ON THE ASYMPTOTIC EXPANSION OF THE QUANTUM SU(2) INVARIANT AT $\zeta = e^{4\pi i/r}$ FOR LENS SPACE $L(p, q)$

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(Received 13 May 2019 and revised 11 March 2020)

Abstract. We give a formula for the quantum SU(2) invariant at $\zeta = e^{4\pi i/r}$ for Lens space $L(p, q)$, and we prove that the asymptotic expansion is represented by a sum of contributions from SL$_2\mathbb{C}$ flat connections whose coefficients are square roots of the Reidemeister torsions.

1. Introduction

When $\zeta = e^{2\pi i/r}$, the asymptotic expansions of the quantum SU(2) invariants of 3-manifolds $M$ have been studied by some researchers [1–4, 6–9, 14, 15], and they are represented by a sum of contributions from SL$_2\mathbb{C}$ flat connections on $M$. Jeffrey [9] gave a formula for the quantum SU(2) invariant of Lens space $L(p, q)$ and proved that the asymptotic expansion is represented by a sum of contributions from SU(2) flat connections whose coefficients are square roots of the Reidemeister torsions.

When $\zeta = e^{4\pi i/r}$ with odd $r$, Chen and Yang [5] recently observed that the quantum SU(2) invariant is of exponential order as $r \to \infty$ for some hyperbolic 3-manifolds and conjectured that the quantum SU(2) invariant for a closed hyperbolic 3-manifold is of exponential order as $r \to \infty$ whose growth is given by the complex volume of $M$. Ohtsuki and the first author [11] showed that when $\zeta = e^{4\pi i/r}$, the asymptotic expansion of the quantum SU(2) invariant of some Seifert 3-manifolds and Lens space $L(p, 1)$ with odd $p$ is represented by a sum of contributions from some SL$_2\mathbb{C}$ flat connections on them, and square roots of the Reidemeister torsions appear as coefficients of such contributions.

We note that the quantum SU(2) and SO(3) invariants for a rational homology 3-sphere are equal when $\zeta = e^{4\pi i/r}$ with odd $r$ (see [11, Appendix]), and we denote by $\tau_r(M)$ the quantum SO(3) invariant of a 3-manifold of a $M$ in this paper.

In this paper, we give a formula for the quantum SO(3) invariant of Lens space $L(p, q)$ at $\zeta = e^{4\pi i/r}$ and prove that the asymptotic expansion is represented by a sum of contributions from SL$_2\mathbb{C}$ flat connections whose coefficients are square roots of the Reidemeister torsions. Since the fundamental group of Lens space $L(p, q)$ is the abelian group $\mathbb{Z}/p\mathbb{Z}$, the SL$_2\mathbb{C}$ flat connections correspond one-to-one to the SU(2) flat connections, and so the asymptotic expansion is represented by a sum of contributions from SU(2) flat connections.

Let $\rho_k$ be a representation of $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$ to SL$_2\mathbb{C}$ such that the eigenvalues of $\rho_k(g)$ for a generator $g$ of $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$ are $e^{\pm 2\pi ik/p}$. The main theorem is the following.

2010 Mathematics Subject Classification: Primary 57M25; Secondary 57M27.

Keywords: quantum invariant; Chern–Simons invariant; Reidemeister torsion.

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THEOREM 1.1. Let $p$ and $q$ be coprime integers with $p \geq 3$ and $p > q > 0$. Then, the quantum invariant $\tau_r(L(p, q))$ of $L(p, q)$ for odd $r$ is expanded as $r \to \infty$ in the form

$$\forall \langle \tau_r(L(p, q)) \rangle \sim e^{\pi i/2} e^{((\pi i)/4)(n_0(p, q)−3)t} e^{(\pi i/2)(1−1)/2n_1(p, q)} c(p)r$$

where $n_0(p, q)$ and $n_1(p, q)$ are defined in Proposition 3.6 and $p/q = [m_1, \ldots, m_t]$ with $m_i \geq 2$,

$$c(p) := \begin{cases} 1 & \text{if } p \text{ is odd}, \\ \frac{1}{2} & \text{if } p \equiv 0(4), \\ \frac{1}{2}(1 + (-1)^{p/4}) & \text{if } p \equiv 2(4), \end{cases}$$

$CS(L(p, q); ad \circ \rho_k)$ is the Chern–Simons invariant, and we put

$$\omega(L(p, q); \rho_k) = \frac{1}{\pi \sqrt{p}} \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}.$$ 

Further, we have that

$$\omega(L(p, q); \rho_k)^2 = \pm \frac{1}{32\pi^2} \text{Tor}(L(p, q); ad \circ \rho_k),$$

where $\text{Tor}(L(p, q); ad \circ \rho_k)$ is the Reidemeister torsion.

This paper is organized as follows. In Section 2, we give a formula for the quantum $SO(3)$ invariant at $\zeta = e^{4\pi i/r}$ for Lens space $L(p, q)$. In Section 3, using our formula, we calculate the asymptotic expansion of it and prove the main theorem.

2. A Formula for $\tau_r(L(p, q))$

In this section, we give a formula for the quantum $SO(3)$ invariant at $\zeta = e^{4\pi i/r}$ for Lens spaces $L(p, q)$.

We briefly recall the definition of the quantum $SU(2)$ and $SO(3)$ invariants of a closed oriented 3-manifold $M$, following the construction of [10].

For a surface $F$ and a complex number $A$, the linear skein $S(F)$ of $F$ is the vector space over $\mathbb{C}$ spanned by link diagrams on $F$ subject to the isotopy and the relations

(i) $\bigotimes_{A} = A$ 

(ii) $D \bigcirc = (−A^2 − A^{-2})D$ for any diagram $D$.

We have that the linear skein of the plane, $S(\mathbb{R}^2)$, is isomorphic to a one-dimensional vector space with the empty diagram $\emptyset$ as a base. That is, for any diagram $D$ in $\mathbb{R}^2$, we can write

$$D = \langle D \rangle \emptyset \quad \text{in } S(\mathbb{R}^2),$$

where $\langle D \rangle$ denotes the Kauffman bracket of $D$, with $\langle \emptyset \rangle = 1$. 

We define the linear skein $\mathcal{S}(D^2, 2n)$ of a disc $(D^2, 2n)$ with $2n$ points on its boundary as follows. A diagram on $(D^2, 2n)$ is a diagram on the disc $D^2$ such that the boundary of the diagram is the union of the $2n$ fixed points on $\partial D$. The linear skein $\mathcal{S}(D^2, 2n)$ is the vector space spanned by diagrams on $(D^2, 2n)$ subject to the relations (i) and (ii).

We introduce the box over $n$ strands in $\mathcal{S}(D^2, 2n)$ recursively by

\begin{align}
\begin{array}{cccc}
\begin{array}{ccc}
\text{n} & \text{n} \\
\end{array} & \begin{array}{ccc}
\begin{array}{ccc}
\text{n-1} & \text{n-1} & -\Delta_{n-2} \\
\Delta_{n-1} & 1 & 1 \\
\end{array} & \begin{array}{ccc}
\text{n-2} & \text{n-1} & 1 \\
\end{array} & \begin{array}{ccc}
\text{n-1} & 1 \\
\end{array}
\end{array}
\end{align}

(1)

\begin{align}
\begin{array}{cccc}
\begin{array}{ccc}
\text{1} & \text{1} \\
\end{array} & = & \begin{array}{ccc}
\begin{array}{ccc}
\text{n} & \text{n} \\
\end{array} & \begin{array}{ccc}
\begin{array}{ccc}
\text{n-1} & \text{n-1} & -\Delta_{n-2} \\
\Delta_{n-1} & 1 & 1 \\
\end{array} & \begin{array}{ccc}
\text{n-2} & \text{n-1} & 1 \\
\end{array} & \begin{array}{ccc}
\text{n-1} & 1 \\
\end{array}
\end{array}
\end{array}
\end{align}

(2)

where a strand with an integer $n$ implies the union of $n$ parallel copies of the strand, and $\Delta_n = (-1)^n(A^{2(n+1)} - A^{-2(n+1)})/(A^2 - A^{-2})$.

We put

$$\mu_0 = \frac{A^2 - A^{-2}}{\sqrt{r}}.$$ 

For a fixed odd integer $r \geq 3$, we define the elements $\omega$ and $\omega_0$ in $\mathcal{S}(S^1 \times I)$ by

$$\omega = \mu_0 \sum_{n=0}^{r-2} \Delta_n \begin{array}{ccc}
\begin{array}{ccc}
\text{n} & \text{n} \\
\end{array} & \begin{array}{ccc}
\begin{array}{ccc}
\text{n-1} & \text{n-1} & -\Delta_{n-2} \\
\Delta_{n-1} & 1 & 1 \\
\end{array} & \begin{array}{ccc}
\text{n-2} & \text{n-1} & 1 \\
\end{array} & \begin{array}{ccc}
\text{n-1} & 1 \\
\end{array}
\end{array}
\end{array}, \quad \omega_0 = \mu_0 \sum_{n=0}^{(r-3)/2} \Delta_{2n} \begin{array}{ccc}
\begin{array}{ccc}
\text{n} & \text{n} \\
\end{array} & \begin{array}{ccc}
\begin{array}{ccc}
\text{n-1} & \text{n-1} & -\Delta_{n-2} \\
\Delta_{n-1} & 1 & 1 \\
\end{array} & \begin{array}{ccc}
\text{n-2} & \text{n-1} & 1 \\
\end{array} & \begin{array}{ccc}
\text{n-1} & 1 \\
\end{array}
\end{array}
\end{array}.$$

Further, for link diagram $D$ on a surface $F$, we define $\langle \omega, \ldots, \omega \rangle_D \in \mathcal{S}(F)$ by substituting $\omega$ into each component of $D$ and $\langle \omega_0, \ldots, \omega_0 \rangle_D \in \mathcal{S}(F)$ by substituting $\omega_0$ into each component of $D$.

