Chromatic-choosability of the power of graphs

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Abstract

The kth power $G^k$ of a graph G is the graph defined on $V(G)$ such that two vertices $u$ and $v$ are adjacent in $G^k$ if the distance between $u$ and $v$ in $G$ is at most $k$. Let $\chi(H)$ and $\chi_l(H)$ be the chromatic number and the list chromatic number of $H$, respectively. A graph $H$ is called chromatic-choosable if $\chi_l(H) = \chi(H)$. It is an interesting problem to find graphs that are chromatic-choosable. A natural question raised by Xuding Zhu [11] is whether there exists a constant integer $k$ such that $G^k$ is chromatic-choosable for every graph $G$.

Motivated by the List Total Coloring Conjecture, Kostochka and Woodall [7] asked whether $G^2$ is chromatic-choosable for every graph $G$. Kim and Park [8] answered the Kostochka and Woodall’s question in the negative by finding a family of graphs whose squares are complete multipartite graphs with partite sets of equal and unbounded size. In this paper, we answer Zhu’s question by showing that for every integer $k \geq 2$, there exists a graph $G$ such that $G^k$ is not chromatic-choosable. Moreover, for any fixed $k$ we show that the value $\chi_l(G^k) - \chi(G^k)$ can be arbitrarily large.

Keywords: List coloring, chromatic-choosable, power of graphs

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1 Introduction

For any graph $G$ and for any positive integer $k$, the $k$th power $G^k$ of a graph $G$ is the graph defined on $V(G)$ such that two vertices $u$ and $v$ are adjacent in $G^k$ if the distance between $u$ and $v$ in $G$ is at most $k$. In particular, $G^2$ is called the square of $G$.

A proper $k$-coloring $\phi : V(G) \rightarrow \{1, 2, \ldots, k\}$ of a graph $G$ is an assignment of colors to the vertices of $G$ so that any two adjacent vertices receive distinct colors.

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The chromatic number $\chi(G)$ of a graph $G$ is the least $k$ such that there exists a proper $k$-coloring of $G$. A list assignment $L$ is an assignment of lists of colors to vertices. A graph $G$ is said to be $k$-choosable if for any list $L(v)$ of size at least $k$, there exists a proper coloring $\phi$ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. The least $k$ such that $G$ is $k$-choosable is called the list chromatic number $\chi_l(G)$ of a graph $G$. Clearly $\chi_l(G) \geq \chi(G)$ for every graph $G$.

A graph $G$ is called chromatic-choosable if $\chi_l(G) = \chi(G)$. It is an interesting problem to find graphs that are chromatic-choosable. There are several famous conjectures that some classes of graphs are chromatic-choosable including the List Coloring Conjecture [1] and the List Total Coloring Conjecture [2], which say that the line graph and the total graph of any graph are chromatic-choosable, respectively. Motivated by the List Total Coloring Conjecture, Kostochka and Woodall [7] proposed the List Square Coloring Conjecture which states that $G^2$ is chromatic-choosable for every graph $G$.

It was noted in [7] that the List Total Coloring Conjecture is true if the List Square Coloring Conjecture is true.

If the diameter of a graph $G$ is $m$, then $G^m$ is chromatic-choosable since $G^m$ is a complete graph. Thus given a graph $G$, there exists an integer $k$ such that $G^k$ is chromatic-choosable. A natural question raised by Xuding Zhu [11] is whether there exists a constant $k$ such that $G^k$ is chromatic-choosable for every graph $G$. The question looks trivially true if we do not think carefully. But, it is not easy to answer the question whether such constant $k$ exists or not. On the other hand, it is an interesting problem to find a class $F$ of graphs such that for every integer $k \geq k_0$ for some constant $k_0$, $H^k$ is chromatic-choosable for every graph $H$ in $F$. In this direction, Wang and Zhu [10] show that $C_n^k$ is chromatic-choosable for any positive integer $k$ and for any cycle $C_n$.

The List Square Coloring Conjecture proposed the smallest $k$ such that $G^k$ is chromatic-choosable for every graph $G$ is 2. Also it proposed that $G^{2t}$ is chromatic-choosable for any positive integer $t$. However, recently, Kim and Park [8] disproved the List Square Coloring Conjecture by showing that for any prime $n$, there exists a graph $G$ such that $G^2$ is $K_{n^{(2n-1)}}$, where $K_{n^{(2n-1)}}$ denotes the complete multipartite graph with $(2n - 1)$ partite sets in which each partite set has size $n$. Note that the gap between $\chi_l(G^2)$ and $\chi(G^2)$ can be arbitrarily large since the gap between the list chromatic number and chromatic number of a complete multipartite graph can be arbitrarily large.

