Topological properties of multilayer magnon insulators

Stephen Hofer,1 Trinjanan Datta,2,* Sumanta Tewari,3 and Dipanjan Mazumdar1,†

1Physics Department, 1245 Lincoln Drive, Southern Illinois University, Carbondale, IL 62901, USA
2Department of Chemistry and Physics, Augusta University, 1120 15th Street, Augusta, Georgia 30912, USA
3Department of Physics and Astronomy, Clemson University, Clemson, South Carolina 29634, USA

(Dated: August 19, 2021)

Two-dimensional magnetic insulators can be promising hosts for topological magnons. In this study, we show that ABC-stacked honeycomb lattice multilayers with alternating Dzyaloshinskii-Moriya interaction (DMI) reveal a rich topological magnon phase diagram. Based on our bandstructure and Berry curvature calculations, we demonstrate jumps in the thermal Hall behavior that corroborate with topological phase transitions triggered by adjusting the DMI and interlayer coupling. We connect the phase diagram of generic multilayers to a bilayer and a trilayer system. We find an even-odd effect amongst the multilayers where the even layers show no jump in thermal Hall conductivity, but the odd layers do. We also observe the presence of topological proximity effect in our trilayer. Our results offer new schemes to manipulate Chern numbers and their measurable effects in topological magnonic systems.

I. INTRODUCTION

The discovery of two-dimensional magnetic crystals in the past few years [1–6] has raised the prospect of realizing topologically protected magnons (spin-wave excitations) [7, 8]. Since topological materials exhibit robustness against disorder [9–11], compared to their electronic counterpart, the existence of topologically protected magnonic edge states can potentially lead to the realization of much lower power consumption spintronic devices [12–21] and applications in quantum information science [22]. Recently, it has been theoretically predicted [20, 23, 24] and experimentally demonstrated [21, 25] that it is possible to harbor topological magnon edge states in realistic geometrically frustrated magnets. At present various materials have the potential to host topological magnonic states [20, 21, 24, 26–28], including the honeycomb magnetic halide CrI2 [26], spin-1/2 Heisenberg antiferromagnets Na3Cu2SbO6 [27] and β-Cu2V2O7 [28]. In addition to the honeycomb lattice, topological magnon excitations have been proposed to exist in the kagomé magnet system Cu3(1–3, bdc) [29] and the square lattice geometry [30]. Topological phase transition induced by magnetic proximity effect in CrI3/SnI3/CrI3 trilayer has been proposed [31]. Einstein-de Haas effect of topological magnons has also been predicted [32].

A topological magnon insulator (TMI) is the bosonic analog of the quantum spin Hall state [33–35]. This phase is fundamentally different from topological magnetic insulators wherein topological electronic insulators are doped with magnetic 3d atoms [36]. The topological origins of the bosonic TMI phase can be traced to spin-orbit coupling interaction which manifests itself in the form of Dzyaloshinskii-Moriya interaction (DMI) [25] and/or pseudodipolar interaction [37–40]. Typically, the later interaction occurs in compounds with heavy ions, a class of material which is beyond the scope of our current investigation [8].

The experimental realization of monolayer, bilayer and few-layer CrI3 with tunable magnetic properties [1, 41–44] provides materials science motivation to pursue a study of few-layer coupled bosonic topological magnon system. It has been shown that protected magnon states in the AB-stacked bilayer honeycomb propagate in the same (opposite) direction for ferromagnetically (antiferromagnetically) coupled layers [45]. Furthermore, Andreas et al. [12] demonstrated through numerical calculations that the edge currents are robust against disorder compared to the bulk current in normal metal/TMI/normal metal heterostructure.

We investigate the thermal transport properties of ferromagnetically coupled TMI multilayers with different DMI strength in adjacent layers, as shown in Fig. 1. Such topologically distinct layers lead to the possibility of observing several TMI phases. The presence of DMI interaction in a magnetic system without inversion center will create band gaps in the magnon dispersion relation [20] and impart non-trivial topological nature to the system. The topological texture of these bands give rise to a non-vanishing Berry curvature. The physical consequence is a nonzero topological invariant (Chern number and winding number) that directly influences thermal Hall conductivity [20, 46]. The emergence of TMI phases are characterized by jumps in the thermal Hall conductance that are analogous to the electrical Hall conductance jumps in Quantum Hall systems.

Using spin wave theory we compute the topological band structure and its edge states, Chern number, and transverse thermal hall conductance $\kappa_{xy}$ behavior. We show that the multilayer supports a rich phase diagram which can be explored by tuning the strength of the intermediate layer’s DMI ($D_2$ in Fig. 1) relative to the top and the bottom layers or by adjusting the interlayer interaction strength $J_z$. We investigate and discuss the variation in thermal Hall conductance with changing interlayer DMI strength ratio $D_2/D_1$ and for different interlayer coupling relative to the DMI interaction $J_z/D_1$. Furthermore, we show that the physics of few-layered topological multilayer has its own characteristic transport properties. The presence of an uncompensated topological layer in odd layered configuration leads to non-trivial behavior in the thermal Hall conductance behavior. As a result, we show that there is

* Corresponding author:tdatta@augusta.edu
† Corresponding author:dmazumdar@siu.edu
an odd-even layering effect which manifests itself as a jump or not in the transverse thermal Hall conductance behavior. Additionally, the trilayer exhibits a topological proximity effect which can be induced by external pressure. Overall, we put forward the design and characterization of a finite number of layered topological magnon insulator systems (odd or even) with several interesting effects directly related to the topology of the system.

This article is organized as follows. In Sec. II we present the model and the method. In Sec. III we perform the topological characterization of our multilayer. In Sec. IV we present our thermal Hall response results of the multilayer system. Finally, in Sec. V we present our conclusions.

