ON HOLOMORPHIC SELF-MAPPINGS OF THE UNIT DISK

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Abstract. Using a symmetrization technique, we prove two distortion theorems for holomorphic mappings of the unit disk into itself taking into account the boundary behavior of these mappings.

Keywords: holomorphic functions, distortion theorems, symmetrization of condensers.

1. Introduction

The study of holomorphic mappings of a disk into itself constitutes a significant part of geometric function theory (see, e.g., the papers [1,2] and the bibliography in them). We are interested in the distortion theorems for such mappings that take into account their boundary behavior [3]. The first result of this kind belongs, apparently, to Unkelbach [4]. Let a function \( w = f(z) \) be holomorphic in the disk \( U = \{ z : |z| < 1 \} \) and satisfy the conditions: \( f(0) = 0 \) and \( |f(z)| < 1 \) when \( z \in U \). If \( E \) is an arc of the circle \( |z| = 1 \) such that the set of the limit values of the function \( f \) with respect to \( E \) belongs to the circle \( |w| = 1 \), then this set is also an arc \( f(E) \) of the circle \( |w| = 1 \). According to [4], the lengths of these two arcs satisfy the inequality

\[
\text{length}(f(E)) \geq \frac{2}{1 + |f'(0)|} \text{length}(E).
\]  

If at a point \( b \in E \) the derivative \( f'(b) \) exists, then by the passage to the limit in (1) we obtain the boundary Schwarz Lemma:

\[
|f'(b)| \geq \frac{2}{1 + |f'(0)|}.
\]
The above inequality is often called the inequality of Osserman. Other versions of the Schwarz inequality on the boundary, as well as analogues of inequality (1), can be found in [5-7]. Some of these results can be used in the proofs of the inequalities for the complex polynomials [8]. In this note we complement the theorems from the articles [6, 9] taking into account the boundary distortion. The main tool employed in the proofs is Pólya’s circular symmetrization (see [10, Sec. 4.1]). The following section is auxiliary.

2. Preliminaries

We will apply the capacity approach and symmetrization of condensers. The notions from the book [10] will be tacitly used below. In this paper, for the most part, we will consider the condensers with two plates of the form

\[ C = (U(R), \{E_0, E_1\}, \{0, 1\}) \equiv (U(R), E_0, E_1), \]

where \( U(R) = \{z : |z| < R\} \), \( 0 < R \leq 1 \), and \( E_0, E_1 \) are closed disjoint nonempty sets \( E_0 \subset \overline{U}(R) \), \( E_1 \subset U(R) \). The capacity \( \text{cap} C \) of \( C \) is defined as the infimum of the Dirichlet integrals

\[ I(v, U(R)) := \int_{U(R)} |\nabla v|^2 \]

over all admissible functions \( v \), that is, real functions \( v \) which are continuous in \( \overline{U}(R) \), Lipschitz on compact subsets of \( U(R) \) and equal to \( k \) on \( E_k \), \( k = 0, 1 \). If the class of admissible functions is reduced to the subclass of functions \( v \), \( 0 \leq v \leq 1 \), equal to \( k \) in a neighbourhood of the plate \( E_k \), \( k = 0, 1 \), then the capacity of a condenser \( C \) does not change [10, Lemma 1.2].

Denote by \( \gamma(\rho) \) the circle \( |z| = \rho \), \( 0 \leq \rho \leq 1 \). Let \( B \) be an open subset of \( \overline{U}(R) \). The circular symmetrization (with respect to the positive real half-axis) assigns to a set \( B \) the "circularly symmetric" set

\[ \text{Cr } B = \{re^{i\theta} : B \cap \gamma(\rho) \neq \emptyset, 2|\theta|\rho < \text{meas}(B \cap \gamma(\rho))\} \cup \{-r : \gamma(\rho) \subset B\}, \]

where \( \text{meas}(\cdot) \) is the Lebesgue linear measure. In a similar way, we define the symmetrization of a closed subset \( E \) of \( \overline{U}(R) \) as follows:

\[ \text{Cr } E = \{re^{i\theta} : E \cap \gamma(\rho) \neq \emptyset, 2|\theta|\rho \leq \text{meas}(E \cap \gamma(\rho))\}. \]

For a condenser \( C = (E_0, E_1) \) we set

\[ \text{Cr } C = (\overline{U}(R) \setminus \text{Cr } (\overline{U}(R) \setminus E_0) \setminus \text{Cr } E_1). \]

The following statement goes back to Pólya’s [11] (cf. [10, Theorem 4.2]).

