Determinantal facet ideals for smaller minors

AYAH ALMOUSA AND KELLER VANDEBOGERT

Abstract. A determinantal facet ideal (DFI) is generated by a subset of the maximal minors of a generic $n \times m$ matrix where $n \leq m$ indexed by the facets of a simplicial complex $\Delta$. We consider the more general notion of an $r$-DFI, which is generated by a subset of $r$-minors of a generic matrix indexed by the facets of $\Delta$ for some $1 \leq r \leq n$. We define and study so-called lcm-closed and unit interval $r$-DFIs, and show that the minors parametrized by the facets of $\Delta$ form a reduced Gröbner basis with respect to any term order for an lcm-closed $r$-DFI. We also see that being lcm-closed generalizes conditions previously introduced in the literature, and conjecture that in the case $r = n$, lcm-closedness is necessary for being a Gröbner basis. We also give conditions on the maximal cliques of $\Delta$ ensuring that lcm-closed and unit interval $r$-DFIs are Cohen-Macaulay. Finally, we conclude with a variant of a conjecture of Ene, Herzog, and Hibi on the Betti numbers of certain types of $r$-DFIs, and provide a proof of this conjecture for Cohen-Macaulay unit interval DFIs.

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1. Introduction. Let $M = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ be an $n \times m$ matrix of indeterminates where $n \leq m$, and let $S = k[M]$ be a polynomial ring over an arbitrary field $k$ with variables in $M$. The study of the ideal generated by all minors of a given size of $M$ has a long history, and such ideals are well understood (see, for instance, [3]). In a similar vein, one can instead consider the ideal generated by some of the minors of a given size of $M$; these are known as determinantal facet ideals (DFIs) and were introduced by Ene, Herzog, Hibi, and Mohammadi in [8]. The study of DFIs turns out to be much more subtle and has seen comparably less attention, even though such ideals arise naturally in algebraic statistics (see [6] and [10]). In [11], the linear strand of DFIs is constructed...
in terms of a generalized Eagon-Northcott complex. In particular, the linear Betti numbers of such ideals may be computed in terms of the $f$-vector of an associated simplicial complex. Likewise, in [17], explicit Betti numbers of certain classes of DFIs are computed in all degrees; for arbitrary DFIs, higher degree Betti numbers have proven to be very nontrivial to compute.

DFIs for the case $n = 2$ were originally introduced as binomial edge ideals independently by Ohtani [14] and Herzog et al. [10]; this generalized work of Diaconis, Eisenbud, and Sturmfels in [6]. To study binomial edge ideals, one can associate each column of $M$ with a vertex of a graph $G$, and one can associate a minor of $M$ involving two columns $i$ and $j$ with an edge $\{i, j\}$ in the graph. For example, the ideal generated by all maximal minors of a $2 \times m$ matrix corresponds to a complete graph on $m$ vertices. The relationship between homological invariants of ideals generated by some maximal minors of $M$ and combinatorial invariants of the associated graph $G$ has been widely studied; see the survey paper [12] for a compilation of such results. DFIs naturally extend this idea by instead associating a pure simplicial complex $\Delta$ on $m$ vertices to the ideal $J_\Delta$, where each $(n - 1)$-dimensional facet of $\Delta$ corresponds to a maximal minor in the set of generators of $J_\Delta$. Mohammadi and Rauh further generalized this notion to that of a determinantal hypergraph ideal, which associates a minor to each hyperedge of a graph, allowing for an ideal that is generated by minors of different sizes.

A particularly interesting class of DFIs is that for which the standard minimal generating set forms a Gröbner basis. It is well-known that the set of maximal minors of a generic matrix is a Gröbner basis for the ideal generated by them with respect to any total monomial order [1,16]. In the case of binomial edge ideals, there is a known necessary and sufficient condition on a graph $G$ for the generators of $J_G$ indexed by $G$ to form a reduced Gröbner basis with respect to a diagonal term order [10, Theorem 1.1]. For a DFI where the maximal cliques of $\Delta$ overlap by $n - 1$ or fewer vertices, a necessary and sufficient condition for $\Delta$ to index the generators of a reduced Gröbner basis for $J_\Delta$ with respect to a diagonal term order is also known [8, Theorem 1.1].

