Noncommutative Algebras, Nano-Structures, and Quantum Dynamics Generated by Resonances

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Abstract

We observe “quantum” properties of resonance equilibrium points and resonance univariant submanifolds in the phase space. Resonances between Birkhoff or Floquet–Lyapunov frequencies generate quantum algebras with polynomial commutation relations. Irreducible representations and coherent states of these algebras correspond to certain quantum nano-structure near the classical resonance motion. Based on this representation theory and nano-geometry, for equations of Schrödinger or wave type in various regimes and zones (up to quantum chaos borders) we describe the resonance spectral and long-time asymptotics, resonance localization and focusing, resonance adiabatic and spin-like effects. We discuss how the mathematical phase space nano-structures relate to physical nanoscale objects like dots, quantum wires, etc. We also demonstrate that even in physically macroscale Helmholtz channels the resonance implies a specific quantum character of classical wave propagation.

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The paper consists of several parts. Part I follows the material of the author lectures at Petrovskii seminar & Moscow Math. Society Conference (May, 2004) and St.-Petersburg University & Steklov Math. Institute Conference (June, 2004).

Part II contains the systematic description of resonance algebras. In the next parts, we shall analyze the nano- and micro- phase space structures and discuss various types resonance phenomena for the Schrödinger type and wave equations, and also establish a bridge to real physical scales.

Part I

0 Introduction

Equations of mathematical physics describing propagation of waves admit solutions (or regimes, or states) of very different kinds. Many important physical applications deal with solutions which are not completely chaotic but follow distinguished “integrable” motions, say, equilibria. These solutions can be considered as certain excitations around the integrable classical core motion. The leading part of excitations is described by a model equation. By resolving this model equation, it is then possible to compute solutions of the original problem via the perturbation theory. Such a scheme goes back to Laplace, Rayleigh, Poincare, Ehrenfest, Birkhoff, Bogolyubov.

Usually, in this approach one presupposes to obtain the model equation to be as simple as possible, say, reducible to trivial scalar operators or to first-order differential operators of classical type. But, in many crucial cases, the model equation occurs to be of a nontrivial quantum type, that is, it carries a certain noncommutative algebra structure. The most typical reason for these quantum algebras to appear is the degeneracy of spectrum of the core integrable motion.

Such a quantum behavior can arise not only in nanoscale problems of atomic physics, but even in purely classical wave propagation problems at usual macrophysical scales. Thus one can claim that there are quantum effects in nonquantum wave systems as well. This very interesting phenomenon is the main motive of our present work.
Studying wave equations very often uses the analogy with classical mechanical systems and exploits the phase space geometry. The quantization technique, developed to obtain an operator representation of the phase space geometric structures, can be effectively applied to construct approximate or exact solutions of wave equations. This general claim is supported, in particular, by the progress in the semiclassical asymptotics [1].

The ray method [2] and the general Maslov’s canonical operator theory [3] were developed to construct asymptotic (semiclassical) solutions of PDE localized at points, trajectories, tori, or other invariant submanifolds in a phase space.

In multidimensional case all approaches of the semiclassical approximation theory have the well-known stumbling block: the incommensurability condition for frequencies of the classical Hamiltonian dynamics over the invariant submanifolds.

If the resonance between frequencies takes place then the usual methods fail. The resonance situation is an old open problem in the theory of wave and quantum equations. Of course, this is related to the resonance problems in classical mechanics [4], for instance, to the resonance theory of averaging and normal forms [4, 5]. However, in the wave mechanics, at least in stable case, the resonance problems are simpler, since they deal with a discrete spectrum and a finite-dimensional degeneracy.

An example can be presented by the Schrödinger (or Helmholtz) operator whose potential (or index of refraction) has a nondegenerate minimum (maximum) point and square roots of eigenvalues of the second derivative matrix at this point, in Euclidean coordinates, are commensurable. The semiclassical asymptotics of the spectrum near such a bottom (or maximum) is an intriguing question unsolved until now.

Another known unsolved problem: spectral asymptotics corresponding to stable trajectories, e.g., geodesics, in the case of a resonance between Lyapunov frequencies or Floquet frequencies [2, 6].

One more interesting question is the long-time evolution of wave packets localized at a resonance stable equilibrium point, or trajectory, or torus.

We suggest a way to solve these resonance problems by studying some noncommutative algebraic structures in micro- and nano-zones near the resonance core motion, and by applying and developing general methods of quantum geometry. We observe that each resonance proportion between frequencies generates an algebra with polynomial commutation relations (polynomial Poisson tensor) and its irreducible representations are given by hyper-
geometric Kählerian structures. The model equation is an equation over this resonance algebra. It is not of classical type, i.e., it has an order greater than one (in the irreducible representation). Therefore the nanozone becomes a purely quantum one. In microzones the model equation is reduced to the first order and resolved by the semiclassical technique. We discuss some important physical examples and detect interesting effects generated by resonances in classical and quantum wave equations.

The given first part of the paper describes mostly the material of the author’s lectures [7, 8].

1 Correlation of modes in resonance clusters

Let us consider the operator

\[ H = H_0 + v \quad \text{in} \quad L^2(\mathbb{R}^2), \]  

(1)

where

\[ H_0 = -\frac{\hbar^2}{2} \Delta + V_2, \quad v = V_3 + V_4 + \ldots, \]  

(2)

and by \( V_j \) we denote a \( j \)-linear form on \( \mathbb{R}^2 \). For instance, \( V_2 \) is a quadratic form, and the operator \( H_0 \) is just the Hamiltonian of a harmonic oscillator.

Let \( A > 0 \) and assume that the domain in \( \mathbb{R}^2 \) where the potential \( V_2 + v \) does not exceed the value \( A \) is a connected neighborhood of zero. We are interested in spectral properties of the operator \( H \) on the energy interval \((0, A)\).

Denote by \( \alpha, \beta \) the frequencies of the oscillator \( H_0 \). Then the spectrum of \( H_0 \) consists of the numbers

\[ \lambda_{m,l} = \hbar \alpha \left( m + \frac{1}{2} \right) + \hbar \beta \left( l + \frac{1}{2} \right) \]  

(3)

with known eigenfunctions \( |m,l\rangle \in L^2(\mathbb{R}^2) \) given by Hermite polynomials multiplied by the Gaussian exponent.

The resonance occurs when the following condition holds:

\[ \frac{\alpha}{\beta} = \text{rational number}. \]  

(4)

In this case, the spectrum of \( H_0 \) is degenerate, that is, there are nontrivial clusters of pairs \((m, l) \sim (m', l')\) such that \( \lambda_{m,l} = \lambda_{m',l'} \).
To understand what is the effect of resonance, let us consider the matrix elements

\[ E(t) = \langle m', l' | e^{-\frac{it}{\hbar}H} | m, l \rangle \]

which control the correlation (the transition probability) between the modes \(|m, l\rangle\) and \(|m', l'\rangle\).

In the nonresonance case where the ratio \(\alpha/\beta\) is irrational, for any \(t \sim h^{-s}\), one has

\[ E(t) \sim h^{1/2} \quad \text{as} \quad h \to 0. \]

But in the resonance case, if \(t \sim h^{-s}\) (for some \(s\)) and the pairs \((m, l) \sim (m', l')\) are inside a cluster, then

\[ E(t) \sim O(1) \quad \text{as} \quad h \to 0. \]

Thus, the resonance implies a strong correlation of modes inside the cluster, or a possibility of transition along modes in the cluster, that is, a dynamics. A phase space geometry which underlies this dynamics is of compact type, since the cluster is finite.

These are analytic and geometric consequences of the resonance. But behind all there is an algebraic phenomenon generated by the resonance.

2 Noncommutative resonance algebras

Let us consider a neighborhood of the origin in \(\mathbb{R}^2\) of order \(h^{1/N}\). We call this domain an \(N\)th microzone if \(N > 2\). If \(N = 2\), then we use a specific term a nanozone.

In micro or nanozones, the potential \(v\) in (1), (2) can be considered as a perturbation with respect to the leading part \(H_0\).

Denote by \(M_0\) the algebra of integrals of motion for \(H_0\), or the commutant of \(H_0\). Thus, each element from \(M_0\) commutes with \(H_0\). In the \(N\)th microzone, one can find a unitary operator \(U\) such that

\[ H = h^{2/N}U^{-1}(H_0 + h^{1/N}F_1 + h^{2/N}F_2 + \cdots + h^F_{LN})U + O(h^{L+3/N}), \quad (5) \]

where \(F_j \in M_0\) for all \(j\). For instance, the operator \(F_1\) is just the projection of \(V_3\) onto \(M_0\), the operator \(F_2\) is determined (explicitly) by \(V_3\) and \(V_4\), etc. The representation (5) is the result of application of the quantum averaging method developed in a similar framework in [9 10 11 12]. As we see from
the study of the operator $H$ in the $N$th microzone can be reduced to the study of operators from the algebra $M_0$.

In the nonresonance case the algebra $M_0$ is commutative.

In the resonance case (1), the algebra $M_0$ is noncommutative.

Thus, the resonance implies the “quantum” behavior of the problem under study. The word “quantum” we use as a synonym of “noncommutative.” The “quantum ray method,” the “quantum characteristics,” the “quantum geometry” – all this appears as a consequence of noncommutativity of the algebra $M_0$ under the resonance.

The commutation relations in the algebra $M_0$ are of the following type:

$$[A_j, A_k] = -i\hbar' \Psi_{jk}(A).$$

(6)

Here $\hbar' = \hbar^{1-2/N}$, and $A = \langle (A_j) \rangle$ is a finite set of generators, $\Psi_{jk}$ is a Poisson (quantum) tensor. One can choose the generators in such a way that the components $\Psi_{js}$ be polynomial, and so we can say that $M_0$ is an algebra with polynomial commutation relations. In general, these relations do not belong to the class of Lie algebras and present more complicated finitely generated algebras whose study began not so long ago (see the reviews in [13, 14] and the references therein).

In the case of the simplest isotropic resonance $1:1$ (that is, $\alpha = \beta = 1$), one has the following relations in $M_0$:

$$[A_1, A_2] = -i\hbar' A_3,$$

$$[A_2, A_3] = -i\hbar' A_1,$$

$$[A_3, A_1] = -i\hbar' A_2.$$  

(6a)

So, in this case, $M_0$ is the enveloping of the Lie algebra $\text{su}(2)$. The spectral analysis of the operator (5) in this case is reduced to the study of a Hamiltonian over $\text{su}(2)$, which can be done by the standard technique (see, for instance, [15, 16, 17]).

Much more interesting case is represented by anisotropic resonances, say, the resonance $1:2$ (where $\alpha = 1$, $\beta = 2$). Denote by $x', y'$ rescaled Cartesian coordinates adapted to the $N$th microzone. Introduce the annihilation operators

$$\eta = x' + \hbar' \frac{\partial}{\partial x'}, \quad \zeta = y' + \hbar' \frac{\partial}{\partial y'}.$$
Then the algebra $M_0$ is generated by self-adjoint operators

$$A_1 = \frac{1}{4} \eta^* \eta, \quad A_2 = \frac{1}{12} (\eta^* \eta - 4 \zeta^* \zeta), \quad A_3 = \frac{1}{8} (\zeta^* \eta^2 + \eta^* \zeta^2), \quad A_4 = \frac{1}{8} i (\zeta^* \eta^2 - \eta^* \zeta^2).$$

(7)

The commutation relations (6) in this case (resonance 1 : 2) are the following:

$$[A_1, A_2] = 0, \quad [A_1, A_3] = -i \hbar' A_4, \quad [A_1, A_4] = i \hbar' A_3,$$
$$[A_2, A_3] = -i \hbar' A_4, \quad [A_2, A_4] = i \hbar' A_3,$$
$$[A_3, A_4] = -3 i \hbar' \left(A_1 A_2 - \frac{\hbar'}{4} A_1 + \frac{\hbar'}{4} A_2\right).$$

(8)

The Casimir elements of this non-Lie algebra are

$$C_1 = A_1 - A_2,$$
$$C_2 = 3 A_1^2 A_2 - A_1^3 + A_3^2 + A_4^2 - \frac{3 \hbar'}{2} A_1^2 + \frac{3 \hbar'}{2} A_1 A_2 + \frac{3 \hbar'^2}{4} A_2 + \frac{\hbar'^2}{4} A_1.$$

(8a)

Relations (8) remind quadratic algebras appearing in the theory of infinite-dimensional integrable systems (but now without the Hopf axiom, see [18, 19]).

Each resonance proportion $q : r$ in (4) generates an algebra $M_0$ with polynomial commutation relations (6). The coefficients of the polynomial $\Psi_{jk}$ in (6) can be made all be integer numbers determined by $q, r$. The same is true for more than two resonance frequencies as well.

From (5) it follows that, in the micro- and nanozone near the bottom of the potential, the operator (1) is approximately reduced to the Hamiltonian

$$\hbar^{2/\nu} (H_0 + \hbar^{1/\nu} f_1(A) + \hbar^{2/\nu} f_2(A) + \ldots),$$

(9)

with some polynomials $f_1, f_2, \ldots$ in the generators $A$ of the algebra $M_0$. The leading part $H_0$ in (9) is a Casimir element in $M_0$ and can be replaced by a scalar (in the irreducible representation).

For example, consider the case of the resonance 1 : 2. The operator

$$H = -\frac{\hbar^2}{2} \Delta + \frac{1}{2} x^2 + 2 y^2 + x^2 y + \gamma x^4, \quad \gamma \geq \frac{1}{8},$$

(10)
in the $N$th nanozone is reduced to
\[
\hbar\left(n + \frac{3}{2}\right) + \frac{\hbar^{3/N}}{\sqrt{2}}A_3 + O(\hbar^{4/N}),
\]
where $A_3$ is the generator of the algebra $(8)$ given by the third formula in $(7)$. Thus, one needs to study the spectrum of the element $A_3$ in the $n$th irreducible representation of the algebra $(8)$ in order to find the bottom part of the spectrum of the operator $(10)$.

The following important question arises: how to construct irreducible representations of algebras of the type $(5)$?

For the case of Lie algebras (in particular, for the resonance case $1 : 1$), one has the Kirillov orbit method [20] and the general Kostant–Souriau geometric quantization [21, 22]. For anisotropic resonances and algebras with nonlinear commutation relations like $(6)$, one needs an extension of this quantization scheme.

### 3 Quantum nano-geometry

The directing idea of the geometric quantization is to use the classical Poisson (symplectic) geometry and polarizations in order to determine quantum objects. The classical analog of the algebra $M_0$ is a Poisson algebra $\mathcal{M}_0$ of functions on the phase space commuting with the symbol of the operator $H_0$. The set $\mathcal{M}_0$, of course, was considered in classical mechanics dealing with resonance systems, see, e.g., [23], but the Poisson structure on $\mathcal{M}_0$ was not described and studied.

The classical analog of the quantum commutation relations $(10)$ is given by a Poisson tensor of a polynomial type. For example, in the case of resonance $1 : 2$, we obtain the following quadratic Poisson algebra:

\[
\begin{align*}
\{A_1, A_2\} &= 0, \quad \{A_1, A_3\} = A_4, \quad \{A_1, A_4\} = -A_3, \\
\{A_2, A_3\} &= A_4, \quad \{A_2, A_4\} = -A_3, \\
\{A_3, A_1\} &= 3A_1A_2.
\end{align*}
\]

The Casimir functions in this algebra are

\[
C_1 = A_1 - A_2, \quad C_2 = 3A_1^2A_2 - A_1^3 + A_3^2 + A_4^2.
\]
Generic symplectic leaves are two-dimensional surfaces

\[ \Omega = \{ C_1 = \text{const}, C_2 = \text{const} \} \subset \mathbb{R}^4 \]  

(13)
diffeomorphic to \( S^2 \). The leaves are Kählerian manifolds with respect to the complex structure

\[ z = \frac{A_3 + iA_4}{c - A_1}, \quad c = \text{const}. \]  

(14)
The symplectic form \( \omega_0 \) generated by brackets (12) on the leaf (13) can be expressed by a Kählerian potential \( F_0 \) in the standard way

\[ \omega_0 = i \bar{\partial} \partial F_0 = ig_0 dz \wedge \bar{z}. \]  

(15)
Note that the differential 2-form

\[ \rho = i \bar{\partial} \partial \log g_0 \]  

(16)
is the Ricci form on \( \Omega \) corresponding to the complex structure (14).