II. MODEL AND METHOD

We analyze an ABC-stacked multilayer honeycomb lattice which is consistent with the low temperature (rhombohedral) experimental structure of CrI$_3$ [6, 47]. To connect with conventional experimental sandwich structures, the DMI strength alternates between two values (for example, $D_1 = D_3$ and $D_2$ in Fig. 1). The individual layers are chosen to be ferromagnetic, and the trilayer transition to the ferromagnetic state can be induced using external pressure [42]. While experimental evidence suggests that even layered CrI$_3$ shows a net antiferromagnetic configuration in the ground state [42], it has also been demonstrated that transition to the ferromagnetic state can be induced using external pressure [43, 44]. Later, we will show that the trilayer forms the basic building block for all odd layered configurations (five, seven, etc) while the bilayer is the basic building block of all even layered structures (four, six, etc).

We model our two dimensional multi-layer FM topological insulator using the Heisenberg exchange term $H_{FM}$ and the DMI term $H_{DMI}$. To model our few layer system we add an interlayer interaction $H_{int}$ to stack the monolayers, as seen in Fig. 1a. The generic multilayer Hamiltonian can be written as

$$H = H_{FM} + H_{DMI} + H_{int},$$

where the individual terms are given by the following expressions

$$H_{FM} = -\sum_{\langle \alpha, \beta \rangle} J_{\tau} S_{\tau,\alpha} \cdot S_{\tau,\beta},$$

$$H_{DMI} = \sum_{\langle \alpha, \beta \rangle} D_{\tau} \hat{\text{z}} \cdot (S_{\tau,\alpha} \times S_{\tau,\beta}),$$

$$H_{int} = -\sum_{\langle \alpha, \beta \rangle} J_{\tau}^{\tau+1} S_{\tau+1,\alpha} \cdot S_{\tau,\beta}.$$

The trilayer bulk and edge configuration bandstructure is shown in Figs. 1b and 1c. The TMI bandstructure with edge states has different energies. No new band crossings are observable as a result of this interaction, so it is omitted to simplify the model.

Next, we apply linear spin wave theory transformation to Eqs. (2) and Fourier transform the Hamiltonian. Thus, the momentum space Hamiltonian can be written as $H = \sum_k \Psi_k^\dagger \mathcal{H}(k) \Psi_k$, with the basis vector $\Psi_k^\dagger = (b_{A,1,k}^\dagger, b_{B,1,k}^\dagger, \ldots)$.
FIG. 1. Trilayer configuration, bandstructure, and edge states. a Lattice crystal structure with ferromagnetic spin ordering. Spin sites A(B) are denoted with red(blue) spheres. $J$ denotes intralayer nearest-neighbor ferromagnetic Heisenberg exchange interaction. $D_i$ denotes the layer specific next-nearest neighbor DMI, where $\tau \in (1,2,3)$ indexes the layer. Although we show distinct DMI interactions for each layer, for our calculations we will take $D_1 = D_3$ (reason explained in main text). $J_z$ denotes the interlayer Heisenberg exchange interaction. b–c Bulk bands (solid dashed lines) with edge states (thin blue lines) for the trilayer plotted along $k_y = 0$. The parameters are $J = 2J_z = 4D_1 = 4D_3$, $D_2/D_1 = -0.426$, and $D_3/D_1 = 0.34$, respectively.

$\pm K = (\pm 4\pi/3 \sqrt{3}, 0)$ in the Brillouin zone. At this momentum point, the nearest-neighbor structure factor $f(k)$ becomes zero. This eliminates the contribution of our strongest interaction $J$. Thus, the energy scale of the problem is governed by $D_1$, leaving $D_2$ and $J_z$ as the tuning parameters by which we can explore the various topological phases of our system.

We define a multilayer tuning ratio $\delta = D_2/D_1$. This will serve as a control knob to study TPTs. As we show later the $\delta = 1$ configuration is of particular interest because of its feasibility to be naturally realized in an experimental setup. The Chern numbers are rearranged at a TPT. Since, the interband edge states are a consequence of these Chern numbers, a change in them implies that the number of edge states will alter across a transition. This is clearly visible in Figs. 1b and 1c. For example, the number of interband edge states in Fig. 1b is one, while in Fig. 1c the number is three. The main physical property that emerges from the TMI phase is the existence of these chiral magnonic edge states which contribute to the non-vanishing thermal Hall conductivity [25, 45, 46, 48–50]. In the next section, we will study the nature of these TPTs in more detail.

### III. TOPOLOGICAL CHARACTERIZATION

The trilayer topological phase diagram is shown in Fig. 2. For convenience, the phases are color coded so that we can compare the two panels (a) and (b). The result depends on two interaction ratios, one is $\delta$ and the other $J_z/D_1$. The feasibility of tuning $J_z$ using pressure has already been experimentally demonstrated in a hexagonal lattice system [43, 44]. Based on our studies, we show that there can be further motivation to tune the DMI interactions, too. For suitable parameter ranges we observe a quantum Hall behavior in our proposed bosonic system. To track the TPTs we compute the energy eigenvalues at $\pm K$ high symmetry points. The analytical expression for the energy eigenvalues calculated at $+K$ yields

$$E^{(3)}_{\eta} = \begin{cases} 
\frac{\sqrt{2 J_z} + (-1)^{\eta+1} \sqrt{3} D_1}{2}, & \eta = 1, 2 \\
\frac{\sqrt{2} J_z + \sqrt{2} D_2 + (-1)^{\eta+1} \Delta}{2} + \frac{2 \sqrt{3}}{2 D_1 (1 - \delta)}, & \eta = 3, 4 \\
\frac{\sqrt{2} J_z + \sqrt{2} D_2 + (-1)^{\eta+1} \Delta - \frac{3 \sqrt{3}}{2 D_1 (1 - \delta)}}{2}, & \eta = 5, 6
\end{cases}$$

where $E^{(3)}_{\eta}$ are the trilayer eigenvalues and we have defined $2\Delta = \sqrt{2 J_z^2 + 27D_1^2 (1 + \delta)^2}$. Eigenvalues for the system solved at $-K$ result in the same solutions as above, except with a sign change which relabels $\eta = 3, 4$ to $\eta = 5, 6$ and vice versa. Just for comparison purposes, we list the energy solutions for the bilayer problem in Appendix A, see Table II. Next, the TPTs are obtained from the real solutions of $E^{(3)}_{i} = E^{(3)}_{j}$ with $i \neq j$ using the above expressions. The topological phase boundaries can be defined as

$$\delta_n = \begin{cases} 
\frac{6 \sqrt{3} \eta + \sqrt{2}}{6 \sqrt{3} \eta + \sqrt{2}}, & n = 5 \text{ where } \delta_5 \in (1, 3) \\
1, & n = 4 \\
\frac{6 \sqrt{3} \eta + \sqrt{2}}{6 \sqrt{3} \eta + \sqrt{2}}, & n = 3 \text{ where } \delta_3 \not\in (1, 3) \\
\frac{-J_z}{27D_1}, & n = 2 \text{ where } \delta_2 \in (-\infty, 0) \\
-1, & n = 1
\end{cases}$$

The number $n$ signifies the boundaries of the different phases. In the limit of zero interlayer interaction we can set $J_z = 0$. In
this case there are three phase boundaries separated by $\delta_n = -1, 0, \text{and } 1$.

