Parametricity, automorphisms of the universe, and excluded middle

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Abstract

It is known that one can construct non-parametric functions by assuming classical axioms. Our work is a converse to that: we prove classical axioms in dependent type theory assuming specific instances of non-parametricity. We also address the interaction between classical axioms and the existence of automorphisms of a type universe. We work over intensional Martin-Löf dependent type theory, and in some results assume further principles including function extensionality, propositional extensionality, propositional truncation, and the univalence axiom.

Keywords. relational parametricity, dependent type theory, univalent foundations, homotopy type theory, excluded middle, classical mathematics, constructive mathematics.

1 Introduction: Parametricity in dependent type theory

Broadly speaking, parametricity statements assert that type-polymorphic functions definable in some system must be natural in their type arguments, in some suitable sense.

Reynolds’ original theory of relational parametricity [9] characterizes terms of the polymorphically typed λ-calculus System F. This theory has since been extended to richer and more expressive type theories: to pure type systems by Bernardy and Jansson [2], and to dependent type theory by Atkey et al. [1].

Most parametricity results are meta-theorems about a formal system and make claims only about terms in the empty context. For instance, Reynolds’ results show that the only term of System F with type \( \forall \alpha. \alpha \to \alpha \) definable in
the empty context is the polymorphic identity function \( \Lambda \alpha \lambda (x : \alpha).x \). Similarly, Atkey et al. \([1, \text{Thm. 2}]\) prove that any term \( f : \prod_{X} X \to X \) definable in the empty context of MLTT must satisfy \( e(f_X(a)) = f_Y(e(a)) \) for all \( e : X \to Y \) and \( a : X \) in their model; it follows that, \( f \) acts as the identity on every type in their model, and hence no such closed term \( f \) can be provably not equal to the polymorphic identity function. Keller and Lasson \([4]\) further consider a type theory with parametricity included as part of the language.

Parametricity meta-theorems depend crucially on the constructive nature of type theory. Indeed, given excluded middle (and perhaps some extensionality axioms), it is easy to construct a function \( f : \prod_{X} X \to X \) whose value at the type \( 2 \) of booleans is the negation function \( \text{flip} : 2 \to 2 \) (see Theorem 1 below), as well as other violations of parametricity. Therefore, parametricity meta-theorems imply in particular that excluded middle is not provable.

In this paper, we give various converse results to emphasize this dependence: failures of parametricity and related principles imply excluded middle and related constructive taboos. These are (like the converse construction of a non-parametric function from excluded middle, but by contrast with most positive parametricity results) theorems of dependent type theory, so they apply not only to closed terms but in any context, and the violations of parametricity are expressed using negations of Martin-Löf’s identity type rather than judgemental (in-)equality of terms. Similarly, we show that excluded middle also follows from certain kinds of non-trivial automorphisms of the universe.

We work throughout in intensional Martin-Löf type theory, with at least \( \Pi-, \Sigma-, \text{identity}, \text{finite}, \text{and natural numbers types, and a universe closed under these type-formers. For concreteness, this may be taken to be the theory of } \[8\], or of \([10, \text{A.2}]\). When results require further axioms—e.g. functional extensionality, or univalence of the universe—we include these as explicit assumptions, to keep results as sharp as possible.

By the law of excluded middle, we mean always the version from univalent foundations \([10, \text{3.4.1}]\), namely that \( P + \neg P \) for all propositions \( P \), where a type is called a proposition if it has at most one element, meaning that any two of its elements are equal in the sense of the identity type.

By a logical equivalence of types \( X \) and \( Y \) we mean two functions \( X \to Y \) and \( Y \to X \) subject to no conditions at all.

By an equivalence of types \( X \) and \( Y \) we mean a function \( e : X \to Y \) that has both a left and a right inverse, i.e. functions \( s, r : Y \to X \) with \( e(s(y)) = y \) for all \( y : Y \) and \( r(e(x)) = x \) for all \( x : X \). This notion of equivalence is logically equivalent to having a single two-sided inverse, which is all that we will need in this paper. But the notion of equivalence is better-behaved in univalent foundations (see \([10, \text{Chapter 4}]\)); the reason is that the type expressing “being an equivalence” is a proposition, in the presence of function extensionality, whereas the type expressing “having a two-sided inverse” may in general have more than one inhabitant, in particular affecting the consistency of the univalence axiom.

2 Classical axioms from non-parametricity

In this section, we give a number of ways in which classical axioms can be derived from specific violations of parametricity.
2.1 Polymorphic endomaps

Say that a function \( f : \prod_{X: \mathcal{U}} X \to X \) is natural under equivalence if for any two types \( X \) and \( Y \) and any equivalence \( e : X \to Y \), we have \( e(f_X(x)) = f_Y(e(x)) \) for any \( x : X \), where we have written \( f_X \) as a shorthand for \( f(X) \) and used the equality sign \( = \) to denote identity types.

