Modeling Transit Dark Energy in \( f(R, L_m) \)-gravity

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Abstract

This research paper deals with a transit dark energy cosmological model in \( f(R, L_m) \)-gravity with observational constraints. For this, we consider a flat FLRW space-time and have taken a cosmological constant-like parameter \( \beta \) in our field equations. The model has two energy parameters \( \Omega_{m0} \) and \( \Omega_{\beta0} \), which govern the mechanism of the universe, in particular its present accelerated phase. To make the model cope with the present observational scenario, we consider three types of observational data set: 46 Hubble parameter data set, SNe Ia 715 data sets of distance modulus and apparent magnitude, and 40 datasets of SNe Ia Bined compilation in the redshift \( 0 \leq z < 1.7 \). We have approximated the present values of the energy parameters by applying \( R^2 \) and \( \chi^2 \)-test in the observational and theoretical values of Hubble, distance modulus, and apparent magnitude parameters. Also, we have measured the approximate present values of cosmographic coefficients \( \{H_0, q_0, j_0, s_0, l_0, m_0\} \). It is found that our approximated value-based model fits best with the observational module. We have found that as \( t \to \infty \) (or \( z \to 0 \)) then \( \{q, j, s, l, m\} \to \{-1, 1, 1, 1, 1\} \). The cosmic age of the present universe is also approximated and comes up to the expectation. Our model shows a transit phase of the present accelerating universe with a deceleration in the past and has a transition point.

Keywords: Flat FLRW Universe; Modified \( f(R, L_m) \)-gravity; Dark Energy; Observational Constraints

1 Introduction

Cosmic observational studies like type Ia supernovae \(^1\)\(^2\)\(^3\), Baryonic Acoustic Oscillations \(^4\)\(^5\), Wilkinson Microwave Anisotropy Probe \(^6\), Large scale Structure \(^7\)\(^8\) and the Cosmic Microwave Background Radiation \(^9\)\(^10\) suggest that the expanding universe is in accelerating expansion phase. After that, among the theoretical cosmologists, a thirst originated from developing cosmological models with accelerating phase expansion by either modifying Einstein’s field equations or providing an alternative theory of gravity. The responsible components of the universe behind this acceleration have been known as “Dark Energy” (DE) among researchers due to its
mystery. In the literature, several dark energy cosmological models were proposed to explain this acceleration. Conventional cosmology also strongly supports these ideas. The cosmological constant $\Lambda$-term, which is also researched as vacuum quantum energy, is considered to be the most likely candidate for dark energy [11]. Even while the $\Lambda$-term closely matches the observed data, it still has two significant flaws, the fine-tuning problem and the cosmological constant problem (due to its origin)[12]. The $\Lambda$ term acquired from particle physics has a value over 120 orders of magnitude greater than that needed to match space observations. Imagining that Einstein’s general theory of relativity model collapses on a vast cosmic scale, with more general factors defining the gravitational field, is another intriguing approach to explaining recent discoveries about the cosmic expansion scenario. The Einstein-Hilbert action of general relativity can be generalized in various ways.

The model introduced in [13, 14, 15] where the default action has been changed the generic function $f(R)$, where $R$ is a Ricci scalar. $f(R)$ gravity may be used to describe the late expansion scenario [16], and [17, 18] studied the limitations of a viable space model. There exists a good $f(R)$ gravity models for solar system testing in [19]-[22]. [23]-[27] describe the observational aspects of the $f(R)$ DE model, together with the solar system constraints and the equivalence principle of $f(R)$ gravity. In [28, 29, 30], another $f(R)$ model that combines early inflation with dark energy and undergoes local testing is detailed. In addition, one can consult the references [31, 32, 33, 34, 35, 36, 37] to learn more about the $f(R)$ gravity model’s cosmic implications.

In [38], an extension of the $f(R)$ theory of gravity was presented, incorporating an explicit conjunction of the matter Lagrangian density $L_m$ and the generic function $f(R)$. During the non-geodesic motion of the heavy particles, extra forces perpendicular to the four-velocity vectors are created due to this matter-geometric connection. This model has been expanded to the situation of arbitrary matter-geometry pairings. In [40]-[44], the cosmological and astrophysical ramifications of non-minimal matter-geometric couplings are examined in depth. Recently, Harko and Lobo [45] introduced the $f(R, L_m)$ gravity theory, where $f(R, L_m)$ is an arbitrary function of the Lagrangian matter density $L_m$ and the Ricci scalar $R$. The $f(R, L_m)$ gravitational theory is the most expansive of all gravitational theories developed in the Riemann space. In $f(R, L_m)$ gravity theory, the motion of the test particle is non-geodesic, with extra forces orthogonal to the four-velocity vectors. The gravitational model $f(R, L_m)$ permits explicit breaches of the equivalence principle, which are tightly confined by the solar system test [46, 47]. Wang and Liao [48] have recently examined the energy condition of $f(R, L_m)$ gravity. Gonclaves and Moraes employ the $f(R, L_m)$ gravity model [49]. We explored cosmology by combining non-minimum matter geometries with a specific version of the $f(R, L_m)$ function [50].

