Exactly Solvable Dielectrics and the Abraham-Minkowskii Controversy

Clifford Chafin
Department of Physics, North Carolina State University, Raleigh, NC 27695 *

June 20, 2014

Abstract

We present an exactly solvable model of a classical dielectric medium that gives an unambiguous local decomposition of field and charge motion and their contribution to the conserved quantities. This is done with special care to the forces that exist at surfaces, coatings and the ends of packets. The result is a mathematically simpler and more intuitive understanding of causality in media than the Brillouin and Sommerfeld theories and an understanding of the Kramers-Kronig relations in terms of dynamics and conservation laws. The Abraham-Minkowskii paradox is clarified from this point of view and the export of such notions to realistic media and metamaterials are discussed. This model can be extended to manifestly maintain these features as general nonlinear and time and space dependent changes in medium response are introduced and provides a universal description for all dielectrics.

The Abraham-Minkowskii paradox has led to a century long debate on the proper assignment of momentum to electromagnetic waves in media [16]. Some now argue that this is now resolved and both assignments are correct when boundary conditions are correctly applied [16]. The purpose of this article is not so much to dispute that position but to view this problem and general dielectric response from a different and more intuitive point of view that does support a particular decomposition. The physical questions we can ask of an electromagnetic wave interacting with a dielectric medium are 1. What are the forces at the surface? 2. What stresses exist due to a steady state beam traversing the medium? 3. What are the impulses (and transient losses) associated with a wave packet traversing the medium? 4. How many photons are involved? 5. What are causal, damping and nonlinear (and nonlocal) effects on propagation? To be able to answer such questions, hopefully easily and with physical intuition, is a sign that we truly do understand the system and are in a good position to extend the theory to more challenging configurations that can arise from nonlinearity, confined geometries and quantum effects.

In this article, we will see that the energy and momentum in the photons in a medium can be unambiguously specified and separated from that in the charge and core motions. Internal stresses can be identified with phase shifts/time lags at the charges and induced standing waves in the medium. The phase velocity turns to out be the rate at which the spatially oscillating part of the energy density advances. The group velocity will be the velocity that the total energy of the packet, including the energy of the charges, propagates which will involve conversion of charge kinetic and

*cechafin@ncsu.edu
potential energy at the ends. In the nondissipative case, the resonant limit gives diverging stress and ratio of charge energy to electromagnetic wave energy. Elastic medium response can absorb large fractions of the electromagnetic momentum that is typically returned when the field leaves the medium, but, for rapidly changing packets, intense fields or large media, the induced acoustic motions can drain energy from the waves in an unrecoverable fashion. This is just one sort of nonlinearity that media can display and will lead us to a general discussion of nonlinearity and causality in media from a physical point of view rather than the formal approaches used in the Kramers-Kronig relations.

One might wonder why one should pursue such a reformulations of the established treatments of dielectric response. One reason is the frustrating and persistent problems that pseudomomentum and pseudostress introduce into physics [13]. Even those bent on resolving these conflicts often add new elements to the confusion. By providing a system that can be exactly parsed locally in terms of conserved quantities one can clarify exactly where the real momentum is, how it is transferred and what the possibilities are for real stresses. Additionally, there is pedagogical value. One should not confuse agreement with experiment as a correct derivation. It is often joked that graduate students can derive whatever you ask them to by some means. However, none of us are beyond finding overly clever and hopeful derivations based on pleasant seeming abstraction to obtain a known or suspected result. To be able to attack a problem from multiple directions is a great comfort that our results are valid even when some arguments seem too clever and might belong more in the domain of mathematics than physics.

Many students express frustration that the treatment of dielectric response suddenly involves the tools of complex analysis, often their first introduction to it, and wonder why an explanation must require such a formal approach. Physicists and mathematicians may differ on what the more “fundamental” sort of treatment means. Most physics students would consider a microscopic description of the medium to be the most illuminating and natural since it has a clear connection to the fundamental laws of motion. However, the complexity of such treatments is generally out of reach of their time and ability and have their own approximations that leave room for doubt. Furthermore, one has the feeling that this is a simple classical problem by nature and it should not require hard tools or wild leaps from the case of single driven oscillators to have an intuitive answer to medium response.

In the following, we will introduce an exactly solvable model that shows that the energy transfer entirely consists of the electromagnetic flux where the ends of packets act as sources and sinks of this in terms of charge kinetic and elastic bond energy. This then gives a very nice consistency condition with conservation laws and the allowed dispersion relations that are related to a similar result for surface waves, a much more complicated system. Delightfully, this approach is not just intuitive but also mathematically rudimentary, brief and leads to some new insight on the ways nonlinearity must essentially enter these systems. In this model, momentum effects can be exactly separated and classified as: reflection impulses, pure electromagnetic flux, longitudinal plate momentum, radiative stress, forces from the static/velocity fields and bond distortion and conversion impulses due to the “rings up” time of the medium to create an equilibrium state with the wave.

The linearized theory of medium response gives a consistent causal theory even though some aspects of induced stress and conservation laws are obscured. How to handle nonlinearities as extra interactions on a basis built from a linearized theory is not always obvious. Formal procedures exist in the form of perturbation theory and path integrals [8] but the domain where this is valid and applicable is not always clear. As an example, the case of water surface waves is quite troublesome. A broad enough spectral distribution can lead to local fluctuations large enough to induce surface
breaking which introduce vorticity and remove energy, momentum and angular momentum from the waves and convert it to heat, nonirrotational flow and angular momentum of the gravitational source. The end-of-packet contributions of optical signals in media will be seen to necessarily give nonlinear contributions as a function of the medium’s elasticity independent of the extent to which the restoring forces are becoming nonlinear. The internal properties of a medium can be induced to change rapidly in real time so that no well defined basis of eigenstates exists. To this end, we will present a general local theory that includes damping and nonlinearity for which causality manifestly holds. This allows the introduction of any local damping law and medium response restricted only by conservation laws that are clear from the initial form of the equations.

1 Comments on Classical Constitutive Theory

Generally, discussions of theory of dielectric response begin very formally and derive the Lorentz-Drude model by introducing a complex dielectric function that gives an out-of-phase damping term. In real space this corresponds to a spatially and temporally local damping term. Often there is an appeal to the transfer functions of a localized driven damped oscillators as a strong analogy. However, the driving and damping are due to fields that are being changed by the motion of the charges and it is easy to get lost in the rather formal definitions of the “macroscopic” variables $D$ and $H$.

If we were to construct a complete basis of the system one might wonder how there can be any damping at all. The radiational degrees of freedom combined with the electron oscillations and core vibrations are all that there are. Quantum statistical mechanics has never adequately reconciled this problem and the Kubo formula is a formal approach to derive results. Classical electrodynamics is the coherent limit of quantum electrodynamics. Losses can take the form of a transition to fields and crystal and collective electronic oscillations that have no classical meaning. This suggests that the losses that we describe with the imaginary part of the dielectric constant have a purely quantum meaning. This makes the classical inclusion of damping an essentially ad hoc process but we will find a basis of temporally damping spatially harmonic solutions that are consistent with Lorentz-Drude based on a local sink of energy that is linear in the charge displacement velocity. One cannot preclude variations in this based on the temperature of the medium or transverse losses out of the medium that are neglected by the description of a signal as a coherent beam. If these can be described locally and linearly they must give similar results.

There is a long history behind the differences between $E, B, D$ and $H$ and which are viewed as fundamental [?]. Originally, $E, H$ were considered fundamental because of our use of magnets to generate fields. Now we consider $E, B$ as the fundamental microscopic fields and $D, H$ as some measure of their macroscopic response (although more general mixing of linear responses than this are possible). We will confine ourselves to the electric case.

In the case of electrostatics, we define the displacement vector $D = \epsilon_0 E + P = \epsilon E$ where $\epsilon = \epsilon_0$, the “permeability of free space” for vacuum and larger values for media. This quantity is chosen for the property that $\nabla \cdot D = \rho_f$ so that only the free charges act as sources. In general solving for the electric field and polarization of the medium would require an iterative self-consistent approach of finding the polarization including the fields from the surface and other uncanceled fields from internal bound charges. The use of $D$ allows many highly symmetric problems to be quickly solved.

---

1Nonlocality due to a sparse phonon spectrum will not be included since this will create a kind of incoherence among photons that is not describable by a classical theory.
by boundary condition constraints and special functions. We can show that the internal energy density stored in the material is $\frac{1}{2}D \cdot E$. Beyond this, its meaning is unclear. It is certainly not the local spatial average of the electric field in a medium. It might best be thought of as an intermediary step to finding the polarization as $P = (\epsilon - \epsilon_0)E$ which is a more physically meaningful quantity.

When we seek a response to a time changing field, we generally elevate the dielectric constant to a function of frequency: $P(\omega) = (\epsilon(\omega) - \epsilon_0)E$. This implies that 1. there has been a relaxation of the medium to a state where $P$ and $E$ obey a constitutive relation (and there is only one such branch for a given $\omega$) and 2. harmonic motion exists as solutions and linear combinations of these give general solutions. We know that electrostatics is not the low frequency limit of electrodynamics. Nonlinear effects at the edges of packets appear which are essential to any discussion of the Fourier transformed fields and media response when it comes to momentum conservation. Linear combinations are limited in their ability to capture this aspect of the physics. While these nonlinear effects can be locally made arbitrarily small by gentler packet gradients, the contributions are additive so cannot be neglected this way. This suggests we will ultimately need to work with purely real space fields to answer such questions thus limiting the value of working with the eigenstate basis.

