Numerical radius orthogonality in $C^\ast$-algebras

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Abstract
In this paper we characterize the Birkhoff–James orthogonality with respect to the numerical radius norm $v(\cdot)$ in $C^\ast$-algebras. More precisely, for two elements $a$, $b$ in a $C^\ast$-algebra $\mathfrak{A}$, we show that $a \perp^v_B b$ if and only if for each $\theta \in [0, 2\pi)$, there exists a state $\varphi_\theta$ on $\mathfrak{A}$ such that $|\varphi_\theta(a)| = v(a)$ and $\text{Re} \left(e^{i\theta} \varphi_\theta(a)\varphi_\theta(b)\right) \geq 0$. Moreover, we compute the numerical radius derivatives in $\mathfrak{A}$. In addition, we characterize when the numerical radius norm of the sum of two (or three) elements in $\mathfrak{A}$ equals the sum of their numerical radius norms.

Keywords Birkhoff–James orthogonality · $C^\ast$-algebra · Numerical radius · State

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1 Introduction and preliminaries
Throughout this paper, let $\mathfrak{A}$ be a unital $C^\ast$-algebra with unit denoted by $e$, and $\mathbb{B}(\mathcal{H})$ be the $C^\ast$-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. We denote by $\mathfrak{A}'$ the dual space of $\mathfrak{A}$. A linear functional $\varphi \in \mathfrak{A}'$ is said to be positive, and write $\varphi \geq 0$, if $\varphi(a^*a) \geq 0$ for all $a \in \mathfrak{A}$. By $S(\mathfrak{A})$ we denote the set of all normalized states of $\mathfrak{A}$, that is,

$$S(\mathfrak{A}) = \{ \varphi \in \mathfrak{A}' : \varphi \geq 0 \text{ and } \varphi(e) = \|\varphi\| = 1 \}.$$
The numerical range of an element \( a \in \mathcal{A} \) is \( V(a) = \{ \varphi(a) : \varphi \in \mathcal{S}(\mathcal{A}) \} \). It is a non-empty compact and convex set of the complex plane \( \mathbb{C} \), and its maximum modulus is the numerical radius \( v(a) \) of \( a \), that is,
\[
v(a) = \sup \{ |\xi| : \xi \in V(a) \}.
\]

This definition generalizes the classical numerical radius in the sense that the numerical radius \( v(A) \) of a Hilbert space operator \( A \) (considered as an element of a \( C^* \)-algebra \( \mathbb{B}(H) \)) coincides with classical numerical radius
\[
w(A) = \sup \{ |\langle Ax, x \rangle| : x \in H, \|x\| = 1 \}.
\]

It is well known that \( v(\cdot) \) defines a norm on \( \mathcal{A} \), which is equivalent to the \( C^* \)-norm \( \| \cdot \| \). In fact, the following inequalities hold for every \( a \in \mathcal{A} \):
\[
\frac{1}{2} \|a\| \leq v(a) \leq \|a\|.
\] (1)

For more material about the numerical radius, we refer the reader to [7, 9].

The usual way to define the orthogonality in \( \mathcal{A} \) is by means of the \( C^* \)-valued inner product: for elements \( a, b \) of \( \mathcal{A} \) we say that \( a \) is orthogonal to \( b \), and we write \( a \perp_B b \), if \( a^*b = 0 \). Another concept of orthogonality in \( \mathcal{A} \) is the Birkhoff–James orthogonality (see [6, 10]). Recall that, an element \( a \in \mathcal{A} \) is said to be Birkhoff–James orthogonal to another element \( b \in \mathcal{A} \), in short \( a \perp_B b \), if \( \|a + \lambda b\| \geq \|a\| \) for all \( \lambda \in \mathbb{C} \).

As a natural generalization of the notion of Birkhoff–James orthogonality in \( C^* \)-algebras, the concept of strong Birkhoff–James orthogonality was introduced in [2]. When \( a \) and \( b \) are elements of \( \mathcal{A} \), \( a \) is orthogonal to \( b \) in the strong Birkhoff–James sense, in short \( a \perp_B b \), if \( \|a + bc\| \geq \|a\| \) for all \( c \in \mathcal{A} \).

The characterizations of the (strong) Birkhoff–James orthogonality for elements of a \( C^* \)-algebra by means of the states are known. For elements \( a, b \) of \( \mathcal{A} \) the following results were obtained in [2, 3, 5]:
\[
a \perp_B b \iff (\exists \varphi \in \mathcal{S}(\mathcal{A}) : \varphi(a^*a) = \|a\|^2 \text{ and } \varphi(a^*b) = 0)
\]
and
\[
a \perp_B^s b \iff (\exists \varphi \in \mathcal{S}(\mathcal{A}) : \varphi(a^*a) = \|a\|^2 \text{ and } \varphi(a^*bb^*a) = 0).
\]

In the next section, inspired by the numerical radius parallelism in [16], we discuss the Birkhoff–James orthogonality with respect to the numerical radius norm in \( \mathcal{A} \). We show that this relation can be characterized in terms of states acting on \( \mathcal{A} \) (Theorem 3). Some other related results are also discussed. Particularly, we prove that \( v(a + b) = v(a) + v(b) \) if and only if there exists a state \( \varphi \) on \( \mathcal{A} \) such that \( \varphi(a)\varphi(b) = \varphi(a)v(b) \). In addition, we compute the numerical radius derivatives in \( \mathcal{A} \) (Theorem 2).
2 Main results

We start our work with the following definition.

