The Modeling of Pickup Ion or Energetic Particle Mediated Plasmas

G P Zank[^1,2], P Mostafavi[^1,2] and P Hunana[^1,2]

[^1] Center for Space Plasma and Aeronomic Research (CSPAR), University of Alabama in Huntsville, Huntsville, AL 35805, USA
[^2] Department of Space Science, University of Alabama in Huntsville, Huntsville, AL 35805, USA

E-mail: garyp.zank@gmail.com

Abstract. Suprathermal energetic particles, such as solar energetic particles (SEPs) in the inner heliosphere and pickup ions (PUIs) in the outer heliosphere and the very local interstellar medium, often form a thermodynamically dominant component in their various environments. In the supersonic solar wind beyond $>10$ AU, in the inner heliosheath (IHS), and in the very local interstellar medium (VLISM), PUIs do not equilibrate collisionally with the background plasma. Similarly, SEPs do not equilibrate collisionally with the background solar wind in the inner heliosphere. In the absence of equilibration between plasma components, a separate coupled plasma description for the energetic particles is necessary. Using a collisionless Chapman-Enskog expansion, we derive a closed system of multi-component equations for a plasma comprised of thermal protons and electrons, and suprathermal particles (SEPs, PUIs). The energetic particles contribute an isotropic scalar pressure to leading order, a collisionless heat flux at the next order, and a collisionless stress tensor at the second-order. The collisionless heat conduction and viscosity in the multi-fluid description results from a non-isotropic energetic particle distribution. A simpler single-fluid MHD-like system of equations with distinct equations of state for both the background plasma and the suprathermal particles is derived. We note briefly potential pitfalls that can emerge in the numerical modeling of collisionless plasma flows that contain a dynamically important energetic particle component.

1. Introduction

Most of the thermal plasma in the heliosphere and the very local interstellar medium (VLISM) is not equilibrated with the ubiquitously present energetic particle component. In the inner heliosphere, for example, energetic particles related to impulsive and gradual solar energetic particle events are pervasive, especially during solar maximum. Recently, there have been a number of reports describing observations of interplanetary shock waves that have an energetic particle component with a pressure that exceeds considerably both the corresponding thermal plasma (comprising thermal protons and electrons) and the compressed interplanetary magnetic field pressure [1–5]. Whereas in the inner heliosphere, such mediation of the thermal plasma by energetic particles appears to be associated largely with shock events, it is well-known that the outer heliosphere beyond the ionization cavity (i.e., $\geq 8$ AU) is dominated thermally by pickup ions (PUIs) (e.g., [6–10]). As reported by Decker et al. [11; 12], the inner heliosheath pressure contributed by energetic PUIs and anomalous cosmic rays far exceeds that of the thermal background plasma and magnetic field. The VLISM is also very interesting. It was
noted already by Zank et al. [13] that energetic neutral H created via charge exchange in the inner heliosheath (IHS) and fast solar wind could “splash” back into the VLISM where they would experience a secondary charge exchange. The secondary charge exchange of hot and/or fast neutral H with cold (∼ 6300 K [14]) VLISM protons leads to the creation of a hot or suprathermal PUI population in the VLISM. The heating of the VLISM has been discussed in detail by Zank et al. [15], who pointed out that the heating of the VLISM plasma would result in an increase of the sound speed with a concomitant weakening or even elimination of the bow shock (yielding instead a bow wave). Zirnstein et al. [16] extended the Zank et al. [17] model by including the multiple PUI populations that contribute to the heating of the VLISM [15]. Here, we restrict out attention to neutrals created in both the IHS and supersonic solar wind, i.e., with typical speeds of ∼ 100 km/s or ∼ 400 km/s, that experience secondary charge exchange in the VLISM. The PUIs form a tenuous \( n_p \approx 5 \times 10^{-15} \text{cm}^{-3} \) suprathermal component in the VLISM.

Coulomb collisions can equilibrate a background thermal plasma and energetic protons. Assume that the background thermal plasma proton and electron distributions are Maxwellian. If we restrict our attention to suprathermal particles with energies greater than 1 - several keV (PUIs for example, and certainly the dominant component of solar energetic particles (SEPs)), then energetic particles satisfy the ordering

\[
v_{ts} < v_p < v_{te},
\]

where \( v_{ts/e} \) denotes the background proton/electron thermal speed respectively and \( v_p \) the energetic particle speed. For energetic particles scattering collisionally off a Maxwellian distribution of background protons, the collision frequency is given by

\[
\nu_{ps}^{scat} = \frac{n_e e^4 \ln \Lambda}{2 \pi \varepsilon_0 m_p v_p^2} s^{-1},
\]

illustrating the well-known \( v^{-3} \) dependence with fast particle speed. By contrast, energetic particle collisional scattering off a Maxwellian electron background yields a larger collision frequency which is given by

\[
\nu_{pe}^{scat} = \frac{n_e e^4 \ln \Lambda m_e^{1/2}}{2(2\pi)^{3/2} \varepsilon_0^2 (kT_e^{3/2} m_p).}
\]

If the collisional time scale exceeds the characteristic flow time of the plasma region of interest, \( \tau_f \approx L/U \), where \( L \) is the size of the region and \( U \) the characteristic velocity, then the energetic particle distribution will not equilibrate with the background plasma. Expressions (1) and (2) should be used to determine whether one needs to introduce a plasma model that distinguishes energetic protons from background plasma protons.

Zank et al. [9] present detailed estimates for the equilibration times for PUIs in the supersonic solar wind of the outer heliosphere, the subsonic solar wind (the inner heliosheath), and the VLSIM using appropriate plasma parameters. In all three regions, the plasma does not equilibrate and cannot therefore be described as a magnetized single-component plasma and at least some elements of a multi-component description are necessary.

