Short Distance Operator Product Expansion of the 1D, $\mathcal{N} = 4$ Extended $\mathcal{GR}$ Super Virasoro Algebra by Use of Coadjoint Representations

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ABSTRACT

Using the previous construction of the geometrical representation ($\mathcal{GR}$) of the centerless 1D, $\mathcal{N} = 4$ extended Super Virasoro algebra, we construct the corresponding Short Distance Operation Product Expansions for the deformed version of the algebra. This algebra differs from the regular algebra by the addition of terms containing the Levi-Civita tensor. How this addition changes the super-commutation relations and affects the Short Distance Operation Product Expansions (OPEs) of the associated fields is investigated. The Method of Coadjoint Orbits, which removes the need first to find Lagrangians invariant under the action of the symmetries, is used to calculate the expansions. Finally, an alternative method involving Clifford algebras is investigated for comparison.

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1 Introduction

One of the fundamental mathematical objects of String Theory (ST) is the Virasoro algebra. It is used in the description of simple open/closed strings and is well-developed in Conformal Field Theory (CFT), a primary tool for probing strings. A familiar technique from CFT commonly used in this context is the Operator Product Expansion (OPE) as it is closely related to the calculation of two-point correlation functions which themselves are related to the propagation and interaction of fields represented in ST.

In many discussions, the beginning of such constructions involves first finding an action (containing appropriate fields) that is invariant under a realization of the (super)conformal symmetry group. The solutions of the fields equations of motion are expanded in terms of Fourier series. The Noether charges associated with the generators are, using their expressions in terms of the fields, also then expressed in terms of such Fourier series. Finally OPE’s are then calculated. Clearly the role of the action is prominent, both in determining the Noether Charges and the field equations of motion.

Instead the method to be used in this work for calculating these OPE’s is the Coadjoint Orbit method developed by Kirillov [1] and built upon the elements of Lie algebras and their realizations. One goes from the closed algebra of operators to elements of a vector space. This vector space is then expanded by the addition of a dual space of covectors and a bilinear metric between the two. These objects are then used to find the coadjoint orbits which can be used in the OPE.

In this paper, the algebra used is the extended $\mathcal{GR}$ Super Virasoro Algebra$^3$ ($\mathcal{GR}$ SVA), a much larger algebra than the one related to the Virasoro algebra. The Virasoro algebra in this case contains familiar time-space operators from the Poincare algebra and the conformal algebra. The extended $\mathcal{GR}$ Super Virasoro algebra has many parts. It is the Super Virasoro algebra because of the inclusion of supersymmetric elements and thus an enlarged symmetry. $\mathcal{GR}$ stands for “Geometrical Representation” which means that methods involve groups and algebras acting on the operators described in a representation as derivations on the superspace. It is extended because it can be considered on the basis of the coordinates of $\mathcal{N}$-extended superspace. In the special case of $\mathcal{N} = 4$, there is a deformation due to the addition of certain terms in the derivations as a function of a variable $\ell$. These terms change the number of operators in the algebra for certain values of $\ell$.

$^3$It is perhaps more accurate to describe our starting point as a ‘Witt algebra’ as it is ‘centerless’ and constructed from the vector fields associated with superspace coordinates.
The calculations to be done in this work are in the context of one temporal dimension (1D) and four fermionic or Grassmann dimensions ($\mathcal{N} = 4$). There is a relationship between 4D, $\mathcal{N} = 1$ theories and 1D, $\mathcal{N} = 4$ theories but that will not be discussed in this paper.

The outline of the paper is as follows. The first section will describe the extended $\mathcal{GR}$ Super Virasoro algebra with a focus on under what conditions does it close and the corresponding number of operators. The next section will explain how the Coadjoint Orbit method is used and its application on the algebra. The third section brings in the OPE and it will be used to calculate various short distance Operator Product Expansions for the algebra. In Section 5, a different look at the whole method from the use of Clifford algebras as an alternative to derivations. Finally, there will be a discussion of some of the implications of the results of the paper.

2 Realization of the 1D, $\mathcal{N} = 4$ Extended GR Super Virasoro Algebra

One can get to the Super Virasoro algebra by starting with the $SO(\mathcal{N})$ algebra using $T_{IJ}$ as generators. Adding translations generated by momentum generators $P$, the dilations, $\Delta$, special conformal transformations, $K$, the supersymmetry generators $Q_I$, and finally and $S_I$, make up the superconformal algebra. The total algebra can be represented as derivations with respect to the superspace. It has some peculiar properties that still remain a mysterious. For $\mathcal{N} \leq 4$, it is known how to close this algebra without additional generators. It is clear however, that for $\mathcal{N} > 4$ closure requires the presence of additional operators. In any event, the operators can be represented by derivations of the one-dimensional time variable and its derivative, $\tau$ and $\partial_\tau$, and the $\mathcal{N} = 4$ superspace variables and their derivatives, $\zeta^I$ and $\partial_\zeta$. The time variable and its derivative are real and commute with everything. The superspace coordinates are real Grassmann variables, anti-commuting ($\zeta^I \zeta^J = -\zeta^J \zeta^I$) and squaring to zero ($[\zeta^I]^2 = 0$). The algebra is defined by its commutation relations. There are 36 possible combinations but only thirteen are nonzero:

\begin{align}
[\Delta, P] &= -iP \\
[\Delta, Q_I] &= -i\frac{1}{2}Q_I \\
[\Delta, S_I] &= i\frac{1}{2}S_I \\
[Q_I, Q_J] &= 4\delta_{IJ}P \\
[S_I, S_J] &= 4\delta_{IJ}K \\
[P, K] &= -i2\Delta
\end{align} (2.1)
\[
\begin{align*}
\{Q_I, S_j\} & = 4\delta_{ij}\Delta + 2T_{ij}, \quad (2.4) \\
\{T_{ij}, Q_K\} & = -i\delta_{IK}Q_j + i\delta_{JK}Q_i, \quad (2.5) \\
\{T_{ij}, S_K\} & = -i\delta_{IK}S_j + i\delta_{JK}S_i, \quad (2.6) \\
\{T_{ij}, T_{KL}\} & = i\delta_{JK}T_{iL} - i\delta_{jL}T_{iK} + i\delta_{iL}T_{jK} + i\delta_{iK}T_{jL}, \quad (2.7)
\end{align*}
\]

The generators and their corresponding symmetries are listed in Table I.

