Bethe ansatz solution of the anisotropic correlated electron model associated with the Temperley-Lieb algebra

A. Lima–Santos,1 Itzhak Roditi2 and Angela Foerster3

1Departamento de Física, Universidade Federal de São Carlos, Caixa Postal 676, 13569-905 São Carlos, Brazil

2Centro Brasileiro de Pesquisas Físicas - CBPF, Rua Dr. Xavier Sigaud 150, 22290-180, Rio de Janeiro, RJ - Brazil

3Instituto de Física da UFRGS, Av. Bento Gonçalves 9500, Porto Alegre, RS - Brazil

A recently proposed strongly correlated electron system associated to the Temperley Lieb algebra is solved by means of the coordinate Bethe ansatz for periodic and closed boundary conditions.

I. INTRODUCTION

Integrable highly correlated electron systems have been attracting increasing interest due to their potential applications in condensed matter physics. The prototypical examples of such systems are the Hubbard and t-J models as well as their supersymmetric generalizations [1]. Recently many other correlated electron models have been formulated [2,3,4,5,6,7,8,9]. Among these an interesting subclass corresponds to models associated to the Temperley-Lieb (TL) algebras [10]. For such models there exists a well established method to construct a series of spin Hamiltonians as representations of the TL algebra and of quantum groups, the R matrix associated with the XXZ chain being the simplest example [11]. Later on, this approach was generalized by Zhang [12] to construct graded representations of the TL algebra using Lie superalgebras and quantum supergroups. Along these lines a new isotropic strongly correlated electron model was obtained [13], as well as its anisotropic version with periodic and closed boundary conditions [14]. In addition, it was shown in [14] that this last
choice of boundary generates a quantum group invariant model, in contrast to the traditional periodic one.

Models with quantum group invariance and closed boundary conditions were first discussed by Martin [13] from representations of the Hecke algebra. More recently, by means of a generalized algebraic Bethe ansatz, Karowski and Zapletal [16] presented a class of quantum group invariant n-state vertex models with closed boundary conditions. Within the framework of the coordinate Bethe ansatz closed spin chains invariant under $U_q(sl(2))$ were investigated by Grosse et al. [17]. Also, an extension of the algebraic approach to the case of graded vertex models [18] was analysed in [19] where a $U_q(sp\ell(2,1))$ invariant susy t-J model with closed boundary conditions was presented.

In this paper we obtain through the coordinate Bethe ansatz approach the solution of the anisotropic, or q-deformed, electronic model proposed in [14], for periodic and closed boundary conditions. Here, the meaning of closed is that an operator, coupling the first and last sites, is introduced into the expression of the Hamiltonian such that we obtain a quantum algebra invariant closed system (see [20] for more details). In particular, for the closed case the Bethe ansatz equations are derived by extending the systematic procedure recently developed in [21] to solve the quantum group invariant closed spin 1 chain associated to the TL algebra for the case of a graded vertex model.

The paper is organized as follows. In section 2 we describe the correlated electron system associated with the TL algebra. In section 3 we find through the coordinate Bethe ansatz the spectra of the model with usual periodic boundary conditions. In section 4 the Bethe ansatz solution is presented for closed boundary conditions. A summary of our main results is presented in section 5.

II. THE MODEL

The starting point for building the model is a 4-dimensional module $V$ of the Lie superalgebra $U_q(g\ell(2/1))$ utilized to obtain a representation of the TL algebra. Let $\{|x\rangle\}^4_{x=1}$ be
an orthonormal basis of $V$ which carries the following parity,

$$[[1]] = [[4]] = 0 \quad [[2]] = [[3]] = 1.$$  \hspace{1cm} (1)

Everywhere we shall use the graded-tensor product law, defined by,

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} (ac \otimes bd)$$

and also the rule,

$$(|x\rangle \otimes |y\rangle)\dagger = (-1)^{|x||y|}\langle x| \otimes \langle y|.$$  

It is then possible to construct the following unnormalized vector of $V \otimes V$

$$|\Psi\rangle = q^{-1/2} |4\rangle \otimes |1\rangle + q^{1/2} |1\rangle \otimes |4\rangle + q^{-1/2} |3\rangle \otimes |2\rangle - q^{1/2} |2\rangle \otimes |3\rangle$$

$$\langle \Psi| = q^{-1/2} \langle 4| \otimes \langle 1| + q^{1/2} \langle 1| \otimes \langle 4| + q^{-1/2} \langle 3| \otimes \langle 2| - q^{1/2} \langle 2| \otimes \langle 3|$$ \hspace{1cm} (2)

Next, to arrive at a hermitian Hamiltonian we consider the operator

$$T = |\Psi\rangle \langle \Psi|$$

A straightforward calculation shows that

$$T^2 = [2(q + q^{-1})]T$$

$$(T \otimes I)(I \otimes T)(T \otimes I) = T \otimes I$$ \hspace{1cm} (3)

$$(I \otimes T)(T \otimes I)(I \otimes T) = I \otimes T,$$

such that $T$ provides a representation of the $TL$ algebra. Following the approach of ref. [11] to obtain solutions of the Yang-Baxter equation through the TL algebra, a local Hamiltonian can be defined by (see ref. [14] for more details).

$$H_{i,i+1} = T_{i,i+1},$$  \hspace{1cm} (4)

where on the N-fold tensor product space we denoted
\[ T_{i,i+1} = I^\otimes(i-1) \otimes T \otimes I^\otimes(N-i-1). \] (5)

