Anderson localization on the Cayley tree: multifractal statistics of the transmission at criticality and off criticality

Cécile Monthus and Thomas Garel

Institut de Physique Théorique, CNRS and CEA Saclay 91191 Gif-sur-Yvette Cedex, France

E-mail: cecile.monthus@cea.fr

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Abstract
In contrast to finite dimensions where disordered systems display multifractal statistics only at criticality, the tree geometry induces multifractal statistics for disordered systems also off criticality. For the Anderson tight-binding localization model defined on a tree of branching ratio $K = 2$ with $N$ generations, we consider the Miller–Derrida scattering geometry (1994 J. Stat. Phys. 75 357), where an incoming wire is attached to the root of the tree, and where $K^N$ outgoing wires are attached to the leaves of the tree. In terms of the $K^N$ transmission amplitudes $t_j$, the total Landauer transmission is $T \equiv \sum_j |t_j|^2$, so that each channel $j$ is characterized by the weight $w_j = |t_j|^2 / T$. We numerically measure the typical multifractal singularity spectrum $f(\alpha)$ of these weights as a function of the disorder strength $W$ and we obtain the following conclusions for its left termination point $\alpha_+(W)$. In the delocalized phase $W < W_c$, $\alpha_+(W)$ is strictly positive $\alpha_+(W) > 0$ and is associated with a moment index $q_+(W) > 1$. At criticality, it vanishes $\alpha_+(W_c) = 0$ and is associated with the moment index $q_+(W_c) = 1$. In the localized phase $W > W_c$, $\alpha_+(W) = 0$ is associated with some moment index $q_+(W) < 1$. We discuss the similarities with the exact results concerning the multifractal properties of the directed polymer on the Cayley tree.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
Since its discovery 50 years ago [1], Anderson localization has remained a very active field of research (see for instance the reviews [2–7]). According to the scaling theory [8], there is no delocalized phase in dimensions $d = 1, 2$, whereas there exists a localization/delocalization at finite disorder in dimension $d > 2$. To gain some insight into this type of transition, it is
natural to consider Anderson localization on the Cayley tree which is expected to represent some mean-field limit. The tight-binding Anderson model on the Cayley tree has thus been studied by various techniques over the years [9–14]. Other studies have focused on random-scattering models on the Cayley tree [15–17]. For the version of the model defined on a random regular graph of fixed degree, we refer to the recent work [18] and references therein.

The motivation to study Anderson localization on the Cayley tree has been revived recently by the question of many-body localization [19], because the geometry of the Fock space of many-body states was argued to be similar to a Cayley tree [20–26]. But of course, the questions on many-body localization are much more difficult and are still debated in the recent studies [22, 25–31].

In quantum coherent problems, the most appropriate characterization of transport properties consists in defining a scattering problem where the disordered sample is linked to incoming wires and outgoing wires and in studying the reflection and transmission coefficients. This scattering theory definition of transport, first introduced by Landauer [32], has been much used for one-dimensional systems [33–35] and has been generalized to higher dimensionalities and multi-probe measurements (see the review [36] and references therein). For the Anderson model on the Cayley tree, an appropriate scattering geometry has been introduced by Miller and Derrida [13] to perform weak-disorder expansions and numerical computations: an incoming wire is attached to the root, and \( K^N \) outgoing wires are attached to the leaves of a tree of branching ratio \( K \) with \( N \) generations. In a previous work [14], we have used this scattering geometry to study numerically the statistical properties of the total Landauer transmission \( T = \sum |t_j|^2 \) as a function of the number \( N \) of generations and of the disorder strength and to measure its critical behavior. The aim of this paper is to characterize the spatial inhomogeneity between the various channels \( j \): the weights \( |t_j|^2 / T \) of the \( K^N \) channels turn out to present a multifractal statistics, not only at criticality but also in the localized and delocalized phases as a consequence of the tree geometry. So we analyze how the singularity spectrum \( f(\alpha) \) changes as a function of the disorder strength.

The paper is organized as follows. In section 2, we introduce the Anderson localization tight-binding model on the Cayley tree and the scattering geometry that we consider to study the multifractal statistics of the Landauer transmission. Our numerical results concerning the multifractal statistics in various phases are described in sections 3 and 4 for the box distribution and for the Cauchy distribution of disorder respectively. Our conclusions are summarized in section 5. In appendix A, we recall the exactly known results concerning the multifractality for the directed polymer on the Cayley tree, as a classical model which is useful to consider as a comparison, both for conceptual and numerical purposes. Appendix B explains how the numerical singularity spectra presented in figures have been obtained.

