Null-plane Quantum Poincaré Algebras
and their Universal $R$-matrices

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Abstract
A non-standard quantum deformation of the Poincaré algebra is presented in a null-plane framework for 1+1, 2+1 and 3+1 dimensions. Their corresponding universal $R$-matrices are obtained in a factorized form by choosing suitable bases related to the $T$-matrix formalism.

1 Introduction

Among quantum deformations of the Poincaré algebra we find three remarkable Hopf structures:

• Two standard deformations as quantizations of coboundary Lie bialgebras coming from skew-solutions of the modified classical Yang–Baxter equation (YBE). Physically these are obtained in a purely kinematical framework encoded within the usual Poincaré basis. They are the $\kappa$-Poincaré algebra [1, 2, 3] where the deformation parameter can be interpreted as a fundamental time scale and a $q$-Poincaré algebra [4] where the quantum parameter is a fundamental length.

• One non-standard deformation as a quantization of a (triangular) coboundary Lie bialgebra coming from a skew-solution of the classical YBE. It is closely related to the Jordanian or $h$-deformation for $sl(2, \mathbb{R})$ introduced in [5, 6, 7]. This structure has been called ‘null-plane’ quantum Poincaré algebra [8] since it is constructed in a null-plane context where the Poincaré invariance splits into a kinematical and dynamical part [8, 10].

On the problem of obtaining universal $R$-matrices for the above quantum Poincaré algebras the results, as far as we know, are as follows:
For standard Poincaré deformations, a universal $R$-matrix has been only found in 2+1 dimensions \[11\] by means of a contraction procedure starting from $so(4)_q$. There are not known universal $R$-matrices neither for the 1+1 case nor for the 3+1 case, although for 1+1 there are partial results \[12\].

For non-standard Poincaré deformations some results have been obtained recently by following different methods:

1+1 case. i) By an explicit construction \[13, 14\] leading to a expression similar to the one obtained in \[15\] for a Hopf subalgebra of $U_h sl(2, \mathbb{R})$. ii) Following a universal $T$-matrix construction \[16\] giving rise to a factorized form \[17\]. iii) A contraction process from the universal $R$-matrix of $U_h sl(2, \mathbb{R})$ can also be applied.

2+1 case. By means of a contraction method, taking as starting point the non-standard deformation of $so(2, 2)$ \[18, 19\].

3+1 case. By applying a non-linear change of basis inspired in the results of the above cases and by using a universal $T$-matrix approach; the resulting $R$-matrix is factorized \[20\].

Factorized universal $R$-matrices are given by ordered (usual) exponentials of the elements appearing within the corresponding classical $r$-matrix, thus they adopt a simple form. This kind of factorized expressions are relevant when $T$ is interpreted as a transfer matrix in quantum field theory \[21\]. These $R$-matrices could be useful in order to construct new integrable examples linked to the null-plane evolution scheme (e.g., the infinite momentum frame approach \[22\], gauge field theory quantized on null-planes \[23\] and applications in Hadron spectroscopy \[24\]).

In the sequel we present the results concerning the factorized universal $R$-matrices for the non-standard quantum Poincaré algebra in a null-plane framework up to 3+1 dimensions, paying special attention to the $T$-matrix method. There is a main point in our procedure: the Poincaré generators involved in the non-standard classical $r$-matrix provide after quantization a Hopf subalgebra $U_z S$ and it is possible to obtain its corresponding universal $R$-matrix, this is, a solution of the quantum YBE which fulfills the property

\[
\mathcal{R} \Delta(X) \mathcal{R}^{-1} = \sigma \circ \Delta(X), \quad \forall X \in U_z S. \tag{1}
\]

The remarkable fact is that (1) is also verified by the Poincaré generators out of the Hopf subalgebra, thus $\mathcal{R}$ is a universal $R$-matrix for the complete quantum Poincaré algebra.

