Isoperimetric inequalities for nilpotent groups

S. M. Gersten, D. F. Holt and T. R. Riley

January 16, 2002

Abstract

We prove that every finitely generated nilpotent group of class $c$ admits a polynomial isoperimetric function of degree $c+1$ and a linear upper bound on its filling length function.

1991 Mathematics Subject Classification: 20F05, 20F32, 57M07
Key words and phrases: nilpotent group, isoperimetric function, filling length

1 Introduction

The main result of this article is a proof of what has been known as the “$c + 1$-conjecture” (see [3] or 5A′′ of [12]).

Theorem A ($c + 1$-conjecture). Every finitely generated nilpotent group $G$ admits a polynomial isoperimetric inequality of degree $c+1$, where $c$ is the nilpotency class of $G$.

An isoperimetric inequality for a finite presentation $\mathcal{P} = \langle A \mid R \rangle$ of a group $G$ concerns null-homotopic words $w$, that is words that evaluate to the identity in $G$. It gives an upper bound (an “isoperimetric function”), in terms of the length of $w$, on the number of times one has to apply relators from $R$ to $w$ in a process of reducing it to the empty word. (More details are given in §2.) The following couple of remarks are important in making sense of the statement of Theorem [A]. Nilpotent groups are coherent, by which we mean that their finitely generated subgroups are finitely presentable. In particular, all finitely generated nilpotent groups have a finite presentation. Further (see §2) an isoperimetric function $f_{\mathcal{P}}(n)$ concerns a fixed finite presentation $\mathcal{P}$ for $G$; however if $\mathcal{Q}$ is another finite presentation for $G$ then there is an
isoperimetric function $f_Q(n)$ for $Q$ that is that satisfies $f_P \simeq f_Q$, where $\simeq$ is a well-known equivalence relation (see Definition 2.6). Moreover, if $f_P$ is polynomial of some given degree $\geq 1$, then we can always take $f_Q$ to be a polynomial of the same degree.

Our strategy will be to prove the $c+1$-conjecture by an induction on the class $c$. However we use an induction argument in which we keep track of more than an isoperimetric function. In fact, we prove the following stronger theorem.

**Theorem B.** Finitely generated nilpotent groups $G$ of class $c$ admit $(n^{c+1}, n)$ as an (Area, FL)-pair.

The terminology used above is defined carefully in §2, but essentially, what this theorem says is the following. Suppose $w$ is a null-homotopic word $w$ of length $n$ in some finite presentation for $G$. We can reduce $w$ to the empty word by applying at most $O(n^{c+1})$ relators from the presentation, and in such a way that in the process the intermediate words have length at most $O(n)$.

Let

$$G = \Gamma_1 > \Gamma_2 > \cdots > \Gamma_{c+1} = \{1\}$$

be the lower central series for $G$ defined inductively by $\Gamma_1 := G$ and $\Gamma_{i+1} := [G, \Gamma_i]$.

We will use a generating set $A$ for $G$ which will be a disjoint union of sets $A_i$, where, for each $i$, $A_i$ is a generating set for $\Gamma_i$ modulo $\Gamma_{i+1}$. For $A_1$, we take inverse images in $G$ of generators of cyclic invariant factors of the abelian group $G/\Gamma_2$. Then, inductively, for $i > 0$, we define $A_{i+1} := \{[x, y] \mid x \in A_1, y \in A_i\}$.

The idea of the proof of Theorem B is to start by reducing the word $w$ to the identity modulo $\Gamma_c$. The quotient $G/\Gamma_c$ is nilpotent of class $c$ and so, by induction hypothesis, we can reduce $w$ to the identity in a presentation for $G/\Gamma_c$, by applying $O(n^c)$ defining relators. When we carry out the corresponding reduction in a presentation for $G$ we introduce $O(n^c)$ extra generators from $A_c$. We use the definition of these generators as commutator words $z$ to compress their powers $z^m$ with $m = O(n^c)$ to words of length $O(n)$. This compression process is handled in Proposition 3.1 and Corollary 3.2, and we show that it can be accomplished by applying $O(n^{c+1})$ relators.
In addition to proving the $c+1$-conjecture, Theorem B yields the following corollary.

**Corollary B.1.** If $\mathcal{P}$ is a finite presentation for a nilpotent group then there exists $\lambda > 0$ such that the filling length function $\FL$ of $\mathcal{P}$ satisfies $\FL(n) \leq \lambda n$ for all $n \in \mathbb{N}$.

This result was proved by the third author in [17] via the rather indirect technique of using asymptotic cones: finitely generated nilpotent groups have simply connected asymptotic cones (see Pansu [15]) and groups with simply connected asymptotic cones have linearly bounded filling length. This article provides a direct combinatorial proof. Note, also, that the filling length function for a finite presentation $\mathcal{P}$ is an isodiametric function for $\mathcal{P}$. So we also have a direct combinatorial proof that finitely generated nilpotent groups admit linear isodiametric functions.

There is a computer science reinterpretation of Theorem B because Area and FL can be recognised to be measures of computational complexity in the following context.

Suppose $\mathcal{P}$ is a finite presentation for a group $G$. One can attempt to solve the word problem using a non-deterministic Turing machine as follows. Initially the input word $w$ of length $n$ is displayed on the Turing tape. A step in the operation of the machine is an application of a relator, a free reduction, or a free expansion (see §2). The machine searches for a *proof* that $w = 1$ in $G$ – that is, a sequence of steps that reduces $w$ to the empty word. The running time of a *proof* is the number of steps, and its space is the number of different entry squares on the Turing tape that are disturbed in the course of the *proof*.

The time $\Time(w)$ for a word $w$ such that $w = 1$ in $G$ is the minimum running time amongst *proofs* for $w$, and the time function $\Time : \mathbb{N} \rightarrow \mathbb{N}$ is defined by $\Time(n) := \max \{\Time(w) \mid \ell(w) \leq n \text{ and } w = 1 \text{ in } G\}$. Similarly we define the space function $\Space$ by setting $\Space(w)$ to be the minimal space amongst all proofs for $w$.

On comparing with the definitions of §2, we quickly recognise that the space function $\Space$ is precisely the same as the filling function $\FL$. Also the time function $\Time$ is closely related to the function Area. (In fact $\Time$ is precisely the function $\Height$ of Remark 2.5, where it is explained how $\Height$ relates to Area.) Thus we have the following corollary to Theorem B.
Corollary B.2. Given a finite presentation for a nilpotent group \( G \) of class \( c \), the non-deterministic Turing machine described above solves the word problem with Space(\( n \)) bounded above by a linear function of \( n \) and Time(\( n \)) bounded above by a polynomial in \( n \) of degree \( c+1 \). Moreover, given an input word \( w \) such that \( w = 1 \) in \( G \), there is a proof for \( w \) that runs within both these bounds simultaneously.

