VIBRATIONS OF A BEAM BETWEEN STOPS: COLLISION EVENTS AND ENERGY BALANCE PROPERTIES

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Abstract. We consider the model problem of an elastic beam vibrating between two stops. More precisely the beam is clamped at its left end while its right end may undergo contact and collision events with two stops. We model the interaction between the beam and the stops either with Signorini complementarity conditions when the stops are perfectly rigid or with a normal compliance contact law allowing some penetration within the stops and given by a linear relationship between the shear stress and the penetration at some positive power \( \beta \) when contact occurs.

Motivated by computational issues we study the evolution of the energy functional defined as the sum of the kinetic energy and the potential energy of elastic deformation of the beam. When contact is modelled with a normal compliance law we prove an energy conservation property. Then we interpret the relationship between the shear stress and the penetration in case of contact as a penalization of the non-penetration condition. We show that the solutions of the penalized problems converge to a strong solution of the problem with Signorini conditions as defined in [26] and we prove that the limit satisfies an energy conservation property through instantaneous collision events.

1. Introduction. In many industrial devices vibrations lead to unilateral contact and impacts between the different parts of complex mechanical systems, inducing unwanted noise and/or untimely wear. Thus a good prediction of such phenomena is of crucial importance and the mathematical study of this kind of problems has been a very active research field since the beginning of the twentieth century.

For the quasi-static case, existence, uniqueness and regularity properties are now quite well understood and we refer the reader to the following monographs and the references therein ([21, 35, 19]).

On the contrary in the dynamic case a lot of open questions still remain. Indeed, the Signorini complementary conditions lead to a formulation of the problem as a hyperbolic variational inequality ([36]) and in the case of a purely elastic material the existence of a solution has been established only for the Laplace operator ([2, 10, 11, 3, 4, 5, 13, 32, 33, 34, 14, 15, 6, 24, 22]) or for some specific geometrical

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settings, namely beams or plates, for which we may use compactness results due to Sobolev embeddings (see for instance [23, 1, 16, 8, 28, 26]).

For most of these results the proof strategy consists in building of a sequence of approximate solutions by using a penalization of the non-penetration condition. Then a priori estimates are derived and compactness properties allow to pass to the limit. Such an approach seems very natural from the mathematical point of view and has also a nice interpretation from the mechanical point of view since obstacles are never perfectly rigid. Hence it is interesting to study also which properties are preserved by the convergence process. Among them energy balance properties play of course a crucial role. Despite its importance for computational issues (see for instance [1, 17] or [27] and the references therein) very few papers have addressed this challenging question, and only for the Laplace operator by using either the method of characteristics ([12, 32, 33, 34, 6]) or by reformulating the problem as a problem at the boundary ([24]). Unfortunately these two methods rely deeply on the properties of the Laplace operator (and also on the geometry in [24]) and do not seem easy to extend to other dynamic unilateral contact problems.

In this paper we consider the model problem of an elastic beam vibrating between two stops. The interaction between the beam and the stops is modelled either with Signorini complementarity conditions when the stops are perfectly rigid or with a normal compliance contact law allowing some penetration within the stops otherwise. We introduce very general normal compliance laws given by a linear relationship between the shear stress and the penetration at some positive power $\beta$ when contact occurs. In Section 2 we describe the two models and derive the corresponding variational formulations of the problem. In Section 3 we consider the problem with normal compliance and we prove that the solutions satisfy an energy conservation property. Then in Section 4 we interpret the normal compliance contact law as a convex penalization of the non-penetration condition and we show that the solutions of the penalized problems converge to a strong solution of the problem with Signorini complementarity conditions as defined in [26]. Finally we prove that the solutions of the Signorini problem obtained as limits of this penalty approach satisfy also an energy conservation property through instantaneous collision events.

2. The model problem of the vibrating beam: Notations and mathematical background. We consider the model problem of an elastic beam vibrating between two stops. The physical setting is presented at Figure 1, namely the beam is clamped at its left end while its right end may undergo contact and collision events with two stops. We assume that the motion is planar.

![Figure 1. The mechanical setting.](image-url)
We denote by \( L \) the length of the beam and by \( u(t, x), (t, x) \in [0, T] \times [0, L] \) \((T > 0)\) the vertical displacement of a point \( x \) belonging to the beam axis. Throughout this paper we will denote with subscripts the partial derivatives with respect to the time or space variables.

Under the assumption of small displacements the shear stress of the beam is given by

\[
\sigma = -k^2 u_{xxx}, \quad k^2 = \frac{EI}{\rho S}
\]

where \( E \) and \( \rho \) are the Young modulus and the density of the material, \( I \) and \( S \) are the inertia momentum and the cross section of the beam. The motion is described by

\[
u_{tt} - \sigma_x = f \quad \text{in} \quad (0, T) \times (0, L)
\]

where \( f \) is the density of external forces.

At \( x = 0 \) the beam is clamped so we have

\[
u(t, 0) = \nu_x(t, 0) = 0 \quad \text{in} \quad (0, T).
\]

At \( x = L \) the interaction between the beam and the stops is modelled either with Signorini conditions if the stops are assumed to be perfectly rigid or by a normal compliance law if some penetration within the stops is possible.

More precisely, if the stops are perfectly rigid, we have a non-penetration condition at \( x = L \) given by

\[
g_1 \leq \nu(t, L) \leq g_2 \quad \text{in} \quad (0, T)
\]

with \( g_1 < 0 < g_2 \) and the shear stress satisfies the following complementarity conditions

\[
\begin{align*}
\sigma(t, L) &\geq 0 \quad \text{if} \quad \nu(t, L) = g_1, \\
\sigma(t, L) &\leq 0 \quad \text{if} \quad \nu(t, L) = g_2.
\end{align*}
\]

Moreover no moments are applied to the right end of the beam and we have

\[
u_{xx}(t, L) = 0 \quad \text{in} \quad (0, T).
\]

On the contrary if the stops are not perfectly rigid some penetration may occur and the contact is modelled with a normal compliance law

\[
\sigma(t, L) = -c \left( \left(\nu(t, L) - g_2\right)_+^{\beta} - \left(\nu(t, L) - g_1\right)_+^{\beta} \right)
\]

with \( c > 0 \) and \( \beta \geq 1 \). Once again no moments are applied to the right end of the beam and we have

\[
u_{xx}(t, L) = 0 \quad \text{in} \quad (0, T).
\]

Of course when \( c \) tends to \(+\infty\), (6) reduces formally to (3)-(4) and (6) coincide with a penalty approach of (3)-(4) via the convex function \( \psi_{NC} \) given by

\[
\psi_{NC}(y) = \frac{c}{\beta + 1} \left( (y - g_2)_+^{\beta + 1} + (g_1 - y)_+^{\beta + 1} \right) \quad \text{for all} \quad y \in \mathbb{R}.
\]

We may observe that the stops behave as linear springs if \( \beta = 1 \) or non-linear springs if \( \beta \neq 1 \) when penetration occurs.
Remark 2.1. The complementarity conditions (3)-(4) can be summarized with the following single non-linear subdifferential boundary condition
\[-\sigma(t, L) \in \partial \Psi_{[g_1, g_2]}(u(t, L)) \quad \text{in} \ (0, T)\]
where \(\Psi_{[g_1, g_2]}\) is the indicator function of the real interval \([g_1, g_2]\) and \(\partial \Psi_{[g_1, g_2]}\) is its subdifferential in the sense of convex analysis (see [30]). If \(\beta = 1\) the normal compliance contact law (6) can be rewritten as
\[-\sigma(t, L) = A_{1/c}(u(t, L)) \quad \text{in} \ (0, T)\]
where \(A_{\lambda}\) is the Yosida approximation of the maximal monotone operator \(A = \partial \Psi_{[g_1, g_2]}\) for all \(\lambda > 0\) (see [9]).

With \(\beta = \frac{3}{2}\) we recover Hertz contact model (see [20]).

Let \(H = L^2(0, L)\) and \(V\) be defined as
\[V = \{v \in H^2(0, L); \ v(0) = v_2(0) = 0\}.

We introduce the convex subset \(K\) as
\[K = \{v \in V; \ g_1 \leq v(L) \leq g_2\}\]
and we let \(K = [g_1, g_2]\). Let \(f \in L^2(0, T; H)\).

