Rediscovery of Cooper pair factor

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In superconductivity, the Cooper pair is a bound state of a pair of weakly interacting electrons in a metal which forces us to insert a factor of $2e$ (instead of $e$) in the phase of the wave function in the Aharonov-Bohm (AB) effect. We claim that the existence of such a factor is universal. In [2], we proposed an interpretation of the AB effect (or equivalently, a supersymmetric Dirac monopole) based on SUSY. It is shown that this model opens a path to recapture this factor without the need of arguments related to attractive pairs. The idea is also applied to the case of a nonlinear sigma model when it is coupled to $N = 1$, supergravity. The paper closes in the fourth section, with a brief concluding remark.

I. INTRODUCTION

A bound state of Cooper pair is an elementary object which causes the superconductivity. Cooper showed that the attraction of the pair is originated by an electron-phonon interaction which causes the energy of the pair lies lower than the Fermi energy. One year later, Bardeen, Cooper and Schrieffer developed the theory of the interacting Cooper pairs to describe the superconductivity which is known as the BCS theory [3].

From the theoretical perspective, one can ask: can we prove the existence of the Cooper pair which implies that there is something more profound out there? Is the Cooper factor a universal notion or it is just restricted to the superconductivity context? We will try to find an answer to these type of questions in the next sections.

The wave function of an electron which moves along a closed path in the presence of a gauge connection $A(r)$, is accompanied by a phase factor:

$$\psi(r) = \psi_0(r) e^{-i(e/\hbar) \oint dr \cdot A(r)},$$

where $\psi_0(r)$ denotes the wave function at point $r$, at the beginning and $\psi(r)$ denotes the wave function at the same point after traveling around the closed path. The integral which appears in the exponent, represents the magnetic flux $\Phi$ enclosed by the loop. Note that in this paper, we will work in the SI units with the Weber convention. The wave function at $r$ should be single valued so we obtain the magnetic flux quantization:

$$\Phi = \frac{2\pi \hbar}{e} n \quad n = 0, \pm1, \pm2, \ldots \quad (2)$$

But in a superconductor, due to the Cooper pairs we should replace the charge $e$ by the charge of the pair namely, $2e$ to get the correct result.

In the next section, we will consider a supersymmetric Dirac monopole and show that in this model the Cooper factor arises automatically. In the third section, we will verify this prescription in the case of a nonlinear sigma model which is coupled to the $d = 4$, $N = 1$ supergravity. The paper closes in the fourth section, with a brief concluding remark.

II. A SUPERSYMMETRIC DIRAC MONOPOLE

In this section, first we will show that a supersymmetric Dirac monopole admits a geometrical phase which can be expressed in terms of FI term and after that, we will derive the Cooper factor which is nicely in agreement with the statement that we made below the equation (2). It is an intersecting D3-D3’ system in which we can put some charged matters in the intersection region. The magnetic property of this set-up was discovered by Mintun, Polchinski and Sun in 2015 [4]. For our purposes, it is not necessary to use the brane interpretation of the system because we just want to focus on its gauge properties, carefully. The world-volume theory of each brane is a usual 4d super Yang-Mills theory with a compact gauge group $U(1)$ that each of which contains a region in which the charges are localized namely, the intersection region. We will develop the formulation from the gauge theory point of view. However, in some cases such as amounts of supercharges and the number of chiral multiplets, we will mention the constraints arises from the string theory to make the discussion more clearer.

We want to construct a supersymmetric Dirac monopole. The Dirac monopole is a point-like particle in a certain position in space e.g., the origin. Knowing the position of the monopole is a crucial fact to analyse its behaviour because as you may know it is a topological object i.e., the Maxwell equation $\nabla B = 0$, holds everywhere in the space except at the position of the monopole. Therefore, we should remove this point from the space (the $\mathbb{R}^3$ manifold). The remained manifold is homotopy equivalent to a $S^2$ surrounds the origin. To understand the deep topology behind the AB effect and the Dirac monopole, see the famous paper [5].