2 $U_z P(1 + 1)$: via universal $T$-matrix

We recall first the main features of the $T$-matrix approach. Let $U_z G$ be a quantum algebra and $Fun_z(G)$ its associated dual Hopf algebra or quantum group. The universal $T$-matrix of $U_z G$ is the Hopf algebra dual form $T = \sum_\mu X^\mu \otimes p_\mu$ where $\{X^\mu\}$ is a basis for $U_z G$ and $\{p_\mu\}$ its dual in $Fun_z(G)$, this is, $\langle p_\nu, X^\mu \rangle = \delta_\nu^\mu$.
In general, the $T$-matrix can be interpreted as the universal $R$-matrix for the quantum double linked to $U_z G$\cite{23,26,27} (within this quantum double, $U_z G$ is a Hopf subalgebra). As a consequence, if there exists an algebra isomorphism and coalgebra anti-isomorphism $\Phi$ between $U_z G$ and $Fun_z(G)$, the $T$-matrix can be completely written in terms of the generators of $U_z G$, in such a way that

$$R = (\text{id} \otimes \Phi)T,$$

with $\Phi$ acting on the generators of $Fun_z(G)$, is a universal $R$-matrix for $U_z G$.

We illustrate now the above ideas working out the ‘toy example’ corresponding to the 1+1 Poincaré algebra $P(1 + 1)$. We choose as Poincaré generators the boost $K$, and the translations along the light-cone $P_+$, $P_-$; they satisfy

$$[K, P_+] = P_+, \quad [K, P_-] = -P_-, \quad [P_-, P_+] = 0. \quad (3)$$

The non-standard classical $r$-matrix $r = 2z K \wedge P_+$ provides the cocommutators by means of $\delta(X) = [1 \otimes X + X \otimes 1, r]$:

$$\delta(P_+) = 0, \quad \delta(K) = 2z K \wedge P_+, \quad \delta(P_-) = 2z P_- \wedge P_+. \quad (4)$$

The coproduct and commutators of the Hopf algebra $U_z P(1 + 1)$ which deform this Poincaré bialgebra are

$$\Delta(P_+) = 1 \otimes P_+ + P_+ \otimes 1, \quad \Delta(P_-) = 1 \otimes P_- + P_- \otimes e^{2zP_+},$$

$$\Delta(K) = 1 \otimes K + K \otimes e^{2zP_+},$$

$$[K, P_+] = \frac{1}{2z}(e^{2zP_+} - 1), \quad [K, P_-] = -P_-, \quad [P_-, P_+] = 0. \quad (5)$$

Let us focus on the Hopf subalgebra $U_z S$ generated by $K$ and $P_+$; its quantum dual group $Fun_z(S)$ has coordinates $\hat{\chi}$ and $\hat{a}_+$ and its structure is given by:

$$\Delta(\hat{\chi}) = \hat{\chi} \otimes 1 + 1 \otimes \hat{\chi}, \quad \Delta(\hat{a}_+) = \hat{a}_+ \otimes 1 + e^{\hat{\chi}} \otimes \hat{a}_+, \quad [\hat{\chi}, \hat{a}_+] = 2z(e^{\hat{\chi}} - 1). \quad (7)$$

The associated universal $T$-matrix is \cite{14}:

$$T = \exp\{P_+ \otimes \hat{a}_+\} \exp\{K \otimes \hat{\chi}\}. \quad (8)$$

By taking into account \cite{3}, \cite{4} and \cite{7} it can be easily checked that the map $\Phi$ defined by

$$\Phi(\hat{\chi}) = 2z P_+, \quad \Phi(\hat{a}_+) = -2z K, \quad \Phi(1) = 1, \quad (9)$$

is an algebra isomorphism and coalgebra anti-isomorphism between $U_z S$ and $Fun_z(S)$, so that

$$R = (id \otimes \Phi) T = \exp\{-2z P_+ \otimes K\} \exp\{2z K \otimes P_+\}, \quad (10)$$

is a universal $R$-matrix for $U_z S$. Moreover, it can be proven that $P_-$ verifies \cite{14} with respect to \cite{17}, hence we conclude that this is a universal $R$-matrix for the whole $U_z P(1 + 1)$. 3
3 \: U_wP(2 + 1): via contraction

