Relativistic BEC-BCS Crossover in a magnetized Nambu-Jona-Lasinio Model

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Abstract. The BEC-BCS crossover in the NJL model is studied in the presence of an external magnetic field. Particular attention is given to two different regularization schemes used in the literature and we show how they compare to each other. The comparison is made for the case of a cold and magnetized two color NJL model. We also make a brief discussion about the $N_c = 3$ case without magnetic fields, as an extension of this work in the future.

1. Introduction
The phase structure of quantum chromodynamics (QCD) is of relevance in many different contexts, for example, in the study of relativistic heavy ion collisions, compact stars and in the early universe. Strong magnetic fields is expected to be produced in noncentral heavy-ion collisions, which motivates the study of their effects in the phase diagram of strongly interacting matter (see [1] and references). Perturbation theory of QCD is applicable in asymptotic regimes of temperatures and densities, while the study of the phase transitions at moderate temperatures and densities cannot be implemented on lattice QCD ($N_c = 3$) calculations due to the fermion sign problem. These facts have then motivated the study of many low energy effective models, such as chiral perturbation theory, linear sigma model, the Nambu-Jona-Lasinio (NJL) model among others models that are used to study the phase structure of strongly interacting matter [2].
It is generally expected that there should exist a crossover from Bose-Einstein condensation (BEC) to Bardeen-Cooper-Shriffer condensation (BCS) for diquarks at finite baryon density. This crossover can be observed in different ways, such as increasing the coupling constant of the attractive interactions or changing the charge number through the variation of the chemical potential [3]. In this work we review the investigations of the BEC-BCS crossover in cold QCD at finite baryon chemical potential using the $SU(2)$ version of the NJL model for $N_c = 2$ and $N_c = 3$ in the mean field approximation. We study in details the effect that an external magnetic field on the crossover. We hope that our results will help to explain the role played by a magnetic field in the crossover BEC-BCS in real systems.
2. Thermodynamic potential for $N_c = 2$ QCD in the mean field approximation

2.1. Zero magnetic field case

We study the diquark condensation in a two color QCD. The starting point is from the Lagrangian density for the NJL model:

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m_0) \psi + G_S \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i\gamma_5 \tau \psi)^2 \right] + G_D \left( \bar{\psi} i\gamma_5 \tau_2 t_2 C \psi^T \right) \left( \psi^T C i\gamma_5 \tau_2 t_2 \psi \right)$$

(1)

where $m_0$ is the current mass, $C = i\gamma_0\gamma_2$ is the charge conjugation matrix, $\tau_i$ and $t_i$ are the Pauli matrices in flavor and color spaces, $G_S$ and $G_D$ are the coupling constants, connected by a Fierz transformation in the color space, leading to $G_S = G_D = G$ in the two color case. The advantage of studying the two color QCD is that the confinement is less important than in three color QCD. The diquarks are colorless baryons and the diquark condensation breaks the baryon symmetry $U_B(1)$ rather than the color symmetry $SU_C(2)$ [3].

In this work our aim is to study the crossover BEC-BCS. It is well known that this phenomenon can happen at low temperatures. Thus, we start from the thermodynamic potential at zero temperature:

$$\Omega_0 = \frac{(m - m_0)^2 + \Delta^2}{4G} - 2N_c N_f \int \frac{d^3k}{(2\pi)^3} \left[ \frac{E^+_\Delta}{2} + \frac{E^-_\Delta}{2} \right]$$

(2)

