Edge states in a two-dimensional non-symmorphic semimetal

P.G. Matveeva, D.N. Aristov, D. Meidan, and D.B. Gutman

1Department of Physics and Research Center Optimas, University of Kaiserslautern, 67663 Kaiserslautern, Germany
2“PNPI” NRC “Kurchatov Institute”, Gatchina 188300, Russia
3Department of Physics, St.Petersburg State University, Ulanskovskaya 1, St.Petersburg 198504, Russia
4Institute for Nanotechnology, Karlsruhe Institute of Technology, 76021 Karlsruhe, Germany
5Department of Physics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
6Department of Physics, Bar-Ilan University, Ramat Gan, 52900, Israel

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Dirac materials have unique transport properties, partly due to the presence of surface states. A new type of Dirac materials, protected by non-symmorphic symmetries was recently proposed by Young and Kane [1]. By breaking of time reversal or inversion symmetry one can split the Dirac cones into Weyl nodes. The later are characterized by local Chern numbers, that makes them two-dimensional analogs of Weyl semimetals. We find that the formation of the Weyl nodes is accompanied by an emergence of one-dimensional surface states, similar to Fermi arcs in Weyl semimetals and edge states in two-dimensional graphene. We explore these states for a quasi-one-dimensional non-symmorphic ribbon. The type and strength of applied deformation control the location and Weyl nodes and their composition. This determines the properties of emerging edge states. The sensitivity of these edge states to the external deformations makes non-symmorphic materials potentially useful as a new type of electromechanical sensors.

I. INTRODUCTION

Topological phases of matter are attracting a growing attention in recent years. Due to their unique band structure, these gapped phases are known to support protected surface states. Weyl semimetals [2–6] are not topological in a strict sense, as they do not possess a non-zero Chern number, yet some of their properties have topological origins. This happens due to the special points in a Brillouin zone, known as Weyl nodes, that can be attributed to a local Chern number. Their presence gives rise to a number of interesting effects associated, for example, with a chiral anomaly [7, 8]. One of the striking features of the Weyl semimetals is an emergence of the Fermi arcs, two-dimensional surface states that are robust against the disorder and a surface preparation. The position of the Fermi arcs in the k-space is unambiguously determined by the position of the Weyl nodes by a projection of these point onto a surface [9]. Such states were numerically predicted in different classes of realistic materials [9–16] and were experimentally observed via angle-resolved photoemission spectroscopy [3, 17–20].

In three dimensional materials the Weyl nodes are stable [21] and can not be removed by any (sufficiently weak) perturbation. The situation in two dimensions is more fragile as robust Weyl semimetals in two dimension do not exist. The closest analog was recently proposed in [1], who considered two-dimensional lattice with non-symmorphic symmetries, that combine half-translations with mirror reflections or rotations. By breaking the time reversal (TR) and inversion symmetries one splits the Dirac points into Weyl nodes. The stability of Weyl nodes (against opening a gap) is protected by a non-symmorphic lattice symmetries [22]. The spectrum of an emergent two dimensional Weyl semimetal is somewhat similar to graphene. It exhibits an even number of band touching points with Dirac type singularity, similar to K and K′ points in graphene. Near these points the Berry curvature has $1/k^2$ singularity and they are associated with a local Chern number $±1$.

The presence of the Dirac cones in graphene leads the formation of edge states [23–25]. For finite graphene ribbons, an edge mode emerges between the projections of the K and K′ points on the direction of the boundary. This yields the longest (in a k-space) edge in a zig-zag type of boundary, that smoothly shrinks as the direction of the boundary changes, while vanishing completely for an armchair boundary, where the points K and K′ are projected on top of each other. In this case the edge state has a zero length, i.e. disappears. For a semi-infinite graphene ribbon in the presence of the chiral symmetry, the edge modes are flat. The breaking of this symmetry turns graphene into an insulator (trivial or topological) [26] and induces a dispersion of the edge state.

