REALIZATION OF MEASUREMENT
AND THE STANDARD QUANTUM LIMIT

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1 Introduction

What measurement is there? It is a difficult question but the importance of this question has increased much in connection with the effort to detect gravitational radiation. For monitoring the position of a free mass such as the gravitational-wave interferometer [1], it is usually supposed that the sensitivity is limited by the so-called standard quantum limit (SQL) [2, 3]. In the recent controversy [4]–[8], started with Yuen’s proposal [4] of a measurement which beats the SQL, the meaning of the SQL has been much clarified. In order to settle this controversy, rigorous treatment of the question on what measurement there is seems to be the key point.

Usually, the first approach to such an ontological question is very mathematical. Fortunately, in the last two decades, mathematical theory for describing wide possibilities of quantum measurement was developed in the field of mathematical physics [9]–[18]. Indeed, the question on what measurement there is was given a solution by the present author [15] under a physically reasonable mathematical formulation. Unfortunately, this result and the mathematics for this result have not been familiar with physicists who need that result.

The purpose of this paper is two folds. The first purpose is to present the results from the mathematical theory of quantum measurement in a form accessible for physicists. As a consequence of this theory, I shall settle the controversy of the SQL in two ways; by a general consideration giving new vistas concerning such a nonstandard measurement and by giving a model of measuring interaction which breaks the SQL [19]. The second purpose is, of course, to present this result.
2 Standard Quantum Limit for Free-Mass Position

The uncertainty principle is a physicist’s wisdom which gives correct answers to many quantum mechanical problems without so much cumbersome analysis of the problem. An application of this wisdom to analysis of the performance of interferometric gravitational-wave detector leads to the limit for sensitivity of monitoring the free-mass position, which has long been a topic of controversy within the quantum optics and general relativity community. This limit — referred to as the standard quantum limit (SQL) for monitoring the position of a free mass — is usually stated as follows: In the repeated measurement of its position \( x \) of a free mass \( m \) with the time \( \tau \) between two measurements, the result of the second measurement cannot be predicted with uncertainty smaller than \( (\hbar \tau / m)^{1/2} \).

2.1 Yuen’s proposal of breaching SQL

In the standard argument \cite{2,3} deriving this limit, the uncertainty principle

\[
\Delta x(0) \Delta p(0) \geq \hbar / 2 \tag{1}
\]

is applied to the position uncertainty \( \Delta x(0) \) and the momentum uncertainty \( \Delta p(0) \) at the time \( t = 0 \) just after the first measurement so that by the time \( \tau \) of the second measurement the squared uncertainty (variance) of \( x \) increases to

\[
\Delta x(\tau)^2 = \Delta x(0)^2 + \Delta p(0)^2 \tau^2 / m^2 \geq 2\Delta x(0)\Delta p(0)\tau / m \geq \hbar \tau / m. \tag{2}
\]

The SQL is thus obtained as

\[
\Delta x(\tau) \geq (\hbar \tau / m)^{1/2}, \tag{3}
\]

and it is usually explained as a straightforward consequence of the uncertainty principle (1).

Yuen \cite{4} pointed out a serious flaw in the standard argument. Since the evolution of a free mass is given by

\[
\hat{x}(t) = \hat{x}(0) + \hat{p}(0)t / m, \tag{4}
\]

the variance of \( x \) at time \( \tau \) is given by

\[
\Delta x(\tau)^2 = \Delta x(0)^2 + \Delta p(0)^2 \tau^2 / m^2 + \langle \Delta \hat{x}(0) \Delta \hat{p}(0) + \Delta \hat{p}(0) \Delta \hat{x}(0) \rangle \tau / m \tag{5}
\]

where \( \Delta \hat{x} = \hat{x} - \langle \hat{x} \rangle \) and \( \Delta x^2 = \langle \Delta \hat{x}^2 \rangle \), etc. Thus the standard argument implicitly assumes that the last term — we shall call it the correlation term — in Eq. (5) is non-negative. Yuen’s assertion \cite{4} is that some measurements of \( x \) leave the free mass in a state with the negative correlation term.

In probability theory, it is an elementary fact that the variance of the sum of two random variables is the sum of their variances plus the correlation term which is twice their covariances. The covariance may be negative and it vanishes if these random variables are independent. In quantum mechanics, if the state at \( t = 0 \) is a minimum-uncertainty one then the correlation term vanishes. However, there are states with negative correlation terms. The ratio of the covariance to the product of the standard deviations is called the correlation coefficient in probability theory. The negative correlation expresses the tendency that the larger than the mean one variable, the less than the mean the other. From this, one can expect that in such a state the momentum works as attracting the free mass around the mean position...
and that the free evolution narrows the wave packet of the mass. Gaussian states [26] with this property are analyzed as follows [4].

Let $a$ be the following operator in the Hilbert space of the mass states:

$$\hat{a} = (m\omega/2\hbar)^{1/2}\hat{x} + 1/(2\hbar m\omega)^{1/2}i\hat{p}, \quad [\hat{a}, \hat{a}^\dagger] = 1,$$

(6)

where $\omega$ is an arbitrary parameter with unit sec$^{-1}$. The twisted coherent state (TCS) $|\mu\nu\alpha\omega\rangle$ is the eigenstate of $\mu\hat{a} + \nu\hat{a}^\dagger$ with eigenvalue $\mu\alpha + \nu\alpha^*$:

$$(\mu\hat{a} + \nu\hat{a}^\dagger)|\mu\nu\alpha\omega\rangle = (\mu\alpha + \nu\alpha^*)|\mu\nu\alpha\omega\rangle, \quad |\mu|^2 - |\nu|^2 = 1.$$  

(7)

The free-mass Hamiltonian is expressed by

$$\hat{H} = \hat{p}^2/2m = (\hbar\omega/2)(\hat{a}^\dagger\hat{a} + 1/2 - \frac{1}{2}\hat{a}^2 - 1/2\hat{a}^\dagger)^2).$$

(8)

Within the choice of a constant phase the wave function $\langle x|\mu\nu\alpha\rangle$, where $\hat{x}|x\rangle = x|x\rangle$, is given by

$$\langle x|\mu\nu\alpha\rangle = \left(\frac{m\omega}{\pi\hbar|\mu - \nu|^2}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}|\mu - \nu|^2(x - x_0)^2 + ip_0(x - x_0)\right).$$

(9)

where

$$\xi = \text{Im}(\mu^*\nu), \quad \alpha = (m\omega/2\hbar)^{1/2}x_0 + 1/(2\hbar m\omega)^{1/2}ip_0, \quad x_0, p_0 \quad \text{real}.$$  

(10)

When $\xi = 0$, the wave function (9) is the usual minimum-uncertainty state. In the context of oscillators, the squeezed states are the wave functions of the form (9) with $\mu \neq 0$. The first two moments of $|\mu\nu\alpha\rangle$ are

$$\langle x \rangle = x_0, \quad \langle p \rangle = p_0,$$

$$\Delta x^2 = \hbar|\mu - \nu|^2/2m\omega, \quad \Delta p^2 = \hbar m\omega|\mu + \nu|^2/2, \quad \Delta\hat{x}\Delta\hat{p} + \Delta\hat{p}\Delta\hat{x} = -\hbar\xi, \quad \Delta x\Delta p = \hbar(1 + \xi^2)^{1/2}/4,$$

$$\langle H \rangle = \langle (p)^2 + \Delta p^2 \rangle/2m.$$  

(11)

(12)

(13)

(14)

When $\xi > 0$ the $x$-dependent phase in (9) leads to a narrowing of the $\Delta x(t)$ from $\Delta x(0)$ during free evolution. Because of this behavior, Yuen called mass states (9) with $\xi > 0$ as contractive states.

The position fluctuation for a free-mass starting in an arbitrary TCS (9) is immediately obtained from Eqs. (5) and (12)–(13):

$$\Delta x(t)^2 = (\hbar/2m\omega)|\mu - \nu|^2 - 4\xi\omega t + |\mu + \nu|^2(\omega t)^2$$

$$= (1/4\xi)(\hbar\tau/m) + (2\hbar/m\omega)|\mu + \nu|\omega/2|^2(t - \tau)^2,$$

(15)

(16)

where

$$\tau = 2\xi/|\omega||\mu + \nu|^2 = \xi\hbar m/\Delta p(0)^2.$$  

(17)

For $\xi > 0$, the position fluctuation decreases during time $t = 0$ to $t = \tau$. The minimum fluctuation achieves $1/(2\xi^{1/2})$ times the SQL at the time $\tau = 2\xi/(|\omega||\mu + \nu|^2)$. Thus the minimum fluctuation $\Delta x(\tau)$ is related only to the momentum uncertainty as follows:

$$\Delta x(\tau) = \hbar/2\Delta p(0) = \Delta x(0)/(1 + 4\xi^2)^{1/2}.$$  

(18)
This shows that $\Delta x(\tau)$ can be arbitrarily small for arbitrarily large $\tau$ with sufficiently large $\xi$. It should be noted that the minimum uncertainty product is realized between the momentum uncertainty at $t = 0$ and the position uncertainty at $t = \tau$.

Thus the SQL formulated by Eq. (3) can be breached if there is a measurement of free-mass position which leaves the free-mass in a contractive state just after the measurement.

