METASTABLE BEHAVIOR OF REVERSIBLE, CRITICAL 
ZERO-RANGE PROCESSES

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Abstract. We prove that the position of the condensate of reversible, critical 
zero-range processes on a finite set $S$ evolves, in a suitable time scale, as a 
continuous-time Markov chain on $S$ whose jump rates are proportional to the 
capacities of the underlying random walk which describes the jumps of particles 
in the zero-range dynamics.

1. Introduction

Evans [14] introduced, twenty years ago, a class of zero-range processes which 
exhibit condensation. These dynamics describe the evolutions of particles on a finite 
or countably infinite set $S$. The dynamics is conservative, particles are not created 
or annihilated, and it condensates in the sense that a macroscopic proportion of 
particles tend to accumulate on a single site.

The condensation can be observed at the level of the stationary states. As the 
total number of particles is conserved and the dynamics is irreducible, for each 
density $\rho$ in a set of possible densities $I$, there exists a unique stationary state, 
indexed by $\rho$. It has been shown, in several different contexts [14, 17, 16, 18, 3], that 
the stationary state is concentrated on sets of configurations in which a macroscopic 
proportion of particles sit on a single site.

When the set $S$ is finite this result reads as follows. For each $N \geq 1$, representing 
the total number of particles, denote by $\mu_N$ the unique stationary state of the zero-
range dynamics with $N$ particles evolving on $S$. Fix a sequence $(\ell_N : n \geq 1)$ such 
that $\ell_N \to \infty$, $\ell_N/N \to 0$. Denote by $E_x^N$, $x \in S$, the set of configurations given by 

$$E_x^N = \{ \eta : \eta_x \geq N - \ell_N \}.$$

In this formula $\eta = (\eta_x)_{x \in S}$ represents a configuration of particles and $\eta_x$ the 
number of particles at site $x$ for the configuration $\eta$. In particular, $E_x^N$ represents 
the set of configurations with at least $N - \ell_N$ particles at site $x$, that is, the 
configurations in which a condensate has been formed at site $x$.

It has been proved (cf. Theorems 2.1 and 2.3 below and references there) that 
for a family of dynamics, there exists a sequence $\ell_N$ with the above properties and 
such that $\mu_N(E_x^N) \to 1/|S|$. Therefore, under the stationary state, essentially all 
particles sit on a single site.

Once this result has been established, it becomes natural to consider the time 
evolution of the model. One expects to observe two different regimes. As particles 
accumulate on a single site in the stationary state, in a certain time-scale, starting 
from a uniform distribution of particles among all sites, nucleation should occur, 
and particles should gradually concentrate on less and less sites, until almost all of
them sit on a single site. This is called the nucleation phase of the dynamics and it has been established in [5] for super-critical zero-range dynamics.

Consider a configuration in which all particles sit on the same site. Call condensate the site at which this occurs. On a longer time-scale, one expects to observe an evolution of the condensate. This has been quantitatively analyzed for super-critical zero-range processes evolving on a finite set in [8, 18, 25, 23] and in [4] when the number of sites increases together with the number of particles.

In this article, we examine the evolution of the condensate on a finite set for critical zero-range dynamics. There is an important difference between super-critical models, considered previously, and the critical one studied here. In super-critical zero-range dynamics, starting from a configuration in a well $E_x^N$, the process visits all configurations of $E_x^N$ before hitting a new well $E_y^N$, $y \neq x$. These dynamics are said to “visit points”.

A general theory has been proposed in [6, 7] to derive the metastable behavior of dynamics which visit points. It has been successfully applied to super-critical zero-range processes in the articles mentioned above and to inclusion processes [10].

The main advantage of dynamics which visit points lies in the fact that there are robust tools, based on potential theory, to prove that the process equilibrates in each well before it attains a new one.

Critical zero-range dynamics do not visit points because the wells are too large. For super-critical dynamics, to prove that $\mu_N(E_x^N) \to 1/|S|$ one can take any sequence $\ell_N$ such that $\ell_N \to \infty$, $\ell_N/N \to 0$. In contrast, for critical dynamics, one needs $\ell_N$ to be large enough; in addition to the previous conditions, $\ell_N$ needs to fulfill $\log \ell_N / \log N \to 1$.

There are two main difficulties in the proof of the metastable behavior of a stochastic dynamics. One has to prove a replacement lemma which permits to substitute time-averages of functions by time-averages of their mean value (with respect to the stationary state) over the wells, and one has to guarantee that the process does not leave a well too quickly. The first property is needed in the proof of the uniqueness of limit points, and the second one in the proof of tightness.

The replacement lemma relies on two results. First, one has to show that the process equilibrates inside a well before it visits another well. This is done (see Theorem 3.1) by showing that the inverse of the spectral gap of the generator restricted to a well is much smaller than the transition times between wells. The proof of this result borrows ideas from [4].

Then, one has to prove that the solution of the Poisson equation, introduced in (7.1), is almost constant inside the wells. This is the first major difficulty faced in this article. When the process visit points, one can derive this property from condition (H1) of [6] and PDE’s estimates, see [23, 24]. Since this condition does not hold for critical zero-range processes, we introduce a method to derive this property from the local spectral gap and a careful analysis of the solution of the Poisson equation.

The proof that the process remains in a well a reasonable amount of time is the second major challenge encountered in this article. This is solved by showing, through the construction of an explicit super-harmonic function, that, starting from a configuration inside the well, the process visits a deep region of the well before it reaches another well. Then, we show that starting from this deep region, the
process visits all points of this deep region before attaining another well. This property and potential-theoretic bounds permit to complete the argument.

All estimates are delicate in the critical case due to the small difference between time scales. While nucleation occurs in the diffusive scale $N^2$, the evolution of the condensate is observed in the time scale $N^2 \log N$, and the equilibration inside the wells in a time scale $(N/\log N)^2$.

To our knowledge, this is the first model which does not visit points and for which the metastable behavior is derived. Interesting problems, left for future investigations, are the description of the nucleation phase of this model and the behavior of the condensate in the case where the number of sites increases together with the number of particles.

2. Notation and Main Results

In this section, we present the main results of the article. All new notation introduced in the text is presented in blue.

2.1. Condensing zero-range processes. Fix a finite set $S$ and let $\kappa = |S|$, which is assumed to be larger than or equal to 2. Elements of $S$ are denoted by the letters $x$, $y$, etc.

In this article, $\{X(t)\}_{t \geq 0}$ represents a continuous-time, irreducible Markov chain on the set $S$. The jump rates are represented by $r$ and the generator by $L_X$ so that

$$(L_X f)(x) = \sum_{y \in S} r(x, y) \left[ f(y) - f(x) \right]$$

for all $f : S \to \mathbb{R}$. For convenience, we set $r(x, x) = 0$ for all $x \in S$.

Denote by $P_x$, $x \in S$, the law of the random walk $X$ starting from $x$, and by $m$ its unique invariant probability measure.

Let $N = \{0, 1, 2, \ldots \}$, fix $\alpha > 0$, and define $a = a_\alpha : \mathbb{N} \to \mathbb{R}^+$ as

$$a(0) = 1 \quad \text{and} \quad a(n) = n^\alpha \quad \text{for} \quad n \geq 1.$$

Denote by $g = g_\alpha : \mathbb{N} \to \mathbb{R}^+$ the function given by

$$g(0) = 0, \quad g(1) = 1 \quad \text{and} \quad g(n) = \frac{a(n)}{a(n-1)} = \left( \frac{n}{n-1} \right)^\alpha, \quad n \geq 2.$$

Denote by $a, g : \mathbb{N}^S \to \mathbb{R}$ the functions given by

$$g(\eta) = \prod_{x \in S} g(\eta_x) \quad \text{and} \quad a(\eta) = \prod_{x \in S} a(\eta_x).$$

For each $x \neq y \in S$ and $\eta \in \mathbb{N}^S$, denote by $\sigma^{x,y} \eta$ the configuration obtained from $\eta$ by moving a particle from $x$ to $y$:

$$\sigma^{x,y} \eta = \begin{cases} 
\eta_x - 1 & \text{if } z = x, \\
\eta_y + 1 & \text{if } z = y, \\
\eta_z & \text{otherwise,}
\end{cases}$$

if $\eta_x \geq 1$, and $\sigma^{x,y} \eta = \eta$ if $\eta_x = 0$.

The zero-range process with parameters $\alpha$ and $r$ is the continuous-time Markov chain $\{\eta_N(t)\}_{t \geq 0}$ on $\mathbb{N}^S$ whose generator, denoted by $\mathcal{L}_N$, is given by

$$(\mathcal{L}_N F)(\eta) = \sum_{x, y \in S} g(\eta_x) r(x, y) \left[ F(\sigma^{x,y} \eta) - F(\eta) \right], \quad (2.1)$$
for all functions $F : \mathbb{N}^S \to \mathbb{R}$. The subscript $N$ refers to the total number of particles, the unique quantity conserved by the dynamics. The ergodic components correspond to the subsets of configurations with a fixed number of particles. Denote by $\mathcal{H}_N = \mathcal{H}_{S,N}$, $N \geq 1$, the set given by

$$\mathcal{H}_N = \left\{ \eta = (\eta_x)_{x \in S} \in \mathbb{N}^S : \sum_{x \in S} \eta_x = N \right\}.$$ 

2.2. Condensation and metastable behavior in the super-critical case. In this subsection, we review the results for the super-critical case $\alpha > 1$ obtained in [8, 13, 25, 23].

Condensation phenomenon. To simplify the presentation, we assume that the invariant probability measure $m$ of the underlying random walk is the uniform measure:

$$m(x) = \frac{1}{\kappa}, \quad x \in S.$$  \hspace{1cm} (2.2)

For the general result without this assumption, we refer to [8, 25].

The invariant probability measure for the zero-range process can be written as

$$\mu_N(\eta) = \frac{1}{\hat{Z}_N} \frac{1}{a(\eta)},$$  \hspace{1cm} (2.3)

where $\hat{Z}_N$ is the normalizing constant given by

$$\hat{Z}_N = \sum_{\eta \in \mathcal{H}_N} \frac{1}{a(\eta)}.$$

Let $(\ell_N)_{N \in \mathbb{N}}$ be a sequence satisfying

$$1 \ll \ell_N \ll N.$$ 

Here, for two sequences $(a_N)_{N \in \mathbb{N}}$ and $(b_N)_{N \in \mathbb{N}}$ of positive real numbers, $a_N \ll b_N$ stands for $\lim_{N \to \infty} a_N/b_N = 0$.

Denote by $\mathcal{E}_N^x$, $x \in S$, the set of configurations with at least $N - \ell_N$ particles at site $x$:

$$\mathcal{E}_N^x = \{ \eta : \eta_x \geq N - \ell_N \},$$  \hspace{1cm} (2.4)

and write

$$\mathcal{E}_N = \bigcup_{x \in S} \mathcal{E}_N^x, \quad \Delta_N = \mathcal{H}_N \setminus \mathcal{E}_N \quad \text{so that} \quad \mathcal{H}_N = \mathcal{E}_N \cup \Delta_N.$$ 

The sets $\mathcal{E}_N^x$, $x \in S$, are called the wells. To stress the dependence of $\Delta_N$ on the set $S$, we sometimes write $\Delta_N$ as $\Delta_{S,N}$.

Next result asserts that the dynamics tend to concentrate particles on a single site. This is called the condensation phenomenon.

**Theorem 2.1** (Condensation in the super-critical case). For $\alpha > 1$,

$$\lim_{N \to \infty} \mu_N(\mathcal{E}_N^x) = \frac{1}{\kappa} \quad \text{for all} \quad x \in S.$$ 

It follows from this result that $\lim_{N \to \infty} \mu_N(\Delta_N) = 0$. Versions of this result have been obtained in [13, 17, 16, 8, 1, 2, 3]. We refer to [8, Section 3] for a proof without the assumption $(2.2)$.
Time-scale. Since the transition time between two wells is of order $N^{1+\alpha}$, we speed-up the process by this amount: Let

$$
\theta_N = N^{1+\alpha}, \quad \xi_N(t) = \eta_N(t\theta_N).
$$

(2.5)

The process $\xi_N(t)$ is the $\mathbb{N}^S$-valued, Markov chain whose generator, denoted by $\mathcal{L}_N^\xi$, is given by $\mathcal{L}_N^\xi = \theta_N \mathcal{L}_N$, where the later one has been introduced in (2.1).

Let $D(\mathbb{R}_+, \mathbb{N}^S)$ be the path space of right-continuous functions $e : \mathbb{R}_+ \to \mathbb{N}^S$ with left-limits, endowed with the Skorohod topology. Denote by $P_N^\eta$, $\eta \in \mathbb{N}^S$, the probability measure on $D(\mathbb{R}_+, \mathbb{N}^S)$ induced by the speed-up zero-range dynamics $\xi_N(t)$ starting from the configuration $\eta$. Expectation with respect to $P_N^\eta$ is represented by $E^\eta$.

Trace process and order process. To describe the metastable behavior, we first introduce the trace process $\xi_N^\mathcal{E}_N(\cdot)$ of $\xi_N(\cdot)$ on $\mathcal{E}_N$. Informally, this is a continuous-time Markov process on $\mathcal{E}_N$ obtained from $\xi_N(\cdot)$ by turning off the clock when the process is not in $\mathcal{E}_N$.

To define the trace process, denote by $T_N^\mathcal{E}_N(t)$ the total time the process $\xi_N(\cdot)$ spent on $\mathcal{E}_N$ in the time-interval $[0, t]$

$$
T_N^\mathcal{E}_N(t) = \int_0^t \chi_{\mathcal{E}_N}(\xi_N(s)) \, ds, \quad t \geq 0.
$$

In this formula, $\chi_A$ is the indicator function of the set $A$. Let $S_N^\mathcal{E}_N(t)$ be the generalized inverse of $T_N^\mathcal{E}_N(t)$:

$$
S_N^\mathcal{E}_N(t) = \sup \{ s \geq 0 : T_N^\mathcal{E}_N(s) \leq t \}.
$$

(2.6)

The trace process $\xi_N^\mathcal{E}_N(\cdot)$ is given by

$$
\xi_N^\mathcal{E}_N(t) = \xi_N(S_N^\mathcal{E}_N(t)), \quad t \geq 0.
$$

(2.7)

One can verify that $\xi_N^\mathcal{E}_N(t)$ is an $\mathcal{E}_N$-valued, continuous-time Markov chain [3, Section 6]. In this article, the trace process always refers to the process $\xi_N^\mathcal{E}_N$.

Let $\Psi_N : \mathcal{E}_N \to S$ be the projection defined by

$$
\Psi_N(\eta) = \sum_{x \in S} x \chi_{\mathcal{E}_N^x}(\eta), \quad \eta \in \mathcal{E}_N,
$$

and denote by $Y_N(t)$ the projection on $S$ of the trace process $\xi_N^\mathcal{E}_N$,

$$
Y_N(t) := \Psi_N(\xi_N^\mathcal{E}_N(t)), \quad t \geq 0.
$$

The process $Y_N(\cdot)$ is non-Markovian. It represents the position of the condensate and is called the order process. For a probability measure $\nu_N$ on $\mathbb{N}^S$, denote by $Q_N^\nu$, the probability measure on $D(\mathbb{R}_+, S)$ induced by the process $Y_N(t)$, with $\xi_N(0)$ distributed according to $\nu_N$.

The limiting process $Y(\cdot)$ in the super-critical case. Denote by $D_X(\cdot)$ the Dirichlet form associated to the random walk $X$: For $f : S \to \mathbb{R}$,

$$
D_X(f) = \frac{1}{2} \sum_{x, y \in S} m(x) r(x, y) [f(y) - f(x)]^2.
$$

Denote by $\tau_C$, $C \subset S$, the hitting time of the set $C$:

$$
\tau_C = \inf\{ t \geq 0 : X(t) \in C \}.
$$
Fix two non-empty, disjoint subsets $A, B$ of $S$. The equilibrium potential $h_{A,B} : S \to \mathbb{R}$ between $A$ and $B$ is defined by

$$h_{A,B}(x) = \mathbb{P}_x[\tau_A < \tau_B], \quad x \in S.$$  \hfill (2.8)

It is well-known that $h_{A,B}$ is the unique solution to the boundary equation:

$$
\begin{cases}
(L_X h)(x) = 0 & x \in S \setminus \{A \cup B\}, \\
h(x) = 1 & x \in A, \\
h(x) = 0 & x \in B.
\end{cases}
$$

The capacity $\text{cap}_X(A, B)$ between $A$ and $B$ is given by

$$\text{cap}_X(A, B) = D_X(h_{A,B}).$$  \hfill (2.9)

If $A = \{x\}$ is a singleton we write $\text{cap}_X(x, B)$ instead of $\text{cap}_X(\{x\}, B)$.

The limiting process $Y(\cdot)$ is the continuous-time Markov chain on $S$ whose generator, represented by $L_Y$, is given by

$$(L_Y f)(x) = \frac{\kappa}{\Gamma_\alpha I_\alpha} \sum_{y \in S} \text{cap}_X(x, y) \left[ f(y) - f(x) \right], \quad f : S \to \mathbb{R}.$$  \hfill (2.10)

In this formula,

$$\Gamma_\alpha = \sum_{j=0}^{\infty} \frac{1}{\alpha(j)} = 1 + \sum_{n \geq 1} \frac{1}{n^\alpha} \quad \text{and} \quad I_\alpha = \int_0^1 u^\alpha (1-u)^\alpha du.$$  \hfill (2.10)

Remark that $\Gamma_\alpha$ is finite because $\alpha > 1$.

Denote by $Q^Y_x$ the probability measure on $D(\mathbb{R}_+, S)$ induced by the Markov chain associated to the generator $L_Y$ starting from $x$.

**Metastable behavior.** We may now describe the evolution of the condensate, characterizing the metastable behavior of super-critical zero range processes.

**Theorem 2.2.** Suppose that $\alpha > 1$ and that the invariant probability measure $m$ of $X(t)$ is the uniform measure \(2.2\). Fix $\delta > 0$ small and let the sequence $\ell_N$, introduced in \(2.4\), be given by $\ell_N = N^\delta$. Fix $x \in S$ and a sequence $\{\eta_N : N \geq 1\}$ such that $\eta_N \in E_N$ for all $N \geq 1$. Then,

1. As $N \to \infty$, the sequence of measures $Q^N_{\eta_N}$ converges to $Q^Y_x$.
2. The amount of time the zero-range process remains in $\Delta_N$ is negligible in the sense that

$$\lim_{N \to \infty} \sup_{q \in E_N} \mathbb{E}^N_q \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) \, ds \right] = 0.$$  \hfill (2.10)

This result has been proven, without the uniformity assumption \(2.2\), in \[8\] for reversible dynamics. It has been extended in \[18, 25\] to the general case.

The first part of the theorem describes the evolution of the condensate after we remove from the trajectories all excursions to $\Delta_N$. The second part states that the process spends a negligible amount of time in this set.
2.3. Critical zero-range processes. We turn to the case $\alpha = 1$, under the uniformity condition (2.2). We adopt the same notation as in the previous subsection. The only and important difference lies on the definition of $\ell_N$, which defines the wells, and of $\theta_N$, which describes the time-scale.

The case $\alpha = 1$ is called critical because a phase transition is observed at this value of the parameter $\alpha$. This phenomenon can be observed at the level of the stationary states, cf. [16]. For $\alpha \leq 1$, under the grand-canonical stationary states, as the density of particles $\rho \to \infty$, there is an infinite number of particles at every site with probability one, as it should be for homogeneous systems. In contrast, for $1 < \alpha \leq 2$, as $\rho \to \infty$, the probability to have a fixed number of particles at a given site (say no particles at one site) converges to a strictly positive value.

This phase transition explains why no condensation is observed for $\alpha < 1$ and why, for $\alpha > 1$, particles are expected to concentrate on one site as the density increases.

Condensation of particles. We first describe the condensation. Let $\ell_N$ be the sequence given by

$$\ell_N = \frac{N}{\log N}.$$  \hspace{1cm} (2.11)

**Theorem 2.3.** The assertions of Theorem 2.1 hold for $\alpha = 1$ provided $(\ell_N)_{N \in \mathbb{N}}$ is chosen as in (2.11).

The proof of this result, given in Section 4, is similar to the one of the super-critical case, presented in [8, Section 3]. The assumption (2.2) can be removed, at the cost of heavy notation, which we preferred to avoid.

It follows from the proof of Theorem 2.3 that the sequence $\ell_N$ needs only to satisfy the conditions

$$\lim_{N \to \infty} \frac{\ell_N}{N} = 0, \quad \lim_{N \to \infty} \frac{\log \ell_N}{\log N} = 1.$$  

In particular, we can select $\ell_N = N/(\log N)^h$ for any $h > 0$. In contrast, the result fails for $\ell_N = N^\delta$, $\delta \in (0, 1)$, see Proposition 2.11.

Time-scale. The transition time between two wells can be easily guessed by examining the case with two sites, where the zero-range becomes a random walk on $\{0, \ldots, N\}$. In this situation, one can compute explicitly the capacities between two wells and deduce from them the time-scale. In the critical case, it is of order $N^2 \log N$. In particular, in the critical case, in the definition of the process $\xi_N(t)$, introduced in (2.5), we take $\theta_N = N^2 \log N$.

Metastable behavior. The statement of the metastable behavior of the condensate requires further notation and hypotheses.

We assume in this article that the underlying random walk is reversible with respect to the invariant (uniform) measure $m$:

$$r(x, y) = r(y, x) \quad \forall x, y \in S.$$  \hspace{1cm} (2.12)

The evolution of the condensate is described by the $S$-valued Markov chain, denoted by $Z(t)$, whose generator $L_Z$ is given by

$$(L_Z f)(x) = 6\kappa \sum_{y \in S} \text{cap}_X(x, y) \{f(y) - f(x)\}, \quad f : S \to \mathbb{R}.$$  \hspace{1cm} (2.13)
The factor 6 in this formula represents $1/I_1$, where $I_n$ is defined in (2.10). Denote by $Q_Z^x$ the probability measure on $D(\mathbb{R}_+,S)$ induced by the Markov chain associated to the generator $L_Z$ starting from $x$.

Recall the definition of the measure $\mu_N$ introduced in (2.3). Denote by $\mu_N^\mathcal{E}$ the measure $\mu_N$ conditioned on $\mathcal{E}$:

$$
\mu_N^\mathcal{E}(\eta) = \frac{\mu_N(\eta)}{\mu_N(\mathcal{E})}, \quad \eta \in \mathcal{E}.
$$

The main result of the article reads as follows.

**Theorem 2.4.** Assume that $\alpha = 1$ and that the conditions (2.12) are in force.

Fix $x \in S$ and a sequence of probability measures $\{\nu_N : N \geq 1\}$ on $\mathcal{H}_N$ such that $\nu_N(\mathcal{E}) = 1$ for all $N \geq 1$. Assume, furthermore, that there exists a finite constant $C_0$ such that

$$
E_{\mu_N^\mathcal{E}} \left( \frac{d\nu_N}{d\mu_N^\mathcal{E}} \right)^2 = \sum_{\eta \in \mathcal{E}} \frac{\nu_N(\eta)^2}{\mu_N^\mathcal{E}(\eta)} \leq C_0
$$

for all $N \geq 1$. Then,

1. As $N \to \infty$, the sequence of measures $Q_{\nu_N}^x$ converges to $Q_x^Z$.
2. The total time spent at $\Delta_N$ is negligible in the sense that

$$
\lim_{N \to \infty} E_{\nu_N}^\mathcal{E} \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) \, ds \right] = 0.
$$

Denote by $\mu_N^\mathcal{E}$ the measure $\mu_N$ conditioned on $\mathcal{E}$:

$$
\mu_N^\mathcal{E}(\eta) = \frac{\mu_N(\eta)}{\mu_N(\mathcal{E})}, \quad \eta \in \mathcal{E}.
$$

Fix $x \in S$ and a sequence of probability measures $\{\nu_N : N \geq 1\}$ on $\mathcal{H}_N$ satisfying the hypotheses of the previous theorem. Then,

$$
E_{\mu_N^\mathcal{E}} \left( \frac{d\nu_N}{d\mu_N^\mathcal{E}} \right)^2 = \frac{\mu_N(\mathcal{E})}{\mu_N(\mathcal{E})} E_{\mu_N^\mathcal{E}} \left( \frac{d\nu_N}{d\mu_N^\mathcal{E}} \right)^2 \leq C_0.
$$

**Remark 2.5.** In the cases $|S| = 2, 3$, one can prove that the process visits all configurations in a well before hitting a new valley in the sense of condition (H1) of [6]. In particular, in these low dimensions, one can repeat the approach presented in [8] to derive the metastable behavior of the critical zero-range process.

**Remark 2.6.** The result should hold without the assumptions that the stationary measure of the random walk $X(t)$ is uniform and that the process is reversible. The first hypothesis should not be difficult to remove. It is a minor technical difficulty. The second one provides some symmetry in the construction of a super-harmonic function in Section 10. In the general case, another test function has to be created.

**Remark 2.7.** On the diffuse scale $N^2$, as in the super-critical case [5], we expect the density of particles to converge to a diffusion which is absorbed at the boundary. This is an open problem which deserves to be considered.

**Remark 2.8.** Another interesting open problem, in the critical case $\alpha = 1$, is to prove that the condensate evolves as a Lévy process when $S$ is the one-dimensional torus with $\kappa$ points, $X(t)$ a finite-range, symmetric random walk on $S$ and $\kappa/N \to \rho > \rho_c$, where $\rho_c$ is the critical density above which condensation occurs. This result has been proven in [4] in the super-critical case $\alpha > 20$. 

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Remark 2.9. In a future work, we examine the metastable behavior of critical zero-range processes starting from a configuration, instead of starting from a measure.

Remark 2.10. With a little more effort, one can prove that the finite-dimensional distributions of the process \( \Psi_N(t) \) converge to the ones of the process \( Y(t) \), applying Proposition 2.1 of [21]. This is left to the future work alluded to in the previous remark.

3. Sketch of the proof, the main ingredients

The proof that the sequence of measures \( Q^N \) converges to \( Q^Z \) is divided in two parts: we show that the sequence is tight and that the limit point is unique.

Uniqueness. The uniqueness part relies on the uniqueness of solutions of martingale problems. Fix a function \( f : S \to \mathbb{R} \). All assertions below on \( G_N \) and \( F_N \) are proved in Section 7. Let \( G_N : \mathcal{H}_N \to \mathbb{R} \) be given by

\[
G_N(\eta) = \sum_{x \in S} (L_Z f)(x) \chi(x)(\eta) .
\]

The function \( G_N \) is equal to \( (L_Z f)(x) \) on \( \mathcal{E}_N^x \), \( x \in S \), it vanishes on \( \Delta_N \) and it has mean-zero with respect to \( \mu_N \).

Denote by \( F_N : \mathcal{H}_N \to \mathbb{R} \) a solution, unique up to an additive constant, of the Poisson equation

\[
\theta_N \mathcal{L}_N F_N = G_N \text{ on } \mathcal{H}_N . \tag{3.1}
\]

Under the measure \( \mathbb{P}^N_{\nu_N} \),

\[
M_N(t) = F_N(\xi_N(t)) - F_N(\xi_N(0)) - \int_0^t (\theta_N \mathcal{L}_N F_N)(\xi_N(s)) \, ds
\]

is a martingale. By definition of \( F_N \), we may replace \( (\theta_N \mathcal{L}_N F_N) \) by \( G_N \), which vanishes on \( \Delta_N \). Hence,

\[
M_N(t) = F_N(\xi_N(t)) - F_N(\xi_N(0)) - \int_0^t G_N(\xi_N(r)) \chi(x)(\xi_N(r)) \, ds .
\]

Recall the definition of the time-change \( S_N^x(t) \) introduced in (2.6). The process \( \tilde{M}_N(t) := M(S_N^x(t)) \) is a martingale with respect to a different filtration. By definition of the trace process and a change of variables [details given in Section 7],

\[
\tilde{M}_N(t) = F_N(\xi_N^x(t)) - F_N(\xi_N^x(0)) - \int_0^t G_N(\xi_N^x(s)) \, ds .
\]

By definition of \( G_N \) and \( Y_N \), we may rewrite this identity as

\[
\tilde{M}_N(t) = F_N(\xi_N^x(t)) - F_N(\xi_N^x(0)) - \int_0^t (L_Z f)(Y_N(s)) \, ds .
\]

According to Proposition 7.3, we can choose the free additive constant in the definition of \( F_N \) so that

\[
\lim_{N \to \infty} \mathbb{E}^N_{\nu_N} \left[ |F_N(\xi_N^x(t)) - f(Y_N(t))| \right] = 0 . \tag{3.2}
\]

Hence,

\[
f(Y_N(t)) - f(Y_N(0)) - \int_0^t (L_Z f)(Y_N(s)) \, ds + o_N(1)
\]
is a martingale. From this, we conclude that
\[ f(Y(t)) - f(Y(0)) - \int_0^t (L_Z f)(Y(s)) \, ds \tag{3.3} \]
is a martingale for any limit point \( Y \) of the sequence \( Y_N \). Uniqueness follows from the unicity of solutions of martingale problems for finite state Markov chains.

The idea of using martingale problems to prove the metastable behavior of Markov chains is the main point of the approach proposed in [6, 7]. The one of proving that (3.3) is a martingale for any limit point of the sequence \( (Y_N(t))_{N \geq 1} \) from properties of the solutions of Poisson equations comes from PDE and has been introduced and developed in the context of Markov processes in [26, 22, 24, 23].

This argument shows that (3.2) is the crucial point in the proof of uniqueness. If condition (H1) of [6] is in force, as well as an equivalent estimate from PDE, one can prove that \( F_N \) and \( f \) are close in \( L^\infty \) [23, 24].

For critical zero-range processes, however, (H1) does not hold and new arguments are required. In Section 7, we present a method to show that \( F_N \) and \( f \) are close in \( L^2 \) based on a spectral gap for the dynamics of the zero-range process restricted to a valley. The method relies on a careful analysis of the solution of the Poisson equation (3.1), which reminds the one performed in [8, 18, 25] to compute capacities between wells. Condition (2.15) is critically used in this computation.

Local spectral gap. For \( x \in S \), let
\[ \hat{E}^x \subset E^x. \tag{3.4} \]
Thus, \( E^x \subset \hat{E}^x \).

Denote by \( \hat{\eta}^x_N(t) \) the zero-range process restricted to the set \( \hat{E}^x_N \). This is the \( \hat{E}^x_N \)-valued Markov chain obtained from the original zero-range process by setting to 0 the rates of all jumps from \( \hat{E}^x_N \) to its complement. Clearly, as \( \eta_N(t) \) is reversible, \( \hat{\eta}^x_N(t) \) is irreducible, reversible and its stationary state is the measure \( \hat{\mu}^x_N(\cdot) \) defined as
\[ \hat{\mu}^x_N(\eta) = \frac{\mu_N(\eta)}{\mu_N(\hat{E}^x_N)}, \quad \eta \in \hat{E}^x_N. \tag{3.5} \]

In Section 6, we prove that the spectral gap of the generator of \( \hat{\eta}^x_N(t) \) is larger than or equal to \( c_0 \ell_N^{-2} \) for some positive constant \( c_0 \). Of course, we expect that a similar holds for the process restricted to \( E^x_N \), but it is easier to prove it in \( \hat{E}^x_N \), and this result is enough for our purposes.

Define the conditional variance on \( \hat{E}^x_N \) of a function \( F : \mathcal{H}_N \to \mathbb{R} \) as
\[ \text{Var}_{\hat{\mu}^x_N}(F) = \mathbb{E}_{\hat{\mu}^x_N} \left[ \left( F - E_{\hat{\mu}^x_N}(F) \right)^2 \right]. \tag{3.6} \]
The Dirichlet form associated to the zero-range process is defined by
\[ D_N(F) = \frac{1}{2} \sum_{x, y \in S} \sum_{\eta \in \mathcal{H}_N} \mu_N(\eta) g(\eta_x) r(x, y) \left[ F(\sigma^x y \eta) - F(\eta) \right]^2 \]
for \( F : \mathcal{H}_N \to \mathbb{R} \).