Now if one follows the geometric quantization ideas, the linear bundle over \( \Omega \) with the curvature \( i \omega_0 \) must be introduced. In the Hilbert space of sections of this bundle, the operators of irreducible representation would act.

However, there are two principal problems. First, we do not know the measure on \( \Omega \) with respect to which the Hilbert norm in the space of sections has to be defined. This measure must satisfy a reproducing property \[24\]. For the inhomogeneous case (where relations (6) are not linear) the existence of such a reproducing measure is, in general, unknown.

Secondly, even if one knows the reproducing measure, the problem is that the operators of irreducible representation of the algebra (6) constructed canonically by the geometric quantization scheme would be pseudodifferential, but not differential operators.

That is why we modify the quantization scheme and from the very beginning replace the symplectic form \( \omega_0 \) by another “quantum” form \( \omega \) in a way that guarantees the existence of the reproducing measure and the existence of irreducible representations of the algebra (6) by differential operators. This approach is explained in \[24\].

Note that the opportunity to obtain irreducible representations of the algebra \( M_0 \) by differential operators is exactly the reason why the polynomial structure of the right-hand side of (6) is so critical.
4 Irreducible representations and coherent states of resonance algebra

Here we demonstrate calculations for algebra $\mathfrak{G}$ related to the resonance $1:2$.

Let us consider the following hypergeometric equation

$$2\hbar' \frac{d^2 K}{ds^2} + (s + \varepsilon_n \hbar') \frac{dK}{ds} - \lfloor n/2 \rfloor K = 0, \quad s > 0$$

$K(0) = 1,$

where $n \geq 0$ is an integer, $\varepsilon_n = 1$ or $\varepsilon_n = 3$ if $n$ is even or odd, and the brackets $\lfloor \cdots \rfloor$ denote the integer part. The solution is given by a hypergeometric function of $1F1$ type, or more precisely,

$$K(s) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{\lfloor n/2 \rfloor!}{j! (2j - 2 + \varepsilon_n)! (\lfloor n/2 \rfloor - j)!} \left( \frac{s}{\hbar'} \right)^j.$$

Here the double factorial $!!$ denotes the product over odd numbers, starting from 1.

Also consider the “dual” hypergeometric equation

$$2\hbar' \frac{d^2 L}{ds^2} - (s + \varepsilon_n \hbar' - 4\hbar') \frac{dL}{ds} - (\lfloor n/2 \rfloor + 2) L = 0, \quad s > 0,$$

$$\frac{1}{\hbar'} \int_0^{\infty} L(s) \; ds = 1.$$ Introduce the Hilbert space $H_n$ of polynomials square integrable over $\mathbb{C}$ with respect to the measure

$$d\mu = L(|z|^2) \; d\bar{z} \; dz, \quad z \in \mathbb{C}.$$ 

Lemma 4.1. The operators

$$\tilde{A}_1 = \frac{\varepsilon_n - 1}{4} \hbar' + \hbar' \bar{z} \partial, \quad \text{where} \quad \partial = \partial/\partial \bar{z},$$

$$\tilde{A}_2 = \frac{\varepsilon_n - 1}{12} \hbar' - \frac{2\hbar' \lfloor n/2 \rfloor}{3} + \frac{5\hbar'}{12} \bar{z} \partial,$$

$$\tilde{A}_3 = \hbar'^2 \bar{z} \partial^2 - \frac{\hbar'}{2} \left( \bar{z}^2 - \varepsilon_n \hbar' \right) \partial + \frac{\hbar' \lfloor n/2 \rfloor}{2} \bar{z},$$

$$\tilde{A}_4 = -i\hbar'^2 \bar{z} \partial^2 - \frac{i\hbar'}{2} \left( \bar{z}^2 + \varepsilon_n \hbar' \right) \partial + \frac{i\hbar' \lfloor n/2 \rfloor}{2} \bar{z}.$$
realize the Hermitian irreducible representation of the algebra (8) in the Hilbert space $H_n$. In this representation the Casimir elements (8a) are $\hat{C}_1 = n\hbar/3$ and $\hat{C}_2 = 0$.

**Lemma 4.2.** Let us define the “vacuum” vector $|0\rangle$ as the solution of the equations

$$
A_1|0\rangle = a_1|0\rangle, \quad A_2|0\rangle = a_2|0\rangle, \quad (A_3 + iA_4)|0\rangle = 0,
$$

where $a_1, a_2$ are constants given by $a_j = \hat{A}_j \rangle$. Also define the coherent states in $L^2(\mathbb{R}^2)$:

$$
|z\rangle \overset{\text{def}}{=} K\left(z(A_3 - iA_4)\right)|0\rangle.
$$

Then the integral transformation

$$
T(\varphi) = \frac{1}{2\pi \hbar'} \int_{\mathbb{C}} \varphi(\overline{z}) |z\rangle d\mu(\overline{z}, z), \quad T : H_n \rightarrow L^2(\mathbb{R}^2),
$$

intertwines the representation (7) of the algebra (8) and the irreducible representation (18), i.e.,

$$
A_j \circ T = T \circ \hat{A}_j \quad (j = 1, \ldots, 4).
$$

**Lemma 4.3.** The hypergeometric polynomial $K$ (17) is the reproducing kernel for the space $H_n$. Moreover,

$$
K(|z|^2) = \langle z|z\rangle, \quad \frac{1}{2\pi \hbar'} \int_{\mathbb{C}} |z\rangle\langle z| d\mu = P_n,
$$

where $P_n$ is the projection in $L^2(\mathbb{R}^2)$ onto the $n$th irreducible component.

**Lemma 4.4.** Let the quantum Kählerian form on $\Omega$ be defined by

$$
\omega = i\hbar' \partial\overline{\partial} \ln K,
$$

where $K$ is the hypergeometric polynomial (17), and $\hbar' = \hbar^{1-2/N}$, $N \geq 2$. Then the reproducing measure on $\Omega$ is given by

$$
dm = K(|z|^2) L(|z|^2) \overline{d\overline{z}} dz,
$$

where $L$ is the solution of the dual hypergeometric equation (17a).
In a micro-zone, where $N > 2$ and $\hbar'$ is a small parameter, $n \sim 1/\hbar'$, the following asymptotics hold:

$$
\omega = \omega_0 + \frac{\hbar'}{2} \rho + O(\hbar'^2), \quad dm = dm_0(1 + O(\hbar')).
$$

Here $dm_0 = g_0 d\pi dz$ is the Liouville measure corresponding to the form $\omega_0$ on $\Omega$, and $\rho$ is the Ricci form, see (15), (16). Moreover, one has

$$
K = e^{F_0 / \hbar'} \sqrt{g_0}(1 + O(\hbar')), \quad L = e^{-F_0 / \hbar'} \sqrt{g_0}(1 + O(\hbar')).
$$

Also the following identities hold:

$$
\frac{1}{2\pi \hbar'} \int_\Omega \omega = \lfloor n/2 \rfloor, \quad \frac{1}{2\pi \hbar'} \int_\Omega dm = \lfloor n/2 \rfloor + 1.
$$

From this lemma one can clearly see that there is an essential difference between the classical and quantum Kählerian structures on symplectic leaves $\Omega$. In micro-zones, where $N > 2$, this difference is just asymptotical: the quantum structure is an $\hbar'$-perturbation of the classical one. But in the nanozone, where $N = 2$ and $\hbar' = 1$, the quantum structure is not a perturbation of the classical structure. Thus, one can talk about a specific quantum nano-geometry accompanying the resonance.

5 Spectrum asymptotics in resonance clusters

Now coming back to the spectral problem for the Hamiltonian (10), we can apply the coherent transform (20) to the operator (11). Then our problem in the $N$th micro zone is reduced to studying the operator

$$
\hbar \left(n + \frac{3}{2} + \frac{\hbar^{3/N}}{\sqrt{2}} \tilde{A}_3 + O(\hbar^{4/N})\right),
$$

where $\tilde{A}_3$ is given by (18).

Recall that $\hbar' = \hbar^{1-2/N}$ in (18). In particular, at the nanozone near the bottom of the potential, we have $N = 2$, $\hbar' = 1$, and (22) becomes

$$
\hbar \left(n + \frac{3}{2} + \frac{\hbar^{3/2}}{\sqrt{2}} \left(\sqrt{2} \tilde{z} - \frac{1}{2} (\sqrt{2} - \varepsilon) \tilde{\varpi} + \frac{\lfloor n/2 \rfloor}{2} \tilde{z} \right) + O(\hbar^2)\right).
$$

(23)
The model ordinary differential operator staying in \( \text{term } \hbar^{3/2} \) determined the asymptotical properties of the original Hamiltonian (10). By resolving the spectral problem in the space \( \mathcal{H}_n \):

\[
\frac{1}{\sqrt{2}} \left( z \overline{\partial}^2 - \frac{1}{2}(z^2 - \varepsilon_n) \overline{j} + \frac{[n/2]}{2} \overline{j} \right) \varphi(z) = \nu(z) \varphi(z),
\]
we obtain the eigenvalues \( \nu = \nu_{n,k} \) and the eigenfunctions (polynomials) \( \varphi = \varphi_{n,k} \), where \( k = 0, \ldots, [n/2] \).

**Theorem 5.1.** The asymptotics of the near bottom eigenvalues \( \lambda \) and the eigenfunctions \( \psi \) of the Hamiltonian \( \text{(10)} \) is the following:

\[
\lambda = \hbar(n + \frac{3}{2}) + \hbar^{3/2} \nu_{n,k} + O(\hbar^2),
\]

\[
\psi = T(\varphi_{n,k}) + O(\hbar^{1/2}).
\]

Here \( n = 0, 1, \ldots \) and \( 0 \leq k \leq [n/2] \); the numerating numbers \( n, k \) are of order \( O(1) \) as \( \hbar \rightarrow 0 \).

Equation (24), which gives the corrections \( \nu_{n,k} \), and the coherent transform \( T \) in (25) present the main difference of this resonance asymptotics from the standard oscillatory approach.

Of course, all the higher corrections of the asymptotics (25) are calculated explicitly by simple perturbation series.

If we go away from the nanozone near the bottom of the potential (10) to some \( N \)th microzone, then we must replace the model equation (24) by the equation

\[
\frac{1}{\sqrt{2}} \left( \hbar' z \overline{\partial}^2 - \frac{\hbar'}{2}(z^2 - \varepsilon_n \hbar') \overline{j} + \frac{\hbar'[n/2]}{2} \overline{j} \right) \varphi = \nu(z) \varphi. \tag{26}
\]

Here \( \hbar' = \hbar^{1-2/N} \) and \( n \sim O(1/\hbar') \). The number \( N \geq 3 \), so \( \hbar' \rightarrow 0 \) and \( n \gg 1 \). In this situation, we can approximately solve Eq. (26) using the technique of geometric coherent states over Lagrangian submanifolds (classical trajectories) in the symplectic leaf \( \Omega \) developed in [25, 26, 27].

We now briefly describe the result.

In view of (22), the classical Hamiltonian over \( \Omega \) is given by the coordinate function \( A_3 \) in the Poisson algebra (12). Let us consider the energy levels of this Hamiltonian, that is the closed curves \( \Lambda \subset \Omega \) defined by

\[
\Lambda = \{ A_3 = \nu \}. \tag{27}
\]
Denote by Σ a part of Ω bonded by the curve Λ and consider the quantization condition
\[
\frac{1}{2\pi \hbar'} \int_{\Sigma} \omega_0 = k + \frac{1}{2}, \quad k \in \mathbb{Z}.
\] (28)

This condition determines the discreet values \( \nu = \nu_{n,k} \) in (27). The numbers \( \nu_{n,k} \) give the leading part of the eigenvalue asymptotics in problem (26). The asymptotics of the corresponding eigenfunctions \( \varphi \) is given by the integral over Λ:
\[
\varphi_{n,k}(z) = \frac{1}{\sqrt{2\pi \hbar'}} \int_{\Lambda} \left( \sqrt{\dot{z}(t)} + O(\hbar') \right) e^{i \int_{t_0}^{t} \theta(t') \frac{dz}{dz(t')}} K(\overline{z}(t)) \, dt.
\]

Here \( \{ z = z(t) \} \) is the parametrization of points of Λ by the time \( t \) in the Hamiltonian system over Ω generated by \( A_3 \), the one-form \( \theta \) is the primitive of the classical Kählerian form \( \omega_0 \), namely, \( \theta = i\partial F_0 \), and \( K \) is the hypergeometric polynomial (17).

So, finally, we obtain the asymptotics of higher energy levels of the original operator \( H \).

**Theorem 5.2.** Let \( \hbar' = \hbar^{1-2/N} \), \( N \geq 3 \). The asymptotics of the eigenvalues of the operator (10) with the numerating numbers \( n \sim 1/\hbar' \) is given by
\[
\lambda = \hbar \left( n + \frac{3}{2} \right) + \hbar^{3/N} \nu_{n,k} + O(\hbar^{4/N}).
\] (29)

Here the values \( \nu_{n,k} = \nu_{n,k}(\hbar') \) are determined by the quantization condition (28), where \( 0 \leq k \leq [n/2] \). The asymptotics of the eigenfunctions corresponding to the eigenvalues (29) is the following:
\[
\psi = \frac{1}{\sqrt{2\pi \hbar'}} \int_{\Lambda} \sqrt{\dot{z}(t)} \exp \left\{ i \hbar' \int_{0}^{t} \left( \theta + O(\hbar^{1/(N-2)}) \right) \right\} |z(t)| \, dt,
\] (30)

where \( |z \rangle \) are coherent states of the algebra (8) in the space \( L^2(\mathbb{R}^2) \) given by (19). All the corrections in the remainders in (29) and (30) are controlled by the higher-order terms in (11) and are calculated explicitly.

By increasing the number \( N = 3, 4, \ldots \), we can go further and further away from the bottom point of the spectrum of \( H \). But in any case the quantum number \( n \sim \hbar^{-(1-2/N)} \) in our asymptotics can never reach the
order $O(h^{-1})$. Such large numbers $n \sim h^{-1}$ correspond to the energy levels of $H$ which are at the distance $O(1)$ from the bottom level. In this area, the behavior of the operator $H$ is completely chaotic.

The semiclassical parameter $h'$ in (28) and (30) depends on the index $N$ of the microzone, and $h' \gg h$. One could take $h' = h$ only in the chaos zone where the asymptotics (29) and (30) fail.

Note that arriving closer to the chaos zone, i.e., taking larger values of $N$, one needs to take into account higher-order terms in expression (29) at least up to $f_{2N-2}$. All these terms give a contribution to the asymptotics of eigenvalues considered with the accuracy $o(h^2)$. This means that all the terms $V_j (j = 3, \ldots, N)$ of the Taylor expansion of the potential at the resonance bottom point contribute to the $o(h^2)$-asymptotics of eigenvalues of the Schrödinger operator in the $N$th microzone. But the geometry of the curve and the leading quantization condition are determined by the term $V_3$ only.

### 6 Resonance long-time evolution

As a simple application of the above results, we describe the solution of the long-time evolution problem for the Schrödinger operator in the presence of resonance. Let us consider the Cauchy problem for the operator (10):

$$i\hbar \frac{\partial \chi}{\partial t} = -\frac{\hbar^2}{2} \Delta \chi + \left(\frac{1}{2} x^2 + 2y^2 + x^2y + \gamma x^4\right) \chi,$$

where $\chi^0 \in S(\mathbb{R}^2)$. The initial data are localized in a nanozone $O(\sqrt{\hbar})$ near the bottom point $x = y = 0$ of the potential. Let us choose the following time values:

$$t = \tau/\sqrt{\hbar}, \quad \tau \sim 1.$$

From the results described above, we obtain the asymptotics

$$\chi \approx \sum_{n \geq 0} \exp\{-i(n + 3/2)\tau/\sqrt{\hbar}\} \chi^*_n(x/\sqrt{\hbar}, y/\sqrt{\hbar}),$$

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where $\chi^\tau_n$ is the solution of the evolution equation in the $n$th irreducible representation of the algebra $\mathfrak{g}$:

$$
\left( -i \frac{\partial}{\partial \tau} + \frac{1}{\sqrt{2}} A_3 \right) \chi^\tau_n = 0, \quad \chi^\tau_n \bigg|_{\tau=0} = P_n(\chi^0).
$$

(34)

Formula (33) demonstrates that the Schrödinger evolution of the wave packet (31) in the long-time interval (32) still keeps the packet to be localized.

