ON THE REPRESENTATION OF \( k \)-FREE INTEGERS BY BINARY FORMS

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Abstract. Let \( F \) be a binary form with integer coefficients, non-zero discriminant and degree \( d \) with \( d \geq 3 \) and let \( r \) denote the largest degree of an irreducible factor of \( F \) over the rationals. Let \( k \) be an integer with \( k \geq 2 \) and suppose that there is no prime \( p \) such that \( p^k \) divides \( F(a, b) \) for all pairs of integers \( (a, b) \). Let \( R_{F,k}(Z) \) denote the number of \( k \)-free integers of absolute value at most \( Z \) which are represented by \( F \). We prove that there is a positive number \( C_{F,k} \) such that \( R_{F,k}(Z) \) is asymptotic to \( C_{F,k}Z^{\frac{2}{d}} \) provided that \( k \) exceeds \( \frac{7}{18}r \) or \( (k, r) \) is \((2, 6)\) or \((3, 8)\).

1. Introduction

Let \( F \) be a binary form with integer coefficients, non-zero discriminant \( \Delta(F) \) and degree \( d \) with \( d \geq 3 \). For any positive number \( Z \) let \( \mathcal{R}_F(Z) \) denote the set of non-zero integers \( h \) with \( |h| \leq Z \) for which there exist integers \( x \) and \( y \) such that \( F(x, y) = h \). Denote the cardinality of a set \( \mathcal{S} \) by \( |\mathcal{S}| \) and let \( R_F(Z) = |\mathcal{R}_F(Z)| \). In \cite{37} Stewart and Xiao proved that there exists a positive number \( C_F \) such that

\[
R_F(Z) \sim C_FZ^{\frac{2}{d}}.
\]

Such a result had been obtained earlier by Hooley in \cite{16}, \cite{22}, \cite{23} and \cite{24} when \( F \) is an irreducible binary cubic form, when \( F \) is a quartic form of the shape

\[
F(x, y) = ax^4 + bx^2y^2 + cy^4.
\]

and when \( F \) is the product of linear forms with integer coefficients. In addition, a number of authors including Bennett, Dummigan, and Wooley \cite{1}, Browning \cite{5}, Greaves \cite{11}, Heath-Brown \cite{13}, Hooley \cite{19}, \cite{20}, \cite{21}, Skinner and Wooley \cite{34} and Wooley \cite{40} obtained asymptotic estimates for \( R_F(Z) \) when \( F \) is a binomial form.

Let \( k \) be an integer with \( k \geq 2 \). An integer is said to be \( k \)-free if it is not divisible by the \( k \)-th power of a prime number. For any positive number \( Z \) let \( \mathcal{R}_{F,k}(Z) \) denote the set of \( k \)-free integers \( h \) with \( |h| \leq Z \) for which there exist integers \( x \) and \( y \) such that \( F(x, y) = h \) and put \( R_{F,k}(Z) = |\mathcal{R}_{F,k}(Z)| \). Extending work of Hooley \cite{16}, \cite{18}, Gouvêa and Mazur \cite{8} in 1991 proved that if there is no prime \( p \) such that \( p^2 \) divides \( F(a, b) \) for all pairs of integers \( (a, b) \), if all the irreducible factors of \( F \) over \( \mathbb{Q} \) have degree at most 3 and if \( \varepsilon \) is a positive real number then there are positive numbers \( C_1 \) and \( C_2 \), which depend on \( \varepsilon \) and \( F \), such that if \( Z \) exceeds \( C_1 \) then

\[
R_{F,2}(Z) > C_2Z^{\frac{2}{d}}.
\]

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This was subsequently extended by Stewart and Top in \cite{36}. Let \( r \) be the largest degree of an irreducible factor of \( F \) over \( \mathbb{Q} \). Let \( k \) be an integer with \( k \geq 2 \) and suppose that there is no prime \( p \) such that \( p^k \) divides \( F(a,b) \) for all integer pairs \((a,b)\). They showed, by utilizing work of Greaves \cite{10} and Erdős and Mahler \cite{6}, that if \( k \) is at least \( (r - 1)/2 \) or \( k = 2 \) and \( r = 6 \) then there are positive numbers \( C_3 \) and \( C_4 \), which depend on \( k \) and \( F \), such that if \( Z \) exceeds \( C_3 \) then

\[
R_{F,k}(Z) > C_4 Z^{\frac{r}{2}}.
\]

The estimates \( (1.2) \) and \( (1.3) \) were used by Gouvêa and Mazur \cite{8} and Stewart and Top \cite{36} in order to estimate, for any elliptic curve defined over \( \mathbb{Q} \), the number of twists of the curve for which the rank of the Mordell-Weil group is at least 2.

For any real number \( x \) let \( \lceil x \rceil \) denote the least integer \( u \) such that \( x \leq u \). In 2016 \cite{41} Xiao extended the range for which \( (1.3) \) holds by generalizing the determinant method of Heath-Brown \cite{14} and Salberger \cite{31}, \cite{32} to the setting of weighted projective space. He proved that if

\[
k > \min \left\{ \frac{7r}{18}, \left\lceil \frac{r}{2} \right\rceil - 2 \right\},
\]

and \((k, r)\) is not \((3, 8)\) then \( (1.3) \) holds. In addition, the related problem of estimating \( B_{F,k}(Z) \), the number of pairs of integers \((x, y)\) with \( \max\{|x|, |y|\} \leq Z \) for which \( F(x, y) \) is \( k \)-free, has been studied by Browning \cite{4}, Filaseta, \cite{7}, Granville \cite{9}, Greaves \cite{10}, Helfgott \cite{15}, Hooley \cite{25}, \cite{26}, Murty and Pasten \cite{28}, Poonen \cite{30} and Xiao \cite{41}. Recently Bhargava \cite{2} and Bhargava, Shankar and Wang \cite{3} have extended these estimates to the case of discriminant forms.

By building on the method used to prove \( (1.1) \) we are now able to give an asymptotic estimate for \( R_{F,k}(Z) \) provided that \( k \) satisfies \( (1.4) \). Such an estimate has not previously been established for any integer \( k \geq 2 \) and any binary form \( F \) with integer coefficients, degree at least 3 and non-zero discriminant. We are able to prove the following result.

\begin{theorem}
Let \( F \) be a binary form with integer coefficients, non-zero discriminant and degree \( d \) with \( d \geq 3 \) and let \( r \) denote the largest degree of an irreducible factor of \( F \) over the rationals. Let \( k \) be an integer with \( k \geq 2 \) and suppose that there is no prime \( p \) such that \( p^k \) divides \( F(a,b) \) for all pairs of integers \((a,b)\). Suppose that \( (1.4) \) holds. Then there exists a positive number \( C_{F,k} \) such that

\[
R_{F,k}(Z) = C_{F,k} Z^{\frac{r}{2}} + O_{F,k} \left( Z^{\frac{r}{2}} / g_{k,r}(Z) \right)
\]

where

\[
g_{k,r}(Z) = \begin{cases} 
\log Z \log \log Z & \text{if } (k, r) \neq (2, 6) \text{ or } (3, 8) \\
(\log Z)^{(d-2)(0.7043)} / d & \text{if } (k, r) = (2, 6) \\
\left( \frac{\log \log Z}{\log \log \log Z} \right)^{1-\frac{r}{2}} & \text{if } (k, r) = (3, 8).
\end{cases}
\]

\end{theorem}
Throughout this article we make use of the standard notation \( O \), \( o \) and \( \sim \), for instance as in Section 1.6 of \[12\], with the convention that the implicit constant denoted by the symbol \( O \) may be determined in terms of the subscripts attached to it.

For a positive number \( Z \) we put
\[
N_F(Z) = \{(x, y) \in \mathbb{Z}^2 : 1 \leq |F(x, y)| \leq Z\}
\]
and
\[
N_F(Z) = |N_F(Z)|.
\]
We also put
\[
A_F = \mu(\{(x, y) \in \mathbb{R}^2 : |F(x, y)| \leq 1\})
\]
where \( \mu(\cdot) \) denotes the area of a set in \( \mathbb{R}^2 \). In 1933 Mahler \[27\] proved that if \( F \) is a binary form with integer coefficients and degree \( d \) with \( d \geq 3 \) which is irreducible over \( \mathbb{Q} \) then
\[
N_F(Z) = A_F Z^{\frac{d}{2}} + O_F \left( Z^{\frac{1}{d-1}} \right).
\]
The assumption that \( F \) is irreducible may be replaced with the weaker requirement that \( F \) have non-zero discriminant; see \[39\].

Let \( k \) be an integer with \( k \geq 2 \). For a positive number \( Z \) we put
\[
N_{F,k}(Z) = \{(x, y) \in \mathbb{Z}^2 : F(x, y) \text{ is } k\text{-free and } 1 \leq |F(x, y)| \leq Z\}
\]
and
\[
N_{F,k}(Z) = |N_{F,k}(Z)|.
\]
For each positive integer \( m \) we put
\[
\rho_F(m) = |\{(i, j) \in \{0, \ldots, m-1\}^2 : F(i, j) \equiv 0 \pmod{m}\}|
\]
and
\[
\lambda_{F,k} = \prod_p \left( 1 - \frac{\rho_F(p^k)}{p^{2k}} \right),
\]
where the product is taken over the primes \( p \). Observe that the product converges since \( k \geq 2 \) and \( \rho_F(p^k) \) is at most \( p^{2k-2} + dp^k \) provided that \( p \) does not divide the discriminant \( \Delta(F) \), see \[35\]. Further \( \lambda_{F,k} = 0 \) whenever there is a prime \( p \) such that \( p^k \) divides \( F(a, b) \) for all \( (a, b) \) in \( \mathbb{Z}^2 \). Next we put
\[
c_{F,k} = \lambda_{F,k} A_F.
\]
In order to establish Theorem 1.1 we require the following extension of Mahler’s result.

