Quantum Aspects of Ergoregion Instability

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Abstract

It has been known classically that a star with an ergoregion but no event horizon is unstable to the emission of scalar, electromagnetic and gravitational waves. This classical ergoregion instability is characterized by complex frequency modes. We show how to canonically quantize a neutral scalar field in the presence of such unstable modes by considering a simple model for a rapidly rotating star. Some of interesting results is that there exists a physically meaningful mode decomposition including unstable normal mode solutions whose representation turns out to be a non-Fock-like Hilbert space. A “particle” detector model placed in the in-vacuum state also shows that stars with ergoregions give rise to a spontaneous energy radiation to spatial infinity until ergoregions disappear.

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1 Introduction

In general relativity, inertial frames around rotating objects are dragged in the sense of the rotation. If the object is rotating rapidly, this dragging effect can be so strong that in some region no physical object can remain at rest relative to an inertial observer at spatial infinity. This region of spacetime is called an ergoregion or an ergosphere. For a stationary asymptotically flat spacetime, the ergosphere is the region in which the Killing vector field \( \xi = \partial / \partial t \) that corresponds to time translations at spatial infinity fails to be timelike. The most common example of ergoregions would be the outside of the event horizon of any rotating black hole. Ergoregions also arise in models of dense, rotating stars in which no event horizon exists inside the ergosurface \([1, 2]\). The question we want to answer in this paper is whether or not and how the presence of ergoregions only gives rise to quantum instability.

For massless field perturbations in rotating black holes, it has been shown that no unstable mode occurs classically \([3, 4]\). However, quantum fields in this background spacetime where both horizon and ergoregion are present reveal vacuum instability, the so-called Starobinskii-Unruh effect, which is indeed the quantum counter part of the classical phenomena such as Penrose process for particles and superradiance for waves \([5, 6, 7, 8]\). On the other hand, spacetimes with ergoregions but without horizons such as rapidly rotating stars are known to be unstable to scalar, electromagnetic and gravitational perturbations \([9, 10]\). As explained in Refs. \([11, 9, 10, 8]\), such classical instability, so-called ergoregion instability, can be understood heuristically by looking at a spherical wave packet in a “superradiant” mode incident to the ergoregion. The reflected wave will be amplified and carry positive energy up to spatial infinity. The transmitted part within the ergoregion is also amplified but now carries negative energy with respect to an observer at infinity. This wave will pass through the center of the rotating object and get back to the ergosurface, again giving transmission as well as reflection there with energies amplified, respectively. This process will repeat as long as the ergoregion remains, resulting in presumably “exponential” radiation of positive energy to infinity and accumulation of negative energy within the ergoregion in such a way that the total energy is conserved. It turns out that this instability is characterized by complex frequency modes in normal mode solutions of classical fields.

The quantum counterpart of such ergoregion instability has been studied by Ashtekar and Magnon \([12, 8]\), and Dray, Kulkarni and Manogue \([13]\) in algebraic approach, and by Matacz, Davies and Ottewill in canonical approach \([14]\) of quantum fields. Ashtekar and Magnon have given a general argument in Ref. \([12]\) that in a rotating spacetime with an ergoregion but without a horizon there does not exists a natural complex structure which gives no “mixing of positive and negative frequencies” under time evolution, and hence no spontaneous creation of particles. In other words, no matter which complex structure is chosen, the resulting quantum field theory should predict a spontaneous particle creation. However, their algebraic approach is somewhat qualitative and it has not been shown what the detailed quantitative pictures of this quantum instability are. On the other hand, Dray, Kulkarni and Manogue have claimed that there exists a natural complex structure if we
quantize the system with respect to ZAMO observers rather than with respect to Killing observers.

In the canonical quantization approach, Matacz, Davies and Ottewill have recently investigated whether quantum vacuum instability occurs near rapidly rotating stars. They have considered a simple spacetime model where the outside of a star is described by the Kerr metric, and classical Klein-Gordon fields satisfy a mirror boundary condition (i.e., $\phi(x) = 0$) near the surface of the rotating star. They found that the quantum instability, the Starobinskii-Unruh effect, is absent provided that only real frequencies occur in normal mode solutions of the Klein-Gordon equation. As mentioned above, however, all asymptotically flat stationary spacetimes with ergoregions but without horizons are unstable to scalar wave perturbations. Consequently, complex frequency modes should exist in such a background spacetime. Therefore, in order to conclude whether or not the quantum instability occurs near stars with ergoregions, one must include such unstable complex frequency modes as well and needs to understand their physical roles in the quantization procedure.

After the discovery of the occurrence of complex frequency modes for a charged scalar field interacting with some strong electrostatic potential, the so-called Schiff-Snyder-Weinberg effect, the quantization of fields including such instability modes has been studied by several authors, all in the Minkowski flat spacetime; a charged scalar field interacting with strong stationary scalar potential, a neutral scalar field with tachyonic mass, a charged scalar field in an electrostatic potential, and a neutral scalar field with time-varying mass or interacting with an external square-well potential.

In this paper we carry out, for the first time, this quantization procedure for a massless scalar field in a certain model of curved spacetime with an ergoregion but without a horizon. We find that it is possible to quantize the system, but mode operators for complex frequencies satisfy unusual commutation relations and do not admit a usual Fock-like representation or a ground state as in other models in Minkowski flat spacetime. Consequently, the results imply that a rotating star with an ergoregion gives a spontaneous energy radiation to spatial infinity until the ergoregion disappears. This quantum ergoregion instability is very much analogous to a laser amplification.

In Sec. 2, we define the inner product for field solutions and show its peculiar properties in the presence of complex frequency modes. We also explain the spacetime model to be considered and construct a complete set of normal mode solutions including instability modes due to the presence of ergoregion. In Sec. 3, the canonical quantization is carried out, which turns out to give a non-Fock-like representation for the quantum field. In Sec. 4, a “particle” detector model is used to show the spontaneous energy radiation to infinity related to the appearance of the ergoregion. Some open questions and possible applications of our results to other physical systems are discussed in Sec. 5.
2 Classical field

We consider a system of a massless real scalar field \( \phi(x) \) minimally coupled to gravitational fields as follows,

\[
L = \int d^3x \sqrt{-g} L = -\frac{1}{2} \int d^3x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi,
\]

(1)

where the integration is performed on a \( x^0 = t = \text{const.} \) spacelike hypersurface and \( g^{\mu\nu} \) is the metric of an arbitrary background spacetime which will be specified below. The conjugate momentum \( \pi \) to the field \( \phi \) is defined by

\[
\pi(x) = \frac{\partial L}{\partial (\partial \phi / \partial x^0)} = -g^{0\nu} \partial_\nu \phi = -\nabla_0 \phi,
\]

(2)

and the Hamiltonian is

\[
H = \int_{x^0=\text{const.}} d^3x \sqrt{-g}(\pi \nabla_0 \phi - L)
= \frac{1}{2} \int d^3x \sqrt{-g}(-g^{00} \nabla_0 \phi \nabla_0 \phi + g^{ij} \nabla_i \phi \nabla_j \phi).
\]

(3)

The Klein-Gordon equation

\[
\Box \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0
\]

(4)

is equivalent to Hamiltonian equations given by

\[
\hat{H}\Phi = i \frac{\partial \Phi}{\partial x^0},
\]

(5)

where \( \Phi = \left( \begin{array}{c} \phi \\ \pi \end{array} \right) \) is the two-component field, and \( \partial \phi / \partial x^0 = \partial H / \partial \pi \) and \( \partial \pi / \partial x^0 = -\partial H / \partial \phi \).

