CONFLUENT HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS RELATED TO THE RATIONAL QUANTUM CALOGERO SYSTEM WITH HARMONIC CONFINEMENT

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Abstract. Two families (type $A$ and type $B$) of confluent hypergeometric polynomials in several variables are studied. We describe the orthogonality properties, differential equations, and Pieri type recurrence formulas for these families. In the one-variable case, the polynomials in question reduce to the Hermite polynomials (type $A$) and the Laguerre polynomials (type $B$), respectively. The multivariable confluent hypergeometric families considered here may be used to diagonalize the rational quantum Calogero models with harmonic confinement (for the classical root systems) and are closely connected to the (symmetric) generalized spherical harmonics investigated by Dunkl.

1. Introduction

In this paper multivariable orthogonal polynomials are studied associated to the weight functions

Type $A$ (Hermite)

\[ \Delta^A(x) = \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2g_0} \prod_{1 \leq j \leq n} e^{-\omega x_j^2}, \]

Type $B$ (Laguerre)

\[ \Delta^B(x) = \prod_{1 \leq j < k \leq n} |(x_j - x_k)(x_j + x_k)|^{2g_0} \prod_{1 \leq j \leq n} |x_j|^{2g_1} e^{-\omega x_j^2}. \]

In the one-variable case ($n = 1$), the type $A$ polynomials become Hermite polynomials ($\Delta^A(x) = \exp(-\omega x^2)$) and the type $B$ polynomials reduce to Laguerre polynomials of a quadratic argument ($\Delta^B(x) = |x|^{2g_1} \exp(-\omega x^2)$).

For both families we will exhibit (systems of) differential equations and Pieri type recurrence relations as well as the normalization constants that convert the polynomials into an orthonormal system. Multiplication of the polynomials by the square root of the weight function, yields an orthogonal basis of eigenfunctions for the rational quantum Calogero model with harmonic confinement and its generalizations associated to (the classical) root systems [Ca, OP]. The connection between this orthogonal basis and the conventional (non-orthogonal) basis of eigenfunctions for the confined rational Calogero model (found by separating the quantum eigenvalue
problem in a ‘radial’ and a ‘spherical’ part) is described using the theory of Dunkl’s generalized spherical harmonics with reflection group symmetry [Du1, Du2].

The multivariable Hermite and Laguerre families associated to the weight functions $\Delta^A(x)$ (1.1a) and $\Delta^B(x)$ (1.1b) were introduced by Macdonald [M3] and Lassalle [La2, La3] as a generalization (more accurately a deformation) of the previously known special case in which the parameter $g_0$ is being fixed at the value $1/2$ [He, Co, J, Mu]. Recently, further insight regarding the properties of the polynomials considered by Macdonald and Lassalle was obtained in the context of a renewed study of the eigenvalue problem for the rational quantum Calogero model with harmonic confinement [D3, UW, BF]. As it turns out, some of the results reported in the present work may also be obtained by combining results from previous literature. For example, our evaluation formulas for the (squared) norms of the polynomials (cf. Theorem 2.2) can also be gleaned from [M3], [La2, La3] and [BF], where expressions for these norms in a modified guise were obtained by different methods. (Specifically, if we make the norm formulas in [M3, La2, La3] and [BF] explicit with the aid of known evaluation formulas for the Jack symmetric functions at the identity due to Stanley [St, M4], then they are seen to be in correspondence with the expressions derived below in a completely different manner.) In all instances where overlap of this kind occurs (see the notes in Section 3.4), our approach provides an alternative, independent method of proof for the statements of interest.

The paper is organized as follows. First, the confluent hypergeometric families associated to the weight functions $\Delta^A(x)$ (1.1a) and $\Delta^B(x)$ (1.1b) are defined in Section 2 and their main properties (orthogonality relations, orthonormalization constants, differential equations, and Pieri type recurrence relations) are formulated. In Section 3, we comment in some detail on the precise relation between our results and those obtained in previous literature. We will—in particular—take the opportunity to detail the connection between the multivariable Hermite/Laguerre families and the Calogero eigenfunctions as well as the relation to Dunkl’s generalized spherical harmonics. In Section 4 we provide the proofs for the statements in Section 2 by viewing the multivariable confluent hypergeometric Hermite and Laguerre families of the present work as a degeneration (viz. a limiting case) of certain families of multivariable hypergeometric orthogonal polynomials that were introduced in [D3] and investigated in more detail in [D4]. The multivariable hypergeometric polynomials relevant to us here are generalizations of the one-variable continuous Hahn polynomials [A2, AtSu, KS] (this corresponds to type A) and of the one-variable Wilson polynomials [W, KS] (this corresponds to type B) to the case of several variables. From a physical viewpoint, the multivariable hypergeometric polynomials in question are connected to the eigenfunctions of a difference (or ‘relativistic’) counterpart of the rational Calogero models with harmonic confinement [D2, D3, R]. The transition from the hypergeometric to the confluent hypergeometric level corresponds to sending the difference step-size to zero. In this (‘nonrelativistic’) limit the difference Calogero model reduces to the ordinary Calogero model. Some technicalities needed to perform this transition at the level of the polynomials (which is established by controlling the convergence of the respective weight functions) are relegated to an appendix at the end of the paper.

Note. Below we will always assume (unless explicitly stated otherwise) that the parameters $g_0$ and $g_1$ entering through the weight functions $\Delta^A(x)$ (1.1a) and $\Delta^B(x)$
(1.1b) are nonnegative and, similarly, that the scale factor $\omega$ is positive. In principle it is possible to rescale the variables $x_1, \ldots, x_n$ so as to reduce to the case that $\omega$ is fixed at the value 1 (say). However, we have found it useful to keep the dependence on $\omega$ explicit in order to have a check on the scaling properties of our expressions and so as to suppress the emergence of numerical constants.

2. Multivariable Hermite and Laguerre polynomials

In this section the multivariable confluent hypergeometric families associated to the weight functions $\Delta^A(x)$ (1.1a) and $\Delta^B(x)$ (1.1b) are defined and the main properties of the polynomials are stated. The proof of these properties can be found in Section 4.

2.1. Definition and orthogonality properties. Let $m_\lambda, \lambda \in \Lambda$ denote the basis of symmetric monomials

$$m_\lambda(x) = \sum_{\mu \in S_n(\lambda)} x_1^{\mu_1} \cdots x_n^{\mu_n}, \quad \lambda \in \Lambda$$

with

$$\Lambda = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}.$$ 

Here the summation in (2.1) is over the orbit of $\lambda$ with respect to the action of the permutation group $S_n$ (which permutes the vector components $\lambda_1, \ldots, \lambda_n$). We will also use the notation

$$m_{2\lambda}(x) = \sum_{\mu \in S_n(\lambda)} x_1^{2\mu_1} \cdots x_n^{2\mu_n}, \quad \lambda \in \Lambda$$

(2.3)

to indicate the basis of the symmetric monomials that are even in the variables $x_1, \ldots, x_n$. The monomial bases (2.1) and (2.3) inherit a partial ordering from the dominance type partial ordering of the cone $\Lambda$ (2.2) that is defined for $\lambda, \mu \in \Lambda$ by

$$\lambda \leq \mu \quad \text{iff} \quad \sum_{1 \leq j \leq k} \lambda_j \leq \sum_{1 \leq j \leq k} \mu_j \quad \text{for} \quad k = 1, \ldots, n$$

(2.4)

($\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$).

Let $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_B$ denote the $L^2$ inner products over $\mathbb{R}^n$ with weight function $\Delta^A(x)$ (1.1a) and $\Delta^B(x)$ (1.1b), respectively. So, explicitly we have

$$\langle f, g \rangle_A = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x) \Delta^A(x) \, dx_1 \cdots dx_n$$

for $f, g$ in $L^2(\mathbb{R}^n, \Delta^A \, dx_1 \cdots dx_n))$ and

$$\langle f, g \rangle_B = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x) \Delta^B(x) \, dx_1 \cdots dx_n$$

for $f, g$ in $L^2(\mathbb{R}^n, \Delta^B \, dx_1 \cdots dx_n))$. After these notational preliminaries, we are now in the position to define the multivariable confluent hypergeometric families associated to the weight functions $\Delta^A(x)$ (1.1a) and $\Delta^B(x)$ (1.1b).

Definition. The type $A$ (or multivariable Hermite) polynomials $p^A_\lambda(x), \lambda \in \Lambda$ are the polynomials determined (uniquely) by the conditions

A.1 $p^A_\lambda(x) = m_\lambda(x) + \sum_{\mu \in \Lambda, \mu < \lambda} c^A_{\lambda, \mu} m_\mu(x), \quad c^A_{\lambda, \mu} \in \mathbb{C}$

A.2 $\langle p^A_\lambda, m_\mu \rangle_A = 0$ if $\mu < \lambda$. 

Similarly, the type $B$ (or multivariable Laguerre) polynomials $p^B_{\lambda}(x)$, $\lambda \in \Lambda$ are the polynomials determined (uniquely) by the conditions

$$p^B_{\lambda}(x) = m_{2\lambda}(x) + \sum_{\mu \in \Lambda, \mu < \lambda} c^B_{\lambda,\mu} m_{2\mu}(x), \quad c^B_{\lambda,\mu} \in \mathbb{C}$$

$$\langle p^B_{\lambda}, m_{2\mu} \rangle_B = 0 \quad \text{if} \quad \mu < \lambda.$$

The type $A$ polynomials $p^A_{\lambda}(x)$, $\lambda \in \Lambda$ constitute a basis for the space of permutation invariant polynomials in the variables $x_1, \ldots, x_n$, and the type $B$ polynomials $p^B_{\lambda}(x)$, $\lambda \in \Lambda$ form a basis for the even subsector of this space (i.e., the subspace of symmetric polynomials in $x_1^2, \ldots, x_n^2$). The following theorem says that the bases in question are orthogonal with respect to the inner products $\langle \cdot, \cdot \rangle_A$ (2.5) and $\langle \cdot, \cdot \rangle_B$ (2.51), respectively.

**Theorem 2.1 (Orthogonality).** Let $\lambda, \mu \in \Lambda \ (2.2)$. We have

$$\langle p^C_{\lambda}, p^C_{\mu} \rangle_C = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p^C_{\lambda}(x) p^C_{\mu}(x) \Delta_C(x) \, dx_1 \cdots dx_n$$

$$= 0 \quad \text{if} \quad \lambda \neq \mu,$$

where $C$ stands for $A$ or $B$.

For weight vectors $\lambda$ and $\mu$ that are comparable with respect to the partial order (2.4) the orthogonality of $p^C_{\lambda}$ and $p^C_{\mu}$ follows immediately from the definition of the polynomials. Theorem 2.1 states that the orthogonality relations in fact hold for general weight vectors $\lambda, \mu \in \Lambda \ (2.2)$ (not necessarily comparable with respect to the partial order (2.4)).

In order to orthonormalize the bases $\{p^A_{\lambda}\}_{\lambda \in \Lambda}$ and $\{p^B_{\lambda}\}_{\lambda \in \Lambda}$, it is needed to evaluate the integrals for the (squared) norms of the polynomials.

**Theorem 2.2 (Norm formulas).** Let $\lambda \in \Lambda \ (2.2)$. We have

$$\langle p^A_{\lambda}, p^A_{\lambda} \rangle_A = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |p^A_{\lambda}(x)|^2 \Delta_A(x) \, dx_1 \cdots dx_n$$

$$= \frac{(2\pi)^{n/2}n!}{(2\omega)^{\lfloor \lambda \rfloor + gn(n-1)/2 + n/2}} \prod_{1 \leq j < k \leq n} \Gamma((k-j+1)g_0 + \lambda_j - \lambda_k) \Gamma(1 + (k-j-1)g_0 + \lambda_j - \lambda_k)$$

$$\times \prod_{1 \leq j \leq n} \Gamma(1 + (n-j)g_0 + \lambda_j)$$

and

$$\langle p^B_{\lambda}, p^B_{\lambda} \rangle_B = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |p^B_{\lambda}(x)|^2 \Delta_B(x) \, dx_1 \cdots dx_n$$

$$= \frac{n!}{\omega^{\lfloor \lambda \rfloor + gn(n-1) + (g_1+1/2)n}} \prod_{1 \leq j < k \leq n} \Gamma((k-j+1)g_0 + \lambda_j - \lambda_k) \Gamma(1 + (k-j-1)g_0 + \lambda_j - \lambda_k)$$

$$\times \prod_{1 \leq j \leq n} \Gamma(1 + (n-j)g_0 + \lambda_j) \Gamma((n-j)g_0 + g_1 + 1/2 + \lambda_j)$$

(where $|\lambda| = \lambda_1 + \cdots + \lambda_n$ and $\Gamma(\cdot)$ denotes the gamma function).
For \( \lambda = 0 \) the polynomials \( p^A_\lambda \) and \( p^B_\lambda \) reduce to the unit polynomial \( (p^0_0(x) = 1) \). The formulas in Theorem 2.2 are then seen to simplify to

\[
\langle 1, 1 \rangle_A = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Delta^A(x) \, dx_1 \cdots dx_n
\]

\[
= \frac{(2\pi)^n/2}{(2\omega)^n \Gamma(n/2 + \eta/2)} \prod_{1 \leq j \leq n} \frac{\Gamma(1 + jg_0)}{\Gamma(1 + g_0)}
\]

and

\[
\langle 1, 1 \rangle_B = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Delta^B(x) \, dx_1 \cdots dx_n
\]

\[
= \frac{1}{\omega^g n(n-1)+(c_1/2)^n} \prod_{1 \leq j \leq n} \frac{\Gamma((j - 1)g_0 + g_1 + 1/2) \Gamma(1 + g_0)}{\Gamma(1 + g_0)}
\]

The integrals in (2.7a) and (2.7b) amount to integrals evaluated by Mehta and Macdonald [Me, M1]. More precisely, Mehta conjectured the closed expression for the value of the integral \( \langle 1, 1 \rangle_A \) and Macdonald generalized this conjecture (in terms of root systems) therewith also including the integral \( \langle 1, 1 \rangle_B \). Both the integration formulas for \( \langle 1, 1 \rangle_A \) and \( \langle 1, 1 \rangle_B \) were then proven in [M1] by viewing them as limiting cases of an integration formula due to Selberg [Sc] (cf. also the introduction of [A1]).