**Definition 1** An element $a \in \mathcal{A}$ is called the *numerical radius Birkhoff–James orthogonality* to another element $b \in \mathcal{A}$, denoted by $a \perp^v_B b$, if $v(a + \lambda b) \geq v(a)$ for all $\lambda \in \mathbb{C}$.

Notice that the relations $\perp_B$ and $\perp^v_B$ are not comparable, in general. As an example, one can take the $C^*$-algebra $\mathcal{A}$ of all complex $2 \times 2$ matrices and let $a = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$, $b = \begin{bmatrix} -2i & 0 \\ 1 + \sqrt{5} & 0 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}$, and $d = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. Then simple computations show that $a \perp_B b$ but $a \not\perp^v_B b$ and also, $c \perp^v_B d$ but $c \not\perp_B d$.

Note that these relations are coincident for certain elements in $C^*$-algebras. For example, if $a \in \mathcal{A}$ is normal, then $v(a) = \|a\|$ (see [7, p. 44]) and hence the condition $a \perp^v_B b$ implies $a \perp_B b$ for all $b \in \mathcal{A}$. Indeed, for every $\lambda \in \mathbb{C}$, by (1), we have $\|a + \lambda b\| \geq v(a + \lambda b) \geq v(a) = \|a\|$.

Furthermore, if $a^2 = 0$, then by [16, Corollary 2.5] $v(a) = \frac{1}{2}\|a\|$ and so the condition $a \perp_B b$ implies $a \perp^v_B b$ for all $b \in \mathcal{A}$. Indeed, for every $\lambda \in \mathbb{C}$, again by (1) it follows that $v(a + \lambda b) \geq \frac{1}{2}\|a + \lambda b\| \geq \frac{1}{2}\|a\| = v(a)$.

**Remark 1** Let $a \in \mathcal{A}$. Define $f : \mathcal{A}' \longrightarrow \mathbb{C}$ by the formula $f(\varphi) = \varphi(a)$. Then the function $f$ is weak*-continuous. Therefore the function $g : \mathcal{A}' \longrightarrow \mathbb{C}$, given by $g(\varphi) := |f(\varphi)| = |\varphi(a)|$, is also weak*-continuous. Moreover, the set of normalized states $\mathcal{S}(\mathcal{A})$ is weak*-compact. Since the function $g$ is weak*-continuous, it follows that the function $g|_{\mathcal{S}(\mathcal{A})} : \mathcal{S}(\mathcal{A}) \longrightarrow \mathbb{C}$ attains its maximum. Therefore, for a given element $a \in \mathcal{A}$ there is $\varphi \in \mathcal{S}(\mathcal{A})$ such that $|\varphi(a)| = v(a)$.

The following proposition states some basic properties of the relation $\perp^v_B$.

**Proposition 1** Let $a, b \in \mathcal{A}$. Then the following statements are equivalent:

(i) $a \perp^v_B b$.
(ii) $a^* \perp^v_B b^*$.
(iii) $aa^* \perp^v_B bb^*$ for all $a, \beta \in \mathbb{C}$.
(iv) $ac \perp^v_B bc$ for every unitary element $c$ in the center of $\mathcal{A}$.

If $a, b$ are self-adjoint, then each one of these assertions is also equivalent to
(v) $v(a + rb) \geq v(a)$ for all $r \in \mathbb{R}$.

**Proof** It is a basic fact that the norm $v(\cdot)$ is self-adjoint (i.e., $v(c^*) = v(c)$ for every $c \in \mathcal{A}$) and so the equivalence (i) $\iff$ (ii) is trivial. The equivalence (i) $\iff$ (iii) immediately follows from the definition of the relation $\perp_B^r$. The implication (iv) $\Rightarrow$ (i) is also trivial. It is therefore enough to prove the implication (i) $\Rightarrow$ (iv).

Suppose that (i) holds. Let $c$ be a unitary element in the center of $\mathcal{A}$. By the first part of the proof of [16, Theorem 3.4], it follows that $v(dc) = v(d)$ for all $d \in \mathcal{A}$. So, we conclude that

$$v(ac + \lambda bc) = v((a + \lambda b)c) = v(a + \lambda b) \geq v(a) = v(ac),$$

for all $\lambda \in \mathbb{C}$. Thus $ac \perp_B^r bc$.

Now, let $a, b$ be self-adjoint. Suppose (v) holds. Let $\lambda = t + is \in \mathbb{C}$ and let $\psi$ be a state on $\mathcal{A}$ such that $|\psi(a + tb)| = v(a + tb)$. We have

$$v^2(a + \lambda b) \geq |\psi(a + \lambda b)|^2 = |\psi(a + tb) + i\psi(sb)|^2$$

$$= |\psi(a + tb)|^2 + |\psi(sb)|^2$$

$$\geq |\psi(a + tb)|^2 = v^2(a + tb) \geq v^2(a),$$

and so $v(a + \lambda b) \geq v(a)$. Thus $a \perp_B^r b$. The converse, that is, (i) implies (v), is obvious. \qed

In the following result we characterize a positive-real version of the numerical radius Birkhoff–James orthogonality. Our approach is similar to the one given in [12].

**Theorem 1** Let $a, b \in \mathcal{A}$. Then the following statements are equivalent:

(i) $v(a + rb) \geq v(a)$ for all $r \in \mathbb{R}^+$.

