HIGH-ORDER COMPACT SCHEMES FOR PARABOLIC PROBLEMS WITH MIXED DERIVATIVES IN MULTIPLE SPACE DIMENSIONS

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Abstract. We present a high-order compact finite difference approach for a rather general class of parabolic partial differential equations with time and space dependent coefficients as well as with mixed second-order derivative terms in n spatial dimensions. Problems of this type arise frequently in computational fluid dynamics and computational finance. We derive general conditions on the coefficients which allow us to obtain a high-order compact scheme which is fourth-order accurate in space and second-order accurate in time. Moreover, we perform a thorough von Neumann stability analysis of the Cauchy problem in two and three spatial dimensions for vanishing mixed derivative terms, and also give partial results for the general case. The results suggest unconditional stability of the scheme. As an application example we consider the pricing of European Power Put Options in the multidimensional Black-Scholes model for two and three underlying assets. Due to the low regularity of typical initial conditions we employ the smoothing operators of Kreiss et al. to ensure high-order convergence of the approximations of the smoothed problem to the true solution.

1. Introduction. In the last decades, starting from early efforts of Gupta et al. [9, 10] high-order compact finite difference schemes were proposed for the numerical approximation of solutions to elliptic [19, 1] and parabolic partial differential equations [20, 12]. These schemes are able to exploit the smoothness of solutions to such problems and allow to achieve high-order numerical convergence rates (typically strictly larger than two in the spatial discretisation parameter) while generally having good stability properties. Compared to finite element approaches the high-order compact schemes are parsimonious and memory-efficient to implement and hence prove to be a viable alternative if the complexity of the computational domain is not an issue. It would be possible to achieve higher-order approximations also by increasing the computational stencil but this leads to increased bandwidth of the discretisation matrices and complicates formulations of boundary conditions. Moreover, such approaches sometimes suffer from restrictive stability conditions and spurious numerical oscillations. These problems do not arise when using a compact stencil.

Although applied successfully to many important applications, e.g. in computational fluid dynamics [18, 16, 15, 8] and computational finance [5, 6, 22, 2, 4], an even wider breakthrough of the high-order compact methodology has been hampered by the algebraic complexity that is inherent to this approach. The derivation of high-order compact schemes is algebraically demanding, hence these schemes are often tailor-made for a specific application or a rather smaller class of problems (with some notable exceptions as, for example Lele’s paper [14]). The algebraic complexity is even higher in the numerical stability analysis of these schemes. Unlike for standard second-order schemes, the established stability notions imply formidable algebraic problems for high-order compact schemes. As a result, there are relatively few stability results for high-order compact schemes in the literature. This is even more pronounced in higher spatial dimension, as most of the existing studies with analytical stability results for high-order compact schemes are limited to a one-dimensional setting.

Most works focus on the isotropic case where the main part of the differential operator is given by the Laplacian. Another layer of complexity is added when the

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anisotropic case is considered and mixed second-order derivative terms are present in
the operator. Few works on high-order compact schemes address this problem, and
either study constant coefficient problems [7] or specific equations [2].

Consequently, our aim in the present paper is to establish a high-order compact
methodology for a rather general class of parabolic partial differential equations with
time and space dependent coefficients and mixed second-order derivative terms in
arbitrary spatial dimension. Problems of this type arise frequently in computational
fluid dynamics and computational finance. We derive general conditions on the co-
efficients which allow to obtain a high-order compact scheme which is fourth-order
accurate in space and second-order accurate in time. Moreover, we perform a thor-
ough von Neumann stability analysis of the Cauchy problem in two and three spatial
dimensions for vanishing mixed derivative terms, and also give partial results for the
general case. As an application example we consider the pricing of European Power
Put Basket options with two and three underlying assets in the multidimensional
Black-Scholes model. The pricing partial differential equation features second-order
mixed derivative terms and, as an additional difficulty, is supplemented by an initial
condition with low regularity. We use the smoothing operators of Kreiss et al. [13] to
restore high-order convergence.

The rest of this paper is organised as follows. In the next section, we state the
general parabolic partial differential equation in \( n \) spatial dimensions and give the
central difference approximation for the associated elliptic problem. We then derive
auxiliary relations for the higher-order derivatives appearing in the truncation error of
the central difference approximation in Section 3. In Section 4 we give conditions on
the coefficients of the partial differential equation under which a high-order compact
scheme is obtainable. Semi-discrete high-order compact schemes in \( n = 2 \) and \( n = 3 \)
space dimensions are derived in Section 5. Section 6 discusses the time discretisation.
A thorough von Neumann stability analysis of the Cauchy problem in \( n = 2 \) and
\( n = 3 \) space dimensions is performed in Section 7. In Section 8 we apply the schemes
to option pricing problems for European Basket Power Put options and report results
of our numerical experiments in Section 9. Section 10 concludes.

2. Parabolic problem and its central difference approximation. We con-
sider the following parabolic partial differential equation with mixed derivative terms
in \( n \) spatial dimensions for \( u = u(x_1, \ldots, x_n, \tau) \),

\[
 u_\tau + \sum_{i=1}^{n} a_i \frac{\partial^2 u}{\partial x_i^2} + \sum_{i,j=1}^{n} b_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} c_i \frac{\partial u}{\partial x_i} = g \quad \text{in } \Omega \times \Omega_\tau,
\]

with initial condition \( u_0 = u(x_1, \ldots, x_n, 0) \) and suitable boundary conditions, with
space- and time-dependent coefficients \( a_i = a_i(x_1, \ldots, x_n, \tau) \), \( b_{ij} = b_{ij}(x_1, \ldots, x_n, \tau) \),
\( c_i = c_i(x_1, \ldots, x_n, \tau) \) and \( g = g(x_1, \ldots, x_n, \tau) \). The spatial domain \( \Omega \subset \mathbb{R}^n \) is of \( n \)-
dimensional rectangular shape with \( \Omega = \Omega_1 \times \ldots \times \Omega_n \) and \( x_i \in \Omega_i = [x_{\min}^{(i)}, x_{\max}^{(i)}] \)
with \( x_{\min}^{(i)} < x_{\max}^{(i)} \) for \( i \in \{1, \ldots, n\} \). The temporal domain is given by \( \Omega_\tau = [0, \tau_{\max}] \)
with \( \tau_{\max} > 0 \). The functions \( a(\cdot, \tau) \), \( b(\cdot, \tau) \), \( c(\cdot, \tau) \) and \( g(\cdot, \tau) \) are assumed to be in
\( C^2(\Omega) \) for any \( \tau \in \Omega_\tau \), \( u(\cdot, \tau) \in C^0(\Omega) \) and \( u \) is assumed to be differentiable with
respect to \( \tau \). Introducing \( f := -u_\tau + g \) we can rewrite (2.1) as

\[
 \sum_{i=1}^{n} a_i \frac{\partial^2 u}{\partial x_i^2} + \sum_{i,j=1}^{n} b_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} c_i \frac{\partial u}{\partial x_i} = f.
\]
We start by defining a grid on $\Omega$,

\begin{equation}
G^{(n)} := \left\{ \left( x^{(1)}_i, \ldots, x^{(n)}_n \right) \in \Omega \mid x^{(k)}_i = x^{(k)}_i + i_k \Delta x_k, 1 \leq i_k \leq N_k, k = 1, \ldots, n \right\},
\end{equation}

where $\Delta x_k = (x^{(k)}_{\text{max}} - x^{(k)}_{\text{min}})/(N_k - 1) > 0$ are the step sizes in the $k$-th direction with $N_k \in \mathbb{N}$ for $k = 1, \ldots, n$. We use $\partial G^{(n)}$ for the interior of $G^{(n)}$. On this grid we denote by $U_{i_1, \ldots, i_n}$ the discrete approximation of the continuous solution $u$ at the point $(x^{(1)}_{i_1}, \ldots, x^{(n)}_n) \in \partial G^{(n)}$ and time $\tau \in \Omega_\tau$. Using the central difference operator $D^c_k$ and the standard second-order central difference operator $D^c_k$ in $x_k$-direction we get

\begin{align}
\frac{\partial^2 u}{\partial x^2_k} &= D^2_k U_{i_1, \ldots, i_n} - \frac{(\Delta x_k)^2}{12} \frac{\partial^4 u}{\partial x^4_k} + O((\Delta x_k)^4), \\
\frac{\partial u}{\partial x_k} &= D^c_k U_{i_1, \ldots, i_n} - \frac{(\Delta x_k)^2}{6} \frac{\partial^3 u}{\partial x^3_k} + O((\Delta x_k)^4), \\
\frac{\partial^2 u}{\partial x_k \partial x_p} &= D^c_k D^c_p U_{i_1, \ldots, i_n} - \frac{(\Delta x_k)^2}{6} \frac{\partial^4 u}{\partial x^3_k \partial x_p} - \frac{(\Delta x_p)^2}{6} \frac{\partial^4 u}{\partial x_k \partial x^3_p} + O((\Delta x_k)^4) \nonumber \\
&\quad + O((\Delta x_k)^2(\Delta x_p)^2) + O((\Delta x_p)^4) + O\left(\frac{(\Delta x_k)^6}{\Delta x_p}\right),
\end{align}

for $k, p \in \{i_1, \ldots, i_n\}$ and $k \neq p$ on the grid points $(x^{(1)}_{i_1}, \ldots, x^{(n)}_n) \in \partial G^{(n)}$. The error terms contain derivatives of $u$ up to sixth order, thus we require $u(\cdot, \tau) \in C^6(\Omega)$ for all $\tau \in \Omega_\tau$. Using the discretisations given in (2.4) on (2.2) gives

\begin{align}
f = \sum_{i=1}^{n} a_i D^c_i u + \sum_{i,j=1}^{n} b_{ij} D^c_i D^c_j u + \sum_{i=1}^{n} c_i D^c_i u - \sum_{i=1}^{n} \frac{a_i(\Delta x_i)^2}{12} \frac{\partial^4 u}{\partial x^4_i} \\
&\quad - \sum_{i,j=1}^{n} b_{ij} \left[ \frac{(\Delta x_j)^2}{6} \frac{\partial^4 u}{\partial x^3_j \partial x_i} + \frac{(\Delta x_j)^2}{6} \frac{\partial^4 u}{\partial x_i \partial x^3_j} \right] - \sum_{i=1}^{n} \frac{c_i(\Delta x_i)^2}{6} \frac{\partial^4 u}{\partial x^3_i} + \varepsilon,
\end{align}

where $\varepsilon \in O(h^4)$ if $\Delta x_i \in O(h)$ for $i = 1, \ldots, n$ for a step size $h > 0$. If the consistency error is in $O(h^4)$, we call the scheme high-order. In order to achieve a high-order scheme we have to find second-order discretisations of the derivatives $\frac{\partial u}{\partial x^2_i}$, $\frac{\partial^2 u}{\partial x^4_i}$ and $\frac{\partial^3 u}{\partial x^3_i \partial x_j}$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$. We call the scheme high-order compact, if we can achieve this using only points from a compact computational stencil for $x = (x^{(1)}_{i_1}, \ldots, x^{(n)}_n) \in \partial G^{(n)}$. With $U_{i_1, \ldots, i_n} \approx u(x^{(1)}_{i_1}, \ldots, x^{(n)}_n)$, we have

\begin{equation}
\tilde{U}(x) = \{U_{i_1 + k_1, \ldots, i_n + k_n} \mid k_m \in \{-1,0,1\} \text{ for } m = 1, \ldots, n\}
\end{equation}

as the compact computational stencil.

