The High-Temperature Expansion of the Hierarchical Ising Model: From Poincaré Symmetry to an Algebraic Algorithm

Y. Meurice

Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa 52242, USA

ABSTRACT

We show that the hierarchical model at finite volume has a symmetry group which can be decomposed into rotations and translations as the familiar Poincaré groups. Using these symmetries, we show that the intricate sums appearing in the calculation of the high-temperature expansion of the magnetic susceptibility can be performed, at least up to the fourth order, using elementary algebraic manipulations which can be implemented with a computer. These symmetries appear more clearly if we use the 2-adic fractions to label the sites. We then apply the new algebraic methods to the calculation of quantities having a random walk interpretation. In particular, we show that the probability of returning at the starting point after \( m \) steps has poles at \( D = -2, -4, ..., -2m \), where \( D \) is a free parameter playing a role similar to the dimensionality in nearest neighbor models.
1. Introduction

The hierarchical model\textsuperscript{[1]} played an important role in the development of the ideas of the renormalization group,\textsuperscript{[2]} because the renormalization group transformation for this model reduces to a simple integral equation which has been studied in great detail.\textsuperscript{[3]} In a recent paper,\textsuperscript{[4]} we calculated numerically the magnetic susceptibility of the hierarchical Ising model as a function of the temperature and for two values the parameter used in the $\epsilon$-expansion. Calculations have been carried with up to $2^{18}$ sites. We found that the numerical data can be fitted precisely with a simple power law of the form $(1 - \beta/\beta_0)^{-g}$ in the whole high-temperature region, i.e for $\beta \in [0, \beta_c]$. As a consequence, it is possible to obtain a simple approximate formula for the high-temperature coefficients of the susceptibility in terms of $g$ and $\beta_0$. This approximate formula implies approximate relations\textsuperscript{[4,5]} among the coefficients which deserve a systematic investigation. Unfortunately, due to the non-locality of the hierarchical model, the calculation of the high-temperature expansion is quite non-trivial even for the very first orders.

The goal of this article is to explain in a pedestrian way how the symmetries of the hierarchical model can be used to reduce the convoluted sums appearing in the high temperature expansion, at least up to order 4 in $\beta$, to simple algebraic manipulations which can be easily implemented with a computer. In section 2, we show that even at finite volume, the hamiltonian of the hierarchical model is invariant under a group of transformation which can be decomposed into translations and rotations as the familiar Poincaré groups.

One important reason to work at finite volume is that high-temperature coefficients of arbitrary order can be calculated with an arbitrary precision using a computer, if the number of sites is not too large. This can be done by introducing a $\beta$ expansion within the exact method of integration described in Ref. [4]. Since it is very easy to make combinatoric mistakes in calculating high-temperature coefficients, formulas valid for arbitrary volume allow unambiguous checks if we can compare them with sufficiently precise numerical results obtained with a finite
number of site. A numerical program performing this calculation is provided in Appendix 2. This program can be used to check the validity of the analytic results presented hereafter with at least 10 significant figures, when the number of sites does not exceed $2^9$. More fundamentally, the renormalization group approach requires techniques of integration at finite volume and the results obtained here can be used in this context.

In section 3, we use extensively the translation invariance to calculate the first four coefficients of the magnetic susceptibility. The final result is that each of these coefficients can be written a sequence of products and Fourier transforms. In section 4, we show that these sequences of operations can be reduced to simple algebraic manipulations which can be implemented on a computer. A ready-to-use Mathematica program is provided in Appendix 1. The methods presented here can be used for higher order calculations, however it is not clear that the calculation of coefficients of an arbitrary order can be completely reduced to simple algebraic manipulations. This issue is briefly discussed in the conclusions.

The results of section 2 and 4 could have been derived more elegantly if we had labeled the sites using 2-adic fractions as in Ref. [7]. This is explained in section 5 which, unlike section 2, assumes some familiarity with the mathematics involved in this reformulation. However, section 5 is not essential to the understanding of the other sections.

The results obtained in section 4 take a more compact form in the infinite volume limit. In section 6, we give the explicit form of each terms appearing in the calculation of the first four coefficients in this limit. These expressions have poles at unphysical values of $D$, a parameter controlling the strength of the bilinear couplings among the spins (see section 2), and playing a role similar to the dimension in nearest neighbor models. By unphysical values of $D$, we mean values such that changing the sign of one spin in one of the configuration where all the spins are aligned causes an infinite increase in energy. Incidentally, all the quantities calculated in that section have a random walk interpretation, provided that a
proper normalization is used. In general, the calculation the $m$-th coefficient of the high temperature expansion involves the calculation of the probability of returning at the starting point after $m$ steps and the probability of visiting less than two sites in $m$ steps. In the infinite volume limit, these quantities can be calculated for arbitrary $m$. Their poles are located at $D = -2, -4, ..., -2m$ and $D = -\frac{2m}{m-1}$ respectively.