This study examines an observationally constrained transit dark energy cosmology model in $f(R, L_m)$-gravity. We consider a flat FLRW space-time for this and have included a term in our field equations called $\beta$ that resembles a cosmological constant. $\Omega_{m0}$ and $\Omega_{b0}$, two energy parameters in the model, determine the universe’s mechanism, particularly its current accelerated phase. We take into account three different types of observational data sets to help the model adapt to the current state of observable phenomena: 46 Hubble parameter data set, SNe Ia 715 data sets of distance modulus and apparent magnitude and 40 datasets of SNe Ia Bined compilation in the redshift $0 \leq z < 1.7$. Using $R^2$ and $\chi^2$-test, the observational and theoretical values of the Hubble, distance modulus, and apparent magnitude parameters, we have approximated the current values of the energy parameters. We have also calculated the estimated present values of the cosmographic coefficients $\{H_0, q_0, j_0, s_0, l_0, m_0\}$. Our approximation-based model was found to fit the observational module the best. As $t \rightarrow \infty$ (or $z \rightarrow 0$), we have discovered that $\{q, j, s, l, m\} \rightarrow \{-1, 1, 1, 1, 1\}$. In our scenario, there is a transition point and a transit phase of an accelerating cosmos that decelerated in the past. The current universe’s cosmic age is likewise approximated and meets expectations. Our model has a transition point and depicts a transit phase of the now accelerating world with a previously decelerating universe.

This investigation is divided into the following sections: In section 2, we have mentioned the formulation of $f(R, L_m)$ gravity. In section 3, field equations are obtained for a flat FLRW space-time universe using a perfect-fluid stress-energy momentum tensor. In section 4, we have obtained the cosmological solutions for $f(R, L_m) =
$R + L_m - \beta$ model. In section 5, we made an observational constraint on Hubble parameter and distance modulus function with observational $H_0$ data sets, SNe Ia 715 data sets, and SNe Ia 40 Bined data sets. Section 6 contains result analysis with cosmographic coefficients and the age of the present universe. Finally, in the last section, conclusions are included.

2 Modified $f(R, L_m)$-gravity theory

The action principle for $f(R, L_m)$-gravity is taken as

$$I = \int f(R, L_m) \sqrt{-g} d^4x \quad (1)$$

where $L_m$ is the matter Lagrangian of the perfect fluid, $R$ is the Ricci scalar curvature and these are related by an arbitrary function $f(R, L_m)$. The Ricci-scalar $R$ is defined in terms of metric tensor $g_{ij}$ and Ricci-tensor $R_{ij}$ as given below

$$R = g^{ij} R_{ij} \quad (2)$$

where the Ricci-tensor is given by

$$R_{ij} = -\frac{\partial^2}{\partial x^i \partial x^j} \log \sqrt{|g|} + \partial_k \Gamma_{ij}^k - \Gamma_{ik}^\alpha \Gamma_{\alpha j}^k + \Gamma_{ij}^\alpha \Gamma_{\alpha k}^k \quad (3)$$

in terms of well-known Levi-Civita connection $\Gamma_{\alpha \beta \gamma}^i$ which is defined as

$$\Gamma_{\beta \gamma}^\alpha = \frac{1}{2} g^{\alpha k} \left( \frac{\partial g_{\gamma k}}{\partial x^\beta} + \frac{\partial g_{k \beta}}{\partial x^\gamma} - \frac{\partial g_{\beta \gamma}}{\partial x^k} \right) \quad (4)$$

On the variation of action (1) over metric tensor $g_{ij}$, one can find the following equations

$$f_R R_{ij} + (g_{ij} \nabla_k \nabla^k - \nabla_i \nabla_j) f_R - \frac{1}{2} (f - f_{L_m} L_m) g_{ij} = \frac{1}{2} f_{L_m} T_{ij} \quad (5)$$

where $f_R = \frac{\partial f}{\partial R}$, $f_{L_m} = \frac{\partial f}{\partial L_m}$ and $T_{ij}$ is the Stress-energy momentum tensor for perfect-fluid, defined by

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{ij}} \quad (6)$$

Now, contracting the field equation (5), we get the relation between Ricci-scalar curvature $R$, matter Lagrangian density $L_m$ and $T$ the trace of the stress-energy-momentum tensor $T_{ij}$ as

$$R f_R + 3 \Box f_R - 2 (f - f_{L_m} L_m) = \frac{1}{2} f_{L_m} T \quad (7)$$

where $\Box F = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{\alpha \beta} \partial_\beta F)$ for any function $F$.

Taking covariant derivative of Eq. (5), we obtain

$$\nabla^i T_{ij} = 2 \nabla^i \log (f_{L_m}) \frac{\partial L_m}{\partial g^{ij}} \quad (8)$$

3 Field Equations

For the spatial isotropic and homogeneous universe, we consider the following flat Friedman-Lamatre-Robertson-Walker (FLRW) metric [51] to analyzed the present cosmological model,

$$ds^2 = -dt^2 + a(t)^2 [dx^2 + dy^2 + dz^2 ] \quad (9)$$
Where, $a(t)$ is known as the scale factor which represents the characteristics of the expanding universe at any instant. For the metric (9), the non-zero components of Christoffel symbols are

$$
\Gamma_{ij}^0 = \frac{1}{2} g^{00} \frac{\partial g_{ij}}{\partial x^0}, \quad \Gamma_{0j}^k = \Gamma_{j0}^k = \frac{1}{2} g^{k\lambda} \frac{\partial g_{j\lambda}}{\partial x^0}
$$

where, $i, j, k = 1, 2, 3$.

