LENGTH SPECTRA AND $P$-SPECTRA OF COMPACT FLAT MANIFOLDS

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ABSTRACT. We study the length, weak length and complex length spectrum of closed geodesics of a compact flat Riemannian manifold, comparing length-isospectrality with isospectrality of the Laplacian acting on $p$-forms. Using integral roots of the Krawtchouk polynomials, we give many pairs of $p$-isospectral flat manifolds having different lengths of closed geodesics and in some cases, different injectivity radius and different first eigenvalue.

We prove a Poisson summation formula relating the $p$-eigenvalue spectrum with the lengths of closed geodesics. As a consequence we show that the spectrum determines the lengths of closed geodesics and, by an example, that it does not determine the complex lengths. Furthermore we show that orientability is an audible property for flat manifolds. We give a variety of examples, for instance, a pair of isospectral (resp. Sunada isospectral) manifolds with different length spectra and a pair with the same complex length spectra and not $p$-isospectral for any $p$, or else $p$-isospectral for only one value of $p \neq 0$.

INTRODUCTION

The $p$-spectrum of a compact Riemannian manifold $M$ is the collection of eigenvalues, with multiplicities, of the Laplacian acting on $p$-forms. It will be denoted by $\text{spec}_p(M)$. Two manifolds $M$ and $M'$ are said to be $p$-isospectral —or isospectral on $p$-forms— if $\text{spec}_p(M) = \text{spec}_p(M')$. The word isospectral is reserved for the function case, i.e. it corresponds to 0-isospectral.

Let $\Gamma$ be the fundamental group of $M$. It is well known that the free homotopy classes of closed paths in $M$ are in a one to one correspondence with the conjugacy classes in $\Gamma$. Furthermore, in each such free homotopy class there is at least a closed (i.e. periodic) geodesic —namely the closed path of smallest length in the class. In the case when the sectional curvature of $M$ is nonpositive, if two closed geodesics are freely homotopic then they can be deformed into each other by means of a smooth homotopy through a flat surface in $M$, hence they have the same length $l$. This length is called the length of $\gamma$, denoted $l(\gamma)$, where $\gamma \in \Gamma$ is any representative of this class. The complex length of $\gamma$ is the pair $l_c(\gamma) := (l(\gamma), [V])$, where $V \in O(n-1)$ is determined by the holonomy of $\gamma$ (see Section 2) and $[V]$ denotes the conjugacy class. The multiplicity of a length $l$ (resp. of a complex length $(l, [V])$) is defined to be the number of free homotopy classes having length $l$ (resp. $(l, [V])$). The weak length spectrum (resp. weak complex length spectrum) of $M$, denoted $L$-spectrum (resp. $L_c$-spectrum), is defined as the set of all lengths (resp. complex lengths) of closed geodesics in $M$, while the length spectrum (resp. complex length spectrum), denoted $[L]$-spectrum (resp. $[L_c]$-spectrum), is the set of lengths (resp.
complex lengths) of closed geodesics, with multiplicities. Two manifolds are said to be
$L$-isospectral or length isospectral if they have the same $[L]$-spectrum. (The notions of
$L$, $L_c$, and $[L_c]$-isospectrality are defined similarly.) They are said to be marked length
isospectral if there exists a length-preserving isomorphism between their fundamental
groups.

The relationship between length spectrum and eigenvalue spectrum of $M$ has been
studied for some time. For flat tori and for Riemann surfaces, it is known that the
length spectrum and the eigenvalue spectrum determine each other (see [Hu1,2]). Also,
it has been proved that “generically” $\text{spec}(M)$ determines the length spectrum of $M$ (see
[ CdV]). In [DG], Introduction, an asymptotic formula (see (4.6)) is stated that indicates
that an analogous result holds for $\text{spec}_p(M)$, for any $p \geq 0$.

All the known examples of isospectral compact Riemannian manifolds are $L$-isospec-
tral. In [Go], C. Gordon gave the first example of a pair of Riemannian manifolds —they
are Heisenberg manifolds— that are isospectral but not $[L]$-isospectral. R. Gornet in
[Gt1,2] gave, among other illuminating examples, the first example of pairs of manifolds
—they are 3-step nilmanifolds— having the same marked length spectrum, isospectral
but not 1-isospectral. For other recent work on the length spectra of nilmanifolds see
[GoM] and [GtM]. The complex length spectrum has been considered in ([Re], [Me]) for
hyperbolic manifolds of dimension $n = 3$ and in [Sa] in the case of locally symmetric
spaces of negative curvature.

The goal of this paper is to study the various length spectra for compact flat Riemann-
ian manifolds (flat manifolds for short) and to compare the different notions of length
isospectrality with $p$-isospectrality, for $p \geq 0$.

We will determine the complex lengths of closed geodesics for general flat manifolds.
We will see that, in general, the $p$-spectrum does not determine the weak length spectrum.
For manifolds of diagonal type (see Definition 1.2) this can happen only when $K^n_p(x)$, the
(binary) Krawtchouk polynomial of degree $n$ has integral roots. Using such roots, we give
many pairs of $p$-isospectral flat manifolds having different lengths of closed geodesics and
in some cases, different injectivity radius (Ex. 2.3(i)-(vi)). These seem to be the first such
examples in the context of compact Riemannian manifolds. These examples might be
considered quite odd since they do not follow the generic behavior and seem to contradict
the wave trace formula. An explanation on how they are consistent with the heat and
wave trace formulas is given in Remark 4.11.

We give several pairs, most of them with different fundamental groups (Examples 3.3
through 3.7), comparing length isospectrality with other types of isospectrality. The ex-
amples are obtained by an elementary construction —they are flat tori of low dimensions
($n \geq 4$), divided by free actions of $\mathbb{Z}_r^2$, $r \leq 3$, $\mathbb{Z}_4$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$— and in particular, it is
quite easy to compute their length spectra and real cohomology. For instance, we give
a pair of isospectral, not Sunada isospectral manifolds that are not $[L]$-isospectral (Ex.
3.6) and a pair of 4-dimensional flat manifolds that are Sunada isospectral (see Remark
1.3) but not $[L]$-isospectral (Ex. 3.4). This last example seemed unlikely to exist in the
context of flat manifolds.

Example 3.5 shows a pair of manifolds both having the same complex length spectrum
and not isospectral to each other. They can be chosen so that they are not $p$-isospectral
for any $p$, or else, isospectral for some $p > 0$. Examples 3.4 and 3.5 show clear differences
with the situation for hyperbolic manifolds, where such examples cannot exist (see [GoM],
[Sa]). There are other examples summarized in Table 3.9. Example 3.8 shows pairs of flat
manifolds having the same lengths and/or complex lengths of closed geodesics but which
are very different from each other; for instance, manifolds having different dimension, or
so that one of them is orientable and the other not. In a different direction, we prove that two flat manifolds having the same marked length spectrum are necessarily isometric (Prop. 3.10).

One of the ways to connect the eigenvalue spectrum with the length spectrum is via Poisson summation formulas or via the Selberg trace formula (see [Bl], [CdV], [Hu1,2], [Pe] and [Su] for instance). In the case of flat manifolds, Sunada gave one such formula in the function case (see [Su]) and as a consequence he showed that if two flat Riemannian manifolds are isospectral, then the corresponding tori must be isospectral. In Section 4 we give a Poisson summation formula for vector bundles that is related but is different from Sunada's. In the proof we use the formula for multiplicities of eigenvalues obtained in [MR2, Theorem 3.1]. As a consequence, we show that the spectrum determines the $L$-spectrum. Example 3.6 shows that the spectrum does not determine the $L_c$-spectrum.

An open general question is whether orientability is an audible property, that is, whether isospectral Riemannian manifolds should be both orientable or both nonorientable. Using the Poisson formula we show this question has a positive answer for flat manifolds. This is not the case for $p$-isospectral flat manifolds; in [MR2] we give several pairs of flat manifolds that are $p$-isospectral for only some values of $p$, one of them orientable and the other not. Also, P. Bérard and D. Webb ([BW]) have constructed pairs of 0-isospectral surfaces with boundary that are Neumann isospectral, but not Dirichlet isospectral, one of them orientable and the other not. Using the Poisson formula, we show that $p$-isospectrality of two flat manifolds for some $p \geq 0$ (or $\tau$-isospectrality, for any representation $\tau$ of $O(n)$), implies that the corresponding tori are isospectral and, furthermore, that the orders of the holonomy groups are the same. This is a natural extension of the result of Sunada for 0-isospectral flat manifolds.

Another application of the formula is concerned with flat manifolds of diagonal type. This is a restricted family but is still a rich and useful class. For instance, all of the examples constructed in sections 2 and 3, with the exception of Ex. 3.6 and Ex. 2.3(iii), are of diagonal type (see also the examples in [MR1,2,3]). Bieberbach groups in this class are more manageable, for instance it is quite straightforward to compute combinatorially all the Betti numbers of the associated manifold. We show that for manifolds of diagonal type, isospectrality —and also $p$-isospectrality when $K_p^n(x)$ has no integral roots— implies Sunada isospectrality, hence $q$-isospectrality for every $q$. This extends to all $n$, a result proved by very different methods in [MR3] for $n \leq 8$.

An outline of the paper is as follows. In Section 1 we recall briefly some basic facts on Bieberbach groups and some results from [MR2]. In Section 2 we give a formula for the complex lengths of closed geodesics and show that, if $p > 0$, the $p$-spectrum does not determine the lengths of closed geodesics. In Section 3 we state a criterion for $[L]$ and $[L_c]$-isospectrality, giving several illustrative examples and counterexamples together with a table showing many different possibilities. Section 4 is devoted to the Poisson summation formula and the consequences described above.

The authors wish to thank Carolyn Gordon for very useful comments on the contents of this paper. The second author was at Dartmouth College while part of this paper was done and would like to thank the great hospitality of the Department of Mathematics, specially of Carolyn Gordon and David Webb.

§1 Preliminaries

We shall first recall some standard facts on flat Riemannian manifolds (see [Ch]). A discrete, cocompact subgroup $\Gamma$ of the isometry group of $\mathbb{R}^n$, $I(\mathbb{R}^n)$, is called a crys-
tallographic group. If furthermore, $\Gamma$ is torsion-free, then $\Gamma$ is said to be a Bieberbach group. Such $\Gamma$ acts properly discontinuously on $\mathbb{R}^n$, thus $M_\Gamma = \Gamma \backslash \mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group $\Gamma$. Any such manifold arises in this way. Any element $\gamma \in I(\mathbb{R}^n)$ decomposes uniquely $\gamma = BL_b$, with $B \in O(n)$ and $b \in \mathbb{R}^n$. The translations in $\Gamma$ form a normal, maximal abelian subgroup of finite index, $L_\Lambda$, $\Lambda$ a lattice in $\mathbb{R}^n$ which is $B$-stable for each $BL_b \in \Gamma$. The quotient $F := \Lambda \backslash \Gamma$ is called the holonomy group of $\Gamma$ and gives the linear holonomy group of the Riemannian manifold $M_\Gamma$. The action of $F$ on $\Lambda$ defines an integral representation of $F$, usually called the holonomy representation. Denote $n_B = \dim \ker(B - \Id)$. If $BL_b$ in $\Gamma$, then it is known that $n_B > 0$. The next lemma contains some facts that we will need.

**Lemma 1.1.** Let $\Gamma$ be a Bieberbach group, and let $\gamma = BL_b \in \Gamma$. Let $L$ be any lattice stable by $B$ and let $p_B$ denote the orthogonal projection onto $\ker(B - \Id)$. Then we have

1. $p_B(b) \neq 0$. Furthermore $\ker(B - \Id) = \text{Im}(B - \Id)^\perp$.
2. $L^B := L \cap \ker(B - \Id)$ is a lattice in $\ker(B - \Id)$.
3. Let $L^* = \{\mu \in \mathbb{R}^n : \langle \mu, \lambda \rangle \in \mathbb{Z}, \text{ for any } \lambda \in L\}$, the dual lattice of $L$. Then

   $$((L^B)^*)^B = p_B(L^*).$$

**Proof.** If $B$ has order $m$, then we have that $(BL_b)^m = LC_b$, where $C = \sum_{j=0}^{m-1} B^j$. Since $B \in O(n)$, then $C = C^t$. Furthermore, since $(B - \Id)C = B^m - \Id = 0$ and $B$ is of order $m$, it follows that $\ker(B - \Id) = \text{Im} C = (\ker C)^\perp$. Now $(BL_b)^m \neq \Id$, so $Cb \neq 0$, hence $b \notin \ker(B - \Id)^\perp$, as claimed. Now $\text{Im}(B - \Id)^\perp = \ker(B - \Id)^* = \ker(B - \Id)$, hence the second assertion in (i) is clear.

