OSCILLATING SINGULAR INTEGRAL OPERATORS ON GRADED LIE GROUPS REVISITED

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Abstract. In this work we extend the Euclidean theory of oscillating singular integrals due to Fefferman and Stein in [18, 19] to arbitrary graded Lie groups. Our approach reveals the strong compatibility between the geometric measure theory of a graded Lie group and the Fourier analysis associated with Rockland operators. Our criteria are presented in terms of the Hörmander condition of the kernel of the operator, and its group Fourier transform. One of the novelties of this work, is that we use the infinitesimal representation of a Rockland operator to measure the decay of the Fourier transform of the kernel.

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1. Introduction

The aim of this paper is to consolidate the theory of oscillating singular integrals on graded Lie groups, in a compatible way with the modern techniques of their non-commutative geometric analysis, namely, the geometric quantization on nilpotent Lie groups [21] and the corresponding theory of pseudo-differential operators, built

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on the fundamental techniques of the harmonic analysis of homogeneous Lie groups developed by Folland and Stein in [24].

To accomplish our goal, we will extend the theory of oscillating singular integrals developed by Fefferman and Stein in [18, 19] to the setting of graded Lie groups. Our setting then, covers a variety of groups of interest in analysis of PDE, namely, the Euclidean space, Heisenberg type groups, stratified groups, etc. For our analysis we will use the Fourier analysis associated to Rockland operators. These are hypoelliptic partial differential operators in view of the Helffer and Nourrigat solution of the Rockland conjecture, and they could be of arbitrary order. In the particular case of stratified Lie groups, they absorb the family of Hörmander sub-Laplacians and also their integer powers.

Now, we are going to the historical aspects on the subject in the Euclidean setting inspiring our approach. As it was pointed out by Stein in [44], there is probably no work in the last seventy years which has had the widespread influence in analysis as the historic memoir [5], “On the existence of certain singular integrals” by Calderón and Zygmund published in 1952 in Acta Mathematica. Using the methods of real interpolation (e.g. using Marcinkiewicz interpolation theorem) one of the main problems of the Calderón-Zygmund theory is to identify the sharp conditions on a distribution $K$ in order that, the convolution operator $f \mapsto f * K$ can be extended to a weak $(1,1)$ type operator, that is

$$
\| f * K \|_{L^{1,\infty}(\mathbb{R}^n)} \leq C \| f \|_{L^1(\mathbb{R}^n)}, \ f \in C_0^\infty(\mathbb{R}^n).
$$

For instance, the singular integrals were important predecessors of the so called pseudo-differential operators, whose impact in analysis, partial differential equations and differential geometry was recognised through the works of Kohn and Nirenberg, Hörmander, Stein and Fefferman, and Atiyah and Singer, among others, see Hörmander [33, Page 178] for the historical details. In terms of convolution operators, a remarkable kernel class $H_\infty$ was introduced by Hörmander, consisting of all distribution satisfying the kernel estimate

$$
[K]_{H_\infty} := \sup_{0 < R < 1} \| \int_{|x| \geq 2R} |K(x - y) - K(x)| \, dx \|_{L^\infty(B(0,R), dy)} < \infty. \quad (1.1)
$$

Then, in [32] Hörmander proved that a convolution operator $Tf = K * f$, with a kernel satisfying the “smoothness” condition in (1.1) and bounded on $L^2(\mathbb{R}^n)$ (that is, the Fourier transform of the kernel $\hat{K} \in L^\infty(\mathbb{R}^n)$) is of weak $(1,1)$ type.

The theory of oscillating singular integrals by Fefferman and Stein in [18, 19] extends in some aspects the one by Calderón and Zygmund allowing the analysis of oscillating multipliers. Indeed, generalising the Hörmander condition, Fefferman in [18] and Fefferman and Stein in [19] have considered distributions satisfying the condition

$$
[K]_{H_\infty, \theta} := \sup_{0 < R < 1} \| \int_{|x| \geq 2R^{1-\theta}} |K(x - y) - K(x)| \, dx \|_{L^\infty(B(0,R), dy)} < \infty. \quad (1.2)
$$

Indeed, roughly speaking, if $K$ satisfies (1.2) and its Fourier transform has order

$$
|\hat{K}(\xi)| = O((1 + |\xi|)^{-\frac{a_0}{2}}), \quad 0 \leq \theta < 1,
$$

as stated in (1.3).
Fefferman’s theorem says that $T$ admits a bounded extension of weak (1,1) type if the support of $K$ is small enough. The condition that the support of $K$ is small enough does not appear in [18, Theorem 2'], but it is assumed in the proof of such a statement in [18, Page 24]. This is not a restrictive assumption because a similar analysis as the one done in [18, Page 23] allows to pass from distributions $K$ with a support of arbitrary size to distributions with small support.

Obviously, with $\theta = 0$, Fefferman’s condition agrees with the one by Hörmander. The boundedness from the Hardy space $H^1(\mathbb{R}^n)$ into itself was proved in [19].

From the point of view of the harmonic analysis, our main Theorem 1.1 establishes a critical estimate on the $L^p$-spaces for $p = 1$ of the oscillating singular integrals, namely, their weak (1,1) boundedness property, by providing from the point of view of the representation theory of such groups, deep connections between the Fourier analysis on graded Lie groups and its geometric measure theory. One of the instrumental tools for this work is the existence of hypoelliptic left-invariant linear partial differential operators on (graded) Lie groups via the solution of the Rockland conjecture by Helffer and Nourrigat in [30].

One of the key tools in the proof of our critical estimate is, as expected, the Calderón-Zygmund decomposition for topological spaces of homogeneous type developed by Coifman and Weiss in [14]. Indeed, to combine the Calderón-Zygmund decomposition theorem of Coifman and Weiss of an integrable function $f = g + b = g + \sum_j b_j$, (into its good and bad part $g$ and $b$, respectively), the geometric properties of the supports of the pieces $b_j$, (that typically are balls $I_j$ generated via the Whitney extension algorithm), with the Fourier analysis of Rockland operators is one of the fundamental developments of this work (encoded in the proof of Lemma 3.2).

The main theorem of this work is the following. Here $\widehat{G}$ denotes the unitary dual of $G$, $\mathcal{R}$ is a positive Rockland operator on $G$, that is $\mathcal{R}$ is a positive, left-invariant, hypoelliptic partial differential operator on $G$. We denote by $\pi(\mathcal{R}) = \widehat{k}_\mathcal{R}(\pi)$ the Fourier transform of its right-convolution kernel $k_\mathcal{R}$, that is characterised by the identity, $\mathcal{R}f = f \ast k_\mathcal{R}$, $f \in C^\infty_0(G)$.

