Modelling static impurities

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Abstract

A simple model is presented for the calculation of the quenched average over impurities which are rendered static by setting their mass equal to infinity. The path integral formalism of the second quantized theory contains annealed averages only. The similarity with the Gaussian quenched potential model is discussed.

I. INTRODUCTION

Disordered systems can be characterized by two time scales, the time $t$ spent between the preparation of the impurity distribution within the sample and the measurement, and the impurity diffusion time, $t_i$. There is actually a third characteristic time, the time span of the measurement, $t_m$ but it is far from the other time scales, $t_m \ll t, t_i$. We are interested in the dynamics in the limit when $t, t_i \to \infty$ but $R = t/t_i$ stays finite. The initial distribution of the impurities within a given sample determined by the preparation remains unchanged when $R \ll 1$. For $R \gg 1$ the impurities reach equilibrium with the faster degrees of freedom, electrons. The self averaging quantities measured on large samples can be obtained by means of averaging over the impurity distributions. The latter, the impurity distribution is determined by the preparation method when $R \ll 1$ or by the equilibrium properties for $R \gg 1$.

The usual modelisation methods were developed for $R \ll 1$. The time available for the impurity motion in this case is insufficient to establish the feedback of the fast degrees of freedom on the impurity dynamics. Such a suppression of a part of the dynamics can be realized by the replica method \cite{1}, \cite{2} the introduction of spurious supersymmetric particles

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A simple model to describe electrons in the presence of quenched disorder is based on the partition function\(^2\)

\[
Z_n = \int D[v] D[\psi] D[\psi^\dagger] e^{-\frac{1}{2} \int dx v^2(x) + i \sum_j \int dx dt \left[ \psi^\dagger_j(x,t) (i \partial_t + \frac{\bar{h}}{2m} \Delta - gv(x)) \psi_j(x,t) \right]},
\]

where \(v(x)\) is a static potential representing the disorder, \(\psi_j\) denotes the electron field, and \(j = 1, \ldots, n\) is the replica index. The limit \(n \to 0\) is made formally at the end of the computation of the observables.

Although these methods have already led to a number of important results they are not completely satisfactory. The strategy based on supersymmetry can not cope with the Coulomb interactions. The Keldysh formalism, given in terms of the propagators, is inherently perturbative. The replica method is not always reliable due to its formal nature.

Apart from glassy materials or fast quenching, the usual preparation of samples gives \(R >> 1\) and there is time for the electron dynamics to influence the impurity distribution seen during the measurement. One can easily incorporate the feedback of the electron dynamics into the impurity distribution by avoiding the replica method in the model\(^1\), i.e. by setting \(n = 1\). But the simple Gaussian distributed static field which is sufficient for the modelisation of the forces due to decoupled impurities might be too simple for the reproduction of the full impurity dynamics with its non-trivial time dependence.

We describe in this paper a simple model for \(R >> 1\), when the slowly moving impurities are in equilibrium with the electrons. Instead of the static potential \(v(x)\) and the replica method used in\(^1\), we return to the more detailed impurity dynamics. The impurities will be made static as far as the observables are concerned, by setting their mass equal to infinity. Their equilibrium with the electrons will be established by considering the partition function at finite density or temperature. Our model involves particle degrees of freedom, the impurities and the electrons. The elimination of the former produces an effective model similar to\(^1\) with \(n = 1\). We find that the salient feature of quenched disorder, the possibility of generating localisation, is realised in the same manner for \(R >> 1\) as for \(R << 1\). In particular, the perturbative solution of our model for the electronic observables obtained by resumming the two-loop self energy can be mapped onto a subset of graphs coming from\(^1\), namely those graphs which have no more than two impurity lines ending at the same space location (our impurities obey fermion statistics). This result implies that the conductivity, when computed through the resummation of the maximally crossed diagramms in the particle-hole channel, agrees in the two models. Since our model is given in terms of an annealed partition function this result opens the way for the application of different non-perturbative numerical methods to deal with quenched disorder when \(R >> 1\).

The organization of the paper is the following. Section II introduces our model which contains impurities with infinite mass, i.e. zero mobility. The infinite mass non-relativistic particles have a singular Fermi sphere, reflecting the high level of degeneracy of the ground state. This makes the use of the grand canonical ensemble questionable in this case. The canonical ensemble is not spoiled by the degeneracy of the free Hamiltonian. In section III we present a method to compute averages in this ensemble. The propagators are computed in section IV and section V describes the setting up of the perturbation expansion in the framework of the path integral formalism. The self energy and the propagator for a particle-hole pair are computed in section VII. Finally conclusions are drawn in section VIII.
Technical points concerning the regularization and the perturbation expansion are developed in the Appendices.

II. THE MODEL

Consider the system of electrons and impurities described by the fields $\psi$ and $\phi$, respectively. The hamiltonian is of the form

$$H = \int dx \left[ -\frac{\hbar^2}{2m} \psi^\dagger_a(x) \Delta \psi_a(x) + \frac{\lambda}{4} \psi^\dagger_a(x) \psi^\dagger_b(x) \psi_b(x) \right. $$

$$\left. -\frac{\hbar^2}{2M} \phi^\dagger(x) \Delta \phi(x) + g \psi^\dagger_a(x) \psi_a(x) \phi^\dagger(x) \phi(x) \right],$$

(2)

where $a$ and $b$ stand for the spin indices. Electrons and impurities are present with finite density, what is usually achieved by using the grand canonical ensemble, the modification of the Hamiltonian,

$$H \rightarrow H - \int dx \left[ \mu_e \psi^\dagger_a(x) \psi_a(x) + \mu_i \phi^\dagger(x) \phi(x) \right].$$

(3)

Here $\mu_e$ and $\mu_i$ are the chemical potential for the electrons and the impurities, respectively. The spinless impurity field is supposed to be anticommuting, a reasonable approximation even for integer spin atoms in solids. In Eq. (3) $H$ is considered as an effective hamiltonian for the conducting band with a spatial resolution which is so low that the inhomogeneity of the crystalline structure is not seen.

We render the impurities static by setting $M = \infty$. Then the lagrangian possesses an additional, space dependent U(1) gauge symmetry

$$\phi(x,t) \rightarrow e^{i\theta(x)} \phi(x,t),$$

(4)

which indicates the absence of spatial correlations for the impurities. This is not equivalent to the space-dependent part of the electromagnetic U(1) gauge invariance, it only characterizes the difference between the slowly moving and the truly quenched impurities.

What happens in the limit $M \rightarrow \infty$? To understand this limit better it is useful to introduce the free diffusion constant $D = \hbar/2M$ for the impurities. The impurities generate an interaction on the distance scale $\sqrt{DT}$ during the elapse of time $T$. When this distance shrinks well below the spacelike cutoff, the lattice spacing, the model reaches the quenched limit as far as the observables up to time $T$ are concerned, and the local symmetry (4) is approximatively realized. The distribution of these apparently static impurities is not uniform, it is governed by the ensemble used in the computation. Since we shall use the canonical ensemble, our results refer to a thermally equilibrated and after then quenched impurity distribution.

It is easy to see that the limit $M \rightarrow \infty$ enhances the large momentum and the low frequency contributions of the interactions. The perturbative short distance contributions are strengthened by the absence of the kinetic energy in the denominator of the propagator. This increase of the short range fluctuations is the impact of the static nature of the impurities on the spatial disorder. The slowing down of the impurity motion induces non-local
interactions in the time rather than the space direction \[6\]. The increased sensitivity at low frequencies arises from the suppression of the hopping between neighboring spatial locations, the reduction of the effective spatial dimensionality of the impurity dynamics \(d_{eff} \to 0\) as \(M \to \infty\). This reduces the impurity dynamics to the sum of non-interacting Quantum Mechanical single site problems. The result of the strong correlations generated in time is a possible non-local effect in time within the electron sector.

The dependence of the static observables on \(M/m\) is not continuous at \(M/m = \infty\), e.g. the conductivity diverges for \(M/m < \infty\) because the electrons, moved by the external electric field drag the impurities along. The conductivity is finite only in the absence of recoil, when \(M/m = \infty\). The discontinuity at \(M/m = \infty\) is reflected in the existence of the gauge symmetry (4) and the singularity of the Fermi surface. In fact, the Fermi sphere for \(M < \infty\) consists of the one particle states \(p^2/2M \leq \mu_i\). The Fermi momentum is \(p_F \approx \rho^{1/d}\) in \(d\) dimensions where \(\rho\) is the particle density which yields \(\mu_i \approx \rho^{2/d}/M\). Hence the Fermi sphere becomes a highly singular point in the momentum space as \(M \to \infty\) and the construction of the grand canonical ensemble is not obvious. The same problem is seen by the vanishing Fermi velocity, \(v_F \approx \rho^{1/d}/M\). The ground state is highly degenerate and admits the gauge symmetry (4). Due to this complication we shall carry out the computations below for \(M = \infty\) in the canonical ensemble for the impurities.

