INTEGRAL STRUCTURES ON $p$-ADIC FOURIER THEORY

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Abstract. In this article, we study integral structures on $p$-adic Fourier theory by Schneider and Teitelbaum. As an application of our result, we give a certain integral basis of the space of $K$-locally analytic functions for any finite extension $K$ of $\mathbb{Q}_p$, generalizing the basis of Amice of locally analytic functions on $\mathbb{Z}_p$. We also use our result to prove congruences of Bernoulli-Hurwitz numbers at supersingular primes originally investigated by Katz and Chellali.

1. Introduction

In [ST], Schneider and Teitelbaum developed a theory of $p$-adic distributions on an integer ring $\mathcal{O}_K$ of a finite extension $K$ of $\mathbb{Q}_p$, which is a natural generalization of the classical theory of $p$-adic measures and distributions on $\mathbb{Z}_p$. In particular, they showed that there exists a natural one-to-one correspondence between $\mathbb{C}_p$-valued distributions on $\mathcal{O}_K$ and rigid analytic functions on the open unit disc in $\mathbb{C}_p$. The aim of this paper is to control denominators or more precisely, integral structures which appear in this correspondence. In the classical case on $\mathbb{Z}_p$, these are established by the works of Mahler and Amice, and their theory is applied to obtain the Kummer congruence between special values of Riemann zeta function as well as the construction of cyclotomic $p$-adic $L$-functions for elliptic curves defined over $\mathbb{Q}$. In this paper, we give an explicit construction by elementary calculations of Schneider-Teitelbaum’s $p$-adic Fourier theory. The advantage of our method is that we can control the integral structures on the Schneider-Teitelbaum correspondence. As an application of our result, we determine an integral structure on the ring of locally analytic functions on $\mathcal{O}_K$. We also use our result to prove the congruences investigated by Katz and Chellali ([Ka2], [Ch]) of Bernoulli-Hurwitz numbers at supersingular primes.

We now give the exact statements of our theorems. Let $p$ be a rational prime. Let $|\cdot|$ be the absolute value of $\mathbb{C}_p$ such that $|p| = p^{-1}$. Let $\mathcal{G}$ be the Lubin-Tate group of $K$ corresponding to a uniformizer $\pi$. Let $R$ be a subring of $\mathbb{C}_p$ containing $\mathcal{O}_K$ and let $LA_N(\mathcal{O}_K, R)$ be the space of locally analytic functions on $\mathcal{O}_K$ of order $N$ which take values in $R$. Namely, $f(x) \in LA_N(\mathcal{O}_K, R)$ if and only if $f(x)$ is defined as a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ on $a + \pi^N \mathcal{O}_K$ for any $a \in \mathcal{O}_K$. We let $\|f\|_{a,N} := \max_n \{|a_n \pi^{nN}|\}$. The space $LA_N(\mathcal{O}_K, R)$ is a Banach space induced by the norm $\max_{a \in \mathcal{O}_K} \{\|f\|_{a,N}\}$.

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Let \( \varpi_p \in \mathbb{C}_p \) be a \( p \)-adic period of \( \mathcal{G} \). We define \( \gamma(k) \) and \( \gamma(k) \) to be any elements such that
\[
|\gamma(k)| = \max_{0 \leq m \leq k} \{ |m!/\varpi_p^m| \}, \quad \gamma(k) = \min_{0 \leq m \leq k} \{ |m!/\varpi_p^m| \}.
\]

**Theorem 1.1.** Let \( f \) be a \( K \)-locally analytic function in \( \operatorname{LAN}(\mathcal{O}_K, \mathbb{C}_p) \). Let \( \varphi(t) = \sum_{k=0}^{\infty} c_k t^k \) be a rigid analytic function on the open unit disc and let \( \mu_{\varphi} \) be the distribution corresponding to \( \varphi \) by Schneider-Teitelbaum’s \( p \)-adic Fourier theory. Then we have
\[
\left| \int_{\alpha + \pi N \mathcal{O}_K} f(x) \, d\mu_{\varphi} \right| \leq |\gamma(0)| \left\| f \right\|_{a,N} \left\| \varphi \right\|_N
\]
where
\[
\left\| \varphi \right\|_N := \max_{k} \left\{ |c_k| \left| \gamma \left( \left[ \frac{k}{q^N} \right] \right) \right| \right\}
\]
and \([x]\) is the integral part of \( x \).

Since \( \left| \gamma \left( \left[ \frac{k}{q^N} \right] \right) \right| \sim p^{-kr} \) where \( r = 1/eq^N(q - 1) \), the value \( \left\| \varphi \right\|_N \) is approximated by
\[
\left\| \varphi \right\|_{B(p^{-r})} = \max_{x \in B(p^{-r})} \{ |\varphi(x)| \}
\]
where \( B(p^{-r}) \subset \mathbb{C}_p \) is the closed disc of radius \( p^{-r} \) centered at the origin. Finer versions of the above theorem is given in Theorem 4.3.

As an application of our main theorem, we obtain an estimate of the Fourier coefficients of Mahler like expansion of functions in \( \operatorname{LAN}(\mathcal{O}_K, \mathbb{C}_p) \). Let \( \lambda(t) \) be the formal logarithm of \( \mathcal{G} \), and following [ST], we define the polynomial \( P_n(x) \) by
\[
\exp(x\lambda(t)) = \sum_{n=0}^{\infty} P_n(x)t^n.
\]

**Theorem 1.2.** i) The series \( \sum_{n=0}^{\infty} a_n P_n(x\varpi_p) \) converges to an element of \( \operatorname{LAN}(\mathcal{O}_K, \mathcal{O}_{\mathbb{C}_p}) \) if \( a_n \) is of the form
\[
a_n = \gamma \left( \left[ \frac{n}{q^N} \right] \right) b_n
\]
with \( |b_n| \leq 1 \) and \( b_n \to 0 \) when \( n \to \infty \). Conversely, if \( f(x) \in \operatorname{LAN}(\mathcal{O}_K, \mathcal{O}_{\mathbb{C}_p}) \), then it has an expansion
\[
f(x) = \sum_{n=0}^{\infty} a_n P_n(x\varpi_p)
\]
of the form
\[
a_n = \gamma \left( \left[ \frac{n}{q^N} \right] \right) b_n
\]
for some \( b_n \) satisfying \( |b_n| \leq c \left\lfloor \frac{\pi}{q} \right\rfloor^N \) and \( b_n \to 0 \) when \( n \to \infty \), where \( c = 1 \) if \( e \leq p - 1 \), and \( c = |\mathfrak{p}(0)| \), otherwise.

**Corollary 1.3.** We let

\[
\varepsilon_{N,n} := \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) P_n(x\varpi_p), \quad (n = 0, 1, \cdots).
\]

We denote by \( L_N \) the \( \mathcal{O}_{C_p} \)-module topologically generated by \( \varepsilon_{N,n} \), then

\[
|\mathfrak{p}(0)|^{-2} \left\lfloor \frac{q}{\pi} \right\rfloor^N L_{AN}(\mathcal{O}_K, \mathcal{O}_{C_p}) \subset L_N \subset L_{AN}(\mathcal{O}_K, \mathcal{O}_{C_p}).
\]

In particular, \( L_N \otimes \mathbb{Q}_p = L_{AN}(\mathcal{O}_K, \mathbb{C}_p) \), namely, the functions \( \varepsilon_{N,n} \) form a Banach basis of \( L_{AN}(\mathcal{O}_K, \mathbb{C}_p) \).

Moreover, if \( e \leq p - 1 \), then

\[
\left\lfloor \frac{q}{\pi} \right\rfloor^{N+1} L_{AN}(\mathcal{O}_K, \mathcal{O}_{C_p}) \subset L_N \subset L_{AN}(\mathcal{O}_K, \mathcal{O}_{C_p}).
\]

In particular, if \( \mathcal{O}_K = \mathbb{Z}_p \), we recover Amice’s result \([Am]\), namely,

\[
\left\lfloor \frac{n}{p^N} \right\rfloor! \left( \frac{x}{n} \right), \quad (n = 0, 1, \cdots)
\]

form a topological basis of \( L_{AN}(\mathbb{Z}_p, \mathcal{O}_{C_p}) \). (Actually, we can show that it is a basis of \( L_{AN}(\mathbb{Z}_p, \mathbb{Z}_p) \).)

As another application, we derive from our estimate of the integral the congruence of Bernoulli-Hurwitz numbers \( BH(n) \) at supersingular primes established by Katz and Chellali ([Ka2], [Ch]). For a fixed \( b \in \mathcal{O}_K \) prime to \( p \), we put

\[
L(n) = \frac{(1 - b^{n+2})(1 - p^n)}{\gamma^np^{[p/\gamma]}(p^2-1)} \frac{BH(n+2)}{n+2}
\]

where \( \gamma \) is some \( p \)-adic unit explicitly given.

**Corollary 1.4.** Let \( l \) be a non-negative integer and \( p \) is a inert prime in \( K \).

i) We have \( L(n) \in \mathcal{O}_K \).

ii) Suppose that \( m \equiv n \mod p^l(q - 1) \). Then

\[
L(m) \equiv L(n) \mod p^l.
\]

Furthermore, if \( n \not\equiv 0 \mod q - 1 \), then

\[
L(m) \equiv L(n) \mod p^{l+1}.
\]

If \( n \equiv 0 \mod q - 1 \) and \( n \not= 0 \), then for \( L'(n) = L(n)/n \) we have

\[
L'(m) \equiv L'(n) \mod p^{l+1}.
\]
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2. Schneider-Teitelbaum’s $p$-adic Fourier theory.

