REGULARITY OF CURVES IN ABELIAN VARIETIES

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Abstract. Inspired by a theorem of Gruson-Lazarsfeld-Peskine bounding the Castelnuovo- Mumford regularity of curves in projective spaces, we bound the Theta-regularity of curves in polarized abelian varieties.

1. Introduction

A coherent sheaf $\mathcal{F}$ on a projective space $\mathbb{P}^n$ is Castelnuovo-Mumford $k$-regular if for all $i > 0$ the spaces $H^i(\mathbb{P}^n, \mathcal{F}(k-i)) = 0$. One can read off some geometric properties of a subvariety $Y \subset \mathbb{P}^n$ just by looking at the regularity of its ideal sheaf $\mathcal{I}_Y$. For instance, if $\mathcal{I}_Y$ is Castelnuovo-Mumford $k$-regular, then $Y$ is cut-out by hypersurfaces of degree $k$ (cf. [La] Theorem 1.8.3). In this direction, a theorem of Gruson-Lazarsfeld-Peskine (cf. [GLP] Theorem 1.1), answering and extending a classical question of Castelnuovo, turns out to be very useful: if $C$ is a reduced, irreducible, non-degenerate curve of degree $d$ in $\mathbb{P}^n$, then $\mathcal{I}_C$ is Castelnuovo-Mumford $(d+2-n)$-regular.

In analogy to Castelnuovo-Mumford regularity, Pareschi and Popa introduced a notion of regularity for sheaves on a polarized abelian variety $(X, \Theta)$, the so called $\Theta$-regularity (cf. [PP1] Definition 6.1). It is defined as follows. Given a coherent sheaf $\mathcal{F}$ on $X$, we denote by

$$V^i(\mathcal{F}) := \{ \alpha \in \text{Pic}^0(X) | h^i(X, \mathcal{F} \otimes \alpha) > 0 \}$$

the cohomological support loci of $\mathcal{F}$. Then we say that $\mathcal{F}$ is Mukai-regular (or $M$-regular for short) if $\text{codim}_{\text{Pic}^0(X)} V^i(\mathcal{F}) > i$ for all $i > 0$ and that

$\mathcal{F}$ is $k$-\(\Theta\)-regular if $\mathcal{F} \otimes \Theta^{(k-1)}$ is $M$-regular.

The systematic study of Pareschi and Popa on $\Theta$-regularity shows that $\Theta$-regular sheaves enjoy analogous properties to Castelnuovo-Mumford regular sheaves on projective spaces (cf. [PP1] Theorem 6.3). In particular, if $\mathcal{I}_C$ is $k$-\(\Theta\)$\text{-regular}$, then $C$ is cut-out by $k$-\(\Theta\)$\text{-equations}$.

The aim of these notes is to provide a bound for the $\Theta$-regularity of curves in a polarized abelian variety.

Theorem 1.1. Let $X$ be a complex abelian variety of dimension $n$ and $\Theta$ be an ample and globally generated line bundle on $X$. Let $\iota : C \hookrightarrow X$ be a reduced and irreducible curve and let $\nu : \tilde{C} \to C$ be its normalization. Set $f := \iota \circ \nu$ and $d := \deg f^*\Theta$. Then the ideal sheaf

$$\mathcal{I}_C \ 	ext{is} \ (n+d+1)\text{-}\Theta\text{-regular}.$$
the dimension of the ambient space and on the degree of the curve. On the other hand it is not sharp, as previous computations of Θ-regularity show that both Abel-Jacobi and Abel-Prym curves are 3-Θ-regular ([PP1] Theorem 4.1, [PP2] Theorem 7.17 and [CMLV] Corollary B).

We point out that other bounds for Θ-regularity have been worked out in [PP1] Theorem 6.5 for subvarieties of a polarized abelian variety \((X, \Theta)\) defined by \(d\)-Θ-equations.

The proof of Theorem 1.1 goes as follows. As in [GLP] Theorem 1.1, we use Eagon-Northcott complexes to resolve \(I_C\) with a complex of locally free sheaves such that: 1) it is exact away from \(C\); 2) the Theta-regularity of its terms is easily computable. However, the methods to establish this resolution differ considerably from the ones used by Gruson-Lazarsfeld-Peskine as they mainly rely on the generic vanishing theory developed in [PP1] and [PP2].

