Anderson localization and momentum-space entanglement

Eric C Andrade\textsuperscript{1,2}, Mark Steudtner\textsuperscript{1} and Matthias Vojta\textsuperscript{1}

\textsuperscript{1} Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany
\textsuperscript{2} Instituto de Física Teórica, Universidade Estadual Paulista, Rua Dr. Bento Teobaldo Ferraz, 271-Bl. II, 01140-070, São Paulo, SP, Brazil
E-mail: matthias.vojta@tu-dresden.de

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Abstract. We consider Anderson localization and the associated metal–insulator transition for non-interacting fermions in $D = 1, 2$ space dimensions in the presence of spatially correlated on-site random potentials. To assess the nature of the wave function, we follow a recent proposal to study momentum-space entanglement. For a $D = 1$ model with long-range disorder correlations, both the entanglement spectrum and the entanglement entropy allow us to clearly distinguish between extended and localized states based upon a single realization of disorder. However, for other models, including the $D = 2$ case with long-range correlated disorder, we find that the method is not similarly successful. We analyze the reasons for its failure, concluding that the much desired generalization to higher dimensions may be problematic.

Keywords: disordered systems (theory), entanglement in extended quantum systems (theory)
1. Introduction

The possibility that electronic bound states can be formed in the presence of a random potential was first discussed in Anderson’ pioneering work [1] where it was shown that, for a sufficiently strong randomness, single-particle wave functions can become exponentially localized, leading to a sharp metal–insulator transition [2] at $T = 0$.

For non-interacting fermions with uncorrelated random potentials, and in the absence of spin–orbit coupling, the scaling theory of localization [3] predicts that all single-particle states are localized for dimensions $D \leq 2$ for any amount of disorder, and thus the Anderson metal–insulator transition only takes place in $D > 2$. One way to circumvent the scaling predictions and to observe an Anderson transition in low-dimensional systems is to consider systems with spatially correlated random potentials [4–10]. Such correlated disorder potential could emerge, for example, as an effective description of an interacting electronic system in the presence of impurities, where the Fermi liquid readjusts itself, producing a spatially inhomogeneous pseudopotential ‘seen’ by quasiparticles [11, 12].

Over the years, several quantities which probe the extension of the electronic wave function were employed to theoretically study the Anderson metal–insulator transition. These include the local density of states [13], the participation ratio [14, 15], the Lyapunov exponent [16], the localization length [17] and the conductance [18], among
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others [19]. More recently, the concept of quantum entanglement has been successfully applied to the study of the Anderson transition [20, 21]. This success is mainly due to the fact that entanglement and the spatial extension of the electronic wave functions are closely related: extended states are entangled in position space, whereas localized states are not.

While the use of entanglement in position space appears natural, the authors of [22] recently showed that the entanglement in momentum-space [23, 24] also appears to be a useful tool for studying the presence of extended states. As an example, the authors revisited the so-called random dimer model [4] in $D = 1$ and pointed out that a direct analysis of the entanglement spectrum, constructed from a momentum-space partition, provides a sharper distinction between the extended and localized states as compared to the position-space partition. Remarkably, useful and accurate results could be obtained from a single realization of disorder. If generalizable to $D > 1$, this would make the method particularly useful considering that the numerical effort increases drastically for higher $D$ on top of which conventional methods to study localization require extensive averaging over different realizations of disorder.

Motivated by the success of [22], we ask whether momentum-space entanglement as a probe of localization is of wider applicability. To this end, we apply this concept to other low-dimensional non-interacting models with correlated disorder. First, we show that, for a $D = 1$ model with long-range correlated disorder, momentum-space entanglement does indeed provide a clear characterization of the Anderson transition. Second, we study a model of coupled chains which displays coexisting extended and localized states, and we also investigate the Anderson transition for correlated disorder in $D = 2$. In both cases we find that the proposed method fails to capture the presence of extended states, casting doubts on the general utility of momentum-space entanglement in the context of localization phenomena. We suggest two reasons for the failure, one related to the specific construction of entanglement measures employed in [26] and another one related simply to the topology of higher-dimensional momentum space.

