Abstract. We present an elementary computational scheme for the moduli spaces of rational pseudo-holomorphic curves in the symplectizations of lens spaces, which are equipped with Morse-Bott contact forms induced by the standard Morse-Bott contact form on $S^3$. As an application, we prove the classical classification statement for lens spaces in the contact category by means of holomorphic curve techniques instead of classical topological tools. Namely, we show that for $p$ prime and $1 < q, q' < p - 1$, if there is a contactomorphism between lens spaces $L(p, q)$ and $L(p, q')$, where both spaces are equipped with their standard contact structures, then $q \equiv (q') \pm 1 \mod p$. For the proof we study the moduli spaces of pair of pants with two non-contractible ends in detail and establish that the standard almost complex structure that is used is regular. Then the existence of a contactomorphism enables us to follow a neck-stretching process, by means of which we compare the homotopy relations encoded at the non-contractible ends of the pair of pants in the symplectizations of $L(p, q)$ and $L(p, q')$.

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1. Introduction

The objects of the study in this paper are 3-dimensional lens spaces and the standard Reeb dynamics on them. These dynamical systems are given as finite quotients of the Hopf fibration on $S^3$. To be more precise, we consider the standard contact form

\[ \alpha_0(z)[w] = \text{Im}(w, z)_\mathbb{C} \]

where $z = (z_1, z_2) \in S^3 \hookrightarrow \mathbb{C}^2$, $w = (w_1, w_2) \in T_z S^3 \subset T_z \mathbb{C}^2 \cong \mathbb{C}^2$ and $(\cdot, \cdot)_\mathbb{C}$ is the standard hermitian product. Then the standard contact structure $\xi_0$ at $z \in S^3$ is given by the hermitian complement of the complex line along $z \in \mathbb{C}^2$. The Reeb vector field and its flow are given by

\[ R_0(z_1, z_2) = (iz_1, iz_2), \quad \phi_t(z_1, z_2) = (e^{it}z_1, e^{it}z_2). \]
We see that all Reeb orbits are closed and we have $2\pi$ as common minimal period. The fibration via Reeb orbits leads to the Hopf fibration

\[ \pi : S^3 \to \mathbb{C}P^1, (z_1, z_2) \mapsto (z_1 : z_2). \]

In order to define our objects of study, we consider the $\mathbb{Z}_p$-action on $S^3$ generated by

\[ \sigma(z_1, z_2) = (e^{i\theta}z_1, e^{iq\theta}z_2) \]

where $0 < q < p$ are integers with $(p, q) = 1$ and $\theta = 2\pi/p$. This is a free action and gives rise to the lens space $L(p, q)$ as the quotient space, where

\[ p : S^3 \to L(p, q) \]

is the quotient/covering map. We fix the integer $0 < v < p$ such that

\[ vq \equiv 1. \]

Since $\sigma$ is a complex linear map on $\mathbb{C}^2$, it preserves the standard contact form. Hence we have an induced standard contact form $\alpha$ on $L(p, q)$, given by

\[ p^*\alpha = \alpha_0, \]

which defines the standard contact structure $\xi = \ker \alpha = p_\sharp \xi_0$.

We also note that $p_*R_0 = R$, where $R$ is the Reeb field of $\alpha$.

**Remark 1.1.**

1. In this paper, we restrict ourselves to the case where $1 < q < p-1$. In the remaining case, the Reeb flow still induces an $S^1$-bundle structure and it is relatively less interesting, see [1].

2. We also assume that $p$ is prime. This assumption simplifies the computational aspect of our work but we claim that most of the statements we present this paper may be generalized after some extra care.

3. In what follows, the expressions like $k \equiv l$ and $k \not\equiv l$ should be understood in mod $p$.

The dynamics of the Reeb field $R$ is of Morse-Bott type and is easy to describe since $\sigma$ commutes with the flow of $R_0$. There are two closed orbits of $R_0$ in the cover which are invariant under the group action, namely

\[ \gamma_\infty = \{(z_1, 0) \mid |z_1| = 1\} = \{(e^{it}, 0) \mid t \in [0, 2\pi]\} \]

and

\[ \gamma_0 = \{(0, z_2) \mid |z_2| = 1\} = \{(0, e^{it}) \mid t \in [0, 2\pi]\}. \]

These orbits are mapped to the points $\infty$ and $0$ in $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$ respectively via $\tau_\infty$ and $\tau_0$. These simple orbits have period $2\pi/p$. We note that the simple orbits of $R_0$ besides $\gamma_0/\infty$ are permuted by $\sigma$. In fact we have an induced action on the orbit space

\[ \sigma_\infty : \mathbb{C}P^1 \to \mathbb{C}P^1, (z_1 : z_2) \mapsto (e^{i\theta}z_1 : e^{iq\theta}z_2), \]

which gives an orbifold structure to the orbit spaces of $R$ via the quotient map

\[ p_\infty : \mathbb{C}P^1 \to \mathbb{C}P^1. \]

The fixed points 0 and $\infty$ correspond to the (iterations of) orbits $\tau_0$ and $\tau_\infty$ respectively. The isotropy group of both singularities is $\mathbb{Z}_p$.

Our aim is to study punctured holomorphic curves in the symplectizations of lens spaces. As pointed above, the standart Reeb flow is perfectly symmetric on $S^3$ and if one chooses an almost complex structure on $\xi_0$, which is invariant under the
Reeb flow, the resulting SFT-type almost complex structure on $\mathbb{R} \times S^3$ leads to a description of the moduli space of punctured curves in terms of the closed curves in the orbit space $\mathbb{C}P^1$ paired with meromorphic sections above them since $\mathbb{R} \times S^3$ may be viewed as a complex (in fact holomorphic in this case) line bundle without its zero section. This is a particular example of a phenomenon observed for the prequantization bundles [1].

If such a symmetric almost complex structure on $\mathbb{R} \times S^3$ is also invariant under then $\mathbb{Z}_p$-action, then it descends to $\mathbb{R} \times L(p, q)$ and one may study punctured curves in $\mathbb{R} \times L(p, q)$ in terms of their lifts to $\mathbb{R} \times S^3$ explicitly. This idea was proposed in [7] and executed in detail in [15] for lens spaces and their unit cotangent bundles. It turns out that it is easy to determine whether a given moduli space of rational curves is non-empty and has the correct dimension. We present a systematic treatment of this idea in the next section. For an application of the resulting computational scheme, we prove the following.

**Theorem 1.1.** Let $p$ be a prime number and $1 < q, q' < p - 1$. Let $\alpha$ and $\alpha'$ be the standard contact forms on $L(p, q)$ and $L(p, q')$ respectively. Suppose that there is a contactomorphism

$$\varphi : (L(p, q), \xi = \ker \alpha) \to (L(p, q'), \xi' = \ker \alpha').$$

Then $q \equiv (q')^{\pm 1} \mod p$.

For the proof we study the moduli spaces of pair of pants with non-contractible positive ends in detail. We show that non-empty components of the moduli spaces of pair of pants are cut out transversally by comparing the Fredholm index of the pair of pants with the dimension of the equivariant perturbations of their lifts. This enables us to perturb the data and perform a standard neck-stretching argument after a given contactomorphism. Noting the homotopy/homology relation $[\gamma_\infty] = [q\gamma_0]$, the result follows from the comparison of the homotopy relation that is encoded by the positive ends of certain pair of pants in $\mathbb{R} \times L(p, q)$ with the corresponding relation encoded by the pair of pants in $\mathbb{R} \times L(p, q')$, which is obtained after stretching the neck, see Figure 4.

We note that the statement of Theorem 1.1 is in fact straightforward, in the sense that two lens spaces $L(p, q)$ and $L(p, q')$ are simple homotopy equivalent/homeomorphic/diffeomorphic if and only if $q \equiv \pm (q')^{\pm 1}$. The non-trivial part, namely the only if part, of this classical statement can be proven by means of classical topological tools like Reidemeister torsion [3]. The idea behind Theorem 1.1 is to provide a symplectic/contact topological proof, which is motivated by [7] and studied in [15] along with the case of the unit cotangent bundles of lens spaces.

We note that if we orient lens spaces via the standard contact forms, then due to the dimensional reasons any contactomorphism is orientation preserving, independent of being positive or negative. In that sense, it is not surprising that we recover only the half of the relation $q \equiv \pm (q')^{\pm 1}$. We also note that it is enough to prove Theorem 1.1 under the assumption of the positivity of the contactomorphism, see Theorem 1.1'. In fact, if a given contactomorphism is negative, then one may compose it with the negative contactomorphism of $L(p, q')$, which is induced by the conjugation $(z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2})$ on $S^3$, and obtain a positive contactomorphism.

The study of holomorphic curves in the symplectizations of lens spaces with respect to the Morse-Bott data is not new. We note that these curves can be studied directly as orbi-curves in the orbit space $\mathbb{C}P^1$ paired with meromorphic sections of orbi-bundles. This leads to a formal relation between the orbifold Gromov-Witten potential of the orbit space $\mathbb{C}P^1$ and the SFT hamiltonian of $L(p, q)$, see [14]. Our approach is rather elementary and serves well since we do not aim to compute the SFT hamiltonian. We just provide a computational scheme and utilizing a small
portion of the SFT hamiltonian, namely pair of pants with positive non-contractible ends, we show that holomorphic curve methods provide a proof of the classification statement in the contact category.

Another work that aligns with our paper is presented in [4]. In [4] it is proven that if there is a symplectic cobordism between two lens spaces, which are equipped with the standard contact structures, then the cobordism is necessarily trivial. This statement is an application of the adjunction formula for the orbi-curves in 4-dimensional symplectic orbifolds, which is the main result of [4]. Given a symplectic cobordism between two lens spaces, one compactifies both ends of the symplectic cobordism and studies a particular moduli space of curves in the resulting symplectic orbifold and shows that the parameters of the lens spaces satisfies the diffeomorphism classification condition. We note that our result follows from the statement in [4] since a positive contactomorphism leads to a symplectic cobordism immediately. We note that due to the Morse-Bott setting, the punctured curves we use can be thought of closed orbi-curves in suitable singular compactifications of the symplectizations or symplectic cobordisms. Nevertheless our treatment stays in the smooth category. On the other hand, in order to achieve the necessary control on the neck-stretching process, we lift punctured curves to suitable coverings and apply the classical 4-dimensional tools to the extensions of these lifts, which are now smooth curves. In that sense, our approach simplifies the proof in [4] since it avoids singular objects. We also stress that we only use the standard geometric transversality results in our proof.

Acknowledgements. I thank Richard Siefring for sharing his work in preparation, for very helpful discussions and for reading parts of this manuscript. This work is part of a project in the SFB/TRR 191 ‘Symplectic Structures in Geometry, Algebra and Dynamics’, funded by the DFG.

2. Holomorphic curves in the symplectizations of lens spaces: the general scheme

Let \((M, \alpha)\) be a closed, \((2n - 1)\)-dimensional contact manifold and let \(R_\alpha\) be the associated Reeb vector field with the flow \(\phi^t\), which is of Morse-Bott type. Namely, the action spectrum of \(\alpha\) is discrete and for any action value (period) \(T\), \(N_T := \{x \in M \mid \phi^T(x) = x\} \subseteq M\) is a closed submanifold such that the rank of \(d\alpha|_{N_T}\) is locally constant and \(T_x N_T = \ker(\phi^T - \text{id})|_x\) for all \(x \in N_T\). In the case that \(N_T\) consists of a single orbit \(\gamma : \mathbb{R} \to M\), we say that \(\gamma\) is non-degenerate. We note that in this case \(\phi^T(\gamma(0))\) does not admit 1 as an eigenvalue. We note that for any period \(T\), the Reeb flow defines an \(S^1\)-action on \(N_T\) and the resulting orbit space \(S_T = N_T/S^1\) is an orbifold, which consists of a single point if \(N_T\) is geometrically a single orbit.

Given an almost complex structure \(J\) on \(\xi = \ker \alpha\) that is compatible with \(d\alpha\), one extends it to the symplectization \((\mathbb{R} \times M, d(e^{\phi} \lambda))\) in such a way that \(J\) is invariant under translations along \(\partial_\alpha\) and \(J\partial_\alpha = R_\alpha\). We note that if \(\gamma : \mathbb{R} \to M\) is a periodic orbit of \(R_\alpha\) with period \(T\), then the trivial cylinder over \(\gamma\)

\[
\begin{align*}
  u : \mathbb{R} \times S^1 &\to \mathbb{R} \times M, \\
  (s, t) &\mapsto (Ts, \gamma(Tt))
\end{align*}
\]

is a \(J\)- holomorphic map, where \(\mathbb{R} \times S^1 := \mathbb{C}/\mathbb{Z}\).

Let \((\Sigma, j)\) be a closed riemann surface and let \(\Gamma \subset \Sigma\) be a finite set of punctures. We consider the maps that satisfy the non-linear Cauchy-Riemann equation, namely

\[
  u : \Sigma \setminus \Gamma \to \mathbb{R} \times M, \\
  du \circ j = J \circ du,
\]
with finite \textit{Hofer energy}, that is

\[ E(u) := \sup_{f \in \mathcal{F}} \int_{\Sigma \setminus \Gamma} u^* d(e^f \alpha) < \infty, \]

where \( \mathcal{F} = \{ f : \mathbb{R} \to (-1, 1) \mid f' > 0 \} \). It turns out that under the Morse-Bott assumption, such a \textit{holomorphic curve} \( u \) with finite Hofer energy converges to a trivial cylinder near a puncture unless its image is bounded near that puncture and in the latter case \( u \) extends holomorphically over a puncture \[1\]. More precisely, for each honest puncture \( z \in \Gamma \), one fixes a holomorphic coordinate chart indentified with an open disk \( \mathbb{D} \) centered at \( z \). The puncture set splits as \( \Gamma = \Gamma^+ \cup \Gamma^- \) and one fixes the following cylindrical coordinates.

- \( Z_+ := [0, +\infty) \times S^1 \to \mathbb{D} \setminus \{0\} \subset \Sigma \setminus \{z\}, (s, t) \mapsto e^{-2\pi(s+it)} \) if \( z \in \Gamma^+ \).
- \( Z_- := (-\infty, 0] \times S^1 \to \mathbb{D} \setminus \{0\} \subset \Sigma \setminus \{z\}, (s, t) \mapsto e^{2\pi(s+it)} \) if \( z \in \Gamma^- \).

Then for every \( z \in \Gamma^\pm \), there is a \( T \)-periodic Reeb orbit \( \gamma \) such that

\[ u(s, t) = \exp_{(T_s, T\gamma)} h(s, t), (s, t) \in Z_\pm \]

for \( |z| \) large, where \( h \) is a vector field along the trivial cylinder \[13\] such that \( h(s,.) \to 0 \) uniformly as \( |s| \to \infty \) and \( \exp \) is the exponential map defined via an \( \mathbb{R} \)-invariant metric on \( \mathbb{R} \times M \). We say that

- \( z \) is a \textit{positive puncture} if \( z \in \Gamma^+ \). We write \( u(z) = (+\infty, \gamma) \) if the the asymptotic end is a non-degenerate orbit \( \gamma \) and \( u(z) \in \{+\infty\} \times S_T \) if the asymptotic end belongs to a non-trivial orbit space \( S_T \).
- \( z \) is a \textit{negative puncture} if \( z \in \Gamma^- \). We write \( u(z) = (-\infty, \gamma) \) if the the asymptotic end is a non-degenerate orbit \( \gamma \) and \( u(z) \in \{-\infty\} \times S_T \) if the asymptotic end belongs to a non-trivial orbit space \( S_T \).

We want to study the moduli space of holomorphic curves for a fixed asymptotic data. We pick a collection of orbit spaces \( S_{1}^{\pm}, ..., S_{n}^{\pm} \) and let

\[ (\Sigma, j, \Gamma, u) | u(z_i^\pm) \in \{\pm \infty\} \times S_{j_i}^{\pm} \text{ for all } z_i^\pm \in \Gamma^\pm \} / \sim \]

be the moduli space of equivalence classes \( [\Sigma, j, \Gamma, u] \) of holomorphic curves, where \( (\Sigma, j, \Gamma, u) \sim (\Sigma', j', \Gamma', u') \) if there exists a biholomorphism \( h : \Sigma \to \Sigma' \) such that \( h \) restricts to a sign and ordering preserving bijection on the corresponding puncture sets and \( u = u' \circ h \).

\textbf{Remark 2.1.} In what follows, we will be mostly interested in moduli spaces of rational curves with at least three punctures. We note that due to the uniqueness of the complex structure on \( \mathbb{C}P^1 \), when \( \Sigma = \mathbb{C}P^1 \) and \( |\Gamma| = 3 + k, k \geq 0 \), the above moduli space \( M \) is identified with the space of pairs \( (u, (z_1, ..., z_k)) \) where

\[ u : \mathbb{C}P^1 \setminus \{0, 1, \infty, z_1, ..., z_k\} \to \mathbb{R} \times M \]

is a holomorphic curve with prescribed sign of punctures and asymptotic ends.

In general it is hard to carry a hand on study of moduli spaces of holomorphic curves. Under nice circumstances, a generic choice of \( J \) on \( \xi \) leads to a smooth structure on the moduli space. But such a generic choice makes hard to grasp the moduli space itself even in the Morse-Bott case. But in certain perfectly symmetric Morse-Bott situations like pre-quantization bundles, these moduli spaces can be described as rather elementary objects \[1\].

The aim of this section is to study holomorphic curves in \( \mathbb{R} \times L(p, q) \), where the setting is a finite quotient of the perfectly Morse-Bott case \( \mathbb{R} \times S^3 \). Hence our strategy is to study curves in \( \mathbb{R} \times L(p, q) \) through their lifts to \( \mathbb{R} \times S^3 \). We want to describe the solution spaces of the Cauchy-Riemann equation on \( \mathbb{R} \times L(p, q) \) as the subspaces of the solution spaces of the lifted problems, which consist of equivariant solutions. To this end we need first to understand the setting of \( \mathbb{R} \times S^3 \).
With real coordinates $z_j = x_j + iy_j$, \((1)\) reads as
\[
\alpha_0 = -y_1 dx_1 + x_1 dy_1 - y_2 dx_2 + x_2 dy_2.
\]
The symplectic form $d\alpha_0$ on $\xi_0$ given by
\[
d\alpha_0 = 2(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)|_{\xi_0}.
\]
The principal $S^1$-bundle given by \((3)\) is in fact the $S^1$-bundle associated to the tautological line bundle
\[
L \rightarrow \mathbb{CP}^1, \quad L(z_1, z_2) = \text{span}_\mathbb{C}\{(z_1, z_2)\} \subset \mathbb{C}^2
\]
and the hermitian metric on $L$, which is induced by the standard hermitian metric on $\mathbb{C}^2$. Hence the Euler clas of \((3)\) is given by
\[
e(\pi) = c_1(L) = -[\pi^{-1}\omega_{FS}],
\]
where $\omega_{FS}$ is the Fubini-Study form on $\mathbb{CP}^1$ s.t. $<\omega_{FS}, [\mathbb{CP}^1]> = \pi$ and
\[
\pi^*\omega_{FS} = (dx_1 \wedge dy_1 + dx_2 \wedge dy_2)|_{\mathbb{S}^3}.
\]
Since the $S^1$-action is generated by the Reeb field $R_0$ and the period is $2\pi$, being the associated contact form, $\alpha_0$ satisfies $L_{R_0}\alpha_0 = 0$ and $\alpha_0(R_0) = 1$. So $\alpha_0$ is a connection 1-form and from the equality above we have $\pi^*\omega_{FS} = \frac{1}{2}d\alpha_0$, that is $2\omega_{FS}$ is the curvature form.

Consider the symplectization $\mathbb{R} \times S^3$. We pull the standard complex structure on $\mathbb{CP}^1$ back to the contact distribution as an $S^1$-invariant complex structure and we extend it to the symplectization as described above. We call this almost complex structure as the standard almost complex structure and denote it by $J_0$. We note that the extension $\pi : \mathbb{R} \times S^3 \rightarrow \mathbb{CP}^1$ of \((3)\) is $J_0$-holomorphic. We consider the diffeomorphism
\[
\Phi : \mathbb{R} \times S^3 \rightarrow L^*, \quad (u, (z_1, z_2)) \mapsto e^a(z_1, z_2)
\]
where $L^*$ is the total space of the line bundle without the zero section. We note that once conjugated by $\Phi$, $J_0$ coincides the complex structure on the fibres of $L$.

Since $\Phi$ also covers holomorphic bundle projections on both its domain and target, it is a biholomorphism.

