Abstract

We introduce orbital graphs and discuss some of their basic properties. Then we focus on their usefulness for search algorithms for permutation groups, including finding the intersection of groups and the stabilizer of sets in a group.

1. Introduction

Combinatorial objects related to groups have been widely studied, in particular there are several ways to define graphs related to groups. Orbital graphs are just one way to define a directed graph coming from a permutation group; they are mentioned for example in [1] and [2].

In this article we prove theoretical results about orbital graphs, but they are motivated from a computational perspective. Orbital graphs have been considered by Heiko Theißen in [5] for the computation of normalizers of subgroups of permutation groups, but our work does not build on his – partly because our hypothesis is more general and partly because his results have not been published except for in his PhD thesis. We use orbital graphs in [3] for refinements of partitions in order to improve search algorithms that are based on the method of partition backtracking. However, it turns out that sometimes our graph refinement does not give any additional information and therefore does not reduce the size of the search that is performed.

Understanding this phenomenon lead to the present article, where we classify the orbital graphs that are not useful for our graph refinement.

We refer to [1] and [2] for the relevant notation, also for orbits, point stabilizers etc., but we introduce everything that might not be standard. After defining orbital graphs and proving some basic properties, we quickly move on to our main result and conclude the article with some open questions and comments on ongoing work.

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2. Basic Properties of orbital graphs

We begin by defining orbital graphs, by explaining some examples and by proving some basic properties. Our notation is standard, and we point out that by a **proper digraph** we mean a digraph that has at least one arc such that there its reverse arc is not in the graph. All digraphs considered here have no multiple arcs and no loops.

**Definition 2.1.** Let $H$ be a group of permutations on a set $\Omega$ and let $\alpha, \beta \in \Omega$ be distinct elements. Then the **orbital graph of $H$ with base-pair $(\alpha, \beta)$** is defined in the following way:

The vertex set is $\Omega$ and the arc set is $\{(\alpha^h, \beta^h) \mid h \in H\}$, where for all $h \in H$ and $\omega \in \Omega$ we denote by $\omega^h$ the image of $\omega$ under the permutation $h \in H$.

Once $H$ and its action on $\Omega$ are given, we denote the orbital graph of $H$ with base-pair $(\alpha, \beta)$ by $\Gamma(H, \Omega, (\alpha, \beta))$.

Just for the purpose of introduction of our notation for group actions, we collect some more definitions:

**Definition 2.2.** Let $\Gamma = (V, E)$ be a graph and let $H$ be a group. We say that $H$ acts on the graph $\Gamma$ if and only if there exists a group homomorphism from $H$ into the automorphism group Aut($\Gamma$) of $\Gamma$. We omit the homomorphism in our notation, so for all $h \in H$ and $x \in V$ we write $x^h$ for the image of $x$ under the image of $h$ in Aut($\Gamma$).

In a similar way we write the images of arcs, connected components, subsets of vertices etc.

If $v \in V$, then we denote by $v^H$ the orbit of $v$ under the action of $H$. Similarly, for an arc $(x, y)$, we write $(x, y)^H$ for the orbit (which is a set of arcs).

An **isolated vertex** of a digraph is a vertex with no arcs going into it or coming out of it.

Following [1] and [2] we say that an orbital graph is **self-paired** if and only if, for all $\gamma, \delta \in \Omega$, it is true that $(\gamma, \delta)$ is an arc if and only if $(\delta, \gamma)$ is an arc.

**Example 2.3.** Let $G := S_3$ and $\Omega := \{1, 2, 3\}$.

Starting with the base-pair $(1, 3)$ and applying all group elements to it, we obtain the arcs $(1, 3), (2, 3), (3, 1), (1, 2), (2, 1)$ and $(3, 2)$. These are all possible arcs, in particular $\Gamma$ is self-paired. The arguments are independent of the base-pair: We always obtain a complete (self-paired) digraph.

The reason for this is the high transitivity of the symmetric groups, as we will see in a lemma later on.

