GLOBAL ANALYTIC HYPOELLIPTICITY AND SOLVABILITY OF CERTAIN OPERATORS SUBJECT TO GROUP ACTIONS

GABRIEL ARAÚJO, IGOR A. FERRA, AND LUIS F. RAGOGNETTE

Abstract. On $T \times G$, where $T$ is a compact real-analytic manifold and $G$ is a compact Lie group, we consider differential operators $P$ which are invariant by left translations on $G$ and are elliptic in $T$. Under a mild technical condition, we prove that global hypoellipticity of $P$ implies its global analytic-hypoellipticity (actually Gevrey of any order $s \geq 1$). We also study the connection between the latter property and the notion of global analytic (resp. Gevrey) solvability, but in a much more general setup.

INTRODUCTION

Two notoriously difficult problems in PDE theory are to determine whether a general linear differential operator $P$ defined, say, on a compact manifold $M$, is globally hypoelliptic or globally solvable. In very general terms, given a reasonable space of functions $\mathcal{F}$ on $M$, the first problem means to determine if we can infer from the information $Pu \in \mathcal{F}$ that $u$ itself belongs to $\mathcal{F}$, where $u$ is a priori taken in some larger space of (generalized) functions. The second one means to solve the equation $Pu = f$ for “every” $f \in \mathcal{F}$, with a solution $u$ also in $\mathcal{F}$ – but a subtlety soon arises, for this is generally impossible as the global geometry of both $M$ and $P$ impose natural constraints on the right-hand side $f$; we compromise by asking instead if we can always solve at least for those “admissible” $f \in \mathcal{F}$.

In order to make the situation more manageable, extra geometrical hypotheses may be imposed to the problem. A traditional one is to assume $M$ endowed with a smooth action of a Lie group $G$ which leaves $P$ invariant; the action may be further assumed transitive (see e.g. [23, Chapter 5] and related works in the references therein) or free, in which case the operator induced by $P$ on the orbit space $M/G$ plays a key role.

Here we are interested in the latter possibility, and deal with the simplest such situation: $M$ is a product $T \times G$, where $T$ is a compact manifold and $G$ is a compact Lie group, which acts freely on $M$ by left-multiplication on the second factor alone (where $G$ acts on itself). Invariance of $P$ under this action allows us to put it in a global canonical form with separate variables (2.2), where the operator $P_0$ induced by $P$ on the orbit space $T$ can be easily read off; this one will be further required to be elliptic in $T$. Operators $P$ satisfying these properties essentially are said to belong to class $\mathcal{T}$ (Definition 2.2) and are our main focus. When $M = T^1_N \times T^1_m$, where $T^N$ is the $N$-dimensional torus, the invariance of $P$ under the action of $T^m$ means that $P = P(t, D_t, D_\xi)$ and there is a vast literature about global hypoellipticity and solvability of such operators in this ambient. Requiring the ellipticity condition on $P_0$ imposes certain restrictions, but still there are important classes of such operators, for instance sums of squares of certain real vector fields (see e.g. [11, 13, 1]).

As for the remaining ingredient – the space of functions $\mathcal{F}$ on which our questions will be posed – we are mainly interested in the space of real-analytic functions (for which we of course need to require further regularity of all the objects involved): global analytic hypoellipticity and solvability of $P$ are then addressed. Actually, our approach applies to the more general Gevrey classes of functions $\mathcal{G}^s$ of order $s \geq 1$ (see e.g. [21]) of which real-analytic functions are a special case ($s = 1$), and one can even guess that the program could be carried forward for other ultradifferential classes of Roumieu type (even quasianalytic ones).

As far as the regularity problem is concerned, we consider the following question: when does global hypoellipticity (i.e. in the smooth setup) implies global analytic hypoellipticity? Positive answers for

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2020 Mathematics Subject Classification. 35A01, 35H10 (primary), 35R01, 35R03 (secondary).
Key words and phrases. Gevrey regularity, global hypoellipticity, global solvability, invariant operators.
This work was supported by the São Paulo Research Foundation (FAPESP, grants 2016/13620-5 and 2018/12273-5).

arXiv:2111.07165v1 [math.AP] 13 Nov 2021
this kind of question first appeared in [11], where the authors proved, in contrast to the famous example due to Baouendi and Goulaouic, that the Hörmander condition (so local hypoellipticity – which is much stronger), together with additional hypotheses (that for us reads as the ellipticity condition for the class $\mathcal{T}$), ensure global analytic hypoellipticity. Similar results in a more general framework can be found in [10, 8]. Our main result about this subject is Theorem 4.2, which ensures that every operator $P$ in class $\mathcal{T}$ that is globally hypoelliptic is also globally analytic hypoelliptic (actually globally $\mathcal{G}^s$-hypoelliptic for every $s \geq 1$). Theorem 4.2 is more properly related to [14, Theorem 2.2] and [20, Theorem 1.10] (and indeed a kind of generalization of them) which deal with the case when both $T$ and $G$ are tori. Their results were extended for more general classes of functions in [2].

The proof of Theorem 4.2 is obtained by first retrieving global estimates from microlocal information (Section 3) and then combining them with some results (Section 4) related to the notion of Gevrey vectors of the partial Laplace-Beltrami operator $\Delta_G$ (associated with a suitable metric on $G$), a business of interest in PDE on its own.

In Section 5 we tackle the issue of solvability from a far more abstract viewpoint. We forget the existence of symmetries and prove (Theorem 5.6) that for a general operator $P$ on a compact manifold $M$ satisfying a regularity property much weaker than global $\mathcal{G}^s$-hypoellipticity we have that the map $P : \mathcal{G}^s(M) \to \mathcal{G}^s(M)$ has closed range – which, we argue, is the correct notion of global $\mathcal{G}^s$-solvability (an analogous result in the smooth category has been recently proved to be true [5]). This result follows from a theory of regularity for abstract operators acting on certain pairs of topological vector spaces started in [3] whose development we continue here. The reader will notice that, thanks to the generality of our approach, the proofs here apply equally well to many other classes of operators: certain pseudodifferential operators acting on $\mathcal{G}^s(M)$ (not necessarily of finite order), and so on; either scalar or vector-valued.

Finally, we turn back to the situation with symmetries linking our class $\mathcal{T}$ with the general property addressed in the abstract results of Section 5: a converse of Theorem 5.6 is proved for operators in that class (Proposition 5.7).

1. Preliminaries

Let $M$ be a real-analytic manifold, assumed throughout to be compact, connected and oriented. The space $\mathcal{G}^s(M)$ of globally defined Gevrey functions [21] can be characterized by means of the powers of a suitable real-analytic elliptic operator [18, 7]: here, we endow $M$ with a real-analytic Riemannian metric (which is always possible [12]) and denote by $\Delta_M$ the underlying Laplace-Beltrami operator acting on functions, in which case a smooth $f$ belongs to $\mathcal{G}^s(M)$ if and only if there exist $C, h > 0$ such that

$$\| (I + \Delta_M)^k f \|_{L^2(M)} \leq C h^k k!^{2s}, \quad \forall k \in \mathbb{Z}_+.$$ 

The case $s = 1$ describes the space of real-analytic functions. As in [3], we furnish topologies to these spaces as follows: for each $h > 0$ we let

$$\mathcal{G}^{s,h}(M) = \left\{ f \in \mathcal{G}^\infty(M) : \sup_{k \in \mathbb{Z}_+} h^{-k} k!^{-2s} \| (I + \Delta_M)^k f \|_{L^2(M)} < \infty \right\}$$

which is a Banach space; as the parameter $h$ increases, these are contained into one another in a continuous and compact fashion, meaning that their union $\mathcal{G}^s(M)$, now endowed with the injective limit topology, is what one calls a DFS space.

