Integrable Kondo impurities in one-dimensional extended Hubbard models

Huan-Qiang Zhou, Xiang-Yu Ge, Jon Links and Mark D. Gould
Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia

Three kinds of integrable Kondo problems in one-dimensional extended Hubbard models are studied by means of the boundary graded quantum inverse scattering method. The boundary K matrices depending on the local moments of the impurities are presented as a nontrivial realization of the graded reflection equation algebras acting in a $(2s_\alpha + 1)$-dimensional impurity Hilbert space. Further, these models are solved using the algebraic Bethe ansatz method and the Bethe ansatz equations are obtained.

PACS numbers: 71.20.Fd, 75.10.Jm, 75.10.Lp

I. INTRODUCTION

The study of integrable models of correlated electrons with open boundary conditions has been the subject of considerable attention [1–9]. Recently it has become apparent that for models on open chains it is possible to obtain integrable impurity boundary conditions as operators which need not be expressed in terms of the (super)symmetry of the bulk model. A very important application of this procedure is in the context of Kondo; i.e. spin impurities in models of correlated electrons. For the case of the supersymmetric $t-J$ model boundary spin-$\frac{1}{2}$ impurities were introduced in [10] and the resulting model solved by means of the co-ordinate Bethe ansatz method. A reformulation of this model in the context of the Quantum Inverse Scattering Method (QISM) was given in [11], demonstrating that the model could be obtained via a family of commuting transfer matrices and thus establishing integrability. Central to this approach is the representations of the reflection equation algebras originally introduced by Sklyanin [12]. Such a solution guarantees that boundary terms may be applied to any model whose bulk integrability is associated with a solution of the Yang-Baxter equation. An interesting observation made in [11] was that the necessary solution of the reflection equation was not regular in the sense that it is not obtained by “dressing”; i.e. it cannot be factorized into a product of local monodromy matrices and a $c$-number matrix.

By utilizing the underlying algebraic structure it was subsequently shown in [13] that a more general classes of integrable $t-J$ models with Kondo impurities exist. These were derived from both $gl(2|1)$ and $gl(3)$ invariant solutions of the Yang-Baxter equation and the solution of the reflection equation was extended to accommodate arbitrary spin $s$ impurities situated on the boundaries. Again, the new solutions of the reflection equation are not regular. Moreover it was also demonstrated in [13] that the algebraic Bethe ansatz is applicable for these models and explicit solutions were given.

Recently, the work of Frahm and Slavnov [14] has provided a representation theoretic explanation for the existence of these non-regular solutions of the reflection equation. In essence, such solutions are obtained by suitable projection onto a subspace of the impurity Hilbert space for a regular solution. A consequence of this projection method is that the remaining (super)symmetry in the new boundary operator on the impurity site corresponds to a subalgebra of the (super)symmetry of the original regular solution. As examples, this was illustrated in [14] for the case of $gl(m)$ impurities coupled to an open $gl(n)$ invariant chain for $m < n$ and a reproduction of the integrable $t-J$ model with Kondo impurities given in [13].

It is immediately evident in view of these results that integrable spin impurities, being characterized by the simplest Lie algebra $su(2)$, can be readily obtained from regular solutions coming from the larger (super)symmetry associated with the model in the bulk. In particular, it is possible to obtain integrable boundary Kondo impurity models associated with the Lie algebra $gl(4)$ and superalgebras $gl(3|1)$ and $gl(2|2)$ which we investigate here. In each case, the bulk Hamiltonian can be expressed in the form of an extended Hubbard model and thus is worthy of investigation in terms of the physical properties that are exhibited. The bulk Hamiltonian associated with the $gl(2|2)$ solution is well known from previous works of Essler et. al. [15]. However the other two cases give rise to bulk Hamiltonians which are apparently new.

In the next section we introduce the three forms of extended Hubbard models with integrable boundary Kondo impurities. Following this we undertake an algebraic Bethe ansatz approach to solve each case. In the last section we conclude with some final remarks.
II. INTEGRABLE NON-C-NUMBER BOUNDARY K-MATRICES AND KONDO IMPURITIES IN ONE-DIMENSIONAL EXTENDED HUBBARD MODELS

Let $c_{j,\sigma}$ and $c_{j,\sigma}^\dagger$ denote fermionic creation and annihilation operators for spin $\sigma$ at site $j$, which satisfy the anticommutation relations $\{c_{j,\sigma}^\dagger, c_{j',\tau}\} = \delta_{ij}\delta_{\sigma\tau}$, where $i, j = 1, 2, \cdots, L$ and $\sigma, \tau = \uparrow, \downarrow$. We consider the following Hamiltonian which describes two impurities coupled to the supersymmetric extended Hubbard open chain of Essler et al. [1],

$$H = -\sum_{j=1}^{L-1} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + \text{H.c.})(1 - n_{j,-\sigma} - n_{j+1,-\sigma})$$

$$-\sum_{j=1}^{L-1} (c_{j,\sigma}^\dagger c_{j+1,\sigma}^\dagger + \text{H.c.}) - 2\sum_{j=1}^{L-1} (S_j \cdot S_{j+1} - \frac{1}{4}n_j n_{j+1})$$

$$+ J_0 S_1 \cdot S_a + V_a n_1 + U_a n_{1\uparrow} n_{1\downarrow} + J_b S_L \cdot S_b + V_b n_L + U_b n_{L\uparrow} n_{L\downarrow},$$

(1)

where $J_\alpha, V_\alpha$ and $U_\alpha(\alpha = a, b)$ are the Kondo coupling constants, the impurity scalar potentials and the boundary Hubbard-like interaction constants, respectively; $S$ is the vector spin operator for the conduction electrons; $S_\alpha(\alpha = a, b)$ are the local moments with spin-$\frac{1}{2}$ located at the left and right ends of the system respectively; $n_{j,\sigma}$ is the number density operator $n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$, $n_j = n_{j,\uparrow} + n_{j,\downarrow}$.

The supersymmetry algebra underlying the bulk Hamiltonian of this model is $gl(2|2)$. It is quite interesting to note that although the introduction of the impurities spoils the supersymmetry, there still remains $u(2) \otimes u(2)$ symmetry in the Hamiltonian [1] whose representation contains the spin and $\eta$-pairing realizations. As a result, one may add some terms like $U \sum_{j=1}^{L-1} n_{j,\uparrow} n_{j,\downarrow} \mu \sum_{j=1}^{L-1} n_j$ and $h \sum_{j=1}^{L-1} (n_j - n_{\mu,j})$ to the Hamiltonian [1], without spoiling the integrability. Below we will establish the quantum integrability of the Hamiltonian [1] for a special choice of the model parameters $J_\alpha, V_\alpha$ and $U_\alpha$

$$J_\alpha = -\frac{2}{c_a(c_a + 2s_a + 1)} V_\alpha = -\frac{c_a^2 + 2c_a s_a - s_a}{c_a(c_a + 2s_a + 1)}, U_\alpha = -\frac{2s_a - 3c_a - c_a(2s_a - 1)}{c_a(c_a + 2s_a + 1)}. \quad (2)$$

This is achieved by showing that it can be derived from the (graded) boundary quantum inverse scattering method [5, 6]. Here we emphasize that a special case of this model, corresponding to $s_a = \frac{1}{2}$, has been studied in [10].

The second choice of couplings which leads to an integrable model is given by

$$H = -\sum_{j=1}^{L-1} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + \text{H.c.})(1 - n_{j,-\sigma} - n_{j+1,-\sigma})$$

$$-\sum_{j=1}^{L-1} (c_{j,\sigma}^\dagger c_{j+1,\sigma}^\dagger + \text{H.c.}) - 2\sum_{j=1}^{L-1} (S_j \cdot S_{j+1} + \frac{3}{4}n_j n_{j+1})$$

$$-2\sum_{j=1}^{L-1} n_{j,\uparrow} n_{j,\downarrow}(n_{j+1,\uparrow} n_{j+1,\downarrow} - n_{j+1,\uparrow} - n_{j,\uparrow} - n_{j,\downarrow})$$

$$+ J_a S_1 \cdot S_a + V_a n_1 + U_a n_{1\uparrow} n_{1\downarrow} + J_b S_L \cdot S_b + V_b n_L + U_b n_{L\uparrow} n_{L\downarrow}.$$ \quad (3)

In this case we can introduce integrable Kondo impurities on the boundary by choosing

$$J_\alpha = \frac{8}{(2c_a + 2s_a + 1)(2c_a - 2s_a - 1)} V_\alpha = -\frac{4c_a^2 + 4c_a - 4s_a(s_a + 1) - 3}{(2c_a + 2s_a + 1)(2c_a - 2s_a - 1)}, U_\alpha = \frac{4c_a^2 + 8c_a - 4s_a(s_a + 1) - 5}{(2c_a + 2s_a + 1)(2c_a - 2s_a - 1)}. \quad (4)$$

A third choice of couplings which leads to an integrable model is

$$H = -\sum_{j=1}^{L-1} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + \text{H.c.})(1 - n_{j,-\sigma} - n_{j+1,-\sigma})$$

$$-\sum_{j=1}^{L-1} (c_{j,\sigma}^\dagger c_{j+1,\sigma}^\dagger + \text{H.c.}) - 2\sum_{j=1}^{L-1} (S_j \cdot S_{j+1} - \frac{1}{4}n_j n_{j+1}) - 2\sum_{j=1}^{L-1} n_{j,\uparrow} n_{j,\downarrow}(n_{j+1,\uparrow} n_{j+1,\downarrow} - n_{j,\uparrow} - n_{j,\downarrow})$$

$$+ J_a S_1 \cdot S_a + V_a n_1 + U_a n_{1\uparrow} n_{1\downarrow} + J_b S_L \cdot S_b + V_b n_L + U_b n_{L\uparrow} n_{L\downarrow}.$$ \quad (5)
where integrable Kondo impurities on the boundary are obtained by the choice
\[
J_\alpha = \frac{8}{(2c_\alpha + 2s_\alpha + 1)(2c_\alpha - 2s_\alpha - 1)},
\]
\[
V_\alpha = \frac{(2c_\alpha^2 - 1)^2 - 4s_\alpha(s_\alpha + 1)}{(2c_\alpha + 2s_\alpha + 1)(2c_\alpha - 2s_\alpha - 1)},
\]
\[
U_\alpha = -\frac{4(c_\alpha^2 - 1)^2 - (2s_\alpha + 1)^2}{(2c_\alpha + 2s_\alpha + 1)(2c_\alpha - 2s_\alpha - 1)}.
\]

Let us recall that the Hamiltonian of the 1D supersymmetric extended Hubbard model with periodic boundary conditions commutes with the transfer matrix, which is the supertrace of the monodromy matrix \( T(u) \)
\[
T(u) = R_{0L}(u) \cdots R_{01}(u).
\]
Here the quantum R-matrix \( R(u) \) comes from the fundamental representation of \( gl(2|2) \) and takes the form
\[
R(u) = \begin{pmatrix}
    u - 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & u - 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & u - 2 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u + 2 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u + 2
\end{pmatrix},
\]

It should be noted that the supertrace is carried out for the auxiliary superspace \( V \). The elements of the supermatrix \( T(u) \) are the generators of an associative superalgebra \( A \) defined by the relations
\[
R_{12}(u_1 - u_2) \cdots T(u_1) \cdots T(u_2) = T(u_2) \cdots T(u_1) R_{12}(u_1 - u_2),
\]
where \( \frac{1}{X} \equiv X \otimes 1, \frac{2}{X} \equiv 1 \otimes X \) for any supermatrix \( X \in End(V) \). For later use, we list some useful properties enjoyed by the R-matrix: (i) Unitarity: \( R_{12}(u)R_{21}(-u) = \rho(u) \) and (ii) Crossing-unitarity: \( R_{12}^{st}(u) = R_{21}^{st}(u) = \tilde{\rho}(u) \) with \( \rho(u), \tilde{\rho}(u) \) being some scalar functions.

