SUPERDIFFUSION OF COSMIC RAYS: IMPLICATIONS FOR COSMIC RAY ACCELERATION

A. Lazarian\textsuperscript{1} and Huirong Yan\textsuperscript{2}

\textsuperscript{1}Department of Astronomy, University of Wisconsin, 475 North Charter Street, Madison, WI 53706, USA
\textsuperscript{2}KIAA, Peking University, 5 Yi He Yuan Rd, Beijing, 100871, China

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ABSTRACT

Diffusion of cosmic rays (CRs) is the key process for understanding their propagation and acceleration. We employ the description of spatial separation of magnetic field lines in magnetohydrodynamic turbulence in Lazarian & Vishniac to quantify the divergence of the magnetic field on scales less than the injection scale of turbulence and show that this divergence induces superdiffusion of CR in the direction perpendicular to the mean magnetic field. The perpendicular displacement squared increases, not as the distance \(x\) along the magnetic field, which is the case for a regular diffusion, but as the \(x^{3/2}\) for freely streaming CRs. The dependence changes to \(x^{3/2}\) for the CRs propagating diffusively along the magnetic field. In the latter case, we show that it is important to distinguish the perpendicular displacement with respect to the mean field and to the local magnetic field. We consider how superdiffusion changes the acceleration of CRs in shocks and show how it decreases efficiency of the CRs acceleration in perpendicular shocks. We also demonstrate that in the case when the small-scale magnetic field is generated in the pre-shock region, an efficient acceleration can take place for the CRs streaming without collisions along the magnetic loops.

\textbf{Key words:} acceleration of particles – cosmic rays – galaxies: magnetic fields – magnetohydrodynamics (MHD) – turbulence

\textit{Online-only material:} color figures

1. INTRODUCTION

Cosmic ray (CR) acceleration and diffusion are long standing problems that have often been described in the literature (see monographs by Ginzburg 1974; Schlickeiser 2002 and references therein). Turbulence plays a key role for both processes and, as a result, advances in understanding magnetohydrodynamic (MHD) turbulence advance both fields.

The past few decades have been marked by substantial progress in understanding MHD turbulence and this has already shifted some of the CR paradigms. For instance, the discovery of the scale-dependent anisotropy of Alfvénic turbulence (Goldreich & Sridhar 1995, henceforth GS95), isotropy of fast modes (Cho & Lazarian 2002, 2003), as well as the quantitative studies of mode coupling (GS95; Cho & Lazarian 2002) resulted in the identification of fast modes as the major scattering agent in the typical interstellar medium (ISM) conditions (Yan & Lazarian 2002, 2004).

In this paper, we explore the consequences of another advance of MHD turbulence theory, namely, the understanding of the magnetic line separation in Alfvénic turbulence (Lazarian & Vishniac 1999, henceforth LV99), which presents a close analog of the separation of particles in turbulent media due to the well known process of Richardson diffusion. This type of explosive growth in the separation between particles, i.e., as (time)\(3/2\), on scales less than the turbulence injection scale \(L\) was inferred from fluids experiments many decades ago (see Richardson 1926). The turbulent wandering of magnetic fields was quantified in LV99 which showed that the separation of magnetic field lines increases as (distance)\(3/2\), where distance is measured along the magnetic field lines. The intimate connection between this and the Richardson diffusion was revealed in Eyink et al. (2011, henceforth ELV11). The LV99 expression for the field wandering was also numerically confirmed (see Lazarian et al. 2004; Maron et al. 2004) and was employed for the description of magnetized plasma thermal conduction (Narayan & Medvedev 2001; Lazarian 2006) and CR propagation (Yan & Lazarian 2008, henceforth YL08). A recent numerical confirmation of the Richardson diffusion for magnetized fluids can be found in Eyink et al. (2013).\textsuperscript{3}

In what follows, we will refer to the magnetic field divergence on scales less than the injection scale \(L\) as the \textit{spatial Richardson diffusion} or, when it does not cause confusion, simply as the \textit{Richardson diffusion}. The intrinsic relation between the magnetic field spatial and time super-diffusion is demonstrated and discussed in detail in ELV11 and this justifies the use of the same term for the two closely related processes.

The spatial Richardson diffusion substantially changes the perpendicular diffusion for CRs both streaming and diffusing along magnetic field lines and this entails important consequences for the CR transport and acceleration. The effect of the Richardson diffusion is important as the difference in the acceleration efficiency of parallel and perpendicular shocks has been the subject of intensive discussions in the literature; e.g., see Jokipii (1987). In the paper, we show that the superdiffusion of CRs arising from the Richardson diffusion of magnetic field lines can substantially modify the arguments.

The notion of superdiffusive behavior can be traced back to the paper by Jokipii (1973) as well as that by Skilling et al. (1974). These papers reported the fast deviations of magnetic field lines in the context of the problem of CR diffusion. The quantitative results in the aforementioned papers are, as we discuss further in this paper, inconsistent with the prediction of Richardson diffusion. More recently, superdiffusion was reported in the analysis of solar wind data (see Perri & Zimbardo 2009) where the phenomenon was attributed to the ballistic behavior of CRs or Levi flights. In contrast, in our study we do not appeal to the hypothetical Levi flights, the nature of which is not clear for CRs.

\textsuperscript{3} We may note parenthetically that the Richardson diffusion in magnetized fluid also demonstrates the violation of flux freezing in turbulent fluids of arbitrary conductivity. This is related to the theory of fast turbulent reconnection presented in LV99 to which the numerical study in Eyink et al. (2013) provides an additional testing (see also Kowal et al. 2009, 2012b).
This paper is a continuation of our exploration of the perpendicular diffusion of CRs. In our earlier paper, namely Yan & Lazarian (2008, henceforth YL08), we mostly dealt with CR diffusion on scales larger than the injection scale \(L\). That paper also included a short discussion of particle transport on scales less than \(L\) for which we considered scale-dependent diffusion coefficients, which, as we argue here, reflect superdiffusive behavior. The predictions in YL08 have been confirmed by numerical simulations that employed results of three-dimensional MHD simulations (Xu & Yan 2013). We discuss the results of the latter testing in view of our theoretical study within this work.

The transport of CRs at scales less than \(L\) that we deal with in this paper is an important regime for many astrophysical applications. The injection scale \(L\) can be quite large, for instance, for ISMs \(L \sim 100\) pc (see Elmegreen & Scalo 2004; Chepurnov et al. 2010), it is around 50 pc for M51 (Fletcher et al. 2011), and about 20 pc in the fan region in the outskirts of the Galaxy (Iacobelli et al. 2013). It is generally accepted that the acceleration processes in shocks happen on scales comparable to or smaller than that. While we show that superdiffusive behavior can alter some of the popular ideas about CR acceleration in shocks, we also discuss a process that avoids the limitations of superdiffusive behavior. This process is related to the acceleration of CRs streaming along small-scale magnetic fields generated in the pre-shock and post-shock regions. Within this process the effective mean free path of the particles is determined by the entangled magnetic field structure (see Lazarian 2006) rather than the scattering of particles at magnetic field perturbations.

In what follows, we discuss in Section 2 some basic properties of MHD turbulence which determines particle transport, the Richardson diffusion of magnetic fields in Section 3, and briefly discuss the process of subdiffusion in Section 4. The modifications of the shock acceleration in the presence of superdiffusion are presented in Section 5. Section 6 provides a quantitative study of the effects of superdiffusion for a few idealized astrophysically motivated settings. In Section 7 we discuss a process of CR acceleration while they stream along the small-scale magnetic field generated in the pre-shock and post-shock regions. We show that a very fast and efficient acceleration is possible. The discussion of our findings and the summary are provided in Sections 8 and 9, respectively.

2. MHD TURBULENCE AS THE KEY FACTOR
FOR COSMIC RAY PROPAGATION

It is generally accepted that CRs follow magnetic field lines and are scattered by magnetic perturbations. The statistics of magnetic field lines and the nature of perturbations are determined by magnetic turbulence. Therefore it is essential to use a model of turbulence that has solid theoretical foundations and agrees with the results of numerical simulations for describing CR propagation and acceleration.\(^4\)

For decades, interaction of CRs with Alfvénic perturbations was the textbook picture for describing the propagation of CRs. While this changed as fast modes were identified as the major agent for resonance scattering (Yan & Lazarian 2002, 2004), in this work we confirm their major role in diffusion perpendicular to the mean field.\(^5\)

The possibility of discussing Alfvén modes separately is based on both numerical and theoretical arguments. The numerical study by Cho & Lazarian (2002) demonstrated that in compressible MHD turbulence the Alfvénic modes develop an independent cascade, which is marginally affected by the fluid compressibility. This observation corresponds to theoretical expectations of the GS95 theory that we briefly describe below (see also Lithwick & Goldreich 2001; Cho & Lazarian 2002). The corresponding numerical studies of the decomposition of MHD turbulence into Alfvén, slow, and fast modes is performed in Cho & Lazarian (2003) and Kowal & Lazarian (2010) with the Fourier technique and wavelets, respectively.

The theory of MHD turbulence has become testable due to the advent of numerical simulations (see Biskamp 2003) that confirmed (see Lazarian et al. 2009, 2012b; Brandenburg & Lazarian 2013 and references therein) the general expectation that magnetized Alfvénic eddies are elongated in the direction of the magnetic field (see Shebalin et al. 1983; Higdon 1984) and provided results in agreement with the quantitative predictions for the variations of eddy elongation obtained in GS95.

The hydrodynamic counterpart of the MHD turbulence theory is the famous Kolmogorov (1941) theory of turbulence. In the latter theory, energy is injected at large scales, creating large eddies that correspond to large Re numbers and therefore do not dissipate energy through viscosity\(^6\) but transfers energy to smaller eddies. The process continues until the cascade reaches eddies that are small enough to dissipate energy over eddy turnover time. In the absence of compressibility, the hydrodynamic cascade of energy is \(\sim v_\perp^2/\tau_{\text{cas,}\perp} = \text{const}\), where \(v_\perp\) is the velocity at the scale \(\ell\) and the cascading time for the eddies of size \(\ell\) is \(\tau_{\text{cas,}\perp} \approx \ell/v_\perp\). From this the well-known relation, \(v_\perp \sim \ell^{1/3}\), follows.

It is easy to see why magnetic turbulence is anisotropic. One can imagine eddies mixing magnetic field lines perpendicular to the direction of the local magnetic field. For such eddies, the original Kolmogorov treatment is applicable, resulting in perpendicular motions scaling as \(v_\perp \sim \ell^{1/3}\), where \(\ell_{\perp}\) denotes eddy scales measured perpendicular to the magnetic field. These mixing motions induce Alfvénic perturbations that determine the parallel size of the magnetized eddy. The keystone of the GS95 theory is \textit{critical balance}, i.e., the equality of the eddy turnover time \(\ell_{\parallel}/v_{\parallel}\) and the period of the corresponding Alfvén wave \(\sim \ell_{\parallel}/V_A\), where \(\ell_{\parallel}\) is the parallel eddy scale and \(V_A\) is the Alfvén velocity. Making use of the earlier expression for \(v_\perp\), one can easily obtain \(\ell_{\parallel} \sim \ell_{\perp}^{2/3}\), which reflects the tendency of eddies to become increasingly more elongated as energy cascades to smaller scales.

It is important to stress that the scales \(\ell_{\perp}\) and \(\ell_{\parallel}\) are measured with respect to the system of reference related to the direction

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\(^4\) A frequently used magnetospheric and heliospheric research model of Alfvénic turbulence, the so-called slab + 2D model (Bieger et al. 1988), was introduced to empirically represent the scattering of energetic particles in the solar wind. This model is not confirmed by numerical simulations and presents an approximate empirical treatment of a particular turbulent system. As we discuss in this section, the existing numerical simulations instead support the GS95 theory and its compressible MHD extensions.