**Theorem 1.2.** Let \( F \) be a binary form with integer coefficients, non-zero discriminant and degree \( d \) with \( d \geq 3 \) and let \( r \) denote the largest degree of an irreducible factor of \( F \) over \( \mathbb{Q} \). Let \( k \) be an integer with \( k \geq 2 \) and suppose that (1.4) holds. Then, with \( c_{F,k} \) defined by (1.8), we have
\[
N_{F,k}(Z) = c_{F,k} Z^{\frac{d}{2}} + O_{F,k} \left( Z^{\frac{2}{d-1}} / g_{k,r}(Z) \right)
\]
with \( g_{k,r}(Z) \) given by (1.6).
Let $A$ be an element of $\text{GL}_2(\mathbb{Q})$ with
\[
A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.
\]

Put $F_A(x, y) = F(a_1 x + a_2 y, a_3 x + a_4 y)$. We say that $A$ fixes $F$ if $F_A = F$. The set of $A$ in $\text{GL}_2(\mathbb{Q})$ which fix $F$ is the automorphism group of $F$ and we shall denote it by $\text{Aut} F$. Let $G_1$ and $G_2$ be subgroups of $\text{GL}_2(\mathbb{Q})$. We say that they are equivalent under conjugation if there is an element $T$ in $\text{GL}_2(\mathbb{Q})$ such that $G_1 = T G_2 T^{-1}$. There are 10 equivalence classes of finite subgroups of $\text{GL}_2(\mathbb{Q})$ under $\text{GL}_2(\mathbb{Q})$-conjugation to which $\text{Aut} F$ might belong, see [29] and [37], and we give a representative of each equivalence class together with its generators in Table 1 below.

| Group | Generators | Group | Generators |
|-------|------------|-------|------------|
| $C_1$ | \[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\] | $D_1$ | \[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] |
| $C_2$ | \[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\] | $D_2$ | \[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\] |
| $C_3$ | \[
\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}
\] | $D_3$ | \[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}
\] |
| $C_4$ | \[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\] | $D_4$ | \[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\] |
| $C_6$ | \[
\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}
\] | $D_6$ | \[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}
\] |

Let $\Lambda$ be the sublattice of $\mathbb{Z}^2$ consisting of $(u, v)$ in $\mathbb{Z}^2$ for which $A(u)$ is in $\mathbb{Z}^2$ for all $A$ in $\text{Aut} F$.

When $\text{Aut} F$ is conjugate to $D_3$ it has three subgroups $G_1, G_2$ and $G_3$ of order 2 with generators $A_1, A_2$ and $A_3$ respectively, and one, $G_4$ say, of order 3 with generator $A_4$. Let $\Lambda_i = \Lambda(A_i)$ be the sublattice of $\mathbb{Z}^2$ consisting of $(u, v)$ in $\mathbb{Z}^2$ for which $A_i(u)$ is in $\mathbb{Z}^2$ for $i = 1, 2, 3, 4$.

When $\text{Aut} F$ is conjugate to $D_4$ there are three subgroups $G_1, G_2$ and $G_3$ of order 2 of $\text{Aut} F/\{\pm I\}$ where $I$ denotes the $2 \times 2$ identity matrix. Let $\Lambda_i$ be the sublattice of $\mathbb{Z}^2$ consisting of $(u, v)$ in $\mathbb{Z}^2$ for which $A_i(u)$ is in $\mathbb{Z}^2$ for $A$ in a generator of $G_i$ for $i = 1, 2, 3$.

Finally when $\text{Aut} F$ is conjugate to $D_6$ there are three subgroups $G_1, G_2$ and $G_3$ of order 2 and one, $G_4$ say, of order 3 in $\text{Aut} F/\{\pm I\}$. Let $A_i$ be in a generator of $G_i$ for $i = 1, 2, 3, 4$. Let $\Lambda_i$ be the sublattice of $\mathbb{Z}^2$ consisting of $(u, v)$ in $\mathbb{Z}^2$ for which $A_i(u)$
is in $\mathbb{Z}^2$ for $i = 1, 2, 3, 4$.

Let $L$ be a sublattice of $\mathbb{Z}^2$. We define $c_{F,k,L}$ in the following manner. For any basis $\{\omega_1, \omega_2\}$ of $L$ with $\omega_1 = (a_1, a_3)$ and $\omega_2 = (a_2, a_4)$ we define $F_{\omega_1, \omega_2}(x, y) = F(a_1x + a_2y, a_3x + a_4y)$. Notice that if $\{\omega'_1, \omega'_2\}$ is another basis for $L$ then it is related to $\{\omega_1, \omega_2\}$ by a unimodular transformation. As a consequence,

$$c_{F_{\omega_1, \omega_2}, k} = c_{F_{\omega'_1, \omega'_2}, k}$$

and so we may define $c_{F,k,L}$ by putting

$$c_{F,k,L} = c_{F_{\omega_1, \omega_2}, k}.$$

Observe that if $L = \mathbb{Z}^2$ then $c_{F,k,L} = c_{F,k}$. For brevity, we shall write

$$(1.10) \quad c(L) = c_{F,k,L}.$$ 

We are now able to determine the positive number $C_{F,k}$ in (1.5) of Theorem 1.1 explicitly in terms of $\text{Aut} F$ and the lattices described above.

**Theorem 1.3.** The positive number $C_{F,k}$ in the statement of Theorem 1.1 is given by the following table:

| Rep($F$) | $C_{F,k}$          | Rep($F$) | $C_{F,k}$          |
|----------|--------------------|----------|--------------------|
| $C_1$    | $c_{F,k}$         | $D_1$    | $c_{F,k} - \frac{c(\Lambda)}{2}$ |
| $C_2$    | $\frac{c_{F,k}}{2}$ | $D_2$    | $\frac{1}{2} \left( c_{F,k} - \frac{c(\Lambda)}{2} \right)$ |
| $C_3$    | $c_{F,k} - \frac{2c(\Lambda)}{3}$ | $D_3$    | $c_{F,k} - \frac{c(\Lambda_1)}{2} - \frac{c(\Lambda_2)}{2} - \frac{c(\Lambda_3)}{2} - \frac{2c(\Lambda_4)}{3} + \frac{4c(\Lambda)}{3}$ |
| $C_4$    | $\frac{1}{2} \left( c_{F,k} - \frac{c(\Lambda)}{2} \right)$ | $D_4$    | $\frac{1}{2} \left( c_{F,k} - \frac{c(\Lambda_1)}{2} - \frac{c(\Lambda_2)}{2} - \frac{c(\Lambda_3)}{2} + \frac{3c(\Lambda)}{4} \right)$ |
| $C_5$    | $\frac{1}{2} \left( c_{F,k} - \frac{2c(\Lambda)}{3} \right)$ | $D_6$    | $\frac{1}{2} \left( c_{F,k} - \frac{c(\Lambda_1)}{2} - \frac{c(\Lambda_2)}{2} - \frac{c(\Lambda_3)}{2} - \frac{2c(\Lambda_4)}{3} + \frac{4c(\Lambda)}{3} \right)$ |

Here $\text{Rep}(F)$ denotes a representative of the equivalence class of $\text{Aut} F$ under $\text{GL}_2(\mathbb{Q})$-conjugation, $\Lambda$ and $\Lambda_i$’s are defined above, $c_{F,k}$ is as in (1.8), and $c(\Lambda)$ and $c(\Lambda_i)$ as in (1.10).

We conjecture that the estimates for $R_{F,k}(Z)$ in Theorem 1.1 and for $N_{F,k}(Z)$ in Theorem 1.2 apply without hypothesis (1.4).

**Conjecture 1.4.** Let $F$ be a binary form with integer coefficients, non-zero discriminant and degree $d$ with $d$ at least 3. Let $k$ be an integer larger than 1. Then either $c_{F,k} = 0$ or

$$(1.11) \quad N_{F,k}(Z) \sim c_{F,k}Z^{\frac{d}{2}}$$
where $c_{F,k}$ is defined by (1.3). If there is no prime $p$ such that $p^k$ divides $F(a,b)$ for all pairs of integers $(a,b)$ then

$$(1.12) \quad R_{F,k}(Z) \sim C_{F,k} Z^{\frac{d}{2}}$$

where $C_{F,k}$ is the positive number given by Theorem 1.3.

Let $F$ be a binary form with integer coefficients, non-zero discriminant and degree $d$ with $d \geq 3$. Granville [9] established an asymptotic estimate for $B_{F,2}(Z)$, the number of pairs of integers $(x, y)$ with absolute value at most $Z$ for which $F(x, y)$ is squarefree subject to the abc conjecture, see eg. [38]. Let $k$ be an integer with $k > 1$. The same analysis allows one to give an asymptotic estimate for $B_{F,k}(Z)$, the number of pairs of integers $(x, y)$ with absolute value at most $Z$ for which $F(x, y)$ is $k$-free. We may use such an estimate in conjunction with the arguments used to prove Theorem 1.1 and Theorem 1.2 in order to prove Conjecture 1.4, see the final paragraph of Section 6. In particular, Conjecture 1.4 follows from the abc conjecture.

