INTEGRALS OF LOGARITHMIC FORMS ON SEMI-ALGEBRAIC SETS 
AND A GENERALIZED CAUCHY FORMULA 
PART II: GENERALIZED CAUCHY FORMULA (DETAILED VERSION)

MASAKI HANAMURA, KENICHIRO KIMURA AND TOMOHIDE TERASOMA

Abstract. This paper is the continuation of the paper [arXiv:1509.06950] which is Part I under the same title. In this paper, we prove a generalized Cauchy formula for the integrals of logarithmic forms on products of projective lines, and give an application to the construction of Hodge realization of mixed Tate motives.

1. Introduction

This paper is the continuation of the paper [10], which is Part I under the same title. In this paper, we prove a generalized Cauchy formula for the integrals of logarithmic forms on products \( P^n = (\mathbb{P}^1)^n \) of projective lines \( \mathbb{P}^1 \). As an application, we define a variant of the Hodge realization functor for the category of mixed Tate motives defined by Bloch and Kriz [4]. In the sequel to this paper, we prove that our construction coincides with the original one defined by Bloch and Kriz. The motivation of our series of papers is to understand the Hodge realization functor via integral of logarithmic differential forms.

Before going into the detail, we describe a simple example of the generalized Cauchy formula. Let \( \omega_2 = (2\pi i)^{-2} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \), \( \omega_1 = (2\pi i)^{-1} \frac{dz_2}{z_2} \) be a holomorphic two resp. one form on \((\mathbb{C} \setminus \{0\})^2\) resp. \(\mathbb{C} \setminus \{0\}\). Let \(0 < a < b\) be real numbers and \(\overline{D} = \{z_1 \in \mathbb{C} \mid |z_1| \leq 1\}\)

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be the closed unit disk. Let $\sigma = \overline{D} \times [a, b]$ be a 3-chain in $\mathbb{C}^2$. Its topological boundary is denoted by $\partial \sigma$. Then we have the following identity, called the generalized Cauchy formula:

\[(1.0.1) \quad \int_{\delta \sigma} \omega_2 = \int_{\overline{D} \times [a, b]} \omega_2 + \int_{(\overline{D} \times \{b\}) - (\overline{D} \times \{a\})} \omega_2 = \int_{[a, b]} \omega_1 = \int_{\sigma \cap (\{0\} \times \mathbb{C})} \omega_1.\]

In the leftmost side of the above equality, although the differential form $\omega_2$ is not defined on $\delta \sigma \cap (\{0\} \times \mathbb{C})$, the integral is defined as an improper integral. To show the second equality we use Fubini’s theorem and the classical Cauchy formula. To generalize the above formula, we define

(1) suitable subspaces $AC_3(\mathbb{C}^2)$ and $AC_2(\mathbb{C}^2)$ of 3-chains and 2-chains in $\mathbb{C}^2$ and a sub

space $AC_1(\mathbb{C})$ of 1-chains in $\mathbb{C}$, and

(2) a “face map” $\partial : AC_3(\mathbb{C}^2) \to AC_1(\mathbb{C})$ which generalizes the above operation $\sigma \mapsto$ $\sigma \cap (\{0\} \times \mathbb{C})$ taking multiplicities into account.

satisfying the following properties.

(1) The topological boundary map $\delta$ induces the map $\delta : AC_3(\mathbb{C}^2) \to AC_2(\mathbb{C}^2)$.

(2) The improper integrals $\int_{\gamma_2} \omega_2$ and $\int_{\gamma_1} \omega_1$ converge for $\gamma_2 \in AC_2(\mathbb{C}^2)$ and $\gamma_1 \in AC_1(\mathbb{C})$.

Using the above setting, the generalized Cauchy formula (1.0.1) is the equality

$I_1(\partial \gamma) + I_2(\delta \gamma) = 0$

where the map $I_2$ (resp. $I_1$) is defined as the improper integral $\gamma_2 \mapsto -\int_{\gamma_2} \omega_2$ (resp. $\gamma_1 \mapsto \int_{\gamma_1} \omega_1$) for $\gamma_2 \in AC_2(\mathbb{C}^2)$ (resp. $\gamma_1 \in AC_1(\mathbb{C})$).

In this paper, we study the generalized Cauchy formula for $P^n = (\mathbb{P}^1)^n$. Let $(z_1, \ldots, z_n)$ be the coordinates of $P^n$, and $D^n$ be the divisor of $P^n$ defined by $\prod^n_i (z_i - 1) = 0$. For a semi-algebraic triangulation $K$ of $P^n$ such that $D^n$ is a subcomplex of $K$, the relative chain complex of $(K, D^n)$ with the coefficients in $\mathbb{Q}$ is denoted by $C_\bullet (K, D^n; \mathbb{Q})$. In §2 we define a subcomplex $AC_\bullet (K, D^n; \mathbb{Q})$ of $C_\bullet (K, D^n; \mathbb{Q})$ consisting of elements satisfying certain admissibility conditions (see Definition 2.3). By taking the inductive limit of $AC_\bullet (K, D^n; \mathbb{Q})$ of all semi-algebraic triangulations under subdivisions, we get a complex $AC_\bullet (P^n, D; \mathbb{Q})$.

The moving lemma Proposition 2.8 asserts that imposing the admissibility condition does not change the homology groups.

In §3 we define a face map

$\partial_{i, \alpha} : AC_i(P^n, D^n; \mathbb{Q}) \to AC_{i-2}(P^{n-1}, D^n; \mathbb{Q}) \quad (1 \leq i \leq n, \alpha = 0, \infty)$

with respect to the hypersurface $H_{i, \alpha} = \{z_i = \alpha\}$. Roughly this map is taking the intersection with the face $H_{i, \alpha}$. More precisely, it is defined by the cap product with a Thom cocycle $T$ of the face $H_{i, \alpha}$. The cubical differential $\partial$ is defined to be the alternating sum of the face maps $\partial_{i, \alpha}$.

In §4 we prove a generalized Cauchy formula. Let $\gamma = \sum a_\sigma \sigma$ be an element of $AC_n(P^n, D; \mathbb{Q})$, where $\sigma$’s are $n$-simplexes in a triangulation $K$ of $P^n$ and $a_\sigma \in \mathbb{Q}$. Let $\omega_n$ be the rational
differential form on $P^n$ defined by

$$\omega_n = \frac{1}{(2\pi i)^n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}. $$

By a result of Part I, the integral $\int_\sigma \omega_n$ converges absolutely if $\sigma$ is admissible (Theorem 4.1) and we define a homomorphism $I_n$ as the following integration:

$$I_n : AC_n(P^n, D^n; \mathbb{Q}) \to \mathbb{C} : \sum a_\sigma \sigma \mapsto (-1)^{n(n-1)/2} \sum_\sigma \int_\sigma a_\sigma \omega_n.$$ 

Then the generalized Cauchy formula (Theorem 4.3) asserts that, for an element $\gamma \in AC_{n+1}(P^n, D^n; \mathbb{Q})$, the equality

$$(1.0.2) \quad I_{n-1}(\partial \gamma) + (-1)^n I_n(\delta \gamma) = 0$$

holds. If the chain $\gamma$ does not meet any of the faces, then we have $\partial \gamma = 0$, and (1.0.2) holds by the Stokes formula. In general, the correction term for the Stokes formula arising from the logarithmic singularity is computed in terms of $\partial \gamma$. The main results of [10] are used in the proof of Theorem 4.3 which is fairly complex. If the reader is mainly interested in its applications and is willing to accept this fact, it is possible to read §5 separately.

We construct a variant of the Hodge realization functor for the category of mixed Tate motives in §5. We briefly recall the construction of the category of mixed Tate motives given in the paper of Bloch and Kriz ([4]). Let $k$ be a subfield of $\mathbb{C}$. Bloch defines a graded DGA $N_k$ of algebraic cycle complexes of $k$. The 0-th cohomology $H = H^0(B(N_k))$ of the bar complex $B(N_k)$ of $N_k$ becomes a commutative Hopf algebra with a grading $H = \oplus_r H_r$. They define the category of mixed Tate motives as that of graded comodules over the Hopf algebra $H$. They also define the $\ell$-adic and the Hodge realization functors from the category of mixed Tate motives over Spec($k$) to that of $\ell$-adic Galois representations of the field $k$, and that of mixed Tate Hodge structures. The papers [9] is one of the works based on this construction.

In [4], they also present an alternative construction of the Hodge realization functor using integral of logarithmic differential forms $\omega_n$ on $P^n$. For this they assume the existence of a certain DGA $\mathcal{D}P$ of topological chains satisfying the following conditions.

(a) The DGA $\mathcal{D}P$ contains the DGA $N_k$.

(b) The integral of the form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ on elements in $\mathcal{D}P$ converges.

(c) The generalized Cauchy integral formula holds for the integral in (b).

(d) The natural map $\tau^* : H^*(B(N)) \to H^*(B(\mathcal{D}P, N, \mathbb{Q}))$ is 0, which implies the $E_1$-degeneracy of the spectral sequence obtained from a certain filtration on $B(\mathcal{D}P, N, \mathbb{Q})$ (for the precise statement see [4] (8.6)).

The cubical differential $\partial$ and the topological differential $\delta$ make the direct sum of $AC_i(P^n, D^n; \mathbb{Q})^{alt}$ a double complex. Here $^{alt}$ indicates that we take the subspace on which a certain group $G_i$ acts by a certain character sign. The associated simple complex of this, which we denote by $AC^\bullet$, is a cohomological complex. Let $I$ be the map from $AC^\bullet$ to $\mathbb{C}$ defined by $I = \sum_n I_n$. The equality $I_{n-1}(\partial \gamma) + (-1)^n I_n(\delta \gamma)$ implies that the map $I$ is a homomorphism of complexes. We use $AC^\bullet$ in place of $\mathcal{D}P$, which enjoys the following properties:
(1) There exists a natural injection \( N_k \to AC^\bullet \). Via this map, we define an \( \mathcal{H} \) comodule \( \mathcal{H}_B \) in Definition [5,3].

(2) There is a canonical map \( \mathbb{Q} \to AC^\bullet \) which is a quasi-isomorphism.

(3) The above homomorphism \( I \) induces a quasi-isomorphism from \( AC^\bullet \otimes \mathbb{C} \) to \( \mathbb{C} \).

Using the properties (2) and (3), we show the \( E_1 \)-degeneracy of the spectral sequence obtained from a similar filtration on \( B(\mathbb{Q}, N, AC^\bullet) \) as in (d).

We give a rough sketch of our construction of the Hodge realization. We define an \( \mathcal{H} \) comodule \( \mathcal{H}_B \) resp. \( \mathcal{H}_{dR} \) as \( H^0(B(\mathbb{Q}, N, AC^\bullet)) \) resp. \( H^0(B(\mathbb{Q}, N, \mathbb{C})) \). The homomorphism \( I \) in (3) yields a comparison isomorphism \( c : \mathcal{H}_B \otimes \mathbb{C} \to \mathcal{H}_{dR} \), and via this comparison map, we construct a “universal” mixed Tate Hodge structure \( \mathcal{H}_{Hg} = (\mathcal{H}_B, \mathcal{H}_{dR}, c) \) with a left “coaction” \( \Delta_{Hg} \) of \( \mathcal{H} \) (see [5,5,5]). We define a functor \( \Phi \) from the category of graded right \( H^0(B(N_k)) \)-comodules \( (V, \Delta_V) \) to that of mixed Tate Hodge structures by the “twisted cotensor product”

\[
\Phi(V) = \ker \left( \bigoplus_i V_i \otimes \mathcal{H}_{Hg}(-i) \xrightarrow{\Delta_V \otimes \text{id} - \text{id} \otimes \Delta_{Hg}} \bigoplus_{i,j} V_i \otimes \mathcal{H}_j \otimes \mathcal{H}_{Hg}(-i-j) \right).
\]

More precisely we consider graded complexes \( AC^\bullet \otimes \mathbb{Q}(*) \) resp. \( \mathbb{C}(*) \) in stead of \( AC^\bullet \) resp. \( \mathbb{C} \). See [5,4,3] for details.

In the sequel to this paper, we will prove that the the above functor \( \Phi \) is isomorphic to that defined by Bloch-Kriz.

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2. Admissible chain complexes

2.1. Semi-algebraic triangulation. Let \( k \) be a non-negative integer. A \( k \)-simplex in an Euclidean space \( \mathbb{R}^n \) is the convex hull of affinely independent points \( a_0, \ldots, a_k \) in \( \mathbb{R}^n \). A finite simplicial complex of \( \mathbb{R}^n \) is a finite set \( K \) consisting of simplexes such that (1) for all \( s \in K \), all the faces of \( s \) belong to \( K \) and (2) for all \( s, t \in K \), \( s \cap t \) is either the empty set or a common face of \( s \) and \( t \). We denote by \( K_p \) the set of \( p \)-simplexes of \( K \). For a simplex \( \sigma \) in \( K \), the relative interior of \( \sigma \) (=the complement of the union of the proper faces of \( \sigma \)) is denoted by \( \sigma^\circ \). For a finite simplicial complex \( K \), the union of simplexes in \( K \) as a subset of \( \mathbb{R}^n \) is denoted by \( |K| \).

As for the definition of semi-algebraic set and their fundamental properties, see [5].

Theorem 2.1 ([5], Theorem 9.2.1). Let \( P \) be a compact semi-algebraic subset of \( \mathbb{R}^m \). The set \( P \) is triangulable, i.e. there exists a finite simplicial complex \( K \) and a semi-algebraic homeomorphism \( \Phi_K : |K| \to P \). Moreover, for a given finite family \( S = \{ S_j \}_{j=1, \ldots, q} \) of semi-algebraic subsets of \( P \), we can choose a finite simplicial complex \( K \) and a semi-algebraic homeomorphism \( \Phi_K : |K| \to P \) such that every \( S_j \) is the union of a subset of \( \{ \Phi_K(\sigma^\circ) \}_{\sigma \in K} \).

Remark 2.2.

(1) By [5] Remark 9.2.3 (a), the map \( \Phi_K \) can be taken so that the map \( \Phi_K \) is facewise regular embedding i.e. for each \( \sigma \in K \), \( \Phi_K(\sigma^\circ) \) is a regular submanifold of \( \mathbb{R}^m \).

(2) The pair \((K, \Phi_K)\) as in Theorem 2.1 is called a semi-algebraic triangulation of \( P \); we will then identify \(|K|\) with \( P \). A projective real or complex variety \( V \) is regarded
as a compact semi-algebraic subset of an Euclidean space by [5] Theorem 3.4.4, thus the above theorem applies to $V$.

Notation 2.3. For a subcomplex $L$ of $K$, the space $|L|$ is a subspace of $|K|$. A subset of $|K|$ of the form $|L|$ is also called a subcomplex. If a subset $S$ of $|K|$ is equal to $|L|$ for a subcomplex $M$ of $K$, then $M$ is often denoted by $K \cap S$.

2.2. The complex $AC_\ast(P^n, D^n; \mathbb{Q})$. Let $(z_1, \cdots, z_n)$ be the coordinates of $P^n = (\mathbb{P}_C^1)^n$. For $i = 1, \cdots, n$ and $\alpha = 0, \infty$, we denote by $H_{i, \alpha}$ the subvariety of $P^n$ defined by $z_i = \alpha$. The intersection of several $H_{i, \alpha}$’s is called a cubical face. We set

$$H^n = \bigcup_{1 \leq i \leq n} H_{i, \alpha}, \quad D^n = \bigcup_{i=1}^n \{z_i = 1\}, \quad \square^n = P^n - D^n. \tag{2.2.1}$$

In the following we suppose that each triangulation $K$ of $P^n$ is semi-algebraic, and the divisor $D^n$ is a subcomplex of $K$. Let $K$ be a triangulation of $P^n$. We denote by $C_\ast(K; \mathbb{Q})$ resp. $C_\ast(K, D^n; \mathbb{Q})$ the chain complex resp. the relative chain complex of $K$. An element of $C_p(K; \mathbb{Q})$ for $p \geq 0$ is written as $\sum a_\sigma \sigma$ where the sum is taken over $p$-simplexes of $K$. By doing so, it is agreed upon that an orientation has been chosen for each $\sigma$. By abuse of notation, an element of $C_\ast(K, D^n; \mathbb{Q})$ is often described similarly.

Definition 2.4. For an element $\gamma = \sum a_\sigma \sigma$ of $C_p(K; \mathbb{Q})$, we define the support $|\gamma|$ of $\gamma$ as the subset of $|K|$ given by

$$|\gamma| = \bigcup_{\sigma \in K_p, a_\sigma \neq 0} \sigma. \tag{2.2.2}$$

Under Notation 2.3. For an element $\gamma = \sum a_\sigma \sigma$ of $C_p(K; \mathbb{Q})$, $|\gamma|$ is sometimes regarded as a subcomplex of $K$.

Definition 2.5. Let $p \geq 0$ be an integer.

1. (Admissibility) A semi-algebraic subset $S$ of $P$ is said to be admissible if for each cubical face $H$, the inequality

$$\dim(S \cap (H - D^n)) \leq \dim S - 2 \text{ codim } H$$

holds. Here note that $\dim(S \cap (H - D^n))$ and $\dim S$ are the dimensions as semi-algebraic sets, and $\text{ codim } H$ means the codimension of the subvariety $H$ of $P^n$.

2. Let $\gamma$ be an element of $C_p(K, D^n; \mathbb{Q})$. Then $\gamma$ is said to be admissible if the support of a representative of $\gamma$ in $C_p(K; \mathbb{Q})$ is admissible. This condition is independent of the choice of a representative.

3. We set

$$AC_p(K, D^n; \mathbb{Q}) = \{ \gamma \in C_p(K, D^n; \mathbb{Q}) \mid \gamma \text{ and } \delta \gamma \text{ are admissible} \}$$
2.3. Subdivision and inductive limit.

**Definition 2.6.** Let \((K, \Phi_K : \lvert K \rvert \to P)\) be a triangulation of a compact semi-algebraic set \(P\). Another triangulation \((K', \Phi_{K'} : \lvert K' \rvert \to P)\) is a subdivision of \(K\) if:

1. The image of each simplex of \(K'\) under the map \(\Phi_{K'}\) is contained in the image of a simplex of \(K\) under the map \(\Phi_K\).
2. The image of each simplex of \(K\) under the map \(\Phi_K\) is the union of the images of simplexes of \(K'\) under \(\Phi_{K'}\).

If \(K'\) is a subdivision of a triangulation \(K\), there is a natural homomorphism of complexes \(\lambda : C_\bullet(K, D^n; \mathbb{Q}) \to C_\bullet(K', D^n; \mathbb{Q})\) called the subdivision operator. See for example [14] Theorem 17.3 for the definition. For a simplex \(\sigma\) of \(K\), the chain \(\lambda(\sigma)\) is carried by \(K' \cap \sigma\), and so that the map \(\lambda\) sends \(AC_\bullet(K, D^n; \mathbb{Q})\) to \(AC_\bullet(K', D^n; \mathbb{Q})\). By Theorem 2.1 two semi-algebraic triangulations have a common subdivision. Since the map \(\lambda\) and the differential \(\delta\) commute, the complexes \(C_\bullet(K, D^n; \mathbb{Q})\) and \(AC_\bullet(K, D^n; \mathbb{Q})\) form inductive systems indexed by triangulations \(K\) of \(P^n\).

**Definition 2.7.** We set

\[
C_\bullet(P^n, D^n; \mathbb{Q}) = \lim_{\kappa \to \infty} C_\bullet(K, D^n; \mathbb{Q}), \quad AC_\bullet(P^n, D^n; \mathbb{Q}) = \lim_{\kappa \to \infty} AC_\bullet(K, D^n; \mathbb{Q}).
\]

Here the limit is taken on the directed set of triangulations.

A proof of the following Proposition will be given in Appendix A.

**Proposition 2.8 (Moving lemma).** The inclusion of complexes

\[
\iota : AC_\bullet(P^n, D; \mathbb{Q}) \to C_\bullet(P^n, D; \mathbb{Q})
\]

is a quasi-isomorphism.

