Abstract

This is the first of two related papers analysing and explaining the origin, manifestations and paradoxical features of the quantum potential (QP) from the non-relativistic and relativistic point of view. QP arises in the quantum Hamiltonian, under various procedures of quantization of the natural systems, i.e. the Hamilton functions of which are the positive-definite quadratic forms in momenta with coefficients depending on the coordinates in \((n\)-dimensional) configurational space \(V_n\) endowed so by a Riemannian structure. The result of quantization may be considered as quantum mechanics (QM) of a particle in \(V_n\) in the normal Gaussian system of reference in the globally-static space-time \(V_{1,n}\). Contradiction of QP to the Principles of General Covariance and Equivalence is discussed.

It is found that actually the historically first Hilbert space based quantization by E. Schrödinger (1926), after revision in the modern framework of QM, also leads to QP in the form that B. DeWitt had been found 26 years later. Efforts to avoid QP or reduce its drawbacks are discussed. The general conclusion is that some form of QP and a violation of the principles of general relativity which it induces are inevitable in the non-relativistic quantum Hamiltonian. It is shown also that Feynman (path integration) quantization of natural systems singles out two versions of QP, which both determine two bi-scalar (independent on choice of coordinates) propagators fixing two different algorithms of path integral calculation.

In the accompanying paper under the same general title and the subtitle "The Relativistic Point of View", relation of the non-relativistic QP to the quantum theory of the scalar field non-minimally coupled to the curved space-time metric is considered.

**Keywords:** Riemannian space-time; Quantization; Path Integration; Quantum Potential; Principle of Equivalence; General Covariance; Problem of Measurement.
1 Introduction

In the present paper, the different procedures of the Hilbert space based quantization of the non-relativistic natural mechanical systems will be analyzed and compared. The (classical) natural systems (the term originated by E. Whittaker and re-animated by V. N. Arnold and A. B. Givental [3]) are those whose Hamilton functions are non-uniform quadratic forms in momenta $p_a$ with coefficients $\omega^{ab}(q)$ depending on coordinates $q^a, \quad a, b, \cdots = 1, \ldots, n$ of configurational space $V_n$:

$$H^{(nat)}(q, p; \omega) = \frac{1}{2m}\omega^{ab}(q)p_ap_b + V^{(ext)}(q). \quad (1)$$

It provides $V_n$ by a Riemannian structure

$$ds^2_{(\omega)} = \omega_{ab}(q)dq^a dq^b. \quad (2)$$

(Henceforth, subscripts $(\omega)$ and $(g)$ will denote objects related to metric tensors $\omega_{ab}$ and $g_{\alpha \beta}, \quad \alpha, \beta, \cdots = 0, 1, \ldots, n$ of $V_n$ and of $n + 1$-dimensional space-time $V_{1,n}$ respectively.)

Thus, $H^{(nat)}$ determines the dynamics of a natural system as a motion of a point-like particle in $V_n$, to which a potential $V^{(ext)}(q)$ acts in addition. So, the actual motion of a neutral point-like particle in the external gravitation (including description of the motion in curvilinear coordinates without gravitation) treated general-relativistically as a curved space is a representative case of a natural system.

However, there is a principal alternative way to construct the non-relativistic QM of this simplest physical system coupled to the geometrized gravitation. Namely, to extract it from
the general-relativistic quantum theory of scalar field as the non-relativistic \((c^{-1} = 0)\) asymptotic of its one-particle sector. The one-particle subspace in the particle-interpretable Fock representation of the canonically quantized field can be defined (in the asymptotical sense) even when the metric is time-dependent, i.e. \(\omega^{ab} = \omega^{ab}(t,q)\). This approach will be considered in the companion paper [II] under the same title and subtitle "The Field-Theoretic Point of View" and compared with conclusions of the present paper. It should be noted at once that these two approaches lead, in general, to QM’s which do not coincide completely. This is one of interesting results of the work as a whole.

Despite that this basic problem may be considered as of little "practical" interest for physics, there is an important aspect of it. The theory, which is more general and geometrically transparent than the standard QM in a potential field, can serve as an instrument for a deeper insight on foundations of QM. E. Schrödinger was guided just by such an idea when he had proposed a method of construction of a quantum Hamiltonian for the generic natural system in the third [4] of his five papers [5] of 1926, by which the wave mechanics had been founded. Apparently, it was the first attempt of quantization in the sense which is close to the modern meaning of the term in theoretical physics. While this step, which had not received a deserving attention, Schrödinger did not even mentioned gravitation or general relativity at all.

In the present paper, we shall, on contrary, analyze whether the two basic general-relativistic principles - the Principle of Equivalence (referred further as PE) and the Principle of General Covariance, which hold on the classical level for any natural system, are satisfied in a sense in the corresponding QM. It is a paradox that the both of the principles does not hold in the ordinary sense in QM constructed by quantization of the natural systems, but satisfy them in a restricted sense in QM extracted from the general-relativistic quantum theory of scalar field.

All various procedures of quantization of Hamiltonian systems with finite degrees of freedom are ambiguous or problematic to be mathematically rigorous. Therefore, it seems more correct to speak on a paradigm of quantization rather than on an well-established theory. See, e.g., a discussion of the topic by M. J. Gotay in [1]. As concerns the level of mathematical rigor of the present discussion, the best way to characterize it is the following amusing citation taken from [6]:

"...as Sir Michael Atiyah said in his closing lecture of the 2000 International Congress in Mathematical Physics,..., Mathematics and Physics are two communities separated by a common language."