We denote by $\sigma(\Delta_M)$ the spectrum of $\Delta_M$. By ellipticity of $\Delta_M$, its $\lambda$-eigenspace $E^M_\lambda$ is a finite dimensional subspace of $\mathcal{G}^1(M)$; connectedness of $M$ ensures that $E^M_0 = \mathbb{C}$. Denoting by $F^M_\lambda : L^2(M) \to E^M_\lambda$ the orthogonal projection we have

$$f = \sum_{\lambda \in \sigma(\Delta_M)} F^M_\lambda(f)$$

with convergence in $L^2(M)$; this is an abstract analog of Fourier series. As such, we extend it to Schwartz distributions: given $f \in \mathcal{G}^1(M)$ – which we identify with a continuous linear functional on $\mathcal{G}^\infty(M)$ using the underlying volume form – we let $F^M_\lambda(f)$ be the unique element in $E^M_\lambda$ such that

$$\langle F^M_\lambda(f), \phi \rangle_{L^2(M)} = \langle f, \phi \rangle, \quad \forall \phi \in E^M_\lambda.$$
In this situation (1.1) still holds, but with convergence in \( \mathcal{S}^\prime(M) \); when \( f \) is smooth this convergence takes place in \( \mathcal{S}^\infty(M) \), and so on. Thanks to Weyl’s asymptotic estimates [9, p. 155], a distribution \( f \) belongs to \( \mathcal{S}^s(M) \) if and only if there exist \( C, h > 0 \) such that
\[
\| \mathcal{F}_\lambda^M(f) \|_{L^2(M)} \leq Ce^{-h(1+\lambda)}^\frac{1}{2}, \quad \forall \lambda \in \sigma(\Delta_M).
\]
This allows us to consider alternatively the adapted norms
\[
\| f \|_{\mathcal{S}^s,h(M)} \doteq \left( \sum_{\lambda \in \sigma(\Delta_M)} 2^{h(1+\lambda)} \| \mathcal{F}_\lambda^M(f) \|_{L^2(M)}^2 \right)^{\frac{1}{2}}
\]
and the spaces
\[
\mathcal{S}^s,h(M) \doteq \{ f \in \mathcal{S}^\infty(M) ; \| f \|_{\mathcal{S}^s,h(M)} < \infty \}
\]
which are better suited to some applications (e.g. in the proof of Lemma 3.3); as \( h \) increases, this produces new Banach spaces compactly contained in one another and forming a directed system equivalent to the previous one, hence with the same injective limit \( \mathcal{S}^s(M) \) – both set-theoretically and topologically. On time, it is also convenient to introduce the following adapted Sobolev norms:
\[
\| f \|_{\mathcal{S}^t(M)} \doteq \left( \sum_{\lambda \in \sigma(\Delta_M)} (1 + \lambda)^{2t} \| \mathcal{F}_\lambda^M(f) \|_{L^2(M)}^2 \right)^{\frac{1}{2}}, \quad t > 0.
\]

When \( M \) is a product \( T \times G \) of two such manifolds and carrying the product metric, an abstract theory of partial Fourier series can also be developed (for details see e.g. [4, 5]). In that case one proves that \( \Delta_M = \Delta_T + \Delta_G \) as differential operators on \( M \), that any \( \alpha \in \sigma(\Delta_M) \) is of the form \( \alpha = \mu + \lambda \) for some \( \mu \in \sigma(\Delta_T) \) and \( \lambda \in \sigma(\Delta_G) \) (and vice versa), and
\[
E^M_\alpha = \bigoplus_{\mu \in \sigma(\Delta_T), \lambda \in \sigma(\Delta_G)} E^T_\mu \otimes E^G_\lambda.
\]
Moreover, given \( \mu \in \sigma(\Delta_T) \) and \( \lambda \in \sigma(\Delta_G) \) we will fix, whenever necessary, bases for \( E^T_\mu \) and \( E^G_\lambda \)
\[
\{ \psi^\mu_i ; 1 \leq i \leq d^T_\mu \}, \quad \text{where} \quad d^T_\mu \doteq \dim E^T_\mu,
\]
\[
\{ \phi^\lambda_j ; 1 \leq j \leq d^G_\lambda \}, \quad \text{where} \quad d^G_\lambda \doteq \dim E^G_\lambda,
\]
which are orthonormal w.r.t. the inner products inherited from \( L^2(T) \), \( L^2(G) \), respectively, in which case
\[
\mathcal{S} \doteq \{ \psi^\mu_i \otimes \phi^\lambda_j ; 1 \leq i \leq d^T_\mu, 1 \leq j \leq d^G_\lambda, \mu \in \sigma(\Delta_T), \lambda \in \sigma(\Delta_G) \}
\]
is a Hilbert basis for \( L^2(T \times G) \). Now given \( f \in \mathcal{S}(T \times G) \) and \( \lambda \in \sigma(\Delta_G) \) one defines an object \( \mathcal{F}^G_\lambda(f) \in \mathcal{S}(T; E^G_\lambda) \cong \mathcal{S}(T) \otimes E^G_\lambda \) that in terms of our choice of basis can be concretely written as
\[
\mathcal{F}^G_\lambda(f) = \sum_{j=1}^{d^G_\lambda} \mathcal{F}^G_\lambda(f)_j \otimes \phi^\lambda_j,
\]
where \( \mathcal{F}^G_\lambda(f)_j \in \mathcal{S}(T) \) is defined by
\[
(\mathcal{F}^G_\lambda(f)_j, \psi) = (f, \psi \otimes \bar{\phi}^\lambda_j), \quad \forall \psi \in \mathcal{S}^\infty(T).
\]
The “total” Fourier projection of \( f \) can then be recovered from the partial ones as
\[
\mathcal{F}^M_\alpha(f) = \sum_{\mu + \lambda = \alpha} \mathcal{F}^T_\mu \mathcal{F}^G_\lambda(f), \quad \alpha \in \sigma(\Delta_M).
\]

Gevrey functions can also be described by the “partial” Fourier projections:

**Proposition 1.1.** If \( f \in \mathcal{S}^s(T \times G) \) then \( \mathcal{F}^G_\lambda(f) \in \mathcal{S}^s(T; E^G_\lambda) \) for every \( \lambda \) and
\[
f = \sum_{\lambda \in \sigma(\Delta_G)} \mathcal{F}^G_\lambda(f)
\]
with convergence in \( \mathcal{S}^s(T \times G) \).
2. A class of invariant operators

Let \( T \) be a real-analytic Riemannian manifold, assumed compact, connected and oriented, and let \( G \) be a compact, connected Lie group, carrying a Riemannian metric which is ad-invariant \([16, Proposition 4.24]\). We denote by \( \mathfrak{g} \) the Lie algebra associated with \( G \). Let \( P \) be a LPDO on \( T \times G \) that is invariant by the left action of \( G \), i.e., \( L_g : T \times G \rightarrow T \times G \) is defined by \( L_g(t, x) = (t, gx) \) then for every \( u \in \mathcal{C}^\infty(T \times G) \) we have

\[
(L_g)^*(Pu) = P[(L_g)^* u], \quad \forall g \in G. \quad (2.1)
\]

We call such operators \( G \)-invariant. By choosing a basis \( X_1, \ldots, X_m \in \mathfrak{g} \) – which we regard as a global frame for the tangent space of \( G \) – we may write \( P \) as follows:

\[
P = \sum_{|\alpha| \leq r} P_{\alpha} X^\alpha \quad (2.2)
\]

where \( P_{\alpha} \) is a LPDO on \( T \) and \( X^\alpha = X_1^{\alpha_1} \cdots X_m^{\alpha_m} \) for each multi-index \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_+^m \).

