HERNANDEZ-LECLERC MODULES AND SNAKE GRAPHS

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ABSTRACT. In 2010, Hernandez and Leclerc studied connections between representations of quantum affine algebras and cluster algebras. In 2019, Brito and Chari defined a family of modules over quantum affine algebras, called Hernandez-Leclerc modules. We characterize the highest $\ell$-weight monomials of Hernandez-Leclerc modules. We give a non-recursive formula of $q$-characters of Hernandez-Leclerc modules using snake graphs, which involves an explicit formula for $F$-polynomials. We also give a new recursive formula of $q$-characters of Hernandez-Leclerc modules.

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1. Introduction

Quantum groups were introduced independently by Drinfeld [13, 14] and Jimbo [26]. Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{C} \) and let \( U_q(\hat{\mathfrak{g}}) \) be the corresponding untwisted quantum affine algebra with quantum parameter \( q \in \mathbb{C}^\times \) not a root of unity. Denote by \( \mathcal{C} \) the category of finite dimensional representations of \( U_q(\hat{\mathfrak{g}}) \). It is well-known that \( \mathcal{C} \) is not semisimple but an abelian tensor category [23]. The isomorphism classes of finite-dimensional simple \( U_q(\hat{\mathfrak{g}}) \)-modules can be parametrized by Drinfeld polynomials in [3–5]. Equivalently, simple \( U_q(\hat{\mathfrak{g}}) \)-modules can also be parametrized by dominant monomials [22].

Cluster algebras were introduced by Fomin and Zelevinsky in their seminal work [18]. A cluster algebra is a commutative ring with a set of distinguished generators called cluster variables which are defined through iterative processes known as mutations.

A finite-dimensional \( U_q(\hat{\mathfrak{g}}) \)-module is said to be prime if it admits no nontrivial tensor factorization [6]. A simple \( U_q(\hat{\mathfrak{g}}) \)-module is said to be real if its tensor square is also simple [28]. In [23], Hernandez and Leclerc introduced the notion of monoidal categorification of cluster algebras. An abelian tensor category is said to be a monoidal categorification of a cluster algebra if the Grothendieck ring of the category is isomorphic to the cluster algebra and the classes of the real (respectively, real prime) simple modules correspond to cluster monomials (respectively, cluster variables).

For each \( \ell \in \mathbb{Z}_{\geq 0} \), Hernandez and Leclerc [23] introduced a full monoidal subcategory \( \mathcal{C}_\ell \) of \( \mathcal{C} \) whose objects are characterized by certain restrictions on the roots of the Drinfeld polynomials of their composition factors.

1.1. Highest \( \ell \)-weight monomials of Hernandez-Leclerc modules. As a generalization of \( \mathcal{C}_1 [23, 24] \), in [1] Brito and Chari introduced a subcategory \( \mathcal{C}_\xi \) and a quiver \( Q_\xi \) depending on a choice of a height function \( \xi \). These real prime simple \( U_q(\hat{\mathfrak{g}}) \)-modules in \( \mathcal{C}_\xi \) are called Hernandez-Leclerc modules by Brito and Chari. Brito and Chari proved that \( \mathcal{C}_\xi \) is a monoidal categorification of the cluster algebra \( \mathcal{A}(x, Q_\xi) \) with coefficients of type \( A \) [1, Section 1.3]. Hernandez-Leclerc modules are in bijection with cluster variables including frozen variables, among the initial cluster variables and frozen variables correspond to some Kirillov-Reshetikhin modules. Let \( \mathcal{K}_0(\xi) \) be the Grothendieck ring of \( \mathcal{C}_\xi \). Denote by \( \iota \) the algebraic isomorphism \( \mathcal{A}(x, Q_\xi) \to \mathcal{K}_0(\xi) \) [1]. Moreover, Brito and Chari gave a non-recursion for \( q \)-characters of Hernandez-Leclerc modules in term of the known \( q \)-characters of initial simple modules [1, Proposition 2.5].

In this paper, we use snake graphs to study \( q \)-characters of Hernandez-Leclerc modules. Let \( I = \{1, 2, \ldots, n\} \). We first give a combinatorial characterization of the highest \( \ell \)-weight monomials of Hernandez-Leclerc modules.

**Theorem 1.1 (Theorem 3.1).** An Hernandez-Leclerc module corresponding to a cluster variable (excluding frozen variables) is a simple \( U_q(\hat{\mathfrak{g}}) \)-module with the highest \( \ell \)-weight
monomial

\[ Y_{i_1,a_1} Y_{i_2,a_2} \cdots Y_{i_k,a_k}, \]

where \( k \in \mathbb{Z}_{\geq 1}, i_j \in I, a_j \in \mathbb{Z} \) for \( j = 1, 2, \ldots, k \), and

(i) \( i_1 < i_2 < \cdots < i_k \),

(ii) \( (a_j - a_{j-1})(a_{j+1} - a_j) < 0 \) for \( 2 \leq j \leq k - 1 \),

(iii) \( |a_j - a_{j-1}| = i_j - i_{j-1} + 2 \) for \( 2 \leq j \leq k \).

1.2. Hernandez-Leclerc modules, snake graphs, and \( q \)-characters. In [29, 30], Mukhin and Young gave a purely combinatorial \( q \)-character formula for snake modules of types \( A_n \) and \( B_n \) via path descriptions. A completely different approach to compute \( q \)-characters of Kirillov-Reshetikhin modules was developed by Hernandez and Leclerc [25]. In fact, Hernandez and Leclerc proposed a geometric \( q \)-character formula conjecture for real simple modules, which implies that the truncated \( q \)-character formula will play an important role similar to the cluster character formula. Recently, Duan and Schiffler in [16] proved the geometric \( q \)-character formula for snake modules of types \( A_n \) and \( B_n \), and characterized the general kernel associated to an arbitrary real simple module. In [2], the authors gave an explicit \( q \)-character formula of simple \( U_q(\widehat{sl}_n) \)-modules. The formula involves Kazhdan-Lusztig polynomials [27] which are hard to compute when the length of the highest \( \ell \)-weight monomial of a simple module is large.

Our aim is to give a non-recursive formula of \( q \)-characters of Hernandez-Leclerc modules using snake graphs. In [17], Fomin, Shapiro, and Thurston introduced an important class of cluster algebras from surfaces with or without punctures. Each cluster variable (cluster) corresponds to a tagged arc (ideal triangulation) in the surface. In [31, 32], Musiker, Schiffler, and Williams constructed a combinatorial object associated to a non-initial arc, called a labeled snake graph, to compute the Laurent expansion of any non-initial cluster variable in a cluster algebra from a surface. Later Canakci and Schiffler studied abstract snake graphs in a series of papers [7, 8, 10] and established interesting connections among cluster variables, snake graphs, and continued fractions [9, 11]. In particular, identities in the cluster algebra have been expressed in terms of snake graphs. Rabideau used continued fractions to give a combinatorial formula for \( F \)-polynomials [33] in cluster algebras with principal coefficients from surfaces. Rabideau and Schiffler proved the constant numerator conjecture for Markov numbers via snake graphs and continued fractions [34].

Given a height function \( \xi \), we construct a unique labeled snake graph for a non-initial Hernandez-Leclerc module. Our idea is summarized by the following diagram.

Fix a simple root system \( \{ \alpha_i \mid i \in I \} \) of type \( A_n \), let \( \alpha_{i,j} = \alpha_i + \cdots + \alpha_j \) be a positive root for \( 1 \leq i \leq j \leq n \). Then the set of all cluster variables in \( \mathcal{A}(x,Q_\xi) \) is \( \{ x[-\alpha_i] \mid i \in I \} \cup \{ x[\alpha_{i,j}] \mid 1 \leq i \leq j \leq n \} \). Denote by \( \text{Match}(\mathcal{G}) \) the set of all perfect matchings of a snake graph \( \mathcal{G} \). For any \( P \in \text{Match}(\mathcal{G}) \), let \( x(P) \) (respectively, \( y(P) \)) be
the weight (respectively, height) monomial in the sense of Musiker, Schiffler, and Williams [31, 32].

By applying the theory of cluster algebras, we give a $q$-character formula of an arbitrary Hernandez-Leclerc module corresponding to a non-initial cluster variable (excluding frozen variables) by perfect matchings of snake graphs.

**Theorem 1.2** (Theorems 3.8 and 3.14). Let $G$ be the labeled snake graph associated to $x[\alpha_{i,j}]$. Then

$$x[\alpha_{i,j}] = \frac{1}{\prod_{\ell=1}^{l} x_{\ell}} \left( \sum_{P \in \text{Match}(G)} x(P)y(P) \right),$$

where the sign $\oplus$ appearing in the denominator refers to the addition of a tropical semifield.

In particular,

$$\chi_q(\iota(x[\alpha_{i,j}])) = \frac{1}{\prod_{\ell=1}^{l} \chi_q(\iota(x_{\ell}))} \chi_q \left( \iota \left( \sum_{P \in \text{Match}(G)} x(P)y(P) \right) \right).$$

There is a unique perfect matching $P \in \text{Match}(G)$ such that the highest or lowest $\ell$-weight monomial in $\chi_q(\iota(x[\alpha_{i,j}]))$ occurs in

$$\frac{1}{\prod_{\ell=1}^{l} \chi_q(\iota(x_{\ell}))} \chi_q \left( \iota \left( \sum_{P \in \text{Match}(G)} x(P)y(P) \right) \right).$$

In [1, Proposition 2.5], Brito and Chari gave a non-recursive formula of $x[\alpha_{i,j}]$ using two sets $\Gamma_{ij}$ and $\Gamma'_{ij}$. The set $\Gamma_{ij}$ consists of $(0, 1)$ sequences with $(j-i+2)$ length subject to four conditions, and $\Gamma'_{ij}$ is determined by $\Gamma_{ij}$ consisting of $(-1, 0, 1)$ sequences with the same length, see Remark 3.10.

On the other hand, we associate to $x[\alpha_{i,j}]$ perfect matchings of the corresponding labeled snake graph.

### 1.3. A recursive formula for Hernandez-Leclerc modules

We give a new recursive formula for Hernandez-Leclerc modules by an induction on the length of the highest $\ell$-weight monomials of HL-modules.
Theorem 1.3 (Theorem 4.1). Let \( j \in I \) be a source or sink vertex and \( j > i \in I \). Then
\[
x[\alpha_{i,j}]x[\alpha_{i+1,j+1}] = x[\alpha_{i,j+1}] + x[\alpha_{i,\text{max}(i-1,j-1)}]^{1-\delta_{i,j}}x^{\min\{1,(1-\delta_{i,j} \cdot j \cdot)\} d_{j \cdot -1} + \delta_{j \cdot ,.i}} x[-\alpha_{j+2}]^{d_{j+1}}x^{1-d_{j+1}},
\]
where \( \delta_{ij} \) is the Kronecker symbol and \( j_* \) is the maximum source or sink vertex strictly less than \( j \) in \( Q_\xi \).

Comparing with a recursive formula given by Brito and Chari in [1, Proposition in Section 1.5], our recursive formula needs one more step mutation for a height function \( \xi \) such that \( j \) is source or sink, see Remark 4.2 (2).

The formula (1.1) provides a possibility of a combinatorial path formula in which we allow overlapping paths, generalizing Mukhin and Young’s path formula for snake modules \([29,30]\).

The content of this paper is outlined as follows. In Section 2, we recall some background on cluster algebras of geometric type, Hernandez-Leclerc modules, snake graphs, and \( F \)-polynomials. In Section 3, we characterize the highest \( \ell \)-weight monomials of Hernandez-Leclerc modules and give a \( q \)-character formula in terms of perfect matchings of snake graphs. In Section 4, a new recursive formula of \( q \)-characters of Hernandez-Leclerc modules is introduced.