For SEPs in the inner heliosphere with energies of e.g., 5 keV and greater, the collisional frequency of thermal and energetic protons at 1 AU (using a solar wind number density ∼ 10 cm\(^{-3}\)) is \( \nu_{ps}^{scat} \approx 7 \times 10^{-11} \text{ s}^{-1} \). Assuming a characteristic scale length \( L \sim 1 \text{ AU} \) for the inner heliosphere and a characteristic speed \( \sim 400 \text{ km/s} \), the dynamical time scale is \( \tau_f \approx 3.75 \times 10^5 \text{ s} \), implying that \( \tau_{scat} = (\nu_{ps}^{scat})^{-1} \gg \tau_f \). Similar results hold for fast protons scattering off electrons. Hence SEPs cannot equilibrate in the inner heliosphere and need to be treated as a distinct and separate component if the energy density becomes dynamically important. As discussed above, this is certainly the case for a number of shocks observed in the inner heliosphere [1–5].
Energetic particles such as SEPs and PUIs both drive streaming instabilities in one form or another, and experience pitch-angle scattering from both self-excited and pre-existing Alfvénic fluctuations. In the case of SEPs, a streaming instability [18–24] is responsible in part for their scattering towards isotropy (i.e., the leading order SEP velocity distribution function can be represented as isotropic and independent of particle pitch-angle with smaller pitch-angle dependent corrections, but the distribution is not equilibrated with the background thermal plasma distribution) and for PUIs, the initial ring-beam distribution is unstable to the generation of Alfvénic fluctuations that scatter PUIs towards isotropy [7; 25–27]. As we show below, pitch-angle scattering serves to introduce both a collisionless heat flux and a non-isotropic pressure tensor into the transport equations describing the energetic particles. The pressure tensor modification is expressed as a collisionless viscosity tensor.

Below, we construct an appropriate multi-component plasma description for a thermal background plasma comprising electrons and protons and a non-equilibrated energetic particle component that is subject to pitch-angle scattering by turbulence and Alfvénic fluctuations. By making various approximations, we derive successively simpler models. In so doing, we place on a more formal footing the derivation of the well-known two-fluid model of cosmic ray magnetohydrodynamics [28; 29], showing, somewhat unexpectedly and contrary to perceived wisdom, that the cosmic ray number density is in fact included implicitly in the total number density.

One conclusion that emerges from this work is that there are potential pitfalls in the numerical modeling of collisionless plasma flows that contain a dynamically important energetic particle component. Specifically, as we show below, the energetic particle transport equations introduce dissipative terms though their collisionless heat flux and viscosity and these terms can dominate those associated with the thermal background plasma. However, numerical schemes that deal with e.g., cosmic rays typically solve the gas dynamic or MHD equations and then introduce the cosmic rays through source terms. The origin of numerical dissipation is typically via the thermal equations and so discontinuities, for example, are resolved on the basis of gas dynamic dissipation rather than perhaps the more physically correct dissipation associated with energetic particles.

2. The multi-component model

In deriving a multi-component plasma model that includes energetic particles, we shall assume that the distribution functions for the background protons and electrons are each Maxwellian, which ensures the absence of heat flux or stress tensor terms for the background plasma. The exact continuity, momentum, and energy equations governing the thermal electrons and protons are therefore given by

\[ \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e u_e) = 0; \]
\[ m_e n_e \left( \frac{\partial u_e}{\partial t} + u_e \cdot \nabla u_e \right) = -\nabla P_e - e n_e (E + u_e \times B); \]
\[ \frac{\partial P_e}{\partial t} + u_e \cdot \nabla P_e + \gamma_e P_e \nabla \cdot u_e = 0, \]

for the electrons, and

\[ \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s u_s) = 0; \]
\[ m_p n_s \left( \frac{\partial u_s}{\partial t} + u_s \cdot \nabla u_s \right) = -\nabla P_s + e n_s (E + u_s \times B); \]
\[ \frac{\partial P_s}{\partial t} + u_s \cdot \nabla P_s + \gamma_s P_s \nabla \cdot u_s = 0, \]
for the protons. Here \( n_{e/s}, u_{e/s}, \) and \( P_{e/s} \) are the macroscopic fluid variables for the electron/proton number density, velocity, and pressure respectively, \( \gamma_{e/s} \) the electron/proton adiabatic index, \( E \) the electric field, \( B \) the magnetic field, and \( \epsilon \) the charge of an electron.

The streaming instability for SEPs and the unstable ring-beam distribution for PUIs excite Alfvénic fluctuations. The self-generated fluctuations and in situ turbulence serve to scatter energetic particles in pitch-angle. The Alfvén waves and magnetic field fluctuations both propagate and convect with the bulk velocity of the system \( \mathbf{U} = U(u_e, u_p, n_e, n_p, n_{e/s}, m_e, m_p) \), where \( n_p \) and \( u_p \) refer to energetic particle variables. The energetic particles are governed by the Fokker-Planck transport equation with a collisional term \( \delta f/\delta t \),

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m_p} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = \frac{\delta f}{\delta t} \bigg|_{\text{ws}} ,
\]

for average electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \). We assume that the velocity \( \mathbf{v} \) of energetic particles is always non-relativistic; the generalization to relativistic energies is straightforward [30]. On transforming the transport equation (9) into a frame that ensures there is no change \( \partial f/\partial t \) of the number density \( n \) and \( \nabla \cdot \mathbf{u} \) are zero. The zeroth moment of (12) yields the continuity equation

\[
\frac{\partial n}{\partial t} + (U_i + c_i) \frac{\partial f}{\partial x_i} = 0 .
\]

The velocity \( \mathbf{U} \) is still unspecified so we choose \( \mathbf{U} \) such that \( \mathbf{E}' \equiv \mathbf{E} + \mathbf{U} \times \mathbf{B} = 0 \). This assumption corresponds to choosing

\[
\mathbf{U}_\perp = \mathbf{U} - \mathbf{U}_\parallel = \frac{\mathbf{E} \times \mathbf{B}}{\mathbf{B}^2} \equiv \mathbf{U} ,
\]

since we choose \( \mathbf{U}_\parallel = 0 \) (\( \mathbf{U}_\parallel \) is parallel to \( \mathbf{B} \) and therefore arbitrary). The use of the velocity \( \mathbf{U} \) then yields

\[
\frac{\partial f}{\partial t} + (U_i + c_i) \frac{\partial f}{\partial x_i} + \left( m_p \right) (U_j + c_j) \frac{\partial f}{\partial c_i} = \frac{\delta f}{\delta t} \bigg|_{\text{ws}} .
\]