2. Preliminary lemmas

Let $F(x, y)$ be a binary form with integer coefficients, non-zero discriminant and degree $d$ with $d \geq 3$. Suppose that $F$ factors over $\mathbb{C}$ as

$$(2.1) \quad F(x, y) = \prod_{i=1}^{d} (\gamma_i x + \beta_i y)$$

and put

$$\mathcal{H}(F) = \prod_{i=1}^{d} \sqrt{\gamma_i^2 + \beta_i^2}.$$ 

Then $\mathcal{H}(F)$ does not depend on the factorization in (2.1).

A special case of Theorem 3 of Thunder [39] is the following explicit version of a result of Mahler [27].

**Lemma 2.1.** Let $F$ be a binary form with integer coefficients, non-zero discriminant and degree $d \geq 3$. Then, with $A_F$ defined by (1.7), we have

$$\left| N_{F}(Z) - A_F Z^{\frac{d}{2}} \right| = O \left( Z^{\frac{d}{2+1}} \mathcal{H}(F)^{d-2} \right),$$

where the implied constant is absolute.

We may write

$$(2.2) \quad F(x, y) = a \prod_{i=1}^{d} (x - \alpha_i y)$$

where $a$ is a positive integer and $\alpha_1, \ldots, \alpha_d$ are the roots of $F(x, 1)$ provided that $y$ is not a factor of $F(x, y)$. In the latter case, since the discriminant of $F$ is non-zero, we have

$$(2.3) \quad F(x, y) = ay \prod_{i=1}^{d-1} (x - \alpha_i y).$$
Put
\[ E_F = \frac{2 \max_{1 \leq j \leq k} (1, |\alpha_j|)}{\min(1, \min_i |\alpha_i - \alpha_j|)} \]
where \( k = d \) if (2.2) holds and \( k = d - 1 \) if (2.3) holds.

**Lemma 2.2.** Let \( F \) be a binary form with integer coefficients, non-zero discriminant and degree \( d \) with \( d \geq 3 \). Let \( Z \) be a positive real number. For any positive real number \( \beta \) larger than \( E_F \) the number of pairs of integers \((x, y)\) with
\[ 0 < |F(x, y)| \leq Z \]
for which
\[ \max\{|x|, |y|\} > Z^{\frac{1}{d}} \beta \]
is
\[ O_F \left( Z^{\frac{2}{d}} \log Z + \frac{Z^{\frac{2}{d}}}{\beta^{d-2}} \right). \]

**Proof.** We shall follow Heath-Brown’s proof of Theorem 8 in [14]. Accordingly put
\[ S(Z; C) = |\{(x, y) \in \mathbb{Z}^2 : 0 < |F(x, y)| \leq Z, C < \max\{|x|, |y|\} \leq 2C, \gcd(x, y) = 1\}|. \]
Suppose that
\[ C \geq Z^{\frac{1}{d}} E_F. \]
Heath-Brown observes that by Roth’s theorem \( S(Z; C) = 0 \) unless \( C \ll FZ^2 \).

Suppose that we are in the case when (2.2) holds and that \((x, y)\) is a pair of integers with \( \gcd(x, y) = 1 \),
\[ 0 < |F(x, y)| \leq Z \]
and
\[ C < \max\{|x|, |y|\} \leq 2C \]
Further suppose that \( i_0 \) is an index for which
\[ |x - \alpha_{i_0} y| \leq |x - \alpha_j y| \]
for \( j = 1, \ldots, d \). Note that then
\[ |x - \alpha_{i_0} y| \leq Z^{1/d}. \]
We have two cases to consider. The first case is when \( \max\{|x|, |y|\} = |y| \). In this case we have, for \( j \neq i_0 \),
\[ |x - \alpha_j y| = |(x - \alpha_{i_0} y) + (\alpha_{i_0} - \alpha_j) y| \geq |\alpha_{i_0} - \alpha_j||y| - |x - \alpha_{i_0} y| \]
and, by (2.5), (2.6) and (2.7),
\[ \frac{1}{2} |\alpha_{i_0} - \alpha_j||y| - |x - \alpha_{i_0} y| \geq \frac{1}{2} |\alpha_{i_0} - \alpha_j|Z^{1/d} E_F - Z^{1/d} \geq 0. \]
Thus, by (2.8) and (2.9),
\[ |x - \alpha_j y| \geq \frac{1}{2} |\alpha_{i_0} - \alpha_j||y| \geq \frac{1}{2} |\alpha_{i_0} - \alpha_j| C. \]
The second case is when $\max(|x|, |y|) = |x|$. Then

$$(2.11) \quad |\alpha_{i_0}(x - \alpha_j y)| = |(\alpha_{i_0} - \alpha_j)x + \alpha_j(x - \alpha_{i_0}y)| \geq |\alpha_{i_0} - \alpha_j||x| - |\alpha_j||x - \alpha_{i_0}y|,$$

and, by $(2.4)$, $(2.10)$ and $(2.11)$,

$$(2.12) \quad \frac{1}{2}|\alpha_{i_0} - \alpha_j||x| - |\alpha_j||x - \alpha_{i_0}y| \geq \frac{1}{2}|\alpha_{i_0} - \alpha_j|Z^{1/d}E_F - |\alpha_j|Z^{1/d} \geq 0.$$

Thus, by $(2.11)$ and $(2.12)$,

$$(2.13) \quad |x - \alpha_j y| \geq \frac{1}{2|\alpha_{i_0}|}|\alpha_{i_0} - \alpha_j|C.$$

It now follows from $(2.6)$, $(2.10)$ and $(2.13)$ that

$$(2.14) \quad C \ll_F |x - \alpha_j y| \ll_F C.$$

We obtain $(2.14)$ in a similar fashion when $(2.3)$ holds.

Thus, by $(2.14)$,

$$(2.15) \quad |x - \alpha_{i_0}y| \ll_F Z/C^{d-1}.$$

The number of coprime integer pairs $(x, y)$ satisfying $(2.6)$ and $(2.15)$ for some index $i_0$ is an upper bound for $S(Z; C)$ and therefore, by Lemma 1, part (vii) of [14],

$$(2.16) \quad S(Z; C) \ll_F 1 + \frac{Z}{C^{d-2}}.$$

Put

$$(2.17) \quad S^{(1)}(Z; C) = |\{(x, y) \in \mathbb{Z}^2 : 0 < |F(x, y)| \leq Z, C < \max\{|x|, |y|\}, \gcd(x, y) = 1\}|.$$

Therefore on replacing $C$ by $2^j C$ in $(2.16)$ for $j = 1, 2, \cdots$ and summing we find that

$$(2.18) \quad S^{(1)}(Z; C) \ll_F \log Z + \frac{Z}{C^{d-2}}.$$

Next put

$$(2.19) \quad S^{(2)}(Z; C) = |\{(x, y) \in \mathbb{Z}^2 : 0 < |F(x, y)| \leq Z, C < \max\{|x|, |y|\}\}|.$$

Then

$$S^{(2)}(Z; C) \ll_F \sum_{h \leq Z^{1/d}} S^{(1)}\left(\frac{Z}{h^{d-1}}, \frac{C}{h}\right)$$

and since $C > Z^{\frac{1}{d}} E_F$ we see that

$$\frac{C}{h} > \left(\frac{Z}{h^{d}}\right)^{\frac{1}{d}} E_F,$$

hence

$$S^{(2)}(Z; C) \ll_F \sum_{h \leq Z^{1/d}} \left(\log Z + \frac{Z}{h^{2C^{d-2}}}\right) \ll_F Z^{\frac{1}{d}} \log Z + \frac{Z}{C^{d-2}}.$$

Our result now follows on taking $C = Z^{\frac{1}{d}} \beta$ since $\beta > E_F$. \qed
For any positive real numbers $Z$ and $\beta$ put
\[
N_F(Z, \beta) = |\{(x, y) \in \mathbb{Z}^2 : |F(x, y)| \leq Z, \max\{|x|, |y|\} \leq Z^{\frac{1}{d}}\beta\}|.
\]

**Lemma 2.3.** Let $F$ be a binary form of degree $d \geq 3$ with integer coefficients and non-zero discriminant. Let $E_F$ be as in (2.4) and suppose that $\beta$ is a real number with $\beta > E_F$. Then
\[
N_F(Z, \beta) = A_F Z^{\frac{1}{d}} + O_F \left(Z^{\frac{1}{d-1}} + Z^{\frac{2}{d}}\beta + Z^{\frac{3}{d}}\beta^{-(d-2)}\right).
\]

**Proof.** This follows from Lemma 2.1 and Lemma 2.2 on noting that the number of pairs of integers $(x, y)$ with $\max\{|x|, |y|\} \leq Z^{\frac{1}{d}}\beta$ for which $F(x, y) = 0$ is at most $O_F \left(Z^{\frac{1}{d}}\beta\right)$. \(\square\)

Next we introduce the quantity
\[
\tilde{N}_F(Z, \beta) = |\{(x, y) \in \mathbb{Z}^2 : |F(x + \theta_1, y + \theta_2)| \leq Z, 0 \leq \theta_1 < 1, 0 \leq \theta_2 < 1, \max\{|x + \theta_1|, |y + \theta_2|\} \leq Z^{\frac{1}{d}}\beta\}|
\]
in order to facilitate the determination of the main terms in Theorems 1.1 and 1.2.