2.2. Differential equations. First we introduce two families of commuting difference operators \( D^A_{r, \beta}, \ldots, D^A_{n, \beta} \) and \( D^B_{1, \beta}, \ldots, D^B_{n, \beta} \). The type A operators are given by

\[
D^A_{r, \beta} = \sum_{J_+ \subseteq \{1, \ldots, n\}} U^A_{J_+ \cap J_-} \cdot r - |J_+| - |J_-| V^A_{J_+ \cap J_-; J_+ \cap J_-} e^{\frac{\beta}{2} (\partial_{J_+} - \partial_{J_-})}
\]

\( r = 1, \ldots, n \), with

\[
e^{\frac{\beta}{2} (\partial_{J_+} - \partial_{J_-})} = \prod_{j \in J_+} e^{\left(\frac{\beta}{2} w^{A}_j \right)} \prod_{j \in J_-} e^{-\left(\frac{\beta}{2} w^{A}_j \right)}.
\]

\[
V^A_{J_+, J_-; K} = \prod_{j \in J_+} w^{A}(x_j) \prod_{j \in J_-} w^{A}(-x_j) \prod_{j \in J_+, j' \in J_-} v^{A}(x_j - x_{j'}) v^{A}(x_j - x_{j' - i\beta})
\]

\[
\times \prod_{j \in J_+, k \in K} v^{A}(x_j - x_k) \prod_{j \in J_+, k \in K} v^{A}(x_k - x_j),
\]

\[
U^A_{K, \rho} = (-1)^\rho \sum_{L_+, L_- \subset K; L_+ \cap L_- = \emptyset; |L_+| + |L_-| = \rho} \left( \prod_{l \in L_+} w^{A}(x_l) \prod_{l \in L_-} w^{A}(-x_l) \right)
\]

\[
\times \prod_{l \in L_+, l' \in L_-} v^{A}(x_l - x_{l'}) v^{A}(x_{l'} - x_l + i\beta)
\]

\[
\times \prod_{k \in K \setminus L_+ \cup L_-} v^{A}(x_k - x_l),
\]

\( k \in K \setminus L_+ \cup L_- 
\)
and
\[ v^A(z) = \left(1 + \frac{\beta g_0}{iz}\right), \quad w^A(z) = (1 + i\beta \omega z). \]

The type \( B \) operators read
\[
(2.8b) \quad D^{R,\beta}_{r,J} = \sum_{J \subset \{1, \ldots, n\}, |J| \leq r, j \in J} U^{R,\beta}_{J, r-|J|} V^{B}_{J, r, J, r} \epsilon_j \partial J, \]
\( r = 1, \ldots, n, \) with
\[
\epsilon_j \partial J, = \prod_{j \in J} e^{\varepsilon_j (\frac{\beta}{r} x_j)} \quad (\varepsilon_j \in \{-1, 1\}),
\]
\[
V^{B}_{J, r, J} = \prod_{j \in J} w^{B}(\varepsilon_j x_j) \prod_{j \neq j'} w^{B}(\varepsilon_j x_j + \varepsilon_j x_j - i\beta)
\times \prod_{j \in J} w^{B}(\varepsilon_j x_j + x_k) w^{B}(\varepsilon_j x_j - x_k),
\]
\[
U^{B}_{J, r, J} = \sum_{L \subset K, |L| = 0, J \subset L} \left( \prod_{l \in L} w^{B}(\varepsilon_l x_l) \prod_{l \neq l'} w^{B}(\varepsilon_l x_l + \varepsilon_l x_l) w^{B}(\varepsilon_l x_l - \varepsilon_l x_l + i\beta)
\times \prod_{k \in L \setminus K} w^{B}(\varepsilon_l x_l + x_k) w^{B}(\varepsilon_l x_l - x_k),
\]
and
\[
v^{B}(z) = \left(1 + \frac{\beta g_0}{iz}\right), \quad w^{B}(z) = (1 + i\beta \omega z).
\]

In (2.8a) the sum is meant over all disjoint pairs of index sets \( J_+, J_- \in \{1, \ldots, n\} \) with the sum of the cardinalities being \( \leq r \). In (2.8b) the sum is over all index sets \( J \subset \{1, \ldots, n\} \) of cardinality \( \leq r \) and all configurations of signs \( \varepsilon_j \in \{+1, -1\}, \)
\( j \in J \). We have furthermore assumed the conventions that empty products are equal to one and that \( U^{A}_{K, p} \) and \( U^{B}_{K, p} \) are equal to one for \( p = 0 \). The exponentials \( \exp\left(\pm \frac{\beta}{r} \frac{\partial}{\partial x_j}\right) \) act on analytic functions of \( x_1, \ldots, x_n \) by means of a shift of the \( j \)th argument in the (for \( \beta \) real) imaginary direction
\[
\left(\exp\left(\pm \frac{\beta}{r} \frac{\partial}{\partial x_j}\right) f\right)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, x_j \mp i\beta, x_{j+1}, \ldots, x_n).
\]

Thus, the operators \( D^{A}_{r,\beta} \) (2.8a) and \( D^{B}_{r,\beta} \) (2.8b) are analytic difference operators of order \( 2r \) in \( \exp\left(\pm \frac{\beta}{r} \frac{\partial}{\partial x_j}\right), \ldots, \exp\left(\pm \frac{\beta}{r} \frac{\partial}{\partial x_n}\right) \).

If we act with \( D^{A}_{r,\beta} \) \((C = A, B)\) on an (arbitrary) analytic function \( f \) of the variables \( x_1, \ldots, x_n \), then we end up with an expression that is holomorphic in the step size parameter \( \beta \) around \( \beta = 0 \). The coefficients of the Taylor expansion in \( \beta \) around zero can be written in terms of partial differential operators applied to the function \( f \). We will call the first nonzero differential operator in this expansion the leading differential part of the difference operator. One obtains the leading differential part \( D^{A}_{0,\beta} \) of \( D^{A}_{\beta} \) \((C = A, B)\) by expanding the analytic difference operator in powers of \( \beta \) using the formal identity \( \exp(\pm \frac{\beta}{r} \frac{\partial}{\partial x_j}) = \sum_{m=0}^{\infty} (\pm \frac{\beta}{r} \frac{\partial}{\partial x_j})^m / m! \).
The result is a formal expansion of the difference operator in terms of differential operators that gets its precise meaning (as a Taylor expansion) upon acting with both sides on an (arbitrary) analytic function.

The following theorem describes the structure of the highest order symbol of the leading differential parts $D_{1,0}^A, \ldots, D_{n,0}^A$ and $D_{1,0}^B, \ldots, D_{n,0}^B$ for the difference operators $D_r^A$ (2.8a) and $D_r^B$ (2.8b), and states that the differential operators of interest are diagonal with respect to the bases $\{p_A^\lambda\}_{\lambda \in A}$ and $\{p_B^\lambda\}_{\lambda \in A}$, respectively.

**Theorem 2.3 (Differential equations).** i. The formal expansion of the difference operator $D_{r,\beta}^C (r = 1, \ldots, n; C = A, B)$ in the step size parameter $\beta$ has the form

$$D_{r,\beta}^C = D_{r,0}^C \beta^{2r} + O(\beta^{2r+1})$$

with

$$D_{r,0}^C = (-1)^r \sum_{J \subseteq \{1, \ldots, n\} \atop |J| = r} \prod_{j \in J} \frac{\partial^2}{\partial x_j^2} + l.o.$$  

(where l.o. stands for the terms of lower order in the derivatives $\partial/\partial x_1, \ldots, \partial/\partial x_n$).

ii. The leading differential parts $D_{1,0}^C, \ldots, D_{n,0}^C$ commute and are simultaneously diagonalized by the multivariable Hermite (type A) and Laguerre (type B) polynomials

$$D_{r,0}^C p_A^\lambda = E_r^C(\lambda) p_A^\lambda, \quad \lambda \in \Lambda,$$

with the corresponding eigenvalues given explicitly by

$$E_r^A(\lambda) = (2\omega)^r \sum_{J \subseteq \{1, \ldots, n\} \atop |J| = r} \prod_{j \in J} \lambda_j \quad \text{and} \quad E_r^B(\lambda) = (4\omega)^r \sum_{J \subseteq \{1, \ldots, n\} \atop |J| = r} \prod_{j \in J} \lambda_j.$$

**Corollary 2.4.** We have that

$$\lim_{\beta \to 0} \beta^{-2r} (D_{r,\beta}^C p_A^\lambda)(x) = E_r^C(\lambda) p_A^\lambda(x), \quad \lambda \in \Lambda.$$

**Corollary 2.5 (Symmetry).** The leading differential operators $D_{1,0}^C, \ldots, D_{n,0}^C$ map the space of permutation-invariant polynomials in $x_1, \ldots, x_n$ (type A) or $x_1^2, \ldots, x_n^2$ (type B) into itself and are symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_C$ (2.5a), (2.5b)

$$\langle D_{r,0}^C m_A^\lambda, m_B^\mu \rangle_C = \langle m_A^\lambda, D_{r,0}^C m_B^\mu \rangle_C, \quad \lambda, \mu \in \Lambda$$

(with $m_A^\lambda(x) \equiv m_A(x)$ (2.1) and $m_B^\lambda(x)$ (2.3)).

The eigenvalue equations $\langle D_{r,0}^C p_A^\lambda(x) \rangle = \lim_{\beta \to 0} \beta^{-2r} (D_{r,\beta}^C p_A^\lambda)(x) = E_r^C(\lambda) p_A^\lambda(x), \quad \lambda \in \Lambda$. It is instructive to exhibit these differential equations in a more explicit form for the case $r = 1$ (which amounts to the order of the equation being equal to two). For $r = 1$, the difference operators $D_{1,\beta}^A$ (2.8a) and $D_{1,\beta}^B$ (2.8b) reduce to

$$D_{1,\beta}^A = \sum_{1 \leq j \leq n} \left( w^A(x_j) \prod_{1 \leq k \leq n, k \neq j} v^A(x_j - x_k)(e^{\frac{\beta}{x_j}} - 1) \right)$$

$$w^A(-x_j) \prod_{1 \leq k \leq n, k \neq j} v^A(-x_j + x_k)(e^{\frac{-\beta}{x_j}} - 1) \right)$$
with \( v^A(\cdot) \), \( w^A(\cdot) \), and \( v^B(\cdot) \), \( w^B(\cdot) \) taken the same as in (2.8a) and (2.8b), respectively. A formal expansion in \( \beta \) reveals that \( D_{1,\beta} = D_{1,0} \beta^2 + O(\beta^3) \) with

\[
(2.10a) \quad D_{1,0}^A = \sum_{1 \leq j \leq n} \left( -\frac{\partial^2}{\partial x_j^2} + 2\omega x_j \frac{\partial}{\partial x_j} \right) - 2g_0 \sum_{1 \leq j < k \leq n} \frac{1}{x_j - x_k} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right),
\]

\[
(2.10b) \quad D_{1,0}^B = \sum_{1 \leq j \leq n} \left( -\frac{\partial^2}{\partial x_j^2} - 2g_1 \frac{\partial}{\partial x_j} + 2\omega x_j \frac{\partial}{\partial x_j} \right)
-2g_0 \sum_{1 \leq j < k \leq n} \left( \frac{1}{x_j - x_k} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) + \frac{1}{x_j + x_k} \left( \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_k} \right) \right).
\]

Formula (2.10a) and (2.10b) combined with the corresponding expressions for the eigenvalues taken from Theorem 2.3

(2.11) \( E_1^A(\lambda) = 2\omega(\lambda_1 + \cdots + \lambda_n) \), \( E_1^B(\lambda) = 4\omega(\lambda_1 + \cdots + \lambda_n) \),

render the second order differential equations \( D_{1,0}^A p_\lambda^A = E_1^A(\lambda)p_\lambda^A \) and \( D_{1,0}^B p_\lambda^B = E_1^B(\lambda)p_\lambda^B \) in a fully explicit form.

In principle it is possible to determine for given \( r \) the differential operator \( D_{r,0}^C \) algorithmically as the leading part of the (explicit) difference operator \( D_{r,\beta}^C \) (2.8a) \( (2.8b) \) (via the formal expansion in the step size parameter \( \beta \)). It seems a rather nontrivial combinatorial exercise, however, to derive along these lines a closed expression for the differential operator \( D_{r,0}^C \) for general \( r \).

### 2.3. Pieri type recurrence formulas.

To describe the recurrence relations for the multivariable Hermite and Laguerre polynomials it is convenient to pass from monic polynomials to a different normalization. Let

\[
(2.12) \quad P_\lambda^A(x) \equiv c_\lambda^A p_\lambda^A(x), \quad P_\lambda^B(x) \equiv c_\lambda^B p_\lambda^B(x)
\]

with

\[
(2.13a) \quad c_\lambda^A = \prod_{1 \leq j < k \leq n} \frac{[(k - j)g_0]_{\lambda_j - \lambda_k}}{[(1 + k - j)g_0]_{\lambda_j - \lambda_k}},
\]

\[
(2.13b) \quad c_\lambda^B = (-\omega)^{|\lambda|} \prod_{1 \leq j < k \leq n} \frac{[(k - j)g_0]_{\lambda_j - \lambda_k}}{[(1 + k - j)g_0]_{\lambda_j - \lambda_k}} \prod_{1 \leq j \leq n} \frac{1}{[(n - j)g_0 + g_1 + 1/2]_{\lambda_j}},
\]

where we have employed the Pochhammer symbol defined by \( [a]_0 \equiv 1 \) and \( [a]_l \equiv a(a + 1) \cdots (a + l - 1) \) for \( l = 1, 2, 3, \ldots \).
Theorem 2.6 (Normalization). The normalization of the polynomial $P^C_\lambda(x)$ is such that
\[
\lim_{\alpha \to \infty} \alpha^{-|\lambda|} P^A_\lambda(\alpha \mathbf{1}) = 1 \quad (\text{with } \mathbf{1} \equiv (1, \ldots, 1))
\]
and
\[
P^B_\lambda(\mathbf{0}) = 1 \quad (\text{with } \mathbf{0} \equiv (0, \ldots, 0)).
\]

The next theorem describes an expansion formula for the product of $P^C_\lambda(x)$ and the first elementary symmetric function in $x_1, \ldots, x_n$ (type $A$) or in $x_1^2, \ldots, x_n^2$ (type $B$). Formulas of this type are often referred to as Pieri formulas. (More generally, Pieri formulas are relations in a commutative algebra describing the expansion (in terms of a basis) of products between the basis elements and a set of generators for the algebra.) For $n = 1$ the formulas in the theorem reduce to classical three-term recurrence relations for the one-variable Hermite and Laguerre polynomials (cf. Section 3.1).

Theorem 2.7 (Pieri formulas: simplest case). The (renormalized) multivariable Hermite and Laguerre polynomials $P^C_\lambda(x)$ satisfy the recurrence relations ($e_j$ denotes the $j$th unit vector in the standard basis of $\mathbb{R}^n$)

\[
\left( \sum_{1 \leq j \leq n} x_j \right) P^A_\lambda(x) = \sum_{1 \leq j \leq n} \left( \hat{V}^A_j P^A_{\lambda + e_j}(x) + \hat{V}^A_{-j} P^A_{\lambda - e_j}(x) \right)
\]

\[
\left( -\omega \sum_{1 \leq j \leq n} x_j^2 \right) P^B_\lambda(x) = \sum_{1 \leq j \leq n} \left( \hat{V}^B_j P^B_{\lambda + e_j}(x) - (\hat{V}^B_j + \hat{V}^B_{-j}) P^B_{\lambda}(x) + \hat{V}^B_{-j} P^B_{\lambda - e_j}(x) \right)
\]

with

\[
\hat{V}^A_j = \prod_{1 \leq k \leq n, k \neq j} \left( 1 + \frac{g_0}{(k-j)g_0 + \lambda_j - \lambda_k} \right),
\]

\[
\hat{V}^A_{-j} = \frac{(n-j)g_0 + 1 + \lambda_j}{2\omega} \prod_{1 \leq k \leq n, k \neq j} \left( 1 - \frac{g_0}{(k-j)g_0 + \lambda_j - \lambda_k} \right)
\]

and

\[
\hat{V}^B_j = \frac{(n-j)g_0 + 1/2 + \lambda_j}{\omega} \prod_{1 \leq k \leq n, k \neq j} \left( 1 + \frac{g_0}{(k-j)g_0 + \lambda_j - \lambda_k} \right),
\]

\[
\hat{V}^B_{-j} = \frac{(n-j)g_0 + \lambda_j}{\omega} \prod_{1 \leq k \leq n, k \neq j} \left( 1 - \frac{g_0}{(k-j)g_0 + \lambda_j - \lambda_k} \right).
\]

One word of caution is at place here. It may of course happen that for certain $\lambda \in \Lambda$ and $j \in \{1, \ldots, n\}$ the vector $\lambda + e_j$ (or $\lambda - e_j$) does not lie in the cone $\Lambda$. For such boundary situations the polynomial $P^C_{\lambda + e_j}$ (or $P^C_{\lambda - e_j}$) is not defined and it might a priori seem that r.h.s. of the recurrence relation does not make sense in this case. It is not difficult to verify, however, that in these situations the coefficient $\hat{V}^C_j$ (or $\hat{V}^C_{-j}$) in front of $P^C_{\lambda + e_j}$ (or $P^C_{\lambda - e_j}$) vanishes. (Indeed, for $\lambda \in \Lambda$ we have that $\lambda + e_j \not\in \Lambda$ if $\lambda_j = \lambda_j$ and that $\lambda - e_j \not\in \Lambda$ if $\lambda_j = \lambda_j$ or if $j = n$ and $\lambda_n = 0$. In the former case we pick up a zero in $\hat{V}^C_j$ from the factor...
1 + g_0((k - j)g_0 + \lambda_j - \lambda_k)^{-1} with k = j - 1 and in the latter case we have a zero in \( \hat{V}_j^C \) from the factor \( 1 - g_0((k - j)g_0 + \lambda_j - \lambda_k)^{-1} \) with k = j + 1 or from the factor \((n - j)g_0 + \lambda_j\) with j = n, respectively.

