Sets of Uniqueness, Weakly Sufficient Sets and Sampling Sets for Weighted Spaces of Holomorphic Functions in the Unit Ball

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Received: 28 October 2020 / Accepted: 17 July 2021 / Published online: 29 November 2021
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Abstract
We consider, in a quite general setting, inductive limits of weighted spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$. The relationship between sets of uniqueness, weakly sufficient sets and sampling sets in these spaces is studied. In particular, the equivalence of these sets, under some conditions of the weights, is obtained.

Keywords  Weighted spaces · Holomorphic functions · Unit ball · Sets of uniqueness · Weakly sufficient sets · Sampling sets

Mathematics Subject Classification  32A38 · 46A13

Communicated by H. Begehr.

This article is part of the topical collection “Higher Dimensional Geometric Function Theory and Hypercomplex Analysis” edited by Irene Sabadini, Michael Shapiro and Daniele Struppa.

B. Hu: Supported in part NSF grant DMS 1600458 and NSF Grant 1500162.
L. H. Khoi: Supported in part by MOE’s AcRF Tier 1 Grant M4011724.110 (RG128/16).

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1 Introduction

Let $B$ be the unit ball in $\mathbb{C}^n$ ($n \geq 1$) and $O(B)$ be the collection of holomorphic functions on $B$ with the usual compact-open topology. A function $\varphi$ defined on $[0, 1)$ is said to be a weight if it takes the values in $(1, +\infty)$. We define the associated weighted space $A_\varphi$ as

$$A_\varphi := \left\{ f \in O(B) : \| f \|_\varphi := \sup_{z \in B} \frac{|f(z)|}{\varphi(|z|)} < +\infty \right\}.$$ 

In this paper, we are interested in the case where the function space is defined by the inner inductive limit of $A_\varphi$’s. More precisely, let $\Phi = (\varphi_p)_{p=1}^{\infty}$ denote an increasing sequence of weights. For simplicity, we use $\| f \|_p$ instead of $\| f \|_{\varphi_p}$, and $A^{-p}_\varphi$ instead of $A_{\varphi_p}$. Then clearly $A^{-p}_\varphi \hookrightarrow A_{\varphi}^{-(p+1)}$ (here $\hookrightarrow$ means a continuous embedding), and hence, we let

$$A_{\Phi}^{-\infty} := \bigcup_{p \geq 1} A^{-p}_\varphi$$

with the topology induced by the inner inductive limit of $A^{-p}_\varphi$, namely

$$(A_{\Phi}^{-\infty}, \tau) = \lim \text{ind} A^{-p}_\varphi.$$ 

Now let $S$ be a subset set of $B$. Define

$$A_{\varphi}^{-p, S} := \left\{ f \in A_{\varphi}^{-\infty} : \| f \|_{p, S} = \sup_{z \in S} \frac{|f(z)|}{\varphi_p(|z|)} < \infty \right\}.$$ 

Notice that the inclusion relations $A_{\varphi}^{-p} \hookrightarrow A_{\varphi}^{-p, S} \hookrightarrow A_{\Phi}^{-\infty}$ always hold. Hence, it follows immediately that

$$A_{\Phi}^{-\infty} = \bigcup_{p \geq 1} A^{-p}_\varphi \subseteq \bigcup_{p \geq 1} A_{\varphi}^{-p, S} \subseteq A_{\Phi}^{-\infty},$$

which implies

$$A_{\Phi}^{-\infty} = \bigcup_{p \geq 1} A^{-p}_\varphi = \bigcup_{p \geq 1} A_{\varphi}^{-p, S}.$$ 

Therefore, we can endow $A_{\Phi}^{-\infty}$ with another weaker inner inductive limit topology of seminormed spaces $A_{\varphi}^{-p, S}$:

$$(A_{\Phi}^{-\infty}, \tau_S) := \lim \text{ind} A_{\varphi}^{-p, S}.$$
We note that this type of spaces appeared as duals to Fréchet-Schwartz (FS) spaces and play an important role in the study of representing holomorphic functions in series of simpler functions, such as exponential functions, or rational functions, which have many applications in functional equations and approximations of functions (see, e.g., [3–6,15,16,19] and references therein).

1.1 Three Special Cases of $S$

We are interested in three important cases of a subset $S \subset \mathbb{B}$.

Firstly, in function spaces there is a quite well-known notion of a set of uniqueness. For the weighted space $A_{\Phi}^{-\infty}$ the sets of uniqueness are precisely those sets which are not subsets of zero-sets for $A_{\Phi}^{-\infty}$ and these sets can be defined as follows.

**Definition 1.1** A set $S \subseteq \mathbb{B}$ is called a set of uniqueness for $A_{\Phi}^{-\infty}$ if $f \in A_{\Phi}^{-\infty}$ and $f(z) = 0$ for all $z \in S$ imply that $f = 0$.

Secondly, for the weighted space $A_{\Phi}^{-\infty}$, the topology $\tau_S$ introduced above is in general much weaker than the global topology $\tau$. However, there are cases when $\tau_S$ is the same as $\tau$. For these cases, a set $S$ can have a special name.

**Definition 1.2** A set $S \subseteq \mathbb{B}$ is said to be weakly sufficient for the space $A_{\Phi}^{-\infty}$ if two topologies $\tau$ and $\tau_S$ are equivalent.

Below we can see that the uniqueness property is a necessary for a set to be weakly sufficient in $A_{\Phi}^{-\infty}$.

Thirdly, let $\psi : [0, 1) \rightarrow (0, \infty)$ be a continuous function satisfying $\lim_{r \rightarrow 1} \psi(r) = \infty$. For $f \in A_{\Phi}^{-\infty}$, we put

$$T_{f, \psi} = \limsup_{|z| \rightarrow 1} \frac{\log |f(z)|}{\psi(|z|)},$$

and for $S \subset \mathbb{B}$,

$$T_{f, \psi, S} = \limsup_{|z| \rightarrow 1, z \in S} \frac{\log |f(z)|}{\psi(|z|)}.$$

In particular, if the closure of $S$ is compact in $\mathbb{B}$, then $T_{f, \psi, S} = 0$. Heuristically, these quantities can be regarded as “types” of a function $f$ with respect to $\psi$.

It is clear from the definition that $T_{f, \psi, S} \leq T_{f, \psi}$ for any $S \subset \mathbb{B}$ and any $f \in A^{-\infty}(\mathbb{B})$. One can ask for which subsets $S$ of $\mathbb{B}$ the reverse inequality may hold. In fact the following notion. It is clear that the inequality always holds. It can happen that for specific choice of $S$, both quantities are equal for all functions of the weighted space $A_{\Phi}^{-\infty}$. This choice of $S$ dominates other cases and hence we can give $S$ a name.