Next we let $H := \langle (23), (46) \rangle \leq S_7$ and $\Omega := \{1, 2, 3, 4, 5, 6, 7\}$.

If we choose the base-pair $(1, 7)$, then this is the only arc in the orbital graph and the points $2, 3, 4, 5, 6$ are isolated.
The base-pair \((1, 3)\) gives an orbital graph with arcs \((1, 3)\) and \((1, 2)\), and we obtain a maximum number of arcs by choosing the base-pair \((3, 4)\). Then we have four arcs in total, namely \((3, 4), (2, 4), (3, 6)\) and \((2, 6)\).

The examples suggest that transitivity properties of \(H\) influence the variety of possible orbital graph structures. In [1] and [2] some properties of orbital graphs are proved, but we give arguments for everything in the next lemma in order to make this article more self-contained.

**Hypothesis 2.4.** Let \(\Omega\) be a finite set, let \(H \leq G := \text{Sym}(\Omega)\) and let \(\alpha, \beta \in \Omega\) be distinct. Let \(\Gamma := \Gamma(H, \Omega, (\alpha, \beta))\) and let \(A\) denote the set of arcs of \(\Gamma\).

**Lemma 2.5.** Suppose that Hypothesis 2.4 holds. Then we have the following:

(i) \(\Gamma = \Gamma(H, \Omega, (\gamma, \delta))\) if and only if \((\gamma, \delta) \in A\).

(ii) \(\Gamma\) is self-paired if and only if some \(h \in H\) interchanges \(\alpha\) and \(\beta\).

(iii) \(\alpha^H\) is precisely the set of vertices of \(\Gamma\) that are the starting point of some arc.

(iv) \(\beta^H\) is precisely the set of vertices of \(\Gamma\) that are the end point of some arc.

(v) The number of arcs starting at \(\alpha\) is \(|\beta^{H_\alpha}|\) and the number of arcs going into \(\beta\) is \(|\alpha^{H_\beta}|\).

**Proof.** (i) If \(\Gamma = \Gamma(H, \Omega, (\gamma, \delta))\), then by definition \((\gamma, \delta)\) is an arc in \(\Gamma\).

Conversely, suppose that \((\gamma, \delta)\) is an arc in \(\Gamma\). Then there exists some \(h \in H\) such that \((\alpha^h, \beta^h) = (\gamma, \delta)\). Hence, if \(\omega \in \Omega\), then the statements \(\omega \in \alpha^H\) and \(\omega \in \gamma^H\) are equivalent. A similar statement holds for \(\delta\), so any arc in \(\Gamma\) can be obtained by mapping \((\gamma, \delta)\) with all elements in \(H\). This means that \((\gamma, \delta)\) can be chosen as a base-pair instead of \((\alpha, \beta)\).

(ii) By (i) \(\Gamma\) coincides with \(\Gamma(H, \Omega, (\beta, \alpha))\) if and only if the arc \((\beta, \alpha)\) exists in \(\Gamma\). Suppose that \(h \in H\) is such that \((\alpha^h, \beta^h) = (\gamma, \delta)\). Then the arc \((\beta, \alpha)\) exists in \(\Gamma\) by definition of the orbital graph. Conversely, if the arc \((\beta, \alpha)\) exists, then by definition of \(\Gamma\) there is some \(h \in H\) such that \((\alpha^h, \beta^h) = (\gamma, \delta)\) and hence \(\gamma = \alpha^h \in \alpha^H\).

(iii) Let \(\gamma \in \alpha^H\) and let \(h \in H\) be such that \(\gamma = \alpha^h\). Then \((\gamma, \beta^h)\) is an arc with starting point \(\gamma\).

Conversely, if \(\delta \in \Omega\) is such that \((\gamma, \delta)\) is an arc in \(\Gamma\), then there exists some \(h \in H\) such that \((\alpha^h, \beta^h) = (\gamma, \delta)\) and hence \(\gamma = \alpha^h \in \alpha^H\).