2. Preliminary

2.1. Cluster algebras. We recall the definition of cluster algebras \([18,20]\).

Let \( \mathcal{F} \) be the field of rational functions in \( n \) independent variables over \( \mathbb{Q}P \), where \( P \) is a semifield. For \( u_1, u_2, \ldots, u_r \) in \( P \), we denote by \( F|_P(u_1, \ldots, u_r) \) the evaluation of a subtraction-free rational expression \( F \) at \( u_1, \ldots, u_r \).

A tropical semifield \((P, \oplus, \cdot)\), where \( P = \text{Trop}(u_1, u_2, \ldots, u_r) \), is an abelian multiplicative group freely generated by \( u_1, u_2, \ldots, u_r \), the (auxiliary) addition \( \oplus \) is defined by
\[
\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j,b_j)}.
\]

An important choice for coefficients or frozen variables in a cluster algebra is the tropical semifield, in this case, the corresponding cluster algebra is said to be of geometric type. In this paper, we pay attention to skew-symmetric cluster algebras of geometric type.

Let \( Q \) be a finite quiver without loops or 2-cycles. Assume without loss of generality that the vertex set of \( Q \) is \( \{1, \ldots, m\} \), among vertices \( 1, \ldots, n \) (\( n \leq m \)) are mutable vertices and vertices \( n+1, \ldots, m \) are frozen variables. Mutating \( Q \) at a mutable vertex \( k \), one obtains a new quiver \( \mu_k(Q) \) defined as follows:

(i) add a new arrow \( i \to j \) for each \( i \to k \to j \), excluding that both \( i \) and \( j \) are frozen variables;
reverse the orientation of each arrow incident to $k$; and

(iii) remove all maximal pairwise disjoint 2-cycles.

**Definition 2.1.** (Seeds) A seed in $\mathcal{F}$ is a pair $(\tilde{x}, Q)$, where

(i) $\tilde{x} = \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\}$ is a free generating set of $\mathcal{F}$, called an extended cluster. The subset $x = \{x_1, \ldots, x_n\}$ is called a cluster and each element in $x$ is called a cluster variable; $\{x_{n+1}, \ldots, x_m\}$ is a set of elements in $\mathbb{P}$, called a coefficient tuple, among them each element is called a coefficient or frozen variable; and

(ii) $Q$ is a quiver as above.

**Definition 2.2.** (Seed mutations). Let $(\tilde{x}, Q)$ be a seed. The seed mutation $\mu_k$ in direction $k \in \{1, \ldots, n\}$ transforms $(\tilde{x}, Q)$ into the seed $(\tilde{x}', Q')$ defined as follows.

(i) The extended cluster $\tilde{x}' = \{x'_1, \ldots, x'_m\}$ is defined by $x'_j = x_j$ for $j \neq k$, whereas $x'_k$ is defined by the exchange relation

$$x'_k = \frac{\prod_{i : i \to k} x_i + \prod_{i : k \to i} x_i}{x_k};$$

(ii) $Q' = \mu_k(Q)$ defined as above.

A cluster algebra is a $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by all cluster variables obtained from seed mutations.

Cluster algebras of finite type are classified by the Dynkin diagrams [19], that is, there is an exchange matrix such that the Cartan counterpart of its principle part is one of Cartan matrices of finite type. Under this correspondence, cluster variables in a cluster algebra of finite type are in bijection with almost positive roots in the root system associated to the corresponding Cartan matrix.

**Theorem 2.3** ([19, Theorem 1.9]). In a cluster algebra of finite type, the cluster variable $x[\alpha]$ for $\alpha = \sum_{i \in I} a_i \alpha_i$ is expressed in term of the initial cluster $x_0$ as

$$x[\alpha] = \frac{P_\alpha(x_0)}{\prod_{i \in I} x_i^{a_i}},$$

where $P_\alpha$ is a polynomial over $\mathbb{ZP}$ with nonzero constant term. In particular, $x[-\alpha_i] = x_i$.

In order to deal with coefficients or frozen variables, we need the following theorem from [20].

**Theorem 2.4** ([20, Theorem 3.7]). Let $\mathcal{A}$ be a cluster algebra over an arbitrary semifield $\mathbb{P}$, with a seed at an initial extended cluster $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Then the cluster variables in $\mathcal{A}$ can be expressed as follows:

$$x_\ell = \frac{X_\ell|_\mathcal{F}(x_1, \ldots, x_n, y_1, \ldots, y_n)}{F_\ell|_\mathbb{P}(y_1, \ldots, y_n)};$$
where $X_\ell$ is the Laurent expression of $x_\ell$ in the case of the principle coefficient, and $F_\ell$ is the specialization of $X_\ell$ evaluating at $x_1 = \cdots = x_n = 1$, called $F$-polynomial.

2.2. Category $\mathcal{C}_\xi$ and Hernandez-Leclerc modules. Let $I = \{1, 2, \ldots, n\}$. Following [1], given a height function $\xi : I \to \mathbb{Z}$ with $|\xi(i) - \xi(i+1)| = 1$ for $1 \leq i \leq n-1$, we often extend $\xi$ to $[0, n+1]$ by defining $\xi(0) = \xi(2)$ and $\xi(n+1) = \xi(n-1)$, and associate a quiver $Q_\xi$ with the vertex set $I \cup I'$ and arrows defined by

\[ i - 1 \xrightarrow{\delta_{i,i_\circ}} i \xleftarrow{1-\delta_{i,i_\circ}} i + 1 \]

\[ 1 - \delta_{i,i_\circ} \xrightarrow{\delta_{i,i_\circ}} (i+1) \]

\[ i' \xrightarrow{\delta_{i,i_\circ}} i \xleftarrow{1-\delta_{i,i_\circ}} (i+1)' \]

if $\xi(i) = \xi(i+1) + 1$ and otherwise reverse all orientations, where $i_\circ$ is the minimum integer $\ell \in [i, n]$ such that $\xi(\ell) = \xi(\ell+2)$ if $i < n$ and otherwise $n_\circ = n$.

A vertex $i \in [1, n]$ is said to be a source or sink if $i = n$ or $\xi(i) = \xi(i+2)$, where we ignore all frozen vertices in $Q_\xi$ and require that 1 is a source or sink if and only if $\xi(1) = \xi(3)$.

Let $j_\circ = 0$ for $1 \leq j \leq 1$ and $j_\bullet$ be the maximal source or sink of $Q_\xi$ satisfying $j_\bullet < j$ for $j > 1$. For $k \geq 1$ let

\[ \mathbf{k} = (k+1)(1-\delta_{k,k_\circ}) + (k_\bullet + 1)\delta_{k,k_\circ}. \]

**Example 2.5.** In $A_9$, let $\xi(1, 2, 3, 4, 5, 6, 7, 8, 9) = (-4, -5, -6, -5, -4, -3, -4, -5, -6)$. Then we have the following table.

| $I$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|
| $\xi$ | -4 | -5 | -6 | -5 | -4 | -3 | -4 | -5 | -6 |
| $i_\circ$ | 2 | 2 | 5 | 5 | 5 | 8 | 8 | 8 | 9 |
| $i_\bullet$ | 0 | 0 | 2 | 2 | 2 | 5 | 5 | 5 | 8 |
| $i$ | 2 | 1 | 4 | 5 | 3 | 7 | 8 | 6 | 9 |

By definition, $Q_\xi$ is the following quiver

\[ 1 \xleftarrow{\delta_{1,2}} 2 \xrightarrow{\delta_{2,3}} 3 \xleftarrow{\delta_{3,4}} 4 \xrightarrow{\delta_{4,5}} 5 \xleftarrow{\delta_{5,6}} 6 \xrightarrow{\delta_{6,7}} 7 \xleftarrow{\delta_{7,8}} 8 \xrightarrow{\delta_{8,9}} 9. \]

Let $\mathbb{P} = \text{Trop}(x'_j : j \in I)$ be the tropical semifield generated by $x'_j : j \in I$. Denote by $\mathcal{A}(x, Q_\xi)$ the cluster algebra with an initial seed $(x, Q_\xi)$, where

\[ x = \{x_1, \ldots, x_n, x'_1, \ldots, x'_n\}, \]
$x_1, \ldots, x_n$ are cluster variables and $x'_1, \ldots, x'_n$ are coefficients.

In this paper, we fix $a \in \mathbb{C}^\times$. For simplicity of notation, we write $Y_{i,r}$ for $Y_{i,aq^r}$ for $i \in I$ and $r \in \mathbb{Z}$. Let $P^+$ be the free abelian monoid generated by variables $Y_{i,\xi(i)\pm 1}$ for $i \in I$. Let $C_\xi$ be the full subcategory of $C$ consisting of objects all of whose Jordan-Hölder constituents are indexed by elements of $P^+$. Simple modules in $C_\xi$ are of the form $L(m)$, where $m \in P^+$ and $m$ is called the highest $\ell$-weight of $L(m)$. The element in $P^+$ are called dominant monomials.

Following [1], for $1 \leq i < j \leq n$, let $i_2 < \cdots < i_{k-1}$ be an ordered enumeration of the subset 

$$\{ p : i < p < j \mid \xi(p-1) = \xi(p+1) \},$$

and $i_1 = i, i_k = j$. Define an element $\omega(i, j) \in P^+$ by

$$\omega(i, j) = Y_{i_1, a_1} Y_{i_2, a_2} \cdots Y_{i_k, a_k},$$

where $a_1 = \xi(i) \pm 1$ if $\xi(i+1) = \xi(i) \mp 1$ and $a_\ell = \xi(i_\ell) \pm 1$ if $\xi(i_\ell) = \xi(i_\ell - 1) \pm 1$ for $\ell \geq 2$.

Let $\text{Pr}_\xi = \{ Y_{i, \xi(i)\pm 1} \mid i \in I \} \cup \{ \omega(i, j) \mid i, j \in I, i \neq j \}$ and $f = \{ f_i = Y_{i, \xi(i)} Y_{i, \xi(i)\pm 1} \mid i \in I \}$. A simple module $L(m)$ for $m \in \text{Pr}_\xi \cup f$ is called an Hernandez-Leclerc module by Brito and Chari [1], which are precisely all the prime objects in this category. The simple modules $L(f_i), i \in I$, are Kirillov-Reshitikhin modules and $L(Y_i, i), i \in I$, are fundamental modules. We are interested in $L(\omega(i, j))$ for $1 \leq i < j \leq n$.

Following Fomin and Zelevinsky’s result [19], the set of cluster variables in $A(x, Q_\xi)$ is in bijection with the set $\Phi_{\geq -1}$ of almost positive roots in the root system of type $A_n$. Let $\{ \alpha_i \mid i \in I \}$ be a set of simple roots of type $A_n$ and let $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$ for $1 \leq i \leq j \leq n$. Denote all cluster variables (including coefficients) by

$$\{ x_i := x[-\alpha_i], x[\alpha_{i,j}], x'_i \mid 1 \leq i \leq j \leq n \}.$$ 

Following [1], the cluster variable $x[\alpha_{i,j}]$ is obtained by mutating the sequence $i, i+1, \ldots, j$ at the initial cluster $\{ x_i \mid 1 \leq i \leq n \}$.

Let $K_0(\xi)$ be the Grothedieck ring of $C_\xi$. Denote by $[M] \in K_0(\xi)$ the isomorphic class of $M \in C_\xi$. Brito and Chari proved that $C_\xi$ is a monoidal category of the cluster algebra $A(x, Q_\xi)$.

