Casimir Energy in the Axial Gauge

Giampiero Esposito,1,2 * Alexander Yu. Kamenshchik3,4 † and Klaus Kirsten5 ‡
1Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio N’, 80126 Napoli, Italy
2Dipartimento di Scienze Fisiche, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio N’, 80126 Napoli, Italy
3L.D. Landau Institute for Theoretical Physics, Russian Academy of Sciences, Kosygina Str. 2, Moscow 117334, Russia
4Landau Network–Centro Volta, Villa Olmo, Via Cantoni 1, 22100 Como, Italy
5The University of Manchester, Department of Physics and Astronomy, Theory Group, Schuster Laboratory, Oxford Road, Manchester M13 9PL, England

Abstract

The zero-point energy of a conducting spherical shell is studied by imposing the axial gauge via path-integral methods, with boundary conditions on the electromagnetic potential and ghost fields. The coupled modes are then found to be the temporal and longitudinal modes for the Maxwell field. The resulting system can be decoupled by studying a fourth-order differential equation with boundary conditions on longitudinal modes and their second derivatives. The exact solution of such equation is found by using a Green-function method, and is obtained from Bessel functions and definite integrals involving Bessel functions. Complete agreement with a previous path-integral analysis in the Lorenz gauge, and with Boyer’s value, is proved in detail.

*Electronic address: giampiero.esposito@na.infn.it
†Electronic address: landau@icil64.cilea.it
‡Electronic address: klaus@a13.ph.man.ac.uk

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I. INTRODUCTION

In recent work by the authors [1] the zero-point energy of a perfectly conducting spherical shell, a problem first investigated by Boyer [2], has been studied from the point of view of path-integral quantization, with the associated boundary conditions on modes for the potential and the ghost. Since there are good reasons for regarding path integrals as a basic tool in the quantization of gauge fields [3], such a re-derivation of a quantum effect is a lot more than a useful exercise in field theory. In particular, one may hope to be able to use similar investigations to prove the gauge independence of path-integral calculations on manifolds with boundary, and also to make predictions which can be tested against observations [4,5]. The key elements of the analysis in Ref. [1] that we need are as follows.

(i) The perfect-conductor boundary conditions for a spherical shell, according to which tangential components of the electric field should vanish at the boundary \( \partial M \), are satisfied if

\[
\begin{align*}
[A_t]_{\partial M} &= 0, \quad (1.1) \\
[A_\theta]_{\partial M} &= 0, \quad (1.2) \\
[A_\phi]_{\partial M} &= 0. \quad (1.3)
\end{align*}
\]

With our notation, \( A_t \) is the temporal component of the electromagnetic potential \( A_\mu \), while \( A_\theta \) and \( A_\phi \) are its tangential components. In the classical theory, the boundary conditions (1.1)–(1.3) are gauge-invariant if and only if the gauge function \( \varepsilon \) occurring in the gauge transformations

\[
\varepsilon A_\mu \equiv A_\mu + \nabla_\mu \varepsilon
\]

(1.4)

vanishes at \( \partial M \):

\[
[\varepsilon]_{\partial M} = 0. \quad (1.5)
\]

In the quantum theory, Eq. (1.5) is a shorthand notation for the vanishing at the boundary of two independent ghost fields [1,6]. At this stage, the only boundary condition whose preservation under gauge transformations (1.4) is again guaranteed by Eq. (1.5) is the vanishing of the gauge-averaging functional at the boundary:

\[
[\Phi(A)]_{\partial M} = 0. \quad (1.6)
\]

The latter is a map which associates to a one-form \( A_\mu dx^\mu \) a real number, and reflects the freedom of choosing supplementary conditions in the classical theory. The square of \( \Phi(A) \), divided by \( 2\alpha \) (see below), should be added to the Maxwell Lagrangian to obtain an invertible operator on perturbations of \( A_\mu \) in the quantum theory [6].

(ii) If \( \Phi(A) \) is chosen to be of the axial type: \( \Phi(A) = N^\mu A_\mu \), which is quite relevant for the quantization program in noncovariant gauges [7], the potential is found to obey the equation
\[ P_\mu^\nu A_\nu = 0, \]  

(1.7)

where \( P_\mu^\nu \) is the second-order operator

\[ P_\mu^\nu = -\delta_\mu^\nu \Box + \nabla_\mu \nabla_\nu + \frac{1}{\alpha} N_\mu N_\nu, \]  

(1.8)

\( \alpha \) being a gauge parameter with dimension length squared. Moreover, \( A_\nu \) obeys the equations

\[ \nabla_\mu (P_\mu^\nu A_\nu) = 0, \]  

(1.9)

\[ N^\mu P_\mu^\nu A_\nu = 0. \]  

(1.10)

