Kaon electroweak form factors in the light-front quark model

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We investigate the form factors and decay rates for the semileptonic decays of the kaon($K_{l3}$) using the light-front quark model. The form factors $f_{\pm}(q^2)$ are calculated in $q^+ = 0$ frame and analytically continued to the time-like region, $q^2 > 0$. Our numerical results for the physical observables, $f_-/f_+|_{q^2=m_l^2} = -0.38$, $\lambda_+ = 0.025$ (the slope of $f_+$ at $q^2 = m_l^2$), $\Gamma(K_{\mu3}) = (7.30 \pm 0.12) \times 10^6 s^{-1}$, and $\Gamma(K_{\mu3}) = (4.57 \pm 0.07) \times 10^6 s^{-1}$ are quite comparable with the experimental data and other theoretical model calculations. The non-valence contributions from $q^+ \neq 0$ frame are also estimated.
I. INTRODUCTION

Even though there have been a lot of analyses on the heavy-to-heavy and heavy-to-light form factors for weak transitions from a pseudoscalar meson to another pseudoscalar meson within the light-front quark model (LFQM) \[1–8\], the light-to-light weak form factor analysis such as \(K_{l3}\) has not yet been studied in LFQM. However, the analysis of semileptonic \(K_{l3}\) decays comparing with the experiment \[9\] has been provided by many other theoretical models, e.g., the chiral perturbation theory (CPT) \[10,11\], the effective chiral Lagrangian approach \[12\], the vector meson dominance \[13\], the extended Nambu-Jona-Lasinio model \[14\], Dyson-Schwinger approach \[15\] and other quark model \[16,17\]. Thus, in this work, we use LFQM to analyze both form factors of the \(K_{l3}\) decays, i.e., \(f_+\) and \(f_-\), and compare with the experimental data as well as other theoretical models.

In the LFQM calculations presented in Refs. \[4–7\], \(q^+ \neq 0\) frame has been used to calculate the weak decays in the time-like region \(m_l^2 \leq q^2 \leq (M_i - M_f)^2\), with \(M_i[f]\) and \(m_l\) being the initial[final] meson mass and the lepton(l) mass, respectively. However, when the \(q^+ \neq 0\) frame is used, the inclusion of the non-valence contributions arising from quark-antiquark pair creation (“Z-graph”) is inevitable and this inclusion may be very important for heavy-to-light and light-to-light decays. Nevertheless, the previous analyses \[4–7\] in \(q^+ \neq 0\) frame considered only valence contributions neglecting non-valence contributions. In this work, we circumvent this problem by calculating the processes in \(q^+ = 0\) frame and analytically continuing to the time-like region. The \(q^+ = 0\) frame is useful because only valence contributions are needed. However, one needs to calculate the component of the current other than \(J^+\) to obtain the form factor \(f_-(q^2)\). Since \(J^-\) is not free from the zero-mode contributions even in \(q^+ = 0\) frame \[19,20\], we use \(J_\perp\) instead of \(J^-\) to obtain \(f_-\). The previous works in Refs. \[1–3\] have considered only the “+”-component of the current which was not sufficient to obtain the form factor \(f_-(q^2)\). Furthermore, the light-to-light decays such as \(K_{l3}\) have not yet been analyzed, even though the calculation of \(f_-\) for heavy-to-heavy and heavy-to-light decays has been made in Ref. \[8\] using the dispersion formulations. Thus, we analyze both currents of \(J^+\) and \(J_\perp\) for \(K_{l3}\) decays using \(q^+ = 0\) frame and analytically continue to the time-like region. Our method of changing \(q_\perp\) to \(iq_\perp\) is not only simple to use in practical calculations for the exclusive processes but also provides the identical results obtained by the dispersion formulations presented in Ref. \[8\].

The calculation of the form factor \(f_-(q^2)\) is especially important for the complete analysis of \(K_{l3}\) decays, since the \(f_-(q^2)\) is prerequisite for the calculation of the physical observables \(\xi_A = f_-/f_+|_{q^2=m_l^2}\) and \(\lambda_-\), the slope of \(f_-(q^2)\) at \(q^2 = m_l^2\). We also estimate the non-valence contributions from \(q^+ \neq 0\) frame by calculating only valence contributions from \(q^+ \neq 0\) frame and comparing them with those obtained from \(q^+ = 0\) frame. Including the lepton mass effects for the \(d\Gamma/dq^2\) spectrum of \(K_{l3}\), we distinguish the decay rate of \(K_{\mu3}\) from that of \(K_{e3}\), where the contribution from \(f_-\) is found to be appreciable for \(\mu\) decays.

