Quantum Field Theory
on the Noncommutative Plane with $E_q(2)$ Symmetry

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Abstract

We study properties of a scalar quantum field theory on the two-dimensional noncommutative plane with $E_q(2)$ quantum symmetry. We start from the consideration of a firstly quantized quantum particle on the noncommutative plane. Then we define quantum fields depending on noncommutative coordinates and construct a field theoretical action using the $E_q(2)$-invariant measure on the noncommutative plane. With the help of the partial wave decomposition we show that this quantum field theory can be considered as a second quantization of the particle theory on the noncommutative plane and that this field theory has (contrary to the common belief) even more severe ultraviolet divergences than its counterpart on the usual commutative plane. Finally, we introduce the symmetry transformations of physical states on noncommutative spaces and discuss them in detail for the case of the $E_q(2)$ quantum group.

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1 Introduction

It is generally believed that the picture of space-time as a manifold $\mathcal{M}$ should break down at very short distances of the order of the Planck length. One possible approach to the description of physical phenomena at small distances is based on noncommutative geometry of space-time. There have been investigations in the context of Connes’ approach \cite{1} to gravity and the Standard Model of electroweak and strong interactions \cite{2,3} and in the framework of the string theory \cite{4}. Another approach starting from study of a relation between measurements at very small distances and black hole formations has been developed in the pioneering works \cite{5}. One more possibility is based on Quantum Group theory (see, e.g., \cite{6}).

The essence of the noncommutative geometry consists in reformulating first the geometry in terms of commutative algebras and modules of smooth functions, and then generalizing them to their noncommutative analogs. If the notions of the noncommutative geometry are used directly for the description of the space-time, the notion of points as elementary geometrical entity is lost and one may expect that an ultraviolet cutoff appears.

As is well known from the standard quantum mechanics, a quantization of any compact space, in particular a sphere, leads to finite-dimensional representations of the corresponding operators, so that in this case any calculation is reduced to manipulations with finite-dimensional matrices and thus there is simply no place for UV-divergences (see \cite{7,8,9} and refs. therein). Things are not so easy in the case of noncompact manifolds. The quantization leads to infinite-dimensional representations and we have no guarantee that noncommutativity of the space-time coordinates removes UV-divergences. In our preceding paper \cite{10} we have shown that ultraviolet behaviour of a field theory on a noncommutative space-time is sensitive to the topology of the space-time, namely to its compactness. We considered theories on a two-dimensional plane with Heisenberg-like commutation relations among coordinates (see also \cite{5,11}) and on a noncommutative cylinder. While the former retains the divergent tadpoles (as an ordinary QFT), the latter proves to be UV-finite. We argued that the underlying reason for such a UV-behaviour of the models is related to the properties of the complete coordinate-momentum quantum mechanical algebra and to the fact that the momenta degrees of freedom are associated to the fully noncompact Heisenberg-Weyl group manifold in the first case and to the cylinder in the second case (the cylinder has one compact dimension).

Using these qualitative arguments, we supposed that the quantum field theory constructed on the $q$-deformed plane \cite{12,13,14} with $E_q(2)$-symmetry also has UV-divergences. We have proved that indeed there are no kinematical reasons for this model to be UV-finite: the Green function of the free theory on the $q$-plane is singular. Moreover, we have shown that the interaction with an external field does produce divergent tadpole. However, in the paper \cite{10} we used decomposition of the fields on the $q$-plane in the so-called distorted plane waves ($q$-deformed exponential functions). This makes difficult matching the $q$-deformed field theory with the corresponding firstly quantized quantum mechanics of particles on the $q$-deformed plane and due to the absence of the additivity property for the $q$-exponentials, makes an explicit calculation of nontrivial (e.g., $\varphi^4$-) vertices impossible. Thus the results of \cite{10} have left open the possibility that the complete interacting theory on the $q$-plane is UV-finite because of (dynamical) properties of the corresponding $\varphi^4$-vertices.
In this paper we use another decomposition of the fields, namely, the decomposition in partial waves, similar to the recently proposed “spherical field theory” on commutative spaces. This decomposition together with the Haar \( (E_q(2))-\text{invariant} \) measure and \( q \)-deformed integral used for the definition of the field theoretical action, allows to present the field theory on the \( q \)-deformed plane as a lattice theory of infinite number of interacting one-dimensional fields (partial waves). The resulting field theoretical degrees of freedom are in transparent correspondence with the spectrum of operators in the firstly quantized version of the model. The calculation of the tadpole with the account of the \( \phi^4 \)-vertex shows that UV-properties of the theory on the \( q \)-deformed plane are even worse than those on the ordinary commutative plane. This fact confirms the conclusion of the paper \([10]\) that the very transition to the noncommutative space-times does not guarantee UV-finiteness.

The example of the plane with the most simple and natural Heisenberg-like commutation relations among coordinates was used in \([10]\) also for study of symmetry transformations of noncommutative space-times with Lie algebra commutation relations for coordinates. The noncommutative coordinates prove to be tensor operators, and we considered concrete examples of the corresponding transformations of localized states (analog of space-time point transformations). In this paper, we extend this consideration to the much more involved case of quantum group coaction on noncommutative space-times. More precisely, we derive the rules of transformations of particle states induced by the coaction of a quantum group.

The paper is organized as follows. In section 2 we consider firstly quantized theory of particles on the \( q \)-deformed plane \( P_q^{(2)} \) with \( E_q(2) \)-symmetry. We derive representations of the algebras of coordinates and momenta on the \( q \)-plane and find spectra of the relevant operators. In section 3 the field theory (second quantization) on \( P_q^{(2)} \) is introduced and presented in the form of infinite number of interacting partial waves defined on a one-dimensional lattice, the partial wave at the sites of the lattice (interpreted as creation and annihilation operators) being in one-to-one correspondence with spectra of the quantum mechanical operators found in section 2. Calculation of a tadpole diagram shows that the model has even more severe UV-divergences than the standard two-dimensional scalar \( \phi^4 \)-theory. In section 4 we are interested in transformation properties of a system on \( P_q^{(2)} \) under the coaction of the quantum group \( E_q(2) \). The point is that now the coordinates \( \bar{z}, z \) are noncommuting operators and \( E_q(2) \) provides only existence of coaction, \( i.e., \) homomorphism of the algebra of functions on \( P_q^{(2)} \) into the direct product \( E_q(2) \otimes P_q^{(2)} \) of algebras of functions on the quantum group and plane. Then the question is: how does this coaction influence states of a quantum system on \( P_q^{(2)} \)? In other words, if a system is in some state \( \psi \) (say, with a definite value of one of the coordinate operators, \( z \) or \( \bar{z} \) or some their combination) we are interested in determination of the state after the \( E_q(2) \)-group coaction. In subsection 4.1 we clarify a general formulation of this problem and then (in subsection 4.2) give an explicit answer for the \( E_q(2) \) group. Section 5 is devoted to the summary of the results.
Quantum mechanics on the noncommutative plane with quantum $E_q(2)$ group symmetry

In this paper we consider Quantum Mechanics induced by a quantum group structure. Recall that in the case of ordinary Lie group $G$, the group structure defines a unique symplectic structure on the cotangent bundle $T^*_G$ to the group manifold $G$ (see, e.g., [16]) and, hence, the corresponding canonical quantization (via substitution of Poisson brackets by the corresponding commutators). A similar construction with necessary generalizations, can be carried out for Lie-Poisson groups, which after the quantization procedure become quantum groups (see, e.g., review in [6] and refs. therein).

In fact, the quantization of a system on a Lie group cotangent bundle $T^*_G$ corresponds to choice of the group manifold as a configuration space (i.e., group parameters as space coordinates) and left- (or right-)invariant vector fields on $G$ (elements of the corresponding Lie algebra) as quantum mechanical momenta. Instead of using a whole group $G$, one can start from some of its coset (homogeneous) space $G/H$, where $H$ is a subgroup $H \subset G$. In this approach the basic problem of Quantum Mechanics, i.e., determination of possible representations of canonical operators, is reduced to mathematically well-developed problem of construction of regular representation (or quasi-regular, if one deals with a homogeneous space) and its decomposition into irreducible parts (see, e.g., [17]). In some particular cases this general construction becomes rather simple and quite familiar from elementary course on Quantum Mechanics. For example, let us consider a two-dimensional Euclidean group $E(2) = U(1) \rtimes T_2$ containing rotations and translations of a two-dimensional plane. Its homogeneous space $P^{(2)} = E(2)/U(1)$ is the Euclidean plane with the metric $\eta_{ij} = \text{diag} \{+1, +1\}$ which is invariant with respect to $E(2)$-transformations. This configuration space is parameterized by two coordinates $x_1, x_2$, while left-invariant fields tangent to this homogeneous space are nothing but usual derivatives which up to the factor $-i\hbar$ correspond to the standard momentum operators. As is well-known, any representation of the algebra of coordinates and left-invariant vector fields on $P^{(2)}$ is unitary equivalent to this representation by the coordinate functions and derivatives in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2)$ of square integrable functions. States $\psi(x) \in \mathcal{H}$ are transformed according to representations of $E(2)$ in the Hilbert space $\mathcal{H}$. Since the coordinate operators are commuting, their eigenvalues are transformed under an action of $E(2)$-group as their classical counterparts: in the convenient complex notation

$$z = x_1 + ix_2 \ , \ \bar{z} = x_1 - ix_2 \ ,$$

the $E(2)$ transformations read as

$$z \rightarrow z' = vz + t$$

$$\bar{z} \rightarrow \bar{z}' = \bar{v}\bar{z} + \bar{t}$$

where $v, \bar{v}$ subjected to the constraint $\bar{v}v = 1$, define two-dimensional rotation group $U(1)$ and $t = t_1 + it_2, \bar{t} = t_1 - it_2$ parameterize translations.

For the simple case of the quantum mechanical systems on the Euclidean plane the underlying mathematics related to cotangent bundle structures, regular representations etc. seems to be redundant. But for generalizations to more complicated homogeneous
spaces, in particular, with non-zero curvature and nontrivial topology, the group theoretical methods become quite actual and powerful.