| Generators | Symmetry      | Derivation                                      | No. of generators |
|------------|---------------|------------------------------------------------|------------------|
| \(P\)     | Translations  | \(i\partial_\tau\)                             | 1                |
| \(\Delta\) | Dilations     | \(i(\tau\partial_\tau + \frac{1}{2}\zeta^4\partial_4)\) | 1                |
| \(K\)     | Special Conformal | \(i(\tau^2\partial_\tau + \tau\zeta^4\partial_4)\) | 1                |
| \(Q_I\)   | Supersymmetry | \(i(\delta_4 - i2\zeta\partial_\tau)\)          | \(4 = [N]\)     |
| \(S_i\)   | S-supersymmetry | \(i\tau\partial_\tau + 2\zeta\partial_\tau + \zeta^2\partial_4\) | 4 = \([N]\)     |
| \(T_{ij}\) | \(\text{SO}(N)\) | \(i(\zeta^i\partial_j - \zeta^j\partial_i)\) | \(6 = [N(N-1)/2]\) |

Table 1

This algebra can be deformed in \(\mathcal{N} = 4\) with the addition of a Levi-Civita tensor, \(\epsilon_{ijkl}\), and a parameter, \(\ell\), that measures the deformation. It only affects three of the six operators:

\[
\begin{align*}
S_1(\ell) & \equiv i\tau\partial_\tau + 2\zeta_4\partial_\tau + 2\zeta_4\zeta^4\partial_4 + \ell\epsilon_{ijkl}(\zeta^i\partial_j - \zeta^j\partial_i - \frac{1}{3}\zeta^i\zeta^k\partial_k) \quad (2.8) \\
K(\ell) & \equiv i(\tau^2\partial_\tau + \tau\zeta^4\partial_4 - i2\ell\epsilon_{ijkl}[\frac{1}{2}\zeta^i\zeta^k\partial_l + \zeta^j\zeta^k\zeta_l\partial_\tau]) \quad (2.9) \\
T_{ij}(\ell) & \equiv i\zeta^i\partial_j - i\ell\epsilon_{ijkl}\zeta^k\partial_l \quad (2.10)
\end{align*}
\]

This changes the last three of the commutation relations

\[
\begin{align*}
\{T_{ij}, Q_K\} & = -i\delta_{IK}Q_j + i\delta_{JK}Q_i + i\ell\epsilon_{ijkl}Q_l \quad (2.11) \\
\{T_{ij}, S_K\} & = -i\delta_{IK}S_j + i\delta_{JK}S_i + i\ell\epsilon_{ijkl}S_l \quad , \quad (2.12) \\
\{T_{ij}, T_{KL}\} & = \frac{1}{2}(\ell^2 + 3)[i\delta_{JK}T_{iL} - i\delta_{jL}T_{iK} + i\delta_{iL}T_{jK} + i\delta_{iK}T_{jL}] \\
& \quad + \frac{1}{2}(\ell^2 - 1)[i\delta_{IK}Y_{iL} - i\delta_{jL}Y_{iK} + i\delta_{iL}Y_{jK} + i\delta_{iK}Y_{jL}] \quad (2.13)
\end{align*}
\]

with \(Y_{ij} \equiv i\zeta^i\partial_j + i\ell\epsilon_{ijkl}\zeta^k\partial_l\). For \(\ell = 1\), there are no \(Y_{ij}\) terms in the last commutation relation.

The next step is to recast the previous generators in form in which the relationship to the super Virasoro algebra is more obvious. This is done by choosing the forms

\[
\begin{align*}
L_m & \equiv -[\tau^{m+1}\partial_\tau + \frac{1}{2}(m + 1)\tau^m\zeta\partial_\zeta], \quad H_r \equiv -[\tau^{r+1}\partial_\tau + \frac{1}{2}(r + 1)\tau^r\zeta\partial_\zeta] \quad (2.14) \\
F_m & \equiv i\tau^{m+\frac{3}{2}}[\partial_\zeta - i2\zeta\partial_\tau], \quad G_r \equiv i\tau^{r+\frac{3}{2}}[\partial_\zeta - i2\zeta\partial_\tau] \quad (2.15)
\end{align*}
\]
where \( m \in \mathbb{Z} \) and \( r \in \mathbb{Z} + \frac{1}{2} \). The \( L \) and \( H \) are the same except \( L \) takes integers and \( H \) takes half integers. The \( F \) and \( G \) forms follow the same pattern. \( H \) is fermionic and \( L \) is bosonic because \( L \) exists in the \( \mathcal{N} = 0 \) case. If one looks at the lowest level of the set of \((L, H, F, G)\) generators, some of the previous generators are now represented:

\[
P \to L_{-1}, \Delta \to L_0, K \to L_{+1}, Q \to G_{-\frac{1}{2}}, S \to G_{+\frac{1}{2}}
\]

The \( T_{1,1} \) generators remain the same.

These new generator pairs can be combined using a different notation with simple commutation relations:

\[
\begin{align*}
(L_A, H_r) &\to [L_A, L_B] = (A - B) L_{A+B} \\
(G_A, G_r) &\to [G_A, G_B] = -i 4 L_{A+B} \\
[L_A, G_B] &\to (\frac{4}{2} A - B) G_{A+B}
\end{align*}
\]

with \( A, B \) taking values in \( \mathbb{Z} \) and \( \mathbb{Z} + \frac{1}{2} \). For \( \mathcal{N} = 1 \), this pair of generators is closed under graded commutation. In the 1D \( \mathcal{N} = 4 \) exceptional Super Virasoro algebra, an index \( I \) for the supersymmetric levels has to be added and the \( \ell \)-deformed terms must be put in properly, including a \( \ell \)-deformed supersymmetric \( T_{1,1}(\ell) \) generator. For the 1D \( \mathcal{N} = 4 \) exceptional Super Virasoro algebra, the set of generators \((L_A(\ell), G_A^{I}(\ell), T_{A}^{I,J}(\ell))\) closes under graded commutation. These generators are

\[
L_A \equiv -[\tau A^{\ell} t_r + \frac{1}{2}(A + 1)^{\ell} \tau A \zeta_1 t_1] + i\ell A (A + 1)^{\ell - 1} c^\ell \zeta^{(4)} t_r
\]

\[
G_A^I \equiv \tau A^\ell \zeta^{(1)} t_r - i 2 c^I \zeta^{(1)} t_1 + 2(A + \frac{1}{2}) \tau A^\ell \zeta^{(1)} \zeta^K t_K
\]

\[
T_A^{I,J} \equiv \tau A^\ell \zeta^{(1)} t_r - c^I \zeta^K t_K - i 2 c A \tau A^\ell \zeta^{(1)} t_r - c^I \zeta^K t_L
\]

