Determination of coupling patterns by parallel searches for
\[ \mu^- \rightarrow e^+ \text{ and } \mu^- \rightarrow e^- \text{ in muonic atoms} \]

Joe Sato,\(^1\)\(^∗\) Kohei Sugawara,\(^2\) Yuichi Uesaka,\(^3\)\(^†\) and Masato Yamanaka\(^4,5,\)\(^§\)

\(^1\)Department of Physics, Graduate School of Engineering Science, Yokohama National University, Yokohama, 240-8501, Japan
\(^2\)Physics Department, Saitama University, Saitama 338-8570, Japan
\(^3\)Faculty of Science and Engineering, Kyushu Sangyo University, 2-3-1 Matsukadai, Higashi-ku, Fukuoka 813-8503, Japan
\(^4\)Department of Mathematics and Physics, Osaka City University, Osaka 558-8585, Japan
\(^5\)Nambu Yoichiro Institute of Theoretical and Experimental Physics (NITEP), Osaka City University, Osaka 558-8585, Japan

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Abstract

We investigate a possibility that the \(\mu^- \rightarrow e^+ \) conversion is discovered prior to the \(\mu^- \rightarrow e^- \) conversion, and its implications to the new physics search. We focus on the specific model including the mixing of the \(SU(2)\)_L doublet- and singlet-type scalar leptoquarks, which induces not only the lepton flavor violation but also the lepton number violation. Such a structure is motivated by R-parity violating (RPV) supersymmetric models, where a sbottom mediates the conversion processes. We formulate the \(\mu^- \rightarrow e^+ \) rate in analogy with the muon capture in a muonic atom, and numerically evaluate it using several target nuclei. The lepton flavor universality test of pion decay directly limits the \(\mu^- \rightarrow e^+ \) rate, and the maximally allowed \(\mu^- \rightarrow e^+ \) branching ratio is \(\sim 10^{-18} \) under the various bounds on RPV parameters. We show that either \(\mu^- \rightarrow e^- \) or \(\mu^- \rightarrow e^+ \) signals can be discovered in near future experiments. This indicates that parallel searches for these conversions will give us significant information on the pattern of coupling constants.

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I. INTRODUCTION

The standard model (SM), where all neutrinos are left-handed and massless, has the accidental global $U(1)$ symmetries which ensure to conserve the lepton flavor numbers, $L_e$, $L_\mu$, and $L_\tau$. Nonetheless, the lepton flavor violation (LFV) was established by the discovery of neutrino oscillation, which implies that the three global symmetries are broken and the SM should be extended to include LFV sources.

In lots of extended models, LFV sources cause not only the flavor violation among charged leptons (called CLFV) but also the lepton number violation (LNV). One may presume that the LNV processes are minor compared with CLFV, because, aside from the flavor number, the particle number must be violated. However, we know situations where it does not hold. A well-known example is the Majorana mass of the neutrinos; the branching ratio of an LNV process $\mu^- \rightarrow e^+$ in nuclei could be much larger than that of LFV process $\mu \rightarrow e \gamma$ due to the GIM suppression in the flavor changing neutral current [1–3]. Therefore, both the LFV and LNV processes should be investigated.

The muonic atom is a good probe to both the LFV and LNV; an LFV process $\mu^- \rightarrow e^-$ conversion, $\mu^-(Z,A) \rightarrow e^-(Z,A)$, and an LNV process $\mu^- \rightarrow e^+$ conversion, $\mu^-(Z,A) \rightarrow e^+(Z-2,A)$. See Ref. [4] for the recent review of the $\mu^- \rightarrow e^+$ conversion. The experimental signals of these modes is single monoenergetic electron (positron), which is highly clean signal with little SM background. In near future experiments, the searches for these modes are planned by using a number of muonic atoms (COMET [5], Mu2e [6], and PRISM/PRIME [7]).

In this article, we investigate a possibility that the $\mu^- \rightarrow e^+$ conversion could be discovered prior to the $\mu^- \rightarrow e^-$ conversion. An interesting example to address the possibility is leptoquarks with the mixing of $SU(2)$ doublet and singlet. The condition is satisfied by sbottoms in R-parity violating (RPV) supersymmetric (SUSY) model [8]. When the sbottom $\tilde{b}$ has the RPV interaction $\tilde{b}\ell q$ and the mixing of $SU(2)$ doublet $\tilde{b}_L$ and singlet $\tilde{b}_R$, the lepton number is not conserved and the $\mu^- \rightarrow e^+$ conversion can be induced at tree level. We formulate the $\mu^- \rightarrow e^+$ conversion rate for the sbottom mediation, and numerically evaluate it under the experimental bounds on RPV parameters. We see that importance to search for and analyze the non-standard reactions of muonic atoms without prejudice that the LFV reactions are always leading compared with the LNV ones.

