Inertial endomorphisms of an abelian group

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Abstract

We describe inertial endomorphisms of an abelian group $A$, that is endomorphisms $\varphi$ with the property $|(\varphi(X) + X)/X| < \infty$ for each $X \leq A$. They form a ring containing multiplications, the so-called finitary endomorphisms and non-trivial instances. We show that inertial invertible endomorphisms form a group, provided $A$ has finite torsion-free rank. In any case, the group $IAut(A)$ they generate is commutative modulo the group $FAut(A)$ of finitary automorphisms, which is known to be locally finite. We deduce that $IAut(A)$ is locally-(center-by-finite). Also we consider the lattice dual property, that is $|X/(X \cap \varphi(X))| < \infty$ for each $X \leq A$. We show that this implies the above one, provided $A$ has finite torsion-free rank.

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1 Introduction and statement of main results

Recently there has been interest for totally inert (TIN) groups, i.e. groups whose all subgroups are inert (see [1], [4], [7], [10]). A subgroup is said inert if it is commensurable to each conjugate of its. Two subgroup $X,Y$ of any group are told commensurable iff $X \cap Y$ has finite index in both $X$ and $Y$ (see [11]).

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When dealing with soluble TIN-subgroups of groups one is concerned with automorphisms with the following property. As in [5] and [6], an endomorphism $\varphi$ of an abelian group $A$ (from now on always in additive notation) is said (right-) inertial iff:

$$(\text{RIN}) \quad \forall X \leq A \mid \varphi(X) + X /X \mid < \infty.$$  

Consideration of endomorphisms instead of automorphisms is due to the following.

**Fact** Inertial endomorphisms of any abelian group $A$ form a ring, say $IE(A)$, containing the ideal $F(A)$ of endomorphisms with finite image.

To prove this notice that if $\varphi_1$ and $\varphi_2$ both have RIN, then $\forall X \leq A \mid (X + \varphi_1(X))/X \mid < \infty$ and $\mid (X + \varphi_1(X) + \varphi_2(X) + \varphi_1\varphi_2(X))/(X + \varphi_1(X))\mid < \infty$.

In Theorem A below we give a rather satisfactory characterization of inertial endomorphisms of an abelian group, from which we deduce useful consequences. In Proposition A we exhibit non-trivial instances of inertial endomorphisms.

**Corollary A** The ring $IE(A)/F(A)$ is commutative.

In [3] we considered inertial automorphisms of an abelian group $A$ generalizing previous results from [2] and [8]. From results in [3] it follows that for an automorphism $\varphi$ of a periodic abelian group $A$ above (RIN) is equivalent to its lattice dual condition (left-inertial):

$$(\text{LIN}) \quad \forall X \leq A \mid X/(X \cap \varphi(X))\mid < \infty.$$  

Note that we will say just inertial for RIN. Endomorphisms which have both LIN and RIN are those mapping subgroups to commensurable ones. Notice that $x \mapsto 2x$ is an automorphism of $\mathbb{Q}^\omega$ with RIN but not LIN. Clearly, the inverse of an automorphism with RIN has LIN and vice-versa.

We consider the group $IAut(A)$ generated by inertial automorphisms of $A$. From Theorem A we have:

**Corollary B** Let $\varphi$ be an endomorphism of an abelian group $A$.

1) If $A$ has finite torsion-free rank, then LIN implies RIN and the two properties are equivalent if $\varphi$ is an automorphism. Thus inertial automorphisms form the group $IAut(A)$.

2) If $A$ has not finite torsion-free rank, then $IAut(A)$ is formed by the products $\gamma_1\gamma_2^{-1}$, where $\gamma_1, \gamma_2$ are both inertial automorphisms.

Recall that the rank $r_0(A)$ of any free abelian subgroup $F$ of $A$ such that $A/F$ is periodic is said torsion-free rank of $A$. From Proposition 7 we will have that when $r_0(A) = \infty$ an endomorphism is both LIN and RIN iff it acts as the identity or the inversion map on a subgroup with finite index.
Automorphisms acting as the identity map on a finite index subgroup form a group, say $\text{FAut}(A)$, which is locally finite (see [12]) and is contained in $\text{IAut}(A)$. Actually, $\text{IAut}(A)$ is not periodic but we have a corresponding statement, which also follows from Theorem A.

**Theorem B** Let $\Gamma = \text{IAut}(A)$ be the group generated by the inertial automorphisms of an abelian group $A$. Then:
1) $\Gamma' \leq \text{FAut}(A)$ is locally finite;
2) $\Gamma$ is locally central-by-finite.

Thus periodic elements of $\text{IAut}(A)$ form a subgroup containing the derived subgroup. However, there are non-elementary instances of periodic non-finitary inertial automorphisms. To see this consider the $p$-group $B \oplus D$ where ($p \not= 2$) $B$ is infinite bounded and $D$ is divisible with finite rank and the automorphism acting as the identity on $B$ and the inversion map on $D$. On the other side, the abelian group $\text{IAut}(A)/\text{FAut}(A)$ may be rather large as in next statement.

**Proposition A** There exists a countable abelian group $A$ with $r_0(A) = 1$ such that $\text{IAut}(A)$ has a subgroup $\Sigma \simeq \prod_p \mathbb{Z}(p)$ with $\Sigma \cap \text{FAut}(A) = T(\Sigma) \simeq \bigoplus_p \mathbb{Z}(p)$, where $p$ ranges over the set of all primes.

We state now the main result of the paper. By multiplication of an abelian group $A$ we mean the (componentwise) action by $p$-adics, if the group is periodic; otherwise we mean the natural action by rational numbers (when possible and uniquely defined). Multiplications form a ring and commute with any endomorphism, clearly, and are often inertial, as in Proposition B.

**Theorem A** Let $\varphi_1, \ldots, \varphi_t$ finitely many endomorphisms of an abelian group $A$. Then each $\varphi_i$ is inertial if and only if there is a finite index subgroup $A_0$ of $A$ such that (a) or (b) holds:

(a) each $\varphi_i$ acts as multiplication by $m_i \in \mathbb{Z}$ on $A_0$;

(b) $A_0 = B \oplus D \oplus C$ and there exist finite sets of primes $\pi \subseteq \pi_1$ such that:
   i) $B \oplus D$ is the $\pi_1$-component of $A_0$ where $B$ is bounded and $D$ is a divisible $\pi'$-group with finite rank,
   ii) $C$ is a $\mathbb{Q}^r[\varphi_1, \ldots, \varphi_t]$-module, with a submodule $V \simeq \mathbb{Q}^r \oplus \cdots \oplus \mathbb{Q}^s$ (finitely many times) such that $C/V$ is a $\pi_1$-divisible $\pi'$-group,
   iii) each $\varphi_i$ acts by (possibly different) multiplications on $B, D, V, C/V,$
   iv) each $\varphi_i = \frac{m_i}{n_i} \in \mathbb{Q}$ on $V$ and on all $p$-components of $D$ such that the $p$-component of $C/V$ is infinite and $\pi = \pi(n_1 \cdot \cdots \cdot n_t)$. 

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Notice that when \( A \) is periodic, the above statement applies with \( V = 0 \) and amounts to Proposition 5 below. On the other side, if \( A \) is torsion-free, then inertial endomorphisms are just multiplications, see Proposition 2. Further, for details on LIN condition see Propositions 5 and 7.

2 Preliminaries and Terminology

As a standard reference on abelian groups we use [9]. Letter \( A \) always denote an abelian group, we regard as a left \( E(A) \)-module, where \( E(A) \) denotes the ring of endomorphism of \( A \). Letters \( \varphi, \gamma \) denote endomorphisms of \( A \), while \( m, n, r, s, t \) denote integers, \( p \) always a prime, \( \pi \) a set of primes, \( \pi(n) \) the set of prime divisors of \( n \). We denote: by \( T = T(A) \) the torsion subgroup of \( A \), by \( A_\pi \) the \( \pi \)-component of \( A \), by \( A[n] := \{ a \in A \mid na = 0 \} \). If \( nA = 0 \), we say that \( A \) is bounded by \( n \). Further, we say that \( A \) is bounded, if it is bounded by some \( n \). Denote by \( D = \text{Div}(A) \) the largest divisible subgroup of \( A \).