Their supercommutation relations are

\[
\begin{align*}
[L_A, L_B] &\equiv (A - B) L_{A+B} + \frac{1}{8} c (A^3 - A) \delta_{A+B,0} \\
[L_A, G_B^I] &\equiv \frac{4}{2} - B G_{A+B}^I \\
[L_A, T_B^{I,J}] &\equiv -B T_{A+B}^{I,J} \\
\{G_A^I, G_B^K\} &\equiv -i 4 \delta^{IJ} L_{A+B} - i 2 (A - B) T_{A+B}^{I,J} - i c (A^2 - \frac{1}{2}) \delta_{A+B,0} \delta^{I,J} \\
[T_A^{I,J}, G_{B}^K] &\equiv 2(\delta^{JK} G_{A+B} - \delta^{IK} G_{A+B}^J) \\
[T_A^{I,J}, T_B^{K,L}] &\equiv T_{A+B}^{I,K} \delta^{J,L} - T_{A+B}^{I,L} \delta^{J,K} + T_{A+B}^{I,K} \delta^{J,L} - T_{A+B}^{I,L} \delta^{J,K} - 2c(A - B) (\delta^{I[K],L])
\end{align*}
\]
A number of interesting points can be found here. In previous papers [2] [3], the non-deformed ($\ell = 0$) 1D $\mathcal{N} = 4$ GR Super Virasoro algebra is used to generate OPEs. This algebra is the “large” $\mathcal{N} = 4$ algebra which has a 16-dimensional representation. It does not close unless two more sets of generators (U’s and R’s, which are related to the T’s,) are added. The $\ell = \pm 1$ cases of the $\ell$-extended algebra map the generators to a 8-dimensional representation which does not need the other generators to close. This can be easily seen when instead of using derivations to represent the generators, an appropriately sized Clifford algebra is used [4]. The use of a Clifford algebra may allow more insight into the whole process. This and the difference between using the “small” and the “large” $\mathcal{N} = 4$ algebras will be discussed in Section 5.

Another point is whether the central extension should be dropped in the equations. From [5], the closure of the algebra is found to be related to the existence of a central extension, specifically the central extension is eliminated for $\mathcal{N} > 2$. Because $\mathcal{N} = 4$ closes also, it is a valid question to ask if a central extension may exist too. The Jacobi Identity on $(G^I_A, U^I_J, G^K_F)$ was used before to answer this question. Because the supercommutators have the same form as the $\mathcal{N} > 2$, it would seem that the answer would be true. But there are no longer $U^I_J$ generators in the algebra. The Jacobi identity for the other generators must be analyzed to check if a central extension is allowed. Although this could be addressed now, this question will be revisited later when the Clifford representation of the generators is presented. For now, $c$ will be set to zero.

### 3 Description of the Coadjoint Orbit Method

A compact description of the Coadjoint Orbit method can be found in a paper by Witten[6] but to go into more detail and understanding, the work of Kirillov [1] provides more insight. To fully understand the Coadjoint Method, one must go to its foundation in Lie groups and algebras then build from there. A Lie group $G$ is a set of elements with certain topological and algebraic properties, namely continuity and analyticity. One can think of it as both a group of elements and a smooth manifold. It has a multiplication law which can be represented as a smooth map. The group can act on itself and there is a special map defined for every point in the group:

$$A_g(h) \equiv h \rightarrow ghg^{-1} : \forall g, h \in G$$

(3.1)

Related to the Lie group is its Lie algebra $\mathfrak{g}$, a vector space that can be understood as the tangent space of the manifold at the unit point in the group, denoted by $e$
The unit point is a fixed point of the previous map, meaning that it is mapped into itself. Around this fixed point, the derivative of \( A_g \) acts to map elements of the Lie group to other elements in the same Lie group. This derivative is called the adjoint map of the Lie group:

\[
Ad_g : g \rightarrow g' \text{ with } g, g' \in G
\]  

(3.2)

Since elements of the Lie algebra \( \mathfrak{g} \) changes elements of the group to other elements, we find that the previous map can be considered a mapping of elements of the Lie algebra to other elements of the algebra. Thus the map from \( g \) to \( Ad(g) \) can also be seen as

\[
Ad : (g \rightarrow Ad(g)) \simeq (g \rightarrow g'), g \in G, g, g' \in G
\]  

(3.3)

This is the adjoint representation of the Lie group \( G \). By taking the derivative of this map, one gets the adjoint representation of the Lie algebra \( \mathfrak{g} \), which has the following property:

\[
ad_g(h) = [g, h], g, h \in G
\]  

(3.4)

where the right-hand side is the Lie bracket defined for the Lie algebra.

Since \( G \) is also a vector space, we can talk about the dual linear space \( G^* \). The dual space \( G^* \) consists of dual elements \( g^* \) of elements \( g \) in \( G \). The dual elements belong to the space of linear functions of the algebra element \( g \). With the definition of a bilinear form on both types of elements \( \langle g^*, g \rangle \), there also exists the space \( G^\perp \) of elements \( g^\perp \) orthogonal to the element \( g \) defined by the bilinear form, \( \langle g^\perp, g \rangle = 0 \). Let \( P \) be a projection operator that projects into \( G^\perp \). Then one can construct a coadjoint representation \( K(g) \) that sends elements of the dual space into a space of other elements orthogonal to the first:

\[
K(g) = \{ h \rightarrow g^\perp \equiv P(ghg^{-1}) \text{ such that } \langle g^\perp, g \rangle = 0, h \in G^*, g^\perp \in G^\perp, g \in G \}. \]  

(3.5)

Once one has a realization of the appropriate algebra, the coadjoint orbit method can be applied. First, an adjoint vector consisting of all the generators and a central extension is constructed. From there, a corresponding coadjoint vector can be formed and calculated using the ideas that

1. an adjoint vector can act on another adjoint vector to give an adjoint vector, and

2. an inner product of an adjoint vector and its dual coadjoint vector should be “orthogonal” in the sense that it gives delta functions in indices.
Once the action of the adjoint vector is understood on the different elements (which is equivalent to the first statement,) then the action of the adjoint vector on an arbitrary coadjoint vector can be calculated. This now defines how the fields in the coadjoint vector transform with respect to the elements of the adjoint vector which are related to the underlying algebra. The coadjoint orbit is the space of all coadjoint vectors that can be reached by application of the action of the algebra.

