TOWARDS THE FULL MORDELL-LANG CONJECTURE FOR DRINFELD MODULES

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Abstract. Let $\phi$ be a Drinfeld module of generic characteristic, and let $X$ be a sufficiently generic affine subvariety of $G_a^g$. We show that the intersection of $X$ with a finite rank $\phi$-submodule of $G_a^g$ is finite.

1. Introduction

In [18], McQuillan proved the Mordell-Lang conjecture in its most general form.

Theorem 1.1 (The full Mordell-Lang theorem). Let $G$ be a semi-abelian variety defined over a number field $K$. Let $X \subset G$ be a $K_{\text{alg}}$-subvariety, and let $\Gamma \subset G(K_{\text{alg}})$ be a finite rank group (i.e. $\Gamma$ lies in the divisible hull of a finitely generated subgroup of $G(K_{\text{alg}})$). Then there exist algebraic subgroups $B_1, \ldots, B_l$ of $G$ and there exist $\gamma_1, \ldots, \gamma_l \in \Gamma$ such that

$$X(K_{\text{alg}}) \cap \Gamma = \bigcup_{i=1}^l (\gamma_i + B_i(K_{\text{alg}})) \cap \Gamma.$$ 

We note that in Theorem 1.1 if $X$ does not contain any translate of a positive dimensional algebraic subgroup of $G$, then the full Mordell-Lang theorem says that $X(K_{\text{alg}}) \cap \Gamma$ is finite. Also, a particular case of the full Mordell-Lang theorem (in the case $\Gamma$ is the torsion subgroup $G_{\text{tor}}$ of $G$) is the Manin-Mumford theorem, which was first proved by Raynaud [19].

Faltings [8] proved the Mordell-Lang conjecture for finitely generated subgroups $\Gamma$ of abelian varieties $G$. His proof was extended by Vojta [23] to finitely generated subgroups of semi-abelian varieties $G$. Finally, McQuillan [18] extended Vojta’s result to finite rank subgroups $\Gamma$ of semi-abelian varieties $G$. Later, Rössler [20] provided a simplified proof of McQuillan’s extension in which he used uniformities for the intersection of translates of a fixed subvariety $X \subset G$ with the torsion subgroup of the semi-abelian variety $G$. Essentially, Rössler showed that the full Mordell-Lang conjecture follows from the Mordell-Lang statement for finitely generated subgroups, combined with a uniform Manin-Mumford statement as proved by Hrushovski [17].

It is important to note that the exact translation of the Mordell-Lang conjecture to semi-abelian varieties in characteristic $p$ is false due to the presence of isotrivial varieties. However, Hrushovski [16] saved the Mordell-Lang theorem for finitely generated subgroups of semi-abelian varieties in characteristic $p$ by treating isotrivial varieties as special. The isotrivial case was treated by Rahim Moosa and the author in [12] where it was obtained a full Mordell-Lang statement for isorivial semi-abelian varieties in characteristic $p$. On the other hand, if we replace $G$ by a power $G_a^g$ of the additive group scheme, then the exact translation of the Mordell-Lang conjecture either fails (in characteristic 0) or it is trivially true (in characteristic $p$).

Inspired by the analogy between abelian varieties in characteristic 0 and Drinfeld modules of generic characteristic, Denis [6] proposed that analogs of the Manin-Mumford and Mordell-Lang theorems hold for such Drinfeld modules $\phi$ acting on $G_a^g$ (in characteristic $p$). Denis conjectures ask for describing the intersection of an affine subvariety $X \subset G_a^g$ with a finite rank $\phi$-submodule $\Gamma$ of $G_a^g$. Using methods of model theory, combined with some clever number theoretical arguments, Scanlon [21] proved the Denis-Manin-Mumford conjecture. In [9], the author proved the Denis-Mordell-Lang conjecture for finitely generated $\phi$-modules $\Gamma$ under two mild technical assumptions. In this paper, we extend our result from [9] to finite rank $\phi$-submodules $\Gamma$.

