The theoretical onset of $e^+e^- \rightarrow \tau^+\tau^-$ at threshold revisited.

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Abstract

The precise knowledge of the onset of the cross section $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$ at the threshold is necessary for improving the accuracy of determination of the $\tau$ mass from the threshold measurements. The QED radiative corrections of relative order $\alpha$ and $\alpha^2$, additional to the well known Coulomb factor, are considered in the threshold region. The correction terms of order $\alpha^2$ are calculated, which contain coefficients enhanced by large parameters. As a result it is argued that the known $O(\alpha)$ corrections provide the accuracy of the description of the cross section close to $10^{-4}$, rather than $10^{-3}$ as claimed in a recent literature. Also analytical expressions are provided for some limiting cases of the corrections, previously calculated numerically.
1 Introduction

The production in the electron-positron annihilation of slow $\tau^+\tau^-$ pairs near the threshold provides a valuable tool for a precision measurement of the $\tau$ lepton mass\[1\]. Such measurement is aided by that the threshold onset of the cross section starts with a finite step due to the Coulomb attraction between the produced $\tau$ leptons\[2\]. Namely, when expressed in terms of the ratio $R = \sigma(e^+e^- \rightarrow \tau^+\tau^-)/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$, the familiar ‘bare’ threshold behavior

$$R_0 = v \frac{3 - v^2}{2}$$

in terms of the velocity $v$ of each of the produced $\tau$ leptons in the c.m. frame, is multiplied by the Coulomb factor\[3\]

$$F_c = \frac{\pi\alpha/v}{1 - \exp(-\pi\alpha/v)} .$$

The product $R_0 F_c$ clearly has a finite limit equal to $3\pi\alpha/2$ at $v \rightarrow 0$. The factor $F_c$ sums up all the graphs with exchange between the $\tau$ leptons of Coulomb quanta and expands in power series in the parameter $\alpha/v$, rather than in powers of the QED coupling $\alpha$. For this reason at the near threshold energies, where $v$ is not parametrically larger than $\alpha$, the Coulomb interaction has to be taken into account exactly. The dependence of $F_c$ on $\alpha/v$ also implies that the higher order corrections can arise both from the QED radiative effects as powers of $\alpha$ as well as from the relativistic expansion through terms with extra powers of $v^2$. The latter terms, being modified by the Coulomb effects, are generally equivalent to the ones with extra powers of $\alpha^2$. The leading QED radiative corrections of order $\alpha$ to the discussed process were analysed in Ref.\[3\], where it was shown that these corrections arise from two sources: a velocity independent factor due to the form factor of the $\tau$ electromagnetic vertex at the threshold, and the Uehling-Serber modification of the Coulomb potential due to the electron vacuum polarization, whose effect has a quite nontrivial dependence on $v$. It should be emphasized that the QED radiative corrections discussed here are those which are on top of the Coulomb factor $F_c$. In other words all the terms of the form $(\alpha/v)^n$ are accounted for exactly, and the discussed corrections are the terms of the form $\alpha (\alpha/v)^n$: the $O(\alpha)$ corrections, and $\alpha^2 (\alpha/v)^n$: the $O(\alpha^2)$ corrections.

Recently an attempt has been made\[4\] at evaluating the $O(\alpha^2)$ corrections, which could potentially contain large factors that would make them numerically significant. It was concluded in \[4\] that the coefficients in front of the $\alpha^2$ terms are not extraordinarily large and

\[F_c\] is also known as the Sommerfeld-Sakharov factor.
that the $O(\alpha)$ radiative corrections\cite{3} are sufficient for describing the excitation curve for the $\tau^+\tau^-$ pair near the threshold with the relative precision not worse than $10^{-3}$. The calculation in Ref.\cite{4} was based on essentially an adaptation of the NRQCD methods used to describe the QCD effects near a heavy quark flavor threshold (see the references in \cite{4}). The known results for QCD radiative effects within these methods are based on the $\overline{MS}$ renormalization scheme, and were used in such form in Ref.\cite{4}. It should be noted however that in QED, unlike in QCD, the renormalization is constrained by the simple requirement that the asymptotic behavior of the Coulomb potential at long distances is given by $\alpha/r$ with no corrections in terms of the physical fine structure constant $\alpha$, tabulated\cite{5} as $\alpha^{-1} = 137.036$. This property, inherent in the “on shell” scheme, is lost in the $\overline{MS}$ scheme, in which the Coulomb potential does receive radiative corrections at asymptotically long distances. Clearly this effect is entirely spurious, since it goes away once the QED constant in the $\overline{MS}$ scheme, $\alpha_{\overline{MS}}$, is expressed in terms of the physical constant $\alpha$. However the necessity of including and subsequently eliminating the spurious terms somewhat complicates, if not obscures, intermediate calculations in the $\overline{MS}$ scheme. As an illustration of a difficulty arising in the latter scheme even at the level of the leading corrections it can be noticed that no distinction has been made in Ref.\cite{4} between the electron and the muon vacuum polarization in the Uehling-Serber type correction to the Coulomb potential. In reality, however, these two effects result in corrections to the cross section of completely different magnitude: the effect of the electron loop at the threshold is an enhanced by a factor containing $\ln(m_\tau \alpha/m_e)$ correction of order $\alpha$, while that of the muon loop can rather be classified as an enhanced by the factor $m_\tau/m_\mu$ correction of order $\alpha^2$ (or, alternatively, as an $O(\alpha)$ correction, suppressed by $m_\tau \alpha/m_\mu$). Numerically, at the threshold the former correction is almost 40 times larger than the latter\cite{3}.