3. Auxiliary relations for higher derivatives. In this section we calculate auxiliary relations for the higher derivatives appearing in (2.5). These relations for the higher derivatives can be calculated by differentiating (2.2). In doing so no additional error is introduced. Differentiating equation (2.2) with respect to $x_k$ and then solving
for $\frac{\partial^4 u}{\partial x_k^4}$ leads to

$$\frac{\partial u}{\partial x_k^n} = - \sum_{i=1}^{n} \frac{a_i}{a_k} \frac{\partial^3 u}{\partial x_i^3 \partial x_k} - \sum_{i=1}^{n} \frac{1}{a_k} \frac{\partial a_i}{\partial x_k} \frac{\partial^2 u}{\partial x_i^2} - \frac{1}{a_k} \frac{\partial a_k}{\partial x_k} \frac{\partial^2 u}{\partial x_k^2} - \sum_{i,j=1}^{n} \frac{b_{ij}}{a_k} \frac{\partial^3 u}{\partial x_i \partial x_j^3} - \frac{1}{a_k} \frac{\partial a_k}{\partial x_k} \frac{\partial^3 u}{\partial x_k^3} \tag{3.1}$$

for $k = 1, \ldots, n$. The relation for $A_k$ can be discretised using the central difference operator with consistency order two on the compact stencil (2.6), as all derivatives of $u$ in the above equation are only differentiated up to twice in each direction.

When we differentiate (2.2) twice with respect to $x_k$ and then solve for $\frac{\partial^4 u}{\partial x_k^4}$, we obtain

$$\frac{\partial^4 u}{\partial x_k^4} = - \sum_{i=1}^{n} \left[ \frac{a_i}{a_k} \frac{\partial^4 u}{\partial x_i^4} + \frac{2}{a_k} \frac{\partial a_i}{\partial x_k} \frac{\partial^3 u}{\partial x_i^3} + \frac{1}{a_k} \frac{\partial^2 a_i}{\partial x_k^2} \frac{\partial^2 u}{\partial x_i^2} + \frac{2}{a_k} \frac{\partial b_{ij}}{\partial x_{ij}} \frac{\partial^2 u}{\partial x_j^2} + \frac{1}{a_k} \frac{\partial^2 b_{ij}}{\partial x_k^2} \frac{\partial^2 u}{\partial x_i \partial x_j} \right]$$

$$- \frac{1}{a_k} \frac{\partial^2 a_k}{\partial x_k^2} \frac{\partial^2 u}{\partial x_k^2} - \sum_{i,j=1}^{n} \left[ \frac{b_{ij}}{a_k} \frac{\partial^4 u}{\partial x_i \partial x_j^3} + \frac{2}{a_k} \frac{\partial b_{ij}}{\partial x_{ij}} \frac{\partial^3 u}{\partial x_j \partial x_k^2} + \frac{1}{a_k} \frac{\partial^2 b_{ij}}{\partial x_k^2} \frac{\partial^3 u}{\partial x_i \partial x_j} \right]$$

$$- \sum_{i=1}^{n} \frac{b_{ik}}{a_k} \frac{\partial^4 u}{\partial x_i^4} - \sum_{i=1}^{n} \left[ \frac{2}{a_k} \frac{\partial b_{ik}}{\partial x_i} \frac{\partial^3 u}{\partial x_k^3} + \frac{1}{a_k} \frac{\partial^2 b_{ik}}{\partial x_k \partial x_k} \frac{\partial^3 u}{\partial x_k \partial x_k} \right] \tag{3.2}$$

We can discretise $B_k$ with second order consistency on the compact stencil (2.6), when using the central difference operator and the auxiliary relations for $A_k$ in (3.1) for $k = 1, \ldots, n$. Differentiating equation (2.2) once with respect to $x_k$ and once with respect to $x_p$ leads to

$$a_k \frac{\partial^4 u}{\partial x_k^4} + a_p \frac{\partial^4 u}{\partial x_p^4}$$

$$= - \sum_{i=1}^{n} \left[ \frac{a_i}{a_k} \frac{\partial^4 u}{\partial x_i^4} + \frac{\partial a_i}{\partial x_k} \frac{\partial^3 u}{\partial x_i^3} + \frac{\partial a_i}{\partial x_p} \frac{\partial^3 u}{\partial x_i^3} + \frac{\partial^2 a_i}{\partial x_k^2} \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 a_i}{\partial x_p^2} \frac{\partial^2 u}{\partial x_i^2} \right] - \frac{\partial a_p}{\partial x_k} \frac{\partial^3 u}{\partial x_k^3}$$

$$- \frac{\partial a_p}{\partial x_p} \frac{\partial^3 u}{\partial x_p^3} - \frac{\partial a_k}{\partial x_k} \frac{\partial^3 u}{\partial x_k^3} - \frac{\partial a_k}{\partial x_p} \frac{\partial^3 u}{\partial x_k^3} - \frac{\partial^2 a_k}{\partial x_k^2} \frac{\partial^2 u}{\partial x_k^2} - \frac{\partial^2 a_k}{\partial x_p^2} \frac{\partial^2 u}{\partial x_k^2}$$
at each point $\text{(5.1)}$

sizes in the discretisation process.

\[ \text{dimensions of the spatial domain completely free, whereas in the other possible cases} \]

\[ \text{the semi-discrete high-order compact schemes in spatial dimensions} \]

\[ b_{ij} \text{can conclude that in order to achieve a high-order compact scheme, we need either} \]

\[ \text{we derive conditions on the coefficients of the partial differential equation} \]

\[ \text{we only want} \]

\[ \text{equation (3.1), and the central difference operator for} \]

\[ C_{kp} \text{can be discretised on the compact stencil (2.6) using} \]

\[ \text{This can be written as} \]

\[ \text{(3.3)} \]

\[ \frac{\partial^4 u}{\partial x_k^4 \partial x_p} = \frac{C_{kp}}{a_k} - \frac{a_p}{a_k} \frac{\partial^4 u}{\partial x_k \partial x_p^3}. \]

\textbf{4. Conditions for obtaining a high-order compact scheme.} In this section we derive conditions on the coefficients of the partial differential equation (2.1) under which achieving a high-order compact scheme is possible, meaning that we only want to use points of the $n$-dimensional compact stencil (2.6) for discretisation and achieve a fourth-order scheme for $\Delta x_i \in O(h)$ for $j = 1, \ldots, n$ for a given step size $h$. Using equations (3.1) and (3.2) and then (3.3) in (2.5) leads to

\[ f = \sum_{i=1}^{n} a_i D_i^2 u + \sum_{i,j=1}^{n} b_{ij} D_i^2 D_j^2 u + \sum_{i=1}^{n} c_i D_i^3 u - \sum_{i=1}^{n} a_i (\Delta x_i)^2 B_i + \varepsilon \]

\[ \text{(4.1)} \]

\[ - \sum_{i,j=1}^{n} \frac{b_{ij} (\Delta x_i)^2 C_{ij}}{12 a_i} - \sum_{i,j=1}^{n} \frac{b_{ij}}{6} \frac{\partial^4 u}{\partial x_i \partial x_j^3} \left[ (\Delta x_j)^2 - \frac{a_j (\Delta x_j)^2}{a_i} \right] - \sum_{i=1}^{n} \frac{c_i (\Delta x_i)^2 A_i}{6}, \]

\[ \text{where} \varepsilon \in O(h^4), \text{if} \Delta x_i \in O(h) \text{for} i = 1, \ldots, n \text{for the step size} h > 0. \]

\[ \text{From this we can conclude that in order to achieve a high-order compact scheme, we need either} \]

\[ b_{ij} = 0 \quad \text{or} \quad (\Delta x_j)^2 = \frac{a_j (\Delta x_j)^2}{a_i} \]

\[ \text{for all pairs} (i, j) \in \{1, \ldots, n\} \text{with} i \neq j. \]

\[ \text{This means that in the case} b_{ij} \equiv 0 \text{for all} \]

\[ \text{it is possible to choose the step size of the discretisations of the different dimensions of the spatial domain completely free, whereas in the other possible cases for a high-order compact scheme there are interdependencies for at least some step sizes in the discretisation process.} \]

\textbf{5. Semi-discrete high-order compact schemes.} In this section we present the semi-discrete high-order compact schemes in spatial dimensions $n = 2, 3$. We consider the case where the cross derivatives do not vanish, hence we assume $a_i = a$ and $\Delta x_i = h > 0$ for $i = 1, \ldots, n$ to satisfy condition (4.2). Our aim in this section is to derive the semi-discrete schemes of the form

\[ \sum_{x \in C_h^{(n)}} [M_x(\hat{x}, \tau) \partial_x U_{i_1, \ldots, i_n}(\tau) + K_x(\hat{x}, \tau) U_{i_1, \ldots, i_n}(\tau)] = \tilde{g}(x, \tau), \]

\[ \text{at each point} x \in C_h^{(n)} \text{and at time} \tau, \text{where the function} \tilde{g} : C_h^{(n)} \times \Omega_\tau \to \mathbb{R} \text{depends} \]

\[ \text{on the function} g \text{given in (2.1).} \]
5.1. Semi-discrete two-dimensional scheme. In this section we derive the high-order compact discretisation of (2.1) in spatial dimension \( n = 2 \). Considering the grid points \((x_{i_1}, x_{i_2}) \in \mathcal{G}_h\) and time \( \tau \in \Omega \), we are able to obtain the coefficients \( \hat{K}_{l,m} \) of \( U_{l,m}(\tau) \) for \( l \in \{i_1 - 1, i_1, i_1 + 1\} \) and \( m \in \{i_2 - 1, i_2, i_2 + 1\} \) on the compact stencil by employing the central difference operator in (4.1). To streamline notation we denote by \([\cdot]'_{k} \) the first derivative with respect to \( x_k \) and by \([\cdot]''_{kp} \) the second derivative, once in \( x_k \)- and once in \( x_p \)-direction with \( k, p \in 1, 2 \). Note that in the following the functions \( a, b_{1,2}, c_1, c_2 \) and \( g \) are evaluated at \((x_{i_1}, x_{i_2}) \in \mathcal{G}_h\) and \( \tau \in \Omega \). We omit these arguments for the sake of readability. The coefficients are given by:

\[
\begin{align*}
\hat{K}_{i_1,i_2} &= - \frac{b_{12}[a]_{12}}{3a} - \frac{b_{12}[c_2]_1}{6a^2} + \frac{b_{12}[a][c_1]}{6a^2} + \frac{2b_{12}[a][a]_2}{3a^2} - \frac{[a]_{11}}{3} - \frac{10a}{3h^2} - \frac{[c_2]_2}{3} - \frac{b_{12}[c_1]_2}{6a} + \frac{2[a]_2}{3a} - \frac{c_1}{3} + \frac{2[a]_2}{3a} = 2.
\end{align*}
\]

\[
\hat{K}_{i_1,i_2} = \frac{b_{12}[a]_{12}}{3a} - \frac{b_{12}[c_2]_1}{6a^2} + \frac{b_{12}[a][c_1]}{6a^2} + \frac{2b_{12}[a][a]_2}{3a^2} - \frac{[a]_{11}}{3} - \frac{10a}{3h^2} - \frac{[c_2]_2}{3} - \frac{b_{12}[c_1]_2}{6a} + \frac{2[a]_2}{3a} - \frac{c_1}{3} + \frac{2[a]_2}{3a} = 2.
\]