3. Face map and cubical differential

Recall that a subcomplex \(L\) of a simplicial complex \(K\) is called a full subcomplex, if all the vertices of a simplex \(\sigma\) in \(K\) belong to \(L\), then \(\sigma\) belongs to \(L\).

**Definition 3.1 (Good triangulation).** We define a family \(\mathcal{L}\) of subsets of \(P^n\) by

\[
\mathcal{L} = \{H_{J_1} \cup \cdots \cup H_{J_k}\}_{(J_1, \ldots, J_k)},
\]

where \(H_{J_i}\) are cubical faces of \(P^n\). In short, a member of \(\mathcal{L}\) is the union of several cubical faces. A finite semi-algebraic triangulation \(K\) of \(P^n\) is called a good triangulation if \(K\) satisfies the following conditions.

1. The divisor \(D^n\) is a subcomplex of \(K\).
2. The map \(\Phi_K : \lvert K \rvert \to P^n\) is facewise regular embedding. cf. Remark 2.4.
3. Each element \(L_i \in \mathcal{L}\) is a full subcomplex of \(K\) i.e. there exists a full subcomplex \(M_i\) of \(K\) such that \(L_i = \lvert M_i \rvert\).
4. For \(i = 1, \ldots, n\), the subsets \(\{|z_i| \leq 1\}\) are subcomplexes of \(K\).

In particular, if \(K\) is a good triangulation, then for any simplex \(\sigma\) of \(K\) and \(L_i \in \mathcal{L}\), the intersection \(\sigma \cap L_i\) is a (simplicial) face of \(\sigma\). This is the primary reason to consider the condition (3).
Remark 3.2. For a simplicial complex $K$, we denote by $sd K$ its barycentric subdivision. If $L$ is a subcomplex of $K$, then $sd L$ is a full subcomplex of $sd K$ (See for example Exercise 3.2 of [15]). It follows that if $K$ is a semi-algebraic triangulation of $P^n$ which is a facewise regular embedding, and such that $D^n$ and each $L_i \in L$ are subcomplexes of $K$, then $sd K$ is a good triangulation.

In the following, each triangulation $K$ of $P^n = (\mathbb{P}_C^1)^n$ is assumed to be good in the sense of Definition 3.1.

3.1. Thom cocycles.

3.1.1. Definition of Thom cocycles. Let $K$ be a good triangulation of $P^n = (\mathbb{P}_C^1)^n$.

Let $\Delta$ be the subset $\{|z_1| < 1\}$ of $P^n$. We set $L_1 = K \cap H_{1,0}$ and

$$N = \bigcup_{\sigma \in K, \sigma \cap H_{1,0} \neq \emptyset} \sigma, \quad W = \bigcup_{\sigma \in K, \sigma \cap H_{1,0} = \emptyset} \sigma.$$ 

For spaces $X$ and $Y$ such that $X \supset Y$, we denote by $H_{\text{sing}}^i(X; \mathbb{Q})$ (resp. $H_{\text{sing}}^i(X, Y; \mathbb{Q})$) the singular cohomology (resp. the relative singular cohomology) of $X$ (resp. $(X, Y)$) with $\mathbb{Q}$-coefficients. Under the comparison isomorphism $H_{\text{sing}}^1(\Delta - H_{1,0}; \mathbb{C}) \simeq H_{dR}^1(\Delta - H_{1,0})$, the de Rham class $\left[\frac{dz_1}{2\pi i z_1}\right]$ of $\frac{dz_1}{2\pi i z_1}$ is contained in the subgroup $H_{\text{sing}}^1(\Delta - H_{1,0}; \mathbb{Q})$ of $H_{\text{sing}}^1(\Delta - H_{1,0}; \mathbb{C})$. Since $L_1$ is a full subcomplex of $K$, $W$ is a deformation retract of $P^n - H_{1,0}$. (This can be seen as follows. Each simplex $\sigma$ in $N$ not contained in $L_1$ is the join $\tau \ast \eta$ where $\tau \in L_1$ and $\eta \in W$. The linear projection $\pi_{\sigma, \eta} : \sigma - \tau \rightarrow \eta$ in Definition 4.12 is glued to a deformation retract from $P^n - H_{1,0}$ to $W$.) Therefore we have isomorphisms

$$H^2(K, W; \mathbb{Q}) \xrightarrow{\simeq} H_{\text{sing}}^2(P^n, P^n - H_{1,0}; \mathbb{Q}) \xrightarrow{\simeq} H_{\text{sing}}^2(\Delta, \Delta - H_{1,0}; \mathbb{Q}).$$

Here the second isomorphism holds by excision.

Definition 3.3. A simplicial cocycle $T$ in $C^2(K, W; \mathbb{Q})$ is a Thom cocycle if its cohomology class is equal to $\delta \left[\frac{dz_1}{2\pi i z_1}\right]$ in $H_{\text{sing}}^2(\Delta, \Delta - H_{1,0}; \mathbb{Q})$ via the isomorphism (3.1.1). Here $\delta$ denotes the connecting homomorphism

$$\delta : H_{\text{sing}}^1(\Delta - H_{1,0}; \mathbb{Q}) \rightarrow H_{\text{sing}}^2(\Delta, \Delta - H_{1,0}; \mathbb{Q}).$$

A $\mathbb{C}$-valued Thom cocycle in $C^2(K, W; \mathbb{C})$ is defined similarly.

We will give some examples of Thom cocycles.

3.1.2. Thom form. Let $\epsilon$ be a positive real number and let $\rho$ be a $[0, 1]$-valued $C^\infty$-function on $\Delta$ such that

1. $\rho = 0$ on $\Delta \cap \{|z_1| < \frac{1}{2}\epsilon\}$.
2. $\rho = 1$ on $\Delta \cap \{\epsilon < |z_1|\}$.

Then $c_\rho = \frac{1}{2\pi i} \rho \frac{dz_1}{z_1}$ defines an element of $C_{\text{sing}}^1(P^n - H_{1,\infty}; \mathbb{C})$. We set $T_\rho = dc_\rho = \frac{1}{2\pi i} d\rho \wedge \frac{dz_1}{z_1}$ for $\epsilon$ sufficiently small we have $c_\rho = \frac{1}{2\pi i} \frac{dz_1}{z_1}$ on $W - H_{1,\infty}$, and $T_\rho$ defines
the same class as $\delta \left( \frac{dz}{2\pi i z} \right)$ in $H^2_{\text{sing}}(P^n - H_{1,\infty}, W - H_{1,\infty}; \mathbb{C}) \simeq H^2_{\text{sing}}(P^n, W; \mathbb{C})$, so it is a rational class.

**Definition 3.4.** The cocycle $T_\rho \in C^2(K, W; \mathbb{C})$ is a $\mathbb{C}$-valued Thom cocycle. We call the above cocycle $T_\rho$ a Thom form.

3.1.3. **Singular Thom cocycle $T^B_{H_{1,0}}$.** We first note that this cocycle is not used in the rest of the paper, and we omit the proof that this is a Thom cocycle. Suppose that $\Delta$ is a subcomplex of $K$ and $\Delta \cap W \subset \Delta - \{0\}$ is a deformation retract. For a 1-simplex $\sigma \in C^1(\Delta)$, we set

$$L^B(\gamma) = \begin{cases} 0 & \text{if } \sigma \not\subset W \\ \frac{1}{2\pi} \left( \text{Im} \int_{\sigma} \frac{dz}{z} + \text{arg}(\gamma(0)) \right) & \text{if } \sigma \subset W. \end{cases}$$

Here $[r]$ denotes the Gauss symbol of a real number $r$ and $\text{arg}(z)$ is the argument of a complex number $z$ in $[0, 2\pi)$. Note that the cochain $L^B$ counts the intersection number (with sign) of $\sigma$ and the positive part of real axis. Then $T^B = dL^B \in C^2(\Delta, \Delta \cap W)$ becomes a Thom cocycle.

We remark that the cocycle $T^B$ counts the winding number of the boundary of relative 2-cycle.

3.2. **The cap product with a Thom cocycle.**

3.2.1. **Simplicial cap product.**

**Definition 3.5 (Ordering of complex, good ordering).** Let $K$ be a good triangulation of $P^n$.

1. A partial ordering on the set of vertices in $K$ is called an ordering of $K$, if the restriction of the ordering to each simplex is a total ordering.
2. Let $H_J$ be a cubical face of $P^n$. An ordering of $K$ is said to be good with respect to $H_J$ if it satisfies the following condition. If a vertex $v$ is on $H_J$ and $w \geq v$ for a vertex $w$, then $w \in H_J$.

We denote by $[a_0, \ldots, a_k]$ the simplex spanned by $a_0, \ldots, a_k$. Let $\mathcal{O}$ be a good ordering of $K$ with respect to $H_{1,0}$. We recall the definition of the cap product $\cap : C^p(K) \otimes C_k(K) \to C_{k-p}(K)$. For a simplex $\alpha = [v_0, \ldots, v_k]$ such that $v_0 < \cdots < v_k$ and $u \in C^p(K)$, we define

$$(3.2.2) \quad u \cap \alpha = u([v_0, \ldots, v_p])[v_p, \ldots, v_k].$$

One has the boundary formula

$$(3.2.3) \quad \delta(u \cap \alpha) = (-1)^p(u \cap (\delta \alpha) - (du) \cap \alpha)$$

where $du$ denotes the coboundary of $u$, see [12], p.239 (note the difference in sign convention from [14]). Thus if $u$ is a cocycle, $\delta(u \cap \alpha) = (-1)^p u \cap (\delta \alpha)$.

**Proposition 3.6.** Let $T$ be a Thom cocycle and $\mathcal{O}$ a good ordering of $K$ with respect to $H_{1,0}$.

1. The map $T \cap \mathcal{O}$ and the differential $\delta$ commute.
The image of the homomorphism $T^\cap$ is contained in $C_{k-2}(L_1; \mathbb{Q})$, where $L_1 = K \cap H_{1,0}$. As a consequence, we have a homomorphism of complexes

$$
T^\cap : C_k(K, D^n; \mathbb{Q}) \to C_{k-2}(L_1, D^{n-1}; \mathbb{Q}).
$$

Proof. (1) Since $T$ is a cocycle of even degree, we have $\delta(T^\cap \sigma) = T^\cap (\delta \sigma)$ for a simplex $\sigma$.

(2) If $v_2 \notin H_{1,0}$, then $[v_0, v_1, v_2] \cap H_{1,0} = \emptyset$ and $T([v_0, v_1, v_2]) = 0$ since the cochain $T$ vanishes on $W$. If $v_2 \in H_{1,0}$, then the vertices $v_2, \ldots, v_k$ are on $H_{1,0}$, and we have $[v_2, \ldots, v_k] \subset H_{1,0}$ since $H_{1,0}$ is a full subcomplex of $K$. Thus the assertion holds.

3.2.2. Independence of $T$ and ordering. Let $K$ be a good triangulation, and let $L_1$ be the subcomplex $K \cap H_{1,0}$.

Proposition 3.7. Let $T \in C^2(K, W; \mathbb{Q})$ be a Thom cocycle of the face $H_{1,0}$ and let $\gamma$ be an element of $AC_k(K, D^n; \mathbb{Q})$. Then we have the following.

(1) The chain $T^\cap \gamma$ is an element in $AC_{k-2}(L_1, D^{n-1}; \mathbb{Q})$.

(2) The chain $T^\cap \gamma$ is independent of the choice of a Thom cocycle $T$ and a good ordering $\cap$. Thus the map

$$
T^\cap : AC_k(K, D^n; \mathbb{Q}) \to AC_{k-2}(L_1, D^{n-1}; \mathbb{Q})
$$

induced by (3.2.4) is denoted by $T^\cap$.

(3) Let $K'$ be a good subdivision of $K$. We set $W' = \bigcup_{\sigma \in K'} \sigma'$. Let $T' \in C^2(K', W'; \mathbb{Q})$ be a Thom cocycle, and $\cap'$ a good ordering of $K'$ with respect to $H_{1,0}$. We set $L'_1 = K' \cap H_{1,0}$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
AC_k(K, D^n; \mathbb{Q}) & \xrightarrow{T^\cap} & AC_{k-2}(L_1, D^{n-1}; \mathbb{Q}) \\
\downarrow \lambda & & \downarrow \lambda \\
AC_k(K', D^n; \mathbb{Q}) & \xrightarrow{T'^\cap} & AC_{k-2}(L'_1, D^{n-1}; \mathbb{Q})
\end{array}
$$

where the vertical maps $\lambda$ are subdivision operators.

Proof. (1) For an element $z \in C_k(K, D^n; \mathbb{Q})$, we have $T^\cap z \in C_{k-2}(L_1, D^{n}; \mathbb{Q})$ by Proposition 3.3 (2). By the definition of the cap product, we see that the set $|T^\cap z| \subset |z| \cap H_{1,0}$. It follows that if $z$ is admissible i.e. $|z| - D^n$ meets all the cubical faces properly, then $|T^\cap z| - D^{n-1}$ meets all the cubical faces of $H_{1,0}$ properly. Similarly, if the chain $\delta z$ is admissible, then $T^\cap (\delta z)$ is admissible in $H_{1,0}$. By Proposition 3.3 (1) we have the equality $\delta(T^\cap z) = T^\cap (\delta z)$.

(2) A proof will be given in Section 3.2.

(3) A proof will be given in Section 3.3.
By taking the inductive limit of the homomorphism

$$T : AC_\bullet(K, D^n; \mathbb{Q}) \to AC_{\bullet-2}(L_1, D^{n-1}; \mathbb{Q}).$$

for subdivisions, we get a homomorphism

$$(3.2.6) \quad T : AC_\bullet(P^n, D^n; \mathbb{Q}) \to AC_{\bullet-2}(P^{n-1}, D^n; \mathbb{Q}).$$

**Definition 3.8.** The map $T$ of (3.2.6) is denoted by $\partial_{1,0}$, and called the face map of the face $H_{1,0}$.

By the symmetry, we have maps $\partial_{i,\alpha}$ for $i = 1, \ldots, n$ and $\alpha = 0, \infty$.

### 3.3. Cubical differentials.

**Definition 3.9 (Cubical differential).** The map $\partial$ is defined by the equality

$$(3.3.7) \quad \partial = \sum_{i=1}^{n} (-1)^{i-1} (\partial_{i,0} - \partial_{i,\infty}) : AC_\bullet(P^n, D^n; \mathbb{Q}) \to AC_{\bullet-2}(P^{n-1}, D^n; \mathbb{Q}).$$

**Proposition 3.10.** The composite

$$\partial^2 : AC_\bullet(P^n, D^n; \mathbb{Q}) \to AC_{\bullet-4}(P^{n-2}, D^n; \mathbb{Q})$$

is the zero map.

A proof of this proposition will be given in Section B.4.

## 4. The Generalized Cauchy Formula

### 4.1. Statement of the generalized Cauchy formula.

Let $K$ be a good triangulation of $P^n$. We define the rational differential form $\omega_n$ on $P^n$ by

$$\omega_n = \frac{1}{(2\pi i)^n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$ 

By applying Theorem 4.4 [10] to the case where $A = \sigma$, $m = \dim \sigma = n$ and $\omega = \omega_n$ we have the following theorem.

**Theorem 4.1.** Let $\sigma$ be an admissible $n$-simplex of $K$. The integral

$$(4.1.1) \quad \int_\sigma \omega_n$$

converges absolutely.

Using Theorem 4.1, the following is well-defined.

**Definition 4.2.** Let $\gamma$ be an element in $AC_n(K, D^n; \mathbb{Q})$, and let $\sum_{\sigma} a_\sigma \sigma$ be a representative of $\gamma$ in $C_n(K; \mathbb{Q})$. We define $I_n(\gamma) \in \mathbb{C}$ by

$$(4.1.2) \quad I_n(\gamma) = (\frac{-1}{2})^{n(n-1)/2} \sum_{\sigma} \int_\sigma a_\sigma \omega_n$$

where the sum is taken over the n-simplexes $\sigma$ not contained in $D^n$. The right hand side of (4.1.2) does not depend on the choice of a representative of $\gamma$. The map $I_n$ is compatible with subdivisions of triangulations, and we obtain a map

$$I_n : AC_n(P^n, D^n; \mathbb{Q}) \to \mathbb{C}.$$
In this section, we prove the following theorem.

**Theorem 4.3** (Generalized Cauchy formula). Let $\gamma$ be an element in $AC_{n+1}(P^n, D^n; \mathbb{Q})$. Then we have the equality

\begin{equation}
I_{n-1}(\partial \gamma) + (-1)^n I_n(\delta \gamma) = 0.
\end{equation}

Let $\gamma = \sum a_\sigma \sigma$ be an element of $AC_{n+1}(K, D^n; \mathbb{Q})$ for a good triangulation $K$. By setting $\partial \gamma = \sum c_\tau \tau$ and $\delta \gamma = \sum b_\nu \nu$, the equality (4.1.3) can be written as

\begin{equation}
\sum_\nu \int_\nu b_\nu \omega_n = \sum_\tau \int_\tau c_\tau \omega_{n-1}.
\end{equation}

Here it is understood that the integral of $\omega_n$ on a simplex contained in $D^n$ is zero by definition.

**Remark 4.4.** For a $n$-simplex $\nu$ contained in $D^n$, the integral $\int_\nu \omega_n$ is defined to be zero. This will be justified as follows. Let $\sigma$ be an admissible $(n+1)$-simplex of a good triangulation $K$. If $\sigma \subset D^n$, then $\partial \sigma$ and $\delta \sigma$ are contained in $D^n$, and $\sigma$ is irrelevant to the problem. Suppose that $\sigma \not\subset D^n$ and let $\nu$ be an $n$-simplex contained in $\sigma \cap D^n$. If $\nu \not\subset H^n$, then the integral $\int_\nu \omega_n$ should be zero since the restriction of $\omega_n$ to $D^n - H^n$ is zero. The critical case is that $\nu \subset H^n$. In this case we have $\nu = \sigma \cap H^n$ since $K$ is a good triangulation, $\sigma$ is admissible and $\sigma \not\subset D^n$. This case will be considered in Proposition 4.5, and it will be clear that our definition is correct.

4.2. Outline of the proof of Theorem 4.3. Let $\gamma$ be an element in $AC_{n+1}(K, D^n; \mathbb{Q})$ and $\sum_\sigma a_\sigma \sigma$ be a representative of $\gamma$ in $C_{n+1}(K; \mathbb{Q})$. We define elements $\gamma_D$ and $\gamma_{D^c}$ in $C_{n+1}(K, D; \mathbb{Q})$ by

\begin{equation}
\gamma_D = \sum_{\sigma \cap H^n \subset D^n} a_\sigma \sigma,
\end{equation}

\begin{equation}
\gamma_{D^c} = \gamma - \gamma_D.
\end{equation}

Then $\gamma_D$ is an element in $AC_{n+1}(K, D^n; \mathbb{Q})$, and as a consequence $\gamma_{D^c}$ is also an element in $AC_{n+1}(K, D^n; \mathbb{Q})$. Theorem 4.3 is a consequence of the following Proposition 4.5 and Proposition 4.6.

**Proposition 4.5.** Let $\gamma$ be an element in $AC_{n+1}(K, D^n; \mathbb{Q})$. Then we have $I_n(\delta \gamma_D) = 0$.

Since $\partial \gamma_D = 0$ by definition, Theorem 4.3 holds for $\gamma_D$. The proof of Proposition 4.5 is given in §4.5.

**Proposition 4.6.** Let $\gamma$ be an element in $AC_{n+1}(K, D^n; \mathbb{Q})$. Suppose that $\gamma$ has a representative $\sum_\sigma a_\sigma \sigma$ in $C_{n+1}(K; \mathbb{Q})$ such that $a_\sigma = 0$ if $\sigma \cap H^n \subset D^n$. Then Theorem 4.3 holds for $\gamma$.