Equation (3) allows us to determine the Ricci curvature tensor’s non-zero components as

$$
R_{00} = 3 \frac{\ddot{a}}{a}, \quad R_{11} = R_{22} = R_{33} = \frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2
$$

As a result, the line element’s associated Ricci-scalar ($R$) is found to be

$$
R = 6 \frac{\ddot{a}}{a} + 6 \left( \frac{\dot{a}}{a} \right)^2 = 6(\dot{H} + 2H^2)
$$

where $H$ is the Hubble parameter defined by $H = \frac{\dot{a}}{a}$.

The stress-energy momentum tensor for the perfect-fluid matter filled in the universe corresponding to the line element (9) is taken as

$$
T_{ij} = (\rho + p) u_i u_j + pg_{ij}
$$

or

$$
T_i^i = diag[-\rho, p, p, p]
$$

where $p$ is the pressure of the cosmic fluid, $\rho$ is the energy density, $g^{ij}$ is the metric tensor and $u^i = (1, 0, 0, 0)$ components of co-moving four velocity vectors in cosmic fluid such that $u_i u^i = -1$.

The modified Friedmann equations that describe the universe’s dynamics in $f(R, L_m)$ gravity are as follows:

$$
R_{00} f_R - \frac{1}{2} (f - f_{L_m} L_m) + 3H f_R = \frac{1}{2} f_{L_m} T_{00}^0
$$

and

$$
R_{11} f_R - \frac{1}{2} (f - f_{L_m} L_m) - \dot{f}_R - 3H f_R = \frac{1}{2} f_{L_m} T_{11}^1
$$

4 Cosmological Solutions for $f(R, L_m)$-Gravity

In this study, we have considered the $f(R, L_m)$-gravity [51] of the form

$$
f(R, L_m) = \frac{R}{2} + \alpha L_m^n - \beta
$$

where $n, \alpha$ and $\beta$ are arbitrary constants. $\alpha$ is coupling constant. Its dimension is that of $L_m^{n-1}$.

Thus, for this particular form of $f(R, L_m)$-gravity model, we have considered dusty universe with $L_m = \rho$ [53] and thus, for the matter-dominated universe, the Friedmann equations (14) and (15) become

$$
3H^2 = (2n - 1)\alpha \rho^n + \beta
$$

and

$$
2\dot{H} + 3H^2 = (n - 1)\alpha \rho^n + \beta
$$

After taking a trace of the field equations, one can also get the following matter conservation equation

$$
(2n - 1)\dot{\rho} + 3H \rho = 0
$$
In particular, one can obtain the standard Friedmann equations of GR for \( n = 1, \alpha = 1 \) and \( \beta = 0 \)

Solving Eq. (19), we get energy density of matter fluid as

\[
\rho = \rho_0 \left( \frac{a_0}{a} \right)^{\frac{3}{2n-1}} \tag{20}
\]

where \( a_0 \) and \( \rho_0 \) are present values of scale factor and energy density respectively.

From equation (17), we can find

\[
\Omega_m + \Omega_\beta = \frac{1}{2n-1} \tag{21}
\]

where \( \Omega_m = \frac{\alpha \rho_0}{3H_0^2} \) and \( \Omega_\beta = \frac{\beta}{3(2n-1)H_0^2} \) are called as matter energy density parameter and dark energy density parameter respectively.

Using Eq. (20) in (21), we get the Hubble parameter as

\[
H = H_0 \sqrt{\frac{2n-1}{\Omega_{m0} \left( \frac{a_0}{a} \right)^{\frac{3}{2n-1}} + \Omega_{\beta0}}} \tag{22}
\]

and in terms of redshift \( z \), using \( \frac{a_0}{a} = 1 + z \) as given in [54] in (22), we get

\[
H(z) = H_0 \sqrt{\frac{2n-1}{\Omega_{m0}(1+z)^{\frac{3}{2n-1}} + \Omega_{\beta0}}} \tag{23}
\]

with \( \Omega_{m0}, \Omega_{\beta0} \) and \( H_0 \) are the present values of corresponding parameters \( \Omega_m, \Omega_\beta \) and \( H \) respectively i.e \( \Omega_{m0} = \frac{\alpha \rho_0}{3H_0^2} \)

and \( \Omega_{\beta0} = \frac{\beta}{3(2n-1)H_0^2} \).

Eq. (22) is integrated, and the scale-factor is obtained as

\[
a(t) = k_0 \left[ sinh(k_1t + k_2) \right]^{\frac{2(2n-1)}{3n}} \tag{24}
\]

where \( k_0 = a_0 \left[ \frac{\Omega_{m0}}{\Omega_{\beta0}} \right]^{\frac{2n-1}{3n}} \), \( k_1 = \frac{2}{3n} H_0 \sqrt{\Omega_{\beta0}(2n-1)^3} \) and \( k_2 \) is an integrating constant.