The next step is to generalize the expansion formulas of Theorem 2.7 to (multiplication by) arbitrary elementary symmetric functions. To this end we introduce (for \( r = 1, \ldots, n \))

\[
\begin{align*}
(2.14a) \quad \hat{E}_r^A(x) &= \sum_{j<\{1, \ldots, n\}, |J|=r} \prod_{j\in J} x_j, \\
(2.14b) \quad \hat{E}_r^B(x) &= (-\omega)^r \sum_{j<\{1, \ldots, n\}, |J|=r} \prod_{j\in J} x_j^2.
\end{align*}
\]

It is clear that the products \( \hat{E}_r^C(x)P_C^\lambda(x) \) can be written as a linear combination of \( P_\mu^C(x) \) with \( \mu \leq \lambda + e_1 + \cdots + e_r \) (this is immediate from the structure of the monomial expansion of \( P_\lambda^C(x) \) and the fact that such expansion formulas for these products evidently hold if we replace the polynomials \( P_\lambda^C(x) \) by their leading monomials \( m_\lambda(x) \) (\( C = A \)) and \( m_{2\lambda}(x) \) (\( C = B \)).

It turns out that many of the coefficients \( c_\mu \) in the expansion \( \hat{E}_r^C P_C^\lambda = \sum_{\mu \leq \lambda} c_\mu P_\mu^C \) are in fact zero (for \( r = 1 \) this is of course apparent from Theorem 2.7). The following theorem provides detailed information on the structure of the terms entering the Pieri type expansion of the product between the basis element \( P_\lambda^C(x) \) and an arbitrary elementary symmetric function \( \hat{E}_r^C(x) \) [2.14a], [2.14b].

**Theorem 2.8** (Pieri formulas: general structure and leading coefficients).
The (renormalized) multivariable Hermite and Laguerre polynomials \( P_\lambda^C(x) \) [2.12] satisfy a system of recurrence relations of the form (\( e_j \equiv \sum_{j\in J} e_j \))

\[
\hat{E}_{r}^C(x)P_\lambda^C(x) = \sum_{J_+, J_- \subset \{1, \ldots, n\}, |J_+| + |J_-| = r, J_+ \cap J_- = \emptyset, |J_+| + |J_-| \leq r} h_{J_+, J_-}^C(\lambda) \frac{\partial^{r}}{\partial x_j^{r}} P_\lambda^C(x)
\]

The coefficients \( h_{J_+, J_-}^C(\lambda) \) that correspond to index sets \( J_+, J_- \) with the sum of the cardinalities \( |J_+| + |J_-| \) being equal to \( r \) are explicitly given by

\[
\hat{E}_{r}^C(x)P_\lambda^C(x) = \hat{V}_{J_+, J_-}^C(\lambda, J_+ \cup J_-)^c
\]

where

\[
\hat{V}_{J_+, J_-}^C(\lambda, J_+ \cup J_-)^c = \prod_{j \in J_-} \frac{(n - j)g_0 + \lambda_j}{2\omega} \\
\times \prod_{j, j' \in J_-} (1 + \frac{g_0}{(j' - j)g_0 + \lambda_j - \lambda_{j'}})(1 + \frac{g_0}{1 + (j' - j)g_0 + \lambda_j - \lambda_{j'}}) \\
\times \prod_{j \in J_k, k \in K} (1 + \frac{g_0}{(k-j)g_0 + \lambda_j - \lambda_k})(1 - \frac{g_0}{(k-j)g_0 + \lambda_j - \lambda_k})
\]
and

\[
\hat{V}^B_{J_+,J_-;K} = \prod_{j \in J_+} ((n - j)g_0 + g_1 + 1/2 + \lambda_j) \prod_{j \in J_-} ((n - j)g_0 + \lambda_j) \\
\times \prod_{j \in J_+,\ j' \in J_-} \left( 1 + \frac{g_0}{j' - j}g_0 + \lambda_j - \lambda_j' \right) \left( 1 + \frac{g_0}{j' - j}g_0 + \lambda_j - \lambda_j' \right) \\
\times \prod_{\substack{j \in J_+,\ k \in K \atop k \neq K}} \left( 1 + \frac{g_0}{k - j}g_0 + \lambda_j - \lambda_k \right) \prod_{\substack{j \in J_-,\ k \in K \atop k \neq K}} \left( 1 - \frac{g_0}{k - j}g_0 + \lambda_j - \lambda_k \right)
\]

(with the convention that empty products are equal to one).

Theorem 2.8 constitutes a partial generalization of Theorem 2.7. For \( r = 1 \) the structure described in Theorem 2.8 is compatible with that of the formulas in Theorem 2.7 and we furthermore recover the coefficient in the r.h.s. of \( P^C_{\lambda + \epsilon_j} \) (viz. \( \hat{V}^C_{(\{j\},\emptyset;\{1,\ldots,n\}\setminus\{j\})} \)) and \( P^C_{\mu - \epsilon_j} \) (viz. \( \hat{V}^C_{(\emptyset;\{1,\ldots,n\}\setminus\{j\})j} \)) but not that of \( P^C_{\Lambda} \) (which—according to Theorem 2.7—happens to be zero for \( \Lambda = \{C\} \)).

Even though Theorem 2.8 is not completely explicit, as it does not tell us the expansion coefficients \( \hat{W}^C_{J_+,J_-;r}(\lambda) \) for \( |J_+| + |J_-| < r \), it is still useful in its present form. For instance, the theorem implies (together with the orthogonality) that

\[
(\hat{E}^C_r P^C_{\lambda},P^C_{\mu}) = \begin{cases} \\
0 \quad \text{if } \mu \neq \lambda + \epsilon_j - \epsilon_j_- \text{ with } |J_+| + |J_-| \leq r \\
\hat{V}^C_{J_+,J_-;\{(J_+ \cup J_-)^c\}} \quad \text{if } \mu = \lambda + \epsilon_j - \epsilon_j_- \text{ with } |J_+| + |J_-| = r
\end{cases}
\]

where \( J_+,J_- \in \{1,\ldots,n\} \) such that \( J_+ \cap J_- = \emptyset \). When applying (2.15) to the identity \( (\hat{E}^C_r P^C_{\lambda},P^C_{\lambda+\epsilon_1,\ldots,\epsilon_r})_{\lambda} = (P^C_{\lambda},\hat{E}^C_r P^C_{\lambda+\epsilon_1,\ldots,\epsilon_r})_{\lambda} \) one arrives at a system of recurrence relations for the squared norm of \( P^C_{\lambda} \)

\[
\hat{V}^C_{\{1,\ldots,r\};\{\{1,\ldots,n\}\setminus\{1,\ldots,r\}\}}(\lambda) \langle P^C_{\lambda+\epsilon_1,\ldots,\epsilon_r},P^C_{\lambda+\epsilon_1,\ldots,\epsilon_r} \rangle_{\lambda} = \\
\hat{V}^C_{\emptyset;\{1,\ldots,r\};\{\{1,\ldots,n\}\setminus\{1,\ldots,r\}\}}(\lambda + \epsilon_1,\ldots,\epsilon_r) \langle P^C_{\lambda},P^C_{\lambda} \rangle_{\lambda}
\]

\((r = 1,\ldots,n)\). The recurrence relations in (2.16) determine \( \langle P^C_{\lambda},P^C_{\lambda} \rangle_{\lambda} \) uniquely in terms of \( (1,1)_{\lambda} \) (because the (fundamental weight) vectors \( e_{\{1,\ldots,r\}} \), \( r = 1,\ldots,n \) positively generate the cone \( \Lambda \) (2.3) and the coefficient \( \hat{V}^C_{\{1,\ldots,r\};\{\{1,\ldots,n\}\setminus\{1,\ldots,r\}\}}(\lambda) \neq 0 \) for \( \lambda \in \Lambda \)). This observation gives rise to an alternative (constructive) proof of the norm formulas in Theorem 2.3 different from the proof presented in Section 4.1.

Indeed, by using the property \( e^C_{\lambda} = c^C_{\lambda + \epsilon_1,\ldots,\epsilon_r} \hat{V}^C_{\{1,\ldots,r\};\{r,\ldots,n\}}(\lambda) \) one rewrites (2.16) in the monic form

\[
\langle p^C_{\lambda+\epsilon_1,\ldots,\epsilon_r},p^C_{\lambda+\epsilon_1,\ldots,\epsilon_r} \rangle_{\lambda} = \\
\hat{V}^C_{\{r,\ldots,n\};\{1,\ldots,r\}}(\lambda) \hat{V}^C_{\emptyset;\{1,\ldots,r\};\{r,\ldots,n\}}(\lambda + \epsilon_1,\ldots,\epsilon_r) \langle p^C_{\lambda},p^C_{\lambda} \rangle_{\lambda},
\]

which upon iteration and matching of the initial conditions so as to reduce for \( \lambda = 0 \) to the Mehta-Macdonald formulas (2.7a), (2.7b) (cf. Section 4.1) leads to the norm formulas of Theorem 2.3.

The last theorem (below) provides a complete (explicit) description of the expansion coefficients \( \hat{W}^C_{J_+,J_-;r}(\lambda) \) for the type B case (thus including the coefficients
corresponding to index sets with $|J_+| + |J_-| < r$. This renders the system of recurrence relations of the type given by Theorem 2.8 in a fully explicit form for the multivariable Laguerre family.

**Theorem 2.9** (Pieri formulas: explicit expansion coefficients Laguerre case). The coefficients in the recurrence relations of the type described by Theorem 2.8 are for the renormalized multivariable Laguerre polynomials $P_n^B(x)$ (3.12) given by

$$
\hat{W}_{J_+,J_-;r}(\lambda) = \hat{V}_{J_+,J_-;r}(\lambda) + \hat{U}_{J_+,J_-;r}(\lambda),
$$

with $\hat{V}_{J_+,J_-;r}(\lambda) \equiv 1$ for $r = 0$.

3. Comments

3.1. **The special case** $n = 1$. In the case of one single variable ($n = 1$), the weight functions reduce to

$$
\Delta^A(x) = e^{-\omega x^2}, \quad \Delta^B(x) = |x|^{2q_1} e^{-\omega x^2}.
$$

The polynomials then become monic Hermite polynomials (type A) and monic Laguerre polynomials of a quadratic argument (type B), which can be written explicitly in terms of a terminating confluent hypergeometric series [AbSt]

\[
\begin{align*}
(3.2a) \quad p^A_\lambda(x) &= \begin{cases}
\frac{[1/2]_{\lambda/2}}{\omega^{\lambda/2}} \binom{x}{\lambda/2} 
\frac{\Gamma(-\lambda/2)}{\Gamma(-\lambda/2)} 1F_1 \left( \frac{-\lambda/2}{1/2}; \omega x^2 \right) & \text{for } \lambda \text{ even} \\
\frac{[3/2]_{(\lambda-1)/2}}{\omega^{(\lambda-1)/2}} x \binom{x}{\lambda-1/2} 1F_1 \left( \frac{-\lambda-1/2}{3/2}; \omega x^2 \right) & \text{for } \lambda \text{ odd}
\end{cases}
\end{align*}
\]

\[
(3.2b) \quad p^B_\lambda(x) = \frac{[g_1 + 1/2]_\lambda}{(-\omega)^\lambda} 1F_1 \left( -\lambda; \frac{g_1 + 1/2}{\omega x^2} \right)
\]

(with $1F_1 \left( \frac{a}{b} ; z \right) = \sum_{m=0}^{\infty} \frac{[a]_m}{[b]_m m!} z^m$ and $\lambda = 0, 1, 2, \ldots$). The norm formulas, differential equations and recurrence relations reduce in this special situation to classical formulas for the one-variable Hermite and the Laguerre polynomials (notice however that the scale parameter $\omega$ is usually taken to be equal to one and that our normalization differs from the standard one).
forms the second order differential operators $D^C$ con-
jugation with the square root of the weight function $\Delta$ exactly solvable quantum mechanical

Most of the results stated in Section 2 admit an interpretation in terms of certain
3.2. Theorem 2.3 that the functions $E_\alpha$ with
rational Calogero model with harmonic term were introduced in [D3, BF] and also
(for type $A$) in [FW]. The orthogonality (in $L^2(\mathbb{R}^n, dx_1, \ldots, dx_n)$) of the basis

Norm formulas (cf. Theorem 2.2)

\begin{align}
(3.3a) \quad & \int_{-\infty}^{\infty} |p_\lambda^A(x)|^2 \Delta^B(x) \, dx = \frac{\lambda \sqrt{\pi}}{2 \omega^{\lambda+1/2}} \\
(3.3b) \quad & \int_{-\infty}^{\infty} |p_\lambda^B(x)|^2 \Delta^B(x) \, dx = \frac{\lambda \Gamma(g_1 + 1/2 + \lambda)}{\omega^{2\lambda+g_1+1/2}}
\end{align}

Differential equations (cf. Theorem 2.3 and Eqs. (2.10a), (2.10b) and (2.11))

\begin{align}
(3.4a) \quad & - \frac{d^2}{dx^2} p_\lambda^A + 2\omega x \frac{d}{dx} p_\lambda^A = 2 \omega \lambda p_\lambda^A \\
(3.4b) \quad & - \frac{d^2}{dx^2} p_\lambda^B - \frac{g_1}{x} \frac{d}{dx} p_\lambda^B + 2\omega x \frac{d}{dx} p_\lambda^B = 4 \omega \lambda p_\lambda^B
\end{align}

Recurrence relations (cf. Theorem 2.7)

\begin{align}
(3.5a) \quad & x p_\lambda^A = p_{\lambda+1}^A + \frac{\lambda}{2\omega} p_{\lambda-1}^A \\
(3.5b) \quad & - \omega x^2 p_\lambda^B = (g_1 + 1/2 + \lambda) p_{\lambda+1}^B - (g_1 + 1/2 + 2\lambda) p_\lambda^B + \lambda p_{\lambda-1}^B
\end{align}

with $p_\lambda^A(x) = p_\lambda^A(x)$ and $p_\lambda^B(x) = \frac{\lambda}{g_1 + 1/2 + \lambda} p_\lambda^B(x)$. The normalization properties
for $p_\lambda^A(x)$ and $p_\lambda^B(x)$ in Theorem 2.4 state that $\lim_{\alpha \to \infty} a^{-\lambda} P_\lambda^A(\alpha) = 1$ and that
$p_\lambda^B(0) = 1$. In the present situation these properties are immediate from the explicit

3.2. Eigenfunctions for the rational Calogero model in an harmonic well.

Most of the results stated in Section 2 admit an interpretation in terms of certain exactly solvable quantum mechanical $n$-particle models on the line. Specifically, conjugation with the square root of the weight function $\Delta^C(x)$ (1.1a), (1.1b) transforms the second order differential operators $D^C$ (2.10a), (2.10b) into Hamiltonians
for the rational quantum Calogero models with harmonic confinement associated
to the classical root systems $[Ca, OP]$.

\begin{align}
(3.6a) \quad & H_1^A = (\Delta^A)^{1/2} D_1^A(\Delta^A)^{-1/2} = \\
& \sum_{1 \leq j \leq n} \left( -\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2 \right) + 2g_0(g_0 - 1) \sum_{1 \leq j < k \leq n} (x_j - x_k)^{-2} - E_0^A,
\end{align}

\begin{align}
(3.6b) \quad & H_1^B = (\Delta^B)^{1/2} D_1^B(\Delta^B)^{-1/2} = \\
& \sum_{1 \leq j \leq n} \left( -\frac{\partial^2}{\partial x_j^2} + g_1(g_1 - 1)x_j^{-2} + \omega^2 x_j^2 \right) \\
& + 2g_0(g_0 - 1) \sum_{1 \leq j < k \leq n} \left( (x_j - x_k)^{-2} + (x_j - x_k)^{-2} \right) - E_0^B
\end{align}

with $E_0^A = \omega n(1 + g_0(n - 1))$ and $E_0^B = \omega n(1 + 2g_0(n - 1) + 2g_1)$. It is clear from
Theorem 2.3 that the functions

\begin{align}
(3.7) \quad & \psi_\lambda^C(x) = (\Delta^C(x))^{1/2} p_\lambda^C(x), \quad \lambda \in \Lambda \quad (C = A, B)
\end{align}

constitute a basis of eigenfunctions for $H_1^C$ (3.6a), (3.6b) with the corresponding
eigenvalues given by $E_1^C(\lambda)$ (2.11). In its present form these eigenfunctions for the
rational Calogero model with harmonic term were introduced in [D3, BF] and also
(for type $A$) in [FW]. The orthogonality (in $L^2(\mathbb{R}^n, dx_1, \ldots, dx_n)$) of the basis
Historically, the study of the eigenvalue problem for the type A Hamiltonian \( H^A \) was initiated by Calogero, who computed the spectrum and determined the structure of the corresponding eigenfunctions to be a product of the ground-state wave function \( \psi_0(x) = (\Delta^A(x))^{1/2} \) and certain symmetric polynomials in \( x_1, \ldots, x_n \). To be precise, Calogero considered a translationally symmetric \( n \)-particle system with a potential of the form

\[
V(x) = \sum_{1 \leq j < k \leq n} \left( G_0(x_j - x_k)^2 + G_1(x_j - x_k)^2 \right),
\]

which is seen to be equivalent to the type A system above up to a simple center of mass motion \( (\sum_{j<k}(x_j - x_k))^2 = n \sum j^2 - (\sum j)^2) \). Explicit expressions for the Calogero eigenfunctions were found for small particle number \( (n \leq 5) \) by Perelomov and Gambardella [PG]. More recently, it was observed [BH], [BHK] that for arbitrary particle number \( n \) it is possible to construct a basis of eigenfunctions for \( H^A \) with the aid of certain creation and annihilation operators \( a_j^+ \) and \( a_j^- \) \( (j = 1, \ldots, n) \) that are built from the differential-reflection operators introduced by Dunkl [Du3]. In a nutshell: the relevant operators \( \Delta^A \) are obtained starting from the usual creation/annihilation operators \( (\pm \partial_j + \omega x_j) \) for a system of uncoupled (bosonic) harmonic oscillators (this corresponds to \( g_0 = 0 \)) by replacing the partial derivatives by the corresponding Dunkl differential-reflection operators (associated to the root system \( A_{n-1} \)). If one acts on the ground-state wave function \( \Psi_0^A(x) = (\Delta^A(x))^{1/2} \) (which is annihilated by \( a_1^- \), \ldots, \( a_n^- \)) with an arbitrary symmetric homogeneous polynomial of degree \( l \) in the creation operators \( a_1^+, \ldots, a_n^+ \), then one winds up with a symmetric eigenfunction of \( H^A \) with eigenvalue \( 2l\omega \). By taking for the symmetric polynomial in question the Jack symmetric function \( J_\lambda(x; 1/g_0) \) \( (|\lambda| = 1, \ldots, n) \), one arrives (up to a normalization factor) precisely at the eigenfunction \( \Psi_\lambda^A(x) \).

