The proton-neutron symplectic model of nuclear collective motions

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Abstract. The proton-neutron symplectic model of nuclear collective motion is presented. It is shown that it appears as a natural multi-major-shell extension of the generalized proton-neutron $SU(3)$ scheme which includes rotations with intrinsic vortex as well as monopole, quadrupole and dipole giant resonance vibrational degrees of freedom.

1. Introduction
Symmetry plays an important role in our understanding of collective structure of atomic nuclei. In nuclear physics there are two approaches which exploit the symmetry. The first, standard, approach appears in the form of exact or dynamical symmetries in which the basis states and physical observables like the Hamiltonian and the transition operators are expressed as tensor operators with respect to the symmetry under consideration. This yields exact solutions for the eigenvalues and eigenfunctions and allows the calculation of required matrix elements by using a generalized Wigner-Eckart theorem. The most popular examples of this approach are the $SU(3)$ Elliott model [1] and the Interacting Boson Model [2]. There is, however, another non-standard approach for exploiting the symmetry in which one first identifies the generators of possible collective flows and then the algebra they close under commutation [3, 4, 5, 6]. This reveals directly the dynamical content of a certain algebraic model. The Hamiltonian of nuclear system is then assumed to be a function of these operators. Along these lines, some algebraic models of collective motions in nuclei have been proposed based on the algebras of $SL(3, R)$ [7], $Rot(3)$ [8], $CM(3)$ [9] and $Sp(6, R)$ [10] groups, respectively. The $Rot(3)$ algebra is just the rigid rotor algebra proposed by Ui [8]. The $SL(3, R)$ model of Weaver and Biedenharn takes into consideration, beyond the rotational, also vibrational degrees of freedom. The $CM(3)$ model has been shown to be a microscopic version of the Bohr-Mottelson model [11] augmented by the vortex spin degrees of freedom. Finally, in order to make the $CM(3)$ model compatible with the microscopic shell model structure, Rowe and Rosensteel extended the latter to the one-component symplactic model with $Sp(6, R)$ dynamical symmetry [10]. The $Sp(6, R)$ group is shown to be the full dynamical group of collective excitations in the one-component many-particle nuclear system.

Recently, the non-standard symmetry approach to nuclear collective motion was applied to the two-component many-particle nuclear systems. As a result, the proton-neutron symplectic model was formulated by considering the symplectic geometry and possible collective flows in the many-particle nuclear system [12]. This allowed to reveal some new features and forms of collective excitations which are missing in the microscopic theory of the one-component nuclear systems [12, 13].
2. The Proton-Neutron Symplectic Model

The collective observables of the proton-neutron symplectic model (PNSM) are spanned by the following one-body operators [12]:

\[ Q_{ij}(\alpha, \beta) = \sum_{s=1}^{m} x_{is}(\alpha)x_{js}(\beta), \]
\[ S_{ij}(\alpha, \beta) = \sum_{s=1}^{m} \left( x_{is}(\alpha)p_{js}(\beta) + p_{is}(\alpha)x_{js}(\beta) \right), \]
\[ L_{ij}(\alpha, \beta) = \sum_{s=1}^{m} \left( x_{is}(\alpha)p_{js}(\beta) - x_{js}(\beta)p_{is}(\alpha) \right), \]
\[ T_{ij}(\alpha, \beta) = \sum_{s=1}^{m} p_{is}(\alpha)p_{js}(\beta), \]

which are expressed in terms of position \( x_{is}(\alpha) \) and momentum \( p_{jt}(\beta) \) \((i, j = 1, 2, 3; \alpha, \beta = p, n; s, t = 1, \ldots, m)\) coordinates of the \( m = A - 1 \) relative Jacobi vectors. The mass quadrupole tensor operators (1) determine the shape and orientation of the nucleus. The shear momentum operators (2) are the infinitesimal generators of deformation and irrotational-flow rotations, which are related to the high-lying excitations associated with the giant resonance vibrational degrees of freedom. The operators (3) are the infinitesimal operators of rigid rotations in the 6-dimensional space and correspond to the low-lying nuclear excitations. Finally, the generators (4) are the momentum tensor operators among which are the kinetic energies of the proton and neutron systems. All these operators close under commutation the non-compact symplectic algebra \( sp(12, R) \), which is the dynamical algebra of collective excitations in the two-component many-particle nuclear system.