Note that if $\gamma$ has a representative which satisfies the above condition, then it is uniquely determined. Therefore we denote $|\sum_\sigma a_\sigma \sigma|$ by $|\gamma|$.

Let $\{H_{i, \alpha}\}$ be the set of codimension one cubical faces defined in §2.2. We define $H_h$ as the union of higher codimensional cubical faces, i.e.

$$H_h = \bigcup_{1 \leq i < i' \leq n, \alpha \in \{0, \infty\}, \beta \in \{0, \infty\}} (H_{i, \alpha} \cap H_{i', \beta}).$$

In §4.3 we prove the following theorem.
Proposition 4.7 (Generalized Cauchy formula for codimension one face). Let $\gamma$ be an element in $AC_{n+1}(K, D^n; \mathbb{Q})$. Suppose that $\gamma$ has the representative $\gamma = \sum_{\sigma} a_{\sigma} \sigma$ in $C_{n+1}(K; \mathbb{Q})$ such that (1) $a_{\sigma} = 0$ if $\sigma \cap H^n \subset D^n$ and (2) $|\gamma| \cap H_h = \emptyset$. Then Theorem 4.3 holds for $\gamma$.

Proposition 4.7 will be proved in §4.3. Proposition 4.6 is a consequence of Proposition 4.7 and a limit argument which will be given in §4.4.

4.3. Proof of Proposition 4.7. First we observe the following fact.

Lemma 4.8. Let $K$ be a good triangulation of $P^n$, and let $\sigma \in K$ be a simplex such that $\sigma \cap H^n \neq \emptyset$. If $\sigma \cap H_h = \emptyset$, then there is a unique codimension one cubical face $H_{i,\alpha}$ such that $\sigma \cap H^n \subset H_{i,\alpha}$. Moreover we have $\sigma \cap H' = \emptyset$ for any other cubical face $H'$.

Proof. This follows from the facts that $H^n = \cup H_{i,\alpha}$, that each $H_{i,\alpha}$ is a subcomplex of $K$ and that $\sigma \cap H^n$ is a simplicial face of $\sigma$. \[\square\]

Let $\gamma$ be an element in $AC_{n+1}(K, D^n; \mathbb{Q})$, and let $\sum_{\sigma} a_{\sigma} \sigma$ be the representative of $\gamma$ satisfying the condition of Proposition 4.7. We set $\gamma_{i,\alpha} = \sum_{\sigma \in H_{i,\alpha}} a_{\sigma} \sigma$. Then by Lemma 4.8 we have the equality

$$\gamma = \sum_{(i,\alpha)} \gamma_{i,\alpha}$$

and $|\gamma_{i,\alpha}| \cap H' = \emptyset$ for any cubical face $H'$ other than $H_{i,\alpha}$. Therefore, each $\gamma_{i,\alpha}$ is an element in $AC_{n+1}(K, D^n, \mathbb{Q})$. It suffices to prove the assertion for each $\gamma_{i,\alpha}$. Therefore to prove Proposition 4.7, we may and do assume that $|\gamma| \cap H^n \subset H = H_{1,0}$.

Notation 4.9. We fix a good ordering of $K$ with respect to $H$. For an $(n+1)$-simplex $\sigma = [v_0, v_1, \ldots, v_{n+1}]$ such that $v_0 < v_1 < \cdots < v_{n+1}$, we denote $\sigma_f = [v_0, v_1, v_2]$ and $\sigma_b = [v_2, \ldots, v_{n+1}]$.

To compute the image $\partial \gamma$ of the face map, we choose a Thom form as follows. Let $\rho : \mathbb{R}_+ \to [0, 1]$ be a $C^\infty$ function such that

$$\rho(r) = \begin{cases} 0 & (r \leq \frac{1}{2}), \\ 1 & (r \geq 1). \end{cases}$$

Let $\epsilon$ be a positive number, and let $\rho_\epsilon$ be the function on $\mathbb{P}^1$ defined by $\rho_\epsilon(z_1) = \rho\left(\frac{|z_1|}{\epsilon}\right)$. The function on $P^n$ given by $(z_1, \ldots, z_n) \mapsto \rho_\epsilon(z_1)$ is also denoted by $\rho_\epsilon$. We take $\epsilon$ small enough so that the set $W = \bigcup_{\sigma \in K, \sigma \cap H = \emptyset} \sigma$ is contained in the set $\{|z_1| \geq \epsilon\}$. Then $T_\epsilon = d\rho_\epsilon \wedge \omega_1$ is a Thom form i.e. $T_\epsilon(\sigma) = \int_{\sigma} T_\epsilon = 0$ for each 2-simplex $\sigma$ in $W$.

We set $\delta \gamma = \sum_\nu b_\nu \nu$. Using the above Thom form, the image of $\gamma$ under the face map is computed as

$$\partial \gamma = \sum_\sigma \left( \int_{\sigma_f} d\rho_\epsilon \wedge \omega_1 \right) a_{\sigma} \sigma_b$$
Note that if $\sigma_f \subset W$, then the above integral is zero. If $\sigma_f \nsubseteq W$, then we have $\sigma_b \subset H$ since $K \cap H$ is a full subcomplex and we take a good ordering. Therefore the assertion (4.1.4) is written as follows:

\[(4.3.7) \sum_\nu \int_\nu b_\nu \omega_n = \sum_{\sigma_f \subset H} a_\sigma \int_{\sigma_f} d\rho_\epsilon \wedge \omega_1 : \int_{\sigma_b} \omega_{n-1}.\]

Let $\sigma$ be an $(n+1)$-simplex of $K$ such that $a_\sigma \neq 0$. Since the form $a_\sigma \rho_\epsilon \omega_n$ is smooth on a neighborhood of $\sigma$, we have the equality

\[(4.3.8) \int_{\sigma} a_\sigma d\rho_\epsilon \wedge \omega_n = \int_{\delta \sigma} a_\sigma \rho_\epsilon \omega_n \]

by the Stokes formula Theorem 2.9 [10]. The summation of the right hand side of (4.3.8) for $\sigma$ is equal to

\[\sum_{\sigma} \int_{\delta \sigma} a_\sigma \rho_\epsilon \omega_n = \sum_\nu \int_\nu b_\nu \rho_\epsilon \omega_n.\]

By Theorem 4.1 and Lebesgue’s convergence theorem, we have

\[\lim_{\epsilon \to 0} \int_{\nu} b_\nu \rho_\epsilon \omega_n = \int_\nu b_\nu \omega_n\]

for each $\nu$ with $b_\nu \neq 0$. By summing up (4.3.8) for all $\sigma$ and taking the limit for $\epsilon \to 0$, we have

\[\lim_{\epsilon \to 0} \sum_{\sigma} \int_{\sigma} a_\sigma d\rho_\epsilon \wedge \omega_n = \sum_\nu \int_\nu b_\nu \omega_n.\]

Comparing with (4.3.7), to prove Proposition 4.7, it is enough to show the equality:

\[(4.3.9) \lim_{\epsilon \to 0} \sum_{\sigma} \int_{\sigma} a_\sigma d\rho_\epsilon \wedge \omega_n = \sum_{\sigma_b \subset H} a_\sigma \int_{\sigma_f} d\rho_\epsilon \wedge \omega_1 : \int_{\sigma_b} \omega_{n-1}\]

We reduce the proof of (4.3.9) to the case where $|\gamma| \cap H$ is a simplex. For this purpose, we prepare the following definition.

**Definition 4.10.** Let $\gamma = \sum a_\sigma \sigma$ be an element in $C_{n+1}(K; \mathbb{Q})$, and let $\tau$ be a simplex such that $\tau \subset H = H_{1,0}$. We define an element $\gamma^{(\tau)}$ in $C_{n+1}(K, D^n; \mathbb{Q})$ by

\[(4.3.10) \gamma^{(\tau)} = \sum_{\sigma \cap H = \tau} a_\sigma \sigma.\]

Suppose that $\gamma \in AC_{n+1}(K, D^n; \mathbb{Q})$ has the representative $\gamma = \sum a_\sigma \sigma$ such that (1) $a_\sigma = 0$ if $\sigma \cap H^n \subset D^n$ and (2) $|\gamma| \cap H^n \subset H = H_{1,0}$. Then we have an equality

\[(4.3.11) \gamma = \sum_{\tau \subset |\gamma| \cap H} \gamma^{(\tau)}.\]

Since we have assumed that $\gamma = \gamma_{1,0}$, we have the equality (4.3.11).

**Proposition 4.11.** Let $\gamma$ be an element of $AC_{n+1}(K, D^n; \mathbb{Q})$ and let $\gamma = \sum a_\sigma \sigma$ be its representative in $C_{n+1}(K; \mathbb{Q})$ such that

1. $a_\sigma = 0$ if $\sigma \cap H^n \subset D^n$,
2. $|\gamma| \cap H^n \subset H = H_{1,0}$ and
(3) $|\gamma| \cap \mathbf{H}_h = \emptyset$. 

Let $\tau$ be a simplex in $|\gamma| \cap H$ not contained in $\mathbf{D}^n$. Then $\gamma^{(\tau)}$ is an element in $\text{AC}_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$.

**Proof.** We prove that $|\delta \gamma^{(\tau)}|$ is admissible. Since $|\gamma| \cap \mathbf{H}_h = \emptyset$, we have $|\gamma| \cap H' = \emptyset$ for any other cubical face $H'$ than $H$ by Lemma 4.8. So it suffices to show that $|\delta \gamma^{(\tau)}| - \mathbf{D}^n$ meets $H$ properly. Since $|\gamma| - \mathbf{D}^n$ meets $H$ properly, we have $\dim \tau \leq n - 1$. If $\dim \tau < n - 1$, then $|\delta \gamma^{(\tau)}| - \mathbf{D}^n$ meets $H$ properly since $|\delta \gamma^{(\tau)}| \cap H \subset \tau$. We consider the case where $\dim \tau = n - 1$. We have

$$
\delta^{(\tau)} \gamma = \sum_{\nu \subset |\gamma^{(\tau)}|} b_\nu \nu, \quad b_\nu = \sum_{\nu \prec |\gamma^{(\tau)}|} [\sigma : \nu] a_\sigma
$$

and

$$
\delta \gamma = \sum_{\nu \subset |\gamma|} c_\nu \nu, \quad c_\nu = \sum_{\nu \prec \sigma} [\sigma : \nu] a_\sigma
$$

Here $\nu \prec \sigma$ means that $\nu$ is a codimension one face of $\sigma$, and $[\sigma : \nu] \in \{\pm 1\}$ is the index of $\sigma$ with respect to $\nu$. To prove the admissibility of $\delta \gamma^{(\tau)}$, it is sufficient to show the following claim.

**Claim.** Let $\nu$ be an $n$-simplex in $K$ such that (1) $\nu \subset |\gamma^{(\tau)}|$, and (2) $\nu - \mathbf{D}^n$ does not meet $H$ properly. Then we have $b_\nu = 0$.

**Proof of the claim.** Let $\nu$ be a simplex with the conditions in the claim. Then $\nu \cap H$ is a face of $\nu$, since $K$ is a good triangulation. By condition (1), we have $\nu \cap H \subset |\gamma^{(\tau)}| \cap H = \tau$ and $\dim(\nu \cap H) \leq \dim(\tau)$. By condition (2), we have $n - 1 \leq \dim \nu \cap H$. Since $\dim \tau = n - 1$, the above inequalities are equalities, and we have $\nu \cap H = \tau$. We consider each term appeared on the right hand side of the equality defining $c_\nu$. Let $\sigma$ be a $n + 1$-simplex such that $\nu \subset \sigma$ and $a_\sigma \neq 0$. The set $\sigma \cap H$ is a face of $\sigma$. By the admissibility condition, we have $\dim \sigma \cap H \leq n - 1$. Since $\dim \tau = n - 1$ and $\tau \subset \nu \subset \sigma$, we have $\sigma \cap H = \tau$. Thus this term appears in the right hand side of the equation defining $b_\nu$. So we have $b_\nu = c_\nu$. Since $c_\nu = 0$ by the admissibility of $\delta \gamma$, we have $b_\nu = 0$.

By Proposition 4.11 the chain $\gamma^{(\tau)} \in \text{AC}_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$ for each $\gamma$ and we assume that $\gamma = \gamma^{(\tau)}$ for a simplex $\tau \subset H$.

Let $\sigma$ be a $(n + 1)$-simplex in $\gamma$ and assume given a smooth $(n - 1)$-form $\varphi$ on a neighborhood $U$ of $\sigma$. The inclusion $i : H \to \mathbf{P}^n$ restricts to an inclusion $i_U : U \cap H \to U$ (when there is no fear of confusion, we abbreviate $i^*_U \varphi$ to $i^* \varphi$). Since $\tau^0 \subset H$ is a smooth submanifold, $i^* \varphi$ restricts to a smooth form on $\tau^0$, denoted by the same $i^* \varphi$ (this is where the facewise regularity is used); it is zero if $\dim \tau < n - 1$.

**Definition 4.12** (Barycentric coordinate, linear projection). Let $\sigma = [a_0, \cdots, a_p]$ be a $p$-simplex. A point $x$ in $\sigma$ is expressed uniquely as $x = \sum_{i=0}^{k} \lambda_i a_i$ with $\sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0$. The vector $(\lambda_0, \ldots, \lambda_p)$ is called the barycentric coordinate of $x$.

Let $\sigma = [v_0, \ldots, v_p]$ be a $p$-simplex and $\tau = [v_k, \ldots, v_{p}]$ be a proper $(p - k)$-face of $\sigma$ $(0 < k \leq p)$. We set $\tau' = [v_0, \cdots, v_{k-1}]$. We define a linear projection $\pi_{\sigma, \tau} : \sigma - \tau' \to \tau$ by

$$
\pi_{\sigma, \tau}(x) = \frac{1}{\sum_{i=k}^{p} \lambda_i} (\lambda_k, \ldots, \lambda_p),
$$
where \((\lambda_0, \ldots, \lambda_p)\) is the barycentric coordinate of \(x\).

Let \(\pi_\sigma = \pi_{\sigma, \tau}\) be the linear projection \(\sigma^0 \rightarrow \tau\) defined in Definition 4.12. The map \(\pi_\sigma\) restricts to a smooth map between submanifolds \(\sigma^0 \rightarrow \tau^0\), thus the pull-back \(\pi_\sigma^*i^*\phi\) defines a smooth form on \(\sigma^0\).

The following proposition will be proved in \(\S 4.3.1\).

**Proposition 4.13.** Let \(\sigma\) be a \((n+1)\)-simplex in \(\gamma\).

1. We have
   \[
   \lim_{\epsilon \to 0} \int_{\sigma} d\rho_\epsilon \wedge \omega_1 \wedge (a_\sigma \omega_{n-1} - \pi_\sigma^* i^*(a_\sigma \omega_{n-1})) = 0
   \]

2. If the dimension of \(\tau < n - 1\), then the equality
   \[
   \lim_{\epsilon \to 0} \int_{\sigma} a_\sigma d\rho_\epsilon \wedge \omega_1 \wedge \omega_{n-1} = 0
   \]
   holds.

We assume Proposition 4.13 for the moment and prove the equality (4.3.9). For an \((n+1)\)-simplex \(\sigma\) in \(\gamma\), if \(\dim \tau = \dim \sigma \cap H < n - 1\), then \(\tau\) is a proper face of \(\sigma_b\), and it follows that the right hand side of (4.3.9) is zero. Therefore by Proposition 4.13 (2), it is sufficient to prove (4.3.9) for the case where \(\gamma = \gamma(\tau)\) and \(\dim \tau = n - 1\). Under this assumption, we have \(\sigma_b = \tau\) for each \((n+1)\)-simplex \(\sigma\) such that \(\sigma \subset |\gamma|\).

**Proposition 4.14.** For a positive \(\epsilon\), we have the equality

\[
\sum_{\sigma} \int_{\sigma} a_\sigma d\rho_\epsilon \wedge \omega_1 \wedge \pi_\sigma^* i^*(\omega_{n-1}) = \sum_{\sigma} a_\sigma \left( \int_{\sigma} d\rho_\epsilon \wedge \omega_1 \right) \cdot \left( \int_{\tau} i^*(\omega_{n-1}) \right)
\]

**Proof.** We recall the formulation of projection formula for integrals of differential forms. Let \(M, N\) be oriented smooth manifolds of dimension \(m, n\), respectively. Then \(M \times N\) is equipped with the product orientation. Let \(\pi : M \times N \rightarrow N\) be the projection to \(N\). For \(\varphi\) an \(m\)-form on \(M \times N\) and \(\psi\) an \(n\)-form on \(N\), we have *projection formula*

\[
\int_{M \times N} \varphi \wedge \pi^* \psi = \int_N (\pi_* \varphi) \psi.
\]

Here \(\pi_* \varphi\) is the function

\[
(\pi_* \varphi)(y) = \int_M \varphi|_{M \times \{y\}}.
\]

(The precise meaning of the equality is that, if the left hand side is absolutely convergent, then the function \(\pi_* \varphi\) is measurable, the right hand side is also absolutely convergent, and the equality holds.) This formula follows from Fubini’s theorem for Lebesgue integrals.

For a \((n+1)\)-simplex \(\sigma = [v_0, \ldots, v_{n+1}]\) in \(\gamma\) with \(v_0 < \cdots < v_{n+1}\), we have \(\tau = [v_2, \cdots, v_{n+1}]\) and for a point \(t \in \tau\), we have \(\pi_\sigma^{-1}(t) = [v_0, v_1, t] - [v_0, v_1]\) where \([v_0, v_1, t]\) is
the simplex spanned by the points \( v_0, v_1, t \). By the projection formula, we have the equality

\[(4.3.14) \quad \sum_{\sigma} a_{\sigma} \int_{\sigma} (\pi_{\sigma}^* \iota (\omega_n) - 1) \]

\[= \sum_{\sigma} \int_{\tau} \left( \int_{[v_0, v_1, t]} a_{\sigma} d\rho \wedge \omega_1 \right) \iota (\omega_n - 1) \]

\[= \int_{\tau} \sum_{\sigma} \left( \int_{[v_0, v_1, t]} a_{\sigma} d\rho \wedge \omega_1 \right) \iota (\omega_n - 1). \]

**Lemma 4.15.** We have the equality

\[(4.3.15) \quad \sum_{\sigma} \int_{[v_0, v_1, t]} a_{\sigma} d\rho \wedge \omega_1 = \sum_{\sigma} \int_{[v_0, v_1, v_2]} a_{\sigma} d\rho \wedge \omega_1 \]

for all \( t \in \tau \).

Note that we have \( \sigma_f = [v_0, v_1, v_2] \) cf. Notation 4.9.