We also note that recently there have been significant progress on establishing additional links between classical diophantine results over number fields and similar statements for Drinfeld modules. The author proved in [10] an equidistribution statement for torsion points of a Drinfeld module, which is similar to the equidistribution statement established by Szpiro-Ullmo-Zhang [22] (which was later extended by Zhang [26] to a full proof of the famous Bogomolov conjecture). Also, Breuer [9] proved a special case of the
André-Oort conjecture for Drinfeld modules, while special cases of this conjecture in the classical case of a number field were proven by Edixhoven-Yafaev \cite{7} and Yafaev \cite{25}. Bosser \cite{2} proved a lower bound for linear forms in logarithms at an infinite place associated to a Drinfeld module (similar to the classical result obtained by Baker \cite{1} for usual logarithms, or by David \cite{4} for elliptic logarithms). Bosser’s result is quite possibly true also for linear forms in logarithms at finite places for a Drinfeld module. Assuming this last statement, Thomas Tucker and the author in \cite{13} to establish certain equidistribution and integrality statements for Drinfeld modules. Moreover, Bosser’s result is quite possibly true also for linear forms in logarithms at an infinite place associated to a Drinfeld module (similar to the classical result obtained by Baker \cite{1} for usual logarithms, or by David \cite{4} for elliptic logarithms). Bosser’s result was used to prove a lower bound for \cite{2} for semi-abelian varieties in characteristic 0.

The plan for our paper is as follows: in Section \ref{sec:notation} we provide the basic notation for our paper, while in Section \ref{sec:main_result} we prove our main result (Theorem \ref{thm:main_result}).

2. The Mordell-Lang theorem for Drinfeld modules

First we note that all subvarieties appearing in this paper are considered to be closed. We define next the notion of a Drinfeld module.

Let \( p \) be a prime and let \( q \) be a power of \( p \). Let \( C \) be a projective non-singular curve defined over \( \mathbb{F}_q \). Let \( A \) be the ring of \( \mathbb{F}_q \)-valued functions defined on \( C \), regular away from a fixed closed point \( \infty \in C \). Let \( K \) be a finitely generated field extension of the fraction field \( \text{Frac}(A) \) of \( A \). We let \( K^{\text{alg}} \) be a fixed algebraic closure of \( K \), and let \( K^{\text{sep}} \) be the separable closure of \( K \) inside \( K^{\text{alg}} \).

We define the operator \( \tau \) as the Frobenius on \( \mathbb{F}_q \), extended so that for every \( x \in K^{\text{alg}} \), we have \( \tau(x) = x^q \). Then for every subfield \( L \subset K^{\text{alg}} \), we let \( L\{\tau\} \) be the ring of polynomials in \( \tau \) with coefficients from \( L \) (the addition is the usual addition, while the multiplication is given by the usual composition of functions).

Following Goss \cite{15}, we call a Drinfeld module of generic characteristic defined over \( K \) a morphism \( \phi: A \to K\{\tau\} \) for which the coefficient of \( \tau^0 \) in \( \phi_a \) is \( a \) for every \( a \in A \), and there exists \( a \in A \) such that \( \phi_a \neq a\tau^0 \). For the remainder of this paper, unless otherwise stated, \( \phi: A \to K\{\tau\} \) is a Drinfeld module of generic characteristic.

A Drinfeld module \( \psi: A \to K^{\text{alg}}\{\tau\} \) is isomorphic to \( \phi \) (over \( K^{\text{alg}} \)) if there exists a nonzero \( \gamma \in K^{\text{alg}} \) such that for every \( a \in A \), we have \( \psi_a = \gamma^{-1} \phi_a \gamma \).

For every field extension \( K \subset L \), the Drinfeld module \( \phi \) induces an action on \( \mathbb{G}_a(L) \) by \( a \cdot x := \phi_a(x) \), for each \( a \in A \). Let \( g \) be a fixed positive integer. We extend diagonally the action of \( \phi \) on \( \mathbb{G}_a^n \).

