Optimization of SQUID Magnetometers and Gradiometers for Magnetic-Field-Fluctuation Thermometers

A Kirste, D Drung, J Beyer and T Schurig
Low-Temperature Thermodynamics and Technology, Physikalisch-Technische Bundesanstalt, Abbestraße 2-12, D-10587 Berlin, Germany
E-mail: Alexander.Kirste@PTB.de

Abstract. We report on the design and optimization of SQUID magnetometers and gradiometers intended for use in compact Magnetic-Field-Fluctuation Thermometers (MFFTs). The MFFT is a novel implementation of a noise thermometer, in which the thermally driven Johnson noise currents are detected by measuring the corresponding fluctuations of the magnetic field. This inductive readout can be realized by means of a classical wire-wound pick-up coil connected to a SQUID current sensor or by using single-chip SQUID magnetometers or gradiometers. In either case the pick-up coil should be as close as possible to the surface of the metallic temperature sensor in order to maximize amplitude and bandwidth of the spectral density of the flux noise. However, the spectral density depends on the geometries of pick-up coil and temperature sensor as well. Here we discuss the particular case of planar magnetometers and gradiometers. By solving the forward problem for an infinite slab we are able to calculate the spectral density of the flux noise in pick-up coils of arbitrary shape. The model is then applied to identify the optimal gradiometer configuration to be used in the MFFT. Finally, we compare calculations of this simple model with experimental data obtained in real setups. A MFFT containing such an optimized SQUID gradiometer was successfully operated down to 7 mK even without a superconducting magnetic shielding.

1. Introduction
The Johnson noise, i.e. the occurrence of thermal fluctuations of the charge carriers in dissipative electrical conductors is a basic effect that must be considered in the design and application of the most sensitive SQUID sensors. While the noise currents are confined to the inside of the dissipative conducting body, the corresponding weak magnetic fields can be sensed outside. For the Magnetic-Field-Fluctuation Thermometer (MFFT) [1,2] it is vital to maximize the intensity of the magnetic flux noise, $\langle \Phi^2 \rangle$, which is picked up by a suitable antenna and transferred into the SQUID for the readout. As a consequence, it is very important to optimize both the coupling into the pick-up coil as well as the inductance of the coil. While the noise generating temperature sensor, characterized by its shape and conductivity, influences the spectral density of the magnetic field noise $\delta B$, but not its intensity $\langle B^2 \rangle$, size and shape of the antenna determine both of the related quantities in the coil, $S_0$ and $\langle \Phi^2 \rangle$.

Qualitatively it may be obvious how to minimize or maximize the magnetic flux in a magnetometer pick-up coil, but this becomes different in case of a gradiometer or in particular if boundary conditions apply. It is difficult to treat these problems in a quantitative manner for the general case since the eddy currents involved require to solve a set of coupled differential equations. Only few studies on the calculation of thermal magnetic noise have been published in recent years. Three different methods...
have been applied to treat this problem: (a) solving the general conductor forward problem [3], (b) solving the forward problem using normal modes [4], (c) using the reciprocity theorem in order to calculate the current density in the conductor caused by the current in a coil and deriving from that the real part of the complex impedance of the coil [5]. While methods (a) and (b) are suitable for simple geometries, the approach (c) can be solved easily numerically by finite element methods even for complicated geometries.

An integrated MFFT, which consists of a SQUID magnetometer in thin-film technology that is placed directly onto the surface of the metallic temperature sensor was introduced in [2]. Gradiometers are preferable to magnetometers as they are less susceptible to external interference. Since the area of the SQUID sensor chip is limited, this results in an optimization problem for the quasi-planar pick-up coils of gradiometer and magnetometer. A simple but practicable model for the calculation of the flux noise is that of a planar antenna that is located parallel to the surface of an infinite conducting slab. This holds in particular when the temperature sensor is very close to and extending far beyond the antenna. Under this condition it is not sufficient to know the flux density noise $S_b$, which was calculated in [3] and [4]. Instead, the flux noise $S_b$ picked up by a coil of contour $c$ must be calculated.

We will develop the theory similar to [3] using method (a) and compare theoretical predictions with experimental results obtained on real systems. By applying the model to simple coil systems, basic principles are derived, which help to maximize the flux noise detected by the coil. It should be pointed out that this is only a partial aspect of the more comprehensive optimization problem, which aims at the best signal-to-noise ratio and includes the simultaneous optimization of the coil inductance. Since the inductance determines the flux transfer coefficient in case of transformer coupled SQUIDs or the intrinsic SQUID noise in case of directly coupled pick-up coils, it must be minimized as well. Finally we will discuss and compare various coil designs for a MFIT including inductance effects.

