Homotopy Operators and Locality Theorems in Higher-Spin Equations

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Abstract

A new class of shifted homotopy operators in higher-spin gauge theory is introduced. A sufficient condition for locality of dynamical equations is formulated and Pfaffian Locality Theorem identifying a subclass of shifted homotopies that decrease the degree of non-locality in higher orders of the perturbative expansion is proven.
1 Introduction

Nonlinear field equations for 4d massless fields of all spins were found in \[1, 2\]. The most symmetric vacuum solution to these equations describes \(AdS_4\). Due to the presence of \(AdS_4\) radius as a dimensionful parameter, higher-spin (HS) interactions can contain infinite tails of higher-derivative terms. This can make the theory non-local in the standard sense, raising the question which field variables lead to the local or minimally non-local setup in the perturbative analysis. Recently, in \[3-4\] it was shown how nonlinear HS equations of \[2\] reproduce current interactions in the lowest order in interactions. It has been then checked in \[5-7\] that the results of \[3, 4\] properly reproduce the holographic expectations thus resolving some of the puzzles of the analysis of HS holography conjectures of \[9, 10, 11\] encountered in \[12, 11, 13\] (and references therein).

The derivation of \[3, 4\] was based on the separation of variables (holomorphic factorization) in the zero-form sector of the 4d HS theory. So far the perturbative analysis of HS equations was based on the conventional homotopy operator technics proposed in \[2\]. In \[5\] it was explicitly checked that, in agreement with \[12, 14\], the field redefinition that brings the results obtained by virtue of the conventional homotopy to the correct local form is non-local. Moreover, in \[5\] it was shown that from the perspective of the full nonlinear HS equations the field redefinition found in \[3\] has distinguished properties indicating that it leads to minimal order of non-locality in the higher orders. However it was not clear how the homotopy technics should be modified to lead directly to the correct local results in the perturbative analysis of HS equations with no reference to field redefinitions.

The main aim of this paper is to generalize the conventional homotopy technics in such a way that it will give immediately correct local results in the lowest order. Based on this generalization we prove a theorem showing how to choose the proper class of homotopy operators to decrease the level of non-locality of HS equations in higher orders as well.

Note that what is interpreted as locality in this paper is probably better to call spin locality as it refers to the form of expressions in the sector of spinor variables underlying the unfolded formulation of HS equations of \[1, 2\]. Its relation to the conventional definition in terms of space-time derivatives is via unfolded equations as we briefly recall now.

Unfolded equations of 4d massless Fronsdal \[15, 16\] fields of all spins \(s = 0, 1/2, 1, 3/2, 2 \ldots \) in \(AdS_4\) are formulated in terms of a one-form \(\omega(Y; K|x) = dx^a\omega_a(Y; K|x)\) and zero-form \(C(Y; K|x)\) \[17\], \(Y = (y, \bar{y})\). The Klein operators \(K = (k, \bar{k})\) satisfy

\[
ky^a = -y^a k, \quad k\bar{y}^a = \bar{y}^a k, \quad \bar{k}y^a = y^a \bar{k}, \quad \bar{k}\bar{y}^a = -\bar{y}^a \bar{k}, \quad kk = \bar{k}\bar{k} = 1, \quad \bar{k}\bar{k} = \bar{k}\bar{k}.
\]

More precisely, to describe massless fields, the one-form \(\omega(Y; K|x)\) should be even in \(k, \bar{k}\) while the zero-form \(C(Y; K|x)\) should be odd. Thus, massless fields are doubled

\[
C(Y; K|x) = C^{1,0}(Y|x)k + C^{0,1}(Y|x)\bar{k}, \quad \omega(Y; K|x) = \omega^{0,0}(Y|x) + \omega^{1,1}(Y|x)k\bar{k}.
\]
Unfolded field equations for free massless fields of all spins in the $AdS_4$ are \[ R_1(Y; K|x) = L(w, C) := \frac{i}{4} \left( \eta H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(0, \bar{y}; K|x) - H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(y, 0; K|x) \right), \tag{1.3} \]

where

\[ R_1(Y; K|x) := D^0 \omega(Y; K|x) := D^L \omega(Y; K|x) + \lambda h^{\alpha\beta} \left( y_\alpha \frac{\partial}{\partial y^\beta} + \frac{\partial}{\partial y^\alpha} \bar{y}_\beta \right) \omega(Y; K|x), \tag{1.5} \]

\[ D^0 f(Y; K|x) := d_x f(Y; K|x) + \left( \omega^{\alpha\beta}_\beta y_\alpha \frac{\partial}{\partial y^\beta} + \bar{\omega}^{\alpha\beta}_\beta y_\alpha \frac{\partial}{\partial \bar{y}^\beta} \right) f(Y; K|x), \quad d_x := d x^n \frac{\partial}{\partial x^n}. \tag{1.7} \]

Background $AdS_4$ space of radius $\lambda^{-1} = \rho$ is described by a flat $sp(4)$ connection $w = (w_{\alpha\beta}, \bar{w}_{\bar{\alpha}\bar{\beta}}, h_{\alpha\bar{\beta}})$ containing Lorentz connection $w_{\alpha\beta}$, $\bar{w}_{\bar{\alpha}\bar{\beta}}$ and vierbein $h_{\alpha\bar{\beta}}$ that obey

\[ d_x w_{\alpha\beta} + w_{\alpha\gamma} w_{\beta}^\gamma - \lambda^2 H_{\alpha\beta} = 0, \quad d_x \bar{w}_{\bar{\alpha}\bar{\beta}} + \bar{w}_{\bar{\alpha}\bar{\gamma}} \bar{w}_{\bar{\beta}}^\bar{\gamma} - \lambda^2 \bar{H}_{\bar{\alpha}\bar{\beta}} = 0, \quad d_x h_{\alpha\bar{\beta}} + w_{\alpha\gamma} h_{\bar{\gamma}\bar{\beta}} + \bar{w}_{\bar{\alpha}\bar{\gamma}} h_{\gamma\beta} = 0, \tag{1.8} \]

where $H_{\alpha\beta} := h^{\alpha\bar{\alpha}} h_{\bar{\beta}\bar{\beta}}$ and $\bar{H}_{\bar{\alpha}\bar{\beta}} := h^{\alpha\bar{\alpha}} h_{\bar{\beta}\bar{\beta}}$ are the frame two-forms (wedge symbol is omitted).

In the massless sector, system (1.3), (1.4) decomposes into subsystems of different spins, with a spin $s$ described by the one-forms $\omega(y, \bar{y}; K|x)$ and zero-forms $C(y, \bar{y}; K|x)$ obeying

\[ \omega(\mu y, \mu \bar{y}; K | x) = \mu^{2(s-1)} \omega(y, \bar{y}; K | x), \quad C(\mu y, \mu^{-1} \bar{y}; K | x) = \mu^{\pm 2s} C(y, \bar{y}; K | x), \tag{1.9} \]

where $+$ and $-$ correspond to helicity $h = \pm s$ selfdual and anti-selfdual parts of the generalized Weyl tensors $C(y, \bar{y}; K|x)$. For spins $s \geq 1$, equation (1.3) expresses the Weyl 0-forms $C(Y; K|x)$ via gauge invariant combinations of derivatives of the HS gauge connections. More precisely, the primary-like Weyl 0-forms are just the holomorphic and antiholomorphic parts $C(y, 0; K|x)$ and $C(0, \bar{y}; K|x)$ which appear on the r.h.s. of Eq. (1.3). Those associated with higher powers of auxiliary variables $y$ and $\bar{y}$ describe on-shell nontrivial combinations of derivatives of the generalized Weyl tensors as is obvious from Eqs. (1.4), (1.6) relating second derivatives in $y, \bar{y}$ to the $x$ derivatives of $C(Y; K|x)$ of lower degrees in $Y$. Hence higher derivatives in the nonlinear system hide in the components of $C(Y; K|x)$ of higher orders in $Y$. To see whether the resulting equations are local or not at higher orders one has to inspect the dependence of vertices on the higher components of $C(Y; K|x)$.