In view of the grading the basis vectors of the module \( V \) can be identified with the eletronic states as follows

\[ |1\rangle \equiv |+\rangle = c^+_i c^-_i |0\rangle, \quad |2\rangle \equiv |\rangle = c^-_i |0\rangle, \quad |3\rangle \equiv |\rangle = c^+_i |0\rangle, \quad |4\rangle \equiv |0\rangle \]

allowing \( H_{i,i+1} \) to be expressed as

\[
H_{i,i+1} = q n_{i,+} n_{i,-}(1 - n_{i+1,+})(1 - n_{i+1,-}) + q^{-1}(1 - n_{i,+})(1 - n_{i,-}) n_{i+1,+} n_{i+1,-} \\
+ q^{-1} n_{i,+}(1 - n_{i,-}) n_{i+1,-}(1 - n_{i+1,+}) + q n_{i,-}(1 - n_{i,+}) n_{i+1,+}(1 - n_{i+1,-}) \\
- S_i^+ S_{i+1}^- - S_i^- S_{i+1}^+ + c^+_i c^-_i c_{i+1,-} c_{i+1,+} + c^+_{i+1,+} c^+_i c_{i,-} c_i \\
+ q c_i^+ c_{i+1,+} n_{i,-}(1 - n_{i+1,-}) + h.c. - c^+_i c_{i+1,-} n_{i,+}(1 - n_{i+1,+}) + h.c. \\
+ c_{i,-} c_i^+ n_{i+1,+}(1 - n_{i,+}) + h.c. - q^{-1} c_i^+ c_{i+1,+} n_{i+1,-}(1 - n_{i,-}) + h.c. 
\] (6)

where the \( c_{i}^{(+)} \) are spin up or down annihilation (creation) operators, the \( S_i \)'s spin matrices and the \( n_i \)'s occupation numbers of electrons at lattice site \( i \). This model describes electron pair hopping, correlated hopping and generalized spin interactions. In the limit \( q \to 1 \) it reduces to the isotropic Hamiltonian introduced by Links in [13], which was shown to be invariant with respect to \( gl(2) \otimes u(1) \).

The global Hamiltonian is given by

\[ H = \sum_{k=1}^{N-1} H_{k,k+1} + b.t. \] (7)

where \( b.t. \) denotes the boundary term. The usual imposition of periodic boundary conditions (PBC), i.e., \( b.t. = H_{N,1} \), has the effect of breaking the \( U_q(gl(2)) \otimes u(1) \) symmetry of the model, since \( H_{N,1} \neq H_{1,N} \), reflecting the non-cocommutativity of the co-product. However, it was shown in [14] following [13,16,17,19] that for a special choice of the boundary term it is in fact possible to recover a quantum algebra invariant Hamiltonian which is in addition periodic in a certain sense. We shall call this type of boundary by closed one (CBC) and denote it by \( b.t. = U_0 \) (see section 4 for details).

In the next sections we will find the spectrum of this Hamiltonian for these two types of boundaries (PBC and CBC) through a modified version of the coordinate Bethe ansatz.
III. BETHE ANSATZ SOLUTION FOR PERIODIC BOUNDARY CONDITIONS

The case with periodic boundary conditions described by the Hamiltonian

\[ H = \sum_{k=1}^{N} T_{k,k+1}, \]  

(8)

can be mapped into a quantum spin chain of \( N \) sites each with spin \( 3/2 \). To verify this we notice that the local Hamiltonian can be rewritten as

\[ T = |\Psi \rangle \langle \Psi| = \begin{pmatrix} U & 0_{4 \times 12} \\ 0_{12 \times 4} & 0_{12 \times 12} \end{pmatrix} \]

where

\[ U = \begin{pmatrix} |14\rangle & |23\rangle & |32\rangle & |41\rangle \\ q & -q & 1 & 1 \\ -q & q & -1 & -1 \\ 1 & -1 & q^{-1} & q^{-1} \\ 1 & -1 & q^{-1} & q^{-1} \end{pmatrix} \]

(9)

and the following correspondence has to be understood

\[ |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

(10)

The spin values are given by the eigenvalues of the operator \( S^z \)

\[ S^z = \begin{pmatrix} 3/2 & & & \\ & 1/2 & & \\ & & -1/2 & \\ & & & -3/2 \end{pmatrix} \]

and the total spin operator commuting with \( H \) is \( S^z_T \).
\[ S_T^z = \sum_{k=1}^{N} I^{\otimes(k-1)} \otimes S^z \otimes I^{\otimes(N-k)}. \] (11)

Following [21] the spectrum of the above Hamiltonian can be classified in sectors which are defined by the eigenvalues of the operator number

\[ r = \frac{3}{2} N - S_T^z. \] (12)

Let us now start to diagonalize \( H \) in every sector, i.e.,

\[ H \Psi = E \Psi \]

In the first sectors \( r = 0, 1, 2 \) the eigenstates do not move under the action of \( H \), i.e., \( H \Psi_{r=0,1,2} = 0 \). For this reason they are called impurities [22]. In sector \( r = 3 \), we encounter the situation where the states \( |\alpha, k\rangle \) and \( |-\alpha, k \pm 1\rangle \), \( \alpha \in S^z = \{ \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2} \} \), occur in neighboring pairs. They move under the action of \( H \), i.e., the sector \( r = 3 \) contains one free pseudoparticle. In general, for a sector \( r \) we may have \( p \) pseudoparticles and \( N_{\frac{1}{2}} \) and \( N_{-\frac{1}{2}} \), impurities of the type \( \frac{1}{2} \) and \( \frac{-1}{2} \), respectively, such that

\[ r = 3p + N_{\frac{1}{2}} + 2N_{-\frac{1}{2}}. \] (13)