In this work we are going to study another generalization: instead of starting from ordinary $E(2)$, we shall use its quantum version $E_q(2)$ [12, 13, 14]. Though in the case of quantum groups and corresponding quantum homogeneous spaces (definition of the latter see, e.g., [14]) group parameters (coordinates) become noncommutative, the general scheme of quantization still can be applied. The role of momentum operators is now attributed the $q$-deformed left- (or right-) invariant generalizations of vector fields (see, e.g., [18]). Thus the Planck constant $\hbar$ enters, as usual, the commutation relations (CR) for momenta and coordinates, while the group deformation parameter $q$ governs nontrivial coordinate-coordinate and momentum-momentum CR. Therefore, first of all we have to construct possible representations of this combined $q$-deformed algebra of noncommuting coordinates and momenta. For the particular case which we consider in this paper ($q$-deformed quantum Euclidean plane $P_q^{(2)}$) this is not a very complicated problem and we shall consider it in this section.

We start from the quantum group $E_q(2)$ generated by elements $\bar{v}, v, \bar{t}, t$ with the defining relations [12]

\[
\bar{v}v = v\bar{v} = 1, \quad t\bar{t} = q^2\bar{t}t, \\
vt = q^2tv, \quad \bar{v}t = q^{-2}t\bar{v}, \\
q \in \mathbb{R}.
\]  

Other commutation relations follow from the involution: $v^\dagger = \bar{v}, t^\dagger = \bar{t}$. The comultiplication has the form

\[
\Delta v = v \otimes v, \quad \Delta \bar{v} = \bar{v} \otimes \bar{v}, \\
\Delta t = v \otimes t + t \otimes 1, \quad \Delta \bar{t} = \bar{v} \otimes \bar{t} + \bar{t} \otimes 1.
\]  

The explicit form of other basic maps for $E_q(2)$ (antipode, counity) will not be used in what follows.

The unitary element $v$ can be parameterized with the help of the symmetric element $\theta$:

\[
v = e^{i\theta}, \quad \theta^\dagger = \theta, \\
\Delta \theta = \theta \otimes 1 + 1 \otimes \theta.
\]  

The corresponding quantum universal enveloping algebra (QUEA) $U_q e(2)$ is generated by the elements $J, T, \bar{T}$ which are dual to the generators $\theta, \bar{t}, t$ of the algebra $E_q(2)$ and, as a result of the duality, satisfy the following commutation relations

\[
[J, T] = iT, \quad [J, \bar{T}] = -i\bar{T}, \\
TT = q^2\bar{T}T
\]  

(comultiplication and the other basic maps are also defined by the duality).

The left action of elements from QUEA $U_q L$ of an arbitrary Lie algebra $L$ on elements of the corresponding quantum group $G_q$ is defined by the expressions

\[
\ell(X)f = (\text{id} \otimes X) \cdot \Delta f = \sum_i f_i^{(1)} \langle \langle X, f_i^{(2)} \rangle \rangle, \\
\lambda(X)f = (S(X) \otimes \text{id}) \cdot \Delta f = \sum_i \langle \langle S(X), f_i^{(1)} \rangle \rangle f_i^{(2)},
\]  

or
where \( X \in \mathcal{U}_q L, \ f, f^{i(1)}_j \in G_q, \ \langle \cdot, \cdot \rangle \) denotes the duality contraction, \( S(X) \) is antipode and where the comultiplication in \( G_q \) is presented in the form \( \Delta f = \sum_i f^{i(1)}_j \otimes f^{i(2)}_j \). An explicit calculation of this left action in the case of \( E_q(2) \) shows that the operators \( T, T \in \mathcal{U}_q e(2) \) act on elements of \( E_q(2) \) generated by \( \tilde{t}, \ t \) exactly in the same way as the \( q \)-deformed derivatives \( \partial_q, \partial_q \). In fact, the elements \( \tilde{t}, \ t \) generate the \( q \)-deformed analog \( P_q^{(2)} = E_q(2)/\mathcal{U}_q(1) \) of the homogeneous space \( P^{(2)} \), i.e., generate the algebra of functions on quantum Euclidean plane \([14]\). We shall denote elements of the algebra \( P_q^{(2)} \) by \( \tilde{z}, \ z \) to distinguish them from elements \( \tilde{t}, \ t \) of the algebra \( E_q(2) \).

The elements \( \tilde{z}, \ z \) and \( \partial_q \), \( \partial_q \) defines the \( q \)-deformed algebra of functions on \( P_q^{(2)} \) together with the \( q \)-deformed left-invariant vector fields (derivatives). Its defining relations read as

\[
\begin{align*}
    z\tilde{z} &= q^2 \tilde{z}z, \quad \partial_q \partial_q = q^2 \partial_q \partial_q \\
    \partial_q z &= 1 + q^{-2} z\partial_q, \quad \partial_q \tilde{z} = 1 + q^2 \tilde{z}\partial_q, \\
    \tilde{\partial}_q z &= q^2 z\tilde{\partial}_q, \quad \tilde{\partial}_q \tilde{z} = q^{-2} \tilde{z}\tilde{\partial}_q, \\
\end{align*}
\] (8)

(the commutation relation for the \( q \)-derivatives is just the rewritten commutation relation for \( \tilde{T}, \ T \) \([3]\) and those for the \( q \)-derivatives and coordinates are derived from \([3]\)). If we put \( q = 1 \) and define \( p = -i\hbar \partial, \ \tilde{p} = -i\hbar \tilde{\partial} \), the relations \([8]\) become the usual canonical commutation relations for a particle in two-dimensional space. The requirement of consistency with antipode dictates the following conjugation rule for the \( q \)-derivatives

\[
\begin{align*}
    \partial_q^\dagger &= -q^2 \partial_q, \quad \tilde{\partial}_q^\dagger = -q^{-2} \tilde{\partial}_q. \\
\end{align*}
\] (9)

We consider the relations \([8]\) as a \( q \)-deformation of the canonical commutation relations and is going to construct their representation in a Hilbert space.

To this aim let us introduce the operators \( N \) and \( \tilde{N} \) defined by the relations

\[
\begin{align*}
    [N; q^{-2}] &= z\partial_q, \quad [\tilde{N}; q^2] = \tilde{z}\tilde{\partial}_q, \\
    [X; q^\alpha] &\equiv \frac{q^\alpha X - 1}{q^\alpha - 1}. \\
\end{align*}
\] (10)

These operators have simple commutation relations

\[
\begin{align*}
    q^{\alpha N}z &= q^\alpha zq^{\alpha N}, \quad q^{\alpha \tilde{N}}\tilde{z} = q^\alpha \tilde{z}q^{\alpha \tilde{N}}, \\
    q^{\alpha N}\partial_q &= q^{-\alpha}\partial_q q^{\alpha N}, \quad q^{\alpha \tilde{N}}\tilde{\partial}_q = q^{-\alpha}\tilde{\partial}_q q^{\alpha \tilde{N}}. \\
\end{align*}
\] (12)

Using \([3]\) and \( \tilde{z}^\dagger = \tilde{z} \), we find

\[
N^\dagger = -\tilde{N} - 1, \quad \tilde{N}^\dagger = -N - 1. \\
\] (13)

The operators \( q^{2\tilde{N}}, q^{2N} \) allow to construct commuting pairs of conjugate operators:

\[
\begin{align*}
    \tilde{Z} &= q^{N-\tilde{N}}z, \quad Z = zq^{N-\tilde{N}}, \\
    \tilde{P} &= qq^{-(N-\tilde{N})}\tilde{\partial}_q, \quad P = -q^{-1}\partial_q q^{-(N-\tilde{N})}, \\
\end{align*}
\] (14)
with the commutation relations
\begin{align*}
P \bar{Z} &= \bar{Z} P , \quad \bar{Z} Z = Z \bar{Z} , \quad Z P = 1 + q^2 P Z , \\
P Z &= Z P , \quad P P = P P , \quad P Z = 1 + q^2 Z P .
\end{align*}

If we were given only the algebra of the operators \( \bar{z}, z, \bar{\partial}_q, \partial_q \), we would reasonably name the commuting operators \( \bar{Z}, Z \) by coordinates and \( P, P \) by the corresponding lattice momenta and then deal with two independent (commuting with each other) one-dimensional algebras on the \( q \)-lattice. However, fields in NC-QFT depend on noncommutative (\( q \)-commuting) coordinates \( \bar{z}, z \) which are more suitable to trace a result of coaction by \( E_q(2) \). We have found convenient to use the hermitian and unitary combination of the coordinate operators:
\begin{align*}
r^2 &\equiv z \bar{z} \quad \text{(hermitian)}, \\
u &\equiv \sqrt{z \bar{z}}^{-1} \quad \text{(unitary),}
\end{align*}

(16)

together with \( q^{(N-N)} \) (hermitian operator) and \( q^{2(N+N+1)} \) (unitary operator) as a basic set of the phase space operators. The commutation relations for this set of operators read as
\begin{align*}
[q^{(N-N)}, r^2] &= 0 , \quad [q^{(N+1)}, u] = 0 , \\
r^2 u &= q^2 u r^2 , \quad [q^{2(N-N)}, q^{(N+1)}] = 0 , \\
q^{2(N-N)} u &= q^2 u q^{2(N-N)} , \quad q^{(N+1)} r^2 = q^2 r^2 q^{(N+1)} ,
\end{align*}

(17)

Now we are ready to construct a representation of this algebra in the space \( \ell^2 \) (i.e., infinite dimensional matrix representation):
\begin{align*}
r^2 | n,m \rangle_{r_0,l_0} &= r_0^2 q^{2n} | n,m \rangle_{r_0,l_0} \\
q^{2(N-N)} | n,m \rangle_{r_0,l_0} &= l_0 q^{2m} | n,m \rangle_{r_0,l_0} .
\end{align*}

(18)
\begin{align*}
u | n,m \rangle_{r_0,l_0} &= | n+1, m+1 \rangle_{r_0,l_0} \\
q^{(N+1)} | n,m \rangle_{r_0,l_0} &= | n+1, m \rangle_{r_0,l_0} .
\end{align*}

(19)