The boundary condition (1.6) and the differential equation (1.9) imply that the normal component of the potential, i.e., \( A_r \equiv \vec{r} \cdot \vec{A} \), vanishes everywhere inside the spherical shell, including the boundary [1]:

\[ A_r(t, r, \theta, \phi) = 0 \ \forall r \in [0, R]. \]  

(1.11)

Moreover, the axial gauge-averaging functional leads to the ghost operator \( Q = -\frac{\partial}{\partial r} \), and hence the boundary condition (1.5) implies that the ghost field vanishes everywhere as well [1]:

\[ \varepsilon(t, r, \theta, \phi) = 0 \ \forall r \in [0, R]. \]  

(1.12)

(iii) One is then left with the temporal component \( A_t \), expanded in harmonics on the two-sphere [1]:

\[ A_t(t, r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l(r) Y_{lm}(\theta, \phi) e^{i\omega t}, \]  

(1.13)

and tangential components (here \( k \) stands for \( \theta \) and \( \phi \))

\[ A_k(t, r, \theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left[ c_l(r) \partial_k Y_{lm}(\theta, \phi) + T_l(r) \varepsilon_{kp} \partial^p Y_{lm}(\theta, \phi) \right] e^{i\omega t}. \]  

(1.14)

Of course, \( c_l \) and \( T_l \) are longitudinal and transverse modes, respectively. On setting \( \omega = iM \) [1], the latter are found to obey the eigenvalue equation

\[ \left[ \frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} \right] T_l = M^2 T_l, \]  

(1.15)

whose regular solution reads

\[ T_l(r) = \sqrt{\pi/2} \sqrt{T_{l+\frac{1}{2}}(Mr)} \ \forall r \in [0, R]. \]  

(1.16)

Section II studies the coupled equations for temporal and longitudinal modes, with the associated fourth-order equation for \( c_l \) modes only. Section III finds the explicit form of \( c_l \) modes by means of a Green-function method, and proves agreement with the Casimir energy found by Boyer. Concluding remarks are presented in Sec. IV.
II. FOURTH-ORDER EQUATION FOR LONGITUDINAL MODES

In the axial gauge on a spherical shell, the temporal and longitudinal modes are known to obey, from the work in Ref. [1], the coupled system

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right] a_l - \frac{MI(l+1)}{r^2} c_l = 0, \quad (2.1)
\]

\[
\left[ \frac{d^2}{dr^2} - M^2 \right] c_l - Ma_l = 0. \quad (2.2)
\]

Although this system cannot be decoupled to find second-order equations with Bessel-type solutions [1], it can however be easily decoupled if one expresses \( a_l \) from Eq. (2.2) and inserts the result into Eq. (2.1). This leads to the following fourth-order equation for \( c_l \):

\[
\left[ \frac{d^4}{dr^4} + \frac{2}{r} \frac{d^3}{dr^3} - \left( M^2 + \frac{l(l+1)}{r^2} \right) \frac{d^2}{dr^2} - \frac{2}{r} M^2 \frac{d}{dr} \right] c_l(Mr) = 0, \quad (2.3)
\]

while \( a_l \) is eventually obtained as

\[
a_l(Mr) = \frac{1}{M} \left[ \frac{d^2}{dr^2} - M^2 \right] c_l(Mr). \quad (2.4)
\]

These equations are dimensionally correct, bearing in mind that, from the rule for taking derivatives of composite functions,

\[
\frac{dc_l}{dr} = M c_l'(Mr)
\]

and so on, where each prime denotes the derivative evaluated at \( y \equiv Mr \). The variable \( y \) is dimensionless, and makes it possible to study Eq. (2.3) in a form more convenient for the following calculations, i.e.,

\[
\left[ \frac{d^4}{dy^4} + \frac{2}{y} \frac{d^3}{dy^3} - \left( 1 + \frac{l(l+1)}{y^2} \right) \frac{d^2}{dy^2} - \frac{2}{y} \frac{d}{dy} \right] c_l(y) = 0. \quad (2.5)
\]

Recall now from the Introduction that, at \( r = R \), both \( a_l \) and \( c_l \) should vanish. By virtue of (2.4), the second derivative of \( c_l \) should then vanish as well on the boundary. Moreover, by regularity at the origin, \( a_l \) and \( c_l \) can be set to zero at \( r = 0 \), which implies that also \( \frac{d^2 c_l}{dr^2} \) vanishes at \( r = 0 \). The full set of boundary conditions for Eq. (2.5) can be therefore taken to be

\[
c_l|_{y=0} = 0, \quad (2.6a)
\]

\[
c_l|_{y=MR} = 0, \quad (2.6b)
\]
\[
\frac{d^2 c_l}{dy^2} \bigg|_{y=0} = 0, \tag{2.7a}
\]

\[
\frac{d^2 c_l}{dy^2} \bigg|_{y=MR} = 0. \tag{2.7b}
\]

At a deeper mathematical level, the boundary conditions (2.6a), (2.6b), (2.7a) and (2.7b) can be proved to determine a domain of self-adjoint extension of a fourth-order differential operator in one dimension on the closed interval \([0, R]\) (see sections 2 and 3 of Ref. [8]).

