Persistent Homology of Morse Decompositions in Combinatorial Dynamics

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Abstract

We investigate combinatorial dynamical systems on simplicial complexes considered as finite topological spaces. Such systems arise in a natural way from sampling dynamics and may be used to reconstruct some features of the dynamics directly from the sample. We study the homological persistence of Morse decompositions of such systems, an important descriptor of the dynamics, as a tool for validating the reconstruction. Our framework can be viewed as a step toward extending the classical persistence theory to “vector cloud” data. We present experimental results on two numerical examples.

1 Introduction

The aim of this research is to provide a tool for studying the topology of Morse decompositions of sampled dynamics, that is dynamics known only from a sample. Morse decomposition of the phase space of a dynamical system consists of a finite collection of isolated invariant sets, called Morse sets, such that the dynamics outside the Morse sets is gradient-like. This fundamental concept introduced in 1978 by Conley \cite{Conley1978} generalizes classical Morse theory to non-gradient dynamics. It has become an important tool in the study of the asymptotic behavior of flows, semi-flows and multivalued flows (see \cite{Conley1978, Conley1981, Mischaikow2008} and the references

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Morse decompositions and the associated Conley-Morse graphs \cite{3, 8} provide a global descriptor of dynamics. This makes them an excellent object for studying the dynamics of concrete systems. In particular, they have been recently applied in such areas as mathematical epidemiology \cite{18}, mathematical ecology \cite{3, 8, 17} or visualization \cite{29, 30}.

Unlike the case of theoretical studies, the methods of classical mathematics do not suffice in most problems concerning concrete dynamics. This is either because there is no analytic solution to the differential equation describing the system or, even worse, the respective equation is only vaguely known or not known at all. In the first case the dynamics is usually studied by numerical experiments. In some cases this may suffice to make mathematically rigorous claims about the system \cite{22}. In the latter case one can still get some insight into the dynamics by collecting data from physical experiments or observations, for instance as a time series \cite{1, 17, 23}. In both cases the study is based on a finite and not precise sample, typically in the form of a data set. The inaccuracy in the data may be caused by noise, experimental error, or numerical error. Consequently, it may distort the information gathered from the data, raising the question whether the information is trustworthy. One of possible remedies is to study the stability of the information with respect to perturbation of the data. This approach to Morse decompositions constructed from samples is investigated in \cite{29} in the setting of piecewise constant vector fields on triangulated manifold surfaces. The outcome of the algorithm proposed in \cite{29} is the Morse merge tree which encodes the zero-dimensional persistence under perturbations of individual Morse sets in the Morse decomposition.

In this paper we study general persistence of Morse decompositions in combinatorial dynamics, not necessarily related to perturbations. By combinatorial dynamics we mean a multivalued map acting on a simplicial complex treated as a finite topological space. This general setting may be applied either to a finite sample of the action of a map on a subspace of $\mathbb{R}^d$ \cite{5, 12} or to a combinatorial vector field \cite{15} and its generalization multivector field \cite{25}. On a theoretical level, the results presented in this paper may be generalized to arbitrary finite $T_0$ topological spaces. From the viewpoint of applications, the finite topological space may be a collection of cells of a simplicial, cubical, or general cellular complex approximating a cloud of sampled points. The multivalued map may be constructed either from the action of a given map on the set of a sample points or from the available vectors of a sampled vector field. The framework for persistence of Morse decompositions in the combinatorial setting developed in this paper is general and may be applied to many different problems.

The language of finite topological spaces (see Appendix A.1) enables us to emphasize differences between the classical and combinatorial dynamics. These differences matter when the available data set is sparse and is difficult to be enriched. In particular, in the classical setting the phase space has Hausdorff topology ($T_2$ topology) (see Appendix A.1) and the Morse sets are compact. Hence, Morse sets are isolated since they are always disjoint. To achieve such isolation in sampled dynamics, one needs data not only in the Morse sets but
also between the Morse sets. This may be a problem if the available data set is sparse and is difficult to be enriched. Fortunately, the finite topological spaces in general are not $T_2$. Every set is compact but compactness does not imply closedness. Consequently, Morse sets need not be closed and may be adjacent to one another. By allowing adjacent Morse sets we can detect finer Morse decompositions. We still can disconnect them by modifying slightly the topology of the space without changing the topology of the Morse sets.

2 Combinatorial dynamics

Simplicial complexes as finite topological spaces. We recall that an abstract simplicial complex [26, Sec. 1.3] is a family $K$ of simplices, that is non-empty subsets of a finite set of vertices, such that any non-empty subset $\sigma$ of a simplex $\tau \in K$, called a face of $\tau$, is in $K$. We begin with an observation that $K$ may be viewed both as a poset and as a finite topological space. The natural partial order on $K$ is the face relation $\sigma \leq \tau$ meaning that $\sigma$ is a face of $\tau$ (also phrased $\tau$ is a coface of $\sigma$). The natural topology on $K$, called Alexandrov topology, is the topology $T_K$ so that $U \subseteq K$ is open in $T_K$ if and only if $U$ is upper with respect to $\leq$, that is, $\sigma \leq \tau$ and $\sigma \in U$ imply $\tau \in U$. Actually, this is a special case of Alexandrov Theorem [2] (see Theorem 11 in Appendix) on a correspondence between finite posets and finite $T_0$ topological spaces. We note that $T_K$ is non-Hausdorff unless $K$ consists of vertices only. It is easy to see that a set $A \subseteq K$ is closed in the Alexandrov topology if and only if all faces of any element of $A$ are also in $A$. Hence, the closure of $A$, denoted $\text{cl} A$, is the collection of all faces of elements in $A$. Since we use more than one topology on the same set, in case of ambiguity, we write a topological space as a pair $(X, T)$ consisting of the space and the selected topology on that space.

