Second order term of cover time for planar simple random walk

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Abstract

We consider the cover time for a simple random walk on the two-dimensional discrete torus of side length $n$. Dembo, Peres, Rosen, and Zeitouni [Ann. Math. 160:433-464, 2004] identified the leading term in the asymptotics for the cover time as $n$ goes to infinity. In this paper, we study the exact second order term. This is a discrete analogue of the work on the cover time for planar Brownian motion by Belius and Kistler [Probab. Theory Relat Fields. 167:461-552, 2017].

MSC: 60J10

Keywords: Cover time; Two-dimensional discrete torus; Simple random walk

1 Introduction

The present paper focuses on the cover time for the simple random walk (SRW) on the two-dimensional discrete torus $\mathbb{Z}_n^2 := (\mathbb{Z}/n\mathbb{Z})^2$ with the origin $o$. Let $S = (S_i, i \geq 0, P_x, x \in \mathbb{Z}_n^2)$ be the discrete-time SRW on $\mathbb{Z}_n^2$. For each $A \subset \mathbb{Z}_n^2$, let $H_A$ be the hitting time of $A$ defined by

$$H_A := \min\{i \geq 0 : S_i \in A\}. \quad (1.1)$$

For each $x \in \mathbb{Z}_n^2$, we will write $H_x$ when $A = \{x\}$. The cover time for the SRW on $\mathbb{Z}_n^2$ is given by

$$\tau_{\text{cov}}^n := \max_{x \in \mathbb{Z}_n^2} H_x. \quad (1.2)$$

Dembo, Peres, Rosen, and Zeitouni [8] obtained the exact leading term of $\tau_{\text{cov}}^n$; they showed that $\tau_{\text{cov}}^n/n^2(\log n)^2$ converges to $4/\pi$ in probability as $n \to \infty$. Ding [10] improved the result by proving that $\sqrt{\tau_{\text{cov}}^n/n^2} - \frac{2}{\sqrt{\pi}} \log n$ is of order $\log \log n$ with probability tending to 1 as $n \to \infty$. Related to the precise estimate of the cover time, geometric properties of the level set (so-called late points) of the form

$$\left\{ x \in \mathbb{Z}_n^2 : H_x > \alpha \frac{4}{\pi} n^2 (\log n)^2 \right\}, \quad \alpha \in (0,1)$$

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were investigated in [9], [16]. Recently, large deviations of the cover time and
the local structure around an unvisited point have been studied via random
interlacement techniques [5], [6], [7], [17]. For a general introduction to the
cover time, we refer the reader to [14, Chapter 11]. See also [11] for close
connections with the discrete Gaussian free field.

The aim of this paper is to obtain the exact second order term of the cover
time of \( Z^2 \) which was conjectured by Belius and Kistler [1]. Our main result is
as follows:

**Theorem 1.1** There exists \( c \in (0, 1) \) such that

\[
2 \log n - \log \log n - (\log \log n)^c \leq \frac{\tau_{cov}^n}{2n^2 \log n} \leq 2 \log n - \log \log n + (\log \log n)^c
\]

(1.3)

holds with \( P_o \)-probability tending to 1 as \( n \to \infty \).

**Remark 1.2** Belius and Kistler [1] have already established a similar estimate
for the \( \varepsilon \)-cover time for the standard Brownian motion on \((\mathbb{R}/\mathbb{Z})^2\). In the proof of Theorem 1.1, we basically follow techniques developed in [1], but it is not
straightforward to apply them. The technical difficulties in our discrete setting
come from lack of rotationally invariance of SRW; it only has an approximately
rotationally invariance, so we have to control the errors very carefully.

**Remark 1.3** It is natural to ask whether \( \frac{\tau_{cov}^n}{2n^2 \log n} - (2 \log n - \log \log n) \) is tight
or not. Recently, Belius, Rosen, and Zeitouni [3] proved that the cover time for
the Brownian motion on the two dimensional sphere is tight.

Let us describe the outline of the paper. In Section 2 we restate Theorem 1.1
in terms of the number of excursions around each vertex, which is called traversal
process. We will recall basic estimates on transition probabilities among circles
(Lemma 2.3). Applying this, we will approximate the law of the traversal process
by that of a (critical) Galton-Watson process in Lemma 2.4 which provides explicit bounds on the approximation errors. As was revealed by Belius and
Kistler [1], so-called "barrier estimates" for the Galton-Watson process play
an important role in studying the traversal process. Recently, Belius, Rosen,
and Zeitouni [2] gave quite general barrier estimates for the Galton-Watson
process. We will recall their result in Lemma 2.5. In Section 3 we obtain the
upper bound of the cover time. In Lemmas 3.1 and 3.2 we will prove that the
traversal process crosses neither the curve \( i \mapsto a_+^n(i) \) nor the curve \( i \mapsto a_-^n(i) \)
(see (3.4) and (3.13) for the definitions). Then, we impose the restriction so
that the traversal process stays in the tube \([a_-^n(\cdot), a_+^n(\cdot)]\). The upper envelope
\( a_+^n(\cdot) \) enables us to apply the transfer lemma (Lemma 2.4) and to make the
approximation errors of order 1. The lower envelope \( a_-^n(\cdot) \) enables us to apply
the barrier estimate (Lemma 2.5). In Section 4 we obtain the lower bound of
the cover time. For each \( x \in \mathbb{Z}^2 \), we will set the event \( A_n(x) \) which says that \( x \)
is an unvisited point and the traversal process corresponding to \( x \) stays in the
tube \([b_-^n(\cdot), b_+^n(\cdot)]\) (see (4.4)–(4.6) for the definitions of \( A_n(x), b_-^n(\cdot) \) and apply
the second moment method to \( Z_n := \sum_{x \in \mathbb{Z}_+^2} 1_{A_n(x)} \). The most difficult part of the proof is to obtain the upper bound of \( E_o([Z_n]^2) \). As is often the case with the study of the two dimensional cover time (e.g. [8], [9], [1]), in order to estimate \( P_o[A_n(x) \cap A_n(y)] \), we will consider branching structure of the two traversal processes corresponding to \( x \) and \( y \). We will take a decreasing sequence of radii \( (r_k)_{k \geq 1} \) and set the level \( k = \ell(x, y) \) at which the balls \( B(x, r_k) \) and \( B(y, r_k) \) of radius \( r_k \) centered at \( x \) and \( y \) become disjoint. See (2.3) and (4.10) for the precise definitions of \( r_k \) and \( \ell(x, y) \). Intuitively speaking, for \( k < \ell(x, y) \), the two balls \( B(x, r_k) \) and \( B(y, r_k) \) overlap and the traversal processes in the balls are heavily correlated, while, for \( k \geq \ell(x, y) \), the two balls \( B(x, r_k) \) and \( B(y, r_k) \) are disjoint and the traversal processes in the balls should be almost independent. Thus, as in Section 3, we can apply the transfer lemma (Lemma (2.4)) and the barrier estimate (Lemma 2.5) to each of the traversal processes in \( B(x, r_k) \) and \( B(y, r_k) \) for \( k \geq \ell(x, y) \). We deal with six cases (see (2.6), (2.8), (3.5) for the definitions of \( w_n, L_n, d_n(s) \)):

\[
(1) \ell(x, y) = 0, \quad (2) 1 \leq \ell(x, y) \leq w_n + 2, \quad (3) w_n + 3 \leq \ell(x, y) \leq d_n(\xi), \\
(4) d_n(\xi) < \ell(x, y) \leq [(1-\varepsilon)L_n], \quad (5) [(1-\varepsilon)L_n] < \ell(x, y) \leq L_n - w_n - 1, \\
(6) \ell(x, y) \geq L_n - w_n
\]

for some \( 0 < \varepsilon < 1 \) and \( \xi > 0 \). We will show that the term corresponding to \( x, y \) in the case (1) forms the main part of the estimate on \( E_o([Z_n]^2) \) (Lemma 4.13) and that other terms corresponding to \( x, y \) in cases (2)–(6) are negligible (Lemmas 4.10, 4.11, 4.12, 4.120). Main ingredients in the estimate of \( P_o[A_n(x) \cap A_n(y)] \) are decoupling inequalities which break dependence of excursions around \( x \) and \( y \) (Lemma 4.2 and (4.54)). In the cases (1)–(3), we will use Lemma 4.2 which claims that trajectories of SRW left in disjoint regions are almost independent. Thanks to Lemma 4.2, in the case (1), \( A_n(x) \) and \( A_n(y) \) are almost independent (see (4.125)) and, in the cases (2)–(3), the barrier conditions in \( A_n(x) \) and \( A_n(y) \) for \( i \geq \ell(x, y) \) are almost independent (see e.g. Lemma 4.5). In the case (4), we use the equation (4.54) which states that excursions inside a ball are independent of excursions outside the ball except the initial and terminal points of the excursions. The Harnack inequality (4.66) removes the dependence of the endpoints. Thanks to this, the barrier conditions in \( A_n(x) \) and \( A_n(y) \) for \( i \geq \ell(x, y) \) are almost independent (Lemma 4.10). In the case (5), thanks to the strong Markov property, \( A_n(x) \) and the barrier condition in \( A_n(y) \) for \( i \geq \ell(x, y) \) are almost independent (see (4.122)). In the case (6), since the number of such pairs is small, we will just bound \( P_o[A_n(x) \cap A_n(y)] \) by \( P[A_n(x)] \) (see (4.120)). In Section 4, we study concentration estimates on excursion lengths. In Sections 3, 4, we give proofs for the transfer lemma (Lemma 2.4) and the barrier estimate (Lemma 2.5).

Throughout the paper, we will write \( c, c', \ldots \) to denote positive constants. Values of \( c, c', \ldots \) will change from line to line. We use \( c_1, c_2, \ldots \) to denote constants whose values are fixed within each argument. Given sequences \( (c_n)_{n \geq 1} \) and \( (c'_n)_{n \geq 1} \), we write \( c'_n = O(c_n) \) if there exists a universal constant \( C \) such that \( |c'_n/c_n| \leq C \) for all \( n \geq 1 \). We will write \( c_n \asymp c'_n \) when \( c_n = O(c'_n) \) and
$c'_n = O(c_n)$. We let $o(1)$ denote a term with $o(1) \to 0$ as $n \to \infty$. For any set $A$, $|A|$ denotes the cardinality of $A$. For $a, b \in \mathbb{R}$, we set $a \lor b := \max\{a, b\}$ and $a \land b := \min\{a, b\}$. Let $d(\cdot, \cdot)$ be the $\ell^2$-distance in $\mathbb{Z}_n^2$. Set $B(x, r) := (y \in \mathbb{Z}_n^2 : d(x, y) < r)$. For each $A \subset \mathbb{Z}_n^2$ with $|A| \geq 2$, we define its boundary by $\partial A := \{y \in \mathbb{Z}_n^2 : y \in \mathbb{Z}_n^2 \setminus A, d(x, y) = 1 \text{ for some } x \in A\}$. When $A = \{x\}$, we set $\partial A := \{x\}$.

## 2 Excursion counts

In this section, we reformulate Theorem 1.1 in terms of the number of excursions between concentric circles. Recall that $(S_i)_{i \geq 0}$ is a SRW on $\mathbb{Z}_n^2$. Let us give a sequence of random times as follows: For each $x \in \mathbb{Z}_n^2$ and $0 < r < R < \frac{1}{2}$, set

$$R_1(x, R, r) := H_{\partial B(x, r)},$$

$$D_k(x, R, r) := \min\{i > R_k(x, R, r) : S_i \in \partial B(x, R)\},$$

$$R_{k+1}(x, R, r) := \min\{i > D_k(x, R, r) : S_i \in \partial B(x, r)\}, \quad k \geq 1. \quad (2.1)$$

We will call the path $(S_{R_k(x, R, r)}, \ldots, S_{D_k(x, R, r)})$ ($(S_{D_k(x, R, r)}, \ldots, S_{R_{k+1}(x, R, r)})$, respectively) excursion from $\partial B(x, r)$ to $\partial B(x, R)$ (from $\partial B(x, R)$ to $\partial B(x, r)$, respectively). Fix sufficiently small $\delta \in (0, \frac{1}{2})$. Pick $\gamma \in (0, 1)$ close enough to 1. Then, take $\alpha, \beta \in (0, 1)$ close enough to zero so that the following holds:

$$2\gamma - 2\beta - \alpha > 1, \quad (1 - 2\delta)\beta > \alpha, \quad \alpha + \beta > \frac{1}{2} + \delta - \alpha \delta. \quad (2.2)$$

We will consider concentric circles of radii

$$r_k = r_k(n) := (\ell_n)^{L_n - k}, \quad k = 0, \ldots, L_n, \quad (2.3)$$

where

$$\ell_n = \ell_n(\alpha) := \exp\{\log \log n^\alpha\},$$

$$w_n = w_n(\beta) := (\log \log n)^\beta,$$

$$L_n = L_n(\alpha, \beta, c_\ast) := \left\lfloor \frac{\log n}{\log(\ell_n)} - c_\ast w_n \right\rfloor. \quad (2.6)$$

Here $c_\ast$ is a sufficiently large positive constant. Set

$$m_n^+ := \left\lfloor \left( 1 - \frac{\log \log n}{2 \log n} + \frac{s_n}{\log n} \right) \frac{2(\log n)^2}{\log(\ell_n)} \right\rfloor, \quad (2.7)$$

$$m_n^- := \left\lceil \left( 1 - \frac{\log \log n}{2 \log n} - \frac{s_n}{\log n} \right) \frac{2(\log n)^2}{\log(\ell_n)} \right\rceil, \quad (2.8)$$

where

$$s_n = s_n(\gamma) := (\log \log n)^\gamma. \quad (2.9)$$

The following indicates that $m_n^+$ and $m_n^-$ are approximations of the number of excursions from $\partial B(x, r_1)$ to $\partial B(x, r_0)$ up to the cover time.
Proposition 2.1 The events that
\[
H_x \leq D_{m_x^+}(x, r_0, r_1), \quad \forall x \in \mathbb{Z}_n^2, \quad (2.10)
\]
\[
H_x \geq D_{m_x^-}(x, r_0, r_1), \quad \exists x \in \mathbb{Z}_n^2 \quad (2.11)
\]
hold with \(P_o\)-probability tending to 1 as \(n \to \infty\).

We will give the proof of (2.10) (respectively, (2.11)) in Section 3 (respectively, in Section 4).

We have uniform controls of \(D_{m_x^+}(x, r_0, r_1)\) and \(D_{m_x^-}(x, r_0, r_1)\) in \(x \in \mathbb{Z}_n^2\):

Proposition 2.2 The events that
\[
D_{m_x^+}(x, r_0, r_1) \leq \frac{4}{\pi} \eta^2 (\log n)^2 \left( 1 - \frac{\log \log n}{2 \log n} + \frac{2s_n}{\log n} \right), \quad \forall x \in \mathbb{Z}_n^2, \quad (2.12)
\]
\[
D_{m_x^-}(x, r_0, r_1) \geq \frac{4}{\pi} \eta^2 (\log n)^2 \left( 1 - \frac{\log \log n}{2 \log n} - \frac{2s_n}{\log n} \right), \quad \forall x \in \mathbb{Z}_n^2 \quad (2.13)
\]
hold with \(P_o\)-probability tending to 1 as \(n \to \infty\).

We will provide the proof of Proposition 2.2 in Section A.

We will use the following basic estimates on the SRW on \(\mathbb{Z}_n^2\). See, for example, [13, Proposition 1.6.7, Exercise 1.6.8] for the proof.

Lemma 2.3 (i) There exists \(c_1 > 0\) such that for all \(0 < r < R < \frac{n}{2}\) and \(x, y \in \mathbb{Z}_n^2\) with \(r < d(x, y) < R\),
\[
\log \left( \frac{R_{d(x, y)}}{\log(R)} \right) - \frac{c_1}{R} \leq P_y[H_{\partial B(x, r)} < H_{\partial B(x, R)}] \leq \log \left( \frac{R_{d(x, y)}}{\log(R)} \right) + \frac{c_1}{R}.
\]

(ii) There exists \(c_2 > 0\) such that for all \(0 < R < \frac{n}{2}\) and \(x, y \in \mathbb{Z}_n^2\) with \(0 < d(x, y) < R\),
\[
\log \left( \frac{R_{d(x, y)}}{\log(R)} \right) - \frac{c_2}{d(x, y)} - \frac{c_2}{\log R} \leq P_y[H_x < H_{\partial B(x, R)}] \leq \log \left( \frac{R_{d(x, y)}}{\log(R)} \right) + \frac{c_2}{d(x, y)} + \frac{c_2}{\log R}.
\]

It is convenient to give specific notation to the probabilities in Lemma 2.3.

Given a decreasing sequence of radii \(\mathbf{R} = (R_i)_{0 \leq i \leq L}\) with \(R_L = 1\), for each \(0 \leq i_1 < i_2 < i_3 < L\), set
\[
p_{i_1, i_3}^{i_2} (\mathbf{R}) := \frac{\log \left( \frac{R_{i_1}}{R_{i_2}} \right) + \frac{2.31}{R_{i_3}}}{\log \left( \frac{R_{i_1}}{R_{i_3}} \right)}
\]
\[
p_{i_1, i_3}^{i_2} (\mathbf{R}) := 1 - p_{i_1, i_3}^{i_2} (\mathbf{R}), \quad (2.14)
\]
Lemma 2.4 (Transfer lemma) Fix any decreasing sequence of radii $\mathbf{R} = (R_l)_{l=0}^L$ with $R_L = 1$, $x, y \in \mathbb{Z}_n^2$, and $m, L \in \mathbb{N}$.

(i) For any $k, \tilde{k} \in \{0, \ldots, L - 1\}$ with $k \leq L - \tilde{k} - 1$, $m_i \geq 0$, and $k \leq i \leq L - \tilde{k} - 1$ with $P^G_m \left[ \{ T_l = m_i, k \leq \forall i \leq L - \tilde{k} - 1 \} \cap \{ T_{L-1} = 0 \} \right] \neq 0$, $P_y \left[ \left\{ T^{x,m}_i(\mathbf{R}) = m_i, k \leq \forall i \leq L - \tilde{k} - 1 \right\} \cap \left\{ T^{x,m}_{L-1}(\mathbf{R}) = 0 \right\} \right] \\
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where $\Delta_1^+$ and $\Delta_+^*$ are defined by
\[
\Delta_1^+ := \left( \Delta_{k+1,0}^{k+1} \lor \Delta_{k,k+1}^0 \right)^m \left( \Delta_{k+1,0}^k \lor \Delta_{k+1,1}^0 \right)^{m_k} 
\times \prod_{i=k+1}^{L-k-1} \left( \Delta_{i+1,0}^{i+1} \lor \Delta_{i,i+1}^{i+1} \right)^{m_{i-1} + m_i},
\] (2.21)
and $\Delta_+^*$ := \left( \Delta_{L-k-1,k}^{L-k-1} \right)^{m_{L-k-1}}. \Delta_1^-$ and $\Delta_+^*$ are defined by replacing “+” and “∨” in the definitions of $\Delta_1^+$ and $\Delta_+^*$ with “−” and “∧”. Moreover (2.20), with \( \{ T_{\ell-1}(R) = 0 \} \) and \( \{ T_{L-1} = 0 \} \) removed and with $\Delta_1^+ \pm$ replaced by 1, holds true.

(ii) For any $\ell \in \mathbb{N}$ and $1 \leq i \leq L-1$,
\[
P_r \left[ T_{\ell-1}(R) = \ell, T_{L-1}(R) = 0 \right] \leq \Delta_+^* \cdot P_{GW} [ T_i \geq \ell, T_{L-1} = 0 ],
\] (2.22)
where
\[
\Delta_+^* := \left( \Delta_{i+1,0}^{i+1} \lor \Delta_{i,i+1}^{i+1} \right)^{m_{i+1}} \left( \Delta_{k+1,0}^k \lor \Delta_{k+1,1}^0 \right)^{\ell} \left( \Delta_{i+1,i}^k \lor \Delta_{i+i,i}^k \right)^{m_{i+1} + m_i} \left( \Delta_{i+1,0}^i \lor \Delta_{i+1,1}^0 \right)^{m_i}.
\] (2.23)

(iii) For any $k, \tilde{k} \in \{ 1, \ldots, L-1 \}$ with $k < L-\tilde{k}-1$, $m_i \geq 0$, $k+1 \leq i \leq L-\tilde{k}-1$, $y \in \partial B(x, R_k),
\[
P_r \left[ \bigcap_{i=k+1}^{L-\tilde{k}-1} \left\{ T_i(x,y,m(R)) = m_i \right\} \cap \left\{ T_{L-1}(R) = 0 \right\} \right.
\]
\[
\bigcap \left\{ R_m(x,R_k, R_{k+1}) < H_{\partial B(x, R_0)} < R_{m+1}(x,R_k, R_{k+1}) \right\}
\]
\[
\leq \Delta_+^* \cdot P_{GW} \left[ \bigcap_{i=1}^{L-k-\tilde{k}-1} \left\{ T_i = m_{k+1} \right\} \cap \{ T_{L-k-1} = 0 \} \right] \cdot P[G = m],
\] (2.24)
where $G$ is a geometric random variable with success probability $\frac{1}{\ell+1}$ and $\Delta_+^*$ is defined by
\[
\prod_{i=k+1}^{L-\tilde{k}-1} \left( \Delta_{i+1,0}^{i+1} \lor \Delta_{i,i+1}^{i+1} \right)^{m_{i-1} + m_i} \left( \Delta_{k+1,0}^k \lor \Delta_{k+1,1}^0 \right)^{m_{k+1} - k-1} \left( \Delta_{k,0}^k \lor \Delta_{k,k+1}^0 \right)^{m} \Delta_{k,0}^{k+1}.
\] (2.25)

We will give the proof of Lemma 2.4 in Section 2.

In the proof of Proposition 2.1 we heavily use so-called barrier estimates for the Galton-Watson process. For $a, b \in \mathbb{R}$ and $L \in \mathbb{N}$, set a linear barrier
\[
f_{a,b}(i; L) := a + \frac{(b-a)i}{L}, \quad 0 \leq i \leq L.
\]
We will consider small perturbations of the linear barrier of the form \( f_{a,b}(i; L) \pm C(i_L)^c \) for some \( c, C > 0 \), where

\[
i_L := i \land (L - i), \quad 0 \leq i \leq L.
\]

The following is a slightly modified version of barrier estimates for the Galton-Watson process by Belius, Rosen, and Zeitouni [2]:

**Lemma 2.5 (Barrier estimate)**

(i) For any \( \delta, C \in (0, \infty) \), \( \varepsilon \in (0, \frac{1}{2}) \), \( \eta > 1 \), \( a, b, x, y \) with \( \sqrt{2} \leq x, y \leq \eta L \), \( \frac{\sqrt{2}}{\eta} \in \mathbb{N} \), \( 0 \leq a \leq x \), \( 0 \leq b \leq y \), \( b \leq a \),

\[
P_{GW} G \left[ \bigcap_{i=1}^{r-1} \left\{ f_{a,b}(i; L) - C_i L^{-\varepsilon} \leq \sqrt{2T_i} \right\} \cap \left\{ \sqrt{2T_L} \in [y, y + \delta] \right\} \right] \leq c_1 e^{c_2 \eta^2 \sqrt{\frac{x}{y} \frac{a - x}{L}}} \left( \frac{\eta + x}{\eta + y} \right),
\]

where \( c_1 \) and \( c_2 \) are positive constants which depend only on \( \delta, C, \varepsilon \).