The stability of the Dirac points in graphene has topological origins. For spinless graphene the existence of Dirac points can be proven by computing the flux of the Berry curvature through the half of the Brillouin zone [27]. As a consequence, while the distortion of graphene layer shifts the positions of the Dirac points in the k-space [28], no gap opens and the spectrum near the degeneracy points has Dirac type singularity. Note, that for the unstrained pristine graphene the degeneracy points are located at the high symmetry points. The spectrum degeneracy at these points is determined by crystal symmetry and follows from the two-dimensional irreducible representation of the little group. For the distorted graphene, on the other hand, the position of the degeneracy points is generic and does not correspond to...
any high symmetry point. However, their existence is

guaranteed by the topological arguments, that allow adi-
abatically connect the "distorted" graphene Hamiltonian

with Dirac point "somewhere" in the Brillouin zone to

the one of ideal graphene, with the edge state following

this evolution. Therefore there is a connection between

the edge state in unstrained graphene and its topological

properties. This connection is well established for a sys-

tem with chiral symmetries\cite{29}. Our work suggests that

this proof can be further generalized to the situations

where this symmetry is weakly violated.

In this manuscript, we study the formation of the edge

states in 2D lattice models with non-symmorphic sym-

metries. We consider the model put forward in Ref. \cite{1}.

The spectrum has three Dirac nodes, protected by crystal

non-symmorphic symmetries, inversion and time reversal

(TR) symmetries. The reduction of some of these sym-

metries splits the Dirac nodes into the Weyl nodes \cite{1}.

We show here that this is inevitably accompanied by the

formation of edge states. However, the nature of these

edge states, such as their spin polarization, depends on

the details of the applied deformation.

The mere presence of Dirac type singularities in the

spectrum leads to the emergence of the edge states. Such

systems fall into a category of distorted graphene Hamil-

tonians and consequently support edge states. Of course,

this approximation overlooks the non-singular parts of

the spectrum, that may lead to the additional surface

states\cite{30}. Moreover, if this Dirac type and the regu-

lar states inhabit the same part of the phase space, the

hybridization of this mode near the boundary can oc-

cur with an unknown outcome \cite{31}. We note that in

the model we consider, there are no regular parts of the

spectrum in the vicinity of the Weyl nodes.

The paper is organized as follows. In section II we re-
derive the results of Ref.\cite{1} with some more details. In sec-

tion III we study a formation of the Weyl nodes protected

by non-symmorphic symmetries and corresponding edge

states in a finite ribbon. In section IV we construct per-

turbation that breaks non-symmorphic symmetries yet

"accidentally" leads to Weyl nodes. We conclude by dis-

cussing the properties of the edge states and their poten-

tial applications.

II. DIRAC SEMIMETAL IN 2D

We focus on the model proposed in \cite{1}. In order to

keep the presentation self-contained we briefly repeat

their findings and explain the relationship between the

symmetry of the system and the spectrum degeneracy.

In 2D, the simplest lattice with a non-symmorphic sym-

metry is shown in Fig\cite{1}. It is a square lattice with two

atoms in the unit cell. One of the atoms is shifted out of

the plane (along \(\hat{z}\) direction). It is accounted by the layer

group P4/\text{mmm}. Within the tight-binding approximation

the Hamiltonian is given by

\[ H_0 = 2t\tau_x \cos \frac{k_x}{2} \cos \frac{k_y}{2} + t^{SO} \tau_z [\sigma_y \sin k_x \sigma_x \sin k_y]. \]

Here \(\tau\) and \(\sigma\) are Pauli matrices in the sublattice and

spin space, \(t\) is an amplitude of nearest neighbor hopping

and \(t^{SO}\) is an amplitude of the next nearest neighbor

spin-orbit interaction. The system posses an inversion

symmetry, represented in the momentum space by \(\mathcal{P} = \{\tau_x |k| \rightarrow -k\}\). It also preserves time-reversal symmetry