To verify (ii) we note that if $L_Q = \mathbb{Q}\text{-span}(L)$, then $BL_Q = L_Q$ and the $\mathbb{Q}$-rank of the matrix of $B - \Id$ on a $\mathbb{Z}$-basis of $L$ equals the $\mathbb{R}$-rank, hence $\dim_{\mathbb{Q}} \ker(B - \Id) \cap L_Q = \dim_{\mathbb{R}} \ker(B - \Id)$. Thus, if $\{v_j : 1 \leq j \leq r\}$ is a $\mathbb{Q}$-basis of $\ker(B - \Id) \cap L_Q$ and if $m_1, \ldots, m_r \in \mathbb{Z} \setminus \{0\}$ are such that $m_j v_j \in L$, for $1 \leq j \leq r$, then $\ker(B - \Id) \cap L$ contains $\sum_{j=1}^r Z m_j v_j$, a lattice in $\ker(B - \Id)$. This implies the assertion.

Relative to (iii), we set $W = \ker(B - \Id)$. Since $W^* = W^{\perp}$ we have:

$$(L \cap W)^* \cap W = (L^* + W^{\perp}) \cap W = p_W(L^*). \quad \Box$$

We now recall from [MR2,3] some facts on the spectrum of Laplacian operators on vector bundles over flat manifolds. If $\tau$ is an irreducible representation of $K = O(n)$ and $G = I(\mathbb{R}^n)$ we form the vector bundle $E_\tau$ over $G/K \simeq \mathbb{R}^n$ associated to $\tau$ and consider the corresponding bundle $\Gamma \backslash E_\tau$ over $\Gamma \backslash \mathbb{R}^n = M_\Gamma$. Let $-\Delta_\tau$ be the connection Laplacian on this bundle. For any $\mu$ a nonnegative real number, let $\Lambda_\mu^b = \{\lambda \in \Lambda^* : ||\lambda||^2 = \mu\}$. In [MR3, Thm. 2.1] we have shown that the multiplicity of the eigenvalue $4\pi^2 \mu$ of $-\Delta_\tau$ is given by

$$d_{\tau,\mu}(\Gamma) = |F|^{-1} \sum_{\gamma = BL_b \in \Lambda \backslash \Gamma} \text{tr} \tau(B) e_{\mu,\gamma} (1.1)$$

where $e_{\mu,\gamma} = \sum_{v \in \Lambda_\mu^b, Bv = v} e^{-2\pi iv \cdot b}$. In the case when $\tau = \tau_p$, the $p$-exterior representation of $O(n)$, we shall write $\text{tr}_p(B)$ and $d_{p,\mu}(\Gamma)$ in place of $\text{tr} \tau_p(B)$ and $d_{\tau_p,\mu}(\Gamma)$ respectively.

For a special class of flat manifolds the terms in this formula can be made more explicit.
**Definition 1.2.** [MR3, Def. 1.3.] We say that a Bieberbach group $\Gamma$ is of *diagonal type* if there exists an orthonormal $\mathbb{Z}$-basis $\{e_1, \ldots, e_n\}$ of the lattice $\Lambda$ such that for any element $BL_b \in \Gamma$, $Be_i = \pm e_i$ for $1 \leq i \leq n$. Similarly, $M_\Gamma$ is said to be of *diagonal type*, if $\Gamma$ is so. We note that it may be assumed that the lattice $\Lambda$ of $\Gamma$ is the canonical lattice.

These manifolds have, in particular, holonomy group $F \simeq \mathbb{Z}_2^r$, for some $r \leq n - 1$. After conjugation by a translation we may assume furthermore that $b \in \frac{1}{2} \Lambda$, for any $BL_b \in \Gamma$ (see [MR3, Lemma 1.4]). In this case we have the following expressions for the terms $e_{\mu, \gamma}$ in the multiplicity formula (1.1):

$$e_{\mu, \gamma} = \sum_{v \in \Lambda \cap Bv = v} (-1)^{|I_v^{odd} \cap I_2b^{odd}|},$$

where $I_v = \{j : v.e_j \neq 0\}$ and $I_v^{odd} = \{j : v.e_j \text{ is odd}\}$. Furthermore, the traces $\text{tr}_p(B)$ are given by integral values of the so called Krawtchouk polynomials $K_p^n(x)$ (see [MR2, Remark 3.6] and also [MR3]; see [KL] for more information on Krawtchouk polynomials). Indeed, if $n_B = \dim \ker(B - Id)$, we have:

$$\text{tr}_p(B) = K_p^n(n - n_B), \quad \text{where } K_p^n(x) := \sum_{t=0}^p (-1)^t \binom{x}{t} \binom{n-x}{p-t}.\ (1.3)$$

We shall also use the notation $I_B := \{1 \leq i \leq n : Be_i = e_i\}$, so $|I_B| = n_B$ and $I_B \cap I_2^{odd} = \{i \in I_B : b.e_i \equiv \frac{1}{2} \mod \mathbb{Z}\}$. We set

$$c_{d,t}(\Gamma) := \left| \{BL_b \in F : n_B = d \text{ and } |I_B \cap I_2^{odd}| = t \}\right|, \quad \text{for } 0 \leq t \leq d \leq n. \ (1.4)$$

We note that by Lemma 1.1 $|I_B \cap I_2^{odd}| > 0$, for any $BL_b \in \Gamma$, except when $B = Id$, $b \in \Lambda$.

**Remark 1.3.** In [MR1,3] we gave combinatorial expressions for the numbers $c_{d,t}$, called *Sunada numbers*. We showed that their equality for $\Gamma$ and $\Gamma'$ is equivalent to have that $M_\Gamma$ and $M_{\Gamma'}$ verify the conditions in Sunada’s theorem, that is, they are *Sunada isospectral* (see [MR3] Def. 3.2, Thm. 3.3 and the discussion following it). In particular $c_{d,t}(\Gamma) = c_{d,t}(\Gamma')$ for every $d, t$ implies that $M_\Gamma$ and $M_{\Gamma'}$ are $p$-isospectral for all $p$. A different proof of this fact will be given in §4 (Theorem 4.5).

## §2 LENGTH SPECTRA OF FLAT MANIFOLDS

Let $M$ be a compact Riemannian manifold of nonpositive curvature. If $\alpha(t)$ is a closed geodesic on $M$ with period $t_\alpha$, the parallel transport $\tau$ along $\alpha(t)$ from $\alpha(0)$ to $\alpha(t_\alpha) = \alpha(0)$ is such that $\tau(\dot{\alpha}(0)) = \dot{\alpha}(0)$, hence it defines an element $V \in O((\mathbb{R}\dot{\alpha}(0))^\perp) \simeq O(n-1)$. One can associate to $V$ a well defined conjugacy class $[V]$ in $O(n-1)$, called the *holonomy* of $\alpha(t)$. By definition, the complex length of $\alpha$ is the pair $l_c(\alpha) := (l(\alpha), [V])$; $l_c(\alpha)$ depends only on the free homotopy class of $\alpha$ and not on $\alpha$.

Now given $l \geq 0$ let $m(l)$ denote the multiplicity of the length $l$, that is, the number of free homotopy classes of closed geodesics in $M$ such that $l(\alpha) = l$. It is known that these multiplicities are finite.

The $L$, $L_c$, $[L]$ and $[L_c]$-spectrum of $M$ and the respective notions of isospectrality have been defined in the Introduction.
Clearly if $M$ and $M'$ are $L_c$-isospectral (resp. $[L_c]$-isospectral) then they are $L$-isospectral (resp. $[L]$-isospectral).

We will start by proving a proposition that gives some basic properties of closed geodesics in a flat manifold $M_f$. Let $\Gamma$ be an $n$-dimensional Bieberbach group and $\overline{\gamma} = BL_b \in \Gamma$. For any $v \in \mathbb{R}^n$ we may write

$$v = v_+ + v' \text{ with } v_+ \in \ker(B - \text{Id}) \text{ and } v' \in \ker(B - \text{Id})^\perp.$$ (2.1)

We note that Lemma 1.1(i) says that $b_+ \neq 0$. We take (2.1) as the definition of $v_+$ and $v'$.

We have that $\|BL_b x - x\|^2 = \|(B - \text{Id}) x + Bb' + b_+\|^2 = \|(B - \text{Id}) x + Bb'\|^2 + \|b_+\|^2$, by Lemma 1.1(i). Since $B - \text{Id}$ is an isomorphism when restricted to $\ker(B - \text{Id})^\perp$, then $x$ can always be chosen so that the first summand is zero. This says that $\inf \{ \text{dist}(\overline{\gamma} x, x) : x \in \mathbb{R}^n \}$ can be attained at some $z$ and it equals $\|b_+\|$. Let $o_\gamma$ be defined uniquely by:

$$(B - \text{Id})o_\gamma = -Bb', \quad o_\gamma \in \ker(B - \text{Id})^\perp.$$ (2.2)

Now, if $z$ is such that $\|BL_b z - z\| = \|b_+\|$, then the first summand in the above expression for $\|BL_b z - z\|^2$ is zero, so we see that $z$ is of the form $o_\gamma + u$ for some $u \in \ker(B - \text{Id})$.

Actually, the next proposition allows to characterize the points $z \in \mathbb{R}^n$ such that $\text{dist}(\overline{\gamma} z, z) = \inf \{ \text{dist}(\overline{\gamma} x, x) : x \in \mathbb{R}^n \} = \|b_+\|$ as the points lying on lines on $\mathbb{R}^n$ stable by $\overline{\gamma}$. These points form the affine space $o_\gamma + \ker(B - I)$ of dimension $n_B$.

For $u \in \ker(B - \text{Id})$, $t \in \mathbb{R}$, we define $\alpha_{\gamma,u}(t) := o_\gamma + u + tb_+$, $t \in \mathbb{R}$, i.e. $\alpha_\gamma = \alpha_{\gamma,0}$.

**Proposition 2.1.** Let $\Gamma$ be a Bieberbach group.

(i) If $\overline{\gamma} = BL_b \in \Gamma$, then $\overline{\gamma}$ preserves the lines $o_\gamma + u + \mathbb{R}b_+$, $o_\gamma$ and $u$ as above. Furthermore $\gamma o_{\gamma,u}(t) = o_{\gamma,u}(t + 1)$. Any line in $\mathbb{R}^n$ stable by $\overline{\gamma}$ is of the form $o_\gamma + u + \mathbb{R}b_+$, for some $u \in \ker(B - \text{Id})$.

(ii) The geodesic $\alpha_{\gamma,u}$ pushes down to a closed geodesic $\overline{\alpha}_{\gamma,u}(t)$ in $M_f$, $t \in \mathbb{Z}\setminus\mathbb{R}$, of length $l(\overline{\alpha}_{\gamma,u}) = \|b_+\|$. Any closed geodesic in $M_f$ is of the form $\overline{\alpha}_{\gamma,u}$ for some $\gamma = BL_b \in \Gamma$ and $u \in \ker(B - \text{Id})$.

(iii) $\overline{\alpha}_{\gamma,u}$ is freely homotopic to $\overline{\alpha}_\gamma$. The holonomy of $\overline{\alpha}_{\gamma,u}$ is given by $[B^\perp]$, where $B^\perp$ denote the restriction of $B$ to $(\mathbb{R}b_+)^\perp$.

(iv) The $L$-spectrum (resp. $L_c$-spectrum) of $M_f$ is the set of numbers $\|b_+\|$ (resp. the set of pairs $(\|b_+\|, [B^\perp])$), where $BL_b$ runs through all elements of $\Gamma$.

(v) The $[L]$-spectrum (resp. $[L_c]$-spectrum) of $M_f$ is the set of numbers $\|b_+\|$ (resp. the set of pairs $(\|b_+\|, [B^\perp])$), counted with multiplicities, where $\gamma = BL_b$ runs through a full set of representatives for the $\Gamma$-conjugacy classes in $\Gamma$.

**Proof.** If $\overline{\gamma} = BL_b$ and $t \in \mathbb{R}$, taking into account the definition of $o_\gamma$, we have

$$\gamma o_{\gamma,u}(t) = b_+ + Bb' + Bo_\gamma + u + tb_+ = o_\gamma + u + (t + 1)b_+ = o_{\gamma,u}(t + 1).$$

Now let $w + \mathbb{R}v$ be a line stable by $\overline{\gamma}$. This happens if and only if $Bb + Bw + \mathbb{R}Bv = w + \mathbb{R}v$, or equivalently

$$\mathbb{R}Bv = \mathbb{R}v \quad \text{and} \quad b_+ + Bb' + (B - \text{Id})w \in \mathbb{R}v.$$

It follows that $Bv = \pm v$, since $B \in O(n)$. Since $b_+ \neq 0$ and $Bb' + (B - \text{Id})w \in \ker(B - \text{Id})^\perp$ we have that $v_+ \neq 0$, thus $v = v_+$. Hence $Bv = v$ and furthermore $\mathbb{R}v = \mathbb{R}b_+$ and $Bb' + (B - \text{Id})w = 0$. Now, by the definition of $o_\gamma \in \ker(B - \text{Id})^\perp$, then clearly $u := w - o_\gamma \in \ker(B - \text{Id})$. This implies the second assertion in (i).
For (ii), it is clear by (i) that $\tilde{\alpha}_{\gamma,u}(t)$ is a closed geodesic in $M_\Gamma$, with length equal to the length of the segment in $\mathbb{R}^n$ from $\alpha_{\gamma,u}(0)$ to $\alpha_{\gamma,u}(1)$, which equals $\|b_+\|$. Since any closed geodesic in $M_\Gamma$ is the push down of a geodesic in $\mathbb{R}^n$ that is translated into itself by some $\gamma \in \Gamma$, then the second assertion in (ii) follows from (i).