In Fig. 2 we plot the six different phases based on the above solutions. The phase diagram depends on the ratio of $D_2/D_1$ (which can be positive or negative) versus $J_z/D_1$ variation. When $\delta$ is positive the DM interactions are aligned in the same direction. In this regime of the tuning parameters we find four phases (marked as 3 – 6 in the phase diagram). Whereas, when $\delta$ is negative, there are three phases (marked as 1 – 3 in the phase diagram). Furthermore, around the $\delta = 0$ line (FM middle layer) an interesting behaviour happens. This phase boundary between 2 and 3 varies as $J_z/D_1$. Hence, When $J_z < D_1$ (weak) the middle layer retains its non-topological behavior because the phase boundary mildly deviate from the $\delta = 0$ line. However, for $J_z > D_1$ (strong) the $D_2$ deviates from zero to acquire a non-zero value. Thus, the FM layer starts to obtain a topological nature. We interpret this to be a signature of topological proximity effect displayed by the multilayer which can be experimentally realized by applying pressure [43, 44]. For positive $\delta$ and for high $J_z > D_1$ we find that there are multiple phases into which the trilayer can transition into. These phases can be classified based on Chern numbers as we describe next, which are calculated from the Berry curvature in the following ways. For the Berry curvature calculation, we employ the following equation

$$\Omega^{\text{B}}_n(k) = -2 \sum_{m \neq n} \text{Im} \left( \frac{\langle n | \partial H(k) | m \rangle \langle m | \partial H(k) | n \rangle}{[E_n(k) - E_m(k)]^2} \right), \quad (8)$$

obtained from standard perturbation theory approach [10].

The Berry curvature calculation will be used later to compute the thermal Hall conductance. The Chern number is then calculated as

$$C_n = \frac{1}{2\pi} \int_{BZ} \Omega^{\text{B}}_n(k) dk_x dk_y. \quad (9)$$

In Fig. 2b we show the variation of the energy eigenvalues for $\delta$ at $J_z/D_1 = 2$ (shown as a dashed vertical line). This ratio choice is motivated by CrI$_3$ experimental parameters reported in Ref. [26], where $J_z/D_1 \approx 2$. We notice that the energy eigenvalues interchange indicating the presence of potential TPTs verified by the reordering of Chern numbers. The values for the Chern numbers given in Table I can be generated by ordering the Chern numbers of each eigenvalue from the lowest to highest energy within each shaded phase. In our multilayer system there are contributions from several underlying bulk bands which can support topologically protected edge states. The Chern numbers determine the character of these edge states based on the winding number, defined as the partial sum $\nu_i = \sum_1^n C_i$. The winding number determines the number and chirality of the edge states which lie between the $i$th and $i + 1$th bulk band. These states (as mentioned earlier) lie above the zero of energy.

If we adopt a fermionic classification scheme for the trilayer, then based on the winding number calculation, $\nu_3 = C_1 + C_2 + C_3$ we should have only two phases. The first two phases will have a winding number in the large gap between the lower and upper grouping of bands of $\nu_3 = 1$. The last four will have $\nu_3 = 3$ as documented in Table I. The band crossing just below $\delta = 0$ also accounts for the winding number.
\[ \sum_{n=1}^{2L} y_n = \sum_{n=1}^{2L} (2L - n)C_n, \]

which are reported in the fourth column of Table I for the trilayer.

To illustrate this concept, we provide an example of the determination of \( \bar{\nu} \) for phase 1 of the trilayer. First, to determine the total number and chirality of the edge states in this phase, we calculate the winding numbers using the Chern numbers from Table I, given as:

- \( \nu_1 = C_1 = -1 \)
- \( \nu_2 = C_1 + C_2 = \nu_1 + 3 = +2 \)
- \( \nu_3 = C_1 + C_2 + C_3 = \nu_2 - 1 = +1 \)
- \( \nu_4 = C_1 + C_2 + C_3 + C_4 = \nu_3 - 3 = -2 \)
- \( \nu_5 = C_1 + C_2 + C_3 + C_4 + C_5 = \nu_4 + 1 = -1 \)
- \( \nu_6 = 0. \)

These numbers represent the number and chirality of the edge states that lie between each consecutive bulk band, with the knowledge that for all systems the final winding number is always zero. Therefore, by summing these numbers together, we get an idea about the net contribution of all the edge states present in that particular topological phase. For phase 1, this summation gives \( \bar{\nu} = -1 + 2 + 1 - 2 - 1 = -1 \), in agreement with the value reported in the table. This process is repeated for each phase as the Chern numbers are rearranged. A comparison of each phase’s net number of edge states has been done to understand the discontinuous behaviors which may appear as a result of the TPT.

