ISOMONODROMIC DEFORMATIONS AND VERY STABLE VECTOR BUNDLES OF RANK TWO

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Abstract. For the universal isomonodromic deformation of an irreducible logarithmic rank two connection over a smooth complex projective curve of genus at least two, consider the family of holomorphic vector bundles over curves underlying this universal deformation. In a previous work we proved that the vector bundle corresponding to a general parameter of this family is stable. Here we prove that the vector bundle corresponding to a general parameter is in fact very stable (it does not admit any nonzero nilpotent Higgs field).

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2010 Mathematics Subject Classification. 14H60, 34M56, 53B15.
Key words and phrases. Logarithmic connections, isomonodromic deformations, very stable bundle, Teichmüller space.

The first author is supported by a J. C. Bose Fellowship. The second author is supported by ANR-13-BS01-0001-01 and ANR-13-JS01-0002-01.
1. Introduction

Let $E \to X$ be a rank two holomorphic vector bundle over a compact Riemann surface $X$ of genus $g$. The Segre-invariant of $E$ is defined as

$$\kappa(E) := \min_L \left( \deg(E) - 2\deg(L) \right),$$

where the minimum is taken over all holomorphic line subbundles $L$ of $E$. This $\kappa(E)$ is bounded above by $g \lfloor g/2 \rfloor$. If $\kappa(E) > 0$ then $E$ is called stable, and it is called maximally stable if $\kappa(E) > g - 2$ with $g \geq 2$. Now let $\pi : \mathcal{X} \to \mathcal{T}$ be a holomorphic family of compact Riemann surfaces of genus $g$, and let $\mathcal{E}$ be a holomorphic vector bundle over $\mathcal{X}$. For every $t \in \mathcal{T}$, set $X_t := \pi^{-1}(t)$ and $E_t := \mathcal{E}|_{X_t}$. We shall further denote

$$\mathcal{T}_k := \{t \in \mathcal{T} \mid \kappa(E_t) \leq k\} \quad (1.1)$$

for every $k \in \mathbb{Z}$. From a theorem of Maruyama and Shatz, [Maru], [Sh], it follows that $\mathcal{T}_k$ is a closed analytic subset of $\mathcal{T}$ for each $k$. The codimension of $\mathcal{T}_k$ in $\mathcal{T}$ will be denoted by $\text{codim}(\mathcal{T}_k, \mathcal{T})$. By convention, the empty set has infinite codimension. The Riemann–Hilbert type question addressed in [He2], [BHH], rephrased for the particular case of rank 2 vector bundles, would be the following:

**What is the maximal Segre-invariant for a family of holomorphic vector bundles that can be endowed with a flat logarithmic connection, inducing a prescribed logarithmic connection on a central parameter $t_0 \in \mathcal{T}$?**

Indeed, the general theme of [BHH], continued to be pursued here, is that the isomonodromic deformations provide a good “transverse” family of deformations of a vector bundle, in which exceptional behaviors, such as instability, seem to occur with an appropriate codimension. Here we consider the stronger condition of very stability. The very stable bundles were introduced by Laumon [La] and play an important role in the study of the Hitchin systems. They are generic but turn out to be extraordinarily hard to produce explicitly. In fact, the existence of very stable bundles is a landmark result [La]; the dominance of the Hitchin map was obtained as a consequence of the existence of very stable bundles.

Let $D = \{x_1, \ldots, x_n\} \subset X$ be a finite subset of $n \geq 0$ distinct points. The corresponding reduced divisor $\sum_{i=1}^n x_i$ on $X$ will also be denoted by $D$. Let $\delta$ be a logarithmic connection, singular over $D$, on a holomorphic vector bundle $E \to X$ (see Section 2.1 for its definition). An isomonodromic deformation of the quadruple $(X, D, E, \delta)$ is given by the following data:

- a holomorphic family of vector bundles $\mathcal{E} \to \mathcal{X} \to \mathcal{T}$ as above,
- a divisor $\mathcal{D}$ on $\mathcal{X}$ given by the sum of the images of $n$ disjoint holomorphic sections $\mathcal{T} \to \mathcal{X}$,
- a flat logarithmic connection $\nabla$ on $\mathcal{E}$ singular over $\mathcal{D}$, and
- an isomorphism of pointed curves $f : (X, D) \to (X_{t_0}, D_{t_0})$, where $t_0 \in \mathcal{T}$ is a base point, together with a holomorphic isomorphism $\psi : E \to f^*\mathcal{E}$ of vector bundles such that $f^*\nabla = \psi_*(\delta)$. 

Any quadruple \((X, D, E, \delta)\) as above with \(3g - 3 + n > 0\) admits a universal isomonodromic deformation, satisfying a universal property for the isomonodromic deformations as above, with respect to a germification \((\mathcal{T}, t_0)\) of the parameter space \([He1]\). This is a consequence of the logarithmic Riemann–Hilbert correspondence \([De]\) combined with Malgrange’s lemma \([Mal]\). For this universal isomonodromic deformation, the parameter space is the Teichmüller space \(\text{Teich}_{g,n}\), with a central parameter corresponding to \((X, D)\), and the family of pointed curves \((\mathcal{X}, \mathcal{D})\) over it being the universal Teichmüller curve. We recall the main theorem of \([He2]\).

