Few-Shot Learning via Learning the Representation, Provably

Simon S. Du*, Wei Hu†, Sham M. Kakade‡, Jason D. Lee§, and Qi Lei¶

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Abstract

This paper studies few-shot learning via representation learning, where one uses \( T \) source tasks with \( n_1 \) data per task to learn a representation in order to reduce the sample complexity of a target task for which there is only \( n_2 (\ll n_1) \) data. Specifically, we focus on the setting where there exists a good common representation between source and target, and our goal is to understand how much of a sample size reduction is possible. First, we study the setting where this common representation is low-dimensional and provide a fast rate of \( O\left(\frac{C(\Phi)}{n_1T} + \frac{k}{n_2}\right) \); here, \( \Phi \) is the representation function class, \( C(\Phi) \) is its complexity measure, and \( k \) is the dimension of the representation. When specialized to linear representation functions, this rate becomes \( O\left(\frac{dk}{n_1T} + \frac{k}{n_2}\right) \) where \( d(\gg k) \) is the ambient input dimension, which is a substantial improvement over the rate without using representation learning, i.e. over the rate of \( O\left(\frac{d}{n_2}\right) \). Second, we consider the setting where the common representation may be high-dimensional but is capacity-constrained (say in norm); here, we again demonstrate the advantage of representation learning in both high-dimensional linear regression and neural network learning. Our results demonstrate representation learning can fully utilize all \( n_1T \) samples from source tasks.

1 Introduction

A popular scheme for few-shot learning, i.e., learning in a data scarce environment, is representation learning, where one first learns a feature extractor, or representation, e.g., the last layer of a convolutional neural network, from different but related source tasks, and then uses a simple predictor (usually a linear function) on top of this representation in the target task. The hope is that the learned representation captures the common structure across tasks, which makes a linear predictor sufficient for the target task. If the learned representation is good enough, it is possible that a few samples are sufficient for learning the target task, which can be much smaller than the number of samples required to learn the target task from scratch.

While representation learning has achieved tremendous success in a variety of applications (Bengio et al., 2013), its theoretical studies are limited. In existing theoretical work, a widely accepted assumption is the existence of a common representation among source and target tasks, on top of which a (different) linear predictor for each task is sufficient for good prediction. However, this assumption itself is not sufficient since the existence alone does not necessarily guarantee one can recover the common representation – it is

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*Institute for Advanced Study. Email: ssdu@ias.edu
†Princeton University. Email: huwei@cs.princeton.edu
‡University of Washington. Email: sham@cs.washington.edu
§Princeton University. Email: jasonlee@princeton.edu
¶University of Texas at Austin. Email: leiqi@ices.utexas.edu
possible that the learned representation is only good on the source tasks and does not transfer to the target task. Therefore, it is necessary to have some condition that characterizes the connection between different tasks.

Existing work often imposes a probabilistic assumption about the connection between tasks: each task is sampled i.i.d. from an underlying distribution. Under this assumption, Maurer et al. (2016) showed an $O(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{n_1}})$ risk bound where $T$ is the number of source tasks, and $n_1$ is the number of samples per source task.\footnote{We only focus on the dependence on $T$ and $n_1$ in this paragraph.} Unsatisfactorily, the bound $O(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{n_1}})$ requires both $T$ and $n_1$ to be large. Ideally, one might expect to obtain $O(\frac{1}{\sqrt{n_1T}})$ (or even $O(\frac{1}{n_1T})$ in the realizable case) rate as opposed to $O(\frac{1}{\sqrt{n_1}} + \frac{1}{\sqrt{n_1T}})$, because $n_1T$ is the total number of training data points from source tasks, which can be potentially pooled to learn the representation.

Unfortunately, as pointed out by Maurer et al. (2016), there exists an example that satisfies these assumptions for which $\Omega(\frac{1}{\sqrt{T}})$ is unavoidable. This means that the i.i.d. assumption alone is not sufficient if we want to take advantage of a large amount of samples per task. Therefore, a natural question is:

*What connections between source and target tasks enable representation learning to utilize all source data?*

In this paper, we obtain the first set of results that fully utilize the $n_1T$ data from source tasks. We replace the i.i.d. assumption over tasks with natural conditions on the linear predictors. These conditions depict that the linear predictors in the source tasks are diverse enough to “cover” the representation space, which guarantees that the representation learned from the source tasks is useful for the target task that share the same representation as the source tasks.

First, we study the setting where there exists a common well-specified low-dimensional representation in source and target tasks, and obtain an $O(\frac{C(\Phi)}{n_1T} + \frac{k}{n_1})$ risk bound on the target task, where $\Phi$ is the representation function class, $C(\Phi)$ is a certain complexity measure of $\Phi$, $k$ is the dimension of the representation, and $n_2$ is the number of data from the target task. When specialized to linear representation functions, this rate becomes $O(\frac{dk}{n_1T} + \frac{k}{n_2})$ where $d(\gg k)$ is the input dimension. Note that this improves the $O(\frac{d}{n_2})$ rate without using representation learning. The term $O(\frac{C(\Phi)}{n_1T})$ indicates that we can fully exploit all $n_1T$ data in the source tasks to learn the representation. Furthermore, to our knowledge, this is the first fast rate result in the representation learning theory literature.

Second, we study the setting where there exists a common linear high-dimensional representation for source and target tasks, and obtain an $O(\frac{R\sqrt{Tr(\Sigma)}}{\sqrt{Tn_1T}} + \frac{R\sqrt{\|\Sigma\|}}{\sqrt{n_2}})$ rate where $R$ is a normalized nuclear norm control over linear predictors, and $\Sigma$ is the covariance matrix of the raw feature. This also improves the baseline rate without using representation learning. We further extend this result to the case of two-layer neural networks with ReLU activation. Again, our results indicate that we can fully exploit $n_1T$ source data.

As we mentioned, our condition requires that the linear predictors in the source tasks can “cover” the representation space. A technical insight coming out of our analysis is that under this condition, any capacity controlled method that gets low test error on the source tasks must also get low test error on the target task by virtue of being forced to learn a good representation. Furthermore, our result on high-dimensional representations shows that the capacity control for representation learning does not have to be through explicit low dimensionality.

**Organization.** The rest of the paper is organized as follows. We review related work in Section 2. In Section 3, we formally describe the setting we consider. In Section 4, we present our main result for low-dimensional linear representation learning. In Section 5, we demonstrate that the result of low-dimensional linear representation learning can be generalized to more general representation classes with certain control
of complexity. In Section 6, we present our main result for high-dimensional linear representation learning. In Section 7, we present our result for representation learning in neural networks. We conclude in Section 8 and leave most of the proofs to appendices.

2 Related Work

The idea of multitask representation learning at least dates back to Caruana (1997), Thrun and Pratt (1998), Baxter (2000). Empirically, representation learning has shown its great power in various domains; see Bengio et al. (2013) for a survey. In particular, representation learning is widely adopted for few-shot learning tasks (Sun et al., 2017, Goyal et al., 2019, Arora et al., 2020). Representation learning is also closely connected to meta-learning (Schaul and Schmidhuber, 2010). Recent work Raghu et al. (2019) empirically suggested that the effectiveness of the popular meta-learning algorithm Model Agnostic Meta-Learning (MAML) is due to its ability to learn a useful representation. In fact, the scheme we analyze in this paper is closely related to Lee et al. (2019), Bertinetto et al. (2018).

On the theoretical side, Baxter (2000) performed the first theoretical analysis and gave sample complexity bounds using covering numbers. Maurer et al. (2016) and follow-up work gave analyses on the benefit of representation learning for reducing the sample complexity of the target task. Besides assuming a common representation for the source and target tasks, they assumed every task is i.i.d. drawn from an underlying distribution and can obtain an \( O\left(\frac{1}{\sqrt{n_1}} + \frac{1}{\sqrt{T}}\right) \) rate. As pointed out in Maurer et al. (2016), the \( \frac{1}{\sqrt{T}} \) dependence is not improvable even if \( n_1 \to \infty \) because \( \frac{1}{\sqrt{T}} \) is the rate of concentration for the distribution over tasks. Our paper uses natural deterministic conditions on the linear predictors and can obtain an \( O\left(\frac{1}{n_1 T}\right) \) fast rate, which can fully utilize all source data.

Another recent line of theoretical work analyzed gradient-based meta-learning methods (Denevi et al., 2019, Finn et al., 2016, Khodak et al., 2019) and showed guarantees for convex losses by using tools from online convex optimization. Lastly, we remark that there are analyses for other representation learning schemes (Arora et al., 2019, McNamara and Balcan, 2017, Galanti et al., 2016, Alquier et al., 2016, Denevi et al., 2018).

3 Notation and Setup

3.1 Notation

Let \( [n] = \{1, 2, \ldots, n\} \). We use \( \|\cdot\| \) or \( \|\cdot\|_2 \) to denote the \( \ell_2 \) norm of a vector or the spectral norm of a matrix. Denote by \( \|\cdot\|_F \) and \( \|\cdot\|_* \) the Frobenius norm and the nuclear norm of a matrix, respectively. Let \( \langle \cdot, \cdot \rangle \) be the Euclidean inner product between vectors or matrices. Denote by \( I \) the identity matrix. Let \( \mathcal{N}(\mu, \sigma^2) \) be the one-dimensional/multi-dimensional Gaussian distribution, and \( \chi^2(m) \) the chi-squared distribution with \( m \) degrees of freedom.

For a matrix \( A \in \mathbb{R}^{m \times n} \), let \( \sigma_i(A) \) be its \( i \)-th largest singular value. Let \( \text{span}(A) \) be the subspace of \( \mathbb{R}^m \) spanned by the columns of \( A \), i.e., \( \text{span}(A) = \{Av \mid v \in \mathbb{R}^n\} \). Denote \( P_A = A(A^\top A)^{1/2} \in \mathbb{R}^{m \times m} \), which is the projection matrix onto \( \text{span}(A) \). Here \(^\dagger\) stands for the Moore-Penrose pseudo-inverse. Note that \( 0 \preceq P_A \preceq I \) and \( P_A^2 = P_A \). We also define \( P_A^{-1} = I - P_A \), which is the projection matrix onto \( \text{span}(A)^\perp \), the orthogonal complement of \( \text{span}(A) \) in \( \mathbb{R}^m \). For a positive semidefinite (psd) matrix \( B \), denote by \( \lambda_{\max}(B) \) and \( \lambda_{\min}(B) \) its largest and smallest eigenvalues, respectively; let \( B^{1/2} \) be the psd matrix such that \((B^{1/2})^2 = B\).

We use the standard \( O(\cdot) \), \( \Omega(\cdot) \) and \( \Theta(\cdot) \) notation to hide universal constant factors. We also use \( a \lesssim b \) or \( b \gtrsim a \) to indicate \( a = O(b) \), and use \( a \asymp b \) or \( b \ll a \) to mean that \( a \geq C \cdot b \) for a sufficiently large universal constant \( C > 0 \).
3.2 Problem Setup

Suppose that there are $T$ source tasks. Each task $t \in [T]$ is associated with a distribution $\mu_t$ over the joint data space $X \times Y$, where $X$ is the input space and $Y$ is the output space. In this paper we consider $X \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}$. For each source task $t \in [T]$ we have access to $n_1$ i.i.d. samples $(x_{t,1}, y_{t,1}), \ldots, (x_{t,n_1}, y_{t,n_1})$ from $\mu_t$. For convenience, we express these $n_1$ samples collectively as an input matrix $X_t \in \mathbb{R}^{n_1 \times d}$ and an output vector $y_t \in \mathbb{R}^{n_1}$.

Multitask learning tries to learn prediction functions for all the $T$ source tasks simultaneously in the hope of discovering some underlying common property of these tasks. The common property we consider in this paper is a representation, which is a function $\phi : X \rightarrow Z$ that maps an input to some feature space $Z \subseteq \mathbb{R}^k$. We restrict the representation function to be in some function class $\Phi$, e.g., neural networks. We try to use different linear predictors on top of a common representation function $\phi$ to model the input-output relations in different source tasks. Namely, for each task $t \in [T]$, we set the prediction function to be $x \mapsto \langle w_t, \phi(x) \rangle$ ($w_t \in \mathbb{R}^k$). Therefore, using the training samples from $T$ tasks, we can solve the following optimization problem to learn the representation:

$$
\min_{\phi \in \Phi, w_1, \ldots, w_T \in \mathbb{R}^k} \frac{1}{2n_1 T} \sum_{t=1}^{T} \sum_{i=1}^{n_1} (y_{t,i} - \langle w_t, \phi(x_{t,i}) \rangle)^2.
$$

We overload the notation to allow $\phi$ to apply to all the samples in a data matrix simultaneously, i.e., $\phi(X_t) = [\phi(x_{t,1}), \ldots, \phi(x_{t,n_1})] \in \mathbb{R}^{n_1 \times k}$. Then (1) can be rewritten as

$$
\min_{\phi \in \Phi, w_1, \ldots, w_T \in \mathbb{R}^k} \frac{1}{2n_1 T} \sum_{t=1}^{T} \left\| y_t - \phi(X_t) w_t \right\|^2.
$$

Let $\hat{\phi} \in \Phi$ be the representation function obtained by solving (2). Now we retain this representation and apply it to future (target) tasks. For a target task specified by a distribution $\mu_{T+1}$ over $X \times Y$, suppose we receive $n_2$ i.i.d. samples $X_{T+1} \in \mathbb{R}^{n_2 \times d}$, $y_{T+1} \in \mathbb{R}^{n_2}$. We further train a linear predictor on top of $\hat{\phi}$ for this task:

$$
\min_{w_{T+1} \in \mathbb{R}^k} \frac{1}{2n_2} \sum_{i=1}^{n_2} \left\| y_{T+1,i} - \hat{\phi}(X_{T+1}) w_{T+1} \right\|^2.
$$

Let $\hat{w}_{T+1}$ be the returned solution. We are interested in whether our learned predictor $x \mapsto \langle \hat{w}_{T+1}, \hat{\phi}(x) \rangle$ works well on average for the target task, i.e., we want the population loss

$$
L_{\mu_{T+1}}(\hat{\phi}, \hat{w}_{T+1}) = \mathbb{E}_{(x,y) \sim \mu_{T+1}} \left[ \frac{1}{2} (y - \langle \hat{w}_{T+1}, \hat{\phi}(x) \rangle)^2 \right]
$$

to be small. In particular, we are interested in the few-shot learning setting, where the number of samples $n_2$ from the target task is small – much smaller than the number of samples required for learning the target task from scratch.

**Data assumption.** In order for the above learning procedure to make sense, we assume that there is a ground-truth optimal representation function $\phi^* \in \Phi$ and specializations $w_1^*, \ldots, w_{T+1}^* \in \mathbb{R}^k$ for all the tasks such that for each task $t \in [T+1]$, we have

$$
\mathbb{E}_{(x,y) \sim \mu_t} [y \mid x] = \langle w_t^*, \phi^*(x) \rangle.
$$

\footnote{We use the $L_2$ loss throughout this paper.}
More specifically, we assume \((x, y) \sim \mu_t\) can be generated by
\[
y = \langle w_t^*, \phi^*(x) \rangle + z, \quad x \sim p_t, z \sim \mathcal{N}(0, \sigma^2),
\]
where \(x\) and \(z\) are independent.

Our goal is to bound the excess risk of our learned model on the target task, i.e., how much our learned model \((\hat{\phi}, \hat{w}_{T+1})\) performs worse than the optimal model \((\phi^*, w_{T+1}^*)\) on the target task:
\[
ER(\hat{\phi}, \hat{w}_{T+1}) = L_{\mu_{T+1}}(\hat{\phi}, \hat{w}_{T+1}) - L_{\mu_{T+1}}(\phi^*, w_{T+1}^*)
= \frac{1}{2} \mathbb{E}_{x \sim p_{T+1}}[(\langle \hat{w}_{T+1}, \phi(x) \rangle - \langle w_{T+1}^*, \phi^*(x) \rangle)^2].
\]

Here we have used the relation (4).

Oftentimes we are interested in the average performance on a random target task (i.e., \(w_{T+1}^*\) is random). In such case we look at the expected excess risk \(\mathbb{E}_{w_{T+1}^*}[ER(\hat{\phi}, \hat{w}_{T+1})]\).

## 4 Low-Dimensional Linear Representations

In this section, we consider the case where the representation is a linear map from the original input space \(\mathbb{R}^d\) to a low-dimensional space \(\mathbb{R}^k\) \((k \ll d)\). Namely, we let the representation function class be \(\Phi = \{x \mapsto B^\top x \mid B \in \mathbb{R}^{d \times k}\}\). Then the optimization problem (2) for learning the representation can be written as
\[
(\hat{B}, \hat{W}) \leftarrow \arg \min_{B \in \mathbb{R}^{d \times k}, W = [w_1, \ldots, w_T] \in \mathbb{R}^{k \times T}} \frac{1}{2n_1T} \sum_{t=1}^T \|y_t - X_tBw_t\|^2.
\]

The inputs from \(T\) source tasks, \(X_1, \ldots, X_T \in \mathbb{R}^{n_1 \times d}\), can be written in the form of a linear operator \(\mathcal{X} : \mathbb{R}^{d \times T} \rightarrow \mathbb{R}^{n_1 \times T}\), where
\[
\mathcal{X}(\Theta) = [X_1\theta_1, \ldots, X_T\theta_T], \quad \forall \Theta = [\theta_1, \ldots, \theta_T] \in \mathbb{R}^{d \times T}.
\]

With this notation, (6) can be rewritten as
\[
(\hat{B}, \hat{W}) \leftarrow \arg \min_{B \in \mathbb{R}^{d \times k}, W \in \mathbb{R}^{k \times T}} \frac{1}{2n_1T} \|Y - \mathcal{X}(BW)\|^2_F,
\]
where \(Y = [y_1, \ldots, y_T] \in \mathbb{R}^{n_1 \times T}\).

