Search techniques for gravitational waves from black-hole ringdowns

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Of all the astronomical sources of gravitational radiation, the ringdown waveform arising from a small perturbation of a spinning black hole is perhaps the best understood: for the late stages of such a perturbation, the waveform is simply an exponentially-damped sinusoid. Searching interferometric gravitational wave antenna data for these should be relatively easy. In this paper, I present the results of a single-filter search for ringdown waveforms arising from a 50 solar mass black hole with 98% of its maximum spin angular momentum using data from the Caltech 40-meter prototype interferometer. This search illustrates the techniques that may be used in analyzing data from future kilometer-scale interferometers and describes some of the difficulties present in the analysis of interferometer data. Most importantly, it illustrates the use of coincident events in the output of two independent interferometers (here simulated by 40-meter data at two different times) to substantially reduce the spurious event rate. Such coincidences will be essential tools in future gravitational wave searches in kilometer-scale interferometers.

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I. INTRODUCTION

The first kilometer-scale gravitational wave observatories, LIGO and VIRGO, are expected to begin performing searches for astrophysical sources of gravitational waves in the year 2002. By this time, it will be necessary to have data analysis software for on-line searches. Because the anticipated signals are expected to be weak, near-optimal strategies for data analysis will be required in order to extract gravitational wave signals from the interferometer noise.

An optimal technique when the waveform of the signal is known in advance is the method of matched filtering. The statistical properties of the matched filter may be characterized when the detector noise is stationary and Gaussian. However, in order to understand the behavior of the matched filter for a real detector, it is necessary to apply the method to a problem of data analysis that resembles, in some way, the expected non-stationary and non-Gaussian LIGO and VIRGO interferometer data. In this paper, I report the results of analyzing data acquired from the Caltech 40-meter prototype interferometer in November 1994. The noise spectrum of the prototype instrument had some of the same characteristics of the spectra expected in LIGO and VIRGO: at low frequency it was dominated by seismic ground motion and at high frequency it became photon shot-noise limited. However the 40-meter prototype experienced many still uncharacterized non-Gaussian transient disturbances. In this respect, the 40-meter prototype is expected to be much worse than the LIGO and VIRGO interferometers. Nevertheless, the prototype forms a useful test-bed for development of robust algorithms that may be used with some degree of confidence in analyzing the data of future instruments.

The data analysis problem I present is a search for ringdown gravitational waves emitted from a distorted spinning black hole. Such systems will form as a final state in the merger of black-hole binaries. These ringdown waves are important candidates for detection by the kilometer-scale interferometers for sufficiently massive (50$M_\odot \lesssim M \lesssim 500M_\odot$) black holes; for such masses, the ringdown waves lie in the frequency band of greatest LIGO sensitivity, $\sim 30$–300 Hz. Flanagan and Hughes have shown that black-hole ringdowns are detectable to distances of $\sim 10$ Mpc–1 Gpc for the initial LIGO interferometers. For the Caltech 40-meter prototype, black-hole ringdown waveforms could be detected in our Galaxy (see the Appendix and Fig. 1). Because the ringdown waveform is an exponentially-damped sinusoid, the construction of filters to search for such a signal is simple. A search for ringdown waveforms, therefore, provides a useful test of the application of matched filtering to interferometer data.

Damped sinusoids are also expected to be produced by a variety of spurious events in the interferometer. For this reason, one would not conclude that any particular event recorded by a matched filter necessarily corresponds to a gravitational wave, no matter how large the event is, unless there is sufficient corroborating evidence. Possible types of corroborating evidence include the presence of a preceding inspiral and merger waveform, which would be expected for the coalescence of a binary system, and a coincident event that is detected in another interferometer. A search for coincident events in detectors with independent noise is an essential technique for discriminating between instrumental effects (which are unlikely to produce coincident events) and gravitational wave signals (which are expected to produce coincident events). In addition to these methods of corroboration, a full-scale search for ringdown waveforms would involve additional vetoes based on environmental monitors and on a catalog of known instrumental effects. In this paper, I will concentrate solely on the use of a coincident strategy to
discriminate between ringdowns arising from spurious instrumental effects and (simulated) gravitational waves.

The purposes of this paper are (i) to present the data analysis for a search for black-hole ringdowns in data obtained from the Caltech 40-meter interferometer, with particular emphasis on the use of coincidences between two detectors to deal with an excess of false alarms produced by the non-Gaussian component of the detector noise; and (ii) to present the details and results of an actual search limited to a single template for a black hole with a particular mass and spin. A more general search would use a bank of filters to cover a range of possible black-hole masses and spins. Furthermore, a more general ringdown search would incorporate many more discriminants based on a better characterization of the detector noise. For this reason, the search presented in this paper should be regarded as an exploratory search that implements the techniques of optimal filtering and coincidence detection.

I start with a discussion of the statistical problem of signal extraction for the anticipated ringdown waveform. For this statistical analysis, it is common to make the assumptions that the detector noise is a stationary and Gaussian random process\(^3\): I will make similar assumptions, though they are not valid for the Caltech 40-meter data. For this reason, I describe a technique one can use to deal with non-Gaussian components of the detector noise, if they are sufficiently infrequent, given two detectors with independent noise. I then present the results of an application of the matched filter data analysis technique to the Caltech 40-meter data and describe the implications of the results for data analysis strategies. I simulated a coincidence search by dividing the 40-meter data into two sets, each set treated as if it came from a separate instrument with independent noise. In the Appendix, I describe some details about the ringdown waveform and the number of ringdown filters that would be required in order to search for all possible ringdowns in the bandwidth of the 40-meter prototype.