**Theorem 1.1.** Let $G$ be a graded Lie group of homogeneous dimension $Q$, and let $T : C^\infty_0(G) \to \mathcal{P}'(G)$ be a left-invariant operator with right-convolution kernel $K \in L^1_{\text{loc}}(G \setminus \{e\})$, $Tf = f \ast K$. Assume that $K$ is a distribution with compact support and that its diameter is small enough. Let us assume that for some $0 \leq \theta < 1$, $K$ satisfies the group Fourier transform condition

$$\sup_{\pi \in \widehat{G}} \| (1 + \pi(\mathcal{R}))^{Q\theta} \widehat{K}(\pi) \|_{\text{op}} < \infty, \quad (1.4)$$

and the oscillating Hörmander condition

$$[K]_{H_{\infty,\theta}(G)} := \sup_{0 < R < 1} \sup_{|y| < R \|x\| \geq 2R^{1-\theta}} |K(y^{-1}x) - K(x)| \, dx < \infty. \quad (1.5)$$

Then $T$ admits a bounded extension of weak (1,1) type.

Now, we briefly discuss our results.

**Remark 1.2.** For $G = \mathbb{R}^n$ we recover the classical weak (1,1) boundedness result proved by Fefferman in [18, Theorem 2']. From the proof of Theorem 1.1 in Section
one can estimate the operator norm of $T : L^1 \rightarrow L^{1,\infty}$, by
\[
\|T\|_{L^1 \rightarrow L^{1,\infty}} \lesssim \|T\|_{L^2 \rightarrow L^2} + [K]_{H^\infty,\theta} = \|\hat{K}\|_{L^\infty} + [K]_{H^\infty,\theta},
\]
in view of the Plancherel theorem.

Remark 1.3. The main contribution of Theorem 1.1 is when $0 < \theta < 1$. For $\theta = 0$, the statement in Theorem 1.1 follows from the general theory of Calderón-Zygmund operators developed by Coifman and Weiss in [14]. Hörmander-Mihlin criteria on graded groups were obtained by the second author and V. Fischer in [22].

Remark 1.4. The boundedness of an operator $T$ satisfying the hypothesis in Theorem 1.1 from the Hardy space $H^1(G)$ into $L^1(G)$ has been proved by the authors in [8], while the $H^1-L^1$-estimate for pseudo-differential operators on graded Lie groups was analysed in [7].

Remark 1.5 (Historical note). Oscillating singular integrals arose as generalisations of the oscillating Fourier multipliers. These are multipliers associated to symbols of the form
\[
\hat{K}(\xi) = \psi(\xi) e^{i|\xi|^a}, \quad \psi \in C^\infty(\mathbb{R}^n), \quad 0 < a < 1,
\]
where $\psi$ vanishes near the origin and is equal to one for $|\xi|$ large. It was proved by Wainger [47] that $K(x)$ is essentially equal to $c_n|x|^{-n-\lambda}e^{i|x|^a}$, where $\lambda = \frac{n(a-a')}{2(1-a)}$, and $a' = \frac{a}{a-1}$. From this one can deduce that $|\nabla K(x)| \lesssim |x|^{-n-\lambda-1+a'}$, from which one can deduce that for $a = \alpha = \theta$ the estimate (1.5) remains valid.

The $L^p$-properties for convolution operators of this form were firstly studied by Hardy [29], Hirschman [31] and Wainger [47]. The sharp version of the $L^\infty$-BMO boundedness for oscillating Fourier multipliers can be deduced from the classical work of Fefferman [20]. Further works on the subject in the setting of manifolds and beyond can be found in Seeger [39, 40, 41], Seeger and Sogge [42] and for the setting of Fourier integral operators, we refer the reader to Seeger, Sogge and Stein [43] and Tao [45].

2. Fourier analysis on graded groups

The notation and terminology of this paper on the analysis of homogeneous Lie groups are mostly taken from Folland and Stein [24]. For the analysis of Rockland operators we will follow [21, Chapter 4].

2.1. Homogeneous and graded Lie groups. Let $G$ be a homogeneous Lie group. This means that $G$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations. We introduce it in the following definition.

Definition 2.1. A family of dilations $D^r_\theta$, $r > 0$, on the Lie algebra $\mathfrak{g}$ is a family of automorphisms on $\mathfrak{g}$ satisfying the following two conditions:

- For every $r > 0$, $D^r_\theta$ is a map of the form
  \[
  D^r_\theta = \text{Exp}[(\ln(r))A]
  \]
  for some diagonalisable linear operator $A \equiv \text{diag}[\nu_1, \cdots, \nu_n]$ on $\mathfrak{g}$.
- $\forall X, Y \in \mathfrak{g}$, and $r > 0$, $[D^r_\theta X, D^r_\theta Y] = D^r_\theta [X, Y]$.  

Remark 2.2. We call the eigenvalues of $A$, $\nu_1, \nu_2, \cdots, \nu_n$, the dilations weights or weights of $G$. 
In our analysis is crucial the notion of the homogeneous dimension of the group. We introduce it as follows.

Definition 2.3. The homogeneous dimension of a homogeneous Lie group $G$ is given by

$$Q = \text{Tr}(A) = \nu_1 + \cdots + \nu_n.$$ 

Definition 2.4 (Dilations on the group). The dilations $D_r^g$ of the Lie algebra $g$ induce a family of maps on $G$ defined via,

$$D_r := \exp_G \circ D_r^g \circ \exp_{G}^{-1}, \quad r > 0,$$

where $\exp_G : g \to G$ is the usual exponential mapping associated to the Lie group $G$. We refer to the family $D_r$, $r > 0$, as dilations on the group.

Remark 2.5. If we write $rx = D_r(x)$, $x \in G$, $r > 0$, then a relation on the homogeneous structure of $G$ and the Haar measure $dx$ on $G$ is given by

$$\int_G (f \circ D_r)(x)dx = r^{-Q} \int_G f(x)dx.$$ 

Remark 2.6. A Lie group is graded if its Lie algebra $g$ may be decomposed as the sum of subspaces $g = g_1 \oplus g_2 \oplus \cdots \oplus g_s$ such that $[g_i, g_j] \subset g_{i+j}$, and $g_{i+j} = \{0\}$ if $i + j > s$.

Examples of graded Lie groups are the Heisenberg group $H^n$ and more generally any stratified groups where the Lie algebra $g$ is generated by $g_1$. Here, $n$ is the topological dimension of $G$, $n = n_1 + \cdots + n_s$, where $n_k = \dim g_k$.

Remark 2.7. A Lie algebra admitting a family of dilations is nilpotent, and hence so is its associated connected, simply connected Lie group. The converse does not hold, i.e., not every nilpotent Lie group is homogeneous although they exhaust a large class, see [21] for details. Indeed, the main class of Lie groups under our consideration is that of graded Lie groups.

Remark 2.8. A graded Lie group $G$ is a homogeneous Lie group equipped with a family of weights $\nu_j$, all of them positive rational numbers. Let us observe that if $\nu_i = \frac{a_i}{b_i}$ with $a_i, b_i$ integer numbers, and $b$ is the least common multiple of the $b_i's$, the family of dilations

$$D_r^g = \text{Exp}(\ln(r^b)A) : g \to g,$$

have integer weights, $\nu_i = \frac{a_i b}{b_i}$. So, in this paper we always assume that the weights $\nu_j$, defining the family of dilations are non-negative integer numbers which allow us to assume that the homogeneous dimension $Q$ is a non-negative integer number. This is a natural context for the study of Rockland operators (see Remark 4.1.4 of [21]).

