THE FORWARD-BACKWARD-FORWARD METHOD FROM DISCRETE AND CONTINUOUS PERSPECTIVE FOR PSEUDO-MONOTONE VARIATIONAL INEQUALITIES IN HILBERT SPACES

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Abstract. Tseng’s forward-backward-forward algorithm is a valuable alternative for Korpelevich’s extragradient method when solving variational inequalities over a convex and closed set governed by monotone and Lipschitz continuous operators, as it requires in every step only one projection operation. However, it is well-known that Korpelevich’s method is provable convergent and thus applicable when solving variational inequalities governed by a pseudo-monotone and Lipschitz continuous operator. In this paper we prove that Tseng’s method converges also when it is applied to the solving of pseudo-monotone variational inequalities. In addition, we show that linear convergence is guaranteed under strong pseudo-monotonicity. We also associate a dynamical system to the pseudo-monotone variational inequality and carry out an asymptotic analysis for the generated trajectories. Numerical experiments show that Tseng’s method outperforms Korplelevich’s extragradient method when applied to the solving of pseudo-monotone variational inequalities and fractional programming problems.

Key words. variational inequalities, pseudo-monotonicity, Tseng’s forward-backward-forward algorithm, dynamical system

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1. Introduction and preliminaries. The object of our investigation is the following variational inequality:

Find $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C,$$

where $C$ is a nonempty, convex and closed subset of the real Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$, and $F : H \to H$ is a Lipschitz continuous operator. We abbreviate the problem (1) as $\text{VI}(F, C)$ and denote its solution set by $\Omega$.

Variational inequalities (VIs) are powerful mathematical models which unify important concepts in applied mathematics, like systems of nonlinear equations, optimality conditions for optimization problems, complementarity problems, obstacle problems, and network equilibrium problems (see, for instance, [12, 14]). In the last decades, various solution methods have been proposed for solving problems of type $\text{VI}(F, C)$ under various conditions on the governing operator $F$ (see [12, 14]). The majority of the solution methods typically require certain monotonicity properties for the operator $F$. When $F$ is the gradient of a differentiable function, this corresponds to generalized convexity properties for the latter (see [18]).

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The most popular algorithm for solving variational inequalities is the so-called gradient projection method, which generates a sequence that approaches the solution set $\Omega$ by
\[ x_{n+1} = P_C(x_n - \lambda F(x_n)) \quad \forall n \geq 0, \]
where $P_C$ is the projection operator onto the convex and closed set $C$ and $\lambda$ is a positive stepsize. It is well-known that the sequence $(x_n)_{n \geq 0}$ generated in this way converges when $F$ is cocoercive (inverse strongly monotone) ([3, 26]) or strongly (pseudo-) monotone ([12, 16]). If $F$ is “only” monotone, then convergence may fail (see [12] for an example). Very recently, Malitsky [19] has introduced a modification of the gradient projection method, called projected reflected gradient method, which reads
\[ x_{n+1} = P_C(x_n - \lambda F(2x_n - x_{n-1})) \quad \forall n \geq 0 \]
and generates a sequence $(x_n)_{n \geq 0}$ which converges to an element in $\Omega$ when $F$ is monotone. Further extensions of this method can be found in [20, 21].

When $F$ is a pseudo-monotone operator, the only known algorithm for solving $\text{VI}(F, C)$ is the extragradient method introduced by Korpelevich in [15], which generates a sequence approaching the solution by carrying out two projections per iteration
\[
\begin{align*}
  y_n &= P_C(x_n - \lambda F(x_n)) \\
  x_{n+1} &= P_T(x_n - \lambda F(y_n))
\end{align*}
\quad \forall n \geq 0.
\]

Although the extragradient method was originally introduced for solving monotone VIs in finite dimensional spaces, it has been shown in [12, Theorem 12.2.11] that it converges even when applied to the solving of pseudo-monotone VIs. Due to its importance in applications, in particular in the context of solving variational inequalities governed by pseudo-monotone operators, the extragradient method has been extensively investigated and extended (see, for instance, [7, 9, 12, 13, 22, 23, 25]). Many efforts have been paid to extend the convergence analysis of the extragradient method from finite to infinite dimensional Hilbert spaces. In [7], Ceng, Teboulle and Yao proved the weak convergence of extragradient method in a infinite dimensional setting by assuming that $F$ is a sequentially weak-to-strong continuous operator. This assumption, which is not even satisfied by the identity operator, has been recently weakened in [25] to the more reasonable sequential weak-to-weak continuity in the context of proving weak convergence for the extragradient method.

A challenging task when designing new efficient algorithms for solving VIs is to minimize the number of projection operations performed at each iteration, since these may be very expensive. Censor, Gibali and Reich proposed in [8, 9] the following numerical scheme, called subgradient extragradient method,
\[
\begin{align*}
  y_n &= P_C(x_n - \lambda F(x_n)) \\
  x_{n+1} &= P_{T_n}(x_n - \lambda F(y_n))
\end{align*}
\quad \forall n \geq 0,
\]
where
\[ T_n = \{ w \in H : \langle x_n - \lambda F(x_n) - y_n, w - y_n \rangle \leq 0 \}. \]

Since the projection onto the half-space $T_n$ can be expressed by an explicit formula (see, for instance, [3]), the subgradient extragradient method requires only one projection per iteration and outperforms from this point of view the extragradient method. The subgradient extragradient method converges for pseudo-monotone VIs
in finite dimensional spaces ([8]) and for monotone VIs in infinite dimensional Hilbert spaces ([9]).

Another numerical method which can be used to solve VI($F,C$) and requires only one projection per iteration is Tseng’s forward-backward-forward algorithm ([24]):

\[
\begin{cases}
  y_n = P_C(x_n - \lambda F(x_n)) \\
  x_{n+1} = y_n + \lambda (F(x_n) - F(y_n))
\end{cases}
\]

\[\forall n \geq 0.\]

It is well-known that the sequence $(x_n)_{n \geq 0}$ converges weakly to a solution of VI($F,C$) when $F$ is a monotone operator. In this paper we will show that the forward-backward-forward method converges even if $F$ is a pseudo-monotone and sequentially weak-to-weak-continuous operator. Consequently, Tseng’s algorithm turns out to be a valuable method for solving pseudo-convex optimization problems. We also show that linear convergence convergence is guaranteed when the pseudo-monotonicity for $F$ is replaced by strong pseudo-monotonicity.

In addition, we attach to VI($F,C$) the following dynamical system of forward-backward-forward-type

\[
\begin{cases}
  y(t) = P_C(x(t) - \lambda F(x(t))) \\
  \dot{x}(t) + x(t) = y(t) + \lambda [F(x(t)) - F(y(t))] \\
  x(0) = x_0,
\end{cases}
\]

and investigate the asymptotic convergence of the generated trajectories to an element in $\Omega$ when $F$ is a pseudo-monotone operator. If $F$ is assumed to be strongly pseudo-monotone, then we obtain the exponential convergence of the trajectories to the unique solution of VI($F,C$). Dynamical systems of this type have been first studied in [2] in the context of approaching from a continuous perspective the set of the zeros of the sum of a maximally monotone operator and a monotone and Lipschitz continuous operator.

