On $\kappa$-Deformation and UV/IR Mixing

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Abstract

We examine the UV/IR mixing property on a $\kappa$-deformed Euclidean space for a real scalar $\phi^4$ theory. All contributions to the tadpole diagram are explicitly calculated. UV/IR mixing is present, though in a different dressing than in the case of the canonical deformation.

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1 Introduction

Divergencies in quantum field theory have been one of the main reasons for introducing non-commutative geometries and non-commutative coordinates. In the simplest case, the commutator of two coordinates is just constant,

\[ [\hat{x}^\mu, \hat{x}^\nu] = i\theta_{\mu\nu}, \]

where \( \theta_{\mu\nu} \in \mathbb{R} \). This is the canonical deformation. There is an enormous amount of literature dealing with field theories built on such spaces. Feynman rules for scalar \( \phi^4 \) theory have been deduced in \([1, 2]\). Unfortunately, non-commutative field theories turned out to be non-renormalisable due to a new property, called ”UV/IR mixing” \([3]\). Although the one-loop integrals for non-planar diagrams are finite for generic external momenta, it diverges for zero external momenta. This causes infrared problems even in massive theories. The insertion into higher-loop contributions gives rise to divergencies which cannot be absorbed by standard procedures.

So far, the only renormalisable model on non-commutative spaces has been provided by R. Wulkenhaar and one of the authors (H.G.) \([4, 5]\).

In this work, we want to study the UV/IR mixing property for real scalar \( \phi^4 \) theory on \( \kappa \)-deformed Euclidean space. In the \( \kappa \)-deformed case \([6–10]\), the space algebra is spanned by coordinates \( \hat{x}^i \), \( i = 1, 2, 3, 4 \) with relations

\[ [\hat{x}^1, \hat{x}^p] = ia\hat{x}^p, \quad [\hat{x}^q, \hat{x}^p] = 0, \quad (1) \]

where \( p, q = 2, 3, 4, \quad a = 1/\kappa \). Elements of the non-commutative space-time algebra and abstract elements of the \( \kappa \)-Poincaré algebra \( \mathcal{U}_a(so(4)) \) are denoted with a hat.

The Klein-Gordon operator is given by \([8]\)

\[ \hat{\Box} = e^{-ia\hat{\partial}_1} \sum_{p=2}^{4} \hat{\partial}_p \hat{\partial}_p + \frac{2}{a^2} (1 - \cos(a\hat{\partial}_1)). \quad (2) \]

In the following, summation over repeated indices is implied. The algebra relations of the symmetry generators are given in \([9]\), for example. The action of the generators on commutative functions - star product representations - are provided in \([10]\), with respect to various different orderings. The ordering defines a basis in the abstract coordinate algebra and therefore a star
product representation on commutative functions. The ordering is not essential, though. The different star products corresponding to different ordering prescriptions are equivalent and related by a (gauge) transformation $D$ [11],
\[ \mathcal{D} f \ast \mathcal{D} g = \mathcal{D}(f \ast' g). \]
Physics should only depend on equivalence classes of star products, not on the representations of a single class. We will concentrate on the symmetrically ordered star product. Its advantage is the hermiticity property,
\[ \bar{f} \ast g(x) = \bar{g} \ast \bar{f}(x). \tag{3} \]
Scalar field theories on $\kappa$-deformed spaces have already been studied in e.g. [12–16]. In [13,14], a functional approach has been applied which we will also adopt here. Nevertheless, no explicit results for Feynman amplitudes have been calculated yet. A main difficulty was the construction of a proper measure [15–18]. For our calculations, we will choose a symmetrically ordered star product and the $\kappa$-Poincaré invariant scalar product introduced in [17].

2 Symmetrically Ordered Star Product

The symmetrically ordered star product is given by [10]
\[ f \ast g(x) = \int d^4k d^4p \tilde{f}(k)\tilde{g}(p) e^{i(\omega_k + \omega_p)x^1} e^{i\vec{x}(\vec{k}e^{a\omega_k}A(\omega_k, \omega_p) + \vec{p}A(\omega_p, \omega_k))}, \tag{4} \]
where $k = (\omega_k, \vec{k})$, and $\vec{x} = (x^2, x^3, x^4)$. We have used the definition
\[ A(\omega_k, \omega_p) \equiv \frac{a(\omega_k + \omega_p)}{e^{a(\omega_k + \omega_p)} - 1} \frac{e^{a\omega_k} - 1}{a\omega_k}. \tag{5} \]
Let us state a very useful identity which we will need a lot in the calculations:
\[ e^{-a\omega_2}A(-\omega_1, -\omega_2) = A(\omega_1, \omega_2). \tag{6} \]
Then, the Klein-Gordon operator acting on commutative functions reads
\[ \square^\ast = \sum_{i=1}^{4} \partial_i \partial_i \frac{2(1 - \cos a\partial_1)}{a^2 \partial_1^2}. \tag{7} \]
2.1 κ–Poincaré Invariant Action