Next result follows from the local spectral gap. Its proof is presented in Section 6.
Theorem 3.1. There exists a finite constant $C_0$ such that
\[
\text{Var}_{\mu_N}(F) \leq C_0 \ell_N^2 D_N(F)
\]
for all $x \in S$ and $F : \mathcal{H}_N \to \mathbb{R}$.

Condition (H1) of [6] asserts that the process equilibrates faster inside the well than globally. This condition is replaced here by the previous theorem which has a similar flavour, as it states that the process equilibrates in the wells at times smaller than $\ell_N^2$, while the transition time between wells is of order $\theta N \gg \ell_N^2$.

Tightness. We turn to the tightness. Here again we are faced with the problem that condition (H1) of [6] does not hold for critical zero-range processes.

Let
\[
\tilde{\mathcal{E}}^x_N = \mathcal{E}_N \setminus \mathcal{E}^x_N = \bigcup_{y : y \neq x} \mathcal{E}^y_N.
\]
We apply Aldous’ criterion to establish tightness. It boils down to show that starting from any point of a well, the process does not visit a new well in a short interval of time. This property is formulated in the next proposition. Denote by $\tau_A$, $A \subset \mathcal{H}_N$ the hitting time of the set $A$:
\[
\tau_A = \inf \{ t \geq 0 : \xi_N(t) \in A \}.
\]

Theorem 3.2. For all $x \in S$,
\[
\limsup_{a \to 0} \limsup_{N \to \infty} \sup_{\eta \in \tilde{\mathcal{E}}^x_N} \mathbb{P}_\eta^N [ \tau_{\tilde{\mathcal{E}}^x_N} < a ] = 0.
\]

This is one of the main technical issues of the paper. We need to estimate an event with respect to a measure on the path space whose initial distribution is concentrated on a configuration. All of our previous estimates involved measures on the path space whose initial distribution were not too far from the stationary measure conditioned to a well (cf. (2.15)).

Denote by $D^x_N$, $x \in S$, the deep wells given by
\[
D^x_N = \{ \eta \in \mathcal{H}_N : \eta_x \geq N - N^\gamma \},
\]
where $0 < \gamma < 1/\kappa$, and by $W^x_N$ the shallow wells given by
\[
W^x_N = \{ \eta \in \mathcal{H}_N : \eta_x \geq N - \frac{N}{(\log N)^\beta} \},
\]
where $0 < \beta < 1$. Clearly,
\[
D^x_N \subset \mathcal{E}^x_N \subset W^x_N.
\]

The proof Theorem 3.2, presented in Section 8, is carried out in two steps. First, in Proposition 8.3, we prove this result starting from the stationary state conditioned on the deeper valley $D^x_N$ (instead of starting from a configuration).

Then, in Proposition 8.6, we prove that, starting from a configuration in $\mathcal{E}^x_N$, the process hits any configuration $\zeta$ in $D^x_N$ before reaching another valley (that is, the set $\tilde{\mathcal{E}}^x_N$). Putting together Propositions 8.3 and 8.6 yield Theorem 3.2.

The proof of Proposition 8.6 relies on the existence of a super-harmonic function on $W^x_N \setminus D^x_N$. This construction is carried out in Sections 9 and 10. It is the second main technical difficulty of the paper.
4. Condensation of Critical Zero-range Process

We prove in this section Theorem 2.3. From now on, \( \alpha = 1 \) and \( \ell_N \) is the sequence introduced in (2.11).

Rewrite the invariant measure \( \mu_N \), introduced in (2.3), as

\[
\mu_N(\eta) := \frac{1}{Z_{N,\kappa}} \frac{N}{(\log N)^{\kappa-1}} \frac{1}{a(\eta)} , \quad \eta \in \mathcal{H}_N ,
\]

where the normalizing constant \( Z_{N,\kappa} \) is given by

\[
Z_{N,\kappa} := \frac{N}{(\log N)^{\kappa-1}} \sum_{\eta \in \mathcal{H}_N} \frac{1}{a(\eta)} .
\]

Sometimes we denote \( Z_{N,|S_0|} \) as \( Z_{N,S_0} \) to stress on which set the sum is carried out.

**Proposition 4.1.** We have that

\[
\lim_{N \to \infty} Z_{N,\kappa} = \kappa .
\]

**Proof.** The proof is carried out by induction in \( \kappa \). For \( \kappa = 2 \), since

\[
Z_{N,2} = \frac{N}{\log N} \left( \frac{2}{N} + \sum_{j=1}^{N-1} \frac{1}{j (N-j)} \right) = \frac{2}{\log N} + \frac{2}{\log N} \sum_{j=1}^{N-1} \frac{1}{j} .
\]

The assertion of the proposition follows.

Assume that \( \lim_{N \to \infty} Z_{N,\kappa} = \kappa \), and write \( Z_{N,\kappa+1} \) as

\[
Z_{N,\kappa+1} = \frac{N}{(\log N)^\kappa} \left\{ \frac{1}{N} + \sum_{j=0}^{N-1} \frac{[\log(N-j)]^{\kappa-1}}{(N-j) a(j)} Z_{N-j,\kappa} \right\} .
\]

The first term inside braces is negligible, as well as the term \( j = 0 \). We divide the remaining ones in four pieces.

Recall from (2.11) the definition of the sequence \( \ell_N \). By the induction hypothesis and the fact that \( N-j \approx N \) for \( j \leq \ell_N \),

\[
\lim_{N \to \infty} \frac{N}{(\log N)^\kappa} \sum_{j=1}^{\ell_N} \frac{[\log(N-j)]^{\kappa-1}}{j (N-j)} Z_{N-j,\kappa} = \kappa \lim_{N \to \infty} \frac{1}{\log N} \sum_{j=1}^{\ell_N} \frac{1}{j} = \kappa .
\]

As \( (Z_{N,\kappa})_{N \geq 1} \) is a bounded sequence (because, by the induction hypothesis, it converges), there exists a finite constant \( C_0 \), independent of \( N \) and whose value may change from line to line, such that

\[
\frac{N}{(\log N)^\kappa} \sum_{j=\ell_N+1}^{N/2} \frac{[\log(N-j)]^{\kappa-1}}{j (N-j)} Z_{N-j,\kappa} \leq \frac{C_0}{\log N} \sum_{j=\ell_N+1}^{N/2} \frac{1}{j} .
\]

Since \( \log \ell_N / \log N \to 1 \), and since

\[
\sum_{j=\ell_N+1}^{N/2} \frac{1}{j} = [1 + o_N(1)] \left( \log \frac{N}{2} - \log \ell_N \right) ,
\]

the assertion follows.
we have that
\[
\lim_{N \to \infty} \frac{N}{(\log N)^\kappa} \sum_{j=\ell_N+1}^{N/2} \frac{[\log(N-j)]^{\kappa-1}}{j(N-j)} Z_{N-j, \kappa} = 0 .
\]

We turn to the third term. By a change of variables,
\[
\frac{N}{(\log N)^\kappa} \sum_{j=N/2+1}^{N-\ell_N-1} \frac{[\log(N-j)]^{\kappa-1}}{j(N-j)} Z_{N-j, \kappa} = \frac{N}{(\log N)^\kappa} \sum_{j=\ell_N+1}^{N/2-1} \frac{[\log j]^{\kappa-1}}{j(N-j)} Z_{j, \kappa} .
\]
This expression is bounded by
\[
\frac{C_0}{\log N} \sum_{j=\ell_N+1}^{N/2-1} \frac{1}{j} .
\]
At this point, we may proceed as for the second term to show that this expression vanishes as \( N \to \infty \).

It remains to consider the sum
\[
\frac{N}{(\log N)^\kappa} \sum_{j=\ell_N}^{N-1} \frac{[\log(N-j)]^{\kappa-1}}{j(N-j)} Z_{N-j, \kappa} = \frac{N}{(\log N)^\kappa} \sum_{j=1}^{\ell_N} \frac{[\log(j)]^{\kappa-1}}{j(N-j)} Z_{j, \kappa} ,
\]
where we performed a change of variables.

Let \((m_N)_{N \geq 1}\) be a sequence such that
\[
\lim_{N \to \infty} m_N = \infty \quad \text{and} \quad \lim_{N \to \infty} \frac{\log m_N}{\log N} = 0 .
\]
Since the sequence \((Z_{N, \kappa})_{N \geq 1}\) is bounded,
\[
\frac{N}{(\log N)^\kappa} \sum_{j=m_N+1}^{\ell_N} \frac{[\log j]^{\kappa-1}}{j(N-j)} Z_{j, \kappa} \leq \frac{C_0}{\log N} \sum_{j=1}^{m_N} \frac{1}{j} .
\]
Thus, by the second property of the sequence \(m_N\), the left-hand side of the previous inequality converges to 0 as \( N \to \infty \).

We turn to the remaining sum. Since \(m_N \to \infty\) and \(Z_{N, \kappa} \to \kappa\), for \(m_N + 1 \leq j \leq \ell_N\), \(Z_{j, \kappa} = \kappa[1 + o_N(1)]\). Hence, as \(\ell_N/N \to 0\),
\[
\frac{N}{(\log N)^\kappa} \sum_{j=m_N+1}^{\ell_N} \frac{[\log j]^{\kappa-1}}{j(N-j)} Z_{j, \kappa} = [1 + o_N(1)] \frac{\kappa}{(\log N)^\kappa} \sum_{j=m_N+1}^{\ell_N} \frac{[\log j]^{\kappa-1}}{j} .
\]
Estimating sums by integrals yields that
\[
\sum_{j=m_N+1}^{\ell_N} \frac{[\log j]^{\kappa-1}}{j} = [1 + o_N(1)] \left( \frac{\log \ell_N}{\kappa} - \frac{\log m_N}{\kappa} \right) .
\]
Thus, since \(\log \ell_N/\log N \to 1\) and \(\log m_N/\log N \to 0\)
\[
\lim_{N \to \infty} \frac{N}{(\log N)^\kappa} \sum_{j=N-\ell_N}^{N-1} \frac{[\log(N-j)]^{\kappa-1}}{j(N-j)} Z_{N-j, \kappa} = 1 .
\]
The assertion of proposition follows from the previous estimates. \(\square\)

We turn to the
Proof of Theorem 2.3. Fix \( x \in S \) and write
\[
\mu_N(E^x_N) = \frac{N}{Z_{N, \kappa} (\log N)^{\kappa-1}} \sum_{\eta \geq N-\ell_N} \frac{1}{a(\eta)},
\]
where the sum is performed over all configurations \( \eta \in H_N \) such that \( \eta_x \geq N - \ell_N \).

We may rewrite this sum as
\[
N \frac{1}{Z_{N, \kappa} (\log N)^{\kappa-1}} \sum_{j=N-\ell_N}^{N-1} \frac{1}{j} \frac{[\log (N-j)]^{\kappa-2}}{N-j} Z_{N-j, \kappa-1} Z_{N-1, \kappa-1}.
\]

By the last part of the proof of the previous proposition,
\[
\lim_{N \to \infty} \frac{N}{Z_{N, \kappa} (\log N)^{\kappa-1}} \sum_{j=N-\ell_N}^{N-1} \frac{1}{j} \frac{[\log (N-j)]^{\kappa-2}}{N-j} Z_{N-j, \kappa-1} Z_{N-1, \kappa-1} = 1.
\]

Therefore, by Proposition 4.1, \( \lim_{N \to \infty} \mu_N(E^x_N) = 1/\kappa \).

The condition \( \log \ell_N / \log N \to 1 \) is crucial in the previous proofs. The next result shows that if it does not hold, the measure of the set \( E^x_N \) is no longer close to \( 1/\kappa \).

Note that the sequence \( p_N = N^\delta \) fulfills the conditions of the next lemma. In particular, in critical zero-range processes the wells are very large. This is in sharp contrast with super-critical dynamics in which the valleys are formed by configurations in which one site contains at least \( N - m_N \) particles, where \( m_N \) is any sequence such that \( m_N \to \infty \), \( m_N / N \to 0 \).

Lemma 4.2. Let \( (p_N)_{N \geq 1} \) be a sequence of positive integers such that
\[
\lim_{N \to \infty} p_N = \infty \quad \text{and} \quad \lim_{N \to \infty} \frac{\log p_N}{\log N} = \delta \in (0, 1] .
\]

Then, for all \( x \in S \),
\[
\lim_{N \to \infty} \mu_N(\{ \eta \geq N - p_N \}) = \frac{1}{\kappa} \delta^{\kappa-1}.
\]

In particular,
\[
\lim_{N \to \infty} \mu_N(D_N^x) = \frac{1}{\kappa} \gamma^{\kappa-1},
\]
where \( D_N^x \) is the deep valley introduced in (3.8).

Proof. The probability \( \mu_N(\{ \eta \geq N - p_N \}) \) can be written as
\[
\frac{1}{Z_{N, \kappa} (\log N)^{\kappa-1}} \left( \frac{1}{N} + \sum_{j=N-p_N}^{N-1} \frac{[\log (N-j)]^{\kappa-2}}{j (N-j)} Z_{N-j, \kappa-1} \right).
\]

At this point, we repeat the steps presented at the end of the proof of Proposition 4.1. Let \( m_N \) be the sequence introduced there and note that \( m_N \ll p_N \) because \( \log m_N / \log p_N \to 0 \).

According to the proof of Proposition 4.1, in the previous displayed equation, the sum of the terms \( N - m_N \leq j \leq N - 1 \) is negligible, while the sum between \( N - p_N \) and \( N - m_N \) is equal to
\[
[1 + o_N(1)] \frac{1}{Z_{N, \kappa} (\log N)^{\kappa-1}} \frac{[\log p_N]^{\kappa-1} - [\log m_N]^{\kappa-1}}{\kappa-1}.
\]

The result now follows from the properties of the sequences \( m_N \) and \( p_N \). □
Proof of part (2) of Theorem \ref{thm:2.4} Fix \( x \in S, \ t > 0 \) and define \( e_N : \mathcal{H}_N \to \mathbb{R} \) by
\[
e_N(\eta) = \mathbb{E}_{\eta}^N \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) \, ds \right].
\]
Clearly, \( e_N \) is uniformly bounded by \( t \).

By Theorem \ref{thm:2.3}, there exists a finite constant \( C_0 \) such that
\[
\sum_{\eta \in \mathcal{E}_k^N} \mu^*_N(\eta) e_N(\eta) = \frac{1}{\mu_N(E_k^N)} \sum_{\eta \in \mathcal{E}_k^N} \mu_N(\eta) e_N(\eta) \leq C_0 \sum_{\eta \in \mathcal{H}_N} \mu_N(\eta) e_N(\eta).
\]
By Fubini theorem, and since \( \mu_N \) is invariant, the last summation is equal to
\[
\mathbb{E}_{\mu_N}^N \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) \, ds \right] = t \mu_N(\Delta_N).
\]
Therefore, by Theorem \ref{thm:2.3}
\[
\lim_{N \to \infty} \sum_{\eta \in \mathcal{H}_N} \mu_N(\eta) e_N(\eta) = 0.
\]

By the Cauchy-Schwarz inequality and since \( e_N \) is uniformly bounded by \( t \), if \( f_N(\eta) = \nu_N(\eta)/\mu_N(\eta) \),
\[
\mathbb{E}_{\nu_N}^N \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) \, ds \right] = \sum_{\eta \in \mathcal{E}_k^N} \mu_N(\eta) f_N(\eta) e_N(\eta)
\leq \sqrt{t} \left( \sum_{\eta \in \mathcal{E}_k^N} \mu_N(\eta) f_N(\eta)^2 \right)^{1/2} \left( \sum_{\eta \in \mathcal{E}_k^N} \mu_N(\eta) e_N(\eta) \right)^{1/2}.
\]
By hypothesis \ref{hyp:2.15}, the first term is bounded, while the second one vanishes as \( N \to \infty \), by the first part of the proof. \( \square \)

5. Proof of Metastable Behavior

In this section, we prove part (1) of Theorem \ref{thm:2.4}. In Section 5.1, we characterize the limit points and in Section 5.2 we examine the tightness. Many technical results are postponed to later sections.

5.1. Identification of the limit points. Recall from \ref{def:2.13} the definition of the generator \( L_Z \) of the \( S \)-valued continuous-time, Markov chain \( Z(t) \), and let \( r_Z(x, y) \) be the jump rates given by
\[
r_Z(x, y) := 6 \kappa \text{cap}_X(x, y), \quad x, y \in S.
\] (5.1)
Note that the invariant measure of \( Z(\cdot) \) is the uniform distribution \( m \) on \( S \) since the capacity is symmetric.

Recall that, for a probability measure \( \nu_N \) on \( N^S \), we denote by \( Q^N_{\nu_N} \) the measure on \( D(\mathbb{R}_+, S) \) induced by the order process \( Y_N(t) \) with \( \xi_N(0) \) distributed according to \( \nu_N \). Expectation with respect to \( Q^N_{\nu_N} \) is represented by the same symbol.

Theorem 5.1. Under the hypotheses of Theorem \ref{thm:2.4}, for any \( 0 \leq s < t, p \geq 1, 0 \leq s_1 < s_2 < \cdots < s_p \leq s, f : S \to \mathbb{R}, h : S^p \to \mathbb{R} \)
\[
\lim_{N \to \infty} Q^N_{\nu_N} \left[ \left\{ f(Y(t)) - f(Y(s)) - \int_s^t (L_Z f)(Y(u)) \, du \right\} h(Y(s_1), \ldots, Y(s_p)) \right] = 0.
\]
The proof of this result is presented in Section 7. It characterizes the limit points of the sequence $Q^N_t$, as the Markov chain with generator $L^Z$, due to the uniqueness of solutions to the martingale problem in the context of finite-state, continuous-time Markov chains.

5.2. Tightness. The proof of the tightness of the process $\{Y_N(t)\}_{t \geq 0}$ is based on part (2) of Theorem 2.4 and on Theorem 3.2. We provide a sketch of proof and refer to [22, Section 7] for more details.

Recall from Section 2 just below (2.5) the definition of the path space $D(\mathbb{R}_+, \mathbb{N}^S)$. Elements of this space are represented by $\xi$. Denote by $\{\mathcal{F}^0_t\}_{t \geq 0}$ the natural filtration of $D(\mathbb{R}_+, \mathbb{N}^S)$, $\{\mathcal{F}^0_t\}_{t \geq 0} = \{\sigma(\xi(s) : s \in [0, t])\}_{t \geq 0}$, and by $\{\mathcal{F}^1_t\}_{t \geq 0}$ its usual augmentation. Let $\mathcal{G}^N_t = \mathcal{F}^S_{E^N(t)}$ for $t \geq 0$, where $E^N(t)$ has been introduced in (2.0).

Next result is [22 Lemma 7.2 and the paragraph below].

**Lemma 5.2.** We have that

1. For every $s \geq 0$, the random time $S^{E^N(S)}(t)$ is a stopping time with respect to the filtration $\{\mathcal{F}^1_t\}_{t \geq 0}$.
2. Let $\tau$ be a stopping time with respect to the filtration $\{\mathcal{G}^N_t\}_{t \geq 0}$. Then, the random time $S^{E^N}(\tau)$ is a stopping time with respect to the filtration $\{\mathcal{F}^1_t\}_{t \geq 0}$.
3. The trace process $\{\xi^N(t)\}_{t \geq 0}$ is a $E^N$-valued, continuous-time Markov chain with respect to the filtration $\{\mathcal{G}^N_t\}_{t \geq 0}$.

Denote by $\mathcal{T}_M$, $M > 0$, the collection of stopping times, with respect to the filtration $\{\mathcal{G}^N_t\}_{t \geq 0}$, bounded by $M$. The proof of the next result is similar to the one of [24 Lemma 5.6]. We present it here in the sake of completeness.

**Lemma 5.3.** Fix $x \in S$. Suppose that the sequence of probability measures $(\nu_N)_{N \in \mathbb{N}}$ satisfies the conditions of Theorem 2.4. Then, for all $M > 0$, we have

$$
\lim_{N \to \infty} \limsup_{a_0 \to 0} \sup_{\tau \in \mathcal{T}_M} \sup_{a \in (0, a_0)} P_{\nu_N}^N \left[ S^{E^N}(\tau + a) - S^{E^N}(\tau) \geq 2a_0 \right] = 0.
$$

**Proof.** We note first that

$$
\{S^{E^N}(\tau + a) - S^{E^N}(\tau) \geq 2a_0\} \subset \left\{ \int_{S^{E^N}(\tau)}^{S^{E^N}(\tau)+2a_0} \chi_{E^N} (\xi^N(t)) \, dt < a \right\}.
$$

Therefore, the probability appearing in the statement of the lemma is bounded by

$$
P_{\nu_N}^N \left[ \int_{S^{E^N}(\tau)}^{S^{E^N}(\tau)+2a_0} \chi_{E^N} (\xi^N(t)) \, dt > 2a_0 - a \right].
$$

This expression is less than or equal to

$$
P_{\nu_N}^N \left[ \int_0^{2M+2a_0} \chi_{E^N} (\xi^N(t)) \, dt > 2a_0 - a \right] + P_{\nu_N}^N \left[ S^{E^N}(\tau) > 2M \right]. \quad (5.2)
$$

By the Chebyshev inequality, the first probability is bounded by

$$
\frac{1}{2a_0 - a} E_{\nu_N}^N \left[ \int_0^{2M+2a_0} \chi_{E^N} (\xi^N(t)) \, dt \right],
$$

and thus by part (2) of Theorem 2.4,

$$
\lim_{N \to \infty} \limsup_{a_0 \to 0} \sup_{\tau \in \mathcal{T}_M} \sup_{a \in (0, a_0)} P_{\nu_N}^N \left[ \int_0^{2M+2a_0} \chi_{E^N} (\xi^N(t)) \, dt > 2a_0 - a \right] = 0.
$$
For the second probability of (5.2), note that $S^{E_N}(\tau) > 2M$ and $\tau \in \mathcal{T}_M$ implies that
\[ \int_0^{2M} \chi_{A_N}(\xi_N(t)) \, dt > M. \]
Thus, the second term in (5.2) can be handled as the previous one, which completes the proof of the lemma. \qed

Now we prove the main result regarding the tightness.

**Theorem 5.4.** Under the assumptions of Theorem 2.4, the sequence of probability measures $\{Q^N_t\}_{t \in \mathbb{N}}$ is tight on $D(\mathbb{R}_+, S)$. Moreover, any limit point $Q^*$ satisfies

\[ Q^*[Y(0) = x] = 1 \quad \text{and} \quad Q^*[Y(t) \neq Y(t-)] = 0 \quad \text{for all} \ t > 0. \]

**Proof.** By Aldous’ criterion, it suffices to verify that for all $M > 0$,

\[ \lim_{a_0 \to 0} \sup_{N \to \infty} \sup_{\tau \in \mathcal{T}_M} \sup_{a \in (0, a_0)} P^{N}_\nu[Y(\tau + a) \neq Y(\tau)] = 0. \]

By Lemma 5.3, it is enough to show that

\[ \lim_{a_0 \to 0} \sup_{N \to \infty} \sup_{\tau \in \mathcal{T}_M} \sup_{a \in (0, a_0)} P^{N}_{\nu_N}[Y(\tau + a) \neq Y(\tau), S^{E_N}(\tau + a) - S^{E_N}(\tau) < 2a_0] = 0. \]

The last probability is bounded from above by

\[ P^{N}_{\nu_N}[\Psi(\xi(S^{E_N}(\tau) + t)) \neq \Psi(\xi(S^{E_N}(\tau))) \text{ for some } t \in (0, 2a_0)]. \]

By part (2) of Lemma 5.2 and the strong Markov property, this expression is less than or equal to

\[ \sup_{\eta \in E_N} P^{N}_\eta[\Psi(\xi(t)) \neq \Psi(\eta) \text{ for some } t \in (0, 2a_0)]. \]

This expression is bounded from above by

\[ \max_{y \in S} \sup_{\eta \in E^R_N} [\tau_{E^R_N} < 2a_0]. \]

To complete the proof of the first assertion of the theorem, it remains to apply Theorem 5.2.

For the second assertion, note that $Q^*[Y(0) = x] = 1$ follows from the fact that $\nu_N$ is concentrated on $E^R_N$. For the last claim of the theorem, it suffices to prove that

\[ \lim_{a_0 \to 0} \sup_{N \to \infty} P^{N}_{\nu_N}[Y(t - a) \neq Y(t) \text{ for some } a \in (0, a_0)] = 0. \]

The proof of this estimate is identical to the one of the first claim. \qed

5.3. **Proof of Theorem 2.4.** Part (2) of this theorem has already been proven at the end of Section 4.

**Proof of part (1) of Theorem 2.4.** Theorem 5.1 and 5.4 combined yield that, for any function $f : S \to \mathbb{R}$, the process $\{M^f(t)\}_{t \geq 0}$ defined by

\[ M^f(t) = f(x(t)) - f(x(0)) - \int_0^t (L_2 f)(x(s)) \, ds \]

is a martingale under any limit point $Q^*$ of the sequence $(Q^N_t)_{t \in \mathbb{N}}$.

By the uniqueness of solutions to the martingale problem for finite-state Markov chains and by the second assertion of Theorem 5.4, which establishes that $Q^*[Y(0) = x] = 1$, the measure $Q^*$ is equal to $Q^Z_x$. 


Since, by Theorem 5.4, the sequence \((\mathbb{Q}^N_{\nu_N})_{N \in \mathbb{N}}\) is tight, it converges to \(\mathbb{Q}^\infty\).

6. Local Spectral Gap

We prove Theorem 3.1 in this section. Fix \(x_0 \in S\), and let \(S_0 = S \setminus \{x_0\}\).

6.1. Restricted process. Recall from (3.4) the definition of the set \(\hat{\mathcal{E}}_N^x\), \(x \in S\). The zero-range process restricted to \(\hat{\mathcal{E}}_N^x\) is the \(\hat{\mathcal{E}}_N^x\)-valued dynamics obtained by removing all jumps from \(\hat{\mathcal{E}}_N^x\) to its complement.

The generator of this process, denoted by \(\mathcal{L}_N^{x_0}\), is given by

\[
(\mathcal{L}_N^{x_0} F)(\eta) = \sum_{z, w \in S} g(\eta_z) r(z, w) \left[ F(\sigma^{z, w} \eta) - F(\eta) \right] \mathbf{1}\{\sigma^{z, w} \eta \in \hat{\mathcal{E}}_N^{x_0}\},
\]

for \(F : \hat{\mathcal{E}}_N^{x_0} \to \mathbb{R}\). Denote by \(\eta_N^{x_0}(t)\) the Markov chain associated to the generator \(\mathcal{L}_N^{x_0}\).

Let

\[
\hat{\mu}_N^{x_0}(\eta) = \frac{\mu_N(\eta)}{\mu_N(\hat{\mathcal{E}}_N^{x_0})}, \quad \eta \in \hat{\mathcal{E}}_N^{x_0}
\]

be the probability measure obtained by conditioning the invariant measure \(\mu_N\) to the set \(\hat{\mathcal{E}}_N^{x_0}\). As \(\mu_N\), this measure fulfills the detailed balance conditions. In particular, it is invariant.

The Dirichlet form associated to the restricted process \(\eta_N^{x_0}(t)\), denoted by \(\mathcal{D}_N^{x_0}\), is given by,

\[
\mathcal{D}_N^{x_0}(F) = \frac{1}{2} \sum_{z, w \in S} \sum_{\eta, \sigma^{z, w} \eta \in \hat{\mathcal{E}}_N^{x_0}} \hat{\mu}_N^{x_0}(\eta) g(\eta_z) r(z, w) \left[ F(\sigma^{z, w} \eta) - F(\eta) \right]^2,
\]

for \(F : \hat{\mathcal{E}}_N^{x_0} \to \mathbb{R}\).

Denote by \(\text{Var}_{\hat{\mu}_N^{x_0}}(F)\) the variance of a function \(F : \hat{\mathcal{E}}_N^{x_0} \to \mathbb{R}\) with respect to the measure \(\hat{\mu}_N^{x_0}(\cdot)\):

\[
\text{Var}_{\hat{\mu}_N^{x_0}}(F) = \mathbb{E}_{\hat{\mu}_N^{x_0}} \left[ \left( F - \mathbb{E}_{\hat{\mu}_N^{x_0}}[F] \right)^2 \right].
\]

The next result establishes a lower bound for the spectral gap of the generator \(\mathcal{L}_N^{x_0}\).

**Theorem 6.1.** There exists a finite constant \(C_0 > 0\) such that, for all \(F : \hat{\mathcal{E}}_N^{x_0} \to \mathbb{R}\),

\[
\text{Var}_{\hat{\mu}_N^{x_0}}(F) \leq C_0 \ell_N^2 \mathcal{D}_N^{x_0}(F).
\]

The proof of the local spectral gap is based on an idea presented in [4, Section 4]. It consists in comparing the restricted process with a collection of independent birth-and-death dynamics whose spectral gap is of order \(\ell_N^{-2}\).

6.2. Proof of Theorem 3.1. The argument relies on the next result.

**Lemma 6.2.** We have that \(\mu_N(\hat{\mathcal{E}}_N^{x_0}) = \left[1 + o_N(1)\right] (1/\kappa)\).

**Proof.** Let \(\hat{\mathcal{F}}_N^{x_0} = \{\eta \in \mathcal{H}_N : \eta_{x_0} \geq N - (\kappa - 1)\ell_N\}\) so that

\[
\hat{\mathcal{E}}_N^{x_0} \subset \hat{\mathcal{F}}_N^{x_0} \subset \hat{\mathcal{F}}_N^{x_0}.
\]

By the proof of Theorem 2.3, \(\mu_N(\hat{\mathcal{F}}_N^{x_0}) = \left[1 + o_N(1)\right] (1/\kappa)\). The assertion of the lemma follows from this observation and Theorem 2.3. □
Proof of Theorem 3.1. Fix $F : \mathcal{H}_N \to \mathbb{R}$. Since $\mathcal{D}_N^\infty (F) \leq \mathcal{D}_N (F)$, it suffices to show that there exists a finite constant $C_0$ such that
\[
\text{Var}_{\mu_N^\infty} (G) \leq C_0 \text{Var}_{\mu_N^\infty} (G)
\]
for all functions $G : \mathcal{H}_N \to \mathbb{R}$.

Write $\overline{c} = \sum_{\xi \in \mathcal{E}_N^\infty} \hat{\mu}_N^\infty (\xi) G(\xi)$. Then,
\[
\begin{align*}
\text{Var}_{\mu_N^\infty} (G) &= \min_{c \in \mathbb{R}} \frac{1}{\mu_N (\mathcal{E}_N^\infty)} \sum_{\eta \in \mathcal{E}_N^\infty} \mu_N (\eta) \left[ G(\eta) - c \right]^2 \\
&\leq \frac{1}{\mu_N (\mathcal{E}_N^\infty)} \sum_{\eta \in \mathcal{E}_N^\infty} \mu_N (\eta) \left[ G(\eta) - \overline{c} \right]^2.
\end{align*}
\]

Since $\mathcal{E}_N^\infty \subset \overline{E}_N^\infty$, this expression is bounded by
\[
\frac{\mu_N (\mathcal{E}_N^\infty)}{\mu_N (\overline{E}_N^\infty)} \sum_{\eta \in \overline{E}_N^\infty} \hat{\mu}_N^\infty (\eta) \left[ G(\eta) - \overline{c} \right]^2 = \left[ 1 + o_N (1) \right] \text{Var}_{\mu_N^\infty} (G),
\]
where the last identity follows from Theorem 2.3 and Lemma 6.2.

6.3. A birth-and-death process. Consider a birth-and-death process $\{w(t)\}_{t \geq 0}$ on $\mathcal{X} = \mathcal{X}_N = \{0, 1, \ldots, \ell_N\}$ with jump rates given by
\[
R(i, j) = \begin{cases} 
1 & \text{if } j = i + 1 \text{ and } j \leq \ell_N, \\
g(i) & \text{if } j = i - 1 \text{ and } j \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

The invariant probability measure, denoted by $\varphi(\cdot) = \varphi_N(\cdot)$, is given by
\[
\varphi(k) = \frac{1}{z_N} \frac{1}{a(k)}, \quad k \in \mathcal{X},
\]
where $z_N$ is the normalizing constant satisfying
\[
z_N = \sum_{k=0}^{\ell_N} \frac{1}{a(k)} = \lceil 1 + o_N (1) \rceil \log N.
\]

The process is actually reversible with respect to $\varphi(\cdot)$.