IV. THERMAL HALL EFFECT

Thermal Hall conductance is a useful response function to accurately characterize the topological nature of 2D magnonic materials [23, 46, 49] and it given by

\[ \kappa_{xy} = -\frac{k_B^2 T}{(2\pi)^2 \hbar} \int_{BZ} (\ln \frac{\rho}{\rho_{c2}})^2 - (\ln \rho)^2 - 2Li_2(-\rho), \]

with \( c_2(\rho) = (1 + \rho) \left( \ln \frac{1 + \rho}{\rho} \right)^2 - (\ln \rho)^2 - 2Li_2(-\rho) \), where \( k_B \) is the Boltzmann constant, \( h \) is the Planck’s constant, \( T \) is the temperature, \( \rho \) is the Bose-Einstein distribution, and \( Li_2(\rho) \) is the polylogarithm function. We notice that the magnitude of the conductance is governed by both the weight function \( c_2(\rho) \), where \( \rho \) is the Bose-Einstein distribution, and the Berry curvature as calculated in Eq. 8. While the Berry curvature is primarily a function of the variables \( J_z/D_1 \) and \( D_2/D_1 \), \( c_2(\rho) \) is a function of temperature \( T \). Fig. 2a shows the parameter values of \( J_z/D_1 \) and \( D_2/D_1 \) over which we explore the topological properties of the multilayer. Thus, we can ask the question.
at what value of the temperature should the conductance be evaluated such that the non-trivial (if any) nature of the TPTs may be accurately captured? Because each band in our energy spectrum has a non-vanishing Chern number, and therefore a non-trivial Berry curvature, for every phase considered in our parameter space we would like to ensure that $\nu = 0$ as a function of $J$ and $D$ does little to effect this. In Fig. 4 we show how the conductance varies as the system evolves through its topological phases. These TPTs can be explored by either tuning $J_\perp/D_1$ or $D_2/D_1$. First, we plot phase changes as a function of $J_\perp/D_1$ in Figs. 4a – Fig. 4c. Each TPT is associated with a jump in the conductance, reminiscent of the Quantum Anomalous Hall effect present in electronic systems [51]. The relative increase or decrease in magnitude of the conductance due to these jumps can be explained by considering the number of edge states available on either side of the transition. Generally, more edge states yield a higher magnitude of the conductance, while fewer edge states result in a lower contribution to the magnitude. This is particularly observable in Fig. 4b, where the conductance shows a sharp decrease in magnitude. This can be explained by the difference in $\bar{v}$ between phases three and four, as shown in Table I. On the left side of the transition, phase 3 hosts nine different edge states, while on the right side of the transition phase 4 hosts seven, thus a difference of two. This decrease in available edge states coincides with the decrease in magnitude of the conductance, as fewer edge states are available to transport thermal energy. Furthermore intuitively, we can conclude that the conductance is sensitive to both the Berry curvature, from which $\bar{v}$ is derived by way of the Chern numbers, as well as the energy spectrum simultaneously. Thus, the exact value of the jump will depend on the rearrangement of the energy spectrum of the bands across the phase transition in addition to the change in the Chern numbers.

In Fig. 4d we plot the conductance as the system passes through a multitude of TPTs by varying $D_2/D_1$ for the bilayer and the trilayer. The general trend is that the magnitude of the conductance increases as $D_2$ increases. For both layers the jumps can be characterized by $\bar{v}$, as done before. The relative increase or decrease in magnitude of the conductance at each TPT coincides directly with the relative increase or decrease of the number of edge states within each phase. To compare the results of the bilayer to the trilayer, we divided the conductance of each by the number of layers present in the system to determine the per layer contribution to the conductance. The number of jumps for the bilayer is different compared to the trilayer. We can attribute this fact to the differences in the topological phase diagram of the two systems. As shown in Fig. 3, the bilayer displays seven TPTs while the trilayer has only five. This is a consequence of the symmetries imposed upon the system by the choice of the stacking arrangement, explained below. In particular, we observe that the bilayer shows no TPT at the isotropic $\delta = 1$ point, but the trilayer does. By exploring this particular value of $\delta$ for different layering numbers $L$, we were able to determine the general nature of the TPT in even and odd layered structures with regards to the thermal conductance.
layer. It is clear that the even numbered layers show no jump in the conductance, while the odd numbered layers do. To highlight the sharpness of the jump across the TPT, for the odd layers, we chose a denser set of points near $\delta = 1$. From Figs. 5(b), 5(d), and 5(f) it is clear that the jump gets sharper as $L$ increases. Thus, within the limit of a few-odd layered systems, this effect is real and will survive. For even layers beyond $L = 2$, band gap closings do occur for the same values of $\delta$ as the odd layers. However, these gap closings do not correspond to TPTs.

The generalization of these results from the bi- and trilayer to any layer can be shown by inspecting the analytically solved eigenvalues at $\pm K$. For $L$ layers, the Hamiltonian at $\pm K$ can be reduced to $L + 1$ independent subspaces containing two $1 \times 1$ subspaces and $L - 1 2 \times 2$ subspaces, which can be solved for their eigenvalues. Since the layers within each multilayer are structured such that their DMI strength alternates between the values of $D_1$ and $D_2$, the $2 \times 2$ subspaces will also repeat according to this pattern. Thus, beyond $L = 3$ no unique subspaces occur, and subsequently no unique eigenvalues will be found. Therefore, the only distinguishable feature between multilayers will be the solution of the $(L+1)^{th}$ subspace, a $1 \times 1$ subspace which depends on the DMI value of the $L^{th}$ layer, $D_1$ (odd $L$) or $D_2$ (even $L$). By this reasoning we can categorize every multilayer by its even- or odd-ness. This generalization is shown in more detail in Appendix A.

The even-odd effect displayed in Figs. 4d and 5 is a result of the intrinsic spin orientation and the stacking direction which is imposed upon the structure by the choice of the stacking arrangement. The cartoon picture of bi- and tri-layer arrangement shown in Fig. 6 demonstrates this principle. In the case of the odd-layered configuration the presence of a mirror symmetry imposed by the stacking arrangement works to preserve the invariance of the system under a time-reversal (TR) operation, which flips the spin orientations, as well as exchanging $+K$ and $-K$ in the Brillouin zone. This ensures that any gap closings must happen at $+K$ and $-K$ simultaneously. In contrast, even-layered configurations do not display this mirror symmetry, and therefore their solutions will not necessarily be TR-invariant. Therefore the even-layered configurations will host a higher number of topological band crossings as the crossings at $+K$ and $-K$ must be considered separately.