At the linearized level, Eq. (1.6) implies that $\frac{\partial^2}{\partial y^\alpha \partial y^\beta}$ is equivalent to $\frac{\partial^2}{\partial \bar{y}^\alpha \partial \bar{y}^\beta}$. Hence, at this level the analysis of spin locality in terms of $y, \bar{y}$ variables is equivalent to that in terms of space-time derivatives. However in higher orders Eq. (1.6) acquires nonlinear corrections. This makes the relation between spin locality in terms of $y, \bar{y}$ variables and space-time locality
less straightforward. Since the spinor sector of HS equations is of fundamental importance all concepts in HS theory including locality have to be originally defined in these terms. Therefore, we regard the spin locality of the HS theory as the fundamental concept. Relation to the space-time locality at higher orders is not straightforward being somewhat analogous to the effect of current exchange contribution in the space-time formulation.

A related comment is that space-time covariant derivatives $D^E$ do not commute in presence of non-zero cosmological constant which is of order one in the absence of other dimensionful parameters like $\alpha'$ in string theory. This raises a nontrivial question of the choice of the ordering prescription in the expansions in higher space-time derivatives. We believe that the concept of spin locality in terms of spinor variables provides an appropriate solution to this problem which may be hard to guess directly in the space-time approach.

Let us explain the idea of the analysis of spin locality in some more detail. As explained in Section 3, general exponential representation for the order-$n$ corrections in the zero-forms $C$ is

$$\sum_{pp} \int d\tau \hat{P}^{pp}_n(y, \bar{y}, p, \bar{p}, \tau) \hat{E}^{pp}_n(y, \bar{y}, p, \bar{p}, \tau) C(Y_1; K) \ldots C(Y_n; K) |_{Y_j=0},$$

(1.10)

where

$$p^j_\alpha := -i \frac{\partial}{\partial y^j_\alpha}, \quad \bar{p}^{\bar{j}}{}_{\dot{\alpha}} := -i \frac{\partial}{\partial \bar{y}^{\bar{j}}{}_{\dot{\alpha}}},$$

(1.11)

$\hat{P}^{pp}_n(y, \bar{y}, p, \bar{p}, \tau)$ is some polynomial of $y, \bar{y}, p^j$ and $\bar{p}^{\bar{j}}$ with coefficients being regular functions of some homotopy integration parameters $\tau$ and

$$\hat{E}^{pp}_n = \hat{E}^{p}_n \hat{E}^{\bar{p}}_n; \quad \hat{E}^{p}_n(\hat{B}, \hat{p}, p|z, y) = \exp i(-\hat{B}_j(\tau)p^j_\alpha y^\alpha + \frac{1}{2} \hat{P}_{ij}(\tau)p^j_\alpha p^{i}{}_{\dot{\alpha}}) k^p,$$

(1.12)

where $p = 0, 1$ and parameters $\hat{B} \in \mathbb{C}^n$, $\hat{P}_{ij} = -\hat{P}_{ji} \in \mathbb{C}^n \times \mathbb{C}^n$ may be $\tau$-dependent.

Spin locality of HS interactions is governed by the coefficients $\hat{P}_{ij}$ in $\hat{E}^{p}_n$ (1.12) and their complex conjugates $\hat{\bar{P}}_{ij}$ in $\hat{E}^{\bar{p}}_n$ that determine contractions between, respectively, undotted and dotted spinorial arguments of different factors of $C(Y; K|x)$. Since the contribution of $\hat{P}_{ij}$ and $\hat{\bar{P}}_{ij}$-dependent terms is via exponential it gives rise to a non-polynomial expansion in $p^i{}_{\alpha} p^j_{\dot{\alpha}}$ and $\bar{p}^{\bar{i}}{}_{\dot{\alpha}} \bar{p}^{\bar{j}}{}_{\dot{\alpha}}$ and, hence, via (1.4) and (1.6), to non-local expansion in space-time derivatives. In fact, nonlinear corrections to HS equations have the form (1.10), (1.12) where at least one of the coefficients $\hat{P}_{ij}(\tau)$ and $\hat{\bar{P}}_{ij}(\bar{\tau})$ is nonzero. This is a manifestation of the fact that HS theory is in a certain sense non-local in agreement with the well-known property that higher spins carry higher derivatives and, hence, in presence of an infinite tower of HS fields the full theory must contain infinite tower of higher derivatives as well.

A less trivial question is on the locality of vertices involving particular spins $s_1, \ldots s_n$. In accordance with (1.3), for fixed helicities, the degree in $y_i$ variables in $C(Y_i; K|x)$ is related to that in $\bar{y}_i$. In that case the degree in $p^i{}_{\alpha} p^j_{\dot{\alpha}}$ gets related to that in $\bar{p}^{\bar{i}}{}_{\dot{\alpha}} \bar{p}^{\bar{j}}{}_{\dot{\alpha}}$ in a particular vertex. As a result, for vertices with fixed spins polynomiality in $p^i{}_{\alpha} p^j_{\dot{\alpha}}$ implies polynomiality in $\bar{p}^{\bar{i}}{}_{\dot{\alpha}} \bar{p}^{\bar{j}}{}_{\dot{\alpha}}$ and vice versa. Hence spin locality for a any fixed set of spins will be achieved if,
for instance, one of the coefficients $\hat{P}_{ij}$ or $\hat{\bar{P}}_{ij}$ is zero. If it happened in all orders, this would imply all-order spin locality of HS equations.

One of the main results of this paper consists of the proof of Pfaffian Locality Theorem in Section 6 stating that, in the sector of equations on zero-forms, there exist such particular homotopy procedures that the antisymmetric matrices $\hat{P}_{ij}(\tau)$ and $\hat{\bar{P}}_{ij}(\ BAR{\tau})$ are degenerate. For the lowest order bilinear corrections associated with $2 \times 2$ antisymmetric matrices this implies that at least one of the matrices $\hat{P}_{ij}$ and $\hat{\bar{P}}_{ij}$ is zero thus implying spin locality of the lowest-order corrections. In fact, this result allows us to speculate that, by a proper choice of homotopy operators, HS contact interactions can be brought to the local form in higher-orders for every fixed set of spins in a vertex. (Note that some higher-order local vertices are constructed in [18].) If nevertheless this does not happen then it makes sense to look for a minimally non-local setup such that, being non-local, it is minimally non-local leading to the fastest decrease of the coefficients in front of higher powers of $p^i_\alpha p^j_\alpha$ and $\bar{p}^{i\dot{\alpha}} \bar{p}^{j\dot{\alpha}}$. Pfaffian Locality Theorem indicates that such minimization should be possible.

Let us stress that there are many reasons why it is important to elaborate the intrinsic analysis of the HS gauge theory in the bulk with no reference to the holographic duals. The simplest is that apart from free boundary theories dual to particular HS gauge theories in the bulk, the latter equally well describe interacting Chern-Simons boundary theories [10, 11] where the computation of amplitudes is more involved. More general background solutions of HS theories with more complicated boundary duals like for instance massive deformations can also be of interest. Another point is that the approach proposed in this paper is applicable to a much more general class of Coxeter HS theories some of which were conjectured in [19] to be related to String Theory upon spontaneous breakdown of HS symmetries. In the latter case, application of holographic duality is more tricky because it becomes strong-weak duality. Hence, independent formulation of the underlying bulk HS theory is of great importance in that case as well.