Relative to (iii), we see that $\tilde{\alpha}_{\gamma,su}$ with $s \in [0,1]$ is a continuous family of closed geodesics in $M_\Gamma$, which shows that $\tilde{\alpha}_{\gamma,u}$ and $\tilde{\alpha}_{\gamma} = \tilde{\alpha}_{\gamma,0}$ are freely homotopic.

To determine $l_\gamma(\gamma)$, we note that the parallel transport along $\tilde{\alpha}_{\gamma,u}$ from $\pi(o_\gamma + u)$ to itself, is given by $B$ and since $Bb_+ = b_+$, then $B$ preserves $(\mathbb{R}b_+)^\perp$. This implies that the holonomy of $\gamma$ is $[B^\perp]$.

Finally, assertions (iv) and (v) follow immediately from (ii),(iii). □

In the sequel we shall denote by $\text{diag}(a_1, \ldots, a_n)$, the $n \times n$ diagonal matrix with $a_j$ in the $j^{th}$ diagonal entry.

Example 2.2. The Klein bottle. As a warm-up, we will first look at the simplest case of the Klein-bottle group. We will determine the conjugacy classes in $\Gamma$ and the closed geodesics, computing their lengths and their respective multiplicities. We let $\Gamma = (BL_h, \Lambda)$ with $B = \text{diag}(-1,1), b = \frac{e_2}{2}, \Lambda = Ze_1 + Ze_2$. Thus, $M_\Gamma$ is a flat Klein bottle. We have

$$\Gamma = \{L_\Lambda : \Lambda \in \Lambda\} \cup \{BL_{b+\lambda} : \lambda \in \Lambda\},$$

a disjoint union. We first compute the conjugacy classes in $\Gamma$. We have:

$$BL_hL_\lambda(BL_h)^{-1} = L_{B\lambda}, \quad L_\mu BL_hL_{-\mu} = BL_{b+(B-I)d\mu},$$

for any $\lambda, \mu \in \Lambda$. Thus

$$L_{m_1e_1+m_2e_2} \sim L_{-m_1e_1+m_2e_2}, \quad BL_h \sim BL_{b+2ke_1},$$

for any $m_1, m_2, k \in \mathbb{Z}$, where $\sim$ means $\Gamma$-conjugate.

Thus, a full set of representatives for the $\Gamma$-conjugacy classes is

$$\{L_{m_1e_1+m_2e_2} : m_1 \in \mathbb{N}_0, m_2 \in \mathbb{Z}\} \cup \{BL_hL_{m_1e_1+m_2e_2} : m_1 = 0,1, m_2 \in \mathbb{Z}\}.$$

The corresponding lengths are given by:

$$l(L_{m_1e_1+m_2e_2}) = (m_1^2 + m_2^2)^{\frac{1}{2}}, \quad l(BL_{b+m_1e_1+m_2e_2}) = |\frac{1}{2} + m_2|.$$

Now, if $x \in \mathbb{R}^2, \gamma = BL_{b+\lambda} \in \Gamma$, with $\lambda = m_1e_1 + m_2e_2$, then the segment in $\mathbb{R}^2$ joining $x$ to $\gamma x = Bx + B\lambda + \frac{e_2}{2}$ has length $\|\gamma x - x\| = \sqrt{\|B(I)\| \cdot \|x - m_1e_1\|^2 + (\frac{1}{2} + m_2)^2})^{\frac{1}{2}}$. This segment pushes down to a closed (periodic) geodesic in $M_\Gamma$ if and only if it has minimal length, that is, $x = -\frac{m_1}{2}e_1 + se_2$, $s \in \mathbb{R}$, and the length is $l(\gamma) = \frac{1}{2} + m_2$. Thus, the closed geodesics in $M_\Gamma$ are the pushdowns of the vertical segments joining $-\frac{m_1}{2}e_1 + se_2$ and $\frac{m_1}{2}e_1 + (\frac{1}{2} + m_2 + s)e_2$, for any $m_1, m_2 \in \mathbb{Z}, s \in [0,1)$, together with the pushdowns of the segments joining $x$ to $x + \lambda$, for any $x \in \mathbb{R}^2$ and any $\lambda \in \Lambda$.

We may also see this in a different way, by noticing that by Proposition 2.1, $BL_{b+\lambda}$ stabilizes the line $\alpha_{\gamma}(t) = -\frac{m_1}{2}e_1 + t(m_2 + \frac{1}{2})e_2$, since $o_{BL_{b+\lambda}} = -\frac{m_1}{2}e_1$. Hence $\alpha_{\gamma}(t)$, the pushdown of $\alpha_{\gamma}(t)$, is a closed geodesic in $M_\Gamma$. We note that $L_{e_1}\alpha_{\gamma}(t) = \alpha_{\gamma L_{-2e_1}}(t)$, for all $t$, hence $\alpha_{\gamma} = \alpha_{\gamma L_{-2e_1}}$, thus we may assume that $m_1 \in \{0,1\}$. 
Using the above set of representatives for the $\Gamma$-conjugacy classes we see that a set of representatives for the free homotopy classes of closed paths are the pushdowns of the segments joining $(0,0)$ to $\lambda \in \Lambda$ with $m_1 \geq 0$, together with the pushdowns of the segments joining $(0,0)$ to $(m_2 + \frac{1}{2})e_2$ and those joining $\frac{d}{2}$ to $\frac{d}{2} + (\frac{1}{2} + m_2)e_2$, $m_2 \in \mathbb{Z}$.

The multiplicities of the lengths of closed geodesics are as follows. If $l^2 \in \mathbb{N}_0$ then $m(l)$ equals the number of solutions of the equation $m_1^2 + m_2^2 = l^2$, with $m_1 \in \mathbb{N}_0$, $m_2 \in \mathbb{Z}$. On the other hand, if $l = \frac{1}{2} + k$, $k \in \mathbb{N}_0$, we find that $m(l) = 4$. Indeed, the conjugacy classes with length $\frac{1}{2} + k$ correspond to the segments joining, either $(0,0)$ with $\pm (\frac{1}{2} + k)e_2$, or $\frac{d}{2}$ with $\frac{d}{2} \pm (\frac{1}{2} + k)e_2$.

Regarding the holonomy of $\gamma$, it equals 1 for $\gamma$ such that $l(\gamma)^2 \in \mathbb{N}_0$, i.e. $\gamma \in \Lambda$, and it equals $-1$ if $l(\gamma) \notin \frac{1}{2} + \mathbb{N}_0$.

We note that a more general Klein-bottle group $\Gamma_{\alpha,\beta}$ for, $\alpha, \beta \in \mathbb{R}_{>0}$, can be defined by taking $\Lambda := \Lambda_{\alpha,\beta} = \mathbb{Z}\alpha e_1 + \mathbb{Z}\beta e_2$, $b = \frac{\beta}{2}e_2$ and with $B$ as before (see [BGM]). The lengths of closed geodesics in $M_{\Gamma_{\alpha,\beta}}$ are in this case given by $|m_2 + \frac{1}{2}|\beta$ and $(m_1^2\alpha^2 + m_2^2\beta^2)^{1/2}$, for $m_1, m_2 \in \mathbb{Z}$.

**Example 2.3.** The goal of this example is to show that the $p$-spectrum for some $0 < p \leq n$ does not determine the lengths of closed geodesics. We will construct several pairs of $p$-isospectral manifolds having different $L$-spectrum. In some cases, the smallest lengths of closed geodesics are distinct for $M_\Gamma$ and $M_{\Gamma'}$. Hence, the injectivity radii $(\frac{1}{2}$ of the smallest length of closed geodesics) are distinct for $M_\Gamma$ and $M_{\Gamma'}$ (see Ex.3(ii),(iii),(iv),(vi)). Also, some examples have different first eigenvalue on functions and, among them, some have different injectivity radius. In Remark 4.11 we explain how these examples are consistent with the heat and wave trace formulas.

(i) We take $\Gamma$ and $\Gamma'$ having holonomy group $\mathbb{Z}_2$. The nontrivial elements in $F$, $F'$ are given by $BL_b$ with $b = \frac{\alpha}{2}$, $B = \text{diag}(1,-1,-1,-1)$ for $\Gamma$ and $B' = \text{diag}(1,1,1,-1)$ for $\Gamma'$. Then $\Gamma$ and $\Gamma'$ are nonisomorphic Bieberbach groups (see [MR2, Ex. 4.1]). By using formula (1.1), we get that the multiplicities of the eigenvalue $4\pi^2$ equal $d_{0,1}(\Gamma) = \frac{1}{2}(8 - 2) = 3$ and $d_{0,1}(\Gamma') = \frac{1}{2}(8 + (-2 + 2 + 2)) = 5$, hence both manifolds are not isospectral.

Furthermore $\text{tr}_p(\text{diag}(1,-1,-1,-1)) = K_p^4(3)$, $\text{tr}_p(\text{diag}(1,1,1,-1)) = K_p^4(1)$ and one has that $K_p^4(3) = K_p^4(1)$ if and only if $p = 2$ (see (1.3)). It follows from (1.1) that $M_\Gamma$ and $M_{\Gamma'}$ are $p$-isospectral if and only if $p = 2$ ([MR2, Ex. 4.1]).

On the other hand, the lengths $l$ such that $l^2 \notin \mathbb{Z}$ have the form $|\frac{1}{2} + m|$ for $M_\Gamma$ and $(|\frac{1}{2} + m_1|^2 + m_2^2 + m_3^2)^{1/2}$ for $M_{\Gamma'}$, for arbitrary $m, m_i \in \mathbb{Z}$. This implies that $M_\Gamma$ and $M_{\Gamma'}$ are not $L$-isospectral. For instance, $\frac{\sqrt{5}}{2}$ is a length of a closed geodesic for $M_\Gamma$, but not for $M_{\Gamma'}$.

Regarding complex lengths we note for instance that if we take $\gamma = BL_b, \gamma' = B'L_b$ we have $l(\gamma) = l(\gamma') = \frac{1}{2}$, the minimal length, however the complex lengths of $\gamma, \gamma'$ are different, since $l_c(\gamma) = (\frac{1}{2}, \text{diag}(-1,-1,-1))$ and $l_c(\gamma') = (\frac{1}{2}, \text{diag}(1,1,-1))$.

(ii) We consider a variation of the previous example (see [MR2, Ex. 4.2]). Again $\Gamma$ and $\Gamma'$ both have holonomy groups $\mathbb{Z}_2$ and we let $B = \text{diag}(1,1,-1,-1)$ for both $\Gamma$ and $\Gamma'$, taking $b = \frac{\alpha}{2}$ for $\Gamma$ and $b' = \frac{\alpha_1 + \alpha_2}{2}$ for $\Gamma'$. In this case $\Gamma$ and $\Gamma'$ are isomorphic but the corresponding manifolds are not isometric ($\Gamma'$ is obtained by conjugating $\Gamma$ by a $C \in GL(4,\mathbb{R})$). Since $K_p^4(2) = 0$ if and only if $p = 1, 3$, it follows that the associated flat manifolds are isospectral only for these values of $p$. They are not isospectral nor $L$-isospectral. Indeed, the lengths that do not correspond to lattice elements have the form $((\frac{1}{2} + m_1)^2 + m_2^2)^{1/2}$ for $M_\Gamma$, and $((\frac{1}{2} + m_1)^2 + (\frac{1}{2} + m_2)^2)^{1/2}$ for $M_{\Gamma'}$, respectively,
for arbitrary \( m_1, m_2 \in \mathbb{Z} \). In particular \( \frac{1}{2} \) is a length for \( \Gamma \) but not for \( \Gamma' \). Thus \( M_{\Gamma} \) and \( M_{\Gamma'} \) have injectivity radius equal to \( \frac{1}{2} \) and \( \frac{\sqrt{2}}{4} \) respectively.

We note that (i) and (ii) can be generalized to any even dimension \( n \geq 4 \), as done in [MR2], with the same properties. In particular, in (ii) we obtain manifolds that are \( p \)-isospectral for any \( p \) odd, \( 0 < p < n \), and are not \( L \)-isospectral.