The extension of the permittivity to complex values is done to consider linear responses that include dissipation. This could equivalently be done with a real response function that is just 90° out of phase from the electric field. This distinction matters because extension to the nonlinear domain is not necessarily able to be done using complex fields where real parts are later taken. We won’t be interested in such strong fields for this paper but when the nonlinearities are very small there are some simple workarounds [2].

The Kramers-Kronig relations assume that the general response function is in this linear domain [9]. The assumption of causality used in this derivation is not the relativistic one but a local one considering the polarization as response of the driving electric field and that this “response” temporally follows the driving. The motivation of this derivation seems to be the response function of a driven damped oscillator. Such an oscillator is a spatially localized system where no space-time relativistic causality problems enter i.e. there is no evolving “front” to observe. Radiation has this as an intrinsic feature and the response radiates out from each point. Furthermore, these fields are constantly getting absorbed and reemitted by radiators to which the “driver” of the response, medium or field, is ambiguous. First, we will consider a dissipationless continuum model of an electromagnetic wave in a medium which makes no such distinctions and incorporates the full degrees of freedom available to the system then consider damping effects later.

### 2 An Exactly Solvable Model

We now seek an exactly solvable model based on an idealized solid. Realistic solids are composed of many atoms with essentially fixed cores and outer electronic shells that can oscillate. The actual displacements of these electrons at microwave frequencies and higher is very small for almost all typical radiation strengths. This is why the linear regime dominates. Even the nonlinear regime generally exhibits relatively small signals and this is often treated by quantum optical methods where the equations again become linear.

The problem that we run into in modeling the field in realistic solids is that there is already a standing electric field from the cores holding the electrons in place at equilibrium. This has a complicated structure even for a crystal. On top of this is the radiation field that will be passing

\(^{2}\)Note that E and B fields must both coexist in electromagnetic waves as $\omega \rightarrow 0$. 


through with possible evanescent local contributions. The Clausius-Mossatti relation gives an expression that relates the local atomic polarizability to the mean dielectric response of the medium. As the wavelengths in the medium become shorter and approach the interparticle separation, the derivation of it becomes less convincing. To get around these complications we introduce a model made of vertical charged plates so that we have a 2D translational symmetry in our solutions. As a first model we choose the arrangement in Fig. 1.

Figure 1: Alternating positive and negative plates of mass \( dm \) and charge \( \pm dq \). The negative plates are attached to springs with constant \( d\kappa \). The positive plates are fixed by constraint.

This has a restoring force given by the springs with constant \( d\kappa \). There is some ambiguity as to the state of the static electric field between the plates. The plate pairs have alternately uniform field and zero field between them. This inhomogeneity does not seem very physical. Closer inspection reveals that displacement of the plates gives strong fringe fields. These will contribute additional forces to restoring the plates. It is however causally problematic to have spring forces and fields playing a role at distances to the plates longer than \( \lambda \). We are looking to mock up the role of restoring forces in a realistic solid. These are due to the deformation energy of deformed orbitals and the static electric fields of the cores on the electrons. For this reason we consider the modified plate arrangement as in Fig. 2.

This confines the fields between the plates so the large gap regions between them contain no field. We interpret the springs to be acting locally. The fringe fields that occur from displacements are spread among the many gaps in the material on a scale much less than \( \lambda \) of any to-be-considered radiation field. The restoring forces from displacements are due to the springs and the displacement fields. These we combine into a net effective force constant \( d\kappa \). Note that we have not taken the dielectric constant \( \epsilon \) or index of refraction \( n \) as basic here. The mass and charge densities per area, \((\sigma_m, \sigma_e)\), of the moving charges and the elastic response density per area \( K\sigma \) are the primitive microscopic descriptors of the medium. Note that the ratios of of surface densities per plate and corresponding ratios of volume densities, \((m\rho, q\rho, K\rho)\) with \( \rho \) the density of oscillators, will be the same regardless of \( d \), the plate separation since they each satisfy relations of the form \( \rho = \sigma / d \).

The advantages of this model are both its symmetry and its ability to let us separate the radiational and nonradiative internal fields in a convenient way. Because of the symmetries of the
system the values of the fields are unambiguous in between the plates. The parcel averaged net field in a medium is not obviously of much value and one has to wonder if the fields will have different values in different media with the same macroscopic dielectric properties. If so, conservation laws may be the only universal way to describe such systems. The energy is a quadratic function of the fields so using the regional average of a strongly changing field can miss the correct energy by a large amount. In our model, we never need to consider these restoring fields even when radiation is passing through the system. In this case, they are not static but restoring fields yet, assuming the vertical gaps and their separation between each other on the same plate is much smaller than the separation of the plate pairs, they still are clearly distinguishable from the radiation field in these gaps. In the case of a boosted medium the restoring field picks up magnetic components but the decomposition between these two types of fields is unchanged. This facilitates a simple consideration of the form of the final dielectric response functions under relativistic transformations.

2.1 Equations of Motion

Averaging methods have the problem that the medium is held up by a balance of electrostatic attraction and wavefunction curvature where the unshielded fields inside the electron shells become extremely large. The internal field of a disturbed solid has both radiative and velocity fields, only the former of which we tend to think of as radiation. However, when we try to compute how the energy in the system is stored and the response we do it in terms of the field strengths and it then become ambiguous if we should use the net or some nonradiationally subtracted local fields to average and if we should average over all of space or to subtract some interior region of the atoms. When it comes to the phase of the radiation it will get strongly distorted from a well defined plane wave on the scale of the atoms.

In this model system, the symmetry of the problem allows a precise decomposition $E_{\text{Net}} = E_{\text{restoring}} + E_{\text{rad}}$ and we can give equations of motion for the $E_{\text{rad}}$ henceforth to be called simply

---

3We often talk about dielectric response as linear and, in the sense that the elastic response of the charges is in the linear regime it is, but it is not meant to imply that the fields are only slightly changed from the vacuum values. The changes can be quite large and vary rapidly over short distances.
The phase distortion is replaced by a discrete jump and the singular charge surfaces so that the wavelength and frequency can be well defined between the plates even when the separation \( d \ll \lambda \). Let us generally assume this plate separation is much smaller than the wavelength \( \lambda \) of a wave in this vacuum cavity between the plates. The displacement of the charged plates from equilibrium will be labelled \( Y \) so that the polarization density is \( P = q\sigma Y \). The current density is related to \( Y \) through \( J = \dot{P} = q\sigma \dot{Y} \).

We assume that a sinusoidal wave is propagating in the \( x \)-direction with E-field polarized in the \( y \)-direction. The equations of motion are

\[
dm \cdot \ddot{Y} = de \cdot E - dk \cdot Y
\]

(1)

where \( dm = m\sigma A \), \( dk = K\sigma A \) etc. for a plates of area \( A \).

Assuming a sinusoidal solution for the electric field at a given location \( x = 0 \) (so we can ignore any \( kx \) terms for now), \( E = E_0 \cos(-\omega t) \) and \( Y = Y_0 \cos(-\omega t) \) we find

\[
Y(t) = \frac{qE_0}{K - \omega^2 m} \cos(-\omega t)
\]

(2)

The current density \( J \) is

\[
J = q\rho \dot{Y} \hat{y} = \frac{q^2 \rho \omega E_0}{K - \omega^2 m} \sin(-\omega t) \hat{y}
\]

We can now apply Maxwell’s equations to obtain

\[
\nabla^2 \mathbf{E} = \mu_0 \mathbf{j} + \frac{1}{c^2} \partial_\tau^2 \mathbf{E}
\]

and using the full space and time dependent ansatz \( \mathbf{E} = E_0 \cos(kx - \omega t)\hat{y} \) we find the dispersion relation (see Fig. 3):

\[
k^2 = \mu_0 \frac{\rho q^2 \omega^2}{K - \omega^2 m} + \frac{1}{c^2} \omega^2 = \frac{\omega^2}{c^2} \left(1 + \frac{\rho}{\epsilon_0 K} - \frac{q^2}{c^2} \right)
\]

(3)

gives a phase velocity of

\[
\nu_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1 + \mu_0 \rho q^2 c^2 (K - \omega^2 m)}}
\]

and, by the definition \( \nu_{ph} = c/n \) and index of refraction

\[
n = \sqrt{1 + \frac{\rho q^2}{\epsilon_0 K - \omega^2 m}}
\]

which is often written in terms of the plasma frequency: \( \omega^2_p = \rho q^2 / m\epsilon_0 \) so that

\[
n = \sqrt{1 + \frac{\omega^2_p}{(K/m - \omega^2)}}
\]
Figure 3: The dispersion relation.

For completeness we should find the B-field and show consistency of the fields solution. Note that since E and B are perpendicular to the direction of propagation, \( \hat{x} \), and the plates are constrained to move in this plane, we do not need to worry about \( v \times B \) forces. These forces are periodic, so cancel\(^{4}\) are additionally an order of \( c \) smaller than those of \( E \) but when we wish to consider coupling to phonon oscillations of the medium and damping we should include them. (In the case of compact packets these magnetic forces will impart net end-of-packet impulses that distinguish between the Abraham, total packet including electronic and core, and Minkowskii, local internal electromagnetic, definitions of momentum.)

