Fermionic dark matter interaction with the photon and the proton in the quantum field-theoretical approach and generalizations

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Mass dimension one quantum fields, constructed upon eigenspinors of the charge conjugation operator with dual helicity (ELKO), are prime candidates to describe dark matter. Their interaction with the photon and the proton, in scattering processes, are here explored and discussed, using both the standard QFT with a Maxwell gauge field and a generalized QFT also involving a Podolsky gauge sector. Renormalization and radiative corrections are analyzed in the ELKO setup, in the context of a generalized spinor dual and the twisted conjugation, both yielding unitarity. Several applications are scrutinized, involving the inherent darkness underlying ELKO construction and galaxies rotation curves, the non-relativistic potential regime, and the Møller-like scattering involving ELKO.

I. INTRODUCTION

Quantum fields constructed upon mass dimension one quantum spinor fields in QFT are prime candidates to describe dark matter, since they are, by construction, neutral fermions under gauge interactions, therefore implementing darkness [1–3]. Neutrality under gauge fields is naturally implemented when one employs eigenspinors of the charge conjugation operator. Additionally demanding that the right- and left-handed components of these spinor fields have dual helicity, the so-called eigenspinors of the charge conjugation operator with dual helicity (ELKO) set in. Despite its inherent darkness, eventual couplings remain and might lead to peculiar experimental signatures of ELKO in high-energy running experiments [4–14]. ELKO resides in non-standard Wigner spinor classes, when the unitary irreducible representations of the Poincaré group are extended to encode discrete symmetries [15–19]. Field-theoretical developments and applications of mass dimension one quantum fields were presented in Refs. [20–52]. In addition, fermionic aspects of AdS/CFT correspondence, emulating mass dimension one quantum fields, have been introduced in Refs. [53–58], whereas the pivotal role of ELKO in the membrane paradigm was investigated in Refs. [59–65]. Other aspects of ELKO in gravity can be found in Ref. [66]. The Hawking radiation of ELKO...
across the horizon of several black hole solutions was studied in Refs. [67–69]. Cosmological aspects of ELKO were addressed in Refs. [70–76].

Some limitations of using a QFT for ELKO, when Maxwell gauge fields are employed, can be circumvented when electrodynamics with a Podolsky sector is taken into account in the ELKO paradigm, together with a generalized spinor dual. As Podolsky himself asserted, the only way that Maxwell electrodynamics can be generalized consists of allowing the Lagrangian to contain terms involving higher-order derivatives of the electromagnetic field strength [77]. Introducing it naturally imposes a cutoff procedure that precludes undesirable effects in QFT due to higher frequencies. Also, in the QFT setup with a Podolsky sector, there is an additional gauge freedom arising from higher-order field equations, allowing to remove any singularity that is intrinsic to the standard approach, avoiding divergences like the one occurring due to the vacuum polarization current and the electron self-energy as well [78]. The main goal of this work is to investigate ELKO scattering processes, including the ELKO-photon and ELKO-proton interaction, in standard QFTs with Maxwell fields and, subsequently, also involving a Podolsky sector, yielding generalized QFTs. Renormalization and radiative corrections are also implemented in the ELKO setup, being unitarity also constructed. Several possible applications of this approach are also pointed out and explored, involving the inherent darkness underlying ELKO construction and galaxies rotation curves. This work is organized as follows: Sec. II is devoted to revisit ELKO setup as eigenspinors of the charge conjugation operator with dual helicity. ELKO quantum fields are also introduced, using a generalized spinor dual and the associated creation and annihilation operators. The spin sums are constructed through an odd-parity operator. The ELKO Hamiltonian structure is also presented. Cross-sections and scattering processes involving ELKO are still reviewed in this section. In Sec. III, the spin sums and the Feynman rules for ELKO-photon interaction, in both Podolsky and Maxwell QFTs, are studied and discussed. Some intricacies involving the polarization tensor and bubble diagrams involving ELKO and anti-ELKO are presented, considering the Dirac spinor dual. Unitarity aspects are scrutinized. In Sec. IV the Podolsky sector in a QFT for ELKO is introduced, being the associated propagator explored, together with radiative corrections. Renormalization features are also discussed. Sec. V is dedicated to reanalyzing the content in Secs. III and IV using the ELKO generalized spinor dual, instead of the Dirac spinor dual. The unitarity is analyzed and the comparison between different spinor field conjugations is implemented, using the new Hermitian-like conjugation prescription [1], the so-called twisted conjugation. Within this setup, the polarization tensor regarding ELKO is computed and divergences are controlled by renormalization procedures. The 3-point function is also calculated, ELKO self-interaction is
included and decay rates are studied, also employing the Podolsky propagator. The associated beta functions are also computed. Sec. VI discusses Podolsky QFT and unitarity aspects involving ELKO scattering processes, regarding an effectively combined propagator. In Sec. VII, the vertexes, the gauge propagator, and the external ELKO, are studied in the low-energy approach, to obtain the non-relativistic potential. A Podolsky gauge field sector and the ELKO fermionic sector with the generalized dual are employed, yielding the associated scattering amplitude to be derived. The potential is shown to be a Yukawa-type one. Relevant aspects are then discussed, as the dark particle system does not interact at large distances. Sec. VIII scrutinizes the corrected Møller scattering with the generalized ELKO spinor dual as well as the twisted conjugation. Sec. IX is devoted to furnishing useful tools for evaluating the relic density and the freeze-out temperature, with the scattering amplitude for the ELKO pair annihilation being computed, and in Sec. X the ELKO-proton scattering is addressed, with the analysis of an effective vertex. In Sec. XI, a linear term arising from the self-energy corrections for the model composed of ELKO spinors with the Dirac dual is employed to study a solution exhibiting a galaxy flat rotation curve. It is implemented by analyzing the ELKO fermionic response associated with the exotic theory constructed with the Dirac spinor dual instead of the generalized ELKO dual, with the addition of a Podolskyan propagator. Some interesting conclusions are made by comparing our results and structures with some analogous content for topological insulator description in condensed matter. Finally in Sec. XII the concluding remarks are presented as well as additional discussion and perspectives. The metric signature $(+,−,−,−)$ is used throughout.

II. ELKO UNDERLYING FRAMEWORK AND RAMIFICATIONS

To approach QFT, in general one takes a spinor $\psi(p^\mu)$ in the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ irreducible representation of the Lorentz group to constitute spin-1/2 quantum fields, which in the Weyl representation of the gamma matrices reads

$$\psi(p^\mu) = \begin{pmatrix} \phi_+(p^\mu) \\ \phi_-(p^\mu) \end{pmatrix}. \quad (1)$$

The right- and left-handed Weyl spinors, respectively carrying the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ Lorentz group irreducible representations, are eigenspinors of the helicity operator,

$$(\sigma \cdot \hat{p})\phi_\pm (k^\mu) = \pm \phi_\pm (k^\mu), \quad (2)$$
where, in spherical coordinates,
\[
\begin{align*}
\Phi_+(k^\mu) &= \sqrt{m} \begin{pmatrix} \cos \left( \frac{\theta}{2} \right) e^{-i\phi/2} \\ \sin \left( \frac{\theta}{2} \right) e^{i\phi/2} \end{pmatrix}, & \Phi_-(k^\mu) &= \sqrt{m} \begin{pmatrix} -\sin \left( \frac{\theta}{2} \right) e^{-i\phi/2} \\ \cos \left( \frac{\theta}{2} \right) e^{i\phi/2} \end{pmatrix},
\end{align*}
\]
(3)
for \( k^\mu = \lim_{p^\mu \to 0} p^\mu \) [1]. To introduce mass dimension one fermions [2, 16, 17, 79], one must better investigate the spinor dual, defined as \( \bar{\psi}(p^\mu) = \psi^\dagger(p^\mu)\eta \). The spin-1/2 matrix \( \eta \) is determined from the constraint that bilinears constructed from spinors are covariant. The matrix \( \eta \) commutes with the generators of rotation and anticommutes with the boost generators, supporting the local gauge transformations of the Standard Model. For the Dirac spinor field, one usually takes \( \eta = \gamma^0 \). For ELKO, the definition is distinct, as dark matter fields have to carry irreducible representations of the Lorentz group, with also well-defined properties under the parity (\( \mathcal{P} \)), charge conjugation (\( \mathcal{C} \)), and time reversal (\( \mathcal{T} \)) operators. ELKO do not support (local) gauge transformations of the Standard Model. ELKO are defined as eigenspinors of charge conjugation operator [2, 16, 17, 79],
\[
\lambda_\alpha(p^\mu) = \begin{pmatrix} \zeta_\lambda \Theta [\phi_\alpha(p^\mu)]^* \\ \phi_\alpha(p^\mu) \end{pmatrix},
\]
(4)
where \( \Theta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) is the Wigner time reversal operator satisfying \( \Theta (\sigma_i) \Theta^{-1} = -\sigma_i^* \), for each Pauli matrix \( \sigma_i \). Here \( \alpha \) (and throughout this paper upgreek indexes also) denotes the helicity index. Their right- and left-hand components present opposite helicity,
\[
\sigma \cdot \hat{p} \left[ \Theta \phi_\pm(p^\mu) \right] = \mp \left[ \Theta \phi_\pm(p^\mu) \right].
\]
(5)
As posed in Ref. [2], the charge conjugation operator reads
\[
\mathcal{C} = \begin{pmatrix} \mathbb{I} & i\Theta \\ -i\Theta & \mathbb{I} \end{pmatrix} K,
\]
(6)
where the \( K \) operator implements complex conjugation. Choosing \( \zeta_\lambda = +i \) \([-i]\) yields a self-conjugate \( \lambda^S(p^\mu) \) [anti-self-conjugate \( \lambda^A(p^\mu) \)] ELKO,
\[
\mathcal{C} \lambda^S_\pm(p^\mu) = +\lambda^S_\pm(p^\mu), & \mathcal{C} \lambda^A_\pm(p^\mu) = -\lambda^A_\pm(p^\mu).
\]
(7)
Hence the spinors at rest, \( \lambda_\alpha(k^\mu) \), read [2]
\[
\lambda^S_\pm(k^\mu) = \begin{pmatrix} +i\Theta [\phi_\pm(k^\mu)]^* \\ \phi_\pm(k^\mu) \end{pmatrix}, & \lambda^A_\pm(k^\mu) = \begin{pmatrix} -i\Theta [\phi_\pm(k^\mu)]^* \\ \phi_\pm(k^\mu) \end{pmatrix},
\]
(8)
For arbitrary momentum, ELKO are obtained from the rest spinors by the action of the boost transformation as \( \lambda^{S/A}_\pm(p^\mu) = D(L(p)) \lambda^{S/A}_\pm(k^\mu) \). One defines spin-1/2 quantum fields with ELKO as its expansion coefficients \([1, 2, 18, 80]\),

\[
\tilde{f}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(p)}} \sum_\alpha \left[ a_\alpha(p) \lambda^S_\alpha(p)e^{-ip\cdot x} + b_\alpha(p) \lambda^A_\alpha(p)e^{ip\cdot x} \right].
\]  

(9)

The spinor field \( f(x) \) has mass dimension one and fermionic statistics. Therefore ELKO fields cannot partake Standard Model doublets, therefore constituting first-principle candidates to describe dark matter \([1–3, 15]\). To construct the Feynman–Dyson propagator associated with ELKO, a twisted spinor dual is necessary, since the norm of ELKO, with respect to the Dirac dual, equals zero. In fact, \( \lambda^{S/A}_\pm(p^\mu)\lambda^{S/A}_\pm(p^\mu) = 0 \). Therefore a new spinor dual was introduced in Ref. [2],

\[
\tilde{\lambda}^{S/A}_\pm(p^\mu) = \mp i \left[ \lambda^{S}_\pm(p^\mu) \right]^\dagger \gamma_0,
\]  

(10)

yielding the following orthonormality relations

\[
\tilde{\lambda}^S_\alpha(p^\mu)\lambda^S_\alpha(p^\mu) = 2m\delta_{\alpha\alpha'} = -\tilde{\lambda}^A_\alpha(p^\mu)\lambda^A_\alpha(p^\mu), \quad \tilde{\lambda}^S_\alpha(p^\mu)\lambda^A_\alpha(p^\mu) = 0 = \tilde{\lambda}^A_\alpha(p^\mu)\lambda^S_\alpha(p^\mu),
\]  

(11)

with the associated spin sums

\[
\sum_\alpha \lambda^{S}_\alpha(p^\mu)\tilde{\lambda}^S_\alpha(p^\mu) = m \left[ \mathbb{I} + \mathfrak{S}(p^\mu) \right], \quad \sum_\alpha \lambda^{A}_\alpha(p^\mu)\tilde{\lambda}^A_\alpha(p^\mu) = -m \left[ \mathbb{I} - \mathfrak{S}(p^\mu) \right],
\]  

(12)

taking into account the odd-parity operator

\[
\mathfrak{S}(p^\mu) = i \begin{pmatrix} 0 & 0 & 0 & -e^{-i\phi} \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & -e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \end{pmatrix}.
\]  

(13)

The obtained spin sums lead to the completeness relation

\[
\frac{1}{2m} \sum_\alpha \left[ \lambda^S_\alpha(p^\mu)\tilde{\lambda}^S_\alpha(p^\mu) - \lambda^A_\alpha(p^\mu)\tilde{\lambda}^A_\alpha(p^\mu) \right] = \mathbb{I}.
\]  

(14)

The matrix \( \mathfrak{S}(p^\mu) \) yields a preferred direction, therefore violating locality \([81]\), which can be circumvented by redefining the ELKO dual \([1–3]\),

\[
\tilde{\lambda}^S_\alpha(p^\mu) \mapsto \tilde{\lambda}^S_\alpha(p^\mu) = \tilde{\lambda}^S_A(p^\mu) \mathfrak{A}, \quad \tilde{\lambda}^A_\alpha(p^\mu) \mapsto \tilde{\lambda}^A_\alpha(p^\mu) = \tilde{\lambda}^A_S(p^\mu) \mathfrak{B},
\]  