Instead of developing the \(T\)-matrix procedure, we show in this case another method: the contraction process. We take as the starting point the coproduct, commutation relations and universal \(R\)-matrix of the non-standard quantum \(U_z\,sl(2, \mathbb{R}) = \langle A, A_+, A_- \rangle\) \[28\]:

\[
\begin{align*}
\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1, & \quad \Delta(A) = 1 \otimes A + A \otimes e^{2zA_+}, \\
\Delta(A_-) = 1 \otimes A_- + A_- \otimes e^{2zA_+},
\end{align*}
\]

\[\tag{11}\]

\[
[A, A_] = \frac{e^{2zA_+} - 1}{z}, \quad [A, A_-] = -2A_- + zA^2, \quad [A_+, A_-] = A. \quad \tag{12}\]

At this point, notice that \(U_z\,P(1 + 1)\) (and its universal \(R\)-matrix) can be recovered by contracting \(U_z\,sl(2, \mathbb{R})\) as the limit \(\varepsilon \to 0\) of the mapping defined by:

\[
K = A/2, \quad P_+ = \varepsilon A_+, \quad P_- = \varepsilon A_-, \quad w = z/\varepsilon. \quad \tag{14}\]

A non-standard quantum Poincaré algebra \(U_wP(2 + 1)\) can be obtained in two steps \[19\]:

- Take two quantum \(sl(2, \mathbb{R})\) algebras: \(U_z^{(1)}sl(2, \mathbb{R}) = \langle A^1, A_+^1, A_-^1 \rangle\) and \(U_z^{(2)}sl(2, \mathbb{R}) = \langle A^2, A_+^2, A_-^2 \rangle\). In this way the following generators

\[
K = \frac{1}{2}(A^1 - A^2), \quad D = \frac{1}{2}(A^1 + A^2), \quad C_1 = -A_+^1 - A_-^1, \\
H = A_+^1 + A_+^2, \quad P = A_+^1 - A_+^2, \quad C_2 = A_+^1 - A_+^2,
\]

\[\tag{15}\]

give rise to \(U_z so(2, 2) \simeq U_z^{(1)}sl(2, \mathbb{R}) \oplus U_z^{(2)}sl(2, \mathbb{R})\) in a conformal basis.

- Apply the contraction defined by:

\[
\begin{align*}
P_+ &= \varepsilon \frac{1}{\sqrt{2}} P, & P_1 &= \varepsilon K, & P_- &= -\varepsilon \frac{1}{\sqrt{2}} C_2, \\
E_1 &= -\frac{1}{\sqrt{2}} H, & F_1 &= \frac{1}{\sqrt{2}} C_1, & K_2 &= D, & w &= \frac{1}{\sqrt{2} \varepsilon}. \quad \tag{16}\end{align*}
\]

After the limit \(\varepsilon \to 0\), the contracted generators \(\{P_+, P_-, P_1, E_1, F_1, K_2\}\) and deformation parameter \(w\) close a 2+1 quantum Poincaré algebra \(U_wP(2 + 1)\) in a null-plane basis:

\[
\begin{align*}
\Delta(P_+) = 1 \otimes P_+ + P_+ \otimes 1, & \quad \Delta(E_1) = 1 \otimes E_1 + E_1 \otimes 1, \\
\Delta(P_-) = 1 \otimes P_- + P_- \otimes e^{2wP_+}, & \quad \Delta(P_1) = 1 \otimes P_1 + P_1 \otimes e^{2wP_+}, \\
\Delta(F_1) = 1 \otimes F_1 + F_1 \otimes e^{2wP_+} - 2wP_+ \otimes e^{2wP_+} E_1, & \quad \Delta(K_2) = 1 \otimes K_2 + K_2 \otimes e^{2wP_+} - 2wP_1 \otimes e^{2wP_+} E_1, \\
\end{align*}
\]

\[\tag{17}\]