A κ-Poincaré invariant integral is given by [17]

\[
(\phi, \psi) = \int d^4x \phi(K \bar{\psi}),
\]

where

\[
K = \left( \frac{-ia\partial_1}{e^{ia\partial_1} - 1} \right)^3.
\]

In momentum space, this amounts to

\[
(\phi, \psi) = \int d^4q \left( \frac{-a\omega_q}{e^{-a\omega_q} - 1} \right)^3 \tilde{\phi}(q) \bar{\tilde{\psi}}(q).
\]

And therefore, the action for a scalar field with φ⁴ interaction is given by

\[
S[\phi] = -(\phi, (\Box^* - m^2)\psi)
\]

\[
+ \frac{g}{4!} (b(\phi \ast \phi, \phi \ast \phi) + d(\phi \ast \phi \ast \phi \ast 1)).
\]

In momentum space, the action has the following form:

\[
S[\phi] = \int d^4q \left( \frac{-a\omega_q}{e^{-a\omega_q} - 1} \right)^3 \tilde{\phi}(q) \left( \frac{q^2 2(\cosh a\omega_q - 1)}{a^2 \omega_q^2} + m^2 \right) \bar{\tilde{\phi}}(q)
\]

\[
+ \frac{g}{4!} \int d^4z \prod_{i=1}^4 d^4k_i \left( \frac{a(\omega_{k_1} + \omega_{k_2})}{e^{a(\omega_{k_1} + \omega_{k_2})} - 1} \right)^3 \tilde{\phi}(k_1) \bar{\tilde{\phi}}(k_2) \phi(k_3) \phi(k_4)
\]

\[
\times e^{i\sum \omega_{k_i}} \exp \left( i\sum \omega_{k_i} \left[ k_1 e^{a\omega_{k_2}} A(\omega_{k_1}, \omega_{k_2}) + k_2 A(\omega_{k_2}, \omega_{k_1})
\right.
\]

\[
\left. + k_3 e^{-a\omega_{k_3}} A(-\omega_{k_3}, -\omega_{k_4}) + k_4 A(-\omega_{k_4}, -\omega_{k_3}) \right] \right)
\]

\[
+ \frac{d}{4!} \int d^4z \prod_{i=1}^4 d^4k_i e^{i\sum \omega_{k_i}} \left( \bar{\tilde{\phi}}(k_1) \phi(k_2) \tilde{\phi}(k_3) \phi(k_4) \right)
\]

\[
\times e^{i\sum (k_1 e^{a\omega_{k_2}} A(\omega_{k_1}, \omega_{k_2}) + k_2 A(\omega_{k_2}, \omega_{k_1})) e^{a(\omega_{k_3} + \omega_{k_4})} A(\omega_{k_1} + \omega_{k_2}, \omega_{k_3} + \omega_{k_4})}
\]

\[
\times e^{i\sum (k_3 e^{a\omega_{k_4}} A(\omega_{k_3}, \omega_{k_4}) + k_4 A(\omega_{k_4}, \omega_{k_3})) A(\omega_{k_1} + \omega_{k_2}, \omega_{k_3} + \omega_{k_4})}.
\]

Note that \( \tilde{\phi}(k) = \tilde{\phi}(-k) \), for real fields \( \phi(x) \). The \( x \)-dependent phase factors are a direct result of the star product \([11]\), \( b \) and \( d \) are real parameters. In the case of canonical deformation, the phase factor is independent of \( x \). In \([11]\), one could also imagine an interaction term proportional to \( \phi \ast \phi \ast \phi \). This term will be discussed later on. It leads to a somewhat different and peculiar behaviour.
3 Tadpole Diagram

The generating functional can be defined as

\[ Z_\kappa[J] = \int D\phi e^{-S[\phi] + \frac{1}{2}(J,\phi) + \frac{1}{2}(\phi,J)}. \]  

(13)

The \( n \)-point functions \( \tilde{G}_n(p_1, \ldots, p_n) \) are given by functional differentiation:

\[ \tilde{G}_n(p_1, \ldots, p_n) = \frac{\delta^n}{\delta \tilde{J}(-p_1) \cdots \delta \tilde{J}(-p_n)} Z_\kappa[J] \bigg|_{J=0}. \]  

(14)

Let us first consider the free case. For the free generating functional \( Z_{0,\kappa} \) we obtain from Eq. (13)

\[ Z_{0,\kappa}[J] = \int D\phi \exp \left[ -\frac{1}{2} \int d^4 k \left( \frac{-a\omega_k}{e^{-a\omega_k} - 1} \right)^3 \tilde{\phi}(k)(M_k + m^2)\tilde{\phi}(-k) \right. \]

\[ + \frac{1}{2} \int d^4 k \left( \left( \frac{-a\omega_k}{e^{-a\omega_k} - 1} \right)^3 + \left( \frac{a\omega_k}{e^{a\omega_k} - 1} \right)^3 \right) \tilde{J}(k) \tilde{\phi}(-k) \right], \]  

