Construction of Realistic Standard–like Models in the Free Fermionic Superstring Formulation

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ABSTRACT

I discuss in detail the construction of realistic superstring standard–like models in the four dimensional free fermionic formulation. The analysis results in a restricted class of models with unique characteristics: (i) Three and only three generations of chiral fermions with their superpartners and the correct Standard Model quantum numbers. (ii) Proton decay from dimension four and dimension five operators is suppressed due to gauged $U(1)$ symmetries. (iii) There exist Higgs doublets from two distinct sectors, which can generate realistic symmetry breaking. (iv) These models explain the top–bottom mass hierarchy. At the trilinear level of the superpotential only the top quark gets a non vanishing mass term. The bottom quark and the lighter quarks and leptons get their mass terms from non renormalizable terms. This result is correlated with the requirement of a supersymmetric vacuum at the Planck scale. (v) The models predict the existence of small hidden gauge groups, like $SU(3)$, with matter spectrum in vector representations.
1. Introduction

The quest of theoretical physics in recent years has been the unification of all known fundamental interactions into one, consistent, theoretical formulation. Although the main prediction of Unified Theories, proton decay, has not yet been observed, calculations of $\sin^2 \theta_W$ and of the mass ratio $\frac{m_b}{m_\tau}$ support their validity. Recent calculations [1] seem to favor supersymmetric unification versus non supersymmetric unification. Superstring theory [2] is a unique candidate for the consistent unification of gravity with the gauge interactions, but lacks experimental support for its existence.

Initially it was believed that for its consistency the superstring had to be embedded in ten space–time dimensions and then the extra dimensions had to be compactified on a Calabi-Yau [2] manifold or on an orbifold [3]. Further study revealed that one could formulate a consistent string theory directly in four space–time dimensions by identifying the extra degrees of freedom as either bosonic [4] or fermionic [5,6] internal degrees of freedom.

On the other hand the Standard–Model agrees with all experimental observations to date, but leaves many questions unresolved. Among them are the fermion mass hierarchy, the number of chiral generations, the origin of fundamental scales, etc. These problems find natural explanations in superstring theories. Therefore, an important task is to connect the superstring with the Standard–Model. This task is obscured by the enormous number of candidate string models and our ignorance of the mechanism which selects the unique model. Two approaches can be followed to connect the superstring with the Standard–Model. One is to use a GUT model with an intermediate energy scale. Many attempts have been made
from superstring theory were made in the free fermionic formulation. However, all these attempts consist of isolated examples and a systematic presentation is still lacking. In this paper I try to fill this gap. Lacking a dynamical mechanism which singles out the unique string model, it is naive to expect that a particular example will turn out to be the correct model. However, by investigating a whole class of models we can extract the general properties of these models and their low energy phenomenological characteristics. The free fermionic formulation is chosen due to its unique properties. First, it is formulated directly in four space–time dimensions. Second, it is an exact conformal field theory which gives us the advantage of using the powerful calculational tools of conformal field theory, yielding highly predictive models. Finally, it is formulated at the self–dual point in the compactified space which enhances space–time gauge symmetries from $U(1)$ to $SU(2)$.

I present a detailed discussion of the spin structure basis vectors and the implications on low energy phenomenology. I impose the following phenomenological constraints on a possible superstring standard–like model:

1. The gauge group is $SU(3)_C \times SU(2)_L \times U(1)^n \times \text{hidden}$, with $N = 1$ space-time supersymmetry.

2. Three generations of chiral fermions and their superpartners, with the correct quantum numbers under $SU(3)_C \times SU(2)_L \times U(1)_Y$.

3. The spectrum should contain Higgs doublets that can produce realistic gauge symmetry breaking.

4. Anomaly cancellation, apart from a single “anomalous” $U(1)$ which is canceled by application of the Dine–Seiberg–Witten (DSW) mechanism.
are usually not found [13]. These dimension four operators are forbidden if the
gauge symmetry of the Standard Model is extended by a $U(1)$ symmetry, which
is a combination of, $B - L$, baryon minus lepton number, and $T_{3R}$, and is exactly
the additional $U(1)$ that is derived in the superstring standard–like models. The
dimension four operators may still appear from the nonrenormalizable terms

$$
\eta_1(u^C_L d^C_L d^C_N L^C)\Phi + \eta_2(d^C_L Q L N^C_L)\Phi,
$$

where $\Phi$ is a combination of fields which fixes the string selection rules and gets
a VEV of $O(m_{pl})$. The ratio $\frac{\langle N^c_L \rangle}{m_{pl}}$ controls the rate of proton decay. While in the
standard–like models this problem can be evaded either by keeping $B - L$ gauged
down to low energies [14], or by simply keeping $\langle N^c_L \rangle = 0$, in superstring models
based on an intermediate GUT symmetry, the problem is more difficult as $N^c_L$ is
necessarily used to break the GUT symmetry [15].

The paper is organized as follows. In section 2, I review the basic tools needed
for the construction of models in the free fermionic formulation. In section 3, I
emphasize the special role played by the first five vectors in the basis that spans
the models. I argue that the important functions of this set make it a unique set.
In sections 4 and 5 I discuss the construction of standard–like models and their
unique characteristics. In section 6, I discuss some of the phenomenology which is
expected to arise from these models.

2. Basic tools for model building

In the free fermionic formulation of the heterotic string in four dimensions all
world–sheet degrees of freedom required to cancel the conformal anomaly are
represented in terms of free fermions propagating on the string world–sheet. For the
left–movers (world–sheet supersymmetric) one has the usual space–time fields
$X^\mu$, $\psi^\mu$, ($\mu = 0, 1, 2, 3$), and in addition the following eighteen real free fermion fields:
$\chi^I$, $y^I$, $\omega^I$ ($I = 1, \ldots, 6$), transforming as the adjoint representation of
$SU(2)^6$. 

The supercurrent is given in terms of these fields as follows

\[ T_F(z) = \psi^\mu \partial_z X_\mu + \sum_{i=1}^{6} \chi^i y^i \omega^i. \]  

(1)

For the right movers we have \( \bar{X}^\mu \) and 44 real free fermion fields: \( \bar{\phi}^a, \ a = 1, \cdots, 44 \). Under parallel transport around a noncontractible loop the fermionic states pick up a phase. A model in this construction is defined by a set of basis vectors of boundary conditions for all world–sheet fermions. These basis vectors are constrained by the string consistency requirements (e.g. modular invariance) and completely determine the vacuum structure of the model. The physical spectrum is obtained by applying the generalized GSO projections. The low energy effective field theory is obtained by S–matrix elements between external states. The Yukawa couplings and higher order nonrenormalizable terms in the superpotential are obtained by calculating correlators between vertex operators. For a correlator to be nonvanishing all the symmetries of the model must be conserved. Thus, the boundary condition vectors determine the phenomenology of the model.

The class of spin structure models which I investigate here are generated by a basis of \( \mathbb{Z}_2^7 \times \mathbb{Z}_4 \). The basis generates an additive group \( \Xi = \sum_k n_k b_k \), where \( n_k = 0, \ldots, N_{Z_k} - 1 \). The physical states in the Hilbert space, of a given sector \( \alpha \epsilon \Xi \), are obtained [6] by acting on the vacuum \( |0\rangle_\alpha \) with bosonic, and fermionic operators with frequencies \( \nu_f = \frac{1+\alpha(f)}{2} \), and \( \nu_{f^*} = \frac{1-\alpha(f)}{2} \), for \( f \) and \( f^* \), respectively. The states satisfy the Virasoro condition:

\[ \mathcal{M}^2 = -\frac{1}{2} \alpha_L \cdot \alpha_L \cdot N_{X_L} = -\frac{1}{2} \alpha_R \cdot \alpha_R \cdot N_{X_R} = \mathcal{M}^2 \text{,...,} (2) \]
satisfy the generalized GSO projections [6],

\[
\left\{ e^{i\pi (b_i F_\alpha)} - \delta_\alpha c^* \left( \frac{\alpha}{b_i} \right) \right\} |s\rangle_\alpha = 0 \quad (3a)
\]

with

\[
(b_i F_\alpha) \equiv \left\{ \sum_{\text{real+complex left}} - \sum_{\text{real+complex right}} \right\} (b_i(f) F_\alpha(f)), \quad (3b)
\]

where \( F_\alpha(f) \) is a fermion number operator counting each mode of \( f \) once (and if \( f \) is complex, \( f^* \) minus once). For periodic fermions the vacuum is a spinor in order to represent the Clifford algebra of the corresponding zero modes. For each periodic complex fermion \( f \), there are two degenerate vacua \(|+\rangle, |-\rangle\), annihilated by the zero modes \( f_0 \) and \( f_0^* \) and with fermion numbers \( F(f) = 0, -1 \), respectively. The \( U(1) \) charges, \( Q(f) \), with respect to the unbroken Cartan generators of the four dimensional gauge group, which are in one to one correspondence with the \( U(1) \) currents \( f^* f \) for each complex fermion \( f \), are given by:

\[
Q(f) = \frac{1}{2} \alpha(f) + F(f) \quad (4)
\]

where \( \alpha(f) \) is the boundary condition of the world–sheet fermion \( f \) in the sector \( \alpha \).