2. The Hierarchical Ising Model and its Poincaré Group

In this section we describe the hierarchical Ising model and its invariance under a group of transformation which can be decomposed into translations and rotations in a way similar to the Poincaré group used in relativistic quantum field theory. Hierarchical models \[2\] are specified by a non-local hamiltonian bilinear in the spin variables and a local measure of integration which will not be specified in this section. The specific choice of an Ising measure where the spins take only the values $\pm 1$ will only be made in the next sections. We first recall the form of the non-local hamiltonian.

The hierarchical models considered here require the number of sites to be $2^n$. For convenience, we label the sites with $n$ indices $x_n.....x_1$, each index being 0 or 1. In order to give a concrete meaning to this notation, one can divide the $2^n$ sites into two boxes, each containing $2^{n-1}$ sites. If $x_n = 0$, the site is in the first box, if $x_n = 1$, the site is in the second box. Repeating this procedure $n$ times (for the two boxes, their respective two sub-boxes, etc.), we obtain an unambiguous labeling for each of the sites. In the following, we often use the symbol $x$ as a short notation for the sequence $x_n.....x_1$.

The hamiltonian of the hierarchical model reads

$$H = -\frac{1}{2} \sum_{l=1}^{n} \left( \frac{c}{4} \right)^l \sum_{x_n.....x_{l+1}} \left( \sum_{x_1.....x_{l+1}} \sigma_{(x_n.....x_1)} \right)^2 .$$  \hspace{1cm} (2.1)
The model has a free parameter $c$ for which we shall use the parametrization

$$c = 2^{1 - \frac{D}{2}}. \quad (2.2)$$

The parameter $D$ plays a role similar to the dimension in nearest neighbor models. The parameter of the epsilon-expansion can be defined as $\epsilon = 4 - D$. When $D \geq 4$, the model has a trivial continuum limit. When $D \leq 2$, the model does not have a phase transition at finite temperature. These two rigorous results can be understood heuristically in terms of the self-intersection properties of the random walk associated with $H$, by noticing that the Hausdorff dimension of this random walk is $2/D$.\(^{[8]}\)

For a given $l$ in Eq. (2.1), the first sum can be interpreted as a sum over boxes of size $2^l$, while the second sum is over the spins inside each of these boxes. Given two distinct sites $x$ and $y$, the corresponding spins have an interaction whenever $l$ is large enough to have $x$ and $y$ within the same box. It is clear that if $x$ and $y$ belong to the same box of size $2^l$, they also belong to the same boxes of larger size, until the maximal size is reached. Since the respective contributions to the interaction are $(\frac{c}{4})^l$ in Eq. (2.1), the total interaction can be expressed as a truncated geometrical series. In order to describe quantitatively this situation, we define a function $v(x, y)$ which indicates the “level” $l$ at which $x$ and $y$ start to differ. More precisely, if $x$ and $y$ are distinct, $v(x, y) = l$ when $x_m = y_m$ for all $m$ such that $n \geq m > l > 0$ and $x_l \neq y_l$. At coinciding arguments, we define $v(x, x) = 0$. The hamiltonian function $H$ can then be rewritten as

$$H = -\frac{1}{2}(\sum_{x,y} K_{xy} \sigma_x \sigma_y + L \sum_x \sigma_x^2) \quad (2.3)$$

where

$$K_{xy} = \begin{cases} \left(\frac{c}{4}\right)^{v(x,y)} - \left(\frac{c}{4}\right)^{n+1}(1 - \frac{c}{4})^{-1} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \quad (2.4)$$
and
\[ L = \left( \frac{C}{4} - \left( \frac{C}{4} \right)^{n+1} \right) \left( 1 - \frac{C}{4} \right)^{-1} \] \tag{2.5}

As made clear by the above equations, the strength of the interaction between two spins \( \sigma_x \) and \( \sigma_y \) depends only on the value of \( v(x,y) \). Consequently, the invariance of \( v(x,y) \) under a group of transformation (see Eq. (2.10)) implies the invariance of \( H \) under corresponding transformations, in a way which will be made precise at the end of this section. The invariance of \( v(x,y) \) under a group of transformation appears rather clearly in the reformulation of the hierarchical model proposed in Ref. [7], where the sites are associated with 2-adic fractions which can be transformed into each other using addition and multiplication. However, it is possible to obtain the results that will be needed later using only a few basic results in arithmetic modulo \( 2^n \). The reader familiar with the 2-adic numbers will notice that since the 2-adic integers can be defined as the projective limit of the integers modulo \( 2^n \), the finite volume presentation given here is implicit in the formulation of Ref. [7].