The body of the paper is organized as follows: In section 2 we briefly present the tight-binding model studied by Anderson [1] and discuss the calculation of entanglement spectrum and entropy. In section 3 we investigate the metal–insulator transition in a $D = 1$ model with long-range correlated disorder. In section 4 we study a particular model of coupled chains with correlated disorder within a unit cell. We then return to long-range correlated disorder, in section 5, but now considering $D = 2$. A discussion of methodological aspects and possible reasons for the apparent failure of momentum-space entanglement is in section 6. A short summary concludes the paper.

2. Disordered Hamiltonian and entanglement

To study the Anderson transition, we consider a single-band tight-binding model of spinless fermions on a lattice with $N$ sites

$$
\mathcal{H} = \sum_i \varepsilon_i n_i - t \sum_{\langle ij \rangle} \left( c_i^\dagger c_j + c_j^\dagger c_i \right),
$$

(1)
where $t$ is the hopping matrix element between nearest-neighbor sites $\langle ij \rangle$, $c_i^\dagger (c_i)$ is the creation (annihilation) operator of an electron at site $i$, $n_i = c_i^\dagger c_i$ is the number operator, and $\varepsilon_j$ are the site energies forming the disorder potential. We measure all our energies in units of $t$ and consider periodic boundary conditions.

### 2.1. Entanglement via correlation matrix

To investigate Anderson localization, we use the notion of quantum entanglement and follow [26]. We begin by partitioning the system into two regions $\mathcal{A}$ and $\mathcal{B}$ which may correspond either to a subset of sites (real-space partitions) or a subset of single-particle momenta (momentum-space partitions). The next step is to construct a reduced density operator $\rho_{\mathcal{A}}$ which only acts in the many-body Hilbert space defined by $\mathcal{A}$ and, by construction, reproduces all expectation values in region $\mathcal{A}$. From $\rho_{\mathcal{A}}$ we obtain the entanglement spectrum and entropy, which quantify how much information region $\mathcal{A}$ contains about the physics in region $\mathcal{B}$.

Any reduced density operator can be written in terms of an entanglement Hamiltonian $\mathcal{H}_{\mathcal{A}}$, $\rho_{\mathcal{A}} = \exp[-\mathcal{H}_{\mathcal{A}}]/Z$, where $Z$ is a normalization constant. For a model of non-interacting particles such as equation (1) it can be shown that the entanglement Hamiltonian $\mathcal{H}_{\mathcal{A}}$ has the form of a free-fermion Hamiltonian [25, 26]. Its eigenvalues $\varepsilon_{\mathcal{A}}^i$ are in one-to-one correspondence with the eigenvalues $\zeta_i$ of the two-point correlation matrix $C_{ij}$. According to [25, 26] $\zeta_i = \frac{1}{2} \left( \exp[\varepsilon_{\mathcal{A}}^i] + 1 \right)^{-1}$ where

$$C_{ij} = \langle \Psi | c_i^\dagger c_j | \Psi \rangle,$$

with $|\Psi\rangle$ being a free-fermion ground state, i.e. the Fermi sea filled up to the Fermi energy $E_F$ and, for simplicity, we consider $i, j \in \mathcal{A}$. Since the entanglement Hamiltonian $\mathcal{H}_{\mathcal{A}}$ has a free-fermion form, the entanglement entropy is then simply given by

$$S = - \sum_i (\zeta_i \log \zeta_i + (1 - \zeta_i) \log (1 - \zeta_i)),$$

and, from now on, we refer to $\zeta_i$ as the entanglement spectrum. We see that $\zeta_i = 0, 1$ corresponds to no entanglement, whereas $\zeta_i = 1/2$ corresponds to maximum entanglement. We also point out that $S$ does not have a fixed upper bound in the present scheme, since we construct $C_{ij}$ using all states up to the Fermi level.

To numerically evaluate $C_{ij}$, we first write the operator $c_i$ in terms of the eigenstates $|\nu\rangle$ of the single-particle problem, $c_i = \sum_\nu \langle i | \nu \rangle a_\nu$, with $\mathcal{H} = \sum_\nu \varepsilon_\nu a_\nu^\dagger a_\nu$, and then

$$C_{ij} = \sum_\nu \langle \nu | i \rangle \langle j | \nu \rangle \theta (E_F - \varepsilon_\nu),$$

where $\theta(x)$ is the usual step-function and $E_F$ is the Fermi energy.

