Anisotropic Special Relativity

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Anisotropic Special Relativity (ASR) is the relativistic theory of nature with a preferred direction in space-time. By relaxing the full-isotropy constraint on space-time to the preference of one direction, we get a perturbative modification of the Minkowski metric as $g_{\mu\nu} = \eta_{\mu\nu} - b\epsilon_{\mu\nu}$ leading to an extension to the geometrical objects such as line element. The symmetry group of ASR is obtained to have six generators satisfying the full Lorentz group algebra. However, the generators are deformed using the perturbation parameter $b$. So, ASR retains the same representations of Special Relativity (SR) but allows for Lorentz-invariant violation at the same time. A procedure to make the anisotropic quantum field theory is provided where the Lorentz-invariant Lagrangians are replaced with their ASR version in which the inner product of any pair of covariant/contravariant indices is mediated by the anisotropic metric $g_{\mu\nu}$.

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INTRODUCTION

One of the widely accepted fundamental theories of physics is Special Relativity (SR) in which Quantum Field Theory (QFT) and Standard Model (SM) of the universe are grounded. In SR, some quantities, like the line element $ds^2 = (dt^2 - dx^2 - dy^2 - dz^2)$, are invariant under the Lorentz group transformations. Despite the very strict restrictions on departure from Lorentz symmetry \[4, 5\], the Lorentz Invariance Violation (LIV) idea \[1–3\] is still popular. One of the scenarios for LIV is Standard Model Extension (SME) \[6\] in which the Lorentz violation is allowed by adding non-relativistic terms to the Lagrangian. On the other side, there are theories wherein the LIV does not require the complete breakdown of relativistic symmetry. The examples of latter are perturbation approach of Coleman and Glashow \[7\] and Very Special Relativity (VSR) proposed by Cohen and Glashow \[8\].

There are two more theories allowing for LIV, Finslerian structure of the space-time \[9\] and Non-Commutative Quantum Field Theory \[10\], in context of which VSR can be realised \[11, 12\]. In fact, a deformation of VSR symmetry can be seen in the case of a Finsler space-time in which the fundamental metric $g$ can be obtained from a function $F = (\eta_{\mu\nu}dx^\mu dx^\nu)^{(1-b)/2}(n_\alpha dx^\alpha)^b$ using $g_{\mu\nu} = \frac{1}{2}\partial_\mu \partial_\nu F^2$, where $\eta_{\mu\nu}$ is the Minkowski metric of SR, $n^\alpha = (1, 0, 0, 1)$ is a null vector and $b$ is a constant parameter. VSR suggests a subgroup of the Lorentz group, SIM(2), in addition to the space-time translations to be the maximal symmetry of nature wherein one of the three spatial directions is preferred. The anisotropy of space is one of the possible scenarios for LIV, whose upper bound has already been investigated by analysing the cosmological data \[13\] as well as performing terrestrial experiments \[14, 15\]. On the track of building an anisotropic version of SR, Bogoslovsky suggested the Finslerian line element $ds = (\eta_{\mu\nu}dx^\mu dx^\nu)^{(1-b)/2}(n_\alpha dx^\alpha)^b$ which is not SR invariant but VSR invariant \[16, 17\].

In this letter, we ask if there may be any other possible form of line element that is invariant under a deformed or reduced Lorentz symmetry but not under the transformations of the full Lorentz group? We introduce a novel form of line element that not only satisfies the anisotropy requirements but also leads to the same Lorentz algebra as SR does. So, on the one hand our theory allows for departure from Lorentz invariance while on the other hand it results in the same representations of Lorentz group. The lack of non-trivia representations is the main issue in the other above-mentioned anisotropic theories, which is resolved completely in ASR.

RESULTS AND DISCUSSION

We obtain an anisotropic line element by easing the full-isotropy constraint on the Minkowski metric, but considering a preferred direction, $N^\mu$, in space-time instead. An anisotropic metric could be obtained either by varying the magnitude of the diagonal elements or introducing off-diagonal elements to $\eta_{\mu\nu}$. The minimal changes that can make the $z$-direction preferred is to have non-unity diagonal $t$- and $z$- elements. A modification of Minkowski metric as

$$
\eta_{\mu\nu} \rightarrow g_{\mu\nu} = \begin{pmatrix}
1 - b & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & \frac{-1}{1+b}
\end{pmatrix}
$$