Let $M$ be a closed oriented 3-manifold obtained from $S^3$ by surgery along a framed link $L$ and $D$ a diagram of $L$. Then, the quantum SU(2) invariant $\tau^{\text{SU}(2)}_r(M)$ and the quantum SO(3) invariant $\tau_r(M)$ at $\xi = A^4 = e^{4\pi i/r}$ with $A = e^{\pi i/r}$ are defined by

$$\tau^{\text{SU}(2)}_r(M) = \langle \omega_{U_+}^{-b_+} \omega_{U_-}^{-b_-} \omega, \ldots, \omega \rangle_D,$$

$$\tau_r(M) = \langle \omega_0_{U_+}^{-b_+} \omega_0_{U_-}^{-b_-} \omega_0, \ldots, \omega_0 \rangle_D,$$

where $U_\pm$ denotes the trivial knot with $\pm 1$ framing, and $b_+$ and $b_-$ denote the numbers of positive and negative eigenvalues of the linking matrix of $L$. We remark that these two invariants are equivalent for rational homology 3-spheres [11, Section A.2].

Now we calculate the quantum SO(3) invariant $\tau_r(L(p, q))$ of Lens space $L(p, q)$. The Lens space $L(p, q)$ is specified by a pair $p, q$ of coprime integers with $0 < |q| < p$, and $L(p, q)$ is obtained from $S^3$ by surgery along a framed link with the diagram $D(p, q)$ shown.
in Figure 1, where $m_1, m_2, \ldots, m_t$ are integers determined by
\[
\frac{p}{q} = m_t - \frac{1}{m_{t-1} - \frac{1}{\cdots - \frac{1}{m_2 - \frac{1}{m_1}}} =: [m_1, \ldots, m_t]
\]
and $m_i$ denotes framing.

In what follows, we assume $0 < q < p$ and all $m_i \geq 2$; then the continued fraction expansion $[m_1, \ldots, m_t]$ of $p/q$ is determined uniquely.

The linking matrix $W$ of the framed link presented by the diagram $D(p, q)$ is given by
\[
W = \begin{pmatrix}
m_1 & 1 & & \\
1 & m_2 & 1 & \\
& \ddots & \ddots & \ddots \\
& & \cdots & 1 & \\
& & & 1 & m_t
\end{pmatrix}
\]
Since $m_i \geq 2$, we see that $W$ has $t$ positive eigenvalues. Therefore, the quantum SO(3) invariant of Lens space $L(p, q)$ is given by
\[
\tau_r(L(p, q)) = \langle \omega_0 \rangle_{U_t}^{-t} \langle \omega_0, \ldots, \omega_0 \rangle_{D(p, q)}.
\]

First we calculate the term $\langle \omega_0, \ldots, \omega_0 \rangle_{D(p, q)}$. To calculate it, we use the following lemma.

**Lemma 2.1.** [10]

\begin{enumerate}
\item \[
\Delta_n = \begin{pmatrix}
& \\
& n
\end{pmatrix} = (-1)^n A^{2(n+1)} - A^{-2(n+1)}.
\]
\item \[
\begin{pmatrix}
& \\
& n
\end{pmatrix} = (-1)^n A^{n+2}.
\]
\item \[
\begin{pmatrix}
& \\
& n
\end{pmatrix} = (-1)^n A^{2(n+1)(a+1)} - A^{-2(n+1)(a+1)}
\]
\end{enumerate}
From Lemma 2.1, we have that
\[ \mu_0 \left( \frac{A^2(2j+1)(2l+1)}{\sqrt{-r}} - A^{-2(2j+1)(2l+1)} \right), \]
\[ \left( \frac{(2j+1)(2l+1)}{A} \right) = A^{(2j+1)^2-1} \left( \frac{(2j+1)(2l+1)}{A} \right). \]

For \( 0 \leq j, l \leq (r - 3)/2 \), we put
\[ S_{j,l} := \frac{1}{\sqrt{-r}} \{ A^2(2j+1)(2l+1) - A^{-2(2j+1)(2l+1)} \}, \]
\[ T_j := A^{(2j+1)^2-1}. \]

We have the following lemma.

**Lemma 2.2.** These have the following symmetries:
\[ S_{j,l} = S_{j,r+l} = S_{j,r-1-l}, \]
\[ T_j = T_{r+j} = T_{r-1-j}, \]
\[ S_{(r-1)/2,l} = S_{j,(r-1)/2} = 0. \]

From Lemma 2.1 and the definitions of \( S_{j,l} \) and \( T_j \), it is easy to see the following.

**Lemma 2.3.**
\[ \langle \omega_0, \ldots, \omega_0 \rangle_{D(p,q)} = \mu_0^{-1} \sum_{k_1, \ldots, k_{(r-3)/2}} S_{0,k_1} T_{k_1}^{m_1} S_{k_1,k_1-1} T_{k_1-1}^{m_1} \cdots S_{k_2,k_1} T_{k_1}^{m_1} S_{k_1,0}. \]

We introduce new notation and give some lemmas concerning the Gauss sum and PSL(2, \( \mathbb{Z} \)) for calculating the sum given in Lemma 2.3.

We put
\[ e(\alpha) := \exp(2\pi i \alpha), \]
\[ e_n(\alpha) := \exp \left( \frac{2\pi i \alpha}{n} \right). \]

**Lemma 2.4.** (Gauss sum reciprocity formula in one dimension, [9])
\[ \sum_{k \mod n} e_{2n}(mk^2)e(\psi k) = \sqrt{\frac{in}{m}} \sum_{k \mod m} e_{2m}(-n(k + \psi)^2), \]
if \( k \in \mathbb{Z}, n, m \in \mathbb{Z}, nm \) is even and \( n \psi \in \mathbb{Z} \).

**Definition 2.1.** Suppose \( U \in \text{PSL}(2, \mathbb{Z}) \). A continued fraction expansion \([m_1, \ldots, m_t]\) for \( U \) is a sequence of integers \( m_1, \ldots, m_t \) such that
\[ U = T^{m_t} S \cdots T^{m_1} S, \]
where $S$ and $T$ are generators of $\text{PSL}(2, \mathbb{Z})$ with
\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
satisfying the relations
\[
S^2 = (ST)^3 = 1.
\]

Such an expansion always exists, and is unique if each $m_i \geq 2$.

**Lemma 2.5.** \[9\] Suppose
\[
U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}),
\]
and suppose $[m_1, \ldots, m_t]$ is a continued fraction expansion for $U$. Then

(i) \[ \frac{a}{c} = [m_1, \ldots, m_t], \]

(ii) \[ \frac{b}{a} = -\left( \frac{1}{a_1} + \frac{1}{a_2a_1} + \cdots + \frac{1}{a_ta_{t-1}} \right). \]

Here we define $a_i, b_i, c_i$ and $d_i$ by
\[
\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = T^{m_t}S \cdots T^{m_1}S \quad \text{for } i \geq 1,
\]
with the convention that
\[
a_0 = d_0 = 1, \quad b_0 = c_0 = 0.
\]

Then these satisfy the recurrence relation for $i \geq 2$:

(iii) \[ a_i = m_i a_{i-1} - c_{i-1}, \quad c_i = a_{i-1}, \]

(iv) \[ b_i = m_i b_{i-1} - d_{i-1}, \quad d_i = b_{i-1}, \]

and if $m_j \geq 2$ for $1 \leq j \leq i$, then

(v) \[ a_i > 0. \]

The right-hand side of the formula (4) is a special case of the following proposition.

