We consider a Schrödinger equation
\[ i\partial_t u(x, t) - u_{xx}(x, t) = p(t)q(x) + f(x, t), \quad 0 < t \leq T, \quad 0 < \rho < 1, \]
with the Riemann–Liouville derivative. An inverse problem is investigated in which, parallel with \( u(x, t) \),
a time-dependent factor \( p(t) \) of the source function is also unknown. To solve this inverse problem, we
use an additional condition \( B[u(\cdot, t)] = \psi(t) \) with an arbitrary bounded linear functional \( B \). The existence
and uniqueness theorem for the solution to the problem under consideration is proved. The stability
inequalities are obtained. The applied method makes it possible to study a similar problem by taking,
instead of \( d^2/dx^2 \), an arbitrary elliptic differential operator \( A(x, D) \) with compact inverse.

1. Introduction

The fractional integration of order \( \sigma < 0 \) of a function \( h(t) \) defined on \([0, \infty)\) has the form (see, e.g., [1,
p. 14; 2, Chap. 3])
\[
J_t^\sigma h(t) = \frac{1}{\Gamma(-\sigma)} \int_0^t \frac{h(\xi)}{(t - \xi)^{\sigma+1}} d\xi, \quad t > 0,
\]
provided that the right-hand side exists. Here, \( \Gamma(\sigma) \) is Euler’s gamma function. By using this definition, we can
define the Riemann–Liouville fractional derivative of order \( \rho \) as follows:
\[
\partial_t^\rho h(t) = \frac{d}{dt}J_t^{\rho-1}h(t).
\]

Note that if \( \rho = 1 \), then the fractional derivative coincides with the ordinary classical derivative of the first
order: \( \partial_t h(t) = (d/dt)h(t) \).

Let \( \rho \in (0, 1) \) be a fixed number and let \( \Omega = (0, \pi) \times (0, T] \). Consider the following initial-boundary-value
problem for the Shrödinger equation:
\[
\begin{align*}
&i\partial_t u(x, t) - u_{xx}(x, t) = p(t)q(x) + f(x, t), \quad (x, t) \in \Omega, \\
&u(0, t) = u(\pi, t) = 0, \quad 0 \leq t \leq T, \\
&\lim_{t \to 0} J_t^{\rho-1}u(x, t) = \varphi(x), \quad 0 \leq x \leq \pi,
\end{align*}
\]

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where \( t^{1-\rho} p(t), \ t^{1-\rho} f(x, t), \ \varphi(x), \) and \( q(x) \) are continuous functions in the closed domain \( \overline{\Omega} \). This problem is also called the forward problem.

If \( p(t) \) is a known function, then, under certain conditions imposed on the given functions, a solution to problem (1.1) exists and is unique (see, e.g., [3]).

We note the following property of the Riemann–Liouville integrals, which simplifies the verification of the initial condition in problem (1.1) (see, e.g., [1, p. 104]):

\[
\lim_{t \to +0} J_t^{\alpha-1} h(t) = \Gamma(\alpha) \lim_{t \to +0} t^{1-\alpha} h(t). \quad (1.2)
\]

This, in particular, implies that the solution of the forward problem may have a singularity at zero \( t = 0 \) of order \( t^{\alpha-1} \).

Let \( C[0, l] \) be the set of continuous functions defined on \([0, l] \) with the standard max-norm \( \| \cdot \|_{C[0,l]} \). The purpose of this paper is not only to find a solution \( u(x, t) \) but also to determine the time-dependent part \( p(t) \) of the source function. To solve this time-dependent source identification problem, we need an extra condition. Following the papers by A. Ashyralyev, et al. [4–6], we consider an additional condition in a fairly general form:

\[
B[u(\cdot, t)] = \psi(t), \quad 0 \leq t \leq T, \quad (1.3)
\]

where \( B : C[0, \pi] \to R \) is a given bounded linear functional

\[
\| B[h(\cdot, t)] \|_{C[0,T]} \leq b \| h(x, t) \|_{C(\overline{\Omega})},
\]

and \( \psi(t) \) is a given continuous function. Thus, as the functional \( B \), we can take either

\[
B[u(\cdot, t)] = u(x_0, t), \quad x_0 \in [0, \pi],
\]

or

\[
B[u(\cdot, t)] = \int_0^\pi u(x, t) dx,
\]

or a linear combination of these two functionals.

The initial-boundary-value problem (1.1) with the additional condition (1.3) is called the inverse problem.

When solving the inverse problem, we investigate the Cauchy and initial-boundary-value problems for various differential equations. In this case, as the solution of the problem, we understand its classical solution, i.e., we assume that all derivatives and functions appearing in the equation are continuous with respect to the variables \( x \) and \( t \) in an open set. As an example, we present the following definition of solution to the inverse problem:

**Definition 1.1.** A pair of functions \( \{ u(x, t), p(t) \} \) with the properties:

1. \( \partial_t^\rho u(x, t), \ u_{xx}(x, t) \in C(\Omega), \)
2. \( t^{1-\rho} u(x, t) \in C(\overline{\Omega}), \)
3. \( t^{1-\rho} p(t) \in C[0, T], \)

and satisfying conditions (1.1) and (1.3) is called the solution of the inverse problem.
Note that condition (3) in this definition is used to cover a broader class of functions regarded as a function of \( p(t) \). In this connection, it should be emphasized that, to the best of our knowledge, the time-dependent source identification problem for equations with the Riemann–Liouville derivative is studied for the first time.

In view of the boundary conditions in problem (1.1), it seems to be convenient to introduce the Hölder classes as follows: Let \( \omega_g(\delta) \) be the modulus of continuity of a function \( g(x) \in C[0, \pi] \), i.e.,

\[
\omega_g(\delta) = \sup_{|x_1 - x_2| \leq \delta} \left| g(x_1) - g(x_2) \right|, \quad x_1, x_2 \in [0, \pi].
\]

If \( \omega_g(\delta) \leq C\delta^a \) is true for some \( a > 0 \), where \( C \) does not depend on \( \delta \) and \( g(0) = g(\pi) = 0 \), then we say that \( g(x) \) belongs to the Hölder class \( C^a[0, \pi] \). We denote the smallest possible constant \( C \) of this kind by \( \|g\|_{C^a[0, \pi]} \).