For any punctured curve $u : \Sigma \setminus \Gamma \rightarrow \mathbb{R} \times S^3$ with finite energy, $c = \pi \circ u$ extends over the punctures and gives a closed holomorphic curve in $\mathbb{CP}^1$. The map $u$ then corresponds to a meromorphic section $f$ of the holomorphic line bundle $c^*L \rightarrow \Sigma$, see \((11)\). The positive ends of $u$ correspond to the poles of $f$ and negative ends of $u$ corresponds to the zeros of $f$ since the complex structure on the symplectization fits to the complex structure on the fibres of $L$.

Note that if $\Sigma = \mathbb{CP}^1$ then the first Chern number of the bundle $c^*(L)$ is given by $-d$ where $d \geq 0$ is the degree of the map $c$. Then the necessary and sufficient condition for the existence of a meromorphic section $f$ is both the divisor of the section and the bundle to have the same degree, namely
\[
\#f^{-1}(0) - \#f^{-1}(\infty) = -d.
\]

We note that the above formula says that there is no punctured curve with only negative ends, which is consistent with the maximum principle.

We now want to add the $\mathbb{Z}_p$ action into the setting. This action is free on $L^*$ but has two fixed points on the zero section $\mathbb{CP}^1$, see \((11)\). Note that the standard almost complex structure $J_0$ on $\mathbb{R} \times S^3$ is $\mathbb{Z}_p$-invariant so we identify $\mathbb{R} \times L(p, q)$ with $\overline{L} := L^*/\mathbb{Z}_p$ where the former space is equipped with the quotient almost complex structure denoted by $J_0$. We call $J_0$ as the standard almost complex structure on $\mathbb{R} \times L(p, q)$. 

We consider a rational $J_\alpha$-holomorphic curve
\[ \pi : \mathbb{C}P^1 \setminus \Gamma \to \mathbb{R} \times L(p, q) \cong \mathcal{T}^* . \]
Such a curve lifts to the cover $L^*$ if and only if $u^* = 0$ on $\pi_1(\mathbb{C}P^1 \setminus \Gamma)$. We note that $\mathbb{C}P^1 \setminus \Gamma$ is homotopy equivalent to the bouquet of $(\# \Gamma - 1)$-many circles so its fundamental group is the free group with $(\# \Gamma - 1)$ generators. Hence the image of the fundamental group is trivial if and only if the image of each generator is trivial, that is, if all of the asymptotics are contractible. In this paper, our main concern is about curves with non-contractible ends. But since a curve $u$ with a non-contractible end does not lift to the cover immediately, we need to precompose the map $\pi$ with a suitable covering map
\[ p : \Sigma \setminus \tilde{\Gamma} \to \mathbb{C}P^1 \setminus \Gamma. \]

Once we pick a suitable cover [17], we have the commutative diagram given by Figure 1. The problem is then to study the lifted $J_0$-holomorphic curves $u : \Sigma \setminus \tilde{\Gamma} \to \mathbb{R} \times S^3 \cong L^*$ which are equivariant with respect to the action of the group of Deck transformations $G$ on the punctured surface $\Sigma \setminus \tilde{\Gamma}$ and $\mathbb{Z}_p$-action on $L^*$.

2.1. Equivariant curves: the necessary conditions for the existence. In this section, our aim is to determine a suitable minimal covering [17] and establish a correspondence between a given moduli problem in $\mathcal{T}^*$ and a lifted moduli problem in $L^*$, which is determined by [17].

We fix a sets of punctures $\Gamma \subset \mathbb{C}P^1$, which partitions as follows.

- $\Gamma = \Gamma_{nc} \cup \Gamma_c$ with cardinalities $n_{nc}$ and $n_c$.
- $\Gamma_{nc} = \Gamma_0 \cup \Gamma_\infty$ with cardinalities $n_0$ and $n_\infty$ so that $n_{nc} = n_0 + n_\infty$.
- $\Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$ with cardinalities $n_0^+$ and $n_0^-$ so that $n_0 = n_0^+ + n_0^-.$
- $\Gamma_\infty = \Gamma_\infty^+ \cup \Gamma_\infty^-$ with cardinalities $n_\infty^+$ and $n_\infty^-$ so that $n_\infty = n_\infty^+ + n_\infty^-.$
- $\Gamma_c = \Gamma_c^+ \cup \Gamma_c^-$ with cardinalities $n_c^+$ and $n_c^-$ so that $n_c = n_c^+ + n_c^-.$
- $\Gamma_{0}^{\pm} = \{ z_{0}^{0, \pm}, \ldots, z_{n_{0}^{0, \pm}}^{0, \pm} \}$, $\Gamma_{\infty}^{\pm} = \{ z_{1}^{\infty, \pm}, \ldots, z_{n_{\infty}^{\infty, \pm}}^{\infty, \pm} \}$ and $\Gamma_{c}^{\pm} = \{ w_{1}^{\pm}, \ldots, w_{n_{c}^{\pm}}^{\pm} \}$.

Suppose that we have a holomorphic curve
\[ \pi : \mathbb{C}P^1 \setminus \Gamma \to \mathcal{T}^* \]
with asymptotics
(a1) \( \overline{\nu}(z^0_i, \pm) = (\pm \infty, k^0_i, \pm \tau_i) \) where \( k^0_i \neq 0 \) for \( i = 1, \ldots, n^0_p \), that is \( \overline{\nu} \) has a positive/negative puncture at \( z^0_i, \pm \) with positive/negative non-contractible asymptotic end \( k^0_i, \pm \tau_i \) for \( i = 1, \ldots, n^0_p \).

(a2) \( \overline{\nu}(z^\infty_i, \pm) = (\pm \infty, k^\infty_i, \pm \tau_i) \) where \( k^\infty_i \neq 0 \) for \( i = 1, \ldots, n^\infty_p \), that is \( \overline{\nu} \) has a positive/negative puncture at \( z^\infty_i, \pm \) with positive/negative non-contractible asymptotic end \( k^\infty_i, \pm \tau_i \) for \( i = 1, \ldots, n^\infty_p \).

(a3) \( \overline{\nu}(w^\pm_i) \in \{ \pm \infty \} \times \mathbb{S}^k_i \) for \( i = 1, \ldots, n^\pm_p \), that is \( \overline{\nu} \) has a positive/negative puncture at \( w^\pm_i \) with positive/negative contractible asymptotic end, which lies in the orbit space \( \mathbb{S}^k_i \) of orbits with action \( 2\pi k^\pm_i \).

We fix a point \( z \in \mathbb{C}P^1 \setminus \Gamma \) and consider the map

\[ \pi_* : \pi_1(\mathbb{C}P^1 \setminus \Gamma, z) \to \pi_1(\mathcal{L}, x_{\pi}) \cong \mathbb{Z}_p, \]

where \( x_{\pi} = \overline{\nu}(z) \). Since \( p \) is prime it is clear that \( \pi_* \) is surjective if and only if \( n_{nc} \geq 1 \). We know that \( K := \ker \pi_* \) is a normal subgroup of \( \pi_1(\mathbb{C}P^1 \setminus \Gamma, z) \) and there exists a covering space

\[ p : \Sigma \setminus \tilde{\Gamma} \to \mathbb{C}P^1 \setminus \Gamma, \]

where \( \Sigma \setminus \tilde{\Gamma} \) is smooth punctured surface and \( p_*(\pi_1(\Sigma \setminus \tilde{\Gamma}, \tilde{z})) = K \) where \( \tilde{z} \) is a fixed lift of \( z \). This covering is (up to isomorphism) determined by \( K \) and the group of Deck transformations \( G \) is given by

\[ G \cong \pi_1(\mathbb{C}P^1 \setminus \Gamma, z)/K \cong \mathbb{Z}_p. \]

We endow \( \Sigma \setminus \tilde{\Gamma} \) with pull-back the complex structure and get a punctured Riemann surface so that \( p \) is holomorphic. By the removal of singularities theorem, \( p \) extends to a holomorphic branched covering \( p : \Sigma \to \mathbb{C}P^1 \).

We note that \( \pi \circ p : \Sigma \setminus \tilde{\Gamma} \to \mathcal{L} \) satisfies

\[ (\pi \circ p)_* (\pi_1(\Sigma \setminus \tilde{\Gamma}, \tilde{z})) = \pi_* p_* (\pi_1(\Sigma \setminus \tilde{\Gamma}, \tilde{z})) = \{ 0 \} = p_*(\pi_1(\mathcal{L}, x_{\pi})) \]

where \( x_{\pi} \) is a fixed lift of \( x_{\pi} \). Hence we have the unique lift

\[ u := \pi_{\circ p} : \Sigma \setminus \tilde{\Gamma} \to \mathcal{L}, \quad u(\tilde{z}) = x_{\pi}. \]

We note that \( u \) is equivariant with respect to the action of \( G \) on \( \Sigma \setminus \tilde{\Gamma} \) and the action of \( \mathbb{Z}_p \) on \( L \setminus \{ 0 \} \). We fix a generator \( \tau_\gamma \) of \( G \) such that

\[ u \circ \tau_\gamma = \sigma \circ u. \]

Now we take a closer look at the branched covering \( p : \Sigma \to \mathbb{C}P^1 \). It is clear that \( \tilde{\Gamma} = p^{-1}(\Gamma) \). We put \( \tilde{\Gamma}_{nc} := p^{-1}(\Gamma_{nc}) \) and \( \tilde{\Gamma}_c := p^{-1}(\Gamma_c) \).

**Lemma 2.1.** Each point in \( \Gamma_{nc} \) is a branch point with exactly one preimage and each point in \( \Gamma_c \) has exactly \( p \) preimages. \( \tilde{\Gamma}_{nc} \) forms the fixed point set of the extended action of \( G \) over \( \Sigma \). Moreover, the genus of \( \Sigma \) is given by

\[ g = \frac{(p-1)(n_{nc} - 2)}{2}. \]

**Proof.** For any \( z_j^{i, \pm} \in \Gamma_{nc} \), the cardinality of the set \( p^{-1}(z_j^{i, \pm}) \) is either \( p \) or 1. In fact, any element in \( p^{-1}(z_j^{i, \pm}) \) has a local isotropy group with respect to the extended action of \( G \) on \( \Sigma \), which is a subgroup of \( G \cong \mathbb{Z}_p \). Since \( p \) is prime, this subgroup is either trivial or \( \mathbb{Z}_p \). We note also that at least one point in \( p^{-1}(z_j^{i, \pm}) \) is a puncture of \( u \) with an asymptotic, which descends to a non contractible asymptotic \( \overline{\nu} \) at \( z_j^{i, \pm} \). Hence at least one point in \( p^{-1}(z_j^{i, \pm}) \) has isotropy group \( \mathbb{Z}_p \) and therefore \( p^{-1}(z_j^{i, \pm}) \) consists of a single point. We conclude that \( p \) branches points over \( \Gamma_{nc} \), so that each branch point has a single preimage with ramification number \( p \). Similarly,
for each \( w_j^\pm \in \Gamma_c \) there is a contractible end of \( \pi \), which lifts to \( p \)-many contractible ends of \( u \). Hence the punctures corresponding to these ends are precisely the preimages of \( w_j^\pm \). Applying Hurwicz formula, we get

\[
2 - 2g = 2p - n_{nc}(p - 1) \Rightarrow g = \frac{(p - 1)(n_{nc} - 2)}{2}.
\]

It is clear that \( \tilde{\Gamma}_{nc} \) the fixed point set of the extended \( G \)-action. \( \square \)

We abuse the notation and call preimages of \( z_i^0, z_i^\pm, z_i^\infty \) under \( p \) by the same letters. For the preimages of \( w_i^\pm \), which is given by \( G = \langle \tau \rangle \)-orbit of any point in the preimage, we write \( w_{i,j}^\pm \), \( j = 1, \ldots, p \). Now we have an equivariant curve

\[
(23) \quad u : \Sigma \setminus \tilde{\Gamma} \to L^*
\]

with asymptotics

- (la1) \( u(z_i^{0,\pm}) = (\pm \infty, k_i^0, \gamma_0) \) for \( i = 1, \ldots, n_i^0 \),
- (la2) \( u(z_i^{\infty,\pm}) = (\pm \infty, k_i^\infty, \gamma_0) \) for \( i = 1, \ldots, n_i^\infty \),
- (la3) \( u(w_{i,j}^\pm) \in \{\pm \infty\} \times S_{k_i^\pm} \) for \( i = 1, \ldots, n_i^\pm, j = 1, \ldots, p \),

where now \( S_{k_i^\pm} \) denotes the space of orbits in \( S^3 \) with the action \( 2\pi k_i^\pm \).

**Figure 2.** An example of a curve in \( \mathbb{R} \times L(p,q) \) and its lift to \( \mathbb{R} \times S^3 \).

**Remark 2.2.** We note the lift \( u \) depends on the choice of the lift \( \tilde{\pi} \). In fact, imposing that the lift maps \( \tilde{\pi} \) to \( \sigma^k \tilde{\pi} \), leads to the unique lift, say \( u' \), which satisfies \( u' = u \circ \tau_k^b \). More precisely, there are \( p \) lifts of \( \pi \), which are distinct as maps, given by \( u \circ \tau_k^b, k = 0, \ldots, p - 1 \) or alternatively by \( \sigma^k \circ u, k = 0, \ldots, p - 1 \). We note that any
such lift \( u' \) satisfies \( u' \circ \tau = \sigma \circ u' \). In particular the image of any lift is invariant under \( \sigma \) and the images of all lifts coincide.

Now we want to understand to what extent the above description depends on \( \pi \).

**Remark 2.3.** By (22) the topology of \( \Sigma \) depends only on the number of the non-contractible ends of \( \pi \).

Since \( \pi_1(\mathbb{C}P^1, z) \) is abelian the homomorphism \( \pi_* \) is invariant under conjugations and therefore depends only on non-contractible asymptotic ends of \( \pi \). In fact, given any representation of \( \pi_1(\mathbb{C}P^1, z) \) via loops, each being the concatenation of a small simple loop around a puncture and a path joining a point on it to the base point \( z \), image of a generator (associated to a puncture) under \( \pi_* \) is equal to the image of the generator (associated to the same puncture) represented by the small simple loop itself, which comes from a representation where the base point \( z \) is taken on that very same loop. Hence \( \pi_* \) and therefore \( K \) is determined up to conjugacy by the data \([a1]\) and \([a2]\). In particular, if \( \pi \) is contained in a continuous family of parametrized curves, one can fix the same subgroup \( K \) for the whole family. We also note that the generator \( \tau \) of \( G \) can be chosen constantly for such a family of curves.

**Remark 2.4.** We note that the set of punctures \( \Gamma \) is fixed so far but in general, one needs to let punctures move. Although the covering \( p \) topologically depends only on \( n_{nc} \), the complex structure on \( \Sigma \) does depend on the positions of the punctures in \( \mathbb{C}P^1 \).

Now we want to understand the picture for the base curves. We fix a lift \( u \) given by (20) and get a holomorphic map \( c := \pi \circ u : \Sigma \setminus \tilde{\Gamma} \rightarrow \mathbb{C}P^1 \), see Figure 1. After extending \( c \) over \( \tilde{\Gamma} \), we have a closed curve

\[
c : \Sigma \rightarrow \mathbb{C}P^1 \text{ s.t. } c \circ \tau = \sigma \circ c
\]

in the view of (11) and (21). We call such a curve \( c \) as a *lifted base curve*. Since \( c \) is equivariant, one can pull back the action on the total space of \( L \rightarrow \mathbb{C}P^1 \) uniquely to the total spaces of \( c^* L \rightarrow \Sigma \) in such a way that the bundle map \( c^* L \rightarrow L \), which covers the map \( c \), is equivariant. We fix a generator \( \tau \) of this \( \mathbb{Z}_2 \)-action on \( c^* L \) so that \( u \) correspond to a meromorphic section \( u : \Sigma \rightarrow c^* L \) such that

\[
u \circ \tau = \tau \circ u.
\]

Moreover, the section \( u \) has poles/zeros at \( z_i^{0,\pm}, z_i^{\infty,\pm}, w_i^{\pm} \)'s. We note that

\[
c(z_i^{0,\pm}) = (0 : 1) = 0 \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\},
\]
\[
c(z_i^{\infty,\pm}) = (1 : 0) = \infty \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}
\]

for all \( i \), which explains the notation for \( z_i^{0,\infty,\pm} \).

We want to understand the consequences of (25) in terms of the multiplicities \( k_i^{0,\infty,\pm} \). To this end, we need to understand the \( \langle \tau \rangle \)-action on \( c^* L \) around the fixed points of \( \langle \tau \rangle \)-action on \( \Sigma \). We note that since we do not know much about the global behaviour of the map \( c \), it is hard to describe \( \langle \tau \rangle \)-action globally. But since \( c \) behaves as a monomial around any point, which is determined by the ramification number at that point, a local description is still easy to get.

We first fix local trivializations of \( L \) and compute the \( \langle \sigma \rangle \)-action on these trivializations.
• Near $0 = (0 : 1)$, we fix the local trivialization

$$(26) \quad \tilde{\phi}_0 : \mathbb{C} \times \mathbb{C} \to L, (z, \lambda) \mapsto \left(\lambda \frac{z}{\sqrt{1+|z|^2}}, \lambda \frac{1}{\sqrt{1+|z|^2}} \right)$$

which covers the chart $\phi_0 : \mathbb{C} \to \mathbb{CP}^1, z \mapsto (z : 1)$. Hence $\sigma$ reads as

$$(27) \quad \sigma_0(z, \lambda) := (\tilde{\phi}_0^{-1} \sigma \tilde{\phi}_0)(z, \lambda) = (e^{i(1-q)\theta} z, e^{i\theta} \lambda).$$

• Near $\infty = (1 : 0)$, we fix the local trivialization

$$(28) \quad \tilde{\phi}_\infty : \mathbb{C} \times \mathbb{C} \to L, (z, \lambda) \mapsto \left(\lambda \frac{1}{\sqrt{1+|z|^2}}, \lambda \frac{z}{\sqrt{1+|z|^2}} \right)$$

which covers the chart $\phi_\infty : \mathbb{C} \to \mathbb{CP}^1, z \mapsto (1 : z)$. Hence $\sigma$ reads as

$$(29) \quad \sigma_\infty(z, \lambda) := (\tilde{\phi}_\infty^{-1} \sigma \tilde{\phi}_\infty)(z, \lambda) = (e^{i(q-1)\theta} z, e^{i\theta} \lambda).$$

We recall that $\tilde{\Gamma}_{\text{nc}}$ is the fixed point set of the $G$ action on $\Sigma$. So using the above trivializations, we may determine the local behavior of an equivariant meromorphic section of $c^* L$ near $\Gamma_{\text{nc}}$.

• Around $z_j^{0, \pm}$: We take holomorphic coordinates on $\Sigma$ centered at $z_j^{0, \pm}$ so that $c(z) = z^r$ for some $r$, and we trivialize $c^* L$ above this coordinate neighbourhood using (26). Locally, $\tau_r = e^{im\theta}$ for some $0 < m < p$. By (24) and (27), we have

$$c(\tau_r(z)) = \sigma c(z) \Rightarrow e^{imr\theta} z^r = e^{i(1-q)\theta} z^r \Rightarrow m \equiv (1-q)r^{-1}. $$

Hence by (27), $\tau$ reads locally as

$$\tau(z, \lambda) = (e^{i\theta} z, e^{iq\theta} \lambda).$$

For a local section $z \mapsto (z, f(z))$ of $c^* L$ to be equivariant, it has to satisfy

$$f(e^{im\theta} z) = e^{i\theta} f(z).$$

Writing the meromorphic section $f(z) = \sum a_k z^k$ around $z_j^{0, \pm}$, we get

$$\sum_k a_k e^{imk\theta} z^k = \sum_k a_k e^{iqk\theta} z^k$$

and we note that

$$(30) \quad a_k \neq 0 \Leftrightarrow mk \equiv q \Leftrightarrow k \equiv qm^{-1} \Leftrightarrow k \equiv rq(1-q)^{-1}.$$ 

• Around $z_j^{\infty, \pm}$: Similarly, we have $c(z) = z^r$ for some $r$ and we put $\tau_r = e^{im\theta}$ for some $0 < m < p$. By the equivariance of $c$ and (29), we have

$$c(\tau_r(z)) = \sigma c(z) \Rightarrow e^{imr\theta} z^r = e^{i(q-1)\theta} z^r \Rightarrow m \equiv (q-1)r^{-1}$$

and locally

$$\tau(z, \lambda) = (e^{i\theta} z, e^{i\theta} \lambda).$$

Hence for an equivariant meromorphic section $f = \sum a_k z^k$, one gets

$$a_k \neq 0 \Leftrightarrow mk \equiv 1 \Leftrightarrow k \equiv m^{-1} \Leftrightarrow k \equiv r(q-1)^{-1}.$$ 

The above observations lead to the following lemma.