As shown in Ref. [6], the operator \( \hat{H} \) defined via Eq. (5) is Hermitian with respect to the following inner product

\[
<\phi_1, \phi_2> = <\Phi_1, \Phi_2>
= \frac{i}{2} \int d^3x \sqrt{-g} (\phi_1^* \pi_2 - \pi_1^* \phi_2)
= \frac{i}{2} \int \phi_1^* \hat{\partial}_\mu \phi_2 \, d\Sigma^\mu.
\]

(6)

In other words,

\[
<\Phi_1, \hat{H}\Phi_2> = <\hat{H}\Phi_1, \Phi_2>
\]

(7)

for any given solutions \( \phi_1 \) and \( \phi_2 \) of Eq. (5) or, equivalently, Eq. (4), satisfying suitable conditions on the spatial boundary. Consequently, the inner product \( <\phi_1, \phi_2> \) is independent of the “time” \( x^0 = t \) at which the spatial integration is performed.
We now assume that the background spacetime possesses a Killing vector field \( \xi = \partial_t \) in some regions, for instance, in the early and late stages of its evolution. Then normal mode solutions can be defined by

\[
\mathcal{L}_\xi \phi = -i\omega \phi.
\]  

(8)

Here \( \mathcal{L}_\xi \) is the Lie-derivative along a Killing vector \( \xi \). Thus the time dependence of normal mode solutions is \( \sim e^{-i\omega t} \). Since \( \mathcal{L}_\xi \phi = \partial \phi / \partial \xi^0 \), normal mode solutions are indeed eigenfunctions of the operator \( \hat{\mathcal{H}} \)

\[
\hat{\mathcal{H}} \Phi = \omega \Phi.
\]

(9)

From the Hermiticity of \( \hat{\mathcal{H}} \) in Eq. (7), one obtains

\[
(\omega_2 - \omega_1^*) \langle \phi_1, \phi_2 \rangle = 0
\]

for any given two normal mode solutions \( \phi_1 \) and \( \phi_2 \). Thus the inner product is zero unless \( \omega_2 = \omega_1^* \). Since our inner product defined in Eq. (4) is not positive definite in general, the normal mode frequency \( \omega \) is not necessarily always real. In the remote past where the spacetime is almost flat, the inner product is positive definite and hence \( \omega \) is real. In the far future where an ergoregion arises, however, it is possible that there exist bounded solutions with complex frequencies as shown in Refs. [11, 10, 9] for certain cases of spacetime. Then, from Eq. (10), the norm of such complex frequency modes should be zero. Therefore, our inner product is not positive definite in the late epoch of star evolution. In other words, we are confronted with the problem of quantizing fields in an indefinite inner product space.

If one defines the classical energy associated with a solution \( \phi \) as

\[
E = \langle \Phi, \hat{\mathcal{H}} \Phi \rangle,
\]

complex frequency modes give vanishing classical energy due to null norm, \( E_\omega = \omega \langle \phi, \phi \rangle = 0 \), whereas real frequency modes give \( E_\omega = \pm \omega \) after suitable normalization of fields. It happens probably because the negative energy spread over within the ergoregion exactly cancels the positive energy outside. As shall explicitly be shown below, it should be pointed out that, by linearly combining complex frequency mode solutions, one can construct a solution whose energy defined in Eq. (11) or, equivalently, in Eq. (3) is negative. From Eq. (3) one can check that, although the Hamiltonian is positive definite in cases that spacetimes are almost flat in the past infinity, it could be negative for certain solutions if ergoregions appear in the future infinity. The above mentioned properties related to instability modes are also satisfied for other models studied in the flat Minkowski spacetime in Refs. [17, 18, 19, 20].

We now specify the background spacetime in more detail. We are interested in a dynamically evolving spacetime which starts from an almost flat spacetime in the past infinity and ends up to a stationary rotating star with an ergoregion in the future infinity. As far as we know, however, there is no known analytic solution representing such spacetime. First of all, we should point out that there is no equivalent to Birkhoff’s theorem for an axially symmetric rotating star. So the outside region of axially symmetric rotating stars may not
be described by the Kerr metric and presumably depends on the details of the star inside. There is a spacetime solution found by Lindblom and Brill [21] for the case of the free-fall collapse of a rotating dust shell where an ergoregion develops at the late epoch of its collapse. Unfortunately, however, the solution, which is based on the first order approximation in the angular velocity of the shell, ceases to be valid near this stage. For our purpose of showing quantum instability of ergoregions, however, it is sufficient to consider any spacetime model which possesses an ergoregion at the late stage of its evolution as will be shown below.

We assume that our background spacetime is described by the Minkowski flat metric in the past infinity and by the Kerr metric with mirror boundary condition on the field $\phi$, which is used in Refs. [14, 11], in the future infinity. Instead of considering the detailed dynamics inside the star, we simply assume that all classical solutions $\phi$ of Eq. (4) vanish on the surface of some sphere inside the ergoregion, e.g., a totally reflecting mirror boundary. The quantization of the field in the past infinity will be straightforward; it will have a Fock representation with a vacuum state $|0\rangle$. To carry out the canonical quantization in the future infinity, let us first construct normal mode solutions of Eq. (4).

In Boyer-Linquist coordinates, the Kerr metric outside the star is given by

$$ ds^2 = \frac{\Delta \Sigma}{(r^2 + a^2)^2} \Delta a^2 \sin^2 \theta dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta (d\varphi - \Omega dt)^2, $$

(12)

where $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Omega = 2aMr/[(r^2 + a^2)^2 - \Delta \sin^2 \theta]$, and $M$ and $a$ are the total mass and the angular momentum per unit mass of the star, respectively. This metric has a rotational Killing vector field $\xi_\varphi = \partial_\varphi$ commuting with $\xi_t = \partial_t$. Thus one can simultaneously define angular eigenmodes by

$$ \mathcal{L}_{\xi_\varphi} \phi = i m \phi, $$

(13)

where $m$ is an integer. As is well known, the Klein-Gordon equation is separable [22] and admits a complete set of normal mode solutions of the form

$$ \phi(x) = \frac{R(r)}{\sqrt{r^2 + a^2}} S(\theta) e^{-i \omega t + i m \varphi}. $$

(14)

Here $S(\theta)$ is the oblate spheroidal harmonics with eigenvalue $\lambda$ satisfying

$$ \left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta \frac{d}{d\theta} + (\omega^2 a^2 \cos^2 \theta - \frac{m^2}{\sin^2 \theta}) \right] S(\theta) = -\lambda S(\theta). $$

(15)

$\lambda$ is a separation constant, a function of $\omega$, $m$ and some integer $l$ with $|m| \leq l$, and determined by requiring $S(\theta)$ to be regular at $\theta = 0, \pi$. The radial function $R(r)$ satisfies

$$ \left[ \Delta \frac{d}{dr} \Delta \frac{d}{dr} + (\omega (r^2 + a^2) - ma)^2 - (\omega a (\omega a - 2m) + \lambda) \Delta \right] \frac{R(r)}{\sqrt{r^2 + a^2}} = 0. $$

(16)
Defining a “generalized” tortoise coordinate \( \tilde{r} \) by \( d\tilde{r}/dr = (r^2 + a^2)/\Delta \), Eq. \((16)\) becomes

\[
\frac{d^2 R}{d\tilde{r}^2} - V_{\omega lm}(\tilde{r}) R = 0.
\]

