ELEMENTARY CRITERIA FOR IRREDUCIBILITY OF \( f(X^n) \)

NATALIO H. GUERSENZVAIG

Abstract. Very simple sufficient conditions for the irreducibility of \( f(X^n) \) over an arbitrary unique factorization domain \( Z \) are established via a generalization of a well known theorem of A. Capelli.

1. Introduction

We fix throughout this work a unique factorization domain \( Z \) with field of fractions \( Q \). The group of units of \( Z \) will be denoted by \( U \).

Let \( f(X) \) be any polynomial in \( Z[X] \) of positive degree that is irreducible in \( Z[X] \). Using the well known Eisenstein’s Criterion we can easily show that, in some cases, \( f(X^n) \) will also be irreducible in \( Z[X] \) for any positive integer \( n \). However, this is not true in general. For example, \( f(X) = X^3 - X^2 - 2X - 1 \) is irreducible in \( Z[X] \) for \( Z \in \{ \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_5 \} \), while \( f(X^2) = (X^3 - X^2 - 1)(X^3 + X^2 + 1) \).

In the main result of this work we will establish sufficient conditions for irreducibility of \( f(X^n) \) in \( Z[X] \) for any integer \( n > 1 \) that, besides \( U \), only depend on \( n \), the degree of \( f(X) \) and the leading and constant coefficients of \( f(X) \). (These conditions can be easily checked if \( Z \) is an effective unique factorization domain.) Elementary necessary and sufficient conditions will also be given. (The adjective “elementary” refers to the fact that these conditions will be stated without using proper algebraic extensions of \( Q \).) The cases \( f(X) = X - a \) and \( f(X) = aX^2 + bX + c \) are considered in [1, pp. 63-74]. Related results for polynomials over finite fields can be found in [7, pp. 93-95].

Henceforth we will use, for \( S \subseteq Q \) and \( t \in \mathbb{N} \), the following notations:

\[
S^* = S \setminus \{0\}, \quad S^t = \{s^t : s \in S\}, \quad tS = \{ts : s \in S\}.
\]

Given \( a, b \in Z \) and \( m, n \in \mathbb{N} \) we denote \( C(m, a, b, n) \) the following condition:

For each prime \( p \) dividing \( n \) and any unit \( u \) in \( U \) at least one of the two following statements is true:

(A) \( ua \notin Z_p \);

(B) (i) \((-1)^m ub \notin Z_p \) and (ii) \( ub \notin Z^2 \), if \( 4 | n \).

Our main result is the following theorem (it will be proved, together with an equivalent dual version, in the last section of this paper).

THEOREM 1.1. Let \( n \) be any integer greater than 1, and let \( f(X) \) be an arbitrary polynomial in \( Z[X] \) of positive degree \( m \), leading coefficient \( a \) and nonzero constant term \( b \), that is irreducible in \( Z[X] \). Assume that at least one of the conditions \( C(m, a, b, n), C(m, b, a, n) \) holds. Then

\[ f(X^n) \text{ is irreducible in } Z[X]. \]
2. Basic facts

In this section we review some results that we will use later without specific reference. First we remind the reader that the characteristic of $\mathbb{Z}$, say $\chi(Z)$, is the only nonnegative integer that satisfies the following two conditions:

(a) $\chi(Z) \cdot 1 = 0$;  
(b) if $k \in \mathbb{Z}$ and $k \cdot 1 = 0$, then $\chi(Z)|k$.

The following fact is needed to prove Corollary 4.6 below:

- either $\chi(Z) = 0$, or $\chi(Z) = p$ is a prime number in which case we have 
  $(x_1 + \cdots + x_n)^p = x_1^p + \cdots + x_n^p$ for all $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in \mathbb{Z}$.

We also recall that a nonzero polynomial $f(X) \in \mathbb{Z}[X]$ if there exist nonzero polynomials $g(X), h(X)$ in $\mathbb{Z}[X] \setminus U$ such that $f(X) = gh(X)$. Otherwise $f(X)$ is called irreducible in $\mathbb{Z}[X]$. The content of $f(X)$, say $c(f)$, is the greatest common divisor of their coefficients (modulo units of $\mathbb{Z}$), and $f(X)$ is called primitive if $c(f) = 1$. Replacing $\mathbb{Z}$ by $\mathbb{Q}$ in this definition yields (since in this case $U = \mathbb{Q}^*$) that $f(X)$ is irreducible in $\mathbb{Q}[X]$ if and only if $f(X)$ has positive degree and there are no polynomials $g(X), h(X)$ in $\mathbb{Q}[X]$ of positive degree such that $f(X) = gh(X)$.

It is also well known the following result (see, for example, [6]):

- if $f(X) \in \mathbb{Z}[X]$ has positive degree, then, $f(X)$ is irreducible in $\mathbb{Z}[X]$ if and only if $f(X)$ is primitive (in $\mathbb{Z}[X]$) and irreducible in $\mathbb{Q}[X]$.

As a consequence, when $f(X) \in \mathbb{Z}[X]$ has positive degree and it is irreducible in $\mathbb{Z}[X]$ we can replace $\mathbb{Z}[X]$ by $\mathbb{Q}[X]$ without risk in any of the expressions, “$f(X)$ is reducible in $\mathbb{Z}[X]$”, “$f(X)$ is irreducible in $\mathbb{Z}[X]$”. To simplify, in these situations we will write “$f(X)$ is reducible” or “$f(X)$ is irreducible”, respectively. For the same reason, except where the contrary is explicitly stated, the terms “primitive” and “prime” will be understood to apply to the sets $\mathbb{Z}[X]$ and $\mathbb{N}$, respectively.

