ALUMINUM ABUNDANCES, DEEP MIXING, AND THE BLUE-TAIL SECOND-PARAMETER EFFECT IN THE GLOBULAR CLUSTERS M3 AND M13

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ABSTRACT

We analyze high-resolution, high signal-to-noise ratio spectra of six red giant branch (RGB) stars in the globular cluster M3 (NGC 5272) and three in M13 (NGC 6205) that were obtained with the 4 m Mayall Telescope and echelle spectrometer on Kitt Peak. The spectra include lines of O, Na, Mg, Al, Si, Ca, Ti, V, Mn, Fe, and Ni. We also analyze the [Al/Fe] values of 96 RGB stars in M13 covering the brightest 3.5 mag, which include 66 measurements that were derived from moderate-resolution, low signal-to-noise ratio spectra obtained with the WIYN 3.5 m telescope and Hydra multiobject spectrograph, also on Kitt Peak. In addition, we compile from the literature and inspect the [Na/Fe] values of 119 RGB stars in M13. We test for bimodality in the [Al/Fe] and [Na/Fe] distributions using the KMM algorithm and find that the [Al/Fe] values in M13 are distributed bimodally at all points along the RGB that were observed, while the [Na/Fe] values are bimodal only over the brightest 2 mag. The ratios of Al-enhanced to Al-normal and Na-enhanced to Na-normal giants increase toward the tip of the RGB in M13, which is suggestive of deep mixing in this cluster. The limited M3 data exhibit a bimodal distribution of [Al/Fe] values and are suggestive of no deep mixing; however, they are too few to be conclusive. We further test for a relationship between deep mixing on the RGB and a second parameter that can create the extended blue tail seen along the horizontal branches of some clusters by using an "instantaneous" mixing algorithm, which we develop here. We conclude that the data for both clusters are consistent with deep mixing as a "blue-tail second parameter," and we suggest future observations to further constrain the results. Finally, we offer a solution to the problem of overproducing sodium during deep mixing that is based on the depletion of $^{22}$Ne in asymptotic giant branch stars and suggest that pollution might best be traced by $s$-process elements in the Sr-Y-Zr peak.

Key words: globular clusters: individual (M3, M13) — stars: abundances — stars: horizontal-branch — stars: late-type — stars: Population II

1. INTRODUCTION

According to canonical stellar evolution models, the by-products of the nuclear processing around the hydrogen-burning shell (H shell) of low-mass red giant branch (RGB) stars should remain confined to the stellar interior; however, observations over the past 25 years have shown star-to-star variations in the elements C, N, O, Na, Mg, and Al, among others, on the surfaces of globular cluster red giants (for detailed reviews of the observations, see Kraft 1994; Briley et al. 1994; Cavallo 1998a). In particular, the data show evidence of the CNO cycle that dominates the energy production in such stars: C and O are anticorrelated while the $^{12}$C/$^{13}$C ratio is near the equilibrium value of 4 in many clusters (Suntzeff & Smith 1991; Shetrone 1996b; Briley et al. 1996b, 1997; Zucker, Wallerstein, & Brown 1996). While the first dredge-up phenomenon (Iben 1967) does alter the carbon and nitrogen abundances slightly, it cannot account for the observed large variations of these elements and their isotopic ratios, nor can it account for the variations of the other elements. In addition, some elements show evidence for gradual changes along the RGB, indicating that something is occurring during the course of evolution to facilitate these alterations. For example, C becomes more depleted with decreasing $V$ in the clusters M15, M55, M92, and NGC 6397 (Bell, Dickens, & Gustafsson 1979; Carbon et al. 1982; Trefzger et al. 1983; Briley et al. 1990).

Two separate approaches have been developed to address the observations. One assumes that some form of noncanonical mixing occurs along the RGB, which gradually builds material from around the H shell to the stellar surface (Sweigart & Mengel 1979, hereafter SM79). Models by SM79, Denissenkov & Denisenkova (1990), Langer, Hoffman, & Sneden (1993), Cavallo, Sweigart, & Bell (1996), Denissenkov & Weiss (1996), and Cavallo, Sweigart, & Bell (1998, hereafter CSB98) have shown that most variations along the brighter part of the RGB can be explained by nuclear processing around the H shell combined with mixing. The source of mixing is generally assumed to be rotationally induced meridional circulation currents (SM79), although other theories abound (Langer, Hoffman, & Zaidins 1997; Fujimoto, Aikawa, & Kato 1999). The observations by Peterson (1983) that show the horizontal-branch (HB) stars in M13, a cluster with large variations of oxygen and aluminum on the RGB, rotating nearly a factor
TABLE 1

| Star Name (mag) | Date Observed (UT) | Exposure (minutes) | S/N |
|-----------------|--------------------|--------------------|-----|
| vZ 238 ……. | 1998 Mar 21–23b | 180 | 110 |
| vZ 752 ....... | 1998 Jun 1 | 120 | 115 |
| vZ 205 ....... | 1998 May 31 | 120 | 95 |
| vZ 297 ....... | 1998 May 30 | 120 | 105 |
| vZ 1127 ...... | 1998 May 31 | 180 | 135 |
| vZ 1000……. | 1998 May 31 | 180 | 90 |
| L324 ………. | 1998 Jun 1 | 120 | 160 |
| L414 ………. | 1998 May 30 | 120 | 140 |
| L262 ………. | 1998 May 31 | 120 | 140 |

Notes.—Star names taken from the catalogs of (vZ) von Zeipel 1908 and (L) Ludendorff 1905. Alternate names are from Sandage 1953 (AA, III-28), Arp 1955 (III-56), Sawyer Hogg 1973 (V11), (CM) Cudworth & Monet 1979, (SK) Sandage & Katem 1982, (F) Ferraro et al. 1997, and (MB) M. Bolte 1998, private communication.

* Estimated.

b Observed by M. Briley.

of 2 faster than the HB stars in M3, a cluster with a composition similar to M13, but with less extreme abundance variations along its RGB, support the SM79 hypothesis.

The second approach assumes that some of the variations, particularly those of the heavier elements, are primordial in nature, perhaps originating in the processed envelopes of intermediate-mass asymptotic giant branch (AGB) stars that were shed into the nascent cluster environment (Cottrell & Da Costa 1981). While it has been shown that this scenario cannot account for all the variations (Denissenkov, Weiss, & Wagenhuber 1997), some aspects of it are plausible in light of the data. For example, observations of CN band strength and sodium variations on the upper main sequence of 47 Tuc, sodium enhancements on the subgiant branch of M92, and enhancements in the neutron-capture elements in some clusters all point to primordial origins (Briley et al. 1989; Briley, Hesser, & Bell 1991; Shetrone 1996a; Briley et al. 1996a; Cannon et al. 1998; King, Stephens, & Boesgaard 1998; Ivanov et al. 1999). The most likely solution to the abundance anomaly problem probably involves a combination of both scenarios, in which primordial pollution is present in the cluster but mixing later plays a role in adjusting the abundance patterns (see, e.g., Denissenkov et al. 1998; Briley, Grundahl, & Andersen 1999), an approach we examine here.

This paper focuses on determining the chemical abundances in the red giants of the globular clusters M3 (NGC 5272) and M13 (NGC 6205) from high-resolution, high signal-to-noise ratio echelle spectra obtained with the 4 m Mayall Telescope on Kitt Peak. We choose these two clusters because they are often considered a classical "second
The observations were made on the nights of 1998 May 13–15 using the echelle spectrograph on the 4 m Mayall Telescope. The echelle setup used—echelle grating 31.6–63", cross grating 226–1, long-focus camera, and the T2KB CCD—resulted in continuous spectral coverage between 5500 and 8800 Å at a dispersion of 0.08 Å per CCD pixel, i.e., resolution \( R \approx 50,000 \) over 4070 to 10500 Å with \( R \approx 50,000 \) over a resolution element.

A log of all the observations with estimates of the signal-to-noise ratio around the \( \text{Al} \); 6696, 6698 lines is given in Table 1. The photometry is from Ferraro et al. (1997) for M3 and from Cudworth & Monet (1979) for M13. The locations of the program stars in their respective color-magnitude diagrams are given in Figure 1 for M3 and Figure 2 for M13. In the following discussions, we refer to each star by its most commonly used identification, which in some cases is its alternate name.

### 3. CCD PROCESSING AND SPECTRAL EXTRACTION

The data were reduced using standard IRAF\(^4\) (Tody 1986) tasks (version 2.1.1.1), following the reduction procedure outlined by Willmarch & Barnes (1994).\(^5\) Zero frames were taken on each night, but since over 99.9% of the pixels had zero values within the 5 e\(^{-}\) rms noise of the T2KB CCD, we only zero-corrected “hot” pixels (zero values greater than 5 e\(^{-}\)). Quartz-flat exposures were taken on each of the three nights and used to flat-field the target stars and to determine “dead” pixels.

After initial processing of the CCD target star data, we used the IRAF task APSCATTER to correct for scattered light between the orders. The orders were then extracted to single-dimensional spectra using the task APSUM with variance weighting, and then wavelength-calibrated using the closest (in time) ThAr spectrum, taken at the same telescope position and slit position angle as the target star. Orders that contained telluric lines were corrected using the task TELLURIC and a comparison spectrum of a fast-rotating B star that was observed at an air mass similar to the program star. Finally, each spectrum was shifted into the rest frame and flattened by fitting a spline through the continuum.