By taking moments of (12), we can derive the evolution equations for the macroscopic energetic particle variables, such as the number density \( n_p = \int f d^3c \), momentum density \( n_{p} u_{p} = \int c_{j} f d^3c \), and energy density. Although unspecified for now, we shall assume that moments of the collisional term \( \delta f/\delta t |_{c} \) are zero. The zeroth moment of (12) yields the continuity equation for energetic particles,

\[
\frac{\partial n}{\partial t} + \left( \frac{\partial}{\partial x_i} \right) (n_p (U_i + u_{p_i})) = 0,
\]

where \( u_p \) is the energetic particle bulk velocity in the guiding center frame. For the first moment, we multiply (12) by \( c_j \) and integrate over velocity space. This yields, after a little algebra, the momentum equation for energetic particles,

\[
\frac{\partial}{\partial t} \left( n_p (U_j + u_{p_j}) \right) + \nabla \cdot \left[ n_p U (U_j + u_{p_j}) + n_p u_p U_j \right] + \frac{\partial}{\partial x_i} \left( c_{i} c_{j} f d^3c \right) = \frac{e}{m_p} n_{p} \varepsilon_{ijk} u_{p_k} B_l ,
\]

where \( \varepsilon_{ijk} \) is the Levi-Civeta tensor. Note the presence of the term \( \int c_{i} c_{j} f d^3c \), which is the momentum flux or pressure tensor.

To close equation (14), we need to evaluate the momentum flux, which requires that we solve (12) for the energetic particle distribution function \( f \). In solving (12), we assume 1) that
the energetic particle distribution is gyrotropic, and 2) that scattering of energetic particles is sufficiently rapid to ensure that the energetic particle distribution is nearly isotropic. We can therefore average (12) over gyrophase, obtaining the “focused transport equation” for non-relativistic particles [31]. Details of the derivation can be found in Ch. 5 of Zank [30]. To solve the gyrophase-averaged transport equation requires that we specify the scattering or collisional operator. We make the simplest possible choice, which is the isotropic pitch-angle diffusion operator,

$$\frac{\partial}{\partial \mu} \left( \nu_{ws}(1 - \mu^2) \frac{\partial f}{\partial \mu} \right), \quad (15)$$

where \( \mu = \cos \theta \) is the cosine of the particle pitch-angle \( \theta \), and \( \nu_{ws} = \tau_{ws}^{-1} \) is the scattering frequency. The form of the scattering operator (15) allows us to solve the focused transport equation using a Legendre polynomial expansion of the distribution function \( f \). The second-order correct solution to the gyrophase-averaged Chapman-Enskog reduction of equation (12) is

$$f \simeq f_0 + \frac{1}{2} (3\mu^2 - 1) f_2; \quad (16)$$

$$f_0 = f_0(x, c, t); \quad (17)$$

$$f_1 = -\frac{c \tau_{ws}}{3} b_i \frac{\partial f_0}{\partial x_i} + \frac{D U_i}{\partial t} \frac{\tau_{ws}}{3} b_i \frac{\partial f_0}{\partial c}; \quad (18)$$

$$f_2 = \frac{c \tau_{ws}}{15} \left( b_i b_j \frac{\partial^2 U_j}{\partial x_i} - \frac{1}{3} \frac{\partial U_i}{\partial c} \right) \frac{\partial f_0}{\partial c}, \quad (19)$$

where \( c = |c| \) is the particle random speed, \( b = B/B \) is a directional unit vector defined by the magnetic field, and \( D/Dt = \partial / \partial t + U_i \partial / \partial x_i \) is the convective derivative. The expansion terms \( f_0, f_1, \) and \( f_2 \) are functions of position, time, and particle random speed \( c \) i.e., independent of pitch-angle \( \mu \) (and of course gyrophase \( \phi \)). Of particular importance is the retention of the large-scale acceleration, and shear terms. These terms are often neglected in the derivation of the transport equation describing \( f_0 \) (for relativistic particles, the transport equation is the familiar cosmic ray transport equation). The velocity derivative term in equation (18) is known as the relativistic heat inertia term [32; 33]. In the context of deriving a multi-fluid model, retaining the various flow velocity terms is essential to derive the correct multi-fluid formulation for energetic particles. We need to evaluate

$$\int c_i c_j f d^3c = \int (c_i - u_{pi})(c_j - u_{pj}) f d^3c + n_p u_{pi} u_{pj} \equiv \int c_i c'_j f d^3c + n_p u_{pi} u_{pj},$$

from which we find the zeroth- and first-order expressions,

$$\int c'_i c'_j f_0 d^3c = \frac{1}{m_p} \left( \delta_{ij} P_p \right), \quad \int c'_i c'_j \mu f_1 d^3c = 0, \quad P_p \equiv \frac{4\pi}{3} \int c^2 f_0 d^2c. \quad (20)$$

Consequently, the first-order energetic particle stress tensor is identically zero and the pressure is isotropic, \( \delta_{ij} P_p \).

