TRANSITIVE 2-REPRESENTATIONS
OF FINITARY 2-CATEGORIES

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Abstract. In this article, we define and study the class of simple transitive 2-representations of finitary 2-categories. We prove a weak version of the classical Jordan-Hölder Theorem where the weak composition subquotients are given by simple transitive 2-representations. For a large class of finitary 2-categories we prove that simple transitive 2-representations are exhausted by cell 2-representations. Finally, we show that this large class contains finitary quotients of 2-Kac-Moody algebras.

1. Introduction

This article, for the first time, proves a general classification result for an axiomatically defined class of 2-representations of a large class of 2-categories covering most examples studied in the area of categorification.

More specifically, we study finitary 2-categories over an algebraically closed field which include the 2-category of Soergel bimodules associated to a finite Coxeter system (see [BG, So, EW]), an exhaustive family of quotients of 2-Kac-Moody algebras (see [BFK, KL, Ro1, CL, We]), quiver 2-categories constructed in [Xa] and the 2-category of projective functors on the module category of a finite dimensional algebra (see [MM1]). We define a new class of 2-representations for such 2-categories which we call simple transitive 2-representations and which we believe serves as the correct 2-analogue for the class of irreducible representations of an algebra. Our definition of simple transitive 2-representations comes in two layers, the first being a discrete transitive action of the multisemigroup of 1-morphisms (this alone is called transitivity), the second being the absence of categorical ideals in the representation invariant under the 2-action (this is what we refer to as simplicity).

For simple transitive 2-representations we obtain, for arbitrary finitary 2-categories, a weak version of the classical Jordan-Hölder Theorem, see Theorem 18 in which simple transitive 2-representations appear as weak composition subquotients of general finitary 2-representations. It turns out that any finitary 2-representation of a finitary 2-category has a filtration with subquotients being transitive 2-representations. In contrast to classical representation theory, transitive 2-representations do not seem to admit any natural filtration, however, they do have a well-defined simple top which is our weak composition subquotient. A different approach to the Jordan-Hölder theory for 2-Kac-Moody algebras is outlined in [Ro1 Subsection 5.1].

Our main result is Theorem 18 which provides a classification of simple transitive 2-representations for a large class of finitary 2-categories. The latter includes the 2-category of Soergel bimodules in type $A$, all of the above mentioned finitary quotients of 2-Kac-Moody algebras and the 2-category of projective functors on the
module category of a finite dimensional self-injective algebra. Moreover, it also
includes all variations of the latter 2-category which constitute a list of finitary
2-categories from [MM3] satisfying a 2-analogue of simplicity for a finite dimen-
sional algebra. The classification result states that for this class of 2-categories
simple transitive 2-representations are precisely the cell 2-representations studied
in [MM1, MM2, MM3]. In particular, this implies uniqueness of categorification
of simple integrable modules for finite dimensional simple Lie algebras. The only
comparable statement in the literature, for the 2-categorical analogue of $U(\mathfrak{sl}_2)$ and
for a special class of 2-representations categorifying simple $\mathfrak{sl}_2$-modules, was proved
in [CR, Proposition 5.26].

The proof can be divided into two major parts. One of these (the proof of Theo-
rem 18) reduces the problem to the case of the 2-category of projective functors on
the module category of a finite dimensional self-injective algebra. The latter case
is treated in Theorem 15 and relies on a detailed study of endomorphism algebras
of certain bimodules and, crucially, on a classical result of Perron and Frobenius on
the structure of real matrices with positive coefficients.

The article is organized as follows. In Section 2 we recall notions developed in
[MM1, MM2, MM3] and state the Perron-Frobenius Theorem. In Section 3 we
introduce transitive and simple transitive 2-representations and gather examples
and preliminary results. Section 4 presents the statement and proof of our weak
Jordan-Hölder Theorem. Section 5 is devoted to the proof of our main result
in the case of the 2-category of projective functors on the module category of a
finite dimensional self-injective algebra. Section 6 establishes the main result in the
general case. Finally, in Section 7 we provide and study examples, including our
family of quotients of 2-Kac-Moody algebras.

Acknowledgment. A substantial part of the paper was written during mutual
visits of the authors to the University of East Anglia respectively Uppsala Uni-
versity, whose hospitality is gratefully acknowledged. Both visits were supported
by EPSRC grant EP/K011782/1. The first author is partially supported by the
Swedish Research Council. The second author is partially supported by EPSRC
grant EP/K011782/1. We thank Anne-Laure Thiel, Qimh Xantcha and Ben Web-
ster for stimulating discussions. We thank the referee for very useful comments and
explanations.

2. Preliminaries

2.1. Notation. Throughout, we let $\mathbb{k}$ denote an algebraically closed field.

A 2-category is a category enriched over the category of small categories. A
2-category $\mathcal{C}$ consists of objects (denoted $i, j, k, \ldots$), 1-morphisms (denoted
$F, G, H, \ldots$) and 2-morphisms (denoted $\alpha, \beta, \gamma, \ldots$). For $i \in \mathcal{C}$, the identity 1-
morphism is denoted $1_i$ and, for a 1-morphism $F$, the corresponding identity 2-
morphism is denoted $\text{id}_F$. Composition of 1-morphisms is denoted by $\circ$, hori-
zontal composition of 2-morphisms is denoted by $\circ_0$ and vertical composition of
2-morphisms is denoted by $\circ_1$. We let $\textbf{Cat}$ denote the 2-category of small cate-
gegories.
2.2. **Finitary 2-categories.** An additive \(k\)-linear category is called **finitary** if it is idempotent split, has finitely many isomorphism classes of indecomposable objects and finite dimensional \(k\)-vector spaces of morphisms. Denote by \(\mathcal{A}_k\) the 2-category whose objects are finitary additive \(k\)-linear categories, 1-morphisms are additive \(k\)-linear functors and 2-morphisms are natural transformations of functors.

A **finitary** 2-category (over \(k\)) is a 2-category \(\mathcal{C}\) with the following properties:

- it has a finite number of objects;
- for any pair \(i, j\) of objects in \(\mathcal{C}\), the category \(\mathcal{C}(i, j)\) is in \(\mathcal{A}_k\) and horizontal composition is both additive and \(k\)-linear;
- for any \(i \in \mathcal{C}\), the 1-morphism \(\mathbb{1}_i\) is indecomposable.

We refer to [Le, McL] for more general details on abstract 2-categories and to [MM1, MM2, MM3, MM4] for more information on finitary 2-categories.

2.3. **2-representations.** Let \(\mathcal{C}\) be a finitary 2-category. By a **2-representation** of \(\mathcal{C}\) we mean a strict 2-functor from \(\mathcal{C}\) to \(\text{Cat}\). By a **finitary** 2-representation of \(\mathcal{C}\) we mean a strict 2-functor from \(\mathcal{C}\) to \(\mathcal{A}_k\). Our 2-representations are generally denoted by \(M, N, \ldots\) with one exception: for \(i \in \mathcal{C}\) we have the principal 2-representation \(P_i := \mathcal{C}(i, -)\). Finitary 2-representations of \(\mathcal{C}\) form a 2-category, denoted \(\mathcal{C}_{af\text{-mod}}\), whose 1-morphisms are 2-natural transformations and whose 2-morphisms are modifications (see [Le, MM3]).

Two 2-representations \(M\) and \(N\) of \(\mathcal{C}\) are called **equivalent** if there exists a 2-natural transformation \(\Phi : M \to N\) such that \(\Phi_i\) is an equivalence for each \(i\).

Let \(M\) be a 2-representation of \(\mathcal{C}\). Assume that \(M(i)\) is an idempotent split additive category for each \(i \in \mathcal{C}\). For any collection of objects \(X_i \in M(i)\), where \(i \in I\), the additive closure of all objects of the form \(F X_i\), where \(i \in I\) and \(F\) runs through all 1-morphisms of \(\mathcal{C}\) is stable under the action of \(\mathcal{C}\) and hence inherits the structure of a 2-representation by restriction. This 2-representation will be denoted \(G_M(\{X_i : i \in I\})\).

To simplify notation, we will often write \(F X\) for \(M(F) X\) where \(F\) is a 1-morphism.

2.4. **Combinatorics of finitary 2-categories.** Let \(\mathcal{C}\) be a finitary 2-category. Denote by \(\mathcal{S}(\mathcal{C})\) the multiset of isomorphism classes of 1-morphisms in \(\mathcal{C}\), see [MM2 Section 3]. As usual, we define the left preorder \(\geq_L\) on \(\mathcal{S}(\mathcal{C})\) as follows: for two 1-morphisms \(F, G\) we set \(G \geq_L F\) provided that there is a 1-morphism \(H\) such that \(G\) is isomorphic to a direct summand of \(H \circ F\). Equivalence classes for \(\geq_L\) are called **left cells**. Right and two-sided preorders \(\geq_R\) and \(\geq_J\) and respective cells are defined analogously.

2.5. **Weakly fiat and fiat 2-categories.** For a 2-category \(\mathcal{C}\) there are three ways of creating an opposite 2-category.

- We can reverse both 1- and 2-morphisms.
- We can reverse only 1-morphisms.
We can reverse only 2-morphisms.

In the present paper we let \( C^{\text{op}} \) denote the first of the three choices above.

A finitary 2-category \( C \) is called weakly fiat provided that

- there is a weak equivalence \(* : C \to C^{\text{op}}\);

- for any pair \( i, j \in C \) and every 1-morphism \( F \in C(i, j) \) we have 2-morphisms \( \alpha : F \circ F^* \to i \) and \( \beta : i \to F^* \circ F \) such that \( \alpha_F \circ_1 F^*(\beta) = \text{id}_F \) and \( F^*(\alpha) \circ_1 F^* = \text{id}_{F^*} \).