The purpose of the present paper is to demonstrate that the calculation of the $O(\alpha^2)$ radiative corrections is significantly more transparent and simple in the “on shell” renormalization scheme and to present an evaluation of the terms in these corrections, enhanced by parametrically ‘large’ coefficients. The latter ‘large’ coefficients include all the factors singular in the limit $m_e \to 0$ as $\ln^2 m_e$ and $\ln m_e$, the already mentioned correction due the muon vacuum polarization, and also an enhanced $O(\alpha^2)$ effect of relativistic corrections, which is proportional to the factor $\ln v$ (becoming $\ln \alpha$ at $v \ll \alpha$). As will be shown, the combined effect of the enhanced $O(\alpha^2)$ corrections is only about $10^{-4}$ in terms of the relative magnitude of their contribution to the cross section, while the non-enhanced terms are
proportional to \((\alpha/\pi)^2 \approx 5 \cdot 10^{-6}\), thus allowing one to argue that factually the corrections calculated in Ref. [3] already provide the theoretical accuracy of \(10^{-4}\) in the cross section, rather than \(10^{-3}\) as estimated in Ref. [4]. In practical terms, this implies that the theoretical accuracy is sufficient for a measurement of the \(\tau\) mass down to at least \(O(1\, keV)\), provided that similar experimental accuracy can be achieved in measurements at the \(\tau\) threshold.

In Ref. [3] the effect of the \(O(\alpha)\) correction due to the vacuum polarization loop, both the electron and the muon, was presented in a form of a two-dimensional integral, which then was calculated numerically. This effect is also revisited in the present paper, and explicit analytical expressions will be given for the electron loop contribution in the limits \(v \gg \alpha\) (but still \(v \ll 1\)), and \(v \to 0\) (\(v \ll \alpha\)), as well as for the muon loop contribution, applicable for all values of the velocity below approximately \(m_\mu/m_\tau\).

2 Types of radiative corrections to \(\sigma(e^+e^- \rightarrow \tau^+\tau^-)\).

Generally the QED radiative corrections in the actual cross section of the process \(e^+e^- \rightarrow \tau^+\tau^-\) arise from the following sources[3, 4]:

- \(i\) – radiation from the initial electron and positron,
- \(ii\) – vacuum polarization in the time-like photon,
- \(iii\) – corrections to the spectral density \(\rho(q^2) = -\frac{1}{3} \sum_X \langle 0 | j_\mu(-q) | X \rangle \langle X | j_\mu(q) | 0 \rangle\) of the electromagnetic current \(j_\mu = (\bar{\tau} \gamma_\mu \tau)\) of the tau leptons.
- \(iv\) – interference between the effects \(i – iii\) which starts from the (relative) order \(\alpha^2\).

The actual cross section at the ‘nominal’ energy \(W = \sqrt{s}\) in c.m. of the electron-positron collision can thus be written in the form:

\[
\sigma(W) = \int_W^\infty r(W, w) |1 - \Pi(w)|^{-2} \bar{\sigma}(w) \, dw + (\text{interference terms}) \, .
\] (3)

The weight function \(r(W, w)\) describes the radiation from the initial state[3] and \(|1 - \Pi(w)|^{-2}\) is the factor for the vacuum polarization[3] in the time-like photon. These two effects are standard and are automatically accounted for in the data analyses, while the dynamics of the final state is encoded in the cross section \(\bar{\sigma}(w) = 8\pi^2\alpha^2\rho(w^2)/w^4\). The last term in eq.(3) arises from the graphs, where the lines of the initial electron and positron and of the \(\tau\) leptons are connected by more than one photon propagators.

The subject of primary interest in the previous studies as well as in the present one of the discussed process is the spectral density \(\rho(q^2)\). However before proceeding to a detailed
discussion of the QED corrections in $\rho$ few remarks are in order concerning the leading contribution of the interference. Clearly, this contribution can arise at the order $\alpha^2$ (relative to the ‘bare Coulomb’ cross section), and is thus within the scope of the discussion in the present paper.

One potentially possible contribution in that order could arise from the square of box-type graphs, where the $\tau$ pair is produced through two photons. However two photons produce the $\tau$ pair in a $C$-even state. For non relativistic heavy leptons the states can be classified in the standard (total spin) - (angular momentum) terms: $2S+1L$ with the production amplitude behaving near the threshold as $v^L$ (modulo the Coulomb effects that ‘convert’ powers of $v$ into powers of $\alpha$ at $v \sim \alpha$). The production of the $C$-even $S$-wave state $^1S_0$ by the $e^+e^-$ is suppressed for chirality reasons by the factor $m_\tau$ in the amplitude, which for all practical purposes makes it totally negligible. The amplitude of production of the allowed by chirality $C$-even $P$-wave states $^3P_1$ and $^3P_2$ contains an extra power of the velocity, and thus the contribution of the box graphs to the cross section near the threshold is additionally suppressed by the factor $v^2[8]$. It can be noticed however that the amplitude for production of the $^3P_2$ state actually contains a somewhat enhancing factor $\ln v[9]$. Thus the relative magnitude of the correction due to the box graphs can in fact be estimated as $\delta\sigma/\sigma \sim \alpha^2 v^2 \ln^2 v \sim \alpha^4 \ln^2 \alpha$, which is still far too small.

