On the Roots of Characteristic Equations of Delay Differential Systems

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Abstract

We prove that characteristic equations of certain types of delay differential systems, under some mild conditions on their coefficients, can possess infinitely many complex roots.

A. Preliminary

Our motivation comes from the linear (single, complex, constant) time-delay complex differential system:

\[ \dot{x}(t) = Ax(t) + Bx(t - \tau), \quad x(t) \in \mathbb{C}^n \]

(1)

where A and B are n-by-n matrices over \( \mathbb{C} \) and \( \tau \in \mathbb{C} \setminus \{0\} \) is a complex time-delay. The stability of the zero solution is determined to the real parts of roots of the characteristic equation:

\[ f(\lambda) := \det (\lambda \text{id} - A - e^{-\tau \lambda} B) = 0, \]

(2)

after the exponential ansatz \( x(t) = e^{\lambda t} x_0 \) is applied. We are interested in the question whether there exist infinitely many complex roots of \( f \).

Our main observations are the following:

(i) \( f \) is an entire function;

(ii) for any \( \epsilon > 0 \), the growth rate of \( f \) is bounded by \( e^{\lambda^{1+\epsilon}} \) for all \( \lambda \in \mathbb{C} \) with \(|\lambda| \) sufficiently large.

We note that (ii) follows directly by using triangle inequality.

Definition. Let \( f \) be an entire function, the order of \( f \), denoted by \( \text{ord}(f) \), is the infimum of \( \alpha > 0 \) such that there exists \( R > 0 \) such that \(|f(\lambda)| \leq e^{\lambda^\alpha} \) holds for all \( \lambda \in \mathbb{C} \) with \(|\lambda| \geq R \).

Hence the observation (ii) indicates that \( \text{ord}(f) \leq 1 \). Now finiteness of \( \text{ord}(f) \) reminds us a dichotomy.

Lemma 1 (Theorem 16.13 in [BaNe10]). Let \( f \) be an entire function and of finite order, then

(i) either \( f(\lambda) = 0 \) possesses infinitely many roots in \( \mathbb{C} \),
(ii) or there exist complex polynomial $g(\lambda)$ and $h(\lambda)$ such that $h(0) = 0$ and

$$f(\lambda) = g(\lambda)e^{h(\lambda)}$$

holds for all $\lambda \in \mathbb{C}$.

Furthermore, in the case (ii), we have $\deg(h) = \text{ord}(f)$.

Thus, our strategy is to give an indirect proof: according to Lemma 1, if $f(\lambda) = 0$ possesses at most finitely many roots in $\mathbb{C}$, then $\text{ord}(f) \leq 1$ implies

$$f(\lambda) = g(\lambda)e^{c\lambda}$$

holds for all $\lambda \in \mathbb{C}$ where $g(\lambda)$ is a complex polynomial and $c \in \mathbb{C}$. Then the main task is to seek conditions on the coefficients $A$ and $B$ to reach a contradiction.

### B. Single Complex Constant Delay

In the following Proposition we apply our strategy carefully.

**Proposition 1.** Suppose $\text{tr}(B) \neq 0$, then for each $\tau \in \mathbb{C} \setminus \{0\}$, the equation

$$f(\lambda) := \det(\lambda \text{id} - A - e^{-\tau \lambda}B) = 0$$

(3)

possesses infinitely many roots in $\mathbb{C}$.

