NC Geometry and Fractional Branes

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Abstract

Considering complex n-dimension Calabi-Yau homogeneous hyper-surfaces \( \mathcal{H}_n \) with discrete torsion and using Berenstein and Leigh algebraic geometry method, we study Fractional D-branes that result from stringy resolution of singularities. We first develop the method introduced in [hep-th/0105229] and then build the non commutative (NC) geometries for orbifolds \( \mathcal{O} = \mathcal{H}_n / \mathbb{Z}_{n+2} \) with a discrete torsion matrix \( t_{ab} = \exp \left[ \frac{2\pi i}{n+2} (\eta_{ab} - \eta_{ba}) \right] \), \( \eta_{ab} \in SL(n, \mathbb{Z}) \). We show that the NC manifolds \( \mathcal{O}^{(nc)} \) are given by the algebra of functions on the real \( (2n+4) \) Fuzzy torus \( T_{\beta_{ij}}^{2(n+2)} \) with deformation parameters \( \beta_{ij} = \exp \left[ \frac{2\pi i}{n+2} (\eta_{ij}^{-1} - \eta_{ji}^{-1}) q_a^i q_j^b \right] \) with \( q_a^i \)'s being charges of \( \mathbb{Z}_{n+2} \). We also give graphic rules to represent \( \mathcal{O}^{(nc)} \) by quiver diagrams which become completely reducible at orbifold singularities. It is also shown that regular points in these NC geometries are represented by polygons with \( (n+2) \) vertices linked by \( (n+2) \) edges while singular ones are given by \( (n+2) \) non connected loops. We study the various singular spaces of quintic orbifolds and analyze the varieties of fractional D branes at singularities as well as the spectrum of massless fields. Explicit solutions for the NC quintic \( \mathcal{Q}^{(nc)} \) are derived with details and general results for complex n dimension orbifolds with discrete torsion are presented.

Key words: Non Commutative Geometry and type II string compactification, Calabi-Yau Orbifolds with discrete torsion, Fractional D Branes, Fuzzy Torus fibrations.

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1 Introduction

Recently some interest has been given to build non commutative (NC) extensions of Calabi-Yau orbifolds $O$ with discrete torsion [11]-[10]. These NC manifolds, which are involved in the twisted sector of type II string compactification, go beyond the standard non commutative $R^5_0$ and NC $T^k_0$ torii examples considered in brane physics [11]-[10]. They offer a manner to resolve non geometric stringy singularities [17], present a natural framework to study fractional $D$ branes at singularities. We show that the Berenstein and Leigh (BL) construction is just a NC torus fibration with base the CY orbifold in complete agreement with the results obtained in [2, 24]-[17]; but also explore new issues such the connection between NC geometry and fractional $D$ branes or the link between discrete torsion and rational tori. More precisely we will study other aspects of NCCY orbifolds with discrete torsion completing the partial results obtained in [2, 24]-[17]: also explore new issues such the connection between NC geometry and fractional $D$ branes or the link between discrete torsion and rational tori.

From the algebraic geometry point of view, the NC manifold $O^{nc}$ is covered by a finite set of holomorphic matrix coordinate patches $U_{(\alpha)} = \{Z^\alpha_i; 1 \leq i \leq n \alpha = 1, 2, \ldots \}$ and holomorphic transition functions mapping $U_{(\alpha)}$ to $U_{(\beta)}$; $\phi_{(\alpha,\beta)}: U_{(\alpha)} \rightarrow U_{(\beta)}$. In other words $A^{nc}$ is covered by a collection of NC local algebras $A^{nc}_{(\alpha)}$, with analytic maps $\phi_{(\alpha,\beta)}$ on how to glue $A^{nc}_{(\alpha)}$ and $A^{nc}_{(\beta)}$. These $A^{nc}_{(\alpha)}$ local algebras have centers $Z_{(\alpha)} = Z(A^{nc}_{(\alpha)})$: when glued together give precisely the commutative target space manifold $O$. In this way a commutative singularity of $O \sim Z(A^{nc})$ can be made smooth in the NC space $A^{nc}$ [11][2]. This idea was successfully used to build NC ALE spaces and some realizations of Calabi-Yau threefolds (CY3) such as the quintic threefolds $Q$. In this regards it was shown that the NC quintic $Q^{nc}$ extending $Q$, when expressed in the coordinate patch $Z_5 = I_{id}$, is given by the following special algebra,

\[
\begin{align*}
Z_1Z_2 &= \alpha Z_2Z_1, & Z_3Z_4 &= \beta\gamma Z_4Z_3, & (a) \\
Z_1Z_4 &= \beta^{-1} Z_4Z_1, & Z_2Z_3 &= \alpha\gamma Z_3Z_2, & (b) \\
Z_2Z_4 &= \gamma^{-1} Z_4Z_2, & Z_1Z_3 &= \alpha^{-1}\beta Z_3Z_1, & (c) \\
Z_iZ_5 &= Z_5Z_i, & i = 1, 2, 3, 4;
\end{align*}
\]

where $\alpha, \beta$ and $\gamma$ are fifth roots of the unity, the parameters of the $Z^5_0$ discrete group. The $Z_i$ ’s are the generators of $A^{nc}$. One of the main features of this non commutative algebra is that its centre $Z(O^{nc})$ coincides exactly with $Q$, the commutative quintic threefolds. In [2], a special solution for this algebra using $5 \times 5$ matrices has been obtained and in [24] a class of solutions for eqs(1.1), depending on the orbifold group charge vectors, has been worked out and some partial results regarding higher dimensional CY hypersurfaces were given.

In this paper, we study other aspects of NCCY orbifolds with discrete torsion completing the partial results obtained in [2, 24]-[17]: but also explore new issues such the connection between NC geometry and fractional $D$ branes or the link between discrete torsion and rational tori. More precisely we will compute the explicit dependence of NCCY orbifolds in the discrete torsion of the orbifold group and study the varieties of the fractional $D$ branes at singularities. We show that the Berenstein and Leigh (BL) construction is just a NC torus fibration with base the CY orbifold in complete agreement with the idea of emergent dimension developed recently in [46], see also [37]. The result concerning NC torus fibration can be directly seen on eqs (1.1) which, for the case of generic complex $d$ dimension CY hypersurfaces, can be rewritten formally as $Z_iZ_j = \alpha_{ij} Z_jZ_i$. A careful inspection of the solution of these eqs, shows that they describe a fibration whose base is indeed the $H_d$ commutative space and a
fiber given by a NC rational torus defined as,
\[ \alpha_i^{n+2} = 1, \quad Z_i^{n+2} \sim I_{id}. \] (1.2)

Throughout this paper, we will also show that the origin of these rational torii is due to a nice property of \( Z_{n+2} \) orbifolds, which induce a NC torus fibration on \( H_d \). In particular, we show that the NC structure has two sources: (i) either induced by quantum symmetries as usual or (ii) by considering NC complex one cycles as in the CDS solution for Matrix model compactification of M theory \([11]\). Both solutions lead to a finite number of fractional \( D \) branes and provide a new way to think about the B field.

The organization of this paper is as follows: In section 2, we review some general features of the commutative quintic and develop its non commutative extensions using the constrained method. We also complete partial results obtained in literature. The analysis we will develop in this section applies as well to all CY hypersurfaces and moreover to CY manifolds embedded in toric varieties \([33]\). In section 3, we develop non commutative geometry induced by discrete torsion and work out the full class of torsion dependent solutions for NC quintic. In section 4, we use this result to derive general solutions for the NCCY orbifolds. This analysis recovers the results of \([17]\) as special cases. We also study the symmetries of the moduli space of NC solutions we have obtained and discuss the varieties of fractional \( D \) branes as well as the full spectrum of massless fields on the \( D \) branes. In section 5, we collect some general results on NC complex \( d \)-dimensional CY orbifolds with a discrete torsion matrix
\[ t_{ab} = \exp \left( \frac{i\pi}{d+2} (\eta_{ab} - \eta_{ba}) \right), \quad \eta_{ab} \in SL(d,\mathbb{Z}) \] and study the various classes of varieties of fractional \( D \) branes at singularities. The NC manifolds are given by the algebra of functions on the Fuzzy torus \( T_d^{2d} \), where \[ \beta_{ab} = \exp \left( \frac{i\pi}{d+2} [(\eta_{cd}^{-1} - \eta_{dc}^{-1}) q_a^c q_b^d] \right). \] In section 6, we give our conclusion.

## 2 NC Quintic \( Q^{nc} \)

In this section, we study the algebraic geometry approach for building NC quintic by using constraint eqs method. This analysis applies as well to any complex \( d \) dimension CY homogeneous hypersurface \( P_{d+2}(z_1, \ldots, z_{d+2}) = 0 \) embedded in \( \mathbb{CP}^{d+1} \). Among the results we will derive here, we prove that the NC quintic obtained by Berenstein and Leigh is a special torus fibration based on \( Q \). We show that orbifolds of the quintic are characterized by a \( SL(3,\mathbb{Z}) \) matrix \( \eta_{ab} \) whose antisymmetric part encodes discrete torsions of the \( Z_3^5 \) orbifold group.

### 2.1 The Quintic \( Q \)

To begin consider the complex analytic homogeneous hypersurface \( P_5(z_1, \ldots, z_5) \) given by,
\[ z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + a_0 \prod_{i=1}^{5} z_i = 0, \] (2.1)

where \((z_1, z_2, z_3, z_4, z_5)\) are commuting homogeneous coordinates of \( \mathbb{CP}^4 \), the complex four dimensional projective space, and \( a_0 \) a non zero complex moduli parameter. This polynomial describes a well known CY3s namely the commutative quintic denoted in this paper by \( Q \). Besides invariance under permutations of the \( z_i \) variables, \( Q \) has a set of geometric discrete isometries acting on the homogeneous coordinates \( z_i \) as:
\[ z_i \rightarrow z_i \omega q_i^\alpha, \] (2.2)
with $\omega^5 = 1$ and $q_i^a$ integer charges to which we will refer here below to as the CY charges\(^1\). These are the entries of the following 5d vectors $q^1 = (q_1^1, q_2^1, q_3^1, -q_1^1 - q_2^2 - q_3^3, 0), q^2 = (q_1^2, q_2^2, q_3^2, -q_1^1 - q_2^2 - q_3^3, 0)$ and $q^3 = (q_1^3, q_2^3, q_3^3, -q_1^1 - q_2^2 - q_3^3, 0)$. In these relations we have set $q_i^5 = 0$, a useful feature which correspond to working in $U_a [z_1, \ldots, z_4, z_5 = 1]$, the local coordinate patch of $Q$ where $z_5 = 1$. For illustrating applications, we will mainly use the following special choice $q_1^1 = 1, q_2^1 = q_3^1 = q_4^1 = -1$; all remaining ones are equal to zero. In other words;

\[
q^1 = (1, -1, 0, 0, 0); \quad q^2 = (1, 0, -1, 0, 0); \quad q^3 = (1, 0, 0, -1, 0).
\]  

(2.3)

The $q_i^a$ charges in eqs (2.2) and eqs (2.3) are defined modulo five; $q_i^a \equiv q_i^a + 5Z$, and satisfy naturally the identities

\[
\sum_{i=1}^{5} q_i^a = 0 \mod(5); \quad a = 1, 2, 3.
\]

(2.4)

This constraint eq is a necessary condition required by invariance under eqs (2.2) of the $a_0 \prod_{i=1}^{5} z_i$ monomials of eq (2.1). It ensures that the holomorphic hypersurface of CY is indeed a CY manifold. Note in passing that the constraint eq (2.3) is just the analogue of the vanishing of the first Chern class of the quintic ($c_1 (Q) = 0$) in differential geometry language. It is also the condition under which the underlying 2d effective field theory flows in the infrared to a CFT \(^{[3]}\) [34, 35]. Orbifolds of the quintic with respect to $Z_5^a$ are as usual obtained by identifying points that are related under the transformations \([1, 2, 3]\). Here we will show that such orbifolds are completely characterized by the $q_i^a$ charges and an SL(3,$Z$) matrix $\eta_{ab}$ and for general hypersurfaces by matrices in SL$(n, Z)$. For symmetric matrices $\eta_{ab}$, that is $\eta_{ab} = \eta_{ba}$, one gets orbifolds without discrete torsion; while for non-symmetric $\eta_{ab}$, there is a discrete torsion. The idea behind this classification is that together with eqs (2.2) and (2.3), there exist extra symmetries of eqs (2.1) acting as $z_i \rightarrow z_i \omega^p_i$ where now the $p_i^a$ charges are given by

\[
p_i^a = \eta_{ab} q_i^b,
\]