With the learned representation \(\hat{B}\) from (7), for the target task, we further find a linear function on top of the representation:
\[
\hat{w}_{T+1} \leftarrow \arg \min_{w \in \mathbb{R}^k} \frac{1}{2n_2} \|y_{T+1} - X_{T+1}^\top \hat{B}w\|^2.
\]

As described in Section 3, we assume that all \(T + 1\) tasks share a common ground-truth representation specified by a matrix \(B^* \in \mathbb{R}^{d \times k}\) such that a sample \((x, y) \sim \mu_t\) satisfies \(x \sim p_t\) and \(y = (w_t^*)^\top (B^*)^\top x + z\) where \(z \sim \mathcal{N}(0, \sigma^2)\) is independent of \(x\). Here \(w_t^* \in \mathbb{R}^k\), and we assume \(\|w_t^*\| = \Theta(1)\) for all \(t \in [T + 1]\). Denote \(W^* = [w_1^*, \ldots, w_T^*] \in \mathbb{R}^{k \times T}\). Then we can write \(Y = \mathcal{X}(B^*W^*) + Z\), where the noise matrix \(Z\) has i.i.d. \(\mathcal{N}(0, \sigma^2)\) entries.

Assume \(\mathbb{E}_{x \sim p_t}[x] = 0\) and let \(\Sigma_t = \mathbb{E}_{x \sim p_t}[xx^\top]\) for all \(t \in [T + 1]\). Note that a sample \(x \sim p_t\) can be generated from \(x = \Sigma_t^{-1/2} \bar{x}\) for \(\bar{x} \sim \bar{p}_t\) such that \(\mathbb{E}_{x \sim \bar{p}_t}[x] = 0\) and \(\mathbb{E}_{x \sim \bar{p}_t}[xx^\top] = I\). \(\bar{p}_t\) is called the whitening of \(p_t\). In this section we make the following assumptions on the input distributions \(p_1, \ldots, p_{T+1}\).
Assumption 4.1 (subgaussian input). There exists $\rho > 0$ such that, for all $t \in [T + 1]$, the random vector $\tilde{x} \sim \tilde{p}_t$ is $\rho^2$-subgaussian.\(^3\)

Assumption 4.2 (covariance dominance). There exists $c > 0$ such that $\Sigma_t \succeq c \cdot \Sigma_{T+1}$ for all $t \in [T]$.

Assumption 4.1 is made due to the probabilistic tail bounds used in our proof. It may be replaced with other moment or boundedness conditions if we use different tail bounds in the analysis.

Assumption 4.2 is a technical condition that makes it possible to transfer the representation learned from the source tasks to the target task. We remark that we do not necessarily need $\Sigma_t \succeq c \cdot \Sigma_{T+1}$ for all $t \in [T]$; as long as this holds for a constant fraction of $t$’s, our result is unaffected.

We also make the following assumption that characterizes the diversity of the source tasks.

Assumption 4.3 (diverse source tasks). The matrix $W^* = [w^*_1, \ldots, w^*_T] \in \mathbb{R}^{k \times T}$ satisfies $\sigma^2(W^*) \geq \Omega(\frac{1}{k})$.

Recall that $\|w^*\| = \Theta(1)$, which implies $\sum_{j=1}^k \sigma_j^2(W^*) = \|W^*\|_F^2 = \Theta(T)$. Thus, Assumption 4.3 is equivalent to saying that $\frac{\sigma_j(W^*)}{\|w^*\|} = O(1)$. Roughly speaking, this means that $\{w^*_t\}_{t \in [T]}$ covers all the directions in $\mathbb{R}^k$ evenly. As an example, Assumption 4.3 is satisfied with high probability when $w_t^*$’s are sampled i.i.d. from $\mathcal{N}(0, \Sigma)$ with $\frac{\lambda_{\max}(\Sigma)}{\lambda_{\min}(\Sigma)} = O(1)$.

Finally, we make the following assumption on the distribution of the target task.

Assumption 4.4 (distribution of target task). Assume that $w^*_{T+1}$ follows a distribution $\nu$ such that

$$\left\|\mathbb{E}_{w \sim \nu}[ww^T]\right\| \leq O\left(\frac{1}{k}\right).$$

Recall that we typically have $\|w^*_{T+1}\| = \Theta(1)$. Thus the assumption $\left\|\mathbb{E}_{w \sim \nu}[ww^T]\right\| \leq O\left(\frac{1}{k}\right)$ means that the distribution of $w^*_{T+1}$ does not align with any direction significantly more than average. It is useful to think of the uniform distribution on the unit sphere as an example, though we can allow a much more general class of distributions. This is also compatible with Assumption 4.3 which says that $w^*_t$’s cover all the directions evenly.

Assumption 4.4 can be removed at the cost of a slightly worse risk bound. See Remark 4.1.

Our main result in this section is the following theorem.

Theorem 4.1 (main theorem for linear representations). Fix a failure probability $\delta \in (0, 1)$. Under Assumptions 4.1, 4.2, 4.3 and 4.4, we further assume $2k \leq \min\{d, T\}$ and that the sample sizes in source and target tasks satisfy $n_1 \gg \rho^4(d + \log \frac{T}{\delta})$, $n_2 \gg \rho^4(k + \log \frac{1}{\delta})$, and $cn_1 \geq n_2$. Define $\kappa = \frac{\max_{t \in [T]} \lambda_{\max}(\Sigma_t)}{\min_{t \in [T]} \lambda_{\min}(\Sigma_t)}$.

Then with probability at least $1 - \delta$ over the samples, the expected excess risk of the learned predictor $x \mapsto \hat{w}_{T+1}^T \hat{B}x$ on the target task satisfies

$$\mathbb{E}_{w_{T+1} \sim \nu}[\text{ER}(\hat{B}, \hat{w}_{T+1})] \lesssim \sigma^2 \left( \frac{kd\log(kn_1)}{cn_1 T} + \frac{k + \log \frac{1}{\delta}}{n_2} \right).$$

Theorem 4.1 shows that it is possible to learn the target task using only $O(k)$ samples via learning a good representation from the source tasks, which is better than the baseline $O(d)$ sample complexity for linear regression, thus demonstrating the benefit of representation learning.

\(^3\)A random vector $\tilde{x}$ is called $\rho^2$-subgaussian if for any fixed unit vector $\nu$ of the same dimension, the random variable $\nu^T \tilde{x}$ is $\rho^2$-subgaussian, i.e., $\mathbb{E}[e^{\nu^T (\tilde{x} - \mathbb{E}[\tilde{x}] )}] \leq e^{\nu^2 \rho^2/2}$ (forall $\nu \in \mathbb{R}$).
Notice that low-dimensional linear representations can be described by $kd$ parameters. Therefore the bound in (9) has the form $\frac{C(\Phi)}{n_1T} + \frac{k}{n_2}$ (ignoring logarithmic factors) where $C(\Phi)$ is the complexity of $\Phi$ from parameter counting. In Section 5 we will see that it is possible to get this type of rate for general representation functions.

Below are several additional remarks on Theorem 4.1.

Remark 4.1 (deterministic target task). We can drop Assumption 4.4 and easily obtain the following excess risk bound for any deterministic $w^*_{T+1}$ by slightly modifying the proof of Theorem 4.1:

$$\text{ER}(\hat{B}, \hat{w}_{T+1}) \lesssim \sigma^2 \left( \frac{k^2 d \log(\kappa n_1)}{cn_1 T} + \frac{k^2 + k + \log \frac{1}{\delta}}{n_2} \right),$$

which is only at most $k$ times larger than the bound in (9).

Remark 4.2 (v.s. the i.i.d. assumption). Our assumptions replace the i.i.d. assumptions on the tasks in previous work (Maurer et al., 2016). Assumptions 4.3 depicts the diversity of the source tasks, which ensures that all the $n_1T$ samples from the source tasks can be utilized collectively. Assumption 4.2 characterizes the allowed level of input distribution shift in order to transfer representation to the target task.

Remark 4.3 (with nuclear norm regularization). Theorem 4.1 can be easily extended to the computationally efficient nuclear norm constrained estimator in lieu of the rank-constrained estimator in (6) with the same statistical rates. However, the rates will depend inversely on the minimum restricted eigenvalue (Negahban and Wainwright, 2011), which is potentially very small. Furthermore, restricted eigenvalue conditions require identifiability of the parameters which is unrealistic in learning applications, so we do not pursue this further. Section 6 instead gives a computationally efficient algorithm that does not rely on minimum eigenvalue conditions with a very different analysis.

4.1 Proof Sketch of Theorem 4.1

The full proof of Theorem 4.1 is given in Appendix A. Here we give a sketch of its proof.

An important step is to prove an in-sample guarantee of the representation learned from (7), i.e., to show that it achieves small risk on the source training data. Specifically, we will prove the following bound in Claim A.3:

$$\frac{1}{n_1T} \|X(\hat{B}\hat{W} - B^*W^*)\|^2_F \lesssim \frac{\sigma^2 (kT + kd \log(\kappa n_1) + \log(1/\delta))}{n_1T}. \quad (10)$$

Notice that this roughly corresponds to the first term in (9).

To prove (10), we let $\hat{\Theta} = \hat{B}\hat{W}$ and $\Theta^* = B^*W^*$, and from the optimality of $\hat{B}$ and $\hat{W}$ for (7) we have $\|Y - X(\hat{\Theta})\|^2_F \leq \|Y - X(\Theta^*)\|^2_F$. Plugging in $Y = X(\Theta^*) + Z$, this becomes

$$\|X(\hat{\Theta} - \Theta^*)\|^2_F \leq 2(Z, X(\hat{\Theta} - \Theta^*)).$$

Notice that rank($\hat{\Theta} - \Theta^*$) $\leq 2k$. In Appendix A we will prove $\langle Z, X(\Delta) \rangle \lesssim \sigma \sqrt{kT} \|X(\Delta)\|_F$ with high probability for any $\Delta$ independent of $Z$ with rank no more than $2k$. However, $\hat{\Theta} - \Theta^*$ depends on $Z$, so we will further apply an $\epsilon$-net argument to cover all the possible low-rank matrices.

After proving (10), we need to connect the source in-sample risk to the expected excess risk on the target task. We will prove the following upper bound on the excess risk:

$$\mathbb{E}_{w^*_{T+1} \sim p}[\text{ER}(\hat{B}, \hat{w}_{T+1})] \lesssim \frac{1}{n_2k} \left\| P_{X_{T+1}^* B} X_{T+1} B^* \right\|_F^2 + \frac{\sigma^2 (k + \log \frac{1}{\delta})}{n_2} \quad (11)$$
On the other hand, we will lower bound the in-sample risk as follows:

\[
\frac{1}{n_1 T} \| \mathcal{X}(\hat{B}\hat{W} - B^*W^*) \|_F^2 \geq \frac{c}{n_2 T} \left\| P_{X_{T+1}^\perp} X_{T+1} B^* \right\|_F^2 \sigma_k^2(W^*). \tag{12}
\]

The proof of (12) relies on Assumption 4.2.

Finally, combining (10), (11) and (12), and using \( \sigma_k^2(W^*) \geq \frac{T}{k} \) (Assumption 4.3) and \( cn_1 \geq n_2 \), we finish the proof of Theorem 4.1.

5 General Low-Dimensional Representations

Now we return to the general case described in Section 3 where we allow a general representation function class \( \Phi \). We still assume that the representation is of low dimension \( k \). The goal is to obtain a result similar to Theorem 4.1. In this section we assume that inputs from all the tasks follow the same distribution, i.e., \( p_1 = \cdots = p_{T+1} = p \), but each task \( t \) still has its own specialization function \( w_t^\star \) (c.f. (4)). We remark that despite this restriction, our result in this section still applies to many interesting and nontrivial scenarios – consider the case where the inputs are all images from ImageNet and each task asks whether the image is from a specific class.

We overload the notation from Section 4 and use \( \mathcal{X} \) to represent the collection of all the training inputs from \( T \) source tasks \( X_1, \ldots, X_T \in \mathbb{R}^{n_1 \times d} \). We can think of \( \mathcal{X} \) as a third-order tensor of dimension \( n_1 \times d \times T \).

To characterize the complexity of the representation function class \( \Phi \), we need the standard definition of Gaussian width.

**Definition 5.1** (Gaussian width). Given a set \( K \subset \mathbb{R}^m \), the Gaussian width of \( K \) is defined as

\[
\mathcal{G}(K) = \mathbb{E}_{z \sim N(0,I)} \sup_{v \in K} \langle v, z \rangle.
\]

We will measure the complexity of \( \Phi \) using the Gaussian width of the following set that depends on the input data \( \mathcal{X} \):

\[
\mathcal{F}_\mathcal{X}(\Phi) = \{ A = [a_1, \ldots, a_T] \in \mathbb{R}^{n_1 \times T} : \|A\|_F = 1, \exists \phi, \phi' \in \Phi \text{ s.t. } a_t \in \text{span}([\phi(X_t), \phi'(X_t)]), \forall t \in [T] \}. \tag{13}
\]

We also need the following definition.

**Definition 5.2** (covariance between two representations). Given a distribution \( q \) over \( \mathbb{R}^d \) and two representation functions \( \phi, \phi' \in \Phi \), define the covariance between \( \phi \) and \( \phi' \) with respect to \( q \) to be

\[
\Sigma_q(\phi, \phi') = \mathbb{E}_{x \sim q} \left[ \phi(x) \phi'(x)^\top \right] \in \mathbb{R}^{k \times k}.
\]

Also define the symmetric covariance as

\[
\Lambda_q(\phi, \phi') = \begin{bmatrix} \Sigma_q(\phi, \phi) & \Sigma_q(\phi, \phi') \\ \Sigma_q(\phi', \phi) & \Sigma_q(\phi', \phi') \end{bmatrix} \in \mathbb{R}^{2k \times 2k}.
\]

It is easy to verify \( \Lambda_q(\phi, \phi') \succeq 0 \) for any \( \phi, \phi' \) and \( q \).\(^4\)

We make the following assumptions on the input distribution \( p \), which ensure concentration properties of the representation covariances.

\(^4\)See the proof of Lemma B.1.
Assumption 5.1 (point-wise concentration of covariance). For \( \delta \in (0, 1) \), there exists a number \( N_{\text{point}}(\Phi, p, \delta) \) such that if \( n \geq N_{\text{point}}(\Phi, p, \delta) \), then for any given \( \phi, \phi' \in \Phi \), \( n \) i.i.d. samples of \( p \) will with probability at least \( 1 - \delta \) satisfy
\[
0.9 \Lambda_p(\phi, \phi') \leq \Lambda_{\hat{p}}(\phi, \phi') \leq 1.1 \Lambda_p(\phi, \phi'),
\]
where \( \hat{p} \) is the empirical distribution over the \( n \) samples.

Assumption 5.2 (uniform concentration of covariance). For \( \delta \in (0, 1) \), there exists a number \( N_{\text{unif}}(\Phi, p, \delta) \) such that if \( n \geq N_{\text{unif}}(\Phi, p, \delta) \), then \( n \) i.i.d. samples of \( p \) will with probability at least \( 1 - \delta \) satisfy
\[
0.9 \Lambda_p(\phi, \phi') \leq \Lambda_{\hat{p}}(\phi, \phi') \leq 1.1 \Lambda_p(\phi, \phi'), \quad \forall \phi, \phi' \in \Phi,
\]
where \( \hat{p} \) is the empirical distribution over the \( n \) samples.

Assumptions 5.1 and 5.2 are conditions on the representation function class \( \Phi \) and the input distribution \( p \) that ensure concentration of empirical covariances to their population counterparts. Typically, we expect \( N_{\text{unif}}(\Phi, p, \delta) \gg N_{\text{point}}(\Phi, p, \delta) \) since uniform concentration is a stronger requirement. In Section 4, we have essentially shown that for linear representations and subgaussian input distributions, \( N_{\text{unif}}(\Phi, p, \delta) = \tilde{O}(d) \) and \( N_{\text{point}}(\Phi, p, \delta) = \tilde{O}(k) \) (see Claims A.1 and A.2).

Our main theorem in this section is the following:

Theorem 5.1 (main theorem for general representations). Fix a failure probability \( \delta \in (0, 1) \). Suppose \( n_1 \geq N_{\text{unif}}(\Phi, p, \frac{1}{n_1}) \) and \( n_2 \geq N_{\text{point}}(\Phi, p, \frac{1}{n_2}) \). Under Assumptions 4.3 and 4.4, with probability at least \( 1 - \delta \) over the samples, the expected excess risk of the learned predictor \( x \mapsto \hat{w}_{T+1}^T \hat{\phi}(x) \) on the target task satisfies
\[
\mathbb{E}_{\hat{w}_{T+1} \sim \nu} [\text{ER}(\hat{\phi}, \hat{w}_{T+1})] \lesssim \sigma^2 \left( \frac{\mathcal{G}(\mathcal{F}_X(\Phi))^2 + \log \frac{1}{\delta}}{n_1 T} + \frac{k + \log \frac{1}{\delta}}{n_2} \right). \tag{14}
\]

Theorem 5.1 is very similar to Theorem 4.1 in terms of the result and the assumptions made. In the bound (14), the complexity of \( \Phi \) is captured by the Gaussian width of the data-dependent set \( \mathcal{F}_X(\Phi) \) defined in (13). Data-dependent complexity measures are ubiquitous in generalization theory, one of the most notable examples being Rademacher complexity. Similar complexity measure also appeared in existing representation learning theory (Maurer et al., 2016). Usually, for specific examples, we can apply concentration bounds to get rid of the data dependency, such as our result for linear representations (Theorem 4.1).

Our assumptions on the linear specification functions \( \hat{w}_t \)'s are the same as in Theorem 4.1. The probabilistic assumption on \( \hat{w}_{T+1}^T \) can also be removed at the cost of an additional factor of \( k \) in the bound – see Remark 4.1.

The proof of Theorem 5.1 is given in Appendix B. Here we prove an important intermediate result on the in-sample risk, which explains how the Gaussian width of \( \mathcal{F}_X(\Phi) \) arises.

