Remez Type Inequalities and Morrey-Campanato Spaces on Ahlfors Regular Sets

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This paper is dedicated to our friend Michael Cwikel with respect and sympathy.

Abstract. The paper presents several new results on Remez type inequalities for real and complex polynomials in $n$ variables on Ahlfors regular subsets of Lebesgue $n$-measure zero. As an application we prove an extension theorem for Morrey-Campanato spaces defined on such sets.

1. Introduction

Recently there has been a considerable interest in multi-dimensional analogs of the classical Remez polynomial inequality in connection with various problems of Analysis, see, e.g., surveys [BB] and [G] and references therein. The first result of this type is the Yu.Brudnyi-Ganzburg inequality [BG]. In its formulation, we let $V \subset \mathbb{R}^n$ be a convex body, $\omega \subset V$ a measurable subset, and $\mathcal{L}_n$ be the Lebesgue measure on $\mathbb{R}^n$.

Suppose that $p \in \mathbb{R}[x_1, \ldots, x_n]$ is a real polynomial of degree $k$ on $\mathbb{R}^n$. Then the following inequality holds:

\begin{equation}
\sup_V|p| \leq T_k \left(\frac{1 + \beta_n(\lambda)}{1 - \beta_n(\lambda)}\right) \sup_\omega|p|.
\end{equation}

Here $T_k$ is the Chebyshev polynomial of degree $k$, $\beta_n(\lambda) := (1 - \lambda)^{1/n}$ and $\lambda := \frac{\mathcal{L}_n(\omega)}{\mathcal{L}_n(V)}$.

This inequality is sharp and for $n = 1$ coincides with the classical Remez inequality.

Most of the known applications of this inequality in Analysis use the following corollary of (1.1):

\begin{equation}
\sup_V|p| \leq \left(\frac{4n}{\lambda}\right)^k \sup_\omega|p|
\end{equation}

which is easily derived from (1.1) using properties of the Chebyshev polynomials.

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In an actively developing field of modern mathematics, analysis on fractal sets, see, e.g., [Tr] and references therein, one requires a generalization of inequality (1.2) for fractal sets. In such a generalization, $\omega$ is a subset of Lebesgue measure 0 in a Euclidean ball $B \subset \mathbb{R}^n$. Since zero sets of real polynomials on $\mathbb{R}^n$ have Hausdorff dimension $\leq n - 1$, to obtain a finite bound for $\sup_B |p|/\sup_\omega |p|$ one assumes also that the Hausdorff dimension of $\omega$ is more than $n - 1$. Further, it is natural to estimate the above ratio by a function depending on the Hausdorff measures of $B$ and $\omega$. Specifically, let $\mathcal{H}_s$ denote the $s$-Hausdorff measure on $\mathbb{R}^n$, $0 < s \leq n$; in particular, $\mathcal{H}_n$ coincides with $\mathcal{L}_n$ up to a factor depending only on $n$. In the present paper we study Remez type inequalities of the following form
\begin{equation}
\sup_B |p| \leq \phi(\lambda) \sup_\omega |p|,
\end{equation}
where $p$ is a real polynomial on $\mathbb{R}^n$ or a holomorphic polynomial on $\mathbb{C}^n$, $B$ is a Euclidean ball in $\mathbb{R}^n$ or $\mathbb{C}^n$, respectively, and $\omega \subset B$ is a subset of finite Hausdorff $s$-measure with $n - 1 < s \leq n$ in the real case and $2n - 2 < s \leq 2n$ in the complex one. Also,
\begin{equation}
\lambda := \left\{ \mathcal{H}_s(\omega) \right\}^{m/s} \mathcal{H}_m(B)^{-1},
\end{equation}
where $m = n$ in the real case and $m = 2n$ in the complex case.

For many applications (related, e.g., to reverse Hölder inequalities or BMO-properties of functions) it is crucial that $\phi$ in (1.3) is a power function in $\lambda$. Inequalities of the form (1.3) with such a function will be referred to as strong Remez type inequalities. However, in applications related to trace and extension theorems for classical spaces of differentiable functions, see, in particular, [Ca], [YBr], [J], it suffices to use inequalities of the form (1.3) with a function $\phi$ whose dependence of $\lambda$ is not specified. In this case the only required information is the monotonicity of $\phi$ in $\lambda$. Such inequalities will be referred to as weak Remez type inequalities.

The existence of inequalities (1.3) for $n = 1$ was first demonstrated in [ABr] where strong Remez type inequalities were proved for Ahlfors $s$-regular sets $\omega$ in $\mathbb{R}$ or $\mathbb{C}$ with $0 < s \leq 1$ for real $p$ and with $0 < s \leq 2$ for holomorphic ones. Moreover, it was proved in [ABr, Prop. 3] that $s$-regularity is necessary for the validity of such an inequality.

Let us recall the definition of Ahlfors regular sets.

For a subset $K \subset \mathbb{R}^n$ and a point $x \in K$ by $B_r(x; K)$ we denote the intersection with $K$ of an open Euclidean ball in $\mathbb{R}^n$ centered at $x$ of radius $r$.

**Definition 1.1.** A subset $K \subset \mathbb{R}^n$ is said to be (Ahlfors) $s$-regular if there is a positive number $a$ such that for every $x \in K$ and $0 < r \leq \text{diam}(K)$
\begin{equation}
\mathcal{H}_s(B_r(x; K)) \leq ar^s.
\end{equation}

The class of these sets will be denoted by $\mathcal{A}_n(s, a)$.

**Definition 1.2.** A subset $K \subset \mathcal{A}_n(s, a)$ is said to be an $s$-set if there is a positive number $b$ such that for every $x \in K$ and $0 < r \leq \text{diam}(K)$
\begin{equation}
br^s \leq \mathcal{H}_s(B_r(x; K)).
\end{equation}

We denote this class by $\mathcal{A}_n(s, a, b)$.

The class of $s$-sets, in particular, contains compact Lipschitz $s$-manifolds (with integer $s$), Cantor type sets and self-similar sets (with arbitrary $s$), see, e.g., [JW, p. 29] and [Ma, Sect. 4.13].
In the present paper we establish inequalities of form (1.3) for s-regular sets \( \omega \subset A_n(s,a) \) with \( \phi \) depending also on \( s, n, k := \deg p \) and \( a \). We prove strong Remez type inequalities for holomorphic polynomials using a technique of Algebraic Geometry. For the real case, strong Remez type inequalities are true for dimensions \( n = 1, 2 \) but the problem is open for \( n > 2 \). On the other hand, weak Remez type inequalities are valid in this case, the proof is outlined in [BB]. For the convenience of the reader we present this proof below.

In the final section of the paper we present an extension theorem for Morrey-Campanato spaces defined on \( s \)-sets with \( n - 1 < s \leq n \). For \( s = n \) this result is proved in [YBr]. Observe that a weak (or strong) Remez type inequality implies the, so-called, Markov inequality for \( s \)-regular sets stating that for some \( c = c(F, n, k) \)

\[
\max_{F \cap B} |\nabla p| \leq c \max_{F \cap B} |p|
\]

(1.6) where \( F \in A_n(s,a), n - 1 < s \leq n, B \) is a closed Euclidean ball of radius \( r \) centered at \( F \) and \( p \in \mathbb{R}[x_1, \ldots, x_n] \) is a polynomial of degree \( k \). For \( s \)-sets this result was established by other methods in [JW, Sec. II.1.3].

### 2. Formulation of Main Results

#### 2.1. We start with strong Remez type inequalities for holomorphic polynomials on \( \mathbb{C}^n \).

Let \( X \subset \mathbb{C}^n \) belong to \( A_{2n}(s,a), s = 2n - 2 + \alpha, \alpha > 0 \). Let \( p \) be a holomorphic polynomial on \( \mathbb{C}^n \) of degree \( k \).

**Theorem 2.1.** For any Euclidean ball \( B \subset \mathbb{C}^n \) and an \( \mathcal{H}_s \)-measurable subset \( \omega \subset X \cap B \) one has

\[
\sup_B |p| \leq \left( \frac{c_1 \mathcal{H}_{2n}(B)}{(\mathcal{H}_s(\omega))^{2n/s}} \right)^{c_2 k} \sup_{\omega} |p|
\]

where \( c_1 \) depends on \( a, n, k, \alpha \) and \( c_2 > 0 \) depends on \( \alpha \).

**Corollary 2.2.** Let \( X \in A_{2n}(s,a,b) \). Let \( B = B_r(x;X) \), \( x \in X, r > 0 \), and \( \omega \subset B \) be \( \mathcal{H}_s \)-measurable. Then for a holomorphic polynomial \( p \) of degree \( k \) the following is true:

\[
\sup_B |p| \leq \left( \frac{c_1 \mathcal{H}_s(B)}{(\mathcal{H}_s(\omega))^{s}} \right)^{c_2 k} \sup_{\omega} |p|
\]

where \( c_1 \) depends on \( a, b, n, k, \alpha \) and \( c_2 \) depends on \( \alpha \).

**Corollary 2.3.** Let \( X \subset \mathbb{C}^n \) be an \( s \)-set with \( s \) as above. Then for any holomorphic polynomial \( p \) the function \( \ln |p| \in BMO(X, \mathcal{H}_s) \).

Another corollary is the following reverse Hölder inequality.

**Corollary 2.4.** Under assumptions of Theorem 2.1 for \( 1 \leq l \leq \infty \) one has

\[
\left( \frac{1}{\mathcal{H}_s(B_r(x;X))} \int_{B_r(x;X)} |p|^l \, d\mathcal{H}_s \right)^{1/l} \leq C \left( \frac{1}{\mathcal{H}_s(B_r(x;X))} \int_{B_r(x;X)} |p| \, d\mathcal{H}_s \right)
\]

where \( C \) depends on \( k, n, \alpha, a \) and \( b \).

**Remark 2.5.** The results of this subsection for \( n = 1 \) were proved in [ABr] with \( c_1 \) independent of \( k \). An interesting open question is whether \( c_1 \) is independent of \( k \) in the general case, as well.
2.2. In this part we present a general form of weak Remez type inequalities for real polynomials on $\mathbb{R}^n$.

Theorem 2.6. Assume that $U \subset \mathbb{R}^n$ is a bounded open set and $\omega \subset U$ belongs to $A_n(s, a)$ with $n - 1 < s \leq n$. Assume also that
\[ \lambda := \frac{\mathcal{H}_s(\omega)}{\mathcal{H}_n(U)} > 0. \]
Then there is a constant $C > 1$ such that for every polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $k$
\[ \left( \frac{1}{\mathcal{H}_n(U)} \int_U |p|^r \, d\mathcal{H}_n \right)^{1/r} \leq C \left( \frac{1}{\mathcal{H}_s(\omega)} \int_\omega |p|^q \, d\mathcal{H}_s \right)^{1/q}. \]
Here $0 < q, r \leq \infty$ and $C$ depends on $U$, $n$, $q$, $r$, $s$, $k$, $a$ and $\lambda$ and is increasing in $1/\lambda$. In particular, for $q = r = \infty$ we obtain the weak Remez type inequality of the form (1.3).

2.3. Let $X \subset \mathbb{R}^n$ be a measurable set of positive Hausdorff $s$-measure. By $\mathcal{K}_X$ we denote the family of closed cubes in $\mathbb{R}^n$ with centers at $X$ and “radii” ($:= \frac{1}{4}$ length sided) at most $4 \text{diam } X$. We write $Q_r(x)$ for the cube of radius $r$ and center $x$ and denote by $X_r(x)$ the set $Q_r(x) \cap X$ for $x \in X$.