2.2. Fourier analysis on nilpotent Lie groups. Let $G$ be a simply connected nilpotent Lie group. Then the adjoint representation $\text{ad} : g \to \text{End}(g)$ is nilpotent. Next, we define unitary and irreducible representations.

Definition 2.9. We say that $\pi$ is a continuous, unitary and irreducible representation of $G$, if the following properties are satisfied,
for every $x \hat{\pi}$ is defined via the Fourier transform of $f$.

The map $(x, v) \mapsto \pi(x)v$, from $G \times H_{\pi}$ into $H_{\pi}$ is continuous.

For every $x \in G$, and $W_{\pi} \subset H_{\pi}$, if $\pi(x)W_{\pi} \subset W_{\pi}$, then $W_{\pi} = H_{\pi}$ or $W_{\pi} = \emptyset$.

**Definition 2.10** (The unitary dual). Let $\text{Rep}(G)$ be the set of unitary, continuous and irreducible representations of $G$. The relation,

$$\pi_1 \sim \pi_2 \text{ if and only if, there exists } A \in \mathcal{B}(H_{\pi_1}, H_{\pi_2}), \text{ such that } A\pi_1(x)A^{-1} = \pi_2(x),$$

for every $x \in G$, is an equivalence relation and the unitary dual of $G$, denoted by $\hat{G}$ is defined via $\hat{G} := \text{Rep}(G)/\sim$. Let us denote by $d\pi$ the Plancherel measure on $\hat{G}$.

Next, we define the main object for our further analysis: the Fourier transform.

**Definition 2.11.** The Fourier transform of $f \in \mathcal{S}(G)$, (this means that $f \circ \exp_G \in \mathcal{S}(\mathfrak{g})$, with $\mathfrak{g} \simeq \mathbb{R}^\dim(G)$) at $\pi \in \hat{G}$, is defined by

$$\hat{f}(\pi) = \int f(x)\pi(x)^*dx : H_{\pi} \to H_{\pi}, \text{ and } \mathcal{F}_G : \mathcal{S}(G) \to \mathcal{S}(\hat{G}) := \mathcal{F}_G(\mathcal{S}(G)).$$

**Remark 2.12.** If we identify one representation $\pi$ with its equivalence class, $[\pi] = \{\pi' : \pi \sim \pi'\}$, for every $\pi \in \hat{G}$, the Kirillov trace character $\Theta_\pi$ defined by

$$(\Theta_\pi, f) := \text{Tr}(\hat{f}(\pi)),$$

is a tempered distribution on $\mathcal{S}(G)$. In particular, the identity $f(e_G) = \int (\Theta_\pi, f)d\pi,$

implies the Fourier inversion formula $f = \mathcal{F}_G^{-1}(\hat{f})$, where

$$(\mathcal{F}_G^{-1}(\sigma))(x) := \int \text{Tr}(\pi(x)\sigma(\pi))d\pi, \text{ } x \in G, \text{ } \mathcal{F}_G^{-1} : \mathcal{S}(\hat{G}) \to \mathcal{S}(G),$$

is the inverse Fourier transform. In this context, the Plancherel theorem takes the form $\|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(\hat{G})}$, where $L^2(\hat{G}) := \int H_{\pi} \otimes H^*_{\pi}d\pi$, is the Hilbert space endowed with the norm: $\|\sigma\|_{L^2(\hat{G})} = (\int_{\hat{G}} \|\sigma(\pi)\|^2_{HS}d\pi)^{\frac{1}{2}}$.

### 2.3. Homogeneous linear operators and Rockland operators.

There is family of continuous linear operators observing the actions of the dilations of the group. These are called homogeneous linear operators. We introduce them in the following definition.

**Definition 2.13.** A linear operator $T : C^{\infty}(G) \to \mathcal{D}'(G)$ is homogeneous of degree $\nu \in \mathbb{C}$ if for every $r > 0$ the equality

$$T(f \circ D_r) = r^{\nu}(Tf) \circ D_r$$

holds for every $f \in \mathcal{D}(G)$.

Now, we introduce the main class of differential operators in the context of nilpotent Lie groups. The existence of these operators classify the family of graded Lie groups. We call them Rockland operators.
**Definition 2.14.** If for every representation \( \pi \in \hat{G} \), \( \pi : G \to U(H_\pi) \), we denote by \( H_\pi^{\infty} \) the set of smooth vectors, that is, the space of elements \( v \in H_\pi \) such that the function \( x \mapsto \pi(x)v, x \in \hat{G} \) is smooth, a Rockland operator is a left-invariant differential operator \( \mathcal{R} \) which is homogeneous of positive degree \( \nu = \nu_\mathcal{R} \) and such that, for every unitary irreducible non-trivial representation \( \pi \in \hat{G} \), \( \pi(R) \) is injective on \( H_\pi^{\infty} \). \( \sigma_\mathcal{R}(\pi) = \pi(\mathcal{R}) \) is the symbol associated to \( \mathcal{R} \). It coincides with the infinitesimal representation of \( \mathcal{R} \) as an element of the universal enveloping algebra.

**Remark 2.15.** It can be shown that a Lie group \( G \) is graded if and only if there exists a differential Rockland operator on \( G \).

Next, we record for our further analysis some aspects of the functional calculus for Rockland operators.

**Remark 2.16.** If the Rockland operator is formally self-adjoint, then \( \mathcal{R} \) and \( \pi(\mathcal{R}) \) admit self-adjoint extensions on \( L^2(G) \) and \( H_\pi \), respectively. Now if we preserve the same notation for their self-adjoint extensions and we denote by \( E \) and \( E_\pi \) their spectral measures, we will denote by

\[
f(\mathcal{R}) = \int_{-\infty}^{\infty} f(\lambda)dE(\lambda), \quad \text{and} \quad \pi(f(\mathcal{R})) \equiv f(\pi(\mathcal{R})) = \int_{-\infty}^{\infty} f(\lambda)dE_\pi(\lambda),
\]

the functions defined by the functional calculus. In general, we will reserve the notation \( \{dE_A(\lambda)\}_{0 \leq \lambda < \infty} \) for the spectral measure associated with a positive and self-adjoint operator \( A \) on a Hilbert space \( H \).

We now recall a lemma on dilations on the unitary dual \( \hat{G} \), which will be useful in our analysis of spectral multipliers. For the proof, see Lemma 4.3 of [21].

**Lemma 2.17.** For every \( \pi \in \hat{G} \) let us define

\[
D_r(\pi)(x) \equiv (r \cdot \pi)(x) := \pi(r \cdot x) \equiv \pi(D_r(x)), \quad (2.1)
\]

for every \( r > 0 \) and all \( x \in G \). Then, if \( f \in L^\infty(\mathbb{R}) \) then \( f(\pi(r)(\mathcal{R})) = f(r^n\pi(\mathcal{R})) \).

**Remark 2.18.** For instance, for any \( \alpha \in \mathbb{N}_0^n \), and for an arbitrary family \( X_1, \cdots, X_n \), of left-invariant vector-fields we will use the notation

\[
[\alpha] := \sum_{j=1}^n \nu_j \alpha_j, \quad (2.2)
\]

for the homogeneity degree of the operator \( X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n} \), whose order is \( |\alpha| := \sum_{j=1}^n \alpha_j \).