The contents of this article are as follows: In Sec. [II] we introduce leptoquarks inspired by sbottom in RPV SUSY and discuss current constraints on the coupling constants. We show the formula for the rate of the $\mu^- \rightarrow e^-$ and $\mu^- \rightarrow e^+$ conversions in a muonic atom in Sec. [III]. The results are shown in Sec. [IV] and finally, the article is summarized in Sec. [V].
II. BENCHMARK MODEL

We introduce a benchmark SUSY model wherein the reaction rates of $\mu^- \rightarrow e^+$ conversion and $\mu^- \rightarrow e^-$ conversion are comparable to each other.

The gauge invariant superpotential contains the RPV terms $\mathcal{W}_{\text{RPV}} = \lambda'_{ijk} L_i Q_j D_k$. Here $D_i$ is a $SU(2)_L$ singlet superfield, and $L_i$ and $Q_i$ are $SU(2)_L$ doublet superfields. Indices $i, j,$ and $k$ represent the generations. The interaction terms related with LFV and LNV processes are

$$
\mathcal{L}_\mathcal{L} = \lambda'_{ijk} \left[ d_{jL} d_{kR} \nu_{iL} - \tilde{d}_{kR}^* (e_{iL}^c u_{jL}) \right] + \text{H.c.},
$$

where $\tilde{d}_j$ is the SUSY partner of down-type quark $d_j$. We assume the simple situation that only the lighter sbottom contributes to low-energy observables, which is motivated by that, in many SUSY scenarios, it is lighter than the first and second generation squarks $[12]$. Thus, $j (k)$ in $\tilde{d}_{jL} (\tilde{d}_{kR})$ must be 3. The left- and right-handed sbottom ($\tilde{b}_L$ and $\tilde{b}_R$) are mixed each other after the $SU(2)_L$ symmetry breaking, and it could be large as $m_{2R} \propto m_\tilde{b}(A_0 - \mu \tan \beta)$. Here $A_0$ is so-called the trilinear scalar coupling, $\mu$ is the higgsino mass parameter, and $\tan \beta$ is the ratio of Higgs field vevs. The mixing is parametrized through the diagonalization of sbottom mass as

$$
-\mathcal{L}_{\tilde{b}-\text{mass}} = \left( \begin{array} {c} \tilde{b}_L^* \tilde{b}_R^* \\ \tilde{b}_L \tilde{b}_R \end{array} \right) \left( \begin{array} {cc} m_1^2 & m_{2R}^2 \\ m_{2R}^2 & m_2^2 \end{array} \right) \left( \begin{array} {c} \tilde{b}_L \\ \tilde{b}_R \end{array} \right) = \left( \begin{array} {c} \tilde{b}_1^* \tilde{b}_2^* \\ \tilde{b}_1 \tilde{b}_2 \end{array} \right) \left( \begin{array} {cc} m_1^2 & 0 \\ 0 & m_2^2 \end{array} \right) \left( \begin{array} {c} \tilde{b}_1 \\ \tilde{b}_2 \end{array} \right),
$$

where we set $m_1 \leq m_2$ and take the mixing angle $\theta_\tilde{b}$ as

$$
\left( \begin{array} {c} \tilde{b}_1 \\ \tilde{b}_2 \end{array} \right) = \left( \begin{array} {cc} \cos \theta_\tilde{b} & -\sin \theta_\tilde{b} \\ \sin \theta_\tilde{b} & \cos \theta_\tilde{b} \end{array} \right) \left( \begin{array} {c} \tilde{b}_L \\ \tilde{b}_R \end{array} \right).
$$

Thus the RPV interaction Lagrangian in terms of mass eigenstates is

$$
\mathcal{L}_\mathcal{L}' \supset \tilde{\lambda}'_{331} \tilde{b}_L \tilde{d}_{RiL} + \tilde{\lambda}'_{1313} (\tilde{e}_{iL}^c \tilde{u}_{jL}) + \text{h.c.}
$$

where we define $\tilde{\lambda}'_{331} = \lambda'_{331} \cos \theta_\tilde{b}$ and $\tilde{\lambda}'_{1313} = \lambda'_{1313} \sin \theta_\tilde{b}$.

The lepton flavors are no longer defined as conserved quantities with the interactions in Eq. (1). Then, the $\mu^- \rightarrow e^-$ conversion in nuclei is induced by the exchange of $\tilde{b}_L$ as shown in Fig. 1.

When the $\tilde{b}_L$-$\tilde{b}_R$ mixing exists in addition to the RPV interactions, the lepton number conservation is violated: if the mixing is absent, the lepton number $-1 \ (+1)$ can be assigned to $\tilde{b}_L \ (\tilde{b}_R)$. The $\mu^- \rightarrow e^+$ conversion in nuclei arises via the LFV vertex and the $\tilde{b}_L$-$\tilde{b}_R$ mixing (Fig. 2). It is important to emphasize that, when either $\lambda'_{213}$ or $\lambda'_{113}$ is zero, the $\mu^- \rightarrow e^-$ conversion rate goes to zero, but the $\mu^- \rightarrow e^+$ conversion could be observable.