To analyze the massless spectrum, I have written a FORTRAN program. The program takes as input the basis vectors \( B = \{b_1, \ldots, b_8\} \), and the GSO coefficients \( c \left( \frac{b_i}{b_j} \right) \), \( (i, j = 1, \ldots, 8) \). The program checks the modular invariance rules, spans the additive group \( \Xi = \sum n_i b_i ; \ (j = 1, \ldots, 8) \), selects the sectors in
standard techniques for evaluating non vanishing correlators and renormalization group equations, it provides powerful machinery for studying the phenomenology of the superstring models.

3. The NAHE set

The first five vectors (including the vector 1) in the basis are

\[
S = (1, \ldots, 1, 0, \ldots, 0|0, \ldots, 0).
\]

\[
b_1 = (1, \ldots, 1, 0, \ldots, 0|1, \ldots, 1, 0, \ldots, 0).
\]

\[
b_2 = (1, \ldots, 1, 0, \ldots, 0|1, \ldots, 1, 0, \ldots, 0).
\]

\[
b_3 = (1, \ldots, 1, 0, \ldots, 0|1, \ldots, 1, 0, \ldots, 0).
\]

with the choice of generalized GSO projections

\[
c \left( \begin{array}{c} b_i \\ b_j \end{array} \right) = c \left( \begin{array}{c} b_i \\ S \end{array} \right) = c \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = -1,
\]

and the others given by modular invariance. This set is referred to as the NAHE* set. The NAHE set is common to all the realistic models constructed in the free fermionic formulation [7,10,16,11,12] and is a basic set common to all the models which I present. The sector S generates \(N = 4\) space–time supersymmetry, which is broken to \(N = 2\) and \(N = 1\) space–time supersymmetry by \(b_1\) and \(b_2\), respectively.
NAHE set is $SO(10) \times E_8 \times SO(6)^3$ with $N = 1$ space-time supersymmetry. The three $SO(6)$ symmetries are horizontal, generational dependent, symmetries. The vectors $b_1, b_2$ and $b_3$ of the NAHE set perform several functions:

1. They produce the chiral generations.

2. They perform a “chirality operation”. To obtain from a given sector $b_j$ a full spinorial 16 of $SO(10)$ with the same chirality, a second vector is needed in the basis. $\psi^\mu, \bar{\psi}^{1\cdots5}$ are periodic in both vectors and the intersection between the remaining boundary conditions is empty.

3. They separate the hidden sector from the observable sector.

At the level of the NAHE set, each sector $b_1, b_2$ and $b_3$ give rise to 16 spinorial 16 of $SO(10)$. The internal 44 right–moving fermionic states are divided in the following way: $\bar{\psi}^{1\cdots5}$ are complex and produce the observable $SO(10)$ symmetry; $\bar{\phi}^{1\cdots8}$ are complex and produce the hidden $E_8$ gauge group; $\{\bar{\eta}^1, \bar{y}^{3\cdots6}\}, \{\bar{\eta}^2, \bar{y}^{1\cdots2}, \bar{\omega}^{5\cdots6}\}, \{\bar{\eta}^3, \bar{\omega}^{1\cdots4}\}$ give rise to the three horizontal $SO(6)$ symmetries. The left–moving $\{y, \omega\}$ states are divided to, $\{y^{3\cdots6}\}, \{y^{1\cdots2}, \omega^{5\cdots6}\}, \{\omega^{1\cdots4}\}$. The left–moving $\chi^{12}, \chi^{34}, \chi^{56}$ states carry the supersymmetry charges.

The Neveu–Schwarz sector produces the massless scalar states

$$\chi^{12}_{\frac{1}{2}} \bar{\psi}_{\frac{1}{2}}^{1\cdots5} \{\bar{\eta}^1_{\frac{1}{2}}, \bar{y}^{3\cdots6}_{\frac{1}{2}}\}$$

$$\chi^{34}_{\frac{1}{2}} \bar{\psi}_{\frac{1}{2}}^{1\cdots5} \{\bar{\eta}^2_{\frac{1}{2}}, \bar{y}^{1\cdots2}_{\frac{1}{2}} \bar{\omega}^{5\cdots6}_{\frac{1}{2}}\}$$

$$\chi^{56}_{\frac{1}{2}} \bar{\psi}_{\frac{1}{2}}^{1\cdots5} \{\bar{\eta}^3_{\frac{1}{2}}, \bar{\omega}^{1\cdots4}_{\frac{1}{2}}\}$$

$$\chi^{12}_{\frac{1}{2}} \bar{\psi}_{\frac{1}{2}}^{1\cdots5} \{\bar{\eta}^1_{\frac{1}{2}}, \bar{y}^{3\cdots6}_{\frac{1}{2}}\}$$

$$\chi^{34}_{\frac{1}{2}} \bar{\psi}_{\frac{1}{2}}^{1\cdots5} \{\bar{\eta}^2_{\frac{1}{2}}, \bar{y}^{1\cdots2}_{\frac{1}{2}} \bar{\omega}^{5\cdots6}_{\frac{1}{2}}\}$$

$$\chi^{56}_{\frac{1}{2}} \bar{\psi}_{\frac{1}{2}}^{1\cdots5} \{\bar{\eta}^3_{\frac{1}{2}}, \bar{\omega}^{1\cdots4}_{\frac{1}{2}}\}$$
symmetry to \( E_6 \). Adding the vector

\[
X = (0, \ldots, 0| 1, \ldots, 1, 0, \ldots, 0) \tag{7}
\]

to the NAHE set, extends the gauge symmetry to \( E_6 \times U(1)^2 \times SO(4)^3 \). The sectors \((b_1; b_1 + X), (b_2; b_2 + X)\) and \((b_3; b_3 + X)\) each give eight 27 of \( E_6 \). The \((NS; NS + X)\) sector gives in addition to the vector bosons and spin two states, three copies of scalar representations in \( 27 + \bar{27} \) of \( E_6 \).

In this model the only internal fermionic states which count the multiplets of \( E_6 \) are the real internal fermions \( \{y, w|\bar{y}, \bar{\omega}\} \). This is observed by writing the degenerate vacuum of the sectors \( b_j \) in a combinatorial notation. The vacuum of the sectors \( b_j \) contains twelve periodic fermions. Each periodic fermion gives rise to a two dimensional degenerate vacuum \(|+\rangle \) and \(|-\rangle \) with fermion numbers 0 and \(-1\), respectively. The GSO operator, Eq. (3), is a generalized parity, operator which selects states with definite parity. From Eq. (3) and after applying the GSO projections, we can write the degenerate vacuum of the sector \( b_1 \) in combinatorial form

\[
\left[ \begin{array}{c}
\binom{4}{0} + \binom{4}{2} + \binom{4}{4} \\
\binom{2}{0} + \binom{5}{0} + \binom{5}{2} + \binom{5}{4} \\
\binom{2}{1} + \binom{5}{1} + \binom{5}{3} + \binom{5}{5}
\end{array} \right] \left\{ \begin{array}{c}
\binom{1}{0} \\
\binom{1}{1}
\end{array} \right\} \tag{8}
\]

where \( 4 = \{y^3y^4, y^5y^6, \bar{y}^3\bar{y}^4, \bar{y}^5\bar{y}^6\} \), \( 2 = \{\psi^\mu, \chi^{12}\} \), \( 5 = \{\bar{\psi}^{1,\ldots,5}\} \) and \( 1 = \{\bar{\eta}^1\} \). The combinatorial factor counts the number of \(|-\rangle\) in a given state. The two terms in the curly brackets correspond to the two components of a Weyl-spinor. The \( 10 + 1 \) in the 27 of \( E_6 \) are obtained from the sector \( b_j + X \). From Eq. (8) it is observed that the states which count the multiplicities of \( E_6 \) are the internal fermionic states \( \{y_3, \ldots, 6| \bar{y}_3, \ldots, 6\} \). A similar result is obtained for the sectors \( b_2 \) and \( b_3 \) with \( \{y_1, 2, \omega_5, \ldots, 6| \bar{y}_1, 2, \bar{\omega}_5, \ldots, 6\} \) and \( \{\omega_1, \ldots, 4| \bar{\omega}_1, \ldots, 4\} \) respectively, which suggests that
these twelve states correspond to a six dimensional compactified orbifold with Euler characteristic equal to 48.