(ii) There exists \( r_0 > 0 \) such that for any \( C > 0 \), \( \eta > 1 \), \( \varepsilon \in (0, \frac{1}{2}) \), \( \mu \in (0, 1) \), \( 4r^{\frac{1}{2} + 2\varepsilon} \leq \mu L \leq a \leq x \leq \eta L \), \( \frac{\sqrt{2}}{\eta} \in \mathbb{N} \), \( C r^{\frac{1}{2} - \varepsilon} \geq \eta \), \( L > 2r > r_0 \),

\[
P_{GW} G \left[ \bigcap_{i=r}^{L-1-r} \left\{ f_{a,b}(i; L) + C_i L^{-\varepsilon} \leq \sqrt{2T_i} \leq f_{x,0}(i; L) + \tilde{C}_i L^{\frac{1}{2} + \varepsilon} \right\} \cap \left\{ \sqrt{2T_L-1} = 0 \right\} \right] \geq c_3 \left( \frac{r}{L - 2r} \right) \left( 1 - \frac{1}{L} \right)^{\frac{r^2}{2L}},
\]

where \( c_3 \) is a positive constant which depends only on \( r_0, C, \varepsilon, \mu \).

We will give the proof of Lemma 2.5 in Section C.

**Remark 2.6** Unfortunately, we cannot directly apply [2, Theorem 1.1] for the following reason: Let \( x, L, \eta \) be the constants in [2, Theorem 1.1]. In our setting, we will typically take \( x \asymp \frac{\log n}{\log(\ell_n)} \) and \( L \asymp \frac{\log n}{\log(\ell_n)} \). Thus, we need to take \( \eta \asymp \sqrt{\log(\ell_n)} \). Since this \( \eta \) depends heavily on \( n \), we need explicit information on how the constant \( c \) in [2, Theorem 1.1] depends on \( \eta \).

### 3 Upper bound of cover time

In this section, we prove (2.10). We will use the same notation as in Section 2. To simplify notation, we will write

\[
D_m^{x,i} := D_m(x, r_i, r_{i+1}), \quad R_m^{x,i} := R_m(x, r_i, r_{i+1}), \quad x \in \mathbb{Z}_n^2, \quad i \geq 0, \quad m \in \mathbb{N}.
\]
For \( m \in \mathbb{N} \), \( x \in \mathbb{Z}_n^2 \), \( 1 \leq i \leq L_n - 1 \), set
\[
\bar{T}^{x,m}_i := \max\{\ell \geq 1 : R^{x,i}_\ell \circ \theta_{H\delta_B(x,r_1)} < D^{x,0}_m\},
\]
where \( \theta_s, s \geq 0 \) are shift operators. We will work on the traversal processes
\((\bar{T}^{x,m}_i)_{k \in \{0,1,\ldots,L_n-1\}}, x \in \mathbb{Z}_n^2 \). We first prove that the traversal process \( \bar{T}^{x,m}_i \) stays in the tube \([a^+_n(\cdot), a^+_n(\cdot)]\) with high probability in Lemmas \( 3.1 \) and \( 3.2 \) (see \( 3.3 \)) and \( 3.4 \) (for the definitions of \( a^+_n(\cdot) \) and \( a^+_n(\cdot) \)). Using this together with the transfer lemma (Lemma \( 2.4 \)) and the barrier estimate (Lemma \( 2.5 \)), we will prove \( 2.10 \).

We need control of the traversal process from above to deal with the approximation errors which appear in applying the transfer lemma.

**Lemma 3.1** There exists \( \kappa > 0 \) such that
\[
\lim_{n \to \infty} P_o \left[ \bigcup_{x \in \mathbb{Z}_n^2} \bigcup_{i=1}^{L_n-1} \left\{ \bar{T}^{x,m}_i \geq a^+_n(i), H_x > D^{x,0}_m \right\} \right] = 0,
\]
where for each \( i \in \{1, \ldots, L_n-1\} \)
\[
a^+_n(i) := \left\lceil \sqrt{m^+_n} \left( 1 - \frac{i}{L_n} \right) + \kappa \sqrt{\frac{(i+1)(L_n-i)}{L_n+1}} \sqrt{\log \log n} \right\rceil. \tag{3.4}
\]

**Proof.** To simplify notation, we will write
\[
d_n(s) := \left\lceil \frac{s \log \log n}{\log(\ell_n)} \right\rceil, \quad s > 0.
\]
Fix \( x \in \mathbb{Z}_n^2 \) and \( i \in \{1, \ldots, L_n - d_n(2) - 1\} \). By the transfer lemma (Lemma \( 2.4 \) ii) with \( \mathbf{R} = (r_i)_{i=0}^{\ell_n-1} \), we have
\[
P_o \left[ \bar{T}^{x,m}_i \geq a^+_n(i), H_x > D^{x,0}_m \right] \leq \Delta^+_n P^{GW}_{m^+_n} [T_i \geq a^+_n(i), T_{L_n-1} = 0], \tag{3.6}
\]
where \( \Delta^+_n \) is defined by \( 2.23 \) with \( m = m^+_n \), \( \ell = a^+_n(i) \), \( L = L_n \). Recalling the definitions of \( \Delta^{i_1,i_2}_{n+1} \), \( \Delta^{i_3,i_4}_{n+1} \) from \( 2.10 \), by a simple calculation, we have
\[
\Delta_{i_1,0}^{i_1+1} \lor \Delta_{i_0,0}^{0+1} = 1 + O \left( \frac{1}{(\log n)^2 \log(\ell_n)} \right),
\]
\[
\Delta_{i_1,i_1+1}^{L_n+1} \lor \Delta_{i_0,i_0}^{i_0+1} = 1 + O \left( \frac{1}{(L_n - i)^2 (\log(\ell_n))^2} \right),
\]
\[
\Delta_{i_1+1,0}^{L_n+1} = 1 + O \left( \frac{1}{(\log \log n)^2} \right), \quad \Delta_{i_0,0}^{L_n+1} = 1 + O \left( \frac{1}{(\log n)^2} \right).
\]
Since $m_n^+ \asymp \frac{(\log n)^2}{\log(\ell_n)}$ and $a_n^+(i) \leq c(L_n - i)^2 \log(\ell_n) + c(L_n - i) \log \log n$, we have \( \Delta_3^+ = 1 + o(1) \). By the Markov property, the probability on the right-hand side of (3.6) is equal to

$$
\sum_{m = a_n^+(i)}^{\infty} P_{m_n}^{GW}[T_i = m] P_{m_n}^{GW}[T_{L_n - i - 1} = 0].
$$

(3.7)

By \([2, (5.3)]\), the first probability of the $m$-th term in (3.7) is bounded from above by $e^{-(\sqrt{m} - \sqrt{m_n^-})^2/(i+1)}$. By a simple calculation, the second probability of the $m$-th term in (3.7) is equal to $(1 - \frac{L_n}{L_n + 1})^m$ and this is bounded from above by $e^{-\frac{m}{m_n^+}}$. Thus, the right of (3.7) is bounded from above by

$$
\sum_{m = a_n^+(i)}^{\infty} e^{-\frac{m}{m_n^+}} \exp \left\{ -\frac{L_n + 1}{i+1} \left( \sqrt{m} - \frac{L_n - i}{L_n + 1} \sqrt{m_n^+} \right)^2 \right\}
$$

$$
\leq n^{-2}(\log n) \sum_{m = a_n^+(i)}^{4m_n^+} e^{-(\kappa_+)^2 \log \log n} + n^{-2}(\log n) \sum_{m = 4m_n^+ + 1}^{\infty} e^{-\frac{m}{m_n^+}}
$$

$$
\leq c_1 n^{-2}(\log n)^{-(\kappa_+)^2 + 3} + c_1 n^{-4}(\log n)^3,
$$

(3.8)

where we used the definition of $a_n^+(i)$ and the inequality $\sqrt{m} - \frac{L_n - i}{L_n + 1} \sqrt{m_n^+} \geq \frac{\sqrt{m}}{2}$ for each $m > 4m_n^+$ in the first inequality. By this and the union bound, taking $\kappa_+$ large enough, we have the desired result. \( \square \)

We need some control of the traversal process from below to apply the barrier estimate.

Lemma 3.2 There exists $\kappa_- > 0$ such that

$$
\lim_{n \to \infty} P_0 \left[ \bigcup_{x \in \mathbb{Z}_n^2} \bigcup_{i = 1}^{L_n - 1} \left\{ \sqrt{\tilde{\tau}_i^x, m_n^+} < \sqrt{m_n^+} \left( 1 - \frac{i}{L_n} \right) - \kappa_- \frac{\log \log n}{\log(\ell_n)} \right\} \right] = 0.
$$

(3.9)

Proof. Fix $i \in \{1, \ldots, L_n - 1\}$. Set $r_i^+ := (1 + \frac{\sqrt{\kappa}}{(\log n)^2}) r_i$, $r_i^- := (1 - \frac{\sqrt{\kappa}}{(\log n)^2}) r_i$. Set

$$
F_i := \left\{ (k \cdot \tilde{r}_i, \ell \cdot \tilde{r}_i) : k, \ell \in \{0, \ldots, \left\lfloor \frac{n}{\ell} \right\rfloor \} \right\},
$$

(3.10)

where $\tilde{r}_i := \left\lfloor \frac{r_i}{\sqrt{\kappa}(\log n)^2} \right\rfloor$. Fix any $x \in \mathbb{Z}_n^2$. There exists $y \in F_i$ such that $x \in (y + [0, \tilde{r}_i]^2) \cap \mathbb{Z}_n^2$ mod $n\mathbb{Z}^2$. One can easily check the following:

$$
B(y, r_i^+ - 1) \subset B(x, r_{i+1}) \subset B(x, r_i) \subset B(y, r_i^+),
$$

$$
B(x, r_i) \subset B(y, r_i^+) \subset B(y, r_i^-) \subset B(x, r_0).
$$
Thus, we have
\[ \hat{T}^{x, m_n^-} \leq T_i \]
where for \( z \in \mathbb{Z}^2_n, m \in \mathbb{N} \), we set
\[ \hat{T}^{z, m} := \max \{ \ell \geq 0 : R^\ell(z, r^{+}_{i}, r^{-}_{i+1}) \circ \theta_{H_{SB(z, r^{+}_{i})}} < D_m(z, r^{-}_{i}, r^{+}_{i+1}) \} \]
(3.11)
Thus, for sufficiently large \( \kappa_- \), the probability in (3.9) is bounded from above by
\[ \sum_{i=1}^{L_n - d_n(\sqrt{\kappa_-})-1} \sum_{y \in F_i} P_o \left[ \hat{T}^{y, m_n^+} \leq a_n^- (i) \right], \]
(3.13)
where
\[ a_n^- (i) := \left\lfloor \left( \sqrt{m_n^-} \left( 1 - \frac{i}{L_n} \right) - \frac{\log \log n}{\sqrt{\log (L_n)}} \right) \sqrt{1} \right\rfloor \]
(3.14)
and we have used the fact that for sufficiently large \( \kappa_- \), \( \sqrt{m_n^-} (1 - \frac{i}{L_n}) - \frac{\log \log n}{\sqrt{\log (L_n)}} < 0 \) for all \( i \geq L_n - d_n(\sqrt{\kappa_-}) \). Fix \( 1 \leq i \leq L_n - d_n(\sqrt{\kappa_-}) - 1 \) and \( y \in F_i \). By the transfer lemma (Lemma 2.4(ii)) with any sequence of radii \( (R_k)_{k=0}^{L_n} \) with \( R_0 = r^-_0, R_1 = r^+_1, R_i = r^+_i, R_{i+1} = r^-_{i+1} \), the \( (i, y) \)-th term in (3.13) is bounded from above by
\[ (1 + o(1)) P_{m_n^+} \left[ T_i < a_n^- (i) \right]. \]
(3.15)
By [2, (5.3)], the probability in (3.15) is bounded from above by
\[ e^{-\left( \sqrt{m_n^-} - \sqrt{a_n^- (i)} \right)^2}. \]
(3.16)
By the definitions of \( m_n^+ \), \( a_n^- (i) \), and the condition \( i < L_n - d_n(\sqrt{\kappa_-}) \), (3.16) is bounded from above by \( (L_n)^{2(i+1)} (\log n)^{-2\sqrt{\kappa_-}+1} \). One can check that \( |F_i| \leq c_1 (\ell_n)^{2(i+1)} (\ell_n)^{2c_2 w_n} (\log n)^4 \). Thus, the sum in (3.13) is bounded from above by \( c_2 e^{3c_3 (\log \log n)^{\alpha+\beta} (\log n)^{-2\sqrt{\kappa_-}+6}} \). This goes to 0 as \( n \to \infty \) for sufficiently large \( \kappa_- > 0 \) since \( \alpha + \beta < 1 \) by the assumption (2.2).

Proof of (2.10). Recall the constants \( \kappa_- \) and \( \kappa_+ \) from Lemmas 3.2 and 3.1. Recall \( a_n^- (k) \) and \( a_n^+ (k) \) from (3.14) and (3.4). Recall the notation \( d_n(\cdot) \) from
To simplify notation, we set \( L_n^* := L_n - d_n(2\kappa) - 1 \). We have

\[
P_o \left[ \bigcup_{x \in \mathbb{Z}_n^2} \left\{ H_x > D_{x,0}^{m,0} \right\} \right]
\leq P_o \left[ \bigcup_{x \in \mathbb{Z}_n^2} \left\{ a_n^- (i) \leq \tilde{T}_i^{x,m_n^+} \leq a_n^+ (i) \right\} \cap \left\{ H_x > D_{x,0}^{m,0} \right\} \right]
\leq P_o \left[ \bigcup_{x \in \mathbb{Z}_n^2} \bigcup_{i=1}^{L_n^*} \left\{ \sqrt{\tilde{T}_i^{x,m_n^+}} < \sqrt{m_n^+} \left(1 - \frac{i}{L_n} \right) - \kappa \frac{\log \log n}{\log(\ell_n)} \right\} \right]
\leq P_o \left[ \bigcup_{x \in \mathbb{Z}_n^2} \bigcup_{i=1}^{L_n^*} \left\{ \tilde{T}_i^{x,m_n^+} > a_n^+ (i), H_x > D_{x,0}^{m,0} \right\} \right],
\]  

(3.17)

where we have used the fact that \( \sqrt{m_n^+} \left(1 - \frac{i}{L_n} \right) - \kappa \frac{\log \log n}{\log(\ell_n)} > 1 \) for all \( i \in \{1, \ldots, L_n^*\} \). The first term on the right of (3.17) is the probability of the event that the traversal process \( \tilde{T}_i^{x,m_n^+} \) stays in the tube \([a_n^-, a_n^+]\) and equals zero at \( L_n - 1 \) for some \( x \). The second term (resp. the third term) on the right of (3.17) is the probability of the event that the traversal process \( \tilde{T}_i^{x,m_n^+} \) hits the lower curve \( a_n^- \) (resp. the upper curve \( a_n^+ \)) for some \( x \). By Lemmas 3.2 and 3.4, the second and third terms of (3.17) go to 0 as \( n \to \infty \). Thus, we only need to deal with the first term on the right-hand side of (3.17).

Fix \( x \in \mathbb{Z}_n^2 \). By the transfer lemma (Lemma 2.4), we have

\[
P_o \left[ H_x > D_{x,0}^{m,0}, a_n^- (i) \leq \tilde{T}_i^{x,m_n^+} \leq a_n^+ (i), 1 \leq i \leq L_n^* \right]
\leq (1 + o(1)) P_{GW}^{m_n^*} \left[ a_n^- (i) \leq T_i \leq a_n^+ (i), 1 \leq i \leq L_n^*, T_{L_n^* - 1} = 0 \right],
\]  

(3.18)

where \( o(1) \to 0 \) as \( n \to \infty \) uniformly in \( x \). By conditioning on \( T_{L_n^*} \), the probability on the right of (3.18) is bounded from above by

\[
\sum_{m = a_n^-(L_n^*)}^{a_n^+(L_n^*)} P_{GW}^{m_n^*} \left[ \bigcup_{i=1}^{L_n^* - 1} \left\{ \sqrt{T_i} \geq \sqrt{m_n^+} \left(1 - \frac{i}{L_n} \right) - \kappa \frac{\log \log n}{\log(\ell_n)} - 1 \right\} \cap \left\{ T_{L_n^*} = m \right\} \right]
\times P_{GW}^{m_n^*} \left[ T_{d_n(2\kappa)} = 0 \right].
\]  

(3.19)

To estimate the first probability of the \( m \)-th term in (3.19), we will apply the barrier estimate (Lemma 2.5) with \( x = \sqrt{2m_n^+}, y = \sqrt{2m_n^+}, a = \sqrt{2} (\sqrt{m_n^+} - \kappa \frac{\log \log n}{\log(\ell_n)} - 1), b = \sqrt{2} (\sqrt{m_n^+} (1 - \frac{L_n^*}{L_n}) - \kappa \frac{\log \log n}{\log(\ell_n)} - 1) \) (we can take \( \eta = c \sqrt{\log(\ell_n)} \)) for some positive constant \( c \). Then, the probability is bounded from
above by
\[
c_1 \left( \frac{\ell_n}{L_n} \right)^2 e^{-\frac{\sqrt{m_n L_n}}{L_n}}. \tag{3.20}
\]
By a simple calculation, the second probability of the \( m \)-th term in (3.19) is equal to \( (1 - \frac{1}{d_n(2\kappa - 1)} + 1)^m \) and this is bounded from above by \( e^{-\frac{m_n^+}{L_n}} \). The product of this and the exponential factor in (3.20) is bounded from above by \( e^{-\frac{m_n^+}{L_n}} \). By this, (3.18) is bounded from above by
\[
c_5 (\ell_n)^c n^{-2} e^{-2\kappa n}. \tag{3.21}
\]
Therefore, the first term on the right-hand side of (3.17) is bounded from above by \( c_5 (\ell_n)^c n^{-2} e^{-2\kappa n} \) and this goes to 0 as \( n \to \infty \). Therefore, the right of (3.17) goes to 0 as \( n \to \infty \). \( \square \)

**Proof of the upper bound of Theorem 1.1 via (2.12).** (2.10) and (2.12) immediately yield the upper bound. \( \square \)

### 4 Lower bound of cover time

In this section, we prove (2.11). Recall some notation from (2.1)-(2.9) and (3.1).

For each \( x \in \mathbb{Z}_n, 1 \le i \le L_n - 1, \) and \( m \in \mathbb{N}, \) set
\[
T_{x,m}^i := \max\{k \ge 0 : R_{x,i}^k < D_{x,m}^0\}. \tag{4.1}
\]
As in Section 3, we will study the traversal process \((T_{x,m}^i)_{i \le L_n - 1}, x \in \mathbb{Z}_n^2.\) Set
\[
f_n(s) := \min\{s^{1/2 - \delta}, (L_n - 1 - s)^{1/2 - \delta}\}, \ s \in [0, L_n - 1], \tag{4.2}
g_n(s) := \min\{s^{1/2 + \delta}, (L_n - 1 - s)^{1/2 + \delta}\}, \ s \in [0, L_n - 1], \tag{4.3}
\]
where \( \delta \) is the one in (2.2). For each \( x \in \mathbb{Z}_n, \) set
\[
A_n(x) := \left\{ b^{-}_n(i) \le T_{x,m}^i \le b^{+}_n(i), \ w_n \le \forall i \le L_n - 1 - w_n, \ H_x > D_{x,m}^0 \right\}, \tag{4.4}
\]
where
\[
b^{-}_n(s) := \left\{ \left( 1 - \frac{s}{L_n} \right) \sqrt{m_n} + f_n(s) \right\}^2, \ s \in [0, L_n - 1], \tag{4.5}
b^{+}_n(s) := \left\{ \left( 1 - \frac{s}{L_n} \right) \sqrt{m_n} + g_n(s) \right\}^2, \ s \in [0, L_n - 1]. \tag{4.6}
\]
We will apply the second moment method to
\[
Z_n := \sum_{x \in \mathbb{Z}_n^2 \setminus B(s, r_n)} 1_{A_n(x)} \tag{4.7}
\]
and we need a lower bound of $E_o[Z_n]$ and an upper bound of $E_o[Z_n^2]$. To estimate $E_o[Z_n]$, we need a lower bound of $P_o(A_n(x)), x \in \mathbb{Z}_n^2$.

**Lemma 4.1** There exist $c_1, c_2 \in (0, \infty)$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $x \in \mathbb{Z}_n^2 \setminus B(o, r_0)$,

$$P_o(A_n(x)) \geq c_1 n^{-2} \log(\ell_n) e^{2w_n(\ell_n)^{-2w_n-c_2} c_2}.$$  \hspace{1cm} (4.8)

**Proof.** Fix $x \in \mathbb{Z}_n^2 \setminus B(o, r_0)$. By the transfer lemma (Lemma 2.4(i)), we have

$$P_o[A_n(x)] \geq (1 + o(1)) P_{GW}^{m_n} \left[ \bigcap_{i=w_n}^{\ell_n-1-w_n} \{ b_n^i(i) \leq T_i \leq b_n^i+i(i) \} \cap \{ T_{\ell_n-1} = 0 \} \right].$$  \hspace{1cm} (4.9)

By the barrier estimate (Lemma 2.5(ii)), the right of (4.9) is bounded from below by $c_1 w_n (1 - \frac{1}{L_n}) m_n$, which yields the desired result. \qed

From now, to obtain an upper bound of $E_o[Z_n^2]$, we estimate $P_o[A_n(x) \cap A_n(y)]$, $x, y \in \mathbb{Z}_n^2 \setminus B(o, r_0)$. When two balls centered at $x$ and $y$ overlap, the traversal processes in those balls are heavily correlated, while, when two balls are disjoint, the traversal processes in the balls should be almost independent. We will use the term “decoupling” to describe such procedure to break dependence of the traversal processes. Based on this observation, we define the branching level by

$$\ell(x, y) := \min \{ k \geq 0 : B(x, r_k) \cap B(y, r_k) = \emptyset \}. \hspace{1cm} (4.10)$$

Recall the definition of $d_n(\cdot)$ from (3.3). We will consider six cases:

- $(1) \ell(x, y) = 0$,
- $(2) 1 \leq \ell(x, y) \leq w_n + 2$,
- $(3) w_n + 3 \leq \ell(x, y) \leq d_n(\xi)$,
- $(4) d_n(\xi) < \ell(x, y) \leq [(1-\varepsilon)L_n]$,  \hspace{1cm} (5) $[(1-\varepsilon)L_n] < \ell(x, y) \leq L_n - w_n - 1$,
- $(6) \ell(x, y) \geq L_n - w_n.$

In the case (4), following the argument in [1 Section 6.2], we will decouple the traversal processes by conditioning on excursions outside the balls and applying the Harnack inequality to remove the dependence of the endpoints of the excursions. In the case (5), following the argument in [1 Section 6.1], we use a recursion argument based on the strong Markov property of SRW to obtain a decoupling inequality. In the case (6), we do not need decoupling estimates since the number of the pairs $(x, y)$ is small. In cases (1)–(3), to decouple the traversal processes, we use the following lemma which states that excursions in disjoint regions are almost independent.

**Lemma 4.2** Let $n \in \mathbb{N}$ and $0 < r < R < R' < \tilde{R} < n/4$. Let $x, y \in \mathbb{Z}_n^2$ with $B(x, R) \cap B(y, R) = \emptyset$ and $B(y, R) \subset B(x, R')$. We define the space $\mathcal{W}$ of nearest-neighbor paths of finite length on $\mathbb{Z}_n^2$ by

$$\mathcal{W} := \left\{ \omega : \exists L_\omega \in \mathbb{N}, \forall i \leq L_\omega, \omega(i) \in \mathbb{Z}_n^2, \ d(\omega(i), \omega(i-1)) = 1, \ 1 \leq \forall i \leq L_\omega \right\}.$$
For each \( z \in \{ x, y \} \), we define the space \( W_z \) of paths from \( \partial B(z, r) \) to \( \partial B(z, R) \) by
\[
W_z := \left\{ \omega \in \mathcal{W} : \begin{array}{l}
\omega(0) \in \partial B(z, r), \\
\omega(I_\omega) \in \partial B(z, R), \\
\forall i < I_\omega
\end{array} \right\}.
\]
Similarly, we define the space \( \widetilde{W}_z \) of paths from \( \partial B(z, \tilde{R}) \) to \( \partial B(z, R') \) by
\[
\widetilde{W}_z := \left\{ \omega \in \mathcal{W} : \begin{array}{l}
\omega(0) \in \partial B(z, \tilde{R}), \\
\omega(I_\omega) \in \partial B(x, R'), \\
\forall i < I_\omega
\end{array} \right\},
\]
where for \( A \subset \mathbb{Z}_n^2 \), we set \( \overline{A} := A \cup \partial A \).