Indeed, let \( U \subset T \) and \( V \subset \tilde{G} \) be coordinate open sets, i.e., there exist \( \tilde{U} \subset \mathbb{R}^n \) and \( V \subset \mathbb{R}^m \) and real-analytic charts

\[
\phi : U \rightarrow \tilde{U}, \quad \psi : V \rightarrow \tilde{V}.
\]

We can write, in \( U \times V \),

\[
P = \sum_{|\alpha| + |\beta| \leq r} a_{\alpha, \beta}^U \partial_{t}^2 X_{\alpha}, \quad a_{\alpha, \beta}^U \in \mathcal{C}^\infty(U \times V)
\]

and first we are going to prove that if \( \psi_1 : V_1 \rightarrow \tilde{V}_1 \) and \( \psi_2 : V_2 \rightarrow \tilde{V}_2 \) are two charts with \( V_1 \cap V_2 \neq \emptyset \) then

\[
a_{\alpha, \beta}^{U, V_1} = a_{\alpha, \beta}^{U, V_2} \quad \text{in} \quad U \times (V_1 \cap V_2) \quad (2.3)
\]

for every \( |\alpha| + |\beta| \leq r \). It is enough to prove that

\[
a_{\alpha, \beta}^{U, V_1} \circ (\phi \times \psi_1)^{-1} = a_{\alpha, \beta}^{U, V_2} \circ (\phi \times \psi_1)^{-1} \quad \text{in} \quad \tilde{U} \times \psi_1(V_1 \cap V_2).
\]

If we denote the transition map

\[
\xi \equiv (\phi \times \psi_2) \circ (\phi \times \psi_1)^{-1} = \text{Id}_{\tilde{U}} \times (\psi_2 \circ \psi_1^{-1}) : \tilde{U} \times \psi_1(V_1 \cap V_2) \rightarrow \tilde{U} \times \psi_2(V_1 \cap V_2)
\]

then the pullback \([((\phi \times \psi_1)^{-1})^* P] \) of \( P \) to the open set \( \tilde{U} \times \psi_1(V_1 \cap V_2) \) is given by

\[
[(\phi \times \psi_1)^{-1}]^* P = \xi^* [((\phi \times \psi_2)^{-1})^* \left( \sum_{|\alpha| + |\beta| \leq r} a_{\alpha, \beta}^{U, V_2} \partial_{t}^2 X_{\alpha} \right)]
\]

\[
= \xi^* \left( \sum_{|\alpha| + |\beta| \leq r} a_{\alpha, \beta}^{U, V_2} \circ (\phi \times \psi_2)^{-1} \partial_{t}^2 (\psi_2^{-1})^* X_{\alpha} \right)
\]

\[
= \sum_{|\alpha| + |\beta| \leq r} a_{\alpha, \beta}^{U, V_2} \circ (\phi \times \psi_2)^{-1} \circ \xi \partial_{t}^2 (\psi_2^{-1})^* X_{\alpha}.
\]

On the other hand

\[
[((\phi \times \psi_1)^{-1})^* P = [((\phi \times \psi_1)^{-1})^* \left( \sum_{|\alpha| + |\beta| \leq r} a_{\alpha, \beta}^{U, V_1} \partial_{t}^2 X_{\alpha} \right)]
\]

\[
= \sum_{|\alpha| + |\beta| \leq r} a_{\alpha, \beta}^{U, V_1} \circ (\phi \circ \psi_1)^{-1} \circ \partial_{t}^2 (\psi_1^{-1})^* X_{\alpha}.
\]

\[\footnote{Notice the ambiguity in the notation: we are denoting by \( \partial_{t_1}, \ldots, \partial_{t_n} \) both the standard Euclidean partial derivatives on \( U \subset \mathbb{R}^n \) and their pullbacks to \( U \) via \( \phi \).} \]
Since \( \{ \phi^* \partial_{t_1}, \ldots, \phi^* \partial_{t_n}, X_1, \ldots, X_m \} \) is a frame in \( U \times (V_1 \cap V_2) \) then
\[
\{ \partial_{t_1}, \ldots, \partial_{t_n}, (\psi_1^{-1})^* X_1, \ldots, (\psi_1^{-1})^* X_m \}
\]
is a frame in \( \tilde{U} \times \psi_1(V_1 \cap V_2) \), hence by [22, Theorem 1.1.2] we conclude that
\[
a_{a,\beta}^U \circ (\phi \times \psi_2)^{-1} \circ \xi = a_{a,\beta}^V \circ (\phi \times \psi_1)^{-1}
\]
for every \( |\alpha| + |\beta| \leq r \), which in turn gives (2.3). Hence for a given coordinate chart \((U, \phi)\) in \( T \) as above, if we set \( a_{a,\beta}^U(t, x) = a_{a,\beta}^U(t, x) \) for \( x \) in the domain of some coordinate chart \((V, \psi)\) then \( a_{a,\beta}^U \in \mathscr{C}^\infty(U \times G) \) and we can write
\[
P = \sum_{|\alpha| + |\beta| \leq r} a_{a,\beta}^U \partial_t^\beta X^\alpha \quad \text{ in } U \times G.
\]

The next step is to prove that \( a_{a,\beta}^U \) does not depend on the \( x \)-variable. We employ (2.1); notice that both \( \partial_t^\beta \) and \( X^\alpha \) are \( G \)-invariant as well – the latter by left-invariance of \( X_1, \ldots, X_m \). For \( u \in \mathscr{C}^\infty(U \times G) \) we have
\[
\sum_{|\alpha| + |\beta| \leq r} a_{a,\beta}^U(t, g x) \partial_t^\beta X^\alpha u(t, g x) = (L_g)^* (P u)(t, x)
\]
\[
= \sum_{|\alpha| + |\beta| \leq r} a_{a,\beta}^U(t, x) \partial_t^\beta X^\alpha [(L_g)^* u](t, x)
\]
\[
= \sum_{|\alpha| + |\beta| \leq r} a_{a,\beta}^U(t, x) [(L_g)^* \partial_t^\beta X^\alpha u](t, x)
\]
\[
= \sum_{|\alpha| + |\beta| \leq r} a_{a,\beta}^U(t, x) \partial_t^\beta X^\alpha u(t, g x).
\]

Since \( u \) is arbitrary this shows that, as LPDOs in \( U \times G \),
\[
\sum_{|\alpha| + |\beta| \leq r} a_{a,\beta}^U(t, g x) \partial_t^\beta X^\alpha = \sum_{|\alpha| + |\beta| \leq r} a_{a,\beta}^U(t, x) \partial_t^\beta X^\alpha, \quad \forall g \in G.
\]

Thanks to [22, Theorem 1.1.2] again we conclude that \( a_{a,\beta}^U(t, g x) = a_{a,\beta}^U(t, x) \) for every \( t \in U \) and \( x, g \in G \), which proves that \( a_{a,\beta}^U \) does not depend on \( x \).

In short, we have proved that given a coordinate chart \((U, \phi)\) in \( T \), there exist functions \( a_{a,\beta}^U \in \mathscr{C}^\infty(U) \) such that
\[
P = \sum_{|\alpha| + |\beta| \leq r} a_{a,\beta}^U \partial_t^\beta X^\alpha \quad \text{ in } U \times G.
\]

Thus if we write \( P_{U, a} = \sum_{|\beta| \leq r-|\alpha|} a_{a,\beta}^U \partial_t^\beta \) then \( P_{U, a} \) is a LPDO in \( U \),
\[
P = \sum_{|\alpha| \leq r} P_{U, a} X^\alpha \quad \text{ in } U \times G.
\]

and it remains to prove that \( P_{U, a} \) is in fact globally defined, that is if \( \phi_1 : U_1 \longrightarrow \tilde{U}_1 \) and \( \phi_2 : U_2 \longrightarrow \tilde{U}_2 \) are two charts in \( T \) then \( P_{U_1, a} = P_{U_2, a} \) in \( U_1 \cap U_2 \). The latter equality means that
\[
P_{U_1, a}(v) = P_{U_2, a}(v), \quad \forall v \in \mathscr{C}^\infty(U_1 \cap U_2).
\]

If we fix \( v \in \mathscr{C}^\infty(U_1 \cap U_2) \) and \( t_0 \in U_1 \cap U_2 \) then
\[
Q_1 = \sum_{|\alpha| \leq r} [P_{U_1, a}(v)(t_0)] X^\alpha \quad \text{ and } \quad Q_2 = \sum_{|\alpha| \leq r} [P_{U_2, a}(v)(t_0)] X^\alpha
\]
are LPDOs in \( G \) and satisfy
\[
Q_1(h)(x) = P(v \otimes h)(t_0, x) = Q_2(h)(x), \quad \forall h \in \mathscr{C}^\infty(G),
\]
so \( Q_1 = Q_2 \). Then we can use one more time [22, Theorem 1.1.2] to conclude that \( P_{U_1, a}(v)(t_0) = P_{U_2, a}(v)(t_0) \) for every \( |\alpha| \leq r \); since \( t_0 \in U_1 \cap U_2 \) and \( v \in \mathscr{C}^\infty(U_1 \cap U_2) \) are arbitrary our proof is complete.
It is then clear that \([P, \Delta_G] = 0\) (since the \(P_\alpha\) are operators in \(T\) only and all the vector fields \(X_1, \ldots, X_m\) commute with \(\Delta_G\) – thanks to the general remark that left-invariant vector fields on \(G\) always commute with the Laplace-Beltrami operator associated to an ad-invariant metric). As such, \(P\) acts as an endomorphism of both \(C^\infty(T; E^G_\lambda)\) and \(\Delta^r_G(T; E^G_\lambda)\) for each \(\lambda \in \sigma(\Delta_G)\), which induces by restriction a differential operator \(\hat{\Delta}_\lambda\) on \(T \times E^G_\lambda\), the latter regarded as a trivial vector bundle over \(T\). These operators are expressed as follows: given \(\psi \in C^\infty(T; E^G_\lambda)\), written as