**Definition 1.3** A set $S \subseteq \mathbb{B}$ is called a $\psi$-sampling set for $A_{\Phi}^{-\infty}$, if $T_{f, \psi} = T_{f, \psi, S}$ for every $f \in A_{\Phi}^{-\infty}$.
It should be noted that for the classical weights
\[ \Phi = \{ \varphi_p(r) = (1 - r)^p, \ p \in \mathbb{N} \}, \]
and \( \psi(r) = |\log(1 - r)| \) we are led to the well-known Koreblum spaces \( A^{-\infty} = \bigcup_{p \geq 1} A^{-p} \) of all the holomorphic functions on \( B \) with polynomial growth.

We notice also that in the one-dimensional case the notion of sampling sets in the space \( A^{-\infty}(D) \), \( D \) being the open unit disc in \( \mathbb{C} \), was has been introduced and studied intensively for some time, by many people (see, e.g., [8], and references therein, also [7,14] ...). It seems that this notion is the same as for effective sets introduced by Iyer in [10,11].

### 1.2 Aim of the Paper

The property of sets of uniqueness, weakly sufficient sets and sampling sets for the space \( A^{-\infty}(D) \) was carefully studied in the 1998 paper [14]. This study was developed for higher dimensions (see [2,12], respectively). More precisely, the following results have been obtained in these papers.

**Theorem 1.4** Let \( S \) be a subset of the unit disc \( B \) (respectively, \( D \)). Consider the following assertions

(i) \( S \) is classical sampling for \( A^{-\infty}(B) \) (respectively, \( D \));
(ii) \( S \) is weakly sufficient for \( A^{-\infty}(B) \) (respectively, \( D \));
(iii) \( S \) is a set of uniqueness for \( A^{-\infty}(B) \) (respectively, \( D \)).

Then (i) \( \iff \) (ii) \( \iff \) (iii).

Note that the inverse implications, in general, are not true. In [14], two counterexamples are provided to show a failure of both inverse implications. Nevertheless, in [12] it was showed, for the classical \( \psi \), that (iii) \( \implies \) (ii), with an additional assumption.

A careful study of proofs of the results above led us to the thought that (ii) does not imply (i) due to “too independent” growths of \( \Phi \) and \( \psi \), while (iii) does not imply (ii) because the assumption of \( S \) to be a set of uniqueness is not strong enough to ensure an inclusion to become a continuous embedding.

So a question to ask is for what \( \Phi \) and \( \psi \), we can have

\[ (i) \iff (ii) \iff (iii) ? \]

In the recent paper [1], the affirmative answer was given for the case \( \varphi_p(r) = e^{pg(r)}, 0 < p < \infty \) and \( \psi(r) = g(r) \), where \( g \) is a so-called everywhere quasi-canonical weight (see, [1, Section 4]).

In this paper, we solve the question above for a general \( \Phi \) and \( \psi \), with the “minimal” assumptions imposed on them.

The structure of this paper is as follows. In Sect. 2, we show that (ii) \( \implies \) (iii) always. Then we introduce conditions \((C_1) - (C_2)\) for \( \Phi \), which are satisfied by the
classical weights and show that with the same additional assumption as for the classical case, \((iii) \implies (ii)\) (Theorem 2.5). The proof follows the scheme in [12]. Section 3 deals with the equivalence \((ii) \iff (i)\). We introduce conditions \((C_3) - (C_5)\) which “relate” growths of \(\Phi\) and \(\psi\)” and, together with \((C_1), (C_2)\), allow us to establish each of implications \((i) \implies (ii)\) and \((ii) \implies (i)\).

## 2 Weakly Sufficient Sets and Sets of Uniqueness

The following characterization of weakly sufficient sets is needed in the sequel.

**Lemma 2.1** [13, Proposition 2.2] For the space \(A_{\Phi}^{-\infty}\), the following statements are equivalent:

(a) \(S\) is weakly sufficient set.

(b) For any \(p \geq 1\), there exists \(m = m(p)\), such that

\[
A_{\psi}^{-p,S} \hookrightarrow A_{\psi}^{-m},
\]

i.e., for any \(p \geq 1\), there exist \(m = m(p)\) and \(C = C(p)\), such that

\[
\|f\|_m \leq C\|f\|_{p,S}, \text{ for all } f \in A_{\psi}^{-p,S}.
\]

(c) For any \(p \geq 1\), there exist \(m = m(p)\) and \(C = C(p)\), such that

\[
\|f\|_m \leq C\|f\|_{p,S}, \text{ for all } f \in A_{\phi}^{-\infty}.
\]

The following proposition is an easy modification of [12, Proposition 3.1], with replacing the factor \((1 - |z|)^p\) by \(\varphi_p(|z|)^{-1}\), and hence we omit the proof here.

**Proposition 2.2** Any weakly sufficient set for \(A_{\phi}^{-\infty}\) is a set of uniqueness for this space.

**Remark 2.3** We give an example of a set of uniqueness which is not a weakly sufficient set. Let \(S = \{z \in \mathbb{B} : |z| \leq 1/2\}, \varphi_p(r) = (1 - r)^{-p}, p \in \mathbb{N}\), and the test functions

\[
f_k(z) = 3^{-k}(1 - z_1)^{-k}, k \in \mathbb{N},
\]

where \(z = (z_1, \ldots, z_n) \in \mathbb{B}\). Clearly, \(S\) is a set of uniqueness. Moreover, for a fixed \(p_0 > 0\), a direct calculation shows \(\|f_k\|_{p_0,S}\) is uniformly bounded in \(k\), but for any \(p > 0\), \(\|f_k\|_p = \infty\) for \(k\) sufficiently large, which implies \(S\) fails to be a weakly sufficient set.

Now we proceed the other direction, and we are inspired by the idea from [12]. For a given \(\Phi = (\varphi_p)_{p=1}^{\infty}\), we consider the following conditions

\[
\lim_{r \to 1} \frac{\varphi_{p+1}(r)}{\varphi_p(r)} = \infty, \text{ for all } p \geq 1, \quad (C_1)
\]
for all \( p \geq 1 \), \( \frac{\varphi_{p+1}}{\varphi_p} \) is bounded on any compact subset of \([0, 1]\). \( \text{(C}_2\text{)} \)

Clearly, if all \( \varphi_p \) are continuous on \([0, 1]\), then \( \text{(C}_2\text{)} \) is satisfied.