The abstract simplicial complex $K$ with its generally non-Hausdorff Alexandrov topology should not be confused with the polytope $|K|$ of a geometric realization of $K$ ([26, Sec. 1.2,1.3]). This polytope is a subset of the Euclidean space with metric topology. It is unique up to a homeomorphism. An open cell $\hat{\sigma}$ associated with a simplex $\sigma \in K$ is the set of points $x$ in the polytope $|K|$ whose barycentric coordinates $t_v(x)$ are strictly positive for every vertex $v \in \sigma$. The solid of a set of simplices $A \subseteq K$ is $|A| := \bigcup \{ \hat{\sigma} | \sigma \in A \}$. Note that if $A$ is a subcomplex of $K$, then its solid is homeomorphic to the polytope of a geometric realization of $A$. This is why we use $|\cdot|$ to denote both solids and polytopes. It is not difficult to verify that $A \subseteq K$ is open (respectively closed) in the Alexandrov topology if and only if its solid is open (respectively closed) in $|K|$.

Combinatorial dynamical systems. By a combinatorial dynamical system on $K$ (cds in short) we mean a multivalued map $F : K \rightharpoonup K$, that is a map which sends each simplex in
$K$ into a family of simplices in $K$. A solution of $F$ in $A \subseteq K$ is a partial map $\rho : \mathbb{Z} \to A$ whose domain, denoted $\text{dom} \rho$, is either the set of all integers or a finite interval of integers and for any $i, i + 1 \in \text{dom} \rho$ the inclusion $\rho(i + 1) \in F(\rho(i))$ holds. The solution passes through $\sigma \in K$ if $\sigma = \rho(i)$ for some $i \in \text{dom} \rho$. The solution $\rho$ is full if $\text{dom} \rho = \mathbb{Z}$, otherwise it is partial. In the latter case, if $\text{dom} \rho = \mathbb{Z} \cap [m, n]$, then $\rho(m)$ and $\rho(n)$ are called respectively the left and right endpoint of $\rho$. The set $A$ is invariant if for every $\sigma \in A$ there exists a full solution in $A$ passing through $\sigma$.

![Figure 1](image.png)

**Figure 1:** Left: A map $f : [0, 1] \ni x \mapsto 3x^2 - 2x^3 \in [0, 1]$ and a sample of 20 points with a large Gaussian noise. A constructed cds with threshold $\mu = 0.3$ (Middle) and $\mu = 0.4$ (Right).

The cds $F$ may be viewed as a digraph $G_F$ whose vertices are simplices in $K$ with a directed edge from $\sigma$ to $\tau$ if and only if $\tau \in F(\sigma)$. Figure 1(middle) and Figure 1(right) show digraph presentations of two cds’s on $K$ consisting of three vertices $A$, $B$, $C$ and two edges $AB$, $BC$. The digraph interpretation of a cds means that some concepts in dynamics may be translated into concepts in digraphs and vice versa. In this translation a solution to $F$ in $A \subseteq K$ corresponds to a walk in $G_F$ through vertices in $A$ and the set $A$ is invariant if every vertex in $A$ is incident to a bi-infinite walk in $G_F$ through vertices in $A$. For instance, in Figure 1(middle), the set $\{AB, B, BC\}$ and all it subsets are invariant. However, $F$ is more than just the digraph $G_F$ because $K$, the set of vertices of $G_F$, is a topological space. In particular, the concept of isolating neighborhood defined in the next section cannot be formulated in the language of digraphs only.

**A cds from a sampled map.** The two cds’s in Figure 1(middle,right) are constructed from a noisy sample of a map $f : [0, 1] \to [0, 1]$ presented in Figure 1(left). We will explain now such a construction in the case of a map $f : X \to X$ on a fixed polytope $X := |K| \subseteq \mathbb{R}^d$. 

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Let $K_{\text{top}}$ denote the family of toplexes in $K$, that is, simplices with no non-trivial coface in $K$. For this construction we assume that each toplex is $d$-dimensional, that is each $\tau \in K_{\text{top}}$ has precisely $d + 1$ elements. Moreover, we assume that $X$ is convex in $\mathbb{R}^d$. For $A \subseteq K$ we write $\text{co} A := \bigcap A$ where $A := \{ B \mid A \subseteq B \subseteq K, |B| \text{ is convex in } \mathbb{R}^d \}$. In words, $\text{co} A$ is the intersection of all supersets of $A$ in $K$ whose solid is convex in $\mathbb{R}^d$. Consider a noisy sample of $f$ by which we mean a non-empty collection of points $\{ (x_i, y_i) \}_{i=1}^n$ satisfying $x_i, y_i \in X$ and $y_i$ equals $f(x_i)$ perturbed by some noise. For toplexes $\tau, \bar{\tau}$ let $n_{\tau, \bar{\tau}}$ denote the number of pairs $(x_i, y_i)$ such that $x_i \in \tau$ and $y_i \in \bar{\tau}$ and let $n_{\text{max}}$ be the maximum of all these numbers. Given a frequency threshold $\mu$ we define a cds $F_\mu : K \rightarrow K$ by $F_\mu(\sigma) := \text{co} \bigcup_{\tau \in K_{\text{top}}, \sigma \leq \tau} \{ \bar{\tau} \in K_{\text{top}} \mid \frac{n_{\tau, \bar{\tau}}}{n_{\text{max}}} \geq \mu \}$. The cds in Figure 1(middle) is $F_{0.3}$ and the cds in Figure 1(right) is $F_{0.4}$.