The constants \( r_0 \) and \( l_0 \) mark different representations and from the eigenvalues of \( r^2 \) and \( q^{2(N-N)} \) it follows that in the ranges \( [r_0, q^4 r_0] \) and \( [l_0, q^4 l_0] \) the representations are nonequivalent. The matrices \( r^2, q^{2(N-N)} \) are hermitian and \( u, q^{(N+1)} \) are unitary with respect to the scalar product defined by
\[ r_0 \delta_{nm} \delta_{mm'} . \]

Thus we have obtained that states of a particle on the quantum plane are characterized by discrete values of its radius-vector and discrete values of the operator \( q^{2(N-N)} \) which is obviously related to deformation of the angular momentum operator. Indeed, from (14) we conclude that the operator
\begin{align*}
J_q \equiv [\bar{N} - N; q^2] &= \frac{q^{2(N-N)} - 1}{q^2 - 1} ,
\end{align*}

(20)
(which differs from $q^{2(N-N)}$ by multiplication and shifting by the constants) in the continuum limit $q \to 1$ becomes the ordinary angular momentum operator. Therefore it is natural to consider $J_q$ as an appropriate deformation of the latter. Of course, discreteness of values of an angular momentum operator is not peculiar feature of $q$-deformed systems but general property of all quantum systems. Analogously, the natural $q$-deformation of the dilatation operator reads as

$$D_q \equiv [(\bar{N} + N + 1); q^2] = \frac{q^{2(\bar{N}+N+1)} - 1}{q^2 - 1}, \quad (21)$$

Another possibility for the construction of representations of the algebra (17) which will be convenient for us in the next section is to construct the representation in the basis of the unitary operators

$$u \equiv \bar{z}z^{-1}, \quad \text{and} \quad q^{2(\bar{N}+N+1)}, \quad (22)$$

which commute with each other and, hence, have common eigenvalues. This basis, certainly, less suitable for construction of a matrix representation of the kind presented above since the two other (hermitian) operators do not shift an eigenvector of the operators (22) exactly into another eigenvector. However, this basis proves to be more suitable for the study of transformations of the states under the coaction of the quantum group $E_q(2)$ which we shall carry out in the section 4.

3 Quantum field theory on $P_q^{(2)}$ as a one-dimensional lattice theory for an infinite set of interacting fields

In this section we shall introduce the scalar $\varphi^4$-field theory on the noncommutative plane $P_q^{(2)}$ and present it in the form of infinite number of interacting partial waves defined on a one-dimensional lattice.

3.1 Preliminaries on “the spherical field theory”

The starting idea of the spherical field theory [15] in a usual commutative space-time is the representation of a $d$-dimensional Euclidean field theory as a theory for an infinite set of one-dimensional interacting fields. In what follows we shall confine ourselves with the simplest case of two-dimensional scalar theory. The initial action is quite standard:

$$S = \int d^2x \left[ (\partial_i \bar{\varphi})(\partial_i \varphi) + \mu^2 \bar{\varphi}\varphi + \frac{\lambda}{2}(\bar{\varphi}\varphi)^2 - j\bar{\varphi} - \bar{j}\varphi \right]. \quad (23)$$

Decomposing $\varphi(x)$ and $j(x)$ into partial waves

$$\varphi(x) = \varphi(r, \alpha) = \frac{1}{\sqrt{2\pi}} \sum_{N=-\infty}^{\infty} \varphi_N(r)e^{iN\alpha}, \quad (24)$$

$$j(x) = j(r, \alpha) = \frac{1}{\sqrt{2\pi}} \sum_{N=-\infty}^{\infty} j_N(r)e^{-iN\alpha}, \quad (25)$$
Figure 1: The tadpole diagram in the spherical field theory

one can rewrite (23) as

$$S = \sum_{N=-\infty}^{\infty} \int_{0}^{\infty} dr \left[ r \frac{\partial \bar{\varphi}_N}{\partial r} \frac{\partial \varphi_N}{\partial r} + \frac{\mu^2 r^2 + N^2}{r} \bar{\varphi}_N \varphi_N - r j_N \bar{\varphi}_N - r \bar{j}_N \varphi_N \right]$$

$$+ \frac{\lambda}{2} \sum_{N,M,K,L=-\infty}^{\infty} \int_{0}^{\infty} dr \left( \bar{\varphi}_N \varphi_M \bar{\varphi}_K \varphi_L \delta_{N-M+K-L,0} \right).$$

(26)

Let \( \tilde{G}(k) \) denote the usual Green function in the momentum representation

$$\tilde{G}(k) = \int d^2 x e^{i\vec{k} \cdot \vec{x}} \langle 0 | \bar{\varphi}(x) \varphi(0) | 0 \rangle.$$

(27)

Then the propagator for the \( N \)-th partial wave proves to be

$$\langle 0 | \bar{\varphi}_N(r_1) \varphi_N(r_2) | 0 \rangle = \int dk k J_{|N|}(kr_1)J_{|N|}(kr_2)\tilde{G}(k).$$

(28)

Here \( J_{|N|}(kr) \) is the Bessel function of the first kind and \( k \equiv \sqrt{k_1^2 + k_2^2} \). For the scalar field theory the propagator has the form

$$\tilde{G}(k) = \frac{1}{k^2 + \mu^2},$$

(29)

so that (23) gives

$$G_N(r_1, r_2) \equiv \langle 0 | \bar{\varphi}_N(r_1) \varphi_N(r_2) | 0 \rangle$$

$$= \int dk k J_{|N|}(kr_1)J_{|N|}(kr_2) \frac{1}{k^2 + \mu^2}$$

$$= \theta(r_1 - r_2)K_{|N|}(\mu r_1)I_{|N|}(\mu r_2) + \theta(r_2 - r_1)K_{|N|}(\mu r_2)I_{|N|}(\mu r_1),$$

(30)

where \( \theta(r) \) is the step-function; \( I_N, K_N \) are the modified Bessel functions of the first and second kind respectively.

The principal aim of the “spherical field theory” (SQFT) is the development of a nonperturbative approach to calculations in the standard QFT. We are interested in UV-behaviour of perturbation expansion in the quantum field theory on the noncommutative plane (NC-QFT) which we are going to present in the form similar to the SQFT. Thus to make a comparison, let us first find out how the UV-divergences of the ordinary (two-dimensional, scalar) field theory reveal themselves in SQFT. To this aim, we consider the tadpole diagram depicted in figure [1]. This diagram is proportional to the factor
\[
\sum_{N=-\infty}^{\infty} G_N(r,r) = \sum_{N=-\infty}^{\infty} K_{|N|}(\mu r)I_{|N|}(\mu r) \\
= K_0(\mu r)I_0(\mu r) + 2 \sum_{N=1}^{\infty} K_{|N|}(\mu r)I_{|N|}(\mu r) .
\]

It is seen that the Green function \(G_N(r_1,r_2)\) for a fixed partial wave is not singular at coinciding arguments: \(G_N(r,r)\) has well-defined values for any \(r\) and \(N = 0, \pm 1, \pm 2, \ldots\). However, the tadpole diagram is still divergent: the divergence appears in the summation over the angular momentum numbers \(N\). Indeed, let us for simplicity consider small values of \(\mu r \ll 1\), so that we can use the asymptotic expressions:

\[
I_N(\mu r)|_{\mu r \ll 1} \approx \frac{1}{N!} \left(\frac{\mu r}{2}\right)^N , \quad N = 0, \pm 1, \pm 2, \ldots ,
\]

\[
K_N(\mu r)|_{\mu r \ll 1} \approx \frac{(N-1)!}{2} \left(\frac{2}{\mu r}\right)^N , \quad |N| \geq 1 ,
\]

\[
K_0(\mu r)|_{\mu r \ll 1} \approx \ln \frac{2}{\gamma \mu r} , \quad (\gamma \text{ is the Euler constant}) .
\]

Thus for \(\mu r \ll 1\) the tadpole is proportional to

\[
\sum_{N=-\infty}^{\infty} G_N(r,r) = \ln \frac{2}{\gamma \mu r} + \sum_{N=1}^{\infty} \frac{1}{N} \longrightarrow \infty ,
\]

so that it is (logarithmically) divergent as it should be. Of course, the same is true for any values of \(\mu r\), though an explicit demonstration of this fact becomes more involved.

In order to circumvent such a calculation with the special (Bessel) functions, we can present the action (26) in a modified form. First, if we are interested only in UV-properties of the model, we can drop out the mass term. However, the massless theory in two dimension has the infra-red divergences which are also logarithmic for the tadpole diagram and may distort the true picture of the UV-behaviour of the model. Thus, in the massless case we need some IR-regularization and to achieve it, let us introduce into the (massless) action (26) the additional term of the form

\[
S^{(IR)} = \int_{0}^{\infty} dr \frac{\sigma^2}{r} \varphi_N(r) \varphi_N(r) .
\]

Now the free action reads as

\[
S^{(0)} = \sum_{N=-\infty}^{\infty} \int_{0}^{\infty} dr \left[ r \frac{\partial \varphi_N}{\partial r} \frac{\partial \varphi_N}{\partial r} + \frac{N^2 + \sigma^2}{r} \varphi_N \varphi_N \right] ,
\]

and after an introduction of the new coordinate \(y\) defined by the relation

\[
y = \ln(\mu r) , \quad -\infty < y < \infty ,
\]

it acquires the simple form of the standard one-dimensional scalar action:

\[
S_0 = \sum_{N=-\infty}^{\infty} \int_{-\infty}^{\infty} dy \left[ \frac{\partial \varphi_N}{\partial y} \frac{\partial \varphi_N}{\partial y} + (N^2 + \sigma^2) \varphi_N \varphi_N \right] .
\]
The free Green function for this action can be easily found by the use of the Fourier transform and proves to be the following

$$G_N(y_1 - y_2) = \frac{1}{2M} e^{-M|y_1 - y_2|}, \quad M = \sqrt{N^2 + \sigma^2}, \quad (36)$$

or, in terms of the initial radial coordinates,

$$G_N(r_1, r_2) = \frac{1}{2M} \left[ \theta(r_1 - r_2) \left( \frac{r_2}{r_1} \right)^M + \theta(r_2 - r_1) \left( \frac{r_1}{r_2} \right)^M \right]. \quad (37)$$

This explicit expression shows immediately that the tadpole diagram in figure 1 is proportional to the sum

$$\sum_{N=-\infty}^{\infty} G_N(r, r) = \sum_{N=-\infty}^{\infty} \frac{1}{\sqrt{N^2 + \sigma^2}}, \quad (38)$$

and, hence, logarithmically divergent. The IR-regularization parameter $\sigma$ is inessential for large $N$ and does not influence on the UV-behaviour of the model (as it should be).

