Nonlinear $n$-pseudo fermions

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Received 2 April 2012
Published 23 October 2012
Online at stacks.iop.org/JPhysA/45/444032

Abstract
Nonlinear pseudo-fermions of degree $n$ ($n$-pseudo-fermions) are introduced as (pseudo) particles with creation and annihilation operators $b$ and $a$, $b \neq a^\dagger$, obeying the simple nonlinear anticommutation relation $ab + b^\dagger a^\dagger = 1$. The $(n+1)$-order nilpotency of these operators follows from the existence of unique (up to a bi-normalization factor) $a$-vacuum. Supposing appropriate $(n+1)$-order nilpotent para-Grassmann variables and integration rules the sets of $n$-pseudo-fermion number states, ‘right’ and ‘left’ ladder operator bi-overcomplete sets of coherent states are constructed. Explicit examples of $n$-pseudo-fermion ladder operators are provided, and the relation of pseudo-fermions to finite level pseudo-Hermitian systems is briefly considered.

This article is part of a special issue of Journal of Physics A: Mathematical and Theoretical devoted to ‘Quantum physics with non-Hermitian operators’.

PACS numbers: 03.65.-w, 03.65.Aa, 05.30.Fk, 02.30.Tb

1. Introduction

In the last decade or so, a great deal of attention has been paid in the literature to quantum systems with non-Hermitian (quasi-Hermitian, crypto-Hermitian, $PT$-symmetric and pseudo-Hermitian) Hamiltonians (see review papers [1, 2] and references therein). More recently considerable attention has been paid to an alternative formalism for the description of non-Hermitian systems, based on the concept of the so-called pseudo-bosons (PB) [3–10] and pseudo-fermions (PF) [11–15]. PB were originally introduced in [3], where the first bi-overcomplete sets of PB coherent states (CS) are constructed in the example of the one-parameter family of PB. Mathematical refinement and further relevant examples of PB are due to Bagarello [4–8]. Para-Grassmann CS (nonnormalized ladder-operator CS) for pseudo-Hermitian finite level Hamiltonian systems are constructed and discussed in [14, 15].

The notion of pseudo-Hermitian fermion (phermion) was introduced by Mostafazadeh [11]. A physical example of phermions is given in [12], where (phermions were called pseudo-fermions and) the first bi-overcomplete family of PF coherent states was established. The
standard fermions and the PF so far considered [11, 12] are defined through linear in terms of anticommutation relations of the corresponding creation and annihilation operators.

In this paper we introduce **nonlinear pseudo-fermions** of degree of nonlinearity \( n \) (abbreviated \( n \)-PF), with \( n \) being a positive integer. These are a non-Hermitian extension of the nonlinear fermions (\( n \)-fermions) described in the previous paper [16], and are relevant for the description of finite level non-Hermitian quantum systems. In the next section we provide a brief summary of [16]. In the third section, the \( n \)-PF are introduced and several examples are presented. In the fourth section, ‘left’ and ‘right’ ladder operator CS are constructed. The \( n \)-PF in finite level pseudo-Hermitian systems are briefly considered in the fifth section.

### 2. Nonlinear \( n \)-fermions

The nonlinear \( n \)-fermions are defined [16] as particles with annihilation and creation operators \( A(n) \) and \( A^\dagger(n) \) satisfying the following nonlinear anticommutation relation\(^1\):

\[
A(n)A^\dagger(n)+A^\dagger n A^n(n)=1, \quad (1)
\]

with \( n \) being a positive integer. At \( n = 1 \), the standard fermionic relations \( aa^\dagger + a^\dagger a = 1 \) are recovered, i.e. \( A(1) = a \). In this terminology the standard fermions are ‘1-fermions’, or linear fermions.