\[
\begin{align*}
\hat{K}_{i_1,i_2} &= \frac{b_{12}[a]_{12}}{3a} - \frac{b_{12}[c_2]_1}{6a^2} + \frac{b_{12}[a][c_1]}{6a^2} + \frac{2b_{12}[a][a]_2}{3a^2} - \frac{[a]_{11}}{3} - \frac{10a}{3h^2} - \frac{[c_2]_2}{3} - \frac{b_{12}[c_1]_2}{6a} + \frac{2[a]_2}{3a} - \frac{c_1}{3} + \frac{2[a]_2}{3a} = 2.
\end{align*}
\]
Additionally, we obtain the coefficients $\hat{M}_{i,m}$ of $\partial_x U_{i,m}(r)$ for $l \in \{i_1 - 1, i_1, i_1 + 1\}$ and $m \in \{i_2 - 1, i_2, i_2 + 1\}$ at each point $(x^{(1)}_{i_1}, x^{(2)}_{i_2}) \in G^{(2)}_h$ and time $r \in \Omega_r$,

\[
\hat{M}_{i_1 \pm 1, i_2 \pm 1} = \frac{1}{12} + \frac{b_{12} h[a]_2}{24a^2} + \frac{h[a]_2}{12a} + \frac{b_{12} h[a]_1}{24a^2} + c_2 h, \\
\hat{M}_{i_1 \pm 1, i_2} = \frac{1}{12} + \frac{b_{12} h[a]_2}{24a^2} + \frac{h[a]_2}{12a} + \frac{b_{12} h[a]_1}{24a^2} + \frac{b_{12} h[a]_2}{24a^2}, \\
\hat{M}_{i_1, i_2 \pm 1} = \frac{1}{12} + \frac{b_{12} h[a]_2}{24a^2} + \frac{h[a]_2}{12a} + \frac{b_{12} h[a]_1}{24a^2} + \frac{b_{12} h[a]_2}{24a^2}.
\]

Additionally, for $x \in G^{(2)}_h$, $r \in \Omega_r$,

\[
\hat{g}(x, r) = \frac{(h^2 a^2 c_1 - 2 h^2 a^2[a]_1 - b_{12} h^2[a]_2 a) [g]_1}{12a^3} + \frac{h^2[g]_{11}}{12} + \frac{b_{12} h^2[g]_{12}}{12a} + \frac{b_{12} h^2[g]_{12}}{12a} + \frac{b_{12} h^2[g]_{12}}{12a} + \frac{b_{12} h^2[g]_{12}}{12a} + \frac{b_{12} h^2[g]_{12}}{12a} + \frac{b_{12} h^2[g]_{12}}{12a} + \frac{b_{12} h^2[g]_{12}}{12a} + \frac{b_{12} h^2[g]_{12}}{12a} + \frac{b_{12} h^2[g]_{12}}{12a} + \frac{b_{12} h^2[g]_{12}}{12a} + g
\]

holds. Thus we have $K_1(x^{(1)}_{i_1}, x^{(2)}_{i_2}, r) = \hat{K}_n_{1, n_2}$ and $M_2(x^{(1)}_{i_1}, x^{(2)}_{i_2}, r) = \hat{M}_{n_1, n_2}$ in (5.1) with $n_1 \in \{i_1 - 1, i_1, i_1 + 1\}$ and $n_2 \in \{i_2 - 1, i_2, i_2 + 1\}$ for $x = (x^{(1)}_{i_1}, x^{(2)}_{i_2}) \in G^{(2)}_h$ and $r \in \Omega_r$. $K_1$ and $M_2$ are zero otherwise and the discretisation only uses points of the compact stencil.

**5.2. Semi-discrete three-dimensional scheme.** In this section we derive the high order compact discretisation of (2.1) in spatial dimension $n = 3$. Considering the conditions in (4.2) we observe that in the three-dimensional case we have three different possibilities to satisfy the conditions and thus create a high-order compact scheme. We focus on the case $a = a_1 = a_2 = a_3$, and $h = \Delta x_1 = \Delta x_2 = \Delta x_3$.

Considering an interior grid point $(x^{(1)}_{i_1}, x^{(2)}_{i_2}, x^{(3)}_{i_3}) \in G^{(3)}_h$ and time $r \in \Omega_r$ we are able to obtain the coefficients $\hat{K}_{k, l, m}$ of $U_{k, l, m}(r)$ for $k \in \{i_1 - 1, i_1, i_1 + 1\}$, $l \in \{i_2 - 1, i_2, i_2 + 1\}$, and $m \in \{i_3 - 1, i_3, i_3 + 1\}$ by employing the central difference operator in (4.1). Again, to streamline the notation we denote by $[\cdot]_k$ and $[\cdot]_x$, the first and second derivative of the coefficients with respect to $x_k$, and with respect to $x_p$, respectively. Note again that in the following a, $b_{12}, b_{13}, b_{23}, c_1, c_2, c_3$ and $g$ are evaluated at $(x^{(1)}_{i_1}, x^{(2)}_{i_2}, x^{(3)}_{i_3}) \in G^{(3)}_h$ and $r \in \Omega_r$. We omit these arguments for the sake of readability. Due to the size of the coefficients $\hat{K}_{k, l, m}$, they are given in the appendix.

In a similar way we define $\hat{M}_{k, l, m}$ as the coefficient of $\partial_x U_{k, l, m}(r)$ for $k \in \{i_1 - 1, i_1, i_1 + 1\}$, $l \in \{i_2 - 1, i_2, i_2 + 1\}$, and $m \in \{i_3 - 1, i_3, i_3 + 1\}$ by

\[
\hat{M}_{i_1 \pm 1, i_2 \pm 1, i_3 \pm 1} = \frac{1}{12} + \frac{b_{12} h[a]_2}{24a^2} + \frac{h[a]_2}{12a} + \frac{b_{12} h[a]_1}{24a^2} + c_2 h, \\
\hat{M}_{i_1 \pm 1, i_2, i_3 \pm 1} = \frac{1}{12} + \frac{b_{12} h[a]_2}{24a^2} + \frac{h[a]_2}{12a} + \frac{b_{12} h[a]_1}{24a^2} + \frac{b_{12} h[a]_2}{24a^2}, \\
\hat{M}_{i_1, i_2 \pm 1, i_3 \pm 1} = \frac{1}{12} + \frac{b_{12} h[a]_2}{24a^2} + \frac{h[a]_2}{12a} + \frac{b_{12} h[a]_1}{24a^2} + \frac{b_{12} h[a]_2}{24a^2} + \frac{b_{12} h[a]_2}{24a^2}.
\]
For the right hand side of (5.1) we have for \( x = (x_{i_1}^{(1)}, x_{i_2}^{(2)}, x_{i_3}^{(3)}) \in \mathbb{G}_h^{(3)}, \tau \in \Omega_\tau, \)
\[
\dot{g}(x, \tau) = \left( c_1 h^2 a - 2h^2[a]_{11} - b_{12} h^2[a]_{2} - b_{13} h^2[a]_{3} \right) g_1 + \frac{b_{13} h^2 g_{13}}{12a^2} + \frac{b_{33} h^2 g_{23}}{12a^2} + \frac{b_{33} h^2 g_{23}}{12a^2} + \frac{b_{33} h^2 g_{23}}{12a^2} + g.
\]

We have \( K_x(x_{i_1}^{(1)}, x_{i_2}^{(2)}, x_{i_3}^{(3)}, \tau) = K_{n_1,n_2,n_3} \) and \( M_x(x_{i_1}^{(1)}, x_{i_2}^{(2)}, x_{i_3}^{(3)}, \tau) = M_{n_1,n_2,n_3} \) with \( n_1 \in \{i_1 - 1, i_1, i_1 + 1\}, n_2 \in \{i_2 - 1, i_2, i_2 + 1\} \) and \( n_3 \in \{i_3 - 1, i_3, i_3 + 1\} \) for each point \( x = (x_{i_1}^{(1)}, x_{i_2}^{(2)}, x_{i_3}^{(3)}) \in \mathbb{G}_h^{(3)} \) and \( \tau \in \Omega_\tau. \) \( K_x \) and \( M_x \) are zero otherwise. Thus the discretisation only uses points of the compact stencil (2.6).

### 6. Fully discrete scheme.

The semi-discrete scheme presented in the previous sections can be extended to a fully discrete scheme for the parabolic problem (2.1) by additionally discretising in time. Any time integrator can be implemented to solve the problem as in [20]. We consider only the most common class of methods involving two time steps and, in particular, employ a Crank-Nicolson type time-discretisation with constant time step \( \Delta\tau \) to obtain a fully discrete scheme. Let

\[
A_x(\hat{x}, \tau_{k+1}) = M_x(\hat{x}, \tau_{k+1}) + \frac{\Delta\tau}{2} K_x(\hat{x}, \tau_{k+1}), \quad B_x(\hat{x}, \tau_k) = M_x(\hat{x}, \tau_k) - \frac{\Delta\tau}{2} K_x(\hat{x}, \tau_k),
\]

where \( K_x(\hat{x}, \tau) \) and \( M_x(\hat{x}, \tau) \) are defined through a semi-discrete finite difference scheme with fourth-order consistency using only points of the compact stencil (2.6).

Then, a fully discrete high-order compact finite difference scheme for (2.1) with \( n \in \mathbb{N} \) on the uniform time grid \( \tau_k = k \Delta\tau \) for \( k = 0, \ldots, N_\tau \) is given by

\[
(6.1) \quad \sum_{\hat{x} \in \hat{U}(x)} A_x(\hat{x}, \tau_{k+1}) U^{k+1}_{\hat{x} l_1, \ldots, l_n} = \sum_{\hat{x} \in \hat{U}(x)} B_x(\hat{x}, \tau_k) U^k_{l_1, \ldots, l_n} + \dot{g}(x, \tau_k, \tau_{k+1}),
\]

at each point \( x = (x_{i_1}^{(1)}, \ldots, x_{i_n}^{(n)}) \in \mathbb{G}_h^{(n)} \) with \( h > 0, \hat{x} = (\hat{x}_{i_1}^{(1)}, \ldots, \hat{x}_{i_n}^{(n)}) \in \hat{U}(x). \) This scheme is second-order consistent in time and fourth-order consistent in space.

### 7. Stability analysis for the Cauchy problem in dimensions \( n = 2, 3. \)

In this section we consider the stability analysis of the high-order compact scheme for the Cauchy problem associated with (2.1) in the case \( n = 2, 3. \) The coefficients of the semi-discrete scheme are given in Section 5.1 for two spatial dimensions and in Section 5.2, when three spatial dimensions occur. Those coefficients are non-constant, as the coefficients of the parabolic partial differential equation (2.1) are non-constant.