2. Scattering geometry for Anderson localization on the Cayley tree

2.1. Anderson tight-binding model on the Cayley tree

We consider the Anderson tight-binding model

\[
H = \sum_i \epsilon_i |i\rangle\langle i| + \sum_{\langle i,j \rangle} |i\rangle\langle j|, \tag{1}
\]

where the hopping between nearest neighbors \( \langle i, j \rangle \) is a constant \( V = 1 \) and where the on-site energies \( \epsilon_i \) are independent random variables drawn from the ‘box’ distribution

\[
p_{\text{Box}}(\epsilon) = \frac{1}{W} \theta \left( -\frac{W}{2} \leq \epsilon \leq \frac{W}{2} \right). \tag{2}
\]
Thus, the parameter $W$ represents the disorder strength. We have also studied the case of the Cauchy disorder

$$p_{\text{Cauchy}}(\epsilon) = \frac{W}{\pi(\epsilon^2 + W^2)}. \quad (3)$$

### 2.2. Miller–Derrida scattering geometry

We consider the scattering geometry introduced in [13] and shown in figure 1: the finite tree of branching ratio $K = 2$ is attached to one incoming wire at its root (generation $n = 0$) and to $K^N$ outgoing wires at generation $N$. One is interested in the eigenstate $|\psi\rangle$ that satisfies the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle \quad (4)$$

inside the disorder sample and in the wires where one requires the plane-wave forms

$$\psi(n \leq 0) = e^{ikn} + r e^{-ikn}$$

$$\psi_j(n \geq N) = t_j e^{ik(n-N)} \quad (5)$$

These boundary conditions define the reflection amplitude $r$ of the incoming wire and the transmission amplitudes $t_j$ of the $j = 1, 2, \ldots, K^N$ outgoing wires. To satisfy the Schrödinger Equation of equation (4) within the wires with the forms of equation (5), one has the following relation between the energy $E$ and the wave vector $k$:

$$E = 2 \cos k. \quad (6)$$

To simplify the discussion, in this paper we will focus on the case of zero energy $E = 0$ and wave-vector $k = \pi/2$, because the zero energy $E = 0$ corresponds to the center of the band.
where the delocalization first appears when the strength $W$ of the disorder is decreased from the strong disorder localized phase. From the conservation of energy, the total transmission $T$ is related to the reflection coefficient $|r|^2$

$$ T = \sum_j |t_j|^2 = 1 - |r|^2. \quad (7) $$

We refer to [13] for the results of a weak disorder expansion within this framework, and for a numerical Monte Carlo approach to determine the mobility edge in the plane $(E, W)$. In [14] we have studied the statistical properties over the disordered samples of the total Landauer transmission $T_N$ at zero energy $E = 0$ as a function of the disorder strength $W$ and of the number $N$ of generations. In the localized phase $W > W_c$, the typical transmission $T_N^{typ} \equiv e^{\ln T_N}$ decays exponentially with the number $N$ of generations

$$ \ln(T_N^{typ}) = \ln T_N(W > W_c) \sim N \to \infty - N \xi_{loc}(W). \quad (8) $$

where $\xi_{loc}$ represents the localization length. In the delocalized phase, the typical transmission remains finite in the limit where the number of generations $N$ diverges:

$$ T_N^{typ} \equiv e^{\ln T_N(W < W_c, N)} \sim N \to \infty T_\infty(W < W_c) > 0. \quad (9) $$

The total Landauer transmission $T$ is thus an appropriate order parameter of the localization transition at the mobility edge $W_c$. We refer to [14] for more details on the critical behaviors of the localization length $\xi_{loc}$ and of the asymptotic value $T_\infty(W < W_c)$. In this paper, we wish to analyze the statistics of the contributions $t_j$ of the various channels to the total transmission of equation (7) as we now explain.

2.3. Statistical properties of the weights of the outgoing channels

In each disordered sample, we consider the $K^N$ weights

$$ w_j \equiv \frac{|t_j|^2}{T} = \frac{|t_j|^2}{\sum_j |t_j|^2} \quad (10) $$

and the ‘analogs’ of inverse participation ratios (IPRs)

$$ I_q(M = K^N) \equiv \sum_{j=1}^{M} w_j^q = \frac{\sum_{j=1}^{M} |t_j|^2 q}{(\sum_{j=1}^{M} |t_j|^2)^q}. \quad (11) $$

It is useful to introduce the multifractal formalism with respect to $M = K^N$ (or equivalently the large deviation formalism with respect to the variable $N = (\ln M)/(\ln K)$): one defines the typical exponents $\tau_{typ}(q)$ as the exponents governing the decays of the typical values

$$ I_q^{typ}(M = K^N) \equiv e^{\ln I_q(M)} \xrightarrow{M \to \infty} M^{-\tau_{typ}(q)} = e^{-N(\ln K)\tau_{typ}(q)}. \quad (12) $$