\(\mathcal{T} = \{i\sigma_y K |k| \rightarrow -k\}\). \(\mathcal{T}^2 = -1\). In addition the model

respects the non-symmorphic symmetries that combine a

translation by a half of the lattice constant with rota-

tion and mirror reflections. To account for these sym-

metries we define the operators \(\hat{t}_x\) and \(\hat{t}_y\) that describe

half-translations along the axes \(\hat{x}\) and \(\hat{y}\). They act on

the Bloch states as \(\hat{t}_x |\Psi_n\rangle = e^{i\frac{k_y}{2}x} |\Psi\rangle\), \(\hat{t}_y |\Psi_n\rangle = e^{i\frac{k_x}{2}y} |\Psi\rangle\), where \(n\) is the a band number. The rotation by \(\pi\) around

the \(x\) axes acts on the orbital part of the wave function as

\(\hat{S}_x (k_x, k_y) = (k_x, -k_y)\), i.e. \(k_y \rightarrow -k_y\) (and similarly for the rotation around \(y\)-axes). In terms of these operators

the three non-symmorphic symmetries can be written as follows:

1. Rotation around \(\hat{x}\) axis together with a half-

    translation along this axis

    \(g_1 = \{C_{2x} |\frac{1}{2} 0\} = i \sigma_y \tau_x \hat{S}_x \hat{t}_x\).

2. Rotation around \(\hat{y}\) axis together with a half-

    translation along this axis:

    \(g_2 = \{C_{2y} |0\frac{1}{2}\} = i \sigma_y \tau_x \hat{S}_y \hat{t}_y\).

3. Mirror reflection around \(z\) plane together with two

    half-translations:

    \(g_3 = \{M_z |\frac{1}{2} \frac{1}{2}\} = i \sigma_z \tau_x \hat{t}_x \hat{t}_y\).

Band crossing must happen at any \(g_i\)-invariant line.

Indeed, on this line the Hamiltonian and the correspond-

ing symmetry operator can be diagonalized simultane-

ously, so \(g_i |\Psi_n\rangle (k) = \pm e^{i\theta_i} |\Psi_n\rangle (k)\). Since \(e^{i\theta_1} = -1\), the two eigenstates switch places as one moves from the
point $k$ in the Brillouin zone to the equivalent point $k \rightarrow k + G$ along the $g_i$ invariant line. That means that the two eigenstates must switch places an odd number of times, generating an odd number of crossing points on this line in the Brillouin zone.

As long as the time-reversal and inversion symmetry are preserved, the spectrum is doubly degenerate. Since the point, $k = G/2$ remains invariant under the reversal of time it must be a crossings point. Moreover, the spectrum near this point must be symmetric and any of the crossing points of the spectrum is fourfold degenerate. Therefore in the presence of time reversal, inversion and any non-symmorphic symmetries Dirac semimetal is formed. The arguments so far were quite general and they hold for any lattice consistent with these symmetries. Now we focus on the lattice shown in Fig.

It has the following $g_i$ invariant lines in the Brillouin zone: $g_1$ invariant lines $k_y = 0, \pm \pi$, $g_2$ invariant lines $k_x = 0, \pm \pi$. The Dirac points $X_1 = \{\pi, 0\}$, $X_2 = \{0, \pi\}$ and $M = \{\pi, \pi\}$ lie on the intersection of those lines.

The Hamiltonian may be reduced to a block diagonal form by the unitary transformation

$$U = \begin{pmatrix} \sigma_0 & \sigma_x \\ \sigma_z & -i\sigma_y \end{pmatrix}$$

(2)

In this basis the Hamiltonian $H_0$ reads

$$\tilde{H} = U^{-1}HU = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}.$$ 

(3)

The diagonal elements

$$H_- = d \cdot \sigma^*, \quad H_+ = d \cdot \sigma .$$

(4)

and

$$d = \left\{ -t^{SO} \sin k_y, -t^{SO} \sin k_x, 2t \cos \frac{k_x}{2} \cos \frac{k_y}{2} \right\}.$$ 

(5)

Close to the Dirac points $M, X_1$ and $X_2$ the Hamiltonian [3] is mathematically equivalent to a graphene-like Hamiltonian [32]. Indeed, by expanding the Hamiltonian [3] near the Dirac points up to linear order in $k$ and performing an additional unitary transformation one finds

$$\tilde{H} = -t^{SO} \begin{pmatrix} \sigma \cdot k & 0 \\ 0 & \sigma^* \cdot k \end{pmatrix}.$$ 

(6)

The resulting Hamiltonian [6] is formally equivalent to the Hamiltonian of spinless graphene in the vicinity of $K$ and $K'$ points in graphene, with coinciding $K$ and $K'$ points. Using the continuous description, one can analyse the change of the spectrum in the presence of a generic perturbation:

$$H = \begin{pmatrix} (\sigma \cdot (k - a) + m_1(k)\sigma_z) & \tilde{m}(k) \\ \tilde{m}^*(k) & (\sigma^* \cdot (k + a) + m_2(k)\sigma_z) \end{pmatrix},$$

(7)

with $\tilde{m}(k) = m\sigma + m_0\sigma_0$.