For any given initial data \(u_0 \in \mathcal{K}\) and \(v_0 \in H\), the variational formulation of the problem (1) with the boundary conditions (2), (5) and the normal compliance contact law (6) is given as
\[\begin{align*}
\text{(P}_{\text{beam}}^{NC}) : & \quad \text{Find } u^{NC} \in L^2(0, T; V) \text{ such that } u^{NC}_t \in L^2(0, T; V'), \ u^{NC}(0, \cdot) = u_0, \ u^{NC}_t(0, \cdot) = v_0 \text{ and } \\
& \quad (u^{NC}_t(t, \cdot), w)V,\nu + k^2 \int_0^L u^{NC}_{xx}(t, x) w_{xx}(x) \, dx \\
& \quad + c \left( ((u^{NC}(t, L) - g_2)_+)^\beta - ((g_1 - u^{NC}(t, L))_+)^\beta \right) w(L) \\
& \quad = \int_0^L f(t, x) w(x) \, dx \quad \text{for all } w \in V, \text{ for a.a. } t \in [0, T].
\end{align*}\]

The variational formulation of the problem (1) with the boundary conditions (2), (5) and the Signorini complementarity conditions (3)-(4) is given as
\[\begin{align*}
\text{(P}_{\text{beam}}^{NC}) : & \quad \text{Find } u^{S} \in L^2(0, T; K) \text{ such that } u^{S}_t \in L^2(0, T; H), \ u^{S}(0, \cdot) = u_0 \text{ and } \\
& \quad - \int_0^T \int_0^L u^{S}_t(t, x)(u_t(t, x) - u^{S}_t(t, x)) \, dxdt \\
& \quad + k^2 \int_0^L \int_0^L u^{S}_{xx}(t, x)(w_{xx}(t, x) - u^{S}_{xx}(t, x)) \, dxdt \\
& \quad \geq \int_0^L v_0(x) (w(0, x) - u_0(x)) \, dx + \int_0^T \int_0^L f(t, x) (w(t, x) \\
& \quad - u^{S}(t, x)) \, dxdt \text{for all } w \in L^2(0, T; K) \text{ such that } w_t \in L^2(0, T; H) \text{ and } \\
& \quad w(T, \cdot) = u^{S}(T, \cdot).
\end{align*}\]

Let us emphasize that \(u^{NC}\) and \(u^{S}\) belong to \(W^{1,2}(0, T; H)\) and \(u^{NC}_t\) belongs to \(W^{1,2}(0, T; V')\). Thus the initial condition for \(u^{NC}, u^{S}\) and \(u^{NC}_t\) make sense in \(H, H\) and \(V',\) respectively.
The existence of a solution to problem \( P_{\text{beam}}^{NC} \) can be easily obtained by using a Galerkin method for instance. On the contrary the existence of a solution to problem \( P_{\text{beam}}^{S} \) requires more involved analysis techniques (see \[23, 16, 26]\).

3. Energy balance properties for \( P_{\text{beam}}^{NC} \). When contact is modelled through a normal compliance contact law, energy balance properties may appear as a trivial question since \( u_{t}^{NC} \) belongs to the Sobolev space \( L^{2}(0, T, V') \). Nevertheless, when the material is purely elastic the regularity of \( u_{t}^{NC} \) does not allow us to simply consider \( u_{t}^{NC} \) as a test-function in \( P_{\text{beam}}^{NC} \). Moreover we cannot derive energy balance properties (i.e. energy equalities) from the Galerkin approximation. Roughly speaking the difficulty comes from the lack of coercivity with respect to \( u_{t}^{NC} \) in \( V \). Indeed, if \( (u_{t}^{NC, m})_{m \geq 1} \) denotes the sequence of solutions of the approximate problems built with the Galerkin method, we obtain a priori estimates of \( (u_{t}^{NC, m})_{m \geq 1} \) and \( (u_{t}^{NC, m})_{m \geq 1} \) in \( L^{\infty}(0, T; V) \) respectively and weak convergences in these spaces lead only to an energy estimate (i.e. only an inequality) at the limit, namely

\[
\frac{1}{2} \| u_{t}^{NC}(\tau, \cdot) \|_{H}^{2} + \frac{k^{2}}{2} \| u_{xx}^{NC}(\tau, \cdot) \|_{H}^{2} + \frac{c}{\beta + 1} \left( ((u^{NC}(\tau, L) - g_{2})^{+})^{\beta+1} + ((g_{1} - u^{NC}(\tau, L))^{+})^{\beta+1} \right)
\leq \frac{1}{2} \| v_{0} \|_{H}^{2} + \frac{k^{2}}{2} \| u_{t,xx}^{NC} \|_{H}^{2} + \int_{0}^{T} \int_{0}^{L} f(t, x) u_{t}^{NC}(t, x) \, dx dt \quad \forall \text{a.a. } \tau \in (0, T).
\]

Similarly if we construct a sequence of approximate solutions to \( P_{\text{beam}}^{NC} \) by applying a vanishing viscosity technique (see \[23\]) we meet the same difficulty and we derive only an energy estimate (i.e. an inequality) at the limit. Nevertheless we may obtain

**Theorem 3.1.** Let \( f \in L^{2}(0, T; H) \), \( u_{0} \in K \) and \( v_{0} \in H \). Let us assume that \( g_{1} < 0 < g_{2} \), \( c > 0 \) and \( \beta \geq 1 \). Then the following equality holds:

\[
\frac{1}{2} \| u_{t}^{NC}(\tau_{2}, \cdot) \|_{H}^{2} + \frac{k^{2}}{2} \| u_{xx}^{NC}(\tau_{2}, \cdot) \|_{H}^{2} + \frac{c}{\beta + 1} \left( ((u^{NC}(\tau_{2}, L) - g_{2})^{+})^{\beta+1} - ((g_{1} - u^{NC}(\tau_{2}, L))^{+})^{\beta+1} \right)
= \frac{1}{2} \| u_{t}^{NC}(\tau_{1}, \cdot) \|_{H}^{2} + \frac{k^{2}}{2} \| u_{xx}^{NC}(\tau_{1}, \cdot) \|_{H}^{2} + \frac{c}{\beta + 1} \left( ((u^{NC}(\tau_{1}, L) - g_{2})^{+})^{\beta+1} - ((g_{1} - u^{NC}(\tau_{1}, L))^{+})^{\beta+1} \right) + \int_{\tau_{1}}^{\tau_{2}} \int_{0}^{L} f(t, x) u_{t}^{NC}(t, x) \, dx dt
\]

for almost all \( (\tau_{1}, \tau_{2}) \in (0, T)^{2} \) such that \( \tau_{1} < \tau_{2} \).

**Proof.** The key idea consists in choosing a finite difference approximation of \( u_{t}^{NC} \) as a test-function in \( P_{\text{beam}}^{NC} \). For the sake of notational simplicity let us introduce

\[
H_{NC}(y) = c \left( ((y - g_{2})^{+})^{\beta} - ((g_{1} - y)^{+})^{\beta} \right)
\]

and

\[
\psi_{NC}(y) = \frac{c}{\beta + 1} \left( ((y - g_{2})^{+})^{\beta+1} + ((g_{1} - y)^{+})^{\beta+1} \right)
\]

for all \( y \in \mathbb{R} \). The mapping \( \psi_{NC} \) is of class \( C^{1} \) on \( \mathbb{R} \) and \( \psi'_{NC}(y) = H_{NC}(y) \) for all \( y \in \mathbb{R} \).
Let $(\tau_1, \tau_2) \in [0, T]^2$ such that $\tau_1 < \tau_2$. For all $h \in \left(0, \min \left(\frac{\tau_2 - \tau_1}{4}, \frac{T - \tau_2}{2}\right)\right)$ and for all $w \in V$ we have

$$\langle u_{tt}^{NC}(t+h, \cdot) + u_{tt}^{NC}(t-h, \cdot), w \rangle_{V', V} + k^2 \int_0^L \left( u_{xx}^{NC}(t+h, x) + u_{xx}^{NC}(t-h, x) \right) dx$$

$$+ (H_{NC}(u^{NC}(t+h, L)) + H_{NC}(u^{NC}(t-h, L))) w(L) = \int_0^{T-h} (f(t+h, x) + f(t-h, x)) w(x) dx \quad \forall \text{ a.a. } t \in (h, T-h).$$

