Active-Passive Brownian Particle in Two Dimensions

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Abstract

This paper presents a model for active particles in two dimensions with time-dependent self-propulsion speed undergoing both translational and rotational diffusion. Usually, for modeling the motion of active particles, the self-propulsion speed is assumed to be constant as in the famous model of active Brownian motion. This assumption is far from what may happen in reality. Here, we generalize active Brownian motion by considering stochastic self-propulsion speed \( v(t) \). In particular, we assume that \( v(t) \) is a two-state process with \( v = 0 \) (passive state) and \( v = s \) (active state). The transition between the two states is also modeled using the random telegraph process. It is expected that the presented two-state model where we call it active-passive Brownian particle has the characteristics of both pure active- and pure passive-Brownian particle. The analytical results for the first two moments of displacement and the effective diffusion coefficient confirm this expectation. We also show that a run-and-tumble particle (such as a motile bacterium) can be mapped to our model so that their diffusivities at large scales are equal.

1 Introduction

Cellular motility has an essential role in biological processes [1]. For instance, wounds cannot heal without the motility of various kinds of cells such as fibroblast [2]. This is not limited only to the cells of the human body. Unicellular microorganisms like bacteria could also move on surfaces or swim in fluid mediums [3,5]. One of the characteristics of cell movement is the temporal changes in the cell speed and its direction of motion. These changes can be due to fluctuations at the cellular scale or due to the decision-making processes inside the cell. The latter happens for microorganisms in response to chemical/physical stimuli in their environment [6,7]. Regardless of why the cell speed changes, studying the mechanics of cells and the effect of variable speed are important because it would help to better understand the biological mechanisms which depend on cell motility.

Any microscopic particle from cells and microorganisms to synthetic particles with the ability to move by the self-propulsion mechanism is known as self-propelled particle or active particle [8,10]. Active particles take up energy from their environment and convert it into motion. Thus, active particles are intrinsically out of equilibrium. Because of the simplicity

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of active systems from theoretical and experimental point of view and at the same time their rich physics, studying active particles has received much attention in recent years [11–16]. In order to describe the motion of active particles several models have been developed among which active Brownian motion is the most famous one [17]. This model is the generalization of the Brownian motion model by adding a self-propulsion force and orientational diffusivity for the direction of motion, (see [18] and references therein). In this model, the self-propulsion force (and consequently the self-propulsion speed) is assumed to be constant. Although, this assumption is true and/or enough for many applications, however it could be violated in reality. In this regard, some studies have investigated temporal changes in the self-propulsion speed. Peruani et. al., [19] studied the general aspects of fluctuations in the speed and the direction of motion of self-propelled particles. They derived the characteristic time scales and different temporal regimes of motion. Babel et. al., [20] also studied statistical properties of an active Brownian motion with time-dependent self-propulsion for three deterministic forms of the speed.

In this paper, we generalize active Brownian motion by considering time-dependent self-propulsion speed but in a stochastic fashion. We then derive analytical solutions for the first two moments of displacement. Different temporal regimes are investigated and the effective diffusion coefficient is obtained. We illustrate that the behavior of the active-passive Brownian model presented in this study includes both the characteristics of standard Brownian motion and active Brownian motion simultaneously. Finally, we study run-and-tumble particles (RTPs) which is a famous model for studying motile bacteria such as Escherichia coli. It is proven that our model can describe the diffusion characteristic of non-interacting RTPs.

2 Model and Results

Consider an active particle in position \( \mathbf{r} = (x, y) \) moving with time-dependent self-propulsion speed \( v(t) \) in direction \( \mathbf{u}(\varphi) = (\cos \varphi, \sin \varphi) \) which undergoes translational and rotational diffusion, see Fig. 1 for a schematic illustration. The motion of this particle is described by two-dimensional Langevin equations

\[
\dot{\mathbf{r}} = v(t)\mathbf{u}(\varphi) + \sqrt{2D_t}\xi(t) \tag{1}
\]

\[
\dot{\varphi} = \sqrt{2D_r}\eta(t), \tag{2}
\]