(iii) We now let \( \Gamma \) and \( \Gamma' \) be as in Example 5.8 in [MR2]. Then \( \Gamma \) and \( \Gamma' \) are nonisomorphic Bieberbach groups and \( M_{\Gamma} \) and \( M_{\Gamma'} \) have dimension 4 and holonomy groups \( \mathbb{Z}_2^2 \) and \( \mathbb{Z}_4 \) respectively. Let \( \Gamma = \langle B_1 L_{b_1}, B_2 L_{b_2}, \Lambda \rangle \), and \( \Gamma' = \langle B' L_{b'}, \Lambda \rangle \), where \( \Lambda = \mathbb{Z}_4^4, b_1 = \frac{e_1 + e_2}{2}, b_2 = \frac{e_1 + e_3}{2}, b' = \frac{e_1}{4} \), \( B_1 = \text{diag}(1,1,-1,-1), B_2 = \text{diag}(1,-1,-1,1) \) and \( B' = \text{diag}(\tilde{J},-1,1,1) \), where \( \tilde{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

The following table lists the nontrivial elements in \( F \) and \( F' \) in a convenient notation, writing the rotational parts in columns, together with subindices that indicate the nonzero translational components: for instance, since \( b_1 = \frac{e_1 + e_2}{2} \), we write \( \frac{1}{2} \) as a subindex of the first diagonal element of \( B_1 \).

| \( B_1 \) | \( B_2 \) | \( B_1B_2 \) |
|---|---|---|
| \( \frac{1}{2} \) | 1 | 1 \( \frac{1}{2} \) |
| 1 | -1 | -1 |
| -1 | -1 | 1 |
| -1 | \( 1 \frac{1}{2} \) | -1 \( \frac{1}{2} \) |

| \( B' \) | \( B'^2 \) | \( B'^3 \) |
|---|---|---|
| \( \tilde{J} \) | -Id_2 | -J |
| -1 | 1 | -1 |
| \( 1 \frac{1}{4} \) | \( 1 \frac{1}{4} \) | \( 1 \frac{1}{4} \) |

We have that \( \text{tr}_p(\text{Id}) = \binom{n}{2} \) and, for the three nontrivial elements \( B \) in \( F \), \( \text{tr}_p(B) = K^4_p(2) \) equals 0 if \( p = 1, 3 \), it equals 1 if \( p = 0, 4 \), and it equals -2 if \( p = 2 \).

In the case of \( \Gamma' \) there are some differences. We have \( n_{B'} = n_{B'^3} = 1 \) and \( n_{B'^2} = 2 \). Also, \( b' = b'_+ = \frac{e_1}{4} \). For \( C := B', B'^3 \), we have that \( \text{tr}_p(C) = 0 \) if \( p = 1, 2, 3, 4 \), and \( \text{tr}_p(C) = 1 \), if \( p = 0, 4 \). Furthermore, \( \text{tr}_p(B'^2) = K^4_p(2) \). Using this information we get that \( M_{\Gamma} \) and \( M_{\Gamma'} \) are \( p \)-isospectral if and only if \( p = 1, 3 \). Also, \( M_{\Gamma} \) is orientable, while \( M_{\Gamma'} \) is not. They are not \( L \)-isospectral, since \( M_{\Gamma'} \) has closed geodesics with length \( \frac{1}{4} \), while \( M_{\Gamma} \) does not. Actually, they have different injectivity radius.

(iv) Let \( \Gamma \) and \( \Gamma' \) of dimension 4, with holonomy groups \( \mathbb{Z}_2^2 \). Set \( B_1 = B_1' = \text{diag}(1,1,-1,-1), b_1 = \frac{e_1 + e_2 + e_3 + e_4}{4}, b'_1 = \frac{e_1}{4} \), \( B_2 = B_2' = \text{diag}(1,1,-1,1), b_2 = \frac{e_3}{4}, b'_2 = \frac{e_2}{4} \). The lattice is \( \mathbb{Z}_4^4 \). By a verification of the conditions (i) and (ii) in [MR2, Prop. 2.1], one has that \( \Gamma \) and \( \Gamma' \) are nonisomorphic Bieberbach groups.

It is not difficult to see, by an argument similar to that in (i)-(iii), that these manifolds are 2-isospectral, they are not \( L \)-isospectral, and the first (nonzero) eigenvalue is \( 4\pi^2 \) for \( M_{\Gamma'} \) and \( 8\pi^2 \), for \( M_{\Gamma} \).

These manifolds have the same injectivity radius; but if we change in \( \Gamma \) the value of \( b_2 \) into \( \frac{e_2 + e_3 + e_4}{2} \), then we obtain manifolds having the same spectral properties as before but now they do not have the same injectivity radius, namely \( \frac{\sqrt{2}}{4} \) and \( \frac{1}{4} \) respectively. In column notation the last groups are as follows.
(v) For each value of $p$ one can give pairs $\Gamma, \Gamma'$ of dimension $n = 2p$, with holonomy group $\mathbb{Z}_2$, that are $q$-isospectral only for $q = p$ and such that they are not $L$-isospectral. More generally, following Proposition 3.15 in [MR3], it is possible to construct, for each integral zero of the Krawtchouk polynomial, namely $K_p^n(j) = 0$ with $j \neq \frac{p}{2}$, groups $\Gamma, \Gamma'$ of dimension $n$, with holonomy group $\mathbb{Z}_2$, that are $p$ and $(n-p)$-isospectral and not $L$-isospectral. They are $q$-isospectral if and only if $K_p^n(j) = 0$ and this happens generically, if and only if $q = p$ or $q = n - p$.

(vi) Each example in (v) can be extended to give large families of $p$-isospectral flat manifolds of dimension $n$, having pairwise different $L$-spectrum. We sketch this construction in a particular case, since all cases are similar. Take for $1 \leq j \leq k < n$, even, the Bieberbach groups $\Gamma_{k,j} := \langle C_k L_{V_1 + \ldots + V_j}, \mathbb{Z}^n \rangle$, where $C_k := \text{diag}(1,1,\ldots,1,-1,\ldots,-1)$.

For fixed $k$, these groups are isomorphic but the corresponding manifolds are not isospectral. For $k$ odd, they are $\frac{n}{2}$-isospectral (see [MR2, Ex. 4.2] for more details). One has that if $j_1 < j_2$, both $\not\equiv 0 \mod 4$, then $\Gamma_{k,j_1}$ and $\Gamma_{k,j_2}$ are not $L$-isospectral, since $\frac{3k}{4}$ is a length for $\Gamma_{k,j_1}$ but not for $\Gamma_{k,j_2}$. For fixed $k$, the family $\Gamma_{k,j}^n$ with $1 \leq j \leq k$, $j \not\equiv 0 \mod 4$ has cardinality approximately equal to $\frac{3k}{4}$. The corresponding manifolds are all $p$-isospectral to each other for $p = \frac{n}{2}$, but they have different lengths of closed geodesics. Observe that it is possible to take $k = n - 1$, giving a family of cardinality approximately equal to $\frac{3n}{4}$.

§3 Length spectrum and $p$-spectra

In this section we will consider the question of $[L]$-isospectrality of flat manifolds. The next criterion will be useful. If $\beta \in \Gamma, \Gamma'$ let $[\beta]$ denote the conjugacy class of $\beta$.

**Proposition 3.1.** Let $\Gamma, \Gamma'$ be Bieberbach groups with translation lattices $\Lambda, \Lambda'$ respectively. Suppose there exist partitions $\mathcal{P}$ and $\mathcal{P}'$ of $\Lambda \setminus \Gamma$ and $\Lambda' \setminus \Gamma'$ respectively and a bijection $\phi : \mathcal{P} \to \mathcal{P}'$ such that for every $c \in \mathbb{R}_{>0}$, $O \in O(n-1)$ and $P \in \mathcal{P}$, the cardinality of $\{[\gamma L_{\lambda}] : \lambda \in \Lambda, \gamma \in P, l(\gamma L_{\lambda}) = c \text{ (resp. } l_c(\gamma L_{\lambda}) = (c, [O])) \}$ equals the cardinality of $\{[\gamma' L_{\lambda'}] : \lambda' \in \Lambda', \gamma' \in \phi(P), l(\gamma' L_{\lambda'}) = c \text{ (resp. } l_c(\gamma' L_{\lambda'}) = (c, [O])) \}$, then $\Gamma \setminus \mathbb{R}^n$ and $\Gamma' \setminus \mathbb{R}^n$ are length (resp. complex length) isospectral.

When applying this criterion in Examples 3.3 and 3.7 below we shall use the point partition, that is, each class in $\mathcal{P}$ and $\mathcal{P}'$ will have one element, hence $\phi$ will be a bijection from $\Lambda \setminus \Gamma$ to $\Lambda' \setminus \Gamma'$. Example 3.5 has less standard spectral properties and will require a less obvious partition of $F$ and $F'$.

In order to be able to compute the $[L]$-spectrum and $[L_c]$-spectrum of a general flat manifold $M$, one needs a parametrization of the conjugacy classes of $\Gamma$. This is in general complicated but it becomes much simpler when the Bieberbach group $\Gamma$ is of diagonal
type. For a general $\Gamma$, we have that if $\gamma_i = B_i L_{b_i}, \gamma_j = B_j L_{b_j} \in \Gamma, \lambda, \mu \in \Lambda$, conjugation of $L_\lambda$ by $\gamma_i$, and $\gamma_i L_\lambda$ by $L_\mu$, yield the following relations:

$$L_\lambda \sim L_{B_i \lambda}, \quad \gamma_i L_\lambda \sim \gamma_i L_\lambda L_{(B_i^{-1} - Id)\mu}. \quad (3.1)$$

Furthermore, if the holonomy group is abelian, conjugation of $\gamma_i L_\lambda$ by $\gamma_j, j \neq i$, yields

$$B_j L_{b_j} B_i L_{b_i + \lambda} L_{-b_j} B_j^{-1} = B_i L_{B_j((B_i^{-1} - Id)b_i + b_i + \lambda)}$$

$$= B_i L_{b_i + \lambda} L_{B_j((B_i^{-1} - Id)b_j + (B_j - Id)(b_i + \lambda)}$$

$$= B_i L_{b_i + B_j\lambda} L_{B_j((B_i^{-1} - Id)b_j + (B_j - Id)b_i)}.$$ 

We thus get:

$$\gamma_i L_\lambda \sim \gamma_i L_{B_j \lambda} L_{(B_i - Id)b_i + B_j((B_i^{-1} - Id)b_j}, \text{ for } j \neq i. \quad (3.2)$$

**Remark 3.2.** We now mention some simple facts that are rather direct consequences of (3.1), (3.2) and the definitions.

(i) We note that $\lambda_1, \lambda_2 \in \Lambda$ are $\Gamma$-conjugate if and only if there is $BL_b \in \Gamma$ such that $B\lambda_1 = \lambda_2$. If $\Gamma$ and $\Gamma'$ have the same lattices and the same integral holonomy representations, then the multiplicities of the lengths corresponding to lattice elements are the same. (In many of the examples these are exactly the lengths $l$ such that $l^2 \in \mathbb{N}$.)

(ii) If $M_\Gamma$ and $M_{\Gamma'}$ are Sunada isospectral then they are $L_c$-isospectral.

Indeed, if $\beta = C L_c \in I(\mathbb{R}^n)$ and $\gamma = BL_b \in \Gamma$ then $\beta \gamma \beta^{-1} = CBC^{-1}L_{((B^{-1} - Id)c + b)}$ and furthermore, $(C((B^{-1} - Id)c + b))_+ = Cb_+$. Thus, $l(\beta \gamma \beta^{-1}) = \|Cb_+\| = \|b_+\|$. The holonomy component of $\beta \gamma \beta^{-1}$ is $[CBC^{-1}((\mathbb{R}Cb_+)^\perp)] = [B_{((\mathbb{R}b_+)^\perp)}].$ Hence, this shows that $l_c(\beta \gamma \beta^{-1}) = l_c(\gamma)$. Note that this also shows that the complex length is an invariant of the conjugacy class of $\gamma$ in $I(\mathbb{R}^n)$, in particular the complex length spectrum is well defined.

Thus, using that $l_c(\beta \gamma L_{\lambda} \beta^{-1}) = l_c(\gamma L_{\lambda})$ it follows that $M_\Gamma$ and $M_{\Gamma'}$ are $L_c$-isospectral.

(iii) In the notation of Prop. 2.1, assume that for each $P \in \mathcal{P}$ (resp. $\mathcal{P}'$) and for any $\gamma_1, \gamma_2 \in P$ the holonomy components of $\gamma_1, \gamma_2$ are the same. Suppose that there exists $\phi$ as in Prop. 2.1 satisfying the conditions for $[L]$-isospectrality and suppose also that $\phi$ preserves holonomy components. Then $\Gamma \setminus \mathbb{R}^n$ and $\Gamma' \setminus \mathbb{R}^n$ are $[L_c]$-isospectral.

**Example 3.3.** We let $\Gamma$ and $\Gamma'$ as in [MR2, Ex. 4.5]. In column notation:

| $B_1$ | $B_2$ | $B_1 B_2$ |
|-------|-------|----------|
| 1 | 1 | 1 |
| $1_\frac{1}{2}$ | $-1$ | $-1_\frac{1}{2}$ |
| $-1$ | $1_\frac{1}{2}$ | $-1_\frac{1}{2}$ |
| $-1_\frac{1}{2}$ | $-1$ | $1_\frac{1}{2}$ |

| $B'_1$ | $B'_2$ | $B'_1 B'_2$ |
|-------|-------|----------|
| 1 | $1_\frac{1}{2}$ | $1_\frac{1}{2}$ |
| $1_\frac{1}{2}$ | $-1$ | $-1_\frac{1}{2}$ |
| $-1$ | 1 | $-1$ |
| $-1$ | $-1$ | 1 |

Then $\Gamma$ and $\Gamma'$ both have the same holonomy representation of diagonal type, with holonomy group $\mathbb{Z}_2^2$. In [MR2] it was shown that $M_\Gamma$ and $M_{\Gamma'}$ are isospectral, actually they are Sunada isospectral but they are not diffeomorphic. They are $[L]$-isospectral and actually $[L_c]$-isospectral, as we shall see by using Proposition 3.1.
We take \( P \) the point partition and the bijection \( \phi \) as the identity. Since \( B_i = B'_i \) for each \( i \), \( \phi \) preserves the holonomy components of the complex lengths \( L(\gamma), \gamma \in \Gamma \).