So far we have seen the importance of the position of the monopole. Now, in a 4d Lorentzian manifold we want to put some $U(1)$ charges such that the configuration preserves some amounts of supersymmetry in addition to the location of charges should be seen directly. The simplest submanifold corresponding to the SO(1,3) alge-
bra to put charges on it in this way, corresponds to the SO\((1,1)\) algebra. So inside the 4d space with coordinates \(x^\mu, \mu = 0, 1, 2, 3\), we find a 2d Lorentzian submanifold with coordinates \(x^\nu, \mu = 0, 1\), too convinient to couple a gauge field (belongs to a vector multiplet) to charged matter (belongs to a chiral multiplet), supersymmetrically. Adding this 2d action to the 4d sourceless gauge theory gives the full theory. At this visit, we still do not know if the configuration contains BPS solutions for the magnetic monopole or not. Note that the supersymmetry imposes a strong constraint that if the Dirac monopole exists, it is no longer a point-like particle in the space, it is a line or more literally, it is a line defect.

As explained in [4], firstly, the world-volume theory of each brane contains magnetic solutions of the BPS equations when it is coupled to a nonlinear sigma model. Secondly, one can construct the full theory by using some beautiful techniques. We do not want to get involved to this procedure in details because one can make a subtle guess to predict the general form of the most important sector of the final theory. For those who are interested in technical problems, we would say that the base of the procedure is as follows: one can start with suitable sector of the final theory. For those who interested in technical problems, we would say that the base of the procedure is as follows: one can start with suitable sector of the final theory. For those who interested in technical problems, we would say that the base of the procedure is as follows: one can start with suitable sector of the final theory.

Secondly, one can construct the full theory by using some bottom-top approach [6] to make a supermanifold in the higher dimensions. So putting charges in some subregion is totally under controlled. If we consider this system as a tiny part of the string theory, then the number of supercharges is forced to be 8.

So on the manifold \(\mathbb{R}^3\{\text{z-axis}\}\), we have: \(dF = 0\). But this time the two hemispheres should be infinitely large in order to cover the infinite line. Therefore, we can still conclude that \(\mathbb{R}^3\{\text{z-axis}\}\), is homotopy equivalent to \(\mathbb{S}^2\), but this time it should be an imaginary infinite sphere on which we can locally define: \(F = dA\). It is in agreement with the relation between the redifined and the usual field strengths \(F\) and \(dF\), which as we mentioned earlier can be obtained by focusing only on the D-term of the full theory:

\[
\begin{align*}
F_{23} &\equiv F_{23} + \phi \delta^{(2)}(x_2, x_3), \\
F_{\mu\nu} &\equiv F_{\mu\nu} \quad \text{for} \quad \mu\nu \neq 23,
\end{align*}
\]

where \(\phi\) is the contribution of some real scalar fields in the D-term which are localized at the line. Out side of the line, the equation of motion \(3\) is invariant under a 4d Lorentz transformation namely, an element of the group SO\((1,3)\). All in all, we conclude that our set-up has a geometrical phase.

According to our discussions in [2], this phase has something to do with the factor of: \(\exp(\frac{i}{\hbar} S_{FI})\), where \(S_{FI}\) is the Fayet-Iliopoulos (FI) D-term contributing to the intersection part of the full theory. It is because we have:

\[
\frac{S_{FI}}{\hbar} = 2\pi n. \quad \text{We have:} \quad S_{FI} = \frac{\chi}{2\hbar} \int d\phi^0 d\phi^1 D(\phi^0, \phi^1, 0, 0),
\]

where \(\chi\) denotes the FI parameter and \(D\) denotes the real nonpropagating auxiliary scalar field in the vector multiplet. In the QFT of this model, the dimension of \(\chi\) is \([\sqrt{\hbar}]\), but classically, we can get rid of this factor beautifully. In this limit, we can verify the dimension of the one-form gauge connection \(A\), either directly from the classical EOM \(3\), or more conveniently from a suitable Wilson loop in the presence of an electric probe charge \(e\):

\[
\exp\left(\frac{ie}{\hbar} \oint \text{d}r A\right), \tag{6}
\]

where the closed loop is placed in the plane on which we have: \(x^0 = x^1 = \text{cte}\) (which we call it \(\mathbb{C}\)-plane), so the contributing components of \(A\), are only \(A_{2,3}\). We obtain: \([A] = [h/LQ] = [\text{Wb}/L]\). Since \(D \sim \frac{d}{dx} A\), So
we have $[D] = [h/L^2 Q]$. Therefore, the integral in the equation (5) has the dimension of the monopole strength $|g| = [Wb]$, so we have: $|\chi| = |Q|$. It looks nice but not enough yet. To see more about the dimensional analysis see the appendix.