**Remark 2.19.** By considering the dilation \( r \cdot x = D_r(x), x \in G, r > 0 \), then a relation on the homogeneous structure of \( G \) and the Haar measure \( dx \) on \( G \) is given by (see [21, Page 100])

\[
f(f \circ D_r)(x)dx = r^{-Q} \int_G f(x)dx.
\]

Note that if \( f_r := r^{-Q}f(r^{-1}) \), then

\[
\widehat{f_r}(\pi) = \int_G f_r(-x)\pi(x)^{*}dx = \int_G f(y)\pi(r \cdot y)^{*}dy = \widehat{f}(r \cdot \pi), \quad (2.3)
\]

for any \( \pi \in \hat{G} \) and all \( r > 0 \), with \( (r \cdot \pi)(y) = \pi(r \cdot y), y \in G \), as in (2.1).
The following lemma presents the action of the dilations of the group $G$ into the kernels of bounded functions of a Rockland operator $\mathcal{R}$, see [21, Page 179].

**Lemma 2.20.** Let $\kappa \in L^\infty(\mathbb{R}_0^+)$ and let $r > 0$. Then,

$$\kappa(r^\nu \mathcal{R})\delta(x) = r^{-Q}[\kappa(\mathcal{R})\delta](r^{-1} \cdot x),$$

for all $x \in G$.

### 3. Proof of the Main Theorem

#### 3.1. Part 1: The Calderón-Zygmund decomposition

Let $G$ be a graded Lie group, $Q$ is its homogeneous dimension, and let $\mathcal{R}$ be a positive Rockland operator on $G$ of homogeneous degree $\nu > 0$. In this section we are going to prove that for a singular integral operator $T$ satisfying (1.4) and (1.5) there is a constant $C > 0$, such that

$$\|Tf\|_{L^{1,\infty}(G)} := \sup_{\alpha > 0} \alpha |\{x \in G : |Tf(x)| > \alpha\}| \leq C \|f\|_{L^1(G)},$$

with $C$ independent of $f$.

We start by analysing the condition in (1.4).

**Remark 3.1.** Observe that in view of the Plancherel theorem, the hypothesis (1.4) in Theorem 1.1 implies that the operator $(1 + \mathcal{R})^{Q\theta} T$ admits a bounded extension on $L^2(G)$. Indeed, the Plancherel theorem implies that

$$\|(1 + \mathcal{R})^{Q\theta} Tf\|_2^2_{L^2(G)} = \int_\hat{G} \|(1 + \pi(\mathcal{R}))^{Q\theta} \hat{K}(\pi)\widehat{f(\pi)}\|_{HS}^2 d\pi \leq \int_\hat{G} \|(1 + \pi(\mathcal{R}))^{Q\theta} \hat{K}(\pi)\|_{op}\|\widehat{f(\pi)}\|_{HS}^2 d\pi,$$

and in view of (1.4) we have that

$$\|(1 + \mathcal{R})^{Q\theta} Tf\|_2^2_{L^2(G)} \leq C f \|\widehat{f(\pi)}\|_{HS}^2 d\pi = C^2 \|f\|_2^2_{L^2(G)},$$

proving the boundedness of $(1 + \mathcal{L}_G)^{Q\theta} T$ on $L^2(G)$. Because $(1 + \mathcal{R})^{-Q\theta}$ we have that $T = (1 + \mathcal{R})^{-Q\theta} (1 + \mathcal{L}_G)^{Q\theta} T$ is also bounded on $L^2(G)$.

For the proof of (3.1), fix $f \in L^1(G)$, and let us consider its Calderón-Zygmund decomposition, see Coifman and Weiss [14, Pages 73-74]. So, for any $\gamma, \alpha > 0$, we have the decomposition

$$f = g + b = g + \sum_j b_j,$$

where the following properties are satisfied.

1. $\|g\|_{L^\infty} \lesssim_G \gamma \alpha$ and $\|g\|_{L^1} \lesssim_G \|f\|_{L^1}$.
2. The $b_j$’s are supported in open balls $I_j = B(x_j, r_j)$ where they satisfy the cancellation property
   $$\int_{I_j} b_j(x) dx = 0.$$  

(3.2)

3. Any component $b_j$ satisfies the $L^1$-estimate
   $$\|b_j\|_{L^1} \lesssim_G (\gamma \alpha)|I_j|.$$  

(3.3)
(4) The sequence \( \{ |I_j| \}_j \in \ell^1 \) and
\[
\sum_j |I_j| \lesssim_G (\gamma \alpha)^{-1} \|f\|_{L^1}.
\]

(5)
\[
\|b\|_{L^1} \leq \sum_j \|b_j\|_{L^1} \lesssim_G \|f\|_{L^1}.
\]

(6) There exists \( M_0 \in \mathbb{N} \), such that any point \( x \in G \) belongs at most to \( M_0 \) balls of the collection \( I_j \).

So, by fixing \( \alpha \gamma > 0 \), note that in terms of \( g, b \) and \( f \) one has the trivial estimate
\[
|\{ x : |T f(x)| > \alpha \}| \leq |\{ x : |T g(x)| > \alpha/2 \}| + |\{ x : |T b(x)| > \alpha/2 \}|.
\]

The estimates \( \|g\|_{L^\infty} \lesssim \gamma \alpha \) and \( \|g\|_{L^1} \lesssim \|f\|_{L^1} \), imply that
\[
\|g\|_{L^2}^2 \leq \|g\|_{L^\infty} \|g\|_{L^1} \lesssim (\gamma \alpha) \|f\|_{L^1}.
\]

So, by applying the Chebyshev inequality and the \( L^2 \)-boundedness of \( T \), we have
\[
|\{ x : |T g(x)| > \alpha/2 \}| \leq 2^2 \alpha^{-2} \|T g\|_{L^2}^2 \leq (2 \|T\|_{\mathcal{B}(L^2)})^2 \alpha^{-2} \|g\|_{L^2}^2
\]
\[
\leq (2 \|T\|_{\mathcal{B}(L^2)})^2 \alpha^{-2} (\gamma \alpha) \|f\|_{L^1} \lesssim \|T\|_{\mathcal{B}(L^2)}^2 \alpha^{-1} \|f\|_{L^1}
\]
\[
\lesssim \gamma \alpha^{-1} \|f\|_{L^1}.
\]

In what follows, let us denote \( I^* = \bigcup I_j^* \), where
\[
I_j^* = B(x_j, 2r_j) = \{ x \in G : |x^{-1} x_j| < 2R_j \},
\]
and let us make use of the doubling condition in order to have the estimate
\[
|I_j^*| \sim 2^j |I_j|,
\]
from which it follows that
\[
|I^*| \lesssim \sum_j |I_j^*| \lesssim \sum_j |I_j| \lesssim \gamma^{-1} \alpha^{-1} \|f\|_{L^1}.
\]

Consequently, we have the estimates
\[
|\{ x : |T b(x)| > \alpha/2 \}| \leq |I^*| + |\{ x \in G \setminus I^* : |T b(x)| > \alpha/2 \}|
\]
\[
\lesssim \gamma^{-1} \alpha^{-1} \|f\|_{L^1} + |\{ x \in G \setminus I^* : |T b(x)| > \alpha/2 \}|.
\]
\[
\lesssim \gamma \alpha^{-1} \|f\|_{L^1} + |\{ x \in G \setminus I^* : |T b(x)| > \alpha/2 \}|.
\]

So, to conclude the inequality (3.1) we have to prove that
\[
\sup_{\alpha > 0} \alpha |\{ x \in G \setminus I^* : |T b(x)| > \alpha/2 \}| \leq C \|f\|_{L^1},
\]
with \( C \) independent of \( f \). So, the proof of Theorem 1.1 consists of estimating the term
\[
|\{ x \in G \setminus I^* : |T b(x)| > \alpha/2 \}|.
3.2. Part 2: proof of the oscillating case. The relevant case in Theorem 1.1 is the case where $\theta \in (0, 1)$. We start the proof of the main theorem by explaining this claim.