Claim 5.3 (analogue of (10)). Let \( \hat{\phi} \) and \( \hat{w}_1, \ldots, \hat{w}_T \) be the optimal solution to (2). Then with probability at least \( 1 - \delta \) we have
\[
\sum_{t=1}^{T} \left\| \hat{\phi}(X_t)\hat{w}_t - \phi^*(X_t)\hat{w}_t^* \right\|^2 \lesssim \sigma^2 \left( \mathcal{G}(\mathcal{F}_X(\Phi))^2 + \log \frac{1}{\delta} \right).
\]

Proof. By the optimality of \( \hat{\phi} \) and \( \hat{w}_1, \ldots, \hat{w}_T \) for (2), we know
\[
\sum_{t=1}^{T} \left\| y_t - \hat{\phi}(X_t)\hat{w}_t \right\|^2 \leq \sum_{t=1}^{T} \left\| y_t - \phi^*(X_t)\hat{w}_t^* \right\|^2.
\]
Plugging in \( y_t = \phi^*(X_t)w_t^* + z_t \) (\( z_t \sim \mathcal{N}(0, I) \) is independent of \( X_t \)), we get
\[
\sum_{t=1}^{T} \left\| \phi^*(X_t)w_t^* + z_t - \hat{\phi}(X_t)\hat{w}_t \right\|^2 \leq \sum_{t=1}^{T} \| z_t \|^2,
\]
which gives
\[
\sum_{t=1}^{T} \left\| \phi(X_t)\hat{w}_t - \phi^*(X_t)w_t^* \right\|^2 \leq 2 \sum_{t=1}^{T} \langle z_t, \hat{\phi}(X_t)\hat{w}_t - \phi^*(X_t)w_t^* \rangle.
\]
Denote \( Z = [z_1, \ldots, z_T] \in \mathbb{R}^{n_1 \times T} \) and \( A = [a_1, \ldots, a_T] \in \mathbb{R}^{n_1 \times T} \) where \( a_t = \hat{\phi}(X_t)\hat{w}_t - \phi^*(X_t)w_t^* \). Then the above inequality reads \( \| A \|_F^2 \leq 2 \langle Z, A \rangle \). Notice that \( \frac{A}{\| A \|_F} \in \mathcal{F}_\chi(\Phi) \) (c.f. (13)). It follows that
\[
\| A \|_F \leq 2 \left\langle Z, \frac{A}{\| A \|_F} \right\rangle \leq 2 \sup_{A \in \mathcal{F}_\chi(\Phi)} \langle Z, \bar{A} \rangle. \tag{15}
\]
By definition, we have \( \mathbb{E}_Z \left[ \sup_{A \in \mathcal{F}_\chi(\Phi)} \langle \sigma^{-1} Z, \bar{A} \rangle \right] = \mathcal{G}(\mathcal{F}_\chi(\Phi)) \). Furthermore, since the function \( Z \mapsto \sup_{A \in \mathcal{F}_\chi(\Phi)} \langle Z, \bar{A} \rangle \) is \( 1 \)-Lipschitz in Frobenius norm, by the standard Gaussian concentration inequality, we have with probability at least \( 1 - \delta \),
\[
\sup_{A \in \mathcal{F}_\chi(\Phi)} \langle \sigma^{-1} Z, \bar{A} \rangle \leq \mathbb{E} \left[ \sup_{A \in \mathcal{F}_\chi(\Phi)} \langle \sigma^{-1} Z, \bar{A} \rangle \right] + \sqrt{\log \frac{1}{\delta}} = \mathcal{G}(\mathcal{F}_\chi(\Phi)) + \sqrt{\log \frac{1}{\delta}}.
\]
Then the proof is completed using (15). \( \square \)

6 High-Dimensional Linear Representations

In this section, we consider the case where the representation is a general linear map without an explicit dimensionality constraint, and we will prove a norm-based result by exploiting the intrinsic dimension of the representation. Such a generalization is desirable since in typical applications the representation dimension is larger than \( n_2 \).

Without loss of generality, we let the representation function class be \( \Phi = \{ x \mapsto B^\top x \mid B \in \mathbb{R}^{d \times T} \} \). We note that a dimension-\( T \) representation is sufficient for learning \( T \) source tasks and any choice of dimension greater than \( T \) will not change our argument.

We assume that inputs of all tasks come from the same distribution: \( p_1 = \cdots = p_{T+1} = p \), and we use the same notation from Section 4 unless otherwise specified.

Since we do not have a dimensionality constraint, we modify (7) by adding \( \ell_2 \) regularization:
\[
(\hat{B}^\Lambda, \hat{W}^\Lambda) \leftarrow \arg \min_{B \in \mathbb{R}^{d \times T}, W \in \mathbb{R}^{T \times T}} \frac{1}{2n_1} \| Y - \mathcal{X}(BW) \|_F^2 + \frac{\lambda}{2} \| B \|_F^2 + \frac{\lambda}{2} \| W \|_F^2. \tag{16}
\]
For the target task, we also modify (8) by adding a norm constraint:
\[
\hat{w}_{T+1} \leftarrow \arg \min_{\| w \| \leq r} \frac{1}{2n_2} \| X_{T+1} \hat{B}^\Lambda w - y_{T+1} \|^2. \tag{17}
\]
We will specify the choices of \( \lambda \) and \( r \) in Theorem 6.1.

Similar to Section 4, the source task data relation is denoted as \( Y = \mathcal{X}(\Theta^*) + Z \), where \( \Theta^* \in \mathbb{R}^{d \times T} \) is the ground truth and \( Z \) has i.i.d. \( \mathcal{N}(0, \sigma^2) \) entries. Suppose that the target task data satisfy \( y_{T+1} = X_{T+1} \theta_{T+1}^* + z_{T+1} \in \mathbb{R}^{n_2} \). Similar to the setting in Section 4, we assume the target task data is subgaussian as in Assumption 4.1.
Theorem 6.1 (main theorem for high-dimensional representations). Fix a failure probability $\delta \in (0, 1)$. Under Assumption 4.1, we further assume $n_1 \gg \sigma^2 \rho^4 R^2 T \log(1/\delta)$, $n_2 \gg \sigma^2 \rho^4 R^2 \|\Sigma\| \log(1/\delta)$ and $n_1 \geq n_2$, where $\tilde{R} = \|\Theta\|_*/\sqrt{T}$. Let the target task model $\theta^*_{T+1}$ be coherent with the source task models $\Theta^*$ in the sense that $\theta^*_{T+1} \sim \nu = N(0, \Theta^*(\Theta^*)^T/T)$. Set $r = 2\sqrt{\|\Theta^*\|_*/T}$ and $\lambda = \Theta \left(\frac{1}{\sqrt{n_1}}(\log 1/\delta)^{2/3} \cdot (\sqrt{T} \|\Sigma\|_2 + \sqrt{T} \text{Tr}(\Sigma)) \log(n + T)\right)$. Then with probability at least $1 - \delta$ over the samples, the expected excess risk of the learned predictor $x \mapsto (\hat{w}^T_{T+1} (\hat{B}^T)^T x$ on the target task satisfies:

$$\mathbb{E}_{\theta^*_{T+1} \sim \nu} [\text{ER}(\hat{B}, \hat{w}_{T+1})] \leq \sigma \tilde{R} \tilde{O} \left(\frac{\sqrt{\text{Tr}(\Sigma)}}{\sqrt{n_1 T}} + \frac{\sqrt{\|\Sigma\|_2}}{\sqrt{n_2}}\right).$$

(18)

Here $\tilde{O}$ hides logarithmic factors.

Note that $\|\Theta^*\|_F = \sqrt{T}$ when each $\theta^*_i$ is of unit norm. Thus $\tilde{R} = \|\Theta^*\|_*/\sqrt{T}$ should generally be regarded as $O(1)$ for a well-behaved $\Theta^*$ that is nearly low-dimensional. In this regime, Theorem 6.1 indicates that we are able to exploit all $n_1 T$ samples from the source tasks, similar to Theorems 4.1 and 5.1.

With a good representation, the sample complexity on the target task can also improve over learning the target task from scratch. Consider the baseline of regular ridge regression directly applied to the target task data:

$$\hat{\theta}^\lambda \leftarrow \arg \min_{\theta} \frac{1}{2n_2} \|X_{T+1} \theta - y_{T+1}\|^2 + \frac{\lambda}{2} \|\theta\|^2. \quad (19)$$

Its standard excess risk bound is

$$\text{ER}(\hat{\theta}^\lambda) \lesssim \sigma \sqrt{\frac{\|\Theta^*\|_F^2 \text{Tr}(\Sigma)}{n_2}}.$$  

(See e.g. Hsu et al. (2012b).) Taking expectation over $\theta^*_T \sim \nu = N(0, \Theta^*(\Theta^*)^T/T)$, we obtain

$$\mathbb{E}_{\theta^*_{T+1} \sim \nu} [\text{ER}(\hat{\theta}^\lambda)] \lesssim \sigma \frac{\|\Theta^*\|_F}{\sqrt{T}} \sqrt{\frac{\text{Tr}(\Sigma)}{n_2}}. \quad (20)$$

Compared with (20), our bound (18) is an improvement as long as $\frac{\|\Theta^*\|_F^2}{\|\Theta^*\|_F^2} \ll \frac{\text{Tr}(\Sigma)}{\|\Sigma\|_F}$. The left hand side $\frac{\|\Theta^*\|_F^2}{\|\Theta^*\|_F^2}$ is always no more than the rank of $\Theta^*$, and we call it the intrinsic rank. Hence we see that we can gain from representation learning if the source predictors are intrinsically low dimensional.

To intuitively understand how this is achieved, we note that a representation $B$ is reweighing linear combinations of the features according to their “importance” on the $T$ source tasks. We make an analogy with a simple case of feature selection. Suppose we have learned a representation vector $b$ where $b_i$ scales with the importance of $i$-th feature, i.e., the representation is $\phi(x) = x^T b$. Then ridge regression on the target task data $(X, y)$:

$$\text{minimize} \quad \frac{1}{2n_2} \|X \cdot \text{diag}(b) \cdot w - y\|^2$$

is equivalent to

$$\text{minimize} \quad \frac{1}{2n_2} \|X v - y\|^2.$$  

From the above equation, we see that the features with large $|b_i|$ (those that were useful on the source tasks) will be more heavily used than the ones with small $|b_i|$ due to the reweighed $\ell_2$ constraint. Thus the important features are learned from the source tasks, and the coefficients are learned from the target task.
Remark 6.1 (The non-convex landscape). Although the optimization problem (16) is non-convex, its structure allows us to apply existing landscape analysis of matrix factorization problems (Haeffele et al., 2014) and to show that it has the nice properties of no strict saddles and no spurious local minima. Therefore, randomly initialized gradient descent or perturbed gradient descent are guaranteed to converge to a global minimum of (16) in polynomial-time (Ge et al., 2015, Lee et al., 2016, Jin et al., 2017).

6.1 Proof Sketch for a Fixed Design

The proof of Theorem 6.1 is given in Appendix C. As a warm-up, now we give a proof sketch for the simpler case of a fixed design, i.e., \( X_1 = X_2 = \cdots = X_{T+1} = X \in \mathbb{R}^{n \times d} \). The test error is also measured on \( \hat{X} \).

For notational convenience, let \( B \) be the representation learned from the source tasks. For fixed design we can use the same regularized form when training the target model (instead of the constrained form (17)):

\[
\begin{align*}
\mathbf{w}_{T+1} &\leftarrow \arg \min_{\mathbf{w}} \left\{ \frac{1}{2n} \| XB \mathbf{w} - y_{T+1} \|_2^2 + \frac{\lambda}{2} \| \mathbf{w} \|_2^2 \right\}.
\end{align*}
\]

Plugging in the solution \( \mathbf{w}_{T+1} \), we decompose the excess risk as follows:

\[
\mathbb{E}_{\Theta_{T+1} \sim \nu} \left[ \text{ER}(B, \mathbf{w}_{T+1}) \right]
= \mathbb{E}_{\Theta_{T+1} \sim \nu} \left[ \frac{1}{2n} \| (S_{XB}^\lambda - I) X \Theta_{T+1}^* + S_{XB}^\lambda z_{T+1} \|_2^2 \right]
\leq \mathbb{E}_{\Theta_{T+1} \sim \nu} \left[ \frac{1}{n} \| (S_{XB}^\lambda - I) X \Theta_{T+1}^* \|_2^2 + \frac{1}{n} \| S_{XB}^\lambda z_{T+1} \|_2^2 \right]
= \frac{1}{nT} \| (S_{XB}^\lambda - I) X \Theta^* \|_F^2 + \frac{1}{n} \| S_{XB}^\lambda z_{T+1} \|_2^2,
\]

where \( S_{XB}^\lambda = \frac{1}{n} XB(\frac{1}{n} B^T XB + \lambda I)^{-1} B^T X^T \).

Meanwhile, from the optimality of \( B \) on data from source tasks, we can ensure that as long as \( \lambda \geq \frac{2}{n} \| X^T Z \|_2 \), we have

\[
\frac{1}{nT} \| (S_{XB}^\lambda - I) X \Theta^* \|_F^2 \leq \frac{\lambda \| \Theta^* \|_2}{T},
\]

\[
\frac{1}{n} \| S_{XB}^\lambda \|_F^2 \leq \frac{\| X \|_2^2 \| \Theta^* \|_2}{n^2 \lambda} = \frac{\| \Sigma \|_2 \| \Theta^* \|_2}{n \lambda}.
\]

The first inequality above settles the representation error in (21), and the second one can be used to bound the noise term in (21) using a standard concentration bound since \( S_{XB}^\lambda \) is independent of \( z_{T+1} \).

7 Neural Networks

In this section, we show that pre-training + fine-tuning neural nets with weight decay regularization provably learns good representations.

Consider two-layer ReLU neural network \( f_{B, \mathbf{w}}(\mathbf{x}) = \mathbf{w}^T (B^T \mathbf{x})_+, \) where \( \mathbf{w} \in \mathbb{R}^d, B \in \mathbb{R}^{d_0 \times d} \) and \( \mathbf{x} \in \mathbb{R}^{d_0} \). Here \((\cdot)_+\) is the ReLU activation \((z)_+ = \max\{0, z\}\) defined element-wise. Namely, we let the representation function class be \( \Phi = \{ \mathbf{x} \mapsto (B^T \mathbf{x})_+ | B \in \mathbb{R}^{d_0 \times d} \} \). On the source tasks we use the square
loss with weight decay regularizer\(^5\):

\[
\hat{B}, \hat{W} = \arg\min_{B \in \mathbb{R}^{d_0 \times d}, \quad W = [w_1, \ldots, w_T] \in \mathbb{R}^{d \times T}} \frac{1}{2n_1 T} \sum_{t=1}^{T} \|y_t - (X_t B)_+ w_t\|^2 + \frac{\lambda}{2} \|B\|_F^2 + \frac{\lambda}{2} \|W\|_F^2. \tag{22}
\]

The inputs from \(T\) source tasks, \(X_1, \ldots, X_T \in \mathbb{R}^{n_1 \times d_0}\), and future data \(X_{T+1} \in \mathbb{R}^{n_2 \times d_0}\) for the target task, are assumed to share a same distribution \(p\). We write the covariance operator \(\Sigma(u, v) = \mathbb{E}_{x \sim p}[(u^\top x)_+ (v^\top x)_+]\).

On the target task, we simply re-train the last layer while fixing the hidden layer weights:

\[
\hat{w}_{T+1} = \arg\min_{\|w\| \leq \rho} \frac{1}{2n_2} \|y_{T+1} - (X_{T+1} \hat{B})_+ w\|^2.
\tag{23}
\]

**Assumption 7.1** (teacher network). Assume for the source tasks that \(y_t = (X_t B^*)_+ w^*_t + z_t\) is generated by a teacher network with parameters \(B^* \in \mathbb{R}^{d_0 \times d}, \quad W^* = [w^*_1, \ldots, w^*_T] \in \mathbb{R}^{d \times T}\), and noise term \(z_t \sim \mathcal{N}(0, \sigma^2 I)\). Similarly for the target task \(y_{T+1} = (X_{T+1} B^*_T)_+ w^*_{T+1} + z_{T+1}\) where \(z_{T+1} \sim \mathcal{N}(0, \sigma^2 I)\).

Let \(\nu = \mathcal{N}(0, B^* W^*(W^*)_\top (B^*)\top / T)\).

**Theorem 7.1.** Fix a failure probability \(\delta \in (0, 1)\). Under Assumption 4.1 and 7.1, we further assume \(n_1 \gg \rho^4 T R^2 \sigma^2 \text{Tr}(\Sigma) \log(1/\delta), n_2 \gg \rho^4 \sigma^2 R^2 \|\Sigma\| \log(1/\delta)\) and \(n_1 \geq n_2\), where \(R = (\frac{1}{2} \|B\|_F^2 + \frac{1}{2} \|W\|_F^2)/\sqrt{T}\). Let the target task model \(w^*_{T+1}\) be coherent with the source task models \(B^*, \quad w^*_t, \quad t \in [T]\) in the sense that \(B^*_T w^*_{T+1} \sim \nu\). Set \(\tau^2 = 2(\|B\|_F^2 + \|W\|_F^2)/T\) and \(\lambda = \Theta\left(\sqrt{n_1} (\log(1/\delta))^{2/3}\right)\cdot (\sqrt{T}\sqrt{\|\Sigma\|_2 + \text{Tr}(\Sigma)} \log(n + T))\). Then with probability at least \(1 - \delta\) over the samples, the expected excess risk of the learned predictor \(x \mapsto \hat{w}^\top_{T+1}(\hat{B} x)_+\) on the target task satisfies:

\[
\mathbb{E}_{B^*_T, w^*_{T+1} \sim \nu}[\mathbb{E}(f_B, \hat{w}_{T+1})] \leq \sigma \tilde{R} \Omega \left(\frac{\text{Tr}(\Sigma)}{\sqrt{n_1 T}} + \frac{\sqrt{\|\Sigma\|_2}}{\sqrt{n_2}}\right). \tag{24}
\]

To highlight the advantage of representation learning, we compare to training a neural network with weight decay directly on the target task:

\[
\hat{B}, \hat{w} = \arg\min_{B, w} \frac{1}{2n_1} \sum_{t=1}^{T} \|y_{t+1} - (X_{t+1} B)_+ w\|^2 + \frac{\lambda}{2} \|B\|_F^2 + \frac{\lambda}{2} \|w\|_2^2. \tag{25}
\]

The error of the baseline method is

\[
\mathbb{E}[\mathbb{E}(f_B, \hat{w})] \lesssim \sigma \frac{1}{\sqrt{n_1}} \left(\|B^*_T\|_2^2 + \|w^*_{T+1}\|_2^2\right) \mathbb{E}_{x_i \sim \mathcal{N}(0, \nu)} \mathbb{E}_{z \sim \mathcal{N}(0, \sigma^2 I)} \|\Phi(X)^\top z\|_{\infty}
\lesssim \sigma \tilde{R} \sqrt{\frac{\text{tr}(\Sigma)}{n_2}}. \tag{26}
\]

We see that Equation (24) is always smaller than Equation (26) whenever \(n_1 \geq n_2\).

See Appendix D for the proof of Theorem 7.1 and the calculation of (26).