(15)

where \( \mathfrak{A} \) and \( \mathfrak{B} \) are operators such that the self-conjugate ELKO are eigenspinors of \( \mathfrak{A} \), whereas anti-self-conjugate ELKO are eigenspinors of the \( \mathfrak{B} \) operator, both corresponding to unity eigenvalues. Also, the conditions

\[
\tilde{\lambda}^S_\alpha(p^\mu) \mathfrak{A} \lambda^A_\alpha(p^\mu) = 0 = \tilde{\lambda}^A_\alpha(p^\mu) \mathfrak{B} \lambda^S_\alpha(p^\mu)
\]  

(16)
hold as well. With these two operators, the orthonormality relations read
\[ \bar{\chi}^S_\alpha(p^\mu)\chi^S_\alpha(p^\mu) = 2m\delta_{\alpha\alpha'}, \quad \bar{\chi}^A_\alpha(p^\mu)\chi^A_\alpha(p^\mu) = -\bar{\chi}^A_\alpha(p^\mu)\chi^S_\alpha(p^\mu), \tag{17} \]
whereas the spin sums for both the self- and anti-self-conjugate ELKO read
\[ \sum_\alpha \lambda^S_\alpha(p^\mu)\bar{\lambda}^S_\alpha(p^\mu) = m[1 + \mathcal{G}(p^\mu)] \mathfrak{A}, \quad \sum_\alpha \lambda^A_\alpha(p^\mu)\bar{\lambda}^A_\alpha(p^\mu) = -m[1 - \mathcal{G}(p^\mu)] \mathfrak{B}. \tag{18} \]
Ref. [2] showed that the \( \mathfrak{A} \) and \( \mathfrak{B} \) operators can be derived from \( \mathcal{G}(p^\mu) \) by
\[ \mathfrak{A} = 2 \lim_{\tau \to 1} [1 + \tau\mathcal{G}(p^\mu)]^{-1}, \quad \mathfrak{B} = 2 \lim_{\tau \to 1} [1 - \tau\mathcal{G}(p^\mu)]^{-1}, \tag{19} \]
where \( \tau \) is a real parameter. This leads to spin sums
\[ \sum_\alpha \lambda^S_\alpha(p^\mu)\bar{\lambda}^S_\alpha(p^\mu) = 2mI = -\sum_\alpha \lambda^A_\alpha(p^\mu)\bar{\lambda}^A_\alpha(p^\mu), \tag{20} \]
with completeness relation \( \sum_\alpha \left[ \lambda^S_\alpha(p^\mu)\bar{\lambda}^S_\alpha(p^\mu) - \lambda^A_\alpha(p^\mu)\bar{\lambda}^A_\alpha(p^\mu) \right] = 4mI. \)

ELKO are well known not to satisfy Dirac equations, but the following coupled system of first-order field equations [2, 82, 83],
\[ \gamma_\mu p^\mu \lambda^S_\pm(p^\mu) = \pm im\lambda^S_\mp(p^\mu), \quad \gamma_\mu p^\mu \lambda^A_\pm(p^\mu) = \mp im\lambda^A_\mp(p^\mu), \tag{21} \]
satisfying the Klein–Gordon equations,
\[ (g_{\mu\nu}p^\mu p^\nu - m^2I)\lambda^{S/A}_\alpha(p^\mu) = 0. \tag{22} \]
To prospect the statistics satisfied by the ELKO field \( \varepsilon(x) \) in Eq. (9), its adjoint is naturally defined as
\[ \bar{\varepsilon}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(p)}} \sum_\alpha \left[ a^{\dagger}_\alpha(p)\bar{\chi}^S_\alpha(p)e^{ipx} + b_\alpha(p)\bar{\chi}^A_\alpha(p)e^{-ipx} \right]. \tag{23} \]
To preclude any causal paradox, Ref. [84] showed that the ELKO and its adjoint have a vanishing anticommutator, \( \{ \bar{\varepsilon}(x), \varepsilon(x') \} = 0 \), with fermionic statistics to the creation and annihilation operators, yielding a consistent local QFT. Ref. [2] showed that the Feynman–Dyson propagator reads
\[ S_{FD}(x' - x) = -\frac{i}{2} \langle |\mathcal{T}[\varepsilon(x')\varepsilon(x)]| | \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{1}{\Box + m^2 + ie^{ip(x-x')}} \tag{24} \]
with mass dimension one and free field Lagrangian density given by
\[ \mathcal{L}_{\text{free}} = \frac{1}{2} \left( g_{\mu\nu} \partial^\mu \bar{\varepsilon}(x) \partial^\nu \varepsilon(x) - m^2 \bar{\varepsilon}(x)\varepsilon(x) \right). \tag{25} \]
The \( f(x) \) field contributes to the zero-point field energy as

\[
H^f_{\text{free}} = -4 \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} E(p). \tag{26}
\]

Using the parity operator \( \mathcal{P} = m^{-1} \gamma \mu P^\mu \) \[85\], and the time-reversal operator \( \mathcal{T} = i\gamma^5 \mathcal{C} \), the properties of Elko under \((\mathcal{C} \mathcal{P} \mathcal{T})^2 = \mathbb{I}\) and \(\{\mathcal{C}, \mathcal{P}\} = 0\), indicate that ELKO quantum fields belong to the non-standard Wigner spinor classes \[16, 17\]. Lourenço’s spinor field classification unites all spinor fields in 4-dimensional spacetime into six disjoint classes, according to the values attained by the bilinear covariants computed from the respective spinor field. This classification is based upon the U(1) gauge symmetry of the first-order equations of motion that rule spinor fields in each spinor class. Lourenço’s classification devises mass dimension one spinors as a class of singular spinors \[86\]. A more general classification has been proposed in Ref. \[87\] encompassing spinor multiplets as realizations of (non-Abelian) gauge fields. A reciprocal classification, generalizing Lourenço’s idea, was implemented in Refs. \[88, 89\]. Mass dimension one quantum fields were investigated under the prism of non-standard Wigner classes in Refs. \[90–99\], also paving the way for other generalized spinor field classifications \[100–103\], including the spinor classification encompassing higher order gauge groups \[104\]. Flipping phenomena between spinor field classes were analyzed in Ref. \[105\].

In QFT, a transition probability can be computed when Hermitian conjugation is employed, to certify a positive-definite probability. Ref. \[4\] showed that the optical theorem is violated at 1-loop, implementing quantum corrections to field theory, and coming from the non-Hermitian character of the ELKO mass dimension one fermion. Therefore, Ref. \[1\] introduced the twisted conjugation \( ^\dagger \), required to be an involution, to compute transition probabilities and observables. The ELKO dual \( \tilde{\lambda} \) can be recast as

\[
\tilde{\lambda}^{S/A}_\alpha(p) = \left[ \lambda^{S/A}_\alpha(p) \right]^\dagger \gamma_0 = -i\alpha \left[ \lambda^{-S/A}_{-\alpha}(p) \right]^\dagger \gamma_0. \tag{27}
\]

Besides, Ref. \[1\] showed that

\[
\left[ \lambda^{S/A}_\alpha(p)^\dagger \lambda^{S/A}_{\alpha'}(p') \right]^\dagger = i\alpha \left[ \lambda^{S/A}_\alpha(p') \right]^\dagger \left[ \lambda^{S/A}_{-\alpha}(p) \right]^\dagger. \tag{28}
\]

Employing the twisted conjugation \( ^\dagger \), the expansion of the ELKO quantum field and its dual can be rewritten as

\[
f(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2mE(p)}} \sum_\alpha \left[ a_\alpha(p) \lambda^{S/A}_\alpha(p) e^{-ip \cdot x} + b_\alpha^\dagger(p) \lambda^{A}_{-\alpha}(p) e^{-ip \cdot x} \right], \tag{29a}
\]

\[
\tilde{f}(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2mE(p)}} \sum_\alpha \left[ a_\alpha^\dagger(p) \tilde{\lambda}^{S/A}_{-\alpha}(p) e^{ip \cdot x} + b_\alpha(p) \tilde{\lambda}^{A}_\alpha(p) e^{-ip \cdot x} \right]. \tag{29b}
\]
The annihilation and creation operators satisfy the canonical anticommutators

\[
\left\{ a_\alpha(p), a_\alpha^+(p') \right\} = \left\{ b_\alpha(p), b_\alpha^+(p') \right\} = (2\pi)^3 \delta_{\alpha\alpha'} \delta^3(p-p'),
\]

with \( a_\alpha^+(p) \equiv a_\alpha(p) \) and \( b_\alpha^+(p) \equiv b_\alpha(p) \). Although the Lagrangian is non-Hermitian, it satisfies \( L^\dagger_{\text{free}}(x) = L_{\text{free}}(x) \). For mass dimension one fermions, the usual demand of Hermiticity should be replaced by the demand of invariance under the twisted conjugation \( \dagger \).

One can therefore compute observables employing the \( \dagger \) operator. Ref. [1] considers scattering processes regarding ELKO for \( \mathcal{M}_{ba} \) denoting the scattering amplitude. Denoting by \( \mathcal{U}_{ba} \) a unimodular phase function, the 2-body cross-section for the \( 12 \rightarrow 34 \) scattering process reads

\[
\sigma_{12 \rightarrow 34} = \frac{(2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2)}{16 \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int \frac{d^3p_3}{(2\pi)^3 E_3} \int \frac{d^3p_4}{(2\pi)^3 E_4} \left[ \mathcal{U}_{(34)(12)} \mathcal{M}^\dagger_{(34)(12)} \mathcal{M}_{(34)(12)} \right].
\]  

(31)

Interactions set in by regarding renormalizable interaction potentials, denoting by \( \phi(x) \) a scalar field,

\[
V_\phi(t) = \frac{g_\phi}{2} \int d^3x [\bar{f}(x)\phi^2(x)], \quad V_\psi(t) = \frac{g_\psi}{2} \int d^3x [\bar{\psi}(x)\psi^2(x)].
\]

(32)

The transition probability regarding ELKO as external states (the capital letter indexes below labeling self-conjugate and anti-self-conjugate ELKO),

\[
K_{IJ} \equiv \left[ \bar{\lambda}_I^\dagger(\alpha)(p)\lambda^I_{\alpha'}(p') \right] \left[ \bar{\lambda}_J^\dagger(\alpha')(p')\lambda^J_{\alpha}(p) \right].
\]

(33)

When \( I = J [I \neq J] \) two external fermions or antifermions [an external fermion-antifermion pair] manifest, whereas \( K_{AA} = K_{SS} \) are non-negative quantities and \( K_{AS} = K_{SA} \) are non-positive ones. Therefore \( \mathcal{M}_{ba} \mathcal{M}_{ba}^\dagger \) is positive-definite [negative-definite] for even [odd] number of external antifermions, yielding \( \mathcal{U}_{ba} = (-1)^{\hat{n}_{ba}} \), for \( \hat{n}_{ba} \) being the total number of incoming and outgoing antifermions.

It is worth to mention that for the Dirac dual the following expression holds,

\[
\sum_\alpha \lambda_{\alpha}^{S/A}(p)\bar{\lambda}_{\alpha}^{S/A}(p) = \bar{p}.
\]

(34)

There is a relation between ELKO under parity, as

\[
\lambda_{\alpha}^{S/A}(p^\mu) = i\epsilon_{\alpha\alpha'}\lambda_{\alpha'}^{A/S}(-p^\mu),
\]

(35)

implying that

\[
\sum_\alpha \lambda_{\alpha}^{S/A}(p^\mu)\bar{\lambda}_{\alpha}^{S/A}(p^\mu) = \sum_\alpha \lambda_{\alpha}^{A/S}(-p^\mu)\bar{\lambda}_{\alpha}^{A/S}(-p^\mu),
\]

(36)

\footnote{The Levi-Civita symbol regarding helicity indexes reads \( \epsilon_{+-} = +1 = -\epsilon_{-+} \).}
which is also valid for the spin sum of the generalized ELKO dual (15).

Taking the free Lagrangian (25) into account and regarding $\mathcal{L}_{\text{free}}$ in (25), the conjugate momentum to $f(x)$ reads [1, 2]

$$
p(x) = \frac{\partial \mathcal{L}_{\text{free}}(x)}{\partial \dot{f}(x)} = \frac{1}{2} \frac{\partial}{\partial t} \tilde{f}(x), \quad \tilde{p}(x) = \frac{\partial \mathcal{L}_{\text{free}}(x)}{\partial \dot{\tilde{f}}(x)} = \frac{1}{2} \frac{\partial}{\partial t} \tilde{f}(x).
$$

The locality of ELKO is regulated by

$$
\{f(t, x), p(t, x')\} = i\delta^3(x - x'), \quad \{\tilde{f}(t, x), \tilde{p}(t, x')\} = i\delta^3(x - x'),
$$

$$
0 = \{f(t, x), f(t, x')\} = \{p(t, x), p(t, x')\} = 0.
$$

The on-shell Hamiltonian density can be obtained [16]. Here, $f$ and $\tilde{f}$ are considered to be independent variables as well as in the Dirac particle case,

$$
\mathcal{H} = \tilde{f} \tilde{p} + \tilde{p} \tilde{f} - \mathcal{L}_{\text{free}} = 2 \tilde{f} \dot{f}.
$$

The Hamiltonian operators reads

$$
H = 2 \int \frac{d^3p}{(2\pi)^3 2mE(p)} E^2(p) \sum_{\alpha, \beta} \left( \hat{\lambda}^\dagger_\alpha(p) \lambda^\dagger_\beta(p) a^\dagger_\alpha a_\beta + \hat{\lambda}^\dagger_\alpha(p) \lambda^\dagger_\beta(p) b_\alpha b^\dagger_\beta \right),
$$

$$
= 2 \int \frac{d^3p}{(2\pi)^3} E(p) \sum_{\alpha, \beta} i\epsilon_{\alpha\alpha'} \left( a^\dagger_\alpha a_{\alpha'} + b^\dagger_\alpha b_{\alpha'} \right),
$$

being Hermitian, with eigenstates

$$
A^\dagger_\alpha |0\rangle = \left( a^\dagger_\alpha + i\epsilon_{\alpha\alpha'} a^\dagger_{\alpha'} \right) |0\rangle, \quad B^\dagger_\alpha |0\rangle = \left( b^\dagger_\alpha + i\epsilon_{\alpha\alpha'} b^\dagger_{\alpha'} \right) |0\rangle.
$$

The second eigenstate has negative energy. In order to circumvent it, we consider $\{a^\dagger_\alpha, a_{\alpha'}\} = \{b^\dagger_\alpha, b_{\alpha'}\} = \delta_{\alpha\alpha'}$ and, for the antiparticle, we flip the interpretation to consider $b^\dagger_\alpha$ as the annihilation operator and $b_\alpha$ as the creation one. Therefore, the antiparticle eigenstate becomes $B_\alpha |0\rangle = (b_\alpha + i\epsilon_{\alpha\alpha'} b_{\alpha'}) |0\rangle$. In this case, both have positive energy and also positive norm. Since the eigenstates are of the form $2E(p)$, one can normalize the ELKO field (9) as $\frac{1}{\sqrt{2}} \tilde{f}$ to $E(p)$ be the Hamiltonian eigenvalue.