### III. CANONICAL ENSEMBLE FOR THE IMPURITIES

Let us consider first a system of fermionic impurities without spin. In order to avoid singularities in the static limit we place them on a lattice with \(N\) lattice points (see Appendix A) but keep the continuous notation whenever it is not misleading. The Hamiltonian with an external source \(J\) is chosen to be time dependent,

\[
H_0(t) = \int dx \left[ -\frac{\hbar^2}{2M} \phi^\dagger(x,t) \Delta \phi(x,t) + J(x,t) \phi^\dagger(x,t) \phi(x,t) \right]
\]

in the Heisenberg representation\[1\]. We introduce the real time expectation values of the operator \(O\) in the canonical ensemble corresponding to the density \(\rho\) as

\[
\langle O \rangle_\rho = \frac{1}{Z_\rho} \text{Tr} P_\rho T \left[ e^{-\frac{i}{\hbar} \int_{-T}^T dt H_0(t)} O \right],
\]

where

\[
Z_\rho = \text{Tr} P_\rho T \left[ e^{-\frac{i}{\hbar} \int_{-T}^T dt H_0(t)} \right],
\]

and \(P_\rho\) is the projection operator onto the subspace of \(n = \rho V\) particles (\(V\) denotes the volume),

\[1\] By anticipation of the perturbation expansion in the source \(J\) the expressions for the free propagator will be given for \(J = 0\).
\[ P_\rho = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{i\alpha} \int dx (\phi^d \phi^\dagger - \rho) \]  

(8)

which guarantees finite density for arbitrary mass. The finite temperature averages can be obtained by performing a Wick rotation in the time evolution,

\[ \langle O \rangle_{\rho,\beta} = \frac{1}{Z_{\rho,\beta}} Tr P_\rho T \left[ e^{-\int_0^\beta dt H_0(t)} O \right], \]

(9)

with

\[ Z_{\rho,\beta} = Tr P_\rho T \left[ e^{-\int_0^\beta dt H_0(t)} \right]. \]

(10)

The trace is evaluated by using localized states either in momentum or in real space for \( M < \infty \) or \( M = \infty \), respectively,

\[ Tr O = \sum_{\{n\}} \langle n | O | n \rangle, \]

(11)

where the states \( |n\rangle \) are the Fock states determined by the occupation number configuration \( n \),

\[ |n\rangle = \begin{cases} \prod_p (a_p^\dagger)^{n_p} |0\rangle & \text{for } M < \infty, \\ \prod_x (a_x^\dagger)^{n_x} |0\rangle & \text{for } M = \infty, \end{cases} \]

(12)

and

\[ [a_p^\dagger, a_q^\dagger]_+ = \delta_{p,q}, \quad [a_x^\dagger, a_y^\dagger]_+ = \delta_{x,y}. \]

(13)

The products of creation operators are taken in an arbitrarily chosen but fixed order in (12). We obviously have the relations

\[ \langle a_p^\dagger a_q \rangle_\rho = \hat{\rho} \delta_{p,q}, \quad \langle a_x^\dagger a_y \rangle_\rho = \hat{\rho} \delta_{x,y} \]

(14)

for a translational invariant system, where \( \hat{\rho} = a^d \rho \) stands for the dimensionless impurity density. These results allow us to obtain the free propagator in the canonical ensemble. The causal propagator is defined as (cf. Appendix [B])

\[ iG(x, t, x', t') = \langle T[\phi(x, t)\phi^\dagger(x', t')] \rangle \]

(15)

giving, for a fixed volume \( V \),

\[ iG(x, t, x', t') = \frac{1}{V} \sum_p e^{ip(x-x')-ie_p(t-t')} \begin{cases} 1 - \hat{\rho} & \text{for } t > t', \\ -\hat{\rho} & \text{for } t < t', \end{cases} \]

(16)

where \( \epsilon_p \) is the single particle excitation spectrum for \( J = 0 \) and

\[ iG(x, t, x', t') = \begin{cases} (1 - \hat{\rho})\delta_{x,x'} & \text{for } t > t', \\ -\hat{\rho}\delta_{x,x'} & \text{for } t < t', \end{cases} \]

(17)
for \( M < \infty \) and \( M = \infty \), respectively. The Fourier transformation

\[
G(p, q, \omega) = \delta_{p, q} G(p, \omega)
\]

\[
iG(p, \omega) = (1 - \hat{\rho}) \int_0^\infty e^{it(\omega - \epsilon_p)} dt - \hat{\rho} \int_0^\infty e^{-it(\omega - \epsilon_p)} dt
\]

(18)

is performed with the usual choice of the boundary condition

\[
\int_0^\infty e^{i\omega t} dt \rightarrow \int_0^\infty e^{i\omega t - \delta t} dt
\]

(19)

with \( \delta \rightarrow 0^+ \),

\[
G(p, \omega) = \frac{1 - \hat{\rho}}{\omega - \epsilon_p + i\delta} + \frac{\hat{\rho}}{\omega - \epsilon_p - i\delta}
\]

\[
= \frac{1}{\omega - \epsilon_p + i\delta} + 2\pi i\hat{\rho} \delta(\omega - \epsilon_p),
\]

\[
G(x, x', \omega) = \frac{\delta_{x, x'}}{\omega + i\delta} + 2\pi i\hat{\rho} \delta(\omega).
\]

(20)

The retarded Green function can be written as

\[
G_R(p, \omega) = \frac{1}{\omega - \epsilon_p + i\delta}, \quad G_R(x, x', \omega) = \frac{1}{\omega + i\delta}
\]

(21)

and the advanced one is \( G_A = G^*_R \).

In order to gain an insight into the impact of the impurities on the dynamics consider the coupled impurity-electron system defined by the Hamiltonian

\[
H = \int dx \left[ \frac{\hbar^2}{2m} \psi_a^\dagger(x, t) \Delta \psi_a(x, t) + g \phi^\dagger(x, t) \phi(x, t) \psi_a^\dagger(x, t) \psi_a(x, t)
\]

\[
+ \frac{\lambda}{4} \left( \psi_a^\dagger(x, t) \psi_a(x, t) \right)^2 \right].
\]

(22)

Since the impurities located at different space locations decouple it is easy to obtain the effective theory for the electrons where the impurity degrees of freedom appear as local variables. Leaving the actual computation to Appendix D the final result for the expectation value of an observable \( \mathcal{O} \) between the electronic states \( \psi_i, \varepsilon \) and \( \psi_f, \varepsilon \) in the canonical ensemble for the static impurities can easily be obtained as a summation over the impurity occupation numbers \( n_\bar{x}, \sum_x n_\bar{x} = n = \hat{\rho} N, \ N = V/a^d \),

\[
\langle \langle \psi_f, \varepsilon | \mathcal{O} | \psi_i, \varepsilon \rangle \rangle = \frac{1}{Z_\rho} \sum_{\{n_\bar{x}\}} \langle \psi_f, \varepsilon | T \left[ e^{-\frac{i}{\hbar} \int_{-T}^T dt H_{eff}(t; \{n_\bar{x}\})} \right] \mathcal{O} | \psi_i, \varepsilon \rangle, \]

(23)

where

\[
Z_\rho = \sum_{\{n_\bar{x}\}} \langle \psi_f, \varepsilon | T \left[ e^{-\frac{i}{\hbar} \int_{-T}^T dt H_{eff}(t; \{n_\bar{x}\})} \right] | \psi_i, \varepsilon \rangle,
\]

(24)

and
\[ H_{\text{eff}}(t; n_{\hat{x}}) = \int dx \left[ -\frac{\hbar^2}{2m} \psi_a^\dagger(x, t) \psi_a(x, t) + gn(x, t) \psi_a^\dagger(x, t) \psi_a(x, t) + \frac{\lambda}{4} \left( \psi_a^\dagger(x, t) \psi_a(x, t) \right)^2 \right]. \] (25)

This result shows clearly the connection with model (I), namely the impurities are static both models but they reach equilibrium with the electrons in our case.

**IV. AUXILIARY PROPAGATORS**

The expressions (20) give the correct propagator in the canonical ensemble but are not well suited for a perturbation expansion. In order to generate more conventional Feynman rules we look into the detailed way the averages (14) were achieved and introduce some auxiliary quantities. These are obtained by postponing the integration in the projection operator (8) and by putting the operator of the exponent in (8) into the hamiltonian. This amounts to the usual strategy of gauge models, treating the integral variables of the constraints in the path integral as dynamical variables. With this in mind we introduce the impurity hamiltonian

\[ H_\alpha(t) = \int dx \left[ -\frac{\hbar^2}{2M} \phi^\dagger(x, t) \phi(x, t) + (\alpha + J(x, t)) \phi^\dagger(x, t) \phi(x, t) \right], \] (26)

with the source \( J \) being kept vanishing in this Section and the expectation values given by

\[
\langle O \rangle_\rho = \frac{\int_{-\pi}^{\pi} d\alpha e^{i\alpha \rho V} Tr T \left[ e^{-\frac{i}{\hbar} \int_{-T}^{T} dt H_\alpha'(t) O} \right]}{\int_{-\pi}^{\pi} d\alpha e^{i\alpha \rho V} Tr T \left[ e^{-\frac{i}{\hbar} \int_{-T}^{T} dt H_\alpha'(t) } \right]},
\] (27)

where \( \alpha' = \alpha / 2T \). It is furthermore useful to introduce the propagator corresponding to a given value of \( \alpha \),

\[ G_\alpha(x, t, x', t') = Tr T \left[ e^{-\frac{i}{\hbar} \int_{-T}^{T} dt H_\alpha'(t) T[\phi(x, t) \phi^\dagger(x', t')] \right]. \] (28)