Supposing the existence of a normalized vacuum state \( |0\rangle \) that is annihilated by \( A(n) \), one can construct \( n \) excited orthonormalized states (Fock states),

\[
|k\rangle = A^k(n)|0\rangle, \quad k = 0, 1, \ldots, n, \quad (2)
\]

and deduce that \( A(n) \) are nilpotent of order \( n + 1 \): \( A^{n+1} = 0 \). The operators \( A(n) \) and \( A^\dagger(n) \) act on \( |k\rangle \) as raising and lowering operators with step 1:

\[
A(n)|k\rangle = |k-1\rangle, \quad A^\dagger(n)|k\rangle = |k+1\rangle. \quad (3)
\]

The corresponding number operator \( N \), \( N|k\rangle = k|k\rangle \), reads

\[
N(n) = A^\dagger(n)A(n) + A^{2\dagger}(n)A^2(n) + \cdots + A^{n\dagger}(n)A^n(n), \quad (4)
\]

\[
[A(n), N(n)] = A(n), \quad [A^\dagger(n), N(n)] = -A^\dagger(n). \quad (5)
\]

In this way the state \( |k\rangle \), equation (2), can be regarded as a normalized state with number \( k \) of \( n \)-fermions, \( k = 0, 1, \ldots, n \). There are no states with more than \( n \) such particles. So the degree of nonlinearity \( n \) is the order of statistics of our \( n \)-fermions. The algebra spanned by \( A(n) \), \( A^\dagger(n) \) and \( N \), satisfying (1) and (5), could be called the \( n \)-fermion algebra. At \( n = 1 \) it coincides with the (standard) fermion algebra.

**Matrix realization.** One can check that the \( n \)-fermion algebra (1) admits the following \((n+1) \times (n+1)\) matrix representation,

\[
A(n) = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}. \quad (6)
\]

\(^1\) Such a relation has been suggested (and realized for \( n = 2, 3 \)) in [19] in the context of polynomial relations for the generators of the \( su(2) \) Lie algebra.
In order to construct eigenstates of \( A(n) \) we adopt the following para-Grassmann algebra for the eigenvalues \( \zeta \),
\[
\zeta^* \zeta^* + \zeta^* \zeta = 0, \quad \zeta^{n+1} = 0 = \zeta^{*n+1}.
\]
and the relations
\[
\zeta A(n) + A(n)\zeta = 0 = \zeta A^\dagger(n) + A^\dagger(n)\zeta, \quad \zeta |0\rangle = |0\rangle\zeta.
\]
These relations are most simple and direct generalizations of those for the standard fermion operators and their Grassmann eigenvalues [20]. They differ from para-Grassmann relations used, e.g., in [14, 15, 17]. The para-Grassmann algebra (7) admits the \((n+1) \times (n+1)\) matrix representation [16].

Using (7) and (8), the ‘right’ and ‘left’ (nonnormalized) eigenstates of \( A(n) \) can be constructed [16]:
\[
A(n)|\zeta; n\rangle_{r} = \zeta |\zeta; n\rangle_{r},
\]
\[
||\zeta; n\rangle_{r} = \sum_{k=0}^{n} (-1)^{k} \zeta^{k} |k\rangle,
\]
\[
A(n)|\zeta; n\rangle_{l} = ||\zeta; n\rangle_{l}\zeta,
\]
\[
||\zeta; n\rangle_{l} = \sum_{k=0}^{n} (-1)^{(k+1)} \zeta^{k} |k\rangle.
\]
The normalized states are \(|\zeta; n\rangle = N_{r}||\zeta; n\rangle_{r}\) and \(|\zeta; n\rangle_{l} = N_{l}||\zeta; n\rangle_{l}\), where \(N_{r} = N_{l} = \sqrt{1 - \zeta^* \zeta}\). However, in view of (7) and (8), the normalized states \(|\zeta; n\rangle_{l}\), unlike the ‘right’ \(|\zeta; n\rangle_{r}\), cease being eigenstates of \( A(n) \). This unsatisfactory property (and several others [16]) of \(|\zeta; n\rangle_{l}\) occurs for other types of para-Grassmann ‘left’ eigenstates [14, 15, 17, 18] as well.