**Proof of Theorem 1.1.** We start by considering only the case $0 < \theta < 1$. Indeed, the statement for $\theta = 0$ in Theorem 1.1 follows from the fundamental theorem of singular integrals due to Coifman and Weiss, see [14, Theorem 2.4, Page 74]. Indeed, a graded Lie group satisfies the doubling condition and then, is a homogeneous topological space in the sense of Coifman and Weiss [14]. For $0 < \theta < 1$ we require some geometric transformations on the support of $K$, that we present in the next subsection. For this, we will use two modern techniques: the stability of the powers of Rockland operators inside of the Hörmander classes $S^{m}_{1,0}(G \times \hat{G})$, and the Calderón-Vaillancourt theorem on graded Lie groups, see [21, Section 5.7, Theorem 5.7.1].

By following our hypothesis, from now, let us suppose that the diameter of the support of $K$ is small, for instance, that

$$\text{diam} (\text{supp}(K)) < c/1000$$

where $0 < c < 1$ is viable for our purposes. With this in mind, that follows is to construct a good replacement $\tilde{b}$ of $b$, that allows us to exploit the hypothesis in (1.5).

3.3. Part 3: The function $\phi$. New family of supports. Let us consider a function $\phi$ on $G$ such that

$$\int_{G} \phi(x)dx = 1, \text{ and } \phi \in C^\infty_0(G, [0, \infty)) \cap \text{Dom}[\mathcal{R}^{-\frac{Q}{\nu}}]. \quad (3.6)$$

For $\varepsilon > 0$, define

$$\phi(y, \varepsilon) := \varepsilon^{-Q}\phi(\varepsilon^{-1} \cdot x). \quad (3.7)$$

Now, for any $j$, define

$$\phi_j(y) := \phi(y, 2^{-\frac{1}{1-\theta}} \text{diam}(I_j)^{\frac{1}{1-\theta}}), \quad (3.8)$$

$$\tilde{b}_j := b_j(\cdot) \ast \phi_j, \quad (3.9)$$

and

$$\tilde{b} := \sum_j \tilde{b}_j. \quad (3.10)$$

Note that

$$Tb(x) = \sum_j Tb_j(x), \quad (3.11)$$

for a.e. $x \in G$. It is important to mention that in (3.11) the sums on the right hand side only runs over $j$ with $\text{diam}(I_j) < c/1000$. Indeed, for all $x \in G \setminus I^*$, the property of the support $\text{diam} (\text{supp}(K)) < c/1000$, implies that for all $j$ with $\text{diam}(I_j) \geq c/1000$, we have that

$$Tb_j = b_j * K = 0.$$

So we only require to analyse the case where $\text{diam}(I_j) < c/1000$. Indeed, for $x \in G \setminus I^*$, and $j$ such that $\text{diam}(I_j) \geq c/1000$,

$$b_j * K(x) = \int_{I_j} K(y^{-1}x)b_j(y)dy.$$
Because in the integral above \( x \in G \setminus I^* \) and \( y \in I_j \),
\[
|y^{-1} x| = \text{dist}(x, y) > \text{diam}(I_j) > c/1000,
\]
we have that the element \( y^{-1} x \) is not in the support of \( K \) and then the integral vanishes.

Now, going back to the analysis of (3.5), note that
\[
\left\vert \left\{ x \in G \setminus I^* : |Tb(x)| > \frac{\alpha}{2} \right\} \right. \left| \left| \begin{array}{c}
\leq \left\vert \left\{ x \in G \setminus I^* : |Tb(x) - T\tilde{b}(x)| > \frac{\alpha}{4} \right\} \right| + \left\vert \left\{ x \in G \setminus I^* : |T\tilde{b}(x)| > \frac{\alpha}{4} \right\} \right| \leq \frac{4}{\alpha} \|T(b - \tilde{b})\|_{L^1(G \setminus I^*)} + \left\vert \left\{ x \in G \setminus I^* : |T\tilde{b}(x)| > \frac{\alpha}{4} \right\} \right|
\]
and let us take into account the estimate:
\[
\|T(b - \tilde{b})\|_{L^1(G \setminus I^*)} = \int_{G \setminus I^*} |Tb(x) - T\tilde{b}(x)| dx \leq \sum_j \int_{G \setminus I^*} |Tb_j(x) - T\tilde{b}_j(x)| dx.
\]
We are going to prove that \( T\tilde{b} \) and \( T\tilde{b}_j \) are good replacements for \( Tb \) and \( Tb_j \), respectively, on the set \( G \setminus I^* \). Observe that
\[
\int_{G \setminus I^*} |Tb_j(x) - T\tilde{b}_j(x)| dx = \int_{G \setminus I^*} |b_j \ast K(x) - \tilde{b}_j \ast K(x)| dx
\]
\[
= \int_{G \setminus I^*} |b_j \ast K(x) - \phi_j \ast K(x)| dx
\]
\[
= \int_{G \setminus I^*} \left| \int_{I_j} K(y^{-1} x) b_j(y) dy - \int_{I_j} (\phi_j \ast K)(y^{-1} x) b_j(y) dy \right| dx
\]
\[
\leq \int_{I_j \setminus |z| > \text{diam}(I_j)} \left| K(z) - \phi_j \ast K(z) \right| dz \int_{I_j} |b_j(y)| dy
\]
where, in the last line we have used the changes of variables \( x \mapsto z = y^{-1} x \), and then we observe that \( |z| > \text{diam}(I_j) \) when \( x \in G \setminus I^* \) and \( y \in I_j \). Using that \( \phi_j \) is supported in a ball of radius
\[
R_j := 2^{-1/\sigma} \text{diam}(I_j)^{1/\sigma}
\]
and that \( \|\phi_j\|_{L^1} = 1 \), we have that
\[
\int_{|z| > \text{diam}(I_j)} \left| K(z) - \phi_j \ast K(z) \right| dz
\]
\[
= \int_{|z| > \text{diam}(I_j)} \left| K(z) - \int_{|y| < 2^{-1/\sigma} \text{diam}(I_j)^{1/\sigma}} \phi_j(y) dy - \int_{|y| < 2^{-1/\sigma} \text{diam}(I_j)^{1/\sigma}} K(y^{-1} z) \phi_j(y) dy \right| dz
\]
In the case of $F$, we have used that $2^{1-\theta}$