In order to introduce the basic concept, Morrey-Campanato space on $X$, we denote by $L_q(X)$, $1 \leq q \leq \infty$, the linear space of $\mathcal{H}_s$-measurable functions on $X$ equipped with norm
\[ \|f\|_q := \left( \int_X |f|^q \, d\mathcal{H}_s \right)^{1/q}, \quad 0 \leq q \leq \infty, \]
and use the following

Definition 2.7. The local best approximation of order $k \in \mathbb{Z}_+$ is a function $\mathcal{E}_k : L_q(X) \times \mathcal{K}_X \to \mathbb{R}_+$ given for $Q = Q_r(x)$ by
\[ \mathcal{E}_k(f; Q) := \inf_p \left\{ \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} |f - p|^q \, d\mathcal{H}_s \right\}^{1/q} \]
where $p$ runs over the space $\mathcal{P}_{k-1} \subset \mathbb{R}[x_1, \ldots, x_n]$ of polynomials of degree $k - 1$.

For $k = 0$ we let $\mathcal{P}_{k-1} := \{0\}$; hence $\mathcal{E}_0(f; Q)$ is the normalized $L_q$-norm of $f$ on $X_r(x)$.

Let now $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone function on $\mathbb{R}_+ := (0, \infty)$ (it may be a constant).

Definition 2.8. The (generalized) Morrey-Campanato space $\dot{C}^{k, \omega}_q(X)$ is defined by seminorm
\[ \|f\|_{\dot{C}^{k, \omega}_q(X)} := \sup \left\{ \frac{\mathcal{E}_k(f; Q)}{\omega(r_Q)} : Q \in \mathcal{K}_X \right\} \]
where $r_Q$ denotes the radius of $Q$.

For $X$ being a domain in $\mathbb{R}^n$ and $s = n$ this space coincides with the Morrey space $\mathcal{M}_q^\lambda$ [Mo] (for $k = 0$, $\omega(t) = t^\lambda$, $-n < \lambda < 0$), the BMO-space [JN] (for $k = 1$, $\omega(t) = \text{const}$) and the Campanato space $[Ca]$ (for $k \geq 1$, $\omega(t) = t^\lambda$, $\lambda > 0$).

To formulate the main result we also need
DEFINITION 2.9. Let \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) be nondecreasing such that
\[
\omega(+0) = 0 \quad \text{and} \quad t \to \frac{\omega(t)}{t^k} \quad \text{be nonincreasing.}
\]
The \( \omega \) is said to be a quasipower \( k \)-majorant if
\[
C_\omega := \sup_{t > 0} \left\{ \frac{1}{\omega(t)} \int_0^t \frac{\omega(u)}{u} \, du \right\} < \infty.
\]
The Lipschitz space \( \dot{\Lambda}^{k,\omega}(\mathbb{R}^n) \) of order \( k \geq 1 \) consists of locally bounded functions \( f : \mathbb{R}^n \to \mathbb{R} \) such that the seminorm
\[
|f|_{\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)} := \sup \left\{ \frac{|\Delta_h^k f(x)|}{\omega(|h|)} : x, h \in \mathbb{R}^n \right\}
\]
is finite.

Here \( |h| \) is the Euclidean norm of \( h \) and
\[
\Delta_h^k f(x) := \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(x + jh).
\]

EXAMPLE 2.10. Choosing in this definition \( \omega(t) := t^\lambda \), \( 0 < \lambda \leq k \), we obtain the (homogeneous) Besov space \( B^\lambda_k(\mathbb{R}^n) \). Let us recall that it coincides with the Sobolev space \( \dot{W}^k_\infty(\mathbb{R}^n) \) for \( \lambda = k \), the Hölder space \( C^{l,\alpha}(\mathbb{R}^n) \) for \( \lambda = l + \alpha \), \( l \) is an integer and \( 0 < \alpha < 1 \), and with the Marchaud-Zygmund space for \( \lambda \) integer and \( 0 < \lambda < k \). In the last case, the corresponding seminorm is
\[
|f|_{B^\lambda_k(\mathbb{R}^n)} := \max_{|\alpha| = \lambda - 1} \sup_h \frac{||\Delta_h^k(D^\alpha f)||_{C(\mathbb{R}^n)}}{|h|}.
\]

THEOREM 2.11. Let \( X \subset \mathbb{R}^n \) be an \( s \)-set with \( n - 1 < s \leq n \) and \( \omega \) be a quasipower \( k \)-majorant. Then there is a linear continuous extension operator \( T_k : \dot{C}^{k,\omega}_q(X) \to \dot{\Lambda}^{k,\omega}(\mathbb{R}^n) \).

In particular, \( \dot{C}^{k,\omega}_q(X) \) is isomorphic to the trace space \( \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X \).

For \( \omega(t) = t^\lambda \), \( 0 < \lambda \leq k \), and \( n \)-sets this result was proved in [YBr] in a different way.

REMARK 2.12. (a) Theorem 2.11 is also true for the nonhomogeneous Morrey-Campanato space \( \dot{C}^{k,\omega}_q(X) \) defined by (quasi)-norm
\[
||f||_{\dot{C}^{k,\omega}_q(X)} := ||f||_q + ||f||_{\dot{C}^{k,\omega}_q(X)}
\]
The target space of the extension operator is now the Banach space \( \Lambda^{k,\omega}(\mathbb{R}^n) \) defined by norm
\[
||f||_{\Lambda^{k,\omega}(\mathbb{R}^n)} := \sup_{Q_0} |f| + ||f||_{\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)}
\]
where \( Q_0 := [0,1]^n \).

(b) For \( 0 < q < 1 \) the extension operator exists also but it is only nonlinear (homogeneous and bounded).

(c) Let \( X_j \subset \mathbb{R}^n \) be \( s_j \)-sets, \( n_j - 1 < s_j \leq n_j \), \( 1 \leq j \leq k \). Then under the same assumptions for \( \omega \), Theorem 2.11 is also valid for \( X := X_1 \times \cdots \times X_k \subset \mathbb{R}^n \), \( n := n_1 + \cdots + n_k \). Here the space \( \dot{C}^{k,\omega}_q(X) \) is defined with respect to the tensor product of \( \mathcal{H}_{s_j} \)-measures. This can be proved similarly to the proof of Theorem 2.11 based on the corresponding Remez type inequalities for such \( X \).
3. Holomorphic Polynomials

3.1. Complex Algebraic Varieties and s-sets.

3.1.1. In this section we gather some standard facts of Complex Algebraic Geometry. For the background and the proofs see, e.g., books [M] and [GH].

By \( \mathbb{CP}^n \) we denote the \( n \)-dimensional complex projective space with homogeneous coordinates \((z_0 : \cdots : z_n)\). The complex vector space \( \mathbb{C}^n \) is a dense open subset of \( \mathbb{CP}^n \) defined by \( z_0 \neq 0 \). The hyperplane at \( \infty \), \( H := \{(z_0 : \cdots : z_n) \in \mathbb{CP}^n : z_0 = 0\} \), can be naturally identified with \( \mathbb{CP}^{n-1} \) and \( \mathbb{CP}^n = \mathbb{C}^n \cup H \).

A closed subset \( X \subset \mathbb{C}^n \) defined as the set of zeros of a family of holomorphic polynomials on \( \mathbb{C}^n \) is called an affine algebraic variety. By \( \dim_{\mathbb{C}} X \) we denote the (complex) dimension of \( X \), i.e., the maximum of complex dimensions of complex tangent spaces at smooth points of \( X \).

Assume that an affine algebraic variety \( X \subset \mathbb{C}^n \) has pure dimension \( k \geq 1 \), i.e., dimensions of complex tangent spaces at smooth points of \( X \) are the same. Then its closure \( \overline{X} \) in \( \mathbb{CP}^n \) is a projective variety of pure dimension \( k \), and \( \dim_{\mathbb{C}}(H \cap \overline{X}) = k - 1 \).

Any linear subspace of dimension \( n - k \) in \( \mathbb{CP}^n \) meets \( \overline{X} \), but there is a linear subspace \( L \subset H \) of dimension \( n - k - 1 \) such that \( L \cap \overline{X} = \emptyset \). Moreover, for a generic \((n - k)\)-dimensional subspace of \( \mathbb{CP}^n \) its intersection with \( \overline{X} \) consists of a finite number of points. The number of these points is called the degree of \( \overline{X} \) and is denoted \( \deg \overline{X} \). For instance, if \( X \) as above is defined as the set of zeros of holomorphic polynomials \( p_1, \ldots, p_{n-k} \) on \( \mathbb{C}^n \) of degrees \( d_1, \ldots, d_{n-k} \), respectively, then by the famous Bezout theorem \( \deg \overline{X} \leq d_1 \cdots d_{n-k} \).

Let \( L \subset H \) be a linear subspace of dimension \( n - k - 1 \) which does not intersect \( \overline{X} \). This subspace defines a projection \( \phi_L : \mathbb{CP}^n \to \mathbb{CP}^k \) as follows.

Fix a linear subspace of dimension \( k \) in \( \mathbb{CP}^n \) disjoint from \( L \). We will simply call it \( \mathbb{CP}^k \). If \( w \in \mathbb{CP}^n \setminus L \), then \( w \) and \( L \) span an \((n - k)\)-dimensional linear subspace which meets \( \mathbb{CP}^k \) in a unique point \( \phi_L(w) \). The map \( \phi_L \) sends \( w \) to \( \phi_L(w) \). Further, \( \mathbb{C}^n \subset \mathbb{CP}^n \setminus L \), and, with a suitable choice of linear coordinates, \( \phi_L|_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{C}^k \) is the standard projection: \((z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_k)\).

The map \( \phi_L|_X : X \to \mathbb{C}^k \) is a surjection and is a branched covering over \( \mathbb{C}^k \) whose order \( \mu \), i.e., the number of points \( \phi_L^{-1}(y) \cap X \) for a generic \( y \in \mathbb{C}^k \), is \( \deg \overline{X} \). Then \( X \) is a complex subvariety of a pure \( k \)-dimensional algebraic variety \( \overline{X} \) defined as the set of zeros of holomorphic polynomials \( p_i \), \( 1 \leq i \leq n-k \), of the form

\[
  p_i(z_1, \ldots, z_n) = z_{k+i}^\mu + \sum_{1 \leq l \leq \mu} b_{il}(z_1, \ldots, z_k)z_{k+i}^{\mu-l}
\]

where \( b_{il} \) is a holomorphic polynomial of degree \( \leq l \) on \( \mathbb{C}^k \). Moreover, let \( S \subset \mathbb{C}^k \) be the branch locus of \( \phi_L|_X \). If \( w \in \mathbb{C}^k \setminus S \), then \( b_{il}(w) \) is the \( l \)-th elementary symmetric function in \( z_{k+i}(w^{(1)}), \ldots, z_{k+i}(w^{(\mu)}) \), where \( \phi_L^{-1}(w) \cap X = (w^{(1)}, \ldots, w^{(\mu)}) \).

(Recall that the elementary symmetric functions \( s_t \) in \( x_1, \ldots, x_s \) are defined from the identity \( \prod_{1 \leq i \leq s}(t - x_i) = t^s + s_1t^{s-1} + \cdots + s_s \) of polynomials in variable \( t \).)

Since \( \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} \overline{X} = k \), \( X \) is the union of some irreducible components of \( \overline{X} \).

Next, the Fubini-Studzi metric on \( \mathbb{CP}^n \) is a Riemannian metric defined by the associated \((1,1)\)-form \( \omega := \sum_{0 \neq i} \partial \overline{\partial} \ln(|z_0|^2 + \cdots |z_n|^2) \), \((z_0 : \cdots : z_n) \in \mathbb{CP}^n \). For a \( k \)-dimensional projective variety \( \overline{X} \) as above the \((k,k)\)-form \( \wedge^k \omega \) determines a
Borel measure $\mu_X$ on $X$,

$$
\mu_X(U) := \int_U \wedge^k \omega
$$

where $U \subset X$ is a Borel subset. Moreover,

$$
\mu_X(X) = \text{deg } X,
$$

see, e.g., [GH, Ch. 1.5].