**Theorem 1.** If there is a function \( f : \prod_{X: \mathcal{U}} X \to X \) such that \( f_2 \) is not pointwise equal to the identity (i.e. \( \neg \prod_{x: 2} f_2(x) = x \)) and \( f \) is natural under equivalence, then the law of excluded middle holds. Assuming functional extensionality, the converse also holds.

**Proof.** First we derive excluded middle from \( f \). To begin, note that if \( \neg \prod_{x: 2} f_2(x) = x \), then we cannot have both \( f_2(\textsf{tt}) = \textsf{tt} \) and \( f_2(\textsf{ff}) = \textsf{ff} \), since then we could prove \( \prod_{x: 2} f_2(x) = x \) by case analysis on \( x \). But then by case analysis on \( f_2(\textsf{tt}) \) and \( f_2(\textsf{ff}) \), we must have \( (f_2(\textsf{tt}) = \textsf{ff}) + (f_2(\textsf{ff}) = \textsf{tt}) \). Without loss of generality, suppose \( f_2(\textsf{tt}) = \textsf{ff} \).

Now let \( P \) be an arbitrary proposition. We do case analysis on \( f_{P+1}(\text{inr}(\star)) : P + 1 \).

1. If it is of the form \( \text{inl}(p) \) with \( p : P \), we conclude immediately that \( P \) holds.

2. If it is of the form \( \text{inr}(\star) \), then \( P \) cannot hold, for if we had \( p : P \), then the map \( e : 2 \to P + 1 \) defined by \( e(\textsf{ff}) = \text{inl}(p) \) and \( e(\textsf{tt}) = \text{inr}(\star) \) would be an equivalence, and hence \( e(f_2(x)) = f_{P+1}(e(x)) \) for all \( x : 2 \) and so \( \text{inl}(p) = e(\textsf{ff}) = e(f_2(\textsf{tt})) = f_{P+1}(e(\text{tt})) = \text{inr}(\star) \), which is a contradiction.

Therefore \( P \) or not \( P \).

For the converse, [10] Exercise 6.9], suppose excluded middle holds, let \( X : \mathcal{U} \) and \( x : X \), and consider the type \( \sum_{x'.X}(x' \neq x) \), where \( a \neq b \) means \( \neg(a = b) \). By excluded middle, this is either contractible or not. (A type \( Y \) is contractible if \( \sum_{y.Y} \prod_{y'.Y} (y = y') \). Assuming function extensionality, this is a proposition.) If it is contractible, define \( f_X(x) \) to be the center of contraction (the point \( y \) in the definition of contractibility); otherwise define \( f_X(x) = x \).

**Remark.**

1. Naturality of any such \( f \) under equivalence is implied directly by the internal parametricity of Keller and Lasson [4]. Hence this theorem implies that excluded middle is (provably) false in their type theory.

2. If we assume univalence, any \( f : \prod_{X: \mathcal{U}} X \to X \) is automatically natural under equivalence, so that assumption can be dispensed with. And, of course, if function extensionality holds (which follows from univalence) then the hypothesis \( \neg \prod_{x: 2} f_2(x) = x \) is equivalent to \( f_2 \neq \lambda(x: 2).x \).

3. We do not know whether the converse direction of Theorem 1 is provable without function extensionality.

The preceding proof can be generalized as follows. We say that a point \( x : X \) is isolated if the type \( x = y \) is decidable for all \( y \), i.e. if we have \( \prod_y.X(x = y) + (x \neq y) \).

**Lemma 2.** A point \( x : X \) is isolated if and only if \( X \) is equivalent to \( Y + 1 \), for some type \( Y \), by a map that sends \( x \) to \( \text{inr}(\star) \).
Proof. Since \text{inr}(\ast) is isolated, such an equivalence certainly implies that \(x\) is isolated. Conversely, from \(\prod_{y:X}(x = y) + (x \neq y)\) we can construct a function \(d : X \to 2\) such that \(d(y) = \ast\) if \(x = y\) and \(d(y) = \#\) if \(x \neq y\). Let \(Y\) be \(\sum_{y:X}(d(y) = \#)\); it is straightforward to show \(X \simeq Y + 1\).