Consider independent, birth-and-death processes $\zeta_x(t), x \in S_0$, each one having the same law as $w(\cdot)$. Denote by $\zeta(t)$ the continuous-time Markov chain on $\mathcal{X}^{S_0}$ given by $\zeta(t) = (\zeta_x(t))_{x \in S_0 \setminus \{x_0\}}$.

Here and below, elements of $\mathcal{X}^{S_0}$ are represented by $\xi = (\xi_x)_{x \in S_0 \setminus \{x_0\}}$. For each $x \in S_0$, let $\delta^x \in \mathcal{X}^{S_0}$ be the configuration consisting of only one particle at site $x$:
\[
(\delta^x)_y = \mathbb{1} \{x = y\}, \quad y \in S \setminus \{x_0\}.
\]

The next assertions about the process $\zeta(t)$ are elementary. The invariant measure is the product measure $\varphi^{S_0}(\cdot)$, defined by
\[
\varphi^{S_0}(\xi) = \prod_{x \in S_0} \varphi(\xi_x), \quad \xi \in \mathcal{X}^{S_0}.
\]

Actually, $\zeta(\cdot)$ is reversible with respect to $\varphi^{S_0}$.
The generator of the process \( \zeta(\cdot) \), denoted by \( L_{BDP}^N \), is given by

\[
(L_{BDP}^N G)(\xi) = \sum_{x \in S_0} [G(\xi + \delta^x) - G(\xi)] \mathbf{1}\{\xi + 1 \in X\}
+ \sum_{x \in S_0} g(\xi_x) [G(\xi - \delta^x) - G(\xi)] \mathbf{1}\{\xi - 1 \in X\},
\]

for \( G : X^{S_0} \to \mathbb{R} \), and the Dirichlet form by

\[
D_{BDP}^N(G) = \frac{1}{2} \sum_{x \in S_0} \sum_{\xi \in X^{S_0}} \varphi_{S_0}(\xi) [G(\xi + \delta^x) - G(\xi)]^2 \mathbf{1}\{\xi + 1 \in X\}.
\]

Denote by \( \text{Var}_{BDP}^N(G) \) the variance of \( G : X^{S_0} \to \mathbb{R} \):

\[
\text{Var}_{BDP}^N(G) = E_{\varphi_{S_0}} \left[ (G - E_{\varphi_{S_0}}[G])^2 \right].
\]

Next result is [12, Theorem 1.2]. The lower bound is sharp. It can be shown that there exists constants \( 0 < C_1 < C_2 < \infty \) such that

\[
C_1 \ell_N^{-2} \leq \lambda_{BDP}^N \leq C_2 \ell_N^{-2},
\]

where \( \lambda_{BDP}^N \) represents the spectral gap of the generator \( L_{BDP}^N \). We provide a simple proof of Proposition 6.3 based on the Efron-Stein inequality.

**Proposition 6.3.** There exists a finite constant \( C_0 \) such that

\[
\text{Var}_{BDP}^N(F) \leq C_0 \ell_N^2 D_{BDP}^N(F)
\]

for all \( N \geq 1 \), \( F : X^{S_0} \to \mathbb{R} \).

Next result is [11, Theorem 6, page 214] and follows from the Efron-Stein inequality [13].

**Lemma 6.4.** Let \( X_1, X_2, \ldots, X_n \) be independent random variables, and let \( f : \mathbb{R}^n \to \mathbb{R}, f_1, f_2, \ldots, f_n : \mathbb{R}^{n-1} \to \mathbb{R} \) be measurable, bounded functions. Define the random variables

\[
Z = f(X_1, X_2, \ldots, X_n),
Z_i = f_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n), \quad 1 \leq i \leq n.
\]

Then,

\[
\text{Var}(Z) \leq \sum_{i=1}^n E[(Z - Z_i)^2].
\]

The proof below is similar to the one of [4, Lemma 4.4].

**Proof of Proposition 6.3.** For \( \xi \in X^{S_0} \) and \( x \in S_0 \), denote by \( \xi^{x,k} \) the configuration obtained from \( \xi \) by replacing \( \xi_x \) with \( k \):

\[
(\xi^{x,k})_y = \begin{cases} 
\xi_y & \text{if } y \neq x \\
k & \text{if } y = x.
\end{cases}
\]

Observe that \( G(\xi^{x,0}), x \in S_0, \) is function of \( \xi_y, y \neq x \). Hence, by Lemma 6.4,

\[
\text{Var}_{BDP}^N(G) \leq \sum_{x \in S_0} \sum_{\xi \in X^{S_0}} \varphi_{S_0}(\xi) [G(\xi) - G(\xi^{x,0})]^2.
\]
By the Cauchy-Schwarz inequality,
\[
\varphi^S_0(\xi) \left[ G(\xi) - G(\xi^{x,0}) \right]^2 \leq \varphi^S_0(\xi) \xi x \sum_{k=0}^{\xi_x-1} \left[ G(\xi^{x,k+1}) - G(\xi^{x,k}) \right]^2.
\]
Since \( k \leq a(k) \) and \( \varphi^S_0(\xi) a(\xi_x) = \varphi^S_0(\xi^{x,k}) a(k) \), the previous expression is less than or equal to
\[
\sum_{k=0}^{\xi_x-1} \varphi^S_0(\xi^{x,k}) a(k) \left[ G(\xi^{x,k} + \cdot) - G(\xi^{x,k}) \right]^2.
\]
Up to this point we proved that
\[
\Var^\text{BDP}_N(G) \leq \sum_{x \in S_0} \sum_{\xi \in \mathbb{S}_{x,0}} \sum_{k=0}^{\xi_x-1} \varphi^S_0(\xi^{x,k}) a(k) \left[ G(\xi^{x,k} + \cdot) - G(\xi^{x,k}) \right]^2.
\]
Changing variables \( \zeta = \xi^{x,k} \), yields that this sum is equal to
\[
\sum_{x \in S_0} \sum_{\xi \in \mathbb{S}_{x,0}} \varphi^S(\zeta) a(\zeta_x) (\ell_N - \zeta_x) \left| G(\zeta + \cdot) - G(\zeta) \right|^2 1\{\xi_x + 1 \in \mathbb{X}\}.
\]
To complete the proof, it remains to observe that \( a(\zeta_x) (\ell_N - \zeta_x) \leq \ell_N^2 \).

6.4. Proof of Theorem 6.1. There exists a natural bijection between \( \mathbb{X}^S_0 \) and \( \tilde{\mathcal{E}}^\text{so}_N \) given by

\[
\xi \in \mathbb{X}^S_0 \leftrightarrow \tilde{\xi} = (N - |\xi|, \xi) \in \tilde{\mathcal{E}}^\text{so}_N,
\]
where \( (N - |\xi|, \xi) \in \mathcal{H}_N \) represents the configuration with \( N - |\xi| \) particles at the site \( x_0 \), and \( \xi_x \) particles at the site \( x \in S_0 \). Therefore, we can identify a function \( G : \mathbb{X}^S_0 \to \mathbb{R} \) with \( \tilde{G} : \tilde{\mathcal{E}}^\text{so}_N \to \mathbb{R} \) by

\[
\tilde{G}(\tilde{\xi}) = G(\xi).
\]

The map \( G \leftrightarrow \tilde{G} \) is a bijection between the space of real-valued functions on \( \mathbb{X}^S_0 \) and on \( \tilde{\mathcal{E}}^\text{so}_N \).

**Proposition 6.5.** There exists a finite constant \( C_0 \) such that,

\[
\Var_{\tilde{\mu}^\text{so}_N}(\tilde{G}) \leq C_0 \Var_N^\text{BDP}(G)
\]
for all \( N \geq 1 \) and \( G : \mathbb{X}^S_0 \to \mathbb{R} \).

**Proof.** We first claim that there exists a finite constant \( C_0 \) such that

\[
\tilde{\mu}_N^{\text{so}}(\tilde{\xi}) \leq C_0 \varphi^S(\xi) \quad \text{for all } N \in \mathbb{N} \text{ and } \tilde{\xi} \in \tilde{\mathcal{E}}^\text{so}_N.
\]

Indeed, since \( |\xi| \leq \ell_N \), by Proposition 4.1 and Lemma 6.2,

\[
\tilde{\mu}_N^{\text{so}}(\tilde{\xi}) = \frac{1}{\mu_N(\mathcal{E}^\text{so}_N)} \frac{1}{Z_{N,S} (\log N)^{\ell_N-1}} \frac{1}{(N - |\xi|)} \frac{1}{a(\xi)} \left[ 1 + o_N(1) \right] \prod_{x \in S_0} \frac{1}{\log N} \frac{1}{a(\xi_x)}.
\]

At this point, (6.5) follows from (6.1) and (6.2).

Fix \( G : \mathbb{X}^S_0 \to \mathbb{R} \). Since the expectation minimizes the square distance,

\[
\Var_{\tilde{\mu}_N^{\text{so}}}(\tilde{G}) \leq \sum_{\xi \in \tilde{\mathcal{E}}^\text{so}_N} \left( \tilde{G}(\xi) - E_{\varphi^S_0} [G] \right)^2 \tilde{\mu}_N^{\text{so}}(\xi).
\]
By (6.3), (6.4), and (6.5), this expression is bounded from above by

$$
C_0 \sum_{\xi \in \mathcal{X}^{G_0}} (G(\xi) - E_{\varphi^{G_0}}[G])^2 \varphi^{G_0}(\xi) = C_0 \text{Var}_{\varphi^{G_0}}(F),
$$

which completes the proof of the proposition. \(\square\)

**Proposition 6.6.** There exists a finite constant \(C_0\) such that

$$
\mathcal{D}^{\text{RDP}}_N(G) \leq C_0 \mathcal{D}^G_N(\tilde{G})
$$

for all \(N \geq 1\) and \(G : \mathcal{X}_{\mathcal{S}_0} \to \mathbb{R}\).

The proof of this result relies on a technical lemma. We say that two configurations \(\eta, \eta'\) are neighbours if \(\eta' = \sigma^x y \eta\) for some \(x, y \in S\) with \(r(x, y) > 0\).

**Lemma 6.7.** For all \(\eta \in \hat{\mathcal{E}}^{\mathcal{S}_0}_N\) and \(x \in S_0\) such that \(\sigma^{x_0} x \eta \in \hat{\mathcal{E}}^{\mathcal{S}_0}_N\), there is a path \(\sigma(\eta, x) = (\eta^{(0)} = \eta, \eta^{(1)}, \ldots, \eta^{(m)}) = \sigma^{x_0} x \eta\) in \(\hat{\mathcal{E}}^{\mathcal{S}_0}_N\) from \(\eta\) to \(\sigma^{x_0} x \eta\) such that

1. \(m \leq \kappa\)
2. \(\eta^{(i)}\) and \(\eta^{(i+1)}\) are neighbours for all \(0 \leq i < m\),
3. \(\mu_N(\eta) \leq 4 \mu_N(\eta^{(i)})\) for all \(0 \leq i \leq m\),
4. Each pair \((\eta^{(i)}, \eta^{(i+1)})\) of neighbouring configurations appears as a consecutive pair in no more than \(2\kappa^3\) paths \(\sigma(\eta, x)\).

**Proof.** Fix \(x \in S_0\). As the random walk is irreducible, there exists \(m < \kappa\) and a sequence

$$
x_0 = v_0, v_1, \ldots, v_m = x
$$

such that \(r(v_k, v_{k+1}) > 0\) for all \(0 \leq k < m\). This sequence depends only on \(x\). It is fixed and will be the same for all configurations \(\eta \in \hat{\mathcal{E}}^{\mathcal{S}_0}_N\).

Fix \(\eta \in \hat{\mathcal{E}}^{\mathcal{S}_0}_N\) such that \(\sigma^{x_0} x \eta \in \hat{\mathcal{E}}^{\mathcal{S}_0}_N\). The natural definition of the path \(\sigma(\eta, x)\) is to set \(\eta^{(k)} = \sigma^{v_k} v_{k+1} \eta\). However, if \(\eta_{v_k} = \ell_N\) for some \(k\), this path leaves the set \(\hat{\mathcal{E}}^{\mathcal{S}_0}_N\), which is not permitted. We modify the natural path to keep it in the set \(\hat{\mathcal{E}}^{\mathcal{S}_0}_N\).

Note that \(\eta_{v_m} < \ell_N\) because \(v_m = x\) and \(\sigma^{x_0} x \eta \in \hat{\mathcal{E}}^{\mathcal{S}_0}_N\). If \(\eta_{v_k} < \ell_N\) for \(1 \leq k < m\), the path \(\sigma(\eta, x)\) is the one above.

If this is not the case, let \(p\) be the first integer such that \(\eta_{v_k} = \ell_N\):

$$
p = \min \{ 1 \leq k \leq m : \eta_{v_k} = \ell_N \};
$$

Let \(q \geq p\) be the last one with the property that all sites in between are occupied by \(\ell_N\) particles:

$$
q = \max \{ p \leq k \leq m : \eta_{v_j} = \ell_N, p \leq j \leq k \}.
$$

Note that \(q < m\) because \(\eta_{v_m} < \ell_N\) and that \(\eta_{v_{q+1}} < \ell_N\).

The path is constructed as follows: We first move a particle from \(x_0 = v_0\) to \(v_{q}\), then we move it from \(v_1\) to \(v_2\), until we reach \(v_{p-1}\). At this point, we may not move it to \(v_p\). To remove a particle from \(v_p\), we move a particle from \(v_q\) to \(v_{q+1}\), then from \(v_{q+1}\) to \(v_{q-1}\) and so on, until we move one from \(v_p\) to \(v_{p+1}\). At this point we move a particle from \(v_{p-1}\) to \(v_p\).

Up to this point, a particle has been displaced from \(x_0 = v_0\) to \(v_{q+1}\). If all sites between \(v_{q+2}\) and \(v_m\) have less than \(\ell_N\) particles, we continue to move the particle up to the end. Otherwise, we repeat the surgery to avoid leaving the set \(\hat{\mathcal{E}}^{\mathcal{S}_0}_N\). This defines the path \(\sigma(\eta, x)\).
Note that the path \( s(\eta, x) \) does not visit the same configuration twice: \( \eta^{(i)} \neq \eta^{(j)} \) for \( i \neq j \).

It is clear that the conditions (1) and (2) are fulfilled. By definition of the path, for each \( 1 \leq k \leq m \), there exists \( w_1, \ldots, w_k \) [which depend on \( k \), naturally], such that \( \eta^{(k)} = \sigma^{w_0, w_1} \eta \) or \( \eta^{(k)} = \sigma^{w_0, w_4} \eta \). Since, for every \( x \neq y \),

\[
\frac{\mu_N(\eta)}{\mu_N(\sigma^{y,x}\eta)} = \frac{a(\eta_x + 1)}{a(\eta_e)} \frac{a(\eta_y - 1)}{a(\eta_y)} \leq 2,
\]

condition (3) is proved.

We turn to (4). Suppose that a pair \( (\eta', \eta'') = \sigma^{w_1', w_2'} \eta' \) appears in the path \( s(\eta, x) \) for some \( \eta \) and \( x \). Then, as we have seen above, either \( \eta' = \sigma^{w_0, w_3} \eta \) or \( \eta' = \sigma^{w_0, w_2} \sigma^{w_3, w_4} \eta \) for some \( w_1, \ldots, w_4 \). Hence, either \( \eta = \sigma^{w_1, w_0} \eta'' \) or \( \eta = \sigma^{w_4, w_3} \sigma^{w_2, x_0} \eta'' \). Therefore, there are at most \( \kappa + \kappa^3 \leq 2\kappa^3 \) possible configurations \( \eta \) and \( \kappa \) possible choices for \( x \), making the total number of possible pairs \( (\eta, x) \) in which neighbours \( (\eta, \eta'') \) appear to be bounded by \( 2\kappa^4 \).

Since a pair \( (\eta', \eta'') \) of neighbour configurations appears only once in a path \( s(\eta, x) \), there are at most \( 2\kappa^4 \) different paths in which a fixed pair \( (\eta', \eta'') \) may appear. This completes the proof of the lemma. \( \square \)

**Proof of Proposition 6.6.** Note that the bijection \( \xi \leftrightarrow \tilde{\xi} \) given in (6.3) satisfies \( \xi + \vartheta x \leftrightarrow \sigma^{x_0, x_0} \xi \). Thus, we can write \( D_N^{\text{BDP}}(G) \) as

\[
D_N^{\text{BDP}}(G) = \frac{1}{2} \sum_{x \in S_0} \sum_{\xi \in X^{S_0}} \varphi^{S_0}(\xi) \left[ G(\xi + \vartheta x) - G(\xi) \right]^2 1\{\xi + \vartheta x \in X^{S_0}\}.
\]

By (6.5) and since the map \( \xi \leftrightarrow \tilde{\xi} \) is bijection, it follows from the previous equation that there exists a finite constant \( C_0 \), independent of \( N \), such that

\[
D_N^{\text{BDP}}(G) \leq C_0 \sum_{x \in S_0} \sum_{\eta \in \bar{E}_N^{S_0}} \tilde{\mu}_N^{S_0}(\eta) \left[ \tilde{G}(\sigma^{x_0, x_0} \eta) - \tilde{G}(\eta) \right]^2 1\{\sigma^{x_0, x_0} \eta \in \bar{E}_N^{S_0}\}. \quad (6.6)
\]

Recall from Lemma 6.7 the definition of the path \( s(\eta, x) = (\eta^{(0)}, \ldots, \eta^{(m)}) \) for \( \eta \in \bar{E}_N^{S_0} \) and \( x \in S_0 \) such that \( \sigma^{x_0, x_0} \eta \in \bar{E}_N^{S_0} \). By the Cauchy-Schwarz inequality and conditions (1) and (3) of that lemma,

\[
\tilde{\mu}_N^{S_0}(\eta) \left[ \tilde{G}(\sigma^{x_0, x_0} \eta) - \tilde{G}(\eta) \right]^2 \leq m \sum_{k=0}^{m-1} \tilde{\mu}_N^{S_0}(\eta) \left[ \tilde{G}(\eta^{(k+1)}) - \tilde{G}(\eta^{(k)}) \right]^2
\]

\[
\leq 4 \kappa \sum_{k=0}^{m-1} \tilde{\mu}_N^{S_0}(\eta^{(k)}) \left[ \tilde{G}(\eta^{(k+1)}) - \tilde{G}(\eta^{(k)}) \right]^2.
\]

Inserting this bound in (6.6), changing the order of summations and applying part (4) of Lemma 6.7 yield that \( D_N^{\text{BDP}}(G) \) is bounded above by

\[
C_0(\kappa) \sum_{(x,y)} \sum_{\eta \in \bar{E}_N^{S_0}} \tilde{\mu}_N^{S_0}(\eta) \left[ \tilde{G}(\sigma^{x_0, x_0} \eta) - \tilde{G}(\eta) \right]^2 1\{\sigma^{x_0, y_0} \eta \in \bar{E}_N^{S_0}\},
\]

where the first sum is carried over all pairs \( (x, y) \) such that \( r(x, y) > 0 \). The last summation is bounded above by \( C_0(\kappa) D_N^{S_0}(G) \) because \( g(k) \geq 1 \) for all \( k \geq 1 \). \( \square \)
Proof of Theorem 6.1. Fix $F: \hat{\mathbf{E}}_{N}^{x_0} \to \mathbb{R}$. By the bijection introduced in Subsection 6.4 there exists $G: \mathbb{X}^{S_0} \to \mathbb{R}$ such that $F = \tilde{G}$ in the sense of (6.4). By Propositions 6.3, 6.5, and 6.6 there exists a finite constant $C_0 = C_0(\kappa)$, independent of $N$, such that
\[
\text{Var}_{\hat{\mathbf{E}}_{N}^{x_0}}(\tilde{G}) \leq C_0 \text{Var}^{\text{BDP}}_N(G) \leq C_0 \ell^2_N D^{\text{BDP}}_N(G) \leq C_0 \ell^2_N D^{\text{BDP}}_N(\tilde{G}).
\]
This proves Theorem 6.1. \qed

7. The Poisson Equation

In the first section of this chapter, we deduce Theorem 5.1 from two properties of the solutions of the Poisson equation (7.1). In Section 7.2, we prove the first property and, in Section 7.3, we sketch the proof of the second one. The remaining part of this chapter is devoted to this proof.

7.1. The equation. Recall the definition of the Markov chain $Z$ introduced in (2.13). Fix $f: S \to \mathbb{R}$, and define $G_N: \mathcal{H}_N \to \mathbb{R}$ as
\[
G_N(\eta) = \sum_{x \in S} (L_Z f)(x) \chi_{\mathcal{E}_x^x}(\eta).
\]
Hence, the function $G_N$ is constant and equal to $(L_Z f)(x)$ on $\mathcal{E}_x^x$, $x \in S$, and vanishes on $\Delta_N$. Note that
\[
\sum_{\eta \in \mathcal{H}_N} G_N(\eta) \mu_N(\eta) = \sum_{x \in S} (L_Z f)(x) \mu_N(\mathcal{E}_x^x) = \mu_N(\mathcal{E}_x^0) \sum_{x \in S} (L_Z f)(x) = 0,
\]
because the invariant measure for the Markov chain $Z$ is the uniform measure. Therefore, as the zero-range process is irreducible, there exists a function $F_N: \mathcal{H}_N \to \mathbb{R}$ such that
\[
\mathcal{E}_x^x F_N = G_N \quad \text{on} \quad \mathcal{H}_N. \tag{7.1}
\]
The solution of this Poisson equation is unique up to a constant.

The proof of Theorem 5.1 relies on two properties of the solution $F_N$ of the Poisson equation (7.1). The first one, stated in Proposition 7.1, asserts that the Dirichlet form of $F_N$ is uniformly bounded. The second one, stated in Proposition 7.2, asserts that the average, with respect to the stationary measure $\mu_N$, of $F_N$ on the well $\mathcal{E}_x^x$ is asymptotically close to $f(x)$, provided we select appropriately the free additive constant of the solution $F_N$.

Proposition 7.1 is proved in Section 7.2 while Proposition 7.2 is the object of the rest of this chapter.

Proposition 7.1. There exists a finite constant $C_0$ such that
\[
\theta_N D_N(F_N) \leq C_0 \quad \text{for all} \quad N \geq 1.
\]

Let $f_N: S \to \mathbb{R}$ be given by
\[
f_N(x) = \mathbb{E}_{\mu_N}[F_N] = \frac{1}{\mu_N(\mathcal{E}_x^x)} \sum_{\eta \in \mathcal{E}_x^x} F_N(\eta) \mu_N(\eta). \tag{7.2}
\]
Hence, $f_N(x)$ is the conditional average of $F_N$ on the valley $\mathcal{E}_x^x$ with respect to the invariant measure $\mu_N$. As observed above, $F_N$ is unique up to an additive constant. We choose the constant to have the identity
\[
\sum_{x \in S} f_N(x) = \sum_{x \in S} f(x). \tag{7.3}
\]
Proposition 7.2. Under the assumption \((7.3)\),
\[
\lim_{N \to \infty} \max_{x \in S} |f_N(x) - f(x)| = 0 .
\]

Combining Propositions \((7.1), (7.2)\) and the local spectral gap estimate obtained in Section 6 yields the next result.

Proposition 7.3. Suppose that the sequence \((\nu_N)_{N \in \mathbb{N}}\) satisfies the condition of Theorem 2.4. Then, for all \(t \geq 0\),
\[
\lim_{N \to \infty} \mathbb{E}^N \left[ \left| f_N(\xi_N^E(t)) - f(Y_N(t)) \right| \right] = 0 .
\]

Proof. Define \(F_N : \mathcal{E}_N \to \mathbb{R}\) as
\[
F_N(\eta) = \sum_{x \in S} f_N(x) \chi_S(x) ,
\]
where \(\chi_S\) represents the indicator function of the set \(\mathcal{A}\). Note that \(f_N(Y_N(t)) = F_N(\xi_N^E(t))\).

Recall the definition of the measure \(\mu_N^{\xi_N^E}\) introduced in \((2.16)\). Denote by \(\nu_N(t)\), \(t \geq 0\), the distribution of \(\xi_N^E(t)\) on \(\mathcal{E}_N\) when the zero-range process starts from \(\nu_N\). With these notations,
\[
\mathbb{E}^N \left[ \left| F_N(\xi_N^E(t)) - f(Y_N(t)) \right| \right] = \mathbb{E}_{\nu_N(t)} \left[ \left| F_N(\eta) - F_N(\eta) \right| \right] .
\]

By the Cauchy-Schwarz inequality, the square of the right-hand side is bounded above by
\[
\mathbb{E}_{\mu_N^{\xi_N^E}} \left[ \left| F_N - F_N \right|^2 \right] \mathbb{E}_{\mu_N^{\xi_N^E}} \left[ \left( \frac{\nu_N(t)}{\mu_N^{\xi_N^E}} \right)^2 \right] .
\]

By [7 Proposition 6.3] \(\mu_N^{\xi_N^E} (\cdot) = \mu_N (\cdot | \mathcal{E}_N)\) is the invariant measure for the trace process \(\nu_N^E (\cdot)\). Therefore, as the \(L^2\)-energy of a Markov chain with respect to the invariant distribution decreases in time, by \((2.17)\) we get
\[
\mathbb{E}_{\mu_N^{\xi_N^E}} \left[ \left( \frac{\nu_N(t)}{\mu_N^{\xi_N^E}} \right)^2 \right] \leq \mathbb{E}_{\mu_N^{\xi_N^E}} \left[ \left( \frac{\nu_N}{\mu_N^{\xi_N^E}} \right)^2 \right] \leq C_0 .
\]

On the other hand, by definition of \(F_N\), and since \(f_N(x)\), introduced in \((7.2)\), is the mean of \(F_N\) with respect to the measure \(\mu_N^\xi\), the first expectation in \((7.4)\) can be written as
\[
\sum_{x \in S} \frac{\mu_N(\mathcal{E}_N^x)}{\mu_N(\mathcal{E}_N)} \mu_N^{\xi_N^E} \left[ \left| F_N - F_N \right|^2 \right] = \frac{1}{\kappa} \sum_{x \in S} \text{Var}_{\mu_N}(F_N) .
\]

In the last term, \(F_N\) is regarded as a function defined only on \(\mathcal{E}_N^x\). By the local spectral gap estimate, stated in Theorem 3.1 and by Proposition 7.1 there exists a finite constant \(C_0\), independent of \(N\), such that
\[
\text{Var}_{\mu_N}(F_N) \leq C_0 \ell_N^3 \mathcal{D}_N(F_N) \leq \frac{C_0 \ell_N^3}{\theta_N} = \frac{C_0}{(\log N)^3} .
\]

Putting together the previous estimates yields that
\[
\mathbb{E}_{\nu_N} \left[ \left| F_N(\xi_N^E(t)) - f_N(Y_N(t)) \right| \right] \leq \frac{C_0}{(\log N)^{3/2}}
\]
for some finite constant \(C_0\).
Proposition 7.2 permits to replace \( f_N(Y_N(t)) \) by \( f(Y_N(t)) \), which completes the proof of the proposition.

**Proof of Theorem 5.1.** Fix a function \( f : S \to \mathbb{R} \). Let \( F_N \) be the function introduced in Proposition 7.1. Under the measure \( \mathbf{P}_{\nu_N}^N \) on \( D([0, \infty), \mathbb{N}^S) \), the process \( M_N(t) \) given by

\[
M_N(t) = F_N(\xi(t)) - F_N(\xi(0)) - \int_0^t (\mathcal{L}_N^\xi F_N)(\xi(r)) \, dr
\]

is a martingale with respect to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) introduced in Section 5.2. By Proposition 7.2, we may replace \( (\mathcal{L}_N^\xi F_N) \) by \( G_N \), a function which vanishes on \( \Delta_N \). Hence,

\[
M_N(t) = F_N(\xi(t)) - F_N(\xi(0)) - \int_0^t G_N(\xi(r)) \chi_{\Delta_N}(\xi(r)) \, dr.
\]

Recall the definition of the time-change \( S^{\xi_N}(t) \) introduced in (2.6) [here, in the path space \( D([0, \infty), \mathbb{N}^S) \)]. The process \( \widehat{M}_N(t) := M(S^{\xi_N}(t)) \) is a martingale with respect to the filtration \( \{\mathcal{G}_t^N\}_{t \geq 0} \). By definition of the trace process,

\[
\widehat{M}_N(t) = F_N(\xi_N(t)) - F_N(\xi_N(0)) - \int_0^{S^{\xi_N}(t)} G_N(\xi(r)) \chi_{\Delta_N}(\xi(r)) \, dr.
\]

The presence of the indicator of the set \( \Delta_N \) permits to perform the change of variables \( r' = T^{\xi_N}(r) \) which yields that

\[
\widehat{M}_N(t) = F_N(\xi_N(t)) - F_N(\xi_N(0)) - \int_0^t G_N(\xi_N(r')) \, dr'.
\]

By definition of \( G_N \) and \( Y_N \), we may rewrite this identity as

\[
\widehat{M}_N(t) = F_N(\xi_N(t)) - F_N(\xi_N(0)) - \int_0^t (L_Z f)(Y_N(r')) \, dr'.
\]

Fix \( 0 \leq s < t, \ p \geq 1, \ 0 \leq s_1 < s_2 < \cdots < s_p \leq s \) and a function \( h : S^p \to \mathbb{R} \). As \( \widehat{M}_N(t) \) is a martingale,

\[
\mathbf{E}_{\nu_N}^N \left[ \left( F_N(\xi_N^s(t)) - F_N(\xi_N^s(s)) - \int_s^t (L_Z f)(Y_N(r)) \, dr \right) H_N \right] = 0,
\]

where \( H_N = h(Y_N(s_1), \ldots, Y_N(s_p)) \). Finally, by Proposition 7.3,

\[
\lim_{N \to \infty} \mathbb{Q}_{\nu_N}^N \left[ \left( f(Y(t)) - f(Y(s)) - \int_s^t (L_Z f)(Y(r)) \, dr \right) h(Y(s_1), \ldots, Y(s_p)) \right] = 0.
\]

This completes the proof. \( \square \)

### 7.2 Energy estimate.

We prove in this section Proposition 7.1, the argument is borrowed from [23] who consider the super-critical case.

Denote by \( \langle \cdot, \cdot \rangle_{\mu_N} \) the scalar product in \( L^2(\mu_N) \):

\[
\langle F_1, F_2 \rangle_{\mu_N} = \sum_{\eta \in \mathcal{H}_N} F_1(\eta)F_2(\eta)\mu_N(\eta).
\]

With this notation, we can write the Dirichlet form as

\[
\mathcal{D}_N(F) = \langle F, -\mathcal{L}_N F \rangle_{\mu_N}.
\]
Lemma 7.4. There exists a finite constant \( C_0 = C_0(\kappa) \) such that for all \( x, y \in S \) and \( F : H_N \to \mathbb{R} \),
\[
\sum_{\eta \in H_N} \mu_N(\eta) [F(\sigma^x, y, \eta) - F(\eta)]^2 \leq C_0 D_N(F).
\]

Proof. Suppose first that \( r(x, y) > 0 \). Then,
\[
\mu_N(\eta) [F(\sigma^x, y, \eta) - F(\eta)]^2 \leq \frac{1}{r(x, y)} \mu_N(\eta) g(\eta_x) r(x, y) [F(\sigma^x, y, \eta) - F(\eta)]^2
\]
because \( g(\eta_x) \geq 1 \) if \( \eta_x \geq 1 \), and both sides are 0 when \( \eta_x = 0 \). Summing this over \( \eta \in H_N \) yields the assertion of the lemma.

If \( r(x, y) = 0 \), by the irreducibility of the Markov chain \( X \), there exists a sequence \( x = z_0, z_1, \ldots, z_m = y \) such that \( r(z_i, z_{i+1}) > 0 \) for \( 0 \leq i < m \). Hence, by the Cauchy-Schwarz inequality
\[
[F_N(\sigma^{x, y}, \eta) - F_N(\eta)]^2 \leq m \sum_{i=0}^{m-1} [F_N(\sigma^{z_0, z_i, y}, \eta) - F_N(\sigma^{z_0, z_i, \eta})]^2.
\]
Applying the previous argument to each term at the right-hand side completes the proof since there exists a finite constant \( C_0 \) such that
\[
\mu_N(\sigma^{x, w, \eta}) \leq C_0 \mu_N(\eta)
\]
for all \( z, w \in S, \eta \in H_N, N \in \mathbb{N} \).