(ii) There exists a state $\varphi$ on $\mathcal{A}$ such that

$$|\varphi(a)| = v(a) \quad \text{and} \quad \text{Re}(\overline{\varphi(a)}\varphi(b)) \geq 0.$$

**Proof** (i) $\Rightarrow$ (ii) Let $v(a + rb) \geq v(a)$ for all $r \in \mathbb{R}^+$. We may assume that $v(a) \neq 0$ otherwise (ii) trivially holds. Thus there is $\varepsilon_o \in (0, 1)$ such that $v(a) - \varepsilon^2 \geq 0$ for all $\varepsilon \in (0, \varepsilon_o)$. So, it follows that

$$v(a + \varepsilon b) \geq v(a) \geq v(a) - \varepsilon^2 \geq 0 \quad (2)$$

for all $\varepsilon \in (0, \varepsilon_o)$. On the other hand, there exists a state $\varphi_\varepsilon$ on $\mathcal{A}$ such that $|\varphi_\varepsilon(a + \varepsilon b)| = v(a + \varepsilon b)$. So, by (2) it follows that

$$v(a) + \varepsilon v(b) \geq |\varphi_\varepsilon(a)| + \varepsilon |\varphi_\varepsilon(b)| \geq |\varphi_\varepsilon(a + \varepsilon b)| = v(a + \varepsilon b) \geq v(a).$$
Since the set $S(\mathcal{A})$ is weak*-compact, we may assume that $\varphi_\varepsilon \xrightarrow{w^*} \varphi_o$ for some $\varphi_o \in S(\mathcal{A})$, where $\varepsilon \to 0^+$. Now, letting $\varepsilon \to 0^+$, we get $|\varphi_o(a)| = v(a)$.

Furthermore, from (2) it follows that
\[
v^2(a) + 2\varepsilon \text{Re} (\varphi_\varepsilon(a)\varphi_\varepsilon(b)) + \varepsilon^2 v^2(b) \geq |\varphi_\varepsilon(a)|^2 + 2\varepsilon \text{Re}(\varphi_\varepsilon(a)\varphi_\varepsilon(b)) + \varepsilon^2 |\varphi_\varepsilon(b)|^2 = |\varphi_\varepsilon(a + \varepsilon b)|^2 = v^2(a + \varepsilon b) \geq v^2(a) - 2\varepsilon^2 v(a) + \varepsilon^4,
\]
and hence
\[
\text{Re}(\varphi_\varepsilon(a)\varphi_\varepsilon(b)) \geq \frac{\varepsilon^3}{2} - \varepsilon v(a) - \frac{\varepsilon^2}{2} v^2(b).
\]

Thus, by letting $\varepsilon \to 0^+$, we obtain $\text{Re}(\varphi_o(a)\varphi_o(b)) \geq 0$.

(ii) $\Rightarrow$ (i) Suppose (ii) holds. Therefore, for every $r \in \mathbb{R}^+$, we have
\[
v^2(a + rb) \geq |\varphi(a + rb)|^2 = |\varphi(a)|^2 + 2r \text{Re}(\varphi(a)\varphi(b)) + r^2 |\varphi(b)|^2 \geq v^2(a),
\]
and so $v(a + rb) \geq v(a)$.

In what follows, we get a very tractable characterization of the numerical radius Birkhoff–James orthogonality in the positive cones of $C^*$-algebras. Recall that the positive elements of $\mathcal{A}$ are the elements of the form $a^*a$, where $a \in \mathcal{A}$.

**Corollary 1** Let $a$, $b$ be positive elements of $\mathcal{A}$. Then the following statements are equivalent:

(i) $a \perp^v_B b$.

(ii) There exists a state $\varphi$ on $\mathcal{A}$ such that $\varphi(a) = v(a)$ and $\varphi(b) = 0$.

**Proof** (i) $\Rightarrow$ (ii) Let $a \perp^v_B b$. By Proposition 1, we have $a \perp^v_B (-b)$. So, by Theorem 1 there exists a state $\varphi$ on $\mathcal{A}$ such that $|\varphi(a)| = v(a)$ and $\text{Re} (\varphi(a)\varphi(-b)) \geq 0$. Since $a$, $b$ are positive, we reach that $\varphi(a) = v(a)$ and
\[
0 \leq \varphi(b) = \frac{-\text{Re}(\varphi(a)\varphi(-b))}{\varphi(a)} \leq 0.
\]

Thus $\varphi(b) = 0$.

(ii) $\Rightarrow$ (i) Let $\varphi$ be a state on $\mathcal{A}$ such that $\varphi(a) = v(a)$ and $\varphi(b) = 0$. Then for every $\lambda \in \mathbb{C}$, we have
\[
v(a + \lambda b) \geq |\varphi(a + \lambda b)| = |\varphi(a) + \lambda \varphi(b)| = \varphi(a) = v(a),
\]
and hence $a \perp^v_B b$. \qed
Remark 2 For positive elements $a, b$ of a unital $C^*$-algebra $\mathcal{A}$, Komure et al. [11, Lemma 2.3] proved that $a \perp_B b$ if and only if there exists a state $\varphi$ on $\mathcal{A}$ such that $\varphi(a) = \|a\|$ and $\varphi(b) = 0$. Therefore, by Corollary 1, we conclude that the relations $\perp_B$ and $\perp_B^*$ are coincident in the positive cones of $C^*$-algebras.