Since there exists a graph $G$ such that $G^2$ is not chromatic-choosable, next direction is to find the smallest $k$ such that $G^k$ is chromatic-choosable for every graph $G$, if such $k$ exists. In this paper, we answer Zhu’s question by showing that there is no constant $k$ such that $G^k$ is chromatic-choosable for every graph $G$. We show the following main
Theorem 1 For any positive integer \( k \geq 2 \) and for any positive integer \( s \), there exists a graph \( G \) such that

\[
\chi_l(G^k) \geq \frac{10}{9} \cdot 3^{3sk-1} - 1 \geq 3^{3sk-1} - 1 = \chi(G^k).
\]

This implies that for any integer \( k \geq 2 \), there exists a graph \( G \) such that \( G^k \) is not chromatic-choosable. Moreover, since

\[
\chi_l(G^k) - \chi(G^k) \geq (\frac{10}{9} - 1) \cdot 3^{3sk-1} - 1 = 3^{3sk-3} - 1,
\]

for any fixed \( k \), the value \( \chi_l(G^k) - \chi(G^k) \) can be arbitrarily large as the integer \( s \) goes to infinity.

2 Construction

In this section, first we define a Cayley graph \( G_{3n} \), and study the properties of \( G_{3n} \). And then we will define a graph \( H_{3n} \) which will be used in the proof of Theorem 1. First, we define the Cayley graph \( G_m \) where \( m \) is an integer at least 2.

Construction 1 For any positive integer \( m \geq 2 \), let \( \mathbb{Z}_3^m \) be an (usual) additive abelian group of order \( 3^m \). Namely \( \mathbb{Z}_3^m = \{(a_1, a_2, \ldots, a_m) : a_i \in \{0, 1, 2\} \text{ for all } 1 \leq i \leq m\} \). Note that the identity element of \( \mathbb{Z}_3^m \), denoted by \( 0 \), is \((0, \ldots, 0)\).

For any \( i, j \in \{1, \ldots, m\} \) with \( i \neq j \), let \( x_{i,j} \) be the vector such that the \( i \)th coordinate of \( x_{i,j} \) is 1, the \( j \)th coordinate of \( x_{i,j} \) is 2, and all other coordinates of \( x_{i,j} \) are 0. For example, \( x_{1,3} = (1, 0, 2, 0, \ldots, 0) \). For any positive integer \( m \geq 2 \), let \( X_m \) be the subset of \( \mathbb{Z}_3^m \) such that

\[
X_m = \{x_{i,j} : 1 \leq i, j \leq m, \text{ and } i \neq j\}.
\]

Let

\[
\Gamma_m = \{z \in \mathbb{Z}_3^m : \sum_{i=1}^{m} z_i = 0 \pmod{3}\},
\]

where \( z_i \) is the \( i \)th coordinate of \( z \) for any \( i \in \{1, \ldots, m\} \). Now \( \Gamma_m \) is a subgroup of \( \mathbb{Z}_3^m \) of index 3, since \( y - z \in \Gamma_m \) for any two \( y, z \in \Gamma_m \). We define a graph \( G_m \) as follows:

\[
V(G_m) = \Gamma_m, \\
E(G_m) = \{yz : y - z \in X_m\}.
\]

Note that \( |V(G_m)| = |\Gamma_m| = 3^{m-1} \) and \( G_m \) is a connected graph. Actually, the graph \( G_m \) is the Cayley graph on \( \Gamma_m \) with the symmetric generating set \( X_m \).
From now on, $G_{3n}$ denotes the graph defined in Construction 1 for $m = 3n$. Let $a_{3n}, b_{3n}$ be the vectors in $\mathbb{Z}_3^{3n}$ defined by

$$a_{3n} = (1, 1, \ldots, 1)$$
$$b_{3n} = (2, 2, \ldots, 2).$$

First, we study basic properties of the graph $G_{3n}$ in Lemma 2 and Lemma 3.