**Theorem 1.1** (\([He2]\)). Let \((X, D, E, \delta)\) be as above with \(3g - 3 + n > 0\) and \(\delta\) irreducible. Let \(\mathcal{E} \to (\mathcal{X}, \mathcal{D}) \to \mathcal{T}\) be the holomorphic family of vector bundles over pointed curves underlying the universal isomonodromic deformation of \((X, D, E, \delta)\). Then for the filtration
\[
\cdots \subset \mathcal{T}_k \subset \cdots \subset \mathcal{T}_{g-1} \subset \mathcal{T}_g = \mathcal{T}
\]
by closed analytic subsets defined in (1.1), the inequality
\[
\text{codim}(\mathcal{T}_k, \mathcal{T}) \geq g - 1 - k
\]
holds for all \(k\), in particular \(\text{codim}(\mathcal{T}_{g-2}, \mathcal{T}) > 0\), implying that the vector bundle \(E_t\) is maximally stable for general \(t \in \mathcal{T}\).

Let again \(E\) be a holomorphic vector bundle on \(X\) of rank two. A **Higgs field** on \(E\) is a holomorphic section of \(\text{End}(E) \otimes K_X = E \otimes E^* \otimes K_X\), where \(K_X\) denotes the holomorphic cotangent bundle of \(X\). Given a Higgs field \(\theta \in H^0(X, \text{End}(E) \otimes K_X)\), we have
\[
\text{trace}(\theta^i) \in H^0(X, K_X^\otimes i), \quad i = 1, 2.
\]
A Higgs field \(\theta\) on \(E\) is called **nilpotent** if \(\text{trace}(\theta) = 0 = \text{trace}(\theta^2)\), or equivalently, if \(\theta^2 = 0\); the nilpotent cone plays a very important role in Geometric Langlands program (see \([Fr]\), \([KW]\), \([GW]\), \([DP]\) and references therein).

A holomorphic vector bundle \(E\) of rank two is called **very stable** if it does not admit any nonzero nilpotent Higgs field. For any \(g \geq 1\), there are maximally stable bundles which are not very stable. But a very stable vector bundle \(E\) is always maximally stable. Indeed, if \(L \subset E\) is a holomorphic line subbundle with \(\deg(E) - 2 \cdot \deg(L) \leq g - 2\), then
\[
\chi(\text{Hom}(E/L, L \otimes K_X)) = h^0(\text{Hom}(E/L, L \otimes K_X)) - h^1(\text{Hom}(E/L, L \otimes K_X))
\]
\[
= \deg(\text{Hom}(E/L, L \otimes K_X)) = -(\deg(E) - 2 \cdot \deg(L)) + g - 1 \geq 1,
\]
so \(H^0(X, \text{Hom}(E/L, L \otimes K_X)) \neq 0\). Now, for any nonzero element
\[
\theta' \in H^0(X, \text{Hom}(E/L, L \otimes K_X)),
\]
the Higgs field \(\theta\) on \(E\) defined by the composition
\[
E \to E/L \xrightarrow{\theta'} L \otimes K_X \to E \otimes K_X
\]
has the property that \(\theta^2 = 0\). By a similar argument one can show that \(E\) is very stable if and only if \(H^0(X, \text{Hom}(E/L, L \otimes K_X)) = 0\) for every holomorphic line subbundle \(L\) of \(E\).
From now on, we are going to assume that the genus $g$ of $X$ is at least two. Note that then the maximally stable bundles are stable. Let $\mathcal{E} \rightarrow \mathcal{X} \rightarrow \mathcal{T}$ be a holomorphic family of rank two vector bundles over curves of genus $g$. The subset
\[ \mathcal{T}^{\text{nil}} := \{ t \in \mathcal{T} \mid E_t \rightarrow X_t \text{ is not very stable} \} \] 
(1.2)
is a closed analytic subset of $\mathcal{T}$. The following is a natural question to ask:

Is the subset $\mathcal{T}^{\text{nil}}$ of the parameter space of a given universal isomonodromic deformation proper?

Our aim here is to prove the following stronger version of Theorem 1.1:

**Theorem 1.2.** Let $(X, D, E, \delta)$ be as above with $g \geq 2$. Assume that $\delta$ is irreducible. Let $\mathcal{E} \rightarrow (\mathcal{X}, D) \rightarrow \mathcal{T}$ be the holomorphic family of vector bundles over pointed curves underlying the universal isomonodromic deformation of $(X, D, E, \delta)$. Then the closed analytic subset $\mathcal{T}^{\text{nil}}$, defined in (1.2), is a proper closed subset of $\mathcal{T}$. In particular, the vector bundle $E_t$ is very stable for general $t \in \mathcal{T}$.

In view of Theorem 1.1 it is enough to consider the case where the holomorphic vector bundle $E$ corresponding to the central parameter is stable but not very stable. Following Donagi and Pantev [DP], stable vector bundles that are not very stable will be called *wobbly* vector bundles.

Let $\mathcal{E} \rightarrow \mathcal{X} \rightarrow \mathcal{T}$ be a family of wobbly rank two vector bundles over compact Riemann surfaces of genus $g$. Then there is a polydisc $\mathcal{U}$ in $\mathcal{T}$ and a section $\Theta$ of $(\text{End}(\mathcal{E}) \otimes \Omega^1_{\mathcal{X}})|_{\mathcal{U}}$ such that for each $t \in \mathcal{U}$, the restriction $\Theta|_{X_t}$ defines a nonzero nilpotent Higgs field on $E_t$. Indeed, since one has a family of stable vector bundles on $X$, the family of possible Higgs fields for $\mathcal{E}$ is a holomorphic vector bundle $V$ over $\mathcal{T}$. The constraint $\theta^2 = 0$ of nilpotence then gives us a cone $\mathcal{N}$ in $V$, which maps surjectively onto $\mathcal{T}$ since the family is wobbly. One can then choose a smooth point of $V$ for which the tangent plane to $\mathcal{N}$ surjects to that of $\mathcal{T}$, and get our section $\Theta$ from there.