Similar arguments show (iv).

(v) We just calculate that the number of arcs starting at \(\alpha\) is \(|\{(\alpha, \gamma) \mid \gamma \in \Omega\}| = |\{(\alpha^h, \beta^h) \mid h \in H, \alpha^h = \alpha\}| = |\beta^{H_\alpha}|\).

The number of arcs going into \(\beta\) is, by the same reasoning, \(|\{(\delta, \beta) \mid \delta \in \Omega\}| = |\{(\alpha^h, \beta^h) \mid h \in H, \beta^h = \beta\}| = |\alpha^{H_\beta}|\).  
\[\square\]
Remark 2.6. Some comments:
(a) By (i) we may choose any arc in the orbital graph as base-pair. Then (i) and (ii) together imply that, if we want to decide whether or not $\Gamma$ is self-paired, then it is sufficient to look at any arc in $A$ and check whether $A$ also contains the reverse arc.
(b) Parts (iii) and (v) of the lemma, together, give the total number of arcs in $\Gamma$. The number of arcs starting at $\alpha$ is exactly $|\beta^{H_\alpha}|$, so we obtain $|A| = |\alpha^H| \cdot |\beta^{H_\alpha}|$.
(c) Warning! In (ii) it is not true that $H$ must contain the transposition $(\alpha, \beta)$. Counterexample: $H := \langle (12)(34) \rangle \leq S_4$ and $\Omega := \{1, 2, 3, 4\}$. Then the orbital graph with base-pair $(1, 2)$ has arcs $(1, 2), (2, 1)$ and two isolated vertices 3 and 4, and we obtain the same graph starting with base-pair $(2, 1)$. But $H$ does not contain (12).

We did not find the following lemma in the literature and therefore we state it here, for future reference.

Lemma 2.7. Suppose that Hypothesis 2.4 holds. Then the following are equivalent:
(a) $\Gamma$ has no isolated vertices.
(b) $\Omega \subseteq \alpha^H \cup \beta^H$.
In particular, if $H$ acts transitively on $\Omega$, then $\Gamma$ has no isolated vertices.

Proof. If $\Gamma$ has some isolated vertex $\omega \in \Omega$, then there is no arc starting at $\omega$ and no arc ending there. By Lemma 2.5 (ii) and (iii) it follows that $\gamma / \in \alpha^H \cup \beta^H$. Hence (b) implies (a).
Conversely we suppose that (a) holds and we let $\gamma \in \Omega$. As $\gamma$ is not isolated, it is the starting point of some arc or the end point. In the first case $\gamma \in \alpha^H$ and in the second case $\gamma \in \beta^H$ by Lemma 2.5 We chose $\gamma$ arbitrarily, so it follows that $\Omega \subseteq \alpha^H \cup \beta^H$ as stated in (b).
The last comment refers to the special case where $\Omega = \alpha^H = \beta^H$. □

Lemma 2.8. Suppose that Hypothesis 2.4 holds. Then $H$ acts on $\Gamma$ as a group of graph automorphisms.

Proof. Of course $H$ acts faithfully on the set $\Omega$ which is the vertex set of $\Gamma$. Let $\gamma, \delta \in \Omega$.
If $(\gamma, \delta)$ is an arc, then there exists some $h \in H$ such that $(\gamma, \delta) = (\alpha^h, \beta^h)$ by definition of $\Gamma$. Hence $(\gamma^g, \delta^g) = (\alpha^{hg}, \beta^{hg})$ is an arc.
Conversely, if $(\gamma^h, \delta^h)$ is an arc, then there exists some $a \in H$ such that $(\gamma^h, \delta^h) = (\alpha^a, \beta^a)$ and hence $(\gamma, \delta) = (\alpha^{ah^{-1}}, \beta^{ah^{-1}})$ is an arc.
As $H$ is a group, the induced maps are bijective and hence every $h \in H$ induces a graph automorphism on $\Gamma$. □

Lemma 2.9. Suppose that Hypothesis 2.4 holds and let $\Delta$ denote the connected component that contains $(\alpha, \beta)$. Then every connected component of $\Gamma$ that has size at least 2 is isomorphic to $\Delta$. 