The problem (34) after the coherent transform is reduced to the evolution problem for the second order ordinary differential operator on the left-hand side of (24). This model problem does not have any small or large parameters and must be resolved exactly.

This is the quantum nano-dynamics describing the Schrödinger evolution of wave packets localized at the resonance bottom point.

For smaller time interval

$$
t \sim O(h^{-1/N}), \quad N \geq 3,
$$

(35)

we have, instead of (34), the problem with a small parameter

$$
\left( -i \hbar' \frac{\partial}{\partial \tau} + \frac{1}{\sqrt{2}} A_3 \right) \chi^\tau_n = 0, \quad \hbar' = h^{1-2/N},
$$

(36)

where $A_3$ is the $n$th irreducible representation is given by the left-hand side of (26). The evolution equation (36) can be asymptotically solved by using the semiclassical approximation theory. So, in the time intervals like (35), the asymptotics of the solution of the Cauchy problem (31) is computed explicitly.

Let us remark that the equations of the form (36) and more general equations which involve algebras of the type (6) are related to some hidden geometry (classical and quantum). The global geometric analysis of these equations leads to constructions of symplectic and quantum paths, to the symplectic and quantum holonomy and curvature, to the translocation operation, see [28, 29, 30].

It may be relevant to mention that in the absence of resonance, the Cauchy problem like (31) is asymptotically solved without any difficulties on the time interval $t \sim O(1/h^\infty)$. 

17
7 Different types of resonance algebras

It is known that there are many different types of resonances. Their variety strongly depends on stability or unstability of the first variation of the dynamical system generated by the leading part of the Hamiltonian. Resonances can be stable, unstable, neutral and combinations of these types. Respectively, the resonance algebra $M_0$ can be of compact, noncompact, nilpotent type or be a mixture (direct product) of these types. We just briefly mention three simple examples.

Example 7.1 (Landau model). The well-known example of the neutral resonance is given by the Hamiltonian of a charged particle moving along a plane in a homogeneous magnetic field which is perpendicular to the plane. In this case the first variation matrix of the Hamiltonian field has zero eigenvalue which is twice degenerate. The resonance algebra $M_0$ of functions commuting with the Hamiltonian is nilpotent and just reduced to the three-dimensional Heisenberg algebra.

Example 7.2 (Inverted oscillator). An example of the unstable resonance is given by the inverted oscillator, i.e., the oscillator (2) which has not a potential quadratic well, but a potential quadratic hill. In this case the oscillator frequencies $\alpha, \beta$ are imaginary, say, $\alpha = \beta = i$. Then the first variation matrix has two real eigenvalues $\pm 1$ which are twice degenerate. The resonance algebra $M_0$ in this case is the enveloping of the Lie algebra $su(1,1)$ which is of noncompact type.

In the anisotropic version, where $\alpha = i, \beta = 2i$, the resonance algebra $M_0$ is given by quadratic commutation relations similar to (8).

Example 7.3 (Artificial magneto-atoms). The resonances described in Section 3 are, of course, stable. Their algebras are of compact type. One of physical models, where such resonances appear, is the so-called artificial atom (or 2-dimensional quantum dot surrounded by electrons moving in the plane). The resonance means the degeneracy of energy levels of such an “atom.” This imply “electron shells,” “filling numbers,” etc. and make such a system highly stable.

It is interesting to consider the artificial atom combined with a homogeneous magnetic field which is perpendicular to the configuration plane. Let the potential well created by the central dot be just quadratic with frequencies $\alpha = \beta = \omega_0$. Then we simply have the 2-dimensional Fock model:
oscillator plus magnetic field. Let us denote by $\omega_L$ half the Larmor frequency which is equal to the magnetic field strength multiplied by the electron charge and divided by the electron mass. One can claim that the energy levels of the Fock model are degenerate if and only if

$$\left(\frac{\omega_L}{\omega_0}\right)^2 = \frac{s^2}{(k^2 - s^2)}, \quad (37)$$

where $s, k$ are some integers, $0 \leq s < k$.

Under this condition we obtain a stable artificial magneto-atom. Its algebra of integrals of motion $M_0$ has four generators with polynomial commutation relations of type (8), and two Casimir functions. All components of the Poisson tensor (the right-hand side of the commutation relations), except one, are linear. The nonlinear component is a polynomial of degree $d_0 = l + m - 1$, where $l, m$ are coprime numbers such that

$$\frac{k + s}{k - s} = \frac{l}{m}. \quad (38)$$

For different realizations of the magneto-atom, that is for different ratios $\omega_0/\omega_L$, one can have arbitrary numbers $l, m$ in (38) and make the degree $d_0$ of the algebra $M_0$ be any number. For instance:

- if $\omega_0 = 2\sqrt{2}\omega_L$, then $d_0 = 2$,
- if $\omega_0 = \sqrt{3}\omega_L$, then $d_0 = 3$,
- if $\omega_0 = 2\sqrt{6}\omega_L$, then $d_0 = 4$,
- if $\omega_0 = \frac{4}{3}\omega_L$, then $d_0 = 4$,
- if $\omega_0 = \frac{1}{2}\sqrt{5}\omega_L$, then $d_0 = 5$, etc.

If there are higher terms $V_3, V_4, \ldots$ in the potential of the central dot, or a kind of an external field, then they determine a Hamiltonian over the algebra $M_0$ like (9). The spectrum and eigenfunctions of this Hamiltonian can be derived using the method described in Sections 5 and 6. Thus one can calculate the splitting of the energy levels of this artificial magneto-atom with an arbitrary accuracy in $\hbar$.

Note that if one replaced the central dot potential (the quadratic well) by the central Coulomb potential, i.e., considered the usual atom rather than the artificial one, then its magnetic version might not exist at all. Indeed, it is well known that the model “Coulomb potential plus magnetic field” is chaotic; there is no spectral degeneracy and therefore no “magneto-atom.”

In this example we obtain resonance algebras with polynomial commutation relations of an arbitrarily high degree. The same growth of the degree is
observed in *multi-dimensional* resonances. For three-dimensional oscillator with the resonance 1 : 2 : 3 commutation relations in the algebra $M_0$ are of the third degree, and so on.

8 **Nano-structures over resonance trajectories and tori**

Another class of problems which can be solved by the methods described in Sections 5–7 is the asymptotics of spectral series corresponding to closed stable trajectories or invariant isotropic tori of the Hamiltonian dynamics in the presence of a resonance between transversal Lyapunov frequencies or a resonance between elements of the Floquet holonomy group. The Maslov complex isotropic bundle theory [31, 6] in this situation has to be supplemented with an additional noncommutative algebraic and geometric nano-structures like (6), (6a), (8), (12).

After application of the quantum averaging method, we again reduce the problem to studying the commutant $M_0$ of the leading part of the Floquet–Lyapunov transversal hamiltonian. The resonance means that the algebra $M_0$ is noncommutative, and we again can apply the quantum geometry technique to calculate irreducible representations of $M_0$, coherent states, etc. Here not only the hypergeometric type but also the theta type [21, 32] quantum deformations can arise.

In nano- and microzones near the trajectory (torus), in addition to the usual geometry [33], we obtain a bundle with fiber-polynomial Poisson structures like (12), and the bundle of their symplectic leaves. The problem is reduced to model differential equations over these bundles. In the nanozone of order $O(\sqrt{\hbar})$ these equations do not contain a small parameter and must be resolved exactly, but in microzones of order $O(\hbar^{1/N})$, $N \geq 3$, the model equations can be effectively analyzed and explicitly solved using the global geometric semiclassical technique.

**Example 8.1 (Nano-electrodynamics).** This analysis can be applied, for instance, to the Helmholtz equation describing propagation of classical electromagnetic waves in nonhomogeneous medium. Assume that the index of refraction of the medium has an extremal, i.e., it takes the maximum value along a certain curve and does not degenerate in directions transversal to the curve. Then a neighborhood of this extremal becomes a wave channel.
The Gaussian-type electromagnetic waves can propagate through this channel preserving their semiclassical localization near the extremal (the channel axis).

If the transversal frequencies are constant and are in resonance, then in nano- and microzones near the axis there appear certain noncommutative algebraic (quantum) structures and Poisson structures of the type described above. One can consider this phenomenon as a generation of a new type “quantum” polarization of the wave. The evolution of this polarization is described by a model differential equation in the quantum bundle over the extremal. Since the extremal is assumed to be stable, this polarization occurs to be of spin type. The “spin” variables are running along symplectic leaves of algebras like (6a) and (8) which are diffeomorphic to a sphere.

Of course, these “spin” variables have nothing to do with the actual physical spin. By the way, the situation is not at the atomic scale, the scale of the wave channel can be even macroscopic.

If the channel axis is not straight, but curvilinear, then the “spin” degrees of freedom correlate with its curvature and torsion. In this case one can speak about a specific “spin–orbit” interaction in the curvilinear resonance wave channel. We detect existence of something like quantum nano-electrodynamics near resonance extremals of the index of refraction.

9 Resonant quantization of wave mechanics

One observes the following:

– the resonance of Birkhoff or Lyapunov, or Floquet frequencies implies the appearance of specific noncommutative algebras, in general, they are not Lie algebras;

– such a noncommutative structure appears in equations of wave mechanics of various type, for instance, in the Helmholtz and Maxwell equations, describing propagation of classical electromagnetic waves along resonance channels, and so the quantum behavior can be inherent in classical wave systems too;

– there is a nontrivial Poisson and symplectic geometry in nano- and micro- phase spaces near the resonance motion, this resonance geometry is invisible in the usual classical limit;

– the structure of the phase space in nanozones near resonances losses its classical behavior and occurs to be pure quantum, in particular, the resonance
algebras have no classical Poisson limit in these zones, and so, the classical geometry must be replaced by a quantum one;

– to construct irreducible representations and coherent states for the resonance algebras, one needs to introduce some new concepts to the quantization procedure, in particular, the quantum Kählerian structures of hypergeometric- or theta-types;

– in nanozones near the resonances, the given spectral problem is reduced to a model differential equation of order more than 1, in general;

– the structure of the resonance algebra and of the accompanying quantum geometry are completely determined by the arithmetical proportion between frequencies, one can consider these structures as a presentation of arithmetic in wave mechanics;

– the micro- and nano- phase spaces related to the resonance carry certain hidden dynamical geometry, classical and quantum, in particular, quantum connections, quantum paths and translocations;

– this dynamic geometry opens a new way to describe the correlation and transition of modes in resonance clusters;

– the general algebraic and geometric technique allows one to resolve the old problem of resonances in the semiclassical approximation theory, in particular, in the spectral and in the long-time analysis.

On this way, we observe something interesting in spectral properties of the Schrödinger operator at resonance. For instance, in two dimensions, in the case of anisotropic resonance, say 1 : 2, the spectrum is given by a series in fractional powers \( \hbar^{1/2} \) of the “Planck constant” (see (25)).

But, perhaps, the main find in the study of the resonance problem is a class of very simple and fundamental physical systems whose description involves algebras with polynomial commutation relations like (8). This means that algebras of such a non-Lie type are actually very common, although they were not clearly visible and therefore their importance was not enough appreciated until now.

We refer to [34], and to the paper by Karasev & Novikova in the given Collection, for another class of basic physical examples where such types of algebras appear.

The ideas described above can imply a change in the viewpoint on the role of resonances in wave and quantum problems. First of all, systems with resonances possibly have to be considered not as exceptional (like in the KAM theory), but as the basic ones with additional perturbations which control deviations. From the quantum attitude the following is clear: resonances im-
ply high spectral multiplicities, this makes gaps between excited energy levels much wider, and therefore such systems become much more stable under perturbations. One can call this phenomenon a resonant stabilization. Note that all the fundamental atomic objects are resonant stabilized. The world on the whole is supplied with enough amount of such a resonant stabilizers.

Secondly, the resonances can be a source of models with noncommutative degrees of freedom (like spin or even with non-Lie commutation relations). The notion of resonant stabilizer might be useful for some extended physical concepts like the string theory, as well as for models of quantum computers and models used in modern micro- and nano-technologies.

Among possible new applications there is the opportunity to construct noncommutative optical channels (see Example 4) with an interference between “spin” modes, with “spin–orbit” effects, and with “spin” level tunneling or crossing. Another application is the artificial magneto-atoms (see Example 3) with the opportunity to change their “chemistry” just by varying resonance values of the magnetic field via formula (37). The artificial atoms are also interesting from the viewpoint of separation of their electron shells, which is closely related to the phenomenon of quantization of the configuration surface where they are located.

Next parts of the paper contain more detailed analysis of resonance noncommutative algebras, as well the description of several effects such as the resonance localization and focusing, resonance traps, resonance adiabatic and “spin” phenomena, secondary resonances, etc.

**Part II**

**0 Introduction**

The resonance between frequencies of a Hamiltonian (or quantum) motion is a source of interesting structures and effects in classical and wave mechanics [4, 5]. It was observed in [7, 8] that resonances generate noncommutative algebras which determine a specific “quantum” behavior of the system near a distinguished motion (or equilibria). Here we investigate the resonance algebras in a more systematic way.

The first two sections below are purely algebraic. We describe in detail the structure of the algebra of integrals of motion for the multidimensional
harmonic oscillator. If the frequencies of the oscillator are in a resonance, this algebra is noncommutative and commutation relations are, in general, not of the Lie type.

In Section 3, we consider examples of resonance algebras for two-dimensional oscillators, including the inverted (unstable) oscillator, and the oscillator in a magnetic field.

In Section 4, we describe the phenomenon of resonance precession in the case of two-dimensional oscillator. In Section 5, we discuss a general procedure of projecting the Poisson brackets and the appearance of the triple algebra structure. We consider the examples of 1:1 and 1:2 resonances, describe their triple algebras over the configuration plane, and demonstrate the way of solving the resonance precession system in noncommuting space–time coordinates.

The Appendix contains a short list of formulas of the operator averaging method.

1 Commutant of the oscillator

Let us consider a quadratic function \( H_0 \) on \( \mathbb{R}^{2M} = \mathbb{R}^M \times \mathbb{R}^M \) and the corresponding Hamiltonian vector field \( JdH_0 \), where \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \). Assume that the first variation matrix \( JD^2H_0 \) is stable, that is, it has only imaginary eigenvalues. Denote them by \( \pm i\omega_l \) (\( l = 1, \ldots, M \)). The function \( H_0 \), in suitable symplectic coordinates \( q, p \), can be written as

\[
H_0 = \frac{1}{2} \sum_{l=1}^{M} \omega_l (q_l^2 + p_l^2).
\]  

One can call \( H_0 \) the oscillator Hamiltonian, and call \( \omega_l \) its frequencies.

We consider the algebra \( F \) of all differential operators with polynomial coefficients on \( \mathbb{R}^{M} \). An element of this algebra is represented by the quantum oscillator

\[
\hat{H}_0 = \frac{1}{2} \sum_{l=1}^{M} \omega_l \left( q_l^2 - h^2 \frac{\partial^2}{\partial q_l^2} - \hat{h} \right),
\]  

where \( h > 0 \) is a fixed number. Denote by \( F_\omega \) the commutant of the oscillator (1.2) in the algebra \( F \). By definition, \( F_\omega \) consists of all elements from \( F \) commuting with \( \hat{H}_0 \). Obviously, \( F_\omega \) is an algebra.
We also consider the Poisson algebra $F_\omega$ of functions on $\mathbb{R}^{2M}$ which are integrals of motion of the Hamiltonian vector field $JdH_0$. Each element from $F_\omega$ has zero Poisson bracket with the function $H_0$ (1.1), i.e., Poisson commutes with $H_0$. The Poisson algebra $F_\omega$ is the classical analog of the associative algebra $F_\omega$.