(2.5)

with $\eta_{ab}$ the above mentioned $3 \times 3$ matrix of $SL(3; Z)$. These dual charges satisfy $\sum_{i=1}^{5} p_i^a = 0$, following naturally from eq (2.3). The $\eta_{ab}$ matrix, which to our knowledge was not known before; turns out to play an important role in building NC geometries à la BL. It appears here as encoding matrix of the automorphisms of characters of the orbifold group. We will show later that $\eta_{ab}$ is the carrier of the discrete torsion of the orbifold symmetry of CY Hypersurfaces. It antisymmetric part ($\eta_{ab} - \eta_{ba}$) is related to the logarithm of the $t_{ab}$ torsion matrix of $Z_5^{a+2}$. Moreover, as far as $Q$ is concerned, it interesting to note that there are different kinds of orbifolds one can build depending on the orbifold group. If we denote by $R_5 [[z_1, \ldots, z_5]]; R_5$ for short, the ring of complex holomorphic and homogeneous polynomials of degree five on $\mathbb{CP}^4$ and by $G_{[a]}$, a generic subgroup of $Z_5^3$, then one can build various orbifolds of the quintic as $Q^{[a]} = R_5 / G_{[a]}$. In addition to $R_5 / Z_5^3$, we have also the two following examples of the quintic orbifolds $Q^{[1]}$ and $Q^{[2]}$ associated respectively with $G_{[1]} = Z_5$ and $G_{[2]} = Z_5^2$ subgroups of $G = Z_5^3$.

\[
Q^{[1]} = R_5 / Z_5, \quad Q^{[2]} = R_5 / Z_5^2.
\]

(2.6)

Throughout this study, we will always stay in the coordinate patch $U_{(a)} [z_1, \ldots, z_4, z_5 = 1]$; the move to an other patch of the quintic, say $U_{(\beta)} [w_1, \ldots, w_4, w_5 = 1]$ with CY charges $r_i^a$, is ensured by

\(^1\)By CY charges we intend the $q_i^a = (q_i^a)$ vectors that define a basis for phase symmetries $Z_5^{d+2}$ of the complex $d$ dimension CY hypersurface $z_1^{d+2} + \ldots + z_{d+2}^{d+2} + \prod_{i=1}^{d+2} z_i = 0$. Since the $a$-th $Z_5^{d+2}$ factor acts on the $z_i$ local variables as $z'_i = z_i \omega^{a_i}$, invariance of the above polynomial requires that $\omega = \exp i \frac{2\pi a_i}{d+2}$ and moreover $\sum_{i=1}^{d+2} a_i = 0$. This $q_i^a$ constraint relation is known to be equivalent to the vanishing condition of the first Chern class of CY manifolds.
holomorphic transition functions $\phi^{(\alpha, \beta)}_{\omega} \rightarrow \phi^{(\alpha, \beta)}_{\omega}$ carrying appropriate $Z_3^n$ charges. Note that on the coordinate patch $U_{(\alpha)} [z_1, \ldots, z_4, z_5 = 1]$, the full $Z_3^n$ orbifold symmetry has no fixed point; the only stable one, namely $(0, 0, 0, 0, 1)$, does not belong to $R_5/Z_3^n$. However thinking of the quintic orbifold $R_5/Z_3^n$ as either a $Z_3$ orbifold of $R_5/Z_5^n$, that is $R_5/Z_3^n \sim (R_5/Z_5^n) / Z_5 \sim R_5/Z_3^n$ or again as a $Z_3$ orbifold of $R_5/Z_5^n$, one can consider the fixed points of the respective singular spaces of $R_5/Z_3^n$ and $R_5/Z_3^n$. This procedure is also equivalent to set to zero some of the CY vector charges associated with $Z_3^n$ symmetry. For example, the orbifold $R_5/Z_3^n$ may also be linked to $R_5/Z_5$ just by setting $q^2 = q^3 = 0$. In section 5, we shall use both the $R_5/Z_3^n$ and $R_5/Z_3^n$ spaces to study fractional branes.

2.2 NC Quintic

A way to get NC extensions of Q by using discrete torsion is to start from the complex homogeneous hypersurface $2.1$ and choose the coordinate patch $z_5 = 1$ and $q_5^a = 0$. Then associate to the $\{z_1, z_2, z_3, z_4\}$ local variables, the set of $5 \times 5$ matrix operators $\{Z_1, Z_2, Z_3, Z_4\}$ and $Z_5$ with the identity matrix $I_{5d}$. The NC quintic $Q^{nc}$, associated to eqs $2.1$ and eqs $2.5$, is a NC algebra generated by the $Z_i$’s. This is a special subalgebra of the ring of functions on the space of matrices $Mat(5, C)$; it reads in term of the $Z_i$ matrix generators as:

$$Z_i Z_j = \beta_{ij} Z_j Z_i, \quad Z_i Z_5 = Z_5 Z_i, \quad i, j = 1, \ldots, 4.$$  \hfill (2.7)

In these eqs, $\beta_{ij}$ is an invertible matrix constrained as,

$$\beta_{ji} = \beta_{ij}^{-1}, \quad \beta_{ii} = \beta_{ij}^2 = \beta_{ij}^3 = \beta_{ij}^4 = \beta_{ij} \beta_{ij} \beta_{ij} \beta_{ij} = 1, \quad \forall i.$$  \hfill (2.8)

These constraint eqs reflect just the property that the commutative $Q$ should be in the centre $Z(Q^{nc})$ of eqs $2.7$. In other words $Q = Z(Q^{nc})$ or equivalently;

$$[Z_j, Z_i^a] = 0, \quad \left[Z_j, \prod_{i=1}^4 Z_i \right] = 0.$$  \hfill (2.9)

A class of solutions of eqs $2.8$ is obtained as follows: First parameterize $\beta_{ij}$ as $\beta_{ij} = \omega^{L_{ij}}$ with $\omega = \exp i \frac{2\pi}{5}$ and $L_{ij}$ is a $5 \times 5$ antisymmetric matrix satisfying $\sum_{i=1}^5 L_{ij} = 0$. This constraint relation comes from invariance of the term $a_0 z_1 z_2 z_3 z_4 z_5$ exactly as for the CY condition eq $2.4$. This means that a possible solution for $L_{ij}$ is as,

$$L_{ij} = \nu_{ab} (q_i^a q_j^b - q_j^a q_i^b)$$  \hfill (2.10)

where $\nu_{ab}$ is an arbitrary $3 \times 3$ matrix of integer entries. The matrix $L_{ij}$ is built as bi-linears of the $q_k^a$ charge vectors ensuring automatically $\sum_{i=1}^5 L_{ij} = 0$. This is why we shall still refer to the constraint eq $\sum_{i=1}^5 L_{ij} = 0$ as the CY condition. Moreover, since eq $2.10$ can also be written as $L_{ij} = m_{ab} q_i^a q_j^b$, with $m_{ab} = \nu_{ab} - \nu_{ba}$, one suspects that $m_{ab}$ matrix should be linked to the structure constants of the underlying $Z_3^n$ orbifold symmetry. We will show in section 6, that $L_{ij}$ should read in fact as;

$$L_{ij} = p_i^a q_j^a - p_j^a q_i^a,$$  \hfill (2.11)

where $p_i^a$ are as in eqs $2.3$. These integers are the charges of a hidden invariance induced by discrete torsion. By analogy with the geometric symmetry $Z_3$, this symmetry can be thought of as acting on the $z_i$ variables as $z_i \rightarrow z_i \omega^{p_i^a}$, with $p_i^a = \nu_{ab} q_i^b$. To have an idea on how the formulas we have derived above work in practice, let us go the local patch $z_5 = 1$ with $q_i^a$ charges as in eq $2.3$ and perform some
generated by a strong constant external magnetic field such as in the Chern Simons model of the fractional quantum Hall effect \[48, 49\], NC geometry is interpretive \[36, 11, 12\]; see also \[37, 38, 39\]. In effective field theoretical models at very low energies, in quantum physics, non-commutativity appears in different ways and has various origins and different interpretations \[36, 11, 12\].

3 NC Geometry and Discrete Torsion

\[ \prod_{i} Z_{i} = \omega^{k_{1}-k_{3}} Z_{2} Z_{1}, \quad Z_{i} Z_{3} = \omega^{-k_{1}+k_{2}} Z_{3} Z_{i}, \quad Z_{1} Z_{4} = \omega^{k_{3}-k_{2}} Z_{4} Z_{1}, \quad Z_{2} Z_{4} = \omega^{-k_{3}+k_{2}} Z_{4} Z_{2}, \quad Z_{3} Z_{4} = \omega^{k_{2}} Z_{4} Z_{3}. \] (2.13)

Putting \( \alpha = \omega^{k_{1}-k_{3}}, \beta = \omega^{k_{1}-k_{2}} \) and \( \gamma = \omega^{k_{3}} \), one discovers the algebra of \[2\] given by eqs(2.1). Moreover the solution of these eqs read, up to a normalization factor, as:

\[ Z_{1} = x_{1} P_{\omega^{k_{1}+k_{2}+k_{3}}} Q^{3}, \quad Z_{2} = x_{2} P_{\omega^{k_{1}}} Q^{-1}, \quad Z_{3} = x_{3} P_{\omega^{k_{2}}} Q^{-1}, \quad Z_{4} = x_{4} P_{\omega^{k_{3}}} Q^{-1}, \] (2.14)

where \( x_{i} \) are as in eqs(2.1), \( \omega \) is the complex conjugate of \( \omega \) and where

\[ P_{\alpha} = \text{diag}(1, \alpha, \alpha^{2}, \alpha^{3}, \alpha^{4}), \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \] (2.15)

with \( \alpha \) standing for \( \omega^{k_{1}}, \omega^{k_{2}}, \omega^{k_{3}} \) and their products. This solution shows clearly that \( Z_{i}^{5} \), the product \( \prod_{i=1}^{4} Z_{i} \) and their linear combination are all of them in the centre \( Z(Q^{nc}) \) of the NC algebra \( Q^{nc} \).

3 NC Geometry and Discrete Torsion

In quantum physics, non-commutativity appears in different ways and has various origins and different interpretations \[36, 11, 12\]; see also \[37, 38, 39\]. In effective field theoretical models at very low energies, such as in the Chern Simons model of the fractional quantum Hall effect \[48, 49\], NC geometry is generated by a strong constant external magnetic field \( B \) which couple the two position vectors \( x^{i}(t) \) of electrons as \( B \varepsilon_{ij} x^{j}(t) \partial x^{i}(t)/\partial t \). Heisenberg quantization rule leads to a non vanishing commutator for these position vectors; i.e \( [x^{i}, x^{j}] = i e^{ij}_{\nu L} \), where \( \nu L = \frac{1}{2} \) is the Laughlin filling factor. For large value of \( B (\sim 15 \text{ Tesla}) \), the quantum properties of the system of electrons are described by a NC Chern-Simons gauge theory on a D2 brane. Such a two dimensional condensed matter phase has received recently an important interest due to similarities with solitons built up with systems of D branes of type IIA string theory. At very high energies, say around the Planck scale as in string theory, non-commutativity is generated by the NS \( B_{\mu \nu} \) antisymmetric field and is linked to the existence of open strings ending on D branes with a dynamics governed by a boundary conformal invariance \[12\]. This issue has been subject to much interest during the few last years in connection with the derivation
of non commutative ADHM solitons 10 and the study of the tachyon condensation by following the GMS method 27 28. In M theory, NC geometry comes as a non trivial solution in the study of the matrix model torus compactification. Here also NC geometry is generated by an antisymmetric field; the eleven dimensional gauge three form field $C_{\mu\nu\rho}$.11 Elements $g_1$ and $g_1$ of the group of automorphism symmetries of the matrix model on a two torus $T^2$ are in general governed by the central relation $g_1g_2g_1^{-1}g_2^{-1}$ taken to be proportional to the identity operator; that is $g_1g_2g_1^{-1}g_2^{-1} = \lambda I_{id}$ with $\lambda \in \mathbb{C}^*$ as required by the Schur lemma 11 13.

