GLOBAL GORENSTEIN DIMENSIONS

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Dedicated to our Advisor Salah-Eddine Kabbaj.

Abstract. In this paper, we prove that the global Gorenstein projective dimension of a ring $R$ is equal to the global Gorenstein injective dimension of $R$, and that the global Gorenstein flat dimension of $R$ is smaller than the common value of the terms of this equality.

1. Introduction

Throughout this paper, $R$ denotes a non-trivial associative ring with identity, and all modules are, if not specified otherwise, left $R$-modules. All the results, except Proposition 2.6, are formulated for left modules and the corresponding results for right modules hold as well. For an $R$-module $M$, we use $\text{pd}_R(M)$, $\text{id}_R(M)$, and $\text{fd}_R(M)$ to denote, respectively, the classical projective, injective, and flat dimension of $M$. We use $\text{l.gldim}(R)$ and $\text{r.gldim}(R)$ to denote, respectively, the classical left and right global dimension of $R$, and $\text{wgldim}(R)$ to denote the weak global dimension of $R$. Recall that the left finitistic projective dimension of $R$ is the quantity $\text{l.FPD}(R) = \sup\{\text{pd}_R(M) \mid M \text{ is an } R\text{-module with } \text{pd}_R(M) < \infty\}$.

Furthermore, we use $\text{Gpd}_R(M)$, $\text{Gid}_R(M)$, and $\text{Gfd}_R(M)$ to denote, respectively, the Gorenstein projective, injective, and flat dimension of $M$ (see [3, 4, 8]). The main result of this paper is an analog of a classical equality that is used to define the global dimension of $R$, see [12, Theorems 9.10]. For Noetherian rings the following theorem is proved in [4, Theorem 12.3.1].

Theorem 1.1. The following equality holds:

$$\sup\{\text{Gpd}_R(M) \mid M \text{ is an } R\text{-module}\} = \sup\{\text{Gid}_R(M) \mid M \text{ is an } R\text{-module}\}.$$  

We call the common value of the quantities in the theorem the left Gorenstein global dimension of $R$ and denote it by $\text{l.Ggldim}(R)$. Similarly, we set

$$\text{l.wGgldim}(R) = \sup\{\text{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}$$

and call this quantity the left weak Gorenstein global dimension of $R$.

Corollary 1.2. The following inequalities hold:

1. $\text{l.wGgldim}(R) \leq \sup\{\text{l.Ggldim}(R), \text{r.Ggldim}(R)\}$.
2. $\text{l.FPD}(R) \leq \text{l.Ggldim}(R) \leq \text{l.gldim}(R)$.
3. $\text{l.wGgldim}(R) \leq \text{wgldim}(R)$.

Equalities hold in (2) and (3) if $\text{wgldim}(R) < \infty$.

The theorem and its corollary are proved in Section 2.

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2. PROOFS OF THE MAIN RESULTS

The proof uses the following results:

**Lemma 2.1.** If \( \sup \{ \text{Gpd}_R(M) \mid M \text{ is an } R \text{-module} \} < \infty \), then, for a positive integer \( n \), the following are equivalent:

1. \( \sup \{ \text{Gpd}_R(M) \mid M \text{ is an } R \text{-module} \} \leq n \);
2. \( \text{id}_R(P) \leq n \) for every \( R \)-module \( P \) with finite projective dimension.

**Proof.** Use [6, Theorem 2.20] and [12, Theorem 9.8]. □

The proof of the main theorem depends on the notions of strong Gorenstein projectivity and injectivity, which were introduced in [1] as follows:

**Definition 2.2.** ([1, Definition 2.1]). An \( R \)-module \( M \) is called strongly Gorenstein projective, if there exists an exact sequence of projective \( R \)-modules

\[
P = \cdots \rightarrow P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots
\]

such that \( M \cong \text{Ker } f \) and such that \( \text{Hom}_R(\_, Q) \) leaves the sequence \( P \) exact whenever \( Q \) is a projective \( R \)-module.

Strongly Gorenstein injective modules are defined dually.

**Remark 2.3.** It is easy to see that an \( R \)-module \( M \) is strongly Gorenstein projective if and only if there exists a short exact sequence of \( R \)-modules \( 0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0 \), where \( P \) is projective, and \( \text{Ext}^i_R(M, Q) = 0 \) for some integer \( i > 0 \) and for every \( R \)-module \( Q \) with finite projective dimension (or for every projective \( R \)-module \( Q \)).

Strongly Gorenstein injective modules are characterized in similar terms.

The principal role of these modules is to characterize the Gorenstein projective and injective modules, as follows:

**Lemma 2.4.** ([1, Theorems 2.7]). An \( R \)-module is Gorenstein projective (resp., injective) if and only if it is a direct summand of a strongly Gorenstein projective (resp., injective) \( R \)-module.

**Proof of Theorem[7]** For every integer \( n \) we need to show:

\( \text{Gpd}_R(M) \leq n \) for every \( R \)-module \( M \) \( \iff \) \( \text{Gid}_R(M) \leq n \) for every \( R \)-module \( M \).

We only prove the direct implication; the converse one has a dual proof.