In order to describe the invariance of \( v(x,y) \), we shall associate with the sequence of 0’s and 1’s \( x_n \ldots x_1 \), a rational number of the form
\[ x = \sum_{m=1}^{n} x_m 2^{-m} \] \tag{2.6}

Since this defines a one-to-one correspondence, we shall use the same symbol \( x \) for the sequence of 0 and 1 and the rational number. We can now define additive and multiplicative operations which will be used to obtain a group of transformation.

If two numbers \( x \) and \( y \) have the form given in Eq. (2.6), then \( x + y \) can also be written as a rational number where the denominator is \( 2^m \) with \( m \leq n \). If we drop the integer part of \( x + y \), we obtain a rational number having the form of Eq. (2.6). Equivalently, we can write \( x = q/2^n \) and \( y = r/2^n \) with \( q \) and \( r \) integers between 0 and \( 2^n - 1 \) and add \( q \) and \( r \) modulo \( 2^n \). Since the integer modulo \( 2^n \)
form an additive group, the set of fractions associated with the sites form a group for the addition modulo 1. In order to clarify the ideas, let us give an example: the additive inverse of $2^{-n}$ modulo 1 can be written as $2^{-n} + 2^{-n+1} + \ldots + 2^{-1}$. This makes clear that the addition modulo 1 is distinct from the addition obtained by adding each of the indices $x_m + y_m$ modulo 2. Furthermore, the odd integers modulo $2^n$ form a multiplicative group. We can pick a canonical form for the representatives of such integers as

$$u = 1 + 2z$$

(2.7)

where $z$ is a positive integer between 0 and $2^{n-1} - 1$. Obviously, if $x$ has the form (2.6), then $ux$ has also the form (2.6) after discarding its integer part.

We are now in position to define a group of transformation acting on the fractions associated with the sites. If $x$ and $a$ have the form of Eq. (2.6) and $u$ has the form of Eq. (2.7), we define a transformation of $x$ depending on $a$ and $u$ and denoted $x[u, a]$ which reads

$$x[u, a] = ux + a,$$

(2.8)

where the r.h.s is understood modulo 1. It is clear that $u = 1, a = 0$ gives the identity transformation. From the composition law $(x[u, a])[v, b] = x[uv, va + b]$, one sees that an inverse transformation is obtained for $v = u^{-1}$ and $b = -u^{-1}a$. Consequently these transformations form a (non-abelian) group.

We can interpret $x[0, a]$ as a translation and $x[u, 0]$ as a rotation like in the usual Poincaré groups. In that sense, this is a “global” group of transformation. This is in contrast with the symmetries noted by Dyson [1] which consists in interchanging $x_m \ldots x_{l+1} 1 x_{l-1} \ldots x_1$ and $x_m \ldots x_{l+1} 0 x_{l-1} \ldots x_1$ “locally”.

In the following, we shall frequently use the fact that $x[u, a]$ with $u$ and $a$ fixed is a one-to-one map. The proof of this statement is simple. If $ux + a = uy + a$ modulo 1, then $u(x - y) = 0$ modulo 1, writing $x - y$ as $q/2^n$, we obtain that $q = 0$ modulo $2^n$ and consequently, $x = y$ modulo 1.
We can now make a connection between \( v(x, y) \) and \( x - y \) through the following result:

\[
v(x, y) = l \iff x - y = q/2^l \mod 1 \text{ for some odd integer } q \tag{2.9}
\]

In order to prove this, we write \( x = r/2^n \) and \( y = s/2^n \) with \( r \) and \( s \) integers between 0 and \( 2^n - 1 \) and we see that r.h.s of the equivalence relation means that \( r \) and \( s \) are equal \( \mod 2^{n-l} \) but differ \( \mod 2^{n-l+1} \). Returning to the canonical form of Eq. (2.6), we see that is equivalent to \( v(x, y) = l \). Noticing that \( x - y \) is translation invariant and that the product of odd integers is still an odd integer, we obtain the invariance of \( v(x, y) \) under the transformations (2.8) in the following way:

\[
v(x[u, a], y[u, a]) = v(x, y) \tag{2.10}
\]

Since \( K_{xy} \) depends only on \( v(x, y) \), this immediately implies that

\[
K_{xy} = K_{x[u, a]y[u, a]} \tag{2.11}
\]

It is time to state the main result of this section: The hamiltonian \( H \) defined by Eq. (2.1) is invariant under the transformation

\[
\sigma_x \longrightarrow \sigma_{x[u, a]} \tag{2.12}
\]

for any \( a \) as in Eq. (2.6) and any \( u \) as in Eq. (2.7).