\(^5\text{Wei et al. (2019)}\) show that (22) can be minimized in polynomial iteration complexity using perturbed gradient descent, though potentially requires exponential width.
8 Conclusion

We give the first statistical analysis showing that few-shot learning via representation learning can fully exploit all data points from source tasks to enable few-shot learning on a target task. This type of results are shown for both low-dimensional and high-dimensional representation function classes.

There are many important directions to pursue in representation learning and few-shot learning. Our results in Section 6 and 7 indicate that explicit low-dimensionality is not necessary, and norm-based capacity control also forces the classifier to learn good representations. This gives a theoretical justification for the common practice of pre-training on ImageNet then fine-tuning on the target task. Open questions include whether this is a general phenomenon in all deep learning models, whether other capacity control can be applied, and how to optimize to attain good representations.

Further directions include investigating unsupervised representation learning via contrastive methods (Arora et al., 2019) and the effectiveness of predicting multiple auxiliary tasks (Doersch and Zisserman, 2017).

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A Proof of Theorem 4.1

We first prove several claims and then combine them to finish the proof of Theorem 4.1. We will use technical lemmas proved in Section A.1.

Claim A.1 (covariance concentration of source tasks). Suppose \( n_1 \gg n^4(d + \log(T/\delta)) \) for \( \delta \in (0, 1) \). Then with probability at least \( 1 - \frac{1}{10T} \) over the inputs \( X_1, \ldots, X_T \) in the source tasks, we have

\[
0.9\Sigma_t \preceq \frac{1}{n_1} X_t^\top X_t \preceq 1.1\Sigma_t, \quad \forall t \in [T].
\]

Proof. According to our assumption on \( p_t \), we can write \( X_t = \hat{X}_t \Sigma_t^{1/2} \), where \( \hat{X}_t \in \mathbb{R}^{n_1 \times d} \) and the rows of \( \hat{X}_t \) hold i.i.d. samples of \( \tilde{p}_t \). Since \( \tilde{p}_t \) satisfies the conditions in Lemma A.6, from Lemma A.6 we know that with probability at least \( 1 - \frac{\delta}{10T} \),

\[
0.9I \preceq \frac{1}{n_1} \hat{X}_t^\top \hat{X}_t \preceq 1.1I,
\]

which implies

\[
0.9\Sigma_t \preceq \frac{1}{n_1} \Sigma_t^{1/2} \hat{X}_t^\top \hat{X}_t \Sigma_t^{1/2} = \frac{1}{n_1} X_t^\top X_t \preceq 1.1\Sigma_t.
\]

The proof is finished by taking a union bound over all \( t \in [T] \).

Claim A.2 (covariance concentration of target task). Suppose \( n_2 \gg \rho^4(k + \log(1/\delta)) \) for \( \delta \in (0, 1) \). Then for any given matrix \( B \in \mathbb{R}^{d \times 2k} \) that is independent of \( X_{T+1} \), with probability at least \( 1 - \frac{\delta}{10T} \) over \( X_{T+1} \) we have

\[
0.9B^\top \Sigma_{T+1} B \preceq \frac{1}{n_2} B^\top X_{T+1}^\top X_{T+1} B \preceq 1.1B^\top \Sigma_{T+1} B.
\]

Proof. According to our assumption on \( p_{T+1} \), we can write \( X_{T+1} = \hat{X}_{T+1} \Sigma_{T+1}^{1/2} \), where \( \hat{X}_{T+1} \in \mathbb{R}^{n_2 \times d} \) and the rows of \( \hat{X}_{T+1} \) hold i.i.d. samples of \( \tilde{p}_{T+1} \). We take the SVD of \( \Sigma_{T+1}^{1/2} B = UDV^\top \), where \( U \in \mathbb{R}^{d \times 2k} \) has orthonormal columns. Now we look at the matrix \( \hat{X}_{T+1} U \in \mathbb{R}^{n_2 \times 2k} \). It is easy to see that the rows of \( \hat{X}_{T+1} U \) are i.i.d. \( 2k \)-dimensional random vectors with zero mean, identity covariance, and are \( \rho^2 \)-subgaussian. Therefore, applying Lemma A.6, with probability at least \( 1 - \frac{\delta}{10T} \) we have

\[
0.9I \preceq \frac{1}{n_2} U^\top \hat{X}_{T+1}^\top \hat{X}_{T+1} U \preceq 1.1I,
\]

which implies

\[
0.9VDDV^\top \preceq \frac{1}{n_2} VDU^\top \hat{X}_{T+1}^\top \hat{X}_{T+1} UDV^\top \preceq 1.1VDDV^\top.
\]

Since \( \frac{1}{n_2} VDU^\top \hat{X}_{T+1}^\top \hat{X}_{T+1} UDV^\top = \frac{1}{n_2} B^\top \Sigma_{T+1}^{1/2} \hat{X}_{T+1}^\top \hat{X}_{T+1} \Sigma_{T+1}^{1/2} B = \frac{1}{n_2} B^\top X_{T+1}^\top X_{T+1} B \) and \( VDDV^\top = VDU^\top UDV^\top = B^\top \Sigma_{T+1} B \), the above inequality becomes

\[
0.9B^\top \Sigma_{T+1} B \preceq \frac{1}{n_2} B^\top X_{T+1}^\top X_{T+1} B \preceq 1.1B^\top \Sigma_{T+1} B.
\]

Claim A.3 (guarantee on source training data). Under the setting of Theorem 4.1, with probability at least \( 1 - \frac{\delta}{5} \) we have

\[
\| \mathcal{A}(\hat{B}_\hat{W} - B^*W^*) \|_F^2 \lesssim \sigma^2 (kT + kd(kn_1) + \log(1/\delta)).
\]

(29)
Proof. We assume that (27) is true, which happens with probability at least 1 − \frac{\delta}{kT} according to Claim A.1.

Let \( \hat{\Theta} = \hat{B}\hat{W} \) and \( \Theta^* = B^*W^* \). From the optimality of \( \hat{B} \) and \( \hat{W} \) for (7) we have \( \|Y - \mathcal{X}(\hat{\Theta})\|_F^2 \leq \|Y - \mathcal{X}(\Theta^*)\|_F^2 \). Plugging in \( Y = \mathcal{X}(\Theta^*) + Z \), this becomes

\[
\|\mathcal{X}(\hat{\Theta} - \Theta^*)\|_F^2 \leq 2\langle Z, \mathcal{X}(\hat{\Theta} - \Theta^*) \rangle.
\] (30)

Let \( \Delta = \hat{\Theta} - \Theta^* \). Since \( \text{rank}(\Delta) \leq 2k \), we can write \( \Delta = VR = [V_1, \ldots, V_T] \) where \( V \in \mathcal{O}_{d,2k} \) and \( R = [r_1, \ldots, r_T] \in \mathbb{R}^{2k \times T} \). Here \( \mathcal{O}_{d_1,d_2} \) is the set of orthonormal \( d_1 \times d_2 \) matrices (i.e., the columns are orthonormal). For each \( t \in [T] \) we further write \( X_tV = U_tQ_t \) where \( U_t \in \mathcal{O}_{n_1,2k} \) and \( Q_t \in \mathbb{R}^{2k \times 2k} \). Then we have

\[
\langle Z, \mathcal{X}(\Delta) \rangle = \sum_{t=1}^{T} z_t^\top X_tV r_t
= \sum_{t=1}^{T} z_t^\top U_tQ_tr_t,
\leq \sum_{t=1}^{T} \|U_t^\top z_t\| \cdot \|Q_tr_t\|
\leq \sqrt{\sum_{t=1}^{T} \|U_t^\top z_t\|^2} \cdot \sqrt{\sum_{t=1}^{T} \|Q_tr_t\|^2}
= \sqrt{\sum_{t=1}^{T} \|U_t^\top z_t\|^2} \cdot \sqrt{\sum_{t=1}^{T} \|Q_t^\top r_t\|^2}
= \sqrt{\sum_{t=1}^{T} \|U_t^\top z_t\|^2} \cdot \sqrt{\sum_{t=1}^{T} \|X_tV r_t\|^2}
= \sqrt{\sum_{t=1}^{T} \|U_t^\top z_t\|^2} \cdot \|\mathcal{X}(\Delta)\|_F.
\] (31)

Next we give a high-probability upper bound on \( \sum_{t=1}^{T} \|U_t^\top z_t\|^2 \) using the randomness in \( Z \). Since \( U_t^\top \)'s depend on \( V \) which depends on \( Z \), we will need an \( \epsilon \)-net argument to cover all possible \( V \in \mathcal{O}_{d,2k} \). First, for any fixed \( \tilde{V} \in \mathcal{O}_{d,2k} \), we let \( X_t\tilde{V} = U_t\tilde{Q}_t \) where \( U_t \in \mathcal{O}_{n,2k} \). The \( U_t^\top \)'s defined in this way are independent of \( Z \). Since \( Z \) has i.i.d. \( \mathcal{N}(0, \sigma^2) \) entries, we know that \( \sigma^{-2} \sum_{t=1}^{T} \|U_t^\top z_t\|^2 \) is distributed as \( \chi^2(2kT) \). Using the standard tail bound for \( \chi^2 \) random variables, we know that with probability at least \( 1 - \delta' \) over \( Z \),

\[
\sigma^{-2} \sum_{t=1}^{T} \|U_t^\top z_t\|^2 \leq kT + \log(1/\delta').
\]

Therefore, using the same argument in (31) we know that with probability at least \( 1 - \delta' \),

\[
\langle Z, \mathcal{X}(\tilde{V}R) \rangle \leq \sigma \sqrt{kT + \log(1/\delta') \|\mathcal{X}(\tilde{V}R)\|_F}.
\]

Now, from Lemma A.5 we know that there exists an \( \epsilon \)-net \( \mathcal{N} \) of \( \mathcal{O}_{d,2k} \) in Frobenius norm such that \( \mathcal{N} \subset \mathcal{O}_{d,2k} \) and \( |\mathcal{N}| \leq \left( \frac{6\sqrt{d}}{\epsilon} \right)^{2kd} \). Applying a union bound over \( \mathcal{N} \), we know that with probability at least \( 1 - \delta'|\mathcal{N}| \),

\[
\langle Z, \mathcal{X}(\tilde{V}R) \rangle \leq \sigma \sqrt{kT + \log(1/\delta') \|\mathcal{X}(\tilde{V}R)\|_F}, \quad \forall \tilde{V} \in \mathcal{N}.
\] (32)
Choosing $\delta' = \frac{\delta}{20(n_1T)}$, we know that (32) holds with probability at least $1 - \frac{\delta}{20}$.

We will use (27), (30) and (32) to complete the proof of the claim. This is done in the following steps:

1. Upper bounding $\|Z\|_F$.
   
   Since $\sigma^{-2} \|Z\|_F^2 \sim \chi^2(n_1T)$, we know that with probability at least $1 - \frac{\delta}{20}$,
   
   $$\|Z\|_F^2 \lesssim \sigma^2(n_1T + \log(1/\delta)).$$

2. Upper bounding $\|\Delta\|_F$.
   
   From (30) we have $\|\mathcal{X}(\Delta)\|_F \leq 2\|Z\|_F \|\mathcal{X}(\Delta)\|_F$, which implies $\|\mathcal{X}(\Delta)\|_F \leq 2\|Z\|_F \lesssim \sigma \sqrt{n_1T + \log(1/\delta)}$.
   
   On the other hand, letting the $t$-th column of $\Delta$ be $\delta_t$, we have
   
   $$\|X_t(\Delta)\|_F = \sum_{t=1}^T \|X_t\delta_t\|_F^2 = \sum_{t=1}^T \delta_t^\top X_t^\top X_t \delta_t$$
   
   $$\geq 0.9n_1 \sum_{t=1}^T \delta_t^\top \Sigma_t \delta_t$$
   
   (using (27))
   
   $$\geq 0.9n_1 \sum_{t=1}^T \lambda_{\min}(\Sigma_t) \|\delta_t\|_2^2$$
   
   $$\geq 0.9n_1 \lambda \|\Delta\|_F^2,$$

   where $\lambda = \min_{t \in [T]} \lambda_{\min}(\Sigma_t)$. Hence we obtain
   
   $$\|\Delta\|_F^2 \lesssim \frac{\|\mathcal{X}(\Delta)\|_F^2}{n_1 \lambda} \lesssim \frac{\sigma^2(n_1T + \log(1/\delta))}{n_1 \lambda}.$$

3. Applying the $\epsilon$-net $\mathcal{N}$.

   Let $\tilde{V} \in \mathcal{N}$ such that $\|V - \tilde{V}\|_F \leq \epsilon$. Then we have
   
   $$\|\mathcal{X}(VR - \tilde{V}R)\|_F$$
   
   $$= \sum_{t=1}^T \|X_t(V - \tilde{V})r_t\|_2^2$$
   
   $$\leq \sum_{t=1}^T \|X_t\|_2^2 \|V - \tilde{V}\|_F \|r_t\|_2^2$$
   
   $$\leq \sum_{t=1}^T 1.1n_1 \lambda_{\max}(\Sigma_t) \epsilon^2 \|r_t\|_2^2$$
   
   (using (27))
   
   $$\leq 1.1n_1 \lambda \epsilon^2 \|R\|_F^2$$
   
   (\(\lambda = \max_{t \in [T]} \lambda_{\max}(\Sigma_t))
   
   $$= 1.1n_1 \lambda \epsilon^2 \|\Delta\|_F^2$$
   
   (\|\Delta\|_F = \|VR\|_F = \|R\|_F)
   
   $$\lesssim n_1 \lambda \epsilon^2 \cdot \frac{\sigma^2(n_1T + \log(1/\delta))}{n_1 \lambda}$$
   
   $$= \kappa \epsilon^2 \sigma^2(n_1T + \log(1/\delta)).$$

(34)
4. Finishing the proof.

We have the following chain of inequalities:

\[
\frac{1}{2} \| \mathcal{X}(\Delta) \|^2_F \\
\leq (Z, \mathcal{X}(\Delta)) = (Z, \mathcal{X}(\overline{V}R)) + (Z, \mathcal{X}(VR - \overline{V}R)) \\
\leq \sigma \sqrt{kT + \log(1/\delta')} \| \mathcal{X}(\overline{V}R) \|_F + \|Z\|_F \| \mathcal{X}(VR - \overline{V}R) \|_F \\
\leq \sigma \sqrt{kT + \log(1/\delta')} (\| \mathcal{X}(VR) \|_F + \| \mathcal{X}(VR - \overline{V}R) \|_F ) + \sigma \sqrt{n_1T + \log(1/\delta')} \| \mathcal{X}(VR - \overline{V}R) \|_F \\
\leq \sigma \sqrt{kT + \log(1/\delta')} \| \mathcal{X}(\Delta) \|_F + \sigma \sqrt{n_1T + \log(1/\delta')} \| \mathcal{X}(VR - \overline{V}R) \|_F \\
\leq \sigma \sqrt{kT + \log(1/\delta')} \| \mathcal{X}(\Delta) \|_F + \epsilon \sigma^2 \sqrt{\kappa(n_1T + \log(1/\delta')).} \\
\leq \sigma \sqrt{kT + \log(1/\delta')} \| \mathcal{X}(\Delta) \|_F + \epsilon \sigma^2 \sqrt{\kappa(n_1T + \log(1/\delta')).} 
\]

Finally, we let \( \epsilon = \frac{k}{\sqrt{\kappa n_1}} \), and recall \( \delta' = \frac{\delta}{20(5\delta^2 + 2k)} \). Then the above inequality implies

\[
\| \mathcal{X}(\Delta) \|_F \\
\leq \max \left\{ \sigma \sqrt{kT + \log(1/\delta')}, \sqrt{\epsilon \sigma^2 \sqrt{\kappa(n_1T + \log(1/\delta'))}} \right\} \\
= \max \left\{ \sigma \sqrt{kT + \log(1/\delta')}, \sigma \sqrt{\frac{k}{n_1}(n_1T + \log(1/\delta'))} \right\} \\
\leq \max \left\{ \sigma \sqrt{kT + \log(1/\delta')}, \sigma \sqrt{kT + \log(1/\delta')} \right\} \\
= \sigma \sqrt{kT + \log(1/\delta')} \\
\leq \sigma \sqrt{kT + kd \log \left( \frac{k}{\epsilon} \right) + \log \left( \frac{1}{\delta} \right)} \\
\leq \sigma \sqrt{kT + kd \log(\kappa n_1) + \log \left( \frac{1}{\delta} \right)}.
\]

The high-probability events we have used in the proof are (27), (32) and (33). By a union bound, the failure probability is at most \( \frac{\delta}{10} + \frac{\delta}{20} + \frac{\delta}{20} = \frac{\delta}{5} \). Therefore the proof is completed. \( \square \)

Claim A.4 (Guarantee on target training data). Under the setting of Theorem 4.1, with probability at least \( 1 - \frac{2\delta}{5} \), we have

\[
\frac{1}{n_2} \left\| P_{X_{T+1}B^*} X_{T+1} B^* \right\|^2_F \leq \frac{\sigma^2 (kT + kd \log(\kappa n_1) + \log \left( \frac{1}{\delta} \right))}{\epsilon n_1 \cdot \sigma^2(W^*)}. 
\]

Proof. We suppose that the high-probability events in Claims A.1, A.2 and A.3 happen, which holds with probability at least \( 1 - \frac{2\delta}{5} \). Here we instantiate Claim A.2 using \( B = [\tilde{B}, B^*] \in \mathbb{R}^{d \times 2k} \).