In quantum mathematics, NC geometry is viewed as an algebraic structure $\mathcal{M}_h [X_1, \ldots, X_N]$ going beyond the usual $C [x_1, \ldots, x_N]$ commutative one with the ideal $\{ x_i x_j = x_j x_i \}$. Formally, the generic commutation relations of the generators of the quantum algebra $\mathcal{M}_h [X_1, \ldots, X_N]$, may be written as $X_I * X_J = r^{KL}_{I J}(X) X_K * X_L + b_{IJ}(x)$, where $r^{KL}_{I J}(X)$ and $b_{IJ}(X)$ are some polynomials in $X_I$ which may be thought of as 20 41:

$$r^{KL}_{I J}(X) = \delta^L_I \delta^K_J + \hbar r^{KL}_{I J} + \ldots; \quad b_{IJ}(X) = \hbar (\Omega_{IJ} + t^K_{I J} X_K + \ldots)$$

(3.1)

In the limit $\hbar \to 0$, $r^{KL}_{I J}(X) \to \delta^L_I \delta^K_J$ and $b_{IJ}(X) \to 0$; one recovers the usual commutative structure of $C [x_1, \ldots, x_N]$. For the general cases, such for instance $(r^{KL}_{I J}(x), b_{IJ}(x))$ equals to $(0, B_{IJ})$ or $(0, t^K_{I J} X_K)$ or again $(R^{KL}_{I J}, 0)$, one gets respectively the canonical commutator, the Lie algebra bracket and the quantum Yang-Baxter spaces 36. The NC structure we are dealing with here corresponds to an other special situation where $b_{IJ}(x) = 0$ and,

$$r^{KL}_{I J}(x) = \beta_{KL} \delta^L_I \delta^K_J,$$

(3.2)

with $\beta_{IJ}$ is a root of unity. This NC geometry is generated by the discrete torsion matrix of the orbifold group $\mathbb{Z}_5^3$ and has much to do with NC Fuzzy tori representations. Since discrete torsion is involved in string compactifications on orbifolds and twisted string sectors, one expects that this NC structure plays some role in string theory on orbifolds and more generally in supersymmetric field theories on orbifolds. As we shall show by explicit analysis, see section 4, NC geometry induced by discrete torsion leads to fractional D branes at orbifold singularities and offers a new way to resolve non geometric singularities. Points of the usual geometry are replaced by polygons in the NC case.

To get the right link between discrete torsion and NC orbifold solutions, in particular quintic ones given above, let us reconsider the solutions eqs(2.14) and explore their structure. To do so we shall first show that the NC solution 21 14 are not so general as claimed in 24 as these solutions form just a special class of a more general set of solutions involving general representations of $\mathbb{Z}_5^3$. We will then begin by giving some results regarding regular representations of $\mathbb{Z}_5^3$; after that we present our general discrete torsion dependent solution.

### 3.1 More on Solutions (2.14)

Eqs(2.14) involve various group elements of the representation of $\mathbb{Z}_5$ namely the $P_{\omega^{k_1}}, P_{\omega^{k_2}}$ and $P_{\omega^{k_3}}$ commuting operators and powers of $Q$. These five dimensional matrices system $\{ P_{\omega^{k_1}}, P_{\omega^{k_2}}, P_{\omega^{k_3}}, Q \}$ have the following torsion matrices,

$$t_{\mu\nu} = \begin{pmatrix} 1 & 1 & 1 & \omega^{k_1} \\ 1 & 1 & 1 & \omega^{k_2} \\ 1 & 1 & 1 & \omega^{k_3} \\ \omega^{-k_1} & \omega^{-k_2} & \omega^{-k_3} & 1 \end{pmatrix}, \quad \theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & 0 & k_2 \\ 0 & 0 & 0 & k_3 \\ -k_1 & -k_2 & -k_3 & 0 \end{pmatrix}$$

(3.3)
where we have set $\theta_{\mu \nu} = \frac{1}{2} \log t_{\mu \nu}$. Therefore discrete torsion exist whenever at least one of the $k_a$ integers is non zero. Moreover since $P_{\alpha} P_{\beta} = P_{\alpha \beta}$, it follows that the expression of the $Z_i$s may also be written as $Z_i = x_i \otimes T_i$, where the $T_i$’s are five dimensional matrices realized as,

$$ T_i = P^r_i Q^{s_i} = \prod_{a=1}^{3} P^{r^a_i} Q^{s^a_i}, \quad T_i T_j = \tau_{ij} T_j T_i, \quad \tau_{ij} = \prod_{a=1}^{3} \omega_a^{s^a_i r^a_j - s^a_j r^a_i} = \omega^{r_i r_j - r_j r_i}, \quad (3.4) $$

where we have set $P = P_\omega$ and where the numbers $r^a_i$ and $s^a_i$ are integers modulo five related to the $k_i$ as $r_1 = \sum_{a=1}^{3} r^a_1 = k_1 + k_2 + k_3$, $s_1 = \sum_{a=1}^{3} s^a_1 = 3$ and $r_i = \sum_{a=1}^{3} r^a_i = -k_i$, $s_i = \sum_{a=1}^{3} s^a_i = -1$ for $i = 2, 3, 4$. One of the remarkable features of the solution (2.14) together with eq (3.4) is that it has only a manifest $Z_5$ sub symmetry and so constitutes a special class of realization of the NC quintic. More general solutions should have a full manifest $Z^3_5$ symmetry. Larger class of solutions corresponds to take the $T_i$ operators of eq (3.4) as $T_i = \prod_{a=1}^{3} E^a_i J^a_i$ with $E_a$s and $J_a$s given by

$$ E_1 = P_1 \otimes I_{id} \otimes I_{id}, \quad E_2 = I_{id} \otimes P_2 \otimes I_{id}, \quad E_3 = I_{id} \otimes I_{id} \otimes P_3, \quad J_1 = Q_1 \otimes I_{id} \otimes I_{id}, \quad J_2 = I_{id} \otimes Q_2 \otimes I_{id}, \quad J_3 = I_{id} \otimes I_{id} \otimes Q_3. \quad (3.5) $$

These are respectively the generators of $Z^3_5$ and the group of automorphisms of their characters. In what follows, we explore further this general solution and give its geometric interpretation in terms of D-branes wrapping the compact manifold.

### 3.1.1 Torsion and NC rational torus

The solution $Z_i = x_i \otimes T_i$ may be given a remarkable geometric interpretation. The $Z_i$s are the local coordinates of a NC fiber bundle whose base is a Chart of $\mathbb{CP}^4$ and its fiber is a NC rational torus $\mathcal{T}^2$. Generic points of this NC variety are then parameterized as $(x_i; P_{\omega_1}, Q_{\omega_1}; P_{\omega_2}, Q_{\omega_2}; P_{\omega_3}, Q_{\omega_3})$ where now $P_{\omega_a}$ and $Q_{\omega_a}$ are viewed as the cycles of a rational torus

$$ P^5_{\omega_a} = Q^5_{\omega_a} = I, \quad Q_{\omega_a} P_{\omega_a} = \omega_{\omega_a} P_{\omega_a} Q_{\omega_a}. \quad (3.6) $$

and where $\omega^3_\omega = 1$. This representation shows clearly that discrete torsion associated with each $Z_5$ subgroup factor induces a 2d NC torus fibration of the quintic. Since this 2d NC rational torus is finite dimensional\(^2\), the original commutative coordinates $z_{i1}$ are now replaced by $5 \times 5$ matrices as

$$ z_{i1} \rightarrow Z_i = \sum_{k,l=1}^{5} Z_i^{kl} |k><l|, \quad (3.7) $$

where we have used the $\{e_{ij} = |i><j|\}$ matrix basis$\{|i>|$ and $\{<j\}$, with $<j|i| = \delta_{ij}$. In the case the three $Z_5$ factors are taken into account, one has a 6d NC torus fibration. In this case, the matrices are of order $5^3$ and then the $k$ and $l$ indices in the expansion (3.7) should be thought of as multi-indices; that is $k = (k_1, k_2, k_3)$ and $l = (l_1, l_2, l_3)$.

### 3.1.2 Branes

Due to discrete torsion, we see form the above eq that the algebraic structure of the D-branes wrapping the compact manifold change. Brane points $\{z_i\}$ of commutative geometry become now fibers based on $\{z_i\}$ and valued in the group representation $\mathcal{D}(Z^3_5)$ as shown on eq (3.7). Following [17], this solution

\(^2\)Note that for the special limit of irrational tori corresponding to taking the $k \rightarrow \infty$ limit of $Z_5 \sim U(1)$, non compact extra dimensions appear. The infinite number of fractional $Dp$ is mapped to a non compact $Dp + 2$ brane in agreement with the result of [10] [17].
has a nice interpretation in terms of quiver diagrams. Associating to each \( e_{kl} \) matrix vector basis, a segment \([k, l]\) oriented from \( k \) to \( l \) (an arrow \( \rightarrow^{kl} \)) and to each \( e_{kk} = \pi_k \equiv |k > < k| \) projector, a loop starting and ending at the position \( k \) as shown on the following table, one may draw a quiver diagram for each \( Z_i \) matrix generator of the NC algebra.

| Operators | Diagrams |
|-----------|----------|
| \( a^+_k \equiv |k > < k + 1| \) | \( k \rightarrow^{(k+1)} \) |
| \( a^-_k \equiv |k + 1 > < k| \) | \( (k+1) \leftarrow \rightarrow^{k} \) |
| \( \prod_{j=0}^{n} a^+_k a^-_{k+j} \equiv |k > < k + n| \) | \( k \rightarrow^{(k+n)} \) |
| \( \prod_{j=0}^{n} a^-_{k+j} \equiv |k + n > < k| \) | \( (k+n) \leftarrow \rightarrow^{k} \) |
| \( \pi_k = a^+_k a^-_k = |k > < k| \) | \( k \equiv \rightarrow^{(k+1)} \equiv \bigcirc \equiv \bullet \) |
| \( \pi_k = a^+_k a^-_{k+1} a^+_k a^-_{k-1} = |k > < k| \) | \( (k-1) \equiv \bullet \equiv \bigcirc \equiv k \equiv \bullet \) |

Using these rules, one sees that for the quintic the generic quiver diagram for the \( Z_i \) operators is given by a polygon with five vertices (a pentagon) and in general twenty links joining the various vertices; see figure 1. Since matrix basis vectors type \( e_{k(k+n)} = |k > < k + n| \) can be usually decomposed as \( e_{k(k+n)} = a^+_k a^-_{k+1} a^+_k a^-_{k+n} \), where \( a^+_k = e_{k(k+1)} \), one concludes that points in NC quintic geometry are roughly speaking described by pentagons. On the NC quintic the D-branes acquire an internal structure and consequently singularities of the original commutative manifolds are resolved by NC geometry.

**3.2 Discrete Torsion Matrix of \( \mathbb{Z}_5^3 \)**

Here we want to study general \( D(\mathbb{Z}_5^3) \) representations in presence of torsion and derive their link with complex three dimension NC rational torus \( T^6_{\omega} \). We also give the quiver diagrams associated with these representations. To that purpose, we shall first consider the simplest situation where all \( \mathbb{Z}_5 \) factors commute amongst themselves; then we discuss the case where they do not commute.

**3.2.1 Free torsion case**

Naively, the geometric symmetry \( \mathbb{Z}_5^3 \) can be seen as the product of three abelian \( \mathbb{Z}_5 \) group factors whose generators may be defined by help of the tensor product as follows:

\[
E_1 = P_1 \otimes I_{id} \otimes I_{id}, \quad E_2 = I_{id} \otimes P_2 \otimes I_{id}, \quad E_3 = I_{id} \otimes I_{id} \otimes P_3,
\]
The $E_a$s are the generators of the three $\mathbb{Z}_5$ factors of $\mathbb{Z}_5^3$; they satisfy the cyclic property $E_a^5 = I_d$. Since they are commuting operators, $E_aE_b = E_bE_a$, they can be diagonalized simultaneously in the same basis $\{|a,i>; 1 \leq a \leq 3; 1 \leq i \leq 5\}$. As such the $P_a$s can be thought of as in eq (2.13) and $E_a$s as diagonal blocks of matrices. Using the convention notation $a_{a,a}^+ = |a,n \rangle \langle a,n+1|$, $a_{a,a}^- = |a,n+1 \rangle \langle a,n|$, $\pi_n = a_{a,a}^+ a_{a,a}^-$ and the graphic representations of figure 2, it is not difficult to see that group elements of $\mathbb{Z}_5$ and their automorphisms can be decomposed as follows,

$$I_{id} = \sum_{n=1}^{5} \pi_n, \quad P_a = \sum_{n=1}^{5} a_{a,n} \pi_n, \quad Q_a = \sum_{n=1}^{5} a_{a,n}^+, \quad Q_a^{-1} = \sum_{n=1}^{5} a_{a,n}^-;$$

(3.10)

For explicit computations, we shall drop out the index $a$ by working in special five dimensional matrix realizations.

