CONSTRUCTION OF A FAMILY OF NAFIL LOOPS OF ODD ORDER
\[ n = 2m + 1 \]

RAOUL E. CAWAGAS

Abstract. The existence of NAFIL loops of every odd order \( n \geq 5 \) is established by construction. These are non-associative finite invertible loops that are simple and power-associative and they form an infinite family. The first member of this family is the NAFIL loop of order \( n = 5 \) which is known to define a Lie algebra with some possible application in particle physics.

1. Introduction

In studying any class of finite algebraic structures (like quasigroups, loops, or groups), the first thing we do is to define the class precisely. After this, the most important task is to show that the class so defined is not empty by showing specific examples of its members. However, a class with only a handful of objects as members is not very interesting. For this reason, we are more interested in a class with a large number of members.

Non-associative finite invertible loops (NAFIL) are loops in which every element has a unique two-sided inverse and they form an interesting class that includes the familiar Moufang, Bol, and IP loops. However, there are other members of this class that have not yet been sufficiently studied. For instance, several NAFIL loops of small order are known to define loop algebras that satisfy the Jacobi identity. Some of these loops are now being studied because of their possible applications in physics [1].

2. On the Existence of NAFIL Loops of Odd Order

In this paper, we shall prove:

Theorem 1. There exists at least one NAFIL loop of every odd order \( n \geq 5 \).

Proof.

To prove this theorem, we shall show how a NAFIL loop \((L_n, \star)\) of order \( n = 2m + 1 \) can be constructed for any value of \( m \geq 2 \). For this, we need two groups (one of order \( m \) and one of order \( k = m + 1 \)) and one quasigroup of order \( k \).

Let \( L_n = \{1, \ldots, m, m+1, \ldots, 2m+1\} \), where \( m \geq 2 \), be a set of order \( n = 2m+1 \) and let \( \star \) be a binary operation over \( L_n \). Next, let \( L(m) = \{1, \ldots, m\} \) be any group of order \( m \) (like the cyclic group \( C_m \)) and let \( L(k) = \{m+1, \ldots, 2m+1\} \) be a group of order \( k = m + 1 \) isomorphic to the cyclic group \( C_k \) of order \( k \). Hence,

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\( L_n = L(m) \cup L(k) \) such that \( L(m) \cap L(k) = \emptyset \). Moreover, let \( C_k = \{1, ..., k\} \) be the counter-cyclic quasigroup \( \mathbb{Z}_k \) of order \( k \) and let \( C_k^T \) be its transpose.

To construct the Cayley table of a system \((L_n, \star)\), we proceed as follows.

- First, we form the Latin square blocks \([L(m)]\), \([L(k)]\) (in normal form) and \([C_k]^T\) of the systems \(L(m)\), \(L(k)\), and \(C_k^T\), respectively.
- Second, using the block \([L(k)]\), we form two blocks \([L(k)]'\) and \([L(k)]''\), where
  - (a) \([L(k)]'\) is a block of the group \(L(k)\) in which row \(k - 1\) has been omitted.
  - (b) \([L(k)]''\) is a block of the group \(L(k)\) in which column \(k\) has been omitted.
- Third, using the block \([C_k]^T\), we form another block \([C_k]^T^\ast\) by replacing each element entry \(k\) of \([C_k]^T\) by elements of the set \(L(k)\) as indicated in Table 2(b).

The block \([L(k)]\) has the general form:

\[
\begin{array}{cccccccc}
  m+1 & m+2 & m+3 & \cdots & 2m-1 & 2m & 2m+1 \\
  m+2 & m+3 & m+4 & \cdots & 2m & 2m+1 & m+1 \\
  m+3 & m+4 & m+5 & \cdots & 2m+1 & m+1 & m+2 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  2m-1 & 2m & 2m+1 & \cdots & 2m-4 & 2m-3 & 2m-2 \\
  2m & 2m+1 & m+1 & \cdots & 2m-3 & 2m-2 & 2m-1 \\
  2m+1 & m+1 & m+2 & \cdots & 2m-2 & 2m-1 & 2m \\
\end{array}
\]

Table 1. General form of the block \([L(k)]\).