The inclusion of the second-order terms yields a non-zero collisionless stress tensor. Since the energetic particle pressure is defined in the frame of the bulk PUI velocity \( \mathbf{u}_p \), the distribution function over which the integral is taken needs to evaluated in this frame. Since the expression (19) for \( f_2 \) is a function of the guiding center velocity \( \mathbf{U} \), we need to transform to the frame.
\[ U_p = U + u_p. \]

On using the solution (19) for \( f_2 \), we obtain

\[
\int c'_x \frac{1}{2} (3\mu^2 - 1) f_2 d^3 c' = \int c'_y \frac{1}{2} (3\mu^2 - 1) f_2 d^3 c' = \int c'_z \frac{1}{2} (3\mu^2 - 1) f_2 d^3 c' = \frac{\eta}{15} \left( b_i b_j \frac{\partial U_{pj}}{\partial x_i} - \frac{1}{3} \frac{\partial U_{pi}}{\partial x_i} \right); \tag{21}
\]

\[
\int c'_x \frac{1}{2} (3\mu^2 - 1) f_2 d^3 c' = -\frac{2\eta}{15} \left( b_i b_j \frac{\partial U_{pj}}{\partial x_i} - \frac{1}{3} \frac{\partial U_{pi}}{\partial x_i} \right); \tag{22}
\]

\[
\int c'_x c'_y \frac{1}{2} (3\mu^2 - 1) f_2 d^3 c' = 0, \quad (i \neq j), \tag{23}
\]

where the coefficient of viscosity \( \eta \) is defined as

\[
\eta = \frac{4\pi}{15} \int \frac{\partial}{\partial c'} (c' \tau_{ws}) f_0 dc' \approx \frac{4\pi}{3} \int c' \tau_{ws} f_0 dc' \approx \frac{P_p \tau_{ws}}{m_p}. \tag{24}
\]

The first equality in (24) is the formal definition of the coefficient of viscosity for the energetic particle gas. If we assume (probably reasonably) that \( |c| \gg |u_p| \), then we obtain the second equality, which may be regarded as an energetic particle pressure moment weighted by the energetic particle scattering time. Finally, if we assume that \( \tau_{ws} \) is independent of \( c \), we then obtain the “classical” form (24) of the viscosity coefficient. The pressure tensor may therefore be expressed as

\[
(P_{ij}) = P_p (\delta_{ij}) + \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{array} \right) \frac{\eta}{15} \left( b_k b_j \frac{\partial U_{pk}}{\partial x_i} - \frac{1}{3} \frac{\partial U_{pi}}{\partial x_m} \right). \]  

(25)

If we introduce a “viscosity matrix,”

\[
(M_{k\ell}) \equiv (\eta_{k\ell}) = \left( \frac{\eta}{15} b_k b_\ell \right) \approx \left( \frac{1}{15} \frac{P_p \tau_{ws} b_k b_\ell}{m_p} \right), \tag{26}
\]

and note that \( \eta_{ij} = \eta_{ji} \) and \( \eta/15 = \eta_{11} + \eta_{22} + \eta_{33} = \eta_{ij} \delta_{ij} \) (since \( b^2 = 1 \)), we can rewrite (25) in the more revealing “classical” stress tensor form,

\[
\frac{\eta}{15} \left( b_k b_\ell \frac{\partial U_{pk}}{\partial x_i} - \frac{1}{3} \frac{\partial U_{pi}}{\partial x_m} \right) = \frac{\eta_{k\ell}}{2} \left( \frac{\partial U_{pk}}{\partial x_\ell} + \frac{\partial U_{pl}}{\partial x_k} - \frac{1}{3} \eta_{k\ell} \delta_{kl} \frac{\partial U_{pm}}{\partial x_m} \right) \tag{27}
\]

The pressure tensor is therefore the sum of an isotropic scalar pressure \( P_p \) and the stress tensor, i.e.,

\[
(P_{ij}) = P_p (\delta_{ij}) + \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{array} \right) \frac{\eta_{k\ell}}{2} \left( \frac{\partial U_{pk}}{\partial x_\ell} + \frac{\partial U_{pl}}{\partial x_k} - \frac{2}{3} \delta_{k\ell} \frac{\partial U_{pm}}{\partial x_m} \right) \equiv P_p I + \Pi_p. \]  

(28)

The stress tensor is a generalization of the “classical” form in that several coefficients of viscosity are present, and of course the derivation here is for a collisionless charged gas of PUU experiencing only pitch-angle scattering by turbulent magnetic fluctuations. Use of the pressure tensor (28) yields a “Navier-Stokes”-like modification of the energetic particle momentum equation,

\[
\frac{\partial}{\partial t} (\rho_p U_p) + \nabla \cdot [\rho_p U_p U_p + \Pi_p] = e n_p (E + U_p \times B) \]

\[
- \nabla \cdot \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{array} \right) \frac{\eta_{k\ell}}{2} \left( \frac{\partial U_{pk}}{\partial x_\ell} + \frac{\partial U_{pl}}{\partial x_k} - \frac{2}{3} \delta_{k\ell} \frac{\partial U_{pm}}{\partial x_m} \right), \]  

\[
= e n_p (E + U_p \times B) - \nabla \cdot \Pi_p \]  

(29)
where we used the transformation $U_p = u_p + U$ for the remaining velocity terms in (14) and $\rho_p = m_p n_p$.

If we introduce $c' \equiv c - u_p$ as before, we can express the heat flux $q(x, t)$ through the definition

$$q_i(x, t) \equiv m_p \int \frac{1}{2} c'^2 c'_i f d^3 c' = \frac{m_p}{2} \int c^2 c_i f d^3 c - \frac{5}{2} u_{p_i} P_p - \frac{1}{2} \rho_p u_{p_i}^2 u_{p_i}. \tag{30}$$

The equation for the total energy of the energetic particles can then be derived from (12), yielding

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_p U_p^2 + \frac{3}{2} P_p \right) + \frac{\partial}{\partial x_i} \left[ \frac{1}{2} \rho_p U_p^2 U_{p_i} + \frac{5}{2} P_p u_{p_i} + \Pi_{ij} U_{p_j} + q_i \right] = \rho_p U_{p_i} \left( E + (U_p \times B) \right)_i, \tag{31}$$

after transforming to $U_p$. To evaluate the heat flux, we have

$$\frac{1}{2} \int c'^2 c'_i f_0 d^3 c = \pi \int c'^3 \mu_i b_i f_0 c'^2 dc' = 0,$$

and

$$\frac{m_p}{2} \int c'^2 c'_i \mu_i f_1 d^3 c' = -\frac{2\pi}{3} m_p \int c'^2 \kappa_{ij} \frac{\partial f_0}{\partial x_j} c'^2 dc' = \frac{1}{2} \tilde{\kappa}_{ij} \frac{\partial P_p}{\partial x_j} = q_i(x, t). \tag{32}$$

In (32), we introduced the spatial diffusion coefficient

$$\kappa_{ij} \equiv \frac{b_i c^2 \tau_{ws} b_j}{3}, \tag{33}$$

together with energetic particle speed-averaged form $\tilde{\kappa}_{ij} \equiv K_{ij}$. The collisionless heat flux for energetic particles is therefore described in terms of the energetic particle pressure gradient and consequently the averaged spatial diffusion introduces a energetic particle diffusion time and length scale into the multi-fluid system.

