Directed Plateau Polyhypercubes

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Abstract

In this paper, we study a particular family of polyhypercubes in dimension $d \geq 3$, the directed plateau polyhypercubes, according to to the width and a new parameter the lateral area. We give an explicit formula and we also propose an expression of the generating function in this case.

Keywords: Polyhypercube, polyomino, lateral area, enumeration, generating function.

1 Introduction

In $\mathbb{Z}^2$, a polyomino is a finite union of cells (unit squares), connected by their edges, without a cut point and defined up to a translation [5]. Polyominoes appear in statistical physics, in the phenomenon of percolation [14]. The number of polyominoes in general is still an open problem.

Polyhypercube are the extension of polyominoes in a dimension $d \geq 3$ [1]. In $\mathbb{Z}^d$, a polyhypercube of dimension $d$ is a finite union of cells (unit hypercubes), connected by their hypercubes of dimension $d-1$, and defined up to translation [7]. Polyhypercubes are also called $d$-polycubes. They are used in an efficient model for real time validation [12] and in representation of finite geometric languages [8]. There is no explicit formula for $A_d(n)$ the number of polyhypercubes in dimension $d$ with $n$ cells. Many algorithms were made and values are given for many dimensions (see [1][13]). Some families of polyhypercubes were enumerated for instance: the directed plateau, the plateau, the espalier and the pyramid polyhyercubes by the hypervolume and width using Dirichlet convolutions [6]. Also, the generating function and asymptotic results were given for the $rs$-directed [9], polyhypercubes that can be split into directed strata. Also for $d = 3$, called polycubes, the directed plateau and the plateau polycubes were enumerated according to the lateral area [2].

In this paper, we introduce a new parameter: the lateral area of a polyhypercube for a dimension $d \geq 3$. Using this parameter and the width, we enumerate the family of directed plateau polyhypercubes.
2 Preliminaries

Let \((0, \vec{i}, \vec{j})\) be an orthonormal coordinate. The area of a polyomino is the number of its cells, its width is the number of its columns and its height is the number of its lines. A polyomino is column-convex if its intersection with any vertical line is connected. A North (resp. East) step is a movement of one unit in \(\vec{i}\)-direction (resp. \(\vec{j}\)-direction). A polyomino is directed if from a distinguished cell of the polyomino called root, we reach any other cell by a path that uses only North or East steps.

Let \((0, \vec{i}_1, \vec{i}_2, ..., \vec{i}_d)\) be an orthonormal coordinate system. The volume of polyhypercube is the number of its hypercubes. The width is the difference between its greatest index and its smallest index according to \(\vec{i}_1\). We define the lateral area of an polyhypercube as the sum of the areas of the polyominoes obtained by its projection on the planes \((\vec{i}_1, \vec{i}_l)\) with \(2 \leq l \leq d\). An elementary step is a positive move of one unit along the axis \(\vec{i}_j\) with \(1 \leq j \leq d\). A polyhypercube is directed, if each cell can be reached from a distinguished cell called root, by a path only made by elementary steps. An stratum is polyhypercube of width one. A plateau is an hyperrectangular stratum. A directed plateau polyhypercube is polyhypercube whose strata are plateaus.

To avoid many steps of calculation we use the following useful convention for binomial coefficient, for \(n \geq 0\),
\[
\binom{n}{k} = 0 \text{ for } k < 0 \text{ or } k > n.
\]

3 Explicit enumeration of directed plateau polyhypercube

To enumerate directed plateau polyhypercubes, we characterize their projections.

Theorem 3.1. For \(k \geq 1\), \(d \geq 3\) and \(2 \leq l \leq d\), the projection of a directed plateau polyhypercube of width \(k\) on a plane \((\vec{i}_1, \vec{i}_l)\) gives a directed column-convex polyomino of width \(k\).

Proof. We have the hypothesis that for each directed plateau polyhypercube we associate a \((d-1)\)-tuple of polyominoes obtained by the projection of the polyhypercube on the planes \((\vec{i}_1, \vec{i}_l)\) for \(2 \leq l \leq d\).

Let \(A\) be a \((d-1)\)-tuple of polyominoes and suppose that we can build two different polyhypercubes. It means that the polyhypercubes are different in at least one plateau on \((\vec{i}_1, \vec{i}_l)\), with \(2 \leq l \leq d\). This implies that the two polycubes have different coordinates in \((\vec{i}_1, \vec{i}_l)\) and their projections on these planes are different, it contradicts the initial hypothesis. \(\blacksquare\)

In order to enumerate the directed plateau polyhypercubes, we use the following lemma.

Lemma 3.1. Let \(c_{k,n}\) be the number of directed column-convex polyominoes having \(k\) columns and area \(n\). Then for \(k \geq 1\) and \(n \geq k\),
\[
c_{k,n} = \binom{n+k-2}{n-k}.
\]

Theorem 3.2. Let \(p_{d,k,n}\) be the number of directed plateau polyhypercubes of dimension \(d\), of width \(k\) and having a lateral area \(n\). Then for \(d \geq 3\) and \(n \geq (d-1)k\),
\[
p_{d,k,n} = \sum_{j_2+j_3+...+j_d=n} \prod_{l=2}^{d} \binom{j_l+k-2}{j_l-k}.
\]
Proof. If a polyhypercube of dimension $d$ has a width $k$ and a lateral area $n$, then from Theorem 3.1 each of its projection on a plane $(\vec{i}_1, \vec{i}_l)$ gives a polyomino of width $k$ and area $j_l$, with $j_l \geq k$, for $l$ such that $2 \leq l \leq d$. And the sum of the areas of all polyominoes obtained from projections is equal to $n$.