The reader may have the following question: Is the state after the measurement always an eigenstate of the position observable? If so, it never has any such contractive character. It is natural to say so in the textbook description of measurement. However, any real measuring apparatus cannot leave the free mass in any eigenstate. This statement has been repeatedly emphasized in the study of measurement of continuous observables (observables with continuous spectrum). In the study of measurement, there is a deep gap between discrete observables and continuous observables. Once von Neumann criticized Dirac’s formulation in this point [20, pp.222–223]: “The division into quantized and unquantized quantities corresponds ... to the division into quantities $\hat{R}$ with an operator $\hat{R}$ that has a pure discrete spectrum, and into such quantities for which this is not the case. And it was for the former, and only for these, that we found a possibility of an absolutely precise measurement — while the latter could be observed only with arbitrarily good (but never absolute) precision. (In addition, it should be observed that the introduction of an eigenfunction which is ‘improper,’ i.e., which does not belong to Hilbert space ... gives a less good approach to reality than our treatment here. For such a method pretends the existence of such states in which quantities with continuous spectra take on certain values exactly, although this never occurs. Although such idealizations have often been advanced, we believe that it is necessary to discard them on these grounds, in addition to their mathematical untenability.)”

### 2.2 Caves’s defense of SQL

After Yuen’s proposal [4] of measurement with a contractive state, Caves published a further analysis [8] of the SQL, where he gave the following improved formulation of the SQL:

Let a free mass $m$ undergo unitary evolution during the time $\tau$ between two measurements of its position $x$, made with identical measuring apparatuses; the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar/2m)^{1/2}$, in average over the possible results of the first measurement.

Caves [8] showed that the SQL holds for a specific model of position measurement due to von Neumann [20, pp.442–445] and he also gave the following heuristic argument for the validity of the SQL. His point is the notion of the imperfect resolution $\sigma$ of one’s measuring apparatus. His argument runs as follows. The first assumption is that the variance $\Delta(\tau)^2$ of the measurement of $x$ is the sum of $\sigma^2$ and the variance of $x$ at the time of the measurement; i.e.,

$$\Delta(\tau)^2 = \sigma^2 + \Delta x(\tau)^2,$$  \hspace{1cm} (19)

and this is the case when the measuring apparatus is coupled linearly to $x$. The second assumption is that just after the first measurement, the free mass has position uncertainty not greater than the resolution:

$$\Delta x(0) \leq \sigma.$$  \hspace{1cm} (20)
Under these conditions, he derived the following estimate for the uncertainty $\Delta(\tau)$ of the second measurement at time $\tau$:

$$\Delta(\tau)^2 = \sigma^2 + \Delta x(\tau)^2 \geq \Delta x(0)^2 + \Delta x(\tau)^2 \geq 2\Delta x(0)\Delta x(\tau) \geq \hbar \tau/m.$$  \hspace{1cm} (21)

According to this argument, the SQL is a consequence from the uncertainty relation

$$\Delta x(0)\Delta x(\tau) \geq |\langle[\hat{x}(0), \hat{x}(\tau)]\rangle|/2 = \hbar \tau/2m,$$  \hspace{1cm} (22)

under assumptions (19)–(20).

However, his definition of the resolution of a measurement is ambiguous and so a critical examination for his argument is necessary. In fact, he used three different definitions in his paper: 1) If the free mass is in a position eigenstate at the time of a measurement of $x$, then $\sigma$ is defined to be the uncertainty in the result. 2) The measurement determines the position immediately after the measurement to be within roughly a distance $\sigma$ of the measured value. 3) The square $\sigma^2$ of the resolution is the variance of the pointer-position just before the system-meter interaction. These three notions are essentially different, although they are the same for von Neumann's model.

The notion of resolution of measurement has been often mentioned in literature but up to now we have not yet reached any satisfactory definition for it. What does the readout value tell one the states of the free mass? There are two principal thoughts about this question. One thinks that the readout tells the position of the free mass just before the measurement, since the prior position is the cause of the effect of the measurement that is the readout value. Another one thinks that the readout tells the position of the free mass just after the measurement, since the measurement changes the position of the free mass so that the two effects of the measurement — the posterior position and the readout — may be in concordance. Which is true is hardly answered. The best way to attack the problem seems to recognize that there are two types of concept of resolution of measurement. From this reason, we make the following distinction: If the free mass is in a position eigenstate at the time of a measurement of $x$ then the precision $\varepsilon$ of the measurement is defined to be the uncertainty in the result and the resolution $\sigma$ is defined to be the deviation of the position of the free mass just after the measurement from the readout just obtained. The mathematical definitions of these concepts including the case of superposition will be given and discussed thoroughly in the later sections.

We shall return to the problem of the validity of the SQL. By the improvement of the formulation of the SQL, the idea of measurement leaving the free mass in a contractive state can not readily vitiate the SQL. However, there is a possibility for circumventing the heuristic proof given by Caves [8], since his assumption $\sigma \geq \Delta x(0)$ is formulated in an ambiguous manner. Nonetheless, it is a heavy burden for one who wants to vitiate the SQL to make a realizable model of measurement which circumvents Caves's assumption. In the next section, we shall give general considerations of realization of measurement.

Before going further, I shall mention certain attempts of breaking the SQL. Recently, Ni [21] proposed a scheme of repeated position measurements for which one can predict the result of the next measurement with arbitrarily small errors. However, a close examination of this scheme leads to the conclusion that the improved formulation of the SQL due to Caves [8] is not broken by this repeated measurement scheme. This scheme uses a combination of the Arthurs-Kelley measurements [22]
which measure the position and momentum simultaneously and approximately. Accordingly, one measurement of this scheme has four meters, two of which measure the free-mass position with a high resolution by one meter and with a low resolution by the other meter, and the other two of which measure the free-mass momentum with lower resolutions. The prediction of the result of the next position measurement is done using these four readouts. If one of these meters is left alone, the prediction cannot have the desired accuracy. This means that one meter reading with the highest position resolution serves the position measurement and the other three meter readings serve the preparation procedure for the next measurement. There are several similar models proposed with preparation procedures for the next measurement and these examples do not vitiate the improved SQL, since it disallows explicitly any tinkering between two identical position measurements. Indeed, in these proposed models (e.g., [21, 23]), there is at least one auxiliary meter with a lower resolution, the reading of which prepares the state for the next position measurement really done by the other high resolution meter-reading. Thus any many-meter systems are excluded out of the scope of the improved SQL. The true problem is thus whether we can make a measurement with only one meter, the reading of which gives the precise position of the mass and simultaneously prepares the state for the next identical position measurement, for instance, in a contractive state.

3 Quantum Mechanics of Measurement

What can one say about quantum measurement from quantum mechanics? Von Neumann is acknowledged to be the first scientist who made a route to this problem. Although his original motivation was to show the consistency of the repeatability hypothesis (usually referred to as the projection postulate) with the axioms of quantum mechanics, his argument opened the way to analyze the quantum measurement within quantum mechanics. However, his method caused the controversy about the difficult problem of interpretation of quantum mechanics. In what follows, I shall attempt to present von Neumann’s method with some elaborations which avoid difficulties of the problem of interpretation and discuss several consequences from the quantum mechanics of measurement.

3.1 Statistics of measurement

By the axioms of quantum mechanics we shall mean the axioms of nonrelativistic quantum mechanics without any superselection rules, which are reduced to (a) the definitions of observables and states as self-adjoint operators and state vectors of a Hilbert space, (b) the Schrödinger time-dependent equation and (c) the Born statistical formula for probability distributions of commuting observables. The projection postulate is excluded from our axioms and its status will be discussed below.

In order to discuss all possible quantum measurements, it is convenient to classify them into two classes. A measurement is of the first kind if it does not destroy the quantum mechanical description of the system to be measured so that we can determine in principle the state just after the measurement corresponding to the result of measurement. A measurement is of the second kind if it destroys the quantum mechanical description of the system. The whole process of a direct interaction with a macroscopic detector such as a photon counter is considered as a measurement of the second kind.

The statistics of a measurement of the first kind is specified by the following
two elements: For the system state $\psi$ just before the measurement, let $P(a|\psi)$ be the probability density of obtaining the result $a$ and let $\psi_a$ be the system state just after the measurement with result $a$. Then the physical design and the indicated preparation of the apparatus determine $P(a|\psi)$ and the transition $\psi \to \psi_a$ for all possible $\psi$. These two elements will be called the statistics of a given measurement of the first kind; $P(a|\psi)$ will be called the measurement probability and $\psi \to \psi_a$ will be called the state reduction, further we shall call the state $\psi$ just before the measurement as the prior state and the state $\psi_a$ just after the measurement as the posterior state. (For notational convenience, $\psi_a$ will be denoted sometimes by $\psi[a]$.)

This specification of measurement statistics implies that if two given measurements are identical then the corresponding two statistics are identical. On the other hand, the statistics of a measurement of the second kind is specified only by its measurement probability, for such a measurement does not allow to describe the system state after the measurement.

3.2 Scheme of measurement

Once we accept the axioms of quantum mechanics, it is natural to accept in principle the following fact as a basis of our discussion.

**Postulate 1** For any observable $A$ with its spectral decomposition

$$\hat{A} = \int x d\hat{A}(x),$$

there is a measurement which may be of the second kind such that its measurement probability $P(a|\psi)$ satisfies the Born statistical formula

$$P(a|\psi) da = \langle \psi | d\hat{A}(a) | \psi \rangle,$$

for all prior state $\psi$.

