Quantum Jacobi fields in Hamiltonian mechanics

G. Giachetta and L. Mangiarotti

Department of Mathematics and Physics, University of Camerino, 62032 Camerino (MC), Italy

G. Sardanashvily

Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

Jacobi fields of classical solutions of a Hamiltonian mechanical system are quantized in the framework of the vertical-extended Hamiltonian formalism. Quantum Jacobi fields characterize quantum transitions between classical solutions.

I. INTRODUCTION

One of the main problems in algebraic quantum theory is to describe transitions between non-equivalent states of an algebra \( A \) of observables of a quantum system. If \( A \) is a \( C^* \)-algebra, one can consider the enveloping von Neumann algebra \( B \) of \( A \) and find an (adjoint) element \( T \) of its center such that non-equivalent states of \( A \) appear to be (generalized) eigenstates of \( T \) with different eigenvalues.\(^1\) The \( T \) exemplifies a superselection operator.\(^2\)

Superselection operators are usually associated to macroscopic (classical) observables because they commute with all elements of an algebra of a quantum system. Furthermore, one may hope that there exists an extended quantum system whose algebra includes both superselection operators and operators which transform non-equivalents states of the algebra \( A \). For instance, let us mention quantization over different classical background fields in quantum field theory. Here, we are concerned with the similar problem in (non-autonomous) Hamiltonian mechanics.

Given a classical Hamiltonian mechanical system, one can associate to its solutions and their Jacobi fields the Hermitian operators \( r \) and \( \dot{r} \) in a Hilbert space such that the eigenvalues of the mutually commutative operators of classical solutions \( r \) are values \( r(t) \) of these solutions at instant \( t \). The Jacobi field operators \( \dot{r} \) perform a transition between the eigenstates of operators of solutions \( r \). Operators of solutions play the role of superselection operators if one considers the standard quantization of linear deviations of the above Hamiltonian system over a classical solution. The Hamiltonian of these deviations depends on the operators of solutions \( r \) seen as \( c \)-numbers with respect to the deviation operators,

\footnotesize
\(^1\)Electronic mail: giachetta@campus.unicam.it
\(^2\)Electronic mail: mangiaro@camserv.unicam.it
\(^3\)Electronic mail: sard@grav.phys.msu.su
but the Jacobi field operators \( \dot{\mathbf{r}} \) acting on this Hamiltonian perform the transition between quantizations over different classical solutions.

The key point of the above quantization scheme is the particular commutation relations of operators of solutions and Jacobi field operators. They result from the Poisson bracket of a classical Hamiltonian system which is extended in order to include Jacobi fields as follows.

A generic momentum phase space of a Hamiltonian mechanical system is a Poisson fiber bundle \( \Pi \rightarrow \mathbb{R} \) over the time axis \( \mathbb{R} \).\(^{3,4}\) We restrict our consideration to mechanical systems which admit a configuration space (see Ref. [5] for opposite examples). This is a smooth fiber bundle \( Q \rightarrow \mathbb{R} \) which is provided with bundle coordinates \((t, q^k)\) where \( t \) is a fixed Cartesian coordinate on \( \mathbb{R} \). Of course, the fiber bundle \( Q \rightarrow \mathbb{R} \) is trivial, but its different trivializations \( Q \cong \mathbb{R} \times M \) correspond to different reference frames. Therefore, we deal with local fiber coordinates \((q^k)\) subject to time-dependent transformations in order to study to what extent the quantization procedure below is frame-independent. The momentum phase space of Hamiltonian mechanics on the configuration bundle \( Q \rightarrow \mathbb{R} \) is the vertical cotangent bundle \( \Pi = \mathbb{V}^*Q \) of \( Q \rightarrow \mathbb{R} \), endowed with the holonomic coordinates \((t, q^k, p_k = \dot{q}_k)\). This momentum phase space admits the canonical exterior three-form

\[
\Omega = dp_k \wedge dq^k \wedge dt,
\]

which provides \( V^*Q \) with the canonical Poisson structure\(^{4,6}\)

\[
\{f, g\} = \partial^k f \partial_k g - \partial_k f \partial^k g, \quad f, g \in C^\infty(V^*Q).
\]

A Hamiltonian system on \( V^*Q \) is defined by a Hamiltonian form

\[
H = p_k dq^k - \mathcal{H}(t, q^k, p_k)dt,
\]

which leads to the Hamilton equations

\[
d_t q^k = \partial^k \mathcal{H}, \quad d_t p_k = -\partial_k \mathcal{H}.
\]

Note that any Poisson bundle \( \Pi \rightarrow \mathbb{R} \) (i.e., the fibration \( \Pi \rightarrow \mathbb{R} \) is a symplectic foliation of \( \Pi \)) is locally isomorphic to the above case \( \Pi = V^*Q \).\(^{4,7}\)

There are different approaches to mathematical definition of Jacobi fields in Lagrangian and Hamiltonian dynamics.\(^{5,8-11}\)

To describe Jacobi fields of solutions of the Hamilton equations,\(^2\) we consider the extension of a Hamiltonian system on \( Q \) to the vertical tangent bundle \( VQ \) of \( Q \rightarrow \mathbb{R} \), viewed as a new vertical-extended configuration space.\(^4,12\) It is provided with the holonomic coordinates \((t, q^k, \dot{q}^k)\). The corresponding momentum phase space is the vertical cotangent bundle \( V^*VQ \) of \( VQ \rightarrow \mathbb{R} \). It is canonically isomorphic to the vertical tangent bundle \( VV^*Q \) of the momentum phase space \( V^*Q \rightarrow \mathbb{R} \), and is coordinated by \((t, q^k, p_k, \dot{q}^k, \dot{p}_k)\).
One obtains easily from the coordinate transformation laws that \((q^k, \dot{p}_k)\) and \((\dot{q}^k, p_k)\) are canonically conjugate pairs.

The above mentioned isomorphism \(V^*VQ \cong VV^*Q\) enables one to extend a Hamiltonian system on \(Q\) to \(VQ\) as its vertical prolongation by means of the the vertical tangent functor

\[
\partial_V = \dot{q}^k \partial_k + \dot{p}_k \partial^k.
\]

Namely, the vertical momentum phase space \(VV^*Q\) admits the canonical three-form

\[
\Omega_V = \partial_V \Omega = [dp_k \wedge dq^k + dp_k \wedge dq^\ast k] \wedge dt.
\]

It provides \(VV^*Q\) with the canonical Poisson structure

\[
\{f, g\}_V = \dot{\partial}_k f \partial_k g + \partial_k f \partial_k g - \partial_k f \dot{\partial}_k g, \quad f, g \in C^\infty(VV^*Q),
\]

where the compact notation \(\dot{\partial}_k = \partial/\partial q^k, \dot{\partial}_k = \partial/\partial \dot{p}_k\) is used. The vertical extension of the Hamiltonian form \(H\) \((\2)\) reads

\[
H_V = \dot{p}_k dq^k + p_k dq^\ast k - \mathcal{H}_V dt, \quad \mathcal{H}_V = \partial_V \mathcal{H} = (\dot{q}^k \partial_k + \dot{p}_k \partial^k)\mathcal{H}.
\]