The form (33) of the action and the UV-behaviour analysis following it will be useful for us in the noncommutative case as well. We are going to show that the $\varphi^4$-theory on the $q$-plane can be rewritten as a theory of the partial waves on a one-dimensional (nonequidistant) lattice with the behaviour in $N$ (angular momentum number) being even worse, so that UV-divergences of the NC-QFT are even more severe than those of the usual scalar theory.

### 3.2 Quantum field theory on the $q$-plane and its partial wave decomposition

Let us consider the generalization of the two-dimensional scalar field theory, induced by the noncommutativity (8) of the space coordinates on the plane $P_q^{(2)}$. The field action for $\varphi(z, \bar{z})$ can be defined with the help of the $q$-deformed Haar (invariant) measure \[14\]. For a function $F_N(z, \bar{z}) = z^N f(z \bar{z})$ on $P_q^{(2)}$ one defines the linear functional ($q$-integral):

$$H_{r_0}[F_N] \equiv \int_q d^2 z z^N f(z \bar{z})$$

$$\equiv \delta_{N,0} r_0^2 (q^2 - 1) - \sum_{k=-\infty}^{\infty} q^{2k} f(q^{2k} r_0^2) \quad (39)$$

($r_0$ labels the nonequivalent representations \[18\] of the $q$-deformed coordinate-momentum algebra). In formula (39) and in what follows we assume, for definiteness, that $q^2 > 1$ (the quite similar construction can be carried out for $q < 1$, cf. \[17\]). In order that the sum on the right hand side of (39) be meaningful, the function $f(z \bar{z})$ must satisfy an appropriate conditions at infinity \[19\]. We shall assume that the set of the fields on the $q$-plane which we consider below does satisfy this condition. Notice that if $N < 0$ in (39), the integrand can be rewritten as $z^{|N|} f'(z \bar{z})$ ($f'(z \bar{z})$ is some modification of the function $f(z \bar{z})$).

Using the $q$-integral (39), we can define the action on the $q$-plane as the straightforward generalization of the usual $\varphi^4$-action \[23\]:

$$S_q = \int_q d^2 z \left[ -\bar{\varphi}(z, \bar{z}) \partial_q \bar{\varphi} \varphi(z, \bar{z}) + \mu^2 \bar{\varphi} \varphi + \lambda \left( \varphi(z, \bar{z}) \right)^2 - j(z, \bar{z}) \bar{\varphi} \varphi - \bar{j}(z, \bar{z}) \varphi \bar{\varphi} \right]. \quad (40)$$
Now our aim is to rewrite the action (40) in the form similar to (26) where the integral over radial variables is substituted by the sum (39) while the sum over the angular momentum numbers remains in the $q$-deformed case too. To achieve this, we decompose a field on $P^{(2)}_q$ into terms with definite eigenvalues of the $q$-deformed angular momentum operator $J_q$ (cf. (21)):

$$\varphi(z, \bar{z}) = \sum_{N=-\infty}^{\infty} z^N r^{-N} \varphi_N(r), \quad r^2 \equiv z\bar{z}.$$  \hspace{1cm} (41)

Again, as in the case of the expression (39), it is worth to notice that the terms with $N < 0$ in the sum (41) can be rewritten in the form with positive powers of $\bar{z}$:

$$z^N r^{-N} = q^{-|N|(|N|-1)} \bar{z}^{|N|-|N|} = q^{-N(N+1)} z^{-N} r^N, \quad (N < 0).$$  \hspace{1cm} (42)

Here we have used the relation:

$$z^N \bar{z}^N = q^{N(N-1)} (z\bar{z})^N \equiv q^{N(N-1)} r^{2N}.$$  \hspace{1cm} (43)

The next step is the substitution of the decomposition (41) into the action (40) and then use of the definition (39) to convert the action into a lattice one. In order to do this, one needs the following commutation relations which are derivable from (8):

$$\sqrt{z} \sqrt{\bar{z}} = \sqrt{q} \sqrt{z} \sqrt{\bar{z}}, \quad z^N r^{-N} = q^{-N(N-1/2)} \bar{z}^{-N/2} r^{N/2},$$

$$z r = q r z, \quad \bar{z} r = q^{-1} r \bar{z},$$

$$\partial_q \sqrt{z} = q^{-1} \sqrt{z} \partial_q, \quad \partial_q \sqrt{\bar{z}} = \frac{1}{q-1 + \sqrt{z}} + q^{-1} \sqrt{z} \partial_q,$$

$$\bar{\partial}_q \sqrt{z} = q \sqrt{z} \bar{\partial}_q, \quad \bar{\partial}_q \sqrt{\bar{z}} = \frac{1}{q + \sqrt{\bar{z}}} + q \sqrt{\bar{z}} \bar{\partial}_q,$$  \hspace{1cm} (44)

$$\partial_q z^{N/2} = \frac{[N; q^{-1}]}{q^{-1} + 1} z^{N/2-1} + q^{-N} z^{-N/2} \partial_q,$$

$$\bar{\partial}_q \bar{z}^{-N/2} = -\frac{[N; q^{-1}]}{q(q + 1)} \bar{z}^{-N/2-1} + q^{-N} \bar{z}^{-N/2} \bar{\partial}_q,$$

$$\partial_q \bar{\partial}_q f(r) = \frac{1}{2 + q + q^{-1}} \left[ \frac{1}{r} D^{(r)}_{q^{-1}} r D^{(r)}_q \right] f(r).$$

The Jackson derivatives in the last line are defined as follows:

$$D^{(r)}_{q^{-1}} f(r) = \frac{f(q^{-1} r) - f(r)}{r(q^{-1} - 1)}, \quad (45)$$

$$D^{(r)}_q f(r) = \frac{f(q r) - f(r)}{r(q - 1)}$$

(These definitions imply that we are working with the representation where the operator $r$ is diagonal).
Use of these relations together with (39) and (41) allows to present the action (40) in the form of the Jackson integral over the radial variable $r^2$:

$$
S_q = \sum_{N=-\infty}^{\infty} \int J \, dq^2 \left\{ q^{-N(N+1)}\bar{\phi}_N(r^2) \left[ -\frac{q^3}{(q+1)^2} \Delta_q + q^{2(N+1)}[N; q^{-1}]^2 - q^{N+1} [N; q^{-1}] \left( q^2 D_{q^2} (r^2) - D_{q^2} (r^2) \right) \right] \phi_N(r^2) + \mu^2 q^{-N(N+1)} \bar{\phi}_N(r^2) \phi_N(r^2) + \frac{\lambda}{2} \sum_{M,K=-\infty}^{\infty} q^{2k} q^{-N^2-K^2-M^2-MK+NK+NM-M-K} \times \bar{\phi}_N(r^2) \phi_N(q^2(N-M)r^2) \bar{\phi}_N(q^2(N-M)r^2) \phi_N(r^2) + q^{-N(N+1)} j_N(r^2) \phi_N(r^2) + q^{-N(N+1)} \bar{j}_N(r^2) \bar{\phi}_N(r^2) \right\} .
$$

(46)

Here the Jackson integral for and arbitrary function $f(r^2)$ is defined in the standard way (see, e.g., [3]):

$$
\int J \, dr^2 \, f(r^2) \overset{\text{def}}{=} r_0^2 |q^2 - 1| \sum_{k=-\infty}^{\infty} q^{2k} f(q^{2k} r_0^2) ,
$$

(47)

and the Jackson derivatives are defined by (45) and by the following similar relations:

$$
D^{(r^2)}_{q^2} f(r^2) = \frac{f(q^{-2} r^2) - f(r^2)}{r^2(q^{-2} - 1)} ,
$$

$$
D^{(r^2)}_{q^2} f(r^2) = \frac{f(q^2 r^2) - f(r^2)}{r^2(q^2 - 1)} .
$$

(48)

The radial part $\Delta_q$ of the $q$-deformed Laplacian reads as

$$
\Delta_q = \frac{1}{r} D^{(r)}_{q^{-1}} r D^{(r^2)}_{q} .
$$

(49)

Notice that the expression (46) for the action $S_q$ in term of the Jackson integral is equally correct both for the case $q > 1$ and for $q < 1$. For definiteness, we continue to discuss the case $q > 1$. The consideration for the case $q < 1$ is essentially the same and we shall present for it only the result (cf. (67)).

Since both the Jackson integral and the Jakson derivatives turn into their nondeformed (continuous) counterparts in the limit $q \to 1$, it is readily seen that the action (46) becomes in this limit the usual action (20) for the two-dimensional scalar theory in the polar coordinates.