In this paper, we introduce the notion of an $r$-DFI (Definition 2.3), which is an ideal generated by a subset of $r \times r$ minors in an $n \times m$ matrix indexed by the facets of some $(r - 1)$-dimensional simplicial complex $\Delta$. This is a natural extension of Ene, Herzog, Hibi, and Mohammadi’s DFIs in [8], but is not as general as Mohammadi and Rauh’s determinantal hypergraph ideals in [13]. We consider two classes of $r$-DFIs which we call lcm-closed and unit interval DFIs (see Definition 2.10). In the case that $r = n$, lcm-closed DFIs are a direct generalization of some important classes of determinantal ideals in the literature, including closed binomial edge ideals and closed DFIs, but is stated for arbitrary term orders. We apply a result of Conca (see Proposition 2.9) to show that the minimal generating set parametrized by the facets of $\Delta$ for any lcm-closed $r$-DFI forms a reduced Gröbner basis with respect to any diagonal term order. We then observe that in the cases that lcm-closed DFIs generalize, these conditions are also necessary for the minimal generating set to form a reduced Gröbner basis. This leads us to pose Question 2.18, which asks
whether or not being lcm-closed is a necessary condition for being a Gröbner
basis when \( r = n \). In the case \( n = 2 \), we prove that Question 2.18 is true.

In Section 3, we apply the results of Section 2 to deduce certain cases
for which lcm-closed and unit interval DFIs must be Cohen-Macaulay. We
start with a result that allows one to deduce when the sum of two Cohen-
Macaulay ideals remains Cohen-Macaulay based on knowledge of their initial
ideals (see Proposition 3.1). This leads to Corollary 3.2, which proves that if
the maximal cliques of \( \Delta \) have sufficiently small pairwise intersections, then
the associated \( r \)-DFI must be Cohen-Macaulay. Lastly, we conclude with a
variant of a conjecture by Ene, Herzog, and Hibi (see Conjecture 3.4) and give
a proof of the conjecture in the case that \( J_\Delta \) is a Cohen-Macaulay unit interval
\( r \)-DFI.

2. Lcm-closed and unit interval determinantal facet ideals. In this section, we
generalize the idea of a determinantal facet ideal to that of an \( r \)-determinantal
facet ideal. We recall examples in the literature, some of which are special-
izations of Definition 2.3, and introduce a sufficient condition for the stan-
dard minimal generating set of these so-called \( r \)-determinantal facet ideals to
be a reduced Gröbner basis (see Definition 2.10). This condition generalizes
conditions considered in the binomial edge ideal case and for that of closed
determinantal ideals (see [8]). Let us set the stage with some notation and
definitions:

Notation 2.1. Fix \( r \) to be a positive integer. Let \( M = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq m} \) be
an \( n \times m \) matrix of indeterminates where \( n \leq m \), and let \( S = k[M] \) be a
polynomial ring over an arbitrary field \( k \) with variables in \( M \). For indices \( a = \{a_1, \ldots, a_r\} \) and \( b = \{b_1, \ldots, b_r\} \) such that \( 1 \leq a_1 < \cdots < a_r \leq n \) and
\( 1 \leq b_1 < \cdots < b_r \leq m \), set

\[
[a|b] = [a_1, \ldots, a_r|b_1, \ldots, b_r] = \det \begin{pmatrix}
x_{a_1,b_1} & \cdots & x_{a_1,b_r} \\
\vdots & \ddots & \vdots \\
x_{a_r,b_1} & \cdots & x_{a_r,b_r}
\end{pmatrix}
\]

where \( [a|b] = 0 \) if \( r > n \). When \( r = n \), use the simplified notation \( [a] = [1, \ldots, n|a] \). The ideal generated by the \( r \)-minors of \( M \) is denoted \( I_r(M) \).

Definition 2.2. Let \( \Delta \) be a pure \( (r-1) \)-dimensional simplicial complex on the
vertex set \([m]\). For an integer \( i \), the \( i \)-th skeleton \( \Delta^{(i)} \) of \( \Delta \) is the subcomplex
of \( \Delta \) whose faces are those faces of \( \Delta \) with dimension at most \( i \). Let \( S \) denote
the set of simplices \( \Gamma \) with vertices in \([m]\) with \( \dim(\Gamma) \geq r-1 \) and \( \Gamma^{(r-1)} \subset \Delta \).