One can easily find its explicit form for \( M = \infty \),

\[ G_\alpha(x, t, x', t) = \delta_{x,x'} e^{-i\alpha t/2} \left( 1 + e^{-i\alpha t} \right)^{N-1} \left\{ 1 \right\} \begin{cases} 1 & \text{for } t > 0, \\ -e^{-i\alpha} & \text{for } t < 0, \\ \end{cases} \] (29)

since this non-normalized expectation value corresponds to the time evolution of a state with a single filled and \( N - 1 \) empty sites, see Appendix [3] for the detailed derivation. As a check we compute the impurity density,

\[
-G(x, 0, 0^+) = \frac{\int_{-\pi}^{\pi} d\alpha e^{i\alpha} (1 + e^{-i\alpha})^{N-1} e^{-i\alpha}}{a d \int_{-\pi}^{\pi} d\alpha e^{i\alpha} (1 + e^{-i\alpha})^N}. \] (30)

The integration over \( \alpha \) selects the particle combinations with the desired particle number,
\[-G(x, 0, x, 0^+ \rangle = \frac{(N - 1)}{a^d} \frac{n-1}{n} = \frac{n}{V} = \rho.\]  

\[(31)\]

When \( M < \infty \) we find

\[ G_\alpha(p, t) = e^{-i(\epsilon_p + \alpha)t} \prod_{q \neq p} \left( 1 + e^{-2i \epsilon_q + \alpha} \right) \left\{ \begin{array}{ll}
1 & \text{for } t > 0, \\
- e^{-2i \epsilon_q} & \text{for } t < 0,
\end{array} \right. \]  

\[(32)\]

where \( \epsilon_p \) denotes the one-particle energy. The finite temperature propagator reads

\[ G(x, t, x', t') = \frac{\int_{-\pi}^{\pi} da e^{ia\rho V} G_{\alpha/\beta}(x, t, x', t')}{\int_{-\pi}^{\pi} da e^{ia\rho V} (1)_{\alpha/\beta}} \]

\[ = \frac{1}{V} \sum_p e^{ip(x-x') - i\epsilon_p(t-t')} \int_{-\pi}^{\pi} da e^{ina} \prod_{q \neq p} \left( 1 + e^{-\beta \epsilon_q - i\alpha} \right) \]

\[(33)\]

which is just the Gibbs average of single particle contributions in the given particle number sector. Note that the parameter \( \alpha \) is purely imaginary in an imaginary time formalism, it appears as the time component of a gauge field.

V. PERTURBATION EXPANSION

We have two different realizations of the canonical averages depending on whether the projection operator is inserted once or after each \( dt \) time step during the time evolution \(-T < t < T\). The first case leads to the path integral expression

\[ \frac{1}{Z_\rho} \int da \int D[\phi] D[\phi^*] e^{i \int dt dx [i\phi^* \phi - \frac{\beta^2}{2M} \phi^* \Delta \phi + \alpha(\phi^* \phi - \rho) + J\phi^* \phi]} \mathcal{O}, \]

\[(34)\]

for the canonical average of the operator \( T[\mathcal{O}] \), where

\[ Z_\rho = \int da \int D[\phi] D[\phi^*] e^{i \int dt dx [i\phi^* \phi - \frac{\beta^2}{2M} \phi^* \Delta \phi + \alpha(\phi^* \phi - \rho) + J\phi^* \phi]} \]

\[(35)\]

In the second case we have a time dependent \( \alpha(t) \) trajectory to integrate over,

\[ \frac{1}{Z_\rho} \int D[\alpha] D[\phi] D[\phi^*] e^{i \int dt dx [i\phi^* \phi - \frac{\beta^2}{2M} \phi^* \Delta \phi + \alpha(\phi^* \phi - \rho) + J\phi^* \phi]} \mathcal{O}, \]

\[(36)\]

and

\[ Z_\rho = \int D[\alpha] D[\phi] D[\phi^*] e^{i \int dt dx [i\phi^* \phi - \frac{\beta^2}{2M} \phi^* \Delta \phi + \alpha(\phi^* \phi - \rho) + J\phi^* \phi]} \]

\[(37)\]

\(^2\) \( dt \) serves as the ultraviolet cutoff needed in the derivation of the path integral formulae. The limit \( dt \to 0 \) is convergent provided that the number of the degrees of freedom is kept finite by a cutoff in space.
Due to the periodicity of the integrand the integration over $\alpha$ can be performed over the whole real axis. The steps leading to (34) and (36) are similar to those giving the path integral expressions in QED for the static temporal gauge, $\partial_0 A_0 = 0$ and for the real temporal gauge $A_0 = 0$, respectively, $A_0(x,t)$ playing the role of the projection operator parameter $\alpha(t)$.

The straight perturbation expansion for (36) yields

$$\langle O \rangle = \frac{1}{\mathcal{Z}_\rho} \int D[\alpha]D[\phi]D[\phi^*] e^{i \int dt dx \left[ i \phi^* \phi - \frac{\hbar^2}{2m} \phi^* \Delta \phi - \alpha \rho \right]}$$

$$\times \mathcal{O} \sum_n \frac{1}{n!} \left( i \int dt dx (\alpha + J) \phi^* \phi \right)^n,$$

(38)

and

$$\mathcal{Z}_\rho = \int D[\alpha]D[\phi]D[\phi^*] e^{i \int dt dx \left[ i \phi^* \phi - \frac{\hbar^2}{2m} \phi^* \Delta \phi - \alpha \rho \right]}$$

$$\times \sum_n \frac{1}{n!} \left( i \int dt dx (\alpha + J) \phi^* \phi \right)^n.$$

(39)

The small parameter of the perturbation expansion is $J$ which stands for the interaction with an external source. The small parameter for the projection operator is $\frac{1}{V}$ for bosons when the expansion is performed around $\phi(x,t) = \sqrt{\rho}$.

For fermions there is no simple way of saturating the path integral and the projection operator must be implemented nonperturbatively. This can be achieved when it is inserted once only in the average,

$$\langle O \rangle = \frac{1}{\mathcal{Z}_\rho} \int d\alpha \int D[\phi]D[\phi^*] e^{i \int dt dx \left[ i \phi^* \phi - \frac{\hbar^2}{2m} \phi^* \Delta \phi + \alpha \phi^* - \alpha^2 \frac{T}{\rho} \right]}$$

$$\times \mathcal{O} \sum_n \frac{1}{n!} \left( i \int dt dx J \phi^* \phi \right)^n,$$

(40)

and

$$\mathcal{Z}_\rho = \int d\alpha \int D[\phi]D[\phi^*] e^{i \int dt dx \left[ i \phi^* \phi - \frac{\hbar^2}{2m} \phi^* \Delta \phi + \alpha \phi^* - \frac{T}{\rho} \right]}$$

$$\times \sum_n \frac{1}{n!} \left( i \int dt dx J \phi^* \phi \right)^n.$$

(41)

In what follows we shall use this formalism, where the hamiltonian is given by (26). There is a vertex corresponding to the interaction term $J \phi^* \phi$, the propagator is $G_\alpha$ and the integration over $\alpha$ is to be done after the loop integrations. The usual Feynman rules are applicable for the computation of the integrand for the integration over $\alpha$ in the numerator and the denominator independently. Due to the independent integrations over $\alpha$ the disconnected diagrams do not always simplify in the expectation values, a remnant of the non-local nature of the canonical ensemble. But one can verify that the grand canonical result where the disconnected contributions simplify is recovered in the thermodynamical limit.

A short discussion is now in order about the use of the lattice regularization in the computation of the loop integrals emerging from the perturbation expansion. Non-relativistic
quantum field theory is non-renormalizable in itself, a problem which is made even more serious by sending the impurity mass to infinity as mentioned in Section II. This is naturally a formal problem only since the cutoff is actually kept finite in effective theories. Nevertheless it is useful to distinguish observables which diverge from those which stay finite when the cutoff is removed because the computation of the latter is simpler. To understand the reason let us start with the remark that the lattice regulated model is more complicated than the one obtained in the continuum due to the trigonometric functions in the propagator. In fact, the rule of generating the lattice propagators from the continuum say for electrons in 2D is

\[
\frac{1}{\omega - \frac{p^2}{2m} + \mu + i\delta_{\mu}^{p_2}} \rightarrow \frac{1}{\omega - \frac{2}{ma^2} \left( \sin^2 \frac{p_x a}{2} + \sin^2 \frac{p_y a}{2} \right) + \mu + i\delta_{\mu}^{p_2}},
\]

where \(a_s\) is the spatial lattice spacing and

\[
\delta_{\mu}^{p_2} = \begin{cases} 
0^+ & \frac{p^2}{2m} > \mu, \\
0^- & \frac{p^2}{2m} < \mu.
\end{cases}
\]

In computing the integral over the spatial momentum on the lattice one usually assumes that the cutoff is far from the internal scale of the model, takes the limit \(a_s \rightarrow 0\) and expands the lattice propagators. We regain the continuum propagator in this manner and the higher order terms in this expansion, the irrelevant, higher order derivative operators are formally suppressed by a positive power of the lattice spacing. One is tempted to ignore them altogether. But this is not always allowed because the ultraviolet divergences of the loop integration might generate so strong singularities that quantities which seemed to disappear turn out to be finite or even divergent as \(a_s \rightarrow 0\).