For a subset \( X \) and an endomorphism \( \varphi \) of a (left) \( R \)-module \( A \), we denote by \( \langle X \rangle \) (resp. \( X^{(\varphi)} = R[\varphi]X \)) the additive subgroup (resp. the \( R[\varphi] \)-submodule) spanned by \( X \), as usual. Note that, if \( X \) is a \( R \)-submodule, then by \( X^{(\varphi)} \) we mean the largest \( R[\varphi] \)-submodule contained in \( X \).

When \( A \) is \( \pi \)-divisible (that is \( pA = A \) for each \( p \in \pi \)) and \( A_\pi = 0 \), we regard it as a \( \mathbb{Q}_\pi \)-module and write \( \varphi = m_n \in \mathbb{Q}_\pi \) for the well-defined map via \( \varphi(nx) = mx \). As usual \( \mathbb{Q}_\pi \) denote the ring of rationals whose denominator is \( \pi \)-number, where \( \mathbb{Q}_\pi^{(n)} := \mathbb{Z}[\frac{1}{n}] \).

We call multiplications of an abelian group \( A \) either the above actions of elements of \( \frac{m}{n} \in \mathbb{Q} \) or, when \( A \) is periodic, the componentwise actions of the elements of the cartesian product of the rings \( \mathbb{Q}_p^* \) of \( p \)-adics for each prime \( p \). Recall that \( \mathbb{Q}_p^* \) contains the ring of rationals whose denominator is coprime to \( p \). Note that we use word “multiplication” in a way different from [9]. Ours are in fact “scalar” multiplications.

When \( A \) is periodic, multiplications are precisely the so-called power endomorphisms which are defined by \( \forall X \subseteq A \varphi(X) \subseteq X \) (see [10]). On the other hand, when \( A \) is not periodic, power endomorphisms are multiplication \( x \mapsto nx \) for fixed \( n \in \mathbb{Z} \).

Let us point out which multiplications have RIN or LIN. Recall that an abelian group \( A \) with the minimal condition (Min) is just a group of the shape \( A = F \oplus D \), where \( F \) is finite and \( D \) is divisible with finite total rank that is the sum of finitely many infinite cocyclic (Prüfer) groups. For short we will write \( A \) has \( \text{FTFR} \) when the torsion-free rank \( r_0(A) \) is finite.
Proposition 1  Let \( \varphi \) be a multiplication of an infinite abelian group \( A \), then

R) \( \varphi \) is inertial iff either \( A \) has FTFR or \( \varphi \) is a multiplication by an integer;
L1) if \( A \) is a \( p \)-group, then \( \varphi \) is LIN iff \( \varphi \) is invertible or \( A \) has Min and \( \varphi \neq 0 \);
L2) if \( 0 < r_0(A) < \infty \), then \( m/n \) is LIN iff \( A_{\pi(m)} \) has Min and \( \varphi \neq 0 \) \((m,n \text{ coprime})\);
L3) if \( r_0(A) = \infty \), then \( \varphi \) is LIN iff \( \varphi = 1/n \).

Proof.  Let us prove part (R). If \( A \) is periodic, the statement is trivial. Otherwise, let \( \varphi = m/n \). Now if \( \varphi \) is inertial, then, arguing in \( \bar{A} := A/T(A) \), we have that for any \( \bar{X} \) free with infinite rank, it results that \((\varphi(\bar{X}) + \bar{X})/\bar{X} \) is infinite.

Conversely, if \( \varphi = m \) acts as an integer, it is trivially inertial. Assume \( A \) has FTFR. For any \( X \leq A \), the section \((\varphi(X) + X)/X \) is a bounded \( \pi(m) \)-group and is finite mod \( T := T(A) \). On the other hand, since \( A \) is a \( \mathbb{Q}^{\pi(m)} \)-module, \( A_{\pi(m)} = 0 \) and therefore \((\varphi(X) + X)/X \) avoids \( T \).

If \( \varphi \) is LIN, then \( A/\varphi(A) < \infty \) implies that \( \varphi \neq 0 \). Further we have:

L1) Let the \( p \)-adic \( \alpha = p^s\alpha_1 \) represent \( \varphi \) on \( A \) with \( \alpha_1 \) invertible. If \( \varphi \) is not invertible, then \( s > 0 \) and \( A[p] \leq ker\varphi \) is finite. Hence \( A \) has Min. Conversely, if \( A \) has Min, then for any \( X \leq A \) we have that \( X/\varphi(X) = X/p^s X \) is finite.

L2) If \( \varphi = m/n \) then \( A_{\pi(m)} \) has Min by (L1). Conversely, for each \( X \leq A \) we have
\[
\frac{X}{X \cap mX} \simeq \frac{nX}{nX \cap mX}
\]
if finite as bounded by \( m \) and the rank of \( A_{\pi(m)} \) and torsion-free rank of \( A \) are finite.

L3) Let \( \varphi = m/n \) and take \( \bar{X} \leq \bar{A} := A/T \) free of infinite rank. As above \( \bar{X}/(\bar{X} \cap m\bar{X}) \) is infinite unless \( |m| = 1 \). On the other hand \( \varphi = 1/n \) is LIN as \( X \leq \varphi(X) \) for each \( X \leq A \).

The other way round, let us see that inertial endomorphisms of a torsion-free abelian group are all multiplications and there is a natural ring monomorphism \( IE(A) \hookrightarrow \mathbb{Q} \).
Proposition 2 Let \( \varphi \) be an endomorphism of a torsion-free abelian group \( A \).

R) \( \varphi \) is RIN iff \( \varphi \) is multiplication by \( \frac{m}{n} \) where if \( n \neq \pm 1 \) then \( r_0(A) < \infty \).

L) \( \varphi \) is LIN iff \( \varphi \) is multiplication by \( \frac{m}{n} \) with \( m \neq 0 \) and if \( m \neq \pm 1 \) then \( r_0(A) < \infty \).

In particular, if \( \varphi \neq 0 \) and \( r_0(A) < \infty \), \( \varphi \) is LIN iff \( \varphi \) is RIN. In the above statement \( m \) and \( n \) are meant to be coprime integers.

Proof. The sufficiency of the condition is clear in both cases (R) and (L).

To prove necessity, we generalize an argument used in [3]. In both cases (R) and (L), if \( a \in A \) there exist \( m, n \in \mathbb{Z} \) such that \( ma = n \varphi(a) \). As \( A \) is torsion-free, \( m, n \) can be chosen coprime. Let us show that \( \frac{m}{n} \) is independent of \( a \). Let \( a_1 \in A \). If \( \langle a_1 \rangle \cap \langle a \rangle \neq \{0\} \), then \( ka_1 = ha \) for some \( h, k \in \mathbb{Z} \). Therefore we may write \( \varphi(a_1) = \frac{h}{k} \varphi(a) = \frac{h}{k} \frac{m}{n} a = \frac{m}{n} a_1 \). Similarly, if \( \langle a_1 \rangle \cap \langle a \rangle = \{0\} \), there will exist \( m_1, m_2, n_1, n_2 \in \mathbb{Z} \) such that

\[
\frac{m}{n} a + \frac{m_1}{n_1} a_1 = \varphi(a) + \varphi(a_1) = \varphi(a + a_1) = \frac{m_2}{n_2} (a + a_1) = \frac{m_2}{n_2} a + \frac{m_2}{n_2} a_1.
\]

It follows \( \frac{m}{n} = \frac{m_2}{n_2} = \frac{m_1}{n_1} \). Thus \( \varphi \) is a multiplication.

For the rank restriction, apply Proposition 1.

Note a sufficient condition for an endomorphism to be inertial. We omit the straightforward proof.

Proposition 3 Let \( \varphi \) be an endomorphism of an abelian group \( A \). If \( \varphi \) acts as an inertial (resp. LIN) endomorphism either on a finite index subgroup of \( A \) or modulo a finite subgroup, then \( \varphi \) is inertial (resp. LIN) on the whole \( A \) indeed.

We will use often the following fact.

Proposition 4 For an endomorphism \( \varphi \) of a periodic abelian group \( A \) the following are equivalent:

MF) \( \varphi \) acts as a multiplication on a finite index subgroup \( A_0 \) of \( A \),

FM) \( \varphi \) acts as a multiplication modulo a finite subgroup \( A_1 \) of \( A \).