If we had function extensionality (for 0-valued functions), we could dispense with \(d\) and define \(Y = \sum_{y:X}(x \neq y)\), since then \(x \neq y\) would be a proposition. In general we use \(d(y) = \#\) as it is always a proposition (since 2 has decidable equality, hence its identity types are propositions by Hedberg’s theorem); this is necessary to show that the composite \(Y + 1 \to X \to Y + 1\) acts as the identity on \(Y\).

\begin{proof}

\textbf{Theorem 4.} In a type theory with propositional truncations, there is an
\textbf{Theorem 3.} If there is a function \(\text{witness}\) induced by a point \(P\)
Therefore either \(x\) direction, assume that we are given \(f\)
extensionality \([6]\), so the converse direction of Theorem 1 applies. For the “only
\text{excluded middle holds. Assuming functional extensionality, the converse also holds.}

\textbf{Proof.} To derive excluded middle from \(f\), let \(Y\) and \(X \simeq Y + 1\) be as in
\text{Lemma 2} and let \(P\) be an arbitrary proposition. We do case analysis on
\(f_{P \times Y + 1}(\text{inr}(\ast)) : P \times Y + 1\).

\begin{enumerate}
\item If it is of the form \(\text{inl}((p, y))\) with \(p : P\), we conclude immediately that \(P\)
holds.
\item If it is of the form \(\text{inr}(\ast)\), then \(P\) cannot hold, for if we had \(p : P\), then
the map \(e : X \to P \times Y + 1\) defined by \(e(x) = \text{inr}(\ast)\) (where \(x\) is the
iso-point) and \(e(y) = \text{inl}((p, y))\) for \(y \neq x\) would be an equivalence, and
hence \(e(f_X(x)) = f_{P \times Y + 1}(e(x))\), and so \(\text{inl}((p, f_X(x))) = e(f_X(x)) =
\text{inr}(\ast)\) which is a contradiction.
\end{enumerate}

Therefore either \(P\) or not \(P\) holds. The converse is proven exactly as in Theorem 1 \(\square\)

Finally, if our type theory includes propositional truncations, axiomatized as in [10] §3.7 and written \(|A|\), we can dispense with isolatedness.

\textbf{Theorem 4.} In a type theory with propositional truncations, there is an
\textbf{Theorem 3} equivalence-natural function \(f : \prod_{X:U}X \to X\) and a type \(X : U\) with a point
\text{a \(x\) such that \(f_X(x) \neq x\) if and only if excluded middle holds.}

\textbf{Proof.} For the “if” direction, note that propositional truncation implies function
extensionality \([6]\), so the converse direction of Theorem 1 applies. For the “only
\text{if}” direction, assume that we are given \(f : \prod_{X:U}X \to X\), a type \(X : U\) and a
point \(x : X\) with \(f_X(x) \neq x\). Let \(P\) be any proposition, and define
\(Z = \sum_{y:X}||x = y|| \lor P, \quad z = (x, |\text{inl}(|\text{refl}_x|)|) : Z, \quad y = \text{pr}_1(f_Z(z)) : X.\)

We write \(A \lor B\) for the truncated disjunction \(||A + B||\), and \(|a| : |A|\) for the
witness induced by a point \(a : A\). Then the second projection \(\text{pr}_2(f_Z(z))\) tells us that
\(|x = y|| \lor P\). However, if \(P\) holds, then \(\text{pr}_1 : Z \to X\) is an equivalence that
maps \(z\) to \(x\). Thus \(f_Z(z) \neq z\) and hence \(x \neq y\). In other words, one of the two
propositions \(|x = y||\) and \(P\) must hold. But they cannot both hold, which is to
say that \(P\) is complemented. \(\square\)
Remark.

1. If \( x : X \) happens to be isolated, then the type \( Z \) defined in the proof of Theorem \( \text{[4]} \) is equivalent to the type \( P \times Y + 1 \) used in the proof of Theorem \( \text{[3]} \).

2. Since propositional truncation implies function extensionality \([6]\), it makes excluded middle into a proposition. Thus, the existence hypothesis of Theorem \( \text{[4]} \) can be truncated or untruncated without change of meaning.

3. The hypothesis can also be formulated as “there is a type \( X \) such that \( f_X \) is apart from the identity of \( X \),” where two functions \( g, h : A \to B \) of types \( A \) and \( B \) are apart if there is \( a : A \) with \( g(a) \neq h(a) \). We don’t know whether it is possible to derive excluded middle from the weaker assumption that \( f_X \) is simply unequal to the identity function of \( X \), or even that \( f \) is unequal to the polymorphic identity function.

The above can be applied to obtain classical axioms from other kinds of violations of parametricity. As a simple example, consider \( f : \prod_{X \in \mathcal{U}} (X \to X) \to (X \to X) \). Parametric elements of this type are Church numerals. Given \( f \), we can define a polymorphic endomap \( g : \prod_{X \in \mathcal{U}} X \to X \) by \( g_X = f_X(\text{id}_X) \), where \( \text{id}_X \) is the identity function. If \( f \) is natural under equivalence, then so is \( g \), and hence the assumption that \( f_2(\text{id}_2) \) is not the identity function gives excluded middle, assuming function extensionality.