**II. DETECTION STRATEGY**

In order to detect the ringdown from a black hole, one must pass the interferometer output through a receiver that will perform some test of the hypotheses “there is a signal present in the data” \((H_1)\) and “there is no signal present in the data” \((H_0)\). In order for the receiver to conclude that there is a signal present, it constructs some statistic and compares the statistic to some pre-assigned threshold. The problem of reception, then, is two-fold: one must choose some optimal statistic, and one must select some threshold. I will explore these problems below. The discussion in this section follows the method presented in Refs.\(^3\)-\(^4\).

In designing the optimal statistic, I will assume, in this section, that the noise present in the detector is a stationary Gaussian process; this assumption simplifies the statistical analysis. However, it is known that the Caltech 40-meter interferometer data have significant non-stationary and non-Gaussian noise components, so the optimal statistic will not have the properties expected of stationary Gaussian noise. The effect of the non-stationary and non-Gaussian noise components will be evident in the observations presented in Sec.\(^3\).

### A. The matched filter

Suppose that the detector strain output \(h(t) = h_{\text{GW}}(t) + n(t)\) consists of stationary Gaussian noise \(n(t)\) and, under hypothesis \(H_1\), a strain caused by a gravitational wave \(h_{\text{GW}}(t) = Aq(t)\) of known form. The waveform \(q(t)\) is the expected gravitational waveform at some fiducial distance so that the amplitude \(A\) represents the inverse distance of the source in units of this fiducial distance. The stationary (colored) Gaussian noise \(n(t)\) can be characterized entirely by its one-sided noise power spectrum \(\tilde{h}(f)\tilde{n}^*(f') = \frac{1}{2} S_h(|f|)\delta(f - f')\) where \(\tilde{n}(f)\) is the Fourier transform of the noise and \(*\) denotes complex conjugation. If the noise is known to be a stationary Gaussian process, then the matched filter statistic

\[
x = (h \mid q) = \int_{-\infty}^{\infty} df \frac{\tilde{h}^*(f)\tilde{q}(f) + \tilde{h}(f)\tilde{q}^*(f)}{S_h(|f|)},
\]

is the optimal statistic. That is, the probability of \(H_1\) given the detector output, \(P(H_1 \mid h)\), depends on the output only through the matched filter statistic \(x\), and the probability is a monotonically increasing function of \(x\). To decide between the two hypotheses, one selects some threshold \(x_*\) and accepts hypothesis \(H_1\) if \(x > x_*\), otherwise one accepts hypothesis \(H_0\). For one to choose the threshold in terms of some desired probability \(P(H_1 \mid h)\) requires that one know the prior probability of hypothesis \(H_1\). The prior probability for \(H_1\) is usually subjective; however, one often wants an objective way of choosing the threshold \(x_*\). In this case, one usually constructs a threshold based on the false alarm probability \(Q_0\) or the true detection probability \(Q_1\). (The true detection probability is the converse of—i.e., one minus—the false dismissal probability.)

Two important properties of the matched filter statistic are the false alarm and true detection probabilities, which are obtained from the probability distribution for the matched filter under the two hypotheses. Under hypothesis \(H_0\), the probability distribution for \(x\) is a Gaussian distribution with zero mean and variance \(\sigma^2 = (q \mid q)\) where the inner product is obtained from replacing \(h\) with \(q\) in Eq.\(^1\)). However, when a signal is present with amplitude \(A\), the mean of this distribution is shifted to \(A\sigma^2\). It will be useful to consider the signal-to-noise ratio \(\rho = |x|/\sigma\) (the absolute value is taken because it is not known whether the signal will have a positive or
negative amplitude). The false alarm and true detection probabilities are

\[ Q_0 = P(\rho > \rho_* \mid H_0) = \text{erfc}(\rho_*/\sqrt{2}) \] (2)

\[ Q_1 = P(\rho > \rho_* \mid H_1) = \frac{1}{2} \text{erfc}((\rho_* - A\sigma)/\sqrt{2}) + \frac{1}{2} \text{erfc}((\rho_* + A\sigma)/\sqrt{2}) \] (3)

where \( \rho_* \) is the signal-to-noise ratio threshold and the complementary error function is defined by \( \text{erfc}(x) = 2\pi^{-1/2} \int_x^\infty e^{-t^2} dt \). Using these equations, one can choose a threshold for a desired false alarm or true detection probability.

In the above discussion, I have made the implicit assumption that one knows the arrival time of the signal. Since this will not be known in general, it is necessary to obtain the matched filter statistic \( x(t_{\text{arr}}) \) as a function of possible arrival times \( t_{\text{arr}} \). This can be accomplished by replacing \( q(t) \) with \( q(t - t_{\text{arr}}) \) or, equivalently, by replacing \( \tilde{q}(f) \) with \( q(f)e^{2\pi ift_{\text{arr}}} \) in Eq. (4). Whenever the signal-to-noise ratio exceeds the threshold, a candidate event has occurred. Because of correlations in the time series \( x(t_{\text{arr}}) \), a single ringdown waveform may cause several threshold crossings at slightly different (but closely spaced) arrival times; thus, it is necessary to maximize the signal-to-noise ratio over events with similar arrival times.