By taking different subsets of symplectic generators ones obtains different subalgebras of \( sp(12, R) \) which reveal directly the dynamical content of the model. For more details see Ref.[12]. In this way, e.g., one obtains the reduction chain:

\[ sp(12, R) \supset gcm(6) \supset gcm(3) \supset rot(3) \supset so(3). \]

Another interesting example is provided by the following reduction chain:

\[ sp(12, R) \supset gcm(6) \supset [R^{21}]so(6) \supset rot_p(3) \oplus rot_n(3) \supset rot(3) \supset so(3), \]

from which one can see the algebraic structure of two coupled rigid rotors (two rotor model). Here, in contrast to the original version of the Two Rotor Model of N. Lo Iudice and F. Palumbo [14], we do not have the restriction to the case of two axial rotors.

Dynamical group \( Sp(12m, R) \) of the whole system allows to separate the many-particle nuclear coordinates, represented by the set of \( m \) translationally-invariant Jacobi vectors, into collective and intrinsic ones. This can be done by the corresponding Zickendraht-Dzublik coordinate transformation which leads to the separation of 21 collective variables \( \xi \equiv \{ \rho^{(ao)}, g_5^+ \} \) and \( 6m - 21 \) intrinsic coordinates \( g_m \) [12]. The six radial variables \( \rho^{(ao)} \) take the values \( 0 \leq \rho^{(ao)} \leq \infty \), \( g_5^+ \) denotes the set of 15 variables parametrizing the group \( SO(6) \), and the set of intrinsic variables \( g_m \) parametrizes the coset space \( O(m)/O(m-6) \). The change of variables induces a corresponding decomposition of the nuclear wave functions into collective and intrinsic parts, respectively. It can be shown that the simplest kinematically correct wave functions can be
classified by the quantum numbers provided by the unitary scheme chain [12, 15]:

$$U(6m) \supset U(6) \otimes U(m)$$

$$[E_0 \ldots 0] [E_1 \ldots E_6] [E_1 \ldots E_6]$$

$$\cup \cup \beta \cup$$

$$G \quad O(m)$$

$$\quad \quad \omega_1 \ldots \omega_6$$

$$\quad \quad \delta \cup$$

$$S_{m+1}$$

$$[f]h$$

which is embedded in the full many-particle dynamical group $Sp(12m, R)$. Then the wave function can be written in the form [12]:

$$\Psi(E_0 \omega_1 \ldots \omega_6 | x^a_1, \ldots, x^a_m) = \Psi(E_0 \omega_1 \ldots \omega_6 | \rho^{(a_0)}, g^+_6, g_m) = \sum_{\rho^0} \Theta(E_0 \omega_1 \ldots \omega_6 | \rho^{(a_0)}, g^+_6) D_{\rho^0}^{(a_0)} \delta|f] (g_m).$$

(8)

The proton-neutron symplectic model is then given a simple expression as a hydrodynamical model with wave functions comprising collective (irrotational-flow) and intrinsic (vortex) components. Here we want to point out that if the $O(m)$ irreducible representation $\omega$ is the scalar representation, then one simply obtains the pure two-fluid irrotational-flow collective model which states are associated with the giant resonance degrees of freedom. For real nuclei $\omega \neq 0$, although we are speaking of collective degrees of freedom, the latter are always coupled to the intrinsic vortex degrees of freedom, as can be seen from (8). The need of consideration of the intrinsic degrees of freedom and their coupling to the irrotational-flow collective dynamics has been realized long time ago. The latter, as we will see further, turns out to be of vital importance for the appearing of the low-lying collective bands.

3. Representations of the $Sp(12, R)$ Lie algebra

Having a given classical collective model with a certain dynamical algebra we can quantize it by constructing its irreducible representations in the many-particle Hilbert space. To construct the irreducible representations of the $Sp(12, R)$ Lie algebra in shell-model terms, it is convenient to introduce the harmonic oscillator raising $b^+_i \alpha, s = \sqrt{\frac{m_i \omega}{2\hbar}} \left( x_{i\alpha} (\alpha) - \frac{i}{m_i \omega} p_{i\alpha} (\alpha) \right)$ and lowering $b_i \alpha, s = \sqrt{\frac{m_i \omega}{2\hbar}} \left( x_{i\alpha} (\alpha) + \frac{i}{m_i \omega} p_{i\alpha} (\alpha) \right)$ operators which satisfy the standard boson commutation relations $[b_i \alpha, s, b^+_j \beta, t] = \delta_{ij} \delta_{\alpha \beta} \delta_{st}$. Then, a basis for the many-particle realization of the $Sp(12, R)$ Lie algebra is given by [13]

$$F_{ij}(\alpha, \beta) = \sum_{s=1}^{m} b^+_i \alpha, s b^+_j \beta, s,$$

$$G_{ij}(\alpha, \beta) = \sum_{s=1}^{m} b_i \alpha, s b_j \beta, s,$$

$$A_{ij}(\alpha, \beta) = \frac{1}{2} \sum_{s=1}^{m} (b^+_i \alpha, s b_j \beta, s + b_j \beta, s b^+_i \alpha, s).$$