I would like to emphasize that the functions 1 and 2 above make the partial set \{1, S, b_1, b_2\} of the NAHE set a completely general set. Indeed, this partial set is common, in one form or another, to all the constructions in the free fermionic formulation. The minimal way to obtain a well defined hidden gauge group [18] is by adding the vector \(b_3\) to this set, which makes the NAHE set a unique set. The analysis of models beyond the NAHE set is reduced, almost entirely, to the study of the boundary conditions of the real fermions \(\{y_1, \cdots, 6, \bar{y}_1, \cdots, 6, \bar{\omega}_1, \cdots, 6\}\), and is simplified considerably. In the language of conformal field theory these real fermions correspond to the left right symmetric internal conformal field theory. As I will show bellow many of the phenomenological implications are determined by the boundary conditions of these real fermions.

4. Beyond the NAHE set

In the following I employ a table notation which emphasizes the division of the internal fermionic states according to their division by the NAHE set. The set of real fermions \(\{y, w|\bar{y}, \bar{\omega}\}\) plays an important role in the low energy properties of the standard–like models. In the table, the real fermionic states \(\{y, w|\bar{y}, \bar{\omega}\}\) are divided according to their division by the NAHE set. The pairing of real fermions into complex fermions or into Ising model sigma operators is noted in the table. The entries in the table represent the boundary conditions in a basis vector for all the fermionic states. The basis vectors in a given table are the three basis vectors which extend the NAHE set.
set. A strong constraint on the possible gauge group comes from the absence of adjoint representations in the massless spectrum of level one Kac–Moody algebra [17]. Therefore the $SO(10)$ symmetry has to be broken to one of its subgroups $SU(5) \times U(1)$, $SO(6) \times SO(4)$ or $SU(3) \times SU(2) \times U(1)_{B-L} \times U(1)_{T_R}$. This is achieved by the assignment of boundary conditions to the set $\bar{\psi}_1^{1\ldots5}$:

1. $b\{\bar{\psi}_1^{1\ldots5}\} = \{1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\} \Rightarrow SU(5) \times U(1)$,

2. $b\{\bar{\psi}_1^{1\ldots5}\} = \{1\ 1\ 1\ 0\ 0\} \Rightarrow SO(6) \times SO(4)$.

To break the $SO(10)$ symmetry to $SU(3) \times SU(2) \times U(1)_{C} \times U(1)_{L^*}$ both steps, 1 and 2, are used, in two separate basis vectors $\dagger$. The $SO(10)$ symmetry has to be broken in at least two of the three vectors which extend the NAHE set. Models in which the $SO(10)$ symmetry is broken in all three vectors are possible.

The weak hypercharge is given by the combination $U(1)_Y = \frac{1}{3}U(1)_C + \frac{1}{2}U(1)_L$. The orthogonal combination is given by $U(1)_{Z'} = U(1)_C - U(1)_L$.

The number of horizontal $U(1)$ symmetries depends on the assignment of boundary conditions and differs between models. All models have at least three horizontal $U(1)$ symmetries, denoted by $U(1)_{r_j}$ ($j = 1, 2, 3$), which correspond to the right-moving world-sheets currents $\bar{\eta}_2^1 \bar{\eta}_2^1$, $\bar{\eta}_2^2 \bar{\eta}_2^2$, and $\bar{\eta}_2^3 \bar{\eta}_2^3$. These complex fermionic states are twisted by a $Z_4$ twist. This twist is necessary to keep the two Weyl spinor components of the chiral fermions in the spectrum. Additional horizontal $U(1)$ symmetries, denoted by $U(1)_{r_j}$ ($j = 4, 5, ...$), arise by pairing two real fermions from the sets $\{\bar{y}^{3\ldots6}\}$, $\{\bar{y}^{1\ldots2}, \bar{\omega}^{5\ldots6}\}$ and $\{\bar{\omega}^{1\ldots4}\}$. The final observable gauge group depends on the number of such pairings.

For each of these complexified right-moving fermions correspond a left-moving
to them, is further constrained by the world–sheet supercurrent, Eq. (1). In
the fermionic formulation and with the supersymmetry generator of the NAHE
set, the boundary conditions of any \((\chi^I, y^I, \omega^I)\) triplet can belong only to
\(\{(1,1,0); (1,0,1); (0,1,1); (0,0,0)\}\) for space–time bosons and to
\(\{(1,0,0); (0,1,0) ; (0,0,1); (1,1,1)\}\) for space–time fermions. Each complexified left–moving fermion
gives rise to a global \(U(1)\) symmetry, denoted by \(U(1)_{\ell_j}\) \((j = 4,5,...)\). As will be
shown below these additional horizontal symmetries play an important role in
the phenomenology of the massless spectrum. Three additional left–moving global
\(U(1)\) symmetries, denoted by \(U(1)_{\ell_j}\) \((j = 1,2,3)\), arise from the charges of the
supersymmetry generator: \(\chi^{12}, \chi^{34}\) and \(\chi^{56}\).

If all right–moving (and hence all left–moving) fermions were complex, the gauge
group would have rank 22. The rank is reduced by pairing a left–moving fermion \((f)\)
with a real right–moving fermion \(\bar{f}\) to form an Ising model sigma operator. These
are denoted by \(\sigma^i_{\pm}\) and \(\tilde{\sigma}^i_{\pm}\) for \((y^i\bar{y}^i)_\pm\) and \((\omega^i\bar{\omega}^i)_\pm\), respectively. For a correlator
between vertex operators to be non vanishing, the real fermions must produce non
zero Ising model correlators. The symmetries of the Ising model correlators and
of the left moving charges must be checked after all picture changing have been
done. The rules for obtaining the non vanishing correlators are given in Ref. [19].

4.2 The number of generations

The question of the number of generations is discussed in detail in Ref. [20].
It is argued that the NAHE set leads to three generations as the most natural
number of generations. After the NAHE set, each sector \(b_1, b_2\) and \(b_3\) give rise to
sixteen chiral generations. The number of generations is determined by the set of
real fermions \(\{y, \omega|\bar{y}, \bar{\omega}\}\) (the vertical line separates left from right movers). The
real part of the sectors \( b_1, b_2 \) and \( b_3 \) and reduces the combinatorial factor of Eq. (8) by a half. Thus, we obtain one generation from each sector \( b_1, b_2 \) and \( b_3 \).

It is important to note that if the final gauge group is \( SU(5) \times U(1) \) or \( SO(6) \times SO(4) \) two of the additional sectors give rise to \( 16 + \bar{16} \) of \( SO(10) \). The net chirality of three generations is not spoiled. If the \( SO(10) \) symmetry is broken to \( SU(3) \times SU(2) \times U(1) \), constructions with exactly three generations and no mirror generations are obtained.

In the notation of table 1, all the real fermions are paired to form Ising model operators. The states \( \bar{\eta}^{1,2,3} \) are complex and are separated from the real fermions by the \( \frac{1}{2} \) twist in the sector \( \gamma \). At least three additional vectors are needed to break all the horizontal symmetries which arise from the part of the real fermions and at the same time reduce the number of generations to one generation from each of the sectors \( b_1, b_2 \) and \( b_3 \). Thus, the minimal additive group is \( \Xi = Z_2^7 \otimes Z_4 \). In the model of table 1 all the real horizontal symmetries are completely broken and the rank of the final gauge group is 16. In the models which I introduce below, some of the real fermions are paired to form complex fermions and therefore give rise to additional horizontal \( U(1) \) symmetries.

In the free fermionic models the chiral generations, from the sectors \( b_1, b_2 \) and \( b_3 \), carry charges under the three horizontal \( U(1)_j \) \( (j = 1, \cdots, 3) \) symmetries. The sign is determined by the product, \( \gamma \cdot b_j = odd/even \Rightarrow U(1)_j = -\frac{1}{2}/\frac{1}{2} \), respectively. In addition to these symmetries the chiral generations carry charges under the additional horizontal \( U(1)_{r_j} \) \( (j = 4, 5, 6) \) symmetries. For example in the model of table 2 we obtain three chiral generations \( G_j = e^c_{L_j} + u^c_{L_j} + N^c_{L_j} + d^c_{L_j} + Q_j + L_j \) \( (j = 1, \cdots, 3) \) with the following charges. From the sector \( b_1 \) we obtain
and from the sector \( b_3 \)

\[
(e_L^c + u_L^c)_{0,0,\frac{1}{2},0,0,\frac{1}{2}} + (N_L^c + d_L^c)_{0,0,\frac{1}{2},0,0,-\frac{1}{2}} + (L)_{0,0,\frac{1}{2},0,0,-\frac{1}{2}} + (Q)_{0,0,\frac{1}{2},0,0,\frac{1}{2}}.
\]  

(9c)

Where

\[
e_L^c \equiv [(1, \frac{3}{2}); (1, 1)]; \quad u_L^c \equiv [(3, -\frac{1}{2}); (1, -1)]; \quad Q \equiv [(3, \frac{1}{2}); (2, 0)]
\]  

(10a, b, c)

\[
N_L^c \equiv [(1, \frac{3}{2}); (1, -1)]; \quad d_L^c \equiv [(3, -\frac{1}{2}); (1, 1)]; \quad L \equiv [(1, -\frac{3}{2}); (2, 0)]
\]  

(10d, e, f)

of \( SU(3)_C \times U(1)_C \times SU(2)_L \times U(1)_L \). The charges under the six horizontal \( U(1) \) are given in Eqs. (9). Three generations of chiral fermions are common to all the models which I present.

