AN ALGORITHM FOR PRIMARY DECOMPOSITION IN POLYNOMIAL RINGS OVER THE INTEGERS

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Abstract. We present an algorithm to compute a primary decomposition of an ideal in a polynomial ring over the integers. For this purpose we use algorithms for primary decomposition in polynomial rings over the rationals resp. over finite fields, and the idea of Shimoyama–Yokoyama resp. Eisenbud–Hunecke–Vasconcelos to extract primary ideals from pseudo–primary ideals. A parallelized version of the algorithm is implemented in SINGULAR. Examples and timings are given at the end of the article.

1. Introduction

Algorithms for primary decomposition in \( \mathbb{Z}[x_1, \ldots, x_n] \) have been developed by Seidenberg (cf. \cite{Se}), Gianni, Trager, Zacharias (cf. \cite{GTZ}) and Ayoub (cf. \cite{A}). Within this article we present a slightly different approach which mainly uses primary decomposition in polynomial rings over a field and therefore seems to be much more efficient. In particular, it uses primary decomposition in \( \mathbb{Q}[x_1, \ldots, x_n] \) resp. \( \mathbb{F}_p[x_1, \ldots, x_n] \) as well as the computation of the minimal associated primes of an ideal in \( \mathbb{F}_p[x_1, \ldots, x_n] \), pseudo–primary decomposition \cite{EHV}, and the extraction of the primary components. The essential difference compared to the corresponding algorithm proposed in \cite{GTZ} is as follows: the primary decomposition of an ideal \( I \) in \( \mathbb{Z}[x_1, \ldots, x_n] \) with \( I \cap \mathbb{Z} = \langle q \rangle \) such that \( q \neq 0 \) is obtained by computing the minimal associated prime ideals of \( I \mathbb{F}_p[x_1, \ldots, x_n] \) for all primes \( p \) dividing \( q \) and extracting subsequently the primary ideals.

Let \( x = \{x_1, \ldots, x_n\} \) always denote a set of indeterminates and let \( I \subseteq \mathbb{Z}[x] \) be an ideal. We use the following known facts from commutative algebra for our algorithm:

(1) If \( I \cap \mathbb{Z} = \langle 0 \rangle \), then there exists an \( h \in \mathbb{Z} \) such that \( I : h = I \mathbb{Q}[x] \cap \mathbb{Z}[x] \) and \( I = (I : h) \cap \langle I, h \rangle \) (cf. \cite{Se}, Theorem 2).

(2) If \( I \cap \mathbb{Z} = \langle 0 \rangle \) and \( I \mathbb{Q}[x] = \sqrt{Q_1} \cap \ldots \cap \sqrt{Q_s} \) is an irredundant primary decomposition with \( \sqrt{Q_i} = \sqrt{Q_i} \), then \( I \mathbb{Q}[x] \cap \mathbb{Z}[x] = (\sqrt{Q_1} \cap \mathbb{Z}[x]) \cap \ldots \cap (\sqrt{Q_s} \cap \mathbb{Z}[x]) \) is an irredundant primary decomposition and \( \sqrt{Q_i} \cap \mathbb{Z}[x] = \sqrt{Q_i} \cap \mathbb{Z}[x] \) (cf. \cite{Se}, Theorem 3).

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1One can choose one of the modern algorithms, cf. \cite{DGP}, \cite{EHV}, \cite{GTZ}, \cite{SY}.

2An ideal is called pseudo–primary if its radical is prime, cf. \cite{EHV}, \cite{SY}.
(3) If $I \cap \mathbb{Z} = \langle q \rangle$ such that $q \neq 0$ and $q = p_1^{r_1} \cdots p_r^{r_r}$ with $p_1, \ldots, p_r$ pairwise different primes, then $I = \bigcap_{i=1}^r (I, p_i^{r_i})$.

(4) If $I \cap \mathbb{Z} = \langle p^r \rangle$ for some prime $p$ and $\mathcal{P}_1, \ldots, \mathcal{P}_s$ are the minimal associated primes of $IF_p[x]$, then the canonical liftings $\mathcal{P}_1, \ldots, \mathcal{P}_s$ to $\mathbb{Z}[x]$ are the minimal associated primes of $I$.

