Proof of the Bonheure-Noris-Weth conjecture on oscillatory radial solutions of Neumann problems

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Abstract. Let $B_1$ be the unit ball in $\mathbb{R}^N$ with $N \geq 2$. Let $f \in C^1([0, \infty), \mathbb{R})$, $f(0) = 0$, $f(\beta) = \beta$, $f(s) < s$ for $s \in (0, \beta)$, $f(s) > s$ for $s \in (\beta, \infty)$ and $f'(\beta) > \lambda_k^r$. D. Bonheure, B. Noris and T. Weth [Ann. Inst. H. Poincaré Anal. Non Linéaire 29(4) (2012)] proved the existence of nondecreasing, radial positive solutions of the semilinear Neumann problem

$$-\Delta u + u = f(u) \quad \text{in} \ B_1, \quad \partial_{\nu} u = 0 \quad \text{on} \ \partial B_1$$

for $k = 2$, and they conjectured that there exists a radial solution with $k$ intersections with $\beta$ provided that $f'(\beta) > \lambda_k^r$ for $k > 2$. In this paper, we show that the answer is yes.

Keywords. Bonheure-Noris-Weth conjecture; Neumann problem; oscillatory radial solutions; bifurcation.

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1 Introduction

Let $B_1$ be the unit ball in $\mathbb{R}^N$ with $N \geq 2$. Very recently, D. Bonheure, B. Noris and T. Weth [1] proved the existence of nondecreasing, radial positive solutions of the semilinear Neumann problem

$$\begin{cases}
-\Delta u + u = f(u) & \text{in} \ B_1, \\
u > 0 & \text{in} \ B_1, \\
\partial_{\nu} u = 0 & \text{on} \ \partial B_1
\end{cases} \quad (1.1)$$

under the assumptions:

(f1) $f \in C^1([0, \infty), \mathbb{R})$, $f(0) = 0$ and $f$ is nondecreasing;
(f2) $f'(0) = \lim_{s \to 0^+} \frac{f(s)}{s} = 0$;
(f3) $\lim_{s \to +\infty} \frac{f(s)}{s} > 1$;
(f4) there exists $\beta > 0$ such that $f(\beta) = \beta$ and

$$f'(\beta) > \lambda_2^r. \quad (1.2)$$

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Here $\lambda_k^r$ is the $k$-th radial eigenvalue of $-\Delta + I$ in the unit ball with Neumann boundary conditions.

It is easy to see that $u \equiv \beta$ is a constant solution of (1.1), and there exists nonlinearity $f$ satisfying $(f1) - (f3)$ such that the problem (1.1) only admits this constant solution, see [1, Proposition 4.1]. For the existence of nonconstant radial solutions, they obtained the following result by variational argument.

**Theorem A.** Assume $(f1) - (f4)$. Then there exists at least one nonconstant increasing radial solution of (1.1).

They raised the question whether it is possible to construct radial solutions with a given number of intersections with $\beta$ provided that $f'(\beta)$ is sufficiently large. More precisely, they conjectured that there exists a radial solution with $k$ intersections with $\beta$ provided that $f'(\beta) > \lambda_k^r$.

The purpose of the present paper is to show that the answer to the above question is yes! The proof is based upon the unilateral global bifurcation theorem [4, 5, 8]. The condition $f'(0) = 0$ and the monotonic condition in $(f1)$ seem unduly restrictive. We shall make the following assumptions:

(A1) $f \in C^1((0, \infty), \mathbb{R})$, $f(0) = 0$;

(A2) $f_+^\infty := \lim_{s \to +\infty} \frac{f(s)}{s} < \infty$;

(A3) there exists $\beta > 0$ such that $f(\beta) = \beta$, $f(s) < s$ for $s \in (0, \beta)$, $f(s) > s$ for $s \in (\beta, \infty)$

and

$$f'(\beta) > \lambda_k^r,$$ for some $k \geq 2$;

(A4) $[f(s + \beta) - (s + \beta)]s > 0$, $s \in (-\beta, 0) \cup (0, \infty)$.

The main result of this paper is the following

**Theorem 1.1** Assume (A1)-(A3). Then for each $j \in \{2, \cdots, k\}$, (1.1) has two nonconstant radial solutions $u_j^+$ and $u_j^-$ such that $u_j^+ - \beta$ changes sign exactly $k - j + 1$ times in $(0, 1)$ and is positive near $0$, and $u_j^- - \beta$ changes sign exactly $k - j + 1$ times in $(0, 1)$ and is negative near $0$. Moreover, if (A4) holds, then $u_j^+$ is decreasing in $[0, 1]$ and $u_j^-$ is increasing in $[0, 1]$.

For other results on the existence of radial solutions of nonlinear Neumann problems, see [2, 10, 14].
The rest of the paper is organized as follows. In Section 2 we study the spectrum structure of the linear Neumann problem

\[
\begin{cases}
-\Delta u(x) = \mu a(|x|)u(x) & \text{in } B_1, \\
\partial_\nu u = 0 & \text{on } \partial B_1,
\end{cases}
\]

where \(a \in C[0, 1]\) satisfies \(a(r) > 0\) for \(r \in [0, 1]\). In Section 3, we introduce some functional setting and state some preliminary bifurcation results on abstract operator equations. Finally in Section 4 we prove our main results on the existence of nonconstant radial solutions by applying the well-known unilateral bifurcation theorem due to Dancer [4, 5].