For a real-space space partition, extended states are expected to display entanglement whereas localized states are not. A momentum-space partition was only recently investigated in the context of a disordered electronic system [26] and, conversely to its real-space counterpart, when looking for extended states we should seek for the absence of entanglement since an extended state in real space has a well defined (localized) momentum.
2.2. Correlation matrix versus single-particle entanglement

While conventional measures of localization, e.g. the inverse participation ratio, are defined for single-particle states, the eigenvalues of the correlation matrix $C_{ij}$ in equation (4), proposed as an indicator of localization in [22], involve the whole Fermi sea [27]. As will become clear in the remainder of the paper, this has a number of consequences.

First we note that in order to study real-space entanglement there is no need to consider $C_{ij}$. One may instead focus on single-particle entanglement [20, 21] which can be directly obtained from an individual single-particle state.

Second, however, single-particle entanglement can be problematic for a momentum-space partition. To illustrate this point, we consider the clean (disorder-free) limit in $D = 1$. In this case, the real single-particle eigenstates of equation (1) can be written as $\psi_k(x) = \sqrt{\frac{2}{L}} \cos(kx)$ or $\psi_k(x) = \sqrt{\frac{2}{L}} \sin(kx)$. If we now evaluate $C_{ij}$ for a real space partition, the only non-vanishing contribution to the entanglement spectrum is $\zeta_1 = \sum_{i=1}^{L/2} |\psi_k^{\ell}(x)|^2 = 1/2$. This maximally entangled situation simply reflects the fact that the wave functions $\psi_k^{\ell}(x)$ are extended over the entire lattice. Performing a similar calculation for a momentum-space partition, we obtain $\zeta_1,2 = 1/2$, which wrongly suggests an ill-defined momentum state and thus a localized wave function in real space. Behind this failure lies the fact that the Fourier-transformed single-particle states $\psi_k^{\ell}(p)$ are non-zero at both $\pm k$. One way to resolve this problem is to consider two particles, occupying both $\psi_k^1$ and $\psi_k^2$, and to construct $C_{ij}$ for this two-particle system. In this case, we obtain $\zeta_1,2 = 1$ as the non-vanishing contributions to the momentum-space entanglement spectrum, which now correctly translates into extended wave functions in real space. Therefore, one role of the Fermi sea in equation (4) is to account for the presence of states at $\pm k$ in momentum-space partitions.

Ultimately, the fact that one considers the whole Fermi sea introduces interference between different occupied states, which in turn can reveal extra information not present at the level of single-particle entanglement [27–29].

3. Long-range correlated disorder in $D = 1$

We first investigate a chain of length $N = L$. To construct a sequence of correlated site energies, we follow the proposal of [5] and consider that the site energies $\varepsilon_i$ have a spectral density $S(k) \propto 1/k^\alpha$, where $S(k)$ is the Fourier transform of the two-point correlation function $\langle \varepsilon_i \varepsilon_j \rangle$, where $\langle \cdots \rangle$ denotes average over disorder. When the exponent
$\alpha = 0$, we recover the usual Anderson [1] model with uncorrelated disorder, which shows a white noise spectrum and a local two-point correlation function $\langle \varepsilon_i \varepsilon_j \rangle = \langle \varepsilon_i^2 \rangle \delta_{ij}$.

To generate a correlated sequence with a given spectral density we follow a standard procedure [30] and write

$$\sum_{\pi} \varepsilon_{\pi \pi} \phi = \begin{bmatrix} \sum_{n_k=1}^{L/2} \left( \frac{2\pi}{L} \right)^{-\alpha} \frac{1}{n_k^\alpha} \cos \left( \frac{2\pi}{L} in_k + \varphi_k \right) \right]$$

where $\varphi_k$ are $L/2$ independent random variables uniformly distributed in the interval $[0, 2\pi]$ and $k = 2\pi n_k/L$. Due to the randomness of $\varphi_k$, we have $\langle \varepsilon_i \rangle = 0$ and we normalize the energy such that $\sigma_\varepsilon = \varepsilon_1^{1/2}$ in order to keep the same disorder strength for all $\alpha$. Interestingly, the energies generated in this fashion have the property that $\langle \varepsilon_i \varepsilon_{i+L/2} \rangle = \frac{2^{1-\alpha}}{1-\alpha}$, meaning that $\varepsilon_i$ and $\varepsilon_{i+L/2}$ (the two most distant sites) are anti-correlated for $\alpha > 1$. The resulting site energies are shown in figure 1(a) and we see that the energy spatial profile becomes smoother as $\alpha$ increases and that the anti-correlation between the most distant sites is more evident.