**Proof.** Let \( \sigma \) and \( \nu \) be an \((n+1)\)-simplex and an \(n\)-simplex respectively, such that \( \sigma \succ \nu \succ \tau \). We set \( \sigma = [v_0, v_1, v_2, \ldots, v_{n+1}] \), \( \tau = [v_2, \ldots, v_{n+1}] \) and \( \nu = [v, v_2, \ldots, v_{n+1}] \). Here the vertex \( v \) is \( v_0 \) or \( v_1 \). For a point \( t \in \tau \) we set \( \sigma_t = [v_0, v_1, v_2, t] \) and \( \nu_t = [v, v_2, t] \). Then we have \([\sigma : \nu] = [\sigma_t : \nu_t]\). Since \( \dim \nu \cap H = \dim \tau = n - 1 \), \( \nu \) is not admissible. Therefore the coefficient of \( \nu \) in \( \delta \gamma \) is zero by the admissibility of \( \delta \gamma \). It follows that we have

\[0 = \sum_{\{\sigma : \sigma \succ \nu\}} [\sigma : \nu] a_{\sigma} = \sum_{\{\sigma : \sigma \succ \nu\}} [\sigma_t : \nu_t] a_{\sigma}. \]

and obtain the equality

\[(4.3.16) \quad \sum_{\sigma} a_{\sigma} \delta \sigma_t = \sum_{\sigma} a_{\sigma} \left( [v_0, v_1, t] - [v_0, v_1, v_2] \right) + \sum_{\{\nu \succ \tau\}} \left( \sum_{\{\sigma : \sigma \succ \nu\}} [\sigma_t : \nu_t] a_{\sigma} \right) \nu_t \]

\[= \sum_{\sigma} a_{\sigma} \left( [v_0, v_1, t] - [v_0, v_1, v_2] \right). \]
The equality (4.3.16) implies the equality

\[ \sum_{\sigma} a_{\sigma} \left( \int_{[v_0, v_1, t][v_0, v_1, v_2]} d\rho_\epsilon \wedge \omega_1 \right) = \sum_{\sigma} a_{\sigma} \int_{\delta \sigma} d\rho_\epsilon \wedge \omega_1 = 0 \]

by the Stokes formula and we finish the proof. \( \square \)

Therefore the last line of (4.3.14) is equal to

\[ \int_{\tau} \sum_{\sigma} \left( \int_{[v_0, v_1, t]} a_{\sigma} d\rho_\epsilon \wedge \omega_1 \right) i^* (\omega_{n-1}) \]

\[ = \int_{\tau} \sum_{\sigma} \left( \int_{\sigma} a_{\sigma} d\rho_\epsilon \wedge \omega_1 \right) i^* (\omega_{n-1}) \quad (\text{Lemma 4.15}) \]

\[ = \left( \sum_{\sigma} \int_{\sigma} a_{\sigma} d\rho_\epsilon \wedge \omega_1 \right) \cdot \int_{\tau} i^* (\omega_{n-1}) \]

Thus we have proved the assertion. \( \square \)

Since \( \gamma = \sum_{\sigma} a_{\sigma} \in AC_{n+1}(K, D^n; \mathbb{Q}) \), the chain \( T_\epsilon \cap \gamma = \left( \sum_{\sigma} \int_{\sigma} a_{\sigma} d\rho_\epsilon \wedge \omega_1 \right) \tau \) is independent of the choice of a Thom cocycle i.e. the choice of a sufficiently small \( \epsilon \) by Proposition 3.7 (2). By Proposition 4.14 and Proposition 4.13 (1), we have

**Proposition 4.16.** If \( \dim \tau = n - 1 \), then for a sufficiently small real number \( \epsilon_0 > 0 \), we have the equality

\[ \lim_{\epsilon \to 0} \sum_{\sigma} a_{\sigma} \int_{\sigma} d\rho_\epsilon \wedge \omega_1 \wedge \omega_{n-1} = \sum_{\sigma} \left( \int_{\sigma} a_{\sigma} d\rho_{\epsilon_0} \wedge \omega_1 \right) \left( \int_{\tau} i^* (\omega_{n-1}) \right). \]

Here the sum is taken over the \((n + 1)\)-simplexes of \( \gamma \).

The equality (4.3.9) follows from Proposition 4.16.

**4.3.1. Proof of Proposition 4.13.** We prove the following proposition from which Proposition 4.13 follows.

**Proposition 4.17.** Let \( \sigma \) be an \((n + 1)\)-simplex in \( \gamma \), and let \( \varphi \) be a smooth \((n - 1)\)-form on a neighborhood of \( \sigma \).

1. When \( \epsilon \) is sufficiently small, the integral \( \int_{\sigma} d\rho_\epsilon \wedge \omega_1 \wedge (\varphi - \pi_\sigma^* i^* \varphi) \) converges absolutely.

2. We have the equality

\[ \lim_{\epsilon \to 0} \int_{\sigma} d\rho_\epsilon \wedge \omega_1 \wedge (\varphi - \pi_\sigma^* i^* \varphi) = 0. \]  

**Proof of Proposition 4.17.** The form \( \varphi \) is a sum of the forms \( f \, du_1 \wedge \cdots \wedge du_{n-1} \), where \( u_i \) are from the set \( \{x_1, y_1, \cdots, x_n, y_n\} \) (we denote \( z_j = x_j + iy_j \)) and \( f \) is a smooth function. One may thus assume \( \varphi = f \, du_1 \wedge \cdots \wedge du_{n-1} \).
(1) We wish to apply \[10\] Theorem 2.6, which reads as follows: Let $S$ be a compact semi-algebraic set of dimension $m$, $h : S \to \mathbb{R}^k$ be a continuous semi-algebraic map, and $\psi$ be a smooth $m$-form defined on an open set of $\mathbb{R}^k$ containing $h(S)$. Then the integral $\int_S |h^* \psi|$ is convergent.

It is useful note that differential forms on $S$ of the form $h^* \psi$, with $h : S \to \mathbb{R}^k$ continuous semi-algebraic, and $\psi$ a smooth $p$-form $(0 \leq p \leq m = \dim S)$ are closed under wedge product. Indeed, if $h' : S \to \mathbb{R}^{k'}$ is another continuous semi-algebraic map, and $\psi'$ a smooth $p'$-form on a neighborhood of $h'(S')$, then $(h^* \psi) \wedge (h'^* \psi')$ equals the pull-back by the product map $(h, h') : S \to \mathbb{R}^k \times \mathbb{R}^{k'}$ of the smooth forms $(p_1^* \psi) \wedge (p_2^* \psi')$ defined on a neighborhood of $(h, h')(S)$ in $\mathbb{R}^{k+k'}$.

In order to show the absolute convergence of $\int_\sigma d\rho \wedge \omega_1 \wedge \pi_\sigma^* i^* \varphi$, let $S$ be the compact semi-algebraic set obtained from $\sigma$ by removing a small neighborhood of $\tau'$, and note that the integral in question equals $\int_S d\rho \wedge \omega_1 \wedge \pi_\sigma^* i^* \varphi$ for $\epsilon$ sufficiently small. We consider the projection $\pi_\sigma : \sigma - \tau' \to \tau \subset H$ restricted to $S$,

$$
\pi_\sigma : S \to \tau \subset H,
$$
and the smooth form $i_U^* \varphi$ defined on a neighborhood of $\tau$; then $\pi_\sigma^* i_U^* \varphi$ is a form of the above-mentioned shape $h^* \psi$. Also, pull-back by the inclusion $S \hookrightarrow P^n$ of the smooth form $d\rho \wedge \omega_1$ gives us another form of the shape $h^* \psi$. Thus the wedge product of them, $d\rho \wedge \omega_1 \wedge \pi_\sigma^* i^* \varphi$, is also a form of the same kind, and we conclude absolute convergence of $\int_S d\rho \wedge \omega_1 \wedge \pi_\sigma^* i^* \varphi$ by the theorem we recalled.

Similarly (and more easily) the absolute convergence of $\int_\sigma d\rho \wedge \omega_1 \wedge \varphi$ is obtained by applying the same theorem to the inclusion $\sigma \hookrightarrow P^n$ and the smooth form $d\rho \wedge \omega_1 \wedge \varphi$.

(2). We need the following lemma.

**Lemma 4.18.** For a complex number $\zeta_1$, we set $\sigma(\zeta_1) = \sigma \cap \{z_1 = \zeta_1\}$. There exists a closed semi-algebraic set $C$ of $\mathbb{C}$ of dimension $\leq 1$ for which the equality

$$
\lim_{|\zeta_1| \to 0, \zeta_1 \not\in C} \int_{\sigma(\zeta_1)} |\varphi - \pi_\sigma^* i^* \varphi| = 0.
$$

holds.

**Proof of Lemma 4.18.** By Semi-algebraic triviality of semi-algebraic maps as stated in Theorem 9.3.2, there exists a closed semi-algebraic set $C$ of $\mathbb{C}$ of dimension $\leq 1$ such that if $\zeta_1 \not\in C$, then the inequality $\dim \sigma(\zeta_1) \leq \dim \sigma - 2$ holds. We have an equality

$$
\varphi - \pi_\sigma^* i^* \varphi = f du_1 \wedge \cdots \wedge du_{n-1} - \pi_\sigma^* i^* (f du_1 \wedge \cdots \wedge du_{n-1}) = (f - \pi_\sigma^* i^* f) du_1 \wedge \cdots \wedge du_{n-1} + \sum_{k=1}^{n-1} \pi_\sigma^* i^* f du_1 \wedge \cdots \wedge du_{k-1} \wedge (du_k - \pi_\sigma^* i^* du_k) \wedge \pi_\sigma^* i^* (du_{k+1} \wedge \cdots \wedge du_{n-1})
$$

We estimate the integral of the first term on the right hand side of (4.3.19). Let $g$ be the map defined by

$$
\sigma \to \mathbb{C} \times \mathbb{R}^{n-1} : z \to (z_1, u_i).
$$

By Proposition 2.7 of [10], we have the inequality

$$
\left| \int_{\sigma(\zeta_1)} (f - \pi_\sigma^* f) \wedge du_1 \wedge \cdots \wedge du_{n-1} \right| \leq \max_{\sigma(\zeta_1)} |f - \pi_\sigma^* f| \delta(g) \int_{g(\sigma(\zeta_1))} |du_1 \wedge \cdots \wedge du_{n-1}|
$$

Thus the above inequality holds.
Here \( \delta(g) \) denotes the maximal of the cardinalities of finite fibers of \( g \). For the precise definition, see Definition 2.2. of [10]. Note that \( \delta(g|_{\sigma(\zeta_1)}) \leq \delta(g) \).

**Claim.** \( \text{Max}_{\sigma(\zeta_1)} |f - \pi^*_\sigma i^* f| \rightarrow 0 \) as \( \zeta_1 \) tends to 0 (outside \( C \)).

**Proof of the claim.** The function \( f - \pi^*_\sigma i^* f \) is continuous semi-algebraic on \( \sigma(\zeta_1) \), and vanishes on \( \tau \). If the claim were false, there exists an \( \epsilon > 0 \) and a sequence \( P_j \in \sigma \) with \( |z_1(P_j)| \rightarrow 0 \) and \( |(f - \pi^*_\sigma i^* f)(P_j)| \geq \epsilon \). Taking a subsequence we may assume that the sequence converges to a point \( P \in \sigma \). Then \( z_1(P) = 0 \), thus \( P \in \tau \), while \( |(f - \pi^*_\sigma i^* f)(P)| \geq \epsilon \), contradicting the function \( f - \pi^*_\sigma i^* f \) being zero on \( \tau \).

The integral \( \int | \wedge_i du_i | \) is bounded by the volume of \( g(\sigma) \) which is independent of \( \zeta_1 \). We conclude that the integral of the first term on the right hand side of (4.3.19) converges to zero as \( |\zeta_1| \rightarrow 0 \).

We estimate the integral of the second term of the right hand side of (4.3.19). Let \( h \) be the map defined by
\[
\sigma \rightarrow \mathbb{C} \times \mathbb{R}^{n-1} : z \mapsto (z_1, v_i) = (z_1, u_1, \ldots, u_{k-1}, u_k - \pi^*_\sigma i^* u_k, \pi^*_\sigma i^* u_{k+1}, \ldots, \pi^*_\sigma i^* u_{n-1}).
\]

By Proposition 2.7 of [10], we have the inequality
\[
\int_{\sigma(\zeta_1)} | \pi^*_\sigma i^* f(z) \wedge du_1 \wedge \cdots \wedge du_{k-1} \wedge (du_k - \pi^*_\sigma i^* du_k) \wedge \pi^*_\sigma i^* (du_{k+1} \wedge \cdots \wedge du_{n-1}) | \\
\leq \text{Max}_{\sigma(\zeta_1)} |\pi^*_\sigma i^* f| |\delta(h|_{\sigma(\zeta_1)})| \int_{h(\sigma(\zeta_1))} |dv_1 \wedge \cdots \wedge dv_{n-1} |
\]

where \( v_1, \ldots, v_{n-1} \) are the coordinates of \( \mathbb{R}^{n-1} \). Note that \( \delta(h|_{\sigma(\zeta_1)}) \) is bounded by \( \delta(h) \) which is independent of \( \zeta_1 \). The function \( |\pi^*_\sigma i^* f| \) is bounded by \( \max \{|f(z)| | z \in \tau \} \). By the same proof as for the Claim, \( M_{\zeta_1} := \text{Max}_{\sigma(\zeta_1)} |u_k - \pi^*_\sigma i^* u_k| \) tends to zero as \( |\zeta_1| \rightarrow 0 \).

There exist numbers \( a < b \) such that
\[
u_i(\sigma) \subset [a, b] \quad \text{for } 1 \leq i \leq k - 1, \text{ and } \\
\pi^*_\sigma i^* u_i(\sigma) \subset [a, b] \quad \text{for } k + 1 \leq i \leq n - 1,
\]

thus
\[
h(\sigma(\zeta_1)) \subset [a, b]^{k-1} \times [-M_{\zeta_1}, M_{\zeta_1}] \times [a, b]^{n-1-k},
\]

hence
\[
\int_{h(\sigma(\zeta_1))} dv_1 \wedge \cdots \wedge dv_{n-1} \rightarrow 0.
\]

We go back to the proof of Proposition 4.14 (2). One has
\[
\int_\mathbb{C} d\rho \wedge \omega_1 = 1,
\]
as follows from the identity
\[
d\rho \wedge \frac{dz_1}{z_1} = i\rho'(r_1)dr_1 \wedge d\theta_1.
\]

For the form \( d\rho_{\epsilon}(z_1) \wedge \omega_1 \), the change of variables \( z_1' = z_1/\epsilon \) yields
\[
d\rho_{\epsilon}(z_1) \wedge \omega_1 = d\rho(z_1') \wedge \frac{1}{2\pi i} \frac{dz_1'}{z_1'}.
\]
So we have \[ \int_{\mathbb{C}} d\rho \wedge \omega_1 = 1. \]
Also, \( d\rho \wedge \omega_1 \) has support in \( |z_1| \leq \epsilon \). Therefore

\[
\left| \int_{\sigma} d\rho \wedge \omega_1 \wedge (\varphi - \pi_{\sigma}^* i^* \varphi) \right| \leq \max_{|\zeta_1| \leq \epsilon, \zeta_1 \notin C} \left( \int_{\sigma(\zeta_1)} |\varphi - \pi_{\sigma}^* i^* \varphi| \right) \int_{\mathbb{C}} d\rho \wedge \omega_1
\]

and the assertion follows from Lemma 4.18. \( \square \)

4.4. **Proof of Proposition 4.6.** We prove Proposition 4.6 assuming Proposition 4.7 by a limit argument. Let \( \epsilon \) be a positive real number and for \( i = 1, \cdots, n \), set \( z_i^{(0)} = z_i \) and \( z_i^{(\infty)} = z_i^{-1} \). We define a neighborhood \( N_\epsilon \) of \( \mathbb{H}_h \) by

\[
N_\epsilon = \bigcup_{1 \leq i < i' \leq n, \alpha \in \{0, \infty\}, \beta \in \{0, \infty\}} \{ z \in P^n | |z_i^{(\alpha)}| \leq \epsilon, |z_i'^{(\beta)}| \leq \epsilon \}.
\]

We set \( N_\epsilon^* = P^n - N_\epsilon \).

By the semi-algebraic triviality of the semi-algebraic maps (\[\ref{semi-algebraic triviality}\]), the condition (P) holds when \( \epsilon \) is sufficiently small.

(P) For each simplex \( \sigma \) of \( K \) and each pair \( (i, \alpha) \), we have \( \dim \sigma \cap \{ |z_i^{(\alpha)}| = \epsilon \} \leq \dim \sigma - 1 \) if the set \( \sigma \cap \{ |z_i^{(\alpha)}| = \epsilon \} \) is not empty. Here if the dimension of a set is negative, then the set must be empty.

In the rest of this section and \( \ref{realization}\) we assume that \( \epsilon \) satisfies the condition (P). Let \( K_\epsilon \) be a good subdivision of \( K \) such that \( N_\epsilon \) and \( N_\epsilon^* \) are subcomplexes of \( K_\epsilon \). We denote by \( \lambda \) the subdivision operator from \( C_*(K) \) to \( C_*(K_\epsilon) \). Note that by the condition (P) we have the following. If \( \sigma \) is a simplex of \( K \) and \( \lambda(\sigma) = \sum \sigma' \), then each \( \sigma' \) is contained in one of \( \{ N_\epsilon, N_\epsilon^* \} \) exclusively.

**Definition 4.19.**

1. Let \( \sigma \) be a \( m \)-simplex in \( K \) and we set \( \lambda(\sigma) = \sum \sigma' \). We put \( \sigma_{\geq \epsilon} = \sum_{\sigma' \subseteq N_\epsilon^*} \sigma' \). For an element \( \gamma = \sum a_\sigma \sigma \in C_m(K, \mathbb{D}_n; \mathbb{Q}) \), we set

\[
\gamma_{\geq \epsilon} = \sum a_\sigma \sigma_{\geq \epsilon}.
\]

2. We put \( \sigma_{= \epsilon} = (\sigma_{\geq \epsilon}) - (\delta\sigma)_{\geq \epsilon} \).

3. For an element \( \gamma = \sum a_\sigma \sigma \in C_m(K, \mathbb{D}_n; \mathbb{Q}) \), we set \( \gamma_{= \epsilon} = \sum a_\sigma \sigma_{= \epsilon} \).

**Lemma 4.20.** Let \( \epsilon \) be a positive number which satisfies the condition (P).

1. Let \( \sigma \) be an \( m \)-simplex in \( K \). We have \( \sigma_{= \epsilon} \subseteq N_\epsilon^* \cap N_\epsilon \).
2. Let \( \sigma \) be an admissible \( m \)-simplex in \( K \). Then the chain \( \sigma_{= \epsilon} \) is admissible.
3. Let \( \gamma = \sum a_\sigma \sigma \) be an element of \( AC_m(K, \mathbb{D}_n; \mathbb{Q}) \). We have \( (\delta \gamma)_{\geq \epsilon} = \delta(\gamma_{\geq \epsilon}) \).

**Proof.**

1. Let \( \lambda(\sigma) = \sum \sigma' \) and we put \( \sigma_{\leq \epsilon} = \sum_{\sigma' \subseteq N_\epsilon} \sigma' \). By condition (P), we have \( \lambda(\sigma) = \sigma_{\geq \epsilon} + \sigma_{\leq \epsilon} \) and \( \delta(\lambda(\sigma)) = (\delta\lambda)(\sigma) = (\delta\sigma)_{\geq \epsilon} + (\delta\sigma)_{\leq \epsilon} \). Let \( \sigma_{\epsilon} = \delta(\sigma_{\leq \epsilon}) - (\delta\sigma)_{\leq \epsilon} \). Then we have \( \sigma_{= \epsilon} + \sigma_{\epsilon} = 0 \). Since \( \sigma_{= \epsilon} \subseteq N_\epsilon^* \) and \( \sigma_{\epsilon} \subseteq N_\epsilon \), we have the assertion.
(2). We set \( \tau = \sigma \cap H^n \). If \( \tau \subset D^n \), then \( \sigma_{\epsilon} \) is admissible by definition. We assume that \( \tau \not\subset D^n \). If \( \tau \subset H_h \), then \( |\sigma_{\epsilon}| \cap H^n = \emptyset \) by the definition of \( \nu^* \). If \( \tau \not\subset H_h \), then there is a unique codimension one face \( H_{i,\alpha} \) such that \( \tau \subset H_{i,\alpha} \). Since \( |\sigma_{\epsilon}| \cap H_h = \emptyset \), \( \sigma_{\epsilon} \) does not meet other cubical face than \( H_{i,\alpha} \). Hence it suffices to show that \( \sigma_{\epsilon} \) meets \( H_{i,\alpha} \) properly. By the admissibility of \( \sigma \) we have \( \dim \sigma \cap H_{i,\alpha} \leq m - 2 \). Since the set \( \tau = \sigma \cap H_{i,\alpha} \) is a face of \( \sigma \), we have the inequality \( \dim \tau \cap \{ |z^{(\alpha)}| = \epsilon \} \leq m - 3 \) for any \( (i', \alpha') \) by the condition (P). By the assertion (1) we have \( |\sigma_{\epsilon}| \cap H_{i,\alpha} \subset \sigma \cap H_{i,\alpha} \cap \{ |z^{(\alpha)}| = \epsilon \} \). Therefore we have \( \dim(|\sigma_{\epsilon}| \cap H_{i,\alpha}) \leq m - 3 \).