This follows from the fact that \( x[u, a] \) defines one-to-one map, consequently we can rewrite the transformed hamiltonian in the following way

\[
\sum_{x,y} K_{xy} \sigma_{x[u, a]} \sigma_{y[u, a]} = \sum_{x,y} K_{x[u^{-1},-u^{-1}a]y[u^{-1},-u^{-1}a]} \sigma_x \sigma_y \tag{2.13}
\]

and then use the invariance of \( K_{xy} \) shown in Eq. (2.11).
As is well-known, the invariance of $H$ under a group of transformations implies identities for the correlation functions. In order to define these, we need to introduce a local measure of integration. In the following sections, we shall consider the case of an Ising measure and define

$$Z = \sum_{\{\sigma = \pm 1\}} e^{-\beta H} \tag{2.14}$$

and

$$< \sigma_x \sigma_y > = Z^{-1} \sum_{\{\sigma = \pm 1\}} \sigma_x \sigma_y e^{-\beta H} \tag{2.15}$$

We can then prove that

$$< \sigma_{x[u,a]} \sigma_{y[u,a]} > = < \sigma_x \sigma_y > \tag{2.16}$$

This follows from the fact that we can always relabel the integration variables according to Eq. (2.12) and then use the invariance of $H$. This argument is clearly independent of the choice of the local measure.

3. The High-Temperature Expansion of the Hierarchical Ising Model

We now proceed to the calculation of the magnetic susceptibility per site for the hierarchical Ising model. We define the magnetic susceptibility per site as

$$\chi_n(\beta) = \frac{1}{2^n} < (\sum_x \sigma_x)^2 > \tag{3.1}$$

Using translation invariance, we find that

$$\chi_n(\beta) = < \sigma_0(\sum_x \sigma_x) >, \tag{3.2}$$

the choice of 0 being arbitrary. In the case of the Ising model we can use the fact
that $\sigma_0^2 = 1$ and write

$$
\chi_n(\beta) = 1 + \sum_{x : x \neq 0} < \sigma_0 \sigma_x > ,
$$

(3.3)

Using the high-temperature expansion of $< \sigma_0 \sigma_x >$, we can calculate the coefficients of the expansion

$$
\chi_n(\beta) = 1 + b_{1,n} \beta + b_{2,n} \beta^2 + ... 
$$

(3.4)

A brief review of the standard methods used to perform such a calculation with arbitrary long-range Ising models is given at the beginning of section II of Ref. [6]. For the sake of briefness, we shall only recall the final result: $< \sigma_0 \sigma_x >$ can be written as the ratio of two expansions in $tanh(\beta K_{yz})$. The terms of each of them correspond to graphs where a given link $y - z$ can only appear once (in which case we have a factor $tanh(\beta K_{yz})$) or zero times (in which case we have a factor 1). At the denominator, all the sites must be visited an even number of times. At the numerator, all the sites must be visited an even number of times except for 0 and $x$ which are visited an odd number of times. This provide an expansion in $tanh(\beta K_{yz})$ which will require further expansion in order to be recast in the form (3.4). It has to be noted that for hierarchial models, the use of Feynman diagrams in the recursion formula can be reduced to a purely combinatoric problem (no sums or integrals), while the high-temperature expansion requires the evaluation of nested sums - a situation which somehow the opposite of what we encounter for nearest neighbor models. We now proceed order by order to the calculation of the $b_{m,n}$ for $m = 1, 2, 3, 4$ and $n$ arbitrary.

The first order contributions to $< \sigma_0 \sigma_x >$ comes from high-temperature graphs with one link joining 0 to $x$. The contribution of these graphs is

$$
\sum_x tanh(\beta K_{0x}) .
$$

(3.5)

Note that $K_{xy}$ is zero when $x = y$ and we do not need to subtract the contribution
with \( x = 0 \). We thus obtain

\[
\sum_{x} K_{0x} \cdot \quad (3.6)
\]

Proceeding similarly with graphs with two links, we obtain

\[
\sum_{y, x \neq 0} \quad K_{0y} K_{yx} \quad (3.7)
\]

At order three, we need to consider the \( \beta^3 \) terms of Eq. (3.5) in addition to the lowest order contributions from the graphs with three links. The result is

\[
\sum_{y, z, w, x \text{ all distinct}} K_{0y} K_{yz} K_{zw} K_{wx} - \frac{1}{3} \sum_x (K_{0x})^3 \quad (3.8)
\]