**Lemma 1.** With the above notation, we have

\[ \text{cap } C \geq \text{cap } \text{Cr } C. \]

The inner radius of an open set \( B \subset C \) with respect to a point \( z_0 \in B \) is the quantity

\[ r(B, z_0) = \exp\{ \lim_{z \to z_0} [g_B(z, z_0) + \log |z - z_0|] \}, \]

where \( g_B(z, z_0) \) is the Green function of the connected component of \( B \) containing the point \( z_0 \). Let \( B \) be a domain in \( C \) bounded by finitely many piecewise analytic curves, \( \Gamma \) be a nonempty closed subset of \( \partial B \) consisting of finitely many nonsingular arcs such that \( (\partial B) \setminus \Gamma \) is union of open smooth arcs, and \( z_0 \) be a point in \( B \). Then
there exists a function \( g_B(z, z_0, \Gamma) \) which is continuous on \( \overline{B} \setminus \{z_0\} \), harmonic in \( B \setminus \{z_0\} \), and satisfying the following conditions:

\[
g_B(z, z_0, \Gamma) = 0 \quad \text{for} \quad z \in \Gamma;
\]

\[
\frac{\partial}{\partial n} g_B(z, z_0, \Gamma) = 0 \quad \text{for} \quad z \in (\partial B) \setminus \Gamma;
\]

\( g_B(z, z_0, \Gamma) + \log |z - z_0| \) is a bounded harmonic function in a neighbourhood of \( z_0 \) (\( \partial/\partial n \) denotes differentiation along the inward normal to \( \partial B \)). The function \( g_B(z, z_0, \Gamma) \) is called the Robin function of the domain \( B \) and the set \( \Gamma \) with pole at \( z_0 \). By the Robin radius of a domain \( B \) with respect to a point \( z_0 \) and a set \( \Gamma \), we mean the quantity

\[
r(B, \Gamma, z_0) = \exp\{ \lim_{z \to z_0} [g_B(z, z_0, \Gamma) + \log |z - z_0|]\}.
\]

Note that if \( a, b \) and \( c \) are real numbers, \( a < b < c \), \( B_1 = \mathbb{C} \setminus \{z : \text{Im} z = 0, \text{Re} z \geq b\} \), \( B_2 = \mathbb{C} \setminus \{z : \text{Im} z = 0, \text{Re} z \geq c\} \), then

\[
r(B_1, [c, +\infty], a) = r(B_2, a) = 4(c - a).
\]

Theorem 2.2 from the book [10] gives

**Lemma 2.** Let \( B \) be a domain, set \( \Gamma \subset \partial B \), \( z_0 \in B \) as above, and let \( B \subset U \), \( (\partial B) \setminus \Gamma \subset \partial U \). Then

\[
\text{cap}(U, \Gamma, \{z : |z - z_0| \leq \varphi(r)\}) = -\frac{2\pi}{\log r} - 2\pi \left[ \log \frac{r(B, \Gamma, z_0)}{\mu} \right] \left( \frac{1}{\log r} \right)^2 + o\left( \left( \frac{1}{\log r} \right)^2 \right), \quad r \to 0,
\]

where \( \varphi(r) \) is any real function of the form \( \varphi(r) = \mu r(1 + o(1)) \), \( r \to 0 \) and \( \mu > 0 \).

The behaviour of the Robin radius under conformal map is described in the following statement.

**Lemma 3.** Let the domains \( B, G \) and the sets \( \Gamma \subset \partial B, \Lambda \subset \partial G \) be as in definition of the Robin radius, and let \( f \) be a function mapping \( B \) conformally and univalently onto \( G \) so that \( f(\Gamma) = \Lambda \). Then

\[
r(G, \Lambda, f(z_0)) = |f'(z_0)|r(B, \Gamma, z_0)
\]

for any point \( z_0 \in B \).

This lemma is a special case of the majorization principle [7, Theorem 4.2].