Our model parameters summarized in Table I were obtained from our previous analysis of quark potential model \[18\], which provided a good agreement with the experimental data of various electromagnetic properties of mesons.
such as \( f_\pi, f_K \), charge radii of \( \pi \) and \( K \), and rates for radiative meson decays etc. As shown in Ref. [18], the gaussian radial wave function \( \phi(x, k_\perp) \) for our LF wave function \( \Psi^{JJ}_{\lambda_\pi, \lambda_q}(x, k_\perp) = \phi(x, k_\perp)\mathcal{R}^{JJ}_{\lambda_\pi, \lambda_q}(x, k_\perp) \) is given by

\[
\phi(x, k_\perp) = \sqrt{\frac{\partial k_z}{\partial x}} \left( \frac{1}{\pi^{3/2} 2 \beta^3} \right)^{1/2} \exp(-k^2/2\beta^2),
\]

where \( \partial k_z/\partial x \) is the Jacobian of the variable transformation \( \{x, k_\perp\} \to k = (k_\mu, k_\perp) \). The spin-orbit wave function \( \mathcal{R}^{JJ}_{\lambda_\pi, \lambda_q}(x, k_\perp) \) is obtained by the interaction-independent Melosh transformation. The detailed description for the spin-orbit wave function can also be found in previous literatures [1–3, 5–7, 18].

The paper is organized as follows. In Sec.II, we obtain the form factors of \( K_{l3} \) decays in \( q^+ = 0 \) frame and analytically continue to the time-like \( q^2 > 0 \) region by changing \( q_\perp \) to \( iq_\perp \) in the form factors. In Sec.III, our numerical results of the observables for \( K_{l3} \) decays are presented and compared with the experimental data as well as other theoretical results. Summary and discussion of our main results follow in Sec.IV. In the Appendix A, we show the derivation of the matrix element of the weak vector current for \( K_{l3} \) decays in the standard \( q^+ = 0 \) frame. In the Appendix B, the valence contribution in \( q^+ \neq 0 \) frame is formulated.

## II. WEAK FORM FACTORS IN DRELL-YAN FRAME

The matrix element of the hadronic current for \( K_{l3} \) can be parametrized in terms of two hadronic form factors as follows

\[
<\pi|\bar{u}\gamma^\mu s|K> = f_+(q^2)(P_K + P_\pi)\mu + f_-(q^2)(P_K - P_\pi)\mu,
\]

\[
= f_+(q^2) \left[ (P_K + P_\pi)\mu - \frac{M_K^2 - M_\pi^2}{q^2} q^\mu \right] + f_0(q^2) \frac{M_K^2 - M_\pi^2}{q^2} q^\mu,
\]

where \( q^\mu = (P_K - P_\pi)\mu \) is the four-momentum transfer to the leptons and \( m_1^2 \leq q^2 \leq (M_K - M_\pi)^2 \). The form factors \( f_+ \) and \( f_0 \) are related to the exchange of \( 1^- \) and \( 0^+ \), respectively, and satisfy the following relations:

\[
f_+(0) = f_0(0), \quad f_0(q^2) = f_+(q^2) + \frac{q^2 M_K^2 - M_\pi^2}{M_K^2 - M_\pi^2} f_-(q^2).
\]

Since the lepton mass is small except in the case of the \( \tau \) lepton, one may safely neglect the lepton mass in the decay rate calculation of the heavy-to-heavy and heavy-to-light transitions. However, for \( K_{l3} \) decays, the muon(\( \mu \)) mass is not negligible, even though electron mass can be neglected. Thus, including non-zero lepton mass, the formula for the decay rate of \( K_{l3} \) is given by [21]:

\[
\frac{d\Gamma(K_{l3})}{dq^2} = \frac{G_F^2}{24\pi} |V_{us}|^2 K_f(q^2)(1 - \frac{m_l^2}{q^2})^2 \\
\times \left\{ (K_f(q^2))^2 \left[ (1 + \frac{m_\pi^2}{2q^2})(f_+(q^2))^2 + M_K^2 (1 - \frac{M_K^2}{M_K^2})^2 \frac{3}{8} \frac{m_l^2}{q^2} f_0(q^2) \right]^2 \right\},
\]

where \( G_F \) is the Fermi constant, \( V_{us} \) is the element of the Cabbibo-Kobayashi-Maskawa mixing matrix and the factor \( K_f(q^2) \) is given by
\[ K_f(q^2) = \frac{1}{2M_K} \left[ (M_K^2 + M_n^2 - q^2)^2 - 4M_K^2 M_n^2 \right]^{1/2}. \] (5)

Since our analysis will be performed in the isospin symmetry \((m_u=m_d)\) but \(SU_f(3)\) breaking \((m_s \neq m_u(d))\) limit, we do not discriminate between the charged and neutral kaon weak decays, \(i.e., f_{K^0}^+ = f_{K^+}^+\). For \(K_{13}\) decays, the three form factor parameters, \(i.e., \lambda_+, \lambda_0 \) and \(\xi_A\), have been measured using the following linear parametrization [9]:

\[ f_{\pm}(q^2) = f_{\pm}(q^2 = m_f^2) \left( 1 + \lambda_{\pm} \frac{q^2}{M_{+,0}^2} \right), \] (6)

where \(\lambda_{\pm,0}\) is the slope of \(f_{\pm,0}\) evaluated at \(q^2 = m_f^2\) and \(\xi_A = f_- / f_{+|q^2=m_f^2}\).