On the C-plane, we can use the complex coordinates:\[ z = \frac{1}{2}(x_2 + i x_3) .\]

**Theorem 1.** On an infinitesimal loop in the C-plane which surrounds the line, $A_z \propto 1/z$ is always a valid nonzero solution.

**Proof.** It follows from the Poincaré’s lemma for a non-trivial topology that there always exists at least two solutions which at least one of them is nonzero which we denote by $A$, and the other by $\tilde{A}$. First, we claim that for a given solution $\mathcal{A}$ on this loop the below relation

\[
A_z \rightarrow \mathcal{A}_z + \frac{1}{z} \approx \frac{1}{z},
\]

is a gauge transformation. So we must show that $1/z$ is a pure gauge on the loop. It is true because the integrand of the integral of $1/z$ over the loop namely,

\[
\oint d\phi \frac{1}{z} = 1 \oint d\phi,
\]

is a total derivative where $\phi$ is the azimuthal angle which parametrizes the infinitesimal loop. Since at every point out of the line the gauge transformation of any given connection $\mathcal{A}$ gives another connection, so at every point on the infinitesimal loop we find $A_z \propto 1/z$, is also a valid nonzero solution. QED

We can continue this solution to the regions far away from the line. We put these two together in a single real equation

\[
\overline{\partial A_z} + \text{c.c.} \propto 4\pi \delta^{(2)}(z, \bar{z}),
\]

which can be regarded as the BPS equation for $A_z$ which holds all over the C-plane. Moreover, it is a FODE as expected.\[{1}\]

Therefore, $D(x^0, x^1, 0, 0)$ remains fixed. According to this proof, we believe that nothing can keeps us away from regarding the relation (5) as the dimensionless geometrical phase i.e., \[ S_{F/L}/h = 2\pi n \] where: $|\chi| = |Q|$.

But the best thing is yet to come. In an overlap region, the general form of the geometrical phase for either the case of flux (in a line bundle) or monopole (in a principal bundle) is

\[
\frac{e}{\hbar} \phi \quad \text{or} \quad \frac{e}{\hbar} g = 2\pi n,
\]

where $\phi$ is magnetic flux like in the equation (11) and $g$ is the strength of a monopole in the Weber convention. As we explained below the equation (2), in physical measurements it does not yield the correct answer for the magnetic quantity. The integral which appears in the equation (5), represents the homotopy class to which this bundle belongs so without any extra coefficient it should be equal to $g$. Therefore, it satisfies

\[
-2\pi \int_{S^2} c_1(F) = \int_{S^2} F = \int_{\text{on the line}} d^2 x \ D = g,
\]

which as we mentioned in [2], can be regarded as SYM statement of the Stokes’ theorem. In the equation (11), $c_1(F)$, denotes the first Chern class of the principal bundle.\[2\]

Now, the equation

\[
\frac{e}{\hbar} g = \frac{\chi}{2\hbar} g = 2\pi n,
\]

yields: $\chi = 2e$, the fundamental unit of the electric charge appears in the measurement of the phase for a fermionic test particle/object).

In equation (11), we can let $x^0$ transform under a Wick rotation then we should use $F_{\text{math}}$ from the footnote [2]. In the dual theory, $g$ denotes the electric charge $e$ and $\chi/2$ is the quantized magnetic charge.

Now, we are in the position to generalize our results. Suppose that in a 4d supersymmetric field theory with the compact gauge group U(1), one finds a 2d surface $M$ equipped with a closed two-form $\Omega$. In the case of $c_1(\Omega)$ is nonzero, we need at least two coordinate neighbourhoods to cover $M$. The phase in the overlap region is ($\hbar = e = 1$)

\[
-\frac{\chi}{2} \int_M c_1(\Omega) = n.
\]

In string theory, since we have two charged matter multiplets on the line, there are two independent copies of this argument.

