A Complex Hyperbolic Structure for Moduli of Cubic Surfaces

Daniel Allcock, James A. Carlson, and Domingo Toledo

Department of Mathematics
University of Utah
Salt Lake City, Utah, USA.
E-mail: allcock, carlson and toledo at math.utah.edu

Abstract. We show that the moduli space $M$ of marked cubic surfaces is biholomorphic to $(B^4 - \mathcal{H})/\Gamma_0$ where $B^4$ is complex hyperbolic four-space, where $\Gamma_0$ is a specific group generated by complex reflections, and where $\mathcal{H}$ is the union of reflection hyperplanes for $\Gamma_0$. Thus $M$ has a complex hyperbolic structure, i.e., an (incomplete) metric of constant holomorphic sectional curvature.

Une structure hyperbolique complexe pour les modules des surfaces cubiques

Résumé. Nous montrons que l'espace des modules $M$ des surfaces cubiques marquées est biholomorphe à $(B^4 - \mathcal{H})/\Gamma_0$ où $B^4$ est l'espace complexe hyperbolique de dimension quatre, où $\Gamma_0$ est un groupe spécifique généré par des réflexions complexes, et où $\mathcal{H}$ est l'union de l'ensemble d'hyperplans de réflexion de $\Gamma_0$. Donc $M$ admet une structure hyperbolique complexe, c'est à dire une métrique (incomplète) de courbure holomorphe sectionnelle constante.

Version française abrégée

A une surface cubique (marquée) correspond une variété cubique de dimension trois (marquée), à savoir le revêtement de $\mathbb{P}^3$ ramifié le long de la surface. L'application des périodes $f$ pour ces variétés de dimension trois est définie sur l'espace des modules $M$ des cubiques marquées, et cette application $f$ prend ses valeurs dans un quotient de la boule unitaire dans $\mathbb{C}^4$ par l'action du groupe de monodromie projective. Ce groupe $\Gamma_0$ est généré par des réflexions complexes dans un ensemble d'hyperplans dont nous notons la réunion par $\mathcal{H}$. Alors nous avons le résultat suivant:

Théorème. L'application des périodes définit une biholomorphisme

$$f : M \longrightarrow (B^4 - \mathcal{H}) / \Gamma_0.$$ 

De ce théorème on obtient des résultats sur la structure métrique de $M$ et sur son groupe fondamental:

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1. Main results

Corollaires. (1) L’espace $M$ des modules des surfaces cubiques marquées admet une structure hyperbolique complexe: une métrique (incomplète) de courbure holomorphe sectionnelle constante. (2) Le groupe fondamental de $M$ contient un sous-groupe normale qui n’est pas de génération finie. (3) Le groupe fondamental de $M$ n’est pas un réseau dans un groupe de Lie semisimple.

Remarques. (1) Nos méthodes montrent aussi que la complétée métrique de $(B^4 - \mathcal{H})/\Gamma_0$ est l’orbifolde $B^4/\Gamma_0$, isomorphe à l’espace des modules des surfaces cubiques marquées stables. (2) Récemment E. Looijenga a trouvé une présentation remarquable du groupe fondamental orbifoldé de l’espace des modules des cubiques lisses non-marquées.

Afin de préciser la notion de surface cubique lisse marquée, fixons un réseau $L$, un $\mathbb{Z}$-module libre avec une base $e_0, \ldots, e_6$ qui est muni de la forme quadratique telle que la base soit orthogonale et telle que $(e_0, e_0) = 1$, $(e_k, e_k) = -1$ pour $k > 0$. Soit $\eta = 3e_0 - (e_1 + \cdots + e_6)$. Alors une surface cubique marquée est composée d’une surface cubique lisse $S$ et d’une isométrie $\psi : L \rightarrow H^2(S, \mathbb{Z})$ qui envoie $\eta$ sur la classe d’un hyperplan. L’ensemble $M$ des modules d’isométrie des surfaces cubiques marquées porte la structure d’une variété et de plus est un espace de modules finies. Une construction de cette espaces a été donnée dans [9], où on trouve aussi une compactification lisse $C$ de $M$ telle que les points de $C - M$ forment un diviseur à croisements normaux.