It leads to the Hamilton equations

\[
\begin{align*}
d_t q^k &= \dot{\partial}_k \mathcal{H}_V = \partial^k \mathcal{H}, \quad d_t p_k = -\dot{\partial}_k \mathcal{H}_V = -\partial_k \mathcal{H}, \quad (5a) \\
d_t \dot{q}^k &= \partial^k \mathcal{H}_V = \partial_V \partial^k \mathcal{H}, \quad d_t \dot{p}_k = -\partial_k \mathcal{H}_V = -\partial_V \partial_k \mathcal{H}, \quad (5b)
\end{align*}
\]

where the equations (5a) are exactly the Hamilton equations (2) of the original Hamiltonian system. Given a solution \(r(t)\) of the Hamilton equations (5a), let \(\dot{r}(t)\) be a Jacobi field, i.e., \(r(t) + \varepsilon \dot{r}(t), \varepsilon \in \mathbb{R}\), is also a solution of the same Hamilton equations modulo terms of order more than one in \(\varepsilon\). Then the Jacobi field \(\dot{r}(t)\) fulfills the Hamilton equations (5b).

In particular, let \(r\) be a local solution of the Hamilton equations (5a), given by local functions

\[
r^k(t, a^j, b_j), \quad r_k(t, a^j, b_j), \quad a^j = r^j(0), \quad b_j = r_j(0).
\]

Its Jacobi field \(\dot{r}\) is a solution of the Hamilton equations (5b), represented by local functions

\[
\dot{r}^k(t, a^j, b_j, c^j, s_j), \quad \dot{r}_k(t, a^j, b_j, c^j, s_j), \quad c^j = \dot{r}^j(0), \quad s_j = \dot{r}_j(0),
\]

which fulfill the system of linear ordinary differential equations

\[
\partial_t \dot{r}^k = \dot{r}_j (\partial_j \partial^k \mathcal{H})(t, r) + \dot{r}_j (\partial^j \partial_k \mathcal{H})(t, r), \quad \partial_t \dot{r}_k = -\dot{r}_j (\partial_j \partial_k \mathcal{H})(t, r) - \dot{r}_j (\partial^j \partial_k \mathcal{H})(t, r). \quad (8)
\]

These equations can be written in the matrix form

\[
\partial_t (\dot{r}^k, \dot{r}_k) = (\dot{r}^k, \dot{r}_k) M(t, a^j, b_j).
\]
Then the Jacobi field \( \dot{r} \) \((7)\) can be written as the time-ordered matrix exponent

\[
(\dot{r}^k, \dot{r}_k) = (c^k, s_k) T \exp \left[ \int_0^t M(t', a^j, b_j) dt' \right].
\] (9)

**Remark:** The similar vertical extension of Lagrangian mechanics on the jet manifold \( J^1Q \) of \( Q \) to the jet manifold \( J^1VQ \cong VJ^1Q \) of \( VQ \) provides a description of Jacobi fields of solutions of the Euler–Lagrange equations.\(^4\) Given a Lagrangian \( L \) on the velocity phase space \( J^1Q \) coordinated by \( (t, q^k, \dot{q}^k) \), its extension to \( VJ^1Q \) reads

\[
L_V = (\dot{q}^k \partial_k + \dot{q}^k \partial_k^t)L.
\]

The same procedure is appropriate for Lagrangian and Hamiltonian field theory.\(^{13}\) It is also a preliminary step toward the SUSY extension of field theory and time-dependent mechanics.\(^{4,13–15}\)

We aim to quantize the vertical-extended Hamiltonian system when \( Q \to \mathbb{R} \) is a vector bundle and a Hamiltonian \( \mathcal{H} \) is a polynomial of coordinates and momenta. Since \( Q \to \mathbb{R} \) is a vector bundle and \( VV^*Q \to \mathbb{R} \) is so, it is a particular variant of geometric quantization which reduces to the canonical quantization of the Poisson bracket \((3)\). The physical peculiarity of quantization of a vertical-extended Hamiltonian system lies in the form of the Poisson bracket \((3)\). To illustrate this peculiarity, let us follow to naive canonical quantization and assign to quantities \( q^k, p_k, \dot{q}^k, \dot{p}_k \) the operators \( q^k, p_k, \dot{q}^k, \dot{p}_k \) which satisfy the canonical commutation relations

\[
[q^k, q^j] = [p_k, q^j] = 0,
\] (10a)

\[
[q^k, \dot{q}^j] = [\dot{p}_k, q^j] = [\dot{p}_k, \dot{q}^j] = 0,
\] (10b)

\[
[q^k, \dot{q}^j] = [p_k, \dot{p}_j] = 0, \quad [p_k, q^j] = [-i\hbar \delta^k_j \mathbf{1}]
\] (10c)

as the operator form of the Poisson bracket \((3)\). The commutation relations \((10a)\) show that operators \( q^k, p_k \) of the original Hamiltonian system mutually commute and, consequently, characterize compatible observables.

**Remark:** Recall the well-known method of second quantization in field theory, which is applied both to quantization of free fields\(^{16}\) and, especially, to quantization in the presence of a background field.\(^{17–19}\) Given a classical field equation, one considers a complete set of its solutions and associates to each solution at any instant the creation and annihilation operators or canonically conjugate operators which obey the canonical commutation relations. In the case under consideration, the Poisson bracket \((3)\) makes the operators of classical solutions commutative.
Furthermore, the commutation relations (10) show that the Jacobi field operators $\dot{p}_k$, $\dot{q}^k$ are not canonically conjugate, in contrary to the customary quantization of linear deviations of a classical system. Indeed, if $Q \to \mathbb{R}$ is a vector bundle, there is the canonical bundle isomorphism

$$VV^*Q \cong Q \times Q^* \times Q \times Q^*$$

(11)

over $\mathbb{R}$. Therefore, the vertical momentum phase space $VV^*Q$ can admit both the Poisson structure (3) and the Poisson structure

$$\{f, g\}_V = \partial^k f \partial_k g - \partial_k f \partial^k g + \dot{\partial}^k f \dot{\partial}_k g - \dot{\partial}_k f \dot{\partial}^k g.$$ 

This Poisson structure leads to the canonical commutation relations for quantum linear deviations.