(3). We denote the cubical face \( \{ z_1 = 0 \} \) by \( H \). Let \( \sigma \) be a \( m \)-simplex in \( \gamma \). We show that \( \partial(\sigma_{\epsilon}) = (\partial \sigma)_{\geq \epsilon} \). We denote the simplex \( \sigma \cap H \) by \( \tau \). We set \( \lambda(\sigma) = \sum \lambda' \) and \( \lambda(\tau) = \sum \tau' \). Then we have \( \sigma_{\epsilon} = \sum_{\sigma' \subset N_{\epsilon}^*} \sigma' \). Let \( T \) be a Thom cocycle for \( K_{\epsilon} \cap H \) and \( \emptyset \) be a good ordering of \( K_{\epsilon} \) with respect to \( H \). We set \( T \cap \lambda(\sigma) = \sum b_{\tau'} \tau' \) and \( T \cap (\sigma_{\epsilon}) = \sum c_{\tau'} \tau' \). We have \( b_{\tau'} \tau' = \sum_{\sigma' \cap H = \tau'} T \cap \sigma' \) and \( c_{\tau'} \tau' = \sum_{\sigma' \cap H = \tau', \sigma' \subset N_{\epsilon}^*} T \cap \sigma' \).

Suppose first that \( \tau \) is a \( (m - 2) \)-simplex which is not contained in \( D^n \). We will show that for each \( (m - 2) \)-simplex \( \tau' \) contained in \( N_{\epsilon}^* \), we have \( b_{\tau'} = c_{\tau'} \). Such a \( \tau' \) is not contained in \( N_{\epsilon} \) by the condition (P). Each \( m \)-simplex \( \sigma' \) in \( \lambda(\sigma) \) is contained either \( N_{\epsilon} \) or \( N_{\epsilon}^* \). If \( \sigma' \cap H = \tau' \) then we have \( \sigma' \subset N_{\epsilon}^* \). It follows that \( b_{\tau'} = c_{\tau'} \).

Suppose that \( \dim \sigma \cap H \leq m - 3 \). We have \( \dim \sigma' \cap H \leq m - 3 \) for each \( \sigma' \). If \( \sigma' = [v_0, \cdots, v_m] \) with \( v_0 < \cdots < v_m \), then we have \( T \cap \sigma' = \pm T([v_0, v_1, v_2]) [v_2, \cdots, v_m] = 0 \) since \( [v_0, v_1, v_2] \cap H = \emptyset \), and \( T \in C^2(K_\epsilon, W) \) where \( W = \bigcup_{\xi \in K_{\epsilon}} H_{\xi \cap H = \emptyset} \). Therefore we have \( b_{\tau'} = c_{\tau'} = 0 \) for each \( \tau' \). The same argument applies to each codimension one cubical face.

\[ \square \]

**Proof of Proposition 4.6 assuming Proposition 4.7.** Let \( \gamma = \sum a_{\sigma} \sigma \) be the representative of \( \gamma \) such that \( a_{\sigma} = 0 \) if \( \sigma \cap H^n \subset D^n \). We show that \( \gamma_{\geq \epsilon} \) is an element in \( AC_{n+1}(K_{\epsilon}, D^n; \mathbb{Q}) \) as follows. The set \( \{ \gamma_{\geq \epsilon} \} \) is contained in \( \gamma \) which is admissible by the assumption. Hence \( |\gamma_{\geq \epsilon}| \) is admissible. We have \( |(\delta \gamma)_{\geq \epsilon}| = \sum |(\delta \gamma)_{\geq \epsilon}| \). The set \( |(\delta \gamma)_{\geq \epsilon}| \) is admissible by the same reason as above. For an \( \epsilon > 0 \) which satisfies (P), \( |\gamma_{\leq \epsilon}| \) is admissible by Lemma 4.20 (2). Therefore \( \gamma_{\geq \epsilon} \) is an element in \( AC_{n+1}(K_{\epsilon}, D^n; \mathbb{Q}) \).

Since \( N_{\epsilon}^* \cap H_h = \emptyset \), the element \( \gamma_{\geq \epsilon} \) satisfies the condition of Proposition 4.7 and we have \( I_{n-1}(\partial(\gamma_{\geq \epsilon})) + (-1)^n I_n(\delta(\gamma_{\geq \epsilon})) = 0 \). We prove the equality

\[ (4.4.20) \quad \lim_{\epsilon \to 0} (I_{n-1}(\partial(\gamma_{\geq \epsilon})) + (-1)^n I_n(\delta(\gamma_{\geq \epsilon}))) = I_{n-1}(\partial \gamma) + (-1)^n I_n(\delta \gamma). \]

We set \( \delta \gamma = \sum b_{\nu} \nu \) and \( \partial \gamma = \sum c_{\tau} \tau \). By the admissibility of \( \delta \gamma \) and \( \partial \gamma \), the integrals \( \int b_{\nu} \omega_n \) and \( \int c_{\tau} \omega_{n-1} \) converge absolutely by Theorem 4.1. By Lebesgue’s convergence theorem, we have

\[ \lim_{\epsilon \to 0} \int_{\nu \geq \epsilon} b_{\nu} \omega_n = \int b_{\nu} \omega_n \quad \text{and} \quad \lim_{\epsilon \to 0} \int_{\tau \geq \epsilon} c_{\tau} \omega_{n-1} = \int c_{\tau} \omega_{n-1}. \]
Therefore we have
\[
\lim_{\epsilon \to 0} I_n((\delta \gamma)_{\geq \epsilon}) = I_n(\delta \gamma)
\]
\[
\lim_{\epsilon \to 0} I_{n-1}(\partial(\gamma_{\geq \epsilon})) = \lim_{\epsilon \to 0} I_{n-1}((\delta \gamma)_{\geq \epsilon}) = I_{n-1}(\partial \gamma).
\]
By the equality \(\delta(\gamma_{\geq \epsilon}) = (\delta \gamma)_{\geq \epsilon} + \gamma_{=\epsilon}\), to show the equality \((4.4.20)\), it is enough to prove the equality
\[
(4.4.21) \quad \lim_{\epsilon \to 0} I_n(\gamma_{=\epsilon}) = 0.
\]
For a positive real number \(t \geq 1\) and \(1 \leq i \neq j \leq n\), \(\alpha \in \{0, \infty\}\), \(\beta \in \{0, \infty\}\), we set
\[
A_t^{(i,\alpha),(j,\beta)} = \{z \in P^n \mid |z_i^{(\alpha)}| \leq |z_j^{(\beta)}| = t\}.
\]
Let \(\sigma\) be an \((n+1)\)-simplex of \(K\). By Lemma \([4.20] (1)\), the set \(|\sigma_{=\epsilon}|\) is contained in the topological boundary of \(N_\epsilon\). Therefore we have the relation
\[
|\sigma_{=\epsilon}| \subset \bigcup_{1 \leq i \neq j \leq n, \alpha \in \{0, \infty\}, \beta \in \{0, \infty\}} \sigma \cap A_t^{(i,\alpha),(j,\beta)}.
\]
By applying Theorem 4.7 \([10]\) to the case where \(A = \sigma\), \(m = n + 1\) and \(\omega = \omega_n\), we obtain the following proposition.

**Proposition 4.21.** Let \(\sigma\) be an \((n+1)\)-simplex. Assume that \(\sigma\) is admissible. Then for a sufficiently small \(t > 0\), the dimension of \(\sigma \cap A_t^{(i,\alpha),(j,\beta)}\) is equal to or less than \(n\), and we have
\[
\lim_{t \to 0} \int_{\sigma \cap A_t^{(i,\alpha),(j,\beta)}} |\omega_n| = 0.
\]
By Proposition \([4.21]\) we have
\[
\lim_{\epsilon \to 0} \int_{\sigma_{=\epsilon}} |\omega_n| \leq \sum_{i \neq j} \lim_{\epsilon \to 0} \int_{\sigma \cap A_t^{(i,\alpha),(j,\beta)}} |\omega_n| = 0.
\]
and as a consequence, we have the equality \((4.4.21)\). \(\square\)

4.5. **Proof of Proposition 4.5.** Suppose that an element \(\gamma \in AC_{n+1}(K, D^n; \mathbb{Q})\) satisfies the condition \(|\gamma| \cap H^n \subset D^n\). If \(\sum a_\sigma \sigma\) is a representative of \(\gamma\) in \(C_{n+1}(K; \mathbb{Q})\), then for each \(\sigma\) with \(a_\sigma \neq 0\) we have \(\sigma \cap H^n \subset D^n\), and \(\sigma \in AC_{n+1}(K, D^n; \mathbb{Q})\). So it is necessary and sufficient to prove the assertion for each such \(\sigma\). Since \(K\) is a good triangulation, for an \((n+1)\)-simplex \(\sigma\), \(\sigma \cap H^n\) is a face of \(\sigma\). Therefore \(\sigma \cap H^n \subset H_{i,\alpha}\) for some \((i, \alpha)\) because \(H^n\) is the union of the codimension one faces \(H_{i,\alpha}\). We may assume that \(H = H_{1,0}\).

For a positive number \(\epsilon\) which satisfies the condition (P), we set
\[
N_\epsilon^* = \{|z_1| \geq \epsilon\} \subset P^n \quad \text{and} \quad N_\epsilon = \{|z_1| \leq \epsilon\} \subset P^n.
\]
Let \(K_\epsilon\) be a subdivision of \(K\) such that \(N_\epsilon^*\) and \(N_\epsilon\) are subcomplexes of \(K_\epsilon\), and let \(\lambda : C_\bullet(K) \to C_\bullet(K_\epsilon)\) be the subdivision operator. Let \(\lambda(\sigma) = \sum \sigma'\), and put \(\sigma_{\geq \epsilon} = \sum_{\sigma' : \sigma' \subset N_\epsilon^*} \sigma\).

We set \(\sigma_{=\epsilon} = \delta(\sigma_{\geq \epsilon}) - (\delta \sigma)_{\geq \epsilon}\). By the same argument as the proof of Lemma \([4.20] (1)\), we have \(|\sigma_{=\epsilon}| \subset \{|z_1| = \epsilon\}\. 
Since $\sigma \geq \epsilon$ does not meet $H^n$, we have the equality
\[(4.5.22) \quad 0 = \int_{\delta(\sigma \geq \epsilon)} a_{\sigma} \omega_n = \int_{\delta(\sigma) \geq \epsilon} a_{\sigma} \omega_n + \int_{\sigma = \epsilon} a_{\sigma} \omega_n\]
by the Stokes formula. We consider the limit of this as $\epsilon \to 0$. As the chain $\delta \sigma$ is admissible, the integral $\int_{\delta \sigma} a_{\sigma} \omega_n$ converges absolutely. By Lebesgue’s convergence theorem, we have
\[
\lim_{\epsilon \to 0} \int_{\delta(\sigma) \geq \epsilon} a_{\sigma} \omega_n = \int_{\delta \sigma} a_{\sigma} \omega_n
\]
By applying Theorem 4.8 [10] to the case where $A = \sigma$, $\omega = \omega_n$, $m = n + 1$ and $\epsilon = 1$, we have the following proposition.

**Proposition 4.22.** Let $\sigma$ be an $(n+1)$-simplex. Assume that $\sigma \cap H^n \subset D^n$. Then we have
\[
\lim_{\epsilon \to 0} \int_{\sigma \cap \{|z| = \epsilon\}} |\omega_n| = 0.
\]
By Proposition 4.22 we have the equality
\[
\lim_{\epsilon \to 0} \int_{\sigma = \epsilon} a_{\sigma} \omega_n = 0.
\]
By taking the limit of (4.5.22) for $\epsilon \to 0$, we have
\[
\int_{\delta \sigma} a_{\sigma} \omega_n = 0
\]
and we finish the proof of Proposition 4.5.

5. **Construction of the Hodge realization functor.**

In this section, we give a construction of Hodge realization functor for the category of mixed Tate motives.

5.1. **Cycle complexes and graded DGA $N$**. Let $k$ be a field. Following [4], we recall that the cycle complex of $\text{Spec} \ k$ may be viewed as a DGA over $\mathbb{Q}$.

Bloch defined the cycle complex for any quasi-projective variety, but we will restrict to the case of $\text{Spec} \ k$. Let $\square^n = \square_k^n = (P^1_k - \{1\})^n$, which is isomorphic to affine $n$-space as a variety (and which coincides with $\square^n$ of §2 if $k = \mathbb{C}$). As in §2, $(z_1, \cdots, z_n)$ are the coordinates of $\square^n$.

For $n \geq 0$ and $r \geq 0$, let $Z^r(\text{Spec} \ k, n)$ be the $\mathbb{Q}$-vector space with basis irreducible closed subvarieties of $\square^n$ of codimension $r$ which meet the faces properly. The cubical differential $\partial : Z^r(\text{Spec} \ k, n) \to Z^r(\text{Spec} \ k, n - 1)$ is defined by the same formula as Definition 3.9.

The group $G_n = \{\pm 1\}^n \rtimes S_n$ acts naturally on $\square^n$ as follows. The subgroup $\{\pm 1\}^n$ acts by the inversion of the coordinates $z_i$, and the symmetric group $S_n$ acts by permutation of $z_i$’s. This action induces an action of $G_n$ on $Z^r(\text{Spec} \ k, n)$. Let $\text{sign} : G_n \to \{\pm 1\}$ be the character which sends $(\epsilon_1, \cdots, \epsilon_n; \sigma)$ to $\epsilon_1 \cdots \epsilon_n \cdot \text{sign}(\sigma)$. The idempotent $\text{Alt} = \text{Alt}_n := (1/|G_n|) \sum_{g \in G_n} \text{sign}(g) g$ in the group ring $\mathbb{Q}[G_n]$ is called the alternating projector. For a $\mathbb{Q}[G_n]$-module $M$, the submodule
\[M^{\text{alt}} = \{\alpha \in M \mid \text{Alt} \alpha = \alpha\} = \text{Alt}(M)\]
One verifies that the product is graded-commutative:

By definition, one has

Definition 5.1

(The complex

AC

5.2. The complex AC

•

Assume now that k is a subfield of C.

Definition 5.1 (The complex AC•).

For an integer k, let

\[ AC^k = \bigoplus_{n-i=k} AC_i(P^n, D^n; Q)^{alt} \]

Here we set

\[ AC_i(P^0, D^0; Q)^{alt} = \begin{cases} Q & i = 0 \\ 0 & i \neq 0 \end{cases} \]

We define the differential d of AC• by the equality

\[ d(\alpha) = \partial \alpha + (-1)^n \delta \alpha \text{ for } \alpha \in AC_i(P^n, D^n; Q)^{alt}. \]

Here we use the fact that the maps \( \partial \) and \( \delta \) commute with the projector \( \text{Alt} \). cf. Lemma 4.3 [4].

We have a natural inclusion \( Z^r(\text{Spec} k, 2r-i) \to AC_{2r-2i}(P^{2r-i}, D^{2r-i}; Q) \) which induces an inclusion \( i : N \to AC^• \).

Proposition 5.2. (1) Let \( u : Q \to AC^• \) be the inclusion defined by identifying Q with \( AC_0(P^0, D^0; Q)^{alt} \). Then the map \( u \) is a quasi-isomorphism.

(2) The inclusion \( i \) and the product defined by

\[ N_i^j \otimes AC_j(P^n, D^n; Q)^{alt} \to AC_{2r-2i+j}(P^{n+2r-i}, D^{n+2r-i}; Q)^{alt} \]

\[ z \times \gamma \mapsto z \cdot \gamma := \text{Alt}(i(z) \times \gamma) \]

makes \( AC^• \) a differential graded N-module i.e. the product sends \( N_i^j \otimes AC^j \) to \( AC^{i+j} \), and one has the derivation formula; For \( z \in N_i^j \) and \( \gamma \in AC^• \), one has

\[ d(z \cdot \gamma) = \partial z \cdot \gamma + (-1)^i z \cdot d\gamma \]
(3) The map I defined by $I = \sum_{n \geq 0} I_n : AC^* \to \mathbb{C}$ is a map of complexes. Here the map $I$ on $AC_0(P^n, D^n; \mathbb{Q})^\text{alt} = \mathbb{Q}$ is defined to be the natural inclusion of $\mathbb{Q}$ to $\mathbb{C}$. The map $I$ is a homomorphism of $N$-modules; For $z \in N$ and $\gamma \in AC^*$, we have an equality

$$I(z \cdot \gamma) = \epsilon(z)I(\gamma)$$

(4) Let $I_C : AC^* \otimes \mathbb{C} \rightarrow \mathbb{C}$ be the map defined by the composition

$$AC^* \otimes \mathbb{C} \xrightarrow{\otimes \text{id}} \mathbb{C} \otimes \mathbb{C} \xrightarrow{m} \mathbb{C}.$$ Here the map $m$ is the multiplication. Then this map $I_C$ is a quasi-isomorphism.

**Proof.** (1) Since $(P^n, D^n) = (\mathbb{P}^1, \{1\})^n$ (the $n$-fold self product), the Künneth formula tells us

$$H_*(C_*(P^n, D^n; \mathbb{Q})) = H_*(P^n, D^n; \mathbb{Q}) = H_*(\mathbb{P}^1, \{1\}; \mathbb{Q})^\otimes n.$$ It follows that $H_i(P^n, D^n; \mathbb{Q}) = 0$ for $i \neq 2n$, and $H_{2n}(P^n, D^n; \mathbb{Q}) = \mathbb{Q}[P^n]$, where $[P^n]$ denotes the image of the orientation class $[P^n] \in H_{2n}(P^n; \mathbb{Q})$. Since $[P^n]$ is fixed by all $g \in G_n$, the alternating part $H_* (P^n, D^n; \mathbb{Q})^\text{alt}$ is zero for $n > 0$. By Proposition 2.8, the complex $AC_*(P^n, D^n; \mathbb{Q})$ is quasi-isomorphic to $C_*(P^n, D^n; \mathbb{Q})$. It follows that $AC_*(P^n, D^n; \mathbb{Q})$ is an acyclic complex for $n > 0$. Therefore the map $u : \mathbb{Q} \rightarrow AC_*(\mathbb{Q})$ is a quasi-isomorphism.

(2) The proof is similar to that of Lemma 4.3 [4]. The points are that the cubical differential $\partial$ and the topological differential $\delta$ commute with the projector $\text{Alt}$, and that we have the equality

$$\delta(z \times \gamma) = z \times \delta(\gamma)$$

since $z$ is a topological cycle of even dimension.

(3) By Theorem 4.3 the map $I$ is a homomorphism of complexes. The equality holds because the integral of the logarithmic differential form $\omega_n$ on an algebraic cycle is zero for the reason of type.

(4) This follows from the assertion (1) and the fact that the map $I_C \circ (u \otimes \text{id}) : \mathbb{C} = \mathbb{Q} \otimes \mathbb{C} \rightarrow \mathbb{C}$ is the identity map. \qed

5.3. The bar complex. Let $M$ (resp. $L$) be a complex which is a differential left $N$-module (resp. right $N$-module). We recall the definition of the bar complex $B(L, N, M)$ (see [5], [11] for details.)

Let $N_+ = \oplus_{r>0} N_r$. As a module, $B(L, N, M)$ is equal to $L \otimes (\oplus_{s \geq 0} (\otimes^s N_+)) \otimes M$, with the convention $(\otimes^s N_+) = \mathbb{Q}$ for $s = 0$. An element $l \otimes (a_1 \otimes \cdots \otimes a_s) \otimes m$ of $L \otimes (\otimes^s N_+) \otimes M$ is written as $l[a_1] \cdots [a_s]m$ (for $s = 0$, we write $l[m]$ for $l \otimes 1 \otimes m$ in $L \otimes \mathbb{Q} \otimes M$).