**Proof.** Setting $\lambda \mapsto \tau \lambda$, without loss of generality we consider $\tau = 1$. The equation (3) can be expressed as

$$f(\lambda) := \lambda^n + a_1(\lambda)e^{-\lambda} + \ldots + a_n(\lambda)e^{-n\lambda},$$

(4)

where

$$a_1(\lambda) = -(\text{tr}(B))\lambda^{n-1} + \text{lower order terms}$$

is a nonzero polynomial since we assume $\text{tr}(B) \neq 0$. Obviously all other $a_j(\lambda)$ for $j \in \{2, \ldots, n\}$ are (maybe identically zero) complex polynomials. Since $a_1(\lambda)$ is nonzero, there exist $k \in \mathbb{N}$ with $1 \leq k \leq n$ such that $a_k(\lambda)$ is the last (with respect to the order as real numbers in the exponential exponents) nonzero polynomial, i.e.

$$f(\lambda) = \lambda^n + a_1(\lambda)e^{-\lambda} + \ldots + a_k(\lambda)e^{-k\lambda}.$$  

(5)

Obviously $f$ is an entire function. We easily see that $\text{ord}(f) \leq 1$, because for each $\epsilon > 0$, using triangle inequality, the estimates

$$|f(\lambda)| \leq (k + 1)\max_{j=1, \ldots, k} \{1, |a_j(\lambda)|\}e^{k|\lambda|} \leq e^{\lambda^{1+\epsilon}}$$

(6)

hold as $|\lambda|$ is sufficiently large.
Contradiction Part: Suppose the contrary that \( f(\lambda) = 0 \) possesses at most finitely many roots in \( \mathbb{C} \). Since \( f \) is entire and \( \text{ord}(f) \leq 1 \), by Lemma 1,
\[
f(\lambda) = g(\lambda)e^{\lambda}
\]
holds for all \( \lambda \in \mathbb{C} \) where \( g(\lambda) \) is a complex polynomial and \( c \in \mathbb{C} \). We claim that
\[
\text{Re}(c) = -k, \quad \text{Im}(c) = 0.
\]
Let \( z_l \lambda^l \) \((0 \leq l \leq n)\) be the leading term of \( a_k(\lambda) \). Multiplying (5) by \( e^{k\lambda/\lambda^l} \) yields
\[
\frac{g(\lambda)}{\lambda^l}e^{\text{Im}(c)\lambda}e^{(\text{Re}(c)+k)\lambda} = \frac{\lambda^ne^{k\lambda} + a_1(\lambda)e^{(k-1)\lambda} + \ldots + a_{k-1}(\lambda)e^{\lambda} + z_l + \tilde{a}_k(\lambda)}{\lambda^l} \quad (8)
\]
where \( \text{deg}(\tilde{a}_k) < l \). Taking \( \lambda \in \mathbb{R} \) and \( \lambda \to -\infty \), since \( g \), all \( a_j \), and \( \tilde{a}_k \) are polynomials, the right-hand side of (8) converges to \( z_l \), while the left-hand side of (8) diverges to infinity (resp. to zero) if \( \text{Re}(c) + k < 0 \) (resp. \( \text{Re}(c) + k > 0 \)). Thus \( \text{Re}(c) + k = 0 \). We now have
\[
\frac{g(\lambda)}{\lambda^l}e^{\text{Im}(c)\lambda}e^{-\text{Im}(c)\lambda} = \frac{\lambda^ne^{k\lambda} + a_1(\lambda)e^{(k-1)\lambda} + \ldots + a_{k-1}(\lambda)e^{\lambda} + z_l + \tilde{a}_k(\lambda)}{\lambda^l}. \quad (9)
\]
Again we play the same trick by taking \( \text{Re}(\lambda) \to -\infty \) and \( \text{Im}(\lambda) \to \infty \) (or \( -\infty \), it does not matter), we see \( \text{Im}(c) = 0 \). As a result, (5) becomes
\[
g(\lambda)e^{-k\lambda} = \lambda^n + a_1(\lambda)e^{-\lambda} + \ldots + a_k(\lambda)e^{-k\lambda}.
\]
At last taking \( \lambda \in \mathbb{R} \) and \( \lambda \to \infty \) we have
\[
0 = \lim_{\lambda \in \mathbb{R}, \lambda \to \infty} \lambda^n,
\]
which is a contradiction. The proof is complete. \( \square \)