Since the lattices and the integral holonomy representations are the same for both manifolds then by Remark 3.2(i), the multiplicities of the lengths of the elements of the form \( L \) are the same for \( \Gamma \) and \( \Gamma' \).

The remaining lengths are of the form \( l = \left( \left( \frac{1}{2} + k_1 \right)^2 + k_2^2 \right)^{1/2} \) where \( k_1, k_2 \in \mathbb{Z} \). In \( \Gamma \) the elements of length \( l \) with rotational part \( B_1 \) are of the form \( B_1 L b_{1+\lambda} \) with \( \lambda = m_1 e_1 + m_2 e_2 + m_3 e_3 + m_4 e_4 \), \( m_i \in \mathbb{Z} \) for \( 1 \leq i \leq 4 \) and such that \( m_1^2 + \left( \frac{1}{2} + m_2 \right)^2 = l^2 \).

The second relation in (3.1) implies that \( B_1 L b_{1+\lambda} \sim B_1 L b_{1+\lambda+2\delta_3 e_3+2\delta_4 e_4} \) for \( \delta_3, \delta_4 \in \mathbb{Z} \), thus we may assume that \( m_3, m_4 \in \{0, 1\} \) without leaving out any conjugacy class. Indeed (3.1) implies that \( m_3 \) and \( m_4 \) can be taken modulo 2 in this case.

Now by (3.2) \( B_1 L b_{1+\lambda} \sim B_1 L b_{1+m_1 e_1-(m_2+1)e_2+(m_3-1)e_3-(m_4+1)e_4} \), and there are no other relations among elements with rotational part \( B_1 \). Since the second coordinate has the form \( \frac{1}{2} + m_2 \) and changes into \( -\frac{1}{2} - m_2 \) when applying (3.2), this implies that in each conjugacy class in \( \Gamma \) there are exactly two elements related in this way. That is, among the elements \( B_1 L v \) in \( \Gamma \) with \( v = m_1 e_1 + \left( \frac{1}{2} + m_2 \right) e_2 + m_3 e_3 + \left( \frac{1}{2} + m_4 \right) e_4 \), \( m_1, m_2 \in \mathbb{Z} \), \( m_3, m_4 = 0 \) or \( 1 \), and such that \( m_1^2 + \left( \frac{1}{2} + m_2 \right)^2 = l^2 \), the number of conjugacy classes is exactly half the total number of elements.

If we proceed to do the same calculation in \( \Gamma' \) for the elements with rotational part \( B'_1 = B_1 \), we see that the set of elements with length \( l \) are exactly the same as in the previous case; also (3.1) gives the same relations as before, and now by (3.2) the relations are \( B_1 L b'_{1+\lambda} \sim B_1 L b'_{1+m_1 e_1-(m_2+1)e_2+m_3 e_3+m_4 e_4} \). As before we see that the elements of a fixed length \( l \) occurring here are divided by a factor or 2 when we apply (3.2) to take conjugacy classes. This proves the equality of cardinalities as required in Proposition 3.1 for the pair \( B_1 \) and \( \phi(B_1) \).

It is not difficult to check that the elements with rotational parts \( B_2 \) and \( B_3 \) can be handled in a completely similar way. After applying (3.1), by (3.2), the elements go in pairs to form a conjugacy class and in all cases the number of classes with a given length corresponding to \( B_i \), \( i = 2 \) or \( i = 3 \), is exactly the same for both \( \Gamma \) and \( \Gamma' \).

Now the above discussion implies that the conditions in Remark 3.2(iii) are satisfied, hence both manifolds are \([L]\)-isospectral.

**Example 3.4.** We now consider a simple pair of Bieberbach groups \( \Gamma, \Gamma' \), of dimension 4, with holonomy group \( \mathbb{Z}^2_2 \) and of diagonal type. We shall see that \( M_\Gamma \) and \( M_{\Gamma'} \) are Sunada isospectral, but not \([L]\)-isospectral. Let \( \Gamma = \langle B_1 L b_1, B_2 L b_2, \Lambda \rangle \), \( \Gamma' = \langle B_1 L b'_{1}, B_2 L b'_{2}, \Lambda \rangle \), let \( B_3 = B_1 B_2 \), where \( B_i, b_i, b'_i \) are given in the following table.

| \( B_1 \) | \( B_2 \) | \( B_3 \) |
|---|---|---|
| \( 1/2 \) | \( 1/2 \) | 1 |
| 1 | \( 1/2 \) | \( 1/2 \) |
| 1 | \( -1 \) | \( -1 \) |
| \( -1 \) | 1 | \( -1 \) |

| \( B'_1 \) | \( B'_2 \) | \( B'_3 \) |
|---|---|---|
| 1 | 1 | 1 |
| 1 | \( 1/2 \) | \( 1/2 \) |
| \( 1/2 \) | \(-1/2\) | \(-1\) |
| \( -1/2 \) | \(1/2\) | \(-1\) |

The corresponding manifolds are Sunada isospectral (see Theorem 4.5 and Remark 1.3) since the numbers \( c_{d,t}, 0 \leq t \leq d \leq 4 \), are the same for \( \Gamma, \Gamma' \). Indeed, \( c_{2,1} = c_{3,1} = c_{3,2} = c_{4,0} = 1 \) and \( c_{d,t} = 0 \) for the other values of \( d, t \).
The lengths of closed geodesics corresponding to nonlattice elements are of the form:

\[(\frac{1}{2} + m_1)^2 + m_2^2 + m_3^2\] and \[(\frac{1}{2} + m_1)^2 + (\frac{1}{2} + m_2)^2 + m_3^2\] with \(m_i \in \mathbb{Z}, 1 \leq i \leq 3,

for both \(\Gamma\) and \(\Gamma'\). Furthermore, since the manifolds are Sunada isospectral, then the
\(L_c\)-spectra are the same by Remark 3.2(ii). We will now see that \(M_\Gamma\) and \(M_{\Gamma'}\) are not
\([L]\)-isospectral, by showing that the multiplicity of the length \(\frac{1}{2}\) is different for \(M_\Gamma\) and
\(M_{\Gamma'}\).

Among the elements in \(\Gamma\) having length \(\frac{1}{2}\) some are of the form \(B_1 L_{\frac{1}{2}+\lambda}\) with \(\lambda = -m_1 e_1 + m_4 e_4\), where \(m_1 \in \{0,1\}, m_4 \in \mathbb{Z}\). Since, by (3.1), \(B_1 L_{\frac{1}{2}+\lambda} \sim B_1 L_{\frac{1}{2}+2ke_4}\) for any \(k \in \mathbb{Z}\), then we may also take \(m_4 \in \{0,1\}\). We easily see that relations (3.1), (3.2)
do not give any identifications between these elements, so we get 4 different conjugacy
classes with length \(\frac{1}{2}\).

The other elements of length \(\frac{1}{2}\) are \(B_3 L_{\frac{1}{2}+\lambda}\) with \(\lambda = -m_2 e_2 + m_3 e_3 + m_4 e_4\), where
\(m_i \in \{0,1\}\) (here we again use relation (3.1) to have \(m_3, m_4 \in \{0,1\}\)). Furthermore, by
(3.2) we have

\[B_3 L_{\frac{1}{2}+\lambda} \sim B_3 L_{\frac{1}{2}+\lambda} = B_3 L_{\frac{1}{2}-m_2 e_2 \pm m_3 e_3 \pm m_4 e_4}\]

hence (3.2) gives no new relations and these elements lie in 8 different conjugacy classes.
Thus the length \(l = \frac{1}{2}\) has multiplicity 12 in \(M_{\Gamma}\).

For \(\Gamma'\) the elements of length \(\frac{1}{2}\) have the form \(B_1' L_{\frac{1}{2}+\lambda}\) with \(\lambda = -m_3 e_3 + m_4 e_4\),
\(m_3, m_4 \in \{0,1\}\) and \(B_3' L_{\frac{1}{2}+\lambda}\) with \(\lambda = -m_2 e_2 + m_3 e_3 + m_4 e_4\), \(m_i \in \{0,1\}\). We have by
(3.2) the following relations:

\[B_1' L_{\frac{1}{2}+\lambda} + B_3' L_{\frac{1}{2}+\lambda} + B_3' L_{\frac{1}{2}+\lambda} = B_1' L_{\frac{1}{2}+\lambda} + (m_3-1)e_3 + (m_4-1)e_4\]

This implies that these 4 elements determine 2 conjugacy classes in \(\Gamma'\). Similarly one
computes that the remaining 8 elements for \(B_3'\) determine 4 conjugacy classes in \(\Gamma'\).
Hence the length \(l = \frac{1}{2}\) has multiplicity 6 in \(M_{\Gamma'}\).

Examples of manifolds with similar spectral properties are given in [Go, Ex. 2.4(a)]
and in [Gt1, Ex.I], by using 2 and 3-step nilmanifolds, respectively. We note that such
an example cannot exist for hyperbolic manifolds since strongly isospectral implies \([L]\)-
isospectral in this context (see [GoM]).

**Example 3.5.** We will now see that, in the context of flat manifolds, the \([L]\)-spectrum
(and even the \([L_c]\)-spectrum) does not determine the \(p\)-spectrum for any \(p, 0 \leq p \leq n\).
This shows a difference with the case of hyperbolic manifolds, since in this context \([L_c]\)-
isospectral implies strongly isospectral (see [Sa]). Indeed, we will construct two flat
manifolds of dimension 13 (resp. 14), with holonomy group isomorphic to \(\mathbb{Z}_3^2\), which are
\([L_c]\)-isospectral but are not \(p\)-isospectral for any \(0 \leq p \leq 13\) (resp. for any \(0 \leq p \leq 14,\nexcept for \(p = 7\)). The corresponding Bieberbach groups have both the canonical lattice
and the same integral holonomy representation. The translational parts of elements in \(\Gamma\)
with nontrivial rotational part differ from the corresponding elements in \(\Gamma'\) only in the
last nine coordinates, where the rotational parts act as the identity.

We shall represent the elements of \(F, F'\) in column notation, writing on the right
(resp. left) the coordinates corresponding to elements in \(\Gamma\) (resp. \(\Gamma'\)).
One sees that $\Gamma$ and $\Gamma'$ are not isospectral. Indeed, one verifies that the contribution of $\gamma_1 = B_1 L_{b_1}$ to the multiplicity formula (1.1) is different from the contribution of $\gamma'_1 = B_1 L_{b'_1}$ for $\Gamma'$. On the other hand, the total contribution of the remaining elements is the same for both $\Gamma$ and $\Gamma'$, as we can see by taking the bijection $\phi : B_2 L_{b_2} \leftrightarrow B_3 L_{b'_3}$, $B_3 L_{b_3} \leftrightarrow B_2 L_{b'_2}$ and $\phi = Id$ for the remaining elements.

We now check the $[L]$-isospectrality by applying the criterion in Proposition 3.1. We shall use the partitions $P$, $P'$ of $\Lambda \setminus \Gamma$ and $\Lambda \setminus \Gamma'$ respectively, such that any class in $P$ and $P'$ has exactly one element except for $\gamma_1, \gamma_2 \in P$ and $\{\gamma_3, \gamma_4\} \in P'$. We choose the bijection $\phi$ so that it maps $\{\gamma_1, \gamma_2\}$ to $\{\gamma'_3\}$, $\{\gamma_3\}$ to $\{\gamma'_1, \gamma'_2\}$ and it equals the identity, otherwise. We must show that for this choice of $\phi$ the conditions in Proposition 3.1 are satisfied.

It is clear that the lengths corresponding to elements of the form $L_{\lambda}$ have the same multiplicities in both manifolds by Remark 3.2(i). The same is true for the elements of the form $B L_{b_{i+\lambda}}$ with $B = B_1 B_2, B_1 B_3, B_2 B_3$ and $B_1 B_2 B_3$, since they play exactly the same role in $\Gamma$ and in $\Gamma'$.

We claim that the multiplicities of the lengths, when restricted to elements of the form $B_3 L_{b_{3+\lambda}}$, are the same as the combined multiplicities of the lengths when restricted to elements of the form $B_1 L_{b'_1+\lambda}$ and $B_2 L_{b'_2+\lambda}$, $\lambda \in \Lambda$. A similar statement can be made when we compare the combined contributions to the length spectrum of elements in $\Gamma$ of the form $B_1 L_{b_{1+\lambda}}$ and $B_2 L_{b_{2+\lambda}}$ with that of the elements $B_3 L_{b'_3+\lambda}$ in $\Gamma'$.