Now the relationship of the adjoint and coadjoint elements to sympletic structures can be utilized. There is a relationship between coadjoint orbits and symplectic structures. An orbit of a map is like an equivalence class of the map. The coadjoint orbit is the equivalence class of dual linear functions on the Lie group. Having a symplectic structure means that there exists a closed non-degenerate, skew-symmetric differential 2-form. This 2-form is $G$-invariant and exists for each orbit in $g^*$. Having a symplectic structure is also related to Poisson brackets and phase space.

The infinitesimal version of the coadjoint action is

$$\langle K(g)h, g' \rangle = \langle h, -ad_g(g') \rangle = \langle h, [g, g'] \rangle, g, g' \in G, h \in G^*.$$  \hspace{1cm} (3.6)

This is equivalent to the natural skew symmetric bilinear form, $\Omega$, found on each coadjoint orbit. The form $\Omega$ is defined on adjoint elements as

$$\Omega_B(\tilde{B}_1, \tilde{B}_2) = \langle \tilde{B}, [a_1, a_2] \rangle$$  \hspace{1cm} (3.7)

with $a_1$ and $a_2$ as associated fields from the adjoint vector and $\tilde{B}$ the coadjoint vector. The change in $\tilde{B}$ from the specific adjoint fields is given by

$$\delta_{a_i} \tilde{B} = a_i \ast \tilde{B}.$$  \hspace{1cm} (3.8)

So the inner product of a adjoint and coadjoint element also generates the infinitesimal variation of the corresponding coadjoint field with respect to the symmetry generated by the adjoint element.

To get to the calculation of the Operator Product Expansions, one more step must be done. The physical fields and the conjugate momentum must be associated with the adjoint and coadjoint elements.

As the adjoint element generates a symmetry transformation of some kind, associated with that transformation is a charge, $Q_i$. For the adjoint field $a_i$, the charge can be calculated as

$$a_i \to Q_{a_i} = \int d\tau G^i a_i$$  \hspace{1cm} (3.9)
$G^i$ is the generating function for the transformation and comes from the action of the adjoint element on the coadjoint vector represented by the associated fields.

Using some concepts from mechanics, one can see that this charge generates the infinitesimal variation of a function of a field, $f_i$, and its conjugate field, $\pi^i$, through the use of Poisson Bracket:

$$\{Q_{a_i}, F(f_i, \pi^i)\} = \frac{\partial Q}{\partial f_i} \frac{\partial F}{\partial \pi^i} - \frac{\partial Q}{\partial \pi^i} \frac{\partial F}{\partial f_i} = -\delta_{a_i} F(f_i, \pi^i) \quad (3.10)$$

There are three fields in this equation: the adjoint field related to the transformation, $a_i$; a coadjoint field, $f_i$; and the conjugate momentum field to $f_i$, $\pi^i$.

Defining $A$ to be the dual coadjoint field to adjoint field $a_i$, then

$$(a_i, A) = a_i * A = \int a_i(x) A(x) \, dx = \text{const.} \quad (3.11)$$

And it can also be shown that

$$\{Q_{a_i}, F(x)\} = a_i(x) \frac{\delta}{\delta a_i(x)} F(x) = \int dy a_i(y) [A(y) F(x)] \quad (3.12)$$

The quantity in brackets on the RHS gives the short distance OPE between $A$, the dual coadjoint element of $\alpha$, and $F$, a function of the phase space elements. By using this equation, the adjoint action on coadjoint elements can be mapped to the infinitesimal variation of the dual fields by the symmetries generated by the algebra elements. The OPE can almost be read off from the resulting equation.

The actual use of the method flows from the following steps:

1. Choose an coadjoint field and an adjoint action on it. This gives the variation of the physical field with respect to some transformation.

2. Calculate the Poisson bracket of the charge generated by the adjoint action on the physical field. The generating function of the transformation will come from the calculations of the adjoint action on the coadjoint vector done earlier.

3. Compare to the integral form of Poisson bracket. The short distance OPE will be the equivalent expression of the previous step once it has been put in the associated integral form. This will involve the use of delta functions on the space (a line in the 1D case) and its derivatives.

Typically, one needs an action to determine the useful field theory quantities such as correlation functions. However, these quantities are dependent on the symmetries...
found in the theory and not necessarily obvious in the action. The Coadjoint Orbit method allows for these quantities to be calculated without an action and totally based on the underlying symmetries of the theory being studied.

As an aside, one of the uses of coadjoint orbits is relate the classification of the orbits to the classification of another related mathematical structure. For example, if G is the set of all linear $n \times n$ real invertible matrices, then the classification of coadjoint orbits is equivalent to the classification of matrices up to similarity. The analysis of the coadjoint orbits allows one to classify two dimensional conformal field theories (2D-CFT’s).

## 4 Calculation of Short Distance Operator Product Expansions

The Operation Product Expansion (OPE) is an expression of the product of two operators as a sum of singular functions of other operators. This is useful when calculating the product of field operators at the same point. Wilson and Zimmerman [8] have a discussion of the use of OPEs in Quantum Field Theory. In this case, the operators are tensor fields. The general form of an OPE is

$$A(y)B(x) \sim \sum_i C_i(x)(y-x)^{-i} + \text{(non singular terms)}$$

(4.1)

where $C_i$ is a member of a complete set of operators. The non-singular terms are not important because the singular terms determine the properties of the product of operators.

The goal is to express the product of fields that represent the underlying algebra in terms of functions of other fields which represent other elements in the algebra. These products are further related to useful field theory quantities such as propagators and mass terms.