For the case of the generalized ELKO dual (15), the Hamiltonian reads

$$
H = 2 \int \frac{d^3p}{(2\pi)^3} E(p) \sum_{\alpha} \left( a^\dagger_\alpha a_\alpha + b^\dagger_\alpha b_\alpha \right),
$$

which is analogous to the mass dimension three-halves fermionic case.
III. FEYNMAN RULES FOR ELKO-PHOTON INTERACTION AND UNITARITY ASPECTS

This section is devoted to present the Feynman rules regarding the various types of ELKO and to investigate associated aspects regarding unitarity of a QFT describing also ELKO\(^2\). Employing the Dirac dual, the Feynman rules related to self-conjugate and anti-self-conjugate ELKO spinors are respectively given by

\[
\frac{\bar{\lambda}^S_\alpha(p^\mu)}{\sqrt{m}} = \begin{array}{c} \rightarrow \\ p \end{array}, \quad \frac{\lambda^S_\alpha(p^\mu)}{\sqrt{m}} = \begin{array}{c} \rightarrow \\ p \end{array}, \quad \frac{\bar{\lambda}^A_\alpha(p^\mu)}{\sqrt{m}} = \begin{array}{c} \rightarrow \\ p \end{array}, \quad \frac{\lambda^A_\alpha(p^\mu)}{\sqrt{m}} = \begin{array}{c} \rightarrow \\ p \end{array},
\]

whereas the polarizations account for

\[
\epsilon^{\ast \alpha}_\mu(p) = \begin{array}{c} \rightarrow \\ p \end{array}, \quad \epsilon^{\bar{\alpha}}_\mu(p) = \begin{array}{c} \rightarrow \\ p \end{array}.
\]

The ELKO field propagator can be obtained using the methodology of Ref. [16] for the Dirac dual case

\[
k \rightarrow \quad = \frac{i\not{k}}{m(k^2 - m^2 + i\epsilon)}
\]

The photonic sector is represented, up to gauge fixing terms, by

\[
\begin{array}{c} \not{k} \not{\nabla} \\ \not{\gamma} \not{\mu} (k) = -\frac{i g_{\mu\nu}}{k^2 + i\epsilon} + A \frac{i g_{\mu\nu}}{k^2 - M^2 + i\epsilon}.
\end{array}
\]

The cases with \(A = 1\) (Podolskian) and \(A = 0\) (Maxwellian) will be considered.

The vertex for ELKO-photon interaction, governed by the Lagrangian

\[
\mathcal{L}_I = \frac{i}{2} \bar{f}(x) F_{\mu\nu}(x) \sigma^{\mu\nu} f(x) = \mathcal{L}^I,
\]

reads

\[
\begin{array}{c} \rightarrow \\ p \end{array} \quad = igp_\mu \sigma^{\mu\nu} \begin{array}{c} \not{a} \\ \not{b} \end{array}
\]

\(^2\) Hereone Eq. (17) will be expressed as

\[
\bar{\lambda}^S_\alpha(p^\mu) \lambda^S_\beta(p^\nu) = m \delta_{\alpha\beta} = -\bar{\lambda}^A_\alpha(p^\nu) \lambda^S_\alpha(p^\mu), \quad \bar{\lambda}^S_\alpha(p^\nu) \lambda^A_\alpha(p^\mu) = 0 = \bar{\lambda}^A_\alpha(p^\nu) \lambda^S_\alpha(p^\mu).
\]

which is equivalent to considering each type of ELKO divided to a \(\sqrt{2}\) factor.
with \( \sigma_{\mu\nu} \equiv \frac{1}{2} [\gamma_\mu, \gamma_\nu] \). The \( a \) and \( b \) latin letters denote spinor indexes. Considering the Dirac dual, the polarization tensor reads

\[
i\pi_{\mu\nu}(p) = \frac{g^2}{m^2} \int \frac{d^4k}{(2\pi)^4} \frac{Tr(p_\omega \sigma^{\alpha\mu}(\not{p} - \not{k})p_\alpha \sigma^{\alpha\nu}\not{k})}{(k^2 - m^2 + i\epsilon)((p - k)^2 - m^2 + i\epsilon)} = p \rightarrow \begin{array}{c} k \\ \end{array} \rightarrow \begin{array}{c} p \\ \end{array} \rightarrow \begin{array}{c} p - k \\ \end{array} \rightarrow
\]

(50)

The imaginary part of a massive photon that decays into an ELKO and an anti-ELKO which combine again to form a photon,

\[\epsilon^\mu 23[i\pi_{\mu\nu}(p)]\epsilon^{*\nu},\]

is given by

\[-\frac{g^2}{m^2} \int \frac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0)\epsilon_\mu \epsilon_\nu^* Tr(p_\alpha \sigma^{\alpha\mu}(\not{p} - \not{k})p_\beta \sigma^{\beta\nu}\not{k}) \delta((p - k)^2 - m^2) \delta(k^2 - m^2).\]

(52)

To verify the optical theorem, we must find the formal expression to the associated decay rate

\[\lim_{M \rightarrow 0} \sum_{\alpha, \alpha'} \int \frac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0)\delta(k^2 - m^2) \delta((p - k)^2 - M^2) = \]

(53)

which equals \(2M_{\text{photon}}\Gamma\). The photon mass \(M_{\text{photon}}\) equals zero in the case here studied. However, the formal expression must be fulfilled in order to ensure unitarity. The spin sum over the squared amplitudes yields

\[\sum_{\alpha, \alpha'} \left| \begin{array}{c} \alpha \\ \end{array} \right| \rightarrow p \rightarrow \left| \begin{array}{c} \alpha' \\ \end{array} \right| = -\left( \frac{1}{\sqrt{m}} \right)^4 g^2 \sum_{\alpha, \alpha'} \epsilon^\mu \epsilon^{*\nu} \bar{\lambda}_\alpha^S (k)p_\omega \sigma^{\omega\mu} \lambda_\alpha^A (p - k)\bar{\lambda}_\alpha^A (p - k)p_\beta \sigma^{\beta\nu} \lambda_\alpha^S (k)\]

(54)

where the spin sums, the Feynman rules, and the expression \(\sigma^{\mu\nu}_\dagger = -\gamma_0 \sigma_{\mu\nu} \gamma_0\) have been considered. Therefore, the optical theorem is verified. Denoting \(\theta_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\), one can write the imaginary part of the polarization tensor as

\[23[i\pi_{\mu\nu}(p)] = \theta_{\mu\nu} \Pi(p^2),\]

(55)
in which

\[ 3\Pi(p^2) = 2g^{\mu\nu}3\left[ i\pi_{\mu\nu}(p) \right] \]
\[ = -\frac{g^2}{m^2} \int \frac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0)g_{\mu\nu}Tr(p_\xi\sigma^{\xi\mu}(p - \vec{k})p_\beta\sigma^{\beta\nu}(\vec{k}))\delta((p-k)^2 - m^2)\delta(k^2 - m^2). \]  

(56)

Together with gamma matrices identities and the relations implied by the delta functions as well, it yields

\[ 3\Pi(p^2) = -\frac{g^2}{m^2} \int \frac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0)2p^2(2 + m^2)\delta((p-k)^2 - m^2)\delta(k^2 - m^2). \]

(57)

This object has the same sign as the one from QED. This is an important check to ensure unitarity, which will be useful when analyzing the possibility of a bosonic sector composed of the so-called Podolsky model. Therefore, as a consistency check, considering an external photon with polarization \(\alpha\) and the fact that these transverse excitations obey \(\epsilon^{\mu\alpha}\epsilon^{*\alpha}_{\mu} = -1\), the expected positivity of the decay rate becomes manifest, for the case of a massive photon. Otherwise, for a massless one, it vanishes\(^3\). Considering still the Dirac dual, the self-energy has the following form,

\[ i\Sigma(p) = \frac{g^2}{m} \int \frac{d^4k}{(2\pi)^4} \frac{(p-k)\beta\sigma^{\beta\gamma}\bar{k}(p-k)\gamma_\alpha\sigma^{\alpha\gamma}(-M^2)^N}{(k^2 - m^2 + i\epsilon)((p-k)^2 + i\epsilon)((p-k)^2 - M^2 + i\epsilon)^N} \]

\[ = \begin{array}{c}
\text{p} \rightarrow \text{p} \\
\text{k} \rightarrow \\
\text{p' k} \rightarrow \\
\end{array} \]

(58)

with \(N = 0\) for Maxwell and \(N = 1\) for the Podolsky QFT. Now, for \(N = 0\), the imaginary part for amplitude of a process associated to an incoming particle that emits a photon, reabsorbs it and then continues to propagate, reads \(\lambda^S(p)2\Im[\Sigma(p)]\lambda^S(p)\), or equivalently,

\[ \frac{g^2}{m} \int \frac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0)\bar{\lambda}^S(p)(p - k)\omega\sigma^{\omega\mu}\bar{k}(p-k)\beta\sigma^{\beta\mu}\lambda^S(p)\delta((p-k)^2)\delta(k^2 - m^2). \]

(59)

To verify the optical theorem, one must evaluate the total decay rate,

\[ \lim_{M \to 0} \sum_{r,s} \int \frac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0)\delta(k^2 - m^2)\delta((p-k)^2 - M^2) = \begin{array}{c}
\text{p' k' alpha} \\
\text{p' k' alpha'} \\
\end{array} = 2m\Gamma. \]

\(^3\) This feature is going to be useful in the next sections regarding a harmless introduction of the Podolsky field.
The explicit spin sum over the squared amplitude is given by

$$\sum_{\alpha,\alpha'} \left| \frac{p \to \alpha}{\alpha'} \right|^2 = -\frac{g^2}{\sqrt{m}} \sum_{\alpha,\alpha'} \bar{\lambda}^S(p)(p - k)_{\alpha} \sigma^{\omega\mu} \lambda^S_{\alpha'}(k) \bar{\lambda}^S_{\alpha'}(k)(p - k)_{\omega} \sigma^{\omega\nu} \lambda^S(p) \epsilon_{\mu}^{\alpha} \epsilon_{\nu}^{\alpha'}$$

$$= \frac{g^2}{m} \bar{\lambda}^S(p)(p - k)_{\omega} \sigma^{\omega\mu} \bar{k}(p - k)_{\zeta} \sigma^{\zeta}_{\mu} \lambda^S(p).$$ (60)

Here the identities \(\sum_{\alpha} \epsilon_{\mu}^{\alpha} \epsilon_{\nu}^{\alpha'} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}\) have been used. Therefore, the unitarity is verified.

The same conclusion can be obtained for the amplitude of an anti-conjugate ELKO that emits a photon, absorbs it and then continues to propagate. The spin sum relevant to compute the decay rate

$$\sum_{\alpha,\alpha'} \left| \frac{p \to \alpha}{\alpha'} \right|^2$$

represents the term

$$-\frac{1}{\sqrt{m}} g^2 \sum_{\alpha,\alpha'} \bar{\lambda}^A(-p)(p - k)_{\zeta} \sigma^{\zeta}_{\mu} \lambda^A_{\alpha'}(-k) \bar{\lambda}^A_{\alpha'}(-k)(p - k)_{\omega} \sigma^{\omega\nu} \lambda^A(-p) \epsilon_{\mu}^{\alpha} \epsilon_{\nu}^{\alpha'}$$

$$= \frac{g^2}{m} \lambda^A(p)(p - k)_{\omega} \sigma^{\omega\mu} \bar{k}(p - k)_{\zeta} \sigma^{\zeta}_{\mu} \lambda^A(p).$$ (62)

We have used the property relating a self-conjugate spinor evaluated in \(p\) with an anti-self-conjugate spinor at \(-p\). The imaginary part of the amplitude reads

$$\bar{\lambda}^A(-p) 2\Im[\Sigma(p)] \lambda^A(-p),$$ (63)

or, equivalently,

$$\frac{g^2}{m} \int \frac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0) \bar{\lambda}^A(-p)(p - k)_{\omega} \sigma^{\omega\mu} \bar{k}(p - k)_{\beta} \sigma^{\beta}_{\mu} \lambda^A(-p) \delta((p - k)^2) \delta(k^2 - m^2).$$ (64)

Therefore, this process complies with unitarity. It is worth mentioning that the operator expansion coefficients of the quantum field are not the energy eigenstates\(^4\), in opposition to the generalized

\(^4\) In fact, they are a linear combination of these operators.
ELKO dual case, see III. Despite this fact, investigating this exotic Dirac dual possibility is still relevant, due to its associated unitary QFT and the existence of a solution with potential application in the description of dark matter phenomenology [106]. Interestingly, these microscopic and cosmological aspects are intimately related.