**Lemma 2.4.** Let $F$ be a binary form with integer coefficients, non-zero discriminant and degree $d \geq 3$. Let $E_F$ be as in (2.4) and suppose that $\beta$ is a real number with $\beta > E_F$. Then
\[
\tilde{N}_F(Z, \beta) = A_F Z^{\frac{1}{d}} + O_F \left(Z^{\frac{1}{d-1}} + Z^{\frac{2}{d}}\beta^{-(d-2)} + Z^{\frac{3}{d}}\beta^{d-1}\right).
\]

**Proof.** Note that there is a positive number $\kappa$, which depends on $F$, such that for $(\theta_1, \theta_2) \in \mathbb{R}^2$ with $|\theta_i| \leq 1$ for $i = 1, 2$ we have
\[
|F(x + \theta_1, y + \theta_2)| \leq |F(x, y)| + \kappa \max\{|x|, |y|\}^{d-1}.
\]

Put
\[
(2.17) \quad Z_1 = Z - \kappa \left(Z^{\frac{1}{d}}\beta\right)^{d-1}.
\]

Thus if $\max\{|x|, |y|\} \leq Z^{\frac{1}{d}}\beta$ and
\[
|F(x, y)| \leq Z_1
\]
then, for $(\theta_1, \theta_2) \in \mathbb{R}^2$ with $|\theta_i| \leq 1$ for $i = 1, 2$, we have
\[
(2.18) \quad |F(x + \theta_1, y + \theta_2)| \leq Z.
\]

Observe that
\[
(2.19) \quad Z_1^{\frac{2}{d}} = \left(Z - \kappa \left(Z^{\frac{1}{d}}\beta\right)^{d-1}\right)^{\frac{2}{d}} = Z^{\frac{2}{d}} + O_F \left(Z^{\frac{1}{d}}\beta^{d-1}\right).
\]

Thus, since $Z_1 \leq Z$, by Lemma 2.3 the number of pairs of integers $(x, y)$ for which $\max\{|x|, |y|\} \leq Z^{\frac{1}{d}}\beta$ and (2.18) holds is at least
\[
(2.20) \quad A_F Z_1^{\frac{2}{d}} + O_F \left(Z_1^{\frac{1}{d-1}} + Z_1^{\frac{1}{d}}\beta + Z_1^{\frac{2}{d}}\beta^{-(d-2)}\right),
\]
and so by (2.19) and (2.20), at least
\[ A_F Z^{\frac{1}{d}} + O_F \left( Z^{\frac{1}{d} - \frac{2}{d} + \frac{2}{d} \beta^{-2d}} + Z^{\frac{1}{d} \beta^{d-1}} \right). \]

Therefore, by Lemma 2.3 and (2.21),
\[ \tilde{N}_F(Z, \beta) = A_F Z^{\frac{1}{d}} + O_F \left( Z^{\frac{1}{d} - \frac{2}{d} + \frac{2}{d} \beta^{-2d}} + Z^{\frac{1}{d} \beta^{d-1}} \right), \]
as required.

We now put, for a real number $Z$, an integer $k$ with $k \geq 2$ and positive numbers $\gamma$ and $\beta$,
\[ N_{F,k}(Z, \gamma, \beta) = |\{(x, y) \in \mathbb{Z}^2 : |F(x, y)| \leq Z, \max\{|x|, |y|\} \leq Z^{\frac{1}{d} \beta} \text{ and } F(x, y) \text{ is not divisible by } p^k \text{ for any prime } p \text{ with } p \leq \gamma\}|, \]
and
\[ N_{F,k}(Z, \gamma) = |\{(x, y) \in \mathbb{Z}^2 : 0 < |F(x, y)| \leq Z \text{ and } F(x, y) \text{ is not divisible by } p^k \text{ for any prime } p \text{ with } p \leq \gamma\}|. \]

**Lemma 2.5.** Let $F$ be a binary form with integer coefficients, non-zero discriminant and degree $d$ with $d \geq 3$. Then
\[ N_{F,k} \left( Z, \frac{1}{2kd} \log Z \right) = c_{F,k} Z^{\frac{2}{d}} + O_{F,k} \left( Z^\frac{2}{d} / (\log Z \log \log Z) \right) \]
with $c_{F,k}$ given by (1.8).

**Proof.** We have
\[ N_{F,k} \left( Z, \frac{1}{2kd} \log Z \right) = N_{F,k} \left( Z, \frac{1}{2kd} \log Z, (\log Z)^6 \right) + O_{F,k} \left( Z^{\frac{2}{d}} (\log Z)^6 \right) \]
\[ + O_{F,k} \left( |\{(x, y) \in \mathbb{Z}^2 : 0 < |F(x, y)| \leq Z \text{ and } \max\{|x|, |y|\} > Z^{\frac{1}{d} \beta} (\log Z)^6\}| \right). \]

By Lemma 2.2, since $d \geq 3$,
\[ N_{F,k} \left( Z, \frac{1}{2kd} \log Z \right) = N_{F,k} \left( Z, \frac{1}{2kd} \log Z, (\log Z)^6 \right) + O_{F,k} \left( Z^\frac{2}{d} / (\log Z)^6 \right). \]

Next we put
\[ V = V(d, k, Z) = \prod_{p \leq \log Z / (2kd)} p^k, \]
where the product is taken over primes $p$. For each pair of integers $(a, b)$ we define $B(a, b)$ by
\[ B(a, b) = \{(x, y) \in \mathbb{R}^2 : aV \leq x < (a + 1)V, bV \leq y < (b + 1)V\}. \]
Observe that $B(a, b)$ is a square in $\mathbb{R}^2$. We say that $B(a, b)$ is admissible if
\[ |F(x, y)| \leq Z \text{ and } \max\{|x|, |y|\} \leq Z^{\frac{1}{d} \beta} (\log Z)^6 \]
whenever $(x, y)$ is in $B(a, b)$. Let $B_1$ denote the number of admissible squares $B(a, b)$. We have
\[ B_1 = \tilde{N}_F \left( \frac{Z}{V^d}, (\log Z)^6 \right) \]
and so by Lemma 2.4, since \( d \geq 3 \),

\[
B_1 = A_F \frac{Z^{\frac{1}{d}}}{V^2} + O_F \left( \frac{Z^2}{V^d} + \frac{Z^{\frac{2}{d}}}{V^2(\log Z)^6} + \frac{Z^{\frac{1}{d}}}{V}(\log Z)^6(d-1) \right)
\]

Therefore by Lemma 2.3 with \( \beta = (\log Z)^6 \) the number of pairs of integers \((x, y)\) for which (2.23) holds and which are not in an admissible square is

\[
O_F \left( Z^{\frac{1}{d}}V^{\frac{d+2}{d}} + Z^{\frac{2}{d}}/(\log Z)^6 + Z^{\frac{1}{d}}V(\log Z)^6(d-1) \right).
\]

We may now apply the Chinese Remainder Theorem to conclude that within each admissible square the number of integer pairs \((x, y)\) for which \( F(x, y) \) is not divisible by \( p^k \) for any prime \( p \) with \( p \leq \frac{1}{2kd} \log Z \) is precisely

\[
\prod_{p \leq \log Z/(2kd)} \left( 1 - \frac{\rho_F(p^k)}{p^{2k}} \right) V^2.
\]

Thus the number of integer pairs \((x, y)\) in an admissible square with \( F(x, y) \) not divisible by \( p^k \) for any prime \( p \) with \( p \leq \frac{1}{2kd} \log Z \) is

\[
B_1 \prod_{p \leq \log Z/(2kd)} \left( 1 - \frac{\rho_F(p^k)}{p^{2k}} \right) V^2.
\]

Therefore, by (2.24), (2.25) and (2.26),

\[
N_{F,k} \left( Z, \frac{1}{2kd} \log Z \right) = A_F \prod_{p \leq \log Z/(2kd)} \left( 1 - \frac{\rho_F(p^k)}{p^{2k}} \right) Z^{\frac{2}{d}} + O_F \left( Z^{\frac{2}{d}}/(\log Z)^6 + Z^{\frac{1}{d}}V(\log Z)^6 \right).
\]

By the Prime Number Theorem,

\[
V = O \left( Z^{\frac{1}{d}+\frac{1}{d^2}} \right)
\]

and so, by (2.27),

\[
N_{F,k} \left( Z, \frac{1}{2kd} \log Z \right) = A_F \prod_{p \leq \log Z/(2kd)} \left( 1 - \frac{\rho_F(p^k)}{p^{2k}} \right) Z^{\frac{2}{d}} + O_{F,k} \left( Z^{\frac{2}{d}}/(\log Z)^6 \right).
\]

Note that the number of integer pairs \((a, b)\) with \( 0 \leq a < p^k \) and \( 0 \leq b < p^k \) for which \( p \) divides both \( a \) and \( b \) is \( p^{2k-2} \). Further the number of pairs \((a, b)\) for which \( p \) does not divide both \( a \) and \( b \) and for which \( F(a, b) \equiv 0 \pmod{p^k} \) is at most \( dp^k \) provided that \( p \) does not divide \( \Delta(F) \), see [35]. Thus for primes \( p \) which do not divide \( \Delta(F) \), we have

\[
\rho_F(p^k) \leq p^{2k-2} + dp^k \leq (d+1)p^{2k-2},
\]

since \( k \geq 2 \). Put

\[
P = \prod_{p \leq \log Z/(2kd)} \left( 1 - \frac{\rho_F(p^k)}{p^{2k}} \right), \quad P_1 = \prod_{p} \left( 1 - \frac{\rho_F(p^k)}{p^{2k}} \right)
\]
and
\[ t = \sum_{p > \log Z/(2kd)} \log \left( 1 - \frac{\rho_F(p^k)}{p^{2k}} \right). \]

Then
\[ P_1 - P = P(e^t - 1) = -P \left( -t - \frac{t^2}{2!} - \frac{t^3}{3!} - \cdots \right). \]