If \(*\) is involutive, then \( C \) is called fiat, see [MM1, MM2].

2.6. 2-ideals. Let \( C \) be a 2-category. A left 2-ideal \( I \) of \( C \) consists of the same objects as \( C \) and for each pair \( i, j \) of objects an ideal \( I(i, j) \) in \( C(i, j) \) such that \( I \) is stable under the left horizontal multiplication with 1- and 2-morphisms in \( C \). Similarly one defines right 2-ideals and two-sided 2-ideals. The latter will simply be called 2-ideals. For example, each principal 2-representation can be interpreted as a left 2-ideal in \( C \).

Let \( C \) be a 2-category and \( M \) be a 2-representation of \( C \). An ideal \( I \) of \( M \) is a collection of ideals \( I(i) \) in \( M(i) \) for each \( i \in C \) stable under the action of \( C \) in the following sense: for any morphism \( \eta \in I \) and any 1-morphism \( F \) the composition \( M(F)(\eta) \) (if it is defined) is in \( I \). For example, left 2-ideals of \( C \) give rise to ideals in principal 2-representations.

2.7. Abelianization. Let \( A \) be a finitary additive \( k \)-linear category. Then the abelianization \( \overline{A} \) of \( A \) is the category whose objects are diagrams \( X \xrightarrow{\eta} Y \) where \( X, Y \in A \) and \( \eta \in A(X, Y) \) and morphisms are equivalence classes of solid commutative diagrams of the form

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & Y \\
\downarrow{\tau_1} & \nearrow{\tau_2} & \\
X' & \xrightarrow{\eta'} & Y'
\end{array}
\]

modulo the subspace spanned by those diagrams for which there exists \( \tau_3 \) as indicated by the dashed arrow such that \( \eta' \tau_3 = \tau_2 \). The category \( \overline{A} \) is abelian (cf. [ET]) and is equivalent to the category of modules over the finite dimensional \( k \)-algebra

\[
\text{End}_A(P)^{\text{op}} \quad \text{where} \quad P := \bigoplus_{Q \in \text{Ind}(A)/\sim} Q.
\]

Let \( C \) be a 2-category and \( M \) a finitary 2-representation of \( C \). Then the abelianization of \( M \) is the 2-representation \( \overline{M} \) of \( C \) which assigns to each \( i \in C \) the category \( \overline{M}(i) \) with the action of \( C \) defined on diagrams component-wise.

Directly from the definition it follows that the action of each 1-morphism on the abelianization of any finitary 2-representation is right exact.
A finitary 2-representation $\mathbf{M}$ of $\mathcal{C}$ will be called exact provided that $\mathbf{M}(F)$ is exact for any 1-morphism $F$ in $\mathcal{C}$. For example, any finitary 2-representation of a weakly flat 2-category is exact.

2.8. Perron-Frobenius Theorem. We will use the following classical result due to Perron and Frobenius, see the original papers [Fro1, Fro2, Pe] or the detailed exposition in [Me, Chapter 8].

**Theorem 1.** Let $A = (a_{ij})$ be a real $n \times n$ matrix with strictly positive coefficients.

(i) $A$ has a positive real eigenvalue, call it $r$, such that any other (possibly complex) eigenvalue of $A$ has a strictly smaller absolute value.

(ii) The eigenvalue $r$ appears with multiplicity one in the characteristic polynomial of $A$.

(iii) There exists a real eigenvector, call it $\mathbf{v}$, for eigenvalue $r$ with strictly positive coefficients, moreover, any real eigenvector of $A$ with strictly positive coefficients is a multiple of $\mathbf{v}$.

(iv) The eigenvalue $r$ satisfies

$$\min_j \{\sum_i a_{ij}\} \leq r \leq \max_j \{\sum_i a_{ij}\}.$$ 

**Corollary 2.** Assume that $A$ is as in Theorem 1 and has rank one. Then, if either inequality in Theorem 1 is an equality, then both inequalities are equalities and all columns of $A$ coincide.

**Proof.** If $A$ has rank one, then all columns of $A$ are proportional to $\mathbf{v}$ and the trace of $A$ equals $r$. Assume, for example, that $\min_j \{\sum_i a_{ij}\} = \sum_i a_{i1} = r$. Set $\lambda_1 = 1$ and for $j = 2, 3, \ldots, n$ let $\lambda_j$ be the positive real number ($\geq 1$) such that the $j$-th column equals $\lambda_j$ times the first column. Then, we have

$$\sum_i a_{i1} = r = \text{trace}(A) = \sum_i a_{ii} = \sum_i \lambda_i a_{i1} \geq \sum_i a_{i1} = r.$$ 

It follows that $\lambda_j = 1$ for all $j$. The case where the second inequality is an equality is similar. $\square$

3. Transitive 2-representations

In this section, $\mathcal{C}$ will be a finitary 2-category.

3.1. **Definition.** Let $\mathbf{M}$ be a finitary 2-representations of $\mathcal{C}$. We will say that $\mathbf{M}$ is transitive provided that for every $\mathbf{i}$ and for every non-zero object $X \in \mathbf{M}(\mathbf{i})$ we have $G_{\mathbf{M}}(\{X\}) = \mathbf{M}$.

3.2. **Example:** transitive group actions. Let $G = (G, \cdot)$ be a finite group. Consider the finitary 2-category $\mathcal{G} = \mathcal{G}_G$ defined as follows:

- $\mathcal{G}$ has one object $\blacklozenge$;
• 1-morphisms in \( G \) are \( \bigoplus_{g \in G} F^{\otimes k_g} \) where all \( k_g \geq 0 \);

• composition of 1-morphisms is given by \( F_g \circ F_h = F_{gh} \) and extended by biadditivity;

• non-zero 2-morphisms between indecomposable 1-morphisms are just scalar multiples of the identity, 2-morphisms between decomposable 1-morphisms are matrices of morphisms between the corresponding indecomposable summands;

• vertical composition of 2-morphisms is given by matrix multiplication;

• horizontal composition of 2-morphisms is given by tensor product of matrices.

The 2-category \( G \) is finitary by definition. Moreover, it is even a fiat 2-category (where \( * \) is induced by \( g \mapsto g^{-1} \)).

Let \( H \) be a subgroup of \( G \). Let \( A \) be a small category equivalent to \( k\)-mod. Consider the category

\[ G_{H,A} := \bigoplus_{gH \in G/H} A(gH), \]

where \( (gH) \) is a formal index. Now define the 2-representation \( M_{H,A} \) of \( G \)

• on the object by \( M_{H,A}(\bullet) = G_{H,A} \);

• on 1-morphisms by \( M_{H,A}(F_g) = (\varphi_{xH,yH})_{xH,yH \in G/H} \) where

\[ \varphi_{xH,yH} = \begin{cases} 1d_A, & \text{if } gyH = xH; \\ 0, & \text{otherwise}; \end{cases} \]

• on 2-morphisms \( M_{H,A} \) in the obvious way using scalar multiples of the identity natural transformations.

It follows from the definition that \( M_{H,A} \) is a transitive 2-representation of \( G \). This 2-representation categorifies the classical transitive action of \( G \) on \( G/H \).

Note that in the above construction instead of \( A \) we can take any small finitary additive \( k \)-linear category \( B \) with one isomorphism class of indecomposable objects.

This example generalizes, in the obvious way, to finite semigroups. One major difference is that in the latter case the 2-category obtained will not be fiat but only finitary. Another difference is that while any transitive action of a finite group on a finite set is equivalent to the action on some \( G/H \), transitive actions of semigroups are more complicated, see e.g. [GM, Chapter 10].

3.3. Cell 2-representations. Here we use the approach from [MM2] to construct cell 2-representations for arbitrary finitary 2-categories.

Let \( L \) be a left cell in \( \mathcal{C} \). Then there is \( i = i_L \in \mathcal{C} \) such that every 1-morphism in \( L \) has domain \( i \). Consider the principal 2-representation \( P_i \). For \( j \in \mathcal{C} \) let \( N(j) \)
denote the additive closure in $P_1(j)$ of all 1-morphisms $F \in C(i,j) \cap L$ such that $F \geq L$. Then $N$ is a 2-subrepresentation of $P_1$.

**Lemma 3.** There is a unique maximal ideal $I$ in $N$ which does not contain $id_F$ for any $F \in L$.

**Proof.** Being an ideal of an additive category, $I$ is uniquely determined by its morphisms between indecomposable objects. If $F \in L \cap C(i,j)$, then the algebra of 2-endomorphisms of $F$ is local as $F$ is indecomposable. Therefore the part of $\text{End}_{C(i,j)}(F)$ contained in $I$ belongs to the radical of $\text{End}_{C(i,j)}(F)$. As the sum of two subspaces of the radical is contained in the radical, we conclude that the sum of all left ideals in $N$ which do not contain $id_F$ for any $F \in L$ still has the latter property. The claim follows. □

The quotient 2-functor $C_L := N/I$, where $I$ is given by Lemma 3, is called the (additive) cell 2-representation of $C$ associated to $L$. From the definitions, it follows directly that $C_L$ is a transitive 2-representation of $C$.

**3.4. A more exotic example.** Similarly to Subsection 3.2 one defines a 2-category $C$ with one object, indecomposable 1-morphisms $/BD$ and $F$, with the multiplication table

\[
\begin{array}{ccc}
/BD & /BD & F \\
/BD & /BD & F \\
F & F & F + F \\
\end{array}
\]

and only scalar multiples of the identity 2-morphisms for indecomposable 1-morphisms. This 2-category $C$ has two left cells (corresponding to the two indecomposable 1-morphisms), so we have the respective cell 2-representations. These are transitive, see Subsection 3.3. Similarly to Subsection 3.2 one can construct a rather different transitive 2-representation on a category $A \oplus A$, where $A$ is as in Subsection 3.2 by mapping the 1-morphism $F$ to the functor

\[
\begin{pmatrix}
\text{Id}_A & \text{Id}_A \\
\text{Id}_A & \text{Id}_A
\end{pmatrix}.
\]

**3.5. Simple transitive 2-representations.** Let $M$ be a transitive 2-representation of $C$.

**Lemma 4.** There is a unique maximal ideal $I$ in $M$ which does not contain any identity morphisms apart from the one for the zero object.