The spectrum for the star M3 AA was given to us in its extracted form and required only correction for telluric lines in some orders.

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\(^3\) We use the usual notation: \([X/Y] = \log(X/Y)_{\odot} - \log(X/Y)_{\odot}\).

\(^4\) IRAF is distributed by the National Optical Astronomy Observatories.

\(^5\) Available via http://iraf.noao.edu/docs/spectra.html.
Figure 3 shows the spectra around the Al I region for all the M3 and M13 giants. The variation in the Al I line strengths is quite apparent from one star to the next.

4. ABUNDANCE ANALYSIS

4.1. Equivalent Widths, Line Lists, and Line Parameters

The free spectral range for the Mayall 4 m data is less than the width of the CCD, so each line appears at least twice in our spectra, with one line usually toward the center of the chip, where the noise is minimized. Unfortunately, combining adjacent orders to increase the number of photon counts resulted in a lowered signal-to-noise ratio since the edges of the chip contained much lower quality spectra than the center. We measured the equivalent widths of the lines closer to the center of the CCD and present the results in Table 2. The equivalent widths were determined by summing the flux in a line if the line was cleanly separated from other lines or by fitting a Gaussian in more crowded regions. Some data are excluded from the table for one of several reasons: (1) in the case of M3 AA, some lines fell between the orders; (2) the line was too weak to be measured (our minimum measurable equivalent width was about 5 mA, depending on the signal-to-noise ratio); (3) the line was contaminated by either a bad column or a cosmic-ray hit; or (4) in very few cases, the line gave abundance results that were anomalously discordant with the mean and rms deviation of the rest of the lines of the same species for unknown reasons and was rejected.

4.1.1. Fe Lines

The iron lines were chosen from Kurucz’s CD-ROM 23 (Kurucz & Bell 1995) and the solar atlas of Moore, Minnaert, & Houtgast (1966). They were used to determine both the iron abundances and the model atmosphere parameters and were thus selected to have a broad range in excitation potentials and oscillator strengths, which were adopted from the empirically derived tables of Nave et al. (1994) and Biémont et al. (1991) for the Fe I and Fe II lines, respectively. We rejected Fe I lines that were listed by Nave et al. as showing blends with other iron lines or uncertainties in their energy levels of more than 0.005 cm⁻¹. For the sake of comparison, we measured the equivalent widths of our chosen iron lines visible and near-infrared solar spectra7 of Wallace, Hinkle, & Livingston (1993, 1998)8 and folded them through the Holweger & Müller (1974) solar model atmosphere with a microturbulent velocity of 1.0 km s⁻¹. We adjusted the log gf’s to reproduce an assumed solar iron abundance of logₑ(Fe)₀ = 7.52.9 This results in an average difference between the two sets of oscillator strengths (in the sense solar model minus laboratory) of +0.09 ± 0.12 for Fe I and +0.05 ± 0.08 for Fe II (see Table 3 below). Both differences show a slight trend toward lower oscillator strengths than the laboratory measurements (<1 σ), which in turn would cause the determined iron abundances to be overestimated by 0.09 and 0.05 dex from the Fe I and Fe II lines, respectively.

4.1.2. Equivalent Width Comparisons

In Figure 4, we compare the equivalent widths of our iron lines with those from the earlier Lick-Texas studies. The solid line in the figure has a 45° slope and represents perfect agreement between the two data sets. The Lick-Texas data tend toward higher equivalent widths relative to our data, especially above ~100 mA. The average differences between the two data sets (in the sense Lick-Texas minus present work) are 6.9 ± 9.7 mA for Fe I and 5.9 ± 9.7 mA for Fe II. We attribute the differences to several factors. First, our present data have higher signal-to-noise ratios than the Lick-Texas data (90–160 compared with 40–100), which reduce the level of uncertainty in placing the continuum. Second, the Lick-Texas spectra have higher resolution, R ≈ 48,000 compared with R ≈ 30,000, which helps separate lines from the continuum and other lines. Third, the two data sets were reduced using two separate software packages that apply scattered-light corrections differently (see, e.g., Sneden et al. 1991), which can affect the continuum levels and depths of each line. Fourth, measuring equivalent widths is subjective and two different observers can obtain different results from the same data. For example, the inset in Figure 4 shows the equivalent widths for two separate observations of the M13 giant III-56 by the Lick-Texas group and demonstrates how much variability can be present in the data even with consistent reduction techniques (see also Fig. 1 in Kraft et al. 1993). Fifth, the methods of measuring equivalent widths differ: we use both a Simpson’s rule technique (i.e., direct integration) and

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6 See http://cfa-www.harvard.edu/amdata/amdata.html.
7 The NSO/Kitt Peak Fourier transform spectrometer data used here were produced by NSF/NOAO.
8 Available at http://www.nso.noao.edu/diglib/ftp.html.
9 Here logₑ(X) = log N(X)/H ± 12.00, where N is the number abundance. For an informative debate about the preferred solar iron abundance, we refer the reader to the papers of Blackwell, Lina-gray, & Smith (1995a), Holweger, Kock, & Bard (1995), Blackwell, Smith, & Lina-gray (1995b), and Koslik, Shechukina, & Rutten (1996). For consistency with the Lick-Texas studies, we adopt the value 7.52.
### TABLE 2
**Measured Equivalent Widths**

| WAVELENGTH (Å) | EP (eV) | log gf | M3 GIANTS (mA) | M13 GIANTS (mA) |
|----------------|---------|--------|----------------|-----------------|
|                |         |        | AA  vZ 297  | III-28  vZ 1000  | vZ 1127  MB 4  |
| Na i:          |         |        |               |                  |                |
| 6154.23        | 2.10    | -1.66  | 33.5 20.7    | ...              | 34.1          |
| 6160.75        | 2.10    | -1.32  | 51.7 29.8    | ...              | 14.9 52.1     |
| Mg i:          |         |        |               |                  |                |
| 5711.09        | 4.35    | -1.58  | 110.7 93.2  | 73.9              | 93.8          |
| 8717.83        | 5.93    | -0.96  | ...          | 18.8              | 15.7          |
| 8736.02        | 5.95    | -0.04  | 59.1 50.7    | 74.0              | 54.1          |
| Al i:          |         |        |               |                  |                |
| 6696.03        | 3.14    | -1.54  | 69.0 52.4    | ...              | 51.1          |
| 6698.67        | 3.14    | -1.91  | 40.1 24.6    | ...              | 25.9          |
| 7835.31        | 4.02    | -0.72  | 36.0 28.3    | ...              | 26.3          |
| 8736.13        | 4.02    | -0.63  | 59.9 36.3    | ...              | 30.2          |
| Si i:          |         |        |               |                  |                |
| 6142.48        | 5.62    | -1.53  | 10.2          | ...              | ...           |
| 6145.02        | 5.61    | -1.41  | ...          | ...              | ...           |
| 6243.82        | 5.61    | -1.32  | 17.1 14.8    | ...              | 9.0           |
| Ca i:          |         |        |               |                  |                |
| 6161.30        | 2.52    | -1.22  | 87.7 88.0    | 70.5              | 78.2          |
| 6166.44        | 2.52    | -1.04  | 95.5 90.1    | 64.6              | 74.6          |
| 6169.04        | 2.52    | -0.67  | 115.9 112.5 | ...              | 108.0         |
| Ti i:          |         |        |               |                  |                |
| 5866.45        | 1.07    | -0.82  | 154.8 137.2  | 110.1              | 115.4         |
| 6064.63        | 1.05    | -0.99  | 72.4 58.0    | 36.3              | 42.9          |
| 6126.22        | 1.07    | -1.43  | 113.9 95.4  | 75.3              | 77.6          |
| 6258.18        | 1.44    | -0.37  | 149.1 140.9  | 99.9              | 115.3         |
| 6261.10        | 1.43    | -0.46  | 139.1 138.4  | 96.9              | 116.9         |
| 6336.10        | 1.44    | -1.80  | 36.6 35.4    | 17.3              | 28.1          |
| 6497.68        | 1.44    | -2.10  | 29.5 13.0    | 13.5              | 14.0          |
| 6508.12        | 1.43    | -2.18  | 28.9 14.1    | 10.2              | 14.5          |
| V i:           |         |        |               |                  |                |
| 5670.85        | 1.08    | -0.47  | 100.8 78.2   | 50.6              | 67.9          |
| 5727.05        | 1.08    | -0.02  | 142.4 120.3  | 80.2              | 105.3         |
| 5735.06        | 1.06    | -0.81  | 69.6 54.9    | 25.7              | 39.5          |
| 6039.72        | 1.06    | -0.73  | 70.7 57.0    | 33.4              | 43.7          |
| 6058.14        | 1.04    | -1.37  | 24.6 20.1    | 12.4              | 12.8          |
| 6111.65        | 1.04    | -0.81  | 74.5 59.9    | 37.4              | 50.7          |
| 6150.16        | 0.30    | -1.56  | 102.6 84.5   | 40.6              | 58.8          |
| 6199.20        | 0.29    | -1.49  | ...          | 112.3              | 57.1          |
| 6224.53        | 0.29    | -1.89  | 80.5 64.4    | 28.9              | 43.0          |
| 6233.16        | 0.28    | -1.94  | 69.2 59.4    | 21.5              | 41.7          |
| 6251.83        | 0.29    | -1.40  | 127.6 101.6  | 57.4              | 80.4          |
| 6266.31        | 0.28    | -2.27  | 55.0 43.2    | 18.7              | 28.7          |
| 6274.65        | 0.27    | -1.76  | 95.6 79.0    | 40.9              | 53.1          |
| 6285.15        | 0.28    | -1.63  | 102.5 89.7   | 51.7              | 63.9          |
Gaussian fits, while the Lick-Texas results come from Gaussian fits for their earlier papers and both techniques for Kraft et al. (1997).