FIG. 1. The form factor calculation in \(q^+ \neq 0\) frame requires both the usual light-front triangle diagram(a) and the non-valence(pair-creation) diagram(b). The vertical dashed line in (b) indicates the energy-denominator for the non-valence contribution. While the white blob represents our LF wavefunction \(\Psi_{\lambda_0,\lambda_q}(x, k_\perp)\), the modeling of black blob has not yet been made.

As shown in Fig.1, the quark momentum variables for \(q_1 \bar{q} \rightarrow q_2 \bar{q}\) transitions in the standard \(q^+ = 0\) frame are given by

\[
\begin{align*}
p_{1}^+ &= (1-x)P_{1}^+, \quad p_{1}^+ = xP_{1}^+, \\
p_{1\perp} &= (1-x)P_{1\perp} + k\perp, \quad p_{1\perp} = xP_{1\perp} - k\perp, \\
p_{2}^+ &= (1-x)P_{2}^+, \quad p_{2}^+ = xP_{2}^+, \\
p_{2\perp} &= (1-x)P_{2\perp} + k'\perp, \quad p_{2\perp} = xP_{2\perp} - k'\perp.
\end{align*}
\] (7)

which requires that \(p_{1\perp}^+ = p_{1\perp}^+\) and \(p_{2\perp} = p_{2\perp}'\). Our analysis for \(K_{13}\) decays will be carried out using this \(q^+ = 0\) frame where the decaying hadron(Kaon) is at rest. Using the matrix element of the “+”-component of the current, \(J^+\), given by Eq.(2), we obtain the form factor \(f_{+}(q_{\perp}^2)\) as follows

\[ f_{+}(q_{\perp}^2) = A_1 A_2 \frac{\sqrt{A_1^2 + k_\perp^2}}{\sqrt{A_2^2 + k_\perp^2}} \int_0^1 dx \int d^2 k_\perp \phi_2(x,k_\perp) \phi_1(x,k_\perp) \frac{A_1 A_2 + k_\perp \cdot k_\perp'}{\sqrt{A_1^2 + k_\perp^2} \sqrt{A_2^2 + k_\perp^2}}, \] (8)

where \(q_{\perp}^2 = -q^2, A_i = m_i x + m_\bar{q}(1-x)\) and \(k_\perp = k_\perp - xq_\perp\). As we discussed in the introduction, we need the “\(\perp\)”-component of the current, \(J_{\perp}\), to obtain the form factor \(f_{-}(q_{\perp}^2)\) in Eq.(2), viz.,
after multiplying $q_\perp$ on both sides of Eq.(2). The l.h.s. of Eq.(9) is given by

$$< P_2 | \bar{q}_2 (q_\perp \cdot \vec{\gamma}_\perp) q_1 | P_1 > = q_\perp^2 \left[ f_- (q_\perp^2) - f_+ (q_\perp^2) \right],$$

(9)

Using the quark momentum variables given in Eq.(7), we obtain the trace term in Eq.(10) as follows

$$\text{Tr} \left[ \gamma_5 (\not{p}_2 + m_2) (q_\perp \cdot \vec{\gamma}_\perp) (\not{p}_1 + m_1) \gamma_5 (\not{q}_\perp - m_\perp) \right]$$

$$= -2 \left\{ 2 \frac{(A_1^2 + k_\perp^2)}{x(1-x)} (k_\perp - q_\perp) \cdot q_\perp + \left( A_2^2 + k_\perp^2 \right) k_\perp \cdot q_\perp + \left[ (m_1 - m_2)^2 + q_\perp^2 \right] k_\perp \cdot q_\perp \right\}. \quad (11)$$

The more detailed derivation of Eqs.(8) and (10) are presented in appendix A. Since both sides of Eq.(9) vanish as $q^2 \to 0$, one has to be cautious for the numerical computation of $f_-$ at $q^2 = 0$. Thus, for the numerical computation at $q^2 = 0$, we need to find an analytic formula for $f_0$. In order to obtain the analytic formula for the form factor $f_-(0)$, we make a low $q^2$ expansion to extract the overall $q^2$ term from Eq.(10). Then, the form factor $f_-(0)$ is obtained as follows

$$f_-(0) = f_+(0) + \int_0^1 dx \int d^2 k_\perp \frac{x \phi_2 (x, k_\perp) \phi_1 (x, k_\perp)}{\sqrt{A_1^2 + k_\perp^2 + A_2^2 + k_\perp^2}} \times \left\{ C_{T1} (C_{T1} - C_{T2} + C_M + C_R) + C_{T2} \right\} k_\perp^2 \cos^2 \phi + C_{T3},$$