\[
\psi = \sum_{i=1}^{d_G} \psi_i \otimes \phi^i_\lambda, \quad \psi_i \in C^\infty(T),
\]

we have

\[
\hat{\Delta}_\lambda \psi = \sum_{i=1}^{d_G} (P_i \psi_i) \otimes \phi^i_\lambda + \sum_{i=1}^{d_G} \sum_{0 < |\alpha| \leq r} \phi^i_\lambda \otimes (P_\alpha \psi_i) \otimes (X^\alpha \phi^i_\lambda)
\]

which we put on canonical form recalling that

\[
X^\alpha \phi^i_\lambda = \sum_{j=1}^{d_G} \gamma^{\alpha}_{ij} \phi^j_\lambda, \quad \gamma^{\alpha}_{ij} \in C,
\]

hence

\[
\hat{\Delta}_\lambda \psi = \sum_{j=1}^{d_G} P_0 \psi_j + \sum_{i=1}^{d_G} \sum_{0 < |\alpha| \leq r} \sum_{0 \leq |\beta| \leq r} \gamma_{ij}^{\alpha} P_\alpha \psi_i \otimes \phi^j_\lambda.
\] (2.4)

From here on we will assume that \(P\) and \(P_0\) have the same order \(r\). Thus the order of \(P_\alpha\) is at most \(r - |\alpha|\), and expression (2.4) reveals that \(\hat{\Delta}_\lambda\), as a differential operator on the vector bundle \(T \times E^G_\lambda\) over \(T\), also has order \(r\): its principal part is essentially the same as that of \(P_0\). Actually, under such circumstances the following identity between their principal symbols holds:

\[
\text{Symb}_{(t_0, \tau_0)}(\hat{\Delta}_\lambda) = \text{Symb}_{(t_0, \tau_0)}(P_0) : \text{id}_{E^G_\lambda}, \quad \forall (t_0, \tau_0) \in T^* T \setminus 0.
\] (2.5)

Indeed, recall that the principal symbol of \(\hat{\Delta}_\lambda\) may be regarded as a linear map

\[
\text{Symb}_{(t_0, \tau_0)}(\hat{\Delta}_\lambda) : E^G_\lambda \to E^G_\lambda
\]

which can be computed as follows. Given \(\psi \in C^\infty(T; \mathbb{R})\) such that \(d\psi(t_0) = \tau_0\) we have, for every \(\phi \in E^G_\lambda\):

\[
\text{Symb}_{(t_0, \tau_0)}(\hat{\Delta}_\lambda) \phi = \lim_{\rho \to \infty} \rho^{-r} e^{-i\rho \psi} \hat{\Delta}_\lambda(e^{i\rho \psi} 1_T \otimes \phi)\bigg|_{t_0} = \lim_{\rho \to \infty} \rho^{-r} e^{-i\rho \psi} \hat{\Delta}_\lambda(e^{i\rho \psi} \phi)\bigg|_{t_0}.
\]

Here, \(1_T \otimes \phi\) plays the role of a section of \(T \times E^G_\lambda\) whose value at \(t_0\) is \(\phi\) and \(r\) is the order of \(\hat{\Delta}_\lambda\). But

\[
\hat{\Delta}_\lambda(e^{i\rho \psi} \phi) = P_0(e^{i\rho \psi}) \phi + \sum_{0 < |\alpha| \leq r} P_\alpha(e^{i\rho \psi}) X^\alpha \phi
\]

note now that when \(|\alpha| > 0\) we have that \(e^{-i\rho \psi} P_\alpha(e^{i\rho \psi})\) is a polynomial in \(\rho\) of degree strictly less then \(r\), and since the order of \(P_0\) is \(r\) we have

\[
\text{Symb}_{(t_0, \tau_0)}(\hat{\Delta}_\lambda) \phi = \left( \lim_{\rho \to \infty} \rho^{-r} e^{-i\rho \psi} P_0(e^{i\rho \psi})\bigg|_{t_0} \phi + \sum_{0 < |\alpha| \leq r} \left( \lim_{\rho \to \infty} \rho^{-r} e^{-i\rho \psi} P_\alpha(e^{i\rho \psi})\bigg|_{t_0} \right) X^\alpha \phi \right)
\]

thus proving (2.5).

In particular:

**Proposition 2.1.** Suppose that \(P\) as in (2.2) has the property that \(P\) and \(P_0\) have the same order. Then the following are equivalent:

1. \(P_0\) is elliptic in \(T\).
2. \(\hat{\Delta}_\lambda\) is elliptic in \(T \times E^G_\lambda\) for some \(\lambda \in \sigma(\Delta_G)\).
3. \(\hat{\Delta}_\lambda\) is elliptic in \(T \times E^G_\lambda\) for every \(\lambda \in \sigma(\Delta_G)\).
Proof. Ellipticity of $\hat{P}_0$ at $t_0 \in T$ is characterized by injectivity of $\text{Symb}_{(t_0, \tau)}(\hat{P}_0)$ for every $\tau \in T^{\ast}_{t_0} T \setminus 0$, which by (2.5) is equivalent to

$$\text{Symb}_{(t_0, \tau)}(P_0) \neq 0, \quad \forall \tau \in T^{\ast}_{t_0} T \setminus 0$$

i.e. to $P_0$ being elliptic at $t_0$. □

This motivates us to introduce the following class of operators on $T \times G$.

**Definition 2.2.** We say that a LPDO $P$ on $T \times G$ belongs to class $\mathcal{T}$ if

1. $P$ is $G$-invariant,
2. $P$ and $P_0$ in (2.2) have the same order and
3. $P_0$ is elliptic in $T$.

2.1. **Examples.**

2.1.1. **Vector fields.** Consider $Y$ a vector field on $T \times G$ that can be written as

$$Y = W + \sum_{j=1}^{m} a_j(t)X_j \quad (2.6)$$

where $a_1, \ldots, a_m \in \mathcal{C}^\infty(T)$ and $W \in \mathfrak{X}(T)$ is a complex vector field. Notice that $W$ has the same order of $Y$ unless it vanishes identically. Therefore, $Y$ belongs to $\mathcal{T}$ if and only if $W$ is elliptic on $T$.

When $W$ is a real vector field the latter condition forces $T$ to be one-dimensional i.e. $T = S^1$, in which case $W$ is a non-vanishing multiple of $\partial_t$, and $Y$ is equivalent to a vector field of the form

$$\partial_t + \sum_{j=1}^{m} a_j(t)X_j \quad \text{on} \quad S^1 \times G.$$

The case $\dim T = 2$ is also possible but in that case $W$ must be complex with $\text{Re}W$ and $\text{Im}W$ linearly independent everywhere; if we additionally assume that these real vector fields commute then one can endow $T$ with the structure of a Riemann surface such that $W$ is essentially the Cauchy-Riemann operator i.e. $Y$ is of the form

$$\partial_z + \sum_{j=1}^{m} a_j(t)X_j \quad \text{on} \quad T \times G.$$

When $\dim T \geq 3$ no vector field $Y$ on $T \times G$ will belong to $\mathcal{T}$ as in that case $W$ can never be elliptic.

2.1.2. **Certain second-order operators.** Let $Q$ be a second-order operator on $T$ and

$$Y_\ell = W_\ell + \sum_{j=1}^{m} a_{\ell j}(t)X_j, \quad \ell \in \{1, \ldots, N\},$$

be vector fields of the form (2.6). Then

$$P = Q - \sum_{\ell=1}^{N} Y_\ell^2$$

is $G$-invariant, and moreover belongs to $\mathcal{T}$ if and only if

$$Q - \sum_{\ell=1}^{N} W_\ell^2$$

is elliptic of order 2 on $T$. Real operators in this class were investigated e.g. when $Q = \Delta_T$ [4, 5]; or when $Q = 0$ and $W_1, \ldots, W_N$ span the tangent bundle of $T$ everywhere [8] (further references therein).
3. FROM MICROLOCAL ANALYSIS TO GLOBAL GEVREY ESTIMATES

Lemma 3.1. Let $P$ be a real-analytic LPDO on $T \times G$ belonging to class $T$. If $u \in \mathcal{G}(T \times G)$ is such that $Pu \in \mathcal{G}(T \times G)$ then for every $\phi \in \mathcal{G}(G)$ we have that $	ilde{u}(\phi) = \langle u, \cdot \otimes \phi \rangle \in \mathcal{G}(T)$.