Then, the present work is from the side of the Physics community. We will be mainly concerned with the so called the Hilbert-space based canonical quantization, which is meant here as a map

\[
q^a \rightarrow \hat{q}^a, \quad p_b \rightarrow \hat{p}_b, \quad \text{so that} \quad \{q^a, p_b\} = \{\hat{q}^a, \hat{p}_b\};
\]

\[
H^{(nat)} \rightarrow \hat{H},
\]

where all "hatted" objects are assumed to have representation as differential operators in the Hilbert space \(L^2(V_n; \mathbb{C}; \omega^{1/2} dq)\) and \(\{,,\}\) is the Poisson bracket. The quantum Hamilto-
nian $\hat{H}$ is assumed to be constructed of the "primary" quantum observables $\hat{q}^a, \hat{p}_b$ by some substantiated way. In the standard canonical quantization, it is found by the straightforward substitution \(3\) into $H^{(\text{nat})}(q, p)$ and some Hermitizing ordering.

Specifically, conclusions of analysis of the following approaches to QM of the natural systems by the present author will be exposed below descriptively or, at least, noted:

- **Schrödinger’s variational approach** (Section 2);
- **revision of Schrödinger’s variational approach** by the present author (Section 3);
- **canonical quantization** (Schrödinger–DeWitt ordering) \(7\) and its generalization by the present author (Section 4);
- **quasi-classical quantization** by B. S. DeWitt \(8\) (Section 5);
- the Blattner–Costant–Sternberg formalism in the geometric quantization \(10\, 11\) (Section 6);
- **path integration**, or the Feynman quantization \(12\, 8\, 13\, 14\, 15\, 16\, 18\, 19\) (Section 7).

Some intermediate conclusions from this first part of the analysis is given in Section 8 and further they will be compared in the accompanying paper \(11\) with the asymptotic in $c^{-1} \to 0$ of the quantum theory of scalar field in the general globally static Riemannian space-time $V_{1,n}$ and the proper frame of reference in which the metric form of $V_{1,n}$ is \(20\, 19\):

$$d s^2_{(g)} = g_{\alpha\beta}(x)dx^\alpha dx^\beta = c^2 dt^2 - \omega_{ab}(q)dq^a dq^b, \quad \alpha, \beta, \cdots = 0, 1, \ldots n; \quad \{ct, q^a\} \sim x^a \in V_{1,n}. \quad (5)$$

## 2 Variational quantization of natural systems by Schrödinger

Schrödinger \(4\) searched a wave theory which plays the same role w.r.t. the Hamilton mechanics, that the Wave Theory of Light does w.r.t. the Geometrical Optics. In \(4\), entitled "On relation of the Heisenberg–Born–Jordan quantum mechanics to the one of mine", the third of the seminal papers \(5\), he constructed a wave (quantum) counterpart for the natural Hamilton function $H^{(\omega)}(q, p)$ as an extremal of the following functional (Schrödinger considered $\omega_{ab} \equiv \omega_{ab}(q)$):

$$J^{(\text{Sch})}\{\psi\} = \int_{V_n} \omega^{\frac{1}{2}} d^n q \left\{ \frac{\hbar^2}{2m} \left( \frac{\partial \psi}{\partial q^a} \omega^{ab} \frac{\partial \psi}{\partial q^b} \right) + \psi^2 V^{(\text{ext})}(q) \right\} \quad (6)$$

with the additional condition

$$\int_{V_n} \omega^{\frac{1}{2}} d^n q \psi^2 = 1.$$
It is important to note that Schrödinger considered here the real wave functions $\psi(q)$. Variation of $J^{(\text{Sch})}\{\psi\}$ results in an equation for eigenvalues $E$ of a differential operator in the space of functions $\psi(q)$, which may be called the quantum Hamiltonian:

$$\hat{H}^{(\text{Sch})}\psi = E\psi,$$

$$(7)$$

$$\hat{H}^{(\text{Sch})} \overset{\text{def}}{=} -\frac{\hbar^2}{2m}\Delta_{(\omega)} + V^{(\text{ext})},$$

$$(8)$$

It looks as satisfying to the conditions which are implied by GR: it is generally covariant, i.e. a scalar w.r.t. point transformations $q^a \to \tilde{q}^a(q)$ and satisfies to PE which sounds in the formulation by S. Weinberg [21] as follows:

"... at every space-time point in an arbitrary gravitational field it is possible to choose "a locally inertial coordinate system", such that within a sufficient small region of the point of question, laws of nature take the same form as in an unaccelerated Cartesian coordinate system."

According to eq.(5), $\omega_{ab}$ may be, in particular, a relic of a general-relativistically treated gravitation and, in this sense, PE can be applied to $\hat{H}^{(\text{Sch})}$. A more fine question is: are the Schrödinger equation (7) and its time-dependent and, further, general-relativistic generalizations are "laws of nature" which must satisfy PE? We shall return to it in Sec.9. and in the companion paper [II]

It should be emphasized also that Schrödinger himself by no means related his quantization of the natural systems to gravitation or GR. He considered it as an instrument to investigate the quantization procedure itself by application it to mathematically more general classical cases than the simple potential ones. Just this is our main aim but for a more wide variety of quantization procedures and in relation with GR.