The only other contribution of the relative order $\alpha^2$ from the interference graphs can arise from the interference of the three photon production amplitude with the ‘bare’ amplitude mediated by one photon. The graphical representation of this contribution is shown as a unitary cut in the graph of Fig.1a. We are interested here in the terms of the lowest order in the velocity, thus in the production amplitudes the velocity can be set to zero. One can verify that no singularity arises from the electron propagators in the limit $m_e \to 0$ and thus that this term does not contain enhancing factors singular at $m_e \to 0$. In other words, it does not contain factors with powers of $\ln(m_\tau/m_e)$. This nonsingular behavior can be seen e.g. based on the Kinoshita-Lee-Nauenberg (KLN) theorem[10, 11]. Indeed, assume temporarily that the electric charge of the electron, $Q_e$, and of the $\tau$ lepton, $Q_\tau$, are independent parameters. According to the KLN theorem there should be no infrared singular terms in the sum of the probabilities including the emission and absorption of soft photons, and this behavior is valid at arbitrary values of $Q_e$ and $Q_\tau$. Expanding this total probability in powers of $Q_e$ and $Q_\tau$ one finds that in the order $Q_e^4 Q_\tau^4$ the only contribution to this probability in the limit $v \to 0$ comes from the unitary cuts of the types shown in Fig.1, since at $v \to 0$ there
are no cuts that would go across the photon line as well as across the \( \tau \) pair, and also the absorption of soft photons by the \( \tau \) leptons is irrelevant, since at \( v \to 0 \) the heavy leptons neither radiate nor absorb photons. On the other hand the cut across the three photons as shown in Fig.1b does not contain infrared singularity in the limit \( m_e \to 0 \): there is no singularity in the integrals over the energies of the photons, since the amplitude \( \tau^+ \tau^- \to 3\gamma \) has no such singularity in the photon energies\(^2\), while the collinear singularities also do not appear in the limit \( m_e \to 0 \), since there is only one electron propagator factor per each photon.

![Figure 1](image_url)  

Figure 1: A representative graph (a) for the interference correction in the process \( e^+ e^- \to \tau^+ \tau^- \). The sum of the unitary cuts in a and b contains no terms singular in \( m_e \) at \( m_e \to 0 \) at the threshold. (Dashed lines show the unitary cuts.)

The absence of photon radiation by the \( \tau \) leptons in the \( v \to 0 \) limit also guarantees that no terms containing \( \ln v \) arise from the graphs of the type shown in Fig.1a. Thus this contribution can only appear as an \( \alpha^2 \) correction in the cross section with just a numerical coefficient and containing no parametrical enhancement. As such this contribution should be taken into account in a complete calculation of the \( \alpha^2 \) terms, which however is beyond the intended accuracy.

Summarizing the previous discussion, one concludes that all the QED radiative corrections with relative magnitude of the first order in \( \alpha \) and the parametrically enhanced ones of the second order, \( \alpha^2 \), are contained in the corrections to the \( \tau \) pair spectral density \( \rho(w^2) \), i.e. they originate from the source labeled as iii above.

\(^2\)This of course is known since the work [15].
3 First order radiative corrections. Effect of the electron vacuum polarization

In the lowest, ‘zeroth’, order the effective cross section $\bar{\sigma}(w)$ can be expressed in terms of the nonrelativistic Green’s function $G(x, y, E)$ of the motion in the center of mass of the $\tau$ lepton pair at energy $E = w - 2m_\tau^{[3]}$:

$$\bar{\sigma}(w) = \frac{2\pi^2\alpha^2}{m_\tau^4} \text{Im} G(0, 0; m_\tau v^2) \ .$$ (4)

In the absence of QED radiative effects the interaction between the $\tau$ leptons is the Coulomb attraction, thus the Green’s function is the well known one for the Coulomb potential $V(r) = -\alpha/r$: $G_c(x, y, E)$. The imaginary part of the latter at $x = y = 0$ is related to that of the free-motion Green’s function,

$$G_0(x, y; \frac{p^2}{m}) = \frac{m \exp(ip|x - y|)}{4\pi |x - y|} ,$$ (5)

by the Coulomb factor $[2]$: $\text{Im} G_c(0, 0; m_\tau v^2) = F_c \text{Im} G_0(0, 0; m_\tau v^2)$, so that the leading order expression for $\bar{\sigma}$ reads as

$$\bar{\sigma}_0 = \frac{\pi^2\alpha^3}{2m_\tau^2} \frac{1}{1 - \exp(-\pi\alpha/v)} .$$ (6)