The situation is further complicated if the waveform depends on parameters other than the time of arrival. In order to search over these parameters, it is necessary to construct a bank of filters that span the parameter space. Thus, if \( \{\tilde{q}_i(t)\} \) for \( i = 1, \ldots, N_{\text{filters}} \) is a set of \( N_{\text{filters}} \) filters that cover the parameter space, then one has a set of matched filter time series \( \{x_i(t_{\text{arr}})\} \).

Because there are correlations between the different outputs \( \{x_i(t_{\text{arr}})\} \) at a given arrival time as well as within each time series itself, it is difficult to estimate the rate of false alarms for the filter bank. An overestimate can be obtained by assuming that each filter and each arrival time is independent. Then the rate of false alarms is

\[ R_0 = N_{\text{filters}} \Delta^{-1} \text{erfc}(\rho_*/\sqrt{2}), \] (4)

where \( \Delta^{-1} \) is the sampling rate of the detector. A more accurate estimate could be obtained by a Monte Carlo simulation of the detection process in the presence of Gaussian noise alone.

The probability of true detection in the presence of a bank of time-series filters must be estimated by Monte Carlo methods. However, an underestimate of this probability can be made by assuming that the detection can only occur at the correct time and with the correct filter; then the probability given in Eq. (4) can be used. This estimate is good for signals with large amplitudes, but it becomes a significant underestimate when the probability of a false alarm becomes comparable to the probability of true detection.

A calculation of the number of filters \( N_{\text{filters}} \) necessary for a search for all possible black-hole ringdown waveforms within the bandwidth of the Caltech 40-meter prototype is given in the Appendix. For the present work, however, I will report on the computationally easier problem of a single filter search of the 40-meter data, which captures most of the issues that arise due to the non-Gaussian nature of the 40-meter noise. The results of a full filter bank search will be reported elsewhere [1].

B. Dealing with non-Gaussian noise

It is unlikely that all possible sources of non-Gaussian instrumental noise will be eliminated from the interferometers, and even if most of the remaining non-Gaussian components can be removed from the data before analysis, there will likely be some non-Gaussian processes occurring at some (hopefully small) rate. It is important to have relatively robust methods for dealing with such events because they will likely occur more frequently than the desired false alarm event rate. I will discuss a very simple method for “removing” the non-Gaussian events based on coincidence between two detectors in a network. Such coincidences have long been central to data analysis for resonant-mass gravitational-wave detectors (see, e.g., Ref. [2]) and were regarded as essential in the original planning of LIGO [3]. That is why the LIGO network will consist of two full-sized detectors that will have largely independent noise (since they will be separated by 3002 km), plus an additional half-sized detector at one of the two sites. VIRGO could provide another full-sized detector to the network, and since it will be located on a different continent, it may have almost entirely independent noise.

Suppose, for simplicity, that there are only two detectors available, and that the detectors have the same sensitivity and independent noise. Also, suppose that one expects that the arrival times of a signal will agree to within some time \( \pm \Delta \tau \), which will be approximately the light travel time between the two detectors (\( \pm 10\text{ms} \) between the LIGO sites). One can then adopt the following naive detection strategy: accept an event as a candidate signal if it occurs with a signal-to-noise ratio greater than \( \rho_* \) in both detectors, and the times of arrival agree to within \( \pm \Delta \tau \). If the threshold \( \rho_* \) corresponds to a false alarm rate of \( r_0 \) in each detector, and if the noises in the two detectors are independent,

*Clearly, this is a crude strategy which can be improved by a more careful likelihood analysis. Such an analysis has been done by Finn [14], in which the outputs of all detectors are jointly analyzed. The naive strategy might be used as a first stage in a hierarchical strategy that will combine detector output only after first identifying candidate signals.
then the fraction of false alarms that are coincident will be $R_{\text{0 single}} \times 2\Delta \tau$, and the overall rate of coincident false alarms is $R_{\text{0 coincident}} = (R_{\text{0 single}}^2)^2 \times 2\Delta \tau$. (The factor of two accounts for the fact that both positive and negative time delays of up to $\Delta \tau$ are allowed.) In practice, this means that the threshold should be set at a level such that the false alarm rate in the individual detectors is

$$R_{\text{0 single}} = (R_{\text{0 coincident}} / 2\Delta \tau)^{1/2}$$

(5)

for some desired overall coincident false alarm rate. For example, for the two full-sized LIGO interferometers (not including the half-sized interferometer at Hanford) and an overall false alarm rate of one per ten years, the individual thresholds should be set so that the individual detector false alarms occur less frequently than one every 1.2 hours. Such a false alarm event rate for the individual detectors may be larger than the rate of non-Gaussian events; if this is indeed the case, then the thresholds $\rho_*$ can be set to the desired false alarm rates based on the results of Sec. II A, which assumed Gaussian noise alone.

The overall coincident true detection probability will be the product of the true detection probabilities from the two interferometers, i.e., $Q_1^{\text{coincident}} = (Q_1^{\text{single}})^2$ if the two detectors have roughly equal true detection probabilities for a given signal. Thus one requires a smaller signal-to-noise ratio threshold in each of the two detectors for a coincident search than for a single detector search in order to achieve the same true detection probability. A useful measure of the effectiveness of a search strategy is the maximum effective observing time $T_{\text{eff}} = Q_1 / R_0$. This is the amount of time one can observe before encountering a false alarm, times the efficiency of detecting a signal. By multiplying this effective time by the rate of astronomical events, one obtains the expected number of observed events before a false alarm occurs. Thus, the rate of astronomical events should be much greater than $1/T_{\text{eff}}$ in order to cleanly distinguish them from false alarms.

