Abstract

We consider the spreading of a thin two-dimensional droplet on a solid substrate. We use a model for viscous fluids where the evolution is governed by Darcy’s law. At the contact point where air and liquid meet the solid substrate, a constant, non-zero contact angle (partial wetting) is assumed. We show local and global well-posedness of this free boundary problem in the presence of the moving contact point. Our estimates are uniform in the contact angle assumed by the liquid at the contact point. In the so-called lubrication approximation (long-wave limit) we show that the solutions converge to the solution of a one-dimensional degenerate parabolic fourth order equation which belongs to a family of thin-film equations. The main technical difficulty is to describe the evolution of the non-smooth domain and to identify suitable spaces that capture the transition to the asymptotic model uniformly in the small parameter $\varepsilon$.

1. Introduction and Model

Recently, much attention has been devoted to fluid systems involving the evolution of a free boundary. Most of these works are concerned with the situation when the interface (or free boundary) separates two phases of the system. We refer, for example, to [12,13,31,32,40,46] for local existence results, see also [19,44] for global existence results [10], for the study of blow-up [2,33] for asymptotic limits; of course many other results exist. The theory of the well-posedness and regularity of solutions in the presence of a contact line between three-phases is less developed. In this paper, we consider a particular situation of this kind. In particular, we consider the evolution of a two-dimensional flow in a Hele-Shaw cell where the liquid touches the lateral boundary of the Hele-Shaw cell. The core of the difficulties in our analysis is related to the presence of a three-phase contact point, namely a contact point between air, liquid and solid, see Fig. 1.
The Hele-Shaw model describes the evolution of a liquid between the plates of a Hele-Shaw cell. We first recall the situation when the fluid does not touch the lateral boundary of the Hele-Shaw cell. In this case, the surface tension-driven Hele-Shaw flow is given by

\[
\begin{align*}
\Delta p &= 0 \quad \text{in } \Omega(t), \\
p &= \gamma \kappa \quad \text{on } \partial\Omega(t), \\
V &= \nabla p \cdot n \quad \text{on } \partial\Omega(t),
\end{align*}
\]  

(1.1)

where the evolving domain \(\Omega(t) \subset \mathbb{R}^2\) describes the region occupied by the fluid. The velocity of the fluid is described by Darcy’s law \(U = -\nabla p\); the normal velocity \(V\) of the fluid interface is particularly given by \(V = \nabla p \cdot n\). Furthermore, \(\kappa\) is the curvature of the air–liquid interface with the convention that \(\kappa > 0\) if the shape of the liquid is concave. The parameter \(\gamma\) describes the surface tension between air and liquid. In addition to its interpretation as the flow in a Hele-Shaw cell, fluid evolutions governed by Darcy’s law appear in a wide range of physical models. One example is the flow of a liquid through a porous medium, see [6]. Other situations which can be modeled by (1.1) are crystal growth or dissolution, directional solidification or melting, electrochemical machining or forming [34, 39, 43]. In the last two decades, the well-posedness of (1.1) has been investigated. Short-time. The short-time existence and regularity of solutions of (1.1) have been proved in [15, 17, 18, 25] and Prokert [36]. Global existence for initial data close to the sphere has been shown in [11]. The case of zero surface tension, \(\gamma = 0\) has been considered in, for example [1, 41].

We now consider the situation where the fluid touches the lateral boundary of the Hele-Shaw cell. In particular, we assume that the Hele-Shaw cell is described by the half-space \(H = \mathbb{R} \times (0, \infty)\) and that the liquid touches the boundary at a subset of the boundary \(\mathbb{R} \times \{0\}\) of the Hele-Shaw cell. We need to assume additional boundary conditions at the liquid–solid interface and at the contact point. We first note that the normal component of the velocity is zero at the liquid–solid interface. Furthermore, at the point where air, liquid and solid meet, we impose the standard assumption that the liquid assumes a static (microscopic) contact angle \(\theta\), determined by Young’s law [45]. The contact thus depends on the ratio of the surface tensions between the three phases \(\gamma\) (air, liquid), \(\gamma_{SL}\) (solid, liquid) and \(\gamma_{SG}\) (air, solid). We consider the case of partial wetting, when \(|\gamma_{SG} - \gamma_{SL}| \leq \gamma\) and where the angle is determined by the equation \(\gamma \cos \theta = \gamma_{SG} - \gamma_{SL}\). Note that this contact angle is sometimes also called the microscopic contact angle in the engineering literature. The above assumptions lead to the following model:

\[
\begin{align*}
\Delta p &= 0 \quad \text{in } \Omega(t), \\
p &= \gamma \kappa \quad \text{on } \partial\Omega(t) \cap H, \\
V &= \nabla p \cdot n \quad \text{on } \partial\Omega(t), \\
p_y &= 0 \quad \text{on } \partial\Omega(t) \cap \partial H, \\
\gamma \cos \theta &= \gamma_{SG} - \gamma_{SL} \quad \text{on } \partial(\partial\Omega(t) \cap \partial H).
\end{align*}
\]  

(1.2)
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Fig. 1. Darcy flow on solid substrate

see Fig. 1. Note that the evolution can also be interpreted as the spreading of a droplet on a plate. Here and in the following, we refer to the fluid configuration as a droplet if its support is compact. A well-posedness result for (1.2) in Hölder spaces has been given by Bazalyi and Friedman [4,5]. However, in their analysis the conditions on the initial data are too restrictive to allow for movement of the triple point (and thus for spreading of the droplet). Our first main result is a well-posedness result for this free boundary problem in a much wider class of weighted Sobolev spaces. In particular, our result seems to be the first result which allows for movement of the triple point.

The second aim of this work is to show the convergence of solutions to a reduced model in the so called lubrication approximation regime or long wave approximation. More precisely, we assume that typical vertical length scales are of order $\varepsilon$ while typical horizontal length scales are of order 1. In particular, we assume that the contact angle $\theta$ at the contact point is small and of order $\varepsilon$:

$$\frac{\varepsilon^2}{2} \approx \frac{\theta^2}{2} \approx 1 - \cos \theta = \frac{\gamma + \gamma_{SL} - \gamma_{SG}}{\gamma}. \quad (1.3)$$

We will show that in the limit of lubrication approximation, solutions of (1.2) can be described by a single scalar evolution equation in terms of the profile $h(t, x)$ of the evolving liquid drop. In this model, the evolution is given by

$$\begin{cases}
  h_t + \gamma (hh_{xxx})_x = 0 & \text{in } \{h > 0\}, \\
  h = 0, |h_x| = \varepsilon & \text{on } \partial\{h > 0\}, \\
  \tilde{V} = \gamma h_{xxx} & \text{on } \partial\{h > 0\}, \\
  h = h_{in} & \text{for } t = 0,
\end{cases} \quad (1.4)$$

where $\tilde{V}$ is the velocity of the moving contact points $\partial\{h > 0\}$. Furthermore, $h_{in}$ is the initial profile. Formal derivations of lubrication models of type (1.4) have been given by Reynolds [38]. We prove the convergence of solutions of (1.2) to solutions of (1.4). This is the first rigorous lubrication approximation in the case of partial wetting (non-zero contact angle). Furthermore, it is the first convergence result in the framework of classical solutions. A rigorous lubrication approximation in the framework of weak solutions and in the situation of complete wetting has been done by Giacomelli and Otto [24]. Their approach is quite different to ours; in particular, their result does not include well-posedness for the initial model. Instead, the authors prove convergence to the limit model by only minimal energy bounds,
assuming the existence of smooth solutions. Indeed, the techniques used in [24] do not seem to be applicable for the case of a non-zero contact angle at the moving contact line as considered in this work since the norms used in this work cannot capture a non-zero contact angle at the contact point.

The main difficulty lies in the derivation of bounds on the solution, which are uniform in the parameter $\varepsilon > 0$. Let us start by giving a short sketch of the strategy of our proof. We first express (1.2) as a nonlocal evolution problem in terms of the profile function $h(t, x)$. This transformation leads to a nonlocal parabolic evolution problem of third order in terms of the film height $h$. Equation (1.4) on the other hand is a local fourth order degenerate parabolic equation. Notice that the maximum principle cannot be used, since considered models are higher order equations. Instead, we base our argument on the underlying dissipative structure. Indeed, solutions of (1.2) satisfy the dissipation relation (see the Appendix)

$$\frac{d}{dt} \left( (|\partial \Omega \cap H| - \alpha |\partial \Omega \cap \partial H|) \right) = -\frac{1}{\gamma} \iint_{\Omega(t)} |\nabla p|^2 \, dx \, dy, \quad (1.5)$$

where $\alpha = \cos(\arctan \varepsilon)$, see for example [23]. The dissipation relation for solutions of (1.4) is

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}} h_x^2 \, dx + \varepsilon^2 |\{h > 0\}| \right) + \gamma \int_{\mathbb{R}} h h_{xxx}^2 \, dx = 0. \quad (1.6)$$

One of the core issues of the analysis is to find suitable norms which allow for uniform bounds in the limit $\varepsilon \to 0$. In [27], we have investigated the linearized equations related to (1.2) and (1.4). The analysis in this work suggests the use of sums of weighted Sobolev norms of the type

$$[f]_{X_k^\varepsilon} \sim \inf_{f = f_+ + f_-} \left( \left\| \frac{x^k}{\varepsilon^\delta} \partial_x^{4k+\delta} f_+ \right\|_{L^2((0, \infty))} + \left\| \left( \frac{1}{\varepsilon} \right)^k x^\delta \partial_x^{3k+\delta} f_- \right\|_{L^2((0, \infty))} \right), \quad (1.7)$$

where $k, \delta \geq 0$. In the sequel, we will indeed define homogeneous norms $[\cdot]_{X_k^\varepsilon}$ which are equivalent to the homogeneous norms in the right side of (1.7). In the limit $\varepsilon \to 0$, the homogeneous norm on the right hand side of (1.7) turns from a sum of weighted Sobolev norms of order $4k + \delta$ and $3k + \delta$ to a weighted Sobolev norm of order $4k + \delta$ [first term on the right hand side of (1.7)]. This transition in the character of the norm is reflected by a transition in the character of the equation. In the limit $\varepsilon \to 0$, the model [understood as an evolution equation for the profile function, cf. (2.18), (2.24)] changes:

- from a third order operator to a fourth order evolution equation and
- from a non-degenerate parabolic equation in a singular two-dimensional domain to a degenerate parabolic equation.

We will use norms of the type (1.7) with the particular choice $\delta = 1$. This seems to be the smallest integer value, sufficient to control the nonlinearity of the problem [an application of Hardy’s inequality shows that the norms (1.7) are stronger for larger $\delta$]. Note that the representation (1.7) of the norms only works for integer
values \(k, \delta \in \mathbb{N}_0\). However, we will introduce a variant of the norms \((1.7)\) in terms of frequencies variables, based on the Mellin transformation, see \((2.30)\)–\((2.34)\). In fact, most of the estimates in the proof are done in frequency variables. Hence, we believe that it should be possible to extend our results also to the case of fractional derivatives and of general weights \(\delta > 1\).

We also need to choose suitable norms for the pressure \(p\). As we will see, the norms for the pressure do not have a real space representation, but are rather described in terms of the Mellin transformed function. In fact, the choice of suitable norms for the pressure turns out to be delicate in order to obtain uniform bounds in the small \(\varepsilon\) parameter. We first apply a Mellin transform on the pressure with respect to the radial variables, but we keep the representation in physical variables in angular direction. We then apply a supremum norm in angular direction and an \(L^2\)-norm in radial direction (in this order), see also \((2.41)\).

Structure of the paper In Section 2, we transform the problem onto a fixed domain and we define the norms to control the profile and the pressure. The main results of this work are stated and discussed in Section 3. In Section 4, we give an overview for the proofs of the main theorems. In Section 5, we prove estimates for weighted spaces. In Section 6, we derive estimates for the nonlinear operator for a droplet supported in half-space. In Section 7, we derive corresponding localized estimates for compactly supported droplets.

2. Setting and Norms

In this section, we reformulate the problem \((1.2)\) on a fixed domain by a change of dependent and independent coordinates. We then express the model as a nonlocal evolution equation in terms of the profile \(h\). We also introduce suitable norms for the profile \(h\) and the pressure \(p\).

2.1. Transformation onto a Fixed Domain

We define the nonlinear operator \(B^h\) of Dirichlet–Neumann type by

\[
B^h \kappa(x) = \sqrt{1 + h_x^2} \left( \partial_n p \right)_{y=h(t,x)} \quad \text{where} \quad \begin{cases} 
\Delta p = 0 & \text{in } \Omega(t), \\
p = \gamma \kappa & \text{on } \partial_1 \Omega(t), \\
p_y = 0 & \text{at } \partial_0 \Omega(t), 
\end{cases}
\]

where \(\Omega(t)\) is the subgraph of \(h\). Furthermore, the the air–liquid interface \(\partial_1 \Omega(t)\) and the liquid–solid interface \(\partial_0 \Omega(t)\) are denoted by

\[
\partial_1 \Omega(t) := \partial \Omega(t) \cap \{y > 0\}, \quad \partial_0 \Omega(t) := \partial \Omega(t) \cap \{y = 0\}.
\]

Assuming that the free boundary moves with the velocity, we infer that \(h_t = w - vh_x\), where \(u = (v, w)\) is the velocity of the liquid. Suppose that the support of the droplet stays at an interval for some time, that is \(\text{supp} h(t) = (s_-(t), s_+(t))\).
for \( t \in (0, \tau) \). In view of the definition (2.1), the evolution (1.2) can then be equivalently expressed as nonlocal evolution equation in terms of the profile \( h \) by

\[
\begin{align*}
    h_t + B^h \kappa &= 0, & \text{for } & x \in (s_-(t), s_+(t)), \\
    h_{|x=s_{\pm}} &= 0, & |h_x| = \varepsilon, & \text{for } x = s_{\pm} \\
    h &= h_{in} & \text{for } t = 0.
\end{align*}
\] (2.2)

By Darcy’s law, the movement of the triple point is given by

\[
\partial_t s_{\pm} = -p_x|_{x=s_{\pm}} = -\gamma \kappa |_{x=s_{\pm}} = \gamma \left( (1 + \varepsilon^2)^{-\frac{2}{3}} h_{xxx} - 3\varepsilon (1 + \varepsilon^2)^{-\frac{2}{3}} h_{xx}^2 \right)_{|x=s_{\pm}}.
\] (2.3)

Indeed, Equation (2.3) follows since \( p(t, x, h(t, x)) = \gamma \kappa (t, x) \) and hence \( p_x + h_x p_y = \gamma \kappa \) at \( x = s(t) \) and we conclude since \( p_y|_{y=0} = 0 \) and since \( |h_x|_{x=s_{\pm}} = \varepsilon \). For simplicity of notation, we assume that the support of \( h \) at initial time is given by \((0, 1)\). We next rescale in time and space to get quantities of order one in the lubrication approximation regime and we fix the position of the moving contact point \( s(t) \) by using a coordinate frame that moves with the contact points. With the notation

\[
M(t) = \frac{1}{2} (s_+(t) + s_-(t)), \quad D(t) = s_+(t) - s_-(t),
\] (2.4)

we introduce the new independent variables \( \tilde{x}, \tilde{y}, \tilde{t} \) by

\[
\tilde{x} = \frac{x - M(t)}{D(t)} + \frac{1}{2}, \quad \tilde{y} = \frac{y}{\varepsilon D(t)}, \quad \tilde{t} = \varepsilon \gamma \int_0^t \frac{1}{D^2(s)} \, ds,
\] (2.5)

cf. [3, 21]. In particular, \( \partial_x \tilde{x} = \frac{1}{D}, \partial_x \tilde{t} = 0, \partial_t \tilde{t} = \frac{\varepsilon \gamma}{D^2} \) and

\[
\partial_t \tilde{x} = -\frac{M_t}{D} - \frac{D_t (x - M)}{D^2} \quad (2.5) \equiv -\frac{\varepsilon \gamma}{D^3} \left( \dot{M} + \frac{1}{2} \dot{D} \right)
\]

\[
\ddot{\tilde{x}} = -\frac{\varepsilon \gamma}{D^3} \left( 1 - \dot{\tilde{x}} \dot{s}_- + \ddot{\tilde{x}} \dot{s}_+ \right),
\]

where the dot denotes differentiation in \( \tilde{t} \). The rescaled dependent variables \( \tilde{h} \) and \( \tilde{p} \) are defined by

\[
h(t, x) = \varepsilon D(t) \tilde{h}(\tilde{t}, \tilde{x}), \quad D(t) p(t, x, y) = \varepsilon \gamma \tilde{p}(\tilde{t}, \tilde{x}, \tilde{y}).
\] (2.6)

In the rescaled variables, the nonlocal Dirichlet–Neumann type operator turns into

\[
B^{\tilde{h}}_{\tilde{p}} \eta(\tilde{x}) \overset{(2.1)}{=} -\tilde{h}_x \tilde{p}_x + \left( \frac{1}{\varepsilon} \right)^2 \tilde{p}_y \big|_{\tilde{y}=\tilde{h}(\tilde{t}, \tilde{x})}, \quad \text{where } \begin{cases} \Delta_\varepsilon \tilde{p} = 0 & \text{in } \tilde{\Omega}, \\
\tilde{p} = \eta & \text{on } \partial_1 \tilde{\Omega}, \\
\tilde{p}_y = 0 & \text{on } \partial_0 \tilde{\Omega}
\end{cases}
\] (2.7)
and where $\Delta_\varepsilon = \partial_x^2 + (\frac{1}{\varepsilon})^2 \partial_y^2$. Here, $\tilde{\Omega} = \{(x, y) : x \in (0, 1), y \in (0, \tilde{h}(\tilde{r}, x))\}$.

We will also use short notations $\tilde{\Gamma} := \partial_1 \tilde{\Omega}$ and $\Gamma_0 := \partial_0 \tilde{\Omega} = (0, 1)$. The transformed evolution—after multiplication by $\frac{D^2}{\varepsilon y}$—is described by

\[(\tilde{h} D)\tilde{t} - \left((1 - \tilde{x}) \partial_t \tilde{s}_+ + \tilde{x} \partial_t \tilde{s}_+\right) \tilde{h} - \mathcal{B}^\varepsilon_{\tilde{x}} \left(\frac{\tilde{h}_{\tilde{x}\tilde{x}}}{(1 + \varepsilon^2 \tilde{h}_{\tilde{x}}^\varepsilon)}\right) = 0 \quad \text{for } \tilde{x} \in (0, 1)
\]

(2.8)

with boundary conditions $\tilde{h} = 0$ and $|\tilde{h}_{\tilde{x}}| = 1$ for $\tilde{x} \in \{0, 1\}$. In the following, we will only use the rescaled variables and we omit the $\sim$ in the notation for readability. We define $h^*(x) = x(1 - x)$, where we note that $h^* : [0, 1] \rightarrow \mathbb{R}$ is an approximation of the stationary solution $h_{\text{stat}} : [0, 1] \rightarrow \mathbb{R}$ of (2.8). The stationary solution is unique non-negative function with constant curvature and which satisfies the boundary conditions. The graph of $h_{\text{stat}}$ is the set of all points $(x, y)$ in the intersection of $[0, 1] \times [0, \infty)$ and the boundary of the ball with radius $\frac{1}{\sqrt{2}}$ and center point $(0, -\frac{1}{2})$. We also set

\[f := (h - h^*)_x = h_x - f^*\]

and $f_{\text{in}} := h_{\text{in}|x} - f^*$, where $f^* := h^*_x = 1 - 2x$, $f^*_x = -2$ and $f^*_{xx} = 0$. In particular, we have $h_{xxx} = f_{xx}$. We also introduce the notations $B^f_{\varepsilon} : = \mathcal{B}^\varepsilon_{\tilde{x}}$ and $Q_\tau := (0, \tau) \times (0, 1)$. With these notations and with the definition

\[
\mathcal{L}_\varepsilon(f) := ((f + f^*) D)\tilde{t} - \left[(1 - x) \dot{s}_- + x \dot{s}_+\right] (f + f^*) + B^f_{\varepsilon} \left(\frac{f_x + f^*_x}{(1 + \varepsilon^2 (f + f^*)^2)}\right),
\]

(2.9)

where $D = D(t)$ is given by (2.4), we arrive at

\[
\begin{cases}
\mathcal{L}_\varepsilon(f) = 0 & \text{for } (t, x) \in Q_\tau, \\
\int_0^1 f(\cdot, x) \, dx = 0 & \text{for } t \in (0, \tau), \\
f = 0 & \text{for } (t, x) \in (0, \tau) \times \{0, 1\}, \\
f = f_{\text{in}} & \text{for } (t, x) \in \{0\} \times (0, 1).
\end{cases}
\]

(2.10)

The evolution of the position of the contact points for $t \in (0, \tau)$ is determined by the ODE

\[
\dot{s}_\pm(t) \overset{(2.3)}{=} \left.(1 + \varepsilon^2)^{-\frac{3}{2}} f_{xx} - 3 \varepsilon^3 (1 + \varepsilon^2)^{-5/2} (f_x + 2)^2\right|_{x=0,1}
\]

(2.11)

with initial positions given by $s_-(0) = 0, s_+(0) = 1$. We have transformed the equation onto a fixed domain at the cost of the non-local operator $B^f_{\varepsilon}$ and the nonlocal terms $s_{\pm}, D$. 
For the corresponding thin-film equation (1.4), we define
\[ \mathcal{L}_0 f := ((f + f^*) D_t) - \left[ ((1 - x) \dot{s}_- + x \dot{s}_+) (f + f^*) + \left( \int_0^x (f + f^*) d\xi \right) f_{xx} \right]_{xx}, \]
where \( D(t) \) is given by (2.4). The analogous transformations as above then yield
\[
\begin{cases}
\mathcal{L}_0 f = 0 & \text{for } (t, x) \in Q_\tau, \\
\int_0^1 f(\cdot, x) \, dx = 0 & \text{for } t \in (0, \tau), \\
f = 0 & \text{for } (t, x) \in (0, \tau) \times [0, 1], \\
f = f_{\text{in}} & \text{for } (t, x) \in [0] \times (0, 1).
\end{cases}
\] (2.12)

The corresponding evolution for the contact points corresponding to (2.12) are defined by the ODE \( \dot{s}_\pm(t) = f_{xx}(0, 1) \) with initial positions given by \( s_-(0) = 0, s_+(0) = 1. \)

Let us remark on the boundary conditions at the contact point. These need to be handled particularly carefully due to the loss of regularity of the free domain at the contact point. The above transformations for the Darcy flow are such that the boundary conditions \( h = 0 \) and \( |h_x| = \varepsilon \), respectively, are equivalent to the integral/boundary conditions \( \int_0^1 f = 0 \) and \( f = 0 \), respectively, for the transformed function \( f \). In the subsequent part of this work, we construct \( f \) by a Lax–Milgram argument in a space where \( f = 0 \) is satisfied at the boundary. The integral condition \( \int_0^1 f \, dx = 0 \) holds automatically for sufficiently smooth solutions if it is satisfied initially. Indeed, with the analogous calculation as for (2.3), it follows that the solution \( f \) of (2.10) satisfies
\[
\frac{d}{dt} \int_0^1 f(t, \cdot) D(t) \, dx = -\left[ ((1 - x) \dot{s}_- + x \dot{s}_+) (f + f^*) - B_f \left( \frac{f_x - 2}{(1 + \varepsilon^2 (f + f^*)^2)^{3/2}} \right) \right]_0^1 = \dot{s}_+ - \dot{s}_- - \dot{s}_+ + \dot{s}_- = 0.
\]

Therefore, \( \int_0^1 f(t, \cdot) D(t) \, dx = \int_0^1 f_{\text{in}} D(0) \, dx = 0 \) and hence \( \int_0^1 f(t, \cdot) \, dx = 0 \) for \( t \in [0, \tau] \) where \( \tau > 0 \) is chosen sufficiently small such that \( D(t) > 0 \) holds. An analogous calculation applies also for the transformed thin-film equation (2.12).

We describe how the problem of an infinite wedge. Near the moving contact point, the free domain occupied by the liquid approximately has the shape of a wedge. This motivates us to consider the corresponding evolution equation which is linearized around an infinite wedge. Let us therefore assume for the moment that \( h(t, x) \approx \varepsilon (x - s(t)) \) and for \( x \in (s(t), \infty), t \in (0, \infty) \) and for \( s(t) \in R \). We describe how the problem
is transformed onto the wedge, see also [27] for more details of the transformations. Analogously to (2.5), the new variables are defined by

\[
x - s(t) = \tilde{x}, \quad y = \varepsilon \tilde{y}, \quad t = \frac{\tilde{t}}{\varepsilon} \quad \text{and} \quad p = \varepsilon \tilde{p}, \quad h = \varepsilon \tilde{h}.
\]

(2.13)

Analogously to (2.8), this transformation yields

\[
\tilde{h} \frac{\partial}{\partial \tilde{t}} - \varepsilon \frac{\partial}{\partial \tilde{x}} - \frac{B_0}{\varepsilon} \left((1 + \varepsilon^2 \tilde{h}^2)^2 - \frac{3}{2} \tilde{h} \frac{\partial}{\partial \tilde{x}} \right) = 0 \quad \text{for} \quad \tilde{x} \in (0, \infty),
\]

(2.14)

where \(s(t)\) is defined as in (2.3). In the following, we omit the ‘\(\sim\)’ in the notation.

We set

\[
f := h_x - 1.
\]

(2.15)

Taking one spatial derivative, of (2.14), we get

\[
\frac{\partial}{\partial t} f - \frac{\partial}{\partial x} \left[ \frac{1}{\varepsilon^2} f_x (1 + \varepsilon^2 (1 + f)^2) \right] = 0
\]

(2.16)

with the single boundary condition \(f(0) = 0\). We introduce the notations \(K_f := \{(x, y): x > 0, 0 < y < h\}, \Gamma_f := \{(x, y): x > 0, y = h\}\) and \(K := K_f^0\) and \(\Gamma := \Gamma_f^0\). The linear operator \(B_\varepsilon := B_0^0\) is given by

\[
B_\varepsilon \eta(t, x) = \left(-q_x + \left(\frac{1}{\varepsilon^2}\right)^2 q_y\right)_{|y=x}, \quad \text{where} \quad \begin{cases}
\Delta_\varepsilon q = 0 & \text{in} \ K, \\
q = \eta & \text{on} \ \partial_1 K, \\
q_y = 0 & \text{on} \ \partial_0 K.
\end{cases}
\]

(2.17)

Equation (2.16) can then be equivalently expressed as

\[
\begin{cases}
\frac{\partial}{\partial t} f + A_\varepsilon f = N_\varepsilon(f) & \text{for} \ x \in (0, \infty), \\
f = 0 & \text{for} \ x = 0, \\
f = f_{in} & \text{for} \ t = 0.
\end{cases}
\]

(2.18)

The main (linear) part of (2.16) is given by the operator

\[
A_\varepsilon := -(1 + \varepsilon^2)^{-\frac{3}{2}} \partial_x B_\varepsilon \partial_x.
\]

(2.19)

The remaining terms in (2.16) are combined in the nonlinear operator,

\[
N_\varepsilon(\varphi, f) = \frac{\varphi_x}{(1 + \varepsilon^2)^{\frac{3}{2}}} \left[ f_{xx} - \frac{3\varepsilon^2 f_x^2}{2} \right]_{|x=0} + \partial_x B_\varepsilon^\varphi \left( \frac{f_x}{(1 + \varepsilon^2 (1 + f)^2)^{\frac{3}{2}}} \right) + A_\varepsilon f.
\]

(2.20)

We also use the notation

\[
N_\varepsilon(f) := N_\varepsilon(f, f).
\]

(2.21)

The first term on the right hand side of (2.20) is related to the movement of the triple point. The second and third term describe the error which appears by replacing
the domain $K^\phi$ by $K$ and by replacing the curvature with $f_x$. The renormalized interfacial energy in the case of an infinite wedge is given by

$$E_\varepsilon(t) := \frac{1}{\varepsilon^2} \int_{s(t)}^\infty \left( \sqrt{1 + \varepsilon^2 (1 + f)} - \sqrt{1 + \varepsilon^2} - \frac{\varepsilon^2 f}{\sqrt{1 + \varepsilon^2}} \right) \, dx. \quad (2.22)$$

Indeed, in the Appendix we show that solutions of (2.24) satisfy

$$\frac{d}{dt} E_\varepsilon(t) + \frac{1}{\gamma} \int_\Omega |\nabla p|^2 \, dx = 0. \quad (2.23)$$

Analogously, for the thin-film equation we apply the coordinate transform $\tilde{x} = x - s(t)$. In the new coordinates, we obtain

$$\begin{cases} f_t + A_0 f = N_0(f) & \text{for } x \in (0, \infty), \\ f = 0 & \text{for } x = 0, \\ f = f_{in} & \text{for } t = 0. \end{cases} \quad (2.24)$$

The linear and nonlinear parts of the equation are given by

$$A_0 f := -\partial_x B_0 \partial_x f,$$

$$N_0(\varphi, f) := -\left( \left( \int_0^x \varphi \, dx \right) (f_{xx} - f_{xx}|_{x=0}) \right)_{xx}, \quad (2.25)$$

where $B_0 := -\partial_x x \partial_x$. Note that $N_0(\varphi, f)$ is bilinear, while $N_\varepsilon(\varphi, f)$ is neither linear in the first nor in the second argument. Also notice that (2.24) has the scaling invariance $(x, t, f) \mapsto (\lambda x, \lambda^3 t, f)$. The appropriate form of the interfacial energy for the thin-film equation in the case of an infinite wedge is given by

$$E_0(t) := \int_{s(t)}^\infty f^2(x) \, dx. \quad (2.26)$$

Indeed, for this energy we obtain the dissipation relation

$$\frac{d}{dt} E_0(t) + \gamma \int_\Omega h h_{xxx}^2 \, dx = 0. \quad (2.27)$$

For more details, we refer to the Appendix, where a derivation of the corresponding dissipation relations is given for Darcy flow and the thin-film equation both in the droplet case and in the case of an infinite wedge.