From the optimality of \( \tilde{B} \) and \( \tilde{W} \) in (6) we know \( X_t \tilde{B} \tilde{w}_t = P_{X_t \tilde{B}} y_t = P_{X_t \tilde{B}} (X_t B^* w_t^* + z_t) \) for each \( t \in [T] \). Then we have

\[
\sigma^2 (kT + kd \log(\kappa n_1) + \log(1/\delta))
\]
\[ \geq \| \mathcal{X}(\hat{B}\hat{W} - B^*W^*) \|_F^2 \]  
\[ = \sum_{t=1}^{T} \left\| X_t \hat{B} \hat{w}_t - X_t B^* \hat{w}_t^* \right\|^2 \]  
\[ = \sum_{t=1}^{T} \left\| P_{X_t \hat{B}}(X_t B^* \hat{w}_t^* + z_t) - X_t B^* \hat{w}_t^* \right\|^2 \]  
\[ = \sum_{t=1}^{T} \left\| -P_{X_t \hat{B}}^\perp X_t B^* \hat{w}_t^* + P_{X_t \hat{B}} z_t \right\|^2 \]  
\[ = \sum_{t=1}^{T} \left( \left\| -P_{X_t \hat{B}}^\perp X_t B^* \hat{w}_t^* \right\|^2 + \left\| P_{X_t \hat{B}} z_t \right\|^2 \right) \]  
\[ \geq \sum_{t=1}^{T} \left\| P_{X_t \hat{B}}^\perp X_t B^* \hat{w}_t^* \right\|^2 \]  
\[ \geq 0.9n_1 \sum_{t=1}^{T} \left\| P_{\Sigma_t^{1/2} B_T} \sum_{t=1}^{1/2} B^* \hat{w}_t^* \right\|^2 \]  
\[ \geq 0.9cn_1 \sum_{t=1}^{T} \left\| P_{\Sigma_t^{1/2} B_T} \sum_{t=1}^{1/2} B^* \hat{w}_t^* \right\|^2 \]  
\[ = 0.9cn_1 \left\| P_{\Sigma_t^{1/2} B_T} \sum_{t=1}^{1/2} B^* W^* \right\|_F^2 \]  
\[ \geq 0.9cn_1 \left\| P_{\Sigma_t^{1/2} B_T} \sum_{t=1}^{1/2} B^* \right\|_F^2 \cdot \sigma_k^2(W^*). \]

Next, we write \( \hat{B} = \begin{bmatrix} \hat{B}, B^* \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} =: BA \) and \( B^* = \begin{bmatrix} \hat{B}, B^* \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} =: BC. \) Recall that we have \( \frac{1}{n_2} B^\top X_{T+1}^\top X_{T+1} B \leq 1.1 B^\top \Sigma_{T+1} B \) from Claim A.2. Then using Lemma A.7 we can obtain

\[ 1.1 \left\| P_{\Sigma_{T+1} B} \sum_{t=1}^{1/2} B^* \hat{w}_t^* \right\|_F^2 \geq \frac{1}{n_2} \left\| P_{X_{T+1} B} X_{T+1} BC \right\|_F^2. \]

i.e.,

\[ 1.1 \left\| P_{\Sigma_{T+1} B} \sum_{t=1}^{1/2} B^* \right\|_F^2 \geq \frac{1}{n_2} \left\| P_{X_{T+1} B} X_{T+1} B^* \right\|_F^2. \]

Therefore we get

\[ \sigma^2 (kT + kd \log(kn_1) + \log(1/\delta)) \geq \frac{0.9cn_1}{1.1n_2} \left\| P_{X_{T+1} B} X_{T+1} B^* \right\|_F^2 \cdot \sigma_k^2(W^*), \]

completing the proof. \( \square \)

**Proof of Theorem 4.1.** We will use all the high-probability events in Claims A.1, A.2, A.3 and A.4. Here we instantiate Claim A.2 using \( B = \begin{bmatrix} B, B^* \end{bmatrix} \in \mathbb{R}^{d \times 2k}. \) The success probability is at least \( 1 - \frac{4k}{n}. \)

For the target task, the excess risk of our learned linear predictor \( x \mapsto (\hat{B} \hat{w}_{T+1})^\top x \) is

\[ \text{ER}(\hat{B}, \hat{w}_{T+1}) = \frac{1}{2} \mathbb{E}_{x \sim p_{T+1}} \left[ (x^\top (\hat{B} \hat{w}_{T+1} - B^* \hat{w}_{T+1}))^2 \right] \]
Applying Claim A.2 with \( \mathbb{B} = [\hat{B}, B^*] \), we have

\[
0.9B^\top \Sigma_{T+1}B \leq \frac{1}{n_2}B^\top X_{T+1}^\top X_{T+1}B,
\]

which implies

\[
0.9v^\top B^\top \Sigma_{T+1}Bv \leq \frac{1}{n_2}v^\top B^\top X_{T+1}^\top X_{T+1}Bv
\]

for \( v = \begin{bmatrix} \hat{w}_{T+1} \\ w_{T+1}^* \end{bmatrix} \). This becomes

\[
(\hat{B} \hat{w}_{T+1} - B^* w_{T+1}^*)^\top \Sigma_{T+1}(\hat{B} \hat{w}_{T+1} - B^* w_{T+1}^*) \leq \frac{1}{0.9n_2}(\hat{B} \hat{w}_{T+1} - B^* w_{T+1}^*)^\top X_{T+1}^\top X_{T+1}(\hat{B} \hat{w}_{T+1} - B^* w_{T+1}^*).
\]

Therefore we have

\[
\text{ER}(\hat{B}, \hat{w}_{T+1}) \leq \frac{1}{1.8n_2}(\hat{B} \hat{w}_{T+1} - B^* w_{T+1}^*)^\top X_{T+1}^\top X_{T+1}(\hat{B} \hat{w}_{T+1} - B^* w_{T+1}^*) = \frac{1}{1.8n_2} \left\| X_{T+1}(\hat{B} \hat{w}_{T+1} - B^* w_{T+1}^*) \right\|^2.
\]

From the optimality of \( \hat{w}_{T+1} \) in (8) we know \( X_{T+1}\hat{B} \hat{w}_{T+1} = P_{X_{T+1}\hat{B}}y_{T+1} = P_{X_{T+1}\hat{B}}(X_{T+1}B^* w_{T+1}^* + z_{T+1}) \). It follows that

\[
\text{ER}(\hat{B}, \hat{w}_{T+1}) \lesssim \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}}(X_{T+1}B^* w_{T+1}^* + z_{T+1}) - X_{T+1}B^* w_{T+1}^* \right\|_F^2
\]

\[
= \frac{1}{n_2} \left\| -P_{X_{T+1}\hat{B}} X_{T+1}B^* w_{T+1}^* + P_{X_{T+1}\hat{B}} z_{T+1} \right\|_F^2
\]

\[
= \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} X_{T+1}B^* w_{T+1}^* \right\|_F^2 + \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} z_{T+1} \right\|_F^2.
\]

Recall that \( w_{T+1}^* \sim \nu \) and \( \left\| E_{w \sim \nu}[ww^\top] \right\| \leq O(\frac{1}{n}) \). Taking expectation over \( w_{T+1}^* \sim \nu \) and denoting \( \Sigma = E_{w \sim \nu}[ww^\top] \), we obtain

\[
E_{w_{T+1}^* \sim \nu}[\text{ER}(\hat{B}, \hat{w}_{T+1})]
\]

\[
\lesssim \frac{1}{n_2} E_{w_{T+1}^* \sim \nu} \left[ \left\| P_{X_{T+1}\hat{B}} X_{T+1}B^* w_{T+1}^* \right\|_F^2 + \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} z_{T+1} \right\|_F^2 \right]
\]

\[
= \frac{1}{n_2} E_{w_{T+1}^* \sim \nu} \left[ \text{Tr} \left[ P_{X_{T+1}\hat{B}} X_{T+1}B^* w_{T+1}^* \left( P_{X_{T+1}\hat{B}} X_{T+1}B^* \right)^\top \right] \right] + \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} z_{T+1} \right\|_F^2
\]

\[
\lesssim \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} X_{T+1}B^* \Sigma \right\|_{F,1/2}^2 + \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} z_{T+1} \right\|_F^2
\]

\[
\lesssim \frac{1}{n_2 k} \left\| P_{X_{T+1}\hat{B}} X_{T+1}B^* \Sigma \right\|_F^2 + \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} z_{T+1} \right\|_F^2
\]

\[
\lesssim \frac{1}{k} \sigma^2 (kT + kd \log(kn_1) + \log(1/\delta)) \quad \text{(using } \|\Sigma\| \lesssim \frac{1}{k}) \quad \text{(using Claim A.4)}
\]

\[
+ \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} z_{T+1} \right\|_F^2
\]

\[
\lesssim \frac{1}{k} \frac{\sigma^2 (kT + kd \log(kn_1) + \log(1/\delta))}{cn_1 \cdot \sigma_k^2(W^*)} + \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} z_{T+1} \right\|_F^2
\]

\[
\lesssim \frac{1}{k} \frac{\sigma^2 (kT + kd \log(kn_1) + \log(1/\delta))}{cn_1 \cdot \sigma_k^2(W^*)} + \frac{1}{n_2} \left\| P_{X_{T+1}\hat{B}} z_{T+1} \right\|_F^2
\]
\[ \frac{\sigma^2 (kT + kd \log (\kappa n_2) + \log(1/\delta))}{cn_1 T} + \frac{1}{n_2} \left\| P_{X_{T+1} \hat{B}} z_{T+1} \right\|_F^2. \]  

(Using $\sigma^2(W^*) \gtrsim \frac{T}{n}$)

For the second term above, notice that
\[ \frac{1}{n_2} \left\| P_{X_{T+1} \hat{B}} z_{T+1} \right\|_F^2 \sim \chi^2(k), \]
and thus with probability at least $1 - \frac{\delta}{n}$ we have
\[ \frac{1}{n_2} \left\| P_{X_{T+1} \hat{B}} z_{T+1} \right\|_F^2 \lesssim k + \log \frac{1}{\delta}. \]
Therefore we obtain the final bound
\[
\mathbb{E}_{\mathbf{w}_{T+1} \sim \nu(\mathbb{E}(\hat{B}, \mathbf{w}_{T+1}))} \lesssim \frac{\sigma^2 (kT + kd \log (\kappa n_1) + \log(1/\delta))}{cn_1 T} + \frac{\sigma^2 (k + \log \frac{1}{\delta})}{n_2}
= \sigma^2 \left( \frac{kd \log (\kappa n_1)}{cn_1 T} + \frac{k}{cn_1 T} + \frac{\log \frac{1}{\delta}}{n_2} \right)
\lesssim \sigma^2 \left( \frac{kd \log (\kappa n_1)}{cn_1 T} + \frac{k + \log \frac{1}{\delta}}{n_2} \right),
\]
where the last inequality is due to $cn_1 \geq n_2$. 

\section{Technical Lemmas}

\begin{lemma}
Let $\mathcal{O}_{d_1, d_2} = \{ V \in \mathbb{R}^{d_1 \times d_2} \mid V^\top V = I \} (d_1 \geq d_2)$, and $\epsilon \in (0, 1)$. Then there exists a subset $\mathcal{N} \subset \mathcal{O}_{d_1, d_2}$ that is an $\epsilon$-net of $\mathcal{O}_{d_1, d_2}$ in Frobenius norm such that $|\mathcal{N}| \leq \left( \frac{6\sqrt{d_2}}{\epsilon} \right)^{d_1d_2}$, i.e., for any $V \in \mathcal{O}_{d_1, d_2}$, there exists $V' \in \mathcal{N}$ such that $\| V - V' \|_F \leq \epsilon$.
\end{lemma}

\begin{proof}
For any $V \in \mathcal{O}_{d_1, d_2}$, each column of $V$ has unit $\ell_2$ norm. It is well known that there exists an $\frac{\epsilon}{2\sqrt{d_2}}$-net (in $\ell_2$ norm) of the unit sphere in $\mathbb{R}^{d_1}$ with size $(\frac{6\sqrt{d_2}}{\epsilon})^{d_1}$. Using this net to cover all the columns, we obtain a set $\mathcal{N}' \subset \mathbb{R}^{d_1 \times d_2}$ that is an $\frac{\epsilon}{2}$-net of $\mathcal{O}_{d_1, d_2}$ in Frobenius norm and $|\mathcal{N}'| \leq \left( \frac{6\sqrt{d_2}}{\epsilon} \right)^{d_1d_2}$.

Finally, we need to transform $\mathcal{N}'$ into an $\epsilon$-net $\mathcal{N}$ that is a subset of $\mathcal{O}_{d_1, d_2}$. This can be done by projecting each point in $\mathcal{N}'$ onto $\mathcal{O}_{d_1, d_2}$. Namely, for each $V \in \mathcal{N}'$, let $P(V)$ be its closest point in $\mathcal{O}_{d_1, d_2}$ (in Frobenius norm); then define $\mathcal{N} = \{ P(V) \mid V \in \mathcal{N}' \}$. Then we have $|\mathcal{N}| \leq |\mathcal{N}'| \leq \left( \frac{6\sqrt{d_2}}{\epsilon} \right)^{d_1d_2}$ and $\mathcal{N}$ is an $\epsilon$-net of $\mathcal{O}_{d_1, d_2}$, because for any $V \in \mathcal{O}_{d_1, d_2}$, there exists $V \in \mathcal{N}$ such that $\| V - V' \|_F \leq \frac{\epsilon}{2}$, which implies $P(V) \in \mathcal{N}$ and $\| V - P(V) \|_F \leq \| V - V' \|_F + \| V' - P(V) \|_F \leq \| V - V' \|_F + \| V - V' \|_F = 2 \| V - V' \|_F \leq \epsilon$.
\end{proof}

\begin{lemma}
Let $a_1, \ldots, a_n$ be i.i.d. $d$-dimensional random vectors such that $\mathbb{E}[a_i] = 0$, $\mathbb{E}[a_i a_i^\top] = I$, and $a_i$ is $\rho^2$-subgaussian. For $\delta \in (0, 1)$, suppose $n \gg \rho^4(d + \log(1/\delta))$. Then with probability at least $1 - \delta$ we have
\[ 0.9I \preceq \frac{1}{n} \sum_{i=1}^n a_i a_i^\top \preceq 1.1I. \]
\end{lemma}

\begin{proof}
Let $A = \frac{1}{n} \sum_{i=1}^n a_i a_i^\top - I$. Then it suffices to show $\|A\| \leq 0.1$ with probability at least $1 - \delta$.

We use a standard $\epsilon$-net argument for the unit sphere $S^{d-1} = \{ v \in \mathbb{R}^d : \| v \| = 1 \}$. First, consider any fixed $v \in S^{d-1}$. We have $v^\top A v = \frac{1}{n} \sum_{i=1}^n [(v^\top a_i)^2 - 1]$. From our assumptions on $a_i$ we know that $v^\top a_i$ has mean 0 and variance 1 and is $\rho^2$-subgaussian. (Note that we must have $\rho \geq 1$.) Therefore $(v^\top a_i)^2 - 1$ is zero-mean and $16\rho^2$-sub-exponential. By Bernstein inequality for sub-exponential random variables, we have for any $\epsilon > 0$,
\[
\Pr \left[ |v^\top A v| > \epsilon \right] \leq 2 \exp \left( -\frac{n}{2} \min \left\{ \frac{\epsilon^2}{16\rho^2}, \frac{\epsilon}{16\rho^2} \right\} \right).
\]

\[ \Pr \left[ |v^\top A v| > \epsilon \right] \leq 2 \exp \left( -\frac{n}{2} \min \left\{ \frac{\epsilon^2}{16\rho^2}, \frac{\epsilon}{16\rho^2} \right\} \right). \]
Next, take a $\frac{1}{5}$-net $\mathcal{N} \subset S^{d-1}$ of $S^{d-1}$ with size $|\mathcal{N}| \leq cO(d)$. By a union bound over all $v \in \mathcal{N}$, we have

$$\Pr \left[ \max_{v \in \mathcal{N}} |v^T Av| > \epsilon \right] \leq 2|\mathcal{N}| \exp \left( -\frac{n}{2} \min \left\{ \frac{c^2}{(16\rho^2)^2}, \frac{\epsilon}{16\rho^2} \right\} \right) \leq \exp \left( O(d) - \frac{n}{2} \min \left\{ \frac{c^2}{(16\rho^2)^2}, \frac{\epsilon}{16\rho^2} \right\} \right).$$

Plugging in $\epsilon = \frac{1}{20}$ and noticing $\rho > 1$, the above inequality becomes

$$\Pr \left[ \max_{v \in \mathcal{N}} |v^T Av| > \frac{1}{20} \right] \leq \exp \left( O(d) - \frac{n}{2} \cdot \left( \frac{1}{20} \right)^2 \right) \leq \delta,$$

where the last inequality is due to $n \gg \rho^4 (d + \log(1/\delta))$.

Therefore, with probability at least $1 - \delta$ we have $\max_{v \in \mathcal{N}} |v^T Av| \leq \frac{1}{20}$. Suppose this indeed happens.

Next, for any $u \in S^{d-1}$, there exists $u' \in \mathcal{N}$ such that $||u - u'|| \leq \frac{1}{5}$. Then we have

$$||u^T Au|| \leq ||(u')^T Au'|| + 2 ||(u - u')^T Au'|| + ||(u - u')^T A(u - u')||$$

$$\leq \frac{1}{20} + 2 \cdot ||u - u'|| \cdot ||u'|| + ||u - u'||^2 \cdot ||A||$$

$$\leq \frac{1}{20} + 2 \cdot \frac{1}{5} \cdot ||u'|| + (\frac{1}{5})^2 \cdot ||A||$$

$$\leq \frac{1}{20} + \frac{1}{2} \cdot ||A||.$$

Taking a supreme over $u \in S^{d-1}$, we obtain $||A|| \leq \frac{1}{20} + \frac{1}{2} ||A||$, i.e., $||A|| \leq \frac{1}{10}$. \hfill \Box

**Lemma A.7.** If two matrices $A_1$ and $A_2$ (with the same number of columns) satisfy $A_1^T A_1 \succeq A_2^T A_2$, then for any matrix $B$ (of compatible dimensions), we have

$$A_1^T P_{A_1^T A_1} B A_1 \succeq A_2^T P_{A_2^T A_2}^+ B A_2.$$

As a consequence, for any matrices $B$ and $B'$ (of compatible dimensions), we have

$$\left\| P_{A_1^T A_1} B A_1 \right\|_F^2 \geq \left\| P_{A_2^T A_2}^+ B A_2 \right\|_F^2.$$

**Proof.** For the first part of the lemma, it suffices to show the following for any vector $v$:

$$v^T A_1^T P_{A_1^T A_1} B A_1 v \geq v^T A_2^T P_{A_2^T A_2}^+ B A_2 v,$$

which is equivalent to

$$\min_{w} \|A_1 B w - A_1 v\|_2^2 \geq \min_{w} \|A_2 B w - A_2 v\|_2^2.$$

Let $w^* \in \arg\min_{w} \|A_1 B w - A_1 v\|_2^2$. Then we have

$$\min_{w} \|A_1 B w - A_1 v\|_2^2 = \|A_1 B w^* - A_1 v\|_2^2$$

$$= (B w^* - v)^T A_1^T A_1 (B w^* - v)$$

$$\geq (B w^* - v)^T A_2^T A_2 (B w^* - v)$$

$$= \|A_2 B w^* - A_2 v\|_2^2.$$
This means that finishing the proof of the first part.