Let \( \Gamma_1, \ldots, \Gamma_c \) be maximal elements in \( S \) with respect to inclusion, and let
\( \Delta_i := \Gamma_i^{(r-1)} \). Each \( \Gamma_i \) is called a maximal clique, and any induced subcomplex
of \( \Gamma_i \) is a clique. The simplicial complex \( \Delta^{\text{clique}} \) whose facets are the maximal
cliques of \( \Delta \) is called the clique complex associated to \( \Delta \). The decomposition
\( \Delta = \Delta_1 \cup \cdots \cup \Delta_c \) is called the maximal clique decomposition of \( \Delta \).

Definition 2.3. Adopt Notation 2.1, and let \( \Delta \) be an \( (r-1) \)-dimensional simplicial
complex on the vertex set \([m]\). The \( r \)-determinantal facet ideal (or \( r \)-DFI)
$J_\Delta \subseteq S$ associated to $\Delta$ is the ideal generated by determinants of the form $[a|b]$ where $b$ supports an $(r-1)$-face of $\Delta$; that is, the columns of $[a|b]$ correspond to the vertices of some $(r-1)$-face of $\Delta$.

**Notation 2.4.** Let $\Delta$ be a pure $(r-1)$-dimensional simplicial complex on the vertex set $[m]$ with maximal clique decomposition $\Delta = \Delta_1 \cup \cdots \cup \Delta_c$. The notation $J_{\Delta_i}$ will be used to denote the $r$-DFI associated to the simplicial complex $\Delta_i$.

**Remark 2.5.** Definition 2.3 naturally generalizes the notion of a DFI introduced by Ene, Herzog, Hibi, and Mohammadi in [8], who considered only the case when $r = n$. DFIs are, in turn, a generalization of binomial edge ideals introduced in [10], which coincides with the case when $r = n = 2$ and $\Delta$ is a graph $G$. However, $r$-DFIs are not as general as Mohammadi and Rauh’s notion of a *determinantal hypergraph ideal* introduced in [13], which allows for the ideal to be generated by minors of different sizes in $M$.

**Remark 2.6.** Let $J_\Delta$ denote any $r$-DFI. The simplicial complex $\Delta$ associated to an $r$-DFI serves as a combinatorial tool to parametrize the column sets appearing on minimal generators of $J_\Delta$. Maximal cliques in the clique decomposition of $\Delta = \bigcup_{i=1}^c \Delta_i$ correspond to the largest submatrices $M_i$ of $M$ such that the ideal generated by all $r$-minors of $M_i$ is contained in $J_\Delta$.

![Figure 1](image.png)

**Figure 1.** A graph with two maximal cliques given by vertex sets $\{1, 2, 3, 4\}$ and $\{3, 4, 5\}$

**Example 2.7.** Let $G$ be the graph in Figure 1. If $r = n = 2$, then $J_G$ corresponds to a subideal of the ideal of maximal minors of a $2 \times 5$ matrix with generators $[a_1a_2]$ indexed by the $(a_1, a_2)$ in the edge set of $G$.

If, instead, $n = 3$, then $J_G$ corresponds to a subideal of the ideal of $I_2(M)$ where $M$ is a $3 \times 5$ matrix. Now the generators of $J_G$ are of the form $[a_1, a_2|b_1, b_2]$ where $\{a_1, a_2\} \subset [3]$ and $\{b_1 < b_2\}$ is an edge of $G$.

**Notation 2.8.** Let $<$ be a total monomial order in a polynomial ring $S$ over a field $k$. If $f$ is a polynomial in $S$, denote by $\text{in}_<(f)$ the leading term of $f$ with respect to $<$. If $I \subseteq S$ is an ideal, denote by $\text{in}_<(I)$ the initial ideal of $I$ with respect to $<$. Frequently, when the term order $<$ is clear, it will be dropped and leading terms and initial ideals will simply be denoted by $\text{in}(f)$ and $\text{in}(I)$, respectively.
The following proposition, originally due to Conca, will turn out to be very useful in the proofs of Theorems 2.15 and 2.17.

**Proposition 2.9** ([5, Lemma 1.3]). Let $S$ be a polynomial ring over a field $k$, and let $< $ be a term order. Let $I$ and $J$ be homogeneous ideals of $S$. Then

a) $\text{in}(I) + \text{in}(J) \subseteq \text{in}(I + J)$ and $\text{in}(I \cap J) \subseteq \text{in}(I) \cap \text{in}(J)$,

b) $\text{in}(I) + \text{in}(J) = \text{in}(I + J)$ if and only if $\text{in}(I \cap J) = \text{in}(I \cap J)$,

c) let $F$ be a Gröbner basis of $I$ and let $G$ be a Gröbner basis of $J$. Then $F \cup G$ is a Gröbner basis of $I + J$ if and only if, for all $f \in F$ and $g \in G$, there exists $h \in I \cap J$ such that $\text{in}(h) = \text{lcm}(\text{in}(f), \text{in}(g))$.