We will extend this Fefferman’s decomposition for an arbitrary Rockland operator $\mathcal{R}$. Indeed,

$$
\|T(b - \tilde{b})\|_{L^1(G \setminus I^*)} \leq \sum_j \int_{I_j \setminus G \setminus I^*} |K(y^{-1}x) - \phi_j \ast K(y^{-1}x)|dx|b_j(y)|dy
\leq \sum_j \int_{I_j \setminus \overline{\text{diam}(I_j)}} |K(z) - \phi_j \ast K(z)|dz|b_j(y)|dy
\leq [K]_{H_\infty,\theta(G)} \sum_j |b_j(y)|dy \leq [K]_{H_\infty,\theta(G)} \|f\|_L^1
\leq [K]_{H_\infty,\theta(G)} \|f\|_L^1.
$$

Putting together the estimates above we deduce that

$$
\{|x \in G \setminus I^* : |Tb(x)| > \frac{\alpha}{2}\} \leq \frac{4}{\alpha} \|Tb - T\tilde{b}\|_{L^1(G \setminus I^*)} + \{|x \in G \setminus I^* : |T\tilde{b}(x)| > \frac{\alpha}{4}\}
\leq \frac{4}{\alpha} [K]_{H_\infty,\theta(G)} \|f\|_L^1 + \{|x \in G \setminus I^* : |T\tilde{b}(x)| > \frac{\alpha}{4}\}.
$$

Now, we will estimate the second term on the right hand side of this inequality. First, note that

$$
\{|x \in G \setminus I^* : |T\tilde{b}(x)| > \frac{\alpha}{4}\} \leq \{|x \in G \setminus I^* : |T\tilde{b}(x)|^2 > \frac{\alpha^2}{16}\}
\leq \frac{16}{\alpha^2} \|T\tilde{b}\|_L^2.
$$

Now, using (1.4) we deduce that $T(1 + \mathcal{R})^{\frac{\alpha}{2\nu}}$ is bounded on $L^2$ (see Remark 3.1), and then

$$
\|T\tilde{b}\|_L^2 \leq \|T(1 + \mathcal{R})^{\frac{\alpha}{2\nu}}\|_{L^2(G \setminus I^*)}(1 + \mathcal{R})^{\frac{\alpha}{2\nu}}|\tilde{b}|_L^2.
$$

In the case of $G = \mathbb{R}^n$, and of the positive Laplace operator $\mathcal{R} = -\Delta$, it was proved by Fefferman that the function $F = (1 + \Delta)^{-\frac{\alpha}{2\nu}b}$, admits a nice decomposition $F = F_1 + F_2$, where $\|F_2\|_L^2 \leq C\alpha \gamma \|f\|_L^1$, and $F_1 = \sum_{j : \text{diam}(I_j) < 1} F_{1j}$, where $\|F_{1j}\|_L^2 \leq A\alpha^2 |I_j|$. We will extend this Fefferman’s decomposition for an arbitrary Rockland operator $\mathcal{R}$ in Lemma 3.2.
3.4. Part 4: A Fefferman type decomposition for Rockland operators. In order to estimate the $L^2$-norm $\|(1 + \mathcal{R})^{-\frac{Q\theta}{2\nu}} b\|_{L^2}$, let us use the following lemma whose proof we postpone for a moment.

**Lemma 3.2.** The function $F := (1 + \mathcal{R})^{-\frac{Q\theta}{2\nu}} b$ can be decomposed in a sum $F = F_1 + F_2$, where $\|F_2\|_{L^2} \leq C\alpha \gamma \|f\|_{L^1}$, and $F_1$ is also a sum of functions $F_1^j$ with the following property:

- There exists $M_0 \in \mathbb{N}$, and $A' > 0$, such that $F_1 = \sum_{j: \text{diam}(I_j) < 1} F_1^j$, $\|F_1^j\|_{L^2} \leq A'\alpha^2 |I_j|$, and for any $x \in G$, there are at most $M_0$ values of $j$ such that $F_1^j(x) \neq 0$.

Let us continue with the proof of Theorem 1.1. Using Lemma 3.2 and the inequalities in (3.4) and (3.12) we have that

\[
\|Tb\|_{L^2} \lesssim \|F_1\|_{L^2} + \|F_2\|_{L^2} \lesssim \alpha \gamma \|f\|_{L^1} + \sum_j \|F_1^j\|_{L^2} \lesssim \alpha \gamma \|f\|_{L^1} + \sum_j \alpha^2 |I_j| \lesssim \alpha \gamma \|f\|_{L^1} + \alpha^2 (\gamma \alpha)^{-1} \|f\|_{L^1} = (\gamma^{-1} + \gamma) \alpha \|f\|_{L^1}.
\]

Consequently

\[
|\{ x \in G \setminus I^* : |Tb(x)| > \frac{\alpha}{4} \} | \lesssim \frac{16}{\alpha^2} \|Tb\|_{L^2} \lesssim (\gamma^{-1} + \gamma) \alpha^{-1} \|f\|_{L^1}.
\]

Thus, we have proved that

\[
|\{ x \in \mathbb{R}^n : |Tf(x)| > \alpha \} | \leq C_{\gamma,[K]}H_{\infty,\theta}(G) \alpha^{-1} \|f\|_{L^1},
\]

with $C := C_{\gamma,[K]}H_{\infty,\theta}(G)$ independent of $f$. Because $\gamma$ is fixed we have proved the weak $(1,1)$ type of $T$. Thus, the proof is complete once we prove the statement in Lemma 3.2. To do so, let us consider the right-convolution kernel $k_\theta := (1 + \mathcal{R})^{-\frac{Q\theta}{2\nu}} \delta$ of the operator $(1 + \mathcal{R})^{-\frac{Q\theta}{2\nu}} b(x)$ as follows:

\[
(1 + \mathcal{R})^{-\frac{Q\theta}{2\nu}} b(x) = \sum_j (1 + \mathcal{R})^{-\frac{Q\theta}{2\nu}} b_j(x) = \sum_j \tilde{b}_j * k_\theta(x)
\]

\[
= \sum_{j : x \sim I_j} \tilde{b}_j * k_\theta(x) + \sum_{j : x \sim I_j} \tilde{b}_j * k_\theta(x) =: G_1(x) + G_2(x),
\]

where $G_1(x) := \sum_{j : x \sim I_j} \tilde{b}_j * k_\theta(x)$ and $G_2(x) := \sum_{j : x \sim I_j} \tilde{b}_j * k_\theta(x)$. We have denoted by $x \sim I_j$, if $x$ belongs to $I_j$ or to some $I_j'$ with non-empty intersection with $I_j$. By the properties of these sets there are at most $M_0$ sets $I_j'$ such that $I_j \cap I_j' \neq \emptyset$. Also, the notation $x \sim I_j$ will be employed to define the opposite of the previous property.