For any \( k, \ell, m \in \mathbb{N}, \nu \in B(x, R') \setminus (B(x, R) \cup B(y, R)) \), \( F_i \subset \widetilde{W}_z, 1 \leq i \leq k \), \( F^x_\ell \subset W_x, 1 \leq \ell \leq \ell \), \( F^y_m \subset W_y, 1 \leq m \), set the event \( \bar{E}_k \) for \( k \) excursions from \( \partial B(x, \tilde{R}) \) to \( \partial B(x, R') \) by
\[
\bar{E}_k := \bigcap_{i=1}^k \left\{ S_{A \cap B(x, R')} \circ \theta_{D_i(x, \tilde{R}, R')} \in \bar{F}_i \right\},
\]
set the event \( E_\ell(x) \) for \( \ell \) excursions from \( \partial B(x, r) \) to \( \partial B(x, R) \) by
\[
E_\ell(x) := \bigcap_{i=1}^\ell \left\{ S_{A \cap B(x, R)} \circ \theta_{R_i(x, R)} \in F_i^x \right\},
\]
set the event \( E_m(y) \) for \( m \) excursions from \( \partial B(y, r) \) to \( \partial B(y, R) \) by
\[
E_m(y) := \bigcap_{i=1}^m \left\{ S_{A \cap B(y, R)} \circ \theta_{R_i(y, R)} \in F_i^y \right\}.
\]

Then,
\[
P_v \left[ \bar{E}_k \cap E_\ell(x) \cap E_m(y) \right] \leq \prod_{i=1}^k \max_{z \in \partial B(z, \tilde{R})} P_z \left[ S_{A \cap B(z, R')} \in \bar{F}_i \right] \times \prod_{i=1}^\ell \max_{z \in \partial B(x, r)} P_z \left[ S_{A \cap B(x, R)} \in F_i^x \right] \times \prod_{i=1}^m \max_{z \in \partial B(y, y)} P_z \left[ S_{A \cap B(y, R)} \in F_i^y \right].
\]

(4.11)

**Remark 4.3** Note that excursions in the events \( \bar{E}_k, E_\ell(x), \) and \( E_m(y) \) are disjoint. Lemma 4.2 states that the probability of the event for those disjoint excursions is bounded by the product of probabilities of events for each excursion.

**Proof.** The claim \((4.11)\) follows from the strong Markov property of the SRW and the transfinite induction in \((k, \ell, m)\). We omit the details. \(\square\)

We first deal with the case \( w_n + 3 \leq \ell(x, y) \leq d_n(\xi) \).
Lemma 4.4 For any $\xi > 0$, as $n \to \infty$,

$$
\sum_{k=w_n+3}^{d_n(\xi)} \sum_{x,y \in \mathbb{Z}^2_n \setminus B(o, r_n)} P_0[A_n(x) \cap A_n(y)] = o(1) (E_0[Z_n])^2.
$$

(4.12)

Recall the notation $D_m^{z,i}$, $R_m^{z,i}$ from (3.1). Fix $w_n + 3 \leq k \leq d_n(\xi)$. To apply Lemma 4.2, we prepare some notation. For each $z \in \mathbb{Z}^2_n$, $k \leq i \leq L_n - 1$, and $m \in \mathbb{N}$, set

$$
T_i^{k,z,m} := \max \left\{ \ell \geq 1 : R_{i,\ell}^{z,i} < D_m^{z,k} \right\}.
$$

(4.13)

We define the interval $J_m^{k,z}$ by

$$
J_m^{k,z} := \left\{ \begin{array}{ll}
\left( D_m^{z,k}, D_m^{z,k+1} \right) & \text{if } m \neq b_n^+(k), \\
\left( D_m^{z,k}, \infty \right) & \text{if } m = b_n^+(k).
\end{array} \right.
$$

We define $b_m^{(k)}(i)$ by

$$
\left[ \sqrt{m} \left( \frac{i+1-k}{L_n+1-k} \right)^2 \right],
$$

(4.14)

where $\kappa > 2$ is a sufficiently large constant. The curve $\overrightarrow{b_m^{(k)}}(\cdot)$ will play a role of an upper barrier for the traversal process $T_m^{k,z,m}$ which controls approximation errors when we apply the transfer lemma (Lemma 2.4). Take any $x,y \in \mathbb{Z}^2_n \setminus B(o, r_n)$ with $\ell(x, y) = k$. Fix $z \in \{x, y\}$. On the event $A_n(z)$, there exists $b_n(k) \leq m \leq b_n^+(k)$ such that $H_z \in J_m^{k,z}$. By this and the monotonicity of the traversal process $T_m^{k,z,\ell}$ in $\ell$, we have

$$
A_n(z) \subset C_n,k(z, d_n(\xi)) \cap \bigcup_{m=b_n^-(k)}^{d_n-1} \bigcap_{i=k+1}^{b_n^+(k)} \left\{ \overrightarrow{b_n^{(k)}}(i) \leq T_i^{k,z,m} \right\} \cap \{ H_z \in J_m^{k,z} \},
$$

(4.15)

where

$$
C_n,k(z) := \left\{ b_n^{-(k-3)} \leq T_k^{z,m-3} \leq b_n^{+(k-3)} \right\}.
$$

(4.16)

Furthermore, we decompose the $m$-th event in (4.15) into the event where the traversal process $T_m^{k,z,m}$ stays below the curve $\overrightarrow{b_m^{(k)}}(\cdot)$ and the event where $T_m^{k,z,m}$ crosses the curve $\overrightarrow{b_m^{(k)}}(\cdot)$. Then, we have

$$
A_n(z) \subset \bigcup_{m=b_n^+(k)} \left\{ \left( C_n,k(z) \cap A_{n,m}^{(k)} \right) \cup \left( C_n,k(z) \cap B_{n,m}^{(k)} \right) \right\},
$$

(4.17)
Lemma 4.5

Watson process:

\[ M \]

Fix \( B \)

where

By (4.17),

\[ P \]

\[ \text{Remark 4.6} \]

where for each \( b^{-}_{n}(k) \leq m \leq b^{+}_{n}(k) \), set

\[ A^{(k)}_{n,m}(z) := \bigcap_{i=k+1}^{L_{n}-d_{n}(2)-1} \{ b^{-}_{n}(i) \leq T^{k,z,m}_{i} \leq \bar{b}^{(k)}_{m}(i) \} \cap \{ H_{z} \in J^{k,z}_{m} \}, \quad (4.18) \]

\[ B^{(k)}_{n,m}(z) := \bigcap_{i=k+1}^{L_{n}-d_{n}(2)-1} \{ T^{k,z,m}_{i} > \bar{b}^{(k)}_{m}(i) \} \cap \{ H_{z} \in J^{k,z}_{m} \}. \quad (4.19) \]

By (4.17), \( P_{o}[A_{n}(x) \cap A_{n}(y)] \) is bounded from above by

\[ \sum_{M_{x}=b^{-}_{n}(k)}^{b^{+}_{n}(k)} \sum_{M_{y}=b^{-}_{n}(k)}^{b^{+}_{n}(k)} (I_{1} + I_{2} + I_{3} + I_{4}), \quad (4.20) \]

where

\[ I_{1} := P_{o}[C_{n,k}(x) \cap A^{(k)}_{n,M_{x}}(x) \cap A^{(k)}_{n,M_{y}}(y)], \]

\[ I_{2} := P_{o}[C_{n,k}(x) \cap A^{(k)}_{n,M_{x}}(x) \cap B^{(k)}_{n,M_{y}}(y)], \]

\[ I_{3} := P_{o}[C_{n,k}(x) \cap B^{(k)}_{n,M_{x}}(x) \cap A^{(k)}_{n,M_{y}}(y)], \]

\[ I_{4} := P_{o}[C_{n,k}(x) \cap B^{(k)}_{n,M_{x}}(x) \cap B^{(k)}_{n,M_{y}}(y)]. \quad (4.21) \]

Fix \( M_{x}, M_{y} \in \{ b^{-}_{n}(k), \cdots , b^{+}_{n}(k) \} \). Since the estimates of \( I_{1}, I_{3}, I_{4} \) are similar to that of \( I_{2} \), we will mainly focus on \( I_{2} \). In the following lemma, we decompose \( I_{2} \) into the product of probabilities of events corresponding to \( C_{n,k}(x), A^{(k)}_{n,M_{x}}(x), B^{(k)}_{n,M_{y}}(y) \) and transfer the law of the traversal process to that of the Galton-Watson process:

**Lemma 4.5** \( I_{2} \) in (4.21) is bounded from above by \( (1+o(1))P_{1}P_{2}P_{3} \), where

\[ P_{1} := P_{m,n}^{GW} \left[ T_{k-3} \leq b^{+}_{n}(k-3) \right], \]

\[ P_{2} := P_{M_{x}}^{GW} \left[ \bigcap_{i=1}^{L_{n}-d_{n}(2)-k-1} \{ T_{i} \geq b^{-}_{n}(k+i) \} \cap \{ T_{L_{n}-k-1} = 0 \} \right] \times \left( \frac{1}{L_{n}-k} \right)^{1_{\{M_{x} \neq b^{+}_{n}(k)\}}}, \]

\[ P_{3} := \sum_{q=k+1}^{L_{n}-d_{n}(2)-1} P_{M_{y}}^{GW} \left[ T_{q-k} \geq \bar{b}^{(k)}_{M_{y}}(q), T_{L_{n}-k-1} = 0 \right] \left( \frac{1}{L_{n}-k} \right)^{1_{\{M_{y} \neq b^{+}_{n}(k)\}}}. \quad (4.22) \]

**Remark 4.6** \( P_{1}, P_{2}, P_{3} \) in Lemma 4.5 correspond to probabilities of events \( C_{n,k}(x), A^{(k)}_{n,M_{x}}(x), B^{(k)}_{n,M_{y}}(y) \) respectively.
Proof. For each \( m \in \mathbb{N} \), we define the number of traversals between \( \partial B(x, r_1) \) and \( \partial B(x, r_0) \) by \( m \) excursions from \( \partial B(x, r_{k-3}) \) to \( \partial B(x, r_{k-2}) \) by

\[
\bar{T}^{k,x,m} := \max\{ \ell \geq 1 : D^{x,0}_\ell < R^{x,k-3}_m \}.
\]

We first decompose \( B^{(k)}_{n,M_y}(y) \) into events with respect to the first time at which the traversal process crosses the curve \( \bar{b}^{(k)}_{M_y}(\cdot) \):

\[
B^{(k)}_{n,M_y}(y) = \bigcup_{k < q < L_n - d_n(2)} \bigcup_{m < M_y} B_{q,m},
\]

where

\[
B_{q,m} := \left\{ T^{k,y,m}_q < \bar{b}^{(k)}_{M_y}(q) \leq T^{k,y,m+1}_q, H_y \in J^{k,y}_n \right\}.
\]

Then, we decompose \( C_{n,k}(x) \) into events which are measurable to the traversal processes before and after \( R^{x,k-3}_1 \):

\[
C_{n,k}(x) = \bigcup_{m' = 0}^{m_n - 1} \bigcup_{p = b_n^+(k-3) - 1} C^1_{m'} \cap C^2_{m',p},
\]

where \( C^1_{m'} := \left\{ D^{x,0}_{m'} < R^{x,k-3}_1 < D^{x,0}_{m'+1} \right\} \) and

\[
C^2_{m',p} := \left\{ \bar{T}^{k,x,p} \circ \theta_{D^{x,k-3}_1} < m_n^--m' \leq \bar{T}^{k,x,p+1} \circ \theta_{D^{x,k-3}_1} \right\}.
\]

By these, we have

\[
I_2 \leq \sum_{q,m,m',p} I_2(q,m,m',p), \quad (4.23)
\]

where

\[
I_2(q,m,m',p) := E_o \left[ 1_{C^2_{m',p}} P_{S_{n|\partial B(x, r_{k-2})}} \left[ C^2_{m',p} \cap A^{(k)}_{n>M_x}(x) \cap B_{q,m} \right] \right], \quad (4.24)
\]

and the sum is taken over \( q = k+1, \ldots, L_n - d_n(2) - 1, m = 0, \ldots, M_y - 1, m' = 0, \ldots, m_n^--1, p = b_n^+(k-3) - 1, \ldots, b_n^+(k-3) - 1. \) Fix such \( q, m, m', p. \)

In order to apply Lemma 3.2, we decompose the events \( C^2_{m',p}, A^{(k)}_{n,M_x}(x), B_{q,m} \) into events for excursions. First, we decompose \( C^2_{m',p} \) into events for excursions from \( \partial B(x, r_{k-3}) \) to \( \partial B(x, r_{k-2}) \) as follows:

\[
\bigcup_{(j^c_\ell) \in U_c} \bigcap_{\ell=1}^{p} \left\{ \tilde{T}^{k,x,1} \circ \theta_{D^{x,k-3}_\ell} = j^c_\ell \right\} \cap \left\{ \tilde{T}^{k,x,1} \circ \theta_{D^{x,k-3}_{\ell+1}} \geq m_n^- - m' - \sum_{\ell=1}^{p} j^c_\ell \right\},
\]

where

\[
U_c := \left\{ (j^c_\ell)_{\ell=1}^p \in \{0, 1, \ldots\}^p : \sum_{\ell=1}^{p} j^c_\ell < m_n^- - m' \right\}. \quad (4.25)
\]
Then, we decompose $A_{n,M_x}^{(k)}(x)$ into events for excursions from $\partial B(x,r_{k+1})$ to $\partial B(x,r_k)$ as follows:

$$
\bigcup_{(j^{a,i}_\ell) \in U_a} \bigcap_{\ell=1}^{M_x} \bigcap_{i=k+1}^{L_n-d_n(2)-1} \left\{ T_i^{a,i} \circ \theta_{R^{a,i}_\ell} = j^{a,i}_\ell \right\}
\cap \left\{ H_x \circ \theta_{R^{a,i}_\ell} > D^{a,i}_\ell \right\} \cap \left\{ H_x \circ \theta_{R^{a,i}_\ell} \in J_x \right\},
$$

where

$$U_a := \left\{ (j^{a,i}_\ell)_{\ell=1}^{M_x} \in \{0,1,\ldots\}^{M_x} : k+1 \leq i \leq L_n - d_n(2) - 1 \right\}:
\sum_{\ell=1}^{M_x} j^{a,i}_\ell \leq \overline{b}^{(k)}_{M_x}(i),
$$

and $J_x$ is the interval $(0, D^{a,i}_m)$ if $M_x \neq b^+_n(k)$ and is the interval $[0, \infty)$ if $M_x = b^+_n(k)$. Finally, we decompose $B_{q,m}$ into events for excursions from $\partial B(y,r_{k+1})$ to $\partial B(y,r_k)$ as follows:

$$
\bigcup_{(j^{b,y}_\ell) \in U_b} \bigcap_{\ell=1}^{M_y} \bigcap_{i=k+1}^{L_n-d_n(2)-1} \left\{ T_i^{b,y} \circ \theta_{R^{b,y}_\ell} = j^{b,y}_\ell, H_y \circ \theta_{R^{b,y}_\ell} > D^{b,y}_\ell \right\}
\cap \left\{ T_i^{b,y} \circ \theta_{R^{b,y}_\ell} \geq \overline{b}^{(k)}_{M_y}(q) - \sum_{\ell=1}^{M_y} j^{b,y}_\ell, H_y \circ \theta_{R^{b,y}_\ell} > D^{b,y}_\ell \right\}
\cap \left\{ H_y \circ \theta_{R^{b,y}_\ell} \in J_y \right\},
$$

where

$$U_b := \left\{ (j^{b,y}_\ell)_{\ell=1}^{M_y} \in \{0,1,\ldots\}^{M_y} : \sum_{\ell=1}^{M_y} j^{b,y}_\ell < \overline{b}^{(k)}_{M_y}(q) \right\}$$

and $J_x$ is obtained from $J_y$ by replacing $x$ with $y$. By these decompositions and Lemma 4.22, the probability in (4.24) is bounded from above by $\sum_{P_c,P_a,P_b}$, where the sum is taken over $(j^{b}_\ell) \in U_c$, $(j^{a,i}_\ell) \in U_a$, $(j^{b}_\ell) \in U_b$ and $P_c$, $P_a$, $P_b$ are defined as follows:

$$P_c := \left\{ \prod_{\ell=1}^{r_{k+1}-3} \max_{z \in \partial B(x,r_{k+1})} P_z[\tilde{T}^{k,\ell} = j^{b}_\ell] \right\}
\times \max_{z \in \partial B(x,r_{k+1})} P_z \left[ \tilde{T}^{k,\ell} \geq m_n - m' - \sum_{\ell=1}^{r_{k+1}} j^{b}_\ell \right],$$

$$P_a := \left\{ \prod_{\ell=1}^{L_n-d_n(2)-1} \max_{z \in \partial B(x,r_{k+1})} P_z \left[ \tilde{T}^{k,\ell} = j^{a,i}_\ell \right] \cap \left\{ H_x > H_{\partial B(x,r)} \right\} \right\}
\times \max_{z \in \partial B(x,r_{k+1})} P_z[H_x \in J_x],$$

(4.28)
where $\tilde{J}_x$ is the interval $[0, H_{\partial B(x, r_0)})$ if $M_x \neq b_n^+(k)$ and is the interval $[0, \infty)$ if $M_x = b_n^+(k)$.

$$P_b := \prod_{\ell=1}^{m} \max_{z \in \partial B(y, r_{k+1})} P_z \left[ T_q^{k,y,1} = j_{\ell}', \ H_y > H_{\partial B(y, r_k)} \right]$$

$$\times \max_{z \in \partial B(y, r_{k+1})} P_z \left[ T_q^{k,y,1} \geq \tilde{\nu}^{(k)}_M(q) - \sum_{\ell=1}^{m} j_{\ell}' , \ H_y > H_{\partial B(y, r_k)} \right]$$

$$\times \prod_{\ell=m+2}^{M_y} \max_{z \in \partial B(y, r_{k+1})} P_z[H_y > H_{\partial B(y, r_k)}] \cdot \max_{z \in \partial B(y, r_{k+1})} P_z[H_y \in \tilde{J}_y],$$

(4.30)

where $\tilde{J}_y$ is obtained from $\tilde{J}_x$ by replacing $x$ with $y$. Recall the definitions of $P_1, P_2, P_3$ from (4.22). First, applying the transfer lemma (Lemma 2.4) to the first line of (4.29) and Lemma 2.3 to the last lines of (4.29), we bound $\sum_{(j_{\ell}')} \in U_a$ by $(1 + o(1))P_2$ (Recall the definition of $U_a$ from (4.26)). Then, applying the transfer lemma (Lemma 2.4) to the first and second lines of (4.30) and Lemma 2.3 to the third line of (4.30), we bound $\sum_{q, (j_{\ell}')} P_b$ by $(1 + o(1))P_3$, where the sum is taken over $q = k + 1, \ldots, L_n - d_n(2) - 1, \ m = 0, \ldots, M_y - 1, \ (j_{\ell})' \in U_b$ (Recall the definition of $U_b$ from (4.27)). Finally, using Lemma 2.3 we have

$$\sum_{(j_{\ell})', p, m'} P_c[C_{m'}^{1}] P_c \leq (1 + o(1))P_1,$$

(4.31)

where the sum is taken over $(j_{\ell})', p = b_n^-(k - 3) - 1, \ldots, b_n^+(k - 3) - 1, \ m' = 0, \ldots, m_n' - 1$ (Recall the definition of $U_c$ from (4.28)). The proof of (4.31) is not difficult but not straightforward, so we give the proof in Section 13 for the sake of completeness. Then, we have the desired result. □

**Proof of Lemma 4.4** We first estimate $I_2$ in (4.21). (Since $I_3$ is basically $I_2$ with the roles of $M_x$ and $M_y$ changed, we only treat $I_2$.) By Lemma 4.3, we should estimate $P_1, P_2, P_3$ in (4.22). First, we estimate $P_1$. By [2, (5.3)], $P_1$ is bounded from above by

$$c_1 e^{-\left(\sqrt{m_n} - \sqrt{b_n^+(k-3)}\right)^2} \leq c_1(\ell_n)^{-2k+8} e^{c_2 \sqrt{\log(\ell_n)}} \leq c_1(\ell_n)^{-2k+8} e^{c_2 (\log \log n)^{1+4-\delta}}.$$  

(4.32)

To simplify notation, we set

$$v := \sqrt{2b_n(L_n - d_n(2) - 1)}.$$
Next, we estimate $P_2$ in (4.22). By conditioning on $T_{L_n-k-d_n(2)-1}$, $P_2$ is bounded from above by

$$\sum_{j=0}^{\infty} P_{GW}^{T_{d_n(2)}} \left[ \bigcap_{i=1}^{L_n-d_n(2)-k-2} \left\{ \sqrt{2T_i} \geq 2b_n(k+i) \right\} \cap \left\{ \sqrt{2T_{L_n-k-d_n(2)-1}} \in [v+j, v+j+1) \right\} \right]$$

(4.33)

$$\times \sup_{u} P_{GW}^{T_{d_n(2)}} [T_{d_n(2)} = 0],$$

where the supremum is taken over $u \in \left[ \frac{(v+j)^2}{2}, \frac{(v+j+1)^2}{2} \right] \cap \mathbb{N}$. When $j \leq 100L_n\sqrt{\log(\ell_n)}$, to the $j$-th term of the sum in (4.33), we can apply the barrier estimate (Lemma 2.5) with $L = L_n - k - d_n(2) - 1$, $a = (1 - \frac{k}{L_n})\sqrt{2m_n}$, $b = (1 - L_n-d_n(2)-1)\sqrt{2m_n}$, $x = \sqrt{2M_x}$, $y = v+j$, $\eta = c\sqrt{\log(\ell_n)}$ ($c$ is a positive constant). Then, the $j$-th term of the sum in (4.33) is bounded from above by

$$c_3(\ell_n)^{c_4} j + 1 \log n e^{-\frac{(v+j)^2}{2(L_n-k-d_n(2)+1)}}.$$  

(4.34)

The supremum in (4.33) is bounded from above by

$$\left( 1 - \frac{1}{d_n(2)+1} \right)^{\frac{(v+j)^2}{2}} \leq e^{-\frac{(v+j)^2}{2(d_n(2)+1)}}.$$  

(4.35)

Note that the product of exponential factors in (4.34) and (4.35) is equal to $e^{-\frac{M}{n} e^{-\frac{M}{2(L_n-k-d_n(2)-1)(n_n(2)+1)}}(v+j)^2(L_n-k-d_n(2)+1)^2}$. By this, the product of the sum over $j \leq 100L_n\sqrt{\log(\ell_n)}$ and the supremum in (4.33) is bounded from above by

$$c_5(\ell_n)^{c_6} \frac{e^{-\frac{M}{L_n-k}}}{\log n}.$$  

(4.36)

When $j > 100L_n\sqrt{\log(\ell_n)}$, bounding the supremum just by 1 and using [2, (5.3)], the product of the sum over $j > 100L_n\sqrt{\log(\ell_n)}$ and the supremum in (4.33) is bounded from above by

$$P_{GW}^{T_{d_n(2)}} \left[ \sqrt{2T_{L_n-k-d_n(2)-1}} \geq 100L_n\sqrt{\log(\ell_n)} \right] \leq n^{-3}.$$  

(4.37)

Next, we estimate $P_3$ in (4.22). By conditioning on $T_{q-k}$, the $q$-th term of the sum in $P_3$ is equal to

$$\sum_{m=0}^{\infty} P_{GW}^{T_{d_n(2)}} [T_{q-k} = m] P_{GW}^{T_{L_n-q-1}}.$$  

(4.38)
For each $m \geq \overline{b}^{(k)}_{M_y}(q)$, by [2 (5.3)], we have $P^{GW}_{M_y}[T_q-k = m] \leq e^{-\frac{M_y}{L_n-k+1}}$. We also have $P^{GW}_{M_y}[T_{L_n-q-1} = 0] = (1 - \frac{1}{L_n-q})^m \leq e^{-\frac{m}{L_n-q}}$. By these, (4.38) is bounded from above by a constant times

$$e^{-\frac{M_y}{L_n-k+1}} \sum_{m=\overline{b}^{(k)}_{M_y}(q)}^\infty \exp \left\{ -\frac{L_n-k+1}{(q-k+1)(L_n-q)} \left( \sqrt{m} - \frac{L_n-q}{L_n-k+1} \sqrt{M_y} \right)^2 \right\}. \tag{4.39}$$

We decompose the sum in (4.39) into the sums over $\overline{b}^{(k)}_{M_y}(q) \leq m \leq 4M_y$ and over $m > 4M_y$. Recall the definition of $\overline{b}^{(k)}_{M_y}(q)$ from (4.14). In the sum over $\overline{b}^{(k)}_{M_y}(q) \leq m \leq 4M_y$, we use the bound

$$\sqrt{m} - \frac{L_n-q}{L_n-k+1} \sqrt{M_y} \geq \kappa \sqrt{\frac{(q-k+1)(L_n-q)}{L_n+1-k}} \sqrt{\log \log n}.$$ In the sum over $m > 4M_y$, we use the bound

$$\frac{L_n-q}{L_n-k+1} \sqrt{M_y} \leq \frac{\sqrt{m}}{2}.$$ By these, (4.39) is bounded from above by

$$c_7 e^{-\frac{M_y}{L_n-k+1} (\log n)^{-\kappa^2+2} \log(t_n)}. \tag{4.40}$$

By a direct calculation, we have

$$\sum_{M_y=b^{(k)}_n(k)} b^{(k)}_n(k) \cdot e^{-\frac{M_y}{L_n-k+1}} \left( \frac{1}{L_n-k} \right)^{1(\ell \neq \ell^+(k))} \leq c_8 e^{-\frac{b^{(k)}_n(k)}{L_n-k+1}} \leq c_9 (\ell_n)^{-2(L_n-k)-4\epsilon \cdot w_n+2} (\log n) e^{2s_n-c_{10} f_n(k) \sqrt{\log(t_n)}}. \tag{4.41}$$

Note that the number of pairs $x, y \in \mathbb{Z}^d \setminus B(o, r_0)$ with $\ell(x, y) = k$ is bounded from above by $n^2 (2r_{k-1})^2$. By this, (4.32), (4.36), (4.37), (4.40), (4.41), and Lemmas 4.5 and 4.1, we have

$$d_n(\xi) \leq c_{11} (\ell_n)^{-2\epsilon \cdot w_n+c_{12} \epsilon^{2} (\log n)^{\frac{\ell_n}{4}+\alpha \delta}(\log n)^{-\kappa^2+4} (E_o[Z_n])^2. \tag{4.42}$$

Since $\alpha + \beta > 1/2 + \delta - \alpha \delta$ by the assumption (2.2), the right of (4.42) is $o(1)E_o[Z_n]^2$.  