As first shown in [5], this model displays extended states when the exponent $\alpha > 2$, with the location of the mobility edges $E_c$ depending on the value of $\alpha$. Moreover, because of the long-range correlations in the disorder potential, this problem lacks self-averaging [5, 32] and thus the location of the phase boundaries in the phase diagram is particular to the correlated sequence used (see, for instance, figures 1(c) and (d)).

### 3.1. Density of states and inverse participation ratio

We start by investigating the total density of states, identical to the site-averaged local density of states $\rho_i(E)$, $\rho_{tot}(E) = N^{-1} \sum_{i=1}^{N} \rho_i(E)$, with

$$\rho_i(E) = \sum_{\nu=1}^{N} \langle i|\nu \rangle^2 \delta(E-E_{\nu}) \right]$$

where $\langle i|\nu \rangle$ is the amplitude of the $\nu$-eigenvector of equation (1), with energy $E_{\nu}$, at the site located at $r_i$. We numerically evaluate the delta function as $\delta(x) \approx \Gamma^{-1} \theta(\Gamma/2 - |x|)$ and we generally choose the width $\Gamma = 0.1t$. In figure 1(b) we show $\rho_{tot}$ for different values of $\alpha$. Because of the imposed normalization $\langle \varepsilon_i^2 \rangle = 1$ the width of $\rho_{tot}$ is essentially independent of $\alpha$. We also see that $\rho_{tot} \neq 0$ for $|E| \leq 4t$, with no sharp band edges. As $\alpha$ increases, the curves become smoother following the same trend as the site energies. Even though $\rho_i$ undergoes a qualitative change upon localization, because it directly measures the local amplitude of the electronic wave functions, $\rho_{tot}$ shows no sign of an Anderson transition for $\alpha > 2$, since site to site fluctuations are averaged out [13].

We also consider the inverse participation ratio

$$\text{IPR}_\nu = \sum_{i=1}^{N} | \langle i|\nu \rangle |^4 .$$

Generically, we have $\text{IPR}_\nu \sim \mathcal{O}(1)$ for a localized state and $\text{IPR}_\nu \sim \mathcal{O}(1/N)$ for an extended one. The value of the IPR$_\nu$ averaged over disorder is often employed as a powerful tool to investigate the Anderson transition. In the present case, however, due
to the long-range character of the disorder, the position of the mobility edges changes considerably from sample to sample, figures 1(c) and (d), even in the thermodynamic limit [32]. Due to this peculiar behavior, we consider the IPRν only for single disorder realizations. Interestingly, for the current model [32], the positions of the mobility edges in a given sample are determined by sharp peaks in the IPRν, which separate the extended states (N × IPRν constant) from the localized ones (N × IPRν size dependent), figures 1(c) and (d).

### 3.2. Entanglement spectrum and entropy

We now turn to characterize the Anderson transition via entanglement measures. To this end, we calculate the entanglement spectrum ζi and entropy S for a single disorder realization. First, we consider a partition in real space with \( \mathcal{A} \in [1, L/2] \) and \( \mathcal{B} \in [L/2 + 1, L] \), with sample results in figures 2(a) and (d). From previous work [5], we know that extended states at \( E_F = 0 \) occur for \( \alpha > 2.5 \) and, while the entropy saturates to its maximum value in this region, the entanglement spectrum ζi does not display a sharp boundary, since there is appreciable entanglement also for \( \alpha < 2.5 \).
Second, we consider a cut in momentum space \([22]\) with 
\[A \pi \in [0, \pi] \quad \text{and} \quad B \pi \pi \in [\pi, 2\pi]\]
with sample results in figures 2(b)–(d). For the clean case, this partition corresponds
to a separation between the right and left movers. Remarkably, the momentum-space
entanglement spectrum displays a qualitative change as a function of \(\alpha\). Comparing
figures 2(a) and (b) we see that the momentum-space cut clearly shows the presence
of extended states. We recall that for this cut extended states are associated with the
suppression of entanglement, \(\zeta_i = 0, 1\), which in figure 2(b) occurs for \(E_F = 0\). It is also
interesting to notice that the entanglement entropy curve in figure 2(d) is very smooth,
even though no average of disorder was performed. For completeness, in figure 2(c)
we show the momentum-space entanglement spectrum for fixed \(\alpha\) as a function of the
Fermi energy. For \(E_F\) smaller than the mobility edge we again see a suppression
of entanglement; recall that the position of the mobility edge is independently known
through IPR as in figures 1(b) and (d).