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Figure 2: The completely reducible diagram (Fig2a) represents the identity operator of 5d representation $D(\mathbb{Z}_5)$. Fig2b is an oriented pentagon representing $Q$ automorphism operator while fig2c is its inverse.

Note that in absence of torsion generic elements of $\mathbb{Z}_5^3$ are denoted as $g = g_1 \otimes g_2 \otimes g_3$ and similarly for their representations $D(g) = D(g_1) \otimes D(g_2) \otimes D(g_3)$ which read in terms of the $E_a$ generators as follows $D(g) = \prod_{i_1,i_2,i_3=1}^{5} \gamma_{i_1,i_2,i_3} E_{i_1}^{g_1} E_{i_2}^{g_2} E_{i_3}^{g_3}$, where the $\gamma_{i_1,i_2,i_3}$ coefficients are such that $\gamma_{i_1,i_2,i_3} = 1$. Since the group multiplication law $gg'$ of elements $g$ and $g'$ of $\mathbb{Z}_5^3$ is defined as usual by performing multiplications of individual elements; that is $gg' = g_1g'_1 \otimes g_2g'_2 \otimes g_3g'_3$; we have $E_a^d = I_{D(\mathbb{Z}_5)}$, with $I_{D(\mathbb{Z}_5)}$ stands for $I_{id} \otimes I_{id} \otimes I_{id}$; the group representation identity. The above eq tells us that the dimension $d$ of $D(\mathbb{Z}_5^3)$ reads in terms of the $d_i$ dimensions of the three $D(\mathbb{Z}_5)$ factors as $d = d_1d_2d_3$. As all elements of $\mathbb{Z}_5^3$ can be expressed as powers of $E_a$s, we will focus our attention now on the monomials $E_{a}^{n_1}E_{b}^{n_2}E_{c}^{n_3}$; $1 \leq n_a \leq 5$.

### 3.2.2 Discrete Torsion

In presence of discrete torsion\(^3\), the $g_a$ elements of the three $\mathbb{Z}_5$ factors of $\mathbb{Z}_5^3$ do not commute among each others. Geometrically this situation corresponds to the case where the three complex cycles $\mathcal{T}_{2a}$ associated with discrete group factors do not commute. Generic couples $(F_a, F_b)$ of elements of $\mathcal{T}_{2a} \otimes \mathcal{T}_{2b}$ satisfy then,

$$F_a F_b = t_{ab} F_b F_a,$$

(3.11)

where $t_{ab}$ is the torsion matrix between the three $\mathbb{Z}_5$ factors of the orbifold group. Note that this relation is quite similar to the one defining the rational torus $\mathcal{T}_2^2$, eq (3.12). Instead of one $\mathbb{Z}_5$, we have now the full $\mathbb{Z}_5^3$ orbifold group. Eq (3.11) define a complex three dimension NC rational torus $\mathcal{T}_2^6$ where a priori

\(^3\)With one $\mathbb{Z}_5$ factor, one has discrete torsion induced by quantum symmetry. Here we discuss the general case of discrete torsion between the various geometric orbifold subgroup factors of $\mathbb{Z}_5^3$.\]
the six real cycles are non commuting. Moreover, like for the $E_a$, the $F_a$ satisfy equally $F_a^5 = I_{D(Z_5^*)}$ requiring that the matrix torsion should be of the form $t_{ab} = \exp i\frac{2\pi}{5}\theta_{ab}$, where $\theta_{ab}$ is antisymmetric $3 \times 3$ matrix with integer entries.

Representations There are different ways to represent eq. (3.11) by using tensor products or/and direct sums involving the $d_a$ dimension matrix generators $E_a$ of the three $D(Z_5^*)$ and the $Q_a$ automorphisms rotating their characters. In the case where one uses tensor products, the matrix representation we get has dimension $d = d_1d_2d_3$ containing as a particular case the solutions obtained in [22]. To see how this representation is built, we introduce the following parameterization of torsion $\theta_{ab} = \eta_{ab} - \eta_{ba}$ where $\eta_{ab}$ is the invertible $SL(3,Z)$ matrix considered before. Note in passing that the fact that $\eta_{ab}$ belongs to $SL(3,Z)$ appears here as a necessary condition for consistency; but this may have a string interpretation in terms of allowed values of the NS B field along the rational elliptic fibers. This parameterization allows us to rewrite eq. (3.11) as $\omega^{\eta_{ab}} F_a F_b = \omega^{\eta_{ba}} F_b F_a$ whose matrix solution reads as

$$F_1 = P_1 \otimes Q_2^{\eta_{12}} \otimes Q_3^{\eta_{13}}, \quad F_2 = Q_1^{\eta_{21}} \otimes P_2 \otimes Q_3^{\eta_{23}}, \quad F_3 = Q_1^{\eta_{31}} \otimes Q_2^{\eta_{32}} \otimes P_3,$$  \hspace{1cm} (3.12)

where the $P_a$'s and the $Q_a$'s are as in eq. (2.10) with $P_a Q_b = \alpha_a Q_b P_a \delta_{ab}$ and $\alpha_a$ fifth roots of unity. To fix the ideas, we set for simplicity $\alpha_1 = \alpha_2 = \alpha_3 = \omega$ and consider the special case where the three basis of the three representation factors of $D(Z_5)$, namely $\{a, i \geq 1, 1 \leq i \leq d_a; 1 \leq a \leq 3\}$, have the same dimension $d_1 = d_2 = d_3$. In this case $d_1 = d_2 = d_3 = 5$ and so $P_1 = P_2 = P_3 = P$ and $Q_1 = Q_2 = Q_3 = Q$ with a realization as in eqs (3.6). Thus the $F_a$ reduce to:

$$F_1 = P \otimes Q^{\eta_{12}} \otimes Q^{\eta_{13}}, \quad F_2 = Q^{\eta_{21}} \otimes P \otimes Q^{\eta_{23}}, \quad F_3 = Q^{\eta_{31}} \otimes Q^{\eta_{32}} \otimes P.$$  \hspace{1cm} (3.13)

This matrix representation has an a 53 order and satisfies $F_a^5 = I$. It extends eqs (3.9) which appears as special cases. Indeed, the $d$ dimensional generic representations of the $F_a$ shows that it is possible to build different realizations for $F_a$ generators (3.11). The representation eqs (2.10) and (3.4) built in [22], correspond to take $(d_1, d_2, d_3)$ equal to either $(5,1,1)$, $(1,5,1)$, or $(1,1,5)$ respectively obtained by setting $\eta_{a2} = \eta_{a3} = 0$, $\eta_{a1} = \eta_{a3} = 0$ and $\eta_{a1} = \eta_{a2} = 0$. In all of these cases, the fiber of the NC quintic is just the NC rational torus $T^4_d$. This property may be viewed as the geometric interpretation of the codimension one fixed planes considered in [17]. In addition to these examples, there are other special cases such as the 25 dimensional matrix realizations of the $F_a$'s. They correspond to the situations where the fibration is $T^4_d$ and $(d_1, d_2, d_3) = (5,1,1)$ as well as permutations.

Quiver Diagrams Like for the case of one abelian factor, one can also build the projectors for full the $Z_5^*$ group. Using the individual $Z_5$ projectors $\pi_{k_a} = \frac{1}{5} \sum_{i=1}^{5} \omega^{-k_aP_i}$, we can construct various kinds of projectors on the representation space of $Z_5^*$. First, the $\Pi_{k_a}$ projectors on the $Z_5$ representation spaces:

$$\Pi_{k_1} = \pi_{k_1} \otimes I_{d_1} \otimes I_{d_1}, \quad \Pi_{k_2} = \pi_{k_2} \otimes \pi_{k_2} \otimes I_{d_1}, \quad \Pi_{k_3} = I_{d_1} \otimes I_{d_1} \otimes \pi_{k_3},$$  \hspace{1cm} (3.14)

They have quiver diagrams more a less similar to that of $\pi_{k_3}$, except that now we have general realizations coming from the decomposition of the identity operators. The full quiver diagram is given by the cross product of the individual graphs and one ends with higher dimensional lattices. Second, the $\Pi_{(k_a,k_b)}$ and $\Pi_{(k_1,k_2,k_3)}$ projectors on the $Z_5^*$ and $Z_5^*$ representation spaces, respectively obtained by taking tensor products of $\Pi_{k_a}$'s:

$$\Pi_{(k_a,k_b)} = \Pi_{k_a} \Pi_{k_b}, \quad \Pi_{(k_1,k_2,k_3)} = \Pi_{k_1} \Pi_{k_2} \Pi_{k_3}.$$  \hspace{1cm} (3.15)
Accordingly, the identity matrix \( \mathbf{I}_{D(G)} \) can be decomposed in different, but equivalent, ways as shown here below:

\[
\mathbf{I}_{D(G)} = \sum_{k_a=1}^{5} \Pi_{k_a} = \sum_{k_a,k_b=1}^{5} \Pi_{k_a,k_b} = \sum_{k_1,k_2,k_3=1}^{5} \Pi_{k_1} \Pi_{k_2} \Pi_{k_3}.
\] (3.16)

So \( \mathbf{I}_D \) can be represented by a completely reducible quiver diagram with \( d_1 d_2 d_3 = 5 \times 5 \times 5 \) vertices.

Similar expansion to eqs (3.16) may be written down for the generators \( J_a \) of the quantum symmetries. Setting

\[
A_{k_1}^\pm = a_{1,k_1}^\pm \otimes I_{id} \otimes I_{id}, \quad A_{k_2}^\pm = I_{id} \otimes a_{1,k_1}^\pm \otimes I_{id}, \quad A_{k_3}^\pm = I_{id} \otimes I_{id} \otimes a_{k_3}^\pm,
\] (3.17)

we can write for instance \( J_a \) and \( J_{a_1} J_{a_2} \) as follows;

\[
J_a = \sum_{k_a=1}^{5} A_{k_a}^+, \quad J_{a_1} J_{a_2} = \sum_{k_{a_1},k_{a_2}=1}^{5} A_{k_{a_1}}^+ A_{k_{a_2}}^+.
\] (3.18)

While the quiver diagram for the \( J_a \) is similar to that given by figures 2b, 2c and figures 3, the quiver diagrams associated with \( J_{a_1} J_{a_2} \) are obtained by taking cross products and are of type figure 4.

Figure 3: Fig3a represents the diagram of \( Q^2 \); the oriented links define the independent massless chiral fields on the \( D \) brane at singularity. Fig3b represents the quiver of \( Q^3 \). Fig 3c represents the diagram of the \( Q^4 \) operator and Fig3d represents the completely reducible quiver diagram of \( Q^5 \).

4 General Solutions

Here we give our general solutions for NCCY orbifolds extending the ones obtained in [2, 24]. These solutions exhibit manifestly both torsion dependence and the full orbifold geometric symmetry. The solution we will derive provide novel regularizations of NC field theories on orbifolds containing as special models gauge theories embedded in string theory. To avoid repetitions, we will treat simultaneously the example quintic threefolds \( Q \) and more generally all the elements of the class of homogeneous complex \( n \) \( (n > 1) \) dimension hypersurfaces \( \mathcal{H}_n \). To start recall that the algebraic relations defining the NC quintic \( Q^{(nc)} \) as appeared first in [2] reads as,

\[
Z_1 Z_2 = \alpha Z_2 Z_1, \quad Z_1 Z_3 = \alpha^{-1} \beta Z_3 Z_1, \quad Z_1 Z_4 = \beta^{-1} Z_4 Z_1, \quad Z_2 Z_3 = \alpha \gamma Z_3 Z_2,
\]

\[
Z_2 Z_4 = \gamma^{-1} Z_4 Z_2, \quad Z_3 Z_4 = \beta \gamma Z_4 Z_3, \quad Z_i Z_5 = Z_5 Z_i, \quad i = 1, 2, 3, 4; \quad (4.1)
\]
where $\alpha, \beta, \gamma$ are fifth roots of unity. In [24], it was noted that the above relations are very special and can be generalized to any complex $n$-dimension holomorphic homogeneous CY hypersurfaces as $Z_iZ_j = \beta_{ij}Z_jZ_i$, $i, j = 1, \ldots, (n+1)$ and $Z_iZ_{d+2} = Z_{d+2}Z_i$ with $i = 1, \ldots, (n+1)$ and $\beta_{kl}$s are realized as,

$$\beta_{ij} = \exp i \left( \frac{2\pi}{n+2} m_{ab} \eta^a \eta^b j_{ij} \right) = \omega^{m_{ab} \eta^a \eta^b j_{ij}}. \quad (4.2)$$

In this relation $\omega = \exp i \frac{2\pi}{n+2}$ and $m_{ab}$ is some given matrix with integer entries which remained without interpretation in [24]. Here, we will prove that $m_{ab}$ is equal to

$$m_{ab} = \eta_{ab}^{-1} - \eta_{ba}^{-1} \quad (4.3)$$

where $\eta_{ab}$ is as before. The antisymmetric part of the $\eta_{ab}$ matrix encodes then torsion and it is required to belong to $\text{SL}(n,Z)$. 