3. Distortion theorems

We will start with an analogue of the assertions [6, Theorem 2] and [9, Theorem 1]. Set

\[
E(\alpha) = \{z = e^{i\theta} : |\theta| < \alpha\}, \quad 0 \leq \alpha < \pi.
\]

**Theorem 1.** Let \( f \) be a holomorphic function in the disk \( U \), \( f(U) \subset U \), and let \( f(E(\alpha)) \subset E(\beta) \) for some \( \alpha \) and \( \beta \), \( 0 \leq \alpha < \pi \), \( 0 \leq \beta < \pi \) (i.e. for each sequence of points \( z_n \in U \), approaching the set \( E(\alpha) \), the corresponding sequence \( f(z_n) \) →

\(1\)If \( \alpha = 0 (\beta = 0) \) then the boundary condition is ignored.
Suppose that $\gamma(p) \not\subseteq f(U)$ for all $p, \tau \leq p \leq 1$, and some fixed $\tau$, $0 \leq \tau < 1$. Then for any real points $z_1, z_2$ such that $-1 < z_1 < z_2 < 1$ and $|f(z_1)| \neq |f(z_2)|$
\begin{equation}
\frac{|k(e^{i\beta}) - k(m)|k(M) - k(-\tau)|}{k(M) - k(m)} \geq \frac{|k(e^{i\alpha}) - k(z_1)|k(M) - k(-\tau)|}{k(z_2) - k(z_1)}
\end{equation}
where $k(z) = (1 + z)^{-2}$ is the Koebe function and $m = \min\{|f(z_1)|, |f(z_2)|\}$, $M = \max\{|f(z_1)|, |f(z_2)|\}$. Equality in (3) is attained for any points $-1 < z_1 < z_2 < 1$ and any conformal map $f : U \to \mathbb{C}$ such that $f(-1) = -\tau$, $f(1) = 1$ and $0 \leq f(z_1) < f(z_2) < 1$.

Proof. It suffices to consider $\alpha \neq 0$, $\beta \neq 0$ and a nonconstant function $f$. Let $C = (U, (\partial U) \setminus E(\alpha), [z_1, z_2])$, and let $u$ be an admissible function for the condenser $C$ which is equal to 0 in a neighbourhood of $(\partial U) \setminus E(\alpha)$ and to 1 in a neighbourhood of $[z_1, z_2]$ and satisfies $0 \leq u(z) \leq 1$ for $z \in \overline{U}$. Let $\gamma$ be a closed circular arc connecting the points $e^{\pm i\alpha}$ and lying in a neighbourhood of $(\partial U) \setminus E(\alpha)$, where $v = 0$. Denote by $B$ the domain in $U$ with the boundary $\partial B = \gamma \cup E(\alpha)$. Note that the set $f(B)$ does not contain the boundary points on the arc $(\partial U) \setminus E(\beta)$. Finally, let $R$ be close to 1, such that $\tau < R < 1$ and $f([z_1, z_2]) \subset U(R)$. Let us inspect the following function on $U(R)$:
\[ u(w) = \begin{cases} 
\max\{v(z) : f(z) = w\}, & w \in U(R) \cap f(B) \\
0, & w \in U(R) \setminus f(B). 
\end{cases} \]
The function $f$ takes each value in $f(B)$ on a finite or countable infinite set of points in $B$, which accumulate at the boundary of $B$. Hence, the maximum in the definition of $u$ is taken over finitely many values of $v$. It is easy to see that $u$ is continuous in $U(R)$ and Lipschitz in a neighbourhood of each point in $U(R)$, with a possible exception of finitely many points $w$ such that $f(z) = w$ and $f'(z) = 0$. From this we conclude that
\[ I(v, U) = I(v, B) \geq I(u, U(R)) \geq \text{cap} C(R), \]
where
\[ C(R) = (U(R), U(R) \setminus f(B), f([z_1, z_2])). \]
By Lemma 1
\[ \text{cap} C(R) \geq \text{cap} C^*(R). \]
Set
\[ C^*(R) = (U(R), [-R, -\tau] \cup (\partial U(R)) \setminus \{w : w/R \in E(\beta(R))\}, [m, M]), \]
where $\beta(R)$ is defined by $2\beta(R) = \text{meas}(f(B) \cap \gamma(R))$. In view of the hypotheses of Theorem 1 and monotonicity property of capacity [10, Theorem 1.15] we have
\[ \text{cap} C^*(R) \geq \text{cap} C^*(R). \]
Thus
\[ I(v, U) \geq \text{cap} C^*(R). \]
Note that
\[ \lim_{R \to 1} \beta(R) \leq \beta. \]
By passage to the limit we obtain
\[ I(v, U) \geq \text{cap} C^*, \]
where
\[ C^* := (U, [-1, -\tau] \cup (\partial U) \setminus E(\beta), [m, M]). \]
Hence,
\begin{equation}
\cap C \geq \cap C^*.
\end{equation}
The function
\[ \xi(z) = \frac{k(z) - k(z_1)}{k(z_2) - k(z_1)} \]
maps the condenser \( C \) onto the condenser \( \xi(C) \) with two plates on the Riemann sphere [10, Sec. 1.2]:
\[ \xi(C) = ([\xi(e^{i\alpha})], +\infty], [0, 1]). \]
The conformal invariance of capacity gives
\[ \cap C = \cap \xi(C). \]
Similarly, the function
\[ \eta(w) = \frac{(k(w) - k(m))(k(M) - k(-\tau))}{(k(w) - k(-\tau))(k(M) - k(m))} \]
maps the condenser \( C^* \) onto the condenser
\[ \eta(C^*) = ([\eta(e^{i\beta})], +\infty], [0, 1]), \]
and
\[ \cap C^* = \cap \eta(C^*). \]
From inequality (4) we have
\[ \xi(e^{i\alpha}) \leq \eta(e^{i\beta}). \]
This yields the inequality in Theorem 1. If \( f \) is a conformal map, \( f : U \to U \setminus [-1, -\tau] \) such that \( f(-1) = -\tau, f(1) = 1 \) and \( 0 \leq f(z_1) < f(z_2) < 1 \) for some points \( z_1, z_2, -1 < z_1 < z_2 < 1 \), then equality in (4) holds. Hence, we have equality in (3). This completes the proof of Theorem 1.