Despite all the clear evidence for the presence of extended states in figures 2(b)–(d),
it is difficult to determine the precise location of the critical \(\alpha\), or of the mobility edges,
with the current method, a situation which remains unchanged even for larger system.

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Figure 2. Entanglement spectra \(\zeta_i\) and entropy \(S\) for the disordered tight-binding Hamiltonian equation (1) with long-range correlated site disorder given by equation (5), all obtained for a single realization of disorder and \(L = 4096\).

(a) Entanglement spectrum \(\zeta_i\) obtained from a real-space cut as a function of the exponent \(\alpha\) for \(E_F = 0\).
(b) Entanglement spectrum \(\zeta_i\) obtained from a cut in momentum space as a function of the exponent \(\alpha\) for \(E_F = 0\).
(c) Entanglement spectrum \(\zeta_i\) obtained from a cut in momentum space now as a function of \(E_F\) for \(\alpha = 3.0\).
(d) Entanglement entropy \(S\), divided by its maximum value \(S_{\text{max}}\), as a function of the exponent \(\alpha\) for cuts both in real and momentum space at \(E_F = 0\).

At \(E_F = 0\) extended states are known to exist for \(\alpha \gtrsim 2.5\), [5], indicated by dashed lines in (a), (b) and (d). For \(\alpha = 3.0\) extended states exist for Fermi energies in the interval \(-0.45 \lesssim E_F \lesssim 0.45\), indicated by dashed lines in (c).
sizes. As an example, we return to figures 2(b) and (d) to point out that a critical value of $\alpha = 2.4$ or 2.6 is as plausible as the quoted value of 2.5 [5]. Thus, when investigating models for which the precise localization of the transition is unknown, the current method may have to be complemented by different measures of localization [19].

4. Short-range correlated ladders

Encouraged by the sharp distinction between localized and extended states for $D = 1$ systems using the entanglement spectrum and entropy, as shown in [26] and the previous section, we study now a correlated random ladder model [9].

This tight-binding model consists of two coupled chains of size $L$, with $N = 2L$. Taking advantage of the quasi-1D nature of the problem we now write the site energies as $2 \times 2$ matrices

$$
\varepsilon_i = \begin{pmatrix}
\varepsilon_{i,1} & -\gamma_i \\
-\gamma_i & \varepsilon_{i,2}
\end{pmatrix},
$$

(8)

where $\gamma_i$ is the disordered interchain hopping amplitude $\gamma_i$ and $\varepsilon_{i,1,2}$ is the on-site energy at site $i$ and chain 1(2). We consider disorder-free intrachain hopping amplitude $t = t_1$, where 1 is the $2 \times 2$ identity. Interestingly, this model shows extended states when the site energies and the interchain hopping amplitude have a particular correlation: [9, 33] $\varepsilon_{i,1} = \varepsilon_{i,2} = \gamma_i$, with $\varepsilon_{i,1}$ randomly distributed. The emergence of these extended states is easily understood: For this particular choice, the eigenvalues of the $\varepsilon_i$ in equation (8) are simply $\lambda_i = 0, 2\varepsilon_{i,1}$. Therefore, half of the eigenstates correspond to those of a clean chain and the other half to those of a chain with uncorrelated on-site disorder given by $2\varepsilon_{i,1}$ [9]. Because of this effective decoupling of the ladder into two chains, we have the unusual situation where extended states coexist with localized ones inside the conduction band $-2t \leq E \leq 2t$.