**Definition 2.10.** Adopt Notation 2.1 and Notation 2.8. Let $< $ be any term order. Let $\Delta$ be a pure $(r - 1)$-dimensional simplicial complex on $m$ vertices with maximal clique decomposition $\Delta = \bigcup_{i=1}^{c} \Delta_i$. The $r$-DFI $J_{\Delta}$ is **lcm-closed** if the following condition holds:

(*) For all $[a|b] \in J_{\Delta_1}$, $[a'|b'] \in J_{\Delta_j}$ with non-coprime lead terms and $[a|b]$, $[a'|b'] \notin J_{\Delta_1 \cap \Delta_j}$, there exists $[c|d] \in J_{\Delta_1 \cap \Delta_j}$ such that $\text{in}_<( [c|d])$ divides $\text{lcm}(\text{in}_<([a|b]), \text{in}_<([a'|b']))$.

The $r$-DFI $J_{\Delta}$ is a unit interval DFI if each $\Delta_i$ may be written as an interval $[a_i, b_i] = \{a_i, a_i + 1, \ldots, b_i - 1, b_i\}$ for integers $a_i < b_i$.

**Example 2.11.** Let $\Delta$ be a 2-dimensional simplicial complex on 5 vertices with clique decomposition $\Delta_1 = \{1, 2, 3, 4\}$ and $\Delta_2 = \{2, 3, 4, 5\}$, and let $r = n = 3$. Let $< $ be any diagonal term order. For any 2-dimensional faces $a \in \Delta_1$ and $a' \in \Delta_2$ such that $a, a' \notin \Delta_1 \cap \Delta_2$, observe that $\text{in}_<([a|b])$ and $\text{in}_<([a'|b'])$ are co-prime except for the case where $a = [1, 3, 4]$ and $a' = [2, 3, 5]$. In this case,

$$\text{lcm}(\text{in}_<([1, 3, 4]), \text{in}_<([2, 3, 5])) = \text{lcm}(x_{11}x_{23}x_{34}, x_{12}x_{23}x_{35}) = x_{11}x_{12}x_{23}x_{34}x_{35}$$

which is divisible by $x_{12}x_{23}x_{34} = \text{in}_<([2, 3, 4])$, and $\text{in}_<([2, 3, 4]) = J_{\Delta_1 \cap \Delta_2}$. Therefore, $J_{\Delta}$ is lcm-closed and, in particular, is a unit interval 3-DFI.

The following classes of $r$-DFIs are lcm-closed. Each of these classes forms a Gröbner basis when $< $ is the standard diagonal term order.

**Example 2.12** (Closed binomial edge ideals). When $r = n = 2$, $\Delta$ may be associated with a graph $G$ and $J_G$ is called a binomial edge ideal. Denote by $E(G)$ the edge set of $G$. A graph $G$ (or its respective binomial edge ideal $J_G$) is closed with respect to a given labeling if, for all distinct pairs of edges $\{i, j\}$ and $\{k, \ell\}$ with $i < j$, $k < \ell$, one has $\{j, \ell\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = \ell$. This condition is equivalent to $J_G$ having a quadratic Gröbner basis [10, Theorem 1.1].

If two generators $\text{in}_<([a|b])$ and $\text{in}_<([c|d])$ of $J_G$ are not relatively prime, then $J_G$ satisfies (*) if and only if they correspond to edges $a$ and $b$ in the same clique of $G$. This is equivalent to the closed condition for a binomial edge ideal, so $J_G$ is lcm-closed if and only if it is closed.

**Example 2.13** (Closed DFIs). Let $r = n$ and assume that a pure $n$-dimensional simplicial complex $\Delta$ satisfies the following condition:
No two maximal cliques of $\Delta$ share more than $n - 1$ vertices. A DFI satisfying $\triangleright$ is said to be closed if for all $i \neq j$ and all $\{a_1 < \cdots < a_n\} \in \Delta_i$ and $\{b_1 < \cdots < b_n\} \in \Delta_j$, the monomials $\text{in}_<(a_1, \ldots, a_n)$ and $\text{in}_<(b_1, \ldots, b_n)$ are relatively prime. A DFI $J_\Delta$ satisfying $\triangleright$ is closed if and only if the generating set of $J_\Delta$ indexed by the facets of $\Delta$ forms a Gröbner basis with respect to $<$ [8, Theorem 1.1].