where \( \xi(t) = (\xi_x(t), \xi_y(t)) \) and \( \eta(t) \) are independent Gaussian white noises of zero mean and unit variance. The noises can be originated by the local thermal fluctuations of the bath containing the particle. The constants \( D_t \) and \( D_r \) are the translational and rotational diffusion constants, respectively. The above equations show the simple generalization of the model of active Brownian particle (ABP) in which the speed is constant. The solution of Eqs. (1) and (2) depends on the functional form of the self-propulsion speed \( v(t) \). Here, we consider the special case that \( v(t) \) at any moment takes only values 0 or \( s \) in a random fashion. Under this assumption, the above Langevin equations describe a Brownian particle with two states: active and passive (inactive). We call this model an active-passive Brownian particle. The reason behind such assumption for \( v(t) \) comes from this observation that a typical biological microorganism is not always active and there are instances of its inactivity. Since the transition from one state to the other state occurs randomly, we also assume the following transition
relations for the self-propulsion speed
\[ v(t) : 0 \overset{\alpha}{\leftrightarrow} \overset{\beta}{\rightleftharpoons} s, \]
where \( \alpha \) and \( \beta \) are transition rates. The propulsion speed \( v(t) \) defined by Eq. (3) is a well-known discrete-state Markov process called random telegraph process.

In order to characterize the dynamics of an active-passive Brownian particle, we determine the first two displacement moments \( \langle r(t) - r_0 \rangle \) and \( \langle (r(t) - r_0)^2 \rangle \) where \( r_0 \equiv r(t = 0) \). The second moment is also known as mean square displacement (MSD). One way to derive these two moments is to derive their components along the coordinate axes. For this purpose, we rewrite the vector translational Langevin equation, Eq. (1), along the coordinate axes

\[ \dot{x}_i = v(t) e_i \cdot u(\varphi) + \sqrt{2D_t} e_i \cdot \xi(t), \quad i = 1, 2 \]

where \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). The first two moments for both directions are then readily obtained as follows

\[ \langle x_i(t) - x_i(0) \rangle = \int_0^t dt_1 \langle v(t_1) \rangle \langle e_i \cdot u(\varphi(t_1)) \rangle, \]

\[ \langle [x_i(t) - x_i(0)]^2 \rangle = 2D_t t + \int_0^t dt_1 \int_0^t dt_2 \langle v(t_1)v(t_2) \rangle \langle e_i \cdot u(\varphi(t_1)) e_i \cdot u(\varphi(t_2)) \rangle, \]
for \( i = 1, 2 \). Note that the prerequisite for writing the integrands of Eqs. (5) and (6) as the product of two averages is the independence of speed \( v(t) \) from angle \( \varphi(t) \). To compute these moments, we need six averages \( \langle v(t) \rangle, \langle v(t_1)v(t_2) \rangle, \langle \cos \varphi(t) \rangle, \langle \sin \varphi(t) \rangle, \langle \cos \varphi(t_1) \cos \varphi(t_2) \rangle \), and \( \langle \sin \varphi(t_1) \sin \varphi(t_2) \rangle \) (see appendices A and B for the details of deriving these averages). After substituting the expressions \( \langle v(t) \rangle, \langle \cos \varphi(t) \rangle, \) and \( \langle \sin \varphi(t) \rangle \) into Eq. (5), the mean position is obtained as

\[ \langle r(t) - r_0 \rangle = s u_0 \left[ \frac{\alpha}{\alpha + \beta} \left( 1 - e^{-D_r t} \right) + \left( \delta_{\varphi_0,s} - \frac{\alpha}{\alpha + \beta} \right) \left( 1 - e^{-(\alpha + \beta + D_r) t} \right) \right], \]

where \( u_0 = (\cos \varphi_0, \sin \varphi_0) \) is the initial direction of motion of the active particle at \( t = 0 \). Notice that different behavioral regimes come from the exponential terms in Eq. (7).
For the short-time regime, $t << (\alpha + \beta + D_r)^{-1}$,

$$\langle r(t) - r_0 \rangle \approx s u_0 \left[ \frac{1}{2} \alpha t^2 + \delta_{v_0,s} \left( t - \frac{1}{2} (\alpha + \beta + D_r) t^2 \right) + O(t^3) \right]. \tag{8}$$

This equation shows different temporal behavior depending on the initial speed $v_0$ so that for $v_0 = 0$ we have $\langle r(t) - r_0 \rangle \sim t^2$ while for $v_0 = s$ we have $\langle r(t) - r_0 \rangle \sim t$.