In the last part of the paper we compare by means of different numerical experiments the behavior of Tseng’s method with the others algorithms for solving pseudo-monotone VIs and pseudo-convex optimization problems.

We close this section by recalling some notions and results which will be useful within this paper.

**Definition 1.1.** Let $C$ be a nonempty subset of the real Hilbert space $H$. The mapping $F : H \to H$ is said to be

(a) **pseudo-monotone on $C$**, if for all $x, y \in C$ it holds

\[\langle F(x), y - x \rangle \geq 0 \Rightarrow \langle F(y), y - x \rangle \geq 0;\]

(b) **monotone on $C$**, if

\[\langle F(y) - F(x), y - x \rangle \geq 0 \quad \forall x, y \in C;\]

(c) **$\gamma$-strongly pseudo-monotone on $C$** with $\gamma > 0$, if for all $x, y \in C$ it holds

\[\langle F(x), y - x \rangle \geq 0 \Rightarrow \langle F(y), y - x \rangle \geq \gamma \|x - y\|^2;\]

(d) **$\gamma$-strongly monotone on $C$** with $\gamma > 0$, if for all $x, y \in C$ it holds

\[\langle F(y) - F(x), y - x \rangle \geq \gamma \|x - y\|^2.\]
We recall that the operator \( F : H \to H \) is called \textit{Lipschitz continuous with Lipschitz constant} \( L > 0 \), if for all \( x, y \in H \)
\[ \|F(x) - F(y)\| \leq L\|x - y\|. \]
The operator \( F \) is called \textit{sequential weak-to-weak continuous}, if for every sequence \((x_n)_{n \geq 0} \) that converges weakly to \( x \) it holds that \((F(x_n))_{n \geq 0} \) converges weakly to \( F(x) \).

For a nonempty, convex and closed set \( C \subseteq H \) and every arbitrary \( x \in H \), there exists a unique element in \( C \), denoted by \( P_C(x) \), such that
\[ \|x - P_C(x)\| \leq \|x - y\| \quad \forall y \in C. \]
The operator \( P_C : H \to C \) is the so-called \textit{projection operator} onto \( C \). For all \( x \in H \) and \( y \in C \) it holds
\[
\langle x - P_C(x), y - P_C(x) \rangle \leq 0.
\]
One can also easily see that, for all \( \lambda > 0 \), \( x^* \) is a solution of \( \text{VI}(F, C) \) if and only if \( x^* = P_C(x^* - \lambda F(x^*)) \). We recall the following characterization of the solution set of \( \text{VI}(F, C) \), when \( F : H \to H \) is a pseudo-monotone operator ([10, Lemma 2.1]).

**Proposition 1.1.** Let \( C \) be a nonempty, convex and closed set of the real Hilbert space \( H \) and \( F : H \to H \) an operator which is pseudo-monotone on \( C \) and continuous. Then for every \( x^* \in C \) we have
\[
\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C \iff \langle F(x), x - x^* \rangle \geq 0 \quad \forall x \in C.
\]

2. **Convergence analysis.** In this section we analyze the convergence of Tseng’s forward-backward-forward method in the context of solving pseudo-monotone VIs.

**Algorithm 2.1.** \textbf{Initialization:} Choose \( x_0 \in H, \lambda > 0 \) and set \( n = 0 \).

\textbf{Step 1:} Compute
\[ y_n = P_C(x_n - \lambda F(x_n)). \]

If \( y_n = x_n \) or \( F(y_n) = 0 \), then STOP: \( y_n \) is a solution.

\textbf{Step 2:} Set
\[ x_{n+1} = y_n + \lambda(F(x_n) - F(y_n)), \]

update \( n \) to \( n+1 \) and go to \textbf{Step 1}.

For the convergence analysis we assume that Algorithm 2.1 does not terminate after a finite number of iterations. In other words, for all \( n \geq 0 \) it holds \( x_n \neq y_n \) and \( F(y_n) \neq 0 \).

**Proposition 2.1.** Assume that the solution set \( \Omega \) is nonempty and \( F \) is pseudo-monotone on \( C \) and Lipschitz continuous with constant \( L \). Then for every solution \( x^* \in \Omega \) and every \( n \geq 0 \) it holds
\[
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda^2L^2)\|y_n - x_n\|^2.
\]

**Proof.** Let \( x^* \) be an arbitrary element in \( \Omega \) and \( n \geq 0 \) be fixed. Then we have
\[ \langle F(x^*), y - x^* \rangle \geq 0 \quad \forall y \in C. \]
Substituting \( y := y_n \in C \) into this inequality it yields
\[ \langle F(x^*), y_n - x^* \rangle \geq 0. \]
From the pseudo-monotonicity of $F$ on $C$ it follows
\begin{equation}
\langle F(y_n), y_n - x^* \rangle \geq 0.
\end{equation}

Since $y_n = P_C(x_n - \lambda F(x_n))$, according to (3), we get
\begin{equation}
\langle y - y_n, y_n - x_n + \lambda F(x_n) \rangle \geq 0 \quad \forall y \in C,
\end{equation}
which yields
\begin{equation}
\langle x^* - y_n, y_n - x_n + \lambda F(x_n) \rangle \geq 0.
\end{equation}

Multiplying both sides of (6) by $\lambda > 0$ and adding the resulting inequality to (7) it yields
\begin{equation}
\langle x^* - y_n, y_n - x_n + \lambda F(x_n) - \lambda F(y_n) \rangle \geq 0
\end{equation}
or, equivalently,
\begin{equation}
\langle x^* - y_n, x_{n+1} - x_n \rangle \geq 0.
\end{equation}

This implies that
\begin{equation}
\langle x_{n+1} - x^*, x_{n+1} - x_n \rangle \leq \langle x_{n+1} - y_n, x_{n+1} - x_n \rangle
\end{equation}
\begin{align*}
&= \|x_{n+1} - x_n\|^2 + \langle x_n - y_n, x_{n+1} - x_n \rangle \\
&= \|x_{n+1} - x_n\|^2 + \langle x_n - y_n, y_n + \lambda (F(x_n) - F(y_n)) - x_n \rangle \\
&= \|x_{n+1} - x_n\|^2 - \|y_n - x_n\|^2 + \lambda \langle x_n - y_n, F(x_n) - F(y_n) \rangle.
\end{align*}