We first consider the case when the terms that open a gap are absent, i.e. $m_1(k) = m_2(k) = 0$. Keeping $m \equiv (m_x, m_y, m_z) = (0, 0, m_z)$, both the vector $a$ and $m_z, m_0$ account for the splitting of the Dirac node at $k = 0$ into two Weyl nodes located at $k_{\pm} = \{\text{Re}[C], \pm \text{Im}[C]\}$, with $C = i\sqrt{(a_x + ia_y)^2 + m_z^2 - m_0^2}$. When these perturbations are absent, $a = 0, m_0 = 0, m_z = 0$, but $m_x \neq 0, m_y \neq 0$, nodal lines emerge. The position of these lines in $k$ space is determined by the equations $k = |m_x + im_y|, k = |m_y - m_x|$, which trace an ellipse in $k$ space. For $m_{1,2}(k) \neq 0$ a gap opens, unless $m_{1,2}(k) = 0$ vanishes either at $k_{\pm}$ or at some point on the nodal lines, where a Weyl node emerges. Due to the Nielsen-Ninomiya theorem, the number of emergent Weyl nodes must be even [33]. In terms of our model [7], this implies that the functions $m_1(k)$ and $m_2(k)$ are not independent, but vanish in a way that an even number of Weyl node arises and the total topological charge remains zero. Notice also, that in [7] we do not consider the perturbations proportional to $\sigma_2 \otimes I_2$; even though they lift the degeneracy of the Dirac point, the resulting nodes with the opposite topological charge remain in the same point in $k$-space.

In the presence of non-symmorphic symmetries the functions $m_1(k)$ and $m_2(k)$ vanish in at least one point along the corresponding high symmetry line[13]. Therefore the formation of Weyl nodes, in this case, is guaranteed by the symmetry. We will refer to such points symmetry protected Weyl nodes. We name Weyl nodes accidental if their emergence cannot be established solely on the basis of lattice symmetries.

We next consider the case when Weyl nodes emerge. Since they have the opposite Chern numbers, the total Chern number at the Dirac point is zero. Indeed, the Berry flux through a cross-section $S$ in the $k$-space is given by an integral of the Berry curvature

$$\phi_n = \int_S B_n(k) \cdot dS,$$

(8)

where $n$–band number. Berry curvature is defined as the curl of the Berry connection $A_n(k)$:

$$B_n(k) = \nabla_k \times A_n(k),$$

(9)

where the Berry connection of the $n$th band is given by

$$A_n(k) = i\langle \Psi_n | \nabla_k | \Psi_n \rangle.$$ 

(10)

In two dimensions Berry curvature has only one component

$$B_n(k) = -2\text{Im} \langle \partial_k \Psi_n | \partial_k \Psi_n \rangle.$$

(11)

The Berry connection for Eq. (11) is shown in the Fig. 2. It demonstrates, that the field $A_n(k)$ has the opposite sign for different $n$ ($A_1(k) = -A_2(k)$). Therefore the total Berry curvature and phase vanish, $B(k) = \sum_n B_n(k) = 0$, $\phi = \sum_n \phi_n = 0$, everywhere in the Brillouin zone.

We next discuss the lowering of degeneracy at the Dirac node which splits it into two Weyl nodes. As is well
known [21], the existence of the Weyl nodes requires violation of time reversal or inversion symmetry (or both). However, unlike in the three-dimensional case, where a Weyl node is stable against any (sufficiently weak) perturbation, the degeneracy in $d = 2$ can be completely lifted by a generic perturbation. To guarantee that the spectrum remains gapless one has to keep at least one of non-symmorphic symmetries intact. We now discuss several representative examples of such perturbations. One of them is consistent with the discussed symmetry-related arguments, and another one results in the accidental Weyl points. In the latter case, the emergence of the Weyl nodes is not guaranteed by the symmetry arguments and happens accidentally. In both cases, we show that the emergence of Weyl nodes is accompanied by the edge states.