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Recall the definition of $I_3$ from (4.21). As in Lemma 4.5, $I_3$ is bounded by $(1 + o(1))P_1P_2P_3$, where $P_1$ is the one in (4.22) and $P_2$, $P_3$ are $P_2$, $P_3$ in (4.22) with $x$ and $y$ swapped. Thus, $\sum_{k = w_n + 3}^{d_n(\xi)} \sum_{x, y \in \mathbb{Z}^2 \setminus B(o, r_0)} \sum_{I(x, y) = k} b_n^+(k) I_3$ is $o(1)E_o[Z_n]^2$.

Recall the definition of $I_1$ from (4.21). Since the event in $I_1$ is just the one in $I_2$ with $B_{n, M_n}(y)$ replaced by $A_{n, M_n}(y)$, one can repeat the proof of Lemma 4.5 (e.g. replace the equations around (4.27) and (4.30) by those around (4.26) and (4.29) (with $x$ swapped for $y$), respectively) and bound $I_1$ from above by $(1 + o(1))P_1P_2P_3$, where $P_1$ and $P_2$ are the ones in (4.22) and $P_3$ is $P_2$ with $x$ replaced by $y$. By this together with (4.32), (4.36), (4.37), (4.41), and Lemma 4.1, we have

\[
\sum_{k = w_n + 3}^{d_n(\xi)} \sum_{x, y \in \mathbb{Z}^2 \setminus B(o, r_0)} \sum_{I(x, y) = k} b_n^+(k) I_1 
\leq c_{13} (\ell_n)^{-2c_\alpha w_n + c_{14} e^{c_2 (\log \log n)^{\frac{1}{2} + \delta - \alpha \delta}}} \{E_o[Z_n]\}^2. \tag{4.43}
\]

Since $\alpha + \beta > 1/2 + \delta - \alpha \delta$ by the assumption (2.2), the right of (4.43) is $o(1)E_o[Z_n]^2$.

Recall the definition of $I_4$ from (4.21). Since the event in $I_4$ is just the one in $I_2$ with $A_{n, M_n}(x)$ replaced by $B_{n, M_n}(x)$, one can repeat the proof of Lemma 4.5 (e.g. replace equations around (4.26) and (4.29) by those around (4.27) and (4.30) (with $y$ swapped for $x$), respectively) and bound $I_4$ from above by $(1 + o(1))P_1P_3P_3$, where $P_1$ and $P_3$ are the ones in (4.22) and $P_3$ is $P_2$ with $y$ replaced by $x$. By this together with (4.32), (4.40), (4.41), and Lemma 4.1, we have

\[
\sum_{k = w_n + 3}^{d_n(\xi)} \sum_{x, y \in \mathbb{Z}^2 \setminus B(o, r_0)} \sum_{I(x, y) = k} b_n^+(k) I_4 
\leq c_{15} (\ell_n)^{-2c_\alpha w_n + c_{16} e^{c_2 (\log \log n)^{\frac{1}{2} + \delta - \alpha \delta}}} (\log n)^{-2c_3 + 8} \{E_o[Z_n]\}^2. \tag{4.44}
\]

Since $\alpha + \beta > 1/2 + \delta - \alpha \delta$ by the assumption (2.2) and $\kappa$ is sufficiently large, the right of (4.44) is $o(1)E_o[Z_n]^2$. Therefore, we have (4.12). □

Next, we deal with the case $1 \leq \ell(x, y) \leq w_n + 2$.

**Lemma 4.7**

\[
\sum_{x, y \in \mathbb{Z}^2 \setminus B(o, r_0), 1 \leq \ell(x, y) \leq w_n + 2} P_o[A_n(x) \cap A_n(y)] = o(1) (E_o[Z_n])^2. \tag{4.45}
\]
Proof. The proof is almost the same as that of Lemma 4.4 with \( k = w_n + 3 \) except that we do not consider the event \( C_{n,k}(x) \), so the proof is simpler. We omit the detail. □

Recall the definition of \( d_n(\cdot) \) from (3.3). Next, we deal with the case \( d_n(\xi) < \ell(x,y) \leq [(1-\varepsilon)L_n] \).

Lemma 4.8 Fix \( \varepsilon \in (0,1/2) \). There exists \( \xi_0 > 0 \) such that for any \( \xi \geq \xi_0 \),

\[
\sum_{k=d_n(\xi)+1}^{[(1-\varepsilon)L_n]} \sum_{\ell(x,y)=k} P_n[A_n(x) \cap A_n(y)] = o(1) \left( E_o[Z_n] \right)^2. \tag{4.46}
\]

In the proof of Lemma 4.8, we cannot apply the argument in the proof of Lemma 4.4 because the factor \( e^{c_n(k-3)\sqrt{\log(\ell_n)}} \) in (4.32) can be too large. Instead, we will follow the argument in [1 Section 6.2]. We first prove the following decoupling lemma:

Lemma 4.9 There exist \( c_1, c_2 \in (0, \infty) \) and \( \xi_0 > 0 \) such that for any \( \xi \geq \xi_0 \), \( d_n(\xi) < k \leq [(1-\varepsilon)L_n] \), and \( x, y \in \Z^2_n \setminus B(o,r_0) \) with \( d(x,y) = k \),

\[
P_o[A_n(x) \cap A_n(y)] \leq c_1(\ell_n)^{-2} g_n (d_n(\eta))^2 (\log n)^{-2} P_o \left[ B_{o,n}(k) \right], \tag{4.47}
\]

where \( d_n(\eta) := k + d_n(\eta) + 1 \) and for each \( m \in \mathbb{N} \), \( B_m \) is defined by

\[
B_m := \left\{ H_x > D_{m,n}^{z,0}, \ H_y > D_{m,n}^{y,k} \right\}. \tag{4.48}
\]

Remark 4.10 Roughly speaking, the equation in (4.47) says that the barrier events in \( A_n(x) \) and \( A_n(y) \) for \( i \geq d_n, k(\eta) \) are almost independent conditioned on the event that \( x \) and \( y \) are avoided. Thanks to the strong Markov property and the fact that \( B(x,r_k) \) and \( B(y,r_k) \) are disjoint, excursions inside \( B(x,r_k) \) and \( B(y,r_k) \) are almost independent conditioned on excursions outside the balls except the dependence on endpoints of the excursions. In order to get rid of the dependence on endpoints, we will apply the Harnack inequality.

Proof of Lemma 4.9 Take any \( d_n(\xi) < k \leq [(1-\varepsilon)L_n] \) and any \( x, y \in \Z^2_n \setminus B(o,r_0) \) with \( \ell(x,y) = k \). (Note that the constants appearing in the proof do not depend on \( x, y \).) Fix \( \eta > 2 \) large enough. To simplify notation, for \( z \in \{x,y\} \) and \( \ell \geq 1 \), set

\[
\bar{R}_{i,k}^{z,k} := R_k(z,r_k, r_{d_n(\eta)}), \quad \bar{D}_{i,k}^{z,k} := D_k(z,r_k, r_{d_n(\eta)}). \tag{4.49}
\]

Recall the definitions of \( R_{i,k}^{z,i}, D_{i,k}^{z,i} \) from (3.1). For each \( m \geq 1 \) and \( k \leq i \leq L_n - 1 \), we define \( M_m^z \) and \( T_{i,k}^{z,m} \) by

\[
M_m^z := \max \left\{ j \geq 1 : \bar{D}_{j,k}^{z,k} < D_{m,n}^{z,0} \right\}, \tag{4.50}
\]
\[ \widehat{T}^{k,z,m}_i := \max \left\{ \ell \geq 1 : R^{x,i}_\ell < \widehat{D}^{z,k}_m \right\}. \] (4.51)

Then, \( P_o[A_n(x) \cap A_n(y)] \) is bounded from above by

\[ \sum_{m_x,m_y=1}^{b^+_n(k+d_n(\eta))} P_o \left[ A_n(x) \cap A_n(y) \cap \left\{ M^z_n = m_z \right\} \right]. \] (4.52)

For each \( z \in \{x,y\} \), let \( G_z \) be the \( \sigma \)-algebra generated by \( S_{\wedge \widehat{R}_{i,k}^z} \) and excursions from \( \partial B(z,r_k) \) to \( \partial B(z,r_{d_n,\eta}(\eta)) \). Fix \( m_x, m_y \in \{1, \cdots, b^+_n(k + d_n(\eta))\} \). The term in (4.52) corresponding to \( m_x, m_y \) is bounded from above by

\[ E_o \left[ 1_B P_o \left[ \bigcup_{q=b^-_n(d_n,\eta)}^{b^+_n(d_n,\eta)} \widehat{A}_{n,q}(y) \bigg| G_y \right] \right], \] (4.53)

where

\[
B := \bigcap_{i=d_n,\eta(\eta)}^{L_n-w_{n-1}} \left\{ b^-_n(i) \leq \widehat{T}^{k,x,m_x}_i \leq b^+_n(i) \right\} 
\]

\[
\bigcap \left\{ H_x > \widehat{D}^{x,k}_m, M^x_n = m_x, M^y_n = m_y, T^y_{d_n,\eta(k)} \geq b^-_n(k) \right\},
\]

\[
\widehat{A}_{n,q}(y) := \left\{ \widehat{T}^{k,y,m_y}_i = q \right\} \bigcap_{i=d_n,\eta(\eta)+1}^{L_n-w_{n-1}} \left\{ b^-_n(i) \leq \widehat{T}^{k,y,m_y}_i \leq b^+_n(i) \right\}
\]

\[
\bigcap \left\{ H_y > \widehat{D}^{y,k}_m \right\}.
\]

By the strong Markov property, we have the following: For any \( M \in \mathbb{N} \),

\[
P_o \left[ \bigcap_{\ell=1}^{M} \left\{ S_{\wedge H_{\partial B(y,r_k)}} \circ \theta_{\widehat{R}_{i,k}^y} \in \cdot \right\} \bigg| G_y \right] = \prod_{\ell=1}^{M} P_{S_{\wedge H_{\partial B(y,r_k)}}} \left[ S_{\wedge H_{\partial B(y,r_k)}} \in \cdot \bigg| S_{H_{\partial B(y,r_k)}}' = S_{\widehat{R}_{i,k}^y} \right], \] (4.54)

where \( S' \) denotes a SRW independent from \( S \).

Fix any \( q \in \{b^-_n(d_n,\eta(\eta)), \cdots, b^+_n(d_n,\eta(\eta))\} \). We can decompose the event \( \widehat{A}_{n,q}(y) \) into events for excursions from \( \partial B(y,r_{d_n,\eta}(\eta)) \) to \( \partial B(y,r_k) \) as follows:

\[
\widehat{A}_{n,q}(y) = \bigcup_{\ell=1}^{m_y} \left\{ \widehat{T}^{k,y,1}_{d_n,\eta(\eta)} = q_\ell \right\} \bigcap_{i=d_n,\eta(\eta)+1}^{L_n-w_{n-1}} \left\{ \widehat{T}^{k,y,1}_{i} \circ \theta_{\widehat{R}_{i,k}^y} = j^{y,i}_\ell \right\}
\]

\[
\bigcap \left\{ H_y \circ \theta_{\widehat{R}_{i,k}^y} > \widehat{D}^{y,k}_\ell \right\},
\] (4.55)
where the union is taken over all sequences of nonnegative integers \((q_t)_{t=1}^{m_y}\) and \((j_t^{y,i})_{t=1}^{m_y}\), \(d_{n,k}(\eta) + 1 \leq i \leq L_n - w_n - 1\) with
\[
\sum_{t=1}^{m_y} q_t = q, \quad b_n^+(i) \leq \sum_{t=1}^{m_y} j_t^{y,i} \leq b_n^+(i), \quad d_{n,k}(\eta) + 1 \leq i \leq L_n - w_n - 1. \tag{4.56}
\]

By (4.56), the probability of the big intersection in (4.55) conditioned on \(G_y\) is equal to
\[
\prod_{t=1}^{m_y} P_{S_{\hat{R}_t}^{y,k}} \left[ \left\{ \hat{T}_{d_{n,k}(\eta)}^{y,i} = q_t \right\} \cap \bigcap_{i=d_{n,k}(\eta)+1}^{L_n-w_n-1} \left\{ \hat{T}_{d_{n,k}(\eta)}^{y,i} = j_t^{y,i} \right\} \cap \left\{ H_y > H_{\partial B(y,r_k)}, S_{H_{\partial B(y,r_k)}}' = S_{\hat{R}_t}^{y,k} \right\} \right] (4.57)
\]
x \prod_{t=1}^{m_y} P_{S_{\hat{R}_t}^{y,k}} \left[ S_{H_{\partial B(y,r_k)}}' = S_{\hat{R}_t}^{y,k} \right]^{-1}.

Fix any \(1 \leq \ell \leq m_y\). Recall the notation \(T_{d_{n,k}(\eta)}^{y,q_t} \) from (4.13). By the strong Markov property at \(D_{\hat{R}_t}^{y,d_{n,k}(\eta)}\), the \(\ell\)-th probability in the first line of (4.57) is equal to
\[
E_{S_{\hat{R}_t}^{y,k}} \left[ \left\{ T_{d_{n,k}(\eta)}^{y,q_t} < H_{\partial B(y,r_k)} \right\} \cap \bigcap_{i=d_{n,k}(\eta)+1}^{L_n-w_n-1} \left\{ T_{d_{n,k}(\eta)}^{y,i} = j_t^{y,i} \right\} \cap \left\{ H_y > D_{\hat{R}_t}^{y,d_{n,k}(\eta)} \right\} \right] \times P_{S_{d_{\hat{R}_t}^{y,d_{n,k}(\eta)}}} \left[ S_{H_{\partial B(y,r_k)}}'' = S_{\hat{R}_t}^{y,k}, H_{\partial B(y,r_k)} < H_{\partial B(y,d_{n,k}(\eta)+1)} \right], \tag{4.58}
\]
where \(S', S''\) denote SRWs independent from \(S\) and from each other.

We will estimate the probability in (4.58). To do so, we fix \(z \in \partial B(y,r_{d_{n,k}(\eta)})\) and \(w \in \partial B(y,r_k)\). By the strong Markov property, we have
\[
P_z \left[ S_{H_{\partial B(y,r_k)}} = w, H_{\partial B(y,r_{d_{n,k}(\eta)+1})} < H_{\partial B(y,r_k)} \right] = E_z \left[ \left\{ H_{\partial B(y,r_{d_{n,k}(\eta)+1})} < H_{\partial B(y,r_k)} \right\} P_{S_{d_{\hat{R}_t}^{y,d_{n,k}(\eta)}}} \left[ S_{H_{\partial B(y,r_k)}}' = w \right] \right]. \tag{4.59}
\]
We use the Harnack inequality from [3] Lemma 2.1: Uniformly in \(\delta' < 1/2\), \(u, u' \in B(0, \delta' N)\), \(u'' \in \partial B(0, N)\),
\[
P_u \left[ S_{H_{\partial B(0,N)}} = u'' \right] = (1 + O(\delta') + O(1/N)) P_{u'} \left[ S_{H_{\partial B(0,N)}} = u'' \right]. \tag{4.60}
\]
By (4.58), the \(P_{S_{d_{\hat{R}_t}^{y,d_{n,k}(\eta)}}}\)-probability in (4.59) is equal to
\[
\left(1 + O \left( \frac{r_{d_{n,k}(\eta)}}{r_k} \right) \right) \frac{P_z \left[ S_{H_{\partial B(y,r_k)}} = w \right]}{P_z \left[ S_{H_{\partial B(y,r_k)}} = w \right]} \tag{4.61}
\]
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By (4.59) and Lemma 2.3, the \( P_{\mathcal{S}_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k}(\eta)}^{y,dn,k} \) probability in (4.58) is bounded from above by \( (1 + \frac{\log \log n}{c_1}) \) times

\[
P_{\mathcal{S}_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k}(\eta)}^{y,dn,k} \left[ H_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k} < H_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k} + 1 \right] P_{\mathcal{S}_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k}(\eta)}^{y,dn,k} \left[ S'_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k} = S_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k} \right].
\]

(4.62)

By this, (4.58) is bounded from above by \( (1 + \frac{\log \log n}{c_1}) \) times

\[
P_{\mathcal{S}_{\mathcal{D}_{y,k}}^{y,k}(\eta)}^{y,k} \left[ \begin{array}{l}
\{ R^{y,dn,k,(\eta)} < H_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k} < R^{y,dn,k,(\eta)} + 1 \} \\
\cap \int_{i=d_{n,k}(\eta)+1}^{d_{n,k}(\eta)-1} \{ T^{d_{n,k}(\eta),y,q \ell} = y^{i+1} \} \}
\end{array} \right] \}
\]

\[
\cap \left\{ H_{y} > D_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k} \right\} \}
\]

\[
\times \max_{z \in \partial B(y,d_{n,k}(\eta))} P_{z} \left[ S'_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k} = S_{\mathcal{D}_{y,dn,k}(\eta)}^{y,dn,k} \right].
\]

(4.63)

By (4.63), the Harnack inequality (4.60), and the transfer lemma (Lemma 2.4(iii)) with \( L = L_n - k, m = q \ell, R_i = r_{k+i}, \tilde{k} = w_n, k = d_n(\eta) + 1 \), the conditional probability in (4.53) is bounded from above by \( (1 + o(1)) \) times

\[
\sum_{q} P_{q}^{GW} \int_{L_n-w_n-1}^{L_n-d_{n,k}(\eta)} \int_{i=1}^{d_{n,k}(\eta)-1} \{ b_i d_n(\eta) + i \}
\]

\[
\cap \left\{ T_{L_n-d_{n,k}(\eta)-1} = 0 \right\} \}
\]

\[
\times \sum_{(q \ell)_{\ell=1}^{m_q}(d_n(\eta))} \left( \frac{1}{d_n(\eta) + 2} \right)^{m_q},
\]

(4.64)

where the sums in the first and the second lines of (4.64) are taken over \( q \in \{ b_i d_n(\eta), \ldots, b_i d_n(\eta) \} \) and \( (q \ell)_{\ell=1}^{m_q} \) satisfying the first condition in (4.60).

By a direct calculation, one can show that the sum in the second line of (4.64) is equal to

\[
P \left[ \sum_{\ell=1}^{m_q} G_{\ell} = q \right],
\]

(4.65)

where \( G_{\ell}, \ell \geq 1 \) are independent geometric random variables with success probability \( \frac{1}{d_n(\eta)+2} \). By conditioning on \( T_{L_n-d_{n,k}(\eta)-w_n-1} \), the \( q \)-th term of the sum
in the first line of (4.64) is bounded from above by

\[ b_n^*(L_n - w_n - 1) \sum_{v = b_n^*(L_n - w_n - 1)} P_{GW}^{\mathbb{G}} \left[ \bigcap_{i=1}^{L_n - d_n,k(\eta) - w_n - 2} \{ T_i \geq b_n^*(d_n,k(\eta) + i) \} \right. \]

\[ \left. \bigcap \{ T_{L_n - d_n,k(\eta) - w_n - 1} = v \} \right] \times P_{GW}^{\mathbb{G}} [ T_{w_n} = 0 ] . \]  

(4.66)

Applying the barrier estimate (Lemma 2.5) with \( x = \sqrt{2q} \), \( y = \sqrt{2v} \), \( a = L_n - d_n,k(\eta) \sqrt{2m_n} \), and \( b = w_n + 1 \sqrt{2m_n} \) (note that we can take \( \eta = c \sqrt{\log(\ell_n)} \) for some positive constant \( c \)), we can bound the first probability of the \( v \)-th term in (4.66) from above by

\[ (\ell_n)^c g_n(d_n,k(\eta)) \frac{1}{\log n} e^{- \frac{(\sqrt{2q} - \sqrt{2v})^2}{4m_n - 4d_n,k(\eta) - w_n - 1}}. \]  

(4.67)

For each \( v \in \mathbb{N} \), we have

\[ P_{GW}^{\mathbb{G}} [ T_{w_n} = 0 ] = \left( 1 - \frac{1}{w_n + 1} \right)^v \leq e^{-\frac{v}{w_n + 1}}. \]  

(4.68)

We can bound the product of the exponential factors in (4.67) and (4.68) by \( e^{-\frac{v}{L_n - d_n,k(\eta)}} \) and this is bounded from above by \((1 + o(1))(1 - \frac{1}{L_n - d_n,k(\eta)})^q\). Thus, (4.64) is bounded from above by

\[ c_3(\ell_n)^c g_n(d_n,k(\eta)) \frac{1}{\log n} \left[ \left( 1 - \frac{1}{L_n - d_n,k(\eta)} \right)^{\sum_{\ell=1}^{m_y} G_\ell} \right] . \]  

(4.69)

Using the geometric distribution of \( G_\ell \), one can compute the expectation in (4.69) and show that (4.69) is equal to

\[ c_3(\ell_n)^c g_n(d_n,k(\eta)) \frac{1}{\log n} \left( 1 - \frac{d_n(\eta) + 1}{L_n - k} \right)^{m_y} . \]  

(4.70)

Next task is to prove that the last factor \( (1 - \frac{d_n(\eta) + 1}{L_n - k})^{m_y} \) in (4.70) is bounded from above by

\[ (1 + o(1)) P_o \left[ H_y > \tilde{D}^{\mathbb{G}}_{m_y} | G_y \right] . \]  

(4.71)

Since the event in (4.71) can be written by

\[ \bigcap_{\ell=1}^{m_y} \left\{ H_y \circ \theta_{\tilde{R}_{\ell,k}} > H_{\theta B(y,r_k) \circ \theta_{\tilde{R}_{\ell,k}}} \right\} , \]

1Note that these are notations used in Lemma 2.5. Please do not confuse these \( x, y \) with the points on \( \mathbb{Z}_n^2 \).
by (4.54), the probability in (4.71) is equal to
\[
\prod_{\ell = 1}^{m_y} \frac{P_{S_{R_y}^\ell,k} \left[ H_y > H_{\partial B(y, r_k)} \right], S_{H_{\partial B(y, r_k)}} = S_{D_y}^\ell}{P_{S_{R_y}^\ell,k} \left[ S_{H_{\partial B(y, r_k)}} = S_{D_y}^\ell \right]}.
\] (4.72)

We will estimate the probabilities in the numerator in (4.72). Fix any \( z \in \partial B(y, r_{d_n,k}(\eta)) \) and \( z' \in \partial B(y, r_k) \). By the strong Markov property, we have
\[
P_z \left[ H_y < H_{\partial B(y, r_k)}, S_{H_{\partial B(y, r_k)}} = z' \right] = P_z \left[ H_y < H_{\partial B(y, r_k)} \right] E_y \left[ P_{S_{H_{\partial B(y, r_k)}} (z')} \left[ S_{H_{\partial B(y, r_k)}} = z' \right] \right].
\] (4.73)

By the Harnack inequality (4.60), the right in (4.73) is equal to
\[
(1 + O((\log n)^{-\eta})) P_z \left[ H_y < H_{\partial B(y, r_k)} \right] P_z \left[ S_{H_{\partial B(y, r_k)}} = z' \right].
\] (4.74)

Note that we can relate probabilities in (4.72) and (4.73) by
\[
P_z \left[ H_y > H_{\partial B(y, r_k)}, S_{H_{\partial B(y, r_k)}} = z' \right] = P_z \left[ S_{H_{\partial B(y, r_k)}} = z' \right] - P_z \left[ H_y < H_{\partial B(y, r_k)}, S_{H_{\partial B(y, r_k)}} = z' \right].
\] (4.75)

By (4.74), (4.75), and Lemma 2.3 (4.72) is bounded from below by
\[
(1 + o(1)) \prod_{\ell = 1}^{m_y} \frac{P_{S_{R_y}^\ell,k} \left[ H_y > H_{\partial B(y, r_k)} \right]}{P_{S_{R_y}^\ell,k} \left[ S_{H_{\partial B(y, r_k)}} = S_{D_y}^\ell \right]}. \] (4.76)

By (4.76), Lemma 2.3 and the condition \( k \leq \lceil (1 - \varepsilon) L_n \rceil \), we have shown that the last factor in (4.70) is bounded from above by (4.71).