Proof. This is very easy. If (MF) holds and \( \varphi = \alpha \in J \) on \( A_0 \), where \( J \) is the cartesian product of the rings \( \mathbb{Q}_p^* \) of the \( p \)-adics for each prime \( p \). Then consider the natural action of \( \alpha \) on \( A \). Thus \( A_1 := im(\varphi - \alpha) \) is an epimorphic image of \( A/A_0 \). The converse is similar.
3 Inertial endomorphisms of a periodic abelian group

This section is devoted to prove the periodic case of the main Theorem A of the paper. In fact we prove a more detailed characterization of inertial endomorphisms of torsion abelian groups. We handle LIN-endomorphisms as well.

Proposition 5 Let \( \varphi_1, \ldots, \varphi_t \) be finitely many endomorphisms of an abelian periodic group \( A \). Then:

R) each \( \varphi_i \) is inertial iff there is a finite index subgroup \( A_0 = B \oplus D \oplus C \) of \( A \) such that:
   i) \( B \oplus D \) and \( C \) are coprime,
   ii) \( B \) is bounded and \( D \) is divisible with \( \text{Min} \),
   iii) each \( \varphi_i \) is multiplication on \( B \), \( D \) and \( C \).

If the above holds and \( \Phi := \mathbb{Z}[\varphi_1, \ldots, \varphi_t] \), then (FS) \( \exists m \forall X \leq A \) \( |X^\Phi/X_\Phi| \leq m \).

L) each \( \varphi_i \) is LIN iff it is inertial and there are subgroups \( A_0, B, D, C \) as above such that \( \varphi_i \) is a non-zero multiplication on each non-zero primary component of \( D \) and an invertible multiplication on \( B \) and \( C \).

Recall that, for each \( i \), \( X^\Phi \) and \( X_\Phi \) are \( \varphi_i \)-invariant and \( X^\Phi \leq X \leq X_\Phi \).

Before proving the Proposition we state the following easy but fundamental fact, which reduces the proof to the case \( A \) is a \( p \)-group.

Proposition 6 An endomorphism of an abelian torsion group \( A \) is inertial (resp. LIN) iff it is such on all primary components and multiplication (resp. invertible multiplication) on all but finitely many of them.

Proof. It is trivial that the condition is sufficient. Concerning necessity, we only deal with case LIN, the case RIN being similar. Let \( \pi \) be the set of primes \( p \) such that \( \varphi \) is not invertible multiplication on \( A_p \). If \( p \in \pi \), then either \( \varphi \) is not a multiplication on \( A_p \) or \( \varphi \) is a non-invertible multiplication. In the former case there is a cyclic subgroup \( X_p \) of \( A_p \) such that \( \varphi(X_p) \not\subseteq X_p \), and hence \( |X_p \cap \varphi(X_p)| < |\varphi(X_p)| \leq |X_p| \). In the latter case there is a cyclic subgroup \( X_p \) of \( A_p \) such that \( \varphi(X_p) \) is properly contained in \( X_p \). In both cases \( |X_p/(X_p \cap \varphi(X_p))| > 1 \). It is now clear that if \( \varphi \) is LIN, then \( \pi \) is finite, as \( |X/(X \cap \varphi(X))| \) must be finite for \( X := \bigoplus_{p \in \pi} X_p \). \( \square \)

We prove a couple of Lemmas. The first one extends Proposition 4.3 in [10].
Lemma 1 Let $A$ be an abelian $p$-group, $a \in A$ and $\varphi \in E(A)$.
1) If $\varphi$ either RIN or LIN, then the cyclic $\varphi$-submodule $\mathbb{Z}[\varphi](a) := \langle a \rangle^{(\varphi)}$ of $A$ generated by $a$ is finite.
2) If $|X/X_{(\varphi)}| < \infty$ for all $X \leq A$, then $|X^{(\varphi)}/X| < \infty$ for all $X \leq A$.
3) If $|X/X_{(\varphi)}| \leq p^m$ for all $X \leq A$, then $|X^{(\varphi)}/X| \leq p^{m^2}$ for all $X \leq A$.

Proof. (1) We may assume $A = \langle a \rangle^{(\varphi)}$. Suppose first $a$ has order prime $p$ and consider the natural epimorphism of $\mathbb{Z}_p[x]$-modules mapping 1 to $a$ and $x$ to $\varphi(a)$ (regard $A$ as $\mathbb{Z}_p[x]$-module where $x$ acts as $\varphi$):

$$F : \mathbb{Z}_p[x] \to A.$$ 

If $F$ is injective, we can replace $A$ by $\mathbb{Z}_p[x]$ and $\varphi$ by multiplication by $x$. If $H := \mathbb{Z}_p[x^2]$, then $\varphi(H) = xH$ is infinite, while $H \cap xH = 0$, a contradiction. Then $F$ is not injective and $A$ is finite as it is isomorphic to a proper quotient of $\mathbb{Z}_p[x]$. If now $a$ has (any) order $p^k$, then $A/pA$ is finite by the above. Moreover, $pA = \langle pa \rangle^{(\varphi)}$ is finite by induction on $k$.

(2) This can be proved in a similar way as case (3)

(3) We claim that if $a \in A$ has order $p^k$, then $|\langle a \rangle^{(\varphi)}| \leq p^{(m+1)\epsilon}$.
Assume first $\epsilon = 1$, that is $a$ has order $p$ and $A_0 := \langle a \rangle^{(\varphi)}$ is elementary abelian. Suppose, by contradiction, the above $F$ is injective. As above, let $H := \mathbb{Z}_p[x^2]$. Then $H_{(\varphi)} = (g(x^2))$ for some polynomial $g$. Since $|H/H_{(\varphi)}| = p^m < \infty$, we have $g \neq 0$. Then $(g(x^2)) \nsubseteq H$, a contradiction. Therefore, for some $f \in \mathbb{Z}_p[x]$ with degree say $n$, we have

$$\frac{\mathbb{Z}_p[x]}{(f)} \simeq _{\varphi} \langle a \rangle^{(\varphi)} = A_0.$$ 

Thus the minimal $\varphi$-invariant subgroups of $A_0$ correspond 1 to 1 to the irreducible monic factors of $f$, which are at most $n$. Consider a $\mathbb{Z}_p$-basis $X$ of $A$ containing an element in each subgroup of them. The the hyperplane $H$ of equation $x_1 + x_2 + \cdots + x_n = 0$ has index $p$ in $\langle a \rangle^{(\varphi)}$ and $H_{(\varphi)} = 0$ as $H \cap X = \emptyset$. Therefore $|\langle a \rangle^{(\varphi)}| \leq p^{m+1}$.

If $\epsilon > 1$, by induction $B := \langle p^{\epsilon-1}a \rangle^{(\varphi)}$ has order at most $p^{(m+1)(\epsilon-1)}$ and $\langle a \rangle^{(\gamma)}/B$ has order at most $p^{m+1}$ by case $\epsilon = 1$. Therefore $|\langle a \rangle^{(\varphi)}| \leq p^{(m+1)\epsilon}$, as claimed.

In the general case let $X$ be any subgroup of $A$ and $X_{(\varphi)} = 0$. Thus $|X| =: p^e \leq p^m$. Write $X = \langle a_1 \rangle \oplus \cdots \oplus \langle a_r \rangle$ with $a_i$ of order $p^{\epsilon_i}$ and $\epsilon_1 + \cdots + \epsilon_r = \epsilon$. Since $|\langle a_i \rangle^{(\varphi)}| \leq p^{(m+1)\epsilon_i}$ by the above, we have $|X^{(\varphi)}| \leq p^{(m+1)\epsilon}$. So that $|X^{(\varphi)}/X| \leq p^{(m+1)\epsilon - \epsilon} \leq p^{m^2}$. \qed
Lemma 2 Let $D$ be a divisible periodic subgroup of an abelian subgroup $A$ and $\varphi \in E(A)$. If $\varphi$ is either RIN or LIN, then $\varphi$ is multiplication on $D$.

Proof. Without loss of generality, we may assume $D$ and $\varphi(D)$ are Prüfer groups. If $\varphi$ is LIN, then $D \leq \varphi(D)$ and thus $D = \varphi(D)$. Therefore in both cases RIN or LIN, we have $\varphi(D) \leq D$. □

Proof of Proposition 5 Note that $(FS)$ implies trivially that each $\varphi_i$ is inertial. By Proposition 6 we may assume $A$ is a $p$-group (see details below). We proceed by a sequence of claims and start by considering the case of a single endomorphism $\varphi := \varphi_1$.