In order to describe integrable models on open chains, we introduce two associative superalgebras \( T_- \) and \( T_+ \) defined by the R-matrix \( R(u_1 - u_2) \) and the relations
\[
R_{12}(u_1 - u_2) \frac{1}{T_-} (u_1) R_{21}(u_1 + u_2) \frac{2}{T_-} (u_2) = \frac{2}{T_-} (u_2) R_{12}(u_1 + u_2) \frac{1}{T_-} (u_1) R_{21}(u_1 - u_2),
\]
\[
R_{21}^{st1}(-u_1 + u_2) \frac{1}{T_+} (u_1) \left[ R_{21}^{st1}(u_1 + u_2) \right]^{-1} \frac{1}{T_+} (u_1) R_{21}^{st1}(u_1 + u_2) \frac{1}{T_+} (u_1) R_{21}^{st1}(u_1 - u_2),
\]
respectively. Here the supertransposition \( st_\alpha \) \((\alpha = 1, 2)\) is only carried out in the \( \alpha \)-th factor superspace of \( V \otimes V \), whereas \( ist_\alpha \) denotes the inverse operation of \( st_\alpha \). By modifying Sklyanin’s arguments, one may show that the quantities \( \tau(u) \) given by \( \tau(u) = st(\tau_1(u) \tau_-(u)) \) constitute a commutative family, i.e., \( [\tau(u_1), \tau(u_2)] = 0 \).

One can obtain a class of realizations of the superalgebras \( T_+ \) and \( T_- \) by choosing \( T_\pm(u) \) to be the form
\[
T_- (u) = T_-(u) T_-(u) T_-^{-1}(-u), \quad T_+ (u) = T_+(u) T_+^{st}(u) T_+^{-1}(-u)^{st}
\]
with
\[ T_-(u) = R_{0M}(u) \cdots R_{01}(u), \quad T_+(u) = R_{0L}(u) \cdots R_{0M+1}(u), \quad \tilde{T}_\pm(u) = K_\pm(u), \tag{13} \]

where \( K_\pm(u) \), called boundary K-matrices, are representations of \( T_\pm \) in some representation superspace.

We now solve (10) and (11) for \( K_-(u) \) and \( K_+(u) \). For the quantum \( R \)-matrix \( \hat{R} \), one may check that the matrix \( K_-(u) \) given by

\[
K_-(u) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & A_-(u) & B_-(u) \\
0 & 0 & C_-(u) & D_-(u)
\end{pmatrix},
\tag{14}
\]

where

\[
A_-(u) = \frac{u^2 - 2u - 4c_2^2 - 4c_4(2s_a + 1) + 4uS_a^z}{(u - 2c_4)(u - 2c_a - 4s_a - 2)},
\]
\[
B_-(u) = \frac{4uS_a^-}{(u - 2c_4)(u - 2c_a - 4s_a - 2)},
\]
\[
C_-(u) = \frac{4uS_a^+}{(u - 2c_4)(u - 2c_a - 4s_a - 2)},
\]
\[
D_-(u) = \frac{u^2 - 2u - 4c_2^2 - 4c_4(2s_a + 1) - 4uS_a^z}{(u - 2c_4)(u - 2c_a - 4s_a - 2)},
\tag{15}
\]

satisfies (10). Here \( S^\pm = S^x \pm iS^y \). The matrix \( K_+(u) \) can be obtained from the isomorphism of the superalgebras \( T_- \) and \( T_+ \). Indeed, given a solution \( T_- \) of (11), then \( \tilde{T}_+(u) \) defined by

\[
\tilde{T}_+^d(u) = \tilde{T}_-(-u)
\tag{16}
\]
is a solution of (11). The proof follows from some algebraic computations upon substituting (10) into (11) and making use of the properties of the R-matrix. Therefore, one may choose the boundary matrix \( K_+(u) \) as

\[
K_+(u) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & A_+(u) & B_+(u) \\
0 & 0 & C_+(u) & D_+(u)
\end{pmatrix}
\tag{17}
\]

with

\[
A_+(u) = -\frac{u^2 - 2u - 4c_2^2 - 4c_4(2s_b - 1) + 8s_b + 4uS_b^z}{(u - 2c_b + 2)(u - 2c_b - 4s_b)},
\]
\[
B_+(u) = -\frac{4uS_b^-}{(u - 2c_b + 2)(u - 2c_b - 4s_b)},
\]
\[
C_+(u) = -\frac{4uS_b^+}{(u - 2c_b + 2)(u - 2c_b - 4s_b)},
\]
\[
D_+(u) = -\frac{u^2 - 2u - 4c_2^2 - 4c_4(2s_b - 1) + 8s_b - 4uS_b^z}{(u - 2c_b + 2)(u - 2c_b - 4s_b)},
\tag{18}
\]

Now it can be shown that Hamiltonian (11) is related to the second derivative of the boundary transfer matrix \( \tau(u) \) with respect to the spectral parameter \( u \) at \( u = 0 \) (up to an unimportant additive constant)

\[
H = \frac{\tau''(0)}{4(V + 2W)} = \sum_{j=1}^{L-1} h_{j,j+1} + \frac{1}{2} K' - (0) + \frac{1}{2(V + 2W)} \left[ str_0 \left( \begin{smallmatrix} 0 \\ K_+(0) \end{smallmatrix} \right) G_{L0} \right] + 2 str_0 \left( \begin{smallmatrix} 0 \\ K_+(0) \end{smallmatrix} \right) H^R_{L0} + str_0 \left( \begin{smallmatrix} 0 \\ K_+(0) \end{smallmatrix} \right) \left( H^R_{L0} \right)^2 \right],
\tag{19}
\]

with

\[
h = -\frac{1}{2} \frac{d}{du} PR(u) \]
where \( P \) denotes the graded permutation operator, and the subscript 0 denotes the 4-dimensional auxiliary superspace \( V = C^{2,2} \) with the grading \( P[i] = 0 \) if \( i = 1, 2 \) and \( 1 \) if \( i = 3, 4 \), and

\[
V = \text{str}_0 K^R_+(0), \quad W = \text{str}_0 \left( \begin{array}{c} 0 \\ K^+_+(0) H^R_{+0} \end{array} \right), \\
H^R_{i,j} = P_{i,j} R^R_{i,j}(0), \quad G^R_{i,j} = P_{i,j} R^R_{i,j}(0).
\]

This implies that this model, as with the following two model we will study, admits an infinite number of mutually commuting conserved currents, thus assuring its integrability.

The second choice of integrable couplings results from use of an \( R \)-matrix obtained by imposing \( Z_2 \) grading associated with two bosonic and two fermionic states to the fundamental \( su(4) \) \( R \)-matrix which reads

\[
R(u) = \left( \begin{array}{cccccccccccccccc}
u - 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
u & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & u & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 2 & 0 & 0 & 0 \\
\end{array} \right)
\]

We now solve (11) and (14) for \( K_-(u) \) and \( K_+(u) \). For (21), we find that the matrix \( K_-(u) \) given by

\[
K_-(u) = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & A_-(u) & B_-(u) \\
0 & 0 & C_-(u) & D_-(u) \\
\end{array} \right),
\]

where

\[
A_-(u) = \frac{u^2 - 2u - 4c_a^2 + 4s_a(s_a + 1) + 1 - 4uS^+_a}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]

\[
B_-(u) = \frac{4uS^-_a}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]

\[
C_-(u) = \frac{4uS^+_a}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]

\[
D_-(u) = \frac{u^2 - 2u - 4c_a^2 + 4s_a(s_a + 1) + 1 + 4uS^-_a}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]

satisfies (11). The matrix \( K_+(u) \) can again be obtained from the isomorphism of the superalgebras \( \mathcal{T}_- \) and \( \mathcal{T}_+ \) through

\[
\mathcal{T}^{st}_+(u) = \mathcal{T}_- (u - 4).
\]

Therefore, one choose the boundary matrix \( K_+(u) \) as

\[
K_+(u) = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & A_+(u) & B_+(u) \\
0 & 0 & C_+(u) & D_+(u) \\
\end{array} \right)
\]
with
\[
A_+(u) = \frac{u^2 - 6u - 4c_a^2 - 8c_b + 4s_a(s_b + 1) + 5 - 4(u - 4)S_b^+}{(u + 2c_a - 2s_a - 3)(u + 2c_b + 2s_b - 1)},
\]
\[
B_+(u) = -\frac{4(u - 4)S_b^+}{(u + 2c_a - 2s_a - 3)(u + 2c_b + 2s_b - 1)},
\]
\[
C_+(u) = -\frac{4(u - 4)S_b^+}{(u + 2c_a - 2s_a - 3)(u + 2c_b + 2s_b - 1)},
\]
\[
D_+(u) = \frac{u^2 - 6u - 4c_a^2 - 8c_b + 4s_a(s_b + 1) + 5 + 4(u - 4)S_b^+}{(u + 2c_a - 2s_a - 3)(u + 2c_b + 2s_b - 1)}.
\]

For this example it can be shown that the Hamiltonian (23) is related to the logarithmic derivative of the transfer matrix \( \tau(u) \) with respect to the spectral parameter \( u \) at \( u = 0 \) (up to an additive chemical potential term)
\[
H = \sum_{j=1}^{L-1} h_{j,j+1} + \frac{1}{2} K_-^{-1}(0) + \frac{str_0 K_+(0)}{str_0 K_+(0)} H_{L,0},
\]
with
\[
h = -\frac{1}{2} \frac{d}{du} PR(u)
\]
and subject to the constraints (21).