**Lemma 2.2.** Let $u$ be a lift of $\pi$ and let $c : \Sigma \to \mathbb{CP}^1$ be the corresponding lifted base curve. Let $r_i^{0/\infty, \pm}$ denote the local degree of $c$ at $z_i^{0/\infty, \pm} \in \tilde{\Gamma}_{\text{nc}}$ and $m_i^{0/\infty, \pm}$ denote the the local representative of $\tau_r$ at $z_i^{0/\infty, \pm}$, that is $\tau_r = e^{im_i^{0/\infty, \pm}} z_i^{0/\infty, \pm}$. Then we have the following relations.
(1) for the positive punctures $z_i^{0/\infty, +}$, we have
\[ m_i^{0, +} \equiv (1 - q)(r_i^{0, +})^{-1} \quad \text{and} \quad k_i^{0, +} \equiv r_i^{0, +}(1 - v)^{-1}, \]
\[ m_i^{\infty, +} \equiv (q - 1)(r_i^{\infty, +})^{-1} \quad \text{and} \quad k_i^{\infty, +} \equiv r_i^{\infty, +}(1 - q)^{-1}. \]

(2) for the negative punctures $z_i^{0/\infty, -}$, we have
\[ m_i^{0, -} \equiv (1 - q)(r_i^{0, -})^{-1} \quad \text{and} \quad k_i^{0, -} \equiv r_i^{0, -}(v - 1)^{-1}, \]
\[ m_i^{\infty, -} \equiv (q - 1)(r_i^{\infty, -})^{-1} \quad \text{and} \quad k_i^{\infty, -} \equiv r_i^{\infty, -}(q - 1)^{-1}. \]

Proof. As a meromorphic section of $c^*L$, $u$ satisfies $u \circ \tau = \tau \circ u$. We write $u$ locally as $(z, f(z))$ so that $f$ has poles at $z_i^{0, +}/z_i^{\infty, +}$ of order $k_i^{0, +}/k_i^{\infty, +}$ and has zeros at $z_i^{0, -}/z_i^{\infty, -}$ of order $k_i^{0, -}/k_i^{\infty, -}$. Hence at any $z_i^{0, +}$ or $z_i^{0, -}$, the Laurent expansions terminates at degrees $-k_i^{0, +}$ and $-k_i^{0, -}$ respectively, while at any $z_i^{\infty, +}$ and $z_i^{\infty, -}$ at degrees $k_i^{\infty, +}$ and $-k_i^{\infty, -}$ respectively. Making these adjustments in above local descriptions, we get the required relations. □

Remark 2.5. We note that the relations in the above lemma, depends only on the multiplicities of non-contractible ends of $\pi$. Hence given a moduli space of curves (parameterized or unparameterized) with fixed non-contractible ends, the multiplicities of these ends determine, in mod $p$, local degrees of lifted base curves at fixed points of $\tau$, on $\Sigma$ and the behaviour of $\tau$ around these fixed points.

We point out two immediate necessary conditions for the existence of $\pi$ in terms of the multiplicities of its asymptotic ends.

Lemma 2.3. Given $\pi$ as with asymptotics $[\text{a1}]$ $[\text{a2}]$ and $[\text{a3}]$, one has
\[ -d = \sum_{i=1}^{n_0} k_i^{0, +} + \sum_{i=1}^{n_\infty} k_i^{\infty, +} + p \sum_{i=1}^{n_0} k_i^{0, -} + \sum_{i=1}^{n_\infty} k_i^{\infty, -} - p \sum_{i=1}^{n_\infty} k_i^{\infty, +} - p \sum_{i=1}^{n_\infty} k_i^{\infty, +} \equiv 0, \]
where $d$ is the degree of $c$.

Proof. Let $u$ be a lift of $\pi$ as in (20). Then it has the asymptotics $[\text{a1}]$ $[\text{a3}]$ Viewing $u$ as a meromorphic section of the bundle $c^*L$ leads to the desired equation since the left hand side of the is the degree of the bundle $c^*L$ and the right hand side is the degree of the divisor of the section $u$. □

Lemma 2.4. Given $\pi$ as with asymptotics $[\text{a1}]$ $[\text{a2}]$ and $[\text{a3}]$, one has
\[ \sum_{i=1}^{n_0} k_i^{0, -} + q \sum_{i=1}^{n_\infty} k_i^{\infty, -} - \sum_{i=1}^{n_0} k_i^{0, +} + q \sum_{i=1}^{n_\infty} k_i^{\infty, +} - p \sum_{i=1}^{n_\infty} k_i^{\infty, +} + q \sum_{i=1}^{n_\infty} k_i^{\infty, +} \equiv 0. \]

Proof. Since $\pi$ is a rational curve, the sum of the homotopy classes of positive ends is equal to the sum of the homotopy classes of negative ends. Writing all homotopy classes in terms of $[\tau_0]$, leads to the above equation since $[T_\infty] = q[\tau_0]$ in $\pi_1$. □

We now have a closer look at the singular base curve
\[ \tau := \pi \circ \pi : \mathbb{C}P^1 \setminus \Gamma \to \mathbb{C}P^1, \]
where $\pi : \mathbb{R} \times L(p, q) \to \mathbb{C}P^1$ is the projection along the Reeb orbits. Using the extension of $c$, we get the orbicurve $\pi : \mathbb{C}P^1 \to \mathbb{C}P^1$. We remove singularities $\{\bar{u}, \infty\} \subset \mathbb{C}P^1$ and consider the map
\[ (31) \quad \bar{\tau} : \mathbb{C}P^1 \setminus \{\bar{u}, \infty\} \to \mathbb{C}P^1, \quad P := \bar{\tau}^{-1}(\{\bar{u}, \infty\}) \]
which is a holomorphic branched covering. One can biholomorphically identify $\mathbb{C}P^1 \setminus \{\bar{u}, \infty\}$ with $\mathbb{C}^*$ and $\tau$ can be viewed as a holomorphic branched covering of $\mathbb{C}^*$. Extending over $P$, we get a holomorphic branched covering
\[ \hat{\tau} : \mathbb{C}P^1 \to \mathbb{C}P^1, \]

(32)
which we call as the smoothened base curve of $\overline{\pi}$. We note that $P = \hat{c}^{-1}(\{0, \infty\})$ and $P$ is in general larger then $\Gamma_{nc}$. We put $\tilde{P} := c^{-1}(\{0, \infty\}) = p^{-1}(P) \subset \Sigma$.

**Lemma 2.5.** The degrees of the maps $[31], [32]$ and $[24]$ coincide.

**Proof.** It is clear that degrees of $\overline{\pi}$ and $\hat{c}$ coincide. The last part of the statement follows since $\overline{\pi} \circ p = p_3 \circ c$ and both $p$ and $p_3$ are $p : 1$ coverings. \hfill $\square$

**Lemma 2.6.** The ramification profiles of $\hat{c}$ and $c$ satisfies the followings.

1. For all $i$, the ramification number of $\hat{c}$ at $z_i^{0, \pm} \Gamma_{nc}$ coincides with the ramification number $\overline{r}_i^{0, \pm}$ of $c$ at $z_i^{0, \pm} \in \Gamma_{nc}$ (see Lemma 2.4).
2. $|\tilde{P} \setminus \hat{\Gamma}_{nc}| = p|P \setminus \Gamma_{nc}|$.
3. For any $z \in P \setminus \Gamma_{nc}$, the ramification numbers of $c$ at $p$-many preimages of $z$ are all the same.
4. For any $z \in P \setminus \Gamma_{nc}$, the ramification number of $\hat{c}$ at $z$ is $p$ times the ramification number of $c$ at any preimage of $z$.

**Proof.** We note that the ramification number of $\hat{c}$ at any $z \in P$ corresponds to the local covering number of $[31]$ around $z$. Hence the remaining statements follow from the local description of the equation $\overline{\pi} \circ p = p_3 \circ c$. \hfill $\square$

### 2.2. Equivariant curves: the sufficient conditions for the existence.

In this section, we discuss the sufficient conditions for the existence of equivariant meromorphic sections of $L$. It turns out that when $n_{nc} = 2$, due to Lemma 2.1 the necessary conditions given above are also sufficient. But if $n_{nc} \geq 3$, then there may be an a priori obstruction due to the genus of $\Sigma$. We postpone the treatment of the first case to the next section and concerning the second case, we discuss the problem for $n_{nc} = 3$ for the sake of presentation. We further assume $n_c = 0$ since contractible ends do not essentially change the problem and we can omit the ambiguity between parametrized or unparametrized curves, see Remark 2.1. At the end of the section, we point out the necessary modifications for more general types of curves.

Let $\mathcal{M}$ denote the moduli space of $J_{\sigma^*}$ holomorphic curves

$$\pi : \mathbb{C}P^1 \setminus \Gamma \to \tilde{L}, \quad \Gamma = \{z_0^+, z_{\infty^+}, z_{0^-}\}$$

with the asymptotics

- $\mathcal{M}(z_0^+): (+\infty, k_0^+, \tau_0)$,
- $\mathcal{M}(z_{\infty^+}): (+\infty, k_{\infty^+}, \tau_0)$,
- $\mathcal{M}(z_{0^-}): (-\infty, k_{0^-}, \tau_0)$

where the necessary conditions given by Lemma 2.4 and Lemma 2.3 are satisfied, namely

$$d := k_0^+ + k_{\infty^+} - k_{0^-} > 0, \quad k_0^+ + qk_{\infty^+} - k_{0^-} \equiv 0.$$

Now given the above data, we define $\tilde{\mathcal{M}}$ to be the moduli space of curves

$$\hat{c} : \mathbb{C}P^1 \to \mathbb{C}P^1; \quad \hat{c}(z_0^+) = 0, \quad \hat{c}(z_{\infty^+}) = \infty, \quad \hat{c}(z_{0^-}) = 0$$

such that

1. The degree of $\hat{c}$ is $d$.
2. The ramification numbers of $\hat{c}$ satisfies
   - at $z_0^+$: $r_0^+ \equiv k_0^+(1 - v)$
   - at $z_{\infty^+}$: $r_{\infty^+} \equiv k_{\infty^+}(1 - q)$
   - at $z_{0^-}$: $r_{0^-} \equiv k_{0^-}(1 - v)$
   - for any $z \in P \setminus \Gamma$, the ramification number at $z$ is divisible by $p$, where $P := \hat{c}^{-1}(\{0, \infty\})$. 

\end{document}
Given the moduli problem $\mathcal{M}$, we fix the covering $(\Sigma, \tilde{\Gamma}, p)$, which exists even if $\mathcal{M}$ is empty, see Remark (2.3). We note that for each $u \in \mathcal{M}$, there corresponds a smoothened base curve $\hat{c}$ and due to Lemma 2.2, Lemma 2.1 and Lemma 2.6 we know that $\hat{c} \in C\mathcal{M}$. Now the question is determine which curves in $C\mathcal{M}$ provide a curve in $\mathcal{M}$. Hence, we need to reverse the procedure given in the previous section. Now given $\hat{c} \in C\mathcal{M}$, it corresponds to a non-singular branched covering $c: \mathbb{P}^1 \rightarrow \mathbb{P}^1 \{0, \infty\}$ where $P := c^{-1}(\{\infty\})$. We first need to construct the lifted base curve $\tilde{c}$. Namely, we should check the diagram in Figure 3 is valid.

\begin{center}
\begin{tikzcd}
\Sigma \setminus \hat{P} \arrow{r}{c} \arrow{d}{p} & \mathbb{C}P^1 \setminus \{0, \infty\} \arrow{d}{\overline{\mathbb{C}P^1} \setminus \{0, \infty\}} \\
\mathbb{C}P^1 \setminus \tau^{-1}(\{\infty\}) \arrow{r}{\tau} & \overline{\mathbb{C}P^1} \setminus \{0, \infty\}
\end{tikzcd}
\end{center}

**Figure 3.** Lifting diagram for $\tau$.

\textbf{Lemma 2.7.} For any $\hat{c} \in C\mathcal{M}$, the corresponding (non-singular) branched covering $\tilde{c}$ lifts through 

$p : \Sigma \setminus \hat{P} \rightarrow \mathbb{C}P^1 \setminus P$

where $\hat{P} := p^{-1}(P)$. We first need to construct the lifted base curve $c$. Namely, we should check the diagram in Figure 3 is valid.

\textbf{Proof.} As in the previous section, we fix $z \in \mathbb{C}P^1 \setminus P$ and put $w := \tau(z)$. We have the induced homomorphism

$\tau_* : \pi_1(\mathbb{C}P^1 \setminus P, z) \rightarrow \pi_1(\overline{\mathbb{C}P^1} \setminus \{0, \infty\}, w) \cong \mathbb{Z}$

where we fix the generator $\eta := \{\text{we}^t : t \in [0, \theta]\}$ for the latter group. The covering $p_* : \mathbb{C}P^1 \setminus \{0, \infty\} \rightarrow \overline{\mathbb{C}P^1} \setminus \{0, \infty\}$ induces the monomorphism

$\mathbb{Z} \rightarrow \mathbb{Z}$, $1 \rightarrow p$

on the fundamental group where we fix a generator of the former group as a lift of $p\eta$. Let

$\rho : \mathbb{Z} \rightarrow \mathbb{Z}/\text{im } (p_*)_* = \text{coker}(p_*)_* \cong \mathbb{Z}_p \cong \langle [\eta] \rangle$

denote the the quotient homomorphism. Then $\tau \circ p$ lifts if and only if

(36) $\rho \circ \tau_* \circ p_* : \pi_1(\Sigma \setminus \hat{P}, \tilde{z}) \rightarrow \mathbb{Z}_p$

is trivial for some $\tilde{z} \in p^{-1}(z)$. We note that by the last statement of (cm2) and the fact that $\mathbb{Z}_p$ is abelian, we have the following commutative diagram

$\pi_1(\mathbb{C}P^1 \setminus P, z) \xrightarrow{i_*} \pi_1(\mathbb{C}P^1 \setminus \Gamma, z) \xrightarrow{\rho \circ \tau_*} \mathbb{Z}_p$

where the upper horizontal arrow is induced by the inclusion $i : \mathbb{C}P^1 \setminus P \hookrightarrow \mathbb{C}P^1 \setminus \Gamma$. Combining this with the with the commutative diagram induced by
we conclude that if
\begin{equation}
(37) \quad \rho \circ \tau_\ast \circ p_\ast : \pi_1(\Sigma \setminus \tilde{\Gamma}, \tilde{z}) \to \mathbb{Z}_p
\end{equation}
vanishes then \((36)\) vanishes as well. Once we fix an isomorphism \(\varphi : \pi_1(\mathcal{L}^\ast) \to \text{coker } (p_\ast)_* \cong \mathbb{Z}_p\) such that \(\varphi([\eta]) = [\eta]\), then it is not hard to see that by \([\text{cm2}]\)
\(\rho \circ \tau_\ast\) coincides with the homomorphism \(\pi_1(\mathbb{C}P^1 \setminus \Gamma, z) \to \pi_1(\mathcal{L}^\ast)\) determined by \(\mathcal{M}\) (see Remark 2.3) up to multiplication by a \((1 - q)\). But the kernel of the latter homomorphism is precisely the image of \(p_\ast\). Hence \((37)\) is trivial. \(\square\)

**Remark 2.6.** As in Remark 2.3 there are \(p\)-many lifts of given \(\tau\). Once we fix a lift \(c\), the other lifts are given by \(\sigma^k \circ c\), \(k = 1, ..., p - 1\).

Given \(\hat{c} \in \mathcal{C}_\mathcal{M}\), we consider the extension \(c : \Sigma \to \mathbb{C}P^1\) of a lift of \(\tilde{\pi}\) such that \(cz\tau = \sigma_0 \circ c\), where \(\sigma_0\) is a fixed generator of the group \(G\) acting on \(\Sigma\). The question is now to determine whether there is a meromorphic section \(u : \Sigma \to c^*L\) such that \(u \circ \tau = \tau \circ u\) where \(\tau\) is the corresponding generator of the \(\mathbb{Z}_p\) action on \(c^*L\) and zeros and poles of \(u\) are given by the moduli problem \(\mathcal{M}\). Namely we ask for a pole at \(z^{0,+}\) of order \(k^{0,+}\), a pole at \(z^{\infty,+}\) of order \(k^{\infty,+}\) and a zero at \(z^{0,-}\) of order \(k^{0,-}\) (as in the previous section, \(z^{\pm}\) denotes the punctures in \(\tilde{\Gamma}\) corresponding to punctures in \(\Gamma_{nc}\)).

**Lemma 2.8.** Assume that there is meromorphic section \(u\) of \(c^*L\) with zeros and poles determined by \(\mathcal{M}\) and \(c\) is equivariant. Then \(u\) is also equivariant.

**Proof.** Given fixed zeros and poles, one has a \(\mathbb{C}^*\)-family of meromorphic sections given by \(\lambda u\), \(\lambda \in \mathbb{C}^*\). Note that since the scaling along fibres commutes with \(\tau\)-action, if there is some equivariant meromorphic section \(u\), then all meromorphic sections of the form \(\lambda u\) are equivariant.

Let \(u\) be a meromorphic section of \(c^*L\) with fixed zeros and poles, where \(c\) is equivariant. Then \(u' := \tau^{-1} \circ u \circ \tau\) is also a meromorphic section of \(c^*L\) having same zeros and poles with \(u\). Hence there is some \(\lambda \in \mathbb{C}^*\) such that \(u' = \lambda u\). In fact, \(\lambda \in S^1\) since all the actions we have are unitary. Both \(u\) and \(u'\) have a pole at \(z^{\infty,+}\) of order \(k^{\infty,+}\). We know that the ramification number of \(c\) at \(z^{\infty,+}\) coincides with \(\hat{c}\) and satisfies
\begin{equation}
(38) \quad k^{\infty,+} \equiv r^{\infty,+}(1 - q)^{-1}.
\end{equation}

by the definition of \(\mathcal{C}_\mathcal{M}\). We take the local trivialization of \(c^*L\) around \(z^{\infty,+} \in \Sigma\) as in the previous section (see Lemma 2.2) so that around \(z^{\infty,+}\), we have \(\tau(z, \lambda) = (e^{i\theta z}, e^{i\theta \lambda})\) and \(\tau_\ast(z) = e^{i\theta z}\) where
\begin{equation}
(39) \quad r^{\infty,+}\lambda \equiv (q - 1).
\end{equation}

With respect to the above trivialization, we write \(u(z) = (z, f(z))\) and \(u'(z) = (z, f'(z))\) where \(f\) and \(f'\) are meromorphic functions. With polar coordinates \(z = \rho e^{it}\), we have
\[\lim_{\rho \to 0} \frac{f(\rho e^{it})}{|f(\rho e^{it})|} = e^{-ik^{\infty,+}t}, \quad \lim_{\rho \to 0} \frac{f'(\rho e^{it})}{|f'(\rho e^{it})|} = \lambda e^{-ik^{\infty,+}t}.\]
On the other hand, definition of $u'$ gives

$$
\lim_{\rho \to 0} \frac{f'(p e^{i\theta})}{f'(p e^{i\theta})} = \lim_{\rho \to 0} \frac{\tau^{-1} \circ f(\tau_{\rho}(p e^{i\theta}))}{\tau^{-1} \circ f(\tau_{\rho}(p e^{i\theta}))} = \lim_{\rho \to 0} \frac{e^{-i\theta} f(p e^{i\theta+im\theta})}{f(p e^{i\theta+im\theta})} = e^{-i\theta} (e^{-ik\gamma + (l+m)\theta}) = e^{-ik\gamma + t} e^{-i(k\gamma_0 + m+1)\theta} = e^{-ik\gamma + t}
$$

where the last equation follows from (38) and (39). Hence $\lambda = 1$ and therefore $\tau \circ u = u \circ \tau_y$. \hfill \Box

We want to understand the moduli space $C_{\mathcal{M}}$ and possible obstructions on the existence of meromorphic sections over the lifts of singular curves, which correspond to the elements of $C_{\mathcal{M}}$. To this end we recall some basic notions in the theory of Riemann surfaces.

Given a closed Riemann surface $\Sigma$ with genus $g$ and $[c] \in H_1(X, \mathbb{Z})$, there is a functional on the (vector) space $\Omega^1(\Sigma)$ consisting of holomorphic 1-forms on $\Sigma$,

$$
\int_{[c]} : \Omega^1(\Sigma) \to \mathbb{C}, \quad \omega \mapsto \int_{c} \omega,
$$

which is well-defined since any holomorphic 1-form on $\Sigma$ is necessarily closed. An element of $\Omega^1(\Sigma)^*$ is called a priori if it is of the above form. The Jacobian of $\Sigma$ is the quotient space

$$
J(\Sigma) := \Omega^1(\Sigma)^*/\Lambda,
$$

where $\Lambda$ is the space of periods. It turns out that $J(\Sigma)$ is isomorphic to the complex torus of dimension $g$.