Let $K$ be a finite extension of $\mathbb{Q}_p$ and $k = \mathbb{F}_q$ the residue field. Let $e$ be the absolute ramification index of $K$. We fix a uniformizer $\pi$ of $K$ and let $G$ be the Lubin-Tate formal group of $K$ associated to $\pi$. Let $R$ be a subring of $\mathbb{C}_p$ containing $\mathcal{O}_K$. For a natural number $N$ and an element $a$ of $\mathcal{O}_K$, we define the space $A(a + \pi^N\mathcal{O}_K, R)$ of $K$-analytic functions on $a + \pi^N\mathcal{O}_K$ as follows.

$$\{f : a + \pi^N\mathcal{O}_K \to R \mid f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, a_n \in \mathbb{C}_p, \pi^N a_n \to 0\}.$$

We equip the space $A(a + \pi^N\mathcal{O}_K, R)$ with the norm

$$\|f\|_{a,N} := \max_n \{|\pi^N a_n|\} = \max_{x \in a + \pi^N\mathcal{O}_p} \{|f(x)|\}.$$

We also define the space $L_{A_N}(\mathcal{O}_K, R)$ of $R$-valued locally $K$-analytic functions on $\mathcal{O}_K$ of order $N$ by

$$\{f : \mathcal{O}_K \to R \mid f|_{a + \pi^N\mathcal{O}_K} \in A(a + \pi^N\mathcal{O}_K, R) \text{ for any } a \in \mathcal{O}_K\}$$

which is a Banach space by the norm $\max_a \{\|f\|_{a,N}\}$. We put

$$L_A(\mathcal{O}_K, R) = \bigcup_N L_{A_N}(\mathcal{O}_K, R)$$

and equip it with the inductive limit topology. A continuous $R$-linear function $L_A(\mathcal{O}_K, R) \to R$ is called an $R$-valued distribution on $\mathcal{O}_K$. We denote the space of $R$-valued distributions on $\mathcal{O}_K$ by $D(\mathcal{O}_K, R)$, namely,

$$D(\mathcal{O}_K, R) = \varinjlim_{N} \text{Hom}_{\text{cont}}^R(L_{A_N}(\mathcal{O}_K, R), R).$$

We write an element of $D(\mathcal{O}_K, \mathbb{C}_p)$ symbolically as

$$\int d\mu : L_A(\mathcal{O}_K, \mathbb{C}_p) \to \mathbb{C}_p, \ f \mapsto \int f d\mu = \int_{\mathcal{O}_K} f(x) d\mu(x).$$

The space $D(\mathcal{O}_K, R)$ has a product structure by the convolution product:

$$\int_{\mathcal{O}_K} f(x)d(\mu * \nu)(x) := \int_{\mathcal{O}_K} \left(\int_{\mathcal{O}_K} f(x + y) d\mu(x)\right) d\nu(y).$$

For a compact open set $U$ of $\mathcal{O}_K$, we let

$$\int_U f(x) d\mu(x) := \int_{\mathcal{O}_K} f(x) \cdot 1_U(x) d\mu(x).$$
where $1_U$ is the characteristic function of $U$.

The structure of $D(O_K, \mathbb{C}_p)$ is well-known for the case $K = \mathbb{Q}_p$ and described through the so called Amice transform. We denote by $R^{\text{rig}}$ the ring of rigid analytic functions on the open disc of radius 1, namely, the ring of power series of the form $\varphi(T) = \sum_{n=0}^\infty c_n T^n$ such that $|c_n| r_0^n \to 0$ for any $0 < r_0 < 1$. Then there exists an isomorphism of topological $\mathbb{C}_p$-algebras

$$D(\mathbb{Z}_p, \mathbb{C}_p) \cong R^{\text{rig}}, \quad \mu \mapsto \varphi$$

that is characterized by the equation

$$c_n = \int_{\mathbb{Z}_p} \frac{x^n}{n!} \, d\mu(x)$$

or equivalently

$$\varphi(T) = \int_{\mathbb{Z}_p} (1 + T)^x \, d\mu(x).$$

For the Mahler expansion

$$f(x) = \sum_{n=0}^\infty a_n \frac{x^n}{n!}$$

of $f \in LA(\mathbb{Z}_p, \mathbb{C}_p)$, Amice showed that $|a_n| r^n \to 0$ for some $r > 1$ and hence we can compute the integral as

$$\int_{\mathbb{Z}_p} f(x) \, d\mu = \sum_{n=0}^\infty a_n c_n.$$

Schneider-Teitelbaum [ST] constructed isomorphism analogous with (3) for a general local field $K$.

Let $\varpi_p$ be a $p$-adic period of $\mathcal{G}$. Namely, by Tate’s theory of $p$-divisible groups and the Lubin-Tate theory we have

$$\text{Hom}_{\mathcal{O}_p}(\mathcal{G}, \hat{\mathcal{G}}_m) \cong \text{Hom}_{\mathbb{Z}_p}(T_\mathcal{G}, T_\hat{\mathcal{G}}_m) \cong \mathcal{O}_K.$$ 

(The last isomorphism is non-canonical.) Hence there exists a generator of the $\mathcal{O}_K$-module $\text{Hom}_{\mathcal{O}_p}(\mathcal{G}, \hat{\mathcal{G}}_m)$, which is written in the form of the integral power series $\exp(\varpi_p \lambda(t)) \in \mathcal{O}_p[[t]]$ where $\lambda(t)$ is the logarithm of $\mathcal{G}$. The element $\varpi_p \in \mathcal{O}_p$ is determined uniquely up to an element of $\mathcal{O}_K^\times$. We fix such a $\varpi_p$ and call it the $p$-adic period of $\mathcal{G}$. (If the height of $\mathcal{G}$ is equal to 1, the inverse of $\varpi_p$ is often called a $p$-adic period of $\mathcal{G}$, for example, see [15].) It is known that $|\varpi_p| = p^{-s}$, where $s = \frac{1}{p-1} - \frac{1}{e(q-1)}$ (see Appendix of [ST] or an elementary proof in [Box1] when $K/\mathbb{Q}_p$ is unramified). We define the polynomials $P_n(X) \in K[X]$ by the formal expansion

$$\exp(X\lambda(t)) = \sum_{n=0}^\infty P_n(X) t^n.$$
Note that in the case \( G = \hat{G}_m, \pi = p \) and \( \lambda(t) = \log(1 + t) \), the polynomial \( P_n(X) \) is no other than the binomial polynomial \( \binom{X}{n} \). By construction, \( P_n(x\varpi_p) \) is in \( \mathcal{O}_{\mathbb{C}_p} \) if \( x \in \mathcal{O}_K \).

**Theorem 2.1** (Schneider-Teitelbaum [ST]).

i) The series \( \sum_{n=0}^{\infty} a_n P_n(x\varpi_p) \) converges to an element of \( LA(\mathcal{O}_K, \mathbb{C}_p) \) if \( \lim_n |a_n|^{1/n} < 1 \). Conversely, any locally \( K \)-analytic function \( f(x) \) on \( \mathcal{O}_K \) has a unique expansion

\[
f(x) = \sum_{n=0}^{\infty} a_n P_n(x\varpi_p)
\]

for some sequence \((a_n)_n\) in \( \mathbb{C}_p \) such that \( \lim_n |a_n|^{1/n} < 1 \).

ii) There exists an isomorphism of topological \( \mathbb{C}_p \)-algebras

\[
D(\mathcal{O}_K, \mathbb{C}_p) \cong R_{\text{rig}}
\]

having the following characterization property: if \( \varphi(T) = \sum_{n=0}^{\infty} c_n T^n \) corresponds to a distribution \( \mu \), then

\[
c_n = \int_{\mathcal{O}_K} P_n(x\varpi_p) \, d\mu(x)
\]

or equivalently

\[
\varphi(t) = \int_{\mathcal{O}_K} \exp(x\varpi_p \lambda(t)) \, d\mu(x).
\]

### 3. Power sums

In this section, we give an estimate of the absolute value of the power sum

\[
\partial^n G \sum_{t_N \in \mathcal{G}[\pi^N]} (t \oplus t_N)^k |_{t=0}
\]

where \( \partial_G \) is the differential operator \( \lambda'(t)^{-1}(d/dt) \) and \( \mathcal{G}[\pi^N] \) is the kernel of the multiplication \( \pi^N \) of \( \mathcal{G} \). This estimate is crucial for everything in this paper. We use Newton’s method to compute this value.

We let \( \gamma[l,n] \) and \( \underline{\gamma}[l,n] \) be any elements of \( \mathbb{C}_p \) such that

\[
|\gamma[l,n]| = \max_{t \leq m \leq n} \{|m!/\varpi_p^m|\}, \quad |\underline{\gamma}[l,n]| = \min_{t \leq m \leq n} \{|m!/\varpi_p^m|\}
\]

for \( l \leq n \). For \( l > n \), we put \( |\gamma[l,n]| = 0 \) and \( |\underline{\gamma}[l,n]| = \infty \). For simplicity, we put \( \overline{\gamma}(k) = \overline{\gamma}[k,\infty] \) and \( \underline{\gamma}(k) = \underline{\gamma}[0,k] \). Note that since approximately \( |m!/\varpi_p^m| \sim p^{-m(m-1)/2} \) for large \( m \), the value \( \overline{\gamma}(k) \) is well-defined.

**Proposition 3.1.**

i) The absolute values of \( \overline{\gamma}(k) \) and \( \underline{\gamma}(k) \) are decreasing for \( k \).

ii) We have

\[
|\underline{\gamma}(k)| \leq |\overline{\gamma}(k)|, \quad |\overline{\gamma}(k)| \leq |\overline{\gamma}(0)| \cdot |\underline{\gamma}(k)|.
\]

iii) We have

\[
|\underline{\gamma}(k_1 + \cdots + k_n)| \leq |\underline{\gamma}(k_1)| \cdots |\underline{\gamma}(k_n)|
\]
and
\[ |\gamma(k_1 + \cdots + k_n)| \leq |\gamma(0)||\gamma(k_1)| \cdots |\gamma(k_n)| \leq |\gamma(0)||\gamma(k_1)| \cdots |\gamma(k_n)|.\]

iv) We have
\[ \frac{1}{p^{\nu - \frac{k}{\epsilon(q-1)}}} \leq |\gamma(k)| \leq 1.\]

**Proof.** i) is clear. For ii), first we have \(|\gamma(k)| \geq |k!/\varpi_p^k| \geq |\gamma(k)|). Suppose \(\gamma(k) = k_1!/\varpi_p^{k_1}\) and \(\gamma(k) = k_2!/\varpi_p^{k_2}\). Then \(k_1 \geq k \geq k_2\) and
\[ \left| \frac{k_1!}{\varpi_p^{k_1}} \cdot \frac{k_2!}{\varpi_p^{k_2}} \right| = \left| \frac{(k_1 - k_2)!}{k_2} \right| \leq |\gamma(0)|.\]

For iii), suppose that \(\gamma(k_i) = l_i!/\varpi_p^{l_i}\) for \(l_i \leq k_i\). Then the assertion for \(\gamma\) follows from
\[ |\gamma(k_1 + \cdots + k_n)| \leq \left| \frac{(l_1 + \cdots + l_n)!}{\varpi_p^{l_1 + \cdots + l_n}} \right| \leq \left| \frac{(l_1 + \cdots + l_n)!}{l_1! \cdots l_n!} \right| \cdot \left| \frac{l_1!}{\varpi_p^{l_1}} \right| \cdots \left| \frac{l_n!}{\varpi_p^{l_n}} \right|.\]

The assertion for \(\gamma\) follows from that for \(\gamma\) and ii). For iv), suppose that \(\gamma(k) = l!/\varpi_p^{l}\) for \(l \leq k\). Then
\[ \frac{1}{p^{\nu - \frac{k}{\epsilon(q-1)}}} \leq \frac{1}{p^{\nu - \frac{i}{\epsilon(q-1)}}} \leq \left| \frac{l!}{\varpi_p^l} \right| = |\gamma(k)|.\]

\[ \square \]

If \(e \leq p - 1\), then we can determine \(|\gamma(k)|\) and \(|\gamma(k)|\) explicitly.