where the dispersion energies are $E^\pm_\Delta = \sqrt{(E_k \pm \mu)^2 + \Delta^2}$ and $E_k = \sqrt{k^2 + m^2}$. Also, note that $E^+_{\Delta}$ corresponds to particles and $E^-_{\Delta}$ corresponds to the antiparticles excitations. The NJL model is nonrenormalizable. Thus, some regularization scheme has to be used to account for the divergences. The most common procedure in the $B = 0$ regimes is to use a three dimensional momentum cutoff $\Lambda$. This is valid since it is well known that the crossover happens away from the asymptotic regions. In this way, all physical quantities become dependent on this parameter $\Lambda$. Besides the cutoff, we have the coupling constants $G_S$ and $G_D$ and the current mass $m_0$ to be specified. In three color QCD case these parameters are fixed such that the empirical values of the pion mass $m_\pi$, the pion decay constant $f_\pi$ and chiral condensate in the vacuum $\langle \bar{\psi} \psi \rangle_0$ might be obtained. On the other hand, to determinize the parameters used in numerical calculations for the $N_c = 2$ case we can follow the procedure proposed in reference [4], rescaling the physical quantities by $N_c$. It is well known that $f_\pi$ is proportional to $N_c^{1/2}$ and $\langle \bar{\psi} \psi \rangle_0$ is proportional to $N_c$; therefore, we rescale the $N_c = 3$ values of these quantities by the factors $\sqrt{2/3}$ and and $2/3$, respectively. Therefore, for $N_c = 2$ the model parameters $\Lambda, G$ and $m_0$ must be such that we would reproduce the rescaled values $f_\pi = 75.45$ MeV, $m_\pi = 140$ MeV and $\langle \bar{q}q \rangle_{0}^{1/3} = -218$ MeV.

In anticipation to the case of including a magnetic field, we call the three dimensional cutoff regularization $\Lambda^{3d}$, and also propose a smooth cutoff, with a Gaussian factor form $f_\Lambda(p) = e^{-p^2/\alpha^2}$, implemented through the substitution

$$\int \frac{d^3k}{(2\pi)^3} \rightarrow \int \frac{d^3k}{(2\pi)^3} f_\Lambda(p)$$

(3)

where $p = k$ in $B = 0$ case. In this work we refer to this regularization based in the Gaussian factor form as GR scheme. For each type of regularization used, we evaluate the corresponding model parameters, that are given in table 1.

It is important to observe that the values obtained for the $\Lambda^{3d}$ scheme were obtained by using the $N_c$ rescaling mentioned before (see reference [4]), but for the GR scheme, it is not possible
Table 1. Parameter sets for the NJL SU(2) model.

| Parameter set | \( m(0) \) (MeV) | \( m_0 \) (MeV) | \( G \) (GeV\(^{-2}\)) | \( \Lambda \) (MeV) |
|--------------|-----------------|----------------|-----------------|----------------|
| GR           | 305.385         | 3.557          | 4.77            | 777            |
| \( \Lambda^{3d} \) | 305.385 | 5.400 | 7.23 | 657 |

to obtain good values for the physical quantities in \( N_c = 2 \), using this rescaling. Therefore, we fix the vacuum mass and evaluate \( G, \Lambda \) and \( m_0 \) that reproduce \( f_\pi = 75.45 \) MeV, \( m_\pi = 140 \) MeV. The value obtained for the quiral condensate is \( \langle \bar{q}q \rangle_0^{1/3} = -250.96 \) MeV and the other parameters are shown in Table 1.

Minimizing the potential with respect to \( \Delta \) and \( m \), we obtain the respective Gap equations. Namely

\[
\frac{\partial \Omega_0}{\partial \Delta} = 0, \quad \frac{\partial \Omega_0}{\partial m} = 0
\]

To study the BEC-BCS crossover we decide to follow the same procedure of the reference [3], defining a reference chemical potential \( \mu_N = \mu - m \), or equivalently, \( \mu_N = \frac{\mu}{2} - m \) for the two-color QCD case. It is possible to note that for small values of \( \mu_B \) we have \( \mu_N < 0 \), and the minimum of the dispersion is located at \( |\vec{k}| = 0 \) with particle gap energy \( \sqrt{\frac{\mu_N^2}{4} + \Delta^2} \). Otherwise, at larger values of \( \mu_B \) we have \( \mu_N > 0 \), the minimum of the dispersion is shifted to \( |\vec{k}| = \mu_B/2 \), and the particle gap is \( \Delta \). This is a signal of the BEC-BCS crossover that happens due to the changes in the quark mass during the chiral symmetry restoration, when the chemical potential increases.