Fix \( x \neq y \in S \). We represent a configuration \( \eta \) in \( H_N \) as \( \eta = (\eta_x, \eta_y, \zeta) \) where \( \zeta = (\zeta_z)_{z \in S \setminus \{x, y\}} \in \mathbb{N}^{S \setminus \{x, y\}} \) stands for the configuration \( \eta \) on \( S \setminus \{x, y\} \): \( \zeta_z = \eta_z \) for \( z \in S \setminus \{x, y\} \).

On the other hand, for \( \zeta \in \mathbb{N}^{S \setminus \{x, y\}} \) with \( |\zeta| \leq N \), define \( \eta^{(i)}_\zeta \in H_N, 0 \leq i \leq N - |\zeta| \), the configuration on \( S \) with \( N - i - |\zeta| \) particles at site \( x \), \( i \) particles at site \( y \), and \( \zeta \) particles at \( z \in S \setminus \{x, y\} \):
\[
\eta^{(i)}_\zeta = (N - i - |\zeta|, i, \zeta).
\]

Proof of Proposition 7.1. Fix \( x, y \in S \) and consider the case where
\[
(L_z f)(z) = 1\{z = x\} - 1\{z = y\}, \quad z \in S.
\]
Since the mean of the right-hand side with respect to the uniform measure vanishes, there exists a function \( f \) which satisfies this identity.

Recall the definitions of the functions \( F_N \) and \( G_N \). Multiply both sides of (7.1) by \( -\mu_N(\eta)F_N(\eta) \) and then sum over \( \eta \in H_N \) to get
\[
\theta_N D_N(F_N) = -\langle F_N, G_N \rangle_{\mu_N} = \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta)F_N(\eta) - \sum_{\eta \in \mathcal{E}_N^y} \mu_N(\eta)F_N(\eta)
\]
in view of the definition of \( G_N \) and \( f \).

For a configuration \( \eta \in \mathcal{E}_N^x \), let \( \hat{\eta} \in \mathcal{E}_N^y \) be the configuration obtained from \( \eta \) by interchanging the values of \( \eta_x \) and \( \eta_y \):
\[
\hat{\eta}_z = \begin{cases} \eta_y & \text{if } z = x, \\ \eta_x & \text{if } z = y, \\ \eta_z & \text{otherwise}. \end{cases}
\]
Clearly, $\mu_N(\eta) = \mu_N(\tilde{\eta})$. Thus, we can rewrite (7.7) as
\[
\theta_N D_N(F_N) = \sum_{\eta \in \mathcal{E}_N} \mu_N(\eta) [F_N(\tilde{\eta}) - F_N(\eta)].
\]

By the Cauchy-Schwarz inequality, since $\mu_N(\mathcal{E}_N) \leq 1$, the square of the right-hand side is bounded by
\[
\sum_{\eta \in \mathcal{E}_N} \mu_N(\eta) [F_N(\tilde{\eta}) - F_N(\eta)]^2.
\]

Let $\mathcal{C}_N = \{\zeta \in \mathbb{N}^{S_N(x,y)} : |\zeta| \leq \ell_N\}$, and rewrite $\mathcal{E}_N$ as
\[
\mathcal{E}_N = \left\{\eta^{(k)}_\zeta : \zeta \in \mathcal{C}_N \text{ and } 0 \leq k \leq \ell_N - |\zeta|\right\}.
\]

Fix $\eta \in \mathcal{E}_N$ and write $\eta$ as $\eta^{(k)}_\zeta$ for some $\zeta \in \mathcal{C}_N$, $0 \leq k \leq \ell_N - |\zeta|$. By definition, $\tilde{\eta} = \eta^{(N-|\zeta|-k)}_\zeta$. By the Cauchy-Schwarz inequality
\[
[F_N(\tilde{\eta}) - F_N(\eta)]^2 \leq \sum_{i=k}^{N-|\zeta|-k-1} \frac{1}{\mu_N(\eta^{(i)}_\zeta)} \sum_{i=k}^{N-|\zeta|-k} \mu_N(\eta^{(i)}_\zeta) \left[F_N(\eta^{(i+1)}_\zeta) - F_N(\eta^{(i)}_\zeta)\right]^2.
\]

By definition of the measure $\mu_N$ and the configurations $\eta^{(i)}_\zeta$, there exists a finite constant $C_0$ such that
\[
\sum_{i=k}^{N-|\zeta|-k-1} \frac{\mu_N(\eta^{(i)}_\zeta)}{\mu_N(\eta^{(i)}_\zeta)} = \frac{1}{a(k)} \frac{1}{a(N - |\zeta| - k)} \sum_{i=k}^{N-|\zeta|-k} a(N - |\zeta| - i) a(i) \lesssim \frac{C_0 N^3}{a(k)}.
\]

Therefore,
\[
\mu_N(\eta) [F_N(\tilde{\eta}) - F_N(\eta)]^2 \leq \frac{C_0 N^2}{a(k)} \sum_{i=0}^{N-|\zeta|-1} \mu_N(\eta^{(i)}_\zeta) \left[F_N(\eta^{(i+1)}_\zeta) - F_N(\eta^{(i)}_\zeta)\right]^2.
\]

Summing over $\eta \in \mathcal{E}_N$ yields that there exists a finite constant $C_0$ such that
\[
\sum_{\eta \in \mathcal{E}_N} \mu_N(\eta) [F_N(\tilde{\eta}) - F_N(\eta)]^2
\]
\[
\leq C_0 \theta_N \sum_{\zeta \in \mathcal{C}_N} \sum_{i=0}^{N-|\zeta|-1} \mu_N(\eta^{(i)}_\zeta) \left[F_N(\eta^{(i+1)}_\zeta) - F_N(\eta^{(i)}_\zeta)\right]^2
\]
\[
\leq C_0 \theta_N \sum_{\zeta \in \mathcal{C}_N} \sum_{i=0}^{N-|\zeta|-1} \mu_N(\eta^{(i)}_\zeta) \left[F_N(\eta^{(i+1)}_\zeta) - F_N(\eta^{(i)}_\zeta)\right]^2
\]

because $\sum_{k=1}^{\ell_N} 1/a(k) \simeq \log \ell_N \simeq \log N$.

Since each $\eta \in \mathcal{H}_N$ can be written as $\eta = \eta^{(i)}_\zeta$ for some $\zeta \in \mathbb{N}^{S_N(x,y)}$ and $0 \leq i \leq N - |\zeta|$ in a unique manner, we actually proved that there exists a finite constant $C_0$ such that
\[
\sum_{\eta \in \mathcal{E}_N} \mu_N(\eta) [F_N(\tilde{\eta}) - F_N(\eta)]^2 \leq C_0 \theta_N \sum_{\xi \in \mathcal{H}_N} \mu_N(\xi) [F_N(\sigma^{\nu,\xi}) - F_N(\xi)]^2.
\]
By Lemma 7.4, this expression is bounded by $C_0 \theta N \mathcal{D}_N(F_N)$.

Up to this point, we proved that there exists a finite constant $C_0$ such that

$$\{ \theta N \mathcal{D}_N(F_N) \}^2 \leq C_0 \theta N \mathcal{D}_N(F_N).$$

This completes the proof of the proposition in the case where $f$ satisfies (7.6). To extend it to the general case, observe that a function with zero mean with respect to the counting measure can be written as a linear combination of functions appearing on the right-hand side of (7.6). Since this is the case of $L_Z f$, for all $f : S \to \mathbb{R}$, the proof is complete. $\square$

7.3. Sketch of the proof of Proposition 7.2

It is shown in [23] that Proposition 7.2 follows from condition (H1) of [6] and the energy estimate derived in Proposition 7.1. Unfortunately, condition (H1) does not hold for critical zero-range dynamics.

We present below a different approach to derive Proposition 7.2, which firstly appeared in [24], in the context of metastable diffusions. For zero-range processes, however, computations are quite different.

Consider the bilinear form in $L^2(\mu_N)$ given by

$$\mathcal{D}_N(F, G) = \frac{1}{2} \sum_{\eta \in \mathcal{H}_N} \sum_{x, y \in S} \mu_N(\eta) g(\eta_x) r(x, y) (T_{x,y} F)(\eta) (T_{x,y} G)(\eta), \quad (7.9)$$

where $(T_{x,y} H)(\eta) = H(\sigma^{x,y} \eta) - H(\eta)$. Clearly, $\mathcal{D}_N(F) = \mathcal{D}_N(F, F)$.

Fix a function $g : S \to \mathbb{R}$. We construct a test function, denoted by $V^g : \mathcal{H}_N \to \mathbb{R}$, with the following properties. On the one hand,

$$\theta N \mathcal{D}_N(V^g, F_N) \leq D_Z(g, f_N) + \text{error}, \quad (7.10)$$

where $D_Z$ is the bilinear energy form associated to the generator $L_Z$, defined below in Section 7.6 and the precise meaning of error term is given in Proposition 7.12.

On the other hand,

$$\theta N \mathcal{D}_N(V^g, F_N) = D_Z(g, f) + \text{error}, \quad (7.11)$$

where the precise meaning of error term is given in (7.24).

It follows from these bounds that

$$D_Z(g, f - f_N) = D_Z(g, f) - D_Z(g, f_N) \leq \text{error}.$$ 

Choosing $g = f - f_N$, we get that $D_Z(f - f_N, f - f_N)$ is asymptotically small, from what we conclude that $\|f - f_N\|_{\infty} = o_N(1)$, as claimed.

In Section 7.4 we present a partition of the set $\mathcal{H}_N$. The idea behind this construction is that in the computation of the form $\mathcal{D}_N(F, G)$, for Lipschitz functions $F, G$, in the sense of Lemma 7.10 only a tiny subspace of $\mathcal{H}_N$, formed by the wells $E_{x}^{N}$ and tubes connecting them, matters.

In Section 7.5 we construct the test function $V^g$ and show that it is Lipschitz. In the next three sections, we prove the two bounds (7.10) and (7.11) on the bilinear form $\mathcal{D}_N(V^g, F_N)$ alluded to above and prove the second property of the solutions of the Poisson equation, i.e., Proposition 7.2.
7.4. Tubes and valleys. Fix $\epsilon > 0$ small. The subsets of $H_N$ constructed in this section may depend on $N$ and $\epsilon$, even if these parameters do not appear in the notation. We also refer to Figure 1 for the illustration of the sets described in this subsection.

Define the enlarged valleys $\mathcal{V}^x$, $\hat{\mathcal{V}}^x$, $x \in S$, by

$$\mathcal{V}^x = \{ \eta \in H_N : \eta_x \geq N(1 - 2\epsilon) \}, \quad \hat{\mathcal{V}}^x = \{ \eta \in H_N : \eta_x \geq N(1 - 4\epsilon) \}.$$

From now on, all the statements may hold only for large enough $N$. More precisely, there exists a constant $N_0 = N_0(\epsilon)$ which is independent of $\eta$ such that the statement holds only for $N > N_0$. With this convention, $E^x_N \subset \mathcal{V}^x \subset \hat{\mathcal{V}}^x$.

For $x, y \in S$, the tubes $\mathcal{T}^{x,y}$ and $\hat{\mathcal{T}}^{x,y}$ connecting the valleys $E^x_N$ and $E^y_N$ are defined by

$$\mathcal{T}^{x,y} = \{ \eta \in H_N : \eta_x + \eta_y \geq N - \ell_N \}, \quad \hat{\mathcal{T}}^{x,y} = \{ \eta \in H_N : \eta_x + \eta_y \geq N(1 - 3\epsilon) \}.$$

Let

$$\mathcal{J}^{x,y} = \mathcal{T}^{x,y} \setminus [\mathcal{V}^x \cup \mathcal{V}^y], \quad \hat{\mathcal{J}}^{x,y} = \hat{\mathcal{T}}^{x,y} \setminus [\hat{\mathcal{V}}^x \cup \hat{\mathcal{V}}^y],$$

and note that the definitions are symmetric: $\mathcal{T}^{x,y} = \mathcal{T}^{y,x}$, $\hat{\mathcal{T}}^{x,y} = \hat{\mathcal{T}}^{y,x}$, $\mathcal{J}^{x,y} = \mathcal{J}^{y,x}$ and $\hat{\mathcal{J}}^{x,y} = \hat{\mathcal{J}}^{y,x}$.

Denote by $\mathcal{G}$ and $\hat{\mathcal{G}}$ the union of wells and tubes:

$$\mathcal{G} = \bigcup_{x,y \in S} (\mathcal{V}^x \cup \mathcal{J}^{x,y}), \quad \hat{\mathcal{G}} = \bigcup_{x,y \in S} (\hat{\mathcal{V}}^x \cup \hat{\mathcal{J}}^{x,y}). \quad (7.12)$$

We present below some properties of these sets.

**Lemma 7.5.** The following holds.

(1) Suppose that $\eta \in \mathcal{J}^{x,y}$ for some $x, y \in S$. Then, $\eta_x, \eta_y \in (N\epsilon, N(1 - 2\epsilon))$.

(2) Suppose that $\eta \in \hat{\mathcal{J}}^{x,y}$ for some $x, y \in S$. Then, $\eta_x, \eta_y \in (N\epsilon, N(1 - 4\epsilon))$. 


(3) For \( \{x, y\} \neq \{x', y'\} \), \( \mathcal{J}^{x, y} \cap \mathcal{J}^{x', y'} = \emptyset \). In particular, the expression \( \text{(7.12)} \) represents a partition of \( \mathcal{G} \).

**Proof.** For part (1), if \( \eta \in \mathcal{J}^{x, y} = \mathcal{J}^{x'} \cap \mathcal{J}^{y'} \), the bound \( \eta_x < N(1 - 2\epsilon) \) is trivial because \( \eta \notin \mathcal{V}^x \). By symmetry this extends to \( \eta_y \). By this bound and since \( \eta \in \mathcal{J}^{x, y} \),
\[
\eta_x + N(1 - 2\epsilon) > \eta_x + \eta_y > N - \ell_N > N - \epsilon N.
\]
Thus, this proves the lower bound. Proof of part (2) is similar.

For part (3), it suffices to show that
\[
\mathcal{T}^{x, y} \cap \mathcal{T}^{x, z} \subset \mathcal{V}^x \quad \text{for all } x, y, z \in S.
\]
To prove this, fix \( \eta \in \mathcal{T}^{x, y} \cap \mathcal{T}^{x, z} \). Since \( \eta_x + \eta_y + \eta_z \leq N \),
\[
2N - 2\ell_N \leq \eta_x + \eta_y + \eta_z \leq \eta_x + N.
\]
Thus, \( \eta_x \geq N - 2\ell_N \), which implies that \( \eta \in \mathcal{V}^x \), proving \( \text{(7.13)} \).

In the remaining part of this subsection, we provide an estimate of the measures \( \mu_N(\hat{\mathcal{G}} \setminus \mathcal{G}) \) and \( \mu_N(\mathcal{J}^{x, y}) \). For \( S_0 \subset S \) and \( k \in \mathbb{N} \), let
\[
\mathcal{H}_{k, S_0} = \left\{ \xi = (\xi_x)_{x \in S_0} \in \mathbb{N}^{S_0} : |\xi| := \sum_{x \in S_0} \xi_x = k \right\}.
\]

We adopt the following convention. Fix \( c : \mathbb{N} \times (0, 1] \to \mathbb{R} \). We write \( c(N, \epsilon) = o_N(1) \) if \( \lim_{N \to \infty} c(N, \epsilon) = 0 \) for all \( \epsilon > 0 \), and \( c(N, \epsilon) = o_\epsilon(1) \) if
\[
\lim_{\epsilon \to 0, N \in \mathbb{N}} |c(N, \epsilon)| = 0.
\]
Mind that we always send \( N \to \infty \) before \( \epsilon \to 0 \).

Hereafter, \( C_0 \) represents a finite constant independent of \( N \), \( \epsilon \) and \( \eta \), and \( C_\epsilon \) a finite one, independent of \( N \) and \( \eta \), but which may depend on \( \epsilon \). The values of \( C_0 \) and \( C_\epsilon \) may change from line to line.

**Lemma 7.6.** We have that
\[
\mu_N(\hat{\mathcal{G}} \setminus \mathcal{G}) \leq \frac{1}{\log N} \left[ o_N(1) + o_\epsilon(1) \right].
\]

**Proof.** For \( x \in S \), define
\[
\mathcal{A}^x = \hat{\mathcal{V}}^x \setminus \mathcal{G} = \hat{\mathcal{V}}^x \setminus \left( \mathcal{V}^x \cup \bigcup_{y \in S \setminus \{x\}} \mathcal{J}^{x, y} \right).
\]

With this notation,
\[
\hat{\mathcal{G}} \setminus \mathcal{G} \subset \bigcup_{x \in S} \mathcal{A}^x \cup \bigcup_{x, y \in S} (\hat{\mathcal{J}}^{x, y} \setminus \mathcal{J}^{x, y}).
\]

Therefore, it enough to show that
\[
\mu_N(\mathcal{A}^x) = \frac{o_N(1)}{\log N} \quad \text{and} \quad \mu_N(\hat{\mathcal{J}}^{x, y} \setminus \mathcal{J}^{x, y}) = \frac{o_\epsilon(1)}{\log N} \quad \text{(7.14)}
\]
for all \( x, y \in S \).

We first consider \( \mathcal{A}^x \). Since \( \hat{\mathcal{V}}^x \cap \mathcal{V}^y = \emptyset \), in the definition of \( \mathcal{A}^x \), we may add \( \mathcal{V}^y \) to the expression inside parenthesis. At this point, we may replace \( \mathcal{J}^{x, y} \) by \( \mathcal{J}^{x, y} \), and then remove \( \mathcal{V}^y \) to get that
\[
\mathcal{A}^x = \hat{\mathcal{V}}^x \setminus \left( \mathcal{V}^x \cup \bigcup_{y \in S \setminus \{x\}} \mathcal{J}^{x, y} \right). \quad \text{(7.15)}
\]
where \( A^x_m = \{ \eta \in A^x : \eta_x = m \} \). Represent a configuration \( \eta \) as \((\eta_x, \xi)\), where \( \xi \in \mathbb{N}^S \backslash \{x\} \) stands for the configuration \( \eta \) on \( S \backslash \{x\} \) \( \xi_y = \eta_y \) for all \( y \in S \backslash \{x\} \).

Note that \( \xi \in \mathcal{H}_{N-m,S \backslash \{x\}} \) if \( \eta \in A^x_m \). Let \( B^x_m \) the subset of configurations \( \xi \in \mathcal{H}_{N-m,S \backslash \{x\}} \) such that \((m, \xi) \in A^x_m \).

Recall the definition of the set \( \Delta_{S,N} \) introduced below (2.4). We claim that \( B^x_m \subset \Delta_{S,N-m} \). Indeed, fix \( \xi \in B^x_m \) and \( y \neq x \). Let \( \eta \) be the configuration \((m, \xi)\), so that \( \eta \in A^x_m \). By (7.15), \( \eta \notin \mathcal{T}^{x,y} \). Hence, as \( \eta_x = m \), \( \eta_y + m = \eta_y < N - \ell_N \) so that \( \eta_y < N - m - \ell_N \leq (N - m) - \ell_{N-m} \) because \( \ell_{N-m} \leq \ell_N \).

Therefore, configurations \( \xi \) in \( B^x_m \) have a total of \( N - m \) particles and each site has strictly less than \((N - m) - \ell_{N-m} \) particles. Thus, by definition of \( \Delta_{S,N} \), \( \xi \) belongs to \( \Delta_{S,N}\backslash \{x\},N-m \), which proves the claim.

By definition of \( \mu_N \) and \( B^x_m \):

\[
\mu_N(A^x) = \frac{N}{Z_{N,S}(\log N)^{\kappa-1}} \sum_{m=N(1-4\epsilon)}^{N(1-2\epsilon)-1} \frac{1}{a(m)} \sum_{\xi \in B^x_m} \frac{1}{a(\xi)}. 
\]

As \( B^x_m \) is contained in \( \Delta_{S,N-m} \), this expression is less than or equal to

\[
\frac{N}{(\log N)^{\kappa-1}} \sum_{m=N(1-4\epsilon)}^{N(1-2\epsilon)-1} \frac{1}{m} \frac{Z_{N,m,N-m-1}(\log(N-m))^{\kappa-2}}{Z_{N,S}} \frac{1}{N-m} \mu_{\kappa-1,N-m}(\Delta_{S,N-m}).
\]

By Proposition 4.1 for each \( p \geq 1 \), \((Z_{N,p})_{N \geq 1} \) is a bounded sequence. Hence, \( Z_{N,m,N-m-1}/Z_{N,S} \leq C_0 \). As \( 2N \leq N - m \leq 4\epsilon N \), \( N/(N-m) \leq C_0/\epsilon \), and \( \log(N-m)/\log N \leq 1 \). The previous expression is thus bounded above by

\[
\frac{C_0}{\log N} \frac{1}{\epsilon} \frac{1}{(1-4\epsilon)N} \sum_{M=2N}^{4N} \mu_{\kappa-1,M}(\Delta_{S,M})/M.
\]

By Theorem 2.3 the sequence \( \mu_{\kappa-1,M}(\Delta_{S,M}) \) vanishes as \( M \to \infty \). This shows that the previous sum is bounded by \( 2\epsilon N o_N(1) \). This proves the first estimate in (7.14).

We turn to the second bound of (7.14). Write \( \tilde{\mathcal{T}}^{x,y} \backslash \mathcal{T}^{x,y} \) as

\[
\tilde{\mathcal{T}}^{x,y} = \bigcup_{m=N(1-4\epsilon)}^{N-\ell_N-1} \mathcal{T}^{x,y}_m, \tag{7.16}
\]

where \( \mathcal{T}^{x,y}_m = \{ \eta \in \tilde{\mathcal{T}}^{x,y} \backslash \mathcal{T}^{x,y} : \eta_x + \eta_y = m \} \). Write \( \eta \in \mathcal{T}^{x,y}_m \) as \( \eta = (\eta_x, \eta_y, \zeta) \), where \( \zeta \in \mathcal{H}_{N-m,S \backslash \{x,y\}} \) represents the configuration of \( \eta \) on \( S \backslash \{x,y\} \).

By part (2) of Lemma 7.5 \( \eta_x, \eta_y > N\epsilon \) for configurations \( \eta \) in \( \tilde{\mathcal{T}}^{x,y} \). Therefore, by Proposition 4.1 there exists a finite constant \( C_0 \) such that

\[
\mu_N(\mathcal{T}^{x,y}_m) \leq \frac{C_0}{(\log N)^{\kappa-1}} \sum_{m=N(1-4\epsilon)}^{m-N\epsilon} \frac{1}{a(i)} \sum_{\xi \in \mathcal{H}_{N-m,S \backslash \{x,y\}}} \frac{1}{a(\zeta)}.
\]
An elementary computation yields that there exists a finite constant $C_0$ such that
\[ \sum_{i=N-\epsilon}^{m-N\epsilon} \frac{1}{a(i) a(m-i)} \leq \frac{C_0}{N} \log \frac{1}{\epsilon} \]
for all $(1-4\epsilon)N \leq m \leq N$.

On the other hand, by Proposition 4.1 and since $m \leq N - \ell_N$, there exists a constant $C_0$ such that
\[ \sum_{\zeta \in H_{N-m, S\setminus \{x,y\}}} \frac{1}{a(\zeta)} \leq C_0 \frac{\log(N-m)^{\kappa-3}}{N-m} \leq C_0 \frac{(\log N)^{\kappa-3}}{\ell_N} \]

Combining the previous estimates yields that
\[ \mu_N(J_{x,y}^m) \leq C_0 \frac{N \log N}{\log 1/\epsilon} \]
and hence by (7.16),
\[ \mu_N(\tilde{J}^{x,y} \setminus J^{x,y}) \leq 4N\epsilon \frac{C_0}{N \log N} \log \frac{1}{\epsilon} = o(1) \]
as claimed.

**Lemma 7.7.** There exists a finite constant $C_0$ such that, for all $x, y \in S$,
\[ \mu_N(J^{x,y}) \leq C_0 \frac{1}{\log N} \log \frac{1}{\epsilon} \]

**Proof.** The proof is similar to the one of the last part of the previous lemma. Fix $x, y \in S$ and write
\[ J^{x,y} = \bigcup_{m=N-\ell_N}^{N} J_{x,y}^m, \]
where $J_{x,y}^m = \{ \eta \in J^{x,y} : \eta_x + \eta_y = m \}$.

Represent a configuration $\eta$ in $J_{x,y}^m$ as $\eta = (\eta_x, \eta_y, \zeta)$ for $\zeta \in H_{N-m, S\setminus \{x,y\}}$. By part (1) of Lemma 7.5, $\eta_x, \eta_y > N\epsilon$ for configurations $\eta$ in $J^{x,y}$. Thus,
\[ \mu_N(J_{x,y}^m) \leq \frac{C_0}{N} \sum_{i=N-\epsilon}^{m-N\epsilon} \frac{1}{a(i) a(m-i)} \sum_{\zeta \in H_{N-m, S\setminus \{x,y\}}} \frac{1}{a(\zeta)} \]
Clearly, there exists a finite $C_0$ such that
\[ \sum_{i=N-\epsilon}^{m-N\epsilon} \frac{1}{a(i) a(m-i)} \leq \frac{C_0}{N} \log \frac{1}{\epsilon} \] (7.17)
for all $N - \ell_N \leq m \leq N$.

By Proposition 4.1,
\[ \sum_{\zeta \in H_{N-m, S\setminus \{x,y\}}} \frac{1}{a(\zeta)} \leq \frac{C_0}{N-m} \frac{\log(N-m)^{\kappa-3}}{N-m} \leq C_0 \frac{(\log N)^{\kappa-3}}{N-m} \]
for $N - \ell_N \leq m < N$. For $m = N$, the sum is bounded by 1.

Putting together the previous estimates yields that
\[ \mu_N(J_{x,y}^m) \leq \frac{C_0}{(\log N)^2} N \frac{1}{N-m} \log \frac{1}{\epsilon} \]
for $N - \ell_N \leq m < N$ and $\mu_N(\mathcal{J}^{x,y}_N) \leq [C_0/(\log N)^{\kappa-1}] \log(1/\epsilon)$.

Summing over $N - \ell_N \leq m \leq N$ gives that

$$\mu_N(\mathcal{J}^{x,y}_m) \leq C_0 \frac{1}{\log N} \sum_{k=1}^{\ell_N} \frac{1}{k} \log \frac{1}{\epsilon} \leq C_0 \log \frac{1}{\epsilon},$$

as claimed. □

Decompose the tube $\mathcal{J}^{x,y}$ as

$$\mathcal{J}^{x,y} = \mathcal{K}^{x,y} \cup \mathcal{L}^{x,y},$$

(7.18)

where

$$\mathcal{K}^{x,y} = \{ \eta \in \mathcal{J}^{x,y} : \eta_x < 6N\epsilon \},$$

$$\mathcal{L}^{x,y} = \{ \eta \in \mathcal{J}^{x,y} : \eta_x, \eta_y \geq 6N\epsilon \}.$$

The next lemma asserts that we can remove the factor $\log(1/\epsilon)$ in the previous lemma replacing $\mathcal{J}^{x,y}$ by $\mathcal{K}^{x,y}$.

**Lemma 7.8.** There exists a finite constant $C_0$ such that, for all $x, y \in S$,

$$\mu_N(\mathcal{K}^{x,y}) \leq \frac{C_0}{\log N}.$$

**Proof.** Assume that $\eta_x \leq 6\epsilon N$, and let $\mathcal{K}^{x,y}_m = \mathcal{K}^{x,y} \cap \mathcal{T}^{x,y}_m$, $N - \ell_N \leq m \leq N$. By (1) of Lemma 7.5, $\eta_x > \epsilon N$. Hence, $\eta_x$ varies from $\epsilon N$ to $6\epsilon N$.

Proceed as in the previous lemma. In the formula for $\mu_N(\mathcal{K}^{x,y}_m)$, let $i$ represent $\eta_x$, so that (7.17) becomes

$$\sum_{i=N\epsilon}^{6N\epsilon} \frac{1}{a(i)} \frac{1}{a(m-i)} \leq \frac{C_0}{N}.$$

The rest of the argument is identical to the one of Lemma 7.7. □

### 7.5. Construction of test functions.

In this section, we introduce functions $U_{x,y} : \mathcal{H}_N \to \mathbb{R}, x, y \in S$, to examine the Poisson equation (7.1). These functions are similar to the ones introduced in the super-critical case in [8] to estimate the capacities between wells.

Fix $x, y \in S$ and a small parameter $\epsilon > 0$. Let $\phi_\epsilon : [0, 1] \to [0, 1]$ be a smooth, non-decreasing, bijective function such that

$$\phi_\epsilon(t) + \phi_\epsilon(1-t) = 1, \quad t \in [0, 1],$$

$$\phi_\epsilon(t) = \begin{cases} 0 & t \in [0, 3\epsilon], \\ (t - 4\epsilon)/(1 - 8\epsilon) & t \in [5\epsilon, 1 - 5\epsilon], \\ 1 & t \in [1 - 3\epsilon, 1]. \end{cases}$$

$$\phi'_\epsilon(t) \leq 1 + \epsilon^{1/2}, \quad t \in [0, 1].$$

Although, the existence of such a function is straightforward, we refer to [25] Section 7.3 for an explicit construction.

Define $\Phi_\epsilon : [0, 1] \to [0, 1] by$

$$\Phi_\epsilon(t) := 6 \int_0^{\phi_\epsilon(t)} u (1 - u) du = 3 \phi_\epsilon(t)^2 - 2 \phi_\epsilon(t)^3.$$
Note that
\[ \Phi_x(t) = \begin{cases} 0 & \text{if } t \in [0, 3\epsilon] \\ 1 & \text{if } t \in [1 - 3\epsilon, 1] \end{cases}. \]

Recall from (2.8) that \( h_{x,y} = h_{\{x\}, \{y\}} : S \to \mathbb{R} \) denotes the equilibrium potential between \( x \) and \( y \) for the random walk \( X(\cdot) \). Let
\[ x = z_1, z_2, \ldots, z_\kappa = y \]
be an enumeration of \( S \) satisfying
\[ 1 = h_{x,x}(z_1) \geq h_{x,y}(z_2) \geq \cdots \geq h_{x,y}(z_\kappa) = 0. \]
Define \( U_{x,y} : \mathcal{H}_N \to \mathbb{R} \) by
\[ U_{x,y}(\eta) = \sum_{j=1}^{\kappa-1} \left( h_{x,y}(z_j) - h_{x,y}(z_{j+1}) \right) \Phi_x \left( \frac{1}{N} \sum_{i=1}^{j} \eta_{z_i} \right). \]
(7.19)
The function \( U_{x,y} \) approximates the equilibrium potential between \( \mathcal{V}^x \) and \( \mathcal{V}^y \) in the tube \( J^{x,y} \).

**Remark 7.9.** Fix \( x, y \in S \), and denote by \( z_1, z_2, \ldots, z_\kappa \) and \( z_1', z_2', \ldots, z_\kappa' \) the sequences \( \{z_i\} \) associated to the functions \( U_{x,y} \) and \( U_{y,x} \), respectively. We assume that \( z_i = z_{i+1} \), \( 1 \leq i \leq \kappa \). Clearly, this condition holds if \( h_{x,y}(z) \neq h_{x,y}(y) \) for all \( z \in S \).

Denote by \( \|u\|_\infty \) the sup-norm of a function \( u : S \to \mathbb{R} \), \( \|u\|_\infty = \max_{x \in S} |u(x)| \).