**Theorem 2.6.** [1, Theorem 1, Corollary in Section 1.3] Let $\xi : I \to \mathbb{Z}$ be a height function. Then there is an isomorphism of rings $\iota : A(x, Q_\xi) \to K_0(\xi)$ such that

$$\begin{align*}
\iota(x[-\alpha_i]) &= [L(Y_{i,\xi(i+1)})], & \iota(x'_i) &= [L(Y_{i,\xi(i)-1}Y_{i,\xi(i)+1})], \\
\iota(x[\alpha_{i,j}]) &= [L(Y_{i,\xi(i)+2})], & \xi(i) &= \xi(i+1) \pm 1, \\
\iota(x[\alpha_{i,k}]) &= [\omega(i, k)], & k &\neq i_0, \\
\iota(x'_i x[\alpha]) &= [x'_i \omega], & i &\in I, \alpha \in \Phi_{\geq -1}, \omega = \iota(x[\alpha]).
\end{align*}$$
Moreover, \( \iota \) sends a cluster variable (respectively, cluster monomial) to a real prime simple object (respectively, real simple object) of \( \CE \). In particular, \( \CE \) is a monoidal categorification of \( \mathcal{A}(x, Q_{\xi}) \).

The proof given by Brito and Chari in [1] used the fact that the set of cluster monomials forms a linear basis of cluster algebras of finite type [12, 15].

2.3. **Snake graphs.** Fix an orthonormal basis of the plane \( \mathbb{R}^2 \). A tile \( G \) is a square of fixed side-length with four vertices and four edges in the plane whose sides are parallel or orthogonal to the chosen basis. Following [7], a snake graph is a connected graph consisting of finitely many tiles \( G_1, G_2, \ldots, G_d \) with \( d \geq 1 \), such that for each \( i = 1, \ldots, d - 1 \)

(i) \( G_i \) and \( G_{i+1} \) share exactly an edge \( e_i \) and the edge is either the north edge of \( G_i \) and the south edge of \( G_{i+1} \) or the east edge of \( G_i \) and the west edge of \( G_{i+1} \).

(ii) \( G_i \) and \( G_j \) have no edge in common whenever \( |i - j| \geq 2 \).

(iii) \( G_i \) and \( G_j \) are disjoint whenever \( |i - j| \geq 3 \).

Let \( G = (G_1, G_2, \ldots, G_d) \) be a snake graph with tiles \( G_1, G_2, \ldots, G_d \). The \( d - 1 \) edges \( e_1, e_2, \ldots, e_{d-1} \) are called interior edges of \( G \) and the rest of edges are called boundary edges. Denote by \( sG \) (respectively, \( wG \)) the south (respectively, west) edge of the first tile of \( G \) and denote by \( G^N \) (respectively, \( G^E \)) the north (respectively, east) edge of the last tile of \( G \).

A snake graph \( G \) is called straight if all its tiles lie in one column or one row, and a snake graph is called zigzag if no three consecutive tiles are straight [9, 10].

A perfect matching \( P \) of a snake graph \( G \) is a subset of the set of edges of \( G \) such that each vertex of \( G \) is exactly in one edge in \( P \). Denote by \( \text{Match}(G) \) the set of all perfect matchings of \( G \). A snake graph \( G \) has precisely two perfect matchings, called the minimal matching \( P_- \) and the maximal matching \( P_+ \) of \( G \), which contain only boundary edges.

A snake graph is called a labeled snake graph if each edge and each tile in the snake graph carries a label or weight [8]. For our snake graphs, these labels are cluster variables. Without confusion, we still use \( G \) to denote a labeled snake graph. Assume that the edges of a perfect matching \( P \) of \( G \) are labeled by \( v_1, v_2, \ldots, v_r \). Following [31], one defines the weight monomial \( x(P) \) of \( P \) by \( x(P) = \prod_{i=1}^{r} x_{v_i} \).

Given a snake graph \( G \), let \( e_0 = sG \) and choose an edge \( e_d \in \{G^N, G^E \} \). In [9], Canakci and Schiffler defined a sign function

\[
 f : \{e_0, e_1, e_2, \ldots, e_{d-1}, e_d \} \to \{\pm\},
\]

such that on every tile in \( G \) the north edge and the west edge have the same sign, the south edge and the east edge have the same sign, and the sign on the north edge is opposite to the sign on the south edge.

Let \( a_i \in \mathbb{Z}_{\geq 1} \) be the number of the sign in a maximal subsequence of constant sign appearing in \( (f(e_0), f(e_1), f(e_2), \ldots, f(e_{d-1}), f(e_d)) \). As an illustrated example, the sign
function of the following snake graph is \((-,-,+,+,-,-,-,-,-,-,-,-,-,-,-,-,-,-,-,-,-,\)\), and \(a_1 = 2, a_2 = 3, a_3 = 5\).

For a snake graph \(G\) with a sign function \((a_1, a_2, \ldots, a_n)\), let \(\ell_i = \sum_{j=1}^{i} a_j\) for \(1 \leq i \leq n\) and we agree that \(\ell_0 = 0\). Following [9], one can define zigzag subsnake graphs \(H_1, \ldots, H_n\) of \(G\) as follows. Let

\[
\begin{align*}
H_1 &= (G_1, G_2, \ldots, G_{\ell_1-1}), \\
\vdots \\
H_i &= (G_{\ell_{i-1}+1}, \ldots, G_{\ell_i-1}), \\
\vdots \\
H_n &= (G_{\ell_{n-1}+1}, \ldots, G_d),
\end{align*}
\]

where \((G_j, \ldots, G_k)\) is the zigzag subsnake graph with the tiles \(G_j, G_{j+1}, \ldots, G_k\) if \(j \leq k\), and \((G_j, \ldots, G_k)\) is the single edge \(e_j\) if \(j > k\). The decomposition is in fact obtained by deleting the sign-changed tiles.

Following [33], once we choose the minimal perfect matching \(P_-\) of a snake graph \(G\), then the minimal matching \(P_-|_{H_i}\) of a subsnake graph \(H_i\) is either the matching which inherits from \(P_-\) or the union of the matching which inherits from \(P_-\) and a unique interior edge.

For an arbitrary perfect matching \(P\) of \(G\), the symmetric different \(P_- \oplus P\) is defined as

\[P_- \oplus P = (P_- \cup P) \setminus (P_- \cap P).\]

**Lemma 2.7.** [31, Lemma 4.8] The set \(P_- \oplus P\) is the set of boundary edges of a (possibly disconnected) subgraph \(G_P\) of \(G\), which is a union of cycles. These cycles enclose a set of ties \(\cup_{j \in J} G_{i_j}\), where \(J\) is a finite index set.

From now on, assume that the label of a tile \(G_i\) is \(\tau_i\) in a labeled snake graph \(G\). Following [31, Definition 4.9], with the notation above, the height monomial \(y(P)\) of a perfect matching \(P\) of \(G\) is defined by

\[y(P) = \prod_{j \in J} y_{\tau_{i_j}}.\]
2.4. Continued fractions and $F$-polynomials. Following [9, 33], a finite continued fraction

$$[a_1, a_2, \ldots, a_n] := a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}$$

is said to be positive if each $a_i$ is a positive integer. Denote by $\mathcal{N}[a_1, a_2, \ldots, a_n]$ the numerator of the continued fraction. Then $\mathcal{N}[a_1, a_2, \ldots, a_n]$ is computed by the recursion

$$\mathcal{N}[a_1, a_2, \ldots, a_n] = a_n \mathcal{N}[a_1, a_2, \ldots, a_{n-1}] + \mathcal{N}[a_1, a_2, \ldots, a_{n-2}],$$

where $\mathcal{N}[a_1] = a_1$ and $\mathcal{N}[a_1, a_2] = a_1 a_2 + 1$. We refer the reader to [34, Proposition 2.1] for more properties of continued fractions.

For a positive continued fraction $[a_1, a_2, \ldots, a_n]$, let $\ell_i = \sum_{s=1}^{i} a_s$ and we agree $\ell_0 = 0$. In [9], Canakci and Schiffler established a bijection between snake graphs and positive continued fractions via the sign function of a snake graph. Fix a positive continued fraction $[a_1, a_2, \ldots, a_n]$, the corresponding snake graph $G[a_1, a_2, \ldots, a_n]$ consists of tiles $G_1, G_2, \ldots, G_{\ell_n-1}$ and has the sign function of the following form

$$(-, \ldots, -, +, \ldots, +, -, \ldots, -, \ldots, \pm, \ldots, \pm).$$

The snake graph $G[a_1, a_2, \ldots, a_n]$ has $\mathcal{N}[a_1, a_2, \ldots, a_n]$ many perfect matchings [9, Theorem 3.4].

For simplicity, let $\prod_{j=i}^{i} x_{\ell_j} = 1$ for $j < i$. The following results from [33] will play an important role in the subsequent proof. In [33] Rabideau required that $sG \in P_-$ for a labeled snake graph $G$.

**Lemma 2.8.** [33, Lemma 3.3] The $F$-polynomial associated to a zigzag snake graph $(G_1, G_2, \ldots, G_d)$ (the label of $G_j$ is $\tau_j$) is

$$\sum_{k=0}^{d} \prod_{j=1}^{k} y_{r_j} \quad \text{or} \quad \sum_{k=1}^{d+1} \prod_{j=k}^{d} y_{r_j},$$

depending on whether the minimal perfect matching $P_-$ contains a pair of opposite boundary edges of $G_1$. 
In [33, Definition 3.1], Rabideau defined a continued fraction of Laurent polynomial $[L_1, L_2, \ldots, L_n]$, where each $L_i = \varphi_i C_i$ and

$$
C_i = \begin{cases} 
\prod_{j=1}^{\ell_i-1} y_{r_j} & \text{if } i \text{ is odd,} \\
\prod_{j=1}^{\ell_i-1} y_{r_j}^{-1} & \text{if } i \text{ is even,}
\end{cases}
$$

$$
\varphi_i = \begin{cases} 
\sum_{k=\ell_i-1}^{k=\ell_i-1+1} \prod_{j=\ell_i-1+1}^{j=k} y_{r_j} & \text{if } i \text{ is odd,} \\
\sum_{k=\ell_i}^{k=\ell_i+1} \prod_{j=\ell_i}^{j=k} y_{r_j} & \text{if } i \text{ is even.}
\end{cases}
$$

Here $\varphi_i$ is in fact the $F$-polynomial associated to a certain zigzag snake graph. Note that we revise the subscripts of the letter $y$, because we agree that the label of $G_j$ is $\tau_j$.

**Theorem 2.9.** [33, Theorem 3.4] The $F$-polynomial associated to the snake graph of the positive continued fraction $[a_1, a_2, \ldots, a_n]$, $a_1 > 1$, is given by the following equation

$$
F(G[a_1, a_2, \ldots, a_n]) = \begin{cases} 
N[L_1, L_2, \ldots, L_n] & \text{if } n \text{ is odd,} \\
C_n^{-1} N[L_1, L_2, \ldots, L_n] & \text{if } n \text{ is even,}
\end{cases}
$$

where $N[L_1, L_2, \ldots, L_n]$ is defined by the recursion

$$
N[L_1, L_2, \ldots, L_n] = L_n N[L_1, L_2, \ldots, L_{n-1}] + N[L_1, L_2, \ldots, L_{n-2}]
$$

where $N[L_1] = L_1$ and $N[L_1, L_2] = L_1 L_2 + 1$.

3. **q-characters of Hernandez-Leclerc modules and snake graphs**

In this section, we give a formula of $q$-characters of Hernandez-Leclerc modules using snake graphs, which involves an explicit formula for $F$-polynomials.