Similarly, if a continuous function \( h(x, t) \) is defined on \( [0, \pi] \times [0, T] \), then the quantity

\[
\omega_h(\delta; t) = \sup_{|x_1 - x_2| \leq \delta} \left| h(x_1, t) - h(x_2, t) \right|, \quad x_1, x_2 \in [0, \pi],
\]

is the modulus of continuity of the function \( h(x, t) \) with respect to the variable \( x \). In case where \( \omega_h(\delta; t) \leq C\delta^a \), \( C \) does not depend on \( t \) and \( \delta \), and \( h(0, t) = h(\pi, t) = 0, \ t \in [0, T] \), we say that \( h(x, t) \) belongs to the Hölder class \( C^a_x(\Omega) \). Similarly, we denote the smallest constant \( C \) by \( \|h\|_{C^a_x(\Omega)} \).

Let \( C^2_{x,a}(\Omega) \) be a class of functions \( h(x, t) \) such that \( h_{xx}(x, t) \in C^2_x(\Omega) \) and \( h(0, t) = h(\pi, t) = 0, \ t \in [0, T] \). Note that the condition \( h_{xx}(x, t) \in C^2_x(\Omega) \) implies that \( h_{xx}(0, t) = h_{xx}(\pi, t) = 0, \ t \in [0, T] \). For a function of one variable \( g(x) \), we introduce the classes \( C^2([0, \pi]) \) in a similar way.

**Theorem 1.1.** Let \( a > 1/2 \) and let the following conditions be satisfied:

1. \( t^{1-\rho}f(x, t) \in C^a_x(\Omega) \),
2. \( \varphi \in C^a[0, \pi] \),
3. \( t^{1-\rho}\psi(t), t^{1-\rho}\partial^\rho_t\psi(t) \in C[0, T] \),
4. \( q \in C^2([0, \pi]), \ B[q(x)] \neq 0 \).

Then the inverse problem has a unique solution \( \{u(x, t), p(t)\} \).

Everywhere in what follows, we denote by \( a \) an arbitrary number greater than \( 1/2: a > 1/2 \).

If we additionally require that the initial function \( \varphi \in C^2([0, \pi]) \), then we can prove the following result on the stability of solution to the inverse problem.

**Theorem 1.2.** Assume that the assumptions of Theorem 1.1 are satisfied and \( \varphi \in C^2([0, \pi]) \). Then the solution to the inverse problem obeys the following stability estimate:

\[
\|t^{1-\rho}\partial^\rho_t u\|_{C(\Omega)} + \|t^{1-\rho}u_{xx}\|_{C(\Omega)} + \|t^{1-\rho}p\|_{C[0, T]}
\]

\[
\leq C_{\rho, q, B} \left[ \|\varphi_{xx}\|_{C^a[0, \pi]} + \|t^{1-\rho}\psi\|_{C[0, T]} + \|t^{1-\rho}\partial^\rho_t\psi\|_{C[0, T]} + \|t^{1-\rho}f(x, t)\|_{C^2(\Omega)} \right],
\]

where \( C_{\rho, q, B} \) is a constant depending only on \( \rho, q, \) and \( B \).
It should be emphasized that the method proposed in the present work and based on the Fourier method is applicable to the equation in (1.1) with an arbitrary elliptic differential operator $A(x, D)$ instead of $d^2/dx^2$ only if the corresponding spectral problem has a complete system of orthonormal eigenfunctions in $L_2(G)$, $G \subset \mathbb{R}^N$.

The interest in the study of the source (right-hand side of the equation $F(x, t)$) identification inverse problems is caused primarily by their relationships with practical requirements in various branches of mechanics, seismology, medical tomography, and geophysics (see, e.g., the survey paper [7]). The identification of $F(x, t) = h(t)$ is appropriate, e.g., in the cases of accidents at nuclear power plants, when it can be assumed that the location of the source is known but the decay of its radiation power with time is unknown, and it is important to estimate this decay. On the other hand, as an example of identification of $F(x, t) = g(x)$, we can mention the problem of detection of illegal wastewater discharges, which is a serious problem in some countries.

The inverse problem of finding the source function $F$ with final time observation was well studied and numerous theoretical results were published for classical partial differential equations (see, e.g., [8, 9]). As for the fractional differential equations, it is possible to construct the theories parallel to [8, 9], and the work is now ongoing. Here, we mention only some of these works (a detailed survey can be found in [7]).

It is worth noting that, for the abstract case of a source function $F(x, t)$, there is currently no general closed theory. Known results deal with the separated source term $F(x, t) = h(t)g(x)$. The appropriate choice of overdetermination depends on the choice of the unknown function: $h(t)$ or $g(x)$.

Relatively fewer works are devoted to the case where the unknown function is $h(t)$ (see the survey works [7] and [10] for the case of subdiffusion equations, and, e.g., [4–6] for the classical heat equation).

The uniqueness questions for the inverse problem of finding a function $g(x)$ in the fractional diffusion equations with the source function $g(x)h(t)$ were studied, e.g., in [11–13].

In many papers, the authors considered an equation in which $h(t) \equiv 1$ and $g(x)$ is unknown (see, e.g., [14–20]). The case of subdiffusion equations whose elliptic part is an ordinary differential expression was considered in [14–19]. In the articles [21–25], the authors studied the subdiffusion equations whose elliptic part is either the Laplace operator or a second-order self-adjoint operator. The paper [26] studied the inverse problem for the abstract subdiffusion equation. In [26] and many other articles, including [21–24], the Caputo derivative was used as a fractional derivative. The subdiffusion equation considered in the recent papers [3, 27] contains the fractional Riemann–Liouville derivative and its elliptic part is an arbitrary elliptic expression of order $m$. In [25, 28], the fractional derivative in the subdiffusion equation is a two-parameter generalized Hilfer fractional derivative. We also note that the works [21, 24, 28] contain a survey of papers dealing with the inverse problems of finding the right-hand side of the subdiffusion equation.

In [25, 29, 30], nonself-adjoint differential operators (with nonlocal boundary conditions) were taken as elliptic part of the equation, and the solutions to the inverse problem were found in the form of biorthogonal series.

In [20], the authors considered the inverse problem of simultaneous finding the order of the Riemann–Liouville fractional derivative and the source function in the subdiffusion equations. By using the classical Fourier method, the authors proved the uniqueness and existence of solution to this inverse problem.

It should be noted that, in all cited works, the Cauchy conditions are considered in time (as an exception, we can mention the work [31], where the integral condition was formulated with respect to the variable $t$). In the recent paper [32], to the best of our knowledge, the inverse problem for a subdiffusion equation with nonlocal condition in time was considered for the first time.

The papers [33, 34] deal with the inverse problems of finding the order of fractional derivative in the subdiffusion equation and in the wave equation, respectively.

The time-dependent source identification problem (1.1) for classical Schrödinger-type equations (i.e., $\rho = 1$) with the additional condition (1.3) was investigated, for the first time, in [4–6]. To study the inverse problem (1.1), (1.3), we borrow some original ideas from these papers.
2. Preliminaries

In this section, we recall some information about the Mittag-Leffler functions, differential and integral equations, which are used in what follows.

