Ground state entanglement constrains low-energy excitations

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For a general quantum many-body system, we show that its ground state entanglement imposes a fundamental constraint on the low-energy excitations. For two-dimensional systems, our result implies that any system that supports anyons must have a nonvanishing topological entanglement entropy. We demonstrate the generality of this argument by applying it to three-dimensional quantum many-body systems, and showing that there is a pair of ground state topological invariants that are associated to their physical boundaries. From the pair, one can determine whether the given boundary can or cannot absorb point-like or line-like excitations.

Introduction— Topological order is an exotic phase of matter that lies outside of Landau’s symmetry breaking paradigm. Systems with nontrivial topological order possess a number of remarkable properties, such as emergent quasi-particles with fractional statistics,\textsuperscript{1} topologically protected ground state degeneracy,\textsuperscript{2} and the long-range entanglement that is present in their ground states.\textsuperscript{3–5} Aside from their importance in fundamental physics, topologically ordered systems are also expected to serve as a useful platform for harnessing quantum information in an intrinsically fault-tolerant manner.\textsuperscript{6}

In this paper, we prove a no-go theorem about topological order that illuminates a general structure of ground state entanglement that is present in such systems. Our theorem can be thought as a variant of the existing theorems that provide conditions under which topological ground state degeneracy can or cannot be present.\textsuperscript{7, 8} The main difference is that our theorem puts a constraint on the low-energy excitations, as opposed to the ground states.

More precisely, we prove that the low-energy excitations of a local gapped Hamiltonian are topologically trivial in the case that the topological entanglement entropy, $\gamma$, vanishes. We show this by proving the following expression

$$\| U V |\psi_0\rangle - V U |\psi_0\rangle \| \leq O(\gamma^{\frac{1}{2}}),$$

for ground state $|\psi_0\rangle$ where $\| |\psi\rangle \| = \langle \psi |\psi \rangle$ is the norm of the vector. The operator $U$ is a unitary operator that creates excitations from the vacuum, and $V$ represents a unitary process of (i) creating particles, (ii) performing some monodromy operation with the quasi-particles created with $U$, and (iii) annihilating the particles created by $V$. We emphasize that $\gamma$ is a property of the quantum state, not the Hamiltonian. Indeed, it is the constant subcorrection term of the ground state entanglement entropy formula

$$S(\rho_A) = \alpha l - n\gamma + \cdots,$$

where $S(\rho_A) = -\text{Tr}(\rho_A \log \rho_A)$ is the entanglement entropy of subsystem $A$, length $l$ is its boundary area, and $n$ is the number of disconnected components of the boundary.\textsuperscript{9, 5} We remark that throughout this Manuscript we consider only regions that are much smaller than the size of the system, but larger than its correlation length.

The result of Eq\textsuperscript{1} may seem unsurprising in view of the rigorously studied two-dimensional(2D) topological phases.\textsuperscript{6, 9} However, the novelty of our method is that we obtain this result without making any assumptions that depend on the microscopic details of the Hamiltonian. We only assume that we can perform a non-trivial monodromy operation between particles using operators $U$ and $V$. This generality enables us to perform a similar analysis in other settings, which in turn enable us to find new topological invariants.

We explicitly demonstrate the power of our framework by proving that certain linear combinations of entanglement entropies cannot vanish on the boundary of certain three-dimensional(3D) topologically ordered systems. Specifically, we find a pair of topological invariants that are defined on the boundary, each of which represent the long-range entanglement associated to the point-like and line-like excitations. If the invariant for the point-like excitations is zero, all such excitations can be condensed at the boundary. Similarly, if the invariant for the line-like excitations is zero, all such excitations can be condensed at the boundary. We give evidence that these numbers are universal by explicit analytical calculation using the 3D toric code.\textsuperscript{10}

Let us sketch the high-level overview of the proof. We begin by considering the creation of the quasi-particles by a string-like operator $U$. Then we identify a condition on $U$ that ought to be true for any anyon models. This condition, which shall be explained shortly, implies that the action of $U$ on the ground state can be approximated by a unitary operator $U'$ which lies only in the vicinity of the quasi-particles, with an approximation error that scales as $O(\gamma^{\frac{1}{2}})$. We show this using the fact that $V$ has no common support with $U'$ and thus commutes with $V$.\textsuperscript{11}
The inequality of Eq. (1) follows from this observation. These arguments make use of the well-known concepts in quantum information theory, and as such, we set the relevant notations first. We use two different distance measures between quantum states $\rho$, $\sigma$, the fidelity, $F(\rho, \sigma) = \| \rho^{\frac{1}{2}} \sigma^{\frac{1}{2}} \|_1$, and the trace distance, $D(\rho, \sigma) = \frac{1}{2} \| \rho - \sigma \|_1$. These two measures can be used interchangeably, due to their well-known relation [11]:

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}.$$  

2D case — We present the proof explicitly for the case of a 2D system. Let us begin by stating the most crucial part of the argument, which is pictorially represented in Fig. 1. To be more specific, consider a pair of quasi-particles created out of the vacuum state $|\psi_0\rangle$ by a string-like unitary operator $U$. We show that

$$\|U |\psi_0\rangle - U' |\psi_0\rangle\| \leq O(\gamma^4) \tag{3}$$

for some $U'$ that lies in the vicinity of the particles, if $U$ is freely deformable; we say that $U$ is freely deformable if the particles can be created by another string-like unitary operator $U_{\text{def}}$ whose support can be continuously deformed into that of $U$. This is a natural assumption that is expected to hold for many anyon models. When $\gamma \approx 0$, the above assertion implies that $U |\psi_0\rangle \approx U' |\psi_0\rangle$. In short, the effective support of $U$ is reduced. We refer to such process as the cleaning process [20].

The cleaning process relies upon two facts about general quantum states. We first lay out these observations and later explain how they can be applied to anyon models. First, for any two bipartite pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ that have identical density matrices over a subsystem can be mapped onto one another by applying a unitary operation only on the complementary subsystem. Second, there is a condition under which one can check the equivalence of two states from their local subsystems [12]. In this paper, we use the second observation to argue that $U |\psi_0\rangle$ and $|\psi_0\rangle$ have the same density matrices over the complement of the support of $U'$ if $\gamma$ is small. Then we use the first observation to argue that there exists a unitary $U'$ which is supported on a smaller region, as explained in Fig. 1. We now elaborate on these observations.

The first observation follows from the celebrated Uhlmann’s theorem [13], which asserts that $F(\rho, \sigma)$ is equal to the maximum overlap over their purifications:

$$F(\rho, \sigma) = \max_{|\psi\rangle} \langle \psi |\psi\rangle \tag{4}$$

In our context, we envision $\rho$ and $\sigma$ to be the reduced states that are inherited from some bipartite pure states $|\psi_\rho\rangle$ and $|\psi_\sigma\rangle$. If the fidelity between $\rho$ and $\sigma$ is 1, the above relation implies that there exists a purification of $\sigma$ that has a unit overlap with $|\psi_\rho\rangle$. In particular, it would imply the existence of a unitary operator acting on the complement of the support of $\rho$, such that it maps $|\psi_\rho\rangle$ to $|\psi_\sigma\rangle$ and vice versa.

The second observation lies on a recently discovered fact: that two locally equivalent many-body quantum states are globally equivalent under a certain condition. If $\rho_{ABC}$ and $\sigma_{ABC}$ are consistent over $AB$ and $BC$, i.e., $\rho_{AB} = \sigma_{AB} \equiv \rho_{BC} = \sigma_{BC}$, the following inequality holds:

$$D(\rho_{ABC}, \sigma_{ABC})^2 \leq I(A : C|B)_\rho + I(A : C|B)_\sigma, \tag{5}$$

where $I(A : C|B)_\rho = S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{B}) - S(\rho_{ABC})$ is the conditional mutual information for density matrix $\rho$ [12].

So far we have discussed two general facts about quantum states. The natural course is to explain what these facts imply for anyon models. Without loss of generality, let us choose $\rho$ to be the ground state, i.e., $|\psi\rangle \langle \psi|$ and $\sigma$ to be the excited state, i.e., $U\rho U^\dagger$. We divide the systems into the regions shown in Fig. 2 for reasons that will soon become apparent. It should be noted that $\rho$ and $\sigma$ must have the same density matrices over $AB$ and $BC$, since $U$ can be deformed to have a support complementary to these regions. Importantly, this implies that we can use Eq. (5).

We estimate the right-hand side of Eq. (5) for the choices we have just made. Quantum entropy obeys strong subadditivity of entropy [14], which implies that $I(A : C|B) \leq I(APQ : C|B)$. Recall that entanglement en-
error of $\gamma$ can be approximated by $\rho$ using Eq.2. We arrive at the conclusion that $\rho$ are equal to each other, since $S$ply that this paper, Eq.1, is simply a sharpening of this observa-

tion, which we explain in the Supplemental Material. Since such systems can host an exotic line-like quasi-
particle excitations that carry nontrivial topological charge. Hence one can ask similar questions to such ex-
citations, and compare it to the result for the point-like excitations. Similar to the 2D example, one can concoct a linear combination of entanglement entropies such that it can vanish only if the excitations can be created locally out of the vacuum.