[Note that these complexity bounds are not the best possible. The torsion subgroup \( T \) of a finitely generated nilpotent group \( G \) is finite, and by a result of S.A. Jennings (see \([13]\) or Theorem 2.5 of \([18]\)), the torsion-free group \( G/T \) can be embedded in an upper unitriangular matrix group \( U \) over \( \mathbb{Z} \). It is easy to see that the integer entries of a matrix in \( U \) representing a word of length \( n \) in \( G/T \) are at most \( O(n^{d-1}) \), where \( d \) is the degree of the matrix. Using this and the fact that two \( k \)-digit numbers can be multiplied in time \( O(k(\log k)^2) \), we can obtain a deterministic solution to the word problem in \( G \) in time \( O(n(\log n)^2) \) and space \( O(\log n) \).

Even this is not the best possible result on the time complexity. In the unpublished manuscript \([4]\), Cannon, Goodman and Shapiro show that finitely generated nilpotent groups admit a Dehn algorithm if we adjoin some extra symbols to the group generators, and this leads to a linear solution of the word problem in \( G \).]

The proof of the \( c+1 \)-conjecture is the culmination of a number of results spanning the last decade or so. The first author proved in \([7]\) that \( G \) admits a polynomial isoperimetric inequality of degree \( 2h \), where \( h \) is the Hirsch length of \( G \). The degree was improved to \( 2 \cdot 3^c \) by G. Conner \([3]\), and then improved further to \( 2c \) by C. Hidber \([14]\).

We also mention Ch. Pittet \([16]\), who proved that a lattice in a simply connected homogeneous Lie group of class \( c \) admits a polynomial isoperimetric function of degree \( c+1 \). (The nilpotent Lie group is called homogeneous if its Lie algebra is graded.) Gromov also suggested a possible reduction to the homogeneous case by perturbing the structure constants \([12]\) 5.A5, but no-one has yet succeeded in carrying out this plan.

The isoperimetric inequality proved in this article is the best possible bound in terms of the class in general. For example if \( G \) is a free nilpotent group of class \( c \) then its minimal isoperimetric function (a.k.a. its Dehn function) is polynomial of degree \( c+1 \) (see \([2]\) or \([3]\) for the lower bound and \([10]\) for the upper bound). However it is not best possible for individual nilpotent groups: for example D. Alcock in \([1]\) gives the first proof that the
2n + 1 dimensional integral Heisenberg groups admit quadratic isoperimetric functions for \( n > 1 \); these groups are all nilpotent of class 2. By way of contrast, the 3-dimensional integral Heisenberg group has a cubic minimal isoperimetric polynomial (\[6\] and \[8\]).

We give two corollaries of Theorem A concerning the cohomology of groups and differential geometry respectively.

The first corollary is about the growth of cohomology classes. Let \( G \) be a finitely generated group with word metric determined by a finite set of generators and denoted \( d(1, g) = |g| \). Recall that a real valued 2-cocycle (for the trivial \( G \)-action on \( \mathbb{R} \)) on the bar construction is a function \( f : G \times G \to \mathbb{R} \) satisfying \( f(x, yz) + f(y, z) = f(x, y) + f(xy, z) \) for all \( x, y, z \in G \).

**Corollary A.1.** Let \( G \) be a finitely generated nilpotent group of class \( c \) and let \( \zeta \in H^2(G, \mathbb{R}) \). Then there is a 2-cocycle \( f \) in the class \( \zeta \) which satisfies \( f(x, y) \leq M(|x| + |y|)^{c+1} \) for all \( x, y \in G \) and constant \( M > 0 \).

The terminology (due to Gromov) of an isoperimetric function for a group is motivated by the analogous notion with the same name from differential geometry. One can draw parallels between van Kampen diagrams (see Remark 2.3) filling edge-circuits in the Cayley 2-complex associated to a finite presentation of a group and homotopy discs for rectifiable loops in the universal cover of a Riemannian manifold.

**Corollary A.2.** Let \( M \) be a closed Riemannian manifold whose fundamental group is nilpotent of class \( c \). Then there is a polynomial \( f \) of degree \( c + 1 \) such that for every rectifiable loop of length \( L \) in the universal cover \( \tilde{M} \), there is a singular disc filling of area at most \( f(L) \).

For the proof one considers a piecewise \( C_1 \)-map of the presentation complex \( K \) of a finite presentation of \( G \) into \( M \) and the induced map of universal covers \( \tilde{K} \to \tilde{M} \). An edge-circuit \( p \) in \( \tilde{K} \) defines a null-homotopic word \( w \) in \( \pi_1 M \), and a van Kampen diagram \( D \) for \( w \) gives a Lipschitz filling for \( p \) in \( \tilde{K} \). For a general Lipschitz loop \( p \) in \( \tilde{M} \), one uses Theorem 10.3.3 of \[4\] to homotop \( p \) by a Lipschitz homotopy to an edge-circuit in \( \tilde{K} \), which is then filling to realise the desired isoperimetric inequality for \( p \).
2 Isoperimetric functions and filling length functions

Let $\mathcal{P} = \langle A \mid R \rangle$ be a finite presentation for a group $G$. A word $w$ is an element of the free monoid $(A \cup A^{-1})^*$. Denote the length of $w$ by $\ell(w)$. We say $w$ is null-homotopic when $w = 1$ in $G$. For a word $w = a_1^{\varepsilon_1}a_2^{\varepsilon_2}\ldots a_s^{\varepsilon_s}$, where each $a_i \in A$ and each $\varepsilon_i = \pm 1$, the inverse word $w^{-1}$ is $a_s^{-\varepsilon_s}\ldots a_2^{-\varepsilon_2}a_1^{-\varepsilon_1}$.

**Definition 2.1.** A $\mathcal{P}$-sequence $S$ is a finite sequence of words $w_0, w_1, \ldots, w_m$ such that each $w_{i+1}$ is obtained from $w_i$ by one of three moves.

1. **Free reduction.** Remove a subword $aa^{-1}$ from $w_i$, where $a$ is a generator or the inverse of a generator.

2. **Free expansion.** Insert a subword $aa^{-1}$ into $w_i$, where $a$ is a generator or the inverse of a generator. So $w_{i+1} = uaa^{-1}v$ for some words $u, v$ such that $w_i = uv$ in $(A \cup A^{-1})^*$.

3. **Application of a relator.** Replace $w_i = \alpha u \beta$ by $\alpha v \beta$, where $uv^{-1}$ is a cyclic conjugate of one of the defining relators or its inverse.

We refer to $m$ as the **height** $\text{Height}(S)$ of $S$ and we define the **filling length** $\text{FL}(S)$ of $S$ by

\[ \text{FL}(S) := \max \{ \ell(w_i) \mid 0 \leq i \leq m \} . \]

The **area** $\text{Area}(S)$ of $S$ is defined to be the number of $i$ such that $w_{i+1}$ is obtained from $w_i$ by an application of a relator move.