For the first nontrivial sector \( r = 3 \), the correspondent eigenspace is spanned by the states \( |k(-\alpha, \alpha)\rangle = |\frac{3}{2}, \frac{3}{2}, \cdots, -\alpha, \frac{3}{2}, \cdots, \frac{3}{2}\rangle \), where \( k = 1, 2, ..., N - 1 \) and \( \alpha \in S^z \). We seek eigenstates of \( H \) which are linear combinations of these vectors. It is very convenient to consider the linear combination

\[ |\Omega(k)\rangle = \left| k\left(\frac{-3}{2}, \frac{3}{2}\right)\right\rangle + \left| k\left(\frac{-1}{2}, \frac{1}{2}\right)\right\rangle - q \left| k\left(\frac{1}{2}, \frac{1}{2}\right)\right\rangle + q \left| k\left(\frac{3}{2}, -\frac{3}{2}\right)\right\rangle \] (14)

which is an eigenstate of \( U_k \)

\[ U_k |\Omega(k)\rangle = (Q + Q^{-1}) |\Omega(k)\rangle = 2(q + q^{-1}) |\Omega(k)\rangle. \] (15)

\[ ^{1} \text{From now on we will adopt the convention that } U_k \equiv U_{k,k+1} \text{ operates in a direct product of complex spaces at positions } k \text{ and } k + 1 \]
and also a highest weight state, i.e., $S^+\Psi = 0$. Moreover, the action of $U_{k\pm1}$ on $|\Omega(k)\rangle$ is very simple

$$U_{k\pm1}|\Omega(k)\rangle = |\Omega(k \pm 1)\rangle$$

$$U_k|\Omega(k \pm 1)\rangle = |\Omega(k)\rangle$$

(16)

$$U_k|\Omega(m)\rangle = 0 \quad k \neq \{m \pm 1, m\}$$

It should be emphasized that the linear combination (14) affords a considerable simplification in the diagonalization of $H$ in comparison with the traditional calculus employing the usual spin basis [21]. In fact, we believe that this type of ansatz is quite general and could be applied to solve a larger class of Hamiltonians derived from representations of the TL algebra.

We will now start to diagonalize $H$ in the sector $r = 3$. Let us consider the non-trivial case of one free pseudoparticle

$$\Psi_3 = \sum_k A(k)|\Omega(k)\rangle.$$  

(17)

Using the eigenvalue equation $H \Psi_3 = E_3 \Psi_3$, one can derive a complete set of equations for the wavefunctions $A(k)$.

When the bulk of $H$ acts on $|\Omega(k)\rangle$ it sees the reference configuration, except in the vicinity of $k$ where we use (15) and (16) to get the following eigenvalue equation

$$(E_3 - Q - Q^{-1})A(k) = A(k - 1) + A(k + 1)$$

$$2 \leq k \leq N - 2$$

(18)

Here we will treat periodic boundary conditions. They demand $U_{N,N+1} = U_{N,1}$, implying $A(k+N) = A(k)$. This permits us to complete the set of equations (18) for $A(k)$ by including the equations for $k = 1$ and $k = N - 1$. Now we parametrize $A(k)$ by plane wave $A(k) = A\xi^k$ to get the energy of one free pseudoparticle as:

$$E_3 = 2(q + q^{-1}) + \xi + \xi^{-1}$$

$$\xi^N = 1$$

(19)
Here $\xi = e^{i\theta}$, $\theta$ being the momenta determined from the periodic boundary to be $\theta = 2\pi l/N$, with $l$ an integer.

Let us consider the state with one pseudoparticle and one impurity of type $\frac{1}{2}$, which lies in the sector $r = 4$. We seek eigenstates in the form

$$\Psi_4(\xi_1, \xi_2) = \sum_{k_1 < k_2} \left\{ A_1(k_1, k_2) \left| \Omega_1(k_1, k_2) \right\rangle + A_2(k_1, k_2) \left| \Omega_2(k_1, k_2) \right\rangle \right\}$$  \hspace{1cm} (20)

We try to build these eigenstates out of translationally invariant products of one pseudoparticle excitation with parameter $\xi_2$ and one impurity with parameter $\xi_1$:

$$\Psi_4(\xi_1, \xi_2) = \left| \frac{1}{2}(\xi_1) \right\rangle \times \Psi_3(\xi_2) + \Psi_3(\xi_2) \times \left| \frac{1}{2}(\xi_1) \right\rangle$$ \hspace{1cm} (21)

Using one-pseudoparticle eigenstate solution (17) and comparing this with (20) we get

$$\left| \Omega_1(k_1, k_2) \right\rangle = \left| k_1\left(\frac{1}{2}\right), k_2\left(-\frac{3}{2}, \frac{3}{2}\right) \right\rangle + \left| k_1\left(\frac{1}{2}\right), k_2\left(-\frac{1}{2}, \frac{1}{2}\right) \right\rangle - q \left| k_1\left(\frac{1}{2}\right), k_2\left(\frac{1}{2}, -\frac{1}{2}\right) \right\rangle + q \left| k_1\left(\frac{1}{2}\right), k_2\left(\frac{3}{2}, -\frac{3}{2}\right) \right\rangle$$

$$\left| \Omega_2(k_1, k_2) \right\rangle = \left| k_1\left(-\frac{3}{2}, \frac{3}{2}\right), k_2\left(\frac{1}{2}\right) \right\rangle + \left| k_1\left(-\frac{1}{2}, \frac{1}{2}\right), k_2\left(\frac{1}{2}\right) \right\rangle - q \left| k_1\left(-\frac{1}{2}, \frac{1}{2}\right), k_2\left(\frac{1}{2}\right) \right\rangle + q \left| k_1\left(\frac{3}{2}, -\frac{3}{2}\right), k_2\left(\frac{1}{2}\right) \right\rangle$$ \hspace{1cm} (22)

and

$$A_1(k_1, k_2) = A_1\xi_1^{k_1}\xi_2^{k_2}, \quad A_2(k_1, k_2) = A_2\xi_2^{k_1}\xi_1^{k_2}.$$ \hspace{1cm} (23)

Periodic boundary conditions $A_1(k_2, N+k_1) = A_2(k_1, k_2)$ and $A_i(N+k_1, N+k_2) = A_i(k_1, k_2)$, $i = 1, 2$ imply that

$$A_1\xi_2^N = A_2, \quad \xi^N = (\xi_1\xi_2)^N = 1$$ \hspace{1cm} (24)