**Lemma 2** Let $n \geq 2$ be an integer. For any vector $y \in \Gamma_{3n}$,

$$d_{G_{3n}}(0, y) \leq \frac{2 \times \text{(the number of nonzero coordinates of } y)}{3},$$

and the equality holds if and only if all nonzero coordinates of $y$ are identical.

**Proof**: We will prove the lemma by the induction on the number of nonzero coordinates of $y$. If $y \neq 0$, then $y$ has at least two nonzero coordinates.

As the basis step, we consider the cases that $y$ has two or three nonzero coordinates. If $y$ has two nonzero coordinates, then one can check $y = x_{i,j}$ for some $i, j \in \{1, \ldots, 3n\}$, and hence the lemma holds. If $y$ has three nonzero coordinates, then all nonzero coordinates of $y$ are identical and $d_{G_{3n}}(0, y) = 2$. So the lemma holds. As the induction step, suppose that the lemma holds for any $y$ which has at most $m$ nonzero coordinates ($m \geq 3$). Now we take a vertex $y$ of $G_{3n}$ which has $(m + 1)$ nonzero coordinates. For convenience, we denote by $y_i$ the $i$th coordinate of $y$ for any $i \in \{1, \ldots, 3n\}$.

**Case 1.** There exist two distinct integers $i_1, i_2 \in \{1, \ldots, 3n\}$ such that $y_{i_1} = 1$ and $y_{i_2} = 2$

Let $w$ the vector such that $y = w + x_{i_1,i_2}$. Note that $w$ has $(m + 1) - 2$ nonzero coordinates. Then by the induction hypothesis,

$$d_{G_{3n}}(0, y) \leq d_{G_{3n}}(0, w) + 1 \leq \frac{2((m + 1) - 2)}{3} + 1 < \frac{2(m + 1)}{3}.$$ 

**Case 2.** All nonzero coordinates of $y$ are identical.

Since $y$ has $(m + 1)$ nonzero coordinates and $m + 1 \geq 4$, there are three distinct integers $i_1, i_2, i_3 \in \{1, \ldots, 3n\}$ such that $y_{i_1} = y_{i_2} = y_{i_3} \neq 0$. First, we suppose that $y_{i_1} = y_{i_2} = y_{i_3} = 1$. Let $w$ the vector such that $y = w + x_{i_1,i_2} + x_{i_2,i_3}$. Note that $w$ has $(m + 1) - 3$ nonzero coordinates and all of the nonzero coordinates of $w$ are identical. Thus by the induction hypothesis,

$$d_{G_{3n}}(0, y) \leq d_{G_{3n}}(0, w) + 2 = \frac{2(m + 1 - 3)}{3} + 2 = \frac{2(m + 1)}{3}.$$ 

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Therefore, \( d_{G_{3n}}(0, y) \leq \frac{2(m+1)}{3} \).

Note that since the sum of all coordinates of \( y \) is 0 modulo 3 and any nonzero coordinate of \( y \) is 1, the number of nonzero coordinates of \( y \) is multiple of 3. Therefore \( \frac{2(m+1)}{3} \) is an integer. Now we will show that \( d_{G_{3n}}(0, y) \geq \frac{2(m+1)}{3} \) to conclude that \( d_{G_{3n}}(0, y) = \frac{2(m+1)}{3} \).

Suppose that \( d_{G_{3n}}(0, y) < \frac{2(m+1)}{3} \). Then there is a subset \( A \) of \( X_{3n} \) such that \( y = \sum_{x_{i,j} \in A} x_{i,j} \) and \( |A| = d_{G_{3n}}(0, y) \). We define a graph \( H \) such that \( V(H) = \{1, 2, \ldots, 3n\} \) and \( E(H) = \{ij \mid x_{i,j} \in A\} \). Note that \( |E(H)| < \frac{2(m+1)}{3} \) since \( |A| < \frac{2(m+1)}{3} \). Let \( W_1, W_2, \ldots, W_s \) be nontrivial connected components of \( H \). Let \( H_0 \) be the union of \( W_1, W_2, \ldots, W_s \). Since each nonzero coordinate \( i \) of \( y \) cannot be an isolated vertex in \( H \), it must belong to \( V(H_0) \), and so \( |V(H_0)| \geq m + 1 \). Note that \( |E(H_0)| = |E(H)| \). On the other hand, it is true that \( |E(H_0)| + s \geq |V(H_0)| \). Thus

\[
s \geq |V(H_0)| - |E(H)| > (m + 1) - \frac{2(m+1)}{3} = \frac{m+1}{3}.
\]