2 Eigenvalues of linear eigenvalue problems

Let us consider the linear eigenvalue problem

\[
\begin{cases}
-\Delta u(x) = \mu a(|x|)u(x) & \text{in } B_1, \\
\partial_\nu u = 0 & \text{on } \partial B_1,
\end{cases}
\]

(2.1)

where \(a \in C[0, 1]\) satisfies

\[
a(r) > 0, \quad r \in [0, 1].
\]

(2.2)

**Theorem 2.1** Assume that (2.2) is fulfilled. Then the radial eigenvalues of (2.1) are as follows:

\[
0 = \mu^r_0 < \mu^r_1 < \mu^r_2 < \cdots \to \infty.
\]

(2.3)

Moreover, for each \(k \in \mathbb{N}^* := \{0, 1, 2, \cdots \}\), the radial eigenvalue \(\mu^r_k\) is simple, and the radial eigenfunction \(\psi_k\), being regarded as a function of \(r\), possesses exactly \(k\) simple zeros in \([0, 1]\), and \(\psi_k\) is radially monotone if and only if \(k \in \{0, 1\}\).

It is easy to see that Theorem 2.1 is an immediate consequence of the following results on singular Sturm-Liouville problems.

**Theorem 2.2** Assume that (2.2) is fulfilled. Then the eigenvalues of the problem

\[
\begin{cases}
- u''(r) - \frac{N - 1}{r} u'(r) = \mu a(r)u(r), & r \in (0, 1), \\
u'(0) = 0 = u'(1)
\end{cases}
\]

(2.4)

are as follows:

\[
0 = \mu^r_0 < \mu^r_1 < \mu^r_2 < \cdots \to \infty.
\]

Moreover, for each \(k \in \mathbb{N}^*\), \(\mu^r_k\) is simple, and the eigenfunction \(\psi_k\) possesses exactly \(k\) simple zeros in \([0, 1]\), and \(\psi_k\) is monotone if and only if \(k \in \{0, 1\}\).
To prove Theorem 2.2, we need several basic lemmas.

**Lemma 2.1** Assume that \( \tilde{f} \in C([0, \infty) \times [0, \infty)) \) is Lipschitz continuous in \( u \) on \([0, \infty)\). Then for given \( \zeta \in (0, \infty) \), the initial value problem

\[
\begin{cases}
- u''(r) - \frac{N-1}{r} u'(r) = \tilde{f}(r, u(r)), & r \in (0, \infty), \\
u'(0) = 0, \\
u(0) = \zeta
\end{cases}
\]  

(2.5)

has a unique solution \( u \) defined on \([0, \infty)\). Moreover, all of zeros of \( u \) are simple.

**Proof.** According to [15, Existence and uniqueness Theorem XIII in §6 of Chapter II], for given \( b > 0 \), the initial value problem

\[
\begin{cases}
- u''(r) - \frac{N-1}{r} u'(r) = \tilde{f}(r, u(r)), & r \in (0, b), \\
u'(0) = 0, \\
u(0) = \zeta
\end{cases}
\]  

has exactly one solution \( u \in C^2[0, b] \). Notice the equation in (2.5) is non-singular for \( r \geq b \), it is evident that \( u \) can be extended to \([0, \infty)\). Since \( \tilde{f} \in C([0, \infty) \times [0, \infty)) \) is Lipschitz continuous in \( u \) on \([0, \infty)\), the uniqueness part can be deduced by the same method in the Appendix in [11].

All of zeros of \( u \) are simple since for any zero point \( \tau \) of \( u \), the initial value problem

\[
\begin{cases}
- u''(r) - \frac{N-1}{r} u'(r) = f(r, u(r)), & r \in (0, \infty), \\
u(\tau) = 0 = u'(\tau)
\end{cases}
\]  

has only trivial solution \( u \equiv 0 \). \qed

**Lemma 2.2** Assume that \( a \in C([0, \infty), (0, \infty)) \) and there exist two positive constants \( a_1 \) and \( a_2 \), such that

\[ a_1 \leq a(r) \leq a_2, \quad r \in [0, \infty). \]

Let \( u \) be a solution of the problem

\[
\begin{cases}
- u''(r) - \frac{N-1}{r} u'(r) = \mu a(r) u(r), & r \in (0, \infty), \\
u'(0) = 0, \\
u(0) = \zeta
\end{cases}
\]  

(2.6)

with \( \mu > 0 \) and \( \zeta > 0 \). Then \( u \) has a sequence of zeros \( \{ \tau_n \} \subset (0, \infty) \) with

\[ \tau_n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty. \]  

(2.7)

**Proof.** From [15, XVIII in §27 of Chapter VI], the solution \( y \) of the initial value problem

\[
(r^{N-1} y')' + r^{N-1} y = 0, \quad y(0) = 1, \quad y'(0) = 0
\]  

(2.8)
oscillates. Denote the zeros of $y$ by $\xi_0 < \xi_1 < \xi_2 < \cdots$. Then
\[ \xi_{n+1} - \xi_n \to \pi, \quad \text{as } n \to \infty. \]  

(2.9)

Let $\gamma_j = \sqrt{\mu a_j}$ for $j = 1, 2$. Let
\[ u_j(r) = y(\gamma_j r), \quad j = 1, 2. \]

Then $u_j$ is the unique solution of the initial value problem
\[ (r^{N-1}u')' + \gamma_j^2 r^{N-1}u = 0, \quad u(0) = 1, \quad u'(0) = 0, \]
and for $j = 1, 2$, $u_j$ oscillates and it has a sequence of zeros $\xi_j < \xi_{j+1} < \cdots$. Combining this with the Sturm-Picone Theorem (see [15, §27 of Chapter VI]), it deduces that the solution $u$ of (2.6) oscillates.