We can solve Eq. (21), (22) and (18) to get deceleration parameter \( q \) as

\[
q(z) = -1 + \frac{3n}{2(2n-1)} - \frac{3n}{2(2n-1)} \Omega_{\beta0} \Omega_{m0}(1+z)^{\frac{2n}{2n-1}} + \Omega_{\beta0} \tag{25}
\]

The transition point \( z_t \) for which \( q(z_t) = 0 \) obtained as

\[
z_t = \left[ \frac{2n - 2 \Omega_{\beta0}}{2 - n \Omega_{m0}} \right]^{\frac{2n-1}{3n}} - 1 \tag{26}
\]

For \( z > z_t \), universe was in a decelerating phase, and for \( z < z_t \), it is accelerating. Thus, as suggested in recent observations [1]-[10], our resulting model depicts an expanding universe in the transit phase (decelerating to accelerating phase).

5 Observational Constraints

To validate and viability of our model, we have compared our results with observational data sets through the Hubble parameter \( H(z) \) with redshift. Now, using Eq. (24) with the relation \( \frac{a_0}{a} = 1 + z \) given in [54], for our model, we have determined the following relationship between cosmic time \( t \) and redshift \( z \):

\[
t(z) = \frac{1}{k_1} sinh^{-1} \left[ \left( \frac{a_0/k_0}{1 + z} \right)^{\frac{3n}{2(2n-1)}} \right] - \frac{k_2}{k_1} \tag{27}
\]

The formulation for the cosmic time \( t(z) \) in terms of redshift \( z \) is represented by equation (27). One might think of this as the current age of the universe, which is \( z \rightarrow \infty \) then \( t \rightarrow 0 \) and as \( z \rightarrow 0 \) then \( t \rightarrow t_0 \).
5.1 Hubble Parameter

At present days cosmological studies expect researchers in this field to validate their theoretical results with observational data sets obtained from various observatories and study centers, viz. [1]-[10]. Therefore, to validate our derived model, we have found the best fit values of model parameters by comparing our model with observational datasets, we have considered 46 Hubble data sets $H(z)$ with high redshift ($0 \leq z \leq 2.36$) estimated by different cosmologists [55]-[61] & [65]-[74] as mentioned in Table 2. With the aid of the $R^2$ formula, we have employed a strategy in the current study to estimate the present values of $\Omega_{m0}, \Omega_{\beta0}, H_0, n$ by comparing the theoretical and observed results.

$$R^2_H = 1 - \frac{\sum_{i=1}^{46} [(H_i)_{ob} - (H_i)_{th}]^2}{\sum_{i=1}^{46} [(H_i)_{ob} - (H_i)_{mean}]^2}$$

Here, $(H_i)_{th}$ is the theoretical value of the Hubble parameter $H(z)$ as given in Eq. (23), $(H_i)_{ob}$ is the observational values of $H(z)$ (as mentioned in Table 2) and $(H_i)_{mean}$ is the mean of Hubble data $(H_i)_{ob}$ (mentioned in Table 2).

To validate our model in front of observational data, we have used the $R^2$-test formula nearest to the ideal case $R^2 = 1$ the condition in which both the data sets (observational and theoretical) are compatible. Hence, we have estimated the best fit values of $\Omega_{m0}, \Omega_{\beta0}, H_0, n$ as mentioned below in Table 1 with $R^2 = 0.945364$ at 95% confidence level of bounds. Thus, we have estimated the present value of the Hubble parameter $H_0 = 68.9596$ km/s/Mpc. In 2018, the Planck Collaboration determined the Hubble constant to be $H_0 = 67.4 \pm 0.5$ km/s/Mpc using the Cosmic Microwave Background (CMB) temperature and polarisation anisotropies and the $\Lambda$CDM cosmological model. In contrast, the SH0ES Collaboration (Riess et al.[63]) determined the Hubble constant to be $H_0 = 73.2 \pm 1.3$ km/s/Mpc using Cepheids and supernovae. Cunha et al. [64] estimated the value of Hubble constant $H_0 \approx 74$ km/s/Mpc. The best fit curve of Hubble parameter $H(z)$ is represented in Figure 1. Figure 2 represents 1σ and 2σ confidence regions for estimated parameters $\Omega_{m0}$ and Hubble constant $H_0$ for our model.

| Parameter | Values |
|-----------|--------|
| $\Omega_{m0}$ | 0.306973$^{+0.0174}_{-0.0107}$ |
| $\Omega_{\beta0}$ | 0.618952$^{+0.908526}_{-0.915226}$ |
| $H_0$ | 68.9596$^{+2.17}_{-0.9538}$ |
| $n$ | 1.04 |
| $R^2$ | 0.945364 |

Table 1: For 46 Hubble parameter $H(z)$ data sets, the best fit values of $\Omega_{m0}, \Omega_{\beta0}, H_0, n$ using the $R^2$-test at the 95% confidence level of bounds.
Figure 1: (a) Hubble parameter $H(z)$ Plot for the best fit values of $\Omega_{m0}, \Omega_{b0}, H_0, n$ with $H_0$ data sets, using $R^2$-test at 95% confidence level of bounds as mentioned in Table 1. (b) $1\sigma$ and $2\sigma$ confidence regions for estimated parameters $\Omega_{m0}$ and Hubble constant $H_0$