In order to give another evidence, let us finish this section by considering the case of $N$ charged chiral multiplets $\Phi_i$. The values of the complex scalar fields $\phi_i$ on the boundaries of the line, determine the total magnetic charge $Q^{\text{tot}}_m$. We denote the boundary values of $\phi_i$ by $\varphi_i$, so we have:

\[
Q^{\text{tot}}_m = Q^1_m + \cdots + Q^N_m = \frac{1}{2} (|\varphi_1|^2 + \cdots + |\varphi_N|^2),
\]

\[
= 2\pi m_1 + \cdots + 2\pi m_N \equiv 2\pi n,
\]

\[2\]

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\[1\] This construction enables us to reproduce the D-term, independently. According to [3], we can define: $D = \partial A_z + \text{c.c.} - 4\pi \delta^{(2)}(z, \bar{z})$. So the BPS equation for $D$, should be: $D|_{\text{out of the line}} = 0$, and $D|_{\text{on the line}}$, should not contain a delta function. Note that the gauge transformations associated with those components which are tangent to the defect, do not affect the D-term so we are in the WZ gauge as expected.

\[2\] In the mathematics texts, you may find it as: $c_1(F) = i F_{\text{Maxwell}}$. The relation between the two is given by: $F_{\text{math}} = i F_{\text{Maxwell}}$. 

where $m_i \in \mathbb{Z}$. Note that all sources belong to the same bundle which is represented by $2\pi n$. For each term in $\Omega$, the FI (large) gauge transformation corresponds to a shift of $\chi_i/2 = 2\pi n_i$. Therefore, in the dual theory we have: $\frac{\chi}{2} = g = 2\pi n$, which implies that in the original theory in the SI units we must have: $\frac{\chi}{2} = e$.

In order to find the BPS equations of the monopole in the Euclidean signature in which we have: $(a_E^0, x_E^1) \subset \mathbb{R}^2$, we should find a suitable Kähler manifold as the target manifold $M$ of the sigma model on which the local complex coordinates $\phi^i$ are defined by the map $\phi^i : \mathbb{R}^2 \rightarrow M$, where: $i = 1, 2$. According to (14), in order to get a correct map, the convenient non-trivial compact manifold should be obtained by performing a one-point compactification on $\mathbb{R}^2$ namely, $\mathbb{R}^2 \cup \{\infty\} \simeq S^2$.

### III. THE CASE OF $\mathcal{N} = 1$, SUPERGRAVITY

The case of $\mathcal{N} = 1$ supergravity in the presence of the FI term has been studied in [7, 8]. Now, we want to investigate it in terms of the arguments presented in the previous section.

In [8], Bagger and Witten found that the $\mathcal{N} = 1$ supergravity can be coupled to a nonlinear sigma model. It is because, despite of the fact that in super Yang-Mills theories (i.e., the case of no gravity) the Kähler transformation:

$$K \rightarrow K + f + f^*,$$

(where $K$ is the Kähler potential and $f$ is an arbitrary holomorphic function), is a symmetry but in supergravity, this transformation is accompanied by some phase factors so it is a transformation between at least two local coordinate neighbourhoods in a non-trivial target space. Therefore, the target space $M$ is a Kähler manifold (i.e., there exists a closed two-form $\Omega$ on $M$ which is called Kähler form) with nonzero first Chern class. Without loss of generality, we can only consider the case in which the (real) dimension of $M$ is two. This is called the Bagger-Witten line bundle which indeed is a canonical line bundle over $\mathbb{C}P^1$.

According to our discussions, in the presence of the FI term we are allowed to use the relation (13) (see table [4]). Note that the only sphere which admits a complex structure is $S^2$ and since $S^2 \simeq \mathbb{C}P^1$, it is obvious that $S^2$ is a Kähler manifold.

In supergravity, $\chi$ does not relate to the space-time $U(1)$ flux but we can still determine it as a free parameter of the theory in the (semi)classical limit. In order to use (13), we should calculate the first Chern class of the line bundle. It can be computed by using the Fubini-Study metric.

### IV. CONCLUSION

It follows from our discussions that the conservation of charge implies that in a classical scattering process we are allowed to anticipate a decay of a magnetic monopole to a pair of leptons with the same electric charge. However, evaluation of such a peculiar vertex in the conventional QFT is indeed a formidable task.