The experimental bounds on the RPV parameters are set by independent measurements.

We summarize the bounds in the rest of this section.
FIG. 1: $\mu^- \to e^-$ conversion via the RPV operator, Eq. (1).

FIG. 2: $\mu^- \to e^+$ conversion by (a) the combination of $\{\lambda_{213}', \lambda_{131}'\}$ and (b) the combination of $\{\lambda_{113}', \lambda_{231}'\}$.

A. Atomic parity violation and parity violating electron scattering

The measurements of atomic parity violation (APV) and parity violating electron scattering (PVES) test the parity violating interaction, and set the bound on $\lambda_{131}'$ [13]. The parity violating interaction is parametrized as $-(G_F/\sqrt{2})C_{1d}\bar{e}\gamma_\mu\gamma_5e\bar{q}_i\gamma^\mu q_i$, where $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$ is the Fermi coupling constant. The sbottom interferes with the photon and $Z$ boson in APV and PVES, and the effective coupling is $1 C_{1d} = \frac{1}{2} - \frac{2}{3} \sin^2 \theta_W + \frac{m_W^2}{g^2 m_t^2} |\tilde{\lambda}_{131}'|$; $C_{1d}$ is obtained by including the APV results in the global fit incorporating the Qweak collaboration result and PVES database, $C_{1d} = 0.3389 \pm 0.0025$ (1σ) [15]. With $\sin^2 \theta_W = 0.2382$

1 We neglect the QED corrections to the $C_{1d}$ because it is small, $|C_{1d}^w - C_{1d}^{w/o}|/C_{1d}^{w/o} \sim O(1)$ [14], and the resultant effect on the $\lambda_{131}'$ bound is negligible.
at the experimental scale, the bound is

$$|\tilde{\lambda}'_{131}| \leq 6.9 \times 10^{-1} \left( \frac{m_{\tilde{t}_L}}{1\text{ TeV}} \right),$$

which depends on the assumption of the stop mass $m_{\tilde{t}_L}$. If the stop is sufficiently heavy, substantially there is no constraint on the coupling. In the analysis of this article, we will set $m_{\tilde{t}_L} = 1\text{ TeV}$ to have a bound, $|\tilde{\lambda}'_{131}| < 0.69$.

### B. Neutrino-nucleon scattering: $\nu_\mu d_R \to \nu_\mu d_R$

The sbottom exchange subprocess via $\lambda'_{231}$ interferes with the SM neutrino deep inelastic scattering (DIS) $\nu_\mu d_R \to \nu_\mu d_R$ [13]. Taking into account the interference, the coupling for the neutral current connecting $\nu_\mu$ and $d_R$ is $g'^2_R = \frac{1}{3} \sin^2 \theta_W + \frac{m^2_{\tilde{\chi}_0}}{m^2_1} |\lambda'_{231}|^2$. The precision measurement of the neutrino DIS provides $g'^2_R = -0.027^{+0.077}_{-0.048}$ [16], which excludes nonzero $\lambda'_{231}$ at the $1\sigma$ level. The bound at the $2\sigma$ level is

$$|\tilde{\lambda}'_{231}| \leq 3.6 \times 10^{-1} \left( \frac{m_1}{200\text{ GeV}} \right).$$

### C. Direct sbottom search

The direct search sets the limits on sbottom mass and RPV couplings. The decay width of RPV channel $\tilde{b}_1 \to e_l L u_L$ is

$$\Gamma(\tilde{b}_1 \to e_l L u_L) = \frac{|\tilde{\lambda}'_{131}|^2}{16\pi m_1} \lambda \left( 1, \frac{m_{e_L}}{m_1}, \frac{m_{u_R}}{m_1} \right) (m^2_1 - m^2_{e_L} - m^2_{u_R}).$$

Here $\lambda(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2yz - 2zx}$. The decay width of R-parity conserving channel $\tilde{b}_1 \to \tilde{\chi}^0 b$ is

$$\Gamma(\tilde{b}_1 \to \tilde{\chi}^0 b) = \frac{g^2_1}{16\pi m_1} \lambda \left( 1, \frac{m_{\tilde{\chi}_0}}{m_1}, \frac{m_b}{m_1} \right) \times \left[ (Y_L^2 \cos^2 \theta_b + Y_R^2 \sin^2 \theta_b) (m^2_1 - m^2_{\tilde{\chi}_0} - m^2_b) - 8Y_L Y_R \sin \theta_b m_b m_{\tilde{\chi}_0} \right].$$

where $Y_L$ and $Y_R$ are the hypercharge for left- and right-handed bottom, $m_{\tilde{\chi}_0}$ is the neutralino mass, and $m_b$ is the bottom mass. Setting the mass scales by maximally small ones $m_1 = 200\text{ GeV}$ and $m_{\tilde{\chi}_0} = 160\text{ GeV}$ [17], the direct search limit $\Gamma(\tilde{b}_1 \to e_l L u_L)/\Gamma(\tilde{b}_1 \to \tilde{\chi}^0 b) < O(10^{-2})$ ($l = e, \mu$) [18, 19] is transferred to the bound on RPV coupling as

$$|\tilde{\lambda}'_{131}| \lesssim 5 \times 10^{-3}.$$
D. Lepton flavor universality of pion decays

