\textbf{\large $\Delta$-Tribrackets and Link Homotopy}

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\section*{Abstract}
We define a type of Niebrzydowski tribracket we call $\Delta$-tribrackets and show that their counting invariants are invariants of link-homotopy. We further identify several classes of tribrackets whose counting invariants for oriented classical knots and links are trivial, including vertical tribrackets satisfying the center-involutory condition and horizontal tribrackets satisfying the late-commutativity condition. We provide examples and end with questions for future research.

\textbf{Keywords:} Tribrackets, Link Homotopy, counting invariants, ternary algebras

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\section{Introduction}
A Niebrzydowski tribracket, also called a knot-theoretic ternary quasigroup, is a set with a ternary operation satisfying axioms encoding the Reidemeister moves in combinatorial knot theory. Specifically, given a finite tribracket $X$ and an oriented knot or link diagram $D$, the set of assignments of elements of $X$ to regions in $D$ satisfying a certain condition at each crossings, known as $X$-colorings of $D$, defines an invariant of links. See [12] for more.

Associated to any oriented link $L$ is a fundamental tribracket $T(L)$ whose isomorphism class is an invariant of links. From a diagram of an oriented link we can obtain a presentation of this fundamental tribracket. The set of $X$-colorings of $D$ can be identified with the set $\text{Hom}(T(L), X)$ of tribracket homomorphisms from the fundamental tribracket $T(L)$ of $L$ to the coloring tribracket $X$. Naturally, different diagrams represent the same homomorphism with different colorings, analogously to how different choices of basis represent the same linear transformation with different matrices. The cardinality of this homset is a non-negative integer-valued invariant of oriented knots and links known as the tribracket counting invariant, denoted $\Phi^X_Z(L)$.

In this paper we identify a few families of tribrackets whose counting invariants are trivial on knots, yielding some easily-checkable conditions on which coloring tribrackets to avoid when using these invariants to distinguish knots. In particular we define $\Delta$-tribrackets whose counting invariants are trivial on knots and yield link-homotopy invariants on links with multiple components. The paper is organized as follows. In Section 2 we recall the basics of tribracket theory. In Section 3 we identify some families of tribrackets with trivial invariants arising from known unknotting moves. We define $\Delta$-tribrackets using the $\Delta$-move, noting that the resulting tribrackets are trivial for knots and define link-homotopy invariants for links. We identify necessary and sufficient conditions for Alexander tribrackets to satisfy the conditions described. In particular the fundamental $\Delta$-tribracket of an oriented link has the potential to yield Alexander ideal- or polynomial-style invariants of link homotopy type, to be the subject of future investigation. In Section 4 we provide some example computations and some tables of invariant values of the counting invariant for some finite $\Delta$-tribrackets on all prime links with up to seven crossings. We conclude in Section 5 with some questions for future research.

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\vspace{1cm}1
2 Tribracket Basics

We begin with a definition; see [8] for more.

**Definition 1.** Let $X$ be a set. A ternary map $[\cdot,\cdot] : X \times X \times X \to X$ defines a horizontal tribracket structure on $X$ if it satisfies the conditions

(i) For all ordered triples $(x, y, z) \in X \times X \times X$, there exist unique elements $u, v, w \in X$ such that

\[
[x, y, u] = z, \quad [x, v, y] = z \quad \text{and} \quad [w, x, y] = z
\]

and

(iii) For all ordered quadruples $(x, y, z, w) \in X \times X \times X \times X$ we have

\[
[y, [x, y, z], [x, y, w]] = [z, [x, y, z], [x, z, w]] = [w, [x, y, w], [x, z, w]].
\]

A ternary map $\langle \cdot, \cdot \rangle : X \times X \times X \to X$ defines a vertical tribracket structure on $X$ if it satisfies the conditions

(i) For all ordered triples $(x, y, z) \in X \times X \times X$, there exist unique elements $u, v, w \in X$ such that

\[
\langle x, y, u \rangle = z, \quad \langle x, v, y \rangle = z \quad \text{and} \quad \langle w, x, y \rangle = z
\]

and

(iv) For all ordered quadruples $(x, y, z, w) \in X \times X \times X \times X$ we have

\[
\langle a, \langle x, y, z \rangle, \langle \langle x, y, z \rangle, z, w \rangle \rangle = \langle x, y, \langle y, z, w \rangle \rangle \quad \text{and} \quad \langle \langle x, y, z, w \rangle, \langle y, z, w \rangle, w \rangle = \langle \langle x, y, z \rangle, \langle y, z, w \rangle \rangle.
\]

Tribracket axiom (i) says that the tribracket operation has left-, center- and right-invertibility.