Finally, we note that recent Raman results suggest that room temperature mechanically exfoliated few-layer samples of CrI$_3$ retain their monoclinic structure even beyond the rhombohedral structural transition associated with the bulk material [52]. In rhombohedral stacking each layer is associated with a shift of $a$ (the unit cell length) in the y-direction. In monoclinic stacking, the layers are shifted by $a/3$ in the x-direction. In Fig. 5a and Fig. 5b we show the results of the $\kappa^{xy}$

---

**FIG. 5.** Thermal Hall conductance near $\delta = 1$ for $L=2$ to $L=7$ multilayers. a - f Plots (a) - (f) are labeled by their value of $L$ as shown in the plots. Even layered systems (top row) show no jump in the conductance, in contrast to odd layered structures (bottom row) which show a clear jump. Red circles indicate the rhombohedral stacking variation, while blue squares show the monoclinic dependence.
for the monoclinic bi- and trilayer. In comparison to the rhombohedral case, neither multilayer shows a jump at $D_1/D_2 = 1$. This means that the odd-layered rhombohedral multilayers are the only configurations which show a jump at this parameter value. This indicates that the jump behavior is not universal.

![Diagram](image)

**FIG. 6.** Bilayer (a) and its time-reversed (TR) partner (b), along with the Trilayer (c) and its TR partner (d). Each layer is labeled and colored by its Dzyaloshinskii-Moriya interaction term. The arrows within each layer denote the orientation of the spins. The time-reversal operation $T$ flips the direction of the spin to produce a TR heterostructure. The stacking direction is denoted by the large vertical black arrow to the left or right of the heterostructure.

### V. CONCLUSION

In summary, we have studied the topological response of a multilayer configuration of hexagonal lattices stacked on top of each other in an ABC arrangement. Our calculation encompasses two different multilayer scenarios as characterized by their DMI interactions, only. While in general there may also be a pseudo-dipolar interaction term in the Hamiltonian, this term is not important for the class of systems considered here, and is therefore left for future study. We can have an all TMI system or another in which there is a combination of TMI-FM-TMI layers. For each of these setup, the observed topological phase transitions (manifested as jumps in the thermal Hall behavior) can be tuned by changing either the interlayer exchange interaction or DMI parameters. In an all TMI configuration the trilayer displays a jump in its thermal Hall conductance, while the bilayer does not. This even-odd jump response holds true beyond the bi- and trilayer. Thus, we propose a topological asymmetry layer experiment (TALE). By performing TALE one can decide whether an asymmetric (even layered) or symmetric (odd layered) has been fabricated during the layering process. Such an experiment could potentially offer device fabrication physicists an additional means to characterize few-layered topological multilayer systems, besides the standard available methods [4]. We observe several topological phase transitions which are experimentally feasible since $J_z$ could be tuned ex-situ through various methods [43, 44] allowing a continuous measurement through the TPT. The possibility to observe topological proximity effect and the presence of jumps distinguishing odd and even layers makes few-layered bosonic topological magnon systems an exciting playground to verify and apply fundamental concepts.

Authors’ note: During the writing of this article the authors become aware of a similar work wherein multilayers of dissimilar DMI were shown to host novel topological states in the form of chiral hinge magnons [53].

### ACKNOWLEDGMENTS

S. H and D.M would like to acknowledge funding from the NSF CAREER grant (ECCS, Award No.1846829) for support of this work. T. D. acknowledges funding support from Sun Yat-Sen University Grant Nos. OEMT-2017-KF-06 and OEMT-2019-KF-04. S. T. acknowledges support from ARO Grant No. W911NF-16-1-0182. T. D. thanks Jun Li and D. X. Yao for several helpful discussions.

### Appendix A: Topological Characterization

Determination of TPTs is done by analyzing band crossings at the high-symmetry points $\pm K = \left(\pm \frac{4\pi}{3\sqrt{3}}, 0\right)$ in the Brillouin zone. By using the facts that $f(\pm K) = 0$ and $m(\pm K) = \mp 3\sqrt{3}$, we can simplify the Hamiltonian and determine the eigenvalues analytically. In this case, Eq. 4 becomes,

$$\mathcal{A}_T(\pm K) = \begin{pmatrix} \Theta_{1}^T & 3\sqrt{3}D_1S & 0 \\ 3\sqrt{3}D_1S & \Theta_{2}^T & 3\sqrt{3}D_2S \end{pmatrix},$$

and Eq. 5 remains unchanged. The general form of the Hamiltonian evaluated at $\mathbf{k} = \pm K$ can be written as,

$$\mathcal{H}_T(\pm K) = \begin{pmatrix} h_1^{(L)} \\ \vdots \\ h_{L}^{(L)} \\ \vdots \\ h_{L+1}^{(L)} \end{pmatrix},$$

resulting in $L + 1$ subspaces. The Hamiltonian can be reduced to two $1 \times 1$ ($h_1^{(L)}$ and $h_{L+1}^{(L)}$ in the above equation) and $L - 1 \times 2 \times 2$ subspaces. Due to the alternating nature of our multilayers, the subspaces will likewise alternate resulting in the following general forms,

$$h_1^{(L)} = 3JS \mp 3\sqrt{3}D_1S, \quad h_{2j}^{(L)} = \begin{pmatrix} 3JS + J_zS \mp 3\sqrt{3}D_1S & -J_zS \\ -J_zS & 3JS + J_zS \mp 3\sqrt{3}D_2S \end{pmatrix}, \quad j = 1, 2 \ldots, L-1,$$

$$h_{2j+1}^{(L)} = \begin{pmatrix} 3JS + J_zS \mp 3\sqrt{3}D_2S & -J_zS \\ -J_zS & 3JS + J_zS \mp 3\sqrt{3}D_1S \end{pmatrix}, \quad h_{L+1}^{(L)} = 3JS \pm 3\sqrt{3}D_2S.$$
where the $2 \times 2$ subspaces are determined by the value of $\eta$ being even or odd. Eigenvalues for the bi- and trilayer systems are reported in Tables II and III. Due to the repetition of the subspaces, no eigenvalues at $\pm K$ beyond the $L = 3$ system are found which are unique. Therefore, the main difference between even and odd layered multilayers is determined by the value of $D_L$, ($D_1$ for odd $L$ and $D_2$ for even $L$), which decides the eigenvalue for the $h_3^{(L)}$ subspace. In Fig. 7 we report the eigenvalues for the 4 and 5 layer configurations near the isotropic point $D_2/D_1 = 1$.