### III. SYMMETRY PROTECTED WEYL NODES

We now consider a perturbation that breaks time-reversal symmetry. It arises if one of the atoms in the elementary cell is displaced in the $\hat{y}$ direction and additionally has a magnetic dipole moment aligned in $\hat{x}$ direction, thus resulting in the spin-dependent nearest-neighbor hopping amplitude. It breaks time-reversal symmetry, but not any of non-symmorphic:

$$V_1 = v_1 \tau_y \sigma_x \sin \frac{k_y}{2} \cos \frac{k_x}{2}. \quad (12)$$

We mention in passing that the spin-dependent hopping in this case happens even in the absence of the lattice deformation, but the resulting term of the Hamiltonian

$$V_{\text{hop}} = t_2 \tau_y \sigma_x \cos \frac{k_y}{2} \cos \frac{k_x}{2} \quad (13)$$

does not affect the splitting of Weyl nodes. This is because it vanishes along $g_1$ invariant lines ($k_x = \pm \pi, k_y = \pm \pi$), where the band crossing happens.

Due to the perturbation (12) the Dirac node $X_2$ splits, as shown in Fig. 3. The resulting Weyl nodes are located at the line $k_y = \pi$, which is protected by $g_1 = \{C_{2z}, \frac{1}{2} \bar{0}\}$. The two Weyl points $X_{2+}$ and $X_{2-}$ have the coordinates:

$$X_{2\pm} = (\pm 2 \arcsin \left[ \frac{v_1}{2 t^{\text{SO}}} \right], \pi) \quad (14)$$

In the bulk spectrum on the $g_1$ invariant line is continuous and is given by

$$\epsilon_{1,2} = \pm t^{\text{SO}} \left[ \frac{v_1}{t^{\text{SO}}} \cos \frac{k_x}{2} - \sin k_x \right]$$

$$\epsilon_{3,4} = \pm t^{\text{SO}} \left[ \frac{v_1}{t^{\text{SO}}} \cos \frac{k_x}{2} + \sin k_x \right]. \quad (15)$$

We now focus on the edge states. Following the discussion above, we expect the edge state to form between the projection of Weyl nodes on momentum axes parallel to the boundary. For a small perturbation $v_1/t^{\text{SO}} \ll 1$ the distance between Weyl points (and the length of the edge state in the $k$-space) scales linearly with the perturbation:

$$\Delta k_x = \frac{2 v_1}{t^{\text{SO}}} \quad (16)$$

Fig. 3 shows the spectrum of the model given by $H_0 + V_1 + V_{\text{hop}}$ for a ribbon infinite in $\hat{x}$ direction and with a finite width $L_y$ in $\hat{y}$ direction. The results show the appearance of a flat band edge state confined to the boundaries in the real space and connecting the projections of $X_{2+}$ and $X_{2-}$ points in $k$-space. We now proceed to a case where non-symmetric symmetries are broken and the existence of Weyl nodes is not determined by symmetry.
The Weyl points $X_{2\pm}$ on the line $k_x = 0$ for $t_{SO}/t < 1$ and $B_x/t_{SO} < 1$ are located at:

$$X_{2+} = \left(0, -\pi + \frac{B_x}{\sqrt{t^2 + (t_{SO})^2}}\right),$$

$$X_{2-} = \left(0, \pi - \frac{B_x}{\sqrt{t^2 + (t_{SO})^2}}\right).$$

For $t > t_{SO}$ the split between $X_{2\pm}$ is parametrically smaller than the distance between the Weyl points $X_{1-}$ and $X_{1+}$ or $M_+$ and $M_-:

$$\Delta k_y = \frac{2B_x}{\sqrt{t^2 + (t_{SO})^2}}.$$  

(22)

