EXISTENCE OF CLOSED GEODESICS THROUGH A REGULAR POINT ON TRANSLATION SURFACES

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Abstract. We show that on any translation surface, if a regular point is contained in a simple closed geodesic, then it is contained in infinitely many simple closed geodesics, whose directions are dense in the unit circle. Moreover, the set of points that are not contained in any simple closed geodesic is finite. We also construct explicit examples showing that such points exist. For a surface in any hyperelliptic component, we show that this finite exceptional set is actually empty. The proofs of our results use Apisa’s classifications of periodic points and of \( \text{GL}(2, \mathbb{R}) \) orbit closures in hyperelliptic components, as well as a recent result of Eskin-Filip-Wright.

Keywords: translation surfaces, simple closed geodesics, periodic points, orbit closure

1. Introduction

Translation surfaces are flat surfaces with conical singularities such that the holonomy of any closed curve (not passing through the singularities) is a translation of \( \mathbb{R}^2 \). On such a surface, one natural question one may ask is whether there exists any simple closed geodesic through a given regular point. This question is particularly relevant in view of applications of the theory to billiards.

In the literature, Masur [23] proved that the set of directions in which there is a simple closed geodesic is a dense subset of \( S^1 \). Later, Boshertzan-Galperin-Krüger-Troubetzkoy [8] improved Masur’s result, they proved that the set of tangent vectors that generate closed geodesics is dense in the unit tangent bundle of the surface (see also [24]). Vorobets ([40], [41]) proved that for any translation surface, almost every regular point on it is contained in infinitely many simple closed geodesics whose directions are dense in the unit circle.

In this paper, we will show

Date: January 9, 2018.

H. Pan and W. Su are partially supported by NSFC No: 11671092 and No: 11631010.
Theorem 1. For any translation surface, every regular point, except for a finite set, is contained in some simple closed geodesic.

Theorem 2. Let \((X, \omega)\) be a translation surface. If a regular point on \((X, \omega)\) is contained in some simple closed geodesic, then

1. it is contained in infinitely many simple closed geodesics \(\{\gamma_n\}_{n \geq 1}\), whose directions are dense in the unit circle;
2. the union \(\bigcup_{n \geq 1} \gamma_n\) is dense in \((X, \omega)\).

For surfaces in the hyperelliptic components we show that the finite exceptional set mentioned in Theorem 1 is actually empty.

Theorem 3. Let \((X, \omega)\) be a translation surface in one of the hyperelliptic components \(H_{\text{hyp}}(\kappa)\) with \(\kappa \in \{(2g - 2), (g - 1, g - 1)\}\) for \(g \geq 2\). Then any regular point of \((X, \omega)\) is contained in infinitely many simple closed geodesics.

It is not difficult to see that regular points that are not contained in any simple closed geodesic do exist (see Section 3). Thus the exceptional subset is not empty in general.

Outline. Here below we will give an outline of the proofs. All the definitions and necessary materials will be introduced in Section 2. It is worth noticing that, even though the statements of these theorems only concern one individual surface, their proof uses in an essential way the classification of its \(\text{GL}(2, \mathbb{R})\)-orbit closure.

(1) The strategy to prove Theorem 1 is to show that any regular point in a translation surface \((X, \omega)\) that is not contained in any simple closed geodesic is a “periodic point” in the sense of Apisa. We then deduce Theorem 1 from a result of Eskin-Filip-Wright [10] (see also [7]) and Lemma 4.1.

(2) Theorem 2 is based on a result of Eskin-Mirzakhani-Mohammadi (Theorem 2.2) which states that for any interval \(I = [a, b) \subset \mathbb{R}/2\pi\) with \(b \neq a\), the sector \(\{a_t r_\theta(X, \omega, p) : t > 0, \theta \in I\}\) is equidistributed in the \(\text{GL}(2, \mathbb{R})\) orbit closure of \((X, \omega, p)\), where \(a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\) and \(r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\).

(3) For Theorem 3, we now assume that \((X, \omega)\) belongs to some hyperelliptic component \(H_{\text{hyp}}(\kappa)\), with \(\kappa \in \{(2g - 2), (g - 1, g - 1)\}\). By Theorem 2 and Theorem 4.2, we only need to show that every periodic point \(p\) in \((X, \omega)\) is actually contained in at least one simple closed geodesic. Note that by a result of Apisa [3, 4], we have a complete list of all the possibilities for \(\mathcal{L} = \text{GL}(2, \mathbb{R}) \cdot (X, \omega)\).
If $L = \mathcal{H}^{hyp}(\kappa)$, then by [4, Th. 1.5], $p$ must be a regular Weierstrass point of $X$. It follows from Proposition 6.1 that $p$ is contained in a simple cylinder of $(X, \omega)$. Thus the theorem is proved for this case. By a simple observation (cf. Lemma 4.5), this also yields the proof for the case $L$ consists of translation covers of surfaces in another hyperelliptic component in lower genus.

If $L$ is a non-arithmetic eigenform locus in genus two then either $p$ is Weierstrass point, or the $\text{GL}(2, \mathbb{R})$ orbit closure of $(X, \omega, p)$ contains a surface with one marked point $(Y, \eta, q)$ where $(Y, \eta)$ is a Veech surface and $q$ is a regular Weierstrass point of $Y$. Note that the latter case occurs only if $L$ is the golden eigenform locus (see [6]). In both cases, Proposition 6.1 allows us to conclude. By Lemma 4.5 this also yields the proof for the case $L$ arises from a non-arithmetic eigenform locus by a covering construction.

Assume now that $L$ is a closed orbit, which means that $(X, \omega)$ is a Veech surface. In this case the theorem follows from Proposition 7.3.

Finally, assume that $L$ consists of translation covers of flat tori branched over two points. In this case, we will show that the $\text{GL}(2, \mathbb{R})$-orbit closure of $(X, \omega, p)$ contains a surface with one marked point $(Y, \eta, q)$, where $(Y, \eta)$ is a Veech surface and $q \in Y$ is a regular point. We then use Proposition 7.3 and Lemma 4.4 to conclude.

**Acknowledgements:** The first-named author thanks Fudan University of Shanghai for its hospitality which helped to initiate the collaboration resulting in this work. The second author would like to thank IMB Bordeaux for the hospitality during his visit. He also wants to thank Alex Wright for explaining his joint work with Paul Apisa.

### 2. Background

In this section, we recall the basic definitions and important results that are used in the proofs of the main theorems.

#### 2.1. Stratum.
A translation surface can be defined as a pair $(X, \omega)$ where $X$ is a compact Riemann surface of genus $g \geq 1$ and $\omega$ is a holomorphic one-form on $X$. In this description, the singularities of the flat metric correspond to the zeros of the one-form. The set of translation surfaces is stratified according to the genus of $X$ and the multiplicities of zeros of $\omega$. Let $\kappa = (k_1, \ldots, k_n)$ be a nonnegative
partition of \(2g - 2\), i.e. \(k_1 + \cdots + k_n = 2g - 2\). Denote by \(\mathcal{H}(k)\) the set of translation surfaces \((X, \omega)\) where \(X\) has genus \(g\) and \(\omega\) has exactly \(n\) zeros with multiplicity \((k_1, \ldots, k_n)\). The space \(\mathcal{H}(k)\) is called a stratum.

A translation surface with marked points is a triple \((X, \omega, \{p_1, \ldots, p_k\})\), where \((X, \omega)\) is a translation surface and \(p_1, \ldots, p_k\) are \(k\) distinct regular points on \(X\). The points \(p_1, \ldots, p_k\) can be considered as zeros of order \(0\) of \(\omega\). Thus, if \((X, \omega)\) belongs to the stratum \(\mathcal{H}(k)\), then the stratum of \((X, \omega, \{p_1, \ldots, p_k\})\) is \(\mathcal{H}(k, 0^k)\).

It is a well-known fact that \(\mathcal{H}(k)\) is a complex orbifold of dimension \(2g + n - 1\). For any \((X, \omega) \in \kappa\), pick a basis \((\gamma_1, \ldots, \gamma_{2g+n-1})\) of the relative homology \(H_1(X, \Sigma; \mathbb{Z})\), where \(\Sigma\) is set of zeros of \(\omega\) (including the ones of order \(0\)). By a slight abuse of notation, for \((X', \omega') \in \mathcal{H}(\kappa)\) close to \((X, \omega)\), we will also denote by \(\gamma_i\) the corresponding element of \(H_1(X', \Sigma'; \mathbb{Z})\), where \(\Sigma'\) is the zero set of \(\omega'\). Then the map

\[
\Phi : (X, \omega) \mapsto (\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{2g+n-1}} \omega) \in \mathbb{C}^{2g+n-1}
\]

is locally homeomorphic, thus defines a local chart for \(\mathcal{H}(\kappa)\) in a neighborhood of \((X, \omega)\). The map \(\Phi\) is called the period mapping, and the associated local coordinates are called period coordinates. For a thorough introduction to the subject, we refer to [24, 46].