Figure 2: Left: A cloud of vectors. Middle: The associated combinatorial multivector field. Right: The associated cds represented as a digraph.

**Combinatorial multivector fields.** When the sampled dynamics constitutes a flow, that is time is continuous, the sampled data often consists of a cloud of vectors. In this case the construction of cds is done in two steps. In the first step the cloud of vectors is transformed into a combinatorial vector field in the sense of Forman [14, 15] or its generalized version of combinatorial multivector fields [25]. This step is delicate. We discuss it in Appendix A.3. In the second step, the combinatorial multivector field is transformed into a cds. In order to explain the second step, we introduce some definitions. We say that $A \subseteq K$ is convex if for any $\sigma_1, \sigma_2 \in A$ and $\tau \in K$ the relations $\sigma_1 \preceq \tau$ and $\tau \preceq \sigma_2$ imply $\tau \in A$. We define a multivector as a convex subset of $K$ and a combinatorial multivector field on $K$ ($\text{cmf}$ in short) as a partition $V$ of $K$ into multivectors. Note that this definition is slightly more general than the one in [25] and encompasses the combinatorial vector field...
of Forman as a special case. Given a cmf $V$, we denote by $[\sigma]_V$ the unique $V$ in $V$ so that $\sigma \in V$. We associate with $V$ a cds $F_V : K \to K$ given by $F_V(\sigma) := \text{cl} \sigma \cup [\sigma]_V$. Also, this formulation is slightly more general than [25]. In particular, each $\sigma \in K$ is a **fixed point** of $F_V$, that is, $\sigma \in F_V(\sigma)$. This may look like a drawback but actually it simplifies the theory and allows detecting and eliminating spurious fixed points by the triviality of their Conley index [25, 19].

A cds from a sampled vector field. Figure 2(left) presents a toy example of a cloud of vectors. It consists of four vectors marked red at four points $P$, $Q$, $R$, $S$. Triangles $PQR$, $QRS$ and its faces constitute the simplicial complex $K$. The associated multivector field $V$ on $K$ consists of multivectors $\{P, PR\}, \{R, QR\}, \{Q, PQ\}, \{PQR\}, \{S, RS, QS, QRS\}$. It is indicated in Figure 2(middle) by orange arrows between centers of mass of simplices sharing the same multivector. The singleton $\{PQR\}$ is marked with an orange circle. The associated cds $F_V$ presented as a digraph is in Figure 2(right).

3 Persistence of Morse decompositions

Isolated invariant sets and Morse decompositions. The closed set $N \subseteq K$ is an **isolating neighborhood** for an invariant set $S \subseteq K$ if $S$ is contained in the interior of $N$ and any partial solution in $N$ with endpoints in $S$ has all values in $S$. If such an isolating neighborhood for $S$ exists, we say that $S$ is an **isolated invariant set**. The invariant set $\{B\}$ in Figure 1(middle) is not an isolated invariant set, because $(B, AB, B)$ is a partial solution in every closed set containing $\{B\}$ in its interior, and its endpoints are in $\{B\}$. The invariant sets $\{BC\}$ and $\{AB, B\}$ are both isolated invariant sets and $\{A, AB, B, BC, C\}$ is an isolating neighborhood for both. The **maximal invariant set** of $F$, denoted $S(F)$, is the set of all simplices $\sigma \in K$ such that there exists a full solution of $F$ in $K$ passing through $\sigma$.

It is straightforward to observe that $S(F)$ is invariant and $K$ is an isolating neighborhood for $S(F)$. Therefore, $S(F)$ is an isolated invariant set. Note that the maximal invariant set $S(F_V)$ for a cmf $V$ is always the whole $K$, because for each $\sigma \in K$ we have $\sigma \in \text{cl} \sigma \subseteq F_V(\sigma)$. This is visible in Figure 2(right) as a loop at every vertex. In contrast, $A$ does not belong to the maximal invariant set in Figure 1(right).

A **connection** from an isolated invariant set $S_1$ to an isolated invariant set $S_2$ is a partial solution with left endpoint in $S_1$ and right endpoint in $S_2$. A family $\mathcal{M}$ consisting of mutually disjoint, non-empty isolated invariant subsets of an isolated invariant set $S$ is a **Morse decomposition** of $S$ if $\mathcal{M}$ admits a partial order $\leq$ such that any connection between elements in $\mathcal{M}$ either has all values in a single element of $\mathcal{M}$ or it originates in $M \in \mathcal{M}$ and terminates in $M'$ such that $M > M'$. If $S$ is not mentioned explicitly, we mean a Morse decomposition of $S(F)$. The elements of $\mathcal{M}$ are called **Morse sets**. Although the definitions
of isolated invariant set and Morse decomposition require topology, there is an important case when they correspond to purely graph-theoretic concepts. An isolated invariant set is minimal if it admits no non-trivial Morse decomposition that is no Morse decomposition consisting of more than one Morse set. A Morse decomposition is minimal if each of its Morse sets is minimal. The following theorem shows that the minimal Morse decomposition of $F$, denoted as $\mathcal{M}(F)$, is unique and consists of the strongly connected components of $G_F$.