The rest of the paper is organized as follows. The form of nonlinear HS equations is sketched in Section 2. Perturbative analysis of HS equations in terms of homotopy operator technics is recalled in Section 3. In Section 4 we introduce modified homotopy operators appropriate for the analysis of locality of HS equations. In Section 5 we prove $Z$-dominance Lemma providing a sufficient criterion for the locality of nonlinear corrections to dynamical field equations. Pfaffian Locality Theorem providing a criterion for the choice of homotopy decreasing the degree of non-locality in higher orders of interactions is proven in Section 6. Section 7 contains brief conclusions.

2 Nonlinear higher-spin equations

4d nonlinear HS equations [2] have the form

\[
\begin{align*}
\text{d}_x W + W \ast W & = i(\theta^A \theta_A + F_\gamma(B) \ast \gamma + \bar{F}_\gamma(B) \ast \bar{\gamma}) , \\
\text{d}_x B + W \ast B - B \ast W & = 0 ,
\end{align*}
\]  
(2.1)  
(2.2)
where

\[ \gamma = \theta^a \theta_\alpha \kappa k, \quad \bar{\gamma} = \bar{\theta}^a \bar{\theta}_{\alpha} \bar{\kappa} k. \]  

(2.3)

\( \mathcal{W} \) and \( B \) are fields of the theory which depend both on space-time coordinates \( x^a \) and on twistor-like variables \( Y^A = (y^\alpha, \bar{y}^\dot{\alpha}) \) and \( Z^A = (z^\alpha, \bar{z}^\dot{\alpha}). \) \( (A = 1, \ldots, 4) \) is a Majorana spinorial index while \( \alpha = 1, 2 \) and \( \dot{\alpha} = 1, 2 \) are two-component ones. The latter are raised and lowered by \( \epsilon_{\alpha\dot{\alpha}} = (-\epsilon_{\dot{\alpha}\alpha}, \epsilon_{12} = 1): A^\alpha = \epsilon^{\alpha\beta} A_\beta, \ A_\alpha = A^\beta \epsilon_{\beta\dot{\alpha}} \) and analogously for dotted indices.)

The \( Y \) and \( Z \) variables provide a realization of HS algebra through the following non-commutative associative star product \( \ast \) acting on functions of two spinor variables

\[ (f \ast g)(Z; Y) = \int \frac{d^4U \, d^4V}{(2\pi)^4} \exp [iU^AVB C_{AB}] f(Z + U; Y + U) g(Z - V; Y + V), \]  

(2.4)

where \( C_{AB} = (\epsilon_{\alpha\beta}, \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}) \) is the 4d charge conjugation matrix and \( U^A, V^B \) are real integration variables. \( 1 \) is a unit element of the star-product algebra, \( i.e., f \ast 1 = 1 \ast f = f. \) Star product (2.4) provides a particular realization of the Weyl algebra

\[ [Y_A, Y_B]_\ast = -[Z_A, Z_B]_\ast = 2iC_{AB}, \quad [Y_A, Z_B]_\ast = 0, \quad [a, b]_\ast := a \ast b - b \ast a. \]  

(2.5)

It is convenient to introduce anticommuting \( Z \)-differentials \( \theta^A, \bar{\theta}^\dot{A} = -\theta^\dot{A} \theta^A. \) \( B \) is a 0-form, while \( \mathcal{W} \) is differential one-form with respect to \( dx^a, \theta^A \) differentials, \( i.e., \mathcal{W} = \{W, S\}, \) where \( W(Z; Y; K|x) \) is a space-time one-form, while \( S = \theta^A S_A(Z; Y; K|x). \)

The Klein operators satisfy relations analogous to (1.1) with \( y^\alpha \to w^\alpha = (y^\alpha, z^\alpha, \theta^\alpha), \) \( \bar{y}^\dot{\alpha} \to \bar{w}^\dot{\alpha} = (\bar{y}^\dot{\alpha}, \bar{z}^\dot{\alpha}, \bar{\theta}^\dot{\alpha}), \) which extend the action of the star product to the Klein operators. Decomposing master-fields with respect to the Klein-operator parity, \( A^\pm(Z; Y; K|x) = \pm A^\pm(Z; Y; -K|x) \), HS gauge fields are \( W^+, S^+ \) and \( B^- \) while \( W^-, S^- \) and \( B^+ \) describe an infinite tower of topological fields with every \( AdS_4 \) irreducible field describing at most a finite number of degrees of freedom. (For more detail see [2] [21].)

\( F_\ast(B) \) is some star-product function of the field \( B. \) The simplest choice of the linear function \( F_\ast(B) = \eta B, \) \( \bar{F}_\ast(B) = \bar{\eta} B, \) where \( \eta \) is a complex parameter \( \eta = |\eta| \exp i\varphi, \varphi \in [0, \pi], \) leads to a class of pairwise nonequivalent nonlinear HS theories. The cases of \( \varphi = 0 \) and \( \varphi = \frac{\pi}{2} \) correspond to so called \( A \) and \( B \) HS models that respect parity [21].

The left and right inner Klein operators

\[ \kappa := \exp iz_\alpha y^\alpha, \quad \bar{\kappa} := \exp iz_\dot{\alpha} \bar{y}^\dot{\alpha}, \]  

(2.6)

which enter Eq. (2.3), change a sign of undotted and dotted spinors, respectively,

\[ (\kappa \ast f)(z, \bar{z}; y, \bar{y}) = \exp iz_\alpha y^\alpha f(y, \bar{z}; z, \bar{y}), \quad (\bar{\kappa} \ast f)(z, \bar{z}; y, \bar{y}) = \exp iz_\dot{\alpha} \bar{y}^\dot{\alpha} f(z, \bar{y}; y, \bar{z}), \]  

(2.7)

\[ \kappa \ast f(z, \bar{z}; y, \bar{y}) = f(-z, \bar{z}; -y, \bar{y}) \ast \kappa, \quad \bar{\kappa} \ast f(z, \bar{z}; y, \bar{y}) = f(z, -\bar{z}; y, -\bar{y}) \ast \bar{\kappa}, \]  

(2.8)

\[ \kappa \ast \kappa = \bar{\kappa} \ast \bar{\kappa} = 1, \quad \kappa \ast \bar{\kappa} = \bar{\kappa} \ast \kappa. \]  

(2.9)

6
3 Perturbative analysis and homotopy operator

3.1 Vacuum

Perturbative analysis of Eqs. (2.1), (2.2) assumes their linearization around some vacuum solution. The simplest choice is

\[
W_0(Z; Y; K|x) = w(Y; K|x), \quad S_0(Z; Y; K|x) = \theta^A Z_A, \quad B_0(Z; Y; K|x) = 0, \tag{3.1}
\]

where \(w(Y|x)\) is some solution to the flatness condition

\[
d_x w + w * w = 0. \tag{3.2}
\]

A flat connection \(w(Y|x)\), that describes \(AdS_4\) via (1.8), is bilinear in \(Y^A\)

\[
w(Y|x) = -\frac{i}{4} (w^{\alpha\beta}(x) y_\alpha y_\beta + w^{\alpha\bar{\beta}}(x) \bar{y}_\alpha \bar{y}_\beta + 2 h^{\alpha\beta}(x) y_\alpha \bar{y}_\beta). \tag{3.3}
\]

Since \(S_0\) has a trivial star-commutator with the Klein operators, a simple computation gives

\[
[S_0, F(Y; Z; K|x)]_* = -2 i d_Z F(Y; Z; K|x), \quad d_Z = \theta^A \frac{\partial}{\partial Z^A}. \tag{3.4}
\]

Let

\[
W(Y, Z|x) = S_0 + w(Y|x) + W'(Y, Z|x). \tag{3.5}
\]

Denoting

\[
\mathcal{D} := \mathcal{D}_x + \mathcal{D}_w, \quad \mathcal{D}_x = -\frac{i}{2} d_x, \quad \mathcal{D}_w A := -\frac{i}{2} [w, A]_\pm, \tag{3.6}
\]

Eqs. (2.1), (2.2) yield

\[
(d_Z - \mathcal{D}) W' + \frac{i}{2} W' * W' = -\frac{1}{2} (\eta B * \gamma + \bar{\eta} B * \bar{\gamma}), \tag{3.7}
\]

\[
(d_Z - \mathcal{D}) B = -\frac{i}{2} [W', B]_*. \tag{3.8}
\]

3.2 Homotopy trick

To eliminate \(Z\)-variables one has to repeatedly solve equations of the form

\[
d_Z f(Z; Y; K|x) = J(Z; Y; K|x). \tag{3.9}
\]

Consistency of HS equations guarantees that \(J(Z; Y; K|x)\) is \(d_Z\)-closed, \(d_Z J(Z; Y; K|x) = 0\), implying formal consistency of Eq. (3.3). However, it admits a solution only if \(J\) is \(d_Z\)-exact.