**Example 2.4** We give some examples to show that these conditions are independent each of other.

1. The classical weights \( \Phi = ((1 - r)^{-p})_{p=1}^\infty \) satisfy both \( \text{(C}_1\text{)} \) and \( \text{(C}_2\text{)} \).
2. For any \( a > 2 \), the weights \( \Phi = (a + r - \frac{1}{p})_{p=1}^\infty \) satisfy \( \text{(C}_2\text{)} \) but not \( \text{(C}_1\text{)} \).
3. For each \( p \in \mathbb{N} \), let \( \varphi_p(r) = \begin{cases} \left(\frac{1}{2} - r\right)^{-p} & r \in \left[0, \frac{1}{2}\right), \\ (1 - r)^{-p} & r \in \left[\frac{1}{2}, 1\right). \end{cases} \)

Then the weights \( \Phi = (\varphi_p)_{p=1}^\infty \) satisfy \( \text{(C}_1\text{)} \) but not \( \text{(C}_2\text{)} \).
4. For \( p \in \mathbb{N} \), consider \( \varphi_p(r) = \begin{cases} \left(\frac{1}{2} - r\right)^{-p} & r \in \left[0, \frac{1}{2}\right), \\ 1 & r \in \left[\frac{1}{2}, 1\right). \end{cases} \)

Then the weights \( \Phi = (\varphi_p)_{p=1}^\infty \) do not satisfy neither \( \text{(C}_1\text{)} \) nor \( \text{(C}_2\text{)} \).

The following is the main result in this section, which is a natural extension of [12, Theorem 3.4].

**Theorem 2.5** Suppose \( \Phi \) is any sequence of weights satisfying conditions \( \text{(C}_1\text{)} \) and \( \text{(C}_2\text{)} \). Then \( S \subset \mathbb{B} \) is a weakly sufficient set for \( A_{\Phi}^{-\infty} \) if and only if

(a) \( S \) is a set of uniqueness for \( A_{\Phi}^{-\infty} \).
(b) For any \( p \geq 1 \), there exists an \( m = m(p) \), such that \( A_{\varphi}^{-p, S} \subset A_{\varphi}^{-m} \).

This result shows that under conditions \( \text{(C}_1\text{)} - \text{(C}_2\text{)} \), a set-inclusion (statement (b)) implies a continuous embedding.

**Proof** In the proof, we make use of equivalent conditions for weakly sufficient sets stated in Lemma 2.1.

Since the necessity follows from Lemma 2.1 and Proposition 2.2, it suffices to show the sufficiency. By assumption (b), for any \( p \geq 1 \) given, there exists an \( m = m(p) \), such that \( A_{\varphi}^{-p, S} \subset A_{\varphi}^{-m} \). Without loss of generality, we may assume that \( m \geq p \).

Let us take and fix an integer \( q > m \). We prove that \( A_{\varphi}^{-p, S} \) is imbedded continuously into some space \( A_{\varphi}^{-q} \) with \( q > m \geq p \).

Denote by \( U_{p, S} \) the unit ball in \( A_{\varphi}^{-p, S} \) and by \( U_q \) the unit ball in \( A_{\varphi}^{-q} \). Since

\[
\sup_{f \in U_{p, S}} \| f \|_q \leq \sup_{f \in U_{p, S} \setminus U_q} \| f \|_q + \sup_{f \in U_q} \| f \|_q,
\]

and
what we need is to show that $U := U_{p,S} \setminus U_q$ is bounded in the space $A_{q}^{-q}$.

We consider the sequence of compact sets $\{ K_s := \{ z \in \mathbb{B} : |z| \leq \frac{s}{s+1} \} \}_{s \in \mathbb{N}}$ which converges to the unit ball $\mathbb{B}$ from inside. The following lemma, which extends [12, Lemma 3.5], plays a key role in the proof of the theorem.

**Lemma 2.6** Let $\ell > m$ be some integer. Then there exists some $s = s(\ell) \geq 1, s \in \mathbb{N}$, such that the $\| f \|_\ell$ is attained on the compact set $K_s := \{ z \in \mathbb{B} : |z| \leq \frac{s}{s+1} \}$ for all $f \in U$.

**Proof** We use the method of contradiction. Assume in contrary that the statement of the lemma is not true. This means that for any $s \in \mathbb{N}$, there exists a $f_s \in U$, such that

$$\sup_{K_s} |f_s(z)| \phi(\|z\|) < \| f_s \|_\ell. \quad (2.1)$$

Note that $A_{p}^{-p,S} \subset A_{q}^{-m} \subset A_{q}^{-(\ell-1)}$ (as $m \leq \ell - 1$). So the proof is done, if we can construct a function $b \in A_{p}^{-p,S}$ but $b \not\in A_{q}^{-(\ell-1)}$.

We construct $b$ in the form of a series

$$b(z) = \sum_{k=1}^{\infty} \varepsilon_k b_k(z),$$

where $b_k \in U := \left\{ \frac{f}{\|f\|_\ell} : f \in U \right\}$ will be defined inductively as follows.

Take an arbitrary $b_1 \in U$. If we have $b_1, b_2, \ldots, b_{t-1} \in U (t \geq 2)$, then $b_t$ is determined in the following way:

**Step I:** By condition $(C_1)$, we can choose $s_t \in \mathbb{N}$ large enough so that

$$\frac{\varphi(\|z\|)}{\varphi_{\ell-1}(\|z\|)} \geq 3 \cdot 4^{t+1} \sum_{k=1}^{t-1} \|b_k\|_{\ell-1}, \quad \text{for all } z \in \mathbb{B} \setminus K_{s_t}. \quad (2.2)$$

Note that here the quantity $\|b_k\|_{\ell-1}$ is well-defined. Indeed, for any $f \in U = U_{p,S} \setminus U_{\ell}$, we have $f \in A_{q}^{-p,S} \subset A_{q}^{-m} \subset A_{q}^{-(\ell-1)} \subset A_{q}^{-\ell}$, which implies the desired claim.

**Step II:** For the $K_{s_t}$ chosen above, there is an $f_t \in U$ such that $(2.1)$ holds, and we define $b_t(z) = \frac{f_t(z)}{\|f_t\|_\ell} \in U$. 

Thus from the above two steps, a sequence \( \{b_k\} \subset \mathcal{U} \) is defined. Taking \( \varepsilon_k = 4^{-k}, k = 1, 2, \ldots \), we have

\[
b(z) = \sum_{k=1}^{\infty} 4^{-k} u_k(z),
\]

which implies

\[
\|b\|_{\ell} \leq \sum_{k=1}^{\infty} \frac{\|b_k\|_\ell}{4^k} = \sum_{k=1}^{\infty} \frac{1}{4^k} < \infty.
\]

This shows that \( b \in A_{\varphi^\ell}^{-\ell} \).