**Lemma 3.2.** Let \(k\) be a non-negative integer and let \(q\) be a power of \(p\).

i) For any integer \(0 \leq r < q\), we have \(\binom{kq+r}{r} \equiv 1 \mod p\).

ii) We have \(\binom{kq}{q} \in [k/q]\mathbb{Z}_p\).

**Proof.** i) is clear. For ii), we write \(k = aq + r\) with \(0 \leq r < q\). We put \((1 + x)^q = 1 + x^q + pf(x)\) for some integral polynomial \(f(x)\). Then
\[ (1 + x)^k = (1 + x^q + pf(x))^a (1 + x)^r \equiv (1 + x^q)^a (1 + x)^r \mod ap\mathbb{Z}_p[x].\]

Hence the coefficient of \(x^q\) in the above is in \(a\mathbb{Z}_p\). \(\square\)

**Proposition 3.3.** Let \(i, e\) and\( b\) be natural numbers. We put \(q = ph\). Then we have
\[ v_p(i!) \geq \frac{i}{p-1} - \frac{i}{e(q-1)} - h + \frac{1}{e} + \left[ \frac{i}{q} \right] \left( \frac{1}{e} - \frac{1}{p-1} + \frac{1}{e(q-1)} \right) + v_p\left( \frac{i}{q}! \right).\]

In the above, the equality holds if and only if \(i \equiv -1 \mod q\). In particular, if \(e \leq p - 1\) or \(i < q\), we have
\[ v_p(i!) \geq \frac{i}{p-1} - \frac{i}{e(q-1)} - h + \frac{1}{e} \]
and the equality holds if and only if \(i = q - 1\). In this case, \(|\gamma(0)| = |\pi/q|\).
Proof. First, we assume that $i < q$. We prove the inequality by induction on $h$. If $h = 1$, then $i < p$. Hence the left hand side is equal to zero, namely $v_p(i!) = 0$, and the right hand side take the maximum value when $i = p - 1$, which is also equal to zero. We assume that the inequality holds for natural numbers less than $h$. Since the right hand side is strictly increasing for $i$, and $v_p(i!)$ strictly increase only when $p$ divides $i$, we may assume that $i$ is of the form $i = kp - 1$ for some natural number $k \leq p^{h-1}$. We have

$$v_p(i!) = v_p((kp)!) - v_p(kp) = k - 1 + v_p((k-1)!).$$

On the other hand, we have

$$\frac{i}{p-1} - \frac{i}{e(q-1)} - h + \frac{1}{e} \leq (k-1) + \frac{k-1}{p-1} - \frac{k-1}{e(p^{h-1}-1)} - (h-1) + \frac{k-1}{e(p^{h-1}-1)} - \frac{kp-1}{e(q-1)} \leq k - 1 + v_p((k-1)!).$$

In the last inequality, we used the inductive hypothesis and $k \leq p^{h-1}$. Hence we have the desire inequality and the equality holds only when $p$ divides $i$, and $v_p(i!)$ strictly increase only when $p$ divides $i$. Hence the left hand side is equal to zero, namely $v_p(i!)$.

From the above argument and the induction, to have the equality, $i$ must be congruent to $0$. On the other hand, if $i \equiv 1 \mod q - 1$, then direct calculations give the equality.

Proposition 3.4. Suppose that $e \leq p - 1$, and that $e > 1$ or $h > 1$.

i) We have $|n!|/\alpha_p^n > 1$ for $0 < n < q$.

ii) For any non-negative integer $n$, $|\gamma(n)| = |n_0|/\alpha_p^{n_0}$ where $n_0 = [n/q]q$.

iii) For $n \equiv -1 \mod q$ and a natural number $i$, we have

$$\left\lfloor \frac{n}{\alpha_p^n} \right\rfloor > \left\lfloor \frac{(n+q)!}{\alpha_p^{n+q}} \right\rfloor > \left\lfloor \frac{(n+i)!}{\alpha_p^{n+i}} \right\rfloor.$$

In particular, for any non-negative integer $n$, we have $|\gamma(n)| = |n_1|/\alpha_p^{n_1}$ where $n_1 = [n/q]q + q - 1$.

Proof. We prove i) by induction for $h$ of $q = p^h$. If $h = 1$, then $n!$ is a p-adic unit and the assertion is clear. For general $q = p^h$, we write as $n = kp + r$ with $0 \leq r < p$. Then

$$\frac{n!}{\alpha_p^n} = \frac{(n)!}{\alpha_p^{kp+r}} \cdot \left(\frac{r!}{\alpha_p^r}\right)^{-1}.$$
and (4.3) i) and the induction for \( n \), we may assume that \( r = 0 \) and \( k \geq 1 \). Then

\[
v_p \left( \frac{(kp)!}{\omega_p^k} \right) = v_p((kp)! - \frac{kp}{p-1} + \frac{kp}{e(q-1)} < v_p(k!) - \frac{k}{p-1} + \frac{k}{e(p^k-1-1)}.
\]

By the inductive hypothesis for \( h \), the right hand side is negative or 0.

Next we prove ii). Suppose that \( m < n_0 \). Then

\[
\left| \frac{n_0!}{\omega_p^{n_0}} \right| \frac{m!}{\omega_p^m} = \left| \frac{n_0}{\omega_p} \right| \frac{(n_0 - 1)!}{m!} \frac{m - 1}{\omega_p^{n_0-m-1}} \leq \left| \frac{n_0}{\omega_p} \right| = |\frac{n_0\pi}{q\omega_p}| < 1.
\]

Suppose that \( n \geq m > n_0 \). We write as \( m = [n/q] + r \) with \( 0 \leq r < q \). Then i) and Lemma 3.2 i) show that

\[
\left| \frac{n_0!}{\omega_p^{n_0}} \right| \frac{m!}{\omega_p^m} = \left( \frac{m}{r} \right)^{-1} \frac{\pi^r}{r!} < 1.
\]

Finally, we show iii). Let \( n \) be such that \( n \equiv -1 \mod q \). We may assume that \( i < q \). We have

\[
\frac{(n+i)!}{\omega_p^{n+i}} \frac{(n+q)!}{\omega_p^{n+q}} = \frac{(n+i)!}{(n+q)!} \frac{q}{\pi} \frac{(i-1)!}{\omega_p^{i-1}} \frac{q^{-i}}{q!}
\]

where \( u = \frac{(n+q)!}{(q-1)!} \) is a \( p \)-adic unit by Lemma 3.2 i). By Proposition 3.3, the \( p \)-adic (additive) valuation of the right hand side is positive. If \( e \leq p - 1 \), then \( v_p(\pi/\omega_p) > 0 \) and hence the \( p \)-adic (additive) valuation of

\[
\frac{(n+q)!}{\omega_p^{n+q}} \frac{n!}{\omega_p^n} = \left( \frac{n+q}{q} \right) \frac{q!}{\pi} \frac{\pi}{\omega_p^{q-1}}
\]

is positive. \( \square \)

Next we investigate the absolute values of the coefficients of a power of the logarithm and the exponential map of the Lubin-Tate group. The case \( k = 1 \) in the proposition below is obtained in [IS].

**Proposition 3.5.** We put \( \partial = d/dt \). Then we have

\[
\left| \omega_p^{k} \frac{\partial^n \lambda(k)^k}{k! n!} \right|_{t=0} \leq |\gamma[k, n]|^{-1}, \quad \left| \partial^n \exp_G^k(t) \right|_{t=0} \leq |\omega_p^{n \pi[k, n]}|.
\]

**Proof.** The case for \( n < k \) or \( k = 0 \) is trivial. Suppose that \( n \geq k \geq 1 \). We first assume that the formal logarithm of \( G \) is given by

\[
\lambda(t) = \sum_{m=0}^{\infty} \frac{t^n}{\pi^m}.
\]

Then it suffices to show inequalities

\[
\left| \partial^n \lambda(t)^k \right|_{t=0} \leq |k! \omega_p^{-n-k}|, \quad \left| \partial^n \exp_G^k(t) \right|_{t=0} \leq |k! \omega_p^{-n-k}|.
\]
When $k = 1$, the inequality for $\lambda(t)$ is proven by direct calculations. We prove the general case by induction on $k$. We have

$$\partial^n \lambda(t)^k|_{t=0} = k \partial^{n-1} (\lambda(t)^k \lambda'(t))|_{t=0}$$

$$= k \partial^{n-1} \sum_{m=0}^{\infty} \frac{(n-1)q^m t^{m-1}}{\pi^m} |_{t=0} = \sum_{m=0}^{\infty} \frac{(n-1)q^m k}{\pi^m} \partial^{n-k} \lambda(t)^k|_{t=0}.$$  

Hence we have $|\partial^n \lambda(t)^k|_{t=0} | \leq |k! \varpi_p^{n-k}|.$

We put $\exp_G^k(t) = \sum a_n t^n$. We prove that $|n! a_n| \leq |k! \varpi_p^{n-k}|$ by induction for $n$. If $n = k$, this is true since $\exp_G^k(t) = t^k + \cdots$. We assume that the assertion is true for integers less than $n$. Since $\exp_G^k(\lambda(t)) = t^k$, we have

$$t^k = a_k \lambda(t)^k + a_{k+1} \lambda(t)^{k+1} + \cdots + a_n \lambda(t)^n + \cdots.$$  

By i) and the inductive hypothesis, we have

$$|a_m \partial^n \lambda(t)^m|_{t=0} \leq |k! \varpi_p^{n-k}|$$

for $m < n$. Since $\partial^n \lambda(t)^n|_{t=0} = n!$, the assertion is also true for $n$.