We now turn to a finite geometry and compute the spectrum of a two-dimensional ribbon, which is infinite in the $y$ direction and has a finite width $L_y$ in the $x$ direction. Note, that in this setup the projections of the Weyl points located around the $M$ and $X_{1,2}$ points, which are associated with an opposite Chern number, do not overlap. The spectrum of the ribbon with $L_y = 60$ is shown in the Fig. 6. The edge states connecting pairs of Weyl points with opposite topological charge are distinctly present. The confinement in the real space depends on a direction of the spin, so that the right edge corresponds to electrons that are mostly in the up and the left edge to the electrons that are mostly in the down spin state (see the Fig. 6(b)).

This effect of spin polarization can be understood at the analytical level. To demonstrate that we study the magnetization density of the edge states for the model in magnetic field in the vicinity of the points $X_{1,\pm}$ on the half-plane (see Appendix B), which is defined as follows:

$$\langle M_z \rangle = \langle \Psi(x, k_y) | \sigma_z | \Psi(x, k_y) \rangle = \rho\uparrow(x, k_y) - \rho\downarrow(x, k_y).$$

(23)

Whereas the magnetization of the edge state with the momentum $k_y$ is:

$$M(k_y) = \int_0^\infty dx \left[ \rho\uparrow(x, k_y) - \rho\downarrow(x, k_y) \right]$$

(24)

We see that the magnetization changes its sign when one moves from one Weyl point to another. So that in the vicinity of the Weyl point $X_{1,+} \simeq (\pi, -\frac{B_x}{t_{SO}})$ the state is almost polarized, $\rho\uparrow(x, k_y) \to 1, \rho\downarrow(x, k_y) \to 0$, and in the vicinity of $X_{1,-} \simeq (\pi, \frac{B_x}{t_{SO}})$ the polarization changes the sign. Distribution of the local magnetization as a function of coordinate is shown on the Fig. 8. It demonstrates different behavior in the vicinity of the different Weyl points and at $k = 0$. Analytical formulas that describe this behavior are derived in the Appendix B (see Eqs. (31)-(36) there).

V. SUMMARY AND DISCUSSION

We studied the formation of edge states in two-dimensional ribbons with non-symorphic symmetries.
By lowering the symmetries one may either open a gap or split the Dirac point into two Weyl nodes. If at least one of non-symmorphic symmetries is preserved, the Weyl nodes appear on the symmetry invariant line in the $k$ space. If non-symmorphic symmetries are broken the outcome is undetermined. In a generic case, the spectrum is gapped however, for some perturbation accidental Weyl nodes may arise.

The type and the strength of the perturbation control the type of emergent Weyl nodes and the composition of the wave function near these points (in terms of sublattices, spins, etc.). It also determines the properties of the edge states emerging at the boundaries of the ribbon.

In this work, we focused on two simple cases. We applied a stress that displaced atoms in one direction, assuming that the atoms have magnetic dipoles aligned in the perpendicular direction. Since this perturbation breaks inversion symmetry but preserves one of the non-symmorphic symmetries, the Weyl nodes are positioned on the corresponding invariant line. Consequently, flat bands form in $k$-space between the points that are projections of the nodes on the boundary.

An accidental type of the Weyl nodes can be obtained by applying in-plane magnetic field. The position of edge states is once again consistent with the projection argument. The spin structure of the Weyl nodes results in spin-polarized edge states, where the direction of polarization of the edge states is controlled by the sign of the magnetic field. The spatial distribution of the polarization depends on the strength of the magnetic field and on the value of momentum along the edge.

While the calculations, that were performed, are specific to this model, some of the emergent features are expected to be general. In particular, for a generic model where Weyl nodes are separated from the bulk by a soft gap (i.e. there are no bulk excitations at these energies) excitations near the Weyl node will give rise to edge states. Therefore the problem in two dimensions is equivalent to a ”deformed graphene” model, e.g. Eq. (7), discussed above and the properties of the edge states can be determined within an approximate Dirac type Hamiltonian.

In the opposite case, where the excitation near the
Weyl node coexists with the regular bulk excitations the sample boundary may lead to the hybridization of those modes. This may (or may not) lead to the destruction of the edge states, but in any case, will change their properties. The precise mathematical criteria for this transition is, to the best of our knowledge, yet to be constructed.