**Claim I:** \( b \in A_{\varphi^{-p}}^{-p} \).

Note that for any \( f \in U = U_{p,S} \setminus U_\ell \), we have \( \|f\|_{p,S} \leq 1 \) and \( \|f\|_\ell > 1 \). Hence, for any \( g \in \mathcal{U} \), it always holds that \( \|g\|_{p,S} < 1 \), which implies

\[
\|b\|_{p,S} \leq \sum_{k=1}^{\infty} \frac{1}{4^k} \|b_k\|_{p,S} \leq \sum_{k=1}^{\infty} \frac{1}{4^k} < \infty,
\]

which implies the first claim.

**Claim II:** \( b \notin A_{\varphi}^{-(\ell-1)} \).

Take and fix any \( t \in \mathbb{N} \). Then from the Step II above, there exists an \( z_t \in \mathbb{B} \setminus K_{s_t} \) such that

\[
\frac{|f_t(z_t)|}{\varphi_{\ell}(|z_t|)} \geq \frac{\|f_t\|_\ell}{2}. \tag{2.3}
\]

Thus

\[
|b(z_t)| = \left| \sum_{k=1}^{\infty} \frac{b_k(z_t)}{4^k} \right| \geq \left| \frac{b_t(z_t)}{4^t} \right| - \left| \sum_{k \neq t} \frac{b_k(z_t)}{4^k} \right|
\]

\[
\geq \left| \frac{b_t(z_t)}{4^t} \right| - \sum_{k \neq t} \left| \frac{b_k(z_t)}{4^k} \right|
\]

\[
= \left| \frac{b_t(z_t)}{4^t} \right| - \sum_{k=1}^{t-1} \left| \frac{b_k(z_t)}{4^k} \right| - \sum_{k=t+1}^{\infty} \left| \frac{b_k(z_t)}{4^k} \right| = J_0 - J_1 - J_2.
\]

**Estimate of** \( J_0 \). Applying (2.3), we have

\[
J_0 \geq \frac{1}{4^t} \cdot \varphi_{\ell}(|z_t|).
\]
**Estimate of $J_1$.** For each $k \in \mathbb{N}$, we have

$$|b_k(z_t)| = \frac{|b_k(z_t)|}{\varphi_{\ell-1}(|z_t|)} \cdot \varphi_{\ell-1}(|z_t|) \leq \|b_k\|_{\ell-1} \varphi_{\ell-1}(|z_t|),$$

and hence

$$J_1 \leq \varphi_{\ell-1}(|z_t|) \cdot \sum_{k=1}^{t-1} \frac{\|b_k\|_{\ell-1}}{4^k}.$$

**Estimate of $J_2$.** Similarly, for each $k \in \mathbb{N}$, we have

$$|b_k(z_t)| = \frac{|b_k(z_t)|}{\varphi_{\ell}(|z_t|)} \cdot \varphi_{\ell}(|z_t|) \leq \varphi_{\ell}(|z_t|),$$

which gives

$$J_2 \leq \varphi_{\ell}(|z_t|) \cdot \sum_{k=t+1}^{\infty} \frac{1}{4^k} = \frac{\varphi_{\ell}(|z_t|)}{3 \cdot 4^t}.$$

Combining the three estimates above yields

$$|u(z_t)| \geq \left(\frac{1}{2} \cdot \frac{1}{4^t} - \frac{1}{3} \cdot \frac{1}{4^t}\right) \cdot \varphi_{\ell}(|z_t|) - \varphi_{\ell-1}(|z_t|) \cdot \sum_{k=1}^{t-1} \frac{\|b_k\|_{\ell-1}}{4^k}
\geq \frac{\varphi_{\ell-1}(|z_t|)}{6 \cdot 4^t} \cdot \frac{\varphi_{\ell}(|z_t|)}{\varphi_{\ell-1}(|z_t|)} - \varphi_{\ell-1}(|z_t|) \cdot \sum_{k=1}^{t-1} \frac{\|b_k\|_{\ell-1}}{4^k}
\geq \frac{\varphi_{\ell-1}(|z_t|)}{6 \cdot 4^t} \cdot 3 \cdot 4^{t+1} \sum_{k=1}^{t-1} \|b_k\|_{\ell-1} - \varphi_{\ell-1}(|z_t|) \cdot \sum_{k=1}^{t-1} \frac{\|b_k\|_{\ell-1}}{4^k}
\geq \varphi_{\ell-1}(|z_t|) \cdot \sum_{k=1}^{t-1} \|b_k\|_{\ell-1}.$$

Noting the easy fact that $\|b_k\|_{\ell-1} \geq \|b_k\|_{\ell} = 1$, for all $k \geq 1$, we have for any $t \geq 2$,

$$\|b\|_{\ell-1} = \sup_{z \in \mathbb{B}} \frac{|b(z)|}{\varphi_{\ell-1}(|z|)} \geq \frac{|b(z_t)|}{\varphi_{\ell-1}(|z_t|)} \geq \sum_{k=1}^{t-1} \|b_k\|_{\ell-1} > t - 1,$$

which implies $\|b\|_{\ell-1} = \infty$ and therefore $b \notin A^{-\ell}(\varphi)$. Therefore, there exists some $s$, such that the $\|f\|_{\ell}$ is attained on $K_s$ uniformly for all $f \in U$. \hfill $\square$
Now return back to the proof of Theorem 2.5. By Lemma 2.6, we take and fix some $s \in \mathbb{N}$ so that $\|f\|_{q-1}$ is attained on $K_s$ for all $f \in U$ (note that $q > m+2$, and hence $q-1 > m$).

By condition (C$_2$), there exists $M = M(q, s) > 0$, such that

$$\frac{\varphi_q(|z|)}{\varphi_{q-1}(|z|)} \leq M, \quad z \in K_s$$

and hence for any $f \in U$,

$$\|f\|_{q-1} = \sup_{z \in \mathbb{B}} \frac{|f(z)|}{\varphi_{q-1}(|z|)} = \sup_{z \in K_s} \frac{|f(z)|}{\varphi_{q-1}(|z|)} = \sup_{z \in K_s} \frac{|f(z)|}{\varphi_q(|z|)} \cdot \frac{\varphi_q(|z|)}{\varphi_{q-1}(|z|)} \leq M \sup_{z \in K_s} \frac{|f(z)|}{\varphi_q(|z|)} \leq M\|f\|_q,$$

which shows that the set

$$\left\{ \frac{f}{\|f\|_q} \right\}_{f \in U}$$

is bounded in $A_q^{-(q-1)}$. An easy application of Montel’s theorem and condition (C$_1$) imply that the identity map $i : A_q^{-(q-1)} \rightarrow A_q^{-q}$ is compact, and hence the set $\left\{ \frac{f}{\|f\|_q} \right\}_{f \in U}$ is relatively compact in $A_q^{-q}$.