The RPV interactions, Eq. \( 1 \), could violate the lepton flavor universality of pion decays (Fig. 3). The RPV contributions \( \delta \Gamma_e \) and \( \delta \Gamma_\mu \) are related to the ratio of decay rates as

\[
R^{\pi}_{e/\mu} = \frac{\Gamma^{SM}_{e} + \delta \Gamma_e}{\Gamma^{SM}_{\mu} + \delta \Gamma_\mu} \approx \beta (1 + \epsilon_e - \epsilon_\mu) ,
\]

where \( \beta = \Gamma^{SM}_{e} / \Gamma^{SM}_{\mu} \), \( \epsilon_e = \delta \Gamma_e / \Gamma^{SM}_{e} \), and \( \epsilon_\mu = \delta \Gamma_\mu / \Gamma^{SM}_{\mu} \). Here we assumed \( \epsilon_e, \epsilon_\mu \ll 1 \). After the straightforward calculation, we obtain

\[
\epsilon_e = \frac{|\tilde{\lambda}^t_{113}|^2}{V_{ud}^2 G_F^2 m_1^1} \left\{ \frac{|\tilde{\lambda}^t_{231}|^2 + |\tilde{\lambda}^t_{131}|^2}{128} \left( \frac{m_\pi}{m_u + m_d} \right)^2 \frac{m^2_e}{m^2_\mu} + \frac{V_{ud} G_F m_1^1}{2\sqrt{2}} + \frac{|\tilde{\lambda}^t_{113}|^2 + |\tilde{\lambda}^t_{213}|^2}{32} \right\} ,
\]

\[
\epsilon_\mu = \frac{|\tilde{\lambda}^t_{212}|^2}{V_{ud}^2 G_F^2 m_1^1} \left\{ \frac{|\tilde{\lambda}^t_{231}|^2 + |\tilde{\lambda}^t_{131}|^2}{128} \left( \frac{m_\pi}{m_u + m_d} \right)^2 \frac{m^2_\mu}{m^2_\mu} + \frac{V_{ud} G_F m_1^1}{2\sqrt{2}} + \frac{|\tilde{\lambda}^t_{113}|^2 + |\tilde{\lambda}^t_{213}|^2}{32} \right\} ,
\]

Here \( V_{ud} \) is the u-d component of the CKM matrix. The first term in the parenthesis comes from the diagrams (a) and (b) in Fig. 3 for \( \epsilon_e \), and from the diagrams (e) and (f) for \( \epsilon_\mu \). These initial states form a scalar state with \( u_L \) and \( d_R \), and their contributions are much bigger than other diagram’s ones by \( m^2_e/(m_u + m_d)^2 \), which is so-called chiral enhancement effect [20]. In the parameter region we are interested in, the first terms dominate \( \epsilon_e \) and \( \epsilon_\mu \). Besides, the direct search limit [9] is more stringent than the limits from the second and third terms of Eqs. (11) and (12). Then the second and third terms are irrelevant in our analysis. The experimental constraint is given by \( R^{\pi, \text{exp}}_{e/\mu} = 1.2327^{(23)} \times 10^{-4} \) according to Ref. [24]. With the SM prediction \( R^{\pi, \text{SM}}_{e/\mu} = 1.2352 \times 10^{-4} \) [22, 23], we set the constraint as

\[
-7 \times 10^{-7} < \beta (\epsilon_e - \epsilon_\mu) < 2 \times 10^{-7} ,
\]
where we allow for a discrepancy of 2σ.

The RPV interactions also affect the decay $\pi^0 \to e^+e^-$. Since the RPV interactions lead to a (pseudo-)vector state for the initial state, this decay mode does not receive the chiral enhancement. It means that, as long as $|\lambda|^2/m_1^2 \ll G_F$, the RPV effects do not appear on this mode. It is because even the $Z^0$ exchange channel is negligible compared with leading channel, i.e., the electromagnetic loop one [24].