**Theorem 1** The family of all strongly connected components of $G_F$ is the unique minimal Morse decomposition of $S(F)$.

**Proof:** Let $\mathcal{S}$ be the family of all strongly connected components of $G_F$. We will show that $K$ is an isolating neighborhood for any $S \in \mathcal{S}$. Obviously, $K$ is closed and $S$ is contained in the interior of $K$. Moreover, any partial solution with endpoints in $S$ must have all values in $S$, because $S$ is a strongly connected component of $G_F$. Hence, each $S \in \mathcal{S}$ is an isolated invariant set and clearly it is a minimal isolated invariant set. For $S_1, S_2 \in \mathcal{S}$ we write $S_1 \geq S_2$ if there exists a connection from $S_1$ to $S_2$. Since $\mathcal{S}$ consists of strongly connected components, this defines a partial order on $\mathcal{S}$. Let $\rho$ be a connection from $S_1$ to $S_2$. Then $S_1 \geq S_2$. Moreover, if the values of $\rho$ are not contained in a single element of $\mathcal{S}$, then $S_1 \neq S_2$. Thus, $S_1 > S_2$. This proves that $\mathcal{S}$ is a minimal Morse decomposition. Assume that $\mathcal{S}'$ is another minimal Morse decomposition and $S' \in \mathcal{S}'$. Then, $S'$ is strongly connected as a subgraph of $G_F$. Hence, $S'$ is contained in a Morse set $S \in \mathcal{S}$. Actually, $S'$ must be equal $S$, because otherwise $S'$ would admit a connection from $S'$ to $S'$ and not entirely contained in $S'$. This proves the uniqueness. □

The minimal Morse decomposition of cds in Figure 1 (middle) consists of two Morse sets: \{AB, B\} and \{BC\} with \{AB, B\} > \{BC\}. The minimal Morse decomposition of cds in Figure 1 (right) consists of three Morse sets: \{AB\}, \{B\} and \{BC\} with \{B\} > \{BC\} and \{B\} > \{AB\}. The minimal Morse decomposition of cds in Figure 2 (right) consists of three isolated invariant sets: $M_1 := \{P, Q, R, PQ, PR, QR\}$, $M_2 := \{S, RS, QS, QRS\}$ and $M_3 := \{PQR\}$ with $M_3 > M_1$ and $M_2 > M_1$.

**Disconnecting topology.** Assume $A$ is a finite family of mutually disjoint non-empty sets and $\mathcal{T}$ is a topology on $\bigcup A$. We say that $A$ is disconnected in $\mathcal{T}$ if each set $A \in A$ is open in the topology $\mathcal{T}$. In the case of a classical Morse decomposition, the union of all Morse sets is always disconnected in the topology induced from the space. This is because Morse sets are disjoint by definition and in this case also compact. But, in finite topological spaces the family of Morse sets generally is not disconnected. Thus, we need a method to disconnect Morse sets. We achieve this by slightly modifying the topology. To explain this, we need the following notation and theorem.
Given a family $\mathcal{A}$ of subsets of a set $X$, we denote the family of unions of elements in $\mathcal{A}$ by $\mathcal{A}^\ast$. If $\mathcal{B}$ is another such family, we write $\mathcal{A} \cap \mathcal{B}$ for the family of intersections of every set in $\mathcal{A}$ with every set in $\mathcal{B}$. We say that $\mathcal{A}$ is inscribed in $\mathcal{B}$ and write $\mathcal{A} \sqsubseteq \mathcal{B}$ if for every $A \in \mathcal{A}$, there exists a $B \in \mathcal{B}$ such that $A \subseteq B$.

In order to shorten the notation we will also write $\langle \mathcal{A} \rangle$ for the union $\cup \mathcal{A}$ of all the sets in $\mathcal{A}$. Note that if $\mathcal{A} \subseteq X$ and $\mathcal{T}$ is a topology on $X$, then the topology induced by $\mathcal{T}$ on $A$ is $A \cap \mathcal{T} := \{A\} \cap \mathcal{T}$.

**Theorem 2** Assume $(X, \mathcal{T})$ is an arbitrary topological space and $\mathcal{A}$ is a finite family of mutually disjoint, non-empty subsets of $X$. Then $\mathcal{T}_\mathcal{A} := (\mathcal{A} \cap \mathcal{T})^\ast$ is a topology on $\langle \mathcal{A} \rangle$. Moreover,

(i) if $\mathcal{T}$ is a $T_0$ topology, then so is $\mathcal{T}_\mathcal{A}$,

(ii) for every $A \in \mathcal{A}$, the topology induced on $A$ by $\mathcal{T}$ coincides with $\mathcal{T}_\mathcal{A}$,

(iii) the family $\mathcal{A}$ is $\mathcal{T}_\mathcal{A}$-disconnected,

(iv) if additionally $\langle \mathcal{A} \rangle = X$ and each set in $\mathcal{A}$ is $\mathcal{T}$-connected, then the connected components with respect to $\mathcal{T}_\mathcal{A}$ coincide with the sets in $\mathcal{A}$.