**Proposition 2.6.** For any $U \in \text{PSL}(2, \mathbb{Z})$ and its continued fraction expansion $[m_1, \ldots, m_t]$ with $m_i \geq 2$, let $\varphi(U)_t$ denote the sum,
\[
\varphi(U)_t := \sum_{k_1, k_2, \ldots, k_0} S_{k_{t+1}, k_t} T^{m_t} S_{k_t, k_{t-1}} T^{m_{t-1}} \cdots T^{m_1} S_{k_1, 0}.
\]

Then we have
\[
\varphi(U)_t = K_t \sum_{\gamma=0}^{4a_t-1} \left\{ x_t(-1)^\gamma + y_7e_8(m_t) e_4((-1)^{(r-1)/2} m_t) (e_4(\gamma) + e_4(-\gamma)) + z_t(-1)^{m_t} \right\}
\]
\[
\times \sum_{\pm} \pm e_8a_t r \left( -c_t \left\{ r\gamma + 2 \left( 2k_{i+1} + 1 \pm \frac{1}{c_t} \right) \right\}^2 \right).
\]
Proof. We will show the proposition by induction. In the case that $t = 1$, we calculate $\varphi(U)_1$:

$$\varphi(U)_1 = \sum_{k_1=0}^{(r-3)/2} S_{k_2,k_1} T_{k_1}^{m_1} S_{k_1,0}$$

$$= \frac{1}{2} \sum_{k_1=0}^{r-1} S_{k_2,k_1} T_{k_1}^{m_1} S_{k_1,0}$$

$$= -\frac{1}{2r} A^{-m_1} \sum_{k_1=0}^{r-1} e_{2r}(m_1(2k_1 + 1)^2)$$

$$\times \left\{ \sum_{\pm} \pm e_{2r}(\pm 2(2k_2 + 1)(2k_1 + 1)) \sum_{\pm} \pm e_{2r}(\pm 2(2k_1 + 1)) \right\}$$

$$= -\frac{1}{2r} A^{-m_1} \sum_{k_1=0}^{r-1} e_{2r}(m_1(2k_1 + 1)^2)$$

$$\times \left\{ \sum_{\pm} \pm e_{2r}(2(2k_1 + 1)(2k_2 + 1) \pm 1) \right\} + \left\{ \sum_{\pm} \pm e_{2r}(-2(2k_1 + 1)(2k_2 + 1) \pm 1) \right\},$$

where the second equality follows from Lemma 2.2. Noting that

$$e_{2r}(m_1(2(r - 1 - k) + 1)^2) = e_{2r}(m_1(2k + 1)^2),$$

$$e_{2r}(-2(2(r - 1 - k) + 1)(2k_2 + 1) \pm 1)) = e_{2r}(2(2k + 1)((2k_2 + 1) \pm 1)),$$

we calculate

$$\varphi(U)_1 = -\frac{1}{r} A^{-m_1} \sum_{k_1=0}^{r-1} e_{2r}(m_1(2k_1 + 1)^2) \left\{ \sum_{\pm} \pm e_{2r}(2(2k_1 + 1)(2k_2 + 1) \pm 1) \right\}$$

$$= -\frac{1}{r} \sum_{k_1=0}^{r-1} e_{2r}(4m_1 k_1^2) \left\{ \sum_{\pm} \pm e_{2r}((4m_1 + 4(2k_2 + 1 \pm 1)k_1)e_{2r}(2(2k_2 + 1 \pm 1)) \right\}.$$
Applying the reciprocity formula in Lemma 2.4, we have that
\[
\varphi(U) = \frac{-1}{r} \sqrt{\frac{ir}{4m_1}} \sum_{k_1=0}^{4m_1-1} \pm e_{8m_1} \left( -r \left\{ 1 + \frac{2(m_1 + 2k_2 + 1 \pm 1)}{r} \right\} e_{2r} (2(2k_2 + 1 \pm 1)) \right) \\
= -\frac{1}{2\sqrt{ra_1}} (\pi^{1/4} A^{-m_1} \sum_{k_1=0}^{4a_1-1} \pm (-1)^{k_1} e_{8a_1 r} (-(r k_1 + 2(2k_2 + 1 \pm 1))^2),
\]
where we use that \( m_1 = a_1 \). This confirms the first step in the induction.

We assume the result of the lemma inductively, and replace by the sum over \( k_t \) from 0 to \( r - 1 \) using the symmetries in Lemma 2.2, and then
\[
\varphi(U)_t = \sum_{k_t=0}^{(r-3)/2} S_{k_t+1,k_t} T_{k_t}^{m_t} \varphi(U)_{t-1}
\]
\[
= \frac{1}{2} \sum_{k_t=0}^{r-1} \left\{ \frac{1}{\sqrt{-r}} \sum_{k_{t-1}=0}^{4a_{t-1}-1} \pm e_{2r} (\pm 2(2k_{t-1} + 1)(2k_t + 1)) \right\} \left\{ e_{2r} (m_t (2k_t + 1)^2 - m_t) \right\} \\
\times \left\{ K_{t-1} \sum_{k_{t-1}=0}^{4a_{t-1}-1} (x_{t-1} (-1)^{k_{t-1}} + \gamma r e_{8a_{t-1} r} (-(r k_{t-1} + 2(2k_t + 1 \pm 1) + 1/c_{t-1}) + 2(2k_t + 1 \pm 1/c_{t-1})^2)) \right\}.
\]
We decompose the sum of \( k_{t-1} \) to that of modulo 4 in the following:
\[
\varphi(U) = \frac{1}{2\sqrt{-r}} e_{2r} (-m_t) K_{t-1} \left\{ x_t \varphi(U)_{t,0} - y_t \varphi(U)_{t,1} + z_t \varphi(U)_{t,2} - y_t \varphi(U)_{t,3} \right\},
\]
where
\[
\varphi(U)_{t,j} = \sum_{k_t=0}^{r-1} \left\{ e_{2r} (m_t (2k_t + 1)^2) \left\{ \sum_{\pm} e_{2r} (\pm 2(2k_{t-1} + 1)(2k_t + 1)) \right\} \\
\times \sum_{k_{t-1}=0}^{a_{t-1}-1} \sum_{\pm} e_{8a_{t-1} r} (-(c_{t-1} (4k_r + jr + 2(2k_t + 1 \pm 1/c_{t-1})^2)) \right\} \right\}. \tag{6}
\]
Using Lemma A.1, Lemma A.2 and Lemma A.3, we get
\[
\varphi(U) = \frac{-1}{2\sqrt{-r}} e_{2r} (-m_t) K_{t-1} e_{2r} \left( -\frac{1}{a_{t-1} c_{t-1}} \right) \sqrt{\frac{a_{t-1} r i}{a_t}} \\
\times \sum_{y=0}^{4a_{t-2}-1} \left\{ (-1)^y x_t + e_{8r} (m_r r) e_{4((-1)^{(r-1)/2} m_r)} (e_{4} (y) + e_{4} (-y)) y_t + (-1)^m z_t \right\} \\
\times \sum_{\pm} e_{8a_r} \left\{ a_{t-1} \left\{ r y + 2(2k_{t-1} + 1 \pm 1/a_{t-1})^2 \right\} \right\}.
Moreover, we have
\[
\frac{i}{2} \sqrt{\frac{a_{t-1} i}{a_t}} A^{-m_t} A^{-1/(a_{t-1} c_{t-1})} K_{t-1}
\]
\[
= \left( \frac{i}{2} \right)^{t-1} \frac{1}{2\sqrt{r a_t}} (e^{\pi i/4})^{t-1} A^{-(m_1 + \cdots + m_t)} A^{b_{t-1}/a_t - 1}
\]
\[
= \left( \frac{i}{2} \right)^{t-1} (e^{\pi i/4})^{t} \left( -\frac{1}{2\sqrt{r a_t}} \right) A^{-(m_1 + \cdots + m_t)} A^{b_t/a_t + 1/a_t c_t}
\]
\[
= K_t,
\]
where the second equality is obtained by Lemma 2.5(ii).

Hence, the claim holds. \(\square\)

To simplify the formula for \(\tau_r(L(p,q))\), we consider the Rademacher phi function and the Dedekind sum.

Definition 2.2. (Rademacher phi function and Dedekind sum [12]) The Rademacher phi function \(\Phi\) is an integer-valued function on \(\text{PSL}(2, \mathbb{Z})\) given by
\[
\Phi\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{cases} (a + d)/c - 12(\text{sign}(c))s(d, |c|), & \text{if } c \neq 0, \\ b/d, & \text{if } c = 0. \end{cases}
\]

Here, for \(c > 0\) the Dedekind sum \(s(d, c)\) is defined by
\[
s(0, 1) = 0,
\]
\[
s(d, c) = \sum_{k=1}^{c-1} \left( \left\langle \frac{k}{c} \right\rangle \right) \left( \left\langle \frac{dk}{c} \right\rangle \right),
\]
where for a real number \(x\),
\[
\langle x \rangle = \begin{cases} 0, & x \in \mathbb{Z}, \\ x - \lfloor x \rfloor - \frac{1}{2}, & \text{otherwise}. \end{cases}
\]

The Dedekind sum \(s(d, c)\) is determined by \(d\) modulo \(c\), and has the properties
\[
s(-d, c) = -s(d, c),
\]
\[
s(d^*, c) = s(d, c),
\]
where \(d^*\) is the inverse of \(d\) (mod \(c\)). The function \(\Phi\) is almost a homomorphism from \(\text{PSL}(2, \mathbb{Z})\) to \(\mathbb{Z}\); more precisely, Rademacher proves that if \(M'' = M'M\), then
\[
\Phi(M'') = \Phi(M) + \Phi(M') - 3 \text{ sign}(cc'c'').
\]

(7)

We define
\[
\Phi_t := \Phi(T^{m_t} S \cdots T^{m_1} S).
\]