IV. RENORMALIZATION PROPERTIES

The use of the Dirac dual leads to a propagator with the high energy behavior as $1/p$ and not $1/p^2$, as in the case with the generalized ELKO dual (15) [16]. It means that the investigation of the renormalization properties is relevant. First of all, one can write

$$i g^\mu\nu \pi_{\mu\nu}(p^2) \equiv \pi(p^2) = \frac{12}{m^2} \int \frac{d^4k}{(2\pi)^4} \frac{x(1-x)}{(k^2 - \Delta(p^2) + i\epsilon)^2},$$

(65)

with $\Delta(p^2) = m^2 - p^2 x(1-x)$. Since $\pi_{\mu\nu}(p^2) = \theta_{\mu\nu} \frac{\pi(p^2)}{3}$, we conclude that to eliminate the logarithmic divergence, a Podolsky sector must be included in the bare Lagrangian. Denoting the electromagnetic field strength by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the Podolsky model can be written as

$$L_{pod} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{a^2}{2} B_\mu B^\mu + a^2 \partial_\mu B_\nu F^{\mu\nu},$$

(66)

with $a^2 = \frac{1}{M^2}$, for $M$ denoting the Podolsky photon mass, and $B_\mu$ denoting the electromagnetic potential in the Podolsky sector. To decouple the fields, one can redefine $A_\mu \mapsto A_\mu + a^2 B_\mu$. Absorbing the factor $a^2$, $B_\mu \mapsto a^2 B_\mu$, yields

$$L_{pod} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} B^\mu (M^2 g_{\mu\nu} + \Box \theta_{\mu\nu}) B^\nu.$$

(67)

The self-energy has the structure given by

$$i \Sigma(p) = \frac{g^2}{m} \int dx \frac{d^4k}{(2\pi)^4} \frac{[p(1-x) - k]_\alpha \sigma^\alpha \gamma (k + px) [p(1-x) - k]_\beta \sigma^\beta \gamma}{(k^2 - \Delta(p^2) + i\epsilon)^2} - \Sigma(p, \tilde{\Delta}(p^2, M^2)),$$

(68)

with

$$\tilde{\Delta}(p) = (1-x)(m^2 - p^2 x), \quad \tilde{\Delta}(p, M) \equiv (1-x)(m^2 - p^2 x) - xM^2.$$

(69)

The subtracted terms account for a possible use of the Podolsky propagator. The divergent parts read

$$i \Sigma_{div}(p) = \frac{g^2}{m} \int dx \frac{d^4k}{(2\pi)^4} \frac{[\frac{3}{2}(x-1)k^2p + 3p^2 px(1-x)^2]}{(k^2 - \tilde{\Delta}(p^2) + i\epsilon)^2} - \Sigma_{div}(p, \tilde{\Delta}(p^2, M^2)).$$

(70)

5 Its well known form can be obtained by simply Gaussian integrating the $B_\mu(x)$ field.
6 As presented in the next sections, the extra massive field associated to the Podolsky sector does not violate the unitarity constraints previously investigated. This is due to the fact that the model is an interacting one leading to a Merlin mode behaviour for $B_\mu(x)$.
If a conventional photon propagator is used, there are quadratic divergences associated with a $p$ term and a logarithmic one associated with $p^2 p$. In the case of a Podolsky propagator, just the linear term in external momenta is logarithmic divergent and the remaining ones are finite. This result differs from the bare Lagrangian and demands an extra renormalization condition. On the other hand, it seems to be the only kind of divergent term, even in higher-order loops. It means that just one extra condition is necessary. Therefore, the fermionic 2-point part is renormalizable, since it needs a finite set of renormalization conditions. Therefore, considering the case of a Podolsky propagator yields the following divergent piece for the linear contribution

$$
\Sigma_{\text{LINEAR}}(p) = -p \frac{(2 - D)D M^2}{m} \int dx \frac{9(x - 1)x}{2} \frac{\mu^{-\epsilon}}{16\pi^2} \left[ -\frac{2}{\epsilon} + \gamma_E \right] 
$$

It is equivalent to an effective action with an extra term $i \bar{f} A \not{\!p} f$. It varies with the energy as

$$
\frac{\mu}{d\mu} = -3 \frac{M^2 g^2}{16\pi^2 m}.
$$

This term may lead to interesting solutions associated with the observed flat rotation curve of the galaxies. Even for the case of the Maxwell propagator, the cubic term in $p$ can be discarded at low energies in a cosmological scenario. Therefore, the linear term is the focus of our attention.

The next important object to investigated is related to the radiative corrections for the interaction vertex. The 1-loop contribution is given by

$$
i \Gamma_\mu = g^3 \int \frac{1}{(2\pi)^4} d^4k dxdydz \delta(x + y + z - 1) \frac{k^3 \sigma^{\gamma\alpha} k_3 (\sigma^{\beta\mu} p_3) k_4 \sigma^{\gamma\beta} k_9}{[k^2 - \Delta'(p)]^3} - i \Gamma_\mu(\Delta')|_M, \quad (73)
$$

where $\Delta'(p) \equiv m^2(1 - z)^2 - xyp^2$ and $\Delta'(p, M) \equiv \Delta'(p) + zM^2$, which can be illustrated by the process

$$
\begin{diagram}
\node{p} \arrow{s} \node{k_1} \arrow{e} \node{q_1} \node{q_2} \arrow{w} \node{k_2}
\end{diagram}
$$

whereas the momentum variables are defined as

$$
k^1_\gamma \equiv k_\gamma - yp_\gamma + (z - 1)q_1^1, \quad k^3_\gamma \equiv k_\gamma - yp_\gamma + zq_1^1, 
\quad k^4_\gamma \equiv k_\gamma - (y + 1)p_\gamma + zq_1^1, \quad k^9_\gamma \equiv k_\gamma - yp_\gamma + (z - 1)q_1^1.
$$
Again, the subtracted term represents the effect of using a Podolsky propagator associated to mixture of massless and massive photons with mass $M$. Considering the Podolsky propagator, the only divergence is of logarithmic kind and proportional to the integrand of

$$
\sum_a \int \frac{d^4k}{(2\pi)^4} \frac{k(\sigma^{\alpha\mu}p_{\alpha'})\hat{k}}{|k^2 - \Delta^A_{\alpha}|^3}.
$$

(76)

This sum is taken over the contributions from the gauge field propagator parts with massless and massive poles, associated to $a = 1, 2$, respectively. If the integration symmetry is considered, $k(\sigma^{\beta\mu}p_{\beta})\hat{k} = \frac{k^2}{2} \gamma^\mu (\sigma^{\beta\mu}p_{\beta})\gamma_\mu = 0$, since $\gamma^\nu (\sigma^{\alpha\mu}p_{\alpha'})\gamma_\nu = 0$. Therefore, no counterterm is necessary for this object.

Regarding the 4-point functions, if the Podolsky model is employed, there is no divergence, and our 1-loop analysis is completed. On the other hand, using the Maxwell propagator leads to divergent terms coming from the 4-point function. These are different from the bare Lagrangian ones. New divergent terms may arise at higher loops implying in a non-renormalizable theory since there are infinite divergent anomalous terms each one demanding its renormalization condition to be eliminated.

V. ELKO GENERALIZED DUAL, UNITARITY AND COMPARISON BETWEEN DIFFERENT SPINOR FIELD CONJUGATIONS

The propagator associated with the generalized ELKO dual (15) reads

$$ k \rightarrow \frac{i}{k^2 - m^2 + i\epsilon} $$

(77)

Regarding the generalized ELKO dual (15), the spin sum associated to the decay rate, calculated via the Hermitian conjugation $\dagger$, related to the polarization tensor, reads

$$
\sum_{a,a'} \begin{vmatrix}
 p \rightarrow & a \\
 & \alpha \\
 & \alpha' \\
 & a' \\
\end{vmatrix}^2 = -\left(\frac{1}{\sqrt{m}}\right)^4 g^2 \sum_{a,a'} \epsilon^\mu \epsilon^{\nu'} \lambda^S_{\alpha'}(k)p_\alpha \sigma^{\mu\nu} \lambda^A_{\alpha}(p-k)i\epsilon_{\alpha\beta} \lambda^A_{\beta}(p-k)p_\omega \sigma^{\omega\nu} \epsilon_{\alpha'\beta'} \lambda^S_{\beta'}(k)
$$

$$ = -\frac{g^2}{m^2} \epsilon^\mu \epsilon^{\nu'} Tr(p_\alpha \sigma^{\alpha\mu}(p-k)\sigma^{\beta\nu}(p-k)\sigma^{\omega\nu} \epsilon_{\alpha'\beta'}). 
$$

(78)

Computing the imaginary part of the associated polarization tensor yields

$$
\epsilon^{\mu} \Im \left[i \pi_{\mu\nu}(p)\right] \epsilon^{\nu'} = g^2 \int \frac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0)\epsilon^\mu \epsilon^{\nu'} Tr(p_\alpha \sigma^{\alpha\mu}p_\beta \sigma^{\beta\nu}) \delta((p-k)^2 - m^2) \delta(k^2 - m^2).
$$

(79)
Therefore, under the Hermitian conjugation †, the theory with the generalized ELKO dual is not compatible with the optical theorem and violates unitarity. A generalized conjugation prescription ‡, as introduced in Sec. II, must be then regarded. It is also important to mention that, considering this generalized prescription ‡, the radiative corrections of the model are in agreement with the optical theorem and lead to a unitary theory.

Renormalization aspects of the model composed of the generalized ELKO dual can be therefore implemented. One can now regard the ‡ twisted conjugation acting on the spinor indices as defined in Eq. (27). The Lagrangian structure is invariant under the ‡ operator,

$$\mathcal{L} = \frac{g}{2} \bar{\psi} \gamma^\mu \gamma^\nu \psi = \mathcal{L}^\dagger.$$  (80)

The explicit divergent parts of the associated polarization tensor reads

$$i\pi^{\mu\nu}(p) = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr(p_\omega \sigma^{\mu\nu} p_\sigma \sigma^{\alpha\nu})}{(k^2 - m^2 + i\epsilon)((p - k)^2 - m^2 + i\epsilon)} = 4ip^2 g^2 \theta_{\mu\nu} \frac{\mu^-}{16\pi^2} \left( -\frac{2}{\epsilon} + \gamma\epsilon \right).$$  (81)

In the last equality just the divergent and constant terms, that are eliminated in a $\overline{\text{MS}}$ renormalization scheme, have been regarded. Differently from the Dirac dual case, there is no necessity of using a Podolsky bosonic sector. On the other hand, its use does not generate any inconsistency and also improves the renormalization properties. First of all, the standard QFT with Maxwell electrodynamics will be analyzed. The divergent part of the polarization tensor can be absorbed by a renormalization of the photonic field. Denoting the square of the photon renormalization constant as $Z_p = 1 + \delta_1$ and noticing that $\pi^{R\mu\nu} = \pi_{\mu\nu} + \delta_1 p^2 \theta_{\mu\nu}$, it reads

$$\delta_1 = -4g^2 \frac{\mu^-}{16\pi^2} \left( -\frac{2}{\epsilon} + \gamma\epsilon \right).$$  (82)

The divergent piece of the self-energy can be expressed as

$$i\Sigma(p) = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{(p - k)_\alpha \sigma^{\alpha\gamma}(p - k)_\beta \sigma^{\beta\gamma}}{(k^2 - m^2 + i\epsilon)((p - k)^2 - m^2 + i\epsilon)}$$

$$= -g^2 \int \frac{d^4k}{(2\pi)^4} \frac{3}{(k^2 - m^2 + i\epsilon)((p - k)^2 - m^2 + i\epsilon)} = 3im^2 g^2 \frac{\mu^-}{16\pi^2} \left( -\frac{2}{\epsilon} + \gamma\epsilon \right).$$  (82)

Therefore, since

$$\Sigma^R(p) = \Sigma(p) + p^2 \delta_2 - m^2(\delta_2 + \delta_m),$$  (83)

then $\delta_2 = 0$ and

$$\delta_m m^2 = 3m^2 g^2 \frac{\mu^-}{16\pi^2} \left( -\frac{2}{\epsilon} + \gamma\epsilon \right).$$  (84)

---

The $\delta_2$ is associated to the ELKO spinor wave function renormalization and $\delta_m$ with the renormalization constant regarding the mass, $Z_m$. 