We choose $w = u^{NC}(t+h, \cdot) - u^{NC}(t-h, \cdot)$ and we integrate in time on $(\tau_1+h, \tau_2-h)$. We obtain

$$\int_{\tau_1+h}^{\tau_2-h} \langle (u_{tt}^{NC}(t+h, \cdot) + u_{tt}^{NC}(t-h, \cdot), u^{NC}(t+h, \cdot) - u^{NC}(t-h, \cdot)) \rangle_{V', V} dt$$

$$+ k^2 \int_{\tau_1+h}^{\tau_2-h} \int_0^L \left( u_{xx}^{NC}(t+h, x) + u_{xx}^{NC}(t-h, x) \right)$$

$$\times \left( u_{xx}^{NC}(t+h, x) - u_{xx}^{NC}(t-h, x) \right) dx dt$$

$$+ \int_{\tau_1+h}^{\tau_2-h} \left( H_{NC}(u^{NC}(t+h, L)) + H_{NC}(u^{NC}(t-h, L)) \right)$$

$$\times \left( u^{NC}(t+h, L) - u^{NC}(t-h, L) \right) dt$$

$$= \int_{\tau_1+h}^{\tau_2-h} \int_0^L \left( f(t+h, x) + f(t-h, x) \right) \left( u^{NC}(t+h, x) - u^{NC}(t-h, x) \right) dx dt. \quad (9)$$

The first term of the left-hand side is rewritten as follows:

$$\int_{\tau_1+h}^{\tau_2-h} \langle u_{tt}^{NC}(t+h, \cdot) + u_{tt}^{NC}(t-h, \cdot), u^{NC}(t+h, \cdot) - u^{NC}(t-h, \cdot) \rangle_{V', V} dt$$

$$= - \int_{\tau_1+h}^{\tau_2-h} \int_0^L \left( u_{tt}^{NC}(t+h, x) + u_{tt}^{NC}(t-h, x) \right)$$

$$\times \left( u_{tt}^{NC}(t+h, x) - u_{tt}^{NC}(t-h, x) \right) dx dt$$

$$+ \int_0^L \left( u_{tt}^{NC}(\tau_2, x) + u_{tt}^{NC}(\tau_2 - 2h, x) \right) \left( u^{NC}(\tau_2, x) - u^{NC}(\tau_2 - 2h, x) \right) dx$$

$$- \int_0^L \left( u_{tt}^{NC}(\tau_1, x) + u_{tt}^{NC}(\tau_1 + 2h, x) \right) \left( u^{NC}(\tau_1 + 2h, x) - u^{NC}(\tau_1, x) \right) dx$$

$$= - \int_{\tau_1+h}^{\tau_2-h} \int_0^L \left( u_{tt}^{NC}(t+h, x) \right)^2 dx dt + \int_{\tau_1+h}^{\tau_2-h} \int_0^L \left( u_{tt}^{NC}(t-h, x) \right)^2 dx dt$$

$$+ \int_0^L \left( u_{tt}^{NC}(\tau_2, x) + u_{tt}^{NC}(\tau_2 - 2h, x) \right) \left( u^{NC}(\tau_2, x) - u^{NC}(\tau_2 - 2h, x) \right) dx$$

$$- \int_0^L \left( u_{tt}^{NC}(\tau_1, x) + u_{tt}^{NC}(\tau_1 + 2h, x) \right) \left( u^{NC}(\tau_1 + 2h, x) - u^{NC}(\tau_1, x) \right) dx$$

$$= \int_{\tau_1+h}^{\tau_2} \int_0^L \left( u_{tt}^{NC}(t, x) \right)^2 dx dt - \int_{\tau_2-2h}^{\tau_2} \int_0^L \left( u_{tt}^{NC}(t, x) \right)^2 dx dt$$

$$+ \int_0^L \left( u_{tt}^{NC}(\tau_2, x) + u_{tt}^{NC}(\tau_2 - 2h, x) \right) \left( u^{NC}(\tau_2, x) - u^{NC}(\tau_2 - 2h, x) \right) dx$$

$$- \int_0^L \left( u_{tt}^{NC}(\tau_1, x) + u_{tt}^{NC}(\tau_1 + 2h, x) \right) \left( u^{NC}(\tau_1 + 2h, x) - u^{NC}(\tau_1, x) \right) dx.$$
Hence
\[
\frac{1}{2h} \int_{\tau_1+2h}^{\tau_2-h} \langle u_{tt}^{NC}(t + h, \cdot) + u_{tt}^{NC}(t - h, \cdot), u^{NC}(t + h, \cdot) - u^{NC}(t - h, \cdot) \rangle_{V'} dt \\
= \frac{1}{2h} \int_{\tau_1}^{\tau_1+2h} \| u_t^{NC}(t, \cdot) \|_H^2 dt - \frac{1}{2h} \int_{\tau_2-2h}^{\tau_2-h} \| u_t^{NC}(t, \cdot) \|_H^2 dt \\
+ \int_0^L \left( u_t^{NC}(\tau_2, x) + u_t^{NC}(\tau_2 - 2h, x) \right) \frac{u_t^{NC}(\tau_2, x) - u^{NC}(\tau_2 - 2h, x)}{2h} dx \\
- \int_0^L \left( u_t^{NC}(\tau_1 + 2h, x) + u_t^{NC}(\tau_1, x) \right) \frac{u_t^{NC}(\tau_1 + 2h, x) - u^{NC}(\tau_1, x)}{2h} dx.
\]

The two first terms of the right-hand side converge respectively to \( \| u_t^{NC}(\tau_1, \cdot) \|_H^2 \) and \( \| u_t^{NC}(\tau_2, \cdot) \|_H^2 \) as \( h \) tends to zero whenever \( \tau_1 \) and \( \tau_2 \) are Lebesgue points of the mapping \( t \mapsto \| u_t^{NC}(t, \cdot) \|_H^2 \) and thus the two first terms of the right-hand side converge respectively to \( \| u_t^{NC}(\tau_1, \cdot) \|_H^2 \) and \( \| u_t^{NC}(\tau_2, \cdot) \|_H^2 \) as \( h \) tends to zero for almost all \( (\tau_1, \tau_2) \in (0, T)^2 \) ([18]). Moreover
\[
\int_0^L \left( u_t^{NC}(\tau_2, x) + u_t^{NC}(\tau_2 - 2h, x) \right) \frac{u_t^{NC}(\tau_2, x) - u^{NC}(\tau_2 - 2h, x)}{2h} dx \\
\longrightarrow_{h \to 0} 2\| u_t^{NC}(\tau_2, \cdot) \|_H^2
\]
for almost all \( \tau_2 \in (0, T) \). Indeed
\[
\int_0^L \left( u_t^{NC}(\tau_2, x) + u_t^{NC}(\tau_2 - 2h, x) \right) \frac{u_t^{NC}(\tau_2, x) - u^{NC}(\tau_2 - 2h, x)}{2h} dx \\
-2 \int_0^L \left( u_t^{NC}(\tau_2, x) \right)^2 dx \\
= \int_0^L \left( u_t^{NC}(\tau_2, x) + u_t^{NC}(\tau_2 - 2h, x) \right) \frac{u_t^{NC}(\tau_2, x) - u^{NC}(\tau_2 - 2h, x)}{2h} dx \\
+ \int_0^L \left( u_t^{NC}(\tau_2 - 2h, x) - u_t^{NC}(\tau_2, x) \right) u_t^{NC}(\tau_2, x) dx.
\]

But
\[
\frac{u_t^{NC}(\tau_2, \cdot) - u^{NC}(\tau_2 - 2h, \cdot)}{2h} = \frac{1}{2h} \int_{\tau_2-2h}^{\tau_2} u_t^{NC}(t, \cdot) dt \\
\longrightarrow_{h \to 0} u_t^{NC}(\tau_2, \cdot) \quad \text{strongly in } H
\]
whenever \( \tau_2 \) is a Lebesgue point of \( t \mapsto u_t^{NC}(t, \cdot) \), i.e. for almost all \( \tau_2 \in (0, T) \).

With (7) we know that \( u_t^{NC} \) belongs to \( L^\infty(0, T; H) \). Hence the first term of the right hand side of (10) tends to zero for almost all \( \tau_2 \in (0, T) \). Next we observe that the second term of the right-hand side of (10) is given by
\[
\left( u_t^{NC}(\tau_2 - 2h, \cdot) - u_t^{NC}(\tau_2, \cdot), u_t^{NC}(\tau_2, \cdot) \right)_H
\]
where \( (\cdot, \cdot)_H \) is the canonical inner product of \( H \). Reminding that \( u_t^{NC} \) belongs to \( W^{1,2}(0, T; V') \cap L^\infty(0, T; H) \) we obtain that (a representant of the class of) \( u_t^{NC} \) belongs to \( C^0([0, T]; V') \cap L^\infty(0, T; H) \) and thus the mapping \( t \mapsto u_t^{NC}(t, \cdot) \) is weakly continuous with values in \( H \) (see [38]). It follows that
\[
\left( u_t^{NC}(\tau_2 - 2h, \cdot) - u_t^{NC}(\tau_2, \cdot), u_t^{NC}(\tau_2, \cdot) \right)_H \longrightarrow_{h \to 0} 0.
\]
Similarly,
\[
\int_0^L \left( u_t^{NC}(\tau_1 + 2h, x) + u_t^{NC}(\tau_1, x) \right) \frac{u^{NC}(\tau_1 + 2h, x) - u^{NC}(\tau_1, x)}{2h} \, dx
\]
\[
\rightarrow_{h \to 0^+} 2\|u_t^{NC}(\tau_1, \cdot)\|_H^2
\]
for almost all \( \tau_1 \in (0, T) \).