(1)
induces such a desirable anisotropy. It is straightforward
to check that the vector \( N^\mu = \left( \frac{1}{\sqrt{1-b}}, 0, 0, \sqrt{1-b} \right) \) is a
null-vector with respect to \( g \) (\( g_{\mu\nu}N^\mu N^\nu = 0 \)); so, we
consider the direction of \( N^\mu \) as the preferred direction
in space-time. As we have not varied any of the \( x \)- or
\( y \)-elements of the metric, the isotropy is still retained in
the \( x-y \) plane. By Anisotropic Special Relativity (ASR)
we mean a relativistic theory in which the inner product
of a pair of covariant/contravariant indices is mediated
by the anisotropic metric \( g \).

Applying metric (1) gives our version of Lorentz viol-
ating line element as
\[
ds^2 = (dt^2 - dx^2 - dy^2 - dz^2) - b(dt^2 + \frac{dz^2}{1-b}) \tag{2}
\]
which can be considered as a perturbation of the Lorentz
invariant line element with \( b \) factor to be a constant pa-
parameter controlling the perturbation. Obviously, setting
\( b = 0 \) turns the situation back to the full Lorentz sym-
metry. The metric \( g \) itself can be also written as a per-
turbation of the Minkowski metric
\[
g_{\mu\nu} = \eta_{\mu\nu} - 2b \epsilon_{\mu\nu} \tag{3}
\]
where
\[
\epsilon_{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{1-b}
\end{pmatrix}, \tag{4}
\]

Moreover, it can be checked that the fundamental metric \( g \)
introduced above is compatible with the full Lorentz
group algebra.

Starting from the line element invariance principal
\( g'_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu \) and considering \( \Lambda^\mu_\nu \) as the
most general transformation of the theory (excluding the
space-time translations) that transforms the contravari-
ant four-vectors including \( dx^\mu \) as \( dx'^\mu = \Lambda^\mu_\nu dx^\nu \), one
obtains \( \Lambda^\mu_\nu \Lambda^\rho_\mu g_{\rho\nu} \Lambda^\nu_\beta \Lambda^\nu_\beta = g_{\alpha\beta} \). This can be simplified
by replacing \( \Lambda^\mu_\nu = e(\omega^\mu_\nu) \) and focusing on infinities
transformations \( \Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \) to obtain \( \omega^\mu_\nu = -\omega^\nu_\mu \).

Since \( \omega \) is antisymmetric, it has only six independent
parameters as:
\[
\omega^\mu_\nu = \begin{pmatrix}
0 & \omega_{01} & \omega_{02} & \omega_{03} \\
-\omega_{01} & 0 & \omega_{12} & \omega_{13} \\
-\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\
-\omega_{03} & -\omega_{13} & -\omega_{23} & 0
\end{pmatrix} \tag{5}
\]
The first covariant index of \( \omega^\mu_\nu \) should now be converted
to a contravariant one using \( g^{\mu\nu} \) (which is the inverse
of \( g_{\mu\nu} \)) as \( \omega^{\mu}_\nu = g^{\mu\rho} \omega^\rho_\nu \). In the matrix form, one need
to calculate the inverse of metric \( g_{\mu\nu} \) given by Eq. (1) and
apply it on the generator \( \omega^{\mu}_\nu \) given by Eq. (5) to obtain:
\[
\omega \sim \omega^{\mu}_\nu = \begin{pmatrix}
0 & \frac{\omega_{01}}{B} & \frac{\omega_{02}}{B} & \frac{\omega_{03}}{B} \\
\frac{\omega_{12}}{B} & 0 & -\omega_{12} & -\omega_{13} \\
\frac{\omega_{03}}{B} & \frac{\omega_{12}}{B} & 0 & -\omega_{23} \\
\omega_{03}B & \omega_{12}B & \omega_{23}B & 0
\end{pmatrix} \tag{6}
\]
where \( B = 1 - b \) has been used. Now, we can define parameters:
\[
\eta_1 = \frac{\omega_{01}}{\sqrt{B}}, \quad \eta_2 = \frac{\omega_{02}}{\sqrt{B}}, \quad \eta_3 = \omega_{03}, \quad \theta_1 = \omega_{12}\sqrt{\frac{B}{\eta_1}}, \quad \theta_2 = -\omega_{12}\sqrt{\frac{B}{\eta_2}}, \quad \theta_3 = \omega_{12} \tag{7}
\]
that allows to rewrite Eq. (6) as \( \omega = -i\eta_1 K_1 - i\eta_2 K_2 - i\eta_3 K_3 - \theta_1 J_1 - \theta_2 J_2 - \theta_3 J_3 \) with \( K_i \) and \( J_i \) being:
\[
K_1 = \begin{pmatrix}
0 & \frac{\omega_{01}}{\sqrt{B}} & 0 & 0 \\
\omega_{02} & 0 & 0 & 0 \\
\omega_{03} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\eta_2}
\end{pmatrix}, \quad J_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{\eta_1} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
\[
K_2 = \begin{pmatrix}
0 & 0 & \omega_{01} & 0 \\
\omega_{02} & 0 & 0 & 0 \\
\omega_{03} & 0 & 0 & 0 \\
\omega_{03} & \omega_{02} & 0 & 0
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\eta_1} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
\[
K_3 = \begin{pmatrix}
0 & 0 & 0 & \omega_{01} \\
0 & 0 & 0 & \omega_{02} \\
0 & 0 & 0 & \omega_{03} \\
\omega_{01} & \omega_{02} & \omega_{03} & 0
\end{pmatrix}, \quad J_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