By (4.69)–(4.71), (4.53) is bounded by
\[
c_5 \left( \ell_n \right) g_n \left( d_n,k(\eta) \right) \frac{1}{\log n} \times \frac{1}{\log n}
\]

Summing over \( m_x, m_y \), we obtain the desired result (4.47). \( \square \)
Proof of Lemma 4.8. Recall the definition of the event $B_m$ from (4.48). To estimate $P_o[B_m]$, we will compare $P_o[B_m]$ with $P_o[B_{m-1}]$. Fix $m \in \mathbb{N}$. Since $R^u, k$ is contained in one of the intervals $(D^x, 0, D^x, 0), 0 \leq \ell \leq m - 1$ or in $(D^x, \infty, \infty)$, $P_o[B_m]$ is equal to

$$
\sum_{\ell=0}^{m-1} P_o \left[ D^x, 0 < R^u, k < D^x, 0, H_x > D^x, m - 1, H_y > D^y, k \right] + P_o \left[ D^x, 0 < R^u, k, H_x > D^x, m - 1, H_y > D^y, k \right]. \tag{4.79}
$$

Fix $\ell \in \{0, \ldots, m - 1\}$. By the strong Markov property at $R^u, k$, the $\ell$-th term of the sum in (4.79) is equal to

$$
E_o \left[ \mathbf{1}_{D^x, 0 < R^u, k < D^x, 0, H_x > D^x, m - 1, H_y > D^y, k} \right] \times Pr_{\partial B(y, r_k)} \left[ H_x > D^x, m - 1, H_y > D^y, k \right]. \tag{4.80}
$$

We will estimate the probability in the expectation in (4.80).

Fix any $z \in \partial B(y, r_{k+1})$. Note that we have $B(y, r_k) \subset B(x, r_{k-2})$. Thus, by the strong Markov property at $H_{\partial B(x, r_{k-2})}$ and at $H_{\partial B(x, r_0)}$, we have

$$
P_z \left[ H_x > D^x, m - 1, H_y > H_{\partial B(y, r_k)} \right] = \sum_{u \in \partial B(x, r_{k-2})} \sum_{v \in \partial B(x, r_0)} P_z \left[ H_y > H_{\partial B(y, r_k)} ; H_x > H_{\partial B(x, r_{k-2})} ; S'_{H_{\partial B(x, r_{k-2})}} = u \right] \times P_u \left[ H_x > H_{\partial B(x, r_0)} ; S_{H_{\partial B(x, r_0)}} = v \right] P_v \left[ H_x > D^x, m - 1 \right]. \tag{4.81}
$$

We will estimate the second factor on the right-hand side of (4.81).

Fix $u \in \partial B(x, r_{k-2})$ and $v \in \partial B(x, r_0)$. By the strong Markov property at $H_x$ and $H_{\partial B(x, r_{k-2})}$, $P_u[H_x < H_{\partial B(x, r_0)} ; S_{H_{\partial B(x, r_0)}} = v]$ is equal to

$$
P_u[H_x < H_{\partial B(x, r_0)}] E_x \left[ P_{S'_{H_{\partial B(x, r_0)}} = v} \right]. \tag{4.82}
$$

By the Harnack inequality (4.60), (4.82) is equal to

$$
\{1 + O(\ell_n)^{-k+2} \} P_u[H_x < H_{\partial B(x, r_0)}] P_u\left[ S_{H_{\partial B(x, r_0)}} = v \right]. \tag{4.83}
$$

By (4.83), Lemma 2.3 and the condition $k \leq \lfloor (1 - \varepsilon) L_n \rfloor$, the first probability in the second line of (4.81) is equal to

$$
\{1 - O(\ell_n)^{-k+2} \} P_u[H_x > H_{\partial B(x, r_0)}] P_u\left[ S_{H_{\partial B(x, r_0)}} = v \right]. \tag{4.84}
$$

By the Harnack inequality (4.60), the second probability in (4.84) is equal to

$$
\{1 + O(\ell_n)^{-k+2} \} P_u\left[ S_{H_{\partial B(x, r_0)}} = v \right]. \tag{4.85}
$$
where \( u_* \) is a fixed vertex on \( \partial B(x, r_{k-2}) \). By Lemma 2.3, the first probability in (4.87) is equal to

\[
(1 + O \left( (\log n)^{-2} \right) ) \frac{L_n - k + 2}{L_n}.
\]

By (4.84)-(4.86), for \( \xi \) large enough, the right of (4.81) is equal to

\[
\{ 1 + O \left( (\log n)^{-2} \right) \} \frac{L_n - k + 2}{L_n} \Pr[H_y > H_{\partial B(y, r_k)}, \ H_x > H_{\partial B(x, r_{k-2})}]
\times \Pr[H_y \circ \theta_{H_{\partial B(y, r_k)}} > D^{x,0}_{m_\infty - l}] .
\]

We will estimate the first probability of (4.87).

By the strong Markov property at \( H_y \) and at \( H_{\partial B(y, r_k)} \circ \theta_{H_y} \), we have

\[
P_z \left[ H_y < H_{\partial B(y, r_k)}, \ H_x > H_{\partial B(x, r_{k-2})} \right]
= P_z \left[ H_y < H_{\partial B(y, r_k)} \right] E_y \left[ P_S_{H_{\partial B(y, r_k)}} \left[ H_x > H_{\partial B(x, r_{k-2})} \right] \right]
\geq \left( 1 - \frac{c_1}{L_n - k} \right) \frac{1}{L_n - k},
\]

where we have used Lemma 2.3 and the fact that \( d(u, x) \geq r_k \) for any \( u \in \partial B(y, r_k) \) in the last inequality. Thus, the first probability in (4.87) is equal to

\[
P_z \left[ H_x > H_{\partial B(x, r_{k-2})} \right] - P_z \left[ H_y < H_{\partial B(y, r_k)}, \ H_x > H_{\partial B(x, r_{k-2})} \right]
\leq P_z \left[ H_x > H_{\partial B(x, r_{k-2})} \right] \left\{ 1 - \left( 1 - \frac{c_1}{L_n - k} \right) \frac{1}{L_n - k} \right\}.
\]

By (4.86), (4.87), (4.89), and applying the strong Markov property at \( R_{m}^{k} \) and Lemma 2.3 to the last term of (4.79), (4.79) is bounded from above by

\[
\{ 1 + c_2 (\log n)^{-2} \} \left\{ 1 - \left( 1 - \frac{c_1}{L_n - k} \right) \frac{1}{L_n - k} \right\}
\times \left\{ \sum_{\ell=0}^{m_n-1} \frac{L_n - k + 2}{L_n} P_{u_*}^{(\ell)} E_\ell + Q \right\},
\]

where \( P_{u_*}^{(\ell)} \) is the probability in the second line of (4.87) and

\[
E_\ell := E_0 \left[ \{ D^{x,0}_{m_n} < R_{m}^{k}, H_x > R_{m}^{k}, H_y > D^{y,0}_{m_n-1} \} P_{G_{m_n}^{k}} \left[ H_x > H_{\partial B(x, r_{k-2})} \right] \right],
\]

\[
Q := P_0 \left[ D^{x,0}_{m_n} < R_{m}^{k}, H_x > D^{y,0}_{m_n}, H_y > D^{y,0}_{m_n-1} \right].
\]

A similar argument implies that \( P_0[B_{m-1}] \) is bounded from below by the factor in the second line of (4.90) multiplied by \( 1 - c_3 (\log n)^{-2} \). Thus, we have

\[
P_0[B_m] \leq \{ 1 + c_4 (\log n)^{-2} \} \left\{ 1 - \left( 1 - \frac{c_4}{L_n - k} \right) \frac{1}{L_n - k} \right\} P_0[B_{m-1}].
\]
In particular, \( P_o \left[ B_{b^*_n}(k) \right] \) is bounded from above by
\[
\left\{ 1 + \frac{c_4}{(\log n)^2} \right\} b^*_n(k) \left\{ 1 - \left( 1 - \frac{c_4}{L_n - k} \right) \frac{1}{L_n - k} \right\} b^*_n(k) \ P_o \left[ H_x > D^{x,0}_{\frac{m}{m_n}} \right]. \tag{4.92}
\]
The first factor of (4.92) is \( 1 + o(1) \). Since \( k \leq \lfloor (1 - \varepsilon)L_n \rfloor \), the second factor of (4.92) is bounded from above by (\( \ell_n \)) \( e^{-c\frac{\ell_n}{\log n}} \) and this is bounded from above by
\[
(\ell_n)^{-2(L_n - k) + c_0} (\log n)^{2s_n - 4c \cdot w_n \log(\ell_n)} e^{-c \ell_n (k) e^{\log(\ell_n)} \sqrt{\log(\ell_n)}} \times \exp \left\{ - (\log \log n + 2s_n - 4c \cdot w_n \log(\ell_n)) \frac{\log(\ell_n)}{\log n} \right\}. \tag{4.93}
\]
Since \( \alpha + \beta < \gamma < 1 \), the last factor of (4.93) is bounded from above by 1. By the strong Markov property and Lemma 2.3, the probability in (4.92) is bounded from above by
\[
(\ell_n)^{-2(L_n - k) + c_0} (\log n)^{2s_n - 4c \cdot w_n \log(\ell_n)} e^{-c \ell_n (k) e^{\log(\ell_n)} \sqrt{\log(\ell_n)}} \times \exp \left\{ - (\log \log n + 2s_n - 4c \cdot w_n \log(\ell_n)) \frac{\log(\ell_n)}{\log n} \right\}. \tag{4.93}
\]
By Lemma 4.12, (4.92), (4.93), \( P_o \left[ A_n(x) \cap A_n(y) \right] \) is bounded from above by
\[
c_0(\ell_n)^{-2(L_n - k) + c_100} \frac{48}{2 \cdot 2 \cdot 2} \frac{2}{4 \cdot s_n - 6c \cdot w_n \log(\ell_n)} g_n(d_n, k(y))^{\ell_n (k)} e^{-c \ell_n (k) e^{\log(\ell_n)} \sqrt{\log(\ell_n)}}. \tag{4.95}
\]
In particular, the sum of \( P_o[A_n(x) \cap A_n(y)] \) over \( x, y \in \mathbb{Z}^2_n \setminus B(o, r_0) \) with \( \ell(x, y) = k \) is bounded from above by (4.95) multiplied by \( n^2 (2r_{k-1})^2 \). By this and Lemma 4.1, the left of (4.46) is bounded from above by
\[
c_{11}(\ell_n)^{\ell_n} e^{-2c \cdot w_n \log(\ell_n)} \{ E_o[Z_n] \}^2. \tag{4.96}
\]
Thus, we have the desired result. \( \square \)

**Remark 4.11** The reader may wonder why we did not apply the argument in the proof of Lemma 4.8 to the proof of Lemma 4.4. This is because, in the case \( k < d_n(\xi) \), the first factor in (4.92) is replaced by (1 + \( c(\ell_n)^{-k+2} b_n^*_n(k) \)) and this is not negligible. In the Brownian motion case, thanks to the rotationally invariance, the error factor does not appear and thus one can apply the same method even in the case \( k < d_n(\xi) \). See [1] Section 6.2 for the details.

Next, we deal with the case \( \lfloor (1 - \varepsilon)L_n \rfloor < \ell(x, y) \leq L_n - w_n - 1 \).

**Lemma 4.12** Let \( \varepsilon \) be the constant in Lemma 4.8. Then,
\[
\sum_{k=\lfloor (1 - \varepsilon)L_n \rfloor + 1}^{L_n - w_n - 1} \sum_{x, y \in \mathbb{Z}^2_n \setminus B(o, r_0)} P_o[A_n(x) \cap A_n(y)] = o(1) \{ E_o[Z_n] \}^2. \tag{4.97}
\]
Proof. We will follow the argument in \cite{1} Section 6.1. Take any \([(1-\varepsilon)L_n] < k \leq L_n - w_n - 1\), any \(x, y \in \mathbb{Z}^2\) \(\setminus B(o, r_0)\) with \(\ell(x, y) = k\), and sufficiently large constant \(\theta > 0\). Recall the notation \(R^{x,i}_\ell, D^{x,i}_\ell, \tau^{x,m}_i, d_n(\cdot)\) from (3.1), (4.1), (4.3). We have

\[
P_o[A_n(x) \cap A_n(y)] \leq P_o \left[ C_{b_n^-(k)} \right],
\]

where for each \(m \in \mathbb{N}\), we set

\[
C_m := \bigcap_{i=w_n}^{k-d_n(\theta)-1} \left\{ b_n^-(i) \leq T^{x,m}_i \leq b_n^+(i) \right\} \cap \left\{ H_x > D^x_{m_n}, H_y > D^y_{m_k} \right\}.
\]

To estimate \(P_o[C_{b_n^-(k)}]\), we will compare \(P_o[C_m]\) with \(P_o[C_{m-1}]\).

Fix any \(m \in \mathbb{N}\). Since \(R^{y,k}_m\) is contained in one of the intervals \((D^{x,0}_\ell, D^{x,0}_{\ell+1}), 0 \leq \ell \leq m_n - 1\) or in \((D^{x,0}_{m_n}, \infty)\), we have \(P_o[C_m] = \) equal to

\[
\sum_{\ell=0}^{m_n-1} P_o \left[ \left\{ D^{x,0}_\ell < R^{y,k}_m < D^{x,0}_{\ell+1} \right\} \cap C_m \right] + P_o \left[ \left\{ D^{x,0}_{m_n} < R^{y,k}_m \right\} \cap C_m \right].
\]

We define the number of traversals before time \(R^y_k_m\) by

\[
T^i_* := \max \left\{ j \geq 1 : D^{x,i}_j < R^{y,k}_m \right\}, w_n \leq i \leq k - d_n(\theta) - 1.
\]

Fix \(\ell \in \{0, \ldots, m_n - 1\}\). By the strong Markov property at \(R^y_k_m\), the \(\ell\)-th term of the sum in (4.100) is equal to

\[
E_o \left[ 1\{D^{x,0}_\ell < R^{y,k}_m < D^{x,0}_{\ell+1}, H_x > R^{y,k}_m, H_y > D^{y,k}_{m-1}\} \right]
\]

\[
\times P_{S^{R^y_k_m}} \left[ \bigcap_{i=w_n}^{k-d_n(\theta)-1} \{ b_n^-(i) \leq T^i_* + T^{x,m^-}_i - \ell \leq b_n^+(i) \} \right] \cap \left\{ H_x > D_{m_n-\ell}, H_y > H_{\partial B(y,r_k)} \right\} \right].
\] (4.102)

Fix any \(z \in \partial B(y, r_{k+1})\) and \(t_i \geq 0\), \(w_n \leq i \leq k - d_n(\theta) - 1\). By the strong Markov property at \(H_{\partial B(x, r_{k-2})}\) and at \(H_{\partial B(x, r_{k-d_n(\theta)})}\), the probability in (4.102) with \(S^{R^y_k_m} = z\) and \(T^i_* = t_i\), \(w_n \leq i \leq k - d_n(\theta) - 1\) is equal to

\[
\sum_{u \in \partial B(x, r_{k-2})} \sum_{v \in \partial B(x, r_{k-d_n(\theta)})} P_z \left[ H_y > H_{\partial B(y,r_k)}, H_x > H_{\partial B(x, r_{k-2})} \right]
\]

\[
\times P_u \left[ H_x > H_{\partial B(x, r_{k-d_n(\theta)})}, S_{H_{\partial B(x, r_{k-d_n(\theta)})}} = u \right] \]

\[
\times P_v \left[ \bigcap_{i=w_n}^{k-d_n(\theta)-1} \{ b_n^-(i) \leq t_i + T^{x,m^-}_i - \ell \leq b_n^+(i) \} \right] \cap \left\{ H_x > D_{m_n-\ell} \right\} \right].
\] (4.103)
Fix \( u \in \partial B(x, r_{k-2}) \) and \( v \in \partial B(x, r_{k-d_n(\theta)}) \). We will estimate the probability in the second line of (4.103). By the strong Markov property at \( H_x \) and at \( H_{\partial B(x, r_{k-2})} \circ \theta_{H_y} \), one can see that

\[
P_u \left[ H_x < H_{\partial B(x, r_{k-d_n(\theta)})}, \, S_{H_{\partial B(x, r_{k-d_n(\theta)})}} = v \right]
\]

is equal to

\[
P_u \left[ H_x < H_{\partial B(x, r_{k-d_n(\theta)})} \right] E_x \left[ P_{S_{H_{\partial B(x, r_{k-2})}}} \left[ S'_{H_{\partial B(x, r_{k-d_n(\theta)})}} = v \right] \right]. \quad (4.104)
\]

By the Harnack inequality (4.60), (4.104) is equal to \( \{ 1 + O \left( (\ell_n)^2 (\log n)^{-\theta} \right) \} \) times

\[
P_u \left[ H_x < H_{\partial B(x, r_{k-d_n(\theta)})} \right] P_u \left[ S_{H_{\partial B(x, r_{k-d_n(\theta)})}} = v \right]. \quad (4.105)
\]

By (4.105) and Lemma 2.3, the probability in the second line of (4.103) is equal to \( \{ 1 + O((\ell_n)^3 (\log n)^{-\theta}) \} \) times

\[
P_u \left[ H_x > H_{\partial B(x, r_{k-d_n(\theta)})} \right] P_u \left[ S_{H_{\partial B(x, r_{k-d_n(\theta)})}} = v \right]. \quad (4.106)
\]

By the Harnack inequality (4.60), the second probability in (4.106) is equal to

\[
\{ 1 + O \left( (\ell_n)^2 (\log n)^{-\theta} \right) \} P_{u_*} \left[ S_{H_{\partial B(x, r_{k-d_n(\theta)})}} = v \right], \quad (4.107)
\]

where \( u_* \) is a fixed vertex on \( \partial B(x, r_{k-2}) \). By Lemma 2.3, the first probability in (4.106) is equal to

\[
\left\{ 1 + O \left( \frac{1}{(\ell_n)^2 (\log(\ell_n))^2} \right) \right\} \frac{L_n - k + 2}{L_n - k + d_n(\theta)}. \quad (4.108)
\]

By these and the condition that \( \theta \) is large enough, (4.103) is bounded from above by

\[
\times P_z \left[ H_y > H_{\partial B(y, r_k)}, \, H_x > H_{\partial B(x, r_{k-2})} \right] \frac{L_n - k + 2}{L_n - k + d_n(\theta)} \times P_{u_*} \left[ \bigcap_{i \in \mathcal{W}_n} \left\{ b_n^-(i) \leq t_i + T_i \right\} \right] \quad (4.109)
\]

\[
\cap \left\{ H_x \circ \theta_{H_{\partial B(x, r_{k-d_n(\theta)})}} > D_{m_n^{-\ell}} \right\} \right].
\]

We will estimate the first probability in (4.109). By the strong Markov property at \( H_y \) and at \( H_{\partial B(y, r_k)} \circ \theta_{H_y} \), we have

\[
P_z \left[ H_y < H_{\partial B(y, r_k)}, \, H_x > H_{\partial B(x, r_{k-2})} \right] = P_z \left[ H_y < H_{\partial B(y, r_k)} \right] E_y \left[ P_{\mathcal{S}_{H_{\partial B(y, r_k)}}} \left[ H_x > H_{\partial B(x, r_{k-2})} \right] \right]. \quad (4.110)
\]
By Lemma 2.3 and the fact that $d(u, x) \geq r_k$ for any $u \in \partial B(y, r_k)$, the right of (4.110) is bounded from below by $\left(1 - \frac{c_1}{L_n - k}\right) \frac{1}{L_n - k}$. Thus, we have

$$P_z \left[H_y > H_{\partial B(y,r_k)}, \ H_x > H_{\partial B(x,r_k-2)}\right] = P_z \left[H_y > H_{\partial B(x,r_k-2)}\right] - P_z \left[H_y < H_{\partial B(y,r_k)}, \ H_x > H_{\partial B(x,r_k-2)}\right] \leq \left\{1 - \left(1 - \frac{c_1}{L_n - k}\right) \frac{1}{L_n - k}\right\} P_z \left[H_x > H_{\partial B(x,r_k-2)}\right]. \quad (4.111)$$

By these and applying the strong Markov property at $R_m^{u,k}$ and Lemma 2.3 to the last term of (4.100), $P_o[C_m]$ is bounded from above by

$$\left\{1 + \frac{c_2}{(L_n - k)^2(\log(\ell_n))^2}\right\} \left\{1 - \left(1 - \frac{c_1}{L_n - k}\right) \frac{1}{L_n - k}\right\} \times \left\{1 + \frac{c_4}{(L_n - k)^2(\log(\ell_n))^2}\right\} \left\{1 - \left(1 - \frac{c_1}{L_n - k}\right) \frac{1}{L_n - k}\right\} P_o[C_m-1]. \quad (4.112)$$

where

$$E_{\ell} := E_o \left[\left\{D_{\tau,0}^o < R_m^{u,k} < D_{\ell+1,0}^o, \ H_x > R_m^{u,k}, \ H_y > D_m^{\omega,k}\right\} \times P_{u,0}^{(\ell)}((T^\ell)_i) P_{S_n^{R_m^{u,k}}} \left[H_x > H_{\partial B(x,r_k-2)}\right]\right], \quad (4.113)$$