2.2. Saddle connection and cylinder. A saddle connection of \((X, \omega)\) is a geodesic segment for the flat metric defined by \(|\omega|\) ending at the zeros of \(\omega\) which does not contain any zero in the interior. A cylinder of \((X, \omega)\) is an open subset of \(X\) which is isometric to \((\mathbb{R}/c\mathbb{Z}) \times (0, h)\), with \(c, h \in \mathbb{R}_{>0}\), and not properly contained in another subset with the same property. The parameters \(c\) and \(h\) are called the circumference and the height of the cylinder respectively. The isometric mapping from \((\mathbb{R}/c\mathbb{Z}) \times (0, h)\) to \((X, \omega)\) can be extended to a map from \((\mathbb{R}/c\mathbb{Z}) \times [0, h]\) to \(X\). The images of \((\mathbb{R}/c\mathbb{Z}) \times \{0\}\) and \((\mathbb{R}/c\mathbb{Z}) \times \{h\}\) under this map are call the boundaries or borders of the cylinder. Each boundary component is a concatenation of some saddle connections in the same direction. Note that the two boundary components are not necessarily disjoint as subsets of \(X\). A cylinder is called a simple if each of its boundary components consists of a single saddle connection.

2.3. \(\text{GL}(2, \mathbb{R})\)-action. There is a natural \(\text{GL}(2, \mathbb{R})\) action on \(\mathcal{H}(k)\), which acts on \((X, \omega) \in \mathcal{H}(\kappa)\) by post-composition with the atlas maps of \((X, \omega)\). More precisely, let \(\{(U_i, \phi_i), i \in I\}\) be an atlas of \((X, \omega)\), covering \(X\) except the zeros of \(\omega\), defining the flat metric \(|\omega|\). Then for \(a \in \text{GL}(2, \mathbb{R})\), \(a \cdot (X, \omega)\) is defined by the atlas \(\{(U_i, a \circ \phi_i), i \in I\}\).
There is a deep connection between the dynamics of $GL(2, \mathbb{R})$ on $\mathcal{H}(\kappa)$ and the dynamics of individual translation surfaces. In more concrete terms, the geometric and dynamical properties of a translation surface is often encoded in its $GL(2, \mathbb{R})$-orbit closure in the moduli space. Studying the $GL(2, \mathbb{R})$-orbit closures has been a central problem of the field. Globally, this problem has been resolved recently by the groundbreaking works of Eskin-Mirzakhani and Eskin-Mirzakhani-Mohammadi. Define an affine invariant submanifold $M$ of a stratum to be an immersed submanifold which is defined locally by homogeneous linear equations with real coefficients in the period coordinates (see [12] and [42] for more details).

**Theorem 2.1** ([11][12]). The $GL(2, \mathbb{R})$ orbit closure of any translation surface is an affine invariant submanifold of its stratum. Any $GL(2, \mathbb{R})$ invariant, closed subset of a stratum is a finite union of affine invariant submanifolds.

Let $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ and $r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. The following theorem describes the equidistribution for sectors.

**Theorem 2.2** ([12]). Let $(X, \omega) \in \mathcal{H}(\kappa)$ be a translation surface. Let $L \subset \mathcal{H}(\kappa, 0)$ be the $GL(2, \mathbb{R})$ orbit closure of $(X, \omega)$. Then for any interval $I = [a, b] \subset \mathbb{R}/2\pi$ with $b \neq a$, the sector $\{a_tr_\theta(X, \omega) : t > 0, \theta \in I \}$ is equidistributed in $L$.

**Remark 2.3.** Theorem 2.2 also holds for translation surface with one marked point.

Note that each stratum is itself a $GL(2, \mathbb{R})$-orbit closure. A surface whose $GL(2, \mathbb{R})$ orbit is dense in its stratum is called generic. This result has been followed by further results by Avila-Eskin-Möller [1], Wright [42, 43], Filip [14] providing more information about the orbit closures. However, obtaining the complete classification of $GL(2, \mathbb{R})$-orbit closures for each stratum remains a major challenge of the field.

2.4. **Veech surfaces.** For a translation surface $(X, \omega)$, let $\text{Aff}(X, \omega)$ be the group of orientation-preserving self-homeomorphisms of $X$ which are given by affine maps in the local charts determined by integrating $\omega$. Elements of $\text{Aff}(X, \omega)$ are called affine automorphisms of $(X, \omega)$ (see [24][18]). To each affine automorphism $f \in \text{Aff}(X, \omega)$, one can associate a matrix $a(f) \in SL(2, \mathbb{R})$ which is the derivative of $f$ in the local charts defined by $\omega$. The image of $\text{Aff}(X, \omega)$ in $SL(2, \mathbb{R})$ under this map is called the Veech group of $(X, \omega)$, denoted by $SL(X, \omega)$. The group $SL(X, \omega)$ is also the stabilizer of $(X, \omega)$ under the action of $SL(2, \mathbb{R})$. If $SL(X, \omega)$ is a lattice of $SL(2, \mathbb{R})$, i.e. $SL(2, \mathbb{R})/SL(X, \omega)$
has finite volume with respect to the Haar measure on $SL(2, \mathbb{R})$, then $(X, \omega)$ is called a Veech surface (or lattice surface).

**Theorem 2.4** ([35] [38]). A translation surface is a Veech surface if and only if its $GL(2, \mathbb{R})$ orbit is closed.

Since the parallel transport on a translation surface does not change the directions of the tangent vectors, for any direction $\theta \in \mathbb{RP}^1$, we have (singular) foliation of the surface by geodesics in this direction. This foliation is said to be uniquely ergodic if each of its leaves is dense and there is a unique transverse measure up to a multiplicative constant. On the other hand, this foliation is said to be periodic if each of its leaves is either a saddle connection, or a (regular) closed geodesic. Veech surfaces are of particular interest because they have more “symmetries” than the others. Moreover, they have optimal dynamical behaviors. Namely, we have

**Theorem 2.5** (Veech dichotomy). Let $(X, \omega)$ be a Veech surface, then for every direction, the directional flow is either periodic or uniquely ergodic.

As a consequence, we get

**Corollary 2.6.** On a Veech surface, the direction of any saddle connection is periodic.

2.5. **Hyperelliptic components.** By the result of Kontsevich-Zorich ([20]), $\mathcal{H}(\kappa)$ has at most three connected components. If $\kappa = (2g - 2)$ or $(g - 1, g - 1)$, then $\mathcal{H}(\kappa)$ contains a special component called hyperelliptic and denoted by $\mathcal{H}^{hyp}(\kappa)$.

For $g \geq 2$, let $\mathcal{Q}(-4 + l, -1^l)$ be the set of meromorphic quadratic differentials on the Riemann sphere with one zero of order $-4 + l$ and $l$ simple poles, where $l = 2g - 3$ if $\kappa = (2g - 2)$, or $2g - 2$ if $\kappa = (g - 1, g - 1)$. Then every translation surface in $\mathcal{H}^{hyp}(\kappa)$ is the canonical double cover of a quadratic differential in $\mathcal{Q}(-4 + l, -1^l)$. The involution of $(X, \omega)$ induced by the double cover is the hyperelliptic involution of $X$. If $\kappa = (2g - 2)$, it fixes the unique zero of $\omega$; if $\kappa = (g - 1, g - 1)$ it exchanges the two zeros of $\omega$.

For $g = 1$, by convention, we consider a flat torus with one or two marked points as a hyperelliptic translation surface and denote the corresponding strata by $\mathcal{H}(0)$ and $\mathcal{H}(0, 0)$ respectively. Surfaces in $\mathcal{H}(0)$ (resp. in $\mathcal{H}(0, 0)$) are canonical double covers of quadratic differentials in $\mathcal{Q}(-1^4)$ and $\mathcal{Q}(0, -1^4)$.

Since all Riemann surfaces of genus two are hyperelliptic, the strata $\mathcal{H}(2)$ and $\mathcal{H}(1, 1)$ contain only the hyperelliptic component. A classification of $GL(2, \mathbb{R})$-orbit closures of translation surfaces in genus two
was obtained by McMullen prior to the works of Eskin-Mirzakhani and Eskin-Mirzakhani-Mohammadi (parts of this classification were also obtained by Calta [9]).

