Completing the complex Poynting theorem: Conservation of reactive energy in reactive time

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The two (disconnected) Poynting theorems

- **Real Poynting theorem** ($\varepsilon_0 = \mu_0 = c = 1$):
  \[
  \frac{1}{2} \partial_t (E^2 + H^2) + \nabla \cdot (E \times H) = -E \cdot H \quad (P)
  \]

- **Complex Poynting theorem** uses **phasors** ($\omega > 0$)
  \[
  X(r, t) = \text{Re} \left( X(r, t) \right) \quad \text{where} \quad X(r, t) = e^{i\omega t}A(r) \quad (1)
  \]
  \[
  \frac{i}{2} \omega (|H|^2 - |E|^2) + \frac{1}{2} \nabla \cdot (E \times H^*) = -\frac{1}{2} E \cdot J^* \quad (P')
  \]

- Real part gives **period-averaged energy conservation**
- Imaginary part is somehow related to **reactive energy**
- **No connection** between the two Poynting theorems
- Extend (P') to time domain, define reactive energy, connect
3 Analytic signals

- Replace phasors by (positive-frequency) analytic signals:
\[ X(r, t) \rightarrow X(r, t) = \frac{1}{\pi} \int_0^\infty d\omega e^{i\omega t} X_\omega(r) \] (2)

- Extends analytically in \( t \rightarrow \tau = t + is, s > 0 \):
\[ X(r, \tau) = \frac{1}{\pi} \int_0^\infty d\omega e^{i\omega \tau} X_\omega(r) = \frac{1}{\pi} \int_0^\infty d\omega e^{i\omega t} e^{-\omega s} X_\omega(r) \] (3)

- \( e^{-\omega s} \) is a filter suppressing high-frequency components

- \( s \rightarrow 0^+ \Rightarrow \Re X \rightarrow X, \Im X \rightarrow \mathcal{H}X \) (Hilbert transform)

- \( X(r, t) = \delta(t)X^o(r) \Rightarrow X(r, \tau) = C(\tau)X^o(r) \), where
\[ C(\tau) = \frac{i}{\pi \tau} = \frac{s + it}{\pi(s^2 + t^2)} \equiv C_s(t) \] (Cauchy kernel) (4)

- Since \( \Delta t > s \) in \( X \), \( s \) is a time resolution parameter and
\[ X(r, t + is) = C_s \ast X(r, t) \] (5)
Complex Poynting theorem in the time-scale domain

- Define the scaled magnetic & electric energy densities

\[ \mathcal{W}_m(r, t, s) = \frac{1}{4} |H|^2, \quad \mathcal{W}_e(r, t, s) = \frac{1}{4} |E|^2 \]  

(6)

\[ \mathcal{U}(r, t, s) = \mathcal{W}_m + \mathcal{W}_e = \text{active energy density} \]  

(7)

\[ \mathcal{X}(r, t, s) = \mathcal{W}_m - \mathcal{W}_e = \text{reactive energy density} \]  

(8)

- \( \mathcal{U} \) and \( \mathcal{X} \) are time averages over \( \Delta t \sim s \) centered at \( t \)

- Scaled complex Poynting theorem:

\[
\partial_t \mathcal{U} - i \partial_s \mathcal{X} + \nabla \cdot \mathbf{S}_c = \mathcal{P}_c \tag{P''}
\]

where \( \mathbf{S}_c = \frac{1}{2} \mathbf{E} \times \mathbf{H}^*, \quad \mathcal{P}_c = -\frac{1}{2} \mathbf{E} \cdot \mathbf{J}^* \)

- \( \text{Re} \ (P'') \Rightarrow \text{active energy conservation in time } t \)

- \( \text{Im} \ (P'') \Rightarrow \text{reactive energy conservation in scale } s \)

- \( s \to 0^+ \Rightarrow \text{Re} \ (P'') \to (P) \), connecting (P) and (P')
5. Electromagnetic inertia and radiation impedance

- The local analog of mass $m = \sqrt{E^2 - p^2} \ (c \equiv 1)$ is the electromagnetic field inertia density

$$I(r, t) \equiv \sqrt{U^2 - S^2} = |Z|, \text{ where}$$

$$Z(r, t) = |E \cdot H| + \frac{i}{2}(H^2 - E^2) \equiv R + iX \quad (10)$$

- $I(r, t)$ represents EM field energy which is *locally at rest*
- $X(r, t)$ is the local version of $X(r, t, s)$
- $X$ and $R$ are the basic invariants of the EM field
- The local conditions for pure radiation are precisely

$$\{R = 0 \text{ and } X = 0\} \iff I = 0 \quad (11)$$

Thus $R$ and $X$ are the impediments to radiation

- $Z$ is the field-density analog of radiation impedance
- Scaled version:

$$Z(r, t, s) = \sqrt{U^2 - |S_c|^2} = |Z|, \text{ where}$$

$$Z(r, t, s) = \frac{1}{2}|E \cdot H| + \frac{i}{4}(|H|^2 - |E|^2) \equiv R + iX \quad (13)$$
The idea of using analytic signals to extend the complex Poynting theorem to the time domain is not new; see

- T D Carozzi, J E S Bergman and R L Karlsson, *Complex Poynting Theorem as conservation law*. Preprint, 2005

However, they did not use analyticity in $\tau$ but kept $s \equiv 0$, so the imaginary part is not a conservation law and no expression be identified as the reactive energy density.

The **local** inertia density $I(r, t)$ was introduced and studied in

- G Kaiser, *Electromagnetic inertia, reactive energy, and energy flow velocity*. J Phys A **44** (2011) 345206
  [http://arxiv.org/abs/1105.4834](http://arxiv.org/abs/1105.4834)

- G Kaiser, *The Reactive Energy of Transient EM Fields*. IEEE/USNC-URSI conference, Chicago, 2012
  [http://arxiv.org/abs/1201.6575](http://arxiv.org/abs/1201.6575)