The internal differential $d_I$ is defined by

$$d_I(l[a_1] \cdots [a_s]m) = dl[a_1] \cdots [a_s]m$$

$$+ \sum_{i=1}^s (-1)^i Jl[J a_1] \cdots [Ja_{i-1}][da_i] \cdots [a_s]m + (-1)^s Jl[J a_1] \cdots [Ja_s] dm$$
where \(Ja = (-1)^{\text{deg}a}a\). The external differential \(d_E\) is defined by
\[
d_E(l[a_1] \cdots [a_s]m) = -(Jl)a_1[a_2] \cdots [a_s]m
+ \sum_{i=1}^{s-1}(-1)^{i+1}Jl|[Ja_1] \cdots |(Ja_i)a_{i+1}| \cdots |a_s]m
+ (-1)^{s-1}Jl|[Ja_1] \cdots |Ja_{s-1}]a_s]m.
\]
Then we have \(d_I d_E + d_E d_I = 0\) and the map \(d_E + d_I\) defines a differential on \(B(L, N, M)\). The degree of an element \(l[a_1] \cdots [a_s]m\) is defined by \(\sum_{i=1}^{s} \text{deg} a_i + \text{deg} l + \text{deg} m - s\).

If \(L = Q\) and the right \(N\)-module structure is given by the augmentation \(\epsilon\), the complex \(B(L, N, M)\) is denoted by \(B(N, M)\) and omit the first factor "1\(\otimes\)". If \(L = M = Q\) with the \(N\)-module structure given by the augmentation \(\epsilon\), we set
\[
B(N) := B(Q, N, Q).
\]
we omit the first and the last tensor factor "1\(\otimes\)" and "\(\otimes\)1" for an element in \(B(N)\).

The complex \(B(N)\) is graded by non-negative integers as a complex: \(B(N) = \oplus_{r \geq 0}B(N)_r\) where \(B(N)_0 = Q\) and, for \(r > 0\),
\[
B(N)_r = \oplus_{r_1 + \cdots + r_s = r, r_i > 0}N_{r_1} \otimes \cdots \otimes N_{r_s}.
\]
Let \(\Delta : B(N) \to B(N) \otimes B(N)\) be the map given by
\[
\Delta([a_1] \cdots [a_s]) = \sum_{i=0}^{s} ([a_1] \cdots [a_i]) \otimes ([a_{i+1}] \cdots [a_s]).
\]
and \(e : B(N) \to Q\) be the projection to \(B(N)_0\). These are maps of complexes, and they satisfy coassociativity \((\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta\) and counitality \((1 \otimes e)\Delta = (e \otimes 1)\Delta = id\), in other words \(\Delta\) is a coproduct on \(B(N)\) with counit \(e\). In addition, the shuffle product (see e.g. [3]) makes \(B(N)\) a DG algebra with unit \(Q = B(N)_0 \subset B(N)\). The shuffle product is graded-commutative. Further, the maps \(\Delta\) and \(e\) are compatible with product and unit. We summarize:

1. \(B(N) = \oplus_{r \geq 0}B(N)_r\) is a DG bi-algebra over \(Q\). (It follows that \(B(N)\) is a DG Hopf algebra, since it is a fact that antipode exists for a graded bi-algebra.)
2. \(B(N) = \oplus_{r \geq 0}B(N)_r\) is a direct sum decomposition to subcomplexes, and product, unit, co-product and counit are compatible with this decomposition.
3. The product is graded-commutative with respect to the cohomological degree which we denote by \(i\).

With due caution one may say that \(B(N)\) is a "graded" DG Hopf algebra over \(Q\), with graded-commutative product; the first "grading" refers to \(r\), and the second grading refers to \(i\), while graded-commutativity of product is with respect to the grading \(i\) (the product is neither graded-commutative or commutative with respect to \(r\)). We recall that graded Hopf algebra in the literature means a graded Hopf algebra with graded-commutative product, so our \(B(N)\) is a graded Hopf algebra in this sense with respect to the grading \(i\), but is not one with respect to the "grading" \(r\).

Let \(\mathcal{H} := H^0(B(N))\). The product, unit, coproduct, counit on \(B(N)\) induce the corresponding maps on \(\mathcal{H}\), hence \(\mathcal{H}\) is a "graded" Hopf algebra over \(Q\) in the following sense:
(1) $\mathcal{H}$ is a Hopf algebra over $\mathbb{Q}$.

(2) With $\mathcal{H}_r := H^0(B(N)_r)$, one has $\mathcal{H} = \bigoplus_{r \geq 0} \mathcal{H}_r$ a direct sum decomposition to subspaces; the product, unit, coproduct and counit are compatible with this decomposition.

We also have the coproduct map $\Delta : H^0(B(Q, N, M)) \to \mathcal{H} \otimes H^0(B(Q, N, M))$ obtained from the homomorphism of complexes $\Delta : B(Q, N, M) \to B(N) \otimes B(Q, N, M)$ given by

$$\Delta([a_1|\cdots|a_s]m) = \sum_{i=0}^{s}([a_1|\cdots|a_i]) \otimes ([a_{i+1}|\cdots|a_s]m).$$

We define the category of mixed Tate motives after Bloch-Kriz [4].

**Definition 5.3** (Graded $\mathcal{H}$-comodules, mixed Tate motives, [4]).

(1) Let $V = \bigoplus_i V_i$ be a graded vector space (to be precise, a finite dimensional $\mathbb{Q}$-vector space equipped with a grading by integers $i$). A linear map

$$\Delta_V : V \to V \otimes \mathcal{H}$$

is called a graded coaction of $\mathcal{H}$ if the following conditions hold.

(a) $\Delta,V(V_i) \subset \bigoplus_{p+q=i} V_p \otimes \mathcal{H}_q$.

(b) (Coassociativity) The following diagram commutes.

$$\begin{array}{ccc}
V & \xrightarrow{\Delta_V} & V \otimes \mathcal{H} \\
\Delta_V \downarrow & & \downarrow \text{id}_V \otimes \Delta_H \\
V \otimes \mathcal{H} & \xrightarrow{\Delta_V \otimes \text{id}_\mathcal{H}} & V \otimes \mathcal{H} \otimes \mathcal{H}
\end{array}$$

(c) (Counitarity) The composite

$$V \to V \otimes \mathcal{H} \xrightarrow{\text{id}_V \otimes e} V,$$

is the identity map, where $e$ is the counit of $\mathcal{H}$.

A graded vector space $V$ with a graded coaction $\Delta_V$ of $\mathcal{H}$ is called a graded right comodule over $\mathcal{H}$. For graded right comodules $V, W$ over $\mathcal{H}$, a linear map $V \to W$ is called a homomorphism of graded right comodules over $\mathcal{H}$ if it preserves the gradings and the coactions of $\mathcal{H}$. The category of graded right comodules over $\mathcal{H}$ is denoted by $(\text{Comgr} \mathcal{H})$.

(2) The category of mixed Tate motives $(MTM) = (MTM_k)$ over $\text{Spec}(k)$ is defined as the category $(\text{Comgr} \mathcal{H})$ of graded right comodules over $\mathcal{H}$.

5.4. Mixed Hodge structure $\mathcal{H}_H$. We recall the definition of the Tate Hodge structure. For an integer $r$, let $\mathbb{Q}(r) = \mathbb{Q}(2\pi i)^r$ with the weight filtration $W$ defined by $\mathbb{Q}(r) = W_{-2r} \supset W_{-2r-1} = 0$, and let $\mathbb{C}(r) = \mathbb{C}$ with the Hodge filtration $F$ defined by $\mathbb{C}(r) = F^{-r} \supset F^{-r+1} = 0$. We define the mixed Tate Hodge structure $Q_H(r)$ of weight $-2r$ by the $\mathbb{Q}$-mixed Hodge structure $(\mathbb{Q}(r), \mathbb{C}(r), F, W)$ where the comparison map $c : \mathbb{Q}(r) \to \mathbb{C}(r)$ is defined by the identity map. This is a Hodge structure of type $(-r, -r)$. For a mixed Hodge structure $H_H$, $H_H \otimes \mathbb{Q}_H(\cdot) = \mathbb{Q}_H(r)$ is denoted by $H_H(\cdot)$. A (finite dimensional) mixed Hodge structure is called a mixed Tate Hodge structure if the weight graded quotients are isomorphic to direct sums of Tate Hodge structures. An inductive limit of mixed Tate Hodge structure is called a ind-mixed Tate Hodge structure.
In this section, we define a mixed Hodge structure \( \mathcal{H}_{H_g} \) with a left coaction of \( \mathcal{H} \). We define the following two homomorphisms of complexes
\[
(5.4.2) \quad N_r \otimes (AC^* \otimes \mathbb{Q}(s)) \to AC^* \otimes \mathbb{Q}(r+s) : z \otimes (\gamma \otimes (2\pi i)^s) \mapsto z \cdot \gamma \otimes (2\pi i)^{r+s},
\]
where the grading is given by
\[
(5.4.3) \quad AC^* \otimes \mathbb{Q}(*) = \bigoplus_r AC^* \otimes \mathbb{Q}(r), \quad \mathbb{C}(*) = \bigoplus_r \mathbb{C}(r),
\]
where the grading is given by \( r \). Then by the product map (5.4.2), \( AC^* \otimes \mathbb{Q}(*) \) and \( \mathbb{C}(*) \) become graded left \( N \) modules. We define the bar complexes \( \mathcal{B}_B \) and \( \mathcal{B}_{dR} \) by
\[
\mathcal{B}_B = B(N, AC^* \otimes \mathbb{Q}(*)) \quad \text{and} \quad \mathcal{B}_{dR} = B(N, \mathbb{C}(*)).
\]
They are equipped with the gradings
\[
\mathcal{B}_B = \bigoplus_r \mathcal{B}_{B,r} \quad \text{and} \quad \mathcal{B}_{dR} = \bigoplus_r \mathcal{B}_{dR,r}
\]
defined by (??). By Proposition 5.2 (3), the map \( I \) induces a morphism of complexes
\[
I_r = I \otimes c : AC^* \otimes \mathbb{Q}(r) \to \mathbb{C}(r),
\]
and the direct sum \( \bigoplus_r I_r : AC^* \otimes \mathbb{Q}(*) \to \mathbb{C}(*) \) of \( I_r \) becomes a homomorphism of graded left \( N \) modules. Therefore the homomorphism \( \bigoplus_r I_r \) induces a homomorphism of graded complexes \( c : \mathcal{B}_B \to \mathcal{B}_{dR} \).

**Definition 5.4.**

1. The homomorphism of complexes \( c : \mathcal{B}_B \to \mathcal{B}_{dR} \) is called the comparison homomorphism.
2. We define the Betti part \( \mathcal{H}_B \) and the de Rham part \( \mathcal{H}_{dR} \) of \( \mathcal{H}_{H_g} \) by \( \mathcal{H}_B = H^0(\mathcal{B}_B) \) and \( \mathcal{H}_{dR} = H^0(\mathcal{B}_{dR}) \). The gradings on \( \mathcal{B}_B \) and \( \mathcal{B}_{dR} \) induces the gradings \( \mathcal{H}_B = \bigoplus_r \mathcal{H}_{B,r} \) and \( \mathcal{H}_{dR} = \bigoplus_r \mathcal{H}_{dR,r} \).

The comparison map \( c \) induces a homomorphism of graded vector spaces
\[
c = \bigoplus_r c_r : \bigoplus_r (\mathcal{H}_{B,r} \otimes \mathbb{C}) \to \bigoplus_r \mathcal{H}_{dR,r}.
\]
We introduce the weight filtration \( W_* \) on \( \mathcal{B}_B \) and \( \mathcal{B}_{dR} \) as follows.
\[
W_n \mathcal{B}_B = \bigoplus_{2r \leq n, s \geq 0} (\otimes^s N_+ \otimes AC^* \otimes \mathbb{Q}(-r)).
\]
\[
W_n \mathcal{B}_{dR} = \bigoplus_{2r \leq n, s \geq 0} (\otimes^s N_+ \otimes \mathbb{C}(-r)).
\]
The Hodge filtration \( F^* \) on \( \mathcal{B}_{dR} \) is defined by
\[
F^p \mathcal{B}_{dR} = \bigoplus_{r \geq p, s \geq 0} (\otimes^s N_+ \otimes \mathbb{C}(-r)).
\]

By Proposition 5.2 (3), the map \( c \) induces a homomorphism
\[
(5.4.4) \quad c : W_n \mathcal{B}_B \otimes \mathbb{C} \to W_n \mathcal{B}_{dR}
\]
The weight and Hodge filtrations are direct sum of the the induced filtrations on each gradings. The weight and Hodge filtrations on \( \mathcal{B}_B \) and \( \mathcal{B}_{dR} \) induce those of \( \mathcal{H}_B \) and \( \mathcal{H}_{dR} \), respectively.
Proposition-Definition 5.5. (1) The map \( c \) induces a filtered quasi-isomorphism \( B_B \otimes \mathbb{C} \rightarrow B_{dR} \) with respect to \( W \).
(2) We have a canonical isomorphism of vector spaces
\[
Gr^W_{2r} \mathcal{H}_B \rightarrow \mathcal{H} \otimes \mathbb{Q}(-r)
\]
(3) Via the isomorphism \( c : \mathcal{H}_B \otimes \mathbb{C} \rightarrow \mathcal{H}_{dR} \), the pair of filtered vector spaces \( \mathcal{H}_{Hg} = (\mathcal{H}_B, \mathcal{H}_{dR}, W, F) \) becomes an ind-mixed Tate Hodge structure with a grading
\[
\mathcal{H}_{Hg} = \oplus_r \mathcal{H}_{Hg,r}.
\]

Proof. (1) One sees that the quotient \( Gr^W_{2r} B_B \) is the tensor product
\[
B_{N} \otimes AC^* \otimes \mathbb{Q}(-r)
\]
as a complex. By Proposition 5.2 (4) the map \( c \) induces a quasi-isomorphism \( Gr^W_{2r} B_B \rightarrow Gr^W_{2r} B_{dR} \).

(2) We consider the spectral sequences for the filtration \( W : \)
\[
E^{p,q}_{1} = H^{p+q}(Gr^W_{2r-p} B_B) \Rightarrow E^{p,q} = H^{p+q}(B_B)
\]
\[
' E^{p,q}_{1} = H^{p+q}(Gr^W_{2r-p} B_{dR}) \Rightarrow ' E^{p,q} = H^{p+q}(B_{dR}).
\]
Since the morphism of complexes \( c : B_B \otimes \mathbb{C} \rightarrow B_{dR} \) is a filtered quasi-isomorphism, the morphism of spectral sequences
\[
E^{*,*}_{s} \otimes \mathbb{C} \rightarrow ' E^{*,*}_{s}
\]
is an isomorphism. Since the complex \( B_{dR} \) is the direct sum \( \oplus \mathbb{C} \oplus (AC^* \otimes \mathbb{Q}(-r)) \), the spectral sequence \( ' E^{*,*}_{s} \) degenerates at \( E^r_1 \)-term, and as a consequence \( E^{*,*}_{s} \) also degenerates at \( E^r_1 \)-term. Therefore the vector space \( Gr^W_{2r} H^0(\mathcal{H}_B) \) is canonically isomorphic to \( H^0(Gr^W_{2r} B_B) \).

By Proposition 5.2 (1) we have
\[
H^0(Gr^W_{2r} B_B) = H^0(B(N)) \otimes H^0(AC^* \otimes \mathbb{Q}(-r)) = \mathcal{H} \otimes \mathbb{Q}(-r).
\]

(3) We need to show the following.
(1) The filtrations \( F \) and \( \overline{F} \) on \( Gr^W_{2r} \mathcal{H}_{dR} \) are 2\( r \)-opposite ([6], (1.2.3)).
(2) \( (F^p \cap \overline{F}^{p-r}) Gr^W_{2r} \mathcal{H}_{dR} = 0 \) for \( p \neq r \).

We denote \( Gr^W_{2r} \mathcal{H}_{dR} = A \). We have
\[
F^p(A) = \begin{cases} A, & p \leq r \\ 0, & p > r \end{cases}
\]
By taking complex conjugate, a similar fact holds for \( \overline{F}^p(A) \). The assertions (1) and (2) follow from this.

\[ \square \]

5.5. Coaction of \( \mathcal{H} \) on \( \mathcal{H}_{Hg} \) and a Hodge realization functor. We have a homomorphism of complexes
\[
\Delta_B : B_B \rightarrow B(N) \otimes B_B, \quad \Delta_{dR} : B_{dR} \rightarrow B(N) \otimes B_{dR}
\]
which are compatible with the gradings and homomorphism \( c \). Here the grading of the target of \( \Delta_B \) (resp. \( \Delta_{dR} \)) is given by tensor product of those of \( B(N) \) and \( B_B \) (resp. \( B_{dR} \)). By taking the cohomology of degree zero, we have a homomorphism

\[
\Delta_{Hg}: \mathbb{H}_{Hg} \to \mathbb{H} \otimes \mathbb{H}_{Hg}
\]

of ind-mixed Tate Hodge structures with gradings, i.e. \( \Delta_{Hg}(\mathbb{H}_{Hg, r}) \subset \oplus_{s+t=r} \mathbb{H}_s \otimes \mathbb{H}_{Hg, t} \).

We define the Hodge realization functor \( \Phi \) from the category \((\text{Com}^{gr}_{\mathcal{H}})\) of graded \( \mathcal{H} \)-comodules to the category \( MTH \) of mixed Tate Hodge structure as follows. Let \( V = \oplus_i V_i \) be a graded comodule over \( \mathcal{H} \) given by the comodule structure

\[
\Delta_V : V \to V \otimes \mathcal{H}, \quad \Delta_V(V_i) \subset \oplus_{j+k=i} V_j \otimes \mathcal{H}_k.
\]

Using the comodule structure \( \Delta_V \) and the coproduct homomorphism \( \Delta_{Hg} \) of (5.5.5), we define the following graded homomorphisms of graded ind-mixed Hodge structures.

\[
\Delta_V \otimes \text{id} - \text{id} \otimes \Delta_{Hg} : V \otimes \mathbb{H}_{Hg} \to V \otimes \mathbb{H} \otimes \mathbb{H}_{Hg}.
\]

The degree zero part of the kernel of a graded homomorphism \( \varphi \) is denoted by \( \ker_0(\varphi) \).

**Definition 5.6 (Realization functor, functor \( \omega_{gr} \)).**

1. We define the functor \( \Phi \) from the category \((\text{Com}^{gr}_{\mathcal{H}})\) of graded \( \mathcal{H} \)-comodules to that of mixed Tate Hodge structures \((MTH)\) by

\[
\Phi(V) = \ker_0(\Delta_V \otimes \text{id} - \text{id} \otimes \Delta_{Hg})
\]

It is called the realization functor.

2. Let \( V \) be a mixed Tate Hodge structure. We define a graded module \( \omega_{gr}(V) \) by

\[
\omega_{gr}(V)_i = \text{Hom}_{MTH}(\mathbb{Q}_{Hg}(-i), \text{Gr}^W_{2i}(V)).
\]

Then \( \omega \) becomes an exact functor from the category \((MTH)\) of mixed Tate Hodge structures to the category \((\text{Vect}^{gr}_\mathbb{Q})\) of finite dimensional graded vector spaces. The category of ind-mixed Tate Hodge structure is denoted by \((\text{ind}-MTH)\) and that of ind-finite dimensional graded vector space is denoted by \((\text{ind}-\text{Vect}^{gr}_\mathbb{Q})\). The functor \( \omega_{gr} \) is extended to the functor from \((\text{ind}-MTH)\) to \((\text{ind}-\text{Vect}^{gr}_\mathbb{Q})\).