The methods used are found in [2, 3, 5, 7]. Applying this process to the algebra of interest, the adjoint vector of the 1D $\mathcal{N} = 4$ $\mathcal{GR}$ SVA is $L = (L_A, G_B^I, T_C^{JK})$. The adjoint acting on this gives

$$\text{ad}((L_M, G_N^K, T_P^{LM}))(L_A, G_B^I, T_C^{JK}) = (L_M, G_N^K, T_P^{LM}) * (L_A, G_B^I, T_C^{JK})$$

$$= (L_{Q,new}, G_{R,new}^H, T_{S,new}^{FG})$$

(4.2)

The coadjoint element is $\tilde{L} = (\tilde{L}_A, \tilde{G}_B^I, \tilde{T}_C^{JK})$ and correspondingly gives

$$\text{ad}((L_M, G_N^K, T_P^{LM}))(\tilde{L}_A, \tilde{G}_B^I, \tilde{T}_C^{JK}) = (L_M, G_N^K, T_P^{LM}) * (\tilde{L}_A, \tilde{G}_B^I, \tilde{T}_C^{JK})$$

$$= (\tilde{L}_{Q,new}, \tilde{G}_{R,new}^H, \tilde{T}_{S,new}^{FG})$$

(4.3)
and the inner product is
\[
\langle (\tilde{L}_M, \tilde{G}_N^L, \tilde{T}_P^{LM}) | (L_A, G_B^I, T_C^{JK}) \rangle = \delta_{M,A} + \delta_{N,B} \delta_{K}^{I} + \delta_{P,C} \delta_{LM}^{JK}
\] (4.4)

To calculate the OPEs, one needs the expression of \( \delta L \tilde{L} = L \ast \tilde{L} \) where \( L \) is an adjoint vector and \( \tilde{L} \) is a coadjoint vector. Using the fact that \( \langle \tilde{L} | L \rangle \) is an invariant and \( L \ast \tilde{L} \) can be calculated from \( \langle L' \ast \tilde{L} | L \rangle \), one can use the Leibnitz rule on the invariant form and get
\[
\langle L \ast \tilde{L} | L \rangle = -\langle \tilde{L} | L \ast L \rangle
\] (4.5)

Since \( L \) and \( \tilde{L} \) are made up of components \( (L, G, T) \), it is easier to calculate pairs of adjoint elements acting on coadjoint elements. This reduces the number of calculations greatly. The list of adjoint/coadjoint pairs are
\[
\begin{align*}
\delta \tilde{L} &= L \ast \tilde{L} + G \ast \tilde{G} + T \ast \tilde{T} \\
\delta \tilde{G} &= L \ast \tilde{G} + G \ast \tilde{L} + G \ast \tilde{T} + T \ast \tilde{G} \\
\delta \tilde{T} &= L \ast \tilde{T} + G \ast \tilde{G} + T \ast \tilde{T}
\end{align*}
\]

This checks against the calculations from [3]. Note that there is no \( \tilde{T} \ast L \) term in the list of changes to the coadjoint vector.

Using a realization of the algebra as tensor fields, the adjoint representation elements are \( F = (\eta, \chi^I, t^{RS}) \), which are general elements of the Virasoro, Kac-Moody, and so(4) algebras respectively. The coadjoint fields are \( B = (D, \psi^I, A^{RS}) \), a rank two pseudo tensor, a set of 4 spin-3/2 fields, and the 6 so(4) gauge fields.

The coadjoint action can be seen as generating the changes in the fields. It acts as
\[
F \ast \tilde{B} = \delta_F \tilde{B} = (\eta, \chi^I, t^{KL}) \ast (D, \psi^I, A^{JK}) = (\delta D, \delta \psi^I, \delta A^{JK}).
\] (4.6)

There are three charges, one for each adjoint element/operator:
\[
\begin{align*}
L_A &\rightarrow \eta \rightarrow \langle Q_\eta \rangle = \int dx \, G_a \eta^a \\
G_A^I &\rightarrow \chi^I \rightarrow \langle Q_{\chi^I} \rangle = \int dx \, G_a (\chi^I)^a \\
T_A^{IJ} &\rightarrow t^{IJ} \rightarrow \langle Q_{t^{IJ}} \rangle = \int dx \, G_a (t^{IJ})^a
\end{align*}
\] (4.7)(4.8)(4.9)

Choosing \( L \ast \tilde{L} \) as an example, the physical field representation is used:
\[
L \ast \tilde{L} \rightarrow \delta_\eta D
\] (4.10)
\[ L_\eta \ast \tilde{L}_D \to \tilde{L}_D : \tilde{D} = -D' \eta - 2D\eta' \] (4.11)

\[ \delta_\eta D = \tilde{D} \] (4.12)

\[ \delta_\eta D = -\{Q_\eta, D\} = \int dy \eta(y)(D(y)D(x)) \] (4.13)

\[ Q_\eta = \int dxG^a_\eta = \int dx(-D' \eta - 2D\eta')\eta_a \] (4.14)

\[ \{Q_\eta, D\} = \int dy(-D'(x)\eta(y) - 2D(x)\eta'(y)). \] (4.15)

Using the 1D formula for the delta function,

\[ \delta(y - x) = \frac{1}{2\pi i(y - x)} \] (4.16)

and integration by parts to separate out \( \eta(x) \) terms,

\[ \{Q_\eta, D\} = \int dy \left( \frac{1}{2\pi i(y - x)} \frac{-1}{\pi i(y - x)} + D(x) \frac{-1}{2\pi i(y - x)} \right) \eta(x) \]

\[ = \int dy \left[ D(y)D(x) \right] \eta(x) \] (4.17)

Thus by taking pairs of individual adjoint elements acting on individual coadjoint elements, the OPE’s can found.

1. \( D(y)O(x) \)

\[ L_\eta \ast \tilde{L}_D = \tilde{L}_D \to \tilde{D} = -D' \eta - 2D\eta' \] (4.18)

\[ L_\eta \ast \tilde{\chi}_Q = \tilde{\chi}_Q \to \tilde{\psi}Q = -(\frac{\chi}{2} \eta' \psi'Q - \eta(\psi'Q')) \] (4.19)

\[ L_\eta \ast \tilde{\chi}^{RS} = \tilde{\chi}^{RS} \to \tilde{\psi}^{RS} = -(A^{RS})' \eta - \eta' A^{RS} \] (4.20)

These expressions yield the following OPEs:

\[ D(y)D(x) = \frac{-1}{\pi i(y - x)^2} D(x) - \frac{1}{\pi i(y - x)} \partial_x D(x) \] (4.21)

\[ D(y)\psi^Q(x) = -\frac{3}{4\pi i(y - x)^2} \psi^Q(x) - \frac{1}{2\pi i(y - x)} \partial_x \psi^Q(x) \] (4.22)

\[ D(y)A^{RS}(x) = \frac{-1}{2\pi i(y - x)^2} A^{RS}(x) - \frac{1}{2\pi i(y - x)} \partial_x A^{RS}(x) \] (4.23)

