THE SOLVABILITY OF BOUNDARY VALUE PROBLEM
FOR NONLINEAR ELLIPTIC-PARABOLIC EQUATIONS

Gunel Guseynova
Institute of Mathematics and Mechanics
of NAS of Azerbaijan
AZ1141, Baku, AZERBAIJAN

Abstract: The nonlinear elliptic-parabolic equations of nondivergent structure is considered. The solvability of the boundary value problems is investigated.

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1. Introduction

The theory of elliptic-parabolic equations ascends to the classical paper by Keldysh [1] in which the correct statements of the boundary value problems for the equations with one space variable were found. G. Ficera [2] has established a weak solvability of the first boundary value problem for a wide class of the second order equations with the non-negative characteristic form (see also [3]). As to strong solvability of the first boundary value problem for elliptic-parabolic equations in the non-divergent form with smooth coefficients, we shall note in this connection the papers [4, 5, 6]. The similar result for the equations in the case when the coefficients satisfy the Cordes condition is obtained in [7].

The theory of nonlinear elliptic-parabolic equation have many applications. For example, the class of nonlinear operators represent the well-known Richards equation, which serves as a basic model for the filtration of water in unsaturated soils (see [9, 10]), see also [11, 12, 13].
Let us consider in $Q_T = \Omega \times (0; T)$, where $Q_T$ be the cylinder, $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with the smooth boundary $\partial \Omega$, following the boundary value problem

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t,u) u_{ij} + \psi(x,t) u_{tt} - u_t = f(x,t),$$

(1)

$$u|_{\Gamma(Q_T)} = 0.$$  

(2)

Here

$$u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \ u_i = \frac{\partial u}{\partial x_i}, \ u_t = \frac{\partial u}{\partial t}, \ u_{tt} = \frac{\partial^2 u}{\partial t^2},$$

$$\Gamma(Q_T) = (\partial \Omega \times (0,T)) \cup \{(x,t): t = 0, x \in \Omega\}$$

is parabolic boundary of $Q_T$ and

$$\psi(x,t) = \omega(t) \lambda(\rho) \varphi(T-t),$$

(3)

where $\lambda(\rho) \geq 0$, $\lambda(\rho) \in C^1[0,diam\Omega]$, $\varphi(z) \geq 0$, $|\lambda(\rho)| \leq \alpha \sqrt{\lambda(\rho)}$, $\varphi'(z) \geq 0$, $\varphi(z) \in C^1[0,T]$, $\varphi(0) = \varphi'(0) = 0$, $\varphi(z) \geq \beta z \varphi'(z)$, $\omega(t) \in C^1[0,T]$, $\omega(t) \geq 0$, and $\alpha, \beta$ are positive constants.

Assume that the coefficients of the equation (1) the following conditions hold: $(a_{ij}(x,t))$ is a real symmetrical matrix with real measurable elements in $Q_T$ for every $(x,t) \in Q_T$, and $\xi \in \mathbb{R}^n$ and satisfies the inequalities hold:

$$\gamma |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t,u) \xi_i \xi_j \leq \gamma^{-1} |\xi|^2,$$

(4)

where $\gamma \in (0,1]$, is a constant.

The purpose of this paper is to obtain solvability boundary problem for nonlinear equations in an appropriate Sobolev spaces.

Before we obtained some a coercive estimations which be used to proving a unique strong solvability of the first boundary value problem (1)-(2) at every $f(x,t) \in L_2(Q_T)$.

The paper is organized as follows. In Section 2 we present some definitions and preliminary results. In Section 3 we give main results.

2. Definitions and preliminary results

For $R > 0$ and $x^0 \in \mathbb{R}^n$ we denote the ball $B_R(x^0) = \{x: |x - x_0| < < R\}$ and a cylinder $B_R(x^0) \times (0,T) = Q^R_T(x_0)$. Let $\overline{B}_R(x^0) \subset \Omega$. We say that $u(x,t) \in$
A \left( Q^R_T(x^0) \right) if u(x,t) \in C^\infty \left( \overline{Q^R_T(x^0)} \right), u|_{t=0} = 0 and \sup \rho u \in \left( \overline{Q^\rho_T(x^0)} \right) for some \rho \in (0, R).

Let us introduce the Banach space of functions u(x,t) given on QT with finite norms
\[ \|u\|_{W^{1,1}(QT)} = \left( \int_{QT} \left( u^2 + \sum_{i=1}^n u^2_{x_i} + u^2_t \right) \, dxdt \right)^{1/2} \]
and
\[ \|u\|_{W^{2,2}(QT)} = \left( \int_{QT} \left\{ \left( u^2 + \sum_{i=1}^n u^2_i + \sum_{i,j=1}^n u^2_{ij} \right) + u^2_t + \psi^2(x,t)u^2_{tt} \right\} \, dxdt \right)^{1/2}. \]

Suppose W^{2,2}_{2,\psi}(QT) is a subspace of the space W^{2,2}_2(Q_T) that contains the set of all functions from C^\infty (\overline{Q_T}) vanishing on the parabolic boundary \Gamma (Q_T).