Assume first that \( M \) is strongly Gorenstein projective. By Remark[2,3] there is a short exact sequence \( 0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0 \) with \( P \) is projective. The Horseshoe Lemma, see [10, Remark page 187], gives a commutative diagram

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* In [1] the base ring is assumed to be commutative. However, for the result needed here, one can show easily that this assumption is not necessary.
where \( I_i \) is injective for \( i = 0, \ldots, n - 1 \). Since \( P \) is projective, \( \text{id}_R(P) \leq n \) (by Lemma 2.1), hence \( E_n \) is injective. On the other hand, from \( [7] \) Theorem 2.2, \( \text{pd}_E(E) \leq n \) for every injective \( R \)-module \( E \). Then, \( \text{Ext}^i_R(E, I_0) = 0 \) for all \( i \geq n + 1 \). Then, from Remark 2.5 \( I_n \) is strongly Gorenstein injective, and so \( \text{GId}_R(M) \leq n \). This implies, from \( [6] \) Proposition 2.19, that \( \text{GId}_R(G) \leq n \) for any Gorenstein projective \( R \)-module \( G \), since every Gorenstein projective \( R \)-module is a direct summand of a strongly Gorenstein projective \( R \)-module (Lemma 2.4).

Finally, consider an \( R \)-module \( M \) with \( \text{Gpd}_R(M) \leq m \leq n \). We can assume that \( \text{Gpd}_R(M) \neq 0 \). Then, there exists a short short exact sequence \( 0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0 \) such that \( N \) is Gorenstein projective and \( \text{Gpd}_R(K) \leq m - 1 \) \( [6] \) Proposition 2.18]. By induction, \( \text{GId}_R(K) \leq n \) and \( \text{GId}_R(N) \leq n \). Therefore, using \( [6] \) Theorems 2.22 and 2.25] and the long exact sequence of \( \text{Ext} \), we get that \( \text{GId}_R(M) \leq n \).

**Proof of Corollary 1.2** (1). We may assume that \( \text{sup}(l.\text{Ggldim}(R), r.\text{Ggldim}(R)) < \infty \). Then, the character module, \( I^* = \text{Hom}_R(I, \mathbb{Q}/\mathbb{Z}) \), of every injective right \( R \)-module \( I \) has finite projective dimension (by \( [7] \) Theorem 2.2) \( [12] \) Theorem 3.52]). Then, similarly to the proof of \( [6] \) Proposition 3.4], we get that every Gorenstein projective \( R \)-module is Gorenstein flat. Therefore, \( l.\text{wGgldim}(R) \leq \text{sup}(l.\text{Ggldim}(R), r.\text{Ggldim}(R)) \).

(2) and (3). The inequality \( l.\text{FPD}(R) \leq l.\text{Ggldim}(R) \) follows from \( [6] \) Theorem 2.28].

The inequalities \( l.\text{Ggldim}(R) \leq l.\text{gldim}(R) \) and \( l.\text{wGgldim}(R) \leq \text{wgdim}(R) \) hold true since every injective (resp., flat) module is Gorenstein projective (resp., Gorenstein flat).

If \( \text{wgdim}(R) < \infty \), then, from \( [10] \) Corollary 3], \( l.\text{FPD}(R) = l.\text{Ggldim}(R) = l.\text{gldim}(R) \) and, from \( [11] \) Corollary 3.8], \( l.\text{wGgldim}(R) = \text{wgdim}(R) \).

**Remark 2.5.** It is well-known that there are examples of rings for which the left and right global dimensions differ (see \( [8] \) pages 74-75] \( [9] \)). Then, by Corollary 1.2 the same examples show that there are also examples of rings for which the left and right Gorenstein global dimensions differ. However, as the classical case \( [12] \) Corollary 9.23], we have \( l.\text{Ggldim}(R) = r.\text{Ggldim}(R) \) if \( R \) is Noetherian \( [4] \) Theorem 12.3.1].

For the case where \( l.\text{Ggldim}(R) = 0 \) or \( r.\text{Ggldim}(R) = 0 \), we have the following result which is \( [2] \) Theorem 2.2] in non-commutative setting. Recall that a ring is called quasi-Frobenius, if it is Noetherian and both left and right self-injective (see \( [11] \)).

**Proposition 2.6.** The following are equivalent:

1. \( R \) is quasi-Frobenius;
2. \( l.\text{Ggldim}(R) = 0 \);
(3) \( r \cdot \text{Ggldim}(R) = 0 \).

Proof. The implications \( 1 \Rightarrow 2 \) and \( 1 \Rightarrow 3 \) are well-known (see, for example, [4] Exercise 5, page 257).

The implication \( 2 \Rightarrow 1 \) follows from Lemma 2.1 and Faith-Walker Theorem [11, Theorem 7.56]. The implication \( 3 \Rightarrow 1 \) is proved similarly. □

We finish with a generalization of a result of Iwanaga, see [4, Proposition 9.1.10].

**Corollary 2.7.** Assume that \( l \cdot \text{Ggldim}(R) \leq n \) holds for some non-negative integer \( n \). If for an \( R \)-module \( M \) one of the numbers \( \text{pd}_R(M), \text{id}_R(M), \) or \( \text{fd}_R(M) \) is finite, then all of them are smaller or equal to \( n \).

Proof. If \( \text{pd}_R(M) \) is finite, then [6, Proposition 2.27] and the assumption give \( \text{pd}_R(M) = \text{Gpd}_R(M) \leq n \). The argument for \( \text{id}_R(M) < \infty \) is similar. Finally, Corollary 1.2(2) and the assumption give \( l \cdot \text{FPD}(R) \leq n \), and then \( \text{fd}_R(M) < \infty \) implies \( \text{pd}_R(M) < \infty \) by [10, Proposition 6]. □

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