The sets of both eigenstates \(|\zeta; n\rangle_{r}\) and \(|\zeta; n\rangle_{l}\) can resolve the identity operator,
\[
\int d\zeta^* d\zeta |\zeta; n\rangle_{r} \langle n; \zeta | = 1,
\]
\[
\int d\zeta^* d\zeta |\zeta; n\rangle_{l} \langle n; \zeta | = 1,
\]
if one adopts the following integration rules [16]:
\[
\int d\zeta^* d\zeta \zeta^{i} \zeta^{j} = \delta_{ij} g_{i}(n),
\]
where \(g_{i}(n)\) are given by
\[
g_{i}(n) = 1 + \sum_{k=1}^{n-k} (-1)^{i+k} \frac{\binom{n}{k}}{k!}.
\]
For \(k = n, n-1, n-2\) and \(n-3\), we have
\[
g_{n} = 1, \quad g_{n-1} = 1 + (-1)^{n}, \quad g_{n-2} = (-1)^{n-1}, \quad g_{n-3} = 0.
\]
Note the different structure of \(g_{i}(n)\) for odd and even \(n\) (i.e. for the even and odd dimensions of the Hilbert space \(\mathcal{H}_{n+1}\)), with the structure for odd \(n\) being simplest. At \(n = 1\), the Berezin rules are reproduced [20]. Thus, the states \(|\zeta; n\rangle_{r}\) and \(|\zeta; n\rangle_{l}\) can be qualified as CS—the \(n\)-fermion ladder operator CS. At \(n = 1\) they reproduce the standard fermionic CS [20]. The \(n\)-fermion displacement-operator-like CS can also be constructed, this time the overcompleteness relation needing the appropriate weight function \(W(\zeta^{*} \zeta)\) [16].
### 3. Nonlinear pseudo-fermions

The nonlinear PF of degree \( n \) (\( n \)-pseudo-fermions) are defined as (pseudo) particles with non-Hermitian annihilation and creation operators \( a(n), b(n), b(n) \neq a^\dagger(n) \), satisfying the \( n \)-nonlinear anticommutation relation

\[
a(n)b(n) + b^\dagger(n)a^\dagger(n) = 1. \tag{16}
\]

At \( n = 1 \) and \( b(1) = a^\dagger(1) \) the phermion relation \( aa^\dagger + a^\dagger a = 1 \) is recovered \([11, 12]\). In [3] a suggestion is made that if a PF operator \( a(1) \equiv a \) admits a vacuum \( a|0\rangle = 0 \), then \( b^\dagger \) also admits a vacuum and \( b \) is \( n \)-pseudo adjoint to \( a \), i.e. \( b = a^\dagger \). It appears that this suggestion could be made for \( n > 1 \) as well. In the above terminology the PF are ‘1-PF’ (or linear PF).

Suppose that \(|\psi_0\rangle\) is annihilated by \( a(n) \). Then, we construct excited states \(|\psi_k\rangle\), \( k = 0, 1, \ldots \),

\[
|\psi_k\rangle = b^k(n)|\psi_0\rangle, \tag{17}
\]
on which \( b \) and \( a \) act as raising and lowering operators with step 1,

\[
b(n)|\psi_k\rangle = |\psi_{k+1}\rangle, \quad a(n)|\psi_k\rangle = |\psi_{k-1}\rangle. \tag{18}
\]
The process is terminated at \( k = n \), i.e. \( b^{n+1}|\psi_0\rangle = 0 \), and this follows from the anticommutation relation (16). Indeed, take \(|\psi_{n+1}\rangle := b^{n+1}|\psi_0\rangle\), and multiply it by \( a(n)b(n) + b^\dagger(n)a^\dagger(n) \). Using (16)–(18), we obtain

\[
|\psi_{n+1}\rangle = (a(n)b(n) + b^\dagger(n)a^\dagger(n))|\psi_{n+1}\rangle = a(n)b(n)|\psi_{n+1}\rangle + b^\dagger(n)a^\dagger(n)|\psi_{n+1}\rangle = 2|\psi_{n+1}\rangle,
\]
which is possible iff \(|\psi_{n+1}\rangle = b^{n+1}|\psi_0\rangle = 0 \). This means that in the space \( \mathcal{H}_{n+1} \) spanned by the \( n + 1 \) vectors \(|\psi_k\rangle\) the operators \( a \) and \( b \) are nilpotent (matrices) of order \( n + 1 \), i.e. \( b^{n+1} = a^{n+1} = 0 \). So the set of \(|\psi_k\rangle\) is a basis in \( \mathcal{H}_{n+1} \), but since \( b \neq a^\dagger \) this basis is not orthogonal.