Thus, the right of (4.98) is bounded from above by

$$\left\{1 + \frac{c_4}{(L_n - k)^2(\log(\ell_n))^2}\right\} \left\{1 - \left(1 - \frac{c_1}{L_n - k}\right) \frac{1}{L_n - k}\right\} P_o[C_m-1]. \quad (4.114)$$

The first factor of (4.114) is $1 + o(1)$. The second factor of (4.114) is bounded from above by $(\ell_n)^{\alpha_2} e^{-\frac{b_n^-(i)}{\ell_n}}$ and this is bounded from above by

$$c_6(\ell_n)^{-2(L_n - k) + c_4\ell_n(L_n - k)} \log \log \frac{\log(\ell_n) + (L_n - k)}{2 \log \log(\ell_n)} \log(\ell_n) \sqrt{\log(\ell_n)}. \quad (4.115)$$
Since \( k > [(1 - \varepsilon)L_n] > \frac{L_n - 1}{2} \), we have
\[
\frac{c_2}{2} f_n(k) \sqrt{\log(\ell_n)} - (L_n - k) \frac{\log \log n}{\log n} \log(\ell_n) > 0.
\]
By this and the condition \( k > [(1 - \varepsilon)L_n] \), (4.115) is bounded from above by
\[
c_6(\ell_n)^{-2(L_n - k) + c_5} e^{2\varepsilon n - (1 - \varepsilon)n - \frac{c_5}{2} f_n(k) \sqrt{\log(\ell_n)}}. \tag{4.116}
\]
By the transfer lemma (Lemma 2.4), the probability in (4.117) is bounded from above by
\[
(1 + o(1)) P_{m_n}^{GW} \left[ k - d_n(\theta) - w_n - 1 \right] \prod_{i=1}^{k - d_n(\theta) - w_n - 2} \left\{ b_n^-(w_n + i) \leq T_i \leq b_n^+(w_n + i) \right\} \cap \{ T_{L_n - 1} = 0 \}. \tag{4.117}
\]
By the Markov property, the probability in (4.117) is equal to
\[
\sum_{m, m'} P_{m_n}^{GW} \{ T_{w_n} = m \}
\times P_{m_n}^{GW} \left[ \mathcal{L}_{m_n} \right] \prod_{i=1}^{k - d_n(\theta) - w_n - 2} \left\{ b_n^-(w_n + i) \leq T_i \leq b_n^+(w_n + i) \right\}
\cap \{ T_{k - d_n(\theta) - w_n - 1} = m' \}
\times P_{m_n}^{GW} \{ T_{L_n - k + d_n(\theta)} = 0 \}, \tag{4.118}
\]
where the sum is taken over \( m \in \{ b_n^-(w_n), \ldots, b_n^+(w_n) \} \) and \( m' \in \{ b_n^-(k - d_n(\theta) - 1), \ldots, b_n^+(k - d_n(\theta) - 1) \} \). By the barrier estimate (Lemma 2.5) with
\[
a = (1 - \frac{w_n}{\sqrt{L_n}}) \sqrt{2m_n}, \quad b = \frac{L_n - k + d_n(\theta) + 1}{L_n} \sqrt{2m_n}, \quad x = \sqrt{2m}, \quad y = \sqrt{2m'} \quad \text{(we can take \( \eta = c\sqrt{\log(\ell_n)} \) for some positive constant), the second probability in (4.118) is bounded from above by}
\[
c_8(\ell_n)^{-2} (L_n - k + d_n(\theta) + 1)^{1/2 + \delta} e^{-\frac{(L_n - k + d_n(\theta) + 1)^2}{\log n}}. \tag{4.119}
\]
The third probability in (4.118) is equal to \( (1 - \frac{1}{L_n - k + d_n(\theta) + 1})^{m'} \) and this is bounded from above by \( e^{-\frac{m'}{L_n - k + d_n(\theta) + 1}} \). The product of this and the exponential factor in (4.119) is bounded from above by \( e^{-\frac{m'}{T_n - w_n}} \). By this, the contribution of the sum over \( m \) in (4.118) comes from
\[
\sum_{m=b_n^-(w_n)}^{b_n^+(w_n)} e^{-\frac{m}{T_n - w_n}} P_{m_n}^{GW} \{ T_{w_n} = m \}
\]
and this is bounded from above by
\[
E_{m_n}^{GW} \left[ 1_{\{ T_{w_n} \geq b_n^-(w_n) \}} e^{-\frac{T_{w_n}}{T_n - w_n}} \right] \leq e^{-\frac{b_n^-(w_n)}{T_n - w_n}}. \tag{4.120}
\]
Therefore, (4.117) is bounded from above by
\[ c_{10} n^{-2} e^{2s_n} (\ell_n)^{-2c_n w_n + c_{11} w_n} \left( L_n - k + d_n(\theta) + 1 \right)^{\frac{1}{2} + \delta}. \]  
(4.121)

Note that the number of pairs of points \( x, y \in \mathbb{Z}_n^2 \setminus B(o, r_0) \) with \( \ell(x, y) = k \) is bounded from above by \( n^2(2r_{k-1})^2 \). Thus, by (4.114), (4.116), (4.121), and Lemma 4.1, the left of (4.97) is bounded from above by
\[ c_{12} (\ell_n)^{c_{13} w_n} e^{-(1-\varepsilon)s_n} \left( E_o[Z_n] \right)^2. \]  
(4.122)

Since \( \gamma > \alpha + \beta \) by the assumption (2.2), we have the desired result. \( \square \)

Next, we deal with the case \( \ell(x, y) = 0 \).

**Lemma 4.13** As \( n \to \infty \),
\[ \sum_{x, y \in \mathbb{Z}_n^2 \setminus B(o, r_0) \atop \ell(x, y) = 0} P_o[A_n(x) \cap A_n(y)] \leq (1 + o(1)) \left( E_o[Z_n] \right)^2. \]  
(4.123)

**Proof.** Fix any \( x, y \in \mathbb{Z}_n^2 \setminus B(o, r_0) \) with \( \ell(x, y) = 0 \). Since \( A_n(z) \) \( (z \in \{x, y\}) \) can be written as an event for \( m_n \) excursions from \( \partial B(z, r_1) \) to \( \partial B(z, r_0) \), we can apply Lemma 4.4 with \( R = r_0, \ r = r_1, \ k = 0, \ \ell = m = m_n \). By this and the transfer lemma (Lemma (2.11)), one can show that \( P_o[A_n(x) \cap A_n(y)] \) is bounded from above by
\[ (1 + o(1)) P_o \left( \bigcap_{i = w_n}^{L_n - w_n - 1} \left\{ b_n^i (i) \leq T_i \leq b_n^i (i) \right\} \cap \{ T_{L_n - 1} = 0 \} \right)^2, \]  
(4.124)

where \( o(1) \to 0 \) as \( n \to \infty \) uniformly in \( x, y \). By (4.9), the right-hand side of (4.124) is bounded from above by
\[ (1 + o(1)) P_o[A_n(x)] P_o[A_n(y)]. \]  
(4.125)

Therefore, we have (4.123). \( \square \)

**Proof of (2.11).** By Lemma 4.1 we have
\[ \sum_{x, y \in \mathbb{Z}_n^2 \setminus B(o, r_0) \atop \ell(x, y) \geq L_n - w_n} P_o[A_n(x) \cap A_n(y)] \leq c_1 (\ell_n)^{2(w_n + 1)} E_o[Z_n] \]
\[ \leq c_1 e^{-2s_n} (\ell_n)^{c_{13} w_n} \left( E_o[Z_n] \right)^2. \]  
(4.126)

Since \( \gamma > \alpha + \beta \) by the assumption (2.2), the right of (4.126) is equal to \( o(1) E_o[Z_n]^2 \). By this and and Lemmas 4.1, 4.4, 4.7, 4.8, 4.12, 4.13 we have
\[ P_o \left[ \exists x \in \mathbb{Z}_n^2 \setminus B(o, r_0), H_x > D_m^{x, 0} \right] \]
\[ \geq P_o[Z_n \geq 1] \geq \frac{\left( E_o[Z_n] \right)^2}{E_o[Z_n^2]} = 1 + o(1), \text{ as } n \to \infty. \]  
(4.127)
Proof of the lower bound of Theorem 1.1 via (2.13). (2.11) and (2.13) immediately yield the lower bound. □

A Excursion length

In this section, we give a proof of Proposition 2.2. We follow arguments in [1, Section 8]. Recall definitions of $R_m(x, R, r), D_m(x, R, r)$ from (2.1).

Lemma A.1 ([18, Lemma 2.1]) Fix $n \in \mathbb{N}$ and $1 < r < R/4 < n/8$. For any $y \in \mathbb{Z}_n$, there exists a pair of probability measures $\mu_{y, R}^{y, R}$ on $\partial B(y, R)$ and $\mu_{y, r}^{y, r}$ on $\partial B(y, r)$ such that

\begin{align*}
\mu_{y, R}^{y, R}(z) &= P_{\mu_{y, R}^{y, R}}[S_{\partial B(y, R)} = z], \quad z \in \partial B(y, R), \quad (A.1) \\
\mu_{y, r}^{y, r}(z) &= P_{\mu_{y, r}^{y, r}}[S_{\partial B(y, r)} = z], \quad z \in \partial B(y, r). \quad (A.2)
\end{align*}

Lemma A.2 Fix $n \in \mathbb{N}$ and $1 < r < R/4 < n/8$. For any $y \in \mathbb{Z}_n$,

\begin{equation}
E_{\mu_{y, R}^{y, R}}[D_1(y, R, r)] = \frac{2}{\pi} n^2 \left\{ \log \left( \frac{R}{r} \right) + O \left( \frac{1}{r} \right) \right\}. \quad (A.3)
\end{equation}

Proof. We define the measure $m$ on $\mathbb{Z}_n^2$ by

\begin{equation}
m(\cdot) := \sum_{v \in \partial B(y, r)} \mu_{y, R}^{y, R}(v) G_{B(y, R)}(v, \cdot) + \sum_{v \in \partial B(y, R)} \mu_{y, r}^{y, r}(v) G_{B(y, r)}(v, \cdot), \quad (A.4)
\end{equation}

where

\begin{equation}
G_{B(y, R)}(u, v) := E_u \left[ \sum_{i=0}^{H_{\partial B(y, R)}-1} 1_{\{S_i = v\}} \right], \quad 0 < R' < n/2, \; u, v \in \mathbb{Z}_n^2.
\end{equation}

One can easily check that $m$ is the stationary measure on $\mathbb{Z}_n^2$. Thus, there exists $c_0 > 0$ such that $m = c_0 \nu$ where $\nu$ is the uniform measure on $\mathbb{Z}_n^2$. By Green’s function estimate (see, for example, [13, Proposition 1.6.7]), we have

\begin{equation}
m(y) = \sum_{v \in \partial B(y, r)} \mu_{y, R}^{y, R}(v) G_{B(y, R)}(v, y) = \frac{2}{\pi} \log \left( \frac{R}{r} \right) + O \left( \frac{1}{r} \right). \quad (A.5)
\end{equation}

By (A.2), we have

\begin{align*}
E_{\mu_{y, R}^{y, R}}[D_1(y, R, r)] &= E_{\mu_{y, R}^{y, R}}[H_{\partial B(y, R)}] + E_{\mu_{y, R}^{y, R}} \left[ E_{S_{\partial B(y, R)}} \left[ H_{\partial B(y, R)} \right] \right] \\
&= E_{\mu_{y, r}^{y, r}}[H_{\partial B(y, r)}] + E_{\mu_{y, r}^{y, r}} \left[ H_{\partial B(y, R)} \right] \\
&= \sum_{u \in \mathbb{Z}_n^2} m(u) = c_0. \quad (A.6)
\end{align*}

By (A.5), (A.6), and the fact that $m(y) = c_0 \nu(y) = c_0 n^{-2}$, we have (A.3). □
Lemma A.3  There exist $c_1, c_2, c_3 \in (0, \infty)$ such that for any $n \in \mathbb{N}$, $\lambda \geq 0$, $x, y \in \mathbb{Z}_n^2$, and $c_1 \leq r < R/4 < n/8$,

$$P_x[D_1(y, R, r) \geq \lambda] \leq c_2 \exp\left\{-c_3 \frac{\lambda}{n^2 \log(n/r)}\right\}. \quad (A.7)$$

Proof. By the exponential Chebyshev inequality, for any $\theta > 0$, the left of (A.7) is bounded from above by

$$e^{-\theta \lambda} E_x \left[ \exp\left\{ \theta D_1(y, R, r) \right\} \right]. \quad (A.8)$$

By the strong Markov property at $H_{\partial B(y, r)}$, (A.8) is bounded from above by

$$e^{-\theta \lambda} E_x \left[ \exp\left\{ \theta H_{\partial B(y, r)} \right\} \right] \cdot \max_{z \in \partial B(y, r)} E_z \left[ \exp\left\{ \theta H_{\partial B(y, R)} \right\} \right]. \quad (A.9)$$

To estimate the exponential moments, we use the following:

- [9, Lemma 3.1] There exist $c, c' \in (0, \infty)$ such that for any $c \leq r < n/6$,

$$\max_{y \in \mathbb{Z}_n^2} \max_{x \in \mathbb{Z}_n^2} E_x \left[ H_{\partial B(y, r)} \right] \leq c' n^2 \log(n/r). \quad (A.10)$$

- [13, (1.21)] For any $z \in B(y, R)$,

$$R^2 - (d(z, y))^2 \leq E_z \left[ H_{\partial B(y, R)} \right] \leq (R + 1)^2 - (d(z, y))^2. \quad (A.11)$$

- Kac’s moment formula [12, (6)] For any first hitting time $T$, $m \in \mathbb{N}$, and $y \in \mathbb{Z}_n^2$,

$$E_y [T^m] \leq m! E_y [T] \left( \max_{z \in \mathbb{Z}_n^2} E_z [T] \right)^{m-1}. \quad (A.12)$$

By (A.10) and (A.12), we have

$$E_x \left[ \exp\left\{ \theta H_{\partial B(y, r)} \right\} \right] = 1 + \sum_{k=1}^{\infty} \frac{\theta^k}{k!} E_x \left[ (H_{\partial B(y, r)})^k \right] \leq 1 + \sum_{k=1}^{\infty} \theta^k \left( \max_{z \in \mathbb{Z}_n^2} E_z [H_{\partial B(y, r)}] \right)^k \leq 1 + \sum_{k=1}^{\infty} \left\{ \theta c_2 n^2 \log(n/r) \right\}^k. \quad (A.13)$$

Taking $\theta \leq \frac{1}{2} (c_2 n^2 \log(n/r))^{-1}$, we have

$$E_x \left[ \exp\left\{ \theta H_{\partial B(y, r)} \right\} \right] \leq 2. \quad (A.14)$$
Fix any \( z \in \partial B(y, r) \). Similarly, by (A.11) and (A.12), we have

\[
E_z \left[ \exp \{ \theta H_{\partial B(y, r)} \} \right] = E_z \left[ \exp \{ \theta H_{B(y, r)} \} \right]
\leq 1 + \sum_{k=1}^{\infty} \left\{ \theta \max_{v \in B(y, R)} E_v \left[ H_{B(y, r)} \right] \right\}^k
\leq 1 + \sum_{k=1}^{\infty} \left\{ \theta (R + 1)^2 \right\}^k. \tag{A.15}
\]

Taking \( \theta \leq \frac{1}{2}(R + 1)^{-2} \), we have

\[
E_z \left[ \exp \{ \theta H_{\partial B(y, r)} \} \right] \leq 2. \tag{A.16}
\]

Note that \( R < n/2 \) and \( r < n/8 \) imply \((R + 1)^2 \leq n^2 \log(n/r)\). Thus, taking \( \theta = \frac{1}{2} \left( (c_2 \lor 1) n^2 \log(n/r) \right)^{-1} \), we have

\[
P_x [D_1(y, R, r) \geq \lambda] \leq 4 \exp \left\{ -\frac{\lambda}{2(c_2 \lor 1) n^2 \log(n/r)} \right\}. \tag{A.17}
\]

**Lemma A.4** There exist \( c_1, c_2, c_3 \in (0, \infty) \) such that for any \( n \in \mathbb{N}, 1 < r < R/4 < n/8, x, y \in \mathbb{Z}_n^2, m \geq 2, \) and \( \delta \in (0, 1) \),

\[
P_x \left[ \left| \sum_{i=2}^{m} \left( D_i(y, R, r) - R_i(y, R, r) \right) \right| \leq c_1 \exp \left\{ -c_2 \delta^2 (m - 1) + c_3 \delta \frac{1}{R} (m - 1) \right\} \right]. \tag{A.18}
\]

**Proof.** By the exponential Chebyshev inequality, for any \( \theta > 0 \), we have

\[
P_x \left[ \sum_{i=2}^{m} (D_i(y, R, r) - R_i(y, R, r)) \geq (1 + \delta)E_{\mu_R^{y, r}} \left[ H_{\partial B(y, R)} \right] (m - 1) \right]
\leq e^{-\theta(1+\delta)E_{\mu_R^{y, r}} \left[ H_{\partial B(y, R)} \right] (m-1)} E_x \left[ \exp \left\{ \theta \sum_{i=2}^{m} (D_i(y, R, r) - R_i(y, R, r)) \right\} \right]. \tag{A.19}
\]

By the strong Markov property at \( R_i(y, R, r), 2 \leq i \leq m \), the expectation on the right of (A.19) is bounded from above by

\[
\left( \max_{z \in \partial B(y, r)} E_z \left[ \exp \{ \theta H_{\partial B(y, r)} \} \right] \right)^{m-1}. \tag{A.20}
\]

By (A.11) and Kac's moment formula (A.12), for any \( z \in \partial B(y, r) \) and \( \theta \leq \frac{1}{2}(R + 1)^{-2} \),
$1/(2(R + 1)^2)$, we have

$$E_z \left[ \exp \{ \theta H_{B(y, R)} \} \right] = E_z \left[ \exp \{ \theta H_{B(y, R')^c} \} \right] \leq 1 + \theta E_z \left[ H_{B(y, R)} \right] + \sum_{k=2}^{\infty} \left( \theta \max_{w \in B(y, R)} E_w \left[ H_{B(y, R)} \right] \right)^k \leq 1 + \theta \{ (R + 1)^2 - r^2 \} + \sum_{k=2}^{\infty} \{ \theta (R + 1)^2 \}^k \leq \exp \{ \theta \{ (R + 1)^2 - r^2 \} + 2\theta^2 (R + 1)^4 \}.$$ (A.21)

Note that (A.11) implies that $E_{\mu_{y, r}^R} \left[ H_{\partial B(y, R)} \right] \geq R^2 - (r + 1)^2$. By this, (A.20), and (A.21), for $\theta = \delta_8 \frac{R^2 - (r + 1)^2}{(R + 1)^4}$, the right of (A.19) is bounded from above by a value of the form of the right of (A.18). By almost the same argument, one can obtain a similar estimate of the left tail. We omit the detail. □

For $y \in \mathbb{Z}_n^2$ and $0 < r < R < n/2$, set

$$q(y, R, r) := \min_{u \in \partial B(y, R), v \in \partial B(y, r)} P_v \left[ S_{H_{\partial B(y, r)}} = u \right] \mu_{v, R}^R(u).$$ (A.22)

**Lemma A.5** There exists $c_1 > 0$ such that for any $n \in \mathbb{N}, x \in \mathbb{Z}_n^2$, and $1 < r < R < n/8$,

$$q(y, R, r) \geq 1 - c_1 \frac{r}{R}.$$ (A.23)

**Proof.** Fix any $v \in \partial B(y, r)$ and $u \in \partial B(y, R)$. By Lemma [A.1] we have

$$\mu_{v, R}^R(u) = \sum_{z \in \partial B(y, r)} \mu_{v, R}^{y, R}(z) P_z \left[ S_{H_{\partial B(y, r)}} = u \right] \leq \max_{z \in \partial B(y, r)} P_z \left[ S_{H_{\partial B(y, r)}} = u \right].$$ (A.24)

By the Harnack inequality [4.60], for any $z \in \partial B(y, r),

$$P_z \left[ S_{H_{\partial B(y, r)}} = u \right] = \left( 1 + O \left( \frac{r}{R} \right) \right) P_v \left[ S_{H_{\partial B(y, r)}} = u \right].$$ (A.25)

By (A.24) and (A.25), we have (A.23). □

For $y \in \mathbb{Z}_n^2$, $1 < r < R/4 < n/8$, let $W(y, R, r)$ be the space of paths from $\partial B(y, R)$ to $\partial B(y, r)$.

**Lemma A.6** Fix $n \in \mathbb{N}, x, y \in \mathbb{Z}_n^2$, and $1 < r < R/4 < n/8$. Then, for the SRW on $\mathbb{Z}_n^2$ starting at $x$, the sequence of excursions $(S_{\wedge H_{\partial B(y, r)}} \circ \theta_D_{i+1}(y, R, r))_{i \geq 0}$ is a $W(y, R, r)$-valued Markov chain with initial distribution

$$P_x \left[ S_{\wedge H_{\partial B(y, r)}} \circ \theta_D_{i+1}(y, R, r) \in \cdot \right].$$
and with transition probabilities

$$K(\omega, \omega') := P_{\omega(I_\omega)} \left[ S_{\Lambda H_{\partial B(y, r)}} \circ \theta_{H_{\partial B(y, r)}} = \omega' \right],$$

where \( I_\omega \) is the terminal time of \( \omega \) defined by

$$I_\omega = \min\{i \geq 0 : \omega(i) \in \partial B(y, r)\}. \tag{A.27}$$

The proof of Lemma A.6 is straightforward, so we omit the proof. Fix \( n \in \mathbb{N} \), \( x, y \in \mathbb{Z}^2 \), and \( 1 < r < R/4 < n/8 \). Assume that the SRW \( S \) starts at \( x \). Recall the definition of \( q(y, R, r) \) from (A.22). For each \( z \in \partial B(y, r) \), we define the probability measure \( \nu_z \) on \( \partial B(y, R) \) by

$$\nu_z(u) := \frac{P_z[S \circ \theta_{D_1} = u]}{1 - q(y, R, r)}, \quad u \in \partial B(y, R). \tag{A.28}$$

We take \( W(y, R, r) \)-valued random variables \( X_\ell, \ell \geq 0 \) and \( \{0, 1\} \)-valued random variables \( I_\ell, \ell \geq 0 \) as follows:

- \( X_0 \) has the law

$$\mathbb{P}[X_0 = \omega] = P_x \left[ S_{\Lambda H_{\partial B(y, r)}} \circ \theta_{D_1} = \omega \right], \quad \omega \in W(y, R, r).$$

- \( I_0 \) is a Bernoulli random variable with success probability \( q(y, R, r) \) which is independent of \( X_0 \).

- Suppose that we have constructed \( X_0, \ldots, X_\ell \) and \( I_0, \ldots, I_\ell \). Then, take \( X_{\ell+1} \) as follows: for each \( \omega \in W(y, R, r) \),

$$\mathbb{P} \left[ X_{\ell+1} = \omega \right| \sigma(X_j, I_j : 0 \leq j \leq \ell) \right] = \begin{cases} P_{X_\ell}^{y,R} \left[ S_{\Lambda H_{\partial B(y, r)}} = \omega \right] & \text{if } I_\ell = 1, \\ P_{\nu_{X_\ell}}^{y,R} \left[ S_{\Lambda H_{\partial B(y, r)}} = \omega \right] & \text{if } I_\ell = 0. \end{cases} \tag{A.29}$$

Take \( I_{\ell+1} \) as a Bernoulli random variable with success probability \( q(y, R, r) \) which is independent of \( X_j, 1 \leq j \leq \ell + 1 \) and of \( I_j, 1 \leq j \leq \ell \).