Our goal in this section is to study the properties of the algebra $F_\omega$ (or $F_\omega$) related to the properties of the frequency system $\omega$.

We consider nonordered systems (sets) of real nonzero numbers $\alpha = \{\alpha_1, \ldots, \alpha_M\}$. We call $M$ the length of the system.

For any two systems $\alpha, \beta$, possibly, of different lengths, there are naturally defined systems $\alpha \cup \beta$ and $\alpha \cap \beta$, as well as the notion of subsystem $\beta \subset \alpha$.

Also for any real number $c \neq 0$, there are naturally defined systems $c \cdot \alpha = \{c\alpha_1, \ldots, c\alpha_M\}$.

A system $\alpha$ is said to be positive (nonnegative, or integer) if all elements $\alpha_i$ are positive (nonnegative, or integer). We call $\alpha$ a prime system if all $\alpha_i$ are integer and coprime.

A system $\alpha$ of length $M$ is called resonance if there is an integer system $m \neq 0$ of length $M$ such that $\sum_{\ell=1}^{M} \alpha_m \ell = 0$.

A system $\alpha$ is said to be pure if all its elements are mutually commensurable, i.e., if $\alpha_i/\alpha_k$ are rational for all $l, k$.

**Lemma 1.1.** Any pure system $\alpha$ can be written uniquely as $\alpha = \alpha_0 \cdot n$, where $n$ is a prime system and $\alpha_0 \in \mathbb{R}$. In particular, any pure system of length $> 1$ is resonance.

The number $\alpha_0$ determined by Lemma 1.1 is called a characteristic of the pure system $\alpha$.

**Lemma 1.2.** Let $\alpha, \beta$ be pure systems. Then the system $\alpha \cup \beta$ is pure if and only if the characteristics $\alpha_0$ and $\beta_0$ are commensurable.

**Definition 1.1.** A pure subsystem of $\alpha$ which cannot be presented as a union of different pure subsystems will be called a component of $\alpha$.

**Lemma 1.3.** Different components do not intersect.

In summary, this results in the following statement.

**Proposition 1.1.** Each system of nonzero numbers can be uniquely resolved into a disjoint union of components. The characteristics of the components are incommensurable. The system is resonance if and only if at least one of its components has length greater than 1.
Now we pass from systems of numbers to algebras.

Let \( B, C \) be two subalgebras in an algebra \( F \). The \textit{algebraic envelope} \( B \hat{+} C \) is a subalgebra in \( F \) generated by all sums and products of elements from \( B \) and \( C \). The \textit{commutator} \([B, C]\) is a subalgebra in \( F \) generated by commutators \([b, c]\) of arbitrary elements \( b \in B \) and \( c \in C \). The subalgebras \( B \) and \( C \) commute if \([B, C] = 0\).

A subalgebra \( B \) is said to be \textit{pure} if it cannot be presented as an algebraic envelope of different commuting subalgebras.

All pure commutative subalgebras are one-dimensional (i.e., generated by a single, non-unity, element).

\textbf{Definition 1.2.} A pure subalgebra in \( F \), which is not contained in any larger pure subalgebra, will be called a \textit{component} of the algebra \( F \).

Now let us return to the commutant \( F_\omega \) of the oscillator \( H_0 \) (1.2). The previous definitions and statements allow us to formulate the following result.

\textbf{Theorem 1.1.} The commutant \( F_\omega \) of the oscillator is uniquely presented as an algebraic envelope of its commuting components

\[ F_\omega = F^{(1)} \hat{+} \ldots \hat{+} F^{(L)}, \quad [F^{(j)}, F^{(s)}] = 0. \quad (1.3) \]

This resolution corresponds to the resolution of the frequency system \( \omega \) into its components (see Proposition 1.1):

\[ \omega = \bigcup_{j=1}^{L} \omega^{(j)} , \quad \omega^{(j)} \cap \omega^{(s)} = \emptyset. \quad (1.4) \]

The characteristics \( \omega_0^{(j)} \) of these components are incommensurable.

The algebra \( F_\omega \) is noncommutative if and only if the frequency system \( \omega \) is resonance. This happens if and only if at least one of the components \( \omega^{(j)} \) has length greater than 1 (and so, the corresponding component \( F^{(j)} \) is greater than one-dimensional).

\textbf{Proof.} Note that the Hamiltonian (1.1) can be written in the form

\[ H_0 = \sum_{l=1}^{M} \omega_l \tilde{z}_l z_l, \quad z_l = \frac{1}{\sqrt{2}} (q_l + ip_l). \quad (1.1a) \]
The oscillator (1.2) is respectively written as

\[ \hat{H}_0 = \sum_{l=1}^{M} \omega_l \hat{z}_l^* \hat{z}_l, \quad \dot{z}_l = \frac{1}{\sqrt{2}} \left( q_l + \hbar \frac{\partial}{\partial q_l} \right), \quad (1.2a) \]

where \( \hat{z}^* \) and \( \hat{z} \) are ordered in the Wick sense (all \( \hat{z}_l \) on the right, all \( \hat{z}_l^* \) on the left), the asterisk means the adjoint operator in \( L^2(\mathbb{R}^M) \).

For any Wick ordered polynomial \( g \) in the operators \( \hat{z}^* \), \( \hat{z} \) we have

\[ [\hat{H}_0, g(\hat{z}^*, \hat{z})] = -i\hbar \{ H_0, g \}(\hat{z}^*, \hat{z}) = \hbar (\partial H_0 \partial \overline{g} - \overline{g} \partial H_0 \partial g)(\hat{z}^*, \hat{z}) \]

\[ = \hbar \sum_{l=1}^{M} \omega_l (\overline{z}_l \partial_l g - z_l \partial_l g)(\hat{z}^*, \hat{z}). \]

Therefore (see [23]) the commutant \( F_\omega \) is generated by the elements

\[ \hat{g}_k = (\hat{z}^*)^{k_-} \hat{z}^{k_+}, \quad \omega \circ k_+ = \omega \circ k_-. \quad (1.5) \]

Here the systems \( k_+, k_- \) are integer nonnegative and have the same length \( M \) as the frequency system \( \omega \), the notation \( \circ \) is used for the scalar product in \( \mathbb{Z}_+^M \):

\[ \alpha \circ \beta = \sum_{l=1}^{M} \alpha_l \beta_l. \quad (1.6) \]

Let the frequency system \( \omega \) be resolved by its components (1.4). By Lemma 1.1, each component can be written as \( \omega^{(j)} = \omega_0^{(j)} \cdot n^{(j)} \), where \( n^{(j)} \) is an integer system. Then scalar products like (1.6) are represented as the sums

\[ \omega \circ \alpha = \sum_{j=1}^{L} \omega_0^{(j)} n^{(j)} \circ \alpha^{(j)}, \]

where \( \alpha^{(j)} \) are disjoint subsystems of the system \( \alpha \), that is, \( \alpha = \bigcup_{j=1}^{L} \alpha^{(j)} \), \( \alpha^{(j)} \cap \alpha^{(s)} = \emptyset \). Thus the basic equation in (1.5) is transformed into

\[ \sum_{j=1}^{L} \omega_0^{(j)} n^{(j)} \circ (k_+^{(j)} - k_-^{(j)}) = 0. \]

Since all the characteristics \( \omega_0^{(j)} \) are mutually incommensurable, this equation is equivalent to

\[ n^{(j)} \circ (k_+^{(j)} - k_-^{(j)}) = 0, \quad \forall j = 1, \ldots, L. \quad (1.7) \]
This means that the $j$th component $F^{(j)}$ of the commutator $F_\omega$ is generated by the operators $\hat{g}_{k^{(j)}}$ of type (1.5), where the integers systems $k_+^{(j)}, k_-^{(j)}$ obey the $j$th equation in (1.7).

From Proposition 1.1, the frequency system is resonance if and only if for some $j$ the system $n^{(j)}$ has length greater than 1. In this case, there are at least two solutions of the $j$th equation in (1.7) and the corresponding monomials $g_k^{(j)}$ and $g_{k'}^{(j)}$ whose mutual Poisson bracket is not zero, and therefore the algebra $F^{(j)}$ is noncommutative. □

**Remark 1.1.** The classical analog of Theorem 1.1 is obvious: just replace the word “algebra” by the words “Poisson algebra” and replace the commutator in (1.3) by the Poisson brackets.

Theorem 1.1 means that in the nonresonance case, where all frequencies $\omega_l$ are incommensurable, the commutant $F_\omega$ is trivial and generated by the commuting elements

$$\hat{g}_l \overset{\text{def}}{=} \hat{z}_l^* \hat{z}_l, \quad l = 1, \ldots, M. \quad (1.8)$$

In the resonance case the study of the algebra $F_\omega$ is reduced to the study of its nontrivial components.

Therefore, in what follows, we consider only the one-component situation, that is, we assume that the frequency system has the form

$$\omega = \{n_1, \ldots, n_M\}, \quad M \geq 2,$$

where $n_l$ are integers. Moreover, without loss of generality, we can assume that $n_l$ are positive, since all minus signs can be included into the systems of numbers $k_+, k_- \in \mathbb{Z}^M.$

Let us consider the oscillator with positive integer frequencies:

$$\hat{H}_0 = \sum_{l=1}^M n_l \hat{z}_l^* \hat{z}_l, \quad n_l \in \mathbb{N}. \quad (1.9)$$

Its commutant $F_n$ will be called a resonance algebra. This algebra is generated by the operators $\hat{g}_k$ (1.5), where

$$n \circ (k_+ - k_-) = 0, \quad k_+, k_- \in \mathbb{Z}^M_+. \quad (1.10)$$
Let us denote by $R_n$ the resonance set of all solutions $k = (k_+, k_-)$ of Eq. (1.10). Evidently, $R_n$ is an Abelian subsemigroup in $\mathbb{Z}_+^M \times \mathbb{Z}_+^M$. We can consider the secondary subsemigroup

$$R_n^{(1)} = R_n + R_n, \quad (1.11)$$

and take its complement

$$R_n \equiv R_n \setminus R_n^{(1)}. \quad (1.12)$$

This later set consists of those solutions of Eq. (1.10) which cannot be presented as a sum of two other solutions. The simplest example of an element from $R_n$ is

$$I_l \equiv \left( 0, \ldots, 0, 1, 0, \ldots, 0; 0, \ldots, 0, 1, 0, \ldots, 0 \right). \quad (1.13)$$

Such $I_l \in R_n$ we call primitive resonance elements. A generic element from $R_n$ will be called a minimal resonance element.

**Theorem 1.2.** The number of minimal resonance elements is finite. Each solution of the resonance equation (1.10) can be represented as a linear combination of minimal resonance elements with integer positive coefficients.

**Proof.** Let $k = (k_+, k_-)$ be a solution of (1.10) and let it be minimal, i.e., $k \in R_n$. We claim that for any $l = 1, \ldots, M$

$$k_{+l} \leq \sum n, \quad k_{-l} \leq \sum n. \quad (1.14)$$

Here the sum $\sum$ is taken over all indices $1, \ldots, M$. To prove (1.14), we, on the contrary, assume that (1.14) does not hold, say,

$$k_{+1} > \sum n. \quad (1.15)$$

Then we have $k_{+1} > \max n$. It follows from this inequality that all $k_{-l}$ do not exceed $n_1$. Indeed, if, on the contrary, there exists some $k_{-l}$ such that

$$k_{-l} > n_1, \quad (1.16)$$

then we obtain two different solutions $k'$ and $k''$ of Eq. (1.10), where

$$k'_+ = (k_{+1} - n_1, k_{+2}, \ldots, k_{+M}), \quad k'_- = (k_{-1}, k_{-2} - n_1, \ldots, k_{-M}),$$
and
\[ k'_n = (n_l, 0, \ldots, 0), \quad k''_n = (0, \ldots, n_1, \ldots, 0). \]
Their sum equals the original solution: \( k' + k'' = k \).

This contradicts the assumption that \( k \in \mathcal{R}_n \). Thus (1.16) fails, and we conclude that
\[ k_{-l} \leq n_1, \quad \forall l = 1, \ldots, M. \]

Then we derive
\[ k_{+1} n_1 \leq k_+ \circ n = k_- \circ n \leq \max_{l} (k_{-l}) \sum n \leq n_1 \sum n, \]
and so \( k_{+1} \leq \sum n \). This contradicts the assumption (1.15), and hence we have proved (1.14).

It follows from (1.14) that the number of elements in \( \mathcal{R}_n \) does not exceed \( 2M \sum n \).

The second statement of the theorem is evident and related to the fact that the multiplicative semigroup \( \mathbb{N} \) acts on \( \mathcal{R}_n \):
\[ \mu \cdot (k_+, k_-) \overset{\text{def}}{=} (\mu k_+ + \mu k_-), \quad \mu = 1, 2, \ldots, \]
and the additive semigroup \( \mathbb{Z}_+^M \) acts too:
\[ m \cdot (k_+, k_-) \overset{\text{def}}{=} (k_+ + m, k_- + m), \quad m \in \mathbb{Z}_+^M. \]

The proof of the theorem is complete.

\[ \square \]

2 Polynomial relations in resonance algebras

The resonance Poisson algebra \( \mathcal{F}_n \) is generated by the functions \( g_k = z^{k_+} \bar{z}^{k_-} \) on \( \mathbb{R}^{2M} \), where systems of nonnegative integers \( (k_+, k_-) \in \mathbb{Z}_+^M \times \mathbb{Z}_+^M \) obey Eq. (1.10). It is easy to derive the Poisson bracket between a pair of generators
\[ \{g_k, g_r\}_{\mathbb{R}^{2M}} = i \sum_{l=1}^{M} [k, r]^l g_{k+r-l}. \quad (2.1) \]

Here \( I_l \) is the primitive element (1.13), the bracket \( [k, r] \in \mathbb{Z}^M \) is defined by
\[ [k; r] = k_+ r_- - k_- r_+, \quad (2.2) \]
where the multiplication in $\mathbb{Z}_+^M$ is naturally determined by its multiplicative semigroup structure.

One can now consider an abstract vector space with the set of complex coordinates $G_k$ such that $G_k = G_k^*$. Here $k \in R_n$ and the involution $k \rightarrow k^*$ is defined by

$$k^* = k_-, \quad k^*_+ = k_+.$$  \hspace{1cm} (2.3)

Let us set the bracket between a pair of these coordinates just by mimicking the bracket (2.1):

$$\{G_k, G_r\} \overset{\text{def}}{=} i \sum_{l=1}^M [k, r]_l G_{k+r-l_l}, \quad k, r \in R_n.$$ \hspace{1cm} (2.4)

**Proposition 2.1.** The bracket (2.4) is skew-symmetric and obeys the Jacobi identity, i.e., it is a Poisson bracket. This Poisson structure is consistent with the involution

$$\{G_k, G_r\} = \{G_k^*, G_r^*\}.$$ \hspace{1cm} (2.5)

**Proof.** The skew-symmetry is evident. To check the Jacobi identity, let us compute the double bracket

$$\{\{G_k, G_r\}, G_s\} = \sum_{l,j=1}^M [k, r]_l [k + r - I_l, s]_j G_{k+r+s-I_l-I_j}.$$ \hspace{1cm} (2.6)

On the left and on the right of this relation, one has to take the cyclic sum $\mathcal{S}$ over permutations of the indices $k, r, s$ and to prove that the right-hand side is identically zero.

On the right, we obtain cyclic sums of two types

$$\sum_{l,j} (\mathcal{S}[k, r]_l [k, s]_j + \mathcal{S}[k, r]_l [r, s]_j) G_{k+r+s-I_l-I_j}$$

and

$$\sum_l (\mathcal{S}[k, r]_l [s, I_l]_j) G_{k+r+s-2I_l}.$$ \hspace{1cm} (2.7)

By the direct computation one can verify that the numerical coefficients in round brackets in both of these expressions are identically zero (in the first expression one even need not open the brackets $[,]$, but in the second expression the brackets $[,]$ must be opened using (2.2) to see that all summands cancel each other). \hfill \square
The linear Poisson tensor on the right-hand side of relations (2.4) determines a Lie–Poisson structure on the resonance algebra $F_n$. But this structure is infinite dimensional. Our goal now is to replace it by a finite dimensional (finitely generated) Poisson algebra structure.