Now we consider a general parameter $s$. Then the logarithm for $G$ and the exponential with parameter $s$ are of the form $\lambda(\phi(s))$ and $\psi(\exp_G(s))$ for some $\phi(s), \psi(s) \in \mathcal{O}_K|[[s]]|^\times$. We put $\lambda(t)^k = \sum_{n=k}^{\infty} c_n^{(k)} t^n$ and $\lambda(\phi(s))^k = \sum \phi_n^{(k)} s^n$. Then we have shown $|c_n^{(k)}| \leq |k! \varpi_p^{n-k}/n!|$. Since $\phi_n^{(k)}$ is a linear sum of $c_l^{(k)} (k \leq l \leq n)$ with integral coefficients, we have

$$\left| \frac{\varpi_p^{n-k} c_n^{(k)}}{k!} \right| \leq \max_{k \leq l \leq n} \left\{ \left| \frac{c_l^{(k)} \varpi_p^{n-k}}{k!} \right| \right\} \leq \max_{k \leq l \leq n} \left\{ \left| \frac{\varpi_p^{n-k}}{l!} \right| \right\} = \gamma[k, n]^{-1}.$$  

Hence we have the inequality for the logarithm. The inequality for the exponential is straightforward. \qed

**Lemma 3.6.** i) Suppose that $f(t) \in \mathcal{O}_K[[t]]$ satisfies $f(t \oplus t_N) = f(t)$ for all $t_N \in \mathcal{G}[\pi^N]$. Then there exists a power series $g(t) \in \mathcal{O}_K[[t]]$ such that $f(t) = g([\pi^N]t)$.

ii) There exists an integral power series $g_k(t) \in \mathcal{O}_K[[t]]$ such that

$$\pi^{-N} \sum_{t_N \in \mathcal{G}[\pi^N]} (t \oplus t_N)^k = g_k([\pi^N]t).$$  

**Proof.** See [Col], Chapter III. \qed

We put

$$F(t, X) = \prod_{t_N \in \mathcal{G}[\pi^N]} (1 - (t \oplus t_N)X) = 1 + \alpha_1(t)X + \cdots + \alpha_q(t)X^q.$$  

For $\partial_X = \partial/\partial X$, we consider the power series

$$\frac{\pi^{-N} \partial_X F(t, X)}{F(t, X)} = -\sum_{k=0}^{\infty} \left( \pi^{-N} \sum_{t_N \in \mathcal{G}[\pi^N]} (t \oplus t_N)^{k+1} \right) X^k.$$  

(6)
By Lemma 3.6 and the above formula, we have \( \pi^{-N} \partial_X F(t, X) \in \mathcal{O}_K[[t]][X] \).

**Proposition 3.7.** Let \( k, n \) be non-negative integers and \( N \) a natural number. Then we have

\[
\left| \sum_{t_N \in \mathcal{G}[[\pi^n]]} \partial_G^n(t \oplus t_N)^k \right|_{t=0} \leq \left| \pi^{Nn+k_0} \pi^n \left( \frac{k}{q^N} \right) \pi(0) \right|
\]

where \( k_0 = \max\{[k/q^N] - n, 0\} \). We also have

\[
\left| \sum_{t_N \in \mathcal{G}[[\pi^n]]} \partial_G^n(t \oplus t_N)^k \right|_{t=0} \leq \left| \pi^{Nn} \pi^n \left( \frac{k}{q^N} \right) \right|.
\]

Moreover, if \( e \leq p - 1 \), we have

\[
\left| \sum_{t_N \in \mathcal{G}[[\pi^n]]} \partial_G^n(t \oplus t_N)^k \right|_{t=0} \leq \left| \pi^{Nn} \pi^n \left( \frac{k}{q^N} \right) \right|.
\]

**Proof.** We put \( G(t, X) = F(0, X) - F(t, X) \), then \( G(0, X) = G(t, 0) = 0 \). We have

\[
\frac{1}{F(t, X)} = \frac{1}{F(0, X) - G(t, X)} = \sum_{l=0}^{\infty} \frac{G(t, X)^l}{F(0, X)^{l+1}} \in \mathcal{O}_K[[t, X]].
\]

Since \( G(0, X) = 0 \) and \( G(t, X) \) is invariant for the translation \( t \mapsto t_N \), it is of the form

\[
G(t, X) = ([\pi^n]t)H([\pi^n]t, X)
\]

for some element \( H \) in \( \mathcal{O}_K[[t]][X] \). Since \( F(0, X) \equiv 1 \mod \pi \), the power series \( F(0, X)^{-l-1} \) is equal to

\[
\sum_{m=0}^{\infty} \binom{-l-1}{m} (F(0, X) - 1)^m = \sum_{m=0}^{\infty} \binom{l+m}{m} \pi^m \left( \frac{1 - F(0, X)}{\pi} \right)^m.
\]

Hence we have

\[
\frac{\pi^{-N} \partial_X F(t, X)}{F(t, X)} = \sum_{l=0}^{\infty} \pi^{-N} \partial_X F(t, X) \cdot G(t, X)^l \cdot F(0, X)^{-l-1}
\]

\[
= \sum_{l=0}^{\infty} \binom{l+m}{m} \pi^m \left( \frac{1 - F(0, X)}{\pi} \right)^m.
\]

To show the assertion for \( k + 1 \), we look the coefficient of \( X^k \) of (12). We consider the coefficients of the terms \( X^a, X^b \) and \( X^c \) with \( a + b + c = k \) of \( \pi^{-N} \partial_X F(t, X), G(t, X)^l \) and \( (1 - F(0, X))^m \pi^{-m} \) respectively. Since \( \deg \partial_X F(t, X) = q^N - 1 \), \( \deg G(t, X) = q^N \) and \( \deg (1 - F(0, X)) = q^N - 1 \) as polynomials for \( X \), we have \( a \leq q^N - 1, b \leq lq^N \) and \( c \leq m(q^N - 1) \). Then
In particular, the product of these coefficients is an integral linear combination of the terms of the form
\[ (l + m) \pi^m G_l([\pi^N]t) \]
where \( G_l(t) \) is a power series in \( t^l \mathcal{O}_K[[t]] \) and \( l, m \) satisfies
\( a + lq^N + m(q^N - 1) \geq a + b + c = k. \)

We estimate the absolute value of
\[ (l + m) \pi^m \partial_l^G G_l([\pi^N]t) |_{t=0}. \]

By Proposition 3.5 we have
\[ (l + m) \pi^m G_l([\pi^N]t) |_{t=0} = |\pi^{Nn} \frac{d^n}{dz^n} \exp_G(z)|_{z=0} \leq |\pi^{Nn} \overline{\omega}_p \gamma(d, n)|. \]

Therefore, we have
\[ |\partial_l^G G_l([\pi^N]t)|_{t=0} | \leq |\pi^{Nn} \overline{\omega}_p \gamma[l, n]|. \]

Hence we have (8). If \( n < l \), then (14) is zero and there is nothing to prove. We assume that \( n \geq l \). We write \( \gamma(l) = l'/\overline{\omega}_p \) for some \( l' \geq l \). Then
\[
\begin{align*}
\left|(l + m) \pi^m \partial_l^G G_l([\pi^N]t) |_{t=0}\right| &\leq \left|(l + m) \pi^{m+Nn} \overline{\omega}_p^m \gamma[l, n]\right| \\
&\leq \left|\pi^{Nn} \overline{\omega}_p^n \frac{(l + m)!}{\overline{\omega}_p^l + m} \frac{l'}{\overline{\omega}_p^{l' - l}} \frac{l'}{m!}\right|.
\end{align*}
\]

First we consider the case \( a \leq q^N - 2 \) or \( m \neq 0 \). Then by (13) we have
\[ l + m \geq \left[\frac{k + 1}{q^N}\right]. \]

In particular, \( m \geq \left[\frac{(k + 1)/q^N}{-n}\right] - n \) and the value (16) is less than or equal to the absolute value of
\[ \frac{\pi^{Nn+k_0} \overline{\omega}_p^{n+k_0}}{k_0!} \gamma\left(\left[\frac{k + 1}{q^N}\right]\right) \gamma(0) \]
where \( k_0 = \max\{\left[(k+1)/q^N\right] - n, 0\} \). Hence in this case we have (7). Suppose that \( c \leq p - 1 \). If \( l' < l + m \), then \( |\overline{\omega}_p^m| < |\overline{\omega}_p^{l' - l}| \) and hence the value (16) is less than that of \( \pi^{Nn} \overline{\omega}_p^n \gamma\left(\left[\frac{k + 1}{q^N}\right]\right) \). If \( l' \geq l + m \), then
\[ |\gamma(l)| = \left|\frac{l'}{\overline{\omega}_p}\right| \leq |\gamma(l + m)| \leq \gamma\left(\left[\frac{k + 1}{q^N}\right]\right)|. \]

Hence the value (15) is also less than or equal to the absolute value of \( \pi^{m+Nn} \overline{\omega}_p^m \gamma\left(\left[\frac{k + 1}{q^N}\right]\right) \). Hence in this case we have (9).
Finally we consider the case when \( a = q^N - 1 \) and \( m = 0 \). Then the coefficient of \( \pi^{-N} \partial_X F(t, X) \) of degree \( a \) is \( (q/\pi)^N \alpha_{q, N}(t) \), which is divisible by \( [\pi^N]t \). Hence in this case the product of the coefficient of \( X^a \) in \( \pi^{-N} \partial_X F(t, X) \), the coefficient of \( X^b \) in \( G(t, X)^l \) and the coefficient of \( X^c \) in \( (1 - F(0, X))^{m/\pi^m} \) is an integral linear combination of terms in the form \( G_{l+1}([\pi^N]t) \) for some \( G_{l+1}(t) \in t^{l+1} \mathcal{O}_K[[t]] \). In this case \( l \) satisfies \( l + 1 \geq \lfloor (k + 1)/q^N \rfloor \). Therefore

\[
|\partial_G^a G_{l+1}([\pi^N]t)|_{t=0} \leq \left| \pi^N \partial^a_{\varpi_p \pi} [l + 1, n] \right| \leq \left| \pi^N \partial^a_{\varpi_p \pi} \left( \left\lfloor \frac{k + 1}{q^N} \right\rfloor \right) \right|.
\]

If \( n < l + 1 \), then \( |a| \) is zero and there is nothing to prove. We assume that \( n \geq l + 1 \). In particular, by (13) we have \( n \geq \lfloor (k + 1)/q^N \rfloor \), and hence \( k_0 = \max\{(k + 1)/q^N - n, 0\} = 0 \). Therefore we have (7) and (9). □

4. Integral structures on \( p \)-adic Fourier theory

In this section, we give an explicit construction of Schneider-Teitelbaum’s \( p \)-adic distribution associated to a rigid analytic function on the open unit disc.