For that reason, in order to prove Theorem 1.2 we may use deformation theory of stable nilpotent Higgs bundles over curves. A *Higgs bundle* on $X$ is a pair of the form $(E, \theta)$, where $E$ is a holomorphic vector bundle on $X$ of rank two and $\theta$ is a Higgs field on $E$. Such a Higgs bundle $(E, \theta)$ is called *nilpotent* if the Higgs field $\theta$ is nilpotent; if $2 \cdot \deg(L) < \deg(E)$ for every holomorphic line subbundle $L \subset E$ with $\theta(L) \subset L \otimes K_X$, then $(E, \theta)$ is called *stable*, so $(E, \theta)$ is stable if $E$ is stable. More specifically, we are going to elaborate, in Section 1.2, the deformation theory of stable nilpotent Higgs bundles over pointed curves.

## 2. ISOMONODROMIC DEFORMATIONS

We begin by recalling a few results from Sections 2 and 4 in [BHH] that would be needed later; note that in [BHH] the Riemann surface is denoted $X_0$ instead of $X$.

### 2.1. Logarithmic connections

Let $(E, X, D)$ be as in the introduction. Let $\text{Diff}^\text{d}(E, E)$ denote the holomorphic vector bundle on $X$ defined by the sheaf of differential operators, of
order at most $i$, from the sheaf of holomorphic sections of $E$ to itself. We have a short exact sequence

$$0 \to \text{Diff}^0(E, E) = \text{End}(E) \to \text{Diff}^i(E, E) \to TX \otimes \text{End}(E) \to 0, \quad (2.1)$$

where $\sigma_1$ is the symbol homomorphism. Define

$$A_{\log} D(E) := \sigma_1^{-1}(TX(-\log D) \otimes \text{Id}_E) \subset \text{Diff}^i(E, E).$$

Since $X$ is one-dimensional, we have $TX(-\log D) = TX(-D) := TX \otimes \mathcal{O}_X(-D)$. From (2.1) it follows that $A_{\log} D(E)$ fits in an exact sequence of holomorphic vector bundles on $X$

$$0 \to \text{End}(E) \to A_{\log} D(E) \to TX(-\log D) \to 0, \quad (2.2)$$

where $\sigma$ is the restriction of the surjection $\sigma_1$ in (2.1).

A logarithmic connection on $E$ singular over $D$ is a holomorphic homomorphism

$$\delta : TX(-\log D) \to A_{\log} D(E)$$

such that

$$\sigma \circ \delta = \text{Id}_{TX(-\log D)}, \quad (2.3)$$

where $A_{\log} D(E)$ and $\sigma$ are constructed in (2.2). We note that (2.3) implies that the image $\delta(TX(-\log D))$ is a holomorphic line subbundle of $A_{\log} D(E)$.

Similarly, we can construct the logarithmic Atiyah bundle $A_{\log} D(E)$ over $X$ for a family $E \to (X, D) \to T$ as in the introduction. Note that $TX(-\log D)$ fits in the short exact sequence

$$0 \to TX(-\log D) \to TX \to N_D \to 0,$$

where $N_D$ is the normal bundle to $D$. So, in this case $TX \otimes \mathcal{O}_X(-D)$ is a subsheaf of $TX(-\log D)$.

Again, a logarithmic connection $\nabla$ on $E$ singular over $D$ is a splitting of the corresponding Atiyah exact sequence. Rather than recalling the definition of flatness for $\nabla$ (see for example [BHH, p. 131, Lemma 4.1]), we just point out the following. A logarithmic connection $\delta$ as in (2.3) is called irreducible if there is no holomorphic line subbundle $L'$ of $E$ such that $\delta(TX(-\log D)(L')) \subset L'$. If $\nabla$ is flat then the connection $\nabla|_{E_t}$ on $E_s \to X_s$ is irreducible for some $s \in T$ if and only if $\nabla|_{E_t}$ is irreducible for all $t \in T$.

2.2. Infinitesimal deformations. As before, $E \to X$ is a rank two holomorphic vector bundle over a compact Riemann surface $X$ of genus $g \geq 2$, and $D$ is a reduced effective divisor of degree $n$ on $X$; let $\delta$ be a logarithmic connection on $E$ with polar divisor $D$. An infinitesimal deformation of the $n$-pointed Riemann surface $(X, D)$ is a family

$$X \xrightarrow{f} X \xrightarrow{\nabla} \mathcal{B} := \text{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$$
together with a divisor $D$ on $\mathcal{X}$ given by $n$ disjoint sections $\mathcal{B} \rightarrow \mathcal{X}$ such that $f^*D = D$. The space of infinitesimal deformations of $(X, D)$ is given by $H^1(X, TX(-D))$, which coincides with the fiber of the holomorphic tangent bundle of $\text{Teich}_{g,n}$ at the point corresponding to $(X, D)$. Any such nontrivial deformation naturally embeds into the universal Teichmüller curve. The space of infinitesimal deformations of $(X, D)$ is given by $H^1(X, T_X(-D))$, which coincides with the fiber of the holomorphic tangent bundle of Teich$_{g,n}$ at the point corresponding to $(X, D)$. Any such nontrivial deformation naturally embeds into the universal Teichmüller curve. The space of infinitesimal deformations of the triple $(X, D, E)$ is given by $H^1(X, \text{At}_D(E))$ [BHH, p. 127, (2.12)]. On the other hand, if we have an infinitesimal deformation of $(X, D, E)$ that can be endowed with a flat logarithmic connection $\nabla$, inducing $\delta$ on the central parameter, then by the universal property of the universal isomonodromic deformation, the deformation of $(X, D, E)$ is already determined by the induced deformation of $(X, D)$. Hence $\delta$ defines a homomorphism of infinitesimal deformations $\delta: H^1(X, T_X(-D)) \rightarrow H^1(X, \text{At}_D(E))$.