Since \( t \) is negative,
\[ 0 \leq P - P_1 \leq -Pt. \]

Further,
\[ -t = O_{F,k} \left( \sum_{p > \log Z/(2kd)} \frac{\rho_F(p^k)}{p^{2k}} \right) \]

and by (2.29),
\[ (2.31) \]
\[ -t = O_{F,k} \left( \sum_{p > \log Z/(2kd)} \frac{1}{p^2} \right). \]

We have
\[ \sum_{p > \log Z/(2kd)} \frac{1}{p^2} = \sum_{j=0}^{\infty} \sum_{2j \log \frac{Z}{2kd} < p < 2j + 1 \log \frac{Z}{2kd}} \frac{1}{p^2} \]

and so, by the Prime Number Theorem,
\[ (2.32) \]
\[ \sum_{p > \log Z/(2kd)} \frac{1}{p^2} = O_k \left( \sum_{j=0}^{\infty} \frac{2^{j+1} \log Z}{(j+1) \log 2 + \log \log Z} \frac{1}{2^{2j} (\log Z)^2} \right) = O_k \left( \frac{1}{\log Z \log \log Z} \right). \]

Therefore, by (2.30), (2.31) and (2.32),
\[ (2.33) \]
\[ P = P_1 + O_{F,k} \left( \frac{1}{\log Z \log \log Z} \right). \]

It now follows from (2.28) and (2.33) that
\[ (2.34) \]
\[ N_{F,k} \left( Z, \frac{1}{2kd} \log \frac{Z}{2kd} \right) = c_{F,k} Z^{\frac{3}{2}} + O_{F,k} \left( \frac{Z^{\frac{3}{2}}}{(\log Z \log \log Z)} \right), \]

as required. \( \square \)

We say that an integer \( h \) is **essentially represented** by \( F \) if whenever \((x_1, y_1), (x_2, y_2)\) are in \( \mathbb{Z}^2 \) and
\[ F(x_1, y_1) = F(x_2, y_2) = h \]
then there exists \( A \) in \( \text{Aut } F \) such that
\[ A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}. \]
We remark that if there is only one integer pair \((x, y)\) for which \(F(x, y) = h\) then \(h\) is essentially represented since \(I\) is in \(\text{Aut } F\).

For any positive number \(Z\) let \(R^{(2)}_F(Z)\) denote the number of integers \(h\) with \(0 < |h| \leq Z\) which are not essentially represented by \(F\). For each binary form \(F\) with integer coefficients, non-zero discriminant and degree \(d\) with \(d \geq 3\) we define \(\beta_F\) in the following way. If \(F\) has a linear factor in \(\mathbb{R}[x, y]\) we put

\[
(2.35) \quad \beta_F = \begin{cases} 
12 \\
19 \\
4 \\
7 \\
5 \\
9 \\
\frac{3}{(d-2)\sqrt{d}+3} \\
\frac{1}{d-1}
\end{cases}
\]

if \(d = 3\) and \(F\) is irreducible over \(\mathbb{Q}\)

if \(d = 3\) and \(F\) has exactly one linear factor over \(\mathbb{Q}\)

if \(d = 3\) and \(F\) has three linear factors over \(\mathbb{Q}\)

if \(4 \leq d \leq 8\)

if \(d \geq 9\).

If \(F\) does not have a linear factor over \(\mathbb{R}\) then \(d\) is even and we put

\[
(2.36) \quad \beta_F = \begin{cases} 
\frac{3}{8} \\
\frac{1}{2\sqrt{6}} \\
\frac{1}{d-1}
\end{cases}
\]

if \(d = 4\)

if \(d = 6\)

if \(d \geq 8\).

In [37], Stewart and Xiao, building on work of Heath-Brown [14], Salberger [31], [32] and Colliot-Thélène [14], proved the following result.

Lemma 2.6. Let \(F\) be a binary form with integer coefficients, non-zero discriminant and degree \(d\) with \(d \geq 3\). Then for each \(\varepsilon > 0\),

\[
R^{(2)}_F(Z) = \mathcal{O}_{F,\varepsilon} (Z^{\beta_F+\varepsilon})
\]

where \(\beta_F\) is given by (2.35) and (2.36).

The proof of Lemma 2.6 is based on the \(p\)-adic determinant method of Heath-Brown as elaborated in [14].

Recall that if \(F\) is a binary form we denote by \(\Lambda\) the sublattice of \(\mathbb{Z}^2\) consisting of integer pairs \((u, v)\) for which \(A_{(u)}(v)\) is in \(\mathbb{Z}^2\) for all \(A\) in \(\text{Aut } F\). Further, if \(\text{Aut } F\) is
conjugate to $\mathbf{D}_3, \mathbf{D}_4$ and $\mathbf{D}_6$ we define $\Lambda_i$ for $i = 1, 2, 3, 4$ as in our discussion following Table 1 in the introduction.

**Lemma 2.7.** Let $F$ be a binary form with integer coefficients, non-zero discriminant and degree $d \geq 3$. If $A$ is an element of order 3 in $\text{Aut } F$ then

$$\Lambda(A^2) = \Lambda(A).$$

If $\text{Aut } F$ is equivalent under conjugation in $\text{GL}_2(\mathbb{Q})$ to $\mathbf{D}_3, \mathbf{D}_4$ or $\mathbf{D}_6$ then

$$\Lambda_i \cap \Lambda_j = \Lambda$$

for $i \neq j$.

Lemma 2.7 is Lemma 3.2 of [37].

### 3. Outline of the proof of Theorem 1.2

Let $N_1$ denote the number of integer pairs $(x, y)$ for which

(i) $0 < |F(x, y)| \leq Z$,

and

(ii) $p^k | F(x, y)$ for $1 \leq p \leq \frac{1}{2kd} \log Z$.

By Lemma 2.5

$$N_1 = c_{F,k} Z^{\frac{d}{2}} + O_{F,k} \left( Z^{\frac{d}{2}} / (\log Z \log \log Z) \right).$$

Our objective is to show that the number $N$ of integer pairs for which (i) holds and

(iii) $p^k | F(x, y)$ for $p$ a prime,

satisfies a similar estimate to (3.1). To that end let $N_2$ denote the number of integer pairs $(x, y)$ for which (i) holds and $p$ divides both $x$ and $y$ for some prime $p > \frac{1}{2kd} \log Z$. Let $N_3$ denote the number of pairs of integers $(x, y)$ for which (i) holds and for some prime $p$ with

$$\frac{1}{2kd} \log Z < p \leq (\log Z)^9$$

we have $p^k | F(x, y)$ and $p \nmid \gcd(x, y)$. Let $N_4$ denote the number of integer pairs $(x, y)$ for which (i) holds and for some prime $p$ with

$$(\log Z)^9 < p \leq \frac{Z^{\frac{d}{2}}}{(\log Z)^9},$$

$p^k | F(x, y)$ and $p \nmid \gcd(x, y)$. Finally let $N_5$ denote the number of integer pairs $(x, y)$ for which (i) holds and for some prime $p$ with

$$\frac{Z^{\frac{d}{2}}}{(\log Z)^9} < p,$$

$p^k | F(x, y)$ and $p \nmid \gcd(x, y)$. Then

$$N = N_1 + O(N_2 + N_3 + N_4 + N_5).$$

In order to establish Theorem 1.2 it suffices, by (3.1) and (3.2), to prove that

$$N_i = O_{F,k} \left( Z^{\frac{d}{2}} / u(z) \right)$$

for $i = 2, 3, 4$ and 5 where

$$u(z) = \log Z \log \log Z.$$
when $k$ and $r$ satisfy (1.4) with $(k, r)$ not $(2, 6)$ or $(3, 8),$

(3.4) $$u(z) = (\log Z)^{\frac{(\delta - 2)\delta}{2}}$$

with $\delta = 0.7043$ when $(k, r)$ is $(2, 6)$ and

(3.5) $$u(z) = (\log \log Z / \log \log \log Z)^{1 - \frac{2}{\delta}}$$

when $(k, r)$ is $(3, 8)$.

We may suppose that $F$ factors over $\mathbb{Q}$ as

(3.6) $$F(x, y) = \prod_{i=1}^{t} F_i(x, y)$$

with $F_i$ in $\mathbb{Z}[x, y]$ and irreducible over $\mathbb{Q}$ for $i = 1, \ldots, t$. Let $r_i$ be the degree of $F_i$ for $i = 1, \ldots, t$ and put

(3.7) $$r = \max\{r_1, \ldots, r_t\}.$$

4. AN ESTIMATE FOR $N_2$ AND FOR $N_3$

Notice that if $p$ divides $a$ and $b$ and $0 < |F(a, b)| \leq Z$ then $|F(a, b)| = p^d |F(a/p, b/p)|$, so $p \leq Z^{\frac{d}{2}}$. As a consequence

(4.1) $$N_2 = O \left( \sum_{\frac{1}{2k+1} \log Z < p \leq Z^{\frac{d}{2}}} \left| \left\{ (x, y) \in \mathbb{Z}^2 : 0 < |F(x, y)| \leq \frac{Z}{p^d} \right\} \right| \right).$$

Further, by Lemma 2.1, for each prime $p$ with $p \leq Z^{\frac{d}{2}}$,

(4.2) $$\left| \left\{ (x, y) \in \mathbb{Z}^2 : 0 < |F(x, y)| \leq \frac{Z}{p^d} \right\} \right| = A_F \frac{Z^\frac{d}{2}}{p^2} + O_F \left( \left( \frac{Z}{p^d} \right)^{\frac{1}{d-1}} \right).$$