Figure 1. Left panel: (Color online) Effective quark mass \( m \) and diquark condensate \( \Delta \), rescaled by the vacuum quark mass \( m(0) \), as functions of \( \mu_B/m_\pi \) for the two regularization schemes used in this work. Continuum lines correspond to \( \Lambda^{3d} \), and dotted lines correspond to the GR scheme. Right panel: Reference chemical potential \( \mu_N = \mu - m \), for \( \Lambda^{3d} \) and GR schemes.

In the first panel of figure 1 we show the numerical solutions for the gap equations (4). Using both parametrizations, based on the \( N_c \) rescaling and for GR scheme, our qualitative results are
similar to the ones obtained in the reference [3], where the authors work with parameters usually taken in the \( N_c = 3 \) case. It is possible to see from the behavior of the diquark condensates that a second order phase transition is happening at \( \mu_B = m_\pi \), where the mass \( m \) begins to decrease from its vacuum value \( m(0) \), while \( \Delta \) becomes nonzero. On the other hand, in the right panel of figure 1 we see that for both regularization schemes the crossover happens at the same value, \( \mu_{B_c} / m_\pi \sim 1.7 \) (corresponding to a \( \mu_{B_c} \) about 240 MeV). The critical baryon chemical potential \( \mu_{B_c} \) is the value where \( \mu_N \) changes its signal. When \( \mu_N < 0 \) the system is in the BEC state and with the increase of the chemical potential \( \mu_N \) becomes positive and the system goes to BCS state.

2.2. Nonzero magnetic field case

In the presence of an external magnetic field, the corresponding Lagrangian density is given by

\[
\mathcal{L} = \bar{\psi} (i\partial - m_0) \psi + G_S \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau_2\psi)^2 \right] + G_D \left( \bar{\psi}i\gamma_5\tau_2t_2C\psi^T \right) \left( \psi^TCi\gamma_5\tau_2t_2\psi \right),
\]

with the same parameters defined in the \( B = 0 \) case; moreover, the coupling of the quarks to the electromagnetic field \( A_\mu \) is implemented through the covariant derivative \( D_\mu = \partial_\mu - iQ_A\mu \), where \( Q \) is the usual quark charge matrix. In this work we set \( q_u = -q_d \) such that the Cooper pairs remain color neutral. In order of include the effects of an external magnetic field in the thermodynamic potential, it is usual to make use of the following prescription in all the \( d^3k \) integrals [5]:

\[
N_f \int \frac{d^3k}{(2\pi)^3} \to \sum_{l=0}^{\infty} \sum_{f=u}^{d} \frac{|q_f|B}{4\pi} \alpha_l \int_{-\infty}^{+\infty} \frac{dk_z}{(2\pi)}
\]

where the factor \( \alpha_l = 2 - \delta_{l,0} \), takes into account the degeneracy of the Landau levels \( l \) that are all doubly degenerate (except for \( l = 0 \)). This procedure is, after all, equivalent to take the \( T = 0 \) limit after perform a Matsubara sum in \( k_0 \to i(\omega_n - i\mu) \) in the thermodynamic potential. The dispersion relation becomes \( E_k \to E_{k,l} = \sqrt{k_z^2 + 2|q_f|B + m^2} \) and \( E_\Delta^{+} \to E_\Delta^{+,l} = \sqrt{(E_{k,l} + \mu)^2 + \Delta^2} \). After this prescription, \( \Omega_0 \) becomes

\[
\Omega_0(B) = \frac{(m - m_0)^2 + \Delta^2}{4G} - N_c \sum_{f=u}^{d} \frac{|q_f|B}{4\pi} \sum_{l=0}^{\infty} \alpha_l \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \left[ E_\Delta^{+} + E_\Delta^{-} \right],
\]

As mentioned before, it is well known that the NJL model is nonrenormalizable. Thus, it is necessary to regularize the momentum integrals. In the \( B \neq 0 \) case we use a similar procedure as in the \( B = 0 \) case studied above, but now we define a cutoff \( \Lambda_l = \sqrt{\Lambda^2 - 2|q_f|B} \), that is a function of the Landau levels, and so the superior limit for the sum over the Landau levels becomes the integer part of \( \Lambda^2/(2|q_f|B) \) [6]. In the Gaussian factor form regularization scheme we now have

\[
\sum_{l=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \to \sum_{l=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} f_\Lambda(p)
\]

where \( p = k_z^2 + 2|q_f|B \) in \( B \neq 0 \) case [7, 8].