**Lemma 2.3** Assume that $a \in C([0, \infty), (0, \infty))$. Let $u$ be a solution of the problem (2.6) with $\mu > 0$ and $\zeta > 0$. Let $r_1, r_2$ be any two consecutive zeros of $u'$ in $[0, \infty)$ with $r_1 < r_2$. Then $u$ has one and only one zero in $(r_1, r_2)$.

**Proof.**\(-u''(r) - \frac{N-1}{r}u'(r) = \mu a(r)u(r)\) can be rewritten as
\[ -(r^{N-1}u'(r))' = \mu a(r)r^{N-1}u(r). \]

(2.11)

Integrating from $r_1$ to $r$, we get
\[ u'(r) = -\mu \int_{r_1}^{r} \frac{a(t)}{t^{N-1}} u(t) dt, \]
and accordingly,
\[ 0 = u'(r_2) = -\mu \int_{r_1}^{r_2} \frac{a(t)}{t^{N-1}} u(t) dt, \]
which implies that $u$ has at least one zero in $(r_1, r_2)$.

Suppose on the contrary that $u$ has two zeros $z_1, z_2 \in (r_1, r_2)$. Then there exists $z^* \in (z_1, z_2) \subset (r_1, r_2)$, such that $u'(z^*) = 0$. However, this is a contradiction.

**Lemma 2.4** Assume that $a \in C([0, \infty), (0, \infty))$. Let $u$ be a solution of (2.6) with $\mu > 0$ and $\zeta > 0$. Let $\tau_1, \tau_2$ be any two consecutive zeros of $u$ in $(0, \infty)$ with $\tau_1 < \tau_2$. Then $u'$ has one and only one zero in $(\tau_1, \tau_2)$.

**Proof.** Obviously, $u'$ has at least one zero $r_1$ in $(\tau_1, \tau_2)$.

Without loss of generality, we may assume that $u(r) > 0$ in $(\tau_1, \tau_2)$. It follows from
\[ -u''(r) - \frac{N-1}{r}u'(r) = \mu a(r)u(r) \]
that
\[ u''(r_1) < 0, \]
which implies that $u'$ has just one zero in $(\tau_1, \tau_2)$.
which implies that $u$ is concave up near $r = r_1$.

Suppose on the contrary that there exists $r_* \in (\tau_1, \tau_2)$ with $r_* \neq r_1$ such that $u'(r_*) = 0$. Then by the same argument, we get

$$u''(r_*) < 0$$

This together with $u''(r_1) < 0$ imply that there exists $\hat{r} \in (\min \{r_1, r_*\}, \max \{r_1, r_*\})$ such that $u$ attains a local minimum at $\hat{r}$, and

$$u(\hat{r}) > 0, \quad u'(\hat{r}) = 0, \quad u''(\hat{r}) \geq 0,$$

which contradicts (2.12). Therefore, $u'$ has only one zero in $(\tau_1, \tau_2)$. \hfill \Box

**Lemma 2.5** Assume that $a \in C([0, \infty), (0, \infty))$ with $a(r) \geq a_0 > 0$ in $(0, \infty)$. Let $u$ be a solution of (2.6) with $\zeta > 0$ and $\mu > 0$. Let $\tau_k(\mu)$ and $r_k(\mu)$ be the $k$-th positive zero of $u$ and $u'$, respectively. Then

1. For given $k \in \mathbb{N}$, $\tau_k(\mu)$ is strictly decreasing in $(0, \infty)$;
2. For given $k \in \mathbb{N}$, $r_k(\mu)$ is strictly decreasing in $(0, \infty)$.

**Proof.** (1) For fixed $k > 1$, $\tau_k(\mu)$ is strictly decreasing in $\mu$, which is an immediate consequence of the well-known Sturm Separation Theorem [15, P. 272] since the differential equation

$$-u'' - \frac{N-1}{r}u' = \mu a(r)u$$$n is non-singular for $r \geq \tau_1(\mu)$. So, we only need to show that $\tau_1(\mu)$ is strictly decreasing in $(0, \infty)$.

Let $\tau_1(\mu)$ be the first zero of the solution $u$ of the initial value problem

$$\begin{cases}
- (r^{N-1}u'(r))' = \mu r^{N-1}a(r)u(r), & r \in (0, \infty), \\
u'(0) = 0, \\
u(0) = \zeta.
\end{cases}$$

(2.13)

Let $\tau_1(\mu^*)$ be the first zero of the solution $v$ of the initial value problem

$$\begin{cases}
- (r^{N-1}v'(r))' = \mu^* r^{N-1}a(r)v(r), & r \in (0, \infty), \\
v'(0) = 0, \\
v(0) = \zeta.
\end{cases}$$

(2.14)

We only need to show that

$$\tau_1(\mu) > \tau_1(\mu^*) \quad \text{if } \mu^* > \mu.$$  

(2.15)

Suppose on the contrary that $\tau_1(\mu) \leq \tau_1(\mu^*)$. Then

$$v(r) > 0, \quad r \in [0, \tau_1(\mu)); \quad v'(\tau_1(\mu)) < 0.$$  

(2.16)
Multiplying the equations in (2.13) and (2.14) by \( v \) and \( u \), respectively, and integrating from 0 to \( \tau_1(\mu) \), we get
\[
-(\tau_1(\mu))^{N-1}v(\tau_1(\mu))u'(\tau_1(\mu)) = (\mu - \mu^*) \int_0^{\tau_1(\mu)} r^{N-1}a(r)u(r)v(r)dr.
\]
However, this is impossible from (2.16) and the fact
\[
u(r) > 0, \quad r \in [0, \tau_1(\mu)]; \quad u'(\tau_1(\mu)) < 0.
\]
Therefore, (2.15) is valid. \( \square \)

(2) Using the similar method to treat (2.15) and the fact \( \tau_k(\mu) \) is strictly decreasing in \((0, \infty)\), it is not difficult to show that \( r_k(\mu) \) is strictly decreasing for \( \mu \in (0, \infty) \).