Let us begin with a choice of regions that are similar to the ones in FIG2 see FIG3. The Roman letters that represent the regions were chosen in such a way that their roles are analogous to the regions with the same letters in FIG2. One can imagine creating a pair of quasi-particles by applying an operator $U$ that penetrates through $CD$. If $S(BC) + S(CD) - S(B) - S(D)$ is small, the action of $U$ on the ground state can be approximated by $U'$ which lies in the vicinity of the quasi-
particles. Such $U'$ exists only if the boundary can absorb any point-like excitations. Conversely, if there is any point-like excitation that cannot be created by such $U'$, $S(BC) + S(CD) - S(B) - S(D)$ cannot vanish. For this reason, we refer to this linear combination as the point topological entanglement entropy, and denote it $\gamma_{\text{point}}$.

Of course, a similar argument can be carried out for the line-like excitations; see FIG2. Here we imagine creating a line-like excitation by applying a unitary operator $U$ that permeates through $CD$. If $S(BC) + S(CD) - S(B) - S(D)$ is small, the action of $U$ on the ground state can be approximated by $U'$ which lies in the vicinity of the line-like excitations. Such $U'$ exists only if the boundary can absorb any line-like excitations. Conversely, if there is any line-like excitation that cannot be created by such $U'$, $S(BC) + S(CD) - S(B) - S(D)$ cannot vanish. For this reason, we refer to this linear combination as the line topological entanglement entropy, and denote it as $\gamma_{\text{line}}$.

In the 2D case, the linear combination was concocted in such a way that the leading term in Eq.2 cancels each other out. Based on a general physical intuition that the leading term is due to the short-range entangle-
ment across the cut, we expect a similar behavior for

![Diagram](image-url)
the regions in FIG. 3 and FIG. 4. It should be noted that the physical boundary does not contribute to such short-range entanglement, since the vacuum that lies beyond the physical boundary is not entangled with the medium. Assuming such a behavior indeed holds, one can easily see that the contributions from the short-range entanglement are canceled out.

The remaining term is invariant under smooth deformation of the regions. Therefore, we expect it to be a topological invariant that characterizes the phase. In particular, we conjecture that the point(line) topological entanglement entropy becomes 0 if and only if all the point-line(excitation) excitations can be condensed at the given boundary. The “only if” part is shown by the previous argument. While we currently do not have a general argument for the “if” part, we give an example which supports this conjecture with the 3D toric code model.

3D toric code—The 3D toric code [10] in the bulk has two-types of excitations; one point-like excitation and one line-like excitation, as shown in FIG. 5 (a) and (b) respectively. The model acquires an $e^{i\pi}$ phase if a point excitation is moved through a line excitation and returned to its initial position, as shown in FIG. 5 (c).

The toric code has two types of boundary, a rough boundary and a smooth boundary. The 3D boundaries generalize straightforwardly from the 2D case [15]. Close to a boundary, the excitations of the model change non-trivially. A rough boundary absorbs point-like excitations, and therefore in the vicinity of a rough boundary we find only line-like excitations. Conversely, a smooth boundary absorbs line-like excitations. Therefore, close to a smooth boundary we find only point-like excitations. We see that the presented diagnostics can distinguish these different boundaries for the considered example.

The bulk entanglement entropy of region $R$ for the 3D toric code [10, 15] is

$$S(\rho_R) = A_R - n \log 2,$$

where $A_R$ is the surface area of the boundary of region $R$, denoted $\partial R$. The term $n$ is the number of disjoint connected surfaces, $\partial R_j$, of $\partial R$, such that $\partial R = \partial R_1 \cup \partial R_2 \cup \ldots \cup \partial R_n$.

To calculate $\gamma_{\text{point}}$ and $\gamma_{\text{line}}$, we generalize Eq. 6 for the toric code to regions that include boundary qubits. These calculations are found explicitly using the method of [19] in Supplemental Material. The method we follow has been used in the context of topologically ordered states in Ref. [20]. Common to the bulk case, we find that the topological contribution for region $R_j$ that includes qubits from a smooth boundary is unchanged, and for such a region, boundary component $\partial R_j$ contributes a single unit to the topological term. Conversely, we find that the $\partial R_j$ that bounds any qubits of the rough boundary will contribute zero to the topological term. This observation is supported by the presented intuition, as for such regions, point particles can be created locally from the rough boundary. We therefore obtain a generalization of Eq. 6 where $n$ counts the disjoint boundaries that enclose no qubits of a rough boundary.