We shall call any measuring apparatus satisfying Eq.(24) as a detector for an observable $A$. It should be noticed that Postulate 1 alone never implies existence of any measurements of the first kind. Our fundamental problem is thus what measurement of the first kind is allowed within our axioms and postulates of quantum mechanics. In order to solve this problem, we adopt the following scheme of measurement instituted by von Neumann.

Suppose that a quantum system $S$ — called the object system — with the unknown system state $\psi$ just before the measurement is to be measured by a measuring apparatus. For the observer — called the first observer — who applies quantum mechanics only to the object system, this measurement is described by the statistics of this measurement specified by the given measuring apparatus. Suppose that this measurement is of the first kind and we shall denote the statistics of this measurement by $P_I(a|\psi)$ and $\psi \to \psi_a$. Then by nondestructive nature of measurement of the first kind, there are other possibilities of application of quantum mechanics to this physical phenomena of the measurement. One possibility of such a quantum mechanical description of measurement arises if one separates the measuring apparatus into two parts. The first part — called the probe system — is a microscopic system which directly interacts with the object system and the second part is a detector which makes a second kind measurement of an observable — usually called the pointer position in somewhat misunderstanding manner — of the probe system.
For the observer — called the second observer — who applies quantum mechanics to the composite system of the object and the probe, this measurement is described as a combination of an object-probe interaction and a second kind of measurement of the probe in the following manner. Let \( P \) be the probe system. By the arrangement of the measuring apparatus the following elements can be specified as controllable elements; the system state \( \varphi \) of the probe system just before the measurement, the time evolution \( \hat{U} \) of the composite system \( S + P \) during the interaction and the observable \( A \) of the probe system to be measured by the detector. Then just before the interaction the state of the composite system is \( \psi \otimes \varphi \) and hence just after the interaction the composite system is in the state \( \hat{U}(\psi \otimes \varphi) \). What can the second observer tell about the statistics of this measurement? The measurement probability \( P_{II}(a|\psi) \) for the second observer is obviously the postulated probability distribution of the observable \( A \) in the state \( \hat{U}(\psi \otimes \varphi) \):

\[
P_{II}(a|\psi) \, da = \langle \hat{U}(\psi \otimes \varphi)|1 \otimes d\hat{A}(a)|\hat{U}(\psi \otimes \varphi) \rangle. \tag{25}
\]

Thus from the consistency of the measurement probabilities of these two observers, we have

\[
P_I(a|\psi) \, da = \langle \hat{U}(\psi \otimes \varphi)|1 \otimes d\hat{A}(a)|\hat{U}(\psi \otimes \varphi) \rangle. \tag{26}
\]

In order to determine the state reduction, we may assume that immediately after the interaction the object system would be subjected to a detector of an arbitrary observable \( X \) of the object system. Since the probe system is also to be subjected to the detection of \( A \) immediately after the interaction in the second-observer description, quantum mechanics predicts the joint probability density \( P_{II}(x,a|\psi) \) of obtaining the result \( A = a \) and \( X = x \) for the second observer by the relation

\[
P_{II}(x,a|\psi) \, da \, dx = \langle \hat{U}(\psi \otimes \varphi)|d\hat{X}(x) \otimes d\hat{A}(a)|\hat{U}(\psi \otimes \varphi) \rangle. \tag{27}
\]

For the first observer, the detection of the observable \( X \) occurs in the system state \( \psi_a \) given the result \( a \) of the first measurement and hence the probability density \( P_I(x,a|\psi) \) of the same event is calculated by the first observer as follows:

\[
P_I(x,a|\psi) \, da \, dx = \langle \psi_a|d\hat{X}(x)|\psi_a \rangle P_I(a|\psi) \, da. \tag{28}
\]

From the consistency of the statistics of these two observers, we have

\[
\langle \psi_a|d\hat{X}(x)|\psi_a \rangle = \frac{\langle \hat{U}(\psi \otimes \varphi)|d\hat{X}(x) \otimes d\hat{A}(a)|\hat{U}(\psi \otimes \varphi) \rangle}{\langle \hat{U}(\psi \otimes \varphi)|1 \otimes d\hat{A}(a)|\hat{U}(\psi \otimes \varphi) \rangle}. \tag{29}
\]

The arbitrariness of \( X \) yields the following relation for any basis \( \{|i\}\),

\[
\langle i|\psi_a \rangle \langle \psi_a|j \rangle = \frac{\langle \hat{U}(\psi \otimes \varphi)|\left(|j\rangle \langle i| \otimes d\hat{A}(a)\right)|\hat{U}(\psi \otimes \varphi) \rangle}{\langle \hat{U}(\psi \otimes \varphi)|1 \otimes d\hat{A}(a)|\hat{U}(\psi \otimes \varphi) \rangle}, \tag{30}
\]

and consequently,

\[
|\psi_a\rangle \langle \psi_a| = \sum_{i,j} |i\rangle \frac{\langle \hat{U}(\psi \otimes \varphi)|\left(|j\rangle \langle i| \otimes d\hat{A}(a)\right)|\hat{U}(\psi \otimes \varphi) \rangle}{\langle \hat{U}(\psi \otimes \varphi)|1 \otimes d\hat{A}(a)|\hat{U}(\psi \otimes \varphi) \rangle} \langle j|. \tag{31}
\]

Thus we have shown that the second observer can calculate the statistics of this measurement from his knowledge about the measuring apparatus and predict what will happen to the first observer. For example, if the first observer observes that
the system state $\psi$ is reduced to the state $\psi_a$ just after the measurement with the readout value $a$, the second observer can calculate just the same state reduction $\psi \rightarrow \psi_a$ by Eq. (31). The difference between those two observers is that the first observer knows only the relation between the readout $a$ and the reduced state $\psi_a$ but the second observer does know the statistical correlation between the object system and the probe system from which he can predict all statistics of this measurement. Some authors have made a misunderstanding at this point. They usually say that the state reduction is a consequence of the observation or of the knowing of the readout. If this would be the case, we would get an obvious contradiction between the first and the second observer; in fact, then the first observer would say that the state reduction occurs after the detection of the probe system contrary to the second observer’s saying that it occurs already before the detection of the probe system just after the object-probe interaction. The point is that there is no causality relation between the readout $a$ and the reduced state $\psi_a$ but there is only a statistical correlation. Usually, statistical correlation between two events does not imply any causality relation. In fact, for another observer who observes the result of succeeding measurement of the object system first, the state reduction of the prove system occurs. We can only say that there is a statistical correlation between the results of successive measurements. Quantum mechanics shows that to get information from one system is to make a statistical correlation with another system by an interaction to be described by quantum mechanics. However, there is another problem — indeed, a different problem. When does the statistical interference between the object system and the probe system vanish? This is the content of Schrödinger’s cat paradox. This is a deep problem. However, in our formalism, we can avoid the difficulty — as promised before — by boldly saying “During the second kind measurement of the probe system.”

3.3 Operation measures and measurement statistics

From the preceding analysis of the process of quantum measurement, the problem as to what measurement there is can be reduced to the problem what statistics can be realized by the measurement scheme. An important step to the mathematical solution of the latter problem is to get a neat mathematical representation of the statistics of measurement. In what follows, we shall show that any plausible description of measurement statistics can be expressed by a single mathematical object called an operation measure. A thorough discussion of operations and effects may be found in [13, 14], and of operation measures and effect measures in [9, 10, 15, 16].

Let $\mathcal{H}$ be a Hilbert space of an object system. Suppose that a statistics of a measurement is given; this means that for any state vector $\psi$ in $\mathcal{H}$ the measurement probability $P(a|\psi)$ and the state reduction $\psi \rightarrow \psi_a$ is presupposed. Our first task is to extend this statistical description to the case that the prior state is a mixture. Let $\hat{\rho}$ be a density operator which represents the prior state with its spectral decomposition

$$\hat{\rho} = \sum_i \lambda_i |\psi^i\rangle \langle \psi^i|.$$  \hspace{1cm} (32)

In this case, the measurement probability — denoted by $P(a|\hat{\rho})$ — is given by

$$P(a|\hat{\rho}) = \sum_i \lambda_i P(a|\psi^i).$$  \hspace{1cm} (33)

Then the posterior state $\hat{\rho}_a$ of the system for the readout $a$ is obviously a mixture of
all $|\psi_a^i\rangle\langle\psi_a^i|$’s, where $\psi_a^i$ is the posterior state for the prior state $\psi^i$, such that their relative frequency is proportional to $\lambda_i$ and $P(a|\psi^i)$. Thus we have

$$\hat{\rho}_a = \frac{\sum_i \lambda_i P(a|\psi^i)|\psi_a^i\rangle\langle\psi_a^i|}{\sum_i \lambda_i P(a|\psi^i)}. \quad (34)$$

Let $\Delta$ be an interval — or more generally a Borel set — in the real line and let $S_\Delta$ the subensemble of the object system selected by the condition that the readout $a$ of this measurement lies in $\Delta$. Then we can ask what is the state of $S_\Delta$ — denoted by $\hat{\rho}_\Delta$ — just after the measurement. It is well known that the state of this ensemble is not the superposition of all $\psi_a$’s with $a$ in $\Delta$ but a mixture of all $|\psi_a\rangle\langle\psi_a|$’s with $a$ in $\Delta$; their relative frequency is proportional to $P(a|\psi)$. Thus the state $\hat{\rho}_\Delta$ is

$$\hat{\rho}_\Delta = \frac{\int_\Delta da P(a|\psi)|\psi_a\rangle\langle\psi_a|}{\int_\Delta da P(a|\psi)}. \quad (35)$$

Further, for the case that the prior state is given by a mixture represented by a density operator (32), the state $\hat{\rho}_\Delta$ of the ensemble $S_\Delta$ just after the measurement is

$$\hat{\rho}_\Delta = \frac{\int_\Delta da P(a|\hat{\rho})\hat{\rho}_a}{\int_\Delta da P(a|\hat{\rho})}, \quad (36)$$

where $\hat{\rho}_a$ is given in Eq.(34).