In a normed linear space $(\mathcal{X}, \|\cdot\|)$, the Gateaux derivatives of the norm are given for $x, y \in \mathcal{X}$ by the two expressions

$$
\rho_{\pm}(x, y) := \lim_{t \to 0^\pm} \frac{\|x + ty\|^2 - \|x\|^2}{2t} = \|x\| \lim_{t \to 0^\pm} \frac{\|x + ty\| - \|x\|}{t}.
$$

If it will not cause a confusion, we will write $\rho_\pm$ instead of $\rho_{\pm}(\|\cdot\|)$. When the norm on $\mathcal{X}$ comes from an inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, we obtain $\rho_+(x, y) = \langle x, y \rangle = \rho_-(x, y)$, i.e., functionals $\rho_+, \rho_-$ are nice generalizations of inner products. By convexity of the norm the above definitions are meaningful. The mappings $\rho_+$ and $\rho_-$ are called the norm derivatives and their following properties, which will be useful in the present note, can be found, e.g., in [8]:

1. $\forall x, y \in \mathcal{X} \quad -\|x\| \cdot \|y\| \leq \rho_-(x, y) \leq \rho_+(x, y) \leq \|x\| \cdot \|y\|;$
2. $\forall x, y \in \mathcal{X} \forall a \geq 0 \quad \rho_\pm(ax, y) = a \rho_\pm(x, y) = \rho_\pm(x, ay)$;
3. $\forall x, y \in \mathcal{X} \forall a < 0 \quad \rho_\pm(ax, y) = a \rho_\pm(x, y) = \rho_\pm(x, ay)$;
4. $\forall x, y \in \mathcal{X} \forall a \in \mathbb{R} \quad \rho_\pm(x, ax + y) = a \|x\|^2 + \rho_\pm(x, y)$.

In a real normed space $\mathcal{X}$, we have for arbitrary $x, y \in \mathcal{X}$:

1. $x \perp_B y \iff \rho_-(x, y) \leq 0 \leq \rho_+(x, y)$;
2. $\rho_-(x, y) = 0 \Rightarrow x \perp_B y, \quad \rho_+(x, y) = 0 \Rightarrow x \perp_B y$.

Moreover, mappings $\rho_+, \rho_-$ are continuous with respect to the second variable, but not necessarily with respect to the first one.

The condition (ND5) shows that the Birkhoff–James orthogonality is connected with the norm derivatives. Therefore, in view of Theorem 1, it seems to be quite natural to compute the numerical radius derivatives, i.e. the norm derivatives in $\mathcal{A}$ equipped with the norm $v(\cdot)$.

**Theorem 2** Let $a, b \in \mathcal{A} \setminus \{0\}$. Then the following statements are true:

1. $\rho_+(a, b) = \max \left\{ \text{Re} \left( \frac{\varphi(a)}{\varphi(b)} \right) : \varphi \in \mathcal{S}(\mathcal{A}), \ |\varphi(a)| = v(a) \right\}$.
2. $\rho_-(a, b) = \min \left\{ \text{Re}(\varphi(a)\varphi(b)) : \varphi \in \mathcal{S}(\mathcal{A}), \ |\varphi(a)| = v(a) \right\}$.

**Proof** Since the proofs are similar we calculate only $\rho_+(a, b)$. It follows from [8, Theorem 15, p.36] that

$$
\rho_+(a, b) = v(a) \sup \left\{ \text{Re}(\varphi(b)) : \varphi \in \mathcal{S}(\mathcal{A}), \ |\varphi| = 1, \ \varphi(a) = v(a) \right\}.
$$

Fix $\varphi \in \mathcal{S}(\mathcal{A})$ such that $|\varphi(a)| = v(a)$. Next we define a linear mapping $\psi : \mathcal{A} \to \mathbb{C}$ by the formula $\psi(\cdot) := \frac{1}{V(a)} \varphi(a)\varphi(\cdot)$. A moments reflection shows that

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\[ \psi \in \mathfrak{A}', \quad \| \psi \| = 1 \quad \text{and} \quad \psi(a) = v(a). \quad (4) \]

Combining (3) and (4), we immediately get
\[ \rho_+^{(c)}(a, b) \geq \sup \left\{ \Re(\varphi(a)\varphi(b)) : \varphi \in \mathcal{S}(\mathfrak{A}), \ |\varphi(a)| = v(a) \right\}. \quad (5) \]

Now we are going to prove the converse inequality. It follows from the property (ND4) that \( \rho_+^{(c)}\left(a, -\frac{\rho_+^{(c)}(a, b)}{v^2(a)} a + b \right) = 0 \). Applying (ND6) we get
\[ v(a + r \left( -\frac{\rho_+^{(c)}(a, b)}{v^2(a)} a + b \right)) \geq v(a), \]

for all \( r \in \mathbb{R} \). Now by Theorem 1 there is a state \( \varphi_o \in \mathcal{S}(\mathfrak{A}) \) such that \( |\varphi_o(a)| = v(a) \) and
\[ \Re\left( \frac{\varphi_o(a)\varphi_o(-\rho_+^{(c)}(a, b))}{v^2(a)} a + b \right) \geq 0. \]

This implies \( \Re(\varphi_o(a)\varphi_o(-\rho_+^{(c)}(a, b)))/v^2(a) + \Re(\varphi_o(a)\varphi_o(b)) \geq 0 \). Since \( \varphi_o(a)\varphi_o(a) = v^2(a) \), we obtain \( \rho_+^{(c)}(a, b) \leq \Re(\varphi_o(a)\varphi_o(b)) \). Further, from this inequality and from (5) we have
\[ \rho_+^{(c)}(a, b) = \sup \left\{ \Re(\varphi(a)\varphi(b)) : \varphi \in \mathcal{S}(\mathfrak{A}), \ |\varphi(a)| = v(a) \right\}. \]