For $0 < \rho < 1$ and an arbitrary complex number $\mu$, by $E_{\rho,\mu}(z)$ we denote the Mittag-Leffler function of complex argument $z$ with two parameters:

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}.$$ (2.1)

If the parameter $\mu = 1$, then we get the classical Mittag-Leffler function: $E_{\rho}(z) = E_{\rho,1}(z)$.

Since $E_{\rho,\mu}(z)$ is an analytic function of $z$, it is bounded for $|z| \leq 1$. On the other hand, the well-known asymptotic estimate for the Mittag-Leffler function has the following form (see, e.g., [35, p. 133]):

**Lemma 2.1.** Let $\mu$ be an arbitrary complex number. Further, let $\alpha$ be a fixed number such that $\frac{\pi}{2} \rho < \alpha < \pi \rho$ and $\alpha \leq |\arg z| \leq \pi$. Then the following asymptotic estimate is true:

$$E_{\rho,\mu}(z) = -\sum_{k=1}^{2} \frac{z^{-k}}{\Gamma(\rho - k\mu)} + O(|z|^{-3}), \quad |z| > 1.$$ 

We can choose a parameter $\alpha$ such that the following estimate is valid:

**Corollary 2.1.** For any $t \geq 0$, the following inequality is true:

$$|E_{\rho,\mu}(it)| \leq \frac{C}{1 + t},$$

where the constant $C$ is independent of $t$ and $\mu$.

We also use a coarser estimate with a positive number $\lambda$ and $0 < \varepsilon < 1$:

$$|t^{\rho-1}E_{\rho,\rho}(-i\lambda t^\rho)| \leq \frac{C t^{\rho-1}}{1 + \lambda t^\rho} \leq C \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}, \quad t > 0,$$ (2.2)

which can be easily verified. Indeed, let $t^\rho \lambda < 1$. Then

$$t < \lambda^{-1/\rho} \quad \text{and} \quad t^{\rho-1} = t^{\rho-\varepsilon\rho} t^{\varepsilon\rho-1} < \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}.$$ 

Further, if $t^\rho \lambda \geq 1$, then

$$\lambda^{-1} \leq t^\rho \quad \text{and} \quad \lambda^{-1} t^{-1} = \lambda^{-1+\varepsilon} \lambda^{-\varepsilon} t^{-1} \leq \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}.$$ 

**Lemma 2.2.** Let $t^{1-\rho}g(t) \in C[0, T]$. Then the unique solution of the Cauchy problem

$$i\partial_t^\rho y(t) + \lambda y(t) = g(t), \quad 0 < t \leq T,$$

$$\lim_{t \to 0} J_t^{\rho-1} y(t) = y_0,$$ (2.3)

where

$$J_t^{\rho-1} y(t) = \frac{1}{\Gamma(\rho-1)} \int_0^t (t-s)^{\rho-2} y(s) \, ds,$$
has the form

\[ y(t) = t^{\alpha-1}E_{\rho,\mu}(i\lambda^\alpha)y_0 - \frac{t}{\alpha} \int_0^t (t-s)^{\alpha-1}E_{\rho,\mu}(i\lambda(t-s)^\alpha)g(s)ds. \]

**Proof.** We multiply equation (2.3) by \((-i)\) and then apply relation (7.2.16) from [36, p. 174] (see also [37, 38]).

By \(A\) we denote the operator \(-d^2/dx^2\) with the domain

\[ D(A) = \{ v(x) \in W_2^2(0, \pi) : v(0) = v(\pi) = 0 \}, \]

where \(W_2^2(0, \pi)\) is the standard Sobolev space. The operator \(A\) is self-adjoint in \(L_2(0, \pi)\) and has the complete set of eigenfunctions \(\{ v_k(x) = \sin(kx) \}\) in \(L_2(0, \pi)\) and the set of eigenvalues \(\lambda_k = k^2, \ k = 1, 2, \ldots.\)

Consider an operator \(E_{\rho,\mu}(itA)\) defined by the spectral theorem of J. von Neumann:

\[ E_{\rho,\mu}(itA)h(x, t) = \sum_{k=1}^\infty E_{\rho,\mu}(it\lambda_k)h_k(t)v_k(x). \]

Here and everywhere in what follows, by \(h_k(t)\) we denote the Fourier coefficients of a function \(h(x, t)\): \(h_k(t) = (h(x, t), v_k)\), and \((\cdot, \cdot)\) stands for the scalar product in \(L_2(0, \pi)\). This series converges in the \(L_2(0, \pi)\)-norm.

However, it is necessary to investigate the uniform convergence of this series in \(\Omega\). To do this, we recall the following statement:

**Lemma 2.3.** Let \(g \in C^\alpha[0, \pi]\). Then, for any \(\sigma \in [0, \alpha - 1/2]\), the following inequality is true:

\[ \sum_{k=1}^\infty k^\sigma |g_k| < \infty. \]

For \(\sigma = 0\), this assertion coincides with the well-known Bernstein theorem on the absolute convergence of trigonometric series and is proved in exactly the same way as this theorem. For the sake of convenience of the readers, we recall the main points of the proof (see, e.g., [39, p. 384]).

**Proof.** In Theorem 3.1 of A. Zygmund [39, p. 384], it is proved that any function \(g(x) \in C[0, \pi]\) such that \(g(0) = g(\pi) = 0\) satisfies the following inequality:

\[ \sum_{k=2}^{2^n} |g_k|^2 \leq \omega_2^2 \left( \frac{1}{2^n+1} \right). \]

Therefore, if \(\sigma \geq 0\), then, by the Cauchy–Bunyakovsky inequality, we get

\[ \sum_{k=2}^{2^n} k^\sigma |g_k| \leq \left( \sum_{k=2}^{2^n} |g_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=2}^{2^n} k^{2\sigma} \right)^{\frac{1}{2}} \leq C 2^{n(\frac{1}{2} + \sigma)\omega_2} \left( \frac{1}{2^n+1} \right), \]

and, finally,

\[ \sum_{k=2}^{\infty} k^\sigma |g_k| = \sum_{n=1}^{\infty} \sum_{k=2^{n-1}+1}^{2^n} k^\sigma |g_k| \leq C \sum_{n=1}^{\infty} 2^{n(\frac{1}{2} + \sigma)\omega_2} \left( \frac{1}{2^n+1} \right). \]
Obviously, if \( \omega_0(\delta) \leq C\delta^a \), \( a > 1/2 \), and \( 0 < \sigma < a - 1/2 \), then the last series is convergent:

\[
\sum_{k=2}^{\infty} k^\alpha |g_k| \leq C \left\| g \right\|_{C^a[0,\pi]}.
\]