(9)

(10)

The $F$’s operators, as we will see, represent the core collective excitations of monopole, quadrupole and dipole type, associated with the giant resonance vibrational degrees of freedom.
The G’s are the corresponding conjugate deexcitation operators. The number-conserving operators (10) generate the maximal compact subalgebra $u(6) \subset sp(12, R)$, related to the internal vortex degrees of freedom, responsible for the appearance of the low-lying collective bands.

An $Sp(12, R)$ unitary irreducible representation is characterized by the $U(6)$ quantum numbers $\sigma = [\sigma_1, \ldots, \sigma_6]$ of its lowest-weight state $|\sigma\rangle$, i.e. $|\sigma\rangle$ satisfies

$$G_{ab}|\sigma\rangle = 0; \quad A_{ab}|\sigma\rangle = 0, \quad a < b; \quad A_{aa}|\sigma\rangle = a|\sigma\rangle \tag{11}$$

for the indices $a \equiv i\alpha$ and $b \equiv j\beta$ taking the values 1, 2, 3, 4, 5, 6. If we introduce the $U(6)$ tensor product operators $P^{(n)}(F) = [F \times \ldots \times F]^{(n)}$, where $n = [n_1, \ldots, n_6]$ is a partition with even integer parts, then by an $U(6)$ coupling of these tensor products to the lowest-weight $U(6)$ state $|\sigma\rangle$, one constructs the whole basis of states for an $Sp(12, R)$ irrep

$$|\Psi(\sigma n \rho E \eta)\rangle = [P^{(n)}(F) \times |\sigma\rangle]^{\rho E}_{\eta}, \tag{12}$$

where $E = [E_1, \ldots, E_6]$ indicates the $U(6)$ quantum numbers of the coupled state, $\eta$ labels a basis of states for the coupled $U(6)$ irrep $E$ and $\rho$ is a multiplicity index. This is shown schematically in Fig.1. In this way we obtain a basis of $Sp(12, R)$ states that reduces the subgroup chain $Sp(12, R) \supset U(6)$.

![Figure 1](image-url)  

**Figure 1.** The construction of the symplectic basis by acting with the $U(6)$-coupled tensor products $P^{(n)}(F) = [F \times \ldots \times F]^{(n)}$ on the lowest-weight state $|\sigma\rangle$.

### 4. The shell-model classification of nuclear collective states

How we can determine the relevant $Sp(12, R)$ irreducible representations appropriate for the description of the low-lying collective states in heavy mass deformed nuclei? The answer is suggested by the shell model, and in particular, from the underlying shell-model structure of the ground state. To reveal this structure we specify the basis of an $Sp(12, R)$ irrep by considering the following reduction of the subgroup $U(6) \subset Sp(12, R)$ [13]:

$$Sp(12, R) \supset U(6) \supset SU_p(3) \otimes SU_n(3) \supset SU(3) \supset SO(3) \supset SO(2). \tag{13}$$

The chain (13) naturally generalizes the Elliott’s $SU(3)$ model [1] by extending the model space to the direct product space $SU_p(3) \otimes SU_n(3)$ of proton and neutron subsystems. The $SU(3)$ irreps of the two subsystems are subsequently strongly coupled to the $SU(3)$ irrep of the combined proton-neutron system to form composite configurations with different possible deformations. The maximum deformation is obtained for the leading $SU(3)$ irrep. The combined $SU(3)$ algebra is generated by the Elliott’s quadrupole $\hat{Q}_M = \hat{Q}_M^p + \hat{Q}_M^n$ and angular momentum $L_M = L_M^p + L_M^n$ operators, respectively. The chain (13) corresponds to the following choice of the index $\eta = \gamma(\lambda_p, \mu_p)(\lambda_n, \mu_n)\varrho(\lambda, \mu)KLM$ labeling the basis states (12) of the $Sp(12, R)$ irrep.
The generators of $Sp(12, R)$ (9)–(10) can be classified as irreducible tensor operators with respect to the different subgroups of the whole chain (13) and hence can be characterized by the quantum numbers determining their irreducible representations. For the raising operators one readily obtains the following tensors:

$$F_{(2,0)(0,0)}^{[2]_6}(L^M(p,p)), \quad F_{(0,0)(2,0)}^{[2]_6}(L^M(n,n)), \quad F_{(1,0)(1,0)}^{[2]_6}(L^M(p,n)), \quad (14)$$

where $L = 0, 2; M = -L, \ldots, M$, and

$$F_{(1,0)(1,0)}^{[2]_6}(0^M(p,n)). \quad (15)$$

We see a multiplication of the standard one-component $Sp(6, R)$ raising generators [10] which for the two-component system correspond to the creation of monopole and quadrupole $pp$ and $pn$ pairs. In addition to the $(2, 0)$ $SU(3)$ raising generators $F_{(2,0)(0,0)}^{[2]_6}(L^M(\alpha, \beta))$ (14) we have also the $(0, 1)$ $SU(3)$ tensor operators $F_{(1,0)(1,0)}^{[2]_6}(M^0(\alpha, \beta)) (15)$, which are new ones compared to the generators of the $Sp(6, R)$ model of Rosensteel and Rowe [10]. The shell model properties of raising operators (14)–(15) can be determined by considering their $U(3) \supset U(1) \otimes SU(3)$ tensor character. Thus, it becomes obvious that the raising operators of $Sp(12, R)$ with $U(1) \otimes SU(3)$ quantum numbers $N(\lambda, \mu) = 2(2, 0)$ and their conjugate lowering ones represent $\pm 2 \hbar \omega$ inter-shell collective excitations of monopole ($L = 0$) and quadrupole ($L = 2$) type. Additionally, in contrast to the $Sp(6, R)$ model, the $Sp(12, R)$ raising operators with $N(\lambda, \mu) = 2(0, 1)$ together with their conjugate lowering operators correspond to the $\pm 2 \hbar \omega$ inter-shell excitations of dipole ($L = 1$) type. Thus, the $Sp(12, R)$ collective dynamics covers the nuclear coherent excitations of monopole, dipole and quadrupole type.

The construction of the shell model representations of the proton-neutron symplectic model is given in detail in Ref.[13]. For example, the shell model considerations give for $^{154}$Sm the symplectic bandhead structure $\sigma = [132, 86, 70, 70, 70, 70]$. The latter will reduce to a huge amount of $SU_p(3) \otimes SU_n(3)$ irreducible representations among which is the $(12, 8) \otimes (34, 8)$ one that is composed by the direct product of the leading $SU(3)$ irreps for the proton and neutron subsystems, which are strongly coupled to the final $SU(3)$ symmetry. The maximum deformation, corresponding to the ground state, is obtained by restricting the direct product irreps $(12, 8) \otimes (34, 8)$ of $SU_p(3) \otimes SU_n(3)$ to the leading irreducible representation (46,16) of $SU(3)$. The other $SU(3)$ irreps of the combined proton-neutron system (with decreasing deformation), contained in the symplectic $Sp(12, R)$ bandhead structure, are obtained from the direct product of the parent $SU(3)$ irreducible representations:

$$(12, 8) \otimes (34, 8) \rightarrow (46, 16), (47, 14), (48, 12), \ldots$$

$$(44, 17), (45, 15), (46, 13), \ldots$$

: : : : : : \quad (16)$$

The fact that one obtains a nonscalar $Sp(12, R)$ irreducible representation $\langle \sigma \rangle \neq 0$ for the real nuclei is a very important feature of the present approach, which is a consequence of the two-component composite character of the nuclear systems. The $Sp(12, R)$ irreducible representation is determined by the symplectic bandhead (or intrinsic) structure, defined by the lowest $U(6)$ irrep $\sigma = [\sigma_1, \ldots, \sigma_6] \neq 0$. The latter, in contrast to the $Sp(6, R)$ case, will contain a plethora of $SU(3)$ multiplets. This is of significant importance in the microscopic nuclear structure theory because the intrinsic base space spanned by the $U(6)$ irrep $[\sigma_1, \ldots, \sigma_6]$, as can be seen from (16), will contain many $SU(3)$ irreducible representations appropriate for the description of different low-lying collective bands in the spectra of heavy even-even deformed nuclei. In this way, in
contrast to the $Sp(6, R)$ model, the intrinsic $Sp(12, R)$ bandhead structure provides us with a framework for the simultaneous shell-model interpretation of the ground state band and the other excited low-lying collective bands without the need of involving the mixing of different symplectic irreps (cf. Ref. [16]).