Up to now, we did not have to take into account the contributions from the denominator because the first non-trivial contributions to \( Z \) come from triangles which at lowest order give order \( \beta^3 \) terms. Since \( \langle \sigma_0 \sigma_x \rangle \) yields contributions which are at least of order \( \beta \) when \( x \neq 0 \), the denominator has to be taken into account at order 4. These contributions cancel the lowest order of the graphs with a triangle at one or the other end and all which is left are minus the graphs where the link \( 0 - x \) is also occupied by one side of a triangle (since such a graphs do not appear at the numerator). Adding to this the lowest order contribution from the graphs joining 5 distinct sites and the order \( \beta^4 \) of the graphs with two links, we obtain

\[
\sum_{y, z, w, x \text{ all distinct}} K_{0y} K_{yz} K_{zw} K_{wx} - \frac{1}{3} \sum_x (K_{0x})^3
\]

\[
- \frac{1}{3} \sum_{x, y} (K_{0y})^3 K_{yx} - \frac{1}{3} \sum_{x, y} K_{0y} (K_{yx})^3 + \frac{2}{3} \sum_x (K_{0x})^4 \quad (3.9)
\]

Note that we have taken into account that \( K_{xy} = K_{yx} \).
From the translation invariance of $K_{xy}$, it is clear that $K_{xy}$ is a function of $x - y$ only. Consequently, we can write
\[
K_{xy} = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{K}(k)e^{-i2\pi k(x-y)}.
\] (3.10)

The explicit form of $\hat{K}(k)$ will be given in Eq. (4.12). Our task in now to rewrite the expressions for the $b_{m,n}$ obtained in the previous section as unrestricted sums over $y$, $z$, ...$x$ with appropriate compensating terms, plug the expansion (4.1) in them and use the well-known results
\[
\sum_{z} K_{xz} G_{zy} = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{K}(k)\hat{G}(k)e^{-i2\pi k(x-y)}.
\] (3.11)

and
\[
\frac{1}{2^n} \sum_{q=0}^{2^n-1} e^{-i\frac{2\pi}{2^n}kq} = \delta_{k,0}.
\] (3.12)

Proceeding this way, we obtain
\[
b_{1,n} = \hat{K}(0)
\] (3.13)

\[
b_{2,n} = (\hat{K}(0))^2 - \hat{K}^2(0)
\] (3.14)

\[
b_{3,n} = (\hat{K}(0))^3 - 2\hat{K}(0)\hat{K}^2(0) + \frac{2}{3}\hat{K}^3(0) - (\hat{K}^3)(0)
\] (3.15)

\[
b_{4,n} = (\hat{K}(0))^4 + \frac{4}{3}\hat{K}(0)\hat{K}^3(0) - \frac{4}{3}\hat{K}^4(0) + 2(\hat{K}^2\hat{K}^2)(0)
- 3\hat{K}^2(0)(\hat{K}(0))^2 + 3(\hat{K}^2(0))^2 - \hat{K}^4(0) - 2\hat{K}(0)((\hat{K})^3)(0)
\] (3.16)

Note that expressions of the same form but with different coefficients would be
obtained if instead of using a Ising measure, we had an arbitrary measure. This can be seen in general by using the random path representation.\textsuperscript{[11]}

4. Explicit Expressions for the High-Temperature Coefficients

Our next task consists in calculating explicitly the above expressions. For this purpose, we first introduce the function $N(k)$ defined over the integers \textit{modulo} $2^n$:

\[
N(k) = \begin{cases} 
2^{-l} & \text{if } k \text{ can be divided by } 2^l \text{ but not by } 2^{l+1} \\
2^{-n} & \text{if } k = 0 \text{ modulo } 2^n
\end{cases}
\]  
(4.1)

This function will appear in $\hat{K}(k)$. We now introduce short notations for some functions of $x$

\[
2^{-sv(x, 0)} \longrightarrow \mathcal{X}_s
\]

\[
\delta_{x, 0} \longrightarrow \delta
\]  
(4.2)

and for some functions of $k$

\[
(N(k))^{2^s} \longrightarrow \mathcal{K}_s
\]

\[
\delta_{k0} \longrightarrow \Delta
\]  
(4.3)

In addition, we also define

\[
P(1 + a) = \frac{1}{1 - 2^{-(1+a)}}
\]  
(4.4)

\[
G(1 + a) = \frac{1 - 2^a}{1 - 2^{-(1+a)}}
\]  
(4.5)

\[
Q(1 + a) = 2^{-(n+1)(a+1)}
\]  
(4.6)
With these notations,

\[ \mathbf{K}_{x0} \rightarrow P(1 + \frac{2}{D})(X_{1+\frac{x}{2}} - Q(1 + \frac{2}{D})X_0 - (1 - Q(1 + \frac{2}{D}))\delta) \]  

(4.7)

It has to be noted that \( Q(1 + a) \) vanishes in the infinite volume limit. Taking this limit simplifies the calculation, but obviously prevents us from checking it using exact results at finite volume.