(15)

where we have defined

\[ M_k := \frac{2k^2(cosh a\omega_k - 1)}{a^2\omega_k^2}. \]  

(16)

The same manipulations as in the classical case yield

\[ Z_{0,\kappa}[J] = Z_{0,\kappa}[0] e^{\frac{1}{2} \int d^4 k \left( \frac{-a\omega_k}{e^{-a\omega_k} - 1} \right)^3 \tilde{J}(k) \tilde{J}(-k) M_k + m^2}. \]  

(17)

We will always consider the normalised functional, which we obtain by dividing with \( Z_{0,\kappa}[0] \). Now, the free propagator is given by

\[ \tilde{G}(k,p) = \frac{\delta^2}{\delta \tilde{J}(-k) \delta \tilde{J}(-p)} Z_{0,\kappa}[J] \bigg|_{J=0} \]

\[ = L(\omega_k) \frac{\delta^{(4)}(k + p)}{M_k + m^2} \equiv \delta^{(4)}(k + p)Q_k. \]  

(18)

For the sake of brevity, we have introduced

\[ L(\omega_k) := \frac{1}{2} \left( \left( \frac{-a\omega_k}{e^{-a\omega_k} - 1} \right)^3 + \left( \frac{a\omega_k}{e^{a\omega_k} - 1} \right)^3 \right). \]  

(19)
Let us switch on the interaction. For the clarity of presentation, we will for now only consider the first interaction term in Eq. (12). The other term will be treated in the next subsection. We make the following observation:

\[
\frac{1}{L(ω_p)} \frac{δ}{δJ(−p)} Z_κ[J] \bigg|_{J=0} = \tilde{φ}(p).
\] (20)

Therefore, we can rewrite the generating functional in the form

\[
Z_κ[J] = e^{-S_1[1/L(ω_k) \frac{δ}{δJ(−k)}]} Z_{0,κ}[J].
\] (21)

The aim of this article is to compute tadpole diagram contributions. In order to do so, we expand the generating functional (21) in powers of the coupling constant \( g \). Using Eq. (12), we obtain

\[
Z_κ[J] = Z_{0,κ}[J] + Z_1^1[J] + O(g^2).
\] (22)

The first order term in this expansion reads

\[
Z_1^1[J] = -\frac{bg}{4} \int \prod_{i=1}^{4} \left( d^4k_i \frac{\left( a(ω_3+ω_4) \right)^3}{L(ω_i)} \frac{δ}{δJ(−k_i)} \right) Z_{0,κ}[J]
\]

\[
\times \delta(\sum_{j=1}^{4} ω_j) \delta^{(3)} \left( \tilde{k}_1 e^{ω_2 A(ω_1, ω_2)} + \tilde{k}_2 A(ω_2, ω_1)
\]

\[
+ \tilde{k}_3 e^{-ω_4 A(ω_3, ω_4)} + \tilde{k}_4 A(−ω_4, ω_3) \right)
\]

\[
= -\frac{bg}{4} \int \prod_{i=1}^{4} \left( d^4k_i \frac{\left( a(ω_3+ω_4) \right)^3}{L(ω_i)} \right) \delta(\sum_{j=1}^{4} ω_j)
\]

\[
\times \delta^{(3)} \left( \tilde{k}_1 e^{ω_2 A(ω_1, ω_2)} + \tilde{k}_2 A(ω_2, ω_1)
\]

\[
+ \tilde{k}_3 e^{-ω_4 A(ω_3, ω_4)} + \tilde{k}_4 A(−ω_4, ω_3) \right)
\]

\[
\times \left\{ \delta^{(4)}(k_1 + k_2) \delta^{(4)}(k_3 + k_4) Q_4 Q_2
\]

\[
+ \delta^{(4)}(k_1 + k_3) \delta^{(4)}(k_2 + k_4) Q_4 Q_3
\]

\[
+ \delta^{(4)}(k_1 + k_4) \delta^{(4)}(k_2 + k_3) Q_4 Q_3
\]

\[
+ Q_4 \tilde{J}(k_4) Q_3 \tilde{J}(k_3) Q_2 \tilde{J}(k_2) Q_1 \tilde{J}(k_1)
\]
the coupling parameter is given by the connected part of the expression. Explicitly, we obtain

\[ + \delta^{(4)}(k_3 + k_4) Q_4 Q_1 \tilde{J}(k_1) Q_2 \tilde{J}(k_2) \]

\[ + \delta^{(4)}(k_2 + k_4) Q_4 Q_1 \tilde{J}(k_1) Q_3 \tilde{J}(k_3) \]