The third choice of integrable couplings results from use of the \( R \)-matrix obtained by imposing \( \mathbb{Z}_2 \) grading to the fundamental \( gl(3|1) \) \( R \)-matrix which reads
\[
R(u) = \begin{pmatrix}
-u - 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & u & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & u & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
Again we solve (21) and (21) for \( K_-(u) \) and \( K_+(u) \). For (21) we obtain
\[
K_-(u) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & A_-(u) & B_-(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & C_-(u) & D_-(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
where
\[
A_-(u) = -\frac{u^2 - 2u - 4c_a^2 + 4s_a(s_a + 1) + 1 - 4uS_a^+}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]
\[
B_-(u) = \frac{4uS_a^-}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]
\[
C_-(u) = \frac{4uS_a^-}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]
\[
D_-(u) = -\frac{u^2 - 2u - 4c_a^2 + 4s_a(s_a + 1) + 1 + 4uS_a^+}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)}.
\]
and
\[ T^*_u = JT_-(u + 2), \quad J = \text{diag}(1, -1, 1, 1), \]  
(31)
giving
\[ K_+(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & A_+(u) & B_+(u) \\ 0 & 0 & C_+(u) & D_+(u) \end{pmatrix}, \]  
(32)
with
\[ A_+(u) = -\frac{u^2 - 2u - 4e_h^2 + 4s_a(s_b + 1) + 1 - 4(u - 2)S^+_b}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}, \]
\[ B_+(u) = \frac{4(u - 2)S^-_b}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}, \]
\[ C_+(u) = \frac{4(u - 2)S^+_b}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}, \]
\[ D_+(u) = -\frac{u^2 - 2u - 4e_h^2 + 4s_a(s_b + 1) + 1 + 4(u - 2)S^-_b}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}. \]  
(33)
The Hamiltonian (3) is related to the logarithmic derivative of the transfer matrix \( \tau(u) \) with respect to the spectral parameter \( u \) at \( u = 0 \) (up to an additive chemical potential term)
\[ H = \sum_{j=1}^{L-1} h_{j,j+1} + \frac{1}{2} K^\ell_1(0) + \frac{\text{str}_0 K_+(0)H_{L,0}}{\text{str}_0 K_+(0)}, \]  
(34)
with
\[ h = -\frac{1}{2} \frac{d}{du} PR(u). \]
For this case we obtain (3) subject to the constraints (3).

III. THE BETHE ANSATZ SOLUTIONS

Having established the quantum integrability of the models, let us now diagonalize the Hamiltonians by means of the algebraic Bethe ansatz method [12,17]. For the first case (1), introduce the ‘doubled’ monodromy matrix \( U(u) \)
\[ U(u) = T(u)K_-(u)\bar{T}(u) = \begin{pmatrix} A(u) & B_1(u) & B_2(u) & B_3(u) \\ C_1(u) & D_{11}(u) & D_{12}(u) & D_{13}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) & D_{23}(u) \\ C_3(u) & D_{31}(u) & D_{32}(u) & D_{33}(u) \end{pmatrix}, \]  
(35)
where \( \bar{T}(u) = T^{-1}(-u) \). Substituting into the reflection equation (33) we may draw the following commutation relations,
\[ \hat{D}_{bd}(u_1)B_c(u_2) = \frac{(u_1 - u_2 - 2)(u_1 + u_2 - 4)}{(u_1 - u_2)(u_1 + u_2 - 2)} r(u_1 + u_2 - 2) A(u_2) - \frac{2(u_1 - 2)u_2}{(u_1 - u_2)(u_1 + u_2 - 1)(u_2 - 1)} r(2u_1 - 2) \gamma_{gh} B_{gh}(u_1)A(u_2) + \frac{2(u_1 - 2)}{(u_1 - 2)(u_1 + u_2 - 1)} r(2u_1 - 2) \gamma_{cd} B_{cd}(u_1)A(u_2) - \frac{2(u_1 - 2)}{(u_1 - 2)(u_1 + u_2 - 1)} r(2u_1 - 2) \gamma_{cd} B_{cd}(u_1)A(u_2), \]  
(36)
\[ A(u_1)B_\beta(u_2) = \frac{(u_1 - u_2 + 2)(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 - 2)} B_{\beta}(u_2)A(u_1) - \frac{2(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 - 2)} B_{\beta}(u_1)A(u_2) + \frac{2}{u_1 + u_2 - 2} B_\alpha(u_1)(\delta_{\alpha,\beta}(u_2) - \frac{1}{u_2 - 1} \delta_{\alpha,\beta} A(u_2)). \]  
(37)
Here $\mathcal{D}_{bd}(u) = \tilde{\mathcal{D}}_{bd}(u) - \frac{1}{u} \delta_{bd} \mathcal{A}(u)$ and the matrix $r(u)$, which in turn satisfies the quantum Yang-Baxter equation, takes the form,

\[ r_{11}^{11}(u) = 1, \quad r_{22}^{12}(u) = r_{33}^{33}(u) = -\frac{u + 2}{u - 2}, \]
\[ r_{12}^{12}(u) = r_{13}^{13}(u) = r_{21}^{21}(u) = r_{31}^{31}(u) = r_{23}^{23}(u) = r_{32}^{32}(u) = -\frac{2}{u - 2}. \]

Next choose Bethe state $|\Omega\rangle$ of the form

\[ |\Omega\rangle = B_{i_1}(u_1) \cdots B_{i_N}(u_N)|0\rangle F^{i_1 \cdots i_N}, \]

with $|0\rangle$ being the pseudovacuum. Acting the transfer matrix $\tau(u)$ on the state $|\Omega\rangle$ we have $\tau(u)|\Omega\rangle = \Lambda(u)|\Omega\rangle$ with the eigenvalue

\[
\Lambda(u) = \frac{u}{u - 1} \frac{(c_b - \frac{u}{2})}{(c_b - \frac{u}{2} - 1)} \prod_{j=1}^{N} \frac{(u + u_j)(u - u_j + 2)}{(u - u_j)(u + u_j - 2)} \left( u - 1 \right) \left( u - 2 \right) \delta(u - u_j)(u - u_j - 2) \Lambda^{(1)}(u; \{u_i\}),
\]

provided the parameters $\{u_j\}$ satisfy

\[
\frac{u_j}{u_j - 2} \frac{(c_b - \frac{u}{2})}{(c_b - \frac{u}{2} - 1)} \prod_{j=1}^{N} \frac{(u + u_j - 2)(u + u_j - 4)}{(u - u_j)(u + u_j - 2)} \Lambda^{(1)}(u; \{u_j\}).
\]

Here $\Lambda^{(1)}(u; \{u_i\})$ is the eigenvalue of the transfer matrix $\tau^{(1)}(u)$ for the reduced problem which arises out of the $r$ matrices from the first term in the right hand side of (39) with the reduced boundary $K$ matrices $K_{\pm}^{(1)}(u)$

\[
K_{\pm}^{(1)}(u) = \begin{pmatrix}
0 & 0 \\
A_{\pm}^{(1)}(u) & B_{\pm}^{(1)}(u) \\
C_{\pm}^{(1)}(u) & D_{\pm}^{(1)}(u)
\end{pmatrix},
\]

where

\[
A_{-}^{(1)}(u) = -\frac{u^2 - 4c_a^2 - 8s_a c_a + 4s_a + 4(u - 1)S_a^2}{(u - 2c_a)(u - 2c_a - 4s_a - 2)},
\]
\[
B_{-}^{(1)}(u) = -\frac{4(u - 1)S_a^2}{(u - 2c_a)(u - 2c_a - 4s_a - 2)},
\]
\[
C_{-}^{(1)}(u) = -\frac{4(u - 1)S_a^2}{(u - 2c_a)(u - 2c_a - 4s_a - 2)},
\]
\[
D_{-}^{(1)}(u) = -\frac{u^2 - 4c_a^2 - 8s_a c_a + 4s_a - 4(u - 1)S_a^2}{(u - 2c_a)(u - 2c_a - 4s_a - 2)},
\]

and

\[
K_{+}^{(1)}(u) = \begin{pmatrix}
0 & 0 \\
A_{+}^{(1)}(u) & B_{+}^{(1)}(u) \\
C_{+}^{(1)}(u) & D_{+}^{(1)}(u)
\end{pmatrix},
\]

where
\[
A_+^{(1)}(u) = - \frac{u^2 - 2u - 4c_b^2 - 4c_b(2s_b - 1) + 8s_b + 4uS_b^+}{(u - 2c_b + 2)(u - 2c_b - 4s_b)}, \\
B_+^{(1)}(u) = - \frac{4uS_b^-}{(u - 2c_b + 2)(u - 2c_b - 4s_b)}, \\
C_+^{(1)}(u) = - \frac{4uS_b^+}{(u - 2c_b + 2)(u - 2c_b - 4s_b)}, \\
D_+^{(1)}(u) = - \frac{u^2 - 2u - 4c_b^2 - 4c_b(2s_b - 1) + 8s_b - 4uS_b^-}{(u - 2c_b + 2)(u - 2c_b - 4s_b)}.
\]

Here \( K_-^{(1)}(u) \), the boundary \( K \) matrix after the first nesting, follows from the relation
\[
\mathcal{D}_{dd}(u)|\Psi\rangle \equiv \frac{u}{u - 1} R_{dd}^{(1)}(u)|\Psi\rangle = \left(K_-^{(1)}(u)_{dd} + \frac{1}{u - 1}\right) \left(\frac{u}{u - 2}\right)^{2L} |\Psi\rangle,
\]
\[
\mathcal{D}_{ab}(u)|\Psi\rangle \equiv \frac{u}{u - 1} R_{ab}^{(1)}(u)|\Psi\rangle = \left(K_-^{(1)}(u)_{ab}\right) \left(\frac{u}{u - 2}\right)^{2L} |\Psi\rangle.
\]