At the next level of approximation, i.e. in the first order in $\alpha$, the corrections to $\bar{\sigma}$ arise from two sources: from the so-called hard correction due to a finite radiative effect in the $\tau$ electromagnetic vertex at the threshold, and from the modification of the Coulomb interaction due to running of the coupling $\alpha$, which is described by the Uehling-Serber radiative correction to the potential. The behavior of the Green’s function at small separations $x$ and $y$ is determined by dynamics of the $\tau$ leptons at characteristic distances $r_c \sim 1/p_c$, where $p_c \sim m_\tau v$ for $v \gg \alpha$ and $p_c \sim m_\tau\alpha$ for $v \sim \alpha$ and $v \ll \alpha$. The hard correction to the vertex comes from distances of order $1/m_\tau$, which are thus point-like on the scale of characteristic distances in the Green’s function. Thus these two effects can be separated in terms of eq.(4) as$[12, 3]$

$$\bar{\sigma}(w) = \frac{2\pi^2\alpha^2}{m_\tau^4} \left(1 - \frac{4\alpha}{\pi}\right) \text{Im} \left[G_c(0, 0; m_\tau v^2) + \delta^{(1)}G(0, 0; m_\tau v^2)\right]$$

$$= \bar{\sigma}_0 \left(1 - \frac{4\alpha}{\pi}\right) \left(1 + \frac{2\alpha}{3\pi} h(v)\right) ,$$ (7)

$^3$For a discussion see Ref. $[3]$ and references therein
where the hard correction factor $1 - 4\alpha/\pi$ is well known in QED (see e.g. in the book [13]), and $\delta^{(1)} G$ is the first-order correction to the Green’s function due to the Uehling-Serber correction $\delta^{(1)} V(r)$ to the Coulomb potential

$$
\delta^{(1)} G(x, y; m_r v^2) = -\int G_c(x, r; m_r v^2) \delta^{(1)} V(r) G_c(r, y; m_r v^2) d^3r ,
$$

with $\delta^{(1)} V(r)$ given by (see e.g. in the textbook [14]):

$$
\delta^{(1)} V(r) = -\frac{2\alpha^2 1}{3\pi} \int_1^\infty e^{-2m_e x} \left( 1 + \frac{1}{2x^2} \right) \frac{\sqrt{x^2 - 1}}{x^2} dx .
$$

The correction term $h(v)$ in eq.(7) due to $\text{Im}\delta^{(1)} G$ can be found by considering the modification of wave function at the origin,

$$
\delta^{(1)} \psi(0) = -\int G_c(0, r; m_r v^2) \delta V(r) \psi_c(r) d^3r ,
$$

where $\psi_c(r)$ is the S-wave wave function at energy $m_r v^2$ in the Coulomb field $-\alpha/r$:

$$
\psi_c(r) = Ce^{-ipr} {\mbox{F}}_1(1 + i\lambda, 2, 2ipr)
$$

with $p = m_r v$, $\lambda = m_r \alpha/(2p) = \alpha/(2v)$ and $C = \psi_c(0)$. Using the representation of the Coulomb Green’s function in the form

$$
G_c(0, r; p^2/m_r) = -i\frac{m_e p}{2\pi} e^{ipr} \int_0^\infty e^{-2ipt} \left( \frac{1 + t}{t} \right)^i\lambda dt ,
$$

and after the integration over $r$ in eq.(10) the result [3] for $h(v)$ at arbitrary $v$ is expressed in terms of a double integral:

$$
h = -2\lambda \text{Im} \int_0^\infty dt \int_1^\infty dx \left( \frac{1 + t}{t} \right)^i\lambda \frac{(t + iz x v^{-1})^{i\lambda-1}}{(t + 1 + iz x v^{-1})^{i\lambda+1}} \left( 1 + \frac{1}{2x^2} \right) \frac{\sqrt{x^2 - 1}}{x^2} \right)
$$

with $z = m_e/m_r$.

The integral in eq.(13) can be readily calculated numerically [3] for an arbitrary relation between $v$ and $\alpha$. However it is instructive to have analytical expressions at least in the limiting cases: $v \gg \alpha$ and $v \ll \alpha$. In either case the discussed correction arises due to the running of the coupling $\alpha$ at distances $r_c \sim p_c^{-1}$, much shorter than the electron Compton wavelength. Thus the structure of term with $h(v)$ can be readily understood by replacing the coupling $\alpha$ in the Coulomb factor $F_c$ by the effective coupling at momentum $p_c$: $\alpha \rightarrow \alpha \left( 1 + \frac{2\alpha}{3\pi} \ln \frac{p_c}{m_e} \right)$, thus finding

$$
h(v) = \frac{1 - (1 + \pi \alpha/v) \exp(-\pi \alpha/v)}{1 - \exp(-\pi \alpha/v)} \ln \frac{p_c}{m_e} .
$$

(14)
In terms of this interpretation the subject of an actual calculation of \( h(v) \) reduces to finding \( p_c \) in terms of \( m_\tau \alpha \) and \( m_\tau v \). The factor in front of \( \ln(p_c/m_e) \) in eq.(14) is obviously given by \((\alpha/F_c)(\partial F_c/\partial \alpha)\).