We have \(\Phi_1 = \Phi(T^{m_1} S) = m_1\), and we find from (7) and Lemma 2.5 that
\[
\Phi_t - \Phi_{t-1} = m_t - 3 \text{ sign}(a_{t-1} c_{t-1}) \quad \text{for } t \geq 2, \quad \Phi_1 = m_1.
\]
Then we have
\[
\Phi_t = (m_1 + \cdots + m_t) - 3\{\text{sign}(a_1a_0) + \cdots + \text{sign}(a_{t-1}a_{t-2})\}.
\]
Since \(a_i > 0\),
\[
\Phi_t = (m_1 + \cdots + m_t) - 3(t - 1).
\]
Also from (7), for \(U = T^{m_1}S \cdots T^{m_t}S\),
\[
\Phi(SU) = \Phi(S) + \Phi(U) - 3 = (m_1 + \cdots + m_t) - 3t \quad \text{(by } \Phi(S) = 0). \tag{8}
\]
For \(U = \begin{pmatrix} p & b \\ q & d \end{pmatrix}\),
\[
SU = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & b \\ q & d \end{pmatrix} = \begin{pmatrix} -q & -d \\ p & b \end{pmatrix},
\]
and by the definition of \(\Phi\),
\[
\Phi(SU) = \frac{b - q}{p} - 12s(b, p). \tag{9}
\]
Hence, from (8) and (9) with \(U = T^{m_1}S \cdots T^{m_t}S = \begin{pmatrix} p & b \\ q & d \end{pmatrix}\),
\[
m_1 + \cdots + m_t = \frac{b - q}{p} - 12s(b, p) + 3t.
\]
As \(-bq = 1 \pmod{p}\),
\[
s(b, p) = -s(q, p).
\]
So we have the following result.

**Lemma 2.7.** Let \(T^{m_1}S \cdots T^{m_t}S = \begin{pmatrix} p & b \\ q & d \end{pmatrix}\). We have
\[
(m_1 + \cdots + m_t) - \left(\frac{b}{p} + \frac{1}{pq}\right) = 12s(q, p) - \frac{q^2 + 1}{pq} + 3t.
\]

We easily obtain the next lemma.

**Lemma 2.8.** It follows that \(\langle a_0 \rangle U_t = -A^{-3}e_8(3 - r)\).

Summing up, we obtain a formula for the quantum \SO(3) invariant of \(L(p, q)\).

**Proposition 2.9.**
\[
\tau_r(L(p, q)) = -\frac{1}{\sqrt{p}(A^2 - A^{-2})}A^{-12s(q,p) + (q^2+1)/pq} \left(\frac{e_8(-3r)}{2}\right)^t \\
\times \sum_{\gamma=0}^{4p-1} \left\{ x_t(-1)^\gamma + y_t e_8(m_t) e_4((-1)^{(r-1)/2}m_t)(e_4(\gamma) + e_4(-\gamma)) + z_t(-1)^{m_t} \right\} \\
\times \sum_{\pm} \pm e_{8r p} \left( -q \left( r \gamma + 2 \pm \frac{2}{q} \right)^2 \right). 
\]
Proof. We recall that
\[ \tau_r(L(p, q)) = \langle \omega_0 \rangle_{U_r}^{-1} \langle \omega_0, \ldots, \omega_0 \rangle_{D(p, q)}. \]
From Proposition 2.6, it follows that
\[ \langle \omega_0, \ldots, \omega_0 \rangle_{D(p, q)} \]
\[ = \mu_0^{-1} \phi(U)_{r_{k, 1=0}} \]
\[ = \mu_0^{-1} K_t \sum_{\gamma=0}^{4p-1} \left\{ x_r(-1)^\gamma + y_r e_8(m_r) e_4((-1)^{r-1} m_r)(e_4(\gamma) + e_4(-\gamma)) + z_r(-1)^{m_r} \right\} \]
\[ \times \sum_{\pm} \pm e_8 p \left( -q \left( r \gamma + 2 \pm \frac{2}{q} \right)^2 \right) \].
From Lemmas 2.7 and 2.8 with \( U = \begin{pmatrix} p & b \\ q & d \end{pmatrix} = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}, \) we can obtain that
\[ \langle \omega_0 \rangle_{U_r}^{-1} K_t = (-1)^t A^{3t} e_8(tr - 3t) \sqrt{\frac{e_4(1)}{2}} \left( A^{-2t} \right) e_8(t) A^{-(m_1 + \ldots + m_t) + (b/p + 1/pq)} \]
\[ = \frac{\sqrt{-1}}{\sqrt{rp}} A^{-12s(q, p) + (q^2 + 1)/pq} \left( \frac{e_8(-3r)}{2} \right)^t. \]
Since \( \mu_0 = (A^2 - A^{-2})/\sqrt{-r}, \) we obtain the proposition.
\[ \square \]

3. The semi-classical limit of \( \tau_r(L(p, q)) \)

In this section, we prove Theorem 1.1.
First we show some propositions.

**Proposition 3.1.** Let \( p \) and \( q \) be odd.

(1)
\[ \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_8 p (-q \gamma^2) e_2 p (-q \pm 1) \gamma \]
\[ = - \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} \sum_{\pm} \pm e_8 p (-q \gamma^2) e_2 p (-q \pm 1) \gamma \]
\[ = -8 e_8(-q r p) \sum_{n \in \mathbb{Z}/p \mathbb{Z}} e_2 p (-q r n^2) \sin \frac{2\pi q n}{p} \sin \frac{2\pi n}{p}. \]

(2)
\[ \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_8 p (-q r \gamma^2) e_2 p (-q \pm 1) \gamma \]
\[ = 16 \sum_{n \in \mathbb{Z}/p \mathbb{Z}} e_2 p (-q r n^2) \sin \frac{2\pi q n}{p} \sin \frac{2\pi n}{p}. \]

**Proof.** We show (1). We note that
\[ e_8 p (-q r (\gamma + 2p)^2) e_2 p (-q \pm 1) (\gamma + 2p) = e_8 p (-q r \gamma^2) e_2 p (-q \pm 1) \gamma (-1)^{q(p+y)}. \]
Since \(p\) and \(q\) are odd, by replacing \(\gamma\) with \(\gamma + 2p\), we see that
\[
\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2)e_{2p}(-(q \pm 1)\gamma) \\
= -\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2)e_{2p}(-(q \pm 1)\gamma),
\]
and so we can restrict this sum for odd \(\gamma\). Therefore, by putting \(\gamma = 2n + p\), we have
\[
\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2)e_{2p}(-(q \pm 1)\gamma) \\
= -e_{8}(-qrp) \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{2p}(-qrn^2)e_{p}(-(q \pm 1)n).
\]
We note that
\[
(-1)^{n+p}e_{2p}(-(qr(n + p)^2)e_{p}(-(q \pm 1)(n + p)) \\
= (-1)^{p(q+1)}(-1)^ne_{2p}(-qrn^2)e_{p}(-(q \pm 1)n).
\]
Then, since \(p(q + 1)\) is even, it follows that
\[
\sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{2p}(-qrn^2)e_{p}(-(q \pm 1)n) \\
= 2 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k \sum_{\pm} \pm e_{2p}(-qrk^2)e_{p}(-(q \pm 1)k) \\
= \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2) \\
\times \{e_{p}(-(q + 1)k) - e_{p}(-(q - 1)k) + e_{p}((q + 1)k) - e_{p}((q - 1)k)\} \\
= \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2)(e_{p}(qk) - e_{p}(-qk))(e_{p}(k) - e_{p}(-k)) \\
= -4 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}, \tag{10}
\]
where we obtain the second equality by making the average of the second line for \(k = \pm n\). Since we have that
\[
e_{2p}(-(qr(p - k)^2)) \sin \frac{2\pi q(p - k)}{p} \sin \frac{2\pi (p - k)}{p} = -e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p},
\]
by replacing even \(k\) with \(p - k\),
\[
\sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{2p}(-qrn^2)e_{p}(-(q \pm 1)n) \\
= 8 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}. \tag{11}
\]
Hence, (1) holds.
We show (2). As in (1), we see that

\[
\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_8p(-qr\gamma^2)e_2p(-(q \pm 1)\gamma)
\]

\[
= \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_8p(-qr\gamma^2)e_2p(-(q \pm 1)\gamma)
\]

\[
= 2 \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_2p(-qrn^2)e_p(-(q \pm 1)n).
\]

Hence, from (11), we obtain (2). \qed

**Proposition 3.2.** Let \( p \) be even and \( q \) be odd.

(1) \[
\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_8p(-qr\gamma^2)e_2p(-(q \pm 1)\gamma)
\]

\[
= \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} \sum_{\pm} \pm e_8p(-qr\gamma^2)e_2p(-(q \pm 1)\gamma)
\]

\[
= \begin{cases} 
-4 \sum_{n \in \mathbb{Z}/p\mathbb{Z}, n \text{ is even}} e_2p(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 0(8), \\
-4 \sum_{n \in \mathbb{Z}/p\mathbb{Z}, n \text{ is odd}} e_2p(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 4(8), \\
-4(1-(-1)^{p/4}) \sum_{n \in \mathbb{Z}/p\mathbb{Z}, n \text{ is odd}} e_2p(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv \pm 2(8).
\end{cases}
\]