Recall that we have to show $U$ is bounded in $A_q^{-q}$, namely

$$\sup_{f \in U} \|f\|_q < \infty. \quad (2.4)$$

We prove it by contradiction. Assume (2.4) does not hold. Then we can take a sequence $\{f_k\} \subset U$ such that

$$\lim_{k \to \infty} \|f_k\|_q = \infty. \quad (2.5)$$

By the remark above, we have the set $\left\{ \frac{f_k}{\|f_k\|_q} \right\}_k$ is sequentially compact, namely, there is a sequence

$$g_\ell := \frac{f_{k_\ell}}{\|f_{k_\ell}\|_q} \rightarrow g_0 \quad (2.6)$$

as $\ell \rightarrow \infty$ in the sense of $A_q^{-q}$. Clearly, $g_0 \in A_q^{-q}$ with $\|g_0\|_q = 1$.

Now we use condition (a) which says that $S$ is a set of uniqueness for $A_q^{-\infty}$. By this condition, we have $\|g_0\|_{q,S} = C > 0$. Moreover, by (2.6), one can also see that
\[ \lim_{\ell \to \infty} \| g_\ell - g_0 \|_{q,S} = 0, \] which implies for sufficient large \( \ell \), we have

\[ \| g_\ell \|_{q,S} \geq \frac{\| g_0 \|_{q,S}}{2} = \frac{C}{2}. \]

On the other hand,

\[ \| g_\ell \|_{q,S} = \frac{\| f_{kI} \|}{\| f_{kI} \|_q} \leq \frac{1}{\| f_{kI} \|_q}, \]

where in the last inequality, we use the fact that \( \| f_{kI} \|_{q,S} \leq \| f_{kI} \|_{p,S} \leq 1 \). Combining the above estimates yields

\[ \| f_{kI} \|_q \leq \frac{2}{C}, \]

which contradicts to the assumption \((2.5)\). The proof is complete. \( \square \)

### 3 Sampling Sets and Weakly Sufficient Sets

In this section, we study the relation between sampling sets and weakly sufficient sets, in particular we investigate under what conditions weakly sufficient sets are sampling, and vice versa.

As said in Introduction, weakly sufficient sets fail to be sampling sets because the growths \( \Phi_1 \) and \( \psi \) are too “independent”. Motivated by this, we introduce some conditions which “relate” growths of the pair \((\Phi, \psi)\).

For each \( p \in \mathbb{N} \), there exists

\[ \lim_{r \to 1} \frac{\log \varphi_p(r)}{\psi(r)} := \alpha_p, \quad (C_3) \]

\[ (\alpha_p) \text{ is strictly increasing}, \quad (C_4) \]

\[ (\alpha_p) \text{ is bounded}. \quad (C_5) \]

In case \((C_3) - (C_5)\) are satisfied, we denote

\[ \alpha := \lim_{p \to \infty} \alpha_p. \]
For each $t > 0$, we define the Banach space
\[
A_{t, \psi} := \left\{ f \in \mathcal{O}(\mathbb{B}) : \| f \|_{t, \psi} := \sup_{z \in \mathbb{B}} |f(z)|e^{-t\psi(|z|)} < \infty \right\},
\]
and introduce the final condition for $(\Phi, \psi)$:

$$(C_6)$$

For any $\xi \in \mathbb{B}$ and $t \in [0, \alpha)$, there exists $f_{\xi, t} \in A_{t, \psi}$, satisfying

(a) $\| f_{\xi, t} \|_{t, \psi} \leq 1$;

(b) $| f_{\xi, t}(\xi) | = e^{t\psi(|\xi|)}$.

Now we can state the main result in this section.

**Theorem 3.1**

(1) Let $\Phi$ and $\psi$ satisfy conditions $(C_1) - (C_4)$. Then every $\psi$-sampling set is a weakly sufficient set.

(2) Let $\Phi$ and $\psi$ satisfy conditions $(C_2) - (C_6)$. Let further, $\varphi_1$ is bounded on any compact subset of $[0, 1)$. Then every weakly sufficient set is a $\psi$-sampling set.

**Remark 3.2**

Let us consider the classical case, that is $\varphi_p(r) = (1 - r)^{-p}, p \in \mathbb{N}$ and $\psi(r) = | \log(1 - r) |$.

It is easy to check that conditions $(C_1) - (C_4)$ are satisfied. So Theorem 3.1 (1) contains the classical result of the implication “sampling sets $\implies$ weakly sufficient sets” as a particular case.

On the other hand, the condition $(C_5)$ is not satisfied, as $\alpha(p) = p$, is unbounded. That is, the classical pair $(\Phi, \psi)$ does not satisfy the assumption in Theorem 3.1 (2).

It should also be noted that for this one variable case, the hypothesis $(C_5)$ is critical, in the sense that, for any given unbounded sequence $(\alpha_p)$, there exist weakly sufficient sets which are not sampling ones.

Indeed, let us take the example of the weights $\varphi_p(r) = (1 - r)^{-\alpha_p}, p \geq 1$ and $\psi(r) = -\log(1 - r)$. We can easily verify that the inner limit inductive space $A_{\varphi, \psi}^{-\infty}$ induced by such a collection of weights is the same as the one induced by the weights $\varphi_k(r) = (1 - r)^k, k \geq 1$, which was considered in [14]. Therefore, the same counterexample in [14] also works for a general unbounded sequence $(\alpha_p)$.

However, for the case of several variables, the answer remains unclear. In the case $n = 1$, a criterion of a symmetric sequence to be sampling ([8, Proposition 3.14]) is used, while its counterpart in higher dimension ([12, Question 4.2]), so far as we know, is still open. So we raise a problem here.

**Conjecture 3.3**

Let $(\alpha_p)$ be sequence in $(C_3)$ with property $(C_4)$. Suppose that $(C_5)$ is not satisfied. Then there are weakly sufficient sets which are not $\psi$-sampling.

**Remark 3.4**

Concerning condition $(C_6)$, it is worth to give some motivations for its appearance.