Starting with this block \([L(k)]\), we form the blocks \([L(k)]'\) (by deleting row \(k - 1\) of \([L(k)]\)) and \([L(k)]''\) (by deleting column \(k\) of \([L(k)]\)).

Next, we take the block \([C_k]^T\) of order \(k\) which has the following general form:

\[
\begin{array}{cccccccc}
  1 & k & k-1 & \cdots & 4 & 3 & 2 \\
  2 & 1 & k & \cdots & 5 & 4 & 3 \\
  3 & 2 & 1 & \cdots & 6 & 5 & 4 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  k-2 & k-3 & k-4 & \cdots & 1 & k & k-1 \\
  k-1 & k-2 & k-3 & \cdots & 2 & 1 & k \\
  k & k-1 & k-2 & \cdots & 3 & 2 & 1 \\
\end{array}
\]

Table 2(a). General form of the block \([C_k]^T\) of order \(k\).
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In this block \( [C_k]^T \), replace the entries \( k \) by the entries of the last column of \( [L(k)] \), that is, \( 2m+1, m+1, ..., 2m-2, 2m-1, 2m \), in this order, from (row 1, column 2), (row 2, column 3) all the way down to (row \( k-1 \), column \( k \)), and ending in (row \( k \), column 1). If this is done, we obtain the following block shown in Table 2(b) which we shall denote by \( [C_k]^T \).*

|     | 1   | 2m+1 | k-1 | \cdot | 4   | 3   | 2   |
|-----|-----|------|-----|------|-----|-----|-----|
| 2   | 1   | m+1  | \cdot| \cdot  | 5   | 4   | 3   |
| 3   | 2   | 1    | \cdot| \cdot  | 6   | 5   | 4   |
| \cdot| \cdot| \cdot | \cdot| \cdot  | \cdot| \cdot| \cdot|
| \cdot| \cdot| \cdot | \cdot| \cdot  | \cdot| \cdot| \cdot|
| k-2 | k-3 | k-4  | \cdot| \cdot  | 1   | 2m-2| k-1 |
| k-1 | k-2 | k-3  | \cdot| \cdot  | 2   | 1   | 2m-1|
| 2m  | k-1 | k-2  | \cdot| \cdot  | 3   | 2   | 1   |

Table 2(b). The block \( [C_k]^T \) obtained from \( [C_k]^T \).

Using the blocks \( [L(m)] \), \( [L(k)]' \), \( [L(k)]'' \), and \( [C_k]^T \) thus formed above, we can now construct the following Cayley table of a NAFIL loop \( (L_n, *) \) of order \( n = 2m + 1 \) as shown in Table 3(a).

| + | 1   | \cdot | m   | m + 1 | \cdot | 2m + 1 |
|---|-----|------|-----|------|------|--------|
| 1 |     |      |     |      |      |        |
| \cdot| \cdot| \cdot | \cdot| \cdot  | \cdot| \cdot  |
| \cdot| \cdot| \cdot | \cdot| \cdot  | \cdot| \cdot  |
| m | \cdot| \cdot | \cdot| \cdot  | [L(m)] | [L(k)]' |
| m + 1| \cdot| \cdot | \cdot| \cdot  | [L(k)]'' | [C_k]^T |

Table 3(a). Cayley table of a NAFIL loop \( (L_n, *) \) of order \( n = 2m + 1 \).

Since \( m \) is finite, the blocks \( [L(m)] \), \( [L(k)]' \), \( [L(k)]'' \), and \( [C_k]^T \) can always be constructed for all values of \( m \geq 2 \) and \( k = m + 1 \). Thus, the Cayley table of the system \( (L_n, *) \) can be constructed for all values of \( m \geq 2 \). This implies that these systems of odd order \( n = 2m + 1 \) form an infinite family.