Let us prove the estimate

\[
\|G_2\|_{L^2} \leq C\alpha \gamma \|f\|_{L^1}.
\]

Observe that

\[
\|G_2\|_{L^1} = \int_G \sum_{j : x \sim I_j} b_j * k_\theta(x) |dx| \leq \sum_{j : x \sim I_j} \int_G b_j * k_\theta(x) |dx|
\]
To do this, let us consider $j$ such that $x \sim I_j$. Since $\tilde{b}_j \ast k_\theta = b_j \ast \phi_j \ast k_\theta$, one has that

$$|b_j \ast \phi_j \ast k_\theta(x)| \leq \int_{I_j} |(\phi_j \ast k_\theta)(y^{-1}x)| |b_j(y)| dy \leq \sup_{y \in I_j} |(\phi_j \ast k_\theta)(y^{-1}x)| \int_{I_j} |b_j(y)| dy$$

$$= \sup_{y \in I_j} |\phi_j \ast k_\theta(y^{-1}x)||I_j| \times \frac{1}{|I_j|} \int_{I_j} |b_j(y)| dy.$$

To continue, we follow as in [18, Page 26] the observation of Fefferman, that in view of the property $x \sim I_j$, we have that $\phi_j \ast |k_\theta|(y^{-1}x)$ is essentially constant over the ball $I_j = B(x, r_j)$ and we can estimate

$$\sup_{y \in I_j} |\phi_j \ast k_\theta(y^{-1}x)||I_j| \leq \sup_{y \in I_j} |\phi_j \ast |k_\theta|(y^{-1}x)||I_j| \lesssim \int_{I_j} |\phi_j \ast |k_\theta|(y^{-1}x)| dy'.$$  \hspace{2cm} (3.16)

On the other hand, using again the positivity of $\phi_j$ leads to

$$\int_{I_j} |\phi_j \ast k_\theta(y^{-1}x)| dy' \frac{1}{|I_j|} \int_{I_j} |b_j(y)| dy \leq \int_{G} (\int b_j(y)|dy) \left( \frac{1}{|I_j|} \int_{I_j} |b_j(y)| dy \right) 1_{I_j}(y') dy'$$

$$= \left( \frac{1}{|I_j|} \int_{I_j} |b_j(y)| dy \times 1_{I_j} \right) * \phi_j \ast |k_\theta|(x).$$

Consequently, we have

$$|G_2(x)| \leq \sum_{j:x \sim I_j} |\tilde{b}_j \ast k_\theta(x)| \lesssim \sum_{j:x \sim I_j} \left( \frac{1}{|I_j|} \int_{I_j} |b_j(y)| dy \times 1_{I_j} \right) * \phi_j \ast |k_\theta|(x)$$

$$\lesssim \sum_{j:x \sim I_j} \gamma \alpha \times 1_{I_j} * \phi_j \ast |k_\theta|(x) = \int_{G} \sum_{j:x \sim I_j} \gamma \alpha \times 1_{I_j} * \phi_j(z)|k_\theta(z^{-1}x)| dz.$$
\[
\|\phi_j * k_\theta\|_{L^2(G)}^2 \leq \gamma \alpha \|k_\theta\|_{L^1} \left\| \sum_{j : x \in I_j} 1_{I_j} * \phi_j \right\|_{L^\infty}.
\]

By observing that the supports of the functions \(1_{I_j} * \phi_j\) have bounded overlaps we have that \(\left\| \sum_{j : x \in I_j} 1_{I_j} * \phi_j \right\|_{L^\infty} < \infty\), and that \(\|G_2\|_{L^\infty} \lesssim \gamma \alpha\).

It remains only to prove that \(\|G_1\|_{L^2}^2 \lesssim \alpha \gamma \|f\|_{L^1}\). Let us define

\[
G_j(x) := \begin{cases} b_j * \phi_j * k_\theta(x), & x \in I_j, \\ 0, & \text{otherwise.} \end{cases}
\]

Then \(G_1 = \sum_j G_j\) and in view of the finite overlapping of the balls \(I_j\)'s, there is \(M_0 \in \mathbb{N}\), such that for any \(x \in G\), \(G_j(x) \neq 0\), for at most \(M_0\) values of \(j\). Therefore, we have that

\[
\int_G |G_1(x)|^2 \, dx \leq M_0 \sum_j \int_G |G_j(x)|^2 \, dx = \sum_j \int_{I_j} |b_j * \phi_j * k_\theta(x)|^2 \, dx
\]

\[
\leq \sum_j \|b_j\|_{L^1}^2 \|\phi_j * k_\theta\|_{L^2}^2 \leq \sum_j \alpha^2 \gamma^2 \|I_j\|^2 \|\phi_j * k_\theta\|_{L^2}^2.
\]

To estimate the \(L^2\)-norm \(\|\phi_j * k_\theta\|_{L^2(G)}^2\), we will use the dilation properties of the spectral calculus of the Rockland operator \(\mathcal{R}\).

3.5. **Final Part.** Now, let us apply the Plancherel theorem:

\[
\|\phi_j * k_\theta\|_{L^2(G)}^2 = \|(1 + \pi(\mathcal{R}))^{-\frac{Q_\theta}{2\pi}} \hat{\phi}_j\|_{L^2(\hat{G})}^2.
\]

Define for any \(j\),

\[
\psi_j(x) := R_j^Q \phi_j(R_j \cdot x), \quad x \in G.
\]

Observe that for any \(j\),

\[
\psi_j(x) := R_j^Q \phi_j(R_j \cdot x) = R_j^Q R_j^{-Q} \phi(R_j \cdot R_j^{-1} \cdot x) = \phi(x).
\]

However, in order to use the dilations properties of the Fourier transform of distributions, let us have in mind the identity for \(\psi_j = \phi\) in (3.18).

Let us remark that we have the following identities, for the function \(\phi_j\),

\[
\phi_j(x) = R_j^{-Q} \psi_j\left(R_j^{-1} \cdot x\right), \quad x \in G.
\]

as well as for its Fourier transform (in view of (2.3))

\[
\hat{\psi}_j(\pi) = \hat{\psi}_j(R_j \cdot \pi), \quad \pi \in \hat{G}.
\]

Also, note that \(\|\psi_j\|_{L^\infty} = 1\), and \(\text{supp}(\psi_j) \subset B(e, 1)\). Using the Plancherel theorem, we have that

\[
\|\phi_j * k_\theta\|_{L^2(G)}^2 = \|(1 + \pi(\mathcal{R}))^{-\frac{Q_\theta}{2\pi}} \hat{\phi}_j\|_{L^2(\hat{G})}^2
\]

\[
= \|(1 + \pi(\mathcal{R}))^{-\frac{Q_\theta}{2\pi}} \hat{\psi}_j(R_j \cdot \pi)\|_{L^2(\hat{G})}^2
\]

\[
= \int_{\hat{G}} \|(1 + \pi(\mathcal{R}))^{-\frac{Q_\theta}{2\pi}} \hat{\psi}_j(R_j \cdot \pi)\|_{\text{HS}}^2 \, d\pi
\]
\[ = R_j^{-Q} \int_\hat{G} \| (1 + (R_j^{-1} \cdot \pi)(\mathcal{R}))^{-\frac{Qa}{2\nu}} \hat{\psi}_j(\pi) \|^2_{HS} d\pi, \]

where we have used the rescaling property in Remark 2.19. Let us write
\[ (R_j^{-1} \cdot \pi)(\mathcal{R}) = \hat{\mathcal{R}} \delta(R_j^{-1} \cdot \pi). \]