4.1.3. Other Lines

We determined the oscillator strengths for the other elements in our study by measuring the equivalent widths in the solar spectrum and adjusting the log \( gf \) values until we derived the Anders & Grevesse (1989) solar abundances using the Holweger & Müller (1974) solar model. Table 3 lists the average differences between the values we determined and the literature values, which are mostly from Thévenin (1990), who performed a similar differential analysis using an older solar spectrum with a MARCS (Gustafsson et al. 1975) solar model and a microturbulent velocity of 0.6 km s\(^{-1}\). With the exception of the Mn I lines, we used our derived log \( gf \)'s to derive the elemental abundances. The two Mn I lines in the solar spectrum (we discard a third line at 6022 Å since it blends with a nearby Fe I feature) suffer from hyperfine splitting effects. To avoid
detailed calculations, we adopt the recommended log gf's from Thévenin (1990) for these two manganese lines.

4.2. Model Parameters: Effective Temperature, Gravity, and Microturbulence

4.2.1. Spectroscopic Models

We constructed models using the Fe I and Fe II lines with the MARCS model atmosphere code and the MOOG abundance analysis code (Sneden 1973). The initial models were built with the parameters determined by the Lick-Texas group and were constructed with the α-elements enhanced by 0.4 dex in accordance with previous observations of cluster giants (see, e.g., Kraft et al. 1997). We iteratively ran MOOG and MARCS to refine the models until the derived abundances were independent of excitation potential, line width, and ionization level. We checked our final choice of model parameters independently using the Ni I lines to determine \( T_{\text{eff}} \), the Ni I and Ti I lines to determine the microturbulent velocity \( v_t \), and the Ti I and Ti II lines to determine \( \log g \). The results were generally in agreement with the more numerous iron lines and allowed us to estimate systematic errors in the model parameters determined from spectroscopy: \( \Delta T_{\text{eff}} \sim \pm 30 \, \text{K} \), \( \Delta \log g \sim \pm 0.2 \, \text{cm s}^{-2} \), and \( \Delta v_t \sim \pm 0.15 \, \text{km s}^{-1} \).

Our final spectroscopically determined model parameters are given in Table 4, along with the original Lick-Texas model parameters, and are the parameters we used to derive the elemental abundances. The effective temperatures generally agree to less than 100 K, while our gravities are typically lower than the Lick-Texas values by 0.15 ± 0.27 dex and our microturbulent velocities are lower by 0.09 ± 0.11 km s\(^{-1}\). The differences are consistent with the error estimates derived from the nickel and titanium lines and are not surprising given the differences in the equivalent widths and the choices of lines and line parameters between the two studies.

### 4.2.2. Photometric Models

We used recent photometry of our cluster giants, given in Tables 5A and 5B, to derive alternative model atmosphere parameters. The \( B-V \) and \( V-I \) data were used to derive effective temperatures based on a 12 Gyr isochrone that was constructed with [Fe/H] = -1.54 and \([\alpha/\text{Fe}] = +0.3 \, \text{dex}\). The luminosities and \( T_{\text{eff}} \)'s for the models in the isochrone were computed by Professor D. VandenBerg, while Professor R. A. Bell performed the luminosity-\( M_V \) and the color-temperature transformations. The age of the iso-

### Table 3

| Species | Average Difference \( \pm \sigma \) | No. Lines | Ref. |
|---------|---------------------------------|----------------|-----|
| Na I    | +0.015 ± 0.021                  | 2             | 1   |
| Mg I    | +0.263 ± 0.349                  | 3             | 1, 2|
| Al I    | +0.035 ± 0.052                  | 6             | 1, 2|
| Si I    | +0.059 ± 0.108                  | 9             | 1, 2|
| Ca I    | +0.203 ± 0.059                  | 8             | 1, 2|
| Ti I    | -0.025 ± 0.113                  | 8             | 1, 2|
| Ti II   | -0.120 ± 0.282                  | 2             | 1   |
| V I     | +0.048 ± 0.084                  | 18            | 1, 3|
| Mn I    | +0.045* ± 0.064*                | 2             | 1   |
| Ni I    | +0.203 ± 0.097                  | 12            | 1   |
| Fe I    | +0.088 ± 0.123                  | 24            | 4   |
| Fe II   | +0.048 ± 0.077                  | 8             | 5   |

* Used values from Thévenin 1990.

### Table 4

| STAR    | \( T_{\text{eff}} \) (K) | \( \log g \) | \( v_t \) (km s\(^{-1}\)) |
|---------|--------------------------|--------------|----------------------------|
| M3:     |                          |              |                            |
| AA      | 4050 ± 0.40              | 2.27         | 4000 ± 0.40                |
| MB 4    | 4060 ± 0.45              | 2.05         | 3925 ± 0.30                |
| III-28  | 4175 ± 0.55              | 1.84         | 4160 ± 0.75                |
| vZ 297  | 4050 ± 0.25              | 1.98         | 4070 ± 0.70                |
| vZ 1127 | 4300 ± 1.00              | 1.98         | 4225 ± 0.90                |
| vZ 1000 | 4200 ± 0.65              | 1.94         | 4175 ± 0.45                |
| M13:    |                          |              |                            |
| L324    | 3990 ± 0.10              | 2.34         | 4050 ± 0.50                |
| III-56  | 4030 ± 0.20              | 2.13         | 4100 ± 0.65                |
| L262    | 4160 ± 0.50              | 1.89         | 4180 ± 0.80                |

### Table 5A

| CM | CFP |
|----|-----|
| V  | B-V | V-K |
| L 324 | 12.60 | 1.60 |
| III-56 | 12.47 | 1.47 |
| L 262 | 12.25 | 1.39 |

### Table 5B

| V | B-V | V-I |
|---|-----|-----|
| AA | 12.72 | 1.58 |
| MB 4 | 12.74 | 1.44 |
| III-28 | 12.75 | 1.44 |
| vZ 297 | 12.84 | 1.44 |
| vZ 1127 | 13.09 | 1.47 |
| vZ 1000 | 13.10 | 1.29 |

**Note:** Abbreviations as in Table 1, plus (Rood) Rood et al. 1999; (vB) von Braun et al. 1998; (Cudworth) Cudworth 1979.
chrones (mass) fixes the gravities, while the microturbulent velocities are still determined from spectroscopy. The $V-K$ calibrations are from Cohen, Frogel, & Persson (1978; "CFP"). We present the results of the photometric calibrations in Tables 6A and 6B. Using the extrema from the photometric and spectroscopic parameters, we derived new model atmospheres from which we determined a range in the abundances allowed by the uncertainties in the models.

### 4.3. Results

Tables 7A, 7B, and 7C present the final results of our abundance analysis for the Fe-peak elements, $\alpha$-elements, and proton-capture elements (those that can be altered in the CNO, NeNa, and MgAl nuclear burning cycles), respectively. The numbers in parentheses are the line-to-line scatter, which we use here as the "error" under the assumption that our models are well determined.

The Fe-peak elements are consistent with the solar ratio with the exception of nickel, which seems to be underabundant in all the giants by 0.22 ± 0.03 dex on average. This would be expected if the oscillator strengths for the Ni lines were overestimated by a similar amount, as indicated in Table 3, which shows our derived oscillator strengths to be 0.20 ± 0.10 dex higher than the literature value. Thus, using the oscillator strengths from the literature would put [Ni/H] closer to zero in our sample. Why it should be that the oscillator strengths for Ni would be inconsistent with the published values while those of the other elements are more agreeable remains uncertain. In fact, the difference in log $gf$'s is even larger since our assumed solar Ni abundance is 0.07 dex higher than what Thévenin (1990) assumes; to force agreement with the lower Ni abundance value would cause us to increase our log $gf$'s by another 0.07 dex.

Figure 5 shows the Fe-peak abundances as a function of $T_{\text{eff}}$ for M3 and M13. The [Fe/H] values for M13 are 0.12 dex lower on average than those for M3, although the difference is only at the 1.5σ level. Despite the marginal disparity in [Fe/H], the trends for [V/Fe], [Mn/Fe], and [Ni/Fe] are the same for each cluster.

Figure 6 shows the $\alpha$-elements as a function of $T_{\text{eff}}$ for M3 and M13. The $\alpha$-element enhancements are consistent with other observations: $[\alpha/\text{Fe}] = +0.28 ± 0.08$ for M3 and $+0.30 ± 0.07$ for M13. The low dispersions probably indicate that the upper and lower limits from the various model atmospheres overestimate the actual errors in the abundances. The two clusters have very similar $\alpha$-enhancements, despite the differences in their iron abundances.