Let $z_0$ be a point in $\Sigma$. We consider the map

$$
A : \Sigma \to \Omega^1(\Sigma)^*; \quad A(z)(\omega) := \int_{\gamma_z} \omega, \quad \omega \in \Omega^1(\Sigma),
$$

where $\gamma_z$ is some path connecting $z_0$ to $z$. Although this map depends on $\gamma_z$, it descends to so called the Abel map $A : \Sigma \to J(\Sigma)$ which is independent of the choice of $\gamma_z$. Note that the Abel map naturally extends to a group homomorphism over the group $Div(\Sigma)$ of divisors via $A(\sum n_z z) := \sum n_z A(z)$. It turns out that once restricted to subgroup $Div_0(\Sigma) = \ker \deg$, where $\deg : Div(\Sigma) \to \mathbb{Z}$ is the degree map, the Abel map is independent of the point $z_0$.

$Div_0(\Sigma)$ has a special subgroup $PDiv(\Sigma)$ consisting of principal divisors, namely the divisors given by meromorphic functions. Namely, given a meromorphic function $f$ on $\Sigma$, then its divisor is given by $D(f) = D_0(f) - D_\infty(f)$ where

$$
D_0(f) := \sum_{f(z) = 0} o_z(f) z, \quad D_\infty(f) := \sum_{f(z) = \infty} o_z(f) z
$$

and $o_z(f) > 0$ stands for the order of zeros/poles. It turns out that the map

$$
A : Div_0(\Sigma) \to J(\Sigma)
$$

is surjective and its kernel is given by $PDiv(\Sigma)$. Hence one gets the isomorphism

$$
Pic(\Sigma) = Div_0(\Sigma)/PDiv(\Sigma) \cong J(\Sigma),
$$

where the Picard group $Pic(\Sigma)$ is the group of isomorphism classes of of degree zero line bundles over $\Sigma$. Moreover if $A([D]) = 0$ for some $[D] \in Pic(\Sigma)$, then the divisor $D$ defines the trivial bundle and we have the following consequence. If $D_1$ and $D_2$ are two divisors, which define a line bundle of the same degree, then these
bundles are isomorphic if and only if $A(D_1 - D_2) = 0$ and this is trivially the case if $D_1 - D_2 = 0$. We let $\mathcal{G}^n(X)$ denote the $n$-fold symmetric product of the set $X$.

Setting the ground, we first give a description of $\mathcal{C}_M$, which is adopted from [12].

**Lemma 2.9.** $\mathcal{C}_M$ is biholomorphic to

$$
(\mathcal{G}^n(\mathcal{C}P^1 \setminus \{z^{\infty,+}\}) \times \mathcal{G}^n(\mathcal{C}P^1 \setminus \{z^{0,-}, z^{0,+}\})) \setminus \Delta \times \mathbb{C}^*,
$$

where $\Delta$ is the subset of pairs $(D_0, D_{\infty})$ in $\mathcal{G}^n(\mathcal{C}P^1 \setminus \{z^{\infty,+}\}) \times \mathcal{G}^n(\mathcal{C}P^1 \setminus \{z^{0,-}, z^{0,+}\})$ with at least one common point and

$$
n_0 := \frac{d - r^{0,+} - r^{0,-}}{p}, \quad n_{\infty} := \frac{d - r^{\infty,+}}{p}.
$$

**Proof.** The moduli space $\mathcal{C}_M$ itself can be seen as the space of meromorphic functions on $\mathcal{C}P^1$, which has a particular distribution of its zeros and poles. We note that in this case the Abel map vanishes identically, that is any degree zero divisor defines a $\mathbb{C}^*$-family of meromorphic functions. Hence it is enough to characterize the set of divisors satisfying the conditions (cm1) and (cm1).

The condition (cm1) tells us that $\deg(D_0(\hat{c})) = \deg(D_{\infty}(\hat{c})) = d$. Namely we have complex $d$-dimensional freedom to choose the zeros or poles. The conditions given by (cm2) translate as follows. We two zeros $z^{0,-}, z^{0,+}$ and a pole $z^{\infty,+}$ whose orders, which corresponds to the ramification numbers, are fixed in $\text{mod } p$. We fix

$$
r^{0,+}, r^{\infty,-}, r^{0,+} \in \{1, \ldots, p - 1\}
$$

such that

$$
r^{0,+} \equiv r^{0,+}, \quad r^{\infty,-} \equiv r^{\infty,+}, \quad r^{0,-} \equiv r^{0,-}.
$$

Now we can perturb away $(r^{0,+} - r^{0,+})$-many zeros at $z^{0,+}$ but any zero different than $z^{0,\pm}$ has to have order divisible by $p$. The same reasoning applies to $z^{0,+}$ and $z^{\infty,+}$. Hence for zeros we have $(d - r^{0,+} - r^{0,-})/p$ many choices among the points in $\mathcal{C}P^1 \setminus \{z^{\infty,+}\}$ and for poles we have $(d - r^{\infty,+})/p$ many choices among the points in $\mathcal{C}P^1 \setminus \{z^{0,+}, z^{0,-}\}$. We note that $d \equiv r^{0,+} + r^{0,-} \equiv r^{\infty,+}$. After taking suitable symmetric products of our domain and removing collections of points which do not result in divisors, we get the above description. \hfill \Box

We note that

$$
\mathcal{G}^n(\mathcal{C}P^1 \setminus \{z^{\infty,+}\}) \times \mathcal{G}^n(\mathcal{C}P^1 \setminus \{z^{0,-}, z^{0,+}\})
$$

is a complex manifold and $\Delta$ is an irreducible subvariety of codimension one [12]. In particular, the dimension of $\mathcal{C}_M$ is given by

$$
\dim_{\mathbb{R}} \mathcal{C}_M = 2 + 2n_0 + 2n_{\infty} = 2 + \frac{2}{p}(2d - r^{0,+} - r^{0,-} - r^{\infty,+}).
$$

Now the question is that given $\hat{c} \in \mathcal{C}_M$, does $c^*L$ admit a meromorphic section with zeros and poles determined by $\mathcal{M}$ for some lift $c$ of corresponding $\hat{c}$. It turns out that by the very nature of the equivariant picture there is no obstruction the existence of such meromorphic sections.

**Proposition 2.1.** Given $\hat{c} \in \mathcal{C}_M$, there exists a meromorphic section of $c^*L$, which leads to a punctured curve in $\mathcal{M}$ where $c$ is a lift of the singular curve $\overline{c}$ corresponding to $\hat{c}$.

**Proof.** We define the divisor $D_M$ of $\Sigma$ by

$$
D_M = k^{0,-}z^{0,-} - k^{0,+}z^{0,+} - k^{\infty,+}z^{\infty,+}.
$$
We note that the isomorphism class of the line bundle $L$ is determined by the
isomorphism class of degree (-1) divisors on $\mathbb{C}P^1$. We fix a divisor in this class of
the form
\[ D_L := l \cdot 0 - (l + 1) \cdot \infty, \quad l > 0 \]
where $l$ is to be chosen. Since the phase shift on $\mathcal{C}_M$ is not relevant for our problem
we ignore it in what follows. Then lifting scheme above provides the following
continuous embedding
\[ (42) \quad \gamma : \mathcal{C}_M \to Div^0(\Sigma), \quad \hat{c} \mapsto D_M - c^* D_L \]
where $c$ is (up to phase shift) the unique lift of $\hat{c}$ corresponding to $\hat{c}$ satisfying
\[ c \circ \tau_c = \sigma_c \circ c \]
and $c^* D_L$ is the pull-back divisor. We note that $\text{deg}(D_M) = \text{deg}(c^* D_L) = d$ so
that the map is well-defined. Now we want to show that there exists some $l > 0$
such that $A \circ \gamma(\hat{c}) = 0$ for a curve $\hat{c}$ satisfying
\[ (43) \quad D_0 = r^{0,+} z^{0,+} + r^{0,-} z^{0,-}, \quad D_\infty = r^{\infty,+} z^{\infty,+}. \]
In this case,
\[ D_M - c^* D_L = (k^{0,-} - l^{-1} \cdot k^{0,-}) z^{0,-} + (-k^{0,+} + l^{-1} \cdot k^{0,+}) z^{0,+} + ((l + 1) r^{\infty,+} - k^{\infty,+}) z^{\infty,+}. \]
We want to find $l$ such that all the coefficients above are zero in mod $p$ and by
[cm2] we see that for $l \equiv (1 - v)^{-1}$ all the coefficients of the above divisor vanishes
in mod $p$. We fix $l > 0$ such that $l \equiv (1 - v)^{-1}$. Then
\[ D_M - c^* D_L = p \left( m^{0,-} z^{0,-} + m^{0,+} z^{0,+} + m^{\infty,+} z^{\infty,+} \right) \]
for some $m^{0/\infty, \pm} \in \mathbb{Z}$. We note that the divisor $m^{0,-} z^{0,-} + m^{0,+} z^{0,+} + m^{\infty,+} z^{\infty,+}$
descends to a divisor on $\mathbb{C}P^1$ where $z^{0/\infty, \pm}$ stands for the images of $z^{0/\infty, \pm}$ under
$p$. Moreover this divisor has degree 0. Then we know that there is a meromorphic
function, say $f$ which realizes this divisor. Then $D_M - c^* D_L$ is precisely the divisor
for the meromorphic function $f \circ p$ on $\Sigma$. Hence $A(D_M - c^* D_L) = 0$. \hfill \square

**Corollary 2.1.** We have $\mathcal{M} \cong \mathcal{C}_M \times \mathbb{C}^*$ and in particular,
\[ \dim_{\mathbb{R}} \mathcal{M} = 4 + \frac{2}{p} (2d - r^{0,+} - r^{0,-} - r^{\infty,+}). \]

**Remark 2.7.** Note that given the moduli problem [33], one reads of the degree $d$ and the quantities $r^{0/\infty, \pm}$ from the multiplicities and immediately gets the dimension of the moduli space. Then one can check the regularity of the almost complex structure $J_\alpha$ immediately by comparing the above dimension with the virtual dimension of the moduli space. Such a comparison will be carried out for pair of pants in the next section and the arguments used for pair of pants case immediately generalizes to other configurations. In fact we claim that $J_\alpha$ is regular for any admissible moduli problem.

3. Computations: pair of pants, cylinders and others

As we saw above, the lifting scheme simplifies drastically in the case $n_{nc} = 2$. One has $g = 0$ and the covering $p$ can be studied directly. In this case it is easy to study all possible lifted base curves and determine non-empty components of the moduli space. In this section, we study the pair of pants with two positive non-contractible ends in detail and comment on other kinds of moduli problems with $n_{nc} = 2$ as well.

After determining the non-empty moduli spaces of pair of pants, we compute the dimensions of these moduli spaces in terms of the dimensions of equivariant moduli spaces in the lift, see Remark [2.7] and compare them with the virtual dimension of
these moduli spaces given by the well-known index formula \[74\]. The observation is that the dimension of the equivariant moduli space coincides with the index of the problem and this establishes the regularity of the almost complex structure $J_\alpha$, see Remark \[2.7\].

The moduli space of pair of pants

We consider the following moduli problem $M_{J_\alpha}$ of $J_\alpha$-holomorphic curves

\[\pi : \mathbb{C}P^1 \setminus \{0, 1, \infty\} \to \mathbb{R} \times L(p, q)\]

with asymptotics

\[\pi(0) = (+\infty, k^0_\tau_0), \quad \pi(\infty) = (+\infty, k^\infty_\tau_\infty), \quad \pi(1) \in \{-\infty\} \times S_k\]

where $k^0_\alpha, k^\infty_\alpha \neq 0$ and $S_k$ denotes the orbit space of contractible orbits of action $2\pi k$.

In order to work with simpler terms, we choose a model for the domain of \[44\] as follows. We consider the $G = \mathbb{Z}_p$-action on $\mathbb{C}P^1$ given by

\[\tau_\flat((z : 1)) = (e^{im\theta}z : 1)\]

where $m \in \{1, \ldots, p - 1\}$ and $\theta = 2\pi/p$. The quotient map leads to the covering map

\[p : \mathbb{C}P^1 \setminus \{0, 1, w_1, \ldots, w_{p-1}, \infty\} \to \mathbb{C}P^1 \setminus \{0, 1, \infty\}\]

We identify the quotient space above with the our domain $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$.

Now we need to determine lifted base curves, namely the equivariant holomorphic maps

\[c : \mathbb{C}P^1 \to \mathbb{C}P^1\]

where the $\langle \sigma_\flat \rangle$-action on the range is given by

\[\sigma_\flat((z_1 : z_2)) = (e^{i(1-q)\theta}z_1 : z_2)\]

Once we parametrize the domain and the range via $z \mapsto (z : 1)$, any non-trivial holomorphic map $c$ is given by $c(z) = \lambda g(z)/h(z)$ where $\lambda \in \mathbb{C}^*$ and $g$ and $h$ are monic polynomials without a common root. Imposing the equivariance leads to the following characterization.

**Lemma 3.1.** A non-trivial holomorphic map $c$ satisfies $c \circ \tau_\flat = \sigma_\flat \circ c$ if and only if it has the following form

\[c(z) = \lambda z^r g(z)/h(z),\]

where

\[mr \equiv 1 - q\]

and

\[g(z) = \prod_{s=1}^{n}(z^p - a_s)^{k_s}, \quad h(z) = \prod_{t=1}^{m}(z^p - b_t)^{l_t}\]

such that $\lambda \in \mathbb{C}^*$, $r \in \{\mp 1, \mp 2, \ldots, \mp(p-1)\}$, $k_s, l_t \in \mathbb{N}^+$ and $a_s, b_t \in \mathbb{C}$ such that $\lambda \in \mathbb{C}^*$.

**Proof.** It is clear that such a map is equivariant. For the other direction, one sees that if $c$ admits $w \in \mathbb{C}P^1 \setminus \{0, \infty\}$ as a zero or pole then it must admit $p - 1$ many distinct zeros/poles which are given by the orbit of $w$. Hence these polynomials have to factorize through terms like $(z^p - a_s)^{k_s}$ and $(z^p - b_t)^{l_t}$ where $k_s$ and $l_t$ are any non zero complex numbers. The only thing that requires attention then is the case where we have zero or infinity as zero or pole. Checking the equivariance one gets the relation above between $m$ and $r$. \[\square\]
Now we restate the equivariance conditions given by Lemma 2.2 in this special case. By (45) we have $c(0) = 0$ and $c(\infty) = \infty$ and by (26) and (28), an equivariant section $u$ locally looks like $(z, f(z))$ where
\[ f(e^{im\theta} z) = e^{i\theta} \text{ around } 0 \]
and
\[ f(e^{-im\theta} z) = e^{i\theta} \text{ around } \infty. \]

Now if $f(z) = \sum a_n z^n$ around 0, we have
\[ a_n \neq 0 \Rightarrow n \equiv m^{-1}q. \]
Similarly, if $f(z) = \sum b_n z^n$ around $\infty$, we have
\[ b_n \neq 0 \Rightarrow n \equiv -m^{-1}. \]

Since we require that $u$ has poles of order $k^0$ at 0 and of order $k^\infty$ at $\infty$, we get
\[ -k^0 \equiv m^{-1}q \text{ and } -k^\infty \equiv -m^{-1}. \]

This means that
\[ k^0 \equiv -m^{-1}q \equiv rq(q - 1)^{-1} \text{ and } k^\infty \equiv m^{-1} \equiv r(1 - q)^{-1}. \]

Since $c(0) = 0$ and $c(\infty) = \infty$, we have $r > 0$ and $r + p\sum k_s > \sum l_t$, so that the degree of $c$ is given by $\deg(c) = r + p\sum k_s$. We note that
\[ -k^0 - k^\infty \equiv m^{-1}q - m^{-1} \equiv -m^{-1}(1 - q) \equiv -r \equiv -\deg(c) \equiv \deg(c^* L). \]

We also require that $u$ has $p$ many zeros of order $k$. Hence for suitable choice of $k$, we have
\[(pk - k^0 - k^\infty = -r - p\sum k_s = \deg(c^* L)) \]
so that the divisor of $u$ has the correct degree. Therefore, in the lift we have a curve with two positive ends asymptotic to $k^* \gamma_0$ and $k^* \gamma_{\infty}$ and $p$ many negative ends with multiplicity $k$. Passing to the quotient, we get pair of pants in the moduli space we look for.

**Remark 3.1.** We remark that in terms of Lemma 2.2 we have $m^{0,+} = m$ and $m^{\infty,+} = -m$. Moreover $r > 0$ implies that the remification numbers $r^{0/\infty, +}$ given in Lemma 2.2 coincides with $r$ in mod $p$ and (48) coincides with the conditions given in Lemma 2.3.

**Remark 3.2.** In the previous section, we constructed the covering $(\Sigma, \tilde{\Gamma}, \mathfrak{p})$ together with a fixed generator $\tau_\gamma$ for the Deck group so that given the moduli space with fixed asymptotics, the equivariant curves are given by $u \circ \tau_\gamma = \sigma \circ u$. In the above treatment, the representation $m$ of $\tau_\gamma$ determines $r$ and hence the homotopy classes of positive ends. Hence different choices of $\tau_\gamma$, equivalently $r$, lead to different homotopy classes of non-contractible ends and therefore different components of the moduli space.

We want to compute the dimension of the moduli space $\mathcal{M}$ using the above description. We note that the formula given in Corollary 2.1 immediately applies here. Nevertheless, we repeat this computation by directly looking at the lifted base curves.

Let $c$ be a curve as in Lemma 3.1. As a lifted base curve, $c$ contributes to the index of the pair of pants in only two ways. One is the freedom of moving the roots of $g$ and $h$ and moving the constant $\lambda$. In fact, if one moves the “roots” of $z^r$, then the resulting non zero root have to appear $p$ many but this is not possible since this makes the degree of the map jump. The contribution of moving “roots” of $g$ and $h$ is little delicate. Given a term like $(z^r - a_s)^{k^s}$ in the factorizations of these polynomials, one has to move roots of $a_s$ simultaneously to keep invariance, i.e. we
only have the freedom moving $a_s$'s and $b_t$'s. More precisely, if $d$ is the degree of a given base curve $c$ then the contribution of the base curve to the index is $4[d/p] + 2$. For the problem $\mathcal{M}$, the lifted base curves satisfies $r > 0$ and $r + p \sum k_s > \sum l_t$, so that the degree of $c$ is given by $r + p \sum k_s$ and therefore $[d/p] = \sum k_s$. Now for fixed $a > 0$, $\mathcal{M} := [d/p] \geq 0$ the index of the problem is given by (compare to Corollary 2.1)

\begin{equation}
(51)
4 + 4\mathcal{M},
\end{equation}

where we add 2-dimensional freedom of rescaling sections.

**The index of holomorphic curves in the Morse-Bott setting**

We first recall generalities about the virtual dimension of the moduli space \([1]\) where $(M, \xi = \ker \alpha)$ is a 3-dimensional Morse-Bott contact manifold. As in the non-degenerate case, the virtual dimension a moduli space is determined by the Conley-Zehnder indices of the asymptotic ends, where the Conley-Zehnder index is non-degenerate case, the virtual dimension a moduli space is determined by the where \((\gamma)\). We consider the generalization given in [13], which is axiomatically described as follows in dimension 2, see [17] and [6].

Let $\Sigma(1)$ be the space of paths $\varphi : [0, 1] \to Sp(1)$ with $\varphi(0) = I$, where $Sp(1)$ is the space of 2-by-2 symplectic matrices and $I$ is the identity matrix. The **Conley-Zehnder index** is the is a unique map $\mu : \Sigma(1) \to \frac{1}{2}\mathbb{Z}$ characterized by the following axioms.

(CZ1) $\mu$ is constant on homotopies $\varphi_s \in \Sigma(1)$ for which $\dim \ker(\varphi_s(1) - I)$ is constant.

(CZ2) If $\varphi \in \Sigma(1)$ and $\psi : \mathbb{R}/\mathbb{Z} \to Sp(1)$ is a loop then

$$\mu(\psi \varphi) = \mu(\varphi) + 2m(\psi)$$

where $m(\psi)$ is the Maslov index of $\psi$.