Without loss of generality, we assume that $\varepsilon_{i,1}$ are uniformly distributed in the interval $[-0.5, 0.5]$ with $\gamma_i = \varepsilon_{i,2} = \varepsilon_{i,1}$. To probe the extent of the wave functions we use the inverse participation ratio, equation (7). For the present case, it is easy to show that, for the extended states, $\text{IPR}_\nu = 3/2N$, since these eigenstates are simply the Bloch states of a clean tight-binding chain. Therefore, for a given realization of disorder, there are $N/2$ extended eigenstates for which $\text{IPR}_\nu = 3/2N$.

Using the IPR as criterion, we calculate the contributions to $\rho_{\text{tot}}$ for both the extended and localized states, figure 3(a). It is then clear that the density of states of the extended states corresponds to half of the density of states of a clean tight-binding chain, as expected. For the localized states, we see that they stretch outside the conduction band ($|E| < 2t$), and that their $\rho_{\text{tot}}$ has a similar value as compared to the density of states for the extended states in the center of the band, illustrating the advertised coexistence. The novel signatures of this coexistence were carefully discussed in [33]. Here, we complement their results by calculating the quantum entanglement.

As in the example of section 3, we calculate the entanglement spectrum $\zeta$ and entropy $S$ for a single disorder realization with partitions both in real and in momentum...
Assuming that the chains run along the $x$-direction, we perform the cuts along the points $x = L/2$ and $k_x = \pi$, in real and momentum space respectively. The resulting entanglement entropy and spectrum are shown in figures 3(b)–(d). Distinct from the case discussed in section 3, here the only signature of extended states is given by a real space cut. This can be clearly seen in figures 3(b) and (c), where there is an enhancement of the entanglement for $|E_F| < 2t$, accompanied by its suppression outside this region. Moreover, we notice that the localized states have minimal effects on this result, since they give rise to spurious entanglement only for $|E_F| < 2t$. For a momentum-space cut, figures 3(b) and (d), there is no suppression of the entanglement for all energies for which $\rho_{tot} > 0$.

Naively interpreted this implies that all states for $|E_F| \lesssim 2.5t$ are localized. The failure of the momentum-space entanglement to detect extended states in the current model obviously comes from the fact that entanglement corresponds to localization and, since localized states exist for all energies where the extended states are present, their contribution to $\zeta_i$ overcomes the lack of entanglement corresponding to extended states. We notice that more conventional measures of localization [33] correctly capture the existence of extended states in the current model.
Finally, we attempt an extension of the entanglement method to $D = 2$. We consider long-range correlated site energies $\varepsilon_{i,j}$ with spectral density $\alpha^\ast_{k,k'} \propto \alpha^{\frac{1}{1}}$ on a $L \times L$ square lattice. We follow [8] and define

$$\varepsilon_{i,j} = \sum_{n_i=1}^{L/2} \sum_{n_j=1}^{L/2} (i^2 + j^2)^{-\alpha/4} \times \cos \left( \frac{2\pi}{L} n_i i + \phi_{i,j} \right) \cos \left( \frac{2\pi}{L} n_j j + \Omega_{i,j} \right),$$

where $\phi_{i,j}$ and $\Omega_{i,j}$ are random phases uniformly distributed in the interval $[0, 2\pi]$ and $k_{n(y)} = 2\pi n_{d(y)}/L$. For each value of $\alpha$ we also shift the energies to ensure that $\langle \varepsilon_{i,j} \rangle = 0$ and normalize them such that $\sigma_{\varepsilon} = \langle \varepsilon^2 \rangle^{\frac{1}{2}} = 1$.

As in the previous cases, we perform cuts both in real and momentum space and consider a torus geometry, i.e. periodic boundary conditions. In real space, we perform a partition along the plane $x = L/2$, whereas in momentum space we cut along the plane $k_x = \pi$.

3 We also considered a cylinder geometry with similar results.
Results are shown in figure 4. For both partitions the distinction between localized and extended states is not apparent in the entanglement spectra and entropy, i.e. unlike for the models discussed in the previous sections, both methods seem to fail. Of course, we cannot exclude that the moderate system sizes are simply too small to probe the nature of the wave functions. It is known that careful finite-size scaling is often necessary to obtain meaningful results on localization properties [3, 13, 17, 20, 34, 35]. However, we suspect there are further difficulties to the use of the entanglement spectrum, to be discussed in the next section.