**Proposition 5.7.**

1. For objects \( M, N \) in \((MTH)\), we have

\[
\omega_{gr}(M \otimes N) = \omega_{gr}(M) \otimes \omega_{gr}(N).
\]

2. We have an equality of graded vector spaces

\[
\omega_{gr}(\mathbb{H}_{Hg}) = \mathbb{H} \otimes \omega_{gr}(\mathbb{Q}_{Hg}(r)),
\]

where \( \mathbb{Q}_{Hg}(\ast) = \oplus_r \mathbb{Q}_{Hg}(r) \). This equality is compatible with the left coaction of \( \mathbb{H} \).

3. Let \( \varphi : (\text{Com}^{gr}_{\mathcal{H}}) \to (\text{Vect}^{gr}_\mathbb{Q}) \) be the functor of forgetting the \( \mathcal{H} \)-comodule structure. Then we have the following equality of functors.

\[
\omega_{gr} \circ \Phi = \varphi : (\text{Com}^{gr}_{\mathcal{H}}) \to (\text{Vect}^{gr}_\mathbb{Q}).
\]

As a consequence, the functor \( \Phi \) is an exact functor.
Proof. (1) The proof will be left to the reader.
(2) By the definition of $\omega_{gr}$, it is a consequence of Proposition 5.5 (1).
(3) Let $V$ be a graded right $\mathcal{H}$-comodule. Since the functor $\omega_{gr}$ is exact (\cite{6}, Theorem 1.2.10 (iv)), we have

$$\omega_{gr}(\ker(\Delta_V \otimes \text{id} - \text{id} \otimes \Delta_{Hg})) \cong \ker(\omega_{gr}(\Delta_V \otimes \text{id} - \text{id} \otimes \Delta_{Hg})) \cong \ker(V \otimes \mathcal{H} \otimes \omega_{gr}(\mathbb{Q}_{Hg}(\ast))) \cong V \otimes \omega_{gr}(\mathbb{Q}_{Hg}(\ast)) = \bigoplus_{r,s} (V \otimes \mathbb{Q}_{Hg}(\ast))_r.$$

By taking the direct summand of the above isomorphisms corresponding to $\Phi(V)$, we have

$$\omega_{gr}(\Phi(V)) = \bigoplus_r (V \otimes \mathbb{Q}_{Hg}(\ast))_r = \varphi(V),$$

Thus we have the statement (3). \qed

Remark 5.8. The functor $\Phi$ is isomorphic to the functor obtained by the map $\text{Real}_{TMH}$ defined in \cite{4}. The proof with slightly different formulation will be given in the sequel paper.

5.6. In the case of dilogarithm. Using the above notations, we describe the Hodge realization of the comodule over $H$ associated to dilogarithm functions after Section 9 of \cite{4}.

We assume $k \subset \mathbb{C}$ and $a \in k^\times - \{1\}$. We define elements $\rho_1(a) \in N^1_1$ and $\rho_2(a) \in N^2_1$ by

$$\rho_1(a) = \{(1 - a) \in \mathbb{P}^1 \}_{\text{alt}},$$
$$\rho_2(a) = \{(x_1, 1 - x_1, 1 - \frac{a}{x_1}) \in (\mathbb{P}^1)^3 \}_{\text{alt}}.$$

Then we have the following relations:

$$\partial \rho_2(a) = -\rho_1(1 - a) \cdot \rho_1(a), \quad \partial \rho_1(a) = \partial \rho_1(1 - a) = 0.$$

Therefore the elements $\text{Li}_1(a), \text{Li}_2(a)$ defined as follows are closed elements in $B(N)$, and thus they define elements in $\mathcal{H}$.

$$\text{Li}_2(a) = [\rho_2] - [\rho_1(1 - a)] \rho_1(a)],$$
$$\text{Li}_1(a) = [\rho_1(a)].$$

Let $V = V_2 \oplus V_1 \oplus V_0$ be the sub $\mathcal{H}$-comodule of $\mathcal{H}$ generated by $e_2 := \text{Li}_2(a)$, $e_1 := \text{Li}_1(1 - a)$ and $e_0 := 1$. We have

$$\Delta_V(e_2) = e_2 \otimes 1 - e_1 \otimes \text{Li}_1(a) + e_0 \otimes \text{Li}_2(a),$$
$$\Delta_V(e_1) = e_1 \otimes 1 + e_0 \otimes \text{Li}_1(1 - a),$$
$$\Delta_V(e_0) = e_0 \otimes 1.$$
Assume that \( a \) is contained in \( \mathbb{R} \) and assume that \( 0 < a < 1 \). We consider elements \( \eta_1(0), \eta_2(1), \eta_2(0) \) in \( AC^0 \) defined by
\[
\eta_1(0) = \{(1 - t_0) \in \mathbb{P}^1 \mid 0 < t_0 < a\}^{alt}, \\
\eta_2(1) = \{(x_1, 1 - x_1, 1 - \frac{t_1}{x_1}) \mid x_1 \in \mathbb{P}^1_{\mathbb{C}} - \{1\}, 0 < t_1 < a\}^{alt}, \\
\eta_2(0) = \{(t_1, 1 - t_0) \mid 0 < t_0 < t_1 < a\}^{alt},
\]
where the orientation of \( \eta_2(0) \) is defined so that \((t_1, t_0)\) are positive coordinates. We have the relations
\[
\delta \eta_1(0) = \rho_1(a), \\
\partial \eta_1(0) = 0, \\
\delta \eta_2(1) = \rho_2(a), \\
\delta \eta_2(0) = -\partial \eta_2(1) + \rho(1 - a) \cdot \eta_1(0), \\
\partial \eta_2(0) = 0.
\]
Let \( \xi_1(a) = \eta_1(0) \), and \( \xi_2(a) = \eta_2(1) + \eta_2(0) \) be chains of \( AC^0 \). We have the equalities
\[
d(\xi_1(a)) = -\rho_1(a), \\
d(\xi_2(a)) = -\rho_2(a) + \rho(1 - a) \cdot \xi_1(a).
\]
Then the elements \( Z_0, Z_1(a), Z_2 \) of \( B_B \) defined by
\[
Z_2 = \text{Li}_2(a) \otimes 1 \otimes (2\pi i)^{-2} - \text{Li}_1(1 - a) \otimes \xi_1(a) \otimes (2\pi i)^{-1} + 1 \otimes \xi_2(a) \otimes (2\pi i)^0 \in B_{B, 0}, \\
Z_1(a) = \text{Li}_1(a) \otimes 1 \otimes (2\pi i)^{-2} + 1 \otimes \xi_1(a) \otimes (2\pi i)^{-1} \in B_{B, -1}, \\
Z_i = 1 \otimes (2\pi i)^i \in B_{B, i} \quad (i = 0, -1, -2)
\]
are cocycles. The elements
\[
v_2 = e_2 \otimes Z_{-2} - e_1 \otimes Z_1(a) + e_0 \otimes Z_2, \\
v_1 = e_1 \otimes Z_{-1} + e_0 \otimes Z_1(1 - a) \cdot (2\pi i), \\
v_0 = e_0 \otimes Z_0
\]
forms a basis of the kernel of the map
\[
\Delta_V \otimes \text{id} - \text{id} \otimes \Delta_B : \bigoplus_i V_i \otimes \mathcal{H}_{B, -i} \to \bigoplus_{i,j} V_i \otimes \mathcal{H}_j \otimes \mathcal{H}_{B, -i-j}.
\]
Similarly, the elements
\[
w_2 = e_2 \otimes 1 \otimes 1 - e_1 \otimes \text{Li}_1(a) \otimes 1 + e_0 \otimes \text{Li}_2(a) \otimes 1 \\
\in V_2 \otimes \mathcal{H}_{dR, -2} \oplus V_1 \otimes \mathcal{H}_{dR, -1} \oplus V_0 \otimes \mathcal{H}_{dR, 0}, \\
w_1 = e_1 \otimes 1 \otimes 1 + e_0 \otimes \text{Li}_1(1 - a) \otimes 1 \\
\in V_1 \otimes \mathcal{H}_{dR, -1} \oplus V_0 \otimes \mathcal{H}_{dR, 0}, \\
w_0 = e_0 \otimes 1 \otimes 1 \in V_0 \otimes \mathcal{H}_{dR, 0}
\]
form a basis of the kernel of the map
\[ \Delta_V \otimes \text{id} - \text{id} \otimes \Delta_{dR} : \bigoplus_i V_i \otimes \mathcal{H}_{dR,-i} \rightarrow \bigoplus_{i,j} V_i \otimes \mathcal{H}_j \otimes \mathcal{H}_{dR,-i-j}. \]

The images of \( Z_i \) under the comparison map \( \mathcal{B}_B \rightarrow \mathcal{B}_{dR} \) are equal to
\[
\begin{align*}
C(Z_2) &= (\text{Li}_2(a) \otimes 1 + \text{Li}_1(1 - a) \otimes \text{Li}_1(a) + 1 \otimes \text{Li}_2(a))(2\pi i)^{-2}, \\
C(Z_1(a)) &= (\text{Li}_1(a) \otimes 1 + 1 \otimes \log(1 - a))(2\pi i)^{-2}, \\
C(Z_0) &= 1 \otimes 1.
\end{align*}
\]

Therefore we have the relation
\[
\begin{align*}
c(v_2) &= (2\pi i)^{-2}(w_2 + w_1 \text{Li}_1(a) + w_0 \text{Li}_2(a)), \\
c(v_1) &= (2\pi i)^{-1}(w_1 + w_0 \log a), \\
c(v_0) &= w_0.
\end{align*}
\]

So the period matrix of the Hodge realization \( \Phi(V) \) of \( V \) with respect to the de Rham basis \( w_0 \in W_0 \cap F^0, w_1 \in F^1 \cap W_1 \) and \( w_2 \in F^2 \cap W_2 \) is equal to
\[
\begin{pmatrix}
(2\pi i)^{-2} & 0 & 0 \\
\text{Li}_1(a)(2\pi i)^{-2} & (2\pi i)^{-1} & 0 \\
\text{Li}_2(a)(2\pi i)^{-2} & \log a(2\pi i)^{-1} & 1
\end{pmatrix}
\]

which is compatible with the one given in [1].

**Appendix A. Proof of the moving lemma**

Before the proof, we recall the following three theorems.

**Theorem A.1.** ([16], Ch.6, Theorem 15) Let \( M \) be a compact PL-manifold and let \( K \) be a PL-triangulation of \( M \). Let \( X, X_0 \) and \( Y \) be subpolyhedra of \( M \) such that \( X_0 \subset X \). Then there exists an ambient PL isotopy \( h : M \times [0, 1] \rightarrow M \) such that \( h(x, t) = x \) for any point \( x \in X_0 \) and \( t \in [0, 1] \) (we refer to this fact by saying that \( h \) fixes \( X_0 \), and such that \( h_1(\vert X \vert - X_0) \) is in general position with respect to \( Y \) i.e. the inequality
\[
\dim(h_1(\vert X \vert - X_0) \cap Y) \leq \dim(\vert X \vert - X_0) + \dim Y - \dim M
\]
holds. Here \( h_t(m) = h(m, t) \) for \( m \in M \) and \( t \in [0, 1] \).

We need the following variant of this theorem.

**Theorem A.2.** Let \( M \) be a compact PL-manifold and let \( K \) be a triangulation of \( M \). Let \( X_0, X \) and \( Y_1, \cdots, Y_n \) be subpolyhedra of \( M \). Then there exists an ambient PL isotopy \( h : M \times [0, 1] \rightarrow M \) which fixes \( X_0 \), and such that \( h_1(\vert X \vert - X_0) \) are in general position with respect to \( Y_i \) for \( 1 \leq i \leq n \).

**Theorem A.3** (Lemma 1.10, [13]). Let \( f : \vert K \vert \rightarrow \vert L \vert \) be a PL map of the realizations of simplicial complexes \( K \) and \( L \). Then there exist subdivisions \( K' \) and \( L' \) of \( K \) and \( L \) respectively, such that \( f \) is induced by a simplicial map \( K' \rightarrow L' \).
Proof of Proposition 2.8. (1) We prove the surjectivity of the map \( \iota \) on homology groups. Let \( \gamma \) be a closed element in \( C_p(K, D^n; \mathbb{Q}) \) for a triangulation \( K \). Let \( \gamma = \sum a_{\sigma} \sigma \) be a representative of \( \gamma \) in \( C_p(K; \mathbb{Q}) \). By applying Theorem A.3 to the case where \( M = P^n, X = |\gamma|, X_0 = D^n \) and \( Y_i \) the set of cubical faces, we have a PL isotopy \( h : P^n \times [0, 1] \rightarrow P^n \) such that

1. \( h \) fixes \( D^n \), and
2. \( h_1(|\gamma| - D^n) \) meets each cubical face properly.

By Theorem A.3 there exists a triangulation \( K \) of \( P^n \times [0, 1] \) and a subdivision \( K' \) of \( K \) such that

1. The map \( h \) is induced by a simplicial map from \( K \) to \( K' \), and
2. \( \sigma \times [0, 1] \), \( \sigma \times \{0\} \) and \( \sigma \times \{1\} \) are subcomplexes of \( K \) for each simplex \( \sigma \) of \( K \).

Let \( \lambda : C_p(K) \rightarrow C_p(K') \) be the subdivision operator. Let

\[
 h_* : C_\bullet(K, D^n \times [0, 1]; \mathbb{Q}) \rightarrow C_\bullet(K', D^n; \mathbb{Q})
\]

be the map of complexes induced by \( h \). For a simplex \( \sigma \in K \), the product \( \sigma \times [0, 1] \), \( \sigma \times \{0\} \) and \( \sigma \times \{1\} \) are regarded elements of \( C_\bullet(K) \). For an element \( \sigma \in K_p \), we set

\[
 h_\sigma = h_*(\sigma \times [0, 1]) \in C_{p+1}(K', D^n),
 h_i(\sigma) = h_*(\sigma \times \{i\}) \in C_p(K', D^n) \quad (i = 0, 1).
\]

Then we have \( h_0(\sigma) = \lambda(\sigma) \). Let \( \theta \) be the chain \( \sum_{\sigma} a_{\sigma} h_\sigma \). Then we have

\[
\delta \theta = \sum_{\sigma} a_{\sigma} \delta h_\sigma = \sum_{\sigma} a_{\sigma} (h_1(\sigma) - h_0(\sigma)) + h_\delta \sigma = \sum_{\sigma} a_{\sigma} h_\delta \sigma + (-1)^p \left( \sum_{\sigma} a_{\sigma} h_1(\sigma) - \sum_{\sigma} a_{\sigma} \lambda(\sigma) \right) = \sum_{\rho} \left( \sum_{\rho \prec \sigma} [\sigma : \rho] a_{\rho} \right) h_\rho + (-1)^p \left( \sum_{\sigma} a_{\sigma} h_1(\sigma) - \sum_{\sigma} a_{\sigma} \lambda(\sigma) \right)
\]

in \( C_p(K'; \mathbb{Q}) \). Here in the last line, the sum in the first term is taken over \((p - 1)\)-simplexes of \( K \), and \( \rho \prec \sigma \) means that \( \rho \) is a face of \( \sigma \). The index of \( \sigma \) with respect to \( \rho \) is denoted by \([\sigma : \rho]\). It is in the set \( \{\pm 1\} \). Since \( \gamma \) is closed, we have \( \sum_{\rho \prec \sigma} [\sigma : \rho] a_{\rho} = 0 \) if \( \rho \not\subset D^n \). Since \( D^n \) is fixed by \( h \), if \( \rho \subset D^n \), then we have \( h_\rho = 0 \). Thus we have the equality

\[
\delta \theta = (-1)^p \left( \sum_{\sigma} a_{\sigma} h_1(\sigma) - \lambda(\gamma) \right).
\]

Since \( |\sum_{\sigma} a_{\sigma} h_1(\sigma)| = h_1(|\gamma|) \) by the construction of \( h \), it follows that \( \sum_{\sigma} a_{\sigma} h_1(\sigma) \in AC_p(K', D^n; \mathbb{Q}) \).

(2) We prove the injectivity of the map \( \iota \) on homology. Let \( \gamma \) be an element in \( AC_p(P^n, D^n; \mathbb{Q}) \) and suppose that \( \gamma \) is the boundary of an element \( \xi \) in \( C_{p+1}(P^n, D; \mathbb{Q}) \). Representatives of \( \gamma \) and \( \xi \) in \( AC_p(K; \mathbb{Q}) \) and \( C_{p+1}(K; \mathbb{Q}) \) are also denoted by \( \gamma \) and \( \xi \). By applying Theorem A.2 to the case where \( M = P^n, X = |\xi|, X_0 = D^n \cup |\gamma| \) and \( Y_i \) the set of cubical faces, we obtain a PL isotopy \( h : P^n \times [0, 1] \rightarrow P^n \) of \( P^n \) such that
(1) $h$ fixes $D^n \cup \{\gamma\}$, and  

(2) $h_1(\{\xi\} - (D^n \cup \{\gamma\}))$ intersects the cubical faces properly.

By a similar argument as the surjectivity, we see that the chain $h_1(\xi)$ is an element of $AC_{p+1}(K', D^n; \mathbb{Q})$ for a subdivision $K'$ of $K$, and that $\delta h_1(\xi) = \lambda(\gamma)$ where  

$$\lambda : C_\bullet(K, D^n; \mathbb{Q}) \to C_\bullet(K', D^n, \mathbb{Q})$$

is the subdivision operator. □

**Appendix B. Properties of cap products**

**B.1. Some facts on homotopy.** The following proposition is known as acyclic carrier theorem (See [14], Theorem 13.4, p76).

**Proposition B.1.** Let $K$ be a simplicial complex and $D_\bullet$ be a (homological) complex such that $D_i = 0$ for $i < 0$. Let $p$ be a non-negative integer, and let $\varphi_\bullet : C_\bullet(K) \to D_{\bullet-p}$ be a homomorphism of complexes. We suppose that there exists a family of subcomplexes \( \{L_\sigma^\bullet\}_{\sigma \in K} \) of $D_\bullet$ indexed by $\sigma \in K$ which satisfies the following conditions.

1. $L_\tau^\sigma \subset L_\tau^\tau$ for $\tau, \sigma \in K$ such that $\tau \subset \sigma$.
2. $\varphi_\bullet(\sigma) \in L_\tau^\sigma - p$ for all $\sigma \in K$.
3. The homology groups $H_k(L_\sigma^\sigma) = 0$ for $k > 0$.
4. The homology class of the cycle $\varphi_p(\sigma)$ in $H_0(L_\sigma^\sigma)$ is zero for each $p$-simplex $\sigma$.

Under the above assumptions, there exist homomorphisms $\theta_{p+q} : C_{p+q}(K) \to D_{q+1}$ ($q \geq 0$), satisfying the following conditions:

1. $\delta \theta_{p+q} + \theta_{p+q-1} \delta = \varphi_{p+q}$ for $q \geq 0$. Here we set $\theta_{p-1} = 0$.
2. $\theta_{q}(\sigma) \in L_\sigma^\sigma$.

**Proof.** We construct maps $\theta_{p+q}$ inductively on $q$. We consider the case where $q = 0$. Let $\sigma$ be a simplex of $K_p$. Since the homology class of $\varphi_p(\sigma)$ in $H_0(L_\sigma^\sigma)$ is zero, there exists an element $t_\sigma \in L_1^\sigma$ such that $\delta t_\sigma = \varphi(\sigma)$. By setting $\theta_p(\sigma) = t_\sigma$, we have a map $\theta_p : C_p(K) \to D_1$.