2.2 Maps of the universe into the booleans

A function \( f : \mathcal{U} \to 2 \) is invariant under equivalence, or extensional, if we have \( f(X) = f(Y) \) for any two equivalent types \( X \) and \( Y \). We say that it is strongly non-constant if we have \( X, Y : \mathcal{U} \) with \( f(X) \neq f(Y) \). Assuming function extensionality, Escardó and Streicher \([3, \text{Thm. 2.2}]\) showed that if \( f : \mathcal{U} \to 2 \) is extensional and strongly non-constant, then the weak limited principle of omniscience holds (any function \( \mathbb{N} \to 2 \) is constant or not). Alex Simpson strengthened this as follows (also reported in \([3, \text{Thm. 2.8}]\)):

**Theorem 5** (Simpson). Assuming function extensionality for \( 0 \)-valued functions, there is an extensional, strongly non-constant function \( f : \mathcal{U} \to 2 \) if and only if weak excluded middle holds (meaning that \( \neg A + \neg \neg A \) for all \( A : \mathcal{U} \)).

**Proof.** In one direction, suppose weak excluded middle, and define \( f : \mathcal{U} \to 2 \) by \( f(A) = \text{ff} \) if \( \neg A \) and \( f(A) = \text{tt} \) if \( \neg \neg A \). Then \( f(\text{ff}) = \text{ff} \) and \( f(\text{tt}) = \text{tt} \), so \( f \) is strongly non-constant. Extensionality follows from the observation that if \( A \simeq B \) then \( \neg A \leftrightarrow \neg B \) and \( \neg \neg A \leftrightarrow \neg \neg B \).

In the other direction, suppose \( f : \mathcal{U} \to 2 \) is extensional, and strongly non-constant witnessed by types \( X, Y : \mathcal{U} \) with \( f(X) \neq f(Y) \). Suppose without loss of generality that \( f(X) = \text{tt} \) and \( f(Y) = \text{ff} \). For any \( A : \mathcal{U} \), define \( Z = \neg A \times X + \neg \neg A \times Y \). If \( A \), then \( \neg A \simeq 0 \) and \( \neg \neg A \simeq 1 \) (using function extensionality), so \( Z \simeq Y \) and \( f(Z) = \text{ff} \). Similarly, if \( \neg A \), then \( Z \simeq X \) and so \( f(Z) = \text{tt} \). On the other hand, \( f(Z) \) must be either \( \text{tt} \) or \( \text{ff} \) and not both. If it is \( \text{tt} \), then it is not \( \text{ff} \), and so \( \neg A \); while if it is \( \text{ff} \), then it is not \( \text{tt} \), and so \( \neg \neg A \).

In Theorem \([6]\) below we reuse Simpson’s argument to establish a similar conclusion for polymorphic functions into the booleans.
2.3 Polymorphic maps into the booleans

A function \( f : \prod_{X : \mathcal{U}} X \to 2 \) is invariant under equivalence if we have \( f_Y(e(x)) = f_X(x) \) for any equivalence \( e : X \to Y \) and point \( x : X \). Parametricity tells us that no closed such \( f \) can be provably non-constant. Consider the violation of parametricity in which there is a type \( X \) with points \( x, y : X \) with \( f_X(x) \neq f_X(y) \). If we additionally assume that there is an automorphism of \( X \) which maps \( x \) to \( y \), we arrive at the contradiction that \( f \) was not invariant under equivalence. So the mere invariance under equivalence shows that this specific violation of parametricity is impossible.

A violation of constancy across types, rather than at a specific type, is equivalent to weak excluded middle.

**Theorem 6.** Assuming function extensionality for \( 0 \)-valued functions, weak excluded middle holds if and only if there is an \( f : \prod_{X : \mathcal{U}} X \to 2 \) that is invariant under equivalence, together with \( X, Y : \mathcal{U} \) with isolated points \( x : X \) and \( y : Y \) such that \( f_X(x) \neq f_Y(y) \).

**Proof.** Assuming weak excluded middle, to show the existence of such an \( f \), let \( X : \mathcal{U} \) and \( x : X \). Then use weak excluded middle to decide \( \neg (\sum_{x', X} x \neq x') + \neg\neg (\sum_{x', X} x \neq x') \). In the left case, expressing that there are no other elements in \( X \) than \( x \), define \( f_X(x) = \texttt{f} \), and in the right case define \( f_X(x) = \texttt{t} \).