Let \( \varphi(t) \) be a rigid analytic function on the open unit disc. We define the integral with respect to the distribution \( \mu_{\varphi} \) associated to \( \varphi(t) \) so that the formula

\[
\int_{\mathcal{O}_K} \exp(x \varpi_p \lambda(t))d\mu_{\varphi} = \varphi(t)
\]

is true. If we had the Mahler like expansion for \( K \)-analytic functions at first, then it is easy to define the integral like as (4), but as in [ST], we first define the integral and then the Mahler like expansion for \( K \)-analytic function is shown by using this integral.

For \( a \in \mathcal{O}_K \) and a natural number \( N \), we let

\[
\int_{a + \pi^N \mathcal{O}_K} (x - a)^n d\mu_{\varphi} := \frac{1}{q^N \varpi_p^n} \left( \partial_G^a \sum_{t_N \in \mathcal{G}^{[\pi^N]}} \varphi_a(t \oplus t_N) \right) \bigg|_{t=0}
\]

where

\[
\varphi_a(t) := \exp(-a \varpi_p \lambda(t)) \varphi(t).
\]

We put \( \varphi(t) = \sum_{k=0}^{\infty} c_k t^k \) and \( \varphi_a(t) = \sum_{k=0}^{\infty} c_k^{(a)} t^k \). Then by Proposition 5.7 we have

\[
\left| \int_{a + \pi^N \mathcal{O}_K} (x - a)^n d\mu_{\varphi} \right| \leq \left| \gamma(0) \right| \left| \pi q \right|^N \left| \pi \right|^{Nn} \max_k \left| c_k^{(a)} \right| \cdot \left| \gamma \left( \left\lfloor \frac{k}{q^N} \right\rfloor \right) \right|
\]

(18)

\[
\left| \int_{a + \pi^N \mathcal{O}_K} (x - a)^n d\mu_{\varphi} \right| \leq \left| \gamma(0) \right| \left| \pi q \right|^N \left| \pi \right|^{Nn} \max_k \left| c_k \right| \cdot \left| \gamma \left( \left\lfloor \frac{k}{q^N} \right\rfloor \right) \right|
\]

(19)
Here for the last estimate, we used the facts that $c_k^{(a)}$ is an integral linear combination of $c_0, \ldots, c_k$ and the function $|\gamma(m)|$ for $m$ is decreasing.

Let $f$ be an element of $LA_N(\mathcal{O}_K, \mathcal{C}_p)$ and on $a + \pi^N \mathcal{O}_K$, we write $f$ in the form $\sum_{n=0}^{\infty} a_n (x - a)^n$ such that $a_n \pi^n \to 0$ if $n \to \infty$. We define the integral of $f$ on $a + \pi^N \mathcal{O}_K$ by

$$
\int_{a+\pi^N \mathcal{O}_K} f(x) \, d\mu := \sum_{n=0}^{\infty} a_n \int_{a+\pi^N \mathcal{O}_K} (x - a)^n \, d\mu_{\varphi}.
$$

We define

$$
\int_{\mathcal{O}_K} f(x) \, d\mu = \sum_{a \mod \pi^N} \int_{a+\pi^N \mathcal{O}_K} f(x) \, d\mu_{\varphi}.
$$

We have to show the well-definedness of the integral.

**Proposition 4.1.** i) The integral (20) is convergent and does not depend on the choice of the representative of $a \mod \pi^N$. The integral (21) does not depend on the choice of $N$.

ii) For a polynomial $f(x)$, we have

$$
\int_{\mathcal{O}_K} f(x) \, d\mu_{\varphi} = f(\varpi_p^{-1} \partial_{\mathcal{O}} T(t)|_{t=0}.
$$

**Proof.** Since $|\gamma(k/q^N)| \leq C k^{-\frac{1}{6}} (q^N)^{-\gamma}$, for some constant $C$ which depends only on $e, q$ and $N$, the value $\max_k \{ |c_k| \cdot |\gamma(k/q^N)| \}$ is finite. Hence the convergence follows from (19). We show that the integral (20) depends only on the class of $a$ modulo $\pi^N$. Since the integral is convergent, we may assume that $f$ is a monomial $(x - a)^n$. For $a'$ such that $a' \equiv a \mod \pi^N$, we put $b = a' - a$. Since

$$(x - a)^n|_{a' + \pi^N \mathcal{O}_K} = \sum_{l=0}^{n} \binom{n}{l} b^{n-l} (x - a')^l|_{a' + \pi^N \mathcal{O}_K},$$

it suffices to show that

$$
\int_{a + \pi^N \mathcal{O}_K} (x - a)^n \, d\mu_{\varphi} = \sum_{l=0}^{n} \binom{n}{l} b^{n-l} \int_{a' + \pi^N \mathcal{O}_K} (x - a')^l \, d\mu_{\varphi}.
$$

However, we have

$$
\varpi_p^{-n} \partial_T^k \mathcal{O}_{\varphi_a(T \oplus T_m)} = \varpi_p^{-n} \partial_T^k \left( \exp(b \varpi_p \lambda(T)) \varphi_a'(T \oplus T_N) \right)
$$

$$
= \exp(b \varpi_p \lambda(T)) \sum_{l=0}^{n} \binom{n}{l} b^{n-l} \varpi_p^{-l} \partial_T^k \left( \varphi_a'(T \oplus T_m) \right).
$$

Hence (22) follows.

Now we show that the integral (21) does not depend on $N$. It is sufficient to show the distribution relation

$$
\int_{a + \pi^N \mathcal{O}_K} f(x) \, d\mu_{\varphi} = \sum_{b \equiv a \mod \pi^N} \int_{b + \pi^{N+1} \mathcal{O}_K} f(x) \, d\mu_{\varphi}.
$$
where the sum runs over a representative \( b \) of \( \mathcal{O}_K/\pi^{N+1} \) such that \( b \equiv a \mod \pi^N \). To show this, replacing \( \varphi \) by \( \varphi_a \), we may assume that \( a = 0 \) and \( f(x) = x^n \). Then

\[
q^{N+1} \omega_p^n \sum_{b \equiv 0 \mod \pi^N} \int_{b+\pi^{N+1} \mathcal{O}_K} x^n \, d\mu_{\varphi}
\]

\[
= \sum_{b \equiv 0 \mod \pi^N} \sum_{i=0}^{\lfloor \log q \rfloor} \binom{n}{k} b^{n-k} \left( \omega_p^{n-k} \varphi_{\mathcal{O}} \sum_{t_{N+1} \in \mathcal{O}[\pi^{N+1}]} \varphi_b(t \oplus t_{N+1}) \right) \bigg|_{t=0}
\]

\[
= \sum_{b \equiv 0 \mod \pi^N} \left( \varphi_{\mathcal{O}} \sum_{t_{N} \in \mathcal{O}[\pi^{N}]} \exp(b \omega_p \lambda(t)) \varphi_b(t \oplus t_{N+1}) \right) \bigg|_{t=0}
\]

\[
= q \left( \varphi_{\mathcal{O}} \sum_{t_{N} \in \mathcal{O}[\pi^{N}]} \varphi(t \oplus t_{N}) \right) \bigg|_{t=0} = q^{N+1} \omega_p^n \int_{\pi^{N} \mathcal{O}_K} x^n \, d\mu_{\varphi}.
\]

The above calculation is also true when \( a = N = 0 \), and hence we have

\[
\omega_p^n \sum_{b \in \mathcal{O}_K/\pi^n} \int_{b+\pi^{N} \mathcal{O}_K} x^n \, d\mu_{\varphi} = \varphi_{\mathcal{O}} \big|_{t=0}.
\]

From this the assertion ii) follows. \( \square \)

For \( \varphi(t) = \sum_{k=0}^{\infty} c_k t^k \in R^{rig} \), we define \( \|\varphi\|_N \) by

\[
(24) \quad \|\varphi\|_N := \max_k \left\{ |c_k| \gamma\left( \left[ \frac{k}{q^N} \right] \right) \right\}.
\]

Since \( \gamma\left( \left[ \frac{k}{q^N} \right] \right) \sim p^{-kr} \) where \( r = 1/eq^N(q-1) \), the value \( \|\varphi\|_N \) is approximately,

\[
\|\varphi\|_{B(p^{-r})} = \max_{x \in B(p^{-r})} \{ |\varphi(x)| \}
\]

where \( B(p^{-r}) \subset C_p \) is the closed disc with radius \( p^{-r} \) at origin.

**Lemma 4.2.** For an element \( a \in \mathcal{O}_K \), we put \( \varphi_a(t) = \exp(-a \omega_p \lambda(t)) \varphi(t) \). as before. Then \( \|\varphi_a\|_N = \|\varphi\|_N \).