Now we proceed to study the UV-behaviour of the field theory on the $q$-deformed plane. Therefore, we again, similarly to the nondeformed case in the preceding subsection (cf. (32)), substitute the mass term with the IR-regularizing term

$$
S^{(IR)}_q = \sum_{N=-\infty}^{\infty} \int J \, dq^2 \frac{q^{-N(N+1)} \sigma^2}{r^2} \bar{\phi}_N(r^2) \phi_N(r^2) .
$$

(50)
The exponential dependence of the fields in (46), (47) on the space (discrete) variable \( k \) inspires to make the substitution similar to that for the nondeformed model (cf. (34)) and to denote:

\[
\bar{\phi}_{Nk} \equiv \bar{\phi}_N(q^{2k}r_0^2), \quad \varphi_{Nk} \equiv \varphi_N(q^{2k}r_0^2),
\]

\[
\bar{j}_{Nk} \equiv \bar{j}_N(q^{2k}r_0^2), \quad j_{Nk} \equiv j_N(q^{2k}r_0^2).
\]

In this notation the action (46) in which the mass term is substituted by (50), acquires the form of an action for infinite number of scalar fields on a one-dimensional lattice:

\[
S_q = S_q^{(0)} + S_q^{(int)} + S_q^{(e.s.)},
\]

\[
S_q^{(0)} = \sum_{N=-\infty}^{\infty} A_N \sum_{k=-\infty}^{\infty} \left[ \frac{(\varphi_{Nk+1} - \varphi_{Nk})^2}{a} + aM_N^2 \bar{\phi}_{Nk}\varphi_{Nk} \right],
\]

\[
S_q^{(int)} = \frac{\lambda r_0^2}{2} (q^2 - 1) \sum_{M,N,K=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{-N^2-k^2-M^2-MK+4NM-Mk} q^{2k} \times \bar{\varphi}_{MK} \varphi_{(k+M-N)} \bar{\varphi}_{K(k+M-N)} \varphi_{(M-N+k)k},
\]

\[
S_q^{(e.s.)} = r_0^2 (q^2 - 1) \sum_{N,k=-\infty}^{\infty} q^{2k} q^{-N(N+1)} \left( \bar{j}_{Nk}\varphi_{Nk} + \bar{\varphi}_{Nk} j_{Nk} \right),
\]

where

\[
A_N = q^{-N^2+4},
\]

\[
M_N^2 = \frac{A_N}{R_N},
\]

\[
R_N = q^{-N(N+1)+4} \left( \frac{|N; q|^2}{(q+1)^2} + \frac{\sigma^2}{q^4} \right),
\]

\[
a = q^2 - 1.
\]

It is obvious that \( A_N > 0, R_N > 0 \) for all \( N = 0, \pm 1, \pm 2, \ldots \). This justifies the definition (54) and shows that the quadratic part of the Euclidean action (40) is positively defined. The latter fact, in turn, provides that the generating functional for Green functions in the model with the action (40) (or, in the lattice form, (51)-(53)) given by the infinite dimensional integral (discrete lattice analog of the path integral):

\[
Z[j] = \frac{\int \prod_{N,k} d\bar{\varphi}_{Nk} d\varphi_{Nk} \exp \{-S\}}{\int \prod_{N,k} d\bar{\varphi}_{Nk} d\varphi_{Nk} \exp \{-S\}_{j=j=0}},
\]

can be calculated by the perturbation expansion. A few remarks are in order:

1. The fields \( \bar{\varphi}_N, \varphi_N \) at points \( q^{2k}r_0, k = 0, \pm 1, \pm 2, \ldots \), i.e., the quantities \( \bar{\varphi}_{Nk}, \varphi_{Nk} \), can be considered as the creation and annihilation operators of particles on the quantum plane \( P_q^{(2)} \) in the states (18). Thus the field model with the action (51) is the secondary quantized theory of the particles on the quantum \( q \)-plane \( P_q^{(2)} \).
Figure 2: Feynman rules (free propagator and the nonlocal vertex) for the scalar theory on the noncommutative plane $P_q^{(2)}$

2. The quadratic part (53) of the action (51) has the standard form of the lattice scalar theory (cf. e.g., [20]), so that we can use the standard method of the Fourier transform in order to diagonalize it.

3. As a result of the nonequidistance of the $q$-lattice, the mass term in (53) (the second line) looks as if the model interacts with the external field, i.e., the mass term contains additional factors $q^{2k}$ under the sign of the sum. This makes diagonalization of the complete quadratic part of the action (46) rather involved problem.

4. It is rather striking result that the action on the $q$-plane is not only lattice-like but also nonlocal, as is seen from the interaction term of the action in the form (53).

The quadratic part $S_q^{(0)}$ of the action can be diagonalized by performing the Fourier transform

$$\varphi_{Nk} = \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} e^{iapk} \varphi_N(p) .$$

Then

$$S_q^{(0)} = \sum_{N=\infty}^{\infty} A_N \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} \varphi_N(p) \left[ \frac{2}{a^2} \left( 1 - \cos(ap) \right) + M_N^2 \right] \varphi_N(p) ,$$

and the free Green function $G_N(k - m)$ has the usual for the lattice field theories form (see, e.g., [20]):

$$G_N(k, m) = \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} A_N \frac{e^{ip(k-m)}}{M_N^2 + \frac{2}{a^2} \left( 1 - \cos(ap) \right)} .$$

Together with the $\varphi^4$-vertex of the action (53), this free propagator defines the Feynman rules for the model under consideration which are depicted in figure 2. We shall not carry out detailed perturbative calculations: because of the nonequidistance of the $q$-lattice such calculations (especially in the case of nonzero mass) prove to be rather cumbersome. Notice, however, that these peculiarities of the $q$-lattice seems to be not a difficulty for computer simulations. In this paper we shall demonstrate only that the UV-divergences retain in the scalar field theory on the $q$-deformed plane. Let us consider the tadpole diagrams presented in figure 3. We confine our consideration to the diagram 3.a because its analysis is a bit easier than that for the diagram 3.b. Using the generating functional (58) and the Feynman rules we find the following expression for the tadpole 3.a:

$$\frac{1}{2} \lambda(q^2 - 1) q^{-M(M+1)} \sum_{l=-\infty}^{\infty} q^{2l} G_M(k, l) G_M(l, m) \sum_{N=-\infty}^{\infty} q^{-N(N+1)} G_N(0) .$$
Figure 3: Two types of the tadpole diagrams in the model under consideration

The partial wave propagator $G_M(0)$ is finite at the coincident arguments (similar to the case of the ordinary $\varphi^4$-model on a commutative plane, cf. (30), (31) and (36), (37)) and is given by the relation (61). The simple integration yields

$$G_M(0) = \frac{1}{\sqrt{R_N(4A_N + a^2 R_N)}}.$$  

The divergence again appears in the summation over the angular momentum numbers: as is seen from (62), the tadpole contribution is proportional to the factor

$$\sum_{N=-\infty}^{\infty} q^{-N(N+1)} G_N(0)$$

$$= \sum_{N=-\infty}^{\infty} \frac{q^2 - 1}{q^4 \sqrt{[(q^N - 1)^2 + (q^2 - 1)^2 \sigma^2/q^4][(q^N + 1)^2 + (q^2 - 1)^2 \sigma^2/q^4]}}.$$  

The terms in this series have the following asymptotics:

$$\frac{q^2 - 1}{q^4 \sqrt{[(q^N - 1)^2 + (q^2 - 1)^2 \sigma^2/q^4][(q^N + 1)^2 + (q^2 - 1)^2 \sigma^2/q^4]}} \quad \xrightarrow{N \to \infty} \quad \frac{q^2 - 1}{q^4 + (q^2 - 1)^2 \sigma^2}.$$  

The series has the linear divergence at the lower limit (the IR-regularization parameter again, as in the nondeformed case, is inessential in the $N \to \pm \infty$ limit). Notice that in the nondeformed limit $q \to 1$ the series (64) turns into the following one

$$\sum_{N=-\infty}^{\infty} \frac{1}{\sqrt{N^2 + \sigma^2}},$$

and coincides with the nondeformed result (cf. (38)) (logarithmic divergence).

Quite similar calculation in the case $q < 1$ shows that the tadpole diagram 3.a is proportional to the series

$$\sum_{N=-\infty}^{\infty} q^4 \sqrt{[(1-q^N)^2 + (1-q^2)^2 \sigma^2/q^4][(q^N + 1)^2 + (1-q^2)^2 \sigma^2/q^4]}.$$
which has the linear divergence at the upper limit.

Thus the perturbation theory for the \( \varphi^4 \)-model on the noncommutative plane \( P^{(2)}_q \) with the action constructed with the help of the \( E_q(2) \)-quantum group invariant measure contains the UV-divergences and, hence, can not be considered as a regularization of the usual scalar field theory on the commutative plane.

4 Transformation of states on the noncommutative plane \( P^{(2)}_q \) induced by the coaction of the quantum Euclidean group \( E_q(2) \)

The central problem of this section is to determine transformation properties of a system on \( P^{(2)}_q \) under a coaction of the quantum group \( E_q(2) \). In the subsection 4.1 we shall clarify a general formulation of this problem and in subsection 4.2 give an explicit answer for \( E_q(2) \) group. In particular, we shall show that this coaction leads to nonlocal transformations of states.

4.1 Transformation of states in noncommutative geometry induced by a quantum group coaction

Let a quantum group \( G_q \) coact on a noncommutative space \( \mathcal{X}_q \), i.e. there exists the homomorphic map

\[
\delta : \text{Fun}_q(\mathcal{X}) \to \text{Fun}_q(G) \otimes \text{Fun}_q(\mathcal{X}) ,
\]

(68)

(the algebra \( \text{Fun}_q(\mathcal{X}) \) of functions on \( \mathcal{X}_q \) is the configuration space subalgebra of the algebra of all operators of the given quantum system). It is natural to say that the system is invariant with respect to the quantum group transformations if all the properties of the system are independent on the coaction map \( \delta \). In other words, the algebra \( \text{Fun}_q(\mathcal{X}) \) can be realized as the subalgebra of multiple tensor product \( \text{Fun}_q(G) \otimes \text{Fun}_q(G) \otimes \ldots \otimes \text{Fun}_q(\mathcal{X}) \) and no measurements can distinguish the description based on the algebras with different numbers of the factors \( \text{Fun}_q(G) \).

At first sight, this definition of symmetry transformations may look unusual but, in fact, it is a direct generalization of commutative transformations. Indeed, usual action of a group \( G \) of transformations of a manifold \( \mathcal{M} \) on a function \( f \in \text{Fun}(\mathcal{M}) \) is defined by the equality

\[
T_g f(x) := f(g^{-1}x) , \quad g \in G, \ x \in \mathcal{M} .
\]

(69)

The right hand side of this definition can be considered as the function defined on \( G \times \mathcal{M} \).