### 4.3 Higgs doublets

The massless spectrum must contain Higgs doublets to give masses to the quarks and leptons. The Higgs doublets in the free fermionic models are obtained from two types of sectors. The first type are scalar doublets from the Neveu–Schwarz sector which arise from the scalar representations Eqs. (6). The presence of this scalar doublets in the massless spectrum is correlated with the additional \( U(1)_{r_j} \) horizontal symmetries which arise from pairing real right–moving fermions. This pairing guarantees that both the chiral family from a sector \( b_j \) (\( j = 1, 2, 3 \)), as well as the corresponding Higgs doublets, \( h_j \) and \( \bar{h}_j \), remain in the massless spectrum. Otherwise an exclusion principle is observed in the application of the GSO projection, \( \alpha \), which breaks the \( SO(10) \) symmetry to \( SO(6) \times SO(4) \). If \( \alpha \cdot b_j = 0 mod 2 \) (\( j = 1, 2, 3 \)), the family from \( b_j \) is in the spectrum and the Higgs doublet \( \chi_j \bar{\psi}^{145} \bar{\eta}_j |0\rangle_0 \) is projected out. If \( \alpha \cdot b_j = 1 mod 2 \) (\( j = 1, 2, 3 \)), the family
To illustrate this dependence I consider the models in tables 1, 2 and 5. In model 2, the three horizontal \((U(1)_\ell; U(1)_r)\) symmetries, which correspond to the world-sheet currents \((y^3 y^6; y^3 \bar{y}^6)\), \((y^1 \omega^6; \bar{y}^1 \bar{\omega}^6)\) and \((\omega^1 \omega^3; \bar{\omega}^1 \bar{\omega}^3)\), guarantee that the Higgs doublets \(h_1, \bar{h}_1, h_2, \bar{h}_2\) and \(h_3, \bar{h}_3\), as well as the chiral generations from the sectors \(b_1, b_2\) and \(b_3\), remain in the massless spectrum. A similar result is obtained in models 3 and 4. In model 1 all the real fermions are paired to form Ising model operators and there are no additional \(U(1)\) symmetries beyond \(U(1)_j\) \((j = 1, 2, 3)\). All the Higgs doublets from the Neveu–Schwarz sector are projected out. In this case the Higgs triplets \(D_1, \bar{D}_1, D_2, \bar{D}_2\) and \(D_3, \bar{D}_3\) from Eqs. (6a,c,e) remain in the massless spectrum. In model 5 we have only one additional horizontal \((U(1)_\ell; U(1)_r)\) symmetry which corresponds to the world-sheet currents \((\omega^2 \omega^3; \bar{\omega}^2 \bar{\omega}^3)\). Therefore in this model only one pair of Higgs doublets from the Neveu–Schwarz sector, \(h_3, \bar{h}_3\), remains in the massless spectrum after the GSO projections. In this case we obtain from Eqs. (6a,c) the Higgs triplets \(D_1, \bar{D}_1\) and \(D_2, \bar{D}_2\). Thus, the extra horizontal \(U(1)_r\) symmetries perform an additional function. They eliminate the dangerous Higgs triplets, \(D\) and \(\bar{D}\), which mediate proton decay through dimension five operators [21].

The horizontal \(U(1)_{\ell,r}\) symmetries also guarantee that the \(SU(5)\) singlets from Eqs. (6b,d,f) remain in the massless spectrum. Thus, in models 2, 3 and 4 we obtain three pairs of singlets \(\Phi_{12}, \bar{\Phi}_{12}, \Phi_{34}, \bar{\Phi}_{34}\) and \(\Phi_{56}, \bar{\Phi}_{56}\); while in model 5 we obtain only one pair of singlets, \(\Phi_{12}, \bar{\Phi}_{12}\).

The second type of Higgs doublets is obtained from a combination of the basis vectors \(\alpha\) and \(\beta\) with some combination of \(b_1, b_2\) and \(b_3\). For example in models 3 and 4 they arise from the combination \(\zeta = b_1 + b_2 + \alpha + \beta\). In this vector, \(\zeta_R \cdot \zeta_R = \zeta_L \cdot \zeta_L = 4\). Therefore the massless states are obtained by acting on the vacuum
imposes an additional strong constraint on the allowed basis vectors. For example, in the model of Ref. [10], it is impossible to obtain such a combination. The reason is the specific pairing of the left–moving real fermions, $y^3 y^6$, $y^1 \omega^6$ and $\omega^1 \omega^3$. These pairings guarantee that both the chiral fermions from the sectors $b_j$ as well as the corresponding Higgs doublets $h_j$, $\bar{h}_j$ are in the massless spectrum. However, the restrictions on the boundary conditions of the left–moving triplets $(\chi^I, y^I, \omega^I)$, forbid the construction of a combination like $\zeta$. Therefore in all the models with this pairing of left–moving fermions, these type of doublets and singlets does not exist. In models 3 and 4 the pairing of left–moving fermions is $y^3 y^6$, $y^1 \omega^5$ and $\omega^2 \omega^4$. In this case a vector of the form of $\zeta$ is obtained. The singlets and doublets from this sector play an important role in the application of the DSW mechanism and in the generation of the fermion mass hierarchy.

4.4 Yukawa couplings

The determination of trilevel Yukawa couplings, for the chiral generations from the sectors $b_1, b_2$ and $b_3$, depends on the assignment of boundary conditions for the set of fermions $\{y^{1\ldots6}, \omega^{1\ldots6}, \bar{y}^{1\ldots6}, \bar{\omega}^{1\ldots6}\}$. To illustrate this dependence I consider the model of table 2. The full massless spectrum of this model is presented in Ref. [10]. The sectors $b_1, b_2$ and $b_3$ give rise to three chiral generations. From the Neveu-Schwarz sector, three pairs of $SU(2)_L$ scalar doublets are obtained.

The basis of table 2 leads to the following trilevel mass terms for the states from the sectors $b_1, b_2$ and $b_3$:

\[\{(u^c_{L1} Q_1 \bar{h}_1 + N^c_{L1} L_1 \bar{h}_1 + d^c_{L2} Q_2 h_2 + e^c_{L2} L_2 h_2 + e^c_{L3} L_3 h_3 + d^c_{L3} Q_3 h_3\}. \tag{11}\]
This asymmetry leads to a non vanishing Yukawa coupling for the $+\frac{2}{3}$ charged quark and for the neutral lepton from the sector $b_1$. On the other hand, examination of the real fermion states from the sectors $b_2$ and $b_3$ reveals that for both sectors the corresponding charges are symmetric in the vector $\gamma$. $\gamma(y^1\omega^6) = \gamma(\bar{y}^1\bar{\omega}^6) = 1$ and $\gamma(\omega^1\omega^3) = \beta(\bar{\omega}^1\bar{\omega}^3) = 0$. This symmetry leads to a non vanishing trilevel Yukawa coupling for the $-\frac{1}{3}$ charged quark and for the charged lepton. In Ref. [22], I prove that in the symmetric case, $|\gamma(U(1)_{\ell_j+3}) - \gamma(U(1)_{r_j+3})| = 0$, trilevel mass terms are possible only for $-\frac{1}{3}$ type quarks while in the asymmetric case, $|\gamma(U(1)_{\ell_j+3}) - \gamma(U(1)_{r_j+3})| = 1$, trilevel mass terms are possible only for $+\frac{2}{3}$ type quarks. The proof is based on showing that, for the states from a sector $b_j$, in the symmetric case only $-\frac{1}{3}$ type quarks form trilevel mass terms which are invariant under $U(1)_j$, while in the asymmetric case only $+\frac{2}{3}$ type quarks form trilevel mass terms which are invariant under $U(1)_j$, $(j = 1, 2, 3)$.