Let us consider the operator L which is arising by problem (1)-(2). Now we like to get some coercive estimates for strong solutions to the problem (1)-(2). First we give the results for the model operator and applying these estimates we obtain the following.

**Lemma 1.** Let condition (3)-(4) for coefficients and weight be fulfilled. Then for any function u(x,t) \in A \left( Q^R_T(x^0) \right), there exists T_1(\psi(x,t),n) such that for T \leq T_1 and following estimate holds:
\[
\int_{Q^R_T(x^0)} \left[ \left( \sum_{i,j=1}^n u^2_{ij} + u^2_t \right) + \psi^2(x,t)u^2_{tt} + \psi(x,t)\sum_{i=1}^n u^2_{it} \right] \, dxdt \leq (1 + D(T)S) \int_{Q^R_T(x^0)} (Lu)^2 \, dxdt,
\]
where S = S(\psi,n) is some constant, D(T) = \sup_{[0,T]} \psi'(t) + \sup_{[0,T]} \varphi'(t).

**Proof.** For proof, we calculate \int_{QT} (Lu)^2 \, dxdt. We have
\[
\int_{QT} (Lu)^2 \, dxdt = i_1 + i_2 + i_3 + i_4 + i_5.
\]
Later we will consider each addend separately. Applying integration by parts with respect to variables $x_i, x_j$ and taking into account $\frac{\partial u}{\partial x_j} \big|_{\partial B_R} = 0$ we obtain estimate of integrals $i_1$. Also we calculate integrals $i_2, i_3, i_4, i_5$.

Let $\delta = \sup_{Q_T} \left( \sum_{i,j=1}^{n} (a_{ij}(x,t,u) - \delta_{ij})^2 \right)^{\frac{1}{2}}$, where $\delta_{ij}$ are the Kronecker symbols.

**Lemma 2.** Let the coefficients of the operators $L$ satisfy conditions (3)-(4). Then there exists $T_2$ such that for every $T \leq T_2$ and $\varepsilon > 0$ the estimate holds:

$$
\|u\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C(\psi, \delta, n, \Omega) \|Lu\|_{L_2(Q_T)} + \varepsilon \|u\|_{W^{2,2}_{2,\psi}(Q_T)} + C(\psi, \delta, n, \Omega) \|u\|_{L_2(Q_T)}
$$

for any function $u(x,t) \in C^\infty(Q_T(x_0))$, $u|_{t=0} = 0$.

**Lemma 3.** Let the conditions of Lemma 2 be satisfied. Then at $T \leq T_2$ for any function $u(x,t) \in W^{2,2}_{2,\psi}(Q_T)$ it holds the estimate

$$
\|u\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C(\psi, \delta, n, \Omega) \|Lu\|_{L_2(Q_T)} + C(\psi, \delta, n, \Omega) \|u\|_{L_2(Q_T)}.
$$

Similarly to Lemma 1, we can proof of Lemmas 2 and 3 with using Friedrich’s inequality.

Now we give the following coercive estimate for solution boundary problem (1)-(2).

**Theorem 1.** Let for strong solutions of the problem (1)-(2) the conditions (3),(4) be fulfilled. Then there exists $T_0(\psi, \delta, n\Omega)$ such that for every $T \leq T_0$ the estimate

$$
\|u\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C(\psi, \delta, n, \Omega) \|Lu\|_{L_2(Q_T)}.
$$

holds for any functions $u(x,t) \in W^{2,2}_{2,\psi}(Q_T)$.

**Proof.** It is enough to prove the estimate (6) for smooth functions from
\( W^{2,2}_{2,\psi}(Q_T) \). We have for any \( t \in (0, T) \) and any \( x \in \Omega \)
\[
    u(x, t) = \int_0^t u_t(x, \tau) d\tau.
\]

Using the Cauchy-Bunyakovsky inequality, we write
\[
    u^2(x, t) = T \int_0^T u^2_t(x, \tau) d\tau.
\]

Then
\[
    \int_{Q^T_R} u^2(x, t) \, dx dt = T^2 \int_{Q^T_R} u^2_t(x, t) \, dx dt.
\]

Thus,
\[
    \|u\|_{L_2(Q^T_R)} \leq T \|u_t\|_{L_2(Q^T_R)} \leq T \|u\|_{W^{2,2}_{2,\psi}(Q_T)}.
\]

Let \( T_0 = \min \{ T_2, \frac{1}{2C} \} \). Then at \( T \leq T_0 \) we obtain estimate (6). The theorem is proved.

\[\Box\]

**Theorem 2.** Let for strong solutions of the problem (1)-(2) the conditions (3),(4) be fulfilled. Then there exists \( T_0(\psi, \delta, n\Omega) \) such that for every \( T \leq T_0 \) problem (1)-(2) is solvable in space \( W^{2,2}_{2,\psi}(Q_T) \) for any \( f(x, t) \in L_2(Q_T) \) and the estimate holds
\[
    \|u\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C(\psi, \delta, n, \Omega) \|f\|_{L_2(Q_T)}.
\]

*Proof.* The estimate (7) and the uniqueness of the solution follow from the coercive estimates. The existence of a solution is proved by considering the family of operators.

\[\Box\]

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