Tribrackets form a category whose objects are tribrackets and whose morphisms are tribracket homomorphisms, i.e., functions $f : X \to Y$ satisfying for all $x, y, z \in X$

\[
f([x, y, z]) = [f(x), f(y), f(z)]
\]

for vertical tribrackets and

\[
f(\langle x, y, z \rangle) = \langle f(x), f(y), f(z) \rangle
\]

for horizontal tribrackets.

The two types of tribrackets are equivalent in the sense that for any vertical tribracket structure on $X$ there is a corresponding horizontal tribracket structure on $X$ and vice-versa satisfying for all $x, y, z \in X$

\[
z = \langle x, y, [x, y, z] \rangle = [x, y, \langle x, y, z \rangle].
\]

Note that we use square brackets to indicate horizontal tribracket operations and angle brackets to indicate vertical tribracket operations without further comment.

**Example 1.** Let $G$ be a group; then the ternary operation $[x, y, z] = yx^{-1}z$ defines a horizontal tribracket known as the Dehn tribracket of the group.

**Example 2.** Let $M$ be a module over the ring to two-variable Laurent polynomials with integer coefficients $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$. Then $M$ is a horizontal tribracket with ternary operation $[x, y, z] = ty + sz -tsx$ known as an Alexander tribracket.
Example 3. We can specify a tribracket structure on a finite set $X = \{1, \ldots, n\}$ with an operation 3-tensor, i.e. an ordered $n$-tuple of $n \times n$ matrices with entries in $\{1, 2, \ldots, n\}$ such that the entry in matrix $i$, row $j$ column $k$ is $[i, j, k]$. For instance, there are two tribrackets of order 2:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$ 

Example 4. Let $G = \{x_1, \ldots, x_n\}$ be a set of generators. A tribracket word in $G$ is either an element of $G$ or a string of symbols of one of the following forms:

$$\{[x,y,z], [\overline{x},y,z], [x,\overline{y},z], [x,y,\overline{z}]\}$$

where $x, y, z \in G$. Then a tribracket presentation is a finite list $G$ of generators and relations, i.e., equations of tribracket words in $G$. The presented tribracket is the set of equivalence classes on tribracket words under the equivalence relation generated by the equivalences

$$x \sim [[x,y,z], y, z]$$
$$y \sim [x, [x, y, z], z]$$
$$z \sim [x, y, [x, y, \overline{z}]]$$
$$[y, [x, y, z], [x, y, w]] \sim [z, [x, y, z], [x, z, w]]$$
$$[z, [x, y, z], [x, z, w]] \sim [w, [x, y, w], [x, z, w]]$$

for all tribracket words $x, y, z, w$ together with the explicitly listed relations.

In particular, the words $[\overline{x},y,z], [x,\overline{y},z]$ and $[x,y,\overline{z}]$ are the left-, center- and right-inverses required by the first tribracket axiom; note that $\overline{x}$ by itself is not a tribracket word.

Definition 2. Let $L$ be an oriented link diagram. The fundamental tribracket of $L$, $\mathcal{T}(L)$ has a presentation with a generator for each region in the planar complement of $L$ and a relation at each crossing of the form

Example 5. The right-handed trefoil knot has fundamental tribracket with presentation

$$\mathcal{T}(3_1) = \langle x, y, z, w \mid [x, y, z] = [x, z, w] = [x, w, y] \rangle.$$

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (1,1);
  \draw (0,1) -- (1,0);
  \node at (0.5,0.5) {$[x, y, z]$};
  %\node at (0.5,0.5) {\textcolor{red}{[x, y, z]}};

  %\draw (2,0) -- (3,1);
  %\draw (2,1) -- (3,0);
  %\node at (2.5,0.5) {$[x, y, z]$};
  %\node at (2.5,0.5) {\textcolor{red}{[x, y, z]}};

  \draw (0,-1) -- (1,-2);
  \draw (0,-2) -- (1,-1);
  \node at (0.5,-1.5) {$[x, z, w]$};
  %\node at (0.5,-1.5) {\textcolor{red}{[x, z, w]}};

  \draw (2,-1) -- (3,-2);
  \draw (2,-2) -- (3,-1);
  \node at (2.5,-1.5) {$[x, y, z]$};
  %\node at (2.5,-1.5) {\textcolor{red}{[x, y, z]}};