Lemma 2.4. Let \( h(x,t) \in C^a_2(\bar{\Omega}) \). Then \( E_{\rho,\mu}(itA)h(x,t) \in C(\bar{\Omega}) \) and

\[
\frac{\partial^2}{\partial x^2} E_{\rho,\mu}(itA)h(x,t) \in C([0,\pi] \times (0, T]).
\]

Moreover, the following estimates are true:

\[
\left\| E_{\rho,\mu}(itA)h(x,t) \right\|_{C(\bar{\Omega})} \leq C \left\| h \right\|_{C^a_2(\bar{\Omega})}; \tag{2.4}
\]

\[
\left\| \frac{\partial^2}{\partial x^2} E_{\rho,\mu}(itA)h(x,t) \right\|_{C([0,\pi])} \leq Ct^{-1} \left\| h \right\|_{C^a_2(\bar{\Omega})}, \quad t > 0. \tag{2.5}
\]

If \( h(x,t) \in C^a_{2,x}(\bar{\Omega}) \), then

\[
\left\| \frac{\partial^2}{\partial x^2} E_{\rho,\mu}(itA)h(x,t) \right\|_{C(\bar{\Omega})} \leq C \left\| h_{xx} \right\|_{C^a_2(\bar{\Omega})}. \tag{2.6}
\]

Proof. By definition, we get

\[
|E_{\rho,\mu}(itA)h(x,t)| = \left| \sum_{k=1}^{\infty} E_{\rho,\mu}(i\lambda_k)h_k(t) v_k(x) \right| \leq \sum_{k=1}^{\infty} |E_{\rho,\mu}(i\lambda_k)h_k(t)|.
\]

Corollary 2.1 and Lemma 2.3 imply that

\[
|E_{\rho,\mu}(itA)h(x,t)| \leq C \sum_{k=1}^{\infty} \left| \frac{h_k(t)}{1 + t\lambda_k} \right| \leq C \left\| h \right\|_{C^a_2(\bar{\Omega})}.
\]

On the other hand,

\[
\left| \frac{\partial^2}{\partial x^2} E_{\rho,\mu}(itA)h(x,t) \right| \leq C \sum_{k=1}^{\infty} \left| \frac{\lambda_k h_k(t)}{1 + t\lambda_k} \right| \leq Ct^{-1} \left\| h \right\|_{C^a_2(\bar{\Omega})}, \quad t > 0.
\]

If \( h(x,t) \in C^a_{2,x}(\bar{\Omega}) \), then \( h_k(t) = -\lambda_k^{-1}(h_{xx})_k(t) \). Therefore,

\[
\left| \frac{\partial^2}{\partial x^2} E_{\rho,\mu}(itA)h(x,t) \right| \leq C \left\| h_{xx} \right\|_{C^a_2(\bar{\Omega})}, \quad 0 \leq t \leq T.
\]

Lemma 2.5. Let \( t^{1-\rho}g(x,t) \in C^a_2(\bar{\Omega}) \). Then there exists a positive constant \( c_1 \) such that

\[
\left| t^{1-\rho} \int_0^t (t-s)^{\rho-1} E_{\rho,\mu}(i(t-s)^{\rho}A)g(x,s)ds \right| \leq c_1 \frac{t^{\rho}}{\rho} \left\| t^{1-\rho}g \right\|_{C^a_2(\bar{\Omega})}. \tag{2.7}
\]
Proof. Applying estimate (2.4), we obtain

$$
\left| t^{1-\rho} \int_0^t (t-s)^{\rho-1} E_{\rho,\rho}(i(t-s)^\rho A)g(x,s)ds \right| \leq C t^{1-\rho} \int_0^t (t-s)^{\rho-1}s^{\rho-1}ds \| t^{1-\rho}g \|_{C^2_2(\Omega)}.
$$

For the integral, we get

$$
\int_0^t (t-s)^{\rho-1}s^{\rho-1}ds = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \leq \frac{2^{(1-\rho)}}{\rho} - t^{2\rho-1}.
$$

(2.8)

Denoting $c_1 = 4C$, we arrive at the assertion of the lemma.

Corollary 2.2. If a function $g(x,t)$ can be represented in the form $g_1(x)g_2(t)$, then the right-hand side of estimate (2.7) takes the form

$$
c_1 \frac{t^\rho}{\rho} \|g_1\|_{C^\alpha[0,\pi]} \| t^{1-\rho}g_2 \|_{C[0,T]}.
$$

Lemma 2.6. Let $t^{1-\rho}g(x,t) \in C^\alpha_2(\Omega)$. Then

$$
\left\| \int_0^t (t-s)^{\rho-1} \frac{\partial^2}{\partial x^2} E_{\rho,\rho}(i(t-s)^\rho A)g(x,s)ds \right\|_{C(\Omega)} \leq C \| t^{1-\rho}g \|_{C^2_2(\Omega)}.
$$

Proof. Let

$$
S_j(x,t) = \sum_{k=1}^{j} \left[ \int_0^t (t-s)^{\rho-1} E_{\rho,\rho}(i\lambda_k(t-s)^\rho)g_k(s)ds \right] \lambda_k v_k(x).
$$

Choosing $\varepsilon$ such that $0 < \varepsilon < a - 1/2$ and applying inequality (2.2), we get

$$
|S_j(t)| \leq C \sum_{k=1}^{j} \int_0^t (t-s)^{\rho-1} s^{\rho-1} \lambda_k^\varepsilon |s^{1-\rho}g_k(s)|ds.
$$

By Lemma 2.3, we find

$$
|S_j(t)| \leq C \| t^{1-\rho}g \|_{C^2_2(\Omega)}.
$$

Since

$$
\int_0^t (t-s)^{\rho-1} \frac{\partial^2}{\partial x^2} E_{\rho,\rho}(i(t-s)^\rho A)h(s)ds = \sum_{j=1}^\infty S_j(t),
$$

the last inequality implies the assertion of the lemma.
Lemma 2.7. Let $t^{1-\rho}G(x, t) \in C^\alpha_x(\bar{\Omega})$ and $\varphi \in C^\alpha[0, \pi]$. Then the unique solution of the following initial-boundary-value problem:

$$i\partial_t^\rho w(x, t) - w_{xx}(x, t) = G(x, t), \quad 0 < t \leq T,$$

$$w(0, t) = w(\pi, t) = 0, \quad 0 < t \leq T,$$

$$\lim_{t \to 0} J_t^{\rho-1} w(x, t) = \varphi(x), \quad 0 \leq x \leq \pi,$$

has the form

$$w(x, t) = t^{\rho-1} E_\rho(it^\rho A)\varphi(x) - i \int_0^t (t-s)^{\rho-1} E_{\rho,\rho}(i(t-s)^\rho A)G(x, s)ds.$$