Given homotopy operator \(\partial\)

\[
\partial^2 = 0, \tag{3.10}
\]

the operator

\[
A := \{d_Z, \partial\} \tag{3.11}
\]
obeys
\[ [\text{d}_Z, A] = 0, \quad [\partial, A] = 0. \tag{3.12} \]

For diagonalizable \( A \), the standard Homotopy Lemma states that cohomology \( H_{d_Z} \) of \( \text{d}_Z \), is in the kernel of \( A \)
\[ H_{d_Z} \subset \text{Ker} A. \tag{3.13} \]

In this case the projector \( h \) to \( \text{Ker} A \)
\[ h^2 = h \tag{3.14} \]

and the operator \( A^* \) can be defined to obey
\[ [h, \text{d}_Z] = [h, \partial] = 0, \quad A^* A = AA^* = \text{Id} - h. \tag{3.15} \]

The resolution operator
\[ \triangle := A^* \partial = \partial A^* \tag{3.16} \]

gives the resolution of identity
\[ \{\text{d}_Z, \triangle\} + h = \text{Id} \tag{3.17} \]

allowing to find a solution to the equation \( \text{d}_Z f = J \) with \( \text{d}_Z \)-closed \( J \) outside of \( H_{d_Z} \), i.e., obeying \( \text{h} J = 0 \), in the form
\[ f = \triangle J + \text{d}_Z \epsilon + g, \tag{3.18} \]

where an exact part \( \text{d}_Z \epsilon \) and \( g \in H_{d_Z} \) remain undetermined.

### 3.3 Perturbative expansion

HS equations reconstruct the dependence on \( Z_A \) in terms of the zero-form \( C(Y; K|x) \in H_{d_Z} \) and one-form \( \omega(Y; K|x) \in H_{d_Z} \) representing the \( d_Z \)-cohomological parts of \( B \) and \( W' \),
\[ B = C(Y; K|x) + \sum_{j=2}^{\infty} B_j(Y, Z; K|x), \quad W' = \omega(Y; K|x) + \sum_{j=1}^{\infty} W_j(Y, Z; K|x), \tag{3.19} \]

where zero-forms \( B_j(Y, Z; K|x) \) and one-forms \( W_j(Y, Z; K|x) \) are of order \( j \) in \( \omega \) and \( C \) and obey
\[ H_{d_Z}(B_j(Y, Z; K|x)) = 0, \quad H_{d_Z}(W_j(Y, Z; K|x)) = 0 \quad \forall j. \tag{3.20} \]

The perturbative analysis goes as follows. Suppose that an order-\( n \) solution
\[ W^{(n)}(Y, Z; K|x) = \omega(Y; K|x) + \sum_{j=1}^{n} W_j(Y, Z; K|x), \tag{3.21} \]
\[ B^{(n)}(Y, Z; K|x) = \sum_{j=1}^{n} B_j(Y, Z; K|x), \quad B_1(Y, Z; K|x) = C(Y; K|x) \tag{3.22} \]
is found. Eqs. (3.7), (3.8) yield at order \( n + 1 \)

\[
\left( (d_Z - D) W^{(n+1)} \right) \bigg|_{n+1} = -\frac{1}{2} \left( i W^{(n+1)} W^{(n+1)} + \eta B^{(n+1)} \gamma + \bar{\eta} B^{(n+1)} \bar{\gamma} \right) \bigg|_{n+1} + \cdots ,
\]

(3.23)

\[
\left( (d_Z - D) B^{(n+1)} \right) \bigg|_{n+1} = -\frac{1}{2} \left( i W^{(n+1)} B^{(n+1)} \right) \bigg|_{n+1} + \cdots ,
\]

where ellipsis denotes higher-order terms, \( A(C, \omega) \big|_k \) is the order-\( k \) part of \( A(C, \omega) \) in \( \omega \) and \( C \), and

\[
A(C, \omega) \big|_{\leq m} := \bigcup_{k \leq m} A(C, \omega) \big|_k .
\]

From (3.23) it follows by virtue of (3.21), (3.22) using (3.6)

\[
-D \omega + \theta (n-2) \frac{i}{2} \omega * \omega + \sum_{m=1}^{n+1} d_Z W_m = D_x W_{n+1} + \sum_{m=1}^{n+1} \left\{ X^W_m + D_w W_m + \left( D_x \sum_{j=1}^{m-1} W_j \right) \right\} ,
\]

(3.24)

\[
-D C + \sum_{m=2}^{n+1} d_Z B_m = D_x B_{n+1} + \sum_{m=2}^{n+1} \left\{ X^B_m + D_w B_m + \left( D_x \sum_{j=2}^{m-1} B_j \right) \right\} ,
\]

(3.25)

where

\[
X^W_m = -\frac{i}{2} \sum_{j=1}^{m-1} \{ W_j , W_{m-j} \} * - \frac{i}{2} \{ \omega , W_{m-1} \} + \frac{1}{2} \eta B_m * \gamma - \frac{1}{2} \bar{\eta} B_m * \bar{\gamma} ,
\]

(3.26)

\[
X^B_m = -\frac{1}{2} \sum_{j=1}^{m-1} [ B_j , W_{m-j} ] .
\]

(3.27)

Let us stress that being of order \( m \) in \( C \) and \( \omega \), \( X^W_m \) and \( X^B_m \) contain \( B_j \) and \( W_j \) with \( j < m \). Also it is used that the order-\( n \) parts of dynamical equations are of the form

\[
d_\omega (Y ; K|x) = \sum_{j=1}^{n} J^0_j (Y ; K|x) ,
\]

\[
d_\omega C (Y ; K|x) = \sum_{j=1}^{n} J^1_j (Y ; K|x) ,
\]

(3.28)

where the two-forms \( J^0_j \in H_{dz} \) and one-forms \( J^1_j \in H_{dz} \) are of order-\( j \) in \( \omega \) and \( C \).

Since, acting trivially on \( \omega \) and \( C \), \( d_Z \) does not mix different perturbation orders, equations (3.24) and (3.25) are \( d_Z \)-closed separately at any \( m \). This allows one to use different homotopy operators for any \( m \) in each of these equations.

### 4 Shifted homotopy

The *conventional* homotopy operator

\[
\partial = Z^A \frac{\partial}{\partial \theta^A} ,
\]

(4.1)
and resolution
\begin{equation}
\triangle J(Z; Y; \theta) = Z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt \frac{1}{t} J(tZ; Y; t\theta) \tag{4.2}
\end{equation}
were used in the perturbative analysis of HS equations since \[2\]. Though being simple and looking natural, they are known to lead to non-localities beyond the free field level \[12, 14, 5\].