For continuous flows, the transport equation for the energetic particle pressure $P_p$ can be derived from (31), yielding

$$\frac{\partial P_p}{\partial t} + U_{p_i} \frac{\partial P_p}{\partial x_i} + \frac{5}{3} P_p \frac{\partial U_{p_i}}{\partial x_i} = \frac{1}{3} \frac{\partial}{\partial x_i} \left( K_{ij} \frac{\partial P_p}{\partial x_j} \right) - \frac{2}{3} \Pi_{ij} \frac{\partial U_{p_j}}{\partial x_i}, \tag{34}$$

illustrating that the energetic particle heat flux yields a spatial diffusion term in the PUI equation of state together with a viscous dissipation term. The energetic particle system of equations is properly closed and correct to the second-order. Note the typo in Zank et al. [9] since we mistakenly omitted the viscous term of equation (34) in the corresponding pressure equation.

The full system of energetic particle equations can be written in the form

$$\frac{\partial \rho_p}{\partial t} + \nabla \cdot (\rho_p U_p) = 0; \tag{35}$$

$$\frac{\partial}{\partial t} (\rho_p U_p) + \nabla \cdot [\rho_p U_p U_p + \Pi_p + \Pi] = en_p (E + U_p \times B); \tag{36}$$

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_p U_p^2 + \frac{3}{2} P_p \right) + \nabla \cdot \left[ \frac{1}{2} \rho_p U_p^2 U_p + \frac{5}{2} P_p U_p + \Pi \cdot U_p - \frac{1}{2} K \cdot \nabla P_p \right] = en_p U_p \cdot E, \tag{37}$$

which is the form we use below.
The full thermal electron-thermal proton-PUI multi-fluid system is therefore given by equations (3) - (8) and (35) - (37) or (34), together with Maxwell’s equations

\[
\begin{align*}
\frac{\partial B}{\partial t} &= -\nabla \times E; \quad (38) \\
\nabla \times B &= \mu_0 J; \quad (39) \\
\nabla \cdot B &= 0; \quad (40)
\end{align*}
\]

\[
J = e (n_s u_s + n_p U_p - n_e u_e), \quad (41)
\]

where \(J\) is the current and \(\mu_0\) the permeability of free space. The diffusion tensor is assumed to be of a simple diagonal form (i.e., we do not include the off-diagonal terms associated with drift and curvature - see the discussion in [30]) and we specify

\[
K = \begin{pmatrix}
\kappa_{\perp} & 0 & 0 \\
0 & \kappa_{\perp} & 0 \\
0 & 0 & \kappa_{\parallel}
\end{pmatrix}; \quad (42)
\]

\[
\kappa_{\perp} = \eta \frac{1}{3} \Omega_p C_0^2, \quad \kappa_{\parallel} = \frac{1}{3} \Omega_p C_0^2. \quad (43)
\]

We parametrize the perpendicular component of the heat conduction tensor by a term \(\eta < 1\). In estimating the diffusion coefficients (43) from (33), we choose a characteristic energetic particle speed for the region of interest and assume that the scattering time can be approximated by the corresponding gyrofrequency.

3. Single-fluid-like model

For many problems, the complete multi-component model derived above is far too complicated to solve. The multi-fluid system (3) - (8) and (35) - (37) or (34), together with Maxwell’s equations can be considerably reduced in complexity by making the key assumption that \(U_p \simeq u_s\). The assumption that \(U_p \simeq u_s\) is quite reasonable since i) the bulk flow velocity of the plasma is dominated by the background protons since the energetic particle component scatters off fluctuations moving with the background plasma speed and ii) the large-scale motional electric field forces both energetic particles and newly created PUIs to essentially co-move with the background plasma flow. Accordingly, we let \(U_p \simeq u_s = U_i\) be the bulk proton (i.e., thermal background protons and PUIs or energetic particles) velocity. The thermal proton and energetic particle/PUI continuity and momentum equations are therefore trivially combined as

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i U_i) = 0; \quad (44)
\]

\[
m_p n_i \left( \frac{\partial U_i}{\partial t} + U_i \cdot \nabla U_i \right) = -\nabla (P_s + P_p) + e n_i (E + U_i \times B) - \nabla \cdot \Pi_p, \quad (45)
\]

where \(n_i = n_s + n_p\). Since the energetic particles are not thermally equilibrated with the background plasma \((T_s \neq T_p)\), we need to deal separately with the \(P_s\) and \(P_p\) equations. These become

\[
\frac{\partial P_s}{\partial t} + U_i \frac{\partial P_s}{\partial x_i} + \gamma_s P_s \nabla \cdot U_i = 0; \quad (46)
\]

\[
\frac{\partial P_p}{\partial t} + U_i \frac{\partial P_p}{\partial x_i} + 5 \frac{P_p}{3} \frac{\partial U_i}{\partial x_i} = \frac{1}{3} \frac{\partial}{\partial x_i} \left( K_{ij} \frac{\partial P_p}{\partial x_j} \right) - \frac{2}{3} \Pi_{ij} \frac{\partial U_j}{\partial x_i}. \quad (47)
\]
We can combine the proton equations (44) - (47) with the electron equations (3) - (5) to obtain an MHD-like system of equations. On defining the macroscopic variables,

\[ \rho \equiv m_e n_e + m_p n_i; \quad q \equiv -e(n_e - n_i); \quad \rho \mathbf{U} \equiv m_e n_e \mathbf{u}_e + m_p n_i \mathbf{u}_i; \quad \mathbf{J} \equiv -e (n_e \mathbf{u}_e - n_i \mathbf{u}_i), \quad (48) \]