**Theorem 2.7** (McMullen [27, 26, 25]). Let \((X, \omega)\) be a translation surface in genus two, and \(\mathcal{L}\) be the closure of its \(\text{GL}(2, \mathbb{R})\) orbit in the corresponding stratum. If \((X, \omega) \in \mathcal{H}(2)\) then either

1. \(\mathcal{L} = \mathcal{H}(2)\), or
2. \(\mathcal{L} = \text{GL}(2, \mathbb{R}) \cdot (X, \omega)\), in this case \(\text{Jac}(X)\) admits a real multiplication by a quadratic order of discriminant \(D\) with \(\omega\) as an eigenform.

If \((X, \omega) \in \mathcal{H}(1, 1)\), then we have the following possibilities

3. \(\mathcal{L} = \mathcal{H}(1, 1)\),
4. \(\mathcal{L}\) equals the locus of pairs \((X, \omega)\) such that \(\text{Jac}(X)\) admits a real multiplication by a quadratic order of discriminant \(D\) with \(\omega\) as an eigenform, in this case \(\dim \mathcal{L} = 3\).
5. \(\mathcal{L} = \text{GL}(2, \mathbb{R}) \cdot (X, \omega)\), in this case \(\mathcal{L}\) must be contained in an orbit closure in case (4).

We refer to [27] for the precise definitions. Orbit closures in case (4) are called **eigenform loci** and indexed by the discriminant \(D\). If \(D = d^2, d \in \mathbb{N}\), then the locus is called **arithmetic**, otherwise it is called **non-arithmetic**. In [26], McMullen showed that a non-arithmetic eigenform locus of discriminant \(D \neq 5\) does not contain any \(\text{GL}(2, \mathbb{R})\) closed orbit. For \(D = 5\), the corresponding eigenform locus, which is called the **golden eigenform locus**, contains a unique \(\text{GL}(2, \mathbb{R})\) closed orbit.

**Definition 2.8** (translation cover). Let \((X_1, \omega_1), (X_2, \omega_2)\) be two translation surfaces. If there exists a (branched) cover \(\pi : X_1 \to X_2\) such that \(\omega_1 = \pi^* \omega_2\), then \((X_1, \omega_1)\) is called a translation cover of \((X_2, \omega_2)\). In the case \((X_2, \omega_2)\) is a flat torus, \((X_1, \omega_1)\) is called a torus cover.

Whenever we mention a (branched) cover in this paper, we mean a translation cover.

Recently, using the results of [11] and [12] together with technical tools developed in [28, 29], Apisa obtained the following classification of \(\text{GL}(2, \mathbb{R})\)-orbit closures in the hyperelliptic components of any genus.

**Theorem 2.9** ([3, 5]). Let \((X, \omega)\) be a translation surface in some hyperelliptic component \(\mathcal{H}^{\text{hyp}}(2g - 2)\) or \(\mathcal{H}^{\text{hyp}}(g - 1, g - 1)\), and \(\mathcal{L}\) the corresponding \(\text{GL}(2, \mathbb{R})\) orbit closure. Then \(\mathcal{M}\) is one of the following.

1. If \(\dim \mathcal{L} = 2\), then \(\mathcal{L}\) is a closed orbit.
(2) If \( \text{dim} L = 3 \) and \( L \) is arithmetic, then \( L \) is a branched construction over \( \mathcal{H}(0, 0) \).

(3) If \( \text{dim} L = 3 \) and \( L \) is nonarithmetic, then \( L \) is a branched construction over some non-arithmetic eigenform locus in \( \mathcal{H}(1, 1) \).

(4) If \( \text{dim} L = 2r \) with \( 1 < r < g \), then \( L \) is a branched construction over \( \mathcal{H}^{\text{hyp}}(2r - 2) \).

(5) If \( \text{dim} L = 2r + 1 \) with \( 1 < r < g \), then \( L \) is a branched construction over \( \mathcal{H}^{\text{hyp}}(r - 1, r - 1) \).

(6) \( L \) is the whole stratum.

The covers are branched over zeros of the holomorphic one-forms and commute with the hyperelliptic involution.

**Remark 2.10.**
- In genus three, this classification was obtained by Nguyen-Wright [34] and Aulicino-Nguyen [2].
- The classification of \( \text{GL}(2, \mathbb{R}) \) closed orbits remains open, even in genus two.

2.6. **Marked points and periodic points.** On a translation surface \((X, \omega)\), a regular point \( p \) is called a periodic point if the orbit closures \( \text{GL}(2, \mathbb{R}) \cdot (X, \omega, p) \) in \( \mathcal{H}(\kappa, 0) \) and \( \text{GL}(2, \mathbb{R}) \cdot (X, \omega) \) in \( \mathcal{H}(\kappa) \) have the same dimension ([4, 6, 7, 15, 30]). For a Veech surface, this is equivalent to say that the orbit of \( p \) under \( \text{Aff}(X, \omega) \) is finite.

When \((X, \omega)\) is a surface in a hyperelliptic component, we have the following result due to Apisa and Möller on the periodic points of \((X, \omega)\).

**Theorem 2.11** ([4, 6, 30]). Let \((X, \omega)\) be a primitive translation surface in some hyperelliptic component \( \mathcal{H}^{\text{hyp}}(\kappa) \), where \( \kappa = (2g - 2) \) or \((g - 1, g - 1)\).

1. If \( \text{GL}(2, \mathbb{R}) \cdot (X, \omega) = \mathcal{H}^{\text{hyp}}(\kappa) \), then the set of periodic points of \((X, \omega)\) are the set of regular Weierstrass points.

2. If \((X, \omega)\) is a primitive Veech surface of genus two, then the set of periodic points of \((X, \omega)\) are the set of regular Weierstrass points.

3. If \((X, \omega) \in \mathcal{H}^{\text{hyp}}(1, 1)\) contains a periodic point \( p \) which is not a Weierstrass point, then \( \text{GL}(2, \mathbb{R}) \cdot (X, \omega) \) is the golden eigenform locus.

Let \((X, \omega)\) be a generic surface in the golden eigenform locus (cf. Section [2.5]), and \( p \in (X, \omega) \) be a periodic point which is not a Weierstrass point. Then it is shown in [6 §2] (see also [19]) that the \( \text{GL}(2, \mathbb{R}) \) orbit closure of \((X, \omega, p)\) in \( \mathcal{H}(1, 1, 0) \) contains a translation surface with one
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marked point \((Y, \eta, q)\), where \((Y, \eta)\) is a Veech surface and \(q\) is a regular Weierstrass point.

3. EXISTENCE OF REGULAR POINTS NOT CONTAINED IN ANY SIMPLE CLOSED GEODESIC

In this section, we will give a family of translation surface which have regular points not contained in any simple closed geodesic.

**Proposition 3.1.** Let \((Y, q)\) be a holomorphic quadratic differential which has only one simple pole and the other singularities are zeros. Let \(\pi : (X, \omega^2) \rightarrow (Y, q)\) be the canonical double cover. Then the pre-image of the simple pole under the double cover is not contained in any simple closed geodesic.

**Proof.** Let \(y \in Y\) be the unique simple pole, and \(\tilde{y} \in X\) be its preimage under the double cover \(\pi\). Let \(i\) be the involution of \(X\) that permutes the preimages under \(\pi\). It is clear that \(\tilde{y}\) is the only regular point which is invariant under \(i\). Suppose to a contradiction that \(\tilde{y}\) is contained in some simple closed geodesic, say \(\gamma\). Then \(\gamma\) is invariant under the involution \(i\), which implies the existence of another fixed point of \(i\) which is contained in \(\gamma\). This contradicts the fact that \(\tilde{y}\) is the only non-singular fixed point of \(i\). \(\square\)

To close this section, we construct an explicit surface on which there is a unique regular point that is not contained in any simple closed geodesic.