For the long-time regime, $t >> D_r^{-1}$,

$$\langle r(t) - r_0 \rangle \approx s u_0 \left[ \frac{\alpha}{\alpha + \beta} + \delta_{v_0,s} \left( \frac{\alpha}{\alpha + \beta} \right) \right]. \tag{9}$$

This constant value depends similarly on the value of the initial speed $v_0$. To show a typical temporal behavior of $\langle r(t) - r_0 \rangle$, we consider a disk-shaped particle of radius $R = 1 \mu m$ moving in water at room temperature. Based on the Stokes-Einstein relation, we have $D_t = k_B T / 6 \pi \eta R$ and $D_r = k_B T / 8 \pi \eta R^3$ where $k_B$ is the Boltzmann constant and $\eta$ is the viscosity of water which is about 0.9 mPa at room temperature. We assume the active state $s = 3 \mu m/s$ with the transition rates $\alpha = \beta = D_r$. The parameters have chosen in the range of real and synthetic active particles \[15\]. Figure 2 shows the different regimes of $\langle r(t) - r_0 \rangle$ once for the particle starts moving from rest and once for the particle with the initial speed $s$. This plot illustrates asymptotic temporal behaviors as we expected.

In a similar manner, after substituting the expressions $\langle v(t_1)v(t_2) \rangle$, $\langle \cos \varphi(t_1) \cos \varphi(t_2) \rangle$, and $\langle \sin \varphi(t_1) \sin \varphi(t_2) \rangle$ into Eq. 6, the mean square displacement is obtained as

$$\langle (r(t) - r_0)^2 \rangle = A_0 + A_1 t + A_2 e^{-D_r t} + A_3 e^{-(\alpha + \beta) t} + A_4 e^{-(\alpha + \beta + D_r) t}, \tag{10}$$

where
translational diffusivity \( D \) in terms of \( \alpha \) and \( \beta \). The effective diffusion coefficient for the active-passive Brownian motion is defined generally as [22]

\[
A_0 = \frac{s^2}{(\alpha + \beta)^2} \left\{ -2\alpha^2 \frac{\alpha \beta}{D_t^2} - 2\frac{\alpha \beta}{(\alpha + \beta + D_r)^2} \right. \\
+ \left( \delta v_0, s(\alpha + \beta) - \alpha \right) \left[ \frac{\alpha}{\alpha + \beta - D_r} \left( \frac{1}{D_r} - \frac{1}{\alpha + \beta} \right) - \frac{\beta}{\alpha + \beta + D_r} \left( \frac{1}{D_r} - \frac{1}{\alpha + \beta} + \frac{1}{D_r} \right) \right],
\]

\[
A_1 = 4D_t + 2\alpha \frac{s^2}{(\alpha + \beta)^2} \left\{ \frac{\alpha}{D_r} + \frac{\beta}{\alpha + \beta + D_r} \right\},
\]

\[
A_2 = 2 \frac{s^2}{(\alpha + \beta)^2} \frac{\alpha}{D_r} \left\{ \frac{\delta v_0, s(\alpha + \beta) - \alpha}{\alpha + \beta - D_r} \right\},
\]

\[
A_3 = \frac{s^2}{(\alpha + \beta)^2} \left( \delta v_0, s(\alpha + \beta) - \alpha \right) \left\{ \frac{\alpha}{\alpha + \beta} \frac{1}{\alpha + \beta - D_r} - \frac{\beta}{\alpha + \beta + D_r} \right\},
\]

\[
A_4 = 2 \frac{s^2}{(\alpha + \beta)^2} \frac{\alpha}{\alpha + \beta + D_r} \left\{ \frac{\delta v_0, s(\alpha + \beta) - \alpha}{D_r} \right\}.
\]  