6. Analysis and discussion

Given that we have encountered only limited success of the momentum-space entanglement method proposed in [26], a more in-depth discussion of its working principles is required, which we attempt here.

6.1. Correlation matrix

As mentioned in section 2, the correlation matrix $C_{ij}$ has contributions from all states below the Fermi level $E_F$. Naively, one would expect that increasing $E_F$ therefore simply adds new eigenvalues to $C_{ij}$. In the presence of a mobility edge at $E_c$, with exclusively localized states below $E_c$, one might therefore expect that these localized states always contribute eigenvalues $0 < \zeta_i < 1$ in the momentum-space partition even for $E_F > E_c$. However, this is clearly not the case, as shown in figure 2(c) where such eigenvalues do not occur in an energy window between the mobility edges.

To settle this apparent contradiction, we have to remember the fact that we are dealing with long-range correlated disorder. While a generic (short-range) disorder spreads single-particle states over all of momentum space, long-range disorder spreads the states only over a much narrower $k$ interval. Suppose now that there are localized states in some energy interval $E_1 < E < E_2$. In the clean case, these energies can be assigned to some momenta $k_1 < |k| < k_2$. If now $E_F$ is sufficiently far above $E_2$ then all momenta near $k_1, k_2$ will be totally filled (despite the presence of localized states), because momentum smearing is moderate. Therefore, the non-vanishing $\zeta_i$ will only be 1, regardless of the fact that there are localized states below $E_F$. Consequently, the energies for which momentum-space entanglement, as calculated from $C_{ij}$, is suppressed do not coincide with the true mobility edges, such that $\zeta_i$ cannot, by construction, sharply determine the position of mobility edges. Notice that this discussion does not apply to the model considered in [26] because this only shows extended states at a particular resonant energy and thus has no mobility edges [4].

We conclude that the entanglement spectrum constructed from $C_{ij}$, which has contributions from all states below $E_F$, renders an energy-resolved detection of localization problematic; this becomes more pressing for short-range correlated disorder and is therefore of relevance for generalizations to $D \geq 2$. The problem may be resolved by a different construction of an appropriate entanglement spectrum. We recall, however,
that the Fermi-sea construction of $C_{ij}$ is vital for momentum-space partitions in order to remove the trivial entanglement between $k$ and $-k$ of extended states.

### 6.2. Spurious entanglement

We believe that the apparent failure of the entanglement measures in the $D = 2$ case, figure 4, is primarily related to spurious entanglement. It is well known that the entanglement entropy has the general property to follow an ‘area law’ [36], i.e. to scale according to $L^{D-1}$ in a system of linear dimension $L$, simply because the cut used to partition the system scales as $L^{D-1}$. In other words, for $D \geq 2$ there is sizable entanglement across the boundary between $A$ and $B$ whose nature is trivial but which contributes eigenvalues $0 < \zeta_i < 1$ of $C_{ij}$ which spoil the naive analysis of the entanglement spectrum.

Hence, the challenge lies in separating this spurious (or short-range) entanglement from the long-range entanglement which contains the physics of localization versus delocalization we are interested in. Such a separation very likely requires a careful and detailed analysis of the system-size dependence of entanglement properties, as has been done in recent numerical work on spin systems [37, 38].

### 7. Summary

Using the recently proposed [22] concept of momentum-space entanglement, we have studied Anderson localization and the associated metal–insulator transition in low-dimensional tight-binding models with spatially correlated disorder.

Similar to the results for a $D = 1$ random-dimer model in [22], we find for a $D = 1$ model with long-range disorder correlations that the momentum-space entanglement calculated via the correlation matrix appears to provide an efficient differentiation between localized and extended states, even for a single realization of disorder. In contrast, for a tight-binding ladder with special short-range disorder correlations, which features coexisting extended and localized states over a range of energies, momentum-space entanglement is not easily able to detect the presence of extended states. Finally, an extension to $D = 2$ which we applied to a model with long-range correlated disorder appears to fail entirely: We found no obvious signatures of the Anderson transition both for a momentum-space and a real-space partition.