For any positive integer $n$ let $\Phi_n(X)$ denote the cyclotomic polynomial of order $n$ over $\mathbb{Q}$, i.e., the polynomial

$$\Phi_n(X) = \prod_{1 \leq k \leq \phi(n)} (X - w^k),$$

where $w$ denotes an arbitrary primitive $n$th root of unity (in some splitting field of $\Phi_n(X)$ over $\mathbb{Q}$), and $\phi$ denotes the Euler function (i.e., $\phi(n)$ is the number of integers in the set $\{1, \ldots, n\}$ that are relatively prime to $n$). The following facts (see [3]) are related to Theorem 4.3, Theorem 4.4 and Corollary 5.2:

- $\Phi_n(X) \in \mathbb{Z}[X]$;
- $\Phi_n(X)$ is irreducible if $\chi(Z) = 0$;
- $X^n - 1 = \prod_{d \mid n} \Phi_d(X)$, if either $\chi(Z) = 0$, or $\chi(Z) \not\mid n$;
- If $\chi(Z) \neq 0$, $\chi(Z) \not\mid n$ and $\#(Z) = q$, then $\Phi_n(X)$ factors into the product of $\phi(n)/d$ distinct monic irreducible polynomials in $\mathbb{Z}[X]$ of degree $d$,
  where $d$ is the least positive integer $d$ such that $q^d \equiv 1 \pmod{n}$.

We will use matrices and determinants as well. With $M_m(\mathbb{Q})$, $|A|$ and $\Delta_A(X)$ we will respectively denote the ring of square matrices of order $m$ with coefficients in $\mathbb{Q}$, the determinant of $A \in M_m(\mathbb{Q})$ and the characteristic polynomial of $A$. In particular we will consider a well known type of matrices associated to polynomials.
Let \( f(X) \) be an arbitrary polynomial in \( \mathbb{Q}[X] \) of positive degree \( m \), say \( f(X) = \sum_{j=0}^{m} a_j X^j \), and let \( f^*(X) \) denote the monic polynomial associate to \( f(X) \), that is, \( f^*(X) = \sum_{j=0}^{m} c_j X^j \), where \( c_j = a_j/a_m \) for \( j = 0, 1, \ldots, m \). The companion matrix of \( f^*(X) \), say \( C_{f^*} \), is the matrix in \( M_m(\mathbb{Q}) \) defined by

\[
C_{f^*} = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
-c_0 & -c_1 & \cdots & -c_{m-2} & -c_{m-1}
\end{bmatrix}
\]

We will freely use the following properties of \( C_{f^*} \):

- \( f^*(X) \) is both the minimum polynomial of \( C_{f^*} \) over \( \mathbb{Q} \) and the characteristic polynomial of \( C_{f^*} \) (so \( f(X) = a_m[XI_m - C_{f^*}] \));
- If \( f(X) \) is irreducible, then the ring \( \mathbb{Q}[C_{f^*}] = \{h(C_{f^*}) : h(X) \in \mathbb{Q}[X]\} \) is an extension field of \( \mathbb{Q} \) of degree \( m \) with \( f(C_{f^*}) = a_m f^*(C_{f^*}) = 0 \).

(In this situation, as usual, we will write \( \mathbb{Q}(C_{f^*}) \) instead of \( \mathbb{Q}[C_{f^*}] \).)

3. Preliminary results

Our irreducibility criteria strongly depend on two beautiful theorems of A. Capelli (which are included in the author’s Ph. D. Thesis, Melbourne University, 1955). The first one gives non-elementary necessary and sufficient conditions for irreducibility of \( g(f(X)) \) in \( \mathbb{Q}[X] \). For the sake of completeness we provide a simple proof of this result.

**CAPELLI’S THEOREM 1.** Let \( f(X), g(X) \) be arbitrary polynomials of \( \mathbb{Q}[X] \) of positive degree. Let \( F \) be any splitting field of \( f(X) \) over \( \mathbb{Q} \), and let \( \alpha \) be any root of \( f(X) \) in \( F \). Then \( f(g(X)) \) is irreducible in \( \mathbb{Q}[X] \) if and only if \( f(X) \) is irreducible in \( \mathbb{Q}[X] \) and \( g(X) - \alpha \) is irreducible in \( \mathbb{Q}(\alpha)[X] \).

**Proof.** We can assume without risk that \( f(X) \) is irreducible in \( \mathbb{Q}[X] \). We also assume that \( f(X) \) and \( g(X) \) have degrees \( m \) and \( n \), respectively, so \( f(g(X)) \) has degree \( s = mn \). Let \( K \) be any splitting field of \( f(g(X)) \) over \( F \). Letting \( \alpha_1 = \alpha \), we have in \( K[X] \) the factorization

\[
f(X) = a(X - \alpha_1) \ldots (X - \alpha_m),
\]

and hence, the factorization

\[
f(g(X)) = a(g(X) - \alpha_1) \ldots (g(X) - \alpha_n) = b(X - \beta_1) \ldots (X - \beta_s)
\]

for certain nonzero \( a, b \in \mathbb{Q} \). Given any \( i \) with \( 1 \leq i \leq m \), there must exist at least one \( j \), \( 1 \leq j \leq s \), such that \( g(\beta_j) = \alpha_i \), for otherwise all \( X - \beta_j \) would be relatively prime to \( g(X) - \alpha_i \), and therefore \( f(g(X)) \) would be relatively prime to \( g(X) - \alpha_i \), a contradiction. This proves that \( \alpha \) is of the form \( g(\beta) \), where \( \beta \) is a root of \( f(g(X)) \) in \( K \). Then we have,

\[
[Q(\beta) : \mathbb{Q}] = [Q(\beta) : Q(\alpha)][Q(\alpha) : \mathbb{Q}] = [Q(\beta) : Q(\alpha)]m.
\]