Finally, since \( \rho_+^{(c)}(a, b) = \Re(\varphi_o(a)\varphi_o(b)) \), the word “sup” can be replaced by the word “max”. The proof is complete. \( \square \)

For \( A, B \in \mathcal{B}(\mathcal{H}) \), Bhatia and Šemrl [4, Remark 3.1] and Paul [13, Lemma 2] independently proved that \( A \perp_B B \) if and only if there exists a sequence \( \{x_n\} \) of unit vectors in \( \mathcal{H} \) such that
\[ \lim_{n \to \infty} \|Ax_n\| = \|A\| \quad \text{and} \quad \lim_{n \to \infty} \langle Ax_n, Bx_n \rangle = 0. \]

Some authors extended the well known result of Bhatia–Šemrl (see [2, 3, 5, 14, 17]). Very recently, the numerical radius Birkhoff–James orthogonality in \( \mathcal{B}(\mathcal{H}) \) has been studied in [12] as our work was in progress. In fact, Mal et al. [12, Theorem 2.3] obtained the following characterization of the numerical radius Birkhoff–James orthogonality for Hilbert space operators: if \( A, B \in \mathcal{B}(\mathcal{H}) \), then \( A \perp_B^w B \) if and only if for each \( \theta \in [0, 2\pi) \), there exists a sequence \( \{x_n\} \) of unit vectors in \( \mathcal{H} \) such that
\[ \lim_{n \to \infty} |\langle Ax_n, x_n \rangle| = w(A) \quad \text{and} \quad \lim_{n \to \infty} \Re \left( e^{i\theta} \langle x_n, Ax_n \rangle \langle Bx_n, x_n \rangle \right) \geq 0. \]

In what follows we shall develop the above result for elements of a C*-algebra.

**Theorem 3** Let \( a, b \in \mathfrak{A} \). Then the following statements are equivalent:
(i) \( a \perp_B^\gamma b. \)

(ii) For each \( \theta \in [0, 2\pi) \), there exists a state \( \varphi_\theta \) on \( \mathfrak{A} \) such that

\[
|\varphi_\theta(a)| = v(a) \quad \text{and} \quad \Re(e^{i\theta} \varphi_\theta(a) \varphi_\theta(b)) \geq 0.
\]

**Proof** (i) \( \Rightarrow \) (ii) Let \( a \perp_B^\gamma b. \) Hence \( v(a + re^{i\theta}b) \geq v(a) \) for all \( \theta \in [0, 2\pi) \) and \( r \in \mathbb{R}^+ \). Fix \( \theta \) and let \( b_\theta = e^{i\theta}b. \) Then we have \( v(a + rb_\theta) \geq v(a) \) for all \( r \in \mathbb{R}^+ \). By Theorem 1 there exists a state \( \varphi_\theta \) on \( \mathfrak{A} \) such that \( |\varphi_\theta(a)| = v(a) \) and \( \Re(e^{i\theta} \varphi_\theta(a) \varphi_\theta(b)) \geq 0. \) From this it follows that \( |\varphi_\theta(a)| = v(a) \) and \( \Re(e^{i\theta} \varphi_\theta(a) \varphi_\theta(b)) \geq 0. \)

(ii) \( \Rightarrow \) (i) Suppose (ii) holds. Let \( \lambda \in \mathbb{C} \). Then there exists \( \theta \in [0, 2\pi) \) such that \( \lambda = |\lambda|e^{i\theta}. \) Therefore, there exists a state \( \varphi_\theta \) on \( \mathfrak{A} \) such that \( |\varphi_\theta(a)| = v(a) \) and \( \Re(e^{i\theta} \varphi_\theta(a) \varphi_\theta(b)) \geq 0. \) Thus

\[
v^2(a + \lambda b) \geq |\varphi_\theta(a + |\lambda|e^{i\theta}b)|^2
\]

\[= |\varphi_\theta(a)|^2 + 2|\lambda| \Re(e^{i\theta} \varphi_\theta(a) \varphi_\theta(b)) + |\lambda|^2 |\varphi_\theta(b)|^2\]

\[\geq v^2(a) + |\lambda|^2 |\varphi_\theta(b)|^2\]

\[\geq v^2(a),\]

and so \( v(a + \lambda b) \geq v(a) \). Hence \( a \perp_B^\gamma b. \)

Recall that (e.g., see [9, p. 63]) the **Crawford number** of \( B \in \mathbb{B}(\mathcal{H}) \) is defined by

\[
c(B) := \inf \big\{ |\langle Bx, x \rangle| : x \in \mathcal{H}, \|x\| = 1 \big\}.
\]

(7)

This concept is useful in studying linear operators (see [9], and further references therein). The **numerical radius Crawford number** of \( b \in \mathfrak{A} \) can be defined by

\[
C(b) := \inf \big\{ |\varphi(b)| : \varphi \in S(\mathfrak{A}) \big\}.
\]

Notice that for \( B \in \mathbb{B}(\mathcal{H}) \), by [16, Remark 2.3], \( C(B) \) coincides with the classical \( c(B) \) given by (7) above.