**Proof.** Mutatis mutandis proof of Lemma 3. □

The main idea of the following definition generalizes [MM2 Subsection 6.5]. A transitive 2-representation $M$ of $C$ is called *simple transitive* provided that its unique maximal ideal given by Lemma 4 is the zero ideal. For a transitive 2-representation $M$ denote by $M$ the quotient of $M$ by the ideal $I$ given by Lemma 4. We will loosely call $M$ the *simple transitive quotient* of $M$. 

3.6. Examples of simple transitive 2-representations. Lemma 3 implies that each cell 2-representation of $\mathcal{C}$ is simple transitive. Furthermore, transitive 2-representations $M_{H,A}$ of $\mathcal{G}$ constructed in Subsection 3.2 are simple transitive (and these are not equivalent to cell 2-representations in general). In fact, the next proposition shows that these exhaust all simple transitive 2-representations of $\mathcal{G}$.

**Proposition 5.** Every simple transitive 2-representations of $\mathcal{G}$ is equivalent to $M_{H,A}$ for some subgroup $H$ of $G$ and a skeletal category $A$ equivalent to $k$-mod.

**Proof.** Let $M$ be a simple transitive 2-representation of $\mathcal{G}$. Invertibility of each $F_g$ implies that $F_g$ sends non-isomorphic objects to non-isomorphic objects, indecomposable objects to indecomposable objects and radical morphisms to radical morphisms. Therefore the ideal $I$ given by Lemma 4 coincides with the radical of $M(\bullet)$. By simple transitivity, we hence obtain that the radical of $M(\bullet)$ is zero and thus $M(\bullet)$ is a semi-simple category.

As each $F_g$ sends indecomposable objects to indecomposable objects, $G$ induces a transitive action on the set of isomorphism classes of indecomposable objects in $M(\bullet)$. Fix an indecomposable object $X \in M(\bullet)$ and set 

$$H := \{ h \in G : F_h X \cong X \}.$$

Let $A$ be a skeletal category equivalent to $k$-mod. Consider the (unique!) functor $\Phi : M(\bullet) \to GH$ which sends an indecomposable object $Y \cong F_g X$ for some $g \in G$ to the unique indecomposable object in $A(gH)$. Then $\Phi$ is easily checked to give an equivalence between $M$ and $M_{H,A}$. The claim follows. 

Note that Proposition 5 does not extend to all transitive 2-representations in an obvious way. For example, let $G$ be the cyclic group of order two. Then $G$ acts by automorphisms on the finite dimensional $k$-algebra $A$ given by the quiver

\[
\begin{array}{c}
1 \rightarrow a \\
\downarrow b \\
2
\end{array}
\]

with relations $ab = ba = 0$ (the non-trivial automorphism is given by the automorphism of the quiver which swaps 1 with 2 and $a$ with $b$). This induces a transitive action of $G$ and hence of the corresponding 2-category $\mathcal{G}$ on any skeletal category equivalent to the category of finite dimensional projective $A$-modules. We refer to [AM, Section 2] for more details.

3.7. Strongly simple 2-representations are (simple) transitive. In parallel to [MM, Subsection 6.2], we call a finitary 2-representation $M$ of $\mathcal{C}$ strongly simple provided that for any $i, j \in \mathcal{C}$ with $M(i)$ nonzero, any simple object $L \in M(i)$ and any pair $P, Q$ of indecomposable projectives in $M(j)$, there exist indecomposable 1-morphisms $F$ and $G$ such that $FL \cong P$, $GL \cong Q$ and the evaluation map $\text{Hom}_{\mathcal{C}(i,j)}(F, G) \to \text{Hom}_{\mathcal{C}(j)}(FL, GL)$ is surjective.

**Proposition 6.** Let $\mathcal{C}$ be a finitary 2-category and $M$ a strongly simple finitary 2-representation of $\mathcal{C}$.

(i) The 2-representation $M$ is transitive.

(ii) If $M$ is exact (in particular, if $\mathcal{C}$ is weakly fiat), then $M$ is simple transitive.
Proof. Let X be a non-zero indecomposable object in some \( M(i) \) and \( L \) be its simple top in \( \overline{M}(i) \). Let Y be a non-zero indecomposable object in some \( M(j) \). By definition of strong simplicity, there is an indecomposable 1-morphism \( F \) such that \( FL \cong Y \). This means that \( Y \) is isomorphic to a direct summand of \( FX \) and hence \( M \) is transitive. This proves claim (i).

Let \( X, Y \in \overline{M}(i) \) be two indecomposable projective objects and \( \eta : X \to Y \) be a non-zero morphism. Denote by \( L \) the simple top of \( X \). Choose two 1-morphisms \( F \) and \( G \) in \( C \) such that \( FL \cong X \) and \( GL \cong Y \). Consider a finite dimensional \( k \)-algebra \( B \) such that \( M(i) \cong B\text{-mod} \). For simplicity, we identify \( M(i) \) and \( B\text{-mod} \).

Let \( e, e' \) be two primitive idempotents of \( B \) such that \( X \cong Be \) and \( Y \cong Be' \). Then, by Lemma 13, the functor \( \overline{M}(F) \) surjects onto the projective functor \( Be \otimes_k eB \otimes_B \). Similarly, the functor \( \overline{M}(G) \) surjects onto the projective functor \( Be' \otimes_k eB \otimes_B \).

Now, for any non-zero map \( \eta' : Be \to Be' \) the induced map
\[
\text{Id}_{Be} \otimes \text{Id}_{eB} \otimes \eta' : Be \otimes_k eB \otimes_B Be \to Be \otimes_k eB \otimes_B Be'
\]
contains, as a direct summand, the identity map on \( Be \). This implies that the ideal \( I \) in \( M \) generated by \( \eta \) contains the identity morphism on \( X \). Therefore \( M \) is simple transitive. \( \square \)

Example 7. The claim of Proposition 6(ii) fails for general finitary 2-representations. Consider the algebra \( D = k[x]/(x^2) \) of dual numbers. Let \( A \) be a small category equivalent to \( D\text{-mod} \) and \( \mathcal{C} \) the finitary category with one object \( \bullet \) which we identify with \( A \), with indecomposable 1-morphisms being endofunctors of \( A \) isomorphic to either the identity functor or tensoring with the \( D\)-\( D \)-bimodule \( D \otimes_k k \), and 2-morphisms being natural transformations of functors. Then the defining 2-representation of \( \mathcal{C} \), i.e. the natural 2-action of \( \mathcal{C} \) on \( A \), is clearly strongly simple. However, as tensoring with \( D \otimes_k k \) annihilates the non-zero nilpotent endomorphism of \( D \), this 2-representation is not simple transitive.

Note also that the example of a transitive 2-representation considered in Subsection 3.4 is, clearly, simple transitive but not strongly simple.

4. Weak Jordan-Hölder theory

In this section, \( \mathcal{C} \) will be a finitary 2-category.

4.1. The action preorder. Let \( M \) be a finitary 2-representation of \( \mathcal{C} \). Consider the (finite) set \( \text{Ind}(M) \) of isomorphism classes of indecomposable objects in all \( M(i) \) where \( i \in \mathcal{C} \). For \( X, Y \in \text{Ind}(M) \) set \( X \geq Y \) provided that there is a 1-morphisms \( F \) in \( \mathcal{C} \) such that \( X \) is isomorphic to a direct summand of \( FY \). Clearly, \( \geq \) is a partial preorder on \( \text{Ind}(M) \) which we will call the action preorder.

Let \( \sim \) be the equivalence relation defined by \( X \sim Y \) if and only if \( X \geq Y \) and \( Y \geq X \). Note that \( M \) is transitive if and only if we have exactly one equivalence class, namely the whole of \( \text{Ind}(M) \). The preorder \( \geq \) induces a genuine partial order on the set \( \text{Ind}(M)/\sim \) which, abusing notation, we will denote by the same symbol.
4.2. 2-subrepresentations and subquotients associated to coideals. Let $Q$ be a coideal in $\text{Ind}(\mathcal{M})/_\sim$. For $i \in \mathcal{C}$ consider the additive closure $M_Q(i)$ in $\mathcal{M}(i)$ of all indecomposable objects $X \in \mathcal{M}(i)$ whose equivalence class belongs to $Q$. Then $M_Q$ has the natural structure of a 2-representation of $\mathcal{C}$ given by restriction from $\mathcal{M}$. This is the 2-subrepresentation of $\mathcal{M}$ associated to $Q$.

Suppose we are given a pair $Q, R$ of coideals in $\text{Ind}(\mathcal{M})/_\sim$ such that $Q \subset R$. For $i \in \mathcal{C}$ let $I(i)$ denote the ideal in $M_R(i)$ generated by the identities on the objects in $M_Q(i)$. Then we can form the quotient category $M_{R/Q}(i) := M_R(i)/I(i)$ and the 2-functor $M_R$ induces the 2-functor $M_{R/Q}$ which sends $i$ to $M_{R/Q}(i)$. This is the 2-subquotient of $\mathcal{M}$ associated to $Q \subset R$. Note that $|R \setminus Q| = 1$ implies that the 2-representation $M_{R/Q}$ is transitive.