(12)

where the angle $\phi$ is defined by $k_\perp \cdot q_\perp = |k_\perp| |q_\perp| \cos \phi$ and the terms of $C'$s are given by

$$C_{J1} = \frac{2 \beta_2^2}{(1-x)(\beta_1^2 + \beta_2^2) M_{10}^2} \left[ \frac{1}{1 - \left( (m_1^2 - m_\perp^2)/M_{10}^2 \right)^2} - \frac{3}{4} \right],$$

$$C_M = \frac{1}{(1-x)(\beta_1^2 + \beta_2^2)} \left[ \frac{M^2_{20} - (m_2 - m_\perp)^2}{M^2_{10} - (m_1 - m_\perp)^2} \right],$$

$$C_R = \frac{-1}{4(1-x)(\beta_1^2 + \beta_2^2)} \left[ \frac{(m_2^2 - m_\perp^2)}{M^2_{20}} - \left( \frac{M^2_{20}}{M^2_{10}} \right)^2 \right],$$

$$C_{T1} = \frac{1}{x(1-x)} \left( A_1^2 + A_2^2 + 2k_\perp^2 \right) + \left( m_1 - m_2 \right)^2,$$

$$C_{T2} = \frac{2(\beta_1^2 - \beta_2^2)}{(1-x)(\beta_1^2 + \beta_2^2)}, \quad C_{T3} = \frac{x \beta_2^2}{\beta_1^2 + \beta_2^2} C_{T1} - \frac{A_1^2 + k_\perp^2}{x(1-x)}.$$

(13)

with

$$M_{10}^2 = \frac{k_\perp^2 + m_\perp^2}{1-x} + \frac{k_\perp^2 + m_\perp^2}{x}.$$

(14)

The form factors $f_+$ and $f_-$ can be analytically continued to the time-like $q^2 > 0$ region by replacing $q_\perp$ by $i q_\perp$ in Eqs.(8) and (9). Since $f_-(0)$ in Eq.(12) is exactly zero in the $SU_f(3)$ symmetry, i.e., $m_u(d) = m_s$ and

1 We note that our numerical results of $f_+$ obtained by the method of replacing $q_\perp$ by $i q_\perp$ in Eq.(8) for any $P \rightarrow P(\rightarrow P)$ semileptonic decays are identical to those obtained from dispersion formulation in Ref. [8].
transfer, excellent agreement with the result of chiral perturbation theory [10], form factor of pion, and variation of $m$ and $\Gamma(K^0)$. We investigated the sensitivity of our results by varying quark masses. For instance, the results of the strange quark mass from observables such as $\lambda_{s}$ can be obtained when $m_s$ takes the values $m_s = 0.48$ GeV to $0.43$ GeV (10% change) for the set 1, which are included in Table II. 

As we discussed in the introduction, we used the same quark model parameters $(m_u(d), m_s, \beta_{u(d)}$, $\beta_{u(d)}$) as in Ref. [18] to predict various observables for $K_{l3}$ decays. These parameters are summarized in Table I. The Sets 1 and 2 in this Table represent the model parameters obtained by the harmonic oscillator and linear confinement potentials, respectively, from Ref. [18].

Our predictions of the parameters for $K_{l3}$ decays in $q^+ = 0$ frame, i.e., $f_+(0)$, $\lambda_+$, $\lambda_0$, $r_{q^+}^2 = 6 f_+^2 (0)/f_+(0) = 6 \lambda_+ / M_{K^0}^2$, and $\xi_A = f_+/ f_+|_{q^2 = m_s^2}$, are summarized in Table II. We do not distinguish $K_{e3}$ from $K_{\mu3}$ in the calculation of the above parameters since the slopes of $f_\pm$ are almost constant in the range of $m_\pi^2 \leq q^2 \leq m_\mu^2$. However, the decay rates should be different due to the phase space factors given by Eq. (4) and our numerical results for $\Gamma(K_{e3})$ and $\Gamma(K_{\mu3})$ in $q^+ = 0$ frame are also presented in Table II. Our results for the form factor $f_+$ at zero momentum transfer, $f_+(0) = 0.961 [0.962]$ for set 1[set 2], are consistent with the Ademollo-Gatto theorem [22] and also in an excellent agreement with the result of chiral perturbation theory [10], $f_+(0) = 0.961 \pm 0.008$. Our results for other observables such as $\lambda_+$, $\xi_A$, and $\Gamma(K_{l3})$ are overall in a good agreement with the experimental data [1]. We have also investigated the sensitivity of our results by varying quark masses. For instance, the results obtained by changing the strange quark mass from $m_s = 0.48$ GeV to $0.43$ GeV (10% change) for the set 1 are included in Table II. As one can see in Table II, our model predictions are quite stable for the variation of $m_s$ except $\lambda_0$, which changes its sign from $-0.007$ to $+0.0027$. The large variation of $\lambda_0$ is mainly due to the rather large sensitivity of $f_-(0)$ (18% change) to the variation of $m_s$. Similar observation regarding on the large sensitivity for $\lambda_0$ compared to other observables has also been reported in Ref. [14] for the variation of quark masses. As discussed in Refs. [15] and [17], $f_-(0)$ is sensitive to the nonperturbative enhancement of the SU(3) symmetry breaking mass difference $m_s - m_u(d)$ since $f_-(0)$ depends on the ratio of $m_s$ and $m_u(d)$.