\end{tikzpicture}
\end{center}
Reidemeister moves on $L$ induce Tietze moves on $\mathcal{T}(L)$; hence it follows that $\mathcal{T}(L)$ is an invariant of oriented knots and links. Since comparing isomorphism classes of objects presented by generators and relations is generally inconvenient, we can use $\mathcal{T}(L)$ to define more computable invariants. One strategy for doing so involves computing sets of tribracket homomorphisms, i.e., maps $f : \mathcal{T}(L) \to X$ from the fundamental tribracket of $L$ to a finite tribracket $X$ satisfying the condition

$$f([x, y, z]) = [f(x), f(y), f(z)]$$

at every crossing. Such homomorphisms can be conveniently represented as region colorings of a diagram of $L$ by elements of $X$; see [1, 7, 9, 10] etc.

### 3 Trivializing Quotients and Trivial Invariants

For other knot-theoretic algebraic structures such as quandles and biquandles, quotients of the fundamental algebraic structure associated to a knot or link can provide interesting invariants. For example, the question of which knots and links have finite fundamental kei (also called fundamental involutory quandle), obtained from the fundamental quandle of the knot or link by including relations $(x \triangleright y) \triangleright y = x$ for all $x, y$, was considered in [13] and later in [4]. Similarly, involutory quotients of knot biquandles were studied in [2] and quotients of fundamental virtual quandles of virtual knots were studied in [11]. Any homomorphism from a knot quandle to an involutory target quandle must factor through the fundamental involutory quandle of the knot. We will now apply this principle to identify some families of tribrackets whose counting invariants are trivial.

**Definition 3.** Say that a horizontal tribracket $X$ is left-, center- or right-involutory respectively if the functions defined by

$$f_{x,y}(a) = [a, x, y], \quad g_{x,y}(a) = [x, a, y] \quad \text{and} \quad h_{x,y}(a) = [x, y, a]$$

respectively are involutions for all $x, y \in X$, i.e., if

$$f_{x,y}(f_{x,y}(a)) = g_{x,y}(g_{x,y}(a)) = h_{x,y}(h_{x,y}(a)) = a.$$ 

Similarly a vertical tribracket is left-, center- or right-involutory if for all $x, y \in X$ the functions

$$f_{x,y}(a) = \langle a, x, y \rangle, \quad g_{x,y}(a) = \langle x, a, y \rangle \quad \text{and} \quad h_{x,y}(a) = \langle x, y, a \rangle$$

respectively are involutions. Say that a tribracket is fully involutory if it is involutory in all three positions.

Let $X$ be a center-involutory vertical tribracket. We then note that the number of colorings of any oriented knot or link by $X$ is unchanged by 2-moves:

Since the 2-move followed by a Reidemeister II move amounts to a crossing change, the fundamental central-involutory vertical tribracket of an oriented knot or link does not change under crossing change, and we have:
Proposition 1. The number of colorings of any oriented link diagram by a center-involutory vertical tribracket is the same as that of the unlink with the same number $c$ of components, namely $\Phi^Z_X(L) = |X|^{c+1}$.

Proof. Let $L$ be an oriented link and consider the fundamental tribracket of $L$. Adding the relation $\langle x, \langle x, y, z \rangle, z \rangle = y$ for all $x, y, z$ yields the fundamental center-involutory tribracket of $L$, denoted $T_{LC}(L)$. The center-involutory condition implies that the presented tribracket is unchanged by crossing-change moves. Hence, the fundamental center-involutory tribracket is invariant under crossing changes; in particular, it is isomorphic to that of the unlink $U_c$ with same number $c$ of components as $L$.

If $X$ is an center-involutory tribracket, then any homomorphism $f : T(L) \to X$ must factor through $T_{LC}(L)$. Since each homomorphism $f : T(L) \to T_{LC}(L)$ is just taking a quotient, it follows that $|\text{Hom}(T(L), X)| = |\text{Hom}(T_{LC}(L), X)|$. Then since $|\text{Hom}(T_{LC}(L), X)| = |\text{Hom}(T_{LC}(U_c), X)| = |X|^{c+1}$, we have the result. \hfill $\square$

Next we have another category of tribrackets with trivial counting invariants.

Definition 4. Let $X$ be a horizontal tribracket. We say $X$ is late-commutative if for all $x, y, z \in X$ we have

$[x, y, z] = [x, z, y]$.

Example 6. An Alexander tribracket is late-commutative if $t = s$, i.e.