3 Revision of Schrödinger approach in framework of modern quantum mechanics

In the modern QM, $\psi(q)$ are actually complex functions from a pre-Hilbertian space $L^2(V_n; \mathbb{C}; \omega^{1/2}d^nq)$ with the scalar product

$$(\psi_1, \psi_2) \overset{\text{def}}{=} \int_{V_n} \overline{\psi}_1 \psi_2 \omega^{1/2}d^nq, \quad \psi \in L^2(V_n; \mathbb{C}; \omega^{1/2}d^nq).$$

$$(9)$$

The physical sense of Schrödinger's functional (6) is the mean value of the energy of the system in the state $\psi(q)$. Instead, today we should take the matrix elements of energy:

$$J^{(\text{modern})}\{\psi_1, \psi_2\} = \int_{V_n} \left\{ \frac{1}{2m} \overline{\psi}_1 \omega^{ab}(\hat{q}) \hat{p}_a \psi_2 + V^{(\text{ext})}(q)\overline{\psi}_1 \psi_2 \right\} \omega^{1/2}d^nq$$

$$(10)$$
where \( \hat{q}^a \overset{def}{=} q^a \cdot \mathbf{\hat{1}} \) are the operators of coordinates in the configurational space \( V_n \) and \( \hat{p}_a \) are the operators of momentum canonically conjugate to \( \hat{q}^a \):

\[
[\hat{q}^a, \hat{p}_b] = i\hbar \delta^a_b.
\]  

(11)

They should be Hermitean(!) w.r.t. the scalar product \( \langle \psi_1, \psi_2 \rangle \). Hermitean momentum operators for \( V_n \) were introduced first by W.Pauli [22] in 1933:

\[
\hat{p}_a \overset{def}{=} -i\hbar \omega^{-1/4} \frac{\partial}{\partial q^a} \omega^{1/4},
\]  

(12)

where “cdot” denotes the operator product. Then, substitution of this expression into \( J^{(\text{modern})}\{\psi, \psi\} \) and Schrödinger’s variational procedure give the eigenvalue equation similar to eq.(7) but with a different quantum hamiltonian \( \hat{H}^{(DW)} \)

\[
\hat{H}^{(DW)} \psi \overset{def}{=} \hat{H}^{(Sch)} \psi + V^{(\text{qm})}(q) \psi = E \psi, \quad V^{(\text{qm})}(q) \overset{def}{=} -\frac{\hbar^2}{2m} \omega^{-\frac{1}{4}} \partial_a (\omega^{ab} \partial_b \omega^{\frac{1}{4}}) (13)
\]

The term \( V^{(\text{qm})} \) was discovered for the first time by DeWitt [7] in 1952 in a different formalism of quantization, who called it the quantum potential, see Section 3 below. Surprising is that it depends on choice of coordinates \( q^a \) (i.e., is not a scalar w.r.t. transformations of \( q^a \)). Also, it violates PE if eq.(13) is taken as a quantum ”law of Nature” for a particle in the gravitational field described by \( \omega_{ab} \) since

\[
V^{(\text{qm})}(y) = -\frac{\hbar^2}{2m} \cdot \frac{1}{6} R(\omega)(y) + O(y). \]

(14)

in the quasi-Cartesian (normal Riemannian) coordinates \( y^a \) with the origin at the point \( q \) under consideration.

Thus, the dogmas of GR and QM are in conflict here! Moreover, non-covariance of the quantum potential implies that the energy spectrum and, after transition to time-dependent version of the Schrödinger equation, the dynamics depend on choice of coordinates in QM so constructed! A heretical thought comes here. Perhaps, it was a good fortune for the early stage of QM that Schrödinger did not realize the conflict: one may suppose, he and his successors in development of the wave mechanics (see [23], Sections 5.3, 6.1) would be embarrassed to proceed!

Returning to expression of \( \hat{H}^{(DW)} \) with account of eqs. (8) and (12) we see that the zero-order term in the quasi-Cartesian coordinates \( y^a \) having been taken separately is ”value” of a scalar object in these particular coordinates by its geometrical sense. However, in the full Hamiltonian \( \hat{H}^{(DW)} \), they are not scalars because the non-invariance of the residual term tangles the situation in other coordinates. We shall call the such terms quasi-scalars in the theory under consideration.
4 Discovery of quantum potential by DeWitt and generalization of his approach

26 years after Schrödinger’s result, B. S. DeWitt [7] had come to the hamiltonian \( \hat{H}^{(DW)} \) by a procedure which may be called the canonical quantization; it is a map:

\[
q_a \rightarrow \hat{q}_a, \quad p^a \rightarrow \hat{p}^a \Rightarrow H^{(\text{nat})}(q, p) \rightarrow H^{(DW)}(\hat{q}, \hat{p}) \overset{\text{def}}{=} \frac{1}{2m} \hat{p}_a \omega^{ab}(\hat{q}) \hat{p}_b + V^{(\text{ext})}(\hat{q}).
\]  

(15)

Here, the von Neumann rule

\[
\hat{f}(q^1, \ldots, q^n) = f(\hat{q}^1, \ldots, \hat{q}^n)
\]

(16)

for definition of the operator corresponding to a function of classical observables, the Poisson brackets of which vanish, is applied for definition \( \omega^{ab}(q) \).

As a differential operator in \( L^2(V; \mathbb{C}; \omega^{1/2} dq) \),

\[
\hat{H}^{(DW)}(\hat{q}, \hat{p}) = \hat{H}^{(DW)} \overset{\text{def}}{=} \hat{H}^{(\text{Sch})} + V^{(\text{qm})}(q)
\]

(17)

Thus, the revised version of the Schrödinger quantization coincides with DeWitt’s canonical quantization! Evidently, DeWitt himself did not know the original Schrödinger work [4].