2. \( \psi(y)O(x) \)

\[ G^{I}_\chi \ast \tilde{L}_D = 4i\tilde{G}^{I}_\chi \to \tilde{\chi}^I = -\chi^I D \] (4.24)

\[ G^{I}_\chi \ast \tilde{\psi}^Q = \frac{\delta^Q}{2} \tilde{\chi}^I \to \tilde{\psi}Q = 2((\chi^I)R^S + \chi^I (t^{RS})') \] (4.25)

\[ G^{I}_\chi \ast \tilde{\psi}^{RS} = \delta^{[RS][IQ]} \tilde{\psi}^{Q} \to \psi^Q = 2((\chi^I)t^{RS} + \chi^I (t^{RS})') \] (4.26)
The OPEs are
\[
\begin{align*}
\psi^I(y)D(x) &= \frac{3}{4\pi(y-x)^2} \psi^I(x) - \frac{i}{4\pi(y-x)} \partial_x \psi^I(x) \quad (4.27) \\
\psi^I(y)\psi^Q(x) &= -\frac{3i}{4(y-x)} \delta^QD(x) \quad (4.28) \\
\psi^A(y)A^{RS}(x) &= \frac{\pi}{4(y-x)} (\delta^{AR}\delta^{L_S} - \delta^{AS}\delta^{L_R}) \psi^L(x) \quad (4.29)
\end{align*}
\]

3. \(A(y)O(x)\)
\[
\begin{align*}
T^I_{tji} \ast \bar{G}^Q_{\psi_Q} &= 2\delta^QI G^I_{\psi_I} - 2\delta^QJ G^I_{\psi_J} \rightarrow \psi^Q = t^{IJ} \psi^Q \\
T^I_{tji} \ast \bar{T}^{RS}_{\tau RS} &= -\delta^{[RS]}(\delta^{JK}_{RS} + \delta^{KJ}_{RS}) \bar{T}^{[JK]}_{(tJ)_\tau RS} - \bar{L}_D \delta^{[RS]J K} \quad (4.30)
\end{align*}
\]

Note that there is no \(T \ast \bar{L}\) term. However the \(A^{JK}(y)D(x)\) and \(A^{JK}(y)A^{RS}(x)\) terms are generated from the \(T \ast \bar{T}\) action. The OPE that follow are
\[
\begin{align*}
A^{JK}(y)D(x) &= \frac{1}{4\pi(y-x)^2} (\delta^{RS}\delta^{JK} - \delta^{RK}\delta^{LS}) A^{RS}(x) \quad (4.32) \\
A^{AB}(y)\psi^C(X) &= \frac{1}{4\pi(y-x)^2} (\delta^{AC}\psi^B(x) - \delta^{AB}\psi^C(x)) \quad (4.33) \\
A^{JK}(y)A^{RS}(x) &= \frac{1}{4\pi(y-x)^2} \delta^{JKRS}_{AB} A^{AB}(x) \quad (4.34)
\end{align*}
\]

In the non-extended version of the algebra \([2, 3, 5, 7]\), there are extra generators that must be added to close the algebra. When the Coadjoint Orbit method is applied, these extra generators correspond to fields and have their own OPEs. The fields \(\omega\) and \(\rho\), which correspond to the U and R operators respectively, have 44 and 11 independent components. The spin of the fields are varied, either being 0 or \(\frac{1}{2}\) depending on the structure of the individual operator. This also true for the general extended \(\ell \neq \pm 1\) case. However, the \(\ell = \pm 1\) case does not have these fields or their OPEs. Thus there is no difference between the regular (\(\ell = 0\)) and extended (\(\ell \neq 0\)) cases except when \(\ell = \pm 1\). These cases reduce the number of operators and fields necessary to fully describe the theory.

5 Reformulation of the Coadjoint Orbit Methods Using Clifford Algebras

Now a different perspective will be investigated using Clifford Algebras instead of derivations. Hasiewicz, Thielemans, and Troost \([4]\) have shown that superconformal Lie superalgebras contain a Clifford algebra structure in them. By exploiting this
structure, new information can be gained by the implications of how the Clifford algebra exists in the larger structure.

Their method starts with a break down of the Lie superalgebra into smaller, relevant subspaces: a Kac-Moody Lie algebra $KM(L)$ with a Lie algebra $L$, a Virasoro algebra $Vir$, and subspaces $Q$ and $G$ with underlying vector spaces respectively, $V$ and $W$. The underlying vectors spaces of these subspaces ( $L$ for $KM(L)$, $R$ for $Vir$, $V$ for $Q$, $W$ for $G$) are important along with a number of mappings that define the properties of each space. For a fixed element $w \in W$, and $w, w' \in W; v, v' \in V; \Sigma, \Sigma' \in L; a \in R$, there are the following mappings:

$$[T_m(\Sigma), T_n(\Sigma')] = T_{m+n}([\Sigma, \Sigma']) - cmK(\Sigma, \Sigma')\delta(m+n)$$  \hspace{1cm} (5.1)

$$[L_m, L_n] = (m-n)L_{m+n} + (m^3 - m)\delta(m+n)c/4$$  \hspace{1cm} (5.2)

$$[L_m, T_n(\Sigma)] = -nT_{m+n}(\Sigma)$$  \hspace{1cm} (5.3)

$$[L_m, Q_n(v)] = -(m/2 + n)Q_{m+n}(v)$$  \hspace{1cm} (5.4)

$$[L_m, G_n(w)] = +(m/2 - n)G_{m+n}(w)$$  \hspace{1cm} (5.5)

$$\{Q_m(v), Q_n(v')\} = -b(v, v')\delta(m+n)c$$  \hspace{1cm} (5.6)

$$[T_m(\Sigma), Q_n(v)] = Q_{m+n}(R(\Sigma)v)$$  \hspace{1cm} (5.7)

$$\{G_m(w), G_n(w')\} = 2B(w, w')L_{m+n} + B(w, w')(m^2 - 1/4)\delta(m+n)c$$  \hspace{1cm} (5.8)

$$-(m-n)T_{m+n}(\varphi(w, w'))$$  \hspace{1cm} (5.9)

$$\{G_m(w), Q_n(v)\} = T_{m+n}(\varphi(w, v))$$  \hspace{1cm} (5.10)

$$[T_m(\Sigma), G_n(w)] = G_{m+n}(\Lambda(\Sigma)w) + mQ_{m+n}(d(\Sigma, w))$$  \hspace{1cm} (5.11)

with $K, b, R, \varphi, \Lambda,$ and $d$ all being mappings and bilinear forms necessary to describe the superconformal Lie superalgebras.