3.1. **A combinatorial description of Hernandez-Leclerc modules.** We recall the definition of Hernandez-Leclerc modules in [1]. Let $I = \{1, 2, \ldots, n\}$. Recall that $C_\xi$ is the full subcategory of $C$ consisting of objects all of whose Jordan-Hölder constituents are indexed by elements of $P^+$, see Section 2.2. Brito and Chari proved that $C_\xi$ is a monoidal categorification of the cluster algebra $A(x, Q_\xi)$. They call these modules in $C_\xi$, which correspond to cluster variables including frozen variables, Hernandez-Leclerc modules.

From now on, we call an Hernandez-Leclerc module an HL-module for simplicity.

**Theorem 3.1.** An HL-module corresponding to a cluster variable (excluding frozen variables) is a simple $U_q(\widehat{gl}_n)$-module with the highest $\ell$-weight monomial

$$
Y_{i_1, a_1} Y_{i_2, a_2} \cdots Y_{i_k, a_k},
$$

where $k \in \mathbb{Z}_{\geq 1}$, $i_j \in I$, $a_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, k$, and

(i) $i_1 < i_2 < \cdots < i_k$,

(ii) $(a_j - a_{j-1})(a_{j+1} - a_j) < 0$ for $2 \leq j \leq k - 1$,

(iii) $|a_j - a_{j-1}| = i_j - i_{j-1} + 2$ for $2 \leq j \leq k$. 
We will prove that every simple $U_q(\mathfrak{g})$-module with the highest $\ell$-weight monomial (3.1) determines a map $\xi : I \to \mathbb{Z}$ and hence it is an HL-module in the sense of Brito and Chari. Indeed, let $m = Y_{i_1,a_1} Y_{i_2,a_2} \cdots Y_{i_k,a_k}$ for $i_1 < i_2 < \cdots < i_k$. Let $i = i_1$, $j = i_k$. We divide the proof into the following two cases.

**Case 1.** If $a_1 < a_2$, we define $\xi(i) = a_1 + 1$, $\xi(i+1) = a_1 + 2$, \ldots, $\xi(i_k) = a_1 + i_2 - i_1 + 1$. Condition (iii) implies that $\xi(i_2) = a_1 + i_2 - i_1 + 1 = a_2 - 1$. Using Condition (ii) and the same argument as before, we define a map $\xi : [i, j] \to \mathbb{Z}$ by

$$\xi(x) = \begin{cases} a_{2\ell-1} + x - i_{2\ell-1} + 1, & \text{if } x \in [i_{2\ell-1}, i_{2\ell}]; \\ a_{2\ell} - x + i_{2\ell-1} - 1, & \text{if } x \in [i_{2\ell}, i_{2\ell+1}]. \end{cases}$$

We extend $\xi$ to the domain $[1, n]$ subject to

$$\begin{align*}
\xi(x) - \xi(x + 1) &= 1 & & \text{if } 1 \leq x \leq i - 1, \\
\xi(x + 1) &= \xi(x - 1) & & \text{if } j \leq x \leq n - 1.
\end{align*}$$

So the vertices $(j - 1), j, \ldots, n$ are sources or sinks. It follows from Theorem 2.6 that

$$\iota(x[\alpha_{ij}]) = \begin{cases} \{L(Y_{i_1,a_1} \}, & \text{if } j = i_0, \\ \omega(i, j) = [L(Y_{i_1,a_1} Y_{i_2,a_2} \cdots Y_{i_k,a_k})] & \text{if } j \neq i_0, \end{cases}$$

which is an HL-module in the sense of Brito and Chari.

**Case 2.** If $a_1 > a_2$, we define $\xi(i) = a_1 - 1$, $\xi(i+1) = a_1 - 2$, \ldots, $\xi(i_k) = a_1 - i_2 + i_1 - 1$. By Condition (iii), we have $\xi(i_2) = a_1 - i_2 + i_1 - 1 = a_2 + 1$. Using Condition (ii) and the same argument as before, we define a map $\xi : [i, j] \to \mathbb{Z}$ by

$$\xi(x) = \begin{cases} a_{2\ell-1} - x + i_{2\ell-1} - 1, & i \in [i_{2\ell-1}, i_{2\ell}]; \\ a_{2\ell} + x - i_{2\ell-1} + 1, & i \in [i_{2\ell}, i_{2\ell+1}]. \end{cases}$$

We extend $\xi$ to the domain $[1, n]$ subject to

$$\begin{align*}
\xi(x + 1) - \xi(x) &= 1 & & \text{if } 1 \leq x \leq i - 1, \\
\xi(x + 1) &= \xi(x - 1) & & \text{if } j \leq x \leq n - 1.
\end{align*}$$

So the vertices $(j - 1), j, \ldots, n$ are sources or sinks. It follows from Theorem 2.6 that

$$\iota(x[\alpha_{ij}]) = \begin{cases} \{L(Y_{i_1,a_1} \}, & \text{if } j = i_0, \\ \omega(i, j) = [L(Y_{i_1,a_1} Y_{i_2,a_2} \cdots Y_{i_k,a_k})] & \text{if } j \neq i_0, \end{cases}$$

which is an HL-module in the sense of Brito and Chari.

Conversely, for $\omega_{i,j: \pm 1} \in \text{Pr}_\xi$, obviously, it can be written into the form (3.1). For any $\omega_{i,j} = \omega_{i_1,a_1} \cdots \omega_{i_k,a_k} \in \text{Pr}_\xi$, by the definition of $\omega_{i,j}$, we have

$$i_1 < i_2 < \cdots < i_k,$$
and the function $\xi$ must be a strictly increasing height function or a strictly decreasing height function on these intervals $[i_1, i_2], \ldots, [i_{k-1}, i_k]$, and the strictly increasing intervals and the strictly decreasing intervals appear alternatively.

Suppose that $\xi$ is a strictly increasing (respectively, decreasing) function on the interval $[i_\ell, i_{\ell+1}]$ for a certain $1 \leq \ell \leq k - 1$. Then

$$a_\ell = \xi(i_\ell) - 1, \quad a_{\ell+1} = \xi(i_{\ell+1}) + 1$$

(respectively, $a_\ell = \xi(i_\ell) + 1, a_{\ell+1} = \xi(i_{\ell+1}) - 1$).

Since $\xi$ is a strictly decreasing (respectively, increasing) function on the interval $[i_{\ell-1}, i_\ell]$, we have

$$(a_\ell - a_{\ell-1})(a_{\ell+1} - a_\ell) = (\xi(i_\ell) - \xi(i_{\ell-1}) - 2)(\xi(i_{\ell+1}) - \xi(i_\ell) + 2) < 0,$$

and

$$|a_{\ell+1} - a_\ell| = |\xi(i_{\ell+1}) - \xi(i_\ell) + 2| = |(i_{\ell+1} - i_\ell) + 2| = i_{\ell+1} - i_\ell + 2,$$

$$|a_\ell - a_{\ell-1}| = |\xi(i_\ell) - \xi(i_{\ell-1}) - 2| = |(i_{\ell-1} + 2 - \xi(i_\ell)| = i_\ell - i_{\ell-1} + 2.$$

Similarly, it holds for the case that $\xi$ is a strictly decreasing (respectively, increasing) function on the interval $[i_{\ell-1}, i_\ell]$. Therefore, every HL-module has its highest $\ell$-weight monomial of the form (3.1). □

**Remark 3.2.** The same HL-module may correspond to different height functions $\xi$, even in the same type.

### 3.2. HL-modules and snake graphs

In a cluster algebra of finite type from a surface without punctures, arcs in an initial triangulation of the surface correspond to initial variables which are parameterized by negative simple roots [19, Theorem 1.9]. The arc crossing initial arcs $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ corresponds to the cluster variable parameterized by $\alpha_{i,j}$, where $i \leq j$. Fix a height function $\xi$, we shall construct a unique labeled snake graph for a non-initial HL-module. In general we have the relationships shown in Figure 1.

From [17], also see [35, Chapter 3], it follows that up to rotation a triangulation of a surface determines a quiver, and the quiver completely reflects the configuration of arcs in the triangulation.

Let $\xi$ be a height function. Denote by $Q$ a connected full subquiver of $Q_\xi$, where we ignore the frozen vertices. Assume without loss of generality that the set of vertices in $Q$ is $\{i, i + 1, \ldots, j\}$ from left to right in order. We define the snake graph

$$G = (G_i, G_{i+1}, \ldots, G_j)$$

associated to $Q$ as follows. The first two tiles are placed in the same horizontal line, that is, the east edge of $G_i$ and the west edge of $G_{i+1}$ are the same. The tile $G_{i+2}$ is placed in the east of $G_{i+1}$ if $(i + 1)$ is a source or sink and otherwise in the north of $G_{i+1}$. The tile
\(G_\ell\) is placed in term of the \((\ell - 1)\)-th vertex for \(i + 2 \leq \ell \leq j\), and so on. From now on, we use a digit “1” to denote the cluster variable \(x_1\), and so on. Label each tile \(G_\ell\) by \(\ell\) in the interior of the tile. For any two consecutive tiles \((G_\ell, G_{\ell+1})\), each edge is labeled near the edge obeying the following rules.

(i) If \(G_\ell\) and \(G_{\ell+1}\) share exactly an edge \(e_\ell\) and the edge is the east edge of \(G_\ell\) and the west edge of \(G_{\ell+1}\), then the north edge of \(G_\ell\) is labeled by \((\ell + 1)\) and the south edge of \(G_{\ell+1}\) is labeled by \(\ell\).

(ii) If \(G_\ell\) and \(G_{\ell+1}\) share exactly an edge \(e_\ell\) and the edge is the north edge of \(G_\ell\) and the south edge of \(G_{\ell+1}\), then the east edge of \(G_\ell\) is labeled by \((\ell + 1)\) and the west edge of \(G_{\ell+1}\) is labeled by \(\ell\).

We leave edges \(W_G\), \(S_G\), \(G_N\), and \(G_E\) no label. In particular, every edge has no label for a snake graph consisting of a tile.

The minimal (respectively, maximal) matching \(P_-\) (respectively, \(P_+\)) of \(G\) is chosen as follows. If there is an arrow \((i + 1) \rightarrow i\) in \(Q\), \(P_-\) is defined as the unique matching which contains only boundary edges and also contains \(W_G\). \(P_+\) is the other matching with only boundary edges. Otherwise \(P_-\) is defined as the unique matching which contains only boundary edges and contains \(S_G\). \(P_+\) is the other matching with only boundary edges.

**Definition 3.3.** The snake graph associated to the HL-module parameterized by \(\alpha_{i,j}\) is the snake graph determined by the connected full subquiver with vertex set \(\{i, i+1, \ldots, j\}\) in \(Q_\xi\).

The following example explains a snake graph associated to an HL-module.