**Example 1.** Consider the example \((X, \omega)\) in Figure 1. It is obtained from a central symmetric polygon \(P\) via edges identification. It is a translation surface in \(\mathcal{H}(1,1,1,1,2)\), tiled by 40 squares. It is also a branched double cover over a quadratic differential \((Y, q) \in Q(1,2,2,-1)\). Let \(O \in (X, \omega)\) be the point represented by the center of the polygon which is the preimage of the simple pole of \(q\). By Proposition 3.1, \(O\) is not contained in any simple closed geodesic of \((X, \omega)\). Moreover, in this case, any other regular point is contained in some simple closed geodesic of \((X, \omega)\). Indeed, by identification, the vertices of the squares are classified into 34 equivalence classes (two vertices belong to the same equivalence class if and only if they represent the same point on \((X, \omega)\)). Five of them are the zeros of \(\omega\), and the remaining 29 classes are regular points of \(\omega\) which are marked out in the figure. The center \(O\) is one of these 29 regular points. Consider the eight cylinders on \((X, \omega)\) which are the images of the two cylinders in Figure 1 under the \(x\)-reflection and \(y\)-reflection. It is not difficult to see that the remaining 28 non-center regular points are contained in the interior of these eight...
Figure 1. A translation surface \((X, \omega) \in \mathcal{H}(1, 1, 1, 1, 2)\). Each labeled edge is identified with the edge of the same label, each unlabeled horizontal (resp. vertical) edge is identified with the horizontal edge in the same vertical strip (resp. vertical edge in the same horizontal strip).

cylinders. Any other regular point which is not a vertex of the unit square is contained in one of the horizontal or vertical cylinders.

4. Proof of Theorem

Our goal in this section is to prove Theorem

4.1. Torus with marked points.

Lemma 4.1. Let \((\mathbb{T}, p_1, \cdots, p_k) \in \mathcal{H}(0^k)\) be a torus with \(k \geq 1\) marked points.

1. If \(1 \leq k \leq 2\), then every point \(p \in \mathbb{T} \setminus \{p_1, \cdots, p_k\}\) is contained in some simple closed geodesic of \(\mathbb{T} \setminus \{p_1, \cdots, p_k\}\).
2. If \(k \geq 3\), then the every regular point in \(\mathbb{T} \setminus \{p_1, \cdots, p_k\}\), except for a finite set, is contained in some simple closed geodesic.
Proof. Without loss of generality, we may assume that \( \mathbb{T} \) is obtained by identifying the opposite sides of \([0, 1] \times [0, 1]\), such that \( p \) corresponds to \((0, 0)\), and \( p_i = (a_i, b_i) \neq (0, 0) \in [0, 1] \times [0, 1], \ 1 \leq i \leq k \).

\((1)\). It suffices to prove the statement for the case \( k = 2 \). Consider the following three simple closed geodesics through \( p \), horizontal geodesic \( \gamma_1 \), vertical geodesic \( \gamma_2 \), and the geodesic \( \gamma \) with slope one. At least one of these three simple closed geodesics does not intersect \( \{ p_1, p_2 \} \).

\((2)\). Let \( H \) (resp. \( V \)) be the union of all the horizontal (resp. vertical) simple closed geodesics passing through \( p_1, \ldots, p_k \). Then every point in \( \mathbb{T} \setminus (H \cap V) \) is contained in at least one simple closed geodesic \( \gamma \) which does not intersect \( \{ p_1, \ldots, p_k \} \). Since \( H \cap V \) is a finite set, this completes the proof. \( \square \)

4.2. Periodic points.

**Theorem 4.2.** Let \((X, \omega)\) be a translation surface. Then any regular point which is not contained in any simple closed geodesic is a periodic point of \((X, \omega)\).

As a consequence of Theorem 4.2, we get

**Corollary 4.3.** Let \((X, \omega)\) be a generic surface in a non-hyperelliptic component. Then every regular point of \((X, \omega)\) is contained in some simple closed geodesic.

*Proof.* By a result of Apisa [4, Cor.1], \((X, \omega)\) does not have any periodic point. Thus, the corollary follows immediately from Theorem 4.2. \( \square \)

Let \( \mathcal{L} = \text{GL}(2, \mathbb{R}) \cdot (X, \omega) \), and \( \mathcal{L}^* \) the set of marked translation surfaces \((Y, \eta, q)\), where \((Y, \eta) \in \mathcal{L} \) and \( q \in (Y, \eta) \) is a regular point. Let \( \mathcal{N} \subset \mathcal{L}^* \) be the set of marked translation surfaces \((Y, \eta, q) \in \mathcal{L}^* \) such that \( q \in (Y, \eta) \) is not contained in simple closed geodesic.

**Lemma 4.4.** The subset \( \mathcal{N} \) is \( \text{GL}(2, \mathbb{R}) \) invariant, closed, and proper.

*Proof.* It is clear that \( \mathcal{N} \) is \( \text{GL}(2, \mathbb{R}) \) invariant. The properness follows from Masur’s result [23] that \((X, \omega)\) contains infinitely many simple closed geodesics. To prove it is closed, it suffices to prove that its complementary \( \mathcal{N}^c \) is open. Let \((Y, \eta, q)\) be an element of \( \mathcal{N}^c \). Then \( q \) is contained in at least one simple closed geodesic of \((Y, \eta)\), which implies that \( q \) is contained in the interior of at least one cylinder. This cylinder persists on nearby marked surfaces in \( \mathcal{L}^* \). As a consequence, for all marked surfaces \((Y', \eta', q')\) sufficiently close to \((Y, \eta, q)\), \( q' \) is contained in at least one simple closed geodesic of \((Y', \eta')\). Therefore, \( \mathcal{N} \) is open in \( \mathcal{L}^* \). \( \square \)
Proof of Theorem 4.2. Let $L_p \subset N$ be the $GL(2, \mathbb{R})$ orbit closure of $(X, \omega, p)$ in $L^*$. By Lemma 4.4, $L_p$ is a proper affine invariant submanifold of $L^*$. Therefore, $\dim L \leq \dim L_p < \dim L^* = \dim L + 1$. Hence, $\dim L = \dim L_p$, and by definition, $p$ is a periodic point of $(X, \omega)$. □

4.3. Proof of Theorem 1

Proof of Theorem 1. Let $J$ be the set of regular points on $(X, \omega)$ each of which is not contained in any simple closed geodesic. By Theorem 4.2, every point $p \in J$ is a periodic point of $(X, \omega)$. By the work of Eskin-Filip-Wright ([10, Theorem 1.5], see also [7, Theorem 1.2]), $(X, \omega)$ has infinitely many periodic points if and only if $(X, \omega)$ is a torus cover. Thus Theorem 1 is proved for surfaces that are not torus covers.

Suppose now that $(X, \omega)$ is a torus cover. Let $\pi : (X, \omega) \to (\mathbb{T}, \{p_1, \ldots, p_k\})$ be a translation cover, with $(\mathbb{T}, \{p_1, \ldots, p_k\}) \in \mathcal{H}(0^k)$ for some $k \geq 1$. By Lemma 4.1, every regular point in $\mathbb{T} \setminus \{p_1, \ldots, p_k\}$, except for a finite set, is contained in some simple closed geodesic. Thus in this case, Theorem 1 is a direct consequence of Lemma 4.5 below. □

Lemma 4.5. Let $(X_1, \omega_1), (X_2, \omega_2)$ be two translation surfaces, and $\pi : (X_1, \omega_1) \to (X_2, \omega_2)$ a translation covering map. Let $\Sigma_2$ be the finite subset of $X_2$ that contains all the zeros of $\omega_2$ and the images of the branched points of $\pi$. Let $\Sigma_1 = \pi^{-1}(\Sigma_2)$. Assume that any point in $X_2 \setminus \Sigma_2$ is contained in some simple closed geodesic which does not intersect $\Sigma_2$. Then any point in $X_1 \setminus \Sigma_1$ is also contained in some simple closed geodesic which does not intersect $\Sigma_1$.

Proof. Observe that by assumption, $\Sigma_1$ contains all the zeros of $\omega_1$. Let $x$ be point of $X_1 \setminus \Sigma_1$, then the image $\pi(x)$ of $x$ is a point of $X_2 \setminus \Sigma_2$. By assumption, $\pi(x)$ is contained in some simple closed geodesic $\gamma$ such that $\gamma \cap \Sigma_2 = \emptyset$. On the other hand, the preimage of $\gamma$ is a finite union of simple closed geodesics of $(X_1, \omega_1)$ which do not intersect $\Sigma_1$, one of which contains $x$, say $\gamma$. This completes the proof. □

5. Proof of Theorem 2

In this section, we prove Theorem 2.