From this result it follows that, depending on the assignment of boundary conditions in the vector $\gamma$, it is possible to construct models with trilevel Yukawa couplings for $+\frac{2}{3}$ charged quarks as well as for $-\frac{1}{3}$ charged quarks and for charged leptons. Apriori, the Yukawa couplings for all the heaviest generation states can be obtained from trilevel terms in the superpotential. I will refer to this type of models as type I models. On the other hand, it is possible to construct models in which only one type of Yukawa coupling is obtained at trilevel. For example, in models 3 and 4, only $+\frac{2}{3}$ charged quarks get a non vanishing trilevel Yukawa coupling. I will refer to this models as type II models. In the next section I argue that the requirement of a supersymmetric vacuum at the Planck scale may indicate that only type II models are allowed.

I now turn to discuss Yukawa couplings from nonrenormalizable terms in these
of real fermions \( \{y, w|\bar{y}, \bar{w}\} \) determines the non vanishing mass terms from higher orders.

The rules for obtaining the non vanishing higher order terms are given in Ref. \[19\]. A non vanishing F term in the superpotential must obey all the string selection rules. It must be invariant under all the gauge and global symmetries. In addition the real fermions must produce non zero Ising model correlators for a non renormalizable term to be non vanishing. The symmetries of the Ising model correlators and of the left–moving global symmetries must be checked after all picture changing have been done \[19\].

Examination of the quartic level terms in the model of table 2 reveals that there are no quartic terms which can give rise to bottom quark and tau lepton mass terms. On the other hand the model of table 3 does give rise to non vanishing quartic level mass terms for the bottom quark and for the tau lepton. These quartic order terms are of the form \[12\],

\[
W_4 = \{ d_{L1}^c Q_1 h_{45}^\prime \Phi_1 + e_{L1}^c L_1 h_{45}^\prime \Phi_1 + d_{L2}^c Q_2 h_{45}^\prime \Phi_2 + e_{L2}^c L_2 h_{45}^\prime \Phi_2\}. \tag{12}
\]

In model 3 nonvanishing mass terms for the bottom quark and for the tau lepton may be obtained from the following non vanishing quintic terms,

\[
W_5 = \{ d_{L1}^c Q_1 h_{45} \Phi_1^\prime \xi_2 + e_{L1}^c L_1 h_{45} \Phi_1^\prime \xi_2 + d_{L2}^c Q_2 h_{45} \Phi_2^\prime \xi_1 + e_{L2}^c L_2 h_{45} \Phi_2^\prime \xi_1\}. \tag{13}
\]

The second type of Higgs doublets, from the vector combination of \( \alpha + \beta \) plus a combination of \( b_1, b_2 \) and \( b_3 \), generate the fermion mass hierarchy in the heaviest generation. They couple to the bottom quark and to the tau lepton to form effective Yukawa couplings from the nonrenormalizable terms. In the application
The massless spectrum of the free fermionic models contains anomaly free and anomalous $U(1)$ symmetries. The boundary condition vectors and the choice of GSO phases determine the anomaly free and anomalous $U(1)$ symmetries. For example in model 2 the following $U(1)$s are anomalous: $\text{Tr}U_1 = -24$, $\text{Tr}U_2 = -30$, $\text{Tr}U_3 = 18$, $\text{Tr}U_5 = 6$, $\text{Tr}U_6 = 6$ and $\text{Tr}U_8 = 12$. Changing $c\left(\begin{array}{c} b_4 \\ 1 \end{array}\right) = +1$ to $c\left(\begin{array}{c} b_4 \\ 1 \end{array}\right) = -1$, changes the anomalous $U(1)$s to: $\text{Tr}U_C = -18$, $\text{Tr}U_L = 12$, $\text{Tr}U_1 = -18$, $\text{Tr}U_2 = -24$, $\text{Tr}U_3 = 24$, $\text{Tr}U_4 = -12$, $\text{Tr}U_5 = 6$, $\text{Tr}U_6 = 6$, $\text{Tr}U_7 = -6$, $\text{Tr}U_8 = 12$ and $\text{Tr}U_9 = 18$.

The anomalous $U(1)$ is broken by the Dine-Seiberg-Witten mechanism, [23] in which a potentially large Fayet-Iliopoulos D term is generated by the VEV of the dilaton field $(\phi_D)$. Such a D term will in general break supersymmetry and destabilize the string vacuum, unless there is a direction in the scalar potential $\phi = \sum_i \alpha_i \phi_i$, which is F flat and also D flat with respect to the nonanomalous gauge symmetries and in which $\sum_i Q_i A |\alpha_i|^2 < 0$. If such a direction exists, it will acquire a VEV, canceling the anomalous D term, restoring supersymmetry and stabilizing the vacuum. Since the fields corresponding to such a flat direction typically also carry charges for the non anomalous D terms, a non trivial set of constraints on the possible choices of VEVs is imposed and will in general break all of these symmetries spontaneously.

The set of constraints is summarized in the following set of equations:

$$D_A = \sum_k Q_k A |\chi_k|^2 = \frac{-g^2 e^{\phi_D}}{192\pi^2} \text{Tr}(Q_A)$$  \hspace{1cm} (14a)$$

$$D_j' = \sum_k Q_k |\chi_k|^2 = 0 \quad j = 1 \cdots 5$$  \hspace{1cm} (14b)
symmetry. The set \( \{ \eta_j \} \) is the set of fields with vanishing VEV. The solution to the set of Eqs.(14) must be positive definite since \( |\chi_k|^2 \geq 0 \).

The set of Eqs. (14) is a non trivial constraint on the allowed models. To illustrate the difficulty in finding solutions to the set of constraints I consider the model of table 5.

The observable gauge group of the model is \( SU(3)_C \times U(1)_C \times SU(2)_L \times U(1)_L \times U(1)^4 \) and the hidden gauge group is \( SU(5)_H \times SU(3)_H \times U(1)^2 \). The horizontal \( U(1) \) symmetries in the observable sector correspond to \( U(1)_j (j = 1, \cdots, 3) \) and to the world–sheet current \( \bar{\omega}^2 \omega^3 \). The \( U(1) \) symmetries in the hidden sector, \( U(1)_7 \) and \( U(1)_{8} \), correspond to the world–sheet currents \( \bar{\phi}^1 \phi^{1*} + \bar{\phi}^8 \phi^{8*} \) and \( -2 \bar{\phi}^j \phi^j + \bar{\phi}^1 \phi^{1*} - 4 \bar{\phi}^2 \phi^{2*} - \bar{\phi}^8 \phi^{8*} \) respectively, where summation on \( j = 5, \cdots, 7 \) is implied.

The massless spectrum in the observable sector contains three chiral generations from the sectors \( b_1, b_2 \) and \( b_3, G_1 \frac{1}{2},0,0 \)\(+G_2 \frac{1}{2},0,0\)\(+\{((e^c + u^c)_{0,0,0,0} \frac{1}{2} + (d^c + N^c)_{0,0,0,0} \frac{1}{2} + (L)_{0,0,0,0} \frac{1}{2} + (Q)_{0,0,0,0} \frac{1}{2}\)}\), the Neveu–Schwarz sector contains in addition to the spin two and spin one states, one pair of Higgs doublets \( h_{3,0,0,1}, \bar{h}_{3,0,0,1} \), two pairs of Higgs triplets \( D_{1,1,0,0}, \bar{D}_{1,1,0,0} \), \( D_{2,1,0,0}, \bar{D}_{2,1,0,0} \), one pair of \( SO(10) \) singlets with charges under the horizontal \( U(1) \) symmetries, \( \Phi_{12,1,1,0,0}, \Phi_{12,1,1,0,0} \) and five singlets which are neutral under all the \( U(1) \) symmetries \( \xi_{1,\cdots,5} \) : \( \chi_{2,2,2,2,2}^{12} y_{2,2,2,2,2}^{12} \omega_{2,2,2,2,2}^{12} |0\rangle_0, \chi_{1,1,1,1,1}^{34} y_{1,1,1,1,1}^{34} \omega_{1,1,1,1,1}^{34} |0\rangle_0 \), \( \chi_{2,2,2,2,2}^{56} y_{2,2,2,2,2}^{56} \omega_{2,2,2,2,2}^{56} |0\rangle_0, \chi_{2,2,2,2,2}^{56} y_{2,2,2,2,2}^{56} \omega_{2,2,2,2,2}^{56} |0\rangle_0 \), \( \chi_{1,1,1,1,1}^{56} y_{1,1,1,1,1}^{56} \omega_{1,1,1,1,1}^{56} |0\rangle_0 \).