2. Calculation of the magnetic flux noise

2.1. Theory for an infinite conducting slab

We consider an arbitrary-shaped linear and isotropic volume conductor with conductivity $\sigma$ at a temperature $T$. According to the Nyquist formula for thermal noise, the mean square of the short-circuit electric noise current in $y$ direction of an infinitesimal small rectangular volume element in a small frequency interval $\Delta f$ is given by $\langle i_y^2 \rangle = (4k_B T / \Delta R_y) \Delta f$, where $k_B$ is the Boltzmann constant and $\Delta R_y = \Delta y'/(\sigma \Delta x' \Delta z')$ is the resistance of the volume element in $y$ direction. By introducing the current dipole component in $y$ direction, $P_y = i_y \Delta y'$, the above equation can be written as $\langle P_y^2 \rangle = \langle i_y^2 \Delta y' \Delta z^2 \Delta f \rangle = 4\sigma k_B T \Delta x' \Delta y' \Delta z' \Delta f$. Due to the isotropy it follows that $\langle P_x^2 \rangle = \langle P_z^2 \rangle = \langle P_y^2 \rangle = 4\sigma k_B T \Delta V' \Delta f$, where $\Delta V' = \Delta x' \Delta y' \Delta z'$ so that $4\sigma k_B T$ is the current dipole spectral density in a unit volume in each coordinate direction.

The mean square value of the magnetic flux noise in a pick-up coil of contour $c$ can then be calculated by means of a frequency-dependent response function $g$, which relates the point current dipole $P$ at a location $r'$ and the magnetic flux $\Phi$ in the coil by a linear transformation $\Phi = P(r') \cdot g(c,r',f)$. Assuming that the noise current components in all volume elements are uncorrelated with each other, the noise contributions of all dipoles can be summed up by integration over the entire conductor volume $V$:

$$\langle \Phi^2(c) \rangle = \int 4\sigma k_B T \Delta f |g(c,r',f)|^2 dV'.$$

(2)

The Green’s function $g(c,r',f)$ will be determined similar to [3]. First it is necessary to find the vector potential $A$ that is caused by a time-varying impressed current density $J(t)$ in a volume
conductor. Starting from the Maxwell’s equations in phasor notation for a time dependence of the form \( \exp(i\omega t) \) a wave equation results for \( \mathbf{A} \), which can be reduced in the quasistatic approximation to a diffusion equation for \( \mathbf{A} \) \[3\]:

\[
\nabla^2 \mathbf{A} - i\mu_0 \omega \mathbf{A} = -\mu(i\mathbf{J} + \phi \nabla \sigma),
\]

where \( \phi \) is the electric potential, \( \mu \) is the magnetic permeability and \( \omega = 2\pi f \).

Although the solution of the coupled differential equations for \( \mathbf{A} \) and \( \phi \) is very difficult to work out in the general case, the problem of a current dipole in an infinite conducting slab can be derived analytically. This is done for a current dipole \( \mathbf{P} = P_x \mathbf{e}_x \) placed at a location \( \mathbf{r}' = (x', y', -z') \) in an infinite slab of thickness \( t \) and constant conductivity \( \sigma \), see figure 1. Then \( \mathbf{J} = P_0 \delta(x-x')\delta(y-y')\delta(z+z')\mathbf{e}_x \) and \( \nabla \sigma = \sigma[\delta(z+t)-\delta(z)]\mathbf{e}_z \). Equation (3) implies now that \( \mathbf{A} \) can have only two nonzero components, namely \( A_x \) and \( A_z \). In the region \( z \geq 0 \), which we are interested in, \( \mathbf{B} \) is uniquely determined by \( A_z \) only, and \( A_x \) does not need to be considered. Inserting \( \mathbf{J} \) into equation (3) gives

\[
\nabla^2 A_y - k^2 A_y = \mu P_0 \delta(x-x')\delta(y-y')\delta(z+z'),
\]

where \( k^2 = i\mu_0 \sigma \omega \). For the solution of equation (4) the space is divided into four regions I-IV. The general solution in region I above the slab \((z \geq 0)\) is

\[
A_{1\mu} = \frac{\mu P_0}{4\pi} \int_{-\infty}^{\infty} A(a,b)e^{ia(x-x')} e^{ib(y-y')} e^{-\sqrt{a^2+b^2}z} \, da \, db.
\]

(5)