The basis of eigenfunctions of the form \( \Psi_\lambda^A(x) \) for \( H^C \) is also very special in that it simultaneously diagonalizes the higher-order quantum integrals for the Calogero model. Specifically, by applying the similarity transformation \( D_r^C \rightarrow H_r^C \) also to the higher-order differential operators \( D_r^C \) in Theorem 2.2, one obtains a complete set of commuting quantum integrals for the Calogero model of the form

\[
(3.8) \quad H_r^C = (\Delta^C)^{1/2} D_r^C (\Delta^C)^{-1/2} = (-1)^r \sum_{j \in \{1, \ldots, n\}} \prod_{|J| = r} \frac{\partial^2}{\partial x_j^2} + \text{lo}. \quad (r = 1, \ldots, n).
\]

It is immediate from Theorem 2.2 that the functions \( \Psi_\lambda^C(x) \) constitute a basis of joint eigenfunctions for the differential operators \( H_r^C \), \( r = 1, \ldots, n \). Since the corresponding eigenvalues \( E_1^C(\lambda), \ldots, E_n^C(\lambda) \) separate the points of the cone \( \Lambda \), it follows that this property actually determines such basis uniquely. In other words, ambiguities in the possible choices for the basis of eigenfunctions of \( H_r^C \), \( r = 1, \ldots, n \) caused by the degeneracy of the spectrum \( \{2j\} \) get eliminated by requiring the eigenfunctions to be a basis of joint eigenfunctions for \( H_1^C, \ldots, H_n^C \). The
other important (and restrictive) property of the basis \( \Psi^C(x) \) \([3.7]\), viz. its orthogonality, implies—together with the real-valuedness of the eigenvalues \( E^C(\lambda) \)—that the differential operators \( H^C \) are (essentially) self-adjoint (since unitarily equivalent to real multiplication operators) in the Hilbert space of permutation-invariant (type \( A \)) or permutation-invariant and even (type \( B \)) functions in \( L^2(\mathbb{R}^n, \, dx_1 \cdots dx_n) \) (cf. Corollary \( 2.5 \)).

One would like to cast the differential operators \( H^C \) for arbitrary \( r \in \{1, \ldots, n\} \) in a more explicit form. Explicit formulas for \( n \)-independent commuting differential operators generating the same algebra as our operators \( H^B_1, \ldots, H^B_n \) appear as special cases of the commuting families presented in \([OOS, OS]\). In order to determine the relation between the operators \( H^B_r \) \((3.8)\) and the type \( B \) differential operators in \([OOS, OS]\) explicitly (for all \( r \)), one would have to know the eigenvalues of the latter operators on the basis \( \Psi^B_1(x) \) \([3.7]\). For the type \( A \) we are not aware that similar explicit formulas for a set of commuting differential operators generating the same (commuting) algebra as \( H^A_1, \ldots, H^A_n \) have been reported in the literature (except for \( \omega = 0 \) \([OOS]\)). However, it is known that such commuting differential operators may be characterized in terms of Dunkl’s differential-reflection operators \([7]\) and recently the eigenvalues of the thus obtained differential operators on the basis \( \Psi^A_1(x) \) \([3.7]\) were determined \([K]\) (thus allowing a comparison with our differential operators \( H^A \)).

### 3.3. Relation to Dunkl’s generalized spherical harmonics

An alternative approach towards the solution of the eigenvalue problem for the second order differential operators \( D^C_1 \) \([2.10a], 2.10b]\) is to separate the eigenvalue equation in a ‘radial’ and a ‘spherical’ component. In essence this is the method used by Calogero \([C]\) to obtain the eigenfunctions for the type \( A \) Hamiltonian \( H^A_1 \) \((3.6a)\) (cf. the comments in the previous subsection). Specifically, if one substitutes an Ansatz function of the form \( R^C(r)Y^C_1(x) \), where \( R^C(r) \) is a function of \( r = \sqrt{x_1^2 + \cdots + x_n^2} \) and \( Y^C_1(x) \) is a permutation symmetric homogeneous polynomial of degree \( l \) in \( x_1, \ldots, x_n \) \((C = A)\) or \( x_1^2, \ldots, x_n^2 \) \((C = B)\), then it is seen that this yields an eigenfunction of \( D^C_1 \) \([2.10a], 2.10b]\) with eigenvalue \( E^C \) if \( R^C(r) \) and \( Y^C_1(x) \) satisfy

\[
(3.9a) - \frac{d^2 R^A}{dr^2} + \left( 2\omega r - \frac{2l + gn(n-1) + n-1}{r} \right) \frac{dR^A}{dr} = (E^A - 2\omega l)R^A,
\]

\[
(3.9b) - \frac{d^2 R^B}{dr^2} + \left( 2\omega r - \frac{4l + gn(n-1) + 2ng_1 + n-1}{r} \right) \frac{dR^B}{dr} = (E^B - 4\omega l)R^B
\]

and

\[
(3.10) \quad L^C Y^C_1 = 0 \quad (C = A, B)
\]

with

\[
(3.11a) \quad L^A = \sum_{1 \leq j \leq n} \frac{\partial^2}{\partial x_j^2} + 2g_0 \sum_{1 \leq j < k \leq n} \frac{1}{x_j - x_k} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right),
\]

\[
(3.11b) \quad L^B = \sum_{1 \leq j \leq n} \left( \frac{\partial^2}{\partial x_j^2} + 2g_1 \frac{\partial}{\partial x_j} \right) + 2g_0 \sum_{1 \leq j < k \leq n} \left( \frac{1}{x_j - x_k} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) + \frac{1}{x_j + x_k} \left( \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_k} \right) \right).
\]
The ‘radial’ equations (3.9a) and (3.9b) are confluent hypergeometric type equations that admit polynomial solutions for \(E^A = 2\omega(l + 2m)\) and \(E^B = 4\omega(l + m)\) (with \(m \in \mathbb{N}\)) given by Laguerre polynomials in \(r^2\) (cf. Eqs. (3.21) and (3.4b))

\[
R_m^A(r) = \frac{[l + (n - 1)(1 + ng_0)/2]_m}{(-\omega)^m} \times \frac{1}{1F1} \left( \begin{array}{c} -m \\ l + (n - 1)(1 + ng_0)/2 : \omega r^2 \end{array} \right),
\]

\[
R_m^B(r) = \frac{[2l + n(1/2 + g_1 + (n - 1)g_0)]_m}{(-\omega)^m} \times \frac{1}{1F1} \left( \begin{array}{c} -m \\ 2l + n(1/2 + g_1 + (n - 1)g_0) : \omega r^2 \end{array} \right).
\]

(Here we have chosen the normalization chosen such that \(R_m^C(r)\) is monic.)

The ‘spherical’ equation (3.10) was studied (for type \(A\)) by Calogero [Ca] and in further detail and more generality (therewith also including the type \(B\)) by Dunkl [Du1]. Let \(P^C_l\) be the space of homogeneous symmetric polynomials of degree \(l\) in \(x_1, \ldots, x_n\) (\(C = A\)) or \(x_1^2, \ldots, x_n^2\) (\(C = B\)) and let \(\mathcal{H}^C_l \subset P^C_l\) be the subspace of polynomials satisfying (3.10). The polynomials in \(\mathcal{H}^C_l\) are referred to as (symmetric) generalized spherical harmonics. For \(g_0, g_1 = 0\) these generalized spherical harmonics reduce to ordinary harmonic polynomials in \(\mathbb{R}^n\). It follows from Dunkl’s theory in [Du1] (see also [Du2] for the extension to the nonsymmetric case) that \(\dim(\mathcal{H}_l^A) = \dim(P^A_l) - \dim(P_{l-2}^A)\) and that \(\dim(\mathcal{H}_l^B) = \dim(P^B_l) - \dim(P_{l-1}^B)\).

The upshot is that each polynomial \(p^C_{\lambda}\) may be written uniquely in the form

\[
(3.13a) \quad p^A_{\lambda}(x) = \sum_{m=0}^{[|\lambda|/2]} R_m^A(r)Y^A_{|\lambda|-2m}(x), \quad Y^A_{|\lambda|-2m} \in \mathcal{H}^A_{|\lambda|-2m},
\]

\[
(3.13b) \quad p^B_{\lambda}(x) = \sum_{m=0}^{[|\lambda|]} R_m^B(r)Y^B_{|\lambda|-m}(x), \quad Y^B_{|\lambda|-m} \in \mathcal{H}^B_{|\lambda|-m}.
\]

(where \([\cdot]\) represents the function that extracts the integer part). Indeed, the functions of the form in the r.h.s. of (3.13a) and (3.13b) are eigenfunctions of \(D^C_l\) (2.10a), (2.10b) corresponding to the eigenvalue \(2\omega|\lambda|\) (type \(A\)) and \(4\omega|\lambda|\) (type \(B\)), respectively. Furthermore, the functions of this form span a space of dimension \(\sum_{m=0}^{[|\lambda|/2]} \dim(\mathcal{H}^C_{|\lambda|-2m}) = \dim(P^C_{|\lambda|})\) and \(\sum_{m=0}^{[|\lambda|]} \dim(\mathcal{H}^B_{|\lambda|-m}) = \dim(P^B_{|\lambda|})\), which is precisely the multiplicity of the eigenvalues \(E^C_1(\lambda)\) (2.11).

The formulas (3.13a), (3.13b) describe the relation between the Calogero type eigenfunctions of the form \(R_m^C(r)Y^C_\lambda(x)\) and the Hermite/Laguerre basis \(p^C_\lambda(x)\). To determine precisely which functions \(Y^C_\lambda(x)\) appear in the decompositions (3.13a) and (3.13b), we pick the leading homogeneous parts on both sides of the equation. It is known from the work of Lassalle [La2] [La3] that the highest-order homogeneous part of the multivariable Hermite (type \(A\)) and Laguerre (type \(B\)) polynomials are (with our normalization monic) Jack polynomials with parameter \(\alpha = 1/g_0\) in \(x_1, \ldots, x_n\) and \(x_1^2, \ldots, x_n^2\), respectively. So, we get upon taking the leading
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homogeneous part

\begin{align}
(3.14a) \quad J_\lambda(x; 1/g_0) &= \sum_{m=0}^{[|\lambda|/2]} r^{2m} Y^A_{|\lambda|-2m}(x), \quad Y^A_{|\lambda|-2m} \in \mathcal{H}^A_{|\lambda|-2m} \\
(3.14b) \quad J_\lambda(x^2; 1/g_0) &= \sum_{m=0}^{[|\lambda|]} r^{2m} Y^B_{|\lambda|-m}(x), \quad Y^B_{|\lambda|-m} \in \mathcal{H}^B_{|\lambda|-m},
\end{align}

where $J_\lambda(x; 1/g_0)$ and $J_\lambda(x^2; 1/g_0)$ denote the monic Jack polynomial in $x_1, \ldots, x_n$ and $x_1^2, \ldots, x_n^2$ with parameter $\alpha = 1/g_0$. In \cite{Du1}, Dunkl provides inversion formulas for decompositions of the form (3.14a), (3.14b) with which one can express the functions $Y^A_{|\lambda|-2m}$ and $Y^B_{|\lambda|-m}$ in terms of the homogeneous symmetric polynomial in the l.h.s. In our case these inversion formulas become

\begin{align}
(3.15a) \quad Y^A_{|\lambda|-2m}(x) &= \Pi^A_{|\lambda|-2m} J_\lambda(x; 1/g_0), \\
(3.15b) \quad Y^B_{|\lambda|-m}(x) &= \Pi^B_{|\lambda|-m} J_\lambda(x^2; 1/g_0)
\end{align}

with

\begin{align}
\Pi^A_{|\lambda|-2m} &= \frac{1}{4^m m! [n/2 + d^A + |\lambda| - 2m]} T^A_{|\lambda|-2m}(L^A)^m \\
T^A_k &= \sum_{j=0}^{[k/2]} \frac{r^{2j}}{4^j j! [-n/2 - d^A - k + 2]_j} (L^A)^j,
\end{align}

d^A = g_0 n (n - 1)/2 and

\begin{align}
\Pi^B_{|\lambda|-m} &= \frac{1}{4^m m! [n/2 + d^B + 2|\lambda| - 2m]} T^B_{|\lambda|-m}(L^B)^m \\
T^B_k &= \sum_{j=0}^{k} \frac{r^{2j}}{4^j j! [-n/2 - d^B - 2k + 2]_j} (L^B)^j,
\end{align}

d^B = g_0 n (n - 1) + nq_1. Formulas (3.13a), (3.13b) combined with (3.15a), (3.15b) render the decomposition of the multivariable Hermite and Laguerre polynomials in terms of Dunkl’s generalized spherical harmonics in a closed form.