On the other hand, we have
\begin{equation}
\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 + \|x_{n+1} - x_n\|^2 = 2 \langle x_{n+1} - x^*, x_{n+1} - x_n \rangle.
\end{equation}
Combining (8) and (9) we obtain
\begin{equation}
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \|x_{n+1} - x_n\|^2 - 2\|y_n - x_n\|^2 \\
+ 2\lambda \langle x_n - y_n, F(x_n) - F(y_n) \rangle.
\end{equation}

Using the Lipschitz continuity of $F$ we obtain
\begin{align*}
\|x_{n+1} - x_n\|^2 &= \|y_n + \lambda (F(x_n) - F(y_n)) - x_n\|^2 \\
&= \|y_n - x_n\|^2 + 2\lambda \langle y_n - x_n, F(x_n) - F(y_n) \rangle + \lambda^2 \|F(x_n) - F(y_n)\|^2 \\
&\leq \|y_n - x_n\|^2 + 2\lambda \langle y_n - x_n, F(x_n) - F(y_n) \rangle + \lambda^2 L^2 \|x_n - y_n\|^2.
\end{align*}

Finally, from (10) and (11) it yields
\begin{equation}
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda^2 L^2) \|y_n - x_n\|^2.
\end{equation}

Remark 2.1. One can notice that in the above proof the pseudo-monotonicity of $F$ is used only in order to obtain relation (6). Thus, as happened in [22, 11], the pseudo-monotonicity of $F$ can be actually replaced by the following weaker assumption
\begin{equation}
\langle F(x), x - x^* \rangle \geq 0 \quad \forall x \in C \forall x^* \in \Omega.
\end{equation}
Remark 2.2. In contrast to the extragradient method, the sequence \((x_n)_{n \geq 0}\) generated by Algorithm 2.1 may not be feasible. This is why we need to ask in the convergence analysis that \(F\) is Lipschitz continuous on the whole space \(H\). However, if the feasible set \(C\) is bounded, then we can weaken this assumption by asking that \(F\) is Lipschitz continuous on the bounded set
\[ D := \{ x + y : x \in C, \|y\| \leq d \}, \]
where \(d\) denotes the diameter of \(C\). Notice that \(C \subseteq D\). In this case, if we start Algorithm 2.1 with an element \(x_0 \in C\) and choose \(\lambda < \frac{1}{L}\), then from (11) we have
\[
\|x_1 - x^*\|^2 \leq \|x_0 - x^*\|^2 - (1 - \lambda^2 L^2) \|y_0 - x_0\|^2,
\]
which implies that \(\|x_1 - x^*\| \leq \|x_0 - x^*\| \leq d\). Since \(x_1 = x^* + x_1 - x^*\), we have \(x_1 \in D\). By induction, we obtain \(\|x_n - x^*\| \leq d\) and therefore \(x_n \in D\) for every \(n \geq 0\).

According to Proposition 2.1 we have that the sequence \((x_n)_{n \geq 0}\) is Féjer monotone with respect to the solution set \(\Omega\). To obtain the convergence of the sequence \((x_n)_{n \geq 0}\) to an element in \(\Omega\), in the light of the Opial Lemma, it is remain to prove that every weak sequential cluster point of the sequence belongs to \(\Omega\).

**Proposition 2.2.** Assume that the solution set \(\Omega\) is nonempty, \(\lambda < \frac{1}{L}\), \(F\) is pseudo-monotone on \(H\), Lipschitz continuous with constant \(L > 0\) and sequentially weak-to-weak continuous. Then every weak sequential cluster point of the sequence \((x_n)_{n \geq 0}\) generated by Algorithm 2.1 is a solution of \(\text{VI}(F,C)\).

**Proof.** Let be \(x^* \in \Omega\) fixed. Since \(\lambda < \frac{1}{L}\), from (5) we have that the sequence \((\|x_n - x^*\|^2)_{n \geq 0}\) is monotonically decreasing and therefore convergent. In addition, we have
\[
\lim_{n \to \infty} \|y_n - x_n\| = 0.
\]
Since \(F\) is Lipschitz continuous on \(H\), we have
\[
\|F(x_n) - F(y_n)\| \leq L\|x_n - y_n\| \quad \forall n \geq 0,
\]
and therefore
\[
\lim_{n \to \infty} \|F(x_n) - F(y_n)\| = 0.
\]
Let \(\hat{x}\) be a weak sequential cluster point of \((x_n)_{n \geq 0}\) and let \((x_{n_k})_{k \geq 0}\) be a subsequence of \((x_n)_{n \geq 0}\) which converges weakly to \(\hat{x}\) as \(k \to \infty\). Since \(\lim_{k \to \infty} \|x_{n_k} - y_{n_k}\| = 0\), \((y_{n_k})_{k \geq 0}\) also converges weakly to \(\hat{x}\) as \(k \to \infty\). Since \((y_n)_{n \geq 0} \subseteq C\) and \(C\) is weakly closed, we have \(\hat{x} \in C\). Our aim is to prove that \(\hat{x} \in \Omega\). We assume that \(F(\hat{x}) \neq 0\), otherwise the conclusion follows automatically.

Let \(y \in C\) be fixed. For every \(k \geq 0\) we have
\[
y_{n_k} = P_C(x_{n_k} - \lambda F(x_{n_k})),
\]
thus
\[
\langle x_{n_k} - \lambda F(x_{n_k}) - y_{n_k}, y - y_{n_k} \rangle \leq 0
\]
or, equivalently,
\[
(12) \quad \frac{1}{\lambda} \langle x_{n_k} - y_{n_k}, y - y_{n_k} \rangle \leq \langle F(x_{n_k}) - F(y_{n_k}), y - y_{n_k} \rangle + \langle F(y_{n_k}), y - y_{n_k} \rangle.
\]
Letting in the last inequality $k \to +\infty$ and taking into account that $\lim_{k \to \infty} \|x_{n_k} - y_{n_k}\| = 0$, $\lim_{k \to \infty} \|F(x_{n_k}) - F(y_{n_k})\| = 0$ and $(y_{n_k})_{k \geq 0}$ is bounded, it follows

$$\lim_{k \to \infty} \inf \langle F(y_{n_k}), y - y_{n_k} \rangle \geq 0.$$  

Let $(\epsilon_k)_{k \geq 0}$ be a positive strictly decreasing sequence which converges to 0 as $k \to \infty$. We can construct inductively a strictly increasing sequence $(N_k)_{k \geq 0}$ with the property that

$$(13) \quad \langle F(y_{n_{N_k}}), y - y_{n_{N_k}} \rangle + \epsilon_k \geq 0 \quad \forall k \geq 0.$$  

For every $k \geq 0$ we have $F(y_{n_{N_k}}) \neq 0$ and, setting

$$z_k := \frac{F(y_{n_{N_k}})}{\|F(y_{n_{N_k}})\|^2},$$

it holds $\langle F(y_{n_{N_k}}), z_k \rangle = 1$. According to (13) we have that

$$\langle F(y_{n_{N_k}}), y + \epsilon_k z_k - y_{n_{N_k}} \rangle \geq 0 \quad \forall k \geq 0.$$  