While the computation of edge states and their composition for realistic materials requires serious numerical analysis, the possibility to control the Weyl nodes by applying various symmetry breaking deformation to the Dirac semimetal constitutes a promising research route. Choosing the type of deformation allows one to control the position for realistic materials requires serious numerical analysis, the possibility to control the Weyl nodes by applying various symmetry breaking deformation to the Dirac semimetal constitutes a promising research route. Choosing the type of deformation allows one to control the position for realistic materials requires serious numerical analysis, the possibility to control the Weyl nodes by applying various symmetry breaking deformation to the Dirac semimetal constitutes a promising research route.

Appendix A \textbf{SPECTRUM ON THE $g_1$ INVARIANT LINE FOR THE MODEL IN MAGNETIC FIELD}

In the presence of external in-plane magnetic field (17), the bulk spectrum of the system on $g_1$ invariant lines $k_x = \pm \pi$ is given by

\begin{equation}
\begin{align*}
\epsilon_{1,2}(k_x = \pm \pi) &= \pm t^{SO} \left[ B_x \frac{t^{SO}}{t} \sin k_y \right] \\
\epsilon_{3,4}(k_x = \pm \pi) &= \pm t^{SO} \left[ \frac{B_x}{t^{SO}} - \sin k_y \right],
\end{align*}
\end{equation}

and on the line $k_x = 0$ by:

\begin{equation}
\begin{align*}
\epsilon_{1,2}(k_x = 0) &= \pm t \left[ \frac{B_x}{t} + \sqrt{2} \cos k_y \frac{t^{SO}}{t} \sqrt{2 + (\frac{t^{SO}}{t})^2 (1 - \cos k_y)} \right] \\
\epsilon_{3,4}(k_x = 0) &= \pm t \left[ \frac{B_x}{t} - \sqrt{2} \cos k_y \frac{t^{SO}}{t} \sqrt{2 + (\frac{t^{SO}}{t})^2 (1 - \cos k_y)} \right]
\end{align*}
\end{equation}

Analyzing Eqs. (25), (26), one finds the position of the Weyl nodes $X_{2,\pm}$ and $M_{\pm}$, see Eqs. (18), (21).

Appendix B \textbf{MAGNETIZATION OF THE EDGE STATES}

Here we derive analytically the polarization of the edge states for the model in magnetic field. Consider the model $H_0 + V$ in the vicinity of $(k_x, k_y) = (\pm \pi, 0)$. In the linear order in $k_x$, $k_y$ the Hamiltonian reads

\begin{equation}
H^{\text{lin}} = t^{SO} \tau_z (\sigma \times k)_z + B_x \sigma_x + t k_x \tau_x
\end{equation}

To account for a finite geometry we replace $k_x \rightarrow -i \frac{\partial}{\partial x}$. The spin and sublattice components can be combined into a spinor $\Psi(k_y, x) = (\Psi^A, \Psi^B)$, where we defined $\Psi^A = (\Psi^A_+, \Psi^A_-)$ and $\Psi^B = (\Psi^B_+, \Psi^B_-)$. In terms of the spinor the Schrödinger equation reads:

\begin{equation}
\begin{align*}
\left\{ t^{SO} (\sigma_z k_y + i \sigma_y \partial_x) + B_x \sigma_x \right\} \Psi^A - it \partial_x \Psi^B &= c \Psi^A \\
\left\{ t^{SO} (-\sigma_z k_y - i \sigma_y \partial_x) + B_x \sigma_x \right\} \Psi^B - it \partial_x \Psi^A &= c \Psi^B,
\end{align*}
\end{equation}

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Where for the brevity of notations we denoted \( \Psi^A/B = \Psi^A/B (x, k_y) \).