So, for example, \( f_1(*) = \texttt{f} \) and \( f_2(\texttt{t}) = \texttt{t} \), showing that we constructed a non-constant \( f \) as required.

For the other direction, without loss of generality, \( f_X(x) = \texttt{t} \) and \( f_Y(y) = \texttt{f} \). By assumption, \( X \) is equivalent to \( 1 + X' \) via an equivalence that sends \( x \) to \( \text{inl}(*) \), and similarly \( Y \) is equivalent to \( 1 + Y' \) via an equivalence that sends \( y \) to \( \text{inl}(*) \). Let \( A : \mathcal{U} \) and define

\[
Z = (1 + \neg A \times X') \times (1 + \neg\neg A \times Y'), \\
z = (\text{inl}(*), \text{inl}(*)).
\]

By the invariance under equivalence of \( f \),

1. if \( \neg A \) then \( Z \simeq X \) via an equivalence that sends \( z \) to \( x \), thus \( f_Z(z) = \texttt{t} \),

2. if \( A \) then \( Z \simeq Y \) via an equivalence that sends \( z \) to \( y \), thus \( f_Z(z) = \texttt{f} \).

The contrapositives of these two implications are respectively

\[
f_Z(z) = \texttt{t} \quad \Rightarrow \quad \neg A, \\
f_Z(z) = \texttt{f} \quad \Rightarrow \quad \neg\neg A.
\]

Hence we can decide \( \neg A \) by case analysis on the value of \( f_Z(z) \).

Provided our type theory includes propositional truncations, we can dispense with isolatedness as in Theorem 6, assuming the types \( x = x \) and \( y = y \) are propositions.

**Theorem 7.** In a type theory with propositional truncations, weak excluded middle holds if and only if there is an \( f : \prod_{X : \mathcal{U}} X \to 2 \) that is invariant under equivalence, together with \( X, Y : \mathcal{U} \) with \( x : X \) and \( y : Y \) such that \( f_X(x) \neq f_Y(y) \), where the types \( x = x \) and \( y = y \) are propositions.
Proof. Assuming weak excluded middle, the existence of such an \( f \) is shown as in the proof of Theorem 4.

For the other direction, without loss of generality, \( f_X(x) = \top \) and \( f_Y(y) = \bot \).

Let \( A : \mathcal{U} \) and define

\[
Z = \left( \sum_{x : X} \| x = x' \| \lor \neg A \right) \times \left( \sum_{y : Y} \| y = y' \| \lor \neg \neg A \right),
\]

\[
z = ((x, \text{inl}(\text{refl})), \langle y, \text{inl}(\text{refl}) \rangle).
\]

By invariance under equivalence of \( f \), we have the following.

1. If \( \neg A \) then \( Z \simeq X \) via an equivalence that sends \( z \) to \( x \), thus \( f_Z(z) = \top \).
   This works because the left factor of \( Z \) becomes equivalent to \( X \), and the right factor equivalent to \( 1 \) by the assumptions that \( y = y \) is a proposition and \( \neg A \).

2. Similarly, if \( A \) then \( Z \simeq Y \) via an equivalence that sends \( z \) to \( y \), thus \( f_Z(z) = \bot \), now using the fact that \( x = x \) is a proposition.

The contrapositives of these two implications are respectively

\[
f_Z(z) = \top \quad \rightarrow \quad \neg A,
\]

\[
f_Z(z) = \bot \quad \rightarrow \quad \neg \neg A.
\]

Hence we can decide \( \neg A \) by case analysis on the value of \( f_Z(z) \).

Remark. In a type theory with pushouts, the assumptions that \( x = x \) and \( y = y \) are propositions can be removed by using the join \((x = x') \ast \neg A\) instead of the disjunction \( \| x = x' \| \lor \neg A \) in the left factor of \( Z \), and similarly for the right factor of \( Z \). (The join \( B \ast C \) of types \( B \) and \( C \) is the pushout of \( B \) and \( C \) under \( B \times C \).) This works since joining with an empty type is the identity, while joining with a contractible type gives a contractible result; see Theorem 9 below for details.

### 2.4 Decompositions of the universe

Theorem 5 can be interpreted as saying that the universe \( \mathcal{U} \) cannot be decomposed into two disjoint inhabited parts without weak excluded middle. In fact, disjointness of the parts is not necessary. All that is needed is that both parts be proper, i.e. not the whole of \( \mathcal{U} \):

**Theorem 8.** In a type theory with propositional truncation and function extensionality for 0-valued functions, suppose we have equivalence-invariant \( P, Q : \mathcal{U} \to \mathcal{U} \) such that for all \( Z : \mathcal{U} \) we have \( P(Z) \lor Q(Z) \), and that we have types \( X \) and \( Y \) such that \( \neg P(X) \) and \( \neg Q(Y) \). Then weak excluded middle holds.