One finds that a coincident search strategy is better—i.e., yields a larger value of $T_{\text{eff}}$—than a single detector search provided that $Q_1^{\text{single}} > R_0^{\text{single}} \times 2\Delta \tau$. This inequality will be satisfied for any realistic search for gravitational waves since it merely requires that the probability of detecting an event, when present, exceeds the probability of a false alarm in the short time $2\Delta \tau$.

Unlike the anticipated LIGO and VIRGO detectors, the Caltech 40-meter prototype is a research and development instrument whose configuration is frequently changed without taking the time to completely eliminate or characterize the sources of non-Gaussian noise. The result is that the Caltech 40-meter data tend to have many non-Gaussian events and also tend to be non-stationary. As will be shown below, this means that the false alarm probability distribution does not agree with the predictions made in Sec. II A. Nevertheless, if two 40-meter-like interferometers with independent noise were available, it would still be possible to achieve a desired overall false alarm rate using the argument above but where the individual false alarm rate distributions would have to be measured in order to set the threshold. Because the required individual false alarm rates $R_0^{\text{single}}$ in LIGO may be on the order of one per hour (so that the coincident false alarm rate would be on the order of one per ten years if the two interferometers were separated by 3002 km), it will be easy to determine the necessary threshold $\rho_*$ by analyzing several hours of data. If one was not able to do a coincident search and if one required a false alarm rate of one per year, then one might hope to set the correct threshold by analyzing several years worth of data in order to be sure to set the correct threshold level. Unfortunately, this could well be insufficient because it is not possible to screen out gravitational wave signals in order to determine the properties of the noise alone. Thus, unless the properties of the detector noise are known exactly, the rate of false alarms cannot be estimated for a single detector, but the rate can be estimated for a coincident search using multiple detectors.

### III. SINGLE FILTER SEARCH

Having reviewed a possible detection strategy and stated the expected distribution of the detection statistic in the presence of stationary Gaussian noise, I now examine the results of applying the detection strategy to real interferometer data. In November 1994, the Caltech 40-meter prototype interferometer was used to collect approximately 46 hours of data. In my analysis of these data, I implemented the detection strategy discussed in the previous section (for a single filter only) using code that is provided in the GRASP data analysis software package [1]. I also made extensive use of the Numerical Recipes library [2] and the FFTW Fourier transform routine [3].

I analyzed the data by correlating them with a single filter function corresponding to the fundamental quadrupole quasinormal mode of a Kerr black hole with a mass $M = 50 M_\odot$ and a spin $\hat{a} = 98\%$ of the maximum spin. This mode is a damped sinusoid with a central frequency of $f \approx 520$ Hz and a quality factor of $Q \approx 12$ (see the Appendix). The central frequency of the ringdown is within the frequency band of the instrument; between approximately 100 and 5000 Hz (see Fig. 2). The filter was constructed in the frequency domain as $\hat{q}(f) / S_h(|f|)$ [the “over-whitened” ringdown waveform required in Eq. (1)]; the filter was then truncated in the time domain so that it had a total duration of $\tau_{\text{filter}} = 1.66$ s. The truncation is necessary because the power spectrum $S_h(|f|)$ possesses narrow line features that would normally cause the filter to have a very long impulse response; by truncating the filter, these line spikes are not so well resolved, but the impulse response
becomes much shorter. It is found that the output from this filter is in good agreement with the output of a filter with a much longer $\tau_{\text{filter}}$; the loss in signal-to-noise ratio caused by the somewhat sub-optimal filter is less than 10%.

The data were filtered in the frequency domain in segments of 6.64 s, or $2^{16}$ samples at the instrumental sampling rate of $\Delta^{-1} \simeq 9.868$ kHz. To remove the effect of the correlation wrap-around, the last $\tau_{\text{filter}} = 1.66$ s (corresponding to the impulse response of the filter) of the correlation output was dropped for each segment; the data segments were overlapped by the same amount to compensate for the dropped output. Only the data from the times in which the instrument was in lock were used, and data segments in which there were outliers (defined as any segment in which there was an interferometer sample that was more than five times the sample standard deviation for the segment, or too many samples more than three times the sample standard deviation for the segment) were discarded. The power spectrum of the instrument that was used to construct the filter was obtained by averaging the spectra of 8 segments of data before and after the start of each segment.

As a stepping-stone towards examining a coincidence-style search for gravitational waves, I describe the results of a single-detector search below. These results show the necessity of devising methods for reducing the false alarm rate caused by non-Gaussian nature of the noise. A coincident event in an independent detector is the most immediate corroboration available (though further corroboration would also be required before one concluded that an event was a gravitational wave). A description of a simulated coincidence search with 40-meter data follows the single detector results.

