Interaction of confining vortices in SU(2) lattice gauge theory

M. Engelhardt, K. Langfeld, H. Reinhardt and O. Tennert

Institut für Theoretische Physik, Universität Tübingen
D–72076 Tübingen, Germany

Abstract

Center projection of SU(2) lattice gauge theory allows to isolate magnetic vortices as confining configurations. The vortex density scales according to the renormalization group, implying that the vortices are physical objects rather than lattice artifacts. Here, the binary correlations between points at which vortices pierce a given plane are investigated. We find an attractive interaction between the vortices. The correlations show the correct scaling behavior and are therefore physical. The range of the interaction is found to be $(0.4 \pm 0.2) \text{ fm}$, which should be compared with the average planar vortex density of approximately $2 \text{ vortices/fm}^2$. We comment on the implications of these results for recent discussions of the Casimir scaling behavior of higher dimensional representation Wilson loops in the vortex confinement picture.

* Supported in part by DFG under contract Re 856/1–3.
1. Introduction

Recently, the $Z(N)$ vortex picture of confinement has attracted renewed interest [1]-[4]. Proposed as early as 1978 [5], this picture assumes vortex type structures to be responsible for the area law behavior of the Wilson loop. These vortices, in the older literature also termed “fluxons” [5],[6], each contribute a factor $(-1)$ to the Wilson loop when they pierce its minimal area. Fluctuations of the number of vortices linked to a given Wilson loop produce a strong cancellation in its expectation value, yielding the desired area law.

Subsequently, many theoretical as well as numerical efforts were devoted to identifying such vortex type confiners and elucidating their nature. On the one hand, a glimpse of such configurations was afforded by the “spaghetti vacuum” [7] induced by the instability of homogeneous chromomagnetic fields. On the other hand, efforts were undertaken to define and classify vortex configurations comprehensively on the lattice [8],[9]. A manifestly gauge invariant description of vortices can be achieved by explicitly separating off the center of the gauge group in the Yang-Mills link variables on the lattice. In such a description, a distinction between “thin” and “thick” vortices [2] arises. The abovementioned factor $(-1)$ contribution to a pierced Wilson loop becomes the defining gauge invariant property of a thick vortex.

A fruitful approach to the investigation of specific infrared degrees of freedom conjectured to be relevant for confinement was pioneered by 't Hooft [10]. One utilizes the gauge freedom to bring an arbitrary gauge field configuration as close as possible to the type of configuration (“confiner”) under scrutiny; subsequently, one neglects residual deviations from the confiner (i.e. one projects onto the latter) in the hope that the gauge fixing procedure has concentrated onto it most of the relevant information contained in the original gauge field configuration. The validity of this projection procedure is difficult to establish a priori and it is more commonly justified a posteriori by the success in reproducing, say, the correct string tension.

In this vein, 't Hooft introduced the so-called Abelian gauges, which induce Abelian monopole singularities in the gauge-fixed fields. Subsequently, only the Abelian monopoles are kept as relevant degrees of freedom (so-called Abelian projection), allowing one to investigate the possibility of confinement as a consequence of a dual Meissner effect resulting from the condensation of the Abelian monopoles.

In complete analogy, one can introduce so-called center gauges which bring the link variables of a given lattice configuration as close as possible to center elements of the gauge group [3]. Vortices are then defined and singled out by center projection (see below for details). The crucial observation of the lattice calculations [3] is that a Wilson loop which is calculated with center projected links gives rise to almost the full string tension (a related conclusion is reached in the gauge invariant approach [2]).
mentioned further above). This implies that the center gauge successfully concentrates the information relevant for confinement onto the vortex degrees of freedom being projected on, a state of affairs sometimes referred to as “center dominance”. By contrast, in quantities other than the Wilson loop, the generic error due to the projection can be quite large [11].

Any physical quantity \( \rho \) (here, of mass dimension two) which is measured on the lattice in units of the lattice spacing \( a \) must display a characteristic dependence on the inverse gauge coupling constant \( \beta = 1/g^2 \), i.e. for sufficiently large \( \beta \) and for a pure SU(2) gauge theory

\[
\rho a^2 \approx \text{const.} \exp \left\{ -\frac{6\pi^2}{11} \beta \right\} \quad \text{(renormalization group)} .
\] (1)

Any violation of the scaling law (1) signals that the field combination under examination is not a physical quantity. Recently, some of us found [11] that the planar density \( \rho \) of vortices piercing a given surface displays the desired scaling law (1), implying that the vortices originating from center projection are physical objects. In this letter, we investigate this type of vortices.