One can see \[7\] that no problem arises if one considers observables in a renormalizable model without anomaly. We can always carry out the substractions in the loop integrands of a renormalizable theory in such a manner that the resulting integrals are finite. An anomaly appears when ”by accident” a graph with non-negative primitive degree of divergence given by the power counting happens to be finite and receives no subtraction during the renormalization process. Thus in a renormalizable, non-anomalous model all loop integrals of the renormalized perturbation expansion are finite and have negative primitive degree of divergence. These integrals converge uniformly as the cutoff is removed and the order of the integration and the limit \(a_s \rightarrow 0\) can be interchanged. By setting \(a_s = 0\) in the integrands one eliminates all lattice artifacts and the loop integrals which follow reproduce the continuum perturbation expansion.

Returning to our non-relativistic model, the lesson of the argument about the suppression of the lattice artifact is that the continuum propagators can safely be used for quantities which stay finite as \(a_s \rightarrow 0\). This is enough to simplify the computation of several important quantities such as the imaginary part of the self energies. For other observables, the cutoff \(a_s\) must be kept finite and the ultraviolet details of the model remain important.

VI. ELECTRON PROPAGATION

In this section we calculate the first two order contributions to the electron self energy in two dimensions. The goal of this calculation is to show that the diagrams with at most
two impurity lines attached to each vertex agree in the present approach and those of 
the Edwards model can be brought into equivalence after the rescaling of the impurity density.

We consider the case $\lambda = 0$ for simplicity when the Feynman rules can be read off from 
the generating functional

$$Z_{\alpha}[J^*, J, j^\dagger, j] = \int D[\psi^\dagger]D[\psi]D[\phi^\dagger]D[\phi]e^{i \int dt dx [\psi^\dagger (i\partial_t - \alpha') \psi + \psi^\dagger (i\partial_t + \frac{\alpha}{2m} \Delta + \mu) \psi]}$$

$$\times e^{i \int dt dx [J^\dagger \phi + \phi^* J + j^\dagger \psi_\alpha + \psi^\dagger J_\alpha]} e^{-\frac{i}{\hbar} \int dt dx \psi_\alpha \phi^* \phi}$$

$$= \det[(iG_\alpha)^{-1}] \det[(iG_0)^{-1}] e^{-\frac{\hbar}{\epsilon} \int dt dx \frac{\delta}{\delta J_\alpha(x,t)} \frac{\delta}{\delta (J_\alpha(x,t))}}$$

$$\times e^{i \int dt dx dt' dx' [J^\dagger (x,t) G_\alpha(x,t,x',t') J(x',t') + j^\dagger (x,t) G_0(x,t,x',t') j(x',t')]}.$$  \hspace{1cm} (44)

The electron self energy in the leading order is computed in Appendix E1 and Eq. (E5) yields

$$\Sigma^{(1)} = \rho g.$$  \hspace{1cm} (45)

The two $O(g^2)$ contributions, shown in Figs. 1 and 2 and computed in Appendix E2 give

$$Im \Sigma^{(2)}_1(\omega, p) = -\frac{1}{\epsilon} g^2 \rho (1 - \hat{\rho}) n_0(\mu),$$  \hspace{1cm} (46)

and

$$Im \Sigma^{(2)}_2(\omega, p) = -g^2 \rho (1 - \hat{\rho}) \frac{m}{2} (2\theta(\omega) - \theta(\omega + \mu)),$$  \hspace{1cm} (47)

respectively where $\epsilon = 0^+$, and

$$n_0(\mu) = \frac{m\mu}{2\pi}. \hspace{1cm} (48)$$

There are peculiarities about the graph in Fig. 1. First, it involves the feedback of 
the electron dynamics on the impurities and it is thereby suppressed in the replica method. 
What is the role of this contribution in our model? The other peculiarity is that the meaningless 
epsilon in Eq. (46) indicates an infrared divergence, expected to appear in zero dimensional 
systems. We show that the solution of the latter problem is the answer for the question 
raised before, namely it leaves this particular graph altogether from our model. The point 
is that the divergence can be removed by the usual ring diagram, RPA resummation, by 
taking into account the ”screening” of the electron density in the impurity propagator so 
that

$$H_\alpha(t) = \alpha' \int dx \phi^\dagger(x,t) \phi(x,t) \rightarrow H_{\alpha + 2Tg_0(\mu)}(t) = (\alpha' + g_0(\mu)) \int dx \phi^\dagger(x,t) \phi(x,t).$$  \hspace{1cm} (49)

This procedure yields

$$iG_{\alpha + 2Tg_0(\mu)}(x,t,x',t') = \delta(x - x') e^{-i(\alpha + 2Tg_0(\mu))(t - t')}$$

$$\times \left\{ \theta(t - t' - \eta) \left( 1 - n_{\alpha + 2Tg_0(\mu)} \right) - \theta(t' - t + \eta) n_{\alpha + 2Tg_0(\mu)} \right\}. \hspace{1cm} (50)$$

The correction to the electron propagator is therefore
Imaginary expression of the Edwards model can be obtained from (52) by the replacement

\[ \tau \rightarrow \rho \]

where \( \tau \) is the relaxation time. Recall that the dimensionless density \( \rho \in [0,1] \) and \( Im\Sigma = \mathcal{O}(\rho) \), the electron relaxation time diverges at low impurity densities. The analogus expression of the Edwards model can be obtained from (52) by the replacement

\[ \rho(1 - \rho) \rightarrow \rho. \]

Our expression reflects the fact that the electrons do not scatter on the completely filled up impurity system (Pauli blocking), a phenomenon which is neglected when the effects of the impurities is represented by a static potential only.

Let us now consider the propagation of a particle-hole pair with energy-momenta \( \omega_+, p_+, \omega_-, p_- \), respectively, and write the corresponding amplitude as

\[ G(\omega_+, p_+, \omega_-, p_-) = iG_0(\omega_+, p_+)iG_0(\omega_-, p_-) \]

\[ \times \delta(p_+ - p'_- - p_- + p'_+) \Xi. \]

In the leading order, before the \( \alpha \) integration we find

\[ \Xi_{\alpha} = (ig)^2 \sum_{\omega,\omega'} iG_\alpha(\omega)iG_\alpha(\omega') \delta(\omega_+ - \omega'_+ + \omega - \omega') \delta(\omega' - \omega + \omega_- \omega_-) \]

\[ = -(ig)^2 2\pi (1 - \rho_\alpha) \rho_\alpha. \]
The $\alpha$ integration yields

$$\Xi = -(ig)^2 2\pi (1 - \hat{\rho}) \rho, \quad (57)$$

a result which agrees with the Edwards model after the replacement $[54]$. The insertion of $\ell$ impurity bubbles at different sites and the $\alpha$ integration yield the factor

$$\frac{\int_{-\pi}^{\pi} e^{i\alpha n} \det [(iG_\alpha)^{-1}] [(1 - n_\alpha)n_\alpha]^{\ell}}{\int_{-\pi}^{\pi} e^{i\alpha n} \det [(iG_\alpha)^{-1}]} \approx [\rho(1 - \hat{\rho})]^{\ell}. \quad (58)$$

Since the particle-hole lines are the usual ones in the rest of the diagram, this leads to the equivalence of the resummation over the ladder or the maximally crossed graphs, as well. In the light of the results of ref. [5] this indicates that the localisation-delocalisation phase structure is similar in these descriptions.

Not all diagrams of the local, Gaussian potential model $[1]$ can be brought into equivalence with our model. There one finds contributions with more than two impurity lines at a coordinate space location. This can not happen in our model where the fermionic statistic of the impurity cancels these diagrams. But it is easy to see that the remaining diagrams, where only two impurity lines are attached at each spatial lattice site agree after the change $[54]$. Furthermore, the disconnected contributions simplify for these graphs in the thermodynamic limit. The point is that the impurity propagators factorise and the disconnected parts drop out as in the usual, annealed averages. The resulting impurity propagators at different lattice sites give the same contributions when the thermodynamic limit is performed. This result is actually expected if the canonical and the grand canonical ensemble are equivalent.

There are furthermore diagrams in our model which have no counterpart in $[1]$. These graph contain internal fermion loops and some of them cancel after resummation. But there are remaining non-vanishing contributions, e.g. the correlation of the potential is ultralocal and unchanged by the electrons for $[1]$ but the electron dynamics generates a non-trivial correlation function for the impurity density $\langle \phi^* \phi \rangle$ in our model, the remnant of the annealed integration over the field $\phi$. Such an electron dynamics generated correlation is absent in the Green functions for the impurities, the correlation functions of the field variable $\phi^*$ or $\phi$ remain local due to the gauge symmetry $[1]$. It remains to be seen if the feedback of the electron dynamics generated by the resummation of the irreducible vertices obtained in the higher loop aproximation cancels or not.