One can check (using the anticommutation relation (16)) that the states \(|\psi_k\rangle\) are eigenstates of the non-Hermitian operator:

\[
N_{pf}(n) = b(n)a(n) + b^\dagger(n)a^\dagger(n) + \cdots + b^\dagger(n)a^\dagger(n), \tag{19}
\]
with the eigenvalue \( k \): \( N_{pf}(n)|\psi_k\rangle = k|\psi_k\rangle \). So \( N_{pf}(n) \) plays the role of a (non-Hermitian) number operator for \( n \)-PF. One can verify that

\[
[a(n), N_{pf}(n)] = a(n), \quad [b(n), N_{pf}(n)] = -b(n). \tag{20}
\]
Since all the \( n + 1 \) distinct eigenvalues of \( N_{pf}(n) \) are real (nonnegative integers \( k \)), the eigenvalues of its Hermitian conjugate \( N_{pf}^\dagger(n) \) are the same (real nonnegative integers \( k \)). Denoting the corresponding eigenvectors as \(|\varphi_k\rangle\), we write

\[
N_{pf}^\dagger|\varphi_k\rangle = k|\varphi_k\rangle. \tag{21}
\]
The nonorthogonal eigenvectors \(|\varphi_k\rangle\) can be similarly constructed from the \( N_{pf}^\dagger(n) \)-lower eigenstate \(|\varphi_0\rangle\) by means of the raising operator \( a^\dagger \) (using \( b^\dagger a^\dagger + a^\dagger b^\dagger n = 1 \) and \( [a^\dagger(n), N_{pf}^\dagger(n)] = -a^\dagger(n), \quad [b^\dagger(n), N_{pf}^\dagger(n)] = b^\dagger(n) \)),

\[
|\varphi_k\rangle = a^k(n)|\varphi_0\rangle. \tag{22}
\]
It is worth noting here that the existence of the nontrivial solution \(|\varphi_0\rangle\) of the equation \( N_{pf}(n)|\varphi_0\rangle = 0 \) follows from the well-known property of systems of (here \( n + 1 \)) linear homogeneous algebraic equations \( AX = 0 \): the nontrivial solution \( x \) exists iff \( \det A \neq 0 \). Indeed, if \( N_{pf}(n)|\varphi_0\rangle = 0 \), \(|\varphi_0\rangle \neq 0 \), then the matrix of \( N_{pf}(n) \) has a vanishing determinant, and so
is the case with the determinant of the matrix \( N_{pf}^j(n) \). Therefore, the equation \( N_{pf}^j(n)|\psi_0\rangle = 0 \) admits a nontrivial solution.

Moreover, the vector \(|\psi_0\rangle\) should be annihilated by the operator \( b^\dagger : b^\dagger|\psi_0\rangle = 0 \). This can be easily proven as follows. Applying \( N_{pf}^j \) to \( b^\dagger|\psi_0\rangle \) and using \( \langle \psi_0 | N_{pf}^j | \psi_0 \rangle = 0 \) and \( [b^\dagger (n), N_{pf}^j (n)] = b^\dagger (n) \), we find that \( b^\dagger |\psi_0\rangle \) is an eigenstate of \( N_{pf}^j \) with the new eigenvalue \(-1\). Therefore, \( b^\dagger |\psi_0\rangle \) should be orthogonal to all eigenstates \(|\psi_k\rangle\) of \( N_{pf} \). The set \(|\psi_k\rangle\) forms a basis in \( \mathcal{H}_{n+1} \); therefore, \( b^\dagger |\psi_0\rangle = 0 \). Thus, if \( a(n)\)-vacuum \(|\psi_0\rangle\) exists, then \( b^\dagger (n)\)-vacuum also exists (and vice versa).