Let us again take a typical example

$$\psi(r) = -\log(1 - r^2), \quad 0 < r < 1.$$  (3.1)
This makes condition \((C_6)\) non-trivial. Indeed, we can prove, for this \(\psi\), a slightly stronger assertion, which is independent of the choice of \(\alpha\) in \((C_6)\): for any \(\xi \in \mathbb{B}\) and \(t > 0\), there exists a \(f_{\xi,t} \in \mathcal{A}_{t,\psi}\), such that

\[
\| f_{\xi,t} \|_{t,\psi} = 1 \quad \text{and} \quad | f_{\xi,t}(\xi) | = e^{t\psi(|\xi|)}.
\]  

(3.2)

One of possible options is

\[
f_{\xi,t}(z) = \frac{(1 - |\xi|^2)^t}{(1 - \langle z, \xi \rangle)^{2t}}.
\]

In this case, since

\[
1 - |\Phi_\xi(z)|^2 = \frac{(1 - |\xi|^2)(1 - |z|^2)}{|1 - \langle \xi, z \rangle|^2}, \quad \xi, z \in \mathbb{B}
\]

(see, [21, Lemma 1.2]), where \(\Phi_\xi\) is the involutive automorphism in \(\mathbb{B}\) associated to \(\xi\), we have

\[
| f_{\xi,t}(z) | = \left( \frac{1 - |\Phi_\xi(z)|^2}{1 - |z|^2} \right)^t \leq \frac{1}{(1 - |z|^2)^t} = e^{-t\psi(|z|)},
\]

which shows that \(f \in \mathcal{A}_{t,\psi}\). Conditions (3.2) then follow from \(\Phi_\xi(\xi) = 0\).

This example leads us to the following.

(1) There are many “\(\psi\)” that satisfy \((C_6)\). One can take, say \(\psi(r) = -3 \log(1 - r^2)\) or \(\psi(r) = -\log 2(1 - r), 0 < r < 1\).

(2) The classical case which is considered in Remark 3.2 satisfies condition \((C_6)\).

(3) The space \(\mathcal{A}_{t,\psi}\) can be thought as of the “growth space” of the classical Bergman space \(A_{n+1-t}^1\). We refer the reader to [21] for detailed information about these growth spaces.

Now we turn to the proof of our main result Theorem 3.1. We divide the proof into two parts.

### 3.1 Proof of (1)

Let \(S \subset \mathbb{B}\) be a \(\psi\)-sampling set for \(A_{-\infty}^\infty\). Since conditions \((C_1) - (C_2)\) are satisfied, we prove that two conditions in Theorem 2.5 are true.

(a) \(S\) is a set of uniqueness.

Suppose \(f \in A_{-\infty}^\infty\) and \(f(z) = 0\) for all \(z \in S\). Then since \(S\) is a \(\psi\)-sampling set, we have

\[
T_{f,\psi} = T_{f,\psi,S} = \limsup_{|z| \to 1, z \in S} \frac{\log |f(z)|}{\psi(|z|)} = -\infty.
\]
In particular, \( T_{f, \psi} = \limsup_{|z| \to 1} \frac{\log |f(z)|}{\psi(|z|)} < -1 \) which implies that there exists a \( \delta > 0 \), such that \( \log |f(z)| \leq -\psi(|z|) \) for all \( z \) with \( 1 - \delta < |z| < 1 \). The desired result then follows from an easy application of the Maximum modulus principle.

(b). For any \( p \in \mathbb{N} \), there exists an \( m = m(p) \), such that \( A_{-p, S} \subset A_{-m} \).

Let \( p \in \mathbb{N} \). For every \( f \in A_{-p, S} \), since

\[
\sup_{z \in S} \frac{|f(z)|}{\varphi_p(|z|)} < \infty,
\]

there exists \( C > 0 \) such that

\[
|f(z)| < C \varphi_p(|z|), \text{ for all } z \in S,
\]

which gives

\[
\frac{\log |f(z)|}{\psi(|z|)} < \frac{\log C}{\psi(|z|)} + \frac{\log \varphi_p(|z|)}{\psi(|z|)}, \text{ for all } z \in S.
\]

Since \( S \) is a \( \psi \)-sampling set, by condition (C3), we have

\[
T_{f, \psi} = T_{f, \psi, S} = \limsup_{|z| \to 1, z \in S} \frac{\log |f(z)|}{\psi(|z|)} \leq \limsup_{|z| \to 1, z \in S} \left( \frac{\log C}{\psi(|z|)} + \frac{\log \varphi_p(|z|)}{\psi(|z|)} \right)
\]

\[
\leq \limsup_{|z| \to 1} \left( \frac{\log C}{\psi(|z|)} + \frac{\log \varphi_p(|z|)}{\psi(|z|)} \right) = \limsup_{|z| \to 1} \frac{\log \varphi_p(|z|)}{\psi(|z|)} = \alpha_p.
\]

Furthermore, by condition (C4), since \( \alpha_p < \alpha_{p+1} \), there exists some \( \delta_1 \in (0, 1) \), such that

\[
\log |f(z)| < \alpha_p + \frac{\alpha_{p+1}}{2} \leq \frac{\log \varphi_{p+1}(|z|)}{\psi(|z|)}, \text{ for all } \delta_1 < |z| < 1.
\]

From this it follows that

\[
\frac{|f(z)|}{\varphi_{p+1}(|z|)} < 1, \text{ for all } \delta_1 < |z| < 1.
\]

On the other hand, since \( \varphi_{p+1}(w) \in (1, \infty) \), we also have

\[
\frac{|f(z)|}{\varphi_{p+1}(|z|)} \leq \sup_{|z| \leq \delta_1} |f(z)| < \infty, \quad |z| < \delta_1.
\]

So we get

\[
\sup_{z \in B} \frac{|f(z)|}{\varphi_{p+1}(|z|)} = \infty,
\]
that is \( f \in A^{−m}_\varphi \), with \( m = p + 1 \).

### 3.2 Proof of (2)

The proof of the assertion (2) is the most technical part of this paper. We prove this result by contradiction. Heuristically, if there is some \( f \in A^{−\infty}_\Phi \) such that

\[
T_{f, \psi, S} < T_{f, \psi}, \tag{3.3}
\]

then using Lemma 2.1, we are able to prove the reverse estimate of (3.3), that is, \( T_{f, \psi, S} \geq T_{f, \psi} \) by studying \( f \) along a sequence \( (z_k) \subset B \), where \( f \) assumes its type (that is, that maximal growth). This clearly leads to a contradiction. We now turn to some details.

Let \( S \) be a weakly sufficient set for \( A^{−\infty}_\Phi \). Assume in contrary that there exists a function \( f \in A^{−\infty}_\Phi \) satisfying (3.3). We take \( d, p \) and \( m \) as follows.

- \( d : 0 < d < T_{f, \psi} - T_{f, \psi, S} \);
- \( p \): By \((C_4) - (C_5)\), there is some \( p \in \mathbb{N} \) sufficient large, such that \( \alpha_p > \alpha - d \) and \( f \in A^{−p}_\varphi \).
- \( m \) and \( C_p \): By Lemma 2.1, there exists some \( m = m(p) \geq p \) and \( C_p > 0 \), such that

\[
\|g\|_m \leq C_p \|g\|_{p, S}, \text{ for all } g \in A^{−\infty}_\Phi. \tag{3.4}
\]

We have the following observation.