### 2.2. Norms for the Profile

The initial problem (1.2) is non–degenerate parabolic on a two-dimensional non-smooth moving domain, the limit problem (1.4) is degenerate parabolic on a 1-d smooth domain. We use weighted Sobolev type spaces to capture the transition between these two problems. Weighted spaces for the analysis of elliptic operators on non-smooth domains have been used in [29]. Weighted spaces have also been
used to analyze degenerate parabolic equations, see for example [14,20,22,26,28].

Our analysis connects these two applications of weighted spaces.

Let \( E = (0, 1) \), \( E = (0, \infty) \) or \( E = (-\infty, 0) \) and let \( d(x) = \text{dist}(x, \partial E) \). For \( \ell \in \mathbb{N} \), we want consider homogeneous norms \( [f]_{X_\ell^\epsilon(E)} \), which are equivalent to the following sum of two weighted Sobolev norms:

\[
[f]_{X_\ell^\epsilon(E)} \sim \inf_{f = f_+ + f_-} \left( \left\| d^{\ell + 1} \partial_x^{4\ell + 1} f_+ \right\|_{L^2(E)} + \left\| e^{\ell \epsilon} \partial_x^{3\ell + 1} f_- \right\|_{L^2(E)} \right), \tag{2.28}
\]

Note that in the limit \( \epsilon \to 0 \), the homogeneous norms (2.28) turn from a norm of order \( 3\ell + 1 \) to a norm of order \( 4\ell + 1 \). Indeed, let us recall Hardy’s inequality which holds for all \( \beta \neq -\frac{1}{2} \) and all \( f \in C_c^\infty((0, \infty)) \):

\[
\left\| x^\beta f \right\|_{L^2((0, \infty))} \leq C_\beta \left\| x^{\beta + 1} f_x \right\|_{L^2((0, \infty))}. \tag{2.29}
\]

In particular, for fixed \( \epsilon > 0 \), the second term on the right hand side of (2.28) is estimated by the first one.

Our estimates require a generalization of the homogeneous norms on the right hand side of (2.28) to the case of fractional derivatives. For this generalization, we will use the Mellin transform, a transformation which has been widely used in the analysis of elliptic problems on conical domains, see for example [29]. For any \( f \in C_c^\infty((0, \infty)) \), its Mellin transform \( \hat{f} \) is defined by

\[
\hat{f}(\lambda) = \int_0^\infty x^{-\lambda} f(x) \frac{dx}{x} = \int_{\mathbb{R}} e^{-\lambda u} F(u) \, du, \tag{2.30}
\]

where here and in the following we use the variables \( u = \ln x \) and \( F(u) = f(x) \). Note that the application of the Mellin transform on \( f \) corresponds to application of the two-sided Laplace transform on \( F \). It is easy to see that \( x^\beta \hat{f}(\lambda) = \lambda^\beta \hat{f}(\lambda) \) and \( \hat{x}^{-\beta} f(\lambda) = \hat{f}(\lambda + \beta) \) for any \( \beta \in \mathbb{R} \). Furthermore, Plancherel’s identity holds

\[
\left\| x^{-\beta} f \right\|_{L^2((0, \infty), \frac{dx}{x})} = \left\| e^{-\beta u} F \right\|_{L^2(\mathbb{R}, du)} = \left\| \hat{f} \right\|_{L^2(\mathbb{R}; \beta)} . \tag{2.31}
\]

The strip of convergence is defined as the set \( (\beta_1, \beta_2) \times \mathbb{R} \subset \mathbb{C} \) where the integrand in (2.30) is absolutely convergent. For any \( \beta \in (\beta_1, \beta_2) \), the inverse Mellin transform of \( f \) is given by

\[
f(x) = \int_{\Re \lambda = \beta} x^\lambda \hat{f}(\lambda) \, d\Im(\lambda) = \int_{\Re \lambda = \beta} e^{\lambda u} \hat{F}(\lambda) \, d\Im(\lambda), \tag{2.32}
\]
the boundary data. We first consider the case of functions, defined on the half-space $(0, \infty)$. Consider $f \in C_c^\infty((0, \infty))$, that is $f$ vanishes for $x \to \infty$. Given $k \geq 0$, let $n_k$ be the largest integer smaller than $3k - \frac{1}{2}$, that is $n_k = \lceil 3k - \frac{3}{2} \rceil$. In particular, if $k \in \mathbb{N}_0$ then $n_k = 3k - 1$. Let $\mathcal{P}_f$ be the Taylor polynomial of order $n_k$ of $f$ at $x = 0$ (if $n_k = -1$, then we choose $\mathcal{P}_f = 0$). We decompose $f = f_1 + f_0$, where $f_1 = \zeta \mathcal{P}_f$ and define
\[
\|f\|_{X^k((0, \infty))} = \|\mathcal{P}_f\| + \|f_0\|_{L^2} + |f_0|_{X^k},
\] (2.33)
where $\|\cdot\|$ is any fixed polynomial norm, for example the $\ell^2$-norm of the coefficients. Here, the homogeneous norm $[\cdot]_{X^k}, \ell \geq 0$, is given by
\[
[f_0]_{X^k} = \left\|\int \lambda^{3\ell + 1} \mu^\ell f_0^0 \right\|_{L^2(\{\lambda = 3\ell - \frac{1}{2}\})},
\] (2.34)
with the notation $\mu = \min(|\lambda|, \frac{1}{\ell^2})$. We turn to the definition of the norms for functions $f \in C_c^\infty((0, 1))$, that satisfy $\zeta = 1$ in $[0, \frac{1}{8}]$, $\zeta = 0$ in $[\frac{1}{4}, \infty)$ and $\|D^k \zeta\|_{L^\infty} \leq C_k$ for all integer $k$. Moreover, we assume that $\zeta(x) + \zeta(1 - x) = 1$ for all $x \in [0, \infty)$. If $E = (0, 1)$ then we set $f^L(x) = f(x)\zeta(x)$ and $f^R(x) = f(1 - x)\zeta(x)$, in particular $\text{supp } f^L, \text{supp } f^R \subseteq [0, \frac{3}{4}]$ and $f^L(x) + f^R(1 - x) = f(x)(\zeta(x) + \zeta(1 - x)) = f(x)$.

With this cut-off function, we define the norm for $f \in C_c^\infty((0, 1))$ by
\[
\|f\|_{X^k((0, 1))} := \|f^L\|_{X^k((0, \infty))} + \|f^R\|_{X^k((0, \infty))},
\] (2.35)
assuming that the norm for functions on the half-space $(0, \infty)$ is defined.

The equivalence of these norms with the characterization (1.7) when $k$ is an integer follows by application of (2.31) and by repeated application of Hardy’s inequality [27, Proposition 2.2]. Note that the definition of the norm (2.33) by the Mellin transform and the definition in physical variables by (2.28) differ by a constant $C_k$ for integer $k$. In the following, by an abuse of notation, we will also use (2.28) as a definition of the homogeneous norms. This does not change the result, since the corresponding norms are equivalent and since all out estimates depend on constants $C_k$.

Let $E = (0, 1)$ or $E = (0, \infty)$. For $k \in [0, \infty)$, we define, the space $X^k(E)$ as completion of the space $C_c^\infty(E)$ with respect to the norm (2.33). Furthermore, the space $\hat{X}^k(E)$ is defined as the completion of the space $\{f \in C^\infty(E): f = 0 \text{ on } \partial E\}$ with respect to the same norm. Note that the trace of $f$ at $\partial E$ is well-defined in $X^k(E)$ if $k > \frac{1}{5}$, in which case $X^k(E) \neq \hat{X}^k(E)$. We also define $\hat{X}^k(E)$ as the completion of $C_c^\infty(E)$ with respect to the norm (2.33). Finally, the space $X^k_{\epsilon, \text{loc}}$ is defined as the space of all functions $f$ such that $f\zeta \in X^k$ for any smooth cut-off function $\zeta \in C_c^\infty([0, \infty))$. The corresponding convergence in $X^k_{\epsilon, \text{loc}}$ is defined in the usual way.

We will also need parabolic time-space norms. For simplicity, we will only define (and use) these time-space norms for integer $k$. Since our analysis is mostly based on estimates in frequency variables, we believe, however, that our results can also be extended in a straightforward way to the more general case of real $k$. Let
the time-space cylinder $Q_{\tau}$ be given by $Q_{\tau} := [0, \tau) \times E$. For $\varepsilon > 0$, $k \in \mathbb{N}_0$, we define
\[
\| f \|_{X^k_{\varepsilon}(Q_{\tau})}^2 := \sum_{1 \leq i + j \leq k} \| \partial_i^j f \|_{L^2(X^j_{\varepsilon}(Q_{\tau}))}^2 + \sum_{0 \leq i + j \leq k-1} \| \partial_i^j f \|_{C^0(X^j_{\varepsilon} \cap \partial E)}^2 .
\]
(2.36)

the corresponding spaces $TX^k_{\varepsilon}(Q_{\tau})$, $TX^k_{\varepsilon}(Q_{\tau})$, $TX^k_{\varepsilon}(Q_{\tau})$ are defined by completion as before, where the superscript ‘\text{oo}’ denotes that the closure is taken over the set of functions vanishing on the spatial boundary $(0, \tau) \times \partial E$ and the superscript ‘\text{ oo}’ denotes the space obtained by taking the closure of $C^\infty_c([0, \tau) \times \overline{E})$. Finally, the space $TX^k_{\varepsilon,\text{loc}}$ is defined as the space of all functions $f$ such that $f \in TX^k_{\varepsilon}$ for any smooth cut-off function $\zeta = \zeta(x) \in C^\infty_c([0, \infty))$. The corresponding convergence in $TX^k_{\varepsilon,\text{loc}}$ is defined as usual.

If the domain is $Q = [0, \infty)^2$ or $E = [0, \infty)$, we sometimes omit the domain in the notation of space and norm, that is we write $X^k_{\varepsilon}$ for $X^k_{\varepsilon}([0, \infty))$, etc..

Note that the choice of norms (2.34) is motivated by the investigation of the linear operator in [27]. In particular, by [27, Theorem 3.2] for $k \geq 0$ and for
\[
\varepsilon \in \left[ 0, \frac{\pi}{3(2k + 1)} \right),
\]
we have the estimate,
\[
c_k [A_{\varepsilon} f]_{X^k_{\varepsilon}} \leq [f]_{X^{k+1}_{\varepsilon}} \leq C_k [A_{\varepsilon} f]_{X^k_{\varepsilon}} .
\]
(2.37)

The restriction (2.37) on the admissible values for $\varepsilon$ is related to an occurrence of non-uniqueness and loss of regularity for the solutions of elliptic operators on non-smooth domains, see also [29]. The constant in (2.37) seems to be optimal in the limit $k \to \infty$ in view of the study based on the Laplace transform performed in [27]. However, the proof of Lemma 6 suggests that if can be improved slightly for fixed $k < \infty$.

2.3. Norms for the Pressure

We introduce norms and spaces $Y^k_{\varepsilon}$ for the pressure $q$ [cf. (2.17)]. Unlike the norms $\| \|_{X^k_{\varepsilon}}$, defined above, for the profile $X^k_{\varepsilon}$, which—at least in the case of integer $k$—can be formulated both in terms of real variables as well as Mellin variables [cf. (1.7) vs. (2.34)], the corresponding norms $\| \cdot \|_{Y^k_{\varepsilon}}$ can only be expressed in terms of Mellin variables. Roughly speaking, we apply the Mellin transform in the radial direction (with respect to the tip of the wedge), but not in the angular variable. The norms for the pressure are then obtained by taking the $L^2$-norm in radial direction in frequency variables, followed by an application of the $L^\infty$-norm in angular direction in physical variables. Using the $L^\infty$-norm in angular variables enables us to obtain estimates which are optimal in $\varepsilon$. A standard approach using an $L^2(L^2)$-norm in the pressure would not capture the optimal $\varepsilon$-dependence which is needed for the convergence to the limit model. There are several technical difficulties related to the
fact that the norm is a supremum norm in an angular direction. One of them is that
the norms cannot be expressed in terms of physical variables. Moreover, complex
interpolation as is possible for the norms for the profile (see Lemma 1) cannot be
directly used for the norms for the pressure.

We first define the space \( Y^k_\varepsilon \) for functions defined on the wedge \( K = \{ (x, y): x > 0, 0 < y < x \} \). For this, we introduce a coordinate transform which maps the wedge
\( K \) onto the infinite strip \( \mathbb{R} \times (0, 1) \): for \( (x, y) \in K \), we define the new variables \( (u, v) \in \mathbb{R} \times (0, 1) \) by

\[
\begin{align*}
  x &= e^u \cos(\varepsilon v), \\
y &= \frac{1}{\varepsilon} e^u \sin(\varepsilon v),
\end{align*}
\]

that is

\[
\begin{pmatrix}
  \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
  \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{pmatrix} = \begin{pmatrix}
  e^u \cos(\varepsilon v) & -\varepsilon e^u \sin(\varepsilon v) \\
  \frac{1}{\varepsilon} e^u \sin(\varepsilon v) & e^u \cos(\varepsilon v)
\end{pmatrix}.
\]

For later reference, we note that \( \mathrm{d}x \mathrm{d}y = e^{2u} \mathrm{d}u \mathrm{d}v \) and

\[
\begin{align*}
  \partial_u &= x \partial_x + y \partial_y, \\
  \frac{1}{\varepsilon} \partial_v &= -\varepsilon y \partial_x + \frac{1}{\varepsilon} x \partial_y, \\
  \partial_x &= e^{-u} \cos(\varepsilon v) \partial_u - \frac{1}{\varepsilon} e^{-u} \sin(\varepsilon v) \partial_v, \\
  \frac{1}{\varepsilon} \partial_y &= e^{-u} \sin(\varepsilon v) \partial_u - \frac{1}{\varepsilon} e^{-u} \cos(\varepsilon v) \partial_v.
\end{align*}
\]

The coordinate transform (2.39) can be understood as a composition of the two
transformations \((x, \varepsilon y) = r(\cos \theta, \sin \theta)\) and \((r, \theta) = (e^u, \varepsilon v)\), see Fig. 2.

The definition of the norms for the pressure is motivated by the fact that the
(homogeneous) linearized pressure equation can be solved explicitly in terms of
frequency variables, see Section 6. For \( q \in C^\infty_c(K \setminus (0, 0)) \), let \( \hat{q} \) be the Laplace
transform of \( q \) with respect to the variable \( u \), where by a slight abuse of notation
we write \( q(u, v) = q(x, y) \). For any (generalized) multi-index \( \alpha = (\alpha_1, \alpha_2) \in [0, \infty) \times \mathbb{N}_0 \), we define \(|\alpha| = \alpha_1 + \alpha_2\) and \( \Lambda^\alpha = \lambda^{\alpha_1} (\frac{1}{\varepsilon} \partial_v)^{\alpha_2} \). For any \( q_0 \in C^\infty_c(K \setminus (0, 0)) \) and for \( \ell \geq 0 \), we set

\[
[q_0]_Y^\ell = \sum_{|\alpha|=\ell} \sup_{v \in (0, 1)} \left\| e^{\frac{1}{\varepsilon}|\lambda|(1-v)} \Lambda^\alpha \mu^\ell \hat{q}_0 \right\|_{L^2(0, 1)}^\ell,
\]

where we recall the definition of the multiplier \( \mu = \inf \{ \frac{1}{\varepsilon}, |\lambda| \} \).

Let \( f \in X^k_\varepsilon \) and let \( q \) be the linearized pressure \( q: K \to \mathbb{R} \), defined as the
solution of (2.17) with right hand side \( \eta := f_\varepsilon \). Since \( q = f_\varepsilon \) on \( \partial_1 K \) and since
by the definition of \( X^k_\varepsilon \) the Taylor polynomial of \( f \) is well-defined up to the order
\( n_k = \lfloor 3k - \frac{3}{2} \rfloor \), we expect to have control on the supremum norm for derivatives of
\( q \) up to order \( n_k - 1 \). Indeed, such an estimate is given in Lemma 4. For \( q \in C^\infty_c(K) \), let \( P_q \) be the Taylor polynomial of \( q \) at \((0, 0)\) of order \( n_k - 1 \) that is \( n_k - 1 = 3k - 2 \)
if \( k \in \mathbb{N} \). Let \( \zeta : K \to \mathbb{R} \) be a cut-off function with \( \zeta = \zeta((x, y)) \) and \( 0 \leq \zeta \leq 1 \) and such that \( \zeta = 1 \) for \( |(x, y)| < \frac{1}{8} \) and \( \zeta = 0 \) for \( |(x, y)| > \frac{1}{4} \). We decompose \( q := q_0 + q_1 \) where \( q_1 := \zeta \mathcal{P}_q \). For \( k, \ell \in \mathbb{R} \) we define

\[
\|q\|_{Y^\ell_k} = \|\mathcal{P}_q\|_{\mathcal{P}} + \sup_{\zeta \leq \epsilon \leq k} [q_0]_{Y^\ell_k}
\]

where \( \| \cdot \|_{\mathcal{P}} \) is any fixed polynomial norm, for example, the Euclidean norm of the coefficients. We do not include the homogeneous norms \( [\cdot]_{Y^\ell_k} \) for \( \ell < \frac{2}{3} \) since this would lead to 'negative derivatives' (and related technical complications) in the definition of the norm \( \| \cdot \|_{\mathcal{Z}^\ell_k} \) which will be defined in (2.46). The spaces \( Y^k_\epsilon \) and \( Y^\omega_\epsilon \) are defined by completion with respect to functions \( C^\infty_c(K) \), respectively \( C^\infty_c(K \setminus (0, 0)) \), as before. By the above considerations, the polynomial \( \mathcal{P}_q \) is uniquely defined and the norm is well-defined. The space \( Y^k_\epsilon(\Omega) \) and its norm are defined by localizing the above definitions, analogously as for the definition of \( X^k_\epsilon(0, 1] \) in (2.35). The parabolic space \( TY^k_\epsilon \) for integer \( k \) and its norm are defined analogously to \( TX^k_\epsilon \), that is

\[
\|q\|^2_{TY^\ell_k(Q)} = \sum_{1 \leq i + j \leq k} \|\partial^i_q\|^2_{L^2(Y^j_\ell(Q))}.
\]

Let us remark that we believe that all homogeneous norms in (2.34) for all real \( \ell \in [\frac{2}{3}, k] \) can be bounded by the two extremal homogeneous norms \( (\ell = \frac{2}{3}, \ell = k) \). However, the proof of this interpolation inequality does not seem to be straightforward, in particular since the analyticity of the expressions involved is destroyed by the supremum in \( v \) in (2.41) so that the theory of complex interpolation does not apply directly. This is the reason that we include the information about all the intermediate homogeneous norms in the definition (2.42). On the other hand, in the definition of the norm for the spaces \( X^k_\epsilon \) in (2.33) we do not include all intermediate norms since we have an interpolation estimate for the corresponding norms \( \| \cdot \|_{X^\ell_k} \), see Lemma 1. We have included \( C^0 \)-type norms in the definition of the space \( TX^k_\epsilon \), but not the space \( TY^k_\epsilon \). The reason is technical: on the one hand, the solution of the parabolic equation yields us control on the supremum type estimates on \( f \), and these estimates are also needed to control, for example, the initial data. On the other hand, such temporal traces are not needed for the pressure, and also these estimates do not follow directly from our elliptic estimates.

We next introduce the space \( Z^k_\epsilon \), which is suitable to describe the regularity of functions \( g : K \to \mathbb{R} \) of type \( g = \Delta_\epsilon q \) where \( q \in Y^k_\epsilon \). In view of (2.41), we thus consider for real \( \ell \in [\frac{2}{3}, k] \) the homogeneous norms of type

\[
[g_0]_{Z^\ell_k} = \sum_{|\alpha| = 3\ell - 2} \left\| \sup_{\nu \in (0, 1)} |e^{1/\nu}|(1-v) \Lambda^\alpha \mu^\ell \hat{g}_0 \right\|_{L^2(\Omega, \lambda = 3\ell - \frac{1}{2})},
\]

where the sums are taken over \( \alpha = (\alpha_1, \alpha_2) \in [0, \infty) \times \mathbb{N}_0 \), using the notation \( |\alpha| = \alpha_1 + \alpha_2 \) and where we recall that \( \Lambda^\alpha = \lambda^\alpha (\frac{1}{v} \partial_v)^{\alpha_2} \). Note that \([g_0]_{Z^\ell_k} < \infty\) implies

\[
\|g_0(x, \cdot)\|_{L^\infty} = o(x^{(3\ell - 7)/2})
\]

(2.45)
for all $g_0 \in Z^k_\varepsilon$. Correspondingly, we say $g \in Z^k_\varepsilon$ if there is a polynomial $\mathcal{P}_g$ in $x, y$ of order $n_k - 3$ (if $k \in \mathbb{N}$ then $n_k - 3 = 3k - 4$) such that $g_0 := g - \zeta \mathcal{P}_g \in Z^k_\varepsilon$ for some radial cut-off $\zeta = \zeta(r)$ with $\zeta = 1$ in $[0, 1]$ and $\zeta = 0$ in $[2, \infty)$. The corresponding norm is given by

$$\|g\|_{Z^k_\varepsilon} = \|\mathcal{P}_g\|_\mathcal{P} + \sup_{\zeta \leq \varepsilon \leq k}[g_0]_{Z^\varepsilon}.$$  

(2.46)

The spaces $Z^\infty_\varepsilon$ and $Z^k_\varepsilon$ are defined by completion as before. The corresponding spaces $Z^k_\varepsilon(\Omega)$ and $Z^\infty_\varepsilon(\Omega)$ for the droplet case and their norm are defined analogously as before by localizing the above definitions. In particular, the corresponding estimate to (2.45) holds near the contact points for all $g_0 \in Z^k_\varepsilon(\Omega)$. Note that the minimal value $\ell = \frac{2}{3}$ in (2.46) is chosen such that the exponent $|\alpha| = 3\ell - 2$ in definition (2.44) stays non-negative.

### 2.4. Compatibility Conditions

Higher regularity for our solution requires compatibility conditions on the initial data (for both Darcy flow and thin-film equation). Indeed, let $f \in T X^{k+1}_\varepsilon$ be a solution of (2.18), respectively (2.24). Note that in view of (2.36) and (2.28), the traces $f|_{x=0} = 0, \ldots, \partial^\varepsilon x f|_{x=0} = 0$ are well-defined for any function $f \in T X^{k+1}_\varepsilon$, while the norm $\|\cdot\|_{TX^{k+1}_\varepsilon}$ is not strong enough to control the trace $\partial^{k+1}_t f|_{x=0} = 0$. Since $f$ solves (2.18), and respectively (2.24), and since $f|_{x=0} = 0$, it follows in particular that $\partial^\varepsilon x f|_{x=0} = 0$ for all $0 \leq \ell \leq k$. This translates to a compatibility condition for the initial data. It is obtained by consecutively replacing the time derivatives in $\partial^\ell f|_{x=0}$ by the spatial operators $N^\varepsilon$ and $A^\varepsilon$, using (2.18) resp. (2.24). The corresponding condition needs to be satisfied for the initial data:

$$f_{\text{in}} \text{ satisfies compatibility conditions ensuring } \partial^\ell x f_{\text{in}}|_{x=0} = 0 \quad \text{at } t = 0 \quad (2.47)$$

for all $0 \leq \ell \leq k$. For example, the condition for $k = 1$ corresponds to $(N^\varepsilon(f_{\text{in}}) - A^\varepsilon f_{\text{in}})|_{x=0} = 0$. We will actually use a slightly stronger version of (2.47) and assume the existence of a lifting of the initial data $\tilde{f}_{\text{in}} \in T X^{k+1}_\varepsilon$ with $\int_0^\infty \tilde{f}_{\text{in}}(t, x) \, dx = 0$ for all $t > 0$ which is a compactly supported extension of $f_{\text{in}}$ to the time interval $[0, 1)$ satisfying $\partial^\ell x \tilde{f}_{\text{in}}|_{x=0} = 0$ for all $0 \leq \ell \leq k$ and such that $\tilde{f}_{\text{in}}(t = 0) = f_{\text{in}}$ and that for $1 \leq \ell \leq k$, we have that

$$\partial^\ell x \tilde{f}_{\text{in}}|_{t=0} = \partial^\ell x^{-1} \left( N^\varepsilon(f_{\text{in}}) - A^\varepsilon f_{\text{in}} \right)|_{t=0}. \quad (2.48)$$

We will also assume that

$$\|\tilde{f}_{\text{in}}\|_{TX^{k+1}_\varepsilon} \leq C \|f_{\text{in}}\|_{X^{k+1/2}_\varepsilon}. \quad (2.49)$$

An analogous compatibility condition needs to be satisfied for the linear evolution $f_t + A^\varepsilon f = g$. In this case, we need $f_{\text{in}}$ and $g$ to satisfy for all $0 \leq \ell \leq k$

$$f_{\text{in}} \text{ and } g \text{ satisfy compatibility conditions ensuring } \partial^\ell x f_{\text{in}}|_{x=0} = 0 \quad \text{at } t = 0. \quad (2.50)$$
This condition is easier to write down explicitly since for $1 \leq l \leq k$, we have
\[
\partial_t^l f = \partial_t^{l-1} g - A_\varepsilon \partial_t^{l-1} f
\]
and hence, by induction, we deduce that for $0 \leq l \leq k$, (2.50) is equivalent to
\[
\left[ \sum_{i=0}^{l-1} (-A)^i \partial_t^{l-i-1} g(t = 0) + (-A)^l f_{\text{in}} \right]_{|x=0} = 0. \tag{2.51}
\]

Now, consider the case of compactly supported initial data. We define $A_\varepsilon$ by
\[
L_\varepsilon(f) := Df_t + A_\varepsilon f,
\]
that is
\[
A_\varepsilon f = Df_t - \partial_x \left[ (1 - x)\dot{s}_- + x\dot{s}_+ \right] (f + (x(1 - x))_x)
\]
\[
+ B_\varepsilon \left( \frac{f_x - 2}{(1 + \varepsilon^2 (f + (x(1 - x))_x)^2)^{3/2}} \right).
\]

Analogously to (2.48) and (2.49), we assume that for given $h_{\text{in}}^\varepsilon \in H^1((0, 1))$, $h_{\text{in}}^\varepsilon > 0$, where $f_{\text{in}}^\varepsilon := [h_{\text{in}}^\varepsilon - x(1 - x)]_x \in X^k_{\varepsilon}$, there exists a compactly supported extension $\tilde{f}_{\text{in}}$ of $f_{\text{in}}$ to the time interval $[0, 1]$ with $\int_{(0, 1)} f(t, x) \, dx = 0$ fuer alle $t > 0$ satisfying $\partial_t^l \tilde{f}_{\text{in}}|_{x=0, 1} = 0$ for all $0 \leq l \leq k$ and such that $\tilde{f}_{\text{in}}|_{t=0} = f_{\text{in}}$ and such that for $1 \leq l \leq k$, we have
\[
\partial_t^{l-1} (D \tilde{f}_{\text{in}, t}) = -\partial_t^{l-1} A_\varepsilon (\tilde{f}_{\text{in}}) \quad \text{for } t = 0 \tag{2.52}
\]
and such that furthermore the estimate
\[
\| \tilde{f}_{\text{in}} \|_{X^k_{\varepsilon} \varepsilon} \leq C \| f_{\text{in}} \|_{X^k_{\varepsilon} \varepsilon} \tag{2.53}
\]
holds. We believe that it should be possible to show that the compatibility condition (2.47) is equivalent to the compatibility conditions (2.48)–(2.49), and respectively (2.52)–(2.53). However, the construction of these extensions seems to require considerable technical effort. For this reason, we prefer to use the stronger versions of the compatibility conditions.