**Example 3.4.** Continue our previous Example 2.5 and consider the cluster variable \(x_{[\alpha_{1,7}]}\) and the corresponding HL-module \(L(Y_{1,-3}Y_{3,-7}Y_{6,-2}Y_{8,-6})\). The full subquiver of \(Q_\xi\) with vertex set \(\{1, 2, 3, 4, 5, 6, 7\}\) is the following quiver

\[
\begin{array}{ccccccccc}
1 & \leftrightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \leftrightarrow & 5 & \leftrightarrow & 6 & \leftrightarrow & 7 & \leftrightarrow & 8 \\
& 1' & & 2' & & 3' & & 4' & & 5' & & 6' & & 7' & & 8'
\end{array}
\]

Following Definition 3.3, the labeled snake graph associated to \(L(Y_{1,-3}Y_{3,-7}Y_{6,-2}Y_{8,-6})\) is shown as follows.

\[
\begin{array}{ccccc}
2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
\]

The same HL-module may correspond to different labeled snake graphs, even in the same type. Consider a height function \(\xi(1, 2, 3, 4, 5, 6, 7, 8, 9) = (-4, -5, -6, -5, -4, -3, -4, -5, -4)\)
in $A_9$. The quiver $Q_\xi$ is the following quiver.

\[
\begin{array}{ccccccccc}
1 & \leftarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \leftarrow & 5 & \leftarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1' & \rightarrow & 2' & \rightarrow & 3' & \rightarrow & 4' & \leftarrow & 5' & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
6 & \rightarrow & 7 & \rightarrow & 8 & \leftarrow & 9 & \\
\end{array}
\]

By Theorem 2.6, we have $\iota(x[\alpha_{1,8}]) = [L(Y_{1,-3}Y_{3,-7}Y_{6,-2}Y_{8,-6})]$. Using Definition 3.3, the labeled snake graph associated to $L(Y_{1,-3}Y_{3,-7}Y_{6,-2}Y_{8,-6})$ is shown as follows.

Lemma 3.5. If a height function $\xi$ is fixed, then an HL-module corresponding to a non-initial cluster variable (excluding frozen variables) determines a unique labeled snake graph.

Proof. Since a height function $\xi$ is fixed, by Theorem 2.6, we have a cluster algebra isomorphism $\iota : A(x, Q_\xi) \rightarrow \mathcal{K}_0(\xi)$ and up to isomorphism HL-modules corresponding to non-initial cluster variables are in bijection with almost positive roots. The result follows from Definition 3.3. \qed

3.3. $q$-characters of HL-modules in term of perfect matchings of snake graphs.

To our purpose, we first give the dual version of Theorem 2.9 by the following theorem whose proof is completely similar to one given by Rabideau in [33, Theorem 3.4] except using the following formula from [7, 8].

$$F(\mathcal{G}[a_1, a_2, \ldots, a_n]) = y_{34}F(\mathcal{G}[a_1, a_2, \ldots, a_{n-1}])F(\mathcal{H}_n) + y_{56}F(\mathcal{G}[a_1, a_2, \ldots, a_{n-2}]),$$

(3.2)

where the variables $y_{34}$ and $y_{56}$ are defined as follows, $y_0 = 1$.

$$y_{34} = \begin{cases} 1 & \text{if } n \text{ is odd}, \\ y_{r_{n-1}} & \text{if } n \text{ is even,} \end{cases} \quad y_{56} = \begin{cases} \prod_{j=r_{n-2}}^{r_{n-1}} y_{r_j} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Theorem 3.6. Suppose that in $\mathcal{G}[a_1, a_2, \ldots, a_n]$ ($a_1 > 1$) the minimal perfect matching contains the west edge of the first tile. Then the $F$-polynomial associated to $\mathcal{G}[a_1, a_2, \ldots, a_n]$ is given by the following equation

$$F(\mathcal{G}[a_1, a_2, \ldots, a_n]) = \begin{cases} C_n^{-1}\mathcal{N}[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n] & \text{if } n \text{ is odd,} \\ \mathcal{N}[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n] & \text{if } n \text{ is even,} \end{cases}$$
where each \( L_i = \varphi_i C_i \) and

\[
C_i = \begin{cases} 
\prod_{j=\ell_1}^{\ell_i-1} y_{r_{ij}} & \text{if } i \text{ is odd}, \\
\prod_{j=\ell_1}^{\ell_i-1} y_{r_{ij}} & \text{if } i \text{ is even}, 
\end{cases}
\quad \varphi_i = \begin{cases} 
\sum_{k=\ell_i-1+1}^{\ell_i} \prod_{j=k}^{\ell_i-1} y_{r_{ij}} & \text{if } i \text{ is odd}, \\
\sum_{k=\ell_i-1+1}^{\ell_i} \prod_{j=k}^{\ell_i-1} y_{r_{ij}} & \text{if } i \text{ is even}.
\end{cases}
\]

**Proof.** We proceed by an induction on \( n \). Consider the case that \( n = 1 \), using Lemma 2.8 and the fact that the number of tiles in \( G[a_1] \) is \( a_1 - 1 \), we have

\[
F(G[a_1]) = \prod_{k=1}^{a_1} y_{r_{kj}} = \varphi_1 = C_1^{-1} C_1 \varphi_1 = C_1^{-1} L_1.
\]

For \( n = 2 \), using (3.2) and the fact that \( \ell_1 = a_1 \) and \( C_1 = 1 \), we have

\[
F(G[a_1, a_2]) = y_{34} F(G[a_1]) F(H_2) + y_{56} = y_{56} (C_1^{-1} L_1) \varphi_2 + 1 \\
= L_1 L_2 + 1 = N[L_1, L_2].
\]

Assume that \( n \) is odd and our formula holds for \( m < n \). The minimal matching of \( H_n \) is the matching which inherits from \( P_- \), shown in Figure 2, so \( F(H_n) = \varphi_n \). Then by (3.2)

\[
F(G[a_1, a_2, \ldots, a_n]) = F(G[a_1, a_2, \ldots, a_{n-1}]) F(H_n) + \prod_{j=\ell_{n-2}}^{\ell_{n-1}} y_{r_{ij}} F(G[a_1, a_2, \ldots, a_{n-2}]) \\
= N[L_1, L_2, \ldots, L_{n-1}] \varphi_n + \prod_{j=\ell_{n-2}}^{\ell_{n-1}} y_{r_{ij}} C_{n-2}^{-1} N[L_1, L_2, \ldots, L_{n-2}] \\
= C_n^{-1} (L_n N[L_1, L_2, \ldots, L_{n-1}] + N[L_1, L_2, \ldots, L_{n-2}]) \\
= C_n^{-1} N[L_1, L_2, \ldots, L_n].
\]

**Figure 2.** The minimal perfect matching of \( H_n \). The dashed tile is a sign-changed tile.

Assume that \( n \) is even and our formula holds for \( m < n \). The minimal matching of \( H_n \) is a union of the matching which inherits from \( P_- \) and \( e_{\ell_{n-1}} \), shown in Figure 3, so
$F(\mathcal{H}_n) = \varphi_n$. Then by (3.2)
\[
F(\mathcal{G}[a_1, a_2, \ldots, a_n]) = y_{\tau_{n-1}} F(\mathcal{G}[a_1, a_2, \ldots, a_{n-1}]) F(\mathcal{H}_n) + F(\mathcal{G}[a_1, a_2, \ldots, a_{n-2}])
\]
\[
= y_{\tau_{n-1}} C_{n-1}^{-1} \mathcal{N}[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-1}] \varphi_n + \mathcal{N}[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-2}]
\]
\[
= \mathcal{N}[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-1}] C_n \varphi_n + \mathcal{N}[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-2}]
\]
\[
= \mathcal{N}[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n] + \mathcal{N}[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-2}]
\]
\[
= \mathcal{N}[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n].
\]

Figure 3. The minimal perfect matching of $\mathcal{H}_n$. The dashed tile is a sign-changed tile.

In practice, we always assume that $a_1 > 1$ for any continued fraction $[a_1, a_2, \ldots, a_n]$. Otherwise, we take the rotation of a snake graph by 180 degrees or the flips at the lines $y = -x$, it follows from [11, Proposition 3.1 (b) and (c)] that $\mathcal{G}[a_1, a_2, \ldots, a_n] \cong \mathcal{G}[a_n, \ldots, a_2, a_1]$. In addition, $[a_1, a_2, \ldots, a_{n-1}, 1] = [a_1, a_2, \ldots, a_{n-1} + 1]$.

**Lemma 3.7.** Let $\mathcal{G}$ be the snake graph associated to the HL-module parameterized by $\alpha_{ij}$ and
\[
F = \sum_{P \in \text{Match}(\mathcal{G})} y(P)
\]
be its $F$-polynomial. Then
\[
F|_P = y(P_+)|_P = (y_i y_{i+1} \cdots y_j)|_P.
\]

*Proof.* Following [31], all perfect matchings in $\text{Match}(\mathcal{G})$ form a poset with the minimal perfect matching $P_-$ and the maximal perfect matching $P_+$. Recall that in [33] a tile $G$ can be turned if two of its edges are in a perfect matching $P$. Let $P'$ be the perfect matching obtained by replacing the two edges of $G$ in $P$ with the other two edges of $G$. We say that $P'$ is obtained from $P$ by turning the tile $G$. Every perfect matching is obtained from $P_-$ or $P_+$ by turning a sequence of tiles.

The height monomial $y(P')$ of $P'$ is defined recursively by $y(P_-) = 1$ and if $P'$ is above $P$ and obtained by turning a tile $G_\ell$ then $y(P') = y_\ell y(P)$. In the following, we will prove that for each step of turning a tile, $y(P)|_P \leq y(P')|_P$, meaning that $y(P')|_P$ contains all possible factors with multiplicities appearing in $y(P)|_P$. As a conclusion, $F|_P = y(P_+)|_P$. 
In our setting, if $P$ is above $P_-$ and obtained by turning a tile $G_\ell$, then $y(P) = y_\ell$ and either $\ell \not\in \{i, j\}$ is a source or $\ell \in \{i, j\}$ such that $\ell \rightarrow l$ for a certain $l \in I$. In general, if $P'$ is above $P$ and obtained by turning a tile $G_\ell$ then $y(P') = y_\ell y(P)$, and at least one of the following two cases occur:

(i) either $\ell \not\in \{i, j\}$ is a source or $\ell \in \{i, j\}$ such that $\ell \rightarrow l$ for a certain $l \in I$, where $y_l$ is not a factor in $y(P)$,

(ii) there exists a certain $l \in I$ (not unique) such that $l \rightarrow \ell$ in $Q_\ell$, where $y_l$ is a factor in $y(P)$.

For the case (i), $y_\ell = x'_\ell$ or all possible factors in the denominator of $y_\ell$ cannot be cancelled by $y(P)$, and hence $y(P)|_P \leq y(P')|_P$. For the case (ii), all possible arrows near the vertex $\ell$ are shown as follows.

So checking case by case from up to down and from left to right

$$y(P')|_P = \left(\frac{x'_k}{x'_\ell}y(P)\right)|_P = y(P)|_P,$$ because $x'_\ell$ is cancelled by the numerator in $y_\ell$,

$$y(P')|_P = \left(\frac{1}{x'_\ell}y(P)\right)|_P = y(P)|_P,$$ because $x'_\ell$ is cancelled by the numerator in $y_\ell$,

$$y(P')|_P = \left(\frac{x'_k}{x'_\ell}y(P)\right)|_P = x'_\ell^{-1}y(P)|_P,$$

$$y(P')|_P = \left(\frac{x'_k}{x'_\ell}y(P)\right)|_P = x'_l^{-1}y(P)|_P,$$

$$y(P')|_P = \left(\frac{1}{x'_\ell}y(P)\right)|_P = x'_l^{-1}y(P)|_P,$$

$$y(P')|_P = \left(\frac{x'_k}{x'_i}y(P)\right)|_P = y(P)|_P,$$ because $x'_i$ is cancelled by the numerator in $y_\ell$,

$$y(P')|_P = \left(\frac{x'_k}{x'_i}y(P)\right)|_P = y(P)|_P,$$ because $x'_i$ is cancelled by the numerator in $y_\ell$,

$$y(P')|_P = \left(\frac{1}{x'_\ell}y(P)\right)|_P = y(P)|_P,$$ because $x'_\ell$ is cancelled by the numerator in $y_k,$

$$y(P')|_P = \left(\frac{1}{x'_\ell}y(P)\right)|_P = x'_\ell^{-1}y(P)|_P,$$
the seventh equation is because both \( G_t \) and \( G_k \) have been turned from \( P_- \) before turning the tile \( G_\ell \).