Thus by (4.1) and (4.2),

(4.3) $$N_2 = O_F \left( \left( \sum_{\frac{1}{2k+1} \log Z < p \leq Z^{\frac{d}{2}}} \frac{1}{p^2} \right) Z^\frac{d}{2} \right).$$

It now follows from (2.32) and (4.3) that

(4.4) $$N_2 = O_{F, k} \left( Z^\frac{d}{2} / (\log Z \log \log Z) \right).$$

The integer pairs $(a, b)$ with $F(a, b) \equiv 0 \pmod{p^k}$ and for which $p$ does not divide both $a$ and $b$ lie in at most $d$ sublattices $L_\theta$ of $\mathbb{Z}^2$, provided that $p$ does not divide the discriminant $\Delta(F)$ of $F$, see [10]. Each sublattice $L_\theta$ is defined by a congruence of the form

$$a \equiv \theta b \pmod{p^k}$$

for some integer $\theta$ with $0 \leq \theta < p^k$. Let $(a_1, a_3)$ and $(a_2, a_4)$ be a basis for $L_\theta$ chosen so that $\max\{|a_1|, |a_2|, |a_3|, |a_4|\}$ is minimized. Then

$$\max\{|a_1|, |a_2|, |a_3|, |a_4|\} \leq p^k.$$
Put
\[ F_{L_\theta}(x, y) = F(a_1 x + a_2 y, a_3 x + a_4 y) \]
and notice that
\[ |N_F(Z) \cap L_\theta| = N_{F_{L_\theta}}(Z). \]
Observe that
\[ \mathcal{H}(F_{L_\theta}) \leq 4^d p^{kd} \mathcal{H}(F). \]
Therefore by Lemma 2.1
\[ N_{F_{L_\theta}}(Z) = A_{F_{L_\theta}} Z^{\frac{d}{p_k}} + O_F \left( p^{kd} Z^{\frac{1}{d^2 - 1}} \right) \]
and, since the lattice \( L_\theta \) has determinant \( p^k \),
\[ N_{F_{L_\theta}}(Z) = \frac{A_F Z^{\frac{d}{p_k}}}{p_k} + O_F \left( p^{kd} Z^{\frac{1}{d^2 - 1}} \right). \]
Thus
\[
N_3 = O_{F,k} \left( Z^{\frac{d}{p_k}} \sum_{\frac{1}{2d} \log Z < p \leq (\log Z)^9} \frac{1}{p^k} \right) \]
\[ = O_{F,k} \left( Z^{\frac{d}{p_k}} \sum_{\frac{1}{2d} \log Z < p} \frac{1}{p^2} \right), \]
and so, by (2.32),
\[
N_3 = O_{F,k} \left( Z^{\frac{d}{p_k}} / (\log Z \log \log Z) \right). \]

5. An estimate for \( N_4 \)

In order to estimate \( N_4 \) we note that
\[ N_4 = O \left( N_4^{(1)} + N_4^{(2)} \right) \]
where \( N_4^{(1)} \) is the number of integer pairs \((x, y)\) for which
\[ \max\{|x|, |y|\} \leq Z^{\frac{d}{p_k}} (\log Z)^{7/2} \]
and for which \( p^k \) divides \( F(x, y) \) for some \( p \) with
\[ (\log Z)^9 < p \leq Z^{\frac{d}{p_k}} / (\log Z)^9 \]
which does not divide both \( x \) and \( y \). Further \( N_4^{(2)} \) is the number of integer pairs \((x, y)\) for which \( 0 < |F(x, y)| \leq Z^d \) and
\[ \max\{|x|, |y|\} > Z^{\frac{d}{p_k}} (\log Z)^{7/2}. \]
By Lemma 2.2 we have, since \( d \) is at least 3,
\[ N_4^{(2)} = O_{F} \left( Z^{\frac{d}{p_k}} / (\log Z)^{\frac{d}{2}} \right). \]
It remains only to estimate \( N_4^{(1)} \) and we shall do so by a modification of an argument of Greaves [10] based on the geometry of numbers.
Recall (3.6). For \( i = 1, \ldots, t \) we let \( N^{(1)}_{4,i} \) be the number of integer pairs \((x, y)\) for which \( F(x, y) \neq 0 \), (5.2) holds and \( p^k \) divides \( F_i(a, b) \) for some prime satisfying (5.3) which does not divide both \( x \) and \( y \). Then

\[
N^{(1)}_4 = O(N^{(1)}_{4,1} + \cdots + N^{(1)}_{4,t}).
\]

Suppose that \((a, b)\) is an integer pair for which \( p^k \) divides \( F_i(a, b) \) for some prime \( p \) satisfying (5.3) which does not divide both \( a \) and \( b \). We may suppose that \( Z \) is sufficiently large that \((\log Z)^9\) exceeds \(|\Delta(F)|\). Then \((a, b)\) belongs to one of at most \( r_i \) lattices \( L_\theta \) defined by a congruence \( a \equiv \theta b \mod{p^k} \).

Following Greaves [10] we let \( M = M(\theta, p^k) \) denote the minimal possible positive value of \( \max\{|a|, |b|\} \) as we range over \((a, b)\) in \( L_\theta \). For any real number \( X \) let \( N_{\theta, k}(X) \) denote the number of pairs \((a, b)\) in \( L_\theta \) for which \(|a| \leq X\) and \(|b| \leq X\). Then, by Lemma 1 of [10],

\[
N_{\theta, k}(X) \leq \frac{4X^2}{p^k} + O\left(\frac{X}{M}\right).
\]

It then follows from (5.6) with \( X = Z^{\frac{1}{2}}(\log Z)^{\frac{7}{2}} \) that

\[
N^{(1)}_{4,i} \leq \sum_{(\log Z)^9 < p \leq Z^{\frac{1}{2}}(\log Z)^{-9}} \sum_\theta N_{\theta, k}(X) \\
\leq \sum_{(\log Z)^9 < p \leq Z^{\frac{1}{2}}(\log Z)^{-9}} \sum_\theta \left(\frac{4Z^{\frac{1}{2}}(\log Z)^{\frac{7}{2}}}{p^k} + O\left(\frac{Z^{\frac{1}{2}}(\log Z)^{\frac{7}{2}}}{M(\theta, p^k)}\right)\right) \cdot
\]

For each prime \( p \) we have at most \( d \) terms \( \theta \) in the inner sum. Thus

\[
N^{(1)}_{4,i} = O_F\left(Z^{\frac{3}{2}}(\log Z)^{\frac{7}{2}} \sum_{(\log Z)^9 < p \leq Z^{\frac{1}{2}}(\log Z)^{-9}} \sum_\theta \frac{1}{p^k}\right) \\
+ O_F\left(Z^{\frac{3}{2}}(\log Z)^{\frac{7}{2}} \sum_{(\log Z)^9 < p \leq Z^{\frac{1}{2}}(\log Z)^{-9}} \sum_\theta \frac{1}{M(\theta, p^k)}\right).
\]

Certainly

\[
\sum_{(\log Z)^9 < p \leq Z^{\frac{1}{2}}(\log Z)^{-9}} \frac{1}{p^k} = O\left(\sum_{(\log Z)^9 < p} \frac{1}{p^k}\right) \\
= O\left(\frac{1}{(\log Z)^{9(k-1)}}\right) \\
= O\left(\frac{1}{(\log Z)^9}\right).
\]
Further, since $M(\theta, p^k)$ is at least 1,

$$\sum_{(\log Z)^9 < p < Z^{\frac{1}{d}} (\log Z)^2} \sum_{\theta} \frac{1}{M(\theta, p^k)} = O_F\left( Z^{\frac{1}{d^2}} (\log Z)^{\frac{9}{2}} \right).$$

(5.9)

It remains to estimate $S$ where

$$S = \sum_{Z^{\frac{1}{d^2}} (\log Z)^{\frac{5}{2}} < p < Z^{\frac{9}{2}} (\log Z)^{\frac{1}{2}}} \sum_{\theta} \frac{1}{M(\theta, p^k)}.$$

Notice that if $r_i = 1$ then $F_i(x, y) = O_F(Z^{\frac{1}{d^2}} (\log Z)^{\frac{5}{2}})$ and so if $p^k$ divides $F_i(x, y)$ then, since $k \geq 2$, $p = O_F(Z^{\frac{1}{d^2}} (\log Z)^{\frac{5}{2}})$. Thus if $r_i = 1$ then

$$S = O_F(1).$$

(5.10)

We shall now estimate $S$ under the assumption that $r_i > 1$. We put $S = S_1 + S_2$ where $S_1$ is the sum over pairs $p, \theta$ with

$$M(\theta, p^k) \geq \frac{Z^{\frac{1}{d}}}{(\log Z)^5}$$

and $S_2$ is the sum over the other pairs $(p, \theta)$. Certainly

$$S_1 = O_F \left( \sum_{p \leq Z^{\frac{9}{2}} (\log Z)^{\frac{5}{2}}} \frac{(\log Z)^{\frac{5}{2}}}{Z^{\frac{1}{d}}} \right)$$

$$= O_F \left( \frac{Z^{\frac{1}{d}}}{(\log Z)^5} \right).$$

(5.11)

