A new formula of Rotational Stiffness Coefficient of pile hinged on top under moment load

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Abstract. The analysis and design of single piles subjected to lateral load are very important. The traditional expression for calculating rotational stiffness coefficient is too complex, and it needs large amount of computation. This paper proposes a new approach for getting rotational stiffness coefficient of pile hinged on top under moment load. It has been proved that the new method is the equivalent of traditional method of calculating rotational stiffness coefficient using power series. A case study shows that the same answer can be gotten in calculating rotational stiffness coefficient by the two methods. However, a large computational cost can be reduced by the new method.

1. Introduction

The piles subjected to lateral load have been widely applied in tall buildings, railways, highways, bridges, offshore platforms, port wharfs and many other engineering. The interaction between lateral loaded pile and soil is complex, and the calculation of internal force and deformation of piles is very important. Many theoretical models and calculation methods for lateral loaded pile are developed. According to different assumptions of horizontal resistance of soil, Zhang's method, m method, k method, c method and p-y curve method are developed [1]. The m method is widely used in engineering because the calculated results are in good agreement with the actual projects. The m method has been recommended by several Chinese Specifications, such as code for design on subsoil and foundation of railway bridge and culvert [2], code for design of foundation and foundation of highway bridges and culverts [3] and technical specification for building pile foundation [4].

Many researchers conducted in-depth research on the application of m method. Zhao Minghua derived a complete set of design theory and calculation method of Inclined loaded pile suitable for Chinese engineering applications by Power series method [5]. Li Hongjiang established a new differential equation for the pile deflection containing the side frictions based the conventional method of analysis to the laterally loaded piles [6]. Mao Jianqiang developed a method, in which m-method model was employed to simulate the interaction between landslide body and pile. Meanwhile, equations were established for the piles above and below the slip surface respectively, and their expressions of displacement and internal force were also solved with power series method [7]. Xenia developed a procedure for determining the dynamic stiffness and damping of a laterally loaded pile in soil exhibiting different types of inhomogeneity with depth [8]. Antônio proposed a new approach for data reduction of horizontal load full-scale tests on piles and pile groups. This approach was developed
on results from tests run on bored concrete piles embedded in homogeneous and nonhomogeneous ground [9].

The improvement of stiffness coefficient of laterally loaded pile receives little attention. This paper proposes a new approach for getting Stiffness Coefficient of pile hinged on top under moment load. It has been proved that the new method is the equivalent of traditional method of calculating Stiffness Coefficient using power series. A case study shows that the same answer can be gotten in calculating Stiffness Coefficient by the two methods. However a large computational cost can be reduced by the new method.

2. Traditional calculation method

2.1 Differential equation of pile

![Figure 1: Laterally loaded pile](image)

Referring to Figure 1 and using the theory of beams on an elastic foundation, we can get

\[
\frac{d^4y_z}{dz^4} + a^5y_z = 0 \tag{1}
\]

\[a = \frac{mb_0}{EI} \tag{2}\]

where \(a\) is deformation coefficient of pile; \(y_z\) is deflection of pile; \(z\) is the depth of computational point; \(m\) is constant of modulus of horizontal subgrade reaction; \(b_0\) is computational width of pile; \(EI\) is bending stiffness of pile.

2.2 Power series solution of Differential Equations

Eq. (1) can be solved by power series expansion. If the deflection of pile top \(y_0\), rotation angle \(\phi_0\), moment \(M_0\) and shear force \(Q_0\) are known, the deflection \(y_z\), rotation angle \(\phi_z\), moment \(M_z\) and shear force \(Q_z\) of pile in any depth can be gotten. The solution of Eq.(1) results in the following expressions:

\[y_z = y_0A_1 + \frac{\phi_0}{\alpha}B_1 + \frac{M_0}{\alpha^2EI}C_1 + \frac{H_0}{\alpha^3EI}D_1\tag{3}\]

\[\frac{\phi_z}{\alpha} = y_0A_2 + \frac{\phi_0}{\alpha}B_2 + \frac{M_0}{\alpha^2EI}C_2 + \frac{H_0}{\alpha^3EI}D_2\tag{4}\]

\[\frac{M_z}{\alpha^2EI} = y_0A_3 + \frac{\phi_0}{\alpha}B_3 + \frac{M_0}{\alpha^2EI}C_3 + \frac{H_0}{\alpha^3EI}D_3\tag{5}\]

\[\frac{Q_z}{\alpha^3EI} = y_0A_4 + \frac{\phi_0}{\alpha}B_4 + \frac{M_0}{\alpha^2EI}C_4 + \frac{H_0}{\alpha^3EI}D_4\tag{6}\]