• If $A$ is elementary abelian and $\varphi$ is either inertial or LIN, then $\varphi$ is FM. Assume by contradiction $\varphi$ is not FM (as in Proposition 4). We prove that: for any finite subgroup $X \leq A$ such that $X \cap \varphi(X) = 0$ there exists a finite subgroup $X' > X$ such that $X' \cap \varphi(X') = 0$ and $\varphi(X') > \varphi(X)$.

Therefore, starting at $X_0 = 0$, by recursion we define $X_{i+1} := X_i'$ and $X_\omega := \bigcup_i X_i$. We get that both $X_\omega$ and $\varphi(X_\omega)$ are infinite and $X_\omega \cap \varphi(X_\omega) = 0$, a contradiction.

First note that $Z := X^{(e)}$ is finite by Lemma 1. Since $\varphi$ does not act as a multiplication on $A/Z$, we can choose $a \in A$ such that $\varphi(a) \not\in \langle a, Z \rangle$ and define $X' := \langle a \rangle + X$. If now $y \in X' \cap \varphi(X')$, then $\exists m, n \in \mathbb{Z}_p$, $\exists x, x_0 \in X$ such that $y = ma + x = n\varphi(a) + \varphi(x_0)$. Thus $n\varphi(a) \in \langle a \rangle + Z$ while $\varphi(a) \not\in \langle a \rangle + Z$. Therefore $n = 0$ and $ma = 0$ as well. It follows $y = x = \varphi(x_0) \in X \cap \varphi(X) = 0$, as claimed.

• If $\varphi$ is LIN then it is inertial, as in both cases it holds:

\[(fs) \quad \forall X \leq A \quad |X^{(e)}/X(X)| < \infty.\]

If $\varphi$ is either inertial or LIN, to show $(fs)$, we may assume $X^{(e)} = 0$. Thus, since $\varphi$ is multiplication on $D_1 := Div(A)$, (see Lemma 2), we have $D_1 \cap X = 0$ and $X$ is reduced. Moreover, by the elementary abelian case, $\varphi$ is multiplication on a subgroup of finite index of $A[p]$ and we get that $X[p]$ is finite. It follows that $X$ is finite. Then $(fs)$ holds by Lemma 1.

• If $\varphi$ is inertial but not FM, then $A$ is critical, that is $D := Div(A)$ has Min and $A/D$ is infinite but bounded. This will be proved by the following steps.
I. \textit{A is not residually finite.} Assume by contradiction it is. As above, we note that if \( \varphi \) is inertial, then \textit{there is no sequence of subgroups} \( X_i \) with the property that if we denote \( Y_i := X_i \cap \varphi(X_i) \) then we have:

1. \( Y_{i+1} \cap X_i = Y_i \)
2. the sequence \( |\varphi(X_i)/Y_i| \) is strictly increasing.

Otherwise there would exists a subgroup \( X_\omega := \cup_i X_i \) with the properties that \( |\varphi(X_\omega)/X_\omega \cap \varphi(X_\omega)| \geq |\varphi(X_i)/Y_i| \geq i \) for each \( i \).

On the other hand, we will construct a prohibited sequence \( X_i \), getting the wished contradiction. Let \( X \) be any finite subgroup of \( A \). By Lemma 1 the subgroup \( K := X^{(\varphi)} \) is finite. Since \( A \) is residually finite, by (fs) there is a \( \varphi \)-subgroup \( A_* \) with finite index in \( A \) such that \( A_* \cap K = 0 \). Now, as \( \varphi \) is not multiplication on \( (A_* + K)/K \), there is \( a \in A_* \) such that \( \varphi(a) \notin \langle a, K \rangle \).

Let \( Y := X \cap \varphi(X) \), \( X' := \langle a \rangle + X \) and \( Y' := X' \cap \varphi(X') \). Let us check that

1. \( \varphi(X) \cap Y' = Y' \)
2. \( \varphi(X') > \varphi(X) + Y' \).

In fact, to prove (1), if \( \varphi(x) \in \varphi(X) \cap Y' \) where \( x \in X \), then \( \varphi(x) = ma + x_0 \) with \( m \in \mathbb{Z}, x_0 \in X \) and \( ma = \varphi(x) - x_0 \in A_* \cap K = 0 \), hence \( \varphi(x) = x_0 \in Y \) and (1) holds.

To prove (2'), note that if \( y' \in Y' = X' \cap \varphi(X') \), then \( \exists n, m \in \mathbb{Z}, \exists x, x_0 \in X \) such that \( y' = ma + x = n\varphi(a) + \varphi(x_0) \). Then \( ma - n\varphi(a) \in A_* \cap K = 0 \). Hence \( x = \varphi(x_0) \in Y := X \cap \varphi(X) \). Since \( \varphi(a) \notin \langle a, K \rangle \), \( p \) divides \( s \), and so \( Y' \leq \langle p\varphi(a) \rangle + Y \not
\neq \varphi(a) \). Therefore (2') holds as \( \varphi(a) \in \varphi(X') \setminus \langle p\varphi(a) \rangle + Y \).

Thus we can define by induction a prohibited sequence as above, since from (1) and (2') it follows \( |\varphi(X')/Y'| > |\varphi(X)/Y| \) and we get a contradiction.

II. \textit{A is not reduced.} Otherwise, let \( R \) be a basic subgroup of \( A \). By (fs), \( R^{(\varphi)}/R \) is finite and so \( H := R^{(\varphi)} \) is residually finite as well. Also, \( A/H \) is divisible. By the above there are a \( p \)-adic \( \alpha \) and a finite \( \varphi \)-invariant subgroup \( A_1 \) of \( H \) such that \( \varphi = \alpha \) on \( H/A_1 \). As the kernel \( K/A_1 \) of \( (\varphi - \alpha)|_{A/A_1} \) contains \( H/A_1 \) and its image is reduced, while \( A/H \) is divisible, it is clear that \( K = A \) and \( \varphi = \alpha \) on \( A/A_1 \), the wished contradiction.

III. \textit{A is critical.} Let \( A = B \oplus D \) with \( B \) infinite but reduced, \( D = \text{Div}(A) \) divisible. As (fs) holds at the expense of substituting \( A \) with a finite index \( \varphi \)-invariant subgroup \( A_0 \) we may assume that \( B \) and \( D \) are both \( \varphi \)-invariant. Further, by the above and Lemma 2 \( \varphi \) is multiplication on both \( D \) and on a finite index subgroup of \( B \). So we may also assume \( \varphi \) is multiplication on \( B \). Let \( \varphi \) act on \( B \) and \( D \) by means of \( p \)-adics \( \alpha_1, \alpha_2 \), resp. As we assumed \( \varphi \) is not FM, we have \( \alpha_1 \) and \( \alpha_2 \) act differently on \( B \).
If by contradiction $B$ is unbounded, then there is a quotient $B/S \simeq \mathbb{Z}(p^\infty)$. By (fs) we can assume $S$ to be $\varphi$-invariant and consider $A := A/S$. This a divisible group on which $\varphi$ acts as a (universal) multiplication by Lemma \([2]\) contradicting the assumption on $\alpha_1$ and $\alpha_2$. So that $B$ is bounded.

If by contradiction $D$ has infinite rank, we may substitute $B$ by $B[p^e]$ where $e$ is the smallest such that $B/B[p^e]$ is finite. By the reduced case above, $\varphi$ is multiplication on a subgroup $A_\ast$ of finite index of $A[p^e]$. Then if $D$ has infinite rank, $\alpha_1 \equiv \alpha_2 \mod p^e$ and $\varphi$ is multiplication on $(B \cap A_\ast) \oplus D$ which has finite index in $A$, a contradiction.

- For finitely many inertial endomorphisms $\varphi_i$ there are subgroups $A_0$, $B$, $C$, $D$ as in the statement. Note that if all $\varphi_i$'s are FM, then clearly there is finite index subgroup $C$ of $A$ such that all $\varphi_i$'s are multiplication on $C$. Otherwise $A = B_0 \oplus D$ is critical, with $B_0$ bounded and $D$ divisible with finite rank. As above, for each $i$ there is a finite index subgroup $B_i$ of $B_0$ such that $\varphi_i$ is multiplication on $B_i$. Let $B := \cap_i B_i$. Then $A_0 := B + D$ is the wished subgroup as in the statement.