**Lemma 3.5** \( T_{f, \psi} \leq \alpha_p \leq \alpha_m < \alpha \).

**Proof** It is clear that \( \alpha_p \leq \alpha_m < \alpha \) since \( m \geq p \) and \( (\alpha_p) \) is strictly increasing by \((C_4)\). It suffices for us to prove the first estimate. Since \( f \in A^{−p}_\varphi \), it is easy to check that

\[
\frac{\log |f(z)|}{\log \varphi_p(|z|)} \leq \frac{\log \|f\|_p}{\log \varphi_p(|z|)} + 1.
\]

Furthermore, note that by \((C_3)\) and the assumption \( \lim_{r \to 1} \psi(r) = \infty \), we have \( \lim_{r \to 1} \log \varphi_p(r) = \infty \). Then

\[
T_{f, \psi} = \limsup_{|z| \to 1} \frac{\log |f(z)|}{\psi(|z|)} = \limsup_{|z| \to 1} \left\{ \frac{\log |f(z)|}{\log \varphi_p(|z|)} \cdot \frac{\log \varphi_p(|z|)}{\psi(|z|)} \right\} \leq \alpha_p.
\]

The lemma is proved. \( \square \)

By Lemma 3.5, we let...
• \( x \): \( x \in (\alpha_m - T_{f,\psi}, \alpha - T_{f,\psi}) \);

• \( y \): \( 0 < y < \min \left\{ \frac{x - \alpha_m + T_{f,m}}{2}, \alpha - T_{f,\psi} - x \right\} \).

• \( \delta \): \( 0 < \delta < x + T_{f,\psi} - \alpha_m - 2y \).

(see, Fig. 1).

In particular, this means

\[
\alpha_m + 2y < \alpha_m + 2m + \delta < x + T_{f,\psi} < \alpha - y. \tag{3.5}
\]

By the definition of \( T_{f,\psi,S} \) and \( T_{f,\psi} \), we have the following assertions.

(a). For all \( z \in B \), \( |f(z)| \lesssim e^{(T_{f,\psi} + y)\psi(|z|)} \);

(b). For all \( z \in S \), \( |f(z)| \lesssim e^{(T_{f,\psi} - d)\psi(|z|)} \);

(c). There exists a sequence \( \{z_k\} \subset B \setminus S \) with \( \lim_{k \to \infty} |z_k| = 1 \), such that

\[
|f(z_k)| \geq e^{(T_{f,\psi} - y)\psi(|z_k|)}, \text{ for every } k = 1, 2, \ldots \tag{3.6}
\]

Here and henceforth, by \( A \lesssim B \), we refer the estimate \( A \leq CB \), where \( C \) is a constant which only depends on

1. all the constants defined in or derived from conditions \((C_2) - (C_6)\);
2. \( T_{f,\psi,S} \) and \( T_{f,\psi} \);
3. \( d, p, m, C_p, \alpha_p, \alpha_m \) and \( \alpha \);
4. \( x, y \) and \( \delta \);
5. any dimensional constants;
6. the \( \mathcal{L}^\infty \)-norm of \( f, \varphi_m \) and \( \psi \) on a ball centered \( 0 \in B \) and radius only depends on the constants derived from (1)–(5).

In particular, this implicit constant is independent of the choice of any particular choice of \( k \), which is defined (3.6) and \( \xi \in B \) (see, (3.7) below). This is crucial in our analysis.

Next we apply condition \((C_6)\) with \( t := x \) defined as above and any fixed \( \xi \in B \). This yields a function \( g_{\xi,x} \in \mathcal{R}(\psi, \xi, x) \), satisfying

\[
|g_{\xi,x}(z)| \leq e^{x\psi(|z|)}, \text{ for all } z \in B. \tag{3.7}
\]

Consider the function \( h_{\xi,x} := f \cdot g_{\xi,x} \in \mathcal{O}(B) \). Here are some important properties for \( h_{\xi,x} \).

Lemma 3.6 Let \( x \) be fixed as above. The following assertions hold

Fig. 1 \( \alpha_m, x, y, \alpha, \delta \) and \( T_{f,\psi} \)
(i). There exists some $p'$ sufficiently large, such that
\[
\sup_{\xi \in \mathcal{B}} \| h_{\xi,x} \|_{p'} \lesssim 1.
\]

(ii). For each $\xi \in \mathcal{B}$, $h_{\xi,x} \in A_{\psi}^{-p,\mathcal{S}}$. More precisely,
\[
\sup_{\xi \in \mathcal{B}} \| h_{\xi,x} \|_{p,\mathcal{S}} \lesssim 1.
\]

Lemma 3.6 is an easy consequence of assertion (a), (b) and the estimate (3.7), and for the reader’s convenience, we include its proof in Sect. 3.3. Let us use it now to finish the proof of Theorem 3.1, (2). By both (i) and (ii) in Lemma 3.6, we are able to apply Lemma 2.1 to the function $h_{\xi,x}$ and conclude that
\[
|h_{\xi,x}(z)| \lesssim \varphi_m(|z|), \quad \text{for all } z \in \mathcal{B}.
\]

By condition (C3), this further implies
\[
|h_{\xi,x}(z)| \lesssim e^{(\alpha_m+\delta)\psi(|z|)}, \quad \text{for all } z \in \mathcal{B}. \tag{3.8}
\]

For each $k \geq 1$, we let $z = \xi = z_k$, where $(z_k)$ are defined in (3.6). Since $g_{z_k,x} \in R(\psi, z_k, x)$, we have
\[
|g_{z_k,x}(z_k)| = e^{x\psi(|z_k|)}.
\]

This together with (3.8) implies
\[
|f(z_k)| = \frac{|h_{z_k,x}(z_k)|}{|g_{z_k,x}(z_k)|} \lesssim e^{(\alpha_m-x+\delta)\psi(|z_k|)} \leq e^{(T_f,\psi-2y)\psi(|z_k|)},
\]

where in the last estimate, we use (3.5). This contradicts to (3.6) if $k$ is sufficiently large.

The proof of Theorem 3.1 is complete. \qed

As a consequence of Theorems 2.5 and 3.1, we have the following result.