The typical singularity spectrum $f^{typ}(\alpha)$ is defined as follows: in a large disordered sample, the number $N_M(\alpha)$ of channels $j$ (among the total of $M$ of channels) that have a weight $w_j$ scaling as $w_j \sim M^{-\alpha}$ scales as

$$ N_M^{typ}(\alpha) \sim M^{f^{typ}(\alpha)}. \quad (13) $$

Then the saddle-point computation of $I_q$ (equations (11) and (12))

$$ I_q^{typ}(M = K^N) = \sum_{j=1}^{M} w_j^q \sim \int d\alpha M^{f^{typ}(\alpha)-q\alpha} \quad (14) $$
leads to the Legendre transform formula
\[ -\tau^{yp}(q) = \max_{\alpha} [ f^{yp}(\alpha) - q\alpha ]. \] (15)

Let us now briefly recall some basic notions about multifractality that will be useful to analyze the numerical results. As a consequence of the weight definition of equation (10), the index \( \alpha \) cannot be negative, so one has \( \alpha \geq 0 \). As a consequence of equation (13), the ‘typical’ singularity spectrum is non-negative: \( f^{yp}(\alpha) \geq 0 \). (Note that this is in contrast with the ‘averaged’ singularity spectrum \( f^{av}(\alpha) \) which can become negative to describe rare events (see [7] for more details), but in this paper we only consider the typical singularity spectrum.)

The termination points \( \alpha_{\pm} \) are defined as the points where the singularity spectrum vanishes \( f(\alpha_{\pm}) = 0 \), whereas the singularity spectrum remains strictly positive in between:
\[ f^{yp}(\alpha) > 0 \quad \text{for} \quad \alpha_{+} < \alpha < \alpha_{-}. \] (16)

The left termination point \( \alpha_{+} \) which represents the smallest possible \( \alpha \) will play an essential role in the following. From the point of view of the Legendre transform formula of equation (15), it is associated with some positive value \( q_{+} > 0 \), where the saddle point \( \alpha(q) \) reaches \( \alpha_{+} \), so that for all higher \( q \), the saddle point remains frozen at this value
\[ \alpha(q > q_{+}) = \alpha_{+} \] (17)
and the typical exponent \( \tau^{yp}(q) \) is simply
\[ \tau^{yp}(q > q_{+}) = q\alpha_{+}. \] (18)

The same discussion can be transposed to the right termination point \( \alpha_{-} \) associated with some negative index \( q_{-} < 0 \), with \( \alpha(q < q_{-}) = \alpha_{-} \) and \( \tau^{yp}(q < q_{-}) = q\alpha_{-} \). The value \( q = 0 \) is associated with the most probable value \( \alpha_{0} \equiv \alpha(q = 0) \) where the singularity spectrum reaches its maximum:
\[ f(\alpha_{0}) = 1. \] (19)

Finally the value \( q = 1 \) is associated with the value \( \alpha_{1} \equiv \alpha(q = 1) \), where the singularity spectrum satisfies
\[ f(\alpha_{1}) = 0 \] (20)
as a consequence of the normalization \( I^{yp}_{q=1} = 1 \) corresponding to \( \tau^{yp}(q = 1) = 0 \).

### 2.4. Comparison with Anderson localization models in finite dimension

We should stress here the similarities and differences with the usual multifractal definitions used for Anderson localization models in finite dimension \( d \) (see the review [7]): for a normalized eigenfunction on the volume \( V = L^{d} \) (i.e. \( \sum_{r \in V = L^{d}} |\psi(r)|^{2} = 1 \)), the IPRs are defined as
\[ P_{q} = \sum_{r \in V = L^{d}} |\psi(r)|^{2q} \] (21)
and the exponents \( \tau(q) \) are defined as
\[ P_{q}^{yp} \propto L^{-\tau(q)}. \] (22)

In finite dimension \( d \), powers of \( L \) and powers of the volume \( V = L^{d} \) correspond to the same scaling (up to a redefinition of the exponents), while on the tree, one should use powers of the number \( M = K^{N} \) in the definitions of equation (12), and not powers of the linear distance \( N \).