Consider a sequence of functions \( g = g_N : S \to \mathbb{R} \), we omit below the dependence of \( g \) on \( N \). We define a function \( V^g : \mathcal{H}_N \to \mathbb{R} \) in few steps. We first construct it on \( \mathcal{G} \), and then extend it to the whole set. Recall from Lemma 7.5-(3) that the set \( \mathcal{G} \) can be represented as a disjoint union of the sets \( \mathcal{V}^x \) and \( J^{x,y} \). Let
\[ V^g(\eta) = \begin{cases} g(x) & \text{if } \eta \in \mathcal{V}^x, \ x \in S, \\ g(y) + [g(x) - g(y)] U_{x,y}(\eta) & \text{if } \eta \in J^{x,y}, \ x, y \in S. \end{cases} \]
(7.21)
By Remark 7.9, we have \( U_{y,x} = 1 - U_{x,y} \) on \( J^{x,y} \). Hence,
\[ g(y) + [g(x) - g(y)] U_{x,y}(\eta) = g(x) + [g(y) - g(x)] U_{y,x}(\eta), \]
and \( V^g \) is well-defined on \( J^{x,y} \).

The function \( V^g \) is smooth enough on \( \mathcal{G} \) in the following sense.

**Lemma 7.10.** There exists a finite constant \( C_0 \) such that,
\[ \max_{\eta \in \mathcal{G}} |V^g(\eta)| \leq 2\|g\|_\infty, \quad |V^g(\sigma_{z,w} \eta) - V^g(\eta)| \leq C_0 \frac{\|g\|_\infty}{N} \]
for all \( N \in \mathbb{N}, \ z, w \in S \), and configurations \( \eta \in \mathcal{G} \) such that \( \sigma_{z,w} \eta \in \mathcal{G} \).

**Proof.** The first bound follows from the definition of \( V^g \) and from the fact that \( U_{x,y} \) is bounded by 1. We turn to the second.

From the definition of the sets \( \mathcal{V}^x, \ J^{x,y} \), for a pair \( (z, w) \) and configurations \( \eta \) and \( \sigma_{z,w} \eta \) in \( \mathcal{G} \), there are three possibilities. Either \( \eta \) and \( \sigma_{z,w} \eta \) belong to some set \( \mathcal{V}^x \), or both to some set \( J^{x,y} \) or \( \eta \) belongs to some \( \mathcal{V}^x \) and \( \sigma_{z,w} \eta \) to some \( J^{x,y} \) [or the opposite]. We consider separately the three cases.

The inequality is trivial if \( \eta, \sigma_{z,w} \eta \in \mathcal{V}^x \) for some \( x \in S \) since in this case \( V^g(\sigma_{z,w} \eta) - V^g(\eta) = 0 \).
By definition of $\Phi$, and the bound on the derivative of $\phi$,

$$
|\Phi'(t)| = 6 |\phi'(t) \phi(t) (1 - \phi(t))| \leq 6 (1 + \epsilon^{1/2}).
$$

Therefore, there exists a finite constant $C_0$ such that,

$$
|U_{x,y}(\sigma^{z,w} \eta) - U_{x,y}(\eta)| \leq \frac{C_0}{N} \tag{7.22}
$$

for all $N \in \mathbb{N}$, $\eta \in \mathcal{H}_N$, and $z, w \in S$. In particular, the inequality stated in Lemma 7.10 holds if $\eta, \sigma^{z,w} \eta \in \mathcal{J}^{x,y}$ for some $x, y \in S$.

Finally, assume that $\sigma^{z,w} \eta \in \mathcal{J}^{x,y}$ and $\eta \in \mathcal{V}^x$ for some $x, y \in S$. The same argument applies to the converse situation. In this case, $U_{x,y}(\eta) = 1$ because $\eta_x \geq (1 - 2\epsilon)N$. Thus, by definition of $V^g$,

$$
V^g(\sigma^{z,w} \eta) - V^g(\eta) = [g(x) - g(y)] \left[ U_{x,y}(\sigma^{z,w} \eta) - U_{x,y}(\eta) \right],
$$

and the assertion of the lemma follows from (7.22). \qed

To extend the function $V^g$ to $\mathcal{H}_N \setminus \mathcal{G}$, let

$$
V^g(\eta) = 0 \quad \text{for } \eta \in \mathcal{H}_N \setminus \mathcal{G}. \tag{7.23}
$$

On $\mathcal{G} \setminus \mathcal{G}$, smoothly interpolate the construction (7.21) and (7.23) in such a way that max$_{\eta \in \mathcal{G} \setminus \mathcal{G}} |V^g(\eta)| \leq 2\|g\|_\infty$ and

$$
|V^g(\sigma^{z,w} \eta) - V^g(\eta)| \leq \frac{C_\epsilon \|g\|_\infty}{N} \quad \text{for all } \eta \in \mathcal{G} \setminus \mathcal{G}, \ z, w \in S,
$$

where $C_\epsilon$ is a constant independent of $N$. This is possible in view of Lemma 7.10 and since the distance between $\mathcal{H}_N \setminus \mathcal{G}$ and $\mathcal{G}$ is of order $N\epsilon$.

Next result summarizes the bounds obtained in the construction.

**Lemma 7.11.** For each $\epsilon$ small, there exists a finite constant $C_\epsilon$ such that,

$$
\max_{\eta \in \mathcal{H}_N} |V^g(\eta)| \leq 2 \|g\|_\infty, \quad |V^g(\sigma^{z,w} \eta) - V^g(\eta)| \leq \frac{C_\epsilon \|g\|_\infty}{N}
$$

for all $N \in \mathbb{N}$, $z, w \in S$, and $\eta \in \mathcal{H}_N$.

7.6. **Proof of Proposition 7.2** The proof is based on Proposition 7.12 stated below. Fix a function $f : S \rightarrow \mathbb{R}$, and recall the definition of $F_N$, introduced in (7.1), and the one of $\mathcal{D}_N$ given in (7.9).

Let $D_Z(u, v)$ be the bilinear form given by

$$
D_Z(u, v) = \frac{1}{N} \sum_{x \in S} u(x) (-L_Z v)(x) = \frac{1}{2N} \sum_{x, y \in S} r_Z(x, y) (u(y) - u(x)) (v(y) - v(x)),
$$

for $u, v : S \rightarrow \mathbb{R}$. Here, $L_Z$ is the generator introduced in (2.13).

The next result is proven in Section 7.7.

**Proposition 7.12.** We have that

$$
\theta_N \mathcal{D}_N(V^g, F_N) \leq D_Z(g, f_N) + \|g\|_\infty \left\{ o_N(1) \|f_N\|_\infty + o_N(1) + o_1 \right\}.
$$

**Proof of Proposition 7.2.** The main idea of the proof is to compute $\theta_N \mathcal{D}_N(V^g, F_N)$ in two different ways. The first one is carried out in Proposition 7.12. The other, and simpler one, is presented below.

Multiply both sides of the Poisson equation (7.1) by $-V^g(\eta) \mu_N(\eta)$ and sum over $\eta \in \mathcal{H}_N$ to obtain that

$$
\theta_N \mathcal{D}_N(V^g, F_N) = \theta_N \langle V^g, -L_N F_N \rangle_{\mu_N} = -\langle V^g, G \rangle_{\mu_N}.
$$
By Theorem 2.3 and Lemma 7.11, since $E^x \subset V^x$,
\[-\langle V^g, G \rangle_{\mu_N} = \frac{1}{\kappa} \sum_{x \in S} g(x)(-L_Z f)(x) + o_N(1) \|g\|_{\infty}.
\] (7.24)

Here, the constant $o_N(1)$ is allowed to depend on $\|f\|_{\infty}$, which is finite and fixed. The first term on the right-hand side is equal to $D_Z(g, f)$. The two previous equations yield that
\[
\theta_N \mathcal{D}_N(V^g, F_N) = D_Z(g, f) + o_N(1) \|g\|_{\infty}.
\]

Thus, by Proposition 7.12,
\[
D_Z(g, f) \leq D_Z(g, f_N) + \|g\|_{\infty} R(N, \epsilon),
\]
where $R(N, \epsilon) = [1 + \|f_N\|_{\infty}] o_N(1) + o_\epsilon(1)$. Subtracting $D_Z(g, f_N)$ and choosing $g = f - f_N$ yields that
\[
D_Z(f - f_N, f - f_N) \leq \|f - f_N\|_{\infty} R(N, \epsilon).
\] (7.25)

Let
\[
c_0 := \frac{1}{\kappa} \min_{x \neq y \in S} r_Z(x, y).
\]
This constant is strictly positive by definition of the rates $r_Z$ and because the capacities between sets, in the context of finite-state, irreducible chains, is strictly positive. For any function $h : S \to \mathbb{R}$,
\[
D_Z(h, h) \geq \frac{c_0}{2} \sum_{x, y \in S} |h(x) - h(y)|^2.
\]
Assume, furthermore, that $\sum_{x \in S} h(x) = 0$. By Poincaré inequality, the previous expression is bounded below by
\[
\frac{c_0 \kappa}{2} \sum_{x \in S} h(x)^2 \geq \frac{c_0 \kappa}{2} \|h\|_{\infty}^2.
\]

By (7.3), $f - f_N$ has mean zero with respect to the counting measure and we may apply the previous bound. Putting together the previous estimate with $h = f - f_N$ and (7.25), we get that
\[
\|f - f_N\|_{\infty} \leq \frac{2}{c_0 \kappa} R(N, \epsilon).
\]

As $\|f - f_N\|_{\infty} \geq \|f_N\|_{\infty} - \|f\|_{\infty}$,
\[
\|f_N\|_{\infty} \leq \|f\|_{\infty} + \frac{2}{c_0 \kappa} R(N, \epsilon),
\]

so that $\|f_N\|_{\infty} \leq C_0$ and
\[
\|f - f_N\|_{\infty} \leq o_N(1) + o_\epsilon(1).
\]
Since both $f$ and $f_N$ do not depend on $\epsilon$, this implies that $\|f - f_N\|_{\infty} = o_N(1)$, which completes the proof. \(\square\)
7.7. Proof of Proposition 7.12. Let $\mathbf{d}^x$, $x \in S$, be the configuration with one particle at $x$ and no particles at the other sites.

For each set $A \subset \mathcal{H}_N$, denote by $A_-, A_+ \subset \mathcal{H}_{N-1}$ the sets defined by

$$A_- = \{ \xi \in \mathcal{H}_{N-1} : \xi + \mathbf{d}^x \in A \ \forall x \in S \},$$

$$A_+ = \{ \xi \in \mathcal{H}_{N-1} : \exists x \in S \text{ s.t. } \xi + \mathbf{d}^x \in A \}.$$

Recall from (7.12) the definition of the subsets $G, \hat{G}$ of $\mathcal{H}_N$. We claim that

$$\mathcal{H}_{N-1} = G_- \cup (\mathcal{H}_N \setminus \hat{G})_- \cup (\hat{G} \setminus \hat{G})_+.$$  \hfill (7.26)

It is clear that the right-hand set is contained in $\mathcal{H}_{N-1}$. Fix $\xi \in \mathcal{H}_{N-1}$. Suppose that $\xi + \mathbf{d}^x$ belongs to $\hat{G} \setminus G$ for some $x \in S$. In this case, $\xi \in (\hat{G} \setminus G)_+$. Suppose, now, that $\xi + \mathbf{d}^x \notin \hat{G} \setminus G$ for all $x \in S$. Fix $x_0 \in S$. Since $\mathcal{H}_N = G \cup (\mathcal{H}_N \setminus \hat{G}) \cup (\hat{G} \setminus G), \xi + \mathbf{d}^{x_0} \in G \cup (\mathcal{H}_N \setminus \hat{G})$. Suppose that $\xi + \mathbf{d}^{x_0} \in G$. The argument applies to the other possibility. Fix $y \in S \setminus \{x_0\}$. Since $\xi + \mathbf{d}^{x_0}$ and $\xi + \mathbf{d}^y$ are neighbors, $\xi + \mathbf{d}^y$ can not belong to $\mathcal{H}_N \setminus \hat{G}$. As it also does not belong to $\hat{G} \setminus G$, $\xi + \mathbf{d}^y$ is in $G$. Hence, $\xi + \mathbf{d}^y \in G$ for all $y \in S$, so that $\xi \in G_-$, as claimed in (7.26).

As the sets on the right-hand side of (7.26) are disjoints, this identity provides a partition of the set $\mathcal{H}_{N-1}$.

**Lemma 7.13.** There exists a finite constant $C_0$ such that

$$\mu_{N-1}(A_+) \leq C_0 \mu_N(A)$$

for all $A \subset \mathcal{H}_N$ and $N \geq 3$.

**Proof.** Note that

$$A_+ = \bigcup_{x \in S} \{ \eta - \mathbf{d}^x : \eta \in A \text{ with } \eta_x \geq 1 \}.$$

Therefore,

$$\mu_{N-1}(A_+) \leq \sum_{x \in S} \sum_{\eta \in A, \eta_x \geq 1} \mu_{N-1}(\eta - \mathbf{d}^x).$$

In particular, it is enough to show that there exists a finite constant $C_0$ such that

$$\frac{\mu_{N-1}(\eta - \mathbf{d}^x)}{\mu_N(\eta)} \leq C_0 \text{ for all } \eta \in \mathcal{H}_N \text{ with } \eta_x \geq 1.$$

By definition of the measure $\mu_N$ and Proposition 4.1,

$$\frac{\mu_{N-1}(\eta - \mathbf{d}^x)}{\mu_N(\eta)} = \frac{N-1}{Z_{N-1,S} [\log(N-1)]^{\kappa-1}} \frac{Z_{N,S} (\log N)^{\kappa-1}}{N} \frac{a(\eta)}{a(\eta - \mathbf{d}^x)} \leq C_0 \frac{1}{[\log(N-1)]^{\kappa-1}}.$$

This proves the bound and the lemma. \hfill $\Box$

Denote by $\mathcal{D}_N(F, G; A)$, $A \subset \mathcal{H}_{N-1}$, the bilinear form given by

$$\frac{a_N}{2} \sum_{\xi \in A} \sum_{x, y \in S} \mu_{N-1}(\xi) r(x, y) \left[ F(\xi + \mathbf{d}^y) - F(\xi + \mathbf{d}^x) \right] \left[ G(\xi + \mathbf{d}^y) - G(\xi + \mathbf{d}^x) \right],$$

for $F, G : \mathcal{H}_N \rightarrow \mathbb{R}$. In this formula,

$$a_N = \frac{Z_{N-1,S} [\log(N-1)]^{\kappa-1}}{N-1} \frac{N}{Z_{N,S} (\log N)^{\kappa-1}} = 1 + o_N(1). \hfill (7.27)$$
Since
\[ \mu_N(\xi + \bar{\Omega}) g(\xi + 1) = a_N \mu_N(\xi) , \]
a change of variables shows that
\[ \mathcal{D}_N(F, G) = \mathcal{D}_N(F, G; \mathcal{H}_{N-1}) . \]

Hence, by Lemma 7.6,
\[ \eta \because \]
and whose absolute value is bounded
\[ \Phi \in \sum \sum \mu_N(\xi) [V^g(\xi + \bar{\Omega}) - V^g(\xi + \bar{\Omega})]^2 \]
\[ \leq C \|g\|_\infty^2 \mu_N(\xi) \mathcal{H}_{N-1}(\bar{\Omega}) \]
Hence, by Lemma 7.6,
\[ \mathcal{D}_N(V^g, V^g; (\bar{\Omega})_+) \leq \eta \mathcal{H}_{N-1}(\bar{\Omega}) \|g\|_\infty^2 \theta N^{-1} , \]
so that
\[ \theta N \mathcal{D}_N(V^g, F_N; (\bar{\Omega})_+) \leq [o_N(1) + o(1)] \|g\|_\infty^2 , \]
which completes the proof of the lemma.

Recall, from (7.18), the definition of the sets \( K^{x,y} \), \( L^{x,y} \), and, from (7.19), the definition of the sequence \((z_i)_{i=1}^\infty\).

**Lemma 7.15.** Fix \( x \neq y \in S \). There exists a constant \( C_0 \) such that for all \( \eta \in \mathcal{J}^{x,y} \) and \( 1 \leq m < \kappa \),
\[ 0 \leq \Phi \left( \frac{1}{N} \sum_{i=1}^m \eta_{z_i} + \frac{1}{N} \right) - \Phi \left( \frac{1}{N} \sum_{i=1}^m \eta_{z_i} \right) \leq \frac{C_0}{N} \left[ \eta_x \eta_y N^2 + o(1) \right] . \]
Moreover, for all \( \eta \in \mathcal{L}^{x,y} \) and \( 1 \leq m < \kappa \),
\[ \Phi \left( \frac{1}{N} \sum_{i=1}^m \eta_{z_i} + \frac{1}{N} \right) - \Phi \left( \frac{1}{N} \sum_{i=1}^m \eta_{z_i} \right) = \frac{6}{N} \left[ \frac{\eta_x \eta_y}{N^2} + o(1) + O(\epsilon) \right] , \]
where \( O(\epsilon) \) is a constant which depends on \( \epsilon \) and whose absolute value is bounded by \( C_0 \epsilon \).
Proof. Fix $\eta \in J^{x,y}$. As $\Phi_\varepsilon$ is non-decreasing, the first inequality holds. We consider the second one. By definition of $\Phi_\varepsilon$ and the mean-value theorem,

$$\Phi_\varepsilon\left(\frac{1}{N}\sum_{i=1}^{m} \eta_{z_i} + \frac{1}{N}\right) - \Phi_\varepsilon\left(\frac{1}{N}\sum_{i=1}^{m} \eta_{z_i}\right) = \frac{6}{N} \phi'_\varepsilon(c) \phi_\varepsilon(c) [1 - \phi_\varepsilon(c)], \quad (7.28)$$

where $c = N^{-1} \sum_{1 \leq i \leq m} \eta_{z_i} + (\delta/N)$ for some $0 \leq \delta \leq 1$. By definition of $\phi_\varepsilon$, $1 - \phi_\varepsilon(c) = \phi_\varepsilon(1 - c)$. Thus, as

$$0 \leq c - \frac{\eta_x}{N} \leq \frac{N - \eta_x - \eta_y + 1}{N} \leq \ell_N + \frac{1}{N},$$

$$0 \leq (1 - c) - \frac{\eta_y}{N} \leq \frac{N - \eta_x - \eta_y}{N} \leq \ell_N,$$

it follows from the uniform bound on $\|\phi'_\varepsilon\|_\infty$, that

$$\Phi_\varepsilon\left(\frac{1}{N}\sum_{i=1}^{m} \eta_{z_i} + \frac{1}{N}\right) - \Phi_\varepsilon\left(\frac{1}{N}\sum_{i=1}^{m} \eta_{z_i}\right) \leq \frac{C_0}{N} \left[ \phi_\varepsilon\left(\frac{\eta_x}{N}\right) + o_N(1) \right] \left[ \phi_\varepsilon\left(\frac{\eta_y}{N}\right) + o_N(1) \right].$$

Since $\phi'_\varepsilon(t) \leq 1 + \varepsilon^{1/2}$ for all $0 \leq t \leq 1$ and $\phi_\varepsilon(0) = 0$, $\phi_\varepsilon(t) \leq (1 + \varepsilon^{1/2})t \leq 2t$. This completes the proof of the first assertion of the lemma, as $\eta_x/N \leq 1$.

We turn to the second one. Fix a configuration $\eta$ in $L^{x,y}$. Since

$$6\varepsilon \leq \frac{\eta_x}{N} \leq \frac{1}{N} \sum_{i=1}^{m} \eta_{z_i} \leq \frac{N - \eta_y}{N} \leq 1 - 6\varepsilon,$$

the constant $c$ belongs to the interval $[6\varepsilon, 1 - (11/2)\varepsilon]$ [provided $1/N \leq \varepsilon/2$], and $\phi'_\varepsilon(c) = 1/(1 - 8\varepsilon)$. On the other hand, since $\phi_\varepsilon$ is linear on the interval $[5\varepsilon, 1 - 5\varepsilon]$ and $\eta_x, \eta_y \geq 6\varepsilon N$,

$$\phi_\varepsilon(c) = \phi_\varepsilon\left(\frac{\eta_x}{N}\right) + o_N(1) = \frac{1}{1 - 8\varepsilon}\left(\frac{\eta_x}{N} - 4\varepsilon\right) + o_N(1),$$

$$\phi_\varepsilon(1 - c) = \phi_\varepsilon\left(\frac{\eta_y}{N}\right) + o_N(1) = \frac{1}{1 - 8\varepsilon}\left(\frac{\eta_y}{N} - 4\varepsilon\right) + o_N(1).$$

To complete the proof of the second assertion, it remains to report these estimates to the right-hand side of (7.28). \(\Box\)

By the definitions of $V^g$, and $U_{x,y}$, given in (7.21), (7.20), respectively, for $\eta \in K^{x,y}$, there exists a finite constant $C_0$ such that for all $z, w \in S$,

$$|V^g(\sigma^z, \omega^w \eta) - V^g(\eta)| \leq \frac{C_0}{N} \|g\|_\infty o_\varepsilon(1). \quad (7.29)$$

Next result asserts that it is enough to estimate the Dirichlet form on the sets $L^{x,y}_-, x, y \in S$.

**Lemma 7.16.** We have that

$$\theta_N D_N(V^g, F_N; G_-) = \sum_{x, y \in S} \theta_N D_N(V^g, F_N; L^-_{x,y}) + \left[ o_N(1) + o_\varepsilon(1) \right] \|g\|_\infty.$$
Proof. An argument, similar to the one presented to derive (7.26), yields that the set $G_-$ can be decomposed as

$$G_- = \bigcup_{x, y \in S} \mathcal{L}^{x, y}_- \cup \bigcup_{x \in S} \mathcal{V}^x_- \cup \bigcup_{x, y \in S} (K^{x, y}_+ \cap G_-).$$

On the one hand,

$$\mathcal{D}_N(V^g, F_N; \mathcal{L}^x_-) = 0$$

because $V^g(\xi + \partial^z) = g(x)$ for all $\xi \in \mathcal{V}^x_-$ and $z \in S$.

On the other hand, by Schwarz inequality and the bound on $a_N$,

$$\mathcal{D}_N(V^g, F_N; K^{x, y}_+ \cap G_-)^2 \leq C_0 \mathcal{D}_N(F_N) \sum_{\xi \in K^{x, y}_+} \sum_{z, w \in S} \mu_{N-1}(\xi) \left[ V^g(\xi + \partial^z) - V^g(\xi + \partial^w) \right]^2$$

for some finite constant $C_0$. By Proposition 7.1 and (7.29), this expression is bounded by

$$\frac{C_0}{\theta N^2} \|g\|_\infty^2 \left[ o_N(1) + o_r(1) \right] \mu_{N-1}(K^{x, y}_+).$$

By Lemmata 7.8 and 7.13, $\mu_{N-1}(K^{x, y}_+) \leq C_0 \mu_N(K^{x, y}) \leq C_0/\log N$ for some finite constant $C_0$.

Putting together the previous estimates yields that

$$\theta_N \mathcal{D}_N(V^g, F_N; K^{x, y}_+ \cap G_-) \leq C_0 \|g\|_\infty \left[ o_N(1) + o_r(1) \right].$$

This completes the proof of the lemma. \(\square\)

It remains to compute the Dirichlet form on $\mathcal{L}^{x, y}_-$. The proof of the next lemma is given in Section 7.8.

**Lemma 7.17.** For $x, y \in S$,

$$\theta_N \mathcal{D}_N(U_{x, y}, F_N; \mathcal{L}^{x, y}_-)$$

$$= \frac{r_Z(x, y)}{\kappa} \left[ f_N(x) - f_N(y) \right] + o_N(1) \left[ 1 + \|f_N\|_\infty \right] + o_r(1).$$

**Proof of Proposition 7.12** By definition (7.21) of $V^g$ on $\mathcal{J}^{x, y}$,

$$\theta_N \mathcal{D}_N(V^g, F_N; \mathcal{L}^{x, y}_-) = \theta_N \left[ g(x) - g(y) \right] \mathcal{D}_N(U_{x, y}, F_N; \mathcal{L}^{x, y}_-).$$

By Lemma 7.17, this expression is equal to

$$\frac{r_Z(x, y)}{\kappa} \left[ g(x) - g(y) \right] \left[ f_N(x) - f_N(y) \right] + \|g\|_\infty \left[ o_N(1) \left[ 1 + \|f_N\|_\infty \right] + o_r(1) \right].$$

It remains to combine this estimate with Lemmata 7.14 and 7.16. \(\square\)

7.8. **Proof of Lemma 7.17** We start with a simple lemma which allows to bound a covariance between two functions $F, G : \mathcal{E}_N \to \mathbb{R}$ in terms of the Dirichlet form of one of them and the $L^\infty$-norm of the other.

**Lemma 7.18.** There exists a finite constant $C_0$ such that, for all $x \in \mathbb{R}$ and $F, G : \mathcal{E}_N \to \mathbb{R}$,

$$\left| E_{\nu_N}[FG] - E_{\nu_N}[F] E_{\nu_N}[G] \right|^2 \leq C_0 \ell_N^2 \|G\|_\infty^2 \mathcal{D}_N(F).$$
Proof. This lemma is a simple consequence of the local spectral gap estimate. By the Cauchy-Schwarz inequality,
\[ \left| \mathbb{E}_{\mu_{\mathbb{S}}}[FG] - \mathbb{E}_{\mu_{\mathbb{S}}}[F]\mathbb{E}_{\mu_{\mathbb{S}}}[G] \right|^2 \leq \|G\|^2 \operatorname{Var}_{\mu_{\mathbb{S}}}(F), \]
where the variance has been introduced in (3.6). To complete the proof, it remains to recall the local spectral gap, stated in Theorem 3.1. \( \square \)

For \( x, y \in S \), define
\[ \mathcal{B}^{x,y}_k = \{ \zeta \in \mathbb{N}^{S \setminus \{x,y\}} : |\zeta| \leq \ell_N - 1 \}. \] (7.30)

**Lemma 7.19.** For \( x, y \in S \),
\[ \sum_{\zeta \in \mathcal{B}^{x,y}_k} \frac{1}{a(\zeta)} = \left[ 1 + o_N(1) \right] (\log N)^{\kappa - 2}. \]

**Proof.** Set \( \xi = (N - |\zeta|, \zeta) \in \mathcal{H}_N,S \setminus \{x\} \). By Theorem 2.3,
\[ \lim_{N \to \infty} N \mathcal{Z}_{N,S \setminus \{x\}}(\log N)^{\kappa - 2} \sum_{\zeta \in \mathcal{B}^{x,y}_k} \frac{1}{a(\zeta)} = \frac{1}{\kappa - 1}. \]
The assertion of the lemma follows from Proposition 4.1. \( \square \)

Recall the definition of the configuration \( \eta^{(i)}_\zeta \) introduced below (7.5).

**Lemma 7.20.** For all \( x \neq y \in S \),
\[ \frac{1}{(\log N)^{\kappa - 2}} \sum_{\zeta \in \mathcal{B}^{x,y}_k} \frac{1}{a(\zeta)} F_N(\eta^{6\epsilon N}_\zeta) = \left[ 1 + o_N(1) \right] f_N(x) + o_N(1), \]
\[ \frac{1}{(\log N)^{\kappa - 2}} \sum_{\zeta \in \mathcal{B}^{x,y}_k} \frac{1}{a(\zeta)} F_N(\eta^{(N^i - |\zeta| - 6\epsilon N)}_\zeta) = \left[ 1 + o_N(1) \right] f_N(y) + o_N(1). \]

**Proof.** We prove the first assertion, as the second one can be obtained by symmetry. Fix \( x, y \in S \). For \( k \in \mathbb{N} \), let
\[ \mathcal{B}^{x,y}_k = \{ \zeta \in \mathbb{N}^{S \setminus \{x,y\}} : |\zeta| = k \}. \] (7.31)

For \( 0 \leq k < \ell_N \), define
\[ c_N(k) = \frac{\log N}{N} \left( \frac{\ell_N - k}{a(N - k - 1)a(1)} \right)^{-1}, \] (7.32)
and set \( c_N(\ell_N) = 0 \). Note that there exists a finite constant \( C_0 \) such that
\[ |c_N(k)| \leq C_0 \log N \text{ for all } 0 \leq k \leq \ell_N. \] (7.33)

Define
\[ \tilde{f}_N(x) = \sum_{\eta \in \mathcal{E}_N} \mu_N(\eta) c_N(N - \eta_x - \eta_y) F_N(\eta). \]

We claim that
\[ \tilde{f}_N(x) = \left[ \frac{1}{k} + o_N(1) \right] f_N(x) + o_N(1), \] (7.34)
and that
\[ \frac{1}{\mathcal{Z}_{N,S}(\log N)^{\kappa - 2}} \sum_{\zeta \in \mathcal{B}^{x,y}_k} \frac{1}{a(\zeta)} F_N(\eta^{6\epsilon N}_\zeta) - \tilde{f}_N(x) = o_N(1). \] (7.35)
The assertion of the lemma follows from these two identities and Proposition 4.1.

To prove the first claim, let

\[ d_N = \sum_{\eta \in \mathcal{E}_N} c_N(N - \eta_x - \eta_y) \mu_N(\eta). \]

By definition of \( \mu_N \) and \( c_N \), as \( c_N(\ell_N) = 0 \), \( d_N \) is equal to

\[
\frac{N}{Z_{N,S}(\log N)^{n-1}} \sum_{k=0}^{\ell_N} \sum_{\zeta \in B_{k}^{x,y}} \frac{c_N(k)}{\log N} \frac{1}{a(\zeta)} \frac{a(N-k)}{a(i)} \frac{a(\zeta)}{a(\zeta)}
\]

\[
= \frac{1}{Z_{N,S}(\log N)^{n-2}} \sum_{\zeta \in B_{N}^{x,y}} \frac{1}{a(\zeta)}.
\]

Hence, by Proposition 4.1 and Lemma 7.19

\[ d_N = \frac{1}{k} + o_N(1). \]

To prove (7.34), it remains to show that

\[ \tilde{f}_N(x) - d_N f_N(x) = o_N(1). \]  

(7.36)

Define \( U : \mathcal{E}_N^x \to \mathbb{R} \) as \( U(\eta) = c_N(N - \eta_x - \eta_y) \), so that

\[
\frac{1}{\mu_N(\mathcal{E}_N^x)} \left\{ \tilde{f}_N(x) - d_N f_N(x) \right\} = E_{\mu_N}^{\mathcal{E}_N^x} [F_N U] - E_{\mu_N}^{\mathcal{E}_N^x} [F_N] E_{\mu_N}^{\mathcal{E}_N^x} [U].
\]

Thus, by Lemma 7.18 and (7.33),

\[
\left[ \tilde{f}_N(x) - d_N f_N(x) \right]^2 \leq C_0 \ell_N^2 (\log N)^2 \mathcal{D}_N(F_N)
\]

for some finite constant \( C_0 \). By Proposition 7.1, this expression is bounded by \( C_0 / \log N \), which proves (7.36) and (7.34).