First we solve for the B-field:

\[
\nabla^2 B + \mu_0 \nabla \times J = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} B
\]

Assuming \( B = B_0 \cos(kx - \omega t)\hat{z} \) we find

\[
k^2 B_0 = \mu_0 \frac{\rho q^2 \omega k}{K - \omega^2 m} E_0 + \frac{1}{c^2} \omega^2 B_0.
\]

With \( E_0 = \frac{\omega}{k} B_0 \) we get consistency with the equation of motion for \( E \) and the original Maxwell’s

\(^{4}\)We properly should consider the elastic coupling of the electrons to the cores and the cores to each other as distinct. The time scale of these oscillating forces and responses determine how much momentum is taken up by electrons alone versus electrons and cores together. This creates a distinction between a plasmon-like and acoustic modes. These vibrations will create an elastic stress on the bonds, hence the medium, despite having no net longitudinal momentum contributions. The inclusion of such modes will increase the number of branches in the dispersion relation.
equations. Collecting our solutions we obtain:

\[ E = E_0 \cos(kx - \omega t)\hat{y} \]  
\[ B = \frac{n}{c} E_0 \cos(kx - \omega t)\hat{z} \]  
\[ Y = \frac{qE_0}{K - \omega^2 m} \cos(kx - \omega t)\hat{y} \]  
\[ J = \frac{\rho q^2 \omega E_0}{K - \omega^2 m} \sin(kx - \omega t)\hat{y} \]

with the dispersion relation \( k = n\omega/c \) where \( n \) is given above.

Eqn. 3 gives a set of four branches of \( \omega \) for each \( k \). Two are the usual radiationally dominated modes that have \( n(k) \to 1 \) as \( k \to \infty \). There are two other modes that have \( \omega \to 0 \) as \( k \to 0 \). These we might be inclined to call “acoustic” modes in analogy with similar results in condensed matter theory. In contrast with the first two modes these are “elastically dominated” (to be made more clear later). If we let the charge density vanish, these modes degenerate to a set of independent oscillators with frequency \( \omega_0 = \sqrt{K/m} \). The phase and group velocity of these modes are all less than \( c \) for the acoustic branch and no singularities occur. However we notice the singular denominators in \( Y \) and \( J \) corresponding to \( \omega = \omega_0 \) the resonant cutoff of the acoustic modes where \( k \) diverges. This means a packet would usually contain negligible amounts of such modes. These give a complete set of variables that allows us to specify any state of the field and medium. Presumably we could expand any packet with vanishing fields, displacements and currents outside a compact set and observe the advance rate of the edge.

Before we move on to more subtle considerations, let us address the question of group velocity and the energy and longitudinal momentum transfer in the medium. Our basis did not allow any longitudinal momentum as the plates were constrained in this direction. This momentum is shared with the elastic modes of the medium which are often very small. However the duration of a packet can be quite long and variations in its intensity can be on periods where acoustic oscillations can be excited. Our basis requires we include these as an ad hoc modification that connects to these acoustic modes to conserve this momentum. It is this consideration that will lead to an understanding of the physical meaning of the Abraham and Minkowskii momenta for this model and will show a very nice built in self-consistency to the theory and a transparency of how momentum and energy are shared between medium and fields and a causal correction for nonlinear and varying damping effects.

### 2.2 Energy, Momentum and the Group Velocity

From this model we can now compute the local energy density and momentum directly for the progressive wave solutions. The energy has three sources

1. KE of plates
2. PE of springs
3. Energy of fields
This gives the total energy density in Eqns. 8-14

\[ E = \frac{1}{2} m \rho y^2 + \frac{1}{2} K \rho y^2 + \frac{1}{2} \left( \epsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \] (8)

\[ = E_q + E_{em} \] (9)

\[ = \frac{1}{2} E_0^2 \left( \frac{\rho q^2 K}{(K - m \omega^2)^2} \cos^2(kx - \omega t) + \frac{\rho q^2 m \omega^2}{(K - m \omega^2)^2} \sin^2(kx - \omega t) \right) \] (10)

\[ + E_0^2 \left( \epsilon_0 + \frac{\rho q^2}{2(K - m \omega^2)} \right) \cos^2(kx - \omega t) \] (11)

\[ = E_0^2 \left( \epsilon_0 + \frac{\rho q^2}{(K - m \omega^2)} \right) \cos^2(kx - \omega t) + \frac{1}{2} E_0^2 \frac{\rho q^2 m \omega^2}{(K - m \omega^2)^2} \] (12)

\[ = E_0^2 \epsilon_0 n^2 \cos^2(kx - \omega t) + \frac{1}{2} E_0^2 \frac{m \omega^2}{K} \epsilon_0 (\eta^2 - 1) \] (13)

\[ = E_{oscill} + E_{static} \] (14)

where

\[ \eta = \sqrt{1 + \frac{K \rho q^2}{(K - \omega^2 m)^2 \epsilon_0}} \] (15)

It is related to the index of refraction by

\[ \frac{\rho K \epsilon_0}{q^2} (n^2 - 1)^2 = \eta^2 - 1 \] (16)

The charge and electromagnetic energy is decomposed here to show that one can view it as static and moving oscillatory components. This will be useful in later consideration of the phase velocity.

We notice that the oscillatory components of the charge energy is exactly equal, and in phase with, the energy of the electromagnetic waves while the potential energy contribution is out of phase with them leading to a net constant energy density component.

The time averaged energy is

\[ < E > = \frac{1}{2} \epsilon_0 n^2 E_0^2 \]

Using the polarization vector defined by \( P = q \rho Y = \chi E \) we have \( \chi = \frac{\rho q^2}{(K - \omega^2 m)} \) so that

\[ < E > = \frac{1}{2} \left( \epsilon_0 + \frac{K \rho q^2}{(K - \omega^2 m)^2} \right) E_0^2 \]

\[ = \frac{1}{2} \epsilon_0 \left( 1 + \frac{K}{\epsilon_0 (K - \omega^2 m)} \chi \right) E_0^2 \]

\[ = \frac{1}{2} \epsilon_0 (1 + \chi) E_0^2 \]

\[ = \frac{1}{2} \epsilon E_0^2 \]

\[ = \frac{1}{2} < E \cdot D > \]
where \( D = \tilde{\epsilon} E \) and \( \tilde{\chi} = \frac{K}{\epsilon_0(K - \omega^2 m)} \chi \). \( D \) and \( \tilde{\chi} \) are contrived here to get the usual relations between energy and displacement.

The Poynting vector is given by

\[
S := \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}
\]

\[
= \frac{n}{c\mu_0} E_0^2 \cos^2(kx - \omega t) \hat{x}
\]

\[
= n\epsilon_0 c E_0^2 \cos^2(kx - \omega t) \hat{x}
\]

(17)

(18)

(19)

(20)

where \( n = \sqrt{1 + \chi} \). The transverse work and longitudinal force on the plates are

\[
W = \mathbf{J} \cdot \mathbf{E} = \frac{n\rho q^2 \omega}{K - \omega^2 m} K - \omega^2 m \sin(kx - \omega t) \cos(kx - \omega t) \]

\[
F = \mathbf{J} \times \mathbf{B} = \frac{n\rho q^2 \omega}{K - \omega^2 m} E_0^2 \sin(kx - \omega t) \cos(kx - \omega t) \hat{x}
\]

(21)

(22)

If we now consider time averages of these quantities we obtain:

\[
< E > = \frac{1}{2} E_0^2 \epsilon_0 \eta^2
\]

\[
< S > = \frac{1}{2} \epsilon_0 c E_0^2
\]

\[
< \mathbf{J} \cdot \mathbf{E} > = 0
\]

\[
< \mathbf{J} \times \mathbf{B} > = 0
\]

(23)

(24)

(25)

(26)

From the dispersion relation we can calculate the group velocity

\[
v_g = \frac{\partial \omega}{\partial k} = \sqrt{\frac{1 + \frac{n\rho q^2}{(K - \omega^2 m)\epsilon_0} c}{1 + \frac{n\rho q^2}{(K - \omega^2 m)^2 \epsilon_0} c}} = \frac{n}{\eta^2} c
\]

which is always less than \( c \).

One can show \[3\] that center of mass and momentum conservation dictate that a packet must move at the group velocity through a uniform medium and carry momentum density \( p = E v_g / c^2 \). This is a universal relation for any disturbance that travels as a bound compact unit and leaves the medium undisturbed after it passes. This does not preclude stresses at the ends of the packet and complicated interactions with surfaces and momentum transfer at boundaries and through antireflective films as such a packet enters the medium.

We can verify this theory in this case by noting that \( p = S / c^2 \) is the momentum density of the electromagnetic field. Consider that the plates carry no net momentum. It is not just that they are laterally constrained. The forces in the x-direction average to zero over a cycle. This means that \( p \) is all the momentum density there is to the packet. \( p = S / c^2 \) is the momentum density of the electromagnetic field and since there is no contribution from the electrons due the lateral constraints on the medium and the fact that these forces average to zero for a plane wave, this is the only momentum in the problem. This lets us immediately verify

\[
< p > = \frac{< E >}{c^2} v_g.
\]

(27)
Next we will consider details of the microscopic solution and how stress builds up in the medium and is shared between the charges and the fields.

3 Impulse and Stress

The Abraham-Minkowskii controversy revolves around the momentum in an electromagnetic wave in media. The use of momentum flux is common in treatments of elasticity and hydrodynamics but unlike mass flux (i.e. momentum density), momentum is not a locally advected quantity (See App. B). The pressure plays the role of a source and sink and incompressibility introduces the ability to transport momentum over large distances apparently acoustically. From this springs and endless list of sins and accidental successes from the use of pseudomomentum. For completeness, and to avoid such pitfalls, we seek to understand all the local forces and momentum densities in a system. These fall into several types: reflection impulses, pure electromagnetic momentum, longitudinal plate momentum (that has been excluded from the degrees of freedom in our basis), radiative stress from “hidden” standing waves, forces from the static/velocity fields and bond distortion that we could term electrostriction, and, most subtle of all, conversion impulses due to momentum absorption of the material while it “rings up” to an equilibrium state with the wave.