We apply these results to find $\gamma_{\text{point}}$ and $\gamma_{\text{line}}$ for both rough and smooth boundaries. We first consider a smooth boundary. Here, we find that $\gamma_{\text{point}} = 1$ and $\gamma_{\text{line}} = 0$, indicative of the existence of topological point particles that cannot be absorbed at the boundary. We compare these results to those obtained on a rough boundary. All the connected boundaries of all the regions used to evaluate $\gamma_{\text{point}}$ touch a boundary, we therefore find $\gamma_{\text{point}} = 0$ at the rough boundary. Conversely, region $C$ used to find $\gamma_{\text{line}}$ is contained in the bulk of the lattice. We therefore find that $\gamma_{\text{line}} = 1$ at the rough boundary, as expected for a region that cannot absorb line-like excitations.

Conclusion—By consideration of the support of quasi-particle creation operators we have shown that we can obtain new entropic invariants for local gapped Hamiltonians using information theoretic arguments. We have used these methods to find two new order parameters for the boundary theories of 3D topological models. We have demonstrated that the proposed measures are effective by studying different boundaries of the 3D toric code. The result we obtain is remarkable given that we cannot distinguish between different excitation structures in the bulk of 3D topological phases using entropic diagnostics [17]. One might consider using the proposed topological invariants to interrogate the structure of more general classes of topologically ordered systems [21] with exotic surface theories, where perhaps the bulk topological entanglement contribution is zero [18]. Another class of models of recent interest in this respect are bosonic topological insulators [22, 23]. One might also consider using the present general proof to find new topological invariants for more exotic phases such as fractal topological quantum field theories [24].

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SUPPLEMENTARY MATERIAL

Remaining part of the proof for Eq[1]

In the main text, we proved that there exists a unitary operator \( U' \) that lies in the vicinity of the particles, such that

\[
\| U |\psi_0\rangle - U' |\psi_0\rangle \| \leq O(\gamma^{\frac{1}{2}}).
\]

Using this inequality, we intend to derive the main result. For the reader’s convenience, we state it again below:

\[
\| UV |\psi_0\rangle - VU |\psi_0\rangle \| \leq O(\gamma^{\frac{1}{2}}).
\]

In order to show this, we make use of two simple facts. First, \( V |\psi_0\rangle = e^{i\phi} |\psi_0\rangle \). This means that the process \( V \) acts trivially on the ground state. Second, \( V \) commutes with \( U' \). This is due to the fact that the support of \( U' \) lies only in the vicinity of the quasi-particles, whereas the support of \( V \) can be made to be far away from the quasi-particles.

Since the norm is invariant under unitary rotation,

\[
\| VU |\psi_0\rangle - VU' |\psi_0\rangle \| = \| U |\psi_0\rangle - U' |\psi_0\rangle \| \leq O(\gamma^{\frac{1}{2}}).
\]

It should be noted that \( VU' |\psi_0\rangle \) is actually equal to \( U'V |\psi_0\rangle \) due to the commutation relation. Since \( V \) acts trivially on the ground state,

\[
\| U'V |\psi_0\rangle - UV |\psi_0\rangle \| = \| U' |\psi_0\rangle - U |\psi_0\rangle \| = O(\gamma^{\frac{1}{2}}).
\]

Applying the triangle inequality to the above two inequalities, we arrive at Eq[1]

The 3D Toric Code

Here we study the bipartite entanglement between simple regions of the ground state of the 3D toric code lattice. We use the method of [19] to find the entanglement entropy of a ball-shaped region in the bulk, and ball-shaped regions that enclose some of the qubits in a rough and a smooth boundary.

The 3D toric code is defined on a square lattice with qubits arranged on its edges. Its degenerate ground space, spanned by \( |\psi_j\rangle \), is described by its (Abelian) stabilizer group, \( S = \{ s \in S | s |\psi_j\rangle = |\psi_j\rangle \} \). The stabilizer group for the 3D toric code contains two types of stabilizers; star and plaquette operators. The stabilizer group \( S \) for the 3D toric code is generated by the star and plaquette operators shown in FIG. 6.

To employ the method [19] we must find a suitable generating set of the stabilizer group that enables us to count the ebits of entanglement shared between two subsystems. Importantly, the generating set is over complete if we include \( B_f \) operators for all the faces. This is seen by taking the product of all the plaquette operators corresponding to the faces that bound a cube. This combination of plaquette operators returns identity, showing an over-complete generating set.