If $\nu = 1$ let $IF_p[x] = \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_s$ be an irredundant primary decomposition with associated primes $\mathcal{P}_1, \ldots, \mathcal{P}_s$ and $Q_1, \ldots, Q_s, P_1, \ldots, P_s$ be the canonical liftings to $\mathbb{Z}[x]$. Then $I = Q_1 \cap \ldots \cap Q_s$ is an irredundant primary decomposition with associated primes $P_1, \ldots, P_s$.

The following result can easily be adapted to $\mathbb{Z}[x]$.

(5) If $P$ is a minimal associated prime of $I$, then $I + P^m$ is a pseudo–primary component of $I$ for a suitable $m \in \mathbb{N}$, i.e. the equidimensional part of $I + P^m$ is the primary component of $I$ associated to $P$. For any $m$ let $Q_m$ be the equidimensional part of $I + P^m$. $Q_m$ is a primary component of $I$ with associated prime $P$ if $Q_m = I\mathbb{Z}[x]_P \cap \mathbb{Z}[x]$ (cf. [EHV]).

Alternatively we can compute a separator $s$ of $I$ w.r.t. $P$ and obtain by $I : s^\infty$ a pseudo–primary component of $I$ (cf. [SY]).

(6) If $Q_1, \ldots, Q_s$ are the primary components of $I$ associated to the minimal associated prime ideals and $J = Q_1 \cap \ldots \cap Q_s$, then there exists a natural number $m$ such that $I = J \cap (I + (I : J)^m)$.

Consequently, by applying (1)–(6), we can reduce the computation of the primary decomposition in $\mathbb{Z}[x]$ to the computation of the primary decomposition in $\mathbb{Q}[x]$, the computation of the minimal associated primes in $\mathbb{F}_p[x]$, and the extraction of the primary components in $\mathbb{Z}[x]$. In this connection, the extraction has to be generalized to polynomial rings over principal ideal domains (cf. Lemma 2.3). In section 2 we state the results used in the algorithm, whereupon in section 3 we explain our algorithm which has been implemented in SINGULAR in a parallel version. Finally we give some examples and the corresponding timings in section 4.

2. Basic definitions and results

**Definition 2.1.** Let $I \subseteq \mathbb{Z}[x]$ be an ideal and $>$ be a monomial ordering on $\mathbb{Z}[x]$. A subset $G \subseteq I$ is called a Gröbner basis of $I$ w.r.t. $>$ if the leading ideal of $G$ equals the leading ideal of $I$. $G$ is called a strong Gröbner basis if for all $f \in I$ there exists a $g \in G$ such that $\text{LT}(g) \mid \text{LT}(f)$.

**Lemma 2.2.** Let $G = \{g_1, \ldots, g_k\} \subseteq \mathbb{Z}[x]$ and $I = \langle G \rangle \mathbb{Z}[x]$. Assume that $I \cap \mathbb{Z} = \langle 0 \rangle$ and $G$ is a Gröbner basis of $I\mathbb{Q}[x]$ w.r.t. some ordering. Let $h = \text{lcm}(\text{LC}(g_1), \ldots, \text{LC}(g_k))$ be the least common multiple of the leading coefficients of $g_1, \ldots, g_k$. Then $I\mathbb{Q}[x] \cap \mathbb{Z}[x] = I : h^\infty$. Moreover, if $I : h^\infty = I : h^m$ for some natural number $m$, then $I = (I : h^m) \cap (I, h^m)$.

The proof of Lemma 2.2 is similar to the corresponding proof for polynomial rings over a field (cf. [GP], Proposition 4.3.1).

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3Choose generators in $\mathbb{F}_p[x]$ and lift the coefficients to non–negative integers smaller than $p$.

4We call $s$ a separator of $I$ w.r.t. $P$ if $s \not\in P$ and $s$ is contained in all other minimal associated primes of $I$.