As mentioned before, a real understanding of the system and a universal theory of dielectrics will involve our ability to keep track of the conserved quantities as they propagate through the medium and exchange forces with it. We will see below that the stress tensor of the combination of medium and electromagnetic field together is not that illuminating but expressing them separately and including the local momentum transfers between them gives the observable impulse induced changes in the medium. The use of packets is essential in this process. Infinite wave trains can provide exactly solvable solutions but, from the standpoint of conservation laws they can harbor hidden inconsistencies. Two examples are the case of electromagnetic momentum of charged infinite plates in a magnetic field and that of the angular momentum of surface waves. In the first case, the net momentum is cancelled by fringe fields at infinity, which we might suspect since no composed collection of charges with no initial momentum can acquire any. In the second case, right moving progressive waves have a ccw angular momentum yet the surface stress on the fluid is clearly cw. Resolution appears in the long range elevations that must arise at the ends of the packet.

In the case of an electromagnetic packet entering a block, fig. 4a, we will see that, even given perfect AR coatings on the surfaces, the medium must acquire part of the momentum of the packet. There is additionally a stress that exists in the solid at the ends of a packet entirely within the block, fig. 4c. This is, however, different from the stresses that exist at the end of an infinite wavetrain, fig. 4b, or long packet that traverses the block. We will compare this with the case of partly in and partly out packet, fig. 4d, to demonstrate how the outwards forces on the surface can be present while the medium picks up a net impulse. We will also investigate the effects of finite time duration of such impulses and how energy can get drained from the packet into acoustic impulses that traverse the block.

The case of electromagnetic waves in media is complicated by the fact that there are two very distinct components: fields and charges. In acoustics and hydrodynamics, this is not the case. Each has other complications that make them more complicated but in this one respect they are simpler. Progressive surface gravity waves have a number of serious complications not the least of which that they carry mass hence have a nonzero momentum density. This is generally considered to be a higher order effect and, when it comes to (the lowest nonrelativistic component of) energy
transport, it can be ignored \cite{17,11}. They will however shed some light on how to consider energy transport and the origins of group and phase velocity from the point of view of conservation laws.

### 3.1 Comparison with Surface Waves

The small amplitude solution of a deep water surface wave is given by the Airy wave with surface profile $\nu = a \sin(kx - \omega t)$ and velocity potential function $\phi = \frac{\omega}{k}ae^{kz} \sin(kx - \omega t)$ with the following dispersion relation: $\omega = \sqrt{gk}$. From this we derive the phase and group velocities: $v_{ph} = \sqrt{g/k}$, $v_g = \frac{1}{2}v_{ph}$. The energy density per unit area is $\frac{1}{2}\rho g a^2$ where the energy is equally divided among the kinetic and potential energy. The kinetic energy comes from the small circular motion that penetrates down to a depth of $\sim \lambda$. There is a small higher order drift that we ignore here. Averaged over depth this is a uniform spatial distribution. The potential energy, however oscillates with position as $U = \frac{1}{2}\rho g a^2 \sin^2(kx - \omega t)$. This gives a total energy density of $E = \frac{1}{2}\rho g a^2 \{\frac{1}{2} + \sin^2(kx - \omega t)\}$.

The energy density is thus of the form of a constant $E_{static} = \frac{1}{2}\rho g a^2$ plus a positive advancing oscillatory function of height $E_{oscill} = \frac{1}{2}\rho g a^2$. It seems that the potential energy is transporting at velocity $v_{ph}$ and the kinetic energy is fixed. The usual interpretation (Rayleigh) in terms of packets indicate we should have a flux of $E_{net} \cdot v_g$ which, interestingly equals $E_{oscill} \cdot v_{ph}$. When the energy reaches the end of a packet, it produces new elevated regions which do not yet have corresponding kinetic motion to propagate them. These crests then drive the flows and an analogous effect happens at the back end of the packet. This gives us a picture of a packet as one where potential energy is transported at $v_{ph}$ and the ends act as sources and sinks that convert half of this back into
kinetic energy. The kinetic energy in the middle is essentially static (except for a small Stokes drift). Packet spreading must arise at the ends due to the fact that the pressure created by the surface distribution will extend down and forwards $\sim \lambda$ from the packet’s end so always leads to some stretching out of surface elevation of the packet.

Let us now apply this mode of thinking to our packets of electromagnetic energy in media. Unfortunately, it is not exactly true that the kinetic and potential energy of the charges is static from Eqn. 10. However, we do have a combination of charge and electromagnetic energy that is constant in Eqn. 13. Here we can decompose the net energy as $\mathcal{E}_{\text{net}} = \mathcal{E}_{\text{oscill}} + \mathcal{E}_{\text{static}}$. Using $< \mathcal{E}_{\text{oscill}} > v_{\text{ph}} = < S >$ we see that a similar interpretation applies here. It seems that there is a static component that is not moved by the traveling oscillatory crests. Unpublished investigation into other systems by the author suggests this is a very general feature when no net mass flux is present. Since $v_{\text{ph}}$ can be greater than $c$ this is clearly an imperfect interpretation. Closer inspection shows that there is a back and forth sloshing of the energy that contributes to the oscillating force on the charges. This oscillating local backwards moving flux of electromagnetic energy explains how the crests of energy above $\mathcal{E}_{\text{static}}$ can advance at greater than $c$ while the total local energy flux never exceeds it. At this point one might wonder why we cannot simply compute the electromagnetic energy between the plates $\mathcal{E}_{\text{em}}$ and use that it advances at the vacuum group velocity, $c$, so that $S = \mathcal{E}_{\text{em}} c \sim n^2$ rather than $n$. The plates themselves can only transfer energy via the em fields so this should be all of the energy flux. As we will see in Sec. 3.3 below, the phase shifts at the plates necessitate a microscopic backwards flux of energy for a macroscopically simple progressive wave. This generates both a stress on the walls of the medium and a reduced effective electromagnetic energy that advances at $c$.

First let us consider the impulses due to the AR coating, specifically, what is the momentum (not internal stress) of these waves in media and what momentum transfer has it exerted on the medium? We will do this two ways. First by an explicit calculation then by an analysis of the microscopic fields of a wavetrain traversing the medium.

### 3.2 Force on an AR layer

It is tempting to simply use the Lorentz force to compute the forces on the surface using the continuum or constitutive model and be done with it. Unfortunately, this will not always be sufficient for reasons already suggested. However, we can directly calculate this force of a steady beam at an AR coating. The details of the AR coating is not important, since the energy flux is constant throughout the system, the impulses at it will not be system dependent. This is in contrast with the stress on it which may be medium dependent. The difficulties with deriving an exactly solvable calculation lies in the asymmetry between the two dynamic Maxwell’s equations. The current term $J_q$ exists in the $\partial_t E$ equation. We can correct this asymmetry by imposing a magnetic monopole current $J_m$. As long as there is no net work done by these current the time averaged impulse will be unchanged. The corresponding Maxwell equations are [15].

---

5 Internal stress is not ignorable here and must be considered in any experimental result. In this model some of this force is hidden at the far off edges of the block in the fringe fields of our plates. Realistic media will generate this from the local velocity fields and changes in the orbitals due to driving. Below we just consider the fraction that arises from global conservation laws.
\[ \partial_t E = \epsilon_0 \nabla \times B - \epsilon_0^{-1} J_e \]  
\[ \partial_t B = -\nabla \times E - \epsilon_0^{-1} J_m \] 

We now match the two solutions with the same phase but discontinuously at \( x = 0 \).

\[ E = E_0 \cos(\omega t) \hat{y} \quad B = \frac{1}{c} E_0 \cos(\omega t) \hat{z} \quad x < 0 \]  
\[ E = \frac{E_0}{\sqrt{n}} \cos(\omega t) \hat{y} \quad B = \frac{\sqrt{n}}{c} E_0 \cos(\omega t) \hat{z} \quad x > 0 \] 

The discontinuities give the implied singular current densities.

\[ J_e = -\sqrt{n} - 1 \epsilon_0 \cos(\omega t) \delta(x) (-\hat{y}) \]  
\[ J_m = \left( \sqrt{n} - 1 \right) \mu_0 \cos(\omega t) \delta(x) (\hat{z}) \] 

The power can be calculated by dotting the midpoint field at \( x = 0 \) with the singular currents \( J_e \cdot E_{\text{mid}} + J_m \cdot B_{\text{mid}} = 0 \). These vanish since they give opposite contributions. The averaged (net outwards) pressure at the surface is given by

\[ P_S = S_{xx}^{AR} = < J_e \times B - J_m \times E > = -\frac{1}{4} \epsilon_0 E_0^2 \frac{(n - 1)^2}{n} \] 

This is consistent with the results of Mansuripur [12].

### 3.3 Microscopic Electromagnetic Stress

We next investigate the stress in the medium by investigating the microscopic decomposition of the fields between the plates and show it is consistent with the above result. Considering the case of fig. 4c, the medium has completed any relaxation and momentum absorption from the advancing action of the packet edge.