Since $F$ is pseudo-monotone on $H$, this yields

$$(14) \quad \langle F(y + \epsilon_k z_k), y + \epsilon_k z_k - y_{n_{N_k}} \rangle \geq 0 \quad \forall k \geq 0.$$  

On the other hand, we have that $\{y_{n_{N_k}}\}$ converges weakly to $\hat{x}$ as $k \to \infty$. Since $F$ is sequentially weak-to-weak continuous, $\{F(y_{n_{N_k}})\}$ converges weakly to $F(\hat{x})$ as $k \to \infty$. Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 < \|F(\hat{x})\| \leq \lim_{k \to \infty} \inf \|F(y_{n_{N_k}})\|,$$

which implies that $\left(\frac{1}{F(y_{n_{N_k}})}\right)_{k \geq 0}$ is bounded and, thus,

$$\lim_{k \to \infty} \|\epsilon_k z_k\| = \lim_{k \to \infty} \frac{\epsilon_k}{\|F(y_{n_{N_k}})\|} = 0.$$  

Taking in (14) the limit as $k \to \infty$ we obtain

$$\langle F(y), y - \hat{x} \rangle \geq 0.$$  

As $y$ was arbitrarily chosen in $C$, it follows from Proposition 1.1 that $\hat{x} \in \Omega$.  

Propositions 2.1 and 2.2 together with the Opial Lemma (see [3, Theorem 5.5]) lead to the following convergence statement.

**Theorem 2.1.** Assume that the solution set $\Omega$ is nonempty, $\lambda < \frac{1}{L}$, and $F$ is pseudo-monotone on $H$, Lipschitz continuous with constant $L > 0$ and sequentially weak-to-weak continuous. Then the sequence $(x_n)_{n \geq 0}$ generated by Algorithm 2.1 converges weakly to a solution of $VI(F, C)$.  

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Remark 2.3. The conclusion of Theorem 2.1 remains valid if we replace in every iteration of Algorithm 2.1 the fixed stepsize $\lambda > 0$ by a variable stepsize $\lambda_n$, where the sequence $(\lambda_n)_{n \geq 0}$ fulfills

$$0 < \inf_{n \geq 0} \lambda_n \leq \sup_{n \geq 0} \lambda_n < \frac{1}{L}.$$  

On the other hand, when (an upper bound of) the Lipschitz constant of $F$ is not available, we can use in Algorithm 2.1 the following adaptive stepsize strategy

$$\lambda_{n+1} := \begin{cases} 
\min \left\{ \frac{\rho \|x_n - y_n\|}{\|F(x_n) - F(y_n)\|}, \lambda_n \right\}, & \text{if } F(x_n) - F(y_n) \neq 0, \\
\lambda_n, & \text{otherwise}, 
\end{cases}$$

where $\rho \in (0,1)$ and $\lambda_0 > 0$. The sequence $(\lambda_n)_{n \geq 0}$ is nonincreasing. If, for $n \geq 0$, $F(x_n) - F(y_n) \neq 0$, then it holds

$$\frac{\rho \|x_n - y_n\|}{\|F(x_n) - F(y_n)\|} \geq \frac{\rho \|x_n - y_n\|}{L \|x_n - y_n\|} = \frac{\rho}{L},$$

which shows that $(\lambda_n)_{n \geq 0}$ bounded from below by $\min \left\{ \lambda_0, \frac{\rho}{L} \right\} > 0$. Notice that, if $\lambda_0 \leq \frac{\rho}{L}$, then $(\lambda_n)_{n \geq 0}$ is a constant sequence, which leads to a fixed stepsize strategy. Consequently, the limit $\lim_{n \to \infty} \lambda_n$ exists and it is a positive real number.

This means that we can adapt the proof of Proposition 2.1 to the new adaptive stepsize strategy and, by taking into consideration (11), we get instead of (5)

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \left( 1 - \frac{\lambda_n^2 \rho^2}{\lambda_{n+1}^2} \right) \|y_n - x_n\|^2 \forall n \geq 0.$$  

Due to $\lim_{n \to \infty} \left( 1 - \frac{\lambda_n^2 \rho^2}{\lambda_{n+1}^2} \right) = 1 - \rho^2 > 0$, there exists $N > 0$ such that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| \forall n \geq N,$$

which implies that $\lim_{n \to \infty} \|x_n - x^*\|$ exists and $\lim_{n \to \infty} \|x_n - y_n\| = 0$. In conclusion, the same convergence analysis as for the fixed stepsize strategy can be carried out.

Remark 2.4. If the operator $F$ is monotone on $C$, then it is not necessary to impose that $F$ is sequentially weak-to-weak continuous. Indeed, for $y \in C$ fixed, relation (12) in the proof of Proposition 2.2 gives rise to

$$\frac{1}{\lambda} \left\langle x_{n_k} - y_{n_k}, y - y_{n_k} \right\rangle \leq \left\langle F(x_{n_k}) - F(y_{n_k}), y - y_{n_k} \right\rangle + \left\langle F(y_{n_k}), y - y_{n_k} \right\rangle$$

$$\leq \left\langle F(x_{n_k}) - F(y_{n_k}), y - y_{n_k} \right\rangle + \left\langle F(y), y - y_{n_k} \right\rangle \forall k \geq 0.$$  

Letting $k \to \infty$ we get

$$\left\langle F(y), y - \hat{x} \right\rangle \geq 0$$

and this leads to the desired conclusion.

In finite dimensional spaces, the conclusion in Theorem 2.1 follows under weaker assumptions.

Theorem 2.2. Let $H$ be a finite dimensional real Hilbert space. Assume that the solution set $\Omega$ is nonempty, $\lambda < \frac{1}{L}$, and $F$ is pseudo-monotone on $C$ and Lipschitz continuous with constant $L > 0$. Then the sequence $(x_n)_{n \geq 0}$ generated by Algorithm 2.1 converges to a solution of VI($F; C$).
Proof. Let be \( x^* \in \Omega \) fixed. Since \( \lambda < \frac{1}{L} \), from (5) it follows that the sequence \( (\|x_n - x^*\|^2)_{n\geq0} \) is monotonically decreasing and therefore convergent. In addition, we have
\[
\lim_{n \to \infty} \|y_n - x_n\| = 0.
\]
As \((x_n)_{n\geq0}\) is bounded, it has subsequence \((x_{n_k})_{k\geq0}\) which converges to an element \( \hat{x} \) as \( k \to \infty \). Since \( \lim_{n \to \infty} \|x_{n_k} - y_{n_k}\| = 0 \), \((y_{n_k})_{k\geq0}\) also converges to \( \hat{x} \) as \( k \to \infty \).