In addition, in the observable sector, the sector \( \zeta = \alpha + \beta \) gives

\[
\begin{align*}
h_{45} & \equiv [(1,0); (2, -1)]_{\frac{1}{2},\frac{1}{2},0,0} \\
D_{45} & \equiv [(3, -1); (1, 0)]_{\frac{1}{2},\frac{1}{2},0,0} \\
\Phi_{45} & \equiv [(1,0); (1,0)]_{\frac{1}{2},\frac{1}{2},-1,0} \\
\Phi^+_{3} & \equiv [(1,0); (1,0)]_{-\frac{1}{2},\frac{1}{2},0,0,\pm1} 
\end{align*}
\]
The sectors $b_i + 2\gamma + (I)$ ($i = 1, ..., 3$) give vector representations which are $SU(3)_C \times SU(2)_L \times U(1)_L \times U(1)_C$ singlets (see Table 6). The vectors with some combination of $(b_1, b_2, b_3, \alpha, \beta)$ plus $\gamma + (I)$ (see Table 7) give representations which transform under $SU(3)_C \times SU(2)_L \times U(1)_L \times U(1)_C$, most of them singlets, but carry either $U(1)_Y$ or $U(1)_{Z'}$ charges. Some of these states carry fractional charges $\pm \frac{1}{2}$ or $\pm \frac{1}{3}$. There are no representations that transform nontrivially both under the observable and hidden sectors. The only mixing which occurs is of states that transform nontrivially under the observable or hidden sectors and carry $U(1)$ charges under the hidden or observable sectors respectively.

The model contains eight $U(1)$ symmetries, six in the observable sector and two in the hidden sector. Out of those four are anomaly free and four are anomalous:

$$\text{Tr}U_1 = 18, \text{Tr}U_2 = 30, \text{Tr}U_3 = 24, \text{Tr}U_4 = 12.$$  \hfill (16)

The two trace $U(1)$s, $U(1)_L$ and $U(1)_C$, are anomaly free. Consequently, the weak hypercharge and the orthogonal combination, $U(1)_{Z'}$, are anomaly free. Likewise, the two $U(1)$s in the hidden sector are anomaly free. Of the four anomalous $U(1)$s, only three can be rotated by an orthogonal transformation and one combination remains anomalous and is uniquely given by: $U_A = k\sum_j [\text{Tr}U(1)_j]U(1)_j$, where $j$ runs over all the anomalous $U(1)$s. For convenience, I take $k = \frac{1}{6}$, and therefore the anomalous combination is given by:

$$U_A = 3U_1 + 5U_2 + 4U_3 + 2U_4, \quad T_{QA} = 318. \hfill (17a)$$

The three orthogonal combinations are not unique. Different choices are related by an orthogonal transformation. One choice is given by:

$$U'_1 = U_1 + U_2 - 2U_3, \quad U'_2 = U_1 - U_2 + U_4, \quad U'_3 = 3U_1 - U_2 + U_3 - 4U_4.$$  \hfill (17b,c,d)
Together with the other four anomaly free $U(1)$s, they are free from gauge and gravitational anomalies. The cancellation of all mixed anomalies among the five $U(1)$s is a non trivial consistency check of the massless spectrum of the model.

The trilevel superpotential is given by

$$W = \left\{ (u_c^L, e_c^L, D_1 + d_c^L, N_{L_1}^c, D_1 + u_{L_2}^c, e_{L_2}^c, D_2 + d_{L_2}^c, N_{L_2}^c, D_1 + u_{L_3}^c, Q_3, h_3 + N_{L_3}^c, L_3, h_3) ight\}$$

$$+ D_1 \bar{D}_2 \Phi_{12} + \bar{D}_1 D_2 \Phi_{12} + \Phi_{12} \Phi_3^+ \Phi_3^- + \Phi_{12} \Phi_3^- \Phi_3^+ + h_3 h_{45} \Phi_{45} + h_{345} \Phi_{45} + \frac{1}{2} \xi_3 (\Phi_{45} \Phi_{45} + h_{45} \bar{h}_{45} + D_{45} \bar{D}_{45} + \phi_1 \bar{\phi}_1 + \phi'_1 \bar{\phi}'_1 + \phi_2 \bar{\phi}_2 + \phi'_2 \bar{\phi}'_2 + \Phi_3^+ \Phi_3^- + \Phi_3^- \Phi_3^+ + H_1 H_2) + \phi_1 (M_3 M_{11} + M_2 M_9) + \bar{\phi}_1 M_6 M_{13} + \bar{\phi}_2 (M_4 M_{10} + M_5 M_{12}) + \phi'_2 (M_7 M_{14} + M_1 M_8) + \phi'_1 M_{17} M_{24} + \bar{\phi}'_1 (M_{16} M_{21} + M_{20} M_{23}) + \bar{\phi}'_2 (M_{15} M_{22} + M_{19} M_{26}) + \phi_2 M_{18} M_{25} + \bar{\phi}_2 H_{13} H_{14}, \right\}$$

(18)

where a common normalization constant $\sqrt{2g}$ is assumed.

The solutions to Eqs. (14) can be divided to two kinds of solutions. Solutions of the first kind keep both $U(1)_C$ and $U(1)_L$ unbroken. Solutions of the second kind keep only the electroweak hypercharge unbroken. Solutions of the first kind are preferred because they are believed to be stable to all orders. For the first kind of solutions the fields $\chi_k$ in Eqs. (14), must be neutral under both $U(1)_C$ and $U(1)_L$. Only the Neveu–Schwarz sector, the sector $\zeta$, and the sectors $b_j + 2\gamma$, produce fields which are neutral under both $U(1)_C$ and $U(1)_L$. By examining the massless states from the Neveu–Schwarz and the $\zeta$ sector, it is observed that the number of fields, with independent charges along the four D constraints is always less than four. The Neveu–Schwarz sector produces only one field, $\Phi_{12}$. The sector $\zeta$ gives $\Phi_{45}$ and $\Phi_3^\pm$ while $\phi_{1,2}$ and $\phi'_{1,2}$ have the same charges, up to a multiplicative constant, as $\Phi_{12}$. The complex conjugate fields can be used to relax the positive definite restriction, however do not add more degrees of freedom. Thus the number of constraints is larger than the number of fields which can be used to solve theim. Adding the states from the sectors $b_j + 2\gamma$ does not resolve the problem, since they always carry positive
charge along the anomalous $U(1)_A^\ast$. Changing the model to include more states from the Neveu–Schwarz sector is possible at the cost of increasing the number of $U(1)$ symmetries with non vanishing trace. Thus, it is found that the number of constraints is always larger than the number of flat directions. It is concluded that solutions of the first kind do not exist in type I models. This result was verified by writing a simple computer program which searches for positive definite solutions. No solutions were found in all type I models. It is therefore concluded that, solutions which keep both $U(1)_C$ and $U(1)_L$ unbroken by the Dine–Seiberg–Witten mechanism, do not exist in type I models.

Turning to the second kind of solutions. These solutions keep only the weak hypercharge unbroken in the application of the Dine–Seiberg–Witten mechanism. The set of fields which can receive a non vanishing VEV is extended to include the states with vanishing weak hypercharge, but with non vanishing $U(1)_{Z'}$ charge. These states include the three right handed neutrinos from the sectors $b_1$, $b_2$ and $b_3$, and the neutral states from the sectors $\pm \gamma + (I)$ plus some combination of $(b_1, b_2, b_3, \alpha, \beta)$ (see Table 7). The number of D flatness constraints in this case is extended to ten equations. To obtain a supersymmetric vacuum we take $W = \frac{\partial W}{\partial \eta} = 0$, where $W$ is the trilevel superpotential. An elaborate computerized search for F and D flat solutions yielded a null result. However at this stage it is not possible to present a definite conclusion whether solutions of the second kind exist or do not exist in type I models. Observation of an additional neutral gauge boson, $Z'$, will exclude this kind of solutions and will therefore exclude type I models.

There is a unique class of type II models [11,12] which admit solutions to the F and D flatness constraints. These models have the following characteristics:
2. The complexification of the left–moving fermions $y^3 y^6$, $y^1 \omega^5$ and $\omega^2 \omega^4$ allows the construction of a vector $\zeta$. The states from this sector are used in the application of the DSW mechanism.

3. These models are constructed at a highly symmetric point in the “compactified space”. This symmetry exhibits itself in the non vanishing $U(1)$ traces [11,12].

5. The Hidden Sector

The hidden sector in the free fermionic standard–like models is determined by the boundary condition of the internal right–moving fermions, $\bar{\phi}^{1\ldots 8}$. A detailed classification is beyond the scope of this paper. However, the following comments are important to note.

The hidden gauge group arises from the states $\bar{\phi}^{1\ldots 8}$. In the NAHE set the contribution to the hidden $E_8$ gauge group comes from the Neveu–Schwarz sector and from the sector $I = 1 + b_1 + b_2 + b_3$. In the standard–like models the hidden gauge group is broken by the vectors which extend the NAHE set.