In the figure 2 we show the numerical results for the gap equations evaluated from the new thermodynamic potential. It is possible to see that there is a magnetic catalysis phenomena happening. As the magnetic field increases, the value of the mass, even at small values of \( \mu_B \),
increases as well. We also note that the phase transition for the diquark condensate remains of second order, even in the presence of the external magnetic fields. The value of the critical chemical potential changes with $qB$. Note that at $qB = 0$ the phase transition happens at $\mu_B = m_\pi$, in agreement with the chiral effective theories [9], for both schemes presented here.

**Figure 2.** (Color online) Effective quark mass $m$ (left panel) and diquark condensate $\Delta$ (right panel), both scaled by the vacuum quark mass, as functions of $\mu_B/m_\pi$, for $B = 0$ and $B = 10m_\pi^2$.

In the figure 3 we have $\mu_N = \mu - m$ for $qB = 10m_\pi^2$ in comparison with $qB = 0$. Remember that negative values of $\mu_N$ corresponds to the BEC region, and positive ones corresponds to the BCS region. We see that the critical baryon chemical potential is also affected by the magnetic field. To determine how the crossover is affected through a large range of magnetic fields, we have the plot shown in the right panel of figure 3. Note that once this phenomenon happens when $\mu_N$ changes its signal, we set $m = \mu$ in the gap equations and solve it for $\mu_B$ and $\Delta$. Here we observe that there are non-physical oscillations when working with the $\Lambda_l$ scheme that are not present in the GR case. From the dashed curve shown in the right panel we may see that the critical baryon chemical potential increases with the magnetic field, i.e., the crossover happens for a higher value of $\mu_B$, favoring the BEC region.

**Figure 3.** (Color online) **Left panel:** $\mu_N = \mu - m$ for $B = 0$ and $qB = 10m_\pi^2$, as a function of the baryon chemical potential (scaled by the pion mass). **Right panel:** Critical chemical potential for the BEC-BCS crossover, as a function of the magnetic field.
3. NJL model with three colors at \( B = 0 \)

In this section we consider a more realistic case, with three color degrees of freedom. The NJL model with \( N_c = 3 \) and including the scalar diquark channel is

\[
\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m_0) \psi + G_S [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2] + G_D \sum_{a=2,5,7} (\bar{\psi}i\gamma_5\tau_2 \lambda_a C\psi^T) (\psi^T C\gamma_5\tau_2 \lambda_a \psi)
\]

(9)

where \( \lambda_a \) are the Gell-Mann matrices in color space. Differently from the two color problem, now the confinement problem is important. Furthermore, the diquarks will no longer be colorless and the requirement of the color neutrality is not satisfied if we just include the diquarks.

The model parameters for the NJL model in the \( N_c = 3 \) case when \( qB = 0 \) are set to be \( m_0 = 5 \text{ MeV}, G_S = 4.93 \times 10^{-6} \text{ MeV}^{-2}, \Lambda = 653 \text{ MeV} \) and we choose to work with the ratio \( \eta = G_D/G_S \) instead of \( G_D \) [3], to reproduce the physical parameters already discussed in previous section. The mean-field thermodynamic potential in the zero temperature limit becomes

\[
\Omega_0 = \frac{(m - m_0)^2}{4G_S} + \frac{\Delta^2}{4G_D} - 4 \int \frac{d^3k}{(2\pi)^3} \left[ E^+_{\Delta} + E^-_{\Delta} + \frac{1}{2} (E^+_k + E^-_k) \right]
\]

(10)

where \( E^\pm_{\Delta} = \sqrt{(E_k \pm \mu_\Delta)^2 + \Delta^2} \) and \( E^\pm_k = E_k \pm 4\mu_B \). Due to the SU(3) color symmetry we may set the gauge \( \Delta_5 = \Delta_7 = 0 \), so that only red and green quarks take place in the condensation [10].