**Lemma 2.1.** The homomorphism $T^\delta$ in (2.4) coincides with the homomorphism $\delta_* : H^1(X, TX(-D)) \rightarrow H^1(X, \text{At}_D(E))$ associated to the connection homomorphism $\delta : TX(-D) \rightarrow \text{At}_D(E)$.

**Proof.** Using the notation of [BHH, p. 131, Lemma 4.1] and the commutative diagram in [BHH, p. 128, Section 2], we have the commutative diagram of homomorphisms of sheaves

$$
\begin{array}{ccccccccc}
0 & \rightarrow & TX(-D) & \rightarrow & f^*TX(-\log D) & \rightarrow & O_X & \rightarrow & 0 \\
& & \downarrow{\delta} & & \downarrow{\nabla} & & \downarrow{\text{id}} & & \downarrow{\text{id}} \\
0 & \rightarrow & \text{At}_D(E) & \rightarrow & f^*\text{At}_D(E) & \rightarrow & O_X & \rightarrow & 0
\end{array}
$$

From (2.5) we have the commutative diagram of cohomologies

$$
\begin{array}{cccccc}
\mathbb{C} = H^0(X, O_X) & \xrightarrow{a} & H^1(X, TX(-D)) & \xrightarrow{\delta_*} & H^1(X, \text{At}_D(E)) \\
& & \downarrow{=} & & \downarrow{=} \\
\mathbb{C} = H^0(X, O_X) & \xrightarrow{b} & H^1(X, \text{At}_D(E))
\end{array}
$$

where $a$ (respectively, $b$) is the connecting homomorphism in the long exact sequence of cohomologies associated to the top (respectively, bottom) exact sequence in (2.5).

Now, $a(1) \in H^1(X, TX(-D))$ is the infinitesimal deformation class for the family of $n$-pointed Riemann surfaces $(\mathcal{X}, D)$ [BHH, p. 125, (2.4)], and $b(1) \in H^1(X, \text{At}_D(E))$ is the infinitesimal deformations of $(X, D, E)$ associated to the isomonodromic deformation of $\delta$ [BHH, p. 132, Lemma 4.2]. Therefore, the proof is completed by the commutativity of (2.6).

### 3. Higgs bundles on a fixed curve

In this section, we recall the deformation theory of Higgs bundles over a fixed Riemann surface.
3.1. Stable Higgs bundles. Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. For any integer $d$, let $\mathcal{M}_X^d$ denote the moduli space of stable vector bundles on $X$ of rank two and degree $d$. It is an irreducible smooth quasi-projective variety defined over $\mathbb{C}$ of dimension $4g - 3$. The locus of very stable bundles in $\mathcal{M}_X^d$ is a nonempty Zariski open subset [La], [BR, p. 229, Corollary 5.6].

Fix an integer $d$. Let

$$\mathcal{M}_X := \mathcal{M}_X^d$$

be the moduli space of stable vector bundles on $X$ of rank two and degree $d$. Let

$$\mathcal{U}_{\text{vs}} \subset \mathcal{M}_X$$

be the moduli space of very stable vector bundles on $X$ of rank two and degree $d$. As noted before, $\mathcal{U}_{\text{vs}}$ is a nonempty Zariski open subset of $\mathcal{M}_X$. Let

$$\mathcal{W} := \mathcal{M}_X \setminus \mathcal{U}_{\text{vs}}$$

be the complement consisting of wobbly bundles, which is a closed sub-scheme of $\mathcal{M}_X$.

For any $E \in \mathcal{M}_X$, we have $T^*_E \mathcal{M}_X = H^0(X, \text{End}(E) \otimes K_X)$, and hence the total space $T^* \mathcal{M}_X$ of the holomorphic cotangent bundle of $\mathcal{M}_X$ in (3.1) is a Zariski open dense subset of the moduli space of stable Higgs bundles of rank two and degree $d$ (openness follows from [Maru]). Let

$$\mathcal{N}_X \subset T^* \mathcal{M}_X$$

be the locus of all $(E, \theta)$ such that $\theta$ is nonzero nilpotent; this $\mathcal{N}_X$ is a locally closed sub-scheme of $T^* \mathcal{M}_X$. The closure of $\mathcal{N}_X$ in $T^* \mathcal{M}_X$ is the union $\mathcal{N}_X \cup \mathcal{W}$, where $\mathcal{W}$ is the wobbly locus defined in (3.2).