We choose an independent generating set that includes all plaquette operators that lie parallel to the \( xy \) and \( yz \) plane, and we only take the plaquette operators parallel to the \( xz \) plane in a single plane at some fixed \( y \). We are free to choose which plane, and for simplicity we always take this plane to be far away from the region of interest for the entropy calculation. For this reason, for all the calculations we make, it is sufficient to consider the only the plaquette generators parallel to the \( xy \) and \( yz \) plane.

In a similar respect, we point out now that we need not account for the logical operators that may appear in the generating set of the stabilizer group. Logical operators can always be deformed away from the regions of the lattice we are interested in, and as such, never contribute to the entanglement in any of the bipartitions we study.

The von Neumann Entropy by finding Canonical Form

We use the method of Fattal et al. [19] to find the entanglement entropy between two subsystems of the 3D toric code. We briefly explain the method by the example of two very simple cases of two and three qubits. We first consider a Bell state \( |\phi_+\rangle = (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)/\sqrt{2} \). This state shares a single ebit of entanglement between subsystem \( A \) and \( B \). The stabilizer group for \( |\phi_+\rangle \) is generated by set

\[
\langle X_A X_B, Z_A Z_B \rangle.
\]

We identify the entanglement by considering the restriction of the generating set on subsystem \( A \),

\[
\{ X_A, Z_A \}.
\]
In this restriction, we identify a pair of anti-commuting operators, which corresponds to a single ebit of entanglement shared between subsystem $A$ and $B$. We compare this with the state $|+\rangle_A|+\rangle_B$, stabilized by group generated by $\langle X_A, X_B\rangle$. The restriction of the generating set onto subsystem $A$ is simply $\{X_A, 1\}$. Here we cannot find any anti-commuting pairs, and as such, no entanglement is shared between $A$ and $B$.

We consider a final, slightly more complicated example to understand the methods we use later. We calculate the entanglement entropy between qubits 2, and qubits 1 and 3 of the state $|\text{GHZ}\rangle = (|0\rangle_1|0\rangle_2|0\rangle_3 + |1\rangle_1|1\rangle_2|1\rangle_3)/\sqrt{2}$.

To illustrate the calculation method we use, we deliberately choose an inconvenient generating set for the GHZ state

$$\langle X_1X_2X_3, Z_1Z_2, Z_2Z_3 \rangle.$$  

(10)

The restriction onto the subsystem of qubits 1 and 3 is

$$\{Z_1, X_1X_3, Z_3\}.$$  

Here, we cannot immediately identify a pair of anti-commuting operators, as both $Z_1$ and $Z_3$ anti commute with $X_1X_3$. We use our freedom to choose the generating set to correct this. We replace the generating set shown at Eq.10, with set $\langle X_1X_2X_3, Z_1Z_3, Z_2Z_3 \rangle$, which generates the same stabilizer group. With this generating set, we have the restriction

$$\{Z_1Z_3, X_1X_3, Z_3\},$$  

where now we have a single pair of anti-commuting operators; $X_1X_3$ and $Z_3$. The operator $Z_1Z_3$ commutes with the other members of the restricted generating set and thus does not contribute to the entanglement. We therefore identify that qubit 2 shares a single ebit of entanglement with qubits 1 and 3. Generating sets where we are able to count pairs of anti-commuting operators when restricted to a subsystem are said to be in canonical form. The result of Fattal et al. shows that it is always possible to find a generating set in canonical form for any subsystem of any stabilizer group.

The von Neumann Entropy of a Ball in the Bulk

We now consider the entropy of a ball in the 3D toric code, see FIG.7. To the left of this Figure, we show the corner of a region, where the region is filled with transparent green ‘jelly’. We show some examples of the restriction of star and plaquette operators outside the green jelly. We seek a canonical generating set.

We simplify this diagram by representing the restricted stabilizers on a graph. We show the graph that corresponds to the corner of the region to the right of FIG.7. In this graph, vertices represent star operators cut by the boundary of the region, and each edge represents a cut plaquette operator. An edge that is incident to a vertex represents a restricted plaquette operator that anti-commutes with the respective restricted star operator, represented by the vertex. The graph shows we are not in canonical form, as there are many edges incident to each vertex.

We show a complete graph for the restriction of a ball-shaped region in FIG.8 where any double edges are removed. We will see why we are free to replace double edges with single edges shortly.