An obvious freedom in the definition of homotopy operator (4.1) is to replace \(Z^A\) by \(Z^A + a^A\) with some \(Z\)-independent \(a^A\),
\begin{equation}
\partial \rightarrow \partial_a = (Z^A + a^A) \frac{\partial}{\partial \theta^A}, \quad \frac{\partial}{\partial Z^A}(a^B) = 0. \tag{4.3}
\end{equation}
Resolution \(\triangle_a\) and cohomology projector \(h_a\) act on \(\phi(Z, Y, \theta)\) as follows
\begin{equation}
\triangle_a \phi(Z, Y, \theta) = \int_0^1 dt \frac{1}{t} (Z + a^A) \frac{\partial}{\partial \theta^A} \phi(tZ - (1-t)a, t\theta), \quad h_a \phi(Z, Y, \theta) = \phi(-a, Y, 0). \tag{4.4}
\end{equation}
\(\triangle_0\) is the conventional resolution (4.2). The resolution of identity has standard form
\begin{equation}
\{d_Z, \triangle_a\} + h_a = Id. \tag{4.5}
\end{equation}

For instance, one can set \(a^A = cY^A\) with some constant \(c\). Naively, this exhausts all Lorentz covariant choices for \(a^A\). This is however not the case since \(a^A\) can also be composed from derivatives with respect to arguments of \(\omega(Y; K)\) and \(C(Y; K)\) in \(J = J(\omega, C)\) in (3.9).

Let
\begin{equation}
\Phi^1(Y; K) = \omega(Y; K), \quad \Phi^0(Y; K) = C(Y; K). \tag{4.6}
\end{equation}
Various terms on the r.h.s. of HS field equations contain ordered products
\begin{equation}
\Phi^a_n(Y; K) = \Phi^{a_1}(Y_1; K)\Phi^{a_2}(Y_2; K) \ldots \Phi^{a_n}(Y_n; K)|_{Y_i = Y_i}, \quad a = \{a_1, \ldots, a_n\}, \quad a_i = 0, 1. \tag{4.7}
\end{equation}

An important feature of system (2.1), (2.2) noticed originally in \[17\] even before this system was obtained in \[2\] is that it remains formally consistent if the fields \(W\) and \(B\) are valued in any associative algebra, for instance, in the matrix algebra \(Mat_N(C)\). As a result, the terms corresponding to different sequences of \(a_i = 1\) or \(0\) like \(\{0, 1, 0, 0, 1, 0, \ldots\} \ etc \), referred to as \(a\), are separately \(d_z\)-closed. Then the homotopy operators (4.3) are allowed to be different for different \(a\). The simplest option is
\begin{equation}
a^n_A = c_0(a)Y_A + \sum_j c_j(a)\partial_j A, \quad a = \{a_1, \ldots, a_n\}. \tag{4.8}
\end{equation}

\(^1\)Retrospectively, one can see that the form of HS equations presented in \[1\] results from such modification of the homotopy operator with \(c = \pm 1\). However, the formulation of \[1\] demanded some non-local field redefinition even at the linear order. This problem was later resolved in \[3\] via application of the conventional homotopy upon introduction of \(Z\)-variables and the fields \(S_A\).
where $\partial_{iA}$ is the derivative with respect to the argument of the $i^{th}$ factor $\Phi_{i}^{\alpha}(Y_{i}; K)$. It is important that the modification via derivative homotopy shift affects locality when two or more arguments are available, i.e., only at the nonlinear level.

This construction provides a broad extension of the class of allowed homotopy operators. In fact it can be further extended by letting the coefficients $c_{j}(a)$ be arbitrary functions of the covariantly contracted combinations $\partial_{iA}$ and $Y_{A}$. Practical analysis shows however that the simplest extension with constant $c_{j}(a)$ is sufficiently general. As shown in more detail in [18], the class of shifted homotopy operators with constant shift coefficients is distinguished by the property of being closed under the elementary operations underlying the perturbative analysis like star products etc. Let us stress that although a number of free parameters in the shifted homotopy operators increases linearly with the order of the vertex in question, this freedom is uncomparably smaller than the functional freedom in general homotopy operators.

5 Z-dominance Lemma

The following evident formula
\[
C(Y) \ast \ldots \ast C(Y) = \exp i \left( \sum_{j} p_{j}^{\alpha} y^{\alpha} - \sum_{j<k} p_{j}^{\alpha} p_{k}^{\alpha} + \sum_{j<k} \bar{p}_{j}^{\alpha} \bar{y}^{\alpha} - \sum_{j<k} \bar{p}_{j}^{\alpha} \bar{p}_{k}^{\alpha} \right) C(Y_{1}) \ldots C(Y_{n}) \bigg|_{Y_{j}=0}
\]  
(5.1)
suggests that, to control locality, it suffices to consider the exponential parts of the operators acting on ordered products $C(Y_{1}; K)C(Y_{2}; K) \ldots C(Y_{n}; K)$ focusing on the derivatives $p_{i}^{\alpha}$, $\bar{p}_{i}^{\alpha}$ ([11]). To simplify analysis it is convenient to define $p^{i}$ and $\bar{p}^{i}$ as respecting the chain rule
\[
p^{i}(C(Y_{1}; K)C(Y_{2}; K)) = p^{i}(C(Y_{1}; K))C(Y_{2}; K) + C(Y_{1}; K)p^{i}C(Y_{2}; K)
\]  
(5.2)
in a way insensitive of the dependence of $C(Y_{1}; K)$ on $K$. (Formally this can be achieved following [22, 23] by introducing additional Clifford elements that anticommute with the Klein operators.) For each factor of $C(Y_{j}; K)$ $p_{i}^{\alpha}$ is defined as the left derivative, i.e., $p_{i}^{\alpha}(C^{ij}(Y)k^{i}\bar{k}^{j}) = -i \frac{\partial}{\partial p_{i}^{\alpha}}(C^{ij}(Y))^{\alpha} k^{i}\bar{k}^{j}$.

Also it is useful to keep track of the extra degree of Klein operators originating from the operators $\gamma$ ([2, 3]). Hence, general exponential representation for, say, the order-$n$ corrections in the zero-forms $C$ is
\[
\sum_{p \bar{p}} \int d\tau P_{n}^{p \bar{p}} E_{n}^{p \bar{p}}(\tau)C(Y_{1}) \ldots C(Y_{n}) \bigg|_{Y_{j}=0},
\]  
(5.3)
where $P_{n}^{p \bar{p}}$ is some polynomial of $z, y$ and $p^{i}$ and their conjugates with coefficients being regular functions of the homotopy parameters $\tau$, and
\[
E_{n}^{p \bar{p}} = E_{n}^{p} E_{n}^{\bar{p}}, \quad E_{n}^{p}(T, A, B, P, p|z, y) = \exp i(Tz_{\gamma}y^{\gamma} - A_{j}p_{j}^{\gamma}z^{\gamma} - B_{j}p_{j}^{\gamma}y^{\gamma} + \frac{1}{2} P_{ij} p^{i\gamma} p^{j\gamma} )k^{p},
\]  
(5.4)
where $p = 0, 1$ and parameters $T \in \mathbb{C}$, $A, B \in \mathbb{C}^{n}$, $P_{ij} = -p_{ji} \in \mathbb{C}^{n} \times \mathbb{C}^{n}$ may be $\tau$-dependent.
For instance, in [3] it was shown that the second-order correction to $B(Z; Y; K)$ that eventually leads to local HS equations in the zero-form sector is $\mathcal{H}_\text{cur}^{loc} = B_{2n}^{loc} + B_{2n}^{loc}$ with

$$B_{2n}^{loc} = \frac{1}{2\eta} \int d^3\tau \left( \delta'(1 - \sum_{i=1}^{3} \tau_i) + iy_\alpha z^\alpha \delta(1 - \sum_{i=1}^{3} \tau_i) \right) \exp(X^{loc})(Y_1; K)^*C(Y_2; K) \bigg|_{Y_1, z = 0} k, \tag{5.5}$$

where $^*$ is the star product with respect to barred variables,

$$d^3\tau := d\tau_1 d\tau_2 d\tau_3 \theta(\tau_1) \theta(\tau_2) \theta(\tau_3), \quad \theta(\tau) = 1(0) \text{ if } \tau \geq 0(\tau < 0) \tag{5.6}$$

and

$$X^{loc} = i\tau_3 z_\alpha y^\alpha + \tau_3 z^\alpha (\partial_1 a + \partial_2 a) + y^\alpha (\tau_2 \partial_2 a - \tau_1 \partial_1 a) + i\tau_3 \partial_1 a \partial_2^\alpha \tag{5.7}.$$

$B_{2n}^{loc}$ is complex conjugated to $B_{2n}^{loc}$.