The subgroups of \( \mathbb{G}_a^n(K^{\text{alg}}) \) invariant under the action of \( \phi \) are called \( \phi \)-submodules. For a \( \phi \)-submodule \( \Gamma \), its full divisible hull is

\[
\Gamma \otimes_A \text{Frac}(A) := \left\{ x \in \mathbb{G}_a^n(K) \mid \text{there exists } 0 \neq a \in A \text{ such that } \phi_a(x) \in \Gamma \right\}.
\]

We define the rank of a \( \phi \)-submodule \( \Gamma \subset \mathbb{G}_a^n(K^{\text{alg}}) \) as \( \dim_{\text{Frac}(A)} \Gamma \otimes_A \text{Frac}(A) \).

**Definition 2.1.** An algebraic \( \phi \)-submodule of \( \mathbb{G}_a^n \) is an algebraic subgroup of \( \mathbb{G}_a^n \) invariant under \( \phi \).

Denis proposed in Conjecture 2 of \cite{8} the following problem.

**Conjecture 2.2** (The full Denis-Mordell-Lang conjecture). Let \( X \subset \mathbb{G}_a^n \) be an affine variety defined over \( K^{\text{alg}} \). Let \( \Gamma \) be a finite rank \( \phi \)-submodule of \( \mathbb{G}_a^n(K^{\text{alg}}) \). Then there exist algebraic \( \phi \)-submodules \( B_1, \ldots, B_l \) of \( \mathbb{G}_a^n \) and there exist \( \gamma_1, \ldots, \gamma_l \in \Gamma \) such that

\[
X(K^{\text{alg}}) \cap \Gamma = \bigcup_{i=1}^l (\gamma_i + B_i(K^{\text{alg}})) \cap \Gamma.
\]

Before stating our result, we need to introduce the following notion.

**Definition 2.3.** We call the modular transcendence degree of \( \phi \) the smallest integer \( d \geq 1 \) such that a Drinfeld module isomorphic to \( \phi \) is defined over a field of transcendence degree \( d \) over \( \mathbb{F}_q \).

In \cite{8} (see Theorem 4.11), the author proved the following result towards Conjecture 2.2.
Theorem 2.4. With the above notation, assume in addition that the modular transcendence degree of $\phi$ is at least 2. Let $X \subset G_a^d$ be an affine subvariety defined over $K^{alg}$ such that there is no positive dimensional algebraic subgroup of $G_a^d$ whose translate lies inside $X$. Let $\Gamma$ be a finitely generated $\phi$-submodule of $G_a^d(K^{alg})$. Then $X(K^{alg}) \cap \Gamma$ is finite.

In Theorem 3.1 we extend the previous result to all finite rank $\phi$-submodules $\Gamma$.

Remark 2.5. We have two technical conditions in Theorem 2.4 which we will keep also in our extension from Theorem 3.1. The condition that $\phi$ has modular transcendence degree at least equal to 2 is a mild technical condition, however necessary due to the methods employed in [9]. The condition that $X$ does not contain any translate of a positive dimensional algebraic subgroup of $G_a^d$ is satisfied by all sufficiently generic affine subvarieties $X$.

3. Proof of our main result

We continue with the notation from Section 2. We define the torsion submodule of $\phi$ as

$$\phi_{tor} = \{ x \in G_a^d(K^{alg}) | \text{there exists } a \in A \setminus \{0\} \text{ such that } \phi_a(x) = 0 \}.$$ 

Next we state our main result.

Theorem 3.1. Let $K$ be a finitely generated field of characteristic $p$ and let $g$ be a positive integer. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic. Assume the modular transcendence degree of $\phi$ is at least 2. Let $X \subset G_a^d$ be an affine subvariety defined over $K^{alg}$ such that there is no positive dimensional algebraic subgroup of $G_a^d$ whose translate lies inside $X$. Let $\Gamma$ be a finitely generated $\phi$-submodule of $G_a^d(K)$, and let $\Gamma' := \Gamma \otimes_A \text{Frac}(A)$. Then $X(K^{alg}) \cap \Gamma'$ is finite.