For $r \in \text{Ind}(\mathcal{M})/_\sim$ let $X_r$ be the maximal coideal in $\text{Ind}(\mathcal{M})/_\sim$ which does not contain $r$. Then $r$ becomes the minimum element in $(\text{Ind}(\mathcal{M})/_\sim) \setminus X_r$ with respect to the induced order. Let $Y_r := X_r \cup \{r\}$. Then $Y_r$ is a coideal in $\text{Ind}(\mathcal{M})/_\sim$. Therefore we have the associated quotient $M_{Y_r/X_r}$ and we set $M_r := M_{Y_r/X_r}$.

4.3. Weak Jordan-Hölder series. Consider a filtration $Q : \varnothing = Q_0 \subsetneq Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_k = \text{Ind}(\mathcal{M})/_\sim$ of coideals such that $|Q_i \setminus Q_{i-1}| = 1$ for all $i$. Such a filtration will be called a complete filtration. With such a filtration we associate a filtration of 2-subrepresentations

\[(1) \quad 0 \subset M_{Q_1} \subset M_{Q_2} \subset \cdots \subset M_{Q_k} = \mathcal{M}\]

and the corresponding sequence

\[(2) \quad M_{Q_1}, M_{Q_2/Q_1}, M_{Q_3/Q_2}, \ldots, M_{Q_k/Q_{k-1}}\]

of simple transitive subquotients. The filtration $\mathcal{(1)}$ is called a weak Jordan-Hölder series of $\mathcal{M}$ and the elements in $\mathcal{(2)}$ are also called weak composition subquotients.

4.4. Weak Jordan-Hölder theorem. The main result of this section is the following weak version of the classical Jordan-Hölder theorem.

**Theorem 8.** Let $\mathcal{C}$ be a finitary 2-category and $\mathcal{M}$ a finitary 2-representation of $\mathcal{C}$. Let further $Q$ and $R$ be two complete filtrations of $\text{Ind}(\mathcal{M})/_\sim$. Let $L_1, L_2, \ldots, L_k$ be the sequence of simple transitive subquotients associated to $Q$ and $L'_1, L'_2, \ldots, L'_l$ be the sequence of simple transitive subquotients associated to $R$. Then $k = l$ and there is a bijection $\sigma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, l\}$ such that $L_i$ and $L'_{\sigma(i)}$ are equivalent for all $i \in \{1, 2, \ldots, k\}$.

**Proof.** Note first that we have $k = l = |\text{Ind}(\mathcal{M})/_\sim|$ by definition. Let $r \in \text{Ind}(\mathcal{M})/_\sim$. Then there are unique $i, j \in \{1, 2, \ldots, k\}$ such that $r = Q_i \setminus Q_{i-1}$ and $r = R_j \setminus R_{j-1}$. To prove the assertion it is enough to show that the 2-representations $M_{L_i}, L_i$ and $L'_j$ are equivalent. By symmetry, it is enough to show that $M_{L_i}$ and $L_i$ are equivalent.

Let $I$ be the ideal in $M_{Y_r}$ used to define $M_{Y_r/X_r}$. Similarly, let $J$ be the ideal in $M_{Q_i}$ used to define $M_{Q_i/Q_{i-1}}$. By construction of $X_r$, we have $Q_{i-1} \subset X_r$ and hence also $Q_i \subset Y_r$. The inclusion $Q_i \subset Y_r$ induces a faithful 2-natural transformation from $M_{Q_i}$ to $M_{Y_r}$, which gives, by taking the quotient, a strong transformation from $M_{Q_i}$ to $M_{Y_r}$.
to $M_{Y/X}$. Since $Q_{i-1} \subset X_r$ for any indecomposable objects $M$ and $N$ whose $\sim$-classes belong to $r$, we have $\mathsf{J}(M, N) \subset \mathsf{I}(M, N)$. Therefore the strong transformation from $M_{Q_i}$ to $M_{Y/X}$ factors through $M_{Q_i/Q_{i-1}}$. This gives a 2-natural transformation from $M_{Q_i/Q_{i-1}}$ to $M_{Y/X}$, which is surjective on morphisms. Note that both 2-representations $M_{Q_i/Q_{i-1}}$ and $M_{Y/X}$ are transitive. Taking now the quotient by the unique maximal ideal given by Lemma 5 induces an equivalence between the corresponding simple transitive quotients, that is between $L_i$ and $M_i$. The claim follows. □

4.5. Example: weak composition subquotients for principal 2-representations. Consider the principal 2-representation $\mathbb{P}_i$ for $i \in \mathscr{C}$. The action preorder $\geq$ for $\mathbb{P}_i$ coincides with the restriction to $\mathbb{P}_i$ of the preorder $\geq_L$. Therefore $\operatorname{Ind}(\mathbb{P}_i)$ coincides with the set of isomorphism classes of 1-morphisms in $\mathscr{C}$ with domain $i$. The set $\operatorname{Ind}(\mathbb{P}_1)/_{\sim}$ thus becomes the set of all left cells with domain $1$. Comparing Subsection 5.3 with Subsection 4.3, we see that weak composition subquotients of $\mathbb{P}_1$ are exactly the cell 2-representations for left cells with domain $1$.

5. Classification of transitive 2-representations for $\mathscr{C}_A$

5.1. The 2-category $\mathscr{C}_A$. Let $A$ be a basic self-injective connected $k$-algebra of finite dimension $m$. Fix a small category $\mathcal{A}$ equivalent to $A$-$\text{mod}$. We assume that $\mathcal{A}$ is not semi-simple. Define the 2-category $\mathscr{C}_A$ as follows (cf. [MM1 Subsection 7.3]):

- $\mathscr{C}_A$ has one object $\bullet$ (which we identify with $\mathcal{A}$);
- 1-morphisms in $\mathscr{C}_A$ are direct sums of functors with summands isomorphic to the identity functor or to tensoring with projective $A$-$A$-bimodules;
- 2-morphisms in $\mathscr{C}_A$ are natural transformations of functors.

Functors isomorphic to tensoring with projective $A$-$A$-bimodules will be called projective functors.

Fix some decomposition $1 = e_1 + e_2 + \cdots + e_n$ of the identity in $A$ into a sum of primitive orthogonal idempotents. The 2-category $\mathscr{C}_A$ has a unique minimal two-sided cell consisting of the isomorphism class of the identity morphism. It has one other two-sided cell $\mathscr{J}$ consisting of the isomorphism classes of functors $F_{ij}$ given by tensoring with the indecomposable bimodules $Ae_i \otimes e_j A$, where $i, j \in \{1, 2, \ldots, n\}$. Left and right cells in $\mathscr{J}$ are

$$\mathcal{L}_j := \{F_{ij} : i \in \{1, 2, \ldots, n\}\} \quad \text{and} \quad \mathcal{R}_i := \{F_{ij} : j \in \{1, 2, \ldots, n\}\},$$

where $i, j \in \{1, 2, \ldots, n\}$. We have

$$F_{ij} \circ F_{kl} \cong F_{ij}^{\dim(e_j A e_i)}.$$ 

Let $\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ be the Nakayama bijection given by requiring $\operatorname{soc} A e_i \cong \top A e_{\sigma(i)}$ which is equivalent to $A e_i \cong \operatorname{Hom}_k(e_{\sigma(i)} A, k)$. Since $\operatorname{Hom}_A(A e_i \otimes_k e_{\sigma(i)} A, -) \cong \operatorname{Hom}_k(e_j A, k) \otimes_k e_i A \otimes_A -$, see e.g. [MM1 Subsection 7.3], we have that $(F_{ij}, F_{\sigma^{-1}(ij)})$ is an adjoint pair of functors. This implies that $\mathscr{C}_A$ is weakly fiat with $*$ defined on 1-morphisms by $F_{ij}^* = F_{\sigma^{-1}(ij)}$. 

\textbf{TRANSITIVE 2-REPRESENTATIONS} 11
We set $F := \bigoplus_{i,j=1}^{n} F_{ij}$. Since $A$ is basic and

$$A \otimes_{k} A \otimes_{k} A \cong A \otimes_{k} A^{\oplus \ell},$$

we have

(3) $F \circ F \cong F^{\oplus \ell}.$

Note that $F^{\ast} \cong F$.

The 2-category $\mathcal{C}_A$ is $\mathcal{J}$-simple in the sense that any nonzero two-sided 2-ideal in $\mathcal{C}_A$ contains the identity 2-morphisms on all indecomposable non-identity 1-morphisms, see [MM2, Subsection 6.2].

Denote by $P$ the full subcategory of $A$ consisting of projective objects. Then the defining action of $\mathcal{C}_A$ on $A$ restricts to $P$. We will denote the latter defining additive 2-representation of $\mathcal{C}_A$ by $D$.

**Proposition 9.** For any $j = 1, \ldots, n$ the 2-representations $D$ and $C_{L_j}$ are equivalent.

**Proof.** It is easy to check that mapping the generator $P_{1\bullet}$ of $P\bullet$ to the simple object in $A$ corresponding to $j$ induces an equivalence from $C_{L_j}$ to $D$. \qed

### 5.2. Matrices in the Grothendieck group

Let $M$ be a finitary 2-representation of $\mathcal{C}_A$. For a 1-morphism $G$ denote by $[G]$ the square matrix with non-negative integer coefficients whose rows and columns are indexed by isomorphism classes of indecomposable objects in $M(\bullet)$ and the intersection of the row indexed by $Y$ and the column indexed by $X$ contains the multiplicity of $Y$ as a direct summand of $G X$.