Proof. Let $\Delta_2$ denote an arbitrary connected component of $\Gamma$ of size at least 2 and let $(\gamma, \delta)$ be an arc in $\Delta_2$. From the definition of orbital graphs let $h \in H$ be such that $(\alpha^h, \beta^h) = (\gamma, \delta)$. Then $h$ induces an automorphism on $\Gamma$ by Lemma 2.8 and it moves all arcs from $\Delta$ to arcs in $\Delta_2$. Conversely, $h^{-1}$ induces an automorphism on $\Gamma$ that moves all arcs of $\Delta_2$ into $\Delta$. Thus it follows that $\Delta$ and $\Delta_2$ are isomorphic as graphs. □

There is work in progress in order to understand better and in more detail the different types of digraphs that can occur as orbital graphs. In particular, we are interested in how exactly the orbit structure of $H$ affects the structure of its orbital graphs and which ones are particularly useful for computational purposes.

3. Usefulness of orbital graphs in algorithms

As was briefly mentioned in the introduction, orbital graphs are used in search algorithms, see [3]. They can be used to refine ordered orbit partitions and hence prune the search tree during partition backtrack search. But there are situations where the automorphism group of an orbital graph is so large that the graph is not useful, and this motivates the following definition:

**Definition 3.1.** Suppose that Hypothesis 2.4 holds and that $P$ is an ordered orbit partition of $H$. We denote the stabilizer of $P$ in $G$ by $G_P$ and we emphasize that $G_P$ stabilizes every $H$-orbit (i.e. every cell of the ordered partition $P$) as a set and that it acts as the full symmetric group on every orbit. This will play a role later on.

We say that the orbital graph $\Gamma$ is **pointless** if and only if $G_P$ is isomorphic to a subgroup of the automorphism group $\text{Aut}(\Gamma)$ of $\Gamma$.

We capture in this definition that some orbital graphs do not give more information than the one that is already contained in the ordered orbit partition for $H$. This means that using such a graph in a search algorithm does not improve the algorithm.

Our search algorithm in [3] builds on Jeffery Leons partition backtrack technique, see [4]. As reasoning about arbitrary permutation groups is computationally extremely expensive, during search Leons partition backtrack algorithm replaces groups with the stabilizer of an ordered orbit partition. This can be seen as an approximation - a group $G$ is a subgroup of the stabilizer of its ordered orbit partition in a supergroup of $G$. In [3] we instead replace a group by the intersection of the stabilizer of an ordered orbit partition with the automorphism group of all orbital graphs. This will provide a smaller group, as long as the automorphism groups of the orbital graphs do not contain some permutations which are in the stabilizer of an ordered orbit partition.
Our main theoretical result on this topic classifies pointless orbital graphs. It explains why in our algorithms in [3], highly transitive groups are not considered and some orbital graphs are never constructed. For a more complete discussion see [3].

**Definition 3.2.** Let $\Gamma = (V, A)$ be a digraph.

$\Gamma$ is called a **complete digraph** if and only if its set of arcs is
$$\{(\omega_1, \omega_2) \mid \omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2\}.$$ 

$\Gamma$ is called a **complete bipartite digraph** if and only if there exist pair-wise disjoint subsets $S, E$ of vertices such that $V$ is the disjoint union of $S$ (the “starting” vertices) and $E$ (the “end” vertices) and the set of arcs is exactly
$$A = \{(\omega_1, \omega_2) \mid \omega_1 \in S, \omega_2 \in E\}.$$ 

We note that the following result does not have any hypothesis about the number of orbits of $H$ on $\Omega$. In particular there could be arbitrarily many isolated points in $\Gamma$.