To provide a unified scheme of quantization both of Jacobi fields and linear deviations of a classical Hamiltonian system, let us consider its second vertical extension to the second vertical tangent bundle $V^2Q$. The $V^2Q$ is a subbundle of the repeated vertical tangent bundle $VVQ$, given by the coordinate relations $\dot{q}^k = Q_k$ where $(t, q^k, \dot{q}^k, \ddot{q}^k)$ are holonomic coordinates on $VVQ$. Due to the canonical isomorphism $V^*V^2Q \cong V^2V^*Q$, the second vertical momentum phase space $V^2V^*Q$ is equipped with the holonomic coordinates $(t, q^k, p_k, \dot{q}^k, \ddot{q}^k, \dot{p}_k, \ddot{p}_k)$, and one can extend a Hamiltonian system from $VQ$ to $V^2Q$ by means of the second vertical tangent functor

$$\partial_{V^2} = \dot{q}^k \partial_k + \dot{p}_k \partial^k + \ddot{q}^k \dot{\partial}_k + \ddot{p}_k \dot{\partial}^k.$$ 

Namely, the second vertical momentum phase space $V^2V^*Q$ admits the canonical three form

$$\Omega_{V^2} = \partial_{V^2} \Omega_V = [dp_k \wedge dq^k + dp_k \wedge \dot{q}^k + 2dp_k \wedge \ddot{q}^k] \wedge dt.$$ 

It provides $V^2V^*Q$ with the Poisson bracket

$$\{f, g\}_{V^2} = \partial^k f \partial_k g + \partial^k f \partial_k g - \partial_k f \partial^k g - \partial_k f \partial^k g + \frac{1}{2}(\partial^k f \partial_k g - \dot{\partial}_k f \dot{\partial}^k g),$$

where $(q^k, \dot{p}_k)$, $(\dot{q}^k, p_k)$, $(Q^k = \sqrt{2q^k}, P_k = \sqrt{2\ddot{p}_k})$ are canonically conjugate pairs. This Poisson bracket leads to the following commutation relations of naive canonical quantization

$$[q^k, q^j] = [p_k, p_j] = [\dot{p}_k, q^j] = 0,$$

$$[\dot{q}^k, \dot{q}^j] = [\dot{p}_k, \ddot{q}_j] = [\ddot{p}_k, \dot{q}^j] = 0,$$

$$[q^k, \ddot{q}^j] = [p_k, \ddot{p}_j] = 0, \quad [p_k, \ddot{q}^j] = [-i\hbar \delta_k^j \mathbf{1},$$

$$[Q^k, Q^j] = [P_k, P_j] = 0, \quad [P_k, Q^j] = [-i\hbar \delta_k^j \mathbf{1}.$$
Comparing these commutation relations with the commutation relations (10a) – (10b), we can treat \( \ddot{q}, \ddot{p} \) as quantum Jacobi fields, while \( P, Q \) obey the canonical commutation relations for quantum deviations. Their common Hamiltonian reads

\[
\mathcal{H}_V = \partial_{V^2} \partial_V \mathcal{H} = (\ddot{q}^k \partial_k + \ddot{p}_k \partial^k) \mathcal{H} + \frac{1}{2} (Q^k \partial_k + P_k \partial^k)^2 \mathcal{H} = \\
\mathcal{H}_1(t, q^k, p_k, \dot{q}^k, \dot{p}_k) + \mathcal{H}_2(t, q^k, p_k, P_k, P_k).
\] (12)

If the original Hamiltonian \( \mathcal{H} \) is quadratic in coordinates \( q \) and momenta \( p \), Jacobi fields with the Hamiltonian \( \mathcal{H}_1 \) and linear deviations with the Hamiltonian \( \mathcal{H}_2 \) are quantized independently. In a general case, since linear deviations are treated as pure quantum objects, the system of classical solutions and their Jacobi fields with the Hamiltonian \( \mathcal{H}_1 \) is quantized at first. Then one quantizes linear deviations whose Hamiltonian contains superselection operators of classical solutions.

This quantization scheme can also be applied to canonical quantization of field theory seen as an instantaneous Hamiltonian system (in the spirit of Ref. [8,20]).

II. THE PREQUANTIZATION ALGEBRA

In this Section, we construct an involutive algebra, called the prequantization algebra, whose Hermitian elements can represent classical solutions of the original Hamiltonian system and quantum Jacobi fields.

Let \( C^\infty(VV^*Q) \) be a ring of smooth real functions on the vertical momentum phase space \( VV^*Q \). It is a real Lie algebra with respect to the Poisson bracket (3), called the Poisson algebra. Quantization on a Poisson manifold usually implies an assignment of a Hermitian operator \( \hat{f} \) to each element \( f \in C^\infty(VV^*Q) \) such that the Dirac condition

\[
[\hat{f}, \hat{f}'] = -i\hbar \{f, f'\}_V
\] (13)

holds. One can follow the geometric quantization procedure. Then, since \( Q \to \mathbb{R} \) is a vector bundle and since the Poisson bracket (3) does not contain a derivation with respect to time, we can easily observe that the geometric quantization of the Poisson algebra \( C^\infty(VV^*Q) \) reduces to the canonical quantization of its Poisson subalgebra \( \mathcal{A} \) of functions which are affine in fiber coordinates \( q^k, p_k, \dot{q}^k, \dot{p}_k \) on \( VV^*Q \to \mathbb{R} \). Therefore, we will simply provide this canonical quantization in a straightforward manner in Section IV.

Since the Poisson algebra \( \mathcal{A} \) can be quantized, let us use its isomorphism as a \( C^\infty(\mathbb{R}) \)-module to the module \( E(\mathbb{R}) \) of global section of the Whitney sum

\[
E = VV^*Q^* \oplus \mathbb{R}^2
\]
of the line bundle $\mathbb{R}^2 \to \mathbb{R}$ and the dual $VV^*Q^*$ of $VV^*Q$. This isomorphism is given by the assignment

$$\mathcal{A} \ni f = a_k(t)q^k + b_k(t)p_k + c_k(t)q^k + d_k(t)p_k + e(t) \mapsto a_k(t)\varphi^k + b_k(t)\chi_k + c_k(t)\dot{\varphi}^k + d_k(t)\dot{\chi}_k + e(t)I \in E(\mathbb{R}),$$

where $I$ is the fiber basis for $\mathbb{R}^2 \to \mathbb{R}$, while $\{\varphi^k\}$ are fiber bases for the dual vector bundle $Q^* \to \mathbb{R}$ coordinated by $(q_k)$, and

$$\{\chi_k = \partial q_k, \quad \varphi^k = \partial/\partial q_k, \quad \dot{\chi}_k = \partial \dot{q}_k\}$$

are holonomic fiber bases for the fiberwise multiplication $VV^*Q^* \to Q^*$. Due to the isomorphism (14), the module $E(\mathbb{R})$ inherits from $\mathcal{A}$ the structure of a Lie $C^\infty(\mathbb{R})$-algebra with local generators $\{\varphi^k, \chi_k, \varphi^k, \dot{\chi}_k, I\}$ subject to the fiberwise Lie algebra multiplication

\begin{align*}
\{\varphi^k, \varphi^j\}_E = \{\chi_k, \chi_j\}_E = \{\chi_k, \varphi^j\}_E &= 0, \quad (15a) \\
\{\varphi^k, \dot{\varphi}^j\}_E = \{\dot{\chi}_k, \chi_j\}_E = \{\dot{\chi}_k, \dot{\varphi}^j\}_E &= 0, \quad (15b) \\
\{\varphi^k, \dot{\varphi}^j\}_E = \{\chi_k, \dot{\chi}_j\}_E = \{\chi_k, \dot{\varphi}^j\}_E = \delta_k^j I, \quad (15c)
\end{align*}