3.2. Infinitesimal deformations of Higgs bundles. Take any Higgs bundle $(E, \theta) \in T^* \mathcal{M}_X$. Consider the $\mathcal{O}_X$–linear homomorphism

$$f_\theta : \text{End}(E) = E \otimes E^* \longrightarrow \text{End}(E) \otimes K_X$$

defined by $A \mapsto \theta \circ A - A \circ \theta$. This produces the following two–term complex $\mathcal{C}^{(E,\theta)}_\cdot$ on $X$

$$\mathcal{C}^{(E,\theta)}_0 = \text{End}(E) \longrightarrow \mathcal{C}^{(E,\theta)}_1 = \text{End}(E) \otimes K_X.$$  (3.4)

The space of all infinitesimal deformations of the Higgs bundle $(E, \theta)$ is parametrized by the first hypercohomology $\mathbb{H}^1(\mathcal{C}^{(E,\theta)}_\cdot)$ [BR], [Mark], [Bo]. In particular, we have

$$T_{(E,\theta)}(T^* \mathcal{M}_X) = \mathbb{H}^1(\mathcal{C}^{(E,\theta)}_\cdot).$$

Now assume that $(E, \theta) \in \mathcal{N}_X$, where $\mathcal{N}_X$ is constructed in (3.3). Note that this implies that $E \in \mathcal{W}$. The subsheaf of $E$ defined by the kernel of $\theta$ is a line subbundle of $E$; this line subbundle of $E$ will be denoted by $L$. Let

$$\text{End}^s(E) \subset \text{End}(E)$$

be the subbundle of rank three given by the kernel of the natural homomorphism

$$\text{End}(E) \longrightarrow \text{Hom}(L, E/L)$$
that sends any locally defined endomorphism $A$ of $E$ to the composition

$$L \hookrightarrow E \xrightarrow{A} E \twoheadrightarrow E/L;$$

in other words, $\text{End}^s(E)$ is the subsheaf of endomorphisms of $E$ that preserve the line subbundle $L$. So we have a homomorphism $\text{End}^s(E) \to \text{Hom}(L, L)$ that sends any locally defined section $A$ of $\text{End}^s(E)$ to $A|_L$. This also implies that we have a homomorphism $\text{End}^n(E) \to \text{Hom}(E/L, E/L)$. Let

$$\text{End}^n(E) \subset \text{End}^s(E) \quad (3.5)$$

be the line subbundle given by the kernel of the homomorphism

$$\text{End}^s(E) \to \text{Hom}(L, L) \oplus \text{Hom}(E/L, E/L)$$

constructed above, so $\text{End}^n(E)$ consists of nilpotent endomorphisms of $E$ with respect to the filtration $0 \subset L \subset E$; the superscripts “$s$” and “$n$” stand for “solvable” and “nilpotent” respectively.

For the homomorphism $f_\theta$ in (3.4) we have

$$f_\theta(\text{End}^s(E)) \subset \text{End}^n(E) \otimes K_X. \quad (3.6)$$

The restriction of $f_\theta$ to $\text{End}^s(E)$ will be denoted by $f_\theta^s$. We have the two-term complex $D_\bullet(E, \theta)$ on $X$

$$D_0^{(E, \theta)} = \text{End}^s(E) \xrightarrow{f_\theta^s} D_1^{(E, \theta)} = \text{End}^n(E) \otimes K_X \quad (3.7)$$

which is a subcomplex of $C_\bullet^{(E, \theta)}$ constructed in (3.4).

Denote the degree of the line bundle $L$ by $\nu$. Consider $\mathcal{N}_X$ in (3.3); let

$$Z^{\nu} \subset \mathcal{N}_X$$

be the locus of all $(E', \theta') \in \mathcal{N}_X$ such that $\text{deg}(\text{kernel}(\theta')) = \nu$. So we have $(E, \theta) \in Z^{\nu}$. The tangent space to $Z^{\nu}$ at this point $(E, \theta)$ has the following description:

$$T_{(E, \theta)} Z^{\nu} = \mathbb{H}^1(D_\bullet^{(E, \theta)}),$$

where $D_\bullet^{(E, \theta)}$ is the complex in (3.7) [BR, p. 228].

4. **Higgs bundles on a family of curves**

We now recall the deformation theory of families of Higgs bundles over moving curves. As noted in Section 2.2, infinitesimal deformations of a triple $(X, D, E)$ is given by the first cohomology space of the logarithmic Atiyah bundle associated to this triple. Now we have to enrich this space with the data corresponding to deformations of a given Higgs field on $E$. Afterwards, we will calculate the obstruction space for an initial nonzero nilpotent Higgs field to extend to a nilpotent Higgs field on a germification of the family.
4.1. Infinitesimal deformation of a \( n \)-pointed curve and Higgs bundle. Let \( E \) be a rank 2 vector bundle on \( X \) of genus \( g \geq 2 \), and let \( D = \sum_{i=1}^{n} x_i \) be a divisor on \( X \).

There is a natural homomorphism

\[
\eta : \text{At}_D(E) \longrightarrow \text{Diff}^1(\text{End}(E) \otimes K_X, \text{End}(E) \otimes K_X),
\]

where \( \text{Diff}^1(\text{End}(E) \otimes K_X, \text{End}(E) \otimes K_X) \) is the vector bundle on \( X \) defined by the sheaf of differential operators of order at most one mapping locally defined holomorphic sections of \( \text{End}(E) \otimes K_X \) to itself. To explain this \( \eta \), if