**VII. CONCLUSIONS**

The quenched averaging over static impurities is achieved within the second quantized formalism by means of an annealed averaging procedure where the impurity motion is slowed down by setting the impurity mass equal to infinity. This description corresponds to a system of electrons and static impurities in equilibrium. Despite the feedback of the electrons on the impurity dynamics the localisation-delocalisation phase structure in the present approach is expected to be similar to the phase structure of the Edwards model.
The problem of the usual methods for disordered systems is their inability to deal with strong interactions. However our model is well suited to non-perturbative methods. The numerical simulation in lattice regularization is feasible since one can easily construct stochastic sampling algorithms in our canonical ensemble at finite temperature. Another natural non-perturbative method available is the functional renormalization group approach in the internal space [11] which can treat the finite density fermionic systems in a simple manner.

VIII. ACKNOWLEDGMENT

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APPENDIX A: LATTICE REGULARIZATION

The regulator applied in this work is a lattice in space-time with spatial and temporal lattice spacing \( a_s \) and \( a_t \), respectively. The action is written as

\[
S = \frac{1}{\hbar} \int dt dx L[\phi^\dagger(x,t), \phi(x,t), \psi^\dagger(x,t), \psi(x,t)] \rightarrow \sum_{\tilde{x},\tilde{t}} \hat{L}[\hat{\phi}^\dagger_{\tilde{x},\tilde{t}}, \hat{\phi}_{\tilde{x},\tilde{t}}, \hat{\psi}^\dagger_{\tilde{x},\tilde{t}}, \hat{\psi}_{\tilde{x},\tilde{t}}],
\]

(A1)

with \( S = S_i + S_e + S' \),

\[
\hbar \int dt dx \phi^\dagger(x,t)(i\partial_t - \alpha)\phi(x,t) \rightarrow \hat{S}_i = \sum_{\tilde{x},\tilde{t}} \hat{\phi}^\dagger_{\tilde{x},\tilde{t}}(i\partial_{\tilde{t}} - \hat{\alpha})\hat{\phi}_{\tilde{x},\tilde{t}},
\]

\[
\int dt dx \psi^\dagger(x,t)(i\partial_t + \frac{\hbar}{2m}\Delta + \mu)\psi(x,t) \rightarrow \hat{S}_e = \sum_{\tilde{x},\tilde{t}} \hat{\psi}^\dagger_{\tilde{x},\tilde{t}}(i\partial_{\tilde{t}} + \frac{\hbar}{2m}\hat{\Delta} + \hat{\mu})\hat{\psi}_{\tilde{x},\tilde{t}},
\]

\[
-g\hbar \int dt dx \psi^\dagger(x,t)\psi(x,t)\phi^\dagger(x,t)\phi(x,t) \rightarrow \hat{S}' = -\hat{g} \sum_{\tilde{x},\tilde{t}} \hat{\psi}^\dagger_{\tilde{x},\tilde{t}}\hat{\psi}_{\tilde{x},\tilde{t}}\hat{\phi}^\dagger_{\tilde{x},\tilde{t}}\hat{\phi}_{\tilde{x},\tilde{t}},
\]

(A2)

where \( \tilde{x}^j = n^j a_s, j = 1, \ldots, d, \tilde{t} = \ell a_t \) and the following lattice quantities were introduced in 2D \( \hat{\phi}_{\tilde{x},\tilde{t}} = a_s \phi(\tilde{x},\tilde{t}) \), (the field variables with space-time coordinates in the subscript are dimensionless) \( \hat{\alpha} = a_t \alpha, \hat{m} = a_s m, \hat{\hbar} = a_t \hbar/a_s, \hat{\psi}_{\tilde{x},\tilde{t}} = a_s \psi(\tilde{x},\tilde{t}), \hat{g} = g/a_s, \) and \( \hat{\mu} = a_t \mu \). The finite difference operators are defined as \( \hat{\partial}_t f_{n,\ell} = f_{n,\ell} - f_{n,\ell-1} \) and \( \hat{\Delta} f_{n,\ell} = \sum_{i=1}^d f_{n+e_i,\ell} + f_{n-e_i,\ell} - 2f_{n,\ell} \).
APPENDIX B: OPERATOR FORMALISM

The dynamics generated by the free impurity Hamiltonian

\[ H_\alpha = \hat{\alpha} \sum_x \hat{n}_x, \tag{B1} \]

where the static impurity density is denoted by

\[ \hat{n}_x = \hat{\phi}^\dagger_{x,t} \hat{\phi}_{x,t} \tag{B2} \]

is simplest to discuss in the operator formalism. The impurity propagator is written as

\[ iG_\alpha(x, t, x', t') = \frac{1}{a_x^2} \hat{G}_{\alpha, \hat{x}, \hat{x}', \hat{t}} = \frac{1}{\mathcal{N}} \sum_{\{n\}} \langle n | T[e^{-\frac{i}{\hbar} \int_{t'}^t dt' H_{\alpha'}(t')} \hat{\phi}(\hat{x}, t) \hat{\phi}^\dagger(\hat{x}', t')]|n \rangle, \tag{B3} \]

where

\[ |n \rangle = \prod_\hat{x} (\hat{\phi}^\dagger_\hat{x})^{\hat{n}_\hat{x}} |0\rangle. \tag{B4} \]

and \( \mathcal{N} \) is a normalization constant. The Hamiltonian \( \text{(B1)} \) gives

\[ i\mathcal{N} \hat{G}_{\alpha, \hat{x}, \hat{x}', \hat{t}} = \sum_{\{n\}} \theta(t - t') \langle n | e^{-i\alpha' \hat{n}(T-t)} \hat{\phi}_\hat{x} e^{-i\alpha'(t-t')} \hat{\phi}^\dagger_\hat{x}' e^{-i\alpha'(t'+T)}|n \rangle \]

\[- \sum_{\{n\}} \theta(t' - t) \langle n | e^{-i\alpha'(T-t')} \hat{\phi}^\dagger_\hat{x}' e^{-i\alpha'(t'-t)} \hat{\phi}_\hat{x} e^{-i\alpha'(t+T)}|n \rangle. \tag{B5} \]

Since the lattice sites are decoupled we find immediately

\[ \sum_{\{n\}} \langle n | e^{-i\alpha'(T-t)} \hat{\phi}_\hat{x} e^{-i\alpha'(t'+T)} \hat{\phi}^\dagger_\hat{x}' e^{-i\alpha'(t'+T)}|n \rangle = e^{-i\alpha'(t-t')} \delta_{\hat{x}, \hat{x}'} (1 + e^{-i\alpha})^{N-1}, \]

\[ \sum_{\{n\}} \langle n | e^{-i\alpha'(T-t')} \hat{\phi}^\dagger_\hat{x}' e^{-i\alpha'(t'-t)} \hat{\phi}_\hat{x} e^{-i\alpha'(t+T)}|n \rangle = e^{-i\alpha'(t-t')} \delta_{\hat{x}, \hat{x}'} (1 + e^{-i\alpha})^{N-1} e^{-i\alpha T}, \tag{B6} \]

where \( N \) denotes the number of sites. The normalization constant is

\[ \mathcal{N} = \det[iG_\alpha]^{-1} = (1 + e^{-i\alpha})^N, \tag{B7} \]

yielding

\[ i\hat{G}_{\alpha, \hat{x}, \hat{x}', \hat{t}} = e^{-i\alpha'(t-t')} \delta_{\hat{x}, \hat{x}'} (1 + e^{-i\alpha})^{-1} \left( \theta(t - t') - \theta(t' - t) e^{-i\alpha} \right) \tag{B8} \]

on the lattice or

\[ iG_\alpha(x, t, x', t') = \delta(x - x') e^{-i\alpha'(t-t')} \left( \theta(t - t') (1 - n_\alpha) - \theta(t' - t) e^{-i\alpha} n_\alpha \right) \tag{B9} \]

in the continuum where

\[ n_\alpha = \frac{1}{1 + e^{i\alpha}}. \tag{B10} \]
APPENDIX C: FUNCTIONAL APPROACH

The interactions are easier to take into account in the path integral representation. To make sure that all singular features of the limit of diverging impurity mass are properly kept we first quickly reproduce the free impurity propagator in this representation. The amplitude

$$Z_{0\alpha} = Tr e^{-i \int_{-T}^{T} dt H_{\alpha}'(t)}$$

(C1)

can be written as a functional integral over Grassmannian coherent state configurations which are antiperiodic in time by inserting the resolution of the identity

$$1 = \int \prod_x d\phi^* (x, t) d\phi (x, t) e^{-\int dx \phi^* (x, t) \phi (x, t)} |\phi (x, t) \rangle \langle \phi (x, t)|,$$