While the vortices defined by center projection are localized to within one lattice spacing, it seems reasonable to assume that the original unprojected gauge configurations associated with these vortices are extended objects. An important consequence of such a finite reach of the underlying configurations is that the vortex vacuum may be able to correctly describe the Casimir scaling behavior of higher dimensional representation Wilson loops, contrary to earlier criticisms of the vortex vacuum picture [3], [4]. For such a mechanism to operate, one needs vortex diameters of one fermi or more. In order to obtain some more information not least concerning this point, we focus in this letter on the interaction between the center vortices. A parallel incentive for such an investigation lies in the observation (see below) that the string tension is too small if correlations between vortices are neglected. We thus measure the binary correlation of vortex points piercing a given plane. We show that the correlation function is a physical quantity, since it scales according to the renormalization group equation (1). Our main result is that the vortex interaction is attractive and has a range of \((0.4 \pm 0.2)\) fm. The implications of our results concerning the Casimir scaling of Wilson loops in higher dimensional representations will be briefly addressed.

### 2. The random vortex vacuum

Let us briefly review the definition of center vortices introduced in [3]. For this purpose, a SU(2) link variable \( U \) is decomposed as

\[
U = \alpha_0 + i\vec{\alpha} \; \vec{\sigma}, \quad \alpha_0^2 + \vec{\alpha}^2 = 1 .
\] (2)
In the Abelian gauge, the magnitudes of the so-called charged components \( \alpha_1, \alpha_2 \) are minimized with the help of gauge transformations; specifically, one maximizes \( \sum_i \text{tr} (U_i \sigma^3 U_i^\dagger \sigma^3) \), where \( i \) is a superindex labeling all the different links on the lattice. The Abelian projected links \( U^A \) are then defined by disregarding the charged components, i.e.

\[
\text{Abelian projection: } U \rightarrow U^A = \frac{\alpha'_0 + i\alpha'_3 \sigma^3}{\sqrt{\alpha'_0^2 + \alpha'_3^2}} .
\]

(3)

The Abelian gauge still allows for U(1) gauge transformations of the type \( \exp(i\eta \sigma^3) \). The center gauge fixes the residual gauge degree of freedom by demanding that the residual gauge transformation maximize \( \sum_i (\text{tr} U_i^A)^2 \). After adopting the center gauge, center projection is defined by disregarding the 3-component, i.e.

\[
\text{Center projection: } U^A \rightarrow U^C = \frac{\alpha''_0}{|\alpha''_0|} \in \{\pm 1\}.
\]

(4)

A plaquette on the lattice is defined to be part of a (center) vortex, if the product of the center projected links which span the plaquette under consideration yields \(-1\). A visualization of these points (for a given time slice) shows that these points are indeed grouped to string-like objects \[11\].

The crucial result of \[3\] was the observation that one recovers almost the full string tension when calculating the Wilson loop with center projected links instead of using full link variables. It should be mentioned that center projection can also be performed after a direct maximal center gauge fixing without preceding Abelian projection. In this case, the string tension agrees even better with the full one \[3\]. Center projection evidently does not strongly truncate the infrared degrees of freedom which are responsible for confinement (center dominance). Resorting to the Stokes theorem, one easily sees that the product of center projected links which lie on the circumference of a Wilson area yields \((-1)^n\) if \( n \) denotes the number of (center) vortices which pierce this area \[11\]. Hence, a vacuum consisting of (center) vortices reproduces, via the relation

\[
\langle W[C] \rangle = \sum_{n=0}^{\infty} (-1)^n P(n) ,
\]

(5)

the approximate expectation value of the Wilson loop obtained with center-projected links, where \( C \) is the Wilson loop, and \( P(n) \) is the probability that \( n \) vortices pierce its minimal surface.

Let us now neglect correlations between the vortices and calculate the string tension obtained in such a random vortex vacuum. Assume that the lattice volume is \( L^4 \), whereas the minimal surface of the Wilson loop under consideration possesses the
area $A, A \ll L^2$. The Wilson loop is embedded in a plane $H$ of area $L^2$. The random vortex model assumes that the probability $p$ that a vortex which pierces $H$ also pierces the Wilson area is $p = A/L^2$. If $N$ vortices pierce the plane $H$, then the probability $P_{\text{rand}}(n)$ that precisely $n \leq N$ vortices pierce the Wilson area is

$$P_{\text{rand}}(n) = \binom{N}{n} p^n (1-p)^{N-n}.$$  \hfill (6)

Hence, the expectation value (5) in the random vortex model gives

$$\langle W[C] \rangle_{\text{rand}} = \sum_{n=0}^{\infty} (-1)^{n} P_{\text{rand}}(n) = (1 - 2p)^N.$$  \hfill (7)