Figure 2: (a,b)Likelihood plots for estimated Hubble $H_0$ and barion energy parameter $\Omega_{m0}$

| S.No. | $z$ | $H(z)$ | $\sigma_H$ | Reference | S.No. | $z$ | $H(z)$ | $\sigma_H$ | Reference |
|-------|-----|--------|------------|-----------|-------|-----|--------|------------|-----------|
| 1     | 0   | 67.77  | 1.30       | [55]      | 24    | 0.4783| 80.9   | 9          | [73]      |
| 2     | 0.07| 69   | 19.6       | [56]      | 25    | 0.48  | 97     | 60         | [57]      |
| 3     | 0.09| 69   | 12        | [72]      | 26    | 0.51  | 90.4   | 1.9        | [59]      |
| 4     | 0.10| 69   | 12        | [57]      | 27    | 0.57  | 96.8   | 3.4        | [74]      |
| 5     | 0.12| 68.6 | 26.2      | [56]      | 28    | 0.593 | 104    | 13         | [71]      |
| 6     | 0.17| 83   | 8         | [57]      | 29    | 0.60  | 87.9   | 6.1        | [65]      |
| 7     | 0.179| 75  | 4         | [71]      | 30    | 0.61  | 97.3   | 2.1        | [59]      |
| 8     | 0.1993| 75  | 5        | [71]      | 31    | 0.68  | 92     | 8          | [74]      |
| 9     | 0.2  | 72.9 | 29.6      | [56]      | 32    | 0.73  | 97.3   | 7          | [65]      |
| 10    | 0.24 | 79.7 | 2.7       | [58]      | 33    | 0.781 | 105    | 12         | [74]      |
| 11    | 0.27 | 77   | 14        | [57]      | 34    | 0.875 | 125    | 17         | [74]      |
| 12    | 0.28 | 88.8 | 36.6      | [56]      | 35    | 0.88  | 90     | 40         | [57]      |
| 13    | 0.35 | 82.7 | 8.4       | [60]      | 36    | 0.9   | 117    | 23         | [57]      |
| 14    | 0.352 | 83  | 14        | [71]      | 37    | 1.037 | 154    | 20         | [58]      |
| 15    | 0.38 | 81.5 | 1.9       | [59]      | 38    | 1.3   | 168    | 17         | [57]      |
| 16    | 0.3802 | 83  | 13.5      | [60]      | 39    | 1.363 | 160    | 33.6       | [67]      |
| 17    | 0.4  | 95   | 17        | [72]      | 40    | 1.43  | 177    | 18         | [57]      |
| 18    | 0.004| 77   | 10.2      | [73]      | 41    | 1.53  | 140    | 14         | [57]      |
| 19    | 0.007 | 87.4 | 16.8      | [61]      | 42    | 1.67  | 203    | 49         | [77]      |

Table 2: Hubble's constant table.
5.2 Luminosity Distance

Cosmology has emerged as a significant area of high-level research. The presence of dark matter and dark energy in abundance in the universe are explored under it. The phenomena of gravitating licensing have roots in dark matter. The redshift-luminosity distance relation, measurements of the distance modulus $\mu$, and apparent magnitude $m_b$ have emerged as powerful observational tools to understand the universe’s development. After accounting for redshift, the luminosity distance ($D_L$) is the distance to a far-off luminous object. The luminosity distance is used to calculate a source’s flux. It is provided as

$$D_L = a_0 r (1 + z).$$  \hfill (28)

where the source’s radial coordinate is represented by $r$. Consider of a light ray’s path as

$$\frac{d\theta}{ds} = 0 \text{ and } \frac{d\phi}{ds} = 0,$$  \hfill (29)

We get the followings from the Geodesic of light photons

$$\frac{d^2 \theta}{ds^2} = 0 \text{ and } \frac{d^2 \phi}{ds^2} = 0.$$  \hfill (30)

Radial directional ray continues to move along the $r$-direction always, and we get the following light ray path

$$ds^2 = c^2 dt^2 - \frac{a^2}{1 + kr^2} dr^2 = 0.$$  \hfill (31)

Taking $k = 0$. we obtain

$$r = \int_0^r dr = \int_0^t \frac{cdt}{a(t)} = \frac{1}{a_0 H_0} \int_0^z \frac{cdz}{h(z)} \hfill (32)$$

where we have used $dt = dz/\dot{z}$, $\dot{z} = -H(1 + z)$ & $h(z) = \frac{H}{H_0}$. This gives the luminosity distance as:

$$D_L = \frac{c(1 + z)}{H_0} \int_0^z \frac{dz}{h(z)}.$$  \hfill (33)

5.3 Apparent Magnitude $m_b$ and Distance Modulus $\mu$

The apparent magnitude $m_b$ and distance modulus $\mu$ are the two most required observational parameters as the leading resources of observational cosmology. SN Ia union 2.1 compilation data set provide $\mu$ and $m_b$ of 715 low and high red shift SN Ia where as pantheon compilation data set provide $m_b$ of 1120 low and high red shift SN Ia. These parameters have following theoretical formulism

$$\mu = m_b - M$$

$$= 5 \log_{10} \left( \frac{D_L}{Mpc} \right) + 25$$

$$= 25 + 5 \log_{10} \left[ \frac{c(1 + z)}{H_0} \int_0^z \frac{dz}{h(z)} \right].$$  \hfill (34)