### III. EXACT SOLUTION

We are interested in solving the fourth-order differential equation (2.5) for \(c_l\). Since the unknown function \(c_l\) never occurs undifferentiated therein, a natural starting point is to put

\[
F_l(y) \equiv c'_l(y), \tag{3.1}
\]

from which the differential equation for \(F_l(y)\) reads

\[
\left[ \frac{d^3}{dy^3} + \frac{2}{y} \frac{d^2}{dy^2} - \left( 1 + \frac{l(l+1)}{y^2} \right) \frac{d}{dy} - \frac{2}{y} \right] F_l(y) = 0. \tag{3.2}
\]

An obvious solution is \(F_l(y) = 0\) and hence \(c_l(y) = \text{const}\). An elegant way to proceed is to point out that (3.2) can be rewritten as

\[
\frac{1}{y^2} \frac{d}{dy} \left[ y^2 \left( \frac{d^2}{dy^2} - \left( 1 + \frac{l(l+1)}{y^2} \right) \right) F_l(y) \right] = 0. \tag{3.3}
\]

This shows that solutions of Eq. (3.2) can be found as solutions of

\[
\left[ \frac{d^2}{dy^2} - \left( 1 + \frac{l(l+1)}{y^2} \right) \right] F_l(y) = \frac{C}{y^2}, \tag{3.4}
\]

where \(C\) is a constant. The general solution of Eq. (3.4) has the form

\[
F_l(y) = CF_{sp}(y) + d_1 F_1(y) + d_2 F_2(y), \tag{3.5}
\]

with \(F_{sp}(y)\) being a special solution of Eq. (3.4) with \(C = 1\), while \(F_1\) and \(F_2\) are the linearly independent integrals of the homogeneous equation

\[
\left[ \frac{d^2}{dy^2} - \left( 1 + \frac{l(l+1)}{y^2} \right) \right] \mathcal{F}(y) = 0. \tag{3.6}
\]

They read, with our conventions,

\[
F_1(y) = \sqrt{\pi/2y} \sqrt{y} I_{l+1/2}(y), \tag{3.7}
\]
$$F_2(y) = \sqrt{\pi/2} \sqrt[4]{I_{l-1/2}(y)}.$$

To find the solution $F_{sp}(y)$ of Eq. (3.4) with $C = 1$ for all values of $l$, a Green-function approach is convenient. On defining

$$L \equiv \frac{d^2}{dy^2} - \left( 1 + \frac{l(l+1)}{y^2} \right),$$

$$f(y) \equiv \frac{1}{y^2},$$

we have first to find the Green function $G(y, \xi)$ which, by definition [9], solves the equation (hereafter $b \equiv MR$)

$$LG = 0 \text{ for } y \in ]0, \xi[ \text{ and } y \in ]\xi, b[,$$

the boundary conditions (see (2.7a), (2.7b) and (3.1))

$$G'(0, \xi) = 0,$$

$$G'(b, \xi) = 0,$$

the continuity condition

$$\lim_{y \to \xi^+} G(y, \xi) = \lim_{y \to \xi^-} G(y, \xi),$$

and the jump condition

$$\lim_{y \to \xi^+} \frac{\partial G}{\partial y} - \lim_{y \to \xi^-} \frac{\partial G}{\partial y} = 1.$$

The general theory of boundary-value problems for second-order equations [9] tells us that, when the operator $L$ is studied with the unmixed boundary conditions of our problem, i.e.

$$F'_l(0) = 0,$$

$$F'_l(b) = 0,$$

the construction of $G(y, \xi)$ involves a nontrivial solution $u_1(y)$ of the homogeneous equation $Lu = 0$ satisfying $u'(0) = 0$, and a nontrivial solution $u_2(y)$ of $Lu = 0$ satisfying $u'(b) = 0$. More precisely, by virtue of (3.11)–(3.13) one finds

$$G(y, \xi) = A(\xi)u_1(y) \text{ if } y \in ]0, \xi[,$$

$$G(y, \xi) = B(\xi)u_2(y) \text{ if } y \in ]\xi, b[.$$
where \( u_1 \) and \( u_2 \) are linearly independent. By virtue of Eqs. (3.14) and (3.15), \( A(\xi) \) and \( B(\xi) \) are obtained by solving the inhomogeneous system