A. Single detector results

After the filtering process, I was left with 20 hours of filter output. From this output, I produced a list of “events” for which the filter output exceeded a signal-to-noise ratio threshold of 5. I next produced a “distinct event” list by maximizing the signal-to-noise ratios over time delays of $\pm \tau_{\text{filter}} = \pm 1.66$ s corresponding to the impulse response time of the filter. A single signal (or noise transient) may cause the filter output to exceed the threshold many times. This maximization accounts for the time-correlations in the filter output caused by the finite impulse response of the filter; thus, any ringdown waveform will cause only one distinct event to be recorded. I then estimated an event rate as a function of signal-to-noise ratio greater than 5. Reasonable estimates of black hole ringdown event rates in our Galaxy and its environs indicates that it is almost certain that there were no gravitational wave signals of this brightness in the 46 hours of data; thus this event rate corresponds to the measured false alarm rate. I plot this false alarm rate as a function of signal-to-noise ratio threshold in Fig. 3. In addition to the false alarm rate observed from the interferometer data, I plot the false alarm rate expected from the same filtering process for simulated stationary Gaussian noise. Figure 3 shows that, for a desired false alarm rate, one would have to set a signal-to-noise ratio threshold greater than the threshold that would be anticipated from an analysis of stationary Gaussian noise. The departure of the false alarm curve from the expected curve assuming stationary Gaussian noise is often seen in real detector data for sufficiently low false alarm rates; however, here it seems that the stationary and Gaussian assumptions are poor even for relatively large false alarm rates.

It should be emphasized that a large number of false alarms is not unexpected when using a ringdown filter because many kinds of non-Gaussian events could produce detector output that resembles an exponentially-damped sinusoid. For example, the ringdown filter that I used, with a central frequency of 520 Hz, could be excited by beating between two lines such as the 500 Hz and the 540 Hz lines seen in Fig. 3; this effect could become important if a large amount of power were present in the lines. A possible remedy would be to remove these lines from the data before filtering it, but it may also be possible to design a discriminator that would reject these events.

The true detection probability of the filtering process can also be estimated, but one requires some assumption of a potential source. For the present purposes, suppose one is interested in detecting a typical black hole ringdown occurring at a distance of the large Magellanic cloud (LMC) (50 kpc away) that emits 1% of the black hole mass in ringdown gravitational waves. Suppose also that one is using the correct filter, i.e., the black hole has the same mass and spin that I assumed when constructing the filter above. Then the true detection probability is the fraction of times that an injected signal will be detected by the filtering process (within $\pm 1$ ms) at a given signal-to-noise ratio threshold. I injected signals into the data with random Poisson-distributed times at a mean
rate of 1 per 10 seconds; only those injections in data that was actually filtered were counted. I plot this true detection probability as a function of signal-to-noise ratio in Fig. 3.

For a given signal, a specification of the desired true detection probability yields a threshold level which can then be used to compute the false alarm rate. For example, if one wanted to detect the above signal with a false dismissal probability of less than 50%, then, from Fig. 3, one would be able to set a threshold signal-to-noise ratio as high as $\rho_* \approx 11$. From Fig. 3, one finds that the false alarm rate will be approximately 4 per hour for this threshold. To reduce this high rate of false alarms, or to improve the true detection probability for a given false alarm rate, it is necessary to develop methods to discriminate between real astronomical events and spurious noise events. The simplest such method is a coincidence experiment. I now describe an application of the coincident detection method of Sec. II B to the 40-meter data from two separate runs where I pretend that the runs were simultaneous output of two interferometers.

B. Simulated coincident detection

To simulate the effect of a coincidence strategy of detection, I used the following procedure: I divided the interferometer data into two streams of (roughly) equal length, and I filtered each of these two streams to obtain two lists of events where the “times” of these events were computed as if the two streams began at the same instant. The amount of filtered output was about 9 hours for each stream. Before maximizing these two streams over the impulse response time of the filter ($\pm \tau_{\text{filter}} = 1.66$ s), I constructed a list of coincident events, i.e., events for which the arrival times agree to within $\pm \Delta \tau = 10$ ms—thus, the two streams of data are treated as if they were produced by two independent 40-meter interferometers running simultaneously and separated by 3000 km. The signal-to-noise ratio of each coincident event is taken to be $\rho = \min(\rho_1, \rho_2)$ where $\rho_1$ was the signal-to-noise ratio recorded in the first interferometer and $\rho_2$ was the signal-to-noise ratio recorded in the second interferometer. The resulting list of events was then maximized over the minimum signal-to-noise ratio $\rho$ for delays up to the impulse response of the filter ($\pm \tau_{\text{filter}} = 1.66$ s) to produce a list of distinct, coincident events.

The number of coincident false alarm events expected is given by $N_{\text{coincident}} = (R_0^{\text{single}})^2 \times 2 \Delta \tau \times T_{\text{obs}}$, where $T_{\text{obs}} = 9$ h is the observation time in each detector and $R_0^{\text{single}}$ is the rate of false alarms expected in each detector for a desired signal-to-noise ratio threshold. For a threshold signal-to-noise ratio of 5, one expects $N_{\text{coincident}}(\rho_0 = 5) \approx 2$ events based on an estimate of $R_0^{\text{single}}(\rho_0 = 5) \approx 5 \times 10^{-2}$ s$^{-1}$ obtained from Fig. 3. However, I obtained 18 coincident events for a threshold $\rho_0 = 5$. The origin of this problem is the following: Because the impulse response time of the filter is $\tau_{\text{filter}} = 1.66$ s, a loud non-Gaussian noise burst can potentially corrupt up to 1.66 s of filter output even if the noise burst duration is far shorter. Thus, although I demand that a coincidence of two events requires that the two events occur within a time $\pm \Delta \tau = 10$ ms, it is possible that two short noise bursts in different detectors and separated in time by up to $2\tau_{\text{filter}}$ could produce a coincident event. This thesis is supported by the following observation: If the two filter outputs are maximized over the filter duration $\pm \tau_{\text{filter}}$ before searching for coincidences, then I observe only 3 coincidences, which is in good agreement with the expected number (2 events). (However, it is preferable to apply discriminators, such as a coincidence requirement, prior to maximization. This is because it is possible that a filter output will contain both a signal and a louder noise event; a discriminator applied after maximization will reject the entire filter output based on the loud noise event, and will miss the signal, while a discriminator applied before maximization will remove the noise event but preserve the signal.)