If $w = w_0$ and $w_m$ is the empty word then we say that $S$ is a null-$\mathcal{P}$-sequence for $w$ (or, more briefly, just a “null-sequence” when $\mathcal{P}$ is clear from the context).

In this article we are concerned with two **filling functions**, that measure “area” and “filling length” – two different aspects of the geometry of the word problem for $G$.

**Definition 2.2.** Let $w$ be a null-homotopic word in $\mathcal{P}$.

We define the **area** $\text{Area}(w)$ of $w$ by

\[ \text{Area}(w) := \min \{ \text{Area}(S) \mid \text{null-}\mathcal{P}\text{-sequences } S \text{ for } w \} , \]
that is, the minimum number of relators that one has to apply to reduce $w$ to the empty word. Similarly, the **filling length** of $w$ is

$$FL(w) := \min \{FL(S) \mid \text{null-$ \mathcal{P}$-sequences } S \text{ for } w \}.$$ 

We define the **Dehn function** $\text{Area} : \mathbb{N} \to \mathbb{N}$ (also known as the **minimal isoperimetric function** and the **filling length function** $\text{FL} : \mathbb{N} \to \mathbb{N}$ by

$$\text{Area}(n) := \max \{\text{Area}(w) \mid \text{null-homotopic words } w \text{ with } \ell(w) \leq n\}$$
$$\text{FL}(n) := \max \{\text{FL}(w) \mid \text{null-homotopic words } w \text{ with } \ell(w) \leq n\}.$$

**Remark 2.3.** The formulations of the definitions above are those we will use in this article, but we mention some equivalent alternatives that occur elsewhere in the literature.

We could equivalently define $\text{Area}(w)$ to be the minimal number of 2-cells needed to construct a van Kampen diagram for $w$, or the minimal $N$ such that there is an equality $w = \prod_{i=1}^{N} u_i^{-r_i} u_i$ in the free group $F(A)$ for some $r_i \in \mathcal{R}^{\pm 1}$ and words $u_i$.

Similarly, $\text{FL}(w)$ can be defined in terms of **shellings** of van Kampen diagrams $D_w$ for $w$. A **shelling** of $D_w$ is a combinatorial null-homotopy down to the base vertex. The filling length of a shelling is the maximum length of the boundary loops of the diagrams one encounters in the course of the null-homotopy. The filling length of $D_w$ is defined to be the minimal filling length amongst shellings of $D_w$ and then $\text{FL}(w)$ can be defined to be

$$\min \{\text{FL}(D_w) \mid \text{van Kampen diagrams } D_w \text{ for } w\}.$$ 

For detailed definitions and proofs we refer the reader to [10].

The following is a technical lemma that we will use in §3.

**Lemma 2.4.** Let $S$ be a $\mathcal{P}$-sequence $w_0, w_1, \ldots, w_m$ as defined above. Let $C := \max \{\ell(r) \mid r \in \mathcal{R}\}$. There is another $\mathcal{P}$-sequence $\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_m$, which we will call $\hat{S}$, such that

- $\hat{w}_0 = w_0, \hat{w}_m = w_m$
- $\text{Area}(\hat{S}) = \text{Area}(S)$
- $\text{FL}(\hat{S}) \leq \text{FL}(S) + C$
and such that every time \( \hat{w}_{i+1} \) is obtained from a word \( \hat{w}_i \) in the sequence \( \hat{S} \) by an application of a relator the whole of a cyclic conjugate of an element of \( \mathcal{R} \) or its inverse is inserted into \( \hat{w}_i \).

Proof. Suppose that \( w_{i+1} \) is obtained from a word \( w_i \) in the sequence \( S \) by an application of a relator: that is, \( w_i = \alpha u \beta \) and \( w_{i+1} = \alpha v \beta \) for some words \( \alpha, \beta, u, v \) where \( uv^{-1} \) is a cyclic conjugate some element of \( \mathcal{R} \pm 1 \). We can obtain \( w_{i+1} \) from \( w_i \) by inserting \( u^{-1}v \) into \( w_i = \alpha u \beta \) to get \( \alpha u u^{-1}v \beta \) and then using at most \( C \) free reductions to retrieve \( \alpha v \beta \).

Remark 2.5. We mention that the area of a null-homotopic word \( w \) is closely related to the height of its null-sequences. Define

\[
\text{Height}(w) := \min \{ \text{Height}(S) \mid \text{null-sequences } S \text{ for } w \}.
\]

Then for all null-homotopic words \( w \),

\[
\text{Area}(w) \leq \text{Height}(w) \leq (C + 1) \text{Area}(w) + \ell(w),
\]

where \( C := \max \{ \ell(r) \mid r \in \mathcal{R} \} \).

The inequality \( \text{Area}(w) \leq \text{Height}(w) \) follows from the definitions since free reductions and expansions do not contribute to the area. To obtain the inequality \( \text{Height}(w) \leq (C + 1) \text{Area}(w) + \ell(w) \) first take a van Kampen diagram \( D_w \) for \( w \) with \( \text{Area}(D_w) = \text{Area}(n) \). Take any shelling

\[
D_w = D_0, D_1, \ldots, D_m = \ast
\]

of \( D_w \) down to its base vertex \( \ast \) in which each \( D_{i+1} \) is obtained from \( D_i \) by a 2-cell collapse or a 1-cell collapse (but never a 1-cell expansion) and let \( w_j \) be the boundary word of \( D_j \). Then \( w_0, w_1, \ldots, w_m \) is a null-sequence for \( w \), where \( w_{i+1} \) is obtained from \( w_i \) by applying of a relator if \( D_{i+1} \) is obtained from \( D_i \) by a 2-cell collapse, and by a free reduction if \( D_{i+1} \) is obtained from \( D_i \) by a 1-cell collapse. We obtain the required inequality by observing that the number of 2-cell collapse moves in the shelling is \( \text{Area}(w) \) and the total number of 1-cell collapse moves is at most the total number if 1-cells in the 1-skeleton of \( D_w \), which is at most \( C \text{Area}(w) + \ell(w) \).

An isoperimetric inequality for \( \mathcal{P} \) is provided by any function \( f : \mathbb{N} \to \mathbb{N} \) such that \( \text{Area}(n) \leq f(n) \) for all \( n \). We refer to \( f \) as an isoperimetric function for \( \mathcal{P} \).
It is important to note that \( \text{Area} : \mathbb{N} \to \mathbb{N} \) and \( \text{FL} : \mathbb{N} \to \mathbb{N} \) are both defined for a fixed finite presentation. However each function is a group invariant in the sense that each is well behaved under change of finite presentation as we now explain.