2. Setting and Proof

Throughout the paper we work in the following setting. Let \(X\) be a complex abelian variety of dimension \(n\) and \(\Theta\) be an ample and globally generated line bundle on \(X\). Let \(\iota : C \hookrightarrow X\) be a reduced and irreducible curve of geometric genus \(g\) and \(\nu : \tilde{C} \to C\) be its normalization. Set \(f = \iota \circ \nu\) and \(d = \deg f^* \Theta\). Let \(\Gamma \subset \tilde{C} \times X\) be the graph of \(f\) and \(p\) and \(q\) be the projections from \(\tilde{C} \times X\) onto the first and second factor respectively. Finally we denote by \(I_C\) and \(I_\Gamma\) the ideal sheaves of \(C\) and \(\Gamma\) respectively.

We recall that a coherent sheaf \(F\) on \(X\) is continuously globally generated if there exists a positive integer \(N\) such that for any general \(\alpha_1, \ldots, \alpha_N \in \text{Pic}^0(X)\) the sum of the twisted evaluation maps

\[
\bigoplus_{i=1}^N H^0(X, F \otimes \alpha_i) \otimes \alpha_i^{-1} \to F
\]

is surjective. For instance, ample line bundles and, in general, \(M\)-regular sheaves are continuously globally generated (cf. [PP1] Proposition 2.13).

In order to compute the Θ-regularity of \(I_C\), we will show that it is enough to check the continuous global generation of sheaves of type \(q_*(I_{\Gamma} \otimes p^* A) \otimes \Theta\) where \(A\) is a globally generated line bundle on \(\tilde{C}\) (cf. in Proposition 2.2). To begin with we present a simple lemma.

**Lemma 2.1.** Let

\[
\mathcal{E}^* : \cdots \to \mathcal{E}_2 \xrightarrow{d_2} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{J} \to 0
\]

be a complex of coherent sheaves on a smooth projective irregular variety \(Y\) such that \(d_0\) is a surjective morphism. If \(\mathcal{E}^*\) is exact away from an algebraic set of dimension \(\leq 1\), then we have an inclusion of cohomological support loci (cf. [11])

\[
V^i(\mathcal{J}) \subset V^i(\mathcal{E}_0) \cup V^{i+1}(\mathcal{E}_1) \cup \ldots \cup V^{\dim Y}(\mathcal{E}_{\dim Y-i}) \quad \text{for any} \quad i \geq 1.
\]

**Proof.** We set \(K_i := \ker d_i, I_i := \text{im} d_i\) and \(H_i := K_i/I_{i+1}\) for \(i \geq 0\). By assumption \(\dim \text{Supp} H_i \leq 1\) for any \(i \geq 0\). Therefore

\[
V^j(H_i) = \emptyset \quad \text{for any} \quad i \geq 0 \quad \text{and} \quad j > 1.
\]

By looking at the exact sequence \(0 \to K_0 \to \mathcal{E}_0 \to \mathcal{J} \to 0\), we have

\[
V^i(\mathcal{J}) \subset V^i(\mathcal{E}_0) \cup V^{i+1}(K_0).
\]
In addition, the exact sequence 0 \rightarrow I_1 \rightarrow K_0 \rightarrow H_0 \rightarrow 0 yields
\[ V^{i+1}(K_0) \subset V^{i+1}(I_1) \cup V^{i+1}(H_0) = V^{i+1}(I_1). \]

Finally, by looking at the exact sequence 0 \rightarrow K_1 \rightarrow E_1 \rightarrow I_1 \rightarrow 0, we deduce that
\[ V^{i+1}(I_1) \subset V^{i+1}(E_1) \cup V^{i+2}(K_1) \]
and therefore that
\[ V^i(J) \subset V^i(E_0) \cup V^{i+1}(E_1) \cup V^{i+2}(K_1). \]

At this point it is enough to iterate the previous argument to obtain the lemma. \(\square\)

**Proposition 2.2.** Let \(A\) be a globally generated line bundle on \(\widetilde{C}\). If \(q_*(I_\Gamma \otimes p^*A) \otimes \Theta\) is continuously globally generated, then
\[ I_C \text{ is } (h^0(\widetilde{C}, A) + n) \cdot \Theta\text{-regular.} \]

**Proof.** Consider the exact sequence defining the graph \(\Gamma\)
\[ 0 \rightarrow I_\Gamma \rightarrow O_{\widetilde{C} \times X} \rightarrow O_\Gamma \rightarrow 0. \]

