AN EXTENSION OF SOLUTIONS TO CONVOLUTION EQUATIONS WITH THE LOSS OF SMOOTHNESS

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Abstract. In the present paper the smoothness loss of an extension of solutions to convolution equations is studied. Also examples for some kinds of convolvers are given.

1. Introduction

The interest in studying the convolution equation can be explained by the fact that this subject is related to many branches of mathematics and its applications. For example, some interrelations between the theory of convolution equations and the integral geometry, complex analysis, theory of differential operators, theory of trigonometric series and their generalizations have been recently found (see [1], [2]).

Results on some differential properties of an extension of solutions to convolution equations under different conditions on the convolver are presented in the paper. This work continues the study of the properties of exponential polynomials and the behavior of extensions of solutions to convolution equations (see [1] – [5]).

2. Notations and Auxiliary Statements

Let $D'(a, b)$ be a class of all distributions. Let $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$, where $\mathcal{E}'(\mathbb{R}^1)$ is the space of compact supported distributions. Let $\text{supp } T = [-r(T), r(T)]$.

Suppose that $-\infty < a < b \leq +\infty, b - a > 2r(T)$.

Let us introduce the following notation.

$(a, b)_T = \{t \in \mathbb{R}^1 : t - \text{supp } T \subset (a, b)\}$.

We denote by $D_T'(a, b)$ the class of all distributions $f \in D'(a, b)$ which are solutions of the convolution equation

\[(1) \quad (f * T)(t) = 0, \quad t \in (a, b)_T.\]

Moreover,

$C^k_T(a, b) = (D_T' \cap C^k)(a, b)$ for $k \in \mathbb{Z}_+$ or $k = \infty$.

Let $\hat{T} = \langle T, e^{-itz} \rangle$ be the Fourier transform of $T$, $\mathcal{Z}(\hat{T})$ be the set of zeroes of $\hat{T}$. For $\lambda \in \mathcal{Z}(\hat{T})$ denote $m(\lambda, T) = n_\lambda(\hat{T}) - 1$, where $n_\lambda(\hat{T})$ $(n_\lambda = n_\lambda(\hat{T}))$ is the multiplicity of the zero $\lambda$ of $\hat{T}$.

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There is a well known fact from the theory of entire functions that for each \( \varepsilon > 0 \)

\[
\sum_{\lambda \in \mathbb{Z}(\hat{T})} \frac{n_\lambda}{|\lambda|^{1+\varepsilon}} < +\infty.
\]  

Let us define the sequence \( \{a_{j}^{\lambda,\eta}(\hat{T})\} \), \( j = 0, \ldots, n_\lambda - 1 \) as follows

\[
a_{0}^{\lambda,\eta}(\hat{T}) = \frac{n_\lambda \delta_{0,\eta}}{T^{(n_\lambda)}(\lambda)},
\]

\[
a_{j}^{\lambda,\eta}(\hat{T}) = \frac{n_\lambda!}{T^{(n_\lambda)}(\lambda)} \left( \frac{\delta_{j,\eta}}{j!} - \sum_{s=0}^{j-1} a_{s}^{\lambda,\eta}(\hat{T}) \frac{T^{(n_\lambda-s-j)}(\lambda)}{(n_\lambda-s-j)!} \right), \quad j \geq 1,
\]

where \( \delta_{j,\eta} \) is the Kronecker symbol.

Denote the following function

\[
\sigma_{\lambda}(\hat{T}) = \sum_{j=0}^{m(\lambda,T)} |a_{j}^{\lambda,0}(\hat{T})|.
\]

In the sequel, we need the following entire function

\[
a^{\lambda,\eta}(\hat{T},z) = \sum_{j=0}^{n_\lambda-1} a_{j}^{\lambda,\eta}(\hat{T}) \frac{T(z)}{(z-\lambda)^{n_\lambda-j}}.
\]

For \( z \in \mathbb{C}, m \in \mathbb{Z}_{+}, t \in \mathbb{R}^{1} \) we denote

\[
e^{z,m}(t) = (it)^{m} e^{itz}.
\]

Let \( T \in \mathcal{E}'(\mathbb{R}^{1}), T \neq 0, \lambda \in \mathbb{Z}(\hat{T}), \eta \in \{0, \ldots, m(\lambda,T)\} \) and \( f \in \mathcal{D'}_{T}(a,b) \). One can show that for some \( c_{\lambda,\eta}(T,f) \in \mathbb{C} \) the following equality is being satisfied

\[
f \ast T_{\lambda,0} = \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta}(T,f) e^{\lambda,\eta},
\]

where the convolution is considered in \( (a,b)_{T} \) and \( T_{\lambda,0} \in \mathcal{E}'(\mathbb{R}^{1}) \) is defined as follows

\[
r(T_{\lambda,0}) = r(T),
\]

and

\[
\hat{T}_{\lambda,0}(z) = a^{\lambda,0}(\hat{T},z), \quad z \in \mathbb{C}.
\]

**Statement 1.** (\cite{2} Theorem 3.9(ii)) Let \( T \in \mathcal{E}'(\mathbb{R}^{1}), T \neq 0 \) and \( f \in \mathcal{D'}(a,b) \).