There are three time scales related to the three different exponents in MSD. Especially, \( D_r^{-1} \) is the characteristic time scale of the rotational diffusion after which correlations in the direction of motion vanish. Interestingly, the transition rates \( (\alpha, \beta) \) appear along with \( D_r \) and not with the translational diffusivity \( D_t \). The value of \( (\alpha, \beta) \) determines the distance between the smallest time scale, \( (\alpha + \beta + D_r)^{-1} \), and the largest time scale, \( D_r^{-1} \). MSD shows a linear asymptotic behavior for times far enough from the interval \( [(\alpha + \beta + D_r)^{-1}, D_r^{-1}] \) while for other times shows a nonlinear behavior. The form of the nonlinear segment depends on the problem parameters. For instance, a ballistic regime, MSD \( \sim t^2 \), could be seen in the nonlinear segment. To show a typical behavior of MSD, we consider the same parameters used for Fig. 2. Figure 3 shows various behavioral regimes of MSD, once the particle starts moving from rest and once it has the initial speed \( s \).

In many experiments, the translational diffusivity \( D_t \) is negligible compared to the self-propelling speed. For example, in wild-type run-and-tumble bacteria, \( D_t \) could be safely set to be zero [21]. If \( D_t = 0 \), there is no linear regime anymore for the short times \( t << (\alpha + \beta + D_r)^{-1} \), and instead we observe ballistic-diffusion and super-diffusion regimes, see Fig 4. It is noteworthy to mention that for \( v_0 = s \) with \( \beta = 0 \), the results of active Brownian motion are re-obtained because the particle starts with a non-zero speed and keeps it since no transition could happen to the zero-speed state.

Among all terms in Eq. (10), the second term, \( A_1 t \), is the most important one, since it is related to the effective diffusion coefficient which is defined generally as [22]

\[
D_{eff} = \lim_{t \to \infty} \frac{(\langle r^2(t) \rangle - \langle r(t) \rangle^2)}{2dt},
\]  

where \( d \) is the spatial dimension. The effective diffusion coefficient for the active-passive Brow-
Figure 3: Temporal behavior of MSD for a typical particle of size $R = 1 \mu m$ with $D_t = 0.24 \mu m^2$, $\alpha = \beta = D_r = 0.18 \, s^{-1}$, and $s = 3 \mu m/s$.

The results of our model can also be linked to the famous class of active particles called run-and-tumble particles (RTPs). The RTP model describes the motion of motile bacteria such as *Escherichia coli* [23]. A single bacteria in a medium swims with almost a constant speed in almost a straight line until a tumbling of its flagella suddenly occurs and changes its orientation. Tumbling decorrelates the orientation and happens randomly with rate $\alpha_0$. For a run-and-tumble particle with the speed $s$ that also undergoes diffusion with coefficients $D_t$ and $D_r$, the effective diffusion coefficient reads [21]

$$D_{eff} = D_t + \frac{s^2}{2(D_r + \alpha_0)}.$$

Comparing $D_{eff}$ in this equation with that of Eq. (13) shows the possibility of finding values for
\[ \frac{\alpha}{(\alpha + \beta)^2} \left( \frac{\alpha + \beta}{\alpha + \beta + D_r} \right) = \frac{D_r}{D_r + \alpha_0}, \]  

(15)

to find \( \alpha \) and \( \beta \). Note that all parameters are non-negative. For any given value of \( D_r \) and \( \alpha_0 \), the above equality turns into an implicit function of the form \( f(\alpha, \beta) = 0 \). It can be shown for \( \alpha > 0 \) that \( f(\alpha, \beta) \) is continuously differentiable and satisfies the relation \( \partial f / \partial \beta \neq 0 \). Thus, according to the implicit function theorem [24], \( \beta \) could be obtained uniquely as \( \beta = g(\alpha) \) where \( g \) is a continuous real function. Consequently, we could say that a run-and-tumble particle can be mapped to an active-passive Brownian particle. Of course, this equivalence is true from the perspective of their diffusion at large time scales and there is still a lack of proof for the strict equivalence between the two models.