The orthogonality of the eigenstates \(|\psi_i\rangle\) and \(|\psi_j\rangle\) of the non-Hermitian operators \( H \) and \( H^\dagger \) with different real eigenvalues \( \varepsilon_i, \varepsilon_j \), used above, is a known fact, but for the sake of completeness let us provide here its short proof: \( \langle \psi_j | H^2 \psi_i \rangle = \langle \psi_j | \psi_i \rangle \varepsilon_i^2 = \langle \psi_j | \psi_i \rangle \varepsilon_i \varepsilon_j \). If \( \varepsilon_i \neq \varepsilon_j \) the last equality is possible iff \( \langle \psi_j | \psi_i \rangle = 0 \). The states \(|\psi_k\rangle\) and \(|\phi_k\rangle\) corresponding to equal eigenvalues can be bi-normalized, so that we have a bi-orthonormalized system of \( n\)-PF states,

\[
\langle \psi_k | \psi_j \rangle = \delta_{kj}.
\]

Herefrom, it follows that \(|\psi_j\rangle = \eta |\psi_j\rangle\), where \( \eta = \sum_k |\psi_k\rangle \langle \psi_k | \), with the inverse operator being \( \eta^{-1} = \sum_k |\psi_k\rangle \langle \psi_k |\). Next, one can readily verify (using (23) and the basis \(|\psi_k\rangle\)) that the sum \( \sum_k |\psi_k\rangle \langle \psi_k | \) acts on any state \(|\psi\rangle\) as a unit operator,

\[
1 = \sum_k |\psi_k\rangle \langle \psi_k | \langle \psi | \psi \rangle.
\]

Finally, we note that the \( n\)-PF creation operator \( b(n) \) is \( \eta\)-pseudo-adjoint to \( a(n) : b(n) = \eta^{-1}a(n)\eta \Rightarrow (a^\dagger)\eta \rangle = b^\dagger \). This can be verified by applying \( \eta^{-1}a(n)\eta \to |\psi_k\rangle \) (the basis vectors in \( \mathcal{H}_{n+1} \)) and see that these actions are the same as those of \( b(n) \).

Three examples of \( n\)-PF:

(1) \( n = 2 \).

\[
a = \alpha A^1 (2) + \beta A^2 (2) A(2), \quad b = \frac{1}{\alpha + \beta} A(2) + \frac{\beta}{\alpha(\alpha + \beta)} A^2 (2) A^1 (2),
\]

where \( A(2) \) and \( A^1 (2) \) are Hermitian ladder operators of 2-fermions (see section 2). One can check the validity of all the required relations for 2-PF with any \( \beta \) and nonvanishing \( \alpha \), \( \alpha \neq -\beta \):

\[
b \neq a^\dagger, \quad ab + b^2 a^2 = 1, \quad a^3 = 0 = b^3,
\]

\[
a|\psi_0\rangle = 0 \longrightarrow \langle \psi_0 | = (0, 0, p^*),
\]

\[
b^\dagger |\psi_0\rangle = 0 \longrightarrow \langle \psi_0 | = (0, 0, 1/p),
\]

with \( p \) being any nonvanishing complex number.

(2) \( n = 4 \). Any nonvanishing \( \alpha, \beta, \gamma, \delta, p \):

\[
a = \begin{pmatrix}
0 & \alpha & 0 & 0 \\
0 & 0 & 0 & 0 \\
\delta \gamma & 0 & \beta / \delta & -\beta / \delta \\
0 & 0 & \beta & -\beta / \delta
\end{pmatrix}, \quad b = \begin{pmatrix}
0 & 0 & 1 / \gamma \delta & -1 / \gamma \delta^2 \\
1 / \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 1 / \beta \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\langle \psi_0 | = (0, 0, p^*, p^* \delta^*), \quad \langle \psi_0 | = (0, 0, 0, 1 / p \delta).
\]
4. n-PF ladder operator eigenstates

The two n-PF lowering and raising ladder operators \( a(n) \) and \( b(n) \) can be diagonalized using the same para-Grassmann variables \( \zeta \) as for the n-fermions, equation (7), and the following relations:

\[
\{a(n), \zeta\} = 0 = \{a(n), \zeta^*\}, \quad \{b(n), \zeta\} = 0 = \{b(n), \zeta^*\},
\]

(25)

\[
\{\zeta, |\psi_0\rangle\} = 0 = \{\zeta^*, |\psi_0\rangle\}, \quad \{\zeta, |\psi_0\rangle\} = 0 = \{\zeta^*, |\psi_0\rangle\}.
\]