Consider the abelianization $\overline{M}$ of $M$. Then the isomorphism classes of simple objects in $\overline{M}(\bullet)$ are in bijection with isomorphism classes of indecomposable objects in $M(\bullet)$. For a 1-morphism $G$ denote by $[\overline{G}]$ the square matrix with non-negative integer coefficients whose rows and columns are indexed by isomorphism classes of simple objects in $\overline{M}(\bullet)$ and the intersection of the row indexed by $Y$ and the column indexed by $X$ contains the composition multiplicity of $Y$ in $G X$. The following generalizes [AM, Lemma 8].

**Lemma 10.** We have $[G^*] = [G]^t$, where $^t$ denotes the transpose of a matrix.

**Proof.** For a projective $P$ and a simple $L$ in $\overline{M}(\bullet)$ we have

$$\text{Hom}_{\overline{M}(\bullet)}(G, P, L) \cong \text{Hom}_{\overline{M}(\bullet)}(P, G^*, L).$$

The inclusion of $M(\bullet)$ to $\overline{M}(\bullet)$ given by $X \mapsto (0 \rightarrow X)$ is an equivalence between $M(\bullet)$ and the category of projective objects in $\overline{M}(\bullet)$. This implies the claim. \qed

**Lemma 11.** Consider the functor $F$ from Subsection 5.1.

(i) The matrix $[F]$ satisfies $[F]^2 = m[F]$.

(ii) If $M$ is transitive, then all entries in $[F]$ are positive.

(iii) If $M$ is transitive, then the rank of $[F]$ equals one.
Proof. Claim (i) follows from (3). Claim (ii) is immediate from the definition of transitivity.

Claim (i) implies that $[F]$ is diagonalizable with eigenvalues $0$ and $m$. By Theorem 5(ii), the eigenvalue $m$ has multiplicity one. Claim (iii) follows. □

5.3. Auxiliary lemmata.

Lemma 12. Let $M$ be a simple transitive $2$-representation of $\mathcal{C}_A$. Then for any $X \in M(\bullet)$ the object $FX$ is projective in $M(\bullet)$.

Proof. Applying $F$ to a minimal projective presentation $P_1 \xrightarrow{\alpha} P_0$ of $FX$ we get a projective presentation $FP_1 \xrightarrow{F(\alpha)} FP_0$ of $F^2X \cong (FX)^\oplus m$.

Consider the split Grothendieck group of the category $W$ of projective objects in $M(\bullet)$. For $i = 0, 1$ let $v_i$ be the vector recording the multiplicities of indecomposable projective objects in $FP_i$. Then, by minimality of the presentation $P_1 \xrightarrow{\alpha} P_0$, we have

\[(4) [F] \cdot v_i = mv_i + w_i \]

for some non-negative vectors $w_i$. Note that $mv_i + w_i$ is a nonzero vector and belongs to the image of $[F]$. Therefore, by Lemma 11(iii), $mv_i + w_i$ is an eigenvector for $[F]$ with eigenvalue $m$. Hence $[F](mv_i + w_i) = m(mv_i + w_i)$. On the other hand,

$[F](mv_i + w_i) = m[F]v_i + [F]w_i \equiv m(mv_i + w_i) + [F]w_i$.

Therefore $[F]w_i = 0$ and since $w_i$ has only non-negative entries and all entries of $[F]$ are positive, we obtain $w_i = 0$.

It follows that $FP_1 \xrightarrow{F(\alpha)} FP_0$ is a minimal projective presentation of $F^2X$, in particular, the morphism $F(\alpha)$ is contained in the radical of $M(\bullet)$.

The category $W$ carries the structure of a $2$-representation of $\mathcal{C}_A$ by restriction. This $2$-representation is equivalent to $M$ (the natural inclusion of $M(\bullet)$ into $W$ is the desired equivalence). In particular, the $2$-representation of $\mathcal{C}_A$ on $W$ is simple transitive. Let $I$ be the ideal of $W$ generated by $F(\alpha)$. This is contained in the radical of $W$ by the above and is $F$-stable by (3). Hence $I$ is $\mathcal{C}_A$-stable as it is stable under all indecomposable non-identity $1$-morphisms. By simple transitivity, we thus get $I = 0$, that is $\alpha = 0$. The claim follows. □

Lemma 13. Let $B$ be a finite dimensional $k$-algebra and $G$ an exact endofunctor of $B$-mod. Assume that $G$ sends each simple object of $B$-mod to a projective object. Then $G$ is a projective functor.

Proof. Consider a short exact sequence of functors $K \hookrightarrow H \twoheadrightarrow G$ where $H$ is a projective functor. This exists because any right exact functor is equivalent to tensoring with some bimodule and is hence a quotient of a projective functor. We assume that $H$ is chosen minimally, that is such that the tops of $H$ and $G$ (viewed as bimodules) agree.

Applying $K \hookrightarrow H \twoheadrightarrow G$ to a short exact sequence $X \hookrightarrow Y \twoheadrightarrow Z$ in $B$-mod we observe that $HX \twoheadrightarrow GX$ and hence the Snake Lemma yields the exact sequence $KX \hookrightarrow KY \twoheadrightarrow KZ$. This implies that $K$ is exact.
Applying $K \hookrightarrow H \twoheadrightarrow G$ to a simple object $L \in \mathcal{B}$-mod we obtain an exact sequence $K L \hookrightarrow H L \twoheadrightarrow G L$. By our choice of $H$, we have $H L = 0$ if and only if $G L = 0$. Furthermore, by assumptions on $G$ we have $H L \cong G L$ whenever $G L \neq 0$. This implies $H L \cong G L$ for all $L$ and hence $K L = 0$. By exactness of $K$ we thus deduce $K = 0$ and hence $H \cong G$.

Lemma 14. Let $A$, $\mathcal{C}_A$ and $F$ be as given in Subsection 5.1. Let further $M$ be a 2-representation of $\mathcal{C}_A$ and $N \in \mathcal{M}(\otimes)$ such that $FN \neq 0$. Then there is an algebra monomorphism from $A$ to $\text{End}_{\mathcal{M}(\otimes)}(FN)$.

Proof. From the definitions we know that the 2-endomorphism algebra of $F$ is isomorphic to $A \otimes_k A^{op}$. We have a natural algebra monomorphism from $A$ to $A \otimes_k A^{op}$ given by $a \mapsto a \otimes 1$. Consider the evaluation homomorphism $\text{Ev}_N : \text{End}_{\mathcal{C}_A}(\otimes)(F) \to \text{End}_{\mathcal{M}(\otimes)}(FN)$.

For a fixed left cell $\mathcal{L}$ consider the corresponding cell 2-representation $\mathcal{C}_\mathcal{L}$ of $\mathcal{C}_A$. By [MM2, Proposition 21], there is a unique maximal left ideal in $\mathcal{C}_A$ which does not contain any identity 2-morphisms for 1-morphisms in $\mathcal{L}$. Now, by [MM2, Subsection 6.5], this left ideal is the annihilator of the sum of all simple objects in $\mathcal{C}_\mathcal{L}$.

From Proposition 9 we know that $\mathcal{C}_\mathcal{L}$ is equivalent to the defining representation which implies that this maximal left ideal is, in fact, $A \otimes \text{rad} A^{op}$. Therefore the kernel of $\text{Ev}_N$, which is a left ideal, must belong to $A \otimes \text{rad} A^{op}$. This implies that the kernel of $\text{Ev}_N$ does not intersect the space $A \otimes 1$ and hence the induced composition $A \to \text{End}_{\mathcal{C}_A}(\otimes)(F) \to \text{End}_{\mathcal{M}(\otimes)}(FN)$ is injective.

5.4. Main result.

Theorem 15. Let $A$ be as given in Subsection 5.1. Then any simple transitive 2-representation of $\mathcal{C}_A$ is equivalent to some cell 2-representation.

Proof. Consider a simple transitive 2-representation $\mathcal{M}$ of $\mathcal{C}_A$ and its abelianization $\bar{\mathcal{M}}$. Let $X_1, X_2, \ldots, X_k$ be a complete and irredundant list of representatives of isomorphism classes of indecomposable objects in $\mathcal{M}(\otimes)$. Denote by $B$ the endomorphism algebra of $\bigoplus_{i=1}^k X_i$. Note that $\mathcal{M}(\otimes)$ is equivalent to $B^{op}$-mod. For $i = 1, 2, \ldots, k$ we let $L_i$ denote the simple quotient in $\mathcal{M}(\otimes)$ of the indecomposable projective object $0 \to X_i$.

Recall the 1-morphism $F$ defined in Subsection 5.1 and the corresponding matrix $[F]$ describing the action of $F$ on the Grothendieck group of $\mathcal{M}(\otimes)$ in the basis of simple modules. By Theorem 14, there is a column,

$$
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_k
\end{pmatrix}
$$

in $[F]$, say with index $j$, such that $v_1 + v_2 + \cdots + v_k \leq m$. By Lemma 12 we have $FL_j \cong \bigoplus_{i=1}^k X_i^{\oplus l_i}$
for some non-negative integers \(l_1, l_2, \ldots, l_k\). Transitivity of \(M\) and \(B\) imply that all \(l_1, l_2, \ldots, l_k\) are, in fact, positive integers. Denote by \(B'\) the endomorphism algebra of \(FL_j\) which is Morita equivalent to \(B\) by the previous sentence. The vector \((l_1, l_2, \ldots, l_k)\) is, by \(B\), an eigenvector of \([F]\). Moreover, by Lemma \([13]\) we have \([F] = [F'] = [F'']\) where the latter equality follows from self-adjointness of \(F\).