Another difficulty with the tree geometry is that sites of different generations are not equivalent in the pure case (see [14] for explicit expressions of wavefunctions that decay
exponentially with the distance $N$ in the pure case). This shows that direct generalizations of $P_q$ where the sum is over all sites of the tree is not appropriate (see again [14] for more detailed discussion on the anomalous behavior of usual IPRs), and this is why we have chosen to consider the weights of the channels in the Miller–Derrida geometry, since they involve the wavefunction weights of the $K^N$ points that are at the same distance $N$ of the origin. In the pure case, all these weights have the same weights $|t_j|^2/T = 1/K^N$ (see again [14] for more details), leading to the mono-fractal behavior of the $I_q$ of equation (11):

$$I^\text{pure}_q(M = K^N) = \frac{\sum_{j=1}^{K^N} |t_j|^{2q}}{\left(\sum_{j=1}^{K^N} |t_j|^2\right)^q} = \frac{1}{K^{N(q-1)}}.$$  

(23)

So the tree geometry has the peculiarity to induce the multifractal behavior of the $I_q$ even in the delocalized phase (the radial symmetry of the pure case is not able to survive even at small disorder), whereas in any finite dimension, the IPRs in the delocalized phase are mono-fractal with the same scaling as the pure case.

Finally in finite dimension, the localized phase is characterized by localized eigenfunctions where some rare sites have finite weights, whereas most sites have exponentially-small weights in the linear size $L$. On the tree, all eigenfunctions have to decay exponentially with the distance $N$, even in the pure case, to fulfill the normalization constraint with an exponentially-growing number of sites with the distance $N$. So in the localized phase, the fluctuations of the weights will also be characterized by a multifractal statistics of the $I_q$ of equation (11).

In summary, in contrast to finite dimensions where disordered systems display multifractal statistics only at criticality, the tree geometry induces multifractal statistics for disordered systems also outside criticality, if one considers the inhomogeneities among the points at a given distance from the center. In the recent mathematical study [37], similar observables have been introduced with a large deviation analysis in $N$.

Since these multifractal properties of disordered models defined on trees are unusual with respect to finite dimensions, it is useful to see how the multifractal analysis of equation (12) works in an exactly solved model: in appendix A we thus recall the case of the directed polymer on the Cayley tree, which is a classical disordered model having the same geometry.

3. Numerical results for the box distribution

In this section, we describe our numerical results for a tree of branching ratio $K = 2$ where the disordered on-site energies are drawn with the box distribution of equation (2). The critical disorder width $W_c$ at the center of the band $E = 0$ has been found to be numerically in the interval [9, 14, 18]

$$16 < W_c < 18.$$  

(24)

3.1. Numerical details

We have studied trees containing $N$ generations with a corresponding number $n_s(N)$ of disordered samples with the values

$$N = 10; 12; 14; 16; 18; 20; 22; 24$$

$n_s(N) = 10^3; 27.10^5; 7.10^5; 17.10^4; 43.10^3; 10^4; 27.10^2; 650.$  

(25)

We have chosen to work only with even $N$, because in the pure case, the total Landauer transmission is perfect $T^\text{pure}_N = 1$ only for even $N$ (see section 2.2 of [14] for more details). The transition amplitudes $t_j$ of the scattering eigenvalue problem of equations (4) and (5) are
computed via the introduction of Riccati variables as explained in detail in [13, 14]. The multifractal spectrum is then obtained via the standard method of [43], where the curve $f(\alpha)$ is obtained parametrically in the parameter $q$ (see appendix B for more details); here we have used values in the range $-5 \leq q \leq +5$. As shown in appendix A, we have checked that the sizes and statistics of equation (25) were sufficient to obtain reliable results for the multifractal properties of the directed polymer model by a direct comparison with exactly known results in various phases.

Let us make some final remark to explain the differences with respect to the numerical method used in our previous work concerning the full transmission $T$. In [14] we had used the so-called pool method (see section 2.3.1 of [14]) which allows us to study much bigger sizes (like $N \sim 10^5$ generations). This was possible because the full transmission $T$ can be directly computed from the reflection coefficient of the incoming wire alone (see equation (17) of [14]), i.e. one does not need to compute the whole set of transmissions $t_j$ of individual channels. So the full transmission $T$ can be obtained directly from the stable probability distribution of the complex Riccati variables, for which the pool method is well adapted. However, the multifractal spectrum is a much more complicated observable: it does not depend only on the one-point distribution of the Riccati variables, but it involves the whole correlations between the Riccati variables along branches (each Riccati variable is computed from its $K$ descendants, see equation (25) of [14]). Of course, one could try to develop some generalized pool method to compute multifractal properties, but one should be very careful to avoid artifacts. In this paper, we have thus chosen to work only with exact numerical data on finite trees to avoid any doubts on the numerical results.

