Lemmas 3.7 through 3.12

#### B.1 Proof of Lemma 3.7

\[
\frac{x_0}{1+\epsilon} + \frac{x_1}{1+\epsilon} + \cdots + \frac{x_m}{1+\epsilon} \geq \frac{1}{1+\epsilon} \left( \frac{x_0}{1+\epsilon} \right)^m E_m \quad \text{(arithmetic mean \geq geometric mean)}
\]

\[
\geq \frac{m}{(1+\epsilon)^m} \left( \frac{x_0}{1+\epsilon} \right)^m E_m \quad \text{(because } (1+\epsilon)^m \geq 1+m\epsilon)\]

\[
\geq \frac{m^m}{(1+\epsilon)^m} \left( \frac{x_0}{1+\epsilon} \right)^m E_m \quad \text{(because } (1+\epsilon)^m \geq 1+m\epsilon)\]

\[
\geq \frac{m^m}{(1+\epsilon)^m} \left( \frac{x_0}{1+\epsilon} \right)^m E_m \quad \text{(because } (1+\epsilon)^m \geq 1+m\epsilon)\]

\[
\geq \frac{m^m}{(1+\epsilon)^m} \left( \frac{x_0}{1+\epsilon} \right)^m E_m \quad \text{(because } (1+\epsilon)^m \geq 1+m\epsilon)\]

\[
\geq \frac{m^m}{(1+\epsilon)^m} \left( \frac{x_0}{1+\epsilon} \right)^m E_m \quad \text{(because } (1+\epsilon)^m \geq 1+m\epsilon)\]
B.2 Proof of Lemma 3.8

By the choice of \( k_n \) and the monotonicity of \( S_w \), we have \( s_{k_n}(\epsilon_n) \geq s_i(\epsilon_n) \) for all \( i = 0, \ldots, k_n \). Hence,

\[
L_n(0) \geq \frac{k_n}{(s_{k_n}(\epsilon_n))^{\epsilon_n}}. \tag{20}
\]

Then, the lemma follows from the facts that \( k_n \to \infty \) and \( \epsilon_n \to 0 \) as \( n \to \infty \) and that by (14) and (20)

\[
C \geq -\epsilon_n + \frac{k_n}{\ln(s_{k_n}(\epsilon_n))^{\epsilon_n}}.
\]

B.3 Proof of Lemma 3.9

First, we have

\[
L_n(1) = \sum_{i=0}^{h_n-1} \frac{s_{i+1}(\epsilon_n)}{(s_i(\epsilon_n))^{1+\epsilon_n}} + \sum_{i=h_n}^{k_n-1} \frac{s_{i+1}(\epsilon_n)}{(s_i(\epsilon_n))^{1+\epsilon_n}}.
\]

Noticing \( s_0(\epsilon_n) = 1 \) and applying Lemma 3.7 to the above two summations, we have

\[
L_n(1) \geq L_n'(1) + L_n''(1), \tag{21}
\]

where

\[
L_n'(1) = \frac{h_n}{(1 + \epsilon_n)^{h_n}} \left( (s_{h_n}(\epsilon_n))^{\frac{h_n}{1+\epsilon_n}} E_{\epsilon_n}(h_n) \right)
\]

and

\[
L_n''(1) = \frac{k_n - h_n}{(1 + \epsilon_n)^{k_n-h_n}} \left( (s_{k_n}(\epsilon_n))^{\frac{h_n-h_n}{1+\epsilon_n}} E_{\epsilon_n}(k_n-h_n) \right).
\]

Now, we can rewrite \( L_n''(1) \) as

\[
L_n''(1) = \frac{k_n - h_n}{(1 + \epsilon_n)^{k_n-h_n}} (s_{k_n}(\epsilon_n))^{\beta''_n},
\]

where

\[
\beta''_n = \left( \frac{1}{1 + \epsilon_n} \right)^{k_n-h_n} - 1 + \nu_n \right) E_{\epsilon_n}(k_n-h_n).
\]

We also rewrite \( L_n'(1) \) as

\[
L_n'(1) = \frac{h_n}{(1 + \epsilon_n)^{h_n}} (s_{k_n}(\epsilon_n))^{\beta'_n}, \tag{22}
\]

where

\[
\beta'_n = (1 - \nu_n) \left( \frac{1}{1 + \epsilon_n} \right)^{h_n} E_{\epsilon_n}(h_n).
\]

By Lemma 3.8 and the fact that by (14) and (21)

\[
C \geq -\epsilon_n + L_n'(1) + L_n''(1),
\]

15
we conclude that for some constant $c$,

$$0 \leq \frac{L_n'(1)}{\ln s_k(\epsilon_n)} \leq c \quad \text{and} \quad 0 \leq \frac{L_n''(1)}{\ln s_k(\epsilon_n)} \leq c$$

for all $n$. \hfill (23)