It follows that
\[
\frac{1}{2h} \int_{\tau_1 + h}^{\tau_2 - h} \left( u_t^{NC}(t + h, \cdot) + u_t^{NC}(t - h, \cdot), u^{NC}(t + h, \cdot) - u^{NC}(t - h, \cdot) \right) \nu_t \, dt
\]
\[
\rightarrow_{h \to 0^+} \|u_t^{NC}(\tau_2, \cdot)\|_H^2 - \|u_t^{NC}(\tau_1, \cdot)\|_H^2
\]
for almost all \((\tau_1, \tau_2) \in (0, T)^2\) such that \(\tau_1 < \tau_2\).

We rewrite the second integral term of (9) as
\[
k^2 \int_{\tau_1 + h}^{\tau_2 - h} \int_0^L \left( u_{xx}^{NC}(t + h, x) + u_{xx}^{NC}(t - h, x) \right)
\times (u_{xx}^{NC}(t + h, x) - u_{xx}^{NC}(t - h, x)) \, dx \, dt
\]
\[
= k^2 \int_{\tau_2 - 2h}^{\tau_2} \int_0^L (u_{xx}^{NC}(t, x))^2 \, dx \, dt - k^2 \int_{\tau_1}^{\tau_2 + 2h} \int_0^L (u_{xx}^{NC}(t, x))^2 \, dx \, dt
\]
and, using the Lebesgue points of \( t \mapsto \|u_{xx}^{NC}(t, \cdot)\|_H^2 \), we obtain
\[
k^2 \int_{\tau_1 + h}^{\tau_2 - h} \int_0^L \left( u_{xx}^{NC}(t + h, x) + u_{xx}^{NC}(t - h, x) \right)
\times (u_{xx}^{NC}(t + h, x) - u_{xx}^{NC}(t - h, x)) \, dx \, dt
\]
\[
\rightarrow_{h \to 0^+} k^2 \left( \|u_{xx}^{NC}(\tau_2, \cdot)\|_H^2 - \|u_{xx}^{NC}(\tau_1, \cdot)\|_H^2 \right)
\]
for almost all \((\tau_1, \tau_2) \in (0, T)^2\) such that \(\tau_1 < \tau_2\).

We consider now the third term of the left-hand side of (9). With a Taylor expansion of \(\psi_{NC}(u^{NC}(t + h, L)) - \psi_{NC}(u^{NC}(t - h, L))\) we get:
\[
\int_{\tau_1 + h}^{\tau_2 - h} \left( H_{NC}(u^{NC}(t + h, L)) + H_{NC}(u^{NC}(t - h, L)) \right)
\times (u^{NC}(t + h, L) - u^{NC}(t - h, L)) \, dt
\]
\[
= 2 \int_{\tau_1 + h}^{\tau_2 - h} \left( \psi_{NC}(u^{NC}(t + h, L)) - \psi_{NC}(u^{NC}(t - h, L)) \right) \, dt
\]
\[
+ \int_{\tau_1 + h}^{\tau_2 - h} \int_0^1 \left( \psi'_{NC}(u^{NC}(t + h, L)) \right)
\times (u^{NC}(t + h, L) - u^{NC}(t - h, L)) \, ds \, dt
\]
\[
+ \int_{\tau_1 + h}^{\tau_2 - h} \int_0^1 \left( \psi'_{NC}(u^{NC}(t - h, L)) \right)
\times (u^{NC}(t + h, L) - u^{NC}(t - h, L)) \, ds \, dt
\]
If \(\beta = 1\) the mapping \(H_{NC}\) is \(c\)-Lipschitz continuous on \(\mathbb{R}\) and if \(\beta > 1\) the mapping \(H_{NC}\) is of class \(C^1\) on \(\mathbb{R}\) and is Lipschitz continuous on any compact subset of \(\mathbb{R}\). With (7) we infer that \(u^{NC} \in L^\infty(0, T; V)\). Thus \(u^{NC}(\cdot, L)\) remains in
a bounded interval of $\mathbb{R}$ and $H_{NC}$ is Lipschitz continuous on this interval. Hence there exists $C_H > 0$ such that

$$\left| \int_{\tau_1+h}^{\tau_2-h} \int_0^1 \left( \psi'_{NC}(u^{NC}(t+h,L)) - \psi'_{NC}(su^{NC}(t+h,L) + (1-s)u^{NC}(t-h,L)) \right) \times \left( u^{NC}(t+h,L) - u^{NC}(t-h,L) \right) ds dt \right|$$

$$+ \left| \int_{\tau_1+h}^{\tau_2-h} \int_0^1 \left( \psi'_{NC}(u^{NC}(t-h,L)) - \psi'_{NC}(su^{NC}(t+h,L) + (1-s)u^{NC}(t-h,L)) \right) \times \left( u^{NC}(t-h,L) - u^{NC}(t-h,L) \right) ds dt \right|$$

$$\leq C_H \int_{\tau_1+h}^{\tau_2-h} \left| u^{NC}(t+h,L) - u^{NC}(t-h,L) \right|^2 dt.$$

Owing that $H^s(0,L)$ can be defined as the interpolation space $[H^2(0,L),L^2(0,L)]_\theta$ with $s = 2(1 - \theta)$ for all $\theta \in (0,1)$ ([25]), we choose $\theta \in \left( \frac{1}{2}, \frac{3}{4} \right)$. Then $s \in \left( \frac{1}{2}, 1 \right)$ and $H^s(0,L)$ is continuously embedded into $C^0([0,L])$. We infer that there exists $C > 0$ such that

$$\int_{\tau_1+h}^{\tau_2-h} \left| u^{NC}(t+h,L) - u^{NC}(t-h,L) \right|^2 dt$$

$$\leq C \int_{\tau_1+h}^{\tau_2-h} \left\| u^{NC}(t+h,\cdot) - u^{NC}(t-h,\cdot) \right\|_{V}^{2(1-\theta)} \left\| u^{NC}(t+h,\cdot) - u^{NC}(t-h,\cdot) \right\|_H^{2\theta} ds dt.$$

It follows that

$$\frac{1}{2h} \int_{\tau_1+h}^{\tau_2-h} \left| u^{NC}(t+h,L) - u^{NC}(t-h,L) \right|^2 dt$$

$$\leq C \frac{2^{2\theta-1}}{2} \int_{\tau_1+h}^{\tau_2-h} \left\| u^{NC}(t+h,\cdot) - u^{NC}(t-h,\cdot) \right\|_{V}^{2(1-\theta)} \left\| u^{NC}(t+h,\cdot) - u^{NC}(t-h,\cdot) \right\|_H^{2\theta} ds dt$$

$$\leq C \frac{2^{2\theta-1}}{2} \left( \left\| u^{NC} \right\|_{L^\infty(0,T;V)} \right)^{2(1-\theta)} \left( \left\| u^{NC}_t \right\|_{L^\infty(0,T;H)} \right)^{2\theta} (\tau_2 - \tau_1 - 2h).$$

By observing that the mapping $t \mapsto u^{NC}(t,L)$ is $\theta$-Hölder continuous on $[0,T]$, we obtain that $t \mapsto \psi_{NC}(u^{NC}(t,L))$ is also continuous on $[0,T]$ and we get

$$\lim_{h \to 0} \frac{1}{2h} \int_{\tau_1+h}^{\tau_2-h} \left( H_{NC}(u^{NC}(t+h,L)) + H_{NC}(u^{NC}(t-h,L)) \right) \times \left( u^{NC}(t+h,L) - u^{NC}(t-h,L) \right) dt$$

$$= \lim_{h \to 0} \frac{1}{2h} \int_{\tau_1+h}^{\tau_2-h} \left( \psi_{NC}(u^{NC}(t+h,L)) - \psi_{NC}(u^{NC}(t-h,L)) \right) dt$$

$$= 2 \psi_{NC}(u^{NC}(\tau_2,L)) - 2 \psi_{NC}(u^{NC}(\tau_1,L))$$

for all $(\tau_1,\tau_2) \in (0,T)^2$ such that $\tau_1 < \tau_2$. 