5We use the notations of [GP] for the basics of Gröbner bases. Especially $\text{LT}(f)$ denotes the leading term (leading monomial with leading coefficient) of $f$ w.r.t. the ordering $>$. The theory of (strong) Gröbner bases over principal ideal domains can be found in [AK], section 4.5.
Remark 2.3. The saturation $I : h^\infty$ can be computed in $\mathbb{Z}[x]$ similarly to the case of a polynomial ring over a field by computing a Gröbner basis of $\langle I, Th - 1 \rangle \mathbb{Z}[x, T]$ w.r.t. an elimination ordering for $T$:

$$I : h^\infty = \langle I, Th - 1 \rangle \mathbb{Z}[x, T] \cap \mathbb{Z}[x].$$

A natural number $m$ satisfying $I : h^m = I : h^m$ can be found by computing the normal form of $h^l g$ w.r.t. $I$ for each generator $g$ of $I : h^\infty$ and increasing $l \in \mathbb{N}$. More precisely, if the normal form of $h^l g$ w.r.t. $I$ is zero for each generator $g$ of $I : h^\infty$ then $h^{l'}(I : h^\infty) \subseteq I$, i.e. $I : h^\infty = I : h^l$.

Lemma 2.4 (cf. [SY]). Let $I \subseteq \mathbb{Z}[x]$ be an ideal with more than one minimal associated prime, $P$ a minimal associated prime and $s \notin P$ a separator, i.e. $s$ is contained in all other minimal associated primes of $I$. Then $I : s^\infty$ is a pseudo–primary component of $I$, and $s$ can be chosen as

$$\prod_{Q \notin P} s_Q$$

where $s_Q$ is an element of a Gröbner basis of $Q$ which is not in $P$.

Lemma 2.5 (Extraction Lemma, cf. [GTZ]). Let $I = Q \cap J$ be pseudo–primary with $\sqrt{J} = P$ and $Q$ be $P$–primary with $\text{ht}(Q) < \text{ht}(J)$. Let $P \cap \mathbb{Z} = \langle p \rangle$ for some prime $p$ and $u \subset x$ be a maximal independent set of variables for $\mathcal{P} = P\mathbb{F}_p[x]$. Let $R := \mathbb{Z}[u]_{\langle p \rangle}$, then the following hold:

1. $IR[x \setminus u] \cap \mathbb{Z}[x] = Q$
2. Let $G$ be a strong Gröbner basis of $I$ w.r.t. a block ordering satisfying $x \setminus u \gg u$. Then $G$ is a strong Gröbner basis of $IR[x \setminus u]$ w.r.t. the induced ordering for the variables $x \setminus u$.
3. Let $G = \{g_1, \ldots, g_k\}$ be as in (2). $\text{LT}_{R[x \setminus u]}(g_i) = p^{\mu_i}a_i(x \setminus u)^{\nu_i}$ with $a_i \in \mathbb{Z}[u] \setminus \langle p \rangle$ for $i = 1, \ldots, k$, and $h = \text{lcm}(a_1, \ldots, a_k)$. Then $IR[x \setminus u] \cap \mathbb{Z}[x] = I : h^\infty$.

Proof.

1. Let $K = \sqrt{J}$ and $\mathcal{K} = K\mathbb{F}_p[x]$ then $\mathcal{K} \supseteq \mathcal{P} = P\mathbb{F}_p$. This implies that $K \cap \mathbb{F}_p[u] \neq \langle 0 \rangle$ since $u \subset x$ is maximally independent for $\mathcal{P}$ and therefore $K \cap (\mathbb{Z}[u] \setminus \langle p \rangle) \neq \emptyset$. Thus it holds $JR[x \setminus u] = R[x \setminus u]$. Finally, because $Q$ is primary, we obtain $IR[x \setminus u] \cap \mathbb{Z}[x] = QR[x \setminus u] \cap \mathbb{Z}[x] = Q$.
2. Let $f \in IR[x \setminus u]$ and choose $s \in \mathbb{Z}[u] \setminus \langle p \rangle$ such that $sf \in I$. Since $G$ is a strong Gröbner basis of $I$ there exists a $g \in G$ such that $\text{LT}_{\mathbb{Z}[u]}(g) \mid \text{LT}_{\mathbb{Z}[x]}(sf)$. As a polynomial in $x \setminus u$ with coefficients in $R$, the element $sf$ can be written as $sf = p^\alpha a(x \setminus u)^\beta$ (terms in $x \setminus u$ of smaller order) with $a \in \mathbb{Z}[u] \setminus \langle p \rangle$. If $p^\tau$ is the maximal power of $p$ dividing the leading coefficient $LC_{\mathbb{Z}[x]}(g)$ of $g$ then $\tau \leq \nu$ since $\text{LT}_{\mathbb{Z}[x]}(sf) = p^\nu \text{LT}_{\mathbb{Z}[x]}(a)(x \setminus u)^\alpha$. Now we can write $g$ as an element of $R[x \setminus u]$ w.r.t. the corresponding ordering, i.e. $g = p^\mu b(x \setminus u)^\beta$ (terms in $x \setminus u$ of smaller order) with $b \in \mathbb{Z}[u] \setminus \langle p \rangle$ and $\mu \leq \tau \leq \nu$. By definition we have $\text{LT}_{R[x \setminus u]}(g) = p^\mu b(x \setminus u)^\beta$ resp. $\text{LT}_{\mathcal{K}[x \setminus u]}(f) = p^\nu b(x \setminus u)^\alpha$ and on the other hand it holds $\text{LT}_{\mathbb{Z}[x]}(g) = p^\mu \text{LT}_{\mathbb{Z}[x]}(b)(x \setminus u)^\beta$ resp. $\text{LT}_{\mathbb{Z}[x]}(sf) = p^\nu \text{LT}_{\mathbb{Z}[x]}(a)(x \setminus u)^\alpha$.
Thus the assumption $LT_{\mathbb{Z}[x]}(g) \mid LT_{\mathbb{Z}[x]}(sf)$ implies $(x \setminus u)^{\beta} \mid (x \setminus u)^{\alpha}$ and consequently $LT_{\mathbb{R}[x \setminus u]}(g) \mid LT_{\mathbb{R}[x \setminus u]}(f)$. This proves (2).