Besides, the 3-point function reads

\[
i\Gamma_\mu = ig^3 \int \frac{d^4k}{(2\pi)^4} \frac{k - q_1}{k^2 - m^2} [k - q_1][k + p]^2 \left( k - q_1 \right) \gamma_\sigma \sigma_\alpha (i\sigma^\beta_\mu p_\beta)(k - q_1) \zeta \sigma_\alpha = ig^3 \int \frac{d^4k}{(2\pi)^4} \frac{i\sigma^\beta_\mu p_\beta}{(k^2 - m^2)[(k + p)^2 - m^2]} = \frac{g^3}{16\pi^2} \left( \frac{\mu - \epsilon}{\epsilon} - \frac{2}{\epsilon} + \gamma E \right),
\]

where

\[
\Gamma_\mu^R = \Gamma_\mu + g \left( \delta_g + \frac{1}{2} \delta_1 \right) \gamma_\beta \mu p_\beta.
\]

Hence one can write

\[
\delta_g + \frac{1}{2} \delta_1 = -g^2 \frac{\mu - \epsilon}{16\pi^2} \left( \frac{2}{\epsilon} + \gamma E \right),
\]

with \( \delta_g \) being associated to the renormalization of the coupling constant. Therefore, we have

\[
\delta_g = g^2 \frac{\mu - \epsilon}{16\pi^2} \left( \frac{2}{\epsilon} + \gamma E \right).
\]

Since the bare coupling can be written as\(^8\) \( g^0 = Z_g g^R \), it yields

\[
\frac{\partial g^R}{\partial \mu} = -\frac{g^3}{8\pi^2}.
\]

Therefore, one can digress about asymptotic freedom for this model. As mentioned, the model complies with unitarity and the optical theorem, if one use the twisted \( \hat{\gamma} \) conjugation, as well as for the Dirac dual case. The imaginary part of the polarization tensor can be written as\(^2\) \( 2\Im[i\pi^\mu_\nu(p)] = \theta_\mu_\nu \Pi(p^2) \). The scalar \( \Pi(p^2) \) is evaluated as

\[
\Pi(p^2) = \frac{2}{3} a^\mu_\nu \Im[i\pi^\mu_\nu(p)]
\]

\[
= -\frac{4g^2}{(2\pi)^2} \int \frac{d^4k}{(2\pi)^2} \theta(k_0) \theta(p_0 - k_0) p^2 \delta((p - k)^2 - m^2) \delta(k^2 - m^2).
\]

As aforementioned, the model is unitary. Hence, associating this result with the formal expression for the decay rate of the photon\(^9\), and considering the polarization identities \( \epsilon^{\alpha\nu} \epsilon_\mu = -1 \) yields a positive \( \Gamma \). For the case of ELKO self-interaction \([1, 16, 17]\),

\[
\left( \tilde{f}(x)f(x) \right)^2,
\]

an analogous decay associated with ELKO and anti-ELKO would be negative and a sign must be included to ensure a probabilistic interpretation. The reason for the absence of an extra sign for

\(^8\) \( Z_g = 1 + \delta_g \)

\(^9\) For a massless photon it is zero, for a massive one, it is indeed positive.
the decay rate is due to the fact that Hermitian conjugating the vertex $\sigma^{\mu\nu}p_{\mu}$ leads also to a minus sign. Therefore, when calculating decay rates, one must consider for the probability density

$$P = \sum_b M_{ab}M_{ba}^\dagger(-1)^N(-1)^V,$$

with $N$ being the number of antifermions in $M_{ab}$ and $V$ denoting the number of vertexes in the amplitude. The fact that the previously mentioned formal decay is positive is important, if one wants to use the Podolsky propagator without any unitarity or causality violation. If one considers a box with four external fermions, there are divergent terms that are different from the bare Lagrangian. Their number may increase if one considers higher loops. The only way to turn the system into a renormalizable one is to consider the Podolsky model for the bosonic sector. In terms of divergences, the use of the Podolsky propagator\(^{10}\) leads to a finite $\Gamma_\mu$, finite external four (or more) legs diagram and a divergent self-energy as

$$i\Sigma_{\text{div}}^\text{pod}(p) = -3(-M^2)g^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2 - m^2)((p - k)^2 - M^2)} = -3M^2g^2\frac{i\mu^-\epsilon}{16\pi^2}\left(-\frac{2}{\epsilon} + \gamma_E\right).$$

(92)

In this case, $\delta_2 = 0$ and

$$m^2(\delta_2 + \delta_m) = -3g^2M^2\frac{\mu^-\epsilon}{16\pi^2}\left(-\frac{2}{\epsilon} + \gamma_E\right).$$

(93)

Therefore,

$$\delta_m = -3\frac{g^2M^2}{m^2}\frac{\mu^-\epsilon}{16\pi^2}\left(-\frac{2}{\epsilon} + \gamma_E\right).$$

(94)

Since the vertex correction is not divergent, the identity $\delta_g + \frac{\delta_1}{2} + \delta_2 = 0$ must hold\(^{11}\), leading to $\delta_g = 2g^2\frac{\mu^-\epsilon}{16\pi^2}\left(-\frac{2}{\epsilon} + \gamma_E\right)$. Therefore, it yields a factor that is the half of the one regarding the previously investigated Maxwellian beta function\(^{12}\)

$$\mu^2\partial g_R/\partial\mu = -\frac{g_R^3}{4\pi^2}.\quad (95)$$

VI. PODOLSKY THEORY AND UNITARITY

This section is devoted to discussing the physical implications that the Podolsky theory which, as we have seen, can be derived when one lowers the order of the derivatives, similarly to a

\(^{10}\) It is given by $-i\left(\frac{1}{p^2} - \frac{1}{p^2 - M^2}\right) = -i\frac{(-M^2)}{p^2(p^2 - M^2)}$.

\(^{11}\) With $\delta_1$ coming from the one loop polarization tensor renormalization. It has the same value for both Maxwell and Podolsky models.

\(^{12}\) For the case of the Podolsky model, the quadratic term in derivatives, proportional to $a^2 \equiv \frac{1}{16\pi^2}$, also contributes to the counterterms associated to the photon 2-point correlation function. Due to the structure of the 1-loop polarization tensor, one must have $a^2(\delta a^2 + \delta_1) = 0$ with $a^2 = Z_a a_R^2$. 


Maxwell theory coupled to a massive Proca model, with a different sign. Therefore, the free theory necessarily leads to unitarity violation [107]. On the other hand, if the model is interacting, the imaginary part of the polarization tensor evaluated at the Podolsky mass does not necessarily vanish. If this object has the right sign, it leads to a kind of resonance called Merlin mode [107]. The normal massless resonance propagates forward in time, with positive energy, whereas the higher-mass pole propagates backward, distinguishing the Merlin mode from the ghost, which has a minus sign in the numerator appearing in the propagator. Faddeev–Popov ghosts present a negative sign in the numerator, however, they carry the usual imaginary unit, in the denominator. In the case of the Merlin mode, a change of sign occurs in the denominator as well, yielding the associated propagator to represent the time-reversed version of the standard resonance propagator. It does not violate unitarity, just microcausality in a small time scale of the form $t \sim 1/|\gamma|$, with $|\gamma|$ being the modulus of the imaginary part evaluated at the propagator pole mass. If it has the opposite sign, this contribution grows indefinitely leading to ill behavior. Therefore, since both theories with ELKO and Dirac duals have a positive definite formal expression for the photon decay, the addition of the Podolsky content can be consistently carried out for these models.

Unitarity is not violated in this case since the lines of a decaying particle are never cut, and do not generate external asymptotic particles. Therefore, we do not consider the external massive photons in our Feynman rules. Up to gauge fixing and longitudinal sectors, the inverse propagator related to the massive sector, associated to $B_\mu(x)$ defined in Sec. IV, evaluated near the renormalized mass, reads

$$P^{-1}_{\mu\nu} = -ig_{\mu\nu}(p^2 - M^2) - i\pi_{\mu\nu}, \quad (96)$$

yielding, up to longitudinal non-physical terms,

$$P_{\mu\nu} = \frac{ig_{\mu\nu}}{(p^2 - M_r^2 - i|\gamma|)}, \quad (97)$$

with $\pi_{\mu\nu} \equiv \theta_{\mu\nu}\Pi$, reflecting the fact that the imaginary part of the trace of the polarization tensor is negative for the QFTs here studied, as well as QED, if they are evaluated for a time-like external momentum as, for example, the case of a massive propagator evaluated near the pole. Then, this massive bosonic excitation becomes a Merlin mode$^{13}$ [107], contributing to improve the renormalizability properties of the system without violating the unitarity. Since both $A_\mu$ and $B_\mu$ appearing in the Podolsky model (66, 67) are coupled to the ELKO, for all internal bosonic lines

$^{13}$ $|\gamma|$ is greater than zero if $M_r > 2m$. 
of the Feynman graphs, the following structure generates the effective combined propagator,

$$\mathcal{G}_{\mu\nu}(p) = -ig_{\mu\nu} \left( \frac{1}{p^2} - \frac{1}{p^2 - M^2} \right) = -ig_{\mu\nu} \frac{(-M^2)}{p^2(p^2 - M^2)}. \quad (98)$$

### VII. THE NON-RELATIVISTIC INTERACTION POTENTIAL

Following Ref. [1], here the vertexes, the gauge propagator, and the external ELKO (chosen to be the self-conjugate ones), are employed in the low-energy approach, to obtain the non-relativistic potential. Here, we consider our fundamental model the one composed of a gauge field in the Podolsky sector and a fermionic sector with the generalized dual. First of all, for obtaining the potential, one should use the expression

$$V(r) = \frac{1}{4E_1E_2} \int e^{ip\cdot r}(-i)\mathcal{M}(p) \frac{d^3p}{(2\pi)^3}. \quad (99)$$

The associated scattering amplitude reads

$$\mathcal{M}_{\alpha,\alpha',\beta,\beta'}(p) = i^2 \frac{g^2}{m^2} \lambda^{S}_{\alpha} (E_q, \vec{q} - \frac{p}{2}) p_{\gamma} \sigma^{\gamma\nu} \lambda^{S}_{\alpha'} (E_q, \vec{q} + \frac{p}{2}) \frac{im^2g_{\nu\mu}}{p^2(p^2 - M^2)} \times \lambda^{S}_{\beta} (E_q, \vec{q} + \frac{p}{2}) - p_{\omega} \sigma^{\omega\mu} \lambda^{S}_{\beta'} (E_q, \vec{q} - \frac{p}{2}). \quad (100)$$

The sign in the momentum of the second vertex is due to the momentum orientation and the Feynman rules as well, in the low energy regime approximation for the spinors.

In order to obtain an expression that do not depend on a specific helicity configuration for external particles, the trace over the helicity space is taken, leading to a potential that is invariant with respect to a change of basis in this internal space

$$\mathcal{M}(p) = \frac{1}{4} \sum_{\alpha,\alpha',\beta,\beta'} \delta^{\alpha'\beta} \delta^{\alpha\beta'} \mathcal{M}_{\alpha,\alpha',\beta,\beta'}(p) = -\frac{1}{4} i^2 Tr (p_{\gamma} \sigma^{\gamma\nu} p_{\omega} \sigma^{\omega\nu}) \frac{iM^2g^2}{p^2(p^2 - M^2)}$$

$$= -3g^2 \frac{iM^2}{(p^2 - M^2)}. \quad (101)$$

Therefore, at low energies, and for $p^\mu = (0, \mathbf{p})$ the expression for the potential yields

$$V(r) = \frac{3g^2}{4m^2} \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot r} \frac{M^2}{(p^2 + M^2)} = -\frac{3g^2M^2}{4m^2} \frac{e^{-mr}}{4\pi r}. \quad (102)$$

---

14. Here the approximation in which the exchanged momentum associated to the propagator line is purely spatial $p^\mu = (0, \mathbf{p})$ with a small norm is employed, such that the particle energy is approximately $\sqrt{m^2 + |\mathbf{q}|^2}$. Then, we consider that the helicity sums, which are associated to spinor bilinears at the same four-momentum point, are valid up to corrections of order $|\mathbf{p}|$.

15. We have not considered the possibility of a product of traces associated to each vertex since it leads to a vanishing result.
After obtaining this Yukawa potential, it is a good moment to point out some important features of the model. The derived calculations show a physical system that does not interact at large distances, being dark for cosmology. On the other hand, due to asymptotic freedom, the intensity of the collisions at very small distances approaches zero. Therefore, this system indeed describes dark matter particles in the sense of macro and micro measurements. Using a Maxwell propagator leads to a delta-like contact interaction following the previously mentioned darkness of the phenomenology. We must demand that even at cosmological distances, the coupling constant is perturbative. Hence, as well as ordinary matter, dark matter has a trilinear coupling with photons organizing its structures. The importance of investigating the photon coupling is that it leads to the possibility of measuring dark matter in experiments, due to the existence of effective vertexes associated with the interaction between electrons and also between protons and other ordinary matter particles that interact with light as well. Therefore, considering the vertex studied here, scattering processes with an electron-muon-like qualitative structure are possible and can be measured.

Beyond the photon, there is also the interaction with the massive intermediate boson that does not show up as external states. It improves quantum properties and also the behavior of the potential. Moreover, we are going to see that at high momentum, the analogous amplitude for the Møller scattering goes to zero instead of being constant as for the QED case. It may be interpreted as another factor that makes it hard to detect it in high-energy colliders. Summing up, it gives another contribution to the darkness of such a system.