DeWitt’s result is related to the particular ordering of non-commuting operators \( \omega^{ab}(\hat{q}), \hat{p}_a \). Other (Hermitean) orderings (Weyl, Rivier, et all) are well known. Then, on the our level of quantization, why not to consider Hermitean linear combinations of different orderings? The simplest class of Hamiltonians so obtained form an one-parametric family:

\[
\hat{H}^{(\nu)} = \frac{2 - \nu}{8m} \omega^{ij}(\hat{q}) \hat{p}_i \hat{p}_j + \frac{\nu}{4m} \hat{p}_i \omega^{ij}(\hat{q}) \hat{p}_j + \frac{2 - \nu}{8m} \hat{p}_i \hat{p}_j \omega^{ij}(\hat{q})
\]

\[
= \hat{H}^{(\text{Sch})} + V^{(\text{qm}; \nu)}(q) \cdot \hat{1}
\]

(18)

\[
V^{(\text{qm}; \nu)}(q) \equiv V^{(\text{qm})}(q) + \frac{\hbar^2 (\nu - 2)}{8m} \partial_a \partial_b \omega^{ab}.
\]

(19)

DeWitt’s ordering corresponds to \( \nu = 2 \). In the quasi-Cartesian coordinates \( y^a \)

\[
V^{(\text{qm}; \nu)} = -\frac{\hbar^2}{2m} \cdot \frac{\nu}{12} R(\omega)(y) + O(y).
\]

(20)

Thus, there is an ordering with \( \nu = 0 \) for which the zeroth-order short distance term vanishes, but the non-zero residual term retains; it means that PE is satisfied in the QM if one considers \( \hat{H}^{(\nu=0)} \) (because there is no curvature term at the point of the particle localization) but it is still not covariant. Besides, it will be seen in [II] that \( \nu = 0 \) does not agree with the requirements of PE to the relativistic propagator.

5 Quasi-classical quantization

DeWitt did not take notice of the non-invariance of \( V^{(\text{qm}; \nu)} \), referring to possibility to transform it from one coordinate system to another, which is, of course, not invariance. However,
evidently he had not been satisfied by the result of the canonical quantization. In 1957, DeWitt [8] determined quantum Hamiltonian as a differential operator in \( L^2(V_n; C; \omega^{1/2}d^nq) \) through construction of quasi-classical propagator \( \mathbb{G}(q'', t''|q', t') \):

\[
\psi(q'', t'') = \int_{V_n} \omega^{1/2}(q')d^nq' \mathbb{G}(q'', t''|q', t') \psi(q', t'), \tag{21}
\]

by generalization of the Pauli construction for a charge in e.m. field [9] to the case of natural systems:

\[
\mathbb{G}(q'', t''|q', t') = \omega^{-1/4}(q'') D^{1/2}(q'', t''|q', t') \omega^{-1/4}(q') \exp \left( -\frac{i}{\hbar}S(q'', t''|q', t') \right), \tag{22}
\]

and \( S(q'', t''|q', t') \) is a solution of the Hamilton-Jacobi equation for \( H^{(\text{nat})}(q, p) \). Using the Hamilton-Jacobi equation DeWitt had found that, in a small neighborhood of space-time point \( \{q', t'\} \), the propagator \( \mathbb{G}(q'' \, t''|q', t') \) "nearly satisfies the Schrödinger equation". (Henceforth, \( V^{(\text{ext})} \equiv 0 \) is taken for simplicity.)

\[
\frac{\partial}{\partial t''} \mathbb{G}(q'', t''|q', t') = -\frac{\hbar^2}{2m} \Delta_{(\omega)} \mathbb{G}(q'', t''|q', t') + \tilde{V}^{(\text{qm})}(q'', t'') \mathbb{G}(q'', t''|q', t'). \tag{23}
\]

where

\[
\tilde{V}^{(\text{qm})}(q'', t''|q', t') \overset{\text{def}}{=} \frac{\hbar^2}{2m} \frac{1}{6} \Delta_{(\omega)} f(q'', t''|q', t') = \frac{\hbar^2}{2m} \frac{1}{6} R_{(\omega)}(q'', t'') + o(q'' - q') + o(t'' - t'), \tag{24}
\]

\[
f(q'', t''|q', t') \overset{\text{def}}{=} \omega^{-\frac{1}{2}}(q'', t'') D^{\frac{1}{2}}((q'', t''|q', t') \omega^{-\frac{1}{2}}(q', t').
\]

So we see that \( \tilde{V}^{(\text{qm})} \) looks as a scalar and yet as violating PE. Actually, \( \tilde{V}^{(\text{qm})}(q'', t''|q', t') \) is a bi-scalar and thus depends on choice of the line connecting points \( q' \) and \( q'' \). If the geodesic lines are chosen, then, in the asymptotic \( q'' \to q' \), it is equivalent to fixation of \( q'' \) as the quasi-Cartesian coordinates \( y^a \) and, thus, the non-invariance of the quantum potential remains.

6 Geometric quantization of natural systems

Geometric quantization is oriented to consider \( V_n \) with non-trivial topologies, see e.g. the monograph by J. Sniatycki [10] and the paper [11]. In the latter paper, expansion by \( c^{-2} \) of the Hamilton function for the the relativistic particle in the proper system of reference:

\[
H^{(\text{rel})}(q, p) = mc^2 \sqrt{1 + \frac{2H^{(\text{nat})}_{(\omega)}(q, p)}{mc^2}} \tag{25}
\]
is considered using the Blattner–Kostant–Sternberg formalism.