Proof of Theorem 2 (1) Let $L$ denote the closure of the $GL(2, \mathbb{R})$ orbit of $(X, \omega)$. Let $L^*$ denote the space of triples $(Y, \eta, q)$, where $(Y, \eta) \in L$ and $q$ is a regular point of $(Y, \eta)$. Let $L_p$ be the $GL(2, \mathbb{R})$ orbit closure of $(X, \omega, p)$ in $L^*$. Without loss of generality, we assume that $p$ is contained in a horizontal simple closed geodesic $\gamma$. By Theorem 2.2,
for any interval $I = [a, b] \subset \mathbb{R}/2\pi$ with $b \neq a$, there exists $t_n \to +\infty$, $\theta_n \in I$, such that

$$a_{tn}r_{\theta_n}(X,\omega, p) \to (X,\omega, p), \text{ as } n \to \infty,$$

where $a_t = (e^t 0 \quad 0 e^{-t})$ and $r_{\theta} = (\cos \theta - \sin \theta \quad \sin \theta \quad \cos \theta)$.

Since $p$ is contained in a horizontal simple closed geodesic, it is contained in a horizontal cylinder $C$. This cylinder persists for near by marked translation surface $(Y, \eta, q)$. Therefore there exists a sequence of simple closed geodesics $\{\gamma_n\}_{n \geq 1}$ on $a_{tn}r_{\theta_n}(X,\omega)$ satisfying the following properties.

(a) For all $n \geq 1$, $p$ is contained in $\gamma_n$.

(b) $\lim_{n \to \infty} \int_{\gamma_n} a_{tn}r_{\theta_n}\omega = \int_{\gamma} \omega$. In particular, the length of $\gamma_n$ on $a_{tn}r_{\theta_n}(X,\omega)$ has an upper bound for all $n \geq 1$.

(c) The direction of $\gamma_n$ on $a_{tn}r_{\theta_n}(X,\omega)$ converges to the direction of $\gamma$ on $(X,\omega)$ as $n \to \infty$.

**Claim.** The direction of $\gamma_n$ on $r_{\theta_n}(X,\omega)$ converges to $\pm \pi/2$, the vertical direction.

**Proof of the claim.** Suppose to a contradiction that there is a subsequence of $\{\gamma_n\}_{n \geq 1}$, still denoted as $\{\gamma_n\}_{n \geq 1}$ for convenience, whose directions converge to some direction $\theta \neq \pm \pi/2$. Recall that the lengths of simple closed geodesics on $r_{\theta}(X,\omega)$ with respect to the flat metric $|r_{\theta_n}\omega| = |\omega|$ has a positive lower bound $\delta$ for all $n \geq 1$. Therefore, the horizontal length of $\gamma_n$ on $r_{\theta_n}(X,\omega)$ has a positive lower bound $\frac{1}{2}|\delta \cos \theta|$, i.e. $|\text{Re} \int_{\gamma_n} r_{\theta_n}\omega| > \frac{1}{2}|\delta \cos \theta| > 0$, for $n$ large enough. As a consequence, the length of $\gamma_n$ on $a_{tn}r_{\theta_n}(X,\omega)$ tends to infinity, since $l(a_{tn}r_{\theta_n}\omega(\gamma_n) = |\text{Re} \int_{\gamma_n} a_{tn}r_{\theta_n}\omega| = |e^{tn}\text{Re} \int_{\gamma_n} r_{\theta_n}\omega| > \frac{1}{2}e^{tn}|\delta \cos \theta| \to \infty$, as $n \to \infty$, where $l(\cdot)$ represents the length. This contradicts the property (b) mentioned above. \qed

It follows from the claim that for any $\epsilon > 0$, there exists $N > 0$, such that the direction of $\gamma_n$, $n > N$ on $(X,\omega)$ is contained in the interval $[\pi/2 - b - \epsilon, \pi/2 - a + \epsilon]$. Recall that $p$ is contained in $\gamma_n$ for all $n \geq 1$. By the arbitrariness of $I$ and $\epsilon$, this completes the proof.

(2) Suppose that the horizontal direction of $(X,\omega)$ is uniquely ergodic. Then the horizontal geodesic $L_p$ through $p$ is dense in $(X,\omega)$. By the density of periodic directions through $p$, there exists $\theta_n > 0$ converging to zero, such that $p$ is contained in a sequence of simple closed geodesics $\gamma_n$ with direction $\theta_n$. It is not difficult to see that $L_p$ is contained in the closure $\overline{\cup_{n \geq 1} \gamma_n}$, which completes the proof. \qed
6. Regular Weierstrass points on surfaces in hyperelliptic components

In this section, we will prove a technical result which will be used in the proof of Theorem 3.

**Proposition 6.1.** Let \((X,\omega)\) be a translation surface belonging to a hyperelliptic component \(\mathcal{H}^{\text{hyp}}(\kappa)\), where \(\kappa\) is either \((2g-2)\) or \((g-1, g-1)\) with \(g \geq 2\). We denote by \(\tau\) the hyperelliptic involution of \(X\). Let \(I\) be a saddle connection on \(X\) that is invariant by \(\tau\). Then there exists a simple cylinder on \(X\) that contains \(I\).

**Proof.** Without loss of generality, we can assume that \(I\) is horizontal. We identify \(I\) with a segment \(PQ\) in \(\mathbb{R}^2\).

We first show that there is an embedded parallelogram in \(X\) invariant under \(\tau\) that contains \(I\) as a diagonal. Using the action of \(\{(1 \ t \ 0 \ 1) , \ t \in \mathbb{R}\}\), we can assume that the vertical flow on \((X,\omega)\) is minimal. Consider the vertical separatrices in the direction \((0, -1)\), that is, the geodesic rays emanating from the singularities of \(X\) in this direction. Since the flow in the vertical direction is minimal, all the rays in this family intersect \(\text{int}(I)\). For each of these rays, we consider the segment from its origin to its first intersection point with \(\text{int}(I)\). We then have a finite family of vertical geodesic segments \(\{\eta_1, \ldots, \eta_m\}\), each of which has one endpoint being a singularity, the other endpoint in \(\text{int}(I)\), and contains no point of \(\text{int}(I)\) in the interior.

Without loss of generality, we can assume that \(|\eta_1| = \min\{|\eta_1|, \ldots, |\eta_m|\}\). Using the developing map, we can identify \(\eta_1\) with a vertical segment in \(\mathbb{R}^2\) whose endpoints are denoted be \(P_1\) and \(R_1\), where \(R_1 \in \text{int}(PQ)\). Let \(Q_1\) denote the image of \(P_1\) under the central symmetry through the midpoint of \(PQ\). Let \(P_1\) denote the parallelogram with vertices \(P, P_1, Q, Q_1\).

By construction, we have a locally isometric map \(\varphi : P_1 \to X\). Note that \(\delta_1^+ := \varphi(P_1P)\) and \(\gamma_1^+ := \varphi(QP)\) are saddle connections of \((X,\omega)\). Since the hyperelliptic involution \(\tau\) fixes the midpoint of \(I\) and permutes its endpoints, this involution is identified with the central symmetry through the midpoint of \(PQ\) via \(\varphi\). It follows in particular that \(\delta_1^- := \varphi(Q_1P)\) and \(\gamma_1^- := \varphi(Q_1Q)\) are also saddle connections of \((X,\omega)\).

Since \(P_1\) is a parallelogram (hence a convex domain in \(\mathbb{R}^2\)), the restriction of \(\varphi\) to \(\text{int}(P_1)\) is actually an embedding. Remark also that if \(\delta_1^+ = \delta_1^-\) and \(\gamma_1^+ = \gamma_1^-\), then the image of \(P_1\) by \(\varphi\) is a closed torus, which is impossible since we have assumed that \(X\) has genus at least two. Therefore, we can always suppose that \(\delta_1^+ \neq \delta_1^-\).
We now consider the following algorithm: assume that we have a locally isometric map \( \varphi_n \) from a parallelogram \( P_n = (PP_n QQ_n) \subset \mathbb{R}^2 \) that contains \( PQ \) as a diagonal to \( X \) such that

(i) \( \varphi_n(PQ) = I \),
(ii) \( \varphi_n \) maps the sides of \( P_n \) to saddle connections of \((X, \omega)\),
(iii) the restriction of \( \varphi_n \) to the interior of \( P_n \) is an embedding,
(iv) \( \delta^+_n := \varphi_n(PP_n) \) and \( \delta^-_n := \varphi_n(QQ_n) \) are two different saddle connections.

Let us denote by \( \gamma^+_n \) and \( \gamma^-_n \) the saddle connections of \((X, \omega)\) which are the images of \( QP_n \) and \( PQ_n \) under \( \varphi_n \) respectively.