The absolute magnitude $M$ of a supernova is obtained as :

$$M = 16.08 - 25 + 5 \log_{10}(H_0/0.026c)$$  \hfill (35)

This provide the apparent magnitude $m_b$ as

$$m_b = 16.08 + 5 \log_{10} \left[ \frac{1 + z}{0.026} \int_0^z \frac{dz}{h(z)} \right].$$  \hfill (36)

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5.4 $\chi^2$ for Apparant Magnitude $m_b$ and Distance Modulus ‘$\mu$’:

We use the most recent compilation of Supernovae pantheon samples which includes 715 SN Ia plus 40 SN Ia bined data in the range of( 0.01 $\leq z \leq$ 1.17 ) [77, 78]. In $(m_b, z)$ pairs, the pantheon compilation of SN Ia data is displayed. Here, we define $\chi^2$ for the parameters with the likelihood given by $\varphi \propto e^{-\chi^2}$ in order to restrict the various parameters of the Universe in the resultant model.

$$\chi^2 = \frac{(m_{b\text{ob}} - m_{b\text{th}})^2}{m_{b\text{err}}^2}$$

and

$$\chi^2 = \frac{(\mu_{\text{ob}} - \mu_{\text{th}})^2}{\mu_{\text{err}}^2}$$

The estimated present values of $H_0$, $\Omega_m$, and parameter ‘$n$’ are displayed in the following table based on the minimum $\chi^2$.

| Parameter | Values |
|-----------|--------|
| $\Omega_m$ | 0.40543 |
| $\Omega_{\beta 0}$ | 0.44068 |
| $n$ | 1.09094 |
| $\chi^2$ | 23506.4 |

Table 3: The best fit values of $\Omega_m$ and ‘$n$’ for the SN Ia data set of Apparent magnitude $m_b$.

| Parameter | Values |
|-----------|--------|
| $H_0$ | 69.5047$^{+0.14846}_{-0.14054}$ |
| $\Omega_m$ | 0.291379$^{+0.0478}_{-0.0448}$ |
| $\Omega_{\beta 0}$ | 0.678478 |
| $n$ | 1.01554 |
| $\chi^2$ | 1220.59 |

Table 4: The best fit values of $\Omega_m$ and ‘$n$’ for the SN Ia data set of Distance modulus $\mu$. 
Figure 3: (a,b) distance modulus ($\mu = M - m_b$) versus redshift ($z$) and Apparent magnitude ($m_b$) versus redshift ($z$). Error bar plots.
Cosmographic Analysis

Of all the rational approaches, cosmography has recently received a lot of attention [80, 81]. This model-independent method relies solely on the observational assumptions of the cosmological principle, allowing the study of dark energy evolution without the need to adopt a specific cosmological model [82, 83, 84]. The usual space flight approach is based on Taylor’s expansion of observations that can be directly compared to the data, and the results of such procedures are independent of the state equations assumed to study the evolution of the universe. For these reasons, cosmography has proven to be a powerful tool for breaking down the cosmological model, and is now widely used to understand the kinematics of the universe [85]–[117].

The cosmological principle requires a scale factor as the only degree of freedom that governs the universe. Therefore, we can expand the current Taylor series of $a(t)$ about present time and hence, we can define the cosmographic series coefficients like Hubble parameter ($H$), deceleration parameter ($q$), jerk ($j$), snap ($s$), lerk ($l$)
and max-out \((m)\) as given in [79]:

\[
H = \frac{1}{a} \frac{da}{dt}, \quad q = -\frac{1}{aH^2} \frac{d^2a}{dt^2}, \quad j = \frac{1}{aH^3} \frac{d^3a}{dt^3}
\]

and

\[
s = \frac{1}{aH^4} \frac{d^4a}{dt^4}, \quad l = \frac{1}{aH^5} \frac{d^5a}{dt^5}, \quad m = \frac{1}{aH^6} \frac{d^6a}{dt^6}
\]

The dynamics of the late universe are studied using these quantities. The Hubble expansion’s shape can be used to determine the physical characteristics of the coefficients. In particular, the sign of parameter \(q\) tells us whether the cosmos is accelerating or decelerating. The sign of \(j\) controls how the dynamics of the universe change, and positive values of \(j\) denote the occurrence of transitional intervals when the universe’s expansion changes. In order to distinguish between developing dark energy theories and cosmological constant behaviour, we also require the value of \(s\).