**Proof.** According to the Fourier method, we seek the solution to this problem in the following form:

$$w(x, t) = \sum_{k=1}^{\infty} T_k(t)v_k(x),$$

where $T_k(t)$ are the unique solutions of the problems

$$i\partial_t^\rho T_k + \lambda_k T_k(t) = G_k(t), \quad 0 < t \leq T,$$

$$\lim_{t \to 0} J_t^{\rho-1} T_k(t) = \varphi_k.$$

Lemma 2.2 implies that

$$T_k(t) = t^{\rho-1} E_\rho(it\lambda_k t^\rho)\varphi_k - i \int_0^t (t-s)^{\rho-1} E_{\rho,\rho}(i\lambda_k(t-s)^\rho A)G_k(s)ds.$$

Hence, the solution to problem (3.1) takes the form

$$w(x, t) = t^{\rho-1} E_\rho(it^\rho A)\varphi(x) - i \int_0^t (t-s)^{\rho-1} E_{\rho,\rho}(i(t-s)^\rho A)G(x, s)ds.$$

Note that the existence of the first term follows from estimate (2.4), while the existence of the second term follows from Lemma 2.5.

By Lemma 2.6 and estimate (2.5), we conclude that $w_{xx}(x, t) \in C(\Omega)$. Since

$$i\partial_t^\rho w(x, t) = -w_{xx}(x, t) + G(x, t),$$

we get $\partial_t^\rho w(x, t) \in C(\Omega)$.

The uniqueness of solution can be proved by using the standard technique based on the completeness of the set of eigenfunctions $\{v_k(x)\}$ in $L_2(0, \pi)$ (see, e.g., [3]).
Let \( t^{1-\rho}F(x, t) \in C(\Omega) \) and \( g(x) \in C^{\alpha}[0, \pi] \). We consider the Volterra integral equation

\[
w(x, t) = F(x, t) + \int_{0}^{t} (t - s)^{\rho - 1} E_{\rho, \rho}(i(t - s)^{\rho}A)g(x)B[w(\cdot, s)]ds.
\]  

(2.9)

**Lemma 2.8.** There exists a unique solution \( t^{1-\rho}w \in C(\Omega) \) to the integral equation (2.9).

**Proof.** Equation (2.9) is similar to the equations considered in the book [40, p. 199] (Eq. (3.5.4)) and it is solved in essentially the same way. We now recall the main points of the proof.

Equation (2.9) is meaningful in any interval \([0, t_1] \in [0, T], 0 < t_1 < T\). We choose \( t_1 \) such that

\[
c_1 b \|g\|_{C^\alpha[0, \pi]} \frac{\rho^\alpha}{\rho} < 1
\]  

(2.10)

and prove the existence of a unique solution \( t^{1-\rho}w(x, t) \in C([0, \pi] \times [0, t_1]) \) to equation (2.9) on the interval \([0, t_1]\) (here, the constant \( c_1 \) is taken from estimate (2.7); see Corollary 2.2). To do this, we use the Banach fixed-point theorem for the space \( C([0, \pi] \times [0, t_1]) \) with the weight function \( t^{1-\rho} \) (see, e.g., [40, p. 68], Theorem 1.9), where the distance is given by

\[
d(w_1, w_2) = \|t^{1-\rho}[w_1(x, t) - w_2(x, t)]\|_{C([0, \pi] \times [0, t_1])}.
\]

We denote the right-hand side of equation (2.9) by \( Pw(x, t) \), where \( P \) is the corresponding linear operator. Applying the Banach fixed-point theorem, it is necessary to prove the following assertion:

(a) if \( t^{1-\rho}w(x, t) \in C([0, \pi] \times [0, t_1]) \), then \( t^{1-\rho}Pw(x, t) \in C([0, \pi] \times [0, t_1]) \);

(b) for any \( t^{1-\rho}w_1, t^{1-\rho}w_2 \in C([0, \pi] \times [0, t_1]) \), the following inequality is true:

\[
d(Pw_1, Pw_2) \leq \delta \cdot d(w_1, w_2), \quad \delta < 1.
\]

Lemmas 2.4 and 2.5 imply the condition (a). On the other hand, in view of (2.7) (see Corollary 2.2), we arrive at the inequality

\[
\left\|t^{1-\rho} \int_{0}^{t} (t - s)^{\rho - 1} E_{\rho, \rho}(i(t - s)^{\rho}A)g(x)B[w(\cdot, s) - w_2(\cdot, s)]ds\right\|_{C([0, \pi] \times [0, t_1])} \leq \delta d(w_1, w_2),
\]

where

\[
\delta = c_1 b \|g\|_{C^\alpha[0, \pi]} \frac{\rho^\alpha}{\rho} < 1
\]

by virtue of condition (2.10).

Hence, by virtue of the Banach fixed-point theorem, there exists a unique solution \( t^{1-\rho}w^*(x, t) \in C([0, \pi] \times [0, t_1]) \) to equation (2.9) on the interval \([0, t_1]\), and this solution is the limit of a convergent sequence

\[
w_n(x, t) = P^n F(x, t) = PP^{n-1} F(x, t),
\]
reader, we present the proof of estimate (2.11). However, estimate (2.11) is proved in the same way as the Gronwall inequality. For the sake of convenience of the following form:

\[
K_{n} \text{ and } w \text{ as above, we conclude that there exists a unique solution }
\参加了
\begin{equation}
\tag{2.11}
\end{equation}

is a known function, since the function \( w(x, t) \) is uniquely defined on the interval \([0, t_1]\). Using the same arguments as above, we conclude that there exists a unique solution \( t^{1-\rho}w^*(x, t) \in C([0, \pi] \times [t_1, t_2]) \) to equation (2.9) on the interval \([t_1, t_2]\). Taking the next interval \([t_2, t_3]\), where \( t_3 = t_2 + l_2 < T \) and \( l_2 > 0 \), and repeating this process (obviously, \( l_n > l_0 > 0 \)), we conclude that there exists a unique solution \( t^{1-\rho}w^*(x, t) \in C([0, \pi] \times [0, T]) \) to equation (2.9) on the interval \([0, T]\) and this solution is the limit of a convergent sequence \( t^{1-\rho}w_n(x, t) \in C([0, \pi] \times [0, T]) \):

\[
\lim_{n \to \infty} \| t^{1-\rho}[w_n(x, t) - w^*(x, t)] \|_{C([\Pi])} = 0,
\]

with the choice of some \( w_n \) on each \([0, t_1], \ldots [t_{L-1}, T]\).