- \((FS)\) holds, if (i), (ii) and (iii) hold. If each $\varphi_i$ is FM, then it is multiplication (by the $p$-adic $\alpha_i$) on a subgroup with finite index $A_i$ and modulo a subgroup with finite order $F_i := \text{im}(\varphi_i - \alpha_i)$. Then all $\varphi_i$ are multiplication on the finite index subgroup $A_0 = \cap_i A_i$ and modulo the finite subgroup $F_0 := F_1 \oplus \ldots \oplus F_i$. Therefore for each $X \leq A$ we have that $X \cap A_0$ and $X + F$ are $\varphi_i$-invariant for each $i$.

If some $\varphi_i$ is not FM, then $A$ is critical. By the above, there is a finite index subgroup $A_0 = B \oplus D$ of $A$ such that each $\varphi_i$ is multiplication on $B$ and $D$, where $B$ is bounded and $D := \text{Div}(A)$ divisible with finite rank.

Let $X_0 := X \cap A_0$. Then $|X/X_0| \leq |A/A_0|$ is finite. Further, $X_* := \langle D \cap X \rangle + \langle B \cap X \rangle$ is $\varphi_i$-invariant and the group $X_0/X_*$ is bounded as $B$ is. Also $X_0/X_*$ has rank $r$ at most the rank of $D$, hence $X_0/X_*$ is finite. Thus $X/X_0$ is finite. Hence each $\varphi_i$ is inertial.

To show that $X^{\Phi}/X$ is finite as well, we can assume $X^{\Phi} = 0$ and that $X$ is finite. Hence there is $m$ such that $X \leq A[m]$. By the above each $\varphi_i$ is FM on $A[m]$, which is not critical, and $X^{\Phi}$ is finite.

- \((L)\) holds. If $A$ is a $p$-group, notice that we have already seen that LIN implies (fs) and therefore RIN. Thus there are subgroups $A_0$, $B$, $D$, $C$ as in part (R) of the statement. If $B \neq 0$ (hence $C = 0$) we can assume $B$ is infinite and, by Proposition \([11, L1]\), each $\varphi_i$ is invertible on $B$ and $\varphi_i \neq 0$ on $D$ (if $D \neq 0$). On the other hand, if some $\varphi_i$ is not invertible on $C \neq 0$ (hence $B \oplus D = 0$), then $C$ has Min and we can put $A_0 := \text{Div}(C)$ and the statement holds.
In the general case apply Proposition 6 and deduce that LIN implies RIN since this is true on the $p$-components. Suppose each $\varphi_i$ is LIN. Then the set $\pi$ of primes such that some $\varphi_i$ is not an invertible multiplication on $A_p$ is finite, by Proposition 6 again. Then for any $p \in \pi$, if the $p$-component $A_p$ is either non-critical or non-Min, then there is a subgroup $C_p$ with finite index in $A_p$ on which $\varphi_i$ is invertible multiplication (see Proposition [L1]). Otherwise there is a subgroup with finite index in $A_p$ of the shape $B_p \oplus D_p$, where $B_p$ is bounded, $D_p$ is divisible of finite rank, such that each $\varphi_i$ is invertible multiplication on $B_p$ and non-zero multiplication on $D_p$ (if $D_p \neq 0$). Then the statement holds with $B := \oplus_{p \in \pi} B_p$, $D := \oplus_{p \in \pi} D_p$ and $C := A_\pi' \oplus \bigoplus_{p \in \pi} C_p$.

Conversely, arguing componentwise, it is enough to show that each $\varphi_i$ is LIN on $B \oplus D$ as in the statement. Then for each $X \leq B \oplus D$ we have $\varphi(X \cap B) = X \cap B$ and $\varphi(X \cap D)$ has finite index in $X \cap D$. Then $X/X_\ast$ and $X_\ast /X_\ast \cap \varphi(X_\ast)$ are finite, where $X_\ast := (X \cap B) + (X \cap D)$.

4 Inertial endomorphisms of a non-periodic abelian group

In order to prove Theorem A, we give a detailed description of inertial and LIN endomorphisms of a non-periodic abelian group. We generalize results from [3] to the more general settings of endomorphisms.

Proposition 7 An endomorphism $\varphi$ of an abelian non-periodic group $A$ is inertial (resp. LIN) if and only if one of the following holds:

(a) there is a $\varphi$-invariant finite index subgroup $A_0$ of $A$ on which $\varphi$ acts as multiplication by $m \in \mathbb{Z}$ (resp. as multiplication by $\frac{1}{n}$, with $n \neq 0$);

(b) there are finitely many elements $a_i$ such that:
   i) the $\varphi$-submodule $V = \mathbb{Z}[\varphi]\langle a_1, \ldots, a_r \rangle$ is torsion-free as an abelian group and $\varphi$ induces on $V$ multiplication by $\frac{m}{n}$ where $m, n$ are coprime integers (resp. $m \neq 0$),
   ii) the factor group $A/V$ is torsion and $\varphi$ induces an inertial (resp. LIN) endomorphism on $A/V$,
   iii) the $\pi(n)$-component of $A$ is bounded.

Notice that $\varphi = \frac{m}{n}$ on both $A/T$ and $V$. Moreover, in case (b) we have $V \simeq \mathbb{Q}^{(n)} \oplus \ldots \oplus \mathbb{Q}^{(n)}$ (finitely many times).
For the proof of the Proposition we need some preliminary result. In the first Lemma we state some well-known facts.

**Lemma 3** Let $A_1$ be a subgroup of an abelian group $A$ and $\pi$ a set of primes.  
1) If $A/A_1$ is a $\pi'$-group, then $A$ is $\pi$-divisible iff $A_1$ is $\pi$-divisible.  
2) If $A$ is torsion-free, $A/A_1$ periodic and $A_1$ is $\pi$-divisible then $A/A_1$ is a $\pi'$-group and $A$ is $\pi$-divisible.  
3) If $A_1$ is torsion-free and $\pi$-divisible while $A/A_1$ is $\pi'$-group, then multiplication by a $\pi$-number is invertible. □

Next Lemma is a generalization of Lemma 1 to non-periodic groups.

**Lemma 4** Let $A$ be an abelian group, $a \in A$ and $\varphi \in E(A)$. If $\varphi$ is either RIN or LIN, then the torsion subgroup $T$ of $\mathbb{Z}[\varphi](a) =: \langle a \rangle^{(\varphi)}$ is finite.

**Proof.** We may assume $A = \langle a \rangle^{(\varphi)}$. If $a$ has finite order, apply Lemma 1(1). Assume $a$ is aperiodic. By Proposition 2, $\varphi = \frac{m}{n}$ on $A/T$ ($m, n$ coprime), that is $(n\varphi - m)(a)$ is periodic. Regard $A$ as $\mathbb{Z}[x]$-module (where $x$ acts as $\varphi$) and consider the natural epimorphism mapping 1 to $a$ and $x$ to $\varphi(a)$:

$$F : \mathbb{Z}[x] \rightarrow A.$$ 

Let $I$ be the inverse image of $T$ via $F$. Then $(nx - m) \subseteq I$ and $\mathbb{Z}[x]/I \simeq A/T$ is torsion-free (as $\mathbb{Z}$-module). Since proper quotients of $\mathbb{Z}[x]/(nx - m) \simeq \mathbb{Z}[1/n] = \mathbb{Q}(n)$ are periodic, then $I = (nx - m)$. Applying $F$ we get that $T = \mathbb{Z}[\varphi]\langle (m\varphi - n)\langle a \rangle\rangle$ is a cyclic $\varphi$-submodule. It is finite by Lemma 1. □

**Proof of Proposition 7** Assume that $\varphi$ is either RIN or LIN. By Proposition 2, $\varphi$ is multiplication by $\frac{m}{n} \in \mathbb{Q}$ on $A/T$, where $T := T(A)$ and $m, n$ are coprime. Let $\pi := \pi(n)$. We proceed by a sequence of claims.