3. Statement and Discussion of the Results

3.1. Statement of Results

The first main result of this paper is the well-posedness for the Darcy flow with moving contact line. Furthermore, in the regime of lubrication approximation, we show convergence of the solutions towards solutions of the thin-film equation. We also establish well-posedness for the limit thin-film equation.
Theorem 1. (Darcy flow on perturbed wedge) Let \( k \geq 1 \) be an integer, \( \varepsilon \in (0, \frac{\pi}{3(2k+1)}) \). Suppose that \( f^\varepsilon_{in} \in X^{k+1/2}_\varepsilon \) satisfies (2.48)–(2.49) and suppose that \( \| f^\varepsilon_{in} \|_{X^{k+1/2}_\varepsilon} + E_{\varepsilon}(0) \leq \alpha_k \) for some (small) constant \( \alpha_k \), which only depends on \( k \). Then there is a unique global in time solution \( f^\varepsilon \in TX^{k+1}_\varepsilon ([0, \infty)^2) \) of (2.18) with initial data \( f^\varepsilon_{in} \). Furthermore,

\[
E_{\varepsilon}(t) + \| f^\varepsilon \|_{TX^{k+1}_\varepsilon} + \| \overline{P}^\varepsilon \|_{TY^{k+1}_\varepsilon} \leq C_k \| f^\varepsilon_{in} \|_{X^{k+1/2}_\varepsilon},
\]

(3.1)

where \( \overline{P}^\varepsilon = p^\varepsilon \circ \Psi^\varepsilon \in Y^k_\varepsilon \) and \( \Psi^\varepsilon : K \to K^{f^\varepsilon} \) is the coordinate transform, defined in Lemma 9; see also (2.33), (2.41) for the definition of the norms. The constant in (3.1) only depends on \( k \) and in particular does not depend on \( \varepsilon \).

The above well-posedness result can also be stated for the Darcy flow in terms of the original variables. If the assumptions of Theorem 1 hold, then there exists a unique classical solution of (1.2). Note also that, if \( f^\varepsilon_{in} \) is sufficiently small, then \( h^\varepsilon_{in} \), defined by \( \partial_3 h^\varepsilon_{in} = 1 + f^\varepsilon_{in} \), satisfies \( h^\varepsilon_{in} > 0 \) for \( x \in (0, \infty) \).

We also have convergence for solutions of the Darcy flow to solutions of the thin-film equation. Furthermore, as suggested by the asymptotic expansion (3.5), in the limit \( \varepsilon = 0 \), the pressure is independent of the vertical direction:

Theorem 2. (Convergence to solutions of the lubrication approximation model) Suppose that the assumptions of Theorem 1 are satisfied. Let \( f^\varepsilon \) be the solution of (2.18) with initial data \( f^\varepsilon_{in} \) and let \( p^\varepsilon \) be the corresponding pressure. Suppose that

\[
\| f^\varepsilon_{in} - f_{in} \|_{X^{k+1/2}_\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0
\]

for some \( f_{in} \in X^{k+1/2}_0 \). Then there exist \( f \), \( p \) and a subsequence \( \varepsilon_j \to 0 \) such that

\[
\| f^\varepsilon_j - f \|_{TX^k_{\varepsilon, loc}} \to 0 \quad \text{and} \quad \| \overline{P}^\varepsilon_j - \overline{P} \|_{TY^k_{\varepsilon, loc}} \to 0 \quad \text{as} \quad j \to \infty,
\]

(3.2)

where \( \overline{P} = p \circ \Psi \in Y^k_\varepsilon \) with \( \Psi : K \to K^{f^\varepsilon} \) defined in (6.34). Furthermore, \( f \in TX^k_0 \) solves (2.24) with initial data \( f_{in} \). The limit pressure \( p \) does not depend on the vertical direction, that is \( p = p(t, x) \).

As a consequence of Theorem 2, the velocity field \( U = (V, W) \) in the limit \( \varepsilon = 0 \) is horizontal and does not depend on \( y \), that is \( U = (V(t, x), 0) \). By Theorem 2 and by Proposition 5(2), the solutions converge also in terms of Sobolev norms. In particular, we have

\[
\| f^\varepsilon - f \|_{L^2_tC^3_yX^{k+1/2}_\varepsilon((0, \infty)^2)} \to 0, \quad \| \nabla \overline{P}^\varepsilon - \nabla \overline{P} \|_{L^\infty_tC^{3k-2}_xX^{k+1/2}_\varepsilon((0, \infty)^2)} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

The next result is concerned with the case when the initial data has the shape of a droplet:

Theorem 3. (Darcy flow of a droplet) Let \( k \geq 1 \) be an integer, \( E = (0, 1) \). Suppose that \( h^\varepsilon_{in} \in H^1(E) \) satisfies \( h^\varepsilon_{in} > 0 \) in \( (0, 1) \) and \( h^\varepsilon_{in} = 0 \), \( |h^\varepsilon_{in,x}| = 1 \) on \( \partial E \). Suppose that \( f^\varepsilon_{in} := [h^\varepsilon_{in} - x(1 - x)]_x \in X^{k+1/2}_\varepsilon(E) \) satisfies the compatibility condition (2.52). Then there is a time \( \tau_k > 0 \), which depends on the initial data and \( k \), such
that for every $\varepsilon \in (0, \frac{\pi}{3(2k+1)})$, there exists a unique short–time solution $h^\varepsilon$ of (2.2) with initial data $h^\varepsilon_{in}$. Furthermore, $f^\varepsilon := [h^\varepsilon - x(1-x)]_x \in TX^k_{\varepsilon}((0, \tau) \times E)$ satisfies

$$
\| f^\varepsilon \|_{TX^{k+1}_{\varepsilon}((0,\tau) \times E)} \leq C_k \| f^\varepsilon_{in} \|_{X^{k+1/2}_{\varepsilon}(E)}.
$$

(3.3)

The solution depends continuously on the initial data. Furthermore, for a subsequence $\varepsilon_j \to 0$ we have

$$
\| f^{\varepsilon_j} - f \|_{TX^k_{\varepsilon}((0,\tau) \times E)} \to 0 \quad \text{as} \quad j \to \infty
$$

and the function $h$, defined by $f = [h - x(1-x)]_x$ and $h|_{\partial E} = 0$, solves (1.4).

Theorem 3 also shows that any solution immediately assumes a regularity of order $\frac{1}{\varepsilon}$ where we recall that $\arctan \varepsilon$ is the opening angle. This is the optimal regularity which can be expected in a non-smooth domain with opening angle of order $\varepsilon$, see [29]:

**Corollary 1.** (Regularity of solutions and local semiflow) Any solution $f^\varepsilon$ as in Theorem 3 satisfies $f(t) \in X^{K+\frac{1}{2}}_{\varepsilon}(E)$ for any positive time $0 < t < \tau$, where $K$ is the largest integer such that $K < \frac{1}{2}(\frac{\pi}{\varepsilon} - 1)$. This allows to define a local in time semiflow from $X^{k+1/2}_{\varepsilon}(E)$ to $X^{K+1/2}_{\varepsilon}(E)$ that maps the initial data $f^\varepsilon_{in}$ to $f^\varepsilon(t)$. Analogously, also the solution for the perturbed wedge can be understood as a semi flow. By standard embedding of Sobolev spaces, we have $X^{K+\frac{1}{2}}_{\varepsilon}(E) \subset C^\ell(E)$ and in particular $f(t) \in C^\ell(E)$ for all $\ell \in \mathbb{N}_0$ with $\ell < 3K - 1$.

Indeed, the regularity assertion in Corollary 1 follows by a bootstrap argument based on (3.3). If $f^\varepsilon_{in} \in X^{k+1/2}_{\varepsilon}(E)$, then it follows that $f \in L^2(X^{k+1}_{\varepsilon}((0, \tau) \times E))$ which in turn yields $f|_{t=t_0} \in X^{k+1}_{\varepsilon}(E)$ for almost every fixed positive time $t_0 > 0$. Application of Theorem 3 then implies $f \in L^2(X^{k+3/2}_{\varepsilon}((t_0, \tau) \times E))$ for all $t > t_0$.

Now, for any $\delta$, we may apply this argument repeatedly for time steps of size $\frac{\delta}{K}$, which eventually yields the assertion of Corollary 1.

We also establish existence, uniqueness and regularity for classical solutions of the limit thin-film equation:

**Theorem 4.** (Thin–film equation) Let $k \geq 1$ be an integer.

1. There is $\alpha_k > 0$ such that for any $f^\varepsilon_{in} \in X^{k+1/2}_{0}$ with $\| f^\varepsilon_{in} \|_{X^{k+1/2}_{0}} + E_0(0) \leq \alpha_k$ and such that (2.48)–(2.49) is satisfied, there is a unique global in time solution $f \in TX^k_{0}$ of (2.24) with initial data $f^\varepsilon_{in}$. Furthermore,

$$
\| f \|_{TX^k_{0}} \leq C_k \| f^\varepsilon_{in} \|_{X^{k+1/2}_{0}}.
$$

2. Let $f^\varepsilon_{in} \in X^{k+1/2}_{0}(E)$ and suppose that the analogous assumptions as in Theorem 3 hold. Then there is a short time solution $f \in TX^{k+1}_{0}((0, \tau) \times E)$ of (2.24). Furthermore,

$$
\| f \|_{TX^{k+1}_{0}((0,\tau) \times E)} \leq C_k \| f^\varepsilon_{in} \|_{X^{k+1/2}_{0}(E)}.
$$
Theorem 4 establishes the first existence and uniqueness result for (1.4) in the partial wetting regime. Transforming back into the original variables, the first part of Theorem 4 also shows that the corresponding solution $h$ converges to the stationary wedge $(x - s_\infty)_+$ for some $s_\infty \in \mathbb{R}$.

Well-posedness for classical solutions with non-zero contact angle for a related model has been shown in [26]. There also exist some results on the existence, but not the uniqueness, of weak solutions [9,35]. The existence and uniqueness of classical solutions in the complete wetting regime, where the droplet assumes a zero contact angle at the contact point, has been shown in [21,22]. Furthermore, the existence, but not the uniqueness, of weak solutions of the thin-film equation (1.4) in the complete wetting regime is well understood, see for example [7,8].

In all our results, $k$ is assumed to be an integer. In particular, this allows us to express the norms $\| \cdot \|_{X^k}$ in physical variables. However, in most parts of the proof, we only use the characterization in frequency variables and we expect therefore that our results can be generalized to the case of general real $k \geq 1$. In order to generalize the results, some care has to be taken when dealing with the polynomial parts of $f$ and $p$.

We believe that the techniques developed in this paper can also be applied to more complicated systems such as the Stokes flow with various slip conditions at the liquid–solid interface or for fluid models where the contact angle condition at the triple point is different.

Notation In the following, we do not explicitly write $k$-dependence of constants, that is we write $C = C_k$.

3.2. Formal Lubrication Approximation

In this section, we show formally how the Darcy flow (2.18) converges to the thin-film equation (2.24). Indeed, we show, formally, convergence of both the linear and nonlinear operator in (2.18), that is $A_\varepsilon \to A_0$ and $N_\varepsilon \to N_0$ as $\varepsilon \to 0$. The argument is based on an asymptotic expansion of the $\varepsilon$-dependent pressure $p_\varepsilon$ in terms of $\varepsilon y$, that is

$$p_\varepsilon(t, x, y) = p_0(t, x) + \varepsilon^2 p_2(t, x, y) + O(\varepsilon^4).$$

Our aim is to solve (2.7) up to first order in $\varepsilon$, that is

$$\begin{cases}
\partial_x^2 p_\varepsilon + (1 + \varepsilon \partial_y)^2 p_\varepsilon = O(\varepsilon^2) & \text{in } \Omega, \\
p_\varepsilon|_{\Gamma} = f_x, \quad p_y|_{\Gamma_0} = 0.
\end{cases} \tag{3.4}$$

Indeed, the solution of (3.4) has the asymptotic expansion

$$p_\varepsilon(t, x, y) = f_x - \frac{\varepsilon^2}{2} \left(y^2 - h^2\right) f_{xxx} + O(\varepsilon^4). \tag{3.5}$$

Inserting this asymptotic expansion into (2.7), we obtain

$$B^p_\varepsilon f_x(t, x) = B^h_\varepsilon f_x(t, x) \overset{(2.7)}{=} -h_x f_{xx} - h f_{xxx} + O(\varepsilon^2) = -(h f_{xx})_x + O(\varepsilon^2) \tag{3.6}$$
where \( h = x + \int_0^x \varphi \). The asymptotic expression of the linear operator \( A_\varepsilon \) follows as a special case of (3.6) by setting \( \varphi = 0 \) or equivalently \( h = x \):

\[
A_\varepsilon f \overset{(2.19)}{=} -(B_\varepsilon f)_x \overset{(3.6)}{=} (xf_{xx})_{xx} + \mathcal{O}(\varepsilon^2) \overset{(2.25)}{=} A_0 f + \mathcal{O}(\varepsilon^2), \tag{3.7}
\]

which implies \( A_\varepsilon \to A_0 \). The argument for the convergence \( N_\varepsilon \to N_0 \) proceeds as follows. With the notation \( \Phi = \int_0^x \varphi \ dx = h - x \), we have

\[
N_\varepsilon(\varphi, f) \overset{(2.20)}{=} \varphi_x f_{xx|x=0} + (B_\varepsilon^0 f_x)_x - (B_\varepsilon f_x)_x + \mathcal{O}(\varepsilon^2) \tag{3.8}
\]

\[
\overset{(3.6)}{=} \Phi_{xx} f_{xx|x=0} - (hf_{xx})_{xx} + (xf_{xx})_{xx} + \mathcal{O}(\varepsilon^2) \tag{3.9}
\]

\[
= (\Phi f_{xx|x=0})_{xx} - (\Phi f_{xx})_{xx} + \mathcal{O}(\varepsilon^2) \tag{3.10}
\]

\[
\overset{(2.25)}{=} N_0(\varphi, f) + \mathcal{O}(\varepsilon^2), \tag{3.11}
\]

which formally shows that \( N_\varepsilon \to N_0 \). Equations (3.7) and (3.8) together formally show that solutions of (2.18) converge to solutions of (2.24) in the limit \( \varepsilon \to 0 \).

### 4. Proof of Theorems 1–4

In this section, we give an overview of the proof of Theorems 1–4.

#### 4.1. Proof of Theorem 1

The proof of Theorem 1 proceeds by an application of a contraction principle. It is based on maximal regularity for the linear operator \( A_\varepsilon \) and corresponding bounds for the nonlinear operator \( N_\varepsilon \). In the following, we drop the superscript \( \varepsilon \) in the notation if the meaning is clear from the context, that is we use, for example, the notation \( f = f^\varepsilon, p = p^\varepsilon \), and so forth.

We will use the following maximal regularity estimate on the linear parabolic operator:

**Proposition 1.** Let \( k \geq 1 \) be an integer and let \( \varepsilon \in (0, \frac{\pi}{4(2k+1)}) \). Suppose that \( f_{\text{in}} \in X_{\varepsilon}^{k+1/2}((0, \infty)) \) and \( g \in TX_{\varepsilon}^k \) satisfy (2.50). Then there is a unique global in time solution \( f^\varepsilon \in TX_{\varepsilon}^{k+1} \) of

\[
\begin{cases}
  f_t + A_\varepsilon f = g & \text{for } (t, x) \in (0, \infty)^2, \\
  f = 0 & \text{for } x = 0, \\
  f = f_{\text{in}} & \text{for } t = 0.
\end{cases} \tag{4.1}
\]

Furthermore, for all \( \tau > 0 \), we have

\[
\| f \|_{TX_{\varepsilon}^{k+1}(Q_{\tau})} \leq C \left( \| g \|_{TX_{\varepsilon}^k(Q_{\tau})} + \| f_{\text{in}} \|_{X_{\varepsilon}^{k}} + \| f \|_{C^0_t L^2_x(Q_{\tau})} \right), \tag{4.2}
\]

where the constant \( C > 0 \) does not depend on \( \varepsilon, \tau > 0 \) and where \( Q_{\tau} = (0, \tau) \times (0, \infty) \).
The proof of Proposition 1 is given in [27, Theorem 2.4]. Note that there is an extra term $\| f \|_{C^0_t L^2_x(Q)}$, on the right-hand side of (4.2), compared with the estimate in [27] due to the slightly different definition of the norm $\| \cdot \|_{TX^{k+1}_\varepsilon(Q)}$.

Our main estimate on the nonlinear operator is stated in the following proposition:

**Proposition 2.** Let $k \geq 1$ be an integer and let $\varepsilon \in (0, \frac{\pi}{3(2k+1)})$. Suppose $f_i \in TX^{k+1}_\varepsilon$ with $\| f_i \|_{TX^{k+1}_\varepsilon} \leq 1$ for $i = 1, 2$. Then for all $\tau > 0$, we have

$$
\| N(\varepsilon f_1) \|_{TX^{k+1} \varepsilon(Q_\tau)} + \| N(\varepsilon f_2) \|_{TX^{k+1} \varepsilon(Q_\tau)} \\
\leq C \| f_1 \|_{TX^{k+1} \varepsilon(Q_\tau)} + \| f_2 \|_{TX^{k+1} \varepsilon(Q_\tau)},
$$

(3.3)

where the constant $C > 0$ does not depend on $\varepsilon, \tau > 0$ and where $Q_\tau = (0, \tau) \times (0, \infty)$.

The proof of Proposition 2 is given in Section 6.

The proof of Theorem 1 now follows by application of a contraction argument. For $\delta > 0$, $\tau > 0$, to be fixed later, we set

$$
\mathcal{E} := \left\{ f \in TX^{k+1}_\varepsilon(Q_\tau) : \| f \|_{TX^{k+1}_\varepsilon(Q_\tau)} < \delta \quad \text{and} \quad \partial^l_t f_{|t=0} = 0, \quad 0 \leq l \leq k \right\}.
$$

(4.4)

We define $S(\varepsilon f)$ for any $f \in \mathcal{E}$ by requiring that $\tilde{f}$ in $S(\varepsilon f)$ is the solution of the linear equation (4.1) with the initial data $\tilde{f}$ in and the right hand side given by $N(\varepsilon \tilde{f} + f)$, that is

$$
\begin{align*}
\left\{ \begin{array}{ll}
(\tilde{f} + S(\varepsilon f))_t + A(\varepsilon \tilde{f} + S(\varepsilon f)) = N(\varepsilon \tilde{f} + f) & \quad \text{for } (t, x) \in Q_\tau, \\
S(\varepsilon f) = 0 & \quad \text{for } x = 0, \\
S(\varepsilon f) = 0 & \quad \text{for } t = 0.
\end{array} \right.
\end{align*}
$$

(4.5)

See (2.48) for the definition of $\tilde{f}$ in. Let us note that due to our choices of $\tilde{f}$ in and of the set $\mathcal{E}$, it is not difficult to check that the compatibility conditions (2.50) of the linear system (4.1) are satisfied and hence $\varphi = \tilde{f} + S(\varepsilon f)$ satisfies

$$
\| \varphi \|_{TX^{k+1}_\varepsilon(Q_\tau)} \leq C \left( \| N(\varepsilon \tilde{f} + f) \|_{TX^{k+1}_\varepsilon(Q_\tau)} + \| f_{\| \cdot \|_{TX^{k+1}_\varepsilon(Q_\tau)}} \right)
$$

(4.6)

where we use the notation $\| \cdot \|_{TX^{k+1}_k} := \| \cdot \|_{TX^{k+1}_\varepsilon(Q_\tau)}$. By standard interpolation, we have for any $f \in \mathcal{E}$,

$$
\| f \|_{C^0_t L^2_x(Q_\tau)} \leq C \tau \| f \|_{TX^{k+1}_\varepsilon(Q_\tau)}.
$$

(4.7)

Hence, there is a small but constant $\tau_0$ such that for $\tau = \tau_0$ we can absorb the last term on the right hand side of (4.6) to get

$$
\| \varphi \|_{TX^{k+1}_\varepsilon(Q_\tau)} \leq C \left( \| N(\varepsilon \tilde{f} + f) \|_{TX^{k+1}_\varepsilon(Q_\tau)} + \| f_{\| \cdot \|_{TX^{k+1}_\varepsilon(Q_\tau)}} \right)
$$

(4.8)
(increasing the constant in the estimate by some universal factor), where we also used (2.49). Here, \( \tau_0 \) only depends on the constant \( C \) appearing in (4.2).

Let \( f_1, f_2 \in \mathcal{E} \) and let \( f := f_1 - f_2 \). In particular, \( S_\varepsilon(f_1) - S_\varepsilon(f_2) \) solves (4.1) with vanishing initial data and the right hand side \( N_\varepsilon(f_{\text{in}} + f_1) - N_\varepsilon(f_{\text{in}} + f_2) \). Hence,

\[
\|S_\varepsilon(f_1) - S_\varepsilon(f_2)\|_{TX^k(Q, t)} \leq C \|N_\varepsilon(f_{\text{in}} + f_1) - N_\varepsilon(f_{\text{in}} + f_2)\|_{(TX^k_t \cap L^2(X^0_t))(Q, t)}. \tag{4.8}
\]

By (4.9) and in view of (4.3), we thus get

\[
\|S_\varepsilon(f_1) - S_\varepsilon(f_2)\|_{TX^k(Q, t)} \leq C\|f_1 - f_2\|_{TX^k(Q, t)} \leq C(\delta + \alpha)\|f_1 - f_2\|_{TX^k(Q, t)}. \tag{4.9}
\]

Hence, \( S_\varepsilon \) is a contraction if \( \alpha, \delta > 0 \) and \( \tau \) are chosen sufficiently small. Similarly, by (4.9), (4.3) and since \( N_\varepsilon(0) = 0 \), we get

\[
\|S_\varepsilon(f_1)\|_{TX^k(Q, t)} \leq C\|N_\varepsilon(f_{\text{in}} + f_1)\|_{(TX^k_t \cap L^2(X^0_t))(Q, t)} + C\alpha \leq C\alpha + C\delta^2 \tag{4.3}
\]

and hence \( S_\varepsilon(\mathcal{E}) \subseteq \mathcal{E} \) for \( \delta \) and \( \alpha = \bar{\alpha}(\delta) \) sufficiently small. Therefore, application of Banach’s fix-point theorem yields the existence and uniqueness of a solution of (2.18) on the time interval \((0, \tau)\). In order to show that the profile \( h \) is positive to the right of the contact point, we use the dissipation of energy instead of (4.7). This is possible to do now since we know that \( f \) solves the full nonlinear system in the time interval \((0, \tau)\). By the dissipation of energy estimate (see the Appendix) and by standard interpolation, we have for sufficiently small \( \alpha \) and for all \( t \in (0, \tau) \),

\[
\|f\|_{L^\infty(Q, t)} \leq C\sup_{t \in (0, \tau)}(E_\varepsilon(t) + \|f_{\text{in}}\|_{L^2}) \leq C\left(E_\varepsilon(0) + \|f_{\text{in}}\|_{X^{1/2}}\right) \leq C\alpha^2 < \frac{1}{2}.
\]

In particular, the profile function is monotonically increasing and positive to the right of the contact point. Furthermore, the calculations in the Appendix in this case also show that

\[
\|f\|_{C^0_t L^2(Q, t)} \leq C\varepsilon_\alpha(t) \leq C\varepsilon_\alpha(0) \leq C\alpha. \tag{4.10}
\]

Note that the constant \( C \) in the estimate above is independent of the time interval.

Using the estimate (4.10), we can get estimates that are independent of the time interval. Indeed, we can assume that \( \delta \) and \( \alpha = \bar{\alpha}(\delta) \) are chosen even smaller in such
a way that \( C\alpha + C\delta^2 \leq \tilde{\alpha}(\delta) \). Hence, we deduce that the solution \( \varphi = \tilde{f}_m + S_\varepsilon(f) \), constructed on the time interval \([0, \tau]\), satisfies the estimate

\[
\|\varphi\|_{TX^{k+1}(\tau)} \leq \tilde{\alpha}(\delta).
\]

Hence, we can use this again as initial data for our problem and construct a solution on the time interval \([\tau, 2\tau]\). Therefore, we have a solution on the time interval \([0, 2\tau]\). The important point now is that all the estimates do not depend on the length of the interval. Using the same estimate as above on the interval \([0, 2\tau]\) it is easy to see that the solution at time \( t = 2\tau \) satisfies the same bound as the solution at time \( t = \tau \), that is \( \|\varphi_{2\tau}\|_{TX^{k+1}_\varepsilon} \leq \tilde{\alpha}(\delta) \). Iteratively, we obtain a uniform bound for the solution, globally in time. This shows the long-time existence and also the uniform bounds, by (5.2) and by (5.12), we have

\[
\text{proof of (3.2).}
\]

4.2. Proof of Theorem 2

By Theorem 1, we have the uniform bound \( \|f^\varepsilon\|_{TX^{k+1}_\varepsilon} \leq C\alpha \) for all \( \varepsilon > 0 \). We put the optimal decomposition into high and low frequencies \( f^\varepsilon = f^\varepsilon_+ + f^\varepsilon_- \) from (5.6). By (5.6), this decomposition commutes with the time derivative. We have uniform bounds on the norms \( \|x^{2j+1} \partial^j_x \partial^{j+1}_x f^\varepsilon_+\|_{L^2((0,\infty)^2)} \) and \( \frac{1}{\varepsilon} \| x \partial^i_x \partial^{i+1}_x f^- \|_{L^2((0,\infty)^2)} \) for all \( i, j \) with \( i + j \leq k + 1 \). Let \( \zeta_1, \zeta_2 \in C^\infty_c((0, \infty)) \) with \( \text{supp } \zeta_2 = 1 \) on \( \text{supp } \zeta_1 \). The standard Lions–Aubin compactness theorem (see [42]) applied to both \( f^\varepsilon_+ \) then shows that, in particular, there is \( f = f^0_+ + 0 \in TX^k_0 \) and a subsequence \( \varepsilon_j \) to \( 0 \) as \( \varepsilon \rightarrow 0 \) such that

\[
\|(f^\varepsilon_j - f)\zeta_1\|_{TX^k_j} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \tag{4.11}
\]

Now, let \( p^\varepsilon_i, p^\varepsilon_j \) be the pressure related to \( f^\varepsilon_i \) and \( f^\varepsilon_j \). By (6.44), we then have

\[
\|(p^\varepsilon_i - p^\varepsilon_j)\zeta_1\|_{TY^k_j} \leq C\|(f^\varepsilon_i - f^\varepsilon_j)\zeta_2\|_{TX^k_j} \left(\|f^\varepsilon_i\zeta_2\|_{TX^k_{j,\text{loc}}} + \|f^\varepsilon_j\zeta_2\|_{TX^k_{j,\text{loc}}}\right) \leq \tag{4.11}
\]

\[
C\alpha\|(f^\varepsilon_i - f^\varepsilon_j)\zeta_2\|_{TX^k_j} \rightarrow 0
\]

as \( i, j \rightarrow \infty, j \leq i \). This shows that \( p^\varepsilon_j \) converges in \( TY^k_j \) thus concluding the proof of (3.2).

It remains to show that \( f \) solves the thin-film equation (2.24). By the above uniform bounds, by (5.2) and by (5.12), we have \( f^\varepsilon \rightarrow f \) in \( L^2(H^4) \), \( f_i^\varepsilon \rightarrow f_i \) in \( L^2(L^2) \) and \( p^\varepsilon \rightarrow p \) in \( L^2(H^1) \) with \( \|p^\varepsilon\|_{L^2(L^2)} \leq C\varepsilon \rightarrow 0 \). The boundary condition \( p^\varepsilon = h_{xx}(1 + \varepsilon^2 h_x^2)^{-\frac{3}{2}} \) thus implies that in the limit \( \varepsilon = 0 \), we get \( p = h_{xx} \). We will show that in the limit \( \varepsilon = 0 \), \( h \) is a solution of the thin-film equation, where we recall that \( f^\varepsilon = h^\varepsilon - 1 \). In particular, \( h^\varepsilon \rightarrow h \) in \( L^2(H^4) \) and \( h^\varepsilon_i \rightarrow h_i \) in \( L^2(L^2) \). Correspondingly, we also have convergence of the velocity \( u^\varepsilon = (v^\varepsilon, w^\varepsilon) = \nabla_x p^\varepsilon \rightarrow u = (v, w) = (p_x, 0) \) in \( L^2(L^2) \). The transition to
the limit can now be conveniently done in terms of the continuity equation. By conservation of mass for (2.18), we have

$$h^\varepsilon_t + \left( \int_0^{h^\varepsilon(t,x)} v^\varepsilon \, d\tilde{x} \right)_x = 0.$$  \hspace{1cm} (4.12)

In the limit $\varepsilon \to 0$ and in view of the above discussion, (4.12) turns into

$$h_t + \left( \int_0^{h(t,x)} v \, d\tilde{x} \right)_x = h_t + (h p_x)_x = h_t + (h h_{xxx})_x = 0.$$  \hspace{1cm} (4.13)

4.3. Proof of Theorem 3

We prove this theorem by an application of the inverse function theorem. For this, we linearize $L^\varepsilon$ at an ‘approximate solution’ $w$, where $w \in TX^{k+1}_\varepsilon$ satisfies

$$\partial_{\varepsilon t}^l w|_{t=0} = \partial_{\varepsilon t}^l f_{\text{in}}|_{t=0}$$

for $l = 1, \ldots, k$, see (2.47). We then show boundedness and differentiability for $L^\varepsilon$ and invertibility and maximal regularity for its linearization $\delta L^\varepsilon w$ at $w$. We keep the details brief and refer to similar arguments in [3, 14, 21].