From the mirror boundary condition, the radial function vanishes at some \( r = r_0 \) (accordingly, \( \tilde{r} = \tilde{r}_0 \)) inside the ergoregion;

\[
R(r_0) = 0.
\]

For simplicity, we assume that \( r_0 \) is very near the “horizon” radius \( r = r_H = M + \sqrt{M^2 + a^2} \). That is, \( \tilde{r}_0 \sim -\infty \). We also require that the field is not singular at spatial infinity, \( \tilde{r} \sim \infty \). The asymptotic behavior of the effective potential \( V_{\omega lm} \) induced through the interaction with gravitational fields is as follows \[23\],

\[
V_{\omega lm}(r) \sim \begin{cases} 
-(\omega - m\Omega_H)^2 & \text{as } \tilde{r} \to \tilde{r}_0, \\
-\omega^2 & \text{as } \tilde{r} \to \infty,
\end{cases}
\]

where \( \Omega_H = a/2Mr_H \). We have \( V_{\omega lm} = \infty \) at \( \tilde{r} \sim \tilde{r}_0 \), corresponding to the mirror boundary condition. Since \( \omega \) could be complex in our model, \( V_{\omega lm}(r) \) is a complex potential in general. For real \( \omega \), we see that, between two asymptotic regions, e.g., \( \tilde{r} \sim \infty \) and \( \tilde{r} \sim \tilde{r}_0 \), there exists a potential barrier which grows as \( l \) increases. Note that the left asymptotic value \( -\omega^2 = -(\omega - m\Omega_H)^2 \) varies from 0 to \( -\infty \) as \( m\Omega_H \) changes. In particular, a deep potential well is formed for a large value of \( m\Omega_H \) which indeed leads to superradiance when \( \tilde{\omega} = \omega - m\Omega_H < 0 \) in the case of the Kerr black hole. Thus, in the asymptotic regions, the form of the radial solution \( R(r) \) will be

\[
R(r) \sim \begin{cases} 
e^{\pm i\tilde{\omega}\tilde{r}} & \text{as } \tilde{r} \to \tilde{r}_0, \\
e^{-i\tilde{\omega}\tilde{r}} & \text{as } \tilde{r} \to \infty.
\end{cases}
\]

Let us now consider normal mode solutions \( u_{\omega lm}(r) \) to Eq. \((17)\) whose asymptotic forms are

\[
u_{\omega lm}(r) \sim \begin{cases} 
B_{\omega lm}(e^{i\tilde{\omega}\tilde{r}} + A_{\omega lm}e^{-i\tilde{\omega}\tilde{r}}) & \text{as } \tilde{r} \to \tilde{r}_0, \\
e^{-i\tilde{\omega}\tilde{r}} + C_{\omega lm}e^{i\tilde{\omega}\tilde{r}} & \text{as } \tilde{r} \to \infty.
\end{cases}
\]

From the boundary condition that they vanish at the mirror surface at \( \tilde{r} = \tilde{r}_0 \),

\[
A_{\omega lm} = -e^{2i\tilde{\omega}\tilde{r}_0}.
\]

That is, as \( \tilde{r} \to \tilde{r}_0 \),

\[
u_{\omega lm}(r) \sim B_{\omega lm}e^{i\tilde{\omega}\tilde{r}_0}(e^{i\tilde{\omega}(\tilde{r} - \tilde{r}_0)} - e^{-i\tilde{\omega}(\tilde{r} - \tilde{r}_0)}) = 2iB_{\omega lm}e^{i\tilde{\omega}\tilde{r}_0}\sin \tilde{\omega}(\tilde{r} - \tilde{r}_0).
\]

If \( \omega \) is complex, \( u_{\omega lm} \) becomes exponentially divergent at spatial infinity and so we exclude this class of solutions from our construction. Thus \( u_{\omega lm} \) represents real frequency normal mode solutions. Now the Wronskian relations from Eq. \((17)\) with the mirror boundary condition give

\[
|A_{\omega lm}| = |C_{\omega lm}| = 1.
\]
Therefore, \( u_{\omega \ell m}(r) \) is a stationary wave without any net ingoing or outgoing flux with respect to ZAMO observers \([15]\). Here \( \omega \) is any continuous real number. In fact, this class of real frequency normal mode solutions is equivalent to the set considered in Ref. \([14]\) as a complete basis.

As mentioned above, however, there exists another class of normal mode solutions with complex frequencies which describe unstable modes in the presence of ergoregions. This class of solutions has not been included in the quantization procedure in Ref. \([14]\). Let \( v_{\omega \ell m}(r) \) be normal mode solutions in such class whose asymptotic behaviors are

\[
v_{\omega \ell m}(r) \sim \begin{cases} e^{i\tilde{\omega}r} + R_{\omega \ell m}e^{-i\tilde{\omega}r} & \text{as } \tilde{r} \to 0, \\ T_{\omega \ell m}e^{i\tilde{\omega}r} & \text{as } \tilde{r} \to \infty. \end{cases}
\]  

(25)

The mirror boundary condition is satisfied if

\[
R_{\omega \ell m} = -e^{2i\tilde{\omega}r_0} = -e^{2i\tilde{\omega}^R r_0} \cdot e^{-2\tilde{\omega}^I r_0},
\]  

(26)

where \( \tilde{\omega} = \omega^R + i\omega^I \). Thus, as \( \tilde{r} \to 0 \), \( v_{\omega \ell m} \sim \sin \tilde{\omega}(\tilde{r} - \tilde{r}_0) \) again. And

\[
\omega^I = \tilde{\omega}^I = -\ln |R_{\omega \ell m}|/2\tilde{r}_0.
\]  

(27)

Note that, since the potential \( V_{\omega \ell m}(r) \) in Eq. \([17]\) is complex, the Wronskian relation does not necessarily give \( |R_{\omega \ell m}| = 1 \). If \( |R_{\omega \ell m}| > 1 \), \( \omega^I > 0 \) and so \( v_{\omega \ell m}(r) \sim e^{-\omega^I r} \) as \( \tilde{r} \to \infty \) and is regular at spatial infinity. From the time-dependence of this solution, i.e., \( \sim e^{-i\omega t} = e^{-i\omega^R t} \cdot e^{i\omega^I t} \), we also notice that it represents an outgoing mode which is exponentially amplifying in time but is exponentially decreasing as \( \tilde{r} \to \infty \). By making a wave packet, as suggested in Ref. \([11]\), we may regard this solution as an outgoing wave packet with \( \tilde{\omega}^R = \omega^R - m\Omega_H < 0 \) starting from near the mirror surface, which will bounce back and forth within the ergoregion, and a part of which is repeatedly transmitted to infinity, resulting in exponential amplification in time in the inside as well as in the outside of the ergosurface. If \( |R_{\omega \ell m}| < 1 \), \( \omega^I < 0 \) and so this solution corresponds to an outgoing decaying mode in time. However, since its radial behavior becomes singular at spatial infinity, we do not include this mode in our construction of normal mode solutions.