(26)

For the non-bi-normalized ‘right’ and ‘left’ eigenstates, we find

\[
|\psi_0\rangle = (p^*, 0, \ldots, 0), \quad |\psi_0\rangle = (1/p, 0, \ldots, 0).
\]

The bi-normalized states satisfying the relations

\[
_{i}(pf; n; \zeta; n, pf)_i = 1, \quad i = l, r,
\]

(31)

are \( |\zeta; n, pf\rangle_l = N_l |\zeta; n, pf\rangle_l, |\zeta; n, pf\rangle_r = N_r |\zeta; n, pf\rangle_r \), where \( N_l N_r = \sqrt{1 - \zeta^* \zeta} \). If \( b = a^l \), they reproduce the normalized n-fermion eigenstates. Note however that the ‘left’ eigenstates \( |\zeta; n, pf\rangle_l, |\zeta; n, pf\rangle_r \) when bi-normalized cease being eigenstates of \( a \) and \( b^l \). This feature is typical for all ‘left’ parafermionic eigenstates.

Next we look for bi-overcompleteness relations. It turned out that the families of both ‘left’ and ‘right’ n-pseudo-fermion eigenstates form the bi-overcomplete sets of states with respect to the same new integration relations (13) and (14). Instead of (11) and (12) we now have the bi-overcompleteness relations

\[
\int d\zeta^* d\zeta |\zeta; n, pf\rangle_{l,r} (pf; n; \zeta) = 1.
\]

(32)
where, to shorten the equations, we have put 

$$\int d\zeta^* d\zeta \ | \zeta; n, pf\rangle_l \langle pf, n | \zeta = 1.$$  

(33)

In view of the above bi-overcompleteness relations, the two sets \{[\zeta; n, pf]_r, [\zeta; n, pf]_s\} and \{[\zeta; n, pf]_l, [\zeta; n, pf]_t\} can be qualified as n-PF ‘left’ and ‘right’ ladder operator CS. At \(n = 1\) they recover the bi-overcomplete sets of pseudo-fermionic CS [12] (1-PF, or linear PF CS in the present terminology).

5. n-PF and non-Hermitian systems

In this section we briefly consider the relations of n-PF to finite level (non-Hermitian) quantum systems. For Hermitian systems, similar relations are discussed in [16].

Consider a system with a finite number of non-degenerate (possibly not equidistant) ‘energy’ levels \((n + 1)\) levels \(\varepsilon_k\), \(k = 0, 1, \ldots, n\), and let \(| \psi_k \rangle\) be the corresponding wavefunctions. Denote the finite-dimensional Hilbert space spanned by \(| \psi_k \rangle\) as \(\mathcal{H}_{n+1}\). If \(| \psi_k \rangle\) are not orthogonal to each other, they could be regarded as eigenstates of some non-Hermitian Hamiltonian \(H\): \(H| \psi_k \rangle = \varepsilon_k| \psi_k \rangle\). In a more general setting the eigenvalues \(\varepsilon_k\) may be complex quantities. If \(\varepsilon_k\) are real or come in complex conjugate pairs, then \(H\) is pseudo-Hermitian [2]: \(H = \eta^{-1}H^\dagger\eta =: H^\rho\), where \(\eta\) is some Hermitian operator. For \(H^\rho\), the eigenvalue relations are \(H^\rho| \psi_k \rangle = \varepsilon_k^\rho| \psi_k \rangle\). For different \(k, k'\), the states \(|\psi_k\rangle\) and \(|\psi_{k'}\rangle\) are orthogonal and for equal \(k\) they can be bi-normalized, \(\langle \psi_k | \psi_k' \rangle = \delta_{kk'}\).