Of special interest, we also observed that the non-valence contributions from $q^+ \neq 0$ frame are clearly visible for $\lambda_+$, $\lambda_0$ and $\xi_A$ even though it may not be quite significant for the decay rate $\Gamma(K_{l3})$. Our predictions with only the valence contributions in $q^+ \neq 0$ frame are $f_+(0) = 0.961 [0.962]$, $\lambda_+ = 0.081 [0.083]$, $\lambda_0 = -0.014 [-0.017]$, $\xi_A = -1.12 [-1.10]$, $\Gamma(K_{e3}) = (8.02 [7.83] \pm 0.13) \times 10^{6} s^{-1}$ and $\Gamma(K_{\mu3}) = (4.49 [4.36] \pm 0.13) \times 10^{6} s^{-1}$ for the set 1[set 2]. Even though the

\[\text{Eq. (4)}\]

2 Even though we show the results only for the set 1, we find the similar variations for the set 2, i.e., the positive sign of $\lambda_0$ can be obtained when $m_s/m_u \leq 1.8$ for both sets 1 and 2. In addition to the observables in this work, our predictions for $f_K$, $r_{K^0}$, and $r_{K^0}^2$ in [18] are changed to $108$ MeV (1% change), $0.385 f^2$ (0.3% change), and $-0.077 f^2$ (15%), respectively.
form factor $f_+(0)$ in $q^+ \neq 0$ frame is free from the non-valence contributions, its derivative at $q^2 = 0$, i.e., $\lambda_+$, receives the non-valence contributions. Moreover, the form factor $f_-(q^2)$ in $q^+ \neq 0$ frame is not immune to the non-valence contributions even at $q^2 = 0$. Unless one includes the non-valence contributions in the $q^+ \neq 0$ frame, one cannot really obtain reliable predictions for the observables such as $\lambda_+, \lambda_0$ and $\xi_A$ for $K_{l3}$ decays.

In Fig. 2, we show the form factors $f_+$ obtained from both $q^+ = 0$ and $q^+ \neq 0$ frames for $0 \leq q^2 \leq (M_K - M_\pi)^2$ region. As one can see in Fig.2, the form factors $f_+$ obtained from $q^+ = 0$ frame(solid lines) for both parameter sets 1 and 2 appear to be linear functions of $q^2$ justifying Eq.(6) usually employed in the analysis of experimental data [9]. Note, however, that the curves without the non-valence contributions in $q^+ \neq 0$ frame(dotted lines) do not exhibit the same behavior. In Fig.3, we show $d\Gamma/dq^2$ spectra for $K_{e3}$(solid line) and $K_{\mu3}$(dotted line) obtained from $q^+ = 0$ frame. While the term proportional to $f_0$ in Eq.(4) is negligible for $K_{e3}$ decay rate, its contribution for $K_{\mu3}$ decay rate is quite substantial(dot-dashed line). Also, we show in Fig.4 the form factors $f_+(q^2)$(solid and dotted lines for the sets 1 and 2, respectively) at spacelike momentum transfer region and compare with the theoretical prediction from Ref. [14](dot-dashed line). The measurement of this observable in $q^2 < 0$ region is anticipated from TJNAF [14].

In addition, we calculated the electromagnetic form factors $F_\pi(q^2)$ and $F_K(q^2)$ in the space-like region using both $q^+ = 0$ and $q^+ \neq 0$ frames to estimate the non-valence contributions in $q^+ \neq 0$ frame. As shown in Figs.5 and 6 for $F_\pi(q^2)$ and $F_K(q^2)$, respectively, our predictions in $q^+ = 0$ frame are in a very good agreement with the available data [23,24] while the results for $q^+ \neq 0$ frame deviate from the data significantly. The deviations represent the non-valence contributions in $q^+ \neq 0$ frame(see Fig.1(b)). However, the deviations are clearly reduced for $F_K(q^2)$(see Fig.6) because of the large suppression from the energy denominator shown in Fig.1 for the non-valence contribution. The suppressions are much bigger for the heavier mesons such as $D$ and $B$. Especially, for the $B$ meson case, the non-valence contribution is almost negligible up to $Q^2 = -q^2 \sim 10$ GeV$^2$.