\[
\begin{align*}
[K_2, P_+] &= \frac{1}{2w}(e^{2wP_+} - 1), & [K_2, P_-] &= \frac{1}{2w}(e^{2wP_+} - 1), \\
[K_2, E_1] &= F_1 e^{2wP_+}, & [K_2, F_1] &= F_1 = 2wP_1 K_2, \\
[E_1, P_+] &= \frac{1}{2w}(e^{2wP_+} - 1), & [F_1, P_1] &= P_+ + wP_1^2, \\
[E_1, F_1] &= K_2, & [P_+, F_1] &= -P_1, & [P_-, E_1] &= -P_1. \quad \tag{18}\end{align*}
\]
The universal $R$-matrix for $U_w \mathcal{P}(2 + 1)$ also comes from the contraction applied onto the one corresponding to $U_2 \mathfrak{so}(2, 2)$ that is $R_z^{(1)} R_z^{(2)}$:

$$R_w = \exp\{2wE_1 \otimes P_1\} \exp\{-2wP_+ \otimes K_2\} \exp\{2wK_2 \otimes P_+\} \exp\{-2wP_1 \otimes E_1\}. \quad (19)$$

The first order in $w$ gives the classical $r$-matrix $r = 2w(K_2 \wedge P_+ + E_1 \wedge P_1)$ which provides the Lie bialgebra underlying this quantum Poincaré algebra.

Obviously this method can be only carried out when the adequate quantum semisimple algebra structure is known. This is not the case for next dimension so we use again the $T$-matrix formalism.

4 $U_z \mathcal{P}(3 + 1)$: via universal $T$-matrix

Let us introduce first the classical Poincaré (bi)algebra in a null-plane basis. We consider the null-plane ‘orthogonal’ to the light-like vector $n = (\frac{1}{2}, 0, 0, \frac{1}{2})$ as our initial surface. A coordinate system well adapted to null-planes is defined in the terms of the usual kinematical ones $\{x^+, x^0, x^3, \}$. The non-vanishing Lie brackets of $\mathcal{P}(3 + 1)$ are given by

$$\{P_+, P_+, P_-, E_i, F_i, K_3, J_3\}, \quad \{P_0, P_1, K_2, J_2\}$$

A point $x$ contained in the null-plane is labelled by the coordinates $(x^+, x^1, x^2)$; the remaining coordinate $x^-$ plays the role of an evolution parameter ('time'). A particular null-plane is $\Pi_n^0, (\tau = 0)$, i.e., $x^- = n \cdot x = 0$ (it is invariant under the action of the boosts generated by $K_3$). A basis of $\mathcal{P}(3 + 1)$ adapted to these coordinates is $\{P_+, P_-, E_i, F_i, K_3, J_3\}$, where the generators $P_+, P_-, E_i$ and $F_i$ are defined in the terms of the usual kinematical ones $\{P_0, P_1, K_2, J_2\}$ by

$$P_+ = \frac{1}{2}(P_0 + P_3), \quad P_- = P_0 - P_3, \quad E_1 = \frac{1}{2}(K_1 + J_2), \quad F_1 = K_1 - J_2, \quad K_2 = K_2 + J_1, \quad E_2 = \frac{1}{2}(K_2 - J_1). \quad (21)$$

The stability group of the plane $\Pi_n^0$ is generated by $\{P_+, P_i, E_i, K_3, J_3\}$. The remaining three generators act on $\Pi_n^0$ as follows: $P_-$ translates $\Pi_n^0$ into $\Pi_n^0$, while both $F_i$ rotate it around the surface of the light-cone $x^2 = 0$. Therefore, if $x^- = \tau$ is considered as an evolution parameter, then $P_-$ and $F_i$ describe the dynamical evolution from the null-plane $x^- = 0$.

The classical $r$-matrix we choose is the natural generalization of the one corresponding to the 2+1 case:

$$r = 2z(K_3 \wedge P_+ + E_1 \wedge P_1 + E_2 \wedge P_2). \quad (23)$$
Therefore the Poincaré cocommutators are:

\[
\begin{align*}
\delta(X) &= 0, \quad \text{for } X \in \{P_+, E_i, J_3\}, \\
\delta(Y) &= 2z(Y \wedge P_+), \quad \text{for } Y \in \{P_-, P_i\}, \\
\delta(F_1) &= 2z(F_1 \wedge P_+ - P_- \wedge E_1 - P_2 \wedge J_3), \\
\delta(F_2) &= 2z(F_2 \wedge P_+ - P_- \wedge E_2 + P_1 \wedge J_3), \\
\delta(K_3) &= 2z(K_3 \wedge P_+ - P_1 \wedge E_1 - P_2 \wedge E_2).
\end{align*}
\]