\[ + \delta^{(4)}(k_2 + k_3) Q_3 Q_1 \tilde{J}(k_1) Q_4 \tilde{J}(k_4) \]

\[ + \delta^{(4)}(k_1 + k_4) Q_4 Q_2 \tilde{J}(k_2) Q_3 \tilde{J}(k_3) \]

\[ + \delta^{(4)}(k_1 + k_3) Q_3 Q_2 \tilde{J}(k_2) Q_4 \tilde{J}(k_4) \]

\[ + \delta^{(4)}(k_1 + k_2) Q_2 Q_3 \tilde{J}(k_3) Q_4 \tilde{J}(k_4) \}ight] Z_{0,\kappa}[J], \]

where \( Q_i := \frac{L(\omega_i)}{M_{k_i + m^2}} \), and \( k_j = (\omega_j, \vec{k}_j) \). The full propagator to first order in the coupling parameter is given by the connected part of the expression

\[ \tilde{G}^{(2)}(p, q) = \frac{\delta^2}{\delta \tilde{J}(-p) \delta \tilde{J}(-q)} Z_\kappa[J] \bigg|_{J=0}. \] (24)

The first three terms of Formula (23) give the disconnected contribution to the 2-point function. The contribution of the fourth term vanishes. What remains, provides us with twelve contributions to the connected 2-point function. Explicitly, we obtain

\[ \tilde{G}^{(2)}(p, q) = \tilde{G}(p, q) - b \frac{g^4}{4!} \int \prod_{i=1}^{4} d^4 k_i \left( \frac{q(\omega_3 + \omega_4)}{\frac{e^q(\omega_3 + \omega_4)}{L(\omega_i)}} \right)^3 \]

\[ \times \delta \left( \sum_{j=1}^{4} \omega_j \right) \delta^{(3)} \left( \vec{k}_1 e^{a\omega_2} A(\omega_1, \omega_2) + \vec{k}_2 A(\omega_2, \omega_1) \right. \]

\[ \left. + \vec{k}_3 e^{-a\omega_4} A(-\omega_3, -\omega_4) + \vec{k}_4 A(-\omega_4, -\omega_3) \right) \]

\[ \times \left\{ \delta^{(4)}(k_2 + p) \delta^{(4)}(k_1 + q) \delta^{(4)}(k_3 + k_4) Q_4 Q_2 Q_1 \right. \]

\[ + \delta^{(4)}(k_2 + p) \delta^{(4)}(k_2 + q) \delta^{(4)}(k_3 + k_4) Q_4 Q_2 Q_1 \]

\[ + \delta^{(4)}(k_1 + p) \delta^{(4)}(k_3 + q) \delta^{(4)}(k_2 + k_4) Q_4 Q_3 Q_1 \]

\[ + \delta^{(4)}(k_3 + p) \delta^{(4)}(k_1 + q) \delta^{(4)}(k_2 + k_4) Q_4 Q_3 Q_1 \]

\[ + \delta^{(4)}(k_4 + p) \delta^{(4)}(k_2 + q) \delta^{(4)}(k_1 + k_3) Q_4 Q_3 Q_2 \]

\[ + \delta^{(4)}(k_4 + p) \delta^{(4)}(k_3 + q) \delta^{(4)}(k_1 + k_3) Q_4 Q_3 Q_2 \]

\[ + \delta^{(4)}(k_3 + p) \delta^{(4)}(k_4 + q) \delta^{(4)}(k_1 + k_2) Q_4 Q_3 Q_2 \]

\[ + \delta^{(4)}(k_3 + p) \delta^{(4)}(k_3 + q) \delta^{(4)}(k_1 + k_2) Q_4 Q_3 Q_2 \]

\[ + \delta^{(4)}(k_4 + p) \delta^{(4)}(k_1 + q) \delta^{(4)}(k_2 + k_3) Q_4 Q_3 Q_1 \]

\[ + \delta^{(4)}(k_4 + p) \delta^{(4)}(k_1 + q) \delta^{(4)}(k_2 + k_3) Q_4 Q_3 Q_1 \]
+\delta^{(4)}(k_1 + p)\delta^{(4)}(k_4 + q)\delta^{(4)}(k_2 + k_3)Q_4Q_3Q_1
+\delta^{(4)}(k_3 + p)\delta^{(4)}(k_2 + q)\delta^{(4)}(k_1 + k_4)Q_4Q_3Q_2
+\delta^{(4)}(k_2 + p)\delta^{(4)}(k_3 + q)\delta^{(4)}(k_1 + k_4)Q_4Q_3Q_2
\}
= \tilde{G}(p, q) + b \sum_{i=1}^{12} \tilde{G}_{c=1}^{(2), b}(p, q), \quad (26)