Assume that

\[
f = \sum_{\lambda \in \mathbb{Z}(\hat{T})} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta} e^{\lambda,\eta},
\]

where \( c_{\lambda,\eta} \in \mathbb{C} \) and the series converges in \( \mathcal{D'}(a,b) \). Then \( f \in \mathcal{D'}_{T}(a,b) \) and \( c_{\lambda,\eta} = c_{\lambda,\eta}(T,f) \).

Let us denote for \( \lambda \in \mathbb{Z}(\hat{T}), R > 0 \) and \( q \in \mathbb{Z}_{+} \) the following.

\[
B(R,\lambda,q) = \begin{cases} R^{m(\lambda,T)} & \text{if } R > 1, \\ m(\lambda,T) + 1 & \text{if } R = 1, \\ \min\{q + 1, m(\lambda,T) + 1\} & \text{if } R < 1. \end{cases}
\]
In [2], the following results were obtained which will be useful in order to prove the main result.

**Statement 2.** ([2] Proposition 2.27(ii)) Let \( r > 0 \) and let

\[
\sum_{\lambda \in \mathcal{Z}(T)} \left( \max_{0 \leq \eta \leq \lambda(T)} \left| c_{\lambda, \eta} \right| \right) B(r, \lambda, k)(|\lambda| + 1)^k e^{-\mu |\lambda|} < +\infty,
\]

for some \( k \in \mathbb{Z}_+ \). Then the series

\[
\sum_{\lambda \in \mathcal{Z}(T)} \sum_{\eta = 0}^{m(\lambda, T)} c_{\lambda, \eta} e^{\lambda \eta}
\]

converges in \( C^k[-r, r] \).

**Statement 3.** ([2] Theorem 3.10] Let \( T \in \mathcal{E}'(\mathbb{R}^1) \), \( T \neq 0 \). Then the following assertions hold.

(i) Let \( f \in \mathcal{D}'_T(a,b) \) and let \( p \) be a nonzero polynomial. Then there exist \( \gamma_1, \gamma_2 > 0 \) independent of \( f \) such that for all \( \lambda \in \mathcal{Z}(T) \), \( |\lambda| > \gamma_1 \) the following estimate holds

\[
\max_{0 \leq \eta \leq m(\lambda(T))} |c_{\lambda, \eta}(T, f)| \leq \frac{\gamma_2}{|p(i \lambda)|} \max_{0 \leq \eta \leq m(\lambda(T))} |c_{\lambda, \eta}(T, p \left( \frac{d}{dt} \right) f)|.
\]

(ii) Let \( k \in \mathbb{Z}_+ \), \( f \in C^k_T(-R, R) \) and \( R > r(T) \). Then there exist \( \gamma_3, \gamma_4, \gamma_5, \gamma_6 > 0 \) independent of \( k \) and \( f \) such that for all \( \lambda \in \mathcal{Z}(T) \) : \( |\lambda| > \gamma_3 \)

\[
\max_{0 \leq \eta \leq m(\lambda(T))} |c_{\lambda, \eta}| \leq \gamma_4^{k+1}|\lambda|^\gamma \sigma_\lambda(T) \left( \int_{r(T)}^{r(T)} |f^{(k)}(t)| dt + \gamma_6 \right),
\]

where \( \gamma_7 > 0 \) is independent of \( k, \lambda \).

We prove the following lemma which we need in the future.

**Lemma 1.** Let \( T \in \mathcal{E}'(\mathbb{R}^1) \), \( T \neq 0 \) and let \( (3) \) be true for some \( k \in \mathbb{N} \) and \( r > 0 \). If for \( R > r \) there exists \( N \in \mathbb{N} : |\lambda| > N \) such that

\[
\sup_{|\lambda| > N} \frac{|\operatorname{Im} \lambda| + m(\lambda, T)}{\ln(2 + |\lambda|)} < \frac{1}{R - r},
\]

then \( (3) \) converges in \( C^{k-1}[-R, R] \).