IV. SUMMARY AND CONCLUSION

In this work, we investigated the weak decays of $K_{l3}$ using the light-front quark model. The form factors $f_\pm$ are obtained in $q^+ = 0$ frame and then analytically continued to the time-like region by changing $q_\perp$ to $iq_\perp$ in the form factors. The matrix element of the $\perp$-component of the current $J^\mu$ is used to obtain the form factor $f_-$, which is necessary for the complete analysis of $K_{l3}$ decays. Using the non-zero lepton mass formula[Eq.(4)] for the decay rate of $K_{l3}$, we also distinguish $K_{\mu3}$ from $K_{e3}$ decay. Especially, for $K_{\mu3}$ decay, the contribution from $f_0$ or $f_-$ form factor is not negligible in the calculation of the decay rate. Our theoretical predictions for $K_{l3}$ weak decays are overall in a good agreement with the experimental data. We also confirmed that our analytic continuation method is equivalent to that of Ref. 8 where the form factors are obtained by the dispersion representations through the (gaussian) wave functions of the initial and final mesons. In all of these analyses, it was crucial to include the non-valence contributions in $q^+ \neq 0$ frame. As we have estimated these contributions in various observables, their magnitudes are not at all
negligible in the light-to-light electroweak form factors. In fact, the non-valence contributions were very large for the most of observables such as $\lambda_+^0$, $\lambda_0$, $\xi_A$, $F_\pi(Q^2)$ and $F_K(Q^2)$.

Finally, we have also estimated the zero-mode contribution by calculating the “−”-component of the current. Our observation in an exactly solvable scalar field theory was presented in Ref. [19]. Using the light-front bad current $J^-$ in $q^+ = 0$ frame, we obtained $f_-(0) = 12.6\{18.6\}$ for the set 1[set 2]. The huge ratio of $f_-(0)/f_+(0)/f_\perp \approx -36\{−40\}$ for the set 1[set 2] is consistent with our observation in Ref. [19]. We also found that the zero-mode contribution is highly suppressed as the quark mass increases. The detailed analysis of heavy-to-heavy and heavy-to-light semileptonic decays is currently underway.

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APPENDIX A: DERIVATION OF THE MATRIX ELEMENT OF THE WEAK VECTOR CURRENT $<P_2|\bar{q}_2\gamma^\mu(=+,-)q_1|P_1>$ IN $q^+ = 0$ FRAME

In this appendix A, we show the derivation of the matrix element of the weak vector current $<P_2|\bar{q}_2\gamma^\mu q_1|P_1>$ given in Eq.(2) for $\mu = +$ and $\perp$, respectively.

In the light-front quark model, the matrix element of the weak vector current can be calculated by the convolution of initial and final light-front wave function of a meson as follows

$$<P_2|\bar{q}_2\gamma^\mu q_1|P_1> = \sum_{\lambda_1,\lambda_2,\lambda} \int dp_i^+ d^2k_\perp \phi_\lambda^1(x, k_\perp) \phi_\lambda^1(x, k_\perp) \times R_{\lambda_2\lambda}^{00}(x, k_\perp) \frac{\bar{u}(p_2, \lambda_2) \gamma^\mu u(p_1, \lambda_1)}{\sqrt{p_2^+} \sqrt{p_1^+}} R_{\lambda_1\bar{\lambda}}^{00}(x, k_\perp), \tag{A1}$$

where the spin-orbit wave function $R_{\lambda Jz}^{00}(x, k_\perp)$ for pseudoscalar meson($J^{PC} = 0^{−+}$) obtained from Melosh transformation is given by

$$R_{\lambda_1\bar{\lambda}}^{00} = \frac{1}{\sqrt{2}\sqrt{M_{\bar{0}0}^2 - (m_i - m_q)^2}} \bar{u}(p_i, \lambda_i) \gamma^5 v(p_q, \bar{\lambda}), \tag{A2}$$

and

$$M_{\bar{0}0}^2 = \frac{k_i^2 + m_i^2}{1 - x} + \frac{k_q^2 + m_q^2}{x}. \tag{A3}$$

Substituting Eq.(A2) into Eq.(A1) and using the quark momentum variables given in Eq.(7), one can easily obtain
< P_2|\bar{q} \gamma^\mu q_1 |P_1 > = - \int dx d^2k_\perp \frac{\phi_1^\dagger(x,k_\perp)\phi_1(x,k_\perp)}{2(1-x)\prod_i^2 \sqrt{M_{i0}^2 - (m_i - m_q)^2}} \\
\times \text{Tr} \left[ \gamma_5(p_2 + m_2)\gamma^\mu(p_1 + m_1)\gamma_5(p_q - m_q) \right], \tag{A4}

where we used the following completeness relations of the Dirac spinors

\sum_{\lambda,1,2} u(p,\lambda)\bar{u}(p,\lambda) = \not{p} + m, \sum_{\lambda,1,2} v(p,\lambda)\bar{v}(p,\lambda) = \not{p} - m. \tag{A5}