First of all, we note that the original brackets (2.1) can be written in the following form

$$\{ g_k, g_r \}_{\mathbb{R}^{2M}} = ig_k g_r \Phi_{kr}(g_I).$$

(2.6)

Here the generators $g_I$ correspond to the primitive elements $I$ (1.13)

$$g_I = |z_l|^2, \quad l = 1, \ldots, M,$$

(2.7)

and the functions $\Phi_{kr}$ are defined by

$$\Phi_{kr}(\lambda) \overset{\text{def}}{=} \sum_{l=1}^{M} \frac{[k, r]^l}{\lambda_l}, \quad \lambda = (\lambda_1, \ldots, \lambda_M).$$

(2.8)

Thus, we see from (2.6) that the bracket of two generators of the algebra $F_n$ is proportional to the product of these generators with a coefficient which is a combination of the inverse primitive generators (2.7). Although this coefficient has singularities, the whole right-hand side of (2.6) is, of course, smooth.

We consider a domain where all $z_l \neq 0$ in order to avoid singularities. Anyway, the final result will not contain any singularities.

Let us consider an abstract set of complex coordinate functions $A_k$, where $k \in \mathbb{R}^n$, which is consistent with the involution (2.3): $\overline{A_k} = A_k^\ast$. The coordinate $A_I$ is said to be primitive if the element $I \in \mathbb{R}_n$ is primitive, and $A_k$ is said to be minimal if the element $k \in \mathbb{R}_n$ is minimal.

Let us introduce a bracket between the coordinates $A_k$ by mimicking the structure of the bracket (2.6):

$$\{ A_k, A_r \} \overset{\text{def}}{=} iA_k A_r \Phi_{kr}(A_I).$$

(2.9)

It follows from (2.9) that the primitive coordinates $A_I$ are in involution with each other:

$$\{ A_{I_l}, A_{I_j} \} = 0, \quad \forall l, j = 1, \ldots, M,$$

(2.10)

since $[I_l, I_j] = 0$ (see definitions (2.2) and (1.13)).

From (2.9) we also obtain

$$\{ A_k, A_{I_j} \} = i(k_+ - k_-) \partial A_k,$$

(2.11)
since $[k, I_{ij}] = (k_+ - k_-) j_{ij}$.

Some other useful properties of the operation (2.2) we collect in the following statement.

**Lemma 2.1.** The operation (2.2) $\mathbb{Z}_+^M \times \mathbb{Z}_+^M \to \mathbb{Z}_+^M$ has the properties

\begin{align*}
(a) \quad [k + s, r] &= [k, r] + [s, r], \\
(b) \quad [\mu \cdot k, r] &= \mu \cdot [k, r], \quad \mu \in \mathbb{Z}_+, \\
(c) \quad [k, r] &= [r^*, k^*], \quad (2.12) \\
(d) \quad [k, r] &= -[r, k], \\
(e) \quad \mathcal{S}_{k, r, s}[k, r](s_+ - s_-) &= 0.
\end{align*}

**Proof.** The only non-evident relation is the last one. Computing the cyclic sum, one obtains

\[
\mathcal{S}_{k, r, s}[k, r](s_+ - s_-) = (k_+ r_- - k_- r_+)(s_+ - s_-) + (r_+ s_- - r_- s_+)(k_+ - k_-) \\
+ (s_+ k_- - s_- k_+)(r_+ - r_-).
\]

This is identical to zero. \hfill \Box

**Corollary 2.1.** The bracket defined by (2.9) is a Poisson bracket. It is consistent with the involution

\[
\{A_k, A_r\} = \{A_k^\dagger, A_r^\dagger\}. \quad (2.13)
\]

**Proof.** The skew-symmetry $\{A_k, A_r\} = -\{A_r, A_k\}$ follows from (2.12) (d). To check the Jacobi identity we derive

\[
\{\{A_k, A_r\}, A_s\} = i\{A_k, A_s\} A_r \Phi_{kr}(A_I) + i\{A_r, A_s\} A_k \Phi_{kr}(A_I) \\
+ i A_k A_r \{\Phi_{kr}(A_I), A_s\}.
\]

Applying once more (2.9), as well (2.11), we obtain

\[
\{\{A_k, A_r\}, A_s\} = -A_k A_r A_s[\Phi_{ks} + \Phi_{rs}] \Phi_{kr} + \Gamma_{krs}],
\]

where

\[
\Gamma_{krs} \overset{\text{def}}{=} \sum_{l=1}^{M} \frac{[k, r]_l (s_+ - s_-)_l}{A^2_{il}}.
\]

It follows from (2.12) (e) that $\mathcal{S}_{k, r, s} \Gamma_{krs} = 0$. From the skew-symmetry of $\Phi_{kr}$ we also obtain $\mathcal{S}_{k, r, s} (\Phi_{ks} + \Phi_{rs}) \Phi_{kr} = 0$. Relation (2.13) follows from (2.12) (c). \hfill \Box

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Now we can prove the key lemma.

**Lemma 2.2.** If an element \( k \in R_n \) obeys the expansion

\[
\sum_{r \in R_n} m_r \cdot r = k, \quad m_r \in \mathbb{Z}_+,
\]

then

\[
\prod_{r \in R_n} A_r^{m_r} = C A_k,
\]

where \( C = C(A) \) is a Casimir function for the Poisson bracket (2.9).

**Proof.** We derive

\[
\left\{ \prod_r A_r^{m_r}, A_s \right\} = \sum_r m_r A_r^{-1} \left\{ A_r, A_s \right\} \prod_{r' \neq r} A_{r'}^{m_{r'}}.
\]

By taking the bracket \( \left\{ A_r, A_s \right\} \) from the definition (2.9) and by using (2.8), (2.12 a), and (2.14), we transform the previous formula as follows:

\[
\left\{ \prod_r A_r^{m_r}, A_s \right\} = i A_s \left( \sum_r m_r \Phi rs \right) \prod_{r'} A_{r'}^{m_{r'}}
= i A_s \left( \sum_{l=1}^M \frac{1}{\lambda_l} \left[ \sum_r m_r \cdot r, s \right] \right) \prod_{r'} A_{r'}^{m_{r'}}
= i A_s \Phi ks \prod_r A_r^{m_r} = \frac{\left\{ A_k, A_s \right\}}{A_k} \prod_r A_r^{m_r}.
\]

Therefore,

\[
\left\{ \frac{1}{A_k} \prod_r A_r^{m_r}, A_s \right\} = 0, \quad \forall s \in R_n,
\]

and hence the function \( C = \frac{1}{A_k} \prod_r A_r^{m_r} \) is a Casimir function. \( \Box \)

Using this lemma, one can transform the right-hand side of (2.9) into a form which does not have a singularity.

From now on, we consider only minimal generators \( A_k, k \in R_n \). The set of minimal elements \( R_n \) can be divided into the disjoint union

\[
R_n = R_n^0 \cup R_n^\#,
\]

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where $\mathcal{R}_0^0$ consists of all primitive elements (1.13) and $\mathcal{R}^\#$ consists of clean elements $k$ which obey

$$k_+k_- = 0. \quad (2.17)$$

Let us fix $l \in (1, \ldots, M)$. For each pair of clean elements $k, r$ such that the numbers $k_+, k_-, r_+, r_-$ are not all zero, there is an expansion of the element $k + r - I_l$ via the minimal ones

$$k + r - I_l = \sum_{s \in \mathcal{R}_n} m_{k,r,s}^l \cdot s, \quad m_{k,r,s}^l \in \mathbb{Z}_+. \quad (2.18)$$

Here we have $m_{k,r,s}^l = m_{r,k,s}^l = m_{k^*,r^*,s^*}^l$.

From Lemma 2.2 we conclude that there are some Casimir functions $C_{k,r}^l$ such that

$$A_k A_r = C_{k,r}^l A_{I_l} \prod_{s \in \mathcal{R}_n} A_{s_{k,r,s}^l}. \quad (2.19)$$

Here we have $\overline{C_{k,r}^l} = C_{k^*,r^*}^l$ and $C_{k^*,r^*}^l = C_{r,k}^l$.

Then the Poisson bracket (2.9) reads

$$\{ A_k, A_r \} = i \sum_{l=1}^M C_{k,r}^l [k, r]_l \prod_{s \in \mathcal{R}_n} A_{s_{k,r,s}^l}. \quad (2.20)$$

for any two clean elements $k, r$.

Formulas (2.20), together with (2.10), (2.11) provide a nonsingular expression for the Poisson brackets of two minimal coordinates via the minimal coordinates. From Theorem 1.2 we know that the number of minimal coordinates $N = \#(\mathcal{R}_n)$ is finite.

In view of the involution property $\overline{A_k} = A_{k^*}$, the real Re$(A_k)$ and imaginary Im$(A_k)$ parts of minimal coordinates belong to the same Poisson algebra. Formula (2.13) demonstrates that the finite dimensional Poisson tensor which one sees in (2.20), (2.10), and (2.11) is actually a real tensor on the constraint surface determined by Eqs. (2.19).

Thus we obtain a finite dimensional Poisson algebra generated by the minimal $A_k (k \in \mathcal{R}_n)$.

In the realization of this algebra by functions on the phase space $\mathbb{R}^{2M}$,

$$A_k \rightarrow g_k = z^k \overline{z}^k, \quad (2.21)$$
all the Casimir elements $C$ in (2.15) are, of course, equal to 1. So, from (2.19) we obtain a distinguished constraint surface given by the equations

$$A_k A_r = A_{l_{ij}} \prod_{s \in R_n} A_s^{m_{k,r,s}}, \quad k, r \in R_n^\#.$$  

(2.22)

On this particular surface we observe the following Poisson structure

$$\{A_k, A_r\} = i \sum_{l=1}^M [k, r]_l \prod_{s \in R_n} A_s^{m_{k,r,s}}, \quad k, r \in R_n^\#.$$  

(2.23)

Note that by a small change in the notation one can make the bracket $[\ , \ ]$ in (2.23) be the actual bracket on the set of integers $Z^M$. To do this, we map $Z^M_+ \times Z^M_+$ into $Z^M$:

$$k \to k_+ - k_-.$$  

This map is one-to-one on the subset of clean elements obeying (2.17). The inverse map is given by

$$\alpha \to (\alpha_+, \alpha_-),$$

where $\alpha_+$ and $\alpha_-$ are the positive and negative parts of the number $\alpha_l$. The resonance equation (1.10) reads

$$n \circ \alpha = 0.$$  

(2.24)

The condition (2.17) is transformed to

$$\alpha_+ \alpha_- = 0.$$  

(2.17a)

The bracket (2.2) and the involution (2.3) are naturally transported to $Z^M$ as

$$[\alpha, \beta] = \alpha_+ \beta_- - \alpha_- \beta_+, \quad \alpha^* = -\alpha.$$  

(2.25)

Properties (2.12) (b,c,d) hold for this bracket as well.

Now the analog of the property (2.12) (e) can be written as

$$G_{\alpha, \beta, \gamma} [\alpha, \beta] \gamma = 0.$$  

(2.26)

The analog of property (2.12) (a) reads

$$[\alpha + \beta, \gamma] = [\alpha, \gamma] + [\beta, \gamma] + (\alpha + \beta) \gamma,$$  

(2.27)
where
\[ \alpha + \beta \overset{\text{def}}{=} \min(\alpha_+ + \beta_+, \alpha_- + \beta_-) \in \mathbb{Z}_+^M. \] (2.28)

Thus the Jacobi-like property (2.26) looks natural, but the linearity-like property (2.27) contains an "anomaly." This anomaly itself obeys the following properties:

\[ \alpha + \beta = \alpha_+ + \beta_+ - (\alpha + \beta)_- = \alpha_- + \beta_- - (\alpha + \beta)_-, \]
\[ (\alpha + \beta) + ((\alpha + \beta) + \gamma) = (\alpha + (\beta + \gamma)) + (\beta + \gamma). \]

The expansion (2.18) is replaced by
\[
\sum_{\gamma \in \Gamma_n} \mu_{\alpha,\beta,\gamma} \cdot \gamma + (\alpha + \beta), \quad \alpha, \beta \in \Gamma_n.
\]

Here \( \mu_{\alpha,\beta,\gamma} \in \mathbb{Z}_+ \), and by \( \Gamma_n \subset \mathbb{Z}_+^M \) we denote the set of solutions of the resonance equation (2.24).

The constraint equations (2.22) are replaced by
\[ A_{\alpha} A_{\beta} = A_{\alpha + \beta} \prod_{\gamma \in \Gamma_n} A_{\gamma}^{\mu_{\alpha,\beta,\gamma}}, \] (2.29)

and the involution condition is written as
\[ \overline{A}_{\alpha} = A_{-\alpha}, \quad \overline{A}_{I} = A_{I}. \]

Now the Poisson brackets are finally read
\[ \{A_{\alpha}, A_{\beta}\} = i\overset{\circ}{\Phi}_{\alpha,\beta}(A_I) \prod_{\gamma \in \Gamma_n} A_{\gamma}^{\mu_{\alpha,\beta,\gamma}}, \quad \alpha, \beta \in \Gamma_n, \] (2.30)
\[ \{A_{\alpha}, A_{I_j}\} = i\alpha_j A_{\alpha}, \quad \{A_{I_l}, A_{I_j}\} = 0, \quad l, j = 1, \ldots, M. \]

Here the functions \( \overset{\circ}{\Phi}_{\alpha,\beta} \) are defined by
\[ \overset{\circ}{\Phi}_{\alpha,\beta}(\lambda) = \lambda^{\alpha + \beta} \sum_{l=1}^{M} \frac{[\alpha, \beta]_l}{\lambda_l}, \quad \lambda \in \mathbb{R}_+^M, \]

where the bracket \( [\alpha, \beta] \) and the operation \( \alpha + \beta \) are given by (2.25) and (2.27).
Note that for $\beta = -\alpha$, the constraint (2.29) and the bracket (2.30) are especially simple:

$$|A_\alpha|^2 = A_\alpha^{\alpha+\alpha-}, \quad (2.29a)$$
$$\{A_\alpha, \overline{A}_\alpha\} = iA_I^{\alpha+\alpha-} \sum_{l=1}^M \frac{\alpha_+^2 - \alpha_-^2}{A_{lI}}. \quad (2.30a)$$

Also note that

$$[\alpha, \beta] = 0 \quad \text{if and only if} \quad \alpha\beta = 0 \quad (\text{where} \quad \alpha, \beta \in \Gamma_n), \quad (2.31)$$

and in this case $\{A_\alpha, A_\beta\} = 0$.

**Theorem 2.1.** The algebra $\mathcal{F}_n$ of functions on $\mathbb{R}^{2M}$ which are in involution with the oscillator $H_0 = \frac{1}{2} \sum_{l=1}^M n_l(q_l^2 + p_l^2)$, $n_l \in \mathbb{N}$, is a finite generated involutive Poisson algebra. After the identification $A_\alpha \leftrightarrow z^{\alpha+\alpha-}$, $A_I \leftrightarrow |z|^{2I}$, the constraints and Poisson brackets of this algebra are of polynomial type and given by (2.29), (2.30). These brackets are arithmetic: all the structural coefficients are integer numbers. The functional dimension $N$ of the algebra $\mathcal{F}_n$ is finite: $N = \#(\Gamma_n) + M$, where $\Gamma_n \subset \mathbb{Z}^M$ is the set of minimal solutions of the resonance equation (2.24).

Thus we conclude with the following.

The resonance algebra $\mathcal{F}_n$ can be considered as a realization of three different Poisson algebras:

— the infinite Lie algebra (2.4),
— the finite generated algebra (2.9) with a singular Poisson tensor,
— the finite generated algebra (2.30) with constraints (2.29).

The first two Poisson algebras are universal and independent of the frequency system $n$. The third Poisson algebra depends on $n$ and has a smooth (polynomial) Poisson tensor.