Before we present the next results, some examples are appropriate. More precisely, the proposition below gives a large family of elements satisfying \( C(b) > 0. \)

**Proposition 2** Let \( a \in \mathfrak{A} \) with \( v(a) < 1 \). If \( b = e + a \), then \( C(b) > 0. \)

**Proof** Since \( v(a) < 1 \), it follows that there is a positive number \( \gamma \) such that \( v(a) < \gamma < 1 \). Fix \( \varphi \in S(\mathfrak{A}) \). Then we obtain

\[
|\varphi(b)| = |\varphi(e) + \varphi(a)| \geq |\varphi(e)| - |\varphi(a)| \geq 1 - v(a) > 1 - \gamma
\]

and passing to the infimum over \( S(\mathfrak{A}) \) we obtain \( C(b) \geq 1 - \gamma. \) So \( C(b) > 0. \)

Now, as an immediate consequence of Theorem 3, we have the following result.

**Corollary 2** Let \( a, b \in \mathfrak{A} \). Then the following statements are equivalent:

\( \Box \)
(i) \( a \perp_B^v b \).
(ii) \( v^2(a + \lambda b) \geq v^2(a) + |\lambda|^2 C^2(b) \) for all \( \lambda \in \mathbb{C} \).

**Proof** If \( a \perp_B^v b \), then for each \( \lambda \in \mathbb{C} \), by (6), there exists a state \( \varphi \) on \( \mathcal{A} \) such that \( v^2(a + \lambda b) \geq v^2(a) + |\lambda|^2 |\varphi(b)|^2 \). Hence \( v^2(a + \lambda b) \geq v^2(a) + |\lambda|^2 C^2(b) \). The converse is obvious. \( \square \)

The following result is a kind of Pythagorean inequality in \( C^* \)-algebras. We are going to apply this inequality in approximation theory.

**Proposition 3** Let \( a, b \in \mathcal{A} \) with \( C(b) > 0 \). Then there exists a unique \( \zeta \in \mathbb{C} \), such that

\[
v^2\left( (a + \zeta b) + \lambda b \right) \geq v^2(a + \zeta b) + |\lambda|^2 C^2(b)
\]

for all \( \lambda \in \mathbb{C} \).

**Proof** Since \( v(a + \lambda b) \) is large for \( |\lambda| \) large, \( \inf \{ v(a + \lambda b) : \lambda \in \mathbb{C} \} \) must be attained at some point, say \( \zeta \) (there may be of course many such points). Therefore, \( v(a + \lambda b) \geq v(a + \zeta b) \) for all \( \lambda \in \mathbb{C} \) and hence \( (a + \zeta b) \perp_B b \). So, by Corollary 2, we have

\[
v^2\left( (a + \zeta b) + \lambda b \right) \geq v^2(a + \zeta b) + |\lambda|^2 C^2(b),
\]

for all \( \lambda \in \mathbb{C} \). Now, suppose that \( \eta \) is another point satisfying the inequality

\[
v^2\left( (a + \eta b) + \lambda b \right) \geq v^2(a + \eta b) + |\lambda|^2 C^2(b),
\]

for all \( \lambda \in \mathbb{C} \). Choose \( \lambda = \zeta - \eta \) to get

\[
v^2(a + \zeta b) = v^2\left( (a + \eta b) + (\zeta - \eta)b \right)
\geq v^2(a + \eta b) + |\zeta - \eta|^2 C^2(b)
\geq v^2(a + \zeta b) + |\zeta - \eta|^2 C^2(b).
\]

Hence \( 0 \geq |\zeta - \eta|^2 C^2(b) \). Since \( C(b) > 0 \), we get \( |\zeta - \eta|^2 = 0 \), or equivalently, \( \eta = \zeta \). This shows that \( \zeta \) is unique. \( \square \)

Now we apply the above result to present a theorem concerning uniqueness of best approximation with respect to the numerical radius norm in \( C^* \)-algebras. Similar investigations have been worked out in compact operators spaces for injective operators (cf. [15, Theorems 5.6, 5.7, 5.8]).

**Theorem 4** Let \( b \in \mathcal{A} \) with \( C(b) > 0 \). Then any \( a \in \mathcal{A}\setminus\text{span}\{b\} \) has a unique best approximation in \( \text{span}\{b\} \) with respect to the numerical radius norm, that is, there exists a unique \( b_a \in \text{span}\{b\} \) such that \( \text{dist}(a, \text{span}\{b\}) = v(a - b_a) \).
\textbf{Proof} Fix \( a \in \mathfrak{A} \setminus \operatorname{span}\{ b \} \). It follows from Proposition 3 that there exists a unique \( \zeta \in \mathbb{C} \) such that \( v^2((a + \zeta b) + \lambda b) \geq v^2(a + \zeta b) + |\lambda|^2 c^2(b) \) for all \( \lambda \in \mathbb{C} \). If \( \lambda \neq 0 \), then, by the inequality \( C(b) > 0 \), we get the following inequality

\[ v((a + \zeta b) + \lambda b) > v(a + \zeta b) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \{0\}. \tag{8} \]

Define \( b_a := -\zeta b \). Now the property (8) becomes

\[ v(a - p) > v(a - b_a) \quad \text{for all} \quad p \in \operatorname{span}\{ b \} \setminus \{ b_a \}, \]

which means that \( \text{dist}(a, \operatorname{span}\{ b \}) = v(a - b_a) \).