For the second part, from $A_1^T P_{A_1} A_1 \geq A_2^T P_{A_2} A_2$ we know

$$(B')^T A_1^T P_{A_1} A_1 B' \succeq (B')^T A_2^T P_{A_2} A_2 B'.$$

Taking trace on both sides, we obtain

$$\|P_{A_1} A_1 B'\|_F^2 \geq \|P_{A_2} A_2 B'\|_F^2,$$

which finishes the proof. \hfill \Box

## B Proof of Theorem 5.1

The proof is conditioned on several high-probability events, each happening with probability at least $1 - \Omega(\delta)$. By a union bound at the end, the final success probability is also at least $1 - \Omega(\delta)$. We can always rescale $\delta$ by a constant factor such that the final probability is at least $1 - \delta$. Therefore, we will not carefully track the constants before $\delta$ in the proof. All the $\delta$‘s should be understood as $\Omega(\delta)$.

We use the following notion of representation divergence.

**Definition B.1** (divergence between two representations). Given a distribution $q$ over $\mathbb{R}^d$ and two representation functions $\phi, \phi' \in \Phi$, the divergence between $\phi$ and $\phi'$ with respect to $q$ is defined as

$$D_q(\phi, \phi') = \Sigma_q(\phi', \phi') - \Sigma_q(\phi', \phi) (\Sigma_q(\phi, \phi))^\top \Sigma_q(\phi, \phi') \in \mathbb{R}^{k \times k}.$$  

It is easy to verify $D_q(\phi, \phi') \succeq 0$, $D_q(\phi, \phi) = 0$ for any $\phi, \phi'$ and $q$. See Lemma B.1’s proof.

The next lemma shows a relation between (symmetric) covariance and divergence.

**Lemma B.1.** Suppose that two representation functions $\phi, \phi' \in \Phi$ and two distributions $q, q'$ over $\mathbb{R}^d$ satisfy $\Lambda_q(\phi, \phi') \geq \alpha \cdot \Lambda_{q'}(\phi, \phi')$ for some $\alpha > 0$. Then it must hold that $D_q(\phi, \phi') \succeq \alpha \cdot D_{q'}(\phi, \phi').$

**Proof.** Fix any $v \in \mathbb{R}^k$. We will prove $v^\top D_q(\phi, \phi') v \geq \alpha \cdot v^\top D_{q'}(\phi, \phi') v$, which will complete the proof of the lemma.

We define a quadratic function $f : \mathbb{R}^k \to \mathbb{R}$ as

$$f(w) = [w^\top, -v^\top] \Lambda_q(\phi, \phi') \begin{bmatrix} w \\ -v \end{bmatrix}.$$  

According to Definition 5.2, we can write

$$f(w) = w^\top \Sigma_q(\phi, \phi) w - 2w^\top \Sigma_q(\phi, \phi') v + v^\top \Sigma_q(\phi', \phi') v$$

$$\quad = \mathbb{E}_{x \sim q} \left( (w^\top \phi(x) - v^\top \phi'(x))^2 \right).$$

Therefore we have $f(w) \geq 0$ for any $w \in \mathbb{R}^k$.\footnote{Note that we have proved $\Lambda_q(\phi, \phi') \succeq 0$.} This means that $f$ must have a global minimizer in $\mathbb{R}^k$. Since $f$ is convex, taking its gradient $\nabla f(w) = 2 \Sigma_q(\phi, \phi) w - 2 \Sigma_q(\phi, \phi') v$ and setting the gradient to 0, we obtain a global minimizer $w^* = (\Sigma_q(\phi, \phi))^\top \Sigma_q(\phi, \phi') v$. Plugging this into the definition of $f$, we obtain\footnote{Note that (35) implies $D_q(\phi, \phi') \succeq 0$.}

$$\min_{w \in \mathbb{R}^k} f(w) = f(w^*) = v^\top D_q(\phi, \phi') v. \quad (35)$$
Similarly, letting \( g(w) = [w, -v]\Lambda_q(\phi, \phi') \begin{bmatrix} w \\ -v \end{bmatrix} \), we have
\[
\min_{w \in \mathbb{R}^k} g(w) = v^\top D_q(\phi, \phi') v.
\]

From \( \Lambda_q(\phi, \phi') \geq \alpha \cdot \Lambda_q(\phi, \phi') \) we know \( f(w) \geq \alpha g(w) \) for any \( w \in \mathbb{R}^k \). Recall that \( w^* \in \arg \min_{w \in \mathbb{R}^k} f(w) \). We have
\[
\alpha v^\top D_q(\phi, \phi') v = \alpha \min_{w \in \mathbb{R}^k} g(w) \leq \alpha g(w^*) \leq f(w^*)
\]
\[
= \min_{w \in \mathbb{R}^k} f(w) = v^\top D_q(\phi, \phi') v.
\]

This finishes the proof. \( \square \)

**Claim B.2** (analogue of Claim A.4). Under the setting of Theorem 5.1, with probability at least \( 1 - \delta \) we have
\[
\frac{1}{n_2} \left\| P_{\hat{\phi}(X_{T+1})} \phi^* (X_{T+1}) \right\|_F^2 \lesssim \frac{\sigma^2 (G(F_X(\Phi))^2 + \log \frac{1}{\delta})}{n_1 \sigma_k^2 (W^*)}.
\]

**Proof**: We continue to use the notation from Claim 5.3 and its proof.

Let \( p_t \) be the empirical distribution over the samples in \( X_t (t \in [T+1]) \). According to Assumptions 5.1 and 5.2 as well as the setting in Theorem 5.1, we know that the followings are satisfied with probability at least \( 1 - \delta \):
\[
\begin{align*}
0.9 \Lambda_p(\phi, \phi') &\leq \Lambda_{p_t}(\phi, \phi') \leq 1.1 \Lambda_p(\phi, \phi'), \quad \forall \phi, \phi' \in \Phi, \forall t \in [T], \\
0.9 \Lambda_p(\hat{\phi}(\phi), \phi^*) &\leq \Lambda_{p_t}(\hat{\phi}(\phi), \phi^*) \leq 1.1 \Lambda_p(\hat{\phi}(\phi), \phi^*).
\end{align*}
\]

Notice that \( \hat{\phi} \) and \( \phi^* \) are independent of the samples from the target task, so \( n_2 \geq N_{\text{point}}(\Phi, p, \hat{\phi}) \) is sufficient for the second inequality above to hold with high probability. Using Lemma B.1, we know that (36) implies
\[
\begin{align*}
0.9 D_p(\phi, \phi') &\leq D_{p_t}(\phi, \phi') \leq 1.1 D_p(\phi, \phi'), \quad \forall \phi, \phi' \in \Phi, \forall t \in [T], \\
0.9 D_p(\hat{\phi}^*(\phi), \phi^*) &\leq D_{p_t}(\hat{\phi}^*(\phi), \phi^*) \leq 1.1 D_p(\hat{\phi}^*(\phi), \phi^*). \tag{37}
\end{align*}
\]

By the optimality of \( \hat{\phi} \) and \( \hat{w}_1, \ldots, \hat{w}_T \) for (2), we know \( \hat{\phi}(X_t) \hat{w}_t = P_{\hat{\phi}(X_t)} y_t = P_{\hat{\phi}(X_t)} (\phi^*(X_t) w^*_t + z_t) \). Then we have the following chain of inequalities:
\[
\sigma^2 \left( G(F_X(\Phi))^2 + \log \frac{1}{\delta} \right)
\]
\[
\gtrsim \sum_{t=1}^T \left\| \hat{\phi}(X_t) \hat{w}_t - \phi^*(X_t) w^*_t \right\|^2 \tag{Claim 5.3}
\]
\[
= \sum_{t=1}^T \left\| P_{\hat{\phi}(X_t)} (\phi^*(X_t) w^*_t + z_t) - \phi^*(X_t) w^*_t \right\|^2
\]
\[
= \sum_{t=1}^T \left\| - P_{\hat{\phi}(X_t)} \phi^*(X_t) w_t^* + \phi^*(X_t) z_t \right\|^2
\]
\[
= \sum_{t=1}^T \left( \left\| P_{\hat{\phi}(X_t)} \phi^*(X_t) w_t^* \right\|^2 + \left\| P_{\phi(X_t)} z_t \right\|^2 \right) \tag{cross term is 0}.
\]

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completing the proof.

Now we can finish the proof of Theorem 5.1.

Proof of Theorem 5.1. The excess risk is bounded as

\[
\text{ER}(\hat{\phi}, \hat{w}_{T+1}) = \frac{1}{2} \mathbb{E}_{x \sim p}\left[ (\hat{w}_{T+1}^\top \hat{\phi}(x) - (w^*_T)^\top \phi^*(x))^2 \right]
\]

\[
= \frac{1}{2} \left[ \begin{array}{c} \omega_{T+1}^* \\ -w^*_T \\ \end{array} \right] ^\top \Lambda_p(\hat{\phi}, \phi^*) \left[ \begin{array}{c} \omega_{T+1}^* \\ -w^*_T \\ \end{array} \right] < \frac{1}{n_2} \left\| \hat{\phi}(X_{T+1}) \omega_{T+1}^* - \phi^*(X_{T+1}) w^*_T \right\|^2
\]

\[
= \frac{1}{n_2} \left\| -P_{\hat{\phi}(X_{T+1})}^\perp \phi^*(X_{T+1}) w^*_T + P_{\hat{\phi}(X_{T+1})} z_{T+1} \right\|^2
\]

\[
\leq \frac{1}{n_2} \left\| P_{\hat{\phi}(X_{T+1})}^\perp \phi^*(X_{T+1}) w^*_T \right\|^2 + \frac{\sigma^2(k + \log \frac{1}{\delta})}{n_2}.
\]

(using \(\chi^2\) tail bound)

Taking expectation over \(w^*_{T+1} \sim \nu\), we get

\[
\mathbb{E}_{w_{T+1} \sim \nu}[\text{ER}(\hat{\phi}, \hat{w}_{T+1})]
\]
Therefore \( \lambda I \) is essentially equivalent to \( \lambda \| W \|_F = \lambda \| \Theta \|_* \).

See reference e.g. Sebbo and Shraibman (2005). At global minimum \( \| \hat{W} \|_F^2 = \| \hat{B} \|_F^2 = \| \hat{\Theta} \|_* \). We first connect the norm of \( \hat{\Theta} \) to \( R = \| \Theta^* \|_* \).

**Claim C.1 (guarantee with source regularization):**

\[
\frac{1}{n_1} \| \mathcal{X}(\Theta^* - \hat{\Theta}) \|_F^2 + \lambda \| \hat{\Theta} \|_* \leq 3\lambda \| \Theta^* \|_*,
\]

for any \( \lambda \geq \frac{2}{3} \| \mathcal{X}^*(Z) \|_2 \).

Here \( \mathcal{X}^* \) is the adjoint operator of \( \mathcal{X} \) such that \( \mathcal{X}^*(Z) = \sum_{t=1}^T X_t^\top(z_t e_t^\top) \).

**Proof of Claim C.1.** With the optimality of \( \hat{\Theta} \) we have:

\[
\frac{1}{2n_1} \| \mathcal{X}(\hat{\Theta} - \Theta^*) - Z \|_F^2 + \lambda \| \hat{\Theta} \|_* \leq \frac{1}{2n_1} \| Z \|_F^2 + \lambda \| \Theta^* \|_*,
\]

Let \( \Delta = \hat{\Theta} - \Theta^* \). Therefore

\[
\frac{1}{2n_1} \| \mathcal{X}(\Delta) \|_F^2 \leq \lambda \| \Sigma(X^*(Z)) \|_* - \| \hat{\Theta} \|_* + \frac{1}{n_1} \| \Delta \|_* + \lambda \| \hat{\Theta} \|_* \leq \lambda \| \Theta^* \|_* + \frac{1}{n_1} \| \Theta^* \|_* \| \mathcal{X}^*(Z) \| + \frac{1}{n_1} \| \hat{\Theta} \|_* \| \mathcal{X}^*(Z) \| - \lambda \| \hat{\Theta} \|_* \leq \lambda \| \Theta^* \|_* + \frac{\lambda}{2} \| \Theta^* \|_* \| \hat{\Theta} \|_* - \lambda \| \hat{\Theta} \|_*
\]

(Let \( \lambda \geq \frac{2}{3} \| \mathcal{X}^*(Z) \| )

\[
= \frac{3}{2} \lambda \| \Theta^* \|_* - \frac{1}{2} \lambda \| \hat{\Theta} \|_*.
\]

Therefore \( \frac{1}{2n_1} \| \mathcal{X}(\Delta) \|_F^2 + \frac{1}{2} \| \hat{\Theta} \|_* \leq \frac{3}{2} \lambda \| \Theta^* \|_* \), and clearly both terms satisfy \( \frac{1}{n_1} \| \mathcal{X}(\Delta) \|_F^2 \leq 3\lambda \| \Theta^* \|_* \) and \( \| \hat{\Theta} \|_* \leq 3 \| \Theta^* \|_* \).
Lemma C.2 (representation error for source task). For any $\lambda' \geq \lambda \geq \frac{2}{n_1}\|\mathcal{X}^*(Z)\|_2$, where $\lambda$ is the parameter we used to get representation matrix $B$. Let $\delta > 0$, with probability $1 - \delta$, when $n_1 \gg \sigma^2/\rho^2 \text{Tr}(\Sigma)\log(1/\delta)$ and under Assumption 4.1, we have:

$$\frac{1}{\sqrt{T}} \|(S_{\Sigma^{1/2}B}^\lambda - I)\Sigma^{1/2}\Theta^*\|_F \lesssim \sqrt{\frac{\lambda'R}{T}}.$$  

Proof of Lemma C.2. We introduce the auxiliary parameter

$$\hat{W}^\lambda = \arg\min_{W} \left\{ \frac{1}{2n_1}\|\mathcal{X}(\Theta^* - BW)\|_F^2 + \frac{\lambda'}{2}\|W\|_F^2 \right\}.$$  

From the definition of $\hat{W}^\lambda$:

$$\frac{1}{2n_1}\|\mathcal{X}(BW^\lambda - \Theta^*)\|_F^2 + \frac{\lambda'}{2}\|W^\lambda\|_F^2 \leq \frac{1}{2n_1}\|\mathcal{X}(B\hat{W}^\lambda - \Theta^*)\|_F^2 + \frac{\lambda'}{2}\|\hat{W}^\lambda\|_F^2 = \left\{ \frac{1}{2n_1}\|\mathcal{X}(\hat{\Theta}^\lambda - \Theta^*)\|_F^2 + \frac{\lambda'}{2}\|\hat{\Theta}^\lambda\|_* \right\} + \frac{\lambda' - \lambda}{2}\|\Theta^*\|_* \leq \frac{3}{2}\lambda R + \frac{3}{2}(\lambda' - \lambda)R = \frac{3}{2}\lambda' R, \quad (38)$$  

for any $\lambda' \geq \lambda$.

From Claim C.5, we get that with probability $1 - \delta/10$, let $W^\lambda$ be the minimizer of $\frac{1}{2}\|\Sigma^{1/2}BW^\lambda - \Sigma^{1/2}\Theta^*\|_F^2 + \frac{\lambda'}{2}\|W^\lambda\|_F^2$, then

$$\frac{1}{2}\|\Sigma^{1/2}BW^\lambda - \Sigma^{1/2}\Theta^*\|_F^2 \leq \frac{1}{2}\|\Sigma^{1/2}BW^\lambda - \Sigma^{1/2}\Theta^*\|_F^2 \leq \frac{3}{2}\lambda' R. \quad \text{(from Claim C.5)}$$

Lemma C.3 (Regularizer Estimation). For $X \in \mathbb{R}^{n \times d}$ drawn from distribution $p$ with covariance matrix $\Sigma$, and noise $Z \sim \mathcal{N}(0, \sigma^2 I_n)$, with high probability $1 - \delta$, we have

$$\frac{1}{n_1}\|X^TZ\|_2 \leq \frac{1}{\sqrt{n_1}}\sigma \left( \log \frac{1}{\delta} \right)^{3/2} \log(T + n)\sqrt{T}\|\Sigma\| + \text{Tr}(\Sigma).$$

Proof of Lemma C.3. We use matrix Bernstein with intrinsic dimension to bound $\lambda$ (See Theorem 7.3.1 in Tropp et al. (2015)).
Write 
\[ A = \frac{1}{\sqrt{n_1}} X^T Z = \frac{1}{\sqrt{n_1}} \sum_{t=1}^{T} X^T z_t e_t^T =: \sum_{t=1}^{T} S_t. \]

\[ E_{X,Z}[AA^T] = E_X \left[ \sum_{t=1}^{T} \frac{1}{n_1} X^T E_Z[z_t z_t^T] X \right] = \sigma^2 T \Sigma \]
\[ E_{X,Z}[^T A] = \sum_{t=1}^{T} \frac{1}{n_2} e E_{X,Z} \left[ z_t^T XX^T z_t e_t^T \right] \]
\[ = \sum_{t=1}^{T} \frac{1}{n_2} E_{X,Z} \left[ z_t^T XX^T z_t e_t e_t^T \right] = \sigma^2 \text{Tr}(\Sigma) I_n. \]

Therefore the matrix variance statistic of the sum \( v(A) \) satisfies: \( v(A) = \sigma^2 \max \{ T \| \Sigma \|, \text{Tr}(\Sigma) \} \). Denote \( V = \text{diag}([T \Sigma, \text{Tr}(\Sigma) I]) \) and its intrinsic dimension \( \| \Sigma \| \leq T + n. \)

Finally from Hanson-Wright inequality, the upper bound on each term is \( \| S_t \|^2 \leq \| X z_t \|^2 \leq \sigma^2 \text{Tr}(\Sigma) + \sigma^2 \| \Sigma \| \log \frac{1}{\delta} + \sigma^2 \| \Sigma \| F \sqrt{\log \frac{1}{\delta}} \) with probability \( 1 - \delta \). Thus using \( \| \Sigma \| F \leq \text{Tr}(\Sigma) \),
\[ \| S_t \| \leq \sigma \sqrt{1 + \sqrt{\log \frac{1}{\delta} \text{Tr}(\Sigma)} + \| \Sigma \| \log \frac{1}{\delta}} \approx L. \]