3.4. Notes. i. Orthogonality. The orthogonality of the multivariable Hermite and Laguerre polynomials was stated in \cite{M3} and \cite{La2, La3}. It was proved by Macdonald for the cases $g_0 = 1/2, 1$ and $2$. Recently, Baker and Forrester \cite{BF} proposed a proof valid for general parameters that exploits the fact that the polynomials may be seen as limiting cases of the multivariable Jacobi polynomials \cite{V, De2, BO, H}. In essence this approach should boil down to extending the transition in \cite{M1} from the Selberg integral (whose integrand is the weight function for the Jacobi polynomials) to the Mehta-Macdonald integrals (2.7a), (2.7b) (whose integrands are the weight functions for the multivariable Hermite and Laguerre polynomials) so as to include also the polynomials of higher degree.

ii. Norm formulas. Norm formulas for the multivariable Hermite and Laguerre polynomials can be found in \cite{M3} and \cite{La2, La3}, with again proofs given by Macdonald for $g_0 = 1/2, 1$ and $2$. Baker and Forrester provide a proof of these norm formulas for general parameters that hinges on a generating function approach (and uses also the orthogonality) \cite{BF}. The expressions for the norms given in \cite{M3, La2, La3, BF} are written in terms of the Mehta-Macdonald integrals (2.7a),
and evaluations of Jack symmetric functions at the identity. It can be verified with the aid of known evaluation formulas for the Jack symmetric functions due to Stanley (see \cite{St, M4}) that our norm formulas in Theorem 2.2 are in agreement with those in \cite{M3, La2, La3, BF}.

iii. Differential equations. The second order differential equation for the multivariable Hermite and Laguerre polynomials of the form \( D_1^C p^C_{\lambda} = E_1^C(\lambda)p^C_{\lambda} \) was already given by Macdonald \cite{M3} and Lassalle \cite{La2, La3}. In \cite{BF} a procedure is described to obtain the complete system of \( n \)-independent differential equations starting from the explicitly known differential equations for the Jack symmetric functions found by Sekiguchi \cite{S} and Debiard \cite{De1} (see also \cite{M2, M4}). Another approach to arrive at such a system of differential equations for \( p^C_{\lambda} \) is to employ Dunkl’s differential-reflection operators; see \cite{Ka} for a treatment along this lines of the Hermite case. Yet another strategy would be to analyze the limit behavior of the system of differential equations for the Jacobi polynomials due to Debiard \cite{De2}. All these methods have in common with our approach in Section 2.2 that it seems a priori difficult (from a computational point of view) to extract further explicit information pertaining to a possible closed form for the higher-order differential equations at the confluent hypergeometric level (cf. also the comments in the last paragraph of Section 3.2).

iv. Recurrence formulas. Recurrence relations of the type given by Theorem 2.7 (i.e. the simplest ones, corresponding to the first elementary symmetric function) were recently derived independently by Baker and Forrester using a generating function for the polynomials \cite{BF}. As it stands, their recurrence formulas are a little less explicit than those obtained here since the coefficients are written in terms of certain implicitly defined generalized binomial coefficients and furthermore contain evaluations of the Jack symmetric functions at the identity. In order make their formulas fully explicit (so as to compare with Theorem 2.7) one again needs Stanley’s expressions for the Jack symmetric functions at the identity in combination with an explicit representation for the specific binomial coefficients at hand that can be found in \cite{La1}.

v. Rodrigues formulas. Recently, Ujino and Wadati derived Rodrigues type formulas for the multivariable Hermite polynomials \cite{UW} following an approach due to Lapointe and Vinet who obtained similar Rodrigues formulas for the Jack symmetric functions \cite{LV1, LV2}. Such Rodrigues formulas are particularly useful when trying to answer questions regarding the structure of the coefficients \( c_{\lambda,\mu} \) that appear in the expansion of the polynomials in terms of monomial symmetric functions. For instance, the Rodrigues formulas allowed Lapointe and Vinet to prove a weak form of the Macdonald-Stanley conjecture saying that (in an appropriate normalization) the expansion coefficients for the Jack symmetric functions in terms of monomial symmetric functions are polynomials in the parameters with integer coefficients \cite{LV1}. (See \cite{St, M4} for the Macdonald-Stanley conjecture and various related conjectures.) A similar statement also holds true for the multivariable Hermite polynomials \cite{UW}.
4. Proofs

In this section the properties of the multivariable confluent hypergeometric families stated in Section 2 are proven by viewing the polynomials as degenerate (limiting) cases of the multivariable hypergeometric continuous Hahn families (type A) and Wilson families (type B) that were investigated in \[ \text{[D3, D4]} \].

4.1. Orthogonality properties. In \[ \text{[D3, D4]} \] multivariable continuous Hahn and Wilson type polynomials were considered that are associated to the weight functions \( \Delta_{\text{H}} \) and \( \Delta_{\text{W}} \) being replaced by \( \Delta_{\text{polynomials}} \) defined by the conditions A. or B. Orthogonality properties.

(4.1a) \( \Delta_{\text{H}}(x) = \prod_{1 \leq j < k \leq n} \left| \frac{\Gamma(g_0 + i(x_j - x_k))}{\Gamma(i(x_j - x_k))} \right|^2 \prod_{1 \leq j \leq n} \left| \frac{\Gamma(a + ix_j) \Gamma(b + ix_j)}{\Gamma(2ix_j)} \right|^2 \)

and

(4.1b) \( \Delta_{\text{W}}(x) = \prod_{1 \leq j < k \leq n} \left| \frac{\Gamma(g_0 + i(x_j - x_k)) \Gamma(g_0 + i(x_j + x_k))}{\Gamma(i(x_j - x_k)) \Gamma(i(x_j + x_k))} \right|^2 \times \prod_{1 \leq j \leq n} \left| \frac{\Gamma(a + ix_j) \Gamma(b + ix_j) \Gamma(c + ix_j) \Gamma(d + ix_j)}{\Gamma(2ix_j)} \right|^2 \)

(with \( g_0 \geq 0 \) and \( \text{Re}(a, b, c, d) > 0 \)). Specifically, the multivariable continuous Hahn polynomials are defined by the conditions A.1, A.2 in Section 2.1 with \( \Delta_{\text{H}}(x) \) (4.1a) replacing the weight function \( \Delta_{\text{A}}(x) \) (1.1a). Similarly, the multivariable Wilson polynomials are defined by the conditions B.1, B.2 in Section 2.1 with \( \Delta_{\text{W}}(x) \) (4.1b) being replaced by \( \Delta_{\text{W}}(x) \) (4.1b).

If we rescale the variables by substituting

(4.2) \( x_j \rightarrow x_j/\beta, \quad j = 1, \ldots, n \)

and simultaneously perform a reparametrization of the form

(4.3) \( a = (\beta^2 \omega)^{-1}, \quad b = (\beta^2 \omega')^{-1}, \quad c = g_1, \quad d = g_1' + 1/2 \)

(with \( \omega, \omega' > 0, g_1, g_1' \geq 0 \) and \( \beta \) real), then the weight functions \( \Delta_{\text{H}}(x) \) (1.1a) and \( \Delta_{\text{W}}(x) \) (4.1b) pass (upon multiplication by the overall normalization constants \( D_{\text{A}}(\beta) \) and \( D_{\text{B}}(\beta) \)) over into

(4.4a) \( \Delta_{\beta}(x) = D_{\text{A}}(\beta) \prod_{1 \leq j < k \leq n} \left| \frac{\Gamma(g_0 + i\beta^{-1}(x_j - x_k))}{\Gamma(i\beta^{-1}(x_j - x_k))} \right|^2 \times \prod_{1 \leq j \leq n} \left| \frac{\Gamma(1/\omega \beta^2 + i\beta^{-1}x_j) \Gamma(1/\omega' \beta^2 + i\beta^{-1}x_j)}{\Gamma(2ix_j)} \right|^2 \)

and

(4.4b) \( \Delta_{\beta}(x) = D_{\text{B}}(\beta) \prod_{1 \leq j < k \leq n} \left| \frac{\Gamma(g_0 + i\beta^{-1}(x_j - x_k)) \Gamma(g_0 + i\beta^{-1}(x_j + x_k))}{\Gamma(i\beta^{-1}(x_j - x_k)) \Gamma(i\beta^{-1}(x_j + x_k))} \right|^2 \times \prod_{1 \leq j \leq n} \left| \frac{\Gamma(g_1 + i\beta^{-1}x_j) \Gamma(g_1' + 1/2 + i\beta^{-1}x_j)}{\Gamma(i\beta^{-1}x_j) \Gamma(1/2 + i\beta^{-1}x_j)} \right|^2 \times \Gamma(1/\omega \beta^2 + i\beta^{-1}x_j) \Gamma(1/\omega' \beta^2 + i\beta^{-1}x_j) \right|^2 \),
respectively. The normalization constants $D^A(\beta)$ and $D^B(\beta)$ are introduced so as to ensure finite limiting behavior of the weight functions for $\beta \to 0$. It is not so difficult to check (see appendix)—using Stirling’s formula for the asymptotics of $\Gamma(z)$ for $|z| \to \infty$ (see e.g. [AbSt])—that if one takes
\begin{align*}
D^A(\beta) &= |\beta|^{\alpha n(n-1)} \delta(\varpi, \beta)^2 n \delta(\varpi', \beta)^2 n, \\
D^B(\beta) &= |\beta|^{2\alpha n(n-1)+2n(g_1+g_1')} \delta(\varpi, \beta)^2 n \delta(\varpi', \beta)^2 n
\end{align*}
where
\begin{equation}
\delta(\alpha, \beta) \equiv \sqrt{\frac{e}{2\pi}} e^{(1+\log(\beta^2 \alpha)) (1/(\beta^2 \alpha)-1/2)},
\end{equation}
then the weight functions $\Delta^A_\beta(x)$ (4.4a) and $\Delta^B_\beta(x)$ (4.4b) converge pointwise to $\Delta^A(x)$ (1.14) and $\Delta^B(x)$ (1.14) for $\beta \to 0$ provided the parameters of $\Delta^C(x)$ are related to those of $\Delta^C_\beta(x)$ by
\begin{equation}
\omega \equiv \varpi + \varpi' \quad \text{and} \quad g_1 \equiv g_1 + g_1'.
\end{equation}
For our purposes, however, pointwise convergence is not sufficient and we need a somewhat stronger convergence result stating that the corresponding measures pass over into each other:
\begin{equation}
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x) \Delta^C(x) \, dx_1 \cdots dx_n = \\
\lim_{\beta \to 0} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x) \Delta^C_\beta(x) \, dx_1 \cdots dx_n \quad (C = A \text{ or } B),
\end{equation}
where $p(x)$ denotes an arbitrary polynomial in the variables $x_1, \ldots, x_n$. A proof of the limit formula (4.7) can be found in the appendix at the end of the paper.

Now, let $\{p^A_{x,\beta}\}_{x \in A}$ and $\{p^B_{x,\beta}\}_{x \in A}$ be the bases determined by the conditions A.1, A.2 and B.1, B.2 in Section 2.1 with the weight functions $\Delta^A(x)$ (1.13) and $\Delta^B(x)$ (1.13) being replaced by $\Delta^A_\beta(x)$ (1.4a) and $\Delta^B_\beta(x)$ (1.4a), respectively. So, up to scaling and reparametrization, the polynomials $p^A_{x,\beta}(x)$ amount to the multivariable continuous Hahn polynomials (multiplied by $\beta^{|L|}$) and the polynomials $p^B_{x,\beta}(x)$ amount to the multivariable Wilson polynomials (multiplied by $\beta^{|L|}$) from [D3, D4]. It then follows from the defining properties for the polynomials (of the form A.1, A.2 and B.1, B.2) and the limit formula (4.7) that
\begin{equation}
p^C_{x,\beta}(x) = \lim_{\beta \to 0} p^C_{x,\beta}(x)
\end{equation}
and that
\begin{equation}
\langle p^C_{x,\beta}, p^C_{\mu,\beta} \rangle^C = \lim_{\beta \to 0} \langle p^C_{x,\beta}, p^C_{\mu,\beta} \rangle^C_{\beta, \beta} = \\
\lim_{\beta \to 0} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p^C_{x,\beta}(x) \overline{p^C_{\mu,\beta}(x)} \Delta^C_\beta(x) \, dx_1 \cdots dx_n
\end{equation}
(of course again assuming a correspondence between the parameters in accordance with (1.6)). Theorem 2.2 is now immediate from limit formula (4.9) and the orthogonality of the multivariable continuous Hahn and Wilson families [D4] (which translates into the orthogonality of the bases $\{p^A_{x,\beta}\}_{x \in A}$ and $\{p^B_{x,\beta}\}_{x \in A}$ with respect to the weight functions $\Delta^A_\beta(x)$ (1.4a) and $\Delta^B_\beta(x)$ (1.4a)).
of the multivariable continuous Hahn resp. Wilson polynomials and the unit polynomial (with respect to the $L^2$ inner product over $\mathbb{R}^n$ with weight function $\Delta^\text{CH}(x)$ resp. $\Delta^\text{W}(x)$)---that were computed in [D4]. We have

$$\frac{\langle p_j^\text{CH}, p_k^\text{CH} \rangle_{\text{CH}}}{\langle 1, 1 \rangle_{\text{CH}}} = \frac{2^{-4|\lambda|}(\rho_j^\text{CH} + \rho_k^\text{CH})\rho_j^\text{CH} + \rho_k^\text{CH}}{\rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}} \rho_j^\text{CH} + \rho_k^\text{CH}}$$

with $\rho_j^\text{CH} = (n - j)g_0 + a + b - 1/2$, $\hat{\rho}_j^\text{CH} = a - b + 1/2$ and $\hat{\rho}_j^\text{W} = a - b + c + d + 1/2$. Here we have adopted the notation

$$\langle a_1, \ldots, a_p \rangle_t \equiv \langle a_1 \rangle_t \cdots \langle a_p \rangle_t$$

$$\langle a_0, [a]_l \equiv [a_0]_l \cdots [a_0]_l$$

where $[a_0]_l = 0, [a]_l = a(a + 1) \cdots (a + l - 1))$ for the Pochhammer symbols (and as usual $|\lambda| \equiv \lambda_1 + \cdots + \lambda_n$). Scaling of the variables and reparametrization in accordance with (4.2) and (4.3) leads us to the corresponding expressions for $p_{\lambda,\beta}^\text{CH}$ and $p_{\lambda,\beta}^\text{W}$, which entail for $\beta \to 0$ (using (4.9) and (4.10))

$$\langle p_{\lambda,\beta}^\text{CH}, p_{\lambda,\beta}^\text{CH} \rangle_{\text{CH}} = \frac{2^{-|\lambda|}}{\langle 1, 1 \rangle_{\text{CH}}} \prod_{1 \leq j < k \leq n} \frac{|(k - j + 1)g_0, 1 + (k - j - 1)g_0|_{\lambda_j - \lambda_k}}{|(k - j)g_0, 1 + (k - j)g_0|_{\lambda_j - \lambda_k}} \prod_{1 \leq j \leq n} [1 + (n - j)g_0]_{\lambda_j}$$

and

$$\langle p_{\lambda,\beta}^\text{W}, p_{\lambda,\beta}^\text{W} \rangle_{\text{W}} = \frac{2^{-|\lambda|}}{\langle 1, 1 \rangle_{\text{W}}} \prod_{1 \leq j < k \leq n} \frac{|(k - j + 1)g_0, 1 + (k - j - 1)g_0|_{\lambda_j - \lambda_k}}{|(k - j)g_0, 1 + (k - j)g_0|_{\lambda_j - \lambda_k}} \prod_{1 \leq j \leq n} [(n - j)g_0 + 1/2, 1 + (n - j)g_0]_{\lambda_j}.$$
Combination of the expressions for the ratios in (4.11a) and (4.11b) with the Mehta-Macdonald formulas for \(\{1,1\}\) and \(\{1,1\}\) (cf. (2.7a), (2.7b)), produces evaluation formulas for \(p^A_\lambda, p^B_\lambda\) and \(p^C_\lambda, p^D_\lambda\) that can be cast in the form given by Theorem 2.2. Indeed, one easily checks that the norm formulas in Theorem 2.2 are in agreement with the ratio formulas (4.11a), (4.11b) (using the relation \(\Gamma(a+b)/\Gamma(a) = [a]_!\) and, furthermore, that they boil down to the Mehta-Macdonald evaluation formulas (2.7a), (2.7b) for \(\lambda = 0\). In order to verify the reduction to (2.7a) and (2.7b) for \(\lambda = 0\), one uses the identity

\[
n! \prod_{1 \leq j < k \leq n} \frac{\Gamma((k-j+1)g_0)}{\Gamma((k-j)g_0)} \frac{\Gamma((k-j)g_0)}{\Gamma((k-j+1)g_0)} = \frac{\Gamma(1+ng_0)}{\Gamma(1+g_0)} \}
\]

(which is derived by canceling common factors in the numerator and denominator of the l.h.s. and some further manipulations involving the standard shift shift property for the gamma function \(\Gamma(z) = \Gamma(z + 1)\)).