E. Neutrinoless double beta decay

We estimate the bound on RPV parameters along with the neutrinoless double beta decay ($0\nu2\beta$) in analogy with that assuming the Majorana neutrinos. Extracting the LNV source part in each amplitude (Fig. 4), we find the relation

$$\left|\tilde{\lambda}_{131}^\prime \tilde{\lambda}_{113}^\prime\right| < 2.6 \times 10^{-9} \left(\frac{m_1}{200 \text{(GeV)}}\right)^2,$$

where $m_{ee}$ is the effective Majorana mass of electron neutrino and $q$ is the momentum of internal neutrino. In our analysis we set $q = 100 \text{ MeV}$, which is evaluated by the typical distance between nucleons in a nucleus. Applying the bound $m_{ee} \lesssim 0.1 \text{ eV}$ [21], above relation (14) leads to the bound on RPV parameters as

$$\left|\tilde{\lambda}_{131}^\prime \tilde{\lambda}_{113}^\prime\right| < 2.6 \times 10^{-9} \left(\frac{m_1}{200 \text{(GeV)}}\right)^2.$$
A. $\mu^- \to e^-$ conversion

The $\mu^- \to e^-$ conversion in nuclei occurs with the combination of $\tilde{\lambda}^{13}_{213}$ and $\tilde{\lambda}^{13}_{113}$ (Fig. 1). Applying the formula for $\mu^- \to e^-$ conversion rate [25], the branching ratio in our scenario is obtained by

$$B(\mu^- \to e^-; N) = \frac{\tilde{\tau}_\mu |\tilde{\lambda}^{13}_{213}\tilde{\lambda}^{13}_{113}|^2}{4m_1^4} (2V^{(p)} + V^{(n)})^2 m_\mu^5.$$  \hspace{1cm} (16)

The dimensionless overlap integral $V^{(p,n)}$ and the muonic-atom lifetime $\tilde{\tau}_\mu$ are listed in Table I. The most stringent bound, $B(\mu^- \to e^-; Au) < 7 \times 10^{-13}$ [26], gives the limit by $|\tilde{\lambda}^{13}_{213}\tilde{\lambda}^{13}_{113}| < 1.6 \times 10^{-7}$ for $m_1 = 200 \text{ GeV}$.

| Nucleus | $V^{(p)}$ | $V^{(n)}$ | $\tilde{\tau}_\mu$ [ns] |
|---------|----------|----------|-------------------|
| $^{27}\text{Al}$ | 0.0161 | 0.0173 | 864.0 |
| $^{197}\text{Au}$ | 0.0974 | 0.146 | 74.3 |

B. $\mu^- \to e^+$ conversion

The combination of the LFV RPV couplings and the $\tilde{b}_L - \tilde{b}_R$ mixing gives rise to the $\mu^- \to e^+$ conversion in nuclei (Fig. 2). The reaction rate of $\mu^- \to e^+$ conversion faces the nuclear transition matrix. For the Majorana-neutrino case, it is evaluated by the nuclear proton-neutron renormalized quasi-particle random phase approximation [28–30] and the shell model calculation [31]. The short-range effective operators inducing the $\mu^- \to e^+$ conversion were discussed in Refs. [32–34]. The conversion rate for other types of operators have not been qualitatively investigated, also for the operator in this work. We therefore estimate the conversion rate in analogy with the muon capture $\mu^- p \to \nu_\mu n$ in muonic atoms.

We adopt the phenomenological parametrization of capture rate for a nucleus of an atomic number $Z$ and of a mass number $A$ [27, 35],

$$\Gamma_{\text{cap}} \simeq Z_{\text{eff}}^4 X_1 \left(1 - X_2 \frac{A - Z}{2A}\right).$$  \hspace{1cm} (17)

Here $Z_{\text{eff}}$ is the effective atomic number for muonic atoms [36]. The $Z_{\text{eff}}$ dependence stems from the effective number of protons in a nucleus ($Z_{\text{eff}}$) and probability of a muon being at the nuclear center ($Z_{\text{eff}}^3$); the latter can also be understood by the expression of the muon wave function, $|\psi_\mu(0)|^2 = (m_\mu Z_{\text{eff}} \alpha)^3 / \pi$. The parameter $X_1$ corresponds to the capture rate for muonic hydrogen, and $X_2$ parametrizes the Pauli blocking effect. The experimental data fit the parameters by $X_1 = 170 \text{ s}^{-1}$ and $X_2 = 3.125$ [27].
The $\mu^- \to e^+$ conversion rate is inferred in an analogy of the muon capture rate as
\[
\Gamma(\mu^- \to e^+; N) \simeq \frac{|\tilde{\lambda}_{2ij} \tilde{\lambda}_{ij}|^2}{m_1^4} \left( \frac{G_F}{\sqrt{2}} \right)^2 \frac{1}{q^2} Q_{\mu^- \to e^+} Z_{\text{eff}}^2 |\psi_\mu(0)|^2 \left( 1 - X_2 \frac{A - Z}{2A} \right)^2 ,
\] (18)
where $N$ presents the initial nucleus, and $(i, j) = (1, 3), (3, 1)$. The factor $1/q^2$ expresses the correlation function of active neutrino of momentum $q$. Since the process associates with the internal conversion $2p \to 2n$, it is expected that the rate is proportional to $Z_{\text{eff}}^2$. Note that the energy scale factor $Q_{\mu^- \to e^+}$ contains the nuclear transition strength in addition to the phase space volume, and its power is determined by the dimensional analysis. The branching ratio is given by $B(\mu^- \to e^+; N) = \sigma_\mu \Gamma(\mu^- \to e^+; N)$. We use $X_2 = 3.125$ as the muon capture, and we take $q = 100 \text{MeV}$, which corresponds to the Fermi momentum of nucleon in the nucleus. The energy scale factor is set by $Q_{\mu^- \to e^+} = m_\mu$.