Let be \( y \in C \) fixed. Then we have that
\[
\langle y - y_{n_k}, y_{n_k} - x_{n_k} + \lambda F(x_{n_k}) \rangle \geq 0 \quad \forall k \geq 0.
\]
Taking the limit as \( k \to \infty \) and using that \( F \) is continuous, we obtain
\[
\langle y - \hat{x}, F(\hat{x}) \rangle \geq 0.
\]
Since \( y \in C \) has been arbitrarily chosen, it follows that \( \hat{x} \) is a solution of VI\((F,C)\).

Replacing in (5) \( x^* \) with \( \hat{x} \), it follows that the sequence \( (\|x_n - \hat{x}\|)_{n\geq0} \) is convergent. Since \( \lim_{k \to \infty} \|x_{n_k} - \hat{x}\| = 0 \), it follows that \( \lim_{n \to \infty} x_n = \hat{x} \).

We give in the following a class of operators which are not monotone but pseudo-monotone and Lipschitz continuous on \( H \) and which we will use later in our numerical experiments.

**Example 2.1.** Let \( H \) be a real Hilbert space, let \( Q, M : H \to H \) be bounded linear operators satisfying
\[
\langle Qx, x \rangle \geq \theta \|x\|^2 \quad \forall x \in H,
\]
where \( \theta > 0 \), and
\[
\langle Mx, x \rangle \geq 0 \quad \forall x \in H.
\]

Let \( F : H \to H \) be defined as
\[
F(x) := (e^{-\langle x, Qx \rangle} + \alpha)(Mx + p),
\]
where \( \alpha \geq 0 \) and \( p \) is an arbitrary vector in \( H \). Then \( F \) is in general not monotone on \( H \) (see the operator plotted in Figure 1 in case \( H = \mathbb{R} \) and the operator in Example 3.1 in case \( H = \mathbb{R}^3 \)). On the other hand, \( F \) is pseudo-monotone on \( H \) and Lipschitz continuous. In order to prove the pseudo-monotonicity, we define \( g : H \to \mathbb{R}, g(x) = e^{-\langle x, Qx \rangle} + \alpha \). Let \( x, y \in H \) be such that \( \langle F(x), y - x \rangle \geq 0 \). Since \( g(x) > 0 \), we have
\[
\langle Mx + p, y - x \rangle \geq 0.
\]
Hence
\[
\langle F(y), y - x \rangle = g(y)\langle My + p, y - x \rangle \geq g(y)(\langle My + p, y - x \rangle - \langle Mx + p, y - x \rangle) = g(y)\langle M(y - x), y - x \rangle \geq 0.
\]

We will show now that \( F \) is Lipschitz continuous. For every \( x, h \in H \) we have
\[
\nabla F(x)(h) = e^{-\langle x, Qx \rangle} \langle -2Qx, h \rangle (Mx + p) + \left(e^{-\langle x, Qx \rangle} + \alpha \right) Mh.
\]
Therefore,
\[
\| \nabla F(x) \| \leq 2\|Q\|e^{-\langle x, Qx \rangle} \left( \|M\| \|x\|^2 + \|p\| \|x\| \right) + \left( e^{-\langle x, Qx \rangle} + \alpha \right) \|M\| \\
\leq 2\|Q\| e^{-\theta \|x\|^2} \left( \|M\| \|x\|^2 + \|p\| \|x\| \right) + \left( e^{-\theta \|x\|^2} + \alpha \right) \|M\| \\
\leq L, \tag{15}
\]
for some \( L > 0 \), where we took into consideration the fact that the function \( t \mapsto e^{-\theta t^2} \left( \|M\| t^2 + \|p\| t \right) \) is bounded from above on \([0, +\infty)\). This shows that \( F \) is Lipschitz continuous on \( H \) with constant \( L \).

\[\text{Fig. 1. The graphs of } F : \mathbb{R} \to \mathbb{R}, F(x) = xe^{-x^2}, \text{ (blue)} \text{ and } \nabla F : \mathbb{R} \to \mathbb{R}, \nabla F(x) = (1 - 2x^2)e^{-x^2}, \text{ (red)}.\]

**Example 2.2.** A differentiable function \( f : E \to \mathbb{R} \), where \( E \subseteq \mathbb{R}^n \) is an open set, is called *pseudo-convex on* \( E \), if for all \( x, y \in E \) it holds
\[
\langle \nabla f(x), y - x \rangle \geq 0 \quad \Rightarrow \quad f(y) \geq f(x).
\]
It is well-known that \( f \) is pseudo-convex on \( E \) if and only if \( \nabla f \) is pseudo-monotone on \( E \) ([18]). Algorithm 2.1 can be used to solve optimization problems of the form
\[
\min_{x \in C} f(x),
\]
where \( f : \mathbb{R}^n \to \mathbb{R} \) is a differentiable function with Lipschitz continuous gradient which is also pseudo-convex on an open set \( E \subseteq \mathbb{R}^n \), and \( C \subseteq E \) is a nonempty, convex and closed set. Recall that when \( E \subseteq \mathbb{R}^n \) is a convex set, \( g : E \to [0, +\infty) \) is a convex function, \( h : E \to (0, +\infty) \) is a concave function, and both \( f \) and \( g \) are differentiable on \( E \), then the function
\[
\frac{f}{g} : E \to [0, +\infty), \quad \frac{f}{g}(x) = \frac{f(x)}{g(x)},
\]
is pseudo-convex on \( E \) ([4]).

In the following we show that when \( F \) is strongly pseudo-monotone on \( C \), then Algorithm 2.1 generates a sequence which converges linearly to the unique solution of \( \text{VI}(F, C) \). We extend in this way a result proved by Tseng in [24] for strongly monotone operators.

**Theorem 2.3.** Assume that \( \lambda < \frac{1}{L} \), and \( F \) is \( \gamma \)-strongly pseudo-monotone on \( C \) with \( \gamma > 0 \) and Lipschitz continuous with constant \( L > 0 \). Then the sequence \((x_n)_{n \geq 0}\) generated by Algorithm 2.1 converges linearly to the unique solution \( x^* \) of \( \text{VI}(F, C) \). In addition, the following estimate holds
\[
\|x_{n+1} - x^*\| \leq \delta \|x_n - x^*\| \quad \forall n \geq 0,
\]
where \( \delta := \left( 1 - (1 - \lambda^2 L^2) \left( \frac{\lambda \gamma}{1 + \lambda L + \lambda \gamma} \right)^2 \right)^{1/2} \in (0, 1) \).