In our proof of Theorem 3.1 we need a uniform version of Scanlon’s result from [21]. He proved the Manin-Mumford theorem for Drinfeld modules in the following form (see his Theorem 1).

Theorem 3.2. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module and let $X \subset G_a^d$ be an affine variety defined over $K^{alg}$. Then there exist algebraic $\phi$-submodules $B_1, \ldots, B_l$ of $G_a^d$ and there exist $\gamma_1, \ldots, \gamma_l \in \phi_{tor}^d$ such that

$$X(K^{alg}) \cap \phi_{tor}^d = \bigcup_{i=1}^l (\gamma_i + B_i(K^{alg})) \cap \phi_{tor}^d.$$ 

In Remark 19 from [21], Scanlon notes that his Manin-Mumford theorem for Drinfeld modules holds uniformly in algebraic families of varieties, i.e. if $X$ varies inside an algebraic family of varieties, then there exists a uniform bound on the degrees of the Zariski closures of $X(K^{alg}) \cap \phi_{tor}^d$. In particular, we obtain the following uniform statement for translates of $X$.

Corollary 3.3. With the notation for $\phi$ and $X$ as in Theorem 3.2, assume in addition that $X$ contains no translate of a positive dimensional algebraic subgroup of $G_a^d$. Then there exists a positive integer $N$ such that for every $x \in G_a^d(K^{alg})$, the set $(x + X(K^{alg})) \cap \phi_{tor}^d$ has at most $N$ elements.

Proof. Because $X$ contains no translate of a positive dimensional algebraic subgroup of $G_a^d$, then for every $x \in G_a^d(K^{alg})$, the algebraic $\phi$-modules $B_i$ appearing in the intersection $(x + X(K^{alg})) \cap \phi_{tor}^d$ are all 0-dimensional. In particular, the set $(x + X(K^{alg})) \cap \phi_{tor}^d$ is finite. Thus, using the uniformity obtained by Scanlon for his Manin-Mumford theorem, we conclude that the cardinality of $(x + X(K^{alg})) \cap \phi_{tor}^d$ is uniformly bounded by some positive integer $N$, independent of $x$. \qed

We will also use the following fact in the proof of our Theorem 3.1.

Fact 3.4. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module. Then for every positive integer $d$, there exist finitely many torsion points $x$ of $\phi$ such that $[K(x) : K] \leq d$.

Proof. If $x \in \phi_{tor}$, then the canonical height $\hat{h}(x)$ of $x$ (as defined in [5] and [24]) equals 0. Also, as shown in [5], the difference between the canonical height and the usual Weil height is uniformly bounded on $K^{alg}$. Actually, Denis [5] proves this last statement under the hypothesis that $\text{trdeg}_{K^{alg}} K = 1$. However, his proof easily generalizes to fields $K$ of arbitrarily finite transcendence degree. For this we need the construction of a coherent good set of valuations on $K$ as done in [11] (see also the similar construction of heights from...
Essentially, a coherent good set $U_K$ of valuations on $K$ is a set of defectless valuations satisfying a product formula on $K$ (for more details, we refer the reader to Sections 2 and 3 of [11]). Then Fact 3.3 follows by noting that there are finitely many points of bounded Weil height and bounded degree over the field $K$ (using Northcott’s theorem applied to the global function field $K$).

Moreover, Corollary 4.22 of [11] provides an effective upper bound on the size of the torsion of $\phi$ over any finite extension $L$ of $K$ in terms of $\phi$ and the number of places of $L$ lying above places in $U_K$ of bad reduction for $\phi$. Because for each field $L$ such that $[L : K] \leq d$, and for each place $v \in U_K$, there are at most $d$ places $w$ of $L$ lying above $v$, we conclude that there exists an upper bound for the size of torsion of $\phi$ over all field extensions of degree at most $d$ over $K$ in terms of $\phi$, $d$ and the number of places in $U_K$ of bad reduction for $\phi$.