**Proof.** Let \( |\lambda| > N \), where \( N \) is such that \( (7) \) is true. Consider the following cases.

1) Let \( R > r > 1 \). If \( q \leq k - \frac{(R-r)|\operatorname{Im} \lambda| + \ln \frac{\rho m(\lambda, T)}{\ln(1+|\lambda|)}}{\ln(1+|\lambda|)} \) then the following is true

\[
\frac{B(R, \lambda, q)(|\lambda| + 1)^q e^{\rho R |\operatorname{Im} \lambda|}}{B(r, \lambda, k)(|\lambda| + 1)^k e^{\rho R |\operatorname{Im} \lambda|}} \leq \frac{R^{m(\lambda, T)}(|\lambda| + 1)^q e^{\rho R |\operatorname{Im} \lambda|}}{r^{m(\lambda, T)}(|\lambda| + 1)^k e^{\rho R |\operatorname{Im} \lambda|}} \leq 1.
\]

2) Let \( R > r = 1 \). If \( q \leq k - \frac{(R-1)|\operatorname{Im} \lambda| + m(\lambda, T) \ln R - \ln(m(\lambda, T) + 1)}{\ln(1+|\lambda|)} \) then the following is true

\[
\frac{B(R, \lambda, q)(|\lambda| + 1)^q e^{\rho R |\operatorname{Im} \lambda|}}{B(r, \lambda, k)(|\lambda| + 1)^k e^{\rho R |\operatorname{Im} \lambda|}} \leq \frac{R^{m(\lambda, T)}(|\lambda| + 1)^q e^{\rho R |\operatorname{Im} \lambda|}}{(m(\lambda, T) + 1)(|\lambda| + 1)^k e^{\rho R |\operatorname{Im} \lambda|}} \leq 1.
\]
3) Let $R = 1 > r$. If $q \leq k - \frac{(1-r)|\text{Im}\lambda| - m(\lambda, T) \ln r}{\ln(1+|\lambda|)}$ then the following is true

$$B(R, \lambda, q)(|\lambda| + 1)^q e^{R|\text{Im}\lambda|} = \frac{m(\lambda, T)(|\lambda| + 1)^q e^{R|\text{Im}\lambda|}}{\min\{k + 1, m(\lambda, T)\}(|\lambda| + 1)^k e^{r|\text{Im}\lambda|}} \leq 1.$$  

4) Let $1 > R > r$. If $q \leq k - \frac{(R-r)|\text{Im}\lambda| + \ln \frac{m(\lambda, T)}{\ln(1+|\lambda|)}}{\ln(1+|\lambda|)}$ then the following is true

$$B(R, \lambda, q)(|\lambda| + 1)^q e^{R|\text{Im}\lambda|} = \frac{(|\lambda| + 1)^q e^{R|\text{Im}\lambda|}}{(|\lambda| + 1)^k e^{r|\text{Im}\lambda|}} \leq 1.$$

In consideration of (7) we can say that if $q = k - 1$ then (8)–(11) hold true. Hence the following transformations are true for arbitrary $R$ and $r$ $(R > r)$

$$\sum_{\lambda \in \mathbb{Z}(T)} \left(\frac{\max_{0 \leq \eta \leq m(\lambda, T)} |c_{\lambda, \eta}|}{\sum_{\eta \leq m(\lambda, T)} |c_{\lambda, \eta}|} \right) B(R, \lambda, q)(|\lambda| + 1)^q e^{R|\text{Im}\lambda|} =$$

$$\sum_{\lambda \in \mathbb{Z}(T)} \left(\frac{\max_{0 \leq \eta \leq m(\lambda, T)} |c_{\lambda, \eta}|}{\sum_{\eta \leq m(\lambda, T)} |c_{\lambda, \eta}|} \right) B(r, \lambda, q)(|\lambda| + 1)^k e^{r|\text{Im}\lambda|} \leq \sum_{\lambda \in \mathbb{Z}(T)} \left(\frac{\max_{0 \leq \eta \leq m(\lambda, T)} |c_{\lambda, \eta}|}{\sum_{\eta \leq m(\lambda, T)} |c_{\lambda, \eta}|} \right) B(r, \lambda, q)(|\lambda| + 1)^k e^{r|\text{Im}\lambda|} < \infty.$$

The statement of the Lemma implies from Statement 2 and (12).

\[\square\]

3. Formulation and Proof of the Main Result.

Theorem 1. Assume that $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$, $f \in C^q_T(-r, r)$, where $r > r(T) > 0$, and there exists $q \in \mathbb{Z}_+$ such that

$$\sum_{\lambda \in \mathbb{Z}(T)} \sigma(\lambda, q) B(r, \lambda, q + 1)(|\lambda| + 1)^{\gamma - k + q + 1} e^{r|\text{Im}\lambda|} < +\infty,$$

where $\gamma > 0$ is independent of $f$, $k$, $\lambda$. Then if there exists $N > 0$ such that for $R > r$ the estimate (7) is true then $f \in C^q_T(-R, R)$, where $q < k - 2 - \gamma$.