The fields in the gaps are made of plane waves with dispersion \( \omega = ck \) where these are presumably of a nearly monochromatic form with the same \( \omega \) as the macroscopic frequency. We can decompose the fields here into right and left moving components by first noting that, for analytic waveforms with \(|E| = c|B|\), we have a traveling wave and removing this component gives a wave moving in the opposite direction. We start by assuming we have primarily a progressive wave in the x-direction and a standard right handed coordinate system. In the following \( E \) and \( B \) are components of these vectors in the induced \((y,z)\) axes. This gives a decomposition in to right and left moving component
waves.

\[
E_\rightarrow = \frac{1}{2} E + \frac{1}{2} cB \\
E_\leftarrow = \frac{1}{2} E - \frac{1}{2} cB \\
B_\rightarrow = \frac{1}{2} B + \frac{1}{2} cE \\
B_\leftarrow = \frac{1}{2} B - \frac{1}{2} cE
\]

(37) (38) (39) (40)

This decomposition eliminates the cross terms in the stress so that we can write \( S = S_\rightarrow + S_\leftarrow \) in terms of the respective crossed right and left moving fields. The corresponding momentum densities are:

\[
< p_\rightarrow > = \frac{1}{2c^2\mu_0} \frac{(1+n)^2}{4} E'^2 \\
< p_\leftarrow > = -\frac{1}{2c^2\mu_0} \frac{(1-n)^2}{4} E'^2
\]

(41) (42)

As a consistency check we see that \( < p > = < p_\rightarrow + p_\leftarrow > = S/c^2 = \frac{1}{2}\epsilon_0 \frac{n}{2} E'^2 = \frac{1}{2}\epsilon_0 \frac{1}{2} E_0'^2 \). This is the net momentum density. The remainder of the momentum is traveling right and left canceling but still generating a stress and force on the boundaries of the material. We can think of this microscopically as having a standing wave component between each plate. The residual flux gets a magnitude of \( |c_{p_\leftarrow}| \) in each direction so an obstruction reflects with twice this magnitude. (Absorption instead of reflection gives half the following result). The stress is therefore:

\[
S_{xx} = < 2c p_\leftarrow > \\
= \frac{1}{2c^2\mu_0} \frac{(1-n)^2}{2} E'^2 = \frac{\epsilon_0 (1-n)^2}{4n} E_0'^2
\]

(43) (44)

Note that \( E_0 \) here is the field in the vacuum before the packet entered the medium and \( E' \) is the field strength inside the medium.

One could alternately view the role of the plates as inducing a phase shift in the local \( E \) and \( B \) fields to the extent that they are well approximated as sections of plane waves of wavevector \( k = c/\omega \). Defining the crest maxima as \( \theta_E \) and \( \theta_B \) we can define the phase shift \( \Delta \theta = \theta_E - \theta_B \) mod(2\( \pi \)). The shifts at each plate gives an infinitesimal shift in the energy and momentum of the waves. This gives an alternate perspective to the above point of view in terms of right and left traveling waves. We can then do a calculation of forces bases on the “phase shift density.” Realistic media have no such singular sheets of charge and this phase shift must be replaces by a stretching of phase near the charges.

3.4 The “Ring-Up” Impulse

Now let us do a careful calculation of the motion of a packet moving into a dielectric slab with perfect AR coatings in the spirit of Balazs \[3\] where the packet begins as in fig. 4a and arrives on the interior as in fig. 4d. Let the packet have length \( L \) (less than the length of the slab), area
A, and roughly monotonic frequency $\omega$. Its energy is $\frac{\omega^2}{2} (AL)E_0^2$. The packet moving into the medium advances at $v_g$ and is contracted by a factor of $v_g/c = n/\eta^2$ so, by the above relations has energy $\frac{\omega^2}{2} (AL)E_0^2 n^2/\eta^2$, where $E'$ is the maximum field intensity in the medium vs. $E_0$ as the field intensity maximum in the vacuum. The yields the relation between the field in vacuum and the medium: $E' = \frac{1}{\sqrt{n}} E_0$. Computing the net Poynting vectors of the packet inside and outside we find they are identical: $S = S' = \frac{1}{2 \mu_0} E_0^2$ so therefore the momentum densities are also equal $p' = p$. Since the packet gets contracted by $n/\eta^2$ we see that there is a deficit of momentum $\Delta p = \frac{1}{2 \mu_0 c^3} E_0^2 AL(1 - \frac{n}{\eta^2})$. This force is inwards and is returned later when the packet leaves the medium. The resulting pressure depends on the the duration of the packet $\Delta \tau = L/c$. The induced pressure is

$$P_B = \frac{\Delta p}{\Delta \tau} = \frac{1}{2 \mu_0 c^2} E_0^2 (1 - \frac{n}{\eta^2})$$  

(45)

Now consider the evolution of the packet midway entering the slab as in fig 4b. There must be a “ring-up” time to build up the standing wave and impart energy to the oscillating charges. We can estimate the lag experienced by the front of the wave. The energy density of the wave is given by Eqn. 23

$$<\mathcal{E}> = \frac{1}{2} E'^2 \left( \epsilon_0 + \frac{K \rho q^2}{(K - \omega^2 m)^2} \right)$$  

(46)

Assuming no reflection from the edge of the advancing front, we find a front velocity that advances at

$$c' = \frac{c_{\text{vacuum}}}{c_{\text{medium}}} c = \frac{n}{\eta^2} c = v_g$$  

(47)

which is always $\leq c$.

### 3.5 Net Stress and Forces

We can now summarize what we know in terms of forces on the surfaces and packet ends in terms of $P_S$ and $P_B$. As the packet enters the medium as in fig. 4b, there is a backwards pressure on the wall $-P_S$ and a forwards pressure on medium at the advancing front $+P_f$ such that the net force is $(P_f - P_S)A = P_B A$ so that $P_f = P_B + P_S$. The electromagnetic standing wave is sitting between the back wall and the front edge of the packet so that we can identify $P_{\text{ring-up}} = P_B$ as the pressure from the ring-up of the medium.

Once the packet is entirely enclosed as in fig. 4d in the medium we have equal forces on the medium on each side due the stress and ring-up and ring-down respectively: $P = P_B + P_S$. In the case of the long traversing wave in fig. 4c we have only the $P_S$ forces at the surface. This gives us a picture of the equilibration of a long beam as transferring a net impulse to the medium then introducing a net stress across the walls. For reflections at an immersed reflector, as in the Ashkin-Dziedzic experiment [1], the net impulse on it is from the momentum flux of the external beam with no $P_B$ forces at all. This explains why the Minkowskii definition of the momentum works for this experiment.
3.6 Acoustic Losses

We have already discussed how damping in media must be an essentially quantum event. However, for the case of modulated beams one can have losses into acoustic modes. In principle such modes simply give new branches of an otherwise lossless dispersion relation. When the relative occupancy of these acoustic modes are small one can view them as a kind of sink and this is the approach we take here. In practice, acoustic modes will be damped to quantum incoherent motions of the media and never recontribute to the beam.

As the front of a half infinite beam advances across the medium it imparts a force at this rapidly advancing layer. The front surface of the block experiences a nearly uniform force $P_{SA}$. The medium, of length $L$, will equilibrate to this on some time scale $\tau \sim L/v_s$, where $v_s$ is the speed of sound in the medium and $Y$ is its Young’s modulus. Let us now modulate our beam into a set of pulses with duration $T_d$ and spacing $T_s$. The impulses create acoustic stress waves of length $l \sim v_s T_d$ and amplitude $A \sim P_{SA} l/Y$. These move through the medium at $v_s$ so that if $T_s \gg T_d$ these packets are well separated. The energy density in these pulses is $E_a \sim A^2 \rho/T_d^2$ so that the power flux lost is

$$P \sim E_a v_s T_d \sim \frac{P_{SA}^2 T_d^2 T_s}{v_s \rho \ T_s}$$

This gives an acoustic loss in the beam energy that is not present in the case of a uniform beam traversing the medium and reduces the power flux from $P_{em} = \frac{1}{2} \epsilon_0 c E_0^2$ to $P_{em} - P_a$.

3.7 Quantum Considerations

The classical fields correspond to coherent states of photons. This means that we do not expect a well defined number of photons to exist in the medium. However, we can consider the case of a single photon entering the medium and ask what is the expected result. We know that a fraction of the photon must now be absorbed into energy of the oscillators. This suggests that the situation is one of a superposition of a system in the zero photon and single photon sector. The waves between the plates are represented by right and left moving plane waves obeying $\omega = ck$ and, although we get the phase shifts at each plane we need a broad distribution of frequencies, we can use that the energy density to momentum density in each right and left moving component is constant by the quantization conditions $E = \hbar \omega$ and $p = \hbar k$. The occupancy of the single photon sector is therefore given by $E_{em}/E_{net}$. The remaining momentum and energy is in some excited state of the medium in the zero photon sector being distributed among various phonon and electronic excitations.

3.8 Stress Tensor

The utility of the stress-energy tensor is in generating forces or, in the case of general relativity, sources for gravity. The bulk stress tensor in the absence of a medium is often of questionable value since when we consider gradients of stress to get forces there must be something to push against. This is the problem with idealized “photon-gas” hydrodynamic models in cosmology [14]. The local photons just travel ballistically regardless of whether or not they are in thermal equilibrium. If there is a medium present then there must be sufficient time to equilibrate and transfer those forces to the material part of the medium and for the medium to reestablish equilibrium with it (either
equilibrium with the charges as in our continuum model or thermal equilibrium). The photon-photon interaction is so weak that we probe material changes, typically with other photons, to determine the response.