**Corollary 3.7** Let $S$ be a subset of $\mathbb{B}$. Let further, $\Phi$ and $\psi$ satisfy the conditions (C1) – (C6). If in addition, the weight $\varphi_1$ is bounded on any compact subset of $[0, 1)$, then the following assertions are equivalent:

(i) $S$ is $\psi$-sampling for $A_{\Phi}^{-\infty}$ relative to $\psi$.
(ii) $S$ is weakly sufficient for $A_{\Phi}^{-\infty}$.
(iii) $S$ is a set of uniqueness for $A_{\Phi}^{-\infty}$.

**Proof** It suffices to recall that the condition (C2) is guaranteed if $\Phi$ consists of continuous weights. \qed
Remark 3.8  The weights which satisfy the condition (C₁)–(C₆) appeared in a study of the so-called weighted-type spaces with normal weights.

Let a positive continuous function $\mu$ on $[0, 1)$ be a normal weight (see, e.g., [17]), i.e. there exist numbers $0 < \alpha < \beta$, and $\delta \in (0, 1)$, such that

$$\begin{align*}
&\frac{\mu(r)}{(1-r)^\alpha} \text{ is decreasing on } [\delta, 1), \quad \lim_{r \to 1^-} \frac{\mu(r)}{(1-r)^\alpha} = 0, \\
&\frac{\mu(r)}{(1-r)^\beta} \text{ is increasing on } [\delta, 1), \quad \lim_{r \to 1^-} \frac{\mu(r)}{(1-r)^\beta} = \infty.
\end{align*}$$

(3.9)

Then the weighted-type space with normal weights in the unit ball $\mathbb{B}$ consists of functions $f \in \mathcal{O}(\mathbb{B})$ for which $\sup_{z \in \mathbb{B}} \mu(|z|) |f(z)| < \infty$. This type of spaces which can be viewed as a generalization of the Bergman-type spaces said above, was studied by many authors (see, e.g., [9,18,20]).

It is worthy to give the way of how to construct a pair $(\Phi, \psi)$ which satisfies the assumptions of our main results in general. First, we start with the conditions (3.9) to obtain a sequence $\Phi$ of normal weights satisfying conditions (C₁) and (C₂). Next, we follow the idea in Remark 3.4 to fix our choice of $\psi$ and this guarantees (C₆) to hold. Finally, we make an adjustment on the choice of each $\varphi_p$ so that (C₃)–(C₅) are fulfilled.

For example, the pair $\Phi = \left( (1 - r^2)^{\frac{-2+1}{p}} \right)_{p=1}^{\infty}$ and $\psi(r) = -\log(1 - r^2)$ works.

3.3 Proof of Lemma 3.6

In the last part of Sect. 3, we prove Lemma 3.6. Note that it is crucial that both of these norms of $h_{\xi,x}$ are uniformly bounded above uniformly in $\xi \in \mathbb{B}$ so that we can apply Lemma 2.1 in a uniform manner. This is mainly guaranteed by the condition (C₆) (namely, (3.7)).

We begin with the proof of (i). Let $\xi \in \mathbb{B}$. From assertion (a) and (3.7), it follows that

$$|h_{\xi,x}(z)| \lesssim e^{(T_f,\psi + x + y)\psi(|z|)}, \quad \text{for all } z \in \mathbb{B}. \quad (3.10)$$

By (3.5), we have $T_f,\psi + x + y < \alpha$, and hence by (C₄), there is $p'$ large enough, such that $T_f,\psi + x + y < \alpha p' - 1 < \alpha p' < \alpha$, which, together with (3.10), gives

$$|h_{\xi,x}(z)| \lesssim e^{\alpha p' - 1 \psi(|z|)}, \quad \text{for all } z \in \mathbb{B}. \quad (3.11)$$

Furthermore, by (C₃), for $0 < \varepsilon' < \frac{\alpha p' - \alpha p' - 1}{3}$, there exists $\delta(p') \in (0, 1)$, such that when $\delta(p') < |z| < 1$,

$$\alpha p' - 1 - \varepsilon' < \frac{\log \varphi_{p' - 1}(|z|)}{\psi(|z|)} < \alpha p' - 1 + \varepsilon'.$$
and
\[ \alpha_{p'} - \varepsilon' < \frac{\log \varphi_{p'}(|z|)}{\psi(|z|)} < \alpha_{p'} + \varepsilon', \]
or equivalently,
\[ \varphi_{p'-1}(|z|) e^{-\varepsilon' \psi(|z|)} < e^{\alpha_{p'-1} \psi(|z|)} < \varphi_{p'-1}(|z|) e^{\varepsilon' \psi(|z|)} \]
and
\[ \varphi_{p'}(|z|) e^{-\varepsilon' \psi(|z|)} < e^{\alpha_{p'} \psi(|z|)} < \varphi_{p'}(|z|) e^{\varepsilon' \psi(|z|)}. \]
Hence, when \( \delta(p') < |z| < 1 \), we have
\[ e^{\alpha_{p'-1} \psi(|z|)} < \varphi_{p'-1}(|z|) e^{\varepsilon' \psi(|z|)} < e^{\left(\alpha_{p'-1} + 2\varepsilon'\right) \psi(|z|)} \]
\[ < e^{\left(\alpha_{p'} - \varepsilon'\right) \psi(|z|)} < \varphi_{p'}(|z|). \]
Thus for \( \delta(p') < |z| < 1 \),
\[ |h_{\xi,x}(z)| \lesssim e^{\alpha_{p'-1} \psi(|z|)} \lesssim \varphi_{p'}(|z|). \]
On the other hand, since \( \psi \) is continuous on \([0, \delta(p')]\), it follows that
\[ \sup_{|z| \leq \delta(p')} e^{\alpha_{p'-1} \psi(|z|)} \lesssim 1. \]
Combining the last two inequalities, we conclude that
\[ |h_{\xi,x}(z)| \lesssim \varphi_{p'}(|z|), \text{ for all } z \in \mathbb{B}. \]
This shows that \( h_{\xi,x} \in A_{\Phi}^{p'} \), and hence \( h_{\xi,x} \in A_{\Phi}^{-\infty} \). Moreover, the above estimate is uniform in \( \xi \in \mathbb{B} \). The proof of Lemma 3.6, (i) is complete.

Now we turn to the proof of the second part of Lemma 3.6. Indeed, the proof of the this part is very similar to the proof of Lemma 3.6, (i). Let us mention the necessary modifications that we need to make to prove the result: first of all, instead of applying assertion (a) and (3.7) at the beginning of the proof, we apply assertion (b), (3.7) and the fact that
\[ T_{f,\psi} - d + x < \alpha - y - d < \alpha - d < \alpha_p. \]
Secondly, we shall consider all \( z \in S \) rather than \( z \in \mathbb{B} \). The rest of the proof for the second part then follows line by line from the proof of (i), and we would like to leave the detail to the interested reader.
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