\(^5\) The inefficiency of Alfvén modes stems from two factors. One of them is a scale-dependent anisotropy that makes perturbations extremely elongated at small scales if the energy is injected, as it usually happens in astrophysical systems, e.g., the ISM or intracluster gas, at large scales. The other is the rapid decrease of energy in terms of \(k_{\parallel}\), where \(k\) is measured in the direction of the local magnetic field on the scale of the Larmor radius of the resonance particle. Those factors are discussed in Chandran (2000) and Yan & Lazarian (2002).

\(^6\) The Reynolds number \(\text{Re} \equiv L_f V/\nu = (V/L_f)/(\nu/L^2_f)\), which is the ratio of an eddy turnover rate \(\tau_{\text{eddy}}^{-1} = V/L_f\) and the viscous dissipation rate \(\nu \cdot L^2_f\). Therefore, large Re corresponds to negligible viscous dissipation of large eddies over the cascading time \(\tau_{\text{cas}}\), which is equal to \(\tau_{\text{eddy}}\) in Kolmogorov turbulence.
of the local magnetic field “seen” by the eddy. This notion was not present in the original formulation of the GS95 theory and was added later (Lazarian & Vishniac 1999; Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002). In terms of mixing motions, it is rather obvious that the free Kolmogorov-type mixing is possible only with respect to the local magnetic field of the eddy rather than the mean magnetic field of the flow.\(^7\)

The quantitative properties of Alfvénic turbulence that we mostly deal with can be expressed for subAlfvénic turbulence using LV99 expressions:

\[ \ell || \approx L_e \left( \frac{\ell_\perp}{L_e} \right)^{2/3} M_A^{-4/3}, \]

\[ \delta u_\ell \approx u_e \left( \frac{\ell_\perp}{L_e} \right)^{1/3} M_A^{1/3}, \]

which, unlike those of GS95, are valid for the arbitrary injection velocity \( V_A \). For subAlfvénic turbulence, the ratio \( M_A \) is

\[ M_A \equiv \frac{V_A}{v_A} = \frac{\delta B}{B} > 1, \]

while, when the injection is superAlfvénic, \( M_A = V_L/V_A > 1 \).

We feel that the claims that the \(-5/3\) slope predicted by the GS95 model is incorrect are not substantiated. For instance, recent work by Beresnyak & Lazarian (2010) shows that present-day numerical simulations are unable to reveal the actual inertial range of MHD turbulence, making discussions of the discrepancies of the numerically measured spectrum and the GS95 predictions rather premature.\(^8\)

The GS95 theory assumes the isotropic injection of energy at the scale \( L \) corresponding to the Alfvén Mach number

\[ M_A \equiv \delta B/B = 1, \]

where \( \delta B \) is the field fluctuation and \( B \) is the mean field. For the incompressible MHD turbulence, \( \delta B/B \) is equal to \( V_L/V_A \), where \( V_L \) is the injection velocity and the Alfvén velocity in the fluid is \( v_A \). Thus it provides the description of transAlfvénic turbulence with \( V_L = v_A \). This model was later generalized for both subAlfvénic, i.e., \( M_A < 1 \), and superAlfvénic, i.e., \( M_A > 1 \), cases (see Lazarian & Vishniac 1999, henceforth LV99; Lazarian 2006, respectively; see also Table 1). Indeed, if \( M_A > 1 \), instead of the driving scale \( L \), one can use another scale, namely, \( l_A = L M_A^{-3} \), which is the scale at which the turbulent velocity is equal to \( V_A \) (see Lazarian 2006). The scale \( l_A \) is called the Alfvénic scale (Lazarian 2006) and is the scale below which magnetic field lines become stiff and their back-reaction becomes essential. For \( M_A \gg 1 \) magnetic fields are not dynamically important at scales larger than \( l_A \) and turbulence at those scales follows the isotropic Kolmogorov cascade \( \delta u_\ell \sim \ell^{1/3} \) over the range of scales \([L, l_A]\). At the same time, if \( M_A < 1 \), the turbulence obeys GS95 scaling (also called “strong” MHD turbulence) not from the scale \( L \), but from a smaller scale \( l_{trans} = L M_A^2 \), while in the range \([L, l_{trans}]\) the turbulence is “weak.”

The properties of weak and strong turbulence are rather different. Weak turbulence is wave-like turbulence with wave packets undergoing many collisions before transferring energy to small scales. Weak turbulence, unlike the strong one, allows an exact analytical treatment (Galtier et al. 2001). On the contrary, the strong turbulence is eddy-like with cascading occurring similar to Kolmogorov turbulence within roughly an eddy turnover time. The strong interactions between wave packets prevent us from using the perturbative approach and do not allow exact derivations. The numerical experiments proved the predicted scalings for incompressible MHD turbulence (see Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002; Beresnyak & Lazarian 2010; Beresnyak 2011) and for the Alfvénic component of the compressible MHD turbulence (Cho & Lazarian 2002, 2003; Kowal & Lazarian 2010).

One also should keep in mind that the notion “strong” should not be associated with the amplitude of turbulent motions but only with the strength of the nonlinear interaction. As the weak turbulence evolves, the interactions of wave packets get stronger, transferring the turbulence into the strong regime. In this case, the amplitude of the perturbations can be very small.

The theories above assume an isotropic injection of turbulence at large scales. This simplest type of energy injection can happen in both media with and without a mean magnetic field (see the simulations of Cho & Lazarian 2003 with a mean magnetic field). Isotropic injection of turbulent energy is also assumed in our paper. An anisotropic injection of energy can occur, e.g., due to CR interaction with compressible turbulence (e.g., Lazarian & Beresnyak 2006; Yan & Lazarian 2011). In this situation, energy is injected at the CRs gyroradii and in the form of the Parker field.\(^7\)

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**Table 1**

| Type of MHD Turbulence | Injection Velocity | Range of Scales | Spectrum | Motion Type | Ways of Study | Magnetic Diffusion | Squared Separation |
|------------------------|--------------------|----------------|----------|-------------|---------------|--------------------|-------------------|
| Weak                   | \( V_L < V_A \)    | \([l_{trans}, L]\) | \( k_\perp^{-5/3} \) | Wave-like    | Analytical    | Diffusion          | \( \sim s L M_A^2 \) |
| subAlfvénic            | \( V_L < V_A \)    | \([l_{min}, l_{trans}]\) | \( k_\perp^{-5/3} \) | Anisotropic  | Numerical      | Richardson         | \( \sim \frac{L^2}{\tau} M_A^2 \) |
| superAlfvénic          | \( V_L > V_A \)    | \([l_{A}, L]\)  | \( k_\perp^{-5/3} \) | Isotropic    | Numerical      | Richardson         | \( \sim \frac{L^2}{\tau} M_A^2 \) |
| Strong                 | \( V_L > V_A \)    | \([l_{min}, l_A]\) | \( k_\perp^{-5/3} \) | Anisotropic  | Numerical      | Richardson         | \( \sim \frac{L^2}{\tau} M_A^2 \) |

**Notes.** \( L \) and \( l_{min} \) are the injection and perpendicular dissipation scales, respectively. \( M_A \equiv \delta B/B, l_{trans} = L M_A^2 \) for \( M_A < 1 \) and \( l_A = L M_A^{-3} \) for \( M_A > 1 \). For weak Alfvénic turbulence \( \ell_\perp \) does not change. \( s \) is measured along magnetic field lines.
of plane waves. These waves will interact and be cascaded by the external Alfvénic turbulence (Yan & Lazarian 2002; Farmer & Goldreich 2004; Beresnyak & Lazarian 2008). Magnetic reconnection provides another example of anisotropic turbulence driving (see Lazarian et al. 2014), but this is not expected to be the dominant source of turbulent energy in the galaxy.

Astrophysical turbulence presents a wide variety of conditions. MHD turbulence in solar wind (Leamon et al. 1998) is different from the interstellar turbulence (see Big Power Law in Armstrong et al. 1995 and Chepurnov & Lazarian 2010; the measured spectra of velocity turbulence in H I and CO are reviewed, e.g., in Lazarian 2009) both in terms of the injection scales and the injection velocities. The media are also different in terms of magnetization (see reviews by Elmegreen & Scalo 2004; McKee & Ostriker 2007; Brandenburg & Lazarian 2013 and references therein). This induces substantial variations of the properties of turbulence including the variations of its Alfvénic Mach number $M_A$. Determining this parameter from observations is not trivial, as this number depends on the ratio of magnetic fluctuation to the mean magnetic field at the injection scale. The value of this scale is still the subject of controversy, but we believe that for interstellar turbulence it should be determined by the supernovae injection and be of the order of 100 pc.

Radio continuum polarization observations find that in external galaxies on the scale of several hundred parsecs, the ordered magnetic field is 1–5 $\mu$G, while the total magnetic field is of the order of 9 $\mu$G (Beck & Wielebinski 2013). The values of the mean and chaotic field that we need to calculate $M_A$ are difficult to obtain from these input data, as the measurements are not done at the turbulence injection scale $L$, which is smaller. The ratio of the chaotic to the total magnetic field on scales larger than $L$ depends on the action of magnetic dynamo, which remains a theory facing many challenges, e.g., related to the helicity conservation (see Vishniac & Cho 2001). We feel that the existing data is compatible with $M_A$ of the order of unity, which corresponds to the energy equipartition of between turbulence and the magnetic field. Naturally, better determination of $L$ and the magnetic field fluctuations on the scale $L$ are required.

3. RICHARDSON DIFFUSION OF MAGNETIC FIELD AND COSMIC RAYS

Above the turbulent injection scale $L$, naturally, magnetic field lines undergo random walk. In what follows, we focus on the field line divergence over scales less than $L$ where the phenomenon of Richardson diffusion takes place. These are important scales for CR propagation and acceleration, as, for instance, the energy injection scale in the ISM is around 100 pc (see Chepurnov et al. 2010). Nevertheless, our treatment below is quite general and it covers various astrophysical environments. Therefore, we consider a variety of possible situations for subAlfvénic, superAlfvénic turbulence, and for situations when the mean free path of CR is larger or smaller than the injection scale.

3.1. Magnetic Field Wandering

Turbulence is frequently visualized as a hierarchy of interacting eddies. In the well-known example of Kolmogorov turbulence with the cascading rate $\epsilon \approx v_\ell^3/\ell = \text{const}$, the relative velocities of particles increase as the scale of motions increases according to the famous $1/3$ law, i.e., $v(\ell) \sim v_\ell(\ell/L)^{1/3}$, where $v_\ell$ is the difference of the velocities of particles separated by distance $\ell$, $v_0$ is the injection velocity at the scale $L$. As $v_\ell \sim (d/dt)\ell(t)$, one can write

$$\frac{d}{dt}\ell(t) \sim (\epsilon\ell)^{1/3},$$

(5)

which gives the solution $\ell^2 \sim \epsilon t^2$, which corresponds to the famous Richardson law (Richardson 1926) of particle diffusion in hydrodynamic turbulence. An interesting feature of this law is the accelerated separation of fluid particles, which arises from the fact that, as time goes on, the particles become enter eddies of larger size which, according to the Kolmogorov theory, have larger velocity dispersions.

In the case of magnetized flows, the Richardson diffusion represents itself in terms of magnetic field wandering (LV99). LV99 quantified the process and its relation to the Richardson diffusion was established in ELV11.