A general discussion of the method indicates two reasons for failure, one related to the many-particle character of the correlation matrix and the other one related to spurious entanglement. We conclude that more elaborate extensions of the method proposed in [22] are required to access localization in $D \geq 2$. Such extensions might require an energy-selective definition of the correlation matrix and a more quantitative analysis of entanglement spectrum and entropy, combined with finite-size scaling, in order to separate spurious (boundary) entanglement from long-range entanglement. It remains to be seen whether such an extended method still provides advantages over more conventional measures of localization.
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References

[1] Anderson P W 1985 Phys. Rev. 109 1492–505
[2] Dobrosavljević V, Trivedi N and Valles J M Jr (ed) 2012 Conductor Insulator Quantum Phase Transitions (Oxford: Oxford University Press)
[3] Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V 1979 Phys. Rev. Lett. 42 673–6
[4] Dunlap D H, Wu H L and Phillips P W 1990 Phys. Rev. Lett. 65 88–91
[5] de Moura F A B F and Lyra M L 1998 Phys. Rev. Lett. 81 3735–8
[6] Izrailev F M and Krokhin A A 1999 Phys. Rev. Lett. 82 4062–5
[7] Garcίa-Garcίa A M and Cuevas E 2009 Phys. Rev. B 79 073104
[8] dos Santos I F, de Moura F A B F, Lyra M L and Coutinho-Filho M D 2007 J. Phys.: Condens. Matter 19 476213
[9] Sil S, Maiti S K and Chakrabarti A 2008 Phys. Rev. B 78 113103
[10] Croy A, Cain P and Schreiber M 2011 Eur. Phys. J. B 82 107–12
[11] Herbut I F 2001 Phys. Rev. B 63 113102
[12] Andrade E C, Miranda E and Dobrosavljević V 2010 Phys. Rev. Lett. 104 236401
[13] Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V 1979 Phys. Rev. Lett. 42 673–6
[14] Wegner F 1980 Z. Phys. B 30 199–14
[15] Johri S and Bhatt R N 2012 Phys. Rev. Lett. 109 076402
[16] Farchioni R, Grosso G and Pastori Parravincini G 1992 Phys. Rev. B 45 6383–9
[17] MacKinnon A and Kramer B 1981 Phys. Rev. Lett. 47 1546–9
[18] Anderson P W, Thouless D J, Abrahams E and Fisher D S 1980 Phys. Rev. B 22 3519–26
[19] Markoš P 2006 Acta Phys. Slovaca 51 581
[20] Jia X, Subramaniam A R, Gruber I A and Chakravarty S 2008 Phys. Rev. B 77 014208
[21] Chen X, Hsu B, Hughes T L and Fradkin E 2012 Phys. Rev. B 86 134201
[22] Mondragon-Shem I, Khan M and Hughes T L 2013 Phys. Rev. Lett. 110 046806
[23] Thomale R, Arovas D and Bernevig B A 2010 Phys. Rev. Lett. 105 116805
[24] Lundgren R, Chua V and Fiete G A 2012 Phys. Rev. B 86 224422
[25] Peschel I 2003 J. Phys. A: Math. Gen. 36 L205
[26] Peschel I and Eisler V 2009 J. Phys. A: Math. Theor. 42 504003
[27] Poupart M and Yang K 2014 Phys. Rev. B 89 115104
[28] Prodan E, Hughes T L and Bernevig B A 2010 Phys. Rev. Lett. 105 115501
[29] Mondragon-Shem I and Hughes T L 2014 arXiv:1403.6129v1
[30] Osborne A R and Provenzale A 1989 Physica D 35 357–81
[31] Petersen G M and Sanders N 2013 Phys. Rev. B 87 195443
[32] Nishino S, Yakubo K and Shimah 2009 Phys. Rev. B 79 033105
[33] de Moura F A B F, Caetano R A and Lyra M L 2010 Phys. Rev. B 81 125104
[34] Slevin K and Ohtsuki T 1999 Phys. Rev. Lett. 82 382–5
[35] Croy A and Schreiber M 2012 Phys. Rev. B 85 205147
[36] Srednicki M 1993 Phys. Rev. Lett. 71 666
[37] Furukawa S and Misguich G 2007 Phys. Rev. B 75 214407
[38] Isakov S V, Hastings M B and Melko R G 2011 Nature Phys. 7 772

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