**Definition 2.6.** For two functions \( f, g : \mathbb{N} \to \mathbb{N} \) we say that \( f \preceq g \) when there exists \( C > 0 \) such that \( f(n) \leq C g(Cn + C) + Cn + C \) for all \( n \), and we say \( f \simeq g \) if and only if \( f \preceq g \) and \( g \preceq f \).

Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two presentations of the same group \( G \), let \( \text{Area}_\mathcal{P} \) and \( \text{Area}_\mathcal{Q} \) be their Dehn functions, and let \( \text{FL}_\mathcal{P} \) and \( \text{FL}_\mathcal{Q} \) be their filling length functions. Then it is proved in [11] that \( \text{Area}_\mathcal{P} \simeq \text{Area}_\mathcal{Q} \) and in [10] that \( \text{FL}_\mathcal{P} \simeq \text{FL}_\mathcal{Q} \).

**Definition 2.7.** We say that a pair \( (f, g) \) of functions \( \mathbb{N} \to \mathbb{N} \) is an \((\text{Area}, \text{FL})\)-pair for \( \mathcal{P} \) when there exists a constant \( \lambda > 0 \) such that for any null-homotopic word \( w \), there exists a null-\( \mathcal{P} \)-sequence \( S \) with

\[
\begin{align*}
\text{Area}(S) &\leq \lambda f(\ell(w)) \\
\text{FL}(S) &\leq \lambda g(\ell(w)).
\end{align*}
\]

The first part of the following lemma justifies the deduction of Theorem [A] and Corollary [B.1] from Theorem [B], and second part justifies the terminology used in the statement of Theorem [B].

**Lemma 2.8.** If \((f, g)\) is an \((\text{Area}, \text{FL})\)-pair for \( \mathcal{P} \) then \( f \) is an isoperimetric function for \( \mathcal{P} \) and \( g \) is an upper bound for its filling length function.

Moreover, if \((n^\alpha, n^\beta)\) is an \((\text{Area}, \text{FL})\)-pair for \( \mathcal{P} \) where \( \alpha, \beta \geq 1 \), then \((n^\alpha, n^\beta)\) is also an \((\text{Area}, \text{FL})\)-pair for any other finite presentation \( \mathcal{Q} \) of the same group \( G \).

**Proof.** The first part of the lemma follows immediately from the definitions.

From the proofs given in [10] and [11] it follows that if \((f, g)\) is an \((\text{Area}, \text{FL})\)-pair for \( \mathcal{P} \), then \((f', g')\) is an \((\text{Area}, \text{FL})\)-pair for \( \mathcal{Q} \), where \( f'(n) = Cf(Cn) + Cn \) and \( g'(n) = Cg(Cn) + Cn \) for some \( C > 0 \); here we have eliminated the additive constants in the definition of the equivalence relation \( \simeq \) by making use of the fact that the empty word is the only one of length 0.

But if \( f(n) = n^\alpha \) with \( \alpha \geq 1 \), then \( Cf(Cn) + Cn \leq C' n^\alpha \) for suitable constant \( C' > 0 \). It follows that if \((n^\alpha, n^\beta)\) is an \((\text{Area}, \text{FL})\)-pair for \( \mathcal{P} \), then it is also an \((\text{Area}, \text{FL})\)-pair for \( \mathcal{Q} \). 

3 Proof of Theorem B

We say that an element of a nilpotent group $G$ has weight $k$ if it lies in $\Gamma_k \setminus \Gamma_{k+1}$, where $\{\Gamma_i\}$ is the lower central series of $G$, as defined in §1. Before we come to the proof of Theorem B we give a proposition and corollary concerning compressing powers of elements of weight $c$ in a finite presentation for a nilpotent group of class $c$.

We use the following conventions for commutator words:

$[a] := a$
$[a, b] := a^{-1} b^{-1} a b$
$[a_1, a_2, \ldots, a_{c-1}, a_c] := [a_1, [a_2, \ldots, [a_{c-1}, a_c]\ldots]]$.

Let $X = \{x_1, x_2, \ldots, x_c\}$ be an alphabet and for $k = 1, 2, \ldots, c$ define

$X_k := \{x_k, x_{k+1}, \ldots, x_c\}$
$R_k := \{[y_k, y_{k+1}, \ldots, y_{c+1}] \mid y_j \in X_k^{\pm 1}, k \leq j \leq c + 1\}$.

Let $P_k := \langle X_k \mid R_k \rangle$, which is a finite presentation for a free nilpotent group $G_k$ of class $c + 1 - k$. Define $z_k$ to be the word $[x_k, x_{k+1}, \ldots, x_c]$. In particular, we have $z_c = x_c$.

One can regard the first part of the following proposition as comparing the use of commutators in a free nilpotent group with the representation of positive integers $s$ by their $n$-ary expansion $s = s_0 + s_1 n + s_2 n^2 + \ldots$. The second part gives upper bounds on the area and filling length of a $P$-sequence between the commutators corresponding to the $n$-ary representations for $s$ and $s + 1$.

**Proposition 3.1.** Suppose $c \in \mathbb{N} \setminus \{0\}$ and $s, n \in \mathbb{N}$ with $0 \leq s \leq n^c - 1$. Express $s$ as a sum:

$s = s_0 + s_1 n + \ldots + s_{c-1} n^{c-1},$

where each $s_i \in \{0, 1, \ldots, n - 1\}$. Then, with the notation above, the word

$\tilde{z}_1^s := z_1^{s_0} [x_1^n, z_2^{s_1} [x_2^n, \ldots, z_{c-1}^{s_{c-2}} [x_{c-1}^n, z_c^{s_{c-1}}] \ldots]]$

equals $z_1^s$ in $G_1$. Also

$\tilde{z}_1^{n^c} := [x_1^n, x_2^n, \ldots, x_c^n]$.
equals $z_1^{n^c}$ in $G_1$.

Moreover, there exists $\kappa > 0$ such that for $0 \leq s \leq n^c - 1$ we can transform $\tilde{z}_1 \tilde{s}$ to $\tilde{z}_1^{s+1}$ via a $P_1$-sequence of filling length at most $\kappa n$ and of area at most $\kappa n^{k+1}$ where $k$ is the integer such that $n^k | (s + 1)$ but $n^{k+1} \not| (s + 1)$.

Proof. Our proof is by induction on the integer $c$. The base case of $c = 1$ is straightforward. We just take $\tilde{z}_1^s := z_1^s = x_1^s$ and $\kappa := 1$.

We now prove the induction step. Suppose $c \in \mathbb{N} \setminus \{0, 1\}$. Since the free reduction of $\tilde{z}_1^0$ is the empty word, we have $\tilde{z}_1^0 = 1$ in $G$. It therefore suffices to show that for $s, n \in \mathbb{N}$ with $1 \leq s \leq n^c - 1$, there is a $P_1$-sequence from $\tilde{z}_1 \tilde{s}$ to $\tilde{z}_1^{s+1}$ within the required filling length and area bounds.