Pour définir le groupe $\Gamma_0$, soit $\mathcal{E}$ l’anneau des entiers d’Eisenstein $\mathbb{Z}[\omega]$ où $\omega = (-1 + \sqrt{-3})/2$ est une troisième racine d’unité, et considérons le product Cartesien $\mathcal{E}^5$ muni d’une forme hermitienne $h(v, w) = -v_1\bar{w}_1 + v_2\bar{w}_2 + v_3\bar{w}_3 + v_4\bar{w}_4 + v_5\bar{w}_5$. Alors $(\mathcal{E}^5, h)$ est l’unique réseau autodual sur les entiers d’Eisenstein qui est de signature $(4,1)$. Donc $\text{Aut}(\mathcal{E}^5, h)$ est un sous-groupe discret du groupe unitaire $U(h)$, qui agit sur $B^4 = \{ \ell \in \mathbb{P}^4 : h|\ell < 0 \}$. Notons que $\mathcal{E}/\sqrt{-3}\mathcal{E} \cong \mathbb{F}_3$ est un corps de trois éléments et notons aussi qu’il y a un homomorphisme naturel $\text{Aut}(\mathcal{E}^5, h) \rightarrow \text{Aut}(\mathbb{F}_3^5, q)$ où $q$ est la forme quadratique obtenue par réduction de $h$ modulo $\sqrt{-3}$. Notons par “$P$” la projectivisation, et définissons un groupe $\Gamma_0$ d’automorphismes de $B^4$ par la suite exacte

$$1 \rightarrow \Gamma_0 \rightarrow \text{PAut}(\mathcal{E}^5, h) \rightarrow \text{PAut}(\mathbb{F}_3^5, q) \rightarrow 1.$$  

Ce groupe est le groupe discret du théorème principal. Les hyperplans de $\mathcal{H}$ sont définis par les équations $h(x, v) = 0$ pour des vecteurs $v$ dans $\mathcal{E}^5$ avec $h(v) = 1$. Notons aussi que $\text{PAut}(\mathbb{F}_3^5, q)$ est isomorphe au groupe de Weyl du réseau $E_6$.

1. Main results

To a (marked) cubic surface corresponds a (marked) cubic threefold defined as the triple cover of $\mathbb{P}^3$ ramified along the surface. The period map $f$ for these threefolds is defined on the moduli space $M$ of marked cubic surfaces and takes its values in the quotient of the unit ball in $\mathbb{C}^4$ by the action of the projective monodromy group. This group $\Gamma_0$ is generated by complex reflections in a set of hyperplanes whose union we denote by $\mathcal{H}$. Then we have the following result:

**Theorem.** The period map defines a biholomorphism

$$f : M \rightarrow (B^4 - \mathcal{H})/\Gamma_0.$$  

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From this identification we obtain results on the metric structure and the fundamental group:

**Corollaries.** (1) The moduli space of marked cubic surfaces carries a complex hyperbolic structure: an (incomplete) metric of constant holomorphic sectional curvature. (2) The fundamental group of the space of marked cubic surfaces contains a normal subgroup which is not finitely generated. (3) The fundamental group of the space of marked cubic surfaces is not a lattice in a semisimple Lie group.

**Remarks.** (1) Our methods also show that the metric completion of $(B^4 - \mathcal{H})/\Gamma_0$ is the complex hyperbolic orbifold $B^4/\Gamma_0$, which is isomorphic to the moduli space of marked stable cubic surfaces. (2) Recently E. Looijenga found a remarkable presentation of the orbifold fundamental group of the moduli space of smooth unmarked cubic surfaces.

To make precise the notion of smooth marked cubic surface, fix the lattice $\mathcal{E}$ as the free complex hyperbolic orbifold $B^4$ endowed with the quadratic form for which the given basis is orthogonal and such that $(e_0, e_0) = 1$, $(e_k, e_k) = -1$ for $k > 0$. Let $\eta = 3e_0 - (e_1 + \cdots + e_6)$. Then a marked cubic surface consists of a smooth cubic surface $S$ and an isometry $\psi : L \rightarrow H^2(S, \mathbb{Z})$ which carries $\eta$ to the hyperplane class. The set $M$ of isomorphism classes of marked cubic surfaces has the structure of a variety and is a finite moduli space. A construction of it is described in [9], and a smooth compactification $C$ is given for which the points of $C - M$ constitute a normal crossing divisor.