### 3 Resonance algebras for 2-frequency systems

In this section, we examine the 2-frequency case. Besides the usual oscillator, we also consider the inverted oscillator and the oscillator in a magnetic field.

**Example 3.1 (Resonance algebra of a two-dimensional oscillator).** In modern physics of nano-structures, the simplest model of a plane quantum
dot or a plane artificial atom became very important because of the opening opportunities to create these objects by new fine technologies. The resonance case in this model is of special interest in view of the effect of resonance stabilization (see [35]). In this example we have the oscillator with two degrees of freedom:

\[
H_0 = \frac{n_1}{2}(q_1^2 + p_1^2) + \frac{n_2}{2}(q_2^2 + p_2^2), \quad \hat{H}_0 = n_1\hat{z}_1^*\hat{z}_1 + n_2\hat{z}_2^*\hat{z}_2, \quad (3.1)
\]

where \(n_1, n_2\) are coprime natural numbers. The resonance set \(\Gamma_n\) in this case consists of only two elements \(\alpha\) and \(-\alpha\), where \(\alpha = (n_2, -n_1) \in \mathbb{Z}^2\). Thus the Poisson algebra \(\mathcal{F}_n\) is realized as the algebra of functions on the constraint surface in \(\mathbb{R}^4\) given by Eq. (2.29a):

\[
|A_\alpha|^2 = A_1^{n_2}A_2^{n_1}. \quad (3.2)
\]

Here we use the simplified notation: \(A_1 \equiv A_{I_1}, \; A_2 \equiv A_{I_2}\). The Poisson brackets are

\[
\{A_\alpha, \overline{A}_\alpha\} = i(n_2^2A_2 - n_1^2A_1)A_1^{n_2-1}A_2^{n_1-1},
\]

\[
\{A_\alpha, A_1\} = in_2A_\alpha, \quad \{A_\alpha, A_2\} = -in_1A_\alpha, \quad \{A_1, A_2\} = 0. \quad (3.3)
\]

In the general situation, for brackets like (2.30) we can claim that the Jacobi condition holds on the constraint surface (2.29). But also, in the general situation, one can claim that it is sufficient to check the Jacobi condition only for a triple of different indices from \(\Gamma_n\). In our example, \(\Gamma_n\) consists of two elements. Thus the Jacobi condition for the bracket (3.3) holds everywhere on \(\mathbb{R}^4\). The constraint (3.2) in this case is actually the Casimir level.

There are two Casimir functions for the bracket (3.3):

\[
C_0 = n_1A_1 + n_2A_2, \quad C_1 = |A_\alpha|^2 - A_1^{n_2}A_2^{n_1}. \quad (3.4)
\]

Note that the first function \(C_0 = \sum n_iA_i\) is a Casimir function in the general case (2.30) as well.

The realization of the algebra (3.3) by functions on \(\mathbb{R}^4\) is

\[
g_\alpha = z_1^{n_2}z_2^{n_1}, \quad g_{-\alpha} = z_1^{-n_2}z_2^{-n_1}, \quad g_1 = |z_1|^2, \quad g_2 = |z_2|^2. \quad (3.5)
\]

In this realization \(C_1 = 0\) and \(C_0 = H_0\) is the Hamiltonian of the initial oscillator.
The quantum version of generators (3.5) is
\[ \hat{g}_\alpha = \hat{z}_2 z_1^n, \quad \hat{g}_{-\alpha} = \hat{z}_1^{*n} \hat{z}_2, \quad \hat{g}_1 = \hat{z}_1^{*n} \hat{z}_1, \quad \hat{g}_2 = \hat{z}_2^{*n} \hat{z}_2. \] (3.6)
They commute with the quantum oscillator (3.1).

The abstract version of these generators we denote by \( \hat{A}_\alpha, \hat{A}^*_\alpha, \hat{A}_1, \hat{A}_2 \). They generate the quantum resonance algebra \( F_\nu \) of the two-dimensional oscillator.

The commutation relations in this algebra are
\[ [\hat{A}_\alpha, \hat{A}^*_\beta] = f(\hat{A}_1, \hat{A}_2), \]
\[ [\hat{A}_\alpha, \hat{A}_1] = \hbar n_2 \hat{A}_\alpha, \quad [\hat{A}_\alpha, \hat{A}_2] = -\hbar n_1 \hat{A}_\alpha, \quad [\hat{A}_1, \hat{A}_2] = 0. \] (3.7)
Here the polynomial \( f \) is defined by
\[ f(A_1, A_2) = \rho(A_1 + \hbar n_2, A_2 - \hbar n_1) - \rho(A_1, A_2), \] (3.8)
where
\[ \rho(A_1, A_2) \overset{\text{def}}{=} A_1(A_1 - \hbar) \ldots (A_1 - (n_2 - 1)\hbar)(A_2 + \hbar) \ldots (A_2 + n_1\hbar). \]

The Casimir elements of the algebra (3.7) are
\[ \hat{C}_0 = n_1 \hat{A}_1 + n_2 \hat{A}_2, \quad \hat{C}_1 = \hat{A}^*_\alpha \hat{A}_\alpha - \rho(\hat{A}_1, \hat{A}_2). \] (3.9)
In representation (3.6), \( \hat{A}_\alpha \to \hat{g}_\alpha, \hat{A}_1 \to \hat{g}_1 \) the Casimir element \( \hat{C}_1 \) takes zero value.

Note that the polynomial \( f \) on the right-hand side of relations (3.7) has degree \( n_1 + n_2 - 1 \). In the case \( n_1 = n_2 = 1 \) (i.e., 1:1 resonance) we obtain the Lie algebra \( \text{su}(2) \). In any other case the polynomial \( f \) is of degree larger than 1, and we have an algebra with nonlinear commutation relations.

The theory of irreducible representations and coherent states for algebras of the type (3.7) was developed in [14]. Representations of the specific resonance algebra (3.7) in the case \( n_1 = 1, n_2 = 2 \) were described in [35].

**Example 3.2 (Resonance algebra of the inverted oscillator).** As the next example, let us consider the inverted (unstable) oscillator
\[ H_0(-) = \frac{1}{2} \sum_{i=1}^{M} n_i (q_i^2 + p_i^2). \] (3.10)
This case is transformed to the previous one by the replacing \( p \to ip \). Thus we claim the set of functions in involution with \( H_0^{(-)} \) is given by the generators

\[
\begin{align*}
\mathcal{A}_\alpha^{(-)} &\sim g_\alpha^{(-)} \equiv \left( \frac{q+p}{\sqrt{2}} \right)^{\alpha_+} \left( \frac{q-p}{\sqrt{2}} \right)^{\alpha_-}, \\
\mathcal{A}_{-\alpha}^{(-)} &\sim g_{-\alpha}^{(-)} \equiv \left( \frac{q+p}{\sqrt{2}} \right)^{\alpha_-} \left( \frac{q-p}{\sqrt{2}} \right)^{\alpha_+}, \\
\mathcal{A}_l^{(-)} &\sim g_l^{(-)} = \frac{q^2 - p^2}{2},
\end{align*}
\]

(3.11)

where \( n \circ \alpha = 0, \alpha \in \mathbb{Z}^M \). The commutation relations in the algebra \( \mathcal{F}_n \) are given by a formula like (2.30) but without the imaginary unit \( i \) factor. For instance, in the case \( M = 2 \) we get the relation (3.3) without the \( i \)-factor.

In order to understand the difference for which the \( i \)-factor is responsible, let us consider the simplest 1:1 resonance. Denote

\[
\begin{align*}
\frac{1}{2}(\mathcal{A}_1 - \mathcal{A}_2) &= b_3, \\
\frac{1}{2i}(\mathcal{A}_\alpha - \mathcal{A}_{-\alpha}) &= b_2, \\
\frac{1}{2}(\mathcal{A}_\alpha + \mathcal{A}_{-\alpha}) &= b_1.
\end{align*}
\]

(3.12)

Then relations (3.3) read

\[
\{b_1, b_2\} = b_3, \quad \{b_2, b_3\} = b_1, \quad \{b_3, b_1\} = b_2.
\]

(3.13)

The Casimir elements (3.4) are reduced to \( C_1 = b_1^2 + b_2^2 + b_3^2 - C_0^2/4 \). In the realization (3.5), \( C_1 = 0 \), and so, \( b_1^2 + b_2^2 + b_3^2 = C_0^2/4 \).

These are the commutation relations and Casimir elements of the Lie algebra \( \text{su}(2) \). Thus, in the case of the 1:1 resonance (for the usual oscillator), the resonance algebra is the envelope of the \( \text{su}(2) \) Lie algebra.

For the inverted oscillator in the 1:1 resonance case, we introduce the generators

\[
\begin{align*}
b_1^{(-)} &= \frac{1}{2}(\mathcal{A}_\alpha^{(-)} + \mathcal{A}_{-\alpha}^{(-)}), \\
b_2^{(-)} &= \frac{1}{2}(\mathcal{A}_\alpha^{(-)} - \mathcal{A}_{-\alpha}^{(-)}), \\
b_3^{(-)} &= \frac{1}{2}(\mathcal{A}_1^{(-)} - \mathcal{A}_2^{(-)}),
\end{align*}
\]

(3.14)

where \( \mathcal{A}_\alpha^{(-)}, \mathcal{A}_{-\alpha}^{(-)}, \mathcal{A}_1^{(-)}, \mathcal{A}_2^{(-)} \) are given by (3.11). Then the relations between them are

\[
\{b_1^{(-)}, b_2^{(-)}\} = b_3^{(-)}, \quad \{b_2^{(-)}, b_3^{(-)}\} = b_1^{(-)}, \quad \{b_3^{(-)}, b_1^{(-)}\} = -b_2^{(-)}.
\]

(3.15)

The Casimir elements in this case are reduced to \( C_1 = (b_1^{(-)})^2 - (b_2^{(-)})^2 + (b_3^{(-)})^2 - C_0^2/4 \). In the realization (3.11), \( C_1 = 0 \), and so, \( (b_1^{(-)})^2 - (b_2^{(-)})^2 + (b_3^{(-)})^2 = C_0^2/4 \).
The algebra (3.15) is the su(1, 1) Lie algebra, and its realization (3.11), (3.14) corresponds to the symplectic leaves \( \{(b_1^(-))^2-(b_2^(-))^2+(b_3^(-))^2 = C_0^2/4\} \), which are one-sheeted hyperboloids.

**Example 3.3 (Resonance algebra of 2D-oscillator in magnetic field).**
To conclude this section, let us consider the 2-dimensional oscillator (the artificial atom) placed into a magnetic field. Let us assume that the magnetic field is homogeneous and perpendicular to the plane where the oscillator sits. Denote by \( \omega_L \) the half of the Larmor frequency (which is the magnetic field strength multiplied by the charge of the particle and divided by its mass). Let the oscillator be isotropic, and let \( \omega_0 \) be its frequency. If we represent the ratio \( \omega_L/\omega_0 \) as
\[
\left( \frac{\omega_L}{\omega_0} \right)^2 = \frac{s^2}{k^2-s^2}, \quad k > s > 0, \tag{3.16}
\]
then the Hamiltonian of this physical model reads
\[
H_0 = k\left( \frac{|q|^2 + |p|^2}{2} \right) + s(q_1p_2 - q_2p_1) = k(|z_1|^2 + |z_2|^2) + is(\overline{z}_2z_1 - \overline{z}_1z_2). \tag{3.17}
\]

A resonance happens if the numbers \( k, s \) are commensurable, and so one can find a pair of coprime numbers \( l, m \) such that
\[
\frac{k + s}{k - s} = \frac{l}{m}. \tag{3.18}
\]

Note that in the coordinates \( z_\pm = \frac{1}{\sqrt{2}}(z_1 \mp iz_2) \) the system is equivalent to the oscillator \( l|z_+|^2 + m|z_-|^2 \). Its resonance algebra was already described above (in Example 3.1). Thus we can claim that the resonance algebra \( F_{l,m} \) of the Hamiltonian (3.17) under condition (3.18) has the generators:
\[
\mathcal{A}_1 = |z_+|^2, \quad \mathcal{A}_2 = |z_-|^2, \quad \mathcal{A}_+ = z_+^{\dagger}z_+^m, \quad \mathcal{A}_- = \overline{z}_+^m. \tag{3.19}
\]

The brackets between them are
\[
\{\mathcal{A}_+, \mathcal{A}_1\} = im\mathcal{A}_+, \quad \{\mathcal{A}_+, \mathcal{A}_2\} = -il\mathcal{A}_+, \quad \{\mathcal{A}_1, \mathcal{A}_2\} = 0, \quad \{\mathcal{A}_+, \mathcal{A}_-\} = i(m^2\mathcal{A}_2 - l^2\mathcal{A}_1)\mathcal{A}_1^{m-1}\mathcal{A}_2^{l-1}. \tag{3.20}
\]

This algebra just coincides with the algebra (3.3) if \( n_1 = l, n_2 = m \). Thus it has the same Casimir elements as in (3.4):
\[
C_0 = l\mathcal{A}_1 + m\mathcal{A}_2, \quad C_1 = \mathcal{A}_+\mathcal{A}_- - \mathcal{A}_1^m\mathcal{A}_2^l.
\]
and $C_1 = 0$ on the realization (3.19).

The quantum version of the algebra (3.20) is easily obtained in the same way as in Example 3.1 (see (3.7)).

Note that the nonlinear character of the Poisson tensor in (3.20) appears automatically if $l \neq m$ or $s \neq 0$ in (3.17), i.e., if the magnetic field is not zero. The magnetic field is responsible for this nonlinearity, in spite of the rotational symmetry of the system (3.17).

We can also consider a non-isotropic oscillator with two different frequencies placed into a magnetic field:

$$H_0 = \frac{1}{2}|p|^2 + \frac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2) + (q_1 p_2 - q_2 p_1).$$

(3.21)

The effective frequencies $\omega_{\pm}$ of the Hamilton flow in this model are given by the formula

$$\omega_{\pm}^2 = \frac{1}{2}\left[\frac{\omega_1^2 + \omega_2^2}{2} \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 8(\omega_1^2 + \omega_2^2)}\right].$$

(3.22)

The resonance happens if $\omega_+ / \omega_- = l/m$, where $l, m$ are certain coprime numbers. One can describe the resonance algebra of this nonsymmetric system in the same way as above.

4 Resonance precession

Now let us discuss some spectral and dynamical effects accompanying the appearance of a resonance.

For simplicity, we consider only the case of two degrees of freedom. Let a Hamiltonian $H$ have the nondegenerate minimum at the point $x = 0$. Around this point, we have the Taylor expansion:

$$H(x) = \text{const} + H_0(x) + H_1(x) + \ldots,$$

(4.1)

where $H_j$ are the homogeneous polynomials of degree $j + 2$. By passing to the new variables

$$x = h^{1/N}x', \quad h' = h^{1-2/N} \quad (N \geq 2),$$

we obtain

$$H = \text{const} + h^{2/N}(H_0(x') + h^{1/N}H_1(x') + h^{2/N}H_2(x') + \ldots).$$

(4.2)
Let the variation matrix of the Hamilton system at the equilibrium point $x' = 0$ have only imaginary eigenvalues $\pm i \omega_1, \pm i \omega_2$. Then in suitable coordinates one can represent the quadratic form $H_0$ as the oscillator

$$H_0 = \omega_1 |z_1'|^2 + \omega_2 |z_2'|^2,$$

(4.3)

where $z_1', z_2'$ are complex coordinates of the point $x' \in \mathbb{R}^4$.

If the frequencies $\omega_1, \omega_2$ are not in resonance, then, applying the averaging method in the $N$th microzone, where $x = O(\hbar^{1/N})$, one can reduce the Hamiltonian to a function in the action variables $A_1 = |z_1'|^2$ and $A_2 = |z_2'|^2$ only:

$$H \sim \text{const} + \hbar^{2/N} H_0 + \hbar^{3/N} f_1(A_1, A_2) + \hbar^{4/N} f_2(A_1, A_2) + \ldots$$

(4.4)

The same is true on the quantum level.