(2) \[
\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_8p(-qr\gamma^2)e_2p(-(q \pm 1)\gamma)
\]

\[
= \begin{cases} 
-8 \sum_{n \in \mathbb{Z}/p\mathbb{Z}, n \text{ is even}} e_2p(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 0(8), \\
8 \sum_{n \in \mathbb{Z}/p\mathbb{Z}, n \text{ is odd}} e_2p(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 4(8), \\
-8(1-(-1)^{p/4}) \sum_{n \in \mathbb{Z}/p\mathbb{Z}, n \text{ is odd}} e_2p(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv \pm 2(8).
\end{cases}
\]

**Proof.** We show (1). We see that

\[
\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_8p(-qr\gamma^2)e_2p(-(q \pm 1)\gamma)
\]

\[
= \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_8p(-qr\gamma^2)e_2p(-(q \pm 1)\gamma)
\]

\[
= \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} \sum_{\pm} \pm e_2p(-qrn^2)e_p(-(q \pm 1)n).
\]
Since \( p(q + 1) \) is even, from the equality (10), it follows that

\[
\sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{2p}(-qrn^2)e_p(-(q \pm 1)n) = -4 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}.
\]

We note that

\[
(-1)^{p/2+k}e_{2p}\left(-qr\left(\frac{p}{2} + k\right)^2\right) \sin \frac{2\pi q}{p} \left(\frac{p}{2} + k\right) \sin \frac{2\pi k}{p} = (1)\rho/2e_8(-qr p)e_{2p}(-qr k^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}.
\]

Therefore, if \( p \equiv 0, 4(8) \), then we get that

\[
\sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p} = \begin{cases} 
\sum_{n \in \mathbb{Z}/p\mathbb{Z}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 0(8), \\
- \sum_{n \in \mathbb{Z}/p\mathbb{Z}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 4(8), 
\end{cases}
\]

and so the equalities in these cases of (1) hold. If \( p \equiv \pm 2(4) \), then, by replacing even \( k \) with \( p/2 + k \), we see that the equality in this case of (1) holds.

Hence, (1) holds.

In the same way as Proposition 3.1 (2), we get (2).

\[\square\]

**Proposition 3.3.** Let \( p \) be odd and \( q \) be even.

(1) \[
\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr \gamma^2)e_{2p}(-(q \pm 1)\gamma)
\]

\[
= \begin{cases} 
-16 \sum_{n \in \mathbb{Z}/p\mathbb{Z}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } q \equiv 0(8), \\
0 & \text{if } q \equiv 4(8), \\
-8(1 + e_8(-qr p)) \sum_{n \in \mathbb{Z}/p\mathbb{Z}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } q \equiv \pm 2(8).
\end{cases}
\]

(2) \[
\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_{8p}(-qr \gamma^2)e_{2p}(-(q \pm 1)\gamma) = 0.
\]
we obtain the second equality by replacing even
\[ k \]
if \( q \equiv 0(8) \),
\[ -8(1 - e_8(-qrp)) \sum_{n \in \mathbb{Z}/p\mathbb{Z}, \ n \ is \ odd} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} \quad \text{if } q \equiv \pm 2(8). \]

**Proof.** We show (1). We see that
\[
\sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-q r \gamma^2)e_{2p}(-(q \pm 1)\gamma)
= 2 \sum_{\gamma \in \mathbb{Z}/2p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-q r \gamma^2)e_{2p}(-(q \pm 1)\gamma).
\]

We note that
\[
(-1)^n e_{8p}(-qr(n+p)^2)e_{2p}(-(q \pm 1)(n+p))
= (-1)^{n+q+1} e_{8p}(-qrp) (-1)^{n+(q/2)n} e_{8p}(-qrn^2)e_{2p}(-(q \pm 1)n).
\]

By replacing \( n \) by \( n + p \), noting that \( p + q + 1 \) is even, we have that
\[
\sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{8p}(-qrn^2)e_{2p}(-(q \pm 1)n)
= e_8(-qrp) \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^{n+(q/2)n} \sum_{\pm} \pm e_{8p}(-qrn^2)e_{2p}(-(q \pm 1)n).
\]

We consider the right-hand side in the three cases.

If \( q \equiv 0(8) \), then we have that \((-1)^{n+(q/2)n} = (-1)^n \) and \( e_8(-qrp) = 1 \). Therefore, from the equality (10), we obtain that
\[
\sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{8p}(-qrn^2)e_{2p}(-(q \pm 1)n)
= -4 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{8p}(-qrk^2) \sin \frac{\pi qk}{p} \sin \frac{\pi k}{p}
= 8 \sum_{k \in \mathbb{Z}/p\mathbb{Z}, \ k \ is \ odd} e_{8p}(-qrk^2) \sin \frac{\pi qk}{p} \sin \frac{\pi k}{p}
= -8 \sum_{k \in \mathbb{Z}/p\mathbb{Z}, \ k \ is \ odd} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p},
\]

where we obtain the second equality by replacing even \( k \) with \( p - k \) and the third equality by replacing odd \( k \) with \( p - 2k \) in the previous line.

If \( q \equiv 4(8) \), then we have that \((-1)^{n+(q/2)n} = (-1)^n \) and \( e_8(- qr p) = -1 \). Therefore, we have that
\[
\sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{8p}(-qrn^2)e_{2p}(-(q \pm 1)n) = 0.
\]
If \( q \equiv \pm 2(8) \), then we have that \((-1)^{n+q/2}n = 1\). By replacing odd \( n \) with \( p + n \),
\[
\sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{8p}(-qr^2)e_{2p}(-q \pm 1)n
\]
\[
= (1 + e_{8}(-qrp)) \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} \sum_{\pm} \pm e_{8p}(-qr^2)e_{2p}(-q \pm 1)n
\]
\[
= (1 + e_{8}(-qrp)) \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} \sum_{\pm} \pm e_{2p}(-qrk^2)e_{p}(-q \pm 1)k
\]
\[
= -4(1 + e_{8}(-qrp)) \sum_{1 \leq k < p} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p},
\]
where we obtain the second equality by replacing \( n \) with \( 2k \) and the last equality in the same way as the equality (10). Hence, we obtain (1). Similarly, (3) holds.

In the same way as Proposition 3.1 (2), we get (2).

We put
\[
X(m) := e_{8}(mr)e_{4}((-1)^{(r-1)/2}m),
\]
for an odd \( r \geq 3 \) and an integer \( m \). From Proposition 2.6, \( x_t, z_t \) and \( y_t \) are determined by
\[
x_t = x_{t-1} + (-1)^{m_{t-1}}z_{t-1} + 2X(m_{t-1})x_{t-2} - (-1)^{m_{t-2}}z_{t-2}, \quad x_1 = 1,
\]
\[
z_t = x_{t-1} + (-1)^{m_{t-1}}z_{t-1} - 2X(m_{t-1})x_{t-2} - (-1)^{m_{t-2}}z_{t-2}, \quad z_1 = 0,
\]
\[
y_t = x_{t-1} - (-1)^{m_{t-1}}z_{t-1}, \quad y_1 = 0,
\]
and it follows that \( x_2 = 1 \) and \( z_2 = 1 \). Then, we have
\[
x_t + z_t = 2(x_{t-1} + (-1)^{m_{t-1}}z_{t-1}), \quad x_t - z_t = 4X(m_{t-1})(x_{t-2} - (-1)^{m_{t-2}}z_{t-2}). \tag{12}
\]

Let \( t \) be an integer with \( t \geq 3 \) and \( m_1, \ldots, m_t \) be integers with \( m_i \geq 2 \). Let \( I_{t}^{\pm} \) and \( I_{t}^{-} \) be empty sets. For \( t \geq 3 \), we define the subsets \( I_{t}^{\pm} \) and \( I_{t}^{-} \) of \( \{2, \ldots, t-1\} \) inductively in the following way: if \( m_i \) is even, then \( I_{t}^+ := \{t-1\} \cup I_{t-2}^- \); and \( I_{t}^+ := I_{t-1}^+ \) if \( m_i \) is odd, then \( I_{t}^- := I_{t-1}^- \cup I_{t-2}^+ \).

We put \( M(I) := \sum_{j=1}^{s} m_{i_j} \) for \( I = \{i_1, \ldots, i_k\} \) with integers \( m_{i_j} \).

**Proposition 3.4.** For \( t \geq 2 \),
\[
x_t \pm (-1)^{m_{t}}z_t = 2^{t-2}X(M(I_t^{\pm}))d_t^\pm,
\tag{13}
\]
where for \( t \geq 3 \),
\[
d_t^\pm := \begin{cases} 2 & \text{if } 2 \in I_t^\pm, \\ 1 + (-1)^{m_2} & \text{if } 2,3 \notin I_t^\pm, \\ 1 - (-1)^{m_2} & \text{if } 2 \notin I_t^\pm, 3 \in I_t^\pm. \end{cases}
\]

and \( d_2^\pm := 1 \pm (-1)^{m_2} \).