What we would like to explain now is that from (5.5) it immediately follows that the final result is located at the lower integration limit $\tau$, this is not necessary since the $\mathcal{H}_\text{cur}^{loc}$-dependence should drop out anyway as a consequence of the previously solved equations implying that $d_{Z}$ of the both sides of equations is zero.

Indeed, the first nontrivial correction to the field equations in the zero-form sector is

$$dC + \omega * C - C * \omega + \mathcal{H}_\text{cur}(w, J) = 0, \tag{5.8}$$

where

$$J(Y_1, Y_2; K|x) := C(Y_1; K|x)C(Y_2; K|x). \tag{5.9}$$

Here $C$ and $\omega$ are $Z^A$-independent and hence the correction $\mathcal{H}_\text{cur}(w, J)$ must be $Z$-independent as well. This happens because consistency of the equations guarantees that the corrections belong to the $d_{Z}$-cohomology. Though in practical computations it is sometimes convenient to set $Z = 0$ to simplify the derivation of the explicit form of the corrections to field equations, this is not necessary since the $Z$-dependence should drop out anyway as consequence of the previously solved equations implying $d_{Z}$ of the both sides of equations is zero.

Practically, this works as follows. The $Z$-dependent term in the exponential contains the integration homotopy parameter $\tau_3$. The fact, that the r.h.s. of (5.8) must be $Z$-independent implies that the integral over $\tau_3$ on the r.h.s. of (5.8) must reduce to integration over such a total derivative that the final result is located at the lower integration limit $\tau_3 = 0$. This however means that not only the $Z$-dependent term $i\tau_3 z_\alpha y^\alpha$ in the exponential in (5.8) disappears but the term $\tau_3 \partial_1 a \partial_2^\alpha$ must disappear as well, so that the final expression will contain at most a finite number of contractions in the preexponential with respect to the undotted variables, leading to a local result.

Generally, we arrive at the following $Z$-dominance Lemma

Lemma 1: All terms in the exponential representation (5.4) dominated by the coefficients in front of the $Z$-dependent terms $T(\tau)$ and $A_j(\tau)$ do not contribute to the field equations on the $d_{Z}$-cohomology-valued dynamical fields.

Note that hatted coefficients $\hat{B}_j(\tau)$ and $\hat{P}_{ij}(\tau)$ in (1.12) coincide with the $d_{Z}$ cohomology reduction of $B_j(\tau)$ and $P_{ij}(\tau)$ while analogous reduction of the coefficients $T(\tau)$ and $A_j(\tau)$ in front of the $Z$-dependent terms in (5.4) is zero.
This simple lemma allows us to show that the level of non-locality of HS equations can be decreased in higher orders by an appropriate choice of shifted homotopy operators (4.3). Let us stress that $Z$-dominance Lemma 1 applies to each term in expressions containing linear combinations of a finite number of exponentials (5.4) as is most easily seen by rewriting the $z$-dependent part of the exponentials in the form

$$\exp i(T(\tau)z_{\gamma}y_{\gamma} - A_j(\tau)p_j^i z_{\gamma}) = \int dt dt_j \delta(T(\tau) - t) \prod_j \delta(A_j(\tau) - t_j) \exp i(tz_{\gamma}y_{\gamma} - t_j p_j^i z_{\gamma})$$

allowing to rewrite a sum of integrals of different exponentials as a sum of terms in the integration measure in front of a single exponential factor $\exp i(tz_{\gamma}y_{\gamma} - t_j p_j^i z_{\gamma})$.

6 Pfaffian Locality Theorem

Here we prove Pfaffian Locality Theorem (PLT) stating that, in the holomorphic sector, there exists such a choice of the shifted resolutions that the matrix $P_{ij}$ of (5.4) is degenerate. In the second order this implies that $P \left( \frac{\partial}{\partial y_i} \right) = 0$ and, hence, $J_2$ is local in agreement with [3, 5]. In higher orders PLT implies at least the decrease of the level of non-locality indicating however that it can be decreased further.

PLT heavily relies on the properties of star product (2.4) and shifted homotopies. In our analysis we focus on the dependence on the zero-forms $C$ discarding the dependence on $\omega$ and $w$ that does not affect spin locality in the 4d HS theory.

It is useful to restrict the representatives of the exponential classes (5.4) as follows. *Even* class

$$\mathcal{E}_n^0: \quad E_n^p(T, A, B, p|z, y), \quad p = n|_2, \quad n \geq 1$$

with parameters satisfying

$$\sum_{j=1}^{n} (-1)^j A_j = -T, \quad \sum_{j=1}^{n} (-1)^j B_j = 0, \quad \sum_{i=1}^{n} (-1)^i P_{ij} = B_j.$$  \hspace{0.5cm} (6.2)

*Odd* class

$$\mathcal{E}_n^1: \quad E_n^p(T, A, B, p|z, y), \quad p = (n + 1)|_2, \quad n \geq 0$$

with

$$\sum_{j=1}^{n} (-1)^j A_j = 0, \quad \sum_{j=1}^{n} (-1)^j B_j = 1 - T, \quad \sum_{i=1}^{n} (-1)^i P_{ij} = -A_j.$$  \hspace{0.5cm} (6.4)

Particular cases include

$$1 \in \mathcal{E}_0^0, \quad \kappa \kappa \in \mathcal{E}_0^0, \quad \exp i(p_{\gamma}^1 y_{\gamma}) \in \mathcal{E}_1^1,$$  \hspace{0.5cm} (6.5)
where $\exp i(p_1^\gamma y^n)$ is generated by the dynamical field $C(y, \bar{y})$ via (5.1) with $n = 1$ while $\kappa k$ is a part of $\gamma$ (2.3).

Perturbative analysis implies that corrections to dynamical equations at any perturbation order can be constructed inductively, starting from $\gamma$ (2.3), $C(y, \bar{y}|x)$ and $\omega(y, \bar{y}|x)$ via application of the star product, shifted resolutions, cohomology projectors, operators $D_w$ (3.6) (not affecting locality) and $d_x$. By Structure Lemma 6 proven in Sections 6.1-6.3, with the proper choice of homotopies (6.14), (6.16), these operations respect the classes $E_{ij}$. This allows us to decrease the level of non-locality in the higher order corrections to dynamical equations in the zero-form (anti) holomorphic sector. Indeed, in this case exponential parts of the order-$n$ deformations $J_n^1(Y; K|x)$ to the field equations in the zero-form sector (3.28) are of the form (5.4) with the parameters obeying (6.4). According to Lemma 1 the coefficients $T$ and $A_i$ trivialize in the $d_{Z}$-cohomology. Hence (6.4) yields
\[
\sum_{i=1}^{n} (-1)^i P_{ij}(\tau) = 0 \quad \text{(6.6)}
\]
in the $d_{Z}$-cohomology proving

**Pfaffian Locality Theorem**: the shifted homotopy can be chosen in such a way that the matrix $P_{ij}$ be degenerate with the null-vector (6.6) in the (anti)holomorphic sector of the dynamical field equations in the zero-form sector.