Let \( u \) be the solution of (2.13) and \( r_k(\mu) \) be the \( k \)-th positive zero of \( u' \). Then
\[
\tau_k(\mu) < r_k(\mu) < \tau_{k+1}(\mu).
\]
Without loss of generality, we may assume that
\[
u'(r) < 0, \quad r \in (r_{k-1}(\mu), r_k(\mu)); \quad u(r) < 0, \quad r \in (\tau_k(\mu), r_k(\mu)). \quad (2.17)
\]
(The other cases can be proved by the similar method.) Let \( v \) be the solution of (2.14) and \( r_k(\mu^*) \) be the \( k \)-th positive zero of \( v' \). Then it follows from (2.17) that
\[
v'(r) < 0, \quad r \in (r_{k-1}(\mu^*), r_k(\mu^*)). \quad (2.18)
\]
Suppose on the contrary that there exist some \( k \) and some \( \mu, \mu^* \) with \( \mu < \mu^* \), such that
\[
r_k(\mu) \leq r_k(\mu^*). \quad (2.19)
\]
Combining this with the fact that \( \tau_k(\mu) \) is strictly decreasing in \( \mu \) and using (2.17), it follows that
\[
v'(r) < 0, \quad r \in [\tau_k(\mu), r_k(\mu)); \quad v(r) < 0, \quad r \in [\tau_k(\mu), r_k(\mu)].
\]
Multiplying the equation in (2.14) by \( u \) and the equation in (2.13) by \( v \) and integrating from \( \tau_k(\mu) \) to \( r_k(\mu) \), we get
\[
(\tau_k(\mu))^{N-1}v(\tau_k(\mu))u'() + (r_k(\mu))^{N-1}v'(r_k(\mu))u(r_k(\mu))
\]
\[
= (\mu - \mu^*) \int_{\tau_k(\mu)}^{r_k(\mu)} r^{N-1}a(r)u(r)v(r)dr.
\]
This together with the signs of \( u, u', v, v' \) at \( \tau_k(\mu) \) and \( r_k(\mu) \) imply that (2.19) is impossible. \( \square \)
Proof of Theorem 2.2 Let $u(r; \zeta, \mu)$ be the unique solution of (2.6). For $k \in \mathbb{N}$. Let $\mu^r_k$ be such that $u'(1; \zeta, \mu^r_k) = 0$ and $u(r; \zeta, \mu^r_k)$ has exactly $k$ zeros in $(0, 1)$.

Let

$$
\psi_k(r) := u(r; \zeta, \mu^r_k), \quad r \in [0, 1].
$$

Then Lemmas 2.1-2.5 guarantee the desired results. In particular,

$$
\psi_0(r) \equiv \zeta; \quad \psi_1(r) \text{ is monotone on } r \in (0, 1).
$$

\[\square\]

Lemma 2.6 Let $\{(\mu_n, y_n)\}$ be a sequence of solutions of the problem

$$
-(r^{N-1}y_n')' = \mu_n r^{N-1} g(y_n), \quad y_n'(0) = y_n'(1) = 0, \tag{2.20}
$$

where $|\mu_n| \leq \hat{\mu}$ ($\hat{\mu}$ is a positive constant), $g: \mathbb{R} \to \mathbb{R}$ satisfies

$$
|g(s)| \leq L|s| \quad \text{for some constant } L > 0.
$$

Then $\|y_n'\|_{\infty} \to \infty$ as $n \to \infty$ implies $\|y_n\|_{\infty} \to \infty$ as $n \to \infty$.

\textbf{Proof.} Assume on the contrary that $\|y_n\|_{\infty} \not\to \infty$ as $n \to \infty$. Then, after taking a subsequence and relabeling, if necessary, it follows that

$$
\|y_n\|_{\infty} \leq M_0 \tag{2.21}
$$

for some $M_0 > 0$. From (2.20), we get

$$
y_n'(r) = -\mu_n \int_0^r \left( \frac{s}{r} \right)^{N-1} g(y_n(s))ds,
$$

which implies that

$$
\|y_n'\|_{\infty} \leq \hat{\mu}L_0 \cdot \|y_n\|_{\infty} \leq \hat{\mu}M_0L_0.
$$

However, this is a contradiction. \[\square\]

3 Functional setting and preliminary properties

The main point to prove Theorem 1.1 consists in using the unilateral global bifurcation theorem of [4, 5, 8]

Let $E$ be a real Banach space with norm $\| \cdot \|$. $\mathcal{E}$ will denote $E \times \mathbb{R}$. Let the mapping $\mathcal{G}: \mathcal{E} \to E$ satisfy

\textbf{Assumption 2:} if $\mathcal{G}(0, \lambda) = 0$ for $\lambda \in \mathbb{R}$, $\mathcal{G}$ is completely continuous and

$$
\mathcal{G}(x, \lambda) = \lambda Lx + H(x, \lambda),
$$
where \( L \) is a completely continuous linear operator on \( E \) and \( \| H(x, \lambda) \| / \| x \| \rightarrow 0 \) uniformly on bounded subsets of \( \mathbb{R} \) as \( \| x \| \rightarrow 0 \).