**Proof.** For any \( A : \mathcal{U} \), let \( Z = \neg A \times X + \neg \neg A \times Y \) as in Simpson’s proof. If \( A \), then \( Z \simeq Y \), and so \( \neg Q(Z) \); thus \( Q(Z) \to \neg A \). But if \( \neg A \), then \( Z \simeq X \), and so \( \neg P(Z) \); thus \( P(Z) \to \neg \neg A \). Hence the assumed \( P(Z) \lor Q(Z) \) implies \( \neg A \lor \neg \neg A \), which is equivalent to \( \neg A + \neg \neg A \) since \( \neg A \) and \( \neg \neg A \) are (by function extensionality) disjoint propositions. □
The proof of Theorem 8 can be similarly adapted.

**Theorem 9.** In a type theory with propositional truncation and 0-valued function extensionality, suppose we have \( P, Q : \prod_{X : \mathcal{U}} X \to \mathcal{U} \) that are invariant under equivalence, i.e. if \( X \simeq Y \) by an equivalence sending \( x : X \) to \( y : Y \), then \( P_X(x) \simeq P_Y(y) \), and likewise for \( Q \). Suppose also that for all \( Z : \mathcal{U} \) and \( z : Z \) we have \( P_Z(z) \lor Q_Z(z) \), and types \( X, Y \) with points \( x : X \) and \( y : Y \) such that \( \neg P_X(x) \land \neg Q_Y(y) \). Finally, suppose either that our type theory has pushouts or that the types \( x = x \) and \( y = y \) are propositions. Then weak excluded middle holds.

**Proof.** For variety in contrast to Theorem 7, suppose we have pushouts; we leave the other case to the reader. Let \( A : \mathcal{U} \) and define

\[
Z = \left( \sum_{x' : X} (x = x') \ast \neg A \right) \times \left( \sum_{y' : Y} (y = y') \ast \neg \neg A \right),
\]

\[
z = ((x, \text{inl(refl)}), (y, \text{inl(refl)})).
\]

Then if \( A, \neg A \simeq 0 \), so \( (x = x') \ast \neg A \simeq (x = x') \), and thus the first factor of \( Z \) is equivalent to \( \sum_{x : X} (x = x') \), which is a “singleton” or “based path space” and hence equivalent to \( 1 \). On the other hand (still assuming \( A \)), \( \neg \neg A \simeq 1 \), so \( (y = y') \ast \neg \neg A \simeq 1 \), and thus the right factor of \( Z \) is equivalent to \( \sum_{y : Y} 1 \) and hence to \( Y \). Thus, \( A \) implies \( Z \simeq Y \), and it is easy to check that this equivalence sends \( z \) to \( y \). Hence \( A \to \neg Q_Z(z) \), and so \( Q_Z(z) \to \neg A \). A dual argument shows that \( \neg A \to \neg P_Z(z) \) and thus \( P_Z(z) \to \neg \neg A \), so the assumption \( P_Z(z) \lor Q_Z(z) \) gives weak excluded middle.

Since a function \( \prod_{X : \mathcal{U}} X \to B \), for any fixed \( B \), is the same as a function \( (\sum_{X : \mathcal{U}} X) \to B \), we can interpret Theorem 9 as saying that the universe \( \sum_{X : \mathcal{U}} X \) of pointed types also cannot be decomposed into two proper parts without weak excluded middle.

The results discussed so far illustrate that different violations of parametricity have different proof-theoretic strength: some violations are impossible, while others imply varying amounts of excluded middle.

### 3 Classical axioms from automorphisms of the universe

There have been attempts to apply parametricity to show that the only automorphism of a universe of types is the identity. Nicolai Kraus observed in the HoTT mailing list that, assuming univalence, automorphisms of a universe \( \mathcal{U} \) living in a universe \( \mathcal{V} \) correspond to elements of the loop space \( \Omega(\mathcal{V}, \mathcal{U}) \). It can be seen that nontrivial elements of the higher loop space \( \Omega^2(\mathcal{V}, \mathcal{U}) \) imply violations of parametricity for \( \prod_{X : \mathcal{U}} X \to X \). This suggests that parametricity may play a role in automorphisms of the universe.

We are not aware of a proof that parametricity implies that the only automorphism of the universe is the identity. However, in the spirit of the above

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1The loop space \( \Omega(X, x) \) of a type \( X \) at a point \( x : X \) is the identity type \( x = x \); see [10] §2.1. 

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development, we can show that automorphisms with specific properties imply excluded middle. First, however, we observe that if we do have excluded middle then we can construct various nontrivial automorphisms of the universe.