**Proof.** The claim for \( t = 2 \) is obvious. We will prove the claim for \( t \geq 3 \) by induction on \( t \). When \( t = 3 \), from the formula (12), we have
\[
x_3 - (-1)^{m_3}z_3 = \begin{cases} x_3 - z_3 = 4X(m_2)(x_1 - (-1)^{m_1}z_1) = 4X(m_2) & \text{if } m_3 \text{ is even}, \\ x_3 + z_3 = 2(x_2 + (-1)^{m_2}z_2) = 2(1 + (-1)^{m_2}) & \text{if } m_3 \text{ is odd}. \end{cases}
\]
On the other hand, if $m_3$ is even, then $I_3^- = \{2\} \cup I_1^- = \{2\}$ and so
\[ 2^{3-2} X(M(I_3^-))d_3^- = 2X(m_2) 2 = 4X(m_2). \]
If $m_3$ is odd, then $I_3^- = I_2^+ = \emptyset$ and so
\[ 2^{3-2} X(M(I_3^-))d_3^- = 2X(0)(1 + (-1)^{m_2}) = 2(1 + (-1)^{m_2}). \]
Since $x_3 + (-1)^{m_3}z_3 = x_3 - (-1)^{m_3+1}z_3$, the claim for $t = 3$ holds.

When $t = 4$, from the formula (12), we have
\[
x_4 - (-1)^{m_4}z_4 = \begin{cases} 
  x_4 - z_4 = 4X(m_3)(x_2 - (-1)^{m_3}z_2) = 4X(m_3)(1 - (-1)^{m_2}) & \text{if } m_4 \text{ is even}, \\
  x_4 + z_4 = 2(x_3 + (-1)^{m_3}z_3) = 2 \cdot 2X(M(I_3^+))d_3^+ & \text{if } m_4 \text{ is odd}.
\end{cases}
\]
On the other hand, if $m_4$ is even, then $I_4^- = \{3\} \cup I_2^- = \{3\}$ and so
\[ 2^{4-2} X(M(I_4^-))d_4^- = 4X(m_3)(1 - (-1)^{m_2}). \]
If $m_4$ is odd, then $I_4^- = I_3^+$, $d_4^- = d_3^+$ and so
\[ 2^{4-2} X(M(I_4^-))d_4^- = 2^2 X(M(I_3^+))d_3^+. \]
Since $x_4 + (-1)^{m_4}z_4 = x_4 - (-1)^{m_4+1}z_4$, the claim for $t = 4$ holds.

We assume the claim up to $t \geq 4$ inductively. If $m_{t+1}$ is even, then from the induction hypothesis, we have
\[
x_{t+1} - (-1)^{m_{t+1}}z_{t+1} = x_{t+1} - z_{t+1} = 4X(m_t)(x_{t-1} - (-1)^{m_{t-1}}z_{t-1}) \\
= 4X(m_t)2^{t-3}X(M(I_{t-1}^-))d_{t-1}^- = 2^{t-1}X(m_t)X(M(I_{t-1}^-))d_{t-1}^-.
\]
On the other hand, we have that $I_{t+1}^- = \{t\} \cup I_{t-1}^-$ and $d_{t+1}^- = d_{t-1}^-$ since $t \geq 4$. Thus, we get
\[ 2^{t+1-2} X(M(I_{t+1}^-))d_{t+1}^- = 2^{t-1}X(m_t)X(M(I_{t-1}^-))d_{t-1}^-.
\]
If $m_{t+1}$ is odd, then from the induction hypothesis, we have
\[
x_{t+1} - (-1)^{m_{t+1}}z_{t+1} = x_{t+1} + z_{t+1} = 2(x_t + (-1)^{m_t}z_t) \\
= 2 \cdot 2^{t-2} X(M(I_t^+))d_t^+ = 2^{t-1}X(M(I_t^-))d_t^+.
\]
On the other hand, we have that $I_{t+1}^- = I_t^+$ and $d_{t+1}^- = d_t^+$. Thus we get
\[ 2^{t+1-2} X(M(I_{t+1}^-))d_{t+1}^- = 2^{t-1}X(M(I_t^+))d_t^+.
\]
Hence, the claim for $x_{t+1} = (-1)^{m_{t+1}}z_{t+1}$ holds. Noting that $x_{t+1} + (-1)^{m_{t+1}}z_{t+1} = x_{t+1} - (-1)^{m_{t+1}+1}z_{t+1}$, the claim for $x_{t+1} + (-1)^{m_{t+1}+1}z_{t+1}$ holds.

\[ \square \]

**Lemma 3.5.** Let $t \geq 3$ and $m_i \geq 2$ be integers. We put $p_\ell / q_\ell := [m_1, \ldots, m_\ell]$. We define the condition $(C_i)$ for $I_{\delta}^\ell$ by
\[
(C_i) \begin{cases} 
  2 \in I_{\delta}^\ell & \text{if } m_1 \text{ is odd}, \\
  2, 3 \notin I_{\delta}^\ell \quad (t \geq 4), \quad I_{\delta}^\ell = I_3^- \quad (t = 3) & \text{if } m_1 \text{ is even, } m_2 \text{ is even}, \\
  2 \notin I_{\delta}^\ell, \; 3 \in I_{\delta}^\ell \quad (t \geq 4), \quad I_{\delta}^\ell = I_2^- \quad (t = 3) & \text{if } m_1 \text{ is even, } m_2 \text{ is odd}.
\end{cases}
\]
Proof. We will show the lemma by induction on $t$.

We consider (a) for $t = 3$. Since $p_3$ and $q_3$ are odd, we have $p_2$ is odd. When $m_3$ is even, $q_2$ is odd, and so $m_1$ is odd and $m_2$ is even. Since $I_3 = \{2\} \cup I_1 = \{2\}$ and $I_2 = \emptyset$, $I_3$ satisfies (C1) and from Proposition 3.4,

$$x_3 - (-1)^{m_3}z_3 = 2^{3-2}X(M(I_3^-))d_3^- = 4X(m_2) \neq 0,$$

$$x_2 - (-1)^{m_2}x_2 = 2^0X(M(I_2^-))d_2^- = 1 - (-1)^{m_2} = 0.$$

When $m_3$ is odd, $q_2$ is even and so $m_1$ is even. Since $I_3^- = I_2^+ = \emptyset$ and $I_2^- = \emptyset$, if $m_2$ is even, then $I_3^-$ satisfies (C1) and from Proposition 3.4,

$$x_3 - (-1)^{m_3}z_3 = 2X(M(I_3^-))d_3^- = 2X(M(I_3^-))(1 + (-1)^{m_2}) = 4,$$

$$x_2 - (-1)^{m_2}x_2 = 1 - (-1)^{m_2} = 0,$$

and if $m_2$ is odd, then $I_2^-$ satisfies (C1) and

$$x_3 - (-1)^{m_3}z_3 = 2X(M(I_3^-))(1 + (-1)^{m_2}) = 0,$$

$$x_2 - (-1)^{m_2}x_2 = 1 - (-1)^{m_2} = 2.$$

Hence, the claim (a) for $t = 3$ holds.

Similarly, the claims (b) and (c) for $t = 3$ hold.

Assume that the claims (a), (b) and (c) hold for $3, \ldots, t$. We consider (a) for $t + 1$. Since $p_{t+1}$ and $q_{t+1}$ are odd, we have that $p_t$ is odd.

When $m_{t+1}$ is even, $q_t$ is odd and from (12),

$$x_{t+1} - (-1)^{m_{t+1}}z_{t+1} = 4X(m_t)(x_{t-1} - (-1)^{t-1}z_{t-1}).$$

Since $p_t$ and $q_t$ are odd, by the induction hypothesis (a), if $I_t^-$ satisfies (C1), then $x_t - (-1)^{m_t}z_t \neq 0$ and $x_{t-1} - (-1)^{m_{t-1}}z_{t-1} = 0$. Then, $I_t^-$ satisfies (C1), $x_t - (-1)^{m_t}z_t \neq 0$, and $x_{t-1} - (-1)^{m_{t-1}}z_{t-1} = 0$. If $I_{t-1}^-$ satisfies (C1), then $x_{t-1} - (-1)^{m_{t-1}}z_{t-1} \neq 0$ and $x_t - (-1)^{m_t}z_t = 0$. Since $I_{t+1}^- = \{t\} \cup I_{t-1}^-$, $I_{t+1}^-$ satisfies (C1), $x_{t+1} - (-1)^{m_{t+1}}z_{t+1} \neq 0$, and $x_t - (-1)^{m_t}z_t = 0$. We note that when $t = 3$, if $I_2^-$ satisfies (C1), then $m_1$ is even and $m_2$ is odd and $I_4^- = \{3\} \cup I_2^- = \{3\}$ satisfies (C1).

When $m_{t+1}$ is odd, $q_{t+1}$ is even and from (12),

$$x_{t+1} - (-1)^{m_{t+1}}z_{t+1} = 2(x_t + (-1)^{m_t}z_t).$$

Since $p_t$ is odd and $q_t$ is even, by the induction hypothesis (c), if $I_t^-$ satisfies (C1), then $x_t - (-1)^{m_t}z_t \neq 0$ and $x_{t+1} - (-1)^{m_{t+1}}z_{t+1} = 0$. Then, $I_t^-$ satisfies (C1), $x_t - (-1)^{m_t}z_t \neq 0$, and $x_{t+1} - (-1)^{m_{t+1}}z_{t+1} = 0$. If $I_{t+1}^+$ satisfies (C1), then $x_t + (-1)^{m_t}z_t \neq 0$ and $x_{t+1} - (-1)^{m_{t+1}}z_{t+1} = 0$. Since $I_{t+1}^- = I_t^+, I_{t+1}^-$ satisfies (C1), $x_{t+1} - (-1)^{m_{t+1}}z_{t+1} \neq 0$, and $x_t - (-1)^{m_t}z_t = 0$.