4.2. Differential equations. In \(\mathbb{D}^4\) systems of difference equations for the multivariable continuous Hahn and Wilson polynomials associated to the weight functions \(\Delta^A(x)\) and \(\Delta^W(x)\) were presented. If we rescale the variables and reparametrize in accordance with (4.2) and (4.3), then the difference equations in question pass over into difference equations of the form

\[
D_{r,\beta}^C p^{C}_{\lambda,\beta} = E_{r,\beta}^C(\lambda) p^{C}_{\lambda,\beta} \quad (r = 1, \ldots, n; C = A, B),
\]

for the polynomials \(p^{A}_{\lambda,\beta}(x)\) and \(p^{B}_{\lambda,\beta}(x)\) associated to the weight functions \(\Delta^A_{\lambda}(x)\) and \(\Delta^B_{\lambda}(x)\). Here the difference operators \(D_{r,\beta}^A\) and \(D_{r,\beta}^B\) in the l.h.s. are the same as in Section 2.2 but with \(w^A(z)\) and \(w^B(z)\) replaced by

\[
\begin{align*}
\text{(4.13a)} & \quad w^A(z) = (1 + \beta xz)(1 + i\beta xz)
\end{align*}
\]

and

\[
\begin{align*}
\text{(4.13b)} & \quad w^B(z) = \left(1 + \frac{\beta g_1}{iz}\right) \left(1 + \frac{\beta g'_1}{i(z + \beta'^2)}\right)(1 + \beta xz)(1 + i\beta x'z).
\end{align*}
\]

The corresponding eigenvalues in the r.h.s. are given by

\[
\begin{align*}
\text{(4.14a)} & \quad E_{r,\beta}^A(\lambda) = (\beta^4 xw'w')^{-1} E_r((\rho^A_1 + \lambda_1)^2, \ldots, (\rho^A_n + \lambda_n)^2) 
\end{align*}
\]

\[
\begin{align*}
\text{(4.14b)} & \quad E_{r,\beta}^B(\lambda) = (\beta^4 xw'w')^{-1} E_r((\rho^B_1 + \lambda_1)^2, \ldots, (\rho^B_n + \lambda_n)^2) 
\end{align*}
\]

with

\[
\begin{align*}
\text{(4.15)} & \quad E_r(\zeta_1, \ldots, \zeta_n; \eta_1, \ldots, \eta_n) = \sum_{\substack{j \in \{1, \ldots, n\} \atop 0 \leq |j| \leq r}} (-1)^{r-|j|} \prod_{j \in j} \zeta_j \sum_{r_1 \leq \cdots \leq r_{r-|j|}} \eta_1 \cdots \eta_{r-|j|}
\end{align*}
\]

and

\[
\begin{align*}
\text{(4.16a)} & \quad \rho^A_j = (n-j)g_0 + \beta^{-2} \left(\frac{1}{\omega} + \frac{1}{\omega'}\right) - 1/2,
\end{align*}
\]

\[
\begin{align*}
\text{(4.16b)} & \quad \rho^B_j = (n-j)g_0 + (g_0 + g'_0)/2 + \beta^{-2} \left(\frac{1}{2\omega} + \frac{1}{2\omega'}\right) - 1/4.
\end{align*}
\]

In order to verify the reduction to \(w^A(z)\) and \(w^B(z)\) when the step size parameter \(\beta\) tends to zero. Essential
is the behavior of the eigenvalues $E^C_{r,\beta}(\lambda)$ (4.14a), (4.14b), which is governed by the following lemma.

**Lemma 4.1.** One has

$$
\lim_{\beta \to 0} \beta^{-2r}E^C_{r,\beta}(\lambda) = E^C_r(\lambda) \quad (r = 1, \ldots, n; \ C = A, B),
$$

where $E^C_r(\lambda)$ is given by the expression in Theorem 2.3 with $\omega = \varpi + \varpi'$. 

**Proof.** The proof hinges on the following decomposition of $E_r(\cdots ; \cdots)$ (4.13) (cf. [D1], Lemma B.2)

$$
E_r(\zeta_1, \ldots, \zeta_n; \eta_1, \ldots, \eta_n) = (\zeta_n - \eta_n) \times \left( \sum_{J \subset \{1, \ldots, n-1\}} (-1)^{r-1-|J|} \prod_{j \in J} \zeta_j \sum_{r \leq l_1 \leq \cdots \leq l_{r-1-|J|} \leq n} \eta_1 \cdots \eta_{r-1-|J|} \right)
+ \sum_{J \subset \{1, \ldots, n-1\}} (-1)^{r-1-|J|} \prod_{j \in J} \zeta_j \sum_{r \leq l_1 \leq \cdots \leq l_{r-|J|} \leq n-1} \eta_1 \cdots \eta_{r-|J|},
$$

with the convention that the sum in the second line equals one if $r = 1$ and that the sum in the third line equals zero if $r = n$. Notice that these sums are of the same type as in the r.h.s. of (4.15) but with one $\zeta$-variable less (there is no longer dependence on $\zeta_n$). With the aid of this formula and induction on the number of variables it is not difficult to infer that

$$
\lim_{\beta \to 0} \frac{E^C_{r,\beta}(\lambda)}{\beta^{2r}(\varpi + \varpi')^r} M^r = \lambda_n \times \left( \sum_{J \subset \{1, \ldots, n-1\}} \prod_{|J|=r-1} \lambda_j \right) + \sum_{J \subset \{1, \ldots, n-1\}} \prod_{|J|=r} \lambda_j
= \sum_{J \subset \{1, \ldots, n\}} \prod_{|J|=r} \lambda_j,
$$

where $M = 2$ for $C = A$ and $M = 4$ for $C = B$. $\square$

It is immediate from Lemma 4.1 and the limit formula (4.8) that $E^C_{r,\beta}(\lambda)p^C_{\lambda,\beta} = \beta^{2r}E^C_r(\lambda)p^C_{\lambda} + o(\beta^{2r})$ (assuming of course the usual identification of the parameters via $\omega = \varpi + \varpi'$ and $g_1 = g_1 + g_1'$). If we plug this asymptotics into the eigenvalue equation (4.12), then we arrive at the following asymptotic behavior of the corresponding l.h.s. for $\beta \to 0$

$$
(D^C_{r,\beta}p^C_{\lambda,\beta})(x) = \beta^{2r}(D^C_rp^C_{\lambda})(x) + o(\beta^{2r}),
$$

with $D^C_r$ a certain partial differential operator. In other words, for $\beta \to 0$ the difference equation for $p^C_{\lambda,\beta}$ passes over (cf. Corollary 2.4) into a differential equation for the polynomial $p^C_{\lambda}$ of the form $D^C_r p^C_{\lambda} = E^C_r(\lambda)p^C_{\lambda}$, where $D^C_r$ is a certain differential operator and $E^C_r(\lambda)$ is of the form given in Theorem 2.3 with $\omega = \varpi + \varpi'$.

To complete the proof of Theorem 2.3 it remains to show that the differential operator $D^C_r$ is indeed of the form described by the first part of the theorem (the commutativity of $D^C_1, \ldots, D^C_n$ is then an immediate consequence of the commutativity (see [D4]) of $D^C_{1,\beta}, \ldots, D^C_{n,\beta}$). More precisely, we should demonstrate that $D^C_r$ is the leading part of the difference operator $D^{C,\beta}_{r,\beta}$ in the formal expansion in $\beta$ and also that its leading symbol is given by the $r$th elementary symmetric function in the partials $-\partial^2/\partial x_1^2, \ldots, -\partial^2/\partial x_n^2$. 

To this end we first observe that we may replace $p_{x,\beta}^C$ in the l.h.s. of (4.17) by $p_{A}^C$ (using once more that $\lim_{x \to 0} p_{x,\beta}^C = p_{A}^C$). Moreover, since the polynomials $\{p_{A}^C\}_{A \in \Lambda}$ constitute a basis for the space of (even) symmetric polynomials, we have in fact that $(D_{r,\beta}^C p)(x) = \beta^{2r}(D_{r}^C p)(x) + O(\beta^{2r+1})$ for arbitrary (even) symmetric polynomial $p(x)$ in the variables $x_1, \ldots, x_n$. But then the same holds true for arbitrary analytic (not necessarily symmetric or even) function of $x_1, \ldots, x_n$ and we have (formally)

\begin{equation}
D_{r,\beta}^C = \beta^{2r}D_{r}^C + O(\beta^{2r+1}).
\end{equation}

Here we have used the fundamental property that the vanishing of a (linear) partial differential operator on the space of (even) symmetric polynomials implies it be zero on an arbitrary (analytic) function (i.e., all its coefficients must be zero). In [D1, Appendix C] a proof of this general property was given in a trigonometric context. (Specifically, we assumed there that the differential operator vanishes on the space of symmetric polynomials in $\sin^2(x_1), \ldots, \sin^2(x_n)$.) The present case follows after an appropriate substitution of the variables turning the relevant space of trigonometric polynomials into the space of (even) symmetric polynomials in $x_1, \ldots, x_n$. (Specifically, the type A case is recovered by the substitution $\sin^2(x_j) \to x_j$ and the type B case by $\sin^2(x_j) \to x_j^2$.) A consequence of this general property is that the leading part of the formal expansion of the difference operator $D_{r,\beta}^C$ in $\beta$ is completely determined by its action on the (even) symmetric polynomials. (This excludes the (a priori) possibility of a lower-order leading part in the formal expansion (4.18) corresponding to a term determined by a nontrivial differential operator that vanishes on the space of (even) symmetric polynomials.) Notice also that this property may be used to arrive at an alternative proof for the commutativity of $D_{1}^C, \ldots, D_{n}^C$. Indeed, the commutators $[D_{r}^C, D_{s}^C]$ obviously vanish on the simultaneous eigenbasis $\{p_{A}^C\}_{A \in \Lambda}$ (and hence on all (even) symmetric polynomials), from which it then follows that they be zero identically.

To determine the highest-order symbol of the leading differential operator $D_{r,\beta}^C$ in (4.18), we use that the functions $v^C$, $w^C$ governing the coefficients of the difference operator $D_{r,\beta}^C$ are of the form $1 + O(\beta)$ and that for $v^C, w^C = 1$

\begin{equation}
D_{r,\beta}^C (v^C, w^C = 1) = \sum_{J \subset \{1, \ldots, n\}: |J| = r} \prod_{j \in J} \left( e^{\frac{\partial}{\partial x_j}} + e^{-\frac{\partial}{\partial x_j}} - 2 \right) = \beta^{2r} \left( (-1)^r \sum_{J \subset \{1, \ldots, n\}: |J| = r} \prod_{j \in J} \frac{\partial^2}{\partial x_j^2} \right) + O(\beta^{2r+1}).
\end{equation}

(Notice in this connection that in the situation where $v^C$ and $w^C$ are taken to be equal to one, the operators $D_{r,\beta}^C$ (2.8a), (2.8b) reduce to

\begin{equation}
D_{r,\beta}^A = \sum_{J_+, J_- \subset \{1, \ldots, n\}: J_+ \cap J_- = \emptyset, |J_+| + |J_-| \leq r} (-2)^{-|J_+| - |J_-|} \left( \frac{n - |J_+| - |J_-|}{r - |J_+| - |J_-|} \right) e^{\frac{\partial}{\partial J_+} - \partial_{J_-}}
\end{equation}
The functions \( \hat{\varphi} \) with the resulting Pieri formula takes the form

\[
\Delta A
\]

for the polynomials \( p \) in accordance with (4.2), (4.3), one arrives at the corresponding Pieri formulas A.1, A.2 and B.1, B.2 of Section 2.1 with the weight functions \( \Delta_A \) able continuous Hahn and Wilson polynomials associated to the weight functions 4.3.

Even difference operators \( D^2 x_j \) polynomials (Corollary 2.5) stems from the fact that the operator in

\[
\mathfrak{d}_x \mathfrak{d}_y \mathfrak{d}_z
\]

which both can be rewritten in the form given by the first line of (4.19). It then follows (i.e. from the asymptotics \( \psi^C \), \( \psi^C = 1 + O(\beta) \) and (4.19)) that the leading part \( D^C \) in (4.18) is of the form

\[
D^C = (-1)^r \sum_{J \in \mathbb{P}^n} \prod \frac{\partial^2}{\partial x_j^2} + \text{l.o.}
\]

as advertised.

Thus far we have shown that the statements of Theorem 2.3 hold when taking difference operators \( D^C_{r,\beta} \) (2.8a), (2.8b) with \( w^C(z) \) given by (1.13a), (1.13b), where \( \omega + \omega' = \omega \) and \( g_1 + g'_1 = g_1 \). The formulation in Theorem 2.3 corresponds to choosing the specialization \( \omega = \omega \), \( \omega' = 0 \) and \( g_1 = g_1 \), \( g'_1 = 0 \).

The symmetry of the differential operator \( D^C \) with respect to the inner product \( \langle \cdot, \cdot \rangle \) in the space of permutation-invariant (type A) or permutation-invariant and even (type B) polynomials (Corollary 2.5) stems from the fact that the operator in question is diagonal on an orthogonal basis (viz. \( \{ \rho^C_\lambda \}_{\lambda \in \Lambda} \) with eigenvalues that are real.

4.3. Pieri type recurrence formulas. In [D4] Pieri formulas for the multivariable continuous Hahn and Wilson polynomials associated to the weight functions \( \Delta^c_H(x) \) (4.1a) and \( \Delta^W(x) \) (4.1b) were introduced. After rescaling and reparametrizing in accordance with (4.2), (4.3), one arrives at the corresponding Pieri formulas for the polynomials \( p^A_{\lambda,\beta} \) and \( p^B_{\lambda,\beta} \) (which—recall—are determined by the conditions A.1, A.2 and B.1, B.2 of Section 2.1 with the weight functions \( \Delta^A(x) \) (1.1a) and \( \Delta^A(x) \) (1.1a) replaced by \( \Delta^A_{\lambda}(x) \) (1.4a) and \( \Delta^B_{\lambda}(x) \) (1.4b)). In the simplest case the resulting Pieri formula takes the form

\[
\mathcal{E}^C_{1,\beta}(x)P^C_{\lambda,\beta}(x) = \sum_{1 \leq j \leq n} \hat{V}_{\pm j,\beta}(\rho^C + \lambda) \left( P^C_{\lambda+e_j,\beta}(x) - P^C_{\lambda,\beta}(x) \right) + \sum_{1 \leq j \leq n} \hat{V}_{\pm j,\beta}(\rho^C + \lambda) \left( P^C_{\lambda-e_j,\beta}(x) - P^C_{\lambda,\beta}(x) \right)
\]

with

\[
\hat{V}_{\pm j,\beta}(\zeta) = \hat{\psi}^C(\pm \zeta_j) \prod_{1 \leq k \leq n, k \neq j} \hat{\psi}^C(\pm \zeta_j + \zeta_k) \hat{\psi}^C(\pm \zeta_j - \zeta_k).
\]