Let $\omega_e := \sqrt{-1} \sum_{1 \leq i \leq n} dz_i \wedge d\bar{z}_i$ be the Euclidean Kähler form determining the Euclidean metric on $\mathbb{C}^n$. Then $\omega$ and $\omega_e$ are equivalent on every compact subset $K \subset \mathbb{C}^n$ where the constants of equivalence depend on $K$ and $n$ only. In particular, the Fubini-Studi and the Euclidean metrics, and the $(k, k)$-forms $\wedge^k \omega_e$ and $\wedge^k \omega$ are equivalent on every such $K$. Let $\mu_{e,X}$ be a Borel measure on a pure $k$-dimensional affine algebraic variety $X$ defined by the formula

$$
\mu_{e,X}(U) := \int_U \wedge^k \omega_e
$$

where $U \subset X$ is a Borel subset. Then for every compact subset $K \subset \mathbb{C}^n$ the measures $\mu_X|_{K \cap X}$ and $\mu_{e,X}|_{K \cap X}$ are equivalent with the constants of equivalence depending on $K$, $k$ and $n$ only.

3.1.2. In this section we establish a relation between complex algebraic varieties and $s$-sets.

**Theorem 3.1.** Let $X \subset \mathbb{C}^n$ be an affine algebraic variety of pure dimension $k \geq 1$ such that $\text{deg } X \leq \mu$. Then $X \in \mathcal{A}_{2n}(2k, a, b)$ where $a$ and $b$ depend on $k$, $\mu$ and $n$ only.

**Proof.** We will prove that

$$
br^{2k} \leq \mu_{e,X}(B_r(x; X)) \leq ar^{2k}
$$

with $a$ and $b$ depending on $k$, $\mu$ and $n$ only, where $X$ satisfies the assumptions of the theorem, $x \in X$ and $\mu_{e,X}$ is the measure on $X$ determined in (3.4). From here applying [JW, Sec. II.1.2, Th. 1] we get the desired statement.

Since $\text{deg } X = \text{deg } x + X$ and $\mu_{e,X}(U) = \mu_{e,x+X}(x + U)$ for all $x \in \mathbb{C}^n$ and all Borel subsets $U \subset X$, without loss of generality we may assume that $0 \in X$ and prove (3.6) for $B_r(0; X)$ only. Since $\mu_{e,X}(\lambda U) = \lambda^{2k} \mu_{e,X}(U)$ and $\text{deg } X = \text{deg } X$ for $\lambda > 0$, $x \in \mathbb{C}^n$, and Borel subsets $U \subset \mathbb{C}^n$, it suffices to prove that

$$
b \leq \mu_{e,X}(B_1(0; X)) \leq a
$$

where $a$ and $b$ depend on $k$, $\mu$ and $n$ only.

First we will prove the left-side inequality in (3.6). Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of affine algebraic varieties containing 0 and satisfying the hypotheses of the theorem such that

$$
\inf_X \mu_{e,X}(B_1(0; X)) = \lim_{i \to \infty} \mu_{e,X_i}(B_1(0; X_i)).
$$

Here the infimum is taken over all $X$ containing 0 and satisfying the conditions of the theorem. Consider the sequence $\{X_i\}_{i \in \mathbb{N}}$ of pure $k$-dimensional projective subvarieties of $\mathbb{C}P^n$. Since $\mathbb{C}P^n$ is a compact manifold, one can choose a subsequence of $\{X_i\}_{i \in \mathbb{N}}$ converging in the Hausdorff metric defined on compact subsets.
of $\mathbb{CP}^n$ to a compact set, say, $Y$. Without loss of generality we may assume that $\{\overline{X}_l\}_{l \in \mathbb{N}}$ itself converges to $Y$.

**Lemma 3.2.** There are a linear subspace $L \subset \mathbb{CP}^n$ of dimension $n - k - 1$ and a number $N \in \mathbb{N}$ such that $L \cap (\{\overline{X}_l\}_{l \geq N} \cup Y) = \emptyset$.

**Proof.** We prove the result by induction on $n - k$, the codimension of $\overline{X}_l$ in $\mathbb{CP}^n$.

For $n - k = 1$ every $\overline{X}_l$ being a projective hypersurface of degree $\leq \mu$ is defined as the set of zeros of a holomorphic homogeneous polynomial $p_l$ of degree $\leq \mu$:

$$\overline{X}_l := \{(z_0 : \cdots : z_n) \in \mathbb{CP}^n : p_l(z_0, \ldots, z_n) = 0\}.$$ 

Without loss of generality we may assume that $l_2$-norms of vectors of coefficients of all $p_l$ are 1. Then we can choose a subsequence $\{p_{s_i}\}_{s_i \in \mathbb{N}}$ that converges uniformly on compact subsets of $\mathbb{C}^{n+1}$ to a nontrivial (holomorphic) homogeneous polynomial $p$ of $\deg p \leq \mu$. Next, if $y \in Y$, then by the definition of the Hausdorff convergence there is a sequence of points $\{x_l\}_{l \in \mathbb{N}}$, $x_l \in X_l$, such that $\lim_{l \to \infty} x_l = y$. In particular, if $y = (y_0 : \cdots : y_n)$ and $x_l = (x_{0l} : \cdots : x_{nl})$ with $\max_{0 \leq i \leq n} |y_i| \leq 1$, $\max_{0 \leq i \leq n} |x_{il}| \leq 1$, $l \in \mathbb{N}$, then

$$p(y_0, \ldots, y_n) = \lim_{s \to \infty} p_{s_i}(x_{0l}, \ldots, x_{nl}) = 0.$$ 

Since $p \neq 0$, the latter implies that $Y$ belongs to a projective hypersurface in $\mathbb{CP}^n$. In particular, $Y$ is nowhere dense in $\mathbb{CP}^n$. Thus there is $z \in \mathbb{CP}^n \setminus Y$. And so there is a neighbourhood $U$ of $Y$ in $\mathbb{CP}^n$ which does not contain $z$. By the definition of the Hausdorff convergence this implies that there is a number $N \in \mathbb{N}$ such that $\{\overline{X}_l\}_{l \geq N} \subset U$ completing the proof of the lemma for $n - k = 1$.

Suppose now that the result is proved for $n - k > 1$ and prove it for $n - k + 1$.

Since every $\overline{X}_l$ is contained in a projective hypersurface in $\mathbb{CP}^n$ of degree $\leq \mu$ (see section 3.1.1), by the induction hypothesis there are a number $N' \in \mathbb{N}$ and a point $y \in \mathbb{CP}^n$ such that $y \notin \{\overline{X}_l\}_{l \geq N'} \cup Y$. The point $y$ determines a projection $\phi_y : \mathbb{CP}^n \setminus \{y\} \to \mathbb{CP}^{n-1}$ as described in section 3.1.1 (with $L := \{y\}$). Set $X'_l = \phi_y(\overline{X}_l)$, $l \geq N'$, and $Y' = \phi_y(Y)$. By the proper map theorem (see, e.g., [GH, Ch. 0.2]) and the Chow theorem (see, e.g., [GH, Ch. 1.3]) $X'_l$ are projective subvarieties of $\mathbb{CP}^{n-1}$. Also, by the construction, cf. section 3.1.1, $\dim_c X'_l = \dim_c X_l$ and $\deg X'_l \leq \mu$ for all $l \geq N'$. Moreover, $\{X'_l\}_{l \geq N'}$ converges in the Hausdorff metric defined on compact subsets of $\mathbb{CP}^{n-1}$ to $Y'$, because $\phi_y$ is continuous in a neighbourhood of $\{\overline{X}_l\}_{l \geq N'} \cup Y$. Since the codimension of $X'_l$ in $\mathbb{CP}^{n-1}$ is $n - k$, by the induction hypothesis there are an integer number $N \geq N'$ and a linear subspace $L' \subset \mathbb{CP}^{n-1}$ of dimension $n - k - 1$ which does not intersect $\{X'_l\}_{l \geq N} \cup Y'$. Then $L = \phi_y^{-1}(L') \cup \{y\}$ is a linear subspace of $\mathbb{CP}^n$ of dimension $n - k$ which does not intersect $\{\overline{X}_l\}_{l \geq N} \cup Y$.

This completes the proof of the lemma. 

Further, since $0 \in \{X_l\}_{l \in \mathbb{N}} \cup Y$, there is a closed Euclidean ball $\overline{B}_{r_0}(0) \subset \mathbb{C}^n$ centered at 0 of radius $0 < r_0 \leq 1$ which does not intersect the $L$ of the above lemma. Clearly,

$$\mu_{e,X_l}(B_l(0; X_l)) \geq \mu_{e,X_l}(B_{r_0}(0; X_l)), \quad l \in \mathbb{N}.$$
(As before, $B_{r_0}(0; X_i) := B_{r_0}(0) \cap X_i$.) Therefore to prove the left-side inequality in (3.8) it suffices to check that

$$\liminf_{l \to \infty} \mu_{e,X_1}(B_{r_0}(0; X_i)) > 0. \tag{3.8}$$

Recall that the Fubini-Studi metric is equivalent to the Euclidean metric on every compact subset $K \subset \mathbb{C}^n$ with the constants of equivalence depending on $K$ and $n$ only. Therefore there is a closed ball $B$ in the Fubini-Studi metric centered at 0 and of radius $s_0 > 0$ depending on $r_0$ and $n$ only such that $B \subset B_{r_0}(0)$. Since $\mu_{e,X_i}$ is equivalent to $\mu_{\mathbb{S}^k}$ on $B_{r_0}(0; X_i)$ with the constants of equivalence depending on $r_0$, $k$ and $n$ only, see section 3.1.1, inequality (3.8) follows from the inequality (3.9) follows from the inequality

$$\liminf_{l \to \infty} \mu_{\mathbb{S}^k}(B \cap X_i) > 0. \tag{3.9}$$

Let us check the last inequality. Diminishing, if necessary, $r_0$ we can find a hyperplane $L' \subset \mathbb{C}P^n$ which contains $L$ from Lemma 3.2 and does not intersect $\overline{B}_{r_0}(0)$. Let $T : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a unitary transformation which induces an isometry $\overline{T} : \mathbb{C}P^n \to \mathbb{C}P^n$ sending $L'$ to the hyperplane at $\infty$, $H$. Then $\overline{T}(B)$ is a closed ball (in the Fubini-Studi metric) in $\mathbb{C}^n = \mathbb{C}P^n \setminus H$. By the definition of $T$, $\deg T(\overline{X}_i) = \deg \overline{X}_i$ and $\mu_{\overline{X}_i}(U) = \mu_{T(\overline{X}_i)}(T(U))$ for a Borel subset $U \subset \overline{X}_i$. These facts and the above equivalence of $\mu_{e,X_i}$ and $\mu_{\overline{X}_i}$ on compact subsets of $\mathbb{C}^n$ show that in the proof of (3.8) without loss of generality we may assume that $L' = H$.

Now, consider the projection $\phi_L : \mathbb{C}^n \to \mathbb{C}^k$ determined as in section 3.1.1. Choosing suitable coordinates on $\mathbb{C}^n$ we may and will assume that $\phi_L$ coincides with the projection $\mathbb{C}^{n} \ni (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_k)$. Then $X_i := \overline{X}_i \setminus H$ are algebraic sub-varieties of algebraic varieties $X_i$ defined as sets of zeros of families of polynomials $p_l$, $1 \leq i \leq n-k$, $l \geq N$, of the form (3.1). Moreover, since $L \cap (\overline{X}_i \setminus \overline{Y}) = \emptyset$, the definition of $p_l$, see section 3.1.1, shows that for every $i$ polynomials $p_l$, $l \geq N$, are uniformly bounded on compact subsets of $\mathbb{C}^n$. Since $\deg p_l \leq \mu$, we can find a subsequence $\{l_s\}_{s \in \mathbb{N}} \subset \mathbb{N}$ such that $\{p_{l_s}\}_{s \in \mathbb{N}}$ converge uniformly on compact subsets of $\mathbb{C}^n$ to polynomials $p_i$, $\deg p_i \leq \mu$, of the form (3.1), $1 \leq i \leq n-k$, and

$$\lim_{s \to \infty} \mu_{\mathbb{S}^k}(B \cap X_{l_s}) = \liminf_{l \to \infty} \mu_{\mathbb{S}^k}(B \cap X_l).$$

This implies easily that $Y \cap \mathbb{C}^n$ with $Y$ from Lemma 3.2 is contained in the pure $k$-dimensional algebraic variety $\tilde{Y}$ defined as the set of zeros of polynomials $p_i$, $1 \leq i \leq n-k$. In what follows by $\Delta^f_i := \{(z_1, \ldots, z_l) \in \mathbb{C}^l : \max_{1 \leq i \leq l} |z_i| < r\}$ we denote the open polydisk in $\mathbb{C}^l$ centered at 0 of radius $r$.