where $I$ commutes with all elements. This Lie algebra structure is coordinate-independent because linear transformations of the fiber bases $\{\varphi^k\}$ for the vector bundle $Q^* \to \mathbb{R}$ and the induced holonomic transformations of the fiber bases $\{\chi_k, \varphi^k, \dot{\chi}_k\}$ for $VV^*Q^* \to Q^*$ maintain the multiplication relations (15a) - (15c). Therefore, this fiberwise multiplication makes $E \to \mathbb{R}$ a fiber bundle of Lie algebras. Its fiber $E_t$ over a point $t \in \mathbb{R}$ is a real Lie algebra given by the multiplication relations (15a) - (15c) with respect to the holonomic frame $\{\varphi^k, \chi_k, \varphi^k, \dot{\chi}_k, I\}$ at $t \in \mathbb{R}$. The Lie algebra $C^\infty(\mathbb{R})$-algebra of global sections of $E$, by construction, is isomorphic to the above mentioned Poisson subalgebra $\mathcal{A}$.

Let us consider the enveloping algebra $\mathfrak{e}_t$ of the Lie algebra $E_t$ for each $t \in \mathbb{R}$. It is the quotient of tensor algebra

$$\mathbb{R} \oplus \bigoplus_{m=1}^m (\otimes E_t)$$

with respect to the two-sided ideal generated by all elements of the form

$$e \otimes e' - e' \otimes e - \{e, e'\}_E, \quad e, e' \in E_t.$$

There is the canonical monomorphism $E_t \to \mathfrak{e}_t$ such that the basis for $E_t$ is also a basis for $\mathfrak{e}_t$. Then, identifying $E_t$ to its image in $\mathfrak{e}_t$, one can write the Lie algebra product in $E_t$ as the commutator

$$\{e, e'\}_E = [e, e'] = ee' - e'e$$

7
with respect to the product in $\mathbb{E}_t$.

Let us complexify $\mathbb{E}_t$ and $E_t$ as $\mathcal{G}_t = \mathbb{E}_t \otimes_\mathbb{R} \mathbb{C}$ and $\mathcal{G}_t = E_t \otimes_\mathbb{R} \mathbb{C}$, respectively. Since the frames $\{\varphi^k\}, \{\chi_k\}, \{\bar{\varphi}^k\}, \{\bar{\chi}_k\}$ are transformed independently due to the splitting (11), one can choose a basis

$$\{\varphi^k, \pi_k = -i\hbar \chi_k, \bar{\varphi}^k, \bar{\pi}_k = -i\hbar \bar{\chi}_k, I\}$$

(16)

for $\mathcal{G}_t$ such that

$$[\varphi^k, \varphi^j] = [\pi_k, \pi_j] = [\bar{\varphi}^k, \bar{\varphi}^j] = 0,$$

(17a)

$$[\varphi^k, \bar{\varphi}^j] = [\pi_k, \bar{\pi}_j] = [\bar{\varphi}^k, \bar{\pi}_j] = 0,$$

(17b)

$$[\varphi^k, \bar{\varphi}^j] = [\pi_k, \pi_j] = 0, \quad [\pi_k, \varphi^j] = [\bar{\pi}_k, \bar{\varphi}^j] = -i\hbar \delta^j_k I.$$  

(17c)

where $I$ commutes with all elements. Of course, the frames $\{\pi_k\}$ and $\{\bar{\pi}_k\}$ possess the same holonomic transformations as $\{\chi_k\}$ and $\{\bar{\chi}_k\}$.

The complex enveloping algebra $\mathcal{G}_t$ can be provided with an involution $\ast$. Given a basis $\{\varphi^k, \pi_k, \bar{\varphi}^k, \bar{\pi}_k, I\}$ (10) of $\mathcal{G}_t$, this operation is defined by the following conditions:

(i) the basis elements $\varphi^k, \pi_k, \bar{\varphi}^k, \bar{\pi}_k, I$ are invariant with respect to the involution $\ast$,

(ii) $\lambda^* = \bar{\lambda}$ for any complex number $\lambda$,

(iii) $(e_1 \cdots e_m)^* = e_m^* \cdots e_1^*$ for all elements $e_1, \ldots, e_m$ of $\mathcal{G}$.

The condition (iii) is well-defined because it together with the condition (i) imply that the involution $\ast$ maintains the commutation relations (17a) – (17c). It also follows that the definition of $\ast$ is coordinate-independent. The involution $\ast$ makes $\mathcal{G}_t$ an involutive algebra generated by the Hermitian basis $\{\varphi^k, \pi_k, \bar{\varphi}^k, \bar{\pi}_k, I\}$. Holonomic transformations of this basis yield automorphisms of $\mathcal{G}_t$.

The involutive algebras $\mathcal{G}_t$, $t \in \mathbb{R}$ make up the fiber bundle $\mathcal{G} \to \mathbb{R}$ of involutive algebras over $\mathbb{R}$. Global sections of this fiber bundle $\mathcal{G}$ constitute an involutive algebra $\mathcal{G}(\mathbb{R})$ over the ring $\mathcal{C}^\infty(\mathbb{R})$ of smooth complex functions on $\mathbb{R}$. There is $\mathcal{C}^\infty(\mathbb{R})$-module monomorphism

$$\mathcal{A} \ni f = a_k(t)q^k + b_k(t)p_k + c_k(t)\bar{q}^k + d_k(t)\bar{p}_k + e(t) \mapsto$$

$$a_k(t)\varphi^k + b_k(t)\pi_k + c_k(t)\bar{\varphi}^k + d_k(t)\bar{\pi}_k + e(t) I = \hat{f} \in \mathcal{G}(\mathbb{R}),$$

(18)

which assigns a Hermitian element $\hat{f}$ of the involutive algebra $\mathcal{G}(\mathbb{R})$ to each element $f$ of the Poisson subalgebra $\mathcal{A}$ in accordance with the prequantization Dirac condition (13).

The monomorphism (13) can be extended as

$$\mathcal{C}^\infty(VV^*Q) \ni \mathcal{C}^\infty(V^*Q) \ni f(t, q_j, p^j) \mapsto f(t, \varphi_j, \pi^j) = \hat{f} \in \mathcal{G}(\mathbb{R})$$

(19)

to functions on $VV^*Q$ which are the pull-back of polynomial functions $f$ of fiber coordinates $q^k, p_k$ on $V^*Q \to \mathbb{R}$. Furthermore, we add to $\mathcal{G}(\mathbb{R})$ the images $\hat{f}$ (13) of functions $f \in \mathcal{C}^\infty$.
$C^\infty(V^*Q)$ which are analytic in fiber coordinates on the vector bundle $V^*Q \to \mathbb{R}$ at points of its canonical zero section $q_j = p^j = 0$.