Our lattice simulations \cite{11} revealed that the planar vortex density $\rho = N/L^2 \approx 2/\text{fm}^2$ is a physical quantity, where the string tension $\kappa = (440 \text{MeV})^2$ was used to fix the renormalization scale. We therefore obtain in the infinite volume limit ($L \to \infty$, i.e. $N \to \infty$, $\rho$ fixed)

$$\langle W[C] \rangle_{\text{rand}} \approx \lim_{N \to \infty} \left( 1 - \frac{2\rho A}{N} \right)^N = \exp \left( -2\rho A \right).$$  \hfill (8)

Eq.(8) yields the desired area law, from which we read off the string tension

$$\kappa_{\text{rand}} = 2\rho \approx (400 \text{MeV})^2,$$  \hfill (9)

which should be compared with the exact (input) value $\kappa = (440 \text{MeV})^2$. While inserting the exact probability distribution $P(n)$ generated in our lattice simulations into (8) yields almost the full string tension, the value in the random vortex vacuum turns out to be 17% too small. The correlations between the vortices are obviously significant.

3. Vortex correlations

In order to study the correlations between the vortex points on the plane $H$, we introduce a field $s_j$, where $j$ is a superindex labeling all the different plaquettes in the lattice: $s_j = 1$, if plaquette $j$ is part of a vortex, and is 0 otherwise. The lattice average of $s_j$ is independent of $j$ due to homogeneity and isotropy. It is directly related to the vortex density, i.e.

$$\rho a^2 = \langle s_j \rangle.$$  \hfill (10)

Consider next the normalized correlator

$$c_{ij} = \frac{\langle s_i s_j \rangle}{\langle s_i \rangle \langle s_j \rangle}$$,  \hfill (11)
where in the following, plaquette $i$ and plaquette $j$ will be considered to lie in the same plane, $l$ lattice spacings (i.e. a distance $r = l \cdot a$) apart in the direction of one of the coordinate axes. Due to homogeneity and isotropy, the corresponding $c_{ij}$ will only depend on $r$ and will thus henceforth be denoted as $c(r)$. This correlator has a very transparent interpretation in terms of a conditional probability: Assuming that one sits on a plaquette which is part of a vortex, $\rho a^2 c(r)$ is the probability of finding another vortex at a distance $r$. This is precisely the algorithm used in practice to extract $c(r)$. The quantity $c(r)$ is normalized such as to give a constant equal to unity if the vortices piercing the plane under consideration are statistically independent; thus, $c(r)$ constitutes what is often termed a (planar) radial distribution function. Interactions produce deviations from unity; the distance scales over which the deviations persist give a rough estimate of the range of the interaction in the medium.

Since the vortices are physical objects, their interaction as revealed in $c(r)$ should also behave as a physical quantity under the renormalization group. In order to verify this, it is necessary to examine the dimensionless function $c(r)$ at different couplings $\beta$, where it is crucial to take into account the running of the lattice spacing $a(\beta)$ entering the physical distance $r = l \cdot a$. In order to estimate the statistical errors as well as the influence of systematic errors, we used three methods to extract $a(\beta)$ in physical units. Firstly, we measured the string tension $\kappa a^2(\beta)$ for $\beta$ values within the scaling window $2 < \beta < 2.8$. Using $\kappa = (440 \text{ MeV})^2$, this procedure directly yields $a(\beta)$ in physical units. Secondly, we fitted the perturbative scaling law (1) to $\kappa a^2(\beta)$, and used the formula (1) to express $l \cdot a = r$ in physical units (“ideal scaling”). Thirdly, we extracted the “running” of $a(\beta)$ from the measured quantity $\rho a^2(\beta)$, and used $\rho \approx 2 \text{ fm}^{-2}$ as physical reference scale.

| measured $\kappa a^2 \rightarrow a(\beta)$ | “measured scaling” |
|------------------------------------------|-------------------|
| fit of $\kappa a^2$ to (1)               | “ideal scaling”   |
| measured $\rho a^2 \rightarrow a(\beta)$| “density scaling” |

Within the statistical error bars, all three methods of extrapolating to the continuum limit should yield the same results. Figure 1 shows our numerical results for the planar radial distribution function $c(r)$ as a function of $r$. We have used lattices consisting of $10^4$ and $12^4$ lattice points in order to estimate the finite size effects. Calculations with both lattice sizes yield the same results within the statistical errors. In the left hand picture, the extrapolation of the data was done with “ideal scaling”. The crucial observation is that the result is indeed renormalization group invariant, i.e. independent of the actual choice of $\beta$. Consequently, $c(r)$ is a physical quantity. We further corroborate this with the right hand picture, in which the different types of scaling mentioned above are confronted with each other, for a $10^4$ lattice.
The shape of the planar radial distribution function $c(r)$ plotted in Figure 1 reveals that an attractive interaction operates between the vortices in the vortex medium. Note that the range of this interaction constitutes a rather vaguely defined notion.