Figure 7 shows the trends in the proton-capture element abundances for all observed stars in M3 and M13 as a function of $T_{\text{eff}}$.
function of $T_{\text{eff}}$. The oxygen abundances were calculated via the synthetic-spectra fitting package in MOOG and are presented without error bars because (1) they come only from the [O I] line at 6300 Å, so there is no line-to-line scatter, and (2) the spectral resolution around the line is too low to accurately deblend the line from the nearby Sc II line, making attempted variations in the models less meaningful. We estimate the error in [O/Fe] to be ±0.15 dex. The magnesium abundances are derived from only three lines, all near the edges of the CCD, where the noise is high, and

\begin{table}[h]
\centering
\caption{Proton-Capture abundances} \label{tab:7c}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Star & [Fe/H]$_{\text{avg}}$ & [O/Fe] & [Na/Fe] & [Mg/Fe] & [Al/Fe] \\
\hline
\textbf{M3:} & & & & & \\
AA & −1.57 & −0.03 & +0.42 (0.13)$^{+0.14}_{-0.12}$ & +0.25 (0.12)$^{+0.13}_{-0.10}$ & +0.87 (0.14)$^{+0.14}_{-0.12}$ \\
MB 4 & −1.55 & −0.05 & +0.42 (0.10)$^{+0.12}_{-0.10}$ & +0.18 (0.13)$^{+0.14}_{-0.13}$ & +0.83 (0.13)$^{+0.14}_{-0.13}$ \\
III−28 & −1.61 & +0.36 & ... & −0.01 (0.11)$^{+0.13}_{-0.12}$ & −0.19 (0.15) \\
vZ 297 & −1.55 & −0.01 & +0.13 (0.15)$^{+0.19}_{-0.16}$ & +0.05 (0.11)$^{+0.16}_{-0.15}$ & +0.71 (0.13)$^{+0.17}_{-0.16}$ \\
vZ 1127 & −1.45 & +0.33 & −0.22 (0.10)$^{+0.10}_{-0.09}$ & +0.01 (0.13)$^{+0.13}_{-0.12}$ & −0.01 (0.11)$^{+0.11}_{-0.10}$ \\
vZ 1000 & −1.49 & −0.01 & +0.17 (0.12)$^{+0.13}_{-0.12}$ & +0.18 (0.13)$^{+0.13}_{-0.12}$ & +0.72 (0.14)$^{+0.14}_{-0.13}$ \\
\textbf{M13:} & & & & & \\
L324 & −1.67 & −0.38 & +0.54 (0.12)$^{+0.12}_{-0.10}$ & −0.02 (0.13)$^{+0.13}_{-0.10}$ & +0.99 (0.12)$^{+0.12}_{-0.10}$ \\
III−56 & −1.70 & −0.05 & +0.50 (0.08)$^{+0.09}_{-0.08}$ & +0.23 (0.16)$^{+0.17}_{-0.16}$ & +0.74 (0.11)$^{+0.11}_{-0.10}$ \\
L262 & −1.61 & +0.11 & +0.34 (0.11)$^{+0.11}_{-0.10}$ & +0.10 (0.14)$^{+0.14}_{-0.13}$ & +0.61 (0.13)$^{+0.13}_{-0.12}$ \\
\hline
\end{tabular}
\end{table}
are susceptible to large uncertainties. For the M3 star III-28, we fitted a synthetic spectrum to the data around the Al I $\lambda\lambda 6696, 6698$ and $\lambda\lambda 7835, 7836$ regions to determine the abundance, and we estimate the error to be $\pm 0.15$ dex.

In M3, $[\text{Al/Fe}]$ spans a range of 1.1 dex over 250 K, and while the three cooler stars appear to be more enhanced than the three hotter ones on average, it must be cautioned that the sample is biased since the stars were chosen from the Lick-Texas studies based on evidence for or against mixing and is not close to being complete. In addition, the oxygen abundances exhibit a strong anticorrelation with aluminum but are not as depleted as in the so-called super-oxygen-poor giants in M13 (Kraft et al. 1992). The sodium abundances likewise show an increasing trend with decreasing $T_{\text{eff}}$ and are correlated with $[\text{Al/Fe}]$. Finally, $[\text{Mg/Fe}]$ seems fairly independent of $T_{\text{eff}}$ and does not appear to show any correlations with the other proton-capture elements.

In M13 it is impossible to make any firm conclusions since the data are so few; however, we note several trends. First, $[\text{O/Fe}]$ and $[\text{Al/Fe}]$ are strongly anticorrelated. Second, the aluminum abundance is high ($>0.6$ dex) for all three stars, while the oxygen varies from slightly enhanced to fairly depleted (but again, not super-oxygen-poor). Third, $[\text{Na/Fe}]$ shows a slight trend of increasing with decreasing $T_{\text{eff}}$. Fourth, the one giant with the strongest aluminum enhancement (L324) has the strongest magnesium depletion and is also the brightest star in the sample.

5. DISCUSSION

5.1. Evidence for Deep Mixing on the RGB

5.1.1. Theoretical Predictions

In Cavallo et al. (1996) and CSB98 we explored the development of the abundance profiles around the H shell of four canonical stellar evolutionary sequences. Although the models were unmixed, we can infer some predictions with regard to deep mixing, which, we remind the reader, is defined as mixing that penetrates the H shell.

1. Carbon, nitrogen, and oxygen are not good tracers of deep mixing, since they are easily altered above the H shell in the CN and ON nuclear reaction cycles.

2. Sodium is altered above the H shell from $^{22}\text{Ne}$ and inside the H shell from $^{20}\text{Ne}$ through the NeNa cycle as shown in Figure 8. The proton-capture rates for the NeNa cycle are still uncertain and the initial neon abundance in real RGB stars is impossible to measure, making the theoretical prediction of actual sodium enhancements difficult.

3. As shown in Figure 8, $^{27}\text{Al}$, the only stable aluminum isotope, is enhanced only inside the H shell at the expense of mostly $^{25}\text{Mg}$ plus $^{26}\text{Mg}$, but also some $^{24}\text{Mg}$ deep inside the H shell for very bright, metal-poor models. The reaction rates for the MgAl cycle are still subject to large uncertainties, although the $^{24}\text{Mg}$ proton capture rates are now well determined (Powell et al. 1999).

4. Aluminum enhancements are temperature sensitive, indicating that they should not be expected until higher luminosities are achieved in lower metallicity giants ($[\text{Fe/H}] \lesssim -1.2$). In addition, the creation of sodium from $^{20}\text{Ne}$ also requires the high temperatures found only inside the H shell of the same bright, metal-poor giants, indicating that large sodium enhancements that originate with this neon isotope occur only toward the RGB tip.

All these inferences are still subject to uncertainties in the source of mixing, the initial abundances, and the nuclear reaction rates; nonetheless, we now venture to compare the observational data with the theoretical predictions.

5.1.2. Observational Results: The Aluminum Data

In addition to the M13 aluminum abundances determined above, we were provided with the equivalent width data of the Al I $6696$ Å line for 78 more giants in this cluster that were obtained from spectra taken with the WIYN tele

![Abundance variations around the H shell of an RGB tip model with $[\text{Fe/H}] = -1.67$ and scaled solar composition. The abscissa is the mass difference between any point and the center of the H shell. Hydrogen and helium are given in mass fractions while the other elements are scaled relative to the total number of metals.](image_url)
In this present study, the WIYN models are rooted in photometry with corrections to WIYN data are small, systematic errors can arise since the differences between the 4 m data and the agreed to within the errors from the line-to-line scatter. Although the differences between the 4 m data and the equivalent widths were folded through models that were built using the MARCS code based on the atmospheric parameters that were initially derived via photometry (Pilachowski et al. 1996). The models were assumed to have [Fe/H] = −1.50, while as with the models from this present study, the [x/Fe] ratio was also assumed to be enhanced by +0.4 dex. As shown in Tables 5, 6, and 7, models based on photometric indices can lead to a wider range of abundances, although we believe the WIYN models to be well determined since they were iteratively corrected with the spectra.

Of the 12 other stars in the WIYN sample that did not have measurable lines, one, II-76, had an aluminum abundance previously determined in the literature: [Al/Fe] = 0.19 (Shetrone 1996a). The other 11 were assumed to have [Al/Fe] = 0 for the purposes of the statistics discussed below, an assumption that seems verified by II-76 having such a low aluminum abundance, although this star is rather bright while the other 11 are much lower magnitude examples. The three M3 stars observed by us were also present in the WIYN sample, and the abundances agreed to within the errors from the line-to-line scatter. Although the differences between the 4 m data and the WIYN data are small, systematic errors can arise since the WIYN models are rooted in photometry with corrections to $T_{\text{eff}}$ and $v_c$ from lower resolution spectra (see Pilachowski et al. 1996 for details). For example, our models are 40–60 K hotter and have gravities that are 0.35–0.45 dex lower and microturbulent velocities that are 0.09–0.13 km s$^{-1}$ higher. In addition, our spectra have a factor of 2 higher resolution than the WIYN data and our signal-to-noise ratios are significantly higher.

Finally, the sample was then augmented with [Al/Fe] values taken from the literature (Wallerstein, Leep, & Oke 1987; Shetrone 1996a; Kraft et al. 1997), bringing the total number of stars with determined [Al/Fe] values to 85. We believe that the systematic errors that might be present in the data are mostly removed before combination since, with the exception of Wallerstein et al. (1987) for two stars, they are derived by the same group of Lick–Texas observers who practice consistent reduction techniques. Small differences will arise as the telescopes and instruments are varied, but the Lick–Texas observers do compare their various observations and show little scatter among their results. If one adds the 11 stars with the assumed low [Al/Fe] values, the total sample size is 96, covering a range from the tip of the RGB at $V = 11.9$ to $V = 15.5$, with complete coverage of the Cudworth & Monet (1979) photometry for $V \leq 13.7$. The data are plotted in Figure 9a as a function of $V$ magnitude, where the 11 stars with [Al/Fe] assumed to be zero are shown as circles. The [Al/Fe] values of giants with multiple measurements were averaged together after being normalized by [Fe/H].