Indeed, applying the monodromy matrix \( T(u) \) and its “adjoint” \( \hat{T}(u) \) to the pseudovacuum, we have
\[
T_{11}(u)|\Psi\rangle = |\Psi\rangle, \quad T_{dd}(u)|\Psi\rangle = \left(\frac{u}{u - 2}\right)^L |\Psi\rangle, \\
T_{1d}(u)|\Psi\rangle \neq 0, \quad T_{dd}(u)|\Psi\rangle = 0, \quad T_{d1}(u)|\Psi\rangle = 0, \\
\hat{T}_{11}(u)|\Psi\rangle = |\Psi\rangle, \quad \hat{T}_{dd}(u)|\Psi\rangle = \left(\frac{u}{u - 2}\right)^L |\Psi\rangle, \\
\hat{T}_{1d}(u)|\Psi\rangle \neq 0, \quad \hat{T}_{dd}(u)|\Psi\rangle = 0, \quad \hat{T}_{d1}(u)|\Psi\rangle = 0.
\]

where \( d \neq b, \quad d, b = 2, 3, 4 \). Then we have
\[
\mathcal{A}(u)|\Psi\rangle = |\Psi\rangle, \\
\mathcal{B}_d(u)|\Psi\rangle \neq 0, \quad \mathcal{C}_d(u)|\Psi\rangle = 0, \\
\mathcal{D}_{dd}(u)|\Psi\rangle = \left(\frac{u}{u - 2}\right)^{2L} K_-^{(1)}(u)_{dd}|\Psi\rangle, \\
\mathcal{D}_{dd}(u)|\Psi\rangle = \left(\frac{u}{u - 2}\right)^{2L} (K_-^{(1)}(u)_{dd} + \frac{1}{u - 1}) |\Psi\rangle - \frac{1}{u - 1} |\Psi\rangle.
\]

\[
(u - 1)T_{21}(u)\bar{T}_{12}(u) - T_{22}(u)\bar{T}_{22}(u) - T_{23}(u)\bar{T}_{32}(u) - T_{24}(u)\bar{T}_{42}(u) \\
= -\bar{T}_{11}(u)T_{11}(u) + (u - 1)\bar{T}_{12}(u)T_{21}(u) - \bar{T}_{13}(u)T_{31}(u) - \bar{T}_{14}(u)T_{41}(u), \\
(u - 1)T_{21}(u)\bar{T}_{13}(u) - T_{22}(u)\bar{T}_{23}(u) - T_{23}(u)\bar{T}_{33}(u) - T_{24}(u)\bar{T}_{43}(u) = u\bar{T}_{13}(u)T_{31}(u), \\
(u - 1)T_{21}(u)\bar{T}_{14}(u) - T_{22}(u)\bar{T}_{24}(u) - T_{23}(u)\bar{T}_{34}(u) - T_{24}(u)\bar{T}_{44}(u) = u\bar{T}_{14}(u)T_{41}(u), \\
T_{31}(u)\bar{T}_{12}(u) - (u - 1)T_{32}(u)\bar{T}_{22}(u) + T_{33}(u)\bar{T}_{32}(u) + T_{34}(u)\bar{T}_{42}(u) = -u\bar{T}_{22}(u)T_{32}(u), \\
T_{31}(u)\bar{T}_{13}(u) - (u - 1)T_{32}(u)\bar{T}_{23}(u) + T_{33}(u)\bar{T}_{33}(u) + T_{34}(u)\bar{T}_{43}(u) = -u\bar{T}_{23}(u)T_{32}(u), \\
T_{31}(u)\bar{T}_{14}(u) - (u - 1)T_{32}(u)\bar{T}_{24}(u) + T_{33}(u)\bar{T}_{34}(u) + T_{34}(u)\bar{T}_{44}(u) = -u\bar{T}_{24}(u)T_{32}(u).
\]

which come from a variant of the (graded) Yang-Baxter algebra \( \mathcal{B} \) with the \( R \) matrix \( \mathcal{S} \),

\[
\frac{1}{T}(u)R(2u)\frac{2}{T}(u) = \frac{2}{T}(u)R(2u)\frac{1}{T}(u).
\]

Noticing the change \( u \rightarrow u - 1 \) with respect to the original problem, one may check that these boundary \( K \) matrices satisfy the reflection equations for the reduced problem. After some algebra the reduced transfer matrix \( \tau^{(1)}(u) \) may
be recognized as that for the inhomogeneous supersymmetric $t-J$ open chain interacting with the Kondo impurities of arbitrary spins, which has been diagonalized in Ref. [13]. The final result is

$$\Lambda^{(1)}(u; \{u_j\}) = \frac{u}{u - 2} \frac{(c_{\beta} - \frac{v}{2})(c_{\beta} - \frac{u}{2} + 2s_{\beta} + 1)}{(c_{\beta} - \frac{u}{2} - 1)} \prod_{\alpha=1}^{M_1} \frac{(u - v_\alpha + 2)(u + v_\alpha - 2)}{(u - v_\alpha)(u + v_\alpha - 4)} \Lambda^{(2)}(u; \{v_\alpha\}),$$

provided the parameters $\{v_m\}$ satisfy

$$\frac{v_\alpha}{v_\alpha - 1} \frac{(c_{\beta} - \frac{v}{2})(c_{\beta} - \frac{u}{2} + 2s_{\beta} + 1)}{(c_{\beta} - \frac{u}{2} - 1)} \prod_{j=1}^{N} \frac{(v_\alpha - u_j - 2)(v_\alpha + u_j - 4)}{(v_\alpha - u_j)(v_\alpha + u_j - 2)} = -\Lambda^{(2)}(v_\alpha; \{u_1\}, \{v_\beta\}).$$

Here $\Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\})$ is the eigenvalue of the transfer matrix $\tau^{(2)}(u)$ for the $M_2$-site inhomogeneous $XXX$ open chain interacting with the Kondo impurities of arbitrary spins,

$$\Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\}) = -\frac{(c_{\beta} - \frac{v}{2})(c_{\beta} - \frac{u}{2} + 2s_{\beta} + 1)}{(c_{\beta} - \frac{u}{2} - 1)} \prod_{\alpha, \beta=1}^{M_2} \frac{c_{\gamma} + \frac{v}{2} + 2s_{\gamma} - 1}{c_{\gamma} - \frac{v}{2} - 2s_{\gamma} + 1} \frac{u - 1}{u - w_{\beta} - 1} \mu_{\beta=1}^{M_2} \frac{(u - w_{\beta} - 3)(u + w_{\beta} - 3)}{(u - w_{\beta} - 1)(u + w_{\beta} - 1)} + \frac{u - 2}{u - 1} \prod_{\gamma=1}^{M_2} \frac{c_{\gamma} + \frac{v}{2} - 1}{c_{\gamma} - \frac{v}{2} - 2s_{\gamma} + 1} \frac{(c_{\gamma} - \frac{v}{2} + 2s_{\gamma})}{(c_{\gamma} - \frac{v}{2} - 2s_{\gamma} - 1)} \prod_{\alpha=1}^{M_2} \frac{(u - v_\alpha)(u + v_\alpha - 4)}{(u - v_\alpha + 2)(u + v_\alpha - 2)} \prod_{\beta=1}^{M_2} \frac{(u - w_{\beta} + 1)(u + w_{\beta} + 1)}{(u - w_{\beta} - 1)(u + w_{\beta} - 1)},$$

provided the parameters $\{w_\alpha\}$ satisfy

$$\prod_{\gamma=1}^{M_2} \frac{(c_{\gamma} - \frac{w}{2} - \frac{1}{2})(c_{\gamma} - \frac{w}{2} + 2s_{\gamma} - \frac{1}{2})}{(c_{\gamma} - \frac{w}{2} - \frac{1}{2})} \prod_{\alpha=1}^{M_2} \frac{(w_{\beta} - v_\alpha + 1)(w_{\beta} + v_\alpha - 3)}{(w_{\beta} - w_{\beta} + 1)(w_{\beta} + w_{\beta} - 1)} = \prod_{\beta=1}^{M_2} \frac{(w_{\beta} - w_{\beta} - 2)(w_{\beta} + w_{\beta} - 2)}{(w_{\beta} - w_{\beta} + 2)(w_{\beta} + w_{\beta} + 2)}.$$

After a shift of the parameters $u_j \rightarrow u_j + 1, v_m \rightarrow v_m + 2$, the Bethe ansatz equations (51), (52) and (54) may be rewritten as follows

$$\frac{(u_j - 1)}{u_j} \prod_{i=1}^{N} \frac{(u_j - u_i + 2)(u_j + u_i + 2)}{(u_j - u_i)(u_j + u_i - 2)} = \prod_{\alpha=1}^{M_1} \frac{(u_j - v_\alpha + 1)(u_j + v_\alpha + 1)}{(u_j - v_\alpha - 1)(u_j + v_\alpha - 1)};$$

$$\prod_{\gamma=1}^{M_2} \frac{(c_{\gamma} - \frac{w}{2} - \frac{1}{2})(c_{\gamma} - \frac{w}{2} + 2s_{\gamma} - \frac{1}{2})}{(c_{\gamma} - \frac{w}{2} - \frac{1}{2})} \prod_{\alpha=1}^{M_2} \frac{(w_{\beta} - v_\alpha + 1)(w_{\beta} + v_\alpha - 1)}{(w_{\beta} - w_{\beta} + 1)(w_{\beta} + w_{\beta} - 1)} = \prod_{\beta=1}^{M_2} \frac{(w_{\beta} - w_{\beta} - 2)(w_{\beta} + w_{\beta} - 2)}{(w_{\beta} - w_{\beta} + 2)(w_{\beta} + w_{\beta} + 2)}.$$

with the corresponding energy eigenvalue $E$ of the model

$$E = -\sum_{j=1}^{N} \frac{4}{u_j^2 - 1}.$$

We now perform the algebraic Bethe ansatz method [12,17] procedure for the second couplings (4). Introducing the ‘doubled’ monodromy matrix $U(u)$,

$$U(u) = T(u)K_{-}(u)T(u) = \begin{pmatrix} A(u) & B_1(u) & B_2(u) & B_3(u) \\ C_1(u) & D_{11}(u) & D_{12}(u) & D_{13}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) & D_{23}(u) \\ C_3(u) & D_{31}(u) & D_{32}(u) & D_{33}(u) \end{pmatrix},$$

(57)
Choosing the Bethe state provided the parameters, where £\tilde{t}\$ions, takes the form,

\[ \Lambda^{(1)}(u) = \prod_{j=1}^{N} \frac{(u - u_j)(u + u_j - 2)}{(u - u_j)(u + u_j - 2)} \Lambda^{(1)}(u; \{u_i\}), \]

provided the parameters \{u_j\} satisfy

\[ \frac{u_j - 4 (c_b + \frac{u_j}{2} + s_b - \frac{1}{2}) (c_b + \frac{u_j}{2} - s_b - \frac{1}{2}) (c_b + \frac{u_j}{2} + s_b - \frac{1}{2}) (c_b + \frac{u_j}{2} - s_b - \frac{1}{2})}{u_j - 2 (c_b + \frac{u_j}{2} + s_b - \frac{1}{2}) (c_b + \frac{u_j}{2} - s_b - \frac{1}{2})} \Lambda^{(1)}(u; \{u_i\}). \]