where \( \int_\Lambda \) denotes the integral regularised by a cut-off \( \Lambda \), see below. The last four terms of Eq. (25) correspond to non-planar diagrams. Let us discuss some of the contributions in detail. As an example of a planar diagram, we will first of all analyse the first term of formula (25), i.e., 
\[
\delta\text{-functions from functional differentiation enable us to integrate over three of the momenta. We obtain the following contribution:}
\]

\[
\tilde{G}_{c=1}^{(2), b}(p, q) = -\frac{g}{4!} \int_\Lambda \prod_{i=1}^{4} \left( d^4k_i \left( \frac{a(\omega_1 + \omega_4)}{e^{\omega_1 + \omega_4} - 1} \right)^3 \right) \delta(\sum_j \omega_j)Q_4Q_2Q_1
\times \delta^{(3)}(\vec{k}_1 e^{\omega_2} A(\omega_1, \omega_2) + \vec{k}_2 A(\omega_2, \omega_1) + \vec{k}_3 e^{a\omega_4} A(-\omega_3, -\omega_4)
+ \vec{k}_4 A(\omega_4, \omega_3)) \delta^{(4)}(k_2 + p)\delta^{(4)}(k_1 + q)\delta^{(4)}(k_3 + k_4)
\]

\[
= -\frac{g}{4!} \delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2 \mathcal{M}_q + m^2} \int_\Lambda d^4k \frac{1}{L(\omega_k) \mathcal{M}_k + m^2}
\times \delta^{(3)}(\vec{k} e^{a\omega_k} A(-\omega_k, \omega_k) - A(\omega_k, -\omega_k))
\]

\[
= -\frac{g}{4!} \delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2 \mathcal{M}_q + m^2} \int_\Lambda d^4k \frac{1}{L(\omega_k) \mathcal{M}_k + m^2}
\times \int_{-\infty}^{\infty} d\omega_k \frac{4\pi}{2(cosh a\omega_k - 1)} \int_0^\Lambda dk \frac{k^2}{k^2 + (\omega_k^2 + \frac{m^2 a^2 \omega_k^2}{2(cosh a\omega_k - 1)})}
\]

\[
= -\frac{g}{4!} \delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2 \mathcal{M}_q + m^2} \int L(\omega_k) \mathcal{M}_k + m^2 \frac{1}{|A(\omega_k, -\omega_k)|^3} \quad (27)
\]

\[
\times \int_{-\infty}^{\infty} d\omega_k \frac{4\pi}{2(cosh a\omega_k - 1)} \int_0^\Lambda dk \frac{k^2}{k^2 + (\omega_k^2 + \frac{m^2 a^2 \omega_k^2}{2(cosh a\omega_k - 1)})}
\]

\[
= -\frac{g}{4!} \delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2 \mathcal{M}_q + m^2} \int L(\omega_k) \mathcal{M}_k + m^2 \frac{1}{|A(\omega_k, -\omega_k)|^3} \quad (28)
\]
This expression is linearly divergent in the cut-off $\Lambda$. The $\omega_k$-integration yields a finite result due to the propagator (18). For the other planar diagrams, we obtain similar results:

\[
\tilde{G}_{e=3}^{(2),b}(p, q) = -\frac{g}{4!} \delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2} \frac{1}{\mathcal{M}_q + m^2} \left( \int_{-\infty}^{\infty} \frac{d\omega_k}{L(\omega_k)} \right) \left( \frac{4\pi}{L(\omega_k)} \frac{a^2\omega_k^2}{2(\cosh a\omega_k - 1)} \right) \times \left( \Lambda - \omega_k \sqrt{1 + \frac{m^2a^2}{2(\cosh a\omega_k - 1)}} \right) \times \arctan \left( \frac{\Lambda}{\omega_k \sqrt{1 + \frac{m^2a^2}{2(\cosh a\omega_k - 1)}}} \right).
\]

\[
\tilde{G}_{e=5}^{(2),b}(p, q) = -\frac{g}{4!} \delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2} \frac{1}{\mathcal{M}_q + m^2} \left( \int_{-\infty}^{\infty} \frac{d\omega_k}{L(\omega_k)} \right) \left( \frac{4\pi}{L(\omega_k)} \frac{a^2\omega_k^2}{2(\cosh a\omega_k - 1)} \right) \times \left( \Lambda - \omega_k \sqrt{1 + \frac{m^2a^2}{2(\cosh a\omega_k - 1)}} \right) \times \arctan \left( \frac{\Lambda}{\omega_k \sqrt{1 + \frac{m^2a^2}{2(\cosh a\omega_k - 1)}}} \right).
\]

\[
\tilde{G}_{e=7}^{(2),b}(p, q) = -\frac{g}{4!} \delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2} \frac{1}{\mathcal{M}_q + m^2} \left( \int_{-\infty}^{\infty} \frac{d\omega_k}{L(\omega_k)} \right) \left( \frac{4\pi}{L(\omega_k)} \frac{a^2\omega_k^2}{2(\cosh a\omega_k - 1)} \right) \times \left( \Lambda - \omega_k \sqrt{1 + \frac{m^2a^2}{2(\cosh a\omega_k - 1)}} \right) \times \arctan \left( \frac{\Lambda}{\omega_k \sqrt{1 + \frac{m^2a^2}{2(\cosh a\omega_k - 1)}}} \right).
\]