### 4.1 More on NC Quintic

First of all note that the NC quintic and more generally NCCY hypersurfaces are no uniquely defined. The following eqs may be also used as definitions for $Q^{nc}$ and $H_n^{(nc)}$,

$$\Phi_i \Phi_j = \Phi_j \Phi_i, \quad i, j = 1, \ldots, n+2$$

$$F_a \Phi_i = \Phi_i F_a \exp i \left( \frac{2\pi}{n+2} \eta^a \right), \quad (4.4)$$

$$F_a F_b = F_b F_a \exp i \left( \frac{2\pi}{n+2} \eta_{ab} \right).$$

These eqs have also a centre that coincide exactly with $H_n$. Therefore the two sets of matrix coordinates $Z_i$ and $\Phi_i$, of eqs (1.1, 4.2) and (4.4) should be linked. In fact as shown in [17], $Z_i$ and $\Phi_i$ are two Morita equivalent coordinates of $H_n^{(nc)}$ and so are related as

$$Z_i = \Gamma_i \Phi_i, \quad (4.5)$$

where $\Gamma_i$s are matrix operators which can be directly derived by comparing eqs (1.2) and (4.4). We will give their explicit expressions later on. For the moment, let us comment the absence of a relation such that $F_a \Phi_i = \Phi_i F_a \exp i \frac{2\pi \eta^a}{3}$ in eqs (1.1, 4.2). At first sight this seems a little bit ambiguous as the naive counting of the degrees of freedom in relations (1.2) and (4.4) do not match. However, this is not a problem since though the $F_a$ group generators do not appear manifestly in eqs (1.1, 4.2); they act as outer automorphisms on these eqs. It turns out that $F_a$ act trivially on $Z_i$; that is $F_a Z_i F_a^{-1} = Z_i$ which can be also rewritten like,

$$F_a Z_i = Z_i F_a \quad (4.6)$$

This relation will play an important role in building the general explicit fiber dependent solutions of eqs (1.2, 4.4). Indeed as $F_a \Phi_i = \Phi_i F_a \omega^{\eta^a}$ is just the Morita transformation of (1.1), one can use it to determine $\Gamma_i$. Acting by $F_a$ on (1.1) and using (1.2), one gets the following constraint on $\Gamma_i$; $F_a \Gamma_i = \omega^{-\nu_{ia}} \Gamma_i F_a$, where $\nu_{ia}$ are integers.

**Solutions of NC algebra** $Z_i Z_j = \beta_{ij} Z_j Z_i$: Using the matrix realization of the $E_a$ and $J_a$ generators of the rational torus fibers eqs (5.5) and (2.13), as well as the relations $E_a^{\eta^a} J_b^{\eta^b} = \delta_{ab} \omega \eta^a \eta^b J_b^{\eta^b} E_a^{\eta^a}$, it is not difficult to check that the $Z_i$s eq (4.2) are solved as,

$$Z_i = \prod_{a=1}^n E_a^{\eta^a} J_a^{\eta^a}. \quad (4.7)$$
Next, solving the constraint eq $F_a Z_i = Z_i F_a$ by using eq (4.7) and the explicit expression of the $F_a$s namely $F_a = \prod_{b=1}^{n} E_a J_b^n$, we find:

$$\sum_{b=1}^{n} \eta_{ab} b_i = q_i^a. \quad (4.9)$$

This relation shows that the $p_i^a$s are related to the $q_i^a$s via the torsion matrix as $p_i^a = \sum_{b=1}^{3} \eta_{ab}^{-1} q_i^b$. Since the $p_i^a$s are integers, eq (4.9) requires that the matrix $\eta_{ab}$ has to belong to $SL(n, \mathbb{Z})$ and shows moreover that $p_i^b$s satisfy themselves the identity,

$$\sum_{i=1}^{n} p_i^a = 0. \quad (4.10)$$

This eq (4.10) tells us whenever torsion is present, the orbifold of the hypersurface $\mathcal{H}_n$ admits an extra hidden discrete symmetry acting on the $z_i$’s as $z_i \rightarrow z_i \exp i \frac{2\pi}{n+2} \eta_{ab}^{-1} q_i^b$. This eq requires $\eta_{ab}$ to be invertible and can be viewed a geometric way to define orbifolds. For $\eta_{ab}^{-1}$ antisymmetric, the orbifolds have then a discrete torsion. Finally using eq (4.9) and comparing eq (4.8) with eq (4.2), one discovers eq (4.3).

**Solution of eqs (4.3):** Since the $\Phi_i$s commute, a natural solution corresponds is to take $\Phi_i$ as depending uniquely of the $E_a$s or again uniquely of $J_a$ generators. For the second case for instance, the $\Phi_i$s are realized as,

$$\Phi_i = z_i \prod_{a=1}^{n} J_a^n. \quad (4.11)$$

In addition to commutativity, this representation fulfills naturally $F_a \Phi_i = \Phi_i F_a \exp i \frac{2\pi q_i^n}{n+2}$ due to the basic relation $E_a^n J_b^n = \delta_{ab} \omega^{-\phi_a \phi_b} J_b^n E_a^n$. Moreover, comparing the two representations (4.7) and (4.11), one gets the expression of the automorphisms $\Gamma_i$ of eq (4.10),

$$\Gamma_i = \prod_{a=1}^{n} E_a^{\pi q_i^n}, \quad (4.12)$$

where the $p_i^a$s are as in eqs (4.9).

**More on Morita equivalence:** We end this discussion by noting that given a set of $q_i^n$ integers, defining the charges of the $z_i$ variables under the $Z_i^{n+2}$, and non symmetric $n \times n$ matrix $\eta_{ab}$ of SL(n; Z); we can build various, but Morita equivalent, realizations of the NC algebra describing $\mathcal{H}_n^{ac}$ where eqs (4.7) and (4.11) appear as two special coordinates basis amongst many others. If we let $\{F_a, W_i\}$ a generic basis of $\mathcal{H}_n^{ac}$ related to $\{F_a, \Phi_i\}$ as $W_i = \Omega_i \Phi_i$ where $\Omega_i = \Omega_i(E_a, J_a)$ are constrained as,

$$\Omega_i \Omega_j = \varepsilon_{ij} \Omega_j \Omega_i, \quad F_a \Omega_i = \kappa_{ai}^{-1} \Omega_i F_a. \quad (4.13)$$

In these eqs, $\varepsilon_{ij}$ and $\kappa_{ai}$ are some given phases satisfying $\varepsilon_{ij}^{n+2} = \kappa_{ai}^{n+2} = 1$. In this basis, the defining relations of the NC algebra for $\mathcal{H}_n^{ac}$ reads as,

$$W_i W_j = \tau_{ij}^{-1} \tau_{ij} W_j W_i, \quad F_a W_i = \kappa_{ai}^{-1} \lambda_a W_i F_a, \quad F_a F_b = F_b F_a \exp i \frac{2\pi \theta_{ab}}{n+2}. \quad (4.14)$$

---

4More general relations use $W_i = \Omega_i \Phi_j$. 

Computing the products $Z_i Z_j$ and $Z_j Z_i$, one gets the explicit expression of the $\beta_{ij}$ parameters namely,

$$\beta_{ij} = \exp i \frac{2\pi}{n+2} \sum_{a=1}^{n} [p_i^a q_j^a - p_j^a q_i^a]. \quad (4.8)$$
where $\tau_{ij}$, $\kappa_{ai}$ and $\lambda_{ai}$ are structure constants of the $H_n$. In these eqs, the $W_i$ generators do no longer commute among themselves nor with the group representation generators $F_a$ as it was the case for eqs (4.9) and (4.11) which are recovered as two extreme situations.

4.2 Example

To fix the ideas, let consider the example of the quintic by choosing $\Omega_4$ as $F_1^{(3)} F_2^{(3)} F_3^{(5)}$ with $r_4^i$ some given integers. In this case, the structure constants appearing in eqs (4.13) and (4.14) read as:

$$\tau_i = \omega^{\sum_a r_a^i q_a^i}, \quad \varepsilon_{ij} = \omega^{\theta_{ab} r_a^i r_b^j}, \quad \kappa_{ai} = \omega^{-\frac{1}{2} r_a^i}.$$ (4.15)

From these relations we see that for $\eta_{ab}$ symmetric, the structure constants $\varepsilon_{ij}$ and $\kappa_{ai}$ are torsion free ($\varepsilon_{ij} = 1$ and $\kappa_{ai} = 1$). Note that in the $\{W_i; F_a\}$ basis, the generators of the NC algebra of the quintic do not commute in general; except for the two following special cases where they take remarkable forms:

(a) $\{F_a; \Phi_i\}$ basis which is recovered by choosing the structure constants as $\tau_i^{-1} \varepsilon_{ij} \tau_j = 1$; i.e,

$$\varepsilon_{ij} = \tau_i \tau_j^{-1}, \quad \kappa_{ai} = 1.$$ (4.16)

In this case, the matrix $\Omega_i$ is just the inverse of $\Gamma_i$; that is $\Omega_i = \Gamma_i^{-1}$ eq (4.12). (b) $\{F_a; Z_i\}$ basis obtained from the generic $\{F_a, W_i\}$ frame by requiring commuting $Z_i$ and $F_a$ operators. This is equivalent to setting $\kappa_{ai} \sigma_{ai} = 1$ which imply in turns,

$$\sigma_{ai} = \kappa_{ai}^{-1}, \quad \beta_{ij} = \tau_i^{-1} \varepsilon_{ij} \tau_j, \quad \Omega_i = \Gamma_i.$$ (4.17)

Since the two sets of matrix generators $\{Z_i\}$ and $\{F_a\}$ decouple completely, the NC quintic $A[Z_i; F_a]$ is then described by a trivial fibration as $A[Z_i; F_a] \equiv A[Z_i] \otimes A[F_a]$.

5 Fractional Branes

The realization of the NC quintic we have studied here above concerns only the regular points of the algebra, that is non singular ones. In this section, we want to complete this analysis by considering the representations for singular points. This is not only important for the study of fractional branes at singularities but also for answering the question regarding the nature of fractional branes on the NC quintic and more generally on NCCY hypersurfaces. To do so, we shall first determine the various sets $S_{(\mu)}$ of singular points of orbifolds of the quintic. Then we give the corresponding singular solutions. At first sight and as far as the full $Z_5$ geometric symmetry is concerned we have only one fixed point under the $Z_5$ actions namely $(z_1, z_2, z_3, z_4, 1) = (0, 0, 0, 0, 1)$. This point belongs however to the $\mathbb{C}P^4$ projective space; but does not belong to the quintic $Q$; no point of the quintic is then fixed by the full symmetry. This property is valid for all CY hypersurfaces; no point of complex $n$ dimensional CY hypersurfaces $\mathcal{P}(z_1, \ldots, z_{n+2})$ is fixed under the full $Z_{n+2}$ geometric invariance. We will therefore consider points of $Q$ that are fixed under subgroups $G_{(a)}$ of $Z_5$. Then we describe the various fractional branes living at these singularities, the corresponding quiver diagrams and the massless chiral fields of the effective theory on the D branes. As there are several subgroups $G_{(a)}$ in $Z_5$, we shall fix our attention on two categories of subsymmetries; those isomorphic to $Z_5$; i.e $G_{[1]} \simeq Z_5$ and those isomorphic to $\mathbb{Z}_2^3$; i.e $G_{[2]} \simeq \mathbb{Z}_2^3$. The CY charges will be taken as in eqs (4.12). Generalization to subgroups of $Z_{n+2}$, though tedious, is a priori straightforward.
5.1 Fixed subspaces of $Q^{[\alpha]}$

We will consider first the spaces $S_{(a)}$ of fixed points under a generic $\mathbb{Z}_5$ factor of $\mathbb{Z}_3^5$. Then we examine the spaces $S_{(ab)}$ of fixed points under $\mathbb{Z}_2^5$ factors. To have an idea on how these spaces look like, it is interesting to think about the quintic homogeneous hypersurface eq(2.1) as a fiber bundle described by the following equation

$$P(z_1, \ldots, z_5) = \sum_{n=0}^{5} b_{n_1 \ldots n_5} z_1^{n_1} \ldots z_5^{n_5} = 0,$$

where $b_{n_1 \ldots n_5} = b_{n_1 \ldots n_5}(z_1, \ldots, z_5)$ are some given monomials in the $z_1, \ldots, z_5$ complex variables, with appropriate degrees. Let us give examples on how this works in practice.