we can express

\[ n_e = \frac{\rho - (m_p/e)q}{m_p(1 + \xi)} \simeq \frac{\rho}{m_p}; \quad n_i = \frac{\rho + \xi (m_p/e)q}{m_p(1 + \xi)} \simeq \frac{\rho}{m_p}; \]

\[ \mathbf{u}_e = \frac{\rho \mathbf{U} - (m_p/e)\mathbf{J}}{\rho - (m_p/e)q} \simeq \mathbf{U} - \frac{m_p}{e} \frac{\mathbf{J}}{\rho} - \frac{\xi (m_p/e)\mathbf{J}}{\rho}, \quad \mathbf{u}_i = \frac{\rho \mathbf{U} + \xi (m_p/e)\mathbf{J}}{\rho} \simeq \mathbf{U}, \quad (49) \]

where the smallness of the mass ratio \( \xi \equiv m_e/m_p \ll 1 \) has been exploited. Use of the approximations (49) allows us to combine the continuity and momentum equations in the usual way and to rewrite the thermal electron and proton pressure in terms of the single-fluid macroscopic variables. Thus,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0; \quad (50) \]

\[ \rho \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = -\nabla (P_e + P_s + P_p) + \mathbf{J} \times \mathbf{B} - \nabla \cdot \Pi; \quad (51) \]

\[ \frac{\partial P_s}{\partial t} + \mathbf{U} \cdot \nabla P_s + \gamma_s P_s \nabla \cdot \mathbf{U} = 0; \quad (52) \]

\[ \frac{\partial P_e}{\partial t} + \mathbf{U} \cdot \nabla P_e + \gamma_P P_e \nabla \cdot \mathbf{U} = \frac{m_p}{e \rho} \mathbf{J} \cdot \nabla P_e + \frac{\gamma_P m_p}{e} P_e \nabla \cdot \left( \frac{\mathbf{J}}{\rho} \right), \quad (53) \]

where

\[ \Pi_{kl} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{\eta}{2} \left( \frac{\partial U_k}{\partial x_l} + \frac{\partial U_l}{\partial x_k} - \frac{2}{3} \delta_{kl} \frac{\partial U_m}{\partial x_m} \right). \]

Since we may assume that the current density is much less than the momentum flux, i.e., \(|\mathbf{J}| \ll |\rho \mathbf{U}|\), we can simplify (53) further by neglecting the RHS. By assuming that \( \gamma_e = \gamma_s = \gamma \), we can combine the thermal proton and electron equations in a single thermal plasma pressure equation with \( P \equiv P_e + P_s \),

\[ \frac{\partial P}{\partial t} + \mathbf{U} \cdot \nabla P + \gamma P \nabla \cdot \mathbf{U} = 0. \quad (54) \]

Note that at this point, no assumptions about either the thermal electron or proton pressures (or temperatures) have been made.

Finally, we need an equation for the electric field \( \mathbf{E} \). To do so, we multiply the respective momentum equations by the electron or proton charge, sum, and use the approximations (49) to obtain

\[ \xi \left( \frac{m_p}{e} \right)^2 \frac{1}{\rho} \left[ \frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{J} \mathbf{U} + \mathbf{U} \mathbf{J}) \right] = \frac{m_p}{e \rho} (\nabla P_e - \mathbf{J} \times \mathbf{B} - \xi \nabla (P_s + P_p) - \xi \nabla \cdot \Pi) + \mathbf{E} + \mathbf{U} \times \mathbf{B}. \]

The generalized Ohm’s law is therefore

\[ \mathbf{E} = -\mathbf{U} \times \mathbf{B} - \frac{m_p}{e \rho} (\nabla P_e - \mathbf{J} \times \mathbf{B} - \xi \nabla P_p), \quad (55) \]

where we have retained the energetic particle pressure since in principle it can be a high temperature component of the plasma system and \( \xi P_p \) may be comparable to the \( P_e \) term.
For typical cases of interest, however, the $P_p$ term can be neglected in Ohm’s law (55). Neglect of the electron pressure and Hall current term then yields the usual form of Ohm’s law. The reduced single-fluid model equations may therefore be summarized as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0; \quad (56)
\]

\[
\rho \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = -\nabla (P + P_p) + \mathbf{J} \times \mathbf{B} - \nabla \cdot \Pi; \quad (57)
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho U^2 + \frac{3}{2} (P + P_p) + \frac{1}{2 \mu_0} B^2 \right) + \nabla \cdot \left[ \frac{1}{2} \rho U^2 \mathbf{U} + \frac{5}{2} (P + P_p) \mathbf{U} + \frac{1}{\mu_0} B^2 \mathbf{U} \right.
\]

\[
- \frac{1}{\mu_0} \mathbf{U} \cdot \mathbf{BB} + \mathbf{\Pi} \cdot \mathbf{U}_p - \frac{1}{2} \mathbf{K} \cdot \nabla P_p \left] = 0; \quad (58) \right.
\]

\[
\frac{\partial P}{\partial t} + \mathbf{U} \cdot \nabla P + \gamma P \nabla \cdot \mathbf{U} = 0; \quad (59)
\]

\[
\mathbf{E} = -\mathbf{U} \times \mathbf{B}; \quad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}; \quad \mu_0 \mathbf{J} = \nabla \times \mathbf{B}; \quad \nabla \cdot \mathbf{B} = 0. \quad (60)
\]

The single-fluid description (56) - (60) differs from the standard MHD model in that a separate description for the energetic particle or PUI pressure is required. Instead of the conservation of energy equation (58), one could use the energetic particle pressure equation (47) for continuous flows. Energetic particles and PUIs introduce both a collisionless heat conduction and viscosity into the system.

The model equations (56) - (60), despite being appropriate to non-relativistic PUIs, are identical to the so-called two-fluid MHD system of equations used to describe cosmic ray mediated plasmas [29]. However, the derivation of the two models is substantially different in that the cosmic ray number density is explicitly neglected in the two-fluid cosmic ray model and a Chapman-Enskog derivation is not used in deriving the cosmic ray hydrodynamic equations. Nonetheless, the sets of equations that emerge are the same indicating that the cosmic ray two-fluid equations do in fact include the cosmic ray number density explicitly.