Since, by the definition, $\tilde{Y}$ is a finite branched covering over $\mathbb{C}^k$ and $0 \in \tilde{Y}$, there is a polydisk $\Delta^f_i = \Delta^e_{n-k} \times \Delta^f_i$ such that $\Delta^f_i \cap \tilde{Y} \subset \tilde{B} \cap \tilde{Y}$ and $\phi_L : \Delta^f_i \cap \tilde{Y} \to \Delta^f_i$ is a finite branched covering over $\Delta^f_i$ (for similar arguments see, e.g., the proof of the preparatory Weierstrass theorem in [GH, Ch. 0.1]). From here using the fact that $\{p_{l_s}\}$ converges uniformly on compact subsets of $\mathbb{C}^n$ to $p_i$ for all $i$ and diminishing, if necessary, $\epsilon$ we obtain analogously that there is a number $N_0 \in \mathbb{N}$ such that $\Delta^f_i \cap \overline{X}_l \subset \tilde{B} \cap \overline{X}_l$ and $\phi_L : \Delta^f_i \cap \overline{X}_l \to \Delta^f_i$ are finite branched coverings over $\Delta^f_i$ for all $s \geq N_0$. But $\Delta^f_i \cap \overline{X}_l$ is a (closed) complex subvariety of $\Delta^f_i \cap \overline{X}_l$, and $\phi_L(\Delta^f_i \cap \overline{X}_l)$ is an open subset of $\Delta^f_i$ (because $0 \in X_l$ and $\phi_L : X_l \to \mathbb{C}^k$ is a finite branched covering). Thus by the proper map theorem, $\phi_L(\Delta^f_i \cap \overline{X}_l) = \Delta^f_i$. (Here we used the fact that the map $\phi_L : \Delta^f_i \cap \overline{X}_l \to \Delta^f_i$ is proper, because $\phi_L : \Delta^f_i \cap \overline{X}_l \to \Delta^f_i$ is proper and $X_l \cap \Delta^f_i$ is a complex subvariety of $\overline{X}_l \cap \Delta^f_i$.)
Let $L_{sk}$ be the Lebesgue measure on $C^k$. Then by the definition of $\mu_{\xi_{ts}}$ there is a constant $c > 0$ depending on $\mu$, $k$ and $n$ only such that

$$\mu_{\xi_{ts}}(B \cap X_{l_s}) \geq cL_{sk}(\phi_L(B \cap X_{l_s})).$$

But for $s \geq N_0$ we have

$$L_{sk}(\phi_L(B \cap X_{l_s})) \geq L_{sk}(\Delta^k) = \pi^k e^{2k} > 0.$$

The combination of the last two inequalities completes the proof of (3.9) and thus the proof of the left-side inequality in (3.6).

The right-side inequality in (3.6) is obtained as follows, see (3.3),

$$\mu_{e,X}(B_1(0; X)) \leq c(n,k)\mu_{\xi}(B_1(0; X)) \leq c(n,k)\mu_{\xi}(X) = c(n,k)\deg X \leq c(n,k)\mu.$$

The proof of Theorem 3.1 is complete. \qed

### 3.2. Strong Remez Type Inequalities.

#### 3.2.1. Covering Lemmas. Our proof is based on a deep generalization of the classical Cartan Lemma [C] discovered by Gorin [GK]. We present a more general version of this result.

Let $X$ be a pseudometric space with pseudometric $d$. By $F := \{ \overline{B}_r(x) \subset X : d(x,y) \leq r, x, y \in X, r \geq 0 \}$ we denote the set of closed balls in $X$. Let $\xi : F \to R_+$ be a function satisfying the following two properties:

1. \( \xi(\overline{B}_{r'}(x)) \leq \xi(\overline{B}_{r''}(x)) \) for all \( x \in X, r' \leq r'' \).

2. There is a numerical constant $A$ such that for any collection of mutually disjoint balls $\{B_i\} \subset F$, \( \sum_{i \geq 1} \xi(B_i) \leq A \)

Consider a continuous strictly increasing nonnegative function $\phi$ on $[0, \infty)$, $\phi(0) = 0$, $\lim_{t \to \infty} \phi(t) > A$ which will be called a majorant.

For each point $x \in X$ we set $\tau(x) = \sup\{ t : \xi(\overline{B}_{\tau(t)}(x)) \geq \phi(t) \}$. It is easy to see that $\xi(\overline{B}_{\tau(x)}(x)) = \phi(\tau(x))$ and $\sup_x \tau(x) \leq \phi^{-1}(A) < \infty$.

A point $x \in X$ is said to be regular (with respect to $\xi$ and $\phi$) if $\tau(x) = 0$, i.e., $\xi(\overline{B}_{\tau(t)}(x)) < \phi(t)$ for all $t > 0$. The next result shows that the set of regular points is sufficiently large for an arbitrary majorant $\phi$.

**Lemma 3.3.** Fix $\gamma \in (0, 1/2)$. There is a sequence of balls $B_k = \overline{B}_{t_k}(x_k)$, $k = 1, 2, \ldots$, which collectively cover all irregular points such that $\sum_{k \geq 1} \phi(\gamma t_k) < A$ (i.e., $t_k \to 0$).

The proof of this lemma for $\xi$ being a finite Borel measure on a metric space $X$ is given by Gorin [GK]. His argument works also in the general case.

**Proof.** Let $0 < \alpha < 1, \beta > 2$ be such that $\gamma < \alpha/\beta$. We set $B_0 = \emptyset$ and assume that the balls $B_0, \ldots, B_{k-1}$ have been constructed. If $\tau_k = \sup\{ \tau(x) : x \notin B_0 \cup \cdots \cup B_{k-1} \}$, then there exists a point $x_k \notin B_0 \cup \cdots \cup B_{k-1}$ such that $\tau(x_k) > \alpha \tau_k$.

We set $t_k = \beta \tau_k$ and $B_k = \overline{B}_{t_k}(x_k)$. Clearly, the sequence $\tau_k$ (and thus also $t_k$) does not increase. The balls $\overline{B}_{\tau_k}(x_k)$ are pairwise disjoint. Indeed, if $l > k$ then $x_l \notin B_k$. 

i.e., the pseudodistance between $x_l$ and $x_k$ is greater than $\beta \tau_k > 2 \tau_k \geq \tau_k + \tau_l$. Thus $\overline{B}_{\tau_k}(x_k) \cap \overline{B}_{\tau_l}(x_l) = \emptyset$ by the triangle inequality for $d$. Now,

\[ \sum_{k \geq 1} \phi(\gamma \tau_k) < \sum_{k \geq 1} \phi(\alpha \tau_k) \leq \sum_{k \geq 1} \phi(\tau(x_k)) = \sum_{k \geq 1} \xi(\overline{B}_{\tau_k}(x_k)) \leq A; \]

consequently, $\tau_k \to 0$, i.e., for each point $x$, not belonging to the union of the balls $B_k$, $\tau(x) = 0$, i.e., $x$ is a regular point. In addition, $t_k = \beta \tau_k \to 0$. \hfill $\square$

**Remark 3.4.** (1) According to the Caratheodory construction, see, e.g., [F, Ch. 2.10], there is a finite Borel measure on $X$ whose restriction to $F$ is $\xi$.

(2) Assume that $\xi$ is the restriction to $F$ of a Borel measure $\mu$ on $X$ with support $\{x_1, \ldots, x_n\}$. Then, as it follows from the proof, the number of the balls $B_k$ in this case is $\leq n$ and the balls $\overline{B}_{\tau_k}(x_k), k \geq 1$, cover the support of $\mu$. For otherwise, there is $\overline{B}_{\tau_k}(x_k)$ which does not meet $\{x_1, \ldots, x_n\}$. Then $\overline{B}_{\tau(x_k)}(x_k)$ does not meet $\{x_1, \ldots, x_n\}$, as well. Consequently, $\mu(\overline{B}_{\tau(x_k)}(x_k)) = 0$, a contradiction with the choice of $x_k$.

Let $X$ be a pseudometric space with pseudometric $d$. For every $x \in X$ we set $S_x := \{y \in X : d(x, y) = 0\}$. Let $\mu$ be a Borel measure on $X$ with $\mu(X) = k < \infty$ such that

\[ \int_X \ln^+ d(x, \xi) \, d\mu(\xi) < \infty \quad \text{for all} \quad x \in X, \]

where $\ln^+ t := \max(0, \ln t)$. Then we define

\[ u(x) = \begin{cases} \int_X \ln d(x, \xi) \, d\mu(\xi), & \text{if} \quad \mu(S_x) = 0 \\ -\infty, & \text{if} \quad \mu(S_x) > 0. \end{cases} \]

By definition, every Lebesgue integral $\int_X \ln d(x, \xi) \, d\mu(\xi)$ exists but may be equal to $-\infty$. In this case we define $u(x) = -\infty$.

**Corollary 3.5.** Fix $\gamma \in (0, 1/2)$. Given $H > 0, s > 0$ there is a family of closed balls $B_j$ with radii $r_j$ satisfying

\[ \sum r_j^s < \frac{(H/\gamma)^s}{s} \]

such that

\[ u(x) \geq k \ln \left( \frac{H}{e} \right) \quad \text{for all} \quad x \in X \setminus \bigcup_j B_j. \]

**Proof.** Let $\phi(t) = (pt)^s$ be a majorant with $p = \frac{(ka)^{1/s}}{H}$. We cover all irregular points of $X$ by closed balls according to Lemma 3.3 (with $\xi = \mu$) and prove that the required inequality is valid for any regular point $x$. This will complete the proof. First, observe that $u(x)$ is finite for every regular point $x$ by the definition of the Lebesgue integral and the regularity condition for the $\phi$. Let $n(t; x) = \mu(\overline{B}_t(x))$ for such $x$. Then, for any $N \geq \max(1, H)$ we have

\[ u(x) \geq \int_{\overline{B}(x, \xi)} \ln d(x, \xi) \, d\mu(\xi) = \int_0^N \ln t \, dn(t; x) = n(t; x) \ln t^H - \int_0^N \frac{n(t; x)}{t} \, dt. \]
Since \( n(t; x) < (pt)^s \), we obtain

\[
u(x) \geq n(N; x) \ln N - \int_0^N \frac{n(t; x)}{t} \, dt.
\]

In addition, \( n(t; x) \leq n(N; x) \) for \( t \leq N \). Therefore,

\[
u(x) \geq n(N; x) \ln N - \int_0^H \frac{(pt)^s}{t} \, dt - \int_H^N \frac{n(N; x)}{t} \, dt =
\]

\[n(N; x) \ln N - \frac{(pH)^s}{s} - n(N; x) \ln N + n(N; x) \ln H = -k + n(N; x) \ln H.
\]

Letting here \( N \to \infty \) and taking into account that \( \lim_{N \to \infty} n(N; x) = k \) we obtain the required result.

We use also the following result proved in [L].

**Corollary 3.6.** Let \( f \) be a holomorphic function in the disk \( |z| \leq 2e^R \) \((R > 0)\) in \( \mathbb{C} \), \( f(0) = 1 \) and \( \eta \) is an arbitrary positive number \( \leq \frac{3}{2}e \). Then inside the disk \( |z| \leq R \) but outside a family of closed disks \( D_{r_i}(z_i) \) centered at \( z_i \) of radii \( r_i \) such that \( \sum r_i \leq 4\eta R \),

\[\ln |f(z)| \geq -H(\eta) \ln M(2eR)\]

where

\[H(\eta) = 2 + \ln \left( \frac{3e}{2\eta} \right)\]

and

\[M(2eR) := \sup_{|z| \leq 2eR} |f|\].