**Proof.** It suffices to show \( \|\varphi_a\|_N \leq \|\varphi\|_N \). This follows from the same argument showing (19). \( \square \)

Then Proposition 3.7 is rewritten as

**Theorem 4.3.** i) Suppose that for \( a \in \mathcal{O}_K \), the function \( f \in LA_N(\mathcal{O}_K, C_p) \) is given by a polynomial of degree \( d \) on \( a + \pi^N \mathcal{O}_K \). For \( \varphi_k(t) = t^k \), we have

\[
(25) \quad \left| \int_{a+\pi^N \mathcal{O}_K} f(x) \, d\mu_{\varphi_k} \right| \leq \left| \frac{1}{q^N} \right| \|f\|_{a,N}.
\]
We also have
\[
\left| \int_{a+\pi N} f(x) \, d\mu_{\varphi} \right| \leq |\gamma(0)| \left| \frac{\pi^{k_0 + N} \varepsilon_0}{k_0 q^N} \|f\|_{a,N} \right| \gamma\left(\left[\frac{k}{q^N}\right]\right),
\]
where $k_0 = \max\{[k/q^N] - d\}$. Moreover, if $e \leq p - 1$, then we have
\[
\left| \int_{a+\pi N} f(x) \, d\mu_{\varphi} \right| \leq |\gamma(0)| \left| \frac{\pi}{q^N} \|f\|_{a,N} \right| \gamma\left(\left[\frac{k}{q^N}\right]\right).
\]
i) We have
\[
\left| \int_{a+\pi N} f(x) \, d\mu_{\varphi} \right| \leq |\gamma(0)| \left| \frac{\pi}{q^N} \|f\|_{a,N} \right| \gamma\left(\left[\frac{k}{q^N}\right]\right).
\]
Moreover, if $e \leq p - 1$, then
\[
\left| \int_{a+\pi N} f(x) \, d\mu_{\varphi} \right| \leq |\gamma(0)| \left| \frac{\pi}{q^N} \|f\|_{a,N} \right| \gamma\left(\left[\frac{k}{q^N}\right]\right).
\]

**Corollary 4.4.** We have
\[
\left| \int_{a+\pi N} f(x) \, d\mu_{\varphi} \right| \leq p^{\frac{p}{p-1} + \frac{1}{q(q-1)}} |\gamma(0)| \left| \frac{\pi}{q^N} \|f\|_{a,N} \right| \gamma\left(\left[\frac{k}{q^N}\right]\right)
\]
where $r = 1/eq^N(q - 1)$ and
\[
\|\varphi\|_{\mathcal{B}(p^{-r})} := \max_k \left\{ |c_k| k^{-q r} \right\}.
\]
Moreover, if $e \leq p - 1$, then
\[
\left| \int_{a+\pi N} f(x) \, d\mu_{\varphi} \right| \leq p^{\frac{p}{p-1} + \frac{1}{q(q-1)}} |\gamma| N \left| \frac{\pi}{q^N} \|f\|_{a,N} \right| \gamma\left(\left[\frac{k}{q^N}\right]\right).
\]

**Proof.** The formula follows from
\[
|\gamma\left(\left[\frac{k}{q^N}\right]\right)| \leq k q^{-N} p^{\frac{p}{p-1} + \frac{1}{q(q-1)}} - eq^N(q - 1),
\]
\[
q \pi \left| \frac{n}{q^N} \right|^{-1} c^{-1} \leq \|P_n(x x_{\varphi})\|_{N} \leq c \left(\left[\frac{n}{q^N}\right]\right) \gamma\left(\left[\frac{n}{q^N}\right]\right),
\]
where $c = 1$ if $e \leq p - 1$ and $c = |\gamma(0)|$, otherwise.

**Proposition 4.5.** For $N \geq 1$, we have
\[
\left| \frac{q}{\pi} \left[ \frac{n}{q^N} \right] \right|^{-1} c^{-1} \leq \|P_n(x x_{\varphi})\|_{N} \leq \left(\left[\frac{n}{q^N}\right]\right) \left| \gamma(0) \right|.
\]

**Proof.** We put $\varphi_n(t) = t^n$ and consider the distribution $\mu_{\varphi_n}$ associated to the power series $\varphi_n$. Then by (26), we have
\[
1 = \left| \int_{O_K} P_n(x x_{\varphi}) \, d\mu_{\varphi_n} \right| \leq \max_k \left\{ \left| \int_{a+\pi N} P_n(x x_{\varphi}) \, d\mu_{\varphi_n} \right| \right\}
\leq \left| \frac{\pi}{q} \|P_n(x x_{\varphi})\|_{N} \cdot \gamma\left(\left[\frac{n}{q^N}\right]\right) \right| \gamma(0).
\]
Similarly, if \( e \leq p - 1 \), then by using (27), we get the lower estimate.

We show the upper estimate. We put \( P_n(x\pi^N x_p) = \sum_{k=1}^{n} a_k^{(n)} x^k \) for \( n \geq 1 \). By definition of \( P_n \), the value \( a_k^{(n)} \) is the coefficient of \( t^n \) of \( x_p^k \lambda([\pi^N] t)^k / k! \).

Since \( \gamma(k) \) is decreasing for \( k \), we may assume that \( \lambda(t) = \sum_{l=0}^{\infty} t^l / \pi^l \).

Therefore by Proposition 3.5 we have

\[
|a_k^{(n)}| \leq \left| \frac{\lambda([\pi^N] t) k!}{\gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right)} \right|^{-1}.
\]

Hence we have \( ||P_n(x\varpi_p)||_{0,N} \leq \left| \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) \right|^{-1} \). Then by the formula before Lemma 4.4 of [ST], for \( a \in \mathcal{O}_K \), we have

\[
||P_n(x\varpi_p)||_{a,N} \leq \max_{0 \leq i \leq n} ||P_i(x\varpi_p)||_{0,N} \leq \left| \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) \right|^{-1}.
\]

Now we prove that our definition of the distribution coincides with that of Schneider-Teitelbaum. Namely, it has the characterization property (5).

**Theorem 4.6.** Let \( \mu_\varphi \) be the distribution associated to a rigid analytic function \( \varphi(t) \) on the open unit disc. Then

\[
\varphi(t) = \int_{\mathcal{O}_K} \exp(x\varpi_p \lambda(t)) d\mu_\varphi.
\]

Conversely, for every distribution \( \mu \), there exists a unique rigid analytic function \( \varphi \) such that \( \mu = \mu_\varphi \). In particular, we have an isomorphism of algebras,

\[
D(\mathcal{O}_K, \mathbb{C}_p) \cong \mathbb{R}^\text{rig}.
\]

**Proof.** We have

\[
P_k(\partial \varphi(t))|_{t=0} = \frac{1}{k!} \partial^k \varphi(t)|_{t=0}
\]

where \( \partial = d/dt \) (for example, formula 6 of Lemma 4.2 of [ST]). We let \( \varphi_n(t) = t^n \) and \( \mu_{\varphi_n} \), the distribution associated to \( \varphi_n(t) \). Then by Proposition 4.1 ii) we have

\[
\int_{\mathcal{O}_K} P_k(x\varpi_p) \, d\mu_{\varphi_n} = \sum_{n=0}^{\infty} P_k(\partial \varphi(t))|_{t=0} = \begin{cases} 1 & (k = n) \\ 0 & (k \neq n). \end{cases}
\]

□
Hence if \( \varphi(t) = \sum_{k=0}^{\infty} c_k t^k \), then
\[
\int_{\mathcal{O}_K} P_k(x \varpi_p) \, d\mu_{\varphi} = c_k.
\]
The first assertion follows from this fact. Conversely, for a given \( \mu \), we put
\[
c_k := \int_{\mathcal{O}_K} P_k(x \varpi_p) \, d\mu.
\]
Since the distribution is a continuous linear operator on the Banach space \( LA_N(\mathcal{O}_K, \mathbb{C}_p) \) for every natural number \( N \), there exists a positive constant \( C \) depending only on \( \mu \) and \( N \) such that
\[
|c_k| = \left| \int_{\mathcal{O}_K} P_k(x \varpi_p) \, d\mu \right| \leq C \|P_k(x \varpi_p)\|_{N} \leq C p^{-\frac{1}{p-1}} q^{\frac{k}{q^{N}(q-1)}}
\]
where for the last inequality, we used Proposition 4.5. Hence for any \( 0 \leq r < 1 \), if we choose sufficiently large \( N \), we have \( |c_k| r^k \to 0 \) when \( k \to \infty \).

Hence \( \varphi(t) = \sum_{k=0}^{\infty} c_k t^k \) is a rigid analytic function on the open unit disc. Then by construction
\[
\varphi(t) = \int_{\mathcal{O}_K} \exp(x \varpi_p \lambda(t)) d\mu.
\]
Since the function \( (x - a)|_{a + \pi \mathcal{O}_K} \) is given by
\[
\frac{1}{q^N x \varpi_p^n} \partial_y^n \left( \sum_{t_N \in \mathcal{O}[\pi^N]} \exp((x - a) \varpi_p \lambda(t)) \right) |_{t=0},
\]
we have
\[
\int_{a + \pi \mathcal{O}_K} (x-a)^n d\mu = \frac{1}{q^N x \varpi_p^n} \partial_y^n \sum_{t_N \in \mathcal{O}[\pi^N]} \varphi_a(t \oplus t_N) |_{t=0} = \int_{a + \pi \mathcal{O}_K} (x-a)^n d\mu_{\varphi}.
\]
Since \( \pi^{-nN}(x-a)^n|_{a + \pi \mathcal{O}_K} \) for \( a \in \mathcal{O}_K \) and \( n = 0, 1, \cdot \cdot \cdot \) are topological generators of \( LA_n(\mathcal{O}_K, \mathbb{C}_p) \), we have
\[
\int_{\mathcal{O}_K} f(x) d\mu = \int_{\mathcal{O}_K} f(x) d\mu_{\varphi}
\]
for all \( f \in LA_N(\mathcal{O}_K, \mathbb{C}_p) \). Hence \( \mu = \mu_{\varphi} \).