The functions \( \hat{\psi}^A \), \( \hat{\psi}^B \) are given by

\[
\hat{\psi}^A(z) = 1 + \frac{g_0}{z}, \quad \hat{\psi}^B(z) = 1 + \frac{g_0}{z},
\]

\[
\hat{\psi}^A(z) = \frac{(\hat{a}^A + z)(\hat{b}^A + z)}{4z}, \quad \hat{\psi}^B(z) = \frac{(\hat{a}^B + z)(\hat{b}^B + z)(\hat{c}^B + z)}{2z(1 + 2z)}.
\]
The vector \( \rho \) \(_{\text{l.h.s.}} \) of (4.20) is given by
\[
\begin{align*}
\hat{a}^A &= \beta^{-2}(\frac{1}{\omega} + \frac{1}{\omega'}) - 1/2, \\
\hat{b}^A &= \beta^{-2}(\frac{1}{\omega} - \frac{1}{\omega'}) + 1/2
\end{align*}
\]
and
\[
\begin{align*}
\hat{a}^B &= (g_1 + g'_1)/2 + \beta^{-2}(\frac{1}{\omega} + \frac{1}{\omega'})/2 - 1/4, \\
\hat{b}^B &= (g_1 + g'_1)/2 - \beta^{-2}(\frac{1}{\omega} + \frac{1}{\omega'})/2 + 3/4, \\
c^B &= (g_1 - g'_1)/2 + \beta^{-2}(\frac{1}{\omega} - \frac{1}{\omega'})/2 + 1/4, \\
d^B &= (g_1 - g'_1)/2 - \beta^{-2}(\frac{1}{\omega} - \frac{1}{\omega'})/2 + 1/4.
\end{align*}
\]
The vector \( \rho^C \) has components given by \([4.16a],[4.16b]\) and the multiplier in the l.h.s. of \([\hat{1}20]\) is given by
\[
\begin{align*}
(4.21a) \quad &\hat{E}^A_{1,\beta}(x) = - \sum_{1 \leq j \leq n} \left( \frac{x_j}{\beta} + \hat{\rho}_j^A \right), \\
(4.21b) \quad &\hat{E}^B_{1,\beta}(x) = - \sum_{1 \leq j \leq n} \left( \frac{x_j^2}{\beta^2} + (\hat{\rho}_j^B)^2 \right)
\end{align*}
\]
with
\[
\hat{\rho}_j^A = (n - j)g_0 + 1/(\beta^2\omega) \quad \text{and} \quad \hat{\rho}_j^B = (n - j)g_0 + g_1.
\]
In the Pieri formula we have furthermore employed the normalization
\[
(4.22) \quad P^C_{\lambda,\beta}(x) = c^C_{\lambda,\beta} p^C_{\lambda,\beta}(x)
\]
with
\[
\begin{align*}
c^A_{\lambda,\beta} &= (-4i/\beta)^{|\lambda|} \prod_{1 \leq j \leq n} \frac{[\rho_j^A]_{\lambda_j}}{1 + \rho_j^A, b + \rho_j^A}_{\lambda_j} \\
&\quad \times \prod_{1 \leq j < k \leq n} \frac{[\rho_j^A + \rho_k^A]_{\lambda_j + \lambda_k}}{[\rho_j^A - \rho_k^A]_{\lambda_j - \lambda_k}} \frac{[g_0 + \rho_j^A + \rho_k^A]_{\lambda_j + \lambda_k}}{[g_0 - \rho_j^A + \rho_k^A]_{\lambda_j - \lambda_k}}, \\
c^B_{\lambda,\beta} &= (-1/\beta^2)^{|\lambda|} \prod_{1 \leq j \leq n} \frac{[2\rho_j^B]_{2\lambda_j}}{[a + \rho_j^B, b + \rho_j^B, c + \rho_j^B, d + \rho_j^B]_{\lambda_j}} \\
&\quad \times \prod_{1 \leq j < k \leq n} \frac{[\rho_j^B + \rho_k^B]_{\lambda_j + \lambda_k}}{[\rho_j^B - \rho_k^B]_{\lambda_j - \lambda_k}} \frac{[g_0 + \rho_j^B + \rho_k^B]_{\lambda_j + \lambda_k}}{[g_0 - \rho_j^B + \rho_k^B]_{\lambda_j - \lambda_k}}.
\end{align*}
\]
The recurrence relations of Theorem 2.7 can be recovered from \([4.20]\) for \( \beta \rightarrow 0 \). To see this, one first observes that
\[
\lim_{\beta \rightarrow 0} \left( \frac{i}{\beta^{\omega'}} \right)^{|\lambda|} P^A_{\lambda,\beta}(x) = P^A_{\lambda}(x), \quad \lim_{\beta \rightarrow 0} P^B_{\lambda,\beta}(x) = P^B_{\lambda}(x)
\]
with \( P^C_{\lambda}(x) \) given by \([2.12]\). These limits follow from \([4.8]\) and the fact that
\[
\lim_{\beta \rightarrow 0} \left( \frac{i}{\beta^{\omega'}} \right)^{|\lambda|} c^A_{\lambda,\beta} = c^A_{\lambda}, \quad \lim_{\beta \rightarrow 0} c^B_{\lambda,\beta} = c^B_{\lambda}
\]
with \( c^A_{\lambda} \) and \( c^B_{\lambda} \) given by \([2.13a] \) and \([2.13b]\). (As usual, we assume an identification of the parameters of the form \( \omega = \omega + \omega' \) and \( g_1 = g_1 + g'_1 \).) For type \( A \),
multiplication of \((1.21)\) by \(i\beta(\frac{1}{\gamma})|L|\) leads for \(\beta \to 0\) to the first recurrence relation of Theorem 2.7. The second (i.e. type \(B\)) recurrence relation of Theorem 2.7 is obtained similarly by sending \(\beta\) to zero in the type \(B\) version of \((1.20)\) after having multiplied both sides by the factor \((\varpi + \varpi')\beta^2\).

To derive these limit transitions for the recurrence relations we have used that for \(\beta \to 0\)
\[
\hat{v}^C(\rho_j^C + \rho_k^C + \lambda_j + \lambda_k), \hat{v}^C(-\rho_j^C - \rho_k^C - \lambda_j - \lambda_k) = 1 + O(\beta^2),
\]
and
\[
\hat{w}^A(\rho_j^A + \lambda_j) = \frac{1}{\beta^2\varpi}(1 + O(\beta^2)),
\]
\[
\hat{w}^A(-\rho_j^A - \lambda_j) = -\varpi\frac{(n - j)g_0 + \lambda_j}{2(\varpi + \varpi')} (1 + O(\beta^2)),
\]
\[
\hat{w}^B(\rho_j^B + \lambda_j) = \frac{(n - j)g_0 + g_1 + g_1' + 1/2 + \lambda_j}{\beta^2(\varpi + \varpi')} (1 + O(\beta^2)),
\]
\[
\hat{w}^B(-\rho_j^B - \lambda_j) = \frac{(n - j)g_0 + \lambda_j}{\beta^2(\varpi + \varpi')} (1 + O(\beta^2)).
\]

Notice to this end also that in the case of type \(A\), a divergent term in the l.h.s. of the recurrence relation \((1.20)\) originating from the factor \(-\sum_j \hat{\beta}_j^A\) (cf. \((1.21a)\)) cancels against a corresponding divergent term in the r.h.s. originating from the factor in front of \(P_{A,\beta}\) of the form \(-\sum_j \hat{V}_{j,\beta}(\rho^A + \lambda)\). (That the divergent terms on both sides indeed cancel is seen using the identity \(\sum_j \prod_{k \neq j} (1 + g_0/|\zeta_j - \zeta_k|) = n\).

In general the recurrence relations for the polynomials \(P_{A,\beta}^C(4.23)\) induced by \([D4]\) become
\[
(4.26) \quad \hat{E}_{\gamma,\beta}^C(x) P_{\gamma,\beta}^C(x) = \sum_{j < \{1, \ldots, n\}, \sum_{j \in \lambda \subset \Lambda} \epsilon_j = \pm 1, j \in J, \lambda + e_{\epsilon, j} \in \Lambda} \hat{U}_{\varepsilon, j - |J|}(\rho^C + \lambda) \hat{V}_{\varepsilon, J', \rho}(\rho^C + \lambda) P_{\lambda + \varepsilon, j, \beta}^C(x),
\]
with
e_{\epsilon, j} = \sum_{j \in J} \epsilon_j e_j \quad (\epsilon_j \in \{+1, -1\}),
\[
\hat{V}_{\varepsilon, J', \rho}(\zeta) = \prod_{j \in J} \hat{v}^C(\epsilon_j \zeta_j) \prod_{j \neq j'} \hat{v}^C(\epsilon_j \zeta_j + \epsilon_{j'} \zeta_{j'}) \hat{v}^C(\epsilon_j \zeta_j + \epsilon_{j'} \zeta_{j'} + 1)
\]
\[
\times \prod_{k \in K} \hat{v}^C(\epsilon_j \zeta_j + \zeta_k) \hat{v}^C(\epsilon_j \zeta_j - \zeta_k),
\]
\[
\hat{U}_{\lambda, \rho, \varepsilon}(\zeta) = (-1)^\rho \sum_{\sum_{L \subset \Lambda \setminus \{\gamma\} = \rho} \sum_{\epsilon_l = \pm 1, l \in L} \prod_{L \subset \Lambda \setminus \{\gamma\}} \hat{w}^C(\epsilon_l \zeta_l) \prod_{\sum_{L \subset \Lambda \setminus \{\gamma\}}} \hat{v}^C(\epsilon_l \zeta_l + \epsilon_{l'} \zeta_{l'}) \hat{v}^C(-\epsilon_l \zeta_l - \epsilon_{l'} \zeta_{l'} - 1)
\]
\[
\times \prod_{k \in K \setminus \Lambda} \hat{v}^C(\epsilon_l \zeta_l + \zeta_k) \hat{v}^C(\epsilon_l \zeta_l - \zeta_k)
\]
and

\[ (4.27a) \quad \hat{E}_{r,\beta}^A(x) = (-1)^r \sum_{J \subset \{1, \ldots, n\}} \prod_{0 \leq |j| \leq r} \frac{ix_j}{\beta} \sum_{r \leq l_1 \leq \cdots \leq l_{r-1} \leq n} \hat{\rho}_{l_1} \cdots \hat{\rho}_{l_{r-1}}, \]

\[ (4.27b) \quad \hat{E}_{r,\beta}^B(x) = (-1)^r \sum_{J \subset \{1, \ldots, n\}} \prod_{0 \leq |j| \leq r} \frac{ix_j}{\beta} \sum_{r \leq l_1 \leq \cdots \leq l_{r-1} \leq n} (\hat{\rho}_{l_1} \cdots \hat{\rho}_{l_{r-1}})^2. \]

For \( r = 1 \) the recurrence formula in (4.26) specializes to that of (4.24).

It is not difficult to see that the recurrence relations for the multivariable Laguerre polynomials characterized by Theorem 2.8 and Theorem 2.9 follow from the type \( B \) version of (4.26) for \( \beta \to 0 \). Indeed, multiplication of (4.26) by the factor \( \beta^{2r}(\varpi + \varpi')^r \) and sending \( \beta \) to zero readily leads to the Laguerre type recurrence relations. The verification of this assertion hinges on the second limit formula of (4.24), the asymptotics for \( \hat{\upsilon}^B \), \( \hat{\upsilon}^B \) displayed above, and the fact that

\[ \lim_{\beta \to 0} \beta^{2r} \hat{E}_{r,\beta}^B(x) = (-1)^r \sum_{J \subset \{1, \ldots, n\}} \prod_{j \in J} x_j^2. \]

For type \( A \) the transition \( \beta \to 0 \) is substantially more complicated due to the singular nature of the terms in (4.24). Specifically, the multiplier \( \hat{E}_{r,\beta}^A(x) \) (4.27a)

consists of a linear combination of the elementary symmetric functions in \( x_1, \ldots, x_n \)

up to degree \( r \). The coefficients in this linear combination have a pole at \( \beta = 0 \), the order of which is reversely proportional to the degree of the elementary symmetric function in question (notice that \( \hat{\rho}_j^A = O(\beta^{-2}) \)). Hence, for \( \beta \to 0 \) the contributions of the lower-degree elementary symmetric functions to \( \hat{E}_{r,\beta}^A(x) \)

become predominant. To get rid of these lower-degree divergent terms, we take an appropriate linear combination of the recurrence relations (4.26) that cast them into a system of the form

\[ (4.28) \quad \left( \sum_{J \subset \{1, \ldots, n\}} \prod_{|J|=r} x_j \right) \hat{P}_{\lambda,\beta}^A(x) = \sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq r} \hat{W}_{x, J; r, \beta}^A \hat{P}_{\lambda + e_{x, J}, \beta}^A(x) \]

with

\[ \hat{P}_{\lambda,\beta}^A(x) = \left( \frac{i}{\beta \varpi} \right)^{|\lambda|} P_{\lambda,\beta}^A(x). \]

Basically, this boils down to passing from Pieri formulas corresponding to the symmetric functions \( \hat{E}_{r,\beta}^A(x) \) (4.27a) to Pieri formulas corresponding to the elementary symmetric functions \( \sum_{|J|=r} \prod_{j \in J} x_j \) by subtracting from the \( r \)th Pieri formula in (4.26) a suitable linear combination of the Pieri formulas corresponding to \( \hat{E}_{s,\beta}^A(x) \) with \( s < r \) (and multiplication by an overall factor). The coefficients of the terms in the r.h.s. of (4.26) labeled by index sets \( J \) with \( |J| = r \) are invariant with respect to such changes in the l.h.s. (up to an overall factor \( (i\beta)^r \) and a factor caused by the change of the normalization \( \hat{P}_{\lambda,\beta}^A \to \hat{P}_{\lambda,\beta}^A \)). More precisely, we obtain that for \( |J| = r \) the coefficient \( \hat{W}_{x, J; r, \beta}^A \) in the r.h.s. of (4.28) is given by

\[ \hat{W}_{x, J; r, \beta}^A = (i\beta)^r (-i\beta \varpi)^{\sum_{\epsilon \in J} \epsilon_j} \hat{V}_{x, J; r, \beta}^A (\rho^A + \lambda). \]

The type \( A \) version of Theorem 2.8 then follows for \( \beta \to 0 \) (using the limit formula (4.24) and the above asymptotics for \( \hat{\upsilon}^A \) and \( \hat{\upsilon}^A \)).
It remains to verify the normalization properties of $P^C_\alpha(x)$ stated in Theorem 2.6. These properties are a consequence of the fact that $P^C_{\lambda,\beta}(i\beta\rho^C) = 1$ (see the remarks in [D4, Sec 6]). Specifically, by sending $\beta$ to zero in the relation $P^C_{\lambda,\beta}(i\beta\rho^C) = 1$ we arrive at the normalization properties of Theorem 2.6. For type $B$ this is immediate from the limit formula (A.2); for type $A$ this is seen by noticing that $\lim_{\beta \to 0} P^A_{\lambda,\beta}(i\beta\rho^A)$ picks up the highest-degree homogeneous part of $P^A_{\lambda,\beta}(x)$ evaluated in $x = 1$ (here we use that $\rho^A\equiv 1 + O(1)$ together with the limit formula (4.22)), which is equal to $\lim_{\alpha \to \infty} \alpha^{-|\lambda|} P^A_\lambda(\alpha 1)$.

APPENDIX: CONVERGENCE OF THE WEIGHT FUNCTIONS

In this appendix it will be shown that the weight functions $\Delta^A_\lambda(x)$ (4.4a) and $\Delta^B_\delta(x)$ (4.4b) converge for $\beta \to 0$ to the weight functions $\Delta^A(x)$ (1.1a) and $\Delta^B(x)$ (1.1b), respectively. More precisely, we will prove the somewhat stronger result that

$$
\lim_{\beta \to 0} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x) \Delta^C_\beta(x) \, dx_1 \cdots dx_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x) \Delta^C(x) \, dx_1 \cdots dx_n \quad (C = A \text{ or } B),
$$

where $p(x)$ denotes an arbitrary polynomial in the variables $x_1, \ldots, x_n$. In (A.1) it is understood that the parameters of $\Delta^C_\beta(x)$ (4.4a) and $\Delta^C_\beta(x)$ (4.4b) are related via the identification $\omega \equiv x + \omega'$ and $g_1 \equiv g_1 + g'_1$.

Let us start by inferring the pointwise convergence of the weight functions. For this purpose we use the limit formula

$$
\lim_{\beta \to 0} \delta(\alpha, \beta) |\Gamma(\frac{1}{\alpha \beta^2} + i\beta^{-1}y)| = \exp(-\alpha y^2 / 2) \quad (\alpha > 0)
$$

with

$$
\delta(\alpha, \beta) \equiv \sqrt{\frac{e}{2\pi}} e^{(1+\log(\alpha \beta^2))(\alpha^{-1} \beta^{-2} - 1/2)}
$$

and the limit formula

$$
\lim_{\beta \to 0} \left| \beta^a \frac{\Gamma(a + b + i\beta^{-1}y)}{\Gamma(b + i\beta^{-1}y)} \right| = |y|^a \quad (a, b \geq 0),
$$

where in both formulas it is assumed that $y$ and $\beta$ are real. By applying (A.2) and (A.3) to the factors of $\Delta^A_\lambda(x)$ (4.4a) and $\Delta^B_\delta(x)$ (4.4b), one readily sees that for $\beta \to 0$ these weight functions converge pointwise to $\Delta^A(x)$ (1.1a) and $\Delta^B(x)$ (1.1b) as indicated. The normalization factors of the form $\delta(\alpha, \beta)$ and $|\beta|^a$ in (A.2) and (A.3) ensure a finite and nontrivial limit; the factors in question have been collected in the weight functions $\Delta^A_\lambda(x)$ (4.4a) and $\Delta^B_\delta(x)$ (4.4b) into the overall normalization constants $D^A(\beta) = |\beta|^{n(n-1)\gamma_0} \delta(\varpi, \beta)^{2n} \delta(\varpi', \beta)^{2n} \delta(\varpi, \beta)^{2n}$ and $D^B(\beta) = |\beta|^{2n(n-1)\gamma_0 + 2n\delta + 2\epsilon} \delta(\varpi, \beta)^{2n} \delta(\varpi', \beta)^{2n}$.