where \(A_1, B_1, \ldots, C_4, D_4\) are dimensionless coefficient, \(A_2, B_2, C_2, D_2\) are the first order derivatives of \(A_1, B_1, C_1, D_1\) and \(A_3, B_3, C_3, D_3\) are the second order derivatives of \(A_1, B_1, C_1, D_1\) and \(A_4, B_4, C_4, D_4\) are the third order derivatives of \(A_1, B_1, C_1, D_1\).
The Equations of $A_1$, $B_1$, $C_1$, $D_1$ are

$$A_1 = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(5k-4)!!}{(5k)!} \left(\frac{az}{\alpha}\right)^k$$

$$B_1 = a \left[ z + \sum_{k=1}^{\infty} (-1)^k \frac{(5k-3)!!}{(5k+1)!} \times \frac{1}{\alpha} \left(\frac{az}{\alpha}\right)^{k+1} \right]$$

$$C_1 = \frac{a^2}{2} \left[ z^2 + \sum_{k=1}^{\infty} (-1)^k \frac{(5k-2)!!}{(5k+2)!} \times \frac{2}{\alpha^2} \left(\frac{az}{\alpha}\right)^{k+2} \right]$$

$$D_1 = \frac{a^3}{6} \left[ z^3 + \sum_{k=1}^{\infty} (-1)^k \frac{(5k-1)!!}{(5k+3)!} \times \frac{6}{\alpha^3} \left(\frac{az}{\alpha}\right)^{k+3} \right]$$

$$\frac{(5k-4)!!}{(5k)!} = \left[5k-4\right]\left[5(k-1)-4\right]…\left[5\times1-4\right] = (5k)(5k-1)…2 \times 1$$

(7) \hspace{1cm} (8) \hspace{1cm} (9) \hspace{1cm} (10)

\[ (11) \hspace{1cm} (12) \]

2.3 Flexibility coefficient

The physical significance of flexibility coefficient is relative displacement due to unit load. The expressions of flexibility coefficients of pile free at pile bottom are

$$\delta_{HH} = \frac{1}{\alpha^2 EI} \times \frac{(B_4D_4 - B_3D_3)}{(A_5B_4 - A_4B_5)}$$

$$\delta_{MM} = \frac{1}{\alpha^2 EI} \times \frac{(A_4D_4 - A_3D_3)}{(A_5B_4 - A_4B_5)}$$

$$\delta_{HM} = \delta_{MH}$$

$$\delta_{MM} = \delta_{MM}$$

where $\delta_{ij}$ is the relative displacement in the i direction due to unit load in the j direction.

The rotation angle of pile top $\phi_0$ is

$$\phi_0 = -\left(H_0\delta_{HH} + M_0\delta_{MM}\right)$$

(13) \hspace{1cm} (14) \hspace{1cm} (15) \hspace{1cm} (16)

The deflection of pile top $y_0$ is

$$y_0 = H_0\delta_{HH} + M_0\delta_{HM}$$

(17) \hspace{1cm} (18)

2.4 Stiffness coefficient

The physical significance of stiffness coefficient is relative internal force due to unit displacement.

Referring to Figure 2, we can get

$$y_0 = 0$$

(19)

Combining Eqs. (18) and (19) yields
\[ H_0 = -M_0 \frac{\delta_{HM}}{\delta_{HH}} \]  

Combining Eqs. (17) and (20) yields

\[ \phi_0 = M_0 \left( \frac{\delta_{MM} - \delta_{HH}^2}{\delta_{HH}} \right) \]  

However, rotation angle of pile top is

\[ \phi_0 = 1 \]

Hence,

\[ \rho_{MM} = M_0 = \frac{1}{\left( \frac{\delta_{MM} - \delta_{HH}^2}{\delta_{HH}} \right)} = \frac{\delta_{HH}}{\delta_{HH} \delta_{MM} - \delta_{HH}^2} \]  

where \( \rho \) is stiffness coefficient which stand for the relative internal force in the i direction due to unit displacement in the j direction. \( \rho_{MM} \) is rotational stiffness coefficient.

Put Eqs.(13), (14), (15) and (16) into Eq.(23), we get

\[ \rho_{MM} = \frac{\alpha EI (A_3D_3 - A_4D_3)(B_4D_3 - B_3D_3)}{(B_3D_3 - B_4D_3)(A_4C_3 - A_3C_3) - (A_3D_3 - A_4D_3)^2} \]  

3. New formula of Rotational Stiffness Coefficient

Pile is free in the bottom, so

\[ M_h = 0 \]
\[ Q_h = 0 \]  

Where \( M_h \) is moment of pile in the bottom, \( Q_h \) is shear force of pile in the bottom.