Because of the finite impulse response time of the filter, a coincidence strategy may not fully achieve the expected reduction in false alarm rate for coincident times much less than the impulse response time unless additional vetoes are used. Nevertheless, the coincident detection strategy already provides a tremendous reduction in the false alarm event rate. At a signal-to-noise ratio threshold of 5, the event rate is about 2 per hour; the true detection probability for my model ringdown would be almost unity at this threshold. Furthermore, for a signal-to-noise ratio threshold of 7, no coincident events were observed and thus the false alarm rate is likely less than about 1 per 4 hours. By injecting signals with the same arrival time into the two data streams, one can compute the true detection probability for a coincidence search (see Fig. 3). For a threshold of 7, the observed true detection probability is $Q_{\text{coincident}}^1 = 94\%$. [One would estimate that this probability would be $Q_{\text{coincident}}^1 = (Q_{\text{single}}^1)^2 = 96\%$ based on the observed value of $Q_{\text{single}}^1 = 98\%$ in Fig. 3 for a threshold of 7.] Thus, for a given false alarm rate, a much lower threshold can be used for the coincident threshold than for the single interferometer, and the false dismissal probability is greatly reduced.

Such occurrences would be easily vetoed by a simple examination of the filter output time series. The burst responses will be much longer than would be expected for a ringdown since a ringdown response will be very short (a few ms).
IV. CONCLUSIONS

Black-hole ringdowns are a promising source of gravitational radiation for detection in the kilometer-scale interferometric gravitational wave detectors. The method of matched filtering is a well-known technique for data analysis and can easily be applied for searches for black-hole ringdown; the computational burden will not be great. Our understanding of the performance of the matched filter is largely based on statistical arguments that assume that the detector noise is stationary and Gaussian. For real detectors, these assumptions will not be true at some level; for the data collected by the Caltech 40-meter prototype interferometer in November 1994, they are relatively poor assumptions.

I have analyzed 20 hours of data obtained from the Caltech 40-meter prototype interferometer in November 1994 using a matched filter for a black-hole ringdown. The sensitivity of the instrument is such that a typical black hole of mass $\sim 50M_\odot$ that emits $\sim 1\%$ of its energy as gravitational waves should be detectable as far out as the LMC. However, the presence of non-stationary and non-Gaussian noise components creates a high rate of false alarms in the single detector. These effects can be dealt with, if one has two separated interferometers with independent noise, by considering a coincidence detection strategy. To simulate a coincidence study, I used the first half of the 40-meter data as if it came from one detector, and the second half as if it came from a second, independent detector. The false alarm rate for a given threshold is significantly reduced (though still not to the level expected from Gaussian noise), while the false dismissal probability at a desired false alarm rate is greatly reduced for a simulated source in the LMC.

The coincidence detection is a simple example of a veto that can be used, if two detectors are available, to discriminate between events that are caused by detector noise and events that are caused by gravitational waves. Because it is possible that two interferometers, though widely separated, may still have correlated noise, it is necessary to supplement a coincidence veto with other vetoes based on environmental monitors at the two sites. A catalog of peculiar “waveforms” produced by vagueries of the interferometers must be generated and events resembling those “waveforms” may need to be vetoed. Other more elaborate vetoes can also be considered: An example would be a parameter-space veto that would make sure that the candidate produces the expected reduction in signal-to-noise for filters with slightly mis-matched central frequencies and quality factor. Such a veto would discriminate between a high amplitude noise spike with a very large quality factor that would trigger all the filters in a filter bank, and a gravitational wave from a ringing black hole with a low quality factor that would only trigger filters with roughly the same quality factor. Because I performed a single filter search, I did not assess the ability of such a veto to “remove” non-Gaussian events.

A multi-filter analysis of the 40-meter data is presently being conducted.

One needs to be somewhat cautious about applying detection vetoes constructed for simulated ringdown signals. Although the signal from the ringdown of a black hole is known exactly for late times, at an earlier time in the ringdown, when the amplitudes are larger and linear perturbation theory is less reliable, the signal is not so well known. This fact does not prevent a matched filter from detecting the late time ringdown waveform, but too-sensitive a veto may reject a real event if the early time waveform is not accurately modeled. Furthermore, exponentially decaying (or growing) sinusoids may model many other potential sources of gravitational radiation; one would prefer not to dismiss these sources by focusing too closely on the black-hole ringdown problem. In this sense, the coincidence veto seems promising for a first pass through the data because it only assumes that the source of the signal is external to the instrument, and that the two instruments have independent internal noise.