**Theorem 4.7.** The series \( \sum_{n=0}^{\infty} a_n P_n(x \varpi_p) \) converges to an element of \( LA_N(\mathcal{O}_K, \mathbb{C}_p) \) if \( a_n \) is of the form
\[
a_n = \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) b_n
\]
with \( |b_n| \leq 1 \) and \( b_n \to 0 \) when \( n \to \infty \). Conversely, if \( f(x) \in LA_N(\mathcal{O}_K, \mathbb{C}_p) \), then it has an expansion
\[
f(x) = \sum_{n=0}^{\infty} a_n P_n(x \varpi_p)
\]
of the form

\[ a_n = \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) b_n \]

with \(|b_n| \leq c \left| \frac{\pi}{q} \right|^N \) and \( b_n \to 0 \) when \( n \to \infty \), where \( c = 1 \) if \( e \leq p - 1 \), and \( c = |\gamma(0)| \), otherwise.

Proof. We proceed as in the proof of Theorem 4.7 of [ST] except the estimate of the Mahler coefficients. Let \( \mu_{\varphi_n} \) be the distribution associated to \( \varphi(t) = t^n \) and put

\[ a_n := \int_{\mathcal{O}_K} f(x) \, d\mu_{\varphi_n}. \]

Then by Theorem 4.3, we have

\[ |a_n| = \left| \int_{\mathcal{O}_K} f(x) \, d\mu_{\varphi_n} \right| \leq c \left| \frac{\pi}{q} \right|^N \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right). \]

We write as \( a_n = \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) b_n \). We show that \( b_n \to 0 \) when \( n \to \infty \). We may assume that \( f(x) = \sum_{i=0}^{\infty} c_i (x - a)^i \) on \( a + \pi^N \mathcal{O}_K \) and \( f(x) = 0 \) outside of \( a + \pi^N \mathcal{O}_K \). For a given \( \epsilon > 0 \), we can take \( N_0 \) so that

\[ \| \sum_{i=N_0}^{\infty} c_i (x - a)^i \|_{\infty,N} < \epsilon. \]

Hence by (26), we have

\[ \left| \int_{a + \pi^N \mathcal{O}_K} \sum_{i=N_0}^{\infty} c_i (x - a)^i \, d\mu_{\varphi_n} \right| \leq \epsilon C_2 \left| \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) \right| \]

where \( C_1 \) is a positive constant independent of \( n \). On the other hand, also by (26), we have

\[ \left| \int_{a + \pi^N \mathcal{O}_K} \sum_{i=0}^{N_0} c_i (x - a)^i \, d\mu_{\varphi_n} \right| \leq C_2 \left| \frac{\pi^{n_0} \varpi^{n_0}}{n_0!} \right| \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) \]

where \( n_0 = \max\{[n/q^N] - N_0, 0\} \) and \( C_2 \) is a positive constant independent of \( n \). Since

\[ \left| \frac{\pi^{n_0} \varpi^{n_0}}{n_0!} \right| \leq p^{-1} q^{-e} \left( 1 - \frac{1}{q^e} \right) \to 0 \quad (n \to \infty), \]

combining with (33) and (34), we have

\[ \left| \int_{a + \pi^N \mathcal{O}_K} f(x) \, d\mu_{\varphi_n} \right| \leq \epsilon C_1 \left| \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) \right| \]
for sufficiently large $n$. Hence $b_n \to 0$ when $n \to \infty$. Then by i), the series $\sum_{k=0}^{\infty} a_n P_n(x\varpi_p)$ converges to a function in $LA_N(\mathcal{O}_K, \mathbb{C}_p)$. We put

$$g(x) = f(x) - \sum_{k=0}^{\infty} a_n P_n(x\varpi_p).$$

Then we have $\int_{\mathcal{O}_K} g(x) d\mu_{\varphi_n} = 0$ for all $n$, and hence $\int_{\mathcal{O}_K} g(x) d\mu = 0$ for all distribution $\mu$. Considering the Dirac distribution $\delta_n : h \mapsto h(a)$, we have $g(a) = 0$ for any $a$. Hence $f(x) = \sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$.

\begin{corollary}
We let $e_{n,n} = \gamma \left( \left[ \frac{n}{q^n} \right] \right) P_n(x\varpi_p), \quad (n = 0, 1, \cdots)$.

If $L_N$ be the $\mathcal{O}_{\mathbb{C}_p}$-module topologically generated by $e_{n,n}$, then

$$\left| \pi(0) \right|^{-2} \left[ \frac{q}{\pi} \right]^N L_{AN}(\mathcal{O}_K, \mathcal{O}_{\mathbb{C}_p}) \subset L_N \subset L_{AN}(\mathcal{O}_K, \mathcal{O}_{\mathbb{C}_p}).$$

In particular, the functions $e_n$ form a topological basis of the Banach space $L_{AN}(\mathcal{O}_K, \mathbb{C}_p)$.

Moreover, if $e \leq p - 1$, then

$$\left[ \frac{q}{\pi} \right]^{N+1} L_{AN}(\mathcal{O}_K, \mathcal{O}_{\mathbb{C}_p}) \subset L_N \subset L_{AN}(\mathcal{O}_K, \mathcal{O}_{\mathbb{C}_p}).$$

In particular, If $\mathcal{O}_K = \mathbb{Z}_p$, we recover Amice’s result, namely,

$$\left[ \frac{n}{p^n} \right] ! \left( \frac{x}{n} \right)$$

for $n = 0, 1, \cdots$ form a topological basis of $L_{AN}(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p})$.

\end{corollary}

5. Relations to Katz’s and Chellali’s results.

As an application, we reprove Katz’s and Chellali’s results (Ch, Ka2) from the $p$-adic Fourier theory.

First we recall results of Katz [Ka2] and Chellali [Ch]. Let $E$ be an elliptic curve with complex multiplication by the ring of integer $\mathcal{O}_K$ of an imaginary quadratic field $K$. For simplicity, we assume that $E$ is defined over $K$ and fix a Weierstrass model

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in \mathcal{O}_K$$

of $E/K$. Let $p$ be an odd prime. We assume that $p$ is inert in $K$ and does not divide the discriminant of the above Weierstrass model, or equivalently, $E$ has good supersingular reduction $p$. Then the Bernoulli-Hurwitz number $BH(n)$ is defined by

$$\varphi(z) = \frac{1}{z^2} + \sum_{n \geq 2} \frac{BH(n+2)}{n+2} \frac{z^n}{n!}$$

where $\varphi(z)$ is the Weierstrass $\varphi$-function for the model. Let $\epsilon$ be a root of unity in $\mathcal{O}_K$ such that the multiplication by $-\epsilon p$ gives the Frobenius
Let \((x, y) \mapsto (x^{p^2}, y^{p^2})\) of \(E\) mod \(p\). Let \(\gamma\) be a unit in the Witt ring \(W = W(F_{p^2})\) such that
\[
\gamma^{p^2-1} = -\epsilon^{-1} \frac{p^2!}{p^{p+1}(p^2-1)}.
\]
For a fixed \(b \in O_K\) prime to \(p\), we put
\[
L(n) = \left(1 - b^{n+2}\right) \left(1 - p^n\right) B_H(n + 2) \gamma^{n[p^{np/(p^2-1)}]/n + 2}.
\]

**Theorem 5.1 (Katz [Ka2]).** The number \(L(n)\) is integral. Let \(l\) and \(n\) be non-negative integers. Then
\[
L(n + p^l(p^2 - 1)) \equiv L(n) \mod p^l.
\]

Later, Chellali [Ch] refined the congruences as follows.

**Theorem 5.2 (Chellali [Ch]).** Let \(l\) and \(n\) be non-negative integers. If \(n \not\equiv 0\) mod \(p^2 - 1\), we have
\[
L(n + p^l(p^2 - 1)) \equiv L(n) \mod p^l + 1.
\]

If \(n \equiv 0\) mod \(p^2 - 1\) and \(n \neq 0\), put \(L'(n) = L(n)/n\), then
\[
L'(n + p^l(p^2 - 1)) \equiv L'(n) \mod p^l + 1.
\]

In the following, let \(K = \text{Frac}(W)\) be the unramified quadratic extension of \(\mathbb{Q}_p\) and let \(G\) be the Lubin-Tate group of height \(h = 2\) associated to the uniformizer \(\pi = -\epsilon p\). We assume that \([\pi]T = \pi T + T^q\) for \(q = p^2\) is an endomorphism of \(G\). It is known that the formal group of \(E\) at \(p\) is isomorphic to \(G\).

**Proposition 5.3.** Let \(\varphi\) be an integral power series and let \(\mu_\varphi\) be the corresponding distribution associated to \(\varphi\).

i) We have
\[
\left| \int_{O_K^\times} x^n \, d\mu_\varphi \right| \leq p.
\]

ii) If \(m \equiv n\) mod \(p^l(q - 1)\), then
\[
\left| \int_{O_K^\times} (x^m - x^n) \, d\mu_\varphi \right| \leq p^{l-1 + \frac{2}{q-1}}.
\]

iii) If \((q - 1)|n\) and \(m \equiv n\) mod \(p^l(q - 1)\), then
\[
\left| \int_{O_K^\times} \left( \frac{x^m - 1}{m} - \frac{x^n - 1}{n} \right) \, d\mu_\varphi \right| \leq p^{-l-1 + \frac{2p}{q-1}}.
\]

**Proof.** We have
\[
\int_{a+\pi O_K} x^n \, d\mu_\varphi = a^n \int_{a+\pi O_K} d\mu_\varphi + \sum_{k=1}^n \left( \begin{array}{c} n \\ k \end{array} \right) (x-a)^k a^{n-k} \, d\mu_\varphi.
\]
Then by the estimate (25) the absolute value of the first integral is less than or equal to $p$. By the estimate (27), the absolute value of the second integral is also less than or equal to $p$ since \[||(x-a)||_{a,1}|\gamma(0)| = 1.\] We put $m - n = k(q - 1)$. Then
\[
x^m - x^n = x^n \sum_{i=1}^{k} \binom{k}{i} (x^{q-1} - 1)^i
\]
\[
= k x^n (x^{q-1} - 1) + x^n \sum_{i=2}^{k} \binom{k-1}{i-1} \frac{(x^{q-1} - 1)^i}{i}
\]
\[
= k \left( c_0 + c_1(x-a) + c_2 \frac{(x-a)^2}{2} + c_3 \frac{(x-a)^3}{3} + \cdots \right)
\]
where $c_i$ are integers satisfying $p | c_0$. Since \[||(x-a)^i/i||_{a,1} \leq p^{-2} \] for $i \geq 2$, the assertion ii) follows from this fact.