Figure 2. The basis states of the $Sp(12, R)$ irreps are in 1-1 correspondence with the coupled product of the two-fluid irrotational-flow collective model and an $U(6)$ model, related to the valence shell proton-neutron degrees of freedom, which contains many $SU(3)$ multiplets appropriate for the description of different low-lying collective bands.

In this way, the $Sp(12, R)$ irreducible representations can be represented as a coupled product of two structures: a symplectic bandhead related to the proton-neutron valence shell degrees of freedom and a collective structure, associated with the giant resonance degrees of freedom. If the symplectic bandhead is represented by the scalar $Sp(12, R)$ representation, then one obtains simply the two-fluid irrotational-flow collective model. The states of the latter are in 1-1 correspondence with the states of the 21-dimensional harmonic oscillator. For non-trivial bandhead symplectic structure, the states of the $Sp(12, R)$ irreps are in 1-1 correspondence with the states of the coupled product of the giant resonance and the proton-neutron valence shell degrees of freedom, as shown in Fig. 2. Moreover, in the limit of large $Sp(12, R)$ representation quantum numbers, the proton-neutron symplectic model contracts to the $U(6)$-phonon model [17] with the semi-direct product structure $[HW(21)]U(6)$ which appears as an unification of the two-fluid irrotational-flow collective model and an $U(6)$ model, containing all the necessary $SU(3)$ multiplets required for the description of the low-lying collective bands. All this becomes more clear, if we express the basis functions in the form

$$\psi_\sigma = \sum_\eta \psi_\eta(F)|\sigma\eta\rangle,$$

where $\psi_\eta$ is a polynomial in the $Sp(12, R)$ raising operators. Eq. (17) can be interpreted as a factoring of an arbitrary wave function into collective and intrinsic parts. The bandhead states $|\sigma\eta\rangle$ can be thought of as intrinsic states and the raising operators $\psi_\eta$ as collective wave functions.

5. Conclusions
The fully-microscopic proton-neutron symplectic model of nuclear collective motion, based on the non-compact $Sp(12, R)$ group, is presented. From a hydrodynamic perspective, it is shown that it appears as a two-fluid irrotational-flow collective model augmented by the intrinsic vortex degrees of freedom. From a shell-model perspective, from other side, the PNSM appears
as a natural multi-major-shell extension of the generalized proton-neutron $SU(3)$ scheme which takes into account the core collective excitations of monopole and quadrupole, as well as dipole type associated with the giant resonance vibrational degrees of freedom.

[1] J. P. Elliott, Proc. R. Soc. London, Ser. A 245, 128 (1958); 245, 562 (1958).
[2] F. Iachello and A. Arima, The Interacting Boson Model (Cambridge University Press, Cambridge, 1987).
[3] G. Rosensteel and D. J. Rowe, Ann. Phys. 96, 1 (1976).
[4] G. Rosensteel and D. J. Rowe, Ann. Phys. 123, 36 (1978).
[5] D. J. Rowe and G. Rosensteel, Ann. Phys. 126, 198 (1980).
[6] O. L. Weaver, R. Y. Cussion and L. C. Biedenharn, Ann. Phys. (N.Y.) 102, 493 (1976).
[7] L. Weaver and L. C. Biedenharn, Nucl. Phys. A185, 1 (1972).
[8] H. Ui, Prog. Theor. Phys. 44, 153 (1970).
[9] L. Weaver, R. Y. Cusson and L. C. Biedenharn, Ann. Phys. (N.Y.) 77, 250 (1973).
[10] D. J. Rowe and G. Rosensteel, Phys. Rev. Lett. 38, 10 (1977).
[11] A. Bohr and B. R. Mottelson, Nuclear Structure (W.A. Benjamin Inc., New York, 1975), Vol. II.
[12] H. G. Ganev, Eur. Phys. J. A50, 183 (2014).
[13] H. G. Ganev, Eur. Phys. J. A51, 84 (2015).
[14] N. Lo Iudice and F. Palumbo, Phys. Rev. Lett. 41, 1532 (1978); Nucl. Phys. A326, 193 (1979).
[15] V. V. Vanagas, Algebraic foundations of microscopic nuclear theory (Nauka, Moscow, 1988) (in Russian).
[16] J. Carvalho et al., Nucl. Phys. A452, 240 (1986).
[17] H. G. Ganev, Int. J. Mod. Phys. E 24, 1550039 (2015).