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APPENDIX: BLACK-HOLE RINGDOWN

1. Ringdown waveform

The ringdown waveform of a distorted spinning black hole can be obtained from the Teukolsky equation for small distortions. The radiative part of the perturbation produces a metric perturbation at large distances that can be expressed as an exponentially-damped sinusoid for the fundamental quadrupole mode (which will dominate the radiation at late times). The central frequency $f$ and quality factor $Q$ depend on the mass $M$ and the spin $S = \dot{a}GM/c$ (where $G$ is Newton’s constant and $c$ is the speed of light) of the perturbed Kerr black hole; they are well approximated by the analytic fit found by Echeverria to Teukolsky-formalism calculations by Leaver:

$$f \simeq 32 \text{kHz} \times [1 - 0.63(1 - \dot{a})^{3/10}] \left(\frac{M_{\odot}}{M}\right)$$

(A1)

and

$$Q \simeq 2(1 - \dot{a})^{-9/20}.$$  

(A2)
The dimensionless spin parameter \( \hat{a} \) lies between zero (Schwarzschild limit) and unity (extreme Kerr limit), so the quality factor is greater than 2.

The metric perturbation caused by the quasinormal ringing of a Kerr black hole will cause a relative strain \( h_{GW}(t) \) in the arms of an interferometric gravitational wave detector; the response of the detector to a gravitational wave is given in Ref. [3]. The response depends on the sky position of the black hole and on the relative orientation of the spin axis of the black hole to the zenith of the detector. The average strain produced by a ringing black hole at some fixed distance can be obtained by rms averaging over the various angles; the result is

\[
h_{GW} \text{ (angle averaged)}(t) = Aq(t),
\]

where \( q(t) \) is an exponentially-damped sinusoid:

\[
q(t) = \begin{cases} (2\pi)^{1/2} e^{-\pi f t / Q} \cos(2\pi f t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}
\]

The amplitude \( A \) depends on the strength of the perturbation and on the distance to the black hole. For a perturbation of a black hole at distance \( r \) that radiates a fraction \( \epsilon \) of the total mass-energy of the hole, the amplitude is

\[
A \approx (2.15 \times 10^{-21} Q^{-1/2} [1 - 0.63(1 - \hat{a})^{3/10}]^{-1/2}) \times \left( \frac{\text{Mpc}}{r} \right) \left( \frac{M}{M_\odot} \right) \left( \frac{\epsilon}{0.01} \right)^{1/2}.
\]

The quality factor measures (roughly) the number of cycles in the waveform, so the amplitude decreases with increasing quality factor for a fixed energy emitted.

The signal-to-noise ratio of a ringdown waveform is \( A\sigma \) where \( \sigma \) is the signal-to-noise ratio that would be observed if the ringdown strain \( q(t) \) were produced. This is

\[
\sigma^2 \approx \frac{2}{S_h(f)} \int_0^\infty q^2(t) dt \approx \frac{Q}{f S_h(f)} \text{ (large } Q)\]

where \( S_h(f) \) is the one-sided noise power spectrum of the detector at the central frequency \( f \) [cf. Eq. (4)] with \( h(t) \) replaced by \( q(t) \), which is nearly monochromatic for large \( Q \). Thus, \( A\sigma \approx \frac{A Q^{1/2}}{h_{\text{rms}}(f)} \) where \( h_{\text{rms}}(f) = \frac{\int S_h(f) f^2}{2} \) is the rms noise strain in the detector.

For example, a black hole of mass 50 \( M_\odot \) and \( \hat{a} = 0.98 \) produces a ringdown with a central frequency \( f \approx 520 \text{ Hz} \) and quality factor \( Q \approx 12 \). At a distance of 10 kpc, the typical ringdown strain on an interferometer would be \( 4 \times 10^{-18} \) if 1% of the mass of the black hole were radiated in the ringdown waves. For the 40-meter prototype interferometer, which has a typical rms strain noise level of \( 3 \times 10^{-19} \) at 520 Hz, the expected signal-to-noise ratio would be about 50. At a distance of the LMC (50 kpc), the same ringdown would produce a signal-to-noise ratio of about 10. For the detectability of ringdown waveforms in the LIGO interferometer, see Ref. [4].

### 2. Number of filters required

Since the different possible waveforms depend on the mass and spin of the black hole (or equivalently on the central frequency and the quality factor of the damped sinusoid), and since these parameters are continuous, it is necessary to choose a discrete set or bank of waveforms to form a “mesh” that covers the parameter space sufficiently finely. By “sufficiently finely,” I mean that the degradation in the signal-to-noise ratio due to having a filter with slightly incorrect parameters should be small. In addition to these parameters, the ringdown is also described in terms of its start time and initial phase. These parameters are closely related: a ringdown will correlate strongly with another ringdown with a different phase and a start time shifted by up to one quarter of a cycle. Thus, by using a ringdown template with an incorrect initial phase, one will make an error in arrival time of less than one quarter of a period and will reduce the observed signal-to-noise ratio by less than the amount that would be accumulated in this quarter cycle. For a high quality-factor ringdown, the signal-to-noise ratio loss will be small. Since one will maximize over arrival times, only the central frequency and the quality factor need be considered when constructing a mesh of ringdown templates.