In each of the limiting cases the asymptotic behavior of \( p_c \) is fixed up to constants: \( p_c \rightarrow \text{const} \cdot m_\tau v \) at \( v \gg \alpha \), and \( p_c \rightarrow \text{const} \cdot m_\tau \alpha \) (with a different constant) at \( v \rightarrow 0 \). In the former limit of large \( v \) in order to find the constant one can use eq.(8) with the Coulomb Green’s function replaced by the free one (eq.(5)), and also make use of the short distance limit of the Uehling-Serber correction:

\[
\delta^{(1)}V(r) = -\frac{\alpha}{r} \left[ 1 + \frac{2\alpha}{3\pi} \left( \ln \frac{1}{m_e r} - \gamma_E - \frac{5}{6} \right) \right] \tag{15}
\]

with \( \gamma_E \) being the Euler’s constant. The integral in eq.(8) can then be readily done, and the result reduces to the replacement

\[
\ln \frac{p_c}{m_e} \rightarrow \ln \left( \frac{2m_\tau v}{m_e} - \frac{5}{6} \right) \tag{16}
\]

in eq.(14) at \( v \gg \alpha \) (but still \( v \ll 1 \)). Taking also the \( v/\alpha \gg 1 \) limit of the factor in front of the logarithm in eq.(14), one finds the expression for \( h(v) \) in this limit as

\[
h(v)|_{v/\alpha \gg 1} = \frac{\pi \alpha}{2v} \left( \ln \left( \frac{2m_\tau v}{m_e} - \frac{5}{6} \right) \right). \tag{17}
\]

Compared to the result of the numerical integration in eq.(13) this expression provides the accuracy sufficient for description of the cross section with a relative error less than \( 10^{-4} \) down to \( v \approx 0.04 \).

One can also find analytically the behavior of \( h(v) \) at \( v \rightarrow 0 \), using eq.(10) and the explicit form of the Coulomb Green’s function at zero energy:

\[
G_c(0, r; E \rightarrow +0) = -\frac{m_e^2 \alpha}{4\sqrt{m_\tau \alpha r}} \left[ Y_1(2\sqrt{m_\tau \alpha r}) + i J_1(2\sqrt{m_\tau \alpha r}) \right] , \tag{18}
\]

where \( J \) and \( Y \) stand for the Bessel functions of respectively the first and the second kind. After inserting in eq.(10) the correction to the potential in the form (9), the integral over \( r \) can be readily done, and the resulting expression for \( h(0) \) reads as

\[
h(0) = \int_{1}^{\infty} \left[ 1 - \frac{m_\tau \alpha}{m_e x} \exp \left( -\frac{m_\tau \alpha}{m_e x} \right) \right] K_1 \left( \frac{m_\tau \alpha}{m_e x} \right) \left( 1 + \frac{1}{2x^2} \right) \frac{\sqrt{x^2 - 1}}{x^2} \, dx \tag{19}
\]

with \( K \) being the standard notation for the modified Bessel function of the second kind.
The integral in eq.(19) can be evaluated using the presence of the large parameter \(m_\tau \alpha/m_e \approx 25.4\). The integration over \(x\) can be split in two intervals: \(1 < x < X\) and \(x > X\), with \(X\) satisfying the conditions \(1 \ll X \ll m_\tau \alpha/m_e\). After making the obvious approximations in these two intervals one finds the result as

\[
h(0) = \ln \frac{m_\tau \alpha}{m_e} + \gamma_E + \frac{1}{6} + O \left[ \left( \frac{m_e}{m_\tau \alpha} \right)^4 \right],
\]

which in terms of eq.(14) corresponds to the replacement

\[
\ln \frac{p_e}{m_e} \to \ln \frac{m_\tau \alpha}{m_e} + \gamma_E + \frac{1}{6}.
\]

Numerically, eq.(20) gives \(h(0) = 3.980\) with a very high accuracy, given that the subsequent terms start with the fourth power of the small parameter \(m_e/(m_\tau \alpha)\).

4 Parametrically enhanced \(O(\alpha^2)\) vertex correction.

The effect of the muon loop arising from its contribution to the Uehling-Serber correction to the potential can be evaluated by replacing \(m_e \to m_\mu\) in eq.(13) and considering there the limit \(z \gg 1\). The resulting correction term \(h(\mu)(v)\) is then found to be essentially constant in \(v\) up to \(v \approx 2m_\mu/m_\tau\) and therefore in this region of \(v\) this term can in fact be approximated by \(h(\mu)(0)\) from eq.(19). Using the latter expression and the fact that \(m_\tau \alpha/m_\mu \ll 1\) one finds

\[
h(\mu)(0) = \frac{m_\tau \alpha}{m_\mu} \int_1^\infty \left( 1 + \frac{1}{2x^2} \right) \frac{\sqrt{x^2 - 1}}{x^3} \, dx = \frac{9\pi}{32} \frac{m_\tau \alpha}{m_\mu}.
\]

It can be noticed that the contribution of the muon vacuum polarization loop, as well as that due to the heavier states, to the discussed correction for the \(\tau\) pair spectral density at small velocity \(v\) of the \(\tau\) leptons behaves quite differently from the contribution of the electron loop. The reason for this difference is that a state contributing to the absorptive part of the vacuum polarization at \(q^2 = s\) modifies the interaction between the \(\tau\) leptons at distances shorter than \(\sim 1/\sqrt{s}\), thus the effect in the dynamics of the produced \(\tau\) leptons can be considered as local, provided that the velocity satisfies the condition \(v \ll \sqrt{s}/m_\tau\). Even for the muon threshold, \(\sqrt{s} = 2m_\mu\), this condition is satisfied in all the region of interest for the velocity. The local effect does not depend on \(v\) in this region and can be calculated by

\[\text{This behavior also agrees with the numerical results of Ref.}\ [3]\]
setting \( v \to 0 \). Therefore in terms of the factorization formula (equations (8) and (9)) the effect of the muon loop and of higher states should rather be written as contribution to a correction, \((\alpha/\pi)^2 \Delta\), in the vertex factor:

\[
\bar{\sigma}(w) = \frac{2\pi^2 \alpha^2}{m_{\tau}^4} \left( 1 - \frac{4\alpha}{\pi} + \frac{\alpha^2}{\pi^2} \Delta \right) \text{Im}G(0, 0; m_{\tau} v^2) . 
\]  

(23)

Being interpreted in terms of the correction term \( \Delta \) in eq.(23) the expression (22) corresponds to \( \Delta(\mu) = 3\pi^2 m_{\tau}/(16 m_{\mu}) \), which although being enhanced by the factor \( m_{\tau}/m_{\mu} \), still gives a correction of only about \( 1.7 \times 10^{-4} \) in eq.(23).