We consider a von Neumann stability analysis. Other approaches which take into account boundary conditions like normal mode analysis [11] are beyond the scope of the present paper. For both \( n = 2 \) and \( n = 3, \) we give a proof of stability in the case of vanishing cross derivative terms and frozen coefficients in time and space, which means that all possible values for the coefficients are considered, but as constants, so the derivatives of the coefficients of the partial differential equation appearing in the discrete schemes are set to zero. This approach has been used as well in [11, 21] and
gives a necessary stability condition, whereas slightly stronger conditions are sufficient to ensure overall stability [17]. This approach is extensively used in the literature and yields good criteria on the robustness of the scheme. In (6.1) we use

\[ U_{j_1, \ldots, j_n}^k = g^k e^{i S_n} \text{ with } S_n = \sum_{m=1}^{n} j_m z_m \]

for \( j_m \in \{i_m - 1, i_m, i_m + 1\} \), where \( I \) is the imaginary unit, \( g^k \) is the amplitude at time level \( k \) and \( z_m = 2\pi h / \lambda_m \) for the wavelength \( \lambda_m \in [0, 2\pi] \) for \( m = 1, \ldots, n \). Then the fully discrete scheme satisfies the necessary von Neumann stability condition for all \( z_1, z_2 \), when the amplification factor \( G = g^{k+1} / g^k \) satisfies

\[ |G|^2 - 1 \leq 0. \]

**7.1. Stability analysis for the two-dimensional case.** In this section we perform the von Neumann stability analysis for the two-dimensional high-order compact scheme of Section 5.1. The analysis of the case with vanishing cross-derivative and frozen coefficients are carried out in detail. In the case of non-vanishing mixed derivatives partial results are given.

**Theorem 7.1.** For \( a = a_1 = a_2 < 0 \) and \( b_{1,2} = 0 \), the fully discrete high-order compact finite difference scheme given in (6.1) with \( n = 2 \), with coefficients defined in Section 5.1, satisfies (for frozen coefficients) the necessary stability condition defined in Section 5.1 can be written as \( |G|^2 - 1 = N_G / D_G \). This discussion the numerator \( N_G \) and the denominator \( D_G \) separately in the following.

The numerator can be written as \( N_G = 8 k a (n_4 h^4 + n_2 h^2) \) where the polynomials

\[ n_2 = 8a^2 f_1(\xi_1, \xi_2) f_2(\xi_1, \xi_2) \quad \text{and} \quad n_4 = f_3(\xi_1) f_4(\xi_1, \xi_2) c_1^2 + f_3(\xi_2) f_4(\xi_2, \xi_1) c_2^2 \]

are non-negative, since

\[
\begin{align*}
  f_1(x, y) &= x^2 + y^2 + 1 \geq 0, \\
  f_2(x, y) &= 2 - x \left( y^2 + \frac{1}{2} \right) - \frac{y^2}{2} \geq 0, \\
  f_3(x) &= x^2 - 1 \leq 0, \\
  f_4(x, y) &= 2x^2 y^2 - x^2 - 1 \leq 0,
\end{align*}
\]

for \( x, y \in [-1, 1] \). Thus, we observe that \( N_G \leq 0 \) holds, as \( \xi_1, \xi_2 \in [-1, 1] \).

Now we consider the denominator \( D_G \), which can be written as

\[ D_G = d_0 h^6 + (d_{1,2} k^2 + d_{4,1} k + d_{4,0}) h^4 + (d_{2,2} k^2 + d_{2,1} k) h^2 + d_0, \]

where

\[
\begin{align*}
  d_0 &= 16a^4 k^2 \left( 2x^2 y^2 + x^2 + y^2 - 4 \right)^2 \geq 0, \\
  d_{2,1} &= 16a^3 f_1(\xi_1, \xi_2) f_5(\xi_1, \xi_2) \geq 0, \\
  d_{2,2} &= 4a^2 \left[ 0 (\xi_1 c_1 + \xi_2 c_2) + 2 f_3(\xi_1) f_6(\xi_1, \xi_2) c_1^2 + 2 f_3(\xi_2) f_6(\xi_2, \xi_1) c_2^2 \right], \\
  d_{4,0} &= 4a^2 f_1(\xi_1, \xi_2) \geq 0, \\
  d_{4,1} &= -4 a_4 \geq 0, \\
  d_{4,2} &= \left[ f_5(\xi_2) c_1^2 - 2 \eta_1 \eta_2 \xi_1 \xi_2 c_1 c_2 + f_3(\xi_2) c_2^2 \right] \geq 0, \\
  d_6 &= (\xi_1 \eta_1 c_1 + \xi_2 \eta_2 c_2)^2 \geq 0,
\end{align*}
\]

because \( a < 0 \) and where

\[
\begin{align*}
  f_5(x, y) &= 2x^2 y^2 + x^2 + y^2 - 4 \leq 0, \\
  f_6(x, y) &= 2x^2 y^4 - 5x^2 - y^2 + 4,
\end{align*}
\]
with \(x, y \in [-1, 1]\). We observe that \(f_6\) (\(x, y\)) changes sign, as, for example \(f_6\) (\(0, 0\)) = 4 and \(f_6\) (\(1, 0\)) = -1. Hence, we cannot determine the sign of \(d_{2,2}\) directly.

If \(c_1 = c_2 = 0\), we have \(d_{2,2} = 0\) and hence \((7.1)\) is satisfied. Since \(d_{2,2}\) is symmetric, we can say without loss of generality that \(c_1 \neq 0\) in the following. Furthermore, as both \(c_1\) and \(c_2\) are frozen coefficients, we set \(m = c_2/c_1\), which leads to

\[
d_{2,2} = 4a^2c_1^2[9(\xi_1\eta_1 + \xi_2\eta_2m)^2 + 2f_3(\xi_1)f_6(\xi_1, \xi_2) + 2f_5(\xi_2)f_6(\xi_2, \xi_1)m^2] = 4a^2c_1^2g(m).
\]

The function \(g(m)\) can be rewritten as

\[
g(m) = \eta_1^2f_7(\xi_1, \xi_2)m^2 + 18\xi_1\xi_2\eta_1\eta_2m + \eta_2^2f_7(\xi_2, \xi_1)
\]

with \(f_7(x, y) = 4x^4y^2 - 2x^2 - y^2 + 8 \geq -2x^2 - y^2 + 8 \geq 5\). In the case \(\eta_2 = 0\) we have \(g(m) = \eta_1^2f_7(\xi_1, \xi_2)m^2 \geq 0\) and thus \(d_{2,2} \geq 0\). Hence, \((7.1)\) is satisfied. In the case \(\eta_2 \neq 0\) we have \(\eta_1^2f_7(\xi_1, \xi_2) > 0\), so the function \(g(m)\) has a global minimum. This minimum is located at

\[
m = \frac{-9\xi_1\xi_2\eta_1}{\eta_2f_7(\xi_1, \xi_2)},
\]

which leads to

\[
g(m) = \frac{2\eta_1^2f_5(\xi_1, \xi_2)f_8}{f_7(\xi_1, \xi_2)},
\]

where \(f_8 = 6\xi_1^4\xi_2^2 + \xi_1^4 + \xi_2^4 - 2\xi_1^4\xi_2^2\eta_1^2 - 2\xi_1^2\eta_1^2\xi_2^4 - 8 \leq 0\). Since \(f_5(\xi_1, \xi_2) \leq 0\) we have \(g(m) \geq 0\) for all \(m \in \mathbb{R}\), and thus we have \(D_{2,2} \geq 0\) as \(a < 0\). Therefore, the von Neumann stability condition \((7.1)\) is satisfied.

In fact, the proof of Theorem 7.1 is a lot stronger than necessary, since we prove that all polynomials appearing in the numerator and denominator of the expression are positive, instead of just their sum. For \(b_{1,2} \neq 0\) the situation becomes much more involved. Many additional terms appear in the expression for the amplification factor \(G\) and we face an additional degree of freedom through \(b_{1,2}\). Since we have proven condition \((7.1)\) holds for \(b_{1,2} \neq 0\) it seems reasonable to assume it also holds at least for values of \(b_{1,2}\) close to zero. In von Neumann stability analysis it is often most difficult to guarantee that stability condition \((7.1)\) holds for extreme values of \(\eta_1\), \(\eta_2\), \(\xi_1\) and \(\xi_2\). We have the following partial result which holds also in the case that \(b_{1,2} \neq 0\).

**Lemma 7.2.** The high-order compact scheme (8.9) with the coefficients for the two-dimensional case defined in Section 5.1 satisfies the stability condition \((7.1)\) at the corner points \(\xi_1 = \pm 1\) and \(\xi_2 = \pm 1\).

**Proof.** Using \(\eta_1 = \sin(z_1/2) = \sqrt{1 - \xi_1^2} = 0\) for \(\xi_1 = \pm 1\) and \(\eta_2 = \sin(z_2/2) = \sqrt{1 - \xi_2^2} = 0\) for \(\xi_2 = \pm 1\), straightforward computation shows that on each corner point \(|G| - 1 = 0\), and condition \((7.1)\) holds.

It is worth mentioning that in a comparable situation in [3] (where a specific partial differential equation of type \((2.1)\) is considered) an additional numerical evaluation of condition \((7.1)\) revealed it to hold also for non-vanishing mixed derivatives. However, the left hand side of \((7.1)\) was very close to zero, and although the inequality was always satisfied, this left little room for analytical estimates. This leads to the conjecture that the stability condition in that case was satisfied also for general parameters, although it would be hard to prove analytically. Lemma 7.2 above suggests the present case is similar. We remark that in our numerical experiments we observe a stable behaviour throughout, also for general choice of parameters.

**7.2. Stability analysis for the three-dimensional case.** In this section we analyse the stability of the high-order compact scheme with coefficients given in Section 5.2 in three space dimensions. We first perform a thorough von Neumann stability
analysis in the case of vanishing cross derivative terms. We observe no additional stability condition in this case. Then we give partial results in the case of non-vanishing cross-derivative terms.

**Theorem 7.3.** For \( a = a_1 = a_2 = a_3 < 0 \) and \( b_{1,2} = b_{1,3} = b_{2,3} = 0 \) the fully discrete high-order compact scheme given in (6.1) with \( n = 3 \), with coefficients given in Section 5.2 and the appendix, satisfies (for frozen coefficients) the necessary stability condition (7.1).

**Proof.** Let \( \xi_i = \cos(z_i/2) \) and \( \eta_i = \sin(z_i/2) \) for \( i = 1, 2, 3 \). The stability condition (7.1) can again be written as \( |G|^2 - 1 = N_G/D_G \). We discuss the numerator \( N_G \) and the denominator \( D_G \) separately in the following.

For the numerator we have \( N_G = -8ak (n_1h^4 + n_2h^2) \leq 0 \), since \( a < 0 \) and the polynomials

\[
n_2 = 4a^2f_1(\xi_1, \xi_2, \xi_3) [f_2(\xi_1, \xi_2) + f_2(\xi_3, \xi_1) + f_2(\xi_2, \xi_3)] \leq 0,
\]

\[
n_4 = [f_3(\xi_1, \xi_2) + f_3(\xi_1, \xi_3)]c_1^2 + [f_3(\xi_2, \xi_1) + f_3(\xi_2, \xi_3)]c_2^2 + [f_3(\xi_3, \xi_1) + f_3(\xi_3, \xi_2)]c_3^2
\]

\[-\eta_3^2(\xi_1\eta_1c_1 + \xi_2\eta_2c_2)^2 - \eta_2^2(\xi_1\eta_1c_1 + \xi_3\eta_3c_3)^2 - \eta_1^2(\xi_2\eta_2c_2 + \xi_3\eta_3c_3)^2 \leq 0,
\]

are non-negative since

\[
f_1(x,y) = x^2 + y^2 + z^2 \geq 0, \quad f_2(x,y) = 2x^2y^2 - x^2 - 1 \leq 0,
\]

\[
f_3(x,y) = 2x^2y^2(1 - x^2) + y^2(x^2 - 1) \leq y^2(1 - x^2) + y^2(x^2 - 1) = 0,
\]

for \( x, y, z \in [-1, 1] \).