For an integer $s$, we have
\[
\frac{(x^{q-1})^s - 1}{s} = \sum_{i=1}^{\infty} \frac{\log_p x^{q-1})^i}{i!} s^{i-1} = \sum_{i=1}^{\infty} \sum_{j+k=i} c_{i,j,k} \frac{x^{q-1} a^j}{k!} s^{i-1}
\]
for some integers $c_{i,j,k}$. If we write $m = s_1(q - 1)$ and $n = s_2(q - 1)$, then
\[
\frac{(x^{q-1})^{s_1} - 1}{s_1} - \frac{(x^{q-1})^{s_2} - 1}{s_2} = \sum_{i \geq j+k} c_{i,j,k} \frac{x^{q-1} a^j}{k!} s^{i-1} - s^{i-1}
\]
By the estimate (25), the integral of \[\frac{x^{q-1} a^j}{k!} s^{i-1}/j!\] is divisible by $p^{1-\frac{2p}{q-1}}$. The assertion iii) follows from this fact. \(\square\)

For $b \in \mathcal{O}_K$ prime to $p$, we put
\[
\tilde{\varphi}_b(z) = (1 - b^2[b]^*) \varphi(z)
\]
and $\varphi(t) = \varphi_b(\varphi)_{|z=\lambda(t)}$. Then $\varphi_b(z)$ has no pole at $z = 0$ and
\[
\varphi_b(z) = \sum_{n \geq 2} (1 - b^{n+2}) \frac{BH(n+2) z^n}{n+2 n!}
\]
It is known that $\varphi(t)$ is an integral power series. Similarly, for $c \in \mathcal{O}_K$ prime to $p$, we put
\[
\zeta_c(z) = (c - [c]^*) \zeta(z), \quad \zeta_{b,c}(z) = (1 - b[b]^*) \zeta_c(z)
\]
where $\zeta(z)$ is the Weierstrass zeta function and $\psi(t) = \zeta_{b,c}(z)_{|z=\lambda(t)}$. Note that $\zeta_c(z)$ is double periodic and $\zeta_{b,c}(z)$ has no pole at $z = 0$. Then
\[
\zeta_{b,c}(z) = \sum_{n \geq 3} (c - c^n)(1 - b^{n+1}) \frac{BH(n+1) z^n}{n+1 n!}
\]
and $\psi(t)$ is an integral power series.
Lemma 5.4. \[
\sum_{z_0 \in \frac{1}{p} \Gamma / \Gamma} \varphi_b(z + z_0) = p^2 \varphi_b(pz), \quad \sum_{z_0 \in \frac{1}{p} \Gamma / \Gamma} \zeta_c(z + z_0) = p \zeta_c(pz).
\]

Proof. It is known that \[
\sum_{z_0 \in \frac{1}{p} \Gamma / \Gamma} \varphi(z + z_0) = p^2 \varphi(pz).
\]
The first formula follows from this. The above formula also show that for a set of \(S\) of representatives of \(\frac{1}{p} \Gamma / \Gamma\), there exists a constant \(A(S)\) such that
\[
\sum_{z_0 \in S} \zeta(z + z_0) = p \zeta(pz) + A(S).
\]
We take \(S = -S\). Then since \(\zeta(z)\) is an odd function, \(A(S)\) should be zero. Therefore,
\[
\sum_{z_0 \in S} \zeta_c(z + z_0) = p \zeta_c(pz).
\]
Since \(\zeta_c(z)\) is an elliptic function, the left hand side does not depend on the choice of \(S\). \(\square\)

Proposition 5.5. We put \(B(n) = BH(n + 2)/(n + 2)\) if \(n \geq 2\) and \(0\) if \(n = -1, 0, 1\). For \(n \geq 0\), we have
\[
\varpi_n \int_{\mathcal{O}_K^\times} x^n d\mu_\phi = (1 - p^n)(1 - b^{n+2})B(n),
\]
\[
\varpi_n \int_{\mathcal{O}_K^\times} x^n d\mu_\psi = (1 - p^{n-1})(c - e^n)(1 - b^{n+1})B(n - 1).
\]

Proof. Since \(\varphi_b(z)\) and \(\zeta_{b,c}(z)\) are double periodic, for \(t_0 \in \mathcal{G}[p]\) we have \(\psi(t \oplus t_0) = \zeta_{b,c}(z + z_0)|_{z = \lambda(t)}\) and \(\phi(t \oplus t_0) = \varphi_b(z + z_0)|_{z = \lambda(t)}\) where \(z_0\) is an image of \(t_0\) by \(\mathcal{G}[p] \to E[p] \to \frac{1}{p} \Gamma / \Gamma\). (See for example, [BK1, Lemma 2.18.]) From this fact and the previous lemma, we have
\[
\phi(t) - \frac{1}{q} \sum_{t_0 \in \mathcal{G}[p]} \phi(t \oplus t_0) = (\varphi_b(z) - \varphi_b(pz)) |_{z = \lambda(t)},
\]
\[
\psi(t) - \frac{1}{q} \sum_{t_0 \in \mathcal{G}[p]} \psi(t \oplus t_0) = (\zeta_{b,c}(z) - p^{-1} \zeta_{b,c}(pz)) |_{z = \lambda(t)}.
\]
Hence
\[
\varpi_n \int_{\mathcal{O}_K^\times} x^n d\mu_\phi = \partial^n_\phi \left( \phi(t) - \frac{1}{q} \sum_{t_0 \in \mathcal{G}[p]} \phi(t \oplus t_0) \right) |_{t = 0}
= \partial_z (\varphi_b(z) - \varphi_b(pz)) |_{z = 0} = (1 - p^n)(1 - b^{n+2})B(n).
\]
The other equality is also shown similarly. \(\square\)
We put
\[ c(n) = (1 - p^n)(1 - b^{n+2})BH(n + 2) \frac{B}{n + 2}. \]

**Corollary 5.6.** i) We have
\[ \left| \frac{c(n)}{\omega_p^n} \right| \leq p. \]
Furthermore, if \( n \equiv 0 \mod q - 1 \), then
\[ \left| \frac{c(n)}{\omega_p^n} \right| \leq p^{\frac{p}{q-1}}. \]

ii) Suppose that \( m \equiv n \mod p^l(q - 1) \). Then
\[ \frac{c(m)}{\omega_p^n} \equiv \frac{c(n)}{\omega_p^n} \mod p^{l + 1 - \frac{2p}{q-1}}. \]
Furthermore, if \( n \not\equiv 0 \mod q - 1 \), then
\[ \frac{c(m)}{m^{\omega_p^n}} \equiv \frac{c(n)}{n^{\omega_p^n}} \mod p^{l + 1 - \frac{2p}{q-1}}. \]

**Proof.** For i), the first inequality follows from Proposition 5.3 i) for \( \mu_\phi \).
The second inequality follows from Proposition 5.3 ii) for \( l = 0 \). Note that \( \int \omega_p^n d\mu_\phi = 0 \). For ii), the first and third congruences follow from Proposition 5.3 for \( \phi \), and the second inequality for \( \psi \). \( \square \)

Next, we compare \( c(n) \) with \( L(n) \).

**Lemma 5.7.** We choose \( u \in \mathbb{C}_p \) so that \( \omega_p^{q-1} = p^r u^{q-1} \). Then \( u \) is a unit of \( \mathcal{O}_\mathbb{C}_p \) and
\[ \left( \frac{u}{\gamma} \right)^{q-1} \equiv 1 \mod p. \]

**Proof.** Simple calculation shows the valuation of \( u \) is zero. We have \( \lambda(t) = t + \theta t^q + \cdots \) with \( \theta = 1/\epsilon(p^q - p) \). The \( q \)-th coefficient of the integral power series \( \exp(\omega_p \lambda(t)) \) is
\[ \frac{\omega_p^q}{q!} + \omega_p \theta = \omega_p \theta \left( \frac{\omega_p^{q-1}}{\theta q!} + 1 \right). \]
Since \( \omega_p \theta \) is not integral, the valuation \( v_p((\omega_p^{q-1}/\theta q!) + 1) \geq 1. \) Thus
\[ \frac{\omega_p^{q-1}}{\theta q!} + 1 \equiv \left( \frac{u}{\gamma} \right)^{q-1} \frac{(1 - p^q)}{(q - 1)} + 1 \equiv - \left( \frac{u}{\gamma} \right)^{q-1} + 1 \mod p. \]
must be congruent to zero. \( \square \)
We write $n = n'(q-1) + r$ with $0 \leq r < q$ and put $c_r = u^{-r}p^{-[pr/(q-1)]}\omega_p^r$. Then
\[ \omega_p^n = c_r p^{[pm/(q-1)]}u^n. \]

Hence we have
\[ L(n) = c_r \left( \frac{u}{\gamma} \right)^n \frac{c(n)}{\omega_p^n}. \]

Therefore by Corollary 5.6 we have $|L(n)| < p$. (Note that if $n \not\equiv 0 \mod q - 1$, then $|c_r| < 1$.) Since $L(n) \in K$, we have $L(n) \in O_K$. Similarly, for $m \equiv n \mod p'(q - 1)$, the fact $L(n) \in O_K$, Lemma 5.7 and
\[ \frac{c(m)}{\omega_p^m} \equiv \frac{c(n)}{\omega_p^n} \mod p^{l - \frac{p}{q - 1}} \]

imply the congruence
\[ L(m) \equiv L(n) \left( \frac{u}{\gamma} \right)^{m-n} \equiv L(n) \mod p^{l - \frac{p}{q - 1}}. \]

Since this is a congruence between elements of $O_K$, we have
\[ L(m) \equiv L(n) \mod p^l. \]

Similarly, from Corollary 5.6 we have Katz’s and Chellali’s congruences.

**Theorem 5.8.** i) We have $L(n) \in O_K$.

ii) Suppose that $m \equiv n \mod p'(q - 1)$. Then
\[ L(m) \equiv L(n) \mod p^l. \]

Furthermore, if $n \not\equiv 0 \mod q - 1$, then
\[ L(m) \equiv L(n) \mod p^{l+1}. \]

If $n \equiv 0 \mod q - 1$, then
\[ L'(m) \equiv L'(n) \mod p^{l+1}. \]

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