Proof of Theorem 3.1. Because $\phi$ has generic characteristic, $\Gamma' \subset \mathbb{G}_a^d(K^\text{sep})$. Hence $X(K^\text{alg}) \cap \Gamma' = X(K^\text{sep}) \cap \Gamma'$.

We note that our theorem is equivalent with showing that if $X$ is a positive dimensional irreducible subvariety of $\mathbb{G}_a^d$ satisfying the hypothesis of Theorem 3.1, then $X(K^\text{sep}) \cap \Gamma'$ is not Zariski dense in $X$. Moreover, at the expense of moding out through the stabilizer Stab($X$) of $X$ (which is a finite group according to the hypothesis of Theorem 3.1), we may assume Stab($X$) is trivial. Note that after moding out through the (finite) group Stab($X$), the variety $X$ is still positive dimensional and irreducible. Finally, at the expense of replacing $K$ by a finite extension, we may assume that $X$ is defined over $K$, and also $\Gamma \subset \mathbb{G}_a^d(K)$.

Our proof follows the argument from [12] (which in turn was inspired by the argument from [20]). We assume by contradiction that $X(K^\text{alg}) \cap \Gamma'$ is Zariski dense in $X$.

We claim that for every $x \in \Gamma'$ and for every $\sigma \in \text{Gal}(K^\text{sep}/K)$, we have $\sigma(x) - x \in \phi_{\text{tor}}^d$. Indeed, because $x \in \Gamma'$, then there exists $0 \neq a \in A$ such that $\phi_a(x) \in \Gamma$. Because $\Gamma \subset \mathbb{G}_a^d(K)$ and $\phi$ is defined over $K$, then $\phi_a(\sigma(x)) = \sigma(\phi_a(x)) = \phi_a(x)$.

Hence $\phi_a(x) - x = 0$, as desired. Therefore, if $x \in X(K^\text{sep})$, then $\sigma(x) - x \in (-x + X) \cap \phi_{\text{tor}}^d$ (we also used that $X$ is defined over $K$, and so, $\sigma(x) \in X$). Using Corollary 3.3, we obtain an upper bound on the number of conjugates of $x$, which gives us an upper bound, say $N$, for $[K(x) : K]$. Implicitly, we also get $[K(\sigma(x)) : K] \leq N$. Because $K(\sigma(x) - x) \subset K(x, \sigma(x))$, we conclude $[K(\sigma(x) - x) : K] \leq N^2$. Using that $\sigma(x) - x \in \phi_{\text{tor}}^d$, and using Fact 3.4, we conclude that the set

$$
\{\sigma(x) - x \mid x \in \Gamma' \text{ and } \sigma \in \text{Gal}(K^\text{sep}/K)\}
$$

is finite.

Assuming that $X(K^\text{sep}) \cap \Gamma'$ is Zariski dense in $X$ (and also using that $X$ is irreducible), then either $X(K) \cap \Gamma'$ is Zariski dense in $X$, or $(X(K^\text{sep}) \setminus X(K)) \cap \Gamma'$ is Zariski dense in $X$. If the former statement holds, then $X$ has a Zariski dense intersection with a finitely generated group, as $\Gamma' \cap \mathbb{G}_a^d(K)$ is a finite rank subgroup of the tame module $\mathbb{G}_a^d(K)$ (see Theorem 1 of [24]). By Theorem 2.4 $X(K) \cap (\Gamma' \cap \mathbb{G}_a^d(K))$ is finite and hence it cannot be dense in $X$ (because $X$ is assumed to be positive dimensional). Therefore the latter case of the above dichotomy should occur. Using [11], we conclude that there exists a nonzero torsion point $y \in \phi_{\text{tor}}^d$ such that the set

$$
\{x \in X(K^\text{sep}) \cap \Gamma' \mid x = y \text{ for some } \sigma \in \text{Gal}(K^\text{sep}/K)\}
$$

is Zariski dense in $X$. Therefore, $y \in \text{Stab}(X)$, contradicting the fact that $X$ has trivial stabilizer. This concludes the proof of Theorem 3.1.

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