Define \( \Phi(\lambda) : E \rightarrow E \) by \( \Phi(\lambda)(x) = x - G(x, \lambda) \) and define \( \mathcal{L} \) to be the closure of \( \{(x, \lambda) \in \mathcal{E} : x = G(x, \lambda), x \neq 0\} \) in \( \mathcal{E} \). Then (cp. Rabinowitz \[13\]) \( \mathcal{L} \cap (\{0\} \times \mathbb{R}) \subseteq \{0\} \times r(L) \), where \( r(L) \) denotes the real characteristic value of \( L \). If \( \mu \in r(L) \), define \( C_{\mu} \) to be the component of \( \mathcal{L} \) containing \((0, \mu)\).

Assume now that \( \mu \in r(L) \) such that \( \mu \) has multiplicity 1. Suppose that \( v \in E \setminus \{0\} \) and \( l \in E^* \) such that
\[
v = \mu Lv, \quad l = \mu L^* l,
\]
(where \( L^* \) is the adjoint of \( L \)) and \( l(v) = 1 \). If \( y \in (0, 1) \), define
\[
K_y = \{(u, \lambda) \in \mathcal{E} : |l(u)| > y \| u \| \},
\]
\[
K^+_y = \{(u, \lambda) \in \mathcal{E} : l(u) > y \| u \| \}, \quad K^-_y = \{(u, \lambda) \in \mathcal{E} : l(u) < -y \| u \| \}.
\]
By \[13, Lemma 1.24\], there exists an \( S > 0 \) such that
\[
(\mathcal{L} \setminus \{(\mu, 0)\}) \cap \mathcal{E}_S(\mu) \subseteq K_y,
\]
where \( \mathcal{E}_S(\mu) = \{(u, \lambda) \in \mathcal{E} : \| u \| + |\lambda - \mu| < S\} \) and \( \mathcal{E}_S(\mu) \) denotes closure of \( \mathcal{E}_S(\mu) \). For \( 0 < \epsilon \leq S \) and \( \nu = \pm \), define \( D_{\mu, \epsilon}^\nu \) to be the component of \( \{(0, \mu)\} \cup (\mathcal{L} \cap \mathcal{E}_\epsilon(\mu) \cap K_y^\nu) \) containing \((0, \mu)\), \( C_{\mu, \epsilon}^\nu \) to be the component of \( \overline{C_{\mu} \setminus D_{\mu, \epsilon}^\nu} \) containing \((0, \mu)\) (where \(-\nu \) is interpreted in the natural way), and \( C_{\mu, \nu} \) to be the closure of \( \bigcup_{S \geq \epsilon > 0} C_{\mu, \epsilon}^\nu \). Then \( C_{\mu, \nu} \) is connected and, by \[5\], \( C_{\mu} = C_{\mu, +} \cup C_{\mu, -} \).

By \[13, Lemma 1.24\], the definition of \( C_{\mu, \nu} \) is independent of \( y \).

**Theorem 3.1** \[5, Theorem 2\] Either \( C_{\mu, +} \) and \( C_{\mu, -} \) are both unbounded or
\[
C_{\mu, +} \cap C_{\mu, -} \neq \{(0, \mu)\}.
\]

### 4 Proof of the Main Results

Let \( X := \{u \in C^1[0, 1] : u'(0) = u'(1) = 0\} \). Then it is a Banach space under the norm
\[
\| u \|_X = \max\{\| u \|_\infty, \| u' \|_\infty\}.
\]

We shall prove that the first choice of the alternative of Theorem 3.1 is the only possibility.

In what follows, we use the terminology of Rabinowitz \[13\]. Let \( S_{k, +} \) denote the set of functions in \( X \) which have exactly \( k - 1 \) interior nodal (i.e. non-degenerate) zeros in \((0, 1)\) and
are positive near \( r = 0 \), set \( S_{k, -} = -S_{k, +} \), and \( S_k = S_{k, +} \cup S_{k, -} \). Finally, let \( \Phi_{k, \pm} = \mathbb{R} \times S_{k, \pm} \) and \( \Phi_k = \mathbb{R} \times S_k \) under the product topology.

Let us consider the problem

\[
\begin{aligned}
&\begin{cases}
  -\Delta u + u = f(u) & \text{in } B_1, \\
  u > 0 & \text{in } B_1, \\
  \partial_\nu u = 0 & \text{on } \partial B_1,
\end{cases} \\
&u > 0 \text{ in } B_1,
\end{aligned}
\]

which is equivalent to

\[
\begin{aligned}
&\begin{cases}
  -u'' - \frac{N - 1}{r} u' + u = f(u), & r \in (0, 1), \\
  u > -\beta, & r \in [0, 1], \\
  u'(0) = u'(1) = 0.
\end{cases}
\end{aligned}
\]

Let

\[
v := u - \beta.
\]

Then (4.2) can be rewritten as

\[
\begin{aligned}
&\begin{cases}
  -v'' - \frac{N - 1}{r} v' + v = f(v + \beta) - \beta, & r \in (0, 1), \\
  v > -\beta, & r \in [0, 1], \\
  v'(0) = v'(1) = 0.
\end{cases}
\end{aligned}
\]

Let

\[
h(s) := \begin{cases} f(s + \beta) - \beta, & s \geq -\beta, \\
-\beta, & s < -\beta.
\end{cases}
\]