There are special mappings that Hasiewicz et. al.[4] use to associate with the Clifford algebra:

$$\varphi_w(w') : w' \in W \rightarrow \varphi(w, w') \in L$$  \hspace{1cm} (5.12)

$$d_w(\Sigma) : \Sigma \in L \rightarrow d(\Sigma, w) \in V$$  \hspace{1cm} (5.13)

$$i_w(a) : a \in R \rightarrow aw \in W.$$  \hspace{1cm} (5.14)

This set gives the exact series

$$\mathbb{R} \xrightarrow{i_w} W \xrightarrow{\varphi_w} L \xrightarrow{d_w} V \rightarrow 0.$$  \hspace{1cm} (5.15)

While this set of mappings and forms

$$\psi_v : v \in V \rightarrow \psi(w, v) \in L$$  \hspace{1cm} (5.16)
\[ \Lambda_w : \Sigma \in L \rightarrow \Lambda(\Sigma)w \in W \] (5.17)
\[ B_w : w' \in W \rightarrow B(w, w') \in \mathbb{R} \] (5.18)

gives the exact series
\[ 0 \rightarrow V \xrightarrow{\psi_w} L \xrightarrow{\Lambda_w} W \xrightarrow{B_w} \mathbb{R}. \] (5.19)

Note that all the mappings resemble adjoint actions, being based on a fixed element \( w \).

One of the most important results is the relationship between the dimensions of the vector spaces:
\[ |W| + |V| = |L| + 1 \] (5.20)

With this relationship, one can categorize the type of algebra possible since there are \( |W| \) symmetries that exist \((\text{dim}(W) = N)\), and \( |L| \) is the dimension of the underlying Lie algebra.

The spaces define a larger space \( S = W \oplus V \oplus L \oplus \mathbb{R} \) of all the elements and an endomorphism \( \Gamma \) that represents a Clifford algebra with the mapping \( B \) above as a definition:
\[ \Gamma_w(w' + v + \Sigma + a) = (aw + \Lambda(\Sigma)w) + d(\Sigma, w) \]
\[ + (\varphi(w, w') + \psi(w, v)) + B(w, w') \]
\[ \Gamma_w \Gamma'_w + \Gamma'_w \Gamma_w = 2B(w, w'). \] (5.21)

\[ S \] is also given a metric by \( \theta \):
\[ \theta(w + v + \Sigma + a, w' + v' + \Sigma' + a') = B(w, w') + b(v, v') - K(\Sigma, \Sigma') - aa' \] (5.23)

At this point, a number of similarities to the Coadjoint Orbit method shown earlier should be apparent. The elements of \( S \) have this particular form because the elements of all the different spaces are now on an equal footing with each other under the Clifford algebra. The metric has the same form (up to some signs) as the action of the dual element on the vectors describe in Section 4.

The superconformal Lie algebra is built from a vector representing the unit element in the space \( \mathbb{R} \). This element is multiplied by the basis elements of the Clifford algebra to get the other spaces \( W, V, \) and \( L \). The previous mappings between spaces allows them to be separated to get the complicated structure needed.

The \( \mathcal{N} = 4 \) case is presented in their paper [4] for a Clifford algebra signature of \((0, 4)\) explicitly and all other signatures by inference. The choice of \( \ell = 1 \) corresponds
to a 16-dimensional representation of $S$ and the Clifford space. The dimensions of the spaces $W$, $V$, and $L$ are 4, 4, and 7 respectively as given by eq. 5.20. The basis vectors for $W$ are

$$w_i = \Gamma_i$$  

(5.24)

and for $V$,

$$v_i = \Gamma_i(\Gamma^5 - \ell)$$  

(5.25)

where $\Gamma^5 = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$, $(\Gamma^2) = 1$, and $\ell$ real, much like defined in the derivation method. The elements of Lie algebra are given by the $\phi$ mapping with the addition of one more element:

$$\phi_{ij} = \phi(w_i, w_j) = \Gamma_i\Gamma_j(i \neq j)$$  

(5.26)

$$\sigma = (\Gamma^5 - \ell).$$  

(5.27)

The mappings and bilinear forms from above now take the form

$$\Lambda(\phi_{ij})w_k = \delta_{jk}w_i - \delta_{ik}w_j + \ell\epsilon_{ijkl}w_l$$  

(5.28)

$$d(\phi_{ij}, w_k) = \epsilon_{ijkl}v_l$$  

(5.29)

$$\psi(w_i, v_j) = \Gamma_i\Gamma_j(\Gamma^5 - \ell) = -\ell\psi_{ij} - \frac{1}{2}\epsilon_{ijkl}\varphi_{kl} - \delta_{ij}\sigma$$  

\(5.30\)

$$d(\sigma, w_k) = v_l$$  

(5.31)

$$R(\phi_{ij})v_k = \delta_{jk}v_l - \delta_{ik}v_j + \ell\epsilon_{ijkl}v_l$$  

(5.32)

Now the correspondence between derivation representation and Clifford algebra representation should be clear:

| Original Notation | Condensed Form | from HTT |
|-------------------|----------------|---------|
| $P, \Delta, K$    | $L$            | $\theta(B, b, K) \in Vir$ |
| $Q, S$            | $G$            | $w_i \in W, v_i \in V$ |
| $T$               | $T$            | $\varphi, \sigma \in L$ |

Table 2

The effects of the extended algebra, which is a function of $\ell$, can be seen in the mapping $b$, the metric term for the vector space $V$, and the mapping $\Lambda$ on the six linear combinations $\phi_{ij} \pm \frac{1}{2}\epsilon_{ijkl}\varphi_{kl}$ :

$$b(v_i, v_j) = -\delta_{ij}(1 - \ell^2)$$  

(5.33)
\[ \Lambda(\varphi_{ij} \pm \epsilon_{ijkl}\varphi_{kl})w_m = (1 \mp \ell)(\delta_{jm}w_i - \delta_{im}w_j \mp \ell \epsilon_{ijmn}w_n) \] (5.34)

The parameter \( \ell \) can be used to categorize all types of 1D \( \mathcal{N} = 4 \) super Virasoro algebras. When \( \ell \neq \pm 1 \), the algebra is the “large” \( \mathcal{N} = 4 \) algebra with \( so(4) = so(3) \otimes so(3) \). At \( \ell = \pm 1 \), it collapses to the “small” 8-dimensional \( \mathcal{N} = 4 \) algebra. It is called small because at \( \ell = \pm 1 \), part of the space is mapped into zero into \( W \). The dimension \( \frac{1}{2} \) fields generated by \( V \) disappear and the corresponding representation now only has 4 dimension-1/2 fields from \( W \) and four dimension-1 fields from the combination of \( L \) and \( \text{Vir} \).