To define the group $\Gamma_0$, let $\mathcal{E}$ denote the ring of Eisenstein integers $\mathbb{Z}[\omega]$ where $\omega = (-1 + \sqrt{-3})/2$ is a cube root of unity, and consider the Cartesian product $\mathcal{E}^5$ endowed with the hermitian inner product $h(v, w) = -v_1\bar{w}_1 + v_2\bar{w}_2 + v_3\bar{w}_3 + v_4\bar{w}_4 + v_5\bar{w}_5$. Then $(\mathcal{E}^5, h)$ is the unique self-dual lattice over the Eisenstein integers with signature $(4,1)$. Thus $Aut(\mathcal{E}^5, h)$ is a discrete subgroup of the unitary group $U(h)$, which acts on $B^4 = \{ \ell \in \mathbb{P}^4 : h|\ell < 0 \}$. Observe that $\mathcal{E}/\sqrt{-3}\mathcal{E} \cong \mathbb{F}_3$ is a field of three elements and that there is a natural homomorphism $Aut(\mathcal{E}^5, h) \rightarrow Aut(\mathbb{F}_3^5, q)$ where $q$ is the quadratic form obtained by reduction of $h$ modulo $\sqrt{-3}$. Let “$\mathcal{P}$” denote projectivization, and define a group $\Gamma_0$ of automorphisms of $B^4$ by

$$1 \rightarrow \Gamma_0 \rightarrow P\text{Aut}(\mathcal{E}^5, h) \rightarrow P\text{Aut}(\mathbb{F}_3^5, q) \rightarrow 1.$$ 

This is the discrete group of the main theorem. The hyperplanes of $\mathcal{H}$ are defined by the equations $h(x, v) = 0$ for vectors $v$ in $\mathcal{E}^5$ with $h(v) = 1$. Note that $P\text{Aut}(\mathbb{F}_3^5, q)$ is isomorphic to the Weyl group of the $E_6$ lattice.

**2. Construction of a period mapping**

To construct the period mapping, we examine in detail the Hodge structures for the cubic threefolds. The underlying lattice $H^3(T, \mathbb{Z})$ is ten-dimensional, carries a unimodular symplectic form $\Omega$, and admits a Hodge decomposition of the form $H^3(T, \mathbb{C}) = H^{2,1} \oplus H^{1,2}$. Choose a generator $\sigma$ for the group of automorphisms of $T$ over $\mathbb{P}^3$, and note that it operates without fixed points on $H^3(T, \mathbb{Z})$. This action gives $H^3(T, \mathbb{Z})$ the structure of a five-dimensional module over the Eisenstein integers. It carries a hermitian form

$$h(x, y) = \frac{1}{2}(\Omega((\sigma - \sigma^{-1})x, y) + (\omega - \omega^{-1})\Omega(x, y))$$
which is unimodular and of signature $(4,1)$.

Now consider the quotient module $H^3(T,\mathbb{Z})/(1-\omega)H^3(T,\mathbb{Z})$ and observe that it can be identified isometrically with $(\mathbb{F}_3^5,q)$. We define a marking of $T$ to be choice of such an isometry, and we claim that a marking of a cubic surface determines a marking of the corresponding threefold. Indeed, if $\gamma$ is a primitive two-dimensional homology class on $S$ then it is the boundary of a three-chain $\Gamma$ on $T$. Since $\Gamma$ and $\sigma \Gamma$ have the same boundary, the three-chain $c(\gamma) = (1-\sigma)\Gamma$ is a cycle. However, it is well-defined only up to addition of elements $(1-\sigma)\Delta$ where $\Delta$ is a three-cycle on $T$. Thus a homomorphism

$$c : H^3_{prim}(S,\mathbb{Z}) \rightarrow H^3(T,\mathbb{Z})/(1-\sigma)$$

is defined. Since a marking of $S$ can be viewed as a basis of $H^3_{prim}(S,\mathbb{Z})$, application of $c$ to the basis elements defines a basis of $H^3(T,\mathbb{Z})/(1-\sigma)$, and this gives the required marking of the threefold.