(3) Follows from (2) similarly to the proof for fields (cf. [GTZ], [GP]).

\[ \square \]

The following Lemma is a consequence of the Lemma of Artin–Rees (cf. [GP]).

**Lemma 2.6.** Let $I \subseteq \mathbb{Z}[x]$ be an ideal and $J$ the intersection of all primary components of $I$ associated to the minimal prime ideals of $I$. Then there exists a natural number $m$ such that $I = J \cap (I + (I : J)^m)$.

**Notation 2.7.** Given an ideal $I \subseteq \mathbb{Z}[x]$ we can always choose a finite set of polynomials $F_I = \{f_1, \ldots, f_k\}$ such that $I = \langle F_I \rangle$ and we denote $F_I^{(m)} := \{f_1^m, \ldots, f_k^m\}$ for $m \in \mathbb{N}$.

**Corollary 2.8.** With the assumptions and notations of Lemma 2.6 there exists a natural number $m$ such that $I = J \cap (I + (F_I^{(m)})_{J})$.

**Proof.** Due to Lemma 2.6 there exists an $m$ such that $I = J \cap (I + (I : J)^m)$. Now we have $I \subseteq J \cap (I + (F_I^{(m)})_{J}) \subseteq J \cap (I + (I : J)^m) = I$ and therefore $I = J \cap (I + (F_I^{(m)}))$. \[ \square \]

**Remark 2.9.** The corollary is very important from a computational point of view because $\langle F_I^{(m)} \rangle$ has fewer generators than $(I : J)^m$.

### 3. The algorithms

In this section we present the algorithm to compute a primary decomposition of an ideal in a polynomial ring over the integers by applying the results of section 2 resp. the introduction (section 1).

Algorithm 1 computes the primary decomposition of an ideal in $\mathbb{Z}[x]$ with the aid of algorithms 2 and 3 which we introduce subsequently in detail.

**Remark 3.1.** Algorithm 1 can easily be parallelized by computing - depending on the prime factorization $q = p_1^{\nu_1} \cdots p_r^{\nu_r}$ of $q$ where $\langle q \rangle = I \cap \mathbb{Z}$ - either the primary decomposition or the set of minimal associated primes in positive characteristic in parallel. If $\nu_i = 1$ we have to compute the primary decomposition whereas, if $\nu_i > 1$, we have to compute the minimal associated primes of $IF_{p_i}[x]$ in $F_{p_i}[x]$. These $r$ computations in positive characteristic are independent from each other such that they can also run separately in parallel on at most $r$ processors if available.

\[ \text{The corresponding procedures are implemented in SINGULAR in the library primdecint.lib.} \]
Algorithm 1 PRIMDEC\(_Z\)

Input: \(F_i = \{f_1, \ldots, f_k\}, I = \langle F_i \rangle \mathbb{Z}[x]\), optional: a test ideal \(T\).