In Figure 9c, we show a histogram of the distribution of [Al/Fe] for the entire sample of 96 M13 giants. From this figure, we see an apparent gap between [Al/Fe] = 0.2 and [Al/Fe] = 0.4, indicating that the distribution might be bimodal. We test for bimodality by applying the KMM algorithm of Ashman, Bird, & Zepf (1994), which tests the null hypothesis that a single Gaussian is a good description of the data by comparing the fit of a single-peaked distribution with one with multiple modes. The algorithm returns a $P$-value that describes the confidence level of the single-mode fit, where $P < 0.05$ indicates that a single Gaussian can be rejected at better than the 95% confidence level, which is generally accepted as strongly consistent with the multimodal distribution. Testing for a bimodal distribution in our sample gives $P = 0.000$ with means in [Al/Fe] of 0.12 ± 0.25 for the “Al normal” peak and 1.03 ± 0.25 for the “Al enhanced” peak. The number of stars in each distribution is 65 and 31 for the Al-enhanced and Al-normal groups, respectively, yielding a ratio of Al-enhanced to Al-normal stars (the “Al ratio”) of 2.10 ± 0.46, where the error is estimated from Poisson statistics.

It is possible that our assumption concerning the actual [Al/Fe] values of stars with no measurable Al lines can introduce a bias into our statistics. For example, Pilachowski et al. (2000) report an upper limit of 20 mÅ for equivalent width measurements of lower luminosity giants. Applying this measurement to the star K272, which has $V = 15.47$, and using the model parameters supplied by Pilachowski et al. (2000), we obtain [Al/Fe] ≤ 0.88, assuming [Fe/H] = −1.49. We test the effect of this bias by subjecting just the 85 giants with actual aluminum measurements to the KMM algorithm, yielding a $P$-value of 0.002, with an Al ratio of 3.72 ± 0.98. This clearly demonstrates that removal of the uncertain data still results in a strongly bimodal distribution. The only real solution to correct this possible bias is to make higher signal-to-noise ratio observations.

If deep mixing is occurring on the RGB, then the Al ratio should be a function of magnitude, increasing with decreasing $V$. To test this hypothesis, we bin the data by magnitude and apply the KMM algorithm to each bin to determine whether the distribution in each bin is bimodal and, if it is, the Al ratio. The KMM algorithm requires that the number of data points be greater than 50, forcing the size of magnitude bins to be rather wide ($\Delta V = 2$) in order to ensure that enough stars are included for reliable statistics. We began our binning at $V = 12.9$, 1 mag lower than the brightest star in the sample, and shifted each bin by 0.1 mag up to $V = 14.5$, 1 mag brighter than the lowest luminosity star in the sample. This choice of bins avoids adding empty points along the RGB into our statistics, although it reaches magnitudes at which the sample is incomplete. According to the KMM algorithm, the aluminum distribution in each magnitude bin is bimodal, with all $P$-values less than 0.013 and most less than 0.004. In Figure 9e we show the Al ratio (with Poisson error bars) as a function of the central magnitude of each bin. The mean of the 15 points between $V = 13.1$ and $V = 14.5$ is 1.99, which is shown as the horizontal line in Figure 9e. The Al ratios in the second brightest and brightest magnitude bins are 2.9 and 3, respectively. The upturn at

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10 We find no evidence of this feature in any of our M3 or M13 spectra.
the brighter magnitudes is due to both an increase in the number of Al-enhanced stars and a decrease in the number of Al-normal stars, as can be seen in Figure 9a, and is consistent with mixing occurring along the RGB, although the error bars do not allow for a definitive conclusion in this regard.

We now compare the M13 data with those from M3, where the number of giants with measured Al abundances is substantially smaller. Although we augment our sample of six giants in M3 with an additional four that were observed by Kraft et al. (1999; none in common with our sample), the numbers are still too small to apply the KMM algorithm; however, as shown in Figure 10a, the M3 [Al/Fe] distribution appears bimodal, with an Al ratio of 1.5 ± 1.0. This is consistent with the presumably unmixed dimmer giants in M13, suggesting that deep mixing is not occurring in M3.

5.1.3. Observational Results: The Sodium Data

We incorporated the [Na/Fe] data from our three M13 giants with the literature values (Lehnert, Bell, & Cohen 1991; Kraft et al. 1992, 1993, 1995; Pilachowski et al. 1996; Shetrone 1996a, 1996b; Kraft et al. 1997) to build a database containing 119 M13 RGB stars with measured sodium abundances, which we show in Figure 9b as a function of V. Again, we believe the systematic errors associated with the combination of various data sets to be minimized for the reasons outlined in the previous section. As demonstrated in Figures 9a and 9b, and in Figure 11 for a subset of stars that have both [Al/Fe] and [Na/Fe] values determined, the range in [Na/Fe] is not as wide as in the [Al/Fe] data, with the [Na/Fe] values being both more negative for "low" sodium stars and not as enhanced for the "high" sodium stars. However, from Figure 9b it is clear that the tip of the RGB does lack sodium-poor giants, if [Na/Fe] = 0 can be considered a high value relative to the rest of the low-Na distribution.

A histogram of the total sodium distribution is shown in Figure 9d, where two peaks are apparent but with no obvious gap between them. However, application of the KMM algorithm does indicate a bimodal distribution (P = 0.001) with two peaks at [Na/Fe] = −0.09 ± 0.13 and [Na/Fe] = +0.29 ± 0.13. The low-sodium group has 33 members while the high-sodium group has 86, for a ratio of Na-enhanced to Na-poor giants (the "Na ratio") of
Since the peaks are not widely separated (2.93 $\sigma$), it is difficult to determine to which group stars between the peaks belong, making the above ratio less certain. The KMM algorithm provides two group membership probabilities (GMPs) for each star, which give the percentage probability that a value belongs to “high” and “low” modes (the sum of the GMPs for each star equals 100%). For the sample of M13 giants, 35 stars (30%) have both GMPs between 10% and 90%, indicating that these stars cannot be assigned to either mode with high confidence.

When we bin the [Na/Fe] values by magnitude, as with the [Al/Fe] data, the bimodality of the distribution within each bin is not as certain as with the entire sample. We demonstrate this in Figure 9, where we show the Na ratio for M13 in bins 2 mag wide. The open triangles in this figure represent ratios where the $P$-values are greater than 0.05, indicating that a unimodal Gaussian is not easily ruled out as the “true” parent distribution. The filled triangles at $V > 14.2$ represent apparently bimodal distributions; however, the stars at these lower magnitudes are undersampled, confounding efforts to determine the nature of the [Na/Fe] distribution in M13.

Despite the uncertainty at lower magnitudes, the [Na/Fe] distribution for the brighter M13 giants is likely bimodal according the KMM statistics and is similar to that of the [Al/Fe] distribution. The mean of the 14 dimmest points is 2.33, represented by the horizontal line in Figure 9f. Relative to the mean, the upturn at $V = 13.1$ appears real and is due to the lack of low-sodium stars with $V \lesssim 12.5$, as seen in Figure 9b.

We compare the distribution of sodium in M3 with that of M13. As with the aluminum data, the sample size is small, with only 14 M3 stars having determined [Na/Fe] values (Kraft et al. 1992, 1993, 1995, 1999), six of which are also determined above. We believe the systematic differences among the Lick-Texas results and between our data and theirs that arise from the use of different telescopes, instruments, and reduction techniques are not significant enough to affect the interpretation of the results. As can be seen in Figure 10b, the distribution is fairly flat, with eight stars having [Na/Fe] $\leq 0.00$ and only three with [Na/Fe] $> +0.3$. This is more consistent with limited mixing, but the numbers are much too small to draw realistic conclusions.

5.1.4. Observational Results: Sodium and Aluminum

Referring again to Figure 11, we look for a correlation between sodium and aluminum in M13 by comparing the 62 giants with measured values of both [Al/Fe] and [Na/Fe]. The data appear correlated, with a linear correlation coefficient of 0.723, which, according to the probability coefficient given in Appendix C of Taylor (1982), is “highly significant” for this sample size. We do note, however, that at [Al/Fe] $\sim +0.5$ the full range of [Na/Fe] values is present, making the correlation suspicious around this narrow range of [Al/Fe] values.

Applying the KMM algorithm to just these 62 stars reveals that both the aluminum and sodium distributions are bimodal, with each having $P = 0.000$, and an Al ratio and a Na ratio both equal to 3.13 $\pm 0.93$. To test whether the identical ratios are just coincidence or if, indeed, a star with high [Al/Fe] is likely to have high [Na/Fe] and vice versa, we examine the difference in the GMPs between the aluminum and sodium data for the “high” modes, as shown in the histogram in Figure 12.