**Theorem 3.3.** Suppose that Hypothesis [2.4] holds. Then $\Gamma$ is pointless if and only if it has a unique connected component $\Delta$ of size at least 2 and moreover one of the following holds:

(a) $\Delta$ is a complete digraph or

(b) $\Delta$ is a complete bipartite digraph.

**Proof.** Let $P$ be an ordered orbit partition of $H$. Of course $G_P$ acts on the set of orbits of $H$ and it acts faithfully on the set of vertices of $\Gamma$. Hence for the question of $\Gamma$ being pointless or not, we only have to consider arcs in $\Gamma$.

**Case 1:** $\Gamma$ is a proper digraph.

Then $\Gamma$ is not self-paired and Lemma [2.5] (i and (ii)) imply that, for all $\omega_1, \omega_2 \in \Omega$, there is at most one arc between them. We also recall Lemma [2.5] (iii) and (iv): All vertices where an arc starts are $H$-conjugate to $\alpha$ and all vertices where an arc ends are $H$-conjugate to $\beta$.

We begin with the hypothesis that $\Gamma$ is pointless.

(1) Suppose that $\gamma, \delta \in \Omega$ are distinct and in the same $H$-orbit. Then they are not on an arc. In particular $\alpha^H \neq \beta^H$.

**Proof.** As $\gamma$ and $\delta$ are in the same $H$-orbit, they lie in the same cell of the partition $P$. The transposition $(\gamma, \delta) \in \text{Sym}(\Omega)$ stabilizes $P$ and interchanges the vertices $\gamma$ and $\delta$. It follows from the pointless property that this transposition induces a graph automorphism on $\Gamma$, and therefore neither $(\gamma, \delta)$ nor $(\delta, \gamma)$ is an arc (otherwise they both are, but $\Gamma$ is not self-paired). From this and the fact that $(\alpha, \beta) \in A$ it follows that $\alpha^H \neq \beta^H$. □

(2) Suppose that $\omega \in \Omega$ is on an arc. Then it is either a starting point or an end point, but not both.

**Proof.** If $\omega$ is a starting vertex of an arc, then by Lemma [2.5] (iii) it is $H$-conjugate to $\alpha$. Then by (1) it cannot be conjugate to $\beta$, so it is not the end vertex of an arc. The same argument works vice versa. □
Let $S := \alpha^H$ and $E := \beta^H$, and let $I \subseteq \Omega$ denote the set of isolated vertices of $\Gamma$.

(3) $\Omega = S \cup E \cup I$. Moreover $S \cup E$ spans the unique connected component of $\Gamma$ of size at least 2, and this component is a complete bipartite digraph.

**Proof.** The first statement follows from (2). Moreover there are no arcs between vertices in $S$ or $E$, respectively, by (1). We show that all elements of $E$ are on an arc with $\alpha$:

For all $\gamma \in E$, we find the transposition $g := (\beta, \gamma) \in G_P$, and it fixes $\alpha^H$ point-wise by (1). Since $\Gamma$ is pointless, $g$ induces an automorphism on $\Gamma$ and hence maps the arc $(\alpha, \beta)$ to the arc $(\alpha, \gamma)$. Now it follows that $A = S \times E$ and hence the digraph spanned by $S \cup E$ is a complete bipartite digraph, and it is the unique connected component of size at least 2 of $\Gamma$. \hfill \Box

This finishes one direction of the statement in Case 1. Next we suppose, conversely, that $\Gamma$ has a unique connected component of size at least 2 and that this component is a complete bipartite digraph. We prove that $\Gamma$ is pointless.

Let $S$ and $E$ denote the subsets of the vertex set of $\Gamma$ such that all arcs start at $S$ and end at $E$. Let $I$ be the set of isolated vertices of $\Gamma$, so that $\Omega = S \cup E \cup I$.