Output: \(L := \{(Q_1, P_1), \ldots, (Q_s, P_s)\}\), \(I = Q_1 \cap \cdots \cap Q_s\) irredundant primary decomposition with \(P_i = \sqrt{Q_i}\).

if \(T\) is not given in the input then
\(T := \langle 1 \rangle\);
\(G := \text{strong Gröbner basis of } I\);
\(q := \text{generator of } I \cap \mathbb{Z}\).
if \(q = 0\) then
\(h \in \mathbb{Z}\) such that \(I : h = I\mathbb{Q}[x] \cap \mathbb{Z}[x]\);
compute \(Q_1, \ldots, Q_s\), an irredundant primary decomposition of \(I\mathbb{Q}[x]\) and \(P_i = \sqrt{Q_i}\) the associated primes;
compute \(Q_i = Q_i \cap \mathbb{Z}[x], P_i = P_i \cap \mathbb{Z}[x]\);\(^8\)
\(L := \{(Q_1, P_1), \ldots, (Q_s, P_s)\};;\)
\(M := \text{PRIMDEC}_{\mathbb{Z}}(I, h)\) & remove redundant primary ideals from \(M\);
return \(L \cup M\);
else
compute \(q = p_1^{\nu_1} \cdots p_r^{\nu_r}\), the prime factorization of \(q\);
for \(i = 1, \ldots, r\) do
if \(\nu_i = 1\) then
compute \(\overline{L_i} = \{(\overline{Q}_1^{(i)}, \overline{P}_1^{(i)}), \ldots, (\overline{Q}_{s_i}^{(i)}, \overline{P}_{s_i}^{(i)})\}\), the primary decomposition of \(I\mathbb{F}_{p_i}[x]\);
\(L_i := \{(Q_1^{(i)}, P_1^{(i)}), \ldots, (Q_{s_i}^{(i)}, P_{s_i}^{(i)})\}, \text{the lifting of } \overline{L_i} \text{ to } \mathbb{Z}[x]\);\(^9\)
else
compute \(\overline{A}_i = \{\overline{P}_1^{(i)}, \ldots, \overline{P}_{s_i}^{(i)}\}\), the set of minimal associated primes of \(I\mathbb{F}_{p_i}[x]\) and independent sets of variables \(\overline{\pi}_1^{(i)}, \ldots, \overline{\pi}_{s_i}^{(i)}\) for \(\overline{P}_1^{(i)}, \ldots, \overline{P}_{s_i}^{(i)}\);
\(A_i := \{P_1^{(i)}, \ldots, P_{s_i}^{(i)}\}\), the lifting of \(\overline{A}_i\) to \(\mathbb{Z}[x]\);
for \(j = 1, \ldots, s_i\) do
\(Q_j^{(i)} := \text{EXTRACT}_{\mathbb{Z}}(I, A_i, P_j^{(i)}, \overline{\pi}_j^{(i)});\)
\(L_i := \{(Q_1^{(i)}, P_1^{(i)}), \ldots, (Q_{s_i}^{(i)}, P_{s_i}^{(i)})\};\)
\(L := L_1 \cup \cdots \cup L_{r_i};\)
compute \(J\), the intersection of all primary ideals in \(L\) and \(T\);
if \(J = I\) then
return \(L\);
calculate \(F_{I,J}\) such that \(\langle F_{I,J} \rangle = I : J\);
calculate \(m\) such that \(J \cap (I + \langle F_{I,J}^{(m)} \rangle) = I\);
\(M := \text{PRIMDEC}_{\mathbb{Z}}(I + \langle F_{I,J}^{(m)} \rangle, J)\) & remove redundant primary ideals from \(M\);
return \(L \cup M\);

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\(^8\) \(q\) is either 0 or the unique element in \(G\) of degree 0.
\(^9\) \(h\) is a suitable power of the least common multiple of all leading coefficients of elements in \(G\), cf. Lemma [22].
\(^{10}\) \(Q_j\) resp. \(P_j\) are primary resp. prime due to (4) of the introduction (section [4]).
\(^{11}\) If \(I = \langle F_i \rangle \subseteq \mathbb{F}_p[x]\) then its lifting is obtained by \((p, F_i)\) with the canonical lifting of \(F_i\).
The algorithm to compute the separators is based on Lemma 2.4.