The action variables $A_1, A_2$ in (4.4) commute with each other and thus the dynamics and the algebra are reduced to purely commutative ones.

Let us assume now that the frequencies $\omega_1, \omega_2$ are in a resonance

$$\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2}, \quad n_1 \text{ and } n_2 \text{ are coprime integers.}$$

(4.5)

Then the application of the averaging method to (4.2) implies (see in Appendix) that the Hamiltonian is reduced to

$$H \sim \text{const} + \hbar^{2/N} H_0 + \hbar^{3/N} f(A_1, A_2, A_1, \bar{A_1}),$$

(4.6)

where $f = f_1 + \hbar^{1/N} f_2 + \ldots$ and $f_j$ are functions in generators of the resonance algebra $\mathcal{F}_n$ with relations (3.3). The same representation exists in the quantum case as well, and in this case, one has to use the resonance algebra (3.7) (with the parameter $\hbar'$ instead of $\hbar$).

Thus under the resonance condition (4.5), instead of the commutative algebra of action variables, we have the noncommutative algebra (3.3). The averaged system (see the definition in [3]) can be represented as a Hamiltonian system on symplectic leaves

$$\Omega = \{C_0 = \text{const}, C_1 = 0\},$$

(4.7)

where $C_0, C_1$ are the Casimir functions (3.4). These leaves are compact surfaces in $\mathbb{R}^4$ diffeomorphic to the sphere $S^2$. 44
The Hamiltonian system corresponding to the perturbing term \( f \) in (4.6) reads
\[
\frac{d}{dt} A = \{ f, A \} \quad (4.8)
\]
or explicitly,
\[
\begin{align*}
\frac{d}{dt} A_\alpha &= i n_1 \frac{\partial f}{\partial A_2} - i n_2 \frac{\partial f}{\partial A_1} + i (n_1^2 A_1 - n_2^2 A_2) A_1^{n_2 - 1} A_2^{n_1 - 1} \frac{\partial f}{\partial A_\alpha}, \\
\frac{d}{dt} A_1 &= i n_2 A_\alpha \frac{\partial f}{\partial A_\alpha} - i n_2 A_\alpha \frac{\partial f}{\partial A_\alpha}, \quad (4.8a) \\
\frac{d}{dt} A_2 &= -i n_1 A_\alpha \frac{\partial f}{\partial A_\alpha} + i n_1 A_\alpha \frac{\partial f}{\partial A_\alpha}.
\end{align*}
\]

In the particular case \( n_1 = n_2 = 1 \) (the isotropic resonance 1:1) the algebra (3.3) is reduced to (3.13), i.e., to \( \text{su}(2) \). Then the leaves (4.7) are exactly the spheres and the system (4.8) is the classical Euler system for the spinning top [36].

In the case of general anisotropic resonance \( n_1 : n_2 \), we obtain something like a generalized top with non-Lie commutation relations (3.3) and the Hamilton dynamics (4.8) on it. This dynamics describes the evolution of the noncommutative integrals of motion of the resonance oscillator.

In the phase space \( \mathbb{R}_2^4 \) on each energy level \( H_0 \equiv C_0 = \text{const} \), we have a fibration by periodic trajectories of \( H_0 \). The whole variety of trajectories is parametrized by the coordinates \( A_1, A_2, A_\alpha \). The evolution (4.8) is similar to the evolution of the rotation axis in the theory of spinning top. That rotation is known as precession. Thus in our case, where we have a resonance analog of the spinning top, one can use the term resonance precession for solutions of the evolution system (4.8).

The quantum version of (4.8) is the Heisenberg type equations:
\[
-i \hbar' \frac{d}{dt} \hat{A}(t) = [\hat{f}, \hat{A}(t)], \quad \hat{A}(t)|_{t=0} = \hat{A}. \quad (4.9)
\]
Here \( \hat{A}(t) \) denotes the evolution of generators \( \hat{A} \) of the algebra (3.7). The Hamiltonian \( \hat{f} \) in (4.9) is a polynomial function in generators of the same algebra. The system (4.9) describes the quantum resonance precession.

Equations (4.9) are related to the \( N \)th microzone with \( N > 2 \). If \( N = 2 \) (this is called the nanozone, [38, 35]), then \( \hbar' = 1 \) and (4.9) has to be read as
\[
-i \frac{d}{dt} \hat{A}(t) = [\hat{f}, \hat{A}(t)]. \quad (4.9a)
\]
Studying the spectral problem for the Hamiltonian $\hat{H}$, we can exploit the representation (4.6) as was demonstrated in [8, 35]. In this situation, instead of the dynamical system (4.9), (4.9a), one has to consider the spectral problem for the operator $\hat{f}$ over the resonance algebra. This problem can be effectively analyzed by applying the coherent states method, see [8, 35].

Note that there are some simple geometrical and topological implications of the compactness of symplectic leaves $\Omega$, where the resonance precession system (4.8) is actually supported. For instance, on $\Omega$, the Hamiltonian $f$ must have stationary points (at least two of them), which provide the equilibria in the $a$-space. The quantum operator $\hat{f}$ near these points can be analyzed using the method of deformed coherent states [34], which generalizes the well-known complex germ method [6] to the case of algebras with non-Heisenberg relations like (3.7).

5 Triple algebras and solving the resonance precession system via noncommuting space–time coordinates

Now we investigate the resonance precession in more detail. First of all, let us analyze the averaging transformation which produces the precession Hamiltonian over the resonance algebra from a Hamiltonian over the original (Heisenberg) algebra.

Let the leading part of the expansion (4.1) be the resonance oscillator $H_0 = \sum_{l=1}^{M} n_l |z_l|^2$. Then the averaging transformation (denoted by $\Pi$) is given by formula (A.7) of the Appendix:

$$F = \sum_k c_k g_k \quad \overset{\Pi}{\rightarrow} \quad F = \sum_{k \in R_n} c_k g_k.$$  \hspace{1cm} (5.1)

Here $c_k \in \mathbb{C}$, the monomials $g_k(z) = z^{k_+} \bar{z}^{k_-}$ are determined by multi-indices $k = (k_+, k_-) \in \mathbb{Z}_+^M \times \mathbb{Z}_+^M$, and the resonance set $R_n$ is determined by solutions of the resonance equation (1.10) $n \circ (k_+ - k_-) = 0$.

Denote by $\mathcal{F}$ the Poisson algebra of all (complex) polynomials over $\mathbb{R}^{2M}$. It is split into the direct sum of subspaces

$$\mathcal{F} = \mathcal{L} \oplus \mathcal{L}^\perp,$$

$$\mathcal{L} = \Pi(\mathcal{F}), \quad \mathcal{L}^\perp = (I - \Pi)(\mathcal{F}).$$

Thus the averaging transformation (5.1) is the projection of $\mathcal{F}$ onto $\mathcal{L}$.
The multiplication operation in $\mathcal{F}$ and the Poisson brackets are naturally projected into $\mathcal{L}$:

\[
F = \sum_{k \in \mathbb{R}^n} c_k g_k, \quad G = \sum_{k \in \mathbb{R}^n} d_k g_k \implies \\
F G = \sum_{k, r \in \mathbb{R}^n} c_k d_r g_{k+r}, \quad \{F, G\}_\mathcal{L} \overset{\text{def}}{=} \sum_{k, r \in \mathbb{R}^n} c_k d_r \{g_k, g_r\}_\mathcal{F}.
\]

(5.2)

The general mechanism of such a projection of the Poisson algebra structure can be easily described.

**Lemma 5.1.** Let $\mathcal{F}$ be an abstract Poisson algebra which admits the splitting into the direct sum of linear subspaces

\[
\mathcal{F} = \mathcal{L} \oplus \mathcal{L}^\perp,
\]

and let $\mathcal{L}^\perp$ be the Poisson $\mathcal{L}$-modulus

\[
\mathcal{L}^\perp \circ \mathcal{L} \subset \mathcal{L}^\perp, \quad \{\mathcal{L}^\perp, \mathcal{L}\} \subset \mathcal{L}^\perp.
\]

(5.3)

Then there is a unique Poisson algebra structure on $\mathcal{L}$ such that

\[
F \circ_\mathcal{L} G = \Pi(F \circ_\mathcal{F} G), \quad \{F, G\}_\mathcal{L} = \Pi(\{F, G\}_\mathcal{F}), \quad \forall F, G \in \mathcal{L},
\]

(5.4)

where $\Pi : \mathcal{F} \to \mathcal{L}$ is the projection on $\mathcal{L}$ along $\mathcal{L}^\perp$.

One has the identity

\[
\Pi(F \circ_\mathcal{F} G) = \Pi(F) \circ_\mathcal{L} \Pi(G) + \Pi((I - \Pi)(F) \circ_\mathcal{F} (I - \Pi)(G)), \\
\Pi(\{F, G\}_\mathcal{F}) = \{\Pi(F), \Pi(G)\}_\mathcal{L} + \Pi(\{(I - \Pi)(F), (I - \Pi)(G)\}_\mathcal{F}).
\]

(5.5)

If $\mathcal{L}$ is a Poisson subalgebra in $\mathcal{F}$, then

\[
F \circ_\mathcal{L} G = F \circ_\mathcal{F} G, \quad \{F, G\}_\mathcal{L} = \{F, G\}_\mathcal{F}, \quad \forall F, G \in \mathcal{L}.
\]

Indeed, one can take (5.4) just as the definition of the multiplication operation and the bracket on $\mathcal{L}$, and check that condition (5.3) implies the associativity and the Jacobi identity for this multiplication and the bracket.
In the case of the averaging projection (5.1), the multiplication and the bracket (5.4) are given by formula (5.2). The identities (5.5) read

\[ \Pi(\{F, G\}) = \Pi(F)\Pi(G) + \sum_{k, r \notin \mathbb{R}_n, k + r \in \mathbb{R}_n} c_k d_r g_{k+r}, \]

\[ \Pi(\{\{F, G\}, F\}) = \{\Pi(F), \Pi(G)\}_\mathcal{L} + \sum_{k, r \notin \mathbb{R}_n, k + r \in \mathbb{R}_n} c_k d_r \sum_{l=1}^M [k, r]_l g_{k+r-I_l}. \] (5.5a)

The last summands in the right-hand sides of (5.5a) are the anomaly which prevents the projection \( \Pi : \mathcal{F} \to \mathcal{L} \) from being a homomorphism. In particular, this anomaly makes the Poisson bracket in \( \mathcal{L} \) be noncommutative in the resonance case (but the associative multiplication in \( \mathcal{L} \) is, of course, Abelian).

In general, the projection allows one to define nontrivial brackets on linear subspaces in \( \mathcal{F} \).

**Lemma 5.2.** Let \( \mathcal{L} \) obey condition (5.3) and \( \mathcal{F}_0 \subset \mathcal{F} \) be a linear subspace such that the projection \( \Pi : \mathcal{F} \to \mathcal{L} \) is injective on \( \mathcal{F}_0 \) and the image \( \mathcal{L}_0 \triangleq \Pi(\mathcal{F}_0) \) is a Poisson subalgebra in \( \mathcal{L} \). Then \( \mathcal{F}_0 \) is endowed with a Poisson algebra structure which makes \( \Pi : \mathcal{F}_0 \to \mathcal{L}_0 \) be a homomorphism.

This trivial statement can be amplified as follows.

**Lemma 5.3.** Let a subspace \( \mathcal{L} \subset \mathcal{F} \) obey (5.3), \( \Pi : \mathcal{F} \to \mathcal{L} \) be injective on a subspace \( \mathcal{F}_0 \subset \mathcal{F} \), and \( \mathcal{L} \) be split into the direct sum of linear subspaces, \( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \) (5.6)

such that \( \mathcal{L}_0 = \Pi(\mathcal{F}_0) \) is the subalgebra with respect to the multiplication in \( \mathcal{L} \), and

\[ \{\mathcal{L}_0, \mathcal{L}_1\}_\mathcal{L} \subset \mathcal{L}_0. \] (5.7)

Then on \( \mathcal{L}_0 \), and so on the subspace \( \mathcal{F}_0 \), there exists the triple Poisson algebra structure with the following triple bracket

\[ \{F, G, E\}_{\mathcal{L}_0} \triangleq \{\{F, G\}_\mathcal{L}, E\}_\mathcal{L}, \quad F, G, E \in \mathcal{L}_0. \] (5.8)

In this framework, let \( \mathcal{F} \) be the algebra of polynomial functions on the phase space \( \mathbb{R}^{2M} = \mathbb{R}_q^M \oplus \mathbb{R}_p^M \), the projection \( \Pi \) be defined by the averaging
transformation (5.1), and \( F_0 \) be a subspace consisting of functions constant along \( \mathbb{R}^M_p \). Then \( \mathcal{L} = \mathcal{F}_n \) is the resonance algebra. If there is a splitting of the resonance algebra like (5.6), (5.7), then the triple bracket (5.8) generates a noncommutative triple algebra structure on \( F_0 \). This might be interpreted as the appearance of a triple noncommutative structure on the configuration space \( \mathbb{R}^M_q \) (since \( F_0 \) consists of functions in \( q \)-coordinates).

In the simple situation, where we know that the averaging of \( F_0 \)-functions forms a Poisson subalgebra in the resonance algebra \( \mathcal{F}_n \), one can apply Lemma 5.2. Then we can claim that the space \( F_0 \) of functions on \( \mathbb{R}^M_q \) is a Poisson noncommutative algebra, and so, the configuration space is noncommutative in the usual sense. The well-known example of coordinate noncommutativity of this type is presented by the Landau model (a charged particle on a plane with a perpendicular magnetic field [37, 38, 39]). In this example, we have the case of a neutral resonance, where the variation matrix of the Hamiltonian system related to \( H_0 \) has the twice degenerate zero eigenvalue, see [35].

The resonance considered in the given paper belongs to the stable (elliptic) class: the eigenvalues \( \pm im \) of the variation matrix are imaginary nonzero numbers. In this situation, Lemma 5.2 cannot be applied any more, but Lemma 5.3 works. We describe now simple examples of the 1:1 and 1:2 resonances, demonstrate the triple brackets over the configuration plane \( \mathbb{R}^2_q \), and show how the resonance precession dynamics can be resolved on this noncommutative plane.

**Example 5.1 (Noncommutative space–time under 1:1 resonance).**

Let us consider the isotropic resonance oscillator \( H_0 = \frac{1}{2}(q_1^2 + p_1^2) + \frac{1}{2}(q_2^2 + p_2^2) \).

The averaging transformation \( \Pi \) acts via the integral

\[
\Pi(F)(q,p) = \frac{1}{2\pi} \int_0^{2\pi} F(q \cos t + p \sin t, p \cos t - q \sin t) \, dt.
\]  

(5.9)

So, the average of any odd function \( F(x) = -F(x) \) is zero: \( \Pi(F) = 0 \). Therefore, we consider the subspace \( F_0 \subset F(\mathbb{R}^2_q \times \mathbb{R}^2_p) \) consisting of all even functions on \( \mathbb{R}^2_q \). This subspace is generated by \( q_1^2, q_2^2, q_1q_2 \). Their averages we denote by

\[
X = q_1^2 = \frac{1}{2}(q_1^2 + p_1^2), \quad Y = q_2^2 = \frac{1}{2}(q_2^2 + p_2^2), \\
Z = q_1q_2 = \frac{1}{2}(q_1q_2 + p_1p_2).
\]  

(5.10)
Also introduce\(^1\) the function \(W = \frac{1}{2}(p_1 q_2 - q_1 p_2)\). All together \(X, Y, Z, W\) generate the subalgebra \(\mathcal{L} \subset \mathcal{F}\), this is exactly the resonance algebra \(\mathcal{L} \equiv \mathcal{F}_{1,1}\) of the isotropic oscillator \(H_0\) with frequencies 1:1. Now, \(\mathcal{L}\) can be split into the sum of the subspace \(\mathcal{L}_0\) enveloping \(X, Y, Z\) and the subspace \(\mathcal{L}_1\) enveloping \(W\).