Then

\[
h(v) = h'(0)v + \xi(v) = f'(\beta)v + \xi(v), \quad h'(0) = f'(\beta),
\]

and

\[
\xi'(0) := \lim_{v \to 0} \frac{\xi(v)}{v} = 0.
\]

Thus, to study the \( S_{k, \nu} \)-solutions of (4.3), let us consider the auxiliary problem

\[
\begin{aligned}
&\begin{cases}
  -v'' - \frac{N - 1}{r} v' + v = \lambda f'(\beta)v + \lambda \xi(v), & r \in (0, 1), \\
  v > -\beta, & r \in [0, 1], \\
  v'(0) = v'(1) = 0.
\end{cases}
\end{aligned}
\]

For \( e \in X \), let \( Te \) be the unique solution of the problem

\[
\begin{aligned}
&\begin{cases}
  -z'' - \frac{N - 1}{r} z' + z = e, & r \in (0, 1), \\
  z'(0) = z'(1) = 0.
\end{cases}
\end{aligned}
\]
Then the map $T : X \to X$ is completely continuous, and

$$r(T) = \{ \lambda_j^r | \lambda_j^r = \mu^r_{j-1} + 1, \ j = 1, 2, \cdots \}.$$  

Here $r(T)$ denotes the real characteristic value of $T$. Obviously (4.5) is equivalent to

$$v = \lambda T(f'(\beta)v) + \lambda T\xi(v),$$  \hspace{1cm} (4.7)

$$v > -\beta.$$  \hspace{1cm} (4.8)

To show that (4.5) has a $S_{k,\nu}$-solution, let us consider the auxiliary problem (4.7)-(4.8) as a bifurcation problem from the trivial solution $v \equiv 0$. Furthermore, we have from (4.4) that

$$\|T\xi(v)\|_X \leq \|T\|_{X \to X} \max \left\{ \|\xi(v)\|_\infty, \|\xi'(v)v'\|_\infty, \|v\|_X \right\} \to 0 \quad \text{as} \quad \|v\|_X \to 0.$$  

Now the Dancer’s unilateral global bifurcation theorem for (4.7) can be stated as follows: Let

$$\mathcal{L} := \left\{ (\lambda, v) \in (0, \infty) \times X : (\lambda, v) \text{ satisfies (4.7), } v \neq 0 \right\}.$$  

For $\lambda_k^r \in r(T)$, define $C_k$ to be the component of $\mathcal{L}$ containing $\left[ \frac{\lambda_k^r}{f'(\beta)}, 0 \right)$. Then

$$C_k := C_{k,+} \cup C_{k,-},$$  

where

$$C_{k,\nu} := C_{\lambda_k^r/f'(\beta), \nu} \quad \nu \in \{+, -\},$$  

see Section 3 for detail. Now the Dancer’s unilateral global bifurcation theorem yields that either $C_{k,+}$ and $C_{k,-}$ are both unbounded or

$$C_{k,+} \cap C_{k,-} \neq \{ (\frac{\lambda_k^r}{f'(\beta)}, 0) \}. \hspace{1cm} (4.9)$$  

From (A1), it follows that if $(\lambda, v)$ is a solution of

$$\begin{cases}  
- v'' - \frac{N-1}{r} v' + v = \lambda h(v), \\
v(\tau) = v'(\tau) = 0
\end{cases}$$

for some $\tau \in (0, \infty)$, then $v \equiv 0$. This implies that

$$C_{k,+} \subset \left( \Phi_{k,+} \cup \{ (\frac{\lambda_k^r}{f'(\beta)}, 0) \} \right).$$

Clearly, if (4.9) holds, then there exists $(\lambda_*, v_*) \in C_{k,+} \cap C_{k,-}$, such that $(\lambda_*, v_*) \neq \left( \frac{\lambda_k^r}{f'(\beta)}, 0 \right)$, and $v_* \in S_{k,+} \cap S_{k,-}$, which contradicts the definition of $S_{k,+}$ and $S_{k,-}$.

Furthermore, we get
Lemma 4.1. For given \( k \geq 2, \) \( C_{k,+} \) and \( C_{k,-} \) are both unbounded, and \( \left( \frac{\lambda}{f'(\beta)}, 0 \right) \) bifurcates two unbounded components \( C_{k,+} \) and \( C_{k,-} \) of solutions to problem (4.7), such that
\[
(C_{k,+} \setminus \{ (\frac{\lambda}{f'(\beta)}, 0) \}) \subseteq \Phi_{k,+}, \quad (C_{k,-} \setminus \{ (\frac{\lambda}{f'(\beta)}, 0) \}) \subseteq \Phi_{k,-}.
\]

Lemma 4.2. Let \((\lambda, v) \in C_{k,\nu}\) with \( \lambda \in [0, 1] \). Then
\[
v(r) > -\beta, \quad r \in [0, 1]. \tag{4.10}
\]

**Proof.** Suppose on the contrary that there exists \( x_0 \in [0, 1] \) such that
\[
v(x_0) = \min_{r \in [0,1]} v(r) = -\beta.
\]
Then there exists \( r_0 \in [0, 1] \) such that either
\[
v(r_0) = 0, \quad v(r) < 0 \text{ for } r \in [x_0, r_0), \quad v'(r) > 0 \text{ for } r \in (x_0, r_0]; \tag{4.11}
\]
or
\[
v(r_0) = 0, \quad v(r) < 0 \text{ for } r \in (r_0, x_0], \quad v'(r) < 0 \text{ for } r \in [r_0, x_0). \tag{4.12}
\]
We only deal with the case (4.11), the case (4.12) can be treated by the similar way.