In the standard \( q^+ = 0 \) frame where the decaying hadron is at rest, the trace terms in Eq.(A4) for the “+” and “\perp”-components of the vector current \( J^\mu = \bar{q} \gamma^\mu q_1 \), respectively, are obtained as follows

\[ \text{Tr} \left[ \gamma_5(p_2 + m_2)\gamma^\mu(p_1 + m_1)\gamma_5(p_q - m_q) \right] \]

\[ = -4 \left[ p_i^\mu(p_2 \cdot p_q + m_2 m_q) + p_{i2}^\mu(p_1 \cdot p_q + m_1 m_q) + p_i^{\mu\prime}(-p_1 \cdot p_2 + m_1 m_2) \right] \]

\[ = -4P^+ \frac{A_i A_2 + \not{k}_\perp \cdot \not{k}'_\perp}{x}, \quad \text{for } \mu = + \tag{A6} \]

\[ = -2 \left[ \frac{(A_i^2 + k_i^2)}{x(1-x)}(k_\perp - q_\perp) + \frac{(A_{i2}^2 + k_{i2}^2)}{x(1-x)}k_\perp + [(m_1 - m_2)^2 + q_{i2}^2]k_\perp \right], \quad \text{for } \mu = \perp \tag{A7} \]

where \( A_i = m_i x + m_q(1-x) \) and \( k'_\perp = k_\perp - xq_\perp \). Our convention of the scalar product, \( p_1 \cdot p_2 = (p_1^+ p_2^+ + p_1^- p_2^-)/2 - p_{1\perp} \cdot p_{2\perp} \) were used to derive Eqs.(A6) and (A7) from the second line of the above equation. Substituting Eqs.(A6) and (A7) into Eq.(A4), we now obtain the matrix element of the weak vector current \( < P_2|\bar{q} \gamma^\mu q_1 |P_1 > \) for \( \mu = + \) (see Eq.(8)) and \( \perp \) (see Eq.(10)) in \( q^+ = 0 \) frame, respectively.

**APPENDIX B: VALENCE CONTRIBUTIONS IN \( q^+ \neq 0 \) FRAME**

For the purely longitudinal momentum transfer, \( i.e., q_\perp = 0 \) and \( q^2 = q^+ q^- \), the relevant quark momentum variables are

\[ p_1^+ = (1-x)P_1^+, \quad p_q^+ = xP_1^+, \]

\[ p_{1\perp} = (1-x)P_{1\perp} + \not{k}_\perp, \quad p_{q\perp} = xP_{1\perp} - \not{k}_\perp, \]

\[ p_2^+ = (1-x')P_2^+, \quad p_{q2}^+ = x'P_2^+, \]

\[ p_{2\perp} = (1-x')P_{2\perp} + \not{k}'_\perp, \quad p_{q2\perp} = x'P_{2\perp} - \not{k}'_\perp, \tag{B1} \]

where \( x(x' = x/r) \) is the momentum fraction carried by the spectator \( \bar{q} \) in the initial(final) state. The fraction \( r \) is given in terms of \( q^2 \) as follows

\[ r_\pm = \frac{M_2}{M_1} \left[ \left( \frac{M_1^2 + M_2^2 - q^2}{2M_1M_2} \right)^{1/2} \pm \sqrt{\left( \frac{M_1^2 + M_2^2 - q^2}{2M_1M_2} \right)^2 - 1} \right], \tag{B2} \]

where the \(+(-)\) signs in Eq.(B2) correspond to the daughter meson recoiling in the positive(negative) z-direction relative to the parent meson. In this \( q^+ \neq 0 \) frame, one obtains

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\[ f_{\pm}(q^2) = \pm \frac{(1 \mp r_-)H(r_+) - (1 \mp r_+)H(r_-)}{r_+ - r_-}, \]  

(B3)

where

\[ H(r) = \int_0^r dx \int d^2 k_\perp \phi_2(x', k_\perp) \phi_1(x, k_\perp) \frac{A_1 A_2' + k_\perp^2}{\sqrt{A_1^2 + k_\perp^2} \sqrt{A_2'^2 + k_\perp^2}}, \]  

(B4)

and \[ A'_i = m_i x' + m_q (1 - x'). \]
TABLE I. Quark masses $m_q$[GeV] and gaussian parameters $\beta$[GeV] used in our analysis. $q=u$ and $d$.

|      | $m_u$ | $m_d$ | $\beta_{q^+}$ | $\beta_{s^2}$ | $\beta_{q^2}$ |
|------|-------|-------|----------------|---------------|--------------|
| Set 1| 0.25  | 0.48  | 0.3194         | 0.3681        | 0.3419       |
| Set 2| 0.22  | 0.45  | 0.3659         | 0.4128        | 0.3886       |

TABLE II. Model predictions for the parameters of $K_{l3}$ decay form factors obtained from $q^+ = 0$ frame. The charge radius $r_{\pi K}$ is obtained by $<r^2>_{\pi K} = \frac{1}{6} f_i^2(q^2 = 0)$. As a sensitivity check, we include the results in square brackets by changing $m_s = 0.48$ to 0.43 GeV for the parameter set 1. The CKM matrix used in the calculation of the decay width(in unit of $10^9 s^{-1}$) is $|V_{us}| = 0.2205 \pm 0.0018[9]$.