The stress-energy tensor can be decomposed as

$$T^{\mu\nu} = T_{\text{em-rad}}^{\mu\nu} + T_{\text{em-restoring}}^{\mu\nu} + T_e^{\mu\nu} + T_{\text{cores}}^{\mu\nu} \quad (49)$$

where, for the progressive wave in the above coordinates,

$$T_{\text{em-rad}}^{\mu\nu} = \begin{pmatrix}
E_{\text{rad}} & p_+ + p_- & 0 & 0 \\
p_+ + p_- & 2cp_- & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

Since the responses we are looking for involve the medium itself as this electromagnetic tensor changes at the boundaries and during propagation in the bulk it is hard to see how a collective stress-energy tensor for the combined system is of much value. Nevertheless, in the local frame of the medium, each component is simple, well defined and local conservation holds.

### 3.9 Microscopic Details of Periodic Solutions

The dispersion relation $\omega(k)$ gives us (Fig. 3) how the frequency $\omega$ relates to the macroscopic wavevectors $k$. Microscopically, however, the waves in the gaps are right and left moving combinations of wave with $k' = c/\omega$. The group velocity is the mean velocity of motion of a packet and is plotted in Fig. 5. We see from the dispersion relation Eqn. 3 that there are two frequencies where the macroscopic $k$ vectors diverge, $\omega_l = \sqrt{K_m}$ and $\omega = \infty$, however now we see that only the first second one contains arbitrarily short microscopic wavelengths. The second critical frequency, $\omega_h = \sqrt{\frac{K_m}{m} + \frac{e^2}{m\epsilon_0}}$, has $k = 0$ but these are again made of microscopic waves of $k' = c/\omega_h$.

We can consider the internal stress of the medium in Fig. 6 and we see that it diverges at both these critical frequencies. This implies the internal standing waves are very large. At $\omega_l$, the resonant frequency, we see that the internal energy diverges (for finite field strength) but $E_q/E_{\text{em}}$ also does so that we can say the system is “elastically dominated.” The other divergence seems to be an artifact of averaging fields since the microscopic $k$ goes to zero and both energies are finite. If we compute it in terms of the internal field strengths the stress is actually finite. In the case of metamaterials one can have a “negative index of refraction” so that the group and phase velocities move in opposite directions. This analysis reassures us that, microscopically, the waves are all primarily moving in the same direction as the groups.

Many approaches to the microscopic analysis of the forces and stresses in a medium use a kind of dispersionless model. This allows the group and phase velocities to be proportional. Clearly they must have the wrong high frequency asymptotics. We have seen in the above analysis that the subtleties of the surface impulses, internal stresses and equilibration impulses have required a distinction between these. Furthermore, a full set of (essentially) nonlinear branches are required to have the freedom to express the full initial data of the system. The microscopic picture of electrodynamic waves near the resonances would not be apparent without the nonlinearity in the dispersion relation. For this reason, we argue that a dispersionless analysis of waves in media is essentially inconsistent.
Figure 5: Group velocity $v_g(\omega)$

Figure 6: Stress $S_{xx}(\omega)$
4 Damping and Causality in a General Medium

So far we have only studied a nondissipative case of a simple medium with one resonance. We could easily add other oscillators to get more complicated dispersion relations but they will always give singular behavior at the resonances. Damping rounds these peaks and this is generally described by an imaginary part in the dielectric function. Since we are seeking a description directly in terms of the microscopic motions instead of constitutive laws we won’t seek a direct analog of the dielectric function. We are also only interested in using real quantities since we want an easy generalization to nonlinear and time changing media.

If one specified a general real dielectric function one cannot even be sure that the imaginary part of the analytic extension will not include source terms (“negative damping”). This is certainly problematic from the point of view of justifying causal evolution. One could simply give a set of frequencies and damping rates for every function \( \cos(kx) \) and enforce a linear evolution however, on Fourier transforming it, one generally finds that the damping function is not local. This locality condition is what enforces the Kramers-Kronig relations for linear solutions. Causality arises from these relations and the various precursors arise\(^5\) as higher velocity components that the rest of the packet during the ring-up time. The usual treatments of causality involves some impressive complex analysis and one might wonder why this should be so. Below we will discuss how conservation laws and positive definiteness of the energy density enforce this easily. There is an analogous causality problem in the case of heat conduction. Since this has led to a number of erroneous attempts at modifying the heat equation including higher order modified or extended schemes that give hyperbolic solutions (in violation of the fundamental degrees of freedom of the system) we include a discussion in the appendix based on a similar microscopic analysis in app. A.

A detailed description of the waves in the gaps showed there is a discontinuity in \( B \) due to the currents at the plates but not in \( E \). For a wave with a well defined \( \omega \) the component waves in the gap region obey the relation \( \omega = ck \) exactly. The plates in this limit now just look like reflectors and energy storage devices. We saw that the distinction between traveling (progressive) and standing waves was that the relative phase of the magnetic field is different relative to the electric one. This suggested that a deviation in the magnetic field due to \( J \) in the previous model is actually introducing a partial reflected wave in the plate gap region. If we initiate a pulse at one plate it causally travels to the next, partially reflects and transmits just as we expect from scattering theory.

The continuum approach has too many advantages in its economy and analytic tools to abandon yet we must choose one that is not averaging away crucial information and retains some intuitive properties connecting it to the microscopic physics. In analogy with the heat equation, it would be nice if we could place some limits on the velocity of the energy flux through the form of the equations themselves. This would 1. manifestly conserve energy and 2. exploit that energy is a positive density function so its vanishing ensures all the other variables vanish at a packet edge.

The electromagnetic part of the dynamics are fixed by Maxwell’s equations. Since we have developed a model where \( E_{rad} \) and \( E_{restoring} \) are nicely separable that worked so well in elucidating the role of conservation laws, we assume that such a decomposition is generally valid or that there is some kind of universality that allows such a model to be equivalently constructed to every realistic medium. The field evolution equations of \( E = E_{rad} \) and \( B = B_{rad} \) are then governed by the microscopic Maxwell’s equations. The interesting parts of the equations that involve ring-up, ring-down of the medium, damping, nonlinearity, time changing media response, etc. are then all in the
details of the medium that manifest as the current and displacement functions.

\[
\partial_t E = \epsilon \nabla \times B - \epsilon_0^{-1} J_e \tag{50}
\]

\[
\partial_t B = -\nabla \times E - \epsilon_0^{-1} J_m \tag{51}
\]

where we have included a magnetic current \( J_m \) as a device to create magnetic polarization later without having to explicitly use dipoles.

Since \( \nabla \cdot E_{rad} = 0 \) we have the wave equation for \( E \) (and similarly for \( B \))

\[
\Box E = \partial_t^2 E - c^2 \nabla^2 E = \frac{1}{\epsilon_0} (\nabla \times J_m + \partial_t J_e) \tag{52}
\]

The fields then propagate at the speed of light with source and sink terms as long as \( J_m \) is not a function of \( \nabla E \) and \( J_e \) is not a function of \( \partial_t E \). This then preserves the characteristic structure of the equations.

In a realistic medium there is an elastic constant for the electron distortions, \( K_e \), and one for the cores, \( K_c \). The electron distortion is measured relative to the cores so that these are coupled. Rapid oscillations will tend to drive the electrons and leave the cores behind as in the case of internal black/gray body radiation. Bulk elastic forces, as \( P_S \), that change slowly will transfer directly to the cores and create acoustic effects. However, the ability of the medium to store and deliver energy through the currents can be quite general and still maintain causality. We can let \( J_e \) and \( J_m \) be local functions of the fields but also of elastic and other local properties of the material and even of external driving forces. The simple linear case of a dielectric gives \( J_e = \frac{q^2}{m} E - \frac{K_q}{m} Y \) where \( J_e \) itself is an independent variable so no constitutive law holds for it in terms of the fields. Rapid damping of acoustic modes will tend to lead to relaxation to the optical modes in eqn. \( \Box \), so that an apparent constitutive law is observed.

As long as the local medium dynamics are a function of the local medium effects and not derivatives of the fields, causality is manifestly preserved. The medium will tend to damp deviations from the optical modes through faster damping losses in the acoustic modes. These lead to the precursors that absorb electromagnetic energy as a packet evolves. The natural time scales associated with the medium that determine how fast energy is absorbed from a nearly monochromatic wave that is not yet in equilibrium with the medium are given by the rate of ring-up to given an equilibrium value of \( E_{eq}/E_{net} \).

As an example, consider the case of the electric field as a monochromatic beam in a dielectric medium. The source terms must conspire to give advance at the phase velocity so that

\[
\partial_t^2 E(\omega) - c^2 \nabla^2 E(\omega) = \frac{1}{\epsilon_0} \partial_t J_e \tag{53}
\]

\[
= (c^2 - v_{ph}^2(\omega)) \nabla^2 E(\omega) \tag{54}
\]

Deviations from this situation as in the case of a free space EM wave inserted in the medium with no medium response yet present, allows the fields to advance at \( c \) while the medium gradually steals energy and momentum from the beam. Such a configuration can be expressed on the basis set using the four branches of the dispersion relation in eqns. \[4,7\]. For an infinite wavetrain, the wavelengths do not change but the frequency is altered from the free space dispersion relation, the amplitude decreases, and backwards components are generated. The resulting four-wave mixing, \( \pm \omega_a(k), \pm \omega_o(k) \) gives an oscillatory change in the fraction of energy storage in the electric field.
For an advancing packet, there is a broad spectral distribution of Fourier components. The Sommerfeld theory \[5\] of precursors states that each component advances at approximately the group velocity of it. In real space, the local absorption of EM energy from a gradually sloping packet by the medium can be described as

\[
\partial_t^2 E(\omega) - c^2 \nabla^2 E(\omega) = (c^2 - \nu^2_{\text{ph}}(\omega)) \nabla^2 E(\omega) + U
\]

where \(U\) is approximately antiparallel to \(\partial_t E\).

