The energy method for calculating the cantilever strip bending flat form stability taking into account its own weight development

S B Yazyev, I M Zotov, A P Lapina, A S Chepurnenko* and D A Vysokovskiy

Don State Technical University, 162 Socialisticheskaya str., Rostov-on-Don, 344000, Russia

E-mail: anton_chepurnenk@mail.ru

Abstract. An effective version of the energy method is recommended when calculating the rectangular cantilever strips for stability of a flat bending shape taking into account its own weight. The essence of this method’s variant is to use the Lagrange variational principle instead of the condition for the potential strain energy equality and the external forces work. The proposed approach makes it possible to perform the calculations’ machine implementation and take into account an arbitrary number of the series members. The solution to the problem for the cantilever beam is presented taking into account its own weight and the concentrated force action.

Introduction

For the first time, the stability problem of a strip flat form bending under the action of the force applied at the end was solved by Professor L. Prandtl [1]. For determining $F_{kr}$ he presented the exact solution of the transcendental equation, the smallest root of which gives the result:

$$F_{kr} = K\sqrt{\frac{EI}{I}}$$

$$K_F = 4,01 \quad K_q = 12,85$$

For determining $F_{kr}$ S.P. Tymoshenko [2] proposed an approximate energy method. The planar shape of the bend will be stable as long as any deviation from it is accompanied by an increase in the energy of the system, in other words, while the torsion energy and the energy of bending in the $x, y$ plane at any deviation from the flat form will be less than the force $F$ work for the same deviation.

A variant of the method proposed by S.P. Tymoshenko, suggests determining the critical load from a condition under which the work of external forces is equal to the potential energy of the transverse bending and the strip’s torsion. The potential strain energy is expressed in terms of the twist angle, and then the function of the twist angle is in the form of a trigonometric series. The form of the energy method proposed by S.P. Tymoshenko to solve the problem is not very convenient for machine implementation, and with the number of members, a series of more than two requires cumbersome calculations. Instead of the condition for equality of the potential energy of deformation and the external forces work, we use the variational Lagrange principle.

1. Method
The calculation method will be demonstrated on the example of a cantilever beam under the action of a load in the form of a concentrated force acting together with the weight of the beam (Figure 1).

**Figure 1.** The design schemes

We consider two cases: 1. \( F = 0 \); 2. \( q = 0 \).

We introduce the dimensionless coordinate \( \xi = x/l \). The bending moment in the beam is determined by the formula:

**1.** \( F = 0 \);

\[
M_y(x) = -\frac{ql^2}{2} + qlx - \frac{qx^2}{2} = \frac{ql^2}{2}(-1 + 2\xi - \xi^2) = ql^2\bar{M}(\xi)
\]

where

\[
\bar{M}(\xi) = \frac{(-1 + 2\xi - \xi^2)}{2}
\]

**2.** \( q = 0 \);

\[
M_y(x) = F(x - l) = Fl(\xi - 1) = Fl\bar{M}(\xi)
\]

where

\[
\bar{M}(\xi) = (\xi - 1);
\]

We determine the critical load from the condition of minimum total energy \( U \), which is defined as:

\[
U = W - A
\]

where \( W \) – is the potential deformation energy, and \( A \) – is the external forces work.

The values \( W \) and \( A \) for a beam with constant stiffness when a load is applied at the center of the cross section gravity are determined by the formulas [3]:

\[
W = \frac{1}{2} \int_0^l \left( EI_z \frac{d^2v}{dx^2} \right)^2 dx + GI_k \int_0^l \left( \frac{d\theta}{dx} \right)^2 dx
\]

\[
A = -\int_0^l M_y\theta \frac{d^2v}{dx^2} dx
\]

where \( v \) – is the lateral beam deviation, \( EI_z \) determines the bending stiffness in the plane of least rigidity, \( GI_k \) determines the torsional rigidity.

The second derivative of the lateral deflection is expressed with the twist angle as follows [4]:

```latex
\[
\frac{d^2v}{dx^2} = M_y\theta / EI_z
\]
```
\[
\frac{d^2v}{dx^2} = -\frac{M_y\theta}{EI_z}
\] (5)

Then
\[A = \int_0^l \frac{M_y^2\theta^2}{EI_z} \, dx\]