(CZ3) If $\varphi \in \Sigma(1)$ and $\varphi^{-1} \in \Sigma(1)$ is the corresponding path of inverses, then

$$\mu(\varphi) + \mu(\varphi^{-1}) = 0.$$ \hfill (CZ4) $\mu(e^{i\pi t}) = 1$ and if $\varphi(t) = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$ then $\mu(\varphi) = \frac{1}{2}$.

Recall that for any $x \in M$ and $t \in \mathbb{R}$, the linearization of the Reeb flow $\Phi_t$ leads to a symplectic map

$$d\Phi_t(x) : (\xi_x, (da)_x) \to (\xi_{\Phi_t(x)}, (da)_{\Phi_t(x)}).$$

Let $\gamma$ be a closed Reeb orbit with period $T > 0$. We fix a symplectic trivialization

$$\Phi : S^1 \times \mathbb{R}^2 \to \gamma(T)^* \xi$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. Then the Conley-Zehnder index of the orbit $\gamma$ with respect to $\Phi$ is given by

\begin{equation}
\mu^\Phi(\gamma) := \mu \left( \{ t \mapsto \Phi^{-1}(t) \circ d\Phi_T(\tilde{\gamma}(0)) \circ \Phi(0) \} \right).
\end{equation}

Let $C = [\Sigma, j, \Gamma, u] \in \mathcal{M}$. We pick a collection $\{ \Phi_{z^j} \}$ of trivializations for the asymptotic ends $\{ \gamma_i^k \}$. Then we consider the complex line bundle $(u^*\xi, J) \to \Sigma \setminus \Gamma$. Let $U \subset \Sigma$ be an open neighbourhood of the puncture set $\Gamma$, consisting of disks centered at each puncture. We endow each such disk with cylindrical coordinates via

$$[0, +\infty) \times S^1 \to \mathbb{D}, \ (s, t) \mapsto e^{-2\pi(s+it)}.$$  
and we extend $\{ \Phi_{z^j} \}$ to a complex trivialization $\Phi : U \times \mathbb{C} \to (u^*\xi, J)_{|U}$ and define first Chern number of $u$ relative to $\Phi$ is defined to be the signed count

\begin{equation}
(53)
c_1^\Phi(u) := \# s^{-1}(0)
\end{equation}
where $s$ is a generic section of $u^*\xi$ such that $\Phi(s) = 1$. Having all the ingredients at hand, the Fredholm index of $C = [\Sigma, j, \Gamma, u] \in \mathcal{M}$ reads as
\begin{equation}
\text{ind}(C) = 2\epsilon_1^\Phi(u) + \sum_{z_i^+ \in \Gamma^+} \mu^\Phi(z_i^+) - \sum_{z_i^- \in \Gamma^-} \mu^\Phi(z_i^-) + \frac{1}{2} \sum_{z_i^\pm \in \Gamma} \dim S_i^\pm + \#\Gamma - \chi(\Sigma),
\end{equation}

see \[11\] and \[17\]. We recall that ind$(C)$ is precisely the index of the Fredholm operator associated to the curve $C$ so that once it is surjective, $\mathcal{M}$ gains a smooth structure near $C$ and the kernel of the operator defines the the tangent space $T_C\mathcal{M}$.

The index of pair of pants

The aim of this section is to compute $\mathcal{Z}(s)$ for any curve in the moduli space $\mathcal{M}$ of pair of pants given by \[44\] and \[45\]. We note that this make sense due to \[2.1\].

Our strategy is utilize the lifting procedure here as well. We first fix global "trivializations" of $\xi_0$ and $\xi$. Using \[1\] we define a non-vanishing section
\begin{equation}
s : S^3 \to C^2; \ (z_1, z_2) \mapsto (z_2, -z_1)
\end{equation}
of $\xi_0$ and get a global complex trivialization
\begin{equation}
\Phi_0 : S^3 \times C \to \xi_0, \ ((z_1, z_2), \lambda) \mapsto \lambda s(z_1, z_2).
\end{equation}

Note that on $\xi_0$, the standard almost complex structure $J_0$ coincides with $i$. We note that $\xi \to L(p, q)$ is non-trivial. Nevertheless, it is convenient to fix a section of it which vanishes along a mild subset. We construct such a section as a quotient of a section of $\xi_0$ as follows. We define a section
\begin{equation}
k(z_1, z_2) := f(z_1, z_2)s(z_1, z_2), \ f(z_1, z_2) := z_1^{q+1} + z_2^{q+1}
\end{equation}
An easy computation shows that the section $k : S^3 \to \xi_0 \subset TS^3$ is equivariant in the sense that
\begin{equation}
k(\sigma(z_1, z_2)) = d\sigma(z_1, z_2)[k(z_1, z_2)]
\end{equation}
where $\langle da \rangle$ is the induced action on $TS^3$ for which $\xi_0$ is invariant. We note that $k$ vanishes along a torus knot $K \subset S^3$, which does not intersect with $\gamma_0$ or $\gamma_\infty$. We define another trivialization of $\xi_0$ away from $K$
\begin{equation}
\Phi : (S^3 \setminus K \times C) \to \xi_0, \ ((z_1, z_2), \lambda) \mapsto \lambda k(z_1, z_2)
\end{equation}
so that $\Phi$ induces a trivialization of $\xi$ away from $p(K)$ given by
\begin{equation}
\bar{\Phi} : (L(p, q) \setminus p(K)) \times C \to \xi \ 	ext{ such that } \bar{\Phi} \circ (p \times \text{id}) = dp \circ \Phi.
\end{equation}

We use this trivialization along the orbits which are away from the vanishing set $p(K)$.

We first compute the Conley-Zehnder indices of the positive ends given by \[45\].

Due to the definition of $\bar{\Phi}$, instead of considering the flow of $R$ along $k^\infty\pi_\infty$, we consider the flow of $R_0$ along the lifted Reeb arc
\begin{equation}
[0, k^\infty\theta] \to S^3, \ t \mapsto (e^{it}, 0); \ \theta = 2\pi/p
\end{equation}
together with the trivialization $\Phi$. The resulting symplectic arc $\varphi_\infty(t)$ is given by
\begin{equation}
\varphi_\infty(t) = \Phi^{-1}(e^{itk^\infty\theta}, 0) \circ d\phi_{(tk^\infty\theta)}(1, 0) \circ \Phi(1, 0).
\end{equation}

Viewing $\varphi_\infty(t) \in C$, we compute
\begin{align*}
\varphi_\infty(t)\Phi(e^{itk^\infty\theta}, 0) &= d\phi_{(tk^\infty\theta)}(1, 0)[f(1, 0)(0, 1)] \\
\varphi_\infty(t)f(e^{itk^\infty\theta}, 0)s(e^{itk^\infty\theta}, 0) &= (0, -e^{itk^\infty\theta}) \\
\varphi_\infty(t)e^{it(k+1)k^\infty\theta}(0, -e^{-itk^\infty\theta}) &= (0, -e^{itk^\infty\theta})
\end{align*}
and get

\[ \varphi_\infty(t) = e^{it2\pi \frac{k_\infty(1-q)}{p}}. \]

A similar computation for \( k_0^0 \) leads to the arc

\[ \varphi_0(t) = e^{it2\pi \frac{k_0^0(1-q)}{p}}. \]

We note that the orbits \( k_0^{0,0} )\) are non-degenerate if and only if \( k \neq 0 \) since \( v,q > 1 \). Hence we can compute the Conley-Zehnder indices of \( k_0^{0,0} )\) using a standard computational recipe, see [6]. We note that

\[ \det( I - \varphi_0(1) ) > 0, \]

that is \( \varphi_0(1) \) and \(-I = e^{i\pi} \) are in the same component of non-degenerate symplectic matrices. We connect \( \varphi_\infty(1) = e^{i2\pi \frac{k_\infty(1-q)}{p}} \) to \(-I = e^{i2\pi \frac{(1-k^\infty(1-q))}{p} + \frac{1}{2}} \) by a rotation that does not hit the Maslov cycle. Squaring the resulting loop and computing the degree leads to

\[ \mu(\varphi, k_\infty) = 2 \left( \frac{k_\infty(1-q)}{p} \right) + 1. \]

A similar argument gives

\[ \mu(\varphi, k_0) = 2 \left( \frac{k_0(1-q)}{p} \right) + 1. \]

**Lemma 3.2.** The Fredholm index of a pair of pants given by (44) and (45) is given by

\[ \text{ind}(\mathfrak{p}) = \mu(\varphi, k_\infty) + \mu(\varphi, k_0) + \frac{2}{p}(d + \frac{k^0}{v} + k^\infty q) - 2k + 2 \]

where \( d = k^0 + k^\infty - pk \).

**Proof.** Let \( \mathfrak{p} \) be given by (44) and (45). We can safely assume that \( \mathfrak{p}(1) = (-\infty, k^\infty) \) and \( \varphi \) is away from the vanishing set of \( \Phi \). Then by (53), the index of \( \mathfrak{p} \) is given by

\[ \text{ind}(\mathfrak{p}) = 2c_1(\mathfrak{p}) + \mu(\varphi, k_0) + \mu(\varphi, k_\infty) - \mu(\varphi, k^\infty) + 2. \]

Now instead of directly computing the remaining terms, we utilize the lifting procedure again. Let \( u \) be a lift of \( \mathfrak{p} \). We consider \( u \) as a representative of an unparametrized curve \( C \) and compute the index via the formula (54) and the trivialization \( \Phi \) given by (57). We get

\[ \text{ind}(C) = 2c_1(\Phi) + \mu(\Phi, k_0) + \mu(\Phi, k_\infty) - \sum_{i=1}^{p} \mu(\Phi, k_{\gamma_i}) \]

\[ + \frac{1}{2} \sum_{i=1}^{p+2} \dim S_i + (p + 2) - 2 \]

\[ = 2c_1(\Phi) + \mu(\Phi, k_0) + \mu(\Phi, k_\infty) - \sum_{i=1}^{p} \mu(\Phi, k_{\gamma_i}) + 2p + 2 \]

where \( \gamma_1, \ldots, \gamma_p \) are \( p \)-distinct lifts of the simple contractible orbit \( \varphi \). Due to the equivariance, we have \( \mu(\Phi, k_{\gamma_i}) = \mu(\Phi, k_{\varphi}) \) for all \( i \) and \( c_1(\Phi) = c_1(\Phi)/p \). Hence we have

\[ 2c_1(\Phi) - \mu(\Phi, k_{\varphi}) = \frac{1}{p}(\text{ind}(C) - \mu(\Phi, k_0) - \mu(\Phi, k_\infty) - 2p - 2). \]
Now we compute the left hand side of the above equation. A computation similar to the one carried out for $k_{0/\infty} \gamma_{0/\infty}$ leads to the symplectic paths

$$\psi(t) = e^{it2\pi k^\infty} e^{(1 - q)}; \quad \psi_0(t) = e^{it2\pi k^0} e^{(1 - v)}$$

for $k^\infty \gamma_{0/\infty}$ and $k^0 \gamma_{0}$ respectively. We first observe that the constant path $I(t) = I$ leads to

$$\mu(I) = \frac{1}{2}(\mu(I) + \mu(I)) = \frac{1}{2}(\mu(I) + \mu(I^{-1})) = 0$$

by (CZ3). For a general symplectic path of the form $\varphi = e^{itk2\pi}, k \in \mathbb{Z}$, viewing the inverse path as a loop and using (CZ2) we get

$$0 = \mu(I) = \mu(\varphi^{-1}) = \mu(\varphi) + 2m(\varphi^{-1}) = \mu(\varphi) - 2k \Rightarrow \mu(\varphi) = 2k.$$ Hence we get

$$\mu^\Phi(k^\infty \gamma_{0/\infty}) = 2k^\infty (1 - q), \quad \mu^\Phi(k^0 \gamma_0) = 2k^0 (1 - v).$$

Now for the term $\text{ind } (C)$, we use the trivialization $\Phi_0$ given by (55). Note that since it is induced by a non-vanishing section, we have $c_1^{(0)}(u) = 0$. For any closed orbit

$$k \gamma(t) = (e^{it}, e^{it}) \in [0, 2\pi k]$$

the associated symplectic path reads as

$$\varphi(t)\Phi_0(e^{itk2\pi} z, e^{itk2\pi} z) = d\phi_{(tk2\pi)}(z_1, z_2) [\Phi_0(z_1, z_2, z_1)]$$

$$\varphi(t)(e^{-itk2\pi} z_1, e^{-itk2\pi} z_2) = (e^{itk2\pi} z_1, e^{itk2\pi} z_2 - z_1)$$

$$\varphi(t) = e^{it(2k)2\pi}.$$ Hence we get

$$\mu^\Phi(0 \gamma_k) = 4k$$ and therefore

$$\text{ind } (C) = 2c_1^{\Phi}(u) + \mu^\Phi(0 \gamma_0) + \mu^\Phi(k^\infty \gamma_{0/\infty}) - \sum_{i=1}^{p} \mu^{\Phi}(k \gamma_i)$$

$$+ \frac{1}{2} \sum_{i=1}^{p+2} S_i + (p + 2) - 2$$

$$= 4k^0 + 4k^\infty - p4k + 2p + 2.$$ Then (63) leads to

$$2c_1^{\pi}(\pi) - \mu^{\pi}(k \pi) = \frac{1}{p} \left(4k^0 + 4k^\infty - p4k - 2k^0 (1 - v) - 2k^\infty (1 - q)\right)$$

$$= \frac{1}{p} \left(d + k^0 v + k^\infty q\right) - 2k.$$ Substituting the above formula in (62) leads to the formula (61).

**Lemma 3.3.** For any $\pi$ given by (44) and (45), we have

$$\text{ind } (\pi) = 4 + 4d^d$$

where $d^d = [d/p]$.

**Proof.** We write

$$k^\infty (1 - q) = l_\infty + n_\infty p, \quad k^0 (1 - v) = l_0 + n_0 p; \quad 0 < l_\infty, l_0 < p$$

so that

$$\mu(k^\infty \gamma_{0/\infty}) = 2n_\infty + 1, \quad \mu(k^0 \gamma_0) = 2n_0 + 1.$$ We note that by (48),

$$k^\infty (1 - q) \equiv k^0 (1 - v) \equiv k^\infty + k^0.$$
By [50] we also have $k^\infty + k^0 = r + p(d^I + k)$ where $0 < r < p$. In particular, $k^\infty + k^0 \equiv r$. Hence we conclude that $r = l_1 = l_2$. Combining all these, we get

$$\text{ind} \left( \bar{\pi} \right) = \frac{2}{p} \left( k_1 q + k_2 u + d \right) + \mu \bar{\pi} \left( k^\infty \gamma_\infty \right) + \mu \bar{\pi} \left( k^0 \gamma_0 \right) - 2k + 2$$

$$= \frac{2}{p} \left( k^\infty - r - n_\infty p + k^0 - r - n_0 p + r + pd^I \right)$$
$$+ (2n_\infty + 1) + (2n_0 + 1) - 2k + 2$$
$$= \frac{2}{p} \left( k^0 + k^\infty - r + pd^I - (n_0 + n_\infty)p \right)$$
$$+ 2(n_0 + n_\infty) - 2k + 4$$
$$= \frac{2}{p} \left( r + pd^I + pk - r + pd^I - (n_0 + n_\infty)p \right)$$
$$+ 2(n_0 + n_\infty) - 2k + 4$$
$$= \frac{2}{p} \left( 2pd^I + pk - (n_0 + n_\infty)p \right) + 2(n_0 + n_\infty) - 2k + 4$$
$$= \left( 4d^I + 2k - 2(n_0 + n_\infty) \right) + 2(n_0 + n_\infty) - 2k + 4$$
$$= 4d^I + 4.$$  

We list the outcomes of above discussion below.

**Remark 3.3.** (Regularity of $J_\alpha$) As a first corollary, we conclude that any non-empty component of the moduli space $\left( 44 - 45 \right)$ is cut out transversally. In fact, the identity above shows that the quotient almost complex structure $J_\alpha$ on $\mathbb{R} \times L(p,q)$ is regular for any $\bar{\pi}$ given by $\left( 44 - 43 \right)$. The holomorphic perturbations of $\bar{\pi}$ are in one to one correspondence with the equivariant holomorphic perturbations of the lift $u$. But the dimension of these perturbations is equal to the equivariant index associated to $\bar{\pi}$ and therefore to $\text{ind}(\bar{\pi})$ by the above lemma. Hence the kernel of the corresponding Cauchy-Riemann operator has the dimension equal to the index and the cokernel is trivial.

**Remark 3.4.** (Components of the moduli space) The equivariant index makes easy to determine the components of the moduli space of pants with two non-contractible ends. In the case above, when $d^I$ and the multiplicity of the contractible end are fixed, we have $p - 1$ components, corresponding to each value of $0 < r < p$ and therefore to each non-trivial homotopy class of ends. We note that the minimal index for the moduli space of pair of pants is 4 and it corresponds to $d^I = 0$. Each component may be identified with $\mathbb{C}^* \times \mathbb{C}^*$, where one factor stands for the freedom of moving $\lambda$ and the other corresponds to the rescaling of the meromorphic section.

We note that once $d^I$ is fixed, one may increase multiplicities of all ends simultaneously in such a way that the degree condition is satisfied and the index is unchanged. Hence, the big moduli space of pair of pants where the asymptotics at punctures $0, \infty$ are fixed geometrically as above, can be written as

$$\bigcup_{d^I \geq 0} \bigcup_{k \geq 0} \bigcup_{a=1}^{p-1} \mathcal{M}_{d^I,k,a}; \quad \dim \mathcal{M}_{d^I,k,a} = 4 + 4d^I,$$

where $k$ is the multiplicity of the contractible end.

**Adding more contractible ends**

The discussion above can be repeated with minor modifications for other curves with only two non-contractible ends.
Adding more contractible ends to the pair of pants configuration does not require any essential change. To be more specific, let us consider the moduli space \( \mathcal{M} \) of unparameterized curves with two non-contractible positive ends, that are fixed geometrically as above, \( s^+ \)-many contractible positive ends and \( (s^- + 1) \)-many negative contractible ends. As noted in Remark 2.1, we interpret \( \mathcal{M} \) as the set of tuples 
\[
(\overline{\pi}, (z_1, ..., z_{s^+}), (w_1, ..., w_{s^-}))
\]
where \( \overline{\pi} \) has the positive ends at 0 asymptotic to \( k_1\overline{\pi}_0 \) and at \( \infty \) asymptotic to \( k_1\overline{\pi}_\infty \), a negative end at 1 of multiplicity \( k \), positive ends at \( z_1, ..., z_{s^+} \) of multiplicities \( l_1^+, ..., l_{s^+}^+ \), negative ends at \( w_1, ..., w_{s^-} \) of multiplicities \( l_1^-, ..., l_{s^-}^- \). That is, we fix tree punctures and let the rest of the punctures move. Given such an object, we have a lifted base curve \( c \) with \( r > 0 \) and \( \deg(c) = r + pd^I \) and 
\[
-k^\infty - k^0 - p \sum l_j^+ + pk + p \sum l_j^- = -(r + pd^I).
\]
The multiplicities \( k_1 \) are determined in \( \text{mod} \, p \) as before. The dimension of the component of \( (\overline{\pi}, (z_1, ..., z_{s^+}), (w_1, ..., w_{s^-})) \) is then 
\[
4 + 4d^I + 2s^+ + 2s^-
\]
since we added the freedom of choosing the places of zeros and poles of the meromorphic section other than 0, \( \infty \) and the lifts of 1. Yet in the lift, lifts of the remaining zeros and poles should be distributed invariantly. Moreover, an analysis similar to above shows that 
\[
\text{ind} (\overline{\pi}, (z_1, ..., z_{s^+}), (w_1, ..., w_{s^-})) = 4 + 4d^I + 2s^+ + 2s^-.
\]
In order to describe the components of the moduli space, one simply adds free zeros and poles to the pair of pants configurations and the multiplicities of the non-contractible ends are adjusted to get the degree condition is satisfied. For fixed values of \( d^I \), \( a \) and \( k \) as before, each component of the moduli space may be identified with 
\[
\mathcal{M}_{d^I, k, a} \times \mathfrak{S}^{(s^+ + s^-)}(\mathbb{C}P^1 \setminus \{0, 1, \infty\})
\]
where \( \mathcal{M}_{d^I, k, a} \) is given in Remark 3.4.