Using the scale-factor \([24]\) in \([37] & [38]\), we have found the cosmographic series coefficients \(q, j, s, l, m\) as

\[
q = -1 + \frac{1}{k} \text{sech}^2(k_1 t)
\]

\[
j(t) = \frac{(k-1)(k-2)}{k^2} + \frac{3k-2}{k^2} \tanh^2(k_1 t)
\]

\[
s(t) = \frac{(k-1)(k-2)(k-3)}{k^3} + \frac{2(k-1)(3k-4)}{k^3} \tanh^2(k_1 t) + \frac{3k-2}{k^3} \tanh^4(k_1 t)
\]

\[
l(t) = \frac{(k-1)(k-2)(k-3)(k-4)}{k^4} + \frac{10(k-1)(k-2)}{k^4} \tanh^2(k_1 t) + \frac{15k^2 - 30k + 16}{k^4} \tanh^4(k_1 t)
\]

\[
m(t) = \frac{(k-1)(k-2)(k-3)(k-4)(k-5)}{k^5} + \frac{5(k-1)(k-2)(k-3)(3k-8)}{k^5} \tanh^2(k_1 t)
\]

\[+ \frac{(k-1)(45k^2 - 150k + 136)}{k^5} \tanh^4(k_1 t) + \frac{15k^2 - 30k + 16}{k^5} \tanh^6(k_1 t)
\]

where \(k = \frac{2(2n-1)}{3n}, \ k_1 = \frac{2}{3n} H_0 \sqrt{\Omega_{\bar{m}}(2n-1)^3}\) and \(t\) is the cosmic time.

| Parameter | For \(H_0\) data | Fo SN Ia 715 data plus 40 bined data | For SN Ia 715 data |
|-----------|------------------|----------------------------------|------------------|
| \(H_0\)   | 68.9596          | -                                | 69.5047          |
| \(q_0\)   | -0.5211          | -0.3366                          | -0.5561          |
| \(j_0\)   | 0.9468           | 0.8469                           | 0.9799           |
| \(s_0\)   | -0.3106          | -0.6178                          | -0.3157          |
| \(l_0\)   | 2.854            | 3.208                            | 2.896            |
| \(m_0\)   | -9.504           | -13.25                           | -9.594           |
| \(t_0\)   | 13.5922 Gyrs     | 12.1960 Gyrs                     | 13.8768          |
| \(z_l\)   | 0.6879           | 0.4553                           | 0.7097           |
| \(t_r\)   | 6.4912 Gyrs      | 9.78 Gyrs                        | 5.8507 Gyrs      |

Table 5: The present values of cosmographic series coefficients \(\{H_0, q_0, j_0, s_0, l_0, m_0\}\) for the best fit values of \(\Omega_{m0}, \Omega_{\bar{m}0}, H_0, n\) with three observational data sets as mentioned in Table 1, 3, 4.

The equation \([39]\) shows the expression for the deceleration parameter (DP) \(q(t)\) in terms of cosmic time \(t\), and Figure 5a represents its geometrical behavior over redshift \(z\). One can see that \(q\) shows a signature-flipping (decelerating to accelerating phase) over its evolution with redshift at \(z_t = 0.6879, 0.4553, 0.7097\) along three data
sets, i.e., the universe expansion is in decelerating phase for $z > z_t$ and it is in accelerating phase for $z < z_t$ which is closed to the recent observational values. The DP $q$ varies over $(-1,0.5)$ and the present value of DP is estimated as $q_0 = -0.5211, -0.3366, -0.5561$ (mentioned in Table 5) which shows that our current universe is in accelerating phase and at a whole evolution of the model, it shows a transit phase universe model which is supported by the recent observations [1]-[10]. Figure 5b shows the geometrical behavior of best-fit Hubble function.

![Graph](image_url)

**Figure 5**: (a) The plot of deceleration parameter $q(z)$ and (b) the plot of Hubble function $H(z)$ versus cosmic redshift $z$ for the best fit values of $\Omega_m, \Omega_\beta, H_0, n$ along three data sets as mentioned in Table 1, 3, 4.

![Graph](image_url)

**Figure 6**: (a) The plot of jerk parameter $j(t)$ and (b) the plot of snap $s(z)$ versus cosmic redshift $z$ for the best fit values of $\Omega_m, \Omega_\beta, H_0, n$ along three data sets as mentioned in Table 1, 3, 4.
The next cosmographic coefficient is the jerk parameter \( j(t) \) represented by the Eq. (40), and its geometrical nature over redshift \( z \) is shown in Figure 6a. One can see that the value of the jerk parameter is always positive \( (j > 0) \), which indicates the occurrence of a transition time during which the universe modifies its expansion phase. The present value of jerk parameter in our model is estimated as \( j_0 = 0.9468, 0.8469, 0.9799 \) along three data sets which is supported by recent observations.

The next cosmographic coefficient is snap parameter \( s \) represented by the Eq. (41), and its geometrical behavior is shown in Figure 6b. The snap parameter shows the dark energy term or cosmological constant behavior of the model. The estimated present value of snap parameter is \( s_0 = -0.3106, -0.6178, -0.3157 \) along three data sets which is supported by \( \Lambda \)CDM values. The other cosmographic coefficients are lerk \( l(t) \), and max-out \( m(t) \), whose expression is given by the Eqs. (42) & (43), respectively, and their geometrical behavior is represented by Figures 7a & 7b respectively. The present values are estimated as \( l_0 = 2.854, 3.208, 2.896 \) and \( m_0 = -9.504, -13.25, -9.594 \) along three data sets and varies with cosmic redshift over the range \([1, 11.65]\) & \([-70.11, 1]\) respectively. Thus, one can see that as \( t \to \infty \) (or \( z \to -1 \)) then \( \{q, j, s, l, m\} \to \{-1, 1, 1, 1, 1\} \) which is a good feature of our derived model.