The most general medium is described by a set of variable \(M_1, M_2 \ldots M_N\) with equations of motion \(\dot{M}_i = h_i(\{M_j\}, E, B)\). The displacement and currents \(Y, J\) are functions of the \(M_j\). Electromagnetic restoring forces are hidden in the material variables so that electrostrictive effects can arise in these equations. Additionally, external forces and sources of energy can be injected that can change the properties of the material, as in the case of electromagnetically induced transparency, or chemically driven medium changes. Therefore \(Y, J\) can be explicit functions of time as well. The momentum effects on the medium can either be computed exactly through the local Lorentz forces or by conservation laws utilizing that the Poynting vector in the medium is unchanged from that of the vacuum.

We can think of this problem as a balance of energy in the equilibrium case where the input energy from the fields \(E \cdot J\) is balanced by the output \(\nabla \cdot S\) for each charge layer. Just as in the radiation reaction case of a point particle \[19\] there are necessary nonlinearities here to give the right damping modification of the the driving force that arise from the relativistic acceleration. For larger field strengths these are unavoidable as the medium response gains nonlinear changes during ring-up and ring-down even if the elastic restoring forces stay in the linear regime.

5 Conclusions

An alternate title for this article could have been “Electrodynamics in Media without Constitutive Laws.” We have presented a simple dielectric model that is easily extendable to the case of multiple resonances and that does not depend on any averaging over localized oscillators and fields. The forces and impulses are now readily apparent by microscopic analysis in terms of the purely radiative field strengths that are not always clearly separable from the restoring fields in the averaging approaches. Some nice byproducts of this have been a complete way to track momentum in the entire system, a measure of the photon number changes as a quanta enters and traverses the media, some understanding of how acoustic losses can be generated and an easy way to see how group velocity corresponds to the microscopic phase velocity in metamaterials.

Nonlinearity and time delay are two possible routes to chaos and at higher intensities and frequencies, media response has both of these. As our ability to generate higher field intensities grow, more exotic dynamics are almost certain to arise. Additionally, as our ability to measure smaller effect improve, the necessity to describe optical impulses and other competing small effects becomes more important. In addition to the case of large bond distortion, the Abraham-Lorentz-Dirac force law ensures that nonlinearity will arise as a function of field gradients. The finite granularity scale of media gives a retardation for propagating changes to advance between separated radiators and to relax to the new mean frequency and field energy. In the case of nonlinear media one can seek a modification of the Kramers-Kronig relations. Generally, a local damping law is assumed but phonons mediate most losses and these are nonlocal objects and we are nowadays frequently concerned with confined systems where discreteness in the phonon spectra is important.
It therefore is advantageous to be able to discuss damping and dielectric response more generally than the linear response method and more intrinsically than a formal nonlinear extension of it.

Granular materials provide the canonical example of a system with no useful continuum limit. Hydrodynamics does not work for granular flows. Static packings are strongly history dependent and focus forces over many orders of magnitude at the scale of individual grains. Frictional torques produce an “indeterminacy” whereby the packing and its boundary forces are inadequate to determine the internal stress. In the case of an electrodynamic field moving in media, one has to make the assumption that the fields are changing slowly on the scale of the separation of charges. For chirped and radically shaped pulses this is not so. X-ray and gamma ray shallow angle reflection will not satisfy this condition either. Having an explicit model that can describe the fields on the scale where the wavelength becomes comparable or smaller than the granularity scale of the medium may give new ways to address such problems where continuum mechanical ideas may no longer be valid.

Precursors have been difficult to detect at a level of accuracy to test the Sommerfeld-Brillouin theory. For this reason people have been hunting for other systems e.g. surface gravity waves, for comparison. Metamaterials are highly tunable. Resistance and real time variations can be easily introduced into the shaped oscillators for microwave frequencies due to their larger scale. This would provide an excellent place to test for precursor shape and damping effects and introduce controls on the flux they generate. Just as importantly, a clear understanding of the microscopic reality may help weed out some of the more fantastical theories and flawed analogs to validate them.

There are other examples of systems with natural velocity limits that are not $c$. The speed of ocean waves, the motion of oscillations on a spring or heat transport in a solid have natural limits in the speed of sound of the underlying medium. It is unclear if there is any universality to the resolution of the dynamics here. Certainly some will seek a sweeping class of equations based on symmetry that ignore the underlying details of the dynamics. The results here and similar ones not included here are suggestive that this is a mistake. Ultimately everything comes from microscopic physics and shortcuts that seem initially successful or are only successful in particular cases can contribute to long lasting confusion.

The quantum use of quasiparticles is widespread yet the form of their dispersion relations imply that they must face such similar constraints. The best form for such corrections is an important challenge. High temperature superconductivity comes from a strongly interacting system utilizing very shallow hole filling bands. It seems that on can have extra energy carried with electrons in the interactions to give larger effective mass but how $m^* < m_e$ can arise seems mysterious. This is only possible due to the fact that the electron-electron interaction is greater at the Brillouin zones so that excitations can actually reduce the net interaction. It would be interesting to see if such a composite model of electron waves modeled on the example given for electromagnetic waves here might lead to new insight on the forces and transport in conducting media.

## A Discrete Walkers

Consider a 1D lattice of points separated by $d$ that each contain $n(i)$ walkers. Every discrete time increment $\tau$ the system is updated and half of the walkers move left and half right one step. (We assume $n$ is always so large that problems posed by odd values do not make important contributions.) There is no a priori reason $n(i)$ should be quasicontinuous but, assuming it is, we
can modify the finite difference equation into an approximate continuum one:

\begin{align}
n(i, k + 1) &= \frac{1}{2} (n(i - 1, k) + n(i + 1, k)) \quad \Rightarrow \quad (56)
n(x, t + \Delta t) &= \frac{1}{2} (n(x - \Delta x, t) + n(x + \Delta x, t)) \quad \Rightarrow \quad (57)
nu(x, t) &\approx \frac{1}{2} \left( \frac{\Delta x^2}{\Delta t} \right) n''(x, t) = Cn''(x, t) = \frac{d}{dx} j(x, t) \quad (58)
\end{align}

where \( j = C \frac{dx}{dt} n \). In the last line we reformulated this into a form reminiscent of \( q = \nabla \cdot j \) where \( j \) is the current of the scalar quantity \( q \).

First note that the discreteness of the system places bounds on the possible gradients of \( n \). Since \( n \) is a positive definite quantity \( j/C = \frac{dx}{dt} n \leq n/\Delta x \). Initial data should remain less than this and the evolved function should preserve this condition.

From here we could attempt to improve the accuracy of the equation by using higher order terms in \( t \) and \( x \) derivatives. Higher order terms in \( t \) violate the sufficiency of \( n(x) \) as initial data. We could attempt to remove these terms by some iteration of lower order approximations to get self consistency (as it done with the radiation reaction (see LL)). It is doubtful if any finite number of spatial derivatives would accomplish the goal of keeping the evolution of the edge bounded by the velocity \( u = \Delta x/\Delta t = 2C/\Delta x \), as is true for the finite difference equation, and any localized initial data. Infinite order equations are unwieldy and can be argued to be disguised nonlocal equations.

To look at this from a different perspective consider the current \( j \) and what it tells us about the velocity of the propagation. We can decompose this current in to right and left moving components

\( j(x) = j_+(x) + j_-(x) \approx (-C/\Delta x)n(x - \frac{1}{2}\Delta x) - (-C/\Delta x)n(x + \frac{1}{2}\Delta x) \). We know from above that the velocity of the flux in each direction is \( v_\pm = \pm \frac{\Delta x}{\Delta t} = \pm 2C/\Delta x = j_\pm/n_\pm \) where \( n_\pm = \frac{1}{2}n \) is just the fraction of \( n \) from a site that moves right or left respectively. By the above physicality condition we have \( |j/n| \leq C/\Delta x \). If the solution approaches this we know we have moved into unphysical territory i.e. \( n(i) \) is “very large” but an adjacent value is now approaching zero. This means the flux from one side is effectively terminated and increasing \( n'' \) does not increase the flux beyond \( n_\pm u \). We can best modify Fick’s law by introducing a scale dependent diffusion constant that drives the diffusion to zero as a function of \( (j/n)^2 - (C/\Delta x)^2 \).

\[ \hat{n}(x, t) = \frac{d}{dx}(\hat{C}n'(x, t)) \quad (59) \]

where \( \hat{C} = Cf((C/\Delta x)^2 - (j/n)^2) \). \( f(s) \) can be as simple as a step function \( \Theta(s) \) or a smoothed version that still vanishes at \( s = 0 \). This introduces some essential nonanalyticity and nonlinearity in the problem. This is the price we have to pay to preserve the possibility of having a continuous and smooth set of descriptors to evolve the system with the correct dynamical degrees of freedom. In the case \( f = \theta \) we can immediately see that causality is preserved because the evolution is just the heat equation until the local velocity reaches \( u \) when it halts until the nearby function changes enough so that it can again evolve at a lower speed. In practice this state is never reached if \( \theta \) is rounded over to a smooth function. A final note, we see energy is conserved both globally and locally by Gauss’s law.