Hence the claim (a) for $t + 1$ holds. Similarly, the claims (b) and (c) for $t + 1$ hold. □
Let us compute the asymptotic expansion of $\tau_c(L(p, q))$. Since, as $r \to \infty$,
\[
\sum_{\pm} \pm e_{8r}p \left( -q \left( r\gamma + 2 \pm \frac{2}{q} \right)^2 \right) \sim \sum_{\pm} \pm e_{8r}(-qr\gamma^2)e_{2p}(-(q \pm 1)\gamma),
\]
we have that
\[
\sum_{\gamma=0}^{4p-1} \left\{ x_t(-1)^{\gamma} + e_{8}(m_l r)e_4((-1)^{(r-1)/2}m_l)(e_4(\gamma) + e_4(-\gamma))y_t + (-1)^{m_l}z_t \right\}
\times \sum_{\pm} \pm e_{8r}p \left( -q \left( r\gamma + 2 \pm \frac{2}{q} \right)^2 \right) \}
\sim F(p, q, r),
\]
where
\[
F(p, q, r) = x_t \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^{\gamma} \sum_{\pm} \pm e_{8r}(-qr\gamma^2)e_{2p}(-(q \pm 1)\gamma) + e_{8}(m_l r)e_4((-1)^{(r-1)/2}m_l)y_t
\times \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_{8r}(-qr\gamma^2)e_{2p}(-(q \pm 1)\gamma)
+ (-1)^{m_l}z_t \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} \sum_{\pm} \pm e_{8r}(-qr\gamma^2)e_{2p}(-(q \pm 1)\gamma).
\]

We give an explicit formula of $F(p, q, r)$.

**PROPOSITION 3.6.** Let $p/q = p_1/q_1 = [m_1, \ldots, m_t]$ and put
\[
G_e(p, q, r) := \sum_{\frac{1 \leq k < p}{k \text{ is even}}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p},
\]
\[
G(p, q, r) := \sum_{\frac{1 \leq k < p}{k \text{ is odd}}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}.
\]

1. If $p$ and $q$ are odd, then
\[
F(p, q, r) = 2^{t+2}e_8(n_0(p, q)r)e_4((-1)^{(r-1)/2}n_1(p, q))G(p, q, r),
\]
where for $t \geq 3$,
\[
(n_0(p, q), n_1(p, q)) = \begin{cases} (4 - pq + M(I^+_t), M(I^+_t)) & \text{if } (C_t) \text{ with } I^+_t = I^+_t, \\ (m_t + M(I^+_{t-1}), m_t + M(I^+_{t-1})) & \text{if } (C_t) \text{ with } I^+_t = I^-_{t-1}. \end{cases}
\]
for $t = 2$, $(n_0(p, q), n_1(p, q)) = (m_2, m_2)$, and for $t = 1$, $(n_0(p, q), n_1(p, q)) = (4 - pq, 0).

2. If $p$ is even and $q$ is odd, then
\[
F(p, q, r) = \begin{cases} 2^{t+1}e_8(n_0(p, q)r)e_4((-1)^{(r-1)/2}n_1(p, q))G_e(p, q, r) & \text{if } p \equiv 0(8), \\ 2^{t+1}e_8(n_0(p, q)r)e_4((-1)^{(r-1)/2}n_1(p, q))G(p, q, r) & \text{if } p \equiv 4(8), \\ 2^{t+1}e_8(n_0(p, q)r)e_4((-1)^{(r-1)/2}n_1(p, q))(1 - (1)^{p/4})G(p, q, r) & \text{if } p \equiv \pm 2(8). \end{cases}
\]
where for $t \geq 3$,

\[
(n_0(p, q), n_1(p, q)) = \begin{cases} 
(4 + M(I^+_t), M(I^-_t)) & \text{if } p \equiv 0(8), (C_1) \text{ with } I^e_8 = I^+_t, \\
(4 + m_t + M(I^-_{t-1}), m_t + M(I^+_t)) & \text{if } p \equiv 0(8), (C_1) \text{ with } I^e_8 = I^-_{t-1}, \\
(4 + M(I^+_t), M(I^-_t)) & \text{if } p \equiv 0(8), (C_1) \text{ with } I^e_8 = I^-_{t-1}, \\
(m_t + M(I^-_{t-1}), m_t + M(I^+_t)) & \text{if } p \equiv 0(8), (C_1) \text{ with } I^e_8 = I^-_{t-1}, \\
(4 + M(I^+_t), M(I^-_t)) & \text{if } p \equiv 0(8), (C_1) \text{ with } I^e_8 = I^-_{t-1}, \\
(-pq + m_t + M(I^-_{t-1}), m_t + M(I^+_t)) & \text{if } p \equiv 0(8), (C_1) \text{ with } I^e_8 = I^-_{t-1},
\end{cases}
\]

for $t = 2,$

\[
(n_0(p, q), n_1(p, q)) = \begin{cases} 
(4 + m_2, m_2) & \text{if } p \equiv 0(8), \\
(m_2, m_2) & \text{if } p \equiv 4(8), \\
(-pq + m_2, m_2) & \text{if } p \equiv \pm 2(8).
\end{cases}
\]

and for $t = 1$, $(n_0(p, q), n_1(p, q)) = (4, 0)$.

(3) If $p$ is odd and $q$ is even, then

\[
F(p, q, r) = 2^{t+2}e_8(n_0(p, q) r) e_4((-1)^{(r-1)/2} n_1(p, q)) G(p, q, r),
\]

where for $t \geq 3$,

\[
(n_0(p, q), n_1(p, q)) = \begin{cases} 
(4 + M(I^+_t), M(I^-_t)) & \text{if } q \equiv 0(8), (C_1) \text{ with } I^e_8 = I^+_t, \\
(4 + M(I^+_t), M(I^-_t)) & \text{if } q \equiv 0(8), (C_1) \text{ with } I^e_8 = I^-_t, \\
(4 + M(I^+_t), M(I^-_t)) & \text{if } q \equiv 0(8), (C_1) \text{ with } I^e_8 = I^-_t, \\
(4 + M(I^+_t), M(I^-_t)) & \text{if } q \equiv 0(8), (C_1) \text{ with } I^e_8 = I^-_t, \\
(-pq + M(I^-_t), M(I^+_t)) & \text{if } q \equiv 0(8), (C_1) \text{ with } I^e_8 = I^-_t,
\end{cases}
\]

and for $t = 2$, when $m_2$ is odd,

\[
(n_0(p, q), n_1(p, q)) = \begin{cases} 
(4, 0) & \text{if } q \equiv 0(8), \\
(0, 0) & \text{if } q \equiv 4(8), \\
(4 - pq, 0) & \text{if } q \equiv \pm 2(8),
\end{cases}
\]

and when $m_2$ is even, $(n_0(p, q), n_1(p, q)) = (4, 0)$.

Proof. From Propositions 3.1, 3.2 and 3.3, we obtain the following formulas. If $p$ and $q$ are odd, then

\[
F(p, q, r) = \{-8e_8(-qr p)(x_t - (-1)^{m_t}z_t) \\
+ 16e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) y_t \} G(p, q, r),
\]

if $p$ is even and $q$ is odd, then

\[
F(p, q, r) = \begin{cases} 
\{-4(x_t + (-1)^{m_t}z_t) - 8e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) y_t \} G_p(p, q, r) & \text{if } p \equiv 0(8), \\
\{-4(x_t + (-1)^{m_t}z_t) + 8e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) y_t \} G(p, q, r) & \text{if } p \equiv 4(8), \\
\{-4(x_t + (-1)^{m_t}z_t) + 8e_8(-q p r) e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) y_t \} \\
\times (1 - (-1)^{(r-1)/4}) G(p, q, r) & \text{if } p \equiv \pm 2(8),
\end{cases}
\]
and if $p$ is odd and $q$ is even, then

$$F(p, q, r) = \begin{cases} -16x_t G(p, q, r) & \text{if } q \equiv 0(8), \\ -16(-1)^{m_i}G(p, q, r) & \text{if } q \equiv 4(8), \\ (-8(x_t + (-1)^{m_i}z_t) - 8e_8(-qrp)(x_t - (-1)^{m_i}z_t))G(p, q, r) & \text{if } q \equiv \pm 2(8). \end{cases}$$

We note that

$$x_t = \frac{1}{2}(x_t + (-1)^{m_i}z_t + x_t - (-1)^{m_i}z_t),$$

$$(-1)^{m_i}z_t = \frac{1}{2}(x_t + (-1)^{m_i}z_t - (x_t - (-1)^{m_i}z_t)).$$

We recall that $y_t = x_{t-1} - (-1)^{m_{i-1}}z_{t-1}$. From Lemma 3.5 for $t \geq 3$ and a direct calculation for $t = 1, 2$, we obtain the claim, using $-1 = e_8(4r)$.

We are now ready to prove Theorem 1.1.