Every element of $\alpha^H$ is the starting point of an arc and is therefore contained in $S$. Then the bipartite structure implies that $\alpha^H = S$. Similarly $\beta^H = E$. Therefore $G_P$ stabilizes the sets $S$, $E$ and $I$. We already know that $G_P$ permutes the vertices of $\Gamma$ faithfully, so now we look at arcs. Let $g \in G_P$ and let $(\omega_1, \omega_2) \in A$.

Then $\omega_1 \in S$, $\omega_2 \in E$ and there exists some $h \in H$ such that $(\alpha^h, \beta^h) = (\omega_1, \omega_2)$. Since $G_P$ stabilizes the sets $S$ and $E$, we see that $\omega_1^g \in S$ and $\omega_2^g \in E$. The completeness property then implies that $(\omega_1^g, \omega_2^g) \in A$.

Conversely, if $(\omega_1^g, \omega_2^g) \in A$, then there exists some $h \in H$ such that $(\alpha^h, \beta^h) = (\omega_1^g, \omega_2^g)$. Now $\omega_1 = \alpha^{hg^{-1}} \in S$ and $\omega_2 = \beta^{hg^{-1}} \in E$ whence $(\omega_1, \omega_2) \in A$ by completeness.

Hence $\Gamma$ is pointless.

**Case 2:** $\Gamma$ is not a proper digraph, which means that it is self-paired.

We begin, once more, with the hypothesis that $\Gamma$ is pointless.

Let $\Delta$ denote the connected component of $\Gamma$ that contains the base-pair $(\alpha, \beta)$ and let $\gamma \in \Omega$ be an arbitrary, non-isolated vertex.

We know that $\beta^H = \alpha^H$ by Lemma [2.5] (iii) and (iv), because $\Gamma$ is self-paired.

Since some arc starts or ends in $\gamma$, we also have that $\gamma \in \alpha^H$ and hence $\alpha, \beta, \gamma$ are all in the same $H$-orbit and hence in a common cell of the partition $P$.

In particular the transposition $g := (\beta, \gamma)$ is contained in $G_P$ and because of the pointless property it induces an automorphism on $\Gamma$.

Then $(\alpha, \beta) \in A$ implies that $(\alpha, \gamma) = (\alpha^g, \beta^g) \in A$. This argument shows that $\Delta$ is a complete digraph and that it is the only connected component of
size at least 2 in $\Gamma$.

This proves one direction of our statement, so now we conversely suppose that $\Gamma$ has a unique connected component of size at least 2 and that it is complete. Together with the definition of orbital graphs (and the fact that arcs always go both ways) this implies that $\alpha^H = \beta^H$ spans the unique connected component of size at least 2 and that the isolated vertices, viewed as elements of $\Omega$, are not contained in $\alpha^H$.

We prove that $\Gamma$ is pointless.

We know that $G_P$ acts faithfully on the vertex set of $\Gamma$. Now let $g \in G_P$ and let $\omega_1, \omega_2 \in \Omega$. We recall that $\alpha^H = \beta^H$ is $G_P$-invariant and then argue as in Case 1:

If $(\omega_1, \omega_2) \in A$, then there exists some $h \in H$ such that $(\alpha^h, \beta^h) = (\omega_1, \omega_2)$. Since $G_P$ stabilizes $\alpha^H$, we know that $\omega_1^g, \omega_2^g \in \alpha^H$ and then $(\omega_1^g, \omega_2^g) \in A$ by completeness. Conversely, if $(\omega_1^g, \omega_2^g) \in A$, then there exists some $h \in H$ such that $(\alpha^h, \beta^h) = (\omega_1^g, \omega_2^g)$. Now $\omega_1 = \alpha^{hg^{-1}} \in \alpha^H$ and $\omega_2 = \beta^{hg^{-1}} \in \alpha^H$ whence, again by completeness, it follows that $(\omega_1, \omega_2) \in A$. Consequently $G_P$ is isomorphic to a subgroup of $\text{Aut}(\Gamma)$, i.e. $\Gamma$ is pointless.

We give an example in order to illustrate that the “pointless” property is not obvious and why further investigations into the usefulness of orbital graphs should be pursued.