3.2. CR Streaming Along Wandering Magnetic Field Lines

Magnetic field scattering by Alfvénic turbulence may be very inefficient if the energy is injected at scales much larger than the gyroradius of the CR (Chandran 2000; Yan & Lazarian 2002). This is due to the scale-dependent anisotropy of the Alfvén waves. Thus, in the absence of the fast modes that were identified as the main scattering agent in most astrophysical environments, e.g., the ISMs (Yan & Lazarian 2002), the scattering may be very inefficient and CRs may freely stream along magnetic field lines. Fast modes can be depleted at small scales due to damping, e.g., the collisionless damping (see the quantitative study in Yan & Lazarian 2004).

If the mean free path of a CR $\lambda_{CR}$ is much larger than the scale over which CR propagation is considered, the CRs stream along the magnetic field and their diffusion perpendicular mean magnetic field is determined by the divergence of magnetic field lines. In this section, we quantify this type of separation.

First we consider the regime of Alfvénic Mach numbers $M_A < 1$, which is important for many astrophysical implications. We describe the divergence of field lines that start at points displaced by vector $\ell$. The displacement increases is considered as one moves a distance $s$ in arc-length along the field lines passing through the two points. Below we use a more formal approach that was suggested in ELV11 and which is complementary to an intuitive approach adopted in LV99. Within the former approach, the corresponding equation describing the change in separation is

$$\frac{d}{ds}\ell(s) = \hat{b}\ell'(s) - \hat{b}\xi(s),$$

(6)

9 Note that the Richardson law was discovered empirically prior to the formulation of the Kolmogorov theory.
10 This field wandering is closely related to the fast magnetic reconnection in a turbulent flow (LV99), which in its turn is related to the violation of magnetic flux freezing in turbulent fluids (ELV11). Indeed, it is easy to see that the wandering of magnetic field lines related to turbulent motions is impossible without fast reconnection and the perpetual changing of magnetic field topology. Without such a reconnection, magnetic field lines in the turbulent fluid would produce a felt-like structure arresting all hydrodynamic-type mixing motions. Although a picture of magnetic field wave-like turbulence with waves propagating in the elastic structure of magnetic interlocked field lines is possible to imagine, it corresponds neither to numerical simulations nor to observations of turbulent diffuse ISMs, molecular clouds, or solar wind (see Armstrong et al. 1995; Leamon et al. 1998; Elmegreen & Scalo 2004; McKee & Ostriker 2007; Lazarian 2009). As we mentioned earlier, fast LV99 reconnection makes the GS95 picture self-consistent, enabling magnetic field lines to change their topology over one eddy turnover time.
where \( \ell(s) = \xi'(s) - \xi(s) \), and \( \hat{b} = B/|B| \) is the unit tangent vector along the magnetic field line.

We leave the details of the derivation to Appendix A, while here we provide the results. For strong Alfvénic turbulence, we get

\[
\frac{d}{ds} \ell^2_\perp \sim D^R_\perp(\ell) \sim (\delta u_\perp/v_A)^2 \ell_\parallel \sim L \left( \frac{\ell_\perp}{L} \right)^{4/3} M^4_A, \quad (7)
\]

while for the weak turbulence regime one should use \( \ell_\parallel = L \) (see Section 2 and Table 1) as a constant, which gives

\[
\frac{d}{ds} \ell^2_\perp \sim D^R_\perp(\ell) \sim L M^4_A, \quad (8)
\]

or

\[
\ell^2_\perp \sim s L M^4_A. \quad (9)
\]

The predicted behavior was tested numerically. Figure 1 illustrates both the Richardson diffusion of magnetic field lines and its transition to the ordinary diffusion at a scale larger than the energy injection one. The results confirm the Richardson scaling or magnetic line separation that follows from Equations (7) and (10). Similar numerical results were also obtained in Maron et al. (2004). We note that for the case of \( \lambda_{\text{CR}} \gg L \), CR trajectories trace magnetic field lines and therefore the divergence of CR trajectories is identical to that of magnetic field.

Thus, we clearly observe two different regimes. For the weak turbulence the solution of Equation (8) induces a usual random walk diffusion law, while solving Equation (7) provides the Richardson-type scaling of magnetic field line separation with \( \ell^2_\perp \sim s^3 \). The Richardson law in terms of magnetic field separation in time was confirmed with numerical simulations involving many time frames of the driven MHD turbulence (Eyink et al. 2013). The Richardson diffusion of magnetic field lines was also observed through tracing the CRs that ballistically follow the turbulent magnetic field (Xu & Yan 2013).

The solution of Equation (7) in terms of \( \ell_\perp \) dependence on \( s \) is

\[
\ell^2_\perp \sim \frac{s^3}{27L} M^4_A, \quad (10)
\]

where we stress the importance of the \( M^4_A \) dependence, which contrasts with the \( M^3_A \) dependence in the classical studies (see Jokipii & Parker 1969). This dependence translates in the corresponding \( M^4_A \) dependences for the perpendicular diffusion of CRs, which means a much stronger suppression of perpendicular diffusion by the magnetic field. The vivid feature of Equation (10) is that \( s_\perp < s \); like the Richardson diffusion, it is required that scales \( s \) are less than the injection scale \( L \).

Figure 2 illustrates the results of numerical simulations of the propagation of CRs in compressible MHD turbulence in Xu & Yan (2013). Unlike many other CR studies, these results are obtained with turbulence produced by direct numerical simulations with the MHD compressible code (for more details on the code, see Cho & Lazarian 2003). Therefore, these results correspond to the propagation of CRs in GS95-like turbulence and they agree well with the theoretical expectations given by Equation (10). We can clearly see that the simulations support the \( M^4_A \) dependence of the perpendicular coefficient instead of the generally accepted \( M^3_A \) one.

The perpendicular displacement given by Equation (8) is an obvious random walk in terms of the path along the magnetic field lines

\[
\ell^2_\perp \sim s L M^4_A, \quad M_A < 1, \quad (11)
\]

where the dependence on \( M^4_A \) is prominently present. In weak turbulence the parallel scale does not change (see LV99; Galtier 2001) and therefore it is always equal to the injection scale. The change happens when the turbulence reaches the scale \( l_{\text{trans}} \) given in Table 1 where at which it transfers to the strong turbulence. In astrophysical situations, there can be more than one source of weak turbulence injection. Therefore, in Equation (11) \( s \gg L \) and this regime corresponds to the usual random walk.

11 The scaling given by Equation (10) can also be obtained by observing that \( \frac{d\ell_\perp}{ds} \sim \frac{\delta u_\perp}{v_A} \sim \frac{s}{L} \). Inserting the turbulent velocity given by Equation (2), one gets \( \ell^2_\perp \sim (s/L) M^4_A \).

12 It is rather difficult to generate turbulence with the scale-dependent anisotropy. Thus, the simulations that use synthetic turbulence cubes without this feature are missing essential physics.

13 Cosmic ray instabilities, e.g., streaming instability, produces Alfvénic waves with wave vectors parallel to the local magnetic field direction. As these waves are being reflected by density inhomogeneities, this creates oppositely moving Alfvén waves, which entails an imbalanced weak cascade (see Lithwick & Goldreich 2001). These inhomogeneities may dominate at sufficiently small scales, which makes the discussed regime of CR diffusion astrophysically relevant.
For Alfvén Mach numbers larger than unity, i.e., $M_A > 1$, at scales larger than the Alfvénic scale $l_A$ given in Table 1, the dynamics of the magnetic field are dominated by hydrodynamic turbulence. The divergence of magnetic field lines at scales $[l_A, L]$ can be estimated with the approach above

$$\frac{d\ell^2}{ds} \approx D(t) \approx \epsilon^{1/3} \ell^{4/3} \quad M_A > 1,$$

(12)

where one does not distinguish between the parallel and perpendicular scale, as this is appropriate for superAlfvénic turbulence. The dependence given by Equation (12) is similar to the one obtained for the separation of particles for pure hydrodynamics. However, the authors of the aforementioned papers assumed $c\mu$ superdiffusion at the scales less than $l_A$, i.e., for scales $l < l_A$ provided that the scale $l_A$ is used instead of $L$. The parallel basis for this is that the magnetic fields at scale less than $l_A$ get stiff and are not pliable to being bend by hydrodynamic turbulence.

Streaming CR moves along magnetic field lines with the velocity $c\mu$, where $\mu$ is the cosine of pitch angle. If we do not consider CRs with the pitch angle close to 90°, the velocity of streaming is of the order of the velocity of light $c$. Ignoring the factors of the order of unity, we can write that in terms of superdiffusion of CRs, for $M_A > 1$ CRs streaming along magnetic field lines experience isotropic superdiffusion $\ell^2 \sim (ct)^3 / L$ over the scales $[l_A, L]$ where the displacement along the magnetic field is $\ell_A$. For scales smaller than $l_A$, i.e., scales $l < l_A$ the superdiffusion is present perpendicular to the magnetic field line and the displacement is isotropic, and the eddies at the scales less than $l_A$ are pseudo-spherical. In the case of superAlfvénic turbulence, we deal with while considering turbulence with $M_A > 1$. In these situations, $d\ell^2 / ds \sim \delta t / V_L$ and the turbulent fluctuation of velocity $\delta v \sim \ell^{(a-1)/2}$, the expected scaling is $d\ell^2 / ds \sim \ell^{(a+1)/2}$, which, for the case of Kolmogorov motions, i.e., $a = 5/3$, provides the result given by Equation (12). However, the authors of the aforementioned papers assumed $d\ell^2 / ds \sim \ell^s$, which provided a different scaling of expected magnetic field wandering.

3.3. CR Diffusion Along Diverging Magnetic Field Lines

The discussion above is applicable for CRs moving ballistically on scales $x$ less than the CR mean free path $\lambda_{CR}$, i.e., $R < \lambda_{CR}$ and $R < L$. The dynamics of superdiffusion of CR changes when multiple scatterings occur on the scale under consideration, i.e., $\lambda_{CR} \ll R < L$.

It is important to note that the CR diffusion has two aspects. As we discussed in Section 2, an important feature of modern understanding of MHD turbulence is that it distinguishes the local and global mean magnetic fields. The parallel scattering behaves in terms of the local magnetic field, as CRs trace the local, rather than the mean, magnetic field. Scattering introduces the displacements with respect to the local magnetic field, as CRs start sampling neighboring magnetic field lines and experience superdiffusion over scales $\sim \lambda_{CR}$. In other words, when the local field direction there will be displacements arising from the diffusion of CRs along the diverging magnetic field lines and additional diffusion due to CR displacements perpendicular to the local direction of magnetic field. Below we compare the two effects.

For the scales $\gg L$ the parallel diffusion coefficient is

$$D_{\parallel, \mathrm{global}} \approx D_1 M_A^4,$$

(15)

where it is taken into account that for $M_A < 1$ the eddies are elongated with the perpendicular dimension $\sim L M_A^2$ and crossing these eddies involves a random walk process with the time step of $\sim L^2 / D_1$. This coincides with the result for the thermal particle diffusion obtained for thermal conduction in Lazarian (2006).