We assume that $\theta_{p+q}$ is constructed and construct $\theta_{p+q+1}$. Let $\sigma$ be a $(p+q+1)$-simplex of $K$ regarded as an element of $C_{p+q+1}(K)$. Using the inductive assumption of the equality (a), we have

$$\delta(\varphi(\sigma) - \theta_{p+q}(\delta\sigma)) = \varphi(\delta\sigma) - \delta\theta_{p+q}(\delta\sigma)$$

$$= \varphi(\delta\sigma) + \theta_{p+q-1}(\delta\delta\sigma) - \varphi(\delta\sigma) = 0.$$

By the inductive assumption of (b) and the assumption (1), we have $\theta_{p+q}(\delta\sigma) \in L_{q+1}^\sigma$. Since we have $\varphi(\sigma) \in L^\sigma$ by the assumption (2), $\varphi(\sigma) - \theta_{p+q}(\delta\sigma)$ is a closed element in $L_{q+1}^\sigma$. By the assumption (3), there exists an element $t_\sigma \in L_{q+q+1}^\sigma$ such that $\delta t_\sigma = \varphi(\sigma) - \theta_{p+q}(\delta\sigma)$. We define a morphism $\theta_{p+q+1}$ to be $\theta_{p+q+1}(\sigma) = t_\sigma$ and the map $\theta_{p+q+1}$ satisfies conditions (a) and (b) for $q + 1$. □

**B.2. Independence of ordering.** Let $K$ be a finite simplicial complex, $L$ a full subcomplexes of $K$, and $\mathcal{O}$ be a good ordering with respect to $L$. We set

$$W = \bigcup_{\sigma \subset L = \emptyset} \sigma.$$

Let $p$ be a positive even integer, and let $T$ be a $p$-cocycle in $C^p(K, W)$ and $\varphi : C_r(K; \mathbb{Q}) \to C_{r-p}(K; \mathbb{Q})$ be the map defined by $\varphi(\alpha) = T \cap \alpha$. Then the map $\varphi$ is a homomorphism.
of complexes, and its image is contained in \( C_{\bullet-p}(L; \mathbb{Q}) \). Thus we have a homomorphism of complexes:

\[
\varphi : C_{\bullet}(K; \mathbb{Q}) \to C_{\bullet-p}(L; \mathbb{Q}).
\]

Let \( T' \) be a \( p \)-cocycle in \( C^p(K,W) \) and set \( \varphi'(\alpha) = T' \cap\alpha \). If \( T - T' \) is the coboundary of \( w \in C^*(K,W) \), i.e. \( dw = T - T' \), then we have

\[
\delta(w \cap\alpha) - w \cap\delta \alpha = -((\varphi - \varphi')(\alpha))
\]

for each \( \alpha \in C_{\bullet}(K) \). Therefore the homomorphism of homologies \( [\varphi] : H_{p+q}(K; \mathbb{Q}) \to H_q(L; \mathbb{Q}) \) induced by \( \varphi \) depends only on the cohomology class \([T]\) of \( T \).

Let \( K^* \) be a subcomplex of \( K \) and set \( L^* = K^* \cap L \). By restricting the homomorphism \( \varphi \) to \( C_{\bullet}(K^*) \), we have a homomorphism of subcomplexes \( C_{p+q}(K^*; \mathbb{Q}) \to C_q(L^*; \mathbb{Q}) \) and a homomorphism of relative homologies

\[
(2.2.1) \quad [\varphi] : H_{p+q}(K^*; \mathbb{Q}) \to H_q(L^*; \mathbb{Q}).
\]

This homomorphism also depends only on the cohomology class \([T]\) of \( T \) in \( H^p(K,W) \).

**Proposition B.2.** The homomorphism \((2.2.1)\) is independent of the ordering \( O \).

**Proof.** The assertion follows from the following lemma.

**Lemma B.3.** Let \( T \) be a cocycle in \( C^p(K,W) \) and \( O \) and \( O' \) be good orderings of \( K \) with respect to \( L \). Then there exists a map \( \theta_{p+q} : C_{p+q}(K) \to C_{q+1}(L) \) \((q \geq 0)\) satisfying the following conditions:

1. \( \delta \theta_{p+q}(x) + \theta_{p+q-1}(\delta x) = T \cap\alpha - T \cap\alpha' \) for \( q \geq 0 \). Here we set \( \theta_{p+q-1} = 0 \).
2. \( \theta_{p+q}(\sigma) \in C_{q+1}(L \cap\sigma) \) for each simplex \( \sigma \in K_{p+q} \).

**Proof.** We apply Proposition B.3 to \( \varphi(x) = T \cap\alpha - T \cap\alpha' \), \( D = C_{\bullet}(L) \) and \( L_{\bullet} = C_{\bullet}(L \cap\sigma) \). Conditions (1), (2) are easily verified. Since the complex \( L \) is a full subcomplex of \( K \), the intersection \( L \cap\sigma \) is a face of \( \sigma \), and condition (3) is satisfied. We check condition (4). Let \( \sigma = [v_0, \ldots, v_p] = \pm[v_0', \ldots, v_p'] \) be a \( p \)-simplex. Here \( v_0 < \cdots < v_p \) for the ordering \( O \) and \( v_0' < \cdots < v_p' \) for the ordering \( O' \). Then we have \( T \cap\sigma = T(\sigma)[v_p] \) and \( T \cap\alpha' = T(\sigma)[v_p'] \). Since \([v_p] \) and \([v_p'] \) are in the same homology class in \( H_0(L \cap\sigma) \), and (4) is proved. Thus we have a map satisfying conditions (1) and (2) in the lemma.

Since the homomorphism \((2.2.1)\) depends only on the choice of cohomology class \([T]\) of \( T \), it is written as \([T]\cap\) .

**Proof of Proposition 3.7 (2).** Let \( \gamma \) be an element in \( AC_k(K, D^n; \mathbb{Q}) \). By the admissibility condition for \( \delta \gamma \), we have \( L_1 \cap|\delta \gamma| \subset (L_1 \cap|\gamma|)^{(k-3)} \cup D^n \), where \((L_1 \cap|\delta \gamma|)^{(k-3)}\) is the \((k-3)\)-skeleton of \( L_1 \cap|\gamma|\). We denote the set \( D^n \cap |\gamma| \) by \( D^n_\gamma \) and the set \( H_{1,0} \cap D^n \cap |\gamma| \)
by $D_{\gamma}^{n-1}$. We have a homomorphism

\[
(2.2.2) \quad \delta \cap : H_k(\{\gamma\}, \delta_\gamma \cup D_{\gamma}^n; \mathbb{Q}) \to H_{k-2}(L_1 \cap \{\gamma\}, (L_1 \cap \delta_\gamma) \cup D_{\gamma}^{n-1}; \mathbb{Q})
\]

The diagram (2.3.4) is commutative.

Lemma B.5. The diagram (2.3.3) induces the following diagram for relative homologies

\[
\begin{align*}
\lambda : C_\bullet(K; \mathbb{Q}) &\to C_\bullet(K'; \mathbb{Q}), \\
\lambda : C_\bullet(L; \mathbb{Q}) &\to C_\bullet(L'; \mathbb{Q}).
\end{align*}
\]

Let $W'$ be the subcomplex of $K'$ defined as $\bigcup_{\sigma' \in K', \sigma \cap L' = \emptyset} \sigma'$ and let $T$ be a closed element in $C_p(K', W')$, i.e. $T(\sigma') = 0$ if $\sigma' \cap L' = \emptyset$. Then the pull back $\lambda^* T$ is contained in $C_p(K, W)$ where $W = \bigcup_{\sigma \in K, \sigma \cap L = \emptyset} \sigma$. We choose a good ordering $\emptyset$ resp. $\emptyset'$ of $K$ resp. $K'$ with respect to $L$ resp. $L'$. Then we have the following (generally non-commutative) diagram.

\[
(2.3.3) \quad \begin{array}{ccc}
C_\bullet(K; \mathbb{Q}) & \xrightarrow{\lambda^* T} & C_{\bullet-p}(L; \mathbb{Q}) \\
\lambda \downarrow & & \downarrow \lambda \\
C_\bullet(K'; \mathbb{Q}) & \xrightarrow{T'} & C_{\bullet-p}(L'; \mathbb{Q})
\end{array}
\]

For a subcomplex $K^*$ of $K$, we set $L^* = K^* \cap L$. The subdivisions of $K^*$ and $L^*$ induced by $K'$ are denoted by $K'^*$ and $L'^*$, respectively. The diagram (2.3.3) induces the following diagram for relative homologies

\[
(2.3.4) \quad \begin{align*}
H_{p+q}(K, K^*; \mathbb{Q}) &\to H_q(L, L^*; \mathbb{Q}), \\
\lambda \downarrow & & \downarrow \lambda \\
H_{p+q}(K', K'^*; \mathbb{Q}) &\to H_q(L', L'^*; \mathbb{Q})
\end{align*}
\]

Proposition B.4. The diagram (2.3.4) is commutative.

Proof. The assertion follows from the following lemma.

Lemma B.5. Consider the following two homomorphism of complexes

\[
\lambda \circ (\lambda^* T) \quad \text{and} \quad (T \cap ) \circ \lambda : C_\bullet(K) \to C_{\bullet-p}(L')
\]

Then there exist maps $\theta_{p-q} : C_{p+q}(K) \to C_{q+1}(L')$ such that

1. $\delta \theta_{p+q}(x) + \theta_{p+q-1}(\delta x) = \lambda(\lambda^* T \cap x) - T \cap \lambda(x)$, $(q \geq 0, \theta_{p-1} = 0)$ and
(2) \( \theta_{p+q}(\sigma) \in C_{q+1}(L' \cap \sigma) \) for each simplex \( \sigma \in K_{p+q} \).

Proof. We apply Proposition B.3 to the case where \( \varphi(x) = \lambda(\lambda^* T \cap \sigma) - T \cap \lambda(x) \), \( D_\bullet = C_\bullet(L') \) and \( L_\bullet^\sigma = C_\bullet(L' \cap \sigma) \). Conditions (1) and (2) are easily verified. Since \( L \) is a full subcomplex of \( K \), for each \( \sigma \in K \), the intersection \( L \cap \sigma \) is a face of \( \sigma \). The complex \( L' \cap \sigma \) is a subdivision of \( L \cap \sigma \), and so the condition (3) is satisfied. We claim that the condition (4) of Proposition B.1 holds for \( \varphi \). Let \( \sigma = [v_0, \ldots, v_p] \in K_p \) and set \( \lambda \sigma = \sum_j \sigma_j = \sum_j \pm[w^j_0, \ldots, w^j_p] \). Here we assume that \( v_0 < \cdots < v_p \) resp. \( w^j_0 < \cdots < w^j_p \) under \( \varnothing' \) resp. \( \varnothing' \) for each \( j \). Then we have
\[
\lambda(\lambda^* T \cap \sigma) = \sum_j T(\sigma_j)[v_p]
\]
and
\[
T \cap (\lambda \sigma) = \sum_j T(\sigma_j)[w^j_p].
\]
Since \( L \cap \sigma \) is a simplex of \( \sigma \), and \( [v_p] \) and \( [w^j_p] \) define the same homology class in \( H_0(L_\bullet^\sigma) \). Thus condition (4) is satisfied. \( \square \)

Proof of Proposition B.4 (3). Let \( \gamma \) be an element in \( AC_k(K, D^n; \mathbb{Q}) \). The homology class of \( \gamma \) in \( H_k(|\gamma|, |\delta \gamma|; \mathbb{Q}) \) is denoted by \( [\gamma] \). We set \( |\gamma|' = K' \cap |\gamma| \) and \( |\delta \gamma|' = K' \cap |\delta \gamma| \). The element \( \lambda(\gamma) \) defines a class \([\lambda(\gamma)]\) in \( H_k(|\gamma|', |\delta \gamma|'; \mathbb{Q}) \). Let \( T' \in C^2(K', W') \) be a Thom cocycle of \( H_{1,0} \). Then \( \lambda^* T' \in C^2(K, W) \) is also a Thom cocycle of \( H_{1,0} \). As in the proof of Proposition B.7 (2), for a chain \( \gamma \in AC_k(K, D^n; \mathbb{Q}) \), the chain \( \lambda^* T \cap \gamma \) resp. \( T \cap \lambda(\gamma) \) is determined by its class
\[
\lambda^* T \cap [\gamma] \in H_{k-2}(L_1 \cap |\gamma|, (L_1 \cap |\gamma|)^{(k-3)} \cup D^n_{\gamma}; \mathbb{Q})
\]
resp.
\[
T \cap [\lambda(\gamma)] \in H_{k-2}(L'_1 \cap |\gamma|', (L'_1 \cap |\gamma|')^{(k-3)} \cup D^n_{\gamma}; \mathbb{Q})
\]
Since these classes are independent of the orderings by Proposition B.7 (2), we forget them. By Proposition B.3 applied to \( K = |\gamma| \), \( K^* = |\delta \gamma| \cup D^n_{\gamma} \) and \( L = L_1 \), we have the following commutative diagram.
\[
\begin{array}{ccc}
H_k(|\gamma|, |\delta \gamma| \cup D^n_{\gamma}; \mathbb{Q}) & \xrightarrow{\lambda^* T \cap} & H_{k-2}(L_1 \cap |\gamma|, (L_1 \cap |\gamma|)^{(k-3)} \cup D^n_{\gamma}; \mathbb{Q}) \\
\downarrow \lambda & & \downarrow \lambda \\
H_k(|\gamma|', |\delta \gamma|' \cup D^n_{\gamma}; \mathbb{Q}) & \xrightarrow{T' \cap} & H_{k-2}(L'_1 \cap |\gamma|', (L'_1 \cap |\gamma|')^{(k-3)} \cup D^n_{\gamma}; \mathbb{Q})
\end{array}
\]
The assertion follows from this. \( \square \)

B.4. Relations between the cap product and the cup product. Let \( K \) be a finite simplicial complex and \( L_1, L_2 \) be subcomplexes in \( K \). Assume that \( L_1, L_2 \) and \( L_1 \cup L_2 \) are full subcomplexes of \( K \). Let \( \varnothing \) be a good ordering with respect to \( L_1 \) and \( L_{12} = L_1 \cap L_2 \). We set
\[
W_i = \bigcup_{\sigma \in L_i = \varnothing} \sigma.
\]
Lemma B.6. Under the above assumptions, we have \( W_1 \cup W_2 = \bigcup_{\sigma \cap L_{12} = 0} \sigma \).

Proof. We will show that \( W_1 \cup W_2 \supseteq \bigcup_{\sigma \cap L_{12} = 0} \sigma \). Suppose that \( \sigma \cap L_{12} = \emptyset \). Since \( L_1 \cup L_2 \) is a full subcomplex of \( K \), \( \sigma \cap (L_1 \cup L_2) \) is a face of \( \sigma \) which we denote by \( \tau \). If \( \tau \in L_1 \), then \( \tau \cap L_2 = \emptyset \).

Definition B.7 (Cup product). For \( T_1 \in C^p(K,W_1) \) and \( T_2 \in C^q(K,W_2) \), we define the cup product \( T_1 \cup T_2 \in C^{p+q}(K) \) by

\[
(T_1 \cup T_2)(\sigma) = T_1(v_0, \ldots, v_p)T_2(v_p, \ldots, v_{p+q})
\]

where \( \sigma = [v_0, \ldots, v_{p+q}] \) with \( v_0 < \cdots < v_{p+q} \). The cup product induces a homomorphism of complexes:

\[
\cup : C^*(K,W_1) \otimes C^*(K,W_2) \rightarrow C^*(K).
\]

Let \( K^* \) be a subcomplex of \( K \) and set \( L_i^* = K^* \cap L_i \) and \( L_{12}^* = K^* \cap L_{12} \). The proof of the following proposition is obvious and is omitted.

Proposition B.8.

1. The restriction of the cup product \( T_1 \cup T_2 \) to \( W_1 \cup W_2 \) vanishes.

2. Let \( T_1 \) and \( T_2 \) be closed elements in \( C^p(K,W_1) \) and \( C^q(K,W_2) \) and set \( T_{12} = T_1 \cup T_2 \). Then the composite of the homomorphisms

\[
T_2 \cap (T_1 \cap \ast) : C_{p+q+r}(K,K^*;\mathbb{Q}) \xrightarrow{T_1 \cap} C_{q+r}(L_1,L_1^*;\mathbb{Q}) \xrightarrow{T_2 \cap} C_r(L_{12},L_{12}^*;\mathbb{Q})
\]

is equal to the homomorphism \( T_{12} \cap \).

As a consequence the composite of the following morphisms coincides with the cap product with \([T_{12}]\).

\[
H_{p+q+r}(K,K^*;\mathbb{Q}) \xrightarrow{[T_1]\cap} H_{q+r}(L_1,L_1^*;\mathbb{Q}) \xrightarrow{[T_2]\cap} H_r(L_{12},L_{12}^*;\mathbb{Q})
\]

Proof of Proposition B.8. Let \( K \) be a good triangulation of \( P^n \), and let \( \gamma \) be an element of \( AC_p(K,D^n;\mathbb{Q}) \). We set \( H_1 = H_{1,0} \), \( H_2 = H_{2,0} \), \( H_{12} = H_1 \cap H_2 \), \( L_1 = K \cap H_{1,0} \), \( L_2 = K \cap H_{2,0} \), \( L_{12} = K \cap H_{12} \), \( W_i = \bigcup_{\sigma \in K, \sigma \cap H_i = \emptyset} \sigma \) (\( i = 1, 2 \)) and \( W_{12} = \bigcup_{\sigma \in K, \sigma \cap H_{12} = \emptyset} \sigma \).

By Lemma B.3 we have \( W_{12} = W_1 \cup W_2 \). The face map \( \partial_i \) (\( i = 1, 2 \)) is denoted by \( \partial_i \). Considering the symmetry on \( H_{i,\alpha} \) (\( 1 \leq i \leq n, \alpha = 0, \infty \)), it is enough to prove the commutativity of the following diagram

\[
AC_p(K,D^n;\mathbb{Q}) \xrightarrow{\partial_2} AC_{p-2}(L_1,D^{n-1};\mathbb{Q}) \xrightarrow{\partial_2} AC_{p-4}(L_{12},D^{n-2};\mathbb{Q})
\]
We denote the complex $D_n^p \cap |\gamma|$ by $D_n^p_\gamma$. Let $T_1 \in C^2(K, W_1; \mathbb{Q})$ resp. $T_2 \in C^2(K, W_2; \mathbb{Q})$ be a Thom cocycle of the face $H_1$ resp. $H_2$. We have the equality

$$[T_1] \cup [T_2] = [T_2] \cup [T_1]$$

in $H^4(K, W_1; \mathbb{Q})$. By Proposition B.8 applied to $K = |\gamma|$ and $K^* = |\delta \gamma| \cup D_n^p_\gamma$, we have a commutative diagram

$$(2.4.7)$$

$$
\begin{array}{c}
H_p(|\gamma|, |\delta \gamma| \cup D_n^p_\gamma; \mathbb{Q})
\quad
\xrightarrow{[T_2] \cap -}

H_p-2(L_1 \cap |\gamma|, L_1 \cap (|\delta \gamma| \cup D_n^p_\gamma); \mathbb{Q})
\quad
\xrightarrow{[T_2] \cap -}

H_p-4(L_{12} \cap |\gamma|, L_{12} \cap (|\delta \gamma| \cup D_n^p_\gamma); \mathbb{Q})
\end{array}
$$

The chain $\gamma$ defines a class $[\gamma]$ in $H_p(|\gamma|, |\delta \gamma| \cup D_n^p_\gamma; \mathbb{Q})$. By the admissibility condition, the complex $L_{12} \cap (|\delta \gamma| \cup D_n^p_\gamma)$ is contained in $(L_{12} \cap |\delta \gamma|)^{(p-5)} \cup (L_{12} \cap D_n^p_\gamma)$ where $(L_{12} \cap |\delta \gamma|)^{(p-5)}$ denotes the $(p-5)$-skeleton of $L_{12} \cap |\delta \gamma|$. The chain $T_2 \cap (T_1 \cap |\gamma|)$ is determined by its class $[T_2] \cap ([T_1] \cap |\gamma|)$ resp. $[T_1] \cap ([T_2] \cap |\gamma|)$ in $H_p-4(L_{12} \cap |\gamma|, (L_{12} \cap |\delta \gamma|)^{(p-5)} \cup (L_{12} \cap D_n^p_\gamma); \mathbb{Q})$. Hence we have the equality $\partial_2 \partial_1(\gamma) = \partial_1 \partial_2(\gamma)$. \hfill \Box

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