### Appendix B: Thermal Hall Weight Function

The transport properties of our heterostructure was characterized by the thermal Hall conductance [49]

$$k_{xy} = -\frac{k_B^2 T}{(2\pi \hbar)^2} \sum_n \int_{BZ} c_2(\rho) \Omega_n^{xy}(\mathbf{k}) dk_x dk_y,$$

where $c_2(\rho) = (1 + \rho)(\ln \frac{1 + \rho}{1 - \rho})^2 - (\ln \rho)^2 - 2Lj_2(-\rho)$, where $k_B$ is the Boltzmann constant, $\hbar$ is the Planck’s constant, $T$ is the temperature, $n$ indexes the bands, $\rho$ is the Bose-Einstein distribution, and $Lj_2(\rho)$ is the polylogarithm function. The weight function $c_2(\rho)$ favors low lying energy bands at low temperatures, while some of the topological phase boundaries in our system are defined by band crossings which occur at the higher end of our energy spectrum. In order to characterize the topological phase diagram using the thermal conductance we must ensure the contribution of each band in the energy spectrum. Therefore, we opt to employ the high temperature limit $k_{xy}^{\lim}$, we write the above equation as

$$k_{xy}^{\lim} = \lim_{T \to \infty} k_{xy} = \lim_{T \to \infty} -\frac{k_B^2 T}{(2\pi \hbar)^2} \sum_n \int_{BZ} c_2(\rho) \Omega_n^{xy}(\mathbf{k}) dk_x dk_y / T.$$

(8)

Since $\lim_{\tau \to \infty} c_2(\rho) = \frac{2}{\tau^2}$ and $C_n = \frac{1}{\pi} \int_{BZ} \Omega_n^{xy}(\mathbf{k}) dk_x dk_y$, the numerator becomes

$$\lim_{T \to \infty} \frac{k_B^2}{(2\pi \hbar)^2} \sum_n \int_{BZ} c_2(\rho) \Omega_n^{xy}(\mathbf{k}) dk_x dk_y = -\frac{k_B^2}{(2\pi \hbar)^2} \frac{\pi^2}{3} \sum_n 2\pi C_n = 0$$

where we have used the fact that $\sum_n C_n = 0$. Additionally, since $\lim_{\tau \to \infty}(1/T) = 0$, we can apply l’Hôpital’s rule such that

$$\lim_{T \to \infty} k_{xy} = \lim_{\tau \to \infty} \frac{k_B^2 T^2}{(2\pi \hbar)^2} \sum_n \int_{BZ} \frac{\partial c_2(\rho)}{\partial T} \Omega_n^{xy}(\mathbf{k}) dk_x dk_y. (B4)$$

To determine the partial derivative $\frac{\partial c_2(\rho)}{\partial T}$, remember that

$$c_2(\rho) = (1 + \rho)(\ln \frac{1 + \rho}{1 - \rho})^2 - (\ln \rho)^2 - 2Lj_2(-\rho).$$

If we define

$$\beta = e^{\frac{E}{k_B T}}, \quad \rho = 1/(\beta - 1),$$

the derivative of the first term with respect to temperature becomes

$$\frac{\partial}{\partial T} (1 + \rho) \left(\ln \frac{1 + \rho}{\rho}\right)^2 = \frac{E(\mathbf{k}) \beta}{k_B T^2} [\rho^2 (\ln \beta)^2 - 2(1 + \rho)(\ln \beta)\beta^{-1}].$$

(B6)

The second term gives us,

$$\frac{\partial(- (\ln \rho)^2)}{\partial T} = -\frac{E(\mathbf{k}) \beta}{k_B T^2} (2\rho(\ln \rho)).$$

(B7)
To determine the partial derivative of the third term we use the definition $L_2(z) = \sum_{k=1}^{\infty} \frac{\rho}{\beta k}$ to obtain,

$$\frac{\partial}{\partial T} (-2L_2(-\rho)) = \frac{E(k)\beta}{k_B T^2} \left[ 2(\rho(1 + \rho)) \right].$$  \hspace{1cm} (B8)

Combining all the terms yields,

$$\frac{\partial c_2(\rho)}{\partial T} = \frac{E(k)\beta}{k_B T^2} \left[ \rho^2(\ln(\beta)^2 - 2(1 + \rho)(\ln(\beta)\beta^{-1}) \right.$$ \hspace{1cm} \(\left. - 2\rho(\ln(\rho) + 2\rho(\ln(1 + \rho))) \right] \right.$$ \hspace{1cm} \(\left. \frac{E(k)}{k_B} \right) \hspace{1cm} \text{(B9)}

Next, using the following limit expressions,

$$\lim_{T \to \infty} \beta = 1; \quad \lim_{\rho \to \infty} \rho(\ln(\rho)) = 0; \quad \lim_{\rho \to \infty} \rho(\ln(1 + \rho)) = 1.$$ \hspace{1cm} (B10)

we have

$$\lim_{T \to \infty} T^2 \frac{\partial c_2(\rho)}{\partial T} = \lim_{T \to \infty} \left\{ T^2 \frac{E(k)\beta}{k_B T^2} \left[ \rho^2(\ln(\beta)^2 \right. \right.$$ \hspace{1cm} \(\left. - 2(1 + \rho)(\ln(\beta)\beta^{-1} + 2\rho(\ln(\beta)) \right[ \right.$$ \hspace{1cm} \(\left. \frac{E(k)}{k_B} \right) \hspace{1cm} \text{(B11)}

Thus we have the final expression as

$$\kappa_{\text{lim}} = \frac{k_B}{(2\pi)^2 \hbar} \sum_{n} \int_{BZ} E_n(k)\Omega_{n}^{\alpha}(k)dk_xdk_y.$$ \hspace{1cm} (B12)

[1] B. Huang, G. Clark, E. Navarro-Moratalla, D. R. Klein, R. Cheng, K. L. Seyler, D. Zhong, E. Schmidgall, M. A. McGuire, D. H. Cobden, W. Yao, D. Xiao, P. Jarillo-Herrero, and X. Xu, Layer-dependent ferromagnetism in a van der waals crystal down to the monolayer limit, Nature 546, 270 (2017).