If $\Delta$ satisfies $\triangleright$, then there are no generators of $J_\Delta$ contained in the intersection of any two maximal cliques, so $J_\Delta$ satisfies (*) if and only if it is closed.

Setup 2.14. Let $\Delta$ be a pure $(r - 1)$-dimensional simplicial complex on the vertex set $[m]$ admitting maximal clique decomposition $\Delta = \bigcup_{i=1}^c \Delta_i$. Let $S = k[x_{ij} \mid 1 \leq i \leq n, \ 1 \leq j \leq m]$ be a polynomial ring over an arbitrary field $k$.

Theorem 2.15. Adopt notation and hypotheses as in Setup 2.14. If the associated $r$-DFI $J_\Delta$ is lcm-closed, then the generators of $J_\Delta$ indexed by the facets of $\Delta$ form a reduced Gröbner basis with respect to any term order $<$.

Proof. Assume $f = [a \mid b]$ is a minimal generator of $J_{\Delta_i}$ and $g = [a' \mid b']$ is a minimal generator of $J_{\Delta_j}$ such that $f, g \notin J_{\Delta_i} \cap J_{\Delta_j}$. By Proposition 2.9, it suffices to check that there exists some $h \in J_{\Delta_i} \cap J_{\Delta_j}$ such that $\text{in}(h) = \text{lcm}(\text{in}(f), \text{in}(g))$. If $\text{in}(f)$ and $\text{in}(g)$ are coprime, then

$$\text{lcm}(\text{in}(f), \text{in}(g)) = \text{in}(f) \cdot \text{in}(g)$$

$$= \text{in}(f \cdot g) \in J_{\Delta_i} \cdot J_{\Delta_j} \subseteq J_{\Delta_i} \cap J_{\Delta_j}.$$ 

Suppose that $\text{in}(f)$ and $\text{in}(g)$ are not coprime. By the definition of lcm-closed, there exists some $h \in J_{\Delta_i} \cap \Delta_j$ such that $\text{in}(h)$ divides $\text{lcm}(\text{in}(f), \text{in}(g))$. Any multiple of $h$ is also contained in both $J_{\Delta_i}$ and $J_{\Delta_j}$, so

$$\text{lcm}(\text{in}(f), \text{in}(g)) \cdot h \in J_{\Delta_i} \cap J_{\Delta_j}$$

and has initial term equal to $\text{lcm}(\text{in}(f), \text{in}(g))$. \hfill \Box

Example 2.16. In the case $r = n$ and $<$ is a diagonal term order, it is clear that unit interval DFIs are lcm-closed. For $r < n$, these notions are distinct. Let $n = 3$ and consider for instance the 2-DFI $J_\Delta$ associated to the simplicial complex $\Delta$ with maximal clique decomposition $[1, 2] \cup [2, 3]$. Notice that $J_\Delta$ is not lcm-closed with respect to a diagonal term order because the generators $[12][12]$ and $[23][23]$ do not have coprime lead terms. However, $J_\Delta$ is a unit interval $r$-DFI and its natural minimal generating set forms a reduced Gröbner basis with respect to any diagonal term order. In particular, one finds that the determinant of

$$\begin{pmatrix}
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23} \\
  x_{31} & x_{32} & x_{33}
\end{pmatrix}$$

is an element of $J_{[1,2]} \cap J_{[2,3]}$ whose lead term divides the lcm of the lead terms of $[12][12]$ and $[23][23]$.

The following theorem can be proved using rather computationally intense methods, but follows much more easily from work of Seccia (see [15]).
Theorem 2.17. Adopt notation and hypotheses as in Setup 2.14. If $J_\Delta$ is a unit interval $r$-DFI, then the generators of $J_\Delta$ indexed by the facets of $\Delta$ form a reduced Gröbner basis with respect to any diagonal term order $<$. 

Proof. See [15, Corollary 2.7].

As shown in Examples 2.12 and 2.13, lcm-closed $r$-DFIs coincide with many known cases of determinantal ideals which form a Gröbner basis in the literature. Moreover, in all of the cases considered previously, the property of being lcm-closed is necessary for the standard minimal generating set to form a Gröbner basis. This leads us to ask Question 2.18.