These considerations lead to the following mathematical definition of the transformation $I(\Delta)$ which maps a density operator $\hat{\rho}$ to a trace class operator $I(\Delta)\hat{\rho}$ given by the relations

$$I(\Delta)\hat{\rho} = \int_\Delta dI(a)\hat{\rho}, \quad dI(a)\hat{\rho} = \hat{\rho}_a P(a|\hat{\rho}) da. \quad (37)$$

An obvious requirement for this transformation $I(\Delta)$ is as follows: If $\hat{\rho}$ is the mixture of $\hat{\sigma}_1$ and $\hat{\sigma}_2$ then $I(\Delta)\hat{\rho}$ is the mixture of $I(\Delta)\hat{\sigma}_1$ and $I(\Delta)\hat{\sigma}_2$. From this $I(\Delta)$ can be extended to the mapping of all trace class operators. Then the following properties can be easily deduced from Eq.(37):

1. For any Borel set $\Delta$, the mapping $\hat{\rho} \rightarrow I(\Delta)\hat{\rho}$ is a linear transformation on the space of trace class operators which maps density operators to a positive trace class operator.

2. For any Borel set $\Delta$ and any density operator $\hat{\rho}$,

$$0 \leq \text{Tr}[I(\Delta)\hat{\rho}] \leq 1 \quad \text{and} \quad \text{Tr}[I(R)\hat{\rho}] = 1. \quad (38)$$

3. For any countable disjoint sequence $\{\Delta_i\}$ of Borel sets and density operator $\hat{\rho}$,

$$\sum_i \text{Tr}[I(\Delta_i)\hat{\rho}] = \text{Tr}[I(\bigcup_i \Delta_i)\hat{\rho}], \quad (39)$$

A mapping $\hat{\rho} \rightarrow I(\Delta)\hat{\rho}$ with properties 1–2 is called an operation and a mapping $\Delta \rightarrow I(\Delta)$ with all properties 1–3 is called an operation measure. Thus we have shown that if we are given a plausible statistical description of measurement we can construct an operation measure. The mathematical importance of the operation measures is that it unifies two statistical data of measurement — the measurement
probability and the state reduction — into a single mathematical object. In fact, the
measurement probability can be recovered by the relation
\[ P(a|\hat{\rho}) da = \text{Tr}[dI(a)|\hat{\rho}], \tag{40} \]
and the state reduction can be retained by the relation
\[ \hat{\rho}_a = \frac{dI(a)|\rho}{\text{Tr}[dI(a)|\hat{\rho}]}. \tag{41} \]
The mathematical justification of the above differentiation is given in [16] for ar-
bitrary operation measure — so that, without any presupposition of measurement
statistics, any operation measure gives a measurement statistics. To put it simply,
integration of any measurement statistics gives an operation measure and differenti-
ation of any operation measure gives a measurement statistics.

Let \( F \) be a (random) variable the values of which shows the readout of a mea-
surement. Then the probability distribution \( P(F \in \Delta|\hat{\rho}) \) that the value of \( F \) is in \( \Delta \) given the prior state \( \hat{\rho} \) is obtained from Eq.(40), i.e.,
\[ P(F \in \Delta|\hat{\rho}) = \int_{\Delta} P(a|\hat{\rho}) da. \tag{42} \]
In this case there is a unique positive operator valued measure \( \Delta \rightarrow \hat{F}(\Delta) \) such that
\[ 0 \leq \hat{F}(\Delta) \leq 1, \quad \text{and} \quad \hat{F}(R) = 1, \tag{43} \]
\[ \text{Tr}[\hat{F}(\Delta)|\hat{\rho}] = P(F \in \Delta|\hat{\rho}). \tag{44} \]
We shall call this \( \hat{F} \) as the effect measure of an operation measure \( I \). If an operation
measure \( I \) represents an exact measurement of an observable \( \hat{X} = \int x d\hat{X}(x) \), then
the corresponding effect measure \( \hat{F} \) satisfies
\[ P(a|\hat{\rho}) da = \text{Tr}[d\hat{X}(a)|\hat{\rho}], \]
\[ \text{Tr}[\hat{F}(\Delta)|\hat{\rho}] = \int_{\Delta} \text{Tr}[d\hat{X}(x)|\hat{\rho}], \tag{45} \]
and hence the effect measure of \( I \) coincides with the spectral measure of operator
\( \hat{X} \), i.e., \( d\hat{F}(x) = d\hat{X}(x) \). In this sense, the notion of an effect measure generalizes
the conventional presupposition that the measurement probability is represented by
a spectral measure to the case of non-exact measurements.

3.4 Characterization of realizable measurement

In the preceding subsections, we have shown the following two facts:

1. Any measurement scheme consisting of an object-probe interaction and a probe
detection determines the unique measurement statistics by means of the second-
observer description.

2. Any plausible measurement statistics of the first-observer description gives an
operation measure which unifies the measurement statistics in a single mathema-
tical object.

However, it is not at all clear whether every operation measure is consistent with the
second-observer description of the measurement — or what operation measures are
consistent with the second-observer description.
This problem has the following rigorous mathematical formulation: Let $\mathcal{H}$ be a Hilbert space describing the object system and let $\mathcal{I}$ be an operation measure for the space $\mathcal{T}(\mathcal{H})$ of the trace class operators on $\mathcal{H}$. The problem is to determine when we can find another Hilbert space $\mathcal{K}$ describing the probe system with a self-adjoint operator $\hat{A}$ in $\mathcal{K}$ describing the observable actually detected, a state vector $\varphi$ in $\mathcal{K}$ describing the preparation of the probe system and a unitary operator $\hat{U}$ on $\mathcal{H} \otimes \mathcal{K}$ describing time evolution of the object-probe composite system during the measurement interaction such that this second-observer description of measurement leads to the same measurement statistics as the first-observer description of the given operation measure $\mathcal{I}$. For the last part of this formulation, recall that the second-observer description of the measurement leads to the statistics given by Eqs.(25) and (31). On the other hand, the operation measure is given by Eq.(37). Thus the condition for these two to give the same statistics is the following relation:

$$\langle i | d\mathcal{I}(a)(|\psi\rangle \langle \psi|)|j \rangle = \langle \hat{U}(\psi \otimes \varphi) | \hat{U}(\psi \otimes \varphi) \rangle$$

(46)

for all $\psi$ and a basis $\{|i\rangle\}$ in $\mathcal{H}$. We shall call any operation measure satisfying Eq.(46) for some $\hat{A}$, $\phi$ and $\hat{U}$ as realizable.

In order to present the solution of the problem which has been obtained in [15], we need one more mathematical concept concerning the positivity property of the operation. Let $\mathcal{I}$ be an operation measure. Then the transformation $\hat{\rho} \rightarrow \mathcal{I}(\Delta)\hat{\rho}$ on the trace class operators is positive, in the sense that for any density operator $\hat{\rho}$ the trace class operator $\mathcal{I}(\Delta)\hat{\rho}$ is a positive operator. It follows that for any vector $\varphi$, $\psi$ we have

$$\langle \varphi | d\mathcal{I}(a)(|\psi\rangle \langle \psi|)|\varphi \rangle \geq 0.$$  

(47)

An operation measure is called completely positive if it has the following stronger positivity property; for any finite sequences of vectors $\xi_1, \xi_2, \ldots, \xi_n$ and $\eta_1, \eta_2, \ldots, \eta_n$,

$$\sum_{i,j=1}^{n} \langle \xi_i | d\mathcal{I}(a)(|\eta_i\rangle \langle \eta_i|)|\xi_j \rangle \geq 0.$$  

(48)

If an operation measure is realizable then by Eq.(46) we obtain

$$\sum_{i,j=1}^{n} \langle \xi_i | d\mathcal{I}(a)(|\eta_i\rangle \langle \eta_i|)|\xi_j \rangle$$

$$= \sum_{i,j=1}^{n} \langle \hat{U}(\eta_j \otimes \varphi) | (|\xi_j\rangle \langle \xi_i| \otimes d\hat{A}(a)) |\hat{U}(\eta_i \otimes \varphi) \rangle$$

$$= \| \sum_{i=1}^{n} (|\xi_i\rangle \langle \xi_i| \otimes d\hat{A}(a)) \hat{U}(\eta_i \otimes \varphi) \|^2$$

$$\geq 0.$$  

Thus every realizable operation measure is completely positive. The converse statement of this has been proved in [15] by mathematical construction of the Hilbert space $\mathcal{K}$ with unit vector $\varphi$, self-adjoint operator $\hat{A}$ on $\mathcal{K}$ and unitary operator $\hat{U}$ on $\mathcal{H} \otimes \mathcal{K}$ which satisfy Eq.(46) for a given completely positive operation measure $\mathcal{I}$ and thus we have