One can check that \( (X_\ell)_{\ell \geq 0} \) is a \( W(y, R, r) \)-valued Markov chain. By a direct calculation of transition probabilities of \( (X_\ell)_{\ell \geq 0} \) and Lemma A.6 we have

$$\left( X_\ell \right)_{\ell \geq 0} \overset{\text{law}}{=} \left( S_{\Lambda H_{\partial B(y, r)}} \circ \theta_{D_{\ell+1}(y, R, r)} \right)_{\ell \geq 0}. \tag{A.30}$$

We define \( J_\ell, \ell \geq 0 \) inductively as follows:

$$J_0 = \min\{i \geq 0 : I_i = 1\}, \quad J_\ell := \inf\{i > J_{\ell-1} : I_i = 1\}, \quad \ell \geq 1. \tag{A.31}$$
Recall the definition of the terminal time $I_\omega$ of $\omega$ from \[A.27\]. Set
\[
G_0 := \sum_{0 \leq \ell \leq J_0} I_X\ell,
\]
\[
G_m := \sum_{J_{m-1} \leq \ell \leq J_m} I_X\ell, \quad m \geq 1.
\] (A.32)

It is straightforward to show that $G_m, m \geq 0$ are independent and $G_m, m \geq 1$ are identically distributed. We need moment estimates of $G_m$:

**Lemma A.7** There exists $c_1 > 0$ such that for any $n \in \mathbb{N}$, $x, y \in \mathbb{Z}^2$, $1 < r < R/4 < n/8$, the following hold:

(i) \[
E[G_1] = \frac{E_{\mu^y,r} [H_{\partial B(y,r)}]}{q(y,R,r)}.
\] (A.33)

(ii) For any $m \in \{0,1\}$ and $k \geq 1$,

\[
E[(G_m)^k] \leq \frac{k!}{(q(y,R,r))^k} \left\{ c_1 n^2 \log \left( \frac{n}{r} \right) \right\}^k.
\] (A.34)

**Proof.** (i) By the definition of $G_1$, we have

\[
E[G_1] = E[I_{X_{J_0+1}}] + \sum_{\ell=2}^{\infty} E[1_{\{J_0+\ell \leq J_1\}} I_{X_{J_0+\ell}}].
\] (A.35)

By the definition of $(X_\ell)_{\ell \geq 0}$ and $J_0$, we have

\[
E[I_{X_{J_0+1}}] = E_{\mu^y,r} [H_{\partial B(y,r)}].
\] (A.36)

By the definitions of $J_0$, $J_1$, and $(X_\ell)_{\ell \geq 0}$, for each $\ell \geq 2$, we have

\[
E[1_{\{J_0+\ell \leq J_1\}} I_{X_{J_0+\ell}}] = E_{\mu^y,r} \left[ E_{\nu^y \mid X_{\ell-1}} [H_{\partial B(y,r)}] \right] (1 - q(y,R,r))^{\ell-1}.
\] (A.37)

By Lemma A.1 and the definition of $(X_\ell)_{\ell \geq 0}$, for any $\ell \geq 2$, we have

\[
E_{\mu^y,r} \left[ E_{\nu^y \mid X_{\ell-1}} [H_{\partial B(y,r)}] \right] = E_{\mu^y,r} [H_{\partial B(y,r)}].
\] (A.38)

By (A.35)-(A.38), we have (A.33).
(ii) We only prove the $m = 0$ case. By Kac's moment formula (A.12),

\[
E[(G_0)^k] = \sum_{j=0}^{\infty} E \left[ 1_{J_0 = j} \left( \sum_{\ell=0}^{j} I_{X_{\ell}} \right)^k \right]
\]

\[
\leq \sum_{j=0}^{\infty} (1 - q(y, R, r))^j q(y, R, r) \sum_{k_0, \ldots, k_j \geq 0, k_0 + \cdots + k_j = k} \frac{k!}{k_0! \cdots k_j!} \prod_{\ell=0}^{j} \max_{v \in \mathbb{Z}_n^2} E_v \left[ (H_{\partial B(y, r)})^k \right] \]

\[
\leq \left\{ k! \sum_{j \geq 0} \binom{k+j}{k} (1 - q(y, R, r))^j q(y, R, r) \right\} \left( \max_{v \in \mathbb{Z}_n^2} E_v \left[ (H_{\partial B(y, r)}) \right] \right)^k.
\]

(A.39)

Let $(E_i)_{i \geq 0}$ be independent standard exponential random variables which are independent of $J_0$. Then, one can check that $E \left[ \left( \sum_{\ell=0}^{j_0} E_{\ell} \right)^k \right]$ is equal to the big bracket on the right-hand side of (A.39). By this and the fact that $\sum_{\ell=0}^{j_0} E_{\ell}$ has the same law as an exponential random variable with mean $q(y, R, r)^{-1}$, $E[\sum_{\ell=0}^{j_0} E_{\ell}]$ is equal to $\frac{k!}{(q(y, R, r))^k}$. By this and (A.10), we have the statement (A.34) with $m = 0$. The proof for the $m = 1$ case is almost the same, so we omit the detail. □

Lemma A.8 There exist $c_1, c_2 \in (0, \infty)$ such that for any $n \in \mathbb{N}$, $x, y \in \mathbb{Z}_n^2$, $1 < r < R/4 < n/8$, $\delta \in (0, 1)$ and $m \geq 1$,

\[
\mathbb{P} \left[ \frac{\sum_{i=0}^{m-1} G_i}{E[G_1]|m} - 1 > \delta \right] \leq c_1 \exp \left\{ -c_2 (m-1)\delta^2 \left( \frac{E[G_1]}{n^2 \log(n/r)} \right)^2 \right\}. \quad (A.40)
\]

Proof. By the exponential Chebyshev inequality, for any $\theta > 0$, we have

\[
\mathbb{P} \left[ \sum_{i=0}^{m-1} G_i > (1 + \delta)E[G_1]|m \right] \leq e^{-(1+\delta)E[G_1]|m} \mathbb{E} \left[ \exp \left\{ -\theta \sum_{i=0}^{m-1} G_i \right\} \right]
\]

\[
= e^{-(1+\delta)E[G_1]|m} \mathbb{E} \left[ e^{\theta G_0} \right] \left( \mathbb{E}[e^{\theta G_1}] \right)^{m-1}, \quad (A.41)
\]

where we have used the fact that $G_m$, $m \geq 0$ are independent and that $G_m$, $m \geq 1$ are identically distributed. By Lemma A.7 for any $0 < \theta \leq \frac{q(y, R, r)}{2 n^2 \log(n/r)}$, we have

\[
E[e^{\theta G_0}] = 1 + \sum_{k=1}^{\infty} \frac{\theta^k}{k!} E[(G_0)^k]
\]

\[
\leq 1 + \sum_{k=1}^{\infty} \left( \frac{\theta c_1 n^2 \log(n/r)}{q(y, R, r)} \right)^k \leq 2. \quad (A.42)
\]
Similarly, by Lemma A.7 for any $0 < \theta \leq \frac{1}{2} \frac{q(y, R, r)}{c_1 n^2 \log(n/r)}$, we have

$$
E[e^{\theta G_1}] = 1 + \theta E[G_1] + \sum_{k=2}^{\infty} \frac{\theta^k}{k!} E[(G_1)^k]
$$

$$
\leq 1 + \theta E[G_1] + \sum_{k=2}^{\infty} \left( \frac{\theta c_1 n^2 \log(n/r)}{q(y, R, r)} \right)^k
$$

$$
\leq \exp \left\{ \theta E[G_1] + 2\theta^2 \left( \frac{c_1 n^2 \log(n/r)}{q(y, R, r)} \right)^2 \right\}. \tag{A.43}
$$

By (A.42) and (A.43), for the optimal value $\theta = \frac{4}{4} E[G_1]$ of \(q(y, R, r)\), the right of (A.41) is bounded from above by a value of the form of the right of (A.40). Repeating a similar argument, we can obtain a similar estimate of the left tail. We omit the detail. \(\square\)

**Lemma A.9** There exist $c_1, c_2 \in (0, \infty)$ such that for any $n \in \mathbb{N}, x, y \in \mathbb{Z}^2$, $1 < r < R/4 < n/8, m > 1 + 4 (q(y, R, r))^{-1}, \delta \in (\frac{4}{(m-1)q(y, R, r)}, 1)$,

$$
P_x \left[ \sum_{i=1}^{m-1} (R_{i+1}(y, R, r) - D_i(y, R, r)) - E_{\mu^x} \left[ H_{\partial B(y, r)} \right] (m-1) \right] > \delta
$$

$$
\leq c_1 \exp \left\{ -c_2 (m-1) q(y, R, r) \theta^2 \left( \frac{E_{\mu^x} \left[ H_{\partial B(y, r)} \right]}{n^2 \log(n/r)} \right)^2 \right\}. \tag{A.44}
$$

**Proof.** Set $m^+ := \left[ \frac{(m-1)q(y, R, r)}{1-\delta/8} \right]$. By (A.30), we have

$$
P_x \left[ \sum_{i=1}^{m-1} (R_{i+1}(y, R, r) - D_i(y, R, r)) > (1 + \delta) E_{\mu^x} \left[ H_{\partial B(y, r)} \right] (m-1) \right]
$$

$$
= \mathbb{P} \left[ \sum_{\ell=0}^{m-2} I_{\chi_{\ell}} > (1 + \delta) E_{\mu^x} \left[ H_{\partial B(y, r)} \right] (m-1) \right]
$$

$$
\leq \mathbb{P} \left[ \sum_{\ell=0}^{m-1} G_{\ell} > (1 + \delta) E_{\mu^x} \left[ H_{\partial B(y, r)} \right] (m-1) \right] + \mathbb{P} [J_{m^+-1} < m-2]. \tag{A.45}
$$

By Lemma A.7 (i), the definition of $m^+$, and the condition $\delta \in (\frac{4}{(m-1)q(y, R, r)}, 1)$, we have

$$
(1 + \delta) E_{\mu^x} \left[ H_{\partial B(y, r)} \right] (m-1) \geq \left( 1 + \frac{\delta}{4} \right) E[G_1]m^+.
$$

By this and Lemmas A.7, A.8, the first term on the right of (A.45) is bounded from above by a value of the form of the right of (A.44). By the definition of
$J_{m+1}$ and the exponential Chebyshev inequality, for any $\theta \in (0, 1)$, the second term on the right of (A.45) is bounded from above by

$$
\Pr \left[ I_0 + \cdots + I_{m-2} \geq m^+ \right] \leq e^{-\theta m^+} \mathbb{E} \left[ e^{\theta \sum_{i=0}^{m-2} I_i} \right].
$$

(A.46)

Since $I_\ell$, $\ell \geq 0$ are i.i.d. Bernoulli random variables with parameter $q(y, R, r)$, the expectation on the right of (A.46) is bounded from above by

$$
\{ e^{\theta q(y, R, r)} + 1 - q(y, R, r) \}^{m-1} \leq \exp \left\{ \left( \theta + 2\theta^2 \right) q(y, R, r) (m - 1) \right\},
$$

(A.47)

where we have used the inequality $e^\theta \leq 1 + \theta + 2\theta^2$ for any $\theta \in (0, 1)$ in the last inequality. Optimizing $\theta$ (take $\theta = \frac{1}{4} \left( \frac{m^+}{(m-1)q(y, R, r)} - 1 \right)$), the right of (A.46) is bounded from above by

$$
\exp \left\{ -c_3(m - 1)q(y, R, r)\delta^2 \right\}.
$$

(A.48)

This is bounded from above by a value of the form of the right of (A.44) since $E_{\mu_y \cdot r}[H_{\partial B(y, r)}] \leq cn^2 \log(n/r)$ by (A.10). Therefore, the right of (A.45) is bounded from above by a value of the form of the right of (A.44). Repeating a similar argument, we can obtain a similar estimate of the left tail. We omit the detail. □

**Proposition A.10** There exist $c_i \in (0, \infty)$, $1 \leq i \leq 8$ such that for any $n \in \mathbb{N}$, $x, y \in \mathbb{Z}^2$, $c_1 \leq r < R/4 < n/8$, $m \geq 1 + 4 (q(y, R, r))^{-1}$, $\delta \in \left(\frac{4}{(m-1)q(y, R, r)}, 1\right)$,

$$
P_x \left[ \left| \frac{D_m(y, R, r)}{E_{\mu_{y \cdot r}}[D_1(y, R, r)]} - 1 \right| \geq \delta \right]
\leq c_2 \exp \left\{ -c_3 \delta E_{\mu_{y \cdot r}}[D_1(y, R, r)](m - 1) \right\}
+ c_4 \exp \left\{ -c_5 \delta^2(m - 1) + c_6 \frac{1}{R} (m - 1) \right\}
+ c_7 \exp \left\{ -c_8 q(y, R, r) \delta^2 \left( \frac{E_{\mu_{y \cdot r}}[H_{\partial B(y, r)}]}{n^2 \log(n/r)} \right) \right\}. \tag{A.49}
$$

**Proof.** By Lemma [A.1], we have

$$
E_{\mu_{y \cdot r}}[D_1(y, R, r)] = E_{\mu_{y \cdot r}}[H_{\partial B(y, r)}] + E_{\mu_R \cdot r} [H_{\partial B(y, R)}].
$$

(A.50)

Note that $D_m(y, R, r)$ is equal to

$$
D_1(y, R, r) + \sum_{i=2}^{m} (D_i(y, R, r) - R_i(y, R, r)) + \sum_{i=1}^{m-1} (R_{i+1}(y, R, r) - D_i(y, R, r)).
$$

(A.51)
By (A.50) and (A.51), the left of (A.49) is bounded from above by

\[ P_x \left[ D_1(y, R, r) \geq \frac{\delta}{2} E_{\mu^x_{\theta}} [D_1(y, R, r)] (m-1) \right. \]
\[ + P_x \left[ \sum_{i=2}^{m-1} \left( \frac{D_i(y, R, r) - R_i(y, R, r)}{E_{\mu^x_{\theta}} [H_{\partial B(y, r)}]} (m-1) \right) \geq \frac{\delta}{2} \right. \]
\[ + P_x \left[ \sum_{i=1}^{m-1} \left( \frac{R_{i+1}(y, R, r) - D_i(y, R, r)}{E_{\mu^x_{\theta}} [H_{\partial B(y, r)}]} (m-1) \right) \geq \frac{\delta}{2} \right] \]  
\[ \geq \sum_{i=2}^{m-1} \left( \frac{D_i(y, R, r) - R_i(y, R, r)}{E_{\mu^x_{\theta}} [H_{\partial B(y, r)}]} (m-1) \right) \]
\[ + \sum_{i=1}^{m-1} \left( \frac{R_{i+1}(y, R, r) - D_i(y, R, r)}{E_{\mu^x_{\theta}} [H_{\partial B(y, r)}]} (m-1) \right) \]  
\[ \geq \frac{\delta}{2} \]  
\[ \geq \frac{\delta}{2} \]  
\[ \geq \frac{\delta}{2} \].  
\[ (A.52) \]

By (A.52) and Lemmas A.3, A.4, A.9, we have (A.49). □

**Proof of Proposition 2.2.** We only prove (2.12). Set \( r_0^- := \left( 1 - \frac{\sqrt{2}}{(\log n)^2} \right) r_1, \)
\[ r_0^+ := \left( 1 + \frac{\sqrt{2}}{(\log n)^2} \right) r_0. \]
We define the set \( F \) by

\[ F := \left\{ \left( i, \left\lfloor \frac{r_0}{\ell_n(\log n)^2} \right\rfloor, \left\lfloor \frac{r_0}{\ell_n(\log n)^2} \right\rfloor : 0 \leq i, j \leq \left\lfloor \frac{n}{\frac{r_0}{\ell_n(\log n)^2}} \right\rfloor \right) \right\}. \]

By a simple observation, one can show that for any \( x \in \mathbb{Z}^2_n \), there exists \( y \in F \) such that
\[ x \in \left( y + \left[ 0, \left[ \frac{r_0}{\ell_n(\log n)^2} \right] \right] \right) \cap \mathbb{Z}^2 \mod n \mathbb{Z}^2 \]
and that
\[ B(y, r_1) \subset B(x, r_0) \subset B(x, r_0^+) \subset B(y, r_1^+) \].

In particular, we have
\[ D_{m_n^+}(x, r_0, r_1) \leq D_{m_n^+}(y, r_0^+, r_1^+) \].

By this observation, we have
\[ P_o \left[ D_{m_n^+}(x, r_0, r_1) > \frac{4}{\pi} n^2 (\log n)^2 \left( 1 - \frac{\log \log n}{2 \log n} + \frac{2 s_n}{\log n} \right) \right] \]
\[ \leq \sum_{y \in F} P_o \left[ D_{m_n^+}(y, r_0^+, r_1^+) > \frac{4}{\pi} n^2 (\log n)^2 \left( 1 - \frac{\log \log n}{2 \log n} + \frac{2 s_n}{\log n} \right) \right]. \]  
\[ (A.53) \]

By Lemma A.2, for sufficiently large \( n \), we have
\[ E_{\mu^0_{r_0^+}} [D_1(y, r_0^+, r_1^-)] (m_n^+ - 1) = 1 + \frac{s_n}{4 \log n} \]
\[ \leq \frac{4}{\pi} n^2 (\log n)^2 \left( 1 - \frac{\log \log n}{2 \log n} + \frac{2 s_n}{\log n} \right). \]  
\[ (A.54) \]  
47
Note that by Lemma A.2, (A.11), and (A.50), we have
\[
E_{\mu_{\nu, \gamma}^+} \left[ H_{\partial B(y, r^{-1})} \right] \geq c_1 n^2 \log(\ell_n).
\]
By this, (A.54), Lemmas A.2, A.5, and Proposition A.10, each term on the right of (A.53) is bounded from above by
\[
c_2 e^{-c_3 (\log \log n)^{2\gamma - \alpha - 2\beta}}.
\]
(A.55)
By the definition of the set \( F \), we have
\[
|F| \leq c_4 (\log n)^4 (\ell_n)^4 e^{2c_1 (\log \log n)^{\alpha + \beta}}.
\]
(A.56)
Note that by the assumption (2.2), we have \( 2\gamma - \alpha - 2\beta > 1 > \alpha + \beta \). By this, (A.53), (A.55), and (A.56), we have (2.12). The proof of (2.13) is almost the same as that of (2.12). So, we omit the detail. \( \square \)

\section*{B Proofs of Lemma 2.4 and (4.31)}

In this section, we prove Lemma 2.4. For simplicity, we drop \( \overline{R} \) from the notation.

Proof of Lemma 2.4(i). We will only prove the upper bound in (2.20). We will keep track of jumps among \( \partial B(x, R_\ell) \), \( \ell \in I_{k_\tilde{k}} \), where \( I_{k_\tilde{k}} := \{0, 1, k, k + 1, \ldots, L - \tilde{k}, L\} \). Note that there are two types of excursions from \( \partial B(x, R_1) \) to \( \partial B(x, R_0) \):

(a) excursions which hit \( \partial B(x, R_{k+1}) \) (we will call them of type (a)),

(b) excursions which do not hit \( \partial B(x, R_{k+1}) \) (we will call them of type (b)).

Let \( e \) be an excursion from \( \partial B(x, R_1) \) to \( \partial B(x, R_0) \). When \( e \) is of type (a), we will define the process \( \sigma_\ell, \ell \geq 0 \), which records subscripts of radii of the circles SRW visits as follows: Set

\[
\sigma_0 := 0, \quad \sigma_1 := \min\{i : e(i) \in \partial B(x, R_{k+1})\}, \quad \ell_{\sigma_1} := k + 1.
\]

Suppose that we have constructed \( \sigma_\ell, \ell_{\sigma_\ell}, 0 \leq \ell \leq i - 1 \) (\( i \geq 2 \)) and that \( \ell_{\sigma_\ell} \neq 0 \) for all \( 0 \leq \ell \leq i - 1 \). Then, we define \( \sigma_i \) and \( \ell_{\sigma_i} \) by

\[
\sigma_i := \min \left\{ j > \sigma_{i-1} : e(j) \in \bigcup_{\ell \in I_{k_\tilde{k}} \setminus \{1\}, \ell \neq \ell_{\sigma_{i-1}}} \partial B(x, R_\ell) \right\},
\]

\[
\ell_{\sigma_i} := \ell \text{ such that } e(\sigma_i) \in \partial B(x, R_\ell).
\]
Stop the construction at the first time when \( \text{tr}^e \) reaches \( 0 \). When \( e \) is of type (b), we define \( \text{tr}^e = (\text{tr}^e(i))_{i \in \{0, 1\}} \) just by \( \text{tr}^e(0) := 1, \text{tr}^e(1) = 0 \).

Let \( e_i \) (1 \( \leq i \leq m \)) be the \( i \)-th excursion from \( \partial B(x, R_1) \) to \( \partial B(x, R_0) \). We define the process \( \text{tr}^S \) by concatenating \( \text{tr}^{e_1}, \ldots, \text{tr}^{e_m} \) in this order. Let \( W \) be the state space of \( \text{tr}^S \). For each \( w \in W \) and \( i, j \in I^L_{\tilde{k}, \tilde{k}} \) with \( i < j \), let \( T_{i \to j}^w \) be the number of traversals from \( i \) to \( j \) by \( w \). Let \( \tilde{W} \) be the set of all \( w \in W \) which satisfies \( T_{i \to i+1}^w = m_i, k \leq \forall i \leq L - \tilde{k} - 1 \) and \( T_{L-k \to L}^w = 0 \). (Recall that \((m_i)_{i = \tilde{k} - 1}^{L-1}\) are integers on the left-hand side of (2.20).

Fix any \( w \in \tilde{W} \). Recall the definition of \( p_{i, i+2}^{\nu_1, \nu_2} \) from (2.14) and (2.15). By the strong Markov property and Lemma 2.3, we have

\[
P_j[\text{tr}^S = w] \\
\leq \left( p_{1,0}^{k+1,+} \right)^{m_i - T_{i \to k+1}^w} \left( p_{0,k+1}^{1,+} \right)^{T_{i \to k+1}^w} \\
\times \left( p_{k,0}^{k+1,+} \right)^{T_{i \to k+1}^w} \left( p_{k+1,k}^{0,+} \right)^{m_i - T_{i \to k+1}^w} \\
\times \prod_{i = k+1}^{L - \tilde{k} - 1} \left\{ \left( p_{i-1,i}^{i+1,+} \right)^{m_i - \tilde{w}_{i+1}} \left( p_{i+1,i}^{i-1,+} \right)^{w_{i+1}} \right\} \\
\times \left( p_{L-k,k}^{L,+} \right)^{m_i - \tilde{w}_{L-k-\tilde{k}-1}} .
\]

Recall the definitions of \( \Delta_{1, i, i+2}^{\nu_1, \nu_2} \), \( \Delta_1^+ \), and \( \Delta_1^+ \) from (2.16) and (2.21). By these definitions and the strong Markov property of the SRW \( Z = (Z_i, i \geq 0, P_k^{D_i}, k \in \{0, \ldots, L\}) \) on \( \{0, \ldots, L\} \), the right-hand side of (B.1) is bounded from above by

\[
\Delta_1^+ \cdot \Delta_1^+ P_1^{1D}[\text{tr}^Z = w] ,
\]

where \( \text{tr}^Z \) is the process which records vertices SRW visits and is defined in the same manner as \( \text{tr}^S \). By (B.1) and (B.2), summing over \( w \in \tilde{W} \), we can bound the numerator on the left-hand side of (2.20) from above by

\[
\Delta_1^+ \Delta_1^+ P_1^{1D} \left[ T_i^m = m_i, k \leq \forall i \leq L - \tilde{k} - 1, T_{L-1}^m = 0 \right] ,
\]

where \( T_i^m \) is the number of traversals from \( i \) to \( i + 1 \) by \( Z \) starting at \( 1 \) up to the \( m \)-th return to \( 0 \). Since \( P_1^{1D} \)-law of \((T_i^m)_{i \geq 0}\) is the same as \( P_{G_i}^{m} \)-law of \((T_i)_{i \geq 0}\), we have obtained the upper bound. Other statements of (i) can be proved similarly. We omit the details.