When $H$ now acts on $\Psi_4$, we will get a set of coupled equations for $A_i(k_1, k_2)$, $i = 1, 2$. We split the equations into far equations, when the pseudoparticle do not meet the impurity and near equations, containing terms when they are neighbors.
Since the impurity is annihilated by $H$, the action of $H$ on (20) in the case far (i.e., $(k_2 - k_1) \geq 3$), can be written directly from (18):

$$(E_4 - Q - Q^{-1}) A_1(k_1, k_2) = A_1(k_1, k_2 - 1) + A_1(k_1, k_2 + 1)$$

(25)

and similar equations for $A_2(k_1, k_2)$. Using the parametrization (23), these equations will give us the energy eigenvalues

$$E_4 = Q + Q^{-1} + \xi_2 + \xi_2^{-1}$$

(26)

To find $\xi_2$ we must consider the near equations. First, we compute the action of $H$ on the coupled near states $|\Omega_1(k, k + 1)\rangle$ and $|\Omega_2(k, k + 2)\rangle$:

$$H |\Omega_1(k, k + 1)\rangle = (Q + Q^{-1}) |\Omega_1(k, k + 1)\rangle + |\Omega_1(k, k + 2)\rangle - |\Omega_2(k, k + 2)\rangle$$

(27)

The last terms in these equations tell us that a pseudoparticle can propagate past the isolated impurity, but in doing so causes a shift in its position by two lattice sites. Substituting (27) into the eigenvalue equation, we get

$$(E_4 - Q - Q^{-1}) A_1(k, k + 1) = A_1(k, k + 2) - A_2(k, k + 2)$$

(28)

These equations, which are not automatically satisfied by the ansatz (23), are equivalent to the conditions

$$A_1(k, k) = -A_2(k, k + 2)$$

(29)

obtained by subtracting Eq. (28) from Eq. (23) for $k_1 = k$, $k_2 = k + 1$. The conditions (29) require a modification of the amplitude relation (24):

$$\frac{A_2}{A_1} = -\xi_1^{-2} = \xi_2^N \Rightarrow \xi_2^N \xi_1^2 = -1 \quad \text{or} \quad \xi_2^{N-2} \xi^2 = -1$$

(30)

In the sectors $3 < r < 6$ we also will find states, which consist of one pseudoparticle with parameter $\xi_{r-2}$ interacting with $r - 3$ impurities, distributing according to (13), with parameters $\xi_i, i = 1, 2..., r - 3$. 

9
The energy of these states is parametrized as in (26) and $\xi_{r-2}$ satisfies the condition (30) with $\xi = \xi_1 \cdots \xi_{r-3} \xi_{r-2}$. It involves only $\xi_{r-2}$ and $\xi_{\text{imp}} = \xi_1 \xi_2 \cdots \xi_{r-3}$, being therefore highly degenerate, i.e.

$$\xi^N_{r-2} \xi_1^2 \xi_2^2 \cdots \xi_{r-3}^2 = (-1)^{r-3}$$

(31)

This is to be expected due to the irrelevance of the relative distances, up to jumps of two positions via exchange with a pseudoparticle. Moreover, these results do not depend on the impurity type.

The sector $r = 6$ contains, in addition to the cases discussed above, states which consist of two interacting pseudoparticles. We seek eigenstates in the form

$$\Psi_6(\xi_1, \xi_2) = \sum_{k_1+1<k_2} A(k_1, k_2) |\Omega(k_1, k_2)\rangle$$

(32)

Applying $H$ to the state of (32), we obtain a set of equations for the wavefunctions $A(k_1, k_2)$. When the two pseudoparticles are separated, $(k_2 - k_1 \geq 3)$ these are the following far equations:

$$(E_6 - 2Q - 2Q^{-1}) A(k_1, k_2) = A(k_1 - 1, k_2) + A(k_1 + 1, k_2)$$

$$+ A(k_1, k_2 - 1) + A(k_1, k_2 + 1)$$

(33)

We already know them to be satisfied, if we parametrize $A(k_1, k_2)$ by plane waves (23). The corresponding energy eigenvalue is

$$E_6 = 2Q + 2Q^{-1} + \xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1}$$

(34)

The real problem arises of course, when pseudoparticles are neighbors, so that they interact and we have no guarantee that the total energy is a sum of single pseudoparticle energies.

Acting of $H$ on the state $|\Omega(k, k + 2)\rangle$ gives the following set of equations for the near states
\[ H |\Omega(k, k + 2)\rangle = 2 (Q + Q^{-1}) |\Omega(k, k + 2)\rangle + |\Omega(k - 1, k + 2)\rangle \] (35)

\[ + |\Omega(k, k + 3)\rangle + U_{k+1} |\Omega(k, k + 2)\rangle \]

Before we substitute this result into the eigenvalue equation, we observe that some new states are appearing. In order to incorporate these new states in the eigenvalue problem, we define

\[ U_{k+1} |\Omega(k, k + 2)\rangle = |\Omega(k, k + 1)\rangle + |\Omega(k + 1, k + 2)\rangle \] (36)

Here we underline that we are using the same notation for these new states. Applying \( H \) to them we obtain

\[ H |\Omega(k, k + 1)\rangle = (Q + Q^{-1}) |\Omega(k, k + 1)\rangle + |\Omega(k - 1, k + 1)\rangle \] (37)

\[ + |\Omega(k, k + 2)\rangle \]

Now, we extend (32), the definition of \( \Psi_6 \), to

\[ \Psi_6(\xi_1, \xi_2) = \sum_{k_1 < k_2} A(k_1, k_2) |\Omega(k_1, k_2)\rangle \] (38)

Substituting (35) and (37) into the eigenvalue equation, we obtain the following set of near equations

\[ \left( E_6 - Q - Q^{-1} \right) A(k, k + 1) = A(k - 1, k + 1) + A(k, k + 2) \] (39)

Using the same plane wave parametrization for these new wavefunctions, the equation (39) gives us the phase shift produced by the interchange of the two interacting pseudoparticles

\[ \frac{A_{21}}{A_{12}} = -\frac{1 + \xi + (Q + Q^{-1})\xi_2}{1 + \xi + (Q + Q^{-1})\xi_1} \] (40)