Proof. Fix $t \in T$ and notice that $WF_s(u)$ does intercept the conormal bundle of $\{t\} \times G$. Indeed, a covector $(t, \tau, x, \xi) \in T^*(T \times G)$ annihilates $T_{(t, x)}((t) \times G)$ if and only if $\xi = 0$, and none of these belongs to the characteristic set of $P$: as one can easily compute,

$$Symb_{(t, \tau, x, 0)}(P) = Symb_{(t, \tau)}(P_0)$$

and $P_0$ is elliptic in $T$ by assumption. Our claim follows since $WF_s(u) \subset \text{Char}(P)$ [15, Theorem 5.1]. We are then allowed to restrict $u$ to $\{t\} \times G$, i.e. to pull it back via the map $x \in G \mapsto (t, x) \in T \times G$, yielding in this way a distribution $u_t \in \mathcal{G}'(G)$, and by [15, Theorem 4.1] the function $t \in T \mapsto \langle u_t, \phi \rangle \in \mathbb{C}$ belongs to $\mathcal{G}(T)$ whatever $\phi \in \mathcal{G}(G)$. That function is, however, none other than the distribution $\psi \in \mathcal{C}^\infty(T) \mapsto \langle u, \psi \otimes \phi \rangle \in \mathbb{C}$. □

Corollary 3.2. Under the hypotheses of Lemma 3.1 we have $\mathcal{F}_\lambda^G(u) \in \mathcal{G}(T; \mathcal{E}_\lambda^G)$ for every $\lambda \in \sigma(\Delta_G)$.

Proof. Apply Lemma 3.1 to a basis of $\mathcal{E}_\lambda^G$. □

For the next result, given $\theta \in (0, 1)$ we define the set

$$\Gamma_\theta = \{ (\mu, \lambda) \in \sigma(\Delta_T) \times \sigma(\Delta_G) : \lambda \leq \theta \mu \}.$$ 

Lemma 3.3. Suppose that $u \in \mathcal{G}(T \times G)$ is such that $\tilde{u}(\phi) \in \mathcal{G}(T)$ for every $\phi \in \mathcal{G}(G)$. Then there exist $C, h > 0$ and $\theta \in (0, 1)$ such that

$$\|F_{\mu}^T F_{\lambda}^G(u)\|_{L^2(T \times G)} \leq C e^{-h(1+\mu+\lambda)^{\frac{1}{\theta}}}, \quad \forall (\mu, \lambda) \in \Gamma_\theta.$$ (3.1)

Proof. We employ the DFS space characterization of the Gevrey spaces as in (1.2)-(1.3). By hypothesis we have $\tilde{u}(\mathcal{G}^{s,1}(G)) \subset \mathcal{G}(T)$ hence the induced linear map $\tilde{u} : \mathcal{G}^{s,1}(G) \to \mathcal{G}(T)$ is continuous by De Wilde’s Closed Graph Theorem [19, p. 57] (whose applicability is granted by the fact that Banach spaces are ultrabornological and DFS spaces are webbed): indeed, notice that its graph is closed thanks to the continuity of $\tilde{u} : \mathcal{C}^\infty(G) \to \mathcal{C}(T)$. As such, $\tilde{u}$ maps bounded sets in $\mathcal{G}^{s,1}(G)$ to bounded sets in $\mathcal{G}(T)$ so by [17, Lemma 3] we may assert the existence of an $h > 0$ such that

$$\tilde{u}(\{ \phi \in \mathcal{G}^{s,1}(G) : \| \phi \|_{\mathcal{G}^{s,1}(G)} \leq 1 \}) \subset \mathcal{G}^{h,1}(T).$$

Of course, we may assume that $h > 2$. By linearity, $\tilde{u}(\mathcal{G}^{s,1}(G)) \subset \mathcal{G}^{h,1}(T)$ and again $\tilde{u} : \mathcal{G}^{s,1}(G) \to \mathcal{G}^{h,1}(T)$ is continuous by the Closed Graph Theorem (the classical one). Therefore, there exists a constant $C > 0$ such that

$$\|\tilde{u}(\phi)\|_{\mathcal{G}^{h,1}(G)} \leq C \|\phi\|_{\mathcal{G}^{s,1}(G)}, \quad \forall \phi \in \mathcal{G}^{s,1}(G).$$

When we take $\phi = \phi_j^\theta$ – an element of our orthonormal basis of $\mathcal{E}_\lambda^G$ – we obtain

$$\|\tilde{u}(\phi_j^\theta)\|_{\mathcal{G}^{h,1}(T)}^2 = \sum_{\mu \in \pi(\Delta_T)} e^{2h(1+\mu)^{\frac{1}{\theta}}} \|F_{\mu}^T [\tilde{u}(\phi_j^\theta)]\|_{L^2(T)}^2 = \sum_{\mu \in \pi(\Delta_T)} e^{2h(1+\mu)^{\frac{1}{\theta}}} \sum_{i=1}^{d^n} \|\langle u, \psi_i^\theta \otimes \phi_j^\theta \rangle\|^2$$

hence

$$\sum_{j=1}^{d^n} \|\tilde{u}(\phi_j^\theta)\|_{\mathcal{G}^{h,1}(T)}^2 = \sum_{\mu \in \pi(\Delta_T)} e^{2h(1+\mu)^{\frac{1}{\theta}}} \sum_{i=1}^{d^n} \sum_{j=1}^{d^n} \|\langle u, \psi_i^\theta \otimes \phi_j^\theta \rangle\|^2 = \sum_{\mu \in \pi(\Delta_T)} e^{2h(1+\mu)^{\frac{1}{\theta}}} \|F_{\mu}^T F_{\lambda}^G(u)\|_{L^2(T \times G)}^2$$

from which we conclude that

$$e^{2h(1+\mu)^{\frac{1}{\theta}}} \|F_{\mu}^T F_{\lambda}^G(u)\|_{L^2(T \times G)}^2 \leq \sum_{j=1}^{d^n} \|\tilde{u}(\phi_j^\theta)\|_{\mathcal{G}^{h,1}(T)}^2 \leq d^n C \|\phi_j^\theta\|_{\mathcal{G}^{1,1}(G)}^2 e^{2(1+\lambda)^{\frac{1}{\theta}}}$$

where we used that

$$\|\phi_j^\theta\|_{\mathcal{G}^{1,1}(G)} = e^{(1+\lambda)^{\frac{1}{\theta}}}.$$
By Weyl’s asymptotic formula \( d_L^G = O(\lambda^{n/2}) \) we have, enlarging \( C \) if necessary,
\[
e^{2k(1+\mu)\frac{\theta}{2}} \| \mathcal{P}_\mu^T \mathcal{F}_\lambda^G(u) \|_{L^2(T \times G)}^2 \leq C^2 e^{4(1+\lambda)\frac{\theta}{2}}, \quad \forall (\mu, \lambda) \in \sigma(\Delta_T) \times \sigma(\Delta_G).
\]

For \((\mu, \lambda) \in \Gamma_\theta\) we then have
\[
1 + \lambda \leq 1 + \theta \mu \leq 1 + \mu
\]
so
\[
\| \mathcal{P}_\mu^T \mathcal{F}_\lambda^G(u) \|_{L^2(T \times G)} \leq C e^{2(1+\lambda)\frac{\theta}{2} - h(1+\mu)\frac{\theta}{2}} \leq C e^{(2-h)(1+\mu)\frac{\theta}{2}}
\]
but also
\[
1 + \mu + \lambda \leq 1 + \mu + \theta \mu \leq 2(1 + \mu)
\]
hence
\[
\| \mathcal{P}_\mu^T \mathcal{F}_\lambda^G(u) \|_{L^2(T \times G)} \leq C e^{-h'(1+\mu+\lambda)\frac{\theta}{2}}
\]
where \( h' = (h - 2)/(2\theta) \).