### 3.1 Automorphisms from excluded middle

Assuming excluded middle and function extensionality, the simplest automorphism of the universe is defined as follows. Given a type $X$, we use excluded middle to decide if it is a proposition (this works because under function extensionality, being a proposition is itself a proposition). If it is, we map $X$ to $\neg\neg X$, and otherwise we map $X$ to itself. Assuming propositional extensionality — that is, that any two logically equivalent propositions are equal — and excluded middle, we have $\neg\neg X = X$ for any proposition, so this is an automorphism. (Propositional extensionality follows from propositional univalence, i.e. univalence asserted only for propositions. The converse holds at least assuming function extensionality; we do not know whether this assumption is necessary.)

The above automorphism swaps the empty type $0$ with the unit type $1$ and leaves all other types unchanged. We can try to construct other automorphisms of the universe by permuting some other subclass of types. For instance, if we have propositional truncation, then given any two non-equivalent types $A$ and $B$, excluded middle implies that for any type $X$ we have $\|X = A\| + \|X = B\| + (X \neq A \land X \neq B)$, so that the universe $U$ decomposes as a sum $U_A + U_B + U_{\neq A,B}$, where

$$U_A = \sum_{X:U} \|X = A\|, \quad U_B = \sum_{X:U} \|X = B\|, \quad U_{\neq A,B} = \sum_{X:U} (X \neq A \land X \neq B).$$

Thus, if $U_A \simeq U_B$ we can switch those two summands to produce an automorphism of $U$.

This leads to the question, when can we have $U_A \simeq U_B$ but $A \not\simeq B$? The simplest example is that both $U_0$ and $U_1$ are contractible, hence equivalent to $1$. Let us call a type $X$ rigid if $U_X$ is contractible. Excluded middle then implies that any permutation of the rigid types yields an automorphism of the universe; our first example above is the simplest case of this.

If we assume UIP, then every type is rigid, so that with UIP and excluded middle there are plenty of automorphisms of the universe. If we instead assume univalence — as we will do for the rest of this subsection — most types are not rigid. For instance, any type with two distinct isolated points, such as $N$, is not rigid, since we can swap the isolated points to give a nontrivial automorphism and hence a nontrivial equality in $U_X$. In particular, if excluded middle holds and $X$ is a set (i.e. its identity types are all propositions), then all points of $X$ are isolated. Thus, with excluded middle and univalence, no set with more than one element (i.e. with points $x, y : X$ such that $x \neq y$) is rigid.

However, there exist types that are connected (i.e. $\prod_{x,y:X} \|x = y\|$), but that are not trivial; indeed, as remarked above, $U_A$ is such a type. Moreover, if we also assume higher inductive types, then from any group $G$ that is a set we can construct a connected type $BG$ such that $\Omega(BG) \simeq G$ [7, §3.2].

This leads us to ask, when is $BG$ rigid for a set-group $G$? Since $BG$ is a 1-type (i.e. its identity types are all sets), $U_{BG}$ is a 2-type (i.e. its identity types are all 1-types). Hence it is contractible as soon as its loop space is connected and its double loop space is contractible. In general, the connected
components of \(\Omega(\mathcal{U}_{BG})\) are the outer automorphisms of \(G\) (equivalence classes of automorphisms of \(G\) modulo conjugation), while \(\Omega^2(\mathcal{U}_{BG})\) is the center of \(G\) (the subgroup of elements that commute with everything). A group with trivial outer automorphism group and trivial center is sometimes known as a complete group (though there is no apparent relation to any topological notion of completeness), and there are plenty of examples.

For instance, the symmetric group \(S_n\) is complete in this sense except when \(n = 2\) or \(6\). Thus, \(BS_n\) is rigid for \(n \notin \{2, 6\}\). (Note also that \(BS_n\) can be constructed without higher inductive types — but with univalence — as \(\mathcal{U}_{[n]}\), where \([n]\) is a finite \(n\)-element type, although of course this type only lives in a larger universe \(V\).) In particular, assuming univalence and excluded middle, there are countably infinitely many rigid types, and hence uncountably many nontrivial automorphisms of \(\mathcal{U}\) (one induced by every permutation of the types \(BS_n\) for \(n \notin \{2, 6\}\)).

This does not exhaust the potential automorphisms of \(\mathcal{U}\). For instance, if \(A\) and \(B\) are any rigid types, then \(\Omega(\mathcal{U}_{A+B}) \simeq 2\), and thus an automorphism of \(\mathcal{U}\) can swap \(A + B\) with \(C + D\) for any two other rigid types \(C\) and \(D\). There might also be rigid types that are not of the form \(BG\), and non-rigid types \(A, B\) such that \(A \not\simeq B\) but \(\mathcal{U}_A \simeq \mathcal{U}_B\). But now we will leave such questions and turn to the converse: when does an automorphism of \(\mathcal{U}\) imply excluded middle?