The algebra $\mathcal{G}(\mathbb{R})$ is provided with local Hermitian derivations

$$\partial_t \phi, \quad \partial_k \phi = \frac{i}{\hbar} [\pi_k, \phi], \quad \partial^k \phi = -\frac{i}{\hbar} [\varphi^k, \phi],$$

$$\dot{\partial}_k \phi = \frac{i}{\hbar} [\pi_k, \phi], \quad \dot{\partial}^k \phi = -\frac{i}{\hbar} [\varphi^k, \phi], \quad \phi \in \mathcal{G}(\mathbb{R}).$$

It is a desired prequantization algebra.

### III. PREQUANTIZATION OF JACOBI FIELDS

In this Section, we aim to represent classical solutions of the Hamiltonian system and quantum Jacobi fields by Hermitian elements of the prequantization algebra $\mathcal{G}(\mathbb{R})$.

Let its Hamiltonian $\mathcal{H}$ be a polynomial $\mathcal{H}(t, q^k, p_k)$ of coordinates $(q^k, p_k)$ on $V^*Q$. This property is coordinate-independent due to the linear transformations of these coordinates. It should be emphasized that, being a part of the Hamiltonian form (1), the Hamiltonian $\mathcal{H}(t, q^k, p_k)$ is not a scalar under coordinate transformations, but has the coordinate transformation law

$$\mathcal{H}'(t, q'^j, p'_j) = \mathcal{H}(t, q^k, p_k) + p'_k(t, p_k)\partial_t q'^j(t, q^k). \quad (20)$$

Let $r$ be a local solution of the classical Hamilton system, given by the local functions (6). Let they be analytic functions of $a^j, b_j$ at values $a^j = b_j = 0$ and at $t$ from an open interval $U$ of $t = 0$. We assign to this solution the Hermitian elements

$$r^k(t, \varphi^j, \pi_j), \quad r_k(t, \varphi^j, \pi_j), \quad r^k(0) = \varphi^k, \quad r_k(0) = \pi_k, \quad (21)$$

of the algebra $\mathcal{G}(U)$. They satisfy the equalities

$$\partial_t r^k = (\partial^k \mathcal{H})(t, r), \quad \partial_t r_k = -(\partial_k \mathcal{H})(t, r), \quad (22)$$

where $(\partial^k \mathcal{H})(t, r^j, r_j)$ and $(\partial_k \mathcal{H})(t, r^j, r_j)$ are also Hermitian elements of the algebra $\mathcal{G}(U)$.

Thus, one can think of (21) as being the prequantization (13) of the classical solution (3) of a Hamiltonian system. Note that classical solutions which differ from each other in the initial values have the same prequantization (21). As will be seen below, they correspond to different mean values of the operators (21).

Turn now to prequantization of Jacobi fields. Let us consider the equations

$$\partial_t \rho^k = \frac{1}{2} [\rho^j (\partial_j \partial^k \mathcal{H})(t, r) + (\partial_j \partial^k \mathcal{H})(t, r) \rho^j]^* + \quad (23a)$$

$$\rho_j (\partial^j \partial^k \mathcal{H})(t, r) + (\partial^j \partial^k \mathcal{H})(t, r) \rho_j^*],$$

$$\partial_t \rho_k = -\frac{1}{2} [\rho^j (\partial_j \partial_k \mathcal{H})(t, r) + (\partial_j \partial_k \mathcal{H})(t, r) \rho^j]^* + \quad (23b)$$

$$\rho_j (\partial^j \partial_k \mathcal{H})(t, r) + (\partial^j \partial_k \mathcal{H})(t, r) \rho_j^*].$$
for elements $\rho^k, \rho_k$ of the prequantization algebra $\mathcal{G}(U)$. They are similar to the equations (8) for classical Jacobi fields. Let

$$
\dot{r}^k(t, \varphi^j, \pi_j, \dot{\varphi}^j, \dot{\pi}_j), \quad r_k(t, \varphi^j, \pi_j, \dot{\varphi}^j, \dot{\pi}_j), \quad \dot{r}^k(0) = \varphi^k, \quad \dot{r}_k(0) = \dot{\pi}_k,
$$

be a local solution of the equations (23a) – (23b) for $t \in V \subset U$. Point out the following two properties of such a solution.

(i) Since the right-hand side of the equations (23a) – (23b) is Hermitian and the initial values are so, any solution (24) of these equations is given by Hermitian elements of the prequantization algebra.

(ii) Using the equalities (22) and (23a) – (23b), one can easily justify that, if the elements $r^k(t), r_k(t)$ (21) and the elements $\dot{r}^k(t), \dot{r}_k(t)$ (24) of $\mathcal{G}(V)$, taken at some instant $t \in V$, obey the commutation relations (17a) – (17c), then the elements

$$
r^k(t) + \partial_t r^k(t) \Delta t, \quad r_k(t) + \partial_t r_k(t) \Delta t, \quad \dot{r}^k(t) + \partial_t \dot{r}^k(t) \Delta t, \quad \dot{r}_k(t) + \partial_t \dot{r}_k(t) \Delta t, \quad \Delta t \in \mathbb{R},
$$

do so modulo terms of order more than one in $\Delta t$. It follows that the elements (21) and (24) of the prequantization algebra $\mathcal{G}(V)$ obey the commutation relations (17a) – (17c) at any $t \in V$.

Note that, for many physical models, the prequantized vertical-extended Hamiltonian

$$
\partial_t \mathcal{H}(t, \varphi^j, \pi_j, \dot{\varphi}^j, \dot{\pi}_j) = \dot{\varphi}^j (\partial_j \mathcal{H})(t, \varphi^i, \pi_i) + \dot{\pi}_j (\partial^j \mathcal{H})(t, \varphi^i, \pi_i)
$$

is Hermitian. A glance at the transformation rule (20) shows that this property is coordinate-independent. In this case, a solution (24) of the equations (23a) – (23b) obey the matrix equality

$$
\partial_t (\dot{r}^k, \dot{r}_k) = (\dot{r}^k, \dot{r}_k) M(t, \varphi^j, \pi_j)
$$

where

$$
M^k_j = (\partial_j \partial^k \mathcal{H})(t, r), \quad M^{jk} = (\partial^j \partial^k \mathcal{H})(t, r),
$$
$$
M_{jk} = (\partial_j \partial_k \mathcal{H})(t, r), \quad M^i_k = (\partial^i \partial_k \mathcal{H})(t, r).
$$

Therefore, one can write such a solution as the time-ordered matrix exponent

$$
(\dot{r}^k, \dot{r}_k) = (\varphi^k, \dot{\pi}_k) T \exp \left[ \int_0^t M(t', \varphi^j, \pi_j) dt' \right].
$$

Moreover, it is readily observed that, when $r$ (21) is the prequantization (19) of the classical solution (3), then $\dot{r}$ (26) is exactly the prequantization (18) – (19) of the Jacobi field (3) if the latter obeys the required analyticity condition.
Thus, the Hermitian elements (21), (24) provide prequantization of solutions (6) of a classical Hamiltonian system and their Jacobi fields (7).