\[
A(\xi)u_1(\xi) - B(\xi)u_2(\xi) = 0, \tag{3.20}
\]

\[
B(\xi)u_2'(\xi) - A(\xi)u_1'(\xi) = 1, \tag{3.21}
\]

which yields

\[
A(\xi) = \frac{u_2(\xi)}{W(u_1, u_2; \xi)}, \tag{3.22}
\]

\[
B(\xi) = \frac{u_1(\xi)}{W(u_1, u_2; \xi)}, \tag{3.23}
\]

where \( W \) is the Wronskian of \( u_1 \) and \( u_2 \). For the operator \( L \), which is formally self-adjoint, the Wronskian reads [9]

\[
W = \gamma, \tag{3.24}
\]

\( \gamma \) being a constant. Thus, on defining as usual \( y_\prec \equiv \min(y, \xi), y_\succ \equiv \max(y, \xi) \), one finds a very simple formula for the Green function, i.e. [9]

\[
G(y, \xi) = \frac{1}{\gamma} u_1(y_\prec) u_2(y_\succ). \tag{3.25}
\]

The particular solution \( F_{sp} \) of the inhomogeneous boundary-value problem given by the equation

\[
LF(y) = f(y), \ y \in [0, b], \tag{3.26}
\]

with boundary conditions (3.16) and (3.17) written for \( F_{sp} \):

\[
F_{sp}'(0) = F_{sp}'(b) = 0, \tag{3.27}
\]

is then given by the integral [9]

\[
F_{sp}(y) = \int_0^b G(y, \xi) f(\xi) d\xi. \tag{3.28}
\]

We may now choose \( (a_1 \) and \( a_2 \) being some parameters)

\[
u_1(y) = \sqrt{\pi/2} \sqrt{y} I_{l+1/2}(y), \tag{3.29}\]

\[
u_2(y) = \sqrt{\pi/2} \sqrt{y} [a_1 I_{l+1/2}(y) + a_2 I_{l-1/2}(y)], \tag{3.30}\]

bearing in mind that \( u_1 \) and \( u_2 \) should be linearly independent and should satisfy Neumann boundary conditions at 0 and at \( b \), respectively, which gives a relation between \( a_1 \) and
However, what is truly essential for us is that the general theory of one-dimensional boundary-value problems ensures that an $F_{sp}$ exists satisfying the conditions (3.27). The general solution (3.5) of Eq. (3.4) can be therefore written in the form (see (3.7) and (3.8))

$$F_l(y) = CF_{sp}(y) + d_1F_1(y) + d_2F_2(y).$$

(3.31)

Regularity at the origin (see (2.6a)) implies that $d_2 = 0$. The vanishing of $c_i$ at the boundary (see (2.6b)) fixes the relation between $C$ and $d_1$, i.e.

$$C = -d_1\int_0^b dz \frac{F_1(z)}{F_{sp}(z)}.$$  

(3.32)

Eventually, the vanishing of $c_i''$ at the boundary (see (2.7b)) leads to

$$I_{l+1/2}(b) + \frac{1}{2b}I_{l+1/2}(b) = 0,$$

(3.33)

because, by construction, $CF_{sp}$ is a particular solution of Eq. (3.4) whose first derivative vanishes at the boundary.

Equation (3.33), jointly with the vanishing at the boundary of the transverse modes given in Eq. (1.16), yields the same set of eigenvalue conditions, with the same degeneracies, found for the interior problem in the Lorenz gauge in Ref. [1]. Thus, complete agreement with Boyer’s value for the Casimir energy is recovered (the exterior problem can be studied on replacing $I_{l+1/2}$ by $K_{l+1/2}$, without any difficulty).

IV. CONCLUDING REMARKS

Although noncovariant gauges break relativistic covariance, they make it possible to decouple Faddeev–Popov ghosts [10] from the gauge field. Thus, ghost diagrams do not contribute to cross-sections and need not be evaluated, and this property has been regarded as the main advantage of noncovariant gauges [7]. In the case of Casimir energies, the ghost field is forced to vanish everywhere by virtue of the boundary conditions appropriate for the axial gauge [1], and the analysis of temporal and longitudinal modes, although rather involved, has been here proved to lead to the same Casimir energy [2] for the interior problem found in the Lorenz gauge [1]. The particular solution of the inhomogeneous equation (3.4) plays a nontrivial role in ensuring that, despite some technicalities, the resulting Casimir energy is the same as in the Lorenz gauge, hence proving explicitly the equivalence of a covariant and a non-covariant gauge for a conducting spherical shell.

A better theoretical understanding of gauge independence in quantum field theory has been therefore gained, after the encouraging experimental progress of recent years in the measurement of Casimir forces [4,5,11,12].

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