Then from intrinsic matrix bernstein (Theorem 7.3.1 in Tropp et al. (2015)), with probability \( 1 - \delta \) we have, \( \| A \| \leq O(\sigma \sqrt{\log \frac{1}{\delta} \| \Sigma \| \log(T + n)} + \sigma \log \frac{1}{\delta} L \log(\| \Sigma \|) \) which gives
\[ \| A \| \leq \sigma \left( \log \frac{1}{\delta} \right)^{3/2} \log(T + n) \sqrt{T \| \Sigma \|} + \text{Tr}(\Sigma). \]

We next bound the error in the new task as follows:

**Lemma C.4** (target task training error bound). For task \( T + 1 \), let \( X_{T+1} \in \mathbb{R}^{n_2 \times d} \) sampled from \( p \) be the training data, and suppose \( y_{T+1} = X_{T+1} \theta^*_{T+1} + z_{T+1} \). We train the model \( w_{T+1} \) as:
\[ \hat{w}_{T+1} = \min_{\| w \| \leq 2 \sqrt{R/T}} \left\{ \frac{1}{2n_2} \| X_{T+1} B w - y_{T+1} \|^2 \right\}. \]

Then its in sample error satisfies:
\[ E_{\theta^*_{T+1}} \frac{1}{2n_2} \| X_{T+1} B \hat{w}_{T+1} - X_{T+1} \theta^*_{T+1} \|^2 \leq \frac{\sigma R}{\sqrt{T}} O \left( \frac{\sqrt{\text{Tr}(\Sigma)}}{\sqrt{T n_1}} + \sqrt{\frac{\| \Sigma \|}{n_2}} \right). \]
Proof of Theorem C.4. We introduce the auxiliary terms $\bar{w}^\lambda$ that is trained without the noise term:

$$\bar{w}^\lambda = \arg\min_{\bar{w}} \left\{ \frac{1}{2n_2} \|X_{T+1}B\bar{w} - X_{T+1}\theta^*_{T+1}\|^2 + \frac{\lambda}{2}\|\bar{w}\|^2 \right\}. $$

First note that our choice of $\lambda$ guarantees that $\|\bar{w}_{T+1}\| \leq 2\sqrt{R/T}$. Notice from (38) in the proof of Lemma C.2, we have when $\lambda \geq \frac{2\sigma}{\rho^4}\|X^TZ\|^2_2$, $\|\bar{W}^\lambda\| \leq 3R$, where $\bar{W}^\lambda \leftarrow \arg\min_{W} \tilde{L}_1(W)$. Also, when $n_1 \gg \sigma^2\rho^4T\sqrt{T} \log(1/\delta)$, and $n_2 \gg \rho^4(1 + \log(1/\delta))$, as shown in Claim C.6, $\mathbb{E}_{\theta^*_{T+1} \sim \nu} \|\bar{w}^\lambda\|^2 \leq 4R/T$, which means it has smaller norm than the constraint we set for $\bar{w}_{T+1}$.

Our final goal is to bound the excess risk $ER(\hat{w}_{T+1}) = \frac{1}{2}\|\Sigma^{1/2}\hat{w}_{T+1} - \Sigma^{1/2}\theta^*_{T+1}\|^2_2$. Instead, we first look at its performance on the training data that will well-approximate our target: $\tilde{L}_{T+1} = \frac{1}{2n_2} \|X_{T+1}B\hat{w}_{T+1} - X_{T+1}\theta^*_T\|^2_2$.

From the definition of $\hat{w}_{T+1}$,

$$\|X_{T+1}B\hat{w}_{T+1} - X_{T+1}\theta^*_T\|^2 \leq \|X_{T+1}B\bar{w}^\lambda - X_{T+1}\theta^*_T + \delta_{\text{apx}} - z_{T+1}\|^2,$$

i.e., $\|X_{T+1}B\hat{w}_{T+1} - X_{T+1}B\bar{w}^\lambda + \delta_{\text{apx}} - z_{T+1}\|^2 \leq \|\delta_{\text{apx}} - z_{T+1}\|^2$.

where we denote $\delta_{\text{apx}} = X_{T+1}B\bar{w}^\lambda - X_{T+1}\theta^*_T$. Therefore

$$T_2 = \frac{1}{2n_2} \|X_{T+1}B\hat{w}_{T+1} - X_{T+1}B\bar{w}^\lambda\|^2_2$$

$$\leq \frac{1}{n_2} \|\delta_{\text{apx}} - z_{T+1}, X_{T+1}B\bar{w} - X_{T+1}B\hat{w}\|$$

$$= \frac{1}{n_2} \|\delta_{\text{apx}}X_{T+1}B\bar{w}^\lambda - X_{T+1}B\hat{w}_{T+1}\| + \frac{1}{n_2} \langle B^\top X_{T+1}z_{T+1}, \hat{w}_{T+1}^\lambda - \hat{w}_{T+1} \rangle$$

$$\leq \sqrt{T_1} \frac{1}{\sqrt{n_2}} \|X_{T+1}B\bar{w}^\lambda - X_{T+1}B\hat{w}_{T+1}\|_2 + \frac{\|\bar{w}^\lambda\| + \|\hat{w}_{T+1}\|_2}{\sqrt{n_2}} \|B^\top \frac{1}{\sqrt{n_2}} X_{T+1}^\top z_{T+1}\|_2$$

$$\leq \sqrt{T_1} \sqrt{T_2} + 4 \sqrt{\frac{R}{n_2}} \|B^\top X_{T+1}^\top z_{T+1}\|_2$$

$$\leq \frac{T_2}{2} + \frac{T_2}{2} + 4 \sqrt{\frac{R}{n_2}} \|B^\top X_{T+1}^\top z_{T+1}\|_2.$$

Therefore $T_2 \approx T_1 + \sqrt{\frac{R}{n_2}} \|B^\top X_{T+1}^\top z_{T+1}\|_2$. From Claim C.7, we get the second term is upper bounded by $\widetilde{O}(\sqrt{\|\Sigma\|})$ with probability $1 - \delta/10$.

Next, we bound $T_1 = \frac{1}{2n_2} \|X_{T+1}B\bar{w}^\lambda - X_{T+1}\theta^*_T\|_2$. We apply Claim C.6 here. When $n_2 \gg \rho^4(1 + \log 1/\delta)$, with probability $1 - \delta/10$, it satisfies

$$\mathbb{E}_{\theta^*_{T+1} \sim \nu} T_1 = \frac{1}{n_2} \|X_{T+1}B\bar{w}^\lambda - X_{T+1}\theta^*_T\|_2^2$$

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\[
\frac{1}{n_2} \mathbb{E}_{\theta_{T+1} \sim \nu} \left\| S_{X_{T+1} B X_{T+1} \theta_{T+1}}^{\lambda} \right\|_2^2 \\
\leq \lambda R/T
\]  
(from Claim C.6).

Therefore we get the in sample error

\[
\hat{L}_{T+1} \leq 2T_1 + \sqrt{\frac{R}{T}} \| B^\top \frac{1}{n_2} X_{T+1}^\top z_{T+1} \|_2
\]

\[
\leq \frac{\sigma R}{\sqrt{T}} \tilde{O} \left( \sqrt{\frac{\| \Sigma \|_2}{n_1}} + \sqrt{\frac{\text{Tr}(\Sigma)}{n_2}} \right)
\]

\[
= \frac{\sigma R}{\sqrt{T}} \tilde{O} \left( \sqrt{\frac{\text{Tr}(\Sigma)}{n_1}} + \sqrt{\frac{\| \Sigma_B \|}{n_2}} \right),
\]

with probability \(1 - \delta/5\).

Finally, back to the proof of Theorem 6.1, our expected excess risk is:

\[
\mathbb{E}_{\theta_{T+1} \sim \nu}[\text{ER}(\hat{B}, \hat{w}_{T+1})],
\]

and we connect to the in sample error as follows: When \(n_2 \gg \sigma^2 \rho^4 \frac{1}{T} \| \Sigma \| \log(1/\delta)\), with probability \(1 - \delta/10\),

\[
\mathbb{E}_{\theta_{T+1} \sim \nu}[\text{ER}(\hat{B}, \hat{w}_{T+1})]
\]

\[
= \mathbb{E}_{\theta_{T+1} \sim \nu} \frac{1}{2} \| \Sigma^{1/2} (B \hat{w}_{T+1} - \theta_{T+1}^*) \|_2^2
\]

\[
\leq 2\hat{L}_{T+1} + \frac{0.1 \sigma R}{\sqrt{T}} \sqrt{\frac{\| \Sigma \|}{n_2}},
\]

(\text{from Claim C.12})

Therefore we get with probability \(1 - \delta/2\),

\[
\mathbb{E}_{\theta_{T+1} \sim \nu}[\text{ER}(\hat{B}, \hat{w}_{T+1})] \leq \tilde{O} \left( \sqrt{\frac{\text{Tr}(\Sigma)}{n_1}} + \sqrt{\frac{\| \Sigma \|}{n_2}} \right).
\]

### C.1 Technical Lemmas

For the ease of proof, we introduce the following auxiliary functions and parameters. Write

\[
L_1(W) = \frac{1}{2} \| \Sigma^{1/2} \Theta^* - \Sigma^{1/2} B W \|_F^2 + \frac{\lambda}{2} \| W \|_F^2,
\]

\[
L_2(w) = \frac{1}{2} \| \Sigma^{1/2} \theta_{T+1}^* - \Sigma^{1/2} B w \|_F^2 + \frac{\lambda}{2} \| w \|_F^2,
\]

\[
\hat{L}_1(W) = \frac{1}{2n_1} \| X(\Theta^* - B W) \|_2^2 + \frac{\lambda}{2} \| W \|_F^2,
\]

\[
\hat{L}_2(w) = \frac{1}{2n_2} \| X_{T+1} \theta_{T+1}^* - X_{T+1} B w \|_2^2 + \frac{\lambda}{2} \| w \|_2^2
\]

Define \(W^{\lambda} \leftarrow \text{arg min}_W \{L_1(W)\}, \bar{W}^{\lambda} \leftarrow \text{arg min}_W \{\hat{L}_1(W)\}, \) and \(\hat{w}^{\lambda} \leftarrow \text{arg min}_w \{\hat{L}_2(w)\}.

**Assumption C.1.**

- We assume the whitening of data distribution \(\hat{p}\) is \(\rho\)-sub-gaussian as specified in Assumption 4.1, with its subgaussian norm equal to \(C_1 \rho\), where \(C_1\) is some constant;

- We choose \(\lambda = \Theta \left( C_1 \frac{\lambda}{\sqrt{n_1}} (\log 1/\delta)^{2/3} T \right)^{2/3} (\sqrt{n_2} \sqrt{\| \Sigma \|_2} + \sqrt{\text{Tr}(\Sigma)}) \log(n + T) \)

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Claim C.5 (source task concentration). For a fixed $\delta > 0$, when Assumption C.1 is satisfied, and $n_1 \geq \sigma^2 \rho^4 R^2 \text{Tr}(\Sigma) \log(1/\delta)$, we have

$$L_1(W^\lambda) \leq L_1(\bar{W}^\lambda) \leq 2\lambda R$$

with probability $1 - \delta/10$.

Proof of Claim C.5. From the definition of $W^\lambda$, we have $L_1(W^\lambda) \leq L_1(\bar{W}^\lambda)$, therefore we only need to concentrate the difference between $L_1(\bar{W}^\lambda)$ and $\hat{L}_1(\bar{W}^\lambda)$. From the proof of Lemma C.2 we see $\|\hat{W}^\lambda\|^2_F \leq 3R$. From Claim C.1 we know $\|\hat{\Theta}^\lambda\|_* \leq 3R$ and since $B = \hat{B}^\lambda$ and $\bar{W}^\lambda$ (recall $\hat{\Theta}^\lambda = \hat{B}^\lambda \bar{W}^\lambda$) has balanced norm, $\|B\|_F = \sqrt{\|\hat{\Theta}^\lambda\|_*} \leq \sqrt{3R}$, therefore $\|B \hat{W}^\lambda\|_* \leq \frac{1}{2} \|B\|^2_F + \frac{1}{2} \|\hat{W}^\lambda\|^2_F \leq 3R$. Therefore let $V = \Theta^* - B \hat{W}^\lambda$, it satisfies $\|V\|_* \leq 2R$. From Lemma C.11 we get $\|\Sigma^{1/2} V\|_F \leq \frac{1}{n_1} \|\mathcal{X}(V)\|_F + 0.1 \sqrt{\lambda R}$.

Therefore we get

$$L_1(\bar{W}^\lambda) = \frac{1}{2} \|\Sigma^{1/2}(\Theta^* - B \hat{W}^\lambda)\|_F^2 + \frac{\lambda}{2} \|\hat{W}^\lambda\|^2_F \\ \leq \frac{1}{2} \left(1 \frac{1}{\sqrt{n_1}} \|\mathcal{X}(\Theta^* - B \hat{W}^\lambda)\|_F + 0.1 \sqrt{\lambda R}\right)^2 + \frac{\lambda}{2} \|\hat{W}^\lambda\|^2_F \quad (\text{from Ineq. (38) from Lemma C.2}) \\ \leq \frac{3}{2} \lambda R + 0.1 \sqrt{3\lambda R} + 0.01 \lambda R$$

$$\leq 2\lambda R.$$ 

\[\bbox]$

Claim C.6 (target task concentration). For a fixed $\delta > 0$, when Assumption C.1 is satisfied, and $n_1 \geq \sigma^2 \rho^4 R^2 \text{Tr}(\Sigma) \log(1/\delta)$, $n_2 \geq \rho^2 (1 + \log(1/\delta))$, we have

$$\mathbb{E}_{\hat{\Theta}^\lambda_{T+1} \sim \nu} \|S_{X_{T+1}B} X_{T+1} \hat{\Theta}^\lambda_{T+1}\|_2^2 \leq \lambda R/T, \quad \text{and} \quad \mathbb{E}_{\hat{\Theta}^\lambda_{T+1} \sim \nu} \|\hat{w}^\lambda\|_2^2 \leq 4R/T,$$

with probability $1 - \delta/10$.

Proof. Let $w^* = \arg \min_w \left\{ \frac{1}{2} \|\Sigma^{1/2}(Bw - \Theta^*_{T+1})\|_2^2 + \frac{\lambda}{2} \|w\|_2^2 \right\}$. 

$$\|\hat{w}^\lambda\|_2^2 \leq \frac{2}{\lambda} \min_w \left\{ \frac{1}{2n_2} \|X_{T+1}^T (Bw - \Theta^*_{T+1})\|_2^2 + \frac{\lambda}{2} \|w\|_2^2 \right\}$$

$$\leq \frac{2}{\lambda} \left\{ \frac{1}{2n_2} \|X_{T+1}^T (Bw^* - \Theta^*_{T+1})\|_2^2 + \frac{\lambda}{2} \|w^*\|_2^2 \right\}$$

$$\leq \frac{2}{\lambda} \left\{ \frac{1}{2} \|\Sigma^T (Bw^* - \Theta^*_{T+1})\|_2^2 + \frac{\lambda}{2} \|w^*\|_2^2 \right\}$$

(\text{from Claim C.13 and the fact that } Bw^* - \Theta^*_{T+1} \text{ is independent of } X_{T+1})

$$\leq \frac{2}{\lambda} \left\{ \frac{1}{2} \|S_{\Sigma^{1/2}B} - I\|_2 \Sigma \Theta^*_{T+1} \|_2^2 + \frac{\lambda}{2} \|\Sigma B + \lambda I\|^{-1} \Sigma \Theta^*_{T+1} \|_2^2 \right\}$$

Therefore when we take expectation over the distribution of $\Theta^*_{T+1}$, we get:

$$\mathbb{E}_{\hat{\Theta}^\lambda_{T+1} \sim \nu} \|\hat{w}^\lambda\|_2^2$$

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\[<\mathbb{E}_{\Theta_{T+1}^*} - \nu, \frac{1}{\lambda} \left\{1.1\| (S_{1/2}^\lambda - I) \Sigma_{T+1}^* \|_2^2 + \lambda \| (\Sigma_B + \lambda I)^{-1} B^T \Sigma \Theta_{T+1}^* \|_2^2 \right\} \]

\[= \frac{1}{T\lambda} \left\{1.1\| (S_{1/2}^\lambda - I) \Sigma_{T+1}^* \|_F^2 + \lambda \| (\Sigma_B + \lambda I)^{-1} B^T \Sigma \Theta_{T+1}^* \|_F^2 \right\} \]

\[= \frac{1}{T\lambda} \left(1.1\| \Sigma^{1/2} (BW^\lambda - \Theta^*) \|_F^2 + \lambda \| W^\lambda \|_F^2 \right) \]

\[\leq \frac{1}{T\lambda} 2.2\lambda R = 2.2R/T \quad \text{(from Claim C.5)} \]

with probability \(1 - \delta/10\). Similarly, we get that the other term is upper bounded by order of \(\lambda R/T\):

\[\mathbb{E}_{\Theta_{T+1}^*} - \nu, \left\| S_{X_{T+1}}^\lambda B X_{T+1}^\lambda \Theta_{T+1}^* \right\|_2^2 \leq \lambda R/T \]

\[\square \]

**Claim C.7** (target noise concentration). For a fixed \(\delta > 0\), with probability \(1 - \delta/10\), \(\frac{1}{\sqrt{n_2}}\| B^T X_{T+1}^T z \|_2 \leq \tilde{O}(\sqrt{\| \Sigma \|_2 R})\).

**Proof of Claim C.7.** Let \(\Sigma_B = B^T \Sigma_B B\). \(v := \frac{1}{\sqrt{n_2}} B^T X_{T+1}^T z\) follows Gaussian distribution with covariance \(\Sigma_B\). Therefore \(\| v \|^2 \leq \text{Tr}(\Sigma_B) + 2\sqrt{\text{Tr}(\Sigma_B^2) \log 1/\delta + \| \Sigma_B \| \log 1/\delta = \tilde{O}(\text{Tr}(\Sigma_B))\) by Proposition 1 of Hsu et al. (2012a). Notice here we used \(\text{Tr}(\Sigma_B^2) = \sum_t \sigma_t^2 \leq (\sum_t \sigma_t^2)^2 = (\text{Tr}(\Sigma_B))^2\). Here \(\sigma_t\) is the eigenvalues of \(\Sigma_B\). Meanwhile \(\text{Tr}(\Sigma_B) = \langle \Sigma, BB^T \rangle \leq \| \Sigma \|_2 \| BB^T \|_* \leq \| \Sigma \|_2 R\). This finishes the proof. \(\square\)

**Definition C.8.** The sub-gaussian norm of some vector \(y\) is defined as:

\[\| y \|_{\psi_2} := \sup_{x \in S^{n-1}} \| \langle y, x \rangle \|_{\psi_2}, \quad (39)\]

where \(S^{n-1}\) denotes the unit Euclidean sphere in \(\mathbb{R}^n\).