• There is a free abelian $F \leq A$ such that $V := \mathbb{Z}[\varphi]F$ is torsion-free and $A/V$ is periodic. In fact, by Zorn’s Lemma, there is a subset $S$ of $A$ which is maximal with respect to “$F := \langle S \rangle$ is free abelian on $S$ and $V := \mathbb{Z}[\varphi]\langle S \rangle$ is torsion-free”. It follows that $A/V$ is periodic. If not, there is an aperiodic $a \in A$ such that $\langle a \rangle \cap V = 0$. By Lemma 1 the torsion subgroup of $\mathbb{Z}[\varphi]\langle a \rangle$ has finite order $s$. Thus $\mathbb{Z}[\varphi]\langle a^s \rangle \simeq \mathbb{Q}(n)$ has rank 1 (see Proposition 2) and $\{a^s\} \cup S$ has the above properties instead of $S$, a contradiction. Then $\varphi = \frac{m}{n}$ on the torsion-free subgroup $V$ and $A/V$ is periodic.
If \( A \) as in the first claim above. By Proposition 2, \( A/V \) is unbounded. Then \( p \) the endomorphism \( \alpha \) of \( Z(p^\infty) \oplus Q^\pi \) and it is enough to check that:

- the endomorphism \( \varphi = \alpha \oplus \frac{m}{n} \) of \( Z(p^\infty) \oplus Q^\pi \) is neither RIN nor LIN (\( \alpha \) a \( p \)-adic). To this aim consider the “diagonal” subgroup

\[
H = \{ [tp^{-i}] \oplus tp^{-i} \mid i \in N, t \in Z, [tp^{-i}] \in Q(p)/Z \}
\]

and note \((m - \nu \varphi)([p^{-i}] \oplus p^{-i}) = (m - \nu \alpha)[p^{-i}] \oplus 0 \). Since \( p \) does not divide \( m \), we have \( Z(p^\infty) = (m - \nu \varphi)(H) \). Therefore \( H + \varphi(H) \) is commensurable neither to \( H \) nor to \( \varphi(H) \), the wished contradiction.

- If \( r_0(A) \) is finite, then (b) holds. This is now clear.

- If \( A \) has not FTFR and \( \varphi \) is inertial, then (a) holds. By Proposition 2 we know that \( \varphi = m \in Z \) on \( V = F \) as above, which is a free abelian group on the basis \( S \). So there exists a (\( \varphi \)-invariant) subgroup \( W \) such that \( V/W \) is a periodic group whose \( p \)-component is divisible with infinite rank for each prime \( p \). By the torsion case, Proposition 5, \( \varphi \) is FM on \( A/W \), since it contains a divisible \( p \)-group with infinite rank for each prime \( p \). Without loss of generality, we can assume \( \varphi \) is a multiplication indeed on \( A/W \) and, for each \( p, \varphi \) is represented by the \( p \)-adic \( \alpha_p \) on the \( p \)-component of \( A/W \). Then \( \alpha_p = m \), as they must agree on the \( p \)-component of \( V/W \), which it is unbounded. Then \( W \geq \text{im}(\varphi - m) \simeq A/\ker(\varphi - m) \) where the former is torsion-free and the latter is periodic, as a factor of \( A/V \). Thus \( \varphi = m \) on \( A \).

- If \( A \) has not FTFR and \( \varphi \) is LIN, then (a) holds. Let \( F \) and \( V = Z[\varphi]F \) as in the first claim above. By Proposition 2 \( m = 1 \), that is \( \varphi = \frac{1}{n} \) on \( V \). Then \( V/F \) is the sum of infinitely many copies of \( Z(p^\infty) \) for each prime \( p \in \pi := \pi(n) \). Take \( F_s \) such that \( F/F_s \) is a periodic \( \pi' \)-group whose \( p \)-component is divisible with infinite rank for each prime \( p \in \pi' \). Let \( V_s/F_s \) be the \( \pi \)-component of \( V/F_s \). Since \( V/V_s \) is a \( \pi' \)-group, \( V_s \) is \( \pi \)-divisible, by Lemma 3(1). Thus \( V_s \leq V \) is \( \varphi \)-invariant and \( V/V_s \simeq \varphi F/F_s \). Let \( A_s/V_s \) be the \( \pi' \)-component of \( A/V_s \).

We claim \( A/A_s \) is finite. It is enough to check that the \( \pi \)-component \( A_1/V_s \) of \( A/V_s \) is finite. To this aim, notice that \( T_1 := T(A_1) = A_\pi \). On one hand \( A_1/(T_1 + V_0) \) is a \( \pi \)-group by definition of \( A_1/V_s \); on the other hand \( A_1/(T_1 + V_0) \) is \( \pi' \)-group as \( (T_1 + V_0)/T_1 \) is \( \pi \)-divisible and \( A_1/T_1 \) is
torsion-free (see Lemma 3(2)). Thus $A_1 = T_1 \oplus V_\pi$ and the claim reduces to show $T_1$ is finite.

Assume by contradiction that $T_1$ has infinite rank (since by (iii) above, $T_1$ is bounded). By Proposition 5, $\varphi$ is FM on $T_1$. Then $\varphi$ is a multiplication by an integer $s$ not multiple of $p$ on a countable $\mathbb{Z}_p$-submodule $B = \bigoplus_i \langle b_i \rangle \leq T_1$. Let $\{a_i \mid i < \omega\}$ be a countable subset of the above basis $S$ for $F$ and set $W := \langle V_i \mid i < \omega \rangle = \bigoplus_i V_i$, where $V_i := \mathbb{Z}[\varphi]a_i$. Also, let $M := B \oplus W$ and $H := \langle a_i + b_i \mid i < \omega \rangle$ its “diagonal” subgroup, which is free on the $\mathbb{Z}$-basis of the $a_i + b_i$’s. Since $\varphi$ is one-to-one on $M$, then $\varphi(H)$ is torsion-free as $H$ is. For all $i$ we have: $H + \varphi(H) \ni (p - n\varphi)(a_i + b_i) = (p - 1)a_i$, as $p$ divides $n$. Since $p a_i \in H$, then $a_i \in H + \varphi(H)$. Thus $B \leq H + \varphi(H)$. Therefore $(H + \varphi(H))/\varphi(H) \geq (B + \varphi(H))/\varphi(H) \simeq B$ is infinite, contradicting $\varphi$ is LIN. Thus $A/A_\pi$ is finite.

Let us show that $\varphi = \frac{1}{n}$ on some $A_0$ with finite index in $A_\pi$. As $A_\pi/V_\pi$ is a $\pi'$-group and, for each prime $p \in \pi'$, its $p$-component contains a divisible $p$-group of infinite rank, by Proposition 5 we have that $\varphi$ is FM on $A_\pi/V_\pi$. Thus $\varphi$ is multiplication on some $A_0/V_\pi$ with finite index in $A_\pi/V_\pi$. On one side $\varphi = \frac{1}{n}$ on $V$. On the other side, by Lemma 3(3) the multiplication by $\frac{1}{n}$ is an endomorphism of the whole $A_0$. Then, as $\ker(\varphi|_{A_0} - \frac{1}{n}) \leq V$ and $(\varphi - \frac{1}{n})(A_0) \leq V$, we have that $A_0/\ker(\varphi|_{A_0} - \frac{1}{n}) \simeq (\varphi - \frac{1}{n})(A_0)$ is both periodic and torsion-free. Therefore $\varphi = \frac{1}{n}$ on $A_0$. Thus (a) holds if $r_0(A)$ is infinite.

Conversely, if $\varphi$ is as in (a), it is trivial that $\varphi$ is RIN (or LIN, resp.). Let then $\varphi$ as in (b). We have to show that for each subgroup $X$ of $A$ the statement $R(X)$ (resp. $L(X)$) below holds.

$$R(X) := \left( \left| \frac{X + \varphi(X)}{X} \right| < \infty \right) \quad L(X) := \left( \left| \frac{X + \varphi(X)}{\varphi(X)} \right| < \infty \right)$$

Let $\pi := \pi(n)$. We proceed by steps:

- $\varphi = \frac{m}{n}$ is inertial on $A/T$ and so $A/T$ is $\pi$-divisible. In fact, if $a \in A$, there is a non-zero integer $s$ such that $sa \in V$. Thus $s(n\varphi - m)(a) = (n\varphi - m)(sa) = 0$ and $(n\varphi - m)(A) \subseteq T$, as claimed.

- If $X$ is periodic, then $R(X)$ (resp. $L(X)$) holds. This is very easy, since $X^{(\varphi)} \cap V = 0$ and one can verify $R(X)$ (resp. $L(X)$) mod $V$.

- It is enough to show that $R(X_0)$ (resp. $L(X_0)$) holds for each torsion-free subgroup $X_0$ of $A$. Let $X$ be any subgroup and $U = T(X)$. By the above $\varphi$
induces on $T$ a RIN (resp. LIN) endomorphism. By Proposition 5 applied to $T$, we have (FS), so that $U/U(\varphi)$ is finite. Since the hypotheses hold modulo $U(\varphi)$ (which is periodic), that is for the endomorphism induced by $\varphi$ on the group $A/U(\varphi)$, we can assume $U(\varphi) = 0$ that is $U = T(X)$ is finite. Therefore $X = X_0 \oplus U$ splits on $U$. Since $X/X_0$ is finite, then $R(X_0)$ (resp. $L(X_0)$) implies straightforward $R(X)$ (resp. $L(X)$).