Let \( f \) be a holomorphic function in the disk \( U, f(U) \subseteq U, f(0) = 0 \). Taking in Theorem 1 \( \alpha = \beta = 0, z_1 = 0 \) and taking the limit as \( z_2 \to 0, \tau \to 1 \) from (3) we deduce the classical Schwarz’s inequality
\[ |f'(0)| \leq 1. \]
In a similar way, letting \( z_1 = 0, z_2 = 1 \) and \( \tau \to 1 \) we obtain the Löwner’s inequality
\[ \alpha \leq \beta. \]
If we set \( \alpha = \beta = \tau = 0, 0 < z_1 < z_2 < 1, |f(z_1)| < |f(z_2)| \) in (3), then we obtain the following estimate:
\[ \frac{(1 + z_2)^2(1 - z_1)^2}{(z_2 - z_1)(1 - z_1 z_2)} \leq \frac{4|f(z_2)|(1 - |f(z_1)|)^2}{(|f(z_2)| - |f(z_1)|)(1 - |f(z_1)f(z_2)|)} \]
(cf. [9, Theorem 1]).

**Theorem 2.** Under the hypotheses of Theorem 1 suppose that a point \( z \in (-1, 1) \). Then
\begin{equation}
\left| \frac{f'(z)k'(|f(z)|)}{k'(z)} \right| \leq \frac{|k(|f(z)|) - k(-\tau)||k(e^{i\beta}) - k(|f(z)|)|}{|k(e^{i\alpha}) - k(z)||k(e^{i\beta}) - k(-\tau)|},
\end{equation}
where \( k(z) = z(1 + z)^{-2} \) is the Koebe function. Equality holds in (5) for any point \( z \in (-1, 1) \) and any conformal map \( f : U \to U \setminus [-1, -\tau] \) such that \( f(-1) = -\tau, f(1) = 1 \) and \( 0 \leq f(z) < 1 \).
Proof. Let \( z_0 \) be a fixed point of the interval \((-1, 1)\), \( f'(z_0) \neq 0 \), and let
\[
C = (U, (\partial U) \setminus E(\alpha), \{z : |z - z_0| \leq r\})
\]
for sufficiently small \( r > 0 \). Repeating the proof of Theorem 1 for the condenser \( C \), we get inequality (4), where this time
\[
C^* = (U, [-1, -\tau] \cup (\partial U) \setminus E(\beta), \{w : |w - |f(z_0)|| \leq \varphi(r)\}),
\]
and \( \varphi(r) = |f'(z_0)|r(1 + o(1)), r \to 0 \). Using Lemma 2 from (4) we obtain
\[
|f'(z_0)|r(U, \gamma, z_0) \leq r(U \setminus [-1, -\tau], \Gamma, |f(z_0)|),
\]
where \( \gamma = (\partial U) \setminus E(\alpha), \Gamma = [-1, -\tau] \cup (\partial U) \setminus E(\beta) \).