The current experimental bound is $B(\mu^- \to e^+; \text{Ti}) < 1.7 \times 10^{-12} (3.6 \times 10^{-11})$ for the transition to the ground (giant dipole resonance) state of calcium $^{37}$Ca. Using Eq. (18), we obtain the limit by $|\tilde{\lambda}_{2ij} \tilde{\lambda}_{ij}|^2 / m_1^4 < 6.5 \times 10^{-21} \text{MeV}^{-4}$.

IV. RESULTS

Numerical analysis is shown in two cases; One is of negligible $\mu^- \to e^-$ conversion rate and the other is more general ones. We adopt $m_1 = 200 \text{GeV}$. Free parameters are the four RPV couplings, $\tilde{\lambda}_{213}^\prime$, $\tilde{\lambda}_{131}^\prime$, $\tilde{\lambda}_{113}^\prime$, and $\tilde{\lambda}_{231}^\prime$.

A. Case of no $\mu^- \to e^-$ conversion

We separately investigate two patterns wherein the $\mu^- \to e^-$ conversion is turned off: (pattern I) $\tilde{\lambda}_{213}^\prime \neq 0$, $\tilde{\lambda}_{131}^\prime \neq 0$, and $\tilde{\lambda}_{113}^\prime = \tilde{\lambda}_{231}^\prime = 0$ (pattern II) $\tilde{\lambda}_{113}^\prime \neq 0$, $\tilde{\lambda}_{231}^\prime \neq 0$, and $\tilde{\lambda}_{213}^\prime = \tilde{\lambda}_{131}^\prime = 0$.

1. Pattern I: $\tilde{\lambda}_{213}^\prime \neq 0$, $\tilde{\lambda}_{131}^\prime \neq 0$, and $\tilde{\lambda}_{113}^\prime = \tilde{\lambda}_{231}^\prime = 0$

We evaluate the maximal $B(\mu^- \to e^+; N)$. Decomposing $B(\mu^- \to e^+; N)$ into the target dependence part $\tilde{B}$ (Table II) and uncertain parts ($q$ and $Q_{\mu^- \to e^+}$), it is rewritten as
\[
B(\mu^- \to e^+; N) = \tilde{B} \left( \frac{100 \text{MeV}}{q} \right)^2 \left( \frac{Q_{\mu^- \to e^+}}{100 \text{MeV}} \right)^8 .
\] (19)
We find $B(\mu^- \to e^+; N) \sim O(10^{-18})$ in the pattern I. The COMET phase-II (Mu2e), PRISM/PRIME, and NUFACT experiments respectively plan to accumulate $O(10^{16})$, $O(10^{18})$, and $O(10^{21})$ muons. Figure 5(a) shows $B(\mu^- \to e^+; \text{Ca}) = 10^{-16}$, $10^{-18}$, and $10^{-21}$ (black solid) corresponding to these muon productions. The calcium (Ca) target would maximize
the $S/N$ ratio \cite{38}. The shaded area shows the excluded parameter region. $\tilde{\lambda}'_{131}$ is unbound unless the stop mass is given. Here we take $m_{\tilde{t}_L} = 1$ TeV. Then the direct search (9) and the measurement of APV-PVES \cite{5} draw the boundaries. The bound on $\tilde{\lambda}'_{131}$ gets looser for the heavier stop mass, and then the LFU test in pion decays makes the boundary. It is testable in near future experiments, and could shed light on the LFV and LNV sources.

It is important to emphasize that the LFU in pion decays tightly correlated with the $\mu^- \rightarrow e^+$ conversion (compare Figs. 2 and 3) in this scenario. When the violation of LFU is observed in pion decays, searches for the $\mu^- \rightarrow e^+$ conversion would provide complementary information for new physics.