Question 2.18. Let $\Delta$ be a pure $(r-1)$-dimensional simplicial complex. Is the property of being lcm-closed not only sufficient, but also necessary, for the generators of $J_\Delta$ indexed by the facets of $\Delta$ to form a reduced Gröbner basis with respect to any term order $<$?

As evidence of a positive answer to Question 2.18, we have the following proposition:

Proposition 2.19. Question 2.18 is true for binomial edge ideals.

Proof. Let $<_1$ be any term order on $S$. By [16, Proposition 1.11], there exists a permutation $\sigma : [m] \to [m]$ such that $\text{in}_{<_1}([a_1,a_2]) = \text{in}_{<_2}([\sigma(a_1),\sigma(a_2)])$, where $\{a_1,a_2\} \in \Delta$ and $<_2$ denotes the standard diagonal term order. Observe that $\sigma$ induces an automorphism of $k$-algebras $\phi : S \to S$ by acting on the second indices of the variables. By [7, Theorem 2.2], the ideal $\phi(J_\Delta) = J_{\sigma(\Delta)}$ is lcm-closed with respect to $<_2$. Since the definition of lcm-closed is an algebraic condition, it follows that $J_\Delta$ is lcm-closed with respect to $<_1$. □

3. Cohen-Macaulayness for certain classes of $r$-DFIs. Recall that an ideal $I$ in a standard graded polynomial ring $S$ over a field $k$ is Cohen-Macaulay if $\text{ht}(I) = \text{pd}_S(S/I)$. In this section, we investigate Cohen-Macaulayness for certain classes of $r$-DFIs. In particular, we give sufficient conditions for the Cohen-Macaulayness of both lcm-closed and unit interval $r$-DFIs in Corollary 3.2. This allows us to prove a variant of a conjecture of Ene, Herzog, and Hibi for Cohen-Macaulay unit interval DFIIs (Corollary 3.3), and we pose a similar conjecture for $r$-DFIs in which each clique has precisely $r$ vertices.

The following proposition is likely well-known, but does not seem to appear explicitly in the literature. We give a complete statement and proof for reference and convenience; it turns out to be a surprisingly effective method for deducing Cohen-Macaulayness of sums of ideals.

Proposition 3.1. Let $S$ be a standard graded polynomial ring over a field $k$ and let $<$ be any term order. Assume that $I$ and $J$ are Cohen-Macaulay ideals with the property that

$$\text{in}_<(I \cap J) = \text{in}_<(I) \cap \text{in}_<(J) = \text{in}_<(I) \cdot \text{in}_<(J).$$

Then $I + J$ is a Cohen-Macaulay ideal.
Proof. It is a standard fact that $\text{ht}(I) = \text{ht}(\text{in}_{<}(I))$ for any ideal $I$. One has the following string of equalities:

\[
\text{ht}(I + J) = \text{ht}(\text{in}_{<}(I + J)) \\
= \text{ht}(\text{in}_{<}(I) + \text{in}_{<}(J)) \quad \text{(by Proposition 2.9)} \\
= \text{ht}(\text{in}_{<}(I)) + \text{ht}(\text{in}_{<}(J)) \quad \text{(since $\text{in}_{<}(I) \cap \text{in}_{<}(J) = \text{in}_{<}(I) \cdot \text{in}_{<}(J)$)} \\
= \text{ht}(I) + \text{ht}(J).
\]

Likewise, there is a string of implications:

\[
\text{in}_{<}(I) \cap \text{in}_{<}(J) = \text{in}_{<}(I) \cdot \text{in}_{<}(J) \\
\implies I \cap J = IJ \\
\iff \text{Tor}_{i}^{S}(S/I, S/J) = 0 \\
\iff \text{Tor}_{i}^{S}(S/I, S/J) = 0 \text{ for all } i > 0 \quad \text{(by [4, Corollary 2.5])} \\
\implies \text{pd}_{S}(S/(I + J)) = \text{pd}_{S}(S/I) + \text{pd}_{S}(S/J).
\]

Since $I$ and $J$ were assumed to be Cohen-Macaulay, the result follows. □

Recall that for a generic $n \times m$ matrix $M$, the ideal of $r$-minors $I_{r}(M)$ is Cohen-Macaulay for any $1 \leq r \leq n$ (see [3]). This implies that for each maximal clique $\Delta_{i}$ appearing in the clique decomposition of $\Delta$, $J_{\Delta_{i}}$ is a Cohen-Macaulay ideal; in particular, one uses this to deduce the following:

**Corollary 3.2.** Adopt notation and hypotheses as in Setup 2.14. Then:

1. If $J_{\Delta}$ is an lcm-closed $r$-DFI with $<$ any term order and $|V(\Delta_{i}) \cap V(\Delta_{j})| \leq r - 1$, then $J_{\Delta}$ is Cohen-Macaulay.