For any given solution

\[
\phi_{\omega \ell m}(x) = \phi_{\omega \ell m}(r, \theta)e^{-i\omega t + i\varphi} = \frac{v_{\omega \ell m}(r)}{\sqrt{r^2 + a^2}}S_{\omega \ell m}(\theta)e^{-i\omega t + i\varphi} \sim e^{\omega^I t}
\]  

(28)

with \( \omega^I > 0 \), we find that there are three linearly independent solutions;

\[
\phi_{\omega \ell m}^*(x) = \frac{v_{\omega \ell m}^*(r)}{\sqrt{r^2 + a^2}}S_{\omega \ell m}^*(\theta)e^{i\omega^I t - i\varphi} \sim e^{\omega^I t},
\]

\[
\phi_{\omega^* \ell m}(x) = \frac{v_{\omega^* \ell m}(r)}{\sqrt{r^2 + a^2}}S_{\omega^* \ell m}(\theta)e^{-i\omega^I t + i\varphi} \sim e^{-\omega^I t},
\]

\[
\phi_{\omega^* \ell m}^*(x) = \frac{v_{\omega^* \ell m}^*(r)}{\sqrt{r^2 + a^2}}S_{\omega^* \ell m}^*(\theta)e^{i\omega^I t} \sim e^{-\omega^I t}. 
\]  

(29)
\( \phi_{\omega l m}^* \) is simply the complex conjugation of \( \phi_{\omega l m} \). \( \phi_{\omega^* l m} \) is obtained simply by taking complex conjugations of Eq. (13) and Eq. (14). Thus we let \( \phi_{\omega^* l m}(r, \theta) = \phi_{\omega l m}^*(r, \theta) \). This mode represents an exponentially decaying wave in time which originates at infinity. In other words, this mode is the same as \( \phi_{\omega l m}(x) \) but backward in time. \( \phi_{\omega^* l m}(x) \) is simply the complex conjugate of \( \phi_{\omega l m}(x) \). These linearly independent modes can be denoted by indices such as \((\omega, l, m), (-\omega^*, l, -m), (\omega^*, l, m)\), and \((-\omega, l, -m)\), respectively.

For these non-stationary modes, \( \omega \) is discrete complex numbers which are determined by the details of the potential and the boundary condition. In fact, by finding poles of the scattering amplitude for a more realistic model of rotating stars, Comins and Schutz [10] have shown that the imaginary part of the complex frequency for a purely outgoing mode is discrete, positive, and proportional to \( e^{-2\beta l m} \), where \( \beta \) is of order unit. For our model, it also can be shown, from Eq. (15) and Eq. (16), that complex eigenfrequencies are confined to a bounded region as follows [3],

\[
0 \leq m \omega^R, \quad 0 \leq |\omega^R| \leq |m| \Omega_H \frac{T_H}{r_0},
\]

and

\[
(\omega^1)^2 \leq (1 + a^4/r_0^4)(\omega^R)^2 + 2mM(m + 2a\omega^R)/r_0^3
\]

for given \( m \) and \( a \).

Now let us consider the norms of these mode solutions constructed above. From our definition of the inner product in Eq. (17), we find

\[
<\phi_{\omega l m}, \phi_{\omega' l m'}> = \frac{i}{2} \int \phi_{\omega l m}^* (\partial_r + \hat{\Omega} \partial_\varphi) \phi_{\omega' l m'} N^{-1} d\Sigma,
\]

\[
= \frac{1}{2} \int [(\omega + \omega^*) - \Omega(m' + m)] \phi_{\omega l m}^* \phi_{\omega' l m'} N^{-1} d\Sigma,
\]

where we have used \( d\Sigma^\mu = n^\mu d\Sigma \), \( n^\mu = N^{-1}(\partial_t + \Omega \partial_\varphi)^\mu \), \( \Omega(r, \theta) = -\partial_t \cdot \partial_\varphi / \partial_\varphi \cdot \partial_\varphi = -g_{t\varphi} / g_{\varphi\varphi} \), and \( N = [-\partial_t + \Omega \partial_\varphi] \cdot (\partial_t + \Omega \partial_\varphi)^{-1/2} = (-g^{tt})^{-1/2} \). Thus,

\[
<u_{\omega l m}, u_{\omega l m}> = \int (\omega - m\Omega) |u_{\omega l m}(x)|^2 N^{-1} d\Sigma
\]

for real frequency modes with \( \omega > 0 \)

\[
u_{\omega l m}(x) = \frac{u_{\omega l m}(r)}{\sqrt{r^2 + a^2}} s_{\omega l m}(\theta)e^{-i\omega t + im\varphi}.
\]

Since \( \Omega(r, \theta) \leq \Omega_H \), this norm is positive if \( \tilde{\omega} = \omega - m\Omega_H > 0 \). When \( \tilde{\omega} < 0 \), the norm could be either positive or negative depending on the behavior of the solution in \( r, \theta \). If the norm of \( u_{\omega l m}(x) \) is negative, we can easily see that \( u_{-\omega l m}(x) \) has the positive norm. Let us define a set \( N^- \) consisting of mode solutions \( u_{\omega l m} \) with \( \omega > 0 \) whose norms are negative. Then, after suitable normalizations, these real frequency modes will satisfy the following orthogonality relations:

\[
<u_{\omega l m}, u_{\omega' l m'}> = \delta(\omega - \omega') \delta_{ll'} \delta_{mm'}, \quad \text{for} \quad u_{\omega l m} \notin N^-,
\]

\[
<u_{-\omega l m}, u_{-\omega' l m'}> = \delta(\omega - \omega') \delta_{ll'} \delta_{mm'}, \quad \text{for} \quad u_{\omega l m} \in N^-.
\]
This set of solutions forms a complete basis of real frequency normal mode solutions having positive norms.

For complex frequency normal mode solutions, the property of inner products is very different. As already mentioned, the norm of

\[ v_{\omega lm}(x) = \frac{v_{\omega lm}(r)}{\sqrt{r^2 + a^2}} S_{\omega lm}(\theta) e^{-i\omega t + im\varphi} \]

is zero. However, the inner product between \( v_{\omega lm}(x) \) and \( v_{\omega^* lm}(x) \) is nonzero

\[ <v_{\omega lm}, v_{\omega^* lm}> = \int (\omega - m\Omega) \left[ \frac{v_{\omega lm}^*(r)}{\sqrt{r^2 + a^2}} S_{\omega lm}^*(\theta) \right]^2 N^{-1} d\Sigma. \]  

After suitably normalizing \( v_{\omega lm}(r) \), we can set it to be unit. Then,

\[ <v_{\omega lm}^*, v_{\omega^* lm}^* > - <v_{\omega lm}, v_{\omega^* lm}>^* = -1. \]  

All other inner products vanish. As mentioned above, by linearly combining \( v_{\omega lm} \) and \( v_{\omega^* lm} \), for example, one can construct a solution whose associated classical energy defined in Eq. (11) is negative.

Finally, it should be pointed out that any normal mode solution with complex frequency can not be expressed by linearly combining the real frequency normal modes \( \{ u_{\omega lm}(x) \} \). It follows because otherwise the complex frequency mode would not have net ingoing or outgoing flux. Therefore, the set of complex frequency normal mode solutions represents new independent degrees of freedom of the system, which can describe field solutions carrying arbitrary values of energy including negative ones by linear combinations.