If \( \gamma^+_n = \gamma^-_n \) then the algorithm stops. In this case, \( \varphi_n(P_n) \) is a simple cylinder (bounded by \( \delta^+_n \) and \( \delta^-_n \)) that contains \( I \) and we are done. Otherwise, we proceed as follows: using the action of \( \{(1 0 \ t), \ t \in \mathbb{R}\} \), we can assume that \( PP_n \) is vertical. We then have two cases:

- **Case 1:** some vertical separatrices intersect \( \text{int}(I) \). In this case, using the construction above, we obtain a locally isometric map \( \varphi_{n+1} \) from parallelogram \( P_{n+1} = (PP_{n+1} QQ_{n+1}) \), that contains \( PQ \) as a diagonal to \( X \), satisfying the properties (i),(ii), (iii). We claim that \( \varphi_{n+1} \) also satisfies (iv). Indeed, let \( \delta^+_{n+1} := \varphi_n(PP_{n+1}) \) and \( \delta^-_{n+1} := \varphi_n(QQ_{n+1}) \). If \( \delta^+_{n+1} = \delta^-_{n+1} \), then \( \varphi_{n+1}^{-1}(\delta^+_{n+1}) \) is a vertical geodesic segment starting from \( Q_{n+1} \) (see Figure 2). This vertical segment must intersect \( \text{int}(PP) \) which contradicts the assumption that the restriction of \( \varphi_n \) to the interior of \( P_n \) is an embedding.

- **Case 2:** no vertical separatrix intersects \( \text{int}(I) \). In this case \( I \) is contained in a vertical cylinder \( V \) whose boundary contains

\[ P_n \] \[ \delta^+_n \] \[ \delta^-_n \] \[ P_{n+1} \] \[ \delta^+_{n+1} \] \[ \delta^-_{n+1} \] \[ Q_n \] \[ Q_{n+1} \]

\[ P'_n \] \[ \delta^+_n \] \[ \delta^-_n \] \[ Q'_n \] \[ \delta^+_{n+1} \] \[ \delta^-_{n+1} \] \[ Q'_{n+1} \] \[ P_{n+1} \]

*Figure 2.* Embedded parallelogram containing \( I \): Case 1 (left) and Case 2 (right).
We can represent $V$ by a rectangle $R = (PQ'P')$ in the plane as shown in Figure 2. The vertical sides of $R$ are decomposed into several segments, each of which is mapped to a saddle connection of $(X, \omega)$. By construction $PP_n$ is the bottom most segment in the left border of $R$. Let $Q'_n$ be the lower endpoint of the topmost segment in the right border of $R$, then $Q'_nQ'_n$ is mapped to the saddle connection $\delta^-_n$.

If $V$ is a simple cylinder then we must have $Q'_n = Q$, and $\gamma^+_n = \gamma^-_n$, which contradicts our assumption. Thus the left (resp. right) border of $V$ must contained other saddle connections than $\delta^+_n$ (resp. than $\delta^-_n$). Let $Q'_{n+1}$ be the lower endpoint of the topmost segment in the left border of $R$, and $P_{n+1}$ be the upper endpoint of the bottommost segment in the right border of $R$. Then the saddle connection which is the image of $PQ'_{n+1}$ is not $\delta^+_n$, and the one which is the image of $QP_{n+1}$ is not $\delta^-_n$.

Let $Q_{n+1}$ be the image of $Q'_{n+1}$ under the translation by $P'P$. Consider the parallelogram $P_{n+1} = (PP_{n+1}QQ_{n+1})$. The map from $R$ to $(X, \omega)$ can be extended to the triangle with vertices $P, Q, Q_{n+1}$, hence we have a locally isometric map $\varphi_{n+1} : P_{n+1} \to (X, \omega)$. It is straightforward to check that $\varphi_{n+1}$ satisfies the properties (i), . . . , (iv).

We now show that the algorithm has to stop after finitely many steps. Let us denote by $\delta^+_{n+1}$ and $\delta^-_{n+1}$ the images under $\varphi_{n+1}$ of $PP_{n+1}$ and $QQ_{n+1}$ respectively. For $i \in \{n, n + 1\}$, since $\delta^+_i$ and $\delta^-_i$ are exchanged by the hyperelliptic involution $\tau$, they cut $X$ into two subsurfaces with boundary, each of which is invariant under $\tau$. Let $X_i$ denote the subsurface that contains $I$. Let $\psi(X_i)$ be the total cone angle at the singularities of $X$ inside $X_i$. Since $X_i$ is bounded by two saddle connections in the same direction, $\psi(X_i)$ must be an integral multiple of $2\pi$. Note that $\psi(X_i) = 2\pi$ if and only if $X_i$ is a simple cylinder.

Observe that $\delta^+_{n+1}$ and $\delta^-_n$ (resp. $\delta^+_{n+1}$ and $\delta^-_n$) can only meet at their endpoints, otherwise $\delta^+_n$ (resp. $\delta^-_n$) would intersect int$(I)$ contradicting our assumption. It follows that $X_{n+1}$ is a proper subsurface of $X_n$, and $\psi(X_{n+1}) < \psi(X_n)$. Therefore, the algorithm has to stop after at most $\psi(X)/(2\pi)$ steps, where $\psi(X)$ is the total cone angle at the singularity of $X$, and we get a simple cylinder that contains $I$. \hfill \Box

**Corollary 6.2.** Let $(X, \omega)$ be a translation surface in some hyperelliptic component $H^{hyp}(\kappa)$ of genus at least 2. Let $W$ be a Weierstrass point of $X$ which is not a zero of $\omega$. Then $W$ is contained in a simple cylinder of $(X, \omega)$. 

**Proof.** Let $J$ be a geodesic segment of that realizes the distance from $W$ to the zero set of $\omega$. Then $I := J \cup \tau(J)$ is a saddle connection invariant by $\tau$. Thus, by Proposition 6.1, there is a simple cylinder $C$ in $X$ that contains $I$. □

7. **Proof of Theorem 3**

In this section, we will prove Theorem 3 for genus at least two (genus one case is already proved in Lemma 4.1). Let $L$ be the GL$(2, \mathbb{R})$ orbit closure of $(X, \omega)$. By Theorem 2.9, there are five possibilities:

(I) $\mathcal{L} = \mathcal{H}^{hyp}(\alpha)$, where $\alpha = (2g - 2)$ or $(g - 1, g - 1)$;

(II) $4 \leq \dim \mathcal{L} < \dim \mathcal{H}^{hyp}(\alpha)$, $\mathcal{L} = \mathcal{H}^{hyp}(\alpha')$ is a branched construction over $\mathcal{H}^{hyp}(\alpha')$, where $\alpha' = (2r - 2)$ or $(r - 1, r - 1)$ with $2 \leq r < g$;

(III) $\dim \mathcal{L} = 3$ and $\mathcal{L}$ non-arithmetic, $\mathcal{L} = \mathcal{E}_{\Omega_D}(1, 1)$ is a branched construction over a non-arithmetic eigenform locus for some square free positive integer $D$;

(IV) $\dim \mathcal{L} = 3$ and $\mathcal{L}$ arithmetic, $\mathcal{L} = \mathcal{H}(0, 0)$ is a branched construction over $\mathcal{H}(0, 0)$;

(V) $\dim \mathcal{L} = 2$, $\mathcal{L}$ is a closed orbit, i.e. $(X, \omega)$ is a Veech surface.

We will prove the theorem for each case listed above.

7.1. **Case I: $\mathcal{L} = \mathcal{H}^{hyp}(\kappa)$**.

**Proof of Theorem 3 for Case I.** Suppose to a contradiction that there is a regular point $p$ of $(X, \omega)$ which is not contained in any closed geodesic. By Theorem 4.2, $p$ is a periodic point of $(X, \omega)$. Therefore, $p$ is a Weierstrass point of $(X, \omega)$ by Theorem 2.11. But by Corollary 6.2, $p$ is contained in a simple cylinder. Thus we have a contradiction, which proves the theorem for this case. □

7.2. **Case II: $\mathcal{L} = \mathcal{H}^{hyp}(\kappa')$ for some $\kappa' = (2r - 2)$ or $(r - 1, r - 1)$ with $2 \leq r < g$**.