On the other hand $S_2$ consists of the sum over pairs $p, \theta$ with

$$1 \leq M \leq \frac{Z^{\frac{1}{d}}}{(\log Z)^5},$$

and $p > Z^{\frac{1}{d^2}} (\log Z)^2$. To each pair $p, \theta$ we may associate a pair of integers $(r, s)$ for which $\max\{|r|, |s|\} = M(\theta, p^k)$. Note that since $r_i > 1$ we have $F_i(r, s) \neq 0$. Further there are at most $O_F(1)$ pairs $(p, \theta)$ with $p > Z^{\frac{1}{d^2}} (\log Z)^2$ which can be associated with a given pair $(r, s)$ since $F_i(r, s) = O_F(Z^{r_i})$. Thus

$$S_2 = O \left( \sum_{1 \leq s \leq Z^{\frac{1}{d}} (\log Z)^{\frac{5}{2}}} \sum_{0 \leq r \leq s} \sum_{p^k | F(r, s)} \sum_{F(r, s) \neq 0} \sum_{p > Z^{\frac{1}{d^2}} (\log Z)^2} \frac{1}{p^k} \right)$$

$$= O_{F, k} \left( Z^{\frac{1}{d}} (\log Z)^{\frac{5}{2}} \right).$$

(5.12)

Therefore, by (5.5), (5.7), (5.8), (5.9), (5.10), (5.11) and (5.12),

$$N^{(1)}_4 = O_{F, k} \left( Z^{\frac{1}{d}} / (\log Z)^{\frac{3}{2}} \right).$$

(5.13)
Further, by (5.1), (5.4) and (5.13),

\[(5.14)\]

\[N_4 = O_{F,k} \left( \frac{Z^2}{(\log Z)^{\frac{3}{2}}} \right). \]

6. An estimate for \(N_5\)

For any real number \(T\) let \(B_{F,k}^*(T)\) denote the number of pairs of integers \((x, y)\) with \(\max(|x|,|y|) \leq T\) and for which \(F(x, y)\) is divisible by \(p^k\) with \(p\) a prime larger than \(T^2/(\log T)^{12}\). Then

\[(6.1)\]

\[B_{F,k}^*(T) = O(B_{F_1,k}^*(T) + \ldots + B_{F_t,k}^*(T)).\]

If \(r \leq 2k + 1\) then Greaves used Selberg’s sieve to prove that

\[(6.2)\]

\[B_{F,k}^*(T) = O_{F,k} \left( T^{2-\frac{3}{10}} \right) \]

for \(i = 1, \ldots, t\). This follows from the proof of Lemma 4 of [10] on taking \(x = T\) and \(\eta = (\log T)^{-16}\); Greaves required the constraint \(\eta \geq (\log T)^{-2}\) but it may be replaced with the weaker constraint \(\eta \geq (\log T)^{-16}\). Xiao dealt with the case when

\[\frac{7}{18} < \frac{k}{r} < \frac{1}{2}\]

in [41] by means of the determinant method applied to weighted projective spaces. It follows from [41] that in this case

\[(6.3)\]

\[B_{F_i,k}^*(T) = O_{F,k} \left( T^{\frac{2}{4}}/(\log T)^{4} \right) \]

for \(i = 1, \ldots, t\). Therefore for \(\frac{k}{r} > \frac{7}{18}\)

\[(6.4)\]

\[B_{F,k}^*(T) = O_{F,k} \left( T^{2}/(\log T)^{4} \right). \]

By a result of Helfgott, see the proof of Theorem 5.2 of [15], when \((k, r)\) is \((2, 6)\)

\[(6.5)\]

\[B_{F,2}^*(T) = O_{F,2} \left( T^{2}/(\log T)^{\delta} \right) \]

where\[\delta = 0.7043.\]

Hooley in 2009 established an asymptotic estimate for the number of integer pairs \((x, y)\) in a box for which \(F(x, y)\) is cubefree when \(F\) is a binary form of degree 8 with integer coefficients which is irreducible over the rationals, see [25] and Theorem 2 of [26]. Xiao [42] extended this work to decomposable forms \(F\) and an examination of his proof yields an explicit error term from which we find that

\[(6.6)\]

\[B_{F,3}^*(T) = O_F \left( T^{2}/(\log \log T/ \log \log \log T) \right) \]

when \((k, r)\) is \((3, 8)\).

Define \(g(T)\) by

\[(6.7)\]

\[g(T) = \begin{cases} 
\left( \log T \right)^4 & \text{if } \frac{k}{r} > \frac{7}{18} \\
\left( \log T \right)^{\delta} & \text{if } (k, r) = (2, 6) \\
\log \log T/ \log \log \log T & \text{if } (k, r) = (3, 8).
\end{cases} \]
Then by (6.4), (6.5) and (6.6),
\[(6.8)\]
\[
B^*_F,k(T) = O_{F,k} \left( T^2 / g(T) \right)
\]
for \((k, r)\) satisfying (1.4).

Put
\[(6.9)\]
\[
f(T) = g \left( T^{\frac{1}{d}} \right)^{\frac{1}{2}}.
\]

Let \(N_5^{(1)}\) be the number of integer pairs \((x, y)\) for which \(F(x, y) \neq 0\) and \(p^k | F(x, y)\) for some prime \(p\) with
\[(6.10)\]
\[
p > Z^{\frac{3}{d}} / (\log Z)^9
\]
which does not divide both \(x\) and \(y\) and for which
\[(6.11)\]
\[
\max\{|x|, |y|\} \leq Z^{\frac{1}{d}} f(Z).
\]

Further, write \(N_5^{(2)}\) for the number of integer pairs \((x, y)\) for which \(0 < |F(x, y)| \leq Z\) and
\[(6.12)\]
\[
\max\{|x|, |y|\} > Z^{\frac{1}{d}} f(Z).
\]

Notice that \(N_5 = O \left( N_5^{(1)} + N_5^{(2)} \right)\). By Lemma 2.2
\[(6.13)\]
\[
N_5^{(2)} = O_{F,k} \left( Z^{\frac{3}{d}} / f(Z)^{d-2} \right) = O_{F,k} \left( Z^{\frac{3}{d}} / g(Z^{\frac{1}{d}})^{d-2} \right).
\]

Furthermore, on taking \(T = Z^{\frac{1}{d}} f(Z)\), we see from (6.8) and (6.9) that
\[
N_5^{(1)} = O_{F,k} \left( Z^{\frac{3}{d}} f(Z)^2 / g(Z^{\frac{1}{d}}) \right).
\]

Since \(g(T)\) is eventually increasing and tends to infinity with \(T\) it follows from (6.9) that \(f(Z)\) is at least 1 for \(Z\) sufficiently large. We then have
\[
g(Z^{\frac{1}{d}}) \leq g(Z^{\frac{1}{d}} f(Z))
\]
and so
\[
N_5^{(1)} = O_{F,k} \left( Z^{\frac{3}{d}} f(Z)^2 / g(Z^{\frac{1}{d}}) \right).
\]

But \(f(Z)^2 / g(Z^{\frac{1}{d}}) = f(Z)^{d+2} / g(Z^{\frac{1}{d}})^{d+2}\), by (6.9), and thus
\[
N_5^{(1)} = O_{F,k} \left( Z^{\frac{3}{d}} / g(Z^{\frac{1}{d}})^{d+2} \right).
\]

Therefore
\[(6.14)\]
\[
N_5 = O_{F,k} \left( Z^{\frac{3}{d}} / g(Z^{\frac{1}{d}})^{d+2} \right).
\]

Theorem 1.2 now follows from (3.1), (3.2), (4.4), (4.6), (5.14), (6.7) and (6.14).

If \(F\) is a binary form with integer coefficients, nonzero discriminant and degree at least 3 and \(k\) is an integer larger than 1 then there exists a positive monotone increasing function \(g_1(t)\) on the positive real numbers with \(0 \leq g_1(t) \leq \log(t + 2)\) for all positive real numbers \(t\) and
\[
\lim_{t \to \infty} g_1(t) = \infty
\]
such that
\[(6.15)\]
\[
B^*_F,k(T) = O_{F,k} \left( T^2 / g_1(T) \right),
\]
subject to the abc conjecture. Granville [9] showed this when \( k = 2 \) and his argument extends readily to the general case. Arguing as above we deduce that Conjecture 1.4 holds for \( N_{F,k}(Z) \). With this estimate for \( N_{F,k}(Z) \) we are then able to establish Conjecture 1.3 for \( R_{F,k}(Z) \) as in the next section.

7. The proof of Theorems 1.1 and 1.3

If \( \text{Aut} F = C_1 \) then every integer pair \((x, y)\) for which \( F(x, y) \) is essentially represented with \( 0 < |F(x, y)| \leq Z \) gives rise to a distinct integer \( h \) with \( 0 < |h| \leq Z \). It follows from Theorem 1.2 and Lemma 2.6 that

\[
R_{F,k}(Z) = c_{F,k} Z^{\frac{2}{2}} + O_{F,k} \left( Z^{\frac{2}{2}} / u(z) \right)
\]

where \( u(z) \) is defined as in (3.3) when \( k \) and \( r \) satisfy (1.4) with \((k, r)\) not \((2,6)\) or \((3,8)\), as in (3.4) when \((k, r)\) is \((2,6)\) and satisfies (3.5) when \((k, r)\) is \((3,8)\). Similarly if \( \text{Aut} F = C_2 \) then (7.1) holds with \( \frac{1}{2} c_{F,k} \) in place of \( c_{F,k} \).

Suppose now that \( \text{Aut} F \) is conjugate to \( C_3 \). Then for \( A \) in \( \text{Aut} F \) with \( A \neq I \) we have, by Lemma 2.7, \( \Lambda(A) = \Lambda(A^2) \). Thus whenever \( (x, y) \) is in \( \Lambda \), \( F(x, y) = h \) and \( h \) is essentially represented there are exactly two other pairs \((x_1, y_1), (x_2, y_2)\) for which \( F(x_i, y_i) = h \) for \( i = 1, 2 \). When \((x, y)\) is in \( Z^2 \) but not in \( \Lambda \) and \( F(x, y) \) is essentially represented then \( F(x, y) \) has exactly one representative.