The cylinders

For the cylinders, we first consider the parameterized cylinders and then modify biholomorphisms that fix the punctures. In our case, this corresponds to removing the contractible end from the configurations of the pair of pants with one positive and one negative non-contractible end and killing the freedom in the domain. More concretely, let us consider the moduli space of cylinders with positive end asymptotic to \( k^0\tau_0 \) and negative end asymptotic to \( k^\infty\tau_\infty \). The equivariant index of parametrized cylinders is again \( 4 + 4d^I \) and modding out reparametrizations means we kill the freedom of moving \( \lambda \) in the definition of the lifted curve \( c \), see Lemma 3.1. Hence the equivariant index reads as \( 2 + 4d^I \). On the other hand, the Fredholm index reads as 
\[
\text{ind} (\overline{\pi}) = 2c_1(\overline{\pi}) + k^0 - k^\infty = -\deg(c)
\]
and similar to Lemma 3.2 we get 
\[
c_1(\overline{\pi}) = k^0 v - k^\infty q + k^\infty - k^0.
\]
Combining all these, one can show that \( \text{ind}(\overline{\pi}) = 4d^I + 2 \) and therefore we have transversality for cylinders as well. Concerning the big moduli space, we have two parameters, namely \( d^I \) and \( a \) that index the components.
4. An application

In this section, we carry out a neck-stretching procedure, which is initiated by a positive contactomorphism

\[ \varphi : (L(p, q), \xi = \ker \alpha) \to (L(p, q'), \xi' = \ker \alpha') \]

We perturb \( J_\alpha \)-holomorphic pair of pants with non-contractible positive ends, which come from a particular component of the moduli space and at the end we would like to obtain a very particular holomorphic building, see Figure 4.

![Figure 4. The outline of the neck-stretching procedure.](image)

As usual, it is very unlikely to achieve such a picture since there are the issues of transversality and compactness. It turns out that the study of the index behaviour of multiples of non-contractible Reeb orbits does not give enough control to handle these issues. The crucial observation is that one can apply well-established 4-dimensional methods to the lifts of punctured curves in the symplectic cobordisms that show up along the way, after extending them to closed curves in a certain completion of these cobordisms. The outcome of this observation is that the total action at the negative ends of punctured curves in cobordisms is not strictly bigger than the total action at the positive ends. This non-negativity of "\( d\alpha \)-energy" provides a strong control over the components of the limiting buildings that emerge along the neck-stretching procedure. As a result we prove the following.
Lemma 3.1. Then the choice of stretching argument, see Lemma 4.5.

Remark 4.2. The conditions on the point \( x \) are easily satisfied and play a significant role in ruling out certain unpleasant configurations at the end of the neck-stretching argument, see Lemma 4.5.

Theorem 4.1. Let \( p \) be prime and \( 1 < q, q' < p - 1 \). Suppose that there is a positive contactomorphism

\[
\varphi : (L(p, q), \xi = \ker \alpha) \to (L(p, q'), \xi' = \ker \alpha').
\]

Then \( q \equiv (q')^{\pm 1} \mod p \).

4.1. The proof of the statement. We take two lens spaces \( L(p, q) \) and \( L(p, q') \), where \( p, q \) and \( q' \) satisfies the assumptions of Theorem 4.1 with the contact structures induced by contact forms \( \alpha \) and \( \alpha' \), which are quotients of \( \alpha_0 \), see Introduction.

We consider the moduli space of \( J_{\alpha} \) holomorphic pair of pants with noncontractible positive ends in the symplectization of \( L(p, q) \), which has the minimal index. By the consideration of equivariant curves in \( \mathbb{R} \times S^3 \cong L^* \), see Section 3, we know that minimal index for this problem is 4 and we have a moduli space with countably many components that can be collected in \( p - 1 \) groups each group being determined by the degree of the underlying closed curve of the lifts of pair of pants to \( L^* \), see Remark 4.4.

Remark 4.1. (Notation) For the rest of the paper we drop the notation \( \pi \) for the curves in \( \mathbb{R} \times L(p, q) \) since we will refer to the lifts of such curves seldom.

We consider the component, denoted by \( \mathcal{M} \), associated to the underlying closed curve with degree one and minimal multiplicity of non-contractible ends. More precisely, \( \mathcal{M} \) is the moduli space of pair of pants

\[
u = (a, v) : CP^1 \setminus \Gamma \to \mathbb{R} \times L(p, q) \cong \mathbb{L}^*
\]

with punctures \( \Gamma = \{0, 1, \infty\} \) and asymptotics

\[
u(0) = (+\infty, k^0\gamma_0), \quad \nu(\infty) = (+\infty, k^\infty\gamma_\infty), \quad \nu(1) \in (-\infty) \times S_1
\]

where \( S_1 \) is the orbit space of the contractible orbits of action \( 2\pi \). We have

\[
k^0 \equiv (1 - v)^{-1}, \quad k^\infty \equiv (1 - q)^{-1}, \quad k^0 + k^\infty = p + 1.
\]

Next, we impose a 4-dimensional constraint on \( \mathcal{M} \) as follows. We pick a point \( x_0 \in L(p, q) \) away from the non-contractible orbits and with the property that \( \varphi(x_0) \) is also away from the non-contractible oritbits in \( L(p, q') \). Given the contactomorphism in (64), we have a positive function

\[
f : L(p, q) \to (0, +\infty) \text{ s.t. } \varphi^*\alpha' = f \alpha.
\]

Now consider the evaluation map

\[
ev : \mathcal{M} \to \mathbb{R} \times L(p, q), \quad ev(u) = u(2).
\]

We cut out a 0-dimesional submanifold of \( \mathcal{M} \) via

\[
\mathcal{M}^0 := ev^{-1}((\log f(x_0), x_0)).
\]

By the description of the curves in \( \mathcal{M} \) given in the previous section, it is easy to see that \( \mathcal{M}^0 \) is cut out transversely and consists of a single curve. In fact one notes that \( x_0 \) lies on some contractible orbit, say \( \gamma \) and considering the equivariant lifts of the curves in \( \mathcal{M} \), \( \gamma \) as a point in the orbit space is the image of the point 2 under the underlying base curve and this choice fixes the parameter \( \lambda \in \mathbb{C} \) given in Lemma 3.1. Then the choice of \( x_0 \in \gamma \) and the quantity \( f(x_0) \) fixes the freedom over the equivariant meromorphic section, which is due to the \( C^* \)-action on \( L \).

Remark 4.2. The conditions on the point \( x_0 \) are easily satisfied and play a significant role in ruling out certain unpleasant configurations at the end of the neck-stretching argument, see Lemma 4.5.
We consider the following exact symplectomorphism
\[
\Phi : (\mathbb{R} \times L(p,q'), d(e^\alpha')) \to (\mathbb{R} \times L(p,q), d(e^\alpha)),
\]
\[ (t, x) \mapsto (t + \log f \circ \varphi^{-1}(x), \varphi^{-1}(x)). \]
Let \( \Sigma := \Phi([0] \times L(p,q')) \) be the contact type hypersurface in \( \mathbb{R} \times L(p,q) \). Then we have
\[
N^+ \cup N^- := (\mathbb{R} \times L(p,q)) \setminus \Sigma
\]
where \( N^+ \) is the upper and \( N^- \) is the lower connected component. We fix \( \varepsilon > 0 \) and put
\[
U := \Phi([-\varepsilon, \varepsilon] \times L(p,q')) \subset \mathbb{R} \times L(p,q).
\]
We fix an open subset \( V \subset \mathbb{R} \times L(p,q) \) such that \( \overline{U} \subset V \). For the later purposes we set
\[
V^+ \cup V^- := V \setminus U
\]
where \( V^+ \) and \( V^- \) are the upper and lower components respectively. Let \( n \) be a positive integer. Following \([8]\), we construct the manifold \( W^n \) as follows. We remove \( \Phi((-\varepsilon/2,\varepsilon/2) \times L(p,q')) \) from \( \mathbb{R} \times L(p,q) \) we glue \([-\varepsilon - n, n + \varepsilon] \times L(p,q) \) in the middle via the following identifications
\[
[-\varepsilon - n, -n - \varepsilon/2] \times L(p,q') \ni (t, x) \sim \Phi(t + n, x) \in U,
\]
\[
[n + \varepsilon/2, n + \varepsilon] \times L(p,q') \ni (t, x) \sim \Phi(t - n, x) \in U.
\]
We consider a smooth function \( \phi_n : [-\varepsilon - n, n + \varepsilon] \to [-\varepsilon, \varepsilon] \) such that
\[
\begin{align*}
\phi'_n > 0, \\
\phi_n(t) &= t + n \text{ for } t \in [-\varepsilon - n, -\varepsilon/2 - n], \\
\phi_n(t) &= t - n \text{ for } t \in [n + \varepsilon/2, n + \varepsilon], \\
\phi_n(0) &= 0.
\end{align*}
\]
Such a function leads to a diffeomorphism
\[
\Phi_n : W^n \to \mathbb{R} \times L(p,q)
\]
where \( \Phi_n = \text{id} \) on \( \Phi\((-\varepsilon/2,\varepsilon/2) \times L(p,q'))\) and
\[
\Phi_n(t, x) = (\phi_n(t) + \log f \circ \varphi^{-1}(x), \varphi^{-1}(x))
\]
on \([-\varepsilon - n, n + \varepsilon] \times L(p,q') \). For later purposes we note that
\[
\Phi_n(0, \varphi(x_0)) = (\phi_n(0) + \log f \circ \varphi^{-1}(\varphi(x_0)), \varphi^{-1}(\varphi(x_0)) = (\log f(x_0), x_0).
\]
We consider the exact symplectic form
\[
\omega_n := \Phi_n^* d(e^\alpha) = d(\Phi_n^* (e^\alpha))
\]
on \( W^n \), which reads as
\[
\omega_n = \begin{cases} 
  d(e^\alpha) & \text{on } U^c \\
  d(\phi_n^* d\alpha') & \text{on } [-\varepsilon - n, n + \varepsilon] \times L(p,q').
\end{cases}
\]
We note that the standard almost complex structure \( J_{\alpha'} \) is compatible with the symplectic form \( d(\phi_n^* \alpha') \). Next we consider an almost complex structure \( J_n \) on \( \mathbb{R} \times L(p,q) \) with the following properties:
\[
\begin{align*}
\text{(Jn1)} & \quad J_n = J_{\alpha} \quad \text{on } V^c, \\
\text{(Jn2)} & \quad (d\Phi_n)^{-1} \circ J_n \circ d\Phi_n = J_{\alpha'} \quad \text{on } [-\varepsilon - n, n + \varepsilon] \times L(p,q'), \\
\text{(Jn3)} & \quad J_n \text{ is regular for any simple curve passing through the open subset } V^+ \cup V^- \\
& \quad \text{and if relevant, satisfying}
\end{align*}
\]
\[
\text{ev}(u) \in \{(\log f(x_0), x_0)\} \subset \mathbb{R} \times L(p,q),
\]
where the evaluation map \( \text{ev} \) is given by \([68]\).
(Jn4) When viewed as an almost complex structure on
\[ (N^+ \setminus U) \cup ((-\infty, n + \varepsilon] \times L(p, q')), \]
\( J_n \) is regular for any simple curve passing through \( V^+ \) and satisfying an
priori constraint. The corresponding statement holds for simple curves
passing through \( V^- \) when \( J_n \) is viewed as an almost complex structure on
\[ \left( [-n - \varepsilon, +\infty) \times L(p, q') \right) \cup (N^- \setminus U). \]

(Jn5) \( J_n \) is compatible with \( d(e^s\alpha) \).

We note that one can find an almost complex structure \( J_n \) without the condition
\([\text{Jn3}]\) being satisfied. In fact, we can construct \( J_n \) on by interpolating between
\( J_n \) and the restriction of \( d\Phi_n \circ J_n' \circ (d\Phi_n)^{-1} \) to \( U \) on the region \( V^+ \sqcup V^- \) via
almost complex structures compatible with \( d(e^s\alpha) \). Since \( J_n' \) is compatible with
\( d(\phi\alpha') \), \( d\Phi_n \circ J_n \circ (d\Phi_n)^{-1} \) is compatible with \( d(e^s\alpha) \). Once such a reference
almost complex structure is fixed, after a generic perturbation of it on the open
set \( V^+ \sqcup V^- \), we get an almost complex structure \( J_n \), for which any simple curve
passing through the above open set is Fredholm regular, see Theorem 7.1 in \([18]\).
Moreover, via an argument similar to the one in Lemma 2.5 of \([16]\), we may assume
that the evaluation map in \([68]\) is transverse for any simple parametrized curve
passing through the the above open set. In fact, one can impose the transversality
condition of the evaluation map at level of universal moduli space and show that
the universal moduli space of simple curves with constraints is a smooth manifold
and then one gets a generic perturbation of the almost complex structure.

Now we consider the moduli space \( \mathcal{M}(J_n) \) of \( J_n \)-holomorphic pair of pants with
the asymptotics given by \([65]\) and \([66]\), satisfying \([78]\).

**Lemma 4.1.** Any \( J_n \)-holomorphic pair of pants satisfying \([65]\) and \([66]\) is simple.

**Proof.** We note that since the negative end of a such a pair of pants \( u \) has action
2\( \pi \), it is a simple orbit unless without loss of generality it is \( p\pi_0 \). Assume that \( u \)
factors through a branched covering \( \psi \) and a simple curve \( v \). By Riemann-Hurwitz
formula \( v \) is rational. Since \( p \) is prime, the degree of \( \psi \) is \( p \) and the negative end
of \( v \) is \( \pi_0 \). It is clear that \( v \) has at most two positive ends. Note that \( v \) can not be
a cylinder since the positive ends of \( u \) are geometrically distinct. Hence \( v \) is also a
pair of pants. In particular, \( \psi \) maps punctures of \( u \) to punctures of \( v \) respectively.
Hence the ramification number of \( \psi \) at each puncture of \( u \) has to be \( p \). In particular \( p \) divides both \( k^0 \) and \( k^\infty \). But this is not possible since \( k^0 + k^\infty = p + 1 \).

By the lemma above and \([\text{Jn3}]\) we conclude that \( \mathcal{M}(J_n) \) is a 0-dimensional
manifold.

Our aim is to show that \( \mathcal{M}(J_n) \) is not empty. To this end, we choose a generic homotopy \( (J_t)_{t \in [0,1]} \) of almost complex structures with the following properties:

- **(Jt1)** \( J_0 = J_n \) and \( J_1 = J_n \),
- **(Jt2)** \( J_t = J_n \) on \( V^c \) for all \( t \),
- **(Jt3)** \( J_t \) is compatible with \( d(e^s\alpha) \) for all \( t \),
- **(Jt4)** \( (J_t)_{t \in [0,1]} \) is a regular homotopy of almost complex structures for any relevant moduli problem concerning simple curves together with the evaluation
condition given by \([78]\) and for which \( J_n \) and \( J_0 \) are regular.

The existence of such a homotopy follows from the parametric version of the geo-
metric transversality statement used for \( J_n \), see Remark 7.4 in \([18]\). In what follows,
we need the regular homotopy property of \( (J_t)_{t \in [0,1]} \) for only finitely many moduli
problems. Hence \([\text{Jt4}]\) is safely assumed.
We recall that $J_0 = J_{\alpha}$ is regular for the pair of pants we consider and Lemma 4.1 applies to $J_t$. Hence the moduli space
\[ \bigcup_{t \in [0,1]} \mathcal{M}^0(J_t) \]
of $J_t$-holomorphic pair of pants with asymptotics given by (65) and (66) together with the evaluation condition is a one-dimensional cobordism between $\mathcal{M}^0$ and $\mathcal{M}^0(J_n)$. We note that if the cobordism $\bigcup \mathcal{M}^0(J_t)$ is compact then $\mathcal{M}^0(J_n)$ is not empty since $\mathcal{M}^0$ consists of an odd number of points.

Compactness of the cobordism

Let $t_0 \in [0,1]$ and $(u_n)$ be a sequence of $J_{\alpha}$-holomorphic pair of pants such that $t_n$ converges to $t_0$. Then by the SFT compactness theorem, there is a subsequence, again denoted by $(u_n)$, converging to a holomorphic building $u_\infty$. A priori the holomorphic building $u_\infty$ is a collection of curves that lie in $\mathbb{R} \times L(p,q)$, which are either $J_{\alpha}$-holomorphic or $J_{\alpha}$-holomorphic. These components fit together along their asymptotic ends and lead to the level structure of the building, see [1], [2].

Since the complex structure on the domain of $u_n$’s is fixed, the components of the building $u_\infty$ emerge only out of bubbling-off. In particular, there is a finite set $P \subset \mathbb{C}P^1 \setminus \Gamma$ such that on $\mathbb{C}P^1 \setminus (\Gamma \cup P)$ the sequence $(u_n)$ has a uniform gradient bound. Hence there exist a component
\[ u_0 : \mathbb{C}P^1 \setminus (\Gamma \cup P) \to \mathbb{R} \times L(p,q) \]
of $u_\infty$ such that one of the followings hold:

- $u_0$ is $J_{\alpha}$-holomorphic and $u_n$ converges to $u_0$ in $C^\infty_{\text{loc}}$ on $\mathbb{C}P^1 \setminus (\Gamma \cup P)$ after a sequence of shifts in $\mathbb{R}$-direction,
- $u_0$ is $J_{\alpha}$-holomorphic and $u_n$ converges to $u_0$ in $C^\infty_{\text{loc}}$ on $\mathbb{C}P^1 \setminus (\Gamma \cup P)$.

We note that the signs of the punctures of $u_0$ that are in $\Gamma$ do not have to match with the signs of the punctures of $u_n$’s but the homotopy classes of corresponding asymptotics do have to match. In particular, $u_0$ is non-constant. The remaining structure of the building is given by so called a bubble tree. Instead of describing the a priori structure of the bubble tree, we immediately utilize a particular control over the components, which is due to the following fact.

**Proposition 4.1.** Given any component of the limiting building, the total action of its positive ends is greater or equal to the total action of its negative ends.

We note that for the components of $u_\infty$ that lie in upper or lower translation invariant levels the above statement is trivial. The non-trivial part of the statement is about $J_{\alpha}$-holomorphic components, namely the ones in the middle level and the proof is given in the next section.

The first implication of Proposition 4.1 is the absence of holomorphic planes. We note that a holomorphic plane requires a positive end of action at least $2\pi$. Together with the action of the very bottom end of the building, a finite energy plane forces the total action of the very top end of the building to be at least $4\pi$. But we know that the total action at the top is $2\pi(1 + 1/p)$.

Now a priori there may be components of the building that are bubbled off at points in $P$. But any collection of such components associated to a given bubble point in $P$ must contain a finite energy plane. Since such planes are ruled out we conclude that $P = \emptyset$. Next we consider the components that are bubbled off at the punctures in $\Gamma$ and note that any such component is cylindrical since the domain of the building has arithmetic genus zero and there are no holomorphic planes. Hence we have the essential component with the puncture set $\Gamma$ (possibly with different signs compared to $u_n$’s) and the remaining components are cylindrical (possibly
with two positive ends), which can be grouped into collections associated to the punctures in \( \Gamma \).

Concerning the signs of the punctures of components, we first note that the puncture 1 is has to stay negative since turning it into a positive puncture requires a cylindrical component associated to the puncture 1, which has two negative punctures. Consequently only one of the punctures among 0 and \( \infty \) may change sign. But it is easy to see that by Proposition 4.1 such a configuration is not possible. Hence the building consists of the essential component \( u_0 \), which is an honest pair of pants and we have honest cylinders.

Concerning the level structure we note the following. Since there is no bubbling off at the marked point 2, we have \( C_{loc}^{\infty} \)-convergence of \( u_n \) to \( u_0 \). Combining this with the fact that \( u_n \)'s satisfy the evaluation condition \( (68) \) we conclude that the \( u_0 \) lies in the middle layer.

Now we need to show that there are no non-trivial cylinders in upper and lower levels. To this end we let \( A^+ \) denote the total action of the positive/negative ends in the middle layer. We note that minimal action of a closed Reeb orbit is \( 2\pi/\rho \) if the orbit is non-contractible and \( 2\pi \) if the orbit is contractible. By Proposition 4.1 we have following possibilities regarding \( A^\pm \):

1. \( A^+ = 2\pi(1 + 1/\rho) \) and \( A^- = 2\pi \),
2. \( A^+ = 2\pi(1 + 1/\rho) \) and \( A^- = 2\pi(1 + 1/\rho) \),
3. \( A^+ = 2\pi \) and \( A^- = 2\pi \).

We first assume that \( A^+ = 2\pi(1 + 1/\rho) \) and \( A^- = 2\pi \). In this case any cylinder in an upper or lower level has trivial \( da \)-energy. Hence any such cylinder is trivial. Now it remains to rule out last two cases.