**Remark 3.7.** The proof is based on a particular case of Corollary 3.5 for \( \mu \) a sum of delta-measures, and the Harnack inequality for positive harmonic functions. According to Remark 3.4 (2), from the proof presented in [L] it follows that the number of disks \( D_{r_i}(z_i) \) does not exceed the number of zeros of \( f \) in the disk \( |z| < 2R \) (which, by the Jensen inequality, is bounded from above by \( \ln M_f(2eR) \)) and, moreover, the disks \( D_{r_i/2}(z_i) \) cover the set of these zeros.

**3.2.2.** In this part we will prove Theorem 2.1

**Proof.** Let \( X \subset \mathbb{C}^n \) be a closed subset of the class \( A_{2n}(s, a) \) where \( s = 2n - 2 + \alpha \), \( \alpha > 0 \). Let \( p \) be a holomorphic polynomial on \( \mathbb{C}^n \) of degree \( k \). Let \( B \subset \mathbb{C}^n \) be a closed Euclidean ball and \( \omega \subset X \cap B \) be an \( H^s \)-measurable subset. We must prove the inequality

\[
\sup_{B} |p| \leq \left( \frac{c_1 H_{2n}(B)}{\{H^s(\omega)\}^{2n/s}} \right)^{c_2 k} \sup_{\omega} |p|,
\]

where \( c_1 \) depends on \( a, n, k, \alpha \) and \( c_2 > 0 \) depends on \( \alpha \).

Since the ratio on the right-hand side of (3.10) is invariant with respect to dilations and translations of \( \mathbb{C}^n \) and the class \( A_{2n}(s, a) \) is also invariant with respect to these transformations, without loss of generality we may assume that \( B \) is the closed unit ball centered at \( 0 \in \mathbb{C}^n \). Then we must prove that

\[
\sup_{B} |p| \leq \left( \frac{\pi}{\lambda^{2n/s}} \right)^{c_2 k} \sup_{\omega} |p|,
\]

where \( \lambda \) is the volume of the unit ball centered at \( 0 \in \mathbb{C}^n \).
where \( \lambda := \mathcal{H}_s(\omega), \tau_1 \) depends on \( a, n, k, \alpha \) and \( c_2 > 0 \) depends on \( \alpha \).

By \( Z_p \subset \mathbb{C}^n \) we denote the set of zeros of \( p \). According to Theorem 3.1, \( Z_p \in \mathcal{A}_{2n}(2n - 2, a, b) \) for some \( a \) and \( b \) depending on \( n \) and \( k \) only. By \( \mathcal{H}_{2n-2, p} \) we denote the Hausdorff \((2n - 2)\)-measure supported on \( Z_p \). Let \( B_1 \subset B_2 \) be closed Euclidean balls centered at \( 0 \in \mathbb{C}^n \) of radii 2 and 10, respectively. Set

\[
\mu := \mathcal{H}_{2n-2, p}|_{B_2}.
\]

Since \( Z_p \in \mathcal{A}_{2n}(2n - 2, a, b) \), we have

\[
(3.12) \quad \mu(B_r(x)) \geq br^{2n-2} \quad \text{for all} \quad x \in B_1, \quad 0 \leq r \leq 5.
\]

Let \( H > 0 \). Consider \( \phi(t) := t^r \) as the majorant in Lemma 3.3. Then a point \( x \in \mathbb{C}^n \) is regular with respect to \( \phi \) and \( \mu \) if \( \mu(B_r(x)) < \frac{r^r}{H} \) for all \( r > 0 \). (Here we consider \( \mathbb{C}^n \) with the Euclidean norm \( | \cdot | \).)

**Lemma 3.8.** There is a sequence of open Euclidean balls \( B_{r_k}(x_k), k = 1, 2, \ldots \), which collectively cover all the irregular points such that

\[
\sum_{k \geq 1} r_k^s \leq 3H \mu(B_2).
\]

Moreover, the distance \( d(x) \) from a regular point \( x \) to the compact set \( K := B_1 \cap Z_p \) is

\[
\geq \min \left\{ 5, \left( \frac{bH}{r} \right)^{1/\alpha} \right\}.
\]

**Proof.** The first statement follows directly from Lemma 3.3. Let \( y \in K \) be such that \( |x - y| = d(x) \). Observe that condition (3.12) implies that \( x \notin K \). For otherwise, we must have

\[
br^{2n-2} < \frac{r^s}{H} \quad \text{for all} \quad 0 < r < 5
\]

which is impossible. Thus \( d(x) > 0 \). Next, the ball centered at \( x \) of radius \( 2d(x) \) contains the ball centered at \( y \) of radius \( d(x) \). Now from the regularity condition for \( x \) by (3.12) we get

\[
b \min \{5, d(x)\}^{2n-2} \leq \mu(B_{2d(x)}(x)) \leq \left( \frac{2d(x)}{H} \right)^{2n-2 + \alpha}.
\]

This implies that

\[
d(x) \geq \min \left\{ 5, \left( \frac{bH}{2^s} \right)^{1/\alpha} \right\}.
\]

Continuing the proof of the theorem observe that by the definition of \( X \),

\[
(3.13) \quad \lambda := \mathcal{H}_s(\omega) \leq a 2^s
\]

(because if \( \omega \subset X \cap B \neq \emptyset \), then \( \omega \) is contained in a closed Euclidean ball of radius 2 centered at a point of \( X \)). Without loss of generality we may assume that \( \lambda > 0 \).

**Lemma 3.9.** The set \( \omega \) cannot be covered by a family \( \{B_j\} \) of open Euclidean balls whose radii \( r_j \) satisfy

\[
\sum r_j^s < \frac{\lambda}{2^s a}.
\]
Proof. Assume to the contrary that there is a family of balls $B_j := B_{r_j}(x_j)$, $j = 1, 2, \ldots$, whose radii satisfy the inequality of the lemma which covers $\omega$. Without loss of generality we may assume that each $B_j$ meets $\omega$. Then for every $x_j$ choose $y_j \in \omega$ so that $|x_j - y_j| \leq r_j$. Clearly, the family of balls $\{B_{2r_j}(y_j)\}$ also covers $\omega$. From here, since $\omega \subset X \in A_{2n}(s, a)$, we obtain

$$\lambda := \mathcal{H}_s(\omega) \leq \sum \mathcal{H}_s(X \cap B_{2r_j}(y_j)) \leq 2^n a \sum r_j^s < \lambda,$$

a contradiction. \hfill \Box

Further, note that $\mu(B_2)$ in Lemma 3.8 is bounded from above by a constant $c$ depending on $n$ and $k$ only (because $Z_p \in A_{2n}(2n-2, a, b)$ with $a, b$ depending on $n, k$ only). Thus choosing in this lemma $H := \frac{\omega}{\sqrt{s}}$ we obtain from Lemma 3.9 for some $\tau$ depending on $n, k$:

Corollary 3.10. There is a point $x \in \omega$ such that

$$\text{dist}(x, Z_p) \geq \min \left\{ 1, (\tau \lambda)_{1/\alpha} \right\}.$$

Proof. From the above lemmas it follows that there is $x \in \omega$ such that

$$\text{dist}(x, Z_p \cap B_1) \geq \min \left\{ 5, (\tau \lambda)_{1/\alpha} \right\}.$$

Moreover, $x \in B$ and so $\text{dist}(x, Z_p \setminus B_1) \geq 1$; this implies the required. \hfill \Box

Let $z \in B$ be a point such that

$$M := \max_B |p| = |p(z)|.$$

Let $l$ be the complex line passing through $z$ and the point $x$ from Corollary 3.11. Without loss of generality we may identify $l$ with $\mathbb{C}$ so that $z$ coincides with $0 \in \mathbb{C}$. Then, in this identification, the point $x$ belongs to $\overline{D}_{2}(0)$, the closed disk of radius 2 centered at 0. Observe also that (under the identification) the set $B_1 \cap l$ contains $\overline{D}_{1}(0)$. Thus, by the classical Bernstein inequality for holomorphic polynomials

$$\max_{|z| \leq 4e} |p| \leq (4e)^k \max_{|z| \leq 1} |p| \leq 4e \max_{B_1} |p| \leq (12e)^k \max_{B} |p| := (8e)^k M.$$

Set $f = p/M$ and apply Corollary 3.4 with $R = 2$. According to this corollary for every $\eta \leq 3e/2$ there is a family of closed disks $D_{r_i}(z_i)$ such that $\sum r_i \leq 8 \eta$ and $\ln |f(z)| > -H(\eta) k \ln(8e)$ for any $|z| \leq 2$ outside the above disks where $H(\eta) = 2 + \ln(3e/2\eta)$. Recall also that the number of these disks is $\leq$ the number of zeros of $f$ in $|z| \leq 4$ and the disks $D_{r_i/2}(z_i)$ cover the set of zeros of $f$ there. In particular, if a point $z \in \overline{D}_{1}(0)$ satisfies $\text{dist}(z, Z_f) \geq 14 \eta$ where $Z_f$ is the set of zeros of $f$ in $\mathbb{C}$, then it cannot belong to the union of disks $D_{r_i}(z_i)$, and therefore $|f(z)|$ satisfies the above inequality. Choose $\eta := \min(1, (\tau \lambda)_{1/\alpha})/14$. Then by Corollary 3.10

$$\text{dist}(x, Z_f) \geq 14 \eta.$$

Thus we have

$$\sup_{z \in \omega} \ln |f| \geq \ln |f(x)| \geq -H(\eta) k \ln(8e).$$

We will consider two cases:

(1) $$(\tau \lambda)_{1/\alpha} \geq 1.$$

Then $\eta = \frac{1}{14}$ and

$$\sup_{\omega} |f| \geq -(3 + \ln 21) k \ln(8e) > -20d.$$
This and (3.13) imply that
\[ \sup_{B} |p| \leq e^{20k} \sup_{\omega} |p| = \left( \frac{e^{20}}{\lambda^{2n/s}} \lambda^{2n/s} \right)^{k} \sup_{\omega} |p| \leq \left( \frac{2^{2n+2n/s} e^{20}}{\lambda^{2n/s}} \right)^{k} \sup_{\omega} |p|. \]
Thus, inequality (3.11) is proved in this case.

(2)
\[ (\overline{c}\lambda)^{1/\alpha} < 1. \]
Then
\[ \sup_{\omega} \ln |f| \geq - (c' - \ln \lambda^{1/\alpha})k \ln(8e) \]
where \( c' \) depends on \( n \) and \( k \) only. This yields
\[ \sup_{B} |p| \leq \left( \frac{\overline{c}_{1}}{\lambda^{2n/s}} \right)^{c_{2}k} \sup_{\omega} |p| \]
where \( \overline{c}_{1} > 0 \) depends on \( k, n \) and \( \alpha \) and \( c_{2} > 0 \) depends on \( \alpha \) only.

The proof of Theorem 2.1 is complete. \( \square \)

3.2.3. Proof of Corollary 2.2. The proof follows directly from the estimates obtained in cases (1) and (2) above and from the fact that \( X \in A_{2n}(s, a, b) \). \( \square \)

3.2.4. Proofs of Corollaries 2.3 and 2.4. The proofs repeat word-for-word proofs of similar statements of Theorems 1 and 3 of [ABr] and are based on the inequality of Corollary 2.2. We leave the details to the readers. \( \square \)

4. Real Polynomials

4.1. Weak Remez Type Inequalities. In this section we will prove Theorem 2.6.

**Proof.** We set for brevity
\[ ||p; \omega||_{q} = \left( \frac{1}{\mathcal{H}_{s}(\omega)} \int_{\omega} |p|^{q} d\mathcal{H}_{s} \right)^{1/q} \]
and
\[ ||p; U||_{r} = \left( \frac{1}{\mathcal{H}_{n}(U)} \int_{U} |p|^{r} d\mathcal{H}_{n} \right)^{1/r}. \]

Since the above functions are invariant with respect to dilations of \( \mathbb{R}^{n} \), without loss of generality we may and will assume that \( \mathcal{H}_{n}(U) = 1 \).