Here \( \Lambda^{(1)}(u; \{u_i\}) \) is the eigenvalue of the transfer matrix \( \tau^{(1)}(u) \) for the reduced problem, which arises out of the \( r \) matrices from the first term in the right hand side of (58), with the reduced boundary \( K \) matrices \( K^{(1)}_{\pm}(u) \) as,

\[
K^{(1)}_{\pm}(u) = \begin{pmatrix}
1 & 0 & 0 \\
0 & A^{(1)}_{\pm}(u) & B^{(1)}_{\pm}(u) \\
0 & C^{(1)}_{\pm}(u) & D^{(1)}_{\pm}(u)
\end{pmatrix},
\]

where

\[
A^{(1)}_{\pm}(u) = \frac{-u^2 - 4c_a^2 - 4c_a + 4s_a(s_a + 1) + 3 - 4(u - 1)S_a^z}{(u + 2c_a - 2s_a + 1)(u + 2c_a + 2s_a + 1)},
\]

\[
B^{(1)}_{\pm}(u) = \frac{4(u - 1)S_a^z}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]

\[
C^{(1)}_{\pm}(u) = \frac{4(u - 1)S_a^z}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]

\[
D^{(1)}_{\pm}(u) = \frac{-u^2 - 4c_a^2 - 4c_a + 4s_a(s_a + 1) + 3 + 4(u - 1)S_a^z}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)}.
\]
and

\[
K_+^{(1)}(u) = \begin{pmatrix}
1 & 0 & 0 \\
0 & A_+^{(1)}(u) & B_+^{(1)}(u) \\
0 & C_+^{(1)}(u) & D_+^{(1)}(u)
\end{pmatrix},
\]

where

\[
A_+^{(1)}(u) = \frac{u^2 - 6u - 4c_5^2 - 8c_6 + 4s_6(u + 1) + 5 - 4(u - 4)S_5^+}{(u + 2c_6 - 2s_6 - 3)(u + 2c_6 + 2s_6 - 1)},
\]

\[
B_+^{(1)}(u) = -\frac{4(u - 4)S_6^-}{(u + 2c_6 - 2s_6 - 3)(u + 2c_6 + 2s_6 - 1)},
\]

\[
C_+^{(1)}(u) = -\frac{4(u - 4)S_6^+}{(u + 2c_6 - 2s_6 - 3)(u + 2c_6 + 2s_6 - 1)},
\]

\[
D_+^{(1)}(u) = \frac{u^2 - 6u - 4c_5^2 - 8c_6 + 4s_6(u + 1) + 5 + 4(u - 4)S_5^+}{(u + 2c_6 - 2s_6 - 3)(u + 2c_6 + 2s_6 - 1)}.
\]

Here $K_-^{(1)}(u)$, the boundary $K$ matrices after the first nesting, follows from the relations,

\[
\tilde{D}_{dd}(u)|\Psi\rangle = \frac{u}{u - 1}K_{dd}^{(1)}(u)|\Psi\rangle = (K_{-}(u))_{dd} + \frac{u}{u - 1}\frac{u}{u - 2}2L|\Psi\rangle,
\]

\[
\tilde{D}_{ab}(u)|\Psi\rangle = \frac{u}{u - 1}K_{ab}^{(1)}(u)|\Psi\rangle = K_{-}(u)_{ab}\frac{u}{u - 2}2L|\Psi\rangle.
\]

Indeed, applying the monodromy matrix $T(u)$ and its “adjoint” $\tilde{T}(u)$ to the pseudovacuum, we have

\[
T_{11}(u)|\Psi\rangle = |\Psi\rangle, \quad T_{dd}(u)|\Psi\rangle = \left(\frac{u}{u - 2}\right)^L|\Psi\rangle,
\]

\[
T_{1d}(u)|\Psi\rangle \neq 0, \quad T_{ab}(u)|\Psi\rangle = 0, \quad T_{dd}(u)|\Psi\rangle = 0,
\]

\[
\tilde{T}_{11}(u)|\Psi\rangle = |\Psi\rangle, \quad \tilde{T}_{dd}(u)|\Psi\rangle = \left(\frac{u}{u - 2}\right)^L|\Psi\rangle,
\]

\[
\tilde{T}_{1d}(u)|\Psi\rangle \neq 0, \quad \tilde{T}_{ab}(u)|\Psi\rangle = 0, \quad \tilde{T}_{dd}(u)|\Psi\rangle = 0.
\]

where $d \neq b, \ d, b = 2, 3, 4$. Then we have

\[
A_d(u)|\Psi\rangle = |\Psi\rangle,
\]

\[
B_d(u)|\Psi\rangle \neq 0, \quad C_d(u)|\Psi\rangle = 0,
\]

\[
D_{ab}(u)|\Psi\rangle = \left(\frac{u}{u - 2}\right)^{2L}K_{-}(u)_{ab}|\Psi\rangle,
\]

\[
D_{dd}(u)|\Psi\rangle = \left(\frac{u}{u - 2}\right)^{2L}(K_{-}(u))_{dd} + \frac{1}{u - 1}|\Psi\rangle - \frac{1}{u - 1}|\Psi\rangle.
\]

\[
(u - 1)T_{21}(u)\tilde{T}_{12}(u) - T_{22}(u)\tilde{T}_{22}(u) - T_{23}(u)\tilde{T}_{32}(u) - T_{24}(u)\tilde{T}_{42}(u) = -\tilde{T}_{11}(u)T_{11}(u) + (u - 1)\tilde{T}_{12}(u)T_{21}(u) - \tilde{T}_{13}(u)T_{31}(u) - \tilde{T}_{14}(u)T_{41}(u),
\]

\[
(u - 1)T_{21}(u)\tilde{T}_{13}(u) - T_{22}(u)\tilde{T}_{23}(u) - T_{23}(u)\tilde{T}_{33}(u) - T_{24}(u)\tilde{T}_{43}(u) = u\tilde{T}_{13}(u)T_{31}(u)
\]

\[
(u - 1)T_{21}(u)\tilde{T}_{14}(u) - T_{22}(u)\tilde{T}_{24}(u) - T_{23}(u)\tilde{T}_{34}(u) - T_{24}(u)\tilde{T}_{44}(u) = u\tilde{T}_{14}(u)T_{41}(u)
\]

\[
T_{31}(u)\tilde{T}_{12}(u) - (u - 1)T_{32}(u)\tilde{T}_{22}(u) + T_{33}(u)\tilde{T}_{32}(u) + T_{34}(u)\tilde{T}_{42}(u) = -u\tilde{T}_{22}(u)T_{32}(u)
\]

\[
T_{31}(u)\tilde{T}_{13}(u) - (u - 1)T_{32}(u)\tilde{T}_{23}(u) + T_{33}(u)\tilde{T}_{33}(u) + T_{34}(u)\tilde{T}_{43}(u) = -u\tilde{T}_{23}(u)T_{32}(u)
\]

\[
T_{31}(u)\tilde{T}_{14}(u) - (u - 1)T_{32}(u)\tilde{T}_{24}(u) + T_{33}(u)\tilde{T}_{34}(u) + T_{34}(u)\tilde{T}_{44}(u) = -u\tilde{T}_{24}(u)T_{32}(u)
\]

\[
T_{31}(u)\tilde{T}_{13}(u) + T_{32}(u)\tilde{T}_{23}(u) - (u - 1)T_{33}(u)\tilde{T}_{33}(u) + T_{34}(u)\tilde{T}_{43}(u) = -u\tilde{T}_{23}(u)T_{32}(u)
\]

\[
T_{31}(u)\tilde{T}_{14}(u) + T_{32}(u)\tilde{T}_{24}(u) - (u - 1)T_{33}(u)\tilde{T}_{34}(u) + T_{34}(u)\tilde{T}_{44}(u) = -u\tilde{T}_{24}(u)T_{32}(u)
\]

\[
(u - 1)\tilde{T}_{21}(u)\tilde{T}_{12}(u) - T_{22}(u)\tilde{T}_{22}(u) - T_{23}(u)\tilde{T}_{23}(u) - T_{24}(u)\tilde{T}_{24}(u) = \tilde{T}_{21}(u)T_{11}(u) + (u - 1)\tilde{T}_{22}(u)T_{21}(u) - \tilde{T}_{23}(u)T_{31}(u) - \tilde{T}_{24}(u)T_{41}(u).
\]

\[
(u - 1)\tilde{T}_{21}(u)\tilde{T}_{13}(u) - T_{22}(u)\tilde{T}_{23}(u) - T_{23}(u)\tilde{T}_{24}(u) - T_{24}(u)\tilde{T}_{24}(u) = u\tilde{T}_{23}(u)T_{31}(u)
\]

\[
(u - 1)\tilde{T}_{21}(u)\tilde{T}_{14}(u) - T_{22}(u)\tilde{T}_{24}(u) - T_{23}(u)\tilde{T}_{24}(u) - T_{24}(u)\tilde{T}_{24}(u) = u\tilde{T}_{24}(u)T_{41}(u)
\]

\[
T_{31}(u)\tilde{T}_{12}(u) - (u - 1)T_{32}(u)\tilde{T}_{22}(u) + T_{33}(u)\tilde{T}_{32}(u) + T_{34}(u)\tilde{T}_{42}(u) = -u\tilde{T}_{22}(u)T_{32}(u)
\]

\[
T_{31}(u)\tilde{T}_{13}(u) - (u - 1)T_{32}(u)\tilde{T}_{23}(u) + T_{33}(u)\tilde{T}_{33}(u) + T_{34}(u)\tilde{T}_{43}(u) = -u\tilde{T}_{23}(u)T_{32}(u)
\]

\[
T_{31}(u)\tilde{T}_{14}(u) - (u - 1)T_{32}(u)\tilde{T}_{24}(u) + T_{33}(u)\tilde{T}_{34}(u) + T_{34}(u)\tilde{T}_{44}(u) = -u\tilde{T}_{24}(u)T_{32}(u)
\]

\[
T_{31}(u)\tilde{T}_{13}(u) + T_{32}(u)\tilde{T}_{23}(u) - (u - 1)T_{33}(u)\tilde{T}_{33}(u) + T_{34}(u)\tilde{T}_{43}(u) = -u\tilde{T}_{23}(u)T_{32}(u)
\]

\[
T_{31}(u)\tilde{T}_{14}(u) + T_{32}(u)\tilde{T}_{24}(u) - (u - 1)T_{33}(u)\tilde{T}_{34}(u) + T_{34}(u)\tilde{T}_{44}(u) = -u\tilde{T}_{24}(u)T_{32}(u).
\]
which come from a variant of the (graded) Yang-Baxter algebra \([1]\) with the \(R\) matrix \([2]\).

\[
\frac{1}{T} (u) R(2u) \frac{2}{T} (u) = \frac{2}{T} (u) R(2u) \frac{1}{T} (u).
\]  \(72\)