We can now manipulate the symbols introduced above by just using their multiplication table and the rules to perform their Fourier transform. The multiplications appearing in Eqs. (3.13-16) can be performed as a multiplication denoted \( \ast \) (respectively \( \ast \) in Fourier transform) in an abstract associative algebra where the basis of the underlying vector space is given by the symbols introduced in Eq. (4.2) (resp. Eq. (4.3)). Since in these two equations \( s \) can be understood as a continuous index, this vector space is infinite dimensional. However, for calculations at finite order, only finite dimensional subspaces are used. The multiplication table for the \( X_s \) and \( \delta \) is

\[
\begin{align*}
X_s \ast X_t &= X_{s+t} \\
X_s \ast \delta &= \delta \\
\delta \ast \delta &= \delta 
\end{align*}
\]  

(4.8)

Similarly for \( K_s \) and \( \Delta \), we have

\[
\begin{align*}
K_s \ast K_t &= K_{s+t} \\
K_s \ast \Delta &= 2^{-ns} \Delta \\
\Delta \ast \Delta &= \Delta 
\end{align*}
\]  

(4.9)

A straightforward but tedious calculation yields the Fourier transforms

\[
\begin{align*}
\hat{X}_s &= (G(s))^{-1}(K_{s-1}\frac{D}{2} - 2^{s-1}K_0) + 2^nQ(s)\Delta \\
\hat{X}_0 &= 2^n\Delta \\
\hat{\delta} &= 1 
\end{align*}
\]  

(4.10)
and their inverse

\[ \hat{\mathcal{K}}_a = G(1 + a)(X_{1+a} - Q(1 + a)X_0) + 2^a \delta \]
\[ \hat{\mathcal{K}}_0 = \delta \]
\[ \hat{\Delta} = 2^{-n}X_0 \]  \hspace{1cm} (4.11)

The basic ingredient in this calculation is the identity (3.12). However, a more organized calculation can be performed using general results given in Ref. [9] (see next section). With these results, one can calculate \( \hat{\mathbf{K}}(k) \) appearing in Eq. (3.10)

\[ \hat{\mathbf{K}}(k) = P(1 + \frac{1}{D})(G(1 + \frac{1}{D}))^{-1} \cdot \]
\[ (\mathcal{K}_1 + (-\frac{1}{2}P(1 + \frac{1}{D}) + Q(1 + \frac{1}{D})G(1 + \frac{1}{D}))\mathcal{K}_0) . \]  \hspace{1cm} (4.12)

This results can also be used for Feynman diagram calculations.

Finally, we need to evaluate all these functions when their argument is 0. This again can be accomplished in a purely algebraic fashion, by multiplying by \( \Delta \) (resp. \( \delta \)) when an odd (resp. even) number of Fourier transform has been performed and retaining the coefficient of \( \Delta \) (resp. \( \delta \)).

Using these results we find (recalling the definition of \( c \) given in (2.2))

\[ b_{1,n} = (1 - \frac{c}{4})^{-1}(\frac{c}{4}(1 - (\frac{c}{2})^n)(1 - \frac{c}{2})^{-1} - (2^n - 1)(\frac{c}{4})^{n+1}) \]  \hspace{1cm} (4.12)

and

\[ b_{2,n} = (b_{1,n})^2 - (1 - \frac{c}{4})^{-2}[(\frac{c}{4})^2(1 - (\frac{c}{2})^n)(1 - \frac{c^2}{8})^{-1}
- 2(\frac{c}{4})^{n+2}(1 - (\frac{c}{2})^n)(1 - \frac{c}{2})^{-1} - (2^n - 1)(\frac{c}{4})^{2(n+1)}] \]  \hspace{1cm} (4.13)

The next two coefficients are more involved and writing them explicitly might not be the most efficient way to communicate the information to the reader. Instead, we give in Appendix 1 a Mathematica program which calculates the first
four coefficients and allows further symbolic or numerical manipulations. This program has been tested in various ways. The most convincing check is the precise comparison between the numerical values obtained by replacing $D$ and $n$ in the expression of the coefficients and the exact values obtained by introducing a high temperature expansion in the recursive integration used in Ref. [4]. A program performing this calculation is given in Appendix 2.

Before closing this section, let us recall that in order to obtain a physically interesting model,[1] it is essential to keep $b_{1,n}$ finite in the limit where $n$ becomes infinite. Otherwise, flipping one spin in one of the configuration where all the spins are aligned causes an infinite change in energy. This requirement imposes $|c| < 2$. If in addition we require ferromagnetic couplings, we see from Eq. (2.4) that $c$ must be real, strictly positive and less than four. Combining the two requirements yields $0 < c < 2$ which incidently corresponds to $D$ real, strictly positive and finite.

5. Remarks Concerning the 2-adic Formulation of the Hierarchical Model

Some of the results presented in the previous sections can derived more elegantly if we had started immediately with the reformulation of Ref. [7]. In this section, we briefly explain how this can be done. Unlike the previous sections, this one assumes some familiarity with the $p$-adic numbers.