The remaining planar contractions $c = 2, 4, 6, 8$ can be obtained from the contributions of $c = 1, 3, 5$ and 7, respectively by interchanging the external momenta $p$ and $q$. 

\[\text{8}\]
Non-planar contributions show a remarkable difference. There we do not have an overall momentum conservation. Remarkably, the components of \( \vec{k} \) of the internal momentum are fixed by the external ones, and the contributions are finite, for generic external momenta. The only exception is the case \( \omega_p = \omega_q = 0 \). There, we get back the UV divergences discussed above. Below, we give the explicit calculation for the non-planar contraction \( c = 9 \):

\[
\tilde{G}^{(2),b}_{c=9}(p, q) = -\frac{g}{4!} \int \prod_{i=1}^{4} \left( d^4k_i \left( \frac{a(\omega_i + \omega_j)}{e^{a(\omega_i + \omega_j)} - 1} \right)^3 \right) \delta(\sum_j \omega_j) Q_4 Q_3 Q_1
\]

\[
\times \delta(3) \left( \vec{k}_1 e^{a\omega_2} A(\omega_1, \omega_2) + \vec{k}_2 A(\omega_2, \omega_1) + \vec{k}_3 e^{-a\omega_4} A(\omega_3, \omega_4)
\]

\[
+ \vec{k}_4 A(\omega_4, \omega_3) \right) \delta^{(4)}(k_4 + p) \delta^{(4)}(k_1 + q) \delta^{(4)}(k_2 + k_3)
\]

\[
= -\frac{g}{4!} \delta(\omega_q + \omega_p) \frac{1}{M_p + m^2 M_q + m^2} \int d^4k \frac{1}{L(\omega_k)} \delta^{(3)}(\vec{k} e^{-a\omega_q} A(-\omega_k, -\omega_q) - A(-\omega_k, -\omega_q))
\]

\[
= -\frac{g}{4!} \delta(\omega_q + \omega_p) \frac{1}{M_p + m^2 M_q + m^2} \int d^4k \left( \frac{a(\omega_k + \omega_q)}{e^{a(\omega_k + \omega_q)} - 1} \right)^3 \frac{a^2 \omega_k^2}{2(\cosh a \omega_k - 1)} \frac{1}{|A(\omega_k, \omega_q)(1 - e^{a\omega_q})|^3}
\]

\[
\times \left[ \omega_k^2 + \left( \frac{\vec{q} + \vec{p} e^{a\omega_q}}{A(\omega_k, \omega_q)} \right)^2 \frac{a^2 \omega_k^2}{2(\cosh a \omega_k - 1)} \frac{1}{2(\cosh a \omega_k - 1)} \right]^{-1}
\]

\[
= -\frac{g}{4!} \delta(\omega_q + \omega_p) \frac{1}{M_p + m^2 M_q + m^2} \int d^4k \left( \frac{a(\omega_k + \omega_q)}{e^{a(\omega_k + \omega_q)} - 1} \right)^3 \frac{a^2 \omega_k^2}{2(\cosh a \omega_k - 1)} \frac{1}{|A(\omega_k, \omega_q)(1 - e^{a\omega_q})|^3}
\]

\[
\times \left[ \omega_k^2 + \left( \frac{\vec{q} + \vec{p} e^{a\omega_q}}{A(\omega_k, \omega_q)} \right)^2 \frac{a^2 \omega_k^2}{2(\cosh a \omega_k - 1)} \frac{1}{2(\cosh a \omega_k - 1)} \right]^{-1}
\]

The above formula is true for generic momenta \( p \) and \( q \). Expression (33) is finite except for \( \omega_q = \omega_p = 0 \). In this case, the \( \delta^{(3)} \)-function in Eq. (32) does
not depend on the internal momentum \( \vec{k} \), and we encounter the same UV singularity as in the planar case. What remains of the \( \delta^{(3)} \)-distribution,

\[
\frac{1}{|A(0, \omega_k)|^3} \delta^{(3)}(\vec{q} + \vec{p} e^{\omega_k}),
\]

gives a contribution only if \( \vec{p} \) and \( \vec{q} \) are parallel to each other.