### 3 Quantization

Based on the analysis of normal mode solutions for the classical scalar field in the previous section, we now proceed the canonical quantization by interpreting the field \( \phi(x) \) as an operator-valued distribution. The neutral scalar field can be expanded in terms of normal mode solutions as follows

\[ \phi(x) = \sum_{l,m} \int_{\mathbb{N}^+} d\omega \frac{1}{\sqrt{2}} [a_\lambda u_\lambda(x) + a^\dagger_\lambda u^*_\lambda(x)] + \sum_{l,m} \int_{\mathbb{N}^-} d\omega \frac{1}{\sqrt{2}} [a_{-\lambda} u_{-\lambda}(x) + a^\dagger_{-\lambda} u^*_{-\lambda}(x)] \]

\[ \sum_{\omega lm} \frac{1}{\sqrt{2}} [b_\lambda v_\lambda(x) + b^\dagger_\lambda v^*_\lambda(x) + b_\lambda v_\lambda(x) + b^\dagger_\lambda v^*_\lambda(x)], \]  

where \( \lambda \) denotes to \( (\omega, l, m) \), \( -\lambda \) to \( (-\omega, l, -m) \), and \( \tilde{\lambda} \) to \( (\omega^*, l, m) \). The expansion coefficients are now operators. We assume the equal-time commutation relations for \( \phi(x) \) and \( \pi(x) \)

\[ [\phi(x) , \pi(y)] = i\delta^{(3)}(x,y) , \quad [\phi(x) , \phi(y)] = [\pi(x) , \pi(y)] = 0 \]  

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at a $x^0 = y^0 = t = \text{const.}$ spacelike hypersurface. $\delta^{(3)}(x, y)$ is the three dimensional Dirac $\delta$ function defined by

$$
\int \delta^{(3)}(x, y)f(y)\sqrt{-g}d^3y = f(x)
$$

at the $x^0 = y^0 = t = \text{const.}$ surface. By using the following relations

$$
a_\lambda = \sqrt{2} < u_\lambda, \phi(x)>, \quad a^\dagger_\lambda = -\sqrt{2} < u_\lambda^*, \phi(x)>,
\quad b_\lambda = \sqrt{2} < v_\lambda, \phi(x)>, \quad b^\dagger_\lambda = -\sqrt{2} < v_\lambda^*, \phi(x)>,
$$

we find commutation relations among mode operators

$$
[a_\lambda, a^\dagger_{\lambda'}] = \delta_{\lambda\lambda'}, \quad [b_\lambda, b^\dagger_{\lambda'}] = \delta_{\lambda\lambda'}, \quad [b_\lambda, b^\dagger_{\lambda'}] = [b_\lambda, b_{\lambda'}] = 0 .
$$

All others vanish. Note that the real frequency mode operators satisfy the usual commutation relations whereas mode operators for complex frequencies have unusual commutation relations as in other models in the Minkowski flat spacetime in Refs. [18, 17, 19, 20]. In particular, $b_\lambda$ does commute with $b^\dagger_\lambda$.

Now the Hamiltonian operator can be expressed in terms of mode operators by using Eq. (3) [24]:

$$
H = \frac{1}{2} \sum_{lm} \int_{\mathbb{R}} d\omega \omega (a^\dagger_\lambda a_\lambda + a_\lambda a^\dagger_\lambda) + \frac{1}{2} \sum_{lm} \int_{\mathbb{R}} d\omega (-\omega) (a^\dagger_{-\lambda} a_{-\lambda} + a_{-\lambda} a^\dagger_{-\lambda})
$$

$$
+ \frac{1}{2} \sum_{\omega lm} [\omega (b^\dagger_\lambda b_\lambda + b_\lambda b^\dagger_\lambda) + \omega^* (b^\dagger_\lambda b_\lambda + b_\lambda b^\dagger_\lambda)],
$$

where $\omega > 0$ for real frequency modes and $\omega^* > 0$ for complex frequency modes. Note first that $H$ is Hermitian, $H^\dagger = H$, as expected. For real frequency modes, let

$$
a_{\pm\lambda} = \frac{1}{\sqrt{2}} (\sqrt{\omega} Q_{\pm \lambda} + i \frac{P_{\pm \lambda}}{\sqrt{\omega}}),
$$

where $Q_{\pm \lambda}$ and $P_{\pm \lambda}$ are Hermitian operators satisfying $[Q_{\pm \lambda}, P_{\pm \lambda}] = i$. Then one can easily see that the Hamiltonian for real frequency modes has a representation of a set of attractive harmonic oscillators as usual. Thus its energy spectrum is discrete. Interestingly, however, the energy associated with the second term in Eq. (44) is always negative and bounded above whereas the first term shows positive and lower bounded energy. Due to this, although a vacuum state can be defined such that $a_{\pm \lambda}|0 >_R = 0$ for all $\lambda$, it is not the state of the lowest energy and in fact there is no such state. All energy eigenstates can be constructed from $|0 >_R$ simply by $(a_{\pm \lambda}^\dagger)^n|0 >_R$. Therefore, real frequency mode operators possess the usual symmetrized Fock representation $H^R$ as well as the particle interpretation.

For complex frequency modes, let

$$
H^C_\lambda = \frac{1}{2} [\omega (b_\lambda b^\dagger_\lambda + b^\dagger_\lambda b_\lambda) + \omega^* (b^\dagger_\lambda b_\lambda + b_\lambda b^\dagger_\lambda)].
$$

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Suppose that it has an energy eigenstate \( |E\rangle \) such that \( H^C_\lambda |E\rangle = E |E\rangle \). Note then that

\[
[H^C_\lambda, b^\dagger_\lambda] = \omega b^\dagger_\lambda, \quad [H^C_\lambda, b_\lambda] = -\omega b_\lambda, \quad [H^C_\lambda, b^\dagger_\lambda] = \omega^* b^\dagger_\lambda, \quad [H^C_\lambda, b^\dagger_\lambda] = -\omega^* b^\dagger_\lambda.
\]

Thus \( b^\dagger_\lambda |E\rangle, b_\lambda |E\rangle \) and \( b^\dagger_\lambda |E\rangle, b_\lambda |E\rangle \) are eigenstates with eigenvalues of \((E \pm \omega)\) and \((E \pm \omega^*)\), respectively, which are no longer real. Since \( H^C_\lambda \) is Hermitian, it presumably implies that energy eigenstates are not normalizable. This property shall be explicitly shown below.

To find the energy spectrum and its representation for \( H^C_\lambda \), let us use some methods developed in Refs. [19, 17, 18]. Let

\[
b_\lambda = \frac{1}{2} \left[ i(\sqrt{\omega} q_{1\lambda} + \frac{1}{\sqrt{\omega}} p_{1\lambda}) + (\sqrt{\omega} q_{2\lambda} + \frac{1}{\sqrt{\omega}} p_{2\lambda}) \right],
\]

\[
b^\dagger_\lambda = \frac{1}{2} \left[ (\sqrt{\omega} q_{1\lambda} - \frac{1}{\sqrt{\omega}} p_{1\lambda}) + i(\sqrt{\omega} q_{2\lambda} - \frac{1}{\sqrt{\omega}} p_{2\lambda}) \right].
\]

Here \( q \) and \( p \) are Hermitian operators satisfying \([q_j \lambda, p_j \lambda] = i, j = 1, 2\). We find then

\[
H^C_\lambda = \frac{1}{2} (p_{1\lambda}^2 - (\omega^1)^2 q_{1\lambda}^2) + \frac{1}{2} (p_{2\lambda}^2 - (\omega^1)^2 q_{2\lambda}^2) + \omega^R (q_{1\lambda} p_{2\lambda} - p_{1\lambda} q_{2\lambda}).
\]

Classical equations of motion corresponding to this Hamiltonian will be

\[
\ddot{q}_1 = |\omega|^2 q_1 - 2\omega^R \dot{q}_2, \quad \ddot{q}_2 = |\omega|^2 q_2 + 2\omega^R \dot{q}_1.
\]

Thus this is a system of two coupled inverted harmonic oscillators with the same frequency \(|\omega|\).