**Example 3.4.** We let $G := S_9$ and we look at the subgroup $H := \langle (12), (13), (45), (46), (14)(25)(36), (789) \rangle$. Let $\Gamma$ be the orbital graph for $H$ with base-pair $(1, 2)$. Then $\Gamma$ has the following shape:

On the vertices $1, 2, 3$ and $4, 5, 6$ we have a complete digraph, respectively, there is no arc between the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$, and the points 7, 8 and 9 are isolated. This might look like a pointless graph, but according to the theorem it is not. So we consider an ordered orbit partition $P := [1, 2, 3, 4, 5, 6] [7, 8, 9]$ of $H$.

The group $G_P$ contains the transposition $(24) \in G$. This element interchanges the vertices 2 and 4 of $\Gamma$ and fixes 1, so this element is not inducing an automorphism on $\Gamma$. (Otherwise the arc $(1, 2)$ would be mapped to the arc $(1, 4)$ which does not exist.)

Hence $G_P$ is not isomorphic to a subgroup of $\text{Aut}(\Gamma)$ and we see that $\Gamma$ is, in fact, not pointless.

We finish with a special case that made us realize in the first place that some orbital graphs are more useful than others.

**Corollary 3.5.** Let $H \leq G := \text{Sym}(\Omega)$ and let $k \in \mathbb{N}$, $k \geq 2$.

If $H$ is $k$-transitive, then all its orbital graphs are pointless.

**Proof.** Suppose that that $H$ is $k$-transitive and let $(\alpha, \beta)$ be a base-pair for an orbital graph. For all $\gamma, \delta \in \Omega$, there exists some $h \in H$ such that $\alpha^h = \gamma$
and $\beta^h = \delta$, because of $k$-transitivity, and then $(\gamma, \delta)$ is an arc. So the orbital graph of $H$ is a complete graph, independently of the base-pair. Now Theorem 3.3 yields the statement. □

From a computational perspective, this is not problematic: Highly transitive groups are well-understood and often our methods will still be useful once the algorithm deals with subgroups that are not as highly transitive.

4. Concluding remarks

As mentioned earlier, this work on orbital graphs is motivated by applications in search algorithms for permutation groups, as explained for example in [3]. This implies that we would like to able to quickly decide (computationally speaking) whether or not an orbital graph is pointless, and ideally we would not even have to compute the graph in order to decide this.

Hence a systematic approach is needed that describes the types of orbital graphs in terms of choice of the base-pair. Lemmas 2.5 and 2.7 and Remark 2.6 indicate that the orbit structure of $H$ on $\Omega$ plays a role. Once the first vertex $\alpha \in \Omega$ of the base-pair is chosen, the orbit structure of the point stabilizer $H_\alpha$ plays a role.

Orbital graphs are an interesting class of graphs in their own right and therefore we phrase some questions, with only some of them being directly related to applications.

- Instead of just separating the pointless graphs from the useful ones for our algorithms, is it possible to create a finer distinction? If so, then is it possible to predict, at low computational costs, which orbital graphs should be constructed during search because they are most useful, and which ones should not be constructed at all? (This extends our current knowledge by which we only detect some pointless graphs before actually constructing them in our algorithms.)
- Higman’s Theorem (see for example p.68 in [2]) says that for transitive groups, primitivity can be detected from the orbital graphs. We have also seen that transitive groups can have connected orbital graphs as well as non-connected ones, so one goal would be a classification result that describes how exactly the connectivity structure depends on the group and on the choice of the base-pair.
- How can we quickly detect blocks if the group action is not primitive? Exercise 3.2.14 in [2] raises a question related to this. Being able to detect blocks quickly and bring them into a “usefulness analysis” of the graph would be beneficial for computational questions.
- The theory of association schemes seems to be closely related to orbital graphs. What applications does this have in computational algebra and how do our results relate to this?

There is work in progress on most of these questions.
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