Let \( \Sigma(a, \lambda), a, \lambda > 0, \) be the class of subsets \( \omega \in \mathcal{A}_{n}(s, a) \) of \( U \) satisfying
\[ \{ \mathcal{H}_{s}(\omega) \}^{n/s} \geq \lambda. \]
We must show that there is a positive constant \( C = C(U, n, q, r, s, k, a, \lambda) \) such that for every real polynomial \( p \) of degree \( k \) on \( \mathbb{R}^{n} \)
\[ ||p; U||_{r} \leq C ||p; \omega||_{q}. \]

**Remark 4.1.** Let \( C_{0} \) be the optimal constant in (4.2). Since the class \( \Sigma(a, \lambda) \) increases as \( \lambda \) decreases, \( C_{0} \) increases in \( 1/\lambda \), as is required in the theorem.
If, on the contrary, the constant in (4.2) does not exist, one can find sequences of real polynomials \( \{ p_j \} \) of degrees \( k \) and sets \( \{ \omega_j \} \subset \Sigma(a, \lambda) \) so that

\[
\| p_j; U \|_r = 1 \quad \text{for all} \quad j \in \mathbb{N},
\]

(4.3)

\[
\lim_{j \to \infty} \| p_j; \omega_j \|_q = 0.
\]

(4.4)

Since all (quasi-) norms on the space of real polynomials of degree \( k \) on \( \mathbb{R}^n \) are equivalent, (4.3) implies the existence of a subsequence of \( \{ p_j \} \) that converges uniformly on \( U \) to a polynomial \( p \in \mathbb{R}[x_1, \ldots, x_n] \) with \( \deg p \leq k \). Assume without loss of generality that \( \{ p_j \} \) itself converges uniformly to \( p \). Then (4.3), (4.4) imply for this \( p \) that

\[
\| p; U \|_r = 1,
\]

(4.5)

\[
\lim_{j \to \infty} \| p; \omega_j \|_q = 0.
\]

(4.6)

From this we derive the next result.

**Lemma 4.2.** There is a sequence of closed subsets \( \{ \sigma_j \} \subset \overline{U} \) such that for every \( j \) larger than a fixed \( j_0 \) the following is true

\[
\{ \mathcal{H}_s(\sigma_j) \}^{n/s} \geq \frac{\lambda}{2^{n/s}}.
\]

Moreover,

\[
\max_{\sigma_j} |p| \to 0 \quad \text{as} \quad j \to \infty.
\]

(4.8)

**Proof.** Let first \( q < \infty \). By the (probabilistic) Chebyshev inequality

\[
\mathcal{H}_s(\{ x \in \omega_j : |p(x)| \leq t \}) \geq \mathcal{H}_s(\omega_j) - \frac{\mathcal{H}_s(\omega_j)}{t_q} \| p; \omega_j \|_q^{q}.
\]

Pick here \( t = t_j := \| p; \omega_j \|_q^{1/2} \). Then by (4.6) the left-hand side is at least \( \frac{1}{2} \mathcal{H}_s(\omega_j) \), for \( j \) sufficiently large. Denoting the closure of the set in the braces by \( \sigma_j \) we also have

\[
\max_{\sigma_j} |p| = t_j \to 0 \quad \text{as} \quad j \to \infty.
\]

If \( q = \infty \), simply set \( \sigma_j := \omega_j \) to produce (4.8). \( \square \)

Apply now the Hausdorff compactness theorem to find a subsequence of \( \{ \sigma_j \} \) converging to a closed subset \( \sigma \subset \overline{U} \) in the Hausdorff metric. We assume without loss of generality that \( \{ \sigma_j \} \to \sigma \). By (4.8) this limit set is a subset of the zero set of \( p \). Since \( p \) is nontrivial by (4.5), the dimension of its zero set is at most \( n - 1 \); hence \( \mathcal{H}_s(\sigma) = 0 \) because \( s > n - 1 \). Then for every \( \epsilon > 0 \) one can find a finite open cover of \( \sigma \) by open Euclidean balls \( B_i \) of radii \( r_i \) at most \( r(\epsilon) \) so that

\[
\sum_i r_i^s < \epsilon.
\]

(4.9)

Let \( \sigma_\delta \) be a \( \delta \)-neighbourhood of \( \sigma \) such that

\[
\sigma_\delta \subset \bigcup_i B_i \quad \text{and} \quad \delta < r(\epsilon).
\]
Pick \( j \) so large that \( \sigma_j \subset \sigma_d \). For every \( B_i \) intersecting \( \sigma_j \) choose a point \( x_i \in B_i \cap \sigma_j \). Consider an open Euclidean ball \( \tilde{B}_i \) centered at \( x_i \) of radius twice that of \( B_i \). Then \( B_i \subset \tilde{B}_i \) and \( \{ \tilde{B}_i \} \) is an open cover of \( \sigma_j \). Hence

\[
\mathcal{H}_s(\sigma_j) \leq \sum_i \mathcal{H}_s(\sigma_j \cap \tilde{B}_i) \leq a 2^s \sum_i r_i^s
\]

because \( \omega_j \in A_w(s, a) \). Together with (4.7) and (4.9) this implies that

\[
\frac{1}{2} \lambda^{s/n} \leq \mathcal{H}_s(\sigma_j) \leq a 2^s \sum_i r_i^s \leq 2^s a \epsilon.
\]

Letting \( \epsilon \to \infty \) one gets a contradiction. \( \square \)

### 4.2. Strong Remez Type Inequalities

Strong Remez type inequalities for real polynomials from \( \mathbb{R}[x] \) and Ahlfors regular subsets of \( \mathbb{R} \) are proved in [ABr]. Inequalities of the form described in Theorem 2.1 are also valid for real polynomials on \( \mathbb{R}^2 \). The method of the proof of such inequalities is very similar to that of Theorem 2.1 and is based on the fact that an analytic compact connected curve in \( \mathbb{R}^n \) is a 1-set. (The detailed proof will be presented elsewhere.) It is still an open question whether similar strong Remez type inequalities are valid for real polynomials on \( \mathbb{R}^n \) for \( n > 2 \).

### 5. Proof of Theorem 2.11

**Proof.** It is well known, see, e.g., [JW, Prop. VIII.1], that the closure \( \overline{X} \) of an \( s \)-set \( X \) is also an \( s \)-set and \( \mathcal{H}_s(\overline{X} \setminus X) = 0 \). Moreover, the spaces \( \dot{C}^{k,\omega}_q(X) \) and \( \dot{C}^{k,\omega}_q(\overline{X}) \) are isometric. Thus without loss of generality we may and will assume in the proof that \( X \) is closed.

Given \( f \in \dot{C}^{k,\omega}_q(X) \) we should find a function \( \overline{f} : X \to \mathbb{R} \) which equals \( f \) modulo zero \( \mathcal{H}_s \)-measure and admits an extension to a function from \( \dot{\Lambda}^{k,\omega}(\mathbb{R}^n) \).

We begin with

**Lemma 5.1.** Let \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) be a quasipower \( k \)-majorant, see Definition 2.9. Let \( t_j := 2^j \), \( j \in \mathbb{Z}_+ \). Then for every pair of integers \( -\infty < i < i' < \infty \) we have

\[
\sum_{j=i}^{i'} \omega(t_j) \leq c(k, \omega) \omega(t_{i'}).
\]

**Proof.** By the monotonicity of \( \omega \)

\[
\omega(t_j) \leq \frac{1}{\ln 2} \int_{t_j}^{t_{j+1}} \frac{\omega(u)}{u} \, du
\]

and therefore the sum in (5.1) is at most

\[
\frac{1}{\ln 2} \int_{t_{i'}}^{t_{i'+1}} \frac{\omega(u)}{u} \, du \leq \frac{1}{\ln 2} C_\omega \omega(t_{i'+1}) \leq \frac{C_\omega}{\ln 2} 2^k \omega(t_{i'}). \]

\( \square \)

Our next result reformulates a theorem of the paper [BSh1], see also [BSh2, Th. 3.5] concerning the trace of the space \( \dot{\Lambda}^{k,\omega}(\mathbb{R}^n) \) to an arbitrary closed subset
$X \subset \mathbb{R}^n$, to adopt it to our situation. The trace space denoted by $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X$ consists of locally bounded functions $f : X \to \mathbb{R}$ and is equipped with seminorm

$$|f|_{\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X} := \inf \{ |g|_{\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)} : f = g|_X \}.$$  

To formulate the result we need

**Definition 5.2.** Let $X \subset \mathbb{R}^n$ and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be as above, and $T_\omega := \{ t_i \}_{i \in \mathbb{Z}^+}$ be the sequence of Lemma 5.1.

A family $\Pi := \{ P_Q \}_{Q \in K_X}$ of polynomials of degree $k-1$ is said to be a $(k, \omega, X)$-chain if for every pair of cubes $Q \subset Q'$ from $K_X$ which satisfy for some $i \in \mathbb{Z}$ the condition

$$t_i \leq r_Q < r_{Q'} \leq t_{i+2}$$

the inequality

$$\max_{x \in Q} |P_Q(x) - P_{Q'}(x)| \leq C\omega(r_{Q'})$$

holds with a constant $C$ independent of $Q, Q'$ and $i$.

The linear space of such chains is denoted by $Ch(k, \omega, X)$. It is equipped with seminorm

$$||\Pi||_{Ch} := \inf C$$

where the infimum is taken over all constants $C$ in $\mathbb{R}^+$. Recall that $K_X$ is the family of closed cubes centered at $X$ and of radii at most $4\text{diam } X$. In the sequel $c_Q$ and $r_Q$ stand for the center and the radius of the cube $Q$.

Using the concept introduced and the related notations we now formulate the desired result.

**Proposition 5.3.** (a) A locally bounded function $f : X \to \mathbb{R}$ belongs to $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X$ if and only if there is a $(k, \omega, X)$-chain $\Pi := \{ P_Q \}_{Q \in K_X}$ such that for every $Q \in K_X$

$$f(c_Q) = P_Q(c_Q).$$

Moreover, the following two-sided inequality

$$||\Pi||_{Ch} \approx |f|_{\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X}$$

holds with constants independent of $f$.

(b) If, in addition, this chain depends on $f$ linearly, then there is a linear extension operator $T_k : \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X \to \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ such that

$$||T_k|| \leq O(1)||\Pi||_{Ch}.$$  

Hereafter $O(1)$ denotes a constant depending only on inessential parameters. It may change from line to line and even in a single line.

**Proof.** In the above cited papers this result is proved under the assumption that inequality (5.4) holds for any pair of cubes $Q \subset Q'$ centered at $X$. The restrictions (5.3) and $r_Q, r_{Q'} \leq 4\text{diam } X$ may be not satisfied for this pair. In the forthcoming derivation we explain how these restrictions can be disregarded to apply the aforementioned Theorem 3.5 of [BSH2] and in this way to complete the proof of the proposition.
Consider first the case of an unbounded $X$. Hence, the only restriction is now inequality (5.5) and we should show that if a $(k, \omega, X)$-chain satisfies condition (5.4), then (5.6) holds for any pair $Q \subset Q'$ from $K_X$. Note that the necessity of conditions (5.4) and (5.5) trivially follows from that in the aforementioned Theorem 3.5 from [BSh2]. So we should only prove their sufficiency.

Assume that $f \in l_{loc}^{1,\infty}(X)$ and conditions (5.4), (5.5) hold. Let $Q \subset Q'$ be a pair of cubes from $K_X$ of radii $r$ and $r'$, respectively. Then for some indices $i \leq i'$

$$t_i \leq r \leq t_{i+1} \quad \text{and} \quad t'_i \leq r' \leq t'_{i+1}.$$ 

If $i = i'$, then by (5.3)

$$\max_Q |P_Q - P_{Q'}| \leq 2|\Pi| \omega(t_{i+1}) \leq 2 \left( \frac{t_{i+1}}{t_i} \right)^k \omega(r')|\Pi| \omega(2^{-k+1}\omega(r'))|\Pi|$$

as is required.