For $f_{\text{in}} \in X^{k+1/2}_\varepsilon$, let $g_{\text{in}} = -A^\varepsilon f_{\text{in}}$ and let $w \in TX^{k+1}_\varepsilon([0, \infty) \times E)$ satisfy for $1 \leq l \leq k$,

$$\partial_{\varepsilon t}^{l-1} L^\varepsilon(w) = 0 \hspace{0.5cm} \text{for} \hspace{0.5cm} t = 0$$  \hspace{1cm} (4.14)

and such that the estimate

$$\|w\|_{TX^{k+1}_\varepsilon} \leq C \|f_{\text{in}}\|_{X^{k+1/2}_\varepsilon}$$  \hspace{1cm} (4.15)

holds. The existence of such a function $w$ follows from (2.52)–(2.53). In particular,

$$L^\varepsilon(w)|_{t=0} = D\partial_{\varepsilon t} w|_{t=0} + A^\varepsilon f_{\text{in}} = g_{\text{in}} - g_{\text{in}} = 0$$  \hspace{1cm} (4.16)

where $L^\varepsilon$ is defined in (2.9). In this sense, $w$ may be called an approximate solution. Let $\delta L^\varepsilon(w)$ be the linearization of $L^\varepsilon$ around $w$. We define the operator

$$\mathcal{M}_\varepsilon : TX^{k+1}_\varepsilon(Q_\tau) \to X^{k+1/2}_\varepsilon(E) \times TX^k_\varepsilon(Q_\tau) \hspace{0.5cm} \text{with} \hspace{0.5cm} f \mapsto \mathcal{M}_\varepsilon(f) := (f|_{t=0}, L^\varepsilon f).$$

We have boundedness and differentiability of $L^\varepsilon$ and boundedness of $(\delta \mathcal{M}(w))^{-1}$ for $\tau$ small enough.

**Proposition 3.** Let $k \geq 1$, $k \in \mathbb{N}$ and suppose that $f_{\text{in}} \in X^{k+1/2}_\varepsilon(E)$ with $\int_E f_{\text{in}} \, dx = 0$ and $g \in TX^k_\varepsilon(Q_\tau)$ satisfy (2.50). Let $w \in X^{k+1}_\varepsilon([0, \infty) \times E)$ be defined as in (4.14). Then for sufficiently small $\tau > 0$ and with the notation $Q_\tau = (0, \tau) \times E$, there exists a unique $f \in TX^{k+1}_\varepsilon(Q_\tau)$, solution of $\delta L^\varepsilon(w) f = g$ in $Q_\tau$ if $f = 0$ on $(0, \tau) \times \partial E$ and with initial data $f = f_{\text{in}}$. Furthermore,

$$\|f\|_{TX^{k+1}_\varepsilon(Q_\tau)} \leq C \left( \|g\|_{TX^k_\varepsilon(Q_\tau)} + \|f_{\text{in}}\|_{X^{k+1/2}_\varepsilon(E)} \right).$$

\hspace{1cm} (4.17)
Note that the condition \( \int_0^1 f(x) \, dx = 0 \) is preserved by the flow generated by \( \delta \mathcal{L}_\varepsilon(w) \cdot f \). We also have differentiability of \( \mathcal{M}_\varepsilon \) in a neighborhood of \( w \).

**Proposition 4.** Suppose that \( f_{\text{in}} \in X^{k+1/2}_\varepsilon(E) \) satisfies (2.47) and \( \int_E f_{\text{in}} \, dx = 0 \) and let \( w \) be defined as in (4.14). Then for sufficiently small \( \tau > 0 \) there is \( \alpha > 0 \) such that \( \mathcal{M}_\varepsilon : TX^{k+1}_\varepsilon(Q_\tau) \to X^{k+1/2}_\varepsilon(E) \times TX^k_\varepsilon(Q_\tau) \) is bounded and continuously differentiable in the \( \alpha \)-neighborhood of \( w \) in \( TX^{k+1}_\varepsilon(Q_\tau) \).

The proof of the above two propositions is given in Section 7. Using the above two propositions, the proof of Theorem 3 follows by application of the inverse function theorem. We claim that the operator \( \mathcal{M}_\varepsilon \) is bounded, continuously differentiable near \( w \) and \( \delta \mathcal{M}_\varepsilon(w) \) is invertible with bounded inverse for \( \tau \) small enough.

We define \( v := \mathcal{L}_\varepsilon(w) \). By the inverse mapping theorem there is a neighborhood of \( w \) and a neighborhood of \( (f_{\text{in}}, v) \) where \( \mathcal{M}_\varepsilon \) is a diffeomorphism. By (4.16) we have \( v|_{t=0} = 0 \). It follows that \( \|v \|_{TX^k_\varepsilon(Q_\tau)} \to 0 \) for \( \tau \to 0 \). Hence, there is \( \tilde{\tau} \in (0, \tau) \) and a function \( \tilde{v} \in Y_\tau \) with \( \tilde{v} = 0 \) for \( t \in (0, \tilde{\tau}) \) and such that \( (0, \tilde{v}) \) is sufficiently near \( (0, v) \). Hence, there is \( f \in TX^k_\varepsilon(Q_\tau) \) with \( \mathcal{M}_\varepsilon(f) = (0, \tilde{v}) \). The function \( f \) is a solution of \( \mathcal{L}_\varepsilon(f) = 0 \) and hence \( h(x) = x(1-x) + \int_0^x f(x') \, dx' \) is a solution of (2.12) for \( t \in (0, \tilde{\tau}) \), thus concluding the proof of Theorem 3.

### 4.4. Proof of Theorem 4

By Theorem 2, we have the existence and regularity of solutions of (2.12) for initial data which are close to the infinite wedge. In order to show Theorem 4(1) it thus remains to prove the uniqueness of solutions. For this, it is enough to show that the corresponding results in Propositions 1 and 2 also hold in the case \( \varepsilon = 0 \) and for the operators \( A_0 \) and \( N_0 \). The estimate for existence, regularity and uniqueness in the case of half-space then follows by the same fix-point argument as for Theorem 1.

The \( \varepsilon = 0 \) version of Proposition 1 has been proved in [27]. The \( \varepsilon = 0 \)-version of Proposition 2 can be obtained by analogous estimates as the ones applied in the proof of Proposition 8.1 in [22]. In fact, the estimate is easier in our situation since we only need an estimate in weighted Sobolev spaces, while the estimates in [22] are based on interpolation spaces. Finally, we note that the local result in Theorem 4 can be obtained by standard localization techniques. The argument can be performed analogously to our localization argument in Section 7; the argument is easier since for \( \varepsilon = 0 \), the norms \( \|\cdot\|_{X^k_\varepsilon} \) are local. We also refer to a similar localization argument for a thin-film equation in [21], in the setting of Hölder norms.

### 5. Estimates in Weighted Spaces

We first state some basic estimates for the spaces \( X^k_\varepsilon \):

**Proposition 5.** Let \( k \in [0, \infty) \).
1. For $0 \leq \varepsilon \leq \varepsilon'$, we have $X^k_\varepsilon \subseteq X^k_\varepsilon$ and
   \[ [f]_{X^k_\varepsilon} \leq [f]_{X^k_\varepsilon} \quad \forall f \in X^k_\varepsilon. \tag{5.1} \]

2. Let $\ell_1, \ell_2 \in \mathbb{N}_0$ with $0 \leq \ell_1 \leq 3k$ and $0 \leq \ell_2 < 3k - \frac{1}{2}$. Then we have $X^k_\varepsilon \subseteq W^{\ell_1,2}((0, \infty)) \cap C^{\ell_2}((0, \infty))$ and
   \[ \| \partial_x^{\ell_1} f \|_{L^2} + \| \partial_x^{\ell_2} f \|_{L^\infty} \leq C \| f \|_{X^k_\varepsilon} \quad \forall f \in X^k_\varepsilon. \tag{5.2} \]

3. If $f \in X^k_\varepsilon$, then $\partial_x f, \partial_x^2 f, \partial_x^3 f \in X^{k-1}_\varepsilon$ and furthermore
   \[ \| f_x \|_{X^k_{\varepsilon-1/2}} + \| f_{xx} \|_{X^k_{\varepsilon-1}} + \| f_{xxx} \|_{X^k_{\varepsilon-1}} \leq C \| f \|_{X^k_\varepsilon}. \tag{5.3} \]

The norm $\| \cdot \|_{X^k_\varepsilon}$ controls a scale of weighted Sobolev spaces of fractional order:

**Lemma 1.** Let $\varepsilon \in [0, 1)$, $k \in [0, \infty)$ and let $f_0 \in X^k_\varepsilon$. Then for every $0 \leq \ell \leq k$, we have
   \[ \left\| \lambda^{3\ell+1} \mu^\ell f_0 \right\|_{L^2(\mathbb{T}^3; \mathbb{C})} + \left\| f_0 \right\|_{L^2(\mathbb{T}^3; \mathbb{C})} \leq C \| f_0 \|_{X^k_\varepsilon}. \tag{5.4} \]

We have the following characterization of the homogeneous norms (2.34):

**Lemma 2.** Let $k \in \mathbb{N}_0$, $\varepsilon \in [0, \frac{\pi}{3(2k+1)})$. Then for $f_0 \in X^k_\varepsilon$ we have
   \[
   c[f_0]_{X^k_\varepsilon} \leq \inf_{f_0 = f_++f_-} \left( \left\| \lambda^{4k+1} f_+ \right\|_{L^2(\mathbb{T}^3; \mathbb{C})} + \left\| (\frac{1}{\varepsilon})^{k+1} f_- \right\|_{L^2(\mathbb{T}^3; \mathbb{C})} \right)
   \leq C[f_0]_{X^k_\varepsilon}, \tag{5.5}
   \]

where $\beta_k = 3k - \frac{1}{2}$. Up to multiplication by a constant that only depends on $k$, for any even integer $M$ with $M \geq k$ the infimum in (5.5) is achieved by
   \[ \widehat{f}_+(\lambda) = (1 - (i \tan))^{M(\varepsilon \lambda)} f_0(\lambda) \quad \text{and} \quad \widehat{f}_-(\lambda) = (i \tan)^{M(\varepsilon \lambda)} f_0(\lambda). \tag{5.6} \]

Furthermore,
   \[ |\lambda|^k |\widehat{f}_+(\lambda)| \leq C \mu^k |\widehat{f}(\lambda)| \quad \text{and} \quad (\frac{1}{\varepsilon})^{k} |\widehat{f}_-(\lambda)| \leq C \mu^k |\widehat{f}(\lambda)|, \tag{5.7} \]

where $\mu = \min\{|\lambda|, \frac{1}{\varepsilon}\}$. In particular, the homogeneous norms (2.34) and (1.7) are equivalent for $k \in \mathbb{N}_0$ and functions in the space $X^k_\varepsilon$.

The proof of Proposition 5, Lemmas 1 and 2 is given in [27, Lemma 3.3].

The next result shows that the space $X^k_\varepsilon$ is an algebra for $k \geq 1$. The algebra property of $X^k_\varepsilon$ for $k \geq 2$ has been shown in [27, Proposition 2.3] with a different proof.

**Lemma 3.** For $k \in \mathbb{N}$ with $k \geq 1$ and $f, g \in X^k_\varepsilon$, we have $fg \in X^k_\varepsilon$ and
   \[ \| fg \|_{X^k_\varepsilon} \leq C \| f \|_{X^k_\varepsilon} \| g \|_{X^k_\varepsilon}. \tag{5.8} \]
Proof. We decompose \( f = f_1 + f_0 \) and \( g = g_1 + g_0 \) where \( f_1 = \mathcal{P}_f \xi, g_1 = \mathcal{P}_g \xi \) and where \( \mathcal{P}_f, \mathcal{P}_g \) are the Taylor polynomials of \( f, g \) of order \( 3k - 1 \). The cut-off function \( \xi \) is the same as in the definition of the norm (2.33). In particular, we have \( f_0, g_0 \in X^k_\varepsilon \). We need to estimate the products \( f_1g_1, f_0g_1, f_1g_0 \) and \( f_0g_0 \). Clearly, we have

\[
\| f_1g_1 \|_{X^k_\varepsilon} \leq C \| f_1 \|_{X^k_\varepsilon} \| g_1 \|_{X^k_\varepsilon}.
\]

As in (5.7), we decompose \( f_0 = f_+ + f_- \) and \( g_0 = g_+ + g_- \). The mixed term \( f_1g_0 = f_1g_+ + f_1g_- \) is estimated as follows

\[
[f_1g_0]_{X^k_\varepsilon} \leq C \| f \|_{X^k_\varepsilon} \| g \|_{X^k_\varepsilon}.
\]

For the first estimate, we use the equivalence of the norms (1.7) and (2.34), stated in Lemma 2. For the second estimate, we note that all derivatives of \( f_1 = \mathcal{P}_f \xi \) are supported in \( (\frac{1}{8}, \frac{1}{2}) \) and that \( x \) is of order 1 in this interval. The estimate then follows by application of (5.2) and Hardy’s inequality. The third estimate follows again by application of Lemma 2. The estimate of \( f_0g_1 \) proceeds analogously.

It remains to show the estimate for the product \( f_0g_0 = f_+g_+ + f_+g_- + f_-g_+ + f_-g_- \). Since \( f_0, g_0 \in X^k_\varepsilon \), we have in particular \( \| f_0 \|, \| g_0 \| \leq Cx^{3k-1} \) for \( x \leq C \). For \( k \geq 1 \), it follows that \( \| f_0g_0 \| \leq Cx^{6k-2} \leq Cx^{3k-2} \) and hence \( f_0g_0 \in X^k_\varepsilon \).

We show the estimate for the high-frequency product \( f_-g_- \). Recall that the Mellin transform for the product of functions can be expressed as a convolution of the Mellin transformed functions, that is

\[
[f_-g_-]_{X^k_\varepsilon} \leq C_k \sum_{i=0}^{3k+1} I_{k,i},
\]

where \( \mu = \min\{\frac{1}{\varepsilon}, |\lambda|\} \) and where \( \alpha \in \mathbb{R} \) is chosen such that the product \( \hat{f}_-(\lambda \cdot \cdot) \hat{g}_-(\cdot) \) is absolutely integrable on the line \( \Re \eta = \alpha \). Since \( \mu \leq \frac{1}{\varepsilon}, \) the right-hand side of (5.9) can be estimated by replacing \( \mu \) by \( \frac{1}{\varepsilon} \). Using the binomial formula \( \lambda^{3k+1} = \sum c_{ki} \eta^i (\lambda - \eta)^{3k+1-i} \) and by (5.9), we have

\[
[f_-g_-]_{X^k_\varepsilon} \leq C_k \sum_{i=0}^{3k+1} I_{k,i},
\]

where

\[
I_{k,i} := \int_{\Re \lambda = 3k-\frac{1}{2}} \int_{\Re \eta = \alpha} \left( \frac{1}{\varepsilon} \right)^k (\lambda - \eta)^{3k+1-i} \hat{f}_-(\lambda - \eta) \eta^i \hat{g}_-(\eta) \ d\lambda(\eta) \ d\eta(\lambda).
\]

(5.10)
By symmetry, it is enough to estimate the terms $I_{k,i}$ with $i \leq \frac{3}{2}k + \frac{1}{2}$. Note that the integrand of the inner integral above is analytic as a function of $\eta$. In particular, the value of the inner integral in (5.10) does not depend on $\alpha$. Note that it is essential for this argument to work that we have avoided replacing $\lambda$ by $|\lambda|$ in our proof. By Young’s inequality for convolutions and by the Cauchy-Schwarz inequality, we have

$$
\left\| \hat{F} \ast \hat{G} \right\|_{L^2(\|\lambda\|=\beta)} \leq C_{\delta} \left( 1 + |\lambda|^{\delta} \right) \left\| \hat{F} \right\|_{L^2(\|\lambda\|=\beta_1)} \left\| \hat{G} \right\|_{L^2(\|\lambda\|=\beta_2)},
$$

(5.11)

which holds for all $\delta > \frac{1}{2}$. With the choice $\alpha = i$ and $\delta = 1$, we get

$$
I_{k,i} \leq C \left( \frac{1}{\beta} \right)^{k - \frac{i}{2}} |\lambda|^{3k - i - \frac{1}{2}} \left\| \hat{f} \right\|_{L^2(\|\lambda\|=3k - i - \frac{1}{2})} \left( \frac{1}{\beta} \right)^{\frac{i}{2}} \left( 1 + |\lambda|^{i+1} \right) \left\| \hat{g} \right\|_{L^2(\|\lambda\|=i)}
$$

(5.12)

The last estimate follows from (5.4) using $0 \leq i \leq \frac{3}{2}k + \frac{1}{2}$ and since $k \geq 1$. The estimate of the terms $f_+g_-, f_-g_+$ and $f_+g_+$ proceeds analogously (see also the related proof in [27, Proposition 2.3]), thus concluding the proof of the lemma. \(\square\)

We next show that also the space $Y^k_\varepsilon$ is embedded into classical Sobolev spaces. For any “multi-index” $\alpha \in [0, \infty) \times \mathbb{N}_0$, we set $|\alpha| = \alpha_1 + \alpha_2$. In the particular case $\alpha \in \mathbb{N}^2_0$, we also use the notation $D^\alpha = \partial^{\alpha_1}_x (\frac{1}{\varepsilon} \partial^{\alpha_2}_y)$.}

**Lemma 4.** Let $k \in [1, \infty)$, $\varepsilon \in (0, 1)$ and let $q \in Y^k_\varepsilon$. Then for all $\ell_1, \ell_2 \in \mathbb{N}_0$ with $\ell_1 < 3k - 1$ and $\ell_2 \leq 3k - \frac{3}{2}$, we have

$$
\sum_{0 \leq |\alpha| \leq \ell_1} \| D^\alpha_\varepsilon q \|_{L^\infty_\varepsilon L^2_\varepsilon(K)} + \sum_{0 \leq |\alpha| \leq \ell_2} \sup_{y \in (0,x)} \| D^\alpha_\varepsilon q \|_{L^\infty(K)} \leq C \| q \|_{Y^k_\varepsilon},
$$

(5.12)

where the sums are taken over multiindices $\alpha \in \mathbb{N}^2_0$.

**Proof.** (Proof of Lemma 4) Let $q = q_0 + q_1$ with $q_0 \in \tilde{Y}^k_\varepsilon$ be the decomposition as in (2.41). By definition, $q_1$ satisfies (5.12). It suffices to show the estimate (5.12) for $q_0 \in \tilde{Y}^k_\varepsilon$. By application of Plancherel’s identity, we have

$$
\sum_{|\alpha|=3\ell-1} \sup_{y \in (0,x)} \| D^\alpha_\varepsilon q_0 \|_{L^2((0,\infty))} \leq C \sum_{|\alpha|=3\ell-1} \sup_{y \in (0,1)} \left\| \partial^{\alpha_1}_\varepsilon (\frac{1}{\varepsilon} \partial^{\alpha_2}_y) \right\|_{L^2((0,\infty))}
$$

(2.41)

$$
\leq C \sum_{|\alpha|=3\ell-1} \sup_{y \in (0,1)} \| A^\alpha \hat{q}_0 \|_{L^2(\|\lambda\|=3\ell - \frac{3}{2})}
$$

$$
\leq C \| q_0 \|_{Y^k_\varepsilon}.
$$

This yields the $L^2$ estimate, the supremum estimate follows by standard interpolation. \(\square\)
For any $f \in TX_{x}^{k+1}$, defined for $t, x \in [0, \infty)^2$, its trace at $t = 0$ is well-defined in $X_{x}^{k+1/2}$:

**Lemma 5.** Let $k, i \in \mathbb{N}_0$. For $f \in TX_{x}^{k+1}$, we have

$$
\|\partial_i^j f\|_{C^0(X_{x}^{k+1/2})} \leq C\|\partial_i^{j+1} f\|_{L^2(X_{x}^{l+1})} + C\|\partial_i^j f\|_{L^2(X_{x}^{k+1})}. 
$$

(5.13)

**Proof.** It suffices to give the proof for $i = 0$ (for $i > 0$ consider $F = \partial_i^j f$ instead). By an approximation argument, it is enough to consider $f \in C_c(0, \infty)$. We decompose $f = f_0 + f_1$ where $f_1 = P_f \xi$ and where $P_f$ is the Taylor polynomial of order $n_{k+1/2} = 3k$ of $f$ at $x = 0$; in particular, $f_0 \in X_{x}^{k+1/2}$. In order to avoid fractional derivatives which appear in the definition of the norm for $X_{x}^{k+1/2}$, we use the equivalence

$$
c[f_0]_{X_{x}^{k+1/2}} \leq \langle A f_0, f_0 \rangle_{X_{x}^{k+1/2}} \leq C[f_0]_{X_{x}^{k+1/2}} \forall f_0 \in X_{x}^{k+1/2}.
$$

A proof of this equivalence is given for $k = 0$ in [27, Lemma 4.6], the argument used there also applies for general $k \in \mathbb{N}$. The estimate of the homogeneous part is then easy: for $f, g \in X_{x}^{k+1/2} \cap X_{x}^{k+1}$, we have $\langle A \varepsilon f_0, g_0 \rangle_{X_{x}^{k}} = \langle f_0, A \varepsilon g_0 \rangle_{X_{x}^{k}}$. Hence, integrating in time from infinity (where $f = 0$), we obtain

$$
sup_t \langle f_0 \rangle_{X_{x}^{k+1/2}} \leq C \sup_t \langle A \varepsilon f_0, f_0 \rangle_{X_{x}^{k}} \leq C \int_0^\infty |\langle A \varepsilon f_0, f_0 \rangle_{X_{x}^{k}}| \ dt \leq C [A \varepsilon f_0]_{L^2(X_{x}^{k})} |f_0|_{L^2(X_{x}^{k})}. 
$$

(5.14)

It remains to give the corresponding estimate for $f_1$, that is, we need to estimate the coefficients of the Taylor polynomial $P_f$. We show the estimate for the highest order coefficient of $P_f$; up to a constant it is given by $F(0)$ where $F := \partial^{3k} f$. We extend $F$ symmetrically as an even function defined for all $t, x \in \mathbb{R}$ by setting $F(t, -x) := F(t, x)$. We claim that

$$
sup_{t \in \mathbb{R}} |F(t)| \leq C \left( \|F_t\|_{L^2(\mathbb{R}^2)} + \|F_{xxx}\|_{L^2(\mathbb{R}^2)} + \|F\|_{L^2(\mathbb{R}^2)} \right). 
$$

(5.15)

Indeed, Equation (5.14) can be derived by taking the Fourier transform $\hat{F}(\eta, \xi)$ of $F(t, x)$

$$
sup_{t \in \mathbb{R}} |F(t)| \leq \left( \iint |\hat{F}|^2 \leq \left( \iint \frac{1}{(1 + |\eta|^2 + |\xi|^6)} \right)^{1/2} \left( \iint (1 + |\eta|^2 + |\xi|^6)|\hat{F}|^2 \right)^{1/2} \right). 
$$

(5.16)

The first integral on the right hand side is bounded. On the one hand, we have

$$
\int_{|\xi| \leq 1 \text{ or } |\eta| \leq 1} \frac{1}{1 + |\eta|^2 + |\xi|^6} \ d\eta \leq C < \infty.
$$

(5.15)

Also, with the coordinate transform $\xi^6 = \eta^2 \lambda^6$ and $d\xi = \eta^{1/3} d\lambda$,

$$
\int_1^\infty \int_1^\infty \frac{1}{|\eta|^2 + |\xi|^6} \ d\xi d\eta \leq C \int_1^\infty \lambda^{-5/3} \int_0^\infty \frac{1}{1 + |\lambda|^6} \ d\lambda d\eta \leq C < \infty.
$$

(5.16)

This concludes the proof of (5.14) and thus of the lemma. □
6. Uniform Estimates for the Operator in the Half-Space

6.1. Linear Pressure Estimates

We derive estimates for the pressure $p \in Y^k_\varepsilon$. Recall that, in general, the constants in the estimates may depend on $k$. All constants are, however, independent of $\varepsilon$.

**Proposition 6.** Let $k \in [0, \infty)$, $\varepsilon \in (0, \frac{\pi}{3(2k+1)})$. Then for any $f \in X^k_\varepsilon$ and any $g \in Z^k_\varepsilon$ there is a unique solution $p \in Y^k_\varepsilon$ of

$$\begin{align*}
\Delta p &= g \quad \text{in} \ K, \\
p &= f_x \quad \text{on} \ \partial_1 K, \\
p_y &= 0 \quad \text{on} \ \partial_0 K.
\end{align*}$$

(6.1)

Furthermore,

$$\|p\|_{Y^k_\varepsilon} \leq C(\|g\|_{Z^k_\varepsilon} + \|f\|_{X^k_\varepsilon}).$$

(6.2)

Before we give the proof of the Proposition in full generality, we first address the situation of homogeneous data $f \in \infty X^k_\varepsilon$ and $g \in \infty Z^k_\varepsilon$. In this case, we can apply the coordinate transform (2.39). In terms of the new variables (6.1) is equivalent to

$$\begin{align*}
\partial^2 u p + \left(\frac{1}{\varepsilon} \partial_v\right)^2 p &= e^{2u} g \quad \text{in} \ \mathbb{R} \times (0, 1), \\
p(u, \cdot) &= e^{-u} f(u) \quad \text{for} \ v = 1, \\
p_v(u, \cdot) &= 0 \quad \text{for} \ v = 0.
\end{align*}$$

(6.3)

By application of the Laplace transform (2.30) in terms of $u$, Equation (6.3) turns into

$$\begin{align*}
\lambda^2 \hat{\hat{p}}(\lambda, v) + \left(\frac{1}{\varepsilon} \partial_v\right)^2 \hat{\hat{p}}(\lambda, v) &= \hat{\hat{g}}(\lambda - 2, v) \quad \text{in} \ \mathbb{C} \times (0, 1), \\
\hat{\hat{p}}(\lambda, \cdot) &= (\lambda + 1) \hat{\hat{f}}(\lambda + 1) \quad \text{for} \ v = 1, \\
\hat{\hat{p}}_v(\lambda, \cdot) &= 0 \quad \text{for} \ v = 0.
\end{align*}$$

(6.4)

The model (6.4) can be explicitly solved if $g = 0$. This yields:

**Lemma 6.** Suppose that the assumptions of Proposition 6 hold with $f \in \infty X^k_\varepsilon$ and $g = 0$. Then there is a unique solution $q := p \in \infty Y^k_\varepsilon$ of (6.1). It’s Mellin transform satisfies

$$\hat{q}(\lambda, v) = \frac{\cos(\varepsilon \lambda v)}{\cos(\varepsilon \lambda)} (\lambda + 1) \hat{\hat{f}}(\lambda + 1).$$

(6.5)

Moreover,

$$\|q\|_{Y^k_\varepsilon} \leq C \|f\|_{X^k_\varepsilon}.$$  

(6.6)
Proof. It is easy to check that $\hat{q}$, defined by (6.5), indeed solves (6.4) with right hand side $g = 0$. In order to prove the estimate (6.6), we note that for $\varepsilon \in (0, \frac{\pi}{3(2k+1)})$, $\Re \lambda \in (-\frac{3}{2}, 3k - \frac{3}{2})$ and $v \in (0, 1)$, we have

$$\varepsilon |\Re \lambda| \leq \left| \frac{\pi (3k - \frac{3}{2})}{6k + 3} \right| \leq \frac{\pi}{2} - c_k, \quad (6.7)$$

for some $c_k > 0$ with $c_k \to 0$ for $k \to \infty$ [the $k$-dependence of the estimate (6.7) will be omitted in the sequel]. With the notation $\lambda = \beta + i \xi$ with $\beta, \xi \in \mathbb{R}$, it follows that

$$|\cos(\varepsilon \lambda)| = \frac{1}{2} |e^{i\varepsilon \beta - \varepsilon \xi} + e^{-i\varepsilon \beta + \varepsilon \xi}| = \frac{1}{2} e^{-\varepsilon \xi} |e^{2i\varepsilon \beta} + e^{2\varepsilon \xi}|, \quad (6.8)$$

$$|\sin(\varepsilon \lambda)| = \frac{1}{2} |e^{i\varepsilon \beta - \varepsilon \xi} - e^{-i\varepsilon \beta + \varepsilon \xi}| = \frac{1}{2} e^{-\varepsilon \xi} |e^{2i\varepsilon \beta} - e^{2\varepsilon \xi}|. \quad (6.9)$$

Using a Taylor expansion for small $|\lambda|$, we have $e^{2i\varepsilon \beta} - e^{2\varepsilon \xi} = 2(i\varepsilon \beta - \varepsilon \xi) + O(\varepsilon^2 |\lambda|^2)$. Hence, we get

$$c \leq |\cos(\varepsilon \lambda)| \leq C \quad \text{and} \quad c|\lambda| \leq |\sin(\varepsilon \lambda)| \leq C|\lambda| \quad \text{for} \quad |\varepsilon \lambda| \leq \frac{1}{10}, \quad (6.10)$$

for some constants which only depend on $k$. By (6.7), we also get

$$ce^{i|\varepsilon \lambda|} \leq |\sin(\varepsilon \lambda)| \leq C e^{i|\varepsilon \lambda|} \quad \text{and} \quad ce^{i|\varepsilon \lambda|} \leq |\cos(\varepsilon \lambda)| \leq C e^{i|\varepsilon \lambda|} \quad \text{for} \quad |\varepsilon \lambda| \geq \frac{1}{10}. \quad (6.11)$$