$[x, y, z] = ty + tz - t^2x$.

Example 7. A Dehn tribracket $G$ is late-commutative if for all $a, b, c \in G$ we have

$ab^{-1}c = cb^{-1}a$.

Then in particular, we must have $ac = ca$ when $b$ is the identity, and $G$ is abelian.

We then have

Proposition 2. If $X$ is a late-commutative tribracket and $K$ is an oriented link, then $\Phi^Z_X(K) = |X|^{c+1}$ where $c$ is the number of components of $L$.

Proof. This is similar to the proof of Proposition 1. \hfill $\square$

A more subtle category of tribrackets with coloring invariants which are trivial on knots but provide link-homotopy invariants for links is what we call $\Delta$-tribrackets:

Definition 5. Let $X$ be a tribracket. We say $X$ is a $\Delta$-tribracket if for all $x, y, z, w \in X$ we have

$[y, [x, y, z], [x, w, y]] = [z, [x, z, w], [x, y, z]] = [w, [x, w, y], [x, z, w]]$.

Example 8. An Alexander tribracket is a $\Delta$-tribracket if it satisfies the property $2st = t^2 + s^2$, since

$[y, [x, y, z], [x, w, y]] = -tsy + t(-tsx + ty + sz) + s(-tsx + tw + sy)$

$= (t^2 + s^2 - ts)y - (t + s)tsx + tsw + tsz$;

$[z, [x, z, w], [x, y, z]] = (t^2 + s^2 - ts)z - (t + s)tsx + tsy + tsw$ and

$[w, [x, w, y], [x, z, w]] = (t^2 + s^2 - ts)w - (t + s)tsx + tsy + tsz$

so the condition we need is

$t^2 + s^2 - st = st$.

Then for example, $Z_8$ is a $\Delta$-tribracket with $t = 3$ and $s = 7$, as is $Z_9$ with $t = 2$ and $s = 5$. 

The name “\(\Delta\)-tribracket” is chosen to reflect that colorings by a \(\Delta\)-tribracket are unchanged \(\Delta\)-moves:

Recall from [6] that

- The \(\Delta\)-move is an unknotting move, and
- Links of several components are link-homotopic iff they are related by \(\Delta\)-moves together with Reidemeister moves.

As a consequence we have:

**Proposition 3.** The counting invariant \(\varphi^\Delta_X\) associated to a \(\Delta\)-tribracket is a link-homotopy invariant.

4 Examples

In this section we collect a few examples.

**Example 9.** Consider the trefoil knot \(3_1\) below.

As we have seen, its fundamental tribracket has presentation

\[
\mathcal{T}(3_1) = \langle x, y, z, w \mid [x, y, z] = [x, z, w] = [x, w, y] \rangle.
\]

As an Alexander tribracket, this has presentation matrix

\[
\begin{bmatrix}
1 & st & -t & -s & 0 \\
1 & st & 0 & -t & -s \\
1 & st & -s & 0 & -t
\end{bmatrix}.
\]
In the case of the fundamental late-commutative tribracket this is
\[
\begin{bmatrix}
1 & t^2 & -t & -t & 0 \\
1 & t^2 & 0 & -t & -t \\
1 & t^2 & -t & 0 & -t \\
\end{bmatrix}
\]
which row-reduces over \(\mathbb{Z}[t^\pm 1]\) to
\[
\begin{bmatrix}
t & 0 & 0 & -2 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\]
with kernel of dimension 2 (the same as the unknot) as expected.

**Example 10.** Continuing with the trefoil from Example 9, first row-reducing the presentation matrix over \(\mathbb{Z}[t^\pm 1, s^\pm 1]\) we have
\[
\begin{bmatrix}
1 & st & -t & -s & 0 \\
1 & st & 0 & -t & -s \\
1 & st & -s & 0 & -t \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & st & -t & -s & 0 \\
0 & 0 & t & t - s & -t \\
0 & 0 & -s & t & s - t \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & st & -t & -s & 0 \\
0 & 0 & t & s - t & -s \\
0 & 0 & -s & t & s - t \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & st & -t & -s & 0 \\
0 & 0 & t & t - s & -t \\
0 & 0 & -s & t & s - t \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & st & -t & -s & 0 \\
0 & 0 & t & s - t & -s \\
0 & 0 & -s & t & s - t \\
\end{bmatrix}
\]
So in the \(\Delta\)-tribracket case, with \(t^2 + s^2 - st = st\), we have
\[
\begin{bmatrix}
1 & st & -t & -s & 0 \\
0 & 0 & st & -s & 0 \\
0 & 0 & 0 & st & -t \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & st & -t & -s & 0 \\
0 & 0 & st & s^2 - ts & -s^2 \\
0 & 0 & 0 & st & t^2 - ts - s^2 - t^2 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & st & -t & -s & 0 \\
0 & 0 & t & st^{-1} - 1 & -st^{-1} \\
0 & 0 & 0 & 1 & -1 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & st & 0 & 0 & -s + t \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\]
again with kernel of dimension 2 as expected.