**Proposition 3.4.** If \( u \in \mathcal{D}'(T \times G) \) is such that

1. there exist \( C, h > 0 \) and \( \theta \in (0, 1) \) such that (3.1) holds and
2. there exist \( C', h' > 0 \) such that
\[
\| \mathcal{F}_\lambda^G(u) \|_{L^2(T \times G)} \leq C' e^{-h'(1+\lambda)\frac{\theta}{2}}, \quad \forall \lambda \in \sigma(\Delta_G)
\]
then \( u \in \mathcal{G}^s(T \times G) \).

*Proof.* For \((\mu, \lambda) \in \sigma(\Delta_T) \times \sigma(\Delta_G)\) not in \( \Gamma_\theta \) we have
\[
1 + \mu + \lambda < 1 + \frac{\lambda}{\theta} + \lambda \leq \frac{2}{\theta}(1 + \lambda)
\]
since \( 1/\theta > 1 \), hence
\[
(1 + \lambda)\frac{\theta}{2} \geq \left( \frac{\theta}{2} \right) (1 + \mu + \lambda)\frac{\theta}{2}.
\]
We conclude that
\[
\| \mathcal{P}_\mu^T \mathcal{F}_\lambda^G(u) \|_{L^2(T \times G)} \leq \begin{cases} 
C' e^{-h'(\theta/2)(1+\mu+\lambda)\frac{\theta}{2}}, & \text{in } \Gamma_\theta^c \\
C e^{-h(1+\mu+\lambda)\frac{\theta}{2}}, & \text{in } \Gamma_\theta
\end{cases}
\]
and therefore \( u \in \mathcal{G}^s(T \times G) \). \( \square \)

4. **Gevrey vectors for the partial Laplacian**

**Proposition 4.1.** Let \( P \) be a real-analytic LPDO on \( T \times G \) that is globally hypoelliptic and commutes with \( \Delta_G \). If \( u \in \mathcal{C}^\infty(T \times G) \) is such that \( Pu \in \mathcal{G}^s(T \times G) \) then \( u \) is a \( s \)-Gevrey vector for \( \Delta_G \), that is, there exist \( C, h > 0 \) such that
\[
\| \Delta_G^k u \|_{L^2(T \times G)} \leq C h^k k^{2s} \quad (4.1)
\]
for every \( k \in \mathbb{Z}_+ \).

*Proof.* We follow an approach similar to that of [8], for which we start with some preliminary remarks. Let \( u \in \mathcal{C}^\infty(T \times G) \) be such that \( Pu \in \mathcal{G}^s(T \times G) \). Since \( P \) is globally hypoelliptic, by standard functional analytic arguments (see e.g. [6, Lemma 3.1]) there exists \( t \in \mathbb{R} \) and \( C_1 > 0 \) such that
\[
\| \Delta_G^k u \|_{L^2(T \times G)} \leq C_1 (\| Pu \|_{\mathcal{G}^t(T \times G)} + \| \Delta_G^k u \|_{\mathcal{G}^{t-1}(T \times G)}), \quad \forall k \in \mathbb{Z}_+.
\]
We make use of the Sobolev norms (1.4). Then the last term in the inequality above can be estimated as follows:
\[
\| \Delta_G^{k+1} u \|_{\mathcal{G}^{t-1}(T \times G)}^2 = \sum_{\mu \in \sigma(\Delta_T), \lambda \in \sigma(\Delta_G) \atop \lambda \leq \sigma(\Delta_G)} \frac{\lambda^2}{(1 + \mu + \lambda)^2} \| \mathcal{P}_\mu^T \mathcal{F}_\lambda^G(\Delta_G^k u) \|_{L^2(T \times G)}^2 \leq \| \Delta_G^k u \|_{L^2(T \times G)}^2, \quad \forall k \in \mathbb{Z}_+.
\]
Moreover, as $P$ commutes with $\Delta_G$ we have
\[
\|P\Delta_G^k u\|^2_{\mathcal{F}_h^s(T \times G)} = \|\Delta_G P u\|^2_{\mathcal{F}_h^s(T \times G)} = \sum_{\mu \in \sigma(\Delta_T) \land \lambda \in \sigma(\Delta_G)} (1 + \mu + \lambda)^{2k} \|F^T \mathcal{F}_h^G(Pu)\|^2_{L^2(T \times G)}
\]
and since $Pu \in \mathcal{G}^s(T \times G)$ there exists $A > 0$ such that
\[
\|F^T \mathcal{F}_h^G(Pu)\|_{L^2(T \times G)} \leq A^{\ell + 1} \ell^{2s} (1 + \mu + \lambda)^{-\ell}, \quad \forall \ell \in \mathbb{Z}_+, \mu \in \sigma(\Delta_T), \lambda \in \sigma(\Delta_G)
\]
hence
\[
\|P\Delta_G^k u\|^2_{\mathcal{F}_h^s(T \times G)} \leq (A^{\ell_0 + k} + 1)^{2s} \sum_{\mu \in \sigma(\Delta_T) \land \lambda \in \sigma(\Delta_G)} (1 + \mu + \lambda)^{2k} \frac{\lambda^{2k}}{(1 + \mu + \lambda)^{2s}}
\]
We choose $\ell_0 \in \mathbb{Z}_+$ such that
\[
B = \left( \sum_{\mu \in \sigma(\Delta_T) \land \lambda \in \sigma(\Delta_G)} (1 + \mu + \lambda)^{2k - 2\ell_0} \right)^{1/2} < \infty
\]
which always exists thanks to Weyl's asymptotic formula, so plugging $\ell = \ell_0 + k$ into (4.2) yields:
\[
\|P\Delta_G^k u\|^2_{\mathcal{F}_h^s(T \times G)} \leq (A^{\ell_0 + k + 1} + 1)^{2s} \sum_{\mu \in \sigma(\Delta_T) \land \lambda \in \sigma(\Delta_G)} (1 + \mu + \lambda)^{2k} \frac{\lambda^{2k}}{(1 + \mu + \lambda)^{2s}} \leq B^2 (A^{\ell_0 + k + 1} + 1)^{2s}
\]
and in particular we have that
\[
\|\Delta_G^{k_0+1} u\|_{L^2(T \times G)} \leq C_1 (BA^{\ell_0 + k_0 + 2s} (k_0 + 1)^{2s} + 1 + \|\Delta_G^{k_0} u\|_{L^2(T \times G)}) \leq C_1 (BA^{\ell_0 + k_0 + 2s} (k_0 + 1)^{2s}) \leq C_1 (BA^{\ell_0 + k_0 + 2s} (k_0 + 1)^{2s} + Ch^{k_0}k_0^{12s})
\]
for every $k \in \mathbb{Z}_+$.

We are in position to assume by induction that there exist $C, h > 0$ such that (4.1) holds up to a certain $k_0 \in \mathbb{Z}_+$. We will check that it also holds for $k = k_0 + 1$. We may assume w.l.o.g. that
\[
C > 2C_1 BA^{\ell_0 + 1} k_0^4 \ell_0^{12s}, \quad h > \max\{2C_1, A4^s\}
\]
By (4.3) we have
\[
\|\Delta_G^{k_0+1} u\|_{L^2(T \times G)} \leq C_1 (BA^{\ell_0 + k_0 + 2s} (k_0 + 1)^{2s} + Ch^{k_0}k_0^{12s})
\]
where
\[
C_1 (BA^{\ell_0 + k_0 + 2s} (k_0 + 1)^{2s} + Ch^{k_0}k_0^{12s}) \leq C_1 (BA^{\ell_0 + 1} A^{4^s} \ell_0^{12s}) \left( \frac{A4^s}{h} \right)^{k_0+1} + \frac{C_1}{h(k_0 + 1)^{2s}}
\]
is less than $1$ thanks to (4.4). This concludes our proof.