By tensoring (2) by \(p^*A\) and then by pushing forward to \(X\) via \(q\), we obtain the exact sequence
\[ 0 \rightarrow q_*(p^*A \otimes I_\Gamma) \rightarrow H^0(\widetilde{C}, A) \otimes O_X \xrightarrow{ev} f_*A. \]

Let \(G\) be the image of the evaluation map \(ev\). By hypotheses, there exists a positive integer \(N\) and line bundles \(\alpha_1, \ldots, \alpha_N \in \Pic^0(X)\) such that the map
\[ \bigoplus_{i=1}^N H^0(X, q_*(p^*A \otimes I_\Gamma) \otimes \Theta \otimes \alpha_i) \otimes \alpha_i^{-1} \rightarrow q_*(p^*A \otimes I_\Gamma) \otimes \Theta \]
is surjective. We set \(W_i := H^0(X, q_*(p^*A \otimes I_\Gamma) \otimes \Theta \otimes \alpha_i)\) for \(i = 1, \ldots, N\). From (3) and (4), we deduce a presentation of \(G\):
\[ \bigoplus_{i=1}^N W_i \otimes \alpha_i^{-1} \otimes \Theta^{-1} \xrightarrow{\varphi} H^0(\widetilde{C}, A) \otimes O_X \rightarrow G \rightarrow 0. \]

We set
\[ E := \bigoplus_{i=1}^N W_i \otimes \alpha_i^{-1} \otimes \Theta^{-1} \quad \text{and} \quad h := \dim H^0(\widetilde{C}, A). \]

Let \(\mathcal{J}\) be the 0-th Fitting ideal of \(G\). Since \(A\) is globally generated and \(I_C\) is a radical ideal, \(\mathcal{J}\) coincides with \(I_C\) away the singular points of \(C\) (see [GLP], p.496). By applying the Eagon-Northcott complex (cf. [GLP] (0.4)) to the morphism \(\varphi\) in (5), we get a complex
\[ E^\bullet : \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow \mathcal{J} \rightarrow 0 \]
which is exact away from \(C\) and whose terms are copies of \(\bigwedge^{h+i} E\) for all \(i \geq 0\). Therefore
\[ E_i \cong \bigoplus_t \Theta^{\otimes (-h-i)} \otimes \beta_t \quad \text{for some} \quad \beta_t \in \Pic^0(X). \]

Moreover we have
\[ V^j(E_i \otimes \Theta^{\otimes (h+n-1)}) \subset V^j(\Theta^{\otimes (n-i+1)}) = \emptyset \quad \text{for any} \quad j > 0 \quad \text{and} \quad i < n - 1 \]
and
\[ V^j(E_{n-1} \otimes \Theta^{\otimes (h+n-1)}) \subset V^j(O_X) = \{O_X\} \quad \text{for any} \quad j > 0. \]
Thus, by Lemma 2.1 we obtain inclusions
\[ V^1(J \otimes \Theta^{(h+n-1)}) \subset V^n(E_{n-1} \otimes \Theta^{(h+n-1)}) \subset \{ \mathcal{O}_X \} \]
and
\[ V^j(J \otimes \Theta^{(h+n-1)}) = \emptyset \quad \text{for any} \quad j > 1. \]
Finally, by noting that
\[ V^j(I_C \otimes \Theta^{(h+n-1)}) \subset V^j(J \otimes \Theta^{(h+n-1)}) \quad \text{for any} \quad j > 0, \]
we conclude then that \( I_C \otimes \Theta^{(h+n-1)} \) is an \( M \)-regular sheaf. \( \square \)

In the next proposition we will give sufficient conditions for the hypotheses of Proposition 2.2 to be satisfied.

**Proposition 2.3.** If \( B \) is a line bundle on \( \mathcal{C} \) such that \( f_*B \) is \( M \)-regular on \( X \), then \( q_*(I_\mathcal{C} \otimes p^*(B \otimes f^*\Theta)) \otimes \Theta \) is \( M \)-regular on \( X \).

**Proof.** The sheaf \( f_*B \otimes \Theta^{\otimes m} \) is globally generated for any \( m \geq 1 \) by [PP1] Proposition 2.12. Moreover, by [PP2] Proposition 3.1, \( f_*B \otimes \Theta^{\otimes m} \) is an \( I.T.0 \) sheaf for any \( m \geq 1 \), i.e. \( V^i(f_*B \otimes \Theta^{\otimes m}) = \emptyset \) for all \( i, m \geq 1 \).