3 Summary

We investigate the motion of an active particle with stochastic self-propulsion speed \( v(t) \). We assume that \( v(t) \) is a two-state process with one passive state and an active state. We called this model an active-passive Brownian particle. The idea behind this choice for \( v(t) \) comes from the fact that motile biological entities do not move continuously and their activity (which e.g., can be due to their swimming) turns on and off randomly. The path statistics of the motion of such a particle is then obtained by calculating the first two moments of displacement. An active-passive Brownian particle exhibits a dual behavior. It diffuses faster than a Brownian particle while diffuses slower than an active Brownian particle with the same parameters. Finally, we have shown that the run-and-tumble motion could be mapped to the active-passive Brownian motion by choosing proper parameters so that they posses the same effective diffusion coefficient.
Appendices

A Random Telegraph Process

Equation (3) is a simple two-state stochastic process that tells us how the propulsion speed \( v(t) \) varies in time. This is a well-known discrete-state Markov process called the random telegraph process. At each instant of time, \( v(t) \) could only have one of the two possible values of 0 (passive state) or \( s \) (active state). After writing the master equation for \( v(t) \), its conditional probability density is obtained, and subsequently the moments \( \langle v(t) \rangle \) and \( \langle v(t_1)v(t_2) \rangle \) are determined. The master equation governs this process is

\[
\frac{\partial}{\partial t} p(0,t|v_0,t_0) = \beta p(s,t|v_0,t_0) - \alpha p(0,t|v_0,t_0) \tag{16}
\]

\[
\frac{\partial}{\partial t} p(s,t|v_0,t_0) = \alpha p(0,t|v_0,t_0) - \beta p(s,t|v_0,t_0), \tag{17}
\]

where \( t_0 \) and \( v_0 \) represents the initial time and speed, respectively. \( p(0,t|v_0,t_0) \) is the probability that the particle is in the passive state at time \( t \) given that \( v(t_0) = v_0 \). Similarly, \( p(s,t|v_0,t_0) \) is the probability that the particle is in the active state at time \( t \) conditioned on \( v(t_0) = v_0 \). The initial speed, \( v_0 \), can also be either 0 or \( s \). Note that these coupled differential equations are not independent of each other because we know in advance that \( p(0,t|v_0,t_0) + p(s,t|v_0,t_0) = 1 \).

The solutions to Eqs. (16) and (17) are

\[
p(0,t|v_0,t_0) = \frac{\beta}{\alpha + \beta} + \left( \delta_{v_0,0} - \frac{\beta}{\alpha + \beta} \right) e^{-(\alpha + \beta)(t-t_0)} \tag{18}
\]

\[
p(s,t|v_0,t_0) = \frac{\alpha}{\alpha + \beta} + \left( \delta_{v_0,s} - \frac{\alpha}{\alpha + \beta} \right) e^{-(\alpha + \beta)(t-t_0)}, \tag{19}
\]

where \( \delta \) is the Kronecker delta function. For simplicity, we take \( t_0 \) equals to zero. Now, we are able to compute the ensemble averages \( \langle v(t) \rangle \) and \( \langle v(t_1)v(t_2) \rangle \). Using the definition of conditional average [25], the desired averages are obtained as

\[
\langle v(t) \rangle = s \ p(s,t|v_0,0), \tag{20}
\]

and

\[
\langle v(t_1)v(t_2) \rangle = s^2 \ p(s,t_2|s,t_1) \ p(s,t_1|v_0,0). \tag{21}
\]

Note that Eq. (21) is only true for \( t_2 > t_1 \), otherwise we should change \( t_1 \) to \( t_2 \) and \( t_2 \) to \( t_1 \). The probability \( p(s,t_2|s,t_1) \) in Eq. (21) has the similar form to \( p(s,t|v_0,t_0) \) in Eq. (19).