The discussed correction due to the muon loop is only a part of the correction to the electromagnetic vertex of the \( \tau \) lepton at the threshold arising from the vacuum polarization. The latter correction can readily be found in full by noticing that the calculation can be formally reduced to a standard calculation of the one-loop vertex correction with a massive photon using the dispersion relation for the vacuum polarization \( P(k^2) \) with subtraction at \( k^2 = 0 \) (the ‘on shell’ renormalization). Indeed, the photon propagator with one insertion of the vacuum polarization can be written as

\[
\frac{P(k^2)}{k^2} = -\frac{1}{\pi} \int \frac{\text{Im}P(s)}{s (k^2 - s)} \, ds ,
\]

(24)

which clearly reduces the calculation to that with formally a massive photon with mass \( \mu = \sqrt{s} \).

At zero velocity, \( v = 0 \), the one loop vertex correction with a massive photon results in the following formula for the term \( \Delta \) in eq.(23)

\[
\Delta = -\int \frac{\text{Im}P(s)}{\alpha} f \left( \frac{\sqrt{s}}{m_{\tau}} \right) \frac{ds}{s} , 
\]

(25)

with the function \( f(\sqrt{s}/m_{\tau}) \) given by

\[
f(\xi) = \frac{2}{3\xi} \left[ (6 + 2 \xi^2 + 2 \xi^4 - \xi^6) \frac{2}{\sqrt{4 - \xi^2}} \arctan \frac{\sqrt{4 - \xi^2}}{\xi} + 2 \xi^5 \ln \xi - 3 \xi - 2 \xi^3 \right] . 
\]

(26)

At \( \sqrt{s} \ll m_{\tau} \) the function \( f(\sqrt{s}/m_{\tau}) \) behaves as

\[
f \left( \frac{\sqrt{s}}{m_{\tau}} \right) = \frac{2\pi m_{\tau}}{\sqrt{s}} - 4 + O \left( \frac{\sqrt{s}}{m_{\tau}} \right) . 
\]

(27)

Thus using the explicit expression for the muon vacuum polarization

\[- \frac{\text{Im}P(s)}{\alpha} = \frac{1}{3} \left( 1 + \frac{2 m_{\mu}^2}{s} \right) \sqrt{1 - \frac{4 m_{\mu}^2}{s}} , \]

10
one finds from the integral in eq.(25) the singular in the ratio $m_\tau/m_\mu$ part of the muon loop contribution to $\Delta$ as

$$\Delta_{(\mu)} = \frac{3\pi^2 m_\tau}{16 m_\mu} - \frac{8}{3} \ln \frac{m_\tau}{m_\mu}. \quad (28)$$

The first term here clearly reproduces the correction given by eq.(22), while the second term describes the running $\alpha$ effect in the hard correction due to the fact that it comes from distances of order $1/m_\tau$ and thus should actually be written as $-4\alpha(m_\tau)/\pi$.

The singular infrared behavior of the vertex correction makes it necessary to return to the discussion of the electron loop effect in the vacuum polarization contribution to the hard correction. For the electron loop, setting $v \to 0$ in the calculation of the vertex correction is in fact not legitimate, since $m_\tau v$ (or $m_\tau \alpha$) is obviously not small as compared to $m_e$, and the leading infrared term has to be accounted for within the Coulomb dynamics of the $\tau$ lepton pair. This has been done in the previous section in terms of the correction factor $h(v)$. On the other hand the contribution to $\Delta$ resulting from the subsequent term of the expansion in eq.(27) and given by $\Delta_{(e)} = -(8/3) \ln(m_\tau/m_e)$ is perfectly correct, and describes the electron loop effect when $\alpha(m_\tau)$ in the hard correction is expressed in terms of the physical constant $\alpha$.

The contribution of the heavier than the $\mu^+\mu^-$ pair states to the discussed correction $\Delta$, in particular of the hadron vacuum polarization, thus contains no large logarithmic factors. Indeed, there is no ‘logarithmic range’ for such masses below the $\tau$ mass, and the effect of the states heavier than $\tau$ rapidly decreases with their mass ($f(\xi) \approx 4 \ln \xi^2/(3 \xi^2)$ at $\xi^2 = s/m_\tau \gg 1$). Thus keeping only the ‘large’ terms one can write the final expression for $\Delta$ as

$$\Delta = -\frac{8}{3} \ln \frac{m_\tau}{m_e} + \frac{3\pi^2 m_\tau}{16 m_\mu} - \frac{8}{3} \ln \frac{m_\tau}{m_\mu}. \quad (29)$$

Numerically these three terms almost cancel each other, resulting in an extremely small value of the vertex correction $(\alpha/\pi)^2 \Delta \approx 1.0 \times 10^{-5}$.