The denominator \( D_G \) can be written as

\[
D_G = d_0h^6 + (d_{4,2}k^2 + d_{4,1}k + d_{4,0}) h^4 + (d_{2,2}k^2 + d_{2,1}k) h^2 + d_0,
\]

where

\[
d_0 = 16a^4k^2 [m_1(\xi_1, \xi_2) + m_1(\xi_3, \xi_1) + m_1(\xi_2, \xi_3)]^2 \geq 0, \quad d_{2,1} = 4an_2 \geq 0,
\]

\[
d_{2,2} = 4a^2m_6(\xi_1, \eta_1, \xi_3)c_1^2 + 2m_7(\xi_3)\xi_1\eta_2\eta_1c_1c_2 + m_6(\xi_2, \eta_2, \xi_1)c_2^2
\]

\[+ m_6(\xi_1, \eta_1, \xi_3)c_3^2 + 2m_7(\xi_2)\xi_1\eta_3\eta_1c_1c_3 + m_6(\xi_3, \eta_3, \xi_1)c_3^2
\]

\[+ m_6(\xi_2, \eta_2, \xi_3)c_2^2 + 2m_7(\xi_1)\xi_2\eta_3\eta_2c_1c_3 + m_6(\xi_3, \eta_3, \xi_2)c_3^2
\]

\[+ m_5(\xi_1, \eta_1, \xi_3)c_1^2 + m_5(\eta_2, \xi_1, \xi_3)c_2^2 + m_5(\eta_3, \xi_1, \xi_3)c_3^2 \]

\[
d_{4,0} = 4a^4n_2^2(\xi_1, \xi_2, \xi_3)^2 \geq 0, \quad d_{4,1} = 4an_4 \geq 0, \quad d_6 = [\xi_1\eta_1c_1 + \xi_2\eta_2c_2 + \xi_3\eta_3c_3]^2 \geq 0,
\]

\[
d_{4,2} = [\eta_1^2c_1^2 + \eta_2^2c_2^2 + \eta_3^2c_3^2 + 2\xi_1\eta_1\xi_2\eta_2c_1c_2 + 2\xi_1\eta_1\xi_3\eta_3c_1c_3 + 2\xi_2\eta_2\xi_3\eta_3c_2c_3]^2 \geq 0,
\]

since \( a < 0 \) and

\[
m_1(x,y) = 2x^2y^2 - x^2 - 1 \leq x^2 - 1 \leq 0, \quad m_2(x,y,z) = x^2 + y^2 + z^2 \geq 0,
\]

\[
m_3(x,y) = 2x^2y^2(1 - x^2) + y^2(x^2 - 1) \leq y^2(1 - x^2) + y^2(x^2 - 1) = 0,
\]

\[
m_4(x,y) = (1 - x^2)(x^2y^2 - 1) + y^2(x^2 - 1) \leq 0,
\]

\[
m_5(x,y,z) = -8x^4y^2z^2 + 4x^2y^2z^2 + 4x^2 \geq -8x^4y^2z^2 + 4x^2y^2z^2 + 4x^2
\]

\[= -4x^2y^2z^2 + 4x^2 \geq -4x^2 + 4x^2 = 0,
\]

\[
m_6(x_1, x_2, y) = 4x_1^2x_2^2y^4 + (8x_2^2x_1^2 + 2x_2^2)y^2 + x_2^2 + \frac{3}{2}x_1^2x_2^2 \in [0, 3],
\]

\[
m_7(x) = 2x^2(x^2 - (1 - x^2)) + 7 \geq 0,
\]
for \( x, y, z \in [-1, 1] \). We still need to show \( d_{2,2} \geq 0 \). Since we cannot determine the sign of \( d_{2,2} \) directly, we consider three different cases.

Having \( \xi_2 ^2 = \xi_3 ^2 = 1 \) leads to

\[
d_{2,2} = 4a^2 \left[ 2(-2.5\xi_1 ^2 \eta_1 ^2 + 3\eta_1 ^2) c_1 ^2 + (-8\eta_1 ^4 + 8\eta_2 ^2) c_1 ^2 \right] \geq 0
\]

as \( \xi_1 ^2 \leq 1 \) and \( \eta_1 ^2 \leq 1 \).

Secondly, we consider \( c_1 = c_2 = c_3 = 0 \). This leads directly to \( d_{2,2} = 0 \).

From now on we have \( (c_1, c_2, c_3) \neq (0, 0, 0) \). Since \( d_{2,2} \) is symmetric with respect to \( c_1, c_2, c_3 \), we assume without loss of generality \( c_1 \neq 0 \). Additionally, we have \( (\xi_2 ^2, \xi_3 ^2) \neq (1, 1) \). Setting \( p_2 := c_2 / c_1 \) and \( p_3 := c_3 / c_1 \) gives

\[
d_{2,2} = 4a^2 c_1 ^2 \left[ m_6 (\xi_1, \eta_1, \xi_2) + 2m_7 (\xi_1) \xi_2 \eta_1 p_2 + m_6 (\xi_2, \eta_2, \xi_1) p_2 ^2 \right.
\]
\[
+ m_6 (\xi_1, \eta_1, \xi_3) + 2m_7 (\xi_1) \xi_3 \eta_1 p_3 + m_6 (\xi_3, \eta_1, \xi_1) p_3 ^2
\]
\[
+ m_6 (\xi_2, \eta_2, \xi_3) p_2 ^2 + 2m_7 (\xi_1) \xi_2 \eta_3 p_2 p_3 + m_6 (\xi_3, \eta_1, \xi_2) p_3 ^2
\]
\[
+ m_5 (\eta_1, \xi_2, \xi_3) + m_5 (\eta_2, \xi_1, \xi_3) p_2 + m_5 (\eta_3, \xi_1, \xi_2) p_3 ^2
\]
\[
=: 4a^2 c_1 ^2 \left[ k_{11} p_2 ^2 + 2k_{22} p_3 ^2 + k_{12} p_2 p_3 + k_1 p_2 + k_2 p_3 + k_0 \right] =: 4a^2 c_1 ^2 g(p_2, p_3).
\]

To calculate the extremum of \( g(p_2, p_3) \),

\[
\nabla g(\hat{p}_2, \hat{p}_3) = \begin{pmatrix} 2k_{11} \hat{p}_2 + k_{12} \hat{p}_3 + k_1 \\ k_{12} \hat{p}_2 + 2k_{22} \hat{p}_3 + k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

is necessary, which leads to

\[
\hat{p}_2 = \frac{2k_1 k_{22} - k_2 k_{12}}{k_{12} - 4k_1 ^2 k_{22}}, \quad \hat{p}_3 = \frac{2k_2 k_{11} - k_1 k_{12}}{k_{12} - 4k_1 ^2 k_{22}}, \quad \text{where} \ k_{12} - 4k_1 ^2 k_{22} = q_1 q_2 q_3
\]

with

\[
q_1 = m_2 ^2 p_3 ^2, \quad q_2 = -2\xi_1 ^2 \xi_2 ^2 - 2\xi_2 ^2 \xi_3 ^2 - 2\xi_1 ^2 \xi_3 ^2 + \xi_1 ^2 + \xi_2 ^2 + \xi_3 ^2 + 3 \in [0, 4],
\]
\[
q_3 = 8\xi_1 ^4 \xi_2 ^2 \xi_3 ^2 + 4\xi_1 ^2 \xi_2 ^4 \xi_3 ^2 + 4\xi_1 ^2 \xi_2 ^2 \xi_3 ^4 + 4\xi_1 ^4 \xi_2 ^4 \xi_3 ^4 - 4\xi_1 ^4 \xi_2 ^2 \xi_3 ^2
\]
\[
- 4\xi_1 ^2 \xi_2 ^2 \xi_3 ^2 - 2\xi_1 ^2 \xi_2 ^2 \xi_3 ^2 - 6\xi_1 ^4 \xi_2 ^2 \xi_3 ^4 + 8\xi_1 ^2 \xi_2 ^2 \xi_3 ^2
\]
\[
+ 8\xi_1 ^2 \xi_3 ^2 + 20\xi_2 ^2 \xi_3 ^2 - 2\xi_1 ^2 - 3\xi_2 ^2 - 3\xi_3 ^2 - 6 \in [-9, 0].
\]

It holds \( q_1 q_2 q_3 \neq 0 \) for \( (\xi_2 ^2, \xi_3 ^2) \neq (1, 1) \). Since this is the unique root of \( \nabla g \), as \( k_{11}, k_{22} \geq 0 \), we have a minimum at \( (p_2, p_3) = (\hat{p}_2, \hat{p}_3) \). We obtain \( g(\hat{p}_2, \hat{p}_3) = q_4 q_5 / q_6 \), where

\[
q_4 = 2\eta_1 ^2 (2\xi_1 ^2 \xi_2 ^2 + 2\xi_1 ^2 \xi_3 ^2 + 2\xi_2 ^2 \xi_3 ^2 - \xi_1 ^2 - \xi_2 ^2 - \xi_3 ^2 - 3) \leq 2\eta_1 ^2 (\xi_1 ^2 + \xi_2 ^2 + \xi_3 ^2 - 3) \leq 0
\]
\[
q_5 = 8\xi_1 ^4 \xi_2 ^2 \xi_3 ^2 + 4\xi_1 ^2 \xi_2 ^4 \xi_3 ^2 + 4\xi_1 ^2 \xi_2 ^2 \xi_3 ^4 + 4\xi_1 ^4 \xi_2 ^4 \xi_3 ^4 - 4\xi_1 ^4 \xi_2 ^2 \xi_3 ^2
\]
\[
- 4\xi_1 ^2 \xi_2 ^2 \xi_3 ^2 - 2\xi_1 ^2 \xi_2 ^2 \xi_3 ^2 - 6\xi_1 ^4 \xi_2 ^2 \xi_3 ^4 + 8\xi_1 ^2 \xi_2 ^2 \xi_3 ^2
\]
\[
+ 8\xi_1 ^2 \xi_3 ^2 + 20\xi_2 ^2 \xi_3 ^2 - 2\xi_1 ^2 - 3\xi_2 ^2 - 3\xi_3 ^2 - 6 \in [0, 9],
\]
\[
q_6 = 8\xi_1 ^4 \xi_2 ^2 \xi_3 ^2 + 4\xi_1 ^2 \xi_2 ^4 \xi_3 ^2 + 4\xi_1 ^2 \xi_2 ^2 \xi_3 ^4 + 4\xi_1 ^4 \xi_2 ^4 \xi_3 ^4 - 4\xi_1 ^4 \xi_2 ^2 \xi_3 ^2
\]
\[
- 4\xi_1 ^2 \xi_2 ^2 \xi_3 ^2 - 2\xi_1 ^2 \xi_2 ^2 \xi_3 ^2 - 6\xi_1 ^4 \xi_2 ^2 \xi_3 ^4 + 8\xi_1 ^2 \xi_2 ^2 \xi_3 ^2
\]
\[
- 6\xi_2 ^2 \xi_3 ^2 + 8\xi_2 ^2 \xi_3 ^2 + 8\xi_1 ^2 \xi_3 ^2 + 20\xi_2 ^2 \xi_3 ^2 - 2\xi_1 ^2 - 3\xi_2 ^2 - 3\xi_3 ^2 - 6 \in [-9, 0],
\]

with \( q_6 \neq 0 \) for \( (\xi_2 ^2, \xi_3 ^2) \neq (1, 1) \). Hence, in all three cases we conclude \( d_{2,2} \geq 0 \), and \( N_G \geq 0 \) holds. Thus, the condition (7.1) is satisfied.
For the more general case with non-vanishing cross-derivatives we have the following result. The comments made in the previous section also apply here.