**Algorithm 2** SEPARATORSZ

**Input:** $B$ a list of prime ideals generated by a Gröbner basis w.r.t. some ordering, not contained in each other, $P \in B$.

**Output:** Polynomial $s$ such that $s \notin P$, $s \in Q$ for all $Q \in B \setminus \{P\}$.

for $Q \in B \setminus \{P\}$ do
  choose $s_Q$ in the Gröbner basis of $Q$ such that $s_Q \notin P$;
return $\prod_{Q \in B \setminus \{P\}} s_Q$;

The algorithm to extract the primary component from the pseudo–primary component is based on the Extraction Lemma 2.5.

**Algorithm 3** EXTRACTZ

**Input:** $I \subseteq \mathbb{Z}[x]$ an ideal, $B$ the list of minimal associated primes of $I$, $P \in B$ with $P \cap \mathbb{Z} = \langle p \rangle$ for some prime $p$, $u \subset x$ an independent set of variables for $P\mathbb{F}_p[x]$.

**Output:** The primary component $Q$ of $I$ associated to $P$.

$s := \text{SEPARATORSZ}(P, B)$;
$I = I : s^\infty$;
compute $G = \{g_1, \ldots, g_k\}$, a strong Gröbner basis of $I$ w.r.t. a block ordering satisfying $x \setminus u \gg u$;
compute $\{a_1, \ldots, a_k\}$ such that $\text{LC}_{\mathbb{Z}[u](p)}[x \setminus u](g_i) = p^{v_i} \cdot a_i$ with $a_i \in \mathbb{Z}[u] \setminus \langle p \rangle$;
compute $h = \text{lcm}(a_1, \ldots, a_k)$, the least common multiple of $a_1, \ldots, a_k$;
return $I : h^\infty$;

**Example 3.2.** Consider $I = \langle 9, 3x, 3y \rangle$, $P = \langle 3 \rangle$, $u = \{x, y\}$ and $B = \{P\}$ in $\mathbb{Z}[x, y]$. Then we obtain $s = 1$, $h = xy$ and thus $I : h^\infty = \langle 3 \rangle$.

4. **Examples and timings**

In this section we provide examples on which we time the algorithm **primdecZ** (cf. section 3) and its parallelization implemented in SINGULAR. Timings are conducted by using the 32-bit version of SINGULAR 3-1-2 on an AMD Opteron 6174 with 48 CPUs, 800 MHz each, 128 GB RAM under the Gentoo Linux operating system. All examples are chosen from The SymbolicData Project (cf. [G]).

**Remark 4.1.** The parallelization of our algorithm is attained via multiple processes organized by SINGULAR library code. Consequently a future aim is to enable parallelization in the kernel via multiple threads.

**Remark 4.2.** In SINGULAR one can compute Gröbner bases not only over fields but also over the rings $\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ (resp. $\mathbb{Z}/2^l\mathbb{Z}$ as a special case of $\mathbb{Z}/m\mathbb{Z}$).

For the integers the implementation is based on the theory for Gröbner bases over integral domains as introduced by Adams and Loustaunau (cf. [AL], chapter 4).

For factor rings further theory needed to be developed by the SINGULAR-Team in Kaiserslautern (cf. [GSW], [W]). Details about the corresponding implementation are presented by Wienand (cf. [W], chapter 3).
We choose the following examples:

**Example 4.3.** Coefficients: integer, ordering: dp, Gerdt-93a.xml (cf. [G]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 181$.

**Example 4.4.** Coefficients: integer, ordering: dp, Gerdt-93a.xml (cf. [G]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 31 \cdot 181$.

**Example 4.5.** Coefficients: integer, ordering: dp, Gerdt-93a.xml (cf. [G]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 37 \cdot 181$.

**Example 4.6.** Coefficients: integer, ordering: dp, Steidel 6.xml (cf. [ES]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$.

**Example 4.7.** Coefficients: integer, ordering: dp, Steidel 6.xml (cf. [ES]) considered with another integer generator $2 \cdot 3^2 \cdot 5 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$.

**Example 4.8.** Coefficients: integer, ordering: dp, Gonnet-83.xml (cf. [BGK]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$.

**Example 4.9.** Coefficients: integer, ordering: dp, Gonnet-83.xml (cf. [BGK]) considered with another integer generator $2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$.