We thus arrive to the (BAE) which fix the values of \( \xi_1 \) and \( \xi_2 \) in the energy equation (34)
In a generic sector $r$ with $l$ impurities parametrized by $\xi_1 \xi_2 \cdots \xi_l$ and $p$ pseudoparticles with parameters $\xi_{l+1} \xi_{l+2} \cdots \xi_{l+p}$, the energy is
\[
E_r = \sum_{n=l+1}^{l+p} \left( Q + Q^{-1} + \xi_n + \xi_n^{-1} \right)
\] (42)
with $\xi_n$ determined by the Bethe ansatz equations
\[
\xi_a^N \xi_1^2 \xi_2^2 \cdots \xi_l^2 = (-1)^l \prod_{b \neq a = l+1}^{l+p} \left( \frac{1 + \xi_b \xi_a + (Q + Q^{-1})\xi_a}{1 + \xi_a \xi_b + (Q + Q^{-1})\xi_b} \right)
\]
a = l + 1, l + 2, \ldots, l + p, \quad p \geq 2
\[
\xi_{l+1}^N \xi_1^2 \xi_2^2 \cdots \xi_l^2 = (-1)^l, \quad p = 1
\]
\[
\xi_c^{N-2p} = (-1)^p, \quad c = 1, 2, \ldots, l
\]
\[
\xi^N = 1, \quad \xi = \xi_1 \xi_2 \cdots \xi_l \xi_{l+1} \xi_{l+2} \cdots \xi_{l+p}.
\] (43)
The energy eigenvalues and the Bethe equations depend on the deformation parameter $q$, through the relation $Q + Q^{-1} = 2q + 2q^{-1}$.

IV. BETHE ANSATZ SOLUTION FOR CLOSED BOUNDARY CONDITIONS

The quantum group invariant closed TL Hamiltonians which can be written as [14]:
\[
H = \sum_{k=1}^{N-1} U_k + U_0
\] (44)
where $U_k$ is a Temperley-Lieb operator and $U_0$ is a boundary term defined through of an operator $G$ which plays the role of the translation operator
\[
U_0 = GU_{N-1}G^{-1} \quad , \quad G = (Q - U_1)(Q - U_2) \cdots (Q - U_{N-1})
\] (45)
satisfying $[H, G] = 0$ and additionally invariance with respect to the quantum algebra. The operator $G$ shifts the $U_k$ by one unit $GU_kG^{-1} = U_{k+1}$ and maps $U_0$ into $U_1$, which manifest the translational invariance of $H$. In this sense the Hamiltonian [14] is periodic. From the physical point of view, this type of models exhibit behavior similar to closed chains with twisted boundary conditions (see for example [23] for the case of the XXZ chain).
The action of the operator $G$ on the states $|\Omega(k)\rangle$ can be easily computed using (14), (15) and (16): It is simple on the bulk and at the left boundary

$$G |\Omega(k)\rangle = -Q^{N-2} |\Omega(k+1)\rangle , \quad 1 \leq k \leq N - 2$$

(46)

but has a non-trivial contribution at the right boundary

$$G |\Omega(N-1)\rangle = Q^{N-2} \sum_{k=1}^{N-1} (-Q)^{-k} |\Omega(N-k)\rangle$$

(47)

Similarly, the action of the operator $G^{-1} = (Q^{-1} - U_{N-1}) \cdots (Q^{-1} - U_1)$ is simple on the bulk and at the right boundary

$$G^{-1} |\Omega(k)\rangle = -Q^{-N+2} |\Omega(k-1)\rangle , \quad 2 \leq k \leq N - 1$$

(48)

and non-trivial at the left boundary

$$G^{-1} |\Omega(1)\rangle = Q^{-N+2} \sum_{k=1}^{N-1} (-Q)^k |\Omega(k)\rangle .$$

(49)

Now we proceed the diagonalization of $H$ as was made for the periodic case. As (44) and (7) have the same bulk, i.e., differences arise from the boundary terms only, we will keep all results relating to the bulk of the periodic case presented in the previous section.

Let us consider one free pseudoparticle which lies in the sector $r = 3$

$$\Psi_3 = \sum_{k=1}^{N-1} A(k) |\Omega(k)\rangle .$$

(50)

The action of the operator $U = \sum_{k=1}^{N-1} U_k$ on the states $|\Omega(k)\rangle$ is:

$$U |\Omega(1)\rangle = (Q + Q^{-1}) |\Omega(1)\rangle + |\Omega(2)\rangle$$

$$U |\Omega(k)\rangle = (Q + Q^{-1}) |\Omega(k)\rangle + |\Omega(k-1)\rangle + |\Omega(k+1)\rangle$$

for $2 \leq k \leq N - 2$

$$U |\Omega(N-1)\rangle = (Q + Q^{-1}) |\Omega(N-1)\rangle + |\Omega(N-2)\rangle .$$

(51)

and using (46)–(49) one can see that the action of $U_0 = GU_{N-1}G^{-1}$ vanishes on the bulk

$$U_0 |\Omega(k)\rangle = 0 , \quad 2 \leq k \leq N - 2$$

(52)
and has the following contributions at the boundaries

\[ U_0 |\Omega(1)\rangle = - \sum_{k=1}^{N-1} (-Q)^k |\Omega(k)\rangle, \quad U_0 |\Omega(N-1)\rangle = - \sum_{k=1}^{N-1} (-Q)^{-N+k} |\Omega(k)\rangle. \]  

(53)

which are connected by

\[ U_0 |\Omega(N-1)\rangle = (-Q)^{-N} U_0 |\Omega(1)\rangle. \]  

(54)

Before we substitute these results into the eigenvalue equation, we will define two new states

\[ |\Omega(0)\rangle = U_0 |\Omega(1)\rangle, \quad |\Omega(N)\rangle = U_0 |\Omega(N-1)\rangle \]  