In even interaction orders condition (6.6) is essential. For instance, from Section 6.2 it follows, that to obtain a local form of dynamical equations to the second order in the holomorphic sector, it is necessary to take the shifted resolution operator
\[
\Delta_{\beta y+\alpha y_{1}-i(1-\alpha)y_{2}}
\]
with arbitrary parameters $\alpha$ and $\beta$. Details of the derivation of the local form of equations are presented in [18] where it is also proven that the resulting equations coincide with those of [4] up to $\beta$-dependent local field redefinitions.

In odd orders the antisymmetric matrix $P_{ij}(\tau)$ is automatically degenerate. However, the additional information following from (6.6) is that it admits a null vector independent of the homotopy parameters $\tau$ on which $P_{ij}(\tau)$ depends. Though we do not know yet whether condition (6.6) increases the degree of degeneracy of $P_{ij}(\tau)$ further or not, its special structure implying that the vertex depends on some linear combinations of helicities associated with different fields suggests that the level of non-locality of the resulting vertices which are rather unusual from the QFT perspective is likely to be further reducible. We interpret this as a possible indication that HS interactions may admit a spin-local form in all orders.

### 6.1 Star-product mapping

Straightforward computation proves

**Lemma 2**: 
\[
\mathcal{E}_n^j \ast \mathcal{E}_m^i \subseteq \mathcal{E}_{m+n}^{(j+i)l_2}. \quad \text{(6.8)}
\]
The proof is by virtue of Eqs. (2.4), (1.1) which give for any \( p, p' = 0, 1 \)

\[
E^p_n(T, A, B, P, p|z, y) \ast E^{p'}_n(T', A', B', P', p'|z, y) = E^{(p+p')/2}_n(T'', A'', B'', P'', p''|z, y),
\]

with

\[
T'' = T(1 - T') + T'(1 - T),
\]

\[
A''_j = (1 - T'A_j - T'B_j), \quad \quad A''_\nu = (-)^P ((1 - T)A'_\nu + T B'_\nu),
\]

\[
B''_j = -T'A_j + (1 - T')B_j, \quad \quad B''_\nu = (-)^P (TA'_\nu + (1 - T)B'_\nu),
\]

\[
P''_{ij} = P_{ij} + P'_{m'i'} - (-)^P (A_i + B_i) (A'_\nu - B'_\nu) + (-)^P (A'_{m'} - B'_{m'}) (A_j + B_j)
\]

\((i, j = 1, \ldots, n; m', l' = 1, \ldots, n')\) from where Lemma 2 follows straightforwardly \( \Box \)

### 6.2 Homotopy mapping

Let

\[
s_n(\mu, v) = v_j p_j + \mu y, \quad \mu \in \mathbb{C}, v \in \mathbb{C}^n.
\]

The mapping

\[
\hat{\Delta}_{s_n(\mu, v)}(E_n(T, A, B, P)) = E_n(T', A', B', P')
\]

with

\[
T' = \tau T, \quad A'_i = \tau A_i, \quad B'_i = B_i + (1 - \tau)Tv_i - (1 - \tau)\mu A_i,
\]

\[
P'_{ij} = P_{ij} + (1 - \tau) (A_jv_i - A_i v_j)
\]

results from the application of the resolution \( \Delta_a \) \((4.4)\) with \( a = s_n(\mu, v) \)

\[
\Delta_{s_n(\mu, v)} \{ \phi(z, y, p, \theta)E_n(T, A, B, P) \}
\]

\[
= \int_0^1 \frac{d\tau}{\tau} (z + s_n(\mu, v))^\alpha \frac{\partial}{\partial \theta^\alpha} \phi(\tau z - (1 - \tau)s_n(\mu, v), y, p, \tau \theta)E_n(T', A', B', P'),
\]

where \( \phi(z, y, p, \theta) \) is some pre-exponential factor containing a finite number of \( p^j \). Elementary calculation yields

**Lemma 3:** If

\[
\sum_{j=1}^n (-1)^j v_j^1 = 1,
\]

then for any \( \tau \) and \( \mu \)

\[
\hat{\Delta}_{s_n(\mu, v)} : \mathcal{E}_n^1 \rightarrow \mathcal{E}_n^1.
\]

Indeed, if parameters \( T, A, B, P \) satisfy \((6.4)\) then, for any \( \tau \), under the assumptions of Lemma 3 parameters \( T', A', B', P' \) of \( \hat{\Delta}_{s_n(\mu, v')} \) \((\mathcal{E}_n(T, A, B, P)) \) \((6.11)\) can be easily shown to satisfy \((6.4)\) by virtue of \((6.12)\) and \((6.14)\). Hence \( \hat{\Delta}_{s_n(\mu, v')} \) \((\mathcal{E}_n(T, A, B, P)) \in \mathcal{E}_n^1 \) \( \Box \)
Analogously, one proves

Lemma 4: If

\[ \sum_{j=1}^{n} (-1)^j v_j^0 = -\mu, \]  

then for any \( \tau \) and \( \mu \)

\[ \tilde{\Delta}_{\tau, \Phi_n(\mu, \nu)} : \mathcal{E}_n^0 \rightarrow \mathcal{E}_n^0. \]  

(6.17)

Note that if \( \mu = -1 \) the conditions (6.14) and (6.16) coincide. This may be important in practical analysis.

6.3 \( d_x \) mapping

It remains to consider the mapping generated by \( d_x \) that acts nontrivially on the dynamical fields \( \omega \) and \( C \). The action on \( w \) does not affect locality. By Eq. (3.28)

\[
d_x \omega(Y; K|x) \longrightarrow \sum_j J_j^0(Y; K|x), \quad d_x C(Y; K|x) \longrightarrow \sum_j J_j^1(Y; K|x).
\]  

(6.18)

Note that the lower label \( j \) of \( J_{i,j} \) equals to the total degree in the dynamical fields, while the respective \( k \)-equipped exponentials depend on the degree in \( C \). For the future convenience we set

\[ J_{i,j} := J_{i,j_\omega,j_c}(Y), \quad i = 0, 1, \]  

(6.19)

where \( j_\omega \) and \( j_c \) are the degrees of \( J_{i,j} \) in \( \omega \) and \( C \), respectively. \( J_{i,j}^{0,1} \) do not depend on \( z \). Hence the \( k \)-equipped exponential \( \tilde{E}_j^p (5.4) \) of \( J_{i,j}^{0,1} \) is

\[ \tilde{E}_j^p (\tilde{B}, \tilde{P}|y) := E_j^p (0, 0, \tilde{B}, \tilde{P}|0, y). \]  

(6.20)

Since \( d_x \omega(Y; K|x) \) (6.18) contributes to the sector of two-forms it does not affect field corrections \( B^{(n)} \) (3.22) and dynamical equations on the zero-form \( C(Y) \) (3.28) for which we obtain schematically

\[
d_x C(Y_1) \ldots C(Y_n) = \sum_i C(Y_i) \ldots \tilde{C}(Y_i) \ldots C(Y_n) \bigg|_{C(Y_i) \rightarrow \sum_j J_j^i(Y_i)}.
\]  

(6.21)

Eq. (6.21) yields

\[ E_n^p(T, A, B, P|z, y) \xrightarrow{d_x} \sum_{j_c} \sum_{i} E_{n+j_c-1}^{i,j_c}(T^{i,j_c}, A^{i,j_c}, B^{i,j_c}, P^{i,j_c}|z, y). \]  

(6.22)

The resulting mapping \( S_{i,\tilde{E}_j^p}(E_n^p) \) generated by \( d_x \) (6.22) for any \( i \) and \( j_c \) is

\[ S_{i,\tilde{E}_j^p}(E_n^p(T, A, B, P, p)) = E_{n+j_c-1}^{(p+\tilde{p})}(T', A', B', P', \tilde{p}', \tilde{p}'), \]  