We turn to (7.35). By definition, \( f_N(x) \) is equal to

\[
\frac{N}{Z_{N,S}(\log N)^{n-1}} \sum_{k=0}^{\ell_N} \sum_{\zeta \in B_{k}^{x,y}} \frac{c_N(k)}{\log N} \frac{1}{a(\zeta)} F_N(\eta^{(i)}_k).
\]

On the other hand, by definitions of \( B_{N,K}^{x,y} \) and \( c_N(k) \), given in (7.30), (7.31) and (7.32), respectively, we have

\[
\frac{1}{Z_{N,S}(\log N)^{n-2}} \sum_{\zeta \in B_{N}^{x,y}} \frac{1}{a(\zeta)} F_N(\eta^{(6_\infty N)}_k)
\]

\[
= \frac{N}{Z_{N,S}(\log N)^{n-1}} \sum_{k=0}^{\ell_N} \sum_{\zeta \in B_{k}^{x,y}} \frac{\log N}{\zeta} \frac{1}{a(\zeta)} F_N(\eta^{(6_\infty N)}_k)
\]

\[
= \frac{N}{Z_{N,S}(\log N)^{n-1}} \sum_{\zeta \in B_{N}^{x,y}} \sum_{i=0}^{\ell_N-k} \frac{c_N(k)}{a(\zeta)} F_N(\eta^{(6_\infty N)}_k).
\]
Therefore, the left-hand side of (7.35) is equal to
\[
\sum_{\zeta \in \mathbb{B}^{r \times y}} \mu_N(\eta^{(i)}_\zeta) \left[ F_N(\eta^{(6\epsilon N)}_\zeta) - F_N(\eta^{(i)}_\zeta) \right].
\]

In view of the previous expression, by the Cauchy-Schwarz inequality and (7.33), the square of the left-hand side of (7.35) is bounded by
\[
C_0 (\log N)^2 \sum_{\zeta \in \mathbb{B}^{r \times y}} \mu_N(\eta^{(i)}_\zeta) \left[ F_N(\eta^{(6\epsilon N)}_\zeta) - F_N(\eta^{(i)}_\zeta) \right]^2
\]
for some finite constant $C_0$. By the Cauchy-Schwarz inequality again, the square inside the previous sum is less than or equal to
\[
C_0 N^2 \sum_{i=0}^{|\zeta|} \mu_N(\eta^{(i)}_\zeta) \left[ F_N(\eta^{(6\epsilon N)}_\zeta) - F_N(\eta^{(i)}_\zeta) \right]^2
\]
for some finite constant $C_0$. The sum (7.37) is thus bounded above by
\[
C_0 (N \log N)^2 \sum_{\zeta \in \mathbb{B}^{r \times y}} \mu_N(\eta^{(i)}_\zeta) \left[ F_N(\eta^{(6\epsilon N)}_\zeta) - F_N(\eta^{(i)}_\zeta) \right]^2
\]
and
\[
\leq C_0 N (\log N)^2 \sum_{\zeta \in \mathbb{B}^{r \times y}} \mu_N(\eta^{(i)}_\zeta) \left[ F_N(\eta^{(6\epsilon N)}_\zeta) - F_N(\eta^{(i)}_\zeta) \right]^2.
\]

Changing the order of summations this expression becomes
\[
C_0 N (\log N)^2 \sum_{\zeta \in \mathbb{B}^{r \times y}} \mu_N(\eta^{(i)}_\zeta) \left[ F_N(\eta^{(6\epsilon N)}_\zeta) - F_N(\eta^{(i)}_\zeta) \right]^2 \frac{1}{a(i)}.
\]

where $A_N(j, \zeta) = \min \{j, \ell_N - |\zeta|\}$. The sum over $i$ is bounded by $C_0 \log \ell_N \leq C_0 \log N$. Hence, by Lemma 7.4 this expression is less than or equal to $C_0 N (\log N)^3 \mathcal{D}_N(F_N)$, which, by Proposition 7.1 is bounded by $C_0 (\log N)^2 N$, which proves (7.35).

\[\square\]

Proof of Lemma 7.17: Fix $x, y \in S$, and recall the definition of $U_{x,y}$ given in (7.20) and the one of the sequence $(z_i)_{i=1}^\infty$ introduced in (7.19). With this notation, we can write $\theta_N \mathcal{D}_N(U_{x,y}, F_N; \mathcal{L}^{x,y})$ as
\[
\frac{\theta_N a_N}{2} \sum_{i,j=1}^\infty \sum_{\zeta \in \mathcal{L}^{x,y}} \mu_N^{-1}(\zeta) r(z_i, z_j) (T_{i,j} U_{x,y})(\zeta) (T_{i,j} F_N)(\zeta),
\]
where $(T_{i,j} G)(\zeta) = G(\zeta + \delta^{z_i}) - G(\zeta + \delta^{z_j})$.

Assume that $i > j$. By definition, we can write $T_{i,j} U_{x,y}(\zeta)$ as
\[
\sum_{n=j}^{i-1} [h_{x,y}(z_n) - h_{x,y}(z_{n+1})] \left[ \Phi(r) \left( \frac{1}{N} \sum_{k=1}^n \xi_{z_k} + \frac{1}{N} \right) - \Phi(r) \left( \frac{1}{N} \sum_{k=1}^n \xi_{z_k} \right) \right].
\]
By the second assertion of Lemma 7.15, this sum is equal to
\[
\frac{6}{N} \sum_{n=j}^{i-1} \left[ h_{x,y}(z_n) - h_{x,y}(z_{n+1}) \right] \left( \frac{\xi \xi_y}{N^2} + o_N(1) + O(\epsilon) \right)
\]
\[
= \frac{6}{N} \left[ h_{x,y}(z_j) - h_{x,y}(z_i) \right] \left( \frac{\xi \xi_y}{N^2} + o_N(1) + O(\epsilon) \right).
\]
A similar identity holds for \( i < j \).

Therefore,
\[
\theta_N D_N(U_{x,y}, F_N; L_{x,y}^-) = I_1 + I_2,
\]
where
\[
I_1 = \frac{3 \theta_N a_N}{N^3} \sum_{\xi \in L_{x,y}^-} \sum_{i,j=1}^{N} \mu_{N-1}(\xi) r(z_i, z_j) \xi \xi_y \left[ h_{x,y}(z_j) - h_{x,y}(z_i) \right] (T_{i,j} F_N)(\xi),
\]
\[
I_2 = \left[ o_N(1) + O(\epsilon) \right] \frac{\theta_N}{N} \sum_{\xi \in L_{x,y}^-} \sum_{i,j=1}^{N} \mu_{N-1}(\xi) \left[ h_{x,y}(z_j) - h_{x,y}(z_i) \right] (T_{i,j} F_N)(\xi).
\]

The second term is easy to estimate. By the Cauchy-Schwarz inequality, its square is bounded by
\[
\left[ o_N(1) + O(\epsilon) \right] \frac{\theta_N}{N^2} \mu_{N-1}(L_{x,y}^-) D_N(F_N).
\]

By definition of \( L_{x,y}^-, L_{x,y}^+ \), and by Lemmata 7.7 and 7.13,
\[
\mu_{N-1}(L_{x,y}^-) \leq \mu_{N-1}(L_{x,y}^+) \leq C_0 \mu_N(L_{x,y}) \leq \frac{C_0}{\log N} \log \frac{1}{\epsilon}
\]
for some finite constant \( C_0 \). Hence, by Proposition 7.11,
\[
I_2 = o_N(1) + o_\epsilon(1).
\]

We turn to \( I_1 \). Write \( \xi \) as \((\xi_x, \xi_y, \zeta)\) for \( \zeta \in \mathbb{N}^{S \setminus \{x,y\}} \). Then, \( I_1 \) is equal to
\[
\frac{3 \theta_N (N-1) a_N}{N^3 Z_{N-1,S} [\log(N-1)]^{N-2}} \times \sum_{\xi \in L_{x,y}^-} \sum_{i,j=1}^{N} \frac{1}{a(\zeta)} r(z_i, z_j) \left\{ h_{x,y}(z_j) - h_{x,y}(z_i) \right\} (T_{i,j} F_N)(\xi).
\]

By Proposition 4.1, by definition of \( \theta_N \) and by (7.27), we may rewrite this expression as
\[
\frac{6 [1 + o_N(1)]}{\kappa (\log N)^{N-2}} \sum_{\xi \in L_{x,y}^-} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{F_N(\xi + \delta_{z_i})}{a(\zeta)} r(z_i, z_j) \left\{ h_{x,y}(z_j) - h_{x,y}(z_i) \right\}
\]
\[
= \frac{6 [1 + o_N(1)]}{\kappa (\log N)^{N-2}} \sum_{\xi \in L_{x,y}^-} \sum_{i=1}^{N} \frac{F_N(\xi + \delta_{z_i})}{a(\zeta)} (-L_X h_{x,y})(z_i).
\]

By (10.1),
\[
(L_X h_{x,y})(z) = \begin{cases} -\kappa \cap_X(x, y) & \text{if } z = x \\ \kappa \cap_X(x, y) & \text{if } z = y \\ 0 & \text{otherwise} \end{cases}
\]
Thus, by the definition \((5.1)\) of \(r_Z(x, y)\),

\[
I_1 = \frac{r_Z(x, y)[1 + o_N(1)]}{\kappa (\log N)^{\kappa - 2}} \sum_{\zeta \in \mathcal{L}_{-}^{x,y}} \frac{1}{a(\zeta)} \left[ F_N(\zeta + \theta^x) - F_N(\zeta + \theta^y) \right].
\]

Recall from \((7.30)\) the definition of the set \(B_{x,y}^{x,y}\) and that \(\eta^{(i)}_\zeta\) represents the configuration \((N - |\zeta| - i, i, \zeta) \in \mathcal{H}_N\). With this notation, the set \(\mathcal{L}_{-}^{x,y}\) can be represented as

\[
\mathcal{L}_{-}^{x,y} = \{(N - 1 - |\zeta| - i, i, \zeta) : \zeta \in B_{x,y}^{x,y}, \quad 6N\epsilon \leq i < N - |\zeta| - 6N\epsilon\}.
\]

Therefore, we can write \(I_1\) as

\[
I_1 = \frac{r_Z(x, y)[1 + o_N(1)]}{\kappa (\log N)^{\kappa - 2}} \sum_{\zeta \in \mathbb{B}_{x,y}^{x,y}} \frac{1}{a(\zeta)} \sum_{i=6N\epsilon}^{N-|\zeta|-6N\epsilon-1} \left[ F_N(\eta^{(i)}_\zeta) - F_N(\eta^{(i+1)}_\zeta) \right].
\]

Thus, by Lemma \(7.20\),

\[
I_1 = \frac{r_Z(x, y)[1 + o_N(1)]}{\kappa (\log N)^{\kappa - 2}} \sum_{\zeta \in \mathbb{B}_{x,y}^{x,y}} \frac{1}{a(\zeta)} \left[ F_N(\eta^{(6N\epsilon)}_\zeta) - F_N(\eta^{(N-|\zeta|-6N\epsilon)}_\zeta) \right],
\]

which completes the proof of the lemma. \(\square\)

8. Proof of Theorem 3.2

The proof of Theorem 3.2 relies on two results presented in this section. The first one, Proposition 8.3, provides a weaker version of Theorem 3.2, in which the initial condition, a configuration, is replaced by the invariant measure conditioned to the set \(\mathcal{D}_x^{x,y}\). The second one, Proposition 8.6, asserts that starting from the well \(\mathcal{E}_x^{y}\), the process visits every configuration of the deep valley \(\mathcal{D}_x^{x,y}\) before it hits a new well \(\mathcal{E}_y^{y}\).

Recall the definition of \(\bar{\mathcal{E}}_x^{x,y}\) in \((3.7)\). The proof of Proposition 8.3 is based on the enlargement of the zero-range process and requires an estimate of the capacity \(\text{cap}_N(\mathcal{E}_x^{x,y}, \bar{\mathcal{E}}_x^{x,y})\).

This estimate is provided in Section 8.1. In Section 8.2, we introduce the enlargement process and present a bound, in terms of capacities, for the probability that the hitting time of a set is small. This general result, stated as Proposition 8.4, can be useful in other contexts.

8.1. Upper bound of the capacity.

Denote by \(h_{A,B} : \mathcal{H}_N \to \mathbb{R}\) the equilibrium potential between two disjoint, non-empty subsets \(A\) and \(B\) of \(\mathcal{H}_N\):

\[
h_{A,B}(\eta) = P_\eta^N [\tau_A < \tau_B].
\]

The capacity between \(A\) and \(B\) is given by

\[
\text{cap}_N(A, B) = \mathcal{D}_N(h_{A,B}). \tag{8.1}
\]

The main result of this subsection reads as follows.

**Proposition 8.1.** There exists a finite constant \(C_0\) such that for all \(x \in S\)

\[
\theta_N \text{cap}_N(\mathcal{E}_x^{x,y}, \bar{\mathcal{E}}_x^{x,y}) \leq C_0.
\]
Proof. Fix \( x \in S \). By the Dirichlet principle [19, equation (B14)],
\[
\text{cap}_N(\mathcal{E}_N^x, \bar{\mathcal{E}}_N^x) \leq \mathcal{D}_N(F)
\]
for any function \( F : \mathcal{H}_N \to \mathbb{R} \) such that \( F \equiv 1 \) on \( \mathcal{E}_N^x \) and \( F \equiv 0 \) on \( \bar{\mathcal{E}}_N^x \).

Let \( \chi_x = \chi_{\{x\}} : S \to \mathbb{R} \) be the characteristic function on \( x \), i.e.,
\[
\chi_x(y) = 1\{x = y\}, \quad y \in S,
\]
and consider the test function is \( F = V^{x^*} \), where \( V^g \) is the function introduced in Section 7.5. Let \( \mathcal{C}_x \) be the subset of \( \mathcal{H}_N \) given by
\[
\mathcal{C}_x = \bigcup_{y \in S \setminus \{x\}} \mathcal{J}^{x,y} \cup (\bar{\mathcal{G}} \setminus \mathcal{G}) .
\]

Since \( F(\sigma_{z,w}^x \eta) = F(\eta) \) unless \( \eta \) or \( \eta^{x,y} \) belongs to \( \mathcal{C}_x \),
\[
\mathcal{D}_N(V^{x^*}) \leq \sum_{z, w \in S} \sum_{\eta} \mu_N(\eta) g(\eta) r(z, w) \left[ V^{x^*}(\sigma_{z,w}^{x^*} \eta) - V^{x^*}(\eta) \right]^2 ,
\]
where the second sum is performed over all \( \eta \) at distance one or less from \( \mathcal{C}_x \). By Lemmata 7.6, 7.7 and 7.11,
\[
\mathcal{D}_N(V^{x^*}) \leq C_0 \frac{\mu_N(\mathcal{C}_x)}{N^2} \leq C_0 \frac{\mu_N(\mathcal{C}_x)}{\theta_N} \log \frac{1}{\epsilon} .
\]
To completes the proof, it remains to fix some \( 0 < \epsilon < 1 \) and observe that the sets \( \mathcal{E}_N^x \) do not depend on \( \epsilon \). \( \square \)

Remark 8.2. Although we do not provide the detailed proof here, we can compute the sharp asymptotics for the capacity and show that
\[
\theta_N \text{cap}_N(\mathcal{E}_N^x, \bar{\mathcal{E}}_N^x) = \left[ 1 + o_N(1) \right] \frac{1}{\kappa} \sum_{y \in S \setminus \{x\}} r_{Z}(x, y) .
\]

8.2. The enlarged process. Recall the definition of the sets \( \mathcal{D}_N^x \), introduced in (3.8). Denote by \( \pi_N^x \) the measure \( \mu_N \) conditioned on \( \mathcal{D}_N^x \):
\[
\pi_N^x(\eta) = \frac{\mu_N(\eta)}{\mu_N(\mathcal{D}_N^x)} , \quad \eta \in \mathcal{D}_N^x .
\]
The main result of this subsection reads as follows.

Proposition 8.3. For all \( x \in S \),
\[
\limsup_{a \to 0} \limsup_{N \to \infty} P_N^{\pi_N^x} \left[ \tau_{\mathcal{E}_N^x} < a \theta_N \right] = 0 .
\]
The proof of this proposition is based on the next result which provides a bound for the transition time in terms of the initial distribution and the capacity. This result is a modification of [9, Corollary 4.2].

Proposition 8.4. For every \( x \in S \), probability measure \( \nu_N \) concentrated on the set \( \mathcal{E}_N^x \), \( \gamma_N > 0 \) and \( N \geq 1 \),
\[
\left( P_{\nu_N}^{\nu_N} \left[ \tau_{\mathcal{E}_N^x} \leq \frac{1}{\gamma_N} \right] \right)^2 \leq \frac{\epsilon^2}{\gamma_N \mu_N} E_{\nu_N} \left[ \left( \frac{\nu_N}{\mu_N} \right)^2 \right] \frac{1}{\mu_N(\mathcal{E}_N^x)} \text{cap}_N(\mathcal{E}_N^x, \bar{\mathcal{E}}_N^x) .
\]
Proof of Proposition 8.4. By the definition of $\pi_N^\tau$,  
$$E_{\mu_N^\tau} \left[ (\pi_N^\tau)^2 \right] = \sum_{\eta \in E_N} \frac{\pi_N^\tau(\eta)^2}{\mu_N(\eta)} = \frac{\mu_N(E_N^\tau)}{\mu_N(D_N^\tau)}.$$ 
Therefore, by Proposition 8.4 with $\gamma_N = \pi_N^\tau$ and $\gamma_N^{-1} = \theta_N$,  
$$\left( P_N^{\pi_N^\tau} \left[ \tau_{E_N^\tau} \leq a \theta_N \right] \right)^2 \leq \frac{e^2 a \theta_N}{\mu_N(D_N^\tau)} \text{cap}_N(E_N^\tau, \tilde{E}_N^\tau).$$

By Propositions 4.2 and 8.1 there exists a finite constant $C(\gamma)$, where $\gamma$ is the parameter appearing in the definition of the set $D_N^\tau$, such that  
$$\left( P_N^{\pi_N^\tau} \left[ \tau_{E_N^\tau} \leq a \theta_N \right] \right)^2 \leq C(\gamma) a.$$ 
This completes the proof. \qed 

Beside Proposition 8.4, the main ingredients of the proof were the strictly positive lower bound for $\mu_N(D_N^\tau)$ and the upper bound for the capacity.

We turn to the proof of Proposition 8.4 which relies on an enlargement of the state space, introduced in [9, Section 2]. Denote by $E_N^\tau : E_N \times E_N \to [0, \infty)$ the jump rates of the trace process $\{\eta_N^\tau(t)\}_{t \geq 0}$ [the trace of the process $\eta_N(t)$ on $E_N$, defined by the equation (2.7)] with $\xi_N(t)$ replaced by $\eta_N(t)$ in the definition of $T^\tau_N(t)$ and in equation (2.7). 

Let $E_N^\tau$ be a copy $E_N$, and denote by $\eta^* \in E_N^\tau$ the copy of $\eta \in E_N$. 

Definition 8.5 (Enlarged process). Fix $N \geq 1$ and $\gamma_N > 0$. The $\gamma_N$-enlarged process $\{\eta_N^\tau(t)\}_{t \geq 0}$ is the continuous-time Markov process on $E_N \cup E_N^\tau$ whose jump rates $R_N^\tau : E_N \cup E_N^\tau \times E_N \cup E_N^\tau \to [0, \infty)$ are given by  
$$R_N^\tau(\eta, \zeta) = \begin{cases} R_N^\tau(\eta, \zeta) & \text{if } \eta, \zeta \in E_N, \\ \gamma_N & \text{if } \zeta = \eta^* \text{ or } \eta = \zeta^*, \\ 0 & \text{otherwise}. \end{cases}$$

Namely, the process $\eta_N^\tau(t)$ at $\eta^* \in E_N^\tau$ only jumps to $\eta$ at rate $\gamma_N$, while at $\eta \in E_N$ it jumps to other points of $E_N$ as in the original dynamics of the trace process, and it jumps to $\eta^*$ at rate $\gamma_N$.

The invariant measure for the $\gamma_N$-enlarged process $\eta_N^\tau(t)$ is given by  
$$\mu_N^*(\eta) = \mu_N^*(\eta^*) = \frac{1}{2} \mu_N(\eta) \text{ for all } \eta \in E_N.$$ 
Actually, the process $\eta_N^\tau(\cdot)$ is reversible with respect to this measure.

Denote by $\text{cap}_N^\tau(A, B)$ the capacity between two disjoint, nonempty subsets $A$, $B$ of $E_N \cup E_N^\tau$, defined in a same manner as (8.1).

Proof of Proposition 8.4. Denote by $P_N^{v_N, \xi}$ the law of the trace process $\xi^\tau_N(t)$ on $E_N$ starting from the measure $v_N$. In view of [9, Corollary 4.2], to prove the proposition, it is enough to show that  
$$P_N^{v_N} \left[ \tau_{E_N^\tau} \leq \frac{1}{\gamma_N} \right] \leq P_N^{v_N, \xi} \left[ \tau_{E_N^\tau} \leq \frac{1}{\gamma_N} \right],$$ 
$$\text{cap}_N \left( E_N^\tau \times E_N^\tau \right) \leq \frac{1}{2\mu_N(E_N^\tau)} \text{cap}_N \left( E_N^\tau \times E_N^\tau \right),$$
where $E_N^\tau$, $\tilde{E}_N^\tau$ represent the copies of $E_N^\tau$, $\tilde{E}_N^\tau$, respectively.
The first estimate holds because the trace process hits the set $\tilde{\mathcal{E}}_N^x$ before the original process, as the latter one may spend some time on $\Delta_N$.
We turn to the second estimate. By [15, Lemma 2.2], the capacity is monotone, so that
$$\text{cap}_N(\mathcal{E}^{x,x}_N \cup \tilde{\mathcal{E}}_N^x) \leq \text{cap}_N(\mathcal{E}^{x,x}_N \cup \mathcal{E}^{x,x}_N \cup \tilde{\mathcal{E}}_N^x).$$
Denote by $\chi^*_x = \chi_{\mathcal{E}^{x,x}_N \cup \tilde{\mathcal{E}}_N^x} : \mathcal{E}_N \rightarrow \mathbb{R}$ the indicator function of the set $\mathcal{E}^{x,x}_N \cup \tilde{\mathcal{E}}_N^x$. Since $\chi^*_x$ is the equilibrium potential between the sets $\mathcal{E}^{x,x}_N \cup \mathcal{E}^{x,x}_N$ and $\tilde{\mathcal{E}}_N^x$ for the $\gamma_N$-enlarged process, the right-hand side of the previous displayed equation is equal to $D^*_N(\chi^*_x)$, where $D^*_N$ represents the Dirichlet form associated to the $\gamma_N$-enlarged process.

By definition of the enlarged process, in the computation of the Dirichlet form of the indicator function $\chi^*_x$ the only terms which do not vanish are those which correspond to jumps between $\mathcal{E}^{x}_N$ and $\tilde{\mathcal{E}}_N^x$. Hence,
$$D_N(\chi^*_x) = \sum_{\eta \in \mathcal{E}^{x}_N, \zeta \in \tilde{\mathcal{E}}_N^x} (\mu_N(\eta) R_N(\eta, \zeta)) [\chi^*_x(\zeta) - \chi^*_x(\eta)]^2.$$
By definition of $\mu_N$, $R_N$ and $\chi^*_x$, this sum is equal to
$$D^*_N(\chi^*_x) = \frac{1}{2} \sum_{\eta \in \mathcal{E}^{x}_N, \zeta \in \tilde{\mathcal{E}}_N^x} (\mu_N(\eta) R_N(\eta, \zeta)) [\chi^*_x(\zeta) - \chi^*_x(\eta)]^2.$$

Denote by $D^*_{EN}$ the Dirichlet form associated to the trace process. The previous sum is equal to $(1/2) D^*_{EN}(\chi^{\mathcal{E}_N^x})$. Since $\chi^{\mathcal{E}_N^x}$ is the equilibrium potential between $\mathcal{E}^{x}_N$ and $\tilde{\mathcal{E}}_N^x$ for the trace process,
$$\frac{1}{2} D^*_{EN}(\chi^{\mathcal{E}_N^x}) = \frac{1}{2} \text{cap}_{EN}(\mathcal{E}^{x}_N, \tilde{\mathcal{E}}_N^x),$$
where $\text{cap}_{EN}$ stands for the capacity for the trace process. By [6, Lemma 6.9],
$$\frac{1}{2} \text{cap}_{EN}(\mathcal{E}^{x}_N, \tilde{\mathcal{E}}_N^x) = \frac{1}{2} \mu(\mathcal{E}^{x}_N, \tilde{\mathcal{E}}_N^x),$$
which completes the proof of the proposition. □

8.3. Visiting points. The proof of Theorem [3.2] relies on the next result, which asserts that, starting from the well $\mathcal{E}^{x}_N$, the process visits every configuration of the deep valley $D_N$ before it hits a new well $\mathcal{E}^{x}_N$.

**Proposition 8.6.** For each $x \in S$,
$$\lim_{N \rightarrow \infty} \inf_{\zeta \in D_N^x} \inf_{\eta \in \mathcal{E}^{x}_N} P^N_\eta[\tau_\zeta < \tau_{\mathcal{E}^{x}_N}] = 1.$$

In most of models, based on the martingale approach developed in [6, 7], this result is proved by verifying condition (H1) of [9], reducing the argument to an estimate of capacities. Condition (H1), however, does not hold in our case because the wells are too large. A new argument is provided in the proof of Proposition [8.6] presented in Section [9]. It relies on the construction of a super-harmonic function on $W_N \setminus D_N$, carried out in Section [10].

**Proof of Theorem 3.2.** Fix $x \in S$, $a > 0$ and $\zeta \in D_N^x$. Clearly,
$$P^N_\eta[\tau_{\mathcal{E}^{x}_N} < a \theta_N] \leq P^N_\eta[\tau_{\mathcal{E}^{x}_N} < a \theta_N, \tau_\zeta < \tau_{\mathcal{E}^{x}_N}] + P^N_\eta[\tau_\zeta > \tau_{\mathcal{E}^{x}_N}].$$
By the strong Markov property, this expression is bounded by
\[ P_N^\eta [\tau_{\mathcal{E}_N^\pi} < a \theta_N] + \sup_{\zeta \in \mathcal{D}_N^\pi} \sup_{\eta \in \mathcal{D}_N^\pi} P_N^\eta [\tau_\zeta > \tau_{\mathcal{E}_N^\pi}]. \]

Multiplying both sides by \( \pi_\zeta(\zeta) \) and summing over \( \zeta \in \mathcal{D}_N^\pi \) yields that
\[ P_N^\eta [\tau_{\mathcal{E}_N^\pi} < a \theta_N] \leq P_{\pi_N^\eta}^\eta [\tau_{\mathcal{E}_N^\pi} < a \theta_N] + \sup_{\zeta \in \mathcal{D}_N^\pi} \sup_{\eta \in \mathcal{D}_N^\pi} P_N^\eta [\tau_\zeta > \tau_{\mathcal{E}_N^\pi}]. \]

At this point the assertion of the theorem follows from Propositions 8.3 and 8.6.

9. Attractor sets in the valleys

The proof of Proposition 8.6 is divided in two steps. We first show that starting from a configuration \( \eta \) in \( \mathcal{E}_N^\pi \), the process hits the set \( \mathcal{D}_N^\pi \) before it leaves the large valley \( \mathcal{W}_N^\pi \). The proof of this result requires the construction of a super-harmonic function on \( \mathcal{W}_N^\pi \setminus \mathcal{E}_N^\pi \), a technical and difficult step presented in the next section. Then, we show that starting from \( \mathcal{D}_N^\pi \), the process visits all configurations of this set before hitting a new well \( \mathcal{E}_N^\pi \).

9.1. Deep valleys are attractors. Next result asserts that starting from \( \mathcal{E}_N^\pi \) the process hits the deep valley \( \mathcal{D}_N^\pi \) before leaving \( \mathcal{W}_N^\pi \).

Proposition 9.1. For all \( x \in S \),
\[ \lim_{N \to \infty} \inf_{\eta \in \mathcal{E}_N^\pi} P_N^\eta [\tau_{\mathcal{D}_N^\pi} < \tau_{\mathcal{W}_N^\pi}] = 1. \]

The proof of this proposition is based on the existence of a super-harmonic function in \( \mathcal{W}_N^\pi \setminus \mathcal{D}_N^\pi \), presented in the next section.

Theorem 9.2. Fix \( x \in S \). There exist positive, finite constants \( c_1, c_2 \) and a function \( G_N^x : \mathcal{H}_N \to \mathbb{R} \) such that,
\[ (\mathcal{L}_NG_N^x)(\eta) \leq 0, \quad c_1 (N - \eta_x) \leq G_N^x(\eta) \leq c_2 (N - \eta_x) \]
for all \( \eta \in \mathcal{W}_N^\pi \setminus \mathcal{D}_N^\pi \) and large enough \( N \).

Proof of Proposition 9.1. Fix \( x \in S \). Since the result holds trivially for \( \eta \in \mathcal{D}_N^\pi \), assume that \( \eta \in \mathcal{E}_N^\pi \setminus \mathcal{D}_N^\pi \).

Let \( G_N^x \) be the function introduced in Theorem 9.2 and let \( \tau = \tau_{\mathcal{W}_N^\pi \setminus \mathcal{D}_N^\pi} \). For every \( t > 0 \),
\[ E_\eta^N \left[ G_N^x(\eta^N(\tau \wedge t)) - G_N^x(\eta) - \int_0^{\tau \wedge t} (\mathcal{L}_NG_N^x)(\eta^N(s)) \, ds \right] = 0. \]

By Theorem 9.2 and since \( N - \eta_x \leq \ell_N \) for \( \eta \in \mathcal{E}_N^\pi \),
\[ E_\eta^N \left[ G_N^x(\eta^N(\tau \wedge t)) \right] \leq G_N^x(\eta) \leq c_2 \ell_N. \]

Letting \( t \to \infty \) and since the hitting time \( \tau \) is finite almost surely, by Fatou’s lemma,
\[ E_\eta^N \left[ G_N^x(\eta^N(\tau)) \right] \leq c_2 \ell_N. \tag{9.1} \]

To obtain a lower bound for this expectation, let
\[ p_N(\eta) = P_\eta^N [\tau_{\mathcal{W}_N^\pi \setminus \mathcal{D}_N^\pi}], \quad \eta \in \mathcal{E}_N^\pi \setminus \mathcal{D}_N^\pi. \]

By definition of the wells \( \mathcal{D}_N^\pi \), \( \mathcal{W}_N^\pi \) and the lower bound of \( G_N^x \),
\[ E_\eta^N \left[ G_N^x(\eta^N(\tau)) \right] \geq p_N(\eta) \frac{N}{(\log N)^\beta} + [1 - p_N(\eta)]c_1 N^\gamma. \tag{9.2} \]
Therefore, by (9.1) and (9.2),
\[ p_N(\eta) \leq \frac{c_2 \ell_N - c_1 N^\gamma}{c_1 N/\log N - c_1 N^\gamma}, \quad \eta \in \mathcal{E}_N^c \setminus \mathcal{D}_N^c. \]
Since \( \ell_N = N/\log N \) and \( 0 < \beta, \gamma < 1 \),
\[ \lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N^c \setminus \mathcal{D}_N^c} p_N(\eta) = 0, \]
as claimed.

9.2. Visiting points in deep valleys. The main result of this section, Proposition 9.4, asserts that starting from a deep valley \( \mathcal{D}_N^c \) the process visits all configurations in \( \mathcal{D}_N^c \) before hitting a new well \( \mathcal{E}_N^c \). This result is a weak version of Proposition 8.6 as it requires the process to start from \( \mathcal{D}_N^c \) instead of \( \mathcal{E}_N^c \).

The proof of Proposition 9.4 is based on a classical bound of equilibrium potentials in terms of capacities. We first provide a lower bound on the capacities between configurations in \( \mathcal{D}_N^c \).

**Lemma 9.3.** Fix \( x \in S \). There exists a positive constant \( c_0 \) such that for all \( x, \eta \in \mathcal{D}_N^c \) and \( N \geq 1 \),
\[ \text{cap}_N(\xi, \eta) \geq \frac{c_0}{N^{\gamma(\log N)^{\kappa-1}}}. \]

**Proof.** Fix \( \xi, \eta \in \mathcal{D}_N^c \). Consider a sequence \( \xi = \zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(p)} = \eta \) in \( \mathcal{D}_N^c \) such that \( \zeta^{(k+1)} = \sigma_{x_k, y_k} \zeta^{(k)} \) for some \( x_k, y_k \in S \) satisfying \( r(x_k, y_k) > 0 \). Since there are at most \( N^\gamma \) particles on \( S \setminus \{x\} \), there exists such a sequence with length bounded by \( C_0 N^\gamma \):
\[ p \leq C_0 N^\gamma \]
for some finite constant \( C_0 \).

Let \( F : \mathcal{H}_N \to \mathbb{R} \) be a function such that \( F(\xi) = 0 \) and \( F(\eta) = 1 \). By Cauchy-Schwarz inequality, there exists a finite constant \( C_0 \) such that
\[ 1 = \left\{ \sum_{k=0}^{p-1} \left[ F(\zeta^{(k+1)}) - F(\zeta^{(k)}) \right] \right\}^2 \leq C_0 D_N(F) \sum_{k=0}^{p-1} \frac{1}{\mu_N(\zeta^{(k)})}. \]
Thus, by the Dirichlet principle,
\[ \text{cap}_N(\xi, \eta) \geq c_0 \left( \sum_{k=0}^{p-1} \frac{1}{\mu_N(\zeta^{(k)})} \right)^{-1}. \]

By definition of the set \( \mathcal{D}_N^c \), \( a(\zeta) \leq N (N^\gamma)^{\kappa-1} = N^{1+(\kappa-1)\gamma} \) for \( \zeta \in \mathcal{D}_N^c \). Hence, by the explicit formula for the invariant measure and Proposition 4.1, there exists a positive constant \( c_0 \) such that
\[ \mu_N(\zeta) \geq c_0 \frac{N}{(\log N)^{\kappa-1}} \frac{1}{N^{1+(\kappa-1)\gamma}} = \frac{1}{N^{\gamma(\log N)^{\kappa-1}}}. \]
To complete the proof, it remains to put together all previous estimates.