[2] C. Gong, L. Li, Z. Li, H. Ji, A. Stern, Y. Xia, T. Cao, W. Bao, C. Wang, Y. Wang, Z. Q. Q. nd R. J. Cava, W. Bao, C. Gong, L. Li, Z. Li, H. Ji, A. Stern, Y. Xia, T. Cao, W. Bao, C. Wang, Y. Wang, Z. Q. Q. nd R. J. Cava, S. G. Louie, K. F. Mak, J. Shan, and D. C. Ralph, Probing and controlling intrinsic ferromagnetism in two-dimensional van der waals crystals, Nature 546, 265–269 (2017).

[3] Y. Deng, Y. Yu, Y. Song, J. Zhang, N. Z. Wang, Z. Sun, Y. Yi, Y. Z. Wu, S. Wu, J. Zhu, J. Wang, X. H. Chen, and Y. Zhang, Gate-tunable room-temperature ferromagnetism in two-dimensional Fe₇GeTe₂, Nature 563, 94 (2018).

[4] K. F. Mak, J. Shan, and D. C. Ralph, Probing and controlling magnetic states in 2d layered magnetic materials, Nature Reviews Physics 1, 646–661 (2019).

[5] M. Gibertini, M. Koperski, A. F. Morpurgo, and K. S. Novoselov, Magnetic 2d materials and heterostructures, Nature Nanotechnology 14, 408–419 (2019).

[6] M. A. McGuire, Crystal and magnetic structures in layered, transition metal dihalides and trihalides, Crystals 7, 121 (2017).

[7] K. Nakata, S. K. Kim, J. Klinovaja, and D. Loss, Magnonic topological insulators in antiferromagnets, Phys. Rev. B 96, 224414 (2017).

[8] X. S. Wang, H. W. Zhang, and X. R. Wang, Topological magnonics: A paradigm for spin-wave manipulation and device design, Phys. Rev. Applied 9, 024029 (2018).

[9] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, Rev. Mod. Phys. 82, 3045 (2010).

[10] Y. Ando, Topological insulator materials, Journal of the Physical Society of Japan 82, 102001 (2013).

[11] J. E. Moore, The birth of topological insulators, Nature 464, 194 (2010).

[12] A. Ruckriegel, A. Brataas, and R. A. Duine, Bulk and edge spin transport in topological magnon insulators, Phys. Rev. B 97, 081106 (2018).

[13] A. V. Chumak, V. I. Vasyuchka, A. A. Serga, and B. Hillebrands, Magnon spintronics, Nature Physics 11, 453–461 (2015).

[14] G. E. W. Bauer, E. Saitoh, and B. J. van Wees, Spin caloritronics, Nature Materials 11, 391–399 (2012).

[15] K. Uchida, J. Xiao, H. Adachi, J. Ohe, S. Takahashi, J. Leda, T. Ota, Y. Kajiwara, H. Umezawa, H. Kawai, G. E. W. Bauer, S. Maekawa, and E. Saitoh, Spin seebeck insulator, Nature Materials 9, 894–897 (2010).

[16] L. J. Cornelissen, J. Liu, R. A. Duine, J. B. Youssef, and B. J. van Wees, Long-distance transport of magnon spin information in a magnetic insulator at room temperature, Nature Physics 11, 1022–1026 (2015).

[17] Y. Wang, D. Zhu, Y. Yang, K. Lee, R. Mishra, G. Go, S.-H. Oh, D.-H. Kim, K. Cai, E. Liu, S. D. Pollard, S. Shi, J. Lee, K. L. Teo, Y. Wu, K.-J. Lee, and H. Yang, Magnetization switching by magnon-mediated spin torque through an antiferromagnetic insulator, Science 366, 1125 (2019).

[18] C. O. Avci, A. Quindeau, C.-F. Pai, M. Mann, L. Caretta, A. S. Tang, M. C. Onbasli, C. A. Ross, and G. S. D. Beach, Current-induced switching in a magnetic insulator, Nature Materials 16, 309 (2017).

[19] B. Li and A. A. Kovalchuk, Chiral topological insulator of magnons, Phys. Rev. B 97, 174413 (2018).

[20] A. Mook, J. Henk, and I. Mertig, Edge states in topological magnon insulators, Phys. Rev. B 90, 024412 (2014).

[21] R. Chisnell, J. S. Helton, D. E. Freedman, D. K. Singh, R. I. Blevins, D. G. Nocera, and Y. S. Lee, Topological magnon bands in a kagome lattice ferromagnet, Phys. Rev. Lett. 115, 147201 (2015).

[22] P. Andrich, C. F. de las Casas, X. Liu, H. L. Bretsch, J. R. Berman, F. J. Heremans, P. F. Nealey, and D. D. Awschalom, Long-range spin wave mediated control of defect qubits in nanodiamonds, npj Quantum Information 3, Article number: 28 (2017).

[23] S. A. Owerre, Topological honeycomb magnon hall effect: A calculation of thermal hall conductivity of magnetic spin excitations, Journal of Applied Physics 120, 043903 (2016).

[24] L. Zhang, J. Ren, J.-S. Wang, and B. Li, Topological magnon insulator in insulating ferrimagnet, Phys. Rev. B 87, 144401 (2013).

[25] Y. Onose, T. Ideue, H. Katsura, Y. Shiomi, N. Nagaosa, and Y. Tokura, Observation of the magnon hall effect, Science 329, 297 (2010).
[26] L. Chen, J.-H. Chung, B. Gao, T. Chen, M. B. Stone, A. I. Kolesnikov, Q. Huang, and P. Dai, Topological spin excitations in honeycomb ferromagnet $\text{cr}_1$, Phys. Rev. X 8, 041028 (2018).