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1 Due to F. Szechtman, University of Regina, Saskatchewan, Canada
Now, \( f(g(X)) \) is irreducible in \( Q[X] \) if and only if \([Q(\beta) : Q] = mn\), that is, if and only if \([Q(\beta) : Q(\alpha)] = n\). Therefore, since \( g(X) - \alpha \) has degree \( n \) and annihilates \( \beta \), \( f(g(X)) \) is irreducible in \( Q[X] \) if and only if \( g(X) - \alpha \) is irreducible in \( Q(\alpha)[X] \). This completes the proof. \( \square \)

The second establishes simple conditions for reducibility of \( X^n - a \) in \( Q[X] \) (see [2] and [7], Theorem 9.1).

**CAPELLI’S THEOREM 2.** Let \( a \) be any nonzero element of \( Q \), and let \( n \) be any integer greater than 1. Then \( X^n - a \) is reducible in \( Q[X] \) if and only if either (i) \( a = c^t \) for some \( c \in Q \) and \( t \nmid n \) with \( t > 1 \), or (ii) \( 4 | n \) and \( a = -4c^4 \) for some \( c \in Q \).

### 4. Necessary and sufficient conditions

In order to prove the main theorem of this section we first establish a result involving primitive polynomials that is interesting in its own right.

**LEMMA 4.1.** Suppose that \( P(X) = \sum_{k=0}^{m} a_k X^k \) is a primitive polynomial of \( Z[X] \) of degree \( m \) with \( a_0 \neq 0 \). Let \( L(X) \) be any monic polynomial in \( Z[X] \) of positive degree \( n \) and nonzero roots \( \lambda_1, \ldots, \lambda_n \) in some extension field of \( Q \) (counting multiplicities). In addition suppose that the constant coefficient of \( L(X) \), say \( c_0 \), is relatively prime to \( a_0 \). Then

\[
\prod_{j=1}^{n} P(\lambda_j X) \text{ is a primitive polynomial of } Z[X].
\]

**Proof.** Since \( L(X) \) is a monic polynomial of \( Z[X] \), from the well known Fundamental Theorem on Symmetric Polynomials it follows that \( \prod_{j=1}^{n} P(\lambda_j X) \) is a polynomial in \( Z[X] \), say

\[
P^*(X) = \sum_{j=0}^{mn} a_j^* X^j.
\]

Looking for a contradiction we suppose \( c(P^*) \neq 1 \). Let \( q \) be any prime of \( Z \) that divides \( c(P^*) \). As \( c(P^*) \) divides \( a_0^* = a_0^q \), \( q \mid a_0 \) and \( q \nmid c_0 \). Thus, since \( P(X) \) is primitive, there is a positive integer \( k \), \( k \leq m \), such that \( q \mid a_j \) for \( 0 \leq j < k \) and \( q \nmid a_k \). Realizing the product \( \prod_{j=1}^{n} P(\lambda_j X) \) we get

\[
a_{nk}^* = a_k^k \lambda_1^k \cdots \lambda_n^k + \sum_{i_1 + \cdots + i_n = nk} a_{i_1} \cdots a_{i_n} \lambda_1^{i_1} \cdots \lambda_n^{i_n}.
\]

Notice that in each summand \( a_{i_1} \cdots a_{i_n} \lambda_1^{i_1} \cdots \lambda_n^{i_n} \) we have \( i_j < k \) for at least one \( j \), which makes each such summand a multiple of \( q \). But \( a_{nk}^* \) is also a multiple of \( q \). This contradicts the fact that \( a_k^k \lambda_1^k \cdots \lambda_n^k = (-1)^{nk-k} a_k^n \) is not divisible by \( q \). \( \square \)

In addition we will use the following result, which is an immediate consequence of two well known identities (see, for example, (3.1.1)-(3.1.4) in [3] and (22)-(25) in [5]).
LEMMA 4.2. Let $F$ be an arbitrary field and let $p$ be any prime number. Let 
$\Psi(Y) = Y^p - 1 \in F[Y]$ and let $w$ be any primitive $p$th root of unity. Let $g(Y) = \sum_{j=0}^{p-1} c_j Y^j \in F(Y)$. Then

$$\prod_{j=0}^{p-1} g(w^j) = |g(C_\Psi)| = \begin{vmatrix} c_0 & c_1 & \ldots & c_{p-2} & c_{p-1} \\ c_{p-1} & c_0 & \ldots & c_{p-3} & c_{p-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1 & c_2 & \ldots & c_{p-1} & c_0 \end{vmatrix}.$$ 

Now the following extension of Capelli’s Theorem 2 can be proved.

THEOREM 4.3. Let $n$ be any integer, $n > 1$, and let $f(X)$ be any irreducible 
polynomial in $Z[X]$ of positive degree $m$ and leading coefficient $a$. The following 
two statements are equivalent.

(a) $f(X^n)$ is reducible.