(6.23)
where parameters are defined straightforwardly via Eq. (5.4). For instance,
\[
\begin{align*}
T' &= T, \\
A'_{k} &= A_{k}, \\
B'_{k} &= B_{k} \quad \text{for } k < i, \\
A'_{k+i-1} &= -A_{i} \tilde{B}_{k}, \\
B'_{k+i-1} &= -B_{i} \tilde{B}_{k} \quad \text{for } 1 \leq k \leq j_{c}, \\
A'_{k+j_{c}-1} &= (-)^{p} A_{k}, \\
B'_{k+j_{c}-1} &= (-)^{p} B_{k} \quad \text{for } k \geq i + 1, \\
\{p'_{1}, \ldots, p'_{j_{c}+n-1}\} &= \{p_{1}, \ldots, p_{i-1}, \tilde{p}_{1}, \ldots, \tilde{p}_{j_{c}}, p_{i+1}, \ldots, p_{n}\}. 
\end{align*}
\]

(6.24)

For odd \(k\)-equipped exponentials \(\tilde{E}^{p}_{j_{c}}\) this gives the following

\textbf{Lemma 5:} For any \(E^{(m+1)\mid 2}_{m}\) and any \(i \in [1, n+1]\)
\[
S_{\tilde{E}^{(m+1)\mid 2}_{m}}(0, 0, B, P, p|0, y) : \mathcal{E}^{a}_{m+1} \rightarrow \mathcal{E}^{a}_{n+1} , \quad a = 0, 1. 
\]
\[
(6.25)
\]

Indeed, since \(E^{(m+1)\mid 2}_{m} \in \mathcal{E}^{1}_{m}\) then by virtue of (5.4)
\[
\sum_{n=1}^{m} (-1)^{n} \tilde{B}_{n} = 1, \quad \sum_{k=1}^{m} (-1)^{k} \tilde{P}_{kn} = 0. 
\]
\[
(6.26)
\]

Hence (6.24) yields
\[
\begin{align*}
\sum_{k=1}^{m+n-1} (-)^{k} A'_{k} &= \sum_{k=1}^{n} (-)^{k} A_{k}, \\
\sum_{k=1}^{m+n-1} (-)^{k} B'_{k} &= \sum_{k=1}^{n} (-)^{k} B_{k} \quad \text{for } a = 0 \text{ and } (5.4) \text{ for } a = 1. 
\end{align*}
\]
\[
(6.27)
\]

satisfying conditions (6.2) for \(a = 0\) and (5.4) for \(a = 1\). Analogously, one can make sure that from (5.20) it follows that the parameters \(P'_{k_{j}}\) on the r.h.s. of (6.23) satisfy the respective conditions (6.2) for \(a = 0\) and (5.4) for \(a = 1\). \(\Box\)

Let us stress that otherwise, if \(\tilde{E}^{p}_{j_{c}}\) (5.20) is even, the resulting \(k\)-equipped exponential \(E^{(p+p)\mid 2}_{n+j_{c}-1}(T', A', B', P', p')\) (6.23) in general has no definite parity.

By induction over perturbation orders, Lemmas 2-5 provide following \textbf{Structure Lemma}

\textbf{Lemma 6:} If the perturbative analysis in the one-form sector contains shifted resolutions \(\Delta\) satisfying (6.10), while that in the zero-form sector contains shifted resolutions satisfying (6.14), then all \(\tilde{B}_{j}\) generate odd \(k\)-equipped exponentials, while all space-time zero- and one-form components of \(W_{j}\), not containing terms resulting from \(d_{x}\omega(Y)\) (6.18), generate even \(k\)-equipped exponentials in the holomorphic sector.

Antiholomorphic sector analysis is analogous up to swap of dotted and undotted spinors.

7 Conclusion

In this paper we explain how to extend the class of homotopy operators in HS theory to make it possible to systematically analyze locality of interactions derived from nonlinear HS equations. It is shown that a number of available homotopy operators increases quickly
with the order of nonlinearity, containing in particular a subclass of homotopy operators that lead directly to the known lower-order local results as shown explicitly in [18]. Also we prove a $Z$-dominance Lemma giving a sufficient condition controlling locality of field equations on dynamical fields and Pfaffian Locality Theorem (PLT) showing how to choose generalized homotopy operators to reach that the Pfaffian matrix of derivatives acting on spinor variables of different fields in multilinear corrections degenerates. As shown in [18], the choice suggested by PLT in the case of bilinear corrections leads to the local results of [3, 5]. In the higher orders the PLT allows us to choose homotopy operators in such a way that the level of higher-order non-locality gets decreased. Indeed, PLT implies that, for the proper homotopy choice, a priori infinite expansion in spinor variables turns out to be finite with respect to at least one their linear combination associated with the null vector (6.6) of the Pfaffian derivative matrix. This result is somewhat analogous to the conclusions of [24].

To appreciate it, the following comments have to be taken into account. The structure of the remaining non-local higher-order interactions obtained by virtue of the homotopy operators satisfying PLT is very special, containing some linear combinations of helicities associated with different fields in a vertex as prescribed by (6.6). Such vertices are rather unusual from the QFT perspective and are anticipated to be further removable by an appropriate homotopy choice. A related comment is that conditions of PLT leave a lot of freedom in the choice of shifted homotopy operators in higher orders to be used to further reduce the level of non-locality of HS interactions. Hopefully, there may exist a specific homotopy choice leading to spin-local higher-order nonlinear corrections at any order.

The conjecture that contact HS interactions can be spin-local should not be interpreted as the claim that all HS interactions are space-time local. Most likely they are not due to the spin-current exchange phenomenon. Indeed, what is proven in our formalism is that contractions with respect to spinorial variables of $C(Y; K|x)$ are suppressed. However, their relation to the space-time derivatives is direct only at the level of free equations (1.6) which, however, receive nonlinear corrections at higher orders. As a result, the relation between space-time and spinor derivatives of $C(Y; K|x)$ becomes nonlinear and also involves higher derivatives. Due to summation over different spins this may eventually lead to further $x$-space non-localities. This mechanism is somewhat analogous to the current exchange mechanism in QFT. Our results indicate that contact HS interactions may be spin-local in all orders. On the other hand at the present stage it is not clear what kind of space-time non-locality is physically admissible in HS theories. The idea is first to identify the spin-local or minimally non-local scheme in HS theory and then investigate its properties.

It should be stressed that the results of this paper are heavily based on the specific form of HS equations (2.1), (2.2) and, in particular, of the star product (2.4). Extension of the analysis of this paper to other cases including the sector of one-forms $\omega$ and mixed (non-holomorphic) sectors needs application of the remarkable properties of the shifted homotopy formalism elaborated in [18] where a number of examples of its applications are presented.

The results of this paper, which are applicable not only to the 4d HS theory of [4] but also to 3d HS theory [22] and Coxeter HS theories proposed recently in [19], provide a step towards complete analysis of the level and role of non-locality in HS gauge theory. Once a
spin-local or minimally non-local formulation of the HS gauge theory is identified this will allow one to analyze such important issues as causality and, in the framework of Coxeter HS theory of [19], relation with analogous aspects of String Theory. An extension of our results to higher orders is also of great importance. Though some progress in that direction is reported in [18] a lot more remains to be done. In particular, it would be extremely interesting to compare predictions of the bulk HS equations against holographic results on the simplest quartic vertices of [25, 24, 26].

Acknowledgements

We would like to thank Slava Didenko and Tolya Korybut for useful discussions and also Slava Didenko for valuable comments on the manuscript. We acknowledge a partial support from the Russian Basic Research Foundation Grant No 17-02-00546. The research was supported in part by the International Centre for Theoretical Sciences (ICTS) during a visit for participating in the program - AdS/CFT at 20 and Beyond. The work of OG is partially supported by the FGU FNC SRISA RAS (theme 0065-2018-0004).

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