Comparison the iterative solvers for large sparse matrix in 3D electromagnetic forward modelling

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Abstract. In 3D electromagnetic (EM) forward modeling, an analytical solution is generally not available. Numerical solution is commonly applied to solve the forward modeling problems, mostly based on iterative solvers. The efficiency of EM forward modeling is critical for the development of practical inversion for EM data. The Krylov subspace solvers are widely used to solve frequency-domain EM forward modeling problems. However, these solvers converge remarkably more slowly as the operating period increases. This can be improved by the use of preconditioner and divergence correction. Multigrid (MG) solver is efficient for solving EM forward modelling problems without the use of preconditioner and divergence correction. In this paper, a MG solver is compared with Bi-Conjugate Gradients Stabilized (BCG) solvers with different preconditioners. They are compared, in terms of iteration number and computing time, indicating the MG solver is much more efficient.

1. Introduction

The EM methods are widely used in all aspect of geophysical applications. Among them, magnetotellurics (MT) method [1] uses natural source signals and covers a broad range of frequencies, which can be used to solve different real problems from shallow surface to great depth. Currently 3D MT survey has been widely applied for various applications [2,3].

As we all known, most time of 3D MT interpretation is spent on the forward modeling, which is typically solved by iterative numerical algorithms. These include finite difference method (FDM) [4], finite element method (FEM) [5], integral equation method (IEM) [6]. For finite difference methods, Yee [7] proposed the staggered grid finite difference method (SFDM) based on the curl-curl equations, which becomes a common practice.

For solving the large and complex linear system equations resulting from either of the above methods (e.g., FDM), Krylov subspace solvers are commonly used, such as BiCGstab (BCG) and quasi-minimal residual (QMR) [8]. However, the resulting linear equation system is usually non-positive definite and non-diagonally dominant. Also, this system has a rich null space. This poses challenge for these iterative solvers to converge. This can be improved by the use of the preconditioner and divergence correction[9]. However, the speed of convergence becomes slow as the period increases.

The MG solver can deal with large modeling problems by smoothly projecting the big problem onto a small scale problem. Its computation just increase linearly as the number of grid nodes increases [10, 11].

In this paper, we compare a MG solver and the BCG solver preconditioned with three different preconditioners. They are compared, in terms of iteration number and computing time. And the three
commonly used preconditioners are Gauss-Seidel (GS) [12], symmetric successive over relaxation (SSOR) and incomplete lower and upper triangular matrix decomposition (ILU) [13].

2. Algorithm

2.1. Forward modelling

At the low frequencies, the displacement currents can be neglected and electromagnetic fields, and , obey the Faraday's law of induction and Ampere's law as:

\[
\nabla \times \mathbf{E} = -i\omega \mu \mathbf{H}
\]

\[
\nabla \times \mathbf{H} = \sigma \mathbf{E}
\]

where \(\mu\), \(\sigma\) and \(\omega\) describe, the magnetic permeability, electrical conductivity parameter and angular frequency, respectively. A time dependence \(e^{i\omega t}\) is considered and SI unit system is used. By taking the curl of equation (1) and substituting it into equation (2), we can obtain

\[
\nabla \times \nabla \times \mathbf{E} + i\omega \mu \sigma \mathbf{E} = 0
\]

For SFDM (see Figure 1), we define the discretized electric field components on cell edges and magnetic field components on cell faces. The discretized second order curl–curl equation (3) can be written as:

\[
[C'C + \text{diag}(i\omega \mu \sigma)]e = 0
\]

where \(C\) is the discrete representation of the curl operator, mapping cell edges to cell faces, \(C^*\) is the adjoint of the discrete curl operator \(C\), mapping cell faces back to cell edges, and \(e\) is the discrete electric field solution. Equation (4) can be expressed as matrix form:

\[
Ae = b
\]

where \(A\) is a curl–curl operators, and the vector \(b\) combines boundary conditions.

2.2. MG solvers

The workflow for MG solver is that at the beginning the residual is smoothed on the finest grid, which can eliminate high frequency error. The second is to project the low frequency residual to the coarse mesh. In this process, the oscillation is enhanced, which can be effectively eliminated by applying additional iterations of smoother. Once coarsest level is reached, the solution can be cheaply obtained.
by either iterative and direct solvers. The solution then can be interpolated back to finer level, and this process is continued until finest level is reached. The above process is repeated, until a converged solution is arrived.

2.3. Numerical result

In this section, we design a low resistivity model, which consists of a low resistivity embedded in the half-space earth. The resistivity of the background is , for the resistivity of the air. Four different periods of 1, 10, 100, and 1000 s are considered. A uniform grid size of uniform cells is designed. The computational domain is 128km long in the horizontal directions and 128km deep in the depth. The boundary conditions used are the constant unit source at the top, zeros at the bottom, 1D solutions is used for the four lateral boundary sides.

![Figure 2. Convergence process of BCG solvers with the divergence correction and three different preconditioners and MG solver without the divergence correction and preconditioner periods from 1 to 1000 s for the 3-D low resistivity anomaly model.](image-url)
Figure 3. Comparison of (a) iteration number and (b) calculation time for the BCG solvers with divergence correction, preconditioned with ILU, SSOR, and GS, with those from the MG solver for the 3-D low resistivity anomaly model.

As shown in figure 2 and 3, the solution from MG is compared with those from BCG solvers, preconditioned with ILU, SSOR, and GS. For BCG solvers, after each preset iterations, one iteration of divergence correction is applied. The MG solver converges much better than other solvers, both in terms of iteration number and computing time.

3. Conclusions
We compare BCG solvers preconditioned with ILU, SSOR and GS with the MG solver in terms of iteration number and computing time for different frequencies for one 3-D low resistivity anomaly model. The results show that the MG solver is most efficient, generally 2-3 times faster. More importantly, the MG solver is less affected by the frequency both for computing time and iterations.

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