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Two-superfluid Model of Two-component Bose-Einstein Condensates; First Sound and Second Sound

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Abstract Superfluid $^4$He at a finite temperature is described by the two-fluid model with the normal fluid component and the superfluid component. We formulate the two-fluid model for two-component BECs, namely two-superfluid model, starting from the coupled Gross-Pitaevskii equations. The two-superfluid model well corresponds to the two-fluid model in superfluid $^4$He. In a special condition, the two sound modes in the two-superfluid model behave like first and second sounds in the two-fluid model of superfluid $^4$He.

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1 Introduction

Superfluid $^4$He has been thoroughly studied theoretically and experimentally in the field of low temperature physics since Kapitsa discovered superfluidity of $^4$He below the transition temperature. Tisza and Landau succeeded in understanding the superfluidity of $^4$He with introducing the two-fluid model, which states that the system consists of normal fluid and superfluid being independent of each other and is described by

$$\rho_n \frac{\partial v_n}{\partial t} + (v_n \cdot \nabla)v_n = \frac{\rho_n}{\rho} \nabla P - \rho_n \sigma \nabla T + \eta_n \nabla^2 v_n,$$

(1)

$$\rho_s \frac{\partial v_s}{\partial t} + (v_s \cdot \nabla)v_s = - \frac{\rho_s}{\rho} \nabla P + \rho_s \sigma \nabla T.$$

(2)

Here $\rho_n$ and $v_n$ are density and velocity of normal fluid, and $\rho_s$ and $v_s$ are those of superfluid. $\sigma$ is entropy per unit mass of the normal fluid, $\rho = \rho_n + \rho_s$ is total

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density and $\eta_n$ is the coefficient of viscosity of the normal fluid. The pressure gradient $\nabla P$ runs both components in the same direction and the thermal gradient $\nabla T$ does in the opposite direction. Thermal counterflow driven by a thermal gradient is characteristic of superfluid $^4$He. When the relative velocity between two components is large, they become dependent through the mutual friction $F_{sn}$, which is added to Eqs. (1) and (2). Other formulations for superfluid $^4$He are derived from the conservation law and equations of motion of mass density and entropy density. The hydrodynamic equations are

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot j,$$

$$\frac{\partial \sigma}{\partial t} = -\frac{\rho_\sigma \rho}{\rho} \nabla \cdot (v_n - v_s),$$

$$\frac{\partial j}{\partial t} = -\nabla P,$$

$$\frac{\partial}{\partial t}(v_n - v_s) = -\frac{\rho \sigma}{\rho_n} \nabla T,$$

where $j = \rho_n v_n + \rho_s v_s$. These equations yield the wave equation of mass density and entropy density, which leads to first sound and second sound. First sound is a mode of oscillation of total density and it exists generally in a usual fluid. While, second sound is a characteristic mode in superfluid $^4$He, in which entropy oscillates without oscillating total density, not existing in a usual fluid.

An atomic BEC is one of the most important subjects in modern physics. Especially, two-component BECs are known to create various exotic structure of quantized vortices and cause some characteristic hydrodynamic instability such as Kelvin-Helmholtz instability and Rayleigh-Taylor instability. In another paper, we investigate counterflow in two-component BECs which has many analogies with thermal counterflow in superfluid $^4$He. For example, when the relative velocity exceeds a critical value, the counterflow becomes unstable and quantum turbulence appears like in thermal counterflow. In this work, we describe two-component BECs following the two-fluid model of superfluid $^4$He and obtain four equations similar to Eqs. (3)-(6). We derive two sound modes from these elementary equations, Two sound modes in two-component BECs were obtained by some other work, but we have them correspond to first and second sounds. This is the main point of this work. Thus we can expect to improve interactive studies in superfluid $^4$He and two-component BECs with investigating common features between these.