In other words, the transformations \( T \) defines the map

\[
T : \text{Fun}(\mathcal{M}) \to \text{Fun}(G) \otimes \text{Fun}(\mathcal{M}) .
\]

More customary map \( \phi : G \otimes \mathcal{M} \to \mathcal{M} \) is defined for points of the manifolds, which play the role of the dual set of states for the commutative algebra of observables (functions) on usual manifolds. Returning to the transformations with noncommutative parameters, let us define the map [21] which is dual to the transformations (68) of observables (operators), i.e.,

\[
S : \mathcal{H}_G \otimes \mathcal{H}_\mathcal{X}_q \to \mathcal{H}_\mathcal{X}_q ,
\]

(70)
where $\mathcal{H}_{G_q}$ and $\mathcal{H}_{X_q}$ are the Hilbert spaces of all representations of the algebras $Fun_q(G)$ and $Fun_q(X)$. The intertwining operator $\mathcal{S}$ is implicitly defined by the equation

$$\langle\langle \delta A| \Psi \otimes \psi \rangle \rangle = \langle\langle A| \mathcal{S}(\Psi, \psi) \rangle \rangle ,$$

where $\Psi$ is arbitrary vector from $\mathcal{H}_G$, $\psi$ is arbitrary vector from $\mathcal{H}_X$ and $\mathcal{S}(\Psi, \psi) \in \mathcal{H}_X$ and the duality relation $\langle\langle A| \psi \rangle \rangle : O \otimes \mathcal{H}_O \rightarrow C$ between an operator $A$ from some algebra $O$ and a vector $\psi$ from the Hilbert space $\mathcal{H}_O$ of the representations of this algebra, is defined by the ordinary mean value of $A$ in the state $\psi : \langle\langle A| \psi \rangle \rangle = \langle \psi| A| \psi \rangle$. In fact, the usual definition (39) of the action of (classical, commutative) transformation groups in the space of functions on some homogeneous manifold $\mathcal{M}$ also has the general form (71). Indeed, in this case the duality relation between the algebra $Fun(\mathcal{M})$ and states, i.e. points of $\mathcal{M}$, is defined as follows

$$\langle\langle f| x \rangle \rangle = f(x), \quad f \in Fun(\mathcal{M}), \; x \in \mathcal{M} .$$

The same is true for the group manifold:

$$\langle\langle T| g \rangle \rangle = T_g, \quad T \in Fun(G), \; g \in G .$$

Thus (71) can be represented in the form

$$\langle\langle \delta f| g \otimes x \rangle \rangle = \langle\langle T \otimes f| g \otimes x \rangle \rangle = T_g f(x) = \langle\langle f| \mathcal{S}(g, x) \rangle \rangle = \langle\langle f| g^{-1} x \rangle \rangle = f(g^{-1} x) ,$$

where the third equality follows from (71) and in this special case $\mathcal{S}(g, x) = g^{-1} x$.

From (71) it follows that the matrix elements of the operator $\mathcal{S}$ in a chosen bases of $\mathcal{H}_{G_q} \otimes \mathcal{H}_{X_q}$ and $\mathcal{H}_{X_q}$ play the role of generalized Clebsch-Gordan coefficients (GCGC). If the multiple index (set of quantum numbers) $\{m\}$ defines basis vectors $\psi_{\{m\}}$ of $\mathcal{H}_{X_q}$, and the set $\{K\}$ defines basis $\Psi_{\{K\}}$ of $\mathcal{H}_{G_q}$, one can write

$$\psi' = \mathcal{S}(\Psi_{\{K\}}, \psi_{\{m\}}) = \sum_{\{l\}} C_{\{m\}\{l\}}^{\{K\}} \psi_{\{l\}} ,$$

where $C_{\{m\}\{l\}}^{\{K\}}$ are the set of GCGC. In this formula the vector $\psi_{\{m\}}$ is a transformed state on the quantum plane, and the vector $\Psi_{\{K\}}$ (analog of a point on a group manifold in the case of ordinary Lie groups) defines “parameters” of the transformation of $\psi_{\{m\}}$.

One can apply analogous consideration to the very quantum group $G_q$ which coacts on itself

$$M_{ji}^N = \Delta M_{ji} = M_{ik}^j \otimes M_k^i ,$$

This leads to the corresponding transformation of vectors in $\mathcal{H}_{G_q}$

$$\Psi' = \mathcal{S}(\Psi_{\{K\}}, \Psi_{\{N\}}) \equiv \mathcal{S}_{\Psi_{\{K\}}} (\Psi_{\{N\}}) = \sum_{\{L\}} C_{\{N\}\{L\}}^{\{K\}} \Psi_{\{L\}} ,$$

Two subsequent coactions of the form (74) induce composition of the transformations (74) and general properties of algebra representations provide its associativity (or, equivalently, this follows from the coassociativity of Hopf algebras). This means that the transformations (74) form the semigroup. The trivial representation $\Psi_{\{0\}} \in \mathcal{H}_{G_q}$ correspond to the identity transformation. However, there is no inverse transformation for arbitrary $\mathcal{S}_{\Psi_{\{K\}}}$, This means that the transformations (74), (72) do not form a group.
The map $S$ satisfies the obvious consistency condition which can be expressed as a requirement of commutativity of the diagram in figure 4 (in other words, an equivalence of the different ways through the diagrams along the arrows). For the generalized Clebsch-Gordan coefficients this consistency relation reads as

$$
\sum_{L,n} C^{L}_{nr} \langle \Psi_{L} | M^{i}_{j} | \Psi_{K} \rangle \langle \psi_{n} | x^{j} | \psi_{m} \rangle = \sum_{l} C^{K}_{ml} \langle \psi_{r} | x^{i} | \psi_{l} \rangle .
$$

(75)

(for the convenience and shortness we use Dirac bracket notation and drop curly brackets indicating that $K, m, ...$ are multi-indices). The equations (75) must be completed by the normalization conditions which follow from the normalization of vectors $|K\rangle$ and $|m\rangle$.

Notice that the equations (75) are an analog of the recursion equations used for determination of ordinary Clebsch-Gordan coefficients of $SU(2)$ group. However, in the case of quantum groups the problem of explicit solution of these equation proves to be much more difficult [21].

4.2 Representations of the algebra of functions on $E_{q}(2)$ and transformations of states on $P_{q}^{(2)}$

In this subsection we shall be interested in construction of the cotransformations of the coordinate subalgebra on a quantum plane and therefore, for shortness, drop the quantum numbers related to the angular momentum $J_{q}$.

According to the discussion in the preceding subsection, a transformation of a state on a quantum plane depends on a vector from representation space for the algebra of functions on the group $E_{q}(2)$. Thus we need explicit construction of representations of the algebra $E_{q}(2)$ (cf. (2)). These representations have been presented in [12]: in slightly different (more "physical") form they read as

$$
t = e^{\sqrt{\eta}a} , \quad \bar{t} = e^{\sqrt{\eta}a^{\dagger}} , \quad q^{2} = e^{\eta} ,
$$

\begin{equation}
v = e^{i\phi} e^{\sqrt{\eta}(a-a^{\dagger})} , \quad \phi \in [0, 2\pi] ,
\end{equation}

(76)

where $a, a^{\dagger}$ are usual creation and annihilation operators

$$
[a, a^{\dagger}] = 1 ,
$$

(77)

and hence can be represented in any well known way (e.g., coordinate, Bargmann-Fock, coherent state, infinite matrix representations). The factor $e^{i\phi}$ in (76) is an eigenvalue of
the central element \( I \) of the algebra \( E_q(2) \)

\[
I = q^{-1} \varepsilon t^{-1} ,
\]

so that different values of \( \phi \in [0, 2\pi] \) separate different (though identical) irreducible representations of the \( E_q(2) \)-algebra. Notice that \( I \) is unitary operator \( I^* I = 1 \), that is why its eigenvalues are parameterized by \( e^{i\phi} \).

For an explicit construction we may use the basis of coherent states

\[
a | \zeta \rangle = \zeta | \zeta \rangle , \quad \zeta \in \mathbb{C} ,
\]

\[
\langle \zeta' | \zeta \rangle = \bar{\zeta} \langle \zeta | \bar{\zeta} \rangle , \quad \zeta \in \mathbb{C} ,
\]

so that

\[
t | \zeta \rangle = e^{\sqrt{\eta} \zeta} | \zeta \rangle ,
\]

\[
\bar{t} | \zeta \rangle = | \zeta + \sqrt{\eta} \rangle ,
\]

\[
v | \zeta \rangle = e^{-\eta/2 + i\phi} e^{\sqrt{\eta} \zeta} | \zeta - \sqrt{\eta} \rangle , \quad \bar{v} = v^{-1} .
\]

The algebra of functions on \( P_q(2) \) generated by \( \bar{z}, z \) has not a central element and its unique representation has the form \( (80) \) where \( \bar{t}, t \) are substituted by \( \bar{z}, z \).

According to the general discussion in the preceding subsection, the coaction of \( E_q(2) \) on \( P_q(2) \)

\[
\bar{z} \rightarrow \zeta' = \delta \bar{z} \equiv \bar{v} \bar{z} + \bar{t} \otimes I ,
\]

\[
z \rightarrow z' = \delta z \equiv v z + t \otimes I ,
\]

induces transformations of states from the representation space \( \mathcal{H}_{P_q}(2) \) of \( P_q(2) \) depending on a state from the representation space \( \mathcal{H}_{E_q}(2) \) of the quantum group \( E_q(2) \):

\[
| \xi \rangle \rightarrow | \xi' \rangle = S \left( | \phi, \zeta \rangle , | \xi \rangle \right) ,
\]

\[
| \phi, \zeta \rangle \in \mathcal{H}_{E_q}(2) , \quad | \xi \rangle , | \xi' \rangle \in \mathcal{H}_{P_q}(2) .
\]

Explicitly this map can be written as follows

\[
| \xi' \rangle = \int d\zeta e^{-\xi \zeta} | \xi \rangle \langle \xi | \phi, \zeta \rangle ,
\]

where the generalized Clebsch-Gordan coefficients for the coaction of \( E_q(2) \) on \( P_q(2) \) are denoted as

\[
\langle \xi | \phi, \zeta ; \xi' \rangle \equiv \langle \xi | S | \phi, \zeta ; \xi' \rangle .
\]

Thus to define transformations properties of the states on \( P_q(2) \) we should calculate the GCGC \((85)\). The consistency relation given by the commutativity of the diagram in figure 3 leads to the set of equations

\[
\langle \xi | \phi, \zeta ; \xi' \rangle = e^{-\sqrt{\eta} \xi} \langle \xi + \sqrt{\eta} | \phi, \zeta ; \xi' \rangle - e^{-\eta/2 + i\phi} e^{\sqrt{\eta} \xi} \langle \xi | \phi, \zeta - \sqrt{\eta} ; \xi' \rangle
\]

\[
\llangle \xi | \phi, \zeta + \sqrt{\eta} ; \xi' \rangle = e^{-\sqrt{\eta} \xi} \langle \xi | \phi, \zeta + \sqrt{\eta} ; \xi' \rangle - e^{-\eta/2 - i\phi} e^{-\sqrt{\eta} \zeta + \xi} \langle \xi | \phi, \zeta + \sqrt{\eta} ; \xi' + \sqrt{\eta} \rangle ,
\]

\[
\llangle \xi + \sqrt{\eta} | \phi, \zeta ; \xi' \rangle = e^{\eta/2} e^{\sqrt{\eta}(\xi + \xi')} \langle \xi | \phi, \zeta - \sqrt{\eta} ; \xi' - \sqrt{\eta} \rangle .
\]
Figure 5: The diagrammatic representation of the consistency relation for the Generalized Clebsch-Gordan map $S$ in case of coaction of the group $E_q(2)$ on itself.

which must be accompanied by normalization conditions. As we mentioned in the preceding subsection, it is not easy to solve this equations straightforwardly. To circumvent the problem it is helpful to consider the basis where the primitive elements $\bar{v}, v \in E_q(2)$ are diagonal. The point is that a cotransformation for primitive elements has the form of a cotransformation for an ordinary Lie algebra, thus the consistency conditions for them also have a most simple form.