\(A(a,b)\) and further five functions occurring in the general solutions of regions II-IV serve to fulfill the boundary conditions for \( \mathbf{B} \): \( A_x \) is continuous at all boundaries, while \( \partial A_y/\partial z \) is continuous at all boundaries except at II-III, where \( \partial A_y/\partial z - \partial A_x/\partial z = \mu P_0 \delta(x-x')\delta(y-y') \). Solving the corresponding 6x6 matrix equation yields

\[
A(a,b) = e^{-\rho z'} (\pi \rho')^{-1} (u + v e^{-2\rho'(r-c)})(u^2 - v^2 e^{-2\rho r})^{-1},
\]

(6)

\[
\rho^2 = a^2 + b^2, \quad \rho'^2 = \rho^2 + k^2, \quad u = 1 + \rho / \rho', \quad v = 1 - \rho / \rho'.
\]

Since \( A_x = 0 \) and \( A_z = \text{const.} \), the flux through the contour \( c = c(x,y,z) \) in region I is given by

\[
\Phi = \oint_c \mathbf{A} \cdot d\mathbf{r} = \oint_c A_z \, dy,
\]

(8)

from which \( g_z \) is extracted:

\[
g_z(c,r',f) = \frac{\mu}{4\pi} \int_{-\infty}^{\infty} A(a,b) \int_{-\infty}^{\infty} e^{ia(x-x')} e^{ib(y-y')} e^{-\sqrt{a^2+b^2}z} \, dy \, da \, db.
\]

(9)

Analogously, \( g_x \) is obtained for a current dipole \( \mathbf{P} = P_x \mathbf{e}_x \), whereas \( g_z = 0 \).
Having established the Green’s function $g$ for a point current dipole in an infinite conducting slab, we can evaluate the magnetic flux noise $\Phi_n = S_n^{1/2} = (\xi/\Delta f)^{1/2}$ using equation (2):

$$\Phi_n = \left\{ 4\pi B_i T \int \left[ |g_x|^2 + |g_y|^2 \right] dv \right\}^{1/2}.$$  

(10)

Applying Parseval’s theorem for the two-dimensional Fourier transform, we obtain the final equation for the magnetic flux noise in an arbitrary contour $c$:

$$\Phi_n = \mu \sqrt{\sigma B_i T} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A(a,b)|^2 dz \left[ \int e^{iax} e^{-ibx} e^{-i2\pi z} dx \right]^2 + \int e^{iax} e^{ibx} e^{-i2\pi z} dy \right\}^{1/2}.$$  

(11)

The integral of $|A(a,b)|^2$ over $z$ can be solved analytically in any case, whereas the two loop integrals depend on the contour $c$ and may be evaluated numerically. This results in the numerical evaluation of two triple integrals for each point in the parameter space $(c, f, \sigma, t)$ of $\Phi_n$. A further simplification is possible if the contour $c$ is planar and parallel to the slab, i.e. $c$ is at constant height $z$ above the surface. In this case equation (11) can be rewritten as

$$\Phi_n = \mu \sqrt{\sigma B_i T} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_A dz \left[ |I_c|^2 + |I_{c,y}|^2 \right] dv \right\}^{1/2},$$  

(12)

$$I_A = \int_0^\infty |A(a,b)|^2 dz, \quad I_{c,x} = \int e^{iax} e^{ibx} dx, \quad I_{c,y} = \int e^{iax} e^{ibx} dy.$$  

(13)

The integral $I_A$ describes the eddy current distribution in the slab, whereas the geometry of the contour $c$ is completely contained in $I_{c,x}$ and $I_{c,y}$. Thus, for a given coil $I_{c,x}$ and $I_{c,y}$ must be evaluated only once in order to examine the parameter space $(z, f, \sigma, t)$ of $\Phi_n$. The exponential function in equation (12) acts as a low pass filter for the spatial frequencies depending on the distance $z$.

Making use of the reciprocity theorem, it is also possible to calculate complex coil systems such as parallel circuits. Any complex (branched) coil can be modeled as simple (branchless) coil provided that the field distributions generated by a current through the coils are equal.

For example, the symmetric multiloop coils consisting of $m$ equally sized partial loops in parallel with (total) noise flux $\Phi_n$ can be modeled easily, since the current through all partial loops is equal. We can create an equivalent, easily calculable, branchless coil having a similar $m$-fold current density and a noise flux $m\Phi_n$. Thus, for calculation, a “cartwheel” multiloop coil [6] with stripline spokes as depicted in figure 2 is replaced by its outer circumference.

2.2. Experimental tests

In order to verify the applicability of the model we performed measurements on our SQUID magnetometers and gradiometers under various conditions. The sensor chips were glued directly onto the surface of different metal bodies providing a distance of $z = 390 \mu m$ between planar pick-up coil and the conducting body. While the model assumes filamentary pick-up coils, real structures have a finite line width $w$. Neglecting the current distribution within the coil conductor, we approximate it by its mean contour $c$. The SQUID sensors are shown in figure 2. Details can be found in [8], with the C3WM being similar to its successor C4WM.