Next, we will show that \( |E(W_i)| \geq 2 \) for each \( i \in \{1, \ldots, s\} \). It is clear \( |E(W_i)| \geq 1 \) since \( W_i \) is nontrivial. If \( |E(W_i)| = 1 \) for some \( i \in \{1, \ldots, s\} \), say \( E(W_i) = \{jk\} \), then one of \( y_j \) and \( y_k \) must be 2, a contradiction to the assumption that any nonzero coordinate of \( y \) is 1. Thus we conclude that \( |E(W_i)| \geq 2 \) for all \( i \in \{1, \ldots, s\} \). Therefore we have

\[
|A| = |E(H)| = |E(H_0)| = \sum_{i=1}^{s} |E(W_i)| \geq 2s > 2 \cdot \frac{m+1}{3},
\]

which is a contradiction. Thus \( d_{G_{3n}}(0, y) \geq \frac{2(m+1)}{3} \), and hence \( d_{G_{3n}}(0, y) = \frac{2(m+1)}{3} \). Therefore the lemma holds for the vector \( y \). For the case that \( y_{i_1} = y_{i_2} = y_{i_3} = 2 \), one can show the lemma holds by a similar argument.

Therefore by Case 1 and Case 2, the lemma holds for the vector \( y \). \( \square \)

Define a relation \( \sim \) on \( V(G_{3n}) \) by

\[
y \sim z \quad \text{if and only if} \quad y - z \in \{0, a_{3n}, b_{3n}\}.
\]

Then one can check the relation \( \sim \) is an equivalence relation. For each vertex \( y \in V(G_{3n}) \), the equivalent class \([y]\) containing \( y \) has three elements,

\[
[y] = \{y, y + a_{3n}, y + b_{3n}\}.
\]

Note that \( K_{n,r} \) denotes the complete multipartite graph with \( r \) partite sets in which each partite set has size \( n \).
Lemma 3 For any integer \( n \geq 2 \), \( G^{2n-1}_{3n} \) is the complete multipartite graph \( K_{3n,3^{3n-2}} \) in which partite sets are the equivalent classes obtained by the relation \( \sim \).

Proof: By Lemma 2, for any two vertices \( y \) and \( z \) in \( G_{3n} \), \( d_{G_{3n}}(y, z) \leq 2n \) and the equality holds only when \( y - z \in \{a_{3n}, b_{3n}\} \). Therefore the lemma holds. \( \square \)

It is proved in [6] that the complete multipartite graph \( K_{3n,3^{3n-2}} \) is not chromatic-choosable. Thus Lemma 3 implies that the \((2n - 1)\)th power of \( G_{3n} \) is not chromatic-choosable. This implies that for every odd integer \( 2k - 1 \), there exists a graph \( G \) such the \((2k - 1)\)th power of \( G \), denoted by \( G^{2k-1} \), is not chromatic-choosable.

Next, we define another graph \( H_{3n} \) which will be used in the proof of Theorem 1. The cartesian product of \( G \) and \( H \), denoted by \( G \Box H \), is the graph with the vertex set \( V(G) \times V(H) \) such that for two vertices \((g_1, h_1), (g_2, h_2) \in V(G) \times V(H)\), \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent in \( G \Box H \) if and only if either \( h_1 = h_2 \) and \( g_1g_2 \in E(G) \), or \( g_1 = g_2 \) and \( h_1h_2 \in E(H) \).

Construction 2 For any positive integer \( n \), let \( H_{3n} \) be the cartesian product \( G_{3n} \Box K_3 \) of \( G_{3n} \) and \( K_3 \).

3 Proof of Theorem 1

For two graphs \( G \) and \( H \), the lexicographic product of \( G \) and \( H \), denoted by \( G[H] \), is the graph with the vertex set \( V(G) \times V(H) \) such that for two vertices \((g_1, h_1), (g_2, h_2) \in V(G) \times V(H)\), \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent in \( G[H] \) if and only if either \( g_1g_2 \in E(G) \), or \( g_1 = g_2 \) and \( h_1h_2 \in E(H) \).

We will show that the \((2n)\)th power of \( H_{3n} \), denoted by \( H^{2n}_{3n} \), is isomorphic to the lexicographic product of the complete graph \( K_{3^{3n-2}}\) and \( K_3 \Box K_3 \).