**Theorem 3.8.** Let \( \mathcal{G}_\ell \) be the full subcategory introduced by Brito and Chari [1] and \( \mathcal{G} \) be the labeled snake graph associated to \( x[\alpha_{i,j}] \). Then

\[
x[\alpha_{i,j}] = \frac{1}{\prod_{\ell=1}^{l'} x_\ell} \left( \sum_{P \in \text{Match}(\mathcal{G})} x(P) y(P) \right),
\]

where the sign \( \oplus \) appearing in the denominator refers to the (auxiliary) addition in a tropical semifield \( \mathbb{T} = \text{Trop}(u_1, u_2, \ldots, u_r) \) \((r \in \mathbb{Z}_{\geq 1})\), and \( y_\ell = \prod_{j \rightarrow \ell} u_j \prod_{\ell \rightarrow j} u_j^{-1} \) is a Laurent monomial in \( r \) variables \( u_1, u_2, \ldots, u_r \) for \( i \leq \ell \leq j \). In particular,

\[
\chi_q(\ell(x[\alpha_{i,j}]))) = \frac{1}{\prod_{\ell=1}^{l'} \chi_q(\ell(x_\ell))} \chi_q \left( \frac{\ell}{\prod_{P \in \text{Match}(\mathcal{G})} x(P) y(P)} \right).
\]

**Proof.** The proof of the first part follows [31, Theorem 4.10] and [20, Theorem 3.7] (see Theorem 2.4). The second one follows from [1, Theorem 1, Corollary] (see Theorem 2.6).

We explain our theorem by the following example.

**Example 3.9.** Continue our previous Example 3.4. Since \( x_8, x_9 \) are not mutated, our coefficients are in \( \text{Trop}(x_1', x_2', x_3', x_4', x_5', x_6', x_7', x_8', x_9', x_8, x_9) \). By Theorem 3.8, we substitute variables by

\[
y_1 = \frac{x_1'}{x_2}, \quad y_2 = x_2', \quad y_3 = \frac{x_3'}{x_3}, \quad y_4 = \frac{x_4'}{x_4}, \quad y_5 = \frac{x_5'}{x_5}, \quad y_6 = \frac{x_6'}{x_6}, \quad y_7 = \frac{x_7'}{x_7}, \quad y_8 = \frac{x_7' x_8}{x_8'}.
\]

All possible perfect matchings of the labeled snake graph associated to the HL-module \( L(Y_{1,-3}Y_{3,-7}Y_{6,-2}Y_{8,-6}) \) are shown in Table 1. In Table 1 we list the monomial \( x(P) y(P) \) in the below of \( P \in \text{Match}(\mathcal{G}) \).

By Theorem 3.6, the associated \( F \)-polynomial is given by

\[
F(y_1, y_2, y_3, y_4, y_5, y_6, y_7) = F(\mathcal{G}[2, 3, 3]) = C_3^{-1} N[\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3],
\]

where

\[
\mathcal{L}_1 = 1 + y_1, \quad \mathcal{L}_2 = y_2(1 + y_3 + y_3 y_4), \quad \mathcal{L}_3 = y_2^{-1} y_3^{-1} y_4^{-1} y_5^{-1} y_6^{-1} y_7^{-1}(1 + y_7 + y_6 y_7).
\]

By computation, we have

\[
F(\mathcal{G}[2, 3, 3]) = (1+y_1) y_2 (1+y_3+y_3 y_4) (1+y_7+y_6 y_7) + y_2 y_3 y_4 y_5 y_6 y_7 (1+y_1) + (1+y_7+y_6 y_7).
\]

Using Lemma 3.7, we have

\[
F|_p(y_1, y_2, y_3, y_4, y_5, y_6, y_7) = (y_1 y_2 y_3 y_4 y_5 y_6 y_7)|_p = \frac{1}{x_3 x_8}.
\]
Therefore by Theorem 3.8,
\[
x[\alpha_1, \ldots, \alpha_7] = \frac{1}{x_1x_2x_3x_4x_5x_6x_7} \sum_{P \in \text{Match}(G)} \frac{x(P)y(P)}{\bigoplus_{t \in \text{Match}(G)} y(P)}
\]
\[
= \frac{x_5x'_3x'_5x'_8}{x_2x_7} + \frac{x_6x_8x'_3x'_5}{x_2x_5x_6} + \frac{x_6x'_3x'_5}{x_2x_5} + \frac{x_6x_5x'_3x'_8}{x_2x_5x_6} + \frac{x_6x_5x'_3x'_8}{x_1x_2x_3x_7} + \frac{x_6x_5x'_3x'_8}{x_1x_2x_3x_6} + \frac{x_6x_5x'_3x'_8}{x_1x_2x_3} + \frac{x_6x_5x'_3x'_8}{x_1x_3x_7} + \frac{x_6x_5x'_3x'_8}{x_1x_3x_6} + \frac{x_6x_5x'_3x'_8}{x_1x_3x_5} + \frac{x_6x_5x'_3x'_8}{x_1x_3x_4} + \frac{x_6x_5x'_3x'_8}{x_1x_3x_3} + \frac{x_6x_5x'_3x'_8}{x_1x_3x_2} + \frac{x_6x_5x'_3x'_8}{x_1x_3x_1} + \frac{x_6x_5x'_3x'_8}{x_1x_3x_0}.
\]
Replacing $x_i, x'_i$ by $q$-characters of the corresponding initial simple modules, we obtain the $q$-character of $L(Y_{1,-3}Y_{3,-7}Y_{6,-2}Y_{8,-6})$.

**Remark 3.10.** In [1, Proposition 2.5], Brito and Chari gave a non-recursive formula of $x[\alpha_{i,j}]$ using two sets $\Gamma_{i,j}$ and $\Gamma'_{i,j}$. The set $\Gamma_{i,j}$ consists of $(0,1)$ sequences with $(j-i+2)$ length subject to four conditions [1, Section 2.3], and $\Gamma'_{i,j}$ is determined by $\Gamma_{i,j}$ consisting of $(-1,0,1)$ sequences with the same length [1, Section 2.4]. Following [1, Section 2.5], one defines

$$m_{i,j}^\varepsilon = x_{i-1}^{(1-\varepsilon_i)} x_i^{\varepsilon_i'} \cdots x_{j+1}^{\varepsilon_{j+1}'} x_j^{\varepsilon_j}, \quad f_{i,j}^\varepsilon = (x_i')^{\varepsilon_i'} \cdots (x_j')^{\varepsilon_j} (x_{j+1}')^{(1-\delta_{j,j}) \varepsilon_j + 1}.$$

For $x[\alpha_{17}]$ in Example 3.9, the sets $\Gamma_{1,7}$ and $\Gamma'_{1,7}$, monomials $m_{1,7}^\varepsilon$ and $f_{1,7}^\varepsilon$ are listed in Table 2. The $i$-th row $(1 \leq i \leq 23)$ in Table 2 corresponds to the $i$-th perfect matching in Table 1 from up to down and from left to right.

### 3.4. The highest and lowest $\ell$-weight monomials of an HL-module.

In this subsection, we determine the highest and lowest $\ell$-weight monomials in the $q$-character of an HL-module using perfect matchings of snake graphs.

**Lemma 3.11.** Let $Q_\xi$ be a quiver associated to a height function $\xi$. Assume that there is no source or sink vertex in the interval $(i, j)$. Then there exists a unique perfect matching $P$ such that

$$\frac{x(P)y(P)}{(x_ix_{i+1} \cdots x_j)^F \prod y_i, y_{i+1}, \ldots, y_j} = \frac{x_{i+1}^{(1-\delta_{j,j})} x_{j+1}^\ell}{x_i} \text{ or } \frac{x_{j+1}^{(1-\delta_{j,j})} x_{j+1}^\ell}{x_j}.$$

**Proof.** If $\xi(i) = \xi(i + 1) + 1$, then there is a full subquiver of $Q_\xi$, which is one of the following quivers

(i - 1) \xleftarrow{} i \xrightarrow{} (i + 1) \xrightarrow{} \cdots \xrightarrow{} (j - 1) \xrightarrow{} j

\xrightarrow{\delta_{j,j}} (j + 1), \quad (3.3)

\begin{align*}
(i - 1) & \xleftarrow{} i \xrightarrow{} (i + 1) \xrightarrow{} \cdots \xrightarrow{} (j - 1) \xrightarrow{} j
\end{align*}

\xrightarrow{\delta_{j,j}} (j + 1), \quad (3.4)
If \( \xi(i) = \xi(i+1) - 1 \), then there is a full subquiver of \( Q_x \), which is one of the following quivers:

\[
\begin{align*}
&\begin{array}{c}
(1+i) \\
(j+1)
\end{array} \\
&\begin{array}{c}
(1+i) \\
(j+1)
\end{array} \\
&\begin{array}{c}
(1+i) \\
(j+1)
\end{array} \\
&\begin{array}{c}
(1+i) \\
(j+1)
\end{array}
\end{align*}
\]

\[\text{Table 2: The sets } H_x \text{ and } \gamma_x, \text{ monomials } m_i^j, \text{ and } f_i, i = 1, \ldots, 11.\]
By our Definition 3.3, the labeled snake graph associated to $\iota(x[\alpha_{i,j}])$ is a zigzag labeled snake graph $G = (G_i, G_{i+1}, \ldots, G_j)$. Let $P_-$ (respectively, $P_+$) be the minimal (respectively, maximal) perfect matching of $G$. Define another labeled perfect matching $P$ of $G$ as follows:

$$P = \begin{cases} P_+ & \text{for (3.3)}, \\ wG \cup e_1 \cup P_- |_{(G_i \setminus e)} & \text{for (3.4)}, \\ P_- & \text{for (3.5)}, \\ wG \cup e_1 \cup P_+ |_{(G_i \setminus e)} & \text{for (3.6)}. \end{cases}$$

Then by Lemma 3.7, we have

$$F|_p(y_i, y_{i+1} \ldots, y_j) = \begin{cases} \frac{1}{x_i x_{i+1}^{x(1-\delta_{j,j+1})} x_{j+1}^{x(1-\delta_{j,j+1})}} & \text{for (3.3)}, \\ \frac{1}{x_i x_{i+1}^{x(1-\delta_{j,j+1})} x_{j+1}^{x(1-\delta_{j,j+1})}} & \text{for (3.4)}, \\ x_i x_{j+1}^{x(1-\delta_{j,j+1})} x_{j+1}^{x(1-\delta_{j,j+1})} & \text{for (3.5)}, \\ \frac{1}{x_i x_{j+1}^{x(1-\delta_{j,j+1})} x_{j+1}^{x(1-\delta_{j,j+1})}} & \text{for (3.6)}. \end{cases}$$

By computation, we have

$$x(P) = \begin{cases} \frac{1}{x_i x_{i+1} \ldots x_j} & \text{for (3.3) and (3.5)}, \\ \frac{1}{x_i x_{i+1} x_{j+1}} & \text{for (3.4) and (3.6)}, \\ \end{cases}$$

$$y(P) = \begin{cases} x_i x_{j+1}^{(1-\delta_{j,j+1})} x_{j+1}^{(1-\delta_{j,j+1})} & \text{for (3.3) and (3.5)}, \\ x_i x_{j+1}^{(1-\delta_{j,j+1})} x_{j+1}^{(1-\delta_{j,j+1})} & \text{for (3.4) and (3.6)}. \end{cases}$$

Therefore

$$\frac{x(P) y(P)}{(x_i x_{i+1} \ldots x_j) F|_p(y_i, y_{i+1} \ldots, y_j)} = \begin{cases} x_i x_{j+1}^{(1-\delta_{j,j+1})} & \text{for (3.3), (3.5)}, \\ x_i x_{j+1}^{(1-\delta_{j,j+1})} x_{j+1}^{(1-\delta_{j,j+1})} & \text{for (3.4), (3.6)}. \end{cases}$$

Finally, the uniqueness of $P$ follows from the uniqueness of the perfect matching with the height monomial $y(P)$, because the set of all perfect matchings in a zigzag snake
graph forms a total order, and a weight monomial does not affect the associated height monomial. □

Denote by $G = (G_i, G_{i+1}, \ldots, G_j)$ the snake graph with sign function $(a_1, a_2, \ldots, a_l)$ associated to the HL-module parameterized by $\alpha_{i,j}$. We have the following observation: The vertices $(i + \ell_1 - 1), (i + \ell_2 - 1), \ldots, (i + \ell_{t-1} - 1)$ are sources or sinks in $Q_\xi$. Hence by Definition 3.3, these tiles $G_{i+\ell_1-1}, G_{i+\ell_2-1}, \ldots, G_{i+\ell_{t-1}-1}$ are sign-changed tiles.