It is important to note that in the standard–like models the hidden $E_8$ gauge group must be broken. This follows from the fact that the vectors which break the $SO(10)$ symmetry always carry an odd number of periodic fermions from the set $\{\bar{\psi}^{1\ldots 5}, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3\}$. The reason is the structure of the NAHE set, which divides the internal fermionic states into three symmetric groups and the requirement of at least one Higgs doublet from the Neveu–Schwarz sector. To obey the modular invariance rule, $\alpha \cdot \gamma = 0 \mod 1$, an odd number of fermions from the set $\{\bar{\phi}^{1\ldots 8}\}$ must be periodic in the vector $\alpha$, and receive boundary condition of $\frac{1}{2}$ in the vector $\gamma$. Therefore, the hidden gauge symmetry is broken in two stages. Typically it is
appear. This offers the possibility of a rich hidden matter spectrum to appear in future colliders. The appearance of small hidden gauge groups may be desirable for generating the breaking of $U(1)_Z'$ as well as for generating supersymmetry breaking at a low scale.

6. Discussion

The construction of free fermionic standard-like models led to a unique class of models. This class of models has unique phenomenological characteristics. They suggest an explanation for the top–bottom mass hierarchy. At the trilinear level of the superpotential only the top quark obtains a non vanishing mass term, while the lighter quarks and leptons get their mass terms from non renormalizable terms. In two recent constructions [11,12], mass terms for the bottom quark and for the tau lepton were found at the quartic and quintic level. These models predict a top quark at $m_t \sim 140 - 180 \text{GeV}$ [22]. The unsuccessful search for solutions to the F and D flatness constraints in type I models, suggests a possible connection between the requirement of a supersymmetric vacuum at the Planck scale and the top quark mass hierarchy. Observation of an additional neutral gauge boson $Z'$ will be further evidence to support this connection.

The standard-like models extend the symmetry of the Standard-Model by one additional, generation independent, $U(1)$ symmetry. This $U(1)$ symmetry is a combination of, $B - L$, baryon minus lepton number, and of $T_{3R}$. The $U(1)_Z'$ may be broken by the application of the DSW mechanism. However, if $U(1)_Z'$ remains unbroken down to $M_{Z'} \leq 10^7 \text{GeV}$, it results in a gauged mechanism to explain the suppression of proton decay from dimension four operators. In this case it may be broken by the running of the renormalization group equations, à la
The underlying $SO(10)$ symmetry of the NAHE set indicates that for every Dirac mass term for a $+\frac{2}{3}$ charged quark, we obtain a Dirac mass term for a neutral lepton, with $m_u = m_\nu$. Therefore, we must be able to construct a see–saw mechanism [25] to suppress the neutrino mass. The entries in the see–saw mass matrix arise from nonrenormalizable terms. For example, in the models of Refs. [11,12] a potential term in the see–saw mass matrix appears at the quartic level $N_c^L H_{17} H_{13} V_9$, where $V_9$ and $H_{13}$ transform as triplets under the hidden $SU(3)$ group.

In this paper I discussed the construction of superstring standard–like models in the free fermionic formulation. To date the free fermionic formulation yielded the most realistic superstring models. This realism may be not accidental but may arise from the fact that the free fermionic formulation is formulated at a highly symmetric point in the moduli space. The question, how does nature choose to have only three generations, finds a simple explanation in free fermionic models [20]. The free fermionic standard–like models have remarkable properties. They have exactly three generations and no mirror generations. They explain the suppression of proton decay via dimension four operators either by a gauged mechanism or by simply not giving a VEV to the neutral singlet in the 10 of $SU(5)$. They explain the suppression of proton decay via dimension five operators by the GSO projection of the dangerous Higgs triplets. The projection of the Higgs triplets is correlated with the appearance of horizontal $U(1)_{\ell,r}$ symmetries. The standard–like models suggest an explanation for one of the most important mysteries of nature, the heaviness of the top quark relative to the lighter quarks and leptons. At trilevel only the top quark obtains a non vanishing mass term. Therefore only the top quark mass is characterized by the electroweak scale. The mass terms
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for 1\alpha specified by modular invariance and space–time supersymmetry. Trilevel Yukawa couplings are obtained
\gamma, \bar{\beta}, \bar{\omega}, \bar{\bar{\omega}}, y, \bar{y}, \chi, \bar{\chi}; \phi, \bar{\phi}, \phi^\dagger, \bar{\phi}^\dagger, \phi^2, \phi^3, \phi^4, \phi^5, \phi^6, \phi^7, \phi^8

| $\psi^\mu$ | $\{\chi^{12}; \chi^{34}; \chi^{56}\}$ | $y^1 \tilde{y}^3, y^2 \tilde{y}^4, y^3 \tilde{y}^5, y^6 \tilde{y}^6$ | $y^1 \tilde{y}^1, y^2 \tilde{y}^2, \omega^0 \omega^5, \omega^b \omega^b$ | $\omega^1 \omega^1, \omega^2 \omega^2, \omega^3 \omega^3, \omega^4 \omega^4$ | $\psi^1, \psi^2, \psi^3, \psi^4, \psi^5, \psi^6, \eta^1, \eta^2, \eta^3$ | $\phi^1, \phi^2, \phi^3, \phi^4, \phi^5, \phi^6, \phi^7, \phi^8$ |
|---|---|---|---|---|---|---|
| $b_4$ | 1 | {0, 0, 0} | 1, 0, 0, 1 | 0, 0, 0, 1 | 1, 1, 1, 0, 0 | 1, 1, 0, 0, 0 |
| $\alpha$ | 1 | {0, 1, 0} | 0, 0, 0, 1 | 0, 1, 0, 1 | 1, 0, 0, 0 | 1, 1, 1, 0, 0 |
| $\gamma$ | 1 | {0, 0, 1} | 1, 1, 0, 0 | 1, 0, 0, 0 | 0, 1, 0, 0 | 0, 0, 0, 0 |

**Table 1.** A three generations $SU(3) \times SU(2) \times U(1)^2$ model without Higgs doublets from the Neveu–Schwarz sector.

| $\psi^\mu$ | $\{\chi^{12}; \chi^{34}; \chi^{56}\}$ | $y^3 \tilde{y}^3, y^4 \tilde{y}^4, y^5 \tilde{y}^5, y^6 \tilde{y}^6$ | $y^1 \omega^0, y^2 \omega^2, \omega^0 \omega^5, \omega^b \omega^b$ | $\omega^1 \omega^1, \omega^2 \omega^2, \omega^3 \omega^3, \omega^4 \omega^4$ | $\psi^1, \psi^2, \psi^3, \psi^4, \psi^5, \psi^6, \eta^1, \eta^2, \eta^3$ | $\phi^1, \phi^2, \phi^3, \phi^4, \phi^5, \phi^6, \phi^7, \phi^8$ |
|---|---|---|---|---|---|---|
| $\alpha$ | 1 | {1, 0, 0} | 1, 0, 0, 1 | 0, 0, 1, 0 | 0, 0, 1, 0 | 1, 1, 1, 1, 0 |
| $\beta$ | 1 | {0, 0, 0} | 0, 0, 0, 1 | 0, 1, 0, 1 | 1, 0, 1, 0 | 1, 1, 1, 0, 0 |
| $\gamma$ | 1 | {0, 0, 1} | 0, 0, 0, 1 | 1, 0, 0, 1 | 0, 1, 0, 0 | 1, 0, 0, 0 |

**Table 2.** A three generations $SU(3) \times SU(2) \times U(1)^2$ model. The choice of generalized GSO coefficients is:
\begin{align*}
e & \begin{pmatrix} \alpha \\ b_4, \beta \end{pmatrix} = -e \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = e \begin{pmatrix} \beta \\ 1 \end{pmatrix} = e \begin{pmatrix} \bar{\beta} \\ 1 \end{pmatrix} = e \begin{pmatrix} \bar{\alpha} \\ b_4 \end{pmatrix} = -e \begin{pmatrix} \bar{\beta} \\ b_4 \end{pmatrix} = e \begin{pmatrix} \gamma \\ b_4 \end{pmatrix} = e \begin{pmatrix} \bar{\gamma} \\ b_4 \end{pmatrix} = -e \begin{pmatrix} \gamma \\ b_4, \beta, \alpha \end{pmatrix} = -1 (j=1,2,3), \end{align*}
with the others specified by modular invariance and space–time supersymmetry. Trilevel Yukawa couplings are obtained for $+\frac{1}{3}$ and $-\frac{1}{3}$ charged quarks as well as $-\frac{1}{3}$ charged quarks and for charged leptons.
charged quarks.