In a conventional quantum $U(1)$ gauge theory with a 4d Minkowski space-time as the base space, the bundle $P = \mathbb{R}^{(1,3)} \times U(1)$, is a trivial one with a single local

| $M$ | Supersymmetric Dirac monopole | Supergravity |
|-----|-----------------------------|-------------|
| $S^2$ | $\mathbb{C}P^1$ |

TABLE I: The relation between the supersymmetric Dirac monopole and $\mathcal{N} = 1$, supergravity when it is coupled to a nonlinear sigma model

Consider $L \rightarrow \mathbb{C}P^1$, as the canonical line bundle over $\mathbb{C}P^1$. By using the Fubini-Study metric, we realize that the closed curvature (1,1)-form is

$$\Omega = i\partial \bar{\partial} \ln(1 + |z|^2)^2 = i \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}. \quad (16)$$

Moving to the real coordinates, one can obtain

$$\Omega = 2 \frac{dr \wedge d\theta}{(1 + r^2)^2} = 2 \frac{r dr \wedge d\theta}{(1 + r^2)^2}. \quad (17)$$

From the relation $c_1(\Omega) = -\frac{1}{2\pi} \Omega$, we have

$$c_1(\Omega) = -\frac{1}{\pi} \frac{r dr \wedge d\theta}{(1 + r^2)^2}. \quad (18)$$

We denote the integral of $c_1(\Omega)$ over $S^2$ by $C_1(L)$, which is an integer:

$$C_1(L) = -\frac{1}{\pi} \int \frac{r dr d\theta}{(1 + r^2)^2} = -\int_1^\infty t^{-2} dt = -1. \quad (19)$$

Putting this into (18), gives the final result:

$$\chi = 2n, \quad (20)$$

which is consistent with the evaluation of $\chi$, derived in [4, 8]. Note that we expect the phase to appear in the fermionic matter field transformation which is also true.
trivialization over base space. In the case of a non-trivial base space, we can use an old technique to build a soft monopole from a singular one: we know that an ordinary Dirac monopole can be submerged in a SU(2) gauge theory. Then, by introducing a scalar field $\phi$ with a constant VEV and performing a special SU(2) gauge transformation, the string singularity disappears and we end up with the usual hedgehog form of the gauge connection of the ‘t Hooft-Polyakov monopole. Ultimately, the monopole source is replaced by the Higgs field. One can show that

$$g_{\text{'t Hooft-Polyakov}} = 2g_{\text{Dirac}}.$$ 

Therefore, replacing $g_{\text{D}}$ by $g_{\text{'t P}}$ in the quantization condition (10), leads to the standard form of the condition that yields the correct result (see below of the equation (2)).

Appendix: More on the dimensional analysis

The free Maxwell action in the vacuum with the dimension of $\hbar$, is given by

$$S = -\frac{1}{4\mu_0} \int dt \, d^4x \, F^{\mu\nu} F_{\mu\nu} = -\frac{1}{4\mu_0 c} \int d^4x \, F^{\mu\nu} F_{\mu\nu}. \quad (A.1)$$

So one may tempt to insert this $\mu_0 c$ factor into the problem. Therefore, the Wilson loop becomes

$$\exp \left( \frac{i\chi}{2\hbar \mu_0 c} \int d^2x \, D \right) = \exp \left( \frac{i\chi}{2\hbar} \int dr \cdot \frac{A}{\mu_0 c} \right). \quad (A.2)$$

Intuitively, it is helpful to use the duality transformation:

$$\frac{1}{c} j_m \longrightarrow \mu_0 j_e, \quad (A.3)$$

from which we find the dimension of the gauge connection:

$$[A] = \left[ \frac{\text{Wb}}{L} \right] = \left[ \frac{\mu_0 c Q}{L} \right]. \quad (A.4)$$

In this sense, we realize that nothing has changed, actually i.e., $\chi/2$, represents $g$ with the dimension of Wb and the integral represents $e$, with the dimension of $Q$.

Note that the relation (A.4), implies that a topological charge $q_m$, is associated with an electric charge $q_e$, by

$$q_m \longrightarrow \mu_0 c q_e \quad \text{(Weber convention)}, \quad (A.5)$$

$$q_m \longrightarrow c q_e \quad \text{(Ampere-meter convention)}.$$