As in Ref. [17], the Hamiltonian operator \( H^C_\lambda \) can be realized as a sum of the infinitesimal generators of dilatations and rotations acting on a function space \( L^2(\mathbb{R}^2) \) in a two-dimensional Euclidean space \( \mathbb{R}^2 \). Energy eigenfunctions are

\[
\psi_{\epsilon k}(\rho, \phi) = (2\pi)^{-1} \rho^{-1} e^{i \epsilon \phi},
\]

where \( \epsilon \) is any continuous real number and \( k \) any integer. \((\rho, \phi)\) are the polar coordinates. These eigenfunctions are orthogonal

\[
\int \psi^*_{\epsilon k}(\rho, \phi) \psi_{\epsilon' k'}(\rho, \phi) \rho d\rho d\phi = \delta(\epsilon - \epsilon') \delta_{kk'},
\]

and form a complete set

\[
\int_{-\infty}^{\infty} \psi^*_{\epsilon k}(\rho, \phi) \psi_{\epsilon k}(\rho, \phi) d\epsilon = \delta^{(2)}(\rho - \rho),
\]

The energy spectrum for \( H^C_\lambda \) is

\[
E_{\epsilon, k_\lambda} = \omega^I \epsilon + \omega^R k_\lambda.
\]

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It shows that the energy eigenvalue is continuous for given $k_\lambda$ and unbounded below. Eq. (52) shows explicitly that energy eigenstates are not normalizable as we expected above. However, we can construct normalizable wave packets from them. These square integrable wave packets, for example, \( |\psi> = \sum_k \int d\varepsilon \lambda <\psi_{\varepsilon, k_\lambda}|\psi> |\psi_{\varepsilon, k_\lambda}> \), will form a Hilbert space \( \mathcal{H}_\lambda^C \) which is isomorphic to \( L^2(\mathbb{R}^2) \).

Any quantum state of the field which is in this Hilbert space will give rise to instability. It follows because, although the total energy of this state is definite and time independent, the energy density outside the ergoregion will be positive and have exponential time dependence whereas the energy density within the ergoregion will have the same behavior but with negative energy, keeping the total energy over the whole space fixed. Therefore, an observer sitting outside the ergoregion will measure time dependent radiation of positive energy. In addition, since the energy spectrum is unbounded below, some external interaction with this system can give energy extraction from the system without bound.

Finally, we complete our quantization of the field \( \phi(x) \) by constructing the total Hilbert space as follows,
\[
\mathcal{H} = \mathcal{H}^R \otimes \prod_\lambda \mathcal{H}_\lambda^C.
\] (55)

Here \( \mathcal{H}^R \) is the usual symmetrized Fock space generated by real frequency modes and \( \prod_\lambda \mathcal{H}_\lambda^C \) is the infinite number of products of Hilbert spaces \( \mathcal{H}_\lambda^C \) generated by complex frequency modes.

### 4 Quantum instability

In this section, let us look at some interesting properties of the Hilbert space \( \mathcal{H} = \mathcal{H}^R \otimes \prod_\lambda \mathcal{H}_\lambda^C \) constructed in the previous section. First of all, one may ask whether or not this Hilbert space still possesses the particle interpretation of the quantum field. For real frequency modes, the energy spectrum is discrete and a vacuum state is defined well in the usual way. \( a_{\pm \lambda}^\dagger \) and \( a_{\pm \lambda} \) can still be interpreted as creation and annihilation operators of energy quanta of \( \pm \hbar \omega \), respectively. Thus real frequency mode operators still have particle interpretation as usual except that the vacuum state defined is not the lowest energy state any more.

For complex frequency modes, however, since the corresponding energy spectrum is continuous, it would not be possible to define some energy quanta whose multiples cover the whole energy spectrum. Therefore mode operators associated with complex frequencies do not have particle interpretation in the usual sense. However, we may expect that there will be the energy quanta of \( \hbar \omega^R \) since the spectrum of the rotation generator \( Q_\lambda = q_{1\lambda} p_{2\lambda} - p_{1\lambda} q_{2\lambda} \) is discrete [26].

Now let us see how the appearance of an ergoregion at the late stage of a dynamically evolving background spacetime starts to give a spontaneous radiation of energy. We expect this spontaneous quantum radiation if the initial vacuum state \( |0>_m \) of the field in the past falls in any state in \( \prod_\lambda \mathcal{H}_\lambda^C \) in the remote future. To see this effect let us consider a “paticle” detector linearly coupled to the field near \( t \sim \infty \) placed in the in-vacuum state.
$|0\rangle_{in}$. The transition probability of the detector is proportional to the response function $F(E)$ \[27\].

$$F(E) = \lim_{t_0 \to \infty} \int_{t_0}^{t_0 + T} dt \int_{t_0}^{t_0 + T} dt' e^{-iE(t-t')} \langle \phi[x(t)]\phi[x(t')] \rangle |0\rangle_{in}. \quad (56)$$

The field operator can also be decomposed as follows

$$\phi(x) = \sum_{lm} \int dk \frac{1}{\sqrt{2}} (c_{\sigma} U_{\sigma}(x) + c_{\sigma}^+ U_{\sigma}^*(x)), \quad (57)$$

where $U_{\sigma}(x) = U_{klm}(x)$ becomes spherical waves in the flat spacetime in the past infinity, and $c_{\sigma}|0\rangle_{in} = 0$ for all $\sigma = (k, l, m)$. Since $\langle in|\phi[x(t)]\phi[x(t')]|in\rangle = \frac{1}{2} \sum_{\sigma} U_{\sigma}(x)U_{\sigma}^*(x')$, we now see

$$F(E) = \lim_{t_0 \to \infty} \frac{1}{2} \sum_{\sigma} | \int_{t_0}^{t_0 + T} dt e^{-iEt} U_{\sigma}(x) |^2. \quad (58)$$

Since $\{U_{\sigma}, U_{\sigma}^*\}$ consists of a complete set, all out normal modes can be expressed by this set. Let the Bogolubov transformations be

$$u_{\lambda} = \sum_{\sigma} (\alpha_{\lambda\sigma} U_{\sigma} + \beta_{\lambda\sigma} U_{\sigma}^*), \quad v_{\lambda} = \sum_{\sigma} (\gamma_{\lambda\sigma} U_{\sigma} + \eta_{\lambda\sigma} U_{\sigma}^*), \quad v_{\lambda} = \sum_{\sigma} (\gamma_{\lambda\sigma} U_{\sigma} + \eta_{\lambda\sigma} U_{\sigma}^*). \quad (59)$$