(b) There exist a prime $p$ that divides $n$, a unit $u$ in $U$ with $ua \in Z_p$, and 
polynomials $S_0(X), S_1(X), \ldots, S_{p-1}(X)$ in $Z[X]$ such that either

$$(-1)^{m(p-1)}u f(X^p) = \begin{vmatrix} S_0(X^p) & XS_1(X^p) & \ldots & X^{p-1}S_{p-1}(X^p) \\ X^{p-1}S_{p-1}(X^p) & S_0(X^p) & \ldots & X^{p-2}S_{p-2}(X^p) \\ \vdots & \vdots & \ddots & \vdots \\ XS_1(X^p) & X^2S_2(X^p) & \ldots & S_0(X^p) \end{vmatrix},$$

or $4|n$ and

$$uf(X^4) = \begin{vmatrix} S_0(X^2) & XS_1(X^2) \\ XS_1(X^2) & S_0(X^2) \end{vmatrix}.$$ 

Proof. Assume (a). When $f(0) = 0$, since $f(X)$ is irreducible, we have $f(X) = aX$ 
with $a \in U$, so (1) follows with any prime $p$ that divides $n$, $u = a^{-1}$, $S_1(X) = 1$ 
and $S_j(X) = 0$ for $j = 0, \ldots, p-1$, $j \neq 1$. Therefore we may also assume $f(0) \neq 0$.

Let $\alpha = C_f$. From Capelli’s Theorem 1 it follows that $X^r - \alpha$ is reducible in 
$Q(\alpha)[X]$. We first assume that condition (i) of Capelli’s Theorem 1 holds. Therefore we have $\alpha = \gamma^t$ for some $\gamma \in Q(\alpha)$ and $t|n$, $t > 1$.

Let $p$ be any prime that divides $t$. Then we can write $\alpha = \beta^p$, where $\beta = \gamma^{t/p} \in Q(\alpha)$. Hence, $X^p - \alpha$ is reducible in $Q(\alpha)[X]$, so $f(X^p)$ is reducible by Capelli’s Theorem 1.

Let $\Psi(X) = X^p - 1$ and let $w$ be an arbitrary primitive $p$th root of unity. From 
$\Psi(X) = \prod_{j=0}^{p-1} (X - w^j)$ we get

$$X^p - \alpha = X^p - \beta^p = \beta^p \Psi(\beta^{-1}X) = \prod_{j=0}^{p-1} (X - w^j \beta)$$

$$= w^{p(p-1)} \prod_{j=0}^{p-1} (w^{-j}X - \beta) = (-1)^{p-1} \prod_{j=0}^{p-1} (w^jX - \beta).$$

Consequently, taking determinants on both sides, we obtain

$$f(X^p) = (-1)^{m(p-1)} a \prod_{j=0}^{p-1} \Delta_\beta(w^jX),$$

where $\Delta_\beta(w^jX)$ is the $j$th elementary symmetric polynomial in the roots of 
$X^p - \beta$.
where $\Delta_\beta(X) = |XI_m - \beta|$, the characteristic polynomial of $\beta$, is a monic polynomial in $Q[X]$ of degree $m$. From unique factorization in $Z$ it follows that there exists $d \in Z$ such that $P(X) = d\Delta_\beta(X)$ belongs to $Z[X]$ and is primitive. Since $P(X)$ has leading coefficient $d$, letting $u = d^p / a$ (so $ua \in Z^p$) we can rewrite (3) as follows:

$$(-1)^{m(p-1)}uf(X^p) = \prod_{j=0}^{p-1} P(w^j X).$$

The right hand side is a primitive polynomial of $Z[X]$, by Lemma 4.1. Hence, since $f(X^p)$ is also primitive (because $f(X)$ is), $u \in U$.

On the other hand, assuming $P(X) = \sum_{k=0}^{m} a_k X^k$ and expressing each index $k$ in the form $k = ip + j$ with $0 \leq j < p$, we can write $a_k = a_{ip+j}X^{ip}X^j$ for $k = 0, \ldots, m$. As a result, grouping the monomials associated to each $X^j$ with $0 \leq j < p$, we obtain the polynomials

$$S_j(X) = \sum_{i \geq 0} a_{ip+j}X^i \in Z[X], \ j = 0, 1, \ldots, p-1,$$

which satisfy

$$P(X) = \sum_{j=0}^{p-1} X^j S_j(X^p).$$

Hence, since $C_\Psi^p$ is the identity matrix of order $p$, we get

$$P(XC_\Psi) = \sum_{j=0}^{p-1} X^j S_j(X^p)C_\Psi^j.$$

Thus (1) follows from the case $F = Q(X)$, $g(Y) = P(XY)$ of Lemma 4.2.

Now suppose that condition (ii) of Capelli’s Theorem 2 applies. Thus $4 \nmid n$ and $\alpha = -4\gamma^4$ for some $\gamma \in Q(\alpha)$. It also follows that $f(X^2)$ is irreducible, because otherwise case (i) applies with $p = 2$. From the identity

$$X^4 + 4\gamma^4 = (X^2 - 2\gamma X + 2\gamma^2)(X^2 + 2\gamma X + 2\gamma^2)$$

it follows

$$f(X^4) = a|X^4 I_m - \alpha| = a|X^2 I_m - 2\gamma X + 2\gamma^2||X^2 I_m + 2\gamma X + 2\gamma^2|.$$ 

Hence, $f(X^4)$ is reducible in $Q[X]$. Consequently, condition (i) of Capelli’s Theorem 2 is satisfied with $n = 2$ and $g(X) = f(X^2)$ instead of $f(X)$. By the first case for $g(X)$ there exist a unit $u$ in $U$ and $S_0(X)$, $S_1(X)$ in $Z[X]$ satisfying (1) with $p = 2$. Since this is the same as (2) (note that the degree of $g(X)$ is $2m$), we have completed the proof of (b).