Let now $i < i'$ and $r_j$ with $i \leq j \leq i' + 1$ are given by

$$r_i := r, \quad r_{i' + 1} = 2r' \quad \text{and} \quad r_j := t_j \quad \text{for} \quad i < j < i' + 1.$$ 

Let $Q_j$ be the cubes centered at $c_{Q_j}$ of radii $r_j$, $i \leq j < i' + 1$, and $Q_{i' + 1}$ be the cube centered at $c_{Q_{i' + 1}}$ of radius $r_{i' + 1}$. (In particular, $\{Q_j\}_{i \leq j \leq i' + 1} \subset K_X$ is an increasing sequence of cubes with $Q_i := Q_j$.) Then

$$(5.6) \quad \max_Q |P_Q - P_{Q'}| \leq \sum_{j=1}^{i'} \max_{Q_{j+1}} |P_{Q_j} - P_{Q_{j+1}}|.$$ 

It is easily seen that (5.3) holds for every pair $Q_j \subset Q_{j+1}$, $i \leq j \leq i'$. Applying (5.4) to each of these pairs and then (5.1) and the definition of $\omega$ we estimate the right-hand side of (5.6) by

$$2|\Pi| \left\| \sum_{j=i}^{i'} \omega(r_{j+1}) \right\| O(1) \left\| \Pi \right\| \omega(t_{i' + 2}) \leq O(1) \left\| \Pi \right\| \omega(r').$$

Thus we conclude that inequality (5.4) holds for every pair $Q \subset Q'$ of cubes centered at $X$.

Let now $\text{diam } X < \infty$. The previous argument proves the required inequality

$$|P_Q - P_{Q'}| \leq C \omega(r_Q')$$

for every pair $Q \subset Q'$ from $K_X$ under the restriction $r_Q' \leq 2\text{diam } X$. Fix a cube $\bar{Q} \subset K_X$ with $r_{\bar{Q}} = 2\text{diam } X$ and introduce a new family of polynomials $\{\overline{P}_Q\}$, where $Q$ runs over the set of all cubes centered at $X$, by setting

$$(5.8) \quad \overline{P}_Q := \begin{cases} 
P_Q, & \text{if} \quad r_Q \leq \text{diam } X \\
\bar{Q} - P_{\bar{Q}}(c_{\bar{Q}}) + f(c_{\bar{Q}}), & \text{if} \quad r_Q > \text{diam } X.
\end{cases}$$ 

We will prove that the new family satisfies the hypotheses of Theorem 3.5 from [BSh2]. This will complete the proof of the proposition in this case.

Clearly, $\{\overline{P}_Q\}$ satisfies condition (5.5), and if the chain $\Pi$ depends linearly on $f$, then $\{\overline{P}_Q\}_Q$ depends linearly on $f$, as well. So we must check only that (5.4) holds for $\{\overline{P}_Q\}$ for every pair $Q \subset Q'$ of cubes centered at $X$. According to (5.4)
and (5.8) inequality (5.4) holds for this family for every pair of cubes $Q \subset Q'$ with $r_{Q'} \leq \text{diam } X$. Assume now that $r_{Q'} \geq r_Q > \text{diam } X$. Then by (5.8) we have

$$\max_Q |P_Q - P_{Q_1}| + \max_{Q_2} |P_{Q_2} - P_{Q_1}| \leq 2C\omega(r_Q) \leq O(1)\omega(r_{Q'}) .$$

Here $Q_1$ and $Q_2$ are some cubes from $\mathcal{K}_X$ centered at $c_Q$ and $c_{Q'}$, respectively, and contained in $Q$. The last two inequalities follow from (5.7) and the definition of $\omega$.

Finally, if $r_Q \leq \text{diam } X < r_{Q'}$, then $Q \subset \tilde{Q}$ and so we have by (5.7) and by the definition of $\omega$

$$\max_Q |P_{Q'} - P_Q| \leq \max_Q |P_{Q'} - P_{Q_1}| + 2C\omega(r_Q) \leq O(1)\omega(r_{Q'})$$

as is required.

Hence, in both of these cases the assumptions of Theorem 3.5 from [BSh2] hold. This completes the proof of the proposition.

Now we outline the proof of Theorem 2.11. Given $f \in \dot{C}^{k,\omega}(X)$ where $X \subset \mathbb{R}^n$ is a closed s-set, $n - 1 < s \leq n$, we will define a new function $\tilde{f} : X \rightarrow \mathbb{R}$ such that

$$(5.9) \quad \tilde{f}(x) = f(x) \quad \mathcal{H}_s - \text{almost everywhere on } X.$$

We then apply Proposition 5.3 to this function to show that $\tilde{f} \in \dot{A}^{k,\omega}(\mathbb{R}^n)|_X$ to construct a linear extension operator from $\dot{C}^{k,\omega}(X)$ to $\dot{A}^{k,\omega}(\mathbb{R}^n)$. To this end we will find for the $\tilde{f}$ a $(k, \omega, X)$-chain linearly depending on $f$. In the definition of the desired chain we will use the following construction. Let $Q := Q_r(x) \in \mathcal{K}_X$. By the Kadets-Snobar theorem [KS] there is a linear projection $\pi_Q$ from $L_1(X_r(x); \mathcal{H}_s)$ onto the subspace of polynomials of degree $k - 1$ restricted to $X_r(x) := Q_r(x) \cap X$ whose norm $||\pi_Q||_1 \leq \sqrt{d_{k,n}}$ where $d_{k,n}$ is the dimension of the space of polynomials of degree $k - 1$ on $\mathbb{R}^n$. Set

$$(5.10) \quad P_Q(f) := \pi_Q(f).$$

Using the definitions of $\tilde{f}$ and $\{P_Q(f)\}_{Q \in \mathcal{K}_X}$ we will show that the following is true.

**Claim 1.** There exists a $(k, \omega, X)$-chain $\tilde{\Pi}(f) := \{\tilde{P}(f)\}_{Q \in \mathcal{K}_X}$ linearly depending on $f$ and such that

$$(5.11) \quad |\tilde{\Pi}(f)|_{\mathcal{C}h} \leq O(1)|f|_{\dot{C}^{k,\omega}(\mathbb{R}^n)|_X}.$$

**Claim 2.** For every $Q \in \mathcal{K}_X$

$$(5.12) \quad \tilde{f}(c_Q) = \tilde{P}_Q(f)(c_Q).$$

Since the operator $f \mapsto \tilde{P}_Q(f)$ is linear, these allow us to apply Proposition 5.3 and to conclude that $\tilde{f} \in \dot{A}^{k,\omega}(\mathbb{R}^n)|_X$, and there is a linear extension operator $T_k : \dot{C}^{k,\omega}(X) \rightarrow \dot{A}^{k,\omega}(\mathbb{R}^n)$ satisfying

$$||T_k|| \leq O(1)$$

completing the first part of the proof of Theorem 2.11. The fact that the restriction to $X$ of every $f \in \dot{A}^{k,\omega}(\mathbb{R}^n)$ belongs to $\dot{C}^{k,\omega}(X)$ follows easily from Proposition 5.3 and Definition 2.8. This proves also the second assertion of the theorem and completes its proof.
To realize this program we need several auxiliary results. The main tool in their proofs is the weak Remez type inequality for $s$-sets, see Theorem 2.6 and (2.1).

**Lemma 5.4.** For every $Q = Q_r(x) \in \mathcal{K}_X$

\[
\left\{ \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} |f - \pi_Q(f)|^q \, d\mathcal{H}_s \right\}^{1/q} \leq O(1)\mathcal{E}_k(f; Q).
\]

**Proof.** Here and below for $Q = Q_r(x) \in \mathcal{K}_X$ by $P_Q$ we denote a polynomial of degree $k - 1$ satisfying

\[
\left\{ \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} |f - P_Q|^q \, d\mathcal{H}_s \right\}^{1/q} = \mathcal{E}_k(f; Q).
\]

Then

\[f - \pi_Q(f) = (f - P_Q) + \pi_Q(f - P_Q),\]

and applying the triangle inequality we estimate the left-hand side in (5.13) as is required but with the factor $(1 + ||\pi_Q||_q)$ instead of $O(1)$. So it remains to show that $||\pi_Q||_q \leq O(1)$. However, for $q = 1$ this norm is bounded by $\sqrt{d_{k,n}}$ by the definition. On the other hand, the weak Remez type inequality, see (2.1), and the fact that $X$ is an $s$-set, imply that

\[||\pi_Q(g)||_1 \approx ||\pi_Q(g)||_q\]

with the constants of equivalence independent of $g$ and $Q$. Thus by the Hölder inequality we have

\[||\pi_Q(g)||_q \leq O(1)||\pi_Q(g)||_1 \leq O(1)||\pi_Q||_1||g||_1 \leq O(1)||g||_q\]

\[\square\]

**Lemma 5.5.** Let $Q = Q_r(x) \in \mathcal{K}_X$. Then there exists the limit

\[
\tilde{f}(x) := \lim_{Q \to x} P_Q(x)
\]

and, moreover,

\[
|\tilde{f}(x) - P_Q(x)| \leq O(1)\omega(r)|f|_{C_{k-1}^s(X)}.
\]

**Proof.** Let $i$ be defined by

\[
t_i < r \leq t_{i+1}
\]

and for $j \leq i$

\[
Q_j := Q_{t_j}(x), \quad P_j := P_{Q_j}.
\]

Recall that $\{t_j\}$ is the sequence of Lemma 5.1. We also set $Q_{i+1} := Q$ and $P_{i+1} := P_Q$. Since $X$ is an $s$-set, the weak Remez type inequality (2.1) implies that

\[|P_{i+1}(x) - P_j(x)| \leq O(1)||P_{j+1} - P_j||_{X_j}||
\]

where for simplicity we set

\[||g; X_j|| := \left\{ \frac{1}{\mathcal{H}_s(X_j)} \int_{X_j} |g|^q \, d\mathcal{H}_s \right\}^{1/q} \quad \text{and} \quad X_j := Q_j \cap X.\]
In particular, in this case and gives the required estimate of 

\[ P \]

This immediately implies that and then Lemma 5.4 and inequality (2.1) yield 

\[
\left( \frac{\mathcal{H}_s(X_{j+1})}{\mathcal{H}_s(X_j)} \right)^{1/q} \mathcal{E}_k(f; Q_j+1) \leq \left( \frac{at_{j+1}}{bt_j} \right)^{1/q} \omega(t_{j+1})|f|_{\mathcal{C}_q^{h,s}(X)},
\]

see Definition 1.2 of \( s \)-sets.

Since, in turn, \( t_{j+1}/t_j \leq 2 \), using the definition of \( \omega \) we finally get 

\[ |P_{j+1}(x) - P_j(x)| \leq O(1)\omega(t_j)|f|_{\mathcal{C}_q^{h,s}(X)}. \]

This, Lemma 5.1 and the choice of \( i \), see (5.17), yield 

\[
\sum_{j \leq i} |P_{j+1}(x) - P_j(x)| \leq O(1)|f|_{\mathcal{C}_q^{h,s}(X)} \sum_{j \leq i} \omega(t_j) \leq O(1)|f|_{\mathcal{C}_q^{h,s}(X)} \omega(t_i) \leq O(1)\omega(r)|f|_{\mathcal{C}_q^{h,s}(X)}.
\]

This implies easily that the limit 

\[ \tilde{f}(x) := \lim_{Q \to x} P_Q(x) = P_{t+1}(x) + \sum_{j \leq i} (P_j(x) - P_{j+1}(x)) \]

exists and, moreover, 

\[ |\tilde{f}(x) - P_Q(x)| \leq O(1)\omega(r)|f|_{\mathcal{C}_q^{h,s}(X)}. \]

\[ \square \]

**Lemma 5.6.** The assertions of the previous lemma hold with the same \( \tilde{f}(x) \) for \( P_Q(f) \) substituted for \( P_Q \).