The function $2^v(x,y)$ introduced in section 2, is a regularized version of the 2-adic distance. Namely,

$$2^v(x,y) = \begin{cases} 
|x - y|_2 & \text{if } |x - y|_2 > 1 \\
1 & \text{if } |x - y|_2 \leq 1
\end{cases} \quad (5.1)$$

Using this, the invariance of $v(x, y)$ given in Eq. (2.10) follows easily.

Similarly, $N(k)$ introduced in section 4, is a regularized version of the 2-adic
norm restricted to the 2-adic integers.

\[ N(k) = \begin{cases} |k|_2 \text{ if } 1 \geq |k|_2 > 2^{-n} \\ 2^{-n} \text{ if } |k|_2 \leq 2^{-n} \end{cases} \quad (5.2) \]

Given this, one can use the results of Ref. [9] to calculate the Fourier transforms. In particular, the function \( G(1 + a) \) is the gamma function associated to a set of multiplicative characters.

6. The Poles in the \( c \)-Complex Plane, in the Infinite Volume Limit

In the infinite volume limit, the expressions for the coefficients simplify significantly. This limit will always be understood in this section and we shall omit the reference to \( n \). In taking this limit, we assume \( 0 < c < 2 \). For instance, for the first coefficient, we obtain

\[ b_1 = \sum_x K_{0x} = \hat{K}(0) = \frac{2c}{(4-c)(2-c)} \quad (6.1) \]

From Eq. (2.4), it is obvious that for any \( m \), \( b_m \) will have a zero of order \( m \) at \( c = 0 \) and a pole of order \( m \) at \( c = 4 \). Also, from the procedure used to calculate the coefficients, one can see that \( b_m \) will have a \( (b_1)^m \) term (which would be the only contribution in the gaussian case). This term has a pole of order \( m \) at \( c = 2 \). In order to get rid of these zeroes and poles, we shall consider \( b_m/(b_1)^m \) in the calculations below. In addition, with this normalization, the quantities calculated have a random walk interpretation when \( 0 < c < 2 \). This is due to the fact that \( K_{xy}/\sum_z K_{0z} \) can then be seen as the probability for going from \( x \) to \( y \) in one step.

A quantity which is easy to calculate is the probability for visiting no more than two sites, including the starting point, after \( m \) step. This quantity reads

\[ \sum_x (K_{0x})^m/(\sum_z K_{0z})^m \]

and appears in the calculation of \( b_m/(b_1)^m \). After reexpressing it in terms of the Fourier transform and using the techniques developed
in section 4, we obtain

\[
\frac{\hat{\mathbf{K}}^m(0)}{(\hat{\mathbf{K}}(0))^m} = \frac{(2 - c)^m}{2 - c^m} \quad (6.2)
\]

Proceeding similarly for the other quantities entering in \(b_3/(b_1)^3\) and \(b_4/(b_1)^4\), we obtain

\[
\frac{\hat{\mathbf{K}}^3(0)}{(\hat{\mathbf{K}}(0))^3} = \frac{6 (-2 + c)^3 c^2}{(8 - c^2) (-16 + c^3)} \quad (6.3)
\]

\[
\frac{\hat{\mathbf{K}}^4(0)}{(\hat{\mathbf{K}}(0))^4} = \frac{(-2 + c)^4(-256 - 160 c^2 - 14 c^5 + 16 c^3)}{(-8 + c^2) (-16 + c^3) (-32 + c^4)} \quad (6.4)
\]

and

\[
\frac{(\hat{\mathbf{K}}^2\hat{\mathbf{K}}^2)(0)}{\hat{\mathbf{K}}^4(0)} = \frac{(-2 + c)^4 (-128 c^2 - 64 c^3 + 12 c^5)}{(-8 + c^2)^2 (8 + c^2) (-32 + c^3)} \quad (6.5)
\]

Note that Eqs. (6.3) and (6.4) can be generalized to the calculation of the probability for a return at the starting point after \(m\) steps. A detailed calculation shows that this quantity can be expressed as

\[
\frac{\hat{\mathbf{K}}^m(0)}{(\hat{\mathbf{K}}(0))^m} = \sum_{l=0}^{m} (-1)^l \binom{m}{l} \frac{(4 - c)^l}{2^{l+1} - c^l} \quad (6.6)
\]

This shows that unless unexpected cancelation occurs, \(b_m\) will have poles at \(c = 2^{1+1}, 2^{1+1/2}, \ldots, 2^{1+1/m}\) or in other words at \(D = -2, -4, \ldots, -2m\). On the other hand, the quantity appearing in Eq. (6.6) has a zero of order \(m\) at \(c = 2\). This is not obvious from the r.h.s, however it can be seen by expanding the denominator about 2 and summing over \(l\), that this is the case for each term of the resulting expression. The location of these poles plays an important role if one tries to use the \(1/D\)-expansion to calculate the corrections to the gaussian contribution.
7. Conclusions

We have shown that the symmetries of the hierarchical model can used extensively to calculate the high temperature expansion of the magnetic susceptibility. Up to order 4, the calculation can be performed using algebraic methods which can be implemented by a computer program. Proceeding this way, we obtain analytical expressions where $D$ and $n$ are arbitrary. Substituting numerical values into these formulas provides very precise numerical comparisons with other numerical methods. The results of the numerical methods at low $n$ are very reliable and can be used to test the analytical results. If the analytical results pass this test, they can provide very accurate results at large $n$ and, in turn, provide a test for the numerical stability of the numerical method.