The quantum deformation \( U_\mathcal{P}(3 + 1) \) of the Lie bialgebra so obtained is given by [20]:

\[
\begin{align*}
\Delta(X) &= 1 \otimes X + X \otimes 1, \quad \text{for } X \in \{P_+, E_i, J_3\}, \\
\Delta(Y) &= 1 \otimes Y + Y \otimes e^{2zP_+}, \quad \text{for } Y \in \{P_-, P_i\}, \\
\Delta(F_1) &= 1 \otimes F_1 + F_1 \otimes e^{2zP_+} - 2zP_- \otimes E_1 e^{2zP_+} - 2zP_2 \otimes J_3 e^{2zP_+}, \\
\Delta(F_2) &= 1 \otimes F_2 + F_2 \otimes e^{2zP_+} - 2zP_- \otimes E_2 e^{2zP_+} + 2zP_1 \otimes J_3 e^{2zP_+}, \\
\Delta(K_3) &= 1 \otimes K_3 + K_3 \otimes e^{2zP_+} - 2zP_1 \otimes E_1 e^{2zP_+} - 2zP_2 \otimes E_2 e^{2zP_+};
\end{align*}
\]

\[
\begin{align*}
[K_3, P_+] &= \frac{1}{2z} (e^{2zP_+} - 1), \quad [K_3, P_-] = -P_- - zP^2_1 - zP^2_2, \\
[K_3, E_i] &= E_i e^{2zP_+}, \quad [K_3, F_i] = -F_i - 2zK_3 P_i, \\
[J_3, P_i] &= -\varepsilon_{ij3} P_j, \quad [J_3, E_i] = -\varepsilon_{ij3} E_j, \quad [J_3, F_i] = -\varepsilon_{ij3} F_j, \\
[E_i, P_j] &= \delta_{ij} \frac{1}{2z} (e^{2zP_+} - 1), \quad [F_i, P_j] = \delta_{ij} (P_+ + zP^2_1 + zP^2_2), \\
[E_i, F_j] &= \delta_{ij} K_3 + \varepsilon_{ij3} J_3 e^{2zP_+}, \quad [P_+, E_i] = -P_i, \\
[F_i, F_j] &= 2z(P_1 F_2 - P_2 F_1), \quad [P_-, E_i] = -P_i.
\end{align*}
\]

The six generators appearing in the classical \( r \)-matrix [23] close a quantum Hopf subalgebra \( U_\mathcal{S} \). The universal \( T \)-matrix for this Hopf subalgebra can be computed and reads [20]:

\[
\mathcal{T} = e^{E_2 \otimes \hat{e}_2} e^{E_1 \otimes \hat{e}_1} e^{P_+ \otimes \hat{a}_+} e^{K_3 \otimes \hat{k}_3} e^{P_1 \otimes \hat{a}_1} e^{P_2 \otimes \hat{a}_2},
\]

where \( \hat{a}_+, \hat{a}_i, \hat{e}_i, \hat{k}_3 \) are dual to the corresponding Poincaré generators. This canonical element leads to a factorized universal \( R \)-matrix for \( U_\mathcal{S} \):

\[
\mathcal{R} = \exp\{2zE_2 \otimes P_2\} \exp\{2zE_1 \otimes P_1\} \exp\{-2zP_+ \otimes K_3\} \times \exp\{2zK_3 \otimes P_+\} \exp\{-2zP_1 \otimes E_1\} \exp\{-2zP_2 \otimes E_2\}.
\]

Furthermore it can be shown that \( \mathcal{R} \) satisfies the property \([1]\) for the four remaining generators: \( P_-, F_1, F_2 \) and \( J_3 \). Therefore, it is a universal \( R \)-matrix for the whole null-plane quantum Poincaré algebra.

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