If the abundances of aluminum and sodium are correlated then the difference between the GMPs will be close to zero, as seems to be the case for most giants since 52 stars fall between $-0.10$ and $+0.10$. However, 10 deviate from zero by more than 0.25, so while Figure 12 indicates that only two or three giants do not fit into the correlation around [Al/Fe] = $+0.5$, the KMM algorithm actually shows that this number is larger and that around 16% of the sample are not statistically correlated. We must also reiterate that the [Al/Fe] values from the WIYN sample are not always well determined and that these numbers are likely to change with better data. In general, the correlation between [Al/Fe] and [Na/Fe] seems fairly constrained;
however, we suggest that when testing for deep mixing, aluminum is a better element to observe than sodium since the data show that the distinction between high and low [Al/Fe] is clearer and the models suggest that aluminum is produced much closer to the H shell than sodium is. According to the models, the appearance of mixing-enhanced aluminum on the surface of a star should imply the existence of extra sodium, while the converse is not necessarily true. The fact that the data show that 84% of the time the abundance of one element is a predictor of the other indicates just how deeply mixed the M13 giants probably are.

5.2. Hot Flashes and Primordial Influences

Enhanced aluminum abundances at magnitudes much lower than the tip of RGB are difficult to explain with mixing models since the peak temperature of the H shell is not high enough to produce a significant amount of aluminum at this stage of evolution (Cavallo et al. 1998). What then is the source of the high aluminum abundances at these lower luminosities? Some have suggested that the H shell might become unstable at lower magnitudes as a consequence of rotation and can undergo flashes that result in peak temperatures near 70 MK or higher (Langer et al. 1997; Fujimoto et al. 1999), as opposed to the canonical temperatures below 60 MK (see, e.g., CSB98). This hot temperature was chosen specifically by Langer et al. (1997) because it reproduces the observed abundance anomalies in some M13 giants, particularly the $^{24}$Mg depletions and aluminum enhancements observed by Shetrone (1996b). Unfortunately, such an exercise depends strongly on the accuracy of the nuclear reaction rates, which, in many cases, are not very well determined (see, e.g., the NACRE compilation; Angulo et al. 1999). Also, in addition to the fact that the H-shell instabilities have yet to be demonstrated in RGB models, it is not clear what the effects of such flashes would be on the structure and evolution of a star. For example, would these flashes have observable consequences that affect the RGB luminosity function, which is generally well reproduced by canonical evolutionary models? Furthermore, the instability scenario of Langer et al. (1997) only applies to the lower RGB (Von Rudloff, VandenBerg, & Hartwick 1988), which is at odds with the data reported here that show aluminum and sodium enhancements occur toward the tip of the RGB; if these elements were being produced on the lower RGB via hot flashes, the Al and Na ratios would vary at lower magnitudes. In the case of the Fujimoto et al. (1999) scenario, which involves continually peeling off layers of the core and completely disrupting the H shell, it is not clear how stars can evolve up the RGB and not have serious consequences for the observed luminosity functions of clusters in which the RGB members experience deep mixing.

We suggest that a more favorable location for hot hydrogen burning is around the H shell of intermediate-mass ($M > 4 \ M_\odot$) AGB stars (referred to as IMSs), which undergo hot bottom burning (HBB), so called because the convective envelope is in contact with the H shell. The IMSs could have shed their nuclearly processed envelopes that include enhanced aluminum and sodium abundances into the early cluster environment (Cottrell & Da Costa 1981; Denissenkov et al. 1997, 1998), creating the observed abundance distributions. For example, the bimodality of [Al/Fe] values on the lower RGB would be created if the ejecta were distributed locally. Likewise, the [Na/Fe] distribution in both clusters is explained if the IMS envelopes were also enriched in sodium. Unfortunately, detailed and accurate aluminum and sodium abundance yields from metal-poor AGB evolution models are nonexistent, but the high temperatures of HBB in IMS, and the observations themselves, lend some weight to this hypothesis.

5.3. Results from a Deep-Mixing Algorithm

We have taken the models described in CSB98 and subjected them to a deep-mixing algorithm that assumes that the mixing is instantaneous, i.e., the mixing timescale is the same as the nuclear burning timescale. A complete derivation of our algorithm is given in the Appendix. While this simplified approach is unable to mimic a realistic mixing timescale, it does have several advantages: (1) it can give an upper limit to the amount of variation an element can experience as a result of nuclear processing, (2) it can show the lowest point on the RGB where an element can be processed, and (3) it can be used to check the effect of the uncertainties in the nuclear reaction rates on the envelope abundances (Cavallo 1998b). We discuss the first two points after a brief description of the algorithm.

The nuclear reaction network employed in the mixing algorithm is the same as the one used in CSB98, with the following modifications: We use updated rates for the $^{24}$Mg($p, \gamma$)$^{25}$Al reaction that have been provided to us by C. Rowland. Her rates are (1) approximately 10–16 times faster than the NACRE rates (Angulo et al. 1999), (2) 1.5 to 4.5 times faster than the rates used in CSB98, and (3) commensurate with the Caughlan & Fowler (1988) tables in the range of $T_\odot = 0.04–0.06$, where $T_\odot$ is in units of $10^9$ K. The $^{24}$Al proton-capture rate has been separated into the short-lived isomeric state, $^{26m}$Al, and the metastable ground state, $^{26g}$Al (this had a negligible impact on the conclusions drawn.

11 Also available at http://pntpm.ulb.ac.be/nacre.htm.
in CSB98). The NACRE compilation shows that the \(^{26}\text{Al}\) proton-capture rate is uncertain by 3 orders of magnitude, the effects of which are discussed in § 6. We use the rates for the \(^{24}\text{Mg}(p, \gamma)^{25}\text{Al}\) reaction that have been updated by Powell et al. (1999), who measured the resonance parameters of the \(E_R = 223\) keV resonance to show that the low-energy contribution to the total rate does not significantly increase this rate as suggested by Zaidins & Langer (1997). The new rates show a 32% increase over the commonly used Caughlan & Fowler (1988) rates in the range \(T_e = 0.04–0.06\), which is not enough to account for the observed depletions of \(^{24}\text{Mg}\) in a handful of M13 and NGC 6752 giants observed by Shetrone (1996b, 1997).

The initial abundances that we put into our algorithm are those of Denissenkov et al. (1998), who suggest using \([^{25}\text{Mg/Fe}] = +1.1\) dex as the result of AGB contamination, while the initial \([^{24}\text{Mg/Fe}]\) and \([^{26}\text{Mg/Fe}]\) both equal zero. This suggestion is further backed by recent results of Lattanzio, Forestini, & Charbonnel (2000), who find the overproduction of \(^{25}\text{Mg}\) and \(^{26}\text{Mg}\) relative to \(^{24}\text{Mg}\) in metal-poor AGB models. In addition, we enhance the other \(x\)-elements by +0.4 dex.

We assume that mixing begins on the part of the RGB where the H shell burns through the hydrogen discontinuity left behind after the first dredge-up, in accordance with the assumption that large \(\mu\)-barriers can prevent mixing at earlier epochs (SM79). This point along the RGB corresponds to the well-known “bump” in the luminosity function (Fusi Pecci et al. 1990). Supporting this choice are the theoretical mixing models by Charbonnel (1994, 1995) that also assumed mixing begins at this point and reproduced the observed variations of the \(^{12}\text{C}/^{13}\text{C}\) ratio, \(^{7}\text{Li}\), and the \(^{12}\text{C}/^{14}\text{N}\) ratio in both open and globular clusters. In addition, the observations of Bell et al. (1979), Suntzeff (1981), Gilroy & Brown (1991), Grundahl (1999), and Carretta et al. (2000) also support this choice for the onset of mixing. We do point out that not all observations support this choice as the onset of mixing (Carbon et al. 1982; Trefzger et al. 1983; Langer et al. 1986; Briley et al. 1990), however, the exact start of deep mixing will little affect our final results.

The time step for nuclear processing fixes the time step for mixing and is controlled by setting a limit on how fast any element above a minimum abundance threshold may vary. Since this time step is much shorter than the time difference between the models used in CSB98, new models were interpolated along the RGB until the He flash was encountered. The free parameters in our code are \(\Delta X\), the mixing depth defined by a change in the H mass fraction, \(X\), within the H shell, and \(\eta\), Reimers’ mass-loss parameter (Reimers 1975). The algorithm was run with different combinations of mixing depths and mass-loss rates for a stellar evolutionary sequence having \([\text{Fe/H}] = -1.67\).

Figure 13 shows the \([\text{Al/Fe}]\) values derived from our algorithm as a function of absolute magnitude, parameterized by different mixing depths and mass-loss rates. The absolute magnitude scale was derived from the bolometric luminosity \(L\) and \(T_{\text{eff}}\) provided by the models. We first converted \(\log (L/L_\odot)\) into the bolometric magnitude, \(M_{\text{bol}}\), using the Sun as a reference with a value of 4.75 for \(M_{\text{bol}}\). Next we used the following relationship between gravity \(g\), mass \(M\), \(T_{\text{eff}}\), and \(L\) to obtain \(\log g\) for each model:

\[
\log g = \log (M/M_\odot) + 4 \log T_{\text{eff}} - \log (L/L_\odot) - 10.61028, \tag{1}
\]

where a value of \(T_{\text{eff}} = 5780\) K is assumed. Mass loss is implicitly accounted for via the first term on the right-hand side of equation 1. Using a 12 Gy isochrone with \([\text{Fe/H}] = -1.67\), also constructed by Professors VandenBerg and Bell, we interpolated according to \(\log g\) to find a bolometric correction and then converted \(M_{\text{bol}}\) into \(M_V\). We show the results for \(\Delta X = 0.05, 0.10, 0.15\), and 0.20, and for \(\Delta X = 0.0, 0.2, 0.4\), and 0.6 at each \(\Delta X\), as described in the figure legend.