### 3.2. Delocalized phase

In the delocalized phase $W < W_c$, we find that the left termination point is strictly positive $\alpha_+ (W) > 0$ and is associated with a moment index $q_+ (W) > 1$. Two examples of our numerical data are shown in figures 2 and 3 corresponding to $W = 5$ and $W = 10$, respectively.
3.3. Critical point

At criticality, the left termination point vanishes $\alpha_+(W_c) = 0$ together with the tangent point $\alpha_1 = f(\alpha_1) = 0$, as shown on figure 4 corresponding to $W = 17$. The corresponding saddle point $\alpha(q)$ saturates at the value $q_+ \simeq 1$.

3.4. Localized phase

In the localized phase $W > W_c$, the vanishing left termination point $\alpha_+(W_c) = 0$ is associated with some moment index $q_+(W) < 1$, as shown in figures 5 and 6 corresponding to $W = 30$ and $W = 40$, respectively.
4. Numerical results for the Cauchy distribution

In this section, we describe our numerical results for a tree of branching ratio $K = 2$ where the disordered on-site energies are drawn with the Cauchy distribution of equation (3). The numerical details are the same as in section 3.1. The critical disorder width $W_c$ at the center of the band $E = 0$ has been previously found to be numerically in the interval \[9 \leq W_c < 4.4.\]

In many areas, the Cauchy distribution whose variance is infinite leads to anomalous results with respect to bounded distributions. In the context of Anderson localization, the Cauchy disorder is of course anomalous from the point of view of weak-disorder expansion which contains explicitly the variance of the disorder (see [35] and references therein). However, from the point of view of Anderson localization transitions at finite disorder, we are not aware of any statement concerning its anomalous behaviors (except old conclusions concerning the
absence of transition that have been shown to be false afterward). On the theoretical side, the Cauchy distribution is well known to have many advantages: it is the only disorder distribution that leads to an exact and simple solution in one dimension (see [35] and references therein), and that leads to an exact and simple solution for the density of states in any dimension [52]. For the Anderson localization on the Cayley tree that we consider in this paper, it is also the only disorder distribution that leads to an exact and simple solution for the stationary distribution of the Riccati variables in the localized phase [9, 13], the only remaining problem being that it is not known in the delocalized phase where the Riccati variables are complex [13]. These theoretical advantages of the Cauchy distribution justify to study numerically its properties and to compare with the case of bounded distributions. In the following, we obtain that the results concerning the multifractal properties of the transmission for the Cauchy distribution are qualitatively the same as the results obtained in the previous section concerning the box distribution.
4.1. Delocalized phase

In the delocalized phase \( W < W_c \), we find that the left termination point is strictly positive \( \alpha_+ > 0 \) and is associated with a moment index \( q_+ > 1 \). Two examples of our numerical data are shown in figures 7 and 8 corresponding to \( W = 0.5 \) and \( W = 1 \), respectively.

4.2. Critical point

At criticality, the left termination point vanishes \( \alpha_+ (W_c) = 0 \) together with the tangent point \( \alpha_1 = f(\alpha_1) = 0 \), as shown in figure 9 corresponding to \( W = 4 \). The corresponding saddle point \( \alpha(q) \) saturates at the value \( q_+ (W_c) \approx 1 \).
4.3. Localized phase

In the localized phase $W > W_c$, the vanishing left termination point $\alpha_+ (W_c) = 0$ is associated with a moment index $q_+ (W) < 1$, as shown in figures 10 and 11 corresponding to $W = 6$ and $W = 10$, respectively.

5. Conclusions

In this paper, we have studied the multifractal properties of the Landauer transmission for the Anderson localization tight-binding model on the Cayley tree within the Miller–Derrida scattering geometry. We have explained why, in contrast to finite dimensions where disordered systems display multifractal statistics only at criticality, the tree geometry induces multifractal statistics for disordered systems also off criticality. As an example, we have recalled in appendix A the exact results concerning the directed polymer on the Cayley tree. We have presented numerical results for the typical multifractal singularity spectrum $f(\alpha)$ of the channels weights as a function of the disorder strength $W$, both for the box distribution and for the Cauchy distribution of disorder. Our main conclusion concerns the left-termination point $\alpha_+ (W)$. In the delocalized phase $W < W_c$, $\alpha_+ (W)$ is strictly positive $\alpha_+ (W) > 0$ and is associated with a moment index $q_+ (W) > 1$. At criticality, it vanishes $\alpha_+ (W_c) = 0$ and is associated with the moment index $q_+ (W_c) = 1$. In the localized phase $W > W_c$, $\alpha_+ (W) = 0$ is associated with some moment index $q_+ (W) < 1$. These properties of the delocalized and localized phases are thus qualitatively similar to the exact results concerning the directed polymer on the Cayley tree.