**Lemma 7.4.** The high-order compact scheme (6.1) with the coefficients for the three-dimensional case defined in Section 5.2 and the appendix satisfies the stability condition (7.1) at the corner points \( \xi_1 = \pm 1, \xi_2 = \pm 1 \) and \( \xi_3 = \pm 1 \).

**Proof.** Using \( \sin (z_1/2) = \sqrt{1 - \xi_1^2} = 0 \) for \( \xi_1 = \pm 1, \sin (z_2/2) = \sqrt{1 - \xi_2^2} = 0 \) for \( \xi_2 = \pm 1 \) and \( \sin (z_3/2) = \sqrt{1 - \xi_3^2} = 0 \) for \( \xi_3 = \pm 1 \), straight-forward computation yields \( |G| - 1 = 0 \) which satisfies condition (7.1). \( \Box \)

8. Application to Black-Scholes Basket options. To illustrate the practicality of the proposed scheme we now consider the \( n \)-dimensional Black-Scholes option pricing PDE (see, e.g. [23]). In the option pricing problem mixed derivatives appear naturally from correlation of the underlying assets. After transformations, the conditions (4.2) are satisfied, and we give the coefficients of the resulting scheme. Then we discuss the boundary conditions as well as the time discretisation.

8.1. Transformation of the \( n \)-dimensional Black-Scholes equation. In the multidimensional Black Scholes model the asset prices follow a geometric Brownian motion,

\[
(8.1) \quad dS_i(t) = (\mu_i - \delta_i)S_i(t)dt + \sigma_iS_i(t)dW_i(t),
\]

where \( S_i \) is the \( i \)-th underlying asset which has an expected return of \( \mu_i \), a continuous dividend of \( \delta_i \), and the volatility \( \sigma_i \) for \( i = 1, \ldots, n \) and \( n \in \mathbb{N} \). The Wiener processes are correlated with \( \langle dW_i, dW_j \rangle = \rho_{ij}dt \) for \( i, j = 1, \ldots, n \) with \( i \neq j \). Application of Itô’s lemma and standard arbitrage arguments show that any option price \( V(S, \sigma, t) \) solves the \( n \)-dimensional Black-Scholes partial differential equation,

\[
(8.2) \quad \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 S_i^2 \frac{\partial^2 V}{\partial S_i^2} + \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} \eta_i S_i \frac{\partial V}{\partial S_i} - rV = 0,
\]

where \( \eta_i = r - \delta_i \). The transformations

\[
(8.3) \quad x_i = \gamma \ln (S_i/K) / \sigma_i, \quad \tau = T - t \quad \text{and} \quad u = e^{\tau} V/K,
\]

for \( i = 1, \ldots, n \), where \( \gamma \) is a constant scaling parameter to assure that the resulting computational domain does not get too large, leads to

\[
(8.4) \quad u_{\tau} - \frac{\gamma^2}{2} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \gamma^2 \sum_{i,j=1}^{n} \rho_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \gamma \sum_{i=1}^{n} \xi_i \frac{\partial u}{\partial x_i} = 0,
\]

where \( \xi_i = \sigma_i / 2 - \eta_i / \sigma_i \). Comparing this equation with (2.1), we identify

\[
(8.5) \quad a_i = -\frac{\gamma}{2}, \quad b_{ij} = -\gamma^2 \rho_{ij}, \quad c_i = \gamma \xi_i, \quad d = 0,
\]

for \( i, j = 1, \ldots, n \) and \( i < j \). We find that the transformed partial differential equation (8.4) with these coefficients satisfies the conditions given by (4.2), if \( \Delta x_i = h \) for a step size \( h > 0 \) is used. Hence, we are able to obtain a high-order compact scheme in any spatial dimension \( n \in \mathbb{N} \).
We consider a European Power-Put Basket option, thus the final condition for (8.2) is given by

\[ V(S_1, \ldots, S_n, T) = \max \left( K - \sum_{i=1}^{n} \omega_i S_i, 0 \right)^p, \]

where \( p \) is an integer and the asset weights satisfy \( \sum_{i=1}^{n} \omega_i = 1 \). Applying the transformations (8.3) leads to the initial condition

\[(8.6) \quad u(x_1, \ldots, x_n, 0) = K^{p-1} \max \left( 1 - \sum_{i=1}^{n} \omega_i e^{\frac{\rho_{ix_i}}{2}}, 0 \right)^p. \]

### 8.2. Semi-discrete two-dimensional Black-Scholes equation
In this section we apply our general two-dimensional semi-discrete scheme, see Section 5.1, to the two-dimensional Black-Scholes model. To obtain the semi-discrete scheme (5.1) we have to apply (8.5) with \( n = 2 \) to the coefficients in Section 5.1, which gives

\[ \hat{K}_{i, i_2} = \frac{\gamma_2^2 (5 - 2 \rho_{i_2}^2)}{3h^2} + \frac{s_1^2 + s_2^2}{3}, \quad \hat{K}_{i_1 \pm 1, i_2} = \frac{\gamma_2^2 \rho_{i_2}^2}{3h^2} \pm \frac{\gamma_2 s_{i_1}}{3h} - \frac{s_1^2}{6} - \frac{\gamma_2^2}{3h^2}, \]

\[ \hat{K}_{i_1, i_2 \pm 1} = \frac{\gamma_2^2 \rho_{i_2}^2}{3h^2} \pm \frac{\gamma_2 s_2}{3h} - \frac{s_1^2}{6} - \frac{\gamma_2^2}{3h^2}, \]

\[ \hat{K}_{i_1 \pm 1, i_2 - 1} = \pm \frac{s_2 s_1}{12} - \frac{\gamma_2 s_2}{12h} + \frac{\gamma_2 s_{i_1}}{12h} - \frac{\gamma_2^{2} \rho_{i_2}^{2}}{6h} - \frac{\gamma_2^{2}}{12h^{2}} \pm \frac{\gamma_2^{2} \rho_{i_2}^{2}}{6h}, \]

\[ \hat{K}_{i_1 \pm 1, i_2 + 1} = \frac{s_2 s_1}{12} + \frac{s_2 s_{i_2}}{12h} + \frac{\gamma_2 \rho_{i_1 s_{i_2}}}{6h} + \frac{\gamma_2 \rho_{i_2 s_{i_1}}}{6h} - \frac{\gamma_2^{2} \rho_{i_2}^{2}}{6h} - \frac{\gamma_2^{2}}{12h^{2}} \pm \frac{\gamma_2^{2} \rho_{i_2}^{2}}{6h}, \]

where \( \hat{K}_{l,m} \) is the coefficient of \( U_{l,m}(\tau) \) for \( l \in \{i_1 - 1, i_1, i_1 + 1\} \) and \( m \in \{i_2 - 1, i_2, i_2 + 1\} \). Similarly, we get

\[ M_{i_1, i_2} = \frac{2}{3}, \quad M_{i_1 + 1, i_2 \pm 1} = M_{i_1 - 1, i_2 \pm 1} = \pm \frac{\rho_{i_2}}{24}, \]

\[ M_{i_1 \pm 1, i_2} = \frac{1}{12} \pm \frac{h s_1}{12 \gamma}, \quad M_{i_1, i_2 \pm 1} = \frac{1}{12} \pm \frac{h s_2}{12 \gamma}, \]

as coefficients of \( \partial_{\tau} U_{l,m}(\tau) \). Additionally, we get \( \tilde{g}(x, \tau) = 0 \). We obtain a semi-discrete scheme of the form (5.1), where \( K_x \) and \( M_x \) are time-independent.

### 8.3. Semi-discrete three-dimensional Black-Scholes equation
In this section we give the semi-discrete scheme (5.1) for the three-dimensional Black-Scholes Basket option. Using (8.5) with \( n = 3 \) in Section 5.1 and the appendix we obtain the coefficients \( \hat{K}_{k,l,m} \) of \( U_{k,l,m}(\tau) \) for \( k \in \{i_1 - 1, i_1, i_1 + 1\}, l \in \{i_2 - 1, i_2, i_2 + 1\} \) and \( m \in \{i_3 - 1, i_3, i_3 + 1\} \), which are

\[ \hat{K}_{i_1, i_2, i_3} = \frac{s_1^2}{3} + \frac{s_2^2}{3} + \frac{s_3^2}{3} - 2 \gamma_2 \rho_{i_2}^2 3h^2 - 2 \gamma_2 \rho_{i_3}^2 3h^2 - 2 \gamma_2 \rho_{i_3}^2 3h^2 + 2 \gamma_2^2 3h^2, \]

\[ \hat{K}_{i_1 \pm 1, i_2, i_3} = \frac{s_1^2}{6} \pm \frac{\gamma_2 s_2}{3h} + \frac{\gamma_2 \rho_{i_2 s_{i_2}}}{3h^2} - \frac{\gamma_2^2 \rho_{i_2}^2}{3h^2} \pm \frac{\gamma_2 \rho_{i_2 s_{i_2}}}{3h^2}, \]

\[ \hat{K}_{i_1, i_2 \pm 1, i_3} = \frac{s_2 s_1}{6} + \frac{s_2 s_{i_3}}{3h} \pm \frac{\gamma_2 \rho_{i_2 s_{i_3}}}{3h^2} - \frac{\gamma_2 \rho_{i_2 s_{i_3}}}{3h^2} + \frac{\gamma_2 \rho_{i_2 s_{i_3}}}{3h^2}, \]

\[ \hat{K}_{i_1, i_2, i_3 \pm 1} = \frac{s_3 s_1}{6} + \frac{s_2 s_{i_3}}{3h} + \frac{\gamma_2 \rho_{i_2 s_{i_3}}}{3h^2} - \frac{\gamma_2 \rho_{i_2 s_{i_3}}}{3h^2} + \frac{\gamma_2 \rho_{i_2 s_{i_3}}}{3h^2}, \]
The coefficients are again given by (8.5) for $f$ with boundaries, again, we are able to obtain (and use) a high-order compact scheme. The $S$ coefficients are given by:

$$
\hat{K}_{i_1,i_2,i_3} = \gamma S_{i_1} \pm \frac{S_{i_2}}{12} \pm \frac{S_{i_3}}{12} - \frac{\gamma^2}{12 h^2} S_{i_1} \pm S_{i_2} \pm S_{i_3} - \frac{\gamma^2}{6 h} \rho_{12} + \frac{\gamma^2}{6 h} \rho_{13} \rho_{23},
$$

$$
\hat{K}_{i_1,i_2,i_3} = \gamma S_{i_1} \pm \frac{S_{i_2}}{12} \pm \frac{S_{i_3}}{12} = \frac{\gamma^2}{12 h^2} S_{i_1} \pm S_{i_2} \pm S_{i_3} + \frac{\gamma^2}{6 h} \rho_{12} + \frac{\gamma^2}{6 h} \rho_{13} \rho_{23},
$$

$$
\hat{K}_{i_1,i_2,i_3} = \gamma S_{i_1} \pm \frac{S_{i_2}}{12} \pm \frac{S_{i_3}}{12} = \frac{\gamma^2}{12 h^2} S_{i_1} \pm S_{i_2} \pm S_{i_3} - \frac{\gamma^2}{6 h} \rho_{12} + \frac{\gamma^2}{6 h} \rho_{13} \rho_{23},
$$

$$
\hat{K}_{i_1,i_2,i_3} = \gamma S_{i_1} \pm \frac{S_{i_2}}{12} \pm \frac{S_{i_3}}{12} = \frac{\gamma^2}{12 h^2} S_{i_1} \pm S_{i_2} \pm S_{i_3} + \frac{\gamma^2}{6 h} \rho_{12} + \frac{\gamma^2}{6 h} \rho_{13} \rho_{23},
$$