C. Multiple Real Constant Delays

We consider the linear (multiple, real, constant) time-delay complex differential systems:
\[
\dot{x}(t) = Ax(t) + \sum_{j=1}^{k} B_j x(t - \tau_j), \quad x(t) \in \mathbb{C}^n \quad (10)
\]
for integer \( j \geq 2 \) and \( -\infty < \tau_1 < \tau_2 < \ldots < \tau_k < \infty \). The characteristic equation is given by
\[
\det(\lambda \text{id} - A - \sum_{j=1}^{k} B_j e^{-\tau_j \lambda}) = 0, \quad (11)
\]
which is a special case of the general quasi-polynomials
\[
f(\lambda) := \sum_{(a_0, a_1, \ldots, a_k) \in \mathbb{N}^{k+1} \cup \{0\}} a_{a_0, a_1, \ldots, a_k} \lambda^{a_0} e^{-(\sum_{j=1}^{k} a_j \tau_j)\lambda} \quad (12)
\]
where only finitely many \( a_{\alpha_0, \ldots, \alpha_k} \in \mathbb{C} \) are nonzero. Denote \( \tau := (\tau_1, \ldots, \tau_k) \) and \( \alpha := (\alpha_1, \ldots, \alpha_k) \). We call \( f \) is \textit{admissible} if there exist \( \alpha^1 \) and \( \alpha^2 \) such that
\[
\alpha^1 \cdot \tau \neq \alpha^2 \cdot \tau
\]
and there exist \( \alpha^1_0, \alpha^2_0 \in \mathbb{N} \cup \{0\} \) such that
\[
a_{\alpha^1_0, \alpha^1} \neq 0, \quad a_{\alpha^2_0, \alpha^2} \neq 0.
\]
In other words, \( f(\lambda) \) possesses two different exponential exponents.

**Proposition 2.** Let \( f \) be defined in (12), then \( f(\lambda) = 0 \) possesses infinitely many roots in \( \mathbb{C} \) if and only if \( f \) is admissible.

**Proof.** Assume \( f \) is not admissible, then \( f(\lambda) = 0 \) is equivalent to a polynomial equation, which possesses at most finitely many roots in \( \mathbb{C} \).

Conversely, assume \( f \) is admissible. Obviously \( f \) is an entire function and \( \text{ord}(f) \leq 1 \). Since all \( \tau_j \) are real, the terms of \( f(\lambda) \) can be sorted by the order as real numbers in the exponential exponents. Hence if \( f \) is admissible, then
\[
f(\lambda) = a_h(\lambda)e^{-(\alpha^h \cdot \tau)\lambda} + \ldots + a_l(\lambda)e^{-(\alpha^l \cdot \tau)\lambda}
\]
holds where \( a_h(\lambda) \) and \( a_l(\lambda) \) are nonzero complex polynomials and \(-(\alpha^h \cdot \tau) > -(\alpha^l \cdot \tau)\) are two different real numbers. Therefore, \( f(\lambda) = 0 \) is equivalent to the equation
\[
\tilde{f}(\lambda) = a_h(\lambda) + \ldots + a_l(\lambda)e^{-(\alpha^l \cdot \tau - \alpha^h \cdot \tau)\lambda} = 0.
\]

**Contradiction Part:** Suppose the contrary that \( \tilde{f}(\lambda) = 0 \) possesses at most finitely many roots in \( \mathbb{C} \), then
\[
\tilde{f}(\lambda) = g(\lambda)e^{c\lambda}
\]
holds for all \( \lambda \in \mathbb{C} \). Now we notice that the claim in the Contradiction Part of the previous Proposition:
\[
\text{Re}(c) = -(\alpha^l \cdot \tau - \alpha^h \cdot \tau), \quad \text{Im}(c) = 0,
\]
holds if we assume all \( \tau_j \) are real. Therefore, taking \( \lambda \in \mathbb{R} \) and \( \lambda \to \infty \) in (14), we have
\[
0 = \lim_{\lambda \in \mathbb{R}, \lambda \to \infty} a_h(\lambda),
\]
a contradiction. The proof is complete. \( \square \)

**Remark.** The assumption \( \text{tr}(B) \neq 0 \) is just a sufficient condition of Proposition 4, but it is the unique sufficient condition that is irrelevant to \( A \).