By (A1)-(A3), there exists \( m \geq 0 \) such that \( h(s) + ms \) is monotone increasing in \( s \) for \( s \in [-\beta, +\infty) \). Then
\[
-v'' - \frac{N - 1}{r} v' + v + \lambda m v = \lambda [h(v) + mv], \quad r \in (0, 1],
\]
and, since
\[
-(v+\beta)'' - \frac{N - 1}{r} (v+\beta)' + (\lambda m + 1)(v+\beta) \leq \lambda [h(v+\beta) + m(-\beta)], \quad r \in (0, 1],
\]
it follows that
\[
-(v+\beta)'' - \frac{N - 1}{r} (v+\beta)' + (\lambda m + 1)(v+\beta) \geq \lambda ([h(v) + mv] - [h(-\beta) + m(-\beta)]) \geq 0, \quad r \in (0, 1].
\]
Denote
\[
w := v + \beta.
\]
Then
\[
w'' + \frac{N - 1}{r} w' - (\lambda m + 1) w \leq 0, \quad r \in (0, 1],
\]
\[
w'(0) = w'(1) = 0.
\]
It follows from [6, Theorem 3.5] or [12, Theorem 3 in Chapter 1] that, \( w \) cannot achieve a non-positive minimum in the interval \((0, 1)\) unless it is constant. From (4.11), it follows that
\[
\inf_{[x_0, r_0]} w(r) = \min\{w(x_0), w(r_0)\} = w(x_0) = 0.
\]
This together with \( w'(x_0) = 0 \) imply that
\[
w(r) \equiv 0, \quad r \in [x_0, r_0].
\]
However, this contradicts the fact \( w'(r) > 0, \ r \in (x_0, r_0) \). Therefore,
\[
v(r) > -\beta, \quad r \in [0, 1].
\]
□

In view of Lemma 4.2, (4.5) is equivalent to (4.7). So, we only need to show that
\[
C_{k, \nu} \cap (\{1\} \times X) \neq \emptyset.
\]
In the following, we only deal with the case \( \nu = + \) since the other case can be treated by the similar way.

Let \( k \geq 2 \) be fixed, and let \( (\eta_n, y_n) \in C_{k, +} \) satisfy
\[
\eta_n + ||y_n||_X \to \infty.
\]
It is easy to check that
\[
\eta_n > 0, \quad n \in \mathbb{N}.
\]
From (A3), it follows that \( \frac{\lambda^r_k}{f'(0)} < 1 \), i.e.
\[
\frac{\lambda^r_k}{f'(\beta)} < 1.
\]
(4.15)

We shall show that
\[
C_{k, +} \cap (\{1\} \times X) \neq \emptyset.
\]
(4.16)

Assume on the contrary that \( C_{k, +} \cap (\{1\} \times X) = \emptyset \). Then
\[
C_{k, +} \subset (0, 1) \times X,
\]
and accordingly,
\[
0 < \eta_n < 1.
\]
Thus
\[
||y_n||_X \to \infty, \quad n \to \infty,
\]
which together with Lemma 2.6 imply that
\[
||y_n||_\infty \to \infty, \quad n \to \infty.
\]
(4.18)

This means that \( C_{k, +} \) is unbounded in \( C[0, 1] \).
We may assume that $\eta_n \to \bar{\eta} \in [0,1]$ as $n \to \infty$. Let

$$z_n := \frac{y_n}{||y_n||_\infty}.$$  

Then $||z_n||_\infty = 1$ and

$$
\begin{cases}
- z''_n - \frac{N-1}{r} z'_n + z_n = \eta_n \frac{h(y_n)}{y_n} z_n, & r \in (0,1), \\
z'_n(0) = z'_n(1) = 0.
\end{cases}
$$  

(4.19)

From (A1)-(A3) and the definition of $h$, it follows that $\frac{h(y_n(r))}{y_n(r)}$ is continuous in $[0,1]$ and is bounded uniformly in $n$. After taking subsequence if necessary, we may assume that

$$(\eta_n, z_n) \to (\bar{\eta}, z^*), \quad \text{in } \mathbb{R} \times X.$$  

(4.20)

Here $||z^*||_\infty = 1$.

As a direct consequence of the Banach contraction mapping principle in a small neighborhood of $\tau$, the initial value problem

$$
\begin{cases}
- z'' - \frac{N-1}{r} z' + z = \bar{\eta} H(r) z, \\
z(\tau) = z'(\tau) = 0
\end{cases}
$$

has a unique solution $z \equiv 0$. Notice that taking subsequence if necessary, we may assume that $\frac{h(y_n(r))}{y_n(r)} \to H$ in $L^2[0,1]$. So, all of zeroes of $z^*$ are simple, and accordingly $(\bar{\eta}, z^*) \in C_{j,+}$ for some $j \in \mathbb{N}$.

Let

$$\tau(1,n) < \cdots < \tau(k-1,n)$$

denote the zeros of $y_n$, and let

$$\tau(0,n) := 0, \quad \tau(k,n) := 1.$$  

Then, after taking a subsequence if necessary,

$$\lim_{n \to \infty} \tau(l,n) := \tau(l,\infty), \quad l \in \{0,1,\cdots,k-1,k\}.$$  

(4.21)

Denote

$$J_l := (\tau(l,\infty), \tau(l+1,\infty)), \quad l \in \{0,1,\cdots,k-1\}.$$  

Claim We claim that

$$J_l = \emptyset \quad \text{if } l \in \{0,1,\cdots,k-1\} \text{ and } l \text{ is odd},$$  

(4.22)
and
\[ \lim_{n \to \infty} y_n(r) = +\infty \text{ uniformly in } [\tau(l, \infty) + \epsilon, \tau(l + 1, \infty) - \epsilon] \text{ if } l \in \{0, 1, \cdots, k - 1\} \text{ and } l \text{ is even}, \]

(4.23)

where \( \epsilon > 0 \) is small constant.