From orthogonality relations among out modes in Eq. \[55\] and Eq. \[58\], we have

$$\sum_{\sigma} (\alpha_{\lambda\sigma}^* \alpha_{\lambda'\sigma} - \beta_{\lambda\sigma}^* \beta_{\lambda'\sigma}) = \delta_{\lambda\lambda'}, \quad \sum_{\sigma} (\alpha_{\lambda\sigma} \beta_{\lambda'\sigma} - \beta_{\lambda\sigma} \alpha_{\lambda'\sigma}) = 0 \quad (60)$$

for real frequency modes, and

$$\sum_{\sigma} (\gamma_{\lambda\sigma}^* \gamma_{\lambda'\sigma} - \eta_{\lambda\sigma}^* \eta_{\lambda'\sigma}) = \sum_{\sigma} (\gamma_{\lambda\sigma}^* \gamma_{\lambda'\sigma} - \eta_{\lambda\sigma}^* \eta_{\lambda'\sigma}) = 0, \quad \sum_{\sigma} (\gamma_{\lambda\sigma} \gamma_{\lambda'\sigma} - \eta_{\lambda\sigma} \eta_{\lambda'\sigma}) = \delta_{\lambda\lambda'} \quad (61)$$

for complex frequency modes. Note the “converted” relations among Bogolubov coefficients for complex frequency modes which are resulted from the unusual form of the orthogonal relations for such modes. Equivalently, we have from Eq. \[59\]

$$U_{\sigma} = \sum_{lm} \int_{\phi N^-} d\omega (\alpha_{\lambda\sigma}^* u_{\lambda} - \beta_{\lambda\sigma}^* u_{\lambda}) + \sum_{lm} \int_{\phi N^-} d\omega (\alpha_{\lambda\sigma}^* u_{\lambda} - \beta_{\lambda\sigma}^* u_{\lambda})
+ \sum_{\omega\lambda m} (\gamma_{\lambda\sigma}^* v_{\lambda} - \eta_{\lambda\sigma} v_{\lambda} + \gamma_{\lambda\sigma}^* v_{\lambda} - \eta_{\lambda\sigma} v_{\lambda}). \quad (62)$$

The response function $F(E)$ in Eq. \[58\] will depend on the polar angle $\theta$, but be independent on $\varphi$ because of the axial symmetry of the background spacetime. For calculational simplicity, we consider the following response function integrated over $\theta; 0 \sim \pi$,

$$F(E) = 2\pi \int_{0}^{\pi} F(E) \sin \theta d\theta. \quad (63)$$
Now, if the detector is at rest near spatial infinity, we obtain, from the asymptotic behavior of normal modes in Eq. (21) and Eq. (25), the transition rate

\[
\frac{F(E)}{T} \sim \frac{1}{2} \sum_{\sigma} \left( \sum_{lm} \int_{\mathcal{N}^-} d\omega |\beta_{\lambda\sigma}|^2 \left| \frac{u_{\lambda}(r)}{\sqrt{r^2 + a^2}} \right|^2 \delta(E - \omega) \right. \\
+ \sum_{lm} \int_{\mathcal{N}^-} d\omega |\alpha_{-\lambda\sigma}|^2 \left| \frac{u_{-\lambda}(r)}{\sqrt{r^2 + a^2}} \right|^2 \delta(E - \omega) \\
- \sum_{\omega lm} 2 \text{Re}[\gamma_{\lambda\sigma}^*] \gamma_{\lambda\sigma} \left( \frac{v_{\lambda}(r)}{\sqrt{r^2 + a^2}} \right)^2 \frac{e^{-i(E + \omega^R)T}}{(E + \omega)^2} \\
+ \eta_{\lambda\sigma} \eta_{\lambda\sigma}^* \left( \frac{v_{\lambda}(r)}{\sqrt{r^2 + a^2}} \right)^2 \frac{e^{-i(E - \omega^S)T}}{(E - \omega^S)^2} \frac{e^{i\omega T}}{T} \right) \quad (64)
\]

for large \( T \gg 1 \). This result in general shows non-vanishing excitations of the particle detector related to complex frequency modes as well as the usual contributions due to the mode mixing in real frequency modes. In particular, the contributions related to complex frequency modes are not stationary, but exponentially increasing in time \( T \). The \( \delta \)-function dependence in the first two terms implies the energy conservation; that is, only the real frequency mode whose quantum energy is the same as that of the particle detector \( (\omega' = E) \) can excite the detector. For complex frequency modes, however, all modes contribute to the excitation possibly because the energy spectrum for any complex frequency mode is continuous.

The first two terms will in general appear because the background spacetime is evolving in time and so it will effectively give time-dependent potential in Eq. (17). However, in the case that there is no mixing in positive real frequency modes, they will vanish. The last term will vanish in the case that both \( \gamma_{\lambda\sigma}^* \gamma_{\lambda\sigma} \) and \( \eta_{\lambda\sigma} \eta_{\lambda\sigma}^* \) vanish for all \( \lambda \) and \( \sigma \). However, we do not expect this case if there is any instability mode due to the presence of the ergoregion since otherwise the last equality in Eq. (61) cannot be satisfied.

## 5 Discussion

In this paper, we have shown how the canonical quantization for a scalar field can be formulated in the presence of unstable modes due to ergoregion in a certain spacetime model. We found that our quantization has essentially the same features as in other models in Minkowski flat spacetime in Refs. [19, 17, 20]. The Hamiltonian operator for complex frequency modes is equivalent to a system of a set of two coupled inverted harmonic oscillators. Thus the energy spectrum is continuous and unbounded below. Consequently, the corresponding Hilbert space is not a Fock-like representation and has no particle interpretation for such complex frequency mode operators. The “particle” detector placed in the in-vacuum state shows that a rotating star with ergoregion but without horizon has the quantum instability as well, leading exponentially time dependent spontaneous energy radiation to spatial infinity. This
quantum instability is possible because negative energy could be accumulated within the ergoregion. Accordingly, our result resolves the contradiction between conclusions in Ref. [14] and Ref. [12].

Although the spacetime model considered in this paper is physically plausible, it is not so realistic. Thus one may ask whether the essential result obtained here would remain in a more realistic model. In the asymptotic region at spatial infinity, there would be no much difference in the analysis. Near the rotating object and inside it, the forms of normal mode solutions will be quite different from ours. As long as an ergoregion is present, however, there will still be two classes of normal mode solutions, that is, one described by real frequencies and the other by complex frequencies. Similar inner product properties shown in Sec. 2 will still hold for this set of normal mode solutions since they do not depend on the details of the form of solutions. Then the rest of the quantization procedure and the structure of the corresponding total Hilbert space will be same. Therefore, the main results in our model will remain in a more realistic spacetime model as well. As pointed out in Ref. [14], the mirror boundary condition that we assumed to avoid the difficulty of obtaining complicated solutions inside the rotating body in fact mimics effectively the center of the star provided that the modes propagate freely through the interior without much interactions with the star body. In the limiting case that $\tilde{r}_0 \to -\infty$, our spacetime model in the future infinity becomes the rotating Kerr black hole. In this case our results should agree with Hawking’s for the case of collapsing rotating black holes [29]. From eq. (27), we see that the imaginary part of the complex frequency becomes zero as $\tilde{r}_0 \to -\infty$, and, accordingly, only real frequency modes $\{u_{\omega lm}(x)\}$ appear in our model. However, it is as yet unclear whether or not this mode decomposition constructed here in the future infinity gives the same vacuum as Hawking’s [30].

It will be straightforward to extend our formalism to other matter fields such as massive charged scalar and electromagnetic fields. For spinor fields, however, it is unclear at the present whether or not rotating stars with ergoregions spontaneously radiate fermionic energy to infinity as well. It is because spinor fields do not give superradiance in the presence of an ergoregion. Hence we cannot apply the heuristic argument in Sec. 1 for the generation of instability modes. In fact, we find that the inner product defined in Ref. [9] for spinor fields is still positive definite in our spacetime model. Then, as explained below Eq. (10), there exists no complex frequency mode and hence no unstable mode for spinor fields classically. However, as rotating black holes give fermion emissions in the quantum theory inspite of no superradiance at the classical level, there might be some quantum process through which the ergoregion gives fermionic spontaneous energy radiation. In addition, the main result obtained in the algebraic approach [12, 8] does not seem to depend on which matter field is considered.