5 **Parametrically enhanced $O(\alpha^2)$ corrections in the Green’s function.**

In this section we consider the radiative effects of the second order in $\alpha$ in the Green’s function in eq.(23). These effects arise from the second order iteration of the first order correction to the potential given by eq.(3), and from the second order radiative correction.
$\delta^{(2)}V(r)$ to the potential. It can be noted in connection with the latter effect that in the on-shell scheme there is no contribution to $\delta^{(2)}V(r)$ due to the diagrams with exchange of two photons between the $\tau$ leptons, and all the corrections are given by the insertions in the single photon propagator, i.e. the two electron loop insertion and the correction in the one loop insertion.

However, in order to find only those terms of order $\alpha^2$ whose coefficients are enhanced by the second and the first power of $\ln(p_c/m_e)$ there is no need for a detailed calculation of each these effects, and the result can be found using the KLN theorem. Indeed, the infrared singularity in $\text{Im } G(0,0,m_\tau v^2)$ in the limit $m_e \to 0$ should be absent, provided that the result is expressed in terms of the effective coupling at the scale $p_c$: $\alpha(p_c)$. Thus all the factors with $\ln(p_c/m_e)$ arise through expressing $\alpha(p_c)$ in terms of the physical constant $\alpha$. Up to the single logarithmic terms of the second order the latter expression is well known:

$$\alpha(p_c) = \alpha + \frac{2\alpha^2}{3\pi} \ln \frac{p_c}{m_e} + \frac{4\alpha^3}{9\pi^2} \ln^2 \frac{p_c}{m_e} + \frac{\alpha^3}{2\pi^2} \ln \frac{p_c}{m_e}. \quad (30)$$

Once $p_c$ is specified in terms of $m_\tau v$ and $v/\alpha$ from the results of the calculation in Section 3, the discussed logarithmically enhanced terms can be found at any velocity in the region of interest. Writing the effect of these terms in the form of a multiplicative factor $[1 + (\alpha/\pi)^2 \Phi(v)]$ in the cross section, one finds from the result in eq.(20) the expression for $\Phi$ in the limit $v \to 0$, where this correction is maximal,

$$\Phi(0) = \frac{4}{9} \ln^2 \frac{m_\tau \alpha}{m_e} + \left( \frac{8}{9} \gamma_E + \frac{35}{54} \right) \ln \frac{m_\tau \alpha}{m_e}. \quad (31)$$

Numerically, the effect of this correction amounts to less than $0.5 \times 10^{-4}$.

The function $\Phi(v)$ decreases with the velocity, and at $v \gg \alpha$ its behavior, as can be readily found from eq.(16), is given by

$$\Phi(v)|_{v \gg \alpha} \approx \left( \frac{\pi \alpha}{2 v} \right) \left( \frac{4}{9} \ln^2 \frac{2m_\tau v}{m_e} - \frac{13}{54} \ln \frac{2m_\tau v}{m_e} \right). \quad (32)$$

6 Relativistic corrections.

Due to the presence of the Coulomb parameter, $\alpha/v$, the relativistic effects, which formally arise from terms of order $v^2$, get converted into corrections of order $\alpha^2$. Such corrections originate from relativistic corrections to the production vertex, and from corrections to the Green’s function arising from the relativistic terms in the Breit-Fermi Hamiltonian.
A detailed study of these effects has been performed\cite{16, 17, 18} in QCD for the threshold production of heavy quarks in $e^+e^-$ annihilation. Since the results of this study in the order $\alpha^2$ are not sensitive to the renormalization scheme, one can directly apply them to the discussed case of the $\tau$ pair production near the threshold. The relativistic correction contains as \textit{‘large’} parameter $\ln p_c/m_\tau$, and keeping only the enhanced terms, it can be written as the following multiplicative factor in the cross section

\begin{equation}
\left\{ 1 + \frac{2}{3} \alpha^2 \left[ \ln \frac{1}{v} - \text{Re} \Psi \left(-i \frac{\alpha}{2v}\right) \right] \right\},
\end{equation}

where $\Psi$ stands for the digamma function.

Numerically, this correction term reaches its largest value \[2 \alpha^2 \ln(2/\alpha)/3 \approx 2.0 \times 10^{-4}\] at $v = 0$ and slowly decreases with velocity (e.g. the numerical value of the correction at $v = 0.1$ is $1.7 \times 10^{-4}$).

It can be also noticed in relation to the expression in eq.(33) that being multiplied by the Coulomb factor (2) it produces a double pole in the cross section at $v = in\alpha/2$ corresponding to the $n$-th bound $^3S_1$ level of the $\tau^+\tau^-$ pair. This double pole correctly reproduces the Breit-Fermi relativistic correction to the energy of the bound state\cite{14}. On the other hand the singularity of the digamma function at $v \to 0$ is canceled by the logarithmic term. These two observations are in principle sufficient to reproduce the structure of the discussed part of the relativistic correction and the coefficient in front of it.