We can check condition (5.7) and evaluate the triple bracket (5.8). Indeed, the brackets between generators of the resonance algebra are

\[
\{X, Y\} = 0, \quad \{X, Z\} = W, \quad \{Y, Z\} = -W, \\
\{X, W\} = -Z, \quad \{Y, W\} = Z, \quad \{Z, W\} = \frac{1}{2}(X - Y).
\]

The Casimir functions are

\[
C_0 = X + Y, \quad C_1 = XY - Z^2 - W^2.
\]

From (5.11) one obtains the triple brackets

\[
\{\{X, Z\}, X\} = Z, \quad \{\{X, Z\}, Y\} = -Z, \quad \{\{X, Z\}, Z\} = \frac{1}{2}(Y - X), \\
\{\{Y, Z\}, X\} = -Z, \quad \{\{Y, Z\}, Y\} = Z, \quad \{\{Y, Z\}, Z\} = \frac{1}{2}(X - Y).
\]

By passing from (5.11) to (5.12) we exclude the momentum coordinate \(W\).

Thus, over the configuration plane \(\mathbb{R}_q^2\) we have the triple Poisson algebra \(\mathcal{F}_0\) (5.12) with the Casimir function \(C_0 = X + Y\). The noncommuting coordinates \(X, Y, Z\) on \(\mathbb{R}_q^2\) are the mean square deviations of the Euclidean coordinates.

Let us write the resonance precession system in these coordinates. Assume that the perturbing Hamiltonians in (4.2) are all coming from the expansion of a potential well \(V\) and so they are \(p\)-independent:

\[
H_j(x) = \sum_{|\alpha| = j+2} \frac{1}{\alpha!} D^\alpha V(0) q^\alpha.
\]

This means that the total Hamiltonian \(H\) is

\[
H = H_0 + V = \frac{1}{2}(|q|^2 + |p|^2) + V(q), \quad q, p \in \mathbb{R}^2, \quad (5.13)
\]

\(^1\)The previous notation \(X \equiv \mathcal{A}_1, Y \equiv \mathcal{A}_2, Z \equiv \frac{1}{2}(\mathcal{A}_\alpha + \overline{\mathcal{A}}\alpha),\) and \(W \equiv \frac{1}{2\sqrt{2}}(\mathcal{A}_\alpha - \overline{\mathcal{A}}\alpha)\) see in Section 3.
where the potential $V$ contains terms of the third and higher orders near the point $q = 0$.

The averaging transformation (5.9) being applied to $H_1, H_3, \ldots$ gives zero. The leading nonzero term is

$$H_2 \equiv \Pi(H_2) = \sum_{|\alpha|=4} \frac{1}{\alpha!} D^\alpha V(0) q^\alpha.$$  

The averages $q^\alpha$ are given in the following list:

$$(5.14) \quad q_4^4 = \frac{3}{2} X^2, \quad q_4^2 = \frac{3}{2} Y^2,$$

$$q_1 q_3 = \frac{3}{2} YZ, \quad q_1^3 q_2 = \frac{3}{2} XZ, \quad q_1^2 q_2^2 = \frac{1}{2} X^2 Y + Z^2.$$  

It is important to mention that there is no generator $W$ in these formulas.

Finally, the average of the perturbing Hamiltonian reads

$$f(X, Y, Z) = \alpha X^2 + \beta Y^2 + \gamma Z^2 + \frac{1}{2} \gamma XY + \delta XZ + \rho YZ + O^4, \quad (5.15)$$

where

$$\alpha = \frac{1}{16} \frac{\partial^4 V}{\partial q_1^4}(0), \quad \beta = \frac{1}{16} \frac{\partial^4 V}{\partial q_2^4}(0), \quad \gamma = \frac{1}{4} \frac{\partial^4 V}{\partial q_1^2 \partial q_2^2}(0),$$

$$\delta = \frac{1}{4} \frac{\partial^4 V}{\partial q_1^2 \partial q_2^2}(0), \quad \rho = \frac{1}{4} \frac{\partial^4 V}{\partial q_1 \partial q_2^3}(0),$$

and $O^4$ denotes terms of order 4 and higher near the origin 0.

Now we can write the resonance precession system (4.8) for the coordinates $X, Y, Z,$ and $W$. Actually, because of the Casimir constraints $C_1 = 0$ and $X + Y = C_0$, it is enough to consider the equation for coordinates $a \equiv X - Y$ and $b \equiv 2Z$ only. These coordinates characterize the shape of the oscillator orbits projected to $\mathbb{R}_q^2$ (the eccentricity and the shear).

The resonance precession system reads

$$\frac{da}{dt} = -4W \frac{\partial f}{\partial b}, \quad \frac{db}{dt} = 4W \frac{\partial f}{\partial a}, \quad \frac{dW}{dt} = a \frac{\partial f}{\partial b} - b \frac{\partial f}{\partial a}. \quad (5.16)$$

Here $f$ is taken from (5.15) and expressed in the $a, b$-coordinates.
In order to exclude $W$, let us introduce a new “time” $\tau$ on the space–time $\mathbb{R}_{a,b}^2 \times \mathbb{R}_t$ by means of the equation

$$\{f, \tau\} = -4W.$$ Taking the second bracket, we obtain

$$\{\{f, \tau\}, f\} = 4\left(a \frac{\partial f}{\partial b} - b \frac{\partial f}{\partial a}\right).$$ (5.17)

Note that the new time $\tau$ does not commute with $a, b$ and we have

$$\{\{f, \tau\}, a\} = -4b, \quad \{\{f, \tau\}, b\} = 4a,$$

$$\{\{a, b\}, b\} = -4a, \quad \{\{a, b\}, a\} = 4b.$$ (5.18)

On the trajectories of (5.16), relation (5.17) becomes an equation for $\tau$

$$\frac{d^2\tau}{dt^2} = 4\left(b \frac{\partial f}{\partial a} - a \frac{\partial f}{\partial b}\right)_{a=a(\tau), \quad b=b(\tau)}.$$ (5.19)

The trajectory $(a(\tau), b(\tau))$ parametrized by the new time is obtained from the first two equations in (5.16):

$$\frac{d\alpha}{d\tau} = \frac{\partial f}{\partial b}, \quad \frac{d\beta}{d\tau} = -\frac{\partial f}{\partial a}.$$ (5.20)

Using (5.20) we resolve (5.19) as follows:

$$\frac{1}{2} \int_0^\tau \frac{d\tau}{\sqrt{C_0^2 - a(\tau)^2 - b(\tau)^2}} = t.$$ (5.21)

Now we note that the function $f$ (5.15) after the change of variables $X = \frac{1}{2}(C_0 + a)$, $Y = \frac{1}{2}(C_0 - a)$, $Z = \frac{1}{2}b$ is quadratic in $a, b$ (in the classical limit in any $N$th microzone near the origin). Thus, the right-hand side of (5.20) is just linear in $a, b$ and so system (5.20) is explicitly integrable in trigonometric functions of time $\tau$. This completely resolves the resonance precession system for the Hamiltonian (5.13).

Note that the reduced system (5.20) can be considered as the Hamiltonian system of the configuration space–time with the triple Poisson structure (5.18).
Example 5.2 (Noncommutative space–time under resonance 1:2). Let us consider the oscillator $H_0 = \frac{1}{2}(q_1^2 + p_1^2) + (q_2^2 + p_2^2)$. Applying the averaging transformation (5.1), we obtain the following coordinates on the configuration plane:

$$X = q_1^2, \quad Y = q_2^2, \quad Z = q_1^2 q_2.$$  

The explicit formulas for them via the coordinates on $\mathbb{R}_q^2 \times \mathbb{R}_p^2$ are

$$X = \frac{1}{2}(q_1^2 + p_1^2), \quad Y = \frac{1}{2}(q_2^2 + p_2^2), \quad Z = \frac{1}{4}(q_1^2 q_2 + 2q_1 p_1 p_2 - q_2 p_1^2).$$

We also introduce $W = \frac{1}{4}(p_1^2 p_2 + 2q_1 q_2 p_1 - q_1^2 p_2)$. Then the mutual brackets between these generators of the resonance algebra $\mathcal{F}_{1,2}$ are the following:

$$\{X,Y\} = 0, \quad \{X,Z\} = 2W, \quad \{Y,Z\} = -W, \quad \{X,W\} = -2Z, \quad \{Y,W\} = Z, \quad \{Z,W\} = \frac{1}{4} X^2 - XY. \quad (5.22)$$

The Casimir functions for this quadratic Poisson bracket are $C_0 = X + 2Y$ and $C_1 = \frac{1}{2} X^2 Y - Z^2 - W^2$.

By excluding $W$ we obtain the triple brackets

$$\\{\{X,Z\},X\} = 4Z, \quad \{\{X,Z\},Y\} = -2Z, \quad \{\{Y,Z\},X\} = -2Z, \quad \{\{Y,Z\},Y\} = Z, \quad (5.23)$$

$$\{\{X,Z\},Z\} = 2XY - \frac{1}{2} X^2, \quad \{\{Y,Z\},Z\} = \frac{1}{4} X^2 - XY. \quad (39)$$

Note that these triple brackets are nonlinear (quadratic) in coordinates. So, this is not a Lie triple system, but its nonlinear generalization.

The resonance precession system in this case also has the form (5.20), where the coordinates $a, b$ are defined by

$$a = X - 2Y, \quad b = Z.$$  

The new time $\tau$ in (5.20) is determined by the equations

$$\frac{d\tau}{dt} = -4W, \quad \frac{dW}{dt} = \left(\frac{1}{4} X^2 - XY\right) \frac{\partial f}{\partial Z} - Z \left(2 \frac{\partial f}{\partial X} - \frac{\partial f}{\partial Y}\right),$$

which are integrated similarly as in (5.21).

In conclusion, we remark that the noncommuting coordinates and the triple brackets which we introduced on the configuration plane $\mathbb{R}_q^2$ admit natural quantum analogs. It is easy to derive quantum versions of (5.12) and (5.23).
Appendix. Operator averaging

Let us consider the family of operators $\hat{H}_0 + \varepsilon \hat{H}_1$, $\varepsilon \to 0$, in the algebra $F$. We are looking for a family of operators $\hat{U}_\varepsilon$ such that

$$(\hat{H}_0 + \varepsilon \hat{H}_1) \hat{U}_\varepsilon = \hat{U}_\varepsilon (\hat{H}_0 + \varepsilon \hat{H}_1 + \varepsilon^2 \hat{H}_2 + O(\varepsilon^3)), \quad (A.1)$$

where

$$[\hat{H}_0, \hat{H}_1] = 0, \quad [\hat{H}_0, \hat{H}_2] = 0. \quad (A.2)$$

(we stopped at the order $O(\varepsilon^3)$ only for simplicity). One can formally choose

$$\hat{U}_\varepsilon = \exp \{-i\varepsilon (\hat{f}_0 + \varepsilon \hat{f}_1 + O(\varepsilon^2))\}. \quad (A.3)$$

In more explicit form,

$$\hat{U}_\varepsilon = I - i\varepsilon \hat{f}_0 - \varepsilon^2 (i\hat{f}_1 + \hat{f}_0^2/2) + O(\varepsilon^3), \quad (A.3)$$

where $\hat{f}_0, \hat{f}_1$ are solutions of the “homological” equations:

$$i[\hat{H}_0, \hat{f}_0] = \hat{H}_1 - \hat{H}_1, \quad i[\hat{H}_0, \hat{f}_1] = \hat{H}_2 - \hat{H}_2. \quad (A.4)$$

Here we denote $\hat{H}_2 \overset{\text{def}}{=} i[\hat{f}_0, \hat{H}_1 + \hat{H}_1]$. Thus one has to solve Eqs. (A.4), (A.2) with additional conditions (A.2).

Assume now that $\hat{H}_0$ is a quadratic form: $\hat{H}_0 = \sum_{l=1}^M \omega_l \hat{z}_l^\dagger \hat{z}_l$ (see (1.1a), (1.2a)) with positive frequencies $\omega_l$.

Any operator from the algebra $F$, say $\hat{H}_1$, can be represented as

$$\hat{H}_1 = \sum_k c_k \hat{g}_k, \quad \hat{g}_k = \hat{z}_k^+ \hat{z}_k^- , \quad (A.5)$$

where $c_k \in \mathbb{C}$, $k_+, k_- \in \mathbb{Z}^M$. Then one solves Eqs. (A.4), (A.2) by the explicit formulas

$$\hat{f}_0 = \frac{1}{i} \sum_{\omega_0(k_+ - k_-) \neq 0} \frac{c_k}{\omega_0(k_+ - k_-)} \hat{g}_k, \quad \hat{H}_1 = \sum_{\omega_0(k_+ - k_-) = 0} c_k \hat{g}_k. \quad (A.6)$$

The second equations in (A.4), (A.2) are solved similarly.

The transformation (A.1) is called the operator averaging. If the operator $\hat{U}_\varepsilon$ (A.3) is invertible, one can conclude from (A.1) that the original operator $\hat{H}_0 + \varepsilon \hat{H}_1$ is equivalent (up to $O(\varepsilon^3)$) to the new one $\hat{H}_0 + \varepsilon \hat{H}_1 + \varepsilon \hat{H}_2$, which contains the perturbations $\hat{H}_1$ and $\hat{H}_2$ commuting with the leading part $\hat{H}_0$. Thus, $\hat{H}_1$ and $\hat{H}_2$ are elements of the commutant $F_\omega$. 

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From Sections 1 and 2 we know how this commutant looks like (see Theorem 1.7) and can represent \( \hat{H}_1, \hat{H}_2 \) as functions in generators of its resonance components \( F^{(1)}, \ldots, F^{(L)} \). This representation follows from (A.6) and (1.7)

\[
\hat{H}_1 = \sum_{k^{(j)} \in R_n^{(j)}, j=1,\ldots,L} c_k \hat{g}_k,
\]

where \( k = (k^{(1)}, \ldots, k^{(L)}) \) and \( R_n^{(j)} \) is the resonance set (1.10) corresponding to the \( j \)th resonance component \( \omega^{(j)} = \omega_0^{(j)} \cdot n^{(j)} \) of the frequency system \( \omega \).

The classical version of (A.1) is the well-known classical averaging method \cite{4, 5}. Namely, in our operator scheme:

— the transformation \( \hat{U}^{-1}_\varepsilon \hat{f} \hat{U}_\varepsilon \) is replaced by \( \gamma_\varepsilon^* f \), where

\[
\gamma_\varepsilon^* = \exp(\varepsilon \text{ad}(f_0) + \varepsilon^2 \text{ad}(f_1) + O(\varepsilon^3))
\]

and \( \text{ad}(\cdot) \) denotes the Hamiltonian vector field;

— the homological equations (A.4) and Eqs. (A.2) are replaced by

\[
\begin{align*}
\{H_0, f_0\} &= H_1 - \underline{H}_1, & \{H_0, f_1\} &= H_2 - \underline{H}_2, \\
\{H_0, H_1\} &= 0, & \{H_0, H_2\} &= 0,
\end{align*}
\]

(A.8)

and the solution of (A.8) is given by (A.6), (A.7), where one has to remove the sign \( \hat{\cdot} \) over all the functions;

— the classical analog of (A.1) is

\[
\gamma_\varepsilon^* (\text{ad}(H_0) + \varepsilon \text{ad}(H_1)) = (\text{ad}(H_0) + \varepsilon \text{ad}(H_1) + \varepsilon^2 \text{ad}(H_2) + O(\varepsilon^3))\gamma_\varepsilon^*,
\]

which means that by the symplectic transformation \( \gamma_\varepsilon^{-1} \) one transforms the Hamiltonian field corresponding to \( H_0 + \varepsilon H_1 \) into the Hamiltonian field corresponding to \( H_0 + \varepsilon H_1 + \varepsilon^2 H_2 \) with \( H_1, H_2 \) being in involution with \( H_0 \).

The flows of these Hamiltonians are related to each other as follows:

\[
\gamma_{H_0 + \varepsilon H_1} = \gamma_\varepsilon \circ \gamma_{H_0} \circ \gamma_{H_1 + \varepsilon H_2} \circ \gamma_{H_1}^{-1} + O(\varepsilon^3).
\]

This operator averaging scheme and its generalizations can be found in \cite{40}; see also more details and applications in \cite{31}.
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