*Proof.* Let \( n \geq 0 \) be fixed and \( x^* \) the unique solution of the problem \( \text{VI}(F, C) \) ([17]). Since \( y_n \in C \) we have
\[
\langle F(x^*), y_n - x^* \rangle \geq 0,
\]
which implies, according to the strong pseudo-monotonicity of \( F \) on \( C \), that
\[
\langle F(y_n), y_n - x^* \rangle \geq \gamma \|y_n - x^*\|^2.
\]
Using the Lipschitz continuity of \( F \) we get
\[
\langle F(x_n), x^* - y_n \rangle = \langle F(x_n) - F(y_n), x^* - y_n \rangle + \langle F(y_n), y_n - x^* \rangle \\
\leq \|F(x_n) - F(y_n)\| \|y_n - x^*\| - \gamma \|y_n - x^*\|^2 \\
\leq L \|x_n - y_n\| \|y_n - x^*\| - \gamma \|y_n - x^*\|^2,
\]
which, in combination with (7), gives
\[
\langle x^* - y_n, x_n - y_n \rangle \leq \lambda \langle F(x_n), x^* - y_n \rangle \\
\leq \lambda L \|x_n - y_n\| \|y_n - x^*\| - \lambda \gamma \|y_n - x^*\|^2
\]
and, further,
\[
\lambda \gamma \|y_n - x^*\|^2 \leq \lambda L \|x_n - y_n\| \|y_n - x^*\| \|x^* - y_n, x_n - y_n\| \\
\leq \lambda L \|x_n - y_n\| \|y_n - x^*\| + \|x^* - y_n\| \|x_n - y_n\| \\
= (1 + \lambda L) \|x_n - y_n\| \|y_n - x^*\|.
\]
This implies
\[
\|y_n - x^*\| \leq \frac{1 + \lambda L}{\lambda \gamma} \|x_n - y_n\|
\]
and, further,

\begin{equation}
\|x_n - x^*\| \leq \|x_n - y_n\| + \|y_n - x^*\| \leq \frac{1 + \lambda L + \lambda \gamma}{\lambda \gamma} \|x_n - y_n\|.
\end{equation}

From (16) and (5) we obtain

\[
\|x_{n+1} - x^*\|^2 \leq \left(1 - (1 - \lambda^2 L^2) \left(\frac{\lambda \gamma}{1 + \lambda L + \lambda \gamma}\right)^2\right) \|x_n - x^*\|^2,
\]

therefore,

\[
\|x_{n+1} - x^*\| \leq \delta \|x_n - x^*\|,
\]

where \(\delta := \left(1 - (1 - \lambda^2 L^2) \left(\frac{\lambda \gamma}{1 + \lambda L + \lambda \gamma}\right)^2\right)^{1/2} \in (0, 1)\). This proves that \((x_n)_{n \geq 0}\) converges linearly to \(x^*\).

**Example 2.3.** If \(M : H \rightarrow H\) is such that

\[
\langle Mx, x \rangle \geq \gamma \|x\|^2 \quad \forall x \in H,
\]

for some \(\gamma > 0\), then one can show that the operator \(F : H \rightarrow H\) in Example 2.1 is \(\alpha \gamma\)-strongly pseudo-monotone on \(H\) and Lipschitz continuous. On the other hand, \(F\) is in general not monotone, as one can see in Figure 2 for an operator defined on \(H = \mathbb{R}\).

**3. A dynamical system of forward-backward-forward type.** In this section we will approach the solution set of \(VI(F,C)\) from a continuous perspective by means of trajectories generated by the following dynamical system of forward-backward-forward type

\[
\begin{cases}
  y(t) = P_C(x(t) - \lambda F(x(t))) \\
  \dot{x}(t) + x(t) = y(t) + \lambda [F(x(t)) - F(y(t))]
\end{cases}
\]

\(x(0) = x_0\),

where \(0 < \lambda < \frac{1}{L}\) and \(x_0 \in H\). The formulation of (17) has its roots in [2], where the continuous counterpart of Tseng’s algorithm has been considered in the more general context of a monotone inclusion problem. The existence and uniqueness of the trajectory \(x \in C^1([0, +\infty), H)\) generated by (17) has been established in [2] in the lines of the global Cauchy-Lipschitz Theorem by making use of the Lipschitz continuity of \(F\). Here, we study the convergence of \(x(t)\) and \(y(t)\) to an element in \(\Omega\) as \(t \rightarrow \infty\) when \(F\) is pseudo-monotone.

To this end we will use the following two results. The first one (see [1, Lemma 5.2]) is the continuous counterpart of a classical result which states the convergence of quasi-Fejér monotone sequences. The second one (see [1, Lemma 5.3]) is the continuous version of the Opial Lemma.

**Lemma 3.1.** If \(1 \leq p < \infty, 1 \leq r < \infty, A : [0, +\infty) \rightarrow [0, +\infty)\) is locally absolutely continuous, \(A \in L^p([0, +\infty)), B : [0, +\infty) \rightarrow \mathbb{R}, B \in L^r([0, +\infty))\) and for almost every \(t \in [0, +\infty)\)

\[
\frac{d}{dt} A(t) \leq B(t),
\]

then \(\lim_{t \rightarrow +\infty} A(t) = 0\).
Lemma 3.2. Let $\Omega \subseteq H$ be a nonempty set and $x : [0, +\infty) \to H$ a given map. Assume that

(i) for every $x^* \in \Omega$ the limit $\lim_{t\to+\infty} \|x(t) - x^*\|$ exists;

(ii) every weak sequential cluster point of the map $x$ belongs to $\Omega$.

Then there exists an element $x^\infty \in \Omega$ such that $x(t)$ converges weakly to $x^\infty$ as $t \to +\infty$.

We start our asymptotic analysis with two preliminary results.

Proposition 3.1. Assume that the solution set $\Omega$ is nonempty, $F$ is pseudo-monotone on $C$ and Lipschitz continuous with constant $L > 0$. Then for every solution $x^* \in \Omega$ it holds

$$\langle \dot{x}(t), x(t) - x^* \rangle \leq - (1 - \lambda L) \|x(t) - y(t)\|^2 \leq 0 \ \forall t \in [0, +\infty).$$

Proof. Since $x^* \in \Omega$ and $y(t) \in C$ it holds

$$\langle F(x^*), y(t) - x^* \rangle \geq 0 \ \forall t \in [0, +\infty).$$

By the pseudo-monotonicity of $F$ it holds

$$\langle F(y(t)), y(t) - x^* \rangle \ \forall t \in [0, +\infty).$$

On the other hand, since $y(t) = P_C(x(t) - \lambda F(x(t)))$, we obtain from (3) that

$$\langle x(t) - \lambda F(x(t)) - y(t), y(t) - x^* \rangle \geq 0 \ \forall t \in [0, +\infty).$$
Combining (18) and (19) we obtain for all \( t \in [0, +\infty) \)
\[
\langle x(t) - y(t) - \lambda [F(x(t)) - F(y(t))], y(t) - x^* \rangle \geq 0
\]
or, equivalently, (see (17))
\[
\langle x(t) - y(t) - \lambda [F(x(t)) - F(y(t))], y(t) - x(t) \rangle - \langle \dot{x}(t), x(t) - x^* \rangle \geq 0.
\]
This implies that
\[
\langle \dot{x}(t), x(t) - x^* \rangle \leq \langle x(t) - y(t) - \lambda [F(x(t)) - F(y(t))], y(t) - x(t) \rangle = -\|x(t) - y(t)\|^2 + \lambda \langle F(x(t)) - F(y(t)), x(t) - y(t) \rangle \leq -(1 - \lambda L) \|x(t) - y(t)\|^2 \quad \forall t \in [0, +\infty).
\]