The non-planar amplitude corresponding to the contraction \( c = 11 \) reads

\[
\tilde{G}_{c=11}^{(2), b}(p, q) = -\frac{g}{4!} \delta(\omega_q + \omega_p) \frac{1}{\mathcal{M}_p + m^2} \frac{1}{\mathcal{M}_q + m^2} \int d^4k \left( \frac{e^{(\omega_q + \omega_k)}}{e^{(\omega_q + \omega_k) - 1}} \right)^3 \delta^{(3)}(\vec{k} A(\omega_k, \omega_q)(e^{\omega_q} - 1) - (\vec{q} e^{\omega_k} + \vec{p}) A(\omega_q, \omega_k))
\]

\[
= -\frac{g}{4!} \delta(\omega_q + \omega_p) \frac{1}{\mathcal{M}_p + m^2} \frac{1}{\mathcal{M}_q + m^2} \int d\omega_k \left( \frac{e^{(\omega_q + \omega_k)}}{e^{(\omega_q + \omega_k) - 1}} \right)^3 \frac{a^2 \omega_k^2}{2(\cosh a\omega_k - 1)} \frac{1}{(1 - e^{\omega_q}) A(\omega_k, \omega_q))^3}
\]

\[
\times \left[ \omega_k^2 + \frac{(\vec{q} e^{\omega_k} + \vec{p})^2 A(\omega_q, \omega_k)^2}{(e^{\omega_q} - 1)^2 A(\omega_k, \omega_q)^2} + \frac{m^2 a^2 \omega_k^2}{2(\cosh a\omega_k - 1)} \right]^{-1}.
\]

For the exceptional situation \( \omega_q = 0 \), we again do not obtain an overall momentum conservation, but a \( \delta^{(3)} \)-distribution fixing the \( \omega_k \) component,

\[
\frac{1}{|A(0, \omega_k)|^3} \delta^{(3)}(\vec{q} e^{\omega_k} + \vec{p}).
\]

As before, the diagram shows a linear UV divergence. Assuming that \( p = q = 0 \), exerts no influence the divergencies, because the \( \omega_k \)-integration is exponentially damped by the modified propagator.

### 3.1 Contributions from \( S_I = \frac{g^4}{4!} (\phi \ast \phi \ast \phi \ast \phi, 1) \)

The connected 2-point function involving all interactions written down in Eq. \( \{11\} \) is given by

\[
\tilde{G}^{(2)}(p, q) = \tilde{G}(p, q) + \sum_{i=1}^{12} \left( b\tilde{G}_{c=1}^{(2), b}(p, q) + d\tilde{G}_{c=1}^{(2), d}(p, q) \right) + \mathcal{O}(g^2).
\]

10
The contributions $\bar{G}_{c=1}^{(2),d}(p, q)$ for the second interaction term in Eq. (31) are obtained in the same way as described in the previous subsection. Also, they display the same characteristic behaviour:

$$
\bar{G}_{c=1}^{(2),d}(p, q) = -\frac{g}{4!}\delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2 \mathcal{M}_q + m^2} \frac{1}{|A(q, -q)|^3} \int_{-\infty}^{\infty} d\omega_k \frac{4\pi}{L(\omega_k)} \frac{1}{2(\cosh a\omega_k - 1)} \times \left( \Lambda - \omega_k \sqrt{1 + \frac{m^2 a^2}{2(\cosh a\omega_k - 1)}} \right),
$$

$$
\bar{G}_{c=3}^{(2),d}(p, q) = -\frac{g}{4!}\delta(q + \omega_p) \frac{1}{\mathcal{M}_p + m^2 \mathcal{M}_q + m^2} \frac{1}{1} \int_{-\infty}^{\infty} d\omega_k \frac{1}{L(\omega_k)} \times \frac{a^2 \omega_k^2}{2(\cosh a\omega_k - 1)} \frac{1}{|A(-\omega_q - \omega_k, \omega_k + \omega_q)|^3} \frac{1}{|A(\omega_k, \omega_q)(1 - e^{\omega_q})|^3} \times \left( \omega_k^2 + \frac{(\tilde{p}e^{\omega_q} + \tilde{q}e^{\omega_k})^2}{a^2 \omega_q^3 (1 - e^{\omega_k})^2} + \frac{m^2 a^2 \omega_k^2}{2(\cosh a\omega_k - 1)} \right)^{-1},
$$

$$
\bar{G}_{c=5}^{(2),d}(p, q) = -\frac{g}{4!}\delta(q + \omega_p) \frac{1}{\mathcal{M}_p + m^2 \mathcal{M}_q + m^2} \frac{1}{1} \int_{-\infty}^{\infty} d\omega_k \frac{1}{L(\omega_k)} \times \frac{a^2 \omega_k^2}{2(\cosh a\omega_k - 1)} \frac{1}{|A(\omega_k + \omega_q, -\omega_k - \omega_q)|^3} \frac{1}{|A(\omega_k, \omega_q)(1 - e^{\omega_q})|^3} \times \left( \omega_k^2 + \frac{(\tilde{p}e^{\omega_q} + \tilde{q}e^{\omega_k})^2}{a^2 \omega_q^3 (1 - e^{\omega_k})^2} + \frac{m^2 a^2 \omega_k^2}{2(\cosh a\omega_k - 1)} \right)^{-1},
$$