Substituting (5) in (4), and then (4) and (3) in (2), we obtain:
\[U = \frac{1}{2} \left( Gl_k \int_0^l \left( \frac{d\theta}{dx} \right)^2 \, dx + El_z \int_0^l \frac{M_y^2\theta^2}{(EI_z)^2} \, dx \right) - \int_0^l \frac{M_y^2\theta^2}{EI_z} \, dx = \]
\[= \frac{1}{2} \left( Gl_k \int_0^l \left( \frac{d\theta}{dx} \right)^2 \, dx - \int_0^l \frac{M_y^2\theta^2}{EI_z} \, dx \right); \]
\[U = \frac{1}{2} \left( \frac{Gl_k}{l^2} \int_0^1 \left( \frac{d\theta}{d\xi} \right)^2 \, d\xi - \frac{q^2l^4}{EI_z} \int_0^1 \left( \frac{M}{\bar{M}} \right)^2 \theta^2 \, d\xi \right); \quad \text{multiplied by} \quad \frac{2l^2}{Gl_k}\]
\[U = \int_0^1 \left( \frac{d\theta}{d\xi} \right)^2 \, d\xi - \lambda \int_0^1 \left( \frac{M}{\bar{M}} \right)^2 \theta^2 \, d\xi\]

where
\[\lambda = \frac{q^2l^6}{Gl_k E I_z}, \quad \bar{M}(\xi) = \frac{M_y(\xi)}{ql^2} = \frac{(-1 + 2\xi - \xi^2)}{2}\]

The twist angle function is represented as a series:
\[\theta(\xi) = \sum_{i=1}^{n} a_i f_i \quad i = 1, 3, 5, ..., n.\] (7)

where \(a_i\) determines the uncertain coefficients, \(f_i\) denotes the basic functions that should satisfy the boundary conditions.

For the problem under consideration, the boundary conditions have the form:
\[\theta(0) = 0; \quad \frac{d\theta}{d\xi} \bigg|_{\xi=1} = 0;\] (8)

Earlier in work \(f_i = 1 - \cos \left( \frac{\pi i\xi}{2} \right)\) was used as a basic function and showed poor convergence. We consider
\[f_i = \sin \left( \frac{\pi i\xi}{2} \right)\] (9)
as a basis function.

Let us define the partial derivatives of the function \(\theta(\xi)\) by the expressions (7):
\[\frac{\partial \theta}{\partial \xi} = \sum_{i=1}^{n} a_i \frac{\partial f_i}{\partial \xi};\]
\[\frac{\partial}{\partial a_j} \left( \left( \frac{d\theta}{d\xi} \right)^2 \right) = 2 \frac{\partial \theta}{\partial \xi} \frac{\partial}{\partial a_j} \left( \frac{d\theta}{d\xi} \right) = 2 \sum_{i=1}^{n} a_i \frac{\partial f_i}{\partial \xi} \frac{\partial f_j}{\partial \xi} \]
\[\frac{\partial}{\partial a_j} (\theta^2) = 2 \theta \frac{\partial \theta}{\partial a_j} = 2 \sum_{i=1}^{n} a_i f_i f_j\]
\[ \frac{\partial \theta}{\partial \xi} = \sum_{i=1}^{n} \pi a_i \cos \pi \xi; \quad \frac{\partial}{\partial a_j} \left( \frac{\partial \theta}{\partial \xi} \right) = \pi j \cos(\pi j \xi); \quad \frac{\partial \theta}{\partial a_j} = \sin(\pi j \xi); \quad \frac{\partial}{\partial a_j} \left( \int_{0}^{1} \left( \frac{\partial \theta}{\partial \xi} \right)^2 d\xi \right) = 2 \sum_{i=1}^{n} a_i \int_{0}^{1} \pi^2 j \cos(\pi \xi) \cos(\pi j \xi) d\xi; \]

The minimization of the functional \( U \) is carried out by the series’ coefficients (7):
\[ \frac{\partial U}{\partial a_j} = 0, \quad j = 1 \ldots n \]

The expression (10) will be equivalent to the following matrix expression:
\[ ([A] - \lambda [B]) (X) \]

where the vector of unknown coefficients, matrix elements \([A]\) and \([B]\) is calculated by the formulas:
\[ A_{ij} = \int_{0}^{1} \frac{\partial f_i}{\partial \xi} \frac{\partial f_j}{\partial \xi} d\xi; \quad B_{ij} = \int_{0}^{1} f_i f_j \left( \bar{M}(\xi) \right)^2 d\xi; \]

After substituting (9) in (12), we obtain:

1. \( F = 0; \)

\[ A_{ij} = \int_{0}^{1} \frac{\partial f_i}{\partial \xi} \frac{\partial f_j}{\partial \xi} d\xi = \frac{\pi^2 i j}{4} \int_{0}^{1} \frac{\sin(\pi \xi) \sin(\pi j \xi)}{2} d\xi = \begin{cases} \frac{\pi^2 i j}{8}, & i = j \\ 0, & i \neq j \end{cases}; \]

\[ B_{ii} = \frac{\pi^4 i^4 - 20 \pi^2 i^2 + 120}{40 \pi^4 i^4}; \]

\[ B_{ij} = \int_{0}^{1} f_i f_j \left( \bar{M}(\xi) \right)^2 d\xi = \int_{0}^{1} f_i f_j \left[ \frac{(-1 + 2 \xi - \xi^2)}{2} \right]^2 d\xi; \]

Integration in formula (14) for the functions (9) is performed numerically. The critical load is calculated from the condition that the system determinant is equal to zero (11):
\[ ||[A] - \lambda [B]] = 0 \]

Thus, the problem is reduced to a generalized secular equation. Table 1 presents the values of the coefficient \( K = \sqrt{\lambda_{\text{min}}} \) with a different number of the series members.