Let $\rho_k$ be a representation of $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$ to $\text{SL}_2\mathbb{C}$ such that the eigenvalues of $\rho_k(g)$ for a generator $g$ of $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$ are $e^{\pm 2\pi ik/p}$.

**Theorem 1.1.** Let $p$ and $q$ be coprime integers with $p \geq 3$ and $p > q > 0$. Then, the quantum invariant $\tau_r(L(p, q))$ of $L(p, q)$ for odd $r$ is expanded as $r \to \infty$ in the form

$$\tau_r(L(p, q)) \sim e^{\pi i/2} e^{(\pi i/4)(n_0(p, q) - 3)r} e^{(\pi i/2)(-1)^{1/2}n_1(p, q)c(p)r}$$

$$\times \begin{cases} \sum_{1 \leq k < p \atop k \text{ is even}} e^{\pi i \text{CS}(L(p, q); ad\rho_k)r} \omega(L(p, q); \rho_k) & \text{if } p \equiv 0(8), \\ \sum_{1 \leq k < p \atop k \text{ is odd}} e^{\pi i \text{CS}(L(p, q); ad\rho_k)r} \omega(L(p, q); \rho_k) & \text{otherwise,} \end{cases}$$

where $n_0(p, q)$ and $n_1(p, q)$ are defined in Proposition 3.6 and $p/q = [m_1, \ldots, m_t]$ with $m_t \geq 2$.

$$c(p) := \begin{cases} 1 & \text{if } p \text{ is odd}, \\ \frac{1}{2} & \text{if } p \equiv 0(4), \\ \frac{1}{2}(1 + (-1)^{p/4}) & \text{if } p \equiv 2(4), \end{cases}$$

$\text{CS}(L(p, q); ad\rho_k)$ is the Chern–Simons invariant, and we put

$$\omega(L(p, q); \rho_k) = \frac{1}{\pi \sqrt{p}} \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}.$$ 

Further, we have that

$$\omega(L(p, q); \rho_k)^2 = \pm \frac{1}{32\pi^2} \text{Tor}(L(p, q); ad\rho_k),$$

where $\text{Tor}(L(p, q); ad\rho_k)$ is the Reidemeister torsion.

**Proof.** We note that, as $r \to \infty$,

$$\frac{1}{A^2 - A^{-2}} = \frac{1}{2i \sin(2\pi/r)} \sim \frac{1}{4\pi i} r.$$
From Proposition 2.9, the definition of $F(p, q, r)$ and Proposition 3.6, we have

$$
\tau_r(L(p, q)) \sim \frac{e_4(1)}{4\pi \sqrt{p}} \left( \frac{e_8(-3r)}{2} \right)^t F(p, q, r)
$$

$$
= \frac{e_4(1)}{\pi \sqrt{p}} e_8(n_0(p, q) - 3t) e_4((-1)^{(r-1)/2} n_1(p, q)) c(p) \ r
$$

$$
\times \begin{cases} 
\sum_{1 \leq k < p \atop k \text{ is even}} e_{2p}(-qrk^2) \sin \left( \frac{2\pi qk}{p} \right) \sin \left( \frac{2\pi k}{p} \right) 
& \text{if } p \equiv 0(8), \\
\sum_{1 \leq k < p \atop k \text{ is odd}} e_{2p}(-qrk^2) \sin \left( \frac{2\pi qk}{p} \right) \sin \left( \frac{2\pi k}{p} \right) 
& \text{otherwise.}
\end{cases}
$$

It is known, see [9], that the Chern–Simons invariant and the Reidemeister torsion of $L(p, q)$ are given by

$$
\text{CS}(L(p, q); ad \circ \rho_k) = -\frac{qk^2}{p},
$$

$$
\text{Tor}(L(p, q); ad \circ \rho_k) = \pm \frac{32}{p} \sin^2 \left( \frac{2\pi qk}{p} \right) \sin^2 \left( \frac{2\pi k}{p} \right).
$$

Hence, we obtain the theorem. \hfill \Box

**Appendix A**

In the appendix, we prove formulas of $\varphi(U)_{t, j}$ defined in (6).

**Lemma A.1.** We have that

$$
\varphi(U)_{t, 0} = -e_{2r} \left( \frac{1}{a_{t-1}c_{t-1}} \right) \sqrt{\frac{a_{t-1}r}{a_t}} \times \sum_{\gamma = 0}^{4a_t - 1} \sum_{\pm} \pm (-1)^\gamma e_{8a_t}( -a_{t-1} \left\{ r\gamma + 2\left( 2k_{t+1} + 1 \pm \frac{1}{c_{t-1}} \right) \right\}^2).
$$

**Proof.** We replace $2kr + 2k_t + 1 = 2\gamma + 1$ to get

$$
\varphi(U)_{t, 0} = \sum_{k_t = 0}^{r-1} \left\{ e_{2r}(m_t(2k_t + 1)^2) \right\} \sum_{\pm} \pm e_{2r}(\pm 2(2k_{t+1} + 1)(2k_t + 1))
$$

$$
\times \sum_{k = 0}^{a_t - 1} \sum_{\pm} \pm e_{8a_t}( -c_{t-1} \left\{ 4kr + 2\left( 2k_t + 1 \pm \frac{1}{c_{t-1}} \right) \right\}^2)\right\}.
$$
Applying the reciprocity formula in Lemma 2.4, we have that

\[
\varphi(U) = e_{2r} \left( \frac{1}{2} \left( \sum_{\gamma = 0}^{a_{t-1} r - 1} e_{2r} \left( \frac{m_t - e_{t-1}}{a_{t-1}} \right) (2\gamma + 1)^2 \right) \right)
\]

By replacing \(a_{t-1} r - \gamma - 1 =: \gamma'\), the second sum can be rewritten as

\[
\sum_{\gamma' = 0}^{a_{t-1} r - 1} e_{2r} \left( \frac{m_t - e_{t-1}}{a_{t-1}} \right) (2(a_{t-1} r - 1 - \gamma') + 1)^2 \]

From Lemma 2.5(iii),

\[
\varphi(U)_{t,0} = -2e_{2r} \left( \frac{1}{a_{t-1} r c_{t-1}} \right) e_{2r} \left( \frac{a_t}{a_{t-1}} \right) \sum_{\gamma = 0}^{a_{t-1} r - 1} e_{2a_{t-1} r} (4a_t \gamma^2) \]

Applying the reciprocity formula in Lemma 2.4, we have that

\[
\varphi(U)_{t,0} = -2e_{2r} \left( \frac{1}{a_{t-1} r c_{t-1}} \right) e_{2r} \left( \frac{a_t}{a_{t-1}} \right) \frac{1}{2} \sqrt{a_{t-1} r i} \frac{a_t}{a_{t-1}} \]

\[
\times \sum_{\gamma = 0}^{4a_{t-1} r - 1} \sum_{\gamma' = 0}^{a_{t-1} r - 1} \left\{ e_{2r} \left( 2 \left( 2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right) \right\} \]

\[
\times e_{8a_t} \left\{ -a_{t-1} r \left\{ \gamma + \frac{2}{r} \left( \frac{a_t}{a_{t-1}} + 2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\} \right\}
\]

\[
= -e_{2r} \left( \frac{1}{a_{t-1} r c_{t-1}} \right) \sqrt{a_{t-1} r i} \frac{a_t}{a_{t-1}} \]

\[
\times \sum_{\gamma = 0}^{4a_{t-1} r - 1} \sum_{\gamma' = 0}^{a_{t-1} r - 1} \left\{ (-1)^\gamma e_{8a_t} \left\{ -a_{t-1} \left\{ r \gamma + 2 \left( 2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\} \right\} \right\}.\]
Hence, we obtain the claim.

**Lemma A.2.** We have that

\[
\varphi(U)_{t,2} = -(-1)^m e_2r \left( \frac{1}{a_{t-1}c_{t-1}} \right) \sqrt{\frac{a_{t-1}r}{a_t}} \times \sum_{\gamma=0}^{4a_t-1} \sum_{\pm} \pm e_{8\alpha,\gamma} \left( -a_{t-1} \left\{ r\gamma + 2 \left( 2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\} \right)^2.
\]

**Proof.** By replacing \(2kr + r + 2k_t + 1 =: 2\gamma\), we get the required formula.

**Lemma A.3.** We have that, for \(j = 1, 3\),

\[
\varphi(U)_{t,j} = e_2r \left( \frac{1}{a_{t-1}c_{t-1}} \right) \sqrt{\frac{a_{t-1}r}{a_t}} \frac{1}{2} e_8(m,j,r) e_4((-1)^{(r-1)/2}m_t) \times \sum_{\gamma=0}^{4a_t-1} \left( e_4(\gamma) + e_4(-\gamma) \right) \sum_{\pm} \pm e_{8\alpha,\gamma} \left( -a_{t-1} \left\{ r\gamma + 2 \left( 2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\} \right)^2.
\]

**Proof.** By replacing \(4kr + jr + 4k_t + 2 =: 4\gamma - (-1)^{(r-j)/2}\), we obtain the required formula.

**Acknowledgements.** The first author is partially supported by JSPS KAKENHI Grant Number JP17K05256. The authors would like to thank the referee for careful reading and valuable suggestions that have enabled them to improve the article.

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