Experimental data were taken at $T = 4.2$ K using the SQUID electronics Magnicon XXF-1 [7] and a spectrum analyzer HP 35665A for the readout. For the simulation we use equation (12) with fixed values for $c, z$ and $t$. The conductors were made from copper of different origin. Since $\sigma$ is unknown, we use a value that provides the best fit to the experimental data. For copper at 4.2 K, a typical conductivity of $\sigma = (1.67 \times 10^8 \, \Omega m / RRR)^{1/4}$ with $RRR = 50...200$ can be expected depending on impurities and treatment.
As can be seen in figure 3, the overall agreement between experimental and calculated data is quite good. In case of the small multiloop magnetometer C4WS (coil diameter ≈1.7 mm, w = 232 µm) on the large OFHC-Cu block of 25 mm x 30 mm, t > 10 mm, the deviation in the frequency range of 0.1 Hz … 50 kHz is below 4% using σ = (1.67 × 10⁻⁸ Ωm / 140)⁻¹. For the medium size multiloop magnetometer C3WM (mean coil diameter ≈2.6 mm, w = 17 µm) on the bottom of a Cu pot of 19 mm diameter, t = 200 µm, the deviation in the frequency range of 5 Hz … 50 kHz is below 5% using σ = (1.67 × 10⁻⁸ Ωm / 75)⁻¹. Compared to the calculation, the experimental noise flux increases for f < 10 Hz due to additional noise modes in the outer ring of the pot. A change in the slope at about 3 kHz occurs at that frequency, where the skin depth δ becomes comparable to t.

Any influence of the intrinsic noise of SQUID and readout electronics on the experimental results can be ruled out because it is about 1 µΦ₀/Hz¹/₂ (white noise). However, the small decrease of the mutual inductance M between SQUID and feedback coil, which occurs at higher frequencies due to eddy currents in the metal body was not considered in the evaluation. The error may be a few percent. It was also found that a superconducting screen may diminish the flux noise at low frequencies if it is very close to the surface of the metal body, whereas higher frequency values are less affected.

Figure 3. Comparison of calculated noise flux and experimental data measured at 4.2 K on different setups: (a) Small magnetometer C4WS on bulk Cu (RRR ≈ 140), (b) Medium size magnetometer C3WM on a 200 µm thin sheet of a Cu pot (RRR = 75). The distance from the surface is z = 390 µm.

3. Application of the theory for planar pick-up coils above an infinite conducting plate

3.1. Correlation of the field noise

In order to establish some basic rules for the optimization of the pick-up coils, it is advantageous to introduce and evaluate the correlation function of the magnetic field noise. In the most general form it can be defined as

\[ \rho = \text{cov}(B_{n,i}(r_1), B_{n,j}(r_2)) \left( \frac{1}{N} \right) \left( \frac{1}{N} \right)^{-1/2} \left( \frac{1}{N} \right)^{-1/2}, \quad i, j = x, y, z, \]  

(14)

where i and j are the components of B_n at r_1 and r_2, respectively (figure 1). Since we are confined to planar coils parallel to the infinite slab, z_1 = z_2 and i = j = z. Equation (14) can be evaluated by means of two infinitesimal small test areas (A_1 = A_2) with distance d, which are wired either as gradiometer or magnetometer. If Φ_{n,1} = Φ_{n,2} is the noise flux from a single test area and Φ_{n,tot,+/-} the noise flux of both connected as magnetometer (+) or gradiometer (-), equation (14) becomes

\[ \rho = \pm \left( \frac{1}{2} \Phi_{n,1}^2 / (2\Phi_{n,1}^2 - 1) \right). \]

(15)

The correlation function ρ(d, z, f, σ, t) is calculated using equations (15), (12) and (13), where the contour integrals I_c are solved analytically for infinitesimal small test areas. Figure 4 shows the normalized noise flux at f = 0 of two test areas in magnetometer or gradiometer configuration as a
function of their separation $d$. Both dependencies correspond to a single curve in figure 5. There is a decrease in correlation with increasing distance, which becomes much steeper for higher frequencies.