2. If $J_{\Delta}$ is a unit interval $r$-DFI with $<$ any diagonal term order and $|V(\Delta_{i}) \cap V(\Delta_{j})| \leq \max\{0, 2r - n - 1\}$, then $J_{\Delta}$ is Cohen-Macaulay.

**Proof.** Both of the conditions on the intersection size of each clique in (1) and (2) ensure that minimal generators coming from any two distinct maximal cliques have coprime lead terms. The result then follows immediately upon combining Theorems 2.15 and 2.17 with Proposition 3.1. □

The following corollary yields a proof of a variant of a conjecture by Herzog and Hibi (see [7]) for a certain class of unit interval DFIs. It states that the graded Betti numbers of $J_{\Delta}$ and its initial ideal with respect to $<$ are always equal. In general, the Betti numbers of the initial ideal are only an upper bound; it is very rare to have equality everywhere.

**Corollary 3.3.** Adopt notation and hypotheses as in Setup 2.14 with $<$ any diagonal term order and assume that $J_{\Delta}$ is a unit interval $r$-DFI with $|V(\Delta_{i}) \cap V(\Delta_{j})| \leq \max\{0, 2r - n - 1\}$. If $|V(\Delta_{i})| = r$ for each $i = 1, \ldots, c$, then

\[
\beta_{ij}(S/J_{\Delta}) = \beta_{ij}(S/\text{in}_{<}(J_{\Delta})) \quad \text{for all } i, j.
\]

**Proof.** Let $F_{*}^{\Delta}$ denote the minimal free resolution of each $S/J_{\Delta_{i}}$. The condition $|V(\Delta_{i}) \cap V(\Delta_{j})| \leq \max\{0, 2r - n - 1\}$ implies that minimal generators coming from any two distinct maximal cliques have coprime lead terms, whence the
minimal free resolution of $S/J_\Delta$ may be obtained as the tensor product complex $F^*_\Delta \otimes \cdots \otimes F^*_c$. Since $|V(\Delta_i)| = r$, one has $\beta_{jk}(S/J_\Delta) = \beta_{jk}(S/\text{in}_{<} J_\Delta)$ for all $j, k$, where $i = 1, \ldots, c$ (see, for instance, [2, Theorem 1.4]). Combining the previous two sentences yields the result. \qed

Corollary 3.3 combined with copious amounts of computational evidence suggests that the following conjecture holds:

**Conjecture 3.4.** Adopt notation and hypotheses as in Setup 2.14 and assume that the standard minimal generating set of $J_\Delta$ forms a reduced Gröbner basis. If $|V(\Delta_i)| = r$ for all $i = 1, \ldots, c$, then

$$\beta_{ij}(S/J_\Delta) = \beta_{ij}(S/\text{in}_{<}(J_\Delta))$$

for all $i, j$.

**Example 3.5.** Let $n = 4$ and consider the 3-DFI associated to the simplicial complex with maximal clique decomposition $[1,3] \cup [2,4]$. Let $<$ be a diagonal term order. It can be shown using Macaulay2 [9] that $J_\Delta$ is not Cohen-Macaulay and that $S/J_\Delta$ and $S/\text{in}_{<}(J_\Delta)$ both have Betti table

|   | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | . | . | . | . |
| 1 | . | . | . | . | . |
| 2 | . | 8 | 7 | . | . |
| 3 | . | . | . | . | . |
| 4 | . | . | 10 | 16 | 6 |

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AYAH ALMOUSA
University of Minnesota
Minneapolis
USA
e-mail: almou007@umn.edu
URL: https://sites.google.com/view/ayah-almousa

KELLER VANDEBOGERT
University of Notre Dame
South Bend
USA
e-mail: kvandebo@nd.edu
URL: https://sites.google.com/view/kellervandebogert/home

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