Assume (b). Adding all other rows to the first row in the determinants of (1) and (2) we get the row $[P(X) \ldots P(X)]$ (with $p = 2$ in (2)), where $P(X) = \sum_{j=0}^{p-1} X^j S_j(X^p)$. Hence we can replace these determinants by $P(X)P^*(X)$, where

$$P^*(X) = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
X^{p-1} S_{p-1}(X^p) & S_0(X^p) & \cdots & X^{p-2} S_{p-2}(X^p) \\
\vdots & \vdots & \ddots & \vdots \\
XS_1(X^p) & X^2 S_2(X^p) & \cdots & S_0(X^p)
\end{vmatrix}.$$
Hence, for \( i \in \{1, 2\} \), if (i) holds we have
\[
(-1)^{im(p-1)} uf(X^{ip}) = P(X)P^*(X),
\]
i.e.,
\[
uf(X^n) = \begin{cases} 
(-1)^mP(X^{n/p})P^*(X^{n/p}) & \text{if (1) holds} \\
P(X^{n/4})P^*(X^{n/4}) & \text{otherwise.}
\end{cases}
\]
On the other hand, with \( \Psi(X) \) and \( w \) as defined above, from Lemma 4.2 we obtain,
\[
P(X)P^*(X) = |P(XC_\Phi)| = \prod_{j=0}^{p-1} P(w^j X),
\]
so that \( P(X) \) and \( P^*(X) \) have positive degrees \( im \) and \( im(p-1) \), respectively. This completes the proof of (a).

Our next theorem provides complementary information about the polynomials \( P(X) \) and \( P^*(X) \) defined in Theorem 4.3.

THEOREM 4.4. Let \( n, f(X), \alpha, w, p \), \( P(X) = \sum_{j=0}^{p-1} X^j S_j(X^p) \) and \( P^*(X) \) be as defined in Theorem 4.3. Then

(I) \( P(X) \) is irreducible.

(II) \( P^*(X) \) is irreducible if and only if the cyclotomic polynomial \( \Phi_p(X) = \sum_{j=0}^{p-1} X^j \) is irreducible and \( Q(w) \cap Q(\alpha) = Q \).

Proof. We shall use (8). In order to prove (I) we suppose that \( P(X) \) is reducible. Therefore, since \( P(X) \) has degree \( im \), \( P(X) \) has a root, say \( \gamma \), in some extension field of \( Q \) of degree \( < im \) over \( Q \). Then \( f(X^i) \) has also a root in \( Q(\gamma) \), namely \( \gamma^p \), contradicting the fact that \( f(X^i) \) is irreducible.

Next we prove (II). This is clear if \( p = 2 \), since \( P(X) \) is irreducible and \( P^*(X) = P(-X) \). Assume \( p \) is odd.

If \( \Phi_p(X) \) is reducible, say \( \Phi_p(X) = g(X)h(X) \), where both \( g(X) \) and \( h(X) \) are monic polynomials of positive degree in \( Z[X] \), from \( \Phi_p(X) = \prod_{j=1}^{p-1}(X - w^j) \) and Lemma 4.2 it follows the nontrivial factorization \( P^*(X) = |P(XC_\Phi)| = |P(XC_g)||P(XC_h)| \).

Assume \( \Phi_p(X) \) is irreducible. Let \( \delta = w^\beta \), where \( \beta \) is defined as in Theorem 4.3. Then, since \( P(\beta) = 0, P^*(\delta) = \prod_{j=2}^p P(w^j \beta) = 0 \). Hence, \( P^*(X) \) is irreducible if and only if the minimum polynomial of \( \delta \) over \( Q \) has degree \( m(p-1) \). On the other hand we have \( \delta \in Q(w, \beta) \subseteq Q(w, \alpha) \). Furthermore, since \( \delta^p = \beta^p = \alpha, \alpha \in Q(\delta) \) and \( Q(\alpha) = Q(\beta) \). Hence, \( w = \beta^{-1}\delta \in Q(\delta) \). This proves \( Q(\delta) = Q(w, \alpha) \), and therefore that the minimum polynomial of \( \delta \) over \( Q \) has degree
\[
[Q(\delta) : Q] = [Q(w, \alpha) : Q(w)][Q(w) : Q] = [Q(w, \alpha) : Q(w)](p-1).
\]
From the well known theorem of natural irrationals (see, for example, [3]) we get \( [Q(w, \alpha) : Q(w)] = [Q(\alpha) : Q(w) \cap Q(\alpha)] \), and hence
\[
[Q(\delta) : Q] = [Q(\alpha) : Q(w) \cap Q(\alpha)](p-1).
\]
Now (II) follows immediately from \( [Q(\alpha) : Q] = m \).
Remark. In particular, $P^*(X)$ is irreducible if $\Phi_p(X)$ is irreducible and $\gcd(m, p - 1) = 1$.

It should be noticed that in the course of the proof of Theorem 4.3 we have also proved, incidentally, the following result.

**COROLLARY 4.5.** Let $f(X)$ be any irreducible polynomial in $\mathbb{Z}[X]$ of positive degree. Let $n$ be any integer, $n > 1$, and let $\sigma(n)$ be the square-free part of $n$. The three following statements are equivalent.