Note that, if the prequantized vertical-extended Hamiltonian (25) is Hermitian, \( r \) (21) and \( \dot{r} \) (24) are solutions of the evolution equations

\[
\begin{align*}
\ih \partial_t r^k &= [r^k, \partial_V \mathcal{H}(t, r, \dot{r})], \\
\ih \partial_t \dot{r}^k &= [\dot{r}^k, \partial_V \mathcal{H}(t, r, \dot{r})],
\end{align*}
\]

with respect to the Hamiltonian \( \partial_V \mathcal{H}(t, r^j, \dot{r}^j, \dot{r}^j) \). Though written in a local coordinate form, these equations are well behaved under coordinate transformation due to the coordinate transformation law (20) of a Hamiltonian. Indeed, one can think of these equations as prequantization of the covariant derivative of a section of the vertical momentum phase bundle \( V^*Q \to \mathbb{R} \) with respect to the Hamiltonian connection

\[
\begin{align*}
\gamma_H &= \partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k + \partial_V \partial^k \mathcal{H} \partial_k - \partial_V \partial_k \mathcal{H} \partial^k, \\
\gamma_H [\Omega_V] &= d\mathcal{H}_V,
\end{align*}
\]

on this fiber bundle.

Remark: Any first order dynamic equation

\[
d_t q^k = \gamma^k, \quad d_t p_k = \gamma_k
\]
on the momentum phase bundle \( V^*Q \to \mathbb{R} \) can be seen as the Hamilton equations (5a) for the Hamiltonian form

\[
H_V = \dot{p}_k (dq^k - \gamma^k) - \dot{q}^k (dp_k - \gamma_k) = \dot{p}_k dq^k - \dot{q}^k dp_k - (\dot{p}_k \gamma^k - \dot{q}^k \gamma_k)
\]
on the vertical momentum phase space \( V^*Q \rightarrow \mathbb{R} \). One can apply the above prequantization construction to this Hamiltonian form, but the property (ii) of a solution of the prequantization Jacobi equations (23a) – (23b) need not take place.

**IV. QUANTUM JACOBI FIELDS**

Now we aim to construct a representation of the elements \( r(t) \) (21) and \( \dot{r}(t) \) (24) of the algebra \( \mathfrak{g}(\mathbb{R}) \), describing prequantization of classical solutions of a Hamiltonian system and their Jacobi fields, by Hermitian operators in a Hilbert space. We use the fact that, in accordance with the property (ii) of a solution of the prequantization Jacobi equations (23a) – (23b), these elements obey the commutation relations (17a) – (17b) at any instant \( t \in V \subset \mathbb{R} \).
Let $G^t$ denote the instant Lie algebra generated by the elements
\[
I, \quad r^k(t, \varphi^k, \pi_k), \quad r_k(t, \varphi^k, \pi_k), \quad \dot{r}^k(t, \varphi^k, \pi_k, \dot{\pi}_k), \quad \dot{r}_k(t, \varphi^k, \pi_k, \dot{\pi}_k) \tag{27}
\]
at $t \in V$. It has two Lie subalgebras $G^t_q$ and $G^t_p$ generated respectively by the elements \{${r^k(t), \dot{r}_k(t), I}$\} and \{${r_k(t), \dot{r}^k(t), I}$\}. These subalgebras are isomorphic to an algebra of canonical commutation relations (a CCR-algebra), but not in a canonical way. These isomorphisms are defined by a fibre metric $g$ on the vector bundle $Q \to \mathbf{R}$ as follows. Given a trivialization $Q|_V = V \times \mathbf{R}^m$ associated with local coordinates $(t, q^k)$, let $\delta_{kj}$ be the Euclidean metric on $\mathbf{R}^m$. Put
\[
{r}^q_k(t) = \frac{1}{\hbar} g_{kj} r^j(t) = \frac{1}{\hbar} r^k(t), \quad \dot{r}^q_k(t) = \frac{1}{\hbar} g_{kj} \dot{r}^j(t) = \frac{1}{\hbar} \dot{r}^k(t).
\]
Then $\{r^q_k(t), \dot{r}^q_k(t), I\}$ and $\{r^k(t), \dot{r}^k(t), I\}$ obey respectively the commutation relations
\[
[r^q_k(t), r^q_j(t)] = [\dot{r}_k(t), \dot{r}_j(t)] = 0, \quad [\dot{r}^q_k(t), r^q_j(t)] = -i \delta_{kj} I
\]
and
\[
[r^q_k(t), r^q_j(t)] = [r_k(t), r_j(t)] = 0, \quad [\dot{r}^q_k(t), \dot{r}^q_j(t)] = -i \delta_{kj} I
\]
which are exactly the commutation relations of the standard Heisenberg–Weyl CCR-algebra modelled over the finite-dimensional Euclidean space $\mathbf{R}^m$. Therefore, one can obtain representations of the instant algebras $G^t_q$ and $G^t_p$ from the well-known representations of this CCR-algebra. Of course, these representations depend on the choice of a coordinate chart on the configuration bundle $Q \to \mathbf{R}$, but they are equivalent by virtue of the well-known Stone-von Neumann uniqueness theorem.

There are different variants of a representation of the Heisenberg–Weyl CCR-algebra (see, e.g., Ref. [23]). We choose its Shrödinger representation where the operators $r^k(t)$ of the instant algebra $G^t_q$ and the operators $r_k(t)$ of the instant algebra $G^t_p$ have a complete set of eigenvectors.\(^{21}\) The representation of the Lie algebra $G^t$ is obtained as the (topological) tensor product of representations of the Lie algebras $G^t_q$ and $G^t_p$. Moreover, it suffices to construct a representation of the instant algebra $G^{t=0} = G_0$ with generators $\{\varphi^k, \pi_k, \dot{\varphi}^k, \dot{\pi}_k, I\}$ which fulfill the commutation relations (17a) -- (17c). Then one can obtain a representation of the instant algebra $G^t$, $t \in V$ as a subalgebra of the enveloping algebra $G_0$ due to the splitting
\[
Q|_V = V \times Q_{t=0} = V \times \mathbf{R}^m.
\]

A desired representation of the Lie algebra $G_0$ is defined in the Hilbert space $L^2(\mathbf{R}^{2m}, \mu_g)$ of complex functions $f(x^k, y_k)$ on $\mathbf{R}^{2m} = Q_0^* \oplus Q_0$ which are square-integrable with respect to the Gaussian measure
\[
d\mu_g = \pi^{-m} \exp \left[-\sum_k (x^k)^2 - \sum_k (y_k)^2\right] d^m x d^m y.
\]
This representation is given by the operators
\[
\varphi^k = i\hbar \frac{\partial}{\partial y_k}, \quad \pi_k = -i\hbar \frac{\partial}{\partial x_k}, \quad \varphi^k = x^k, \quad \pi_k = y_k
\] (28)
on the dense subspace of smooth functions of \(L^2(\mathbb{R}^{2m}, \mu_g)\). It is readily observed that they are Hermitian operators with respect to the Hermitian form
\[
\langle f | f' \rangle = \int f \overline{f'} \, d\mu_g
\]
on \(L^2(\mathbb{R}^{2m}, \mu_g)\). In this representation, the operators \(\varphi^k\) and \(\pi_j\) have the common eigenstates
\[
f_{a,b} = \exp \left[ \frac{i}{\hbar} (b_k x^k - a^k y_k) \right].
\] (29)