Proof. The condition $f \in C^q_T(-r, r)$ means that the function $f$ can be represented as the series (13) (see Statement 1) and for $c_{\lambda, \eta}$ the estimate (9) follows from Proposition 2. Using (9) and (13) we obtain (7) for $q + 1$ and $r$. This means that the conditions of Lemma 1 are satisfied, i.e. $f \in C^q_T(-R, R)$. \[\square\]

4. Examples of an Extension of Solutions for Some Convolvers.

To illustrate the loss of smoothness in some cases we give several examples.

Example 1. Consider the following form of the convolver

$$T = \begin{cases} \frac{(r^2 - t^2)^{\alpha - 1/2}}{|t|} & \text{for } |t| \leq r, \\ 0 & \text{for } |t| > r, \end{cases}$$

where $\alpha > -1/2$. Then

$$\tilde{T}(t) = \frac{c J_\alpha(r^2 t)}{r^\alpha},$$

where $J_\alpha$ is the Bessel function and $c = c(r, \alpha) > 0$. 

The asymptotic behavior of zeros of the Bessel functions is following \( \zeta_m = \pi(m + \frac{2\alpha - 1}{2}) + O\left(\frac{1}{m}\right) \) as \( m \to +\infty \) (see [1, p. 26]). Also there is a well known fact that \( |J'_\alpha(\zeta_m)| \geq |\zeta_m|^{-1/2} \). Hence easy to see that

\[
\sigma(\zeta_m(\hat{T})) = \frac{c_1|\zeta_m(\hat{T})|^2}{J'_\alpha(\zeta_m)} \leq c_2m^{\alpha+1/2}.
\]

**Proposition 1.** If \( f \in C^q_T[-R, R] \), where \( k \in \mathbb{N}, T \) is defined by formula (14) and \( R > r \) then \( f \in C^q_T(\mathbb{R}) \) when \( q < k - (\alpha + \frac{3}{2}) \).

**Proof.** According to the conditions, \( f \in C^q_T[-R, R] \), where \( R > r \). By using Statement (3) we obtain

\[
|c_{\zeta_m}(T, f)| \leq \frac{\gamma_1}{|\zeta_m|^k} |c_{\zeta_m}(T, f^{(k)})| = \frac{\gamma_1}{|\zeta_m|^k} \left| \int_{-r}^{r} T_{\zeta_m}(t) f^{(k)}(t) dt \right| \leq \frac{\gamma_1}{|\zeta_m|^k} \max_{[-r, r]} |T_{\zeta_m}(t)| \int_{-r}^{r} |f^{(k)}(t)| dt \leq \frac{\gamma_2}{|\zeta_m|^k} |\sigma(\zeta_m(\hat{T}))| \int_{-r}^{r} |f^{(k)}(t)| dt,
\]

where \( \gamma_1, \gamma_2 > 0 \) is independent of \( f \) and \( \zeta_m \).

Consider now condition (3) for \( R > R \). In this case it becomes as following

\[
\sum_{m=1}^{\infty} |c_{\zeta_m}| \max \{|\zeta_m| + 1\}^q < +\infty.
\]

We find out now under which conditions on \( q \) and \( k \) (17) holds. From (15) i (16) implies that

\[
\sum_{m=1}^{\infty} |c_{\zeta_m}| \max \{|\zeta_m| + 1\}^q \leq \sum_{m=1}^{\infty} \frac{\gamma_3}{|\zeta_m|^k} |\sigma(\zeta_m(\hat{T}))| (|\zeta_m| + 1)^q \leq \sum_{m=1}^{\infty} \frac{\gamma_3}{m^{\alpha-1/2-q}},
\]

where \( \gamma_3 > 0 \) is independent of \( \zeta_m \). That is the series (17) converges when \( k - \alpha - 1/2 - q > 1 \) for every \( R \). By using Statement (2) we conclude that \( f \in C^q_T(\mathbb{R}) \) for \( q < k - (\alpha + 3/2) \). □