Express $s$ as a sum:

$$s = s_0 + s_1 n + \ldots + s_{c-1} n^{c-1},$$

where each $s_i \in \{0, 1, \ldots, n - 1\}$. Define

$$t := s_1 + s_2 n + \ldots + s_{c-2} n^{c-2}.$$ 

Then $\tilde{z}_1^s = z_1^{s_0} [x_1^n, \tilde{z}_2^t]$ as words.

If $s_0 + 1 < n$ then $\tilde{z}_1 \tilde{s} = \tilde{z}_1^{s+1}$ as words and there is a trivial $P_1$-sequence from $\tilde{z}_1 \tilde{s}$ to $\tilde{z}_1^{s+1}$. If $s_0 + 1 = n$ then we calculate that $\tilde{z}_1 \tilde{s} = \tilde{z}_1^{s+1}$ in $G_1$ as follows:

$$\tilde{z}_1 \tilde{s} = z_1^{s_0} [x_1^n, \tilde{z}_2^t] \quad (1)$$

$$= z_1^n x_1^{-n} (\tilde{z}_2^t)^{-1} x_1^n \tilde{z}_2^t \quad (2)$$

$$= z_1^n x_1^{-n} (\tilde{z}_2^t)^{-1} z_2^{-1} z_2 x_1^n \tilde{z}_2^t \quad (3)$$

$$= x_1^{-n} (\tilde{z}_2^t)^{-1} z_2^{-1} x_1^n z_2 \tilde{z}_2^t \quad (4)$$

$$= [x_1^n, \tilde{z}_2^t] \quad (5)$$

$$= [x_1^n, \tilde{z}_2^{t+1}] \quad (6)$$

$$= \tilde{z}_1^{s+1}.$$ \quad (7)

In the step from (3) to (4) we use the fact that $z_1 = [x_1, z_2]$ and is central. For the step from (5) to (6) we invoke the induction hypothesis to tell us that $\tilde{z}_2 \tilde{z}_2^t = \tilde{z}_2^{t+1}$ in $G_2$, from which it follows that $\tilde{z}_2 \tilde{z}_2^t (\tilde{z}_2^{t+1})^{-1}$ is central in $G_1$ and $[x_1^n, \tilde{z}_2 t^t] = [x_1^n, \tilde{z}_2^{t+1}]$. 
The course of the calculation above dictates how to construct a \( P_1 \)-sequence from \( z_1 \sim z_1^s \) to \( z_1^{s+1} \). The steps that require some further explanation are those at which application of relator moves are used: that is, from (3) to (4) and from (5) to (6).

The step from (3) to (4) is performed by a \( P_1 \)-sequence in which we introduce \( n \) commutator words \( z_1^{-1} = [x_1, z_2]^{-1} \) to move \( z_2 \) past \( x_1^n \). After a word \( z_1^{-1} \) is introduced it is immediately moved through the word using relations from \( R_1 \) and is then cancelled with a \( z_1 \). The number of letters each \( z_1 \) has to be moved past is bounded by \( n \) up to a multiplicative constant that depends only on \( P_1 \) (since \( \ell(\sim z_2^t) \leq \kappa n \), where \( \kappa > 0 \) is a constant depending only on \( P_2 \)). So the area of this \( P_1 \)-sequence is at most \( n^2 \) and its filling length is at most \( n \) up to a multiplicative constant.

Now let us explain how to construct a \( P_1 \)-sequence for the step from (5) to (6). Suppose that \( n^k \mid (s + 1) \) but \( n^{k+1} \nmid (s + 1) \). Then \( n^{k-1} \mid (t + 1) \) but \( n^k \nmid (t + 1) \). By induction hypothesis there is a \( P_2 \)-sequence \( S \) from \( z_2 \sim z_2^t \) to \( \sim z_2^{t+1} \) with area at most \( \kappa n^k \) and filling length at most \( \kappa n \).

Lemma 2.4 allows us to assume (by suitably adjusting the constant \( \kappa \)) that in every instance of an application of a relator move in the sequence \( S \) a whole cyclic conjugate of a relator is inserted.

We now explain how to use \( S \) to induce a \( P_1 \)-sequence that transforms \( [x_1^n, z_2 \sim z_2^t] \) into \( [x_1^n, \sim z_2^{t+1}] \). First define \( S' \) to be the \( P_2 \)-sequence that transforms \( (\sim z_2^t)^{-1} z_2^{-1} \) to \( (\sim z_2^{t+1})^{-1} \) and is obtained by inverting every word in \( S \). Now

\[
[x_1^n, z_2 \sim z_2^t] = x_1^{-n} (\sim z_2^t)^{-1} z_2^{-1} x_1^n z_2 \sim z_2^t.
\]

Consider running the \( P_2 \)-sequences \( S \) and \( S' \) concurrently on the subwords \( z_2 \sim z_2^t \) and \( (\sim z_2^t)^{-1} z_2^{-1} \) respectively in \( [x_1^n, z_2 \sim z_2^t] \): that is, we do the first move in \( S \) and then the first move in \( S' \), then the second move in \( S \) and then the second move in \( S' \), and so on. However we want to construct a \( P_1 \)-sequence, not a \( P_2 \)-sequence.

Suppose a move in \( S \) is the insertion of a word \( r \in R_2 \). Then the corresponding move in \( S' \) inserts the word \( r^{-1} \). Use free expansion moves to insert the word \( r^{-1}r \) in the place where \( S \) dictated that \( r \) was to be inserted; then use relators in \( R_1 \) to move \( r^{-1} \) through the word to the place where \( S' \) dictated \( r^{-1} \) was to be inserted. (Recall that \( r \) represents a central element of \( G_1 \) and the appropriate commutator to move it past letters from \( X_1 \) are in \( R_1 \).)
The number of relators from $R_1$ that have to be applied to move each $r^{-1}$ to its appropriate place is at most $n$ up to a multiplicative constant. It follows that we can find a constant $\kappa > 0$ such that there is a $P_1$-sequence that transforms the word $z_1 \tilde{z}_1^s$ into $z_1^{s+1}$, with area at most $\kappa n^{k+1}$ and filling length at most $\kappa n$. This completes the proof.

Recall from §1 that we define a set of generators $A$ for a finitely generated nilpotent group $G$ of class $c$, as follows. The set $A$ is a disjoint union of sets $A_i$, where, for each $i$, $A_i$ is a generating set for $\Gamma_i$ modulo $\Gamma_{i+1}$. For $A_1$, we take inverse images in $G$ of generators of cyclic invariant factors of the abelian group $G/\Gamma_2$. Then, inductively, for $i > 0$, we define $A_{i+1} := \{[x, y] \mid x \in A_1, y \in A_i\}$.