Given the representation (28) of the algebra \(G^0\), the representation of the instant algebra \(G^t\) with the generators (27) is given in the same Hilbert space \(L^2(\mathbb{R}^{2m}, \mu_g)\) by the operators
\[
r^k(t) = r^k(t, i\hbar \frac{\partial}{\partial y_k}, -i\hbar \frac{\partial}{\partial x_k}), \quad r_k(t) = r_k(t, i\hbar \frac{\partial}{\partial y_k}, -i\hbar \frac{\partial}{\partial x_k}),
\] (30a)
\[
\dot{r}^k(t) = \dot{r}^k(t, i\hbar \frac{\partial}{\partial y_k}, -i\hbar \frac{\partial}{\partial x_k}, x^k, y_k), \quad \dot{r}_k(t) = \dot{r}_k(t, i\hbar \frac{\partial}{\partial y_k}, -i\hbar \frac{\partial}{\partial x_k}, x^k, y_k).
\] (30b)
The operators (30a) also have eigenstates \(f_{a,b}\) (29). Their eigenvalues at the eigenstate \(f_{a,b}\) are exactly the values at the instant \(t\) of the classical solution (6) with the initial values \(a^k\) and \(b_k\).

One can think of the operators (30a) as being the operators of quantum Jacobi fields at the instant \(t\). Indeed, the operators
\[
R(t, \beta_k) = \exp \left\{ \frac{i}{\hbar} \beta_k \dot{r}^k(t) \right\}, \quad R(t, \alpha^k) = \exp \left\{ -\alpha^k \frac{i}{\hbar} \dot{r}_k(t) \right\}
\] (31)
of the Weyl Lie group of the Lie algebra \(G_t\) obey the commutation relations
\[
[r_j(t), R(t, \beta_k)] = \beta_j R(t, \beta_k), \quad [r_j(t), R(t, \alpha^k)] = 0,
\]
\[
[r^j(t), R(t, \alpha^k)] = \alpha^j R(t, \alpha^k), \quad [r^j(t), R(t, \beta_k)] = 0.
\]
It means that, given an eigenstate \(f_{a,b}\) of the operators \(r^j(t)\) and \(r_j(t)\) with the eigenvalues \(r^j(t, a, b)\) and \(r_j(t, a, b)\), the vectors \(R(t, \beta_k) f_{a,b}\), \(R(t, \alpha^k) f_{a,b}\) are also eigenvectors of these operators with the eigenvalues \(r^j(t, a, b) + \alpha^j\) for \(r^j(t)\) and \(r_j(t, a, b) + \beta_j, r_j(t, a, b)\) for \(r_j(t)\). In particular, the operators \(R(0, \beta_k)\) and \(R(0, \alpha^k)\) send the eigenstate \(f_{a,b}\) to the eigenstates \(f_{a,b+\beta}\) and \(f_{a+a, b}\), respectively.

Thus, one can say that the operators (31) of quantum Jacobi fields perform a transition between the classical solution of the original Hamiltonian system.
Now we can easily construct the representation of elements $r(t)$ (21) and $\dot{r}(t)$ (24) of the prequantization algebra $\mathcal{R}(\mathbb{R})$ restricted to the interval $V = (-\varepsilon, \varepsilon)$ of $\mathbb{R}$. This representation is defined in the Hilbert space $L^2((-\varepsilon, \varepsilon) \times \mathbb{R}^{2m}, \mu)$ of complex functions $f(t, x, y)$ on $(-\varepsilon, \varepsilon) \times \mathbb{R}^{2m}$ which are square-integrable with respect to the measure $d\mu = (2\varepsilon)^{-1}d\mu_\mathbb{R}dt$. The Hermitian form on this Hilbert space reads

$$\langle f|f' \rangle = \int f\overline{f'}d\mu.$$ 

The representation operators take the form (30a) – (30b) where $t$ now is a variable. In particular, the functions $f_{a,b}$ (29) are vectors of this representation, but not eigenvectors of the operators $r(t)$.

Remark: If the prequantization operators of classical solutions and Jacobi fields are defined on $\mathbb{R}$, one can construct their representation in the Hilbert space $L^2(\mathbb{R}^{2m+1}, \mu = \mu_\mathbb{R} \times \mu_\mathbb{R})$ by a choice of some measure $\mu_t$ of total mass 1 on $\mathbb{R}$.

V. QUANTUM JACOBI FIELDS OF THE HARMONIC OSCILLATOR

In order to illustrate the above construction, let us provide quantization of Jacobi fields of a one-dimensional harmonic oscillator.

Its configuration space $Q$ is the line bundle $\mathbb{R}^2 \to \mathbb{R}$ coordinated by $(t, q)$. The corresponding momentum phase space $V^*Q$ is the plane bundle $\mathbb{R}^3 \to \mathbb{R}$ coordinated by $(t, q, p)$, while the vertical momentum phase bundle is $\mathbb{R}^5 \to \mathbb{R}$ endowed with holonomic coordinates $(t, q, p, \dot{q}, \dot{p})$. The Hamiltonian form $H$ of a harmonic oscillator reads

$$H = pdq - \mathcal{H}dt, \quad \mathcal{H} = \frac{1}{2}(p^2 + \omega^2q^2),$$

where we put the oscillator mass equal 1 for brevity. The vertical extension (4) of this Hamiltonian form is

$$H_V = pd\dot{q} + \dot{p}dq - \mathcal{H}_V dt, \quad \mathcal{H}_V = \dot{p}p + \omega^2\dot{q}q.$$ 

It leads to the Hamilton equations

$$d_tq = p, \quad d_tp = -\omega^2q, \quad (33a)$$
$$d_t\dot{q} = \dot{p}, \quad d_t\dot{p} = -\omega^2\dot{q}. \quad (33b)$$

The Hamilton equations (33a) have a familiar solution

$$q(t) = q_0 \cos \omega t + p_0\omega^{-1}\sin \omega t, \quad p(t) = -q_0\omega \sin \omega t + p_0 \cos \omega t, \quad (34)$$

14
where $q_0$ and $p_0$ are the initial values. The Hamilton equations (33b) for Jacobi fields have the similar solution

$$
\dot{q}(t) = \dot{q}_0 \cos \omega t + \dot{p}_0 \omega^{-1} \sin \omega t, \quad \dot{p}(t) = -\dot{q}_0 \omega \sin \omega t + \dot{p}_0 \cos \omega t.
$$

The prequantization algebra $\mathcal{G}(\mathbb{R})$ over the ring $\mathbb{C}(\mathbb{R})$ of smooth complex functions on $\mathbb{R}$ is generated by the elements $\{\varphi, \pi, \dot{\varphi}, \dot{\pi}, I\}$ which obey the commutation relations

$$[\varphi, \pi] = [\dot{\varphi}, \dot{\pi}] = [\varphi, \dot{\varphi}] = [\pi, \dot{\pi}] = 0, \quad [\pi, \dot{\varphi}] = [\dot{\pi}, \varphi] = -i\hbar I. \tag{36}\label{36}
$$