Appendix A. Reminder on the directed polymer on the Cayley tree

A.1. Reminder on the thermodynamics

The directed polymer on a Cayley tree with disorder has been introduced in [38] as a mean-field version of the directed polymer in a random medium [39]. The model is defined by the
partition function

\[ Z_N = \sum_{C} e^{-\beta E(C)}, \]  

(A.1)

where the \( K^N \) configurations \( C \) are the paths of \( N \) steps on a Cayley tree with coordination number \( K \). The energy \( E(C) \) of a path is the sum of the energies of the visited bonds. Each bond has a random energy drawn independently, for instance with the Gaussian distribution

\[ \rho(\epsilon) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\epsilon^2}{2}}. \]  

(A.2)

This model presents many similarities [38, 40] with the random energy model, introduced by Derrida in the context of spin glasses [41]. It presents a freezing transition at

\[ T_c = \frac{1}{\sqrt{2 \ln K}}. \]  

(A.3)

The free energy per step \( \phi(T) \) coincides with the annealed free energy above \( T_c \) and is completely frozen below [38, 40]:

\[ \phi(T) = \phi_{ann}(T) = -T \ln K - \frac{T}{2T_c^2} = \frac{1}{2T} \quad \text{for} \quad T \geq T_c \]  

(A.4)

\[ \phi(T) = -\frac{1}{T_c} \quad \text{for} \quad T \leq T_c. \]  

(A.5)

A.2. Reminder on the finite weight statistics in the frozen phase

The configurations weights in the partition function (equation (A.1)) are defined as

\[ w_C = \frac{e^{-\beta E(C)}}{Z_N(\beta)}. \]  

(A.6)

The moments

\[ Y_q(N) = \sum_{i=1}^{K^N} w_C^q \]  

(A.7)

have finite disorder averages in the frozen phase \( \mu(T) = T/T_c < 1 \) for values \( q > \mu(T) = T/T_c \) [42]:

\[ \overline{Y}_q = \Gamma(q - \mu(T)) \Gamma(1 - \mu(T)) \quad \text{with} \quad \mu(T) = \frac{T}{T_c}. \]  

(A.8)

The density \( g(w) \) giving rise to these moments

\[ \overline{Y}_q = \int_0^1 dw w^q g(w) \]  

(A.9)

reads [42]

\[ g(w) = \frac{w^{-1-\mu} (1-w)^{\mu-1}}{\Gamma(\mu) \Gamma(1-\mu)} \]  

(A.10)

and represents the averaged number of terms of weight \( w \). This density is non-integrable as \( w \to 0 \), because in the limit \( N \to \infty \), the number of terms of vanishing weights diverges. The normalization corresponds to

\[ \overline{Y}_{q=1} = \int_0^1 dw w g(w) = 1. \]  

(A.11)
A.3. Reminder on multifractal properties of the weights

In the non-frozen phase, the $Y_q$ of equation (A.7) vanish with the number $M = KN$ of configurations as power-laws with non-trivial exponents for averaged and typical values, and it is convenient to introduce the multifractal formalism of equations (12), (14), (15). In the frozen phase, the finite asymptotic values obtained for $q > \mu(T)$ in equation (A.9) correspond to $\tau_q = 0$, but it is nevertheless interesting to define the multifractal exponents of equation (12) for $|q| < \mu$. The fact that the multifractal formalism is appropriate to describe the weight statistics for all values of $T$, and its exact computation are explained in detail in [44–51]. Here we simply recall the main results, and we refer to [44–51] for more details and discussions.

The main point is that the moments $Y_q$ can be rewritten as in terms of the partition functions at inverse temperatures $\beta$ and $|q|\beta$ :

$$Y_q(M = KN) \equiv \frac{\sum_{i=1}^{KN} e^{-q|\beta|E_i}}{\left(\sum_{i=1}^{KN} e^{-\beta E_i}\right)^q} = \frac{Z_N(|q|\beta)}{(Z_N(\beta))^q} = e^{-\beta(|q|F_N(|q|\beta) - qF_N(\beta))},$$

so that the typical exponents of equation (12) are directly given in terms of the free energy per step of equation (A.5):

$$\tau_{typ}(q) = \frac{\beta}{\ln K} \left[ |q|\phi(|q|\beta) - q\phi(\beta) \right].$$

In the following, we thus quote the final results for the typical singularity spectrum $f(\alpha)$ in the various phases, and we present the numerical results obtained on trees of sizes $10 \leq N \leq 22$ to show that such sizes are sufficient to obtain reliable results by comparison with the exact forms (more details on the numerical procedure can be found in appendix B).