**Theorem 1** Every completely positive operation measure is realizable.
For a particular type of measurement statistics, this problem is the one originally considered by von Neumann. In order to clarify the relation between the conventional approach and our general approach, we shall review his well-known result. Let $X = \sum_i x_i |x_i\rangle \langle x_i|$ be an observable with a simple discrete spectrum \{ $\ldots, x_{-1}, x_0, x_1, \ldots$ \} and unit eigenvectors $|x_i\rangle$. The statistics of the precise measurement of $X$ is usually presupposed as follows:

measurement probability: \[ P(x_i|\psi) = |\langle x_i|\psi \rangle|^2, \quad (49) \]
state reduction: \[ \psi \rightarrow \psi[x_i] = |x_i\rangle. \quad (50) \]

For real numbers $a$ outside the spectrum, we have $P(a|\psi) = 0$ and we can put $\psi[a]$ as arbitrary. This leads to the operation measure $\mathcal{I}$ such that

\[ \mathcal{I}(\Delta)\hat{\rho} = \sum_{x_i \in \Delta} |x_i\rangle \langle x_i| \hat{\rho} |x_i\rangle \langle x_i|. \quad (51) \]

Von Neumann showed that this statistics is consistent with the second-observer description: One can construct the probe system from an arbitrary Hilbert space $\mathcal{K}$ with basis \{ $\ldots, \varphi_{-1}, \varphi_0, \varphi_1, \ldots$ \}. Let $\varphi_0$ be the probe system preparation and $\hat{U}$ the unitary operator on $\mathcal{H} \otimes \mathcal{K}$ such that

\[ \hat{U}(|x_i\rangle \otimes |\varphi_i\rangle) = |x_i\rangle \otimes |\varphi_{i+j}\rangle, \]
describing the measurement interaction. Let $A = \sum_i x_i |\varphi_i\rangle \langle \varphi_i|$ be the probe observable. Suppose that the prior state of the object is $\psi = \sum_k c_k |x_k\rangle$. We have the following time evolution starting with $\psi \otimes \varphi_0$

\[ \hat{U}|\psi \otimes \varphi_0\rangle = \sum_k c_k |x_k\rangle \otimes |\varphi_k\rangle. \quad (52) \]

Then with this setting, the joint probability distribution $P(X = x_j, A = x_i)$ obtaining the result $A = x_i$ and $X = x_j$ in the joint detection just after the object-probe interaction is

\[ P(X = x_j, A = x_i) = |\langle x_j| \otimes \langle \varphi_i| \hat{U}(\psi \otimes \varphi_0)\rangle|^2 \]
\[ = \sum_k |c_k|^2 |\langle x_j| |x_k\rangle|^2 |\langle \varphi_i| \varphi_k\rangle|^2 \]
\[ = |\langle x_j|\psi \rangle|^2 \delta_{i,j}. \quad (53) \]

Since, for the second-observer, the measurement probability $P_{II}(x_j|\psi)$ of this measurement is the probability $P(A = x_j)$ obtaining the result $A = x_j$, we have from Eq.(53),

\[ P_{II}(x_j|\psi) = P(A = x_j) = \sum_i P(X = x_i, A = x_j) \]
\[ = |\langle x_j|\psi \rangle|^2. \quad (54) \]

Thus Eq.(49) holds for the second-observer.

Let $P(X = x_j|A = x_i)$ be the conditional probability of obtaining the result $X = x_j$ given $A = x_i$. Then by Eq.(53) we have

\[ P(X = x_j|A = x_i) = \frac{P(X = x_j, A = x_i)}{P(A = x_i)} \]
\[ = \delta_{i,j}. \quad (56) \]
This enables the second observer to make the following statistical inference: If the first observer were to make the detection of the observable $X$ immediately after the first measurement, then the results of these two measurement always coincides. This is possible only if, for the first observer, the first measurement changes the state as $\psi \rightarrow |x_j\rangle$ depending on the result $X = x_j$ of the first measurement. Thus we obtain Eq. (50) from the second-observer description. Obviously these reasoning is a particular case of general consideration presented in subsection 3.2 and we can thus obtain the operation measure (51) directly from Eqs. (31) and (52) by the following computation.

$$I(\Delta)(|\psi\rangle\langle\psi|) = \int_{\Delta} dI(da)(|\psi\rangle\langle\psi|)$$
$$= \sum_{x_k \in \Delta} |\psi[x_k]\rangle\langle\psi[x_k]|P(x_k|\psi)$$
$$= \sum_{x_k \in \Delta} \sum_{i,j} |x_i\rangle \langle \hat{U}(\psi \otimes \varphi_0)|(x_j\rangle \langle x_i| \otimes |\varphi_k\rangle \langle \varphi_k|) \langle \hat{U}(\psi \otimes \varphi_0)|x_j\rangle$$
$$= \sum_{x_k \in \Delta} \sum_{i,j} |x_i\rangle \langle \hat{U}(\psi \otimes \varphi_0)|x_i\rangle |\varphi_k\rangle \langle x_j| \langle \varphi_k|\hat{U}(\psi \otimes \varphi_0)|x_j\rangle$$
$$= \sum_{x_k \in \Delta} \sum_{i,l} |x_i\rangle \left( \sum_{l} c_l^* |x_l|x_i\rangle \langle \varphi_l| \varphi_k\rangle \right) \left( \sum_{l} c_l |x_j|x_l\rangle \langle \varphi_k| \varphi_l\rangle \right) |x_j\rangle$$
$$= \sum_{x_k \in \Delta} \left( \sum_{i,l} c_l^* |x_i\rangle |x_l|x_i\rangle \langle \varphi_l| \varphi_k\rangle \right) \left( \sum_{l,j} c_l |x_j|x_l\rangle \langle \varphi_k| \varphi_l\rangle \right)$$
$$= \sum_{x_k \in \Delta} |c_k|^2 |x_k\rangle \langle x_k|$$
$$= \sum_{x_k \in \Delta} |x_k\rangle \langle x_k| \psi\rangle \langle \psi| |x_k\rangle \langle x_k|$$

Now I shall give some remarks about the conventional approach to the determination of the state reduction from the second-observer description. In the conventional argument, they apply the so-called projection postulate to the state $\hat{U}(\psi \otimes \varphi)$ (see Eq. (52)) just after the interaction and conclude that if the second observer get the result $A = x_i$ then the state of the object-probe composite system changes into $|x_i\rangle |\varphi_i\rangle$ and the state of the object changes into $|x_i\rangle$. Although this argument has an apparent advantage that it never uses the explicit statistical inference, it has the several definite weak points. First, this argument applies only when the measurement of the probe system is of the first kind and when it satisfies the projection postulate. However, any first kind measurement is subjected to the Schrödinger equation for the object-probe composite system and hence we can never realize the dynamics causing the projection postulate. Thus they need to assume a process of the second kind measurement at some point between the object and the real observer and try to describe dynamics causing the projection postulate in this process. However, this means the contradiction that they can use the projection postulate nowhere since after the second kind measurement the system state cannot be described by the standard quantum mechanics and hence the projection postulate cannot apply to it. Second, their state reduction occurs only after the detection of the probe system contrary to the fact that the measurement interaction finishes before the detection of the probe system. This implies that their argument puts an apparent limitation for the time interval of the successive measurements. When one can perform the second measurement of the same system at the earliest time? We can say that after the
object-probe interaction but they must say that after the macroscopic interaction between the probe and the detector. Thus, from the first point they cannot describe any successive measurements of the one system. Last, their argument cannot be used for the measurement of observables with continuous spectrum such as the position observable, since we have no canonical state reduction postulate for continuous observables. In this case, only statistical inference can apply.

Historically speaking, von Neumann did not explicitly use the projection postulate in the second-observer description. Indeed he only mentioned the probability correlation between the object system and the probe system and wrote “If III [the second observer] were to measure (by process 1. [by the subsequent detection of observables]) the simultaneously measurable quantities A [the object observable], B [the probe observable] ( in I [the object system] or II [the probe system] respectively, or both in I + II ), then the pair of values \( a_n , b_n \) would have the probability 0 for \( m \neq n \), and the probability \( w_n \) [= the measurement probability for \( a_n \)] for \( m = n \). . . . If this is established, then the measuring process so far as it occurs in II, is ‘explained’ theoretically...” [20, p.440]. (The notes in the brackets above are due to the present author.)

4 Measurement Breaking SQL

In the preceding sections, we have discussed what is the problem of the SQL and what we can tell about quantum measurement from quantum mechanics. One conclusion is that every measurement statistics described by a completely positive operation measure is realizable. In this last section, I shall show that a measurement of the free-mass position which breaks the SQL is realizable.

4.1 Statistics of approximate measurement

Let \( \mathcal{I} \) be a completely positive operation measure. Then \( \mathcal{I} \) describes a statistics of a measurement. Now a problem arise — when can one consider \( \mathcal{I} \) as a statistics of a measurement of a given observable? In what follows, we shall consider the problem as to when a given operation measure can be considered as a position measurement of a mass with one degree of freedom. Our analysis will lead to mathematical definitions of precision and resolution of measurement. This subsection will be concluded with the mathematical formulation of the SQL.