To show that \( (L_n, *) \) is indeed a NAFIL loop, note that the resulting block \( [L(n)] \) formed by the blocks \( [L(m)] \), \( [L(k)]' \), \( [L(k)]'' \), and \( [C_k]^T \) is an \( n \times n \) Latin square in standard form over the set \( L_n = \{1, \ldots, m, m+1, \ldots, 2m+1\} \), that is, the entries in the first row and first column of \( [L(n)] \) are the elements of \( L_n \) in natural order. If this block \( [L(n)] \) is now converted into the Cayley table shown in Table 3(a), the element 1 is seen to be a unique identity element. This means that \( (L_n, *) \) is at least a loop.

Clearly, the group \( L(m) \) is a subgroup of \( L(n) \). Being a group, every element of \( L(m) \) has a unique inverse. Moreover, every element of the subset \( L(k) \) is of order 2 (self-inverse) because the
square of such an element is an entry 1 in the diagonal of the block 
\( [C_k]^T \). Since \( L_n = L(m) \cup L(k) \), then every element of \( L_n \) has a 
unique inverse. Therefore, the system \((L_n, *)\) is an invertible loop.

Finally, the order \( m \) of the subgroup \( L(m) \) is not a divisor of the 
order \( n \) of \((L_n, *)\). By Lagrange’s theorem, it follows that \((L_n, *)\) 
in not a group and hence it is a NAFIL. ■

The smallest NAFIL loop \((L_5, *)\) that can be constructed using the above pro-
cedure is of order \( n = 5 \) when \( m = 2 \). This Cayley table defines a non-abelian 
NAFIL loop of order \( n = 5 \) that is simple. Analysis using the software FINITAS 
[3] has shown that it has four subgroups of order \( m = 2 \) and that it satisfies the 
cross-inverse property (CIP), the weak inverse property (WIP), automorphic in-
verse property (AIP), the flexible law (FL), power-associative property (PAP), “A 
sub m” loop property \((A_m)\), and the RIF loop property (RIF).

\[
\begin{array}{c|ccccc}
+ & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 2 & 1 & 5 & 3 & 4 \\
3 & 3 & 4 & 1 & 5 & 2 \\
4 & 4 & 5 & 2 & 1 & 3 \\
5 & 5 & 3 & 4 & 2 & 1 \\
\end{array}
\]

Cayley table of NAFIL loop of order \( n = 5 \).

This loop can be used as the basis of a loop algebra \( A(L_5) \) whose associated 
commutator algebra \( A^-(L_5) \) satisfies the Jacobi identity. Hence \( A^-(L_5) \) is a Lie 
algebra which has a subalgebra that is related to the algebra of the Pauli spin 
matrices in particle physics.

### 3. Sample Construction

We now show how we can construct a loop \((L_n, *)\) of order \( n = 9 \), where \( m = 4 \) 
and \( k = m + 1 = 5 \).

First, we start with the Latin square blocks \([L(4)], L(5)\), and \([C_5]^T \) shown 
below. Second, we form the block \([L(5)]' \) by deleting row 4 of \([L(5)]\). Third, we 
form the block \([L(5)]'' \) by deleting column 5 of \([L(5)]\). And fourth, we form the 
block \([C_5]^T * \) by replacing the entries 5 in \([C_5] \) by the entries \(9, 5, 6, 7, 8 \) in the 
last row of \([L(5)]\). These blocks are shown below.
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If we now put the blocks \([L(4)], [L(5)]', [L(5)]'',\) and \([\overleftarrow{C}_5]^T\) together as indicated in Table 3(a), we obtain the Cayley table shown in Table 3(b) of a NAFIL loop of odd order \(n = 9\).

| *  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|----|----|----|----|----|----|----|----|----|----|
| 1  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| 2  | 2  | 3  | 4  | 1  | 6  | 7  | 8  | 9  | 5  |
| 3  | 3  | 4  | 1  | 2  | 7  | 8  | 9  | 5  | 6  |
| 4  | 4  | 1  | 2  | 3  | 9  | 5  | 6  | 7  | 8  |
| 5  | 5  | 6  | 7  | 8  | 1  | 9  | 4  | 3  | 2  |
| 6  | 6  | 7  | 8  | 9  | 2  | 1  | 5  | 4  | 3  |
| 7  | 7  | 8  | 9  | 5  | 3  | 2  | 1  | 6  | 4  |
| 8  | 8  | 9  | 5  | 6  | 4  | 3  | 2  | 1  | 7  |
| 9  | 9  | 5  | 6  | 7  | 8  | 4  | 3  | 2  | 1  |

Table 3(b). NAFIL loop\((L_9, \ast)\) of order \(n = 9\).