**Proof of Theorem 3 for Case II.** Let $\pi : (X, \omega) \to (X', \omega')$ be the branched cover with $(X', \omega') \in \mathcal{H}^{hyp}(\kappa')$. Let $\Sigma$ and $\Sigma'$ be the zero sets of $\omega$ and $\omega'$ respectively. In this case we have $\Sigma = \pi^{-1}(\Sigma')$. By assumption, the GL$(2, \mathbb{R})$ orbit closure of $(X', \omega')$ is $\mathcal{H}^{hyp}(\kappa')$. By Theorem 3 for Case I, any regular point of $(X', \omega')$ is contained in infinitely many simple closed geodesics. Lemma 4.5 then implies that the same is true for $(X, \omega)$. □
7.3. **Case III**: \( \mathcal{L} = E\Omega_D(1, 1) \), \( D \) is a square free discriminant.

*Proof of Theorem 3 for Case III.* Suppose to a contradiction that \((X, \omega)\) contains a regular point \( p \) which is not contained in any simple closed geodesic. Then, by Theorem 4.2, \( p \) is a periodic point of \((X, \omega)\). If \( p \) is a Weierstrass point, by Corollary 6.2, \( p \) is contained in a simple cylinder and we have a contradiction.

If \( p \) is not a Weierstrass point of \((X, \omega)\), then by Theorem 2.11, \( \text{GL}(2, \mathbb{R}) \cdot (X, \omega) \) is the golden eigenform locus. It follows from [6, Sect. 2], that the \( \text{GL}(2, \mathbb{R}) \) orbit closure of \((X, \omega, p)\) contains a marked translation surface \((Y, \eta, q)\), where \((Y, \eta)\) is a Veech surface and \( q \) is a regular Weierstrass point. By Lemma 4.4, \( q \) is not contained in any simple closed geodesic of \((Y, \eta)\), which again contradicts Corollary 6.2.

**Remark 7.1.** For this case, the theorem can also be shown by more direct arguments, using the fact that surfaces in eigenform loci are completely periodic in the sense of Calta.

7.4. **Case V**: \((X, \omega)\) is a Veech surface. Let us start by the following observation

**Lemma 7.2.** Let \((X, \omega)\) be a translation surface in one of the components \( H_{\text{hyp}}(2g - 2) \) or \( H_{\text{hyp}}(g - 1, g - 1) \). Let \( \gamma \) be saddle connection of \((X, \omega)\) which is invariant under the hyperelliptic involution. Assume that \( \gamma \) is contained in the boundary of a cylinder \( C \). Then \( \gamma \) is contained a simple cylinder \( D \) such that \( D \subset \overline{C} \), and the core curves of \( D \) cross \( C \) once.

*Proof.* We can assume that \( C \) is horizontal. Assume that \( \gamma \) is contained in the top boundary of \( C \). Since \((X, \omega)\) belongs to a hyperelliptic component, it is a well-known fact that \( C \) is invariant under the hyperelliptic involution (see for instance [20, 34]). This implies that \( \gamma \) is also contained in the bottom boundary of \( C \). Therefore, there are (infinitely many) simple closed geodesics in \( \overline{C} \) that join the midpoint of \( \gamma \) to itself and cross every core curve of \( C \) once. The cylinder corresponding to any such closed geodesic has the required properties. \( \square \)

We now show

**Proposition 7.3.** Let \((X, \omega)\) be a Veech surface in one of the components \( H_{\text{hyp}}(\kappa) \) where \( \kappa = (2g - 2) \) or \((g - 1, g - 1)\). Then every regular point is contained in a simple closed geodesic of \((X, \omega)\).

*Proof.*
Step 1. Let $w$ be a regular Weierstrass point, let $J$ be a geodesic segment of that realizes the distance from $w$ to the zero set of $\omega$. Then $\gamma_0 := J \cup \tau(J)$ is a saddle connection invariant by $\tau$. Since $(X, \omega)$ is a Veech surface, it is periodic in the direction of $\gamma_0$. Thus $\gamma_0$ is contained in the boundary of some cylinder $C_0$. By Lemma 7.2 there is a simple cylinder $D_0$ in $\overline{C_0}$ that contains $\gamma_0$. Let $\gamma_1^+, \gamma_1^-$ be the two saddle connections bordering $D_0$. Then the saddle connections $\gamma_0, \gamma_1^+, \gamma_1^-$ bound an embedded parallelogram $P_0$. By construction every the regular point in $P_0$ is contained in a simple closed geodesic.

Step 2. Note that the hyperelliptic involution exchanges $\gamma_1^+$ and $\gamma_1^-$. By construction, the complement of $D_0$ in $(X, \omega)$ is a translation surface bounded by $\gamma_1^+$ and $\gamma_1^-$. Identifying $\gamma_1^+$ with $\gamma_1^-$ on this surface via translation, we obtain a closed translation surface $(X_1, \omega_1)$ in some hyperelliptic component of lower dimension. Let $\gamma_1$ be the saddle connection corresponding to $\gamma_1^+$ and $\gamma_1^-$. Then $\gamma_1$ is invariant under the hyperelliptic involution of $(X_1, \omega_1)$. On the other hand, since $(X, \omega)$ is a Veech surface, it is periodic in the direction of $\gamma_1^\pm$. Therefore, $(X_1, \omega_1)$ is periodic in this direction. It follows that $\gamma_1$ is contained in the boundary of a cylinder $C_1$ of $(X_1, \omega_1)$. Again by Lemma 7.2, $\gamma_1$ is contained in a simple cylinder $D_1 \subset \overline{C_1}$ bounded by two saddle connections $\gamma_2^+, \gamma_2^-$. By construction every point in $\text{int}(\gamma_2^+) \cup \text{int}(\gamma_2^-)$ is contained in $C_1$.

In $X$, $D_1$ corresponds to an embedded parallelogram $P_1$ which is bounded by $\gamma_1^\pm$ and $\gamma_2^\pm$. By construction, every regular point in $\overline{P_1}$ is contained in a simple closed geodesic of $(X, \omega)$.

![Figure 3. Decomposition of $(X, \omega)$ into parallelograms.](image_url)
is contained in at least one simple closed geodesic. Hence every regular point on \((X, \omega)\) is contained in a simple closed geodesic. \qed

**Proof of Theorem 3** Case V. In this case, Theorem 3 is a direct consequence of Proposition 7.3 and Theorem 2. \qed

7.5. Case IV: \(\mathcal{L} = \mathcal{H}(0,0)\).

**Proof of Theorem 3 for Case IV.** Let \(\pi : (X, \omega; \Sigma) \to (\mathbb{T}; \{x_1, x_2\})\) be the branched covering map, where \(\Sigma\) is the set of zeros of \(\omega\). Let \(p \in X\) a regular point. If \(\pi(p) \notin \{x_1, x_2\}\), then the theorem for this case follows from Lemma 4.1 (1) and Lemma 4.3.

Assume now that \(\pi(p) \in \{x_1, x_2\}\), say \(\pi(p) = x_1\). Recall that the set of square-tiled surfaces is dense in \(\mathcal{L} = \mathcal{H}(0,0)\). Suppose to a contradiction that \(p\) is not contained in any simple closed geodesic of \((X, \omega)\). Let \((X', \omega') \in \mathcal{L}\) be a square-tiled surface and \(\pi' : (X', \omega'; \Sigma') \to (\mathbb{T}', \{x'_1, x'_2\})\) be a branched cover for some \((\mathbb{T}', \{x'_1, x'_2\}) \in \mathcal{H}(0,0)\).

Note that the condition \((X', \omega')\) is square-tiled means that \(x'_1 - x'_2\) is a torsion point of \(\mathbb{T}'\).

By assumption, there exist \(a_n \in \text{GL}(2, \mathbb{R})\), \(n \in \mathbb{N}\), such that \(a_n \cdot (X, \omega) \to (X', \omega')\) as \(n \to \infty\).

Let \((X_n, \omega_n, p_n) := a_n \cdot (X, \omega, p), (\mathbb{T}_n; \{x_{1n}, x_{2n}\}) := a_n \cdot (\mathbb{T}; \{x_1, x_2\})\), and \(\pi_n : (X_n, \omega_n; \Sigma_n) \to (\mathbb{T}_n; \{x_{1n}, x_{2n}\})\) the corresponding branched coverings. Then \((\mathbb{T}_n; \{x_{1n}, x_{2n}\}) \to (\mathbb{T}', \{x'_1, x'_2\})\) as \(n \to \infty\), since the map \(\mathcal{L} \to \mathcal{H}(0,0)\) is a continuous map. On the other hand, there exists some positive constant \(\delta\) such that any two points in \((\pi')^{-1}(\{x'_1, x'_2\})\) are of distance at least \(\delta\) away from each other with respect to the flat metric \(|\omega'|\). Therefore, there exists a constant \(L > 0\), such that for all \(n > L, p_n\) is of distance at least \(\delta/2\) away from any zero of \(\omega_n\) with respect to the flat metric \(|\omega_n|\), since \(p_n \in (\pi_n)^{-1}(\{x_{1n}, x_{2n}\})\). As a consequence, after passing to a subsequence if necessary, \((X_n, \omega_n, p_n) \to (X', \omega', p')\) as \(n \to \infty\), where \(p' \in (\pi')^{-1}(\{x'_1, x'_2\})\) is a regular point. By Lemma 4.4, \(p'\) is not contained any simple closed geodesic of \((X', \omega')\), which contradicts Theorem 3 for Case V, since \((X', \omega')\) is a Veech surface. The proof of Theorem 3 is now complete. \qed

**References**

[1] A. Avila, A. Eskin, and M. Möller: Symplectic and Isometric \(\text{SL}(2, \mathbb{R})\)-invariant subbundle of the Hodge bundle, J. Reine Angew. Math. (to appear), arXiv:1209.2854 (2012).