Notice that if $u \in \mathcal{C}^\infty(T \times G)$ is a $s$-Gevrey vector for $\Delta_G$ then
\[
\lambda^k \|F^h_G(u)\|_{L^2(T \times G)} = \|F^h_G(\Delta_G^k u)\|_{L^2(T \times G)} \leq \|\Delta_G^k u\|_{L^2(T \times G)} \leq Ch^{k+1}k!^{2s}
\]
for some $C, h > 0$ independent of both $\lambda \in \sigma(\Delta_G)$ and $k \in \mathbb{Z}_+$. Assuming that $h > 1$ we obtain
\[
(1 + \lambda)^k \|F^h_G(u)\|_{L^2(T \times G)} \leq C(2h)^{k+1}k!^{2s}
\]
which, by standard computations, is equivalent to (3.2) for some constants $C', h' > 0$. Summing up with the results from Section 3 we immediately get:

**Theorem 4.2.** Let $P$ be a real-analytic LPDO on $T \times G$ belonging to class $T$. If $P$ is globally hypoelliptic in $T \times G$ then for each $s \geq 1$ it is also globally $\mathcal{G}^s$-hypoelliptic:
\[
\forall u \in \mathcal{G}^s(T \times G), \quad Pu \in \mathcal{G}^s(T \times G) \implies u \in \mathcal{G}^s(T \times G).
\]
5. A general result and an application on global Gevrey solvability

Our goal in this section is to introduce an abstract notion of hypoellipticity of an operator and show that this notion implies closedness of range of the same operator. As a consequence of this relationship we prove that a very weak notion of Gevrey hypoellipticity is sufficient to global solvability in the corresponding setup.

We take a look at the category of pairs of topological vector spaces: its objects are 2-tuples of topological vector spaces \((E^3, E)\), where \(E\) is a linear subspace of \(E^3\) carrying a topology finer than that inherited from \(E^3\), while its morphisms are maps of pairs \(T : (E^3, E) \to (F^3, F)\), meaning that \(T : E^3 \to F^3\) is a continuous linear map such that \(T(E) \subset F\) and the induced map \(T : E \to F\) is continuous. In this context:

**Definition 5.1.** We shall say that \(T\) satisfies:

- property \((\mathcal{H})\) if for every \(u \in E^3\) such that \(Tu \in F\) we have that \(u \in E\);
- property \((\mathcal{H}')\) if for every \(u \in E^3\) such that \(Tu \in F\) there exists \(v \in E\) such that \(Tv = Tu\).

Clearly \((\mathcal{H})\) holds if and only if \(T\) satisfies both \((\mathcal{H}')\) and \(\ker T \subset E\). Moreover, one has that \((E^3/\ker T, E/(E \cap \ker T))\) is also a pair of topological vector spaces and \(T\) descends to the quotient as a map of pairs

\[
T' : (E^3/\ker T, E/(E \cap \ker T)) \longrightarrow (F^3, F)
\]  

(5.1)

and a simple argument shows that \(T\) satisfies \((\mathcal{H}')\) if and only if \(T'\) satisfies \((\mathcal{H})\).

Our goal is to investigate closedness of the range of \(T : E \to F\) as a continuous linear map.

**Lemma 5.2.** Let \(T : (E^3, E) \to (F^3, F)\) be a map of pairs of topological vector spaces with \(F^3\) Hausdorff. If \(T\) satisfies \((\mathcal{H})\) then the graph of \(T : E \to F\) is closed in \(E^3 \times F\).

**Proof.** We prove that the range of the continuous map

\[
\gamma_T : E \longrightarrow E^3 \times F
\]

\[
u \longmapsto (u, Tu)
\]

is closed. Take \(\{u_\alpha\}\) a net in \(E\) such that \((u_\alpha, Tu_\alpha) \to (u, f)\) in \(E^3 \times F\) for some \((u, f) \in E^3 \times F\). Then:

- \(u_\alpha \to u\) in \(E^3\), and since \(T : E^3 \to F^3\) is continuous we have that \(Tu_\alpha \to Tu\) in \(F^3\);
- \(Tu_\alpha \to f\) in \(F\), hence also in \(F^3\).

Since \(F^3\) is Hausdorff we have that \(Tu = f \in F\): therefore \(u \in E\) thanks to property \((\mathcal{H})\), hence \((u, f) = (u, Tu) \in \text{ran } \gamma_T\). \(\Box\)

From here on we focus on the particular situation in which \(E, E^3, F\) are DFS spaces, and we fix \(\{E_j\}_{j \in \mathbb{Z}_+}, \{E^3_k\}_{k \in \mathbb{Z}_+}, \{F_k\}_{k \in \mathbb{Z}_+}\) injective sequences of Banach spaces with compact inclusion maps whose injective limits are \(E, E^3, F\) respectively. We also assume the following condition (stronger than \(E \hookrightarrow E^3\)):

\[
E \hookrightarrow E^3_k \text{ continuously.}
\]  

(5.2)

**Theorem 5.3.** Let \(E, E^3, F\) be DFS as above and \(F^3\) be a Hausdorff space. If \(T\) satisfies \((\mathcal{H})\) then \(T : E \to F\) has closed range.

**Proof.** By Lemma 5.2 the graph map \(\gamma_T\) has closed range. Since both \(E\) and \(E^3 \times F\) are DFS spaces, and moreover \(\gamma_T\) is injective, the following criterion for closedness of its range applies [3, Lemma 2.3]: for every \(k \in \mathbb{Z}_+\) there exists \(j \in \mathbb{Z}_+\) such that

\[
\forall u \in E, \ \gamma_T(u) \in E^3_k \times F_k \implies u \in E_j
\]

which, thanks to (5.2) and the definition of \(\gamma_T\), is equivalent to

\[
\forall u \in E, \ Tu \in F_k \implies u \in E_j
\]

in turn implying that \(T : E \to F\) has closed range [3, Theorem 2.5]. \(\Box\)

Since the class of DFS spaces is closed under taking closed subspaces and quotients [17], the whole picture above is preserved if we replace \(T\) by \(T'\) (5.1). For instance, \(H \doteq E/(E \cap \ker T)\) is a DFS space;
actually, $H_j \cong E_j/(E_j \cap \ker T)$ is a Banach space, the inclusion map $E_j \hookrightarrow E_{j'}$ ($j < j'$) descends to a compact injection $H_j \hookrightarrow H_{j'}$, and we have

$$H \cong \lim_{j \to \infty} H_j$$

where we regard $H_j$ as a subspace of $H$ by means of the identification

$$u + (E_j \cap \ker T) \in H_j \mapsto u + (E \cap \ker T) \in H$$

(which the reader may check at once to be well-defined, continuous and injective). The same goes for $E^\sharp/\ker T$, and (5.2) descends to a continuous injection $E/(E \cap \ker T) \hookrightarrow E^\sharp_0/(E^\sharp_0 \cap \ker T)$ – the first step in the sequence that naturally defines the injective limit topology on $E^\sharp/\ker T$.

**Corollary 5.4.** If $T$ satisfies $(\mathcal{H}')$ then $T : E \to F$ has closed range.

**Proof.** In that case, $T'$ (5.1) satisfies $(\mathcal{H})$: by our previous digression, we are entitled to apply Theorem 5.3 to it, yielding closedness of the range of $T' : E/(E \cap \ker T) \to F$, which equals that of $T : E \to F$.

\[\square\]

### 5.1. Global Gevrey solvability.

Let $M$ be a compact real-analytic manifold as in Section 1. Given $s \geq 1$, we denote by $\mathcal{D}'_s(M)$ the topological dual of $\mathcal{G}^s(M)$, the so-called space of Gevrey ultradistributions of order $s$ (when $s = 1$ this is the space of hyperfunctions on $M$).

Given $P$ a Gevrey LPDO on $M$, we are interested in solving the equation $Pu = f$ for $f \in \mathcal{G}^s(M)$. If there exists a solution $u \in \mathcal{G}^s(M)$ to this problem then for any $v \in \mathcal{D}'_s(M)$ we have

$$\langle v, f \rangle = \langle v, Pu \rangle = \langle ^tPv, u \rangle$$

where $^tP$ denotes the transpose of $P$. A necessary condition on $f$ to the solvability of $Pu = f$ is then that

$$\langle v, f \rangle = 0 \quad \text{for every } v \in \mathcal{D}'_s(M) \text{ such that } ^tPv = 0. \quad (5.3)$$

This leads us to the following:

**Definition 5.5.** We say that $P$ is globally $\mathcal{G}^s$-solvable if for every $f \in \mathcal{G}^s(M)$ satisfying (5.3) there exists $u \in \mathcal{G}^s(M)$ such that $Pu = f$.