On scales less than the $L$ a different treatment is necessary. One can express the perpendicular displacement with respect to the mean magnetic field as

$$I_{\perp, \mathrm{CR}}^2 \sim (D_1 \delta t)^{3/2} \quad M_A < 1,$$

(16)

where we took into account that the displacement along the magnetic field is governed by the parallel diffusion and therefore

$$I_{\perp, \mathrm{CR}}^2 \approx D_{\parallel} \delta t.$$

(17)

Similarly, superAlfvénic turbulence at scales less than the Alfvénic scale $l_A$, i.e., as shown in Table 1, for $l < l_A$, the parallel and perpendicular diffusion coefficients provide

$$I_{\perp, \mathrm{CR}}^2 \sim (D_1 \delta t)^{3/2} M_A^4, \quad M_A > 1.$$

(18)

We observe that the perpendicular displacements given by Equations (16) and (18) exhibit superdiffusive behavior, although the rate of superdiffusion is reduced compared to that in the case of the Richardson diffusion. Indeed, instead of the Richardson growth in proportion to $t^{3/4}$ for the diffusive propagation of CRs along magnetic field lines, we observe $t^{3/4}$ dependence. This is consistent with the findings in YL08. Figure 3

---

14 In the studies of Jokipii (1973) and Skilling et al. (1974), a model where magnetic field lines follow an isotropic turbulent cascade with the power law $\alpha$, i.e., $E(k) \sim k^{-\alpha}$, was adopted. The adopted isotropic cascade is physically the same as that of superAlfvénic turbulence that we deal with while considering turbulence with $M_A > 1$. In these situations, $d\ell^2 / ds \sim \delta t / V_L$ and the turbulent fluctuation of velocity $\delta v \sim \ell^{(a-1)/2}$, which, for the case of Kolmogorov motions, i.e., $a = 5/3$, provides the result given by Equation (12). However, the authors of the aforementioned papers assumed $d\ell^2 / ds \sim \ell^s$, which provided a different scaling of expected magnetic field wandering.

15 Here streaming is understood in terms of motion without scattering. Note that when cosmic ray stream in one direction they may produce "streaming instability." However, we do not consider such a case.

16 In YL08, we attempted to introduce a diffusion coefficient that changes with the scale of the motions. We feel, however, that this way of describing superdiffusion is not useful since the superdiffusion does not obey a simple Laplacian equation.
presents the numerical confirmation of the superdiffusive regime that corresponds to the expectations given by Equation (16).

The displacements given by Equations (16) and (18) are calculated with respect to the global mean magnetic field. They arise due to the divergence of magnetic field lines as the particle diffuse tracing magnetic field lines.

In reality, particles undergo scattering and this is an additional effect that should be accounted for. To quantify the effect of perpendicular diffusion in the presence of scattering, we consider a sequence of scattering events having CRs with the mean free path of $\lambda_{\text{CR}}$, which is measured along the local direction of the magnetic field. As the CRs trace the divergent field lines they experience the effect of Richardson diffusion and the perpendicular displacement of CRs given by Equation (10) is

$$l_{\perp,\text{elem}} \sim (1/3)^{3/2}L^{-1/2}\lambda_{\text{CR}}^{-1/2}M_A^2, \quad M_A < 1,$$

where it is assumed that $l_{\perp,\text{elem}}$ is much larger than the Larmor radius $r_L$ over which the CR is being shifted perpendicular to the magnetic field as a result of a scattering event.

After each passing of the mean free path the spread increases in the random walk fashion. Therefore, the total spread after $N$ scattering events will be

$$l_{\perp,\text{cumm}}^2 \sim (1/3)^3N\lambda_{\text{CR}}^{-2}M_A^4, \quad M_A < 1.$$  

Due to the random walk nature of the transport in the presence of scattering, one can introduce a perpendicular diffusion coefficient for the diffusion perpendicular to the local magnetic field at scales less than the perpendicular scale of the strong turbulence eddy:

$$D_{\perp,\text{local}} \sim \frac{R^2}{\delta t} \sim \frac{R^2}{(R/l_{\perp,\text{elem}})^2\lambda_{\text{CR}}/v_1} \sim \frac{1}{81} \frac{\lambda_{\text{CR}}}{L} \lambda_{\text{CR}}^2M_A^2,$$

where, to cross the distance $R$ due to random walk with elementary length $l_{\perp,\text{elem}}$, one requires $(R/l_{\perp,\text{elem}})^2$ steps and each step takes time $\lambda_{\text{CR}}/v_1$. Finally, it was assumed that the parallel velocity of the CRs with isotropic distribution is $1/3v_{\text{CR}}$.

The physical meaning of the diffusion described by Equation (21) is very straightforward. If CRs trace magnetic field lines in the case of the free streaming that we discussed in Section 3.1, in the case of their diffusion that we discuss here, CRs, in addition, spread perpendicular to the flux tubes that they follow. This is not the usual perpendicular diffusion of CRs when every scattering event results in a CR perpendicular displacement of around the CR Larmor radius $r_L$. Due to the superdiffusion of magnetic field lines at scales $\lambda_{\text{CR}} < L$, the CR after each scattering is being displaced by a substantially larger scale given in Equation (19).\footnote{The Larmor radius is irrelevant for this process and the same displacements will be present for particles of different energies, including thermal particles. Thus this effect can be important for thermal conductivity of magnetized plasmas.}

If the measurements are done with respect to the local magnetic flux tube, the diffusion given by Equation (21) is the dominant effect. However, in terms of the displacements with respect to the global magnetic field, the additional diffusion presented by Equation (21) is subdominant compared to the rate of particle separation arising from their diffusion along the divergent field lines (see Equation (15)). Similar observations are also true for the comparison of Equation (20) with Equation (16) if one takes into account that the number of the scattering events $N \approx (D_{\parallel}\delta t)^{1/2}/\lambda_{\text{CR}}^2$.

4. SUBDIFFUSION OF COSMIC RAYS

Subdiffusion is the diffusion process along magnetic field lines that undergo diffusion in space. The subdiffusion is an ingredient for a number of models of acceleration and propagation of CRs. In what follows, we extend the arguments in YL08 and define the conditions necessary for the subdiffusion to be important.

The subdiffusion is a process widely discussed in the literature (see, e.g., Kótá & Jokipii 2000; Getmansev 1963; Mace et al. 2000; Qin et al. 2002; Webb et al. 2006). If we introduced for magnetic field lines a spatial diffusion coefficient $D_{\text{spat}} = \delta s^2/\delta s$ and adopt that the transport along the magnetic field lines is diffusive, i.e., $\delta s = (D_{\text{spat}}\delta t)^{1/2}$, we can get the perpendicular diffusion coefficient

$$D_{\perp} = \frac{(\delta l_{\perp}/\delta s)^2}{2} D_{\parallel} = D_{\text{spat}1/2} D_{\parallel} = D_{\text{spat}1/2}(\delta t)^{-1/2}.$$  

Therefore, the perpendicular transposition is $l_{\perp,\text{CR}}^2 = D_{\perp}\delta t = D_{\text{spat}1/2}(\delta t)^{1/2}$ in accordance with the findings in many papers dealing with subdiffusion.

The major implicit assumption in the reasoning above is that the particles trace back their trajectories as they are scattered backward. This seems possible when one considers a toy model of “turbulence” with random motions at a single scale that was described in the Rechester & Rosenbluth (1978) seminal paper (for more information, see Appendix B). There the distance over which the particle trajectories become uncorrelated is comparable with the injection scale and on scales less than the corresponding Rechester & Rosenbluth length, the phenomenon of subdiffusion is expected. The corresponding calculations can be found in Duffy et al. (1995).

The problem with this reasoning is that turbulence is not a process with one scale of random motions. We claim that...
this retracing requires extremely special conditions and, in fact, impossible in the presence of the Richardson diffusion of the magnetic field and the retracing of CR trajectories impossible for scales corresponding to the inertial range of the strong turbulence. In Appendix B, we study what is expected to happen at scales beyond the inertial range of strong MHD turbulence \([l_{\text{min}}, l_{\text{max}}]\), where \(l_{\text{max}} = \min(v_{\text{Larmor}}, L)\). There we show that the process of CR subdiffusion requires very special circumstances and therefore is very unlikely.

5. MODIFICATION OF ACCELERATION MECHANISMS IN THE PRESENCE OF RICHARDSON DIFFUSION

5.1. Difference in Parallel and Perpendicular Shock Acceleration

An accepted picture of CR acceleration involves a shock moving at an angle to the ordered magnetic field. In particular, two limiting cases, the parallel and perpendicular shocks, are considered, where the perpendicular shock means that the angle between the shock velocity and magnetic field is 90°, while in a parallel shock the shock velocity along magnetic field. In the well-known mechanism of shock acceleration (Krymskii et al. 1978; Bell 1978; Blandford & Ostriker 1978), CRs are accelerated by scattering back and forth from upstream to downstream regions. While the shock itself does not affect CRs, the compression associated with the shock induce a regular acceleration. Indeed, for the upstream CRs, the downstream plasmas are approaching with the velocity \(3/4U\), where \(U\) is the velocity of a strong shock. A similar effect is present for the downstream CRs crossing the shock, which results in an energy gain for the particles every time they cross the shock. This is the essence of the efficient first order Fermi acceleration in which the particle energy is increased by a factor \((\Delta p)/p = 4[U - U_2]/(3\nu)\) every time it crosses the shock.

The above simple picture critically depends on the interaction of CRs with magnetic perturbations in the upstream and downstream regions. A usual textbook description of shock acceleration involves parallel shocks. There CRs streaming along the magnetic field become scattered by magnetic perturbations in the upstream and downstream regions when scattered back transverse the shock again (see Longair 1994). The rate of scattering limits the rate at which particles can be returned to the shock and experience another cycle of acceleration. This type of acceleration should work, but the efficiency of it is subject to debate.

Jokipii (1987) noted that the rate of CRs scattering may be insufficient to explain observational data. To remedy this problem, he proposed, instead, the idea of acceleration within perpendicular shocks.

The arguments in Jokipii (1987) are based on the standard kinetic theory (see Axford 1965) and do not take into account the wandering of magnetic field lines. Jokipii (1987) accepts that this is an important effect but does not consider it in view of the existing uncertainties related to the process. The demonstration of the dominance of the perpendicular shocks was then very straightforward. Indeed, the time of CR acceleration from the initial momentum \(p_i\) to \(p_f\) is given by a standard expression (Forman & Morfill 1979)

\[
\tau_{\text{accel}} = \frac{3}{V_{\text{up}} - V_{\text{down}}} \int_{p_i}^{p_f} \left( \frac{D_{\text{up}}}{V_{\text{up}}} + \frac{D_{\text{down}}}{V_{\text{down}}} \right) \frac{dp}{p},
\]

where \(V_{\text{up}}, \kappa_{\text{up}}\) and \(V_{\text{down}}, \kappa_{\text{down}}\) are velocities and diffusion coefficients in the upstream and downstream of the shock, respectively. For the sake of simplicity, while dealing with a model problem of the shock acceleration one can accept that \(D_{\text{up}} = D_{\text{down}} = D_x\), which can be presented as a combination of diffusion coefficients parallel to the mean magnetic fields \(\kappa_{\parallel}\) and perpendicular to it, \(D_{\perp}\)

\[
D_x = D_{\parallel} \cos^2 \theta + D_{\perp} \sin^2 \theta,
\]

where \(\theta\) is the angle between the magnetic field and the shock normal.

The ratio of parallel and perpendicular diffusion coefficients in kinetic theory is given by the ratio

\[
\frac{D_{\perp}}{D_{\parallel}} = \frac{1}{1 + (\lambda_{\text{CR}}/r_L)^2},
\]

where \(\lambda_{\text{CR}}\) is larger and in many cases much larger than the CR Larmor radius \(r_L\). This suggests that a perpendicular shock \(\theta = \pi/2\) provides the fastest acceleration (see Equations (23) and (24)). The possible ratio given by Equation (25) presented in Jokipii (1987) is \(\sim (r_L/\lambda_{\text{CR}})^2\).