The bound produced by this argument in the case where \( \xi \) belongs to \( \mathcal{D}_N^c \) and \( \eta \) to \( \mathcal{E}_N^c \) is too crude to prove Proposition 9.4, below with \( \eta \in \mathcal{E}_N^c \), instead of \( \eta \in \mathcal{D}_N^c \).

**Proposition 9.4.** For all \( x \in S \),
\[ \lim_{N \to \infty} \inf_{\eta \in \mathcal{D}_N^c} \inf_{\xi \in \mathcal{D}_N^c} P_N^\tau \left[ \tau_\xi < \tau_{\mathcal{E}_N^c} \right] = 1. \]
Proof. By [20] equation (3.3)] and the monotonicity of the capacity,
\[ P_N[\tau_\zeta > \tau_{\bar{\epsilon}_N}] \leq \frac{\text{cap}_N(\eta, \bar{\epsilon}_N)}{\text{cap}_N(\eta, \zeta)} \leq \frac{\text{cap}_N(\bar{\epsilon}_N, \bar{\epsilon}_N)}{\text{cap}_N(\eta, \zeta)}. \]
Thus, by Proposition 8.1 and Lemma 9.3,
\[ P_N[\tau_\zeta > \tau_{\bar{\epsilon}_N}] \leq C_0 \frac{N^{\gamma \kappa} (\log N)^{\kappa - 1}}{N^2 \log N}. \]
Since, by hypothesis, \( \gamma < 1/\kappa \), this expression vanishes as \( N \to \infty \), as claimed. \( \square \)

Proof of Proposition 8.6 Fix \( x \in S, \eta \in \mathcal{E}_N^x, \zeta \in \mathcal{D}_N^x \). By the strong Markov property,
\[ P_N[\tau_\zeta < \tau_{\bar{\epsilon}_N}] \geq P_N[\tau_{\mathcal{D}_N^x} < \tau_{\bar{\epsilon}_N}, \tau_\zeta < \tau_{\bar{\epsilon}_N}] \geq P_N[\tau_{\mathcal{D}_N^x} < \tau_{\bar{\epsilon}_N}] \inf_{\xi \in \mathcal{D}_N^x} P_N[\tau_\zeta < \tau_{\bar{\epsilon}_N}] \]
Optimizing over \( \eta \in \mathcal{E}_N^x \) yields that
\[ \inf_{\eta \in \mathcal{E}_N^x} P_N[\tau_\zeta < \tau_{\bar{\epsilon}_N}] \geq \inf_{\eta \in \mathcal{E}_N^x} P_N[\tau_{\mathcal{D}_N^x} < \tau_{(W_N)^c}] \inf_{\xi \in \mathcal{D}_N^x} P_N[\tau_\zeta < \tau_{\bar{\epsilon}_N}] \]
because \( \tau_{(W_N)^c} < \tau_{\bar{\epsilon}_N} \). To complete the proof, it remains to recall the statements of Propositions 9.1 and 9.3 \( \square \)

10. A SUPER-HARMONIC FUNCTION

In this chapter, we prove Theorem 9.2. The super-harmonic function \( G_N^x \) is introduced in Section 10.4. We explain below the ideas behind its construction. To propose candidates, one interprets the zero-range process as a random walk on the simplex \( \Sigma_N = \{ k = (k_x : x \in S) \in \mathbb{N}^S : k_x \geq 0, \sum_{x \in S} k_x = N \} \).

Fix \( x_0 \in S \), and denote by \( \Sigma_N^{x_0} \) the subset of \( \Sigma_N \) of all configurations such that \( N - \alpha_N \leq k_{x_0} \leq N - \beta_N \), where \( \beta_N \ll \alpha_N \ll N \) are two sequences. This means that all coordinates \( k_x \) are much smaller than \( k_{x_0} \) on the set \( \Sigma_N^{x_0} \).

One wishes to show that \( \sum_{x \neq x_0} k_x \) decreases with time in this set. This is done by constructing an increasing function \( F : N \to \mathbb{R} \) such that \( (L_N F)(\sum_{x \neq x_0} k_x) \leq 0 \) on the set \( \Sigma_N^{x_0} \), where \( L_N \) represents the generator of the random walk.

It is not difficult to find functions which are super-harmonic in the interior of \( \Sigma_N^{x_0} \) [the points \( k \) in this set such that \( k_x > 0 \) for all \( x \)]. Indeed, in the interior, it is clear that \( \sum_{x \neq x_0} k_x \) decreases in time because the rate of a jump from \( x_0 \) to \( x \) is strictly smaller than the rate of a jump from \( x \) to \( x_0 \). The problem occurs at the boundary. The sum \( \sum_{x \neq x_0} k_x \) may increase due to a jump from \( x_0 \) to a site \( x \) such that \( k_x = 0 \), and the reverse jumps are forbidden.

In the diffusive scale the random walk should converge to a diffusion on a continuous simplex. Denote by \( x_0 \) the corner of this simplex which corresponds to the configuration in which all particles sit at site \( x_0 \). One can write down the drift of this diffusion and define a \(|S| - 2\)-dimensional manifold with the property that at any point of this manifold the scalar product of the drift of the diffusion with the normal vector to the manifold [which point towards the corner] is positive. For \( |S| = 3 \) or 4, one can draw pictures of the vector field induced by the drift to create an intuition.
A good choice for this manifold is the one given by

\[ \mathcal{M}_A = \left\{ k \in \Sigma_N^0 : \sum_{x,y \in S \setminus \{x_0\}} a_{x,y} k_x k_y + \sum_{x \neq x_0} b_x k_x = A \right\} \]

for appropriate coefficients. Each value of \( A \) gives a different manifold. The corresponding function should be constant on each manifold and a natural candidate emerges: \( F = F(\sum_{x,y \in S \setminus \{x_0\}} a_{x,y} k_x k_y + \sum_{x \neq x_0} b_x k_x) \).

This is how the function \( F \), introduced in Lemma 10.8, emerges. By Proposition 10.7 and the proof of Proposition 10.14,

\[ (\mathcal{L}_N P^{1/2})(\eta) \leq \frac{1}{P(\eta)^{1/2}} \left\{ -1 + \sum_{x \in S_0} 1(\eta_x = 1) \right\} \]

where \( S_0 = S \setminus \{x_0\} \). Thus, \( P^{1/2} \) is super-harmonic except when there is a coordinate with only one particle.

To modify this function at the boundary, we introduce functions \( P^A, A \subset S_0 \), which, by the second assertion of Proposition 10.7, eliminate the positive part of \( (\mathcal{L}_N P^{1/2})(\eta) \) if the configuration \( \eta \) has two or more particles at the sites in \( A^c \).

More precisely, for any constant \( c > 0 \),

\[ (\mathcal{L}_N (P - P^A + c)^{1/2})(\eta) \leq \frac{-1}{(P - P^A + c)(\eta)^{1/2}} \]

if \( \eta_x \geq 2 \) for all \( x \in A^c \).

Therefore, the functions \( (P - P^A + c)^{1/2} \) are super-harmonic in different regions of the space, and the union of these regions contains the annulus \( \Sigma_N^0 \). We use these functions to define one on \( \Sigma_N^0 \). The problem occurs at the boundary of these regions. This obstacle is circumvented by averaging these functions over the free constant \( c \).

10.1. **Potential theory of underlying random walk.** Fix \( x_0 \in S \), and recall that \( S_0 = S \setminus \{x_0\} \). For a subset \( \mathcal{C} \) of \( \mathcal{H}_N \), let

\[ \text{int} \mathcal{C} = \{ \eta \in \mathcal{C} : \sigma^{x,y} \eta \in \mathcal{C} \text{ for all } x, y \text{ with } r(x, y) > 0 \} , \]

\[ \partial \mathcal{C} = \mathcal{C} \setminus \text{int} \mathcal{C} , \]

\[ \overline{\mathcal{C}} = \{ \eta \in \mathcal{H}_N : \eta \in \mathcal{C} \text{ or } \sigma^{x,y} \eta \in \mathcal{C} \text{ for some } x, y \text{ with } r(x, y) > 0 \} . \]

To prove Theorem 0.2 it suffices to construct a function \( \sigma^{x_0} \) on \( \mathcal{W}_N \setminus \mathcal{D}_N \), satisfying the conditions of the proposition in the set \( \mathcal{W}_N \setminus \mathcal{D}_N \), and to extend it arbitrarily to \( \mathcal{H}_N \).

Let

\[ \mathcal{U}_N^{x_0} = \mathcal{W}_N \setminus \mathcal{D}_N \]

so that \( \text{int} \mathcal{U}_N^{x_0} = \mathcal{W}_N \setminus \mathcal{D}_N \).

Recall the definition of equilibrium potential (2.8) and the one of capacity (2.9) for the underlying random walk.

**Lemma 10.1.** Let \( B \) be a non-empty subset of \( S \) and let \( x, y \in S \setminus B \). Then,

\[ \frac{h_{x,B}(y)}{\text{cap}_X(x, B)} = \frac{h_{y,B}(x)}{\text{cap}_X(y, B)} . \]

**Proof.** Recall that we denote by \( P_x \) the probability on the path space \( D(\mathbb{R}_+, S) \) induced by the random walk \( X(t) \) starting from \( x \), and by \( E_x \) the expectation with respect to \( P_x \).
By [6] Proposition 6.10,

\[
\mathbb{E}_x \left[ \int_0^{\tau_B} \chi_y(X(t)) \, dt \right] = \frac{\langle \chi_y, h \rangle}{\text{cap}_X(x, B)} = \frac{m(y) h(x, y)}{\text{cap}_X(x, B)},
\]

\[
\mathbb{E}_y \left[ \int_0^{\tau_B} \chi_x(X(t)) \, dt \right] = \frac{\langle \chi_x, h \rangle}{\text{cap}_X(y, B)} = \frac{m(x) h(y, x)}{\text{cap}_X(y, B)}.
\]

It remains to show that

\[
m(x) \mathbb{E}_x \left[ \int_0^{\tau_B} \chi_y(X(t)) \, dt \right] = m(y) \mathbb{E}_y \left[ \int_0^{\tau_B} \chi_x(X(t)) \, dt \right].
\]

Denote by \((Y(n))_{n \in \mathbb{N}}\) the embedded, discrete-time Markov chain. Recall that \(Y(n)\) is a \(S\)-valued chain which jumps from \(x\) to \(y\) with probability \(p(x, y) = r(x, y)/\lambda(x)\), where \(\lambda(x) = \sum_{y \in S} r(x, y)\), and that its invariant measure, denoted by \(M\), is given by \(M(x) = m(x) \lambda(x)\).

Let \(\epsilon_k, k \geq 0\), be a sequence of independent, mean-one exponential random variables, independent from the chain \(Y(n)\). Denote by \(\mathbb{E}_x^{\epsilon_k}\) the expectation with respect to the chain \(Y(n)\) starting from \(x \in S\) and the sequence \((\epsilon_n)_{n \in \mathbb{N}}\). With this notation,

\[
\mathbb{E}_x \left[ \int_0^{\tau_B} \chi_y(X(t)) \, dt \right] = \mathbb{E}_x^{\epsilon_k} \left[ \sum_{n=0}^{\tau_B-1} \mathbf{1}(Y(n) = y) \frac{\epsilon_n}{\lambda(Y(n))} \right].
\]

Replacing in the denominator \(Y(n)\) by \(y\), and then integrating over \(\epsilon_k\), yields that the right-hand side is equal to

\[
\frac{1}{\lambda(y)} \mathbb{E}_x^{\epsilon_k} \left[ \sum_{n \geq 0} \mathbf{1}(Y(n) = y, n < \tau_B) \epsilon_n \right] = \frac{1}{\lambda(y)} \sum_{n=0}^\infty \mathbb{E}_x^{\epsilon_k} [Y(n) = y, n < \tau_B].
\]

We are left to show that for all \(n \geq 0\)

\[
M(x) \mathbb{E}_x^{\epsilon_k} [Y(n) = y, n < \tau_B] = M(y) \mathbb{E}_y^{\epsilon_k} [Y(n) = x, n < \tau_B],
\]

which follows from the reversibility of the chain \(Y(n)\) with respect to the stationary measure \(M\).

\[\square\]

Note that we did not use in this proof the fact that the stationary measure \(m\) of the random walk \(X\) is the uniform measure. This result holds for general reversible dynamics, and a version for non-reversible ones can be obtained along the same lines.

We conclude this section with an identity used many times in this article. Let \(A, B\) be two non-empty, disjoint subsets of \(S\). Since \(L_X h_{A,B} = -L_X h_{B,A}\), by the last displayed equation in the proof of [19] Lemma B9,

\[
\text{cap}_S(A, B) = -\sum_{x \in A} m(x) (L_X h_{A,B})(x) = \sum_{x \in A} m(x) (L_X h_{B,A})(x).
\]

(10.1)

10.2. Coefficients of a quadratic function. The super-harmonic function is, essentially, the square root of a quadratic function. We introduce in this section the coefficients of this quadratic function.

For each non-empty subset \(A\) of \(S_0\), define the coefficients \((b^A_{x,y})_{x,y \in S}\) by

\[
b^A_{x,y} = \frac{1}{\kappa} \frac{h_{x,A}}{\text{cap}_X(x, A)}, \quad x, y \in A,
\]

(10.2)
and let \( b_{x,y}^A = 0 \) otherwise.

**Lemma 10.2.** For each non-empty subset \( A \) of \( S_0 \) and for all \( x, y \in S \), \( b_{x,y}^A = b_{y,x}^A \).

**Proof.** For \( x, y \in A \) this identity follows from Lemma 10.1. If either \( x \notin A \) or \( y \notin A \) (or both), \( b_{x,y}^A = b_{y,x}^A = 0 \) by definition. \( \square \)

We present below some properties of this sequence.

**Lemma 10.3.** For two non-empty subsets \( A, B \) of \( S_0 \) satisfying \( A \subset B \), \( b_{x,y}^A \leq b_{x,y}^B \) for all \( x, y \in S \).

**Proof.** As the coefficients are non-negative, it is enough to check the inequality for \( x, y \in A \). In this case, since the measure \( m \) is the uniform measure, by [6, Proposition 6.10],

\[
\begin{align*}
    b_{x,y}^A &= \frac{m(y) h_{x,A^c}(y)}{\cap P_X(x,A^c)} = \mathbb{E}_x \left[ \int_0^{\tau_{A^c}} \chi_{\{y\}}(X(t)) \, dt \right], \\
    b_{x,y}^B &= \frac{m(y) h_{x,B^c}(y)}{\cap P_X(x,B^c)} = \mathbb{E}_x \left[ \int_0^{\tau_{B^c}} \chi_{\{y\}}(X(t)) \, dt \right].
\end{align*}
\]

The first expectation is bounded by the second since \( \tau_{A^c} \leq \tau_{B^c} \). \( \square \)

For a non-empty subset \( A \) of \( S_0 \), let \( z_x^A \), \( x \in S \), be given by

\[
z_x^A = \frac{1}{2} \sum_{y \in S} r(x, y) \left[ b_{x,x}^A + b_{y,y}^A - 2 b_{x,y}^A \right]. \tag{10.3}
\]

**Lemma 10.4.** For each non-empty \( A \subset S_0 \), we have that

\[
\frac{1}{2} \sum_{y \in S} r(x, y) (b_{y,y}^A - b_{x,x}^A) = \begin{cases} 
    z_x^A - 1 & \text{for } x \in A, \\
    z_x^A & \text{for } x \in A^c.
\end{cases}
\]

**Proof.** If \( x \in A^c \) the result follows because \( b_{x,y}^A = 0 \) for all \( y \in S \). On the other hand, if \( x \in A \), by (10.1)

\[
\sum_{y \in S} r(x, y) (b_{x,x}^A - b_{x,y}^A) = \frac{m(x)}{\cap P_X(x,A^c)} \sum_{y \in S} r(x, y) (h_{x,A^c}(x) - h_{x,A^c}(y))
\]

\[
= - \frac{m(x)(L_{X} h_{x,A^c})(x)}{\cap P_X(x,A^c)} = 1.
\]

Hence,

\[
z_x^A - 1 = \frac{1}{2} \sum_{y \in S} r(x, y) \left[ b_{x,x}^A + b_{y,y}^A - 2 b_{x,y}^A \right] - \sum_{y \in S} r(x, y) (b_{x,x}^A - b_{x,y}^A)
\]

\[
= \frac{1}{2} \sum_{y \in S} r(x, y) [b_{y,y}^A - b_{x,x}^A],
\]

as claimed. \( \square \)
10.3. Linear and quadratic functions. Define the quadratic function $Q^A : \mathcal{U}^\geq_\mathbb{N} \to \mathbb{R}$, $A \subset S_0$, and the linear function $U^A : \mathcal{U}^\geq_\mathbb{N} \to \mathbb{R}$ as

$$Q^A(\eta) = \frac{1}{2} \sum_{x, y \in S} b_{x, y} \eta_x \eta_y = \frac{1}{2} \sum_{x \in A} b^A_{x, x} \eta_x^2 + \sum_{\{x, y\} \subset A} b_{x, y} \eta_x \eta_y,$$

$$U^A(\eta) = \frac{1}{2} \sum_{x \in S} b^A_{x, x} \eta_x = \frac{1}{2} \sum_{x \in A} b^A_{x, x} \eta_x.$$

In the last sum of the first line, each pair $\{x, y\}$ appears only once. Let

$$P^A(\eta) = Q^A(\eta) - U^A(\eta) = \frac{1}{2} \sum_{x \in A} b^A_{x, x} \eta_x - 1 + \sum_{\{x, y\} \subset A} b_{x, y} \eta_x \eta_y.$$ 

Note that $P^\geq_\mathbb{N}(\eta) = 0$ for all $\eta$.

Fix $A \subset S_0$, $x \in S$ and $\eta \in \text{int} \mathcal{U}^\geq_\mathbb{N}$ such that $\eta_x \geq 1$. An elementary computation yields that

$$U^A(\sigma^x \cdot y \eta) - U^A(\eta) = \frac{1}{2} \left\{ b^A_{x, y} \mathbf{1}\{y \in A\} - b^A_{x, x} \mathbf{1}\{x \in A\} \right\}.$$ 

(10.4)

**Lemma 10.5.** For $x, y \in S \setminus \{x\}$, and $\eta \in \text{int} \mathcal{U}^\geq_\mathbb{N}$,

$$Q^A(\sigma^x \cdot y \eta) - Q^A(\eta) = \sum_{z \in A} \eta_z \left[ b^A_{z, y} - b^A_{z, x} \right] + \frac{1}{2} \left[ b^A_{x, x} + b^A_{y, y} - 2 b^A_{x, y} \right].$$ 

**Proof.** First, fix $y \in A$, $y \neq x$. By definition of $Q_A$, $Q^A(\sigma^x \cdot y \eta) - Q^A(\eta)$ is equal to

$$\frac{1}{2} b^A_{x, x} \left[ (\eta_x - 1)^2 - \eta_x^2 \right] + \frac{1}{2} b^A_{y, y} \left[ (\eta_y + 1)^2 - \eta_y^2 \right]$$

$$+ b^A_{x, y} \left[ (\eta_x - 1)(\eta_y + 1) - \eta_x \eta_y \right] + \sum_{z \in A \setminus \{x, y\}} b^A_{z, z} \left[ (\eta_x - 1) \eta_z - \eta_x \eta_z \right]$$

$$+ \sum_{z \in A \setminus \{x, y\}} b^A_{z, y} \eta_z \left( \eta_y + 1 \right) - \eta_y \eta_z \right].$$

We may rewrite this sum as

$$\frac{1}{2} b^A_{x, x} \left[ 1 - 2 \eta_x \right] + \frac{1}{2} b^A_{y, y} \left[ 2 \eta_y + 1 \right] + b^A_{x, y} \left[ \eta_x - \eta_y - 1 \right]$$

$$+ \sum_{z \in A \setminus \{x, y\}} \eta_z \left[ b^A_{z, y} - b^A_{z, x} \right].$$

Since $b^A_{z, w}$ is symmetric by Lemma 10.2 if $y \in A$,

$$Q^A(\sigma^x \cdot y \eta) - Q^A(\eta) = \sum_{z \in A} \eta_z \left[ b^A_{z, y} - b^A_{z, x} \right] + \frac{1}{2} \left[ b^A_{x, x} + b^A_{y, y} - 2 b^A_{x, y} \right],$$

as claimed.

Assume now that $y$ belongs to $A^c$. In this case, by definition of $Q_A$, $Q^A(\sigma^x \cdot y \eta) - Q^A(\eta)$ is equal to

$$\frac{1}{2} b^A_{x, x} \left[ (\eta_x - 1)^2 - \eta_x^2 \right] + \sum_{z \in A \setminus \{x\}} b^A_{z, z} \left[ (\eta_x - 1) \eta_z - \eta_x \eta_z \right] = - \sum_{z \in A} b^A_{z, x} \eta_z + \frac{1}{2} b^A_{x, x}.$$

To complete the proof, it remains to recall that $b_{z, w} = 0$ for all $z \in S$. ∎
Fix $A \subset S_0$, $x \notin A$, $y \neq x$ and $\eta \in \text{int} \mathcal{U}_N^0$ such that $\eta_x \geq 1$. A similar computation yields that

$$Q^A(\sigma^x, y \eta) - Q^A(\eta) = \left\{ \frac{1}{2} (2\eta_y + 1) b_{y, y}^A + \sum_{z \in A \setminus \{y\}} b_{y, z}^A \eta_z \right\} 1\{ y \in A \} .$$

It follows from (10.4), Lemma 10.5 and the previous estimate that there exists a constant $C_0$ such that

$$|P^A(\sigma^x, y \eta) - P^A(\eta)| \leq C_0 \left\{ 1 + \sum_{z \in A} \eta_z \right\}$$

for all subsets $A$ of $S_0$, $x, y \in S$, $y \neq x$ and $\eta \in \text{int} \mathcal{U}_N^0$ such that $\eta_x \geq 1$.

Let $u_{x, y}^A, x \in A, y \in A^c$, be given by

$$u_{x, y}^A = \frac{m(x) (L_x h_{x, A^c})(y)}{\text{cap} X (x, A^c)}. \quad (10.6)$$

Since $m(x) = m(y)$ and $L_x h_{x, A^c} = -L_x h_{A^c, x}$, by (10.1),

$$\sum_{y \in A^c} u_{x, y}^A = 1, \quad \text{for all } x \in A. \quad (10.7)$$

Observe this identity holds only because $m$ is the uniform measure, as we replaced $m(x)$ by $m(y)$.

**Lemma 10.6.** Fix $A \subset S_0$ and $\eta \in \text{int} \mathcal{U}_N^0$. If $x \in A$ and $\eta_x \geq 1$, then

$$\sum_{y \in S} r(x, y) \left[ P^A(\sigma^x, y \eta) - P^A(\eta) \right] = -\eta_x + 1.$$ 

On the other hand, if $x \in A^c$ and $\eta_x \geq 1$, then

$$\sum_{y \in S} r(x, y) \left[ P^A(\sigma^x, y \eta) - P^A(\eta) \right] = \sum_{z \in A} u_{x, z}^A \eta_z .$$

In particular, for $A = S_0$,

$$\sum_{y \in S} r(x_0, y) \left[ P^{S_0}(\sigma^{x_0}, y \eta) - P^{S_0}(\eta) \right] = \sum_{z \in S_0} \eta_z .$$

**Proof.** We consider $Q^A$ and $U^A$ separately. By Lemma 10.5

$$\sum_{y \in S} r(x, y) \left[ Q^A(\sigma^x, y \eta) - Q^A(\eta) \right] = \sum_{z \in A} \eta_z \left[ b_{z, y}^A - b_{x, z}^A \right] + \frac{1}{2} \left[ b_{z, x}^A + b_{x, y}^A - 2b_{z, y}^A \right].$$

By definition of $b_{x, y}^A$ and $z_{x}^A$ given in (10.2) and (10.3), respectively, and by changing the order of summation yield that the previous expression is equal to

$$\sum_{z \in A} \eta_z \frac{m(z)}{\text{cap} X (z, A^c)} \sum_{y \in S} r(x, y) \left[ h_{z, A^c}(y) - h_{x, A^c}(y) \right] + z_{x}^A .$$

Thus, by definition of $u_{x, y}^A$, introduced in (10.6), we have that

$$\sum_{y \in S} r(x, y) \left[ Q^A(\sigma^x, y \eta) - Q^A(\eta) \right] = \sum_{z \in A} \eta_z u_{x, z}^A + z_{x}^A . \quad (10.8)$$
On the other hand, by definition of \( U^A \), and since \( \eta_x \geq 1 \),
\[
\sum_{y \in S} r(x, y) \left[ U^A(\sigma^x, y \eta) - U^A(\eta) \right] = \frac{1}{2} \sum_{y \in S} r(x, y) \left[ b_{y, x}^A - b_{x, x}^A \right].
\]

Assume that \( x \in A \) and \( \eta_x \geq 1 \). In this case, by definition of \( u_{z, x}^A \) and (10.1),
\[
u_{z, x}^A = 0 \quad \text{for} \quad z \in A \setminus \{ x \} \quad \text{and} \quad u_{x, x}^A = -1.
\]
Therefore, in this case,
\[
\sum_{y \in S} r(x, y) \left[ Q^A(\sigma^x, y \eta) - Q^A(\eta) \right] = -\eta_x + z_x^A.
\]
Moreover, by Lemma 10.4,
\[
\sum_{y \in S} r(x, y) \left[ U^A(\sigma^x, y \eta) - U^A(\eta) \right] = z_x^A - 1.
\]
This completes the first part of the proof, in view of the definition of \( P^A \).
Assume that \( x \in A^c \) and \( \eta_x \geq 1 \). In this case, by Lemma 10.4,
\[
\sum_{y \in S} r(x, y) \left[ U^A(\sigma^x, y \eta) - U^A(\eta) \right] = z_x^A.
\]
This identity together with 10.8 completes the proof of the second assertion of the lemma.

For the last assertion of the lemma, we have to check that \( u_{z, x_0}^{S_0} = 1 \) for all \( z \in S_0 \). By (10.6), and since \( h_{z, x_0} = 1 - h_{x_0, z} \) and \( m(z) = m(x_0) \),
\[
u_{z, x_0}^{S_0} = \frac{m(z) (L_X h_{z, x_0}(x_0))}{\operatorname{cap}_X(z, x_0)} = -\frac{m(x_0) (L_X h_{x_0, z}(x_0))}{\operatorname{cap}_X(z, x_0)}.
\]
By (10.1), this expression is equal to 1, which completes the proof of the lemma.

The next result is a consequence of Lemma 10.6. Fix a function \( J : \mathcal{U}_N^{S_0} \to \mathbb{R} \). In the remaining part of the current section, we write \( J(\eta) = o_N(1) \) if
\[
\limsup_{N \to \infty} \sup_{\eta \in \mathcal{U}_N^{S_0}} |J(\eta)| = 0.
\]

**Proposition 10.7.** Fix a non-empty subset \( A \) of \( S_0 \) and \( \eta \in \mathsf{int} \mathcal{U}_N^{S_0} \). Then,
\[
(\mathcal{L}_N P^A)(\eta) = \sum_{x \in A} g(\eta_x) \left[ 1 - \eta_x \right] + \sum_{x \in \mathcal{A}^c} g(\eta_x) \sum_{z \in A} u_{z, x}^A \eta_z.
\]
If \( \eta_x \geq 2 \) for all \( x \in A^c \), then
\[
(\mathcal{L}_N P^A)(\eta) \geq \sum_{x \in A} 1 \{ \eta_x = 1 \}.
\]
Finally, if \( A = S_0 \),
\[
(\mathcal{L}_N P^{S_0})(\eta) = \sum_{x \in S_0} 1 \{ \eta_x = 1 \} + o_N(1).
\]

**Proof.** The first assertion is a consequence of Lemma 10.6. For the second one, since \( \eta_x \geq 2 \) for all \( x \in A^c \), we obtain from the first part that
\[
(\mathcal{L}_N P^A)(\eta) \geq \sum_{x \in A} g(\eta_x) \left[ 1 - \eta_x \right] + \sum_{x \in \mathcal{A}^c} \sum_{z \in A} u_{z, x}^A \eta_z \quad \text{(10.9)}
\]
because \( g(\eta_x) \geq 1 \) for \( x \in A' \). By \((10.7)\), the second term on the right-hand side is equal to \( \sum_{z \in A} \eta_z \), so that

\[
(L_N P^A)(\eta) \geq \sum_{x \in A} \{ \eta_x - g(\eta_x) [\eta_x - 1] \} = \sum_{x \in A} 1 \{ \eta_x = 1 \},
\]

because \( n - g(n)(n-1) = 1 \{ n = 1 \} \). This proves the second assertion of the proposition.

We turn to the last claim. By the first assertion of this proposition and the last one of Lemma 10.6,

\[
(L_N P^S_0)(\eta) = \sum_{x \in S_0} g(\eta_x) [1 - \eta_x] + g(\eta_{x_0}) \sum_{z \in S_0} \eta_z.
\]

As \( \eta \) belongs to \( \text{int} U_{x_0}^N \), \( \sum_{x \in S_0} \eta_x/|\eta_{x_0} - 1| = o_N(1) \). Hence, writing \( g(\eta_{x_0}) \) as

\[
1 + [\eta_{x_0} - 1] - 1,
\]

the previous identity becomes

\[
(L_N P^S_0)(\eta) = \sum_{x \in S_0} \{ \eta_x - g(\eta_x) [\eta_x - 1] \} + o_N(1).
\]

To complete the argument, it remains to recall that \( n - g(n)(n-1) = 1 \{ n = 1 \} \). \( \square \)

Let us write \( P = P^S_0 \).

**Lemma 10.8.** There exist constants \( c_1, c_2 > 0 \) such that

\[
c_1 \left( \sum_{x \in S_0} \eta_x \right)^2 \leq P(\eta) \leq c_2 \left( \sum_{x \in S_0} \eta_x \right)^2
\]

for all \( \eta \in U_{x_0}^N \).

**Proof.** The upper bound follows from the definition of \( P \). To prove the lower bound, note that there exists constants \( 0 < \lambda < A < \infty \) such that \( \lambda < b_{x_0}^{S_0} < A \) for all \( x \in S_0 \). Thus, by definition of \( P \), since \( b_{x_0}^{S_0} \geq 0 \) for all \( x, y \in S \), and by the Cauchy-Schwarz inequality,

\[
P(\eta) \geq \frac{\lambda}{2} \sum_{x \in S_0} \eta_x^2 - \frac{A}{2} \sum_{x \in S_0} \eta_x \geq \frac{\lambda}{2(\kappa - 1)} \left( \sum_{x \in S_0} \eta_x ^2 \right)^2 - \frac{A}{2} \sum_{x \in S_0} \eta_x.
\]

To complete the proof, it remains to recall that \( \sum_{x \in S_0} \eta_x \geq N^\gamma \gg 1 \) for \( \eta \in U_{x_0}^N \). \( \square \)

**Lemma 10.9.** There exists a positive constant \( c_0 > 0 \) such that

\[
\sum_{x \in S} g(\eta_x) \sum_{y \in S} r(x, y) [P(\sigma^{x_0} y_\eta) - P(\eta)]^2 \geq c_0 P(\eta)
\]

for all \( \eta \in \text{int} U_{x_0}^N \).