We face the task to find edges we are allowed to remove to find a canonical generating set, while still generating the correct stabilizer group. We complete the entropy calculation by introducing rules that enable us to find canonical form, and easily count the ebits of entanglement shared between the region and its complement.

For a ball-shaped region in the bulk of the lattice, $R$, we recover the known result $S(\rho_R) = A_R - 1$ where $A_R$ is the surface area of the boundary or region $R$. To find this result, we propose that the surface area is measured in units of cut star operators, i.e. the number of vertices

FIG. 7: (Left) The corner of a ball-shaped region, labeled $A$, of the 3D toric code. We show the support of one star and two plaquette operators on region $A$, in bright colours. (Right) We represent operators with non-trivial support on region $A$ and $A$ as a graph. Vertex operators are represented as vertices, and plaquette operators are represented as edges. Every edge incident to a vertex represents a plaquette that anti-commutes with a star operator. Clearly, the natural generating set is not in canonical form.

FIG. 8: The graph for a cuboid-like, ball shaped region in the bulk of the 3D toric code with double edges removed. The top face differs from the side faces due to the inhomogeneous generating set. The entanglement entropy does not depend on the choice of the generating set. This will become apparent as we progress through the calculation.
in graph in FIG.8. This elegant definition is particularly suitable for the following reasons; firstly, it is easily checked that each vertex will appear in each boundary the correct number of times, and thus surface area terms will cancel exactly in the topological entanglement entropy schemes we propose in the main text. We also observe that all but one vertex operator will contribute to the entanglement, which gives the desired result known in the literature. Moreover, this definition is robust to deformations in the choice of region boundary.

We now set out to find a canonical generating set. In the first step, we are free to remove double edges, as we have already done in FIG.8. We are free to do this due to the circuit rule.

**Rule 1 (The Circuit rule).** Given any closed circuit of edges on the graph, we can remove any single edge from this circuit without affecting the entanglement of the partition. We give examples of the circuit rule in Fig.9.

To show the circuit rule, we consider the explicit examples of the (Left) and (Middle) cases of circuits shown in FIG.9. For (Left), we see two edges, labeled 1 and 2. Both edges anti-commute with the same two adjacent vertices. To remove the generator of edge 2 from the graph, we replace it with the generator that is the product of all edges 1, 2, 3 and 4, allowing us to remove edge 4 from the graph. (Right) In general, we can always remove a single edge from any closed loop of edges.

FIG. 9: (Left) Double edges can be replaced by single edges, without any contribution to the entanglement of the region. (Middle) We replace restricted generator edge 4 with the product of all edges 1, 2, 3 and 4, allowing us to remove edge 4 from the graph. (Right) In general, we can always remove a single edge from any closed loop of edges.

We apply the circuit rule to the different faces of the graph shown in FIG.8. Using this rule, we obtain the equality shown in FIG.10 between different faces of the cuboid. We reduce all the faces of the cuboid to the form of the right of equality FIG.11 for the next step in the calculation.

The new face we find in FIG.11 has ‘loose ends’ in the graph. The length of these loose ends is proportional to its entropy contribution by the following loose end rule.

**Rule 2 (The loose-end rule).** A one-dimensional string of $x$ vertices where one end connects to an extended graph, and the other end terminates with a vertex, contributes $x$ ebits of entanglement. We show this rule pictorially in FIG.11.

We see the loose-end rule rigorously by enumerating the restricted stabilizers, $S_j$, along the loose end. Here $S_j$ for odd $j$ are restricted $A_v$ operators, represented in the graph by vertices, and restricted $B_j$ operators, edges, have even $j$. The indices take values $1 \leq j \leq 2x + 1$, and $2x + 1$ indexes the operator corresponding to the black vertex at the end of the blue string. We have that $\{S_j, S_{j+1}\} = 0 \forall j \leq 2x$. This is not canonical form. We find a canonical form for all the blue vertices and edges by making the replacement $S_j \rightarrow S'_j = \prod_{0 < k < j} S_k$ for all odd $j$. With this replacement, all the vertices shown in blue are replaced by operators in canonical form with respect to the blue edges of the graph, thus identifying $x$ ebits of entanglement.

We now identify the entanglement of a face of a cuboid, as shown in FIG.12. We use the loose-end rule to see that all the vertices in each face of the graph contribute a single unit of entanglement to the calculation, and thus

FIG. 10: We use the rules we have introduced to show that an face with a square grid of edges is equivalent to a face which contains only vertical edges. The right equality is obtained with further use of the circuit rule.