The general lowering and raising operators between levels are of the form [14]

\[
a(n; \rho) = \sum_{k=0}^{n-1} \sqrt{\rho_k} |\psi_k\rangle \langle \psi_{k+1}|, \quad b(n; \rho) = \sum_{k=0}^{n-1} \sqrt{\rho_k} |\psi_{k+1}\rangle \langle \psi_k|,
\]

(34)

where \(\rho_k\) are arbitrary complex (dimensionless) quantities. Such operators with \(\rho_k = ([k + 1]) = (q^{k+1} - q^{-k-1})/(q - 1/q) \equiv \rho_k^{(q)}, q = \exp(i\pi/(n + 1))\) were considered in [15]. At \(\rho_k = \varepsilon_{k+1}\) we find that \(b(n; \varepsilon)a(n; \varepsilon)|\psi_k\rangle = \varepsilon_k|\psi_k\rangle\), which means that these operators factorize the Hamiltonian

\[
H = b(n; \varepsilon)a(n; \varepsilon)\varepsilon_0.
\]

(35)

If \(\varepsilon_0 \neq 0\), then \(H \equiv b(n; \varepsilon)a(n; \varepsilon) + \varepsilon_0\). Note that here \(H\) is dimensionless, along with \(\varepsilon_k\) and operators \(a(n)\) and \(b(n)\).

At \(\rho_k = 1\) the operators \(a(n; 1), b(n; 1)\) obey relation (16), i.e. \(a(n; 1), b(n; 1)\) are n-PF ladder operators for the \((n + 1)\)-level quantum system: \(a(n; 1) = a(n), b(n; 1) = b(n)\). The general ladder operators \(a(n; \rho), b(n; \rho)\) can be expressed in terms of \(a(n), b(n)\) as

\[
a(n; \rho) = \sigma_0 a(n) + (\sigma_1 - \sigma_0) b(n)a^\dagger(n) + \cdots + (\sigma_{n-1} - \sigma_{n-2}) b^{n-1}(n)a^\dagger(n),
\]

\[
b(n; \rho) = \sigma_0 b(n) + (\sigma_1 - \sigma_0) b^\dagger(n)a(n) + \cdots + (\sigma_{n-1} - \sigma_{n-2}) b^{n-1}(n)a(n),
\]

(36)

(37)

where, to shorten the equations, we have put \(\sqrt{\rho_k} \equiv \sigma_k\).

It is not difficult to check that \(H\) commutes with the n-PF number operator \(N_{pf}(n) = b(n)a(n) + \cdots + b^\dagger(n)a^\dagger(n)\). In view also of the fact that \(|\psi_k\rangle\) are eigenstates of \(N_{pf}(n)\) with eigenvalues \(k\), one could interpret the energy value \(\varepsilon_k\) as a sum of the energies \(\varepsilon_k/k\) of number \(k\) of n-PF. If the spectrum of \(H\) is equidistant, then \(H\) is proportional to \(N_{pf}(n)\).

Following the scheme developed in section 3 (and in [16] for the Hermitian case) one can construct bi-overcomplete ‘left’ and ‘right’ ladder operator eigenstates for \(a(n; \rho)\) and \(b^\dagger(n; \rho)\) using the para-Grassmann algebra (7) and the integration rules (13). Bi-overcomplete sets of para-Grassmann ‘left’ (nonnormalized) eigenstates of \(a(n; \rho)\) and \(b^\dagger(n; \rho)\) were constructed in [14] using different para-Grassmannian variables and integration rules. In
6. Conclusion

We have introduced nonlinear pseudo-fermions of degree $n$ ($n$-pseudo-fermions) as (pseudo) particles with non-Hermitian annihilation and creation operators $a(n), b(n), b(n) \neq a^\dagger(n)$, satisfying the $n$-nonlinear anticommutation relation $a(n)b(n) + b^\dagger(n)a^\dagger(n) = 1$. A pair of non-Hermitian operators $a, b \neq a^\dagger$ could represent $n$-pseudo-fermion if they obey the relation $ab + b^\dagger a^\dagger = 1$ and $a$ admits a nontrivial ground state $|\psi_0\rangle$, $a|\psi_0\rangle = 0$. Then $b^\dagger$-vacuum also exists, and $a$ and $b$ are nilpotent of order $n + 1$, and bi-orthonormalized set of Fock states and bi-overcomplete sets of ‘left’ and ‘right’ CS can be constructed, as we have done this in the paper, using appropriately defined new para-Grassmannian variables and integration rules. At $b = a^\dagger$ (the Hermitian case) $n$-pseudo-fermion CS recover the $n$-fermion CS [16], and at $n = 1$ both ‘left’ and ‘right’ $n$-pseudo-fermion CS reproduce the pseudo-fermionic CS of [12]. Different kinds of (nonnormalized) para-Grassmann ‘left’ CS for finite level pseudo-Hermitian systems are considered in [14, 15].