It is straightforward to check that the above generators
satisfy the full Lorentz group algebra:
\[
[K_i, J_j] = i\epsilon_{ijk} K_k, \quad [J_i, J_j] = i\epsilon_{ijk} J_k, \quad [K_i, K_j] = -i\epsilon_{ijk} K_k \tag{9}
\]
The fact that generators \( K_i \) and \( J_i \) satisfy the full-
Lorentz group algebra warrants the representations of
ASR to be the same as those of SR. On the other hand, the
fact that ASR generators, given by Eq. (8), are deforma-
tions of Lorentz group generators means that ASR is
capable of explaining LIV, see below. Unlike other exist-
ing theories such as VSR, ASR explains LIV without
leading to only trivial \( 1 \times 1 \) representations \( \Gamma \); spinor
and four-vector representations exist in ASR naturally.
However, the equation of motion for such fields could
vary as the fundamental metric \( g^{\mu\nu} \) deviates from the
Minkowski metric.

As a consequence of ASR metric \( g \) given by (1), we can infer to introducing a Lorentz violating term into
the fermionic Lagrangian to explain the neutrino’s mass
in a natural way without violating the lepton number or
adding sterile right-handed neutrinos. This has been
discussed for the first time in [18, 19]; however, we ask if
we can add a different Lorentz violating term as an im-
}
Lagrangian of neutrino and modify it just by applying metric \(g_{\mu\nu}\) to lower the index of one of the contravariant vectors i.e. \(\mathcal{L}^{\text{ASR}} = i\bar{\psi}M^\gamma \gamma^\alpha g_{\alpha\beta} \partial^\beta \psi + \) leading to

\[
\mathcal{L}^{\text{ASR}} = \mathcal{L}^{\text{SR}} - ib\bar{\psi}(\gamma^\alpha \partial^\alpha + \gamma^3 \partial^3 - 1) - \psi
\]

where \(\mathcal{L}^{\text{SR}}\) is the massless Majorana Lagrangian of neutrino in SR given by \(\bar{\psi}M^\gamma \gamma^\alpha \partial^\alpha \psi + \) with \(\gamma^\alpha, \alpha = 0 - 3, \) to be the Dirac matrices. Eq. \((10)\) gives the dispersion relation as \((m^2)^{\text{ASR}} = (p^\alpha p^\alpha)^{\text{SR}} - b((p^0)^2 + (p^3)^2)\) for which \((p^\alpha p^\alpha)^{\text{SR}} = (m^2)^{\text{SR}}\) is assumed to be zero; even in such a case, it is still possible for \((m^2)^{\text{ASR}}\) to be non-zero due to the appearance of the perturbative term \(b((p^0)^2 + (p^3)^2)\). However, for the neutrinos moving purely in the \(x-y\) plane \((p^0 = p^3 = 0)\), the ASR mass yet remains zero. Therefore, the neutrino’s Majorana mass in anisotropic spacetime will depend on the direction of its movement which in turn means that the neutrino flavour oscillation allowed by anisotropic theories like ASR and VSR \([20]\) could be direction-dependent as well.