Let

\[ \mathcal{H}_t = (G_{i+\ell_{t-2}}, G_{i+\ell_{t-1}}, \ldots, G_{i+\ell_{t-1}}), \quad 1 \leq t \leq l, \]

be all the (labeled) subsnake graphs in the decomposition of $G$. Let $P_-$ (respectively, $P_+$) be the minimal (respectively, maximal) perfect matching of $G$. Denote by $P_-|_{\mathcal{H}_t}$ (respectively, $P_+|_{\mathcal{H}_t}$) the minimal (respectively, maximal) perfect matching obtained by restricting $P_-$ (respectively, $P_+$) to $\mathcal{H}_t$, where $1 \leq t \leq l$. Define another (labeled) perfect matching $P$ of $G$ as follows. If $i$ is a source or sink in $Q_\xi$, we define

\[
P|_{\mathcal{H}_t} = \begin{cases} wG \cup e_1 \cup P_-|_{(\mathcal{H}_t\setminus G_i)} & \text{if } i \text{ is a source}, \\ wG \cup e_1 \cup P_+|_{(\mathcal{H}_t\setminus G_i)} & \text{if } i \text{ is a sink}, \end{cases}
\]

and otherwise let

\[
P|_{\mathcal{H}_t} = \begin{cases} P_-|_{\mathcal{H}_t} & \text{if } i \rightarrow i' \text{ in } Q_\xi, \\ P_+|_{\mathcal{H}_t} & \text{if } i' \rightarrow i \text{ in } Q_\xi. \end{cases}
\]

For $1 < t \leq l$, let

\[
P|_{\mathcal{H}_t} = \begin{cases} P_-|_{\mathcal{H}_t} & \text{if } (i + \ell_{t-1}) \rightarrow (i + \ell_{t-1})' \text{ in } Q_\xi, \\ P_+|_{\mathcal{H}_t} & \text{if } (i + \ell_{t-1})' \rightarrow (i + \ell_{t-1}) \text{ in } Q_\xi. \end{cases}
\]

Define $P$ as the gluing of $P|_{\mathcal{H}_t}$ for $1 \leq t \leq l$ and it is a perfect matching of $G$.

**Definition 3.12.** Let $i$ be a source or sink in $Q_\xi$. Define the revised height monomial

\[
y(P|_{\mathcal{H}_t}) = \begin{cases} y_i & \text{if } i \text{ is a source,} \\ y(P|_{\mathcal{H}_t}) & \text{if } i \text{ is a sink and } l = 1, \\ y_{i+\ell_{t-1}}y(P|_{\mathcal{H}_t}) & \text{if } i \text{ is a sink and } l > 1. \end{cases}
\]

Otherwise, for $1 < t \leq l$ we define the revised height monomial

\[
y(P|_{\mathcal{H}_t}) = \begin{cases} 1 & \text{if } (i + \ell_{t-1}) \rightarrow (i + \ell_{t-1})' \text{ in } Q_\xi, \\ y(P|_{\mathcal{H}_t}) & \text{if } (i + \ell_{t-1})' \rightarrow (i + \ell_{t-1}) \text{ in } Q_\xi \text{ and } t = l, \\ y_{i+\ell_{t-1}}y(P|_{\mathcal{H}_t}) & \text{if } (i + \ell_{t-1})' \rightarrow (i + \ell_{t-1}) \text{ in } Q_\xi \text{ and } t \neq l. \end{cases}
\]

We immediately have the following lemma.
Lemma 3.13. With the notation above, there exists a unique perfect matching \( P \in \text{Match}(\mathcal{G}) \) such that

\[
\frac{x(P)y(P)}{(\prod_{s=1}^{j} x_s) F|_{\mathcal{P}}(y_1, y_{i+1}, \ldots, y_j)} = \begin{cases} 
\ell_1 - 2 \prod_{s=2}^{\ell_1-2} x_{i+s} & \text{if } l = 1, \\
\ell_1 - 1 \prod_{s=2}^{\ell_1-1} x_{i+s} & \text{if } l > 1,
\end{cases}
\]

otherwise.

Proof. By the definition of \( P \), if \( i \) is a source or sink in \( Q_\xi \), then we have

\[
x(P|_{H_1}) = \begin{cases} 
\ell_1 - 2 \prod_{s=2}^{\ell_1-2} x_{i+s} & \text{if } l = 1, \\
\ell_1 - 1 \prod_{s=2}^{\ell_1-1} x_{i+s} & \text{if } l > 1,
\end{cases}
\]

\[
\tilde{y}(P|_{H_1}) = \begin{cases} 
y_i & \text{if } i \text{ is a source}, \\
\ell_1 - 2 \prod_{s=1}^{\ell_1-2} y_{i+s} & \text{if } i \text{ is a sink and } l = 1, \\
\ell_1 - 1 \prod_{s=1}^{\ell_1-1} y_{i+s} & \text{if } i \text{ is a sink and } l > 1,
\end{cases}
\]

and otherwise for \( 1 \leq t \leq l \)

\[
x(P|_{H_t}) = \begin{cases} 
\ell_1 - 2 \prod_{s=\ell_1-t+1}^{\ell_1-1} x_{i+s} & \text{if } t = l, \\
\ell_1 - 1 \prod_{s=\ell_1-t+1}^{\ell_1-1} x_{i+s} & \text{if } t < l,
\end{cases}
\]

\[
\tilde{y}(P|_{H_t}) = \begin{cases} 
1 & \text{if } (i + \ell_{t-1}) \rightarrow (i + \ell_{t-1})' \text{ in } Q_\xi, \\
\ell_1 - 2 \prod_{s=\ell_1-t+1}^{\ell_1-1} y_{i+s} & \text{if } (i + \ell_{t-1})' \rightarrow (i + \ell_{t-1}) \text{ in } Q_\xi \text{ and } t = l, \\
\ell_1 - 1 \prod_{s=\ell_1-t+1}^{\ell_1-1} y_{i+s} & \text{if } (i + \ell_{t-1})' \rightarrow (i + \ell_{t-1}) \text{ in } Q_\xi \text{ and } t \neq l.
\end{cases}
\]

By Lemma 3.7, we have

\[
F|_{\mathcal{P}}(y_1, y_{i+1}, \ldots, y_j) = y(P_+)|_{\mathcal{P}} = \left( \prod_{i=1}^{l_1-1} \prod_{s=\ell_{i-1}}^{\ell_i} y_{i+s} \right) \left( \prod_{s=\ell_{l_1-1}}^{\ell_{l_1}-2} y_{i+s} \right).
\]

We consider zigzag snake graphs

\[
(G_{i+\ell_{t-1}}, G_{i+\ell_{t-1}+1}, \ldots, G_{i+\ell_{t-1}}) = H_t \cup G_{i+\ell_{t-1}}
\]
for $1 \leq t \leq l - 1$ and
\[(G_{i+t_1}, G_{i+t_2}, \ldots, G_{i+t_{l-2}}) = \mathcal{H}_t.\]

By checking case by case, for $1 \leq t \leq l - 1$, $x(P|_{\mathcal{H}_t})$ is just the weight monomial on $\mathcal{H}_t \cup G_{i+t_1}$ with the height monomial $\tilde{y}(P|_{\mathcal{H}_t})$, and $x(P|_{\mathcal{H}_t})$ is the weight monomial on $\mathcal{H}_t$ with the height monomial $\tilde{y}(P|_{\mathcal{H}_t})$.

Since these vertices $(i + t_1 - 1), (i + t_2 - 1), \ldots, (i + t_{l-1} - 1)$ are sources or sinks in $Q_\xi$, by Lemma 3.11, we have

\[
\frac{x(P)y(P)}{(\prod_{s=1}^{j} x_s) \prod_{1 \leq t \leq l} x_{i+t}(\prod_{s=t+1}^{\ell_t-1} y_{i+s})(\prod_{s=t+1}^{\ell_t-1} y_{i+s})} = \frac{x(P|_{\mathcal{H}_t})\tilde{y}(P|_{\mathcal{H}_t})}{(\prod_{s=t+1}^{\ell_t-1} x_{i+s}) (\prod_{s=t+1}^{\ell_t-1} y_{i+s})}.
\]

Finally, the uniqueness of $P$ follows from the uniqueness of the perfect matching of each $\mathcal{H}_t$ for $1 \leq t \leq l$.

By Theorem 2.6 and Theorem 3.1, assume without loss of generality that
\[t(x[O_{i,j}]) = [L(Y_{1,1}Y_{2,2} \ldots Y_{k,k})]\]
for $i = i_1 < i_2 < \cdots < i_k$. Denote by $G = (G_i, G_{i+1}, \ldots, G_j)$ the snake graph with a sign function $(a_1, a_2, \ldots, a_l)$ associated to the HL-module $L(Y_{1,1}Y_{2,2} \ldots Y_{k,k})$. Then by Lemma 3.13, either $l = k$ or $|l - k| = 1$, and $l = k$ if and only if $i$ is not a source or sink and $j$ is a source or sink.