In order to estimate how close the templates must be, I follow the procedure of Owen [20] in defining a distance function \( ds^2_{i\bar{j}} = 1 - (\hat{q}_i | \hat{q}_{\bar{j}}) \) corresponding to the mismatch between the two filters \( \hat{q}_i \) and \( \hat{q}_{\bar{j}} \) that are normalized so that \( (\hat{q}_i | \hat{q}_{\bar{i}}) = 1 \). In the continuum limit, this interval can be written in terms of a metric as \( ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \) where \( x^\alpha = (f, Q)^\alpha \) is a coordinate on the two dimensional parameter space. The metric can be evaluated, in the continuum limit, by \( g_{\alpha\beta} = -\frac{1}{2} (\hat{q}_i | \partial_i \hat{q}_{\bar{j}}) \) where \( \partial_i \) is a partial derivative with respect to \( x^\alpha \). One can show that the mismatch between a filter with frequency and quality factor \( (f, Q) \) and a filter with frequency and quality factor \( (f + df, Q + dQ) \) is

\[
ds^2 \approx \frac{1}{8} \frac{dQ^2}{Q^2} - \frac{1}{4} \frac{dQ}{Q} \frac{df}{f} + Q^2 \frac{df^2}{f^2}.
\]

In deriving this equation, I have assumed that the noise power spectrum is approximately constant over the frequency band of the two filters combined, and I have

\[

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kept only the dominant term in the metric coefficients in the limit of high quality factor. The minimum number of filters, \( N_{\text{filters}} \), required to cover the parameter space such that there is no point that is a distance larger than \( ds_{\text{max}}^2 = (1 - \text{“minimal match”})^2 \) from the nearest template can be found by integrating the volume element \( \sqrt{\det g_{\alpha\beta}} \) over the parameter space. For the metric of Eq. (A7) and the parameter ranges \( 0 \leq Q \leq Q_{\text{max}} \) and \( f_{\text{min}} \leq f \leq f_{\text{max}} \), one finds

\[
N_{\text{filters}} \approx \frac{1}{4\sqrt{2}} \left( ds_{\text{max}}^2 \right)^{-1} Q_{\text{max}} \ln\left( f_{\text{max}} / f_{\text{min}} \right)
\approx 460 \left( ds_{\text{max}}^2 / 0.03 \right)^{-1} \left( Q_{\text{max}} / 20 \right)
\times \left\{ 1 + \frac{1}{\log 50} \left[ \log \left( f_{\text{max}} / 5 \text{ kHz} \right) - \log \left( f_{\text{min}} / 100 \text{ Hz} \right) \right] \right\}. \tag{A8}
\]

A value of \( ds_{\text{max}}^2 = 3\% \) is needed in order to ensure that there is a loss of no more than 10% of the expected event rate due to a mismatched template. The frequency ranges given here are suitable ranges for the Caltech 40-meter prototype interferometer (see Fig. [2]): for LIGO and VIRGO, one would choose a different range.

[1] A. Abramovici et al., Science 256, 325 (1992).
[2] B. Caron et al., Class. Quantum Grav. 14, 1461 (1997); C. Bradaschia et al., Nucl. Instrum. Methods A 289, 518 (1990).
[3] C. W. Helstrom, Statistical Theory of Signal Detection, 2nd ed. (Permagon, London, 1968).
[4] A. Abramovici et al., Phys. Lett. A 218 157 (1996); A. D. Gillespie, Ph.D. thesis, California Institute of Technology (1995); T. T. Lyons, Ph.D. thesis, California Institute of Technology (1997).
[5] É. É. Flanagan and S. A. Hughes, Phys. Rev. D 57, 4535 (1998).
[6] L. S. Finn, Phys. Rev. D 46, 5236 (1992).
[7] J. D. Creighton, (research in progress).
[8] P. Kafka, in Topics in Theoretical and Experimental Gravitational Physics, proceedings of the International School of Cosmology and Gravitation, Erice, Italy, 1975, edited by V. De Sabbata and J. Weber (Plenum Press, New York, 1977), p. 161.
[9] R. E. Vogt, R. W. P. Drever, F. J. Raab, K. S. Thorne, and R. Weiss, The Construction, Operation, and Supporting Research and Development of a Laser Interferometric Gravitational-Wave Observatory, proposal to the National Science Foundation (California Institute of Technology, 1989), Vol. 1, Sec. VII.B.
[10] L. S. Finn, (in preparation).
[11] B. Allen, computer code GRASP, LIGO Project, California Institute of Technology, Pasadena, CA, http://www.lsc-group.phys.uwm.edu, 1997.
FIG. 1. Time and frequency representations of the ringdown waveform used in this analysis. This ringdown has a central frequency of 520 Hz and a quality factor of 12, which corresponds to the fundamental quadrupole quasinormal mode of a black hole with mass $50 M_\odot$ and 98% of the maximum spin.

FIG. 2. The rms strain sensitivity $h_{\text{rms}}(f) = [f S_h(f)]^{1/2}$ of the Caltech 40-meter prototype interferometer in November 1994.

FIG. 3. The false alarm rate for a single detector as a function of threshold level $\rho_*$. The filled circles represent the observed false alarm event rate from filtering the Caltech 40-meter prototype interferometer data while the open squares represent the false alarm event rate obtained by filtering simulated stationary Gaussian noise. The solid line is the theoretical rate for stationary Gaussian noise given by Eq. (4) with $N_{\text{filters}} = 1$ filters.

FIG. 4. The true detection probabilities as a function of threshold level $\rho_*$ for a simulated black-hole ringdown with mass $50 M_\odot$ and 98% of maximum spin at a distance of 50 kpc and a gravitational wave luminosity equal to 1% of the black hole mass. The solid line was measured for injected signals in a single detector search, while the dotted line was measured for simultaneously injected signals in a simulated two-detector coincidence search.