Lemma \([14]\) provides an algebra embedding of \(A\) into \(B'\) and hence an embedding \(A_A \hookrightarrow B'_A\) of \(A\)-modules. Since the algebra \(A\) (and hence also \(A^{op}\)) is self-injective, each indecomposable summand of \(A_A\) has simple socle. Therefore the embedding \(A_A \hookrightarrow B'_A\) induces an embedding (of right \(A\)-modules) from \(A_A\) into

\[
\bigoplus_{j=1}^{k} \Hom_{\mathcal{M}(\mathfrak{A})}(X_i, FL_j).
\]

The dimension of the latter equals \(v_1 + v_2 + \cdots + v_k \leq m\), while \(\dim_k A = m\), therefore \(v_1 + v_2 + \cdots + v_k = m\) and by Corollary \([2]\) all columns of \([F]\) are equal. In particular, it follows that \(l_1 = l_2 = \cdots = l_k = l\) for some \(l \in \mathbb{N}\) and thus \(B'\) is isomorphic to the algebra of \(l \times l\) matrices with coefficients in \(B\).

The algebra of \(B'\)-endomorphisms of \((5)\) is isomorphic to \(B\) and embeds into the algebra of \(A\)-endomorphisms of \((5)\) (the latter embedding is due to the fact that \(A\) is a subalgebra of \(B'\)) which is equal to \(A\) by comparing dimensions. Therefore we have

\[
B \hookrightarrow A \hookrightarrow B'.
\]

Next we argue that \(FL_s = (X_1 \oplus X_2 \oplus \cdots \oplus X_k)^{\oplus l}\) for any \(s\). The arguments above imply that \(FL_s = (X_1 \oplus X_2 \oplus \cdots \oplus X_k)^{\oplus l_s}\) for some positive integer \(l_s\). Now \(l = l_s\) since all columns of \([F]\) are equal.

As \(FL_j = (X_1 \oplus X_2 \oplus \cdots \oplus X_k)^{\oplus l}\), it follows that \(\dim_k B' = lm\) and therefore \(\dim_k B = \frac{m^2}{l}\). Set \(\Theta := \mathcal{M}(F)\). Lemma \([13]\) implies that \(\Theta\) is a projective functor which sends each simple to \((X_1 \oplus X_2 \oplus \cdots \oplus X_k)^{\oplus l}\). The dimension of the endomorphism algebra of \(\Theta\) thus equals \(l \cdot \frac{m^2}{l} = m^2\). Note that \(\mathcal{J}\)-simplicity of \(E_A\) gives us a natural inclusion of the algebra \(\End_{\mathcal{M}(\mathfrak{A})}(\Theta) \cong A \otimes A^{op}\) of 2-endomorphisms of \(F\) into the endomorphism algebra of \(\Theta\) in the category of right exact endofunctors of \(\mathcal{M}(\mathfrak{A})\). As both these algebras have dimension \(m^2\), this natural inclusion is, in fact, an isomorphism.

Therefore \(B \cong A \cong B'\) and thus \(\mathcal{M}\) is equivalent to the defining 2-representation of \(E_A\). Now the proof is completed by applying Proposition \([9]\).

5.5. **Generalizations.**

**Remark 16.** Theorem \([15]\) generalizes verbatim and with the same proof to the case where \(A\) is a basic self-injective finite dimensional \(k\)-algebra (not necessarily connected). The technical difficulty in this case is that, in order to be consistent with the requirement for \(\mathfrak{A}\) to be indecomposable, one has to consider a 2-category with several objects indexed by connected components of \(A\).

**Remark 17.** Theorem \([17]\) generalizes verbatim to 2-subcategories of \(E_A\) described in \([MM3\] Subsection 4.5]. These 2-subcategories exhaust all “simple” 2-categories of certain type, see \([MM3\] Theorem 13] and Subsection \([16] below for details. The only difference between those 2-subcategories and \(E_A\) is that the former may contain fewer 2-endomorphisms of the identity 1-morphisms. We did not use 2-endomorphisms of identity 1-morphisms in the above proof.
6. Transitive 2-representations for some general fiat 2-categories

6.1. Strong regularity and a numerical condition. Let \(\mathcal{C}\) be a fiat 2-category and \(\mathcal{J}\) a two-sided cell in \(\mathcal{C}\). We say that \(\mathcal{J}\) is strongly regular, see [MM1 Subsection 4.8], provided that

- different right (left) cells in \(\mathcal{J}\) are not comparable with respect to the right (left) preorder;
- the intersection of a left and a right cell in \(\mathcal{J}\) consists of exactly one isomorphism class of indecomposable 1-morphisms.

For example, the 2-category \(\mathcal{C}_A\) from Subsection 5.1 is strongly regular.

If \(\mathcal{J}\) is strongly regular, we have a well-defined function sending \(F \in \mathcal{J}\) to the number of indecomposable summands in \(F^* \circ F\) which belong to \(\mathcal{J}\). We will say that \(\mathcal{J}\) satisfies the numerical condition provided that this function is constant on right cells. Again, it is easy to check that the 2-category \(\mathcal{C}_A\) from Subsection 5.1 satisfies the numerical condition, see [MM1 Subsection 7.3].

Another example of a 2-category in which each two-sided cell is strongly regular and satisfies the numerical condition is the 2-category \(\mathcal{S}_n\) of Soergel bimodules for the symmetric group \(S_n\), see [MM1 Subsection 7.1] and [MM2 Example 3] for details.

6.2. Another generalization of the main result.

**Theorem 18.** Let \(\mathcal{C}\) be a fiat 2-category such that all two-sided cells in \(\mathcal{C}\) are strongly regular and satisfy the numerical condition. Then any simple transitive 2-representation of \(\mathcal{C}\) is equivalent to a cell 2-representation.

**Proof.** Let \(M\) be a simple transitive 2-representation of \(\mathcal{C}\). First of all, we claim that there is a unique maximal two-sided cell \(\mathcal{J}\) which does not annihilate \(M\). Indeed, assume that we have two maximal two-sided cells \(\mathcal{J}_i\) for \(i = 1, 2\) with this property. Then for any \(F_i \in \mathcal{J}_i\), \(i = 1, 2\), we have \(M(F_1) \circ M(F_2) = 0\) and \(M(F_2) \circ M(F_1) = 0\) whenever the expression makes sense. Therefore the additive closure of objects in all \(M(\mathfrak{a})\) which may be obtained by applying 1-morphisms from \(\mathcal{J}_i\) is, on the one hand, a 2-subrepresentation of \(M\) (by maximality of \(\mathcal{J}_i\)) and, on the other hand, annihilated by all 1-morphisms from \(\mathcal{J}_2\). Due to transitivity of \(M\), we obtain that \(\mathcal{J}_2\) annihilates \(M\), a contradiction.

Now denote by \(\mathcal{J}\) the maximal two-sided cell of \(\mathcal{C}\) which does not annihilate \(M\). Without loss of generality we may assume that \(\mathcal{J}\) is the unique maximal two-sided cell in \(\mathcal{C}\) and that \(M\) is 2-faithful in the sense that it does not annihilate any 2-morphisms. Indeed, we may replace \(\mathcal{C}\) by its quotient modulo the kernel of \(M\) which does not change the structure of the surviving cells.

Denote by \(\mathcal{C}_\mathcal{J}\) the 2-full 2-subcategory of \(\mathcal{C}\) formed by all 1-morphisms in \(\mathcal{J}\) together with their respective identity 1-morphisms. By restriction, \(M\) becomes a 2-representation \(M_\mathcal{J}\) of \(\mathcal{C}_\mathcal{J}\). As the additive closure of 1-morphisms in \(\mathcal{J}\) is stable with respect to left multiplication by 1-morphisms in \(\mathcal{C}\), it follows that \(M\) is a transitive 2-representation of \(\mathcal{C}_\mathcal{J}\).
We claim that $M_J$ is simple transitive. Indeed, assume that $J$ is an ideal of $M$ stable with respect to the action of $C$. Assume that it is nonzero and take any nonzero morphism $\alpha$ in it. As $M$ is a simple transitive 2-representation of $C$, there exists a 1-morphism $G$ in $C$ such that $G(\alpha)$ has an invertible nonzero direct summand. Applying 1-morphisms from $C_J$ we, on the one hand, will map such an invertible direct summand to another invertible morphism (and since $M$ is transitive there is a 1-morphism $F$ in $C_J$ which does not annihilate this invertible direct summand). On the other hand, $F \circ G$ is in $J$ and hence application of it to $\alpha$ cannot produce any invertible direct summands, a contradiction. Therefore $J$ is zero.

By Theorem 15, Remark 17 and [MM3, Theorem 13], $M_J$ is equivalent to a cell 2-representation $C_J$ of $C_J$ where $L$ is a left cell in $J$. By [MM1, Theorem 43] any choice of $L$ yields an equivalent 2-representation. Set $i = i_L$ and let $L$ be a simple object in $C_J(i)$ which is not annihilated by 1-morphisms in $L$. Then we can consider $L$ as an object in $M(i)$.

By Theorem 15, Remark 17 and [MM3, Theorem 13], $M_J$ is equivalent to a cell 2-representation $C_J$ of $C_J$ where $L$ is a left cell in $J$. By [MM1, Theorem 43] any choice of $L$ yields an equivalent 2-representation. Set $i = i_L$ and let $L$ be a simple object in $C_J(i)$ which is not annihilated by 1-morphisms in $L$. Then we can consider $L$ as an object in $M(i)$.

Sending $P_i$ to $L$ gives a 2-natural transformation $\Phi$ from the 2-representation $P_i$ of $C$ to $M$. In the notation of Subsection 3.3, the image of $N(j)$ for $j \in C$ under $\Phi$ is inside the category of projective objects in $M(j)$ and contains at least one representative in each isomorphism class of indecomposable objects, see [MM1, Subsection 4.5]. We also have that $I$ (see Subsection 3.3) annihilates $L$ by construction. It follows that the 2-representation $K$ of $C$ on projective objects in the categories $M(j)$ (for $j \in C$) is equivalent to the cell 2-representation $C_L$ of $C$.