3.2. Basic principles for the optimization of the pick-up coil
From the concept of correlation we can infer the following rules in order to maximize spectral density and intensity of the noise flux picked up in a coil:

(a) Favorable for a magnetometer are compact, coherent coils, i.e. a circle in the ideal case. The increase of the coil area is efficient as long as $\rho$ remains large, i.e. only up to a certain size depending on $z$, $t$, $\sigma$. The noise intensity $\langle \Phi^2 \rangle$ rises monotonously with the (circular) area $A = \pi r^2$, but $\langle \Phi^2 \rangle/A$ as function of $r$ has a maximum at $r^*$, which can be considered as an optimum. In the range examined, $100 \mu m \leq z \leq 1600 \mu m$, this maximum occurs roughly at $r^* = (5...3)z$ for arbitrary $t$ and $\sigma$.

(b) Favorable for a gradiometer are compact, coherent partial coils, which are separated sufficiently from each other. The increase of the separation (or base length) is efficient as long as $\rho$ remains large, i.e. only up to a certain size depending on $z$, $t$, $\sigma$.

Generally, structures with an aspect ratio far above or below 1 should be avoided as it would raise the coil inductance in addition. Both $\langle \Phi^2 \rangle$ and $\Phi_n$ increase and saturate for $z \rightarrow 0$. In accordance with [5] it was found that $\langle \Phi^2 \rangle$ is independent of the properties of the conductor, whereas $t$ and $\sigma$ have an influence on $\Phi_n$.

![Figure 4](image1.png) **Figure 4.** Normalized noise flux of magnetometer and gradiometer made up of two identical single loops at distance $d$.

![Figure 5](image2.png) **Figure 5.** Correlation of the magnetic field noise $B_n$ at distance $d$ for an infinite slab for different $z$ and $t$.

4. Optimization of SQUID based magnetometers and gradiometers for a MFFT
From the above principles it is clear that a multiloop ("cartwheel") SQUID [6] is a very efficient magnetometer as it approximates a circular coil and combines the multiloop principle for superior flux coupling into the SQUID. So it may serve as reference. With regard to $\langle \Phi^2 \rangle/A$, the small magnetometer C4WS is best used at $z \leq 0.25$ mm, while the medium size magnetometer C3WM is appropriate for $z \leq 0.35$ mm.

In order to find the best SQUID gradiometer design including inductance effects, we compared various types: classical transformer coupled gradiometer coils of both first and second order as well as directly coupled, rotational symmetric multiloop coils. It turned out for many designs that varying the coil size has only little net effect on the total signal-to-noise ratio because any increase or decrease in noise flux $\Phi_n(f = 0)$ or intensity $\langle \Phi^2 \rangle$ is compensated largely by a corresponding change of the coil inductance. The multiloop SQUIDs proved to be superior to transformer coupled SQUIDs due to the more efficient flux transfer from the pick-up coil into the SQUID. We found that a concentric multiloop gradiometer, consisting of a small inner cartwheel and an outer ring for compensation is a very efficient configuration.
This design was realized on a 3 mm x 3 mm chip enclosing a total area of about 6 mm$^2$. Assuming $T = 4.2$ K, $\sigma = (1.67 \times 10^{-8} \, \Omega m / 100)^{-1}$, $z = 100 \, \mu m$, $t = 10$ mm, the noise signal in the SQUID is characterized by $\langle \Phi^2 \rangle = 500 \, (m\Phi_0)^2$, $\Phi_n(f = 0) = 670 \, \mu \Phi_0/Hz^{1/2}$ and its cut-off frequency $f_c = 320$ Hz. Thus an intrinsic SQUID noise of $1 \, \mu \Phi_0/Hz^{1/2}$ corresponds to a sensor noise temperature $T_N$ of about $9.4 \, \mu K$. Sensors of this type have successfully been used for practical low temperature thermometry down to 7 mK [9].

References

[1] Netsch A, Hassinger E, Enss C and Fleischmann A 2006 AIP Conf. Proc. 850 1593
[2] Beyer J, Drung D, Kirste A, Engert J, Netsch A, Fleischmann A and Enss C 2007 IEEE Trans. Appl. Supercond. 17 760
[3] Varpula T and Poutanen T 1984 J. Appl. Phys. 55 4015
[4] Roth B J 1998 J. Appl. Phys. 83 635
[5] Harding J T and Zimmermann J E 1968 Phys. Lett. 27A 670
[6] Drung D, Knappe S and Koch H 1995 J. Appl. Phys. 77 4088
[7] Drung D, Hinrichs C and Barthelmess H-J 2005 Supercond. Sci. Technol. 18 S1
[8] Drung D, Aßmann C, Beyer J, Kirste A, Peters M, Ruede F and Schurig T 2007 IEEE Trans. Appl. Supercond. 17 699
[9] Engert J, Beyer J, Drung D, Kirste A and Peters M 2007 Int. J. Thermophys. In print