Theorem 4 If \( H_{3n} \) is the graph defined in Construction 2, then \( H^{2n}_{3n} \) is isomorphic to the lexicographic product \( K_{3^{3n-2}}[K_3 \Box K_3] \) of the complete graph \( K_{3^{3n-2}} \) and \( K_3 \Box K_3 \).

Proof: By Lemma 3, \( G^{2n-1}_{3n} \) is isomorphic to \( K_{3n,3^{3n-2}} \). Let \( P_1, P_2, \ldots, P_{3^{3n-2}} \) be the partite sets of \( G^{2n-1}_{3n} \). For each \( i \in \{1, \ldots, 3^{3n-2}\} \), let \( Q_i \) be a subset of vertices of \( H_{3n} \) defined by

\[
Q_i = P_i \times V(K_3).
\]

Note that \(|Q_i| = 9\), and \( \{Q_i \mid 1 \leq i \leq 3^{3n-2}\} \) is a partition of \( V(H_{3n}) \).
To show that $H_{3n}^{2n}$ is isomorphic to the lexicographic product $K_{3^{3n-2}}[K_3 \square K_3]$, it is sufficient to show the following claims.

**Claim 1.** For any vertex $x \in Q_i$ and $y \in Q_j$ with $i \neq j$, we have $d_{H_{3n}}(x, y) \leq 2n$.

**Claim 2.** For each $i \in \{1, \ldots, 3^{3n-2}\}$, the subgraph of $H_{3n}^{2n}$ induced by $Q_i$ is isomorphic to $K_3 \square K_3$.

Since $V(H_{3n}) = V(G_{3n}) \times V(K_3)$, for any two vertices $x$ and $y$ in $H_{3n}$, we can denote $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $x_1, y_1 \in V(G_{3n})$ and $x_2, y_2 \in V(K_3)$. By Lemma 2, $d_{G_{3n}}(x_1, y_1) \leq 2n$ for any $x_1, y_1 \in V(G_{3n})$ and $d_{G_{3n}}(x_1, y_1) = 2n$ only when $x_1 \neq y_1$ and $x_1 \sim y_1$. It is well-known fact that

$$d_{G_{3n} \sqcup K_3}(x, y) = d_{G_{3n}}(x_1, y_1) + d_{K_3}(x_2, y_2).$$

Thus

$$d_{H_{3n}}(x, y) \leq 2n + 1$$

and $d_{H_{3n}}(x, y) = 2n + 1$ if and only if $d_{G_{3n}}(x_1, y_1) = 2n$ and $d_{K_3}(x_2, y_2) = 1$.

If $x \in Q_i$ and $y \in Q_j$ with $i \neq j$, then $d_{G_{3n}}(x_1, y_1) \leq 2n - 1$ by Lemma 3. Thus

$$d_{H_{2n}}(x, y) \leq 2n.$$

This completes the proof of Claim 1.

Suppose that $x, y \in Q_i$ for some integer $i \in \{1, 2, \ldots, 3^{3n-2}\}$. Then $x_1, y_1 \in P_i$ and $x_1 \sim y_1$. Therefore $d_{H_{3n}}(x, y) = 2n + 1$ if and only if $x_1 \neq y_1$ and $x_2 \neq y_2$. This means that two vertices $x$ and $y$ of $Q_i$ are non-adjacent in $H_{3n}^{2n}$ if and only if $x_1 \neq y_1$ and $x_2 \neq y_2$, which implies that the subgraph of $H_{3n}^{2n}$ induced by $Q_i$ is isomorphic to $K_3 \square K_3$. This completes the proof of Claim 2.

By Claim 1, for any vertex $x \in Q_i$ and $y \in Q_j$ with $i \neq j$, two vertices $x$ and $y$ are adjacent in $H_{3n}^{2n}$. Therefore $H_{3n}^{2n}$ is isomorphic to the lexicographic product $K_{3^{3n-2}}[K_3 \square K_3]$ by Claim 2.

**Corollary 5** For every integer $k \geq 2$ and for any positive integer $s$, there is a graph $G$ such that $G^k$ is isomorphic to the lexicographic product $K_{3^{3k-2}}[K_3 \square K_3]$ of the complete graph $K_{3^{3k-2}}$ and $K_3 \square K_3$.