The zero-order quantum potential is $V^{(qm)}(q) = \frac{\hbar^2}{2m} \frac{1}{6} R^{(\omega)}(q)$, that is a scalar but the geometric quantization is a locally asymptotic theory by construction, and, thus, merely supports DeWitt’s and revised Schrödinger’s local asymptotic quantum potential. This paper is interesting also in that the second order term in the asymptotic expansion in $c^{-2}$, which is quartic in the momenta, is considered. The corresponding potential is a rather complicated scalar expression including derivatives and quadratic expressions of the curvature tensor. Thus, $H^{(nat)} \neq \{\hat{H}^{(nat)}\}^2$ and consequently, the von Neumann rule does not work for the polynomials of $H^{(nat)}$. An interesting problem to study.

7 Feynman quantization of natural systems

There are many papers devoted to construction of the quantum propagator for natural systems by path integration so that the short-time asymptotic of the propagator would reproduce Schrödinger’s original (invariant) Hamiltonian. However, it requires some deformation of the classic Lagrangean with which the path integration starts usually, see, e.g., [14, 15] and, as a method of quantization is equivalent, on my opinion, to mere postulation of the Schrödinger original Hamiltonian. Instead, I shall return to the original idea of Feynman on path integration [12], but with use results of the generalized canonical quantization (Section 4 above) and admit, if necessary, QP in the quantum Hamiltonian generating the original form of the Feynman propagator. Fixation of QP is, in fact, quantization (in the sense accepted here) of the natural mechanics under consideration.

The Feynman propagator $G^{(F)}(q, t|q_0, t_0)$ is constructed by division of finite time interval $t - t_0$ by $N \to \infty$ intervals $[t_I, t_{I+1})$, $I = 0, 1, \ldots, N - 1$, $t_N = t$ of duration $\epsilon = (t - t_0)/N$ as follows:

$$G^{(F)}(q, t|q_0, t_0) = \lim_{N \to \infty} \frac{1}{A^{N+1}} \int \exp \left( \int_{t_0}^{t} L^{(eff)}(q,t') dt' \right) \prod_{I=1}^{N} \omega_I^2 dq_I,$$

(26)

where $A \equiv (2\pi i \hbar)^{-2n}$, $q_I \equiv q(t_I)$, $\omega_I \equiv \omega(q_I)$. A question arises at once what is the effective Lagrangean $L^{(eff)}$?

For the natural systems Feynman’s choice would be

$$L^{(eff)} = L^{(classic)} = \frac{m}{2} \omega_{ab}(q) \dot{q}^a \dot{q}^b,$$

(27)

i.e. the Lagrangean of geodesic motion in $V_n$ (the case of $V(q) \equiv 0$ is taken for simplicity and straightforward comparison with the relativistic field theory in [20]). Then each integration on interval $[t_I, t_{I+1})$ is taken along a geodesic connecting $q_I$ and $q_{I+1}$. However, to have the desired Schrödinger’s result

$$i\hbar \frac{d}{dt} \psi(q, t) = \hat{H}^{(Sch)} \psi(q, t),$$

(28)
according to [8, 15] et al., the choice should be
\[ L_{(\text{eff})} \equiv L_{(\text{classic})} - \frac{\hbar^2}{2m} \frac{1}{12} R(\omega)(q) \] (29)
to compensate the quantum potential term. But then the virtual classical motion between \( q_I \) and \( q_{I+1} \) will be not geodesical. Also, other ambiguities arise in the process of calculation of a Hamilton operator from the path integral \([26]\). Instead of reviewing them, further I expose briefly main points of a special approach the initial idea of which is taken from paper by D’Olivo and Torres [16] but essentially modified in [19] and consists of the following steps:

1. Consider the Hamiltonian representation of \( G_{(F)}(q,t|q_0, t_0) \) as a fold of the short-time propagators in the configuration space representation \([12]\):
\[
G_{(F)}(q,t|q_0, t_0) \overset{\text{def}}{=} \lim_{N \to \infty} \int \prod_{K=1}^{N-1} \omega^\frac{1}{2}(q_K) \, d^nq_K \prod_{J=1}^{N} <q_J|e^{-\frac{i}{\hbar}\hat{H}(\text{eff})(\hat{q}, \hat{p})}|q_{J-1}> 
\]
(30)
where \( q_K = q(t_K) \).

2. It is natural to suggest that the effective Hamiltonian \( \hat{H}(\text{eff})(\hat{q}, \hat{p}) \) as a differential operator in \( L^2(V_q; \mathbb{C}; \omega^{1/2} d^nq) \) is known up to some effective potential \( V(\text{eff})(q) \), i.e.
\[
\hat{H}(\text{eff}) = -\frac{\hbar^2}{2m} \Delta(\omega) + V(\text{eff})(q) \] (31)
Thus, our task is to find \( V(\text{eff})(q) \) which provides the hamiltonian form of propagator \([30]\) with the Lagrangean form \([26]\) so that \( L_{(\text{eff})} \equiv L_{(\text{classic})} \).