Noticing the change \(u \to u - 1\) with respect to the original problem, one may check that these boundary \(K\) matrices satisfy the reflection equations for the reduced problem. After some algebra, the reduced transfer matrix \(\tau^{(1)}(u)\) may be recognized as that for the inhomogeneous \(su(3)\) \(t-J\) open chain interacting with the Kondo impurities of arbitrary spins, which has been diagonalized in Ref. \([3]\). The final result is,

\[
\Lambda^{(1)}(u; \{u_j\}) = \frac{u - 4 (c_b + \frac{u}{2} + s_b + \frac{1}{2}) (c_b + \frac{u}{2} - s_b - \frac{1}{2})}{u - 2 (c_b + \frac{u}{2} + s_b - \frac{1}{2}) (c_b + \frac{u}{2} - s_b + \frac{1}{2})} \prod_{\alpha=1}^{M_1} \frac{(u - v_\alpha + 2)(u + v_\alpha - 2)}{(u - v_\alpha)(u + v_\alpha - 4)} - \frac{u - 1}{u - 2} \prod_{j=1}^{N} \frac{(u - u_j)(u + u_j - 2)(u + u_j - 4)}{(u - u_j)(u + u_j - 4)} \Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\})
\]  \(73\)

provided the parameters \(\{v_m\}\) satisfy

\[
\frac{v_\alpha - 4 (c_b + \frac{u}{2} + s_b + \frac{1}{2}) (c_b + \frac{u}{2} - s_b - \frac{1}{2})}{v_\alpha - 3 (c_b + \frac{u}{2} + s_b - \frac{1}{2}) (c_b + \frac{u}{2} - s_b + \frac{1}{2})} \prod_{j=1}^{N} \frac{(v_\alpha - u_j - 2)(v_\alpha + u_j - 4)}{(v_\alpha - u_j)(v_\alpha + u_j - 2)} \prod_{c=1, c \neq \alpha}^{M_1} \frac{(v_\alpha - v_\zeta + 2)(v_\alpha + v_\zeta - 2)}{(v_\alpha - v_\zeta - 2)(v_\alpha + v_\zeta - 6)}
\]  \(74\)

Here \(\Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\})\) is the eigenvalue of the transfer matrix \(\tau^{(2)}(u)\) for the \(M_2\)-site inhomogeneous XXX open chain interacting with the Kondo impurities of arbitrary spins,

\[
\Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\}) = - \frac{(c_b + \frac{u}{2} + s_b + \frac{1}{2}) (c_b + \frac{u}{2} - s_b - \frac{1}{2})}{(c_b + \frac{u}{2} + s_b - \frac{1}{2}) (c_b + \frac{u}{2} - s_b + \frac{1}{2})} \prod_{j=1}^{M_2} \frac{(u - w_\beta + 2)(u + w_\beta - 4)}{(u - w_\beta)(u + w_\beta - 6)} + \frac{u - 2}{u - 3} \prod_{c=1, c \neq \beta}^{M_1} \frac{(c_\gamma + \frac{u}{2} + s_\gamma + \frac{1}{2})(c_\gamma - \frac{u}{2} - s_\gamma + \frac{1}{2})}{(c_\gamma + \frac{u}{2} - s_\gamma - \frac{1}{2})(c_\gamma - \frac{u}{2} + s_\gamma + \frac{1}{2})} \frac{(u - v_\alpha)(u + v_\alpha - 4)}{(u - v_\alpha)(u + v_\alpha - 4)} \prod_{j=1}^{M_2} \frac{(u - w_\beta - 2)(u + w_\beta - 8)}{(u - w_\beta)(u + w_\beta - 6)}
\]  \(75\)

provided the parameters \(\{w_\beta\}\) satisfy

\[
\prod_{\gamma=a, b} \frac{(c_\gamma + \frac{u}{2} + s_\gamma + \frac{1}{2})(c_\gamma - \frac{u}{2} - s_\gamma - \frac{1}{2})}{(c_\gamma - \frac{u}{2} + s_\gamma - \frac{1}{2})(c_\gamma + \frac{u}{2} - s_\gamma + \frac{1}{2})} \prod_{j=1}^{N} \frac{(v_\alpha - u_j - 2)(v_\alpha + u_j - 4)}{(v_\alpha - u_j)(v_\alpha + u_j - 2)} \prod_{\beta=1}^{M_2} \frac{(w_\beta - w_\delta + 2)(w_\beta + w_\delta - 4)}{(w_\beta - w_\delta - 2)(w_\beta + w_\delta - 8)}
\]  \(76\)

After a shift of the parameters \(u_j \to u_j + 1, v_m \to v_m + 2, w_l \to w_l + 3\), the Bethe ansatz equations \((3), (4)\) and \((7)\) may be rewritten as follows

\[
\prod_{\gamma=a, b} \frac{(c_\gamma + \frac{w_\beta}{2} + s_\gamma - \frac{1}{2})(c_\gamma - \frac{w_\beta}{2} - s_\gamma + \frac{1}{2})}{(c_\gamma - \frac{w_\beta}{2} + s_\gamma + \frac{1}{2})(c_\gamma + \frac{w_\beta}{2} - s_\gamma - \frac{1}{2})} \prod_{j=1}^{N} \frac{(v_\alpha - u_j - 1)(v_\alpha + u_j - 1)}{(v_\alpha - u_j)(v_\alpha + u_j - 1)} \prod_{\beta=1}^{M_2} \frac{(w_\beta - w_\delta + 1)(w_\beta + w_\delta + 1)}{(w_\beta - w_\delta - 1)(w_\beta + w_\delta - 1)}
\]  \(77\)
with the corresponding energy eigenvalue $E$ of the model

$$E = - \sum_{j=1}^{N} \frac{4}{u_j^2 - 1}. \quad (78)$$

We now perform the algebraic Bethe ansatz method \cite{12,13} procedure for the third couplings \cite{10}. Introducing the ‘doubled’ monodromy matrix $U(u)$,

$$U(u) = T(u)K_{-1}(u)\tilde{T}(u) \equiv \begin{pmatrix} A(u) & B_1(u) & B_2(u) & B_3(u) \\ C_1(u) & D_{11}(u) & D_{12}(u) & D_{13}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) & D_{23}(u) \\ C_3(u) & D_{31}(u) & D_{32}(u) & D_{33}(u) \end{pmatrix}. \quad (79)$$

where $\tilde{T}(u) = T^{-1}(-u)$. Substituting into the reflection equation \cite{10}, we may draw the following commutation relations,

$$\begin{align*}
\mathcal{D}_{bd}(u_1)B_c(u_2) &= \frac{(u_1 - u_2 - 2)(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 + 2)}r(u_1 + u_2 + 2)g^b_{\alpha\beta}r(u_1 - u_2)g^c_{\alpha\beta}B_c(u_2)\mathcal{D}_{gi}(u_1) - \frac{2u_1u_2}{u_1 + u_2 + 2}(u_1 + u_2 + 1)A(u_2) + \frac{2u_1}{u_1 + u_2 + 2}r(u_1 + u_2 + 2)g^b_{\alpha\beta}B_g(u_1)\mathcal{D}_{ic}(u_2), \\
A(u_1)B_{\beta}(u_2) &= \frac{(u_1 - u_2 - 2)(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 + 2)}B_{\beta}(u_2)A(u_1) - \frac{2(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 + 2)}B_{\beta}(u_1)A(u_2) - \frac{1}{u_1 - u_2}B_{\alpha}(u_1)\mathcal{D}_{\alpha\beta}(u_2). \quad (80)
\end{align*}$$

Here $\mathcal{D}_{bd}(u) = \mathcal{D}_{bd}(u) + \frac{1}{u_1 - u_2}\delta_{bd}A(u)$ and the matrix $r(u)$, which in turn satisfies the quantum Yang-Baxter equation, takes the form,

$$\begin{align*}
r_{11}^{12}(u) &= r_{22}^{22}(u) = r_{33}^{33}(u) = 1, \\
r_{12}^{12}(u) &= r_{13}^{13}(u) = r_{21}^{21}(u) = r_{31}^{31}(u) = r_{23}^{23}(u) = r_{32}^{32}(u) = r_{33}^{33}(u) = -\frac{2}{u - 2}.
\end{align*} \quad (82)$$

Choosing the Bethe state $|\Omega\rangle$ as

$$|\Omega\rangle = B_{i_1}(u_1) \cdots B_{i_N}(u_N)|0\rangle F^{i_1 \cdots i_N}, \quad (83)$$

with $|0\rangle$ being the pseudovacuum, and acting the transfer matrix $\tau(u)$ on the state $|\Omega\rangle$, we have $\tau(u)|\Omega\rangle = \Lambda(u)|\Omega\rangle$, with the eigenvalue,

$$\begin{align*}
\Lambda(u) &= \frac{u - 2}{u + 1} \frac{c_0 + \frac{u}{2} - s_0 - \frac{1}{2}}{(c_0 + \frac{u}{2} + s_0 + \frac{1}{2})(c_0 + \frac{u}{2} - s_0 - \frac{1}{2})} \prod_{j=1}^{N} \frac{(u + u_j)(u - u_j - 2)}{(u - u_j)(u + u_j + 2)} \left( -\frac{u}{u + 2} \right)^{2L} \prod_{j=1}^{N} \frac{(u + u_j)(u - u_j - 2)}{(u - u_j)(u + u_j + 2)} \Lambda^{(1)}(u; \{u_i\}),
\end{align*} \quad (84)$$

provided the parameters $\{u_j\}$ satisfy

$$\begin{align*}
\frac{u_j - 2}{u_j} \frac{c_0 + \frac{u}{2} - s_0 - \frac{1}{2}}{(c_0 + \frac{u}{2} + s_0 + \frac{1}{2})(c_0 + \frac{u}{2} - s_0 - \frac{1}{2})} \left( \frac{u_j + 2}{u_j} \right)^{2L} &= \prod_{i=1}^{N} \frac{(u_j - u_i + 2)(u_j + u_i + 4)}{(u_j - u_i - 2)(u_j + u_i)} \Lambda^{(1)}(u_j; \{u_i\}). \quad (85)
\end{align*}$$

Here $\Lambda^{(1)}(u; \{u_i\})$ is the eigenvalue of the transfer matrix $\tau^{(1)}(u)$ for the reduced problem, which arises out of the $r$ matrices from the first term in the right hand side of \cite{83}, with the reduced boundary $K$ matrices $K^{(1)}_{\pm}(u)$ as,
where

\[
A_-^{(1)}(u) = -\frac{u^2 - 4c_a^2 + 4c_a + 4s_a(s_a + 1) - 1 - 4(u + 1)S_a^z}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]
\[
B_-^{(1)}(u) = \frac{4(u + 1)S_a^-}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]
\[
C_-^{(1)}(u) = \frac{4(u + 1)S_a^+}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]
\[
D_-^{(1)}(u) = -\frac{u^2 - 4c_a^2 + 4c_a + 4s_a(s_a + 1) - 1 + 4(u + 1)S_a^z}{(u + 2c_a - 2s_a - 1)(u + 2c_a + 2s_a + 1)},
\]

and

\[
K_+^{(1)}(u) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & A_+^{(1)}(u) & B_+^{(1)}(u) \\
0 & C_+^{(1)}(u) & D_+^{(1)}(u)
\end{pmatrix},
\]

where

\[
A_+^{(1)}(u) = -\frac{u^2 - 2u - 4c_b^2 + 4c_b + 4s_b(s_b + 1) + 1 - 4(u - 2)S_b^z}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)},
\]
\[
B_+^{(1)}(u) = \frac{4(u - 2)S_b^-}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)},
\]
\[
C_+^{(1)}(u) = \frac{4(u - 2)S_b^+}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)},
\]
\[
D_+^{(1)}(u) = -\frac{u^2 - 2u - 4c_b^2 + 4c_b + 4s_b(s_b + 1) + 1 + 4(u - 2)S_b^z}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}.
\]