(55)

to include the cases \( k = 0 \) and \( k = N \) into the definition of \( \Psi_3 \), equation (50). Finally, the action of \( H = \mathcal{U} + U_0 \) on the states \( |\Omega(k)\rangle \) is

\[ H |\Omega(0)\rangle = (Q + Q^{-1}) |\Omega(0)\rangle + (-Q)^N |\Omega(N-1)\rangle + |\Omega(1)\rangle \]

\[ H |\Omega(N)\rangle = (Q + Q^{-1}) |\Omega(N)\rangle + |\Omega(N-1)\rangle + (-Q)^{-N} |\Omega(1)\rangle \]  

(56)

Substituting these results into the eigenvalue equation \( H\Psi_3 = E_3 \Psi_3 \) we get a complete set of eigenvalue equations for the wavefunctions

\[ E_3 A(k) = (Q + Q^{-1})A(k) + A(k - 1) + A(k + 1) \]

for \( 1 \leq k \leq N - 1 \)  

(57)

provided the following boundary conditions

\[ (-Q)^N A(k) = A(N + k) \]  

(58)

are satisfied.

The plane wave parametrization \( A(k) = A\xi^k \) solves these eigenvalue equations and the boundary conditions provided that:
\[ E_3 = Q + Q^{-1} + \xi + \xi^{-1} \]
\[ \xi^N = (-Q)^N \quad (59) \]

where \( \xi = e^{i\theta} \) and \( \theta \) being the momentum.

Let us now consider the sector \( r = 6 \), where we can find an eigenstate with two interacting pseudoparticles. We seek the corresponding eigenfunction as products of single pseudoparticles eigenfunctions, \( i.e. \)

\[ \Psi_6 = \sum_{k_1+1<k_2} A(k_1,k_2) |\Omega(k_1,k_2)\rangle \quad (60) \]

To solve the eigenvalue equation \( H\Psi_6 = E_6\Psi_6 \), we recall (16) to get the action of \( U \) and \( U_0 \) on the states \( |\Omega(k_1,k_2)\rangle \). Here we have to consider four cases: \( (i) \) when the two pseudoparticles are separated in the bulk, the action of \( U \) is

\[ U |\Omega(k_1,k_2)\rangle = 2(Q + Q^{-1}) |\Omega(k_1,k_2)\rangle + |\Omega(k_1-1,k_2)\rangle + |\Omega(k_1+1,k_2)\rangle \]
\[ + |\Omega(k_1,k_2-1)\rangle + |\Omega(k_1,k_2+1)\rangle \quad (61) \]

i.e., for \( k_1 \geq 2 \) and \( k_1 + 3 \leq k_2 \leq N - 2 \); \( (ii) \) when the two pseudoparticles are separated but one of them or both are at the boundaries

\[ U |\Omega(1,k_2)\rangle = 2(Q + Q^{-1}) |\Omega(1,k_2)\rangle + |\Omega(2,k_2)\rangle + |\Omega(1,k_2-1)\rangle \]
\[ + |\Omega(1,k_2+1)\rangle \quad (62) \]

\[ U |\Omega(k_1,N-1)\rangle = 2(Q + Q^{-1}) |\Omega(k_1,N-1)\rangle + |\Omega(k_1-1,N-1)\rangle \]
\[ + |\Omega(k_1+1,N-1)\rangle + |\Omega(k_1,N-2)\rangle \quad (63) \]

\[ U |\Omega(1,N-1)\rangle = 2(Q + Q^{-1}) |\Omega(1,N-1)\rangle + |\Omega(2,N-1)\rangle + |\Omega(1,N-2)\rangle \quad (64) \]

where \( 2 \leq k_1 \leq N - 4 \) and \( 4 \leq k_2 \leq N - 2 \); \( (iii) \) when the two pseudoparticles are neighbors in the bulk

\[ U |\Omega(k,k+2)\rangle = 2(Q + Q^{-1}) |\Omega(k,k+2)\rangle + |\Omega(k-1,k+2)\rangle + |\Omega(k,k+3)\rangle \]
\[ +U_{k+1} |\Omega(k,k+2)\rangle \quad (65) \]
for $2 \leq k \leq N-4$ and $(iv)$ when the two pseudoparticles are neighbors and at the boundaries

$$ U |\Omega(1, 3)\rangle = 2(Q + Q^{-1}) |\Omega(1, 3)\rangle + |\Omega(1, 4)\rangle + U_2 |\Omega(1, 3)\rangle $$  \hspace{1cm} (66)

$$ U |\Omega(N-3, N-1)\rangle = 2(Q + Q^{-1}) |\Omega(N-3, N-1)\rangle + |\Omega(N-4, N-1)\rangle $$

$$ + U_{N-2} |\Omega(N-3, N-1)\rangle $$  \hspace{1cm} (67)

Moreover, the action of $U_0$ does not depend on the pseudoparticles are neither separated nor neighbors. It is vanishes in the bulk

$$ U_0 |\Omega(k_1, k_2)\rangle = 0 \quad \text{for} \quad k_1 \neq 1 \text{ and } k_2 \neq N-1, $$  \hspace{1cm} (68)

and different of zero at the boundaries:

$$ U_0 |\Omega(1, k_2)\rangle = - \sum_{k=1}^{k_2-2} (-Q)^k |\Omega(k, k_2)\rangle - (-Q)^{k_2-1} U_{k_2} |\Omega(k_2-1, k_2+1)\rangle $$  \hspace{1cm} (69)

$$ - \sum_{k=k_2+2}^{N-1} (-Q)^{k-2} |\Omega(k_2, k)\rangle $$  \hspace{1cm} (70)

$$ U_0 |\Omega(k_1, N-1)\rangle = (-Q)^{-N+2} U_0 |\Omega(1, k_2)\rangle $$  \hspace{1cm} (71)

where $2 \leq k_1 \leq N-3$ and $3 \leq k_2 \leq N-2$.