Let \( \{\omega_1, \omega_2\} \) be a basis for \( \Lambda \) with \( \omega_1 = (a_1, a_3) \) and \( \omega_2 = (a_2, a_4) \) and such that max\(\{a_1, a_2, |a_3|, |a_4|\}\) is minimized. Recall that

\[ F_{\omega_1, \omega_2}(x, y) = F(a_1 x + a_2 y, a_3 x + a_4 y). \]

Since

\[ |N_{F,k}(Z) \cap \Lambda| = N_{F\omega_1\omega_2,k}(Z), \]

by Theorem 1.2 we have

\[
|N_{F,k}(Z) \cap \Lambda| = c(\Lambda) Z^{\frac{2}{2}} + O_{F,k} \left( Z^{\frac{2}{2}} / u(z) \right).
\]

Note that since \( \omega_1 \) and \( \omega_2 \) are chosen so that max\(\{a_1, a_2, |a_3|, |a_4|\}\) is minimized the implicit constant in the error term may be determined in terms of \( F \) and \( k \). By (7.2) and Lemma 2.6 the number of integer pairs \((x, y)\) in \( \Lambda \) for which \( 0 < |F(x, y)| \leq Z \) and \( F(x, y) \) is \( k \)-free and essentially represented is

\[ c(\Lambda) Z^{\frac{2}{2}} + O_{F,k} \left( Z^{\frac{2}{2}} / u(z) \right). \]

Each pair \((x, y)\) is associated with two other pairs which represent the same integer. These pairs yield

\[
\frac{c(\Lambda)}{3} Z^{\frac{2}{2}} + O_{F,k} \left( Z^{\frac{2}{2}} / u(z) \right)
\]

integers \( h \) with \( 0 < |h| \leq Z \). It now follows from Theorem 1.2 and Lemma 2.6 that there are

\[
(c_{F,k} - c(\Lambda)) Z^{\frac{2}{2}} + O_{F,k} \left( Z^{\frac{2}{2}} / u(z) \right)
\]
integer pairs \((x, y)\) not in \(\Lambda\) for which \(F(x, y)\) is \(k\)-free and essentially represented and each pair gives rise to an integer \(h\) with \(0 < |h| \leq Z\) which is uniquely represented by \(F\). It follows from (7.3) and (7.4) that when \(\text{Aut} F\) is equivalent to \(C_3\) we have
\[
R_{F,k}(Z) = \left( c_{F,k} - \frac{2}{3}c(\Lambda) \right) Z^{\frac{8}{3}} + O_{F,k} \left( Z^{\frac{8}{3}} / u(z) \right).
\]
A similar analysis applies to the case when \(\text{Aut} F\) is equivalent to \(D_1, D_2, C_4\) or \(C_6\). These groups are cyclic with the exception of \(D_2\) but \(D_2/\{\pm I\}\) is cyclic and that is sufficient for our purposes.

We are left with the possibility that \(\text{Aut} F\) is conjugate to \(D_3, D_4\) or \(D_6\). We first consider the case when \(\text{Aut} F\) is equivalent to \(D_4\). In this case (7.2) holds as before and since each \(h\) which is essentially represented by \(F\) and for which \(h = F(x, y)\) with \((x, y)\) in \(\Lambda\) is represented by 8 integer pairs the number of \(k\)-free integers \(h\) with \(0 < |h| \leq Z\) for which there exists an integer pair \((x, y)\) in \(\Lambda\) with \(F(x, y) = h\) is
\[
(\frac{c(\Lambda)}{8}) Z^{\frac{8}{3}} + O_{F,k} \left( Z^{\frac{8}{3}} / u(z) \right).
\]
By Lemma 2.7 \(\Lambda_i \cap \Lambda_j = \Lambda\) for \(1 \leq i < j \leq 3\) and so the number of integer pairs \((x, y)\) in \(\Lambda_1, \Lambda_2\) or \(\Lambda_3\) but not in \(\Lambda\) for which \(F(x, y)\) is essentially represented and \(k\)-free with \(0 < |F(x, y)| \leq Z\) is, by Theorem 1.2,
\[
(c(\Lambda_1) + c(\Lambda_2) + c(\Lambda_3) - 3c(\Lambda)) Z^{\frac{8}{3}} + O_{F,k} \left( Z^{\frac{8}{3}} / u(z) \right).
\]
Each such integer \(F(x, y)\) has precisely four representatives and so the terms in \(\Lambda_1, \Lambda_2, \Lambda_3\) but not in \(\Lambda\) contribute
\[
(\frac{1}{4} (c(\Lambda_1) + c(\Lambda_2) + c(\Lambda_3) - 3c(\Lambda)) Z^{\frac{8}{3}} + O_{F,k} \left( Z^{\frac{8}{3}} / u(z) \right)
\]
terms to \(R_{F,k}(Z)\). Finally the terms \((x, y)\) in \(N_{F,k}(Z)\) but not in \(\Lambda_1, \Lambda_2\) or \(\Lambda_3\) for which \(F(x, y)\) is essentially represented have cardinality
\[
(c_{F,k} - c(\Lambda_1) - c(\Lambda_2) - c(\Lambda_3) + 2c(\Lambda)) Z^{\frac{8}{3}} + O_{F,k} \left( Z^{\frac{8}{3}} / u(z) \right).
\]
Each integer represented by such a term has 2 representations and therefore these terms contribute
\[
(\frac{1}{2} (c_{F,k} - c(\Lambda_1) - c(\Lambda_2) - c(\Lambda_3) + 2c(\Lambda)) Z^{\frac{8}{3}} + O_{F,k} \left( Z^{\frac{8}{3}} / u(z) \right)
\]
terms to \(R_{F,k}(Z)\). It now follows from (7.5), (7.6), (7.7) and Lemma 2.6 that
\[
R_{F,k}(Z) = \frac{1}{2} \left( c_{F,k} - \frac{c(\Lambda_1)}{2} - \frac{c(\Lambda_2)}{2} - \frac{c(\Lambda_3)}{2} + \frac{3c(\Lambda)}{4} \right) Z^{\frac{8}{3}} + O_{F,k} \left( Z^{\frac{8}{3}} / u(z) \right),
\]
as required.

Next suppose that \(\text{Aut} F\) is conjugate to \(D_3\). As before the pairs \((x, y)\) in \(N_{F,k}(Z) \cap \Lambda\) for which \(F(x, y)\) is essentially represented yield
\[
(\frac{c(\Lambda)}{6}) Z^{\frac{8}{3}} + O_{F,k} \left( Z^{\frac{8}{3}} / u(z) \right)
\]
terms in $\mathcal{R}_{F,k}(Z)$. Since $\Lambda_i \cap \Lambda_j = \Lambda$ for $1 \leq i < j \leq 3$ by Lemma 2.7, the pairs $(x, y)$ in $\mathcal{N}_{F,k}(Z) \cap \Lambda_i$ for $i = 1, 2, 3$ which are not in $\Lambda$ and which are essentially represented contribute

$$(7.9) \quad \left(\frac{c(\Lambda_1)}{2} + \frac{c(\Lambda_2)}{2} + \frac{c(\Lambda_3)}{2} - \frac{3c(\Lambda)}{2}\right)Z^2 + O_{F,k}\left(Z^2/u(z)\right)$$

terms to $\mathcal{R}_{F,k}(Z)$. The pairs $(x, y)$ in $\mathcal{N}_{F,k}(Z) \cap \Lambda_4$ which are not in $\Lambda$ and for which $F(x, y)$ is essentially represented contribute

$$(7.10) \quad \left(\frac{c(\Lambda_4)}{3} - \frac{c(\Lambda)}{3}\right)Z^2 + O_{F,k}\left(Z^2/u(z)\right)$$

terms to $\mathcal{R}_{F,k}(Z)$. Finally the pairs $(x, y)$ in $\mathcal{N}_{F,k}(Z)$ which do not lie in $\Lambda_i$ for $i = 1, 2, 3, 4$ contribute, by Lemma 2.7,

$$(7.11) \quad (c_{F,k} - c(\Lambda_1) - c(\Lambda_2) - c(\Lambda_3) - c(\Lambda_4)/3 + 3c(\Lambda))Z^2 + O_{F,k}\left(Z^2/u(z)\right)$$

terms to $\mathcal{R}_{F,k}(Z)$. It then follows from (7.8), (7.9), (7.10), (7.11) and Lemma 2.6 that $R_{F,k}(Z) = (c_{F,k} - c(\Lambda_1) - c(\Lambda_2) - c(\Lambda_3)/2 - \frac{2c(\Lambda_4)}{3} + \frac{4c(\Lambda)}{3})Z^2 + O_{F,k}\left(Z^2/u(z)\right)$, as required.

When $\text{Aut} F$ is equivalent to $D_6$ the analysis is the same as for $D_3$ taking into account the fact that $\text{Aut} F$ contains $-I$ and so the weighting factor is one half of what it is when $\text{Aut} F$ is equivalent to $D_3$.

Finally we note that, since there is no prime $p$ such that $p^k$ divides $F(a, b)$ for all pairs of integers $(a, b)$, $c_{F,k}$ is a positive number. We have

$$C_{F,k} \geq c_{F,k}/|\text{Aut} F|$$

and, since the order of the automorphism group of $F$ is at most 12, the order of $D_6$, we deduce that $C_{F,k}$ is positive. This completes the proof of Theorems 1.1 and 1.3.

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