Three different families of $n$-pseudo-fermion operators have been provided as examples for $n = 2, n = 3$ and any $n > 1$. The $n$-pseudo-fermion operators can be introduced for any pseudo-Hermitian system with non-degenerate finite number of energy levels $\varepsilon_k$. The $n$-pseudo-fermion number operator commutes with the corresponding non-Hermitian Hamiltonian and thereby the energy $\varepsilon_k$ could be regarded as a sum of energies of $k$ number of pseudo-particles (or pseudo-excitation). If the energy levels are equidistant, then the Hamiltonian is proportional to the pseudo-fermion number operator.

References

[1] Bender C M 2007 Making sense of non-Hermitian Hamiltonians Rep. Prog. Phys. 70 947–1018
[2] Mostafazadeh A 2010 Pseudo-Hermitian representation of quantum mechanics Int. J. Geom. Methods Mod. Phys. 7 1191–306
[3] Trifonov D A et al 2009 Pseu-do-boson coherent and Fock states Differential Geometry, Complex Analysis and Mathematical Physics ed K Sekigawa (Singapore: World Scientific) pp 241–50 (arXiv/0902.3744 [quant-ph])
[4] Bagarello F 2010 Pseudo-bosons, Riesz bases and coherent states J. Math. Phys. 50 023531
[5] Bagarello F and Calabrese F 2010 Pseudo-bosons arising from Riesz bases Boll. Dip. Metodi Modelli Matematici 2 15–26
[6] Bagarello F 2010 Construction of pseudo-bosons systems J. Math. Phys. 51 023531
[7] Ali S T, Bagarello F and Gazeau J P 2010 Modified Landau levels, damped harmonic oscillator and two-dimensional pseudo-bosons J. Math. Phys. 51 123502
[8] Bagarello F 2011 (Regular) pseudo-bosons versus bosons J. Phys. A: Math. Theor. 44 015205
[9] Bagarello F and Znojil M 2011 Nonlinear pseudo-bosons versus hidden Hermiticity J. Phys. A: Math. Theor. 44 415305
[10] Calabrese F F G 2011 Pseudo-bosons arising from standard ladder operators J. Math. Phys. 52 072102
[11] Mostafazadeh A 2004 Statistical origin of pseudo-Hermitian supersymmetry and pseudo-Hermitian fermions J. Phys. A: Math. Gen. 37 10193
[12] Cherbal O, Drir M, Maamache M and Trifonov D A 2007 Fermionic coherent states for pseudo-Hermitian two-level systems J. Phys. A: Math. Theor. 40 1835–44
[13] Cherbal O, Drir M, Maamache M and Trifonov D A 2010 Supersymmetric extension of non-Hermitian $su(2)$ Hamiltonian and supercoherent states SIGMA 6 096
[14] Najarbashi G, Fasihi M A and Fakhri H 2010 Generalized Grassmannian coherent states for pseudo-Hermitian $n$-level systems J. Phys. A: Math. Theor. 43 325301
[15] Maleki Y 2011 Para-Grassmannian coherent and squeezed states for Pseudo-Hermitian q-oscillator and their entanglement SIGMA 7 084
[16] Trifonov D A 2012 Nonlinear fermions and coherent states *J. Phys. A: Math. Theor.* **45** 244032

[17] Daoud M and Kibler M 2002 A fractional supersymmetric oscillator and its coherent states *Proc. Int. Wigner Symposium (Istanbul, Turkey, 16–22 Aug. 1999)* (arXiv:math-ph/9912024)

[18] Cabra D C, Moreno E F and Tanasa A 2006 Para-Grassmann variables and coherent states *SIGMA* **2** 087

[19] Chaichian M and Demichev A P 1996 Polynomial algebras and higher spins *Phys. Lett. A* **222** 14–20

[20] Cahill K E and Glauber R J 1999 Density operators for fermions *Phys. Rev. A* **59** 1538