**Proof:** We will show that $\mathcal{A} \cap \mathcal{T}$ is a basis (see [27, Section 13]) for some topology on $\langle \mathcal{A} \rangle$. Let $x \in \langle \mathcal{A} \rangle$. There exists an $A \in \mathcal{A}$ such that $x \in A$. Hence, $x \in A = A \cap X \in \mathcal{A} \cap \mathcal{T}$. Assume that $x \in (A \cap U) \cap (B \cap V)$ for some $A, B \in \mathcal{A}$ and $U, V \in \mathcal{T}$. Then $A = B$ and consequently $(A \cap U) \cap (B \cap V) = A \cap (U \cap V) \in \mathcal{A} \cap \mathcal{T}$. This shows that $\mathcal{A} \cap \mathcal{T}$ is indeed a basis. By [27, 13.1] it follows that $\mathcal{T}_\mathcal{A}$ is a topology. Consider $x, y \in \langle \mathcal{A} \rangle$, $x \neq y$. Let $A \in \mathcal{A}$ be such that $y \in A$. Choose an open neighborhood $U \in \mathcal{T}$ of $y$ such that $x \notin U$. Then $y \in U \cap A$ and $x \notin U \cap A$, hence (i) holds. To prove (ii) we need to show that $A \cap \mathcal{T} = A \cap \mathcal{T}_\mathcal{A}$. Obviously $A \cap \mathcal{T} \subseteq A \cap \mathcal{T}_\mathcal{A}$. To prove the opposite inclusion take a $V \in A \cap \mathcal{T}_\mathcal{A}$. This means that there is a $U \in \mathcal{T}_\mathcal{A}$ such that $V = U \cap A$. Then $U = (U_1 \cap A_1) \cup \ldots \cup (U_k \cap A_k)$ for some $U_i \in \mathcal{T}$ and $A_i \in \mathcal{A}$. Let $W := U_1 \cup \ldots \cup U_k \in \mathcal{T}$. Then $V = U \cap A = W \cap A \in A \cap \mathcal{T}$ and (ii) is proved. Property (iii) is obvious, because $A \in \mathcal{A}$ implies $A = A \cap X \in \mathcal{T}_\mathcal{A}$. To prove (iv) assume $A$ is $\mathcal{T}$-connected. It follows from (ii) that $A$ is $\mathcal{T}_\mathcal{A}$-connected. Let $x \in A$. Then $A$ is contained in $[x]_{\mathcal{T}_\mathcal{A}}$, the $\mathcal{T}_\mathcal{A}$-connected component of $x$. This means that $[x]_{\mathcal{T}_\mathcal{A}} = \bigcup \mathcal{A}'$ for some $\mathcal{A}' \subseteq \mathcal{A}$. Since every set in $\mathcal{A}$ is open in $\mathcal{T}_\mathcal{A}$, the family $\mathcal{A}'$ must contain precisely one element. Consequently $A = [x]_{\mathcal{T}_\mathcal{A}}$ and (iv) holds. □

We note that in general $\langle \mathcal{M} \rangle$ is not a subcomplex of the simplicial complex $K$. Therefore, we cannot take simplicial homology of $\langle \mathcal{M} \rangle$. Moreover, we are interested in the special
Theorem 3
Persistence of Morse decompositions.

Particular, every simplicial map respects continuity. Hence, it suffices to show that for any \( f \) to \( f \) is well defined. Since \( f \) is continuous, we have \( \bar{f} \) and \( \bar{f} \) are arbitrary, the set \( \bar{f} \in T \) is open in \( T \). Fortunately, the singular homology makes sense for any topological space, in particular we can consider \( H(\langle M \rangle, T_M) \). In Section 4 we use McCord’s Theorem 21 to show that \( H(\langle M \rangle, T_M) \) may be computed as simplicial homology of a subcomplex of the barycentric subdivision of \( K \).

**Persistence and zig-zag persistence of Morse decompositions.** Consider two simplicial complexes \( K \) and \( K' \) with cds’s \( F \) on \( K \) and \( F' \) on \( K' \) and a map \( f : K \rightarrow K' \) continuous with respect to Alexandrov topologies \( T \) on \( K \) and \( T' \) on \( K' \). By Alexandrov Theorem \( f \) is continuous if and only if it preserves the face relation in \( K \) and \( K' \). In particular, every simplicial map is continuous.

The following theorem lets us define homomorphisms in homology needed to set up persistence of Morse decompositions.

**Theorem 3** Let \( M \) and \( M' \) be Morse decompositions respectively for \( F \) and \( F' \). Assume that \( f \) respects \( M \) and \( M' \) that is \( f(\langle M \rangle) \subseteq \langle M' \rangle \) where \( f(\langle M \rangle) := \{ f(M) \mid M \in M \} \). Then, the map \( f_{M,M'} : (\langle M \rangle, T_M) \ni \sigma \mapsto f(\sigma) \in (\langle M' \rangle, T'_{M'}) \) is well defined and continuous.