5.1.1 $\mathbb{CP}^2 \bowtie S_1$ Fibration

A simple example for realizing fibrations of the quintic consists to rewrite $P(z_1, \ldots, z_5)$ as,

$$P(z_1, \ldots, z_5) = b_{00} + b_{11} z_1 z_2 + b_{50} z_1^5 + b_{05} z_2^5,$$

where the $b_{mn}$ coefficients are holomorphic functions given by

$$b_{00}(z_3, z_4, z_5) = z_3^5 + z_4^5 + z_5^5, \quad b_{11}(z_3, z_4, z_5) = a_0 z_3 z_4 z_5,$$
$$b_{50}(z_3, z_4, z_5) = 1, \quad b_{05}(z_3, z_4, z_5) = 1.$$

The remaining others are equal to zero. Eqs(5.1,5.2) mean that the quintic may be viewed as a fibration space with base $\mathbb{CP}^2$ and fiber $S_1$ given by,

$$z_1^5 + z_2^5 + b_1 z_1 z_2 = 0.$$  

This relation is invariant under the change $(z_1, z_2) \rightarrow (\omega z_1, \omega^{-1} z_2)$; that is under the $\mathbb{Z}_5$ subsymmetry of charges $q_1^1$; the $B_1$ base is not affected under this change. The symmetry of the $S_1$ fiber has one fixed point namely $(0,0)$ and so $S_1$ is singular at the origin $z_1 = z_2 = 0$. To see what eq(5.3) represents, note that from the $z_1$ and $z_2$ variables, one can build three invariant namely $u = z_1^5, v = z_2^5$ and $w = z_1 z_2$ having an $A_4$ singularity. In terms of the new variables, the equation of the $S_1$ complex curve reads as

$$u + v + b w = 0; \quad uv = w^5.$$  

Therefore near the fixed point $z_1 = z_2 = 0$, the $\mathbb{Z}_5$ orbifold of the commutative quintic $Q$ can be then viewed as given by the fiber bundle $\mathbb{CP}^2 \bowtie S_1$ with a vanishing two cycle at $z_1 = z_2 = 0$. Before going ahead, let us comment briefly the complex resolution of this kind of singularity and give its toric
geometry diagram representation as shown here below,

| A₄ Singularity | uv = w⁵ |
|----------------|---------|
| Complex Resolution of A₄ | uv = w⁵ + α₄w⁴ + α₃w³ + α₂w² + α₁w + α₀ |

Rules

i) White nodes such as are associated to each non compact C variables x and y

ii) Nodes such as are associated with blown up spheres with self intersection (−2)

iii) Each link ←→ represents intersecting spheres with a weight (1)

Quiver Diagrams

i) Quiver diagram for the resolution of A₂:

ii) Quiver diagram for the resolution of A₄

More details on this graphic representation of Kahler and complex resolution of ADE singularities as well as their applications in string compactifications may be found in [42, 45].

5.1.2 B ≪ S₁₂ Fibration

The second example we want to give corresponds to the fibration B ≪ F ≡ CP¹ ≪ S₁₂. In this case, the analogue of the above equations read for S₁₂ as follows:

P(z₁, ..., z₅) = b₀₀₀ + b₁₁₁ z₁z₂z₃ + b₅₀₀ z₁⁵ + b₀₅₀ z₂⁵ + b₀₀₅ z₃⁵, (5.6)

where the bₘₙₙ coefficients are as follows,

b₀₀₀(z₄, z₅) = z₄⁵ + z₅⁵, b₁₁₁(z₄, z₅) = a₀z₄z₅,

b₅₀₀(z₄, z₅) = 1, b₀₅₀(z₄, z₅) = 1, b₀₀₅(z₄, z₅) = 1, (5.7)

and all others are equal to zero. The equation of the singular fiber S₁₂ is given by z₅⁵ + z₂⁵ + z₃⁵ + b z₁z₂z₃ = 0; this is a complex surface which has one fixed fiber point at (0, 0, 0) under the change (z₁, z₂, z₃) → (ω²z₁, ω⁻¹z₂, ω⁻¹z₃) generating the Z₅ ⊗ Z₅ subsymmetry with charges (q₁, q₂). Note that this subsymmetry does not affect the B₁₂ base space. Now introducing the following four invariant

u₁ = z₁⁵, u₂ = z₂⁵, u₃ = z₃⁵ and t = z₁z₂z₃, one sees that they are related as

u₁u₂u₃ = t⁵; (5.8)

while the S₁₂ complex surface reads in terms of these invariants as u₁ + u₂ + u₃ + bt = 0. From this relation, one recognizes the two individual singularities associated with each factor of the Z₅ ⊗ Z₅ symmetry. These are given by the following A₄ eqs;

u₁u₂ = t⁵ / u₃, for u₃ ≠ 0,

u₁u₃ = t⁵ / u₂, for u₂ ≠ 0. (5.9)
Eq.(5.8) describes the case where both of the above singularities collapse; it has a nice description in terms of quiver diagrams.

| Singularity | Equations |
|-------------|-----------|
| $u_1u_2u_3 = t^5$ | |

### Complex Resolution

| Rules | Equations |
|-------|-----------|
| Same rules as in previous table. | $u_1u_2u_3 = t^5 + \alpha_4t^4 + \alpha_3t^3 + \alpha_2t^2 + \alpha_1t + \alpha_0$ |

### Symmetries

The following symmetries are associated with the fixed points of the discrete symmetries reported above:

\[
\begin{align*}
Z_5 &: \quad z_1 \rightarrow \omega z_1, \quad z_3 \rightarrow \omega^{-1}z_3, \quad z_1z_3 \text{ is an invariant.} \\
Z_5 \otimes Z_5 &: \quad z_1 \rightarrow \omega^2 z_1, \quad z_2 \rightarrow \omega^{-1}z_2, \quad z_3 \rightarrow \omega^{-1}z_3, \\
Z_5 &: \quad z_1 \rightarrow \omega z_1, \quad z_2 \rightarrow \omega^{-1}z_2, \quad z_1z_2 \text{ is an invariant.}
\end{align*}
\]

### Quiver Diagram

Here it is represented the three graphs associated to the resolution of the singularities of the discrete symmetries reported above.

Note that the $Z_5 \otimes Z_5$ symmetry has a total charge $q^1 + q^2 = (2, -1, -1, 0, 0)$, behaving then as the $Z_5$ diagonal symmetry. The remaining off diagonal factor has a total charge $q^1 - q^2 = (0, -1, 1, 0, 0)$.

### 5.1.3 Other Fibrations

Following the same method we have used for $S_1$ and $S_{12}$, we can work out the other $B_a \bowtie S_a$ and $B_{(ab)} \bowtie S_{(ab)}$ quintic fibrations associated with the natural subgroups of $Z_5^2$. Denoting the various invariants under the subgroups of $Z_5^3$ as $u = z_1^5$, $w_{ij} = z_i z_j$ and $t_{ijk} = z_i z_j z_k$, one can work out the different equations of the $S_a$ and fibers $S_{(ab)}$: the basis $B_a$ and $B_{(ab)}$ are respectively given by the $\mathbb{CP}^2$ and $\mathbb{CP}^4$ complex projective spaces. The results are collected in the table eq.(5.11).

| Fibers | Equations |
|--------|-----------|
| $S_1$  | $u_1 + u_2 + b_1 w_{12} = 0$ |
|        | $u_1u_2 = u_{12}^5$. |
| $S_2$  | $u_1 + u_3 + b_2 w_{13} = 0$ |
|        | $u_1u_3 = u_{13}^5$. |
| $S_3$  | $u_1 + u_4 + b_3 w_{14} = 0$ |
|        | $u_1u_4 = u_{14}^5$. |

| Fibers | Equations |
|--------|-----------|
| $S_{12}$ | $u_1 + u_2 + u_3 + b t_{123} = 0$, |
|        | $u_1u_2u_3 = t_{123}^5$. |
| $S_{23}$ | $u_1 + u_3 + u_4 + b t_{134} = 0$, |
|        | $u_1u_3u_4 = t_{134}^5$. |
| $S_{13}$ | $u_1 + u_2 + u_4 + b t_{124} = 0$, |
|        | $u_1u_2u_4 = t_{124}^5$. |

Having these results at hand, we turn now to give some details by studying the fixed spaces under the orbifold subgroups $Z_5$ and $Z_5^2$. We first consider the orbifolds $Q^{[1]} \simeq R'/Z_5$ and then the $Z_5^2$ orbifolds $Q^{[2]} \simeq R''/Z_5^2$. These orbifolds correspond also to start from eq.(5.21) and choose either one of the three $q_p^2$ vector charges non vanishing say $q_1^2 = (1, -1, 0, 0, 0)$ while the two others $(q_2^2) = (q_3^2) = 0$; or two vector charges non vanishing while the third is zero such as for instance $q_1^2 = (1, -1, 0, 0, 0)$, $q_2^2 = (1, 0, -1, 0, 0)$ and $(q_3^2) = 0$. 

\[(q_1^2) = 0.\]
5.2 Fractional Branes on $Q^{[1]}$

As there are three manifest $Z_5$ sub-symmetry factors in the orbifold group $Z_5^3$, each one generated by an operator $F_a$, one can write down three corresponding $B_a \triangleright S_a$ fibrations for the commutative quintic. The $B_a$ spaces are the bases of the fiberation and the $S_a$'s their fibers.

**Example:** $Q^{[1]} \simeq B_1 \triangleright S_1$

Consider the $Z_5$ subgroup generated by $E_1$ with $q_i^1$ charges taken as $q_i^1 = (1, -1, 0, 0, 0)$; the $B_1 \triangleright S_1$ fibration of the quintic is just that given by eqs (5.1, 5.3). Since we are working in the coordinate patch $z_5 = 1$, the $B_1$ base is a patch of $\mathbb{CP}^2$; that is $B_1 \sim \mathbb{C}^2$ parameterized by the $z_3$ and $z_4$ complex coordinates. The codimension two fiber $S_1$ is given by the following complex curve with an $A_4$ singularity,

$$u + v + \frac{v^5}{u} = 0. \quad (5.12)$$

**Results**

The $B_a \triangleright S_a$ fibrations of the quintic are completely determined by the $q_i^a$ CY charges. The $B_a$ base manifolds are parameterized by those holomorphic coordinates $z_i$ with $q_i^a = 0$ while the $S_a$ fibers are parameterized by those complex variables with non zero $q_i^a$ charges with fixed points at the origin. For the $q_i^a$'s taken as in eq (5.11), we have the following results collected in the table eq (5.13).

| Generators $F_a$ | $F_1$ | $F_2$ | $F_3$ |
|------------------|-------|-------|-------|
| Fixed Points     | $(0, 0, z_3, z_4, z_5)$ | $(0, z_2, 0, z_4, z_5)$ | $(0, z_2, z_3, 0, z_5)$ |
| $B_a = \mathbb{CP}^2$ | $B_{12} = \{z_3, z_4, z_5\}$ | $B_{23} = \{z_2, z_4, z_3\}$ | $B_{31} = \{z_2, z_3, z_5\}$ |
| $S_a$ Fibers     | $u_1 + w_{12} + \frac{w_{12}}{w_{13}} = 0$ | $u_1 + w_{13} + \frac{w_{13}}{w_{12}} = 0$ | $u_1 + w_{14} + \frac{w_{14}}{w_{13}} = 0$ |
|                  | $u_1 = z_5^1, u_2 = z_5^2$, | $u_1 = z_5^5, u_3 = z_5^3$, | $u_1 = z_5^1, u_4 = z_5^5$, |
|                  | $u_{13} = z_1 z_3$. | $u_{12} = z_1 z_2$. | $w_{14} = z_1 z_4$. |

Having the above features in mind, the singular representations of the NC quintic may be obtained by starting from the regular representation eqs (1.14) and taking the appropriate limits. For the $S_1$ singular space for instance, the field moduli are obtained from eqs (1.14) by setting $q_i^1 = q_i^2 = 0$ and taking the zero limit of $z_1$ and $z_2$. We have for the fiber $S_1$,

$$\Phi_1 = (Q \otimes I_{id} \otimes I_{id}) \lim_{z_1 \to 0} z_1, \quad \Phi_2 = (Q^{-1} \otimes I_{id} \otimes I_{id}) \lim_{z_1 \to 0} z_2, \quad F_1 = P \otimes Q^{n_{12}} \otimes Q^{n_{13}}, \quad (5.14)$$

and for $B_1$ base,

$$\Phi_3 = z_3 (I_{id} \otimes I_{id} \otimes I_{id}), \quad \Phi_4 = z_4 (I_{id} \otimes I_{id} \otimes I_{id}), \quad \Phi_5 = z_5 (I_{id} \otimes I_{id} \otimes I_{id}),$$

$$F_2 = Q^{n_{21}} \otimes P \otimes Q^{n_{23}}, \quad F_3 = Q^{n_{31}} \otimes Q^{n_{32}} \otimes P. \quad (5.15)$$

As one sees, once the limit to the singular point is taken, the non vanishing matrix field moduli $\Phi_i$, $i = 3, 4, 5$ are proportional to the identity $(I_{id} \otimes I_{id} \otimes I_{id}) = I_{D(G)}$ of the group representation $D(G)$; i.e. $\Phi_i = z_i \otimes I_{D(G)}$. This is a very remarkable feature at singularity and has algebraic and brane interpretations.