VIII. CORRECTED MØLLER SCATTERING WITH THE GENERALIZED ELKO DUAL AND THE TWISTED CONJUGATION PRESCRIPTION

The amplitude for the Møller scattering, considering the Podolsky propagator and the generalized Elko spinor dual, reads

\[
\mathcal{M}_{\alpha,\alpha',\beta,\beta'} = \frac{M^2 g^2}{m^2(u - M^2) t} i (i)^2 \overline{\lambda}_\alpha(p_4) \gamma^\nu(p_1 - p_3) \lambda_\beta^S(p_2) \tilde{\lambda}_{\beta'}^S(p_3) \chi^\nu(p_1 - p_3) \lambda_{\alpha'}^S(p_1) - \frac{M^2 g^2}{m^2(u - M^2) u} i (i)^2 \overline{\lambda}_\alpha(p_3) \gamma^\nu(p_1 - p_4) \lambda_\beta^S(p_2) \tilde{\lambda}_{\beta'}^S(p_4) \chi^\nu(p_1 - p_4) \lambda_{\alpha'}^S(p_1). \]

(103)

The relative negative sign is due to fermionic exchange of identical electrons in the u channel diagram. The \( m \) in the denominator is related to the Feynman rules. Using the \( \xi \) twisted conjugation

\[ \chi^\mu(p) = p_\nu \sigma^{\nu \mu}. \]
yields

\[
\mathcal{M}_{\alpha,\alpha',\beta,\beta'}^k = \frac{M^2 g^2}{m^2(t-M^2)t}(-i)^2 \tilde{\Lambda}^S_{\alpha}(p_1)\chi^\nu(p_1-p_3)\lambda^S_{\beta'}(p_3)\tilde{\Lambda}^S_{\beta}(p_2)\chi^\nu(p_1-p_3)\lambda^S_{\alpha}(p_4)
- \frac{M^2 g^2}{m^2(u-M^2)t}(-i)^2 \tilde{\Lambda}^S_{\alpha}(p_1)\chi^\nu(p_1-p_4)\lambda^S_{\beta'}(p_4)\tilde{\Lambda}^S_{\beta}(p_2)\chi^\nu(p_1-p_4)\lambda^S_{\alpha}(p_3).
\] (104)

Performing the sum over the external particle helicities implies that

\[
\sum_{\alpha,\alpha',\beta,\beta'} \mathcal{M}_{\alpha,\alpha',\beta,\beta'} M_{\alpha,\alpha',\beta,\beta'}^k = \frac{M^2 g^4}{(t-M^2)^2 t^2} Tr\{\chi^\gamma(p_1-p_3)\chi^\nu(p_1-p_3)\} Tr\{\chi^\nu(p_1-p_3)\chi^\gamma(p_1-p_3)\}
+ \frac{M^2 g^4}{(u-M^2)^2 u^2} Tr\{\chi^\gamma(p_1-p_4)\chi^\nu(p_1-p_4)\} Tr\{\chi^\nu(p_1-p_4)\chi^\gamma(p_1-p_4)\}
- \frac{2M^2 g^4}{(u-M^2)u(t-M^2)t} Tr\{\chi^\gamma(p_1-p_3)\chi^\nu(p_1-p_3)\chi^\gamma(p_1-p_4)\chi^\nu(p_1-p_4)\}.
\]

Computing the traces, using \(Tr\{\chi^\nu(p)\chi^\mu(p)\} = -4p^2\theta_{\mu\nu}\) and \(\theta^{\mu\nu}\theta_{\mu\nu} = 3\), and averaging, yields

\[
\frac{1}{4} \sum_{\alpha,\alpha',\beta,\beta'} \mathcal{M}_{\alpha,\alpha',\beta,\beta'} M_{\alpha,\alpha',\beta,\beta'}^k = \frac{12M^4 g^4 t^2}{(t-M^2)^2 t^2} + \frac{12M^4 g^4 u^2}{(u-M^2)^2 u^2} + \frac{6M^4 g^4 ((p_1-p_4)^{\mu}(p_1-p_3)_{\mu})^2}{(u-M^2)u(t-M^2)t}
= \frac{12M^4 g^4}{(t-M^2)^2} + \frac{12M^4 g^4}{(u-M^2)^2}.
\] (105)

The crossed term vanishes. It can be easily understood considering the Mandelstam variables. This amplitude vanishes at high energies and tends to be a constant in the low energy regime. It is worth emphasizing that light ELKO coupling, which might evade any collider constraint do not need to be excluded, and the phenomenological necessity, demanding that the Møller scattering must remain unitary up to 8 TeV [7], can be reformulated considering our improvements.

**IX. ELKO PAIR ANNIHILATION**

In order to evaluate the relic density and the associated freeze out temperature, important phenomenological content to constrain the model, the amplitude for the ELKO pair annihilation must be computed. One can use the approach in Ref. [10], however now using the correct and unitary prescription for the generalized spinor dual and for the twisted conjugation. The same model employed in the previous section will be regarded. Then, considering the expected value for this temperature due to cosmology investigations for dark matter, it is possible to find constraints for the coupling \(g\) and the ELKO mass in a future work. The relevant amplitude has contributions
from the $t$ and $u$ channels. It reads\(^{17}\)

\[
M_{\alpha,\alpha',\beta,\beta'} = \frac{g^2}{m(t-m^2)}(i)^2 \tilde{\lambda}_\beta^A(p_2)\chi^\sigma(p_1-p_3)\chi^\nu(p_1-p_3)\epsilon_\nu^\alpha(p_4)\epsilon_\sigma^\nu(p_3)\lambda_\beta^S(p_1) \\
+ \frac{g^2}{m(u-m^2)}(i)^2 \tilde{\lambda}_\beta^A(p_2)\chi^\sigma(p_1-p_4)\chi^\nu(p_1-p_4)\epsilon_\nu^\alpha(p_4)\epsilon_\sigma^\nu(p_3)\lambda_\beta^S(p_1).
\]

The relative positive sign is due to bosonic photon exchange in the $u$ channel diagram. The $m$ in the denominator is related to the Feynman rules. Using the twisted conjugation implies that

\[
M^\dagger_{\alpha,\alpha',\beta,\beta'} = \frac{g^2}{m(t-m^2)}(-i)^2 \tilde{\lambda}_\beta^A(p_1)\chi^\sigma(p_1-p_3)\epsilon_\nu^\alpha(p_4)\epsilon^*\nu(p_3)\lambda_\beta^S(p_2) \\
+ \frac{g^2}{m(u-m^2)}(-i)^2 \tilde{\lambda}_\beta^A(p_1)\chi^\sigma(p_1-p_4)\epsilon_\nu^\alpha(p_4)\epsilon^*\nu(p_3)\lambda_\beta^S(p_2).
\]

Then, summing over the helicities over the product of these terms yields

\[
-\sum_{\alpha,\alpha',\beta,\beta'} M_{\alpha,\alpha',\beta,\beta'} M^\dagger_{\alpha,\alpha',\beta,\beta'} = \frac{g^4}{(t-m^2)^2} Tr\left\{\chi^\sigma(p_1-p_3)\chi^\nu(p_1-p_3)\chi^\sigma(p_1-p_3)\chi^\nu(p_1-p_3)\right\} \\
+ \frac{g^4}{(u-m^2)^2} Tr\left\{\chi^\sigma(p_1-p_4)\chi^\nu(p_1-p_4)\chi^\sigma(p_1-p_4)\chi^\nu(p_1-p_4)\right\} \\
+ \frac{2g^4}{(u-m^2)(t-m^2)} Tr\left\{\chi^\sigma(p_1-p_3)\chi^\nu(p_1-p_3)\chi^\sigma(p_1-p_4)\chi^\nu(p_1-p_4)\right\}.
\]

This extra overall minus sign is due to the latter polarization sum. Therefore, as previously discussed, it can be associated with probability density, including an averaging factor, as

\[
P = \frac{1}{4} \sum_{\alpha,\alpha',\beta,\beta'} M_{\alpha,\alpha',\beta,\beta'} M^\dagger_{\alpha,\alpha',\beta,\beta'} (-1)^N (-1)^V,
\]

with $V = 2$ and $N = 1$, eliminating the minus sign. Therefore it yields

\[
-\frac{1}{4} \sum_{\alpha,\alpha',\beta,\beta'} M_{\alpha,\alpha',\beta,\beta'} M^\dagger_{\alpha,\alpha',\beta,\beta'} = \frac{9g^4t^2}{(t-m^2)^2} + \frac{9g^4u^2}{(u-m^2)^2} - \frac{6g^4(p_1-p_4)(p_1-p_3)^\mu}{(u-m^2)(t-m^2)}.\]

Considering the definition of Mandelstan variables we have $(p_1-p_4)(p_1-p_3)^\mu = 0$. Then, at low energies, since $t$ and $u$ are proportional to $p^2$,

\[
\frac{1}{4} \sum_{\alpha,\alpha',\beta,\beta'} M_{\alpha,\alpha',\beta,\beta'} M^\dagger_{\alpha,\alpha',\beta,\beta'} \sim 0.
\]

\(^{17}\) The definition $\chi_\alpha(p) = p_\beta^\sigma \sigma_\alpha$ is used.
X. ELKO-PROTON SCATTERING, AN EFFECTIVE VERTEX

In order to detect dark matter in laboratory, it must interact with ordinary matter associated to the detector devices. Therefore, by means of the ELKO-photon interaction vertex, a Møller like scattering between dark matter and protons can occur and, at principle, be measured.

For practical calculation, one needs the effective vertex describing the interaction of this fermionic particle with light. At the low transferred energy regime, it reads

\[ V_\mu \equiv \frac{1}{m'} \sigma_{\mu \nu} p'^\nu + \gamma_\mu, \]  

with \( m' \) being the proton mass. The expression for this amplitude is given by

\[ M_{\alpha, \alpha', \beta, \beta'} = \frac{M^2 e g}{m(t - M^2)} i (i)^2 \lambda_\alpha(p_1) \chi_\alpha'(p_1 - p_3) \lambda_\beta(p_2) \bar{u}_\beta'(p_3) V_\nu(p_1 - p_3) u_\alpha'(p_1), \]  

whose twisted conjugation yields

\[ M^\dagger_{\alpha, \alpha', \beta, \beta'} = \frac{M^2 e g}{m(t - M^2)} (-i) (i)^2 \bar{u}_\alpha'(p_1) V_\nu'(p_1 - p_3) u_\beta'(p_3) \lambda_\beta'(p_2) \chi_\alpha'(p_1 - p_3) \lambda_\alpha(p_4), \]  

with \( V_\mu = -\frac{1}{m'} \sigma_{\mu \nu} p'^\nu + \gamma_\mu \) and \( u(p) \) denoting a spinor that represents an incoming proton. Then, summing over the helicities yields

\[
\sum_{\alpha, \alpha', \beta, \beta'} M_{\alpha, \alpha', \beta, \beta'} M_{\alpha, \alpha', \beta, \beta'}^\dagger = \frac{M^4 g^2 e^2}{(t - M^2)^2 t^2} Tr \left\{ \chi_\gamma(p_1 - p_3) \chi_\gamma'(p_1 - p_3) \right\} \times Tr \left\{ V_\nu(p_1 - p_3)(p_1 + m') V_\nu'(p_1 - p_3)(p_3 + m') \right\} \\
= \frac{M^4 g^2 e^2}{(t - M^2)^2 t^2} \left[ -16 t^2 + 8 t \left( 4 m^2 + \frac{t^2}{m^2} \right) \right].
\]  

Regarding this amplitude, \( V = 1 \) and \( N_f = 0 \). Then, according to the established rule, averaging over the helicity sum, the following positive definite probability density is associated to the physical process

\[ P = (-1)^0 (-1)^\frac{1}{4} \sum_{\alpha, \alpha', \beta, \beta'} M_{\alpha, \alpha', \beta, \beta'} M_{\alpha, \alpha', \beta, \beta'}^\dagger = \left[ 4 t^2 - 2 t \left( 4 m^2 + \frac{t^2}{m^2} \right) \right] \frac{M^4 g^2 e^2}{(t - M^2)^2 t^2}. \]  

The XENON1T experiment has the goal of directly detecting dark matter in the laboratory as weakly interacting particles [108]. One can therefore use the Møller and the pair annihilation, together with the ELKO-proton scattering studied in this section, to obtain experimental limits for the model parameters via XENON1T-type experimental cross sections and through freeze-out temperature, using the structures obtained in these last three sections.
XI. LINEAR TERM FROM THE DIRAC DUAL AND A SOLUTION EXHIBITING A GALAXY FLAT ROTATION CURVE

In this section, we are going to investigate the fermionic response associated with the exotic theory constructed with the Dirac and not the generalized ELKO dual (15) with the addition of a Podolskyan propagator. It is associated with the possibility of a solution in which the so-called flat rotation curve for a planar galaxy occurs. These are consequences of the renormalized ELKO coupled system of equations of motion that, at low energies, due to the presence of the radiative linear term from the self-energy\(^{18}\), reads

\[
(iA\gamma^\mu \partial_\mu + \Box + m^2)\Psi(\vec{r},t) = 0
\]

(116)

with \(A\) fixed by the mentioned extra renormalization condition, whose associated experimental input can be of cosmological nature. This kind of differential operator has a Chern number associated with it, which is a functional of this operator in the momentum space calculated in the frame \(p^\mu = (0, \mathbf{p})\). In the context of condensed matter, namely for topological insulators, it leads to interesting localized states on the boundary of the sample [109, 110]. At principle, our goal here would be to emulate it in the context heretofore presented. The mentioned Chern number reads

\[
n_{\text{Chern}} = \frac{1}{2} \left[ \text{SIGN} \left( -\frac{1}{A} \right) + \text{SIGN} \left( \frac{m^2}{A} \right) \right],
\]

(117)

which is non-trivial just for tachyonic particles, which is not our case. Despite this fact, since there are linear and quadratic terms in derivatives, considering large distances and low energy, a fast decaying solution displaying interesting properties can be possibly derived. Considering spherical coordinates with the fixed azimuthal angle \(\theta = \frac{\pi}{2}\), to study planar galaxies in form of a disc, and discarding derivative terms with \(1/r\) factor due to the cosmological distances, a specific solution is given by

\[
[iA\gamma^r (\partial_r e^{\Omega r}) \Lambda(\vec{r},t) - (\partial_r^2 e^{\Omega r}) \Lambda(\vec{r},t) - Am \Psi(\vec{r},t)] = 0,
\]

(118)

with \(\gamma^r = \gamma^1 \sin \theta \cos \phi + \gamma^2 \sin \theta \sin \phi + \gamma^3 \cos \theta\). The ansatz for the composed ELKO is explicitly given by

\[
\Psi(\vec{r},t) = \Lambda(\vec{r},t)e^{\Omega r},
\]

(119)

with \(\Lambda(\vec{r},t)\) being a linear combination of ELKO self-conjugate fermions with different helicities,

\[
\Lambda(p^\mu) = \lambda^S(p^\mu) - i\lambda^L_+(p^\mu).
\]

(120)

\(^{18}\) See Section V.
Taking into account the coupled system (21) that consists of the equations of motion for the types of ELKO, therefore the equation of motion for the composed ELKO (120) reads

$$\mathcal{P}\Lambda(p^\mu) = m\Lambda(p^\mu).$$ \hspace{1cm} (121)