**Proposition 3.2.** Assume that the solution set \( \Omega \) is nonempty, \( F \) is pseudomonotone on \( C \) and Lipschitz continuous with constant \( L > 0 \). Then for every solution \( x^* \in \Omega \) the function \( t \rightarrow \|x(t) - x^*\|^2 \) is nonincreasing and it holds
\[
\int_0^{+\infty} \|x(t) - y(t)\|^2 dt < +\infty \quad \text{and} \quad \lim_{t \to +\infty} \|x(t) - y(t)\| = 0.
\]

**Proof.** Using Proposition 3.1, for every \( t \in [0, +\infty) \) it holds
\[
\frac{1}{2} \frac{d}{dt} \|x(t) - x^*\|^2 = \langle x(t) - x^*, \dot{x}(t) \rangle \leq -(1 - \lambda L) \|x(t) - y(t)\|^2 \leq 0,
\]
which shows that \( t \rightarrow \|x(t) - x^*\|^2 \) is nonincreasing. Let be \( T > 0 \). Integrating the previous inequality from 0 to \( T \) it yields
\[
(1 - \lambda L) \int_0^T \|x(t) - y(t)\|^2 dt \leq \frac{1}{2} \left( \|x(0) - x^*\|^2 - \|x(T) - x^*\|^2 \right) \leq \frac{1}{2} \|x(0) - x^*\|^2.
\]
Letting \( T \to +\infty \), it follows that \( \int_0^{+\infty} \|x(t) - y(t)\|^2 dt < +\infty \).

Since \( P_C \) is nonexpansive and \( F \) is Lipschitz continuous with constant \( L \), we get that \( P_C \circ (I - \lambda F) \) is Lipschitz continuous with constant \( 1 + \lambda L \). Therefore, (17) implies that for almost every \( t \in [0, +\infty) \)
\[
\|\dot{y}(t)\| \leq (1 + \lambda L) \|\dot{x}(t)\|.
\]
On the other hand,
\[
\|\dot{x}(t)\| = \|x(t) - y(t) - \lambda [F(x(t)) - F(y(t))]\| \leq (1 + \lambda L) \|x(t) - y(t)\| \quad \forall t \in [0, +\infty).
\]
Thus, for almost every \( t \in [0, +\infty) \),
\[
\frac{d}{dt} \|x(t) - y(t)\|^2 = 2 \langle x(t) - y(t), \dot{x}(t) - \dot{y}(t) \rangle \leq 2 (\|\dot{x}(t)\| + \|\dot{y}(t)\|) \|x(t) - y(t)\| \leq 2 (1 + \lambda L + (1 + \lambda L)^2) \|x(t) - y(t)\|^2.
\]
and from here, according to Lemma 3.1,
\[
\lim_{t \to +\infty} \|x(t) - y(t)\| = 0.
\]
We can prove now the main theorem of this section.

**Theorem 3.1.** Assume that the solution set $\Omega$ is nonempty, and $F$ is pseudo-monotone on $H$, Lipschitz continuous with constant $L > 0$ and sequentially weak-to-weak continuous. Then the trajectories $x(t)$ and $y(t)$ generated by (17) converge weakly to a solution of $\text{VI}(F,C)$ as $t \to +\infty$.

Proof. Let $\hat{x} \in H$ be a weak sequential cluster point of $x(t)$ as $t \to +\infty$ and $(t_n)_{n \geq 0}$ be a sequence in $[0, +\infty)$ with $t_n \to +\infty$ and $x(t) \to \hat{x}$ as $n \to +\infty$. Since $\lim_{t \to +\infty} \|x(t) - y(t)\| = 0$, we also have $y(t_n) \to \hat{x}$ as $n \to +\infty$. Furthermore, since $F$ is Lipschitz continuous, $\|F(x(t_n)) - F(y(t_n))\| \to 0$ as $n \to +\infty$. We have shown in Proposition 2.2 that, under these auspices, $\hat{x} \in \Omega$. On the other hand, by Proposition 3.2, for all $\alpha > 0$, $\|x(t) - x^\ast\|$ converges as $t \to +\infty$. Thus, according to the Lemma 3.2, $x(t)$ converges weakly to an element of $\Omega$ as $t \to +\infty$. Since, due to Proposition 3.2, we have that

$$\lim_{t \to +\infty} \|x(t) - y(t)\| = 0,$$

it follows that $y(t)$ converges also weakly to the same element of $\Omega$ as $t \to +\infty$. \qed

For the important particular case of strongly pseudo-monotone operators we are able to show exponential convergence of the trajectories to the unique solution of $\text{VI}(F,C)$.

**Theorem 3.2.** Assume that $F$ is $\gamma$-strongly pseudo-monotone on $C$ with $\gamma > 0$ and Lipschitz continuous with constant $L > 0$. Then for every $t \in [0, +\infty)$ we have

$$\| x(t) - x^\ast \|^2 \leq \lambda \gamma\| x(0) - x^\ast \| \exp(\alpha t),$$

where $\alpha = 2(1 - \lambda L) \left( \frac{\lambda \gamma}{1 + \lambda L + \lambda \gamma} \right)^2$ and $x^\ast$ is the unique solution of $\text{VI}(F,C)$.

Proof. As in the proof of Theorem 2.3 (see (16)), one can show that for every $t \in [0, +\infty)$

$$\| x(t) - x^\ast \| \leq \frac{1 + \lambda L + \lambda \gamma}{\lambda \gamma} \| x(0) - y(t) \|,$$

which, in combination with Proposition 3.1, leads to

$$\frac{1}{2} \frac{d}{dt} \| x(t) - x^\ast \|^2 = \langle x(t) - x^\ast, \dot{x}(t) \rangle \leq - (1 - \lambda L) \| x(t) - y(t) \|^2 \leq - (1 - \lambda L) \left( \frac{\lambda \gamma}{1 + \lambda L + \lambda \gamma} \right)^2 \| x(t) - x^\ast \|^2.$$

The relation (20) is a direct consequence of Gronwall’s Lemma. \qed

**Example 3.1.** Let $C = [-5, 5]^3 \subseteq \mathbb{R}^3$ and $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$F(x) = \left( e^{-\|x\|^2 + q} \right) M x,$$

where $q = 0.2$,

$$M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1.5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

and $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^3$. As mentioned in Example 2.3, $F$ is $\gamma$-strongly pseudo-monotone on $\mathbb{R}^3$ with constant $\gamma := q \cdot \lambda_{\min} \approx 0.0764$, where
$\lambda_{\text{min}}$ is the smallest eigenvalue of $M$, and Lipschitz continuous with constant $L := \left(\frac{2}{e} + 1 + q\right) \|M\| \approx 5.0679$. Since for $x = (-1, 0, 0)^T, y = (-2, 0, 0)^T \in \mathbb{R}^3$

$$\langle F(x) - F(y), x - y \rangle = -0.1312 < 0,$$

$F$ is not monotone.