As shown in preceding sections, the Hamiltonian operators associated with unstable modes do not admit a Fock-like representation, a vacuum state, or the particle interpretation of mode operators. Accordingly, the conventional analysis of the vacuum instability based on the uses of asymptotic vacua and appropriately defined number operators no longer applies to our case. However, we have shown that Unruh’s “particle” detector model, which indeed
does not require the particle interpretation of the field, is still applicable for extracting some useful physics in our case. In fact, our case serves as a good example illustrating the point of view that the fundamental object in quantum field theory is the field operator itself, not the “particles” defined in a preferred Fock space [31]. The expectation value of the energy-momentum tensor operator, which is defined by field operators only, should also be a useful quantity in our case. To obtain meaningful expectation value, however, renormalization of the energy-momentum tensor would have to be understood first in the presence of such instability modes [32]. As far as we know, this interesting issue has never been addressed in the literature.

The Penrose diagram for a realistic spacetime in which a stationary rotating star with an ergosphere develops in the remote future will be as in Fig. 1. The solid line is the trajectory of the surface of a rotating object. The dotted lines denote the boundaries of the ergoregion. Based on the analysis in our paper, exponentially time dependent spontaneous energy radiation will occur as soon as an ergoregion is formed. Then the back reaction of the quantum field on the metric will change the gravitational fields of the evolving rotational object itself, depending on the strength and the time scale of the spontaneous radiation. Since the wave trapped inside the ergoregion carries negative energy and the angular momentum in the opposite sense of the rotation, the rotating object will loose its angular momentum and so the ergoregion can disappear at some point of its evolution. Then the spontaneous radiation will also stops to occur. The corresponding Penrose diagram is shown in Fig. 2.

It will also be very interesting to see how our quantization procedure in this paper can
be translated into algebraic approach. To obtain a quantum description in this approach, one defines first the $\ast$-algebra of the field operators and then constructs the Hilbert space of states by choosing an appropriate $\ast$-representation (equivalently, a suitable complex structure) with a set of rules for dynamics. The most difficult part in this prescription is to single out the “correct” representation among all possible $\ast$-representations. For spacetimes such as static or stationary spacetimes [33], certain physically motivated requirements select a unique complex structure and hence the “correct” representation [8]. For a spacetime where an ergoregion is present, on the other hand, Ashtekar and Magnon [12] argue that there appears to exist no obvious way to choose even a specific complex structure since $\mathcal{I}^-$ is not a Cauchy surface in this case [34]. However, the mode decomposition constructed in Sec. 2 and in Sec. 4 in our canonical quantization procedure suggests that there may exist some way to construct the corresponding complex structure in the algebraic approach as well.

Finally, it should be pointed out that there are many other fields in physics in which complex frequency modes play important roles and so our quantization formalism is potentially applicable. Generically, if a system stores some “free” energy which can be released through interactions, then some amplifications occur, revealing complex frequency modes classically. In a system of plasma, for instance, a small perturbation of electric field exponentially increases in time if the phase velocity of the perturbed field is smaller than the velocity of charged particles, and is damped in the opposite case. The energy stored in plasma is released quickly by a small perturbation, giving complex frequency modes [35]. In a tunable laser, the energy stored in dielectric material amplifies an incident light and results in the intensity increment of the output laser beam. In a field theoretic treatment of the
system, the dielectric material plays the role of a source producing an external potential and it is possible for complex frequency modes to occur under suitable conditions. In the theory of linear quantum amplifiers[36], one assumes a time dependent annihilation operator, $a(t) = a(0)e^{Wt/2 - i\omega t}$ with a gain factor $W$. This gain factor $W$ may be interpreted simply as coming from the imaginary part of a complex frequency mode in the second quantization scheme where one does not need to assume the non-unitary evolution of the mode operator. Therefore, the quantization formalism described at the present work may be useful to understand those phenomena in the context of quantum field theory.

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ZAMO observers are locally nonrotating observers whose trajectories are tangent to
\[ \partial_t = \partial_t + \Omega \partial_\phi \]
where \( \Omega = -\partial_t \cdot \partial_\phi / \partial_\phi \cdot \partial_\phi \) and \( \partial_\phi \) is the rotational Killing vector field of stationary axisymmetric spacetimes. The trajectories of Killing observers are tangent to \( \partial_t \) which becomes spacelike inside ergoregions.

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For the detailed functional form of \( V_{\omega lm}(r) \), see Ref. [7].

An easier way to compute the Hamiltonian operator is to use \( \hat{H} \) defined in Eq. (5);
\[
\hat{H} = \langle \Phi, \hat{H} \Phi \rangle
\]
where \( \Phi \) is now the two-component field operator.

In Ref. [17], the dilatation operator \( \tilde{U}(a) \) is defined as \( \tilde{U}(a) \psi(x,y) = e^{-a} \psi(e^{-a}x,e^{-a}y) \).
Our operators are expressed as:
\[
q_{1\lambda} = (x + i \partial/\partial x)/\sqrt{2\omega^4}, \quad p_{1\lambda} = \sqrt{\omega^4/2}(x - i \partial/\partial x),
\]
\[
q_{2\lambda} = (y + i \partial/\partial y)/\sqrt{2\omega^4}, \quad p_{2\lambda} = \sqrt{\omega^4/2}(y - i \partial/\partial y).
\]
We find then \( H_C^\lambda = -i \omega^4 (\rho \partial/\partial \rho + 1) - i \omega^4 \partial/\partial \phi \). The first term is the infinitesimal generator of dilatations and the second that of rotations.

Note that \( B_\lambda^\dag = b_\lambda^\dag b_\lambda^\dag \) and \( B_\lambda \) behave like creation and annihilation of energy quanta \( 2\hbar \omega^4 \) (not \( \hbar \omega^4 \)), respectively.

N.D. Birrell and P.C.W. Davies, Quantum Fields in Curved Space (Cambridge University Press, Cambridge, England, 1982).

Note, however, that it does not depend on the setting time \( t_0 \) of the detector.

S.W. Hawking, Commun. Math. Phys. 43, 199(1975).

The set \( \{ u_{\omega lm}(x) \} \) may not be equivalent to the mode set constructed in Ref. [29] since \( u_{\omega lm} \) are solutions satisfying the mirror boundary condition. Work in progress.
W.G. Unruh, “Particles and Fields” in *Quantum Mechanics in Curved Space-Time* edited by J. Audretsch and V. de Sabbata (Plenum Press, New York, 1990); R.M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1994).

Work in progress.

By stationary spacetimes here we mean there exists a Killing vector field $\xi^t = \partial_t$ which is timelike *everywhere*.

The reason for this is that any data set on $\mathcal{I}^-$ can make only positive contribution to the field energy and so such a data set can not recover a field with negative total energy which possibly exists in presence of an ergoregion in the future [12]. However, this reasoning would not be valid in our case since the spacetime is dynamically evolving and so is time dependent.

T.H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, New York, 1962).

S. Stenholm, Physica Scripta, T12, 56(1986).