By (6) we obtain exact sequences for any \( \alpha \in \text{Pic}^0(X) \)
\[ 0 \longrightarrow q_*(I_\mathcal{C} \otimes p^*(B \otimes f^*\Theta)) \otimes \Theta \otimes \alpha \longrightarrow H^0(\mathcal{C}, B \otimes f^*\Theta) \otimes \Theta \otimes \alpha \longrightarrow f_*B \otimes \Theta^{\otimes 2} \otimes \alpha \longrightarrow 0. \]
We set \( \mathcal{H} := q_*(I_\mathcal{C} \otimes p^*(B \otimes f^*\Theta)) \) so that we only need to check the conditions \( \text{codim}_{\text{Pic}^0(X)} V^i(\mathcal{H} \otimes \Theta) > i \) for all \( i > 0 \). By the Kodaira Vanishing Theorem and by (6), we have
\[ V^i(\mathcal{H} \otimes \Theta) = \emptyset \quad \text{for any} \quad i \geq 3. \]
Furthermore
\[ V^2(\mathcal{H} \otimes \Theta) \cong V^1(f_*B \otimes \Theta^{\otimes 2}) = \emptyset. \]
Now we study the dimension of \( V^1(\mathcal{H} \otimes \Theta) \). By denoting by
\[ m_\alpha : H^0(X, f_*B \otimes \Theta) \otimes H^0(X, \Theta \otimes \alpha) \longrightarrow H^0(X, f_*B \otimes \Theta^{\otimes 2} \otimes \alpha) \]
the multiplication map on global sections induced by (6), we have an identification
\[ V^1(\mathcal{H} \otimes \Theta) = \{ \alpha \in \text{Pic}^0(X) \mid m_\alpha \text{ is not surjective} \}. \]
We claim that the inverse morphism \((-1) : \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)\) taking \( \alpha \) to \( \alpha^{-1} \) maps
\[ V^1(\mathcal{H} \otimes \Theta) \rightarrow V^1(f_*B). \]
This finishes the proof since it implies
\[ \dim V^1(\mathcal{H} \otimes \Theta) \leq \dim V^1(f_*B) \leq n - 2. \]

Now we show (6). Since \( \Theta \otimes \alpha \) is globally generated for any \( \alpha \in \text{Pic}^0(X) \), we have exact sequences
\[ 0 \longrightarrow M_{\Theta \otimes \alpha} \longrightarrow H^0(X, \Theta \otimes \alpha) \otimes \mathcal{O}_X \longrightarrow \Theta \otimes \alpha \longrightarrow 0. \]
Tensoring by \( \Theta \) and then restricting to \( C \) and finally tensoring by \( \nu_*B \), we get exact sequences
\[ 0 \longrightarrow t^*(M_{\Theta \otimes \alpha} \otimes \Theta) \otimes \nu_*B \longrightarrow H^0(X, \Theta \otimes \alpha) \otimes t^*\Theta \otimes \nu_*B \longrightarrow \nu_*B \longrightarrow 0. \]
We note that the map on global sections induced by $ev_{\alpha}$ coincides with the multiplication map (7). Hence by (8)

\begin{equation}
\alpha \in V^1(\mathcal{H} \otimes \Theta) \implies H^1(C, \iota^*(M_{\Theta \otimes \Theta} \otimes \Theta) \otimes \nu_s B) \neq 0.
\end{equation}

Now pick an arbitrary element $\alpha \in V^1(\mathcal{H} \otimes \Theta)$ and set $W := \text{Im}(H^0(X, \Theta \otimes \alpha) \to H^0(C, \iota^*(\Theta \otimes \alpha)))$. We note that $W$ generates $\iota^*(\Theta \otimes \alpha)$ and hence $\dim W \geq 2$ since $\Theta$ is not trivial. Moreover, the preimages $s_1$ and $s_2$ in $H^0(X, \Theta \otimes \alpha)$ of two general sections in $W$ generate $\iota^*(\Theta \otimes \alpha)$. We have then a commutative diagram

\begin{equation}
\begin{array}{cccccc}
0 & \to & \iota^*(\Theta^{-1} \otimes \alpha^{-1}) & \to & \mathcal{O}_C \otimes \mathcal{O}_C & \to & \iota^*(\Theta \otimes \alpha) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \iota^*M_{\Theta \otimes \Theta} & \to & H^0(X, \Theta \otimes \alpha) \otimes \mathcal{O}_C & \to & \iota^*(\Theta \otimes \alpha) & \to & 0.
\end{array}
\end{equation}