In [1], for \( A, B \in \mathbb{B}(\mathcal{H}) \), a necessary and sufficient condition for the equality \( w(A + B) = w(A) + w(B) \) has been given. In fact, it has been shown that \( w(A + B) = w(A) + w(B) \) if and only if there exists a sequence \( \{ x_n \} \) of unit vectors in \( \mathcal{H} \) such that

\[ \lim_{n \to \infty} \langle x_n, Ax_n \rangle \langle Bx_n, x_n \rangle = w(A) w(B). \]

In the following theorem, we give a necessary and sufficient condition for the equality \( v(a + b) = v(a) + v(b) \) in \( C^* \)-algebras.

\textbf{Theorem 5} Let \( a, b \in \mathfrak{A} \). Then the following statements are equivalent:

\begin{itemize}
  \item[(i)] \( v(a + b) = v(a) + v(b) \).
  \item[(ii)] There exists a state \( \varphi \) on \( \mathfrak{A} \) such that \( \overline{\varphi(a)\varphi(b)} = v(a)v(b) \).
\end{itemize}

\textbf{Proof} (i) \( \Rightarrow \) (ii) Let \( v(a + b) = v(a) + v(b) \). By Proposition 4.1 of [3], we get \( a \perp_B v(a)b - v(b)a \). Therefore, by Theorem 1, there exists a state \( \varphi \) on \( \mathfrak{A} \) such that \( |\varphi(a)| = v(a) \) and \( \text{Re} \left( \overline{\varphi(a)\varphi(v(a)b - v(b)a)} \right) \geq 0 \). This implies \( v(a)v(b) \leq \text{Re} \left( \overline{\varphi(a)\varphi(b)} \right) \). Consequently,

\[ v(a)v(b) \leq \text{Re} \left( \overline{\varphi(a)\varphi(b)} \right) \leq |\varphi(a)\varphi(b)| \leq v(a)v(b), \]

which yields \( \text{Re} \left( \overline{\varphi(a)\varphi(b)} \right) = v(a)v(b) \) and \( \text{Im} \left( \overline{\varphi(a)\varphi(b)} \right) = 0 \). Hence \( \overline{\varphi(a)\varphi(b)} = v(a)v(b) \).

(ii) \( \Rightarrow \) (i) Suppose (ii) holds. So, there exists a state \( \varphi \) on \( \mathfrak{A} \) such that \( \varphi(a)\varphi(b) = v(a)v(b) \). From this it follows that \( |\varphi(a)| = v(a) \) and \( |\varphi(b)| = v(b) \). Therefore, we have

\[ \left( v(a) + v(b) \right)^2 = |\varphi(a)|^2 + 2\overline{\varphi(a)\varphi(b)} + |\varphi(b)|^2 \]

\[ = |\varphi(a + b)|^2 \leq v^2(a + b) \leq \left( v(a) + v(b) \right)^2, \]

and so \( v(a + b) = v(a) + v(b) \). \( \square \)
Applying the above result we may prove another theorem.

**Theorem 6** Let \(a, b, c \in \mathfrak{A} \setminus \{0\}\). Then the following statements are equivalent:

(i) \(v(a + b + c) = v(a) + v(b) + v(c)\).

(ii) There exists a state \(\varphi\) on \(\mathfrak{A}\) such that \(\frac{\varphi(a)}{v(a)} = \frac{\varphi(b)}{v(b)} = \frac{\varphi(c)}{v(c)}\) and \(\varphi(a)\varphi(b) = v(a)v(b)\), \(\varphi(a)\varphi(c) = v(a)v(c)\), \(\varphi(b)\varphi(c) = v(b)v(c)\).

(iii) There exists a state \(\psi\) on \(\mathfrak{A}\) such that \(\frac{\psi(a)}{v(a)} = \frac{\psi(b)}{v(b)} = \frac{\psi(c)}{v(c)}\) and \(\|\psi(a)\| = 1\).

**Proof** (i) \(\Rightarrow\) (ii) It is known that the norm equality \(v(a + b + c) = v(a) + v(b) + v(c)\) holds if and only if \(v(\alpha a + \beta b + \gamma c) = v(\alpha a) + v(\beta b) + v(\gamma c)\) for all \(\alpha, \beta, \gamma \geq 0\). We assume (i), so without loss of generality, we may assume that \(v(a) = v(b) = v(c) = 1\) and \(v(a + b + c) = 3\). Since

\[
v(a + b + c) = v(a) + v(b) + v(c) \leq v(a) + v(b) + v(c) = 3,
\]

we have \(v(a + (b + c)) = v(a) + v(b + c)\) and \(v(b + c) = 2\). By Theorem 5, there is a state \(\varphi\) on \(\mathfrak{A}\) such that \(\varphi(a)\varphi(b) + \varphi(a)\varphi(c) = v(a)v(b + c) = 2\). It follows that

\[
\frac{1}{2} \varphi(a)\varphi(b) + \frac{1}{2} \varphi(a)\varphi(c) = 1.
\]

We know that those three numbers \(\varphi(a)\varphi(b), \varphi(a)\varphi(c), 1\) are in \(\{\xi \in \mathbb{C} : |\xi| = 1\}\). Since one of them is a convex combination of the others, they must all be the same scalar. Therefore \(\varphi(a)\varphi(b) = \varphi(a)\varphi(c) = 1\). It follows easily that \(\varphi(a)\varphi(b) = 1\). Multiplying those equalities we have \(\varphi(a)\varphi(b)\varphi(a)\varphi(c) = 1\). Since \(\varphi(a)\varphi(c) = 1\), we get \(\varphi(b)\varphi(c) = 1\). To summarize, it has been shown that \(\varphi(a)\varphi(b) = 1\), \(\varphi(a)\varphi(c) = 1\), \(\varphi(b)\varphi(c) = 1\). If we divide both sides of the equality \(\varphi(a)\varphi(b) = \varphi(a)\varphi(c)\) by \(\varphi(a)\), we obtain \(\varphi(b) = \varphi(c)\). Similarly, since \(\varphi(a)\varphi(c) = \varphi(b)\varphi(c)\), we get \(\varphi(a) = \varphi(b)\). The proof of the implications (i) \(\Rightarrow\) (ii) is complete.