Similarly, one can show (check in particular the case $|\varepsilon \lambda v| \leq 1 \leq |\varepsilon \lambda|$),

$$c \leq |\cos(\varepsilon \lambda v)| \leq C \quad \text{and} \quad c|\lambda v| \leq |\sin(\varepsilon \lambda v)| \leq C|\lambda v| \quad \text{for} \quad |\varepsilon \lambda| \leq \frac{1}{10}, \quad (6.12)$$

$$ce^{i|\varepsilon \lambda v|} \leq |\cos(\varepsilon \lambda v)| \leq C e^{i|\varepsilon \lambda v|} \quad \text{and} \quad |\sin(\varepsilon \lambda v)| \leq C e^{i|\varepsilon \lambda v|} \quad \text{for} \quad |\varepsilon \lambda| \geq \frac{1}{10}. \quad (6.13)$$

In view of these estimates, it follows for $\varepsilon \in (0, \frac{\pi}{3(2k+1)})$, $\Re \lambda \in (-\frac{3}{2}, 3k - \frac{3}{2})$ and $v \in (0, 1)$ that

$$\frac{|\sin(\varepsilon \lambda v)|}{|\cos(\varepsilon \lambda)|} \leq C \left| \frac{\cos(\varepsilon \lambda v)}{\cos(\varepsilon \lambda)} \right| \leq C e^{i|\varepsilon \lambda|(v-1)} \leq C e^{\frac{1}{2}|\varepsilon \lambda|(v-1)}. \quad (6.14)$$

In view of (2.34), (2.41), this proves the estimate (6.6). The uniqueness of the solution follows from (6.6).  \qed

Explicitly solving (6.3) for $f = 0$ yields:
Lemma 7. Suppose that the assumptions of Proposition 6 hold. Suppose that \( f = 0 \) and \( g \in Z^k_{\varepsilon} \). Then there is a unique solution \( w := p \in Y^k_{\varepsilon} \) of (6.1). It is given by

\[
\hat{w}(\lambda, v) = \frac{\varepsilon}{\lambda \cos(\varepsilon \lambda)} \int_v^1 \sin(\varepsilon \lambda (z - 1)) \hat{g}(\lambda - 2, z) \, dz \\
+ \frac{\varepsilon}{\lambda \cos(\varepsilon \lambda)} \sin(\varepsilon \lambda (v - 1)) \int_0^v \cos(\varepsilon \lambda z) \hat{g}(\lambda - 2, z) \, dz.
\]

Moreover,

\[
\|w\|_{Y^k_{\varepsilon}} \leq C \|g\|_{Z^k_{\varepsilon}}.
\]

Proof. The solution \( w \) can be expressed in terms of the Green function \( G(v, z) \) by

\[
\begin{cases}
\varepsilon^2 \lambda^2 G(v, z) + \partial^2_v G(v, z) = \delta_{v=z} & \text{for } v \in (0, 1), \\
G = 0 & \text{for } v = 1, \\
G_v = 0 & \text{for } v = 0.
\end{cases}
\]

Since \( G \) is harmonic away from \( v = z \) and continuous at \( v = z \), it must be of the form

\[
G(v, z) = \begin{cases}
C \sin(\varepsilon \lambda (z - 1)) \cos(\varepsilon \lambda v) & \text{if } v < z, \\
C \cos(\varepsilon \lambda z) \sin(\varepsilon \lambda (v - 1)) & \text{if } z < v,
\end{cases}
\]

for some constant \( C \) to be determined. Taking the derivative of (6.16) in \( v \), we get

\[
\partial_v G(v, z) = \begin{cases}
-C \varepsilon \lambda \sin(\varepsilon \lambda (z - 1)) \sin(\varepsilon \lambda s) & \text{if } v < z, \\
C \varepsilon \lambda \cos(\varepsilon \lambda (v - 1)) \cos(\varepsilon \lambda z) & \text{if } z < v.
\end{cases}
\]

Since the jump of \( \partial_v G(v, z) \) at \( v = z \) is 1, namely \([\partial_v G(v, z)] = 1\), we deduce that

\[
\frac{1}{C} = \cos(\varepsilon \lambda (z - 1)) \cos(\varepsilon \lambda z) + \sin(\varepsilon \lambda (z - 1)) \sin(\varepsilon \lambda z) = \varepsilon \lambda \cos(\varepsilon \lambda),
\]

which implies (6.15). We next give the proof of the estimate. We will use that

\[
|\sin(\varepsilon \lambda (v - 1))| \leq Ce^{[\varepsilon \lambda](1-v)}, \quad |\cos(\varepsilon \lambda v)| \leq Ce^{[\varepsilon \lambda]v} \quad \text{and} \quad \frac{\cos(\varepsilon \lambda v)}{\cos(\varepsilon \lambda)} \leq Ce^{[\varepsilon \lambda](v-1)}.
\]

Hence, using the notation \( \hat{g}(\sigma) = \hat{g}(\lambda - 2, \sigma) \), we infer that

\[
\frac{1}{C} \int_0^1 \left| \int_v^1 \sin(\varepsilon \lambda (\sigma - 1)) \hat{g}(\sigma) \, d\sigma \right| \leq C \int_0^1 e^{[\varepsilon \lambda](1-\sigma)} e^{-\frac{[\varepsilon \lambda]}{2}(1-\sigma)} \sup_{\sigma \in (0, 1)} e^{\frac{[\varepsilon \lambda]}{2}(1-\sigma)} \hat{g}(\sigma) \, d\sigma
\]

\[
\leq \frac{C}{[\varepsilon \lambda]} e^{\frac{[\varepsilon \lambda]}{2}(1-v)} \sup_{\sigma \in (0, 1)} e^{\frac{[\varepsilon \lambda]}{2}(1-\sigma)} \hat{g}(\sigma).
\]
and
\[
\left| \int_0^v \cos(\varepsilon \lambda \sigma) \hat{g}(\sigma) \, d\sigma \right| \leq C \int_0^v e^{\frac{|\varepsilon \lambda|}{2} (1-\sigma)} \sup_{\sigma \in (0,1)} \left| e^{\frac{|\varepsilon \lambda|}{2} (1-\sigma)} \hat{g}(\sigma) \right| \, d\sigma \\
\leq \frac{C}{|\varepsilon \lambda|} e^{\frac{|\varepsilon \lambda|}{2} (3v-1)} \sup_{\sigma \in (0,1)} \left| e^{\frac{|\varepsilon \lambda|}{2} (1-\sigma)} \hat{g}(\sigma) \right|. \tag{6.18}
\]

By (6.17)–(6.18) and in view of (6.15), we deduce that
\[
|\hat{w}(\lambda, v)| \leq \frac{C}{|\lambda|^2} e^{\frac{|\varepsilon \lambda|}{2} (v-1)} \sup_{v \in (0,1)} \left| e^{\frac{|\varepsilon \lambda|}{2} (1-v)} \hat{g}(\lambda - 2, v) \right|. \tag{6.19}
\]

We calculate the first derivative,
\[
\partial_v \hat{w}(\lambda, v) = -\frac{\varepsilon^2}{\cos(\varepsilon \lambda)} \sin(\varepsilon \lambda v) \int_v^1 \sin(\varepsilon \lambda (\tilde{v} - 1)) \hat{g}(\lambda - 2, \tilde{v}) d\tilde{v} \\
- \frac{\varepsilon}{\lambda \cos(\varepsilon \lambda)} \cos(\varepsilon \lambda v) \sin(\varepsilon \lambda (v - 1)) \hat{g}(\lambda - 2, v) \\
+ \frac{\varepsilon^2}{\cos(\varepsilon \lambda)} \cos(\varepsilon \lambda (v - 1)) \int_0^v \cos(\varepsilon \lambda \tilde{v}) \hat{g}(\lambda - 2, \tilde{v}) d\tilde{v} \\
+ \frac{\varepsilon}{\lambda \cos(\varepsilon \lambda)} \sin(\varepsilon \lambda (v - 1)) \cos(\varepsilon \lambda v) \hat{g}(\lambda - 2, v). \tag{6.20}
\]

A similar calculation as before shows that
\[
\left| \frac{1}{\varepsilon} \hat{w}_v(\lambda, v) \right| \leq \frac{C}{|\lambda|} e^{\frac{|\varepsilon \lambda|}{2} (v-1)} \sup_{v \in (0,1)} \left| e^{\frac{|\varepsilon \lambda|}{2} (1-v)} \hat{g}(\lambda - 2, v) \right|. \tag{6.21}
\]

Multiplication of both sides of (6.19)–(6.21) by $|\lambda|^k$ yields higher regularity in the radial variables. Higher regularity in $v$ follows by (6.19), (6.21) and repeated application of
\[
\left| \lambda^k \left( \frac{1}{\varepsilon} \right)^2 \hat{w}_{vv}(\lambda, v) \right| \overset{(6.4)}{\leq} |\lambda|^{k+2} \hat{w}(\lambda, v) + |\lambda|^k \hat{g}(\lambda - 2, v). \tag{6.22}
\]

Estimates (6.19), (6.21) and (6.22) imply
\[
\sum_{|\alpha| = 3k + 2} \sup_{v \in (0,1)} \left| e^{\frac{|\varepsilon \lambda|}{2} (1-v)} \Lambda^{3k+2} \mu^k \hat{w}(\lambda, v) \right| \\
\leq C \sum_{|\alpha| = 3k} \sup_{v \in (0,1)} \left| e^{\frac{|\varepsilon \lambda|}{2} (1-v)} \Lambda^{3k} \mu^k \hat{g}(\lambda - 2, v) \right|. 
\]

Estimate (6.2) follows by taking the $L^2$-norm on the line $\Re \lambda = 3k - \frac{3}{2}$ on both sides. \qed
Proof. (Proof of Proposition 6) Let $P_f$ be the Taylor polynomial of $f$ at $x = 0$ of order $n_k = 3k - 1$; let $P_g$ be the Taylor polynomial of $g$ at $(x, y) = 0$ of order $n_k - 3$. Let $P_p$ be the polynomial solving (6.1) with $g$ and $f$ replaced by $P_g$ and $P_f$. The existence and uniqueness of this polynomial solution follows from a straightforward calculation, and furthermore,

$$\|P_p\|_p \leq C \left( \|P_f\|_p + \|P_g\|_p \right), \tag{6.23}$$

see also the proof of [27, Lemma 4.2] as well as the proof of Lemma 8 below. Let $\zeta : K \to \mathbb{R}$ be a smooth radial cut-off function with $\zeta = 1$ for $|(x, y)| < [0, \frac{1}{4}]$, $\zeta = 0$ for $|(x, y)| \geq [0, \frac{1}{4}]$ and such that $0 \leq \zeta \leq 1$ and let $\tilde{\zeta} : (0, \infty) \to \mathbb{R}$ be given by $\tilde{\zeta} (x) = \zeta (|(x, 0)|)$. We define

$$p_1 := P_f \zeta, \quad g_1 := \Delta_s p_1, \quad f_1 := \tilde{\zeta} \int_0^x P_p \, dx, \tag{6.24}$$

and $g_0 := g - g_1 \in \dot{\gamma}^k_\zeta$ and $f_0 := f - f_1 \in \dot{\gamma}^k_\zeta$. Furthermore, let $p_0 \in Y^k_\zeta$ be the solution of $\Delta p_0 = g_0$ in $K$ with $p_0 = \eta_0$ on $\partial_1 K$ and $\partial_y p_0 = 0$ on $\partial_0 K$. By the previous two lemmas, there exists such a solution $p_0$ satisfying

$$\|p_0\|_{\dot{\gamma}^k_\zeta} \leq C \left( \|g_0\|_{Z^k_{\zeta}} + \|f_0\|_{Z^k_{\zeta}} \right). \tag{6.25}$$

Hence, $p := p_0 + p_1$ is a solution of (6.1) and satisfies the desired estimate. $\Box$

Since $\varepsilon$ has the scaling of vertical length, one could expect that there is only uniform control on the norm $\left(\frac{1}{\varepsilon}(p_y)\right)_{\Gamma}$, but it turns out that even $\left(\frac{1}{\varepsilon}^2(p_y)\right)_{\Gamma}$ is bounded uniformly for $\varepsilon > 0$. The proof of this statement is given in the next lemma:

**Lemma 8.** Let $k \geq 1$, $\varepsilon \in (0, \frac{\pi}{2(2k+1)})$. Then the solution $p \in Y^k_\varepsilon$ of (6.1) satisfies

$$\|\partial_x (p_y)_{\Gamma}\|_{X^k_{\varepsilon}} + \left(\frac{1}{\varepsilon}\right)^2 \|\partial_x (p_y)_{\Gamma}\|_{X^k_{\varepsilon}} \leq C \left( \|f\|_{X^{k+1}_\varepsilon} + \|g\|_{Z^{k+1}_\varepsilon} \right). \tag{6.26}$$

**Proof.** Analogously to the proof of Proposition 6, we decompose $p = p_1 + p_0$ where $p_1$ encodes the expansion at the contact point and where $p_0 \in Y^{k+1}_\varepsilon$ is the solution of (6.1) with corresponding homogeneous data $f_0 \in X^{k+1}_\varepsilon$ and $g_0 \in X^{k+1}_\varepsilon$. Furthermore, let $p_0 = q_0 + w_0$ where $q_0$ is the solution of (6.1) with boundary data $f_0$ and with right hand side $g = 0$. Correspondingly, $w_0$ is the solution with right hand side $g_0$ and with boundary data $f = 0$. In the sequel, we give the corresponding estimates to (6.26) for $q_0$, $w_0$ and $p_1$; together these estimates imply (6.26).

**Estimate for $q_0$** With the transformation (2.40), we need to show

$$\|e^{-2u} q_{0, uu|\Gamma}\|_{X^k_{\varepsilon}} + \left(\frac{1}{\varepsilon}\right)^2 \|e^{-2u} q_{0, vu|\Gamma}\|_{X^k_{\varepsilon}} \leq C \|f_0\|_{X^{k+1}_\varepsilon}. \tag{6.27}$$

Here and in the following, by a slight abuse of notation, we understand $q_0$ as a function of $(u, v)$. By (6.5), since $|\sin(\varepsilon \lambda)/\cos(\varepsilon \lambda)| \leq C \mu$ and for $\Re \lambda \in \left(\frac{1}{2}, 4k - \frac{1}{2}\right)$ we obtain

$$\left| e^{-2u} q_{0, vu}(\lambda, 1) \right| = \left| \varepsilon \lambda^2 \sin(\varepsilon (\lambda + 2))/\cos(\varepsilon (\lambda + 2)) (\lambda + 3) f_0 (\lambda + 3) \right| \leq C |\varepsilon^2 \lambda^3 | \mu f_0 (\lambda + 3).$$
Note that here we use that $|\lambda| \geq 1$; in particular in the complex strip $\Re \lambda \in (\frac{1}{2}, 4k - \frac{1}{2})$, we have $|\lambda + 3| \leq |\lambda|$. Multiplying this identity by $|\lambda|^{3k+1}$ and taking the $L^2$-norm on the line $\Re \lambda = 3k - \frac{1}{2}$, we obtain the estimate for the second term on the left hand side of (6.27). The estimate for the first term proceeds analogously.

**Estimate for $w_0$** With the transform (2.40), we need to show

$$
\|e^{-2\mu}w_{0,uu}\|_{L^2_k} + \left(\frac{1}{\varepsilon}\right)^2 \|e^{-2\mu}w_{0,uu}\|_{L^2_k} \leq C\|g_0\|_{Z^{k+1}_\varepsilon}.
$$

(6.28)

Evaluating (6.20) at $v = 1$ we note that only the third term does not vanish, that is

$$
\partial_v \hat{w}(\lambda, 1) \overset{(6.20)}{=} \frac{\varepsilon^2}{\cos(\varepsilon \lambda)} \int_0^1 \cos(\varepsilon \lambda \hat{v}) \hat{g}(\lambda - 2, \hat{v})d\hat{v}.
$$

(6.29)

With the notation $\hat{\varphi} = \left(\frac{1}{\varepsilon}\right)^2 e^{-2\mu}w_{0,vu}$ and since $\hat{\varphi}(\lambda) = \left(\frac{1}{\varepsilon}\right)^2(\lambda + 2)w_{0,v}(\lambda + 2, 1)$, we get

$$
\hat{\varphi}(\lambda) \overset{(6.29)}{=} \frac{\lambda + 2}{\cos(\varepsilon(\lambda + 2))} \int_0^1 \cos(\varepsilon(\lambda + 2)\hat{v}) \hat{g}_0(\lambda, \hat{v})d\hat{v}.
$$

(6.30)

For $|\varepsilon \lambda| \leq 1$, we have $|\lambda + 2| \leq C|\lambda|, |\lambda| = \mu, |e^{\varepsilon|\lambda|}(1-\varepsilon)| \geq 1$. Furthermore, we have $|\cos(\varepsilon(\lambda + 2))| \geq c_k$ for some constant $c_k > 0$. Indeed, since $\varepsilon \leq 1$ and since $|3\lambda| \leq c_k$, we get $\cos(\varepsilon(\lambda + 2)) \geq |e^{\varepsilon(3|\lambda|+2)} - e^{-i(3|\lambda|+2)}| \geq c_k$ since $|\varepsilon| |\lambda| + 2 \varepsilon < \pi$ for $|\varepsilon \lambda| \leq 1$ and $\varepsilon \in (0, 1)$. Therefore

$$
|\hat{\varphi}(\lambda)| \leq C|\lambda| \sup_{v \in (0, 1)} |\hat{g}_0(\lambda, v)| \leq C\mu \sup_{v \in (0, 1)} \left|e^{\frac{|\varepsilon|}{2}(1-v)}\hat{g}_0(\lambda, v)\right|.
$$

(6.31)

For $|\varepsilon \lambda| \geq 1$, we have $\mu = \frac{1}{\varepsilon}, |\lambda + 2| \leq |\lambda|$ and $|\cos(\varepsilon(\lambda + 2))| \geq c e^{\varepsilon|\lambda|}$. Application of (6.18) hence yields

$$
|\hat{\varphi}(\lambda)| \leq \frac{C}{\varepsilon} \sup_{v \in (0, 1)} \left|e^{\frac{|\varepsilon|}{2}(1-v)}\hat{g}(\lambda, v)\right| = C\mu \sup_{v \in (0, 1)} \left|e^{\frac{|\varepsilon|}{2}(1-v)}\hat{g}(\lambda, v)\right|.
$$

(6.32)

The above two inequalities together imply

$$
|\lambda|^{3\ell+1} \mu^\ell \hat{\varphi}(\lambda) | \leq C \sup_{v \in (0, 1)} \left|e^{\frac{|\varepsilon|}{2}(1-v)}\lambda^{3\ell+1} \mu^\ell \hat{g}_0(\lambda, v)\right|
$$

for all $\ell \geq 0$. Integrating the square of the above estimate on the line $\Re \lambda = 3k - \frac{1}{2}$ yields

$$
|\varphi|_{L^2_{k\varepsilon}} \leq \left|\lambda^{3\ell+1} \mu^\ell \hat{\varphi}\right|_{L^2_{\Re \lambda = 3\ell - \frac{1}{2}}} \leq C \sup_{v \in (0, 1)} \left|e^{\frac{|\varepsilon|}{2}(1-v)}\lambda^{3\ell+1} \mu^\ell \hat{g}_0\right|_{L^2_{\Re \lambda = 3\ell - \frac{1}{2}}}
$$

$$
\leq C\|g_0\|_{Z^{k+1}_\varepsilon}.
$$

which concludes the estimate for the second term on the left hand side of (6.28). The estimate of the first term on the left hand side of (6.28) proceeds similarly.
Estimate for $p_1$ Let $\mathcal{P}_p = \sum_{i+j \leq \ell} a_{ij} x^i y^j$ be the polynomial which solves (6.1) with boundary data $\mathcal{P}_f$ and the right hand side $\mathcal{P}_g$, where $\mathcal{P}_f = \sum_i b_i x^i$ is the Taylor polynomial of $f$ of order $3k - 1$ and where $\mathcal{P}_g = \sum_{ij} g_{ij} x^i y^j$ is the Taylor polynomial of $g$ of order $3k - 4$. Analogously to the proof of Lemma 7, we need to show that the coefficients of the polynomial $(\frac{\epsilon}{\ell})^2 \mathcal{P}_{p, y | \Gamma}$ are bounded by the coefficients of $f$. Indeed, by the boundary condition in (6.1), we have $\partial x_i \mathcal{P}_{p, y | y=0} = 0$, and hence it follows that $a_{i,1} = 0$ for all $i \geq 0$. With the equation

$$\partial_y \partial_x \mathcal{P}_{p, y} = \epsilon^2 \left( \partial_x \mathcal{P}_{g, y} - \partial_y \mathcal{P}_{p, y} \right),$$

(6.33)

we first get $|a_{i,3}| \leq \epsilon^2 \|\mathcal{P}_g\|$ and then iteratively $|a_{i,2j+1}| \leq \epsilon^2 \|\mathcal{P}_g\|$ for all $i, j \geq 0$. Furthermore since $\mathcal{P}_{p, y | \Gamma} = f_x$ and again using (6.33), one can easily deduce that $|a_{i,2j}| \leq C \epsilon^2 (|b_{i,j+1}| + \|\mathcal{P}_g\|)$ for all $i, j \geq 0$ (for example, we have $a_{02} = \epsilon^2 (a_{20} + g_{00})$ and $a_{02} + a_{02} = b_2$). In particular, $|a_{ij}| \leq \epsilon^2 (\|\mathcal{P}_f\| \|\mathcal{P}_p\| + \|\mathcal{P}_g\| \|\mathcal{P}_p\|)$ for all $i \geq 0, j \geq 1$, which yields the desired estimate. \hfill \Box

6.2. Pull-Back onto Wedge

We need to measure the difference $p_1 - p_2$, where $p_1$ is the solution for the pressure on the domain $\tilde{K}_\phi$ and $p_2$ is the corresponding solution on $\tilde{K}_\phi$, see Proposition 7. In order to estimate this difference, for any given profile function $\varphi$, we introduce a pull-back from the perturbed wedge $K^\varphi$ to the unperturbed wedge $K$, see Fig. 3a. The estimates are nonlinear due to the geometry of the domain.

Lemma 9. Let $k \in \mathbb{N}$ with $k \geq 1$, $\epsilon \in (0, \frac{\pi}{3(2k+1)})$ and let $\varphi \in X_k^{\epsilon}$. Then there is $\delta_1 > 0$ such that if $\|\varphi\|_{X_k^{\epsilon}} \leq \delta_1$, then there is a diffeomorphism $\Psi: K \rightarrow K^\varphi$ of the form

$$\Psi(x, y) = (x, y) + (0, \psi(x, y)) = (\hat{x}, \hat{y}).$$

(6.34)

Furthermore, analogous to the definition (2.42), there is a decomposition $\psi = \psi_0 + \psi_1$ such that $\psi_1 = \zeta \mathcal{P}$ and where $\mathcal{P}$ is a polynomial of order $3k$ such that

$$\sup_{0 \leq \ell \leq k} \sum_{|\alpha| = 3\ell + 2} \left\| \sup_{0 \leq \nu \leq 1} \left| \epsilon \frac{\lambda(1-\nu)}{\nu + 1} \lambda^{\ell+1} \mathcal{A}^\nu \mathcal{P}_{\psi_0} \right| \right\|_{L^2(\mathbb{R}_x = 3\ell + 1, 1)} + \|\mathcal{P}\|_p \leq C \|\varphi\|_{X_k^{\epsilon}},$$

(6.35)

where we recall that $\mu = \inf\{(|\lambda|, \frac{1}{\epsilon})\}$. Furthermore, we have

$$\|\epsilon \psi_{x|\Gamma}\|_{X_k^{\epsilon}} + \|\psi_{y|\Gamma}\|_{X_k^{\epsilon}} \leq C \|\varphi\|_{X_k^{\epsilon}} < 1.$$  

(6.36)
Lemma 10. Let $\psi$ be the solution of
\[
\begin{cases}
\Delta_\varepsilon \psi = 0 & \text{in } K, \\
\psi = \int_0^x \varphi \, dx & \text{on } \partial_1 K, \\
\psi = 0 & \text{on } \partial_0 K.
\end{cases} 
\]
(6.37)

As in the previous proofs, the argument relies on a decomposition of the right hand side into a polynomial part and a homogeneous remainder. Since this decomposition proceeds analogously to the proof of Proposition 6, we only consider the case of homogeneous data assuming $\varphi \in X^s_{\varepsilon}$. With the transformation (2.39), (6.37) takes the form $\partial_\varepsilon^2 \psi + (\frac{1}{\varepsilon} \partial_\varepsilon) \psi = 0$ for $(u, v) \in \mathbb{R} \times (0, 1)$. Furthermore, $\psi(u, 1) = \int u \varphi \, e^{\hat{d}\hat{u}} =: H(u)$ and $\psi(u, 0) = 0$. Apply of the Laplace transform in $u$, and since $\lambda \hat{H} = \hat{\varphi} (\lambda - 1)$, the solution can be explicitly calculated as [cf. (6.3)–(6.5)]
\[
\hat{\psi}(\lambda, v) = \frac{\sin(\varepsilon \lambda v)}{\sin(\varepsilon \lambda)} \hat{H}(\lambda) = \frac{1}{\lambda} \frac{\sin(\varepsilon \lambda v)}{\sin(\varepsilon \lambda)} \hat{\varphi}(\lambda - 1).
\]
(6.38)

We have for $v \in (0, 1)$,
\[
\left| \frac{\sin(\varepsilon \lambda v)}{\sin(\varepsilon \lambda)} \right| \leq \left| \frac{\cos(\varepsilon \lambda v)}{\sin(\varepsilon \lambda)} \right| \leq \frac{C}{\varepsilon \mu} e^{\varepsilon |\lambda|(v-1)} \leq \frac{C}{\varepsilon \mu} e^{\frac{1}{\varepsilon} |\lambda|(v-1)}.
\]
(6.39)

Indeed, for $|\varepsilon \lambda| \leq 1$, we have $\mu = |\lambda|$, $|\sin(\varepsilon \lambda v)| \leq |\cos(\varepsilon \lambda v)|$, $\frac{\cos(\varepsilon \lambda v)}{\sin(\varepsilon \lambda)} \leq \frac{C}{\varepsilon \mu} = \varepsilon^{-1}$. For $|\varepsilon \lambda| \geq 1$, we have $\mu = \frac{1}{\varepsilon}$ and hence $\frac{1}{\varepsilon \mu} = 1$. $|\cos(\varepsilon \lambda v)| \leq 4 e^{\varepsilon |\lambda|v}$ and $|\sin(\varepsilon \lambda)| \leq \frac{1}{4} e^{\varepsilon |\lambda|}$. This proves (6.39). Now, we get
\[
\left| e^{\varepsilon |\lambda|(1-v) \mu} |\lambda|^{3 \ell + 2} \varepsilon \hat{\psi}(\lambda, v) \right| \leq \frac{C |\mu| \varepsilon |\lambda|^{3 \ell + 2} \varepsilon \hat{\varphi}(\lambda - 1)}{\sin(\varepsilon \lambda)} \leq \frac{C |\mu| \varepsilon |\lambda|^{3 \ell + 2} \varepsilon \hat{\varphi}(\lambda - 1)}{\sin(\varepsilon \lambda)}.
\]
(6.39)

We now apply the $L^2$ norm on the line $\Re \lambda = 3 \ell + \frac{1}{2}$. This yields the estimate (6.35) for $\alpha = (3 \ell + 2, 0)$. When taking derivatives of (6.38) in $v$, additional factors of $\varepsilon |\lambda| \leq |\lambda|$ are created; furthermore the corresponding multiplier in (6.38) has either the cosinus or sinus in the nominator. In view of (6.39), we can estimate the multiplier in both cases. The estimate then proceeds analogously to the case when there are derivatives in $v$. This concludes the proof of (6.35). The estimate of the first term in (6.36) follows by multiplying (6.38) with $|\lambda|$ and taking the $L^2$-norm over the line $\Re \lambda = 3k + \frac{1}{2}$. The estimate of the second term in (6.36) proceeds similarly. If $\delta_1$ is chosen sufficiently small, the right hand side of (6.36) is bounded by 1. Uniqueness of the solution follows from (6.35). \hfill \Box