From Lemma 3.1 it is known that the number of column-convex polyominoes having $k$ columns and area $j_l$ is equal to \((j_l + k - 2)\). Therefore the number of directed plateau polyhypercubes of dimension $d$, width $k$ and whose projections on the planes $(\vec{i}_1, \vec{i}_l)$ give a polyomino of area $j_l$ is \(\prod_{l=2}^{d} (j_l + k - 2)\), with $2 \leq l \leq d$. Therefore, the formula is obtained by summing for all values of $j_l$, $2 \leq l \leq d$. \[\blacksquare\]

**Lemma 3.2** ([11]). For $x$, $y$, $n$ and $k$ integers,

\[
\sum_{k=0}^{n} \binom{x+k}{k} \binom{y+n-k}{n-k} = \binom{x+y+n+1}{n}.\]

For more properties on Vandermonde’s convolutions see [4].

**Theorem 3.3.** For $d \geq 3$ and $n \geq (d-1)k$,

\[
p_{d,k,n} = \binom{n + (d-1)k - d}{n - (d-1)k}.
\]

Proof. Let us prove the result by induction. In dimension 3, according to [2], the number of directed plateau polycubes of width $k$ and having a lateral area equal to $n$ is equal to,

\[
\binom{n + 2k - 3}{n - 2k}.
\]

Here, this result corresponds to the case of $d = 3$. Let us now suppose that, for a given $d$

\[
p_{d,k,n} = \binom{n + (d-1)k - d}{n - (d-1)k},
\]

and let us prove that

\[
p_{d+1,k,n} = \binom{n + dk - d - 1}{n - dk}.
\]

From Theorem 3.2

\[
p_{d+1,k,n} = \sum_{j_2+j_3+\ldots+j_{d+1}=n}^{d+1} \prod_{l=2}^{d+1} \binom{j_l + k - 2}{j_l - k}
\]

\[
= \sum_{j_{d+1}=k}^{n-(d-1)k} \binom{j_{d+1} + k - 2}{j_{d+1} - k} \sum_{j_2+j_3+\ldots+j_{d+1}=n-j_{d+1}}^{d} \prod_{l=2}^{d} \binom{j_l + k - 2}{j_l - k}
\]

\[
= \sum_{j_{d+1}=k}^{n-(d-1)k} \binom{j_{d+1} + k - 2}{j_{d+1} - k} \binom{n - j_{d+1} + (d-1)k - d}{n - j_{d+1} - (d-1)k}.
\]
Setting \( i = j_{d+1} - k, \ m = n - dk, \ a = 2k - 2 \) and \( b = 2(d-1)k - d, \) we obtain

\[
\sum_{i=0}^{m} \binom{i+a}{i} \binom{m-i+b}{m-i}.
\]

Using Lemma 3.2 and replacing \( m, a \) and \( b \) by their values we get the formula.  

\[
\sum_{i=0}^{m} \binom{i+a}{i} \binom{m-i+b}{m-i}.
\]

4 Generating functions

Let \( P_{d,k}(t) \) be the generating function of the directed plateaus polyhypercubes of dimension \( d \) and width \( k \) according to the lateral area.

\[
P_{d,k}(t) = \sum_{n \geq 1} p_{d,k,n} t^n.
\]

**Proposition 4.1.** For \( k \geq 1, \) we have

\[
P_{d,k}(t) = \frac{t^{k(d-1)}}{(1-t)^{2k(d-1)-(d-1)}}.
\]

**Proof.** Using Theorem 3.2, we get

\[
P_{d,k}(t) = \sum_{n \geq 0} \prod_{l=2}^{d} \binom{j_l+k-2}{j_l-k} t^n.
\]

For \( l \) such that \( 2 \leq l \leq d, \) if \( j_l < k \) or \( j_l > n - (d-1)k \) then \( \prod_{l=2}^{d} \binom{j_l+k-2}{j_l-k} = 0. \) Therefore,

\[
P_{d,k}(t) = \sum_{n \geq 0} \prod_{j_l \geq 0} \binom{j_l+k-2}{j_l-k} t^n.
\]

It is know from Barcucci et al. [3], that

\[
\sum_{n \geq 0} \binom{n+k-2}{n-k} t^n = \frac{t^k}{(1-t)^{2k-1}},
\]

thus we get the result.  

Let

\[
P_d(t, x) := \sum_{k \geq 1} P_{d,k}(t)x^k,
\]

be the generating function of directed plateau polyhypercubes according to the width (coded by \( x \)) and the lateral area (coded by \( t \)).

From Proposition 4.1 then

\[
P_d(t, x) = \frac{xt^{d-1}(1-t)^{d-1}}{(1-t)^{2(d-1)} - xt^{d-1}}.
\]

From this expression we deduce the following theorem.
Theorem 4.1. Let $P_d(t)$ be the generating function of directed plateau polyhypercubes according to the lateral area. Then for $d \geq 3$,

$$P_d(t) = \frac{t^{d-1}(1-t)^{d-1}}{(1-t)^{2(d-1)} - t^{d-1}}.$$

Proof. We set $x = 1$ in equation [1]. ■

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