**Corollary 3.2.** Let $G$ be a finitely generated nilpotent group of class $c$, and let $A$ be the generating set defined above. Then, for any finite presentation $P = \langle A \mid R \rangle$ for $G$, there is a constant $\xi > 0$ depending only on $P$ with the following properties.

Let $n \in \mathbb{N}$. For each $z \in A_c$ there are “compression words” $\tilde{z}^0, \tilde{z}^1, \tilde{z}^2, \ldots, \tilde{z}^n$ such that for $0 \leq s \leq n^c$ there is an equality $\tilde{z}^s = z^s$ in $G$. Moreover, we can transform each $z \tilde{z}_s$ to $\tilde{z}_s^{s+1}$ via a $P$-sequence, which when all are concatenated:

$$z^n \rightarrow z^n \tilde{z}^0 \rightarrow z^n \tilde{z}^1 \rightarrow z^n \tilde{z}^2 \rightarrow \ldots \rightarrow z \tilde{z}^{n^c-1} \rightarrow \tilde{z}^{n^c},$$

gives a $P$-sequence that converts $z^{n^c}$ to $\tilde{z}^{n^c}$, and has area at most $\xi n^{c+1}$ and filling length at most $\xi n$.

**Proof.** First observe that if we can prove the corollary for any one finite set of relations $R$ then it holds true for any finite set of relations.

Since the free reduction of $z^0$ is the empty word, we can transform $z^{n^c}$ to $z^{n^c} \tilde{z}^{0}$ by free expansion moves.

By definition, each $z \in A_c$ is a commutator of length $c$. Take any $R$ such that $P = \langle A \mid R \rangle$ is a finite presentation for $G$ and $R$ includes all commutators involving $A^{\pm 1}$ of length $c + 1$. So the presentation for a free nilpotent group used in Proposition [3.1] is a subpresentation of $P$ and we can invoke that proposition to give us the compression words $\tilde{z}^s$ and the $P$-sequence from $z \tilde{z}^s$ to $\tilde{z}^{s+1}$, for each $s$.

The number of $s \in \{0, 1, \ldots, n^c - 1\}$ such that $n^k \mid (s + 1)$ is at most $n^{c-k}$. For each such $s$ the area of the sequence from $z \tilde{z}^s$ to $\tilde{z}^{s+1}$ is at most
\( \kappa n^{k+1} \) by Proposition 3.1. So, taking \( \xi := (c + 1)\kappa \), the total area of the \( \mathcal{P} \)-sequence in question is at most \( \xi n^{c+1} \).

Before we come to the proof of Theorem B we give two technical lemmas.

**Lemma 3.3.** Suppose that the abelian group \( H \) is generated by the finite set \( \mathcal{Y} \). Then there is a subset of \( \mathcal{Y} \) that freely generates a free abelian group having finite index in \( H \).

**Proof.** This is by induction on \( |\mathcal{Y}| \). There is nothing to prove if \( |\mathcal{Y}| = 0 \). If \( |\mathcal{Y}| > 0 \), let \( \mathcal{Y} = \mathcal{Y}_1 \cup \{y\} \) with \( |\mathcal{Y}_1| = |\mathcal{Y}| - 1 \), and let \( \mathcal{Y}_1 \) generate \( K \). By inductive hypothesis, \( \mathcal{Y}_1 \) has a subset \( \mathcal{Y}_2 \) that freely generates a free abelian subgroup \( L \) with \( |K : L| \) finite. If \( |H : L| \) is finite then we are done. Otherwise \( H/K \) is infinite cyclic and generated by \( yK \), and then \( \mathcal{Y}_2 \cup \{y\} \) has the required property.

**Lemma 3.4.** Suppose the finite group \( H \) of order \( t \) is generated by \( \mathcal{Y} \). Then any word \( w \in (\mathcal{Y} \cup \mathcal{Y}^{-1})^* \) with \( \ell(w) \geq t \) contains a subword \( u \) with \( u = 1 \) in \( H \).

**Proof.** Let \( w_i \) be the prefix of \( w \) of length \( i \), for \( 0 \leq i \leq \ell(w) \). Then, if \( \ell(w) \geq t \), there exist \( i, j \) with \( 0 \leq i < j \leq \ell(w) \) and \( w_i = w_j \) in \( H \), and then \( w_j = w_i v \), where \( v \) is the required subword of \( w \).

We now give the proof of Theorem B. The following argument is expressed in terms of null-\( \mathcal{P} \)-sequences, however the reader familiar with van Kampen diagrams will be able to reinterpret the ideas in geometric terms.

**Proof of Theorem B.** It suffices to find any one finite presentation \( \mathcal{P} \) for \( G \) which admits the required (Area, FL)-pair.

Let \( \mathcal{A} \) be the generating set for \( G \) defined both in §1 and before Corollary 3.2. Define \( \overline{G} := G / \Gamma_c \), which is a finitely generated nilpotent group of class \( c - 1 \).

Suppose \( \mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle \) is a finite presentation for \( G \). Then define \( \overline{\mathcal{P}} = \langle \overline{\mathcal{A}} \mid \overline{\mathcal{R}} \rangle \) to be the finite presentation for \( \overline{G} \) in which \( \overline{\mathcal{A}} \) is obtained from \( \mathcal{A} \) by removing all generators of weight \( c \), and \( \overline{\mathcal{R}} \) is obtained from \( \mathcal{R} \) by deleting all occurrences of these generators from the words in \( \mathcal{R} \).
We prove that \((n^{c+1}, n)\) is an \((\text{Area}, \text{FL})\)-pair for \(G\) by induction on \(c\). The base case \(c = 1\) concerns abelian groups. The process of moving letters through words, collecting together and cancelling every instance of each particular generator, gives the required \((\text{Area}, \text{FL})\)-pair of \((n^2, n)\).

We now prove the induction step. Suppose \(w\) is a null-homotopic word in a finite presentation \(\mathcal{P}\) for \(G\). Define \(n := \ell(w)\). Let \(\overline{w}\) be the word obtained from \(w\) by removing all occurrences of generators in \(\mathcal{A}_c\). By induction hypothesis, \(\overline{\mathcal{P}}\) admits \((n^c, n)\) as an \((\text{Area}, \text{FL})\)-pair. Thus there is a constant \(\bar{\lambda}\) depending only on \(\overline{\mathcal{P}}\) such that there is a null-\(\overline{\mathcal{P}}\)-sequence \(S\):

\[
\overline{w} = \overline{w}_0, \overline{w}_1, \ldots, \overline{w}_m = 1
\]

for \(\overline{w}\) with \(\text{Area}(\overline{S}) \leq \bar{\lambda} n^c\) and \(\text{FL}(\overline{S}) \leq \bar{\lambda} n\). Our intention is to produce a null-\(\mathcal{P}\)-sequence \(S\) for \(w\) from \(\overline{S}\) that demonstrates that \(G\) admits \((n^{c+1}, n)\) as an \((\text{Area}, \text{FL})\)-pair.