The single-fluid-like model may be extended to include e.g., anomalous cosmic rays (ACRs) as well as PUIs. In this case, the ACRs are relativistic particles. As noted above, the same analysis carries over, and one has an obvious extension of the model equations (56) - (60) with the inclusion of the ACR pressure. Thus, the extension of (56) - (60) is

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0; \quad (61)
\]

\[
\rho \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = -\nabla (P + P_p + P_A) + \mathbf{J} \times \mathbf{B} - \nabla \cdot \Pi_p - \nabla \cdot \Pi_A; \quad (62)
\]

\[
\frac{\partial P}{\partial t} + \mathbf{U} \cdot \nabla P + \gamma P \nabla \cdot \mathbf{U} = 0; \quad (63)
\]

\[
\frac{\partial P_p}{\partial t} + \mathbf{U} \cdot \nabla P_p + \gamma_p P_p \nabla \cdot \mathbf{U} = \frac{1}{3} \nabla \cdot (\mathbf{K}_p \cdot \nabla P_p) - (\gamma_p - 1) \Pi_p : (\nabla \mathbf{U}); \quad (64)
\]

\[
\frac{\partial P_A}{\partial t} + \mathbf{U} \cdot \nabla P_A + \gamma A P_A \nabla \cdot \mathbf{U} = \frac{1}{3} \nabla \cdot (\mathbf{K}_A \cdot \nabla P_A) - (\gamma_A - 1) \Pi_A : (\nabla \mathbf{U}); \quad (65)
\]

\[
\mathbf{E} = -\mathbf{U} \times \mathbf{B}; \quad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}; \quad \mu_0 \mathbf{J} = \nabla \times \mathbf{B}; \quad \nabla \cdot \mathbf{B} = 0, \quad (66)
\]

where we have introduced the ACR pressure $P_A$, the corresponding stress tensor $\Pi_A$, the ACR diffusion tensor $\mathbf{K}_A$ and adiabatic index $\gamma_A (4/3 \leq \gamma_A \leq 5/3)$. The coupled system (61) - (66) is the simplest continuum model to describe a non-equilibrated plasma comprising a thermal proton-electron plasma with suprathermal particles (e.g., PUIs or SEPs) and relativistic energy.
(anomalous) cosmic rays. The system includes both the collisionless heat flux and viscosity associated with the suprathermal and relativistic particle distributions.

On reverting to equations (56) - (60), we can recover the standard form of the MHD equations if we set the heat conduction spatial diffusion tensor $K = 0$ and the coefficient of viscosity $(\eta_{kl}) = 0$, which corresponds to assuming $\tau_{ws} \to 0$. If the total thermodynamic pressure $P_{\text{total}} = P + P_p$ is introduced, then we recover the standard MHD equations (dropping the subscript “total”) i.e.,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0;$$  \hspace{1cm} (67)

$$\rho \frac{\partial \mathbf{U}}{\partial t} + \rho \mathbf{U} \cdot \nabla \mathbf{U} + (\gamma - 1) \nabla e + (\nabla \times \mathbf{B}) \times \mathbf{B} = 0;$$  \hspace{1cm} (68)

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho \mathbf{U}^2 + e + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left[ \left( \frac{1}{2} \rho \mathbf{U}^2 + \gamma e \right) \mathbf{U} + \frac{1}{\mu_0} \mathbf{B} \times (\mathbf{U} \times \mathbf{B}) \right] = 0;$$  \hspace{1cm} (69)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}); \quad \nabla \cdot \mathbf{B} = 0,$$  \hspace{1cm} (70)

with an equation of state $e = \alpha n k_B T/(\gamma - 1)$. The choice of $\alpha = 2$ (or greater if incorporating the contribution of cosmic rays etc.) corresponds to a plasma population comprising protons and electrons.

In setting $K = 0$ and $(\eta_{kl}) = 0$, we have implicitly assumed that energetic particles and PUIs are completely coupled to the thermal plasma. With $K \neq 0$, heat conduction reduces the effective coupling of energetic particles to the thermal plasma, and their contribution to the total pressure is not as large. This will have important consequences for numerical models of e.g., the large-scale heliosphere since they incorporate PUIs into the MHD equations, without distinguishing PUIs from thermal plasma and therefore neglect heat conduction. Consequently the total pressure is over-estimated.

4. Conclusions

Energetic particles (solar energetic particles, PUIs) can be dynamically important in numerous settings [1–12]. The energetic particle population is characterized by their scattering in pitch-angle off pre-existing and self-induced Alfvénic and turbulent fluctuations. Pitch-angle scattering drives energetic particle distributions towards a state of near but not complete isotropy, and infrequent intra-particle collisions between the relatively tenuous energetic particles and more populous thermal particles ensure that the energetic particles are not equilibrated with the background plasma. This therefore necessitates distinct descriptions for the different plasma species. Our results can be summarized as follows.

(i) We have presented a systematic derivation of a multi-component plasma model that includes thermal electrons and protons and an energetic particle species such as solar energetic particles or pickup ions, coupled through Maxwell’s equations. The derivation is based on a collisionless Chapman-Enskog expansion [9; 30].

(ii) The Chapman-Enskog expansion yields collisionless forms of the heat conduction tensor and (viscous) stress tensor for the energetic particles. Both the heat conduction and stress tensor expressions bear a formal resemblance to the “classical” expressions found from collisional gas dynamics.

(iii) Even though the energetic particles and the thermal plasma are not equilibrated, the multi-component model can be simplified in the MHD limit of assuming a small mass ratio $m_e/m_p$ and current density. This yields a non-equilibrated single-fluid-like model that is reminiscent of the cosmic ray two-fluid model [28; 29]. However, the derivation presented...
here is fundamentally different and our approach clarifies the somewhat ad hoc derivation of
the two-fluid model, showing for example that the total number density includes the cosmic
ray number density. The single-fluid-like model retains distinct descriptions for energetic
particle and thermal plasma pressure or temperature. We show further how to include
energetic relativistic particles such as anomalous cosmic rays along with suprathermal
particles such as PUIs.