As $K$ is equivalent to $M$ by [MM2, Theorem 11], we deduce that $M$ is equivalent to $C_L$. This completes the proof.

7. Examples

7.1. A non weakly fiat 2-category $\mathcal{C}_A$. In this subsection we give an example of a non weakly fiat 2-category $\mathcal{C}_A$ for which Theorem 15 generalizes to the class of exact simple transitive 2-representations. Taking into account the example considered in Subsection 3.4, the present example is somewhat surprising.

For $A = \mathbb{k}[x,y]/(x^2, y^2, xy)$ consider the 2-category $\mathcal{C}_A$ as defined in Subsection 3.1. Note that $A$ is local but not self-injective which implies that $\mathcal{C}_A$ is not weakly fiat. Let $F$ be an indecomposable 1-morphism in $\mathcal{C}_A$ which is not isomorphic to the identity 1-morphism. The defining 2-representation of $\mathcal{C}_A$ is easily seen to be equivalent to the cell 2-representation $C_L$ for $L = \{F\}$.

Proposition 19. For $A = \mathbb{k}[x,y]/(x^2, y^2, xy)$, any exact simple transitive 2-representation of $\mathcal{C}_A$ is equivalent to a cell 2-representation.

Proof. Let $M$ be an exact simple transitive 2-representation of $\mathcal{C}_A$. Without loss of generality we may assume $M(F) \neq 0$. Then $F \circ F \cong F^{S3}$ and hence $[F]^2 = 3[F]$ by exactness of $M(F)$. Using Theorem 11 it is easy to check that $[F]$ is equal to one of the following matrices:

$$M_1 := \begin{pmatrix} 3 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_3 := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad M_4 := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$
Let $B$ and $B'$ be as in the proof of Theorem 15. Note that both Lemma [12] and Lemma [13] are still applicable in our situation. Despite the fact that $\mathcal{C}_A$ is not weakly flat, it is still $J$-simple, where $J = \{F\}$.

If $[F] = M_4$, then $B \cong k^{\oplus 3}$ and $M(F)$ is the direct sum of nine copies of the identity functors (between the three different copies of $k$-mod). The endomorphism algebra of $M(F)$ has thus dimension nine and is clearly not isomorphic to $A \otimes_k A^{op}$. Hence this case is not possible.

If $[F] = M_3$, then $B = B' \cong k^{\oplus 2}$ and the algebra $A$ does not inject into $B'$. This contradicts Lemma [14] and hence this case is not possible either.

If $[F] = M_2$, then either $B = B'$ is a 3-dimensional algebra which is not local or $B \cong k^{\oplus 2}$ and $B' \cong k \oplus \text{Mat}_{2 \times 2}(k)$. In the first case we again get a contradiction to Lemma [14]. In the second case the endomorphism algebra of $M(F)$ has dimension ten and two direct summands isomorphic to $k$, say this endomorphism algebra is $Q \oplus k \oplus k$. If the local algebra $A \otimes_k A^{op}$ were to inject into the endomorphism algebra of $M(F)$, the algebra $A \otimes_k A^{op}$ would also inject into $Q$ which has strictly smaller dimension, a contradiction. Hence this case is not possible.

If $[F] = M_1$, then either $B \cong k$ and $B' = \text{Mat}_{3 \times 3}(k)$ or $B = B'$ has dimension 3. In the former case the endomorphism algebra of $M(F)$ has dimension nine and is not local, implying a contradiction similarly to the case $[F] = M_4$. In the latter case we again use Lemma [14] to get $B = B' \cong A$ and then we readily deduce that $M$ is equivalent to the cell 2-representation. \qed

7.2. Categorification of finite dimensional 2-Lie algebras. Let $\mathfrak{g}$ denote a simple finite dimensional complex Lie algebra. We fix a triangular decomposition $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. For any $\mathfrak{h}$-weight $\lambda$ denote by $L(\lambda)$ the corresponding simple highest weight module with highest weight $\lambda$. Let $\leq$ denote the natural partial order on $\mathfrak{h}$-weights.

Let $\mathcal{W}$ be the 2-category categorifying the idempotent version $\hat{U}$ of the universal enveloping algebra of $\mathfrak{g}$ as defined in [We] Definition 2.4 (the origins of this 2-category are in [CL], see also [KL] [Ro1] for other variations). The categorification statement is justified by [We] Theorem B.2. For each dominant integral $\mathfrak{h}$-weight $\lambda$ there is a 2-representation of $\mathcal{W}$ given by a functorial action on the direct sum (over $n$) of categories of projective modules over the cyclotomic quiver Hecke algebras (KLR algebras) $R_n^\lambda$ associated with $\mathfrak{g}$ (see [We] Theorem 3.17) for $\mathcal{W}$ and also [KK] [Ka] for a similar statement related to Rouquier’s 2-Kac-Moody algebras). This 2-representation categorifies $L(\lambda)$. We note the following properties of this 2-representation:

- As $L(\lambda)$ is finite dimensional, only finitely many of the algebras $R_n^\lambda$ are non-zero.
- As $L(\lambda)$ is finite dimensional, sufficiently high powers of the generators annihilate our 2-representation. Hence, the commutation relations in $\mathfrak{g}$ imply that only finitely many indecomposable 1-morphisms from $\mathcal{W}$ act as non-zero functors in this 2-representation.
- Each $R_n^\lambda$ is finite dimensional and all involved functors are exact.
• Each 1-morphism in \( \mathcal{U} \) acts as an exact functor and hence can be realized as tensoring with a finite-dimensional bimodule. This implies that the spaces of two morphisms in this 2-representation are finite dimensional.

• Each 1-morphism in \( \mathcal{U} \) has a biadjoint which is again a functor representing the action of some 1-morphism in \( \mathcal{U} \).

• The endomorphism algebra of each identity 1-morphism in \( \mathcal{U} \) is positively graded by the non-degeneracy part of [We, Theorem B.2] and isomorphic to a polynomial ring ([We, Proposition 3.31]). In particular, each finite dimensional graded quotient of this algebra is local.

Let \( I_\lambda \) be the kernel of this 2-representation and set \( \mathcal{U}_\lambda := \mathcal{U} / I_\lambda \). Then the above implies that \( \mathcal{U}_\lambda \) is a flat 2-category. Note that \( I_\lambda \) is, in general, not generated by 2-morphisms of the form \( \text{id}_F \), where \( F \) is some 1-morphism, but it additionally contains some of the 2-morphisms between 1-morphisms which are not in \( I_\lambda \), see [MM2, Remark 31].

Consider a finite set \( \lambda := \{ \lambda_1, \lambda_2, \ldots, \lambda_k \} \) of dominant integral h-weights such that \( \lambda_i \not\leq \lambda_j \) for all \( i \neq j \) and denote by \( \mathcal{X} \) the set of all dominant integral weights \( \mu \) such that \( \mu \leq \lambda_i \) for some \( i \). Note that \( \mathcal{X} \) is a finite set. Define

\[
\mathcal{U}_\lambda := \mathcal{U} / (I_{\lambda_1} \cap I_{\lambda_2} \cap \cdots \cap I_{\lambda_k}),
\]

which is again a flat 2-category.

**Remark 20.** Let \( L \) be the left cell in \( \mathcal{U}_\lambda \) containing the indecomposable 1-morphism \( \mathbb{1}_\lambda \) for \( l \in \{1, 2, \ldots, k\} \). As \( \mathbb{1}_\lambda \) is a genuine idempotent and is, obviously, the unique element in the intersection of its left and right cells, the radical of its endomorphism ring is contained in the ideal \( I \) from Subsection 3.3 used to define the corresponding cell 2-representation \( \mathcal{C}_L \). Consequently, the image of \( \mathbb{1}_\lambda \) in the abelianized cell 2-representation is both simple and projective (this corresponds to a projective module over \( R^0_\lambda \cong \mathbb{C} \)). Moreover, the functor \( \mathcal{C}_L(\mathbb{1}_\lambda) \) is just the identity functor on the category of complex vector spaces, in particular, its endomorphism ring consists only of scalars. Note that our construction of \( \mathcal{C}_L \) differs, in particular, from the construction of the universal categorification of \( L(\lambda) \) in [Ro1, Subsection 5.1.2]. In the latter case the endomorphism of \( \mathbb{1}_\lambda \) is much bigger in general.

**Theorem 21.** For any \( \lambda \) as above every two-sided cell in the 2-category \( \mathcal{U}_\lambda \) is strongly regular and satisfies the numerical condition.

**Proof.** For \( l \in \{1, 2, \ldots, k\} \) consider the two-sided cell \( \mathcal{J} \) of \( \mathcal{U}_\lambda \) containing \( \mathbb{1}_\lambda \). Then, factoring out the maximal 2-ideal containing \( \mathbb{1}_\lambda \) which contains \( \text{id}_F \), and does not contain the identity 2-morphism for any 1-morphism outside \( \mathcal{J} \) (note that such an ideal does not have to be generated by 2-morphisms of the form \( \text{id}_F \), where \( F \) is some 1-morphism), we obtain the 2-category \( \mathcal{U}_\mu \) where \( \mu \) is uniquely defined via \( \mu := \mathcal{X} \setminus \{\lambda_l\} \), cf. [DG] Section 9]. Therefore it is enough to prove that \( \mathcal{J} \) is strongly regular and satisfies the numerical condition.