3. To calculate the matrix elements in configuration representation, one should to express the differential operator \(-\hbar^2/(2m)\Delta(\omega)\) in eq.(31) through operators \( \hat{q}^a, \hat{p}_b \) remaining \( V(\text{eff}) \) still undetermined. The expression depends on the rule of ordering of \( \hat{q}^a, \hat{p}_b \) and \( \omega^{ab}(\hat{q}) \). We take the one-parametric family of linear combinations of different Hermitean orderings introduced in Section 4:
\[
\hat{H}(\text{eff})(\hat{q}, \hat{p}) = \hat{H}^{(\nu)}(\hat{q}, \hat{p}) - V^{(\text{qm}; \nu)}(\hat{q}) + V^{(\text{eff})}(\hat{q}).
\]
(32)
4. Calculation of the matrix elements within the terms linear in \( \epsilon \) using our generalized rule of ordering gives:
\[
G^{(\nu)}(q'', t''|q', t') = \lim_{N \to \infty} \int \left( \frac{1}{2\pi \hbar \epsilon} \right)^{\pi N/2} \prod_{I=1}^{N-1} \sqrt{\omega(q_I)} \, d^nq_I \\
\times \prod_{J=1}^{N-1} \left( \frac{\sqrt{\omega}}{\omega(q_J)\omega(q_{J-1})} \right)^{(\nu)} \exp \left\{ \frac{i}{\hbar} \hat{L}^{(\nu)}(q_{J-1}, q_J; \frac{\Delta q_J}{\epsilon}) \right\}.
\]
(33)
\[
\Delta q_J \equiv \{ \Delta q^J \overset{\text{def}}{=} q^J - q^J_{J-1} \}.
\]
Here \( (\sqrt{\omega})^{(\nu)}(q_{J-1}, q_J), \hat{L}^{(\nu)}(q_{J-1}, q_J, \Delta q_J/\epsilon) \) are the kernels of the corresponding operators in configurational representation. They are expressed, respectively, through functions \( \sqrt{\omega}(q) \) and
\[
L^{(\nu)}(q, \Delta q_J/\epsilon) \overset{\text{def}}{=} L_{(\text{classic})}(q, \Delta q_J/\epsilon) - V^{(\text{eff})}(q) + V^{(\text{qm}; \nu)}
\]
(34)
along the rule:

\[ \tilde{f}^{(\nu)}(q_{J-1}, q_J) = \nu f(\tilde{q}_J) + \frac{1 - \nu}{2} (f(q_{J-1}) + f(q_J)), \quad \tilde{q}_J \overset{\text{def}}{=} \frac{1}{2} (q_J + q_{J-1}) \]  

which follows from the general rules of quantization of Beresin and Shubin, \[17\], Chapter 5, in terms of the kernels of operators.

5. Then, the product enumerated by \( J \) in eq. (33) should be represented as a product of exponentials of the values of the classical action on the intervals \([q_{J-1}, q_J]\), that is as a product of factors of the form

\[ \exp \left\{ \frac{i}{\hbar} L'_{(\text{eff})} (q_J, \Delta q_J/\epsilon) \right\}, \]  

where, in the exponent, the value of some effective Lagrangian \( L'_{(\text{eff})}(q, \dot{q}) \) (in general, it differs from \( L_{(\text{eff})}^{(\nu)} \)) stands, which is taken at a point \( q_J \in [q_{J-1}, q_J] \) chosen so that to represent the exponent as \( L_{(\text{classic})} \). To obtain the representation, all functions of \( q_{J-1}, q_J, \tilde{q}_J \) under the product in \( J \) should be expanded into the Tailor series near the point \( q_J \) up to terms quadratic in \( \Delta q_J \), since only such terms contribute to the integral in eq. (33). Further, one should include the contribution of the pre-exponential factor to the exponent in a form of an additional QP.

6. The problem of evaluation of the integrands is divided into the two principally different choices of the point \( q'_J \in [q_{J-1}, q_J] \):

- **Case A**. The end point evaluation of the integrands: \( q' = q_{J-1} \) or \( q' = q_J \) i.e., \( q' \) is taken at the ends of the segment \([q_{J-1}, q_J]\). However, to avoid appearance of terms proportional to \( \dot{q}_J \) and, thus, of parity violation in the resulting \( L'_{(\text{eff})}(q, \dot{q}) \) and also for agreement with rule (35), the quantum image \( \tilde{f}^{(\nu)}(q_{J-1}, q_J) \) of the generic function \( f(q) \) should depend on the endpoints symmetrically. In the necessary approximation, it is

\[
\tilde{f}^{(\nu)}(q_{J-1}, q_J) = \frac{1}{2} f(q_{J-1}) + \frac{1}{2} f(q_J) + \frac{\nu}{8} (\partial_i f(q_{J-1}) - \partial_i f(q_J)) \Delta q'_J \\
+ \frac{\nu}{16} (\partial_i \partial_j f(q_{J-1}) + \partial_i \partial_j f(q_J)) \Delta q'_i \Delta q'_j + O \left( (\Delta q_J)^3 \right). \tag{37}
\]

- **Case B**. The intermediate point evaluation of the integrands: \( q'_J = (1 - \mu)q_{J-1} + \mu q_J, \quad 0 < \mu < 1 \), i.e., \( q'_J \in (q_{J-1}, q_J) \). For the generic function \( f(q) \) on the interval \((q_{J-1}, q_J)\) one has

\[
\tilde{f}^{(\nu)}(q_{J-1}, q_J) = f(q'_J) + \left( \frac{1}{2} - \mu \right) \partial_i f(q'_J) \Delta q'_J + \frac{1}{2} \left( \frac{2 - \nu}{4} - \mu + \mu^2 \right) \partial_i \partial_j f(q'_J) \Delta q'_i \Delta q'_j. \tag{38}
\]