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Finally we decompose the right-hand side of (9) as
\[
\int_{\tau_1}^{\tau_2-h} \int_0^L (f(t + h, x) + f(t - h, x))(u^{NC}(t + h, x) - u^{NC}(t - h, x)) \, dx \, dt
\]
\[= \int_{\tau_1 + 2h}^{\tau_2} \int_0^L f(t, x)(u^{NC}(t, x) - u^{NC}(t - 2h, x)) \, dx \, dt
\]
\[+ \int_{\tau_2 - 2h}^{\tau_1} \int_0^L f(t, x)(u^{NC}(t + 2h, x) - u^{NC}(t, x)) \, dx \, dt.
\]
Thus
\[
\frac{1}{2h} \int_{\tau_1}^{\tau_2-h} \int_0^L (f(t + h, x) + f(t - h, x))(u^{NC}(t + h, x) - u^{NC}(t - h, x)) \, dx \, dt
\]
\[= 2 \int_{\tau_1}^{\tau_2} \int_0^L f(t, x)u^{NC}_t(t, x) \, dx \, dt
\]
\[+ \int_{\tau_1}^{\tau_2} \int_0^L f(t, x) \left( \frac{u^{NC}(t + 2h, x) - u^{NC}(t - 2h, x)}{2h} - 2u^{NC}_t(t, x) \right) \, dx \, dt
\]
\[- \int_{\tau_1}^{\tau_2} \int_0^L f(t, x) \left( \frac{1}{2h} \int_{t-2h}^{t} u^{NC}_t(s, x) \, ds \right) \, dx \, dt
\]
\[- \int_{\tau_2 - 2h}^{\tau_1} \int_0^L f(t, x) \left( \frac{1}{2h} \int_{t}^{t+2h} u^{NC}_t(s, x) \, ds \right) \, dx \, dt.
\]
The last two integral terms can be estimated as $\sqrt{2h} \|f\|_{L^2(0,T;H)} \|u^{NC}_t\|_{L^\infty(0,T;H)}$.

By observing that
\[
u^{NC}(t + 2h, \cdot) - u^{NC}(t - 2h, \cdot) = \frac{1}{2h} \int_{t-2h}^{t+2h} u^{NC}_t(s, \cdot) \, ds
\]
for any Lebesgue point $t$ of $t \mapsto u^{NC}_t(t, \cdot)$ we obtain that
\[
\int_0^L f(t, x) \left( \frac{u^{NC}(t + 2h, x) - u^{NC}(t - 2h, x)}{2h} - 2u^{NC}_t(t, x) \right) \, dx \rightarrow_{h \to 0^+} 0
\]
for almost all $t \in (\tau_1, \tau_2)$. Moreover, since $u^{NC}_t \in L^\infty(0, T; H)$ we have
\[
\left| \int_0^L f(t, x) \left( \frac{u^{NC}(t + 2h, x) - u^{NC}(t - 2h, x)}{2h} - 2u^{NC}_t(t, x) \right) \, dx \right|
\]
\[= \left| \int_0^L f(t, x) \left( \frac{1}{2h} \int_{t-2h}^{t+2h} u^{NC}_t(s, x) \, ds \right) \, dx \right|
\]
\[\leq 4 \|f(t, \cdot)\|_H \|u^{NC}_t\|_{L^\infty(0,T;H)} \quad \forall \text{ a.a. } t \in (\tau_1, \tau_2).
\]

By using Lebesgue dominated convergence theorem we obtain
\[
\int_{\tau_1}^{\tau_2} \int_0^L f(t, x) \left( \frac{u^{NC}(t + 2h, x) - u^{NC}(t - 2h, x)}{2h} - 2u^{NC}_t(t, x) \right) \, dx \, dt \rightarrow_{h \to 0^+} 0
\]
for all $(\tau_1, \tau_2) \in (0, T)^2$ such that $\tau_1 < \tau_2$, which allows us to conclude.

\[\Box\]

**Remark 3.1.** Let us define the energy functional $E^{NC}$ by
\[
E^{NC}(t) = \frac{1}{2} \|u^{NC}_t(t, \cdot)\|_H^2 + \frac{k^2}{2} \|u^{NC}(t, \cdot)\|_H^2 \quad \forall \text{ a.a. } t \in (0, T).
\]

i.e. $E^{NC}$ is the sum of the kinetic energy and the elastic deformation potential energy associated to $u^{NC}$. 

From (7) we infer that \( E^{NC} \in L\infty(0,T;\mathbb{R}) \). The equality (8) can be rewritten as
\[
E^{NC}(\tau_2) + \psi_{NC}(u^{NC}(\tau_2, L)) = E^{NC}(\tau_1) + \psi_{NC}(u^{NC}(\tau_1, L)) + \int_{\tau_1}^{\tau_2} \int_0^L f(t, x) u^{NC}_t(t, x) \, dx \, dt
\]
for almost all \((\tau_1, \tau_2) \in (0, T)^2\) such that \( \tau_1 < \tau_2 \).

Owing that the stops act smoothly as (non-)linear one-sided springs in case of contact, the terms \( \psi_{NC}(u^{NC}(\tau_i, L)) \) \((i = 1, 2)\) correspond to the potential energy of the spring and can be interpreted as the potential energy of contact at instants \( \tau_i \), \( i = 1, 2 \). The last integral term corresponds to the work of the external forces during the time-interval \([\tau_1, \tau_2]\).

Thus (8) is an energy-conservation property for the solutions of problem \( (P^{NC}_{beam}) \).

4. Energy balance properties for \( (P^S_{beam}) \). As already explained in Section 2 the normal compliance contact law (6) reduces formally to the Signorini complementarity conditions (3)-(4) as \( c \) tends to \(+\infty\) and we may expect that the corresponding solutions tend to a solution of \( (P^S_{beam}) \). This idea is substantiated by the energy estimate (7) which shows that \( u^{NC} \) and \( \psi_{NC}(u^{NC}(\cdot, L)) \) remain bounded in \( W^{1,\infty}(0,T;H) \cap L\infty(0,T;V) \) and \( L\infty(0,T;\mathbb{R}) \) respectively, uniformly with respect to the parameter \( c \).

So we will consider in this section a sequence \((u^\varepsilon)_{\varepsilon > 0}\) of solutions of problem \( (P^{NC}_{beam}) \) for a given \( \beta \geq 1 \) and a sequence of parameters \((c_\varepsilon)_{\varepsilon > 0}\) such that \( \lim_{\varepsilon \to 0} c_\varepsilon = +\infty \). Starting from (7) we may extract a subsequence, still denoted \((u^\varepsilon)_{\varepsilon > 0}\), such that
\[
u^\varepsilon \rightharpoonup u \quad \text{weakly* in } L\infty(0,T;V) \text{ and weakly in } L^2(0,T;V),
\]
and
\[u^\varepsilon_t \rightharpoonup u_t \quad \text{weakly* in } L\infty(0,T;H) \text{ and weakly in } L^2(0,T;H).
\]
With the Simon lemma ([37]), by possibly extracting a subsequence still denoted \((u^\varepsilon)_{\varepsilon > 0}\), we have
\[
u^\varepsilon \longrightarrow u \quad \text{strongly in } C^0([0,T];C^0([0,L])
\]
and (7) implies that \( u(t,L) \in [g_1, g_2] \) for all \( t \in [0,T] \). Hence \( u \in L\infty(0,T;\mathcal{K}) \cap W^{1,\infty}(0,T;H) \) and \( u(0, \cdot) = u_0 \).

Moreover we can prove also that

**Proposition 4.1.** The sequence \( (c_\varepsilon\left(\left(\left(|u^\varepsilon(L, \cdot) - g_2|_+\right)^\beta - \left((g_1 - u^\varepsilon(L, \cdot))_+\right)^\beta\right)\right)_{\varepsilon > 0}\) is bounded in \( L^1(0,T;\mathbb{R}) \).

**Proof.** When \( \beta = 1 \) and \( c_\varepsilon = \frac{1}{\varepsilon} \) the result has been proved in [26] Proposition 3.2 by using the convexity of the mapping \( y \mapsto \frac{1}{2\varepsilon}\left((|y - g_2|_+)^2 + ((g_1 - y)_+)^2\right) \). Since the mapping \( y \mapsto \frac{c_\varepsilon}{\beta + 1}\left((|y - g_2|_+)^{\beta + 1} + ((g_1 - y)_+)^{\beta + 1}\right) \) is convex for any \( \beta \geq 1 \) and \( c_\varepsilon > 0 \), we may reproduce the same computations and the result follows.