We need Gronwall’s inequality of the following kind:

**Lemma 2.9.** Let \( 0 < \rho < 1 \). Assume that a nonnegative function \( h(t) \in C[0, T] \) and positive constants \( K_0 \) and \( K_1 \) satisfy the inequality

\[
h(t) \leq K_0 + K_1 \int_0^t (t-s)^{\rho-1}s^{\rho-1}h(s)\,ds
\]

for all \( t \in [0, T] \). Then there exists a positive constant \( C_{\rho,T} \) depending only on \( \rho \), \( K_2 \), and \( T \) such that

\[
h(t) \leq K_0 C_{\rho,T}.
\]

Usually, Gronwall’s inequality is formulated with a continuous function \( k(s) \) instead of \( K_1(t-s)^{\rho-1}s^{\rho-1} \). However, estimate (2.11) is proved in the same way as the Gronwall inequality. For the sake of convenience of the reader, we present the proof of estimate (2.11).

**Proof.** Iterating the hypothesis of Gronwall’s inequality, we find

\[
h(t) \leq K_0 + K_0 K_1 \int_0^t (t-s)^{\rho-1}s^{\rho-1}\,ds
\]

\[
+ K_1^2 \int_0^t (t-s)^{\rho-1}s^{\rho-1} \int_0^s (s-\xi)^{\rho-1}\xi^{\rho-1}h(\xi)d\xi\,ds
\]
\[
\leq K_{\rho,T} + K_2^2 \int_0^t u(\xi)\xi^{\rho-1} \int_\xi^t (t-s)^{\rho-1}(s-\xi)^{\rho-1}s^{\rho-1}dsd\xi,
\]

where

\[
K_{\rho,T} = K_0 + K_0 K_1 \int_0^T (t-s)^{\rho-1}s^{\rho-1}ds.
\]

For the inner integral, we get (see (2.8))

\[
\int_\xi^t (s-\xi)^{\rho-1}(t-s)^{\rho-1}ds = \int_0^{t-\xi} y^{\rho-1}(t - \xi - y)^{\rho-1}dy \leq \frac{2^{2(1-\rho)}}{\rho}(t-\xi)^{2\rho-1}.
\]

Thus, the hypothesis takes the form

\[
h(t) \leq K_0 + K_2^2 \frac{2^{2(1-\rho)}}{\rho} \int_0^t (t-s)^{2\rho-1}h(s)ds.
\]

Repeating this process so many times that \(k\rho > 1\), we make sure that there is a positive constant

\[
C_\rho = C(\rho, K_2, T) > 0
\]

such that

\[
h(t) \leq K_0 + C_\rho \int_0^t h(s)ds
\]

or

\[
\frac{h(\xi)}{K_0 + C_\rho \int_0^\xi h(s)ds} \leq 1.
\]

Multiplying this by \(C_\rho\), we get

\[
\frac{d}{d\xi} \ln \left( K_0 + C_\rho \int_0^\xi h(s)ds \right) \leq C_\rho.
\]

Integrating from \(\xi = 0\) to \(\xi = t\) and exponentiating, we obtain

\[
K_0 + C_\rho \int_0^t h(s)ds \leq K_0 e^{C_\rho t}.
\]

Finally, note that the left-hand side is \(\geq h(t)\).
3. Auxiliary Problem and the Proof of Theorem 1.1

We now consider the following auxiliary initial-boundary-value problem:

\[
\begin{align*}
    i\partial_t^\rho \omega(x, t) - \omega_{xx}(x, t) &= -i\mu(t)q''(x) + f(x, t), \quad (x, t) \in \Omega, \\
    \omega(0, t) &= \omega(\pi, t) = 0, \quad 0 \leq t \leq T, \\
    \lim_{t \to 0} J_t^{\rho-1}\omega(x, t) &= \varphi(x), \quad 0 \leq x \leq \pi,
\end{align*}
\]  
\tag{3.1}

where the function \( \mu(t) \) is the unique solution to the Cauchy problem

\[
\begin{align*}
    \partial_t^\rho \mu(t) &= p(t), \quad 0 < t \leq T, \\
    \lim_{t \to 0} J_t^{\rho-1}\mu(t) &= 0.
\end{align*}
\]  
\tag{3.2}

Note that the solution to the Cauchy problem (3.2) has the form (see, e.g., [37])

\[\mu(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t - s)^{\rho-1}p(s)ds.\]

**Definition 3.1.** A functions \( \omega(x, t) \) with the properties:

1. \( \partial_t^\rho \omega(x, t), \omega_{xx}(x, t) \in C(\Omega), \)
2. \( t^{1-\rho}\omega_{xx}(x, t) \in C((0, \pi) \times [0, T]), \)
3. \( t^{1-\rho}\omega(x, t) \in C(\overline{\Omega}), \)

satisfying conditions (3.2), is called the solution of problem (3.2).

**Lemma 3.1.** Let \( \omega(x, t) \) be a solution of problem (3.1). Then the unique solution \( \{u(x, t), p(t)\} \) to the inverse problem (1.1), (1.3) has the form

\[
\begin{align*}
    u(x, t) &= \omega(x, t) - i\mu(t)q(x), \\
    p(t) &= \frac{i}{B[q(x)]} \left\{ \partial_t^\rho \psi(t) - B[\partial_t^\rho \omega(\cdot, t)] \right\},
\end{align*}
\]  
\tag{3.3}

where

\[
\mu(t) = \frac{i}{B[q(x)]} \left[ \psi(t) - B[\omega(\cdot, t)] \right].
\]  
\tag{3.4}

**Proof.** We substitute the function \( u(x, t) \) given by equality (3.3) in the equation in (1.1). This gives

\[
i\partial_t^\rho \omega(x, t) + \partial_t^\rho \mu(t)q(x) - \omega_{xx}(x, t) + i\mu(t)q''(x) = p(t)q(x) + f(x, t).
\]
Applying (1.3), we obtain
\[ q_{\mu}(x, t) = \lim_{t \to 0} J_t^{p-1} \omega(x, t) - i \lim_{t \to 0} J_t^{p-1} \mu(t) q(x) = \lim_{t \to 0} J_t^{p-1} \omega(x, t) = \varphi(x). \]

On the other hand, the conditions \( q(0) = q(\pi) = 0 \) imply that \( u(0, t) = u(\pi, t) = 0, \ 0 \leq t \leq T \).