The action of $\sigma$ decomposes $H^3(T,\mathbb{C})$ into eigenspaces $H^3_\lambda$ where $\lambda$ varies over the primitive cube roots of unity. Because $\sigma$ is holomorphic, the decomposition is compatible with the Hodge decomposition and one has

$$H^3_\omega = H^{2,1}_\omega \oplus H^{1,2}_\omega \quad H^3_{\bar{\omega}} = H^{2,1}_{\bar{\omega}} \oplus H^{1,2}_{\bar{\omega}}.$$

The dimensions of the Hodge components can be found with the help of Griffiths’ Poincaré residue calculus [5]. Details for this case are found in [3], section 5. One finds that

$$\dim H^{2,1}_\omega = 4, \quad \dim H^{1,2}_\omega = 1, \quad \dim H^{2,1}_{\bar{\omega}} = 1, \quad \dim H^{1,2}_{\bar{\omega}} = 4,$$

and from the Hodge-Riemann bilinear relations one finds that $h$ has signature $(4,1)$.

Now let $\phi$ be a generator of the one-dimensional space $H^{2,1}_\omega$ and let $\gamma_1, \ldots, \gamma_5$ be a standard basis of $H^3(T,\mathbb{Z})$ considered as an $E$-module. By this we mean that the $\gamma_k$ are orthogonal and that $h(\gamma_1,\gamma_1) = -1$ and $h(\gamma_k,\gamma_k) = 1$ for $k > 1$. Let $v(\phi,\gamma)$ be the vector in $\mathbb{C}^5$ with components

$$v_k = \int_{\gamma_k} \phi.$$

One verifies that $h(v,v) < 0$ where now $h$ is the hermitian form $-|v_1|^2 + |v_2|^2 + \cdots + |v_5|^2$. Thus the line generated by $v(\phi,\gamma)$ defines a point in $B^4 \subset \mathbb{P}^4$, and one checks that $v(\phi,\gamma) \notin \mathcal{H}$. By well-known constructions (the work of Griffiths), the period vector defines a holomorphic map from the universal cover of $M$ to the ball which transforms according to the projectivized monodromy representation for marked cyclic cubic threefolds. The proof that $\Gamma_0$ is the projective monodromy group relies on the work of Libgober [6] and the first author [1]. Thus our construction yields a period map $f : M \rightarrow (B^4 - \mathcal{H})/\Gamma_0$.

### 3. Properties of the period mapping

We must now show that $f$ is bijective. For injectivity, consider once again the period vector $v(\phi,\gamma)$. The vectors $\gamma_k$ can be decomposed into eigenvectors $\gamma'_k$ and $\gamma''_k$ for $\sigma$, with eigenvalues $\omega$ and $\bar{\omega}$, respectively. Let $\hat{\gamma}'_k$ and $\hat{\gamma}''_k$ denote elements of the corresponding
dual basis. Because $\phi$ is an eigenvector with eigenvalue $\bar{\omega}$, its integral over $\gamma_k'$ vanishes, so that

$$\phi = \sum_k \gamma_k'' \int_{\gamma_k'} \phi = \sum_k \gamma_k'' \int_{\gamma_k} \phi.$$ 

Thus the components of $v(\phi, \gamma)$ determine $\phi$ as an element of $H^3_{\bar{\omega}}$. Consequently the line $\mathbb{C}v(\phi, \gamma)$ determines the complex Hodge structure $H^3_{\bar{\omega}}$. Viewing the Hodge components of $H^3_{\bar{\omega}}$ as subspaces of $H^3(T, \mathbb{C})$, we may take their conjugates to determine the complex Hodge structure $H^3_{\omega}$. These two complex Hodge structures determine the Hodge structure on $H^3(T, \mathbb{Z})$. Thus, by the Torelli theorem of Clemens-Griffiths [4], the period vector $v(\phi, \gamma)$ determines the cubic threefold $T$ up to isomorphism. It remains to show that $T$, which perforce is a cyclic cubic threefold, determines its ramification locus uniquely. This follows from the fact that the locus in question is a planar component of the Hessian surface.