**Proof:** Let \( \sigma \in \langle M \rangle \). Then \( \sigma \in M \) for some \( M \in M \). Since \( f \) respects \( M \) and \( M' \), there is an \( M' \in M' \) such that \( f(M) \subseteq M' \). It follows that \( f(\sigma) \in \langle M' \rangle \). Hence, \( f_{M,M'} \) is well defined. Since \( M' \cap T' \) is a basis of topology \( T'_{M'} \), in order to prove continuity it suffices to show that for any \( M' \in M' \) and \( T' \in T' \) the set \( f_{M,M'}^{-1}(M' \cap T') \) is open in \( T_M \). Let \( M_{M'} := \{ M \in M \mid f(M) \subseteq M' \} \). Then \( f^{-1}(M') \cap \langle M \rangle = \langle M_{M'} \rangle \). By continuity of \( f \) we have \( f^{-1}(T') \in T \). Therefore, \( f_{M,M'}^{-1}(M' \cap T') = f^{-1}(M') \cap f^{-1}(T') \cap \langle M \rangle = f^{-1}(T') \cap \langle M_{M'} \rangle = \bigcup \{ M \cap f^{-1}(T') \mid M \in M_{M'} \} \in (\langle M \cap T \rangle^* = T_M \), which completes the proof. \( \square \)

**Corollary 4** The map \( f_{M(F),M(F')} : (\langle \langle M(F) \rangle \rangle, T_{M(F)}) \rightarrow (\langle \langle M(F') \rangle \rangle, T'_{M(F')}) \) is continuous under the assumption that \( f \circ F \subseteq F' \circ f \) that is \( f(F(\sigma)) \subseteq F'(f(\sigma)) \) for any \( \sigma \in K \).

**Proof:** By Theorem 3, it suffices to show that \( f \) respects \( M(F) \) and \( M(F') \). Let \( M \in M(F) \). By Theorem 1 the Morse set \( M \) is a strongly connected component of \( G_F \). Let \( \sigma, \tau \in M \) and let \( \rho \) be a partial solution in \( M \) with endpoints \( \sigma \) and \( \tau \). It follows from the assumption that \( f \circ \rho \) is a solution in \( M(F) \) with endpoints \( f(\sigma) \) and \( f(\tau) \). Since \( \sigma, \tau \in M \) are arbitrary, the set \( f(M) \) must be contained in one strongly connected component of \( G_{F'} \), that is \( f(M) \subseteq M' \) for some \( M' \in M' \). \( \square \)

Assume now that for \( i = 1, 2, \ldots, n \), we have a simplicial complex \( K_i \) with Alexandrov topology \( \mathcal{T}_i \), a cds \( F_i \) on \( K_i \), and a Morse decomposition \( M_i \) of \( F_i \). Let \( \{ f_i : K_i \rightarrow \} \)
be a sequence of continuous maps such that \( f_i \circ F_i \subseteq F_{i+1} \circ f_i \) and \( f_i(\mathcal{M}_i) \sqsubset \mathcal{M}_{i+1} \). Note that by Corollary 4, the latter condition may be dropped if \( \mathcal{M}_i = \mathcal{M}(F_i) \). It follows from Theorem 3 that the maps \( \bar{f}_i := (f_i, \mathcal{M}_i, \mathcal{M}_{i+1}) : (\langle \mathcal{M}_i \rangle, \mathcal{T}_{\mathcal{M}_i}^1) \rightarrow (\langle \mathcal{M}_{i+1} \rangle, \mathcal{T}_{\mathcal{M}_{i+1}}^{i+1}) \) are continuous. Thus, we have homomorphisms induced in singular homology \( H(\bar{f}_i) : H(\langle \mathcal{M}_i \rangle, \mathcal{T}_{\mathcal{M}_i}) \rightarrow H(\langle \mathcal{M}_{i+1} \rangle, \mathcal{T}_{\mathcal{M}_{i+1}}^{i+1}) \). This yields a persistence module

\[
H(\langle \mathcal{M}_1 \rangle, \mathcal{T}_{\mathcal{M}_1}) \xrightarrow{H(\bar{f}_1)} H(\langle \mathcal{M}_2 \rangle, \mathcal{T}_{\mathcal{M}_2}^2) \xrightarrow{H(\bar{f}_2)} \cdots \xrightarrow{H(\bar{f}_{n-1})} H(\langle \mathcal{M}_n \rangle, \mathcal{T}_{\mathcal{M}_n}^n).
\]  

We refer to the persistence diagram of this module as the **persistence diagram of Morse decompositions**. We note that zig-zag persistence diagram of Morse decompositions may be obtained analogously by replacing, whenever appropriate, inclusions \( f_i \circ F_i \subseteq F_{i+1} \circ f_i \) by \( f_i \circ F_i \supseteq F_{i+1} \circ f_i \) and respectively \( \bar{f}_i(\mathcal{M}_i) \sqsubseteq \mathcal{M}_{i+1} \) by \( \mathcal{M}_i \sqsupseteq \bar{f}_i(\mathcal{M}_{i+1}) \).