The equation (20) represents the expression for matter energy density in terms of scale factor and it may be shown in terms of redshift as \( \rho = \rho_0(1 + z)^{\frac{n-1}{2}} \). We see that as \( z \to 0 \) then \( \rho \to \rho_0 \) the present value.

### Age of the present universe

The age of the present universe can be calculated by

\[
t_0 - t = -\int_{t_0}^{t} dt = \int_{0}^{z} \frac{dz}{(1+z)H(z)}
\]

Using (23) in (44) and integrating, we get

\[
H_0(t_0 - t) = \frac{0.6923050296}{\sqrt{\Omega_{\beta 0}(2n-1)}} \left[ \tanh^{-1} \left( \frac{\sqrt{\Omega_{m0} + \Omega_{\beta 0}}}{\sqrt{\Omega_{\beta 0}}} \right) - \tanh^{-1} \left( \frac{\sqrt{\Omega_{m0}(1+z)^{2.8889} + \Omega_{\beta 0}}}{\sqrt{\Omega_{\beta 0}}} \right) \right]
\]

Figure 7: (a) The plot of lerk \( l(z) \) and (b) the plot of max-out \( m(z) \) versus cosmic redshift \( z \) for the best fit values of \( \Omega_{m0}, \Omega_{\beta 0}, H_0, n \) along three data sets as mentioned in Table 1, 3, 4.
Figure 8: The plot of cosmic time $H_0(t_0 - t)$ versus redshift $z$ for the best fit values of $\Omega_{m0}, \Omega_{\beta0}, H_0, n$ along three data sets as mentioned in Table 1, 3, 4.

Figure 8 shows the cosmic age of the universe over redshift $z$ and we have estimated the age of the present universe as $t_0 = 13.5922, 12.1960, 13.8768$ Gyrs respectively along three data sets which is very closed the the recent observational values in [62, 63, 64].

6 Conclusions

This research paper deals with a transit dark energy cosmological model in $f(R, L_m)$-gravity with observational constraints, where $R$ is Ricci scalar curvature and $L_m$ is matter Lagrangian for a perfect fluid. We have derived the field equations using a flat FLRW space-time metric and have found a relationship between energy density parameters $\Omega_{m0}, \Omega_{\beta0}$ through the Hubble function $H(z)$. We have approximated the present values of the energy parameters as $\Omega_{m0}, \Omega_{\beta0}$ and some other parameters by comparing observational and theoretical datasets using $R^2$ and $\chi^2$-test formula. The estimated present values of various parameters in the context of three data sets are as given in below Table 6:

| Parameter | For $H_0$ data | For SN Ia 715 data | For SN Ia 715 data plus 40 bined data |
|-----------|----------------|------------------|-------------------------------------|
| $\Omega_{m0}$ | 0.306973 | 0.40543 | 0.291379 |
| $\Omega_{\beta0}$ | 0.618952 | 0.44068 | 0.678478 |
| $n$ | 1.04 | 1.09094 | 1.01554 |
| $H_0$ | 68.9596 | 68.96 | 69.5047 |
| $q_0$ | -0.5211 | -0.3366 | -0.5561 |
| $j_0$ | 0.9468 | 0.8469 | 0.9799 |
| $s_0$ | -0.3106 | -0.6178 | -0.3157 |
| $l_0$ | 2.854 | 3.208 | 2.896 |
| $m_0$ | -9.504 | -13.25 | -9.594 |
| $t_0$ | 13.5922 Gyrs | 12.1960 Gyrs | 13.8768 |
| $z_t$ | 0.6879 | 0.4553 | 0.7097 |
| $t_r$ | 6.4912 Gyrs | 9.78 Gyrs | 5.8507 Gyrs |

Table 6: The estimated present values of various parameters used in the above derived model over $0 \leq z < 3$.  

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The main features of the model are as follows:

- The derived model shows a signature-flipping point at $z = z_t$ (i.e. at this point the expansion phase of the universe is changed from the decelerating to the accelerating phase). Also, the model is decelerating for $z > z_t$ and it is in accelerating for $z < z_t$ (see Table 6).

- The present value of deceleration parameter is measured as $q_0 = -0.5211, -0.3366, -0.5561$ along three data sets which indicates that our present universe expansion is in accelerating phase supported by [1]-[10].

- Also, we have measured the approximate present values of cosmographic series coefficients $\{H_0, q_0, j_0, s_0, l_0, m_0\}$ as mentioned in above Table 6.

- Thus, we have found that as $t \to \infty$ (or $z \to -1$) then $\{q, j, s, l, m\} \to \{-1, 1, 1, 1, 1\}$ which indicates that our model shows $\Lambda$CDM model in late-time universe.

- The models shows a dark energy model and the $\beta$ behaves just like dark energy candidate and model parameter $n$ as scaling parameter.

- The cosmic age of the present universe is approximated as $t_0 = 13.5922, 12.1960, 13.8768$ Gyrs respectively along three data sets.

- Our universe has been come in accelerating phase $t_r$ Gyrs ago from today and the values of $t_r$ are mentioned in above Table 6.

Thus our derived model shows good features of the recent cosmological models which will attract the researchers in this field for future investigations.

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