The limit formulas (A.2) and (A.3) may be verified with the aid of Stirling’s formula for the asymptotics of the gamma function for large values of the argument, which reads (see e.g. [AbSt1, O1])

$$
\Gamma(z) = (2\pi)^{1/2} e^{-z} z^{z-1/2} \cdot \exp(R(z))
$$

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|
with \( R(z) = O(1/|z|) \) for \( |z| \to \infty \) in the sector \( |\arg(z)| < \pi \). Substitution of \( z = \alpha^{-1}\beta^{-2} + i\beta^{-1}y \) in (A.4) entails that for \( \beta \to 0 \)

\[
|\Gamma(\frac{1}{\alpha\beta^2} + i\beta^{-1}y)| = \frac{1}{\delta(\alpha, \beta)}(1 + \alpha^2\beta^2y^2)\exp\left(-\frac{\pi}{2}\arctan(\alpha\beta y)\right)(1 + O(\beta^2)),
\]

which implies (A.2). In a similar way one concludes from (A.4) that for \( \beta \to 0 \)

\[
|\Gamma(a + b + i\beta^{-1}y)| \leq \frac{|y/\beta|^a(1 + O(\beta))}{\Gamma(b + i\beta^{-1}y)}
\]

which implies (A.3).

Let us next demonstrate that the pointwise convergence of the integrands carries over to the convergence of the integrals by invoking Lebesgue’s dominated convergence theorem. For this purpose it is needed to dominate (the absolute value of) the integrand in the l.h.s. of (A.1) uniformly in \( \beta \) by an integrable function. We will do so by bounding individually the factors comprising the weight function. Specifically, it turns out that factors of the form \( \delta(\omega, \beta) \Gamma(\frac{1}{\omega} + i\beta^{-1}x_j) \) (and \( \delta(\omega', \beta) \Gamma(\frac{1}{\omega'} + i\beta^{-1}x_j) \)) may be dominated by an exponentially decaying function and that the remaining factors—which consist of ratios of the form \( |\beta^g_\gamma \Gamma(g_0 + i\beta^{-1}(x_j \pm x_k))/\Gamma(i\beta^{-1}(x_j \pm x_k))| \), \( |\beta^g_\gamma \Gamma(g_1 + i\beta^{-1}x_j)/\Gamma(i\beta^{-1}x_j)| \) and \( |\beta^g_\gamma \Gamma(g_1' + 1/2 + i\beta^{-1}x_j)/\Gamma(1/2 + i\beta^{-1}x_j)| \)—grow at most polynomially in the variables \( x_1, \ldots, x_n \). Hence, the integrand on the l.h.s. may be dominated by an exponentially decaying function and (A.1) follows from the pointwise convergence of the integrands by the dominated convergence theorem as advertised.

To validate the above claims regarding the bounds on the growth of the factors constituting the weight functions \( \Delta^a(x) \) (4.4a) and \( \Delta^b(x) \) (4.4b), we need a precise estimate for the error term \( R(z) \) appearing in the Stirling formula (1).

\[
|R(z)| \leq \frac{1}{12|z|\cos^2(\theta/2)} \quad (\theta = \arg(z)).
\]

(This estimate is valid in the whole sector \( |\arg(z)| < \pi \)). By substituting \( z = \alpha^{-1}\beta^{-2} + i\beta^{-1}y \) in the Stirling formula (A.4) (where \( \alpha \) is assumed to be positive) and combining with the error estimate (A.7), we find (cf. (A.3))

\[
\delta(\alpha, \beta) |\Gamma(\frac{1}{\alpha\beta^2} + i\beta^{-1}y)| \leq e^{-F_\beta(y)}G_\beta,
\]

with

\[
F_\beta(y) = \frac{y}{\beta} \arctan(\alpha\beta y) - \frac{1}{2\alpha\beta^2} \log(1 + \alpha^2\beta^2y^2)
\]

and \( G_\beta = e^{\beta^2/6} \). (In our situation \( \cos^2(\theta/2) \geq 1/2 \) in view of the fact that the real part of \( z = \frac{1}{\alpha\beta^2} + i\beta^{-1}y \) is positive.) Differentiation of \( F_\beta(y) \) with respect to \( y \) yields

\[
\partial_y F_\beta(y) = \beta^{-1} \arctan(\beta\alpha y),
\]

which shows that \( F_\beta(y) \) is nonnegative as an increasing/decreasing function for \( y \) positive/negative with \( F_\beta(0) = 0 \). From the asymptotics for \( |y| \to \infty \) one further-sees that the factor \( \exp(-F_\beta(y)) \) decays exponentially. A little more precise
analysis reveals that for $0 < \beta < 1$

\[ e^{-F_\beta(y)} \leq \begin{cases} 1 & \text{for } |y| < 1/\alpha \\ e^{-|y|/\beta} & \text{for } |y| \geq 1/\alpha. \end{cases} \]  

(A.11)

To obtain the exponential bound on the tail we have used: (i.) that for $0 < \beta < 1$

\[ F_\beta(1/\alpha) = \frac{1}{\alpha \beta} \arctan(\beta) - \frac{1}{2\alpha \beta^2} \log(1 + \beta^2) \]

\[ > \frac{\pi}{4} \log(\sqrt{2})/\alpha > 1/(3\alpha), \]

(ii.) that for $y \geq 1/\alpha$ and $0 < \beta < 1$

\[ (\partial_y F_\beta(y)) \geq (\partial_y F_\beta)(1/\alpha) = \beta^{-1} \arctan(\beta) > \pi/4 > 1/3, \]

and (iii.) that $F_\beta(y)$ is even in $y$. We conclude from (A.8) and (A.11) that for $0 < \beta < 1$ the factors in the weight function $\Delta_\beta^C(x)$ ($C = A, B$) of the form $\delta(x, \beta) |\Gamma(\frac{1}{\alpha\beta} + i\beta^{-1} x_j)|$ (and $\delta(x', \beta) |\Gamma(\frac{1}{\alpha\beta} + i\beta^{-1} x_j)|$ ) may indeed be dominated uniformly in $\beta$ by an exponentially decaying function of $x_j$.

It remains to check that the rest of the factors—which consist of ratios of gamma functions—can be dominated by a function that grows at most polynomially in the variables. To this end we should find bounds on the gamma function ratios of the type appearing in the l.h.s. of (A.3). Notice that for $a \in \mathbb{N}$ we have

\[ \beta^a \frac{\Gamma(a + b + i\beta^{-1} y)}{\Gamma(b + i\beta^{-1} y)} = \prod_{m=0}^{a-1} |(m + b)\beta + iy|, \]

which is easily dominated uniformly in $0 < \beta < 1$ by a function with polynomial growth in $y$ (take e.g. the function $(a+b)^2 + y^2)^{a/2}$). The case of general positive (not necessarily integer valued) $a$ is a little less straightforward; it will be addressed here with the aid of the integral representation

\[ \frac{\Gamma(a + z)}{\Gamma(z)} = z^a \exp \left( \int_0^\infty \left( a - \frac{1 - e^{-zt}}{1 - e^{-t}} \right) dt \right) \quad \text{Re}(z) > 0, \]

which can be obtained by integrating Gauss' integral formula for the psi function $\psi(z) = \Gamma'(z)/\Gamma(z)$ [AbSt, Ol]

(A.13)

\[ \psi(z) = \int_0^\infty \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}} \right) dt \quad \text{Re}(z) > 0 \]

(usual also that $\log(z) = \int_0^\infty t^{-1}(e^{-t} - e^{-tz}) dt$ for $\text{Re}(z) > 0$). Let us for the moment assume that $b$ is positive. Then substitution of $z = b + i\beta^{-1} y$ in (A.13) entails

\[ \beta^a \frac{\Gamma(a + b + i\beta^{-1} y)}{\Gamma(b + i\beta^{-1} y)} = \]

\[ (\beta^2 b^2 + y^2)^{a/2} \exp \left( \int_0^\infty e^{-bt} \cos(yt/\beta) \left( a - \frac{1 - e^{-at}}{1 - e^{-t}} \right) dt \right). \]

(A.15)

The integral within the exponent is bounded by a constant with value

\[ \int_0^\infty e^{-bt} \left| a - \frac{1 - e^{-at}}{1 - e^{-t}} \right| dt \]
and the factor in front is smaller than \((b^2 + y^2)^{\alpha/2}\) for \(0 < \beta < 1\). The case that \(b\) becomes zero can be reduced to the previous situation with positive \(b\) by means of the identity
\[
\frac{\Gamma(a + i\beta^{-1}y)}{\Gamma(i\beta^{-1}y)} = \frac{\Gamma(a + 1 + i\beta^{-1}y)}{\Gamma(1 + i\beta^{-1}y)} \left( \frac{iy}{a\beta + iy} \right).
\]
(A.16)

The upshot is that for \(0 < \beta < 1\) the factors \(|\beta^{g_0}\Gamma(g_0 + i\beta^{-1}(x_j \pm x_k))/\Gamma(i\beta^{-1}(x_j \pm x_k))|, |\beta^{g_1}\Gamma(g_1 + i\beta^{-1}x_j)/\Gamma(i\beta^{-1}x_j)|\) and \(|\beta^{g_1'}\Gamma(g_1' + 1/2 + i\beta^{-1}x_j)/\Gamma(1/2 + i\beta^{-1}x_j)|\) can be uniformly dominated in \(\beta\) by a function that grows at most polynomially in the variables \(x_1, \ldots, x_n\), which completes the proof of (A.1).

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References

A1. R. Askey, Some basic hypergeometric extensions of integrals of Selberg and Andrews, SIAM J. Math. Anal. 11 (1980), 938–951.

A2. ———, Continuous Hahn polynomials, J. Phys. A: Math. Gen. 18 (1985), L1017–L1019.

AtSu. N. M. Atakishiyev and S. K. Suslov, The Hahn and Meixner polynomials of an imaginary argument and some of their applications, J. Phys. A: Math. Gen. 18 (1985), 1583–1596.

AbSt. M. Abramowitz and I. A. Stegun (eds.), Handbook of mathematical functions, Dover Publications, New York, 1972 (9th printing).

BF. T. H. Baker and P. J. Forrester, The Calogero-Sutherland model and generalized classical polynomials, Preprint Research Institute for Mathematical Sciences, Kyoto, RIMS-1094, 1996.

BO. R. J. Beerends and E. M. Opdam, Certain hypergeometric series related to the root system BC, Trans. Amer. Math. Soc. 339 (1993), 581–609.

BHV. L. Brink, T. H. Hansson, and M. A. Vasiliev, Explicit solution of the n-body Calogero model, Phys. Lett. B 286 (1992), 109–111.

BHKV. L. Brink, T. H. Hansson, S. Konstein, and M. A. Vasiliev, The Calogero model—anonic representation, fermionic extension and supersymmetry, Nucl. Phys. B 401 (1993), 591–612.

Ca. F. Calogero, Solution of the one-dimensional n-body problems with quadratic and/or inversely quadratic pair potentials, J. Math. Phys. 12 (1971), 419–436.

Co. A. G. Constantine, The distribution of Hotelling’s generalized $T^2_0$, Ann. Math. Statist. 37 (1966), 215–225.

De1. A. Debiard, Polynômes de Tchebychev et de Jacobi dans un espace euclidien de dimension $p$, C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), 529–532.

De2. ———, Système différentiel hypergéométrique et parties radiales des opérateurs invariants des espaces symétriques de type $BC_p$, in: Séminaire d’Algèbre Paul Dubreil et Marie-Paule Malliavin (M.-P. Malliavin, ed.), Lecture Notes in Math., vol. 1296, Springer, Berlin, 1988, pp. 42–124.

D1. J. F. van Diejen, Commuting difference operators with polynomial eigenfunctions, Compositio Math. 95 (1995), 183–233.

D2. ———, Difference Calogero-Moser systems and finite Toda chains, J. Math. Phys. 36 (1995), 1299–1323.

D3. ———, Multivariable continuous Hahn and Wilson polynomials related to integrable difference systems, J. Phys. A: Math. Gen. 28 (1995), L369–L374.

D4. ———, Properties of some families of hypergeometric orthogonal polynomials in several variables, Math. preprint University of Tokyo UTMS 96-10, 1996.

Du1. C. F. Dunkl, Orthogonal polynomials on the sphere with octahedral symmetry, Trans. Amer. Math. Soc. 282 (1984), 555–575.

Du2. ———, Reflection groups and orthogonal polynomials on the sphere, Math. Z. 197 (1988), 33–60.

Du3. ———, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc. 311 (1989), 167–183.

G. P. J. Gambardella, Exact results in quantum many-body systems of interacting particles in many dimensions with $SU(1,1)$ as the dynamical group, J. Math. Phys. 16 (1975), 1172–1187.

H. G. J. Heckman, An elementary approach to the hypergeometric shift operator of Opdam, Invent. Math. 103 (1991), 341–350.

He. C. S. Herz, Bessel functions of matrix argument, Ann. Math. 61 (1955), 474–523.

J. A. T. James, Special functions of matrix and single argument in statistics, in: Theory and applications of special functions (R. Askey, ed.), Academic Press, New York, 1975, pp. 497–520.

Ka. S. Kakei, Common algebraic structure for the Calogero-Sutherland models, Preprint 1996.

KS. R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Math. report Delft University of Technology 94-05, 1994.

LV1. L. Lapointe and L. Vinet, A Rodrigues formula for the Jack polynomials and the Macdonald-Stanley conjecture, Internat. Math. Res. Notices 1995, 419–424.

LV2. ———, Exact operator solution of the Calogero-Sutherland model, Commun. Math. Phys. 178 (1996), 425–455.
La1. M. Lassalle, *Une formule du binôme généralisée pour les polynômes de Jack*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), 253–256.

La2. ———, *Polynômes de Laguerre généralisés*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), 725–728.

La3. ———, *Polynômes de Hermite généralisés*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), 579–582.

M1. I. G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Anal. **13** (1982), 988–1007.

M2. ———, *Commuting differential operators and zonal spherical functions*, in: Algebraic groups Utrecht 1986 (A. M. Cohen, W. H. Hesselink, W. L. J. van der Kallen, and J. R. Strooker, eds.), Lecture Notes in Math., vol. 1271, Springer, Berlin, 1987, pp. 189–200.

M3. ———, *Hypergeometric functions*, unpublished manuscript.

M4. ———, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford mathematical monographs, Clarendon Press, Oxford, 1995.

Me. M. L. Mehta, *Random matrices*, 2nd edition, Academic Press, Boston, 1991.

Mu. R. J. Muirhead, *Aspects of multivariate statistical theory*, Wiley, New York, 1982.

OOS. H. Ochiai, T. Oshima, and H. Sekiguchi, *Commuting families of symmetric differential operators*, Proc. Japan Acad. Ser. A Math. Sci. **70** (1994), 62–66.

OP. M. A. Olshanetsky and A. M. Perelomov, *Quantum integrable systems related to Lie algebras*, Phys. Rep. **94** (1983), 313–404.

Ol. F. W. J. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974.

OS. T. Oshima and H. Sekiguchi, *Commuting families of differential operators invariant under the action of a Weyl group*, J. Math. Sci. Univ. Tokyo **2** (1995), 1–75.

Pe. A. M. Perelomov, *Algebraic approach to the solution of a one-dimensional model of n interacting particles*, Theoret. and Math. Phys. **6** (1971), 263–282.

Po. A. P. Polychronakos, *Exchange operator formalism for integrable systems of particles*, Phys. Rev. Lett. **69** (1992), 703–705.

R. S. N. M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys. **110** (1987), 191–213.

S. J. Sekiguchi, *Zonal spherical functions on some symmetric spaces*, Publ. RIMS Kyoto Univ. **12** Suppl. (1977), 455–459.

Se. A. Selberg, *Bemerkninger om et multipelt integral*, Norsk Mat. Tidsskr. **26** (1944), 71–78 (Collected papers, vol. 1, Springer, Berlin, 1989, pp. 204–213).

St. R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. **77** (1989), 76–115.

UW. H. Ujino and M. Wadati, *Rodrigues formula for hi-Jack symmetric polynomials associated with the quantum Calogero model*, J. Phys. Soc. Japan **65** (1996), 2423–2439.

V. L. Vretare, *Formulas for elementary spherical functions and generalized Jacobi polynomials*, SIAM J. Math. Anal. **15** (1984), 805–833.

W. J. A. Wilson, *Some hypergeometric orthogonal polynomials*, SIAM J. Math. Anal. **11** (1980), 690–701.

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