It turns out [3, Lemma 2.2] that $P$ is globally $\mathcal{G}^s$-solvable if and only if the map between DFS spaces $P : \mathcal{G}^s(M) \to \mathcal{G}^s(M)$ has closed range. In [5, Theorem 3.5] we proved that the following property, weaker that global hypoellipticity, implies global solvability of $P$ in the smooth setting (i.e. closedness of the range of $P : \mathcal{G}^\infty(M) \to \mathcal{G}^\infty(M)$):

$$\forall u \in \mathcal{D}'(M), \quad Pu \in \mathcal{G}^\infty(M) \implies \exists v \in \mathcal{G}^\infty(M) \text{ such that } Pv = Pu.$$

Corollary 5.4 gives, in the Gevrey framework, that a weak notion of global Gevrey hypoellipticity implies global solvability.

**Theorem 5.6.** Let $s \geq 1$ and suppose that for some $s_+ > s$ we have

$$\forall u \in \mathcal{G}^{s_+}(M), \quad Pu \in \mathcal{G}^s(M) \implies \exists v \in \mathcal{G}^s(M) \text{ such that } Pv = Pu. \quad (5.4)$$

Then $P$ is globally $\mathcal{G}^s$-solvable.

**Proof.** Let

$$E \doteq \mathcal{G}^s(M), \quad E^\sharp \doteq \mathcal{G}^{s_+}(M), \quad F \doteq \mathcal{G}^s(M), \quad F^\sharp \doteq \mathcal{D}'_s(M).$$

Since with these choices (5.4) is per se property $(\mathcal{H}')$ for $T \doteq P$, all the hypotheses of Corollary 5.4 are automatically fulfilled. \[\square\]

The converse of Theorem 5.6 holds for operators in the class $\mathcal{T}$:

**Proposition 5.7.** Let $P$ be a real-analytic LPDO on $T \times G$ in class $\mathcal{T}$. If $P$ is globally $\mathcal{G}^s$-solvable then

$$\forall u \in \mathcal{D}'(T \times G), \quad Pu \in \mathcal{G}^s(T \times G) \implies \exists v \in \mathcal{G}^s(T \times G) \text{ such that } Pv = Pu.$$
Proof. Let \( u \in \mathcal{D}'(G \times T) \) be such that \( f = Pu \in \mathcal{D}'(G \times T) \). Thanks to Proposition 1.1 we have

\[
f = \sum_{\lambda \in \sigma(E_G)} F^G_\lambda(f) = \lim_{\nu \to \infty} \sum_{|\lambda| \leq \nu} \hat{P}_\lambda F^G_\lambda(u) = \lim_{\nu \to \infty} P \sum_{|\lambda| \leq \nu} F^G_\lambda(u)
\]

with convergence in \( \mathcal{D}'(G \times T) \). Since each \( \hat{P}_\lambda \) is elliptic we have that \( \hat{P}_\lambda F^G_\lambda(u) = F^G_\lambda(f) \in \mathcal{D}'(G; E^G_\lambda) \) implies that \( F^G_\lambda(u) \in \mathcal{D}'(G; E^G_\lambda) \), hence (5.5) ensures that \( f \) belongs to the closure of the range of \( P : \mathcal{D}'(G \times T) \to \mathcal{D}'(G \times T) \). As the latter is closed in \( \mathcal{D}'(G \times T) \) by assumption, there exists \( v \in \mathcal{D}'(G \times T) \) such that \( Pv = f = Pu \).

\[ \square \]

References

[1] A. A. Albanese. On the global \( C^\infty \) and Gevrey hypoellipticity on the torus of some classes of degenerate elliptic operators. \textit{Note Mat.}, 31(1):1–13, 2011.
[2] A. A. Albanese and D. Jornet. Global regularity in ultradifferentiable classes. \textit{Ann. Mat. Pura Appl. (4)}, 193(2):369–387, 2014.
[3] G. Araújo. Regularity and solvability of linear differential operators in Gevrey spaces. \textit{Math. Nachr.}, 291(5-6):729–758, 2018.
[4] G. Araújo, I. A. Ferra, and L. F. Ragognette. Global hypoellipticity of sums of squares on compact manifolds. \textit{J. Anal. Math.}, (to appear), 2022.
[5] A. A. Albanese. On the global \( C^\infty \) and Gevrey hypoellipticity on the torus of some classes of degenerate elliptic operators. \textit{Note Mat.}, 31(1):1–13, 2011.
[6] A. A. Albanese and D. Jornet. Global regularity in ultradifferentiable classes. \textit{Ann. Mat. Pura Appl. (4)}, 193(2):369–387, 2014.
[7] P. Bolley, J. Camus, and C. Mattera. Analyticité microlocale et itérés d’opérateurs. In \textit{Séminaire Goulaouic-Schwartz (1978/1979)}, pages Exp. No. 13, 9. École Polytech., Palaiseau, 1979.
[8] N. Braun Rodrigues, G. Chinni, P. D. Cordaro, and M. R. Jahnke. Lower order perturbation and global analytic vectors for a class of globally analytic hypoelliptic operators. \textit{Proc. Amer. Math. Soc.}, 144(12):5159–5170, 2016.
[9] I. Chavel. \textit{Eigenvalues in Riemannian geometry}, volume 115 of \textit{Pure and Applied Mathematics}. Academic Press, Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.
[10] M. Christ. Global analytic hypoellipticity in the presence of symmetry. \textit{Math. Res. Lett.}, 1(5):559–563, 1994.
[11] P. D. Cordaro and A. A. Himonas. Global analytic hypoellipticity of a class of degenerate elliptic operators on the torus. \textit{Math. Res. Lett.}, 1(4):501–510, 1994.
[12] H. Grauert. On Levi’s problem and the imbedding of real-analytic manifolds. \textit{Ann. of Math. (2)}, 68:460–472, 1958.
[13] A. A. Himonas and G. Petronilho. Global hypoellipticity and simultaneous approximability. \textit{J. Funct. Anal.}, 170(2):356–365, 2000.
[14] A. A. Himonas and G. Petronilho. \textit{C^\infty} and Gevrey regularity of sublaplacians. \textit{Trans. Amer. Math. Soc.}, 358(11):4809–4820, 2006.
[15] L. Hörmander. Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients. \textit{Comm. Pure Appl. Math.}, 24:671–704, 1971.
[16] A. W. Knapp. \textit{Lie groups beyond an introduction}, volume 140 of \textit{Progress in Mathematics}. Birkhäuser Boston, Inc., Boston, MA, 1996.
[17] H. Komatsu. Projective and injective limits of weakly compact sequences of locally convex spaces. \textit{J. Math. Soc. Japan}, 19:366–383, 1967.
[18] T. Kotake and M. S. Narasimhan. Regularity theorems for fractional powers of a linear elliptic operator. \textit{Bull. Soc. Math. France}, 90:449–471, 1962.
[19] G. Köthe. \textit{Topological vector spaces. II}, volume 237 of \textit{Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]} Springer-Verlag, New York-Berlin, 1979.
[20] G. Petronilho. On Gevrey solvability and regularity. \textit{Math. Nachr.}, 282(3):470–481, 2009.
[21] L. Rodino. \textit{Linear partial differential operators in Gevrey spaces}. World Scientific Publishing Co., Inc., River Edge, NJ, 1993.
[22] V. S. Varadarajan. \textit{Lie groups, Lie algebras, and their representations}, volume 102 of \textit{Graduate Texts in Mathematics}. Springer-Verlag, New York, 1984. Reprint of the 1974 edition.
[23] N. R. Wallach. \textit{Harmonic analysis on homogeneous spaces}. Marcel Dekker, Inc., New York, 1973. Pure and Applied Mathematics, No. 19.

\textbf{Universidade de São Paulo, IC\textsc{mc-USP}, São Carlos, SP, Brazil.}
\textbf{Email address: gccsa@icmc.usp.br}

\textbf{Universidade Federal do ABC, CMCC-UFABC, São Bernardo do Campo, SP, Brazil.}
\textbf{Email address: ferra.igor@ufabc.edu.br}

\textbf{Universidade Federal de São Carlos, DM-UFSCar, São Carlos, SP, Brazil.}
\textbf{Email address: luissragognette@dms.ufscar.br}