Referring to more details of rather complicate calculation in \[19\], now only final conclusions will be given, which are important for further discussion. In the both cases, we come again to noninvariant quantum potentials \( V^{(\text{eff})}(q) \) which we will denote \( V^{(\text{eff};A)}(q) \) and \( V^{(\text{eff};B)}(q) \).
In the case A), transformation of the Hamiltonian form (30) of the Feynman propagator $G^F$ to its Lagrangean form (26) with $L_{(\text{eff})}\equiv L_{(\text{classic})}$ is possible only if $\nu=2$ and then the generally non-invariant potential $V^{(\text{eff})}(q)$ has the form

$$V^{(\text{eff})}_A(q) = -\frac{\hbar^2}{12m} (2\omega^{ij}\omega^{kl} + \omega^{ik}\omega^{jl}) \partial_i \partial_j \omega_{kl} - \frac{\hbar^2}{16m} (2\partial_i \omega^{ij}\partial_j \ln \omega + \omega^{ij}\partial_i \ln \omega \partial_j \ln \omega),$$  \hspace{1cm} (39)

and, in the quasi-Cartesian coordinates $y^a$, it is

$$V^{(\text{qm})}(y) = -\frac{\hbar^2}{2m} \cdot \frac{1}{6} R(\omega)(y) + O(y).$$  \hspace{1cm} (40)

That is, in the case A) the quantization map $H_0 \rightarrow \hat{H}_0^{(\text{eff};A)}$, which may be called the Feynman quantization, coincides remarkably with the result of the revised Schrödinger and “canonical” DeWitt quantizations in the zeroth order of local asymptotic. In the complete form, it differs from the latter, what is worth of a further study.

In the case B) the same approach leads to (the result does not depend on the parameter $\mu$)

$$V^{(\text{eff})}_B(q) = \frac{\hbar^2}{4m} \left( \frac{\nu + 2}{4} \omega^{km} \omega^{ln} \omega^{ij} - \frac{\nu - 2}{4} \omega^{im} \omega^{jn} \omega^{kl} - (\nu - 2) \omega^{im} \omega^{j} \omega^{kl} \right) \partial_i \omega_{mn} \partial_j \omega_{kl},$$  \hspace{1cm} (41)

$$V^{(\text{eff})}_B(y) = -\frac{\hbar^2}{2m} \frac{1}{3} R(\omega)(y) + O(y^2).$$  \hspace{1cm} (42)

Asymptotic local expressions (40) and (42) for quantum potentials $V^{(\text{eff})}_A$ and $V^{(\text{eff})}_B$ (but not the complete potentials (39) and (41)) follow also from the study of the Pauli-DeWitt and Feynman quantizations by G. Vilkovyskii [24]. He worked in the framework of the formalism of proper time in relativistic quantum mechanics $V_{1,n}$ (so that his expressions for potentials include $R \equiv R_g$, the scalar curvature of $V_{1,n}$). From his argumentation one may conclude that just the case A) is the preferred one.

However, if the set of points $q_J$ is considered as a lattice in $V_n$, then the case A) corresponds to evaluation of the integrand as the arithmetic mean of its values on the adjacent nodes of the lattice. The case B) corresponds to evaluation at the mean point of the edges. Thus, these two cases fix two different ways of lattice calculation of the path integral (26) with same integrand $L_{(\text{eff})} = L_{(\text{classic})}$, that give, in principle, different propagators.

More important is that the complete expressions for QPs (39) and (41) are defined for any coordinates $q$ whereas the asymptotic expressions (40) and (42) are suited only for quasi-Cartesian coordinates according to argumentation of Section 4. But the most important, though paradoxical, conclusion is that QPs in the both cases are not scalars of the general transformation $q^a = \tilde{q}^a(q)$. Thus, we encountered again with the non-invariance of quantum Hamiltonian despite that the Lagrangean form (26) of the propagator is invariant.

8 Intermediate conclusion

Firstly, we see a deep conflict in QM between the requirements of observability (hermiticity) and invariance with respect to transformations $\tilde{q}^a = \tilde{q}^a(q)$ (general covariance). Apparently, the
non-invariance of QM seems to be a general property of the standard quantization approaches based on a Hilbert space of states. It looks very strange, but conceptually it can be explained by that the quantum operators of observables and, particularly, of coordinates $\hat{q}^a = q^a \cdot \hat{1}$, imply some concrete classical measurements over the quantum natural system which is an open system and depend on choice of them, while the classical $q^a$ are considered as any of abstract arithmetizations of space points. Further, this conception probably is related to the ideas that information on the open quantum system always includes more or less information on the apparatus which provides an information on the system. Discussion of these ideas may be found in papers by M. Mensky [26] and C. Rovelli [27]. However, our consideration perhaps leads further: not only information on the state of object but also information on its dynamics contains a mixture of information on the measuring device (of the particle’s position, in our case) through the additional terms in the Hamilton operators. This thought is supported by that QP is not a specificity of curvature of the space only and related to choice of curvilinear coordinate systems in the flat space, too.

Of course, this issue needs a deep analysis and still I can give nebulous speculations encouraged by the following words of Leon Rosenfeld concerning QM:

"... inclusion of specifications of conditions of observation into description of phenomena is not an arbitrary decision but a necessity dictated by the laws themselves of evolution of the phenomena and mechanism of observation them, which makes of these conditions an integral part of the description of the phenomena"  

From this point of view, propagator $\mathcal{G}(q'', t''|q', t')$, eq.(33), is the amplitude of transition of the particle from point $q'$ to point $q''$, the position of which is subjected to continual observation by means of local coordinates $y^a$ on each infinitesimal section of possible trajectory. To this end, it is sufficient to take the (quasi-)scalar terms of the quasi-classical approximation in the Hamiltonian and therefore the amplitude is a bi-scalar. (Apparently, this paradox is an analog of the one known as "the continually observed kettle boils never", see, e.g. [28].) So, the local quasi-Cartesian coordinates $y^a$ add no information except the scalar curvature.