Table 1 summarizes the results where $\text{primdecZ}^*(k)$ denotes the parallelized version of the algorithm using $k$ processes. All timings are given in seconds.

| Example | $\text{primdecZ}$ | $\text{primdecZ}^*(2)$ | $\text{primdecZ}^*(3)$ | $\text{primdecZ}^*(4)$ |
|---------|-------------------|------------------------|------------------------|------------------------|
| 4.3     | 604               | 383                    | 339                    | 249                    |
| 4.4     | 757               | 480                    | 392                    | 350                    |
| 4.5     | 907               | 542                    | 396                    | 396                    |
| 4.6     | 17                | 9                      | 7                      | 4                      |
| 4.7     | 10                | 6                      | 5                      | 4                      |
| 4.8     | 21                | 14                     | 10                     | 8                      |
| 4.9     | 39                | 35                     | 34                     | 31                     |

Table 1. Total running times for computing a primary decomposition of the considered examples via $\text{primdecZ}$ and its parallelized variant $\text{primdecZ}^*(k)$ for $k = 2, 3, 4$.

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$^{12}$Degree reverse lexicographical ordering: Let $x^\alpha, x^\beta$ be two monomials in $x$, i.e. $\alpha, \beta \in \mathbb{N}^n$. $x^\alpha >_\text{dp} x^\beta \iff \deg(x^\alpha) > \deg(x^\beta)$ or $(\deg(x^\alpha) = \deg(x^\beta)$ and $\exists 1 \leq i \leq n : \alpha_n = \beta_n, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i$), where $\deg(x^\alpha) = \alpha_1 + \ldots + \alpha_n$; cf. [CF].
References

[A] Ayoub, C.W.: The Decomposition Theorem for Ideals in Polynomial Rings over a Domain. Journal of Algebra 76, 99–110 (1982).

[AL] Adams, W.W.; Loustaunau, P.: An Introduction to Gröbner Bases. Graduate Studies in Mathematics, Volume 3, American Mathematical Society (1994).

[BGK] Boege, W.; Gebauer, R.; Kredel, H.: Some Examples for Solving Systems of Algebraic Equations by Calculating Groebner Bases. Journal of Symbolic Computation 1, 83–98 (1986).

[DGP] Decker, W.; Greuel, G.-M.; Pfister, G.: Primary Decomposition: Algorithms and Comparisons. In: Algorithmic Algebra and Number Theory, Springer, 187–220 (1998).

[DGPS] Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: Singular 3-1-1 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2010).

[EHV] Eisenbud, D.; Huneke, C.; Vasconcelos, W.: Direct Methods for Primary Decomposition. Inventiones Mathematicae 110, 207–235 (1992).

[ES] Eisenbud, D.; Sturmfels, B.: Binomial ideals. Duke Mathematical Journal 84 (No. 1), 1–45 (1996).

[GP] Greuel, G.-M.; Pfister, G.: A Singular Introduction to Commutative Algebra. Second edition, Springer (2007).

[GSW] Greuel, G.-M.; Seelisch, F.; Wienand, O.: The Gröbner basis of the ideal of vanishing polynomials. Journal of Symbolic Computation, Article in Press, Corrected Proof, doi:10.1016/j.jsc.2010.10.006, 14 pages (2011).

[GTZ] Gianni, P.; Trager, B.; Zacharias, G.: Gröbner Bases and Primary Decomposition of Polynomial Ideals. Journal of Symbolic Computation 6, 149–167 (1988).

[G] Gräbe, H.-G.: The SymbolicData Project — Tools and Data for Testing Computer Algebra Software. http://www.symbolicdata.org (2010).

[M] Monico, C.: Computing the Primary Decomposition of zero-dimensional Ideals. Journal of Symbolic Computation 34, 451–459 (2002).

[Sa] Sausse, A.: A New Approach to Primary Decomposition. Journal of Symbolic Computation 11, 1–15 (1996).

[Se] Seidenberg, A.: Constructions in a Polynomial Ring over the Ring of Integers. American Journal of Mathematics 100 (No. 4), 685–703 (1978).

[SY] Shimoyama, T.; Yokoyama, K.: Localization and Primary Decomposition of Polynomial Ideals. Journal of Symbolic Computation 22, 247–277 (1996).

[W] Wienand, O.: Algorithms for Symbolic Computation and their Applications. Ph.D. Thesis, Kaiserslautern (2011).

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