Now \(\mathcal{A}_c\) generates the abelian group \(\Gamma_c\), and it follows from Lemma 3.3 above, that we can write \(\mathcal{A}_c = \mathcal{A}_{c1} \cup \mathcal{A}_{c2}\) where \(\mathcal{A}_{c2}\) freely generates a free abelian group \(K\) having finite index \(t\) in \(\Gamma_c\). Let \(\mathcal{A}_{c2} = \{z_1, z_2, \ldots, z_k\}\). For \(q = 0, 1, \ldots, n^c\) let \(\tilde{z}_j^q\) denote the compression word for \(z_j^q\) of Corollary 3.2. Extend this sequence of compression words to all non-negative powers of \(z_j\) by defining

\[
\tilde{z}_j^{A+Bn^c} := \tilde{z}_j^A (\tilde{z}_j^{n^c})^B
\]

for \(0 \leq A \leq n^c - 1\) and \(B \geq 0\).

The elements of \(\mathcal{A}_c\) are all central in \(G\), so we may as well assume that \(\mathcal{R}\) includes the commutators \([x, z]\) for all \(x \in \mathcal{A}^{\pm 1}\) and \(z \in \mathcal{A}_c^{\pm 1}\).

Lemma 2.4 tells us that, by suitably adjusting the constant \(\bar{\lambda}\), we can assume that each time a word \(w_{i+1}\) is obtained from a word \(w_i\) in the sequence \(\overline{S}\) by applying a relator \(r \in \mathcal{R}\), the whole of some cyclic conjugate of \(r^{\pm 1}\) is inserted into \(\overline{w}_i\).

Suppose a word \(u\) of length at most \(t\) in the letters \(\mathcal{A}_{c1}^{\pm 1}\) represents an element of \(K\). Then we can choose a word \(v_u\) in the letters \(\mathcal{A}_{c2}^{\pm 1}\) such that \(u = v_u\) in \(G\). There are only finitely many words of length at most \(t\) so we may as well assume that each \(uw_u^{-1}\) is in \(\mathcal{R}\).

We now explain how to find a null-\(\mathcal{P}\)-sequence for \(w\), by expanding \(\overline{S}\). Suppose that the move from \(\overline{w}_i\) to \(\overline{w}_{i+1}\) is an instance of an application of a relator move in the null-\(\overline{\mathcal{P}}\)-sequence \(\overline{S}\). So \(\overline{w}_i = \alpha \beta\) for some words \(\alpha\) and \(\beta\). Also \(\overline{w}_{i+1} = \alpha r \beta\), where some cyclic conjugate of \(r\) is in \(\mathcal{R}^{\pm 1}\). There is some \(r\) which has a cyclic conjugate in \(\mathcal{R}^{\pm 1}\) and from which one can obtain
by removing all occurrences of generators in \(A_{c}^{\pm 1}\). The idea is to apply an expanded version of the move: instead of inserting \(\mathcal{P}\), we insert \(r\). The newly introduced generators from \(A_{c}^{\pm 1}\) are then collected at the ends of the word and compressed before the \(\mathcal{P}\)-sequence is continued as dictated by \(\overline{\mathcal{P}}\).

More precisely, we construct a null-\(\mathcal{P}\)-sequence \(S\) for \(w\) as follows. Amongst the words in \(S\) will be words of the form

\[
  w_i = (\widetilde{z_k q_i})^{-1} \ldots (\widetilde{z_2 q_i})^{-1} (\widetilde{z_1 q_i})^{-1} \overline{w} \ u_i \ \widetilde{z_1 q_i}^+ \ \widetilde{z_2 q_i}^+ \ldots \ \widetilde{z_k q_i}^+,
\]

for \(0 \leq i \leq m\). In these words each \(q_j^+\) and \(q_j^-\) is a non-negative integer, and \(u_i\) is a word of length less than \(t\) in the generators \(A_{c}^{\pm 1}\).

The \(\mathcal{P}\)-sequence \(S\) starts by transforming \(w\) to \(w_0\) by moving all generators from \(A_{c}^{\pm 1}\), through \(w_i\) using commutator relations (recall that we assumed these to be in \(R\)): move all occurrences of \(z_1, z_2, \ldots, z_k\) towards the (right-hand) end of the word, all occurrences of \(z_1^{-1}, z_2^{-1}, \ldots, z_k^{-1}\) towards the start of the word, and all generators from \(A_{c}^{\pm 1}\) towards the right to form the subword \(u_0\). These generators from \(A_c\) are all collected and compressed as we now explain.

When a letter \(z_j\) is moved towards the (right-hand) end of a word it arrives at some subword word \(\widetilde{z_j q}\), where \(q \geq 0\). We then transform

\[
  z_j \ \widetilde{z_j q} \rightarrow \ \widetilde{z_j q+1}
\]

via a \(\mathcal{P}\)-sequence \(S_{jq}\) of Corollary 3.2. Similarly when a \(z_j^{-1}\) is moved towards the start of the word it meets some \((\widetilde{z_j q})^{-1}\), where \(q \geq 0\), and we then make the transformation

\[
  (\widetilde{z_j q})^{-1} \ z_j^{-1} \rightarrow (\widetilde{z_j q+1})^{-1}
\]

using the \(\mathcal{P}\)-sequence obtained by inverting every word in \(S_{jq}\).

The letters from \(A_{c}^{\pm 1}\) are collected together at the appropriate position in \(w_0\). However in the process of collection, if we create a subword that represents an element of \(K\) we immediately replace this subword by a word in \(A_{c}^{\pm 1}\) (which, by assumption, we can do using a relator in \(R\)). The resulting letters from \(A_{c}^{\pm 1}\) are then moved and collected in their compression words in the manner already explained above. It follows from Lemma 3.4 that the collected words in the letters \(A_{c}^{\pm 1}\) always has length less than \(t\).

The \(\mathcal{P}\)-sequence \(S\) continues with a concatenation of \(\mathcal{P}\)-sequences that transform \(w_i\) to \(w_{i+1}\) for \(i = 0, 1, \ldots, m - 1\). The sequence from \(w_i\) to
\( w_{i+1} \) starts by transforming the subword \( w_i \) as dictated by \( \overline{S} \): if the move is a free expansion or a free reduction then this move is performed and we immediately have \( w_{i+1} \). However if the move is an application of a relator then we apply the \emph{expanded version}. This introduces letters from \( \mathcal{A}_{c \pm 1} \). Move the \( z_1, z_2, \ldots, z_k \) towards the end of the word, the occurrences of \( z_1^{-1}, z_2^{-1}, \ldots, z_k^{-1} \) towards the start of the word, and all generators from \( \mathcal{A}_{c \pm 1} \) to the start of the subword \( u_i \), and \emph{compress} in the manner explained above.