According to Figure 13, the dominant parameter for determining \([\text{Al/Fe}]\) at the RGB tip is the mixing depth. Mass loss plays a secondary role for deeply mixed models but is more important for less deeply mixed models. The importance of \(\eta\) is controlled by a competition between the timescale for mass loss and the timescale for converting magnesium into aluminum: for a given sequence, the mass-loss timescale is fixed by \(\eta\), so with deeper mixing the Mg-burning timescale decreases, bringing the two closer together and limiting the influence of mass loss. Thus, in the limit of instantaneous mixing the distribution of \([\text{Al/Fe}]\) near the tip of the RGB is due primarily to variations in the mixing depth, although, for the deeply mixed sequences, the value of \([\text{Al/Fe}]\) at the tip begins to saturate.

One factor that depends strongly on the mixing depth is the earliest point along the RGB at which mixing-induced aluminum variations can occur. In Figure 13 we draw a vertical line at \([\text{Al/Fe}] = +0.4\) to indicate where the aluminum enhancements cross the into the “high” aluminum distribution for M13 giants. Our results show that for the mixing depths shown in Figure 13, large aluminum enhancements should appear along the brightest \(~1.5\) to 0.5 mag of the RGB. A change in \(\Delta X\) from 0.05 to 0.20 results in a 1 mag difference in where the aluminum abundance rises on the RGB. We apply these estimates to the
M13 sample by binning the data according to the magnitudes at which the different mixing depths predict the aluminum abundances will cross the \([\text{Al/Fe}] = +0.4\) threshold. The results are shown in Table 8, where the first two columns describe the models and the next five discuss the data. The first column gives the mixing depth and the second column describes how far down the RGB the models predict a star will cross into the “high” aluminum group for that mixing depth, while the third column gives the fraction of stars in the bin (out of 96). The fourth and fifth columns give the \(P\)-value and the Al ratio for stars in the bin, respectively, while the final two columns give the \(P\)-value and the Al ratio for the remaining stars outside the bin.\(^{12}\)

If the assumption that mixing is the cause of the aluminum enhancements in the bright giants of M13 holds, then it is apparent from Table 8 that it must turn on somewhere during the brightest magnitude of the RGB, as the Al ratio changes from 67:33 to 88:12 as one approaches the RGB tip. This signifies that at least 21% of the giants are experiencing deep mixing. We call this a lower limit because our technique for measuring aluminum enhancements cannot detect mixing in stars with initially high \([\text{Al/Fe}]\) values. Since two-thirds of the giants in M13 appear to have high aluminum abundances before mixing takes effect, we could be missing a substantial number of stars undergoing deep mixing. If the same relative number of giants with initially high aluminum abundances undergo deep mixing as the relative number of giants with initially low aluminum abundances, the percentage of all stars undergoing deep mixing jumps to 63%.

5.4. Deep Mixing, the Blue-Tail Parameter, and the Signatures of AGB Pollution

Deep mixing in red giants might have an effect on their future evolution. For example, Carretta & Gratton (1996) noted a relationship between the HB morphology and the amount of depletion of oxygen in RGB stars. This does not necessarily imply that oxygen is a second parameter, but rather that whatever is responsible for the oxygen depletions might also be causing the blueward shift in the HB. One such mechanism that can do both is rotation, which has several effects: (1) it can extend the life of a red giant, causing it to lose more mass and ultimately end up on the blue HB, (2) it can drive meridional circulation currents, which can deplete the oxygen, and (3) if fast enough, it can cause the circulation currents to penetrate the H shell and bring helium to the surface. This extra helium causes RGB stars to evolve to the blue HB at brighter luminosities than their unmixed counterparts, mimicking the second-parameter effect and reproducing the upward slope of the HB with decreasing color (Sweigart & Catelan 1998) that is observed in some metal-rich clusters (Piotto et al. 1997; Rich et al. 1997).

In this section, we examine the suggestion by Ferraro et al. (1998) that a so-called blue-tail second parameter (BTP) exists in M13. Such a parameter differs from the more commonly sought after second parameter in the sense that the latter typically deals with difference in HB morphology on the flat part of the HB, whereas the former describes how clusters such as M13, M80, and NGC 6752 develop extended blue tails. We attempt to discover whether or not deep mixing can be a (or the) BTP by adding extra helium into the atmospheres of RGB stars. Unfortunately, helium cannot be measured spectroscopically in cool giants; however, as shown in Figure 8, aluminum is made from magnesium inside the H shell, where helium is being produced, so the mixing of helium is accompanied by the mixing of aluminum; i.e., aluminum can be a good tracer of helium mixing. We conclude that if deep mixing is a BTP and if aluminum traces helium mixing, then there should exist a correlation between the Al ratio and the HB morphology. To describe the HB morphology quantitatively, we suggest using the ratio of blue to red HB stars (the “HB ratio”), which, of course, require definitions of their own. Perhaps a solution can be found in the corollary assumption that if cluster giants do not mix, then the cluster should not have an extended blue tail on the HB. Therefore, by assumption, a cluster such as M3, whose giants appear not to experience deep mixing, defines the “red” HB, so for clusters such as M13, any star on the HB that is hotter than the M3 horizontal branch is defined as “blue,” provided, of course, that the clusters are similar in all other ways (e.g., metallicity, age, environment).

To make the comparison between M3 and M13, we obtained high-quality Hubble Space Telescope photometric data for both clusters from F. Ferraro and shifted the M3 horizontal branch by \(\delta V = -0.6\) and \(\delta(U - V) = -0.03\) to align it with that of M13, as done by Ferraro et al. (1998). We then plotted histograms of the distributions of colors along the HBs for each cluster, as shown in Figure 14, and compared the HB ratio in M13 with its Al ratio. We define stars with \(U - V < -0.3\) as being blue, which yields an HB ratio of 58:42. We note that while this choice of color coincides with the apparent gap in Figure 14, it is chosen because this is where the M3 distribution drops off and not because of the appearance of bimodality in the M13 distribution. To estimate an error in the HB ratio, we fit a Gaussian to the M3 distribution to determine the standard error.
deviation, $\sigma$, and call blue all M13 stars hotter than the mean minus 3 $\sigma$ in M3, resulting in an HB ratio of 74:26. Compared with the 21% to 63% deeply mixed stars on the RGB, the 58% to 74% blue HB stars is consistent with deep mixing as the BTP, although the uncertainties in the number of RGB stars that have undergone deep mixing makes the results less robust than desired.

It would be helpful if we could discriminate between stars that have undergone deep mixing and those that have been polluted. One way to do this might be by using the $s$-process abundances, which are also created in intermediate-mass AGB stars (IMSS) with aluminum, albeit at different locations within the stars (Denissenkov et al. 1998; Lattanzio 1999; Boothroyd & Sackmann 1999). Stars above $M \gtrsim 4 M_\odot$ experience HBB, which, as discussed above, results in the production of $^{27}$Al from magnesium. These same stars also create a neutron exposure during the thermal pulses in the He-burning shell that favors the production of the Sr-Y-Zr peak elements and $^{25}$Mg, all through the $^{22}$Ne($x$,n)$^{23}$Mg reaction (Gallino et al. 2000). If the $n$-rich material is mixed with the HBB by-products and ejected into the cluster medium, then one should trace the other in the polluted stars. Specifically, we expect stars with high $s$-process abundances to also have high [Al/Fe] values, but not vice versa. The key is to choose the best $s$-process elements that trace the Al-rich IMS ejecta. We suggest using zirconium since many lines are available in the optical spectrum and its abundance is easily computed (Cavallaro & Nagar 2000). Conversely, elements near the barium peak would not be good choices to represent the $s$-process/HBB enhancement from IMSs, since they are produced in low-mass AGB stars ($M \sim 2 M_\odot$).

5.5. On the Overproduction of Sodium

Recently, Charbonnel, Denissenkova, & Weiss (2000) pointed out that mixing into the H shell to enrich the stellar atmosphere in aluminum and helium would result in an overproduction of sodium by $\gtrsim 0.3$ dex (see their Fig. 3) relative to the M13 data, essentially precluding deep mixing. This constraint keeps the change in the atmospheric helium abundance to less than 0.06, much less than the 0.20 found by Sweigart (1997a, 1997b) that is needed to account for the most extended HBs seen in clusters such as M13. We submit that the solution to this discrepancy might be found in the initial abundances; i.e., primordial contamination from the IMSs plays a role. Our algorithm shows that the overproduction of sodium is avoided, even with deep mixing, if the initial $^{22}$Ne abundance was depleted as a result of the $^{22}$Ne($x$,n)$^{23}$Mg reaction in the IMSs, which also enhances the initial $^{25}$Mg, as needed to produce the large aluminum enhancements in the RGB stars. The calculations of Gallino et al. (2000) indicate that the $^{22}$Ne abundance is depleted by approximately 30% during the thermal pulses in IMSs, but it is not clear how HBB and interpulse burning affects the net $^{22}$Ne abundance. One would assume that HBB would deplete the $^{22}$Ne reserves in the convective envelope and produce $^{23}$Na as is done on the RGB. In contradistinction, the AGB yields calculated by Denissenkov et al. (1998) actually enhance the net $^{22}$Ne abundance from a series of $x$-captures on $^{14}$N. Clearly, a more complete and detailed look into the yields of all abundances from primordial AGB stars is necessary to determine a more transparent picture of how pollution plays a role on the RGB.