In the textbook description of position measurement, the statistics is so characterized as

| operation measure               | \( P(a|\psi) = |\psi(x)|^2 \)                                                                 | \( \psi \rightarrow \psi = |a| \).                                                                 |
|---------------------------------|------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------|

However, it is proved that there are no operation measures which satisfies these conditions both. In general, it is proved [15] that there are no weakly repeatable operation measures for non-discrete observables, where operation measure \( \mathcal{I} \) is called \textit{weakly repeatable} if

\[
\text{Tr}[\mathcal{I}(B \cap C)\hat{\rho}] = \text{Tr}[\mathcal{I}(B)\mathcal{I}(C)\hat{\rho}],
\]

for all density operators \( \hat{\rho} \) and Borel sets \( B, C \) — Eqs.(57) and (58) lead to this condition. Thus every realizable position measurement is an approximate measurement. In order to clarify the meaning of \textit{approximate} measurement, we shall introduce two error criteria for the preciseness of measurement.
In Subsection 2.2, we have introduced the following distinction: If the free mass is in a position eigenstate at the time of a measurement of $x$ then the precision $\varepsilon$ of the measurement is defined to be the uncertainty in the result and the resolution $\sigma$ is defined to be the deviation of the position of the free mass just after the measurement from the readout just obtained. Now, we shall give precise definitions for the case of superposition.

A difficult step in defining the precision is to extract the noise factor from the readout distribution — this may be the reason why the above distinction has hardly ever discussed in literature. Let $I$ be an operation measure and $\hat{F}$ be its effect measure. Consider the requirements for $I$ to describe some approximate measurement of the position observable. Our first requirement is that $\hat{F}$ is compatible with the position observable, i.e.,

$$[\hat{x}, \hat{F}(\Delta)] = 0, \quad \text{for all Borel sets } \Delta. \quad (60)$$

This condition may be justified by the compatibility of the information obtained from this measurement with the original information of the position. Under this condition, there is a kernel function $G(a, x)$ such that

$$d\hat{F}(a) = da \int dx \, G(a, x)|x\rangle\langle x|. \quad (61)$$

Even if the measuring apparatus measures the position observable approximately, the readout distribution $P(a|\psi)$ is expected to be related to the position distribution $|\psi(x)|^2$ — from Eq.(61), this relation is expressed in the following form

$$P(a|\psi) = \int dx \, G(a, x)|\psi(x)|^2. \quad (62)$$

Note that $G(a, x)$ is independent of a particular wave function $\psi(x)$ and thus it expresses the noise in the readout. Obviously, $P(a|\psi) = |\psi(a)|^2$ for all $\psi$ (i.e. the noiseless case) if and only if $G(a, x) = \delta(a - x)$. From Eq.(9) of [8], in the case of von Neumann’s model, $G(a, x) = |\Psi(a - x)|^2$ where $\Psi$ is the prepared state of the probe (See also [25], where $\Psi(a - x)$ is called the resolution amplitude.) Roughly speaking, $G(a, x)$ is the (normalized) conditional probability density of the readout $a$, given that the free-mass is in the position $x$ at the time of measurement; hence the precision $\varepsilon(x)$ of this case should be

$$\varepsilon(x)^2 = \int da \,(a - x)^2 G(a, x). \quad (63)$$

Thus if for the prior state $\psi$ of the mass, the precision $\varepsilon(\psi)$ of the measurement is given by

$$\varepsilon(\psi)^2 = \int dx \varepsilon(x)^2 |\psi(x)|^2. \quad (64)$$

By the similar reasoning, for the mass state $\psi$ at the time of measurement, the resolution $\sigma(\psi)$ of the measurement is given by

$$\sigma(\psi)^2 = \int da \sigma(a)^2 P(a|\psi), \quad (65)$$

where

$$\sigma(a)^2 = \int dx \,(a - x)^2 |\psi_a(x)|^2. \quad (66)$$
The second requirement is that the noise is *unbiased* in the sense that the mean value of the readout is identical with the mean position in the prior state, i.e.,

$$
\int dx x P(x|\psi) = \langle \psi | \hat{x} | \psi \rangle,
$$

(67)

for all possible $\psi$.

Let $\Delta(\psi)$ be the uncertainty of the readout for the prior state $\psi$ of the mass and $\Delta x(\varphi)$ the uncertainty of the mass position at any state $\varphi$. Then in general we can prove the following

**Theorem 2** Let $\mathcal{I}$ be an operation measure for one-dimensional system. Under conditions of compatibility and unbiasedness expressed by Eqs. (60) and (67), the following relations hold:

$$
\varepsilon(\psi)^2 = \Delta(\psi)^2 - \Delta x(\psi)^2,
$$

(68)

$$
\sigma(\psi)^2 = \int da P(a|\psi) \Delta x(\psi_a)^2 + \int da P(a|\psi) (a - \langle \psi_a | \hat{x} | \psi_a \rangle)^2.
$$

(69)

Let $\varepsilon_{\text{max}}$ be the maximum of $\varepsilon(\psi)$ ranging over all $\psi$. Then we have $\varepsilon_{\text{max}} = 0$ if and only if $P(x|\psi) = |\psi(x)|^2$ for all $\psi$.

The above theorem shows that the conditions of compatibility and unbiasedness are plausible conditions for characterizing approximate position measurements and for those measurements further specification concerning approximation can be done through the precision $\varepsilon$ and the resolution $\sigma$. Thus in the following, we shall say that an operation measure $\mathcal{I}$ is of *approximate position measurement* if it satisfies the compatibility condition (60) and the unbiasedness condition (67).

Now we shall give a mathematical formulation of the SQL. Let $\mathcal{I}$ be an operation measure for a one-dimensional system. Suppose that a free mass $m$ undergo unitary evolution during the time $\tau$ between two identical measurement described by the operation measure $\mathcal{I}$. Let $\hat{U}_\tau$ be the unitary operator of the time evolution. Suppose that the free mass is in a state $\psi$ just before the first measurement. Then just after the measurement (at $t = 0$) the free mass is in the posterior state $\psi_a$ with probability density $P(a|\psi)$. From this readout value $a$, the observer make a prediction $h(a)$ for the readout of the second measurement at $t = \tau$. Then the squared uncertainty of this prediction is

$$
\Delta(\tau, \psi, a)^2 = \int dx (x - h(a))^2 P(x|\hat{U}_\tau \psi_a).
$$

(70)

In literature, the following mean-value-prediction strategy is adopted for determination of $h(a)$:

$$
h(a) = \langle \psi_a | \hat{x}(\tau) | \psi_a \rangle,
$$

(71)

where

$$
\hat{x}(\tau) = \hat{U}_\tau^\dagger x(0) \hat{U}_\tau = \hat{x} + \hat{p}_\tau / m.
$$

(72)

The predictive uncertainty $\Delta(\tau, \psi)$ of this repeated measurement with prior state $\psi$ and time duration $\tau$ is defined as the squared average of $\Delta(\tau, \psi, a)$ over all readouts of the first measurement,

$$
\Delta(\tau, \psi)^2 = \int da \Delta(\tau, \psi, a)^2 P(a|\psi).
$$

(73)
The SQL asserts the relation
\[ \Delta(\tau, \psi)^2 \geq \bar{h}\tau/m, \quad \text{for all prior state } \psi. \] (74)

If the operation measure \( I \) is of approximate position measurement, we have from Eq.(68)
\[ \Delta(\tau, \psi, a)^2 = \Delta(\hat{U}_\tau \psi_a)^2, \]
\[ = \varepsilon(\hat{U}_\tau \psi_a)^2 + \Delta x(\tau)(\psi_a)^2 \] (75)
and hence
\[ \Delta(\tau, \psi)^2 = \int da P(a|\psi) \left( \varepsilon(\hat{U}_\tau \psi_a)^2 + \Delta x(\tau)(\psi_a)^2 \right), \]
\[ = [\varepsilon(\hat{U}_\tau \psi_a)^2] + [\Delta x(\tau)(\psi_a)^2], \] (76)
where the brackets means the average due to \( P(a|\psi) \).

Now from refinement of Caves’s argument in 2.2 (see Eq.(21)), we have the following sufficient condition for the SQL.

**Theorem 3** Let \( I \) be an operation measure of approximate position measurement with precision \( \varepsilon \). If for any prior state \( \psi \) the relation
\[ [\Delta x\psi_a)^2] \leq [\varepsilon(\hat{U}_\tau \psi_a)^2] \] (77)
holds, then the SQL holds for this measurement, i.e,
\[ \Delta(\tau, \psi)^2 \geq \bar{h}\tau/m, \quad \text{for all prior state } \psi. \] (78)

In fact, we have the following estimate using the uncertainty relation (22),
\[ \Delta(\tau, \psi)^2 = [\varepsilon(\hat{U}_\tau \psi_a)^2] + [\Delta x(\tau)(\psi_a)^2] \]
\[ \geq [\Delta x(0)(\psi_a)^2] + [\Delta x(\tau)(\psi_a)^2] \]
\[ \geq [2\Delta x(0)(\psi_a)\Delta x(\tau)(\psi_a)] \]
\[ \geq \bar{h}\tau/m. \]

The above theorem shows that the concept of precision is relevant for derivation of the SQL among other candidates for error of measurement.