The Koebe function \( \xi = k(z) \) maps the disk \( U \) onto the domain \( B_1 = \mathbb{C} \setminus [1/4, +\infty] \) so that \( k(\gamma) = [k(e^{i\alpha}), +\infty] \). By Lemma 3 and equality (2)
\[
|k'(z_0)|r(U, \gamma, z_0) = r(B_1, k(\gamma), k(z_0)) = r(B_2, k(z_0)) = 4[k(e^{i\alpha}) - k(z_0)],
\]
\( B_2 = \mathbb{C} \setminus [k(e^{i\alpha}), +\infty] \). Similarly, the function
\[
\eta(w) = \frac{1}{k(w) - k(-\tau)}
\]
maps the domain \( U \setminus [-1, -\tau] \) conformally and univalently onto the domain \( B_3 = \mathbb{C} \setminus [-\infty, \eta(1)] \), so that \( \eta(\Gamma) = [-\infty, \eta(e^{i\beta})] \). By Lemma 3,
\[
|\eta'|(|f(z_0)|)r(U \setminus [-1, -\tau], \Gamma, |f(z_0)|) = r(B_3, \eta(\Gamma), \eta(|f(z_0)|)) = \eta(B_4, \eta(|f(z_0)|)) = 4[\eta(|f(z_0)|) - \eta(e^{i\beta})],
\]
where \( B_4 = \mathbb{C} \setminus [-\infty, \eta(e^{i\beta})] \). Thus inequality (6) gives
\[
\frac{|f'(z_0)\eta(|f(z_0)|)|}{|k'(z_0)|} \leq \frac{\eta(|f(z_0)|) - \eta(e^{i\beta})}{k(e^{i\alpha}) - k(z_0)}.
\]
This yields the inequality in Theorem 2 for \( z = z_0 \). If \( f \) is a conformal map, \( f : U \to U \setminus [-1, -\tau] \), such that \( f(-1) = -\tau, f(1) = 1 \) and \( 0 \leq f(z_0) < 1 \) for a point \( z_0 \in (-1, 1) \), then by Lemma 3 the equality in (6) holds. Hence, we have equality in (5) for \( z = z_0 \). This completes the proof of Theorem 2. \( \square \)

For a holomorphic self-mapping \( f \) of the unit disk \( U \), \( f(0) = 0 \), the inequality (5) gives the classical Schwarz’s inequality \( |f'(0)| \leq 1 \) again. If we set \( z = 0 \), \( f(0) = 0 \) and \( \tau \to 1 \), then we obtain the estimate
\[
|f'(0)| \cos^2 \frac{\beta}{2} \leq \cos^2 \frac{\alpha}{2},
\]
which is an implication of the Schwarz’s inequality and the Lowner’s inequality. Now let \( f \) be a holomorphic function from Theorem 2, \( \alpha \neq 0, \beta \neq 0 \), and suppose, in addition, that \( f \) has an angular limit \( f(1) = 1 \) and a finite angular derivative \( f'(1) \). Then \( f' \) has a finite angular limit \( f'(1) \) at \( z = 1 \) [3, Proposition 4.7]. Applying Theorem 2 to the function \( f \) and taking \( \tau \to 1, z \to 1 \) we arrive at the new inequality:
\[
|f'(1)| \tan \frac{\alpha}{2} \leq \tan \frac{\beta}{2}.
\]
Setting in Theorem 2 \( \alpha = \beta = \tau = 0 \) we obtain the inequality (11) from [9].
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