B Angular Ensemble Averages

Note that before solving the translational Langevin equation, Eq. (1), one needs to have the orientational probability density \( p(\varphi,t) \). Since \( \eta(t) \) is a Gaussian white noise, \( \varphi(t) \) is a Gaussian process with Markov property. Consequently, in order to determine \( p(\varphi,t) \), it is sufficient to have its first two moments. After integrating Eq. (2) and using the moments of \( \eta(t) \), i.e.
\( \langle \eta(t) \rangle = 0 \) and \( \langle \eta(t_1) \eta(t_2) \rangle = \delta(t_2 - t_1) \), we get \( \langle \varphi(t) - \varphi_0 \rangle = 0 \) and \( \langle (\varphi(t) - \varphi_0)^2 \rangle = 2D_r t \), where \( \varphi_0 \) is the initial angle of the particle at time \( t_0 \). Thus, the one-joint probability density is obtained as follow

\[
p(\varphi, t) = \frac{1}{\sqrt{4\pi D_r (t - t_0)}} \exp \left( -\frac{1}{4D_r} (\varphi - \varphi_0)^2 \right). \tag{22}
\]

Note that this equation is the same as the conditional probability \( p(\varphi, t|\varphi_0, t_0) \). Moreover, since \( \varphi(t) \) is a Markov process, all its joint probability densities can be determined in terms of the transition probability densities of the form \( p(\varphi_2, t_2|\varphi_1, t_1) \).

In general, conditional averages for a continuous-state Markov process \( X(t) \) can be then defined using the transition densities \( p(x, t|x', t') \) \(\text{[25]}\). For instance

\[
\langle f(X(t))|X(t') = x' \rangle = \int_{-\infty}^{\infty} dx f(x) p(x, t|x', t') , \quad t \geq t', \tag{23}
\]

and

\[
\langle g(X(t_1), X(t_2))|X(t') = x' \rangle = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 g(x_1, x_2) p(x_2, t_2|x_1, t_1) p(x_1, t_1|x', t'), \tag{24}
\]

where \( f(.) \) and \( g(., .) \) are any arbitrary univariate and bivariate functions, respectively. Note that in Eq. \(\text{(24)}\) is true for \( t_2 \geq t_1 \geq t' \). For a discrete-state Markov process, the integrals are replaced with sums over all possible values of the process. What we are interested in are the averages \( \langle \cos \varphi(t) \rangle, \langle \sin \varphi(t) \rangle, \langle \cos \varphi(t_1) \cos \varphi(t_2) \rangle \), and \( \langle \sin \varphi(t_1) \sin \varphi(t_2) \rangle \). Using Eqs. \(\text{(23)}\) and \(\text{(24)}\) and the transition probability in the form of Eq. \(\text{(22)}\) one can derive the followings

\[
\langle \cos \varphi(t) \rangle = \int_{-\infty}^{\infty} d\varphi \cos \varphi p(\varphi, t|\varphi_0, 0) = \cos \varphi_0 e^{-D_r t},
\]

\[
\langle \sin \varphi(t) \rangle = \int_{-\infty}^{\infty} d\varphi \sin \varphi p(\varphi, t|\varphi_0, 0) = \sin \varphi_0 e^{-D_r t}. \tag{25}
\]

and

\[
\langle \cos \varphi(t_1) \cos \varphi(t_2) \rangle = \int_{-\infty}^{\infty} d\varphi_1 \int_{-\infty}^{\infty} d\varphi_2 \cos \varphi_1 \cos \varphi_2 p(\varphi_2, t_2|\varphi_1, t_1) p(\varphi_1, t_1|\varphi_0, 0)
\]

\[
= \frac{1}{2} \left( 1 + \cos 2\varphi_0 e^{-4D_r t_1} \right) e^{-D_r (t_2 - t_1)} , \quad t_2 \geq t_1 \tag{26}
\]

\[
\langle \sin \varphi(t_1) \sin \varphi(t_2) \rangle = \int_{-\infty}^{\infty} d\varphi_1 \int_{-\infty}^{\infty} d\varphi_2 \sin \varphi_1 \sin \varphi_2 p(\varphi_2, t_2|\varphi_1, t_1) p(\varphi_1, t_1|\varphi_0, 0)
\]

\[
= \frac{1}{2} \left( 1 - \cos 2\varphi_0 e^{-4D_r t_1} \right) e^{-D_r (t_2 - t_1)} , \quad t_2 \geq t_1. \tag{26}
\]
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