- We may assume $A_\pi = 0$. Recall that by hypothesis $B := A_\pi$ is bounded by some $e$. Let $X \leq A$ be torsion-free. Clearly $X$ has finite rank. If $\varphi$ is RIN on $A/B$, let $\left\lfloor \frac{X + \varphi(X) + B}{X + B} \right\rfloor = s < \infty$. Then $\varphi(X) \leq X + B$. Thus $es\varphi(X) \leq X$. It follows $\varphi(X) + X/X$ is finite, as claimed, as it is bounded and has finite rank. If $\varphi$ is LIN on $A/B$, let $\left\lfloor \frac{X + \varphi(X) + B}{\varphi(X) + B} \right\rfloor = s < \infty$. By an argument as above, $esX \leq \varphi(X)$ and $(\varphi(X) + X)/\varphi(X)$ is finite.

- If $A_\pi = 0$ and $X$ is torsion-free then $R(X)$ (resp. $L(X)$) holds. As the hypotheses on $\varphi$ hold even in $A_1 := \mathbb{Z}[\varphi]X$ with respect to $V_1 := A_1 \cap V$, we can assume $A = A_1$, that is $X$ has maximal torsion-free rank $r$ and $V \simeq \mathbb{Q}^r \oplus \ldots \oplus \mathbb{Q}^r$ ($r$ times).

Let $K/X$ be the $\pi$-component of $A/X$ (which is periodic). By hypothesis $R(K + V)$ holds, thus $R(K)$ holds, since $(K + V)/K \simeq V/(V \cap K)$ is finite as it is a $\pi'$-group. On the other hand, $K$ is torsion-free, as $T$ is a $\pi'$-group. Thus $T(K + \varphi(K))$ is finite.

Let $Y := X + \varphi(X)$, $Y_R := Y \cap (X + T)$ and $Y_L := Y \cap (\varphi(X) + T)$. On one side, $Y_R/X$ and $Y_L/\varphi(X)$ are both finite, as isomorphic to quotients of $Y \cap T \leq T(K + \varphi(K))$, which is finite. On the other side, $|Y/Y_R| = |(Y + T)/(X + T)| < \infty$ because of $R(X + T)$ (resp. $|Y/Y_L| = |(Y + T)/(\varphi(X) + T)| < \infty$ because of $L(X + T)$) which holds as $\varphi$ is inertial on $A/T$ (and $m \neq 0$, resp.) see Proposition 2.

5 Proofs of the main results

We use a Lemma dealing with finitely many inertial endomorphisms. Denote by $\oplus_r \mathbb{Q}^\pi$ the direct sum of $r$ copies of $\mathbb{Q}^\pi$.

Lemma 5 Let $\varphi_1, \ldots, \varphi_t$ be finitely many inertial endomorphisms of an abelian group $A$ with FTFR, where $\varphi_i = \frac{m_i}{n_i} \in \mathbb{Q}$ on $A/T$ ($m_i, n_i$ coprime $\forall i$) and $\pi := \pi(n_1 \ldots n_t)$. Then there is a $\mathbb{Z}[\varphi_1, \ldots, \varphi_t]$-submodule $V \simeq \oplus_r \mathbb{Q}^\pi$ ($r \in \mathbb{N}$) and such that $A/V$ is periodic.
Proof. It is easily seen that for each $p_j \in \pi$ there is $\psi_j \in \mathbb{Z}[\varphi_1, \ldots, \varphi_t]$ such that $\psi_j = r_j/p_j$ on $A/T$ ($r_j$ and $p_j$ coprime). Then there are coprime $m, n$ such that $\varphi := \sum_j \psi_j = m/n$ on $A/T$ where $\pi(n) = \pi$.

By Proposition 5 for each $i$ there is a torsion-free $\varphi_i$-invariant subgroup $V_i$ such that $A/V_i$ is periodic. Pick $\mathbb{Z}$-independent elements $b_1, \ldots, b_r$ of $\cap_i V_i$, where $r = r_0(A)$. By Lemma 4 for each $k$ there exists $a_k \in \langle b_k \rangle$ such that $\mathbb{Z}[\varphi]\langle a_k \rangle$ is torsion-free with rank $1$. Therefore the subgroup $V := \mathbb{Z}[\varphi]\langle a_1, \ldots, a_r \rangle$ is torsion-free and $\pi$-divisible.

We claim $\varphi_i(V) \subseteq V$. Set $W_i := \mathbb{Z}[\varphi_i]\langle a_1, \ldots, a_r \rangle$, which is torsion-free as it is a subgroup of $V_i$ and is contained in $V$, since $\varphi_i = m/n$ on $W_i$. Then $V/W_i$ is a $\pi$-group, as $V/\langle a_1, \ldots, a_r \rangle$ is such.

Let $a$ be any element of $V$ and $e$ be the bound of $A_\pi$ (which is bounded by Proposition 5). Therefore there is a $\pi$-number $t$ such that $ta \in W_i$ hence $(n_i\varphi_i - m_i)(ta) \in T \cap W_i = 0$. Thus $(n_i\varphi_i - m_i)(a) \in A_\pi$, so that $e(n_i\varphi_i - m_i)(a) = 0$. Since $e$ and $n_i$ are $\pi$-numbers and $V$ is $\pi$-divisible, we have $\varphi_i(V) = em_i\varphi_i(V) = em_i V \subseteq V$ as wished. □

Proof of Theorem A. Assume all $\varphi_i$ are inertial. If $A$ has not FTFR, then by Proposition 7 it is clear that (a) holds.

Assume now $A$ has FTFR. As in Lemma 5, there is a $\mathbb{Z}[\varphi_1, \ldots, \varphi_t]$-submodule $V$ such that $V \simeq \bigoplus_r \mathbb{Q}^\pi_r$ ($r \in \mathbb{N}$) and $A/V$ is periodic. Let $\pi_2$ be the set of primes $p$ such that some $\varphi_i$ is not FM on the $p$-component of $A/V$. Note that the definition of $\pi_2$ is independent of $V$, as all possible $V$ are commensurable each other. From Proposition 5 it follows that $\pi_2$ is finite. Let $\pi_1 := \pi \cup \pi_2$. On one side, $A_{\pi_1}$ is bounded, by Proposition 5. On the other side, for each $p \in \pi_2$ the $p$-component $A_p$ of $A$ is the sum of a bounded subgroup and a finite rank divisible subgroup. Thus there is $C^*$ such that $A = A_{\pi_1} \oplus C^*$.

By Proposition 5 and the definition of $\pi_1$, there is a finite index subgroup $B \oplus D$ of $A_{\pi_1}$ such that $B$ is bounded, $D$ is divisible with finite rank hence a $\pi'$-group and each $\varphi_i$ acts by multiplications on both $B$ and $D$, as in the statement.

We may assume $V \leq C^*$. In fact $|V/(V \cap C^*)| =: s$ is finite as $V \simeq \bigoplus_r \mathbb{Q}^\pi_r$ ($r \in \mathbb{N}$) and $V/(V \cap C^*) \simeq (V + C^*)/C^*$ is periodic with bounded $\pi$-component. So we may substitute $V$ with $sV$ and we get $V \leq C^*$.

Use bar notation in $\bar{A} := A/V$. Consider the primary decomposition $\bar{C}^* = \bar{C}_1^* \oplus \bar{C}_0^*$ where $\bar{C}_1^*$ (resp. $\bar{C}_0^*$) is a $\pi_1$-group (resp. $\pi_1'$-group). By (FS) of Proposition 5 and the definition of $\pi_1$, each $\varphi_i$ is multiplication on a subgroup $\bar{C}_0$ with finite index in $\bar{C}_0^*$. On the other side $\bar{C}_1^*$ is torsion-free (with finite rank) hence $\bar{C}_1^*$ has Min, thus $\bar{C}_1^*$ has a divisible finite
index subgroup, say \( C_1 \), on which each \( \varphi_i \) is multiplication (see Lemma \( \text{[2]} \)). Therefore there is a finite index subgroup \( C := C_1 + C_0 \) of \( C^* \) such that \( \varphi_i \) is multiplication on \( \bar{C} \). So conditions (i), (ii), (iii) of the statement hold for \( A_0 := B \oplus D \oplus C \).