In order to demonstrate the effect of polarization of the edge states, consider the problem \([28]\) on the semi-infinite plane \( x > 0 \) (or very wide ribbon \( L_x \gg 1 \)) with the boundary conditions \( \Psi^B(k_y, x = 0) = 0 \). Solving this system of differential equations, we get the following zero-energy solution:

\[
\begin{align*}
\Psi^A_1 &= 0 \\
\Psi^B_1 &= C_1 e^{-k_1 x} + C_2 e^{-k_2 x} \\
\Psi^A_2 &= C_3 (e^{-k_1 x} - e^{-k_2 x}) \\
\Psi^B_2 &= 0,
\end{align*}
\]

where \( C_1, C_2, C_3 \) are functions of the parameters \( t, t^\text{SO}, B_x \) and \( k_y \). The penetration depths of magnetization are

\[
\begin{align*}
\frac{1}{\lambda_1} &= k_1 = \frac{B_x t^\text{SO} + \sqrt{[t^2 + (t^\text{SO})^2] (t^\text{SO})^2 k_y^2 - t^2 B_x^2}}{t^2 + (t^\text{SO})^2}, \\
\frac{1}{\lambda_2} &= k_2 = \frac{B_x t^\text{SO} - \sqrt{[t^2 + (t^\text{SO})^2] (t^\text{SO})^2 k_y^2 - t^2 B_x^2}}{t^2 + (t^\text{SO})^2}.
\end{align*}
\]

Notice that for \( B_x > 0 \) the exponentially decaying solution (i.e., \( k_1, k_2 > 0 \)) exists, provided \( |k_y| < B_x/t^\text{SO} \). This means that the momentum \( k_y \) vary between the two Weyl nodes \( X_{1,\pm} = (\pi, \pm \frac{B_x}{t^\text{SO}}) \). This result was obtained within the linearized model but is expected to hold in general. Now we calculate the local and total magnetization of the model, defined in \([23], [24]\). Around the Weyl point \( X_{1,+} \approx (\pi, -\frac{B_x}{t^\text{SO}}) \) the total magnetization is described by:

\[
M(k_y \approx X_{1,+}) = 1 - \frac{t^2}{8B_x^2} (\delta k_y)^2,
\]

where \( \delta k_y = X_{1,+} - k_y \). Around the Weyl point \( X_{1,-} \) the magnetization changes the sign,

\[
M(k_y \approx X_{1,-}) = -1 + \frac{4t^\text{SO}(t^2 + (t^\text{SO})^2)}{B_x t^2} \delta k_y,
\]

where \( \delta k_y = X_{1,-} - k_y \). The local magnetization also shows different behavior depending on the momentum \( k_y \). Close to the point \( X_{1,+} \) one has:

\[
M_{X_{1,+}}(x) = 2\delta k_y + \left( \frac{t^2 (1 - 2e^{-\frac{x}{\lambda_M}})}{B_x t^\text{SO}} - 4x \right) \delta k_y^2
\]

where \( \lambda_M \) is defined as:

\[
\lambda_M = \frac{t^2 + (t^\text{SO})^2}{2B_x t^\text{SO}},
\]

and the magnetization around the point \( X_{1,-} \) is:

\[
M_{X_{1,-}}(x) = e^{-\frac{x}{\lambda_M}} \left[ 4 \sinh^2 \frac{x}{2\lambda_M} - \frac{8(t^\text{SO})^2}{t^2} e^{-\frac{x}{\lambda_M}} \right] \delta k_y
\]

However, around the point \( k_y = 0 \) the local magnetization oscillates.

\[
M_{k_y=0}(x) = \frac{2B_x t^\text{SO} e^{-\frac{x}{\lambda_M}}}{5t^2 + 8(t^\text{SO})^2} \left( 3 + 5 \cos \frac{x}{\lambda_M} \right)
\]

Our results we derived assuming that \( B_x > 0 \). The magnetization for the negative field acquires an opposite sign. This follows from \([28]\), since the states \( \Psi^A_\uparrow \rightarrow \Psi^A_\downarrow \) and \( \Psi^B_\uparrow \rightarrow \Psi^B_\downarrow \) under reversal of a magnetic field \( (B_x \rightarrow -B_x) \). Also notice that one can perform the similar analysis of the magnetization around the other Weyl points, \( X_{2,\pm} \), and the qualitatively similar results for that case can be obtained by the replacement \( t \rightarrow \frac{t}{2} \).

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