2. Pattern II: $\tilde{\lambda}'_{113} \neq 0$, $\tilde{\lambda}'_{231} \neq 0$, and $\tilde{\lambda}'_{213} = \tilde{\lambda}'_{131} = 0$

Applying $\tilde{B}$ in Table II for the pattern II, the maximal $B(\mu^- \rightarrow e^+; N)$ is obtained by $\mathcal{O}(10^{-21})$. This implies that the discovery of $\mu^- \rightarrow e^+$ conversion at COMET, Mu2e, and PRISM experiments rules out the pattern II. Figure 5 (b) is the same as Fig. 5 (a) but for the $\tilde{\lambda}'_{113}$-$\tilde{\lambda}'_{231}$ plane. The direct search (9) and the LFU test in pion decays (13) draw the boundaries to the excluded region.
TABLE II: Target dependent coefficient $\tilde{B}$ by $(\tilde{\lambda}'_{131}, \tilde{\lambda}'_{213}) = (6.9 \times 10^{-1}, 5.0 \times 10^{-3})$ for the pattern I and $(\tilde{\lambda}'_{113}, \tilde{\lambda}'_{231}) = (5.0 \times 10^{-3}, 8.0 \times 10^{-3})$ for the pattern II which leads to the maximal $B(\mu^- \rightarrow e^+; N)$, effective atomic number $Z_{\text{eff}}$, and lifetime of the muonic atom $\tilde{\tau}_\mu$ [27].

| Nucleus | $Z_{\text{eff}}$ | $\tilde{\tau}_\mu$ [ns] | $\tilde{B}$ (Pattern I) | $\tilde{B}$ (Pattern II) |
|---------|----------------|--------------------------|------------------------|--------------------------|
| $^{27}\text{Al}$ | 11.48 | 864 | $7.0 \times 10^{-19}$ | $9.2 \times 10^{-23}$ |
| $^{32}\text{S}$ | 13.64 | 540 | $1.4 \times 10^{-18}$ | $1.8 \times 10^{-22}$ |
| $^{40}\text{Ca}$ | 16.15 | 333 | $2.0 \times 10^{-18}$ | $2.6 \times 10^{-22}$ |
| $^{48}\text{Ti}$ | 17.38 | 330 | $1.4 \times 10^{-18}$ | $1.8 \times 10^{-22}$ |
| $^{65}\text{Zn}$ | 21.61 | 161 | $2.2 \times 10^{-18}$ | $2.8 \times 10^{-22}$ |
| $^{73}\text{Ge}$ | 22.43 | 167.4 | $1.6 \times 10^{-18}$ | $2.1 \times 10^{-22}$ |

B. General analysis including all four couplings

The bounds on RPV couplings from the searches for $\mu^- \rightarrow e^-$ conversion and $0\nu2\beta$ are also comprehended, in addition to the bounds discussed in Sec. IV A. The bounds derived from relevant observables are summarized in Table III.

TABLE III: Bounds on the RPV couplings applied in Sec. IV B. Here $m_1 = 200$ GeV. $\beta = \Gamma_{e}^{\text{SM}} / \Gamma_{\mu}^{\text{SM}}$, $\epsilon_e$, and $\epsilon_\mu$ are given in Eqs. (11) and (12).

| Observables | Bound | Section |
|-------------|-------|---------|
| APV and PVES | $\tilde{\lambda}'_{131} \leq 0.69$ | II A |
| $\nu_\mu d_R \rightarrow \nu_\mu d_R$ | $\tilde{\lambda}'_{231} \leq 0.36$ | II B |
| Direct sbottom search | $\tilde{\lambda}'_{1i3} \leq 5 \times 10^{-3}$ ($i = 1, 2$) | II C |
| LFU of $\pi^\pm$ decays | $-7 \times 10^{-7} \leq \beta (\epsilon_e - \epsilon_\mu) \leq 2 \times 10^{-7}$ | II D |
| $0\nu2\beta$ | $\tilde{\lambda}'_{1i3} \tilde{\lambda}'_{131} \leq 2.6 \times 10^{-9}$ | II E |
| $\mu^- \rightarrow e^-$ conversion | $\tilde{\lambda}'_{213} \tilde{\lambda}'_{113} \leq 1.6 \times 10^{-7}$ | III A |

Figure 6 shows the excluded region (shaded area) for each combination of RPV couplings. In each panel, the other RPV couplings are set to be zero. It has been already investigated for the $\tilde{\lambda}'_{131}-\tilde{\lambda}'_{213}$ and $\tilde{\lambda}'_{113}-\tilde{\lambda}'_{231}$ planes, wherein both the $\mu^- \rightarrow e^-$ conversion and $0\nu2\beta$ are turned off. The $0\nu2\beta$ search draws the outline of excluded region in the $\tilde{\lambda}'_{113}-\tilde{\lambda}'_{131}$ plane (Fig. III (a)). The $\mu^- \rightarrow e^-$ conversion search draws the outline of excluded region in the $\tilde{\lambda}'_{113}-\tilde{\lambda}'_{213}$ plane (Fig. III (b)). These processes are therefore important ingredients for the analysis in the space of $(\tilde{\lambda}'_{113}, \tilde{\lambda}'_{131}, \tilde{\lambda}'_{213}, \tilde{\lambda}'_{231})$.