FIG. 11: An open end, that is terminated by a vertex can be identified as ebits of entanglement, where the length of the open ended string, measured by the number of vertices, $x$, corresponds to $x$ ebits of entanglement.

FIG. 12: The entanglement of a face of the cuboid graph.
FIG. 13: We use the calculation summarised in FIG.12 to find the entanglement of all the faces of a cuboid graph, where we have that $X = 2[(L_x - 2)(L_y - 2) + (L_x - 2)(L_z - 2) + (L_z - 2)(L_y - 2)]$

far show that all the vertices contribute to the area term of the entropy. We extend this to all the faces of the cube, as shown in FIG.13.

We can continue to use the circuit rule and the loose end rule to arrive at the result of FIG.14 where only a single closed loop of vertices remains in the graph. Once again, all the vertices that have been removed from the graph have contributed to the area term of the entanglement entropy; no vertex has been removed without contributing to the entanglement entropy. To complete the calculation we must assess the entanglement of a single closed loop that remains in the graph.

**Rule 3** (The closed loop rule). A periodic one-dimensional loop of $x$ vertices with no additional incident edges will contribute $x - 1$ ebits of entanglement, shown graphically in FIG.15.

We consider the case of a closed loop carefully, as shown in FIG.15. Like the loose end rule, we give the restricted generators $S_j$ in a closed loop of $x$ vertices the indices $1 \leq j \leq 2x$, where even $j$ is an edge and odd $j$ is vertex. Initially, we have that every $S_j$ anti commutes with two other restricted generators, $S_{j-1}$ and $S_{j+1}$, where $j \pm 1$ is carried out modulo $2x$ to accommodate the periodic structure of the closed loop.

To obtain canonical form, we must replace a single vertex operator with the product of all the vertices in the loop $S_1 \rightarrow S'_1 = \prod_{\text{odd}} S_k$, such that it commutes with all edges in the closed loop. Similarly, we replace an edge with the product of all edges, such that it commutes with all vertices, $S_2 \rightarrow S'_2 = \prod_{\text{even}} S_k$. We are then free to reduce the remaining $x - 1$ vertices into canonical form, corresponding to $x - 1$ ebits.

The closed loop rule removes a single vertex from the graph without contributing to the entanglement, thus giving the topological contribution in the calculation.

It is easy to check that this method extends to any region with a connected boundary, such as an annulus. Ultimately, the calculation will always reduce to a single closed loop of vertices, where we are able to remove a single vertex without contributing to the entanglement, thus always giving the desired result for a connected boundary. Indeed, in general, for more complicated regions that include multiple disjoint boundaries, every connected boundary can be reduced to a single closed loop, enabling us to remove one vertex per connected boundary without contributing to the entanglement. The topological correction will therefore scale with the number of connected boundaries that enclose the region.

### A Ball on a Smooth Surface

The entanglement entropy of a ball-shaped region that includes a topologically trivial patch of qubits on a smooth boundary gives the same topological contribution as the case we previously considered in the bulk. We show the graph of restricted generators of such a region in FIG.16. It is easy to extend the methods of the

FIG. 14: For a cuboid of dimensions of $L_x \times L_y \times L_z$ we have $Y = 4L_y + 2(L_x + L_z) - 12$, where it is the $z$-axis that extends outside of the page. The $x$ and $y$ axis extend horizontally and vertically, as is convention.

FIG. 15: A closed loop of $x$ vertices corresponds to only $x - 1$ ebits of entanglement.

FIG. 16: We show a graph of the restricted generators for a ball-shaped region pressed against a smooth surface, where the smooth surface is at the bottom of the cuboid. The calculation can be performed using the loose-end rule. The calculation finds a single closed loop, giving $S(\rho_R) = A_R - 1$, as in the case in the bulk.
previous Section, and obtain the result $S(\rho_R) = A_R - 1$ where $A_R$ is the number of star operators cut by the boundary. Star operators cut near the surface are not treated differently from those cut in the bulk.

A Ball on a Rough Surface

We now consider the entanglement entropy for the case where the region touches the rough face. Contrary to the cases we have considered previously, we do not find a topological contribution to the entanglement entropy. As such, we describe this calculation in detail. We show a picture of an example restricted stabilizer graph in FIG.17. This differs from the case considered in the previous Section as here the graph terminates at an edge, not at a vertex.

As before, we begin by applying the circuit rule and the loose end rule repeatedly to clean the face of this graph. We show the graph in FIG.18. For clarity, we also flatten the graph in this diagram.

To calculate the entanglement of this region, we introduce two new rules, the extended circuit rule, and the lone-string rule to find the entanglement.