**Definition C.9.** Let \(T \subset \mathbb{R}^d\) be a bounded set, and \(g\) be a standard normal random vector in \(\mathbb{R}^d\), i.e., \(g \sim \mathcal{N}(0, I_d)\). Then the quantities

\[w(T) := \mathbb{E} \sup_{x \in T} \| \langle g, x \rangle \|, \quad \gamma(T) := \mathbb{E} \sup_{x \in T} | \langle g, x \rangle | \]

\[\quad (40)\]

are called the Gaussian width of \(T\) and the Gaussian complexity of \(T\), respectively.

**Theorem C.10** (Restated Matrix deviation inequality from Vershynin (2017)). Let \(A\) be an \(m \times n\) matrix whose rows \(\alpha_i\) are independent, isotropic and sub-gaussian random vectors in \(\mathbb{R}^n\). Let \(T \subset \mathbb{R}^n\) be a fixed bounded set. Then

\[\mathbb{E} \sup_{x \in T} \| A x \|_2 - \sqrt{m} \| x \|_2 \leq C \rho^2 \gamma(T), \quad (41)\]

where \(K = \max_i \| A_i \|_{\psi_2}\) is the maximal sub-gaussian norm of the rows of \(A\). A high-probability version states as follows. With probability \(1 - \delta\),

\[\sup_{x \in T} \| A x \|_2 - \sqrt{m} \| x \|_2 \leq C \rho^2 \gamma(T) + \sqrt{\log(2/\delta)} r(T), \quad (42)\]

where the radius \(r(T) := \sup_{x \in T} \| x \|_2\).
Lemma C.11 (intrinsic dimension concentration). Let \( X, X_t, t \in [T] \) be \( n \times d \) matrix whose rows \( x \) are independent, isotropic and sub-gaussian random vectors in \( \mathbb{R}^d \) that satisfy Assumption 4.1, and the whitening distribution is with sub-gaussian norm \( C_1 \rho \), where \( \mathbb{E}[x] = 0 \) and \( \mathbb{E}[xx^\top] = \Sigma \). For a fixed \( \delta > 0 \), and any \( \nu \in \mathbb{R}^d \), we have

\[
\|\Sigma^{1/2} \nu\|_2 \leq \frac{1}{\sqrt{n}} \|X \nu\|_2 + \frac{C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)\|\Sigma\|} \right) \|\nu\|_2.
\]

For any \( \Theta \in \mathbb{R}^{d \times T} \), we further have

\[
\|\Sigma^{1/2} \Theta\|_F \leq \frac{1}{\sqrt{n}} \sqrt{\sum_{t=1}^T \|X_t \theta_t\|_2^2} + \frac{2C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)\|\Sigma\|} \right) \|\Theta\|_*.
\]

with probability \( 1 - \delta \).

Additionally when \( n \gg \rho^4 R^2 \sigma^2 \text{Tr}(\Sigma) \log(1/\delta) \), and \( \|\Theta\|_* \lesssim R \), we get with probability \( 1 - \delta \),

\[
\|\Sigma^{1/2} \Theta\|_F \leq \frac{1}{\sqrt{n}} \sqrt{\sum_{t=1}^T \|X_t \theta_t\|_2^2} + 0.1 \sqrt{R}.
\]

Proof of Lemma C.11. We use Theorem C.10. Let \( T = \{ \nu : \|\Sigma^{-1/2} \nu\|_2 \leq 1 \} \). Let \( x = \Sigma^{1/2} z, X = Z \Sigma^{1/2} \). Then \( \gamma(T) = \sqrt{\text{Tr}(\Sigma)}, r(T) = \|\Sigma^{1/2}\|. \) We note with probability \( 1 - \delta \),

\[
\sup_{\|\nu\|_2=1} \left| \frac{1}{\sqrt{n}} \|X \nu\|_2 - \|\Sigma^{1/2} \nu\|_2 \right| = \sup_{\|\Theta\|_*} \left| \frac{1}{\sqrt{n}} \|Z \Theta\|_2 - \|\Theta\|_2 \right|
\leq \frac{C \rho^2}{\sqrt{n}} \left( \gamma(T) + \sqrt{\log(2/\delta) r(T)} \right)
\leq \frac{C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)\|\Sigma\|} \right).
\]

Therefore \( \frac{1}{\sqrt{n}} \|X \nu\|_2 - \|\Sigma^{1/2} \nu\|_2 \leq \frac{C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)\|\Sigma\|} \right), \forall \|\nu\| = 1 \). Then by homogeneity of \( \nu \), for arbitrary \( \nu \), we have

\[
\left| \frac{1}{\sqrt{n}} \|X \nu\|_2 - \|\Sigma^{1/2} \nu\|_2 \right| \leq \|\nu\|_2 \frac{C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)\|\Sigma\|} \right).
\]

Notice when \( n \gg C^2 \rho^4 \sigma^2 (\text{Tr}(\Sigma) + \|\Sigma\| \log 1/\delta) \), term I \( \leq 0.1 \sqrt{\lambda} \). Therefore \( \|\Sigma^{1/2} \nu\|_2 - \frac{1}{\sqrt{n}} \|X \nu\|_2 \leq 0.1 \sqrt{\lambda} \|\nu\| \).

Write \( \Theta = UDV^\top \), where \( D = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_T) \).

\[
\frac{1}{n} \sum_{t=1}^T \|X_t \theta_t\|_2^2 = \frac{1}{n} \sum_{t=1}^T \sigma_t^2 \|X_t u_t\|^2
\geq \sum_{t=1}^T \sigma_t^2 \left( \|\Sigma^{1/2} u_t\|_2 - \|u_t\|_2 \frac{C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)\|\Sigma\|} \right) \right)^2
\]

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\[ \sum_{t=1}^{T} \sigma_t^2 \left( \| \Sigma^{1/2} u_t \|_2^2 \right) - 2 \| \Sigma^{1/2} u_t \|_2 \| u_t \|_2 \frac{C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)} \| \Sigma \| \right) \]
\[ = \| \Sigma^{1/2} \Theta \|_F^2 - \frac{2C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)} \| \Sigma \| \right) \sum_t \sigma_t (\| \Sigma^{1/2} u_t \|_2) \]
\[ \geq \| \Sigma^{1/2} \Theta \|_F^2 - \frac{2C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)} \| \Sigma \| \right) \| \Theta \|_* \| \Sigma^{1/2} \Theta \|_F. \]

Therefore \( \| \Sigma^{1/2} \Theta \|_F \leq \frac{1}{\sqrt{n}} \sum_{t=1}^{T} |X_t \theta_t|^2 + \frac{2C \rho^2}{\sqrt{n}} \left( \sqrt{\text{Tr}(\Sigma)} + \sqrt{\log(2/\delta)} \| \Sigma \| \right) \| \Theta \|_* \| \Sigma^{1/2} \Theta \|_F \).

Notice when \( n \gg n_2 \rho^4 R^2 / \rho^2 (\text{Tr}(\Sigma) + \| \Sigma \| \log(1/\delta)) \), and \( \| \Theta \|_* \leq 2R \), we have the second term \( \lambda R \leq 1 \), we get term II \( \leq 0.1 \sqrt{\lambda R} \).

**Claim C.12.** Let \( X \) be a \( n_2 \times d \) matrix whose rows \( x \) are independent, isotropic and sub-gaussian random vectors in \( \mathbb{R}^d \) that satisfy Assumption 4.1, where \( \mathbb{E}[x] = 0 \) and \( \mathbb{E}[xx^T] = \Sigma \). Let \( \Sigma_B = B^T \Sigma B \) for some matrix \( B \) that satisfies \( \| BB^T \|_* \leq R \). Then for a fixed \( \delta > 0 \), and any \( v \in \mathbb{R}^d \) that satisfies \( \| v \| \leq \sqrt{R/T} \), we have when \( n_2 \gg \rho^2 R^2 \| \Sigma \| \log(1/\delta) \), we have: \( \| \Sigma_B^{1/2} v \|^2 \leq \frac{2}{n_2} \| XBv \|^2 + 0.1 \frac{\rho \sqrt{R}}{\sqrt{T}} \sqrt{\| \Sigma \| \}}. \)

**Proof.** This result directly use Lemma C.11 when replace \( X \) by \( XB \). Notice now the subgaussian norm for the whitening distribution for \( B^T x \) remains the same as \( C \rho \). Therefore \( \| \Sigma_B^{1/2} Bv \|_2 \leq \frac{1}{\sqrt{n_2}} \| XBv \|_2 + S \| B \| \| \Sigma_B \| \| v \| \leq \frac{1}{\sqrt{n_2}} \| Xv \|_2 + S \sqrt{R/T} \). Here \( S = C \rho^2 / \sqrt{n} \left( \sqrt{\text{Tr}(\Sigma_B)} + \sqrt{\log(2/\delta)} \| \Sigma_B \| \right) \leq C \rho^2 / \sqrt{n_2} \sqrt{\| \Sigma \| R \log(1/\delta)} =: S' \). By AM-GM \( \| \Sigma_B^{1/2} v \|_2 \leq \frac{2}{n_2} \| Xv \|_2 + (S')^2 R/T. \)

Therefore we need to set proper \( n_2 \) such that \( (S')^2 R/T \leq \frac{\rho \sqrt{R}}{\sqrt{T}} \sqrt{\| \Sigma \| \}}. \) We solve the inequality and finish the proof.

**Claim C.13.** (Covariance concentration of task target). Suppose \( n_2 \gg \rho^4 (1 + \log(1/\delta)) \) for \( \delta \in (0, 1) \). Then for any given vector \( \theta \in \mathbb{R}^d \) that is independent of \( X_{T+1} \), with probability at least \( 1 - \frac{\delta}{10} \) over \( X_{T+1} \) we have
\[ 0.9 \theta^T \Sigma_{T+1} \theta \leq \frac{1}{n_2} \theta^T X_{T+1} X_{T+1} \theta \leq 1.1 \theta^T \Sigma_{T+1} \theta. \] (44)

**Proof.** The proof follows the same as Claim A.2 that replaces width of \( B \) from \( 2k \) to \( 1 \).

**D.** Proof of Theorem 7.1

First, we describe a standard lifting of neural networks to infinite dimension linear regression Wei et al. (2019), Rosset et al. (2007), Bengio et al. (2006). Define the infinite feature vector with coordinates \( \phi(x)_b = (b^T x)_+ \) for every \( b \in \mathbb{S}^{d_{b-1}} \). Let \( \alpha_t \) be a signed measure on \( \mathbb{S}^{d_{b-1}} \). The inner product notation denotes integration: \( \alpha^T \phi(x) \triangleq \int_{\mathbb{S}^{d_{b-1}}} \phi(x) \rho d\alpha(b) \). The \( t^\text{th} \) output of the infinite-width neural network is \( f_{\alpha_t}(x) = \langle \alpha_t, \phi(x) \rangle \). Consider the least-squares problem
\[ \min_{\alpha_1, \ldots, \alpha_t : \| \alpha \|_{2,1}, \sup(\alpha)} \frac{1}{2n} \sum_{t=1}^{T} (y_{it} - \alpha_t^T \phi(x_i))^2 + \lambda \| \alpha \|_{2,1}, \] (45)
where \( \alpha(u) = [\alpha_1(u), \ldots, \alpha_T(u)] \), and \( \| \alpha \|_{2,1} = \int_{\mathbb{S}^{d_{b-1}}} \| \alpha(b) \|_2 d(b) \). The regularizer corresponds to a group \( \ell_1 \) regularizer on the vector measure \( \alpha \).
Proposition D.1. Let $\gamma_d$ be the value of Equation (22) when the network has $d$ neurons and $\gamma_d^*$ be the value of Equation (45). Then

$$\gamma_d = \gamma_d^*. \quad (46)$$

Proof. Let $B, W$ be solutions to Equation (22). Let $\bar{B} = BD_\beta^{-1}$ and $D_\beta$ be a diagonal matrix whose entries are $\beta_j = \|B^\top e_j\|_2$. The network $f_{B,W}(x) = W^\top D_\beta (\bar{B}^\top x)_+$ and the weight decay regularizer has value

$$\lambda \frac{2}{2} \|B\|_F^2 + \lambda \frac{2}{2} \|W\|_F^2 = \lambda \frac{2}{2} \|\beta\|_2^2 + \lambda \frac{2}{2} \|W\|_F^2. \quad (46)$$

We first show that $\gamma_d^* \leq \gamma_d$. Define $\alpha_t(b_j) = W_{ij} \beta_j$. We verify that

$$\alpha_t^\top \phi(x) = \sum_{j=1}^d \alpha_t(b_j) \phi(x) b_j = \sum_{j=1}^d W_{ij} \beta_j (b_j^\top x)_+ = w^T (B^\top x)_+ = f_{B,W}(x).$$

Due to the regularizer, and using the AM-GM inequality, at optimality $\beta_j = \|We_j\|_2$. Next, we verify that the two regularizer values are the same

$$\|\alpha\|_{2,1} = \sum_{j=1}^d \left( \sum_{t} W_{jt}^2 \beta_j^2 \right)^{1/2} = \sum_{j=1}^d \beta_j \|We_j\|_2 = \|W\|^2_F \quad \text{(plugging in $\beta_j = \|We_j\|$)} = \frac{1}{2} \|\beta\|^2 + \frac{1}{2} \|W\|^2_F. \quad \text{(since $\|\beta\| = \|W\|_F$)}$$

Thus the network given by $\alpha_t^\top \phi(x)$ has the same network outputs and regularizer values. Thus $\gamma^* \leq \gamma_d$.

Finally, we show that $\gamma_d \leq \gamma_d^*$. Let $\bar{b}_j$ for $j \in [d]$ be the support of the optimal measure of (45). Define $\beta_j = \sqrt{\|\alpha(b_j)\|_2}$, $B = BD_\beta$ where $\bar{B}$ is a matrix whose rows are $\bar{b}_j$, and $W$ such that $W_{jt} = \alpha_t(b_j)/\sqrt{\|\alpha(b_j)\|_2}$.

We verify that the network values agree

$$e^T f_{B,W}(x) = e^T W^\top D_\beta (\bar{B}^\top x)_+ = \sum_j W_{jt} \beta_j (\bar{b}_j^\top x)_+ = \alpha^\top \phi(x).$$

Finally by our construction $\beta_j = \|We_j\|$, so the regularizer values agree. Thus $\gamma_d = \gamma_d^*$.

Finally, we note that the regularizer can be expressed in a variational form as$^8$

$$\|\alpha\|_{2,1} = \min_{b,W: \alpha_t(b) = \beta(b) w_t(b)} \|\beta\|_2^2 + \|W\|^2_F,$$

where $\|\beta\|_2^2 = \int \beta(\bar{b})^2 d(\bar{b})$ and $\|W\|^2_F = \sum_t \int w_t(\bar{b})^2 d(\bar{b})$. With these in place, we note that Equation (45) can be expressed as Equation (16) with $B$ constrained to be a diagonal operator and $x_{it}$ as the lifted features $\phi(x_{it})$.

$^8$Informally if $\alpha \in \mathbb{R}^{D \times T}$ with $D$ potentially infinite, $\|\alpha\|_{2,1} = \min_{\alpha = \text{diag}(b)W} \frac{1}{2} \|b\|_2^2 + \frac{1}{T} \|W\|_F^2$. 37
Proof of Theorem 7.1. The global minimizer of Equation (45) with \( d = \infty \) may have infinite support, so the corresponding value may not be achieved by minimizing (22). However, Theorem 6.1 only requires that we obtain a learner network with regularized loss less than the regularized loss of the teacher network. Since the teacher network has \( d \) neurons, this value is attainable by (22). Thus the finite-size network does not need to attain the global minimum of (45) for Claim C.1 to apply.

Since Theorem 6.1 has no dependence (even in the logarithmic terms) on the input dimension of the data, it can be applied when the input features the infinite-dimensional feature vector \( \phi(x) \). The only part of the proof of Theorem 6.1 specific to the nuclear norm is that the dual norm is the operator norm. In Lemma C.3, upper bounds \( \frac{1}{T} \| X^\top Z \|_2 \). Since we use the \( \| \cdot \|_{2,1} \) norm, we must upper bound \( \frac{1}{T} \| X^\top Z \|_{2,\infty} \), the dual of the \((2, 1)\)-norm. Since \( \| A \|_* \leq \| A \|_{2,1} \), then \( \| A \|_{2,\infty} \leq \| A \|_2 \) so the upper bound in Lemma C.3 still applies. Thus, Theorem 7.1 follows from Theorem 6.1.

Proof of (26). The test error of (25) is given by

\[
\mathbb{E}[\text{ER}(f_{\hat{B}, \hat{w}})] \lesssim \sigma \frac{1}{2\sqrt{n}} (\| B_{T+1}^* \|^2 + \| w_{T+1}^* \|^2) \mathbb{E}_{x_i \sim p, z \sim N(0, \sigma^2 I)} \| \Phi(X)^\top z \|_{\infty},
\]

via the basic inequality (c.f. proof of Claim C.1). By the matrix Bernstein inequality (c.f. Lemma C.3 or Wei et al. (2019)), \( \mathbb{E}_{x_i \sim p, z \sim N(0, I)} \| \Phi(X)^\top z \|_{\infty} \lesssim \sqrt{\text{tr}(\Sigma)} \). When \( B_{T+1}^*, w_{T+1}^* \) are sampled from the same distribution as the source tasks, then \( \frac{1}{2}(\| B_{T+1}^* \|^2 + \| w_{T+1}^* \|^2) \geq \frac{R}{\sqrt{T}} \). Thus we conclude

\[
\mathbb{E}[\text{ER}(f_{\hat{B}, \hat{w}})] \lesssim \sigma \frac{R}{\sqrt{T}} \sqrt{\frac{\text{tr}(\Sigma)}{n_2}}.
\]