**Fractional branes on $R'/Z_5$**

To fix the ideas, let start from the $D9$ brane of type IIB string wrapped on the quintic $Q$. Let $(x^\mu; y_1, y_2, y_3)$ denote the $D9$ coordinates with $x^\mu = (x^0, x^1, x^2, x^3)$ being the longitudinal non compact variables (representing a $D3$ brane embedded in $D9$) and the $y_i$'s the compact transverse
complex coordinates of the wrapped $D9$ branes ($D9 \sim D3 \times Q$). In the coordinate patch $z_5 = z_4 = 1$, the $y$ coordinates may be imagined as related to those of the quintic as,

$$y_1 = z_1, \quad y_2 = z_2, \quad y_3 = \frac{-a_0 z_1 z_2 z_3}{(2 + z_1^5 + z_2^5 + z_3^5)}.$$  

(5.16)

In the case of $Q^{[1]}$ with discrete torsion, the wrapped $D9$ becomes a NC brane generated by the algebra eqs(4.11). At the fixed point of $Q^{[1]}$, a real two cycle shrinks to zero and the original NC wrapped $D9$ reduces to a NC wrapped $D7$ described by;

$$\Phi_3 = z_3 I_{D(Z_5)}, \quad \Phi_4 = z_4 I_{D(Z_5)}, \quad \Phi_5 = z_5 I_{D(Z_5)},$$

$$F_1 = P, \quad F_2 = Q^{21}, \quad F_3 = Q^{31}.$$  

(5.17)

The singular modes at the orbifold point are carried by the $Q$ operator and its inverse $Q^{-1}$ as shown on the following eqs.

$$\Phi_1 = Q \lim_{z_1 \to 0} z_1, \quad \Phi_2 = Q^{-1} \lim_{z_2 \to 0} z_2.$$  

(5.18)

However due to the complete reducibility property of $I_{D(Z_5)}$ namely $I_{D(Z_5)} = \sum_{n=1}^{5} \Pi_n$, eqs(5.17) describe in fact a set of five commuting Euclidean wrapped $D7$ branes parameterized as

$$\Phi_{3,n} = z_3 \Pi_n, \quad \Phi_{4,n} = z_4 \Pi_n, \quad \Phi_{5} = z_{5,n} \Pi_n.$$  

(5.19)

Therefore at the orbifold point of $\mathcal{R}'/\mathbb{Z}_5$, we have five fractional wrapped $D7$ branes. Moreover since near the singularity, $\Phi_1$ and $\Phi_2$ split as

$$\Phi_1 = \left( \lim_{z_1 \to 0} \sum_{n=1}^{5} a_n^+ \right), \quad \Phi_2 = \left( \lim_{z_2 \to 0} \sum_{n=1}^{5} a_n^- \right),$$

(5.20)

there are also massless modes $\phi_{1,n} \sim a_n^+$ and $\phi_{2,n} \sim a_n^-$ living on the wrapped $D7$ branes. They are propagating modes traveling between $\Phi_{3,n}$ and $\Phi_{4,n} \Pi_n$ fractional branes. The quiver diagram representing the fractional D branes is also a $\Delta_5$ pentagon with wrapped $D7$ sitting at the vertices and massless modes represented by the links. A similar analysis to that we have developed above may be also written down for the $\mathbb{B}_2 \bowtie S_2$ and $\mathbb{B}_3 \bowtie S_3$ quintic fibrations. The five links joining the neighboring $\pi_n$ nodes describe massless fields of the effective field theory on the $D4$ branes at singularity. For each fibration, there are $(2 \times 5)$ massless complex fields which we denote as $\chi_{a,k_a} \sim a_{k_a}^+$ and $\psi_{a,k_a} \sim a_{k_a}^-$. In particular at $z_1 = z_2 = z_3 = 0$, we have therefore triplets of massless fields as shown on table (5.21) where we give the fields spectrum on the fractional wrapped $D7$ branes on $\mathbb{B}_a$ at the various singularities.

**Fields Spectrum**

| Chiral fields $\rightarrow$ | Massless complex fields | Number of massless fields |
|--------------------------|------------------------|---------------------------|
| Singularity $S_1$       | $\chi_{1,k_1}, \psi_{1,k_1}$ | $2 \times 5$ |
| Singularity $S_2$       | $\chi_{2,k_2}, \psi_{2,k_2}$ | $2 \times 5$ |
| Singularity $S_3$       | $\chi_{3,k_3}, \psi_{3,k_3}$ | $2 \times 5$ |
| Locus $z_1 = 0$         | $\chi_{1,k_1}, \chi_{2,k_2}, \chi_{3,k_3}$ | $3 \times 5$ |

(5.21)

Such analysis can also be extended to the case where the orbifold is realized as $Q^{[2]} \sim \mathcal{R}''/\mathbb{Z}_5$. 


5.3 Fractional Branes on $Q^{[2]}$

There are three manifest fibrations of the commutative quintic orbifold $Q^{[2]}$ depending on the nature of the $\mathbb{Z}_5^2$ subgroups of $\mathbb{Z}_5^3$ one is considering. Let us describe them briefly here below.

**Fibration $Q^{[2]} \simeq CP^1 \bowtie S_{12}$:** Here the $\mathbb{Z}_5^2$ orbifold sub symmetry of the $\mathbb{Z}_5^2$ group has the vector charges $q^1_i = (1, -1, 0, 0, 0)$ and $q^2_i = (1, 0, -1, 0, 0)$. The $q^3_i$ vector charge of the third factor may be thought of as being set to zero. As such the $B_{12}$ basis of the fibration is parameterized by the $z_4$ and $z_5$ coordinates while the codimension one fiber $S_{12}$ is given by the following complex surface with an $A_4$ singularity,

$$u + v + t + \frac{\epsilon^5}{uv} = 0. \quad (5.22)$$

Like for the $B_u \bowtie S_u$ fibrations, the $B_{ab}$ bases are parameterized by those holomorphic coordinates $z_i$ with $q^a_i = 0$ and the $S_{ab}$ fibers by the complex variables with non zero $q^a_i$ charges with fixed points at the origin.

**Fibration $Q^{[2]} \simeq CP^1 \bowtie S_{23}$:** In this fibration the $\mathbb{Z}_5^2$ orbifold subgroup has the following vector charges $q^1_i = (1, 0, -1, 0, 0)$ and $q^2_i = (1, 0, 0, -1, 0)$. The $B_{23}$ basis and the $S_{23}$ fiber of the quintic are respectively parameterized by $(z_2, z_5)$ and $(z_1, z_3, z_4)$. Locally $B_{23} \sim \mathbb{C}$ while the $S_{23}$ singular surface with an $A_4$ singularity at the origin.

**Fibration $Q^{[2]} \simeq CP^1 \bowtie S_{31}$:** In this case the CY vector charges of the $\mathbb{Z}_5^2$ orbifold subgroup are given by $q^1_i = (1, -1, 0, 0, 0)$ and $q^2_i = (1, 0, 0, -1, 0)$. The base is parameterized by $z_3$ and $z_5$ coordinates and the $S_{31}$ fiber has a singularity at $z_1 = z_2 = z_4 = 0$. Its complex equation $u_1 + u_2 + t + \frac{\epsilon^5}{u_1u_2} = 0$ has an $A_4$ singularity at $u_1 = u_2 = t = 0$.

The main features of the various $B_{(ab)} \bowtie S_{(ab)}$ fibrations are collected in the following table.

| Fix Points | $F_1 \otimes F_2$ | $F_2 \otimes F_3$ | $F_3 \otimes F_1$ |
|------------|-----------------|-----------------|-----------------|
| $(0,0,0,z_4,z_5)$ | $(0,z_2,0,0,z_5)$ | $(0,0,z_3,0,z_5)$ |

| Fibrations | $B_{ab}$ | $S_{ab}$ | Fields |
|------------|----------|---------|--------|
| $u_1 + u_2 + t + \frac{\epsilon^5}{u_1u_2} = 0$ | $CP^1 = \{z_4, z_5\}$ | $u_1 + u_3 + v + \frac{\epsilon^5}{u_1u_2} = 0$ | $u_1 + u_4 + w + \frac{\epsilon^5}{u_1u_4} = 0$ |
| $u_1 = z_5^1, u_2 = z_2^2$ | $u_1 = z_1^3, u_3 = z_3^3$ | $u_1 = z_5^1, u_4 = z_2^5$ | $w = z_1z_2z_4$ |
| $t = z_1z_2z_3$ | $v = z_1z_3z_4$ | $v = z_1z_3z_4$ | $v = z_1z_3z_4$ |

The singular representations of NC $Q^{nc[2]}$ may be obtained by starting from the regular representation eq.(4.11) and taking the appropriate limits. For the $S_{12}$ singular space, the field moduli are associated with $q^1_i = q^2_i = 0$ and taking $z_1, z_2$ and $z_3$ to zero. We have for the $S_{12}$ fiber,

$$\Phi_1 = (Q \otimes I_{id} \otimes I_{id}) \lim_{z_1 \to 0} z_1, \quad \Phi_2 = (Q^{-1} \otimes I_{id} \otimes I_{id}) \lim_{z_2 \to 0} z_2, \quad \Phi_3 = (I_{id} \otimes Q^{-1} \otimes I_{id}) \lim_{z_3 \to 0} z_3, \quad F_1 = P \otimes Q^{n_{12}} \otimes Q^{n_{13}}, \quad F_2 = Q^{n_{23}} \otimes P \otimes Q^{n_{23}}, \quad (5.24)$$

and for $B_{12}$,

$$\Phi_4 = z_4 \otimes I_{D(G)}, \quad \Phi_5 = z_5 \otimes I_{D(G)}, \quad F_3 = Q^{n_{31}} \otimes Q^{n_{32}} \otimes P. \quad (5.25)$$

Once the limit to the singular point is taken, the non vanishing matrix moduli $\Phi_i, i = 4, 5$, are proportional to the identity $I_{D(G)}$ of $D(G)$. As before this property reflects just the existence of fractional $D$ branes at the orbifold point $z_1 = z_2 = z_3 = 0$. To illustrate the idea, let us reconsider the example of
the D9 brane of type IIB string wrapped on \( Q^{[2]} \); with local coordinates \((x^\mu; y_1, ..., y_3)\): the \( x^\mu \)'s being the longitudinal non compact coordinates of the D3 part of D9, while the \( y \) compact coordinates are as in eqs (5.10). In presence of discrete torsion, the wrapped D9 on \( Q^{[2]} \) becomes a NC brane generated by the algebra eqs (4.4) where the \( q^a \) charges are as indicted above. At the fixed points of \( Q^{[2]} \) where a real four cycle shrinks to zero, the NC wrapped D9 give rise to twenty five fractional wrapped D5 branes on \( B_{12} \sim \mathbb{CP}^1 \). The transverse coordinates of these fractional D5 branes are given by:

\[
\Phi_{4,n,m} = x_4 \Pi_{n,m}, \quad \Phi_{5,n,m} = x_5 \Pi_{n,m},
\]

where \( \Pi_{n,m} \) are the projectors on the \( D (Z_5^2) \) representation states. The singular modes at the orbifold point are carried by the \( Q \otimes Q, Q^{-1} \otimes I_{\mu} \) and \( I_{\mu} \otimes Q^{-1} \) operators of \( D (Z_5^2) \). Moreover since near the singularity the \( \Phi_1, \Phi_2 \) and \( \Phi_3 \) operators may also be split as

\[
\Phi_1 = \lim_{z_1 \to 0} (x_1) \sum_{n_1,n_2=1}^5 A_{n_1}^+ A_{n_2}^+ A_{1,n}, \quad \Phi_2 = \lim_{z_2 \to 0} (x_2) \sum_{n_1,n_2=1}^5 A_{n_1}^- \Pi_{n_2}, \quad \Phi_3 = \lim_{z_3 \to 0} (x_3) \sum_{n_1,n_2=1}^5 \Pi_{n_1} A_{n_2}^-, \quad (5.27)
\]

one gets \( 3 \times 25 \) massless modes \( \phi_{1;n_1,n_2} \sim A_{n_1}^+ A_{n_2}^+ \), \( \phi_{2;n_1,n_2} \sim A_{n_1}^- \Pi_{n_2} \), and \( \phi_{3;n_1,n_2} \sim \Pi_{n_1} A_{n_2}^- \) living on the D5 branes wrapping \( \mathbb{CP}^1 \). They are propagators between the \( \Phi_{1;n_1,n_2} \) and \( \Phi_{i,n_1 \pm 1, n_2 \pm 1} \) fractional D5 branes. The quiver diagram representing the fractional D5 branes wrapping \( \mathbb{CP}^1 \) is a \( \Delta_5 \times \Delta_5 \) polygon with D5 branes sitting at the vertices and the \( \phi_{a;n_1,n_2} \) massless modes propagating along the links; see figure 4.