It can also be noticed that the field theory (FT) obtained in ASR remains local in contrast to the non-local FT emerging from the correction term introduced by Cohen and Glashow in VSR \([18]\). The term they suggested to correct the equations of motion with is \(m^2_{\text{ASR}} = \frac{m^2_{\text{SR}}}{2} \) where the appearance of \(\partial\) operator in the denominator makes the theory non-local. However, ASR-FT remains local as only the lower orders of the derivative operator appear in the Lagrangian.

In general, the mass of any field in ASR can be modified just similar to the line element in Eq. \((2)\) i.e.

\[
(m^2)^{\text{SR}} \rightarrow (m^2)^{\text{ASR}} = (m^2)^{\text{SR}} - b((p^0)^2 + (p^3)^2)\]

where \(m^2\) is the mass of the field in SR given by \((m^2)^{\text{SR}} = ((p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2)\) with \(p^i, i = 0 - 3,\) being the elements of the contravariant four-momentum vector in SR.

In order to make an anisotropic quantum field theory, one can promote any Lagrangian by replacing the isotropic inner product of any pair of covariant/contravariant indices with its anisotropic version. Similar to the Majorana Lagrangian of left-handed neutrinos, the Dirac Lagrangian for the fermions can be modified to

\[
\mathcal{L}^{\text{ASR}} = \mathcal{L}^{\text{SR}} - i\bar{\psi}(\gamma^\alpha \partial^\alpha + \gamma^3 \partial^3 - 1)\psi
\]

a consequence of which is the extension of the dispersion relation as \((11)\). Similarly, in the electrodynamics Lagrangian \(\mathcal{L}^{\text{ASR}} = -\frac{1}{4} \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}\) we just need to replace the Minkowski metric with \(g\) i.e. \(\mathcal{L}^{\text{ASR}} = -\frac{1}{4} g_{\mu\rho} g_{\nu\beta} F^{\mu\alpha} F^{\rho\nu}\) resulting in equations of motion as

\[
(\partial_{\mu} F^{\mu\nu})^{\text{SR}} - b_{\mu\alpha} \partial^{\alpha} F^{\mu\nu} = 0
\]

from which the anisotropically modified Maxwell equations can be derived.

The shape of gauge symmetry in ASR remains the same as in SR

\[
\begin{align*}
\partial_{\mu} \psi & \rightarrow e^{iqA_{\mu}} \psi \\
A_{\mu} & \rightarrow A_{\mu} + \partial_{\mu} \phi
\end{align*}
\]

with the difference that, here, all the covariant indices are made from the contravariant ones by applying the metric \(g\). For instance, the ASR version of \(\partial_{\mu}\) and \(A_{\mu}\) are

\[
\partial_{\mu} = \begin{pmatrix} B \partial^0 \\
-\partial^1 \\
-\partial^2 \\
-\partial^3
\end{pmatrix} ; A_{\mu} = \begin{pmatrix} B A^0 \\
-A^1 \\
-A^2 \\
-A^3
\end{pmatrix}
\]

**CONCLUSION**

In this letter, we proposed an anisotropic relativistic theory, ASR, as an alternative to the special theory of relativity. ASR can be formalised by replacing the Minkowski metric with a perturbative anisotropic metric in which the space-time isotropy constraint has been replaced with the preference of a direction determined by the null vector \(N^\mu\). On the one hand, the perturbative deviation of Lorentz invariance is allowed while on the other hand, the symmetry group’s deformed generators satisfy the same algebra as that of the full-Lorentz group, which guarantees the irreducible representations of ASR to be just the same as those of SR. Despite a direction-dependent modification to the dispersion relation, the QFT built up in ASR remains local.

As one of the interesting applications of ASR, we can infer to redesigning of neutrino oscillation experiments. ASR suggests the neutrino mass, hence the neutrino oscillation, to be direction dependent; so, designing a neutrino oscillation experiment in which the direction of propagation of neutrinos can sweep different spatial directions could be of special interest. In addition, as the transformations of ASR are direction dependent, we expect most of the relativistic phenomena like length contraction, time dilation, Doppler effect and aberration of light, etc. to be direction dependent too.

A possible origin for the space-time anisotropy could be the anisotropic mass/energy distribution in the universe, which can be reflected in cosmic microwave background. This means that different locations in the universe could be expected to experience different level or direction
of anisotropy. In such a case, a universal version of ASR can be achieved by promoting the perturbative parameter $b$ from a constant to a field.

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