2 The two-fluid model in two-component BECs

We consider binary mixture of BECs described by the wave functions $\Psi_j = \sqrt{n_j} \phi^j(r)$ in the mean-field approximation at $T = 0 \text{ K}$, where the index $j$ refers to each component $j (j = 1, 2)$. The wave functions $\Psi_j$ are governed by the coupled Gross-Pitaevskii (GP) equations,

$$i\hbar \frac{\partial \Psi_1}{\partial t} = -\frac{\hbar^2}{2m_1} \nabla^2 \Psi_1 + V(r) \Psi_1 + g_{11} |\Psi_1|^2 \Psi_1 + g_{12} |\Psi_2|^2 \Psi_1,$$
\[ i\hbar \frac{\partial}{\partial t} \Psi_j = -\frac{\hbar^2}{2m_j} \nabla^2 \Psi_j + V(r) \Psi_j + g_{2j} |\Psi_j|^2 \Psi_j + g_{1j} |\Psi_i|^2 \Psi_j, \quad (8) \]

where \( m_j \) is particle mass associated with the species, \( g_{ij} \) is intracomponent interaction and \( g_{12} \) is intercomponent interaction. We insert \( \Psi_j \) into Eqs. (7) and (8) and obtain the hydrodynamic equations,

\[ \frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_1 v_1), \quad (9) \]
\[ \frac{\partial \rho_2}{\partial t} = -\nabla \cdot (\rho_2 v_2), \quad (10) \]
\[ \rho_1 \frac{\partial v_1}{\partial t} = \rho_1 \nabla \left\{ \frac{\hbar^2}{2m_1 \sqrt{\rho_1}} \triangle \sqrt{\rho_1} - \frac{1}{2} v_1^2 - \frac{1}{m_1} V \right\} - \rho_1 \nabla \left( \frac{g_{11}}{m_1^2} \rho_1 - \frac{g_{12}}{m_1 m_2} \rho_2 \right), \quad (11) \]
\[ \rho_2 \frac{\partial v_2}{\partial t} = \rho_2 \nabla \left\{ \frac{\hbar^2}{2m_2 \sqrt{\rho_2}} \triangle \sqrt{\rho_2} - \frac{1}{2} v_2^2 - \frac{1}{m_2} V \right\} - \rho_2 \nabla \left( \frac{g_{22}}{m_2^2} \rho_2 - \frac{g_{12}}{m_1 m_2} \rho_1 \right). \quad (12) \]

where \( \rho_j = m_j n_j \) is mass density and \( v_j = \frac{\hbar}{m_j} \nabla \phi_j \) is superfluid velocity. Equations (9) and (10) are equations of continuity for \( \rho_j \) and Eqs. (11) and (12) are quasi-Euler equations for the superfluid velocity.

We will derive equations similar to Eqs. (1) and (2) to reveal correspondence between superfluid \(^4\)He and two-component BECs. Here we consider a uniform system and apply the long-wavelength approximation, so the potential term and the quantum pressure term in Eqs. (11) and (12) are neglected. The pressure of the whole system is

\[ P = \frac{g_{11} \rho_1^2}{2m_1^2} + \frac{g_{22} \rho_2^2}{2m_2^2} + \frac{g_{12} \rho_1 \rho_2}{m_1 m_2}, \]

since \( P = -\frac{\partial E}{\partial V} \). Then Eqs. (11) and (12) turn into

\[ \rho_1 \left( \frac{\partial v_1}{\partial t} + (v_1 \cdot \nabla) v_1 \right) = -\frac{1}{2} \nabla \tilde{P} - \frac{1}{2} \tilde{\nabla} T, \quad (13) \]
\[ \rho_2 \left( \frac{\partial v_2}{\partial t} + (v_2 \cdot \nabla) v_2 \right) = -\frac{1}{2} \nabla \tilde{P} + \frac{1}{2} \tilde{\nabla} T, \quad (14) \]

with

\[ \tilde{\nabla} T = \frac{g_{11}}{2m_1^2} \rho_1^2 - \frac{g_{22}}{2m_2^2} \rho_2^2 + \frac{g_{12}}{m_1 m_2} (\rho_1 \nabla \rho_2 - \rho_2 \nabla \rho_1). \]