Now
\[
\overline{w_m} = (\overline{z_k q_{1 \pi m}})^{-1} \cdots (\overline{z_2 q_{2 \pi m}})^{-1} \overline{(z_1 q_{1 \pi m})^{-1} u_m z_1 q_{1 \pi m} z_2 q_{2 \pi m} \cdots z_k q_{k \pi m}},
\]
is a null-homotopic word in \( \Gamma_c \), and so in fact \( u_m \) must be the empty word (else it would represent an element of \( K \), and hence it would have already been eliminated from \( w_m \) by the procedure above). So \( q_{j \pi m}^+ = q_{j \pi m}^- \) for \( j = 1, 2, \ldots, k \) and we can complete our \( \mathcal{P} \)-sequence for \( w \) by freely reducing \( u_m \) to the empty word.

All that remains is to explain why there is a constant \( \lambda \), depending only on \( \mathcal{P} \), such that
\[
\text{Area}(S) \leq \lambda n^{c+1} \\
\text{FL}(S) \leq \lambda n.
\]

Let \( M \) be the maximum number of letters from \( \mathcal{A}_{c \pm 1} \) occurring in any one of the relators in \( \mathcal{R} \). At most \( M \text{Area}(S) \) letters from \( \mathcal{A}_{c \pm 1} \) are collected and compressed in the subwords \( u_i \). In the compression process one relator may be used for each of these letters, in the process releasing further letters from \( \mathcal{A}_{c \pm 1} \) to be collected.

Letters from \( \mathcal{A}_{c \pm 1} \) are released in \( S \) on account of each application of a relator in \( \overline{S} \) and also each relator used when compressing the word that collects letters from \( \mathcal{A}_{c \pm 1} \). Thus the powers \( q_{ji}^+ \) and \( q_{ji}^- \) are all bounded by \( 2M \text{Area}(S) \).

Corollary \ref{cor:area} therefore tells us that the length of the compression words for powers of elements of \( \mathcal{A}_{c} \) is \( O(n) \). Moreover the total contribution to the area of \( S \) arising in the compression process is \( O(n^{c+1}) \).

It follows that there is a constant \( \lambda' > 0 \), depending only on \( \mathcal{P} \), such that \( \text{FL}(S) \leq \lambda' n \). For the bound on \( \text{Area}(S) \) notice that the observations above
tell us that the total contributions on account of the compression process is $O(n^{c+1})$, and we need only bound the contribution made by our use of commutators to move letters through words. But each of the $O(n^c)$ letters from $A_{c±1}$ that arise in $S$ is moved a distance at most $\lambda' n$. So indeed, we can find $\lambda \geq \lambda'$, depending only on $P$, such that $\text{Area}(S) \leq \lambda n^{c+1}$ and $\text{FL}(S) \leq \lambda n$. ■

References

[1] D. Allcock. An isoperimetric inequality for the Heisenberg groups. *Geom. Funct. Anal.*, 8(2):219–233, 1998.

[2] G. Baumslag, C. F. Miller, III, and H. Short. Isoperimetric inequalities and the homology of groups. *Invent. Math.*, 113(3):531–560, 1993.

[3] M. R. Bridson. The geometry of the word problem. To appear in *Invitations to Geometry and Topology*, Oxford University Press, 2002.

[4] J. Cannon, O. Goodman, and M. Shapiro, Dehn’s algorithm for non-hyperbolic groups, Preprint.

[5] G. Conner. Isoperimetric functions for central extensions. In R. Charney, M. Davis, and M. Shapiro, editors, *Geometric Group Theory*, volume 3 of *Ohio State University, Math. Res. Inst. Publ.*, pages 73–77. de Gruyter, 1995.

[6] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston. *Word Processing in Groups*. Jones and Bartlett, 1992.

[7] S. M. Gersten. Isodiametric and isoperimetric inequalities in group extensions. Preprint, University of Utah, 1991.

[8] S. M. Gersten. Dehn functions and $l_1$-norms of finite presentations. In G. Baumslag and C. Miller, editors, *Algorithms and classification in combinatorial group theory* (Berkeley, CA, 1989), volume 23 of *Math. Sci. Res. Inst. Publ.*, pages 195–224. Springer-Verlag, 1992.
[9] S. M. Gersten. Isoperimetric and isodiametric functions. In G. Niblo and M. Roller, editors, *Geometric group theory I*, number 182 in LMS lecture notes. Camb. Univ. Press, 1993.

[10] S. M. Gersten and T. R. Riley. Filling length in finitely presentable groups. Preprint, to appear in Geom. Dedicata, http://www.math.utah.edu/~gersten/grouptheory.htm/, 2000.

[11] S. M. Gersten and H. Short. Some isoperimetric inequalities for free extensions. Preprint, to appear in Geom. Dedicata, http://www.math.utah.edu/~gersten/grouptheory.htm/, 2000.

[12] M. Gromov. Asymptotic invariants of infinite groups. In G. Niblo and M. Roller, editors, *Geometric group theory II*, number 182 in LMS lecture notes. Camb. Univ. Press, 1993.

[13] P. Hall. Nilpotent groups. Notes of lectures given at the Canadian Mathematical Congress, University of Alberta, 1957.

[14] C. Hidber. Isoperimetric functions of finitely generated nilpotent groups. *J. Pure Appl. Algebra*, 144(3):229–242, 1999.

[15] P. Pansu. Croissance des boules et des géodesiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems*, 3:415–445, 1983.

[16] Ch. Pittet. Isoperimetric inequalities for homogeneous nilpotent groups. In R. Charney, M. Davis, and M. Shapiro, editors, *Geometric Group Theory*, volume 3 of *Ohio State University, Mathematical Research Institute Publications*, pages 159–164. de Gruyter, 1995.

[17] T. R. Riley. Higher connectedness of asymptotic cones. Preprint, http://www.maths.ox.ac.uk/~rileyt/, 2001.

[18] B. A. F. Wehrfritz. *Infinite Linear Groups*. Springer-Verlag, 1973.

Steve M. Gersten
Mathematics Department, 155S. 1400E., Rm. 233, Univ. of Utah, Salt Lake City, UT 84112, USA
gersten@math.utah.edu, http://www.math.utah.edu/~gersten
Derek F. Holt
Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK
dfh@maths.warwick.ac.uk, http://www.maths.warwick.ac.uk/~dfh

Tim R. Riley
Mathematical Institute, 24-28 St. Giles’, Oxford OX1 3LB, UK
rileyt@maths.ox.ac.uk, http://www.maths.ox.ac.uk/~rileyt