(C2)

into (C1) at different times $t_m$,

$$Z_{0\alpha} = \lim_{N_t \to \infty} \int \prod_{m=-N_t+1}^{N_t} \prod_x d\phi^* (x, t) d\phi (x, t) e^{-\int dx \phi^* (x, t) \phi (x, t)}$$

$$\times \lim_{N_t \to \infty} \prod_{m=-N_t+1}^{N_t} \langle \phi (x, t) | e^{-ia t H_{\alpha}'(t)} | \phi (x, t) \rangle$$

$$= \lim_{N_t \to \infty} \int \prod_{m=-N_t+1}^{N_t} \prod_x d\phi^* (x, t) d\phi (x, t) e^{-\sum_{m=-N_t+1}^{N_t} \int dx \phi^* (x, t) \phi (x, t)}$$

$$\times e^{\sum_{m=-N_t+2}^{N_t} \int dx \phi^* (x, t) (1-ia t \alpha') \phi (x, t)} e^{-\int dx \phi^* (x, t-N_t+1) (1-ia t \alpha') \phi (x, t-N_t)}$$

$$= \prod_x \lim_{N_t \to \infty} \int \prod_{m=-N_t+1}^{N_t} d\phi^* (x, t) d\phi (x, t)$$

$$\times e^{-\sum_{m,n=-N_t+1}^{N_t} \phi^* (x, t) S_{mn} \phi (x, t)}],$$

(C3)

where the matrix $S$ is

$$S = \begin{pmatrix}
1 & 0 & \ldots & \ldots & \ldots & 0 & A \\
-A & 1 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & -A & 1 & 0 & \ldots & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & -A & 1
\end{pmatrix},$$

(C4)

with $A = 1 - ia t \alpha' = 1 - i \alpha / 2 N_t$. The Grassmann integration gives

$$Z_{0\alpha} = \prod_x \left[ \lim_{N_t \to \infty} \det S \right] = \prod_x \lim_{N_t \to \infty} \left[ 1 + A^2 N_t \right] = \left( 1 + e^{-i \alpha} \right)^N.$$

(C5)

The propagator is given for $t_m > t_{m'}$ by
\[ i\hat{G}_{\alpha,\hat{x},\hat{x}'}(t_m, t_{m'}) = \delta_{\hat{x},\hat{x}'} \lim_{N \to \infty} (S^{-1})_{mm'} \]

\[ = \delta_{\hat{x},\hat{x}'} \lim_{N \to \infty} \frac{A^{m-m'}}{1 + A^{2N_t}} \]

\[ = \delta_{\hat{x},\hat{x}'} e^{-i\alpha(t_m - t_{m'})} \left(1 - \frac{1}{1 + e^{i\alpha}}\right), \quad (C6) \]

and for \( t_m < t_{m'} \) by

\[ iG_{\alpha,\hat{x},\hat{x}'}(t_m, t_{m'}) = \delta_{\hat{x},\hat{x}'} \lim_{N_t \to \infty} (S^{-1})_{mm'} \]

\[ = \delta_{\hat{x},\hat{x}'} \lim_{N_t \to \infty} \frac{-A^{2N_t + m - m'}}{1 + A^{2N_t}} \]

\[ = -\delta_{\hat{x},\hat{x}'} e^{-i\alpha(t_m - t_{m'})} \frac{1}{1 + e^{i\alpha}}, \quad (C7) \]

in agreement with (B8).

The Fourier transform of the propagator in time is

\[ G_{\alpha}(x, x', \omega) = \int d(t - t') e^{i\omega(t - t')} G_{\alpha}(x, t, x', t') \]

\[ = \delta(x - x') \lim_{\eta \to 0^+} e^{i\omega \eta} \left(\frac{1 - n_\alpha}{\omega - \alpha' + i\epsilon} + \frac{n_\alpha}{\omega - \alpha' - i\epsilon}\right) \quad (C8) \]

in the continuum and

\[ \hat{G}_{\alpha,\hat{x},\hat{x}'}(\omega_m) = \delta_{\hat{x},\hat{x}'} \lim_{\eta \to 0^+} e^{i\omega_m \eta} \left(\frac{1 - n_\alpha}{\pi m - \hat{\alpha}' + i\epsilon} + \frac{n_\alpha}{\pi m - \hat{\alpha}' - i\epsilon}\right) \quad (C9) \]

on the lattice where the infinitesimal quantity \( \eta \) was introduced to take care about the point splitting.

We shall need later

\[ iG_{\alpha}(x, 0; x, 0^+) = \lim_{\eta \to 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} iG_{\alpha}(x, x; \omega) e^{i\omega \eta} \]

\[ = \frac{i}{a_s^2} \lim_{\eta \to 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{1 - n_\alpha}{\omega - \alpha' + i\epsilon} + \frac{n_\alpha}{\omega - \alpha' - i\epsilon}\right) e^{i\omega \eta} \]

\[ = \frac{i}{a_s^2} \lim_{\eta \to 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{n_\alpha e^{i\omega \eta}}{\omega - \alpha' - i\epsilon} \]

\[ = -\frac{1}{a_s^2} (1 + e^{-i\alpha})^{-1} e^{-i\alpha} \quad (C10) \]

which, after integration over \( \alpha \) leads to

\[ -iG(x, 0, x, 0^+) = \frac{1}{a_s^2} \int_{-\pi}^{\pi} d\alpha e^{i\alpha} \left(1 + e^{-i\alpha}\right)^{N-1} e^{-i\alpha} \]

\[ = \frac{n}{N a_s^2} = \rho. \quad (C11) \]
APPENDIX D: EFFECTIVE THEORY

Consider now the transition amplitudes of the impurity-electron coupled system in the path integral representation

\[
\langle \Psi_{f,e}, \Psi_{i,e} | e^{-i\hat{H}t} | \Psi_{i,e}, \Psi_{i,i} \rangle = \prod_{\hat{x}, \hat{t}} \int d\hat{\psi}_{\hat{x}, \hat{t}, \alpha} d\hat{\hat{\psi}}_{\hat{x}, \hat{t}, \alpha} d\hat{\phi}^*_\hat{x}, \hat{t} d\hat{\hat{\phi}}^*_\hat{x}, \hat{t}
\]

\[
\Psi_{f,e}[\hat{\psi}^*, \hat{\psi}] \Psi_{i,i}[\hat{\phi}^*, \hat{\phi}] \Psi_{i,e}[\hat{\psi}^*, \hat{\psi}] \Psi_{i,i}[\hat{\phi}^*, \hat{\phi}] e^{-\frac{i}{\hbar}S[\hat{\psi}^*, \hat{\psi}, \hat{\phi}^*, \hat{\phi}]},
\]

(D1)

of the model (2) on a discretised space-time lattice, where

\[
\frac{1}{\hbar}S[\hat{\psi}^*, \hat{\psi}, \hat{\phi}^*, \hat{\phi}] = \sum_{\hat{x}, \hat{t}} \left[ i\hat{\psi}^*_{\hat{x}, \hat{t}, \alpha}(\hat{\psi}_{\hat{x}, \hat{t}, \alpha} - \hat{\hat{\psi}}_{\hat{x}, \hat{t}-1, \alpha})
\right.
\]

\[
+ \frac{\hbar}{2m} \sum_i \hat{\psi}^*_{x, \hat{t}, \alpha}(\hat{\psi}_{x+\alpha, \hat{t}-1, \alpha} + \hat{\psi}_{x-\alpha, \hat{t}-1, \alpha} - 2\hat{\hat{\psi}}_{x, \hat{t}-1, \alpha})
\]

\[
- \frac{\lambda}{4}(\hat{\psi}^*_{\hat{x}, \hat{t}, \alpha} \hat{\hat{\psi}}_{\hat{x}, \hat{t}-1, \alpha})^2
\]

\[
+ i\hat{\phi}^*_{\hat{x}, \hat{t}}(\hat{\phi}_{\hat{x}, \hat{t}} - \hat{\phi}_{\hat{x}, \hat{t}-1}) + \hat{\psi}^*_{\hat{x}, \hat{t}, \alpha} \hat{\hat{\phi}}^*_{\hat{x}, \hat{t}, \alpha} \hat{\phi}^*_{\hat{x}, \hat{t}-1, \alpha}
\].

(D2)

The wave functionals of the initial and the final states are \(\Psi_{i,e}[\hat{\psi}^*, \hat{\psi}], \Psi_{i,i}[\hat{\phi}^*, \hat{\phi}], \Psi_{f,e}[\hat{\psi}^*, \hat{\psi}],\) and \(\Psi_{f,i}[\hat{\phi}^*, \hat{\phi}]\).