It immediately follows from the Definition 3.1 of the solution \( \omega(x, t) \) and the property of the functions \( \mu(t) \) and \( q(x) \) that the function \( u(x, t) \) satisfies the following requirements:

\[ \partial_t^\rho u(x, t), u_{xx}(x, t) \in C(\Omega), \quad t^{1-\rho} u(x, t) \in C(\overline{\Omega}). \]

Thus, the function \( u(x, t) \) defined by (3.3) is a solution of the initial-boundary-value problem (1.1).

We now prove equation (3.4). We rewrite (3.3) as follows:

\[ iq(x) \mu(t) = \omega(x, t) - u(x, t). \]

Applying (1.3), we obtain

\[ i \mu(t) B[q(x)] = B[\omega(\cdot, t)] - \psi(t) \]

or, since \( B[q(x)] \neq 0 \), we get (3.5). Finally, by using equality \( \partial_t^\rho = p(t) \), we get

\[ p(t) = \frac{i}{B[q(x)]} \left[ \partial_t^\rho \psi(t) - B[\partial_t \omega(\cdot, t)] \right], \]

which coincides with (3.4). Moreover, it follows from the definition of the solution \( \omega(x, t) \) of problem (3.1) and the property of the function \( \psi(t) \) that \( t^{1-\rho} \psi(t) \in C[0, T] \).

Thus, to solve the inverse problem (1.1), (1.3), it is sufficient to solve the initial-boundary-value problem (3.1).

**Theorem 3.1.** Under the assumptions of Theorem 1.1, problem (3.1) has a unique solution.

**Proof.** Let

\[ G(x, s) = \frac{i}{B[q(x)]} \left( B[\omega(\cdot, s)] - \psi(s) \right) q''(x) + f(x, s). \]  

(3.6)

Suppose that \( s^{1-\rho} G(x, s) \in C_x^a(\overline{\Omega}) \). Thus, by virtue of Lemma 2.7, problem (3.1) is equivalent to the integral equation

\[ \omega(x, t) = t^{\rho-1} E_{\rho}(i t^\rho A) \varphi(x) - i \int_0^t (t - s)^{\rho-1} E_{\rho}(i(t - s)^\rho A) G(x, s) ds. \]

We rewrite this equation as

\[ \omega(x, t) = F(x, t) + \int_0^t (t - s)^{\rho-1} E_{\rho}(i(t - s)^\rho A) \frac{q''(x)}{B[q(x)]} B[\omega(\cdot, s)] ds, \]

(3.7)
where

\[ F(x, t) = t^{\rho-1} E_{\rho}(it^\rho A) \varphi(x) - i \int_0^t (t-s)^{\rho-1} E_{\rho,\rho}(i(t-s)^\rho A) \left[ \frac{iq''(x)}{B[q(x)]} \psi(s) + f(x, s) \right] ds. \]

In order to apply Lemma 2.8 to equation (3.7), we show that \( t^{1-\rho} F(x, t) \in C(\overline{\Omega}) \). Indeed, by estimate (2.4), we have \( E_{\rho}(it^\rho A) \varphi(x) \in C(\overline{\Omega}) \). According to the conditions of Theorem 1.1,

\[ h(x, s) = s^{1-\rho} \left[ -\frac{iq''(x)}{B[q(x)]} \psi(s) + f(x, s) \right] \in C^a_{x}(\overline{\Omega}). \]

Therefore, by virtue of estimate (2.7), the second term of the function \( t^{1-\rho} F(x, t) \) also belongs to the class \( C(\overline{\Omega}) \). Hence, in view of Lemma 2.8, the Volterra equation (3.7) possesses a unique solution \( t^{1-\rho} \omega(x, t) \in C(\overline{\Omega}) \).

We now show that \( \partial^2 \omega(x, t) \), \( \omega_{xx}(t) \in C(\Omega) \). First, we consider \( F_{xx}(x, t) \) and note that, by estimate (2.5), we can write

\[ \frac{\partial^2}{\partial x^2} E_{\rho}(it^\rho A) \varphi(x) \in C([0, \pi] \times (0, T]). \]

Since the function \( h \) defined above belongs to the class \( C^2_{x}(\overline{\Omega}) \), by virtue of Lemma 2.6, the second term of the function \( F_{xx}(x, t) \) belongs to \( C(\Omega) \) and satisfies the estimate

\[
\left\| t^{1-\rho} \int_0^t (t-s)^{\rho-1} \frac{\partial^2}{\partial x^2} E_{\rho,\rho}(i(t-s)^\rho A) \left[ \frac{iq''(x)}{B[q(x)]} \psi(s) + f(x, s) \right] ds \right\|_{C(\overline{\Omega})} \\
\leq C \left[ \left\| t^{1-\rho} \frac{q''(x)}{B[q(x)]} \psi(t) \right\|_{C^2(\overline{\Omega})} + \left\| t^{1-\rho} f(x, t) \right\|_{C^2(\overline{\Omega})} \right] \\
\leq C_{a,q,B} \left[ \left\| t^{1-\rho} \psi \right\|_{C[0,T]} + \left\| t^{1-\rho} f(x, t) \right\|_{C^2(\overline{\Omega})} \right].
\]  

(3.8)

We now pass to the second term on the right-hand side of equality (3.7). Since \( t^{1-\rho} \omega(x, t) \in C(\overline{\Omega}) \), the conditions of Theorem 1.1 imply that

\[ s^{1-\rho} \frac{q''(x)}{B[q(x)]} B[\omega(\cdot, s)] \in C^a_{x}(\overline{\Omega}). \]

Thus, again by Lemma 2.6, this term belongs to \( C(\overline{\Omega}) \) and satisfies the estimate

\[
\left\| t^{1-\rho} \int_0^t (t-s)^{\rho-1} \frac{\partial^2}{\partial x^2} E_{\rho,\rho}(i(t-s)^\rho A) \frac{q''(x)}{B[q(x)]} B[\omega(\cdot, s)] ds \right\|_{C(H)} \\
\leq C \left\| \frac{q''(x)}{B[q(x)]} B[t^{1-\rho} \omega(\cdot, t)] \right\|_{C(\overline{\Omega})} \leq C_{a,q,B} \left\| t^{1-\rho} \omega(x, t) \right\|_{C(\overline{\Omega})}.
\]  

(3.9)
Therefore, $\omega_{xx}(x,t) \in C((0,T]; H)$. On the other hand, by virtue of equation (3.1) and the conditions of Theorem 1.1, we find
\[
\partial_t^\rho \omega(x,t) = \omega_{xx}(x,t) - i\mu(t)q''(x) + f(x,t) \in C(\bar{\Omega}).
\]
Here, the fact that $\mu \in C[0,T]$ also follows from the conditions of Theorem 1.1 and equality (3.5).