Lemma 10. Let $\Psi$ be the coordinate transform from Lemma 9. Then $P : K^\varphi \rightarrow \mathbb{R}$ is a solution of (6.42) if and only if $p = P \circ \Psi : K \rightarrow \mathbb{R}$ satisfies
\[
\begin{cases}
\Delta_\varphi p = R(p, \varphi) & \text{in } K, \\
p = f_x & \text{on } \partial_1 K, \\
p_\gamma = 0 & \text{on } \partial_0 K.
\end{cases} 
\]
(6.40)
where the operator $R(p, \varphi)$ is given by [using the notation $\gamma := (1 + \psi_y)^{-1}$]

$$R(p, \varphi) = -\gamma_x \psi_x p_y - \gamma \psi_{xx} p_y - 2\gamma \psi_x p_{xy} + \gamma \psi_y \psi_x^2 p_y + \gamma^2 \psi_x^2 p_{yy} + \frac{1}{\varepsilon} \gamma \psi_y p_y + \left(\frac{1}{\varepsilon}\right)^2 \gamma^2 (1 - 1)p_{yy}. \tag{6.41}$$

**Proof.** We can write the inverse coordinate transform $\Psi^{-1}: K^\varphi \to K$ as $\Psi^{-1}(\hat{x}, \hat{y}) = (\hat{x}, \hat{y} + \eta(\hat{x}, \hat{y}))$ for some $\eta: K^\varphi \to K$. In particular, $\hat{y} + \eta(\hat{x}, \hat{y}) + \psi(\hat{x}, \hat{y} + \eta(\hat{x}, \hat{y})) = \hat{y}$. By differentiating this equality in $\hat{x}$ and $\hat{y}$, we get $\psi_x + (1 + \psi_y) \eta_{\hat{z}} = 0$ and $(1 + \psi_y)(1 + \eta_z) = 1$. This implies $\frac{\partial \hat{x}}{\partial x} = 1$, $\frac{\partial \hat{y}}{\partial y} = 0$, $\frac{\partial \hat{x}}{\partial y} = -\gamma \psi_x$ and $\frac{\partial \hat{y}}{\partial x} = \gamma$, in particular, $\tilde{P}_{\hat{y}} = 0 \Leftrightarrow P_y = 0$, justifying the boundary condition in (6.42). Equation (6.41) follows from (6.42) together with

$$P_{\hat{x}\hat{x}} = (\partial_x - \gamma \psi_t \partial_y)(p_x - \gamma \psi_x p_y) = p_{xx} - \gamma_x \psi_x p_y - \gamma \psi_{xx} p_y - 2\gamma \psi_x p_{xy} + \gamma \psi_y \psi_x^2 p_y + \gamma^2 \psi_x^2 p_{yy},$$

$$P_{\hat{y}\hat{y}} = \gamma (\gamma \tilde{p}_x) - 2\gamma^2 p_{yy} + \gamma \gamma_y p_y.$$

\[\Box\]

6.3. Estimates on the Pressure

The main result of this section is:

**Proposition 7.** (Shape dependence of $p$) Let $k \in \mathbb{N}$, $k \geq 1$, $\varepsilon \in (0, \frac{\pi}{2(2k+1)})$ and let $\varphi \in X^{k-1/2}_\varepsilon$, $f \in X^k_\varepsilon$. Then there is a constant $0 < \delta < 1$ such that if $\|\varphi\|_{X^{k-1/2}_\varepsilon} \leq \delta$, then there is a unique solution $P: K \to \mathbb{R}$ of

$$\begin{cases}
\Delta_x P = 0 & \text{in } K^\varphi, \\
P = f_x & \text{on } \partial_1 K^\varphi, \\
P_y = 0 & \text{on } \partial_0 K^\varphi. \tag{6.42}
\end{cases}$$

Furthermore, $p = P \circ \Psi \in Y^k_\varepsilon$ satisfies

$$\|p\|_{Y^k_\varepsilon} \leq C \|f\|_{X^k_\varepsilon} \tag{6.43}$$

for some constant $C > 0$ which only depends on $k$. Let $q$ be the solution of (6.1) with data $f$ and with right hand side $g = 0$. Furthermore, for $\tilde{f} \in X^k_\varepsilon$, let $\tilde{P}$, $\tilde{p}$ and $\tilde{q}$ be the corresponding solutions with boundary condition $\tilde{f}$ instead of $f$ and let $w = p - q$ and $\tilde{w} = \tilde{P} - \tilde{q}$. Then

$$\begin{align*}
\|w\|_{Y^k_\varepsilon} & \leq C \|\varphi\|_{X^{k-1/2}_\varepsilon} \|f\|_{X^k_\varepsilon}, \\
\|p - \tilde{p}\|_{Y^k_\varepsilon} & \leq C \|\varphi - \tilde{\varphi}\|_{X^{k-1/2}_\varepsilon} \left(\|f\|_{X^k_\varepsilon} + \|\tilde{f}\|_{X^k_\varepsilon}\right) + C \|f - \tilde{f}\|_{X^k_\varepsilon}, \\
\|w - \tilde{w}\|_{Y^k_\varepsilon} & \leq C \|\varphi - \tilde{\varphi}\|_{X^{k-1/2}_\varepsilon} \left(\|f\|_{X^k_\varepsilon} + \|\tilde{f}\|_{X^k_\varepsilon}\right) + C \|f - \tilde{f}\|_{X^k_\varepsilon} \left(\|\varphi\|_{X^{k-1/2}_\varepsilon} + \|\tilde{\varphi}\|_{X^{k-1/2}_\varepsilon}\right) \tag{6.44}
\end{align*}$$

for some constant $C > 0$ which only depends on $k$.\[\Box\]
Before we address the proof of Proposition 7, we give an estimate of the non-linear term $R(p, \varphi)$ defined in (6.41).

**Proposition 8.** Let $k \in \mathbb{N}$, $k \geq 1$, $\varepsilon \in (0, \frac{1}{3(2k+1)})$. Suppose that $f, \tilde{f} \in X_k^\varepsilon$ and $\varphi, \tilde{\varphi} \in X_k^\varepsilon$. Let $\delta_1 > 0$ be the constant from Lemma 9. Then there is a constant $\delta_2$ with $0 < \delta_2 < \delta_1$ such that if $\|\varphi\|_{X_k^\varepsilon-1/2}, \|\tilde{\varphi}\|_{X_k^\varepsilon-1/2} \leq \delta_2$, the following holds.

Let $p \in Y_k^\varepsilon$ be the solution of (6.42) on $K^\varphi$ with boundary data $f$, let $\tilde{p} \in Y_k^\varepsilon$ be the corresponding solution on $K^{\tilde{\varphi}}$ with boundary data $\tilde{f}$. Then

$$\|R(p, \varphi) - R(\tilde{p}, \tilde{\varphi})\|_{Z_k^\varepsilon} \leq C \|p - \tilde{p}\|_{Y_k^\varepsilon}(\|\varphi\|_{X_k^\varepsilon-1/2} + \|\tilde{\varphi}\|_{X_k^\varepsilon-1/2})$$

(6.45)

for some constant $C > 0$ which only depends on $k$.

**Proof.** We will show that given

$$\|R(p, \varphi)\|_{Z_k^\varepsilon} \leq C \|\varphi\|_{X_k^\varepsilon-1/2} \|p\|_{Y_k^\varepsilon},$$

(6.46)

the proof of (6.45) follows easily, using the multilinear structure of $R$. The proof of (6.46) uses some ideas of the proof of the algebra property in Lemma 3. The main difference in the argument of (6.46) with respect to the previous argument is related to the fact that we have to estimate the fully two-dimensional norm $\|\cdot\|_{Z_k^\varepsilon}$. Indeed, in order to estimate this norm, we need to take a supremum in the angular variable $v$, which causes some technical issues. Furthermore, different from the proof of Lemma 3, in the estimate (6.46) we have to consider different norms on the right hand side of the estimate. Recall the definition of $R$ in (6.41), of the coordinate transform $\psi$ in (6.37), and recall that $\gamma = (1 + \psi_x)^{-1}$. We also note that in view of the definitions of the norms (2.41), (2.46) and by (6.43) we have

$$\|p_x\|_{Z_k^\varepsilon} + \|p_y\|_{Z_k^\varepsilon} + \|p_{xx}\|_{Z_k^\varepsilon} + \|p_{yy}\|_{Z_k^\varepsilon} + \|(\frac{1}{\varepsilon})^2 p_{xy}\|_{Z_k^\varepsilon} \leq C \|p\|_{Y_k^\varepsilon}.$$  

(6.47)

We claim that (6.46) follows by iterative application of (6.47) and the two estimates

$$\|\rho g\|_{Z_k^\varepsilon} + \|(1 - \gamma)g\|_{Z_k^\varepsilon} \leq C \|\varphi\|_{X_k^\varepsilon-1/2} \|g\|_{Z_k^\varepsilon} \quad \text{for } \rho \in \{\varepsilon \psi_x, \psi_y\},$$

(6.48)

$$\|\rho_x w\|_{Z_k^\varepsilon} + \|(\frac{1}{\varepsilon})\rho_y w\|_{Z_k^\varepsilon} \leq C \|\varphi\|_{X_k^\varepsilon-1/2} \|w_x\|_{Z_k^\varepsilon} + \|w_y\|_{Z_k^\varepsilon} \quad \text{for } \rho \in \{\varepsilon \psi_x, \psi_y\}$$

(6.49)

and for any functions $g, w$ such that the right hand sides of the above estimates are finite. Indeed, assuming that (6.48)–(6.49) hold, the proof of (6.46) is easy. We show how the estimate of (6.46) proceeds for the term $\gamma_x \psi_x p_y$ [the first term on the right hand side of (6.41)]. In view of $\gamma_x = -\gamma^2 \psi_{xy}$, we have

$$\|\gamma_x \psi_x p_y\|_{Z_k^\varepsilon} = \|\gamma^2 (\varepsilon \psi_x) \psi_{xy} (\frac{1}{\varepsilon} p_y)\|_{Z_k^\varepsilon}$$

(6.48)

$$\leq C \|\psi_{xy} (\frac{1}{\varepsilon} p_y)\|_{Z_k^\varepsilon} \leq C \|\varphi\|_{X_k^\varepsilon-1/2} \|p\|_{Y_k^\varepsilon}.$$
It can be checked that the estimate of the other terms in (6.41) proceeds analogously. In order to conclude the proof of the Proposition, it thus remains to give the argument for (6.48)—(6.49).

**Proof of (6.48)** We first present the proof of the estimate

$$\|\rho g\|_{Z_k^\epsilon} \leq C \|\varphi\|_{X_k^{\epsilon-1/2}} \|g\|_{Z_k^\epsilon}. \tag{6.50}$$

Let $\mathcal{P}_\psi$ and $\mathcal{P}_g$ be the Taylor polynomials at $(x, y) = (0, 0)$ of $\psi$ and $g$ of order $3k - 1$ and order $3k - 4$, respectively. We decompose $\psi = \psi_1 + \psi_0$ and $g = g_1 + g_0$, where $\psi_1 := \mathcal{P}_\psi \zeta$ and $g_1 := \mathcal{P}_g \zeta$ with $\zeta(x, y) := \zeta((x, y))$ where the cut-off function $\zeta \in C_\infty^\infty([0, \frac{1}{2}])$ satisfies $\zeta = 1$ in $[0, \frac{1}{4}]$ and $\zeta = 0$ in $[\frac{1}{4}, \infty]$. The terms $\rho_0$ and $\rho_1$ are defined correspondingly. It is then enough to show the corresponding estimate to (6.50) for the products $\rho_1 g_1, \rho_1 g_0, \rho_0 g_1$ and $\rho_0 g_0$. In the following, we will present the estimate for $\rho_0 g_0$ since the estimate of the other terms proceeds analogously as in Lemma 3. Furthermore, for simplicity of notation, we give the proof with $\Lambda$ replaced by $|\lambda|$ in (2.44), that is for the case when only radial derivatives appear. The argument in the case of angular derivatives $\frac{1}{\epsilon} \partial_\theta$ proceeds by distributing the derivatives on the two factors using Leibniz’ rule. We will use the notation $K_\epsilon^{[|\lambda|]} = e^{\frac{3}{\epsilon} |\lambda| (1 - v)}$. In particular note that by the triangle inequality we have

$$K_\epsilon^{[|\lambda|]} \leq K_\epsilon^{[|\lambda - \eta|]} K_\epsilon^{[|\eta|]} \tag{6.51}$$

Analogously to (5.6), we decompose $g_0$ into its low and high frequency parts, that is we define $g_+ := g - g_0$, where for some $M \geq k$ the high-frequency part $g_-$ is defined in terms of its Mellin transform by

$$\widehat{g_-}(\lambda, v) := (i \tan)^M (\epsilon \lambda) \hat{g}_0(\lambda, v). \tag{6.52}$$

Correspondingly, we decompose $\psi_0 = \psi_+ + \psi_-$ into its low and high frequency parts. We also define $\rho_\pm$ by $\rho_\pm = \epsilon \varphi_{\pm, x}$ if $\rho = \epsilon \varphi_x$, respectively $\rho_\pm = \varphi_{\pm, y}$ if $\rho = \varphi_y$. In particular, analogously to (5.7), we have the estimate

$$|\lambda|^k |\widehat{g_+}| + \left(\frac{1}{\epsilon}\right)^k |\widehat{g_-}| \leq C \mu^k |\hat{g_0}|, \quad |\lambda|^k |\widehat{\rho_+}| + \left(\frac{1}{\epsilon}\right)^k |\widehat{\rho_-}| \leq C \mu^k |\hat{\rho_0}|. \tag{6.53}$$

We need to estimate the terms $g_+ \rho_+, g_+ \rho_-, g_- \rho_+$ and $g_- \rho_-$. We show the estimate for the high frequency/high frequency product $g_- \rho_-$. The estimate for the other terms proceeds as in the proof of Proposition 2.3 in [27].

In view of (2.44) and (6.53), we need to show for all $\ell \in [\frac{3}{2}, k]$,

$$[\rho_\pm g_{-}]_{Z_\epsilon^\ell} \leq C \left\| \sup_{v \in (0, 1)} \left( \int_{s \eta = v} K_\epsilon^{[|\lambda|]} \left(\frac{1}{\epsilon}\right)^{\ell} \lambda^3 \rho_-(\lambda - \eta) \hat{g_-}(\eta) \, d\eta(\eta) \right) \right\|_{L^2(\theta \lambda = \ell - \frac{\ell}{2})} \leq C \|\varphi\|_{X_\epsilon^{\ell-1/2}} \|g\|_{Z_\epsilon^{\frac{k}{2}}} \tag{6.54}$$
for some $\gamma \in \mathbb{R}$ of our choice such the above integral converges. Let $\kappa \in [0, 1)$ be the smallest number such that $3\ell - 2 + \kappa \in \mathbb{N}_0$. In view of $|\lambda|^{3\ell-2} \leq 1 + |\lambda|^N$, it is enough to show

$$
\left\| \sup_{v \in (0,1)} \left| \int_{\Re \eta = \gamma} K^{[\lambda]}_{\epsilon} \left( \frac{1}{\epsilon} \right)^{\ell} \hat{\rho}_- (\lambda - \eta) \hat{g}_- (\eta) \right| \right\|_{L^2(\Re \lambda = 3\ell - \frac{3}{2})} \leq C \| \varphi \|_{X^{k-1/2}} \| g \|_{Z^{k}}.
$$

(6.55)

$$
\left\| \sup_{v \in (0,1)} \left| \int_{\Re \eta = \gamma} K^{[\lambda]}_{\epsilon} \left( \frac{1}{\epsilon} \right)^{\ell} \lambda^{3\ell-2+\kappa} \hat{\rho}_- (\lambda - \eta) \hat{g}_- (\eta) \right| \right\|_{L^2(\Re \lambda = 3\ell - \frac{3}{2})} \leq C \| \varphi \|_{X^{k-1/2}} \| g \|_{Z^{k}}.
$$

(6.56)

We show the corresponding estimate for the term in (6.56). The estimate for the term in line (6.55) then follows since $|\lambda| \geq \frac{1}{\epsilon}$ for $\Re \lambda = 3\ell - \frac{3}{2}$ and $\ell \in \left[ \frac{3}{2}, k \right]$. The advantage of replacing the multiplier $\lambda^{3\ell-2}$ with the multiplier $\lambda^{3\ell-2+\kappa}$ is that the binomial formula

$$
\lambda^N = \sum_{j=0}^{N} C_{N,j} (\lambda - \eta)^j \eta^{N-j}
$$

with $N := 3\ell - 2 + \kappa \in \mathbb{N}_0$ can be applied. In order to estimate the term in line (6.56), it is therefore enough to estimate for all $i = 0, \ldots, 3\ell - 2 + \kappa$, the terms

$$
\int \sup_{v \in (0,1)} \left| \int_{\Re \eta = \gamma} K^{[\lambda]}_{\epsilon} \left( \frac{1}{\epsilon} \right)^{\ell} (\lambda - \eta)^{3\ell-2+\kappa-i} \hat{\rho}_- (\lambda - \eta) \hat{g}_- (\eta) \right|^2 \, d\eta.
$$

Note that the above inner integrand is analytic in $\eta$ and hence does not depend on the value of $\gamma$ as long as the integral is well-defined; hence we may choose $\gamma = \gamma (i)$ freely as a function of $i$. By (6.51), it is enough to estimate terms of the form

$$
\int \sup_{v \in (0,1)} \left| \int_{\Re \eta = \gamma} K^{[\lambda-\eta]}_{\epsilon} \left( \frac{1}{\epsilon} \right)^{\ell} \lambda^{3\ell+\kappa-2-i} \hat{\rho}_- (\lambda - \eta) \hat{g}_- (\eta) \right|^2 \, d\eta.
$$

(6.57)

for $\ell \in \left[ \frac{3}{2}, k \right]$, all integers $i \in \{0, 3\ell - 2 + \kappa\}$ and with our choice of $\gamma = \gamma (i) \in \mathbb{R}$. We next apply the following variant of (5.11) which says that for all $\delta > \frac{1}{2}$, we have

$$
\left\| \sup_{v \in (0,1)} |\hat{F} \ast \hat{G}| \right\|_{L^2(\Re \lambda = \beta)} \leq C_\delta \left\| \sup_{v \in (0,1)} (1 + |\lambda|^\delta) |\hat{F}| \right\|_{L^2(\Re \lambda = \beta_1)} \left\| \sup_{v \in (0,1)} |\hat{G}| \right\|_{L^2(\Re \lambda = \beta_2)}.
$$

(6.58)

if $\beta_1 + \beta_2 = \beta$ and as long as all integrals are well-defined. We introduce the short notation $\| \hat{\varphi} \|_{\Re \lambda = \beta} := \left\| \sup_{v \in (0,1)} K^{[\lambda]}_{\epsilon} |\hat{\varphi}| \right\|_{L^2(\Re \lambda = \beta)}$. In view of (6.57) and (6.58),
for the proof of (6.54) it suffices to show for all $\ell \in [\frac{2}{3}, k]$ and for all integers $i \in [0, 3\ell - 2 + \kappa],$

$$\|(1 + |\lambda|^{3\ell+\kappa-2-i+\delta})|\hat{\rho}_-|||_{H^\infty,\lambda=\beta_1}|||\lambda|^i\hat{g}_-|||_{H^\infty,\lambda=\beta_2} \leq C_\delta \|\varphi\|_{X^{k-1/2}_\epsilon} \|g\|_{Z^k_\epsilon},$$

(6.59)

where we can arbitrarily choose $\delta > \frac{1}{2}$ and $\beta_1, \beta_2$ with $\beta_1 + \beta_2 = 3\ell - \frac{7}{2}$. With the notation $\delta = \frac{k}{2} + (\kappa - 1) < \delta$, it is equivalent enough to show

$$\|(1 + |\lambda|^{3\ell-1-i+\delta})|\hat{\rho}_-|||_{H^\infty,\lambda=\beta_1}|||\lambda|^i\hat{g}_-|||_{H^\infty,\lambda=\beta_2} \leq C_\delta \|\varphi\|_{X^{k-1/2}_\epsilon} \|g\|_{Z^k_\epsilon},$$

(6.60)

where we can arbitrarily choose $\delta \geq \frac{1}{2}$ and $\beta_1, \beta_2$ with $\beta_1 + \beta_2 = 3\ell - \frac{7}{2}$. Both $\beta_1, \beta_2$ as well as $\delta$ are allowed to depend on $\ell$ and $i$. In fact, in the sequel we will always choose $\delta = \frac{1}{2}$.

Recall that either $\rho_0 = \epsilon \psi_0 x$ or $\rho_0 = \psi_0 y$, and hence either $|\hat{\rho}_0(\lambda)| = |\lambda^i \hat{\psi}_0(\lambda + 1)|$ or $|\hat{\rho}_0(\lambda)| = |\partial_v \hat{\psi}_0(\lambda + 1)|$. By (6.35) and by the same argument as in the proof of Lemma 1, we thus have

$$\|(1 + |\lambda|^{3\ell+1})(1 + \mu^{\ell+1})|\hat{\rho}_0|||_{H^\infty,\lambda=3\ell-\frac{1}{2}} \leq C \|\varphi\|_{X^{k-1/2}_\epsilon}, \quad \forall \ell \in [0, k - \frac{1}{2}].$$

(6.61)

By (6.61) and (6.53), we have

$$\|(1 + (\frac{1}{e})^{\ell+1}|\lambda|^{3\ell+1})|\hat{\rho}_-|||_{H^\infty,\lambda=3\ell-\frac{1}{2}} \leq C \|\varphi\|_{X^{k-1/2}_\epsilon}, \quad \forall \ell \in [0, k - \frac{1}{2}].$$

(6.62)

Furthermore, in view of (2.44) and by (6.53), we also have

$$\|(\frac{1}{e})^{\ell}|\lambda|^{3\ell-2}|\hat{g}_-|||_{H^\infty,\lambda=3\ell-\frac{7}{2}} \leq C \|g\|_{Z^k_\epsilon}, \quad \forall \ell \in [\frac{2}{3}, k].$$

(6.63)

We prove (6.60) for the three extreme cases, that is the corners of the triangle in $(\ell, i)$ where $\ell \in [\frac{2}{3}, k]$ and $i \in [0, 3\ell - 2]$, the estimate of the other terms follows by interpolation of these estimates:

(a1) $\ell = \frac{2}{3}, i = 0$ (and hence $\beta = 3\ell - \frac{7}{2} = -\frac{3}{2}$). With the choice $\beta_1 = 3 \cdot \frac{1}{6} - \frac{1}{2} = 0, \beta_2 = 3 \cdot \frac{2}{3} - \frac{7}{2} = -\frac{3}{2}$ and $\delta = \frac{1}{2}$, we need to estimate

$$\|(1 + |\lambda|^3)|\hat{\rho}_-|||_{H^\infty,\lambda=0}|||\hat{g}_-|||_{H^\infty,\lambda=-\frac{3}{2}} \leq C \|\varphi\|_{X^{k-1/2}_\epsilon} \|g\|_{Z^k_\epsilon}.$$  

(b1) $\ell = k, i = 0$. With the choice $\beta_1 = 3(k - \frac{1}{2}) - \frac{1}{2} = 3k - 2, \beta_2 = 3 \cdot \frac{2}{3} - \frac{7}{2} = -\frac{3}{2}$ and $\delta = \frac{1}{2}$, we need to estimate

$$\|(1 + |\lambda|^{k-\frac{1}{2}}(\frac{1}{e})^{k-\frac{3}{2}})|\hat{\rho}_-|||_{H^\infty,\lambda=3k-2} |||\hat{g}_-|||_{H^\infty,\lambda=-\frac{3}{2}} \leq C \|\varphi\|_{X^{k-1/2}_\epsilon} \|g\|_{Z^k_\epsilon}.$$  

(c1) $\ell = k, i = 3k - 2$. With the choice $\beta_1 = 3 \cdot \frac{1}{6} - \frac{1}{2} = 0, \beta_2 = 3k - \frac{7}{2}$ and $\delta = \frac{1}{2}$, we need to estimate

$$\|(1 + |\lambda|^3)|\hat{\rho}_-|||_{H^\infty,\lambda=0}|||\lambda|^{3k-2}(\frac{1}{e})^{k}|\hat{g}_-|||_{H^\infty,\lambda=3k-2} \leq C \|\varphi\|_{X^{k-1/2}_\epsilon} \|g\|_{Z^k_\epsilon}.$$  


Now, using (6.62) and (6.63), it can be easily checked that the above estimates (a1), (b1) and (c1) hold true for \( k \geq 1 \). This concludes the proof of (6.50) and hence of (6.48).

It remains to give the proof of

\[
\| (1 - \gamma) g \|_{Z^k_*} \leq C \| \varphi \|_{X^{k-1/2}_*} \| g \|_{Z^k_*}, \quad \text{for } \rho \in \{ \varepsilon \psi_x, \psi_y \}. \tag{6.64}
\]

Indeed, in view of the Taylor expansion

\[
\gamma - 1 = \psi_y - \psi_y^2 + \cdots,
\]

the estimate (6.64) follows by (6.50) together with the fact that \( \| \varphi \|_{X^{k-1/2}_*} \leq \delta < 1 \).

**Proof of (6.49)** The proof of (6.49) proceeds similarly to the proof of (6.48). As before, we show the estimate for the crucial high frequency/high frequency case \( \rho_{-w} w_- \). With the same arguments as before, we need to show [correspondingly to (6.60)]:

\[
\left( \frac{1}{\varepsilon} \right)^k | \lambda |^{3\ell - 1 - i + \kappa} | \hat{\rho}_- |_{\| \varphi \|_{X^{k-1/2}_*}} (1 + | \lambda |^{1 + \delta}) | \hat{\varphi}^- |_{\| \phi \|_{X^{k-1/2}_*}} \leq C \| \varphi \|_{X^{k-1/2}_*} \| w_x \|_{Z^k_*} + \| \frac{1}{\varepsilon} w_y \|_{Z^k_*} =: \mathcal{R}, \tag{6.65}
\]

(we have put the \( \delta \) on the second factor). In the above inequality, we can arbitrarily choose \( \beta_1, \beta_2, \delta \) as long as \( \beta_1 + \beta_2 = 3\ell - \frac{5}{2} \) and \( \delta \geq \frac{1}{2} \). Both \( \beta_1, \beta_2 \) as well as \( \delta \) may depend on \( \ell, i \). We note that by (2.44), we have

\[
\left( \frac{1}{\varepsilon} \right)^k | \lambda |^{3\ell - 1} | \hat{\rho}_- |_{\| \varphi \|_{X^{k-1/2}_*}} \leq C \left( \| w_x \|_{Z^k_*} + \| \frac{1}{\varepsilon} w_y \|_{Z^k_*} \right), \quad \forall \ell \in \left[ \frac{1}{2}, k \right]. \tag{6.66}
\]

We prove (6.65) in the case when the maximum number of derivatives fall onto \( \rho \), that is \( \ell = k \) and \( i = 0 \) (in particular, \( \kappa = 0 \) where we recall that \( \kappa \) is defined as the smallest nonnegative integer such that \( 3\ell - 2 \in \mathbb{N} \)). With the choice \( \beta_1 = 3(k - \frac{1}{2}) - \frac{1}{2} = 3k - 2, \beta_2 = 3 \cdot 1 - \frac{5}{2} = \frac{1}{2} \) and \( \delta = 1 \), we thus need to estimate

\[
\left( \frac{1}{\varepsilon} \right)^k | \lambda |^{3k - 1} | \hat{\rho}_- |_{\| \varphi \|_{X^{k-1/2}_*}} (1 + | \lambda |)(\frac{1}{\varepsilon})^\frac{1}{2} | \hat{\varphi}^- |_{\| \phi \|_{X^{k-1/2}_*}} \leq C \mathcal{R}.
\]

This estimate holds true as can easily be checked using (6.62) and (6.66). The estimate of the terms \( \rho_{+w} w_- \), \( \rho_{-w} w_+ \) and \( \rho_{+w} w_+ \) proceeds similarly. This concludes the estimate of (6.49) and hence of the proposition. \( \square \)

We turn to the proof of Proposition 7:

**Proof.** (Proof of Proposition 7) We first show the existence of a solution \( P \) of (6.42) on \( K^\psi \). By Lemma 10, we need to find a solution \( P \) of (6.40). We will solve (6.40) using an iterative argument. We set \( p_0 := 0 \) and iteratively define \( p_{i+1} \) to be the solution of (6.1) with right hand side \( g = R(p_{i-1}, \varphi) \) and boundary data \( f_x \). By (6.2) and (6.45), we get

\[
\| p_i - p_{i-1} \|_{Y^k_*} \leq C \| R(p_{i-1}, \varphi) \|_{Z^k_*} \leq C \| \varphi \|_{X^{k-1/2}_*} \| p_i - p_{i-1} \|_{Y^k_*} \leq C \| p_i - p_{i-1} \|_{Y^k_*} \leq \frac{1}{2} \| p_i - p_{i-1} \|_{Y^k_*}
\]
for \( \delta \) sufficiently small, \( \{ p_i \}_{i \in \mathbb{N}} \) is a Cauchy sequence and converges to a solution \( p \) of (6.40). By (6.2), \( p \) satisfies (6.43). The estimates (6.44) now follow from the representation (6.40) of the solution together with the estimates (6.45) and (6.46).