- \( \alpha \) is a locally defined holomorphic section of \( \text{At}_D(E) \),
- \( \beta \) is a locally defined holomorphic section of \( \text{End}(E) \),
- \( \omega \) is a locally defined holomorphic section of \( K_X \) and
- \( s \) is a locally defined holomorphic section of \( E \),

then

\[
(\eta(\alpha))(\beta \otimes \omega)(s) = \alpha(\beta(s)) \otimes \omega + \beta(s) \otimes (L_{\alpha}(\omega)) - \beta(\alpha(s)) \otimes \omega,
\]

where \( L_{\alpha}(\omega) \) is the Lie derivative of \( \omega \) with respect to the vector field \( \sigma(\alpha) \) (the homomorphism \( \eta \) is defined in \( \text{[2.2]} \)); note that both sides of \( (4.2) \) are sections of \( E \otimes K_X \). To prove that \( (4.2) \) defines \( \eta \), substitute \( (f \beta) \otimes ( \frac{1}{f} \omega) \) in place of \( \beta \otimes \omega \) in \( (4.2) \), where \( f \) is a locally defined nowhere vanishing holomorphic function on \( X \). Then in the right–hand side of \( (4.2) \), the first term becomes \( \alpha(\beta(s)) \otimes \omega + \frac{d(f\alpha)}{f} \cdot \beta(s) \otimes \omega \) while the second term becomes \( \beta(s) \otimes (L_{\alpha}(\omega)) - \frac{d(f\alpha)}{f} \cdot \beta(s) \otimes \omega \). Therefore, we have \( (\eta(\alpha))(\beta \otimes \omega) = (\eta(\alpha))((f \beta) \otimes ( \frac{1}{f} \omega)) \). From this it follows immediately that \( \eta \) is well–defined by \( (4.2) \).

We will give another description of the homomorphism \( \eta \) which is more canonical.

Let \( p : P_{\text{GL}(2)} \longrightarrow X \) be the holomorphic principal \( \text{GL}(2, \mathbb{C}) \)–bundle on \( X \) corresponding to \( E \); so for any \( x \in X \), the fiber \( p^{-1}(x) \) is the space of all \( \mathbb{C} \)–linear isomorphisms from \( \mathbb{C}^{\oplus 2} \) to the fiber \( E_x \). Let

\[
dp : T P_{\text{GL}(2)} \longrightarrow p^*TX \tag{4.3}
\]

be the differential of the above projection \( p \). The kernel

\[
T_{\text{rel}} := \ker(dp) \subset TP_{\text{GL}(2)}
\]

is identified with \( p^*\text{End}(E) \). The pullback \( p^*\text{At}_D(E) \) is identified with \( (dp)^{-1}(TX(-D)) \subset TP_{\text{GL}(2)} \). The image of \( p^*K_X \) under the dual homomorphism \( (dp)^* : p^*K_X \longrightarrow T^*P_{\text{GL}(2)} \) (see \( 4.3 \)) will be denoted by \( \mathcal{S} \). For any

- open subset \( U \subset X \),
- \( \text{GL}(2, \mathbb{C}) \)–invariant holomorphic section \( \alpha \) of \( (dp)^{-1}(TX(-D)) = p^*\text{At}_D(E) \) defined over \( p^{-1}(U) \), and
- \( \text{GL}(2, \mathbb{C}) \)–invariant holomorphic section \( \beta \) of \( T_{\text{rel}} \otimes \mathcal{S} \) over \( p^{-1}(U) \),

the Lie derivative \( L_\alpha \beta \) is again a \( \text{GL}(2, \mathbb{C}) \)–invariant holomorphic section of \( T_{\text{rel}} \otimes \mathcal{S} \) over \( p^{-1}(U) \). The homomorphism \( \text{At}_D(E) \longrightarrow \text{Diff}^1(\text{End}(E) \otimes K_X, \text{End}(E) \otimes K_X) \) defined by \( \alpha \mapsto \{ \beta \mapsto L_\alpha \beta \} \) coincides with \( \eta \).
Let \( \theta \) be a Higgs field on the vector bundle \( E \). Let
\[
\psi_\theta : \operatorname{At}_D(E) \to \operatorname{End}(E) \otimes K_X, \quad \alpha \mapsto \eta(\alpha)(\theta)
\] (4.4)
be the \( \mathcal{O}_X \)-linear homomorphism, where \( \eta \) is the homomorphism in (4.1).

**Lemma 4.1** ([Bi, p. 105, Proposition 2.4]). The space of infinitesimal deformations of a \( n \)-pointed Riemann surface equipped with a Higgs bundle 
\[
\mathfrak{z} = (X, D, (E, \theta))
\]
as above is the first hypercohomology \( H^1(\mathcal{A}_\mathfrak{z}^\mathfrak{e}) \), where \( \mathcal{A}_\mathfrak{z}^\mathfrak{e} \) is the two–term complex
\[
\mathcal{A}_\mathfrak{z}^\mathfrak{e} = \operatorname{At}_D(E) \xrightarrow{\psi_\theta} \mathcal{A}_\mathfrak{z}^\mathfrak{e} = \operatorname{End}(E) \otimes K_X,
\]
where \( \psi_\theta \) is constructed in (4.1).

Note that in the case where \( \theta \) is the zero–Higgs field, the space of infinitesimal deformations in Lemma 4.1 coincides with \( H^1(X, \operatorname{At}_D(E)) \oplus H^0(X, \operatorname{End}(E) \otimes K_X) \); as shown in Section 2.1, the space of infinitesimal deformations of the triple \((X, D, E)\) is \( H^1(X, \operatorname{At}_D(E)) \).