A.3.1. Non-frozen phase $\mu \equiv \frac{T}{T_c} > 1$. In the non-frozen phase, the left and right termination points read

$$\alpha_+ = \left(1 - \frac{1}{\mu}\right)^2$$

$$\alpha_- = \left(1 + \frac{1}{\mu}\right)^2$$

and the typical singularity spectrum is exactly Gaussian in the interval $\alpha_+ < \alpha < \alpha_-$. where it exists:

$$f_{T,T_c}^{typ}(\alpha) = \frac{\mu^2}{4}(\alpha - \alpha_+)(\alpha_- - \alpha).$$

The terminating values $\alpha_\pm$ are associated with the values $q_\pm = \pm \mu$. In the interval $q_- = -\mu \leq q \leq q_+ = +\mu$, the value $\alpha(q)$ dominating the saddle-point calculation of equation (15) is simply linear in $q$:

$$\alpha(q_- = -\mu \leq q \leq q_+ = +\mu) = \left(1 + \frac{1}{\mu^2}\right) - \frac{2q}{\mu^2}.$$  

The typical value corresponding to $q = 0$ in equation (A.16)

$$\alpha_0 = \left(1 + \frac{1}{\mu^2}\right)$$

is the point where the singularity spectrum reaches its maximum $f(\alpha_0) = 1$ (equation (19)). Finally, the value $q = 1$ where the singularity spectrum is tangent to the line $\alpha = f(\alpha)$ (equation (20)) corresponds to (equation (A.16))

$$\alpha_1 = 1 - \frac{1}{\mu^2} = f(\alpha_1).$$
Figure A1. Directed polymer in the non-frozen phase at $\mu \equiv T/T_c = 2$ (see appendix A.3.1). (a) Singularity spectrum $f(\alpha)$: the terminating points are $\alpha_+ = 0.25$ and $\alpha_- = 2.25$, the typical value is $\alpha_0 = 1.25$, the line $\alpha = f(\alpha)$ is tangent at $f(\alpha_1) = \alpha_1 = 0.75$. (b) The saddle point $\alpha(q)$ remains frozen at $\alpha_+$ for $q > q_+ = 2$ and to $\alpha_-$ for $q < q_- = -2$.

The numerical results shown in figure A1 are in agreement with these expressions for $\mu = 2$.

A.3.2. Critical point $\mu \equiv T/T_c = 1$. In the limit $\mu \equiv T/T_c \to 1^*$, the above singularity spectrum has the following properties: the left terminating point $\alpha_+$ of equation (A.14) vanishes

$$\alpha_+ = 0 \quad (A.19)$$

together with the tangent point to the line $\alpha = f(\alpha)$ (equation (A.18)):

$$\alpha_1 = f(\alpha_1) = 0. \quad (A.20)$$

The corresponding numerical results for $\mu = 1$ are shown in figure A2.

A.3.3. Frozen Phase $\mu \equiv T/T_c < 1$. In the frozen phase, the left termination point is zero

$$\alpha_+ = 0 \quad (A.21)$$

and the right termination point is

$$\alpha_- = \frac{4}{\mu}. \quad (A.22)$$

In the interval $\alpha_+ = 0 < \alpha < \alpha_-$ where it exists, the typical singularity spectrum is again exactly Gaussian

$$f_{T<T_c}^{\text{typ}}(\alpha) = \frac{\mu^2}{4} \alpha(\alpha_- - \alpha). \quad (A.23)$$

The left terminating value $\alpha_+ = 0$ is reached for $q_+ = \mu$ and the right terminating value $\alpha_-$ is reached for $q_- = -\mu$. In the interval $q_- = -\mu \leq q \leq q_+ = +\mu$, the value $\alpha(q)$ dominating the saddle-point calculation of equation (15) is again linear in $q$:

$$\alpha(q_- = -\mu \leq q \leq q_+ = +\mu) = \frac{2}{\mu} - \frac{2q}{\mu^2}. \quad (A.24)$$
Figure A2. Directed polymer at the critical point for $\mu = T/T_c = 1$ (see appendix A.3.2). (a) Singularity spectrum $f(\alpha)$: the terminating points are $\alpha_+ = 0$ and $\alpha_- = 4$, the typical value is $\alpha_0 = 2$, the line $\alpha = f(\alpha)$ is tangent at the origin $f(\alpha_0) = \alpha_0 = 0$. (b) The saddle point $\alpha(q)$ remains frozen at $\alpha_+$ for $q > q_+ = 1$ and to $\alpha_-$ for $q < q_- = -1$.