Proceeding in this way, let us construct representation in the basis of the primitive element $v$. This is easy to do taking into account that in the parameterization (4) the commutation relations for the $E_q(2)$ takes the form

\[
[\theta, \rho^2] = -i2\eta \rho^2, \quad \text{(87)}
\]

\[
[\theta, \nu^2] = 0, \quad \text{(88)}
\]

where we have introduced, in analogy with the algebra $P_q^{(2)}$ (cf. (16)) the operators:

\[
\rho^2 \equiv \bar{t} t, \quad \nu^2 \equiv \bar{t} t^{-1}. \quad \text{(89)}
\]

The only nontrivial commutation relations (87) is equivalent to that for $igl(1, \mathbb{R})$ Lie algebra (Lie algebra of translations and dilatations on a line). Representations of this algebra are well known (see, e.g., [22]) and this allows to write immediately the required representation with $v = e^{i\theta}$ being diagonal:

\[
v f_\phi(x) = e^{i\phi} e^{-2i\eta x} f_\phi(x),
\]

\[
\rho^2 f_\phi(x) = e^{-\eta} e^{-i\theta} f_\phi(x), \quad \text{(90)}
\]

\[
\nu^2 f_\phi(x) = e^{\eta/2} e^{2i\eta x} f_\phi(x). \quad \text{(91)}
\]

From the form of the operators it is clear that the variable $x$ takes values on a circle: $x \in [0, 2\pi/\eta]$. Thus the functions $f_\phi(x)$ are defined on the circle and form the Hilbert space with the scalar product

\[
\langle f_1\phi(x), f_2\phi(x) \rangle = \int_0^{2\pi/\eta} dx \bar{f}_1\phi(x) f_2\phi(x). \quad \text{(92)}
\]

The consistency relations (commutativity of the diagram in figure 5) for the generators $t, \bar{t}$ result again in still rather complicated recursion relations:

\[
\Delta t = v \otimes t + t \otimes I \quad \Rightarrow
\]
and algebra: This is the analog of additivity of the magnetic quantum number in the case of $\phi$ representations where according to (95) we have defined resulting representation are connected by the relation

$$
\Delta \tilde{t} = \tilde{\nu} \otimes \tilde{\nu} \quad \Rightarrow
$$

$$
\left[ e^{i \phi_1 e^{-2i(\eta_1 + \eta_2)}} e^{i \partial_2/2} + e^{-i \eta_1 e^{i \partial_1/2}} e^{-i \partial_2/2} \right] \langle \phi, x | \phi_1, x_1; \phi_2, x_2 \rangle = 0 ,
$$

$$
\Delta \tilde{\nu} = \tilde{\nu} \otimes \tilde{\nu} \quad \Rightarrow
$$

$$
e^{-i(\phi_1 - 2\eta_1 + \phi_2 - 2\eta_2)} \langle \phi, x | \phi_1, x_1; \phi_2, x_2 \rangle = e^{-i(\phi - 2\eta)} \langle \phi, x | \phi_1, x_1; \phi_2, x_2 \rangle . \quad (93)
$$

Here $\langle \phi, x | \phi_1, x_1; \phi_2, x_2 \rangle$ denotes GCGC for the algebra $E_q(2)$ in the realization (90) and we used the basis $| \phi, x \rangle$ of eigenfunctions of the operator $e^{2i\eta x}$ which are, of course, $\delta$-functions on the circle:

$$
| \phi, x_0 \rangle = \delta^{(S)}(x - x_0) = \frac{1}{2\pi \eta} \sum_{k=-\infty}^{\infty} e^{2i\eta k(x-x_0)} . \quad (94)
$$

The relation (93) shows that the eigenvalues $x_1, x_2$ of vectors in the representations $\phi_1$ and $\phi_2$ under the tensor product sign and the eigenvalue $x$ in an irreducible part $\phi$ of the resulting representation are connected by the relation

$$
- \phi_1 + 2\eta_1 - \phi_2 + 2\eta_2 = -\phi + 2\eta x + 2\pi n , \quad n = 0, 1, 2, ...
$$

(95)

This is the analog of additivity of the magnetic quantum number in the case of $su(2)$ Lie algebra: $m = m_1 + m_2$ ($m_1, m_2$ are $J_3^{(1)}$, $J_3^{(2)}$-eigenvalues of two spins to be summed up and $m$ is an eigenvalue of $J_3 = J_3^{(1)} + J_3^{(2)}$).

Let us consider the action of the central operator $I$ on the direct product of two representations

$$
\Delta I | \phi_1, x_1 \rangle | \phi_2, x_2 \rangle . \quad (96)
$$

Acting in addition by the intertwining operator $S$ and denoting

$$
| \phi_1, x_1; \phi_2, x_2 \rangle \equiv | \phi_1, x_1 \rangle | \phi_2, x_2 \rangle ,
$$

we have

$$
S \Delta I | \phi_1, x_1; \phi_2, x_2 \rangle = IS | \phi_1, x_1; \phi_2, x_2 \rangle
$$

$$
= I \int d\phi \, dy \, | \phi, y \rangle \langle \phi, y | S | \phi_1, x_1; \phi_2, x_2 \rangle
$$

$$
= I \int d\phi \, | \phi, y \rangle C(\phi; \phi_1, x_1; \phi_2, x_2)
$$

$$
= \int d\phi \, e^{i\phi} | \phi, y \rangle C(\phi; \phi_1, x_1; \phi_2, x_2) , \quad (97)
$$

where according to (93) we have defined

$$
\langle \phi, y | S | \phi_1, x_1; \phi_2, x_2 \rangle \equiv \delta^{(S)}(\phi - \phi_1 - \phi_2 + 2\eta(x_1 + x_2 - y)) C(\phi; \phi_1, x_1; \phi_2, x_2) . \quad (98)
$$
On the other hand, consider the concrete realization of the operators \( I \) and \( \Delta I \) in \( \mathcal{L}^2(S) \) and in the tensor product \( \mathcal{L}^2(S) \otimes \mathcal{L}^2(S) \), respectively. To this aim we need an explicit form of \( \Delta \nu^2 \) and \( \Delta \rho^2 \). Calculation in the representation (90) gives the result:

\[
\Delta \rho^2 = \Phi \otimes \rho^2 + \rho^2 \otimes \Phi + \bar{v} \otimes t + \bar{t} \otimes \bar{v}
\]

\[
= e^{-\eta} e^{-i\tilde{\eta}} \left( e^{-i\tilde{\eta}} + e^{i\tilde{\eta}} + e^{-i\tilde{\eta}} e^{-\eta/2} e^{i\eta x} + e^{i\tilde{\eta}} e^{\eta/2} e^{-i\eta x} \right), \tag{99}
\]

\[
\Delta \nu^2 = \left( \bar{v} t^{-1} \otimes \bar{t} + \nu^2 \otimes I \right) \left( I \otimes I + \nu t^{-1} \otimes t \right)^{-1}
\]

\[
= \frac{e^{\eta/2} e^{i\eta(x-x)}}{1 + e^{-\eta/2} e^{i\eta x} e^{i\tilde{\eta}}} \tag{100}
\]

Here we used the change of variables

\[
x = x_1 + x_2, \quad \bar{x} = x_1 - x_2, \tag{101}
\]

inspired by the equality (97). Now we can easily calculate the comultiplication for the central operator:

\[
\Delta I = q^{-1} \Delta \nu \Delta \nu^2
\]

\[
= e^{i(\phi_1 + \phi_2)} e^{-i\eta x} e^{i\eta x} \frac{1 + e^{-\eta/2} e^{-i\tilde{\eta}} e^{i\eta x} e^{i\tilde{\eta}}}{1 + e^{-\eta/2} e^{i\tilde{\eta}} e^{-i\eta x} e^{i\tilde{\eta}}} \tag{102}
\]

In spite of their rather cumbersome form it is easy to check that the operators \( \Delta \rho^2 \) and \( \Delta \nu^2 \) indeed satisfy the commutation relations of the \( E_q(2) \) algebra. In fact, this immediately follows from their general form which can be written as follows

\[
\Delta \rho^2 = \text{const} \cdot e^{-i\tilde{\eta}} F(x, \tilde{\eta}),
\]

\[
\Delta \nu^2 = \text{const} \cdot \Delta I e^{2i\eta x}
\]

(here \( F(x, \tilde{\eta}) \) is a function of only \( x \) and \( \tilde{\eta} \) explicit form of which is given in (99)).