The bounds and observables in Fig. 6 have actually more complicated correlations with each other. Figure 7 shows an example result. In the parameter space wherein
FIG. 6: Excluded regions (shaded area) in the plane of two couplings. The observables attached on each line draw the boundaries according to Table III. In the investigation for a combination of two RPV couplings, other RPV couplings are set to be zero.

free from all experimental bounds except for the $\mu^- \to e^-$ conversion, first, we lead the maximally allowed $B(\mu^- \to e^+;\text{Ca})$. With these arrangements, the maximally allowed $B(\mu^- \to e^-;\text{Al})$ is evaluated. For the region of $\tilde{\lambda}_{113} \lesssim 4 \times 10^{-9}$, as is close to the setting in Sec. IV A I, $B(\mu^- \to e^-;\text{Al})$ does not reach the PRISM/PRIME sensitivity. In this region, the maximized combination $\tilde{\lambda}_{131} \tilde{\lambda}_{213}$ leads to the large $B(\mu^- \to e^+;\text{Ca})$. For the region of $4 \times 10^{-9} \lesssim \tilde{\lambda}_{113} \lesssim 5 \times 10^{-6}$, since the search for $0\nu2\beta$ limits the combination $\tilde{\lambda}_{113} \tilde{\lambda}_{131}$, $B(\mu^- \to e^+;\text{Ca})$ decreases with $\tilde{\lambda}_{113}$. For the region of $\tilde{\lambda}_{113} \gtrsim 5 \times 10^{-6}$, the $\tilde{\lambda}_{113} \tilde{\lambda}_{213}$ term dominates over the $\tilde{\lambda}_{131} \tilde{\lambda}_{213}$ term in Eq. (18), and $B(\mu^- \to e^+;\text{Ca})$ increases with $\tilde{\lambda}_{113}$. For the region of $\tilde{\lambda}_{113} \gtrsim 4 \times 10^{-5}$, the measurement for of LFU limits the combination $\tilde{\lambda}_{113} \tilde{\lambda}_{231}$ (see Fig. 5), and $B(\mu^- \to e^+;\text{Ca})$ levels off at $\simeq 10^{-22}$.

The largest $B(\mu^- \to e^+;\text{Ca})$ is achieved for $B(\mu^- \to e^+;\text{Ca}) \lesssim 10^{-20}$. Both $\mu^- \to e^-$ and $\mu^- \to e^+$ conversions could be observed in near future experiments. Complementary
We have investigated the possibility that the LNV process $\mu^- \to e^+$ conversion is observed prior to the LFV process $\mu^- \to e^-$ conversion. For a reference scenario of our interest, we have focused on RPV SUSY models wherein the $SU(2)_L$ doublet and singlet sbottom ($\tilde{b}_L$ and $\tilde{b}_R$) mixes each other.

When the conservation of lepton flavors is violated by the RPV interactions, they give rise to the $\mu^- \to e^-$ conversion. The $\tilde{b}_L$-$\tilde{b}_R$ mixing flips the lepton number on the internal sbottom line, and hence the lepton number is no longer conserved. The $\mu^- \to e^+$ conversion arises via the LFV vertex and the $\tilde{b}_L$-$\tilde{b}_R$ mixing. It is important to emphasize that, when either $\lambda'_{213}$ or $\lambda'_{113}$ is zero, the $\mu^- \to e^-$ conversion rate goes to zero, but the $\mu^- \to e^+$ conversion still could be observable.

We have evaluated the rate of $\mu^- \to e^+$ mediated by the sbottom in analogy of the muon capture process in muonic atom. Then we have investigated how could the $\mu^- \to e^+$ rate be large under the experimental bounds on RPV parameters. Bounds come from the $\mu^- \to e^-$ conversion search, the measurement of LFU in pion decays, the direct sbottom search at the LHC, and so on. Especially, we have found that the LFU in pion decays provides the direct constraints for the $\mu^- \to e^+$ rate because they are connected through the same combinations.
of the couplings. The largest $B(\mu^- \rightarrow e^+; \text{Ca})$ is achieved in the parameter region of small $B(\mu^- \rightarrow e^-; \text{Al})$. In some parameter regions, both $B(\mu^- \rightarrow e^+; \text{Ca})$ and $B(\mu^- \rightarrow e^-; \text{Al})$ are experimentally reachable at next-generation experiments. Complementary measurements of these conversions shed light on not only the LFV source and also the origin of LNV in new physics scenarios. It is important to search for and analyze the non-standard reactions of muonic atoms without prejudice that the LFV reactions always are leading compared with the LNV ones.

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