To prove surjectivity we first consider a smooth compactification $C$ of $M$ by a normal crossing divisor $D$, e.g., the one given by Naruki [9], as well as the Satake compactification $\overline{B^4/\Gamma_0}$, obtained by adding forty points, the “cusps,” each corresponding to a null point of $P(\mathbb{P}^3_3, q)$. By well-known results [2] in complex variable theory, the period map has a holomorphic extension to a map $\bar{f}$ from $C$ to the Satake compactification. Since $C$ is compact, $\bar{f}$ is open, and $\overline{B^4/\Gamma_0}$ is connected, we conclude that $\bar{f}$ is surjective.

4. Boundary components

To pass from surjectivity of $\bar{f}$ to surjectivity of $f$, we must show that $\bar{f}$ maps the compactifying divisor $D$ to the complement of $(\overline{B^4} - \mathcal{H})/\Gamma_0$ in the Satake compactification. To this end write $D$ as a sum of irreducible components, $D = \bigcup D_i' \cup \bigcup D_j''$, where $D_i'$ parametrizes nodal cubic surfaces via the map to the geometric invariant theory compactification of the moduli space of smooth cubics, and where in the same way the $D_j''$ parametrize cubics with an $A_2$ singularity.

Now consider a one-parameter family of cubic surfaces with smooth total space acquiring a node. Its local equation near the node has the form $x^2 + y^2 + z^2 = t$ and the corresponding family of cyclic cubic threefolds has the form $x^2 + y^2 + z^2 + w^3 = t$. The local monodromy of the latter has order six, its eigenvalues are primitive sixth roots of unity, and the space of vanishing cycles is two-dimensional. (These facts are well-known and the relevant literature and arguments are summarized in [3], section 6). From [7] we conclude that coefficients of the period vector on vanishing cycles are of the form $A(t)t^{1/6} + B(t)t^{5/6}$ where $A$ and $B$ are holomorphic. Now the space of vanishing cycles is invariant under the action of $\sigma$ and so constitutes a rank one $E$-submodule. One can choose a generator $\delta$ for it so that $h(\delta, \delta) = 1$, and then one has

$$\lim_{t \to 0} \int_{\delta} \phi = 0.$$ 

Thus the limiting value of the period vector lies in the orthogonal complement of $\delta$. In other words, $\bar{f}(D_i')$ lies in $\mathcal{H}/\Gamma_0$, as required.
Consider next a one-parameter family of cubic surfaces with smooth total space whose central fiber acquires an $A_2$ singularity. Its local equation is $x^2 + y^2 + z^3 = t$ and the corresponding family of cyclic cubic threefolds has local equation $x^2 + y^2 + z^3 + w^3 = t$. In this case the local monodromy is of infinite order. After replacing $t$ by $t^3$ one finds an expansion of the form $\phi(t) = A(t)(\log t)^{\hat{\gamma}} + (\text{terms bounded in } t)$, where $A(0) \neq 0$ and where $\hat{\gamma}$ is an integer cohomology class which is isotropic for $h$. Consequently the line $\mathbb{C}\phi(t)$ converges to the isotropic line $\mathbb{C}\hat{\gamma}$ as $t$ converges to zero, hence converges to a cusp in the Satake compactification.

6. The corollaries

Finally, we comment on the collaries. Part (a) is immediate. For part (b) let $K$ denote the kernel of the map $\pi_1(M) \rightarrow \Gamma_0$. Then $K$ is isomorphic to the fundamental group of $B^4 - \mathcal{H}$ and it is easy to see that its abelianization is not finitely generated. We remark that $K$ is not free: there are many sets of commuting elements corresponding to normal crossings of $\mathcal{H}$. For (c) we note first that for lattices in semisimple Lie group of real rank greater than one, the results of Margulis [8] imply finite generation of all normal subgroups. The rank one case can be treated separately, as was shown to us by Michael Kapovich.

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