Notice that the impurity field is coupled in the time direction only and the functional integral over \(\phi^*\) and \(\phi\) decouples into a single site, quantum mechanical problem. This suggests the introduction of an effective theory obtained after integration over the impurity field,

\[
e^{-\frac{i}{\hbar}S_{eff}[\hat{\psi}^*, \hat{\psi}] = \prod_{\hat{x}, \hat{t}} \int d\hat{\phi}^*_{\hat{x}, \hat{t}} d\hat{\phi}_{\hat{x}, \hat{t}} \Psi^*_{f,i}[\hat{\phi}^*, \hat{\phi}] \Psi_{i,i}[\hat{\phi}^*, \hat{\phi}] e^{-\frac{i}{\hbar}S[\hat{\psi}^*, \hat{\psi}, \hat{\phi}^*, \hat{\phi}]}.
\]

(D3)

This is the path integral of a two level system with hamiltonian \(H_t = J_t\), where

\[
J_t = \hat{\psi}^*_{\hat{x}, \hat{t}, \alpha} \hat{\psi}_{\hat{x}, \hat{t}-1, \alpha}.
\]

(D4)

The matrix elements

\[
\langle 0 | e^{-i\hat{H}t} | 0 \rangle = 1,
\]

\[
\langle 1 | e^{-i\hat{H}t} | 1 \rangle = T \left[ \prod_t e^{-iJ_t} \right]
\]

(D5)

agree with the path integral

\[
\langle n | e^{-i\hat{H}t} | n \rangle = (\hat{\phi}_f \hat{\phi}_i)^n \prod_i \int d\phi^*_i d\phi_i e^{-i[\phi^*_i(\phi_{i-1} - \phi_{i-2}) + J_i(\phi^*_i - \phi^*_{i-1})]}.
\]

(D6)

where \(\phi_i\) and \(\phi_f\) stand for the first and the last integration variable in time, the arguments of the initial and final single site wave function
impurity propagators where for the states \{\phi\} expanding the integrand in the quasi-local term \phi_i^*\phi_{i-1}. One finds the effective action

\[ \frac{1}{\hbar} S_{\text{eff}}[\psi^*, \psi] = \sum_{\hat{x}, \hat{t}} \left[ i\psi^*_{\hat{x}, \hat{t}, a}(\hat{\psi}_{\hat{x}, \hat{t}, a} - \hat{\psi}_{\hat{x}, \hat{t}, a}) + \frac{\hbar}{2m} \sum_{\hat{i}} \psi^*_{\hat{x}, \hat{t}, a}(\hat{\psi}_{\hat{x} + \hat{a}, \hat{t} - \hat{a}, a} + \hat{\psi}_{\hat{x} - \hat{a}, \hat{t} - \hat{a}, a} - 2\hat{\psi}_{\hat{x}, \hat{t} - \hat{a}, a}) - \frac{\lambda}{4}(\hat{\psi}_{\hat{x}, \hat{t}, a}^*\hat{\psi}_{\hat{x}, \hat{t} - \hat{a}, a})^2 + \hat{g}n_{\hat{x}}(\hat{\psi}_{\hat{x}, \hat{t}, a}^*\hat{\psi}_{\hat{x}, \hat{t} - \hat{a}, a}) \right] \]  

(D8)

for the states

\[ \Psi_{\hat{x}, \hat{t}}[\phi^*, \phi] = \Psi_{\hat{x}, \hat{t}}[\phi^*, \phi] = \prod_{\hat{x}} \phi_{\hat{x}}^{\nu_{\hat{x}}} \]  

(D9)

which belong to the occupation number configuration \{n_{\hat{x}}\}, \( n_{\hat{x}} = 0, 1 \). Notice that the impurity induced interaction is as long range in time as possible, i.e. time independent, and ultralocal in space due to the conservation laws originating from the symmetry \( \{\} \).

The time ordered expectation value of an observable \( O \) of the electrons can easily be obtained as a summation over the impurity occupational number configurations \( n_{\hat{x}} \), \( \sum_{\hat{x}} n_{\hat{x}} = \rho N \),

\[ \langle \langle \Psi_{\hat{t}, e} | T[O] | \Psi_{\hat{t}, e} \rangle \rangle = \frac{1}{Z_\rho} \sum_{\{n_{\hat{x}}\}} \prod_{\hat{x}, \hat{t}} \int d\psi^*_{\hat{x}, \hat{t}, a} d\hat{\psi}_{\hat{x}, \hat{t}, a} \psi_{\hat{t}, e}^*[\psi^*, \psi] \Psi_{\hat{t}, e}[\psi^*, \psi] \Psi_{\hat{t}, e}[\psi^*, \psi] e^{-\frac{\hbar}{\rho} S_{\text{eff}}[\psi^*, \psi; n_{\hat{x}}]} O, \]  

(D10)

where

\[ Z_\rho = \sum_{\{n_{\hat{x}}\}} \prod_{\hat{x}, \hat{t}} \int d\psi^*_{\hat{x}, \hat{t}, a} d\hat{\psi}_{\hat{x}, \hat{t}, a} \Psi_{\hat{t}, e}[\psi^*, \psi] \Psi_{\hat{t}, e}[\psi^*, \psi] e^{-\frac{\hbar}{\rho} S_{\text{eff}}[\psi^*, \psi; n_{\hat{x}}]}. \]  

(D11)

**APPENDIX E: ELECTRON SELF ENERGY**

1. \( O(g) \)

The \( \alpha \)-dependent electron propagator can be written in terms of the free electron and impurity propagators \( G_0 \) and \( G_\alpha \), respectively as

\[ \langle \psi(x_1)\psi^\dagger(x_2) \rangle_\alpha = \det[(iG_\alpha)^{-1}] \det[(iG_0)^{-1}] \times \left\{ iG_0(x_1, x_2) \left[ 1 - i\frac{g}{\hbar} \int dx iG_\alpha(x, x) iG_0(x, x) \right] \right. \left. + \left( -i\frac{g}{\hbar} \right)^2 \int dx \int dy (iG_0(x, x) iG_0(y, y) iG_\alpha(x, x) iG_\alpha(y, y) \right) \right. \]  

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\[-iG_{\alpha}(x, x)iG_{\alpha}(y, y)iG_{0}(x, y)iG_{0}(y, x)\]
\[-iG_{\alpha}(x, y)iG_{\alpha}(y, x)iG_{0}(x, x)iG_{0}(y, y)\]
\[+i\frac{g}{\hbar}\int dx iG_{0}(x, x)iG_{0}(x, x)\]
\[
\times \left[ 1 - i\frac{g}{\hbar}\int dy iG_{0}(y, y)iG_{\alpha}(y, y) \right]
\]
\[= \left( i\frac{g}{\hbar}\int dx \int dy \left( iG_{0}(x, x)iG_{0}(x, x)iG_{\alpha}(x, y)iG_{\alpha}(y, y)iG_{0}(y, y) \right) \right) + \mathcal{O}(g^3), \tag{E1}\]

where the compact notation \((x, t) \to x\) was introduced. For the normalization we need

\[
Z_{\alpha} = \int D[\psi^\dagger]D[\psi]D[\phi]e^{\frac{i}{\hbar} \int dx dt [\psi (i\hbar\partial_{tx} - \alpha) \psi + i\bar{\psi} (i\hbar\partial_{tx} + \frac{g^2}{2m} \Delta + \mu) \psi]}
\times \left[ 1 - i\frac{g}{\hbar}\int dx dt iG_{0}(x; xt) iG_{\alpha}(x; xt) + \mathcal{O}(g^2) \right]
= \det[iG_{0}]^{-1} \left( 1 + e^{-i\alpha} \right)^N
\times \left\{ 1 - i\frac{g}{\hbar}\int dx dt iG_{0}(x; xt) iG_{\alpha}(x; xt) + \mathcal{O}(g^2) \right\}. \tag{E2}\]

The complete propagator

\[
\langle \psi(x_1)\psi\dagger(x_2) \rangle = \left. \frac{\int_{-\pi}^{\pi} d\alpha e^{i\alpha n} \langle \psi(x_1)\psi\dagger(x_2) \rangle}{\int_{-\pi}^{\pi} d\alpha e^{i\alpha n} Z_{\alpha}} \right|_{\alpha = 0}. \tag{E3}\]

starts as \(iG_{0}(x_1, x_2)\) in \(\mathcal{O}(g^0)\). The non-trivial \(\mathcal{O}(g)\) piece in the numerator is

\[
\int_{-\pi}^{\pi} d\alpha e^{i\alpha n} \det[(iG_{0})^{-1}]i\frac{g}{\hbar}\int dx iG_{0}(x, x)iG_{0}(x, x) \times \left[ 1 - i\frac{g}{\hbar}\int dy iG_{0}(y, y)iG_{\alpha}(y, y) \right]
= i\frac{g}{\hbar}\alpha^2 \left[ \left( \frac{N - 1}{n - 1} \right) + i\frac{g}{\hbar}\alpha^2 Tr[iG_{0}] \left( \frac{N - 2}{n - 2} \right) \right] \int dx G_{0}(x, x)G_{0}(x, x). \tag{E4}\]

Taking into account the normalization we find

\[
\langle \psi(x_1)\psi\dagger(x_2) \rangle = iG_{0}(x_1; x_2)
\]
\[+ i\frac{g}{\hbar}\alpha^2 \int dx G_{0}(x, x) G_{0}(x, x) \left[ \left( \frac{N - 1}{n - 1} \right) + i\frac{g}{\hbar}\alpha^2 Tr[iG_{0}] \left( \frac{N - 2}{n - 2} \right) \right] + \mathcal{O}(g^2)
\]
\[= iG_{0}(x_1; x_2) + i\frac{g}{\hbar}\alpha^2 \int dx G_{0}(x_1; x) G_{0}(x; x_2) + \mathcal{O}(g^2) \tag{E5}\]

in the thermodynamic limit, \((N, n \to \infty, n/N \to \hat{\rho})\). The comparison with the Schwinger-Dyson equation gives the self energy
\[ \Sigma^{(1)} = \rho g, \]  
(E6)
a simple shift in the energy. In general, the diagram with \( \ell \) impurity bubbles produces the factor
\[
\left( \frac{N - \ell}{n - \ell} \right) \frac{N! (n - \ell)!}{n! (N - \ell)!} \to \hat{\rho}^\ell. 
\]  
(E7)