We also have the nonlinear version of the trace estimate in Lemma 8:

**Lemma 11.** Suppose that the assumptions of Proposition 7 are satisfied (in particular, \( k \geq 1 \)). Then with the notation of Proposition 7, we have

\[
|\partial_x (w_x) |_F X_{k-1}^\epsilon + (\frac{1}{\epsilon})^2 |\partial_x (p_x) |_F X_{k-1}^\epsilon \leq C \| f \|_{X_k^\epsilon}.
\]

(6.67)

**Proof.** Indeed, Equation (6.67) follows by application of (6.26) and since \( p \) satisfies (6.40). \( \Box \)

Note that the corresponding ‘bilinear’ estimates for \( w, p - \tilde{p} \) and \( w - \tilde{w} \) also hold (correspondingly as in Proposition 7). The estimates for the case of \( x \) derivative are

\[
|\partial_x (w_x) |_F X_{k-1}^\epsilon \leq C \| \varphi \|_{X_{k-1/2}^\epsilon} \| f \|_{X_k^\epsilon},
\]

\[
|\partial_x (p_x) |_F - |\partial_x (\tilde{p}_x) |_F X_{k-1}^\epsilon \leq C \| \varphi - \tilde{\varphi} \|_{X_{k-1/2}^\epsilon} \left( \| f \|_{X_k^\epsilon} + \| \tilde{f} \|_{X_k^\epsilon} \right) + C \| f - \tilde{f} \|_{X_k^\epsilon},
\]

(6.68)

\[
|\partial_x (w_x) |_F - |\partial_x (\tilde{w}_x) |_F X_{k-1}^\epsilon \leq C \| \varphi - \tilde{\varphi} \|_{X_{k-1/2}^\epsilon} \left( \| f \|_{X_k^\epsilon} + \| \tilde{f} \|_{X_k^\epsilon} \right) + C \| f - \tilde{f} \|_{X_k^\epsilon} \left( \| \varphi \|_{X_{k-1/2}^\epsilon} + \| \tilde{\varphi} \|_{X_{k-1/2}^\epsilon} \right).
\]

These estimates follow analogously to (6.67) but by using the ‘bilinear’ estimates (6.45) for \( R \). The corresponding estimates for (6.68) with \( |\partial_x (w_x) |_F \) and \( |\partial_x (\tilde{p}_x) |_F \) replaced by \( (\frac{1}{\epsilon})^2 |\partial_x (w_y) |_F \) and \( (\frac{1}{\epsilon})^2 |\partial_x (w_y) |_F \), respectively, also hold.

### 6.4. Estimates for the Profile

In this section, we prove Proposition 2. We show that

\[
\| N_\epsilon (f) \|_{TX_k^\epsilon} + \| N_\epsilon (f) \|_{L^2(X_0^\epsilon)} \leq C \| f \|_{TX_k^\epsilon}^2.
\]

(6.69)

The proof of (4.3) then follows by a straightforward extension of this estimate [indeed the estimates (6.45) show that the main part \( \partial_x B_\epsilon \partial_x \) of the operator \( N_\epsilon \) can be estimated like a bilinear operator]. We recall that

\[
N_\epsilon (f) = \frac{f_x}{(1 + \epsilon^2)^{\frac{3}{2}}} \left( \frac{f_{xx} - 3 \epsilon^2 f_x^2}{1 + \epsilon^2} \right)_{|x=0} + \partial_x B_\epsilon^f \left( \frac{f_x}{1 + \epsilon^2(1 + f)^{2\frac{3}{2}}} \right) + A_\epsilon f,
\]

cf. (2.20). The two main estimates in favor of (6.69) are:

\[
|f_{xx}|_{x=0} f_x \|_{TX_k^\epsilon} + |f_{xx}|_{x=0} f_x \|_{L^2(X_k^\epsilon)} \leq C \| f \|_{TX_k^\epsilon}^2,
\]

(6.70)

\[
|\partial_x B_\epsilon^f \partial_x f + A_\epsilon f \|_{TX_k^\epsilon} + |\partial_x B_\epsilon^f \partial_x f + A_\epsilon f \|_{TX_k^\epsilon} \leq C \| f \|_{TX_k^\epsilon}^2.
\]

(6.71)
Two “nonlinear” corrections have been neglected in the estimates (6.70) and (6.71). One correction is related to the $-3\varepsilon^2 f^2_x|_{x=0}$ term, the other correction is related to the term $f_x((1 + \varepsilon^2(1 + f)^2) - 1)$. Indeed, these corrections are easily controlled as lower order terms for sufficiently small $f$; the estimate of these terms is left to the reader. In the following, we will give the proof of (6.70) and (6.71).

Proof of (6.70) In view of the definition (2.36), (6.70) follows from

$$[\partial^i_x (f_{xx}|x=0 f_x)]_{L^2(X^i_{X^j_k})} \leq C\|f\|^2_{T X^{k+1}_X}, \text{ for } 0 \leq i + j \leq k. \quad (6.72)$$

In order to see (6.72), we note that by (5.2), we have

$$\|f_{xx}|x=0 f_x\|_{X^0} = \|f_{xx}|x=0 f_x\|_{L^2} \leq \|f_x\|_{L^2} \|f_{xx}\|_{L^\infty} \leq C\|f\|_{X^{1/2}} \|f\|_{X^1}, \quad (5.2)$$

$$\|f_{xx}|x=0 f_x\|_{X^j_X} \leq \|f_x\|_{X_{X}^j} \|f_{xx}\|_{L^\infty} \leq C\|f\|_{X^{1/2}} \|f\|_{X^{j+1}_X} \text{ for } 1 \leq j \leq k. \quad (6.73)$$

We take the square of (6.73) and integrate the equation in time. We then apply Hölder’s inequality with $L^1$ on the $\|\cdot\|_{X^j}$ term and $L^\infty$ on the $\|\cdot\|_{X^{1/2}}$ term. In view of (5.13), this yields (6.72) for $i = j = 0$. The estimate for $i > 0$ or $j > 0$ follows from (6.74): We take the square of (6.74) and integrate in time. We then apply Hölder’s inequality with $L^1$ on the $\|\cdot\|_{X^{1/2}}$ term and $L^\infty$ on the $\|\cdot\|_{X^j}$ term. In view of (5.13), this yields (6.72) for $i = 0$ and $1 \leq j \leq k$. It remains to prove (6.72) for $i > 0$. In this case we note that the time derivatives on the left hand side of (6.72) are distributed according to Leibniz’ rule on the two factors. The loss of regularity due to the time derivation is compensated by the fact that we only need to estimate the smaller norm $j = \ell - i$. In view of the definition of $\|\cdot\|_{TX^{k+1}_X}$, this yields (6.72) for all $0 \leq i \leq \ell$ and $j \geq 1$. This concludes the proof of (6.70).

Proof of (6.71) With the same argument as before, the proof of (6.71) can be reduced to the following estimate which does not involve time:

$$[\partial^i_x B^j_x \partial^j_x f + A^j_x f]_{X^j_X} \leq C\|f\|_{X^{j+1/2}} \|f\|_{X^{j-1}_X} \text{ for } 0 \leq j \leq k, \quad (6.75)$$

where we may assume that $\|f\|_{X^{k+1/2}_X} \leq \delta \leq 1$ (this corresponds to the estimate $\|f\|_{C^0(X^{k-1/2}_X)} \leq \delta \leq 1$ for the corresponding time-space estimate).

Let $\psi$, $\gamma$ be defined as in Lemma 9 and let $p$, $q$ be defined as in Proposition 7. By (2.19), (2.17) and (2.7), we have

$$A^j_x f_x = (1 + \varepsilon^2)^{-3/2} \partial^j_x \{ q_x - (\frac{1}{\varepsilon})^2 q_y \} |_{\Gamma},$$

$$-\partial^i_x B^j_x f_x = (1 + \varepsilon^2)^{-3/2} \partial^i_x \{ (1 + f) (p_x - \gamma \psi_x p_y) - (\frac{1}{\varepsilon})^2 \gamma p_y \} |_{\Gamma}.$$  

With the notation $w = p - q$ and $C^j_x = (1 + \varepsilon^2)^{-3/2} \leq 1$, we thus get

$$A^j_x f + \partial^i_x B^j_x f_x = C^j_x \partial^i_x \{ (\frac{1}{\varepsilon})^2 w_y - w_x - fp_x + (1 + f) \gamma \psi_x p_y + (\frac{1}{\varepsilon})^2 (\gamma - 1)p_y \} |_{\Gamma}. \quad (6.76)$$
We first note that for all $j \geq 0$, we have
\[
\| (w_x | \Gamma)_x \|_{X^j_\varepsilon} + \|(1 + \varepsilon^2)^{1/2}(w_y | \Gamma)_x \|_{X^j_\varepsilon} \leq C \| f \|_{X^{j+1/2}_\varepsilon} \| f \|_{X^{j+1}_\varepsilon}. \tag{6.68}
\]

For the other terms on the right hand side of (6.76) and for $j \geq 1$ we additionally use the algebra property (5.8). The estimate of the term $f(p_x | \Gamma)_x$ proceeds as follows
\[
\| (fp_x | \Gamma)_x \|_{X^j_\varepsilon} \leq \| f \|_{X^{j+1/2}_\varepsilon} \| p_x | \Gamma \|_{X^j_\varepsilon} \| (p_x | \Gamma)_x \|_{X^j_\varepsilon}, \tag{5.3},(6.67)
\]
where we also used $\| p_x | \Gamma \|_{L^2} \leq C \| p \|_{Y^j_\varepsilon} \leq C \| f \|_{X^j_\varepsilon}$ which follows from (5.12) and (6.43). We also note that in view of (6.36) we have $\| \varepsilon \psi_x | \Gamma \|_{X^{j+1/2}_\varepsilon} \leq C \delta < 1$ if the constant $\delta$ in the assumption of Theorem 1 is chosen sufficiently small. In view of the Taylor expansion $1 - \gamma = \psi_y - \psi_y^2 + \cdots$ and by (6.36), this implies also $\| \gamma | \Gamma - 1 \|_{X^j_\varepsilon} \leq \| \psi_x | \Gamma \|_{X^j_\varepsilon} \leq C$. For $j \geq 1$, the estimate of the remaining terms in (6.76) then follows using these estimates together with (5.8), (6.36) and (6.67).

It remains to give the estimate for $j = 0$, where the algebra property (5.8) does not apply. We show the estimate for the nonlinear term $(fp_x | \Gamma)_x$. Indeed, we have
\[
\| (fp_x | \Gamma)_x \|_{L^2} \leq \| f \|_{L^\infty} \| (p_x | \Gamma)_x \|_{L^2} + \| f_x \|_{L^2} \| p_x | \Gamma \|_{L^\infty} \leq C \| f \|_{X^j_\varepsilon} \| f \|_{X^j_\varepsilon}, \tag{6.67}
\]
where we also used $\| p_x | \Gamma \|_{L^2} \leq C \| p \|_{Y^j_\varepsilon} \leq C \| f \|_{X^j_\varepsilon}$ which follows from (5.12) and (6.43). The estimate of the other terms proceeds analogously. This concludes the proof of (6.75) and hence of the lemma.

### 7. Localization Argument

We prove Propositions 3–4, thus concluding the proof of Theorem 3 in Section 4. We will use the notation used in the proof of this theorem. We first note that the derivative $\delta \mathcal{L}^w_\varepsilon$ of $\mathcal{L}_\varepsilon$ at $w$ is given by
\[
\delta \mathcal{L}^w_\varepsilon f = L^w_\varepsilon f + K^w_\varepsilon f, \tag{7.1}
\]
cf. (2.10), where the operators $L^w_\varepsilon$ (leading order) and $K^w_\varepsilon$ (remainder) are given by
\[
L^w_\varepsilon f = f_t - \frac{1}{(1 + \varepsilon^2)^{3/2}} (B^w_\varepsilon f_x)_x =: f_t + A^w_\varepsilon f \tag{7.2}
\]
and
\[ K^w f = (D^w - 1)f_t + D^w f + \partial_t(D^w f w) \] (7.3)
\[- \left[ ((1 - x)\hat{s}_- + x\hat{s}_+)\hat{f} \right]_x - \left[ ((1 - x)\hat{s}_- + x\hat{s}_+)w \right]_x \] (7.4)
\[ - \left[ B^w_e \left( \frac{3w_x(2\varepsilon^2 w_x f)}{2(1 + \varepsilon^2 (w + f^*)^2)^{\frac{3}{2}}} \right) \right]_x + \left[ \delta B_e (w) \frac{f w_x}{(1 + \varepsilon^2 w^2)^{\frac{3}{2}}} \right]_x. \]

Here, we have introduced the following notation: \( s^w_\delta(t) \) is defined as in (2.11) with \( f \) replaced by \( w \); \( D^w \) is defined as in (2.4) with \( s^\pm_\delta(t) \) replaced by \( s^w_\delta(t) \). Furthermore \( D^w f = \delta D^w(f) \) and \( \sigma_{\pm, w} f = \delta \sigma_{\pm, w}(f) \) are the linearizations of \( D^w \) and \( \sigma_{\pm, w} \). A dot on the top of a symbol denotes the time derivative.

The proof is based on two small parameters \( \delta, \tau > 0 \). The parameter \( \delta \) is used to localize the estimates near the boundary. Note that in the interior where \( C > h > c \) > 0, the operator is uniformly parabolic. The parameter \( \tau \) specifies the time interval where the solution is defined. In the course of the proof, we will choose \( \delta \) and \( \tau = \tau(\delta) \), in this order, to be sufficiently small. In this section, we write \( c, C \) for all constants which depend only on \( f_{in}, k \), but neither depend on \( \delta \) nor \( \tau \).

For \( \delta > 0 \), we define a covering of \( E = (0, 1) \) by setting \( E_{1\delta} := (0, 2\delta) \), \( E_{2\delta} := (\delta, 1 - \delta) \), \( E_{3\delta} := (1 - 2\delta, 1) \). Correspondingly, let \( Q_{i\delta} := (0, \tau) \times E_{i\delta} \). We choose a smooth partition of unity \( \{ \psi_{i\delta} \}_{i=1,2,3} \) subordinate to this covering with \( \psi_{i\delta} \in C^\infty(\overline{E_{i\delta}}, [0, 1]) \), \( \sum_i \psi_{i\delta} = 1 \) on \( E \) and also \( \| \partial_j \psi_{i\delta} \|_{L^\infty} \leq C \delta^{-j} \) for all \( j \geq 0 \). We also define a second set of cut-off functions \( \{ \hat{\psi}_{i\delta} \}_{i=1,2,3} \). Let \( \tilde{E}_{1\delta} := (0, 4\delta) \), \( \tilde{E}_{2\delta} := (3\delta, 1 - 3\delta) \), \( \tilde{E}_{3\delta} := (1 - 4\delta, 1) \). We choose \( \hat{\psi}_{i\delta} \in C^\infty(\overline{\tilde{E}_{i\delta}}, [0, 1]) \) with \( \hat{\psi}_{i\delta} = 1 \) in \( E_{i\delta} \) and in particular, \( \hat{\psi}_{i\delta} \hat{\psi}_{j\delta} = \psi_{i\delta} \). Finally, we also consider the cut-off function \( \hat{\psi}_{1\delta} \), supported on an even larger set such that \( \hat{\psi}_{1\delta} \hat{\psi}_{i\delta} = \hat{\psi}_{i\delta} \). The support of \( \hat{\psi}_{1\delta} = 1 \) is included in \( (0, 5\delta) \), the functions \( \hat{\psi}_{2\delta}, \hat{\psi}_{3\delta} \) are defined analogously.

Since \( \| f_{in} \|_{X^2_{\varepsilon}(E)} < C \) and \( \| w \|_{TX^2_{\varepsilon}(Q\tau)} < C \), it follows that \( f_{in} \) and \( w \) are Hölder continuous in time and space. Therefore, by the boundary condition \( |h_{in, x}(0)| = |h_{in, x}(1)| = 1 \), and recalling that \( h_{in} = x(1 - x) + \int_0^x f_{in}(x')dx' \), there is \( \delta_0 > 0 \) such that for all \( \delta < \delta_0 \) we have \( |\partial_x h_{|0}| \in (\frac{1}{\tau}, 2) \) in \( E_{1\delta} \cap E_{3\delta} \) and \( h_0 \in (c, C) \) in \( E_{2\delta} \). Similarly, there is \( \delta_0 \) and \( \tau_0 \) such that the corresponding estimates hold for \( h_{w}(\tau) = x(1 - x) + \int_0^\tau \int_3^x w(x', \tau')dx' \) for all \( \delta < \delta_0 \) and every fixed time \( \tau < \tau_0 \). In the sequel, we will always assume \( 0 < \delta, \tau < \min(\delta_0, \tau_0, 0.1) \).

7.1. Proof of Proposition 3

Proof. (Proof of Proposition 3) In view of the approximate solution \( w \), defined in (4.14), we only need to consider the case of zero initial data so that in the following we may assume \( f_{in} = 0 \). We begin with the proof of maximal regularity estimate (4.17). That is, we assume that \( f \) satisfies
\[ \delta L^w f = g \text{ in } Q_{\tau} \text{ and } f = 0 \text{ on } (0, \tau) \times \partial E \] (7.5)
and with initial data \( f = 0 \) and we will show
\[
\| f \|_{TX^k_t(Q_t)} \leq C \| g \|_{TX^k_t(Q_t)}. \quad (7.6)
\]

With the partition of unity \( \psi_{i\delta}, i = 1, 2, 3 \) and by the triangle inequality, we have
\[
\| f \|_{TX^k_t(Q_t)} \leq \| \psi_{1\delta} f \|_{TX^k_t(Q_t)} + \| \psi_{2\delta} f \|_{TX^k_t(Q_t)} + \| \psi_{3\delta} f \|_{TX^k_t(Q_t)}. \quad (7.7)
\]

We begin with the estimate for \( \psi_{1\delta} f \) (related to the left boundary of the domain). The idea is to use that on \( Q_{1\delta r} \) (that is near the left boundary), \( \partial L \) (and also \( L^w \)) are approximated by \( L_\varepsilon \), where \( L_\varepsilon f := f_i + A_\varepsilon f \) and where \( A_\varepsilon \) is defined in (2.19). We first claim that
\[
\| \psi_{1\delta} f \|_{TX^k_t(Q_t)} \leq C_\delta \| \tilde{\psi}_{1\delta} L_\varepsilon (\psi_{1\delta} f) \|_{TX^k_t(Q_t)} + C_\delta \| f \|_{TX^k_t(Q_t)} + \frac{1}{10} \| f \|_{TX^k_t(Q_t)}, \quad (7.8)
\]
where we recall \( \tilde{\psi}_{1\delta} \psi_{1\delta} = \psi_{1\delta} \) and hence \( (1 - \tilde{\psi}_{1\delta}) \psi_{1\delta} = 0 \). In order to see (7.8), we first note that by (4.2), we get \( \| \psi_{1\delta} f \|_{TX^k_t(Q_t)} \leq C \| L_\varepsilon \psi_{1\delta} f \|_{TX^k_t((0, r) \times K)} \) (the \( L^2 \)-term is estimated by the other term since the support of the right hand side is bounded). Furthermore,
\[
L_\varepsilon (\psi_{1\delta} f) = \tilde{\psi}_{1\delta} L_\varepsilon (\psi_{1\delta} f) + (1 - \tilde{\psi}_{1\delta}) L_\varepsilon (\psi_{1\delta} f) = \tilde{\psi}_{1\delta} A_\varepsilon (\psi_{1\delta} f) + \psi_{1\delta} f_i + (1 - \tilde{\psi}_{1\delta}) A_\varepsilon (\psi_{1\delta} f).
\]

Furthermore, by (2.19) and since \( \tilde{\psi}_{1\delta} \) depends only on \( x \), by a short calculation we obtain
\[
A_\varepsilon (\psi_{1\delta} f) = \partial_x \left( -p'_x + \frac{1}{\varepsilon^2} p'_y \right),
\]
\[
(1 - \tilde{\psi}_{1\delta}) A_\varepsilon (\psi_{1\delta} f) = \partial_x \left( -\tilde{p}'_x + \frac{1}{\varepsilon^2} \tilde{p}'_y \right) - \partial_{xx} \tilde{\psi}_{1\delta} p' - 2\partial_x \tilde{\psi}_{1\delta} p'_x + \frac{1}{\varepsilon^2} \partial_{x} \tilde{\psi}_{1\delta} p'_y,
\]
where \( p' : K \rightarrow \mathbb{R} \) and \( \tilde{p}' := (1 - \tilde{\psi}_{1\delta}) p' \) are the solutions of
\[
\begin{cases}
\Delta_{\varepsilon} p' = 0 & \text{in } K, \\
p' = (\psi_{1\delta} f)_x & \text{on } \Gamma = \partial_1 K, \\
p'_y = 0 & \text{on } \partial_0 K. 
\end{cases}
\]
\[
\begin{cases}
\Delta_{\varepsilon} \tilde{p}' = -2\partial_x \tilde{\psi}_{1\delta} p'_x - \partial_{xx} ^2 \tilde{\psi}_{1\delta} p' & \text{in } K, \\
\tilde{p}' = 0 & \text{on } \partial_1 K, \\
\tilde{p}'_y = 0 & \text{on } \partial_0 K.
\end{cases}
\quad (7.9)
\]

In the following, we use the following Rellich-type estimate for lower order terms which can be obtained by standard interpolation. For any given \( \delta > 0 \), we have
\[
\| f_{xx} \|_{X^k_{\varepsilon^{-1}}} \leq \delta \| f \|_{X^k_{\varepsilon}} + C_\delta \| f \|_{X^k_{\varepsilon^{-1}}}.
\quad (7.10)
\]

Using this inequality, estimate (7.8) follows by application of Proposition 6.
Let \( [L^w_\varepsilon, \psi_{1\delta}] := L^w_\varepsilon \psi_{1\delta} - \psi_{1\delta} L^w_\varepsilon \) denote the commutator of \( L^w_\varepsilon \) and \( \psi_{1\delta} \), then

\[
\| \tilde{\psi}_{1\delta} L^w_\varepsilon (\psi_{1\delta} f) \|_{TX^k_{\varepsilon}(Q_\tau)} \leq \| \tilde{\psi}_{1\delta} (L^w_\varepsilon - \delta L^w_\varepsilon) (f \psi_{1\delta}) \|_{TX^k_{\varepsilon}(Q_\tau)} + \| \tilde{\psi}_{1\delta} \delta L^w_\varepsilon (\psi_{1\delta} f) \|_{TX^k_{\varepsilon}(Q_\tau)}
\]

\[
\leq \| \tilde{\psi}_{1\delta} (L^w_\varepsilon - \delta L^w_\varepsilon) (f \psi_{1\delta}) \|_{TX^k_{\varepsilon}(Q_\tau)} + \| \tilde{\psi}_{1\delta} [\delta L^w_\varepsilon, \psi_{1\delta}] f \|_{TX^k_{\varepsilon}(Q_\tau)} + \| \tilde{\psi}_{1\delta} \delta \|_{TX^k_{\varepsilon}(Q_\tau)},
\]

where we have used \( \psi_{1\delta} \tilde{\psi}_{1\delta} = \psi_{1\delta} \). The estimate of the first and second term on the right-hand side of the above estimate is given in Lemmas 12 and 13. Choosing \( \delta \) sufficiently small, we thus obtain

\[
\| \psi_{1\delta} f \|_{TX^{k+1}_{\varepsilon}(Q_\tau)} \leq C_\delta \| g \|_{TX^k_{\varepsilon}(Q_\tau)} + C_\delta \| f \|_{TX^k_{\varepsilon}(Q_\tau)} + \frac{1}{6} \| f \|_{TX^{k+1}_{\varepsilon}(Q_\tau)}.
\]

The third term on the right hand side of (7.7) can be estimated analogously. For the middle term (corresponding to the interior of the domain), we note that in the interior, our weighted Sobolev norms are equivalent to standard Sobolev norms (with equivalence depending on \( \delta \)) and an analogous estimate to the one above can be achieved for the middle term using standard parabolic estimates, see also [16,30]. Altogether, these estimates yield

\[
\| f \|_{TX^{k+1}_{\varepsilon}(Q_\tau)} \leq C_\delta \| g \|_{TX^k_{\varepsilon}(Q_\tau)} + C_\delta \| f \|_{TX^k_{\varepsilon}(Q_\tau)} + \frac{1}{2} \| f \|_{TX^{k+1}_{\varepsilon}(Q_\tau)}.
\]

Then using that \( f|_{t=0} = 0 \), we deduce that \( \| f \|_{TX^k_{\varepsilon}(Q_\tau)} \leq C \tau^{1/2} \| f \|_{TX^{k+1}_{\varepsilon}(Q_\tau)} \) and we choose \( \tau \) such that \( C_\delta C \tau^{1/2} < \frac{1}{5} \). Hence,

\[
\| f \|_{TX^{k+1}_{\varepsilon}(Q_\tau)} \leq C_\delta \| \delta L^w_\varepsilon \|_{TX^k_{\varepsilon}(Q_\tau)} + \frac{1}{2} \| f \|_{TX^{k+1}_{\varepsilon}(Q_\tau)},
\]

which yields (7.6) by absorbing on the left hand side.

The existence part is similar to the proof of Lemma 3.4 of [21]. Indeed, the argument used there can be generalized since only very little of the particular structure is used. The main ingredient in this argument is the existence and maximal regularity for the linearized operator at the boundary. The second ingredient is the fact that the existence of a solution together with estimates in the interior follows by standard parabolic theory. We have already proved these properties for our operator. Finally, the argument also requires that the operator can be localized in the sense that the long-range interaction of the solution operator only yields a lower order contribution. Indeed, we have used this idea already in the proof of (7.8). \( \square \)

In the following we give the estimate for the right-hand side of (7.11). We use the notations and assumptions of the proof of Proposition 3.