Let \( L \) denote the left cell of \( \mathbb{1}_\lambda \). Let further \( L \) be an indecomposable object in \( R^0_\lambda \)-proj. Note that \( R^0_\lambda \cong \mathbb{C} \). As \( L \) corresponds to the highest weight vector in \( L(\lambda) \), all 1-morphisms which do not annihilate \( L \) must correspond to the \( U(\mathfrak{n}) \) part of \( \mathfrak{g} \). This means that \( L \) consists of direct summands of powers of the negative generators of \( \mathcal{U} \). Then, from [We, Theorem 3.17] in combination with [Ro1, Theorem 5.7]
and [VV] Theorem 4.4], it follows that mapping an indecomposable 1-morphism \( F \in \mathcal{L} \) to \( F L \) induces a bijection between \( \mathcal{L} \) and the set of isomorphism classes of indecomposable objects in \( \bigoplus_{n \geq 0} R_n^\lambda \)-\text{proj}.

Set
\[
A := \bigoplus_{n \geq 0} R_n^\lambda \quad \text{and} \quad B := \bigoplus_{n \geq 1} R_n^\lambda.
\]

For any \( M \in B\)-\text{proj} we have \( \mathbb{1}_\lambda, M = 0 \) and therefore \( F M = 0 \) for any \( F \in \mathcal{L} \). Consider the abelian 2-representation \( \mathbb{C}_\lambda \).

Since \( \mathcal{K}_\lambda \) is flat, Lemma [13] implies that \( \mathbb{C}_\lambda(F) \) is an indecomposable projective functor from \( \mathcal{C}\text{-mod} \) to \( \mathcal{A}\text{-mod} \). Consequently, for any \( G \in \mathcal{L} \) the functor \( \mathbb{C}_\lambda(F \circ G^*) \) is indecomposable. We claim that this implies that \( F \circ G^* \) is indecomposable. Indeed, if \( F \circ G^* \cong X \circ Y \), then without loss of generality we may assume \( \mathbb{C}_\lambda(Y) = 0 \). Since \( \mathcal{J} \) is a maximal two-sided cell, we have \( Y \in \mathcal{J} \) and hence \( \mathbb{C}_\lambda(Y) \neq 0 \), a contradiction.

The previous paragraph shows that the set \( \{F \circ G^*\} \), where \( F, G \in \mathcal{L} \), consists of indecomposable 1-morphisms and hence coincides with \( \mathcal{J} \). In particular \( |\mathcal{J}| = \mathcal{L}^2 \).

It is now obvious that the left cells in \( \mathcal{J} \) are obtained fixing \( G \) and the right cells in \( \mathcal{J} \) are obtained fixing \( F \). Therefore \( \mathcal{J} \) is strongly regular. To check the numerical condition we note that \( \mathbb{C}_\lambda \) realizes elements of \( \mathcal{J} \) as tensoring with indecomposable projective \( \mathcal{A}\text{-}\mathcal{A} \)-bimodules, so the numerical condition follows from [MM1, Subsection 7.3]. \( \square \)

7.3. Soergel bimodules in type \( B_2 \). Consider the 2-category \( \mathcal{S} \) of Soergel bimodules for a Lie algebra of type \( B_2 \), see [MM1 Section 7.1] and [MM2 Example 20]. We denote by \( \mathbf{a} \) the (unique) object in \( \mathcal{S} \). The Weyl group in this case is given by

\[
W = \{e, s, t, st, ts, sts, tst, stst = tstst\},
\]

where \( s^2 = t^2 = e \), and is isomorphic to the dihedral group \( D_4 \). The group \( D_4 \) has five simple modules over \( \mathbb{C} \): the one-dimensional simple modules \( V_{\varepsilon, \delta} \), for \( \varepsilon, \delta \in \{\pm 1\} \), where \( s \) acts via \( \varepsilon \) and \( t \) acts via \( \delta \); and the 2-dimensional simple module \( V_2 \) (the defining geometric representation). For an additive category \( \mathcal{A} \) we denote by \( K_0(\mathcal{A}) \) the split Grothendieck group of \( \mathcal{A} \). Our aim in this section is to apply previous results in order to prove the following statement which describes simple \( W \)-modules admitting a finitary categorification.

**Proposition 22.** Let \( \mathbb{M} \) be a finitary 2-representation of \( \mathcal{S} \). Assume that the induced action of the algebra \( \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{S}(\mathbf{a}, \mathbf{a})) \) on the vector space \( \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathbb{M}(\mathbf{a})) \) gives a simple \( W \)-module \( V \). Then \( V \cong V_{1,1} \) or \( V \cong V_{-1,-1} \).

**Proof.** We have three two-sided cells
\[
\mathcal{J}_1 = \mathcal{L}_1 = \{e\}, \quad \mathcal{J}_2 = \{s, t, st, ts, sts, tst\}, \quad \mathcal{J}_3 = \mathcal{L}_3 = \{sts\}
\]

and \( \mathcal{J}_2 \) splits into two left cells
\[
\mathcal{L}^{(1)}_2 = \{s, st, sts\} \quad \text{and} \quad \mathcal{L}^{(2)}_2 = \{t, ts, tst\}.
\]

Right cells are obtained using the map \( w \mapsto w^{-1} \).

It is easy to check that the cell 2-representations \( \mathbb{C}_{\mathcal{L}_1} \) and \( \mathbb{C}_{\mathcal{L}_3} \) categorify \( V_{1,1} \) and \( V_{-1,-1} \), respectively.
We identify indecomposable Soergel bimodules \( \theta_w \) for \( w \in W \) with the corresponding elements
\[
\begin{align*}
\theta_c &= e, & \theta_s &= e + s, & \theta_t &= e + t, & \theta_{st} &= e + t + s + st, & \theta_{ts} &= e + t + s + ts, \\
\theta_{sts} &= e + t + s + ts + st + st + s, & \theta_{tst} &= e + t + s + ts + st + t, & \theta_{stst} &= e + t + s + ts + st + st + st + st + st
\end{align*}
\]
in the Kazhdan-Lusztig basis for \( Z[W] \).

Note that the element \( \theta_s \) annihilates \( V_{-1,1} \) while \( \theta_t \) does not annihilate \( V_{-1,1} \). If we had a 2-representation \( M \) decategorifying to \( V_{-1,1} \), then \( M(\theta_s) = 0 \) while \( M(\theta_t) \neq 0 \) which is impossible as \( \theta_s \) and \( \theta_t \) belong to the same two-sided cell. Therefore \( V \not\cong V_{-1,1} \) and, by symmetry, \( V \not\cong V_{1,-1} \). (This argument came up in discussion with Catharina Stroppel.)

It is left to show that \( V \not\cong V_2 \). Note that \( \theta_{stst} \) annihilates \( V_2 \). Assume that \( M \) is a 2-representation of \( \mathcal{I} \) decategorifying to \( V_2 \) and consider \( \overline{M} \). Set \( \Theta := \sum_{w \in J_2} \theta_w \).

Direct computation shows that
\[
(\theta_{st} + \theta_{ts})^2 = 2\Theta \mod J_3, \quad \Theta^2 = 10\Theta + 4(\theta_{st} + \theta_{ts}) \mod J_3.
\]
This implies that the matrix \( X := [\theta_{st} + \theta_{ts}] \) satisfies the polynomial equation \( X^4 - 20X^2 - 16X = 0 \). Consequently, \( X \) is diagonalizable with eigenvalues in \( \{0, -4, 2(1 \pm \sqrt{2})\} \). Clearly, \( X \) is not the zero matrix. As all entries of \( X \) are non-negative, the trace of \( X \) is non-negative which implies that the eigenvalues of \( X \) are \( 2(1 \pm \sqrt{2}) \), each with multiplicity one. Thus the trace of \( X \) is 4 and the determinant is \(-4\), leaving
\[
\begin{align*}
\begin{pmatrix} 4 & 4 \\ 1 & 0 \end{pmatrix}, & \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, & \begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}, & \begin{pmatrix} 3 & 7 \\ 1 & 1 \end{pmatrix}, & \begin{pmatrix} 3 & 1 \\ 7 & 1 \end{pmatrix}, \\
\begin{pmatrix} 2 & 8 \\ 1 & 2 \end{pmatrix}, & \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix}, & \begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix}, & \begin{pmatrix} 2 & 1 \\ 8 & 2 \end{pmatrix}
\end{align*}
\]
as possibilities (up to reordering of the basis).

We have \( \theta_s^2 \cong 2\theta_s \) and \( \theta_t^2 \cong 2\theta_t \), which implies that both \( [\theta_s] \) and \( [\theta_t] \) satisfy the polynomial equation \( x^2 - 2x = 0 \). Similarly to the above, this leads to the list of candidates for \( [\theta_s] \) and \( [\theta_t] \) being given by
\[
\begin{align*}
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \begin{pmatrix} 2 & a \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 2 & 0 \\ a & 0 \end{pmatrix}, & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}
\end{align*}
\]
where \( a \in \{0, 1, 2, \ldots \} \). Note that \( \theta_{st} = \theta_s\theta_t \) and \( \theta_{ts} = \theta_t\theta_s \). Hence, the equation
\[
[\theta_{st} + \theta_{ts}] = [\theta_s][\theta_t] + [\theta_t][\theta_s]
\]
reduces the choice to
\[
(6) \quad [\theta_s] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad [\theta_t] = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}
\]
or vice versa.

Now we may restrict \( M \) to the 2-subcategory \( \mathcal{I} \) of \( \mathcal{I} \) generated by \( \theta_t \) and \( \theta_s \) and adjunction morphisms between them. This 2-category clearly satisfies all hypotheses of Theorem 18. Note that \( \theta_s \) is self-adjoint, hence Lemma 18 implies that this restricted 2-representation is transitive. Let \( N \) be its simple transitive quotient. Then \( N \) gives rise to a simple transitive 2-representation of \( \mathcal{I} \) in which \( \theta_s \) has the matrix described by (6). This, however, contradicts Theorem 18. The obtained contradiction completes the proof. \( \Box \)
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