This condition also implies that $\Lambda(\vec{r},t)$ is an element in the kernel of the Klein–Gordon operator, $(\Box + m^2)\mathbb{I}$. The defining equation (118) was obtained by means of the use of these properties. The explicit matrix form for this equation is given below

$$K_{4\times4}\Psi(\vec{r},t) = 0,$$ \hspace{1cm} (122)

where

$$K_{4\times4} ≡ \begin{bmatrix} -G & 0 & 0 & iAe^{-i\phi}\Omega \\ 0 & -G & iAe^{i\phi}\Omega & 0 \\ 0 & -iAe^{-i\phi}\Omega & -G & 0 \\ -iAe^{i\phi}\Omega & 0 & 0 & -G \end{bmatrix} = -G\mathbb{I} - A\Omega(\sigma_3 \otimes \sigma_3)\mathcal{S}(p^\mu),$$ \hspace{1cm} (123)

with $G = \Omega^2 + Am$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the spin Pauli matrix. It is worth to mention the central role played by the operator $\mathcal{S}(p^\mu)$ in Eq. (13), entering the ELKO spin sums (12) and with the generalized ELKO dual, in Eqs. (15, 18). The freedom in fixing a counterterm for the linear term is used to eliminate divergences as fixing $A$ with a renormalization condition at the low-energy regime, also demanding that $A < 0$, with an extremely small modulus to reproduce the qualitative aspect of the flat rotation curve. The factor $\Omega$ can be found by demanding a vanishing determinant as a necessary condition for this equation to hold. After it, the correspondent ELKO solution can be obtained. To derive $\Omega$, the following identity may be employed,

$$K_{4\times4} = \begin{bmatrix} \tilde{A}_{2\times2} & B_{2\times2} \\ C_{2\times2} & D_{2\times2} \end{bmatrix}, \quad \det K = \det D \det(\tilde{A} - BD^{-1}C).$$ \hspace{1cm} (124)

Particularizing for the case here studied yields

$$\det K = \det G^2 \det \begin{pmatrix} G - G^{-1}\Omega^2A^2 & 0 \\ 0 & G - G^{-1}\Omega^2A^2 \end{pmatrix}.$$ \hspace{1cm} (125)

Therefore, there are four possibilities given by

$$\Omega = \begin{cases} \pm \sqrt{|A|m}, \\ \frac{1}{2} \left( \mp |A| \pm \sqrt{|A|^2 - 4mA} \right) \sim \pm \frac{A}{2} \pm i \sqrt{m|A|}. \end{cases}$$ \hspace{1cm} (126)
The solution can be therefore split into two parts. The first one lies in the range starting from the center of the galaxy up to a critical radius $r_c$. The second range starts from this point. The disc with radius $r_c$ corresponds to the dark matter halo. Therefore, this halo do not disappear far from $r_c$, this is just the region of maximal dark matter density. In the region $I$ it is increasing and, in region $II$, it decreases. A possible solution for the first range, $0 < r < r_c$, is given by

$$\Psi(\vec{r},t) = \left(iAe^{\frac{|A|}{2}(r-r_c)} + Be^{-\frac{|A|}{2}(r-r_c)}\right) \exp\left[i\sqrt{|A|}m(r-r_c)\right] \Lambda(\vec{r},t). \quad (127)$$

It is important to note that

$$\Lambda^\dagger(p^\mu)\gamma_0\Lambda(p^\mu) = 4m, \quad (128)$$

yielding

$$\frac{4m}{(2\pi)^4} \int d^4p = \int d^4x \bar{\Lambda}(x)\Lambda(x). \quad (129)$$

Hence,

$$\frac{4mV_p}{(2\pi)^4V} = \bar{\Lambda}(x)\Lambda(x), \quad (130)$$

with $V_p$ being the volume of the 4-dimensional momentum space and $V$ is the spacetime volume. Considering our solution for cosmological distances, spatial momentum and, therefore, the spatial derivatives ($\hat{p}_k = -i\partial_k$) can be can neglected. The radial one leads just to a contribution of order $\sim |A|$, when acting on the exponential part, and another tiny contribution from the $\Lambda(x)$ structure at large cosmological distances, corresponding to the low momentum regime. The same occurring for the remaining spatial derivatives. In order to derive an expression for the matter density, one must consider these latter approximations in the $T_{00}$ component of the energy momentum tensor of Ref. [1]. Only the scalar-like part of the tensor will be employed, since the fermionic-like sector depends on the four-momentum of the graviton, which is massless and, at low energies, can be discarded. The scalar sector, for the case of an approximately flat geometry, reads

$$T_{00}^{\text{SCALAR}}(x) = \frac{1}{2} \partial_0 \Psi(x)\partial_0 \Psi(x) - \frac{g_{00}}{2} (\partial_i \Psi(x)\partial_i \Psi(x) - m^2 \Psi(x)\Psi(x)). \quad (131)$$

we have discarded the contributions from the spatial derivatives owing to the previously exposed reasons.

Using the equation of motion of the composed ELKO, $\partial \Lambda(x) = im\Lambda$, we can derive, for the approximation employed here $\hat{p} \mapsto \gamma_0 p_0$, the following results

$$\partial_0 \Lambda(x) = im\gamma_0 \Lambda(x), \quad \partial_0 \bar{\Lambda}(x) = -im\bar{\Lambda}(x)\gamma_0. \quad (132)$$
The energy density, in this approximation, reads

\[ T_{00}(x) = \frac{4m^3V_p}{(2\pi)^4V} |\vec{A}|^2e^{\frac{1}{|A|}(r-r_c)} + \frac{4m^3V_p}{(2\pi)^4V} |B|^2e^{-\frac{1}{|A|}(r-r_c)}. \]  

(133)

It is necessary to conveniently choose constants to properly identify this with a matter density. Besides, one can verify that

\[ \Psi(\vec{r},t)\Psi(\vec{r},t) = \frac{4m^3V_p}{(2\pi)^4V} |\vec{A}|^2e^{\frac{1}{|A|}(r-r_c)} + \frac{4m^3V_p}{(2\pi)^4V} |B|^2e^{-\frac{1}{|A|}(r-r_c)}. \]  

(134)

Considering a galaxy in the form of a disc, one can obtain

\[ M_I = 2\pi \int_0^R r \, T_{00}(r) \, dr, \]  

(135)

yielding

\[ M_I \sim \frac{4m^3V_p \, 32\pi}{(2\pi)^4V |A|} \left[ |\vec{A}|^2e^{\frac{1}{|A|}(R-r_c)} - |B|^2e^{-\frac{1}{|A|}(R-r_c)} \right] R, \]  

(136)

where one must demand \(|\vec{A}|^2 > |B|^2\). The following range of model parameters have been considered

\[ |A(R-r_c)| \lesssim 10^{-1} \quad \text{and also} \quad |A(r_c)| \lesssim 10^{-1}. \]  

Then, it leads to a constraint for experimental limits regarding \( A \) in order to derive a galaxy dark matter mass that grows approximately at a linear rate, with \( e^{-|A|(R-r_c)} \sim 1 \) and \( e^{\frac{1}{|A|}(r_c)} \sim 1 \). Therefore, if one wants to reproduce cosmological data, the intensity of the coefficient of the anomalous self energy linear term must be fixed in order to fulfil this condition. Then, the order of magnitude for the halo \( r_c \) and the visible galaxy radius should be taken into account since, for region \( I, R_g < R < r_c \). This procedure can be understood as a renormalization condition for this anomalous self energy contribution.

Considering these constraints and denoting by \( R_g \) the visible galaxy radius, the velocity for \( R_g < R < r_c \) the region in which a unexpected flat rotation curve occurs

\[ v^2(R) = \frac{G}{R} \left[ M_T + \frac{8m^3V_p}{|A|\pi^3V} \left( |\vec{A}|^2 - |B|^2 \right) R \right], \]  

(137)

where \( M_T \) denotes the total visible mass. The constants involved in the previous equations can be fixed by using experimental and observational links to cosmology, as flat rotation curves. Limits can be hence imposed for the constant coupling through this type of data. Ref. [111] imposes experimental limits on the ELKO mass through thermodynamic considerations related to the order of magnitude of the size of a galaxy and the degeneracy pressure to maintain a volume of free ELKO.

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19 In this integration, we considered the term \( e^{\frac{|A|(R-r_c)}} \sim 1 \) in the subtraction on the integration limits.

20 In fact, without considering any approximation, it is proportional to

\[ \frac{|\vec{A}|^2}{|A|} e^{\frac{|A|(R-r_c)}} \left( R + \frac{1}{|A|}e^{\frac{|A|(R-r_c)}} \right) - \frac{|B|^2}{|A|} e^{-\frac{|A|(R-r_c)}} \left( R + \frac{1}{|A|}e^{-\frac{|A|(R-r_c)}} \right). \]
gas for the case of these dimensions. It would be interesting to develop a similar approach for our planar disc galaxy case.

For completeness, we present the solution for the region outside the halo $R > r_c$ with decreasing dark matter density

$$
\Sigma(\vec{r}, t) = \left( iC e^{-\frac{|A|}{2}(r-r_c)} e^{i\sqrt{|A|m}(r-r_c)} + De^{-\frac{|A|}{2}(r-r_c)} e^{-i\sqrt{|A|m}(r-r_c)} \right) \Lambda(\vec{r}, t) \tag{138}
$$

To have a well-defined solution, continuity at $r_c$ must be demanded,

$$
\Sigma(\vec{r}_c, t) = \Psi(\vec{r}_c, t), \quad \frac{\partial}{\partial r} \Sigma(\vec{r}, t)|_{r_c} = \frac{\partial}{\partial r} \Psi(\vec{r}, t)|_{r_c}. \tag{139}
$$

These conditions lead to $i\tilde{A} + B = iC + D$ and $\tilde{A} = -\frac{2B}{|A|}$, respectively. As previously mentioned, physical solutions must have $|\tilde{A}| > |B|$. Finally, it is important to mention that, for region $II$, we have $R > r_c$. Then, for sufficiently large distances $|A(R-r_c)| \geq 1$, the exact result for the integration (135) must be considered, leading to an exponential decaying behaviour describing the expected continuously decreasing mass profile in $II$.

**XII. CONCLUDING REMARKS AND PERSPECTIVES**

We investigated the completely peculiar role played by ELKO in QFT as non-standard quantum fields with new physical signatures. For the ELKO-photon interaction, the spin sums and the Feynman rules were derived, using the Dirac spinor dual. Due to the use of the Dirac spinor dual, some issues involving the polarization tensor and bubble diagrams of ELKO and anti-ELKO are presented and solved. Also, unitarity aspects of ELKO in this context are discussed. The Podolsky sector in a QFT for ELKO was introduced and the associated propagator explored, together with radiative corrections. Renormalization features were also addressed. When taking into account the ELKO generalized spinor dual, instead of the Dirac spinor dual, the unitarity was analyzed and the different spinor field conjugations were analyzed, with the use of the twisted conjugation. Therefore, the polarization tensor regarding ELKO was computed, regulating divergences under renormalization procedures. ELKO self-interaction was then included and decay rates were studied, also employing the Podolsky propagator. The associated beta functions have been also computed. With these tools, one can refine constraints regarding the dark matter described by ELKO in monophoton events at the LHC. Podolsky QFT and unitarity aspects involving ELKO scattering

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21 For these real parameters, the equalities $\tilde{A} = C$ and $D = B$ hold.
processes were addressed, where the vertexes, the gauge propagator, and the external ELKO were scrutinized in the low-energy approach, to obtain the non-relativistic potential. To study the scattering amplitude, the Podolsky gauge sector field and ELKO with the generalized dual were employed, yielding Yukawa-like potential. The interaction between the ELKO and the proton was also investigated. The corrected Møller scattering with the generalized ELKO spinor dual as well as the twisted conjugation was discussed. We also derived an important result for a future investigation of the relic density and the freeze-out temperature, computing the scattering amplitude for the ELKO pair annihilation. When using a linear term arising from the self-energy corrections for the model composed of ELKO spinors with the Dirac dual, a solution exhibiting a galaxy flat rotation curve was studied, implementing the ELKO fermionic response associated with the exotic theory constructed with the Dirac spinor dual instead of the generalized ELKO dual, with the addition of a Podolskyan propagator. The results in the context of a comparison with topological insulators tools in condensed matter were also discussed. Besides, we proved that the Lagrangian (80), which carries the interaction between ELKO and the electromagnetic strength tensor, is invariant under the twisted conjugation.

As ELKO is a dark quantum field, being a prime candidate to describe dark matter particles, the tree level ELKO-Higgs interaction can be also further studied in the context here presented, emulating the previous results in Ref. [5], wherein ELKO yields the relic abundance probed by WMAP. Also, a triple coupling scenario can be scrutinized, for the ELKO mass can be generated by a mechanism related to the electroweak symmetry breaking. The Higgs decay into ELKO/anti-ELKO pair, with monojet production, can be also better studied [6].

**Declaration of competing interest.** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Data Availability Statements:** the datasets generated during and/or analyzed during the current study are available from the corresponding author upon reasonable request.