The arguments above should be modified in the presence of the superdiffusion discussed in this paper. Indeed, in the perpendicular direction, one expects to see that the superdiffusion with the perpendicular transpositions with respect to the local magnetic field is not \(r_L\) that enters in Equation (25), but instead is given by Equation (19). Moreover, in terms of the laboratory system of reference related to the shock, the transpositions in the perpendicular direction are super diffusive (see Equation (18)).

5.2. Example: Anomalous Cosmic Rays

Consider whether the process of superdiffusion may present a problem for accounting for a particular case when fast acceleration is required, i.e., for the anomalous CR acceleration in the termination shock.

The anomalous CRs are CRs accelerated locally within the heliosphere. This problem became a hot topic in view of the failure of the Voyagers to detect the signature of the anomalous acceleration while crossing the termination shock. As argued in Jokipii (1992), the acceleration in the parallel shock is too slow and only the quasi-perpendicular shock has a high enough acceleration rate according to Equation (25). Below we discuss the conditions when the approach in Jokipii (1992) is valid.

To have the rates of acceleration given by Equation (25), one should postulate that the magnetic field wandering over the mean free path \(\lambda_{\text{CR}}\) is less than the CR Larmor radius \(r_L\). By analyzing the factors in Equation (19) that describes an elementary perpendicular transposition of CR at its scattering over the mean free path \(\lambda_{\text{CR}}\), one can observe that if

\[
\frac{L}{\lambda_{\text{CR}}} > \frac{1}{27} \left(\frac{\lambda_{\text{CR}}}{r_L}\right)^2 M_A^4,
\]

then the transposition of the CR due to the field wandering over the mean free path is less than the Larmor radius of the CR \(r_L\). In this situation the original treatment in Jokipii (1987) is applicable. Assuming that the interstellar \(M_A\) outside the termination shock is of the order of unity, we see that the condition given by Equation (26), i.e., \(4.4M_A^4(\lambda_{\text{CR}}/0.2\text{AU})^2(B/8\mu\text{G})^2(\text{MeV}/E_0)(100\text{AU}/L) < 1\), is restrictive and may not necessarily be satisfied for the termination shock.

Irrespective of this particular application, our example with the termination shock illustrates the point that turbulence is
known to be ubiquitous and therefore the Richardson diffusion must be considered while dealing with the problems of CR acceleration.\textsuperscript{18}

As for the acceleration of anomalous CRs in a perpendicular shock, one can see from Equation (25) that the efficiency of parallel and perpendicular acceleration should become comparable in the case of the Bohm diffusion when $\lambda_{CR} \approx r_L$. Therefore, we are confused by the conclusion in Jokipii (1992) that the acceleration in the parallel shock in the Bohm limit is too slow to explain the origin of the anomalous CRs, while the perpendicular shocks should do the job of the acceleration. We find this to be in contradiction with Equation (25). Thus we feel that the issue of the efficiency of the acceleration of anomalous CRs in parts of the termination shock deserves further investigation.\textsuperscript{19}

All in all, we showed the importance of magnetic field wandering for the efficiency of perpendicular shock acceleration. Our study testifies that the difference between the acceleration in parallel and perpendicular shocks is reduced in the presence of the Richardson diffusion of the magnetic field. To achieve much more efficient acceleration in perpendicular shocks, the Alfvénic Mach number must be reduced, meaning that the magnetic field should be only very weakly perturbed. Our quantitative study in Section 6.1 supports this intuitive conclusion.

### 5.3. Acceleration of Particles Streaming Along the Magnetic Field in Shocks

**The original idea.** An innovative idea of particle acceleration without diffusion was suggested in Jokipii & Giacalone (2007, henceforth JG07). They considered an adiabatic compression induced by shock interacting with free streaming particles. They discussed the idea in terms of low-rigidity particles, e.g., electrons, but the very possibility of such a streaming acceleration mechanism deserves a dedicated study.

The idea of such an acceleration can be traced back to the analysis of Northrop (1963), who discussed a possibility of a particle acceleration in a moving magnetic loop. In the case of the shock, the physics of the process is related to the decrease of the length of the magnetic field line due to compression. The shock “shortens” the magnetic field lines and, as a result, the parallel component of the particle velocity increases.\textsuperscript{20}

JG07 demonstrated that if a magnetic field line that is nearly perpendicular to the shock velocity is also subject to random walk excursions, then the magnetic field line will enter the shock front many times, enabling the particles streaming along the line to get many kicks that increase their parallel component of momentum.

**Pre-existing turbulence in the pre-shock.** Within the JG07 mechanism, the accelerated CRs are streaming and are not subject to scattering. However, turbulence is required to induce magnetic field wandering. Consider sub-Alfvénic or trans-Alfvénic turbulence of interstellar origin pre-existing in the upstream region. Assume an idealized situation when the magnetic field in the upstream region is not perturbed by the shock precursor or streaming instabilities. In many astrophysically important cases, e.g., in the case of interstellar turbulence, the injection scale is large, e.g., larger than the radius of the supernovae shock. Thus the magnetic field lines are subject to the Richardson diffusion.

The lines that cross the shock move away from the shock as $s^{3/2}$ (see Equation (10)) and the accelerated character of this deviation makes improbable for the magnetic field line to reenter shock again on scales less than the injection scale $L$. This is very different from the random walk process considered in JG07 and therefore the acceleration of CRs this way is improbable. The fast divergence of the magnetic field from the shock region is illustrated by Figure 4.

For the same example of the supernovae shock in the ISM, the driving scale of ambient pre-existing turbulence is $L \approx 100$ pc (see Chepurnov et al. 2010), which is larger than the size of a supernovae remnant for the Sedov phase. Indeed the turbulent magnetic field is dominant in young supernova remnants (Reynolds et al. 2012). Even if the injection scale of interstellar turbulence is just several parsecs, the return of magnetic field lines due to the random walk is not frequent enough to enable the perpendicular shocks to accelerate particles with the adiabatic JG07 process.

We also note that streaming CRs can increase their perpendicular momentum through the process of drift acceleration. We discuss this process in Section 7 in the framework of shocks in highly chaotic small-scale magnetic fields. In the JG07 original set-up, the perpendicular acceleration is also limited by the effects of Richardson diffusion.

**Small-scale weak turbulence as the driver for magnetic field wandering.** As we discussed in Section 3, the random walk of magnetic field lines is possible in the case of the weak small-scale turbulence (see Equation (9) and Table 1). As we discussed in Section 2, the fluctuations associated with the weak Alfvénic turbulence are not necessarily small. If the small-scale perturbations induced by CR instabilities are substantially larger than the fluctuations of strong turbulence arising from the large scale driving, then the original JG07 idea works. We consider in Appendix C the requirements for this process to take place. We find the requirements to be rather restrictive due to the problems of generating of small-scale weak turbulence in the pre-shock...
region. On the contrary, in Section 7 we discuss how the situation is being modified in the presence of small-scale superAlfvénic turbulence that is naturally generated in the shock precursor (see Beresnyak et al. 2009).

5.4. Importance of Superdiffusion

Our analysis in the present section shows that the superdiffusive magnetic field wandering modifies the popular ideas of perpendicular shock acceleration. While we attempt a limited quantitative study in the next section, it is clear that the Richardson diffusion of magnetic field lines presents a fundamental process that must be accounted for thoroughly. The quantitative description of the process presented in the present paper is intended to contribute to this task.

Partially ionized gas may have an appreciable viscous damping scale and on scales less than this scale, the magnetic field does not exhibit superdiffusion. This is the range of scales where the earlier treatment of the difference of perpendicular and parallel shocks in JG07 is applicable. Thus in what follows we deal with the acceleration of CRs on scales larger than the turbulent damping scale. In the case of fully ionized gas, the Alfvénic turbulence in many cases proceeds up to the proton gyro-radius.

6. QUANTITATIVE TREATMENT OF ACCELERATION IN THE PRESENCE OF SUPERDIFFUSION

We study here different regimes of shock acceleration in the presence of superdiffusion and compare them with the acceleration of usual Diffusive Shock Acceleration (DSA) in parallel shocks.

6.1. Acceleration Time at Perpendicular Shock

In DSA, there is a length scale \( l_{\text{pen}} = D/U_1 \), corresponding to the distance \( \sqrt{Dt} \), that particles can diffuse before they are overtaken by the shock moving with speed \( U_1 \). Dividing the flux of particles \( nU/4 \) by the column density \( nD/U_1 \) gives the mean residence time of particles at the upstream \( 4D/(U_1v) \). Since the average gain of momentum per crossing is \( (\Delta p) = 4(U_1 - U_D)/p(3Uv) \), the acceleration time will be (see Duffy et al. 1995)

\[
\tau_{\text{acc}} = \frac{tr_{\text{res}}}{(\Delta p)} = \frac{3D}{(U_1 - U_D)} \left( \frac{1}{U_1} + \frac{1}{U_2} \right),
\]

(27)

if assuming a similar expression for \( t_{\text{acc}} \) at the downstream. Using a similar argument, we can obtain the acceleration for the case of anomalous transport. For diffusion on scales less than injection scale, the residence time is equal to \( 4ll_{\text{pen}}/v \), where \( l_{\text{pen}} \) can be obtained by equating Equation (16) or Equation (18) with \( (Ut)^2 \).

\[
l_{\text{pen}} = \frac{D^3}{U^2} \left( \frac{M_\Lambda^\xi}{27L} \right)^2,
\]

(28)

and

\[
\tau_{\text{res}} = \frac{4D^3}{vU^2} \left( \frac{M_\Lambda^\xi}{27L} \right)^2,
\]

(29)

where \( \xi = 4 \) for subAlfvénic turbulence and \( \xi = 3 \) for superAlfvénic turbulence. The acceleration time is then

\[
\tau_{\text{acc}} = \frac{tr_{\text{res}}}{(\Delta p)} = \frac{3D^3}{(U_1 - U_D)} \left( \frac{1}{U_1} + \frac{1}{U_2} \right) \left( \frac{M_\Lambda^\xi}{27L} \right)^2.
\]

(30)

Accordingly, the ratio of the acceleration time in a perpendicular shock with superdiffusion (Equation (30)) and parallel diffuse shock acceleration (Equation (27)) is

\[
\frac{\tau_{\text{acc, sup}}}{\tau_{\text{acc, DSA}}} = \left( \frac{DM_\Lambda^\xi}{27LU_1} \right)^2 (r^2 - r + 1)
\]

\[
\approx 0.012M_\Lambda^\xi \left( \frac{\lambda_{\text{cr}} v}{1 \text{ pc} c \ L} \frac{1000 \text{ km s}^{-1}}{U_1} \right)^2 
\times (r^2 - r + 1).
\]

(31)

\( r = U_1/U_2 \) is the compression ratio. For a strong shock with \( r = 4 \), the above ratio is \( \approx 0.16 \), much larger than \( (\lambda_{\text{cr}}/r_1)^2 \) as indicated by Equation (25). This shows that in the presence of superdiffusion, the quasi-perpendicular shocks generically have efficiencies comparable to those of quasi-parallel shocks and this presents the problem for the solution of acceleration at termination shock, for instance, as discussed in Jokipii (1992).

The maximum energy attainable for shock acceleration can be easily obtained by equating the acceleration time with the shock expansion timescale \( \sim R_{\text{sh}}/U_1 \) (see, e.g., Yan et al. 2012).

6.2. Spectra of Accelerated Particles at Perpendicular Shock

For an isotropic particle distribution, the spectrum index of shock acceleration is determined by the escape probability per cycle at the downstream \( P_{\text{esp}} = 4n(-\infty)ur_2/[n(0)v] \), the ratio of the far downstream flux to the isotropic particle flux (Bell 1978),

\[
a = 3 + \frac{p}{(\Delta p)} P_{\text{esp}}.
\]

(32)

Far downstream, the density distribution should relax to an unperturbed state, i.e., \( n(-\infty) = Q_0/U_1 \).