It still remains to find a microscopic derivation of the function \( f(s) \) or even to see what the form of corrections to this model might be. This will have to remain to future work. To experimentally measure this quantity, using the edge of a packet of heat seems extraordinarily difficult. One could
probe rapidly brought together surfaces with large temperature differences. For diffusion, one could look for rapidly moving outliers and large number averages.

Nonlinear equations follow as descriptions of truly linear phenomena when we suppress dimensionality e.g. the Schrödinger equation and HF, DFT and the GP equations. In the case of quantum field theory, nonlinearities show up in the running of the coupling constant as a result of suppressing scales that correspond to energies beyond what is physically relevant for the given problem. Here we have shown that nonlinearities follow from neglected (and apparently intractable to the analytic tools of smoothing and taking limits) small scale and discrete physics when we need to work with a best fit continuum model.

It is interesting to note that equations like the porous medium equation \( \dot{n} = \nabla \cdot (n^m \nabla n) \) with \( m \geq 1 \) give a set of weak solutions (essentially continuous solutions with smoothness discontinuities) that have finite velocity of front propagation. We can see that these also have a vanishing current where \( n \) approaches zero. In contrast we have sought a solution where the velocity of propagation is bounded by a physically fundamental quantity. If we were to extend this to heat transport in a solid we note that the phonons are bounded by the velocity of sound. As long as we are at temperatures where the relevant phonons are from the linear part of the dispersion relation, we can expect such an equation to be relevant. The gas of heat flow in gases is more complicated. The particles can have a very broad distribution of velocities. This above result would have to use contribution over bounds from all these velocities and the sharp edge we see above would become blurred. Even though sound speed is closely related to the thermal mean velocity the “thermal edge” of the heat distribution would creep out beyond it. The speed of light bound will be established once a relativistic distribution of particle speeds is used.

### B The Stress Tensor and Momentum

The stress-energy tensor of a system \( T^{\mu\nu} \) is often described as the flux of \( p^\mu \) momentum across the \( \nu \)th hypersurface composed of normals to the vector \( x^\nu \). This is a common approach in many texts on continuum mechanics. Since we are interested in the microscopic interplay between material and radiation in a way that treats neither as the “driver” or “responder,” as is generally done in derivations of the Kramers-Kronig relations, it is good to pause and look at some specific cases from a microscopic point of view.

In the case of a gas, the above is a good model. All the pressures and stresses in the medium are the results of kinetics. As such, these are the direct result of microscopic transfer of momentum. The case of solids and liquids are different. There is a kinetic component but there is also potential energy in the bonds between atoms that can be strained and do the work of transferring forces across the parcels. This may seem academic but when we look at the some formulations of momentum conservation in continua, we can start to consider the naively implied “momentum flux” from such a \( T^{\mu\nu} \) as though it is a conserved quantity. Specifically \( \partial_i T^{ij} = f_i = -\partial_j P \) implies a globally conserved momentum (since \( \int \partial_j P dV = 0 \)) however the local conservation law \( \partial_i (\rho v^i) + v^j \partial_j (\rho v^i) = 0 \) (or \( \dot{p} + \vec{v} \cdot \nabla \vec{p} = 0 \)) is not generally true even when external forces are zero.\(^6\) This is what we would expect for a conserved momentum flux. Instead we see that \( \nabla P \) plays the role of a source and sink

\(^6\)Note that for constant density fluids \( \partial_i v^i = 0 \) we can rewrite this as \( \partial_i (\rho v^i) + \partial_j (\rho v^i v^j) = 0 = \partial_i \Pi^{ij} \) for \( \Pi^{ij} = \rho v^i v^j \). This is the correct Eulerian momentum flux in the lab frame assuming any microscopic motions can be neglected. We generally refer to these fluids as “incompressible” but one can have fluids with varying density e.g. from varying solute concentration where each parcel is effectively incompressible so “constant density” is more accurate here.
term for it. Momentum is exactly conserved and, absent any electromagnetic contribution, is the same as the mass density flux. The subtleties with momentum flux is that it is not locally conserved and that one often tries to write forces as momentum fluxes that may partially cancel on a given material parcel. During a microscopic evaluation of the sample one only sees bond strain and net motion. The net result is the same net force but momentum flux in this sense has no microscopic meaning.

If we consider a gas we can locally keep track of the momentum of the system and how it shifts at each collision. In fact we can give a locally well defined right and left moving momentum flux. In the case of condensed matter, this is not the case and the elastic forces driven by the potential energy must be included in the microscopic description of the stress. Inspecting a stressed zero temperature solid microscopically will indicate the strained bonds but no momentum is being transferred.

When the fields exist in the presence of media we can lump the net local stress into a stress tensor. It is not clear how valuable this is since the waves might propagate through the medium for some distance before it relaxes. We generally try to use such stress tensors to take gradients to derive local forces. These must be the net force on particles and fields. If this induces reflection and packets do not evolve as we expect from linear combination of the states in a dispersion law it is not clear how much each one responds to this.

When electromagnetic fields are being included, we need to include its momentum. How to decouple these from media is not entirely obvious for realistic media. In the upcoming solvable case it is clear and so is suggestive as to how to decompose the fields into nonradiative and radiational components. Several respectable sources have promoted some confusion about the nature of the “canonical” momentum of a particle in an electromagnetic wave [10] and this ends up playing a role in some discussions of the meaning of the Abraham and Minkowskii momentum in media [4]. Since it is easy to clarify we shall do it here.

The canonical momentum of a particle is stated as \( p^i = mv^i + qA^i \). There are attempts to use this and other pseudomenta to give true forces at gradients or boundaries. Some discussion has taken place as to when this is valid [13]. Let us first consider the gauge problem and this proposed definition. Clearly it is not invariant. We know that these problems go away in the case of the Schrödinger equation because the phase of the wave transforms to cancel any vector potential gauge change. In this case we have \( mj^i = p^i = \rho(\hbar \partial_i \psi + eA_i) \). The gauge transformation of \( A_i \) is now compensated by a stretching in the phase, \( \varphi \), of the wavefunction. By Ehrenfest’s Theorem, a localized packet moving with with velocity \( v \) continues along the Lorentz force law path. To get the original canonical relation to give the true momentum we need to choose and maintain a gauge choice of \( A_i = 0 \) at each particle. The wavefunction packet constitutes a model for the classical charged particle to the extent the (in this case, irremovable) self-forces can be neglected. Since \( mv \) is a gauge invariant quantity, the classical canonical momentum really only coincides with this gauge restricted case. In this model, \( v \) is a function of the phase gradient of the packet and the particular value of \( A^i \) chosen for the given magnetic field. Any modification of this vector potential by a gauge choice \( A^i \rightarrow A^i + \partial^i \Phi \) should leave \( j^i \) unchanged and give no new forces on the particle. This implies that the only classical “canonical” transformation of the momentum that should be allowed obey the same local constraints i.e. \( \partial^i \Phi = 0 \) at each charge.

---

7 Gaussian units are used in this section by tradition. The of this paper is in MKS bowing to the broad engineering applicability of the subject and the growing popularity of these units.
References

[1] A. Ashkin and J. Dziedzic, Radiation pressure on a free liquid surface, Phys. Rev. Lett. 30, 139 (1973).

[2] D. E. Aspnes, Local-field effects and effective-medium theory: A microscopic perspective, Am. J. Phys. 50, 704 (1982).

[3] N. L. Balazs, The Energy-Momentum Tensor of the Electromagnetic Field inside Matter, Phys. Rev. 91, 408 (1953).

[4] S. M. Barnett, Resolution of the Abraham-Minkowski Dilemma, Phys. Rev. Lett. 104, 070401 (2010).

[5] L. Brillouin, Wave Propagation and Group Velocity Academic Press, New York (1960).

[6] C. Chafin, The Quantum State of Classical Matter II: Thermodynamic Equilibrium and Hydrodynamics, arXiv:quant-ph/1309.1111, (2013).

[7] E. Falcon, C. Laroche and S. Fauve, Observation of Sommerfeld Precursors on a Fluid Surface, Phys. Rev. Lett., 91, 68 (2003).

[8] K. Hasselmann, On the non-linear energy transfer in a gravity-wave spectrum. 1. General theory, J. Fluid Mech. 12, 481 (1962).

[9] J. D. Jackson. Classical Electrodynamics. Wiley, New York, (1962).

[10] L. Landau and E.M. Lifshitz, The Classical Theory of Fields, Pergamon, Oxford, (1979).

[11] J. Lighthill, Waves in Fluids, Cambridge, (2001).

[12] M. Mansuripur and A. R. Zakharian, Whence the Minkowski Momentum?, Opt. Comm., 283, 3557, (2010).

[13] M. E. McIntyre, On the ‘wave momentum’ myth, J. fluid Mech. 106, 331 (1982).

[14] C. W. Misner, K. S. Thorne, and J. A. Wheeler. Gravitation. Freeman, (1973).

[15] F. Moulin, Magnetic monopoles and Lorentz force, Nuo. Cim. B, 116, 869 (2001).

[16] R. N. C. Pfeifer, T. A. Nieminen, N. R Heckenberg, and H. Rubinsztein-Dunlop, Momentum of an electro- magnetic wave in dielectric media, Rev. Mod. Phys. 79, 1197 (2007).

[17] O. M. Phillips, The Dynamics of the Upper Ocean, Cambridge, (1966).

[18] E. Purcell, Electricity and Magnetism, Berkeley, Vol.2 (1965).

[19] F. Rohrlich, Classical Charged Particles, World Scientific, (2007).