**Example 11.** Taking the case of the Hopf link \(L2a1\) below,

![Diagram of the Hopf link](image)

we have fundamental Alexander tribracket matrix presented by
\[
\begin{bmatrix}
1 & st & -t & -s \\
1 & st & -s & -t \\
\end{bmatrix}
\]
which row-reduces over \(\mathbb{Z}\) to
\[
\begin{bmatrix}
1 & st & -t & -s \\
0 & 0 & t - s & s - t \\
\end{bmatrix}.
\]
In the case of the late-commutative tribracket we get kernel of dimension 3 as anticipated, the same as the unlink of two components. More generally, for coloring tribrackets in which \( t - s \) is a unit, we will have kernel of dimension 2, distinguishing the link-homotopy type of the Hopf link from that of the unlink. For coloring tribrackets in which \( t - s \) is a zero divisor, the torsion part of the coloring space can also distinguish the link-homotopy type of this link.

Example 12. Using Python code, we computed the numbers of colorings for the prime links with up to seven crossings as listed at the Knot Atlas (www.knotlib.org) with respect the two Alexander \( \Delta \)-tribrackets \( X_1 = (\mathbb{Z}_8, s = 7, t = 3) \) and \( X_1 = (\mathbb{Z}_9, s = 5, t = 2) \); the results are collected in the table.

| \( L \) | \( \Phi_{\bar{X}_1}(L) \) |
|-------|----------------------------------|
| 256   | \( L_{2a1}, L_{6a2}, L_{6a3}, L_{7a5}, L_{7a6} \) |
| 512   | \( L_{4a1}, L_{5a1}, L_{6a1}, L_{7a1}, L_{7a2}, L_{7a3}, L_{7a4}, L_{7n1}, L_{7n2} \) |
| 1024  | \( L_{6a1}, L_{6a1}, L_{7a7} \) |
| 4096  | \( L_{6a4} \) |
| 243   | \( L_{2a1}, L_{4a1}, L_{6a1}, L_{7a2}, L_{7a5}, L_{7a6}, L_{7n1} \) |
| 729   | \( L_{5a1}, L_{6a2}, L_{6a3}, L_{7a1}, L_{7a3}, L_{7a4}, L_{7a7}, L_{7n2} \) |
| 2187  | \( L_{6a5}, L_{6n1} \) |
| 6561  | \( L_{6a4} \) |

These \( \Delta \)-tribracket invariants are fairly quick and easily computable, providing a useful tool for distinguishing link-homotopy classes. Example 12 shows that non-isomorphic finite \( \Delta \)-tribrackets provide different information about link-homotopy type in general. As with all such invariants, the invariant defined by any given finite \( \Delta \)-tribracket is not expected to be a complete invariant of link-homotopy. However, the set of all \( \Delta \)-tribracket invariants could potentially be a complete invariant.

5 Questions

We end with a few questions for future research.

What is the exact relationship between the fundamental \( \Delta \)-tribracket of a link, its homset invariants, and other link-homotopy invariants such as Milnor invariants or the quasi-trivial quandles and biquandle invariants in [3,5]? The fundamental Alexander \( \Delta \)-tribracket of a link seems to potentially play the role of an Alexander invariant for link-homotopy; what ideal-based or polynomial invariants can be extracted from it?

What enhancements of the fundamental \( \Delta \)-tribracket of a link can be defined, and what information about link-homotopy type do they extract?

Is the set of all \( \Delta \)-tribracket counting invariants a complete invariant of link homomotopy? That is, given any two non-link-homotopic links \( L \) and \( L' \), is there always a finite \( \Delta \)-tribracket \( X \) such that

\[
\Phi_{\bar{X}}(L) \neq \Phi_{\bar{X}}(L')
\]

If so, how can we find the smallest such \( X \)? If not, identify under what conditions \( L \) and \( L' \) are not link-homotopic but have isomorphic fundamental \( \Delta \)-tribrackets.

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