If we do not take into account the effect of color neutrality, we may consider the same chemical potential for red, green and blue quarks, \( \mu_r = \mu_g = \mu_b = \mu_B/3 \). In the left panel of figure 4 we show some results for the mass \( m \) and diquark condensate \( \Delta \), without considering effects of color neutrality [3]. Those are the numerical solutions obtained when minimizing the thermodynamic potential with respect to \( \Delta \) and \( m \). In the right panel we have the reference chemical potential \( \mu_N \), as a function of \( \mu_B \), for different values of \( \eta \). The BEC-BCS crossover in \( N_c = 3 \) QCD also happens when the quantity \( \mu_N \) exchange its signal, and the increase of \( \eta \) favors the BCS region, once \( \mu_N \) becomes positive in a smaller value of \( \mu_B \).

When considering the color neutrality, the gauge chosen for \( \Delta \) may be kept, but it is necessary to include a color chemical potential \( \mu_8 \), such that the quark chemical potentials are replaced by \( \mu_r = \mu_g = \mu_B/3 + \mu_8/3, \) and \( \mu_8 = \mu_B/3 - 2\mu_8/3, \) and a new Gap equation, \( -\frac{\partial \Omega_0}{\partial \mu_8} = 0 \), has to be solved for \( \mu_8 \). Our next step is study the influence of the magnetic field on the BEC-BCS crossover in \( N_c = 3 \) case, with and without the effects of color neutrality.

4. Final Remarks

In this work we have studied the effect of the application of an external magnetic field in the BEC-BCS crossover in the NJL model with \( N_c = 2 \). By using the GR scheme, we verify that when increasing the value of the magnetic field, the BCS region is strengthened, once the chemical potential \( \mu_N \) changes its signal at higher values of \( \mu_B \). Interestingly, this result is the opposite of that found by the authors in the reference [8], where they study the effect of \( qB \) in the crossover in a different model, keeping constant the quark mass \( m \). The result obtained in [8] was that at strong magnetic fields the system goes to the BCS regime.

It is important to note that the diquark condensate phase transition happens at \( \mu_B = m_\pi \), in agreement with effective theories predictions, and when including effects of magnetic fields it remains of second order, for both regularization schemes that we have used.

To summarize, we have reached two important conclusions in this work. Firstly, the GR scheme does not reproduce the physical parameters in the vacuum. We have not found a way
Figure 4. (Color online) **Left panel:** Effective quark mass $m$ and diquark condensate $\Delta$ as functions of $\mu_B$, ($\mu = \mu_B/3$ in $N_c = 3$ case, without considering the effects of color neutrality) for the ratio $\eta = G_D/G_S = 1.5$. **Right panel:** $\mu_N = \mu_B/3 - m$ as a function of the baryon chemical potential for different values of the ratio $\eta = G_D/G_S$.

to obtain the correct scaled values of the pion mass, chiral condensate and pion decay constant in the vacuum for $N_c = 2$ case. Secondly, in the $\Lambda_\chi$ scheme there are non-physical oscillations that are not expected in this problem. To avoid this problems a new work is being developed using a different regularization procedure [11], with emphasis on the new scheme purposed in the reference [12]. In [12] it was found a way to extract completely the purely magnetic contributions, such that there are no divergences depending explicitly on the magnetic field. It seems that this method is well parametrized and does not present the non-physical oscillations that are very common to appear in other regularization schemes [6].

The discussion about $N_c = 3$ made in this work is just an introduction, where we reproduce the results obtained in reference [3]. It is a natural continuation of our original work to the more realistic three colors QCD case, with and without including the effects of color and charge neutralities. It will be interesting to see the competition between the magnetic field, that tends to favor the BEC region and the increase of the ratio $G_S/G_D$, that tends to favor the BCS region, and how this combination would affect the BEC-BCS crossover as a whole.

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