**Proof.** By (5.10) 

\[ P_Q - P_Q(f) = \pi_Q(P_Q - f) \]

and then Lemma 5.4 and inequality (2.1) yield 

\[
|P_Q(x) - P_Q(f)(x)| \leq O(1)\max_{Q \cap X} |P_Q - P_Q(f)| |P_Q - P_Q(f); Q \cap X|| \leq O(1)|\mathcal{E}_k(f; Q) + ||f - P_Q; Q \cap X||| \leq O(1)\mathcal{E}_k(f; Q) \leq O(1)\omega(r)|f|_{\mathcal{C}_q^{h,s}(X)}.
\]

This immediately implies that 

\[ \lim_{Q \to x} P_Q(f)(x) = \lim_{Q \to x} P_Q(x) = \tilde{f}(x) \]

and gives the required estimate of \( |\tilde{f}(x) - P_Q(f)(x)| \) by the right-hand side of (5.10). \[ \square \]

Hereafter we assume for simplicity that 

\[ (5.18) \]

\[ |f|_{\mathcal{C}_q^{h,s}(X)} = 1. \]

In particular, in this case 

\[ (5.19) \]

\[ \mathcal{E}_k(f; Q) \leq \omega(r_Q), \quad Q \in \mathcal{K}_X. \]
LEMMA 5.7. Let $Q \subset K$ be cubes from $\mathcal{K}_X$ of radii $r$ and $R$, respectively, $r < R \leq 2 \text{diam } X$. Let $\tilde{K}$ be the cube centered at $c_K$ of radius $2R$. Then it is true that

$$
E_1(f; Q) \leq O(1) \left\{ r \int_r^{2R} \frac{\omega(t)}{t^2} \, dt + \frac{r}{R} \| f; Q \cap \tilde{K} \| \right\}.
$$

PROOF. Choose $J \in \mathbb{N}$ from the condition

$$
R \leq 2^J r < 2R
$$

and let $Q_j$ be the cubes centered at $c_Q$ and of radii $r_j := 2^j r$, $j = 0, 1, \ldots J - 1$, and $Q_J := \tilde{K}$, $r_J := 2R$. Then $\{Q_j\}_{0 \leq j \leq J} \subset \mathcal{K}_X$ is an increasing sequence of cubes. We also set $P_j := P_{Q_j}$, $0 \leq j \leq J$, see (5.14) for the definition of $P_Q \in \mathcal{P}_{k-1}$. Under these notations we get

$$
E_1(f; Q) \leq \left\{ E_1(f - P_Q; Q) + \sum_{j=0}^{J-1} E_1(P_{j+1} - P_j; Q) + E_1(P_{k}; Q) \right\}.
$$

The first summand clearly equals

$$
E_k(f; Q) \leq \omega(r) \leq O(1) r \int_r^{2R} \frac{\omega(t)}{t^2} \, dt
$$

as is required.

To estimate the remaining terms we use two inequalities whose proofs are postponed to the end.

(A) Let $p$ be a polynomial of degree $k - 1$ and $Q \in \mathcal{K}_X$ be a cube of radius $r$.

Then

$$
E_1(p; Q) \leq O(1) r \max_{|\alpha|=1} \|D^\alpha p; Q \cap X\|.
$$

(B) Let, in addition, $\tilde{Q} \in \mathcal{K}_X$ be a cube of radius $\tilde{r}$ containing $Q$. Then

$$
\max_{|\alpha|=1} \|D^\alpha p; Q \cap X\| \leq O(1) \frac{1}{\tilde{r}} \| p; \tilde{Q} \cap X\|.
$$

Using these inequalities to estimate the $j$-th term in (5.21) we get

$$
r^{-1} E_1(P_{j+1} - P_j; Q) \leq O(1) \frac{1}{r_j} \| P_{j+1} - P_j; Q_j \cap X\|.
$$

By the definitions of $s$-sets, $\omega$ and (5.19), the norm on the right-hand side is at most

$$
O(1) \frac{1}{r_j} \left( E_k(f; Q_j) + \left( \frac{\mathcal{H}_s(Q_j \cap X)}{\mathcal{H}_s(Q_j)} \right)^{1/q} E_k(f; Q_{j+1}) \right) \leq O(1) \frac{\omega(r_j)}{r_j}.
$$

Moreover, by the definition of $r_j$ we get

$$
\frac{\omega(r_j)}{r_j} \leq O(1) \int_{r_j}^{r_j+1} \frac{\omega(t)}{t^2} \, dt, \quad 0 \leq j \leq J - 1.
$$

Summing the finally obtained estimates over $j$ we then have

$$
\sum_{j=0}^{J-1} E_1(P_{j+1} - P_j; Q) \leq O(1) r \int_r^{2R} \frac{\omega(t)}{t^2} \, dt.
$$
Using now (5.22) and (5.23) we bound the last summand in (5.21) by
\[ O(1) \frac{\|P_{\bar{K}}; \bar{K} \cap X\|}{R} \leq O(1) r \frac{\|f; \bar{K} \cap X\|}{R} \]
as is required.

To complete the proof of the lemma it remains to prove (5.22) and (5.23). By the hypothesis of (A) we get
\[ E_1(p; Q) \leq \inf_{\tilde{p}} \|p - \tilde{p}\|_{C(Q)} \leq O(1) r \max_{|\alpha|=1} \|D^\alpha p\|_{C(Q)} \]
where \( \tilde{p} \) runs over the space of polynomials of degree 0. The second of these inequalities is proved as follows. Using a homothety of \( \mathbb{R}^n \) we replace \( Q \) by the unit cube \( Q_0 := [0, 1]^n \). The functions in \( p \) of the both parts of this inequality are norms on the finite-dimensional factor-space \( P_{k-1}/P_0 \) and therefore they are equivalent. This implies the desired inequality.

Continuing the derivation we now use the weak Remez type inequality, see (2.1), and the fact that \( X \) is an \( s \)-set to have
\[ \|D^\alpha p\|_{C(Q)} \leq O(1) \|D^\alpha p; Q \cap X\| \]
and this completes the proof of (5.22).

Inequality (5.23) is proved in a similar way by means of the Markov inequality.

**Lemma 5.8.** \( f = \tilde{f} \) modulo \( \mathcal{H}_s \)-measure zero.

**Proof.** Let \( L(f) \) be the Lebesgue set of \( f \), i.e., the set of points \( x \in X \) such that
\[ f(x) = \lim_{r \to 0} \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} f \, d\mathcal{H}_s. \]
Since \( X \) is an \( s \)-set, the family of “balls” \( \{X_r(x) : x \in X, 0 < r \leq 1\} \) satisfies axioms (i), (ii) in [St, p.8]. Therefore the Corollary of Section I.3 from this book can be applied to our case with the measure \( \mu := \mathcal{H}_s|_X \). By this Corollary
\[ \mathcal{H}_s(X \setminus L(f)) = 0. \]
It remains to show that
\[ f(x) = \tilde{f}(x) \quad \text{for} \quad x \in L(f). \]
To this end choose a cube \( Q = Q_r(x) \in \mathcal{K}_X, 0 < r < 1 \), and set
\[ f_r(x) := \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} f \, d\mathcal{H}_s. \]
By the triangle inequality, the weak Remez type inequality for \( f_r(x) - P_Q \), see (2.1), and the fact that \( X \) is an \( s \)-set we obtain
\[ |f_r(x) - P_Q(x)| \leq O(1) \{\|f - f_r(x); Q \cap X\| + \mathcal{E}_k(f, Q)\}. \]
But \( f \mapsto f_r \) is a projection from \( L_1(X_r(x)) \) onto the space \( P_0 \) of polynomials of degree 0 whose norm is 1. Applying an argument similar to that of Lemma 5.4 with this projection substituted for \( \pi_Q \) we obtain that
\[ \|f - f_r(x); Q \cap X\| \leq O(1) \mathcal{E}_1(f; Q), \]
and therefore by Lemma 5.11 and (5.19) for a sufficiently small $r$ the right-hand side of (5.24) is bounded by

$$O(1) \{ E_{k} (f; Q) + E_{k} (P_{Q} (f; Q)) \} \leq O(1) \left\{ r \left( \int_{r}^{2} \frac{\omega (t)}{t^{2}} \, dt + || f ; K \cap X || \right) + \omega (r) \right\}$$

for some fixed cube $K$ of radius 1 containing $Q$. We conclude from here that for every $0 < \epsilon < 2$

$$\lim_{r \to 0} | f_{r} (x) - P_{Q} (x) | \leq$$

$$O(1) \limsup_{r \to 0} \left( \omega (r) + r \left( \int_{r}^{2} \frac{\omega (t)}{t^{2}} \, dt + \int_{r}^{t} \frac{\omega (t)}{t^{2}} \, dt + || f ; K \cap X || \right) \right) =$$

$$O(1) \limsup_{r \to 0} \left( r \int_{r}^{t} \frac{\omega (t)}{t^{2}} \, dt \right) \leq O(1) \omega (\epsilon).$$

Letting $\epsilon \to 0$ and noting that $\lim_{r \to 0} f_{r} (x) = f (x)$ for the Lebesgue point $x$ and $\lim_{Q \to x} P_{Q} (x) = \tilde{f} (x)$ we complete the proof of the lemma.

Now we finalize the proof of Theorem 2.11. For $Q \in \mathcal{K}_{X}$ and the polynomial $P_{Q} (f)$ of degree $k - 1$ defined in (5.10) we set

$$\bar{P}_{Q} (f) := P_{Q} (f) - P_{Q} (f)(c_{Q}) + \tilde{f} (c_{Q}).$$

Then $\bar{P}_{Q} (f)(c_{Q}) = \tilde{f} (c_{Q})$ and Claim 2, see (5.12), is true for the family $\bar{P}_{Q} (f) := \{ \bar{P}_{Q} \}_{Q \in \mathcal{K}_{X}}$. Show that Claim 1 is also true for $\bar{P}_{Q} (f)$.

Let $Q \subset Q'$ be cubes from $\mathcal{K}_{X}$ of radii $r < r'$ satisfying for some $i$ the condition

$$t_{i} \leq r < r' \leq t_{i+2}.$$ 

By the weak Remez type inequality, see (2.1), and Lemma 5.13 we have

$$\max_{Q} | P_{Q} (f) - P_{Q'} (f) | \leq O(1) \max_{X \cap Q'} | P_{Q} (f) - P_{Q'} (f) | \leq$$

$$O(1) || P_{Q} (f) - P_{Q'} (f) ; X \cap Q || \leq O(1) \left\{ E_{k} (f; Q) + \left( \frac{\mathcal{H}_{s} (Q' \cap X)}{\mathcal{H}_{s} (Q \cap X)} \right)^{1/q} E_{k} (f; Q') \right\}. $$

Both of the best approximations are bounded by $\omega (r') | f |_{C_{q,s}^{k} (X)}$ while, since $X$ is an $s$-set, the ratio of $\mathcal{H}_{s}$-measures is at most

$$\left\{ \frac{a}{b} \left( \frac{r'}{r} \right)^{s} \right\}^{1/q} \leq O(1) \left( \frac{t_{i+2}}{t_{i}} \right)^{s/q} \leq O(1).$$

Hence, in this situation, see (5.18),

$$\max_{Q} | P_{Q} (f) - P_{Q'} (f) | \leq O(1) \omega (r').$$

Moreover, by Lemma 5.14 see (5.10),

$$| \tilde{f} (c_{Q}) - P_{Q} (f)(c_{Q}) | \leq O(1) \omega (r).$$

Taking into account the definition of $\bar{P}_{Q} (f)$ we then obtain the inequality

$$\max_{Q} | \bar{P}_{Q} (f) - \bar{P}_{Q'} (f) | \leq O(1) \omega (r')$$

as is required in the definition of a $(k; \omega, X)$-chain.
This completes the proof of Claim 1 and therefore of Theorem 2.11.

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