It is impossible to describe the right hand side by gradient of some scalar potential because \( \rho_1 \) and \( \rho_2 \) are spatially dependent, but we represent it by \( \tilde{\nabla} T \) in order to emphasize the correspondence to \( \nabla T \) in Eqs. (1) and (2). From Eqs. (13) and (14), we can find that two-component BECs are driven by two terms. The pressure gradient \( \nabla \tilde{P} \) runs both components in the same direction, while \( \tilde{\nabla} T \) runs them oppositely. This nature is just the same as one of the two-fluid model in superfluid \(^4\)He.
3 First sound and second sound in two-component BECs

In this section, we will derive two sound modes from the four elementary equations in two-component BECs and let them correspond to first and second sounds. Here we assume that superfluid velocities \( v_j \) are small and the non-linear terms are neglected. By making Eq. (9) \( \pm \) Eq. (10) and Eq. (11) \( \pm \) Eq. (12) we obtain

\[
\frac{\partial}{\partial t} \rho_+ = -\nabla \cdot j_+, \quad (15)
\]

\[
\frac{\partial}{\partial t} \rho_- = -\nabla \cdot j-, \quad (16)
\]

\[
\frac{\partial}{\partial t} j_+ = -\nabla \left\{ \frac{g_{11}}{8m_1^2} (\rho_+ + \rho_-)^2 + \frac{g_{22}}{8m_2^2} (\rho_+ - \rho_-)^2 + \frac{g_{12}}{4m_1m_2} (\rho_+^2 - \rho_-^2) \right\}, \quad (17)
\]

\[
\frac{\partial}{\partial t} j_- = -\nabla \left\{ \frac{g_{11}}{8m_1^2} (\rho_+ + \rho_-)^2 - \frac{g_{22}}{8m_2^2} (\rho_+ - \rho_-)^2 \right\}
\]

\[
+ \frac{g_{12}}{4m_1m_2} \left\{ (\rho_+ + \rho_-) \nabla (\rho_+ - \rho_-) - (\rho_+ - \rho_-) \nabla (\rho_+ + \rho_-) \right\}, \quad (18)
\]

where \( \rho_\pm \equiv \rho_1 \pm \rho_2 \) and \( j_\pm \equiv \rho_1 v_1 \pm \rho_2 v_2 \). Because first sound means oscillation of \( \rho \) with two components in phase and second sound means oscillation of \( \sigma \) with them out of phase, we expect that oscillations of \( \rho_+ \) and \( \rho_- \) correspond respectively to first and second sounds. Now the right hand sides of Eqs. (17) and (18) should be \( \nabla \tilde{P} \) and \( \nabla \tilde{T} \) respectively. We can write \( \nabla \tilde{P} \) and \( \nabla \tilde{T} \) as functional of \( \rho_+ \) and \( \rho_- \) by

\[
\nabla \tilde{P} = A \nabla \rho_+ + B \nabla \rho_-, \quad (19)
\]

\[
\nabla \tilde{T} = C \nabla \rho_+ + D \nabla \rho_-, \quad (20)
\]

where

\[
A = \frac{g_{11}}{4m_1^2} (\rho_+ + \rho_-) + \frac{g_{22}}{4m_2^2} (\rho_+ - \rho_-) + \frac{g_{12}}{2m_1m_2} \rho_+,
\]

\[
B = \frac{g_{11}}{4m_1^2} (\rho_+ + \rho_-) - \frac{g_{22}}{4m_2^2} (\rho_+ - \rho_-) - \frac{g_{12}}{2m_1m_2} \rho_-,
\]

\[
C = \frac{g_{11}}{4m_1^2} (\rho_+ + \rho_-) - \frac{g_{22}}{4m_2^2} (\rho_+ - \rho_-) + \frac{g_{12}}{2m_1m_2} \rho_-,
\]

\[
D = \frac{g_{11}}{4m_1^2} (\rho_+ + \rho_-) + \frac{g_{22}}{4m_2^2} (\rho_+ - \rho_-) - \frac{g_{12}}{2m_1m_2} \rho_+.
\]