Recall now that the elements of the form $B_3 L_{b_{3+\lambda}}$ with $\Lambda \ni \lambda = (m_1, m_2, \ldots, m_{13})$ have length

$$\left( m_2^2 + m_5^2 + m_6^2 + \left( \frac{1}{2} + m_7 \right)^2 + \left( \frac{1}{2} + m_8 \right)^2 + \cdots + \left( \frac{1}{2} + m_{12} \right)^2 + m_{13}^2 \right)^{\frac{1}{2}}. \tag{3.3}$$

By conjugating by $L_{e_i}$ for $i = 1, 2, 4$, we may assume that $m_1, m_2, m_4 \in \{0, 1\}$. According to (3.2) the only extra relation among these elements is given by $B_3 L_{b_{3+\lambda}} \sim$
$B_3 L_{b_3 + \nu}$ where $\nu$ equals $\lambda$ except for a sign change in only one coordinate, namely $\nu = (m_1, m_2, -m_3, m_4, m_5, m_6, \ldots, m_{13})$. Meanwhile, in $\Gamma'$, $B_1 L_{b_1' + \lambda}$ and $B_2 L_{b_2' + \lambda}$, as $\lambda \in \Lambda$ varies, have the same lengths as in (3.3). Here we are using that the holonomy group is abelian. For $B_1 L_{b_1' + \lambda}$ we may assume $m_2, m_3, m_4 \in \{0, 1\}$. The remaining relations among these elements are: $B_1 L_{b_1' + \lambda} \sim B_1 L_{b_1' + \nu}$ where $\nu$ equals $\lambda$ except for a sign change in the first coordinate; $B_1 L_{b_1' + \lambda} \sim B_1 L_{b_1' + \nu'}$ where $\nu' = (-m_1, m_2, 1 - m_3, m_4, m_5, \ldots, m_{13})$; and the composition of these two relations which gives $B_1 L_{b_1' + \lambda} \sim B_1 L_{b_1' + \nu''}$ where $\nu'' = (m_1, m_2, 1 - m_3, m_4, m_5, \ldots, m_{13})$.

By taking into account all these relations, one can check that the multiplicities of a length for the elements of the form $B_3 L_{b_3 + \lambda}$ equal twice the multiplicities of the same length for the elements of the form $B_1 L_{b_1' + \lambda}$. The same is true for $B_2 L_{b_2' + \lambda}$ in place of $B_1 L_{b_1' + \lambda}$. Hence, the contribution of $B_3 L_{b_3 + \lambda}$ to the $[L]$-spectrum of $M_\Gamma$, as $\lambda \in \Lambda$ varies, turns out to be the same as the contribution of $B_1 L_{b_1' + \lambda}$ and $B_2 L_{b_2' + \lambda}$ to the $[L]$-spectrum of $M_{\Gamma'}$, as $\lambda \in \Lambda$ varies.

Similarly, the same happens with $B_3 L_{b_3' + \lambda}$ when compared with $B_1 L_{b_1' + \lambda}$ and $B_2 L_{b_2' + \lambda}$ taken together. Therefore $M_\Gamma$ and $M_{\Gamma'}$ are $[L]$-isospectral. Furthermore the conditions in Remark 3.2(iii) are satisfied, thus $M_\Gamma$ and $M_{\Gamma'}$ are actually $[L_c]$-isospectral.

It is easy to check that $M_\Gamma$ and $M_{\Gamma'}$ are not $p$-isospectral for any $p$, by using Theorem 3.6(ii), since the coefficients $K_p^{13}(3) \neq 0$ for every $p$.

Finally, if we modify a little bit these manifolds enlarging them to dimension 14, we can obtain an example of a pair of manifolds with the same properties as before with the only exception that they become 7-isospectral. The change is achieved by replacing the fourth row by two rows, one of the form $(-1, -1, 1, -1, 1, -1)$ and the other of the form $(1, 1, -1, 1, -1, 1)$. Thus $\text{tr}_p(B_1) = K_p^{14}(4)$, for $1 \leq j \leq 3$ which is zero if and only if $p = 7$. We do not include the verification in this case for brevity.

**Example 3.6.** We now briefly consider $\Gamma$ and $\Gamma'$ as in [MR1, Ex. 4.1]. Here $n = 6$ and $F \simeq F' \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$. In the standard column notation we may represent the nontrivial elements in $F$, $F'$ as follows:

| $B_1$ | $B_1^2$ | $B_1^3$ | $B_2$ | $B_1 B_2$ | $B_2^2$ | $B_1^3 B_2$ |
|-------|---------|---------|-------|-----------|---------|-------------|
| $\bar{J}$ | $-\text{Id}_2$ | $-\bar{J}$ | $-\text{Id}_2$ | $-\bar{J}$ | $\text{Id}_2$ | $\bar{J}$ |
| $\bar{J}$ | $-\text{Id}_2$ | $-\bar{J}$ | $\text{Id}_2$ | $\bar{J}$ | $-\text{Id}_2$ | $-\bar{J}$ |
| $1_\frac{1}{4}$ | $1_\frac{1}{2}$ | $1_\frac{3}{4}$ | $1_\frac{3}{4}$ | $1_\frac{1}{2}$ | $1_\frac{1}{4}$ | $1_\frac{1}{2}$ |
| $1$ | $1$ | $1$ | $1$ | $1_\frac{1}{2}$ | $1_\frac{1}{2}$ | $1_\frac{1}{2}$ |

| $B_1'$ | $B_1'^2$ | $B_1'^3$ | $B_2'$ | $B_1' B_2'$ | $B_1'^2 B_2'$ | $B_1'^3 B_2'$ |
|--------|----------|----------|--------|-------------|-------------|-------------|
| $\bar{J}$ | $-\text{Id}_2$ | $-\bar{J}$ | $-\text{Id}_2$ | $-\bar{J}$ | $\text{Id}_2$ | $\bar{J}$ |
| $1$ | $1$ | $1$ | $1$ | $1_\frac{1}{2}$ | $1_\frac{1}{4}$ | $1_\frac{1}{4}$ |
| $-1$ | $1$ | $-1$ | $1_\frac{1}{2}$ | $-1_\frac{1}{4}$ | $1_\frac{1}{4}$ | $-1_\frac{1}{4}$ |
| $-1$ | $1$ | $-1$ | $-1_\frac{1}{4}$ | $1_\frac{1}{4}$ | $-1_\frac{1}{4}$ | $1_\frac{1}{4}$ |
| $1_\frac{1}{4}$ | $1_\frac{1}{2}$ | $1_\frac{3}{4}$ | $1$ | $1_\frac{1}{2}$ | $1_\frac{1}{4}$ | $1_\frac{1}{4}$ |
We have proved in [MR1] that $M_\Gamma$ and $M_{\Gamma'}$ are 0 and 6-isospectral, but not $p$-isospectral for $p \neq 0, 6$. We now show they are not $[L]$-isospectral by comparing the multiplicities of the length $l = \frac{1}{\sqrt{2}}$.

The only elements in $\Gamma$ with this length are of the form $B_1^2 B_2 L_{\frac{3}{2}+\frac{8}{2}+\lambda}$ where $\lambda = m_3 e_3 + m_4 e_4 - m_5 e_5 - m_6 e_6$, with $m_3, m_4 \in \mathbb{Z}, m_5, m_6 \in \{0, 1\}$. Using relation (3.1) we may assume that $m_3, m_4 \in \{0, 1\}$, thus having 16 elements with length $\frac{1}{\sqrt{2}}$. Now relation (3.2) gives $B_1^2 B_2 L_{\frac{3}{2}+\frac{8}{2}+\lambda} L_{\lambda} \sim B_1^2 B_2 L_{\frac{3}{2}+\frac{8}{2}+\lambda} L_{B\lambda}$, with $B \in F$. This implies —by taking $B = B_1$— that the choice $m_3 = 0, m_4 = 1$ is equivalent to the choice $m_3 = 1, m_4 = 0$, and there are no other identifications. This gives 12 classes with length $l = \frac{1}{\sqrt{2}}$ in $\Gamma$. An entirely similar argument shows that in $\Gamma'$ there are only 8 such classes. This proves the assertion.

We note also that $M_\Gamma$ and $M_{\Gamma'}$ have the same $L$-spectrum but they have different $L_c$-spectrum, since for instance, the holonomy component of $\gamma_1 = B_1 L_{\frac{3}{4}}$ is not the holonomy component of any element in $\Gamma'$.

**Example 3.7.** We will describe two 7-dimensional flat manifolds of diagonal type which are Sunada isospectral, with isomorphic fundamental groups (hence diffeomorphic, by Bieberbach’s second theorem), $[L_c]$-isospectral but not marked length isospectral.

We take $\Gamma = \langle B_1 L_{b_1}, B_2 L_{b_2}, L_{\lambda}; \lambda \in \mathbb{Z}^7 \rangle$, $\Gamma' = \langle B'_1 L_{b'_1}, B'_2 L_{b'_2}, L_{\lambda}; \lambda \in \mathbb{Z}^7 \rangle$ as follows

| $B_1$ | $B_2$ | $B_1 B_2$ |
|-------|-------|-----------|
| $1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ | 1 |
| $1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ | 1 |
| $1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ |
| 1 | $-1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ |
| $-1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ |
| $-1_{\frac{1}{2}}$ | 1 | $-1_{\frac{1}{2}}$ |
| $-1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ |

| $B'_1$ | $B'_2$ | $B'_1 B'_2$ |
|-------|-------|-------------|
| $1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ | 1 |
| $1_{\frac{1}{2}}$ | $-1$ | $1_{\frac{1}{2}}$ |
| $1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ |
| $1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ |
| $-1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ |
| $-1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ |
| $-1_{\frac{1}{2}}$ | $-1_{\frac{1}{2}}$ | $1_{\frac{1}{2}}$ |

By computing the Sunada numbers it is straightforward to check that the manifolds $M_\Gamma$ and $M_{\Gamma'}$ are Sunada isospectral (see Remark 1.3). Furthermore it is not hard to give an explicit isomorphism from $\Gamma$ to $\Gamma'$ by conjugation by an affine motion.

By using Proposition 3.1 one can show that they are indeed $[L_c]$-isospectral, with arguments similar to those in Example 3.3.

We now show that the manifolds are not marked length isospectral and hence not isometric. Indeed, suppose that there exists a length-preserving isomorphism $\phi : \Gamma \rightarrow \Gamma'$. Then we must have $\phi(\Lambda) = \Lambda$ and also $\phi(\text{span}\{e_1, e_2\}) = \text{span}\{e_1, e_2\}$, since this is the space fixed by the action of the holonomy group. Furthermore $\phi^{-1}(e_i) = \pm e_i$ with $i = 1$ or 2. Since $l(B_1 L_{b_1}) = \frac{\sqrt{2}}{2}$ and $\phi$ is length-preserving, it follows that $\phi(B_1 L_{b_1}) = B'_1 L_{c'_j}$ where $B'_j = B'_1$ or $B'_2$ and $c'_j = \pm \frac{\sqrt{2}}{2} + w$ for some $w \in \frac{1}{2}\text{span}\{e_3, e_4, \ldots, e_7\}$ such that $\|c'_j\| = \frac{\sqrt{2}}{2}$.

Now we have $l(B_1 L_{b_1 + e_i}) = \|(b_1 + e_i)\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{6}}{2}$; while
\[ l(\phi(B_1 L_{b_1} + e_i)) = l(\phi(B_1 L_{b_1}) \circ \phi(L_{e_i})) = l(B_1 L_{b_1} L_{e_2}) = \sqrt{(1/2)^2 + 1^2 + (1/2)^2} = \sqrt{7}/2, \]
a contradiction. Hence \( M_\Gamma \) and \( M_\Gamma' \) are not marked length isospectral.

**Example 3.8.** There exist flat manifolds having the same lengths of closed geodesics but which are very different from each other. We will now give several \( L \)-isospectral pairs having, either different dimension, or nonisomorphic fundamental groups, or one of them orientable and the other not. We will make use of the following classical theorem, proved by Lagrange: *Every nonnegative integer can be written as a sum of four squares* (see [Gr] for instance). As a consequence of this fact we see that *all canonical tori \( \mathbb{Z}^n \setminus \mathbb{R}^n, n \geq 4, \) have the same \( L \)-spectrum.*

The same is true for many other flat manifolds. For instance, if we take the Bieberbach groups \( \Gamma_n^k := \langle C_k L_{\frac{n-k}{2}}, \mathbb{Z}^n \rangle, \) where \( C_k = \text{diag}(1,1,\ldots,1,-1,\ldots,-1), \) as in Ex. 2.3(vi) then, for \( 5 \leq k < n, \) all groups are \( L \)-isospectral. This is the case since the \((\frac{1}{2} + m_1)^2 + m_2^2 + \cdots + m_5^2, \) with \( m_i \in \mathbb{Z}, \) represent every number of the form \( \frac{1}{4} + m, m \in \mathbb{N}. \) However, it is clear that \( \Gamma_n^k \) are not \( L_c \) isospectral nor \([L]\)-isospectral to each other. For fixed \( k \) and varying \( n, \) they have different dimensions \( n; \) half of them are orientable and half are nonorientable.

| Some \( p \)-isospectrality? | Sunada isospectral? | Isomorphic fundamental groups? | \([L]/[L_c]\)-isospectral? | \( L/L_c \)-isospectral? | Ex. | dim. |
|-----------------------------|------------------|-------------------------------|-----------------------------|-----------------------------|-----|-----|
| \( p \)-iso., not 0-iso.    | No               | Yes/No                        | No                          | No                          | 2.3 | \( n \geq 4 \) |
| all                         | Yes              | No                            | No                          | Yes                         | 3.4 | \( n \geq 4 \) |
| 0-iso., not \( p \)-iso.   | No               | No                            | Yes/No                      |                            | 3.6 | \( n \geq 6 \) |
| all                         | Yes              | No                            | Yes                         | Yes                         | 3.3 | \( n \geq 4 \) |
| all                         | Yes              | Yes                           | Yes                         | Yes                         | 3.7 | \( n \geq 7 \) |
| none/some \( p \)-iso      | No               | No                            | Yes                         | Yes                         | 3.5 | \( n \geq 13 \) |
| none                        | No               | No                            | Yes                         | Yes                         | 3.8 | \( n \geq 4 \) |

We notice that all the pairs in the table above are not marked length isospectral. Some of the examples in the table have some similar spectral properties as other known examples in the context of nilmanifolds (see [Go] and [Gt1]).