### 4.2. Deformation of a \( n \)-pointed curve with a nilpotent Higgs bundle.

Consider the data \( \mathfrak{z} = (X, D, (E, \theta)) \) in Lemma 4.1. Assume that \((E, \theta) \in \mathcal{N}_X\), where \( \mathcal{N}_X \) is defined in (3.3). As before, the line subbundle of \( E \) defined by the kernel of \( \theta \) will be denoted by \( L \). Let
\[
\operatorname{At}_D(E, L) \subset \operatorname{At}_D(E)
\]
be the holomorphic subbundle of co-rank one generated by the sheaf of differential operators \( \alpha \) such that \( \alpha(L) \subset L \). Note that we have a commutative diagram
\[
\begin{array}{ccccccccc}
0 & \to & \operatorname{End}^\bullet(E) & \to & \operatorname{At}_D(E, L) & \to & \mathcal{L}(D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \operatorname{End}(E) & \to & \operatorname{At}_D(E) & \to & \mathcal{L}(D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}(L, E/L) & \to & \operatorname{Hom}(L, E/L) & \to & 0 & & 0 & & 0 \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\] (4.5)

where all the rows and columns are exact; the bottom exact row is the one in (2.2) and \( \operatorname{End}^\bullet(E) \) is the vector bundle in (3.5). It can be shown that the homomorphism \( \psi_\theta \) in (4.1) satisfies the equation
\[
\psi_\theta(\operatorname{At}_D(E, L)) \subset \operatorname{End}^\bullet(E) \otimes K_X.
\]
Indeed, this follows from the expression in the right–hand side of (4.2).

The restriction of \( \psi_\theta \) to \( \operatorname{At}_D(E, L) \) will also be denoted by \( \psi_\theta \); this should not cause any confusion.
Lemma 4.2. The space of infinitesimal deformations of the \( n \)-pointed Riemann surface equipped with a nilpotent Higgs bundle

\[ z = (X, D, (E, \theta)) \]

such that the Higgs field remains nilpotent, is the first hypercohomology \( \mathbb{H}^1(B^z_\bullet) \), where \( B^z_\bullet \) is the two-term complex

\[ B^z_0 = \mathcal{A}_0 = \text{At}_D(E, L) \xrightarrow{\psi_\theta} B^z_1 = \text{End}^s(E) \otimes K_X. \]

\[ \text{Proof. Consider the short exact sequence of complexes} \]

\[
\begin{array}{cccccccc}
0 & \xrightarrow{} & B^z_0 & \xrightarrow{\psi_\theta} & B^z_1 & \xrightarrow{\psi_\theta} & A^z_0 & \xrightarrow{\psi_\theta} & A^z_1 & \xrightarrow{f_\theta} & \text{End}(E)/\text{End}^s(E) \otimes K_X & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & \mathcal{A}_0 & \xrightarrow{\psi_\theta} & \mathcal{A}_1 & \xrightarrow{\psi_\theta} & \text{End}(E) \otimes K_X & \xrightarrow{} & (\text{End}(E)/\text{End}^s(E)) \otimes K_X & \xrightarrow{} & 0 & \xrightarrow{} & 0
\end{array}
\]

where the middle complex is the one in Lemma 4.1 and the homomorphism \( f_\theta \) is induced by \( f_\theta \) in (3.4); note that from (3.6) it follows immediately that \( f_\theta \) induces such a homomorphism.

Consider the above complex \( \widetilde{C}_\bullet \)

\[ \widetilde{C}_0 = \text{End}(E)/\text{End}^s(E) \xrightarrow{f_\theta} \widetilde{C}_1 = (\text{End}(E)/\text{End}^s(E)) \otimes K_X. \]

The homomorphism \( \mathbb{H}^1(A^z_\bullet) \rightarrow \mathbb{H}^1(\widetilde{C}_\bullet) \) induced by the homomorphism of complexes in (4.6) coincides with the natural homomorphism from the infinitesimal deformations of \( (X, D, (E, \theta)) \) (without assuming that the Higgs field remains nilpotent) to the failure of the Higgs field to be nilpotent. The lemma follows from it. \( \square \)

5. Proof of Theorem 1.2

Let \( E \) be a holomorphic vector bundle on \( X \) of rank two and degree \( d \), which is wobbly. Let \( \theta \) be a nonzero nilpotent Higgs field on \( E \). As before, let \( L \subset E \) be the line subbundle defined by the kernel of \( \theta \). Given a logarithmic connection \( \delta \) on \( E \) with polar divisor \( D \), consider the composition homomorphism

\[ q \circ \delta : TX(-D) \rightarrow \text{Hom}(L, E/L), \]

where \( q \) is the homomorphism in (4.5). Let

\[ (q \circ \delta)_* : H^1(X, TX(-D)) \rightarrow H^1(X, \text{Hom}(L, E/L)) \]

be the homomorphism of cohomologies induced by \( q \circ \delta \) in (5.1).
Proposition 5.1. Let \((X, \mathcal{D}, \mathcal{E})\) be an infinitesimal deformation, with parameter space \(\mathcal{B} = \text{Spec} (\mathbb{C}[\varepsilon]/\varepsilon^2)\), of a stable rank two vector bundle \(E\) over a \(n\)-pointed Riemann surface \((X, D)\). Assume that \(\mathcal{E}\) is endowed with a nonzero nilpotent Higgs field \(\Theta\), inducing a nonzero nilpotent Higgs field \(\theta\) on \(E\). Further assume that \(\mathcal{E}\) is endowed with a flat logarithmic connection \(\nabla\), singular over \(\mathcal{D}\), inducing a logarithmic connection \(\delta\) on \(E\) with polar divisor \(D\). Then

\[(q \circ \delta)_* = 0,\]

where \((q \circ \delta)_*\) is the homomorphism in (5.2).