**Theorem 3.14.** With the notation above, the highest or lowest $\ell$-weight monomial in the $q$-character of an arbitrary HL-module $L(Y_{1,1}Y_{2,2} \ldots Y_{k,k})$ occurs in

\[
\begin{cases}
\frac{\chi_q(t(x_i x_{i+1} x_{i+2} \ldots x_{i+\ell_1} x_{i+\ell_2} \ldots x_{i+\ell_{l-1}} x_{i+\ell})^{(1-\delta_{i,j,o}))}}}{\chi_q(t(x_i x_{i+1} x_{i+2} \ldots x_{i+\ell_{l-1}}))} & \text{if } i \text{ is a source or sink,} \\
\frac{\chi_q(t(x_i x_{i+1} x_{i+2} \ldots x_{i+\ell_1} x_{i+\ell_2} \ldots x_{i+\ell_{l-1}} x_{i+\ell})^{(1-\delta_{i,j,o}))}}}{\chi_q(t(x_i x_{i+1} x_{i+2} \ldots x_{i+\ell_{l-1}}))} & \text{otherwise.}
\end{cases}
\]
Proof. By Theorem 3.8, we have
\[
\chi_q(\prod_{\ell=i}^j \iota(x_\ell))\chi_q(\iota(x[\alpha_{i,j}])) = \chi_q\left(\iota\left(\sum_{P \in \text{Match}(G)} x(P)y(P)\right)\right),
\]
(3.7)

By [21, Corollary 6.9], for an arbitrary simple \(U_q(\hat{\mathfrak{g}})\)-module \(L(m)\), the lowest \(\ell\)-weight monomial in \(\chi_q(L(m))\) is the product of the lowest \(\ell\)-weight monomials of fundamental modules whose highest \(\ell\)-weight monomials are factors of \(m\), and the highest or lowest \(\ell\)-weight monomial is unique. Hence the lowest \(\ell\)-weight monomial in both sides of Equation (3.7) should be same. By Lemma 3.13, there exists a unique perfect matching \(P\) of the snake graph associated to the HL-module \(L(Y_{i_1,a_1}Y_{i_2,a_2} \ldots Y_{i_k,a_k})\) such that
\[
x(P)y(P) = \begin{cases} 
\frac{x'_{i}x'_{i+1}x'_{i+\ell_1}x'_{i+\ell_2} \ldots x'_{i+\ell_{j-1}}x'_{j+1}}{x_ix_{i+1}x_{i+\ell_1}x_{i+\ell_2} \ldots x_{i+\ell_{j-1}}} (\prod_{\delta(i_j,j)}^j x_s) & \text{if } i \text{ is a source or sink}, \\
\frac{x'_{i}x'_{i+1}x'_{i+\ell_1}x'_{i+\ell_2} \ldots x'_{i+\ell_{j-1}}x'_{j+1}}{x_ix_{i+1}x_{i+\ell_1}x_{i+\ell_2} \ldots x_{i+\ell_{j-1}}} (\prod_{\delta(i_j,j)}^j x_s) & \text{otherwise}.
\end{cases}
\]

By Theorem 2.6, for any \(1 \leq \ell \leq k\),
\[
\iota(x_{i_\ell}) = [L(Y_{i_\ell,\xi(i_{\ell+1})})], \quad \iota(x'_{i_\ell}) = [L(Y_{i_\ell,\xi(i_{\ell})}Y_{i_\ell,\xi(i_{\ell+1})})].
\]
Comparing the highest \(\ell\)-weight monomial and the lowest \(\ell\)-weight monomial in both sides of Equation (3.7), and by their uniqueness property, our result holds. \(\Box\)

Example 3.15. Continue our previous Example 3.9, the minimal perfect matching \(P_+\) of \(\mathcal{G}\) and each \(P_+|_{\mathcal{H}_t}\) (\(t = 1, 2, 3\)) are shown in the left and right hand side of the following figure respectively.

By definition, the required perfect matching \(P\) of \(\mathcal{G}\) is
By the definition of $x(P|_{H_t})$ for $t = 1, 2, 3$ and Definition 3.12, we have

\[ x(P|_{H_1}) = x_2, \quad x(P|_{H_2}) = x_4x_5, \quad x(P|_{H_3}) = x_7, \]

\[ \tilde{y}(P|_{H_1}) = y_1y_2, \quad \tilde{y}(P|_{H_2}) = 1, \quad \tilde{y}(P|_{H_3}) = y_6y_7. \]

Using Lemma 3.13, we have

\[
\frac{x(P)y(P)}{(\prod_{s=1}^{7} s)F_{\mathbb{P}}(y_1,y_2,\ldots,y_7)} = \frac{(x_2y_1y_2)(x_4x_5)(x_7y_6y_7)}{(x_1x_2(y_1y_2)||\mathbb{P}) (x_3x_4x_5(y_3y_4y_5)||\mathbb{P}) (x_6x_7(y_6y_7)||\mathbb{P})} = \frac{x'_1x'_3x'_6x_8}{x_1x_3x_6}.
\]

Therefore by Theorem 3.14, the highest $\ell$-weight monomial and the lowest $\ell$-weight monomial in the $q$-character of $L(Y_{1,-3}Y_{3,-7}Y_{6,2}Y_{8,-6})$ occur in

\[
\frac{\chi_q(\ell(x'_1x'_3x'_6x_8))}{\chi_q(\ell(x_1x_3x_6))} = \frac{\chi_q(L(Y_{1,-3}Y_{3,-7}Y_{6,-4})L(Y_{6,-2}L(Y_{8,-6})))}{\chi_q(L(Y_{1,-5}L(Y_{3,-5})L(Y_{6,-4}))}
\]

and they are

\[
\frac{(Y_{1,-3}Y_{3,-7}Y_{6,-4})(Y_{6,-2}L(Y_{8,-6}))}{(Y_{1,-5}Y_{3,-5})} = Y_{1,-3}Y_{3,-7}Y_{6,-2}Y_{8,-6},
\]

\[
\frac{(Y_{9,5}Y_{9,7}^{-1})(Y_{7,3}^{-1}Y_{7,5}^{-1})(Y_{4,6}^{-1}Y_{4,8}^{-1})(Y_{2,4}^{-1})}{Y_{9,5}Y_{7,5}Y_{4,6}^{-1}} = Y_{2,4}^{-1}Y_{4,8}^{-1}Y_{7,3}^{-1}Y_{9,7}^{-1},
\]

respectively.

4. NEW RECURSION OF q-CHARACTERS OF HERNANDEZ-LECLERC MODULES

In this section, we give a recursive formula for HL-modules by an induction on the length of the highest $\ell$-weight monomials of HL-modules.

We have the following theorem.

**Theorem 4.1.** Let $j \in I$ be a source or sink and $j > i \in I$. Then

\[
x[\alpha_{i,j}]x[\alpha_{j+1,j+1}] = x[\alpha_{i,j+1}]
\]

\[+ x[\alpha_{i,\max\{i-1,j+1\}}]^{1-\delta_{i+1,j+1}} x_{\max\{i,j+1\}}^{\min\{1,1-\delta_{j+1,j+1}\}} x[-\alpha_{j+1}]^{d_{j+1}} x_{j+2}^{1-d_{j+1}}. \tag{4.1}
\]
**Proof.** By [1, Lemma 2.2], if there is an arrow \((j - 1) \rightarrow j\) in \(Q_\xi\), then after mutating the sequence \(i, (i + 1), \ldots, (j - 1)\) we obtain the following arrows at vertices \(j\) and \((j + 1)\):

\[
\begin{array}{c}
\text{max}\{i - 1, j_{\bullet} - 1\} \\
\overset{a_j}{\longrightarrow} (j - 1) \overset{b_j}{\longrightarrow} j \overset{d_{j+1}}{\longrightarrow} (j + 1) \overset{1-d_{j+1}}{\longrightarrow} (j + 2), \\
\text{max}\{i, j'\} \overset{1}{\longleftarrow} j' \overset{1-d_{j+1}}{\longleftarrow} (j + 1') \overset{d_{j+1}}{\longleftarrow} (j + 2'),
\end{array}
\]

where \(a_j = 1 - \delta_{i,j_{\bullet}}\) and \(b_j = \min\{1, (1 - \delta_{j_{\bullet}, i})d_{j_{\bullet} - 1} + \delta_{j_{\bullet}, i}\}\).

We mutate vertices \(j\) and \((j + 1)\) in order and obtain the following arrows at vertices \(j\):

\[
\begin{array}{c}
\text{max}\{i - 1, j_{\bullet} - 1\} \\
\overset{a_j}{\longrightarrow} (j - 1) \overset{b_j}{\longrightarrow} j \overset{d_{j+1}}{\longrightarrow} (j + 1) \overset{1-d_{j+1}}{\longrightarrow} (j + 2), \\
\text{max}\{i, j'\} \overset{1}{\longleftarrow} j' \overset{1-d_{j+1}}{\longleftarrow} (j + 1') \overset{d_{j+1}}{\longleftarrow} (j + 2').
\end{array}
\]

Nextly, we mutate the vertex \(j\) and compare the denominators in the exchange relation using Theorem 2.3, then

\[x[\alpha_{i,j}]x[\alpha_{j+1,j+1}] = x[\alpha_{i,j+1}] + x[\alpha_{i,\max\{i-1,j_{\bullet}+1\}}]a_j x^{b_j} x_{\max\{i,j_{\bullet}\}}^{\alpha_{j+2}} x_{j+1}^{1-d_{j+1}} x_{j+2}^{1-d_{j+1}}.\]

If there is an arrow \(j \rightarrow (j - 1)\) in \(Q_\xi\), then we reverse all the orientations and obtain the same equation. \(\square\)

**Remark 4.2.**

1. The \(q\)-character of \(\iota(x[\alpha_{i,j+1}])\) is the product of \(q\)-characters of \(\iota(x[\alpha_{i,j}])\) and \(\iota(x[\alpha_{j+1,j+1}])\) except those terms appearing in the product of \(q\)-characters of \(\iota(x[\alpha_{i,\max\{i-1,j_{\bullet}+1\}}]a_j)\), \(\iota(x^{b_j})\), \(\iota(x_{\max\{i,j_{\bullet}\}}^{\alpha_{j+2}})\), and \(\iota(x_{j+1}^{1-d_{j+1}})\).

2. Our recursive formula (4.1) is different from a recursive formula given by Brito and Chari [1, Proposition in Section 1.5], which are exchange relations corresponding to the mutation sequence \((i, i + 1, \ldots, j, j + 1)\), whereas our recursive formula needs to mutate \((i, i + 1, \ldots, j, j + 1)\) for a height function \(\xi\) such that \(j\) is source or sink.

3. It is possible by Equation (4.1) for us to give a combinatorial path formula in which we allow overlapping paths, generalizing Mukhin and Young’s path formula for snake modules [29, 30].

In practice, given an HL-module, we construct a height function \(\xi\) such that \(\iota(x[\alpha_{i,j+1}])\) is an HL-module (up to isomorphic) for some \(i < j\), and \(j, (j + 1)\) being sources or sinks,
see our construction in Theorem 3.1. Hence Equation (4.1) reduces to
\[ x[\alpha_{i,j+1}] = x[\alpha_{i,j}]x[\alpha_{j+1,j+1}] - x[\alpha_{i,\max\{i-j-1\}}]a_{i,j}x[\alpha_{j+1,j+1}] - x[\alpha_{i,\max\{i,j-1\}}]b_{j+1}x[\alpha_{j+2}]d_{j+1} \]
giving a recursive formula for the $q$-character of an HL-module by an induction on the length of its highest $\ell$-weight monomial.

We end this section with an example illustrating Theorem 4.1.

Example 4.3. Let $L(Y_1,-7Y_2,-4Y_3,-7)$ be an HL-module in type $A_4$. By Theorem 3.1, we take
\[ \xi(1) = -6, \quad \xi(2) = -5, \quad \xi(3) = -6, \quad \xi(4) = -5, \]
and we have a convention that $\xi(0) = -5, \xi(5) = -6$. The quiver $\mathbf{Q}_\xi$ is as follows.

\[
\begin{array}{cccccc}
1 & \quad & 2 & \quad & 3 & \quad & 4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1' & & 2' & & 3' & & 4'
\end{array}
\]

Take $j = 2$ being a source in $\mathbf{Q}_\xi$ and by Theorem 4.1, we have
\[ L(Y_1,-7Y_2,-4Y_3,-7) = L(Y_1,-7Y_2,-4)L(Y_3,-7) - L(Y_1,-5Y_1,-7)L(Y_4,-6). \]
The $q$-character of $L(Y_1,-7Y_2,-4Y_3,-7)$ is the product of $q$-characters of $L(Y_1,-7Y_2,-4)$ and $L(Y_3,-7)$, except those terms which are in the product of $q$-characters of $L(Y_1,-5Y_1,-7)$ and $L(Y_4,-6)$. In the product of $q$-characters of $L(Y_1,-7Y_2,-4)$ and $L(Y_3,-7)$, there are 400 monomials, among them 75 monomials are the same as the monomials in the product of $q$-characters of $L(Y_1,-5Y_1,-7)$ and $L(Y_4,-6)$.

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