Our hypothesis is consistent with the sodium data, which raise two important questions: (1) why does the Na ratio increase only at the same magnitudes as the Al ratio when sodium is very easily produced from $^{22}$Ne above the H shell at luminosities far below the RGB tip, and (2) why does the sodium abundance vary without oxygen abundance variations for “oxygen normal” giants? (See, e.g., Kraft et al. 1992.) The answer to both these questions might be found on the AGB: if $^{22}$Ne is depleted to build up $^{25}$Mg during the He shell flashes and $^{23}$Na during HBB, then sodium will not be produced at lower magnitudes on the RGB but will be made at brighter magnitudes from $^{22}$Ne with deep mixing. The extra sodium produced during HBB could be distributed locally within the cluster, creating the [Na/Fe]-rich stars that are independent of their oxygen abundances. Although oxygen is likely depleted during HBB, this is unlikely to create an oxygen-poor RGB atmosphere since it is easier to enhance elements in an atmosphere than to deplete them by pollution. A primordial pollution scenario is consistent with the data that show aluminum and sodium enhancements on the lower RGB and, for some clusters, on the main sequence and subgiant branch, and it can help prevent the overproduction of sodium during deep mixing.

6. CONCLUSIONS

Before we discuss our final conclusions, we first remind the reader of the number of assumptions that have gone into our analysis. First, there are errors associated with the abundance determinations that we tried to characterize by allowing for significant variations in the model atmosphere parameters, which contribute the most to the uncertainty in the analysis. Second, the inclusion of the WIYN data into our analysis comes at a price: the data have poor signal-to-noise ratios, come from only one line, and require the assumption that, for some stars with indeterminate line strengths, the [Al/Fe] value is “low.” Third, despite the fact that this is the largest compilation of [Al/Fe] and...
[Na/Fe] values in one globular cluster to be analyzed in a single paper, the data are still subject to small number effects, particularly at the RGB tip. Unfortunately, there are only so many tip stars that can be spectroscopically measured from the ground, leaving this problem difficult to solve. We suggest the best way to handle the small numbers is to expand this analysis to other clusters for a broad comparative study. Fourth, the models have many assumptions in them: we assume that canonical evolution holds and add in our mixing algorithm after the fact, we assume that mixing is instantaneous, we assume that the abundances are distributed as per Denissenkov et al. (1998), we assume that our reaction rates are accurate, and we assume that we adequately searched the parameter space allowed by the uncertainties in the initial abundances, nuclear reaction rates, and mass-loss rates. Fifth, we assume that no other second parameter affects the relationship between the M3 and M13 horizontal-branch morphologies.

Sixth, we make no attempt to correct for blending of the AGB with the RGB when performing our analysis. Approximately 20% of the red giants above the point where giant branches merge are supposedly AGB interlopers based on comparative lifetimes: the problem is to determine which ones are really AGB stars. This might not be as much of a problem for the M13 sample, however, since blue HB stars tend to evolve away from the AGB. The best work-around for this problem is also an extension of our analysis to other clusters to look for consistent trends despite this, and the other, uncertainties.

The importance of having accurate nuclear reaction rates cannot be overstated. This is particularly true when using aluminum as a diagnostic of deep mixing. If we were to enhance without increasing the aluminum abundance if the mixing currents do not penetrate the H shell, as seems to be indicated in M3 from the low Al ratio. However, some semblance of a correlation between aluminum and sodium might be set up by primordial effects in this cluster. In addition, the Na ratio increases near the same magnitude as the Al ratio, which is contrary to the previous predictions that sodium should be enhanced further down the RGB from $^{22}\text{Ne}$ (CSB98). Our models show that this would be expected if the $^{22}\text{Ne}$ were depleted in primordial intermediate-mass AGB stars.

When comparing the Al ratio with the HB ratio, it seems that the assumption of deep mixing as a blue-tail parameter is self-consistent; however, the large range allowed in the actual number of mixed RGB stars and the empirical definitions of “blue” and “red” HB stars do not constrain the results enough to be firmly conclusive. Again we suggest that a similar analysis to the one presented here be extended to other clusters to determine the Al ratio as a function of $V$ and to compare this with the HB ratio. If the Al ratio at the RGB tip can be shown to be a predictor of the HB ratio, then helium mixing would certainly be given greater credence as a blue-tail second parameter, supplanting the oft-assumed cluster age differences that have been shown to fail for this classical pair of clusters. In particular, we suggest the study of metal-rich clusters to see whether the aluminum distribution is bimodal, and if it is, whether the Al ratio varies. According to our models it should not vary, since aluminum cannot be produced in metal-rich cluster giants on the same scale as it can in the intermediate-metallicity and metal-poor giants. In addition, we suggest further examination of the sodium abundance in clusters to search for similar behavior to that seen in M13. Also, we suggest a more extensive comparison of the s-process abundances with the aluminum data as a test of primordial contamination.

Finally, we conclude that the problem of abundance anomalies in globular cluster red giants requires detailed study of the abundance yields from primordial AGB stars, as well as an in-depth and complete study of the hydrodynamic evolution of rotating RGB stars. In the meantime, aluminum, and to a lesser extent sodium, gives the best diagnostics of deep mixing during the evolution up the RGB, and the s-process elements near the Sr-Y-Zr peak are the best tracers of AGB pollution from IMRs.

Table 9, provided in the electronic edition only, provides model atmosphere and equivalent width measurements for M13 giants observed with the WIYN telescope.

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APPENDIX

INSTANTANEOUS-MIXING ALGORITHM

This derivation of instantaneous mixing begins with the form of the nuclear reaction equation that involves the proton-capture reactions and \( \beta \)-decays, although it is easily extended to other rates and is applied in its most general form in our code:

\[
\frac{dn_i}{dt} = \sum_j \left( \pm n_i n_{i\beta} \sigma_v^j \pm n_i \frac{\ln 2}{\tau_j} \right), \tag{A1}
\]

where \( n_i \) is the number of nuclei of type \( i \) per cubic centimeter that are being produced or destroyed, \( n_{i\beta} \) is the number per cubic centimeter of nuclei that produce (plus sign) or, when \( j = i \), destroy (minus sign) nuclei of type \( i \), \( n_{i\beta} \) is the number per cubic centimeter of protons, \( \sigma_v^j \) is the velocity-averaged cross section of the proton-capture reaction, and \( \tau_j \) is the mean lifetime of radioactive isotopes that destroy or produce element \( i \).

If we integrate equation (A1) over a mass interval from some mixing depth, \( M_d \), to the surface, \( M \), and substitute the molar fraction \( Y = n/\rho N_A \), where \( \rho \) is the density and \( N_A \) is Avogadro's number, so that \( Y_H = X \), the hydrogen mass fraction, we obtain

\[
\int_{M_d}^{M} \frac{dY_j}{dt} dM_r = \sum_j \left( \pm \int_{M_d}^{M} Y_j X N_A \sigma_v^j \rho dM_r + \frac{\ln 2}{\tau_j} \int_{M_d}^{M} Y_j dM_r \right), \tag{A2}
\]

which is equivalent to spreading the nuclei-processed material over the whole mixing zone. Mass loss can be accounted for by modifying the total integrated mass by some mass-loss recipe, such as that given by Reimers (1975).

Now, assuming that the mixing is instantaneous so that \( dY_j/\rho \), \( Y_j \), and \( X \) vary little over the whole mixing zone, we can rewrite equation (A2) as

\[
\frac{dY_j}{dt} \int_{M_d}^{M} dM_r = \sum_j \left( \pm Y_j X \int_{M_d}^{M} N_A \sigma_v^j \rho dM_r + \frac{\ln 2}{\tau_j} Y_j \int_{M_d}^{M} dM_r \right). \tag{A3}
\]

The first integral on the right-hand side is just the mass-averaged reaction rate, which can be substituted as an effective reaction rate, \( \sigma_v^j \left|_{\text{eff}} \right. \), while the other two integrals in equation (A3) are just the total mixed mass, \( M_{\text{mix}} \). Thus, equation (A3) can now be written

\[
\frac{dY_j}{dt} = \frac{1}{M_{\text{mix}}} \sum_j \left( \pm Y_j X \sigma_v^j \left|_{\text{eff}} \right. + \frac{\ln 2}{\tau_j} Y_j \right), \tag{A4}
\]

the final form of equation (A1) under the assumption of instantaneous mixing.

The implementation of this equation in our nuclear reaction network is straightforward. We average the reaction rates together by weighting the reaction rate determined at each mesh point by the mass contained between that mesh point and the one below it and summing over all the mass intervals. The temperature and density for calculating the reaction rate for each mass interval are taken at the top mesh point (toward the surface). Since the spacing between mesh points becomes closer as the temperature profile steepens, the differences between the temperature and density at the top and bottom of the mass intervals have negligible influence on the effective reaction rates. The effective rates are then applied to the initial abundances and integrated (“mixed”) over some determined mass interval where the mixing depth and mass-loss rate are the free parameters chosen by the user. The burning timescale is controlled by limiting how much the fastest-changing isotope with some chosen minimum abundance can vary in a single time step. New models are interpolated from a sequence at time steps according to this nuclear burning timescale. Each new model contains the output abundances derived from the previous model for its input abundances. The mixing algorithm can begin anywhere on the RGB and proceeds until the helium flash is encountered at the tip. The code outputs the new abundances, the mass-averaged reaction rates, and information regarding the position of the model on the theoretical RGB.

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