[27] Y. Miura, R. Hirai, Y. Kobayashi, and M. Sato, Spin-gap behavior of na3cu2sbo6 with distorted honeycomb structure, Journal of the Physical Society of Japan 75, 084707 (2006).

[28] A. A. Tsirlin, O. Janson, and H. Rosner, $\beta$-cu$_2$y$_2$o$_7$: A spin-$\frac{1}{2}$ honeycomb lattice system, Phys. Rev. B 82, 144416 (2010).

[29] R. Chisnell, J. S. Helton, D. E. Freedman, D. K. Singh, F. Demmel, C. Stock, D. G. Nocera, and Y. S. Lee, Magnetic transitions in the topological magnon insulator cu(1,3-bdc), Phys. Rev. B 93, 214403 (2016).

[30] M. Kawano and C. Hotta, Thermal hall effect and topological edge states in a square-lattice antiferromagnet, Phys. Rev. B 99, 054422 (2019).

[31] Y. Zeng, L. Wang, S. Li, C. He, D. Zhong, and D.-X. Yao, Topological phase transition induced by magnetic proximity effect in two dimensions, Journal of Physics: Condensed Matter 31, 395502 (2019).

[32] J. Li, T. Datta, and D.-X. Yao, Einstein-de Haas Effect of Topological Magnons, , arXiv:2005.10273 (2020).

[33] F. D. M. Haldane, Model for a quantum hall effect without landau levels: Condensed-matter realization of the “parity anomaly”, Phys. Rev. Lett. 61, 1988 (1988).

[34] S. Banerjee, J. Fransson, A. M. Black-Schaffer, H. Ågren, and A. V. Balatsky, Granular superconductor in a honeycomb lattice as a realization of bosonic dirac material, Phys. Rev. B 93, 134502 (2016).

[35] S. K. Kim, H. Ochoa, R. Zarzuela, and Y. Tserkovnyak, Realization of the haldane-kane-mele model in a system of localized spins, Phys. Rev. Lett. 117, 227201 (2016).

[36] Y. Tokura, K. Yasuda, and A. Tsukazaki, Magnetic topological insulators, Nature Reviews Physics 1, 126–143 (2019).

[37] X. S. Wang and X. R. Wang, Topological magnonics, Journal of Applied Physics 129, 151101 (2021).

[38] Y. Su, X. S. Wang, and X. R. Wang, Magnonic weyl semimetal and chiral anomaly in pyrochlore ferromagnets, Phys. Rev. B 95, 224403 (2017).

[39] X. S. Wang, Y. Su, and X. R. Wang, Topologically protected unidirectional edge spin waves and beam splitter, Phys. Rev. B 95, 014435 (2017).

[40] Y. Su and X. R. Wang, Chiral anomaly of weyl magnons in stacked honeycomb ferromagnets, Phys. Rev. B 96, 104437 (2017).

[41] S. Jiang, L. Li, Z. Wang, K. F. Mak, and J. Shan, Controlling magnetism in 2d $\text{cr}_1$ by electrostatic doping, Nature Nanotechnology 13, 549 (2018).

[42] L. Thiel, Z. Wang, M. A. Tschudin, D. Rohner, I. Gutierrez-Lezama, N. Ubrig, M. Gibertini, E. Giannini, A. F. Morpurgo, and P. Maletinsky, Probing magnetism in 2D materials at the nanoscale with single-spin microscopy, Science 364, 1309 (2019).

[43] T. Li, S. Jiang, N. Sivadas, Z. Wang, Y. Xu, D. Weber, J. E. Goldberger, K. Watanabe, T. Taniguchi, C. J. Fennie, K. F. Mak, and J. Shan, Pressure-controlled interlayer magnetism in atomically thin $\text{cr}_1$, Nature Materials 18, 1309 (2019).

[44] T. Song, Z. Fei, M. Yankowitz, Z. Lin, Q. Jiang, K. Hvangebo, Q. Zhang, B. Sun, T. Taniguchi, K. Watanabe, M. A. McGuire, D. Graf, T. Cao, J.-H. Chu, D. H. Cobden, C. R. Dean, D. Xiao, and X. Xu, Switching 2D magnetic states via pressure tuning of layer stacking, Nature Materials 18, 1298 (2019).

[45] S. A. Owerre, Magnon hall effect in ab-stacked bilayer honeycomb quantum magnets, Phys. Rev. B 94, 094405 (2016).

[46] A. Mook, J. Henk, and I. Mertig, Magnon hall effect and topology in kagome lattices: A theoretical investigation, Phys. Rev. B 89, 134409 (2014).

[47] S. Djurdjic-Mijin, A. Solajic, J. Pesic, M. Scepanovic, Y. Liu, A. Baum, C. Petrovic, N. Lazarevic, and Z. Popovic, Lattice dynamics and phase transition in cr$_1$ single crystals, Phys. Rev. B 98, 104307 (2018).

[48] A. A. Kovalev, V. A. Zyzin, and B. Li, Pumping of magnons in a dzyaloshinskii-moriya ferromagnet, Phys. Rev. B 95, 165106 (2017).

[49] R. Matsumoto and S. Murakami, Rotational motion of magnons and the thermal hall effect, Phys. Rev. B 84, 184406 (2011).

[50] R. Matsumoto and S. Murakami, Theoretical prediction of a rotating magnon wave packet in ferromagnets, Phys. Rev. Lett. 106, 197202 (2011).

[51] C.-X. Liu, X.-L. Qi, X. Dai, Z. Fang, and S.-C. Zhang, Quantum anomalous hall effect in Hg$_{1-y}$Mn$_y$Te quantum wells, Phys. Rev. Lett. 101, 146802 (2008).

[52] N. Ubrig, Z. Wang, J. Teyssier, T. Taniguchi, K. Watanabe, E. Giannini, A. F. Morpurgo, and M. Gibertini, Low-temperature monoclinic layer stacking in atomically thin cr$_1$ crystals, 2D Materials 7, 015007 (2019).

[53] A. Mook, S. A. Diaz, J. Klinovaja, and D. Loss, Chiral hinge magnons in second-order topological magnon insulators, Phys. Rev. B 104, 024406 (2021).