![Figure 4: Here is represented the 25 vertices of fractional D branes quiver at \( z_1 = z_2 = z_3 = 0 \) singularity of \( Z^2 \) orbifold subsymmetry. Links between vertices represent massless modes of wrapped D5 branes on \( \mathbb{CP}^1 \)](image)

An analogous analysis may be also made for the \( B_{23} \cong S_{23} \) and \( B_{31} \cong S_{31} \) fibrations. As the results one gets are similar, we will omit this part. In the end of this discussion, let us give the results for orbifold of \( H_n \).

**Extension:** Here we give the results for generic complex \( n \) dimension hypersurface \( z_1^{n+2} + ... + z_{n+2}^{n+2} + a_0 \prod_{i=1}^{n+2} z_i = 0 \) with orbifold group \( Z_{n+2}^\bullet \). In this case the torsion matrix \( \eta_{ab} = \eta_{[ab]} + \eta_{[ab]} \) has to belong to \( SL(n, \mathbb{Z}) \) where the antisymmetric part \( \eta_{[ab]} \) is non zero in presence of discrete torsion. Geometrically, the corresponding NC geometry is an elliptic fibration with base \( H_n \) and fiber a real \((n + 2)\) dimensional Fuzzy torus \( T_{\beta_{1,j}}^{n+2} \). The \( \beta_{ij} \) cocycles are as

\[
\beta_{ij} = \exp \left( - \frac{2\pi}{n+2} \eta_{[ab]} q_i^a q_j^b \right),
\]

(5.28)
where now the $n$ vectors $q^i = (q^i_a)$ have $(n+2)$ integer entries and satisfy the CY condition $\sum_{i=1}^{n+2} q^i_a = 0$. These charge vectors can be chosen as,

\[
\begin{align*}
Z_{n+2} : & \quad q^1 = (1, -1, 0, 0, \ldots, 0, 0), \\
Z_{n+2} : & \quad q^2 = (1, 0, -1, 0, \ldots, 0, 0), \\
& \quad \ldots \\
Z_{n+2} : & \quad q^{n-1} = (1, 0, 0, 0, \ldots, -1, 0), \\
Z_{n+2} : & \quad q^n = (1, 0, 0, 0, \ldots, 0, -1).
\end{align*}
\]

Fractional D-branes at orbifold points depend on the orbifold subgroups $G[\alpha]$ of $Z_{n+2}$ one is considering. Since there are several subgroups in $Z_{n+2}$, we have then several possible representations. The natural ones are those associated with the manifest factors $G[\alpha] = Z_{n+2}$ with $1 \leq \alpha \leq (n-1)$. Let us give some comments regarding this point, but forget for a while about string applications by letting $n$ to be a generic positive integer greater than one and suppose that we have a $p$-brane ($p > n$) wrapping the $H_n$ compact manifold.

(a) Fractional branes on $H_n/Z_{n+2}$ Since there are $n$ manifest $Z_{n+2}$ subsymmetries in $Z_{n+2}$, we have $n$ classes of fractional $(p-2)$-branes at the $n$ kinds of singularities $z_1 = z_{a+1} = 0$. Extending the analysis we have made for the quintic to generic $H_n$’s by thinking about $H_n/Z_{n+2}$ as a fiber bundle $B_3 \cong S_n$ of base $\mathbb{C}P^{n-1}$ and a fiber $S_n$ given by the complex curve $u + v + \frac{(n+2)}{u} = 0$ with an $A_{n-1}$ singularity at the origin $u = v = 0$, we will have $d_1$ fractional branes at each orbifold point $(d_1 = \dim D (Z_{n+2}))$ and $2n$ massless modes living on the $(p-2)$ branes. Points in this NC geometry are represented by polygons $\Delta_{n+2}$ with $(n+2)$ vertices and $(n+2)$ edges.

(b) Fractional branes on $H_n/Z_{n+2}$ Before giving the result concerning $H_n/Z_{n+2}$, note that the action of $Z_{n+2}$ on the $z_i$ variables is additive. If one performs two successive $Z_{n+2}$ actions with charges, say $q^i$ and $q^j$, then the total $Z_{n+2}$ action has the charge $q^i_a + q^j_a$.

\[
\begin{align*}
Z_{n+2} : & \quad z_i \longrightarrow z_i \omega^{q^i}, \\
Z_{n+2} : & \quad z_i \longrightarrow z_i \omega^{q^j}.
\end{align*}
\]

From these relations, one sees that this $Z_{n+2}$ action is equivalent to a $Z_{n+2}$ diagonal sub symmetry with $q^i_a + q^j_a$ charge. The remaining $q^j_a - q^i_a$ charge is associated with the off diagonal sub symmetry which play no role here. Using eqs (5.30), we get,

\[
Z^{n+2}_2 : \quad q^1 + q^2 = (2, -1, -1, 0, \ldots, 0, 0, 0).
\]

The fixed points of (5.31) are at $z_1 = z_2 = z_3 = 0$; but for generic $q^i_a + q^j_a$ vector charge fixed points are at $z_1 = z_{a+1} = z_{b+1} = 0$. This property shows that the $H_n^{nc}$ geometries have the following features: (i) the $B_{ab} \cong S_{ab}$ realizations of $H_n^{nc}$ have a $\mathbb{C}P^{n-2}$ base and a fiber $S_{ab}$ described by the complex surface $u + v + t + \frac{a+b+1}{u} = 0$ with an $A_{n-1}$ singularity at the origin $u = v = t = 0$. (ii) there are $\frac{(n-1)}{2}$ possible fibrations for $H_n/Z_{n+2}$ and $\frac{n(n-1)}{2}$ classes of fractional $(p-4)$-branes at the $z_1 = z_{a+1} = z_{b+1} = 0$ singularities. Each class contains $d_1d_2$ fractional $D (p-4)$-branes and $2 (d_1d_2 - d_1 - d_2)$ massless modes living on thses branes ($d_1 = \dim D (Z_{n+2})$). Points in this NC geometry are given by the crossed product of two polygons $\Delta_{d_1} \times \Delta_{d_2}$.

(c) Fractional branes on $H_n/Z_{n+2}$ Extending the above reasoning to $H_n/Z_{n+2}$, the orbifold symmetry $Z_{n+2}$ with charge vectors $q^1, \ldots, q^{n}$, acts in practice through ist diagonal $Z_{n+2}$ subgroup with a CY charge $Q_{(a_1, \ldots, a_k)} = q^{a_1} + \ldots + q^{a_k}$; it has fixed points located at $z_1 = z_{a_1+1} = \ldots = z_{a_k+1} = 0$. 

For the example of the leading $k$ factors $\mathbb{Z}_{n+2}^k$ of $\mathbb{Z}_{n+2}^n$, the vector charge $Q_{(1,...,k)}$ of $\mathbb{Z}_{n+2}^k$ is given by:

$$Z_{n+2}^k: \quad Q_{(1,...,k)} = (k, -1, -1, ..., -1, 0, ..., 0); \quad (5.32)$$

where the first $(k + 1)$ entries are non zero and all remaining ones are null. A simple counting of the degrees of freedom shows that there are $\frac{n!}{(n-k)k!}$ possible $\mathbb{Z}_{n+2}^k$ subgroups in $\mathbb{Z}_{n+2}^n$. The $B_{(a_1,...,a_k)}$ $d\times S_{(a_1,...,a_k)}$ representations of the manifold $\mathcal{H}_{n}/\mathbb{Z}_{n+2}^k$ have a $\mathbb{C}P^{n-k}$ base and a fiber $S_{(a_1,...,a_k)}$ given by the complex dimension $k$ hypersurface $u_1 + ... + u_k + t + \frac{t^{n+2}}{\varepsilon} = 0$ with $\varepsilon = \prod_{j=1}^k u_j$ and an $A_{n-k}$ singularity at $u_1 = ... = u_k = t = 0$. Together with these realizations, there are $\frac{n!}{(n-k)k!}$ classes of fractional $D(p-2k)$-branes at the $u_1 = ... = u_k = t = 0$ singularities. Each class contains $\prod_{j=1}^k d_j$ fractional $(p-2k)$-branes and $k \prod_{j=1}^k d_j - \sum_{i=1}^k \prod_{j \neq i} d_i$ massless modes living on these.

6 Conclusion

Using results on type II string compactification on CY orbifolds and the algebraic geometry method of Berenstein and Leigh, we have developed the study fractional $D$ branes on generic complex $n$ dimension NC orbifolds of CY hypersurfaces $\mathcal{H}_n$. This is an explicit study which give the general solutions for NC geometry and complete by the way special results obtained previously in [2] [21]. It also allows a more insight in NC geometry induced by discrete torsion and recovers as a particular case stringy constructions in connection with orbifolds of $D = 4$ SYM theory embedded in type II string compactifications on local CY orbifolds with discrete torsion [33]. The general solutions we have obtained have geometric and algebraic interpretations and can moreover be viewed as an explicit verification of Adams and Fabinger conjecture regarding emergent dimensions considered recently in [36].

Geometrically, we have shown that the NC orbifolds with discrete torsion are special elliptic fiber bundles on $\mathcal{H}_n$ and fuzzy torii as fibers. The latter resolve singularities and lead to a fractionalisation of D branes due to the complete reducibility property of the representations of the orbifold group at singularities. In the large $n$ limit our analysis extends naturally: the discrete $\mathbb{Z}_{n+2}$ group factors tends to $U(1)$ and the original $Dp$ branes are mapped to $Dp+2$ ones in accord with the prediction of [40]; but also with the explicit computation made in [2] [21]. In the continuous limit the solutions are quite similar; rational torii fibers considered in the present study are replaced in the continuum by irrational ones; for details see [24] [33].

Algebraically, the general solutions we have derived in this paper offers, amongst others, a remarkable classification of NC orbifolds. This classification is completely characterized by the following: (i) the $q_i^a$ vector charges of the orbifold group $\mathbb{Z}_{n+2}^n$ with $n > 1$ and (ii) a $n \times n$ matrix $\eta_{ab} = \eta_{(ab)} + \eta_{[ab]}$ of the group $SL(n,Z)$ defining the various possible orbifolds. Discrete torsion exists whenever the $\eta_{[ab]}$ antisymmetric part is non zero. In addition to the fact that they go beyond the known ones in literature, our solutions exhibit manifestly the discrete torsion dependence embodied by $\eta_{[ab]}$ and full orbifold geometric invariance carried by the $q_i^a$’s. In this regards it is worthwhile to recall that the $\beta_{ij}$ cocycles appearing in $Z_i Z_j = \beta_{ij} Z_j Z_i$ NC geometry relations read, in terms of $q_i^a$ charges and the matrix $\eta_{ab} = \beta_{ij} = \exp i \frac{2\pi}{\eta_{ab} q_i^a q_j^b}$. Supersymmetric field theoretic aspects of this construction seems to be linked to deformations by $\mathcal{N} = 1$ adjoint matter superpotentials; progress in this issue will be considered in a future occasion.

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