In fact, suppose on the contrary that
\[ J_{l_0} \neq \emptyset \quad \text{for some } l_0 \in \{0, 1, \cdots, k - 1\} \text{ and } l_0 \text{ is odd}. \]

Then we have from Lemma 4.2 that
\[-\beta < y_n(r) < 0, \quad r \in (\tau(l_0, n), \tau(l_0 + 1, n)).\]

Thus, for any \( r \in (\tau(l_0, n), \tau(l_0 + 1, n)) \), it follows from (4.18) and
\[-z''_n - \frac{N - 1}{r} z'_n + z_n = \eta_n \frac{h(y_n)}{|y_n|_{\infty}}, \quad r \in (\tau(l_0, n), \tau(l_0 + 1, n)),\]

that
\[ \begin{cases} 
-z''^* - \frac{N - 1}{r} z'^* + z^* = 0, \quad r \in J_{l_0}, \\
 z^*(\tau) = z'^*(\tau) = 0,
\end{cases} \]

for some \( \tau \in J_{l_0} \). This implies that
\[ z^*(r) = 0 = z'^*(r), \quad r \in J_{l_0}. \]

However, this contradicts the fact the solution \((\bar{\eta}, z^*) \in C_{j,+} \subset [0, 1] \times S_{j,+} \) for some \( j \in \mathbb{N} \).

Therefore, (4.22) is true!

Obviously, (4.23) is an immediate consequence of the fact that all of the zeros of \( z^* \in S_{j,+} \) are simple and \( l \in \{0, 1, \cdots, k - 1\} \) is even.

Therefore, the Claim is true!

In the following, we shall use some idea from the proof of [7, Lemma 3.2] and the proof of main results of [3, 9] to show (4.16) is valid.

Let \((y_n)^-\) be the negative part of \( y_n \). Then it follows from Lemma 4.2 that \( 0 \leq (y_n)^- < \beta \) since \( \eta_n \in (0, 1) \), and consequently,
\[ (z_n)^- \to 0, \quad \text{as } n \to \infty. \]

Combining this with the Claim and using the definition of \( h \), it concludes that
\[ \begin{cases} 
-z'' - \frac{N - 1}{r} z'^* + z^* = \bar{\eta} f_{+\infty}(z^*)^+ , \quad \text{a.e. } r \in (0, 1), \\
z'^*(0) = z'^*(1) = 0,
\end{cases} \]

(4.24)
where $(z^*)^+$ is the positive part of $z^*$. Now, it follows from [6, Theorem 3.5] or [12, Theorem 3 in Chapter 1] that, $z^*$ cannot achieve a non-positive minimum in the interval $(0, 1)$ unless it is constant. Since $z_n(0) > 0$, we get

$$z^*(0) \geq 0.$$ 

If $z^*(0) = 0$, then it follows from (4.24) that

$$z^* \equiv 0, \quad r \in (0, 1).$$

However, this contradicts $||z^*||_\infty = 1$. So $z^*(0) > 0$.

Again, from [6, Theorem 3.5] or [12, Theorem 3 in Chapter 1] that $z^*(r) > 0$ in $[0, 1]$. This means that $z^* \in S_{1,+}$, and therefore, since $S_{1,+}$ is open and $||z_n - z^*||_X \to 0$ as $n \to \infty$, $z_n \in S_{1,+}$ for $n$ large enough. However, this contradicts $z_n \in S_{k,+}$ for all $n \in \mathbb{N}$ and $k \geq 2$.

Therefore, (4.16) is valid, and we may take $v_{k,+} \in (C_{k,+} \cap \{(1) \times X\})$. Similarly, we may take $v_{k,-} \in (C_{k,-} \cap \{(1) \times X\})$.

To show that $v_{2,+}$ is decreasing in $[0, 1]$. Let us denote $t_1 (0 < t_1 < 1)$ be the zero of $v_{2,+}$. Notice that $v_{2,+}$ satisfies (4.3), i.e.,

$$\begin{cases} 
-(r^{N-1}(v_{2,+})')' = r^{N-1}[f(v_{2,+} + \beta) - (v_{2,+} + \beta)], & r \in (0, 1), \\
v_{2,+} > -\beta, \\
v_{2,+}'(0) = v_{2,+}'(1) = 0.
\end{cases}$$

Combining this with (A4) and using Lemma 4.2, it concludes that

$$(r^{N-1}(v_{2,+})')' < 0, \quad t \in (0, t_1); \quad (r^{N-1}(v_{2,+})')' > 0, \quad t \in (t_1, 1).$$

This together with the boundary condition $v_{2,+}'(0) = v_{2,+}'(1) = 0$ imply that

$$(v_{2,+})' < 0, \quad t \in (0, t_1); \quad (v_{2,+})' < 0, \quad t \in (t_1, 1).$$

Therefore, $v_{2,+}$ is decreasing in $[0, 1]$.

Using the same method, with the obvious changes, we may deduce that $v_{2,-}$ is increasing in $[0, 1]$. \[\square\]

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