**Rule 4** (The extended circuit rule). We can remove a single edge from an open circuit if both ends of the open circuit terminate at an open edge. We show this in FIG.19.

We show that the extended circuit rule is true in FIG.19. The edge that is missing on the right hand side of this equality has been replaced by the product of all the solid edges on the left hand side of the Figure, such that the new generator commutes with all the vertices of the graph. The dotted edge on the left of the Figure represents some extended part of the graph. The extended graph will not affect the use of the extended circuit rule.

We implement the extended circuit rule many times, to arrive at the graph shown in FIG.20. We can now find the entanglement contribution of the detached one-dimensional strings using the lone string rule.

**Rule 5** (The lone-string rule). An open one-dimensional string of $x$ vertices, where one end terminates at a vertex, and the other terminates with an edge, will contribute
FIG. 21: An open string of $x$ vertices that has one termination point at a vertex, and one at an edge, will contribute $x$ ebits of entanglement to the graph.

FIG. 22: The remaining graph of $x$ vertices will contribute $x$ ebits of entanglement due to the one lone string rule.

$x$ ebits of entanglement to the graph. We show this in FIG[21]

We leave the proof of the lone-string rule to the reader.

Once again, every vertex removed from the detached legs of the graph by the one lone string rule contributes a single unit to the entanglement entropy, and therefore all the vertices contribute a unit of entanglement to the area term of the calculation.

We complete this calculation with one final application of the circuit rule, giving the graph shown in FIG[22]. The one lone-string rule shows that all the remaining vertices will contribute to the entanglement entropy.

All the vertices removed from the graph have contributed a unit to the entanglement entropy, giving the result

$$S(\rho_R) = A_R,$$

for the case where $R$ includes qubits of a rough boundary of the 3D toric code.

[1] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. 53, 722 (1984).

[2] X. G. Wen and Q. Niu, Phys. Rev. B 41, 9377 (1990).

[3] A. Hamma, R. Ionicioiu, and P. Zanardi, Phys. Rev. A 71, 022315 (2005), quant-ph/0409073.

[4] M. Levin and X. G. Wen, Phys. Rev. Lett. 96, 110405 (2006), cond-mat/0510613.

[5] A. Y. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006), hep-th/0510092.

[6] A. Y. Kitaev, Annals Phys. 303, 2 (2003), quant-ph/0507021.

[7] M. B. Hastings, Phys. Rev. B 69, 104431 (2004).

[8] I. H. Kim, Phys. Rev. Lett. 111, 080503 (2013), 1304.3925.

[9] M. A. Levin and X. G. Wen, Phys. Rev. B 71, 045110 (2005), cond-mat/0404617.

[10] A. Hamma, P. Zanardi, and X.-G. Wen, Phys. Rev. B 72, 035307 (2005).

[11] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, Cambridge Series on Information and the Natural Sciences (Cambridge University Press, 2000), ISBN 9780521635035.

[12] I. H. Kim (2014), 1405.0137.

[13] A. Uhlmann, Rep. Math. Phys. 9, 273 (1976).

[14] E. H. Lieb and M. B. Ruskai, J. Math. Phys. 14, 1938 (1973).

[15] S. B. Bravyi and A. Y. Kitaev (1998), quant-ph/9811052.

[16] C. Castelnovo and C. Chamon, Phys. Rev. B 78, 155120 (2008), 0804.3591.

[17] T. Grover, A. M. Turner, and A. Vishwanath, Phys. Rev. B 84, 195120 (2011), 1108.4038.

[18] C. W. von Keyserlingk, F. J. Burnell, and S. H. Simon, Phys. Rev. B 87, 045107 (2012), 1208.5128.

[19] D. Fattal, T. S. Cubitt, Y. Yamamoto, S. Bravyi, and I. L. Chuang (2004), quant-ph/0406168.

[20] B. J. Brown, S. D. Bartlett, A. C. Doherty, and S. D. Barrett, Phys. Rev. Lett. 111, 220402 (2013).

[21] K. Walker and Z. Wang, Front. Phys. 7, 150 (2012).

[22] A. Vishwanath and T. Senthil, Phys. Rev. X 3, 011016 (2013).

[23] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, Phys. Rev. B 88, 035131 (2013).

[24] B. Yoshida, Phys. Rev. B 88, 125122 (2013).

[25] S. Bravyi and B. Terhal, New J. Phys. 11, 043029 (2008), 0810.1983.

[26] This method of reducing the support of an operator is of the spirit to the ‘cleaning lemma’, presented for stabilizers in 23.