Figure A3. Directed polymer in the frozen phase at $\mu = T/T_c = 0.5$ (see appendix A.3.3). (a) Singularity spectrum $f(\alpha)$: the terminating points are $\alpha_+ = 0$ and $\alpha_- = 8$, the typical value is $\alpha_0 = 4$. (b) The saddle point $\alpha(q)$ remains frozen at $\alpha_+$ for $q > q_+ = 0.5$ and to $\alpha_-$ for $q < q_- = -0.5$.

The typical value corresponding to $q = 0$ in equation (A.24) is

$$\alpha_0 = \frac{2}{\mu}.$$  \hspace{1cm} (A.25)

Note that here the value $q = 1$ is already in the frozen region $1 > q_+ = \mu$, so that the singularity spectrum is completely below the line $\alpha = f(\alpha)$. The numerical results shown in figure A3 are in agreement with these expressions for $\mu = 0.5$. 

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Appendix B. Details concerning the numerical evaluation of $f(\alpha)$

In each sample with $N$ generations, we compute the $M = K^N$ weights of equation (10), from which we obtain immediately the parameters $I_q$ of equation (11). The corresponding typical exponents $\tau^{\text{ty}}(q)$ of equation (12) are then obtained by the following three-parameter fit of the average over disordered samples:

$$\ln I_q(M) \simeq -\tau^{\text{ty}}(q) \ln M + c_0 \ln(\ln M) + c_0',$$

i.e. $\tau^{\text{ty}}(q)$ is obtained as the coefficient of the leading linear term. The presence of the subleading term ($c_0$) is important only in the regions where $\tau_q$ nearly vanishes $\tau_q \sim 0$, whereas in the regions where $\tau_q$ is not small, a direct linear fit could be acceptable and would give nearly the same numerical value for $\tau_q$.

The typical multifractal spectrum $f^{\text{ty}}(\alpha)$ could in principle be obtained from $\tau^{\text{ty}}(q)$ by some numerical procedure to perform the Legendre transform of equation (15), but this method has a lot of numerical drawbacks [43]. Thus, we have followed the standard method of [43], with the simplification that we consider only boxes of size $L = 1$ in the notation of [43], i.e. we do not use any coarse-graining with various box sizes, but we analyze instead the scaling with respect to the total number of boxes $M = K^N$ from our data obtained of various sizes $N$. The main idea of [43] is to construct the following normalized $q$-measures from the initial measure defined by the weights of equation (10):

$$w_j^{(q)}(M) = \frac{[w_j]^{q}}{\sum_j [w_j]^{q}}$$

Of course, $q = 1$ corresponds to the initial measure $w_j^{(q=1)} = w_j$. The denominator corresponds to $I_q$ of equation (11). It is then useful to introduce

$$F_q(M) = -\sum_{j=1}^{M} w_j^{(q)} \ln w_j^{(q)}$$

$$A_q(M) = -\sum_{j=1}^{M} w_j^{(q)} \ln w_j.$$  \(B.3\)

$F_q(M)$ represents the Shannon entropy of the $q$-measure, whereas $A_q(M)$ represents the averaged log of the initial weight $\ln w_j$ with respect to the $q$-measure. From equation (B.2), one obtains immediately the simple relation

$$F_q(M) = q A_q(M) + \ln I_q(M)$$

which, after the division by the scaling factor $(\ln M)$, exactly corresponds to the Legendre transform relation of equation (15). Numerically, one only has to compute $I_q$ and $A_q$, whereas $F_q$ can be immediately obtained from them by equation (B.3). The averages over the disordered samples of the two observables of equation (B.3) can be analyzed by the following three-parameter fits:

$$\frac{F_q(M)}{A_q(M)} \simeq f_q \ln M + c_1 \ln(\ln M) + c'_1$$

$$\frac{A_q(M)}{A_q(M)} \simeq \alpha_q \ln M + c_2 \ln(\ln M) + c'_2.$$  \(B.5\)

The leading coefficients ($\alpha_q$, $f_q$) then constitute a parametric representation of the typical singularity spectrum $f(\alpha)$ as $q$ varies. The presence of the subleading terms ($c_1$, $c_2$) are important only in the regions where $f_q$ nearly vanishes $f_q \sim 0$, whereas in the regions where $f_q$ is not small, direct linear fits could be acceptable and would give nearly the same numerical values for ($\alpha_q$, $f_q$).
All the singularity spectra $f(\alpha)$ shown in the figures correspond to the parametric representation $(\alpha_q, f_q)$ obtained by the procedure just described. We have also presented our corresponding data for $\alpha_q$ to show the freezing phenomena in the parameter $q$. In appendix A, we find that the singularity spectra obtained via this numerical analysis are in agreement with the available exact results.

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