**Remark.** It is interesting to seek sufficient conditions for \( f \) in (11) being admissible. For instance Pontryagin’s condition that \( f \) is without the principal term, see [Po55]. Another sufficient condition is that \( \tau_j \) are linearly independent over \( \mathbb{Z} \), i.e.
\[
\beta \cdot \tau = 0, \quad \beta \in \mathbb{Z}^k \quad \text{implies} \quad \beta = 0.
\]
and one of \( B_j \) is of trace zero.
D. Single Real Distributed Delay

We consider a linear (single, real, distributed) time-delay complex differential equation:

\[ \dot{x}(t) = ax(t) + \int_0^\tau M(\theta)x(t - \theta)d\theta, \quad x(t) \in \mathbb{C}. \]  

(15)

where \( a \in \mathbb{C}, \tau > 0, \) and \( M \in C^\theta([0, \tau], \mathbb{C}). \) The characteristic equation of (15) is given by

\[ f(\lambda) := \lambda - a - \int_0^\tau M(\theta)e^{-\lambda \theta}d\theta = 0 \]

Proposition 3. \( f(\lambda) = 0 \) possesses infinitely many roots in \( \mathbb{C} \) if and only if \( M \) is not identically zero.

Proof. Assume \( M \) is identically zero, then \( f(\lambda) = 0 \) possesses the unique root \( \lambda = a. \)

Conversely, assume \( M \) is not identically zero. Suppose the contrary that \( f \) possesses at most finitely many roots in \( \mathbb{C} \). Obviously \( f \) is an entire function and \( \text{ord}(f) \leq 1 \), then by Lemma 1,

\[ f(\lambda) = g(\lambda)e^{c\lambda} \]

holds for all \( \lambda \in \mathbb{C} \). Define \( \delta := \tau \|M\|_{C^\theta} > 0 \), then using triangle inequality,

\[ |\lambda - a| - \delta e^{\text{Re}(\lambda)} \leq |f(\lambda)| = |g(\lambda)|e^{\text{Re}(c)|\text{Re}(\lambda)| - \text{Im}(c)|\text{Im}(\lambda)|} \leq |\lambda - a| + \delta e^{\text{Re}(\lambda)}. \]  

(16)

Taking \( \lambda \in \mathbb{R} \) and \( \lambda \to -\infty \), we see \( |f(\lambda)| \) cannot grow exponentially, hence \( \text{Re}(c) = 0 \). Similarly, taking \( \text{Re}(\lambda) \to -\infty \) and \( \text{Im}(\lambda) \to \infty \) (or \( -\infty \), it does not matter), we have \( \text{Im}(c) = 0 \). Now that \( c = 0 \), the growth constraint of \( |g(\lambda)| \) also implies that \( g(\lambda) \) is linear. Therefore there exist \( p, q \in \mathbb{C} \) such that

\[ \int_0^\tau M(\theta)e^{-\lambda \theta}d\theta = p\lambda + q \]  

(17)

holds for all \( \lambda \in \mathbb{C} \). To reach a contradiction, we differentiate (17) twice to obtain

\[ \int_0^\tau \theta^2 M(\theta)e^{-\lambda \theta}d\theta = 0. \]

Since \( M \) is continuous, by using Fourier series, we have \( \theta^2 M(\theta) = 0 \) for all \( \theta \in [0, \tau] \).

Thus \( M \) is identically zero, a contradiction. The proof is complete.

References

[BaNe10] J. Bak and D. J. Newman. Complex Analysis. Springer-Verlag New York, 2010.

[Po55] L. S. Pontryagin. On the zeros of some elementary transcendental functions, Amer. Math. Soc. Transl. (2) 1 (1955), p. 95-110.