We use the particle propagator approach in Kirk et al. (1996) to estimate the flux of particles crossing the shock where they experience superdiffusion. Given a source function \( Q_0 \), the particle propagator \( P(x,t) \) is defined as

\[
n(x-U_1,t) = Q_0 \int_0^\infty dt P(x,t),
\]

(33)

where \( n(x-U_1,t) \) is the spatial density distribution at a distance \( x-U_1 \) from the shock front in the upstream.

The propagator is determined by the transport property of particles. For transport with \( \alpha = \infty \), \( a \) is a propagator of the form below may be adopted (Kirk et al. 1996)

\[
P(x,t) = t^{\beta/2} \Phi \left( \frac{x}{t^{\beta/2}} \right).
\]

(34)

Inserting Equation (34) into Equation (33), we can get \( n(0) = Q_0/[U(2-\beta)] \) if we assume that the particle transport properties are the same upstream and downstream.\(^{21}\) Inserting this result into the expression for \( P_{\text{esp}} \) and Equation (32), one gets

\[
a = 3 + \frac{3r}{r-1} \left( 1 + \frac{1}{r} \right).
\]

(35)

Table 2 is an illustration of the effect of different transport regimes for strong shock with \( r = 4 \).

\(^{21}\) This assumption may not hold especially in view of the different turbulence generation mechanism at preshock and postshock. We adopt the simplified model only to illustrate the impact of superdiffusion to shock acceleration.
In the case of partially ionized media, the damping scale of turbulence is relatively large and therefore the case that the transport scale $R$ can be less than the three-dimensional (3D) scale corresponding to the Rochester–Rosenbluth scale $L_{RR}$ (Narayan & Medvedev 2001; Lazarian 2006) where the separations of field lines have not reached the size of the smallest eddy and they are essentially bundled together with only perpendicular displacement occurring through random walk $(\Delta x^2) \propto \Delta z$. Below the damping scale, particles are not scattered and therefore $(\Delta x^2) \propto t$, corresponding to a normal diffusion and the momentum spectrum index of the accelerated particles is $-4$, the same as the standard DSA case (see Table 2).

6.3. Maximum Energy of Accelerated Particles at Parallel Shock of Finite Size

Richardson diffusion also presents a source of loss for shock with a finite spatial extent, which we illustrate on the example of a parallel shock. Indeed, CRs diffusing along magnetic field lines as shown in Figure 5 due to fast deviations of magnetic field lines may leave the part of the volume that is going to be affected by the shock.

Figure 5. Streaming CRs experience Richardson diffusion and the acceleration stops once the diffusion distance becomes larger than the size of the shock $l_{\text{sh}}$. (A color version of this figure is available in the online journal.)

In the case of partially ionized media, the damping scale of turbulence is relatively large and therefore the case that the transport scale $R$ can be less than the three-dimensional (3D) scale corresponding to the Rochester–Rosenbluth scale $L_{RR}$ (Narayan & Medvedev 2001; Lazarian 2006) where the separations of field lines have not reached the size of the smallest eddy and they are essentially bundled together with only perpendicular displacement occurring through random walk $(\Delta x^2) \propto \Delta z$. Below the damping scale, particles are not scattered and therefore $(\Delta x^2) \propto t$, corresponding to a normal diffusion and the momentum spectrum index of the accelerated particles is $-4$, the same as the standard DSA case (see Table 2).

\[
E_{\text{max}} = 32 \left( \frac{U_1}{400 \text{ km s}^{-1}} \right)^2 \left( \frac{L}{100 \text{ Au}} M_A^4 \right)^{4/7} \left( \frac{l_{\text{sh}}}{90 \text{ Au}} \right)^{4/7} \left( \frac{10 \text{ Au}}{\lambda} \right)^4 \text{ MeV nucleon}^{-1}.
\] (38)

7. FAST ACCELERATION WITHOUT SUPERDIFFUSION: STRONG TURBULENCE GENERATED IN THE SHOCK PRECURSOR

In the sections above, we have shown how Richardson diffusion of a magnetic field can modify acceleration. In general, decreases the efficiency of the acceleration in perpendicular shocks, although, as we discuss in Section 6, even in the presence of superdiffusion the perpendicular shocks may be still more efficient than in the parallel ones (see Equation (32)). We also considered how weak Alfvénic turbulence may help to recover the arguments on the acceleration of freely streaming particles (Section 5.2), but noticed that the process is rather restrictive. However, the Richardson diffusion of the magnetic field may be neglected in the case of small integral scale of magnetic perturbations. Indeed, in this case the mean free path $\lambda_{\text{CR}}$ becomes larger than $L$. In this situation, on the scale of $\lambda_{\text{CR}}$ the turbulence induces the random walk if it is superAlfvénic perturbations making the propagation diffusive. Below we show how the arguments related to the adiabatic acceleration (see Section 5.2) can be modified for the shocks propagating through media with a stochastic field.

In the superAlfvénic turbulence, the role of the injection scale is played by $l_A = L/M_A^3$, as it is the scale at which magnetic fields resist to further bending. The diffusion coefficient in the situation $\lambda_{\text{CR}} \gg l_A$, where $\lambda_{\text{CR}}$ is calculated through scattering calculations is (Lazarian 2006; YL08)

\[
D \approx 1/3 l_{\text{es}} U_{\text{CR}},
\] (39)

where $l_{\text{es}}$ is the integral correlation scale of magnetic perturbations, which, in the case of superAlfvénic turbulence, is equal to $l_A$.

The acceleration in the case of $l_{\text{es}} \ll \lambda_{\text{CR}}$ has its own features. First, one can clearly see that the process of increasing the parallel to local magnetic field component of CR momentum considered in JG07 is applicable to this situation. If the time for the magnetic eddy $l_{\text{es}}$ is $t_{\text{conv}} \approx l_{\text{es}}/U$, where $U$ is the shock velocity, then the number of eddies sampled by a CR during the shock compression time is

\[
t_{\text{conv}} \frac{\nu_{\text{CR}}}{l_{\text{es}}} = \frac{\nu_{\text{CR}}}{U},
\] (40)

which is $\gg 1$ for non-relativistic shocks that we consider. This proves that the acceleration is expected to be efficient. The

| $R < L$ | Partially Ionized Medium w. $R < \sqrt{L_{RR}/\xi_e}$ | $R > L$ |
|---|---|---|
| $\beta$ | $3/2$ | 1 |
| $a$ | $-7/2$ | $-4$ |

Notes. $\beta$ and $a$ are the power index of time for the square of displacement (see Section 6.2) and the three-dimensional momentum spectrum index of accelerated particles.
corresponding spectrum of the accelerated particles is (see Jokipii & Giacalone 2007)
\[ f(x, p) = \begin{cases} p^{-\alpha}_\parallel \exp(U_\parallel x/D) & \text{for upstream} \\ p^{-\alpha}_\perp & \text{for downstream}, \end{cases} \tag{41} \]
where \( a = r/(r M_{\text{A,up}}^2 - M_{\text{A,down}}^2) \). For the diffusion coefficient \( D \), we should use Equation (39) instead of the original one in JG07 for the fast acceleration we discuss here.

However, we claim that not only the parallel component of the CR will increase due to the compression induced by a shock. The processes of scatter-free drift acceleration in perpendicular shocks (Armstrong & Decker 1979; Pesses et al. 1979) should be important for the parts of the shock where the magnetic field is parallel to the shock front. Indeed, consider a process of the interaction of a CR moving along the loop of the size \( l_{cs} \) and interacting with a magnetic mirror created by an adjacent loop moving due to the compression. Naturally, this process increases the perpendicular momentum of the CR. As a result, we expect both parallel and perpendicular components of the CR momentum to increase.

The maximal energy available through this process is determined by the condition that the Larmor radius of the CR \( r_L \) is equal to the integral correlation scale of the magnetic field \( l_{cs} \), i.e.,
\[ E_{\text{max}} \approx \frac{l_{cs} B_{\mu G}}{5 \times 10^{11} \text{ cm GeV}}. \tag{42} \]
For CRs of energies larger than given by Equation (42), the acceleration proceeds in the diffusive regime as the turbulence on a scale of less than \( r_L \) and \( \delta B > B \) induces stochastic behavior of energetic particles.

The acceleration rate for the drift acceleration is (Kóta 1979)
\[ \dot{\mathbf{p}}_\perp / p = -\frac{1}{3} \nabla \cdot \mathbf{U}_\perp \]
and the acceleration rate for the parallel momentum increase due to the decrease of the magnetic loop is
\[ \dot{p}_\parallel / p = -\partial(U \langle B_\perp \rangle B^2) / \partial x. \tag{44} \]
In the case of strong turbulence, we see that the rates of the two are comparable.

The generation of a small-scale entangled field in front of the shock can be done through either the non-resonant instability proposed by Bell (2004) or by turbulent dynamo in the precursor as is suggested in Beresnyak et al. (2009, henceforth BJL09). Both mechanisms were proposed to explain the acceleration of high-energy CRs, but here we are interested in the chaotic small-scale structure of the magnetic field that is produced by these processes. The injection scale for the chaotic magnetic fields is much less than the spatial extent of a supernovae shock and therefore particles, e.g., low rigidity particles, streaming along magnetic field lines may enter and cross the shock many times.

Consider the BJL09 process. The precursor is an accepted part of the picture of CR acceleration. It is created by the accelerated CRs streaming ahead of the shock and reflected back (Diamond & Malkov 2007). BJL09 predicts that when the precursor interacts with the inhomogeneities of the turbulent density pre-existing in the upstream, this generates vorticity and turbulent motions which in turn generate magnetic fields in the precursor via turbulent dynamo (see Cho et al. 2009).

The characteristic scale of the largest eddies is limited by the thickness of the precursor. The latter may be much larger than the magnetic field structures at the \( l_{cs} \) scale. The details depend on the properties of the precursor and the density inhomogeneities’ pre-existing turbulence (see BJL09 for details).

Figure 6 shows the acceleration that the shock induces for the particles streaming along magnetic field entangled on the scale \( L \) much smaller than \( \lambda_{\text{CR}} \). In view of parallel acceleration of the acceptance in terms of the particle momentum parallel to the magnetic field, the process is similar to that discussed in JG07, but in the presence of a chaotic field entangled at small scales, there is no difference between the parallel and perpendicular shocks. In fact, the process can be referred to as “shock acceleration in entangled magnetic field.”

8. DISCUSSION

8.1. Superdiffusion and CR Acceleration in Entangled Field

Our study shows that while the diffusive shock acceleration is applicable to parallel shocks, the effects arising from the fact that the Richardson explosive diffusion of magnetic field lines (LV99; ELV11) induces superdiffusive behavior of CRs. Fast deviations of magnetic field lines from the mean direction of the magnetic field allow CRs to diffuse in the direction of the shock velocity faster than in the case of a classical perpendicular shock. This naturally makes the parallel and perpendicular shock acceleration comparable.

The issues related to superdiffusion are not present for scales larger than the scale of entanglement. In this paper, we discuss that the scale of magnetic field entanglement can substitute the mean free path of the CR, decreasing the CR’s diffusivity and making the acceleration more efficient. Moreover, the CRs streaming along entangled magnetic field lines can experience adiabatic acceleration in the shock. All this makes the acceleration in “shocks in an entangled field” a subject that deserves further careful investigation.