It is clear that the addition of the \( \ell \)-terms, which also involved the Levi-Civita tensor, has its basis in the \( Q \) vector space describing the supersymmetric operators and requires the necessary adjustments to the other operators to close the algebra. The original \([T, Q], [T, S],\) and \([T, T]\) supercommutators reflect this relationship and the close ties between the supersymmetric operators and the Lie algebra underneath.

The question of whether the algebra has central charges can now be revisited. In [4], the commutation relations, which are given earlier, contain the central charge \( c \). They make the assumption of a nonzero central charge and show that the set of generators is closed. With some additional work, the central charge can be re-added into the equations.

With the algebra elements written as elements of a Clifford algebra, all of the previous work can be double checked and reanalyzed in a different context. The benefit of going to a Clifford algebra representation is that the Clifford algebras are well-known and well-understood. In [4], there is some discussion about what this would entail and will be investigated for future research.

### 6 Discussions, Interpretations, and Conclusions

There are a number of interesting ideas and directions that this work has brought up:

- **Coadjoint Orbit Method**: The Coadjoint Orbit method has a clear mathematical basis underlying it. There exists a relationship between the equivalence class of linear functions on a Lie group (trajectories) and a natural symplectic structure on the relevant manifold (phase space). The connection between the two seems more obvious in terms of Clifford algebras, which has a foot in both worlds. It may be that a simpler explanation can be found by exploring this direction with
the first step going from the Clifford algebras to the underlying Spin groups and algebras which are closer related to Lie groups.

- Higher-point functions (3-point and 4-point correlators): The methods of this paper describe using any representation of symmetry generators to develop OPEs describing two-point correlation functions. In [8], there is a way to extend this methodology to higher point functions. Thus, it may be possible to totally “skip” Hamiltonian and Lagrangian and just calculate correlation functions from symmetries. Skipping that step, however, does not absolve one from still figuring out the dynamics of the theory, which are contained in the propagation and interaction terms calculated from the OPEs.

- Since the Virasoro and Kac-Moody algebras are Lie algebras, they have interpretations as manifolds. What does the central extension mean in terms of manifolds? A central extension in group representation terms means that there are operators (or combinations of operators) that exist in the center of the group besides the typical identity element. The formal name for this concept is an ideal, a subgroup that maps products between members inside and outside the subgroup into the subgroup. In this case, it represents that elements in the group can be pulled back to “another origin”. The interpretation of the central extension should be important for any work involving Geometrical Representation theory.

- In [4], they discussed the non-existence of a description of superconformal Lie superalgebras with \( \dim W > 4 \). There were a number of restrictions to this statement but they discuss \( \mathcal{N} > 4 \) superconformal superalgebras that were not Lie superalgebras. Further research into this area could provide a possible generalization of supersymmetry algebras.

Our use of super vector fields in order to realize the symmetry generators in a geometrical manner also points in one other direction. Since there is no metric defined on a Salam-Strathdee superspace, the conventional and familiar role of the metric (or a putative super-metric) is taken over by super-frame fields or super vielbeins. Thus a definition of Killing super-vectors must rely on a super vielbein. As such there is a superspace geometry that is naturally associated with the vector fields (realizing the symmetry). This geometry is the conventional one of a flat Salam-Strathdee superspace. This raises a question. One can imagine a super vielbein that does not describe a flat Salam-Strathdee superspace but one with a non-trivial topology. If it
possesses a related set of Killing super vectors. In principle it should be possible to derive short distance expansions in this case.

In conclusion, the short distance OPEs for the extended 1D $\mathcal{N} = 4$ Super Virasoro algebra was calculated and found to be exactly of the same form of the 1D $\mathcal{N} = 2$ case. Further investigation showed the full relationship between the “large” and “small” $\mathcal{N} = 4$ algebras and the deeper relationship between the two through the Clifford algebra. Let us end by noting that the relation to Clifford algebras also suggest that ‘Garden Algebras’ defined in [9] seem likely to provide a starting point for some OPE’s.

“No human investigation can be called real science if it cannot be demonstrated mathematically.”
– Leonardo da Vinci

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References

[1] A. A. Kirillov, “Merits and Demerits of the Orbit Method,” Bull. Amer. Math. Soc. 36, 4 (1999).

[2] S. J. Gates, Jr. and L. Rana, “Superspace geometrical representations of extended super Virasoro algebras,” Phys. Lett. B 438, 80 (1998) [hep-th/9806038].

[3] C. Curto, S. J. Gates, Jr. and V. G. J. Rodgers, “Superspace geometrical realization of the N-extended super Virasoro algebra and its dual,” Phys. Lett. B 480, 33 (2000) [hep-th/0002010].

[4] Z. Hasiewicz, K. Thielemans and W. Troost, “Superconformal algebras and Clifford algebras,” J. Math. Phys. 31 3 (1990).

[5] S. J. Gates, Jr., W. D. Linch III, Joseph Phillips, and V. G. J. Rodgers, “Short Distance Expansion from the Dual Representation of Infinite Dimensional Lie Algebras,” Commun. Math. Phys. 246 2 (2004).

[6] E. Witten, “Coadjoint Orbits of the Virasoro Group,” Commun. Math. Phys. B114 (1998) 1.
[7] I. Bah, “Space-time interactions from Virasoro and Kac-Moody algebra” Thesis from NSF-REU/Summer Theoretical Physics Research Session 2005.

[8] K. Wilson and W. Zimmermann “Operator Product Expansions and Composite Field Operators in the General Framework of Quantum Field Theory,” Commun. Math. Phys. 24, 87-106 (1972).

[9] S. J. Gates, Jr. and L. Rana, “A Theory of Spinning Particles for Large N-extended Supersymmetry,” Phys. Lett. B 352, 50 (1995) [hep-th/9504025]; idem. “A Theory of Spinning Particles for Large N-extended Supersymmetry (II),” Phys. Lett. B 369, 262 (1996) [hep-th/9510151].