If the following simple relation
\[ \sigma(\psi)^2 \leq [\varepsilon(\hat{U}_\tau \psi_a)^2]. \] (79)
between the precision and the resolution holds then assumption (77) of Theorem 3 follows from Eq.(69); and hence condition (79) implies the SQL. Thus it can be said that one’s intuition which leads to the SQL is supported by the following statements; (1) every approximate measurement satisfies the compatibility condition and the unbiasedness conditions for the noise and for the state reduction, and (2) the resolution is no greater than the precision. However, it is not clear at all that every realizable measurement satisfies the last statement even if the first statement is admitted.
4.2 Von Neumann’s model of approximate measurement

In [8], Caves showed that von Neumann’s model of approximate position measurement satisfies the assumption of Theorem 3 and so the SQL holds for this model. Since the object-probe coupling of von Neumann’s model is a simple linear coupling and it illustrates some proposed models of quantum nondemolition measurement, we shall review this result in our framework below; see [20, 8] for original treatment.

This model of measurement is presented by the second-observer description. The probe system is a one dimensional system with coordinate \( Q \) and momentum \( P \) as well as the object system (the free mass) with coordinate \( x \) and momentum \( p \). The object-probe coupling is turned on from \( t = -\tilde{\tau} \) to \( t = 0 \) (\( 0 < \tilde{\tau} \ll \tau \)), it is described by an interaction Hamiltonian \( \hat{K}\hat{x}\hat{P} \) where \( K \) is a coupling constant and it is assumed to be so strong that the free Hamiltonians of the mass and the probe can be neglected; so we choose units such that \( K\tilde{\tau} = 1 \). Then if \( \Psi_0(x, Q) = f(x, Q) \), the solution of the Schrödinger equation is

\[
\Psi_t(x, Q) = f(x, Q - Ktx). \tag{80}
\]

At \( t = -\tilde{\tau} \), just before the coupling is turned on, the unknown free-mass wave function is \( \psi(x) \), and the probe is prepared in a state with wave function \( \Phi(Q) \); for simplicity we assume that \( \langle \Phi|\hat{Q}\Phi \rangle = \langle \Phi|\hat{P}\Phi \rangle = 0 \). The total wave function is \( \Psi_0(x, Q) = \psi(x)\Phi(Q) \). At the end of the interaction (\( t = 0 \)) the total wave function becomes

\[
\Psi(x, Q) = \hat{U}\Psi_0(x, Q) = \psi(x)\Phi(Q - x). \tag{81}
\]

In order to obtain the measurement statistics, recall that the result of this measurement — the inferred value of \( x \) — is the value \( Q \) called the “readout” obtained by the detection of \( Q \) of the probe system turned on at \( t = 0 \) with the subsequent stage called the “detector” in the overall macroscopic measuring apparatus. Thus the measurement probability obtaining the result \( Q \) is given by the Born statistical formula

\[
P(Q) = \int dx |\Psi(x, Q)|^2 = \int dx |\psi(x)|^2|\Phi(Q - x)|^2. \tag{82}
\]

The posterior wave function \( \psi(x|Q) \) of the free mass (at \( t = 0 \)) is obtained (up to normalization) by evaluating \( \psi(x, Q) \) at \( Q = \overline{Q} \):

\[
\psi(x|\overline{Q}) = \frac{|\Psi(x, Q)|}{P(\overline{Q})^{1/2}} = \frac{\psi(x)\Phi(\overline{Q} - x)}{P(\overline{Q})^{1/2}}. \tag{83}
\]

Note that the posterior wave function (83) can be obtained directly from application of Eq.(31) up to normalization.

To write down the operation measure \( \mathcal{I} \) of this measurement, notice that if the prior state of the free mass is a mixture \( \hat{\rho} \) then the posterior state \( \hat{\rho}_{\overline{Q}} \) satisfies the relations

\[
d\mathcal{I}(\overline{Q})\hat{\rho} = \hat{\rho}_{\overline{Q}}P(\overline{Q}|\hat{\rho}) d\overline{Q} = \Psi(\overline{Q}1 - \hat{x})\hat{\rho}\Psi(\overline{Q}1 - \hat{x})\dagger d\overline{Q}. \tag{84}
\]

Thus we have the operation measure \( \mathcal{I} \) of this measurement:

\[
\mathcal{I}(\Delta)\hat{\rho} = \int_\Delta d\mathcal{I}(\overline{Q})\hat{\rho} = \int_\Delta d\overline{Q} \Psi(\overline{Q}1 - \hat{x})\hat{\rho}\Psi(\overline{Q}1 - \hat{x})\dagger. \tag{85}
\]
From this the effect measure of this measurement is such that

\[
\hat{F}(\Delta) = \int_{\Delta} d\hat{F}(\overline{Q}),
\]

\[
d\hat{F}(\overline{Q}) = d\overline{Q} |\Psi(\overline{Q}1 - \hat{x})|^2 = d\overline{Q} \int dx |\Psi(\overline{Q} - x)|^2 |x\rangle\langle x|.
\]  

(86)

From Eq. (86), it is obvious that this measurement satisfies the compatibility condition (60) and the kernel function \(G(\overline{Q}, x)\) representing the noise in the result can be written as

\[
G(\overline{Q}, x) = |\Psi(\overline{Q} - x)|^2.
\]  

(87)

The following computations shows that the unbiasedness condition (67) holds:

\[
\int d\overline{Q} Q P(\overline{Q}|\psi) = \int d\overline{Q} dx |\psi(x)|^2 |\Phi(\overline{Q} - x)|^2 = \int dx |\psi(x)|^2 = \langle \psi|\hat{x}|\psi\rangle.
\]

(88)

Thus this measurement satisfies the conditions for approximate position measurement. The precision \(\varepsilon\) of this measurement is given by

\[
\varepsilon^2 = \int dx |\psi(x)|^2 \int d\overline{Q} (\overline{Q} - x)^2 |\Psi(\overline{Q} - x)|^2
\]

\[
= \int d\overline{Q} a^2 |\Psi(\overline{Q})|^2
\]

\[
= \langle \Psi|\hat{Q}^2|\Psi\rangle.
\]

(89)

From definition and Eqs. (82)–(83), the resolution \(\sigma\) of this measurement is given by

\[
\sigma^2 = \int d\overline{Q} P(\overline{Q}|\psi) \int dx (\overline{Q} - x)^2 |\psi(x)|^2
\]

\[
= \int d\overline{Q} dx (\overline{Q} - x)^2 |\Psi(\overline{Q} - x)|^2 |\psi(x)|^2
\]

\[
= \langle \Psi|\hat{x}^2|\Psi\rangle.
\]

(90)

Let \(\Delta Q\) be the uncertainty of the probe observable just before the measurement, i.e.,

\[
(\Delta Q)^2 = \langle \hat{\Phi}|\hat{Q}^2|\hat{\Phi}\rangle - \langle \hat{\Phi}|\hat{Q}||\hat{\Phi}\rangle^2.
\]

Then in von Neumann’s model we have just obtained

\[
\varepsilon = \varepsilon(\psi) = \Delta Q \quad \text{for all } \psi,
\]

(91)

\[
\sigma = \sigma(\psi) = \Delta Q \quad \text{for all } \psi.
\]

(92)

Thus the assumptions of Theorem 3 hold from the following computation (cf. Eq. (69))

\[
[\varepsilon(\hat{U}_r\psi|\overline{Q})]^2 = \Delta Q = \sigma(\psi) = [\Delta x(\psi|\overline{Q})]^2 + [(\overline{Q} - \langle \psi|\hat{x}|\psi\rangle)^2] \geq [\Delta x(\psi|\overline{Q})]^2.
\]

Thus the SQL holds for von Neumann’s model of approximate position measurement.
4.3 Realization of measurement breaking SQL

We shall now turn to Yuen’s proposal [4]. His observation is that if the measurement leaves the free mass in a contractive state \( \psi_a \) for every readout \( a \) then we can get

\[
\Delta x(\tau)(\psi_a) \ll (\hbar/2m)^{1/2} \ll \Delta x(0)(\psi_a). \tag{93}
\]

Thus the SQL breaks if such a measurement has a good precision

\[
\varepsilon(\hat{U}_\tau \psi_a) \ll (\hbar/2m)^{1/2}. \tag{94}
\]

In fact, from the combination of Eqs.(76) and (93)–(94), we get

\[
\Delta(\tau,\psi)^2 = \left[ \varepsilon(\hat{U}_\tau \psi_a)^2 \right] + \left[ \Delta x(\tau)(\psi_a) \right]^2 \ll \hbar/\tau. \tag{95}
\]

In this subsection, I shall show that such statistics of measurement can be realized by a measurement considered first by Gordon and Louisell [24].