**Example 2.** We generalize a bit the previous example. Now let

\[
T = \begin{cases} 
(r^2 - t^2)^{\alpha-1/2} h(t) & |t| \leq r, \\
0 & |t| > r,
\end{cases}
\]

where \( \alpha > -1/2 \) and \( h(t) \in C^2[-r, r] \) is an even function. Then

\[
\hat{T}(z) = \int_{-r}^{r} (r^2 - t^2)^{\alpha-1/2} h(t) e^{izt} dt,
\]

\[
\hat{T}'(z) = i \int_{-r}^{r} (r^2 - t^2)^{\alpha-1/2} h(t) e^{izt} dt.
\]

The asymptotic behavior of integrals of this kind at \( z \to \infty \) is known (see [1, p. 27]).

\[
c_{k,1} = \frac{(-1)^k \Gamma(k + \alpha)}{k!} \left( \frac{d}{dt} \right)^k ((b - t)^{\beta-1} h(t))|_{t = a},
\]

\[
c_{k,2} = \frac{(-1)^k \Gamma(k + \beta)}{k!} \left( \frac{d}{dt} \right)^k ((t - a)^{\alpha-1} h(t))|_{t = b},
\]
\begin{equation}
\int_a^b (b-t)^{\beta-1}(t-a)^{\alpha-1} h(t) e^{izt} dt \sim e^{i(z+\alpha+\beta)} \sum_{k=0}^{\infty} c_{k,1} (iz)^{\alpha+k} + e^{iz\beta} \sum_{k=0}^{\infty} c_{k,2} (iz)^{\beta+k}.
\end{equation}

In the case of \( \hat{T}(z) \): \( a = -r, b = r, \alpha = \beta = \alpha + 1/2 \). Note that as far as \( h(t) \) is an even function \( c_{0,1} = c_{0,2} \). Then we rewrite the formula (21) as following

\begin{equation}
\hat{T}(z) \sim \frac{(e^{i(-rz+(\alpha+1/2)\pi)} + e^{izr})c_{0,1}}{(iz)^{\alpha+1/2}} + O\left(\frac{1}{|z|}\right).
\end{equation}

For \( \hat{T}'(z) \) in (19) and (20) instead of \( h(t) \) one should substitute \( th(t) \). In that case we get that \( c_{0,1} = -c_{0,2} \), then the formula (21) has the following form

\begin{equation}
\hat{T}'(z) \sim \frac{i(e^{i(-rz+(\alpha+1/2)\pi)} - e^{izr})c_{0,1}}{(iz)^{\alpha+1/2}} + O\left(\frac{1}{|z|}\right).
\end{equation}

In order to find the asymptotics for the zeros of \( \hat{T}(z) \) we need to solve the equation

\[ e^{i(-rz+(\alpha+1/2)\pi)} + e^{izr} = O(1/n). \]

This is equivalent to the next

\[ \cos(rz - (\alpha + 1/2)\pi/2) = O(1/n). \]

Then \( \zeta_n = \frac{\pi(3/2+\alpha+2n)}{2\pi} + O(1/n) \) \( (n \to \infty) \) is the asymptotics in question. Substitute \( \zeta_n \) in formulas (22) and (23). As a result we find that

\begin{equation}
\hat{T}(\zeta_n) \sim \frac{e^{i((\alpha+1/2)\pi/2)c_{0,1}}}{O(n^{\alpha+1/2})} O(1/n) + O\left(\frac{1}{n}\right).
\end{equation}

\begin{equation}
\hat{T}'(\zeta_n) \sim \frac{e^{i((\alpha+1/2)\pi/2)c_{0,1}}}{O(n^{\alpha+1/2})} O(1) + O\left(\frac{1}{n}\right).
\end{equation}

That is for \( n \to \infty \) and in consideration of formulas (24) and (25) \( \hat{T}(\zeta_n) = O(1/n) \) and \( \hat{T}'(\zeta_n) = O(1/n^{\alpha+1/2}) \). Hence easy to see that

\[ \sigma_{\zeta_n}(\hat{T}) = \frac{c_1}{\hat{T}'(\zeta_n)} = O(n^{\alpha+1/2}). \]

**Proposition 2.** If \( f \in C^k([-R,R]) \) \( (k \in \mathbb{N}) \), \( T \) is defined by formula (18) and \( R > r \) then \( f \in C^k(\mathbb{R}) \) for \( q < k - (\alpha + 3/2) \).

**Proof.** The proof is similar to the proof of Proposition 1. \( \square \)

**References**

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