Figure (3) displays the trajectories generated by the dynamical system (17) attached to $\text{VI}(F, C)$, when $x_0 = (-4, 3, 5)^T$ and $\lambda := 0.19 < 1/L$. One can notice that the trajectories converge exponentially to the unique solution $x^* = (0, 0, 0)^T$ of $\text{VI}(F, C)$.

**Fig. 3. Trajectories generated by the dynamical system (17) for $x_0 = (-4, 3, 5)^T$ and $\lambda = 0.19$.**

**4. Numerical experiments.** In this section we discuss two numerical experiments which we carried out in order to compare Algorithm 2.1 with other algorithms in the literature designed for solving pseudo-monotone variational inequalities. We implemented the numerical codes in Matlab and performed all computations on a Linux desktop with an Intel(R) Core(TM) i5-4670S processor at 3.10GHz. In our experiments we considered only pseudo-monotone variational inequalities that are not monotone.

In the first experiment we considered $\text{VI}(F, C)$ with

$$C = \left\{ x \in \mathbb{R}^3 : \sum_{i=1}^{m} x_i \leq 5, 0 \leq x_i \leq 5 \right\}$$
and
\[ F : \mathbb{R}^5 \to \mathbb{R}^5, F(x) = \left( e^{-\|x\|^2} + \alpha \right) (Mx + p), \]
where \(\| \cdot \|\) denotes the Euclidean norm on \(\mathbb{R}^5\), \(\alpha = 0.1\), \(p = (-1, 2, 1, 0, -1)^T\) and

\[
M := \begin{bmatrix}
5 & -1 & 2 & 0 & 2 \\
-1 & 6 & -1 & 3 & 0 \\
2 & -1 & 3 & 0 & 1 \\
0 & 3 & 0 & 5 & 0 \\
2 & 0 & 1 & 0 & 4
\end{bmatrix}
\]

is a positive definite matrix. We computed the unique solution \(x^*\) of the variational inequality \(VI(F, C)\) by making 10000 iterations of Algorithm 2.1. We performed Tseng’s method and the extragradient method with the same stepsize \(\lambda = \frac{0.9}{L}\), after calculating \(L\) by using (15), starting point \(x_0 = (1, 3, 2, 1, 4)^T\) and stopping criterion \(\|x_n - x^*\| \leq 10^{-6}\). We computed the projection on to \(C\) by using the \texttt{quadprog} function in Matlab. Figure 4 shows that Tseng’s method in Algorithm 2.1 outperforms the extragradient method, being at least two times faster. This is something what we expected, since the extragradient method requires two projections onto the set \(C\) at each iteration, while Algorithm 2.1 requires only one. Observe that, if there is an explicit formula for the projection operator, as it is the case in the second experiment, then the difference between Tseng’s forward-backward-forward method and the extragradient method is small.

In the second experiment we considered the quadratic fractional programming problem

\[
\min_{x \in C} f(x) := \frac{x^T M x + a^T x + c}{b^T x + d},
\]
with
\[
C = \{ x \in \mathbb{R}^5 : 1 \leq x_i \leq 3, \ i = 1, 2, 3, 4, 5 \},
\]
\(M\) taken as in (21), \(a = (1, 2, -1, -2, 1)^T\), \(b = (1, 0, -1, 0, 1)^T\), \(c = -2\) and \(d = 20\). According to the discussion in Example 2.2, \(f\) is pseudo-convex on the open set \(E := \{ x \in \mathbb{R}^5 : b^T x + d = x_1 - x_3 + x_5 + 20 > 0 \}\), which implies that
\[ F : \mathbb{R}^5 \to \mathbb{R}^5, F(x) = \nabla f(x) := \frac{\left( b^T x + d \right) (2M x + a) - b (x^T M x + a^T x + c)}{(b^T x + d)^2}, \]
is pseudo-monotone on \(E\). One can also notice that \(C \subseteq E\).

Coming now to the Lipschitz continuity of \(F\), since \(C\) is bounded, according to Remark 2.2 is enough to prove that this property holds on the set
\[
D = \{ x + y \in \mathbb{R}^5 : x \in C, \|y\| \leq 2\sqrt{5} \} = \{ x \in \mathbb{R}^5 : 1 - 2\sqrt{5} \leq x_i \leq 3 + 2\sqrt{5}, \ i = 1, 2, 3, 4, 5 \}.
\]
Notice that \(C \subseteq D \subseteq E\).

One can easily see that \(\| \nabla F(x) \| \leq 148.68 =: L > 0\) for all \(x \in D\), which means that \(F\) is Lipschitz continuous on \(D\) with constant \(L\). We compared Algorithm 2.1 with the extragradient method and the proximal-gradient method for fractional programming proposed in [5, Algorithm 6]. We considered as stepsize \(\lambda = 0.9/L\), as starting point \(x_0 = (3, 1.5, 2, 1.5, 2)^T\) and as stopping criterion \(\|x_n - x^*\| \leq 10^{-6}\),
where the optimal solution of (22) $x^* = (1, 1, 1, 1)^T$ we obtained by making 10000 iterations of Algorithm 2.1. We solved the quadratic subproblem in [5, Algorithm 6] by using the *quadprog* function in Matlab. The numerical performances of these three methods are displayed in Figure 5. One can notice that Tseng’s method and the extragradient method have a similar convergence behaviour, which outperforms the one of the proximal-gradient method. A possible reason is that, while the projection onto $C$ can be computed explicitly, in every iteration of the proximal-gradient method a subproblem has to be solved by using an external solver.

5. Conclusion. We prove the convergence of the Tseng’s forward-backward-forward algorithm when employed to the solving of pseudo-monotone variational inequalities in Hilbert spaces. In addition, we show that linear convergence is guaranteed under strong pseudo-monotonicity. We also associate a dynamical system to the pseudo-monotone variational inequality and carry out an asymptotic analysis for the generated trajectories. Numerical experiments for some pseudo-monotone variational inequalities and fractional quadratic programming problems show that Tseng’s forward-backward-forward method outperforms the extragradient method.

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