**Proof:** For any positive integers $k$ and $s$, let $G = H_{3sk}^{2s}$. Since

$$G^k = (H_{3sk}^{2s})^k = H_{3sk}^{2sk},$$

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$G^k$ is isomorphic to $K_{3^{3k-2}}[K_3 \Box K_3]$ by Theorem 4.

Next, we will compute the chromatic number and the list chromatic number of the lexicographic product $G[H]$ of $G$ and $H$ to complete the proof of Theorem 1. For the chromatic number of the lexicographic product $G[H]$ of $G$ and $H$, the following is known.

**Proposition 6** ([3]) If $\chi(H) = l$, then $\chi(G[H]) = \chi(G[K_l])$ for any graph $G$.

**Lemma 7** If $H = K_{3^{3n-2}}[K_3 \Box K_3]$ is the lexicographic product of the complete graph $K_{3^{3n-2}}$ and $K_3 \Box K_3$, then $\chi(H) = 3^{3n-1}$.

Proof: Note that $\chi(K_3 \Box K_3) = 3$. By Proposition 6 we have $\chi(H) = \chi(K_{3^{3n-2}}[K_3])$. Since $K_{3^{3n-2}}[K_3]$ is a complete graph of order $3^{3n-1}$, we have $\chi(H) = 3^{3n-1}$.

**Lemma 8** If $H = K_{3^{3n-2}}[K_3 \Box K_3]$ is the lexicographic product of the complete graph $K_{3^{3n-2}}$ and $K_3 \Box K_3$, then

$$\chi_l(H) \geq \frac{10}{9} \cdot 3^{3n-1} - 1.$$

Proof: Let $V(K_{3^{3n-2}}) = \{1, \ldots, 3^{3n-2}\}$, and denote $V(K_{3^{3n-2}}[K_3 \Box K_3]) = \{(x, y) : x \in \{1, \ldots, 3^{3n-2}\} \text{ and } y \in V(K_3 \Box K_3)\}$.

Let $V(K_3) = \{0, 1, 2\}$, and we partition $V(K_3 \Box K_3)$ into three subsets $W_1, W_2,$ and $W_3$ such that $W_1 = \{(0,0), (0,1), (0,2)\}$, $W_2 = \{(1,0), (1,1), (1,2)\}$, and $W_3 = \{(2,0), (2,1), (2,2)\}$. Note that for each $1 \leq i \leq 3$, the subgraph of $K_3 \Box K_3$ induced by $W_i$ is a complete graph.

For each $1 \leq i \leq 3$, let

$$R_i = V(K_{3^{3n-2}}) \times W_i.$$

Then it is clear that $R_i$ is a subset of $V(H)$ and $\{R_1, R_2, R_3\}$ is a partition of $V(H)$. Note that the subgraph of $H$ induced by $R_i$ is a complete graph.

Let $t$ be an even integer. Let $A_1, A_2, A_3$ be mutually disjoint sets such that $|A_1| = |A_2| = |A_3| = \frac{t}{3}$, and let $A = A_1 \cup A_2 \cup A_3$. For each vertex $v \in R_i$ with $1 \leq i \leq 3$, we define the list $L(v) = A \setminus A_i$. We will show that $H$ is not $L$-choosable if $t < \frac{10}{9} \cdot 3^{3n-1}$.

First, we define a family of subsets of $V(H)$, denoted by $\{S_i : 1 \leq i \leq 3^{3n-2}\}$. For each $i \in \{1, \ldots, 3^{3n-2}\}$, let

$$S_i = \{(i, y) : y \in V(K_3 \Box K_3)\}.$$
Then \( \{S_i : 1 \leq i \leq 3^{3n-2}\} \) is a partition of \( V(H) \). Note that when \( i \) and \( j \) are distinct, for any two vertices \( u \in S_i \) and \( v \in S_j \), \( u \) and \( v \) are adjacent in \( H \) by the definition of the lexicographic product \( K_{3^{3n-2}}[K_3\square K_3] \).