**Persistence in combinatorial multivector fields.** Let \( V \) be a cmf on a simplicial complex \( K \). We say that \( \mathcal{M} \) is a Morse decomposition of \( V \) if it is a Morse decomposition of the associated cds \( F_V \). We extend this terminology to minimal Morse decompositions. We denote the minimal Morse decomposition of \( V \) by \( \mathcal{M}(V) := \mathcal{M}(F_V) \) and the topology of this Morse decomposition by \( \mathcal{T}_V := \mathcal{T}_{\mathcal{M}(V)} \).

**Theorem 5** Morse decompositions of cmf’s have the following properties.

(i) The minimal Morse decomposition of a cmf \( V \) on \( K \) is a partition of \( K \). In particular, \( \langle \mathcal{M} \rangle = K \).

(ii) Given \( W \), another cmf on \( K \), the family \( V \cap W \) is a cmf on \( K \). It is inscribed both in \( V \) and \( W \). Moreover, If \( V \sqsubseteq W \), then \( F_V \subseteq F_W \).

(iii) If \( V' \) is a cmf on a simplicial complex \( K' \) and \( f : K \rightarrow K' \) is continuous, then \( f^*(V') := \{ f^{-1}(V') : V' \in V' \} \), called the pullback of \( V' \), is a cmf on \( K \).

(iv) The maps \( \kappa := \text{id}_{V \cap f^*(V')} : (K, \mathcal{T}_{V \cap f^*(V')}) \rightarrow (K, \mathcal{T}_V) \) induced by identity and \( \lambda := f_{V \cap f^*(V')}, V' : (K, \mathcal{T}_{V \cap f^*(V')}) \rightarrow (K', \mathcal{T}_{V'}) \) induced by \( f \) are continuous.

**Proof:** Note that by Theorem 4, the Morse sets in the minimal Morse decomposition are the strongly connected components of \( G_{F_V} \). Hence, to prove (i) it suffices to observe that every \( \sigma \in K \) belongs to a strongly connected component. This is obvious because \( \sigma \in \text{cl} \sigma \subseteq F_V(\sigma) \) for any \( \sigma \in K \). Thus, (i) is proved. Since the intersection of two convex sets is easily seen to be convex, each element of \( V \cap W \) is convex. Obviously, \( V \cap W \) is a partition of \( K \) and is inscribed in \( V \) and \( W \). Take \( \sigma \in K \). Assumption \( V \sqsubseteq W \) implies that \( [\sigma]_V \subseteq [\sigma]_W \). It follows that \( F_V(\sigma) = \text{cl} \sigma \cup [\sigma]_V \subseteq \text{cl} \sigma \cup [\sigma]_W = F_W(\sigma) \). Thus, (ii) is also proved. Obviously, \( f^*(V') \)}
is a partition of $K$. To show that for every $V' \in \mathcal{V}'$ the set $f^{-1}(V')$ is convex, take $\sigma, \sigma' \in f^{-1}(V')$ and $\tau \in K$ such that $\sigma \leq \tau \leq \sigma'$. Then $f(\sigma), f(\sigma') \in V'$, $f(\sigma) \preceq f(\tau) \preceq f(\sigma')$, and the convexity of $V'$ implies that $f(\tau) \in V'$. It follows that $\tau \in f^{-1}(V')$ and $f^{-1}(V')$ is convex. This proves (iii).

To prove (iv), we verify that the maps $\kappa$ and $\lambda$ satisfy the assumption of Corollary 1. It follows from (ii) that $\text{id} \circ F_{V_f(V')} = F_{V_f(V')} \subseteq F_{V'} = F_{V} \circ \text{id}$, which proves that $\kappa$ is continuous. Similarly, we get $f \circ F_{V_f(V')} \subseteq f \circ F_{f'(V')}$. Thus, it suffices to prove that $f \circ F_{f'(V')} \subseteq F_{V'} \circ f$. Indeed, for $\sigma \in K$ we get from the continuity of $f$ and the definition of $f^*(V')$ that $(f \circ F_{f'(V')})(\sigma) = f(F_{f'(V')}(\sigma)) = f(\text{cl} \sigma \cup [\sigma]_{f'(V')}) = f(\text{cl} \sigma) \cup f([\sigma]_{f'(V')}) \subseteq \text{cl} f(\sigma) \cup [f(\sigma)]_{V'} = F_{V'}(f(\sigma)) = (F_{V'} \circ f)(\sigma). \square$

We use the diagram of continuous maps $(K, \mathcal{T}_V) \xrightarrow{\kappa} (K, \mathcal{T}_{\mathcal{V}_f(V')}) \xrightarrow{\lambda} (K', \mathcal{T}_{V'})$, referred to as the comparison diagram of cmfs $\mathcal{V}$ and $\mathcal{V}'$, to define the persistence of Morse decompositions for cmfs. To this end, assume that, for $i = 1, 2, \ldots, n$, we have a cmf $\mathcal{V}_i$ on a simplicial complex $K_i$. Moreover, assume that we have a sequence of continuous maps $f_i : K_i \to K_{i+1}$. Putting together the comparison diagrams of $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$ and applying the singular homology functor we obtain the following zig-zag persistence module

$$
H(K_1, \mathcal{T}_{\mathcal{V}_1}) \xrightarrow{H(\kappa_1)} H(K_1, \mathcal{T}_{\mathcal{V}_1 \cap f_2(\mathcal{V}_2)}) \xrightarrow{H(\lambda_2)} H(K_2, \mathcal{T}_{\mathcal{V}_2}) \xrightarrow{H(\kappa_2)} \ldots
$$

$$
\ldots \xrightarrow{H(\kappa_n)} H(K_{n-1}, \mathcal{T}_{\mathcal{V}_{n-1} \cap f_n(\mathcal{V}_n)}) \xrightarrow{H(\lambda_n)} H(K_n, \mathcal{T}_{\mathcal{V}_n}). \quad (2)
$$

We refer to the persistence diagram of this module as the persistence diagram of Morse decompositions of the sequence of cmfs $\mathcal{V}_i$.