$$
\bar{G}_{c=7}^{(2),d}(p, q) = -\frac{g}{4!}\delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2 \mathcal{M}_q + m^2} \frac{1}{1} \frac{1}{1} \int_{-\infty}^{\infty} d\omega_k \frac{4\pi}{L(\omega_k)} \frac{1}{2(\cosh a\omega_k - 1)} \times \left( \Lambda - \omega_k \sqrt{1 + \frac{m^2 a^2}{2(\cosh a\omega_k - 1)}} \right),
$$

$$
\bar{G}_{c=9}^{(2),d}(p, q) = -\frac{g}{4!}\delta^{(4)}(p + q) \frac{1}{\mathcal{M}_p + m^2 \mathcal{M}_q + m^2} \frac{1}{1} \frac{1}{1} \int_{-\infty}^{\infty} d\omega_k \frac{4\pi}{L(\omega_k)} \frac{1}{2(\cosh a\omega_k - 1)}
$$
Due to the different arrangement of fields in the scalar product, planar and non-planar graphs have changed their "position" (with respect to the numbering in \( c \)).

**Remark**

The possible interaction term \((\phi * \phi * \phi, \phi)\) leads to a somewhat different behaviour. The contraction \( c = 1 \), for example, is proportional to

\[
\int d^4k \frac{1}{L(\omega_k)} \frac{1}{M_k + m^2} \left( \frac{-a \omega_k}{e^{-a \omega_k} - 1} \right)^3 \delta^{(3)} \left( \vec{k} - e^{a \omega_k} A(0, \omega_k) A(\omega_q, -\omega_q) (\vec{q} + \vec{p}) \right).
\]

In the limit \( a \to 0 \), the integral in Eq. (43) reduces to

\[
\int d\omega_k \frac{1}{\omega_k^2 + (\vec{p} + \vec{q})^2 + m^2} = \frac{\pi}{\sqrt{(\vec{p} + \vec{q})^2 + m^2}},
\]

contrary to the examples above where the \( \delta^{(3)} \)-distribution did not depend on \( \vec{k} \) and therefore did not act as a regulator.

The contribution \( c = 7 \), in order to give another example, is of the form

\[
\delta(\omega_p + \omega_q) \delta^{(3)}(\vec{p}) \int d^4k \frac{1}{L(\omega_k)} \frac{1}{M_k + m^2},
\]

which is independent of \( q \) and therefore peculiar.
4 Conclusions

Using a generating functional approach, we have deduced the Feynman rules for scalar $\phi^4$ theory on $\kappa$-deformed Euclidean space. We have calculated the tadpole contributions explicitly. As in the canonically deformed theory, we can distinguish between planar and non-planar diagrams. The planar diagrams (28 - 31) and (37, 40 - 42), respectively display a linear UV divergence. The non-planar graphs (33, 34) and (38, 39), respectively are finite for generic external momenta $p, q$. In the exceptional case $\omega_p = \omega_q = 0$, however, the amplitudes also diverge linearly in the UV cut-off $\Lambda$. This is the form of appearance of UV/IR mixing on $\kappa$-deformed spaces. Considering $\kappa$-Minkowski space-time, UV/IR mixing is also expected to show up in a similar way.

So far, we have only discussed the massive case. For the massless case $m = 0$, the divergencies have a richer structure. The planar diagrams also show linear divergences in the cut-off $\Lambda$. The $\omega_k$-integration is also finite. The integrand in the massive case is peaked at $\omega_k = 0$, whereas in the massless case it vanishes there. There are two peaks, one below and one above $\omega_k = 0$. This behaviour displays similarities to a phase transition. In the non-planar case, the generic contribution is again finite. For $\omega_q = 0$, the amplitude diverges as described in the massive case. But there is an additional exceptional configuration, namely $\vec{q} = -\vec{p}$ and $\omega_q \neq 0$. In this case, the divergence structure of the integrand of the $\omega_k$-integration (eg. (33)) has to be studied in more detail.

The basic difference to the case of canonical deformation [1] is the appearance of $x$-dependent phase factors in Eq. (12). Similar $x$-dependent phase factors already occurred in [19], where the UV/IR mixing has been discussed for two other Lie algebra deformations of space-time. But there, no generalised symmetry is present. The star products $*_{RS}$ have an especially simple form,

$$ f *_{RS} g(x) = f(x) \cdot g(x) + \text{total divergence}. $$

Using the usual integration yields unmodified propagators. The only modifications are in the interaction part of the action. They obtain quadratic divergencies for the planar contributions. On the contrary, in our case the necessary modifications of the propagator and of the free action change the divergence of the planar graphs (and of the non-planar ones for exceptional momenta).
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