**Proof.** Since \( g(\eta_{x_0}) \geq 1 \), it suffices to show that

\[
\sum_{y \in S} r(x_0, y) [P(\sigma^{x_0} y_\eta) - P(\eta)]^2 \geq c_0 P(\eta).
\]
By the Cauchy-Schwarz inequality,
\[ \sum_{y \in S} r(x_0, y) \left( \sum_{y \in S} r(x_0, y) \left( P(\sigma^{x_0,y} \eta) - P(\eta) \right) \right)^2 \]
\[ \geq \left( \sum_{y \in S} r(x_0, y) \left( P(\sigma^{x_0,y} \eta) - P(\eta) \right) \right)^2 = \left( \sum_{z \in S_0} \eta_z \right)^2, \]
where the last identity follows from the third assertion of Lemma 10.6. To complete the proof, recall the upper bound of Lemma 10.8. \( \square \)

10.4. A super-harmonic function. Some notations are required. We first claim that for each non-empty subset \( B \) of \( S_0 \), there exist positive constants \( \alpha_B, \beta_B > 0 \) such that
\[ \frac{1}{2} \sum_{x \in B_0} b^B_{x,x} \eta_x (\eta_x - 1) + \sum_{\{x,y\} \in B_0} b^B_{x,y} \eta_x \eta_y < \alpha_B \sum_{x \in B_0} \eta_x (\eta_x - 1) + \beta_B \]
for all \( \emptyset \neq B_0 \subset B \) and for all \( \eta \in \mathcal{U}_N^{S_0} \).

To prove this claim, let \( a = \max \{ b^B_{z,w} \} \), where the maximum is performed over all nonempty subset \( C \) of \( S_0 \), and all \( z, w \in C \). Clearly, there exists a finite constant \( C_0 \), depending only on \( \kappa \), such that the left-hand side of (10.10) is bounded by
\[ C_0 a \sum_{x \in B_0} \eta_x^2 \leq 2C_0 a \sum_{x \in B_0} \eta_x (\eta_x - 1) + C_0 \kappa a, \]
because \( t^2 \leq 2(t - 1) + 1 \). This proves the claim. Clearly, we may assume that
\[ \alpha_B > b^B_{x,x} \quad \text{for all } x \in B. \]

We assign a positive constant \( c_A > 0 \) to each proper, non-empty subset \( A \) of \( S_0 \), i.e., \( \emptyset \subset A \subset S_0 \), as follows.

If \( A \) is a singleton, \( |A| = 1 \), set \( c_A > 0 \) arbitrarily. Fix \( 2 \leq k \leq |S_0| - 1 \), and suppose that \( c_A \) has been assigned to all sets \( A \neq \emptyset \) such that \( |A| < k \). Fix a subset \( B \) of \( S_0 \) such that \( |B| = k \), and let
\[ c^0_B = \max_A \max_{x \in A} \left\{ 2 \alpha_B c_A \frac{b^A_{x,x}}{b^B_{x,x}} + \beta_B \right\}, \]
where the first maximum is performed over all proper, non-empty subsets \( A \) of \( B \). Then, select a constant \( c_B \) larger than \( c^0_B \) and such that
\[ c_A \neq c_B \quad \text{for all } A, B \subset S_0 \text{ with } A \neq B. \]

Fix a positive integer \( \ell \geq 2 \). For each proper, non-empty subset \( A \) of \( S_0 \), let \( P^A_\ell : \mathcal{U}_N^{S_0} \to \mathbb{R} \) be given by
\[ P^A_\ell(\eta) = P^A(\eta) - c_A \ell^2. \]
Clearly,
\[ (\mathcal{L}_N P^A_\ell)(\eta) = (\mathcal{L}_N P^A)(\eta) \quad \text{for all } \eta \in \mathcal{U}_N^{S_0}. \]

Let \( P^{S_0}_\ell(\eta) = 0 \) for all \( \eta \in \mathcal{U}_N^{S_0} \), and define the correction function \( W_\ell : \mathcal{U}_N^{S_0} \to \mathbb{R} \) by
\[ W_\ell(\eta) = \min \{ P^A_\ell(\eta) : A \subset S_0, A \neq S_0 \}. \]

**Lemma 10.10.** There exists a constant \( C_0 < \infty \) such that, for all \( \eta \in \mathcal{U}_N^{S_0} \),
\[ -C_0 \ell^2 \leq W_\ell(\eta) \leq 0. \]
Proof. Since $W_t(\eta) \leq P_\ell^\infty(\eta) \leq 0$, the upper bound is clear. We turn to the lower bound. Since $P^A$, $A \subset S_0$, is a non-negative function, $P_\ell^A(\eta) \geq -c_A \ell^2$. It remains to set $C_0$ as $\max_{A \subset S_0} c_A$.

By Lemmata 10.8 and 10.10, we get $P(\eta) - W_t(\eta) > 0$ for all $\eta \in U_N^\infty$. Let $F_m : U_N^\infty \to \mathbb{R}$, $m \geq 2$, be defined by

$$F_m(\eta) = \sum_{\ell=2}^m \frac{1}{\ell} [P(\eta) - W_\ell(\eta)]^{1/2}.$$  

**Theorem 10.11.** There exists $m \in \mathbb{N}$ and $N_0 \geq 1$, such that for all $N \geq N_0$, $F_m$ is super-harmonic on int $U_N^\infty$.

The proof of this theorem is given in Section 10.8. This result, as well as the majority of the next ones, are asymptotic in $N$. This means that they may fail for small $N$, but that there exists a constant $N_0$, which may depend only on $\kappa$ and $\ell$, such that the assertion holds for $N > N_0$.

**Proof of Theorem 9.2.** Let

$$G_N^\infty(\eta) = \begin{cases} F_m(\eta) & \eta \in U_N^\infty, \\ 0 & \text{otherwise}. \end{cases}$$

The first requirement follows from Theorem 10.11 while the second one follows from Lemmata 10.8 and 10.10.

10.5. The corrector $W_t$. Let $D_\ell(A)$, $A \subset S_0$, be the set given by

$$D_\ell(A) = \{ \eta \in U_N^\infty : P_\ell^A(\eta) = W_\ell(\eta) \}.$$  

(10.15)

Note that some configurations may belong to several $D_\ell(A)$’s.

Let $A_0$ be a proper, non-empty subset of $S_0$ and let $\eta$ be a configuration in $U_N^\infty$ such that $\eta_x = 0$ for all $x \in A_0$. Next lemma states that $W_\ell(\eta) = P_\ell^A(\eta)$ for some set $A$ which contains $A_0$.

**Lemma 10.12.** Fix a proper, non-empty subset $A_0$ of $S_0$ and $\eta$ in $U_N^\infty$ such that $\eta_x = 0$ for all $x \in A_0$. Suppose that

$$P_\ell^A(\eta) = W_\ell(\eta)$$

for some $A \subset S_0$. Then, $A \supset A_0$ provided $N$ is large enough.

**Proof.** Fix $A \subset S_0$, and assume that

$$P_\ell^A(\eta) = W_\ell(\eta) = \min \{ P'_\ell^B(\eta) : B \subset S_0 \}.$$  

In particular, since $\eta_x = 0$ for all $x \in A_0$,

$$P_\ell^A(\eta) \leq P'_\ell^{A_0}(\eta) = -c_{A_0} \ell^2 < 0.$$  

(10.16)

We consider separately three cases.

Suppose that $A \subset A_0$. By definition, $c_{A_0} > c_{A_0}^3$. By (10.12) and Lemma 10.3 this constant is larger than $2\alpha_{A_0} e_A / b_{x,x} \geq 2\alpha_{A_0} c_A / b_{x,x}$. By (10.11), we get $2\alpha_{A_0} c_A / b_{x,x} \geq 2e_A \geq c_A$. This proves that $c_{A_0} > c_A$. Thus, as $\eta_x = 0$ for all $x \in A_0$,

$$P_\ell^A(\eta) = -c_A \ell^2 > -c_{A_0} \ell^2 = P'_\ell^{A_0}(\eta),$$

in contradiction with (10.16).
Assume that $A_0 \not\subset A$, $A \not\subset A_0$ and $A_0 \cup A \neq S_0$. We claim that
\[ P_{A_0 \cup \{A\}}^{A_0 \cup \{A\}}(\eta) < P_{\{A\}}^{A}(\eta). \] (10.17)
Since $\eta_x = 0$ for all $x \in A_0$ and since $A_0 \cup A \neq S_0$,
\[ P_{\{A\}}^{A}(\eta) = \frac{1}{2} \sum_{x \in A \setminus A_0} b_{x,x}^A \eta_x (\eta_x - 1) + \sum_{x, y \in A \setminus A_0} b_{x,y}^A \eta_x \eta_y - c_A \ell^2, \]
\[ P_{A_0 \cup \{A\}}^{A_0 \cup \{A\}}(\eta) = \frac{1}{2} \sum_{x \in A \setminus A_0} b_{x,x}^{A_0 \cup \{A\}} \eta_x (\eta_x - 1) + \sum_{x, y \in A \setminus A_0} b_{x,y}^{A_0 \cup \{A\}} \eta_x \eta_y - c_{A_0 \cup \{A\}} \ell^2. \]
These sums are carried over a set which is not empty because we assumed that $A \not\subset A_0$. Let
\[ M = \max_{x \in A} \frac{\alpha_{A_0 \cup \{A\}}}{b_{x,x}^A}. \] By Lemma 10.3 and \(10.11\), \( M > 1 \).
By \(10.10\) and the explicit formula for $P_{A_0 \cup \{A\}}^{A_0 \cup \{A\}}(\eta)$,
\[ P_{A_0 \cup \{A\}}^{A_0 \cup \{A\}}(\eta) < \alpha_{A_0 \cup \{A\}} \sum_{x \in A \setminus A_0} \eta_x (\eta_x - 1) + \beta_{A_0 \cup \{A\}} - c_{A_0 \cup \{A\}} \ell^2. \]
By definition of $M$, this expression is bounded by
\[ M \sum_{x \in A \setminus A_0} b_{x,x}^A \eta_x (\eta_x - 1) + \beta_{A_0 \cup \{A\}} - c_{A_0 \cup \{A\}} \ell^2. \]
By definition of $P_{\{A\}}^A$, this sum is less than or equal to
\[ 2M P_{\{A\}}^A(\eta) + 2M c_A \ell^2 + \beta_{A_0 \cup \{A\}} - c_{A_0 \cup \{A\}} \ell^2. \]
By definition, $c_{A_0 \cup \{A\}} > c_{A_0 \cup \{A\}}^\ell$. Since $A_0 \not\subset A$, $A \not\subset A_0 \cup A$. Thus, by \(10.12\) and by definition of $M$, $c_{A_0 \cup \{A\}}^\ell \geq 2M c_A + \beta_{A_0 \cup \{A\}}$. Hence, by the previous estimates, and since $\ell \geq 1$,
\[ P_{A_0 \cup \{A\}}^{A_0 \cup \{A\}}(\eta) < 2M P_{\{A\}}^A(\eta) \leq P_{\{A\}}^A(\eta) \]
because $M > 1$ and $P_{\{A\}}^A(\eta) < 0$ by \(10.16\). This proves \(10.17\) and contradicts the fact that $P_{\{A\}}^A(\eta) = W_\ell(\eta)$.
Assume, finally, that $A_0 \cup A = S_0$. Since both are proper subsets of $S_0$, $A_0 \not\subset A$ and $A \not\subset A_0$.
The set $S_0$ can be decomposed into $S_0 = A_0 \cup (A \setminus A_0)$. Since $\eta \in \mathcal{U}_N^0$, and $\eta_x = 0$ for all $x \in A_0$,
\[ \sum_{x \in A \setminus A_0} \eta_x = \sum_{x \in S_0} \eta_x \geq N^{\gamma}. \]
Since $b_{x,x}^A > 0$ for all $x \in A$, a similar computation to the one presented in the proof of Lemma \(10.8\), yields that
\[ \frac{1}{2} \sum_{x \in A \setminus A_0} b_{x,x}^A \eta_x (\eta_x - 1) \geq c_0 N^{2\gamma} \]
for some positive constant $c_0$. Thus,
\[ P_{\{A\}}^A(x) > c_0 N^{2\gamma} - c_A \ell^2 \geq 0 \]
f for large enough $N$, which contradicts \(10.16\).
In conclusion, none of the previous three cases can be in force, so that $A \supset A_0$, as claimed. \( \square \)
Corollary 10.13. Fix a proper, non-empty subset $A$ of $S_0$. Then, for all $x \in S_0 \setminus A$, $\eta \in D_\ell(A)$, we have that $\eta_x \neq 0$.

Proof. Fix a proper, non-empty subset $A$ of $S_0$ and $\eta \in D_\ell(A)$. Let

$$A_0 = \{ x \in S_0 : \eta_x = 0 \} .$$

As $\eta \in D_\ell(A)$, $P_\ell^A(\eta) = W_\ell(\eta)$. Hence, by Lemma 10.12, $A_0 \subset A$, as claimed. \qed

10.6. The set $D_\ell(A)$. The crucial point in the proof of Theorem 10.11 is to estimate $W_\ell$. This is relatively easy in each set int $D_\ell(A)$ because $W_\ell$ is equal to $P_\ell^A$.

In contrast, its behavior at the boundary $\partial D_\ell(A)$ is problematic.

The next result states that $\sum_{x \in A} \eta_x$ can not be too large for configurations $\eta$ in $D_\ell(A)$.

Proposition 10.14. There exists $\gamma_1 > 0$ such that, for all proper, non-empty subset $A$ of $S_0$,

$$D_\ell(A) \subset G_\ell^{\gamma_1}(A) := \{ \eta \in U_N^{x_0} : \eta_x < \gamma_1 \ell \text{ for all } x \in A \} .$$

Proof. Fix $\eta \in D_\ell(A)$. By definition of $D_\ell(A)$,

$$P_\ell^A(\eta) \leq P_\ell^\sigma(\eta) = 0 .$$

On the other hand, by definition of $P_\ell^A$, there exists $\gamma_A > 0$ such that

$$P_\ell^A(\xi) > 0 \text{ if } \xi_x \geq \gamma_A \ell \text{ for some } x \in A .$$

It follows from the two previous remarks that $D_\ell(A) \subset G_\ell^{\gamma_1}(A)$. To complete the proof, it remains to set $\gamma_1 = \max \gamma_A$. \qed

Proposition 10.15. Fix a proper, non-empty subset $A$ of $S_0$ and $\eta \in \text{int } D_\ell(A)$.

Then,

$$(\mathcal{L}_N W_\ell)(\eta) = (\mathcal{L}_N P^A)(\eta) \geq \sum_{x \in S_0} 1\{\eta_x = 1\} .$$

Proof. Fix $\eta \in \text{int } D_\ell(A)$, so that

$$W_\ell(\eta) = P_\ell^A(\eta) \text{ and } W_\ell(\sigma^{x,y}\eta) = P_\ell^A(\sigma^{x,y}\eta)$$

for all $x, y$ in $S$ with $r(x, y) > 0$. Thus,

$$(\mathcal{L}_N W_\ell)(\eta) = (\mathcal{L}_N P^A)(\eta) = (\mathcal{L}_N P^A)(\eta) .$$

We turn to the second assertion. By Corollary 10.13, $\eta_x \neq 0$ for all $x \in S_0 \setminus A$.

If $\eta_x = 1$ for some $x \in S_0 \setminus A$, by Corollary 10.13, $\sigma^{x,y}\eta \notin D_\ell(A)$ for any $y \in S$ with $r(x, y) > 0$, so that $\eta \notin \text{int } D_\ell(A)$ as well. Therefore, $\eta_x \geq 2$ for all $x \in S_0 \setminus A$, and the second claim follows from the second assertion of Proposition 10.7. \qed

Lemma 10.16. Fix $x \neq y \in S$, and proper subsets $A, B$ of $S_0$, $A \neq B$. There exists a constant $C_0 > 0$ such that

$$| P_\ell^B(\eta) - P_\ell^A(\eta) | \leq C_0 \ell \text{ and } | P_\ell^B(\sigma^{x,y}\eta) - P_\ell^A(\sigma^{x,y}\eta) | \leq C_0 \ell$$

for all $\eta \in \text{int } U_N^{x_0}$ such that $\eta \in D_\ell(A)$ and $\sigma^{x,y}\eta \in D_\ell(B)$,
Proof. Since the proof for these two estimates are identical, we only focus on the first one. We regard $P^A_t$ and $P^B_t$ as quadratic functions on $\mathbb{R}^{n-1}$ whose restriction to $\mathbb{N}^{n-1}$ is given by \ref{10.14}.

As $\eta$ belongs to $D_t(A)$ and $\sigma^{x,y}\eta$ to $D_t(B)$,
\begin{align*}
P^A_t(\eta) &\le P^B_t(\eta) , \quad P^A_t(\sigma^{x,y}\eta) \ge P^B_t(\sigma^{x,y}\eta) .
\end{align*}
Hence, by the intermediate value theorem, there exists $w_0 \in \mathbb{R}^{n-1}$ belonging to the line segment connecting $\eta$ and $\sigma^{x,y}\eta$ such that
\begin{align*}
(P^A_t - P^B_t)(w_0) &= 0 .
\end{align*}

Since
\begin{align*}
|\eta - w_0| &\le |\eta - \sigma^{x,y}\eta| = \sqrt{2} ,
\end{align*}
by the Taylor expansion, there exists a finite constant $C_0$ such that
\begin{align*}
|P^A_t(\eta) - P^A_t(w_0)| &\le C_0 \{ |\nabla P^A_t(\eta)| + \|\nabla^2 P^A_t\|_{L^\infty(\mathbb{R}^{n-1})} \} .
\end{align*}
As $P^A_t$ is a quadratic function, $\|\nabla^2 P^A_t\|_{L^\infty(\mathbb{R}^{n-1})} \le C_0$. On the other hand, since $\eta$ belongs to $D_t(A)$, by Proposition \ref{10.14}, $\eta \le C_0 \ell$ for all $z \in A$. Hence, there exists a finite constant $C_0$ such that $|\nabla P^A_t(\eta)| < C_0 \ell$, and the previous displayed equation becomes
\begin{align*}
|P^A_t(\eta) - P^A_t(w_0)| &\le C_0 \ell . \tag{10.18}
\end{align*}

To use the same argument to estimate $P^B_t(\eta) - P^B_t(w_0)$ we only need to show that $\eta \le C_0 \ell$ for all $z \in B$. Since $\sigma^{x,y}\eta \in D_t(B)$, by Proposition \ref{10.14}, $(\sigma^{x,y}\eta)_z \le \gamma_1 \ell$ for all $z \in B$. Thus, as $| (\sigma^{x,y}\eta)_z - \eta_z | \le 1$, $\eta \le C_0 \ell + 1$ for all $z \in B$. This proves \ref{10.18} with $A$ replaced by $B$.

Putting together the previous estimates yields that
\begin{align*}
|P^A_t(\eta) - P^B_t(\eta)| &\le |P^A_t(\eta) - P^A_t(w_0)| + |P^B_t(\eta) - P^B_t(w_0)| \le C_0 \ell ,
\end{align*}
as claimed \hfill $\square$

Lemma 10.17. If $\eta$ belongs to $\partial D_t(A)$, there exists a constant $C_0$ such that
\begin{align*}
| (\mathcal{L}_N W_t)(\eta) - (\mathcal{L}_N P^A_t)(\eta) | &\le C_0 \ell .
\end{align*}

Proof. Assume that $\eta$ belongs to $\partial D_t(A)$. It is enough to show that there exists a constant $C_0$ such that for all $\eta \in \partial D_t(A)$ and $x, y \in S$ with $r(x, y) > 0$,
\begin{align*}
|W_t(\sigma^{x,y}\eta) - P^A_t(\sigma^{x,y}\eta)| &\le C_0 \ell .
\end{align*}
This inequality holds clearly when $\sigma^{x,y}\eta \in D_t(A)$. Assume that $\sigma^{x,y}\eta \in D_t(B)$ for some $B \neq A$. Then,
\begin{align*}
|W_t(\sigma^{x,y}\eta) - P^A_t(\sigma^{x,y}\eta)| & = |P^B_t(\sigma^{x,y}\eta) - P^A_t(\sigma^{x,y}\eta)| .
\end{align*}
By Lemma \ref{10.16} this quantity is bounded by $C_0 \ell$. \hfill $\square$

The following proposition is crucial in the proof of Theorem \ref{10.11}. It is here that condition \ref{10.13} plays a role. Let
\begin{align*}
\partial D_t = \bigcup_{A \subseteq S_0} \partial D_t(A) .
\end{align*}
Proposition 10.18. There exists a constant $\gamma_2 > 0$ such that, for all $\eta \in U_N^{x_0}$,

$$\sum_{\ell \geq 2} 1\{\eta \in \partial \mathcal{D}_\ell\} \leq \gamma_2.$$ 

In other words, each configuration $\eta \in U_N^{x_0}$ belongs to a boundary set $\partial \mathcal{D}_\ell(A)$ at most $\gamma_2$ times.

Proof. Fix $\eta \in \partial \mathcal{D}_\ell(A)$, so that there exists $x, y \in S$ with $r(x, y) > 0$ such that $\sigma^x \eta \in \mathcal{D}_\ell(B)$ for some $B \neq A$. By Lemma 10.16, there exists $C_0 > 0$ such that

$$|P^A_\ell(\eta) - P^B_\ell(\eta)| \leq C_0 \ell.$$ 

Therefore, it suffices to prove that there exists a finite constant $C_1$ such that

$$\sum_{\ell=1}^\infty 1\{|P^A_\ell(\eta) - P^B_\ell(\eta)| \leq C_0 \ell\} \leq C_1.$$ 

Recall that

$$P^A_\ell(\eta) - P^B_\ell(\eta) = P^A(\eta) - P^B(\eta) - (c_A - c_B) \ell^2.$$ 

Since $c_A \neq c_B$, the left-hand side of the penultimate displayed equation can be written as

$$\sum_{\ell=1}^\infty 1\{|\ell^2 - \frac{(P^A - P^B)(\eta)}{c_A - c_B}| \leq \frac{C_0 \ell}{|c_A - c_B|}\}.$$ 

By Lemma 10.19 below, this sum is bounded by a constant which only depends on $c_A$, $c_B$ and $C_0$, as claimed. \qed

Lemma 10.19. For $\alpha > 0$ and $t \in \mathbb{R}$, the set

$$A_{\alpha, t} = \{ x \in \mathbb{R} : x^2 - 2\alpha x + t \leq 0 \leq x^2 + 2\alpha x + t \}$$

is either an empty set or a closed intervals of length at most $2\alpha$.

Proof. If $t > \alpha^2$, the inequality $x^2 + 2\alpha x + t < 0$ cannot hold and the set $A_{\alpha, t}$ is empty. We may, therefore, assume that $t \leq \alpha^2$. In this case, let

$$u^\pm = \alpha \pm \sqrt{\alpha^2 - t}, \quad v^\pm = -\alpha \pm \sqrt{\alpha^2 - t},$$

so that

$$A_{\alpha, t} = [u^-, u^+] \setminus (v^-, v^+).$$

This set is a closed sub-interval of $[v^+, u^+]$ and $u^+ - v^+ = 2\alpha$. This completes the proof. \qed

10.7. The function $h_\ell$. Fix $\ell \geq 2$, and let

$$h_\ell(\eta) := P(\eta) - W_\ell(\eta).$$

Next result is the main step in the construction of a super-harmonic function.

Proposition 10.20. There exist positive constants $c_1$, $c_2$ such that

$$(L_N h_\ell^{1/2})(\eta) \leq \frac{1}{P(\eta)^{1/2}} \left\{ - c_1 + c_2 \ell 1\{\eta \in \partial \mathcal{D}_\ell\} \right\}$$

for all $\ell \geq 2$, large enough $N$ and $\eta \in \text{int } U_N^{x_0}$.

To prove this proposition, we first investigate $L_N h_\ell$. 
Lemma 10.21. There exists a finite constant $C_0$ such that

$$
(\mathcal{L}_N h_\ell)(\eta) \leq C_0 \ell \{ \eta \in \partial D_\ell \} + o_N(1)
$$

for all $\eta \in \text{int } U^0_N$.

Proof. Suppose that $\eta \in \text{int } D_\ell(A)$ for some proper subset $A$ of $S_0$. By Proposition 10.15 and the third assertion of Proposition 10.7

$$(\mathcal{L}_N h_\ell)(\eta) = (\mathcal{L}_N P)(\eta) - (\mathcal{L}_N W^\ell)(\eta) \leq o_N(1).$$

Assume that $\eta \in \partial D_\ell(A)$, for some proper subset $A$ of $S_0$. By Lemma 10.17

$$(\mathcal{L}_N h_\ell)(\eta) \leq (\mathcal{L}_N P)(\eta) - (\mathcal{L}_N P^A)(\eta) + C_0 \ell$$

for some finite constant $C_0$. By the first assertion of Proposition 10.7

$$\mathcal{L}_N P^A(\eta) \geq -C_0 \sum_{x \in A} \eta_x$$

for some finite constant $C_0$. By Proposition 10.14 this expression is bounded below by $-C_0 \gamma_1 \ell = -C_0 \ell$. On the other hand, by the third assertion of Proposition 10.7 $(\mathcal{L}_N P)(\eta) \leq \kappa + o_N(1)$. This completes the proof of the proposition. \(\square\)

Next result is an extension of Lemma 10.9.

Lemma 10.22. There exists a positive constant $c_0$ such that

$$
\sum_{x \in S} g(\eta_x) \sum_{y \in S} r(x, y) \left[ h_\ell(\sigma^x, y \eta) - h_\ell(\eta) \right]^2 \geq c_0 P(\eta)
$$

for all $\eta \in (\text{int } U^0_N) \setminus \partial D_\ell$.

Proof. Since $g(\eta_{x_0}) > 0$, it suffices to show that

$$
\sum_{y \in S} r(x_0, y) \left[ h_\ell(\sigma^{x_0, y} \eta) - h_\ell(\eta) \right]^2 \geq c_0 P(\eta).
$$

By the Cauchy-Schwarz inequality, the square of the left-hand side is bounded below by

$$
\left\{ \sum_{y \in S} r(x_0, y) \right\}^{-1} \left\{ \sum_{y \in S} r(x_0, y) \left[ h_\ell(\sigma^{x_0, y} \eta) - h_\ell(\eta) \right] \right\}^2.
$$

Thus, by Lemma 10.8 it is enough to show that

$$
\sum_{y \in S} r(x_0, y) \left[ h_\ell(\sigma^{x_0, y} \eta) - h_\ell(\eta) \right] \geq c_0 \sum_{x \in S_0} \eta_x. \quad (10.19)
$$

Since $\eta \notin \partial D_\ell$, $\eta$ belongs to int $D_\ell(A)$ for some $A \subset S_0$. In particular, $\eta$ and $\sigma^{x_0, y_0} \eta$ belong to $D_\ell(A)$ for all $y$ such that $r(x_0, y) > 0$. The left-hand side of the previous displayed equation is thus equal to

$$
\sum_{y \in S} r(x_0, y) \left[ (P - P^A_{\ell})(\sigma^{x_0, y} \eta) - (P - P^A_{\ell})(\eta) \right].
$$

If $A = \emptyset$, then $P^A_{\ell}(\sigma^{x_0, y} \eta) = P^A_{\ell}(\eta) = 0$. In this case, (10.19) follows from the third assertion of Lemma 10.6. If $A \neq \emptyset$,

$$
P^A_{\ell}(\sigma^{x_0, y} \eta) = P^A_{\ell}(\eta) = P^A(\sigma^{x_0, y} \eta) - P^A(\eta).
$$
By the second and third statements of Lemma \[10.6\] the left-hand side of \[10.19\] is equal to
\[
\sum_{z \in S_0} \eta_z - \sum_{z \in A} u^A_{z, x_0} \eta_z .
\]

On the set $U^c_{N^0}$, $\sum_{z \in S_0} \eta_z \geq N^\gamma$ and, by Proposition \[10.14\] $\eta_z \leq \gamma_1 \ell$ for all $z \in A$. In particular, the previous expression is greater than $(1/2) \sum_{z \in S_0} \eta_z$ for $N$ large enough. This completes the proof. \[\square\]

**Proof of Proposition \[10.20\]** By definition,
\[
(L_N h^{1/2}_\ell)(\eta) = \sum_{x \in S} g(\eta_x) \sum_{y \in S} r(x, y) \left[ h^{1/2}_\ell(\sigma^x \cdot y \eta) - h^{1/2}_\ell(\eta) \right].
\]

By Lemmata \[10.8\] and \[10.10\] there exists a positive constant $c_0$ such that
\[
h_\ell(\eta) \geq P(\eta) \geq c_0 \left( \sum_{z \in S_0} \eta_z \right)^2
\]
for all $\eta \in \text{int} \, U^c_{N^0}$. On the other hand, by definition of $h_\ell$, \[10.5\] [for $A = S_0$ and $A$ a proper subset of $S_0$] and Lemma \[10.16\] there exists a finite constant $C_0$ such that
\[
|h_\ell(\sigma^x \cdot y \eta) - h_\ell(\eta)| \leq C_0 \left\{ \ell + \sum_{z \in S_0} \eta_z \right\}
\]
for all $x, y \in S$, $y \neq x$ and $\eta \in \text{int} \, U^c_{N^0}$ such that $\eta_x \geq 1$. This expression is bounded by $C_0 \sum_{z \in S_0} \eta_z$ for $N$ sufficiently large. Since $\sum_{z \in S_0} \eta_z \geq N^\gamma$ on $\text{int} \, U^c_{N^0}$, it follows from the two previous estimates that there exists a finite constant $C_0$ such that
\[
|h_\ell(\sigma^x \cdot y \eta) - h_\ell(\eta)| \leq C_0 N^{-\gamma}
\]
(10.20)
for all $x, y \in S$, $y \neq x$ and $\eta \in \text{int} \, U^c_{N^0}$ such that $\eta_x \geq 1$.

A second order Taylor expansion and the previous bound yield that $(L_N h^{1/2}_\ell)(\eta)$ is equal to
\[
\frac{(L_N h_\ell)(\eta)}{2 h_\ell(\eta)^{1/2}} - \left[ 1 + c_N \right] \frac{1}{8 h_\ell(\eta)^{3/2}} \sum_{x \in S} g(\eta_x) \sum_{y \in S} r(x, y) \left[ h_\ell(\sigma^x \cdot y \eta) - h_\ell(\eta) \right]^2,
\]
where $c_N$ is bounded by $C_0 N^{-\gamma}$. Hence, by Lemmata \[10.21\] and \[10.22\] there exist a finite constant $C_0$ and a positive constant $c_0$ such that
\[
(L_N h^{1/2}_\ell)(\eta) \leq \frac{C_0 \ell 1\{\eta \in \partial D_\ell\}}{h_\ell(\eta)^{1/2}} + \frac{\sigma_N(1)}{h_\ell(\eta)^{3/2}} - 1\{\eta \notin \partial D_\ell\} \frac{c_0 P(\eta)}{h_\ell(\eta)^{1/2}}.
\]
Write $1\{\eta \notin \partial D_\ell\}$ as $1 - 1\{\eta \in \partial D_\ell\}$. Since $h_\ell(\eta) \geq P(\eta)$ and, by Lemma \[10.10\] $P(\eta) \geq (1/2) h_\ell(\eta)$.
\[
(L_N h^{1/2}_\ell)(\eta) \leq \frac{C_0 \ell 1\{\eta \in \partial D_\ell\}}{P(\eta)^{1/2}} - \frac{c_0}{P(\eta)^{1/2}},
\]
as claimed. \[\square\]
10.8. **Proof of Theorem 10.11**

**Proof of Theorem 10.11** The function $F_m$ can be written as

$$F_m(\eta) = \sum_{\ell=2}^{m} \frac{1}{\ell} h_\ell(\eta)^{1/2}.$$ 

By Proposition 10.20

$$P(\eta)^{1/2} (L_N F_m)(\eta) \leq -c_0 \sum_{\ell=2}^{m} \frac{1}{\ell} + C_0 \sum_{\ell=2}^{m} \mathbf{1}\{\eta \in \partial D_\ell\}.$$ 

By Proposition 10.18 this expression is bounded by

$$-c_0 \log m + C_0 \gamma_2.$$ 

Thus, taking $m$ large enough yields that $P(\eta)^{1/2} (L_N F_m)(\eta) < 0$ for all $\eta \in \text{int } U_N^m$, as claimed. \qed

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