The implication (ii) \(\Rightarrow\) (iii) is trivial. So we prove (iii) \(\Rightarrow\) (i). Assume that (iii) holds. Again, we may assume that \(v(a) = v(b) = v(c) = 1\). It follows from (iii) that \(|\psi(a)| = |\psi(b)| = |\psi(c)| = 1\). Further, from the condition (iii) we have

\[
3 = 3|\psi(a)| = |\psi(a) + \psi(a) + \psi(a)| = |\psi(a) + \psi(b) + \psi(c)|
\]

\[
= |\psi(a + b + c)| \leq v(a + b + c) \leq v(a) + v(b) + v(c) = 3.
\]

So, the inequalities become equalities and the proof is complete. \(\square\)

It is worth mentioning that investigations with more than two elements have been appeared in [15], but for sum of operators.

**Remark 3** In [16, Theorem 2.2], the following characterization of the numerical radius for elements of a \(C^*\)-algebra has been given,

\[
v(a) = \sup_{\theta \in \mathbb{R}} \Re(e^{i\theta}a).
\]

Then, a refinement of the triangle inequality for the numerical radius in \(C^*\)-algebras has been shown in [16, Theorem 3.6] that for every \(a, b \in \mathfrak{A}\),
\[ v(a + b) \leq \frac{1}{2} \left( v(a) + v(b) \right) + \frac{1}{2} \sqrt{\left( v(a) - v(b) \right)^2 + 4 \sup_{\theta \in \mathbb{R}} \| \operatorname{Re}(e^{i\theta}a) \operatorname{Re}(e^{i\theta}b) \|} \tag{9} \]
\[ \leq v(a) + v(b). \]

Therefore, Theorem 5 implies that, for every \( a, b \in \mathfrak{A} \), if there exists a state \( \varphi \) on \( \mathfrak{A} \) such that \( \varphi(a) \varphi(b) = v(a)v(b) \), then \( v(a + b) = v(a) + v(b) \) and consequently by (9) we get
\[
\sup_{\theta \in \mathbb{R}} \| \operatorname{Re}(e^{i\theta}a) \operatorname{Re}(e^{i\theta}b) \| = v(a)v(b) = \varphi(a)\varphi(b). 
\]

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References

1. Abu-Omar, A., Kittaneh, F.: Notes on some spectral radius and numerical radius inequalities. Stud. Math. 227(2), 97–109 (2015)
2. Arambašić, L.J., Rajić, R.: A strong version of the Birkhoff-James orthogonality in Hilbert \( C^* \)-modules. Ann. Funct. Anal. 5(1), 109–120 (2014)
3. Arambašić, L.J., Rajić, R.: The Birkhoff–James orthogonality in Hilbert \( C^* \)-modules. Linear Algebra Appl. 437, 1913–1929 (2012)
4. Bhatia, R., Šemrl, P.: Orthogonality of matrices and some distance problems. Linear Algebra Appl. 287(1–3), 77–85 (1999)
5. Bhattacharyya, T., Grover, P.: Characterization of Birkhoff–James orthogonality. J. Math. Anal. Appl. 407(2), 350–358 (2013)
6. Birkhoff, G.: Orthogonality in linear metric spaces. Duke Math. J. 1, 169–172 (1935)
7. Bonsall, F.F., Duncan, J.: Numerical Ranges of Operators on Normed Spaces and Elements of Normed Algebras. London Mathematical Society Lecture Note Series, vol. 2. Cambridge University Press, London (1971)
8. Dragomir, S.S.: Semi-Inner Products and Applications. Nova Science Publishers Inc, Hauppauge (2004)
9. Gustafson, K.E., Rao, D.K.M.: Numerical Range. The Field of Values of Linear Operators and Matrices. Universitext. Springer, New York (1997)
10. James, R.C.: Orthogonality in normed linear spaces. Duke Math. J. 12, 291–302 (1945)
11. Komuro, N., Saito, K.S., Tanaka, R.: On symmetry of Birkhoff orthogonality in the positive cones of \( C^* \)-algebras with applications. J. Math. Anal. Appl. 474(2), 1488–1497 (2019)
12. Mal, A., Paul, K., Sen, J.: Orthogonality and numerical radius inequalities of operator matrices. arXiv:1903.06858v1 [math.FA] (2019)
13. Paul, K.: Translatable radii of an operator in the direction of another operator. Sci. Math. 2, 119–122 (1999)
14. Wójcik, P.: The Birkhoff Orthogonality in pre-Hilbert \( C^* \)-modules. Oper. Matrices 10(3), 713–729 (2016)
15. Wójcik, P.: Generalized Daugavet equations, affine operators and unique best approximation. Stud. Math. 238(3), 235–247 (2017)
16. Zamani, A.: Characterization of numerical radius parallelism in \( C^* \)-algebras. Positivity 23(2), 397–411 (2019)
17. Zamani, A.: Birkhoff-James orthogonality of operators in semi-Hilbertian spaces and its applications. Ann. Funct. Anal. 10(3), 433–445 (2019)