Here \(K_-^{(1)}(u)\), the boundary \(K\) matrices after the first nesting, follows from the relations,

\[
\mathcal{D}_{dd}(u)|\Psi\rangle \equiv \frac{u}{u + 1}K_{dd}^{(1)}(u)|\Psi\rangle = (K_-(u))_{dd} - \frac{1}{u + 1}(-\frac{u}{u + 2})^{2L}|\Psi\rangle,
\]
\[
\mathcal{D}_{db}(u)|\Psi\rangle \equiv \frac{u}{u + 1}K_{db}^{(1)}(u)|\Psi\rangle = K_-(u)_{db}(-\frac{u}{u + 2})^{2L}|\Psi\rangle.
\]

Indeed, applying the monodromy matrix \(T(u)\) and its “adjoint” \(\tilde{T}(u)\) to the pseudovacuum, we have

\[
T_{11}(u)|\Psi\rangle = |\Psi\rangle, \quad T_{dd}(u)|\Psi\rangle = (-\frac{u}{u + 2})^L|\Psi\rangle,
\]
\[
T_{1d}(u)|\Psi\rangle \neq 0, \quad T_{ab}(u)|\Psi\rangle = 0, \quad T_{d3}(u)|\Psi\rangle = 0,
\]
\[
\tilde{T}_{11}(u)|\Psi\rangle = |\Psi\rangle, \quad \tilde{T}_{dd}(u)|\Psi\rangle = (-\frac{u}{u + 2})^L|\Psi\rangle,
\]
\[
\tilde{T}_{1d}(u)|\Psi\rangle \neq 0, \quad \tilde{T}_{ab}(u)|\Psi\rangle = 0, \quad \tilde{T}_{d1}(u)|\Psi\rangle = 0.
\]

where \(d \neq b, \quad d, b = 2, 3, 4\). Then we have

\[
A(u)|\Psi\rangle = |\Psi\rangle,
\]
\[
B_d(u)|\Psi\rangle \neq 0, \quad C_d(u)|\Psi\rangle = 0,
\]
\[
D_{db}(u)|\Psi\rangle = (-\frac{u}{u + 2})^{2L}K_-(u)_{db}|\Psi\rangle,
\]
\[
D_{dd}(u)|\Psi\rangle = (-\frac{u}{u + 2})^{2L}((K_-(u))_{dd} - \frac{1}{u + 1})|\Psi\rangle + \frac{1}{u + 1}|\Psi\rangle.
\]
(u + 1)T_{21}(u)\tilde{T}_{12}(u) + T_{22}(u)\tilde{T}_{22}(u) + T_{23}(u)\tilde{T}_{32}(u) + T_{24}(u)\tilde{T}_{42}(u)
= \tilde{T}_{11}(u)T_{11}(u) - (u - 1)\tilde{T}_{12}(u)T_{21}(u) + \tilde{T}_{13}(u)T_{31}(u) + \tilde{T}_{14}(u)T_{41}(u),

(u + 1)T_{21}(u)\tilde{T}_{13}(u) + T_{22}(u)\tilde{T}_{23}(u) + T_{23}(u)\tilde{T}_{33}(u) + T_{24}(u)\tilde{T}_{43}(u) = -u\tilde{T}_{13}(u)T_{21}(u)
(u + 1)T_{21}(u)\tilde{T}_{14}(u) + T_{22}(u)\tilde{T}_{24}(u) + T_{23}(u)\tilde{T}_{34}(u) + T_{24}(u)\tilde{T}_{44}(u) = -u\tilde{T}_{14}(u)T_{21}(u)
T_{31}(u)\tilde{T}_{12}(u) - (u - 1)T_{32}(u)\tilde{T}_{22}(u) + T_{33}(u)\tilde{T}_{32}(u) + T_{34}(u)\tilde{T}_{42}(u) = -u\tilde{T}_{32}(u)T_{32}(u)
T_{31}(u)\tilde{T}_{13}(u) - (u - 1)T_{32}(u)\tilde{T}_{23}(u) + T_{33}(u)\tilde{T}_{33}(u) + T_{34}(u)\tilde{T}_{43}(u) = -u\tilde{T}_{33}(u)T_{32}(u).

(93)

which come from a variant of the (graded) Yang-Baxter algebra [1] with the R matrix [28],

\[ \frac{1}{u} R(2u) \frac{1}{u} = \frac{1}{2} R(2u) \frac{1}{u}. \]

(94)

Noticing the change \( u \to u + 1 \) with respect to the original problem, one may check that these boundary K matrices satisfy the reflection equations for the reduced problem. After some algebra, the reduced transfer matrix \( \tau(1)(u) \) may be recognized as that for the inhomogeneous \( su(3) \) \( t-J \) open chain interacting with the Kondo impurities of arbitrary spins, which has been diagonalized in Ref. [13]. The final result is,

\[ \Lambda^{(1)}(u; \{ u_j \}) = \frac{u - 2}{u} \left( \frac{c_b + \frac{u}{2} - s_b - \frac{1}{2}}{c_b + \frac{u}{2} + s_b - \frac{1}{2}} \right) \left( \frac{c_b + \frac{u}{2} + s_b + \frac{1}{2}}{c_b + \frac{u}{2} - s_b - \frac{1}{2}} \right) M_1 \prod_{\alpha=1}^{M_1} \left( \frac{u - v_\alpha + 2}{u - v_\alpha} \right) \left( \frac{u + v_\alpha}{u + v_\alpha} \right) \frac{u + 1}{u} \prod_{j=1}^{N} \left( \frac{u - u_j}{u - u_j - 2} \right) \left( \frac{u + v_\alpha}{u + v_\alpha} \right) \prod_{\alpha=1}^{M_1} \left( \frac{u - v_\alpha - 2}{u - v_\alpha} \right) \left( \frac{u + v_\alpha}{u + v_\alpha} \right) \Lambda^{(2)}(u; \{ u_j \}, \{ v_\alpha \}) \]

(95)

provided the parameters \( \{ v_\alpha \} \) satisfy

\[ \frac{v_\alpha - 2}{v_\alpha - 1} \left( \frac{c_b + \frac{u}{2} - s_b - \frac{1}{2}}{c_b + \frac{u}{2} + s_b - \frac{1}{2}} \right) \left( \frac{c_b + \frac{u}{2} + s_b + \frac{1}{2}}{c_b + \frac{u}{2} - s_b - \frac{1}{2}} \right) \prod_{j=1}^{N} \left( \frac{v_\alpha - u_j}{v_\alpha - u_j - 2} \right) \left( \frac{v_\alpha + u_j}{v_\alpha + u_j} \right) \prod_{\alpha=1}^{M_1} \left( \frac{v_\alpha - v_\beta + 2}{v_\alpha + v_\beta} \right) = -\Lambda^{(2)}(v_\alpha; \{ u_j \}, \{ v_\beta \}). \]

(96)

Here \( \Lambda^{(2)}(u; \{ u_j \}, \{ v_\alpha \}) \) is the eigenvalue of the transfer matrix \( \tau^{(2)}(u) \) for the \( M_2 \)-site inhomogeneous XXX open chain interacting with the Kondo impurities of arbitrary spins,

\[ \Lambda^{(2)}(u; \{ u_j \}, \{ v_\alpha \}) = \left( \frac{c_b + \frac{u}{2} - s_b - \frac{1}{2}}{c_b + \frac{u}{2} + s_b - \frac{1}{2}} \right) \left( \frac{c_b + \frac{u}{2} + s_b + \frac{1}{2}}{c_b + \frac{u}{2} - s_b - \frac{1}{2}} \right) \prod_{\gamma=a,b} \frac{c_\gamma - \frac{u}{2} + s_\gamma + \frac{1}{2}}{c_\gamma + \frac{u}{2} + s_\gamma + \frac{1}{2}} \prod_{j=1}^{M_2} \left( \frac{u - w_\beta + 2}{u - w_\beta} \right) \left( \frac{u + w_\beta}{u + w_\beta} \right) \prod_{\alpha=1}^{M_2} \left( \frac{u - v_\alpha - 2}{u - v_\alpha} \right) \left( \frac{u + v_\alpha}{u + v_\alpha} \right) \prod_{\beta=1}^{M_2} \left( \frac{u - w_\beta - 2}{u - w_\beta} \right) \left( \frac{u + w_\beta - 2}{u + w_\beta} \right), \]

(97)

provided the parameters \( \{ w_\beta \} \) satisfy

\[ \prod_{\gamma=a,b} \left( \frac{c_\gamma + \frac{u}{2} + s_\gamma + \frac{1}{2}}{c_\gamma - \frac{u}{2} + s_\gamma - \frac{1}{2}} \right) \prod_{\gamma=a,b} \left( \frac{c_\gamma + \frac{u}{2} - s_\gamma + \frac{1}{2}}{c_\gamma - \frac{u}{2} - s_\gamma - \frac{1}{2}} \right) \prod_{\alpha=1}^{M_2} \left( \frac{w_\beta - v_\alpha - 2}{w_\beta - v_\alpha} \right) \left( \frac{w_\beta + v_\alpha}{w_\beta + v_\alpha - 2} \right) = \prod_{\delta \neq \beta} \left( \frac{w_\beta - w_\delta + 2}{w_\beta + w_\delta} \right) \left( \frac{w_\beta + w_\delta - 2}{w_\beta + w_\delta} \right). \]