Put Eqs.(19), (22), (25) into Eq.(5), we get

\[ 0 = B_3 + \frac{M_0}{\alpha^2 EI} C_3 + \frac{H_0}{\alpha^2 EI} D_3 \]  

Put Eqs. (19), (22), (26) into Eq.(6), we get

\[ 0 = B_4 + \frac{M_0}{\alpha^2 EI} C_4 + \frac{H_0}{\alpha^2 EI} D_4 \]

Combining Eqs. (27) and (28) yields

\[ \rho_{MM}^* = -M_0 = \frac{\alpha EI (B_3D_3 - B_4D_3)}{(C_3D_3 - C_4D_3)} \]  

where \( \rho_{MM}^* \) is the new formula of rotational stiffness coefficient which physical significance is as the same as \( \rho_{MM} \).

4. Case study

4.1 Case parameters

A pile hinged on top with cap. Diameter of pile \( d=1.0m \), \( h=10m \). Strength grade of concrete of pile is C30. Thickness of protective layer is 0.05m and reinforcement ratio \( \rho_g = 0.004 \). Constant of modulus of horizontal subgrade reaction \( m=2 \times 10^4 kN/m^4 \), Elastic modulus of steel bar \( E_s = 2.1 \times 10^8 kN/m^4 \), Elastic modulus of concrete \( E_c = 3 \times 10^7 kN/m^4 \). The calculation process is as follows:

Effective diameter of pile is

\[ d_o = 0.9m \]

Section Resistance Moment of pile is
\[
W_0 = \frac{\pi d}{32} \left[ d^2 - 2(\alpha_k - 1)\rho d_0^2 \right] = \frac{3.1416}{32} \left[ 1^2 - 2 \times (7 - 1) \times 0.004 \times 0.9^2 \right] = 0.1020 m^3
\]

Cross-sectional moment of inertia of pile is
\[
I_0 = \frac{W_0 d_0}{2} = 0.1020 \times \frac{0.9}{2} = 0.0459 m^4
\]

Bending Stiffness of pile is
\[
EI = 0.85E, I_0 = 0.85 \times 3 \times 10^7 \times 0.0459 = 1.170 \times 10^8 kN \cdot m^2
\]

Computational Width of pile is
\[
b_0 = 0.9 \times (1.5 + 0.5) = 1.8 m
\]

Horizontal Deformation Coefficient of pile is
\[
\alpha = \sqrt{\frac{mb_0}{EI}} = \sqrt{\frac{2 \times 10^4 \times 1.8}{1.170 \times 10^8}} = 0.4985 m^{-1}
\]

4.2 Result of traditional method

Using traditional method, we get
\[
\delta_{MM} = \frac{1}{\alpha EI} \times \frac{(A_2D_2 - A_1D_1)}{(A_2B_2 - A_1B_1)} = \frac{1}{0.4985 \times 1.17 \times 10^8} \times \left[ (-11.73066) \times (-23.14040) \right] - (-0.35762) \times (-15.07550)
\]
\[
= 1.6840 \times 10^{-5} m / kN
\]

\[
\delta_{MH} = \frac{1}{\alpha^2 EI} \times \frac{(A_2D_2 - A_1D_1)}{(A_2B_2 - A_1B_1)} = \frac{1}{0.4985 \times 1.17 \times 10^8} \times \left[ (-11.73066) \times (-23.14040) \right] - (-0.35762) \times (-15.07550)
\]
\[
= 5.5753 \times 10^{-6} kN^{-1}
\]

\[
\delta_{MH} = \frac{1}{\alpha EI} \times \frac{(A_1C_2 - A_2C_1)}{(A_1B_2 - A_2B_1)} = \frac{1}{0.4985 \times 1.17 \times 10^8} \times \left[ (-11.73066) \times (-23.14040) \right] - (-0.35762) \times (-15.07550)
\]
\[
= 3.0014 \times 10^{-5} (kN \cdot m)^{-1}
\]

\[
\rho_{MM} = \frac{\delta_{MM}}{\delta_{MM} \delta_{MM} - \delta_{MM}^2} = \frac{1.6840 \times 10^{-5}}{1.6840 \times 10^{-5} \times 3.0014 \times 10^{-6} - (5.5753 \times 10^{-6})^2} = 0.8654 \times 10^6 kN \cdot m / m
\]

4.3 Result of new formula

\[
\rho'_{MM} = \frac{\alpha EI (B_2D_2 - B_1D_1)}{(C_2D_2 - C_1D_1)} = 0.4985 \times 1.17 \times 10^8 \times \left[ (-11.73066) \times (-23.14040) \right] - (-0.35762) \times (-15.07550)
\]
\[
= 0.8654 \times 10^6 kN \cdot m / m
\]

Comparing to the results of the two method, we get
\[
\rho_{MM}' = \rho_{MM}
\]
5. Conclusion
The conclusions of this paper can be summarized as follows:

(1) The traditional expression for calculating Rotational Stiffness Coefficient is too complex, and it needs large amount of computation.

(2) The new method is the equivalent of traditional method of calculating Rotational Stiffness Coefficient, and a large computational cost can be reduced by the new method.

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