Similarly, we get the coefficients $\hat{M}_{k,l,m}$ of $\partial_i U_{k,l,m}(\tau)$, given by

$$
\hat{M}_{i\pm1,j,m} = \hat{M}_{i\mp1,j,m+1} = \frac{\rho_{13}}{24}, \quad \hat{M}_{i,j\pm1,m} = \frac{\rho_{13}}{24}, \quad \hat{M}_{i,j\pm1,m+1} = \frac{\rho_{13}}{24},
$$

$$
\hat{M}_{i,j\pm1,m} = \frac{1}{12} + \frac{h_{i2}}{12 \gamma}, \quad \hat{M}_{i,j,m\pm1} = \frac{1}{12} + \frac{h_{i3}}{12 \gamma}, \quad \hat{M}_{i,j,m} = \frac{1}{12},
$$

$$
\hat{M}_{i\pm1,j,m+1} = \hat{M}_{i\pm1,j,m+1} = 0, \quad \hat{M}_{i\pm1,j,m+1} = \hat{M}_{i\pm1,j,m+1} = 0.
$$

Additionally, we have $\hat{g}(x, \tau) = 0$, similar as in the case $n = 2$. We obtain a semi-discrete scheme of the form (5.1), where $K_x$ and $M_x$ are time-independent.

### 8.4. Treatment of the boundary conditions

**8.4.1. Lower boundaries.** The first boundary we discuss is $S_i = 0$ for some $i \in I \subset \{1, \ldots, n\}$ at time $t \in [0,T]$. Once the value of the asset is zero, it stays constant over time, see (8.1). Thus using $S_i = 0$ for $i \in I$ in (8.2) and applying the transformation (8.3) leads to

$$
-\frac{\gamma^2}{2} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\gamma^2}{2} \sum_{i,j=1, i < j}^{n} \rho_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \gamma \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} = f,
$$

with $f = -u_x$. Comparing this differential equation with (2.1) we can see that the coefficients are again given by (8.5) for $i, j \in \{1, \ldots, n\} \setminus I$ with $i < j$. So at these boundaries, again, we are able to obtain (and use) a high-order compact scheme. The case $I = \{1, \ldots, n\}$ leads to the Dirichlet boundary condition $u(x_{(1)}^{(1)}, \ldots, x_{(n)}^{(n)}, \tau) = u(x_{(1)}^{(1)}, \ldots, x_{(n)}^{(n)}, 0)$ at time $\tau \in [0, \tau_{\text{max}}]$, since in that case $u_x = 0$. 


8.4.2. Upper boundaries. Upper boundaries are boundaries with \( S_i = S^\text{max}_i \) for some \( i \in J \subset \{1, \ldots, n\} \) at time \( t \in [0, T] \). For a sufficiently large \( S^\text{max}_i \) for \( i \in J \), we set

\[
\frac{\partial V(S_1, \ldots, S_n, t)}{\partial S_i} \bigg|_{S_i = S^\text{max}_i} = 0,
\]

with \( S_k \in [S^\text{min}_k, S^\text{max}_k] \) for \( k = \{1, \ldots, n\} \setminus \{i\} \) for a European Power Put Basket option. Employing this in (8.2) and using the transformations (8.3), yields

\[
(8.7) \quad -\gamma^2 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \gamma^2 \sum_{i,j=1}^{n} \rho_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \gamma \sum_{i=1}^{n} \gamma_{ij} \frac{\partial u}{\partial x_i} = f,
\]

with \( f = -u_t \). Hence the upper boundaries show the same behaviour as the lower boundaries for a European Power Put Basket and we can obtain a high-order compact scheme for these boundaries as well. As in Section 8.4.1, we have the Dirichlet boundary condition \( u(x^{(1)}_{\text{max}}, \ldots, x^{(n)}_{\text{max}}, \tau) = u(x^{(1)}_{\text{max}}, \ldots, x^{(n)}_{\text{max}}, 0) \) for \( \tau \in [0, \tau_{\text{max}}] \) if \( J = \{1, \ldots, n\} \).

8.5. Time discretisation. With the results from the previous sections we obtain a semi-discrete system of the form

\[
(8.8) \quad \sum_{\hat{x} \in G_h^{(n)}} [M_\hat{x}(\hat{x}) u_\tau(x, \tau) + K_\hat{x}(\hat{x}) u(\hat{x}, \tau)] = \bar{g}(x),
\]

for each point \( x \) of the grid \( G_h^{(n)} \), see (2.3). The functions \( K_\hat{x}, M_\hat{x} \), as well as \( g \) are given through the spatial discretisation process and are time-independent in our example. \( M_\hat{x} \) and \( K_\hat{x} \) are only non-zero on the compact \( n \)-dimensional stencil (2.6). Thus, our equation system given by (8.8) only has up to \( 3^n \) entries on the grid \( G_h^{(n)} \) for \( u_\tau \) and \( u \), respectively. We have defined these non-zero coefficients, as well as \( \bar{g} \), in Sections 8.2 and 8.3 for the cases \( n = 2 \) and \( n = 3 \), respectively.

We use an equidistant time grid of the form \( \tau = k \Delta \tau \) for \( k = 0, \ldots, N_\tau \) with \( N_\tau \in \mathbb{N} \) and a Crank-Nicolson-type time discretisation with step size \( \Delta \tau \), leading to

\[
(8.9) \quad \sum_{\hat{x} \in G_h^{(n)}} \left[ M_\hat{x}(\hat{x}) + \frac{\Delta \tau}{2} K_\hat{x}(\hat{x}) \right] u(\hat{x}, \tau + \Delta \tau) \]

\[
= \sum_{\hat{x} \in G_h^{(n)}} \left[ M_\hat{x}(\hat{x}) - \frac{\Delta \tau}{2} K_\hat{x}(\hat{x}) \right] u(\hat{x}, \tau) + (\Delta \tau) \bar{g}(x)
\]

on each point \( x \) of the grid \( G_h^{(n)} \). This system of equations has to be solved for every time step with \( \tau = k \Delta \tau \) for \( k = 0, \ldots, N_\tau - 1 \). For the Crank-Nicolson time discretisation this fully discrete scheme has consistency order two in time and four in space. Thus, we have fourth-order consistency in terms of \( h \) for \( \Delta \tau \in \mathcal{O}(h^2) \).

9. Numerical experiments for Black-Scholes Basket options. In this section we discuss the numerical experiments for the Black-Scholes Basket Power Puts in spatial dimensions \( n = 2, 3 \). The equation systems which have to be solved over
time have been derived in Section 8. According to [13], we cannot expect fourth-order convergence if the initial condition is not sufficiently smooth. Hence, we have to smooth the initial condition for Power Puts with \( p = 1, 2 \). In [13] suitable smoothing operators are identified in Fourier space. Since the order of convergence of our high-order compact scheme is four, we use the smoothing operator \( \Phi_4 \), given by its Fourier transform

\[
\hat{\Phi}_4(\omega) = \left( \frac{\sin(\omega/2)}{\omega/2} \right)^4 \left[ 1 + \frac{2}{3} \sin^2(\omega/2) \right].
\]

This leads to the smoothed initial condition

\[
\tilde{u}_0(x_1, x_2) = \int_{-3h}^{3h} \int_{-3h}^{3h} \Phi_4 \left( \frac{x}{h} \right) \Phi_4 \left( \frac{y}{h} \right) u_0(x_1 - x, x_2 - y) \, dx \, dy,
\]

in the case \( n = 2 \) for any step size \( h > 0 \), where \( u_0 \) is the original initial condition and \( \Phi_4(x) \) denotes the Fourier inverse of \( \hat{\Phi}_4(\omega) \), see [13]. If \( u_0 \) is smooth enough in the integrated region around \((x_1, \ldots, x_n)\), we have \( \tilde{u}_0(x_1, \ldots, x_n) = u_0(x_1, \ldots, x_n) \). That means that it is possible to identify the points where smoothing is necessary.

![Example of grid points selected for the smoothing procedure in two space dimensions.](image)

Figure 1 shows an example of a two-dimensional grid on the left side and on the right side a graph of the non-differentiable points of the initial condition given in (8.6) together with the identified grid points, where smoothing is necessary. The points are chosen in such a way that we ensure that the non-differentiable points have no influence on \( \tilde{u}_0(x_1, x_2) \) for those points, which are not shown in Figure 1 on the right hand side. This approach reduces the necessary calculations significantly. As \( h \to 0 \), the smooth initial condition \( \tilde{u}_0 \) converges towards the original initial condition \( u_0 \) given in (8.6). The results in [13] guarantee high-order convergence of the approximation of the smoothed problem to the true solution of (8.4).

We use the relative \( L^2 \)-error \( \| U_{\text{ref}} - U \|_{L^2} / \| U_{\text{ref}} \|_{L^2} \), as well as the \( L^\infty \)-error \( \| U_{\text{ref}} - U \|_{L^\infty} \) to examine the numerical convergence rate, where \( U_{\text{ref}} \) denotes a reference solution on a fine grid and \( U \) is the approximation. When identifying the convergence
order of the schemes, we determine it as the slope of the linear least square fit of the individual error points in the loglog-plots of error versus number of discretisation points per spatial direction.

9.1. Numerical example with two underlying assets. In this section we report the numerical results for a two-dimensional Black-Scholes Basket Power Put. We compare the high-order compact scheme (‘HOC’) with the standard scheme (‘2nd’), which is obtained by using the central difference operator directly in (8.4) for $n = 2$ with no further action, leading to a classical second-order scheme. We consider plain European Puts ($p = 1$) as well as European Power Puts with power $p = 2, 3$. For the European Put ($p = 1$) and the European Power Put with power $p = 2$, we use the smoothing procedure outlined above for the initial condition (8.6), whereas we use the original initial condition for $p = 3$. The parameter values

$\sigma_1 = 0.25, \sigma_2 = 0.35, \gamma = 0.25, r = \ln(1.05) , \omega_1 = 0.35 = 1 - \omega_2, K = 10,$

and $\delta_1 = \delta_2 = 0$ are used, unless stated otherwise. The parabolic mesh ratio is fixed to $\Delta\tau/h^2 = 0.4$, although we point out that neither the von Neumann stability analysis nor our numerical experiments revealed any practical restrictions on its choice.

![Figure 2](image)

**Fig. 2.** $l^\infty$- (left) and relative $l^2$-error (right) for two-dimensional Black-Scholes Basket Put and smoothed initial condition.

Figure 2 shows convergence plots for the $l^\infty$-error (left) and for the relative $l^2$-error (right) for a European Put, respectively. The initial condition is smoothed using the procedure outlined above. For both types of errors we observe that the numerical convergence rates agree very well with the theoretical orders of the schemes. The high-order compact scheme yields numerical convergence orders close to four and strongly outperforms the standard second-order scheme. The choice of the correlation parameter $\rho = -0.8, \rho = 0$ and $\rho = 0.8$ has very little influence.