(a) $f(X^n)$ is reducible;
(b) either $f(X^{\sigma(n)})$ is reducible, or else $4|n$ and $f(X^4)$ is reducible;
(c) there exists a positive divisor of $r$, say $t$, with $t$ prime or $t = 4$, such that $f(X^t)$ is reducible.

In particular, for any positive integer $s$ we have:

(i) $f(X^{2s})$ is reducible if and only if either $f(X^2)$ is reducible, or else $s \geq 2$ and $f(X^4)$ is reducible;
(ii) if $p$ is an odd prime, then

$f(X^{p^s})$ is reducible if and only if $f(X^p)$ is reducible.

For example, since $X^p$ can be replaced by $X$ in both sides of (1), we have:

(i) $f(X^2)$ is reducible if and only if there exist $S_0(X), S_1(X)$ in $\mathbb{Z}[X]$ and $u \in U$ with $ua \in Z^2$ such that either

$$(-1)^muf(X) = S_0^2(X) - XS_1^2(X),$$

or else

$$s \geq 2 \text{ and } uf(X^2) = S_0^2(X) - XS_1^2(X).$$

(ii) $f(X^{3^s})$ is reducible if and only if there exist $S_0(X), S_1(X), S_2(X)$ in $\mathbb{Z}[X]$ and $u \in U$ with $ua \in Z^3$ such that

$$uf(X) = S_0^2(X) + XS_1^2(X) + X^2S_2^2(X) - 3XS_0(X)S_1(X)S_2(X).$$

On the other hand, from (II) of Theorem 4.4 it easily follows that (4) yields a factorization of $f(X^p)$ in $\mathbb{Z}[X]$ into $p$ irreducible factors if and only if $w \in Z$. From the case $w = 1$, by using (i) and (ii) of Corollary 4.5 and the fact that

$$uf(X^2) \in Z^2[X] \text{ if and only if } uf(X) \in Z^2[X],$$

the following result can also be easily derived.

**COROLLARY 4.6.** Assume $\chi(Z) = p$ is a prime number. Let $f(X)$ be any irreducible polynomial in $\mathbb{Z}[X]$ of positive degree and let $s$ be any positive integer.

(a) $f(X^{p^s})$ is reducible if and only if there exist $u \in U$ and $P(X) \in \mathbb{Z}[X]$, $P(X)$ irreducible, such that

$$uf(X^{p^s}) = P^p(X);$$

(b) $f(X^{p^s})$ is reducible if and only if there exists $u \in U$ such that

$$uf(X) \in Z^p[X].$$
5. Sufficient conditions

First we derive Theorem 1.1 from Theorem 4.3.

Proof. Assume \( f(X) = \sum_{0 \leq k \leq m} a_k X^k \) (so \( a = a_m, \ b = a_0 \)). It is well known that either \( f(X^n) = \sum_{0 \leq k \leq m} a_k X^{nk} \) and \( \tilde{f}(X^n) = X^{nm} f(1/X^n) = \sum_{0 \leq k \leq m} a_{m-k} X^{nk} \) are both irreducible, or both reducible in \( Z[X] \). Thus we can assume that \( C(m, a, b, n) \) holds. Looking for a contradiction suppose that \( f(X^n) \) is reducible. In this situation, from Theorem 4.3 it follows that there exist a prime \( p \) that divides \( n \) (\( p = 2 \) if (2) holds), a unit \( u \) in \( U \) with \( u a \in \mathbb{Z}^p \) (this contradicts (A)) and polynomials \( P(X), P^*(X) \) in \( Z[X] \) of positive degree that, in particular, satisfy \( P(0) P^*(0) = (P(0))^p \) and (9). Therefore, putting \( X = 0 \) in both sides of (9) we contradict (B) (i) if (1) holds, and (B) (ii) otherwise. \( \square \)

As an immediate consequence of Theorem 1.1 we get the following result.

COROLLARY 5.1. Let \( f(X) \) be any non constant polynomial in \( Z[X] \) with leading coefficient \( a \) and nonzero constant term \( b \) that is irreducible in \( Z[X] \). If \( a \notin U \) or \( b \notin U \), then the set of primes \( p \) such that \( f(X^p) \) is reducible is finite.

When \( U \) is finite we may reformulate Corollary 5.1 in the following way.

COROLLARY 5.2. Let \( f(X) \) be any non constant polynomial in \( Z[X] \) with leading coefficient \( a \) and nonzero constant term \( b \) that is irreducible in \( Z[X] \). Assume \( U \) is finite. If \( f(X) \) does not divides to any cyclotomic polynomial over \( Q \), then the set of primes \( p \) such that \( f(X^p) \) is reducible is finite.

Proof. From Corollary 5.1 it will be sufficient to prove that both \( a, b \) are in \( U \) if and only if \( f(X) \) is a divisor of some cyclotomic polynomial over \( Q \).

First suppose \( f(X) \) divides to some cyclotomic polynomial over \( Q \), say \( \Phi(X) \). Hence, since \( \Phi(X) \) is a monic polynomial in \( Z[X] \) and its constant coefficient is a product of units in \( U \), both \( a \) and \( b \) belong to \( U \).