Let us denote from now on by \( (P^\varepsilon_{beam}) \) the problem \( (P^{NC}_{beam}) \) and by \( H_\varepsilon \) and \( \psi_\varepsilon \) the mappings \( H_{NC} \) and \( \psi_{NC} \) with the parameter \( c = c_\varepsilon \) for any \( \varepsilon > 0 \). Starting
from \((P_{\text{beam}}^\varepsilon)\) we have

\[
\langle u^\varepsilon_t(t, \cdot), w \rangle_{V^\prime, V} + k^2 \int_0^L u^\varepsilon_{xx}(t, x) w_{xx}(x) \, dx
\]

\[+ c \varepsilon \left( \left( (u^\varepsilon(t, L) - g_2) \right)_+^\beta - (g_1 - u^\varepsilon(t, L))_+^\beta \right) w(L)
\]

\[= \int_0^T f(t, x) w(x) \, dx \quad \text{for all } w \in V, \text{ for a.a. } t \in [0, T].
\]

Hence

\[
- \int_0^T \langle u^\varepsilon_t(t, \cdot), w_t(t, \cdot) \rangle_{V^\prime, V} \, dt + \langle u^\varepsilon_i(T, \cdot), w(T, \cdot) \rangle_{V^\prime, V}
\]

\[+ k^2 \int_0^T \int_0^L u^\varepsilon_{xx}(t, x) w_{xx}(t, x) \, dx \, dt
\]

\[+ \int_0^T c \varepsilon \left( \left( (u^\varepsilon(t, L) - g_2) \right)_+^\beta - (g_1 - u^\varepsilon(t, L))_+^\beta \right) w(t, L) \, dt
\]

\[= H_\varepsilon(u^\varepsilon(t, \cdot)) \]

\[= \langle v_0, w(0, \cdot) \rangle_{V^\prime, V} + \int_0^T \int_0^L f(t, x) w(t, x) \, dx \, dt \quad \text{for all } w \in C^1([0, T]; V).
\]

Since \(u^\varepsilon_i \in W^{1, 2}(0, T; V') \cap L^\infty(0, T; H)\) the mapping \(t \mapsto u^\varepsilon_i(t, \cdot)\) is weakly continuous from \([0, T]\) to \(H\) and we may rewrite the previous equation as

\[
- \int_0^T \int_0^L u^\varepsilon(t, x) w_t(t, x) \, dx \, dt + \int_0^T u^\varepsilon_i(T, x) w(T, x) \, dx
\]

\[+ k^2 \int_0^T \int_0^L u^\varepsilon_{xx}(t, x) w_{xx}(t, x) \, dx \, dt + \int_0^T H_\varepsilon(u^\varepsilon(t, L)) w(t, L) \, dt
\]

\[= \int_0^T v_0(x) w(0, x) \, dx + \int_0^T \int_0^L f(t, x) w(t, x) \, dx \, dt
\]

for all \(w \in C^1([0, T]; V)\). By density the same equality holds for any test-function \(w \in W^{1, 2}(0, T; H) \cap L^2(0, T; V)\).

From Proposition 4.1 we infer that the density measure \(\mu^\varepsilon \overset{\text{def}}{=} H_\varepsilon(u^\varepsilon(\cdot, L)) m\), where \(m\) is Lebesgue measure on \([0, T]\), is uniformly bounded in \(M^1([0, T]; \mathbb{R})\) with respect to \(\varepsilon\). Hence there exists \(\mu \in M^1([0, T]; \mathbb{R})\) such that

\[
\mu^\varepsilon = H_\varepsilon(u^\varepsilon(\cdot, L)) m \rightharpoonup \mu \quad \text{weakly* in } M^1([0, T]; \mathbb{R}).
\]

It follows that

\[
\int_0^T H_\varepsilon(u^\varepsilon(t, L)) \omega(t) \, dt
\]

\[= \langle \mu^\varepsilon, \omega \rangle_{M^1([0, T]; \mathbb{R}), C^0([0, T]; \mathbb{R})} \quad \longrightarrow \quad \langle \mu, \omega \rangle_{M^1([0, T]; \mathbb{R}), C^0([0, T]; \mathbb{R})}
\]

for all \(\omega \in C^0([0, T]; \mathbb{R})\). Moreover the sequence \((u^\varepsilon_i(T, \cdot))_{\varepsilon > 0}\) is bounded in \(H\), and possibly extracting another subsequence still denoted as \((u^\varepsilon)_{\varepsilon > 0}\), there exists \(v_* \in H\) such that

\[
u^\varepsilon_i(T, \cdot) \rightharpoonup v_* \quad \text{weakly in } H.
\]
By passing to the limit as $\varepsilon$ tends to zero in (14) we get
\[
- \int_0^T \int_0^L u_t(t, x) w(t, x) \, dx \, dt + \int_0^L v_s(x) w(T, x) \, dx \\
+ k^2 \int_0^T \int_0^L u_{xx}(t, x) w_{xx}(t, x) \, dx \, dt + \langle \mu, w(\cdot, L) \rangle_{M^1([0,T];\mathbb{R}), C^0([0,T];\mathbb{R})}
\]
\[
= \int_0^T v_0(x) w(0, x) \, dx + \int_0^T \int_0^L f(t, x) w(t, x) \, dx \, dt
\]
for all $w \in W^{1,2}(0, T; H) \cap L^2(0, T; V)$ such that $w(\cdot, L) \in C^0([0,T]; \mathbb{R})$.

Next we observe that

**Proposition 4.2.** The measure $\mu$ has the following properties
\[
\langle \mu, \omega - u(\cdot, L) \rangle_{M^1([0,T];\mathbb{R}), C^0([0,T];\mathbb{R})} \leq 0 \quad \forall \omega \in C^0([0,T]; [g_1, g_2])
\]
and
\[
\text{Supp}(\mu) \subset \{ t \in [0,T] : u(t, L) \in \{g_1, g_2\} \}.
\]

**Proof.** For $\beta = 1$ and $c_\varepsilon = \frac{1}{\varepsilon}$ the reader is referred to [26]. For $\beta \geq 1$ and $(c_\varepsilon)_\varepsilon > 0$ being a sequence of positive real numbers tending to $+\infty$, a straightforward adaptation of the proof given in [26] leads to the same conclusion.

Hence
\[
- \int_0^T \int_0^L u_t(t, x) (w(t, x) - u_t(t, x)) \, dx \, dt + \int_0^L v_s(x) (w(T, x) - u(T, x)) \, dx \\
+ k^2 \int_0^T \int_0^L u_{xx}(t, x) (w_{xx}(t, x) - u_{xx}(t, x)) \, dx \, dt \\
\geq \int_0^T v_0(x) (w(0, x) - u_0) \, dx + \int_0^T \int_0^L f(t, x) (w(t, x) - u(t, x)) \, dx \, dt
\]
for all $w \in W^{1,2}(0, T; H) \cap L^2(0, T; V)$ such that $w(\cdot, L) \in C^0([0,T]; [g_1, g_2])$. By density the same inequality holds for any $w \in W^{1,2}(0, T; H) \cap L^2(0, T; K)$ and we may conclude that $u$ is solution of problem ($P_{\text{beam}}^S$).

**Remark 4.1.** As a corollary of Proposition 4.1 we obtain also that the sequence $(u^\varepsilon_t)_\varepsilon > 0$ is bounded in $L^1(0, T; V')$ and the vector-valued measures $u^\varepsilon_t \, m$ admit a weak$^*$ limit in $\mathcal{M}^1([0,T]; V')$ given by the Stieltjes measure $du_t$ ([7]). By denoting $du_t = u_{tt}$ we obtain
\[
\langle u_t, w \rangle_{\mathcal{M}^1([0,T]; V'), C^0([0,T]; V)} + k^2 \int_0^T \int_0^L u_{xx}(t, x) w_{xx}(t, x) \, dx \, dt \\
+ \langle \mu, w(\cdot, L) \rangle_{\mathcal{M}^1([0,T]; \mathbb{R}), C^0([0,T]; \mathbb{R})} = \int_0^T \int_0^L f(t, x) w(t, x) \, dx \, dt
\]
for all $w \in C^0([0,T]; V)$.