The prequantization of the solution (34) of a classical harmonic oscillator and the Jacobi fields (35) by Hermitian elements of the algebra $\mathcal{G}(\mathbb{R})$ reads

$$
q(t) = \varphi \cos \omega t + \pi \omega^{-1} \sin \omega t, \quad p(t) = -\varphi \omega \sin \omega t + \pi \cos \omega t, \tag{37a}\label{37a}
$$

$$
\dot{q}(t) = \dot{\varphi} \cos \omega t + \dot{\pi} \omega^{-1} \sin \omega t, \quad \dot{p}(t) = -\dot{\varphi} \omega \sin \omega t + \dot{\pi} \cos \omega t. \tag{37b}\label{37b}
$$

It is readily observed that they obey the commutation relations

$$
[\dot{q}(t), \dot{p}(t)] = [\dot{q}(t), \dot{p}(t)] = [q(t), \dot{q}(t)] = [\dot{p}(t), \dot{p}(t)] = 0,
$$

$$
[p(t), \dot{q}(t)] = [\dot{p}(t), q(t)] = -i\hbar I.
$$

Note that the prequantized vertical-extended Hamiltonian

$$
\mathcal{H}_V = \dot{\pi} \pi + \omega^2 \dot{\varphi} \varphi
$$

is Hermitian.

Following the general scheme, we construct the representation of the Lie algebra (36) in the Hilbert space $L^2(\mathbb{R}^2, \mu_g)$ of complex functions $f(x, y)$ on $\mathbb{R}^2$ which are square-integrable with respect to the Gaussian measure

$$
d\mu_g = \pi^{-1} \exp[-x^2 - y^2] dx dy.
$$

This representation is given by the operators

$$
\varphi = i\hbar \partial_y, \quad \pi = -i\hbar \partial_x, \quad \dot{\varphi} = x, \quad \dot{\pi} = y.
$$

Accordingly, the elements (37a) – (37b) are represented by the operators

$$
\mathbf{q}(t) = i\hbar \partial_y \cos \omega t - i\hbar \partial_x \omega^{-1} \sin \omega t, \quad \mathbf{p}(t) = -i\hbar \partial_y \omega \sin \omega t - i\hbar \partial_x \cos \omega t, \tag{38a}\label{38a}
$$

$$
\dot{\mathbf{q}}(t) = x \cos \omega t + y \omega^{-1} \sin \omega t, \quad \dot{\mathbf{p}}(t) = -x \omega \sin \omega t + y \cos \omega t. \tag{38b}\label{38b}
$$

Considered at a given instant $t \in \mathbb{R}$, these operators act in the Hilbert space $L^2(\mathbb{R}^2, \mu_g)$, and the operators (38a) have the eigenstates

$$
f_{q_0, p_0} = \exp \left[ \frac{i}{\hbar} (p_0 x - q_0 y) \right].
$$
Their eigenvalues at the eigenstate $f_{q_0,p_0}$ are exactly the values of the classical solution (34) at the instant $t$.

Treated as operator functions of $t \in \mathbb{R}$, the operators (38a) – (38b) act in the Hilbert space $L^2(\mathbb{R}^3, \mu)$ of complex functions $f(t, x, y)$ on $\mathbb{R}^3$ which are square-integrable with respect to the Gaussian measure

$$d\mu = \pi^{-3/2} \exp[-t^2 - x^2 - y^2] dt dx dy.$$ 

It is readily observed that the Hamiltonian $H_2$ (12) for linear deviations of a harmonic oscillator coincides with the quadratic Hamiltonian (32) of this oscillator after the replacement of $q, p$ with $Q, P$. The Hamiltonians $H_1$ and $H_2$ (12) for a harmonic oscillator are independent. Therefore, the quantization of Jacobi fields of a harmonic oscillator and that of linear deviations of a harmonic oscillator (i.e., the standard quantization of a harmonic oscillator itself) are independent. It follows that quantizations of a harmonic oscillator over different classical solutions coincide with each other, and quantum Jacobi fields do not transform the states of a quantum Harmonic oscillator.

[1] J.Dixmier, $C^*$-Algebras (North-Holland, Amsterdam, 1977).

[2] S.Horuzhy, Introduction to Algebraic Quantum Field Theory, Mathematics and its Applications (Soviet Series) 19 (Kluwer Academic Publ. Group, Dordrecht, 1990).

[3] A.Hamoui and A.Lichnerowicz, J. Math. Phys. 25, 923 (1984).

[4] L.Mangiarotti and G.Sardanashvily, Gauge Mechanics (World Scientific, Singapore, 1998).

[5] J.Souriau, Structures des Systemes Dynamiques (Dunod, Paris, 1970).

[6] G.Sardanashvily, J. Math. Phys. 39, 2714 (1998).

[7] I.Vaisman, Lectures on the Geometry of Poisson Manifolds (Birkhäuser Verlag, Basel, 1994).

[8] S.Bażański, Acta Phys. Polon. B7, 305 (1976).

[9] W.Dittrich and M.Reuter, Classical and Quantum Dynamics (Springer–Verlag, Berlin, 1994).
[10] A.Nesterov, Algebras, Groups and Geometry 15, 25 (1998).

[11] H.Núñez-Yépez and A.Salas-Brito, Phys. Lett. A 275 218 (2000).

[12] G.Giachetta, L.Mangiarotti and G.Sardanashvily, J. Math. Phys. 40, 1376 (1999).

[13] L.Mangiarotti and G.Sardanashvily, Connections in Classical and Quantum Field Theory (World Scientific, Singapore, 2000).

[14] L.Mangiarotti and G.Sardanashvily, J. Math. Phys. 41, 2858 (2000).

[15] G.Sardanashvily, Int. J. Mod. Phys. A. 15, 3095 (2000).

[16] N.Bogoliubov and D.Shirkov, Introduction to the Theory of Quantized Fields (Wiley, New York, 1980).

[17] N.Birrel and P.Davies, Quantum Fields in Curved Space (Cambridge Univ. Press, Cambridge, 1982).

[18] A.Grib, S.Mamayev, and V.Mostepanenko, Vacuum Quantum Effects in Strong Fields (Friedmann Laboratory Publishing, St.Peterburg, 1994).

[19] A.Lobashov and V.Mostepanenko, in Gravity, Particles and Space-Time, edited by P.Pronin and G.Sardanashvily (World Scientific, Singapore, 1996), p. 61.

[20] C.Stephens, Ann. Phys. 181 120 (1988).

[21] J.Śniatycki, Geometric Quantization and Quantum Mechanics (Springer-Verlag, Berlin, 1980).

[22] J.Dixmier, Enveloping Algebras, Graduate Studies in Mathematics 11, American Mathematical Societi (Providence, RI, 1996).

[23] M.Floring and S.Summers, Proc. London Math. Soc. 80, 451 (2000)