On contrary, any version of the full Hamiltonian $\hat{H}$ is used to prepare a state $\psi(q)$ with use of curvilinear coordinates $q^a$. Their coordinate lines are determined by $n$ curvatures which are equal to zero only in the case of the Cartesian coordinates which exist only in $E^n$. They affect as forces on motion of the particle, see, e.g., [29], Chapter 1, which are different for different choice of the coordinates $q^a$.

The difference between the versions of quantum Hamiltonian originally formulated in the Cartesian coordinates and thereupon transformed to the the spherical ones and the one which is immediately formulated in the latter coordinates had been noted by B.Podolsky in 1928 [25].

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2Requotation from the Preface "To Soviet readers" for the Russian edition of the book by I.Pigojine and I. Stengers Order out of Chaos. Man’s new dialogue with nature, "Progress" Publishing House, Moscow, 1986. (inverse translation from Russian of the present author (E.T.))
The way out of the problem, which he had proposed, is a particular case of the following more general postulate:

\[ \hat{H}^{(\text{Pod})} \overset{\text{def}}{=} \hat{\omega}^{-1/2} \hat{p}_{\omega} \hat{\omega}^{1/2} \hat{\omega}^{ab} \hat{p}_{\omega} \hat{\omega}^{1/2} \equiv \hat{H}^{(\text{Sch})} = -\frac{\hbar^2}{2m} \Delta_\omega \] \quad (43)

However, this is equivalent to the direct postulation of the desired invariant result while Schrödinger wanted to find the quantum Hamiltonian from a general variational principle. Further, why one may not to dispose any appropriate degrees of \( \hat{\omega}(q) \) between multipliers of \( \hat{H}^{(\text{DW})} \) or even take normalized Hermitean linear combinations of such disposals? There is a continuum of such generalizations of \( \hat{H}^{(\text{Pod})} \) with zero and as well as non-zero QPs. An answer may be that \( \hat{H}^{(\text{Pod})} \) and apparently the the mentioned disposals of degrees of \( \hat{\omega} \) with the resulting zero QPs are discriminated by their invariance with respect to transformations of coordinates. Nevertheless, their multiplicity causes some dissatisfaction and needs of further study. Besides, as it will be seen in the companion paper [II], all these versions of the theory do not satisfy to the Principle of Equivalence from the general-relativistic point of view.

Another way to invariant quantum Hamiltonian for the natural systems, which is based on use of non-holonomic coordinates, have been proposed by H. Kleinert in his monumental monograph [18] on path integration and by M. Mensky [30]. However, this interesting non-metric approach will be out yet of the scope of the present paper where we hold only the metric approaches.

Summing up, the considered or just now mentioned approaches to quantization of the natural systems discriminate three preferred classes of the quantum Hamiltonians which are characterized by the values of the constant \( \xi \) in the quasi-scalar term of the local asymptotics:

\[ V^{(\text{qm})}(y) = -\frac{\hbar^2}{2m} \xi R_\omega(y), \] \quad (44)

These values and formalisms which fix them are:

\( \xi = \frac{1}{6} \) ← \{ canonical and quasi-classical quantization by DeWitt; revised Schrödinger variational approach and Feynman quantization in present author’s version along the evaluation rule on the lattice for functions in integrands:

\[ q' \in [q_J, q_{J+1}], \quad f(q') = \frac{f(q_{J+1}) + f(q_J)}{2}; \] \quad (45)

\( \xi = \frac{1}{3} \) ← \{ quasi-classical quantization by Vilkovisky, Feynman quantization present author’s version along the evaluation rule on the lattice for functions in integrands:

\[ q' \in (q_J, q_{J+1}), \quad f(q') = f\left(\frac{q_{J+1} + q_J}{2}\right); \] \quad (46)

\( \xi = 0 \) ← \{ Schrödinger’s original approach (no QP), Rivier ordering in canonical quantization (there is QP but the quasi-scalar term vanishes), the Podolski postulate and its generalizations, quantization in non-holonomic coordinates [18] \}. 

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There are very interesting problems to study related to each of the three versions of quantization. For example, it would be important to extend them for the algebra of polynomials of momenta \( p \) with coefficients depending on canonically conjugate \( q \). This would transfer the theoretico-physical conception of quantization, which is adopted in the present paper and the most of references therein, to a more mathematically refined level.

It should not be forgotten also that the (revised) Schrödinger variational approach to quantization can apparently be of interest for application to topologically non-trivial cases of \( V_n \), while, in the present work, the triviality is intrinsically implied.

However, still, we shall follow the more pragmatic way and consider the possible values of \( \xi \) from a point of view of general-relativistic quantum theory of the linear scalar field, of which the nonrelativistic asymptotic \( (e^{-1} \to 0) \) of the one-particle sector of which should produce quantum mechanics of the natural systems. In particular, it will be shown there that just the theory with \( \xi = 1/6 \) is in accord with the Principle of Equivalence as it formulated by S. Weinberg, see Sec.3 and supported by the conformal symmetry if \( n = 3 \).

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