Now suppose \( a \in U \) and \( b \in U \). Since the product of all roots of \( f(X) \) (in some extension of \( Q \) is equal to \( \pm ba^{-1} \in U \), all roots of \( f(X) \) in any extension of \( Q \) are actually in \( U \). Hence, for each root \( \alpha \) of \( f(X) \) we have \( \alpha^s - 1 = 0 \), where \( s = \#(U) \). Thus, since \( f(X) \) is irreducible, \( f(X) \) divides \( X^s - 1 \). Therefore, since \( X^s - 1 = \prod_{d \mid s} \Phi_d(X) \), \( f(X) \) divides to some cyclotomic polynomial over \( Q \). \( \square \)

Remark. Hypothesis \( U \) is finite can not be suppressed in Corollary 5.2. Because otherwise, considering for example, \( Z = Q = \mathbb{R} \) and the irreducible polynomial \( f(X) = X^2 + X + 2 \), we have that \( f(X) \) does not divides to any cyclotomic polynomial over \( Q \) (because the absolute values of the roots of \( f(X) \) are distinct of 1) and \( f(X^p) \) is reducible for each prime \( p \).

At this point it should be noted that Theorem 1.1 essentially establishes that for a given positive integer \( n \), if an arbitrary triple \( (m, a, b) \in \mathbb{N} \times \mathbb{Z}^* \times \mathbb{Z}^* \) satisfies \( C(m, a, b, n) \), then, for any \( f(X) = aX^m + \cdots + b \in Z[X], \)

\[ f(X^n) \text{ is irreducible if and only if } f(X) \text{ is irreducible.} \]

It is also of interest to determine, for a given irreducible polynomial \( f(X) = aX^m + \cdots + b \in Z[X] \) of positive degree \( m \), an appropriate set of positive integers, say \( N(m, a, b) \), such that \( f(X^n) \) is irreducible for each \( r \in N(m, a, b) \).
To illustrate the case that \( f(X^n) \) is irreducible for all \( r \in \mathbb{N} \) we consider Schur’s polynomials, which are defined for each positive integer \( m \) by

\[
f_m(X) = 1 + \frac{a_1}{1!}X + \frac{a_2}{2!}X^2 + \cdots + \frac{a_{m-1}}{(m-1)!}X^{m-1} + \frac{1}{m!}X^m \quad \text{(each } a_i \in \mathbb{Z}).
\]

It is well known that all these polynomials are irreducible in \( \mathbb{Q}[X] \) (see [4, pp. 373-374]). It is clear that the polynomial

\[
m!f_m(X) = \pm X^n + ma_{m-1}X^{m-1} + \cdots + \frac{m!a_2}{2!}X^2 + \frac{m!a_1}{1!}X + m! \in \mathbb{Z}[X]
\]

is primitive, so it is irreducible in \( \mathbb{Z}[X] \). Assume \( m \geq 2 \). In some cases (for example, when \( m \) is prime) we get that \( m!f_m(X^n) \) is irreducible for any \( n \in \mathbb{N} \) from Eisenstein’s Criterion, but in general this does not happen (consider, for example, \( m = 2^n > 3 \) and \( a_{m-1} = 1 \)). In any case we have \( \pm m! \notin \mathbb{Z}^p \) for each prime \( p \), because the largest prime not exceeding \( m \) has such a property. Then, since condition (B) of \( C(m, a, b, n) \) is always satisfied, \( m!f_m(X^n) \) is irreducible (i.e., \( f_m(X^n) \) is irreducible in \( \mathbb{Q}[X] \)) for any positive integer \( n \).

In order to include the precedent example in a more general result we assume that \( a, b \) are arbitrary nonzero elements of \( \mathbb{Z} \). First, we define the \((a, b)\)-admissible primes. We shall say that a prime number \( p \) is \((a, b)\)-admissible if there is no unit \( u \) in \( U \) such that both \( ua, ub \) are in \( \mathbb{Z}^p \). Otherwise we shall say that \( p \) is \((a, b)\)-inadmissible.

There is a simple procedure to determine the \((a, b)\)-inadmissible primes. We first define the exponent of \((a, b)\), say \( e(a, b) \). Assume that \( a \) has the factorization \( a = u_0p_1^{\alpha_1}\cdots p_s^{\alpha_s} \) in \( \mathbb{Z} \), where \( u_0 \in U \) and \( (a \notin U) \) \( p_1, \ldots, p_s \) are non-associate primes of \( \mathbb{Z} \) with positive exponents \( \alpha_1, \ldots, \alpha_s \). Let \( e(a) = 0 \) if \( a = u_0 \), and \( e(a) = \gcd(\alpha_1, \ldots, \alpha_s) \) otherwise. Assume a similar factorization for \( b \), and let \( e(a, b) = 0 \) if \( e(a) = e(b) = 0 \) and \( e(a, b) = \gcd(e(a), e(b)) \) otherwise. Then we can establish the following result.

**Lemma 5.3.** Let \( a, b \) be nonzero elements of \( \mathbb{Z} \) and let \( p \) be a prime number. Then

\[
p \text{ is } (a, b)\text{-inadmissible if and only if } p|e(a, b) \text{ and } u_a \equiv u_b \pmod{U^p}.
\]

**Proof.** To begin we express \( a \) and \( b \) in the form

\[
a = u_0^{e(a)}a_0, \quad b = u_0^{e(b)}b_0,
\]

where each one of \( a_0, b_0 \) is either equal 1, or a product of non-associate prime-powers of \( \mathbb{Z} \). Assume \( p|e(a, b) \) and \( u_a^{-1}u_b \in U^p \), say \( u_a^{-1}u_b = u_0^p \). Letting \( e = e(a, b) \) we can write

\[
a = u_0^\alpha a_0, \quad b = u_0^\beta b_0,
\]

where \( \alpha = u_0^{e(a)/e}, \beta = u_0^{e(b)/e} \). Hence,

\[
u_a^{e-1}a = (u_0^\alpha a_0)\varepsilon, \quad u_a^{e-1}b = (u_a^{-1}u_b)(u_a^\beta)^\varepsilon.
\]