Proof. Consider the complex \(\mathcal{B}_z^\bullet\) in Lemma 4.2. The identity map of \(\mathcal{B}_0^z\) produces a homomorphism of complexes

\[
\begin{array}{ccc}
\mathcal{B}_0^z & \xrightarrow{\psi} & \mathcal{B}_1^z \\
\downarrow & & \downarrow \\
\text{At}_D(E, L) & \rightarrow & 0
\end{array}
\]

The \(i\)-th hypercohomology of the bottom complex coincides with \(H^i(X, \text{At}_D(E, L))\). Let

\[
\phi : H^1(\mathcal{B}_0^z) \rightarrow H^1(X, \text{At}_D(E, L))
\]

be the homomorphism of hypercohomologies associated to the above homomorphism of complexes.

Since the nilpotent Higgs field \(\theta\) on \(E\) extends as a nilpotent Higgs field \(\Theta\) on \(\mathcal{E}\), from Lemma 4.2 it follows that the homomorphism \(T^\delta\) in (2.4) factors as

\[
T^\delta = \iota_* \circ \phi \circ T^\delta_0,
\]

where

\[
T^\delta_0 : H^1(X, TX(-D)) \rightarrow H^1(\mathcal{B}_0^z)
\]

is a homomorphism, \(\phi\) is the homomorphism in (5.3) and

\[
\iota_* : H^1(X, \text{At}_D(E, L)) \rightarrow H^1(X, \text{At}_D(E))
\]

is the homomorphism of cohomologies induced by the inclusion \(\iota : \text{At}_D(E, L) \hookrightarrow \text{At}_D(E)\) (see (4.5)).

From Lemma 2.1 we know that

\[(q \circ \delta)_* = q_* \circ T^\delta,\]

where

\[
q_* : H^1(X, \text{At}_D(E)) \rightarrow H^1(X, \text{Hom}(L, E/L))
\]

is the homomorphism of cohomologies induced by the homomorphism \(q\) in (4.5). Now from (5.4) and (5.5) it follows that

\[(q \circ \delta)_* = q_* \circ \iota_* \circ \phi \circ T^\delta_0 .\]

On the other hand, from (4.5) it follows immediately that \(q_* \circ \iota_* = 0\). Hence we conclude that \((q \circ \delta)_* = 0\). \(\square\)
Let $\text{Teich}_{g,n}$ be the Teichmüller space for $n$-pointed surfaces of genus $g \geq 2$. Take $(X, D, E, \delta)$ as before. Let $(X', D', E', \nabla)$ be the universal isomonodromic deformation of $(X, D, E, \delta)$, with parameter space $\mathcal{T} = \text{Teich}_{g,n}$ and central parameter $t_0 \in \mathcal{T}$, together with an isomorphism of $n$-pointed surfaces $f : (X, D) \to (X_{t_0}, D_{t_0})$ and a holomorphic isomorphism $\psi : E \to f^*E$ such that $f^*\nabla = \psi^*(\delta)$ as in the introduction. Note that for any $t_1 \in \mathcal{T}$, the universal isomonodromic deformation $(X', D', E', \nabla)$ is also the universal isomonodromic deformation of $(X', D', E', \nabla)|_{t_1}$. As before, for any $t \in \mathcal{T}$, we shall denote $(X', E')|_{t}$ by $(X_t, E_t)$.

In view of Theorem 1.1 and the openness of the very stability condition, the following theorem implies Theorem 1.2.

**Theorem 5.2.** Let $(X, D, E, \delta)$ and $(X', D, E', \nabla)$ be as above with $\delta$ irreducible. Then there is a point $t \in \mathcal{T}$ such that the vector bundle $E_t \to X_t$ is very stable.

**Proof.** In view of Theorem 1.1 we can take $E$ to be stable. Assume that $E$ is wobbly. Let $\theta$ be a nonzero nilpotent Higgs field on $E$. In view of Proposition 5.1 it suffices to show that

$$
(q \circ \delta)_* \neq 0,
$$

where $(q \circ \delta)_*$ is the homomorphism in (5.2). Note that the homomorphism $q \circ \delta$ in (5.1) is nonzero because the logarithmic connection $\delta$ is irreducible. Consider the short exact sequence of coherent sheaves

$$
0 \to TX(-D) \xrightarrow{q \circ \delta} \text{Hom}(L, E/L) \to \mathbb{T} \to 0,
$$

where $\mathbb{T}$ is a torsion sheaf on $X$, so $H^1(X, \mathbb{T}) = 0$. Hence the long exact sequence of cohomologies associated to it gives a surjection

$$
H^1(X, TX(-D)) \xrightarrow{(q \circ \delta)_*} H^1(X, \text{Hom}(L, E/L)) \to 0.
$$

Therefore, to prove (5.6) it suffices to show that

$$
H^1(X, \text{Hom}(L, E/L)) \neq 0 .
$$

(5.7)

Now, by Serre duality,

$$
H^1(X, \text{Hom}(L, E/L)) = H^0(X, \text{Hom}(E/L, L) \otimes K_X)^*.
$$

Since $\theta$ is nonzero nilpotent and $L \subset E$ is the kernel of $\theta$, it follows immediately that $\theta$ is a nonzero section of $\text{Hom}(E/L, L) \otimes K_X$. Therefore, (5.7) is proved. This completes the proof of the theorem. \qed

When the rank is more than two, the very last part of the argument in the proof of Theorem 5.2 breaks down — the nonzero nilpotent Higgs field $\theta$ no longer implies that $(q \circ \delta)_* \neq 0$.

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