\section{Summary and concluding remarks}

We are now ready to collect all the discussed terms into one formula and to write down the complete expression for $\sigma(e^+e^-) \to \tau^+\tau^-$ near the threshold including all corrections of order $\alpha$ and all the parametrically enhanced terms of order $\alpha^2$:

\begin{equation}
\bar{\sigma} = \frac{\pi^2 \alpha^3}{2m_e^2} \left\{ \frac{1}{1 - \exp(-\pi\alpha/v)} \left\{ 1 - \frac{4\alpha}{\pi} + \frac{2\alpha}{3\pi} h(v) \right. \right. \\
+ \frac{\alpha^2}{\pi^2} \left\{ \Phi(v) - \frac{8}{3} \ln \frac{m_\tau}{m_e} + \frac{3\pi^2 m_\tau}{16 m_\mu} - \frac{8}{3} \ln \frac{m_\tau}{m_\mu} \\
+ \frac{2\pi^2}{3} \left[ \ln \frac{1}{v} - \text{Re} \Psi \left(-i \frac{\alpha}{2v}\right) \right] - \frac{8}{3} h(v) \right\} \right\},
\end{equation}

where the function $h(v)$ is given by the equations (13), (17), and (20), and $\Phi(v)$ is defined in the Section 5 (cf. the equations (31) and (32)). The last term with $h(v)$ in the $O(\alpha^2)$

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correction in eq.(34) obviously arises from the interference of the first order vertex correction with the electron loop effect in the first order (cf. eq.(7)).

The relative magnitude of the discussed QED radiative corrections decreases with $v$. Thus their combined effect can be majorated by their value at $v = 0$. Writing the numerical values at $v = 0$ of the individual terms in the inner curly braces in eq.(34), the $O(\alpha^2)$ term can be evaluated as $(\alpha/\pi)^2 (8.41 - 21.75 + 31.12 - 7.52 + 36.93 - 10.61) = 36.58 (\alpha/\pi)^2 \approx 2.0 \times 10^{-4}$.

It can be also noticed that the $O(v^2)$ relativistic term, present in eq.(1) was intentionally omitted throughout the previous discussion as well as the total energy dependence of the cross section arising from the timelike photon propagator producing the $\tau^+\tau^-$ pair. This is justified, as long as non enhanced $O(\alpha^2)$ corrections are also ignored, in the range of the velocity where $v$ and $\alpha$ are considered as being parametrically of the same order, and where the discussed effects of the interaction between the $\tau$ leptons are most interesting.

For practical reasons however it is desirable to have a description that interpolates with a high accuracy the cross section between the ‘Coulomb’ region of small $v$ and the relativistic region. An interpolating formula of this type was suggested in Ref.[3] in terms of the well known full expression for the cross section up to order $\alpha$ (but without any summation of the Coulomb terms):

$$
\sigma_0(e^+ e^- \rightarrow \tau^+ \tau^-) = \frac{2\pi \alpha^2}{3s} v (3 - v^2) \left(1 + \frac{\alpha}{\pi} S(v)\right), \tag{35}
$$

where $S(v)$ is given by

$$
S(v) = \frac{1}{v} \left\{(1 + v^2) \left[\frac{\pi^2}{6} + \ln \left(\frac{1+v}{2}\right) \ln \left(\frac{1-v}{1+v}\right) + 2 \text{Li}_2 \left(\frac{1-v}{2}\right) + 2 \text{Li}_2 \left(\frac{1+v}{2}\right) - 2 \text{Li}_2 \left(\frac{1-v}{2}\right) - 4 \text{Li}_2(v) + \text{Li}_2(v^2)\right] + \left[\frac{11}{8} (1 + v^2) - 3v + \frac{1}{2} \frac{v^4}{(3-v^2)}\right] \ln \left(\frac{1-v}{1+v}\right) + \frac{3}{4} \frac{v^5}{(3-v^2)}\right\} \tag{36}
$$

with $\text{Li}_2(x) = - \int_0^1 \ln(1-t) dt/t = \sum_{n=1}^{\infty} x^n/n^2$. The interpolation formula, which does include the summation of all the Coulomb terms as well as the $O(\alpha)$ correction to them, has the form

$$
\bar{\sigma}(e^+ e^- \rightarrow \tau^+ \tau^-) = \frac{2\pi \alpha^2}{3s} v (3 - v^2) F_c \left(1 + \frac{\alpha}{\pi} S(v) - \frac{\pi \alpha}{2v} + \frac{2\alpha}{3\pi} h(v)\right). \tag{37}
$$

Near the threshold this formula correctly reproduces the Coulomb enhancement as well as the first radiative corrections to it. At all $v$ the interpolation correctly reproduces the terms of the first order in the expansion in $\alpha$. From these observations it was concluded[3] that the higher corrections to eq.(37) are uniformly of order $\alpha^2$ at any velocity with no enhancement
by inverse powers of $v$ near the threshold. From the calculations in the present paper it is seen that the corrections of the relative magnitude $O(\alpha^2)$ in fact receive a moderate logarithmic enhancement near the threshold which however does not exceed $2 \times 10^{-4}$ numerically.\footnote{Actually, the leading effect of the muon loop (eq.(22)) was included in the analysis of Ref.\cite{3} in the function $h(v)$, thus the numerical correction to that analysis is less than $10^{-4}$.} I believe that this accuracy should be quite sufficient for all foreseeable practical purposes.

**Acknowledgements**

I thank A. Czarnecki, M. Shifman, and A. Vainshtein for enlightening discussions. This work is supported in part by the DOE grant DE-FG02-94ER40823.

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