By defining $\bar{V} := H^0(X, \Theta \otimes \alpha)/\langle s_1, s_2 \rangle$ and by the Snake Lemma, we obtain the exact sequence

\begin{equation}
0 \to \iota^*(\Theta^{-1} \otimes \alpha^{-1}) \to \iota^*M_{\Theta \otimes \Theta} \to \bar{V} \otimes \mathcal{O}_C \to 0
\end{equation}

and hence the exact sequence

\begin{equation}
0 \to \iota^*\alpha^{-1} \otimes \nu_s B \to \iota^*(M_{\Theta \otimes \Theta} \otimes \Theta) \otimes \nu_s B \to \bar{V} \otimes \iota^*\Theta \otimes \nu_s B \to 0.
\end{equation}

Finally, the projection formula yields isomorphisms

$H^1(C, \nu_s B \otimes \iota^*\alpha^{-1}) \cong H^1(X, f_s B \otimes \alpha^{-1})$ and $H^1(C, \nu_s B \otimes \iota^*\Theta) \cong H^1(X, f_s B \otimes \Theta) = 0$

which in turn show that $\alpha^{-1} \in V^1(f_s B)$ by (10).

Before proceeding with the proof of Theorem 1.1, we prove a lemma giving a necessary condition for a sheaf of the form $f_s B$ to be $M$-regular on $X$.

**Lemma 2.4.** Let $D$ be a smooth and irreducible curve of genus $g$ and $\varphi : D \to X$ be a morphism to a complex abelian variety $X$ of dimension $n$. If $B$ is a general line bundle on $D$ of degree $b$, then

$$\dim V^1(\varphi_s B) \leq n + g - b - 2.$$ 

**Proof.** Without loss of generality, we can assume that $\varphi$ in non-constant since in this case $V^1(\varphi_s B) = \emptyset$ for all $B \in \text{Pic}^0(D)$. The algebraic set $V^1(B)$ is irreducible as Serre duality yields an isomorphism $V^1(B) \cong W_{2g-2-b}(D)$ (here $W_{2g-2-b}(D)$ is the image of the Abel-Jacobi map $\text{Sym}^{2g-2-b}(D) \to \text{Pic}^{2g-2-b}(D)$). The algebraic group $\text{Pic}^0(D)$ acts on itself via translations. For any $\gamma \in \text{Pic}^0(D)$, we write $\gamma V^1(B)$ for the image of $V^1(B)$ under the action of $\gamma$. Then, by Kleiman’s Transversality Theorem [Kl] Theorem 2, there exists an open dense subset $V \subset \text{Pic}^0(D)$ such that for all $\alpha \in V$ the fiber product $\gamma V^1(B) \times_{\text{Pic}^0(D)} \text{Pic}^0(X)$ is either empty or of dimension

$$\dim V^1(B) + \dim \text{Pic}^0(X) - \dim \text{Pic}^0(D) = \dim V^1(B) + n - g.$$

We note the isomorphisms of algebraic sets $\gamma V^1(B) \cong V^1(B \otimes \gamma^{-1})$ for any $\gamma \in V$. Moreover, by the universal property of the fiber product and by the projection formula, we obtain closed
immersions $V^1(\varphi_*(B \otimes \gamma^{-1})) \hookrightarrow V^1(B \otimes \gamma^{-1}) \times_{\text{Pic}^0(D)} \text{Pic}^0(X)$. Hence for any $\gamma \in V$ we have

$$\dim V^1(\varphi_*(B \otimes \gamma^{-1})) \leq \dim (\gamma V^1(B) \times_{\text{Pic}^0(D)} \text{Pic}^0(X))$$

$$= \dim V^1(B) + n - g$$

$$= \dim W_{2g-2-b}(D) + n - g$$

$$\leq n + g - b - 2. \quad \Box$$

At this point the proof of Theorem 1.1 easily follows by the previous lemmas and propositions.

Proof of Theorem 1.1 By Lemma 2.4 we can pick a line bundle $B$ of degree $g$ on $\tilde{C}$ such that $f_*B$ is $M$-regular on $X$ and $h^1(\tilde{C}, B \otimes f^*\Theta) = 0$. Hence, the sheaf $q_*(I_\Gamma \otimes p^*(B \otimes f^*\Theta)) \otimes \Theta$ is continuously globally generated by Proposition 2.3 and we conclude then by applying Proposition 2.2 after having noted that $h^0(\tilde{C}, B \otimes f^*\Theta) = d + 1$. \( \Box \)

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