### 3.2 Excluded middle from automorphisms

Our first automorphism of the universe constructed from excluded middle swapped the empty type with the unit type. Conversely, any such automorphism implies excluded middle:

**Theorem 10.** Assuming propositional extensionality, if there is an automorphism of the universe that maps the unit type to the empty type, then excluded middle holds.

To prove Theorem 10 we use the following lemma.

**Lemma 11.** Excluded middle holds if and only if every proposition is logically equivalent to the negation of some type.

**Proof.** Excluded middle is equivalent to \(\neg \neg P \rightarrow P\) for all propositions \(P\), and we always have that \(\neg \neg \neg \neg A \rightarrow \neg A\) for any type \(A\). \(\square\)

Now we can prove Theorem 10. In fact, not even an embedding of \(\mathcal{U}\) into itself that maps the unit type to the empty type is possible without classical axioms:

**Lemma 12.** Assuming propositional extensionality, if there is a left-cancellable map \(f : \mathcal{U} \rightarrow \mathcal{U}\) with \(f(1) = 0\) then excluded middle holds.

**Proof.** For an arbitrary proposition \(P\), we have:

\[
\begin{align*}
P & \leftrightarrow P = 1 & \text{(by propositional extensionality)} \\
\leftrightarrow f(P) & = f(1) & \text{(because } f \text{ is left-cancellable)} \\
\leftrightarrow f(P) & = 0 & \text{(by the assumption that } f(1) = 0) \\
\leftrightarrow \neg f(P) & & \text{(by propositional extensionality).}
\end{align*}
\]
(Note that if $-f(P)$, then $f(P) \leftrightarrow 0$, so $f(P)$ is a proposition and we can apply propositional extensionality to get $f(P) = 0$.) Hence $P$ is logically equivalent to the negation of the type $f(P)$, and therefore, by Lemma 11 excluded middle holds.

This concludes the proof of Theorem 10. If we further assume univalence and propositional truncations, we can generalize this as follows. Say that a type $A$ is inhabited if the unique map $A \to 1$ is surjective. This is equivalent to giving an element of the propositional truncation $\|A\|$.

**Lemma 13.** Assuming univalence and propositional truncations, if $A$ is an inhabited type, then any proposition $P$ is logically equivalent to the identity type $(P \times A) = A$.

*Proof.* If $P$ then $P \simeq 1$, so $(P \times A) \simeq A$, and hence by univalence $(P \times A) = A$. Conversely, assume $(P \times A) = A$. Then $\|P \times A\| = \|A\| = 1$ by univalence, as $A$ is inhabited. So $\|P\| \times \|A\| = 1$, and hence $P = 1$.

Using this, we can weaken the hypothesis of Lemma 12 to the requirement that $f$ maps some inhabited type to the empty type, and get the same conclusion, at the expense of requiring univalence rather than just propositional extensionality:

**Lemma 14.** Assuming univalence and propositional truncations, if there is a left-cancellable map $f : U \to U$ with $f(A) = 0$ for some inhabited type $A$, then excluded middle holds.

*Proof.* For an arbitrary proposition $P$, we have:

$$P \leftrightarrow (P \times A) = A \quad \text{(by Lemma 13)}$$
$$\leftrightarrow f(P \times A) = f(A) \quad \text{(because $f$ is left-cancellable)}$$
$$\leftrightarrow f(P \times A) = 0 \quad \text{(by the assumption that $f(A) = 0$)}$$
$$\leftrightarrow -f(P \times A) \quad \text{(by propositional extensionality)}.$$  

Hence $P$ is logically equivalent to the negation of the type $f(P \times A)$, and therefore, by Lemma 11 excluded middle holds.

**Theorem 15.** Assuming univalence and propositional truncations, if there is an automorphism of the universe that maps some inhabited type to the empty type, then excluded middle holds.

**Corollary 16.** Assuming univalence and propositional truncations, if there is an automorphism $g : U \to U$ of the universe with $g(0) \neq 0$, then the double negation

$$\neg \neg \prod_{P : U} \text{isProp}(P) \to P + \neg P$$

of the law of excluded middle holds.

*Proof.* Let $f$ be the inverse of $g$. If $g(0)$ then $\|g(0)\|$, and because $f$ maps $g(0)$ to $0$, we conclude that excluded middle holds by Theorem 15. But the assumption $g(0) \neq 0$ is equivalent to $\neg \neg g(0)$ by propositional extensionality, and so it implies the double negation of excluded middle.

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