| Observables | Set 1 [$m_q = 0.48 \rightarrow 0.43$] | Set 2 | Other models | Experiment |
|-------------|-------------------------------|-------|--------------|------------|
| $f_+(0)$    | 0.961[0.974]                  | 0.962 | $0.961 \pm 0.008^a, 0.952^b, 0.98^c, 0.93^d$ | 0.0286 $\pm 0.0022[K_{\pi^0}]$ | $0.0300 \pm 0.0016[K_{\pi^0}]$ |
| $\lambda_+$ | 0.025[0.029]                  | 0.026 | $0.031^b, 0.033^b, 0.025^d$ | $0.028^a, 0.018^b, 0.019^g$ | $0.028^a, 0.018^b, 0.019^g$ |
| $\lambda_0$ | $-0.007 [+0.0027]$           | $-0.009$ | $0.017 \pm 0.004^b, 0.013^c, 0.0^d$ | $0.004 \pm 0.007[K_{\pi^0}]$ | $0.025 \pm 0.006[K_{\pi^0}]$ |
| $\xi_A$    | $-0.38 [-0.31]$              | $-0.41$ | $-0.164 \pm 0.047^b, -0.24^c, -0.28^d$ | $-0.35 \pm 0.15[K_{\pi^0}]$ | $-0.11 \pm 0.09[K_{\pi^0}]$ |
| $< r >_{\pi K}$ (fm) | 0.55[0.59]                    | 0.56  | $0.61^b, 0.57^c, 0.47^d, 0.48^g$ | $7.7 \pm 0.5[K_{\pi^0}]$ | $5.25 \pm 0.07[K_{\pi^0}]$ |

| $\Gamma(K_{\mu 3}^0)$ | $7.30 \pm 0.12 [7.60 \pm 0.12]$ | $7.36 \pm 0.12$ | $7.7 \pm 0.5[K_{\pi^0}]$ | $5.25 \pm 0.07[K_{\pi^0}]$ |
| $\Gamma(K_{\mu 3}^0)$ | $4.57 \pm 0.07 [4.84 \pm 0.08]$ | $4.56 \pm 0.07$ | $7.7 \pm 0.5[K_{\pi^0}]$ | $5.25 \pm 0.07[K_{\pi^0}]$ |

$^a$ Ref. [10], $^b$ Ref. [11], $^c$ Ref. [12], $^d$ Ref. [13], $^e$ Ref. [14], $^f$ Ref. [15], $^g$ Ref. [16].

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Fig. 2. The form factors $f_+(q^2)$ for the $K \rightarrow \pi$ transition in time-like momentum transfer $q^2 > 0$. The solid and dotted lines are the results from the $q^+ = 0$ and $q^+ \neq 0$ frames for the parameter sets 1 and 2 given in Table I, respectively. The differences of the results between the two frames are the measure of the non-valence contributions from $q^+ \neq 0$ frame.

Fig. 3. The decay rates $d\Gamma/dq^2$ of $K_{e3}$ (solid line) and $K_{\mu3}$ (dotted line) for the parameter set 1 in $q^+ = 0$ frame. The dot-dashed line is the contribution from the term proportion to $f_0$ in Eq.(4) for $K_{\mu3}$ decay. The results for the set 2 are not much different from those for the set 1.

Fig. 4. The form factors $f_+(q^2)$ for the $K \rightarrow \pi$ transition in spacelike momentum transfer $-q^2 < 0$. The solid and dotted lines are the results from the sets 1 and 2, respectively. The dot-dashed line is the result from Ref. [14].

Fig. 5. The EM form factor of pion for low $Q^2 = -q^2$ compared with data [23]. The solid and dotted lines are the results from the $q^+ = 0$ and $q^+ \neq 0$ frames for the parameter sets 1 and 2, respectively.

Fig. 6. The EM form factor of kaon compared with data [24]. The same line code as in Fig. 5 is used.
Fig. 2

The graph illustrates the behavior of $f_+^{K\pi}(q^2)$ as a function of $q^2 > 0$ (GeV$^2$). Two sets of data are plotted:

- Solid line: $q^+=0$
- Dotted line: $q^+=0$

The graph shows the trend for two sets labeled as set 1 and set 2.
Fig. 3
Fig. 4

\[ -q^2 f_+ (q^2) f_+(0) \text{[GeV}^2] \]

- Set 1
- Set 2
Fig. 5

$|F_{\pi}(Q^2)|^2$ vs $Q^2[\text{GeV}^2]$ for different sets.
Fig. 6

$|F_K(Q^2)|^2$ vs. $Q^2[GeV^2]$ for two sets of data:

- Set 1
- Set 2