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[1] D. V. Ahluwalia, J. M. H. da Silva, C.-Y. Lee, Y.-X. Liu, S. H. Pereira, and M. M. Sorkhi, Phys. Rept. 967, 1 (2022), 2205.04754.
[2] D. Ahluwalia, *Mass Dimension One Fermions*, vol. 229 (Cambridge University Press, 2019), 2007.15098.
[3] D. V. Ahluwalia and S. Sarmah, EPL 125, 30005 (2019), 1810.04985.
[4] C.-Y. Lee and M. Dias, Phys. Rev. D 94, 065020 (2016), 1511.01160.
[5] A. Alves, F. de Campos, M. Dias, and J. M. Hoff da Silva, Int. J. Mod. Phys. A 30, 1550006 (2015), 1401.1127.
[6] A. Alves, M. Dias, and F. de Campos, Int. J. Mod. Phys. D 23, 1444005 (2014), 1410.3766.
[7] A. Alves, M. Dias, F. de Campos, L. Duarte, and J. M. Hoff da Silva, EPL 121, 31001 (2018), 1712.05180.
[8] L. Duarte, M. Dias, and F. de Campos, Eur. Phys. J. ST 229, 2133 (2020).
[9] M. Dias, F. de Campos, and J. M. Hoff da Silva, Phys. Lett. B 706, 352 (2012), 1012.4642.
[10] B. Agarwal, P. Jain, S. Mitra, A. C. Nayak, and R. K. Verma, Phys. Rev. D 92, 075027 (2015), 1407.0797.
[11] R. J. Bueno Rogerio, J. M. Hoff da Silva, M. Dias, and S. H. Pereira, JHEP 02, 145 (2018), 1709.08707.
[12] L. C. Duarte, R. d. C. Lima, R. J. B. Rogerio, and C. H. C. Villalobos, Adv. Appl. Clifford Algebras 29, 66 (2019), 1705.10302.
[13] R. J. B. Rogerio and J. M. Hoff da Silva, EPL 118, 10003 (2017), 1602.05871.
[14] J. M. Hoff da Silva and R. J. Bueno Rogerio, EPL 128, 11002 (2019), 1908.00458.
[15] D. V. Ahluwalia, EPL 131, 41001 (2020), 2008.02630.
[16] D. V. Ahluwalia and D. Grumiller, JCAP 07, 012 (2005), hep-th/0412080.
[17] D. V. Ahluwalia and D. Grumiller, Phys. Rev. D 72, 067701 (2005), hep-th/0410192.
[18] D. V. Ahluwalia, C.-Y. Lee, and D. Schritt, Phys. Lett. B 687, 248 (2010), 0804.1854.
[19] D. V. Ahluwalia, C.-Y. Lee, and D. Schritt, Phys. Rev. D 83, 065017 (2011), 0911.2947.
[20] A. E. Bernardini and R. da Rocha, Phys. Lett. B 717, 238 (2012), 1203.1049.
[21] L. Fabbri, Eur. Phys. J. ST 229, 2117 (2020), 1910.11082.
[22] L. Fabbri, Adv. Appl. Clifford Algebras 28, 7 (2018), [Erratum: Adv.Appl.Clifford Algebras 28, 74 (2018)], 1711.05119.
[23] L. Fabbri and S. Vignolo, Int. J. Mod. Phys. D 23, 1444001 (2014), 1407.8237.
[24] R. da Rocha, A. E. Bernardini, and J. M. Hoff da Silva, JHEP 04, 110 (2011), 1103.4759.
[25] C.-Y. Lee, Eur. Phys. J. C 81, 90 (2021), 1809.04381.
[26] C.-Y. Lee, Phys. Rev. D 93, 045011 (2016), 1512.09175.
[27] C.-Y. Lee, Int. J. Mod. Phys. A 31, 1650187 (2016), 1510.04983.
[28] L. Fabbri and S. Vignolo, Int. J. Theor. Phys. 51, 3186 (2012), 1201.5498.
[29] L. Fabbri, Gen. Rel. Grav. 43, 1607 (2011), 1008.0334.
[30] L. Fabbri, Phys. Rev. D 85, 047502 (2012), 1101.2566.
[31] L. Fabbri and S. Vignolo, Annalen Phys. 524, 77 (2012), 1012.4282.
[32] L. Fabbri, Phys. Lett. B 704, 255 (2011), 1011.1637.
[33] L. Fabbri, Mod. Phys. Lett. A 25, 2483 (2010), 0911.5304.
[34] R. da Rocha and J. M. Hoff da Silva, Int. J. Geom. Meth. Mod. Phys. 6, 461 (2009), 0901.0883.
[35] J. A. Nieto and E. A. León, Int. J. Mod. Phys. A 37, 2250032 (2022).
[36] R. T. Cavalcanti, J. M. Hoff da Silva, and R. da Rocha, Eur. Phys. J. Plus 129, 246 (2014), 1401.7527.
[37] G. P. de Brito, J. M. Hoff Da Silva, and V. Nikoofard, Eur. Phys. J. ST 229, 2023 (2020), 1912.02912.
[38] J. A. Nieto, Mod. Phys. Lett. A 34, 1950211 (2019), 1907.00740.
[39] R. J. B. Rogerio and L. Fabbri, Proc. Roy. Soc. Lond. A A 478, 20210893 (2022), 2203.05992.
[40] R. Dale, A. Herrero, and J. A. Morales-Lladosa (2022), 2206.00639.
[41] J. M. Hoff da Silva, R. J. Bueno Rogerio, and N. C. R. Quinquiolo (2022), 2203.02065.
[42] R. da Rocha and J. M. Hoff da Silva, Adv. Appl. Clifford Algebras 20, 847 (2010), 0811.2717.
[43] L. Fabbri and R. J. B. Rogerio, Mod. Phys. Lett. A 36, 2150124 (2021), 2011.09366.
[44] R. da Rocha and J. M. Hoff da Silva, J. Math. Phys. 48, 123517 (2007), 0711.1103.
[45] L. Fabbri and R. J. B. Rogerio, Eur. Phys. J. C 80, 880 (2020), 2004.14155.
[46] R. J. Bueno Rogerio, C. H. Coronado Villalobos, and A. R. Aguirre, Eur. Phys. J. C 79, 991 (2019), 1911.01742.
[47] J. M. Hoff da Silva, C. H. Coronado Villalobos, R. J. Bueno Rogerio, and E. Scatena, Eur. Phys. J. C 76, 563 (2016), 1608.05365.
[48] R. J. Bueno Rogerio, J. M. Hoff da Silva, S. H. Pereira, and R. da Rocha, EPL 113, 60001 (2016), 1603.09183.
[49] C.-Y. Lee, Phys. Lett. B 760, 164 (2016), 1404.5307.
[50] C.-Y. Lee, Int. J. Mod. Phys. A 30, 1550048 (2015), 1210.7916.
[51] J. Vaz, Int. J. Theor. Phys. 57, 582 (2018).
[52] A. G. Nikitin, Int. J. Mod. Phys. D 23, 1444007 (2014), 1403.0880.
[53] P. Meert and R. da Rocha, Eur. Phys. J. C 78, 1012 (2018), 1809.01104.
[54] K. P. S. de Brito and R. da Rocha, J. Phys. A 49, 415403 (2016), 1609.06495.
[55] L. Bonora and R. a. da Rocha, JHEP 01, 133 (2016), 1508.01357.
[56] L. Bonora, K. P. S. de Brito, and R. da Rocha, JHEP 02, 069 (2015), 1411.1590.
[57] R. Lopes and R. da Rocha, JHEP 08, 084 (2018), 1802.06413.
[58] A. Yanes and R. da Rocha, PTEP 2018, 063B09 (2018), 1803.08282.
[59] D. M. Dantas, R. da Rocha, and C. A. S. Almeida, EPL 117, 51001 (2017), 1512.07888.
[60] X.-N. Zhou, Y.-Z. Du, Z.-H. Zhao, and Y.-X. Liu, Eur. Phys. J. C 78, 493 (2018), 1710.02842.
[61] Y.-X. Liu, X.-N. Zhou, K. Yang, and F.-W. Chen, Phys. Rev. D 86, 064012 (2012), 1107.2506.
[62] M. Moazzen Sorkhi and Z. Ghalenovi, Eur. Phys. J. C 80, 314 (2020).
[63] I. C. Jardim, G. Alencar, R. R. Landim, and R. N. Costa Filho, Phys. Rev. D 91, 085008 (2015), 1411.6962.
[64] X.-N. Zhou, X.-Y. Ma, Z.-H. Zhao, and Y.-Z. Du (2018), 1812.08332.
[65] R. V. Maluf, D. M. Dantas, and C. A. S. Almeida, Eur. Phys. J. C 80, 442 (2020), 1905.04824.
[66] S. Vignolo, S. Carloni, R. Cianci, F. Esposito, and L. Fabbri, Class. Quant. Grav. 39, 015009 (2022), 2112.03628.
[67] R. da Rocha and J. M. Hoff da Silva, EPL 107, 50001 (2014), 1408.2402.
[68] R. da Rocha and R. T. Cavalcanti, Phys. Atom. Nucl. 80, 329 (2017), 1602.02441.
[69] R. T. Cavalcanti and R. da Rocha, Adv. High Energy Phys. 2016, 4681902 (2016), 1507.03714.
[70] S. H. Pereira and T. M. Guimarães, JCAP 09, 038 (2017), 1702.07385.
[71] S. Kouwn, J. Lee, T. H. Lee, and P. Oh, Mod. Phys. Lett. A 28, 1350121 (2013), 1211.2981.
[72] S. H. Pereira, R. F. L. Holanda, and A. P. S. Souza, EPL 120, 31001 (2017), 1703.07636.
[73] S. H. Pereira and R. C. Lima, Int. J. Mod. Phys. D 26, 1730028 (2017), 1612.02240.
[74] A. Basak and S. Shankaranarayanan, JCAP 05, 034 (2015), 1410.5768.
[75] A. Basak, J. R. Bhatt, S. Shankaranarayanan, and K. V. Prasanth Varma, JCAP 04, 025 (2013), 1212.3445.
[76] H. M. Sadjadi, Gen. Rel. Grav. 44, 2329 (2012), 1109.1961.
[77] B. Podolsky, Phys. Rev. 62, 68 (1942).
[78] M. C. Bertin, B. M. Pimentel, and G. E. R. Zambrano, J. Math. Phys. 52, 102902 (2011), 0907.1078.
[79] D. V. Ahluwalia, Adv. Appl. Clifford Algebras 27, 2247 (2017), 1601.03188.
[80] D. V. Ahluwalia, Proc. Roy. Soc. Lond. A 476, 20200249 (2020), 2008.01525.
[81] D. V. Ahluwalia and S. P. Horvath, JHEP 11, 078 (2010), 1008.0436.
[82] V. V. Dvoeglazov, Nuovo Cim. A 108, 1467 (1995), hep-th/9506083.
[83] V. V. Dvoeglazov, Int. J. Theor. Phys. 34, 2467 (1995), hep-th/9504158.
[84] D. V. Ahluwalia and A. C. Nayak, Int. J. Mod. Phys. D 23, 1430026 (2015), 1502.01940.
[85] L. D. Sperançá, Int. J. Mod. Phys. D 23, 1444003 (2014), 1304.4794.
[86] R. da Rocha and W. A. Rodrigues, Jr., Mod. Phys. Lett. A 21, 65 (2006), math-ph/0506075.
[87] L. Fabbri and R. da Rocha, Phys. Lett. B 780, 427 (2018), 1711.07873.
[88] R. T. Cavalcanti, Int. J. Mod. Phys. D 23, 1444002 (2014), 1408.0720.
[89] R. Abłamowicz, I. Gonçalves, and R. a. da Rocha, J. Math. Phys. 55, 103501 (2014), 1409.4550.
[90] J. M. Hoff da Silva and R. da Rocha, Phys. Lett. B 718, 1519 (2013), 1212.2406.
[91] R. T. Cavalcanti and J. M. Hoff da Silva, Eur. Phys. J. C 80, 325 (2020), 2004.00385.
[92] R. J. Bueno Rogerio, Eur. Phys. J. C 80, 299 (2020), 1911.08506.
[93] R. J. Bueno Rogerio, Eur. Phys. J. C 79, 929 (2019), 1911.02386.
[94] J. M. Hoff da Silva and R. T. Cavalcanti, Phys. Lett. A 383, 1683 (2019), 1904.03999.
[95] D. Beghetto, R. J. Bueno Rogerio, and C. H. C. Villalobos, J. Math. Phys. 60, 042301 (2019), 1806.02210.
[96] R. J. Bueno Rogerio, J. M. Hoff da Silva, and C. H. Coronado Villalobos, Phys. Lett. A 402, 127368 (2021), 2010.08597.
[97] R. da Rocha and J. G. Pereira, Int. J. Mod. Phys. D 16, 1653 (2007), gr-qc/0703076.
[98] R. J. B. Rogerio, Mod. Phys. Lett. A 35, 2050319 (2020), 2009.08318.
[99] R. da Rocha, L. Fabbri, J. M. Hoff da Silva, R. T. Cavalcanti, and J. A. Silva-Neto, J. Math. Phys. 54, 102505 (2013), 1302.2262.
[100] M. R. A. Arcodia, M. Bellini, and R. da Rocha, Eur. Phys. J. C 79, 260 (2019), 1902.08833.
[101] R. da Rocha and A. A. Tomaz, J. Phys. A 53, 465201 (2020), 2003.03619.
[102] C. H. Coronado Villalobos, R. J. Bueno Rogerio, A. R. Aguirre, and D. Beghetto, Eur. Phys. J. C 80, 228 (2020), 1906.11622.
[103] J. M. Hoff da Silva and R. T. Cavalcanti, Mod. Phys. Lett. A 32, 1730032 (2017), 1708.06222.
[104] L. Bonora, J. M. Hoff da Silva, and R. da Rocha, Eur. Phys. J. C 78, 157 (2018), 1711.00544.
[105] J. M. Hoff da Silva, R. T. Cavalcanti, D. Beghetto, and R. da Rocha, Eur. Phys. J. C 80, 117 (2020), 1910.00995.
[106] S. H. Pereira, A. P. S. S., J. M. Hoff da Silva, and J. F. Jesus, JCAP 01, 055 (2017), 1608.02777.
[107] J. F. Donoghue and G. Menezes, Phys. Rev. D 104, 045010 (2021), 2105.00898.
[108] E. Aprile et al. (XENON), Phys. Rev. D 100, 052014 (2019), 1906.04717.
[109] M. I. Monastyrsky and V. S. Retakh, Commun. Math. Phys. 103, 445 (1986).
[110] F. Schindler, J. Appl. Phys. 128, 221102 (2020), 2012.05308.
[111] S. H. Pereira and R. S. Costa, Mod. Phys. Lett. A 34, 1950126 (2019), 1807.06944.