Following the same procedure of one-pseudoparticle case we again define new states in order to have consistency between bulk and boundaries terms

$$ U_0 |\Omega(1, k_2)\rangle = |\Omega(0, k_2)\rangle, \quad U_0 |\Omega(k_1, N-1)\rangle = |\Omega(k_1, N)\rangle $$

$$ U_0 |\Omega(1, N-1)\rangle = |\Omega(0, N-1)\rangle + |\Omega(1, N)\rangle $$

$$ U_{k+1} |\Omega(k, k+2)\rangle = |\Omega(k, k+1)\rangle + |\Omega(k+1, k+2)\rangle $$  \hspace{1cm} (72)

Acting with $H$ on these new states, we get

$$ H |\Omega(0, k_2)\rangle = 2(Q + Q^{-1}) |\Omega(0, k_2)\rangle + |\Omega(0, k_2-1)\rangle + |\Omega(0, k_2+1)\rangle $$

$$ + |\Omega(1, k_2)\rangle + (-Q)^{N-2} |\Omega(k_2, N-1)\rangle $$  \hspace{1cm} (73)
\[
H |\Omega(k_1, N)\rangle = 2(Q + Q^{-1}) |\Omega(k_1, N)\rangle + |\Omega(k_1 - 1, N)\rangle + |\Omega(k_1 + 1, N)\rangle \\
+ |\Omega(k_1, N - 1)\rangle + (-Q)^{-N+2} |\Omega(1, k_1)\rangle
\]

(74)

\[
H |\Omega(k, k + 1)\rangle = (Q + Q^{-1}) |\Omega(k, k + 1)\rangle + |\Omega(k - 1, k + 1)\rangle + |\Omega(k, k + 2)\rangle
\]

(75)

Substituting these results into the eigenvalue equation, we get the following equations for wavefunctions corresponding to the separated pseudoparticles.

\[
(E_6 - 2Q - 2Q^{-1}) A(k_1, k_2) = A(k_1 - 1, k_2) + A(k_1 + 1, k_2) \\
+ A(k_1, k_2 - 1) + A(k_1, k_2 + 1)
\]

(76)

i.e., for \( k_1 \geq 1 \) and \( k_1 + 3 \leq k_2 \leq N - 1 \). The boundary conditions read now

\[
A(k_2, N + k_1) = (-Q)^{N-2} A(k_1, k_2).
\]

(77)

The parametrization for the wavefunctions

\[
A(k_1, k_2) = A_{12} \xi_1^{k_1} \xi_2^{k_2} + A_{21} \xi_1^{k_2} \xi_2^{k_1}
\]

(78)

solves the equation (76) provided that

\[
E_6 = 2(Q + Q^{-1}) + \xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1}
\]

(79)

and the boundary conditions (77) provided that

\[
\xi_2^N = (-Q)^{N-2} \frac{A_{21}}{A_{12}} , \quad \xi_1^N = (-Q)^{N-2} \frac{A_{12}}{A_{21}} \Rightarrow \xi^N = (-Q)^{2(N-2)}
\]

(80)

where \( \xi = \xi_1 \xi_2 = e^{i(\theta_1 + \theta_2)} \), \( \theta_1 + \theta_2 \) being the total momenta.

Now we include the new states (72) into the definition of \( \Psi_6 \) in order to extend (60) to

\[
\Psi_6 = \sum_{k_1 < k_2} A(k_1, k_2) |\Omega(k_1, k_2)\rangle.
\]

(81)

Here we have used the same notation for separated and neighboring states. Substituting (53) and (74) into the eigenvalue equation, we get
\[(E_6 - Q - Q^{-1})A(k, k + 1) = A(k - 1, k + 1) + A(k, k + 2)\]  
\[(82)\]

which gives us the phase shift produced by the interchange of the two pseudoparticles

\[\frac{A_{21}}{A_{12}} = \frac{-1 + \xi + (Q + Q^{-1})\xi_2}{1 + \xi + (Q + Q^{-1})\xi_1}.\]  
\[(83)\]

We thus arrive to the Bethe ansatz equations which fix the values of \(\xi_1\) and \(\xi_2\):

\[\xi_2^N = (-Q)^{N-2} \left\{ \frac{-1 + \xi + (Q + Q^{-1})\xi_2}{1 + \xi + (Q + Q^{-1})\xi_1} \right\},\]

\[\xi_1^N \xi_2^N = (-Q)^{2(N-2)}\]  
\[(84)\]

Thus in the sector \(r = 3p\), we expect that the \(p\)-pseudoparticle phase shift will be a sum of two-pseudoparticle phase shifts and the energy is given by

\[E_{3p} = \sum_{n=1}^{p} \{ Q + Q^{-1} + \xi_n + \xi_n^{-1} \} \]  
\[(85)\]

where

\[\xi_a^N = (-Q)^{N-2p+2} \prod_{b \neq a} \left\{ \frac{1 + \xi_a\xi_b + (Q + Q^{-1})\xi_a}{1 + \xi_a\xi_b + (Q + Q^{-1})\xi_b} \right\}, \quad a = 1, \ldots, p\]

\[(86)\]

The corresponding eigenstates are

\[\Psi_r(\xi_1, \xi_2, \ldots, \xi_p) = \sum_{1 \leq k_1 < \ldots < k_p \leq N-1} A(k_1, k_2, \ldots, k_p) |\Omega(k_1, k_2, \ldots, k_p)\rangle\]  
\[(87)\]

where \(|\Omega(k_1, k_2, \ldots, k_p)\rangle = \otimes_{i=1}^{p} |\Omega(k_i)\rangle\) and the wavefunctions satisfy the following boundary conditions

\[A(k_1, k_2, \ldots, k_p, N + k_1) = (-Q)^{N-2p+2} A(k_1, k_2, \ldots, k_p)\]  
\[(88)\]