2. \( O(g^2) \)

The graph shown in Fig.3 is
\[
I_{\alpha} = \int dx dt dy dt' iG_0(x_1, t_1; x, t) iG_0(x, t; x_2, t_2) iG_0(y, t'; y, t') iG_\alpha(x, t; y, t') iG_\alpha(y, t'; x, t) 
\]
\[
= \frac{1}{a_s^2} \int dx \int dt \int dt' iG_0(x_1, t_1; x, t) iG_0(x, t; x_2, t_2) iG_0(y, t'; y, t') 
\]
\[
\times \left[ \theta(t - t') (1 - n_{\alpha}) - \theta(t' - t) n_{\alpha} \right] \left[ \theta(t' - t) (1 - n_{\alpha}) - \theta(t - t') n_{\alpha} \right] 
\]
\[
= \frac{i^5}{a_s^2} \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \int \frac{d\omega_3}{2\pi} \int \frac{dp_1}{(2\pi)^2} \int \frac{dp_2}{(2\pi)^2} \frac{e^{ip_1(x_1 - x_2) - i\omega(t_1 - t_2)}}{\omega_1 - \frac{p_1^2}{2m} + \mu + i\delta_{p_1}^\mu} 
\]
\[
\times \frac{1}{\omega_2 - \frac{p_2^2}{2m} + \mu + i\delta_{p_2}^\mu} e^{2i\omega_3 \eta} \left( \frac{1 - n_{\alpha}}{\omega_3 + i\epsilon} + \frac{n_{\alpha}}{\omega_3 - i\epsilon} \right)^2. 
\]  
(E8)
The integration over \( \omega_3 \) is divergent
\[
\int \frac{d\omega_3}{2\pi} e^{2i\omega_3 \eta} \left( \frac{1 - n_{\alpha}}{\omega_3 + i\epsilon} + \frac{n_{\alpha}}{\omega_3 - i\epsilon} \right)^2 = \frac{(1 - n_{\alpha})n_{\alpha}}{\epsilon}, 
\]  
(E9)
where \( \epsilon \to 0^+ \). The projection onto the subspace with \( n \) impurities is obtained by the integration
\[
\left[ \int_{-\pi}^\pi e^{i\alpha} \det[(iG_{\alpha})^{-1}] \right] \frac{1 - n_{\alpha}}{N - 2} = \left[ \int_{-\pi}^\pi e^{i\alpha} \det[(iG_{\alpha})^{-1}] \right] \frac{1 + e^{-i\alpha}}{N} 
\]
\[
= \frac{N}{n} \left( \frac{N}{n} \right) 
\]
\[
\approx \hat{\rho}(1 - \hat{\rho}). 
\]  
(E10)
The integration over \( \omega_2 \) and \( p_2 \) gives
\[
\int \frac{d^2 p_2}{(2\pi)^2} \int \frac{d\omega_2}{2\pi} \frac{1}{\omega_2 - \frac{p_2^2}{2m} + \mu + i\delta_{p_2}^\mu} = \int \frac{d^2 p_2}{(2\pi)^2} \lim_{t \to 0^-} \int \frac{d\omega_2}{2\pi} \frac{e^{-i\omega_2 t}}{\omega_2 - \frac{p_2^2}{2m} + \mu + i\delta_{p_2}^\mu} 
\]
\[
= \int \frac{d^2 p_2}{(2\pi)^2} i\theta(\mu - \frac{p_2^2}{2m}) 
\]
\[
= in_0(\mu). 
\]  
(E11)
The corresponding contribution to the self-energy, obtained after removing the external legs is

$$\Sigma_1^{(2)}(\omega, p) = \frac{i}{\epsilon} g^2 \rho (1 - \hat{\rho}) n_0(\mu).$$  \hspace{1cm} (E12)

The graph of Fig. 2 gives

$$I_{a2} = \int dx dt dy dt' i G_0(x_1, t_1; x, t) i G_0(x, t; y, t') i G_0(y, t'; x_2, t_2) i G_\alpha(x, t; y, t') i G_\alpha(y, t'; x, t)$$

$$= \frac{1}{a_s^2} \int d\alpha \int dt \int dt' i G_0(x_1, t_1; x, t) i G_0(x, t; x, t') i G_0(x, t; x, t') i G_0(x, t'; x_2, t_2)$$

$$\times [\theta(t - t') (1 - n_\alpha) - \theta(t' - t) n_\alpha] [\theta(t' - t) (1 - n_\alpha) - \theta(t - t') n_\alpha]$$

$$= \frac{i}{a_s^2} \int d\omega_1 \int d\omega_2 \int d\omega_3 \int d^2 p_1 \int d^2 p_2 \frac{1}{(2\pi)^2} \frac{1}{(2\pi)^2}$$

$$\times \frac{1}{\omega_2 - \frac{p_2^2}{2m} + \mu + i\delta_{p_2}^\mu} \left( \frac{1 - n_\alpha}{\omega_3 + i\epsilon} + \frac{n_\alpha}{\omega_3 - i\epsilon} \right)$$

$$\times \left( \frac{1 - n_\alpha}{\omega_3 + \omega_2 - \omega_1 + i\epsilon} + \frac{n_\alpha}{\omega_3 + \omega_2 - \omega_1 - i\epsilon} \right). \hspace{1cm} (E13)$$

The integration over $\omega_3$ of the terms between the brackets yields

$$i(1 - n_\alpha)n_\alpha \left( \frac{1}{\omega_1 - \omega_2 + i\epsilon} - \frac{1}{\omega_1 - \omega_2 - i\epsilon} \right) = 2\pi (1 - n_\alpha)n_\alpha \delta(\omega_1 - \omega_2), \hspace{1cm} (E14)$$

thus

$$I_{a2} = \frac{i}{a_s^2} \int d\omega_1 \int d^2 p_1 \int d^2 p_2 \frac{e^{ip_1(x_1 - x_2) - i\omega_1(t_1 - t_2)}}{(2\pi)^2}$$

$$\times \frac{1}{\omega_1 - \frac{p_1^2}{2m} + \mu + i\delta_{p_1}^\mu} \frac{1}{\omega_1 - \frac{p_2^2}{2m} + \mu + i\delta_{p_2}^\mu} (1 - n_\alpha)n_\alpha. \hspace{1cm} (E15)$$

The integration over $p_2$ gives

$$\int \frac{d^2 p_2}{(2\pi)^2} \frac{1}{\omega_1 - \frac{p_2^2}{2m} + \mu + i\delta_{p_2}^\mu} = \int \frac{d^2 p_2}{(2\pi)^2} \left\{ P.P. \frac{1}{\omega_1 - \frac{p_2^2}{2m} + \mu} - i \frac{\delta_{p_2}^\mu}{(\omega_1 - \frac{p_2^2}{2m} + \mu)^2 + \delta^2} \right\}. \hspace{1cm} (E16)$$

where $P.P.$ stands for the principal part and produces only a shift in the energy. This shift, being divergent as $a_s \to 0$ is not computable without specifying the details of the ultraviolet cutoff. But the more important imaginary part is finite and can be decomposed in the following way

$$-i \int \frac{d^2 p_2}{(2\pi)^2} \frac{\delta_{p_2}^\mu}{(\omega_1 - \frac{p_2^2}{2m} + \mu)^2 + \delta^2} = i \frac{m}{2\pi} \left( \int_0^\infty \right) d\omega \pi \delta(\omega_1 + \mu - \epsilon)$$

$$= i \frac{m}{2} \left\{ [\theta(\omega_1 + \mu) - \theta(\omega_1)] - \theta(\omega_1) \right\}. \hspace{1cm} (E17)$$

The term containing $\theta(\omega_1 + \mu) - \theta(\omega_1)$ represents the contribution of the Fermi sphere. The last term stands for the contributions of the excitations above the Fermi surface. Their sum is proportional to the free electron density of states,
\[ \tilde{N}_0(E) = \frac{m}{2\pi\hbar^2} (2\theta(E - \mu) - \theta(E)) \]  

(E18)

which is positive when \( E \) is above the Fermi level and negative for \( 0 < E < \mu \). The final result

\[ ImI_2(\omega, p) = \frac{i}{a^2} \hat{\rho}(1 - \hat{\rho}) \frac{1}{(\omega - \frac{p^2}{2m} + \mu + i\delta_p^\mu)^2} \frac{i m}{2} (\theta(\omega + \mu) - 2\theta(\omega)) . \]  

(E19)

gives the imaginary part of the self-energy

\[ Im\Sigma_2^{(2)}(\omega, p) = -ig^2 \frac{m}{2} \hat{\rho}(1 - \hat{\rho}) (2\theta(\omega) - \theta(\omega + \mu)) . \]  

(E20)
FIG. 1. Energy independent $O(g^2)$ contribution to the electron self energy.

FIG. 2. Energy dependent $O(g^2)$ contribution to the electron self energy.