Thus \( p \) is \((a, b)\)-inadmissible, because

\[
u_a^{e-1}a = ((u_0^\alpha a_0)\varepsilon)^p \text{ and } u_a^{e-1}b = (u_0^\beta)^{(p/e)}p.
\]

Now assume that \( p \) is \((a, b)\)-inadmissible. Therefore, there exist \( u \in U \) and \( \alpha, \beta \in \mathbb{Z} \) such that \( u\alpha = \alpha^p, u\beta = \beta^p \). Proceeding as previously with \( a \) and \( b \), we
can write \( \alpha = u_0 a^{e(a)} \), \( \beta = u_0 b^{e(b)} \), whence

\[
zu_0 a^{e(a)} = u_0 a^{pe(a)}, \quad uu_0 b^{e(b)} = u_0 b^{pe(b)}.
\]

Hence, from the unique factorization property of \( Z \), it follows \( pe(a) = e(a), pe(b) = e(b) \) and both \( uu_0, uu_0 b \in U^p \). Thus, \( p|e(a, b) \) and \( uu_0^{-1} uu_0 b \in U^p \). \( \square \)

Next we define the \((a, b)\)-admissible odd integers. For convenience we agree that 1 is \((a, b)\)-admissible. Let \( \mathbb{N}_a \) denote the set of odd positive integers. We shall say that \( r \in \mathbb{N}_a \) is \((a, b)\)-admissible if each one of their prime divisors is \((a, b)\)-admissible. Otherwise we shall say that \( r \) is \((a, b)\)-inadmissible.

Let \( \mathbb{N}_a(a, b) \) denote the set of \((a, b)\)-admissible odd integers. The set \( \mathbb{N}(m, a, b) \) of \((m, a, b)\)-admissible integers is defined then as follows:

\[
\mathbb{N}(m, a, b) = \begin{cases} 
\mathbb{N}_a(a, b) & \text{if } 2 \text{ is } (a, (-1)^m b)\text{-inadmissible}, \\
\mathbb{N}_a(a, b) \cup 2\mathbb{N}_a(a, b) & \text{if } 2 \text{ is both } (a, (-1)^m b)\text{-inadmissible and } (a, b)\text{-admissible}, \\
\cup_{k=0}^\infty 2^k \mathbb{N}_a(a, b) & \text{if } 2 \text{ is both } (a, (-1)^m b)\text{-inadmissible and } (a, b)\text{-admissible}.
\end{cases}
\]

Let \( n \) be any integer greater than 1. Writing \( n = 2^s q \), with \( q \) odd and \( s \) a nonnegative integer, we easily get the following:

1. If \( \mathbb{N}(m, a, b) = \mathbb{N}_a(a, b) \), then \( r \in \mathbb{N}(m, a, b) \iff s = 0 \) and \( C(m, a, b, r) \);
2. If \( \mathbb{N}(m, a, b) = \mathbb{N}_a(a, b) \cup 2\mathbb{N}_a(a, b) \), then \( r \in \mathbb{N}(m, a, b) \iff s \leq 1 \) and \( C(m, a, b, r) \);
3. If \( \mathbb{N}(m, a, b) = \cup_{k=0}^\infty 2^k \mathbb{N}_a(a, b) \), then \( r \in \mathbb{N}(m, a, b) \iff s \geq 0 \) and \( C(m, a, b, r) \).

Hence,

\( n \in \mathbb{N}(m, a, b) \iff C(m, a, b, n) \).

Consequently we reformulate Theorem 1.1 as follows.

**Theorem 5.4.** Let \( n \) be any integer greater than 1 and let \( f(X) \) be an irreducible polynomial in \( Z[X] \) of positive degree \( m \), leading coefficient \( a \) and nonzero constant term \( b \). Assume that at least one of the conditions \( n \in \mathbb{N}(m, a, b) \), \( n \in \mathbb{N}(m, b, a) \) holds. Then

\( f(X^n) \) is irreducible in \( Z[X] \).

Finally we use Lemma 5.3 to illustrate Theorem 5.4. Let \( Z = Z[i] \), where \( i = \sqrt{-1} \), and assume \( f(X) = X^m + \cdots + 8i \) is irreducible in \( Z[X] \). We have

\[
U = \{ \pm 1, \pm i \}, \quad a = 1 = u_a, \quad e(a) = 0 \quad \text{and} \quad b = -2(2i)^3 = -(1 + i)^6 \quad \text{with} \quad uu_0 b = -1, \quad e(b) = 6.
\]

Then, since \( e(a, b) = 6 \) and \( uu_0^{-1} uu_0 b = -1 \in U^p \) for each prime \( p \), we have that 2 and 3 are the unique \((1, 8i)\)-inadmissible primes. On the other hand, since \((-1)^m uu_0^{-1} uu_0 b = (-1)^{m+1} \in U^2 \), we also have that 2 is \((1, (-1)^m 8i)\)-inadmissible for all \( m \). Therefore,

\[
\mathbb{N}(m, 1, 8i) = \mathbb{N}_a(1, 8i) = \{ r \in \mathbb{N}_a : 3 \nmid r \}.
\]

which guarantees that \( f(X^n) \) is irreducible for each positive integer \( n \) that is relatively prime to 6.
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Universidad CAECE (retired), Buenos Aires, Argentina.

E-mail address: nguersonz@fibertel.com.ar