One way of defining the range of an attractive interaction is to look for the first crossover of the radial distribution function below unity (note that such a crossover must exist, since an appropriate integral over the radial distribution function must reproduce the total number of vortex points). This crossover happens for the present data at $r = 0.6$ fm, where it must be noted that in this region the statistical errors are already of the same magnitude as the deviation from unity. The value $r = 0.6$ fm can be regarded as an upper limit on the range of the interaction. Another possible definition of the interaction range lies in fitting an exponential decay to the deviation of the radial distribution function from unity and thus extracting a typical screening length. This yields a value of roughly 0.2 fm, which can be regarded as a lower limit on the interaction range. There is an intuitive argument making the appearance of this scale plausible: The planar radial distribution function is by its definition a plaquette-plaquette correlation function, albeit with center-projected links, and shifted by unity. The exponential fall-off of such correlation functions is generically controlled by glueball masses. Thus, the emergence of the relatively high energy scale associated with a screening length of 0.2 fm in $c(r)$ is not too surprising.

Finally, it should be noted that the planar correlations measured here still represent a rather unspecific yardstick for the structure of the vortex vacuum. They subsume a variety of more detailed effects; not only are they sensitive to the actual interaction of segments of neighboring vortices, but also e.g. to the shape distributions of the individual vortices in the directions orthogonal to the plane under consideration. It would be interesting to further disentangle the effects of the actual vortex-vortex interaction and the effects due to, say, curvature terms in the single-vortex action.
4. Conclusions

We have further investigated the center vortices introduced in [3]. These vortices can account for almost the full string tension, and were recently recognized as physical objects (rather than lattice artifacts) [11], since the density $\rho$ of vortices piercing a plane scales according the renormalization group.

Here, we have observed that the random vortex model, in which correlations between the vortices are neglected, qualitatively describes the gross features of confinement, but underestimates the string tension by 17%. In order to study the binary vortex correlations induced by the full Yang-Mills action, we have introduced the planar radial distribution function $c(r)$. Our numerical simulations reveal that $c(r)$ is a renormalization group invariant and therefore a physical quantity. The data show that the vortex interaction is attractive and possesses a range of $(0.4 \pm 0.2)\, \text{fm}$ in the vortex medium.

With regard to the possibility of correctly describing the Casimir scaling of Wilson loops in higher dimensional representations along the lines discussed in [4], our results do not allow very definite conclusions. If we roughly identify the range of the vortex-vortex interaction, as read off from the measured planar radial distribution function, with the diameter of the *unprojected* gauge configurations associated with the center vortices, we reach the conclusion that these configurations are on the average too thin to allow for the mechanism of Casimir scaling discussed in [4]. However, it is entirely possible that the vortices are significantly thicker and through some cancellation only start to feel an appreciable attractive interaction when they already considerably overlap. Therefore, our present results do not necessarily contradict the mechanism of Casimir scaling of Wilson loops in higher dimensional representations proposed in [4].

References

[1] E. T. Tomboulis, Phys. Lett. B303 (1993) 103.

[2] T. G. Kovács and E. T. Tomboulis, hep-lat/9711009.

[3] L. Del Debbio, M. Faber, J. Greensite and Š. Olejník, Nucl. Phys. Proc. Suppl. B53 (1997) 141;
L. Del Debbio, M. Faber, J. Greensite and Š. Olejník, Phys. Rev. D 53 (1996) 5891;
L. Del Debbio, M. Faber, J. Greensite and Š. Olejník, talk presented at the NATO Advanced Research Workshop on Theoretical Physics: New Developments in Quantum Field Theory, Zakopane, Poland, 14-20 June 1997,
[4] M. Faber, J. Greensite and Š. Olejník, hep-lat/9710033.

[5] Y. Aharonov, A. Casher and S. Yankielowicz, Nucl. Phys. B146 (1978) 256.

[6] P. Vinciarelli, Phys. Lett. B78 (1978) 485.

[7] J. Ambjørn and P. Olesen, Nucl. Phys. B170 [FS1] (1980) 265;
   P. Olesen, Nucl. Phys. B200 [FS4] (1982) 381.

[8] G. Mack, in “Recent developments in gauge theories”, eds. G. ’t Hooft et
   al. (Plenum, New York, 1980);
   G. Mack and E. Pietarinen, Nucl. Phys. B205 [FS5] (1982) 141.

[9] R. C. Brower, D. A. Kessler and H. Levine, Nucl. Phys. B205 [FS5] (1982)
   77.

[10] G. ’t Hooft, Nucl. Phys. B190 (1981) 455.

[11] K. Langfeld, H. Reinhardt and O. Tennert, hep-lat/9710068. Phys. Lett. B,
    in press.