The following proposition will show that marked length isospectral implies isometric for flat manifolds. This result adds more information to the table above. The analogous result is known in other contexts, for instance, for flat tori, for closed surfaces of negative curvature (see [Ot] and [Cr]) and for certain two-step nilmanifolds that include Heisenberg manifolds (see [Eb]).

**Proposition 3.10.** If two flat manifolds have the same marked length spectrum then they are isometric.
Proof. By assumption, there is an isomorphism, \( \phi : \Gamma \rightarrow \Gamma' \), between the fundamental
groups preserving lengths, i.e. \( l(\gamma) = l(\phi(\gamma)) \), for any \( \gamma \in \Gamma \).

By Bieberbach’s second theorem \( \phi \) is given by conjugation by \( AL_a \), an affine motion.
Since \( \phi(L_A) = AL_a L_A(AL_a)^{-1} = L_{AA}, \) and on the other hand \( \phi(L_A) = L_{\Lambda'} \) (since \( \phi \)
is an isomorphism), it follows that \( A\Lambda = \Lambda' \). Since \( \phi \) preserves lengths, so does \( A \), hence
\( A \in O(n) \). Thus \( \phi \) is given by conjugation by an isometry, hence \( M_\Gamma \) and \( M_{\Gamma'} \) are
isometric. \( \Box \)

\section{Poisson summation formulas for flat manifolds}

We now consider, for \( \Gamma \) a Bieberbach group and \( \tau \) a finite dimensional representation
of \( O(n) \), the zeta function

\[ Z_\tau(\Gamma)(s) := \sum_{\mu \geq 0} d_{\tau,\mu}(\Gamma) e^{-4\pi^2\mu_s}. \]  

(4.1)

The series is uniformly convergent for \( s > \varepsilon \), for any \( \varepsilon > 0 \). We recall that for any
\( BL_b \in \Gamma \) we have set \( n_B = \dim \ker(B - Id) \) and \( b_+ = p_B(b) \), where \( p_B \) denotes the orthogonal projection onto \( \ker(B - Id) \). In the case when \( \tau = \tau_p \), for some \( 0 \leq p \leq n \), we write \( Z_p(\Gamma)(s) = Z_{\tau_p}(\Gamma) \).

Theorem 4.1. (i) We have

\[ Z_\tau(\Gamma)(s) = |F|^{-1} \sum_{BL_b \in F} \dim \ker(B - Id) (4\pi s)^{-2p} \sum_{\lambda_+ \in p_B(\Lambda)} e^{-\|\lambda_+ s\|_2^2}. \]  

(4.2)

(ii) \( \text{Spec}(M_\Gamma) \) determines the lengths of closed geodesics of \( M_\Gamma \) and the numbers \( n_B \).

(iii) \( \text{Spec}_p(M_\Gamma) \) (in particular \( \text{Spec}_p(M_\Gamma) \), for any \( p \geq 0 \)) determines the spectrum of
the torus \( T_\Lambda = \Lambda \backslash \mathbb{R}^n \) and the cardinality of \( F \). That is, if \( M_\Gamma \) and \( M_{\Gamma'} \) are \( \tau \)-isospectral
then the associated tori \( T_\Lambda \) and \( T_{\Lambda'} \) are isospectral and \( |F| = |F'| \).

Proof. Using expression (1.1) for \( d_{\tau,\mu}(\Gamma) \) we may write

\[ Z_\tau(\Gamma)(s) = |F|^{-1} \sum_{BL_b \in F} \dim \ker(B - Id) \sum_{v \in (\Lambda^*)^B} e^{-2\pi i v.b} e^{-4\pi^2\|v\|^2 s}. \]  

(4.3)

where \( (\Lambda^*)^B := \Lambda^* \cap \ker(B - Id) \). We have that \( (\Lambda^*)^B \) is a lattice in \( \ker(B - Id) \), by
Lemma 1.1(ii).

We shall recall some standard facts on Poisson summation. If \( f \in \mathcal{S}(\mathbb{R}^n) \), the Schwartz
space of \( \mathbb{R}^n \), let \( \hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x.y} dx \), the Fourier transform of \( f \). We then have
(see [Se], for instance):

- If \( h(x) := e^{-\pi\|x\|^2} \), then \( \hat{h} = h \). If \( a > 0 \) and \( g(x) := e^{-a\pi\|x\|^2} \) then \( \hat{g}(y) = a^{-\frac{n}{2}} h\left(\frac{y}{a}\right) \).

- If \( b \in \mathbb{R}^n \) and \( f \in \mathcal{S}(\mathbb{R}^n) \), set \( f_b(x) := f(x) e^{2\pi i x.b} \). Then \( \hat{f}_b(y) = \hat{f}(y - b) \).

- If \( L \) is a lattice in \( \mathbb{R}^d \) and \( \Lambda^* \) is the dual lattice of \( L \) then:

\[ \sum_{\nu \in L} f(\nu) = \text{vol}(L)^{-1} \sum_{\nu' \in \Lambda^*} \hat{f}(\nu'). \]
Now we apply Poisson summation in (4.3) for $Z^\Gamma_\tau(s)$, with $L = (\Lambda^*)^B$, a lattice in $\ker(B - Id)$, observing that in the expression for $e_{\mu,\gamma}$ in (1.1) we may write $b_+$ in place of $b$. By Lemma 1.1(iii) we get:

$$Z^\Gamma_\tau(s) = |F|^{-1} \sum_{BL_b \in F} \text{tr} \tau(B) \text{vol}(\Lambda^*)^{-1} \sum_{\lambda_+ \in p_B(\Lambda)} (4\pi s)^{-\frac{n_B}{2}} e^{-\frac{|\lambda + b_+|^2}{4s}}.$$ 

Now, since $\text{tr}_0(B) = 1$ for any $B$, the standard asymptotic argument implies that the lengths of closed geodesics $l(\gamma L_\lambda) = ||\lambda + b_+||$ and the numbers $n_B$, are determined by $\text{spec}(M_T)$ (see for instance [Bu, §9.2]), hence (ii) follows.

To verify (iii) we note first that by (4.2) the $\tau$-spectrum of $M_T$ determines the zeta function $Z^\Gamma_\tau(s)$. The standard asymptotic argument shows that this determines the following series

$$|F|^{-1} \dim(\tau) \text{vol}(\Lambda^*)^{-1} (4\pi s)^{-\frac{n}{2}} \sum_{\lambda \in \Lambda} e^{-\frac{|\lambda|^2}{4s}}$$

which is the partial sum of the right hand side of (4.2), corresponding just to the element $BL_b = Id$ in $F$. Now, by using Poisson summation for the torus, this expression is equal to

$$|F|^{-1} \dim(\tau) \sum_{v \in \Lambda^*} e^{-4\pi^2 ||v||^2 s} = |F|^{-1} \dim(\tau) Z^\Gamma_1(s).$$

Now we can leave out the factor $\dim(\tau)$ and since the eigenvalue zero of the Laplacian on functions has multiplicity one, $|F|$ is determined and hence the zeta function for the torus. This completes the proof of the theorem. □

**Remark 4.2.** (a) The eigenvalue spectrum does not determine the complex lengths of closed geodesics, as Example 3.6 shows.

(b) Formula (4.2) is a Poisson summation formula for natural vector bundles over flat manifolds. Sunada (see [Su]) has obtained a similar formula in the case of functions, i.e. $\tau$ is the trivial representation, by using the heat kernel on $M_T$ and the Selberg trace formula. As a consequence, Sunada obtains (iii) in the theorem in the function case. The above approach is different since it uses the formula for the multiplicities of eigenvalues obtained in [MR2] and furthermore the final formula is also different.

The $p^{th}$-Betti number of a closed Riemannian manifold gives the multiplicity of the eigenvalue zero for the Hodge-Laplacian acting on $p$-forms. Thus, a closed orientable $n$-manifold cannot be $n$-isospectral to a nonorientable one, since the $n^{th}$-Betti numbers are distinct. In particular such manifolds cannot be strongly isospectral. One can ask whether a closed orientable manifold can be isospectral on functions to a nonorientable one. As a consequence of Theorem 4.1, we shall now show that this cannot happen for flat manifolds. We have:

**Corollary 4.3.** If two flat manifolds are isospectral then they are both orientable or both nonorientable.

**Proof.** Let $M = M_T$, $\Gamma$ a Bieberbach group and let $\gamma = BL_b \in \Gamma$. The possible eigenvalues of $B$ are 1,-1 and a set of complex roots of unity which come in pairs, each one with the conjugate root. Hence, if $d_B^\gamma$ denotes the multiplicity of the eigenvalue $-1$, $\det(B) = (-1)^{d_B^\gamma} = (-1)^{n - n_B}$. Hence, $M_T$ is orientable if and only if $n_B \equiv n$, mod 2, for each $\gamma \in \Gamma$. On the other hand, all $n_B$'s are determined by the spectrum, by Theorem 4.1, thus the corollary follows. □
Remark 4.4. The assertion in the corollary is not true for \( p \)-isospectral manifolds, \( p > 0 \), as shown in Ex. 2.3(iii),(v) and in [MR2,3].

If \( \Gamma \) is of diagonal type it is possible to give a much more explicit formula for the zeta functions. Indeed, using the notations in (1.4) we have:

\[ Z_p^\Gamma(s) = 2^{-r} \sum_{n\in F} K^n_p(n - d) (4\pi s)^{-\frac{d}{2}} \sum_{t=0}^d c_{d,t}(\Gamma) \theta_{d,t}(\frac{1}{4\pi s}) \]

where, for \( \text{Re } z > 0 \),  
\[ \theta_{d,t}(z) = \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} e^{-z(\sum_{j=1}^d (\frac{1}{2} + m_j)^2 + \sum_{j=t+1}^d m_j) + \zeta(t,\theta)} \]

(ii) If \( \Gamma \) and \( \Gamma' \) are Bieberbach groups of diagonal type with holonomy group \( \mathbb{Z}_2'^d \), then \( M_\Gamma \) and \( M_{\Gamma'} \) are \( p \)-isospectral if and only if 

\[ K^n_p(n - d) c_{d,t}(\Gamma) = K^n_p(n - d) c_{d,t}(\Gamma') \]

for each \( 1 \leq t \leq d \leq n \). In particular, if \( c_{d,t}(\Gamma) = c_{d,t}(\Gamma') \) for every \( d, t \), then \( M_\Gamma \) and \( M_{\Gamma'} \) are \( p \)-isospectral for all \( p \).

If \( K^n_p(x) \) has no integral roots and \( M_\Gamma \) and \( M_{\Gamma'} \) are \( p \)-isospectral then they are Sunada isospectral. In particular, isospectrality implies Sunada isospectrality.

Proof. In this case the volumes in (4.2) equal 1. Using the notation \( \tilde{I}_B = I_B \cap I_{2b}^{odd} \) and (1.4) we see that the zeta functions can be written:

\[ Z_p^\Gamma(s) = \left| F \right|^{-1} \sum_{B \in F} K^n_p(n - n_B) (4\pi s)^{-\frac{d}{2}} \sum_{m_j \in \mathbb{Z}^d : j \in I_B} e^{-\frac{1}{4\pi s} \left( \sum_{j \in I_B} \frac{1}{2} m_j^2 + \sum_{j \in I_B^c \setminus I_B} m_j^2 \right)} \]

\[ = 2^{-r} \sum_{d=1}^n K^n_p(n - d) (4\pi s)^{-\frac{d}{2}} \sum_{t=0}^d c_{d,t}(\Gamma) \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} e^{-\frac{1}{4\pi s} \left( \sum_{j=1}^d (\frac{1}{2} + m_j)^2 + \sum_{j=t+1}^d m_j \right)} \]

This implies the expression for \( Z_p^\Gamma(s) \) asserted in (i).