(iv) Finally, in the limit of extremely rapid scattering $\tau_{ws} \to 0$, the single-fluid-like model reduces
to the standard set of MHD equations in which heat conduction and viscous stress tensor
terms are absent.

Several implications of our results should be noted.

(i) The models developed here provide a basic description for modeling non-equilibrated
plasma systems that have distinct thermal and energetic particle components, each carrying
different pressures or temperatures. The most elaborate is the full multi-component model
that contains the thermal electron as well as the thermal proton and energetic particle
physics. As discussed by Zank et al. [9], the presence of an energetic particle component
yields a linear wave description that is much richer than the standard two-fluid description.
However, a “low-frequency” approximation yields a single-fluid like description that is
simpler to study and implement numerically while retaining many of the characteristics
of the non-equilibrated plasma system.

(ii) The momentum and energy transport equations for energetic particles provide physically
motivated collisionless dissipative terms (heat conduction and viscous stress tensor).

(iii) The presence of physically motivated dissipative terms in the energetic particle equations
raises questions regarding the design of numerical algorithms that couple gas dynamic or
MHD equations to an energetic particle component. Is the incorporation of numerical
dissipation in the gas dynamics/MHD solver appropriate given the possibly more dominant
physically based dissipative terms introduced by energetic particles?

Acknowledgments
We acknowledge the partial support of NASA grants NNX08AJ33G, Subaward 37102-2,
NNX14AC08G, NNX14AJ53G, A99132BT, RR185-447/4944336 and NNX12AB30G.

References
[1] Lario D, Decker R B, Roelof E C and Viñas A F 2015 Journal of Physics Conference Series
642 012014
[2] Lario D, Decker R B, Roelof E C and Viñas A F 2015 Ap. J. 813 85 (Preprint 1509.04368)
[3] Russell C T, Mewaldt R A, Luhmann J G, Mason G M, von Rosenvinge T T, Cohen C M S,
Leske R A, Gomez-Herrero R, Klassen A, Galvin A B and Simunac K D C 2013 Ap. J. 770 38
[4] Terasawa T 1999 International Cosmic Ray Conference 6 528
[5] Terasawa T, Oka M, Nakata K, Keika K, Nosé M, McEntire R W, Saito Y and Mukai T
2006 Advances in Space Research 37 1408–1412
[6] Burlaga L F, Ness N F, Belcher J W, Szabo A, Isenberg P A and Lee M A 1994 J. Geophys.
Res. 99 21511
[7] Zank G P 1999 Space Sci. Rev. 89 413–688
[8] Richardson J D, Paularena K I, Lazarus A J and Belcher J W 1995 Geophys. Res. Lett.
22 1469–1472
[9] Zank G P, Humana P, Mostafavi P and Goldstein M L 2014 Ap. J. 797 87
[10] Zank G P 2015 *Ann. Rev. Astron. Astrophys.* **53** 449–500
[11] Decker R B, Krimigis S M, Roelof E C, Hill M E, Armstrong T P, Gloeckler G, Hamilton D C and Lanzerotti L J 2008 *Nature* **454** 67–70
[12] Decker R B, Krimigis S M, Roelof E C and Hill M E 2015 *Journal of Physics Conference Series* **577** 012006
[13] Zank G P, Pauls H L, Williams L L and Hall D T 1996 *J. Geophys. Res.* **101** 21639–21656
[14] McComas D J, Alexashov D, Bzowski M, Fahr H, Heerikhuisen J, Izmodenov V, Lee M A, Möbius E, Pogorelov N, Schwadron N A and Zank G P 2012 *Science* **336** 1291–
[15] Zank G P, Heerikhuisen J, Wood B E, Pogorelov N V, Zirnstein E and McComas D J 2013 *Ap. J.* **763** 20
[16] Zirnstein E J, Heerikhuisen J, Zank G P, Pogorelov N V, McComas D J and Desai M I 2014 *Ap. J.* **783** 129
[17] Zank G P, Heerikhuisen J, Pogorelov N V, Burrows R and McComas D 2010 *Ap. J.* **708** 1092–1106
[18] Lee M A 1983 *J. Geophys. Res.* **88** 6109–6119
[19] Ng C K, Reames D V and Tylka A J 1999 *Geophys. Res. Lett.* **26** 2145–2148
[20] Gordon B E, Lee M A, Möbius E and Trattner K J 1999 *J. Geophys. Res.* **104** 28263–28278
[21] Rice W K M, Zank G P and Li G 2003 *Journal of Geophysical Research (Space Physics)* **108** 1369
[22] Li G, Zank G P and Rice W K M 2003 *Journal of Geophysical Research (Space Physics)* **108** 1082
[23] Zank G P, Li G and Verkhoglyadova O 2007 *Space Sci. Rev.* **130** 255–272
[24] Verkhoglyadova O P, Zank G P and Li G 2015 *Phys. Rep.* **557** 1–23
[25] Lee M A and Ip W H 1987 *J. Geophys. Res.* **92** 11041–11052
[26] Williams L L and Zank G P 1994 *J. Geophys. Res.* **99** 19229–+
[27] Cannon B E, Smith C W, Isenberg P A, Vasquez B J, Murphy N and Nuno R G 2014 *Ap. J.* **784** 150
[28] Axford W I, Leer E and McKenzie J F 1982 *Astron. Astrophys.* **111** 317–325
[29] Webb G M 1983 *Astron. Astrophys.* **127** 97–112
[30] Zank G P 2014 *Transport Processes in Space Physics and Astrophysics (Lecture Notes in Physics, Berlin Springer Verlag* vol 877) (Springer-Verlag)
[31] Isenberg P A 1997 *J. Geophys. Res.* **102** 4719–4724
[32] Webb G M 1985 *Ap. J.* **296** 319–330
[33] Webb G M 1989 *Ap. J.* **340** 1112–1123