**Lemma 12.** (Estimate of difference) For \( 0 < \tau < 1 \), we have

\[
\| \tilde{\psi}_{1\delta} (L^w_\varepsilon - \delta L^w_\varepsilon) (\psi_{1\delta} f) \|_{TX^k_{\varepsilon}(Q_\tau)} \leq C_\delta \| f \|_{TX^k_{\varepsilon}(Q_\tau)} + C_\delta \| f \|_{TX^{k+1}_{\varepsilon}(Q_\tau)}. \quad (7.12)
\]
Proof. Let $p'$ and $P$ be the solutions of

$$\begin{cases}
\Delta_x p' = 0 & \text{in } K, \\
p' = (\psi_{1\delta} f)_x & \text{on } \partial_1 K, \\
p'_y = 0 & \text{on } \partial_0 K,
\end{cases}$$

$$\begin{cases}
\Delta_x P = 0 & \text{in } \Omega^w, \\
P = (\psi_{1\delta} f)_x & \text{on } \partial_1 \Omega^w, \\
P_y = 0 & \text{on } \partial_0 \Omega^w,
\end{cases}$$

(7.13)

where $\Omega^w = \{(x, y) | 0 < x < 1, \text{ and } 0 < y < h^w(x)\}$ and $\partial_1 \Omega^w = \text{graph } h^w$ and $h^w = x(1-x) + \int_0^x w(x')dx'$. The existence of a solution $p'$ follows from Proposition 6, the existence of a solution $P$ can be shown with similar arguments as in the proof of Proposition 7; they will not be detailed here. The reason to introduce these two functions is that we have $L_\varepsilon - L^w_\varepsilon = A_\varepsilon - A^w_\varepsilon$. Furthermore, $A_\varepsilon(\psi_{1\delta} f) = \partial_x (-p'_x + \frac{1}{\varepsilon} p'_y)$ and $A^w_\varepsilon(\psi_{1\delta} f) = \partial_x (-h^w P_x + \frac{1}{\varepsilon} P_y)$. Note that $A_\varepsilon$ and $A^w_\varepsilon$ are not defined on the same interval in $x$. To compare them, we therefore use the cut-off function $\tilde{\psi}_{1\delta}$. Indeed, $\tilde{\psi}_{1\delta} P$ can be seen as a function in the domain $K \tilde{\psi}_{1\delta}(w - \frac{1}{\varepsilon})$ since the domains $K \tilde{\psi}_{1\delta}(w - \frac{1}{\varepsilon})$ and $\Omega^w$ coincide on the support of $\tilde{\psi}_{1\delta} P$ (recall that $\tilde{\psi}_{1\delta} \tilde{\psi}_{1\delta} = \tilde{\psi}_{1\delta})$. More precisely, $\tilde{\psi}_{1\delta} P$ solves

$$\begin{cases}
\Delta_x (\tilde{\psi}_{1\delta} P) = 2\partial_x \tilde{\psi}_{1\delta} \partial_x P + \partial_x^2 \tilde{\psi}_{1\delta} P & \text{in } K \tilde{\psi}_{1\delta}(w - \frac{1}{\varepsilon}), \\
(\tilde{\psi}_{1\delta} P)_y = 0 & \text{on } \partial_0 K \tilde{\psi}_{1\delta}(w - \frac{1}{\varepsilon}),
\end{cases}$$

(7.14)

We use Lemma 9 to construct a coordinate transform $\Psi = (\psi, id)$ from $K$ to $K \tilde{\psi}_{1\delta}(w - \frac{1}{\varepsilon})$. Hence, arguing as in the proof of Proposition 7, we deduce that $p = P \circ \Psi$ solves

$$\begin{cases}
\Delta_x (\tilde{\psi}_{1\delta} p) = R(p, \psi) + \mathcal{K}_1(p, \psi) & \text{in } K, \\
\pi = 0 & \text{on } \partial_1 K, \\
\pi_y = 0 & \text{on } \partial_0 K,
\end{cases}$$

(7.15)

where $R(p, \psi)$ is given in (6.41) and where $\mathcal{K}_1(p, \psi)$ is a lower order term involving at most one derivative of $P$ and such that $\text{supp } \mathcal{K}_1 \subseteq \text{supp } \partial_x \tilde{\psi}_{1\delta}$. Arguing as in the proof of Proposition 7, we deduce that $\tilde{\psi}_{1\delta} p \in Y^1_\varepsilon$ and hence $\tilde{\psi}_{1\delta} P \in Y^1_\varepsilon(\Omega^w)$. Similarly, it can also be shown that $(\tilde{\psi}_{2\delta} + \tilde{\psi}_{3\delta}) P \in Y^1_\varepsilon(\Omega^w)$. One can then start a bootstrap argument and prove that $P \in Y^{k+1}_\varepsilon(\Omega^w), \|P\|_{Y^{k+1}_\varepsilon(\Omega^w)} \leq C(w)\|\psi_{1\delta} f\|_{Y^k_\varepsilon}$ and $\|P\|_{Y^{k+1}_\varepsilon(\Omega^w)} \leq C(w)\|\psi_{1\delta} f\|_{Y^k_\varepsilon}$ and $\|P\|_{Y^{k+1}_\varepsilon(\Omega^w)} \leq C(w)\|\psi_{1\delta} f\|_{X^{k+1}_\varepsilon}$. Note that $\tilde{\psi}_{1\delta} p'$ satisfies the same system as (7.14) when replacing $P$ by $p'$ and $K \tilde{\psi}_{1\delta}(w - \frac{1}{\varepsilon})$ by $K$. Taking the difference, $\pi = p - p'$, we thus obtain

$$\begin{cases}
\Delta_x (\tilde{\psi}_{1\delta} \pi) = R(p, \psi) + \mathcal{K}_2(p, p', \psi) & \text{in } K, \\
\pi = 0 & \text{on } \partial_1 K, \\
\pi_y = 0 & \text{on } \partial_0 K.
\end{cases}$$
where $K_2(p, p', \psi)$ is a lower order term with at most one derivative of $p$ or $p'$. We have

$$\tilde{\psi}_{18}A_w^\epsilon(\psi_{18}f) = \left( -h_x^\epsilon(\tilde{\psi}_{18}P)_x + \frac{1}{\epsilon^2}(\tilde{\psi}_{18}P)_y + h_x^\epsilon P(\tilde{\psi}_{18})_x \right)_x$$

$$- \partial_x \tilde{\psi}_{18} \left( -h_x^\epsilon P_x + \frac{1}{\epsilon^2}P_y \right),$$

$$\tilde{\psi}_{18}A_x(\psi_{18}f) = \left( -(\tilde{\psi}_{18}P')_x + \frac{1}{\epsilon^2} (\tilde{\psi}_{18}P')_y + P'(\tilde{\psi}_{18})_x \right)_x$$

$$- \partial_x \tilde{\psi}_{18} \left( -p'_x + \frac{1}{\epsilon^2}p'_y \right).$$

Hence and since $|h_x^\epsilon - 1| \leq C\delta$ on supp $\tilde{\psi}_{18}$, we obtain

$$\tilde{\psi}_{18}(L_w^\epsilon - L_x)(\psi_{18}f) = \tilde{\psi}_{18}(A_w^\epsilon - A_x)(\psi_{18}f)$$

$$= \left( -(\tilde{\psi}_{18}\pi)_x + \frac{1}{\epsilon^2}(\tilde{\psi}_{18}\pi)_y + \pi(\tilde{\psi}_{18})_x \right)_x$$

$$- \partial_x \tilde{\psi}_{18} \left( -\pi_x + \frac{1}{\epsilon^2}\pi_y \right) + K_3,$$  \hspace{1cm} (7.16)

where the remainder term $K_3$ satisfies $\|K_3\|_{T\mathcal{X}_\epsilon^{k+1}(Q_\tau)} \leq C\delta \|f\|_{T\mathcal{X}_\epsilon^{k+1}(Q_\tau)}$. Notice that the second term on the right-hand side on (7.16) consists of lower order term. Hence, we only need to estimate the first term. We have

$$\|K_2\|_{L^{k+1}_\epsilon} \leq C\delta \|f\|_{X^{k}_\epsilon} + \delta \|f\|_{X^{k+1}_\epsilon} \quad \text{and} \quad \|R(p, \psi)\|_{L^{k+1}_\epsilon} \leq C\delta \|f\|_{X^{k+1}_\epsilon}.$$  

Hence, $\|\pi\|_{Y^{k+1}_\epsilon} \leq C\delta \|f\|_{X^{k}_\epsilon} + C\delta \|f\|_{X^{k+1}_\epsilon}$ and (7.12) follows easily. In a sense we proved that the difference between $A_w^\epsilon$ and $A_x$ comes from terms which are either small or terms which are more regular. This shows the estimate of $L_w^\epsilon - L_x^\epsilon$; the estimate of $K_w^\epsilon$ follows similarly, also using (7.10). This concludes the proof of the lemma. □

**Lemma 13.** (Estimate of commutator) For $0 < \tau < 1$, we have

$$\|\tilde{\psi}_{18}[\delta L_x, \psi_{18}]f\|_{T\mathcal{X}_\epsilon^{k}(Q_\tau)} \leq C\delta \|f\|_{T\mathcal{X}_\epsilon^{k}(Q_\tau)} + C\delta \|f\|_{T\mathcal{X}_\epsilon^{k+1}(Q_\tau)}.$$  

**Proof.** Let $P$ and $Q$ be the solutions of

$$\begin{cases}
\Delta_x P = 0 & \text{in } \Omega^w, \\
P = (\psi_{18}f)_x & \text{on } \partial_1\Omega^w, \\
P_y = 0 & \text{on } \partial_1\Omega^w,
\end{cases} \quad \begin{cases}
\Delta_x Q = 0 & \text{in } \Omega^w, \\
Q = f_x & \text{on } \partial_1\Omega^w, \\
Q_y = 0 & \text{on } \partial_0\Omega^w.
\end{cases}$$  \hspace{1cm} (7.17)

Arguing as above, we can prove that $Q, P \in Y^{k+1}_\epsilon(\Omega^w)$ and that $\|Q\|_{Y^{k+1}_\epsilon(\Omega^w)}$,

$$\|P\|_{Y^{k+1}_\epsilon(\Omega^w)} \leq C(w) \|f\|_{X^{k+i}_\epsilon} \quad \text{for } i = 0, 1. \quad \text{Moreover,}$$

$$\tilde{\psi}_{18}[L_x, \psi_{18}]f = \tilde{\psi}_{18} \left[ (-w_x P_x + \frac{1}{\epsilon^2}P_y)_x - \psi_{18}(-w_x Q_x + \frac{1}{\epsilon^2}Q_y)_x \right]$$

$$= \tilde{\psi}_{18} \left[ (-w_x(P - \psi_{18}Q)_x + \frac{1}{\epsilon^2}(P - \psi_{18}Q)_y)_x \right] + K.$$  \hspace{1cm} (7.18)
where the terms of lower order are collected in $K$. Note that $P - \psi_{1\delta} Q$ solves

\[
\begin{align*}
\Delta \varepsilon (P - \psi_{1\delta} Q) &= -2\partial_x \psi_{1\delta} \partial_x Q - \partial_x^2 \psi_{1\delta} Q \quad \text{in } \Omega^w, \\
P - \psi_{1\delta} Q &= \psi_{1\delta,x} f \quad \text{on } \partial_1 \Omega^w, \\
(P - \psi_{1\delta} Q)_y &= 0 \quad \text{on } \partial_0 \Omega^w.
\end{align*}
\]  

(7.19)

The right-hand side of (7.19) is more regular and allows us to estimate $P - \psi_{1\delta} Q$ by $\|P - \psi_{1\delta} Q\|_{Y^k} \leq C \|f\|_{X^k} + \frac{1}{10} \|f\|_{X^{k+1}}$. Moreover, the operator $K$ on the right hand side of (7.18) only involve one derivative of $Q$ and hence can be estimated similarly. This concludes the proof of the lemma. $\square$

7.2. Proof of Proposition 4

A localization of the estimate in Lemma 5 yields the following estimate: for all $k, i \in \mathbb{N}_0$, $\tau > 0$ and $f_0 \in TX^{k+1}_\varepsilon(Q_\tau)$ we have

\[
\|\partial^i_t f_0\|_{C^0(\mathcal{S}^{k+1/2}(E))} \leq C \left( \|\partial^i_t f_0\|_{L^2(\mathcal{S}^{k}(E))} + \|\partial^i_t f_0\|_{L^2(\mathcal{S}^{k+1}(E))} \right). 
\]  

(7.20)

In view of Lemma 5, it remains to show the boundedness and continuous differentiability of $\mathcal{L}_\varepsilon$, defined in (2.10). The highest order term of $\mathcal{L}_\varepsilon$ is given by the operator $\partial_x B^f_\varepsilon$, see 2.7. Boundness of this operator follows from a localization of the estimates in Proposition 7 and Lemma 8. For $p$ be defined by (2.7), we have

\[
\|\partial_x B^f_\varepsilon f_x\|_{X^k_\varepsilon(E)} \leq C \left( \|\partial_x (p_x)|_\Gamma\|_{X^k_\varepsilon(E)} + \left(\frac{1}{\varepsilon}\right)^2 \|\partial_x (p_y)|_\Gamma\|_{X^k_\varepsilon(E)} \right)
\]

\[
\leq C \left( \|f\|_{X^{k+1}(E)} + \|g\|_{Z^{k+1}} \right),
\]

where $g$ is a lower order term. Indeed, near the left boundary of $E$, $g$ is the same term as in the right hand side of (7.15). In the center and near the right boundary of $E$, $g$ is defined analogously. By a localization of (6.46) and since $\|f\|_{X^{k+1}(Q_\tau)} \leq C$, we hence obtain

\[
\|\partial_x B^f_\varepsilon f_x\|_{X^k_\varepsilon} \leq C \|f\|_{X^{k+1}(E)}.
\]

Furthermore, also using that $\|f\|_{X^{k+1}(Q_\tau)} \leq C$, we have

\[
\|\delta (f + f^*)\|_{X^k_\varepsilon} \leq C \|\delta (f + f^*)\|_{L^\infty} \left( \|f\|_{X^k_\varepsilon} + \|f^*\|_{X^k_\varepsilon} \right)
\]

\[
\leq C \|f\|_{X^k_\varepsilon} \left( \|f\|_{X^k_\varepsilon} + \|f^*\|_{X^k_\varepsilon} \right)
\]

\[
\leq C \|f\|_{X^{k+1}_\varepsilon}.
\]

It remains to show the boundedness and continuity of the first derivative $\delta \mathcal{L}^w_\varepsilon$. In view of (7.1), the estimate of $\delta \mathcal{L}^w_\varepsilon$ leads to terms analogous to the one for the estimate of $\mathcal{L}_\varepsilon$. The estimate thus follows analogously and is not more difficult than the estimate of $\mathcal{L}_\varepsilon$ itself. Similarly, continuity of the first derivative follows by a bound on the second derivative. As before, in view of the structure of the operator it is clear that these terms do not impose any other difficulties than the ones we found already when estimating the operator $\mathcal{L}_\varepsilon$ itself. Hence, a bound on the second derivative can be obtained similarly to the calculations before.
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Appendix

For the convenience of the reader, we present the derivation of the dissipation relations (1.5) and (1.6).

Thin-film equation Suppose that $h$ is a solution of the thin-film equation (1.4) with $\text{supp } h = (s_-(t), s_+(t))$, sufficiently regular up to the free boundary. Using integration by parts, we get

$$
\frac{d}{dt} \frac{1}{2} \int_{s_-}^{s_+} h_x^2 \, dx = \frac{1}{2} \varepsilon^2 (\dot{s}_+ - \dot{s}_-) + \int_{s_-}^{s_+} h_x h_{xt} \, dx (1.4) = \frac{1}{2} \varepsilon^2 (\dot{s}_+ - \dot{s}_-) - \gamma \int_{s_-}^{s_+} h_x (hh_{xxx})_{xx} \, dx
$$

(1.4)

$$
= \frac{1}{2} \varepsilon^2 (\dot{s}_+ - \dot{s}_-) - \gamma \int_{s_-}^{s_+} hh_{xx}^2 \, dx - \gamma h_x (hh_{xxx}) \bigg|_{s_-}^{s_+} + \gamma h_{xx} h_{xxx} \bigg|_{s_-}^{s_+}
$$

$$
= \frac{1}{2} \varepsilon^2 (\dot{s}_+ - \dot{s}_-) - \gamma \int_{s_-}^{s_+} hh_{xx}^2 \, dx - \gamma h_{xx}^2 h_{xxx} \bigg|_{s_-}^{s_+}
$$

The dissipation relation (1.6) for the thin-film equation hence is as the following:

$$
\frac{1}{2} \frac{d}{dt} \left( \int_{s_-}^{s_+} h_x^2 \, dx + \varepsilon^2 \{ h > 0 \} \right) + \gamma \int_{s_-}^{s_+} hh_{xx}^2 \, dx = 0. \quad (7.21)
$$

For the thin-film equation in the case of the infinite wedge, we we recall the definition (2.15) of $f$, that is $\varepsilon f = h_x - \varepsilon$. The energy in this case is given by (2.26),

$$
E_0(t) = \int_{s(t)}^{\infty} f^2 \, dx. \quad (7.22)
$$

Indeed, a straightforward integration by parts as before shows

$$
\frac{d}{dt} E_0(t) = \frac{d}{dt} \left( \int_{s_-}^{s_+} h_x^2 \, dx + \varepsilon^2 \{ h > 0 \} \right) + \gamma \int_{s_-}^{s_+} hh_{xx}^2 \, dx = - \frac{1}{\varepsilon^2} \int_0^\infty hh_{xx}^2 \, dx.
$$

Note that in this case all boundary terms due to the integration by parts vanish.

Darcy flow Now, suppose that $h$ is a solution of the Darcy flow (1.2) with $\text{supp } h = (s_-(t), s_+(t))$. We use the notation $\partial_1 \Omega := \partial \Omega(t) \cap \{ y > 0 \}$ and $\partial_0 \Omega := \partial \Omega(t) \cap \{ y = 0 \}$. For $\varepsilon > 0$ sufficiently small, we may assume that the solution is sufficiently regular, to allow for the integrations by parts, applied in the following. The slope
at the contact line is $\varepsilon$. The angle $\delta$ at the contact line is given by $\tan \delta = \varepsilon$. Furthermore, let $\alpha$ be defined by $\alpha = \cos \delta$. By Green’s first identity, we have

$$\frac{d}{dt} |\partial_1 \Omega| = - \int_{\partial_1 \Omega} \kappa V \, d\sigma + \alpha (\hat{s}_+ - \hat{s}_-)$$

This yields the dissipation relation (1.5), that is

$$\frac{d}{dt} (|\partial_1 \Omega| - \alpha |\partial_0 \Omega|) = - \frac{1}{\gamma} \int_{\partial_1 \Omega} |\nabla p|^2 \, dx \leq 0.$$  

A similar calculation can be done in the situation of a Darcy flow when the initial data approximate the infinite wedge. In this case, we calculate

$$\frac{d}{dt} \int_{s(t)}^\infty \left( \sqrt{1 + h_x^2} - \sqrt{1 + \varepsilon^2} \right) \, dx = \int_{s(t)}^\infty \frac{h_x h_{x,t}}{\sqrt{1 + h_x^2}} \, dx$$

$$= - \int_{s(t)}^\infty \left( \frac{h_x}{\sqrt{1 + h_x^2}} \right)_x h_t \, dx - \frac{\varepsilon}{\sqrt{1 + \varepsilon^2}} h_t(t, s(t))$$

$$= - \int_{s(t)}^\infty \kappa \nabla p \cdot n \sqrt{1 + h_x^2} \, dx + \frac{\varepsilon^2}{\sqrt{1 + \varepsilon^2}} \hat{s}(t)$$

$$= - \frac{1}{\gamma} \int_{\partial_1 \Omega} p \partial_n p \, d\sigma + \frac{\varepsilon^2}{\sqrt{1 + \varepsilon^2}} \hat{s}(t), \quad (7.23)$$

where we used that $h_t = \nabla p \cdot n \sqrt{1 + h_x^2} = V \sqrt{1 + h_x^2}$, where $V$ is the normal velocity of the fluid on $\partial_1 \Omega$ and where $n$ is the normal vector pointing from the fluid into the gas phase. Furthermore, we used $h_t + \varepsilon \hat{s}h_x = 0$ at $x = s(t)$ and hence $h_t = -\varepsilon \hat{s}$. Since $(h(t, x) - \varepsilon x) \to 0$ when $x$ goes to infinity and hence at $x = s(t)$, we have $(h(t, x) - \varepsilon x)|_{x=s(t)} = -\varepsilon s(t)$. We recall the definition (2.15) of $f$, that is $\varepsilon f = h_x - \varepsilon$. From this, we deduce

$$\int_{s(t)}^\infty f \, dx = s(t). \quad (7.24)$$

Also using Green’s first identity, we thus obtain from (7.23)–(7.24),

$$\frac{d}{dt} \int_{s(t)}^\infty \left( \sqrt{1 + h_x^2} - \sqrt{1 + \varepsilon^2} - \frac{\varepsilon^2}{\sqrt{1 + \varepsilon^2}} f \right) \, dx = - \frac{1}{\gamma} \int_{\partial_1 \Omega} |\nabla p|^2 \, dx. \quad (7.25)$$

For the (renormalized) energy in terms of $f$ with $h_x = \varepsilon (1 + f)$, defined as in (2.22), that is

$$E_\varepsilon(t) = \frac{1}{\varepsilon^2} \int_{s(t)}^\infty \sqrt{1 + \varepsilon^2 (1 + f)^2} - \sqrt{1 + \varepsilon^2} - \frac{\varepsilon^2}{\sqrt{1 + \varepsilon^2}} f \, dx \quad (7.26)$$
and get the dissipation relation

\[
\frac{d}{dt} E_\varepsilon(t) = -\frac{1}{\gamma} \int_\Omega |\nabla p|^2 \, dx.
\]

(7.27)

In fact, the energy has a sign as the following calculation shows:

\[
E_\varepsilon(t) = \sqrt{1 + \varepsilon^2} \int_0^\infty \left( \sqrt{1 + \frac{\varepsilon^2}{1+\varepsilon^2} (2f + f^2)} - 1 - \frac{\varepsilon^2}{1+\varepsilon^2} f \right) \, dx.
\]

(7.28)

We define the sets \( G = \{ x \in (0, \infty): |f| \leq \frac{1}{\varepsilon} \} \). By the Taylor expansion

\[
\sqrt{1 + X} \geq 1 + \frac{1}{2} X - \frac{1}{8} X^2 \quad \text{for} \quad X \geq 0,
\]

we get

\[
E_\varepsilon(t) \geq \int_G \frac{1}{2(1 + \varepsilon^2)} f^2 - \frac{\varepsilon^2}{8(1 + \varepsilon^2)^2} (2f + f^2)^2 \, dx
\]

\[
\geq \int_G \frac{1}{2} f^2 - \frac{\varepsilon^2}{4} f^2 - \frac{\varepsilon^2}{4} f^4 \, dx \geq c \int_G f^2 \, dx.
\]

With the notation \( S = (0, \infty) \setminus G \), we also have

\[
E_\varepsilon(t) \geq \frac{\sqrt{1 + \varepsilon^2}}{\varepsilon^2} \int_S \left( \sqrt{1 + \frac{\varepsilon^2}{1+\varepsilon^2} (2f + f^2)} - 1 - \frac{\varepsilon^2}{1+\varepsilon^2} f \right) \, dx \geq \frac{c}{\varepsilon} \int_S |f| \, dx.
\]

These calculations show that we have

\[
E_\varepsilon(t) \geq c \int_G f^2 \, dx + \frac{c}{\varepsilon} \int_S |f| \, dx.
\]

(7.29)

This motivates us to impose the additional (natural) condition on the initial energy:

\[
E_\varepsilon(0) \leq C \alpha^2.
\]

References

1. AMBROSE, D.M.: Well-posedness of two-phase Hele-Shaw flow without surface tension. Eur. J. Appl. Math. 15(5), 597–607 (2004). doi:10.1017/S0956792504005662
2. AMBROSE, D.M., MASMoudi, N.: The zero surface tension limit of two-dimensional water waves. Commun. Pure Appl. Math. 58(10), 1287–1315 (2005)
3. ANGENENT, S.: Analyticity of the interface of the porous media equation after the waiting time. Proc. Am. Math. Soc. 102(2), 329–336 (1988)
4. BAZALIY, B.V., FRIEDMAN, A.: The Hele-Shaw problem with surface tension in a half-plane. J. Differ. Equ. 216(2), 439–469 (2005)
5. BAZALIY, B.V., FRIEDMAN, A.: The Hele-Shaw problem with surface tension in a half-plane: a model problem. J. Differ. Equ. 216(2), 387–438 (2005)
6. BEAR, J.: Dynamics of Fluids in Porous Media. Dover Publications, New York, 1972
7. BERETTA, E., BERTSCH, M., DAL PASSO, R.: Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. Arch. Ration. Mech. Anal. 129(2), 175–200 (1995)
8. BERTOZZI, A.L., PUGH, M.: The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. Commun. Pure Appl. Math. 49(2), 85–123 (1996)
9. Bertsch, M., Giacomelli, L., Karali, G.: Thin-film equations with “partial wetting”
energy: existence of weak solutions. Phys. D 209(1-4), 17-27 (2005)
10. Castro, A., Córdoba, D., Fefferman, C., Gancedo, F., Lopez-Fernandez, M.: 
Rayleigh–Taylor breakdown for the Muskat problem with applications to water waves.
Ann. Math. (2) 175(2), 909–948 (2012). doi:10.4007/annals.2011.173.1.10
11. Constantin, P., Pugh, M.: Global solutions for small data to the Hele-Shaw problem.
Nonlinearity 6(3), 393–415 (1993)
12. Córdoba, A., Córdoba, D., Gancedo, F.: Interface evolution: the Hele-Shaw and
Muskat problems. Ann. Math. (2) 173(1), 477–542 (2011). doi:10.4007/annals.2011.
173.1.10
13. Coutand, D., Shkoller, S.: Well-posedness of the free-surface incompressible Euler
equations with or without surface tension. J. Am. Math. Soc. 20(3), 829–930 (2007).
doi:10.1090/S0894-0347-07-00556-5
14. Daskalopoulos, P., Hamilton, R.: Regularity of the free boundary for the porous
medium equation. J. Am. Math. Soc. 11(4), 899–965 (1998)
15. Duchon, J., Robert, R.: Évolution d’une interface par capillarité et diffusion de vol-
ume. I. Existence locale en temps. Ann. Inst. H. Poincaré Anal. Non Linéaire 1(5),
361–378 (1984)
16. Eidelman, S.D.: Parabolic Systems. Translated from the Russian by Scripta Technica,
London. North-Holland Publishing Co., Amsterdam, 1969
17. Escher, J., Simonett, G.: Classical solutions for Hele-Shaw models with surface ten-
sion. Adv. Differ. Equ. 2(4), 619–642 (1997)
18. Escher, J., Simonett, G.: Classical solutions for the quasi-stationary Stefan problem
with surface tension. Differential Equations, Asymptotic Analysis, and Mathematical
Physics (Potsdam, 1996), Math. Res., Vol. 100. Akademie, Berlin, 98–104, 1997
19. Germain, P., Masmoudi, N., Shatah, J.: Global solutions for the gravity water waves
equation in dimension 3. Ann. Math. (2) 175(2), 691–754 (2012)
20. Giacomelli, L., Gnann, M.V., Knüpfer, H., Otto, F.: Well-posedness for the Navier-
slip thin-film equation in the case of complete wetting. J. Differ. Equ. 257(1), 15–81
(2014). doi:10.1016/j.jde.2014.03.010
21. Giacomelli, L., Knüpfer, H.: A free boundary problem of fourth order: classical solutions
in weighted Hölder spaces. Comm. Partial Differ. Equ. 35(11), 2059–2091 (2010).
doi:10.1080/03605302.2010.494262
22. Giacomelli, L., Knüpfer, H., Otto, F.: Smooth zero-contact-angle solutions to a
thin-film equation around the steady state. J. Differ. Equ. 245(6), 1454–1506 (2008)
23. Giacomelli, L., Otto, F.: Variational formulation for the lubrication approximation of
the Hele-Shaw flow. Calc. Var. Partial Differ. Equ. 13(3), 377–403 (2001)
24. Giacomelli, L., Otto, F.: Rigorous lubrication approximation. Interfaces Free Bound.
5(4), 483–529 (2003)
25. Kawarada, H., Koshigoe, H.: Unsteady flow in porous media with a free surface. Jpn.
J. Ind. Appl. Math. 8(1), 41–84 (1991)
26. Knüpfer, H.: Navier slip thin-film equation for partial wetting. Commun. Pure Appl.
Math. 64(9), 1263–1296 (2011)
27. Knüpfer, H., Masmoudi, N.: Well-posedness and uniform bounds for a nonlocal third
order evolution operator on an infinite wedge. Comm. Math. Phys. 320(2), 395–424
(2013). doi:10.1007/s00220-013-1708-z
28. Koch, H.: Non-euclidean singular integrals and the porous medium equation. Ph.D.
thesis, Dortmund (1999)
29. Kozlov, V.A., Mazya, V.G., Rossmann, J.: Elliptic boundary value problems in do-
mains with point singularities. Mathematical Surveys and Monographs, Vol. 52. Am.
Math. Soc., Providence, 1997
30. Krylov, N.V.: Lectures on elliptic and parabolic equations in Sobolev spaces. Graduate
Studies in Mathematics, Vol. 96. American Mathematical Society, Providence, 2008
31. Lannes, D.: Well-posedness of the water-waves equations. J. Am. Math. Soc. 18(3),
605–654 (2005)
32. Lindblad, H.: Well-posedness for the motion of an incompressible liquid with free surface boundary. Ann. Math. (2) 162(1), 109–194 (2005)

33. Masmoudi, N., Rousset, F.: Uniform regularity and vanishing viscosity limit for the free surface Navier–Stokes equations (2012). arXiv:1202.0657

34. McGeough, J., Rasmussen, H.: On the derivation of the quasi-steady model in electrochemical machining. J. Inst. Math. Appl. 13(1), 13–21 (1974)

35. Otto, F.: Lubrication approximation with prescribed nonzero contact angle. Commun. Partial Differ. Equ. 23(11–12), 2077–2164 (1998)

36. Prokert, G.: Existence results for Hele-Shaw flow driven by surface tension. Eur. J. Appl. Math. 9(2), 195–221 (1998)

37. Ren, W., Hu, D., Weinan, E.: Continuum models for the contact line problem. Phys. Fluids 22(10), 102103 (2010)

38. Reynolds, O.: On the theory of lubrication and its application to mr. beauchamp tower’s experiments, including an experimental determination of the viscosity of olive oil. Proc. R. Soc. Lond. 40, 191–203 (1886)

39. Rubenstein, L.I.: The Stefan problem. Translated from the Russian by A. D. Solomon. Translations of Mathematical Monographs, Vol. 27. American Mathematical Society, Providence, 1971

40. Shatah, J., Zeng, C.: Geometry and a priori estimates for free boundary problems of the Euler equation. Commun. Pure Appl. Math. 61(5), 698–744 (2008). doi:10.1002/cpa.20213

41. Siegel, M., Caflisch, R.E., Howison, S.: Global existence, singular solutions, and ill-posedness for the Muskat problem. Commun. Pure Appl. Math. 57(10), 1374–1411 (2004). doi:10.1002/cpa.20040

42. Simon, J.: Compact sets in the space L^p(0, T; B). Ann. Math. Pura Appl. 146(4), 65–96 (1987). doi:10.1007/BF01762360

43. Tabeling, P.: Growth and Form: Nonlinear Aspe. Springer, New York, 1991

44. Wu, S.: Global wellposedness of the 3-D full water wave problem. Invent. Math. 184(1), 125–220 (2011). doi:10.1007/s00222-010-0288-1

45. Young, T.: An essay on the cohesion of fluids. Philos. Trans. R. Soc. Lond. 95, 65–87 (1805). http://www.jstor.org/stable/107159

46. Zhang, P., Zhang, Z.: On the free boundary problem of three-dimensional incompressible Euler equations. Commun. Pure Appl. Math. 61(7), 877–940 (2008). doi:10.1002/cpa.20226

University of Heidelberg,
Im Neuenheimer Feld 294,
69120 Heidelberg,
Germany.
e-mail: hans.knuepfer@math.uni-heidelberg.de;
knuepfer@uni-heidelberg.de

and

Courant Institute of Mathematical Sciences,
251 Mercer Street,
New York, NY 10012,
USA.
e-mail: masmoudi@cims.nyu.edu

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