Figure 3 shows the behaviour of the relative $l^2$-error for a European Power Put with $p = 2$ in the left plot. Again, smoothing of the initial condition is employed. The high-order compact scheme has convergence rates close to four, whereas the convergence rates of the standard schemes are about two. In the right plot of Figure 3 we see the results for the relative $l^2$-error for a European Power Put with $p = 3$. The initial conditions is not smoothed in this case. Again, the high-order compact schemes behaves very similar for different correlation values, and outperforms the second-order scheme drastically.
Fig. 3. Relative $l^2$-error for two-dimensional Black-Scholes Basket Power Put, with $p = 2$ and smoothed initial condition (left) and $p = 3$ (right).

Fig. 4. Relative $l^2$-error for three-dimensional Black-Scholes Basket Power Put, with $p = 3$ (left) and $p = 4$ (right).

9.2. Numerical example with three assets. In this section we report on numerical experiments with three underlying assets. We choose the parameters

$$\delta_i = 0.01, \quad \sigma_i = 0.3, \quad \omega_i = 1/3, \quad r = \ln(1.05), \quad \gamma = 0.3, \quad T = 0.25, \quad K = 10.$$  

Due to the computational intensity of the three-dimensional problem the number of grid points per spatial dimension is smaller compared to the results in two dimensions reported above. To ensure that at the same time there is a sufficiently large number of grid points in time, we fix the parabolic mesh ratio to $\Delta \tau / h^2 = 0.1$ (not for stability reasons). We perform two types of experiments: without any correlation between the assets (labeled by ‘nc’ in the plots), and with correlation (labeled by ‘c’ in the plots) using the parameter values $\rho_{1,2} = -0.4, \rho_{1,3} = -0.1, \rho_{2,3} = -0.2$.

We compare the standard discretisation to our high-order compact scheme for European Power Put options with $p = 3, 4$. For the European Power Puts with $p = 1, 2$ one would smooth the initial condition, similar as above, to ensure high-order convergence. Figure 4 shows the convergence of the relative $l^2$-error for a European Power Put with $p = 3$ and $p = 4$. We use the original initial conditions, no smoothing is applied here. The numerical convergence rates of the high-order compact scheme are slightly reduced to about three and three and a half, respectively. Additional smoothing, which we omitted here due to limit the computational load, would result in even better results. Still, in the high-order compact scheme outperforms the standard
second-order scheme significantly in all cases.

10. Conclusion. We presented a new high-order compact scheme for a rather general class of parabolic partial differential equations with time and space dependent coefficients, including mixed second-order derivative terms in \( n \) spatial dimensions. The resulting schemes are fourth-order accurate in space and second-order accurate in time. In a thorough von Neumann stability analysis, where we focussed on the case of vanishing mixed derivative terms, we showed that a necessary stability condition holds without further conditions in two and three space dimensions. For non-vanishing mixed derivative terms we were able to give partial results. The results suggest unconditional stability of the scheme. As an application example we considered the pricing of European Power Put options in the multidimensional Black-Scholes model. The typical initial conditions of this problem lack sufficient regularity, therefore a suitable smoothing procedure was employed to ensure high-order convergence. In all numerical experiments performed a comparative standard second-order scheme is significantly outperformed.

Although we derived the scheme in arbitrary space dimension, it was not our aim in this paper to attack the so-called curse of dimensionality. The issue of exponentially increasing number of unknowns with growing spatial dimension on full grids is of course alleviated to some degree by a high-order scheme. To obtain a similar accuracy as a second-order scheme which uses \( O(N^d) \) unknowns on a full grid, our high-order compact approach will "only" require \( O(N^{d/2}) \) unknowns. To really attack very high-dimensional problems one would need to combine our approach with hierarchical approaches, e.g. using sparse grids (typically requiring \( O(N \ln(N)^{d-1}) \) unknowns), which is beyond the scope of the present paper.

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Appendix. Coefficients for semi-discrete scheme in three dimensions.

Considering an interior grid point \((x_{i1}^{(1)}, x_{i2}^{(2)}, x_{i3}^{(3)}) \in \Omega_h\) and time \(\tau \in \Omega_T\), the coefficients \(\hat{K}_{k,l,m}\) of \(U_{k,l,m}(\tau)\) for \(k \in \{i_1 - 1, i_1, i_1 + 1\}\) and \(m \in \{i_3 - 1, i_3, i_3 + 1\}\) of the three-dimensional semi-discrete scheme are given by:

\[
\hat{K}_{11,12,13} = \frac{b_{13}[a]_{12} c_{12}}{6a^2} + \frac{b_{12}[a]_{13} c_{13}}{6a^2} - \frac{c_{11} c_{12}}{3} - \frac{c_{13}}{6a} - \frac{c_{32}}{6a} - \frac{[a]_{11}}{2} - \frac{[a]_{22}}{2} - \frac{[a]_{33}}{2} + \frac{b_{13}[a]_{13} c_{13}}{6a^2}
\]

\[
+ \frac{b_{12}[a]_{12} c_{12}}{6a^2} \frac{4a}{h^2} + \frac{b_{13}[a]_{13} c_{13}}{a^2} + \frac{b_{23}[a]_{13} c_{23}}{6a^2} + \frac{b_{23}[a]_{13} c_{23}}{6a^2} + \frac{b_{12}[a]_{12} c_{12}}{a^2}
\]

\[
+ \frac{b_{12}[a]_{12} c_{12}}{6a^2} - \frac{b_{13}[a]_{13} c_{13}}{6a} - \frac{c_{11} c_{12}}{3} + \frac{c_{13} c_{23}}{6a} + \frac{b_{12}[a]_{12} c_{12}}{2a} - \frac{c_{22} c_{23}}{2a} + \frac{b_{12}[a]_{12} c_{12}}{6a^2} + \frac{b_{12}[a]_{12} c_{12}}{6a^2} + \frac{b_{12}[a]_{12} c_{12}}{6a^2}
\]

\[
\hat{K}_{11,12,-1,13} = \frac{b_{13}[a]_{31} b_{12}}{24a^2 h} + \frac{b_{23}[a]_{32} b_{12}}{24a^2 h} \frac{[b_{12}]_{11}}{48} + \frac{[b_{12}]_{22}}{48} + \frac{[b_{12}]_{33}}{48} + \frac{b_{12}[a]_{12}}{12a h} - \frac{b_{12}[a]_{12}}{12a h} \frac{b_{12}[a]_{12}}{12a h}
\]

\[
+ \frac{b_{12}[a]_{12} c_{12}}{48a^2} + \frac{[b_{12}]_{12}}{48a^2} + \frac{[b_{12}]_{22}}{48a^2} + \frac{[b_{12}]_{33}}{48a^2} + \frac{b_{12}[a]_{12}}{12a h} + \frac{b_{12}[a]_{12}}{12a h} \frac{b_{12}[a]_{12}}{12a h}
\]

\[
+ \frac{b_{12}[a]_{12} c_{12}}{48a^2} + \frac{[b_{12}]_{12}}{48a^2} + \frac{[b_{12}]_{22}}{48a^2} + \frac{[b_{12}]_{33}}{48a^2} + \frac{b_{12}[a]_{12}}{12a h} + \frac{b_{12}[a]_{12}}{12a h} \frac{b_{12}[a]_{12}}{12a h}
\]

\[
+ \frac{c_{11} c_{12}}{12a h} + \frac{c_{12} c_{23}}{12a h} - \frac{b_{12}[a]_{12}}{24a h} + \frac{b_{12}[a]_{12}}{24a h} \frac{b_{12}[a]_{12}}{24a h}
\]

\[
+ \frac{b_{12}[a]_{12} c_{12}}{48a^2} + \frac{[b_{12}]_{12}}{48a^2} + \frac{[b_{12}]_{22}}{48a^2} + \frac{[b_{12}]_{33}}{48a^2} + \frac{b_{12}[a]_{12}}{12a h} + \frac{b_{12}[a]_{12}}{12a h} \frac{b_{12}[a]_{12}}{12a h}
\]

\[
+ \frac{b_{12}[a]_{12} c_{12}}{48a^2} + \frac{[b_{12}]_{12}}{48a^2} + \frac{[b_{12}]_{22}}{48a^2} + \frac{[b_{12}]_{33}}{48a^2} + \frac{b_{12}[a]_{12}}{12a h} + \frac{b_{12}[a]_{12}}{12a h} \frac{b_{12}[a]_{12}}{12a h}
\]
\[ \pm \frac{c_3[b_{23}]_3}{48a} \pm \frac{b_{12}[b_{23}]_{12}}{48a} \pm \frac{b_{13}[c_2]_1}{48a} \pm \frac{b_{13}[b_{23}]_{12}}{48a} \pm \frac{[a]_2[b_{23}]_2}{24a} \pm \frac{[a]_3c_2}{24a} \]
\[ - \frac{b_{12}[a]_1b_{23}}{24a^2h} \pm \frac{c_2}{12h} \pm \frac{a}{6h^2} \pm \frac{6h^2}{12ah} \pm \frac{b_{23}c_3}{12ah} \pm \frac{b_{23}[a]_3}{12ah} \pm \frac{b_{13}[b_{23}]_1}{24ah} \pm \frac{b_{23}[b_{23}]_2}{24ah} \pm \frac{c_2b_{23}}{12ah} \]
\[ + \frac{[a]_2b_{23}^2}{24a^2h} \pm \frac{b_{23}[b_{23}]_{23}}{12ah^2} \pm \frac{c_2[b_{23}]_2}{48a} \pm \frac{c_3[b_{23}]_3}{48a} \pm \frac{c_3[b_{23}]_3}{12ah^2} \pm \frac{c_2[b_{23}]_2}{24a^2h} \pm \frac{c_3[b_{23}]_3}{24a^2h} \pm \frac{b_{23}[a]_2[b_{23}]_3}{48a^2} \pm \frac{b_{23}[a]_2[b_{23}]_3}{48a^2} \pm \frac{b_{12}[a]_1[b_{23}]_2}{48a^2} \pm \frac{b_{12}[a]_1[b_{23}]_2}{48a^2} \pm \frac{b_{13}[a]_3[b_{23}]_1}{48a^2} \pm \frac{b_{13}[a]_3[b_{23}]_1}{48a^2} \pm \frac{b_{13}[a]_3[b_{23}]_1}{48a^2} \pm \frac{b_{12}[a]_2[b_{23}]_1}{48a^2} \pm \frac{b_{12}[a]_2[b_{23}]_1}{48a^2} \pm \frac{b_{12}[a]_2[b_{23}]_1}{48a^2} \]
\[ + \frac{b_{23}[b_{23}]_3}{24ah} \pm \frac{b_{12}[b_{23}]_1}{24ah} \pm \frac{b_{23}[b_{23}]_3}{24ah} \pm \frac{[c_2]_3}{12h} \pm \frac{[c_2]_3}{24} \]

Note that in the above, \( a, b_{12}, b_{13}, b_{23}, c_1, c_2, c_3 \) and \( g \) are evaluated at \((x_{i_1}^{(1)}, x_{i_2}^{(2)}, x_{i_3}^{(3)}) \in \subset^{(3)}_h \) and \( \tau \in \Omega_\tau \). To streamline the notation we used \([\cdot]_k \) and \([\cdot]_{kp} \) to denote the first and second derivative of the coefficients with respect to \( x_k \), and with respect to \( x_k \) and \( x_p \), respectively.