(98)
After a shift of the parameters $u_j \to u_j - 1, w_m \to w_m + 1$, the Bethe ansatz equations (85), (96) and (98) may be rewritten as follows

$$\left(u_j + 1 \right)_{\alpha=1}^M \prod_{i=1}^N \frac{(u_j - u_i + 2)(u_j + u_i + 2)}{(u_j - u_i - 2)(u_j + u_i - 2)} = \prod_{\gamma=a,b} \frac{c_\gamma - \frac{w_\beta}{2} + s_\gamma + \frac{1}{2}}{c_\gamma - \frac{w_\beta}{2} + s_\gamma + \frac{1}{2}} \prod_{\gamma=a,b} \frac{(v_\alpha - u_j - 1)(v_\alpha + u_j - 1)}{(v_\alpha - u_j + 1)(v_\alpha + u_j + 1)} = \prod_{\beta=1}^M \frac{(v_\alpha - w_\beta + 1)(v_\alpha + w_\beta + 1)}{(v_\alpha - w_\beta - 1)(v_\alpha + w_\beta - 1)} \prod_{\gamma=a,b} \frac{(c_\gamma + \frac{w_\beta}{2} + s_\gamma)(c_\gamma - \frac{w_\beta}{2} - s_\gamma)}{(c_\gamma - \frac{w_\beta}{2} + s_\gamma)(c_\gamma + \frac{w_\beta}{2} - s_\gamma)} \prod_{\gamma=a,b} \frac{(w_\beta - v_\alpha + 1)(w_\beta + v_\alpha + 1)}{(w_\beta - v_\alpha - 1)(w_\beta + v_\alpha - 1)} = \prod_{\delta=1}^M \frac{(w_\beta - w_\delta + 2)(w_\beta + w_\delta + 2)}{(w_\beta - w_\delta - 2)(w_\beta + w_\delta - 2)},$$

with the corresponding energy eigenvalue $E$ of the model

$$E = -\sum_{j=1}^N \frac{4}{u_j^2 - 1}. \quad (100)$$

IV. CONCLUSION

In conclusion, we have studied integrable Kondo problems describing two boundary impurities coupled to one-dimensional extended Hubbard open chains. The quantum integrability of these systems follows from the fact that the Hamiltonians in each case are derived from a one-parameter family of commuting transfer matrices. Moreover, the Bethe Ansatz equations and expressions for the energies are derived by means of the algebraic Bethe ansatz approach. We would like to emphasize that the boundary $K$ matrices found here are non-regular in that they can not be factorized into the product of a $c$-number $K$ matrix and the local monodromy matrices. However, similar to the cases discussed in [11],[13], it is possible to introduce a singular local monodromy matrix $\tilde{L}(u)$ to express the boundary

$$K_-(u) = \tilde{L}(u)\tilde{L}^{-1}(-u). \quad (101)$$

where, for example in the case of the superalgebras $gl(2|2)$ model

$$\tilde{L}(u) = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & u + 2c + 2s + 1 + 2S^+ & 2S^- \\ 0 & 0 & 2S^+ & u + 2c + 2s + 1 - 2S^- \end{pmatrix}. \quad (102)$$

which constitutes a realization of the Yang-Baxter algebra (9) when $\epsilon$ tends to 0. The recent work of Frahm and Slavnov [14] confirms the existence of such non-regular solutions by means of a projecting method.

Finally, we would like to stress that here we have only considered the case of Kondo impurities in these extended Hubbard models which are based on the $sl(2)$ subalgebra of the bulk symmetry of the models. It is of course possible to consider other boundary impurities corresponding to different subalgebra embeddings such as $sl(1|1)$ for the $gl(2|2)$, $gl(3|1)$ cases or $sl(3)$ for the $gl(3|1), gl(4)$ models and even $gl(2|1)$ for $gl(3|1), gl(2|2)$. For the case of $t - J$ models such other types of integrable boundary impurities have been studied in [15].

This work is supported by OPRS and UQPRS. Jon Links and Mark D.Gould are supported by an Australian Research Council.
APPENDIX A: DERIVATION OF THE NON-C-NUMBER BOUNDARY K-MATRICES

In this appendix, we sketch the procedure of solving the \((\mathbb{Z}_2\text{-graded})\) RE for \(K-(u)\). To describe integrable Kondo impurities coupled with the one-dimensional supersymmetric extended Hubbard model open chain, it is reasonable to assume that

\[
K-(u) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & A(u) & B(u) \\
0 & 0 & C(u) & D(u)
\end{pmatrix}.
\]

(A1)

Throughout, we have omitted all the subscripts for brevity, reflecting that the fermionic degrees of freedom do not occur, as it should be for a magnetic impurity. For the \(R\)-matrix \([8]\), one may get from the \(\text{RE} \,(\mathbb{H})\) 54 functional equations, of which 14 are identities. After some algebraic analysis, together with the \(su(2)\) symmetry, we may assume that

\[
A(u) = \alpha(u) + \beta(u)S^z, \quad B(u) = \beta(u)S^+, \quad C(u) = \beta(u)S^-, \quad D(u) = \alpha(u) - \beta(u)S^z.
\]

(A2)

There are 10 equations automatically satisfied and 10 same equations, leaving only 20 equations left to be solved

\[
\begin{align*}
A(u_1)B(u_2) + B(u_1)D(u_2) &= A(u_2)B(u_1) + B(u_2)D(u_1), \\
C(u_1)A(u_2) + D(u_1)C(u_2) &= C(u_2)A(u_1) + D(u_2)C(u_1), \\
u_- (A(u_1)B(u_2) + B(u_1)D(u_2)) &= u_+ (B(u_1) - B(u_2)), \\
u_- (A(u_2)B(u_1) + B(u_2)D(u_1)) &= u_+ (B(u_1) - B(u_2)), \\
u_- (C(u_1)A(u_2) + D(u_1)C(u_2)) &= u_+ (C(u_1) - C(u_2)), \\
u_- (C(u_2)A(u_1) + D(u_2)C(u_1)) &= u_+ (C(u_1) - C(u_2)), \\
u_- (A(u_1)A(u_2) + B(u_1)C(u_2) - 1) &= u_+(A(u_1) - A(u_2)), \\
u_- (A(u_2)A(u_1) + B(u_2)C(u_1) - 1) &= u_+(A(u_1) - A(u_2)), \\
u_- (C(u_1)B(u_2) + D(u_1)D(u_2) - 1) &= u_+(D(u_1) - D(u_2)), \\
u_- (C(u_2)B(u_1) + D(u_2)D(u_1) - 1) &= u_+(D(u_1) - D(u_2)) \end{align*}
\]

(20)

with \(u_+ = u_1 + u_2, u_- = u_1 - u_2\). Substituting (A2) into these equations, we find that all these equations are reduced to the following three equations
\[
\begin{align*}
u_+ (\alpha(u_1) - \alpha(u_2)) &= u_- (-1 + \alpha(u_1) \alpha(u_2) + s(s+1)\beta(u_1)\beta(u_2)), \\
u_+ (\beta(u_1) - \beta(u_2)) &= u_- (\alpha(u_1)\beta(u_2) + \alpha(u_2)\beta(u_1) - \beta(u_1)\beta(u_2)), \\
2u_+ (\alpha(u_2)\beta(u_1) - \alpha(u_1)\beta(u_2)) &= 2u_- (\alpha(u_1)\beta(u_2) + \alpha(u_2)\beta(u_1)) - u_-(u_+ + 2)\beta(u_1)\beta(u_2)).
\end{align*}
\] (A4)

Taking the limit \( u_1 \to u_2\), these equations become

\[
\begin{align*}
\frac{d\alpha(u)}{du} &= \frac{1}{2u}(-1 + \alpha(u)^2 + s(s+1)\beta(u)^2), \\
\frac{d\beta(u)}{du} &= \frac{1}{2u}(2\alpha(u)\beta(u) - \beta(u)^2), \\
\alpha(u)\frac{d\beta(u)}{du} - \beta(u)\frac{d\alpha(u)}{du} &= \frac{1}{2u}(2\alpha(u)\beta(u) - (u + 1)\beta(u)^2).
\end{align*}
\] (A5)

Solving the first two equations, we have

\[
\alpha(u) = \frac{(c_1c_2 - u^2)(2s + 1) + (c_2 - c_1)u}{(2s + 1)(c_1 - u)(c_2 - u)}, \quad \beta(u) = \frac{2(c_2 - c_1)u}{(2s + 1)(c_1 - u)(c_2 - u)}.
\] (A6)

where \( c_1 \) and \( c_2 \) are integration constants. Substituting these results into the third equation in (A5), we may establish a relation between \( c_1 \) and \( c_2 \): \( c_2 = c_1 - 4s - 2 \). This is nothing but the non-c-number boundary \( K \) matrix \( \square \) (after a redefinition of the constant: \( c_1 \to 2c + 4s + 2 \)).

A similar construction also works for the quantum \( R \) matrix \( \square \) and \( \square \).

\[\text{[1]} \quad \text{A. Foerster and M. Karowski, Nucl. Phys. B408, 512 (1993).}\
\[\text{[2]} \quad \text{A. González-Ruiz, Nucl. Phys. B424, 468 (1994).}\
\[\text{[3]} \quad \text{D. Arnaudon, C. Chryssomalakos and L. Frappat, J. Math. Phys. 36, 5262 (1995).}\
\[\text{[4]} \quad \text{F.H.L. Essler, J. Phys. A29, 6183 (1996).}\
\[\text{[5]} \quad \text{H.-Q. Zhou, Phys. Lett. A226, 104 (1997); A230, 45 (1997).}\
\[\text{[6]} \quad \text{D. Arnaudon, J. High Energy Phys. 12, 6 (1997).}\
\[\text{[7]} \quad \text{G. Bédürftig and H. Frahm, J. Phys. A30, 4139 (1997).}\
\[\text{[8]} \quad \text{A. J. Bracken, X.-Y. Ge, Y.-Z. Zhang and H.-Q. Zhou, Nucl. Phys. B516, 588 (1998).}\
\[\text{[9]} \quad \text{Y.-Z. Zhang and H.-Q. Zhou, Phys. Rev. B58, 51 (1998); Phys. Lett. A244, 427 (1998).}\
\[\text{[10]} \quad \text{Z.-N. Hu, F.-C. Fu and Y. Wang, J. Phys. A31, 5241 (1998).}\
\[\text{[11]} \quad \text{H.-Q. Zhou and M. D. Gould, Phys. Lett. A251, 279 (1999).}\
\[\text{[12]} \quad \text{E.K. Sklyanin, J. Phys. A21, 2375 (1988).}\
\[\text{[13]} \quad \text{H.-Q. Zhou, X.-Y. Ge, J. Links and M.D. Gould, Nucl. Phys. B546, 779 (1999).}\
\[\text{[14]} \quad \text{H. Frahm and N.A. Slavnov, New solution of the reflection equation and the projecting method, cond-mat/9810312.}\
\[\text{[15]} \quad \text{F. H. Essler, V. E. Korepin and K. Schoutens, Phys. Rev. Lett. 68, 2960, (1992); ibid. 70, 73 (1993).}\
\[\text{[16]} \quad \text{H.-Q. Zhou, X.-Y. Ge and M.D. Gould, Integrable Kondo impurities in the one-dimensional supersymmetric extended Hubbard model, cond-mat/9811044. J. Phys. A, in press.}\
\[\text{[17]} \quad \text{A. González-Ruiz and H. J. de Vega, Nucl. Phys. B424, 553 (1994).}\
\[\text{[18]} \quad \text{G. Bédürftig and H. Frahm, Open t – J chain with boundary impurities, cond-mat/9903202.}\
\]