To prove condition (iv), for each \( i \) let \( \varphi_i = \frac{m_i}{n_i} \in Q \) on \( V \) (see Proposition \( \text{[2]} \)) and \( p \in \pi(D) \) such that the \( p \)-component \( \bar{C}_p \) of \( \bar{C} \) is infinite (hence unbounded). On one hand, as \( C_p \) is torsion-free, we get that \( \varphi_i = \frac{m_i}{n_i} \) on \( C_p \). On the other hand, \( \varphi_i \) acts by the same \( p \)-adic \( \alpha \) on \( D_p \) as on \( \bar{D}_p \). Therefore \( \alpha = \frac{m_i}{n_i} \) by Proposition \( \text{[3]} \) (as \( \bar{D}_p \oplus C_p \) is non-critical).

Conversely, for the sufficiency of the condition, it is clear that if (a) holds, then all \( \varphi_i \) are inertial, so only case (b) is left. By Proposition \( \text{[3]} \) we may assume \( A = A_0 \). Fix \( i \) and \( a_1, ..., a_r \) such that \( V := \mathbb{Z}[\varphi_1, ..., \varphi_i](a_1, ..., a_r) \). Let \( V_i := \mathbb{Z}[\varphi_i](a_1, ..., a_r) \). Then \( V/V_i \) is a divisible \( \pi \)-group with finite rank. By Proposition \( \text{[7]} \) any \( \varphi_i \) is inertial iff it is such on the periodic group \( A/V_i \). Clearly, by Proposition \( \text{[5]} \), \( \varphi_i \) is already inertial on \( A/V \). Use bar notation in \( A := A/V_i \). By Proposition \( \text{[6]} \) it is enough to show that \( \varphi_i \) is inertial on all \( p \)-components of \( \bar{A} \) and even multiplication on all but finitely many.

If \( p \in \pi \) (which is finite), the \( p \)-component of \( \bar{A} \) is \( \bar{A}_p \oplus \bar{C}_p \) where \( \bar{A}_p \) is the \( p \)-component of \( A \) and \( \bar{C}_p \), the \( p \)-component of \( \bar{C} \), contains \( \bar{V}_p \), the \( p \)-component of \( V \). Since \( C \) has no elements of order \( p \), then \( C_p \) is torsion-free and, by Lemma \( \text{[3]} \), \( \bar{C}_p/\bar{V}_p \) is a \( \pi' \)-group. Hence \( \bar{C}_p = \bar{V}_p \) is divisible of finite rank. Therefore \( \varphi_i \) is multiplication on \( \bar{C}_p \). On the other hand, \( \varphi_i \) is multiplication even on \( \bar{A}_p \simeq \varphi_i \), \( A_p \leq B \), which is bounded. Thus \( \varphi_i \) is inertial on \( \bar{A}_p \oplus \bar{C}_p \) by Proposition \( \text{[5]} \).

If \( p \not\in \pi \), then the \( p \)-component of \( \bar{A} \) is \( \varphi_i \)-isomorphic to the \( p \)-component of \( A/V \) as \( V/V_i \) is a \( \pi \)-group. \( \square \)

**Proof of Corollary A.** Apply Theorem A to any couple of inertial endomorphisms \( \varphi_1, \varphi_2 \) of an abelian group \( A \). If (a) holds for both \( \varphi_1 \) and \( \varphi_2 \), then \( \varphi_1 \varphi_2 - \varphi_2 \varphi_1 = 0 \) on a subgroup with finite index of \( A \), since multiplications commute. Otherwise, by Proposition \( \text{[2]} \), \( \varphi_1, \varphi_2 \) commute on \( A/T \) anyway, where \( T = T(A) \). Moreover, there is a subgroup \( A_0 \) with finite index of \( A \) such that \( \varphi_1, \varphi_2 \) commute on \( A_0/V \) for some \( V \) as in Theorem A. As \( T \cap V = 0 \), then \( \varphi_1, \varphi_2 \) commute on \( A_0 \), as wished. \( \square \)

**Proof of Theorem B.** It is enough to prove the statement for a finitely generated subgroup \( \Gamma \) of \( IAut(A) \). Let \( A_0 \) and \( V \) as in Theorem A and \( T_0 := T(A_0) \). Then \( \Gamma' \) acts trivially on both \( A_0/V \) and \( A_0/T_0 \). Thus \( \Gamma' \) acts trivially on \( A_0 \) and (1) holds.
The subgroup $\Gamma_0 := C_T(A/A_0)$ has finite index in $\Gamma$. On the other hand, by the above, $\Sigma := \Gamma' \cap \Gamma_0$ stabilizes the series $0 \leq A_0 \leq A$ and embeds in $\text{Hom}(A/A_0, A)$, which is bounded by $m := |A/A_0|$. Thus $[A, \Sigma] \leq A_0[m] = B[m] \oplus C[m] \oplus D[m]$. As each $\gamma \in \Gamma$ acts by multiplications on $B[m], C[m], D[m]$ and $\Gamma$ is finitely generated, $[\Gamma/\Gamma_2]$ is finite, where $\Gamma_2 = C_{\Gamma_0}([A, \Sigma])$. On the other hand, we have that $[A, \Sigma, \Gamma_2] = 0$ and $[A, \Gamma_2, \Sigma] \leq [A_0, \Sigma] = 0$. Thus, by the Three Subgroup Lemma $[\Gamma_2, \Sigma, A] = 0$, that is $\Sigma$ is contained in the center of $\Gamma_2$ which turns to be nilpotent and finitely generated as well. Therefore $\Gamma$ has the maximal condition (Max) and $\Gamma'$ is finite, being periodic. Finally, as $\Gamma$ is finitely generated, we have $\Gamma$ is central-by-finite.

\textbf{Proof of Corollary B.} In part (1), by Proposition 5 and 7, LIN implies RIN. Also, in any case, if $\varphi$ is invertible, then $\varphi$ has LIN iff the $\varphi^{-1}$ has RIN.

For statement (2), note that if $\gamma_1$ is RIN and $\gamma_2$ is LIN, then $\gamma_1\gamma_2 = \gamma_2\gamma_1[\gamma_1, \gamma_2]$ where $\gamma_1[\gamma_1, \gamma_2]$ is RIN as $[\gamma_1, \gamma_2]$ is finitary.

\textbf{Proof of Proposition A.} Let $A$ be any abelian group. If $V$ is a torsion-free and $A/V$ is periodic, denote $T := T(A)$ and $\tilde{A} := A/(V + T)$, then the stabilizer $\Sigma$ of the series $0 \leq (V + T) \leq A$ is canonically isomorphic to $\text{Hom}(A, V + T) = \text{Hom}(A, T)$. In the particular case that $V$ has finite rank and $A/V$ is locally cyclic, then by Proposition 7, we have $\Sigma \leq I\text{Aut}(A)$ and $\Sigma \cap F\text{Aut}(A)$ corresponds to the subgroup of $\text{Hom}(A, T)$ forming by the homomorphisms with finite image. Therefore, if the abelian group $A$ is such that, for each prime $p$, $A_p$ has order $p$ while the $p$-component of $A/V$ has order $p^2$, we have: $\Sigma \simeq \text{Hom}(A, T) \simeq \prod_p \mathbb{Z}(p)$ and $\Sigma \cap F\text{Aut}(A) \simeq \bigoplus_p \mathbb{Z}(p)$, where $p$ ranges over the set of all primes.

To show the existence of the wished group $A$, let $G := B \oplus C$ where $B := \prod_p \langle b_p \rangle$, $C := \prod_p \langle c_p \rangle$, and $b_p, c_p$ have order $p, p^2$ (resp.). Consider the (aperiodic) element $v := (b_p + p c_p)_p \in G$ and $V := \langle v \rangle$. Note that for each prime $p$ there exists an element $d_{(p)} \in G$ such that $pd_{(p)} = v - b_p$. Define $A := V + \langle d_{(p)} | p \rangle$. Then we have that $A/T \simeq \langle 1/p \mid p \rangle \leq \mathbb{Q}$ as it has torsion free rank 1 and $v + T$ has $p$-height 1 for each prime. Thus

$T = T(B) \simeq \bigoplus_p \mathbb{Z}(p)$, while the $p$-component of $A/V$ is generated by $d_{(p)} + V$ and has order $p^2$ as $pd_{(p)} = v - b_p$. \qed
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