We furthermore note that, for each \( d, t \), as \( s \to 0^+ \),

\[ (4\pi s)^{-\frac{d}{2}} \theta_{d,t}(\frac{1}{4\pi s}) \sim 2^t (4\pi s)^{-\frac{d}{4}} e^{-\frac{s}{4\pi}}. \]

Now,

\[ Z_p^\Gamma(s) \sim 2^{-r} \sum_{d=1}^n \sum_{t=0}^d K^n_p(n - d) 2^t c_{d,t}(\Gamma) (4\pi s)^{-\frac{d}{2}} e^{-\frac{s}{4\pi}} \]

as \( s \to 0^+ \) and furthermore we have that \( (4\pi s)^{-\frac{d}{2}} e^{-\frac{s}{4\pi}} = o \left( (4\pi s)^{-\frac{d}{4}} e^{-\frac{s}{4\pi}} \right) \) if and only if \( t > t' \), or else \( t = t' \) and \( d < d' \). Thus, by a standard asymptotic argument we may conclude from (4.4) that if \( Z_p^\Gamma(s) = Z_p^{\Gamma'}(s) \), then necessarily \( K^n_p(n - d) c_{d,t}(\Gamma) = K^n_p(n - d) c_{d,t}(\Gamma') \), for every \( d, t \). The converse is also clear from (4.4). This proves the first assertion in (ii). Furthermore, if \( K^n_p(x) \) has no integral roots, then \( p \)-isospectrality implies the equality of the numbers \( c_{d,t} \) for \( \Gamma \) and \( \Gamma' \), hence \( M_\Gamma \) and \( M_{\Gamma'} \) are Sunada isospectral by Remark 1.3. \( \square \)
We note furthermore that the isospectrality criterion in [MR2, Thm. 3.1] is actually an equivalence, for flat manifolds of diagonal type. That is,

**Proposition 4.7.** Let $\Gamma, \Gamma'$ be of diagonal type. Then $M_\Gamma$ and $M_{\Gamma'}$ are $p$-isospectral if and only if there is a bijection $\gamma \leftrightarrow \gamma'$ between the holonomy groups $F$ and $F'$, such that for each $\mu$

$$tr_p(B)e_{\mu,\gamma} = tr_p(B')e_{\mu,\gamma'}$$

where $\gamma = BL_b$, $\gamma' = B'L_{b'}$. In particular, $M_\Gamma$ and $M_{\Gamma'}$ are isospectral if and only if there is a bijection $\gamma \leftrightarrow \gamma'$ from $F$ onto $F'$ satisfying $e_{\mu,\gamma} = e_{\mu,\gamma'}$, for all $\gamma \in F$.

**Remark.** We note that the bijection in the case of $p$-isospectrality is only necessary for those elements with nonvanishing trace.

**Proof.** The “if” part is stated in [MR2, Thm. 3.1] and follows directly from (1.1) with $\tau = \tau_p$. To prove the “only if” part we set $F_d(\Gamma) = \{\gamma = BL_b \in F : n_B = d\}$, defining $F'_d(\Gamma')$ similarly for $\Gamma'$. By Thm. 4.5 (ii), $K^n_p(n-d)c_{d,t}(\Gamma) = K^n_p(n-d)c_{d,t}(\Gamma')$ for every $d, t, t \leq d$. We distinguish two cases:

(a) $d$ is such that $K^n_p(n-d) \neq 0$, (b) $d$ is such that $K^n_p(n-d) = 0$.

For $d$ verifying (a) we have that $c_{d,t}(\Gamma) = c_{d,t}(\Gamma')$ for every $t \leq d$. Hence, by (1.4) we can define a bijection $\gamma \leftrightarrow \gamma'$ between $F_d(\Gamma)$ and $F_d(\Gamma')$, such that $n_B = n_{B'}$ and $|I_B \cap I_{2n_B}^{odd}| = |I_{B'} \cap I_{2n_{B'}}^{odd}|$. Since $\Gamma$ is of diagonal type, $e_{\mu,\gamma}$ depends only on $\mu, n_B$ and $|I_B \cap I_{2n_B}^{odd}|$, hence this bijection is such that $e_{\mu,\gamma} = e_{\mu,\gamma'}$ for every $\gamma \in F_d(\Gamma)$. Thus, there is bijection between the elements in $F$ and $F'$ with $n_B = n_{B'} = d$, $d$ verifying (a), such that $tr_p(B)e_{\mu,\gamma} = tr_p(B')e_{\mu,\gamma'}$.

On the other hand, by Theorem 4.1, $|F| = |F'|$, thus the partial bijection defined above implies that the number of elements $BL_b \in F$ with $n_B = d$ of type (b) (i.e. $tr_p(B) = 0$) equals the number of elements $B'L_{b'} \in F'$ with $n_{B'}$ of type (b). For these elements, the equality $tr_p(B)e_{\mu,\gamma} = tr_p(B')e_{\mu,\gamma'}$ holds trivially, hence we can complete the bijection between $F$ and $F'$ as desired. \qed

To conclude this paper, we will compute explicitly the zeta functions $Z^\Gamma_p(s)$ for some pairs of Bieberbach groups. We notice that from the expression of the zeta functions one can read $p$-isospectrality. We consider first the case of the Klein bottle group.

**Example 4.8.** In the notation of Example 2.2 we have

$$2Z^\Gamma_p(s) = \frac{2}{4\pi s} \sum_{(m_1, m_2) \in \mathbb{Z}^2} e^{-\frac{m_1^2 + m_2^2}{4\pi s}} + \frac{tr_p(B)}{\sqrt{4\pi s}} \sum_{m \in \mathbb{Z}} e^{-\frac{(\frac{d}{2} + m)^2}{4\pi s}}. \quad (4.5)$$

We note furthermore that $tr_1(B) = 0$, $tr_2(B) = -1$, thus from (4.5) we get an explicit formula for $Z^\Gamma_p(s)$, $p = 0, 1, 2$. 

**Remark 4.6.** The previous theorem says that, for groups of diagonal type, $p$-isospectrality for one value of $p$ implies the $p$-isospectrality, provided the Krawtchouk polynomial $K^n_p(x)$ has no integral roots. This extends to all dimensions a result obtained in [MR3, Theorem 3.12(d)] for dimensions $\leq 8$ —using a very different approach.

We note that the theorem gives another proof of the fact that if $c_{d,t}(\Gamma) = c_{d,t}(\Gamma')$ for all $d, t$, then $M_\Gamma$ and $M_{\Gamma'}$ are $p$-isospectral for all $p$ (see Remark 1.3).
Example 4.9. If we take $\Gamma$ and $\Gamma'$ as in Example 2.3(i), using the information therein, we get

$$2 Z_p^\Gamma(s) = \frac{(4)^p}{(4\pi s)^2} \sum_{(m_1, \ldots, m_4) \in \mathbb{Z}^4} e^{-\frac{\sum_{j=1}^4 m_j^2}{4s}} + K_p^4(3) \sum_{m \in \mathbb{Z}} e^{-\left(\frac{1}{4s} + m^2\right)}.$$  

$$2 Z_p^\Gamma'(s) = \frac{(4)^p}{(4\pi s)^2} \sum_{(m_1, \ldots, m_4) \in \mathbb{Z}^4} e^{-\frac{\sum_{j=1}^4 m_j^2}{4s}} + K_p^4(1) \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^4} e^{-\left(\frac{1}{4s} + m^2 + m_2^2 + m_3^2\right)}.$$  

These expressions indicate that $\Gamma$ and $\Gamma'$ are $p$-isospectral if and only if $p = 2$, since $K^4_2(1) = K^4_2(3) = 0$ for $p = 2$ only. Furthermore we also see that $M_\Gamma$ and $M_{\Gamma'}$ are neither isospectral nor $L$-isospectral.

Example 4.10. Using the information in Example 2.3(iii), in the case of $\Gamma$ we have that $n_{B_i} = 2$ for $1 \leq i \leq 3$ and in all three cases, $\text{tr}_p(B) = K^4_p(2)$ equals 0 if $p = 1, 3$, it equals $-2$ if $p = 2$ and 1 if $p = 0, 4$.

Thus, we obtain for $\Gamma$:

$$4 Z_p^\Gamma(s) = \frac{(4)^p}{(4\pi s)^2} \sum_{(m_1, \ldots, m_4) \in \mathbb{Z}^4} e^{-\frac{\sum_{j=1}^4 m_j^2}{4s}} + 3 K_p^4(2) \sum_{(m_1, m_2) \in \mathbb{Z}^2} e^{-\frac{1}{4s} + m_1^2}.$$  

In the case of $\Gamma'$, we have that $n_{B'_{i}} = n_{B'^3_{i}} = 1$, $n_{B'^2_{i}} = 2$ and $\text{tr}_p(B') = \text{tr}_p(B'^3)$ equals 0 if $p = 1, 2, 3$ and equals 1 if $p = 0, 4$. Furthermore, $\text{tr}_p(B'^2) = K^4_p(2)$. Thus we obtain:

$$4 Z_p^{\Gamma'}(s) = \frac{(4)^p}{(4\pi s)^2} \sum_{(m_1, \ldots, m_4) \in \mathbb{Z}^4} e^{-\frac{\sum_{j=1}^4 m_j^2}{4s}} + K_p^4(2) \sum_{(m_1, m_2) \in \mathbb{Z}^2} e^{-\frac{1}{4s} + m_1^2} + \frac{\text{tr}_p(B')}{\sqrt{4\pi s}} \sum_{m \in \mathbb{Z}} \left(e^{-\frac{1}{4s} + m^2} + e^{-\frac{1}{4s} + m^2}\right).$$  

These expressions and the values of the $p$-traces imply that $M_\Gamma$ and $M_{\Gamma'}$ are not $L$-isospectral and furthermore, that they are $p$-isospectral if and only if $p = 1, 3$. Indeed, for these values of $p$ only, all $p$-traces are zero for any $B \neq \text{Id}$ for both $\Gamma$ and $\Gamma'$, hence for such $p$ the expression of $Z_p^\Gamma(s) = Z_p^{\Gamma'}(s)$ contains only the contribution of the lattice elements.

Remark 4.11. We now discuss how the $p$-isospectrality in Examples 2.3(i)-(vi), together with the existence of different lengths of closed geodesics are not in contradiction with the wave trace and the heat trace formulas. The basic point in both cases is that when considering the Laplacian acting on $p$-forms, certain coefficients appear multiplying the contributions of each geodesic to the formulas and they are not always positive, so they can cancel out when added up, or they can vanish in some cases. The last situation happens in our examples.

We first look at formula (4.2), that can be viewed as a heat trace formula for the Laplacian on $p$-forms, concentrating on Example 4.9 (i.e. Ex. 2.3(i)). In the comparison between $\Gamma$ and $\Gamma'$ we can see why the case $p = 2$ is different. Indeed, in the expressions of the heat traces $Z_p^\Gamma(s)$ and $Z_p^{\Gamma'}(s)$, we find the coefficients $K_p^4(3)$ and $K_p^4(1)$ respectively in
the right hand side, which vanish for $p = 2$ and this makes both manifolds $2$-isospectral. The lengths of closed geodesics are distinct for $M_\Gamma$ and $M_{\Gamma'}$ but there is no contradiction with the equality of the heat traces for $p = 2$, since these lengths show up in the exponents in the right hand side of the formula with coefficient $0$ (for $p = 2$), so they do not influence the sum. For other values of $p$ the coefficients $K^+_{1,0}(3)$ and $K^+_{1,0}(1)$ do not vanish and we get quite different heat traces for $M_\Gamma$ and $M_{\Gamma'}$ showing that the manifolds are not $p$-isospectral for $p \neq 2$.

A similar phenomenon happens with the singularities of the wave traces, located at lengths of closed geodesics.

For the Laplacian acting on $p$-forms the following residue formula for the wave trace is stated in [DG, Introduction] under a genericity condition on the closed geodesics of period $T$:

$$
\lim_{t \to T} (t - T) \sum_{k \geq 0} e^{i \sqrt{\lambda_k} t} = \sum_{\gamma : \Lambda \to \Lambda} T_{0, \gamma} \frac{2\pi}{2\pi i} |I - P_\gamma|^{1/2} \operatorname{tr}(H_\gamma) \quad (4.6)
$$

Here $T_{0, \gamma}$ is the smallest positive period of $\gamma$, $\sigma_\gamma$ is the Maslov factor, $P_\gamma$ the Poincaré map around $\gamma$ and $H_\gamma : \Lambda^p \mapsto \Lambda^p$ the holonomy along $\gamma$. We observe in (4.6) that the factor $\operatorname{tr}(H_\gamma)$ is in our flat case the $p$-trace of the orthogonal transformation $B$ denoted by $\operatorname{tr}_p(B)$ which vanishes for some values of $p$ (as we have just seen) and, in this case, this implies the vanishing of the r.h.s. in (4.6). For instance in Example 2.3(i), $\frac{\sqrt{2} \pi}{2}$ is the length of a closed geodesic for $M_{\Gamma'}$ but not for $M_\Gamma$. For $M_{\Gamma'}$, it is a singularity of the wave trace in (4.6) for the Laplacian acting on functions, but it is not a singularity for the wave trace of the Laplacian acting on $2$-forms since $\operatorname{tr}_2(B') = 0$, for $\gamma' = B' L_{b'} \in \Gamma'$ corresponding to the geodesics of length $\frac{\sqrt{2} \pi}{2}$ in $M_{\Gamma'}$.

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