[2] D. Aulicino and D.-M. Nguyen: Rank two affine submanifolds in \(H(2,2)\) and \(H(3,1)\). Geom. Topol. 20 (2016), no. 5, 2837-2904.
[3] P. Apisa: GL(2, \mathbb{R}) Orbit Closures in Hyperelliptic Components of Strata, arXiv:1508.05438v2 [math.DS]
[4] P. Apisa: GL(2, \mathbb{R})-Invariant Measures in Marked Strata: Generic Marked Points, Earle-Kra for Strata, and Illumination, arXiv:1601.07894v4 [math.DS]
[5] P. Apisa: Rank one orbit closures in \mathcal{H}^{hyp}(g - 1, g - 1), arXiv:1710.05507v1 [math.DS]
[6] P. Apisa: Periodic Points in Genus Two: Holomorphic Sections over Hilbert Modular Varieties, Teichmuller Dynamics, and Billiards, arXiv:1710.05505v1 [math.DS]
[7] P. Apisa and A. Wright: Marked points on translation surfaces, arXiv:1708.03411v1 [math.DS]
[8] M. Boshernitzan, G. Galperin, T. Krüger, S. Troubetzkoy: Periodic billiard orbits are dense in rational polygons. Trans. Amer. Math. Soc. 350 (1998), no. 9, 3523-3535.
[9] K. Calta: Veech surfaces and complete periodicity in genus two. J. Amer. Math. Soc. 17 (2004), no. 4, 871-908.
[10] A. Eskin, S. Filip and A. Wright: The algebraic hull of the Kontsevich-Zorich cocycle, arXiv:1702.02074v1 [math.DS].
[11] A. Eskin and M. Mirzakhani: Invariant and stationary measures for the SL(2, \mathbb{R}) action on Moduli space, arXiv:1302.3320v4 [math.DS].
[12] A. Eskin, M. Mirzakhani and A. Mohammadi: Isolation, equidistribution, and orbit closures for the SL(2, \mathbb{R}) action on moduli space. Ann. of Math. (2) 182 (2015), no. 2, 673–721.
[13] A. Eskin, M. Mirzakhani, and K. Rafi: Counting closed geodesics in strata, preprint, arXiv:1206.5574
[14] S. Filip: Splitting mixed Hodge structures over affine invariant manifolds, Annals of Math. 183 (2016), no.2, 681-713.
[15] E. Gutkin, P. Hubert and T. Schmidt: Affine diffeomorphisms of translation surfaces: periodic points, Fuchsian groups and arithmeticity, Ann. Scient. Éc. Norm. Sup.(4), 36, (2003), no.6, 847-866.
[16] E. Gutkin and C. Judge: Affine mappings of translation surfaces: geometry and arithmetic. Duke Math. J. 103 (2000), no. 2, 191-213.
[17] U. Hamenstädter: Dynamics of the Teichmüller flow on compact invariant sets, J. Mod. Dyn. 4 (2010), no. 2, 393-418.
[18] R. Kenyon and J. Smillie: Billiards on rational-angles triangles, Comment. Math. Helv. 75 (2000), 65–108.
[19] A. Kumar and R. E. Mukamel: Real multiplication through explicit correspondences. LMS J. Comput. Math. 19 (2016), suppl. A, 29-42.
[20] M. Kontsevich and A. Zorich: Connected components of the moduli spaces of Abelian differentials with prescribed singularities. Invent. Math. 153 (2003), no. 3, 631-678.
[21] S. Lelievre and B. Weiss: Translation surfaces with no convex presentation. Geom. Funct. Anal. 25 (2015), no. 6, 1902-1936.
[22] S. Lelievre, T. Monteil, and B. Weiss: Everything is illuminated, Geometry and Topology 20 (2016), 1737-1762.
[23] H. Masur: Closed trajectories for quadratic differentials with an application to billiards. Duke Math. J. 53 (1986), no. 2, 307-314.
[24] H. Masur and S. Tabachnikov: Rational billiards and flat structures. Handbook of dynamical systems, Vol. 1A, 1015-1089, North-Holland, Amsterdam, 2002.
[25] C.T. McMullen: Teichmüller curves in genus two: Discriminant and spin, Math. Annalen 333 (2005), 87-130.
[26] C. T. McMullen: Teichmüller curves in genus two: torsion divisors and ratios of sines. Invent. Math. 165 (2006), no. 3, 651-672.
[27] C. T. McMullen: Dynamics of SL_2\mathbb{R} over the moduli space in genus two, Ann. of Math. 165 (2007), 397-456.
[28] M. Mirzakhani and A. Wright: The boundary of an affine invariant submanifold, Invent. Math. 209 (2017), no. 3, 927-984.
[29] M. Mirzakhani and A. Wright: Full rank affine invariant submanifolds, Duke Math. Journal (to appear), arXiv:1608.02147
[30] M. Möller: Periodic points on Veech surfaces and the Mordell-Weil group over a Teichmüller curve, Invent. Math. 165, 633-649 (2006).
[31] M. Möller: Affine groups of flat surfaces. Handbook of Teichmüller theory. Vol. II, 369-387, IRMA Lect. Math. Theor. Phys., 13, Eur. Math. Soc., Zürich, 2009.
[32] C. C. Moore: The Mautner phenomenon for general unitary representations, Pacific J. Math., vol. 86 (1980), no.1, 155-169.
[33] D.-M., Nguyen: Parallelogram decompositions and generic surfaces in \( \mathcal{H}(4) \), Geom. Topol. 15 (2011), no. 3, 1707-1747.
[34] D.-M. Nguyen and A. Wright: Non-Veech surfaces in \( \mathcal{H}(4) \) are generic. Geom. Funct. Anal. 24 (2014), no. 4, 1316-1335.
[35] J. Smillie and B. Weiss: Finiteness results for flat surfaces: a survey and problem list. Partially hyperbolic dynamics, laminations, and Teichmüller flow, 125-137, Fields Inst. Commun., 51, Amer. Math. Soc., Providence, RI, 2007.
[36] J. Smillie and B. Weiss: Minimal sets for flows on moduli space. Israel J. Math. 142 (2004), 249-260.
[37] K. Strebel: Quadratic differentials. Springer-Verlag, Berlin, 1984.
[38] W. A. Veech: Geometric realizations of hyperelliptic curves. Algorithms, fractals, and dynamics (Okayama/Kyoto, 1992), 217-226, Plenum, New York, 1995.
[39] W. A. Veech: Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. Invent. Math. 97 (1989), no. 3, 553-583.
[40] Y. Vorobets: Periodic geodesics on translation surfaces, arXiv:math/0307249 [math.DS]
[41] Y. Vorobets, Periodic geodesics on generic translation surfaces, Algebraic and topological dynamics, 205-258, Contemp. Math., 385, Amer. Math. Soc., Providence, RI, 2005.
[42] A. Wright: The field of definition of affine invariant submanifolds of the moduli space of abelian differentials. Geom. Topol. 18 (2014), no. 3, 1323-1341.
[43] A. Wright: Cylinder deformations in orbit closures of translation surfaces, Geometry & Topology 19 (2015), 413-438.
[44] R. Zimmer: Ergodic theory and Semisimple groups, Monographs in Mathematics, Birkhäuser, Boston, 1984.
[45] A. Zorich: Square tiled surfaces and Teichmüller volumes of the moduli spaces of abelian differentials. Rigidity in dynamics and geometry (Cambridge, 2000), 459-471, Springer, Berlin (2002).
[46] A. Zorich: Flat surfaces, Frontiers in number theory, physics, and geometry, 437-583, Springer, Berlin (2006).

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