Using the functional calculus for \( \mathcal{R} \), we obtain
\[ \forall r > 0, \forall x \in G, \quad \mathcal{R}(r^\nu \mathcal{R}) \delta(x) = r^{-Q} \mathcal{R}(\mathcal{R}) \delta(r^{-1} \cdot x), \]

that is equivalent to say that
\[ \forall r > 0, \forall x \in G, \quad (1 + r^\nu \mathcal{R})^{-\frac{Qa}{2\nu}} \delta(x) = r^{-Q} (1 + \mathcal{R})^{-\frac{Qa}{2\nu}} \delta(r^{-1} \cdot x). \]

Taking the Fourier transform in both sides of this equality, and using the functional calculus for Rockland operators, we obtain
\[ \forall r > 0, \forall \pi \in \hat{G}, \quad (1 + r^\nu \pi(\mathcal{R}))^{-\frac{Qa}{2\nu}} = \mathcal{F}[r^{-Q} (1 + \mathcal{R})^{-\frac{Qa}{2\nu}} \delta(r^{-1} \cdot \cdot)](\pi) \]

Thus, we have
\[ \forall r > 0, \forall \pi \in \hat{G}, \quad (1 + r^\nu \pi(\mathcal{R}))^{-\frac{Qa}{2\nu}} = \mathcal{F}[(1 + \mathcal{R})^{-\frac{Qa}{2\nu}} \delta(\cdot)](r \cdot \pi). \]  

(3.20)

By taking \( r = R_j^{-1} \) in (3.20) we have that
\[ \forall j, \forall \pi \in \hat{G}, \quad (1 + R_j^{-1} \cdot \pi)(\mathcal{R})^{-\frac{Qa}{2\nu}} = \mathcal{F}[(1 + \mathcal{R})^{-\frac{Qa}{2\nu}} \delta(J^{-1} \cdot \pi)](1 + (R_j^{-1} \cdot \pi)(\mathcal{R}))^{-\frac{Qa}{2\nu}}. \]

Note that
\[ R_j^{-Q} \int_\hat{G} \| (1 + (R_j^{-1} \cdot \pi)(\mathcal{R}))^{-\frac{Qa}{2\nu}} \hat{\psi}_j(\pi) \|^2_{HS} d\pi = R_j^{-Q} \int_\hat{G} \| (1 + R_j^{-1} \cdot \pi)(\mathcal{R})^{-\frac{Qa}{2\nu}} \hat{\psi}_j(\pi) \|^2_{HS} d\pi \]
\[ = R_j^{-Q} R_j^{Qa} \int_\hat{G} \| (R_j^{-1} + \pi(\mathcal{R}))^{-\frac{Qa}{2\nu}} \hat{\psi}_j(\pi) \|^2_{HS} d\pi. \]

Now, let us prove that for \( a = R_j^\nu \) and \( b = \frac{Qa}{2\nu} \), and \( \tau(\pi) := \hat{\psi}_j(\pi) \), we have the inequality:
\[ \| \pi(a + \mathcal{R})^{-b} \tau(\pi) \|^2_{HS} \leq \| \pi(\mathcal{R})^{-b} \tau(\pi) \|^2_{HS}. \]  

(3.21)

So, let us denote by \( \{dE_\pi(\mathcal{R})(\lambda)\}_{\lambda > 0} \) the spectral measure associated to the operator \( \pi(\mathcal{R}) \). If \( B_\pi = \{e_{\pi,k}\}_{k=1}^\infty \) is a basis of the representation space \( H_\pi \), then,
\[ \| \pi(a + \mathcal{R})^{-b} \tau(\pi) \|^2_{HS} = \sum_{k=1}^\infty \| \pi(a + \mathcal{R})^{-b} \tau(\pi)e_{\pi,k} \|^2_{H_\pi} \]
\[ = \sum_{k=1}^\infty \left\| \int_0^\infty (a + \lambda)^{-b} dE_\pi(\mathcal{R})(\lambda) \tau(\pi)e_{\pi,k} \right\|^2_{H_\pi} \]
\[ = \sum_{k=1}^\infty \int_0^\infty (a + \lambda)^{-2b} d\| E_\pi(\mathcal{R})(\lambda) \tau(\pi)e_{\pi,k} \|^2_{H_\pi} \]
\[ \leq \sum_{k=1}^\infty \int_0^\infty \lambda^{-2b} d\| E_\pi(\mathcal{R})(\lambda) \tau(\pi)e_{\pi,k} \|^2_{H_\pi}. \]
\[
\sum_{k=1}^{\infty} \left\| \int_0^\infty \lambda^{-b} dE_{\pi(R)}(\lambda) \tau(\pi) e_{\pi,k} \right\|_{H_{\pi}}^2
\]
\[
= \sum_{k=1}^{\infty} \left\| \pi(R)^{-b} \tau(\pi) e_{\pi,k} \right\|_{H_{\pi}}^2
\]
\[
= \left\| \pi(R)^{-b} \tau(\pi) \right\|_{HS}^2,
\]
as desired. So, we can estimate
\[
R_j^{-Q} \int_G \left\| (1 + (R_j^{-1} \cdot \pi(R))^{-Q} \widehat{\psi_j}(\pi)) \right\|_{HS}^2 d\pi \lesssim R_j^{-Q} R_j^{QQ/2} \int_G \left\| \pi(R)^{-Q} \widehat{\psi_j}(\pi) \right\|_{HS}^2 d\pi
\]
\[
= R_j^{-Q(1-\theta)} \int_G \left\| \pi(R)^{-Q} \widehat{\psi_j}(\pi) \right\|_{HS}^2 d\pi
\]
\[
= R_j^{-Q(1-\theta)} \| \mathcal{R} \cdot \Delta \psi_j \|_{L^2(G)}^2
\]
\[
\lesssim |I_j|^{-1},
\]
where we have used (3.19) and the estimate $|I_j| \sim R_j^{Q(1-\theta)}$ because $2R_j^{1-\theta} = \text{diam}(I_j)$. Thus, we deduce that
\[
\int_G |G_1(x)|^2 dx \lesssim \sum_j \alpha^2 \gamma^2 |I_j|^2 \| \phi_j \ast k_0 \|_{L^2}^2 \lesssim \sum_j \alpha^2 \gamma^2 |I_j|^2 |I_j|^{-1} = \alpha^2 \gamma^2 \sum_j |I_j| \lesssim \alpha \gamma \| f \|_{L^1} \lesssim \gamma \| f \|_{L^1},
\]
in view of (3.4). Consequently, we can take $F_2 := G_2$, $F_1 := G_1$ and $F_1^j := G_1^j$. Thus, the proof of Lemma 3.2 is complete as well as the proof of Theorem 1.1. \[\square\]

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