The simplicity of the calculations presented here relies on the fact that at the end, the arguments of the functions are set to zero. It will not be necessarily the case for higher order calculations and it is conceivable that some complicated sums will need to be performed numerically. Nevertheless, it is clear that the methods proposed here provide a drastic simplification when compared to a straightforward evaluation of the sums as they appear in Eqs. (3.6-9).

In the infinite volume limit, one obtains more compact formulas. The coefficients can be written as ratios of polynomials in $c$. The location of the poles at each order seems to obey some regularity. For the contributions which are calculable for arbitrary order, we see that when the order increases, some of the poles get closer to $c = 2$ and $c = 4$. This fact has to be kept in mind if one attempts to calculate the corrections to the gaussian approximation using a $1/D$ expansion.
APPENDIX 1: Calculation of the $b_{n,m}$

This is a Mathematica program which calculates the first four coefficients. The explanations between (* ...*) play no role. If this program is stored in a file named `bees.math` for instance, then it can introduced in a Mathematica session with the command `<< bees.math`. When all the instructions are completed (it takes a few minutes), you will see `completed`. You can then use $b_3$ for $b_3$ etc...

In order to check numerical values with the program given in Appendix 2, you can type for instance $N[[b_1, b_2, b_3, b_4]/.\{n \rightarrow 5, d \rightarrow 3\}, 10]$, you should then obtain $\{1.075127131, 0.8953433348, 0.6726561277, 0.4555210038\}$. This program is appended to the tex file.

APPENDIX 2: Checking the Previous Calculation at Finite Volume

This is a Mathematica program which calculates numerically the coefficients up to a given order for $2^n$ sites. It is convenient to take $n < 8$. The numerical values of $n$ and $D = d$ can be changed. The choice of values below, allows to check the numbers mentioned in Appendix 1. After entering the program in a Mathematica session, you get (after a few minutes) $1. + 1.075127131B + 0.8953433348B^2 + 0.6726561277B^3 + 0.4555210038B^4$. This program is appended to the tex file.

Acknowledgements: I would like to thank V.G.J. Rodgers for suggestions concerning the computer program mentioned at the end of the article and G. Ordaz for providing independent checks of the results presented in section 3 and 4. This work has been completed during my stay at Brookhaven National Laboratory, I would like to thank the theory group for its hospitality.
REFERENCES

1. F. Dyson, *Comm. Math. Phys.* **12** (1969) 91 ; G. Baker, *Phys. Rev. B* **5** (1972) 2622 .

2. K. Wilson, *Phys. Rev. B* **4** (1971) 3185 ; K. Wilson and J. Kogut, *Phys. Rep.* **12** (1974) 75 .

3. P. Bleher and Y. Sinai, *Comm. Math. Phys.* **45** (1975) 247 ; P. Collet and J. P. Eckmann, *Comm. Math. Phys.* **55** (1977) 67 and *Lecture Notes in Physics* **74** (1978) ; K. Gawedzki and A. Kupiainen, Les Houches 1985, K. Osterwalder and R. Stora, Editors .

4. Y. Meurice, G. Ordaz and V.G.J. Rodgers, *Jour. Stat. Phys.* **77** (1994) 0000 (in press)

5. Y. Meurice, G. Ordaz, U. of Iowa Preprint 94-15

6. Y. Meurice, *Jour. Math. Phys.* **35** (1994) 769 .

7. E. Lerner and M. Missarov, *Theor. Math. Phys.* **78** (1989) 177 .

8. J.L. Lucio and Y. Meurice, *Mod. Phys. Lett.* **6** (1991) 1199 ; Y. Meurice, *Phys. Lett.* **265B** (1991) 377 .

9. M. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press.

10. J.P. Serre, *A Course in Arithmetics*, (Springer, New York, 1973).

11. K. Symanzik, in *Local Quantum Theory*, R. Jost Editor, Academic Press, New York, 1969 ; D. Brydges, J. Fröhlich and T. Spencer, *Comm. Math. Phys.* **83** (1982) 123 ; C. Itzykson and J.M Drouffe, *Statistical Field Theory*, Cambridge University Press, Cambridge, 1989.