In [24], Gordon and Louisell considered the following statistical description of measurement. Let \( \{\Psi_a\} \) and \( \{\Phi_a\} \) be a pair of families of wave functions with real parameter \( a \). The Gordon-Louisell measurement, denoted by \( \{|\Psi_a\rangle\langle\Phi_a|\} \), is the measurement with the following statistics: For any prior state \( \psi \),

- measurement probability: \( P(a|\psi) = |\langle\Phi_a|\psi\rangle|^2 \), \tag{96}
- state reduction: \( \psi \rightarrow \psi_a = \Psi_a \). \tag{97}

One of the characteristic properties of the Gordon-Louisell measurement is that the posterior state \( \Psi_a \) depends only on the measured value \( a \) and not at all on the prior state \( \psi \). For the condition that Eq.(96) determines the probability density, we assume that \( \{\Phi_a\} \) is so normalized as

\[
\int da |\Phi_a\rangle\langle\Phi_a| = 1. \tag{98}
\]

From Eq.(98), it is provided that \( \Phi_a \) may not be a normalizable vector such as position eigenstate \( |a\rangle \). However, \( \Psi_a \) is assumed to be a normalized vector. The measurement statistics Eqs.(96)–(97) yields the following operation measure \( \mathcal{I} \) and effect measure \( F \):

\[
\mathcal{I}(\Delta)\hat{\rho} = \int da |\Psi_a\rangle\langle\Phi_a|\hat{\rho}|\Phi_a\rangle\langle\Psi_a|, \tag{99}
\]

\[
F(\Delta) = \int da |\Phi_a\rangle\langle\Phi_a|. \tag{100}
\]

The following computation shows that the operation measure \( \mathcal{I} \) in Eq.(99) is completely positive: For any finite sequences of vectors \( \xi_1, \xi_2, \ldots, \xi_n \) and \( \eta_1, \eta_2, \ldots, \eta_n \), we obtain

\[
\sum_{i,j=1}^n \langle \xi_i | \mathcal{I}(a) | \eta_i \rangle \langle \eta_j | \xi_j \rangle = da \sum_{i,j=1}^n \langle \xi_i | \Psi_a \rangle \langle \Phi_a | \eta_i \rangle \langle \eta_j | \Phi_a \rangle \langle \Psi_a | \xi_j \rangle
\]

\[
= da \left| \sum_{i=1}^n \langle \xi_i | \Psi_a \rangle \langle \Phi_a | \eta_i \rangle \right|^2
\]

\[
\geq 0.
\]

Thus from Theorem 1, we get the following

**Theorem 4** Every Gordon-Louisell measurement is realizable.
As mentioned above, Gordon-Louisell measurements control the posterior states independently of the prior states and this property is suitable for our purpose of realization of measurement which leaves the free mass in a contractive state. The following Theorem is an immediate consequence from Theorem 4, which asserts that the state reduction of position measurement can be arbitrarily controlled.

**Theorem 5** For any Borel family \( \{ \Psi_a \} \) of unit vectors, the following statistics of position measurement is realizable:

- measurement statistics: \( P(a|\psi) = |\psi(a)|^2 \)
- state reduction: \( \psi \rightarrow \psi_a = \Psi_a \)

This measurement corresponds to the Gordon-Louisell measurement \( \{|\mu\nu\omega\rangle\langle a|\} \) with the following operation measure \( I \) and effect measure \( \hat{F} \):

\[
I(\Delta)\hat{\rho} = \int_{\Delta} da |\Psi_a\rangle\langle a|\hat{\rho}|\Psi_a|,
\]

\[
\hat{F}(\Delta) = \int_{\Delta} da |a\rangle\langle a|.
\]

Let \( |\mu\nu\omega\rangle \) be a fixed contractive state with \( \langle x \rangle = \langle p \rangle = 0 \) and let \( \Psi_a \) be such that \( \langle x|\Psi_a\rangle = \langle x-a|\mu\nu\omega\rangle \). Then \( \{ \Psi_a \} \) is the family of contractive states \( |\mu\nu\omega\rangle \) with \( \langle x \rangle = a \) and \( \langle p \rangle = 0 \), which satisfies the assumption in Theorem 5. Thus the following measurement statistics of Gordon-Louisell measurement \( \{|\mu\nu\omega\rangle\langle a|\} \) is realizable:

- measurement probability: \( P(a|\psi) = |\langle a|\psi\rangle|^2 \)
- state reduction: \( \psi \rightarrow \psi_a = |\mu\nu\omega\rangle \)

Now we shall examine the predictive uncertainty \( \Delta(\tau) \) of the repeated measurements of this measurement. From Eqs.(104) and (105), this measurement satisfies the compatibility condition (60) and the unbiasedness condition (67). Further, it is an exact measurement of the position observable, i.e., \( G(a,x) = \delta(a-x) \) and \( \varepsilon(\psi) = 0 \) for all \( \psi \). From Eqs.(76) and (16)–(17), we have

\[
\Delta(t,\psi)^2 = \int da P(a|\psi) \left( \varepsilon(\hat{U}_t\psi_a)^2 + \Delta x(t)(\psi_a)^2 \right),
\]

\[
= \int da P(a|\psi) \Delta x(t)(\psi_a)^2,
\]

\[
= (1/4\xi)(\hbar\tau/m) + (2\hbar/m\omega)(|\mu + \nu\omega/2|^2(t-\tau)^2),
\]

where

\[
\tau = 2\xi/(\omega|\mu + \nu|^2) = \xi\hbar m/\Delta p(0)^2.
\]

Thus for \( t = \tau \) and large \( \xi \), we have

\[
\Delta(\tau,\psi)^2 = (1/4\xi)(\hbar\tau/m) \ll \hbar\tau/m.
\]

We have therefore shown that the Gordon-Louisell measurement \( |\mu\nu\omega\rangle\langle a| \) is realizable measurement and it breaks the SQL. In the rest of this subsection, I shall give a realization of this measurement with an interaction Hamiltonian of the object-probe coupling [19].

The model description is parallel with that of von Neumann’s measurement in 4.2. The free mass (the object system) is coupled to a probe system which is a one
A dimensional system with coordinate $Q$ and momentum $P$. The coupling is turned on from $t = -\tilde{\tau}$ to $t = 0$ ($0 < \tilde{\tau} \ll \tau$) and it is assumed to be so strong that the free Hamiltonians of the mass and the probe can be neglected. We choose the following interaction Hamiltonian

$$H = \frac{K\pi}{3\sqrt{3}} \{2(\hat{x}\hat{P} - \hat{Q}\hat{p}) + (\hat{x}\hat{p} - \hat{Q}\hat{P})\}, \quad (107)$$

where $K$ is the coupling constant chosen as $K\tilde{\tau} = 1$. Then if $\Psi_0(x, Q) = f(x, Q)$, the solution of the Schrödinger equation is

$$\Psi_t(x, Q) = f \left( \frac{2}{\sqrt{3}} \left\{ x \sin \left( \frac{(1-Kt)\pi}{3} \right) + Q \sin \left( \frac{Kt\pi}{3} \right) \right\} \right),$$

$$f \left( \frac{2}{\sqrt{3}} \left\{ -x \sin \left( \frac{Kt\pi}{3} \right) + Q \sin \left( \frac{(1+Kt)\pi}{3} \right) \right\} \right). \quad (108)$$

At $t = -\tilde{\tau}$, just before the coupling is turned on, the unknown free-mass wave function is $\psi(x)$, and the probe is prepared in a contractive state $\Phi(Q) = \langle Q | \mu \nu \omega \rangle$, so that the total wave function is $\Psi_0(x, Q) = \psi(x)\Phi(Q)$; expectation values for this state is $\langle \hat{Q} \rangle_0 = \langle \hat{P} \rangle_0 = 0$. At $t = 0$, the end of the interaction, the total wave function becomes

$$\Psi(x, Q) = \psi(Q)\Phi(Q - x). \quad (109)$$

Compare with Eq.(81); the statistics is much different. At this time, a value $\bar{Q}$ for $Q$ is obtained by the detector of the probe observable in the subsequent stage of the measuring apparatus, from which one infers a value for $x$. Thus the probability density $P(\bar{Q}|\psi)$ to obtain the value $\bar{Q}$ as the result of this measurement is given by

$$P(\bar{Q}|\psi) = \int dx |\Psi(x, \bar{Q})|^2 = |\psi(\bar{Q})|^2. \quad (110)$$

The free-mass wave function $\psi(\bar{Q})(x) = \psi(x | \bar{Q})$ just after this measurement ($t = 0$) is obtained (up to normalization) by

$$\psi(x | \bar{Q}) = \left[ 1/P(\bar{Q}|\psi) \right]^{1/2} \Psi(x, \bar{Q})$$

$$= \left[ \psi(\bar{Q})/|\psi(\bar{Q})| \right] \Phi(\bar{Q} - x)$$

$$= C \langle x | \mu \nu \bar{Q} \omega \rangle,$$

where $C (|C| = 1)$ is a constant phase factor.

Thus we have just obtained the measurement statistics of this measurement as follows:

- **measurement probability:** $P(\bar{Q}|\psi) = |\psi(\bar{Q})|^2$, \quad (111)
- **state reduction:** $\psi \rightarrow \psi_{\bar{Q}} = |\mu \nu \bar{Q} \omega \rangle$. \quad (112)

This shows that this measurement is a realization of Gordon-Louisell measurement $\{|\mu \nu \bar{Q} \omega \rangle(\bar{Q})\}$.

Now I have shown everything I promised before. From our analysis, we can conclude that there are no general reasons in physics which limits the accuracy of the repeated measurement of the free-mass position such as the standard quantum limit for monitoring the free-mass position. In [27], Bondurant analyzed the performance of an interferometric gravity-wave detector which has a Kerr cell in each arm used to counter the effects of radiation pressure fluctuation and has a feedback loop used to keep the interferometer operating at the proper null. He succeeded in showing
that this measurement realizes a monitoring the free-mass position which breaks the SQL. It will be an interesting problem to show that the role of the Kerr cell and the feedback loop in his analysis has some corresponding part in the interaction scheme realizing the Gordon-Louisell measurement discussed above.

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