Hence $u$ is a strong solution of the contact problem with Signorini complementarity conditions as defined in [26]. By using the Radon-Nikodym theorem, we may decompose the measure $\mu$ as
\[
\mu = h_{reg} \, m + \mu_s, \quad \mu_s = h_s \, |\mu_s|,
\]
where \( \mu_s \) is a singular measure with respect to \( m \), \( h_{reg} \in L^1(0,T;\mathbb{R},m) \), \( h_s \in L^1((0,T];\mathbb{R},|\mu_s|) \) such that \( |h_s(t)| = 1 \) for \( |\mu_s| \)-almost all \( t \in [0,T] \). From Proposition 4.2 we infer that the measure \( \mu \) satisfies the Signorini complementarity conditions in the following sense

\[
\begin{cases}
    h_{reg}(t) \in \partial\psi_{[g_1,g_2]}(u(t,L)) & \text{for } m\text{-a.a. } t \in [0,T], \\
    h_s(t) \in \partial\psi_{[g_1,g_2]}(u(t,L)) & \text{for } |\mu_s|-\text{a.a. } t \in [0,T] 
\end{cases}
\]

(see [29, 31]).

By using the weak convergences (11)-(12) we obtain that \( u \) satisfies also an energy estimate and we have

\[
\frac{1}{2}\|u_t(\tau,\cdot)\|_H^2 + \frac{k^2}{2}\|u_{xx}(\tau,\cdot)\|_H^2 \leq \frac{1}{2}\|v_0\|_H^2 + \frac{k^2}{2}\|u_{0,xx}\|_H^2 \\
+ \int_0^{\tau_2} \int_0^L f(t,x) u_t(t,x) \, dx \, dt \quad \forall \text{ a.a. } \tau \in (0,T).
\]

But we may wonder if the limit trajectory satisfies some energy balance properties analogous to (8) i.e. do we have

\[
\frac{1}{2}\|u_t(\tau_2,\cdot)\|_H^2 + \frac{k^2}{2}\|u_{xx}(\tau_2,\cdot)\|_H^2 = \frac{1}{2}\|u_t(\tau_1,\cdot)\|_H^2 + \frac{k^2}{2}\|u_{xx}(\tau_1,\cdot)\|_H^2 \\
+ \int_{\tau_1}^{\tau_2} \int_0^L f(t,x) u_t(t,x) \, dx \, dt
\]

for almost all \((\tau_1,\tau_2) \in (0,T)^2\) such that \( \tau_1 < \tau_2 \)? Of course weak convergences prevent us to pass to the limit in (8). Nevertheless we can prove that energy is conserved during free flight motion intervals and through instantaneous collision events i.e.

**Theorem 4.1.** Let \( f \in L^2(0,T;H) \), \( u_0 \in \mathcal{K} \) and \( v_0 \in H \). Let us assume that \( g_1 < 0 < g_2 \) and let \( u \) be a solution of the contact problem with Signorini complementarity conditions obtained as a limit of a sequence of solutions \((u^e)_{e>0}\) of (PNC) for a given \( \beta \geq 1 \) and a sequence of positive real parameters \((c_e)_{e>0}\) tending to \(+\infty\).

Let \( t_0 \in (0,T) \) and assume that there exists \( \delta \in (0,\min(t_0,T-t_0)) \) such that \( u(t,L) \in (g_1,g_2) \) for all \( t \in (t_0-\delta,t_0) \cup (t_0,t_0+\delta) \). Then we have

\[
\frac{1}{2}\|u_t(\tau_2,\cdot)\|_H^2 + \frac{k^2}{2}\|u_{xx}(\tau_2,\cdot)\|_H^2 = \frac{1}{2}\|u_t(\tau_1,\cdot)\|_H^2 + \frac{k^2}{2}\|u_{xx}(\tau_1,\cdot)\|_H^2 \\
+ \int_{\tau_1}^{\tau_2} \int_0^L f(t,x) u_t(t,x) \, dx \, dt
\]

for almost all \((\tau_1,\tau_2) \in (t_0-\delta,t_0+\delta)^2\) such that \( \tau_1 < \tau_2 \).

**Remark 4.2.** If \( u(t_0,L) \in \{g_1,g_2\} \) then contact occurs at \( t_0 \) and we will say that \( t_0 \) is an instantaneous collision instant. Otherwise \( u(t,L) \in (g_1,g_2) \) for all \( t \in (t_0-\delta,t_0+\delta) \) and we will say that \((t_0-\delta,t_0+\delta)\) is a free flight motion interval.

**Proof.** Let us consider \( t_0 \in (0,T) \) such that there exists \( \delta \in (0,\min(t_0,T-t_0)) \) such that \( u(t,L) \in (g_1,g_2) \) for all \( t \in (t_0-\delta,t_0) \cup (t_0,t_0+\delta) \). Let \( t_1 \in (t_0-\delta,t_0) \) and \( t_2 \in (t_0,t_0+\delta) \). Then \( u(t_1,L) \in (g_1,g_2) \) and \( u(t_2,L) \in (g_1,g_2) \) and we may define \( \gamma \in (0,\min(-g_1,g_2)) \) such that \( u(t_i,L) \in (g_1+\gamma,g_2-\gamma) \) for \( i = 1,2 \).

By using the continuity of the mapping \( t \mapsto u(t,L) \) we infer that there exists \( \alpha \in \)
\((0, \min(t_0 - t_1, t_1 - t_0 + \delta, t_0 + \delta - t_2, t_2 - t_0))\) such that \(u(t, L) \in \left( g_1 + \frac{3\gamma}{4}, g_2 - \frac{3\gamma}{4} \right)\) for all \(t \in (t_1 - \alpha, t_1 + \alpha) \cup (t_2 - \alpha, t_2 + \alpha)\). The strong convergence of the sequence \((u^\varepsilon(\cdot, L))_{\varepsilon > 0}\) to \(u(\cdot, L)\) in \(C^0([0, T]; \mathbb{R})\) implies that there exists \(\varepsilon_0 > 0\) such that \(u^\varepsilon(t, L) \in \left( g_1 + \frac{\gamma}{2}, g_2 - \frac{\gamma}{2} \right)\) for all \(t \in (t_1 - \alpha, t_1 + \alpha) \cup (t_2 - \alpha, t_2 + \alpha)\) and for all \(\varepsilon \in (0, \varepsilon_0)\).

Going back to problem \((P^{\varepsilon}_{beam})\) we have

\[
\int_a^b \langle u^\varepsilon_t(t, \cdot), w \varphi_t(t) \rangle_{V^*, V} dt + \langle u^\varepsilon_t(h, \cdot), w \varphi(b) \rangle_{V^*, V} - \langle u^\varepsilon_t(a, \cdot), w \varphi(a) \rangle_{V^*, V} + k^2 \int_a^b \int_0^L u^\varepsilon_{xx}(t, x) w_{xx}(x) \varphi(t) dx dt + \int_a^b H^\varepsilon(u^\varepsilon(t, L)) w(L) \varphi(t) dt = \int_0^b f(t, x) w(x) \varphi(t) dx dt
\]

for all \(w \in V, \varphi \in H^1(0, T)\) and for all \((a, b) \in [0, T]^2\).

We may choose \(a = t_1\) and \(b = t_2\) with a function \(\varphi \in C^\infty([0, T]; \mathbb{R})\) such that \(0 \leq \varphi(t) \leq 1\) for all \(t \in [0, T]\), \(\varphi(t) = 1\) for all \(t \in [t_1, t_2]\) and \(\varphi(t) = 0\) for all \(t \in [0, t_1 - \alpha] \cup [t_2 + \alpha, T]\). Then, for all \(\varepsilon \in (0, \varepsilon_0)\), we have

\[
\int_0^T H^\varepsilon(u^\varepsilon(t, L)) w(L) \varphi(t) dt = \int_0^{t_1 - \alpha} H^\varepsilon(u^\varepsilon(t, L)) w(L) \varphi(t) dt - \int_{t_1 - \alpha}^{t_1} H^\varepsilon(u^\varepsilon(t, L)) w(L) \varphi(t) dt + \int_{t_1}^{t_2 + \alpha} H^\varepsilon(u^\varepsilon(t, L)) w(L) \varphi(t) dt - \int_{t_2 + \alpha}^T H^\varepsilon(u^\varepsilon(t, L)) w(L) \varphi(t) dt
\]

i.e.

\[
\int_{t_1}^{t_2} H^\varepsilon(u^\varepsilon(t, L)) w(L) \varphi(t) dt = \langle \mu^\varepsilon, \varphi \rangle_{M^1([0, T]; \mathbb{R}), C^0([0, T]; \mathbb{R})} w(L).
\]

It follows that

\[
\langle u^\varepsilon_t(t_2, \cdot) - u^\varepsilon_t(t_1, \cdot), w \rangle_{V^*, V} + k^2 \int_{t_1}^{t_2} \int_0^L u^\varepsilon_{xx}(t, x) w_{xx}(x) dx dt + \langle \mu^\varepsilon, \varphi \rangle_{M^1([0, T]; \mathbb{R}), C^0([0, T]; \mathbb{R})} w(L) = \int_{t_1}^{t_2} \int_0^L f(t, x) w(x) dx dt
\]

for all \(w \in V\) and for all \(\varepsilon \in (0, \varepsilon_0)\).