The wave equations derived from Eq.(15)-(18) are reduced to

\[
\frac{\partial^2}{\partial t^2} \rho_+ = AV^2 \rho_+ + BV^2 \rho_- , \quad (21)
\]

\[
\frac{\partial^2}{\partial t^2} \rho_- = CV^2 \rho_+ + DV^2 \rho_- . \quad (22)
\]
Considering the plane waves that $\rho_+$ and $\rho_-$ oscillate around the equilibrium values $\rho_0^+$ and $\rho_0^-$ with the frequency $\omega$ and the wave number $k$ like

$$\rho_+ = \rho_0^+ + \delta \rho_+ \exp[i(k \cdot r - \omega t)],$$
$$\rho_- = \rho_0^- + \delta \rho_- \exp[i(k \cdot r - \omega t)],$$

the sound velocities are

$$c^2 = \frac{\mathcal{g}_{11}}{4m_1^2} (\rho_0^+ + \rho_0^-) + \frac{\mathcal{g}_{22}}{4m_2^2} (\rho_+ - \rho_-)$$
$$\pm \sqrt{\left(\frac{\mathcal{g}_{11}}{4m_1^2} (\rho_0^+ + \rho_0^-) - \frac{\mathcal{g}_{22}}{4m_2^2} (\rho_+ - \rho_-)\right)^2 + \frac{\mathcal{g}_{12}^2}{4m_1^2 m_2^2} (\rho_0^2 - \rho_0^+)}.$$

where $c \equiv \omega / |k|$. These are obtained from the dispersion relation of the Bogoliubov excitations in two-component BECs in the limit of long wavelength.

In superfluid $^4$He first and second sounds are modes that $\rho$ and $\sigma$ independently oscillate. However, Eq. (23) does not necessarily describe first and second sounds because two modes are mixed. We can find that $\rho_+$ and $\rho_-$ oscillate independently when $B$ and $C$ vanish in Eqs. (21) and (22). This conditions are reduced to

$$\rho_0^1 = \rho_0^2,$$
$$\mathcal{g}_{11} = \mathcal{g}_{22}.$$

Then sound velocities of the two modes are

$$c^2_\pm = s^2 \pm \frac{\mathcal{g}_{12} \rho_0^2}{m_1 m_2},$$

where $s = \sqrt{\mathcal{g}_{11} \rho_0^2 / m_1^2} = \sqrt{\mathcal{g}_{22} \rho_0^2 / m_2^2}$ and $\rho_0^0 = \rho_0^1 = \rho_0^2$. The mode of $c^2_+$ is oscillation of $\rho_+$, first sound, and the mode of $c^2_-$ is oscillation of $\rho_-$, second sound. First sound velocity increases with $\mathcal{g}_{12}$ and second sound velocity decreases with $\mathcal{g}_{12}$ (Fig.1). When $|\mathcal{g}_{12}| > g \equiv \sqrt{\mathcal{g}_{11} \mathcal{g}_{22}}$, $c_+$ or $c_-$ becomes imaginary so that the dynamical instability leads to the collapse or the phase separation in the two-component BECs.

4 Summary

We formulated the two-fluid model for two-component BECs, starting from the coupled GP equations. This model well corresponds to the two-fluid model in superfluid $^4$He expect for the mutual friction term. We obtained the condition that two sound modes are independent of each other like first and second sounds in superfluid $^4$He. Second sound has an important role to investigate quantum turbulence in superfluid $^4$He. We are interested in how second sound interacts with a vortex in two-component BECs. In the future, we should investigate "mutual friction" induced by the interaction between vortices and the Bogoliubov excitations in two-component BECs. The details will be reported soon elsewhere.
Fig. 1  Velocity of first and second sounds as a function of $g_{12}$. The solid and dashed line refers to first sound $c_2^-$ and second sound $c_2^+$ respectively.

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