It is not all, in a sector \(r\) we may have \(p\) pseudoparticles and \(N_1, N_{-1}\) impurities of the type \(\frac{1}{2}, -\frac{1}{2}\), respectively. Since \(H\) is a sum of projectors on spin zero, these states are also annihilated by \(U_0\). Therefore the impurities play here the same role as in the periodic case.
It means that for a sector \( r \) with \( l \) impurities with parameters \( \xi_1, ..., \xi_l \) and \( p \) pseudoparticles with parameters \( \xi_{l+1}, ..., \xi_{l+p} \) the energy is given by (86), and the Bethe equations do not depend on impurity type and are given by

\[
\xi_a \xi_1^2 \xi_2^2 \cdots \xi_l^2 = (-1)^l (-Q)^{N-2p+2} \prod_{b=l+1, b \neq a}^{l+p} \left\{ \frac{1 + \xi_a \xi_b + (Q + Q^{-1}) \xi_a}{1 + \xi_a \xi_b + (Q + Q^{-1}) \xi_b} \right\} \tag{89}
\]

with \( a = l + 1, l + 2, ..., l + p \), \( p \geq 1 \), and

\[
\xi^{2p} (\xi_{l+1} \cdots \xi_{l+p})^{N-2p} = (-1)^l (-Q)^{p(N-2p+2)} \tag{90}
\]

where \( \xi = \xi_1 \xi_2 \cdots \xi_{l+1} \cdots \xi_{l+p} \).

Notice in the BAE (89) the presence of a special "q-term" \((-Q)^{N-2p+2}\) in comparison with the corresponding ones with usual periodic boundary conditions (13). In fact, this feature also appeared in other models [16,17,19] and seems to be a peculiarity of quantum group invariant closed spin chains.

V. CONCLUSIONS

We have applied the coordinate Bethe ansatz to find the spectra of the anisotropic correlated electron system associated with the TL algebra. This procedure was carried out for periodic and closed boundary conditions and the differences between both cases have been remarked.

We believe that the methods here presented could also be applied to solve a larger class of Hamiltonians derived from representations of the graded TL algebra, such as the orthosympletic models discussed by Zhang in [12]. This is presently under investigation.

Another interesting extension of this work would be to adapt the methods employed in this paper to solve multiparametric versions of these models. [24,25]

Acknowledgment: The support of CNPq - Conselho Nacional de Desenvolvimento Científico e Tecnológico is gratefully acknowledged. A.L.S also thanks FAPESP - Fundação
de Amparo à Pesquisa do Estado de São Paulo for financial assistance. A.F. would like

to thank the Institute für Theoretische Physik - FUB for its kind hospitality, particu-
larly M. Karowski. She also thanks DAAD - Deutscher Akademischer Austauschdienst and
FAPERGS - Fundação de Amparo à Pesquisa do Estado do Rio Grande do Sul for financial

support.

[1] V.E. Korepin and F.H.L. Essler (ed), Exactly Solvable Models of Strongly Correlated Electrons
(Singapore: World Scientific, 1994)

[2] F.H.L. Essler, V.E. Korepin and K. Schoutens, Phys. Rev. Lett. 70, 73 (1993)

[3] A. Foerster and M. Karowski Nucl. Phys. B408, 512 (1993)

[4] I.N. Karnaukhov, Phys. Rev. Lett. 73, 1130 (1994)

[5] L. Arrachea and A.A. Aligia, Phys. Rev. Lett. 73, 2240 (1994)

[6] A.J. Bracken, M.D. Gould, J.R. Links and Y.-Z. Zhang, Phys. Rev. Lett. 74, 2769 (1995); M.D.
Gould, K.E. Hibberd, J.R. Links and Y-Z Zhang, Phys. Lett. A 212, 156 (1996)

[7] J. de Boer, V.E. Korepin and A. Schadschneider, Phys. Rev. Lett. 74, 789 (1995)

[8] D. Arnaudon, C. Chryssomalakos and L. Frappat, J. Math. Phys 36, 5262 (1995); D. Arnaudon,
Algebraic approach to q-deformed supersymmetric variants of the Hubbard model with pair
hoppings, physics/9711001 (1997)

[9] F.C. Alcaraz and R. Bariev, Exact solution of the biquadratic spin-1 t-J model in one dimension,
cond-mat/9706228 (1997)

[10] H.N.V. Temperley and H. Lieb, Proc. Roy. Soc. 322, 251 (1971)

[11] M.T. Batchelor and A. Kuniba, J. Phys. A 24, 2599 (1991)
[12] R.B. Zhang, J. Math., Phys. 32, 2605 (1991)

[13] J. Links, J. Phys. A29, L69 (1996)

[14] A. Foerster, J. Links and I. Roditi, Mod. Phys. Lett. A12, 1035 (1997)

[15] Martin P P Potts Models and Related Problems in Statistical Mechanics (Singapore: World Scientific, 1991).

[16] M. Karowski and A. Zapletal, Nucl. Phys. B419[FS], 567 (1994); J. Phys. A27, 7419 (1994)

[17] H. Grosse, S. Pallua, P. Prester and E. Raschhofer, J. Phys. A27, 4761 (1994);

[18] P. Kulish and E.K. Sklyanin, J. Sov. Math. 19, 1596 (1982); V.V. Bazhanov and A.G. Shadrikov. Theor. Math. Phys. 73, 1302 (1987)

[19] A. Foerster, J. Phys. A29, 7625 (1996)

[20] J. Links and A. Foerster, J. Phys. A30, 2483 (1997)

[21] A. Lima-Santos and R.C.T. Ghiotto, A Bethe ansatz solution for the closed $U_q(sl(2))$ TL quantum spin chains, accepted for publication in J. Phys. A

[22] R. Koberle and A. Lima-Santos, J. Phys. A29, 519 (1996)

[23] S. Pallua and P. Prester, J. Phys. A29, 1187 (1996)

[24] R.B. Zhang, J. Phys. A24, L535 (1991)

[25] A. Foerster, J. Links and I. Roditi, Integrable multiparametric quantum spin chains, accepted for publication in J. Phys. A