It is clear that if we start from some eigenvector of the operator \( \nu^2 \)

\[
\nu^2 |\phi_2, x_2 \rangle = e^{\eta/2} e^{2i\eta x} |\phi_2, x_2 \rangle \tag{103}
\]

(\textit{cf. (30)}) then, after the comultiplication, we have

\[
\Delta \nu^2 |\phi_1, x_1 \rangle |\phi_2, x_2 \rangle = e^{\eta/2} e^{-i(\phi_1 + \phi_2)} e^{2i\eta(x_1 + x_2)} \Delta I |\phi_1, x_1 \rangle |\phi_2, x_2 \rangle \tag{104}
\]

Since in any irreducible component the central element is proportional to the unity operator

\[
\Delta I P_\phi \left( |\phi_1, x_1 \rangle |\phi_2, x_2 \rangle \right) = e^{i\phi} P_\phi \left( |\phi_1, x_1 \rangle |\phi_2, x_2 \rangle \right) \tag{105}
\]

(\( P_\phi \) is a projector onto the irreducible component of the representation space corresponding to the central element eigenvalue \( e^{i\phi} \)), the relation (104) shows that for the \( \phi \)-component the eigenvalue of \( \Delta \nu^2 \) is

\[
e^{\eta/2} e^{i(\phi_1 + \phi_2 + 2i\eta)} \quad x = x_1 + x_2 \tag{106}
\]
This expression shows how the initial eigenvalue \( e^{\eta/2} e^{i \eta x_2} \) of the operator \( \nu^2 \) is transformed under the coaction of \( E_q^{(2)} \) with the state \( |\phi_1, x_1\rangle \) defining the “parameters” of the transformations (cf. (72), (74)).

The thing which we have to do now is to find the decomposition of an arbitrary function \( f(x_1, x_2) = f(x, \bar{x}) \in \mathcal{H}_{E_q(2)} \otimes \mathcal{H}_{E_q(2)} \) into eigenvectors of \( \Delta I \), i.e., into irreducible components. To this aim we must solve the eigenvalue equation

\[
\Delta I g_\phi(x, \bar{x}) = e^{i \phi} g_\phi(x, \bar{x}) .
\] (107)

The solution written in terms of the Fourier transform

\[
g_\phi(x, \bar{x}) = \sum_{k=-\infty}^{\infty} \tilde{g}_\phi(x, k) e^{i k \bar{x}} ,
\] (108)

has the form

\[
\tilde{g}_\phi(x, k) = e^{i(\phi - \phi_1 - \phi_2 + \eta x)k} e^{i d(x, k)} ,
\] (109)

d\((x, k)\) being defined by the simple recursion relation

\[
d(x, k + 1) = d(x, k) - \lambda(x, k + 1) , \quad d(x, 0) = 0 ,
\] (110)

where

\[
e^{i \lambda(x, k)} = \frac{1 + e^{-\eta/2} e^{-i \phi_1} e^{i n_2} e^{i k}}{1 + e^{-\eta/2} e^{i \phi_1} e^{-i n_2} e^{i k}} .
\] (111)

The solution of this recursion is obvious:

\[
d(x, k) = - \sum_{n=1}^{k} \lambda(x, n) , \quad k > 0 ,
\]

\[
d(x, k) = \sum_{n=0}^{k+1} \lambda(x, n) , \quad k < 0 ,
\]

\[
e^{i d(x, k)} = \prod_{n=1}^{k} \frac{1 + e^{-\eta/2} e^{i \phi_1} e^{-i n_2} e^{i m}}{1 + e^{-\eta/2} e^{i \phi_1} e^{i n_2} e^{i m}} , \quad k > 0 ,
\] (112)

\[
e^{i d(x, k)} = \prod_{n=1}^{k} \frac{1 + e^{-\eta/2} e^{-i \phi_1} e^{i n_2} e^{i m}}{1 + e^{-\eta/2} e^{-i \phi_1} e^{i n_2} e^{i m}} , \quad k < 0 ,
\] (113)

The solution (109), of course, is not square integrable function but a distribution (like momentum eigenvectors in ordinary Quantum Mechanics).

Any function \( f(x, k) \) can be presented as a Fourier integral of the functions \( g_\phi(x, k) \)

\[
f(x, k) = \frac{1}{2\pi} \int d\phi \ c(\phi) g_\phi(x, k)
\]

\[
= \frac{1}{2\pi} e^{-i(\phi_1 + \phi_1 - \eta x - c(x, k))} \int d\phi \ c(\phi) g_\phi(x, k) e^{i \phi k} ,
\] (114)

\[
c(\phi) = \sum_{k} e^{i(\phi_1 + \phi_1 - \eta x - c(x, k))} e^{-i \phi k} f(x, k) .
\] (115)
If we start from eigenvectors of the operator $\nu^2$ (in each component of the tensor product $L^2(S) \otimes L^2(S)$)

$$f(x, \bar{x}) \equiv |\phi_1, x_1\rangle |\phi_2, x_2\rangle = \delta^{(S)}(x_1 - X_1)\delta^{(S)}(x_2 - X_2)$$

$$= \frac{1}{(2\pi \eta)^2} \sum_{m, n = -\infty}^{\infty} e^{i\eta m(x - \bar{x}) + n(\bar{x} - \bar{X})}, \quad X = X_1 + X_2, \quad \bar{X} = X_1 - X_2$$

$(X_1, X_2)$ are the representation variables, while $x_1, x_2$ labels eigenvalues of the operator $\nu^2$: $\nu^2 \delta^{(S)}(x_i - X_i) = \exp \{\eta/2 + 2\eta x_i\} \delta^{(S)}(x_i - X_i)$, its decomposition over the eigenfunctions $g_\phi$ has the form (114) with the coefficients:

$$c(\phi, x, \phi_1, \phi_2) = \delta^{(S)}(x - X) \sum_{k = -\infty}^{\infty} e^{i(\phi_1 + \phi_2 - \eta x)k + id(x, k)} e^{-2i\eta \bar{X}k}.$$  \hspace{1cm} (116)

Thus

$$\Delta I |\phi_1, x_1\rangle |\phi_2, x_2\rangle = \frac{1}{2\pi} \int d\phi e^{i\phi} c(\phi, x, \phi_1, \phi_2) g_\phi(x, \bar{x}).$$  \hspace{1cm} (117)

Comparison of (117) and (117) allows to read off the expression for GCGC of the quantum group $E_q(2)$:

$$\langle \phi, y | S |\phi_1, x_1; \phi_2, x_2\rangle \equiv \delta^{(S)}(\phi - \phi_1 - \phi_2 + 2\eta(x_1 + x_2 - y)) C(\phi; \phi_1, x_1; \phi_2, x_2),$$

$$C(\phi; \phi_1, x_1; \phi_2, x_2) = c(\phi, x, \phi_1, \phi_2),$$  \hspace{1cm} (118)

and $c(\phi, x, \phi_1, \phi_2)$ is given by (116) and (112), (113).

The corresponding coaction on vectors on $P^{(2)}_q$ have quite the same form with the only restriction $\phi_2 = 0$. Now we are ready to answer the question about transformations of the coordinate operators eigenvalues. The operator $u^2 = \bar{z}z^{-1} \in P^{(2)}_q$ (cf. (16)) has the eigenvectors similar to those of $\nu^2$:

$$u^2 |x_2\rangle = e^{\eta/2} e^{2i\eta x_2} |x_2\rangle.$$  \hspace{1cm} (119)

After the coaction $\delta$ by $E_q(2)$-quantum group, we obtain the operator $\delta u^2$ acting on vectors $|\phi_1, x_1\rangle |x_2\rangle \in \mathcal{H}_{E_q(2)} \otimes \mathcal{H}_{P^{(2)}_q}$ and with a structure which is quite similar to $\Delta u^2$ (cf. (104))

$$(\delta u^2) |\phi_1, x_1\rangle |x_2\rangle = e^{\eta/2} e^{-i\phi_1} e^{2i\eta(x_1 + x_2)} \Delta I |\phi_1, x_1\rangle |\phi_2, x_2\rangle.$$  \hspace{1cm} (120)

This implies that the resulting vector can be decomposed into irreducible parts and, simultaneously, into vectors with definite values of the coordinate $u^2$ as follows:

$$(\delta u^2) |\phi_1, x_1\rangle |x_2\rangle = \frac{1}{2\pi} e^{\eta/2} \int d\phi c(\phi, x, \phi_1, \phi_2) e^{i(\phi - \phi_1 + 2\eta x_2)} g_\phi(x, \bar{x}).$$  \hspace{1cm} (121)

The expression (122) presents the form of transformation of position eigenvectors on a quantum plane $P^{(2)}_q$ and shows that the coaction of $E_q(2)$ induces nonlocal transformations of the states.
5 Conclusion

We have shown that transition to a noncommutative $q$-deformed plane does not lead to an ultraviolet regularization of the scalar $\varphi^4$-quantum field theory. We start from the firstly quantized theory of quantum particles on the noncommutative plane. Then we have defined quantum fields depending on noncommutative coordinates and the field theoretical action using the quantum analog of the Haar ($E_q(2)$-invariant) measure on the noncommutative plane. With the help of the partial wave decomposition we have shown that this quantum field theory can be considered as a second quantization of the particle theory on the noncommutative plane and that it has (contrary to the common belief) even more severe ultraviolet divergences than its counterpart on the usual commutative plane.

We have discussed symmetry transformations on the noncommutative spaces and the induced transformations of the states. In the case of Lie algebra-like spaces the coordinates form a tensor operator $\hat{x}_i \to \hat{x}'_i = M_{ij}\hat{x}_j + b_i = \hat{U}_g\hat{x}_i\hat{U}_g^{-1}$ and states of the field system are transformed by the operator $\hat{U}_g$. We considered the example of such transformations for the case of noncommutative Euclidean and Minkowski planes in the preceding paper [10]. In the $q$-deformed case, we have shown that the quantum group coaction on a coordinate algebra induces nonlocal transformations of states in the coordinate space. These transformations are defined by the generalized Clebsch-Gordan coefficients, describing decomposition of tensor products of representations of algebras of functions on quantum spaces and representations of the corresponding quantum group. In other words, the coaction puts in correspondence to a pair of states on a group algebra $G_q$ and on a quantum space $X_q$ some new state on the quantum space. We have considered such transformations for the case of the $q$-deformed plane with the $E_q(2)$-symmetry.

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