by modular invariance and space–time supersymmetry. Trilevel Yukawa couplings are obtained only for
\( \alpha, \beta, \gamma \) \( j=1,2,3 \), with the others specified

| \( \psi^\alpha \) | \( \chi^{1,2,3} \) | \( y^1 y^3, y^1 y^5, y^4 y^6 \) | \( y^1 \omega^3, y^2 \omega^2, \omega^5 \bar{y}^3, y^3 \bar{y}^6 \) | \( \omega^2 \omega^4, \omega^1 \omega^1, \omega^2 \omega^3, \omega^2 \omega^4 \) | \( \phi^1, \phi^2, \phi^3, \phi^4, \phi^5, \phi^6, \phi^7, \phi^8 \) | \( \alpha \) | \( \beta \) | \( \gamma \) |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0, 0, 0 | 1, 0, 0, 0 | 0, 0, 1, 1 | 0, 0, 1, 1 | 1, 1, 1, 1, 0, 0, 0, 0 | 1, 1, 1, 1, 0, 0, 0, 0 |
| 0 | 0, 0, 0 | 0, 0, 1, 1 | 0, 0, 1, 1 | 0, 1, 0, 0 | 1, 1, 1, 0, 0, 0, 0, 0 | 1, 1, 1, 0, 0, 0, 0, 0 |
| 0 | 0, 0, 0 | 0 | 0, 1, 0, 0 | 1, 0, 0, 0 | 1, 0, 0, 0 | 1, 1, 1, 1, 0, 0, 0, 0 | 1, 1, 1, 1, 0, 0, 0, 0 |

**Table 3.** A three generations \( SU(3) \times SU(2) \times U(1)^2 \) model. The choice of generalized GSO coefficients is:

\[
e^{-c(\frac{b_j}{\alpha, \beta, \gamma})} = -e^{-c(\frac{\alpha}{\beta})} = -e^{-c(\frac{\beta}{\gamma})} = -e^{-c(\frac{\gamma}{\alpha, \beta})} = -1 \ (j=1,2,3),
\]

with the others specified by modular invariance and space–time supersymmetry. Trilevel Yukawa couplings are obtained only for \( \frac{\psi}{2} \) charged quarks.

| \( \psi^\alpha \) | \( \chi^{1,2,3} \) | \( y^1 y^3, y^1 y^5, y^4 y^6 \) | \( y^1 \omega^3, y^2 \omega^2, \omega^5 \bar{y}^3, y^3 \bar{y}^6 \) | \( \omega^2 \omega^4, \omega^1 \omega^1, \omega^2 \omega^3, \omega^2 \omega^4 \) | \( \phi^1, \phi^2, \phi^3, \phi^4, \phi^5, \phi^6, \phi^7, \phi^8 \) | \( \alpha \) | \( \beta \) | \( \gamma \) |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0, 0, 0 | 1, 1, 1, 0 | 1, 1, 1, 0 | 1, 1, 1, 1, 0, 0, 0, 0 | 1, 1, 1, 1, 0, 0, 0, 0 |
| 0 | 0, 0, 0 | 0, 1, 0, 1 | 0, 1, 0, 1 | 0, 1, 0, 0 | 1, 1, 1, 0, 0, 0, 0, 0 | 1, 1, 1, 1, 0, 0, 0, 0 |
| 0 | 0, 0, 0 | 0, 0, 1, 1 | 1, 0, 0, 0 | 0, 1, 0, 0 | 1, 1, 1, 0, 0, 0, 0, 0 | 1, 1, 1, 1, 0, 0, 0, 0 |

**Table 4.** A three generations \( SU(3) \times SU(2) \times U(1)^2 \) model. The choice of generalized GSO coefficients is:

\[
e^{-c(\frac{b_j}{\alpha, \beta, \gamma})} = -e^{-c(\frac{\alpha}{\beta})} = -e^{-c(\frac{\beta}{\gamma})} = -e^{-c(\frac{\gamma}{\alpha, \beta})} = -1 \ (j=1,2,3),
\]

with the others specified by modular invariance and space–time supersymmetry. Trilevel Yukawa couplings are obtained only for \( \frac{\psi}{2} \) charged quarks.
Table 5. A three generations $SU(3) \times SU(2) \times U(1)^2$ model with four horizontal $U(1)$ symmetries. The choice of generalized GSO coefficients is: $c\left(b_1, b_2, \alpha, \beta, \gamma\right) = -c\left(b_2, \alpha\right) = c\left(1, b_j, \gamma\right) = -c\left(\gamma, 1, b_1, b_2\right) = c\left(\gamma, b_3\right) = -1$ ($j=1,2,3$), with the others specified by modular invariance and space–time supersymmetry. Trilevel Yukawa couplings are obtained for $+\frac{1}{2}$ charged quarks as well as for $-\frac{1}{2}$ charged quarks and for charged leptons.
Table 6. Massless states in model 5 and their quantum numbers. V indicates that these states form vector representations of the Hidden group.
| F | SEC | $SU(3)_C \times SU(2)_L$ | $Q_C$ | $Q_L$ | $Q_1$ | $Q_2$ | $Q_3$ | $SU(5) \times SU(3)$ | $Q_7$ | $Q_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | $S + b_1 + b_2$ | (3,1) | $\frac{1}{3}$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | (1,1) | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| $M_2$ | $\beta + \gamma + (I)$ | (1,2) | $\frac{1}{3}$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | (1,1) | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $M_3$ | | (1,1) | $-\frac{1}{3}$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | (1,1) | $-\frac{4}{3}$ | $-\frac{1}{3}$ |
| $M_4$ | | (1,1) | $\frac{1}{3}$ | $-\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | (1,1) | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $M_5$ | | (1,1) | $-\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | (1,1) | $-\frac{4}{3}$ | $\frac{1}{3}$ |
| $M_6$ | | (1,1) | $-\frac{1}{3}$ | $\frac{2}{3}$ | $-\frac{2}{3}$ | $-\frac{1}{3}$ | 0 | (5,1) | $-\frac{4}{3}$ | $\frac{1}{3}$ |
| $M_7$ | | (1,1) | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0 | (1,3) | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| $M_8$ | $S + b_1 + b_2$ | (3,1) | $-\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | (1,1) | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $M_9$ | $+\alpha \pm \gamma + (I)$ | (1,2) | $-\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | (1,1) | $-\frac{1}{3}$ | $\frac{1}{3}$ |
| $M_{10}$ | | (1,1) | $-\frac{1}{3}$ | $\frac{1}{3}$ | $-\frac{2}{3}$ | $-\frac{1}{3}$ | 0 | (1,1) | $\frac{1}{3}$ | $-\frac{1}{3}$ |
| $M_{11}$ | | (1,1) | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | (1,1) | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $M_{12}$ | | (1,1) | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | (1,1) | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $M_{13}$ | | (1,1) | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | (5,1) | $\frac{1}{3}$ | $-\frac{1}{3}$ |
| $M_{14}$ | | (1,1) | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | (1,3) | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $M_{15}$ | $1 + b_1 + b_2 + b_3$ | (1,2) | 0 | 0 | $-\frac{1}{3}$ | 0 | 0 | (1,1) | 1 | 0 |
| $M_{16}$ | $+\beta + 2\gamma$ | (1,2) | 0 | 0 | $-\frac{1}{3}$ | 0 | $-\frac{1}{3}$ | (1,1) | 1 | 0 |
| $M_{17}$ | | (1,1) | 0 | $-1$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | (1,1) | 1 | 0 |
| $M_{18}$ | | (1,1) | 0 | $-1$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | (1,1) | 1 | 0 |
| $M_{19}$ | | (1,1) | 0 | $-1$ | $\frac{1}{3}$ | 0 | $-\frac{1}{3}$ | (1,1) | $-1$ | 0 |
| $M_{20}$ | | (1,1) | 0 | $-1$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | (1,1) | $-1$ | 0 |
| $M_{21}$ | $1 + b_1 + b_2 + b_3$ | (1,2) | 0 | 0 | 0 | $\frac{1}{3}$ | 0 | (1,1) | $-1$ | 0 |
| $M_{22}$ | $+\alpha + 2\gamma$ | (1,2) | 0 | 0 | 0 | $\frac{1}{3}$ | $-\frac{1}{3}$ | (1,1) | $-1$ | 0 |
| $M_{23}$ | | (1,1) | 0 | 1 | 0 | $-\frac{1}{3}$ | 0 | (1,1) | $1$ | 0 |
| $M_{24}$ | | (1,1) | 0 | 1 | 0 | $-\frac{1}{3}$ | $-\frac{1}{3}$ | (1,1) | $-1$ | 0 |
| $M_{25}$ | | (1,1) | 0 | 1 | 0 | $\frac{1}{3}$ | $-\frac{1}{3}$ | (1,1) | 1 | 0 |
| $M_{26}$ | | (1,1) | 0 | 1 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | (1,1) | 1 | 0 |