We will show that if there is a proper coloring \( \phi \) such that \( \phi(v) \in L(v) \) for each vertex \( v \), then \( |\{\phi(v) : v \in S_i\}| \geq 5 \) for each \( i \in \{1, \ldots, 3^{3n-2}\} \). Let \( S_i \) be a subset of \( H \) with \( i \in \{1, \ldots, 3^{3n-2}\} \). If there exist \( u, v, w \in S_i \) such that \( \phi(u) = \phi(v) = \phi(w) \), then \( \{u, v, w\} \) forms an independent set of \( H \). Since each \( R_j \) induces a complete graph in \( H \), any two of \( u, v, w \) cannot belong to the same \( R_j \). Thus \( L(u) \cap L(v) \cap L(w) = \emptyset \), which is a contradiction. Hence, each color must be used at most two times in \( S_i \). Thus \( |\{\phi(v) : v \in S_i\}| \geq \left\lceil \frac{9}{2} \right\rceil = 5 \). Therefore \( |\{\phi(v) : v \in S_i\}| \geq 5 \) for each \( i \in \{1, \ldots, 3^{3n-2}\} \).

Note that
\[
A \supseteq \bigcup_{i=1}^{3^{3n-2}} \{\phi(v) : v \in S_i\}.
\]

Note that when \( i \) and \( j \) are distinct, for any two vertices \( u \in S_i \) and \( v \in S_j \), \( u \) and \( v \) are adjacent. Thus \( \{\phi(v) : v \in S_i\} \) and \( \{\phi(v) : v \in S_j\} \) are disjoint for any distinct \( i \) and \( j \). Hence
\[
\frac{3t}{2} = |A| \geq \left| \bigcup_{i=1}^{3^{3n-2}} \{\phi(v) : v \in S_i\} \right| \geq 5 \cdot 3^{3n-2}.
\]

Therefore
\[
t \geq \frac{10}{3} \cdot 3^{3n-2} = \frac{10}{9} \cdot 3^{3n-1}.
\]

This implies that for any even integer \( t \) less than \( \frac{10}{9} \cdot 3^{3n-1} \), we have \( \chi_l(H) > t \). Since there is a possibility that \( \chi_l(H) \) is odd, one can say that \( \chi_l(H) \geq \frac{10}{9} \cdot 3^{3n-1} - 1 \). \( \square \)

When \( n = sk \), we have \( \chi_l(K_{3^{3sk-2}}[K_3\square K_3]) \geq \frac{10}{9} \cdot 3^{3sk-1} - 1 \) by Lemma 8 and \( \chi(K_{3^{3sk-2}}[K_3\square K_3]) = 3^{3sk} - 1 \) by Lemma 7 Therefoe Theorem 4 holds by Corollary 5 and Lemmas 7 and 8.

### 4 Remark

Since the List Square Coloring Conjecture is not true in general, a natural problem is to find the upper bound of the list chromatic number of \( G^2 \) for any graph \( G \). By observing some straightforward bounds on \( \omega(G^2) \) and \( \Delta(G^2) \), one immediately obtains \( \chi_l(G^2) \leq (\chi(G^2))^2 \). Noel [9] proposed the following two problems.

**Question 1** [Noel [9]] Is there a function \( f(x) = o(x^2) \) such that for every graph \( G \),
\[
\chi_l(G^2) \leq f(\chi(G^2))?
\]
The example of Kim and Park shows that the function $f$ in Question \[Q\] must satisfy $f(x) = \Omega(x \log x)$. Noel \[9\] asked whether it is possible to obtain a general upper bound of the same order of magnitude.

**Question 2** [Noel \[9\]] Does there exist a constant $c$ such that every graph satisfies

$$
\chi_l(G^2) \leq c\chi(G^2) \log \chi(G^2).
$$

In this paper, we show that there is no constant $k$ such that $G^k$ is chromatic-choosable for every graph $G$. On the other hand, Gravier and Maffray \[4\] conjectured that every claw-free graph is chromatic-choosable. As a relaxation of the conjecture, it is an interesting problem to answer the following question.

**Question 3** Is there a constant $k$ such that $G^k$ is chromatic-choosable if $G$ is claw-free?

One could expect that if $G$ is a chromatic-choosable, then $G^2$ is also chromatic-choosable. But, there exists a graph $G$ such that $G^2$ is not chromatic-choosable even though $G$ is chromatic-choosable. The smallest example of Kim and Park's \[8\] is one of the graphs that have such property. Therefore it is not clear whether $G^k$ is chromatic choosable for every $k \geq k_0$ even though $G^{k_0}$ is chromatic-choosable. Thus by assumption that Gravier and Maffray’s conjecture is true, it is an interesting problem to answer the following question.

**Question 4** Is $G^k$ chromatic-choosable for every integer $k \geq 2$ if $G$ is claw-free?

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