This non-abelian NAFIL loop is simple and power-associative. It has one subgroup of order 4 and six of order 2.

3.1. Some Remarks. In constructing the NAFIL loop \((L_n, \ast)\), we made use of the groups \(L(m)\) and \(L(k)\) and the counter-clockwise quasigroup \(\overleftarrow{C}_k^T\). The group \(L(m)\) can be any group of order \(m\) while the group \(L(k) = \{m + 1, \ldots, 2m + 1\}\) must be a group of order \(k = m + 1\) that is isomorphic to the cyclic group \(C_k = \{1, \ldots, k\}\) of order \(k\) whose Latin square block \([C_k]\) is shown in Table 4.
Table 4. Latin square block $[C_k]$ of the cyclic group $C_k$.

It is clear that if $(m + i) \in L(k)$ and $i \in C_k$, where $i = 1, ..., k$, then the following one-to-one correspondence between $L(k)$ and $C_k$ is an isomorphism: $(m + i) \leftrightarrow i$. Thus, if we simply rename every element $(m + i)$ of $L(k)$ by $i$, we readily obtain the block $[C_k]$.

If we permute the rows of $[C_k]$ according to the row permutation

$$
\begin{pmatrix}
1 & 2 & 3 & \cdots & k-2 & k-1 & k \\
1 & k & k-1 & \cdots & 4 & 3 & 2 \\
\end{pmatrix}
$$

then we obtain the block $\overleftarrow{C_k}^T$ shown in Table 2(a) which is the transpose of the counter-cyclic block $\overrightarrow{C_k}$. This Latin square block defines a quasigroup of order $k$ denoted by $\overrightarrow{C_k}^T$. Since the cyclic group $C_k$ exists for all values of $k \geq 5$, then $\overrightarrow{C_k}^T$ also exists for all values of $k \geq 5$. Thus, both systems form families.

It is interesting to note that the quasigroup $\overrightarrow{C_k}^T$ satisfies the Left Bol (LBol) property.

4. Summary

In this paper, we proved the existence of at least one NAFIL loop $(L_n, \star)$ of every finite order $n = 2m + 1$, where $m \geq 2$. This was done by actually constructing $(L_n, \star)$ using two groups $L(m)$ of order $m$ and $L(k)$ of order $k = m + 1$, and a special quasigroup $\overrightarrow{C_k}$ of order $k$.

The group $L(m)$ is any group of order $m$ while $L(k)$ is a group isomorphic to the cyclic group $C_k$ of order $k$. On the other hand, the quasigroup $\overrightarrow{C_k}$ (which is combinatorially equivalent to $C_k$) satisfies the Left Bol Property. Using these systems, the NAFIL loop $(L_n, \star)$ is constructed and shown to exist for all values of $n = 2m + 1$, where $m \geq 2$. Thus, these loops form an infinite family.

We also indicated that the loop $(L_5, \star)$ can be used as the basis of a loop algebra whose associated commutator algebra is a Lie algebra with potential applications in particle physics.
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References

[1] P. H. Frampton, S. L. Glashow, T. W. Kephart, R. M. Rohm, Non-Associative Loops for Holger Bech Nielsen, arXiv:hep-th/0111292v1, 30 Nov. 2001

[2] R. E. Cawagas, Latin Square Composition of Factorable Groups and Loops, Matimyas Matematika, Vol. 21, No. 1, pp. 1-11, (1998)

[3] R. E. Cawagas, FINITAS - A Software for the Construction and Analysis of Finite algebraic Structures, PUP Journal of Research and Exposition, Vol. 1, No. 1, 1997.

RAOUL E. CAWAGAS, SciTech R&D Center, OVPRD, Polytechnic University of the Philippines, Manila

E-mail address: raoulec@pacific.net.ph / raoulec@yahoo.com

URL: http://www.geocities.com/raoulec2001/raoulweb.htm