---

2More precisely, \( T_{i \to j}^w \) is defined as follows: set \( R_i^w := \min(\ell : w(\ell) = i), R_{p+1}^w := \min(\ell : D_p^w : w(\ell) = i) \), \( P_{\ell}^w := \min(\ell > R_\ell^w : w(\ell) = i) \), \( P_{\ell-1}^w := \min(\ell : D_{\ell-1}^w : w(\ell) = i) \), \( P_{\ell}^w := \min(\ell : D_{\ell}^w : w(\ell) = i + 1) \), \( P_{\ell-1}^w := \min(\ell : D_{\ell-1}^w : w(\ell) = i + 1) \), \( p \geq 1 \). Then, we define \( T_{i \to j}^w \) by \( T_{i \to j}^w := \min(\ell : R_\ell^w < t_w) \), where \( t_w \) is the terminal time of \( w \).

3More precisely, \( 2) \) is defined as follows: set \( R_i^w := \min(\ell : Z_\ell = i + 1), R_{p+1}^w := \min(\ell : D_p^w : Z_\ell = i), R_{p}^w := \min(\ell : D_p^w : Z_\ell = i + 1) \), \( p \geq 1 \). Then, we define \( T_i^m \) by \( T_i^m := \min(\ell : R_\ell^w < D_m^w) \).
Proof of Lemma 2.4 (ii). We have

\[ P_y \left[ T^{x,m}_1 \geq \ell, T^{x,m}_{L-1} = 0 \right] = \sum_{k=0}^{m-1} P_y \left[ T^{x,k}_1 < \ell \leq T^{x,k+1}_1, T^{x,m}_{L-1} = 0 \right]. \] (B.4)

Fix 0 \leq k \leq m - 1. By the strong Markov property, the k-th term on the right-hand side of (B.4) is bounded from above by

\[ \sum_{j=0}^{\ell-1} P_y \left[ T^{x,k}_1 = j, T^{x,k+1}_1 \geq \ell, T^{x,m}_{L-1} = 0 \right] \leq \sum_{j=0}^{\ell-1} P_y \left[ T^{x,k}_1 = j, T^{x,k}_{L-1} = 0 \right] \cdot \max_{z \in \partial B(x,R_0)} P_z \left[ T^1_1 \geq \ell - j, T^{m-k-1}_{L-1} = 0 \right]. \] (B.5)

By Lemma 2.4 (i), for each 0 \leq j \leq \ell - 1, the first factor of the j-th term on the right of (B.5) is bounded from above by

\[ (\Delta_{i+1,1}^0 \vee \Delta_{i+1,1}^{1+})^k \cdot (\Delta_{i+1,1}^{1+} \vee \Delta_{i+1,1}^{0+})^j \cdot (\Delta_{i+1,1}^{L+} + \Delta_{i+1,1}^{L+})^j \cdot P^{1D}_1 \left[ T^1_1 = j, T^{k}_{L-1} = 0 \right]. \] (B.6)

By Lemma 2.3, the second factor of the j-th term on the right of (B.5) is bounded from above by

\[ (\Delta_{i+1,1}^{0+})^k \cdot (\Delta_{i+1,1}^{L+} \vee \Delta_{i+1,1}^{L+})^j \cdot (\Delta_{i+1,1}^{L+} + \Delta_{i+1,1}^{L+})^j \cdot P^{1D}_1 \left[ T^1_1 \geq \ell - j, T^{m-k-1}_{L-1} = 0 \right]. \] (B.7)

By the definition of \( \Delta_{i+1,1}^{0+} \) (see (2.16)) and the strong Markov property of SRW on \( \{0, \ldots, L\} \), (B.7) is bounded from above by

\[ \Delta_{i+1,1}^{0+} \left( p_{i+1,i}^{0+} \right)^{\ell-j-1} \left( p_{L+}^{L+} \right)^{\ell-j-1} \left( p_{L+}^{L+} \right)^{m-k-1} \] (B.8)

By (B.4)-(B.8) and the strong Markov property, (B.4) is bounded from above by

\[ \Delta_{i+1,1}^{0+} P^{1D}_1 \left[ T^1_1 \geq \ell - j, T^{m-k-1}_{L-1} = 0 \right]. \] (B.9)

Since \( P^{1D}_1 \)-law of \( (T^m_j)_{j \geq 0} \) is the same as \( P^{GW}_m \)-law of \( (T_j)_{j \geq 0} \), we have obtained the desired result. \( \square \)

Proof of Lemma 2.4 (iii). By the strong Markov property, the probability on
the left-hand side of (2.24) is bounded from above by
\[
\sum \prod_{\ell=1}^{m} \max_{z \in \partial B(x,R_{k+1})} P_z \left[ T_i^{k,x,1} = m_i^{(\ell)}, \ k + 1 \leq \forall i \leq L - \tilde{k} - 1, \ T_{L-1}^{k,x,1} = 0 \right] 
\times \left( \max_{z \in \partial B(x,R_k)} P_z \left[ H_{\partial B(x,R_{k+1})} < H_{\partial B(x,R_0)} \right] \right)^m 
\times \max_{z \in \partial B(x,R_k)} P_z \left[ H_{\partial B(x,R_0)} < H_{\partial B(x,R_{k+1})} \right], \tag{B.10}
\]
where the sum is taken over all nonnegative integers \(m_i^{(\ell)}, \ k + 1 \leq i \leq L - \tilde{k} - 1, 1 \leq \ell \leq m\) satisfying \(\sum_{\ell=1}^{m} m_i^{(\ell)} = m_i\) for each \(k + 1 \leq i \leq L - \tilde{k} - 1\). By Lemmas 2.4 (i) and 2.3 (B.10) is bounded from above by
\[
\sum \prod_{\ell=1}^{m} \left\{ (\Delta_{i+1}^L)^{m_i^{(\ell)} + 1} \right\} \left( \Delta_{L-1}^{L-1} \right)^{m_{L-1}} 
\times \prod_{\ell=1}^{m} P_1^{GW} \left[ T_i = m_i^{(\ell)}, 1 \leq \forall i \leq L - k - \tilde{k} - 1, \ T_{L-1} = 0 \right] 
\times \left( \Delta_{k,k+1}^{0,0} \right)^{m_{k,k+1}} \Delta_{k,0}^{k+1} \left( \frac{k}{k+1} \right)^m \left( \frac{k-3}{k-2} \right)^{m_{L-1}-m' - \sum_{\ell=1}^{m} j_{\ell}^{(\ell)}}. \tag{B.11}
\]
(B.11) immediately yields the desired result. \(\square\)

Proof of (4.31). Recall the definitions \(P_c, P_1, C_m, \) from (4.23), (4.22), and above (4.23). By Lemma 2.3 \(P_c\) is bounded from above by \((1 + o(1))\) times
\[
\left\{ \prod_{\ell=1}^{p} \left( \frac{1}{k-2} \right)^{j_{\ell}^{(\ell)}-1} \left( \frac{k-3}{k-2} \right)^{j_{\ell}^{(\ell)}} \right\} \left\{ \left( \frac{1}{k-2} \right)^{m_{L-1}} \left( \frac{k-3}{k-2} \right)^{m_{L-1} - m' - \sum_{\ell=1}^{p} j_{\ell}^{(\ell)}} \right\}. \tag{B.12}
\]
We will relate (B.12) to a law of the 1-dimensional SRW. Recall the definitions of \(D_k^\ell, R_k^\ell\) from the footnote \(n\) For each \(m \in \mathbb{N}\), set
\[
\bar{T}^m := \max \left\{ \ell \geq 1 : D_k^{0} < R_k^{m-3} \right\}.
\]
By the strong Markov property, the right of (B.12) is equal to
\[
(1 + o(1)) P_{k-2}^{1D} \left[ \bigcap_{\ell=1}^{p} \left\{ \bar{T}^{1 \circ \theta_{D_k^{\ell-3}} = j_{\ell}^{(\ell)}} \cap \left\{ \bar{T}^{1 \circ \theta_{D_k^{\ell+1}}} \geq m_{L-1}^{\ell} - m' - \sum_{\ell=1}^{p} j_{\ell}^{(\ell)} \right\} \right\} \right]. \tag{B.13}
\]
By (B.12) and (B.13), we have
\[
\sum_{j_{\ell}^{(\ell)}} \begin{cases} \end{cases} \leq (1 + o(1)) P_{k-3}^{1D} \left[ \bar{T}^{p} < m_{L-1}^{\ell} - m' \leq \bar{T}^{p+1} \right]. \tag{B.14}
\]
where the sum is taken over \((j_c^\ell)_{\ell=1}^p\) satisfying the condition in (4.25). Similarly, by Lemma 2.3 and the strong Markov property, we have
\[
P_0(C_{1m'}) \leq (1 + o(1)) P_1^{1D} [D^0_{m'} < H_{k-2} < D^0_{m'+1}] .
\] (B.15)
By (B.14), (B.15), and the strong Markov property, \(P_0\left[\sum_{(j_c^\ell)} P_0 \right] \leq (1 + o(1)) P_1^{1D} [D^0_{m'} < H_{k-2} < D^0_{m'+1}] \). This probability is equal to \(P_1^{1D} [D^0_{m'} < H_{k-2} < D^0_{m'+1}, T_{m-3} \leq \tilde{T}_{p+1} \circ \theta^{D_{k-3}}] \). Therefore, summing over \(p = b_k - n(k - 3) - 1, \ldots, b_k + n(k - 3) - 1\) and \(m' = 0, \ldots, m_n = 0\), . . . , \(m_{n-1}\) and using the fact that \(P_1^{1D}\)-law of \(T_{m-3}\) is the same as \(P_{GW}^{m_n}\)-law of \(T_{k-3}\), we have the desired result. □

C Barrier estimates

In this section, we will prove Lemma 2.5. The proof is heavily based on arguments in [2] and we use the same notation as in [2].

As mentioned in Remark 2.6, we need to know how the constant \(c\) in [2, Theorem 1.1] depends on \(\eta\). The proof of Lemma 2.5 (i) is almost the same as that of [2, Theorem 1.1], so we only show how we should slightly modify the argument. The constant \(c\) in [2, Lemma 2.5] depends on \(\eta\) as follows:

Lemma C.1 Under the same assumption as in [2, Lemma 2.5],
\[
\begin{align*}
& P_Y \left[ f_{a,b}(\ell; L) - CL^{1/2} \leq Y_\ell, \quad \ell = 1, \ldots, L - 1, \quad Y_L \in H_{y,\delta} \right] \\
& \leq ce^{-\delta \eta} \left( 1 + x - a \right) \left( 1 + y - b \right) \sqrt{\frac{x}{yL}} e^{-\frac{(y-x)^2}{2L}},
\end{align*}
\] (C.1)
where \(c > 0\) depends only on \(\delta, \varepsilon, C\) (not on \(\eta\)).

Proof. As mentioned in [2, Remark 2.2], the second line in [2, (2.3)] is bounded from above by [2, (2.4)] multiplied by a constant depending only on \(\delta, \varepsilon, C\). By the assumption \(|x-y| \leq \eta L\), we have \((x-z)^2 \geq (x-y)^2 - 2\eta L|z| \) for any \(z \in H_{y,\delta}\). Thus, the exponential factor in [2, (2.4)] is bounded from above by \(e^{-\frac{(y-x)^2}{2L}}\). By this and the proofs of [2, Lemmas 2.3, 2.5], we have the desired result. □

Proof of Lemma 2.5 (i). As mentioned before, the proof is almost the same as that of [2 Theorem 1.1], but we need some minor modifications as follows:

- Below [2, (4.4)]: Using our additional condition \(a \geq b\), we have
\[
\sqrt{T_0 + T_1} \geq \sqrt{T_0 + T_1} \geq a' + \frac{x-a}{2} - C.
\]

- [2, (4.7)]: When \(\ell - \ell' \geq 0\), using the condition \(a \geq b\), we have \(f_{a,b}(\ell; L) \leq f_{a,b}(\ell'; L')\). We use this in place of [2, (4.7)] in our proof.
• [2] (4.8): By a simple calculation, for each $2 \leq \ell \leq L - 1$, we have
\[
(\ell - 1)^{\frac{1}{L} - \varepsilon} \leq 2^{\frac{1}{2} - \varepsilon} \ell^{\frac{1}{L} - \varepsilon}, \quad \ell^{\frac{1}{L} - \varepsilon} \leq 2^{\frac{1}{2} - \varepsilon}(\ell - 1)^{\frac{1}{L} - \varepsilon}.
\]
These inequalities imply the following:
\[
\sqrt{\frac{T_{L-1} + T_\ell}{2}} \geq \frac{1}{\sqrt{2}} \left\{ f_{a', b}(\ell - 1; L - 1) - 2^{1 - 2c} C(\ell - 1)^{\frac{1}{L} - \varepsilon} \right\},
\]
Thus, we can replace $\frac{C}{2}$ above [2] (4.9) with $2^{1 - 2c} C$. (In particular, this does not depend on $\eta$.)

• [2] (4.16): By Lemma C.1, we can replace the constant $C$ in [2] (4.16) with $ce^\eta$, where $c > 0$ is independent of $\eta$.

• [2] (4.17): For any $j, k \geq 1$ with $|j - k| \leq 2\eta L$ and $z \in H_j$, we have
\[
(k - z)^2 = ((k - j) + (j - z))^2 \geq (k - j)^2 + 2(k - j)(j - z) \geq (k - j)^2 - 4\eta L.
\]
Thus, we can replace $c$ (the first factor in the first line of [2] (4.17)) with $c'e^{5\eta}$, where $c' > 0$ is independent of $\eta$.

• [2] (4.18): Let us look at the exponential factors in the first sum of [2] (4.17). Since
\[
(k - j)^2 = ((k - y) + (y - x) + (x - j))^2 \geq (y - x)^2 + 2(y - x)(k - y) + 2(y - x)(x - j),
\]
we have
\[
e^{-\frac{(k - j)^2}{2(k - 1)}} \leq e^{-\frac{(y - x)^2}{k - 1}e^{-\frac{y - x}{2c}(k - y)}e^{-\frac{y - x}{2c}(x - j)}}.
\]
Since $\sqrt{2} \leq x, y \leq \eta L$, we have
\[
c(x - j)^2 + \frac{y - x}{L - 1}(x - j) = c\left\{ (x - j) + \frac{y - x}{2c(L - 1)} \right\}^2 - \frac{(y - x)^2}{4c(L - 1)^2} \geq c\alpha^2 - c_1\eta^2,
\]
where we set $\alpha := (x - j) + \frac{y - x}{2c(L - 1)}$ and $c_1 > 0$ is independent of $\eta$. ($c_i, i = 2, \ldots, 10$ below are also positive constants independent of $\eta$.) Similarly, we have
\[
c(k - y)^2 + \frac{y - x}{L - 1}(k - y) \geq c\beta^2 - c_2\eta^2,
\]
where we set $\beta := (k - y) + \frac{y - x}{2c(L - 1)}$. Thus, the product of three exponential factors in the first sum of [2] (4.17) is bounded from above by
\[
e^{(c_1 + c_2)\eta} e^{-c(\alpha^2 + \beta^2)} e^{-\frac{(y - x)^2}{2c}}.
\]
By the assumptions $|a - b| \leq \eta L$ and $|x - y| \leq \eta L$, we have
\[
1 + j - a' = 1 - \alpha + (x - a) + \frac{y - x}{2c(L - 1)} + \frac{a - b}{L} \leq 1 + |a| + (x - a) + c_3\eta.
\]
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Similarly, we have
\[ 1 + k - b \leq 1 + |\beta| + (y - b) + c_4 \eta. \]

Since \( k \geq \frac{y + b}{2} \geq \frac{x}{2} \) and \( \sqrt{2} \leq x, y \leq \eta L \), we have
\[ \sqrt{\frac{j}{k}} = \sqrt{\frac{-\alpha + x + \frac{y - x}{2L-1}}{k}} \leq \sqrt{\frac{2x}{y}} \sqrt{\frac{|\alpha| + 1 + c_5 \eta}{\sqrt{2}}} \leq c_6 \sqrt{\frac{x}{y} \left( \sqrt{|\alpha|} + \eta \right)}. \]

By the above estimates, the first sum in [2, (4.17)] is bounded from above by
\[ c_7 e^{c_8 \eta^2 (\eta + x - a)(\eta + y - b)} \sqrt{x} \frac{L}{2} e^{-c_9 \eta^2 L^2}. \]

By the argument below [2, (4.18)], the second sum in [2, (4.17)] is bounded from above by
\[ c_9 e^{c_{10} \eta^2 L^2}. \]

By the above modifications, we have the desired result. □

Proof of Lemma 2.5 (ii). We will use notation in [2, Section 3]. Set
\[ C' := 3 \cdot 2^{\frac{1}{2} - \epsilon} C, \quad C'' := \left( \frac{1}{2} \right)^{\frac{1}{2} + \epsilon} \bar{C}. \]

Let \( A \) be the event on the left hand side of (2.27). Set
\[ B := \bigcap_{i=r}^{L-r} \left\{ f_{a,0}(i;L) + C'i_L^{\frac{1}{2} + \epsilon} \leq \sqrt{2L} \leq f_{x,0}(i;L) + C'i_L^{\frac{1}{2} + \epsilon} \right\} \cap \{ \mathcal{L}_L = 0 \}. \]

By [2, Lemma 3.1 c)], \( \mathbb{Q}_{1}^{x^2} [A \cap B] \) is equal to
\[ \mathbb{Q}_{1}^{x^2} \left[ \bigcap_{i=r}^{L-r} \left\{ f_{a,0}(i;L) + C'i_L^{\frac{1}{2} + \epsilon} \leq \sqrt{2L} \leq f_{x,0}(i;L) + C'i_L^{\frac{1}{2} + \epsilon} \right\} \bigg| \mathcal{L}_i, \mathcal{L}_{i+1} \right]. \]

Fix \( i \in \{ r, \ldots, L - 1 \} \). Since \( a \leq \eta L \), we have
\[ f_{a,0}(i + 1;L) \geq f_{a,0}(i;L) - \eta, \quad f_{x,0}(i + 1;L) \leq f_{x,0}(i;L). \]

By a simple calculation, we have
\[ (i + 1)^{\frac{1}{2} + \epsilon} \leq 2^{\frac{1}{2} + \epsilon} i_L^{\frac{1}{2} + \epsilon}, \quad (i + 1)^{-\frac{1}{2} - \epsilon} \geq 2^{-\frac{1}{2} - \epsilon} i_L^{\frac{1}{2} - \epsilon}. \]

Recall the definitions of \( C' \) and \( C'' \) from (C.2). By (C.1) and (C.2), and the assumption \( C't^{\frac{1}{2} - \epsilon} > \eta \), under the event \( B \), we have
\[ f_{a,0}(i;L) + 2Ci_L^{\frac{1}{2} - \epsilon} \leq (4\mathcal{L}_i\mathcal{L}_{i+1})^{\frac{1}{2}} \leq f_{x,0}(i;L) + C'i_L^{\frac{1}{2} + \epsilon}. \]
By \((C.6)\) and \([2, \text{Lemma 3.4, c)}\), under the event \(B\), the big product in \((C.3)\) is bounded from below by a positive constant (not depending on \(\eta\)). Thus, we have

\[
Q_1^{x_2} A \geq Q_1^{x_2} A \cap B \geq c \cdot Q_1^{x_2} B = c \cdot Q_1^{x_2} B|\mathcal{L}_L = 0 \cdot Q_1^{x_2} |\mathcal{L}_L = 0. \quad (C.7)
\]

By \([2, \text{Lemma 3.1, e)}\), \(Q_1^{x_2} B|\mathcal{L}_L = 0\) is bounded from below by

\[
P_{x \to 0} \left[ \left\{ f_{a,0}(s; L) + C'(s_L)^{\frac{1}{2} - \varepsilon} \leq Y_s \leq f_{x,0}(s; L) + C''(s_L)^{\frac{1}{2} + \varepsilon}, \forall s \in [r, L - r] \right\} \right.
\]

\[
\left. \cap \left\{ Y_{s'} \geq f_{a,0}(s'; L) - r^{\frac{3}{2} + 2\varepsilon}, \forall s' \in [0, r] \right\} \right], \quad (C.8)
\]

where \(P_{x \to 0} [ \cdot ] := P_Y [ \cdot | Y_L = 0]\). By an argument similar to the proof of \([1, \text{Lemma 7.6)}\), \((C.8)\) is bounded from below by

\[
c' \frac{r}{L - 2r}, \quad (C.9)
\]

where \(c' > 0\) is independent of \(\eta\). By \((C.7), (C.9)\), and \(Q_1^{x_2} |\mathcal{L}_L = 0 = (1 - \frac{1}{L})^{\frac{3}{2}}\), we have the desired result. \(\square\)

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**References**

[1] Belius, D., Kistler, N.: The subleading order of two dimensional cover times. *Probab. Theory Relat. Fields.* **167** 461-552 (2017).

[2] Belius, D., Rosen, J., Zeitouni, O.: Barrier estimates for a critical Galton-Watson process and the cover time of the binary tree. *Ann. Inst. Henri Poincaré.* **55** 127-154 (2019).

[3] Belius, D., Rosen, J., Zeitouni, O.: Tightness for the cover time of the two dimensional sphere. *Probab. Theory Relat. Fields.* **https://doi.org/10.1007/s00440-019-00940-2** (2019)

[4] Bramson, M.: Convergence of solutions of the Kolmogorov equation to traveling waves. *Mem. Amer. Math. Soc.* **44** (1983)
[5] Comets, F., Gallesco, C., Popov, S., Vachkovskaia, M.: On large deviations for the cover time of two-dimensional torus. *Electron. J. Probab.* **18** no. 96 (2013).

[6] Comets, F., Popov, S., Vachkovskaia, M.: Two-dimensional random interlacements and late points for random walks. *Commun. Math. Phys.* **343** 129-164 (2016).

[7] Comets, F., Popov, S.: The vacant set of two-dimensional critical random interlacement is infinite. *Ann. Probab.* **45** 4752-4785 (2017).

[8] Dembo, A., Peres, Y., Rosen, J., Zeitouni, O.: Cover times for Brownian motion and random walks in two dimensions. *Ann. Math.* **160** 433-464 (2004).

[9] Dembo, A., Peres, Y., Rosen, J., Zeitouni, O.: Late points for random walks in two dimensions. *Ann. Probab.* **34** 219-263 (2006).

[10] Ding, J.: On cover times for 2D lattices. *Electron. J. Probab.* **17** no. 45 (2012).

[11] Ding, J., Lee, J. R., Peres, Y.: Cover times, blanket times, and majorizing measures. *Ann. of Math.* **175** 1409-1471 (2012).

[12] Fitzsimmons, P. J., Pitman, J.: Kac's moment formula and the Feynman-Kac formula for additive functionals of a Markov process. *Stochastic Process. Appl.*, **79** 117-134 (1999).

[13] Lawler, G.: Intersections of random walks. Birkhäuser, Boston (1991).

[14] Levin, D. A., and Peres, Y.: Markov Chains and Mixing Times, Second Edition with contributions by Elizabeth L. Wilmer with an Appendix written by James G. Propp and David B. Wilson. American Mathematical Soc, RI (2017).

[15] Lyons, R., Peres, Y.: Probability on Trees and Networks. Cambridge Series in Statistical and Probabilistic Mathematics, 42. Cambridge University Press, New York (2016) Available at [http://pages.iu.edu/~rdlyons/](http://pages.iu.edu/~rdlyons/)

[16] Okada, I.: Geometric structures of late points of a two-dimensional simple random walk. *Ann. Probab.* **47**, no. 5, 2869–2893 (2019).

[17] Rodriguez, P.-F.: On pinned fields, interlacements, and random walk on $(\mathbb{Z}/N\mathbb{Z})^2$. *Probab. Theory Relat. Fields.* **173**, no. 3-4, 1265–1299 (2019).

[18] Ueno, T.: On recurrent Markov processes. *Kōdai Math. Sem. Rep.* **12** 109-142 (1960).

[19] Zhai, A.: Exponential concentration of cover times. *Electron. J. Probab.* **23** no. 32 (2018)