The critical load is determined by the formula:
\[ (ql)_{kr} = \frac{K \sqrt{EIzGk}}{I^2} \]

2. \( q = 0; \)

\[ A_{ij} = \int_{0}^{1} \frac{\partial f_i}{\partial \xi} \frac{\partial f_j}{\partial \xi} d\xi = \frac{\pi^2 i j}{4} \int_{0}^{1} \frac{\sin(\pi \xi) \sin(\pi j \xi)}{2} d\xi = \begin{cases} \frac{\pi^2 i j}{8}, & i = j \\ 0, & i \neq j \end{cases}; \]

\[ B_{ii} = \frac{\pi^2 i^2 - 6}{6 \pi^2 i^2}; \]
\[ B_{ij} = \int_0^1 f_i f_j \left( M(\xi) \right)^2 d\xi = \int_0^1 f_i f_j (\xi - 1)^2 d\xi; \]  

(17)

The integration in formula (17) for the functions (9) is performed numerically.

2. Results and discussion

Coefficient values \( K = \sqrt{\lambda_{\text{min}}} \), obtained by our method for the function under consideration for a different number of the series members are shown in the graph. The convergence depending on the number of the series members is shown in Table 1 and 2. Also, Table 2 shows the results when using other basic functions presented in the work for comparison [5].

In [1], by directly integrating the differential stability equation, the value \( K = 4.0126 \).

Fig. 2 and Fig. 3 show the graphs of the coefficient \( K \) dependence on the number of the series members \( n \).

![Figure 2. The method convergence depending on the number of the series members for \( F = 0 \)](image)

![Figure 3. The method convergence depending on the number of the series members for \( q = 0 \)](image)

1. Blue dash-dotted line - power series
2. Red dotted \( f_i = 1 - \cos\left(\frac{n\pi \xi}{2}\right) \)
3. Black solid \( f_i = \sin\left(\frac{n\pi \xi}{2}\right) \)

Table 1. The coefficient \( K \) values with a different number of row members for a cantilever beam at \( F = 0 \).

| \( n \) | 1  | 2  | 3  | 4  | 5  |
|-------|----|----|----|----|----|
| \( K \) | 15.4966 | 13.2341 | 12.9081 | 12.8633 | 12.8589 |
| \( n \) | 6  | 7  | 8  | 9  | 10 |
| \( K \) | 12.8544 | 12.8540 | 12.8539 | 12.8538 | 12.8538 |

Table 2. The coefficient \( K \) values with a different number of row members for a cantilever beam \( q = 0 \).

| \( n \) | 1  | 2  | 3  | 4  | 5  |
|-------|----|----|----|----|----|
| \( K(\text{authors}) f_i = \sin\left(\frac{n\pi \xi}{2}\right) \) | 4.345 | 4.023 | 4.0132 | 4.0127 | 4.0126 |
| \( K f_i = 1 - \cos\left(\frac{n\pi \xi}{2}\right) \) | 10.04 | 5.31 | 4.67 | 4.47 | 4.37 |
| \( K(\text{power series}) f_i = \xi^{i+1} \) | 11.83 | 5.77 | 4.65 | 4.35 | 4.24 |
| \( n \) | 6  | 7  | 8  | 9  | 10  |
\[ K(\text{authors}) f_i = \sin \left( \frac{\pi i \xi}{2} \right) \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
K f_i = 1 - \cos \left( \frac{\pi i \xi}{2} \right) & 4.0126 & 4.0126 & 4.0126 & 4.0126 & 4.0126 \\
\hline
K(\text{power series}) f_i = \xi^{i+1} & 4.18 & 4.14 & 4.11 & 4.09 & 4.08 \\
\hline
\end{array}
\]

Summary
Analyzing the convergence for option \( F = 0 \), we can say (Table 1) that the result quickly approaches the exact solution, starting with the fourth approximation. The deviation from the exact solution when \( n = 4 \) makes up 0.1047%.

Regarding the option \( q = 0 \), then (Table 2), the deviation from the exact solution for \( n = 10 \) makes up 4.07% in the case of using a trigonometric series with \( f_i = 1 - \cos \left( \frac{\pi i \xi}{2} \right) \), and when using the power series - 1.71%. For functions \( f_i = 1 - \cos \left( \frac{\pi i \xi}{2} \right) \) for the large \( n \), the computational process was stable, but the value 4.01 accurate to the 3rd decimal place was obtained only with the 1000 series members.

Also, at \( n \geq 12 \) for the power series \( f_i = \xi^{i+1} \) there was the computational process stability loss (imaginary eigenvalues).

For the authors’ solution with basic functions (9) for \( n = 3 \) the deviation from the exact solution is only 0.078%. Thus, the method substantially convergence depends on the correct choice of the basic functions.

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