It remains to show that $t^{1-\rho}G(x, t) \in C_x^a(\bar{\Omega})$. However, this fact follows from the conditions of Theorem 1.1 and the already established assertion that $t^{1-\rho} \omega(x, t) \in C(\bar{\Omega})$.

As indicated above, Theorem 1.1 is an immediate consequence of Lemma 3.1 and Theorem 3.1.

4. Proof of Theorem 1.2

First, we prove the following statement on stability of the solution to problem (3.1), (3.2):

**Theorem 4.1.** Let the assumptions of Theorem 1.2 be satisfied. Then the solution to problem (3.1), (3.2) obeys the following stability estimate:

\[
\|t^{1-\rho}\partial_t^\rho \omega\|_{C(\bar{\Omega})} \leq C_{\rho,q,B} \left[ \|\varphi_{xx}\|_{C^a[0,\pi]} + \|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho} f(x,t)\|_{C_x^a(\bar{\Omega})} \right],
\]

(4.1)

where $C_{\rho,q,B}$ is a constant that depends only on $\rho, q, B$.

**Proof.** We begin the proof of inequality (4.1) by establishing an estimate for $\omega_{xx}(x,t)$, and then use this estimate with equation (3.1). To this end, from (2.6), we obtain
\[
\left\| \frac{\partial^2}{\partial x^2} E_{\rho}(it^\rho A)\varphi \right\|_{C(\bar{\Omega})} \leq C\|\varphi_{xx}\|_{C^a[0,\pi]}.
\]

Together with (3.8), this estimate implies that
\[
\|t^{1-\rho}F_{xx}(x,t)\|_{C(\bar{\Omega})} \leq C\|\varphi_{xx}\|_{C^a[0,\pi]} + C_{q,a,B} \left[ \|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho} f(x,t)\|_{C_x^a(\bar{\Omega})} \right].
\]

Thus, by using equality (3.7) and inequality (3.9), we get
\[
\|t^{1-\rho}\omega_{xx}(x,t)\|_{C(\bar{\Omega})} \leq C\|\varphi_{xx}\|_{C^a[0,\pi]}
\]
\[
+ C_{q,a,B} \left[ \|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho} f(x,t)\|_{C_x^a(\bar{\Omega})} + \|t^{1-\rho} \omega(x,t)\|_{C(\bar{\Omega})} \right].
\]

(4.2)

As a result, we arrive at the estimate for $\omega_{xx}(x,t)$ in terms of $\omega(x,t)$. To estimate $\|t^{1-\rho} \omega(x,t)\|_{C(\bar{\Omega})}$, we proceed as follows: Applying estimates (2.4) and (2.7), we get
\[
\|t^{1-\rho}F(x,t)\|_{C(\bar{\Omega})} \leq \|\varphi\|_{C^a[0,\pi]}
\]
\[
+ \frac{T^\rho}{\rho} \left[ C_{q,B} \|q''\|_{C^a[0,\pi]} \|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho} f\|_{C_x^a(\bar{\Omega})} \right].
\]
Further, by using estimate (2.4), we find

$$\left\| t^{1-\rho} \int_0^t (t-s)^{\rho-1} E_{\rho,\rho}(i(t-s)^\rho A) \frac{q''(x)}{B[q(x)]} B[\omega(s)] ds \right\|_{C[0,\pi]} \leq C_q B \| q'' \|_{C^\alpha[0,\pi]} \int_0^t (t-s)^{\rho-1} \| \omega(x,s) \|_{C[0,\pi]} ds$$

Therefore, equation (3.7) yields the following inequality:

$$\left\| t^{1-\rho} \omega(x,t) \right\|_{C[0,\pi]} \leq \| \varphi \|_{C^\alpha[0,\pi]} + C_{q,\rho,B} \left[ \| t^{1-\rho} \psi \|_{C[0,T]} + \| t^{1-\rho} f \|_{C^2_\alpha(\Omega)} \right]$$

$$+ C_{q,B} \int_0^t (t-s)^{\rho-1} s^{\rho-1} \| s^{1-\rho} \omega(x,s) \|_{C[0,\pi]} ds$$

for all $t \in [0,T]$. Finally, the Gronwall inequality (2.11) implies that

$$\left\| t^{1-\rho} \omega(x,t) \right\|_{C(\Omega)} \leq C_{q,\rho,B} \left[ \| \varphi \|_{C^\alpha[0,\pi]} + \| t^{1-\rho} \psi \|_{C[0,T]} + \| t^{1-\rho} f \|_{C^2_\alpha(\Omega)} \right].$$

Substituting this estimate in (4.2) and applying the inequality

$$\varphi \|_{C^\alpha[0,\pi]} \leq C \| \varphi_{xx} \|_{C^\alpha[0,\pi]},$$

we get

$$\left\| t^{1-\rho} \omega_{xx} \right\|_{C(\Omega)} \leq C_{\rho,q,B} \left[ \| \varphi_{xx} \|_{C^\alpha[0,\pi]} + \| t^{1-\rho} \psi \|_{C[0,T]} + \| t^{1-\rho} f \|_{C^2_\alpha(\Omega)} \right].$$

To obtain estimate (4.1), it remains to note that

$$\partial_t^\rho \omega(x,t) = \omega_{xx}(x,t) - i \mu(t) q''(x) + f(x,t)$$

and apply the estimate

$$\left\| t^{1-\rho} \mu \right\|_{C[0,T]} \leq C_{q,B} \left[ \| t^{1-\rho} \psi \|_{C[0,T]} + \| t^{1-\rho} \omega \|_{C(\Omega)} \right],$$

which follows from definition (3.5) and the conditions of Theorem 1.1.

**Proof of Theorem 1.2.** We now apply (3.4) to get

$$\left\| t^{1-\rho} p(t) \right\|_{C[0,T]} \leq C_{q,B} \left[ \| t^{1-\rho} \partial_t^\rho \omega \|_{C(\Omega)} + \| t^{1-\rho} \partial_t^\rho \psi \|_{C[0,T]} \right].$$

Equations (3.3) and (3.2) imply that

$$\partial_t^\rho u(x,t) = \partial_t^\rho \omega(x,t) + p(t) q(x).$$
Hence, from estimates for $\partial_t^\alpha \omega(x, t)$ and $p(t)$ we obtain an estimate for $\partial_t^\alpha u(x, t)$. On the other hand, by virtue of equation (1.1), we have

$$-u_{xx}(x, t) = -i\partial_t^\alpha u(x, t) + p(t)q(x) + f(x, t).$$

Thus, in order to establish estimate (4.1), it suffices to use the statement of Theorem 4.1.

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