**Lemma 4.2.** The case of \( A^+ = 2\pi(1 + 1/\rho) \) and \( A^- = 2\pi(1 + 1/\rho) \) is not possible.

*Proof.* In this case there is no non-trivial cylinder in the upper levels and there has to be a non-trivial cylinder in a lower level. But such a cylinder has to have a contractible positive end with action \( 2\pi(1 + 1/\rho) \) and this is not possible. \( \Box \)

**Lemma 4.3.** The case of \( A^+ = 2\pi \) and \( A^- = 2\pi \) is not possible.

*Proof.* We note that there is no non-trivial component in the lower level and there is only one non-trivial cylinder in the upper level since the minimal period of Reeb orbits is \( 2\pi/\rho \) and this is precisely the action difference between \( A^+ \) and the total action at the top of the building. The index of the non-trivial cylinder is at least 2 (in fact it is precisely 2 in this case) and therefore the index of \( u_0 \) is at most 2. Now if \( u_0 \) is simple then we get a contradiction since \((J_t)_{t \in [0,1]}\) is a regular homotopy and \( u_0 \) satisfies a 4-dimensional constraint.

Next we assume that \( u_0 \) is multiply covered. Without loss of generality we further assume that the positive end of the non-trivial cylinder is \( k_0\gamma_0 \). Then negative end of the non-trivial cylinder is \( (k_0 - 1)\gamma_\infty \) and the positive ends of \( u_0 \) are given by \( (k_0 - 1)\gamma_\infty \) and \( k_\infty\gamma_\infty \). Note that the negative end of \( u_0 \) is not simple and it has to be \( p\gamma_0 \) or \( p\gamma_\infty \). Now let

\[ v : \Sigma \setminus \Gamma \to \mathbb{R} \times L(p, q) \]

be the underlying simple curve and \( \psi : \mathbb{C}P^1 \to \Sigma \) be the branched covering so that \( u_0 = v \circ \psi \) and \( \psi^{-1}(\Gamma) = \{0, 1, \infty\} \). Let \( N > 1 \) be the degree of \( \psi \). By Riemann-Hurwitz formula, \( \Sigma \) is a sphere. Since \( u_0 \) has only one negative end so does \( v \) and this negative end has multiplicity \( p/\rho \). Since \( p \) is prime, we get \( N = p \). It is clear that \( v \) has at most two positive ends.

We first assume that \( v \) has one positive end, namely \( l\gamma_\infty \). Let \( r_{0/\infty} \) be the ramification number of \( \psi \) at the point \( 0/\infty \). Then we get \( k_0 - 1 = r_0l \) and \( k_\infty = r_\infty l \). Since \( k_0 - 1 + k_\infty = r_0 + r_\infty = p \), we get \( l = 1 \). By Lemma 4.1 and the fact that
q \neq \pm 1$, the negative end of $v$ is also $\gamma_\infty$. Hence the Fredholm index of $v$ is 0 as an unparametrized curve. Fixing a parametrization and viewing $v$ as a parametrized curve it has Fredholm index 2 but it satisfies a 4-dimesional constraint induced by $u_0$ and $\psi$. Since $(J_t)_{t \in [0,1]}$ is a regular homotopy and $v$ is simple, this is not possible.

Now we assume that $v$ has two positive ends. In this case the ramification number of the points 0 and $\infty$ are both $p$. Hence $p$ divides both $k_0 - 1$ and $k_\infty$. But this is not possible since $k_0 + k_\infty - 1 = p$. □

Hence we conclude that the only non-trivial component of the limiting building $u_\infty$ is a pair of pants $u_0$ having the asymptotics of $(u_n)$ and satisfying the evaluation condition. This finishes the proof of the compactness of the cobordism.

Stretching the neck
Knowing that the cobordism $\cup_{t \in [0,1]} M_t(\alpha)$ is compact, we get $J_\alpha$- holomorphic pair of pants $u_\alpha$ for each $\alpha$. We look at the limit of the sequence $(u_n)$ as $n \to \infty$. We know that a subsequence of $(u_n)$, again denoted by $(u_n)$, converges to a holomorphic building $u_\infty$. The limiting building has $J_\alpha$- holomorphic components in upper and lower levels, $J_\alpha'$- holomorphic components in middle levels and finally some components in the upper and the lower connecting levels. The connecting levels has the following description. Note that the hypersurface $\Sigma = \Phi(\{0\} \times L(p,q'))$ divides $\mathbb{R} \times L(p,q)$ into two components $N^+$ and $N^-$ each admitting $\Sigma$ as its boundary. The upper connecting level can be seen as the manifold

$$W^+ := N^+ \cup (-\infty, \varepsilon) \times L(p,q')$$

where the open neighbourhood $\Phi([0,\varepsilon) \times L(p,q'))$ of $\Sigma \subset N^+$ is identified with $[0,\varepsilon) \times L(p,q')$ via the symplectomorphism $\Phi$ that is given by (70). $W^+$ is endowed with an almost complex structure $J^+$ such that $J^+ = J_\alpha$ on $N^+ \setminus V^+$ and $J^+ = J_\alpha'$ on $(-\infty,0] \times L(p,q')$. Moreover $J^+$ is regular for any simple curve that passes through the open set $V^+$ together with an a priori constraint if applies. This follows from the fact that $J^+$ coincides with some $J_n$ on $V^+$ and the assumption (Jn4) Similarly the lower connecting level is given by

$$W^- := N^- \cup (-\varepsilon, +\infty) \times L(p,q')$$

FIGURE 5. The case $A^+ = 2\pi$ and $A^- = 2\pi$. 

Hence we conclude that the only non-trivial component of the limiting building $u_\infty$ is a pair of pants $u_0$ having the asymptotics of $(u_n)$ and satisfying the evaluation condition. This finishes the proof of the compactness of the cobordism.
together with the almost complex structure $J^-$ such that $J^-$ is $J_0'$ on $N^- \setminus V^-$ and $J^- = J_\nu'$ on $[0, +\infty) \times L(p, q')$ and and by $(1)$ it is regular for any simple curve passing through $V^-$ and satisfying an a priori constraint.

As before, since the domains of $u_0$’s are fixed, we have a bubble tree structure on the limiting building. We have an essential component $u_0$ with domain $P \setminus (U \cup P')$, which in this case can be $J_\alpha$- holomorphic or $J_\nu'$- holomorphic or $J^+$- holomorphic. In order to rule out unpleasant components of the building, we want to argue in terms of the actions of the asymptotic ends as before. Similar to Proposition 4.1 we have an a priori control over the actions of asymptotic ends of $J^+$- holomorphic components.

**Proposition 4.2.** The total action at the positive ends of any component in the upper or the lower connecting level is greater or equal to the total action at its negative ends.

We postpone the proof of this statement to the next section and continue with the proof of the main statement. We note that replacing Proposition 4.1 with Proposition 4.2, the previous arguments apply to $u_\infty$ word by word since in $L(p, q')$, the minimal action of the Reeb orbits $2\pi/p$ and the action of a contractible orbit is $2\pi$. Hence we have a pair of pants $u_0$ and bunch of cylinders in the building. Moreover, $u_0$ converges to $u_0$ in $C^\infty_{\text{loc}}$ near the marked point 2 and in the light of the equation (75) $u_0$ lies in a middle layer since $u_0$’s satisfies (78). Note that since the contactomorphism (64) induces an isomorphims on the fundamental group, $u_0$ has two positive non-contractible ends and one negative contractible end.

Due to the action window $u_0$ is the only non-trivial component in the middle layer. We let $A^+$ to be the total action of the positive ends of $u_0$ and $A^-$ be the total action of the negative end of $u_0$. By Proposition 4.2 we have following possibilities regarding $A^+$:

1. $A^+ = 2\pi(1 + 1/p)$ and $A^- = 2\pi$,
2. $A^+ = 2\pi(1 + 1/p)$ and $A^- = 2\pi(1 + 1/p)$,
3. $A^+ = 2\pi$ and $A^- = 2\pi$.

We first discuss the unpleasant cases.

**Lemma 4.4.** The case of $A^+ = 2\pi(1 + 1/p)$ and $A^- = 2\pi(1 + 1/p)$ is not possible.

**Proof.** Note that the negative end of $u_0$ has action $2\pi(1 + 1/p)$ but such an orbit cannot be contractible. \qed

**Lemma 4.5.** The case of $A^+ = 2\pi$ and $A^- = 2\pi$ is not possible.

**Proof.** In this case $u_0$ has trivial $\alpha$-energy and this is possible only if $u_0$ is a cover of a trivial cylinder. In this case, the negative end of $u_0$ is either $p^\infty_{\tau_0}$ or $p^\infty_{\tau_\infty}$. Without loss of generality we assume that it is $p^\infty_{\tau_0}$. Then $u_0$ is the $p$-fold cover of the trivial cylinder over $p^\infty_{\tau_0}$. But this is not possible since the image of $u_0$ misses the point $(0, \varphi(x_0))$ in $\mathbb{R} \times L(p, q')$ due to the choice of the point $x_0$, see Figure 6. \qed

Now consider the case $A^+ = 2\pi(1 + 1/p)$ and $A^- = 2\pi$. In this case $u_0$ is a $J_\alpha$- holomorphic pair of pants. Moreover we have cylinders $C_1$ and $C_2$ in $W^+$ such that $C_1$ has the positive end $k^\infty_{\tau_0}$ and $C_2$ has the positive end $k^\infty_{\tau_\infty}$. Following the discussions given in the previous sections we conclude that $u_0$ is pair of pants with a base curve of degree 1. We have two possible profiles for the positive ends of $u_0$:

1. $u_0(0) = (+\infty, l^\infty_{\tau_0})$ and $u_0(\infty) = (+\infty, l^\infty_{\tau_\infty})$,
2. $u_0(0) = (+\infty, l^\infty_{\tau_\infty})$ and $u_0(\infty) = (+\infty, l^\infty_{\tau_0})$. 

In both cases we have $l^0 + l^\infty = p + 1$. Following Lemma 2.2 we have

\[ l^0 \equiv (1 - v')^{-1} \quad \text{and} \quad l^\infty \equiv (1 - q')^{-1} \]

in the first case. Here the integer $v'$ is given by $1 < v' < p - 1$ and $v'q' \equiv 1$. Now the negative end of $C_1$ is $l^0 \pi_0$ and the negative end of $C_2$ is $l^\infty \pi_\infty$. We apply Proposition 4.2 to $C_1$ and $C_2$ and use the fact that $l^0 + l^\infty = k^0 + k^\infty = p + 1$ to conclude $l^0 = k^0$ and $l^\infty = k^\infty$. We finally get

\[ (1 - v')^{-1} \equiv l^0 = k^0 \equiv (1 - v)^{-1} \Rightarrow v' \equiv v \Rightarrow q' \equiv q. \]

In the second case, we have

\[ l^0 \equiv (1 - q')^{-1} \quad \text{and} \quad l^\infty \equiv (1 - v')^{-1}. \]

Considering the cylinder $C_1$ again we get

\[ (1 - q')^{-1} \equiv l^0 = k^0 \equiv (1 - v)^{-1} \Rightarrow q' \equiv v \Rightarrow q' \equiv q^{-1}. \]

This concludes the proof of the theorem.

4.2. Action control via 4-dimensional tools. In this section we prove Proposition 4.1 and Proposition 4.2. The method is to apply intersection theory for closed holomorphic curves to the lifts of relevant punctured curves in symplectic cobordisms associated to lens spaces. Once these curves are lifted to the cobordisms associated to $S^3$, we compactify these cobordisms and extend the punctured curves to closed curves in order to apply the intersection theory.

4.2.1. Basics of intersection theory. We briefly recall the basics of the intersection theory of closed holomorphic curves. For the details and proofs of the following statements, we refer to [10] and [19].

Let $W$ be a closed oriented 4-manifold, $\Sigma$ and $\Sigma'$ be closed oriented surfaces, and let $u : \Sigma \to W$, $v : \Sigma' \to W$ be smooth maps. An intersection $u(z) = v(w) = p$ is transverse if $du(T_z \Sigma) \oplus dv(T_w \Sigma') = T_p W$. We say this intersection is positive if the direct sum of the orientations of the surfaces coincides with the orientation of $W$ and say negative otherwise. We define the local intersection index $i(u, z; v, w)$ as +1 if the intersection is positive and -1 otherwise. We note that if an intersection
is transverse then it is isolated. Hence if all intersections of $u$ and $v$ are transverse, there are finitely many of them and we can define the total intersection number

$$[u] \cdot [v] = \sum_{u(z) = v(w)} i(u, z; v, w).$$

It turns out that $[u] \cdot [v]$ depends only on the homology classes $[u], [v] \in H_2(W)$. Moreover, it defines a bilinear symmetric form on $H_2(W)$, which is non-degenerate.

If an intersection $u(z) = v(w) = p$ is not transverse but still isolated, one can still define a local intersection index as follows. One localizes the intersection via closed discs $D_z$ and $D_w$ around $z$ and $w$. Then one picks $C^\infty$-small perturbation $u_\epsilon$ of $u$ so that when restricted to $D_z$ and $D_w$, $u_\epsilon$ and $v$ have only transverse intersections and $u_\epsilon(\partial D_z) \cap v(D_w) = \emptyset$. Then one defines

$$i(u, z; v, w) = \sum_{u_\epsilon(z') = v(w')} i(u_\epsilon, z'; v, w')$$

where the sum is taken for $(z', w') \in D_z \times D_w$.

Now we assume that $W$ is equipped with an almost complex structure $J$ and it is oriented via $J$. We also assume that $\Sigma$ and $\Sigma'$ carry complex structures $j$ and $j'$ respectively, and they are oriented via $j$ and $j'$. Finally, we assume that $u$ and $v$ are closed $J$-holomorphic curves, that is $du \circ j = J \circ du$ and $dv \circ j = J \circ dv$. A well known fact is that any intersection $u(z) = v(w) = p$ of two such curves is either isolated or there are neighborhoods $z \in U$ and $w \in V$ such that $u(U) = v(V)$. We note that this phenomenon is independent of the dimension of $W$.

The special features of the case $\dim W = 4$ are as follows. It is clear that for any transverse intersection $u(z) = v(w) = p$, we have $i(u, z; v, w) = +1$. The non-trivial fact is that if an intersection is isolated then $i(u, z; v, w) \geq 1$, with equality if and only if the intersection is transverse. This phenomenon is referred as local positivity of intersections and has the following global consequence. By the principle of unique continuation, one can show that if $u$ and $v$ have infinitely many intersections then $\text{im}(u) = \text{im}(v)$, that is either one is a reparametrization of the other or they are multiple covers of the same simple curve. Hence, if $\text{im}(u) \neq \text{im}(v)$, there are finitely many intersections and we have

$$[u] \cdot [v] \geq \# \{(z, w) \in \Sigma \times \Sigma' | u(z) = v(w)\},$$

with equality if and only if all the intersections are transverse. In particular, $[u] \cdot [v] = 0$ if and only if $\text{im}(u) \cap \text{im}(v) = \emptyset$. We refer this fact as global positivity of intersections.

Finally, we want to address the question of self-intersections of a holomorphic curve. Let $u : \Sigma \to W$ be a closed $J$-holomorphic curve. If $u$ is not simple, then it is clear that it has infinitely many double points. But if $u$ is simple, one has only finitely many intersections. For a simple curve $u$, one defines

$$\delta(u) = \frac{1}{2} \sum_{u_\epsilon(z) = u_\epsilon(w), z \neq w} i(u_\epsilon, z; u_\epsilon, w),$$

where $u_\epsilon$ is some $C^\infty$-small perturbation of $u$ that is immersed. It turns out that $\delta(u)$ is non-negative for any such perturbation $u_\epsilon$, and vanishes if and only if $u$ is embedded. In fact, $\delta(u)$ is independent of the chosen perturbation since it satisfies the adjunction formula:

$$[u] \cdot [u] = 2\delta + c_N(u),$$

where the normal Chern number $c_N(u)$ is defined as $c_N(u) = c_1([u]) - \chi(\Sigma)$. We note that $c_N(u) = c_1(N_u)$ when $u$ is immersed, where $N_u$ is the normal bundle of $u$. 
4.2.2. Intersections in the completions of the symplectizations and symplectic cobordisms. We recall the biholomorphic identification between \((\mathbb{R} \times S^3, J_0)\) and \(L^*\), where the latter space is the total space of the tautological line bundle without its zero section. We now consider the compact completion \(\hat{L}\) of \(L\), where we compactify each fibre by turning it into \(\mathbb{C}P^1\). We get the complex manifold \(\hat{L}\) with the complex structure \(J_0\) that extends \(J_0\). We note that \(\hat{L}\) is a sphere bundle over \(\mathbb{C}P^1\) such that each fibre is holomorphic. Moreover \(\hat{L}\) contains two holomorphic embedded spheres \(S_0\) and \(S_\infty\), first being the zero section of \(L\) and second being the section at infinity such that

\[
H_2(\hat{L}) = \mathbb{Z} \cdot [S_0] \oplus \mathbb{Z} \cdot [S_\infty].
\]

Let

\[
u : \Sigma \setminus (\Gamma^+ \cup \Gamma^-) \to L^*
\]

be a finite energy \(J_0\)-holomorphic curve. We recall from the previous sections that \(c := \pi \circ \nu : \Sigma \setminus (\Gamma^+ \cup \Gamma^-) \to \mathbb{C}P^1\), extends to a holomorphic branched covering \(c : \Sigma \to \mathbb{C}P^1\) with degree \(d\) and \(\nu\) is identified with a meromorphic section of the bundle \(c^*L \to \Sigma\), zeros corresponding the negative ends and poles corresponding to the positive ends, with

\[\#\text{zeros} - \#\text{poles} = -d.\]

Now we extend \(\nu\) to \(\hat{L}\) by extending the corresponding meromorphic section over the zeros and poles using the local holomorphic coordinates. We note that the extension is unique and we end up with a closed curve, denoted by \(\hat{\nu}\) in \(\hat{L}\). Viewing a fibre \(F\) of \(L\) as a trivial cylinder, we get a fibre \(\hat{F}\) of \(\hat{L}\). We observe the following.

**Lemma 4.6.** Let \(\nu\) be a punctured curve as above with underlying base curve having degree \(d\). Let \(\hat{F}\) be a fibre of \(\hat{L}\). Then

\[
[\hat{\nu}] \cdot [\hat{F}] = - (\sum k^-_j) + (\sum k^+_j) = d,
\]

where \(k^-_j/ k^+_j\)'s are the multiplicities of positive/negative ends of \(\nu\).

**Proof.** We first observe that \([\hat{\nu}] \cdot [S_0] = \sum k^-_j\) and \([\hat{\nu}] \cdot [S_\infty] = \sum k^+_j\). In fact, if \(\nu\) has a negative end with multiplicity \(k\) at a puncture \(z\) and \(c\) has the ramification number \(r\) at \(z\), then one can locally write \(u(z) = (z^r, z^k)\). In order to compute the intersection number with \([S_0]\) at \(z\), one needs to compute the local intersection number between \(u\) and \(v(w) = (w, 0)\). Since the intersection at 0 is not transverse, we perturb \(u\), and put \(u'_i(z) = (z^r, z^k + \epsilon)\). We see that an intersection \((z, w)\) is a solution of the system \((z^r, z^k + \epsilon) = (w, 0)\) and the second coordinates produce \(k\)-distinct roots \(z_1, \ldots, z_k\) of \(-\epsilon\) and for each \(z_i,\) we have \(w_i = z_i^r\). We note that \(v'(w_i) = (1, 0)\) and \(u'_i(z_i) = (z_i^r, k z_i^{k-1})\) has a non-vanishing second coordinate and hence the intersections are transversal. Same argument applies for a pole. In particular if \(F\) is a fibre, equivalently a trivial cylinder as a punctured curve, then

\[
[\hat{F}] \cdot [S_0] = [\hat{F}] \cdot [S_\infty] = 1.
\]

Next we observe that \([S_0] \cdot [S_0] = c_1(N_0) = -1\). Here \(N_0\) is the normal bundle of \(S_0\) and it can be identified with the bundle \(L\). Similarly, \([S_\infty] \cdot [S_\infty] = c_1(N_\infty) = 1\) since the normal bundle \(N_\infty\) of \(S_\infty\) can be identified with the dual bundle of \(L\). It is clear that \([S_0] \cdot [S_\infty] = 0\). Using the intersection pairing on \(H_2(\hat{L})\), we get

\[
[\hat{\nu}] \cdot [\hat{F}] = -[S_0] + [S_\infty],
\]

\[
\Rightarrow [\hat{\nu}] \cdot [\hat{F}] = - (\sum k^-_j) + (\sum k^+_j) = d.
\]

\(\square\)
Note that the lemma above can be interpreted as the positivity of $\partial\alpha$-energy. Namely, the total action of the positive ends of $u$ is greater or equal to the total action of its negative ends. In fact, since the degree of $c$ is non-negative, we get

$$-(\sum k_j^-) + (\sum k_i^+) = d \geq 0 \Rightarrow \sum 2k_i^+ \pi \geq \sum 2k_j^- \pi.$$ 

Now we want to see how much of the above structure is preserved if one considers an almost complex structure with cylindrical ends.