Since $1 \leq k_n - h_n \leq w - 1$ and $\nu_n \geq 0$, no subsequence of $\{\beta''_n, n \geq 0\}$ can approach $-\infty$ or converge to a finite negative number. On the other hand, by Lemma 3.8 and (23), no subsequence of $\{\beta''_n, n \geq 0\}$ can approach $+\infty$ or converge to a finite positive number. Thus, $\lim_{n \to \infty} \beta''_n = 0$ and consequently, $\lim_{n \to \infty} \nu_n = 0$. By (22),

$$\frac{L_n'(1)}{\ln s_k(\epsilon_n)} = \frac{h_n}{(1 + \epsilon_n) h_n \ln (s_k(\epsilon_n))^{\frac{1}{\epsilon_n}}.$$

Since $1 \leq k_n - h_n \leq w - 1$ and $0 \leq k_n - \frac{1}{\sqrt{\epsilon_n}} \leq w - 1$,

$$\lim_{n \to \infty} \frac{h_n}{1 + \epsilon_n} = 1, \quad \lim_{n \to \infty} k_n = 1, \quad \text{and thus} \quad \lim_{n \to \infty} \frac{L_n'(1)}{\ln s_k(\epsilon_n)} = \lim_{n \to \infty} \frac{s_i(\epsilon_n)}{\ln s_k(\epsilon_n)}^{\frac{1}{\epsilon_n}}.$$

Using Lemma 3.8, (23) and an argument similar to the proof for $\lim_{n \to \infty} \nu_n = 0$, we can show that for some constant $\Delta$

$$\lim_{n \to \infty} (s_k(\epsilon_n))^{\frac{1}{\epsilon_n}} = \Delta > 1.$$

**B.4 Proof of Lemma 3.10**

This lemma follows from Lemma 3.3 and the fact that $1 \leq k_n - h_n \leq w - 1$ and $0 \leq k_n - \frac{1}{\sqrt{\epsilon_n}} \leq w - 1$. The calculations are similar to those for proving Lemma 3.3.

**B.5 Proof of Lemma 3.11**

Since $b_n \geq 1$, it suffices to show that for all $i$,

$$b_n^i \geq s_i(\epsilon_n),$$

where $d_i = \sum_{i'=0}^{i-1} (1 + \epsilon_n)^{i'}$. We can prove this inequality by induction on $i$. The base case follows from the fact that $s_0(\epsilon_n) = 1$ and $b_n \geq 1$. The induction step follows from the fact that by Fact \[ and Lemma 3.3,

$$C \geq \frac{-\epsilon_n + \frac{s_{i+1}(\epsilon_n)}{(s_i(\epsilon_n))\epsilon_n}}{\ln s_k(\epsilon_n)}.$$

**B.6 Proof of Lemma 3.12**

$$L_n(j) = \sum_{u=0}^{j-1} \left( \sum_{0 \leq i < k_n - j \atop i \equiv u \pmod{j}} \frac{s_i+j(\epsilon_n)}{(s_i(\epsilon_n))^{\epsilon_n}} \right)$$
\[
\begin{align*}
= \sum_{u=0}^{j-1} \left( \sum_{i' = 0}^{g(j,u)-1} \frac{s_{u+(i'+1)j}(\epsilon_n)}{(s_{u+i'j}(\epsilon_n))^{\gamma_n}} \right) \\
\geq \sum_{u=0}^{j-1} L'_n(j,u),
\end{align*}
\]

where
\[g(j,u) = \left\lfloor \frac{k_n - u}{j} \right\rfloor\]
and
\[L'_n(j,u) = \frac{g(j,u)}{(1 + \epsilon_n)^{g(j,u)}} \left( \frac{(s_{h_n}(\epsilon_n))^{\frac{1}{1+\epsilon_n}}}{b_n^{(w-1)(1+\epsilon_n)^{w-1}}} \right) E_{\epsilon_n}(g(j,u)).\]

The term \(L'_n(j,u)\) is obtained by applying Lemma 3.7 to the inner summation in the right-hand side of the above equalities. The derivation also uses the fact that because \(k_n - w + 1 \leq u + g(j,u)j \leq k_n\),

\[s_{u+g(j,u)j}(\epsilon_n) \geq s_{h_n}(\epsilon_n)\]
and the fact that by Lemma 3.11

\[b_n^{(w-1)(1+\epsilon_n)^{w-1}} \geq s_u(\epsilon_n).\]

On the other hand, for each \(j = 2, \ldots, w - 1\) and \(u = 0, \ldots, j - 1\),

\[\lim_{n \to \infty} \frac{L'_n(j,u)}{\ln s_{h_n}(\epsilon_n)} = \frac{\Delta^j}{j \ln \Delta},\]

which can be verified using Lemma 3.3, the fact \(1 \leq k_n - h_n \leq w - 1\) and \(0 \leq k_n - \frac{1}{\sqrt{\epsilon_n}} \leq w - 1\), and calculations similar to those in the proof of Lemma 3.9.

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