With \((P^{\varepsilon}_{beam})\) we obtain that \((u^\varepsilon_t)_{\varepsilon > 0}\) is uniformly bounded in \(L^2(0, T; (V \cap H^1_0(0, L))^\prime)\). With the Simon lemma it follows that, possibly extracting another subsequence still denoted \((u^\varepsilon)_{\varepsilon > 0}\), we have

\[
u^\varepsilon_t \rightharpoonup u_t \quad \text{strongly in } C^0([0, T]; (V \cap H^1_0(0, L))^\prime).
\]

Hence, possibly modifying \(u_t\) on a negligible subset of \([0, T]\), we have (a representant of the class of) \(u_t \in C^0_w([0, T]; H)\). Moreover \((u^\varepsilon_t)_{\varepsilon > 0}\) is uniformly bounded in
$L^1(0, T; V')$, which implies that $(u^n_\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in $BV(0, T; V')$. By using Helly theorem ([7]) there exists $v \in BV(0, T; V')$ such that

$$u^n_\varepsilon(t, \cdot) \rightharpoonup v(t, \cdot) \quad \text{weakly in } V', \text{ for all } t \in [0, T].$$

We infer that $v \in L^\infty(0, T; V')$ and with Lebesgue dominated convergence theorem

$$ \int_0^T \langle u^n_\varepsilon(t, \cdot), w \rangle_{V', V} \psi(t) \, dt \rightarrow \int_0^T \langle v(t, \cdot), w \rangle_{V', V} \psi(t) \, dt $$

for all $w \in V, \psi \in \mathcal{D}(0, T; \mathbb{R})$. But, with (12)

$$ \int_0^T \langle u^n_\varepsilon(t, \cdot), w \rangle_{V', V} \psi(t) \, dt = \int_0^T \langle u^n_\varepsilon(t, \cdot), w \rangle_H \psi(t) \, dt \rightarrow \int_0^T \langle u_\varepsilon(t, \cdot), w \rangle_H \psi \, dt $$

which yields $v(t, \cdot) = u_\varepsilon(t, \cdot)$ for almost all $t \in (0, T)$.

Now we pass to the limit as $\varepsilon$ tends to zero in (17). We obtain

$$ \langle v(t_2, \cdot) - v(t_1, \cdot), w \rangle_{V', V} + k^2 \int_{t_1}^{t_2} \int_0^L u_{xx}(t, x) w_{xx}(x) \, dx \, dt $$

$$ + \langle \mu, \varphi \rangle_{\mathcal{M}^1((0, T]; \mathbb{R}), \mathcal{C}^0((0, T]; \mathbb{R})} w(L) = \int_{t_1}^{t_2} \int_0^L f(t, x) w(x) \, dx \, dt $$

for all $w \in V$. With Proposition 4.2 we infer that $\text{Supp}(\mu) \cap \text{Supp}(\varphi) \subset \{t_0\}$. Thus

$$ \langle v(t_2, \cdot) - v(t_1, \cdot), w \rangle_{V', V} + k^2 \int_{t_1}^{t_2} \int_0^L u_{xx}(t, x) w_{xx}(x) \, dx \, dt $$

$$ + \mu(\{t_0\}) \varphi(t_0) w(L) = \int_{t_1}^{t_2} \int_0^L f(t, x) w(x) \, dx \, dt $$

for all $w \in V$. It follows that

$$ \langle u_\varepsilon(t_2, \cdot) - u_\varepsilon(t_1, \cdot), w \rangle_{V', V} + k^2 \int_{t_1}^{t_2} \int_0^L u_{xx}(t, x) w_{xx}(x) \, dx \, dt $$

$$ + \mu(\{t_0\}) w(L) = \int_{t_1}^{t_2} \int_0^L f(t, x) w(x) \, dx \, dt $$

for all $w \in V$ and for almost all $t_1 \in (t_0 - \delta, t_0), \ t_2 \in (t_0, t_0 + \delta)$. We may choose a sequence of such points $(t^n_1)_{n \geq 0}$ and $(t^n_2)_{n \geq 0}$ such that $(t^n_1)_{n \geq 0}$ converges to $t_0$ and $(t^n_2)_{n \geq 0}$ converges to $t_0$. Owing that $t \mapsto u_\varepsilon(t, \cdot)$ is weakly continuous from $[0, T]$ to $H$ we get

$$ \langle u(t^n_2, \cdot) - u(t^n_1, \cdot), w \rangle_{V', V} = \langle u(t^n_2, \cdot) - u(t^n_1, \cdot), w \rangle_H \rightarrow 0 $$

and finally

$$ \mu(\{t_0\}) w(L) = 0 $$

for all $w \in V$. With $w \in V$ such that $w(L) \neq 0$ (for instance $w(x) = \left(\frac{x}{L}\right)^2$ for all $x \in [0, L]$) we get $\mu(\{t_0\}) = 0$ and $\text{Supp}(\mu) \cap (t_0 - \delta, t_0 + \delta) = \emptyset$. 

Going back to (16) and choosing \( a = t_0 - \delta, b = t_0 + \delta \), we have

\[
- \int_{t_0 - \delta}^{t_0 + \delta} \left\langle u_{z}^\varepsilon(t, \cdot), w \varphi_{\varepsilon}(t) \right\rangle_{V', V} dt + k^2 \int_{t_0 - \delta}^{t_0 + \delta} \int_{0}^{L} u_{xx}^\varepsilon(t, x) w_{xx}(x) \varphi(t) dx dt
+ \int_{t_0 - \delta}^{t_0 + \delta} H_{\varepsilon}(u^\varepsilon(t, L)) w(L) \varphi(t) dt = \int_{t_0 - \delta}^{t_0 + \delta} f(t, x) w(x) \varphi(t) dx dt
\]

for all \( w \in V \) and \( \varphi \in D(t_0 - \delta, t_0 + \delta) \). By passing to the limit as \( \varepsilon \) tends to zero we obtain

\[
- \int_{t_0 - \delta}^{t_0 + \delta} \left\langle u_\varepsilon(t, \cdot), w \varphi_{\varepsilon}(t) \right\rangle_{V', V} dt + k^2 \int_{t_0 - \delta}^{t_0 + \delta} \int_{0}^{L} u_{xx}(t, x) w_{xx}(x) \varphi(t) dx dt
+ \left\langle \mu, \varphi \right\rangle_{M^1([0,T];E), C^0([0,T];E)} w(L) = \int_{t_0 - \delta}^{t_0 + \delta} f(t, x) w(x) \varphi(t) dx dt
\]

for all \( w \in V \) and \( \varphi \in D(t_0 - \delta, t_0 + \delta) \). It follows that

\[
\left| \int_{t_0 - \delta}^{t_0 + \delta} \left\langle u_\varepsilon(t, \cdot), w \varphi_{\varepsilon}(t) \right\rangle_{V', V} dt \right| \leq k^2 \| u_{xx} \|_{L^2(0,T;H)} \| w_{xx} \varphi \|_{L^2(t_0 - \delta, t_0 + \delta; H)} + \| f \|_{L^2(0,T;H)} \| w \varphi \|_{L^2(t_0 - \delta, t_0 + \delta; H)}
\]

for all \( w \in V \) and \( \varphi \in D(t_0 - \delta, t_0 + \delta) \). Hence \( u_\varepsilon \) belongs to \( W^{1,2}(t_0 - \delta, t_0 + \delta; V') \) and we may identify \( du_\varepsilon = u_\varepsilon dt \) a density measure with respect to Lebesgue measure on \((t_0 - \delta, t_0 + \delta)\). So we have

\[
\left\langle u_\varepsilon(t, \cdot), w \right\rangle_{V', V} + k^2 \int_{0}^{L} u_{xx}(t, x) w(x) dx = \int_{0}^{L} f(t, x) w(x) dx \quad \forall w \in V, \forall a.a. t \in (t_0 - \delta, t_0 + \delta).
\]

Finally by using the same computations as in Theorem 3.1 we obtain the announced result.

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