We note that the entangled magnetic fields are being naturally produced in the pre-shocked regions through the interaction of the precursor with the density inhomogeneities existing in the ambient media. They also can be generated by various instabilities induced by CRs. Future research should shed more light on the nature of the entangled magnetic field in the pre-shock and post-shock regions in different environments and enable researchers to make detailed quantitative calculations.

8.2. Perpendicular Diffusion of CR and Reconnection of Magnetic Field Lines

A major subject that this paper deals with is the implications of the Richardson diffusion for the CR propagation and
acceleration. The issues of magnetic field wandering corresponding to the Richardson diffusion have been the focus of recent discussion of problems of magnetic reconnection and flux freezing violation in turbulent media (LV99; ELV11; see Lazarian et al. 2014 for a review). Both the theoretical predictions of fast magnetic reconnection and the violation of flux freezing have been supported by numerical simulations (Kowal et al. 2009, 2012b; Eyink et al. 2013). The natural question is to what extent the textbook treatment of magnetic fields in fluids in conducting fluids is valid in the presence of the violation of flux freezing.

The answer to the question above depends on the problem with which one is dealing. We claim that for CRs the complicated dynamics of ever-changing reconnecting magnetic field lines is not essential. A CR samples the instantaneous complicated and stochastic field line that it is moving along. The existence of such a field line is due to the ability of the field lines to reconnect rather than forming small-scale knots at the intersection of field lines. In fact, this provides the necessary condition for the complicated field line wandering required in all the models of efficient perpendicular diffusion.

8.3. Different Regimes of Superdiffusion

If CRs are moving ballistically, their perpendicular displacement grows as $s^{3/2}$, while on scales larger than the mean free path of the CR, i.e., $\lambda_{CR} < s < L$, the perpendicular displacement grow as $s^{3/4}$. The mean free path is determined by the turbulent scattering of CRs. Yan & Lazarian (2002, 2004) identified the fast MHD modes as the major CR scattering agent. Therefore, the efficiency of scattering depends not only on the level of turbulence, but on the fraction of turbulent energy associated with fast modes. The coupling and energy transfer between Alfvén, slow, and fast modes are suppressed (Cho & Lazarian 2002, 2003).

The two regimes of superdiffusion are confirmed by numerical simulations in Xu & Yan (2013) where data cubes obtained by MHD turbulence simulations were used to study particle propagation. We note that superdiffusion cannot be described by a diffusion coefficient. We also note that the while talking about perpendicular diffusion of CRs one should specify whether the diffusion is considered with respect to the mean field or to the local field. We showed in this paper (Section 3.3) that the results can be very different in these two cases.

8.4. Acceleration and Superdiffusion of Charged Dust

Charged dust particles behave similar to CRs, but their velocities and the charge to mass ratios are enormously different. They, unlike CRs, can also be accelerated by hydromagnetic drag (Lazarian & Yan 2002). The theories of dust acceleration by magnetic fluctuations and hydrodynamic turbulent motions have been developed in a number of papers (Yan & Lazarian 2003; Yan et al. 2004; Yan 2009; Hoang et al. 2012). The stochastic acceleration of dust particles by turbulence can pre-accelerate them prior to the acceleration in shocks. Similar to the case of CRs, for dust acceleration and propagation the effects of superdiffusion can be important though the scale range that superdiffusion applies is smaller than the case for CRs because of the larger radii of dust grains.

8.5. CR Acceleration in Shocks and Sites of Turbulent Reconnection

The process of turbulent reconnection induces the first-order Fermi CR acceleration (de Gouveia dal Pino & Lazarian 2005; Lazarian 2005; Kowal et al. 2011, 2012a). CRs in the reconnection region move along the contracting loops gaining energy in a regular way. Unlike the case of shock acceleration, the distribution of particles is required to be anisotropic for the process to proceed efficiently.

The superdiffusion of CRs introduces an additional channel of losses from the reconnection region, similar to the process that we described in Section 6.3. These losses are expected to increase with the increase of the Alfvén Mach number of turbulence within the reconnecting magnetic fluxes. A more detailed discussion of the effect requires further study and is beyond the scope of the present paper.

In this paper, we noted a possible analogy between CR acceleration in reconnection sites and shocks. Indeed, as we discussed in Section 7, CRs freely streaming along entangled magnetic field lines at small scales can experience fast acceleration in the shocks. The parallel component of CR momentum increases due to the effective decrease of a magnetic loop subjected to the shock compression. In reconnection sites, magnetic reconnection also decreases the length of magnetic field lines inducing parallel acceleration. A drift acceleration is present for the perpendicular component in both cases as well. Both processes of acceleration require further study.

9. SUMMARY

In this paper, we considered MHD turbulence driven isotropically at the injection scale $L$ and considered the propagation of CRs at different regimes. Our findings are as follows.

1. At scales less than the turbulence injection scale, the perpendicular dynamics of CRs is super-diffusive, the separation between CRs grows faster than the square root of time. This is not related to the hypothetical Levi flight behavior, but is due to the well-established divergence of magnetic field lines related to the process of Richardson diffusion. As the injection scales of turbulence in galaxies may be equal to or larger than 100 pc, the superdiffusion must be accounted in the models of CR propagation and acceleration.

2. The superdiffusion in the case of ballistic propagation, i.e., on scales less than the mean free path of a CR, induces the CR separation that is similar to the separation of magnetic field lines and grows as time to the power of $3/2$. On scales larger than the mean free path, the CR separation grows as time to the power of $3/4$. At scales larger than the turbulent injection scale CRs exhibit diffusive behavior.

3. In contrast to superdiffusion, the subdiffusion is an extremely special improbable phenomenon that, as we discuss in the paper, requires very special conditions.

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22 The static magnetic field assumption is not true for relativistic environments, but we do not deal with such cases in this paper. For other situations when superdiffusion is important, the dynamics of magnetic field lines can be essential. For instance, for the process of the removal of magnetic flux from molecular clouds and disks via reconnection diffusion (see Lazarian et al. 2012a), the dynamics of magnetic field is essential. In other processes, e.g., thermal conduction of plasma in galaxy clusters, both the motion of electrons along the fast diverging magnetic field lines and the dynamics of magnetized eddies are important (see Lazarian 2006).

23 The effects of compression due to reconnection are subdominant and the calculations in Cho & Lazarian (2006) show that the acceleration efficiency in incompressible flows decreases dramatically in the presence of efficient scattering.
4. The superdiffusion changes the properties of the acceleration of CR acceleration in shocks. In particular, the superdiffusion diminishes the difference that is present between the parallel and perpendicular shocks.

5. The process of CR acceleration for the shocks with the precursor with small-scale magnetic field may be efficient for the CRs ballistically moving along turbulent magnetic field lines.

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APPENDIX A

QUANTITATIVE DESCRIPTION OF THE SPATIAL DIVERGENCE OF MAGNETIC FIELD LINES

To describe the divergence of magnetic field lines, it is convenient to define a “two-line diffusivity” (ELV11)

\[
D_{ij}^B(\ell) = \int_{-\infty}^{0} ds \langle \delta \hat{b}_i(0) \delta \hat{b}_j(\ell, s) \rangle, \tag{A1}
\]

where \(\delta \hat{b}_i(\ell, s) = \delta \hat{b}_i(\xi(s)) - \hat{b}_i(\xi(s))\) with \(\ell = \xi(s) - \xi(0)\). Thus, the corresponding diffusion tensor is

\[
\frac{d}{ds} \langle \ell_i(s) \ell_j(s) \rangle = \langle D_{ij}^B(\ell) \rangle. \tag{A2}
\]

In agreement with LV99 theory of magnetic field wandering, the integrand in Equation (A1) for field-perpendicular increments can be presented as

\[
\langle \delta \hat{b}_i(0) \delta \hat{b}_j(\ell, s) \rangle \sim \frac{\delta u^2(\ell)}{v_A^2} \Re \left[ e^{i\delta \ell / \ell_{ij}} \right], \tag{A3}
\]

where \(\langle \delta \hat{b}_i(0) \delta \hat{b}_j(\ell, s) \rangle \sim \delta B(\ell)/B_0 \sim \delta u(\ell)/v_A\).

Consider the two terms in the \(s\)-dependent part. It is easy to see that the imaginary exponent represents Alfvénic oscillations with the scale \(\ell_{ij}\), where \(\ell_{ij}\) is the parallel length scale of GS95 eddies with the parallel and perpendicular dimensions \(\ell_{\parallel}\) and \(\ell_{\perp}\), respectively (see Section 2).

The other term includes a correlation length \(\lambda(\ell)\) of tangent-vector increments along the field line. This term describes an exponential decay of correlations along the field. In LV99 theory \(\lambda(\ell)\) corresponds to the distance traveled by an Alfvénic perturbation along the field-line with velocity \(v_A\) during the cascading time for turbulence at scale \(\ell_{\perp}\). In other words,

\[
\lambda(\ell) = v_A \tau_\ell = v_A \frac{\delta u^2_{\perp}}{\varepsilon}. \tag{A4}
\]

As Equation (A1) can be written as

\[
D_{ij}^B(\ell) \sim \delta \hat{b}_i(\ell) \delta \hat{b}_j(\ell) s_{\text{int}}(\ell), \tag{A5}
\]

where \(s_{\text{int}}(\ell)\) is an integral correlation length of the increment in the tangent vector along the lines. One should deal with the properties of \(s_{\text{int}}(\ell)\) in order to describe the line separation.

Integrating Equation (A3) in \(s\) gives the following result

\[
s_{\text{int}}(\ell) \sim \frac{1/\lambda(\ell)}{1/\lambda^2(\ell) + 1/\ell_{ij}^2} \sim \frac{\ell_{ij}^2}{\lambda(\ell)} = \frac{\varepsilon}{v_A} \frac{\ell_{ij}^2}{\ell_\perp^2}, \tag{A6}
\]

for \(\lambda(\ell) \gg \ell_{\parallel}\). Substituting into Equation (A5), one obtains

\[
D_{ij}^B(\ell) \sim \frac{\varepsilon \ell_{ij}^2}{v_A^3} = \frac{\ell_{ij}^2}{L M_A^4}, \tag{A7}
\]

where the factor of \(\delta u^2(\ell)\) is canceled.

Consider different regimes of turbulence. In the strong GS95 turbulence regime, the condition of critical balance requires that \(\lambda(\ell) \sim \ell_{\parallel}\) and thus, from Equation (A6), \(s_{\text{int}}(\ell) \sim \ell_{\parallel}\). Thus one can write

\[
\frac{d}{ds} \ell_{ij}^2 \sim D_{ij}^B(\ell) \sim (\delta u/v_A)^2 \ell_{\parallel} \sim L \left( \frac{\ell_{\perp}}{L} \right)^{4/3} M_A^{-4/3}, \tag{A8}
\]

where we have substituted from Equations (A4) and (2) for \(\ell_{\parallel}\) and \(\delta u/v_A\).

In the weak turbulence regime, one should use \(\ell_{\parallel} = L\) as a constant, which, when substituted into Equation (A7), gives

\[
\frac{d}{ds} \ell_{ij}^2 \sim D_{ij}^B(\ell) \sim L M_A^4, \tag{A9}
\]

and this is solved to give the result corresponding to the ordinary diffusion

\[
\ell_{ij}^2 \sim s L M_A^4. \tag{A10}
\]

These results are used in the main part of the paper.