**Lemma 4.7.** Let $J$ be an almost complex structure on $\mathbb{R} \times S^3$ which coincides with $J_0$ outside of a compact set and tamed by a suitable symplectic form. Let $u$ be a finite energy $J$-holomorphic curve. Let $\hat{J}$ be the extension of $J$ to the completion $\hat{L}$. Assume further that the fibre class $[\hat{F}]$ of $\hat{L}$ admits a $\hat{J}$-holomorphic representative $\hat{u}_0$. Then the action difference between the positive ends and the negative ends of $u$ is non-negative.

**Proof.** We note that $J$ extends to an almost complex structure $\hat{J}$ on $\hat{L}$ since it coincides with $J_0$ near the ends. Hence we have $\hat{J}$-holomorphic spheres $S_0$ and $S_\infty$. Now let $z$ be a negative puncture with the asymptotic end of multiplicity $k$. Note that the projection $L \to \mathbb{C}P^1$ is not $J$-holomorphic in general. Nevertheless we consider a punctured disk neighbourhood $D^*$ of $z$ in $\Sigma$ and consider $u : D^* \to L^*$. This map is $J_0$-holomorphic and leads to a holomorphic map $c : D^* \to \mathbb{C}P^1$. Since $u$ has finite energy, $c$ extends over the origin and one gets a section $f : D^* \to c^*L$. Clearly the section $f$ is holomorphic and its singularity at the origin is a zero of order $k$ due to the asymptotic behaviour of $u$. A similar argument applies to the positive punctures and we conclude that $u$ extends to a closed curve $\hat{u}$ in $(\hat{L}, \hat{J})$. Moreover one has $[\hat{u}] \cdot [S_0] = \sum k_j^-$ and $[\hat{u}] \cdot [S_\infty] = \sum k_i^+$ where $k_j^-, k_i^+$’s are the multiplicities of the negative/positive ends of $u$. Similarly one has $[\hat{u}_0] \cdot [S_0] = [\hat{u}_0] \cdot [S_\infty] = 1$.

Hence we get $[\hat{u}] \cdot [\hat{u}_0] = -(\sum k_j^-) + (\sum k_i^+)$ as before.

We first assume that $\text{im}(\hat{u}) \neq \text{im}(\hat{u}_0)$. Then by the positivity of intersections we get

$$[\hat{u}] \cdot [\hat{u}_0] = -(\sum k_j^-) + (\sum k_i^+) \geq 0.$$ 

Multiplying both sides with $2\pi$ gives the desired conclusion. Now if $\text{im}(\hat{u}) = \text{im}(\hat{u}_0)$, then we view $u_0$ as a cylinder with simple ends and note that $u$ factors through $u_0$. Let $\psi$ be the underlying branched covering with degree $N$. Then the ramification number of any puncture of $u$ is precisely the multiplicity of the corresponding asymptotic end since the asymptotics of $u_0$ are simple. In particular $N = \sum k_j^+ = \sum k_i^-$ and therefore $\sum k_j^+ \geq \sum k_i^-$. □
4.2.3. Proof of Proposition 4.4 We want to apply Lemma 4.7 to a lift of given \( J_{t_0} \)-holomorphic component. To this end we need to specify coverings of the cobordisms at hand. We fix a lift

\[ \tilde{\varphi} : S^3 \to S^3 \]

of the contactomorphism \([61]\) such that \( \tilde{\varphi} \circ \sigma = \sigma' \circ \tilde{\varphi} \), where \( \sigma' \) is the standard generator of \( \mathbb{Z}_p \)-action that leads to \( L(p, q') \). We put \( \tilde{\beta}_0 := \tilde{\varphi} \circ \alpha_0 \). Here \( \tilde{f} = f \circ \tilde{p} : S^3 \to (0, +\infty) \) is the lift of \([67]\). We let

\[ \tilde{\Phi} : \mathbb{R} \times S^3 \to \mathbb{R} \times S^3 \]

be the corresponding lift of \([70]\). We have \( \tilde{\Phi} \circ \sigma = \sigma' \circ \tilde{\Phi} \) where \( \sigma \) and \( \sigma' \) acts on the \( \mathbb{R} \)-direction trivially. We put \( \Sigma := \tilde{\Phi}(\{0\} \times S^3) \). Due to the translation invariance of \( \sigma \) and \( \sigma' \), the construction of \( W_n \) given by \([74]\) lifts via \( \tilde{\Phi} \) for any integer \( n \). We have a lifted diffeomorphism

\[ \tilde{\Phi}_n : \tilde{W}_n \to \mathbb{R} \times S^3 \]

and a free \( \mathbb{Z}_p \)-action on \( \tilde{W}_n \) generated by \( \sigma_n := \tilde{\Phi}^{-1} \circ \sigma \circ \tilde{\Phi} \). In particular we have a covering space \( \tilde{W}_n \to W_n \) and \( J_n \) lifts to a \( \sigma_n \)-invariant almost complex structure. Looking at the picture on the other side, we have \( \sigma \)-invariant almost complex structures \( (\tilde{J}_t)_{t \in [0, 1]} \) on \( \mathbb{R} \times S^3 \) given by the lift of the path \( (J_t)_{t \in [0, 1]} \).

Let \( t_0 \in [0, 1] \) be given. In order to make use of Lemma 4.7, we want to construct a \( \tilde{J}_{t_0} \)-holomorphic sphere \( u : \mathbb{C}P^1 \to \tilde{L} \) in the fibre class \( [\tilde{F}] \), where \( \tilde{J}_{t_0} \) is the extension of \( J_{t_0} \). First we want to show that such a curve is embedded. Since \( [\tilde{F}] \) is a primitive class, such a \( u \) is simple. Applying the adjunction formula to \( u \), we get

\[ [u] \cdot [u] = 2\delta(u) + c_1([\tilde{F}]) - 2. \]

We note that

\[ c_1(S_\infty) = [S_\infty] \cdot [S_\infty] + \chi(S_\infty) = 1 + 2 = 3, \]

\[ c_1(S_0) = [S_0] \cdot [S_0] + \chi(S_0) = -1 + 2 = 1. \]

Hence we get \( c_1([\tilde{F}]) = 3 - 1 = 2 \) and \( [u] \cdot [u] = [\tilde{F}] \cdot [\tilde{F}] = 0 \). Combining all these with the adjunction formula we get \( \delta(u) = 0 \) and therefore \( u \) is embedded.

We want to show that the moduli space of \( \tilde{J}_{t_0} \)-holomorphic spheres is non-empty. To this end we consider the moduli space \( M_p(J_{t_0}, p) \) of \( \tilde{J}_{t_0} \)-holomorphic spheres in the class \( [\tilde{F}] \) that passes through the point \( p \in S_\infty \subset \tilde{L} \). By the automatic transversality results due to \([9]\), \( J_{t_0} \) is Fredholm regular for this moduli space, see Theorem 2.46 in \([19]\) for the explicit statement. On the other hand, \( M_p(J_{t_0}, p) \) is the space with a single element being the fibre through the point \( p \), or equivalently it consists of the trivial cylinder over the orbit corresponding to \( p \). It is clear from the geometric picture that \( J_0 \) is also Fredholm regular. Hence we are able to pick a regular homotopy \( (J_s)_{s \in [0, 1]} \) that connects \( J_0 \) to \( J_{t_0} \) and has the property that for all \( s \), \( J_s \) coincides with \( J_0 \) near \( S_{0/\infty} \). We consider the resulting 1-dimensional cobordism \( \cup_{s \in [0, 1]} M_p(J_s, p) \). We claim that the cobordism is compact. To show this we take a sequence of \( \tilde{J}_{t_n} \)-holomorphic spheres \( (u_n) \) such that \( s_n \to s_0 \in [0, 1] \).

By the Gromov compactness a subsequence, still denoted by \( u_n \), converges to a nodal curve \( u_\infty \). Ignoring constant components if any, \( u_\infty \) is necessarily a collection of holomorphic spheres \( \{u_1, \ldots, u_a, v_1, \ldots, v_b, w_1, \ldots, w_c\} \) such that

- \( \text{im}(u_i) \neq S_{0/\infty} \) so that \( [u_i] = -k_i^0[S_0] + k_i^\infty[S_\infty], k_i^0, k_i^\infty \geq 0 \) for \( i = 1, \ldots, a \),
- \( \text{im}(v_i) = S_0 \) and therefore \( [v_i] = n_i[S_0], n_i > 1 \) for \( i = 1, \ldots, b \),
- \( \text{im}(w_i) = S_\infty \) and therefore \( [w_i] = m_i[S_\infty], m_i > 0 \) for \( i = 1, \ldots, c \).
Since \([u_\infty] = -[S_0] + [S_\infty]\) we have
\[
(81) \quad \sum_{i=1}^{a} k_i^\infty + \sum_{i=1}^{c} m_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} k_i^0 - \sum_{i=1}^{b} n_i = 1.
\]
From the first equation we get two cases. In the first case we have \(c = 1, m_1 = 1\) and \(k_i^\infty = 0\) for all \(i = 1, \ldots, a\). But then \(u_\infty\) is disconnected as a nodal curve and this is not possible. In the second case we have \(c = 0, k_i^\infty = 1\) and \(k_i^\infty = 0\) for \(i = 2, \ldots, a\). Now we take a closer look at \(u_1\). Since \(u_1\) has a simple intersection with \(S_\infty\) it is a simple curve. By the adjunction formula
\[
\left((-k_1^0[S_0] + [S_\infty])\right) \cdot \left((-k_1^0[S_0] + [S_\infty])\right) = 2\delta(u_1) + c_1[u_1] - 2.
\]
We note that
\[
c_1(u_1) = -k_1^0 c_1(S_0) + c_1(S_\infty) = 3 - k_1^0.
\]
Hence we have
\[
1 - (k_1^0)^2 = 2\delta(u_1) + 3 - k_1^0 - 2 \Rightarrow k_1^0(1 - k_1^0) = \delta(u_1) \geq 0.
\]
Hence we get \(k_1^0 = 1\). Moreover given any \(u_i\) with \(i \neq 1\), we know that \(\text{im}(u_1) \neq \text{im}(u_i)\) since \(k_i^\infty = 1\) and \(k_i^\infty = 0\). By the positivity of intersections we get
\[
0 \leq [u_1] \cdot [u_i] = (-[S_0] + [S_\infty]) \cdot (-k_1^0[S_0]) = -k_1^0 \Rightarrow k_1^0 = 0.
\]
Finally by the second equation of (81) we have \(b = 0\). To sum up \(u_\infty\) is a collection of spheres \(u_1, \ldots, u_a\) such that none of \(u_2, \ldots, u_a\) intersects with \(u_1\). Again since \(u_\infty\) is connected we get \(a = 1\). This concludes that \(u_\infty\) consists of a single component, which is in the class \([\tilde{F}]\) and satisfies the point constraint. Since the cobordism \( \cup_{s \in [0, 1]} M_F(J_s, p) \) is compact, the moduli space \( M_F(J_s, p) \) is non-empty.

Now let \( \overline{\pi} : CP^1 \setminus \Gamma \to \mathbb{R} \times L(p, q) \) be a \( J_{t_{\infty}} \)-holomorphic component. We know that there is a \( \tilde{J}_{t_{\infty}} \)-holomorphic sphere \( u_0 \) in the fibre class \([\tilde{F}]\) of \( \tilde{L} \). Now we assume that \( \overline{\pi} \) has a non-contractible end and the asymptotic profile of \( \overline{\pi} \) are given by \([a1], [a2]\) and \([a3]\). Then using the scheme given in previous section, which is purely topological, after precomposing it with a suitable \( p \)-fold covering we get a lifted \( \tilde{J}_{t_{\infty}} \)-holomorphic curve \( u : \Sigma \setminus \tilde{\Gamma} \to \mathbb{R} \times S^3 \) with the asymptotics given by \([la1], [la2]\) and \([la3]\). Applying Lemma 4.7 to \( \tilde{u} \) and \( u_0 \), we conclude that
\[
\sum_{i=1}^{n} k_i^{0+} + \sum_{i=1}^{n} k_i^{\infty+} + p \sum_{i=1}^{n} k_i^{+} \geq \sum_{i=1}^{n} k_i^{0-} + \sum_{i=1}^{n} k_i^{\infty-} + p \sum_{i=1}^{n} k_i^{-}.
\]
Dividing both sides by \( p \) and multiplying with \( 2\pi \) leads to the claim. If \( \overline{\pi} \) does not have any non-contractible end, it lifts immediately with precisely same multiplicities at the same ends. Again the result follows from Lemma 4.7. We remark that whether \( \overline{\pi} \) has a non-contractible end or not, a contractible end of the form \( k_{p_{\infty}} \gamma_0/\infty \), lifts to an end of the form \( k_{\gamma_0/\infty} \). Hence the argument that compares the total actions at the ends of \( \overline{\pi} \) and \( u \) is again valid.

4.2.4. Proof of Proposition 4.2. After carrying on the procedure above for the lift \( \tilde{J}_{t_{n}} \) of \( J_{t_{n}} \) we get a sequence \( \tilde{u}_{n} \) of \( \tilde{J}_{t_{n}} \)-holomorphic spheres in the homology class \([\tilde{F}]\). Each of these spheres has a single transverse intersection with \( S_0 \) and \( S_\infty \).

Moreover for each \( n \), the intersection of \( \tilde{u}_{n} \) with \( S_\infty \) takes place at a prescribed point, which is away from \( \gamma_0 \) and \( \gamma_\infty \). Now in this section we view these spheres as \( \tilde{J}_{t_{n}} \)-holomorphic cylinders \( (u_n) \) in the symplectic cobordisms \( \tilde{W}_{n} \) and we look at the SFT-limit of these cylinders. The limit \( u_\infty \) is again a holomorphic building with only rational components and we want to extract information out of its structure.

We study the holomorphic building from top to bottom. We first note that the action at the very top of the building is \( 2\pi \), which is minimal. In particular, there is no non-trivial component in upper translation invariant levels. Let \( \tilde{W}^{+} \) denote
the covering space of $W^+$, which is determined by the lifting scheme given at the proof of Proposition 4.1 and which is precisely the upper connecting layer in this case. Now there is a single positive end for all of the components in $\tilde{W}^+$, hence there is a single component, say $u^+$, in this level, which has only a simple positive end.

Now assume that $u^+$ has $k$-many negative ends with multiplicities $m_1, ..., m_k$. We want to show that $k = 1$ and $m_1 = 0$. To this end we compactify $\tilde{W}^+$ and look at the extension of $u^+$ as follows.

Topologically we view $\tilde{W}^+$ as the the upper connected component $\tilde{N}^+$ of $(\mathbb{R} \times S^3) \setminus \tilde{\Sigma}$. We compactify $\tilde{W}^+$ by collapsing its positive end to a 2-sphere $S_\infty$ via the Reeb flow of $\alpha_0$ and by collapsing its negative end, namely $\tilde{\Sigma}$, to a 2-sphere $S_0$ via the Reeb flow of $\beta_0$. When we view $\tilde{W}^+$ as an almost complex manifold equipped with the lift $\tilde{J}^+$ of $J^+$, the negative end is given by $S^3$ and it is collapsed via the Reeb flow of $S^3$ since $\tilde{J}^+$ coincides with $\tilde{J}_0$ near the negative end. We denote the resulting almost complex manifold by $(\tilde{W}^+, \tilde{J}^+)$. It is not hard to see that $H_2(\tilde{W}^+)$ is generated again by two spherical classes $[S_0]$ and $[S_\infty]$ and clearly the representatives $\tilde{S}_0$ and $\tilde{S}_\infty$ are $\tilde{J}^+$-holomorphic. Since $\tilde{J}^+$ coincides with $\tilde{J}_0$ on a tubular neighbourhood of $S_\infty$ in $\tilde{W}^+$, we get $[S_\infty] \cdot [S_\infty] = 1$ by the adjunction formula. Similarly on a tubular neighbourhood of $S_0$, $\tilde{J}^+$ coincides with $\tilde{J}_0$, we also have $[S_0] \cdot [S_0] = -1$ and clearly $[S_0] \cdot [S_\infty] = 0$. The extension $\tilde{u}^+$ satisfies

$$[\tilde{u}^+] = [S_\infty] - M[S_0], \quad M = \sum_{i=1}^{k} m_i.$$
We get
\[ [\hat{u}^+] \cdot [\hat{u}^+] = ([S_\infty] - M[S_0]) \cdot ([S_\infty] - M[S_0]) = 1 - M^2 \]
and
\[ c_1([\hat{u}^+]) = 3 - M. \]
We know that \( \hat{u}^+ \) is a simple curve hence by the adjunction formula we get
\[ 1 - M^2 - 3 + M + 2 = \delta(\hat{u}^+) \geq 0 \Rightarrow M(1 - M) \geq 0. \]
Hence we get \( M = 1 \). In other words, \( u^+ \) is a cylinder with simple ends.

Now let \( \overline{\pi} \) be a given \( J^+ \)-holomorphic component in \( W^+ \). We note that our lifting scheme applies the covering \( \hat{W}^+ \rightarrow W^+ \) as well since the contactomorphism \( \varphi \) induces an isomorphism on the fundamental group. In particular, independent of whether \( \overline{\pi} \) has a non-contractible end or not, the multiplicities of the ends of \( \overline{\pi} \) and the ends of its lift \( u \) are related as before. Now let \( u : \Sigma \setminus \Gamma \rightarrow \hat{W}^+ \) be a \( J^+ \)-holomorphic lift of \( \overline{\pi} \) such that \( k^+_i/k^-_j \)'s are the multiplicities of positive/negative ends of \( u \). As in the proof of Lemma [17] we extend \( u \) over its punctures and get a closed curve \( \hat{u} \) such that [\( \hat{u} = -(\sum k^-_j)[S_0] + (\sum k^+_i)[S_\infty] \]. Applying the positivity of intersections to \( \hat{u} \) and \( \hat{u}^+ \) we get
\[ \Rightarrow [\hat{u}] \cdot [\hat{u}] = \sum k^+_i - \sum k^-_j \geq 0 \Rightarrow \sum k^+_i \geq \sum k^-_j. \]
This concludes the proof for \( J^+ \)-holomorphic components. Note that if \( \text{im}(\hat{u}) = \text{im}(\hat{u}^+) \), the conclusion again follows since \( u^+ \) is a cylinder with simple ends.

Going back to the limiting building \( u_\infty \) and knowing that \( u^+ \) is a cylinder with simple ends, we note that there is a single orbit at the positive end of the first translation invariant level and the action of this orbit is minimal. Hence we have only trivial cylinders in the middle translation invariant levels.

On the lower transition level \( \hat{W}^- \), we have a single component \( u^- \) with a single positive end, which is simple. Assume that \( u^- \) has \( k \)-many negative ends with multiplicities \( m_1, \ldots, m_k \). Now we study the cobordism \( \hat{W}^- \). We view \( \hat{W}^- \) as the the lower connected component \( \hat{N}^- \) of \( (\mathbb{R} \times S^2) \setminus \hat{\Sigma} \). We compactify \( \hat{W}^- \) by collapsing its positive end, namely \( \hat{\Sigma} \), to a 2-sphere \( S_\infty \) via the Reeb flow of \( \beta_0 \), equivalently via the Reeb flow of \( \alpha_0 \), and by collapsing its negative end to a 2-sphere \( S_0 \) via the Reeb flow of \( \alpha_0 \). We denote the resulting almost complex manifold by \( (\hat{W}^-, J^-) \), where \( J^- \) is induced by the lift \( J^- \) of \( J^- \). Again \( H_2(\hat{W}^-) \) is generated again by two spherical classes \( [S_0] \) and \( [S_\infty] \) and the representatives \( S_0 \) and \( S_\infty \) are \( J^- \)-holomorphic. Moreover they satisfy \([S_0] \cdot [S_0] = -1, [S_\infty] \cdot [S_\infty] = 1 \) and \([S_0] \cdot [S_\infty] = 0 \). The extension \( \hat{u}^- \) of \( u^- \) satisfies \([\hat{u}] = [S_\infty] - M[S_0] \), where \( M = \sum_{i=1}^{k} m_i \) and it is simple. Applying the adjunction formula as before we get \( M = 1 \). Given a lifted curve \( u : \Sigma \setminus \Gamma \rightarrow \hat{W}^- \), we look at the intersection of its extension \( \hat{u} \) and \( \hat{u}^- \) and get the desired inequality as before.

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