APPENDIX B

SUBDIFFUSION AS AN UNLIKELY PROCESS

Consider first the scales less that the viscous dissipation scale. For magnetic field to be present at these scales, the viscosity of the fluid should be larger than the resistivity, i.e., the fluid should have high Prandtl number. At scales less than \(l_{\min}\), the magnetic field is stirred by the turbulence of larger eddies. The most important fast steers at the marginally damped eddies of the size \(l_{\min}\). These are still Alfvénic anisotropic eddies and therefore it is appropriate to characterize them by two distinct scales, the perpendicular scale \(l_{\perp}\) and \(l_{\parallel}\), where \(l_{\perp} < l_{\parallel}\). The eddy-type motions correspond to motions perpendicular to the local direction of magnetic field, and thus \(l_{\perp} \ll l_{\parallel}\), which is confirmed by the single scale of the driving in the Rechester & Rosenbluth theory (Narayan & Medvedev 2001; Lazarian 2006). In that theory, it was calculated the path length of the particles for them to get separated by the driving scale and therefore lose the ability to retrace their trajectories. The corresponding Rechester & Rosenbluth scale is given by the expression in YL08:

\[
L_{RR} = l_{\parallel} \ln(l_{\perp}/r_{CR}), \tag{B1}
\]

where the only difference from the corresponding expression in Lazarian (2006) is its use of the Larmor radius of CR, \(r_{CR}\), instead of the thermal electron Larmor radius in the latter work. Due to the slow growth of the logarithm, \(L_{RR}\) in Equation (B1)
is of the order of $L_{\min}$. Therefore it follows from Equation (B1) that there can be diffusion of magnetic field lines with a spatial diffusion coefficient $D_{\text{spat}} = \delta t_{\min}^2 / \delta s$. However, parallel transport of particles below $L_{\min}$ is unlikely to be diffusive lacking of perturbations to scatter. This makes the subdiffusion process rather exotic and improbable below $L_{\min}$.

Consider now whether the process of subdiffusion is possible for scales larger than $L_{\max}$. Naturally, if one considers CRs with the Larmor radius $r_L$ less than $L_{\max}$, such CRs will trace the super diffusing magnetic field lines and the retraction with such CRs is impossible. However, if $r_L$ is larger than $L_{\max}$ the CR would trace large-scale magnetic fields undergoing random walk (see Figure 1 for large scales). For instance, for $M_s < 1$ a CR with $r_L > L_{\text{trans}}$ would interact with perturbations arising from weak turbulence while moving along the mean magnetic field. Such perturbations, according to Equation (11), produce random walk spatial diffusion with the step $L$. While the CR can also become scattered by magnetic perturbations along the mean field. In this case, we do have a case of subdiffusion. The scattering of the CR may arise from, e.g., CR instabilities, e.g., from a streaming instability.

All in all, due to the Richardson diffusion of magnetic field lines, the process of subdiffusion is not possible over the range of scales corresponding to the inertial range of turbulence. The subdiffusion is extremely unlikely for scales less than the scale of viscosity damped eddies, but still possible for the CRs of scales corresponding to the inertial range of turbulence.

APPENDIX C

WEAK TURBULENCE AND ACCELERATION OF FREE STREAMING PARTICLES IN A SHOCK

The case of the original idea in Jokipii (1987) works if the small-scale weak Alfvénic turbulence is generated with the injection scale much less than the scale of the system. Such weak turbulence would induce random walk displacements according to Equation (11) and can provide multiple crossings of the shock front. CR instabilities, e.g., CR streaming, can produce waves, which, while being scattered back through the parametric instability or through the reflection from density inhomogeneities pre-existing in the turbulent pre-shock environment, produce weak turbulence at small scales.24 As we discussed in Section 2, the weak and strong MHD turbulences do not reflect the amplitude of Alfvénic perturbations, but only the strength of nonlinear interactions. Therefore the amplitude of magnetic perturbations arising from weak turbulence may substantially exceed, on the small scales that we are dealing with, the amplitude of perturbations arising from the large scale strong Alfvénic turbulence.

An interesting feature of this scenario is that the weak turbulence produced by CRs is competing with the strong pre-existing turbulence in the upstream. The latter induces Richardson explosive separation of magnetic field lines according to Equation (10) and this way decreases the possibility of a magnetic field line to re-enter the shock, while the former induces random walk wandering of magnetic field lines according to Equation (11), thus helping the magnetic field line to re-enter the shock many times. Note that the injection scales in Equations (10) and (11) are very different. For the strong pre-existing turbulence the injection scale is determined, e.g., by the large-scale steering of the interstellar gas, while for weak turbulence generated by CRs the injection scale is the scale of the perturbations created by the instabilities may be of the order of the gyro-radius $r_L$.

The above scenario has obvious limitations. A CR’s streaming instability may be the primary source of perturbations. However, if a magnetic field line enters many times the shock, this creates many points of injection over the length of the magnetic field line, which limits imbalance in the flow of oppositely moving CRs. Potentially, the gyro-resonance instability (see Kulsrud 2005) arising from the anisotropic distribution of accelerated particles of the momentum space can also produce waves and thus generate weak turbulence. We do not attempt to quantify these possibilities here, but want to mention that in this scenario the possibility that CRs of low and high energies are experiencing different types of acceleration. The low-energy ones create the instability and get scattered and reflected back as in a usual picture of parallel shock acceleration and at the same time create weak turbulence that induces magnetic field random walk. The instability for higher energy particles may experience turbulence suppresses the streaming instability by the ambient turbulence (Yan & Lazarian 2002; Farmer & Goldreich 2004; Beresnyak & Lazarian 2008).25 Therefore, such particles can stream freely along the magnetic field lines, which, due to weak turbulence, undergo random walk and exhibit multiple entries of the shock as suggested in Jokipii (1987).

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24 We note that to have Alfvénic turbulence rather than Alfvénic waves, the colliding Alfvénic packets should move in the opposite directions.

25 This is another process of a competition between the pre-existing turbulence in the pre-shock and the turbulence generated by cosmic rays.
This is a small correction to the published article in relation to the particular regime of CR diffusion, namely, the diffusion in the superAlfvenic turbulence. This correction makes the discussion in the paper self-consistent.

In the published article, Equations (12) and (13) are wrong. In fact, at scales larger than the Alfvenic scale \( l_A \) magnetic fields are entangled by turbulence and therefore the separation of magnetic field lines is a random walk process with the step \( l_A \). Therefore, the mean squared separation between the magnetic field lines \( l_\perp^2 \) is increasing with the distance tracked along the magnetic field line \( s \) as

\[
l_\perp^2 \sim sl_A.
\]

Therefore, for the scales \([l_A, L]\), magnetic field lines undergo diffusion and the transport of cosmic rays that stream along magnetic field is diffusive on scales larger than \( l_A \). As a result, Table 1 in the paper should be modified as we present below.

This mistake in the paper does not change any conclusions or results of the aforementioned work because cosmic ray propagation and acceleration in the limit of superAlfvenic turbulence considered in Section 7 used the correct diffusion coefficient given by Equation (39), which is consistent with our Equation (1) above. Equation (39) was obtained from physical considerations and not derived from Equations (12) and (13). In other words, the published article contained a contradiction and the present erratum removes this contradiction.

### Table 1

| Type of MHD turbulence | Injection velocity | Range of scales | Spectrum E(k) | Motion type | Ways of study | Magnetic diffusion | Squared separation of lines |
|------------------------|--------------------|----------------|--------------|-------------|---------------|--------------------|-----------------------------|
| Weak                   | \( V_L < V_A \)    | \([l_{\text{min}}, L]\) | \( k_\perp^2 \) | wave-like   | analytical    | diffusion          | \( \sim sL_M^2 \)          |
| Strong subAlfvenic     | \( V_L < V_A \)    | \([l_{\text{min}}, l_{\text{trans}}]\) | \( k_\perp^{-5/3} \) | anisotropic | numerical      | Richardson          | \( \sim sL_M^2 \)          |
| Strong superAlfvenic   | \( V_L > V_A \)    | \([l_{\text{trans}}, L]\) | \( k_\perp^{-5/3} \) | isotropic   | numerical      | diffusion          | \( \sim sL_M^2 \)          |
| Strong superAlfvenic   | \( V_L > V_A \)    | \([l_{\text{min}}, l_A]\) | \( k_\perp^{-5/3} \) | anisotropic | numerical      | Richardson          | \( \sim sL_M^2 \)          |

Note. \( L \) and \( l_{\text{min}} \) are the injection and perpendicular dissipation scales, respectively. \( M_A \equiv \delta B/B, l_{\text{trans}} = L M_A^2 \) for \( M_A < 1 \) and \( l_A = L M_A^3 \). for \( M_A < 1 \). For weak Alfvenic turbulence, \( \xi \) does not change. \( s \) is measured along magnetic field lines.
Erratum: “Superdiffusion of Cosmic Rays: Implications for Cosmic Ray Acceleration” (2014, ApJ, 784, 38)

A. Lazarian\textsuperscript{1} and Huirong Yan\textsuperscript{2,3} \textsuperscript{©}

\textsuperscript{1} Department of Astronomy, University of Wisconsin, 475 North Charter Street, Madison, WI 53706, USA
\textsuperscript{2} DESY, Platanenallee 6, D-15738 Zeuthen, Germany
\textsuperscript{3} Institut für Physik und Astronomie, Universität Potsdam, D-14476 Potsdam-Golm, Germany

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An error is found in Table 1 of the published article describing the regimes of diffusion and superdiffusion. For the weak turbulence when the scales $s$ measured along the magnetic field are larger than the turbulence injection scale $L$, the separation of magnetic field lines increases in a diffusive way with the step $LM_A^4$, where $M_A$ is the Alfvén Mach number, i.e., $M_A = V_L/V_A$, where $V_L$ is the turbulent injection velocity and $V_A$ is the Alfvén velocity. We provide Table 1 with the error corrected here.

The separation of magnetic field lines was studied in Lazarian & Vishniac (1999). There it was found there that for the weak turbulence at scales $s \ll L$, where $s$ is measured along the magnetic field and $L$ is the injection scale of turbulence the separation is growing as a random walk with the step $LM_A^4$, where $M_A$ is the Alfvén Mach number. This result was used in further publications, e.g., in Lazarian (2006) to study the propagation of heat in clusters of galaxies. Unfortunately, due to an error the corresponding entry in our published article is incorrect. We provide the corrected version of the Table 1 here describing the separation of magnetic field lines for different regimes of MHD turbulence.

### Table 1

| Type of MHD Turbulence | Injection Velocity | Range of Scales | Spectrum $E(k)$ | Motion Type | Ways of Study | Magnetic Diffusion | Squared Separation of Lines |
|------------------------|--------------------|----------------|----------------|-------------|---------------|---------------------|--------------------------|
| Weak                   | $V_L < V_A$        | $[l_{\text{trans}}, L]$ | $k_s^{-2}$ | wave-like   | analytical    | diffusion           | $\sim sLM_A^4$          |
| Strong subAlfvénic     | $V_L < V_A$        | $[l_{\text{min}}, l_{\text{trans}}]$ | $k_s^{-5/3}$ | anisotropic | numerical    | Richardson           | $\sim sM_A^4$           |
| Strong superAlfvénic   | $V_L > V_A$        | $[l_A, L]$      | $k_A^{-5/3}$ | isotropic   | numerical    | diffusion           | $\sim s l_A$            |
| Strong superAlfvénic   | $V_L > V_A$        | $[l_{\text{min}}, l_A]$ | $k_A^{-5/3}$ | anisotropic | numerical    | Richardson           | $\sim sM_A^4$           |

Note. $L$ and $l_{\text{min}}$ are the injection and perpendicular dissipation scales, respectively. $M_A \equiv \ell B/B_L$, $l_{\text{trans}} = LM_A^2$ for $M_A < 1$ and $l_A = LM_A^3$ for $M_A < 1$. For weak Alfvénic turbulence $l_\ell$ does not change. $s$ is measured along magnetic field lines.

### References

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