4 Computational considerations and a geometric interpretation

Computational considerations. Singular homology is not very amenable to computations. Therefore, to compute the persistence module (possibly zigzag) in (1) and (2) efficiently, we take a more combinatorial approach. We take the help of McCord’s Theorem in order to convert (1) and (2) to a persistence module where the objects are simplicial homology groups.

Let $(X, \mathcal{T})$ be a finite $T_0$ topological space and let $\leq_\mathcal{T}$ be the partial order associated with $\mathcal{T}$ by Alexandrov Theorem (see Theorem 11 in Appendix). The nerve of this partial order, that is, the collection of subsets linearly ordered by $\leq_\mathcal{T}$ called chains, forms an abstract simplicial complex. We denote it $N(X, \mathcal{T})$ or briefly $N(X)$ if $\mathcal{T}$ is clear from the context. Also
by Alexandrov Theorem, a continuous map \( f : (X, T) \to (X', T') \) of two finite topological \( T_0 \) spaces preserves the partial orders \( \leq_T \) and \( \leq_{T'} \). Therefore, it induces a simplicial map \( N(f) : N(X, T) \to N(X', T') \). The following proposition is straightforward for which we recall that every simplicial map \( f : K \to K' \) of abstract simplicial complexes extends linearly to a continuous map \( |f| : |K| \to |K'| \) on the polytopes of \( K \) and \( K' \).

**Proposition 6** If \( K \) is a simplicial complex, then the barycentric subdivision (cf. [20, Sec. 2.15]) of a geometric realization of \( K \) is a geometric realization of \( N(K) \). In particular, \(|K| = |N(K)|\). Moreover, if \( f : K \to K' \) is continuous, then \(|f| = |N(f)|\).

By McCord’s Theorem [21] (see Theorem [13] in Appendix [A.1]), there is a continuous map \( \mu_{(X, T)} : |N(X, T)| \to X \) which induces an isomorphism \( H(\mu_{(X, T)}) : H(|N(X, T)|) \to H(X, T) \) of singular homologies. Moreover, the map \((X, T) \mapsto H(\mu_{(X, T)})\) is a natural transformation, that is for any continuous map \( f : (X, T) \to (X', T') \) of finite \( T_0 \) topological spaces \( H(\mu_{(X', T')}) \circ H(|f|) = H(f) \circ H(\mu_{(X, T)}) \). Applying McCord’s Theorem to every homology group in (1) we obtain the following proposition.

**Proposition 7** Persistence module (1) is isomorphic to the persistence module

\[
H(|N(\langle M_1 \rangle, T^1_{M_1})|) \xrightarrow{\bar{f}^N_1} H(|N(\langle M_2 \rangle, T^2_{M_2})|) \xrightarrow{\bar{f}^N_2} \ldots H(|N(\langle M_n \rangle, T^m_{M_n})|), \quad (3)
\]

where \( \bar{f}^N_i := H(|N(\bar{f}_i)|) \).

Persistence module (3) is not yet simplicial, but the map which sends each simplex in \( K \) to the associated linear singular simplex in \(|K|\) induces an isomorphism between the simplicial homology of \( K \) and singular homology of \(|K|\). Moreover, this isomorphism commutes with the maps induced in simplicial and singular homology by simplicial maps (see [20, Theorem 34.3,34.4]). Thus, we obtain the following corollary. It facilitates the algorithmic computations of persistence diagrams for Morse decompositions of cds’s.

**Corollary 8** The persistence diagram of (1) is the same as the persistence diagram of the persistence module

\[
H^\Delta(N(\langle M_1 \rangle, T^1_{M_1})) \xrightarrow{f^\Delta_1} H^\Delta(N(\langle M_2 \rangle, T^2_{M_2})) \xrightarrow{f^\Delta_2} \ldots H^\Delta(N(\langle M_n \rangle, T^m_{M_n})), \quad (4)
\]

where \( H^\Delta \) denotes simplicial homology and \( f^\Delta_i := H^\Delta(N(\bar{f}_i)) \). Moreover, an analogous statement holds for the zig-zag persistence module (2).
For computing the persistence diagram of the module in \(4\), we identify the Morse sets in linear time by computing strongly connected components in \(G_{F_i}\). The nerve of these Morse sets can also be easily computed in time linear in input mesh size (assuming the dimension of the complex to be constant). Finally, the mapping of the simplices of the nerve to the adjacent complexes in the sequence can be obtained by the given simplicial maps. One can avail the persistence algorithm in [11] specifically designed for computing the persistence diagram of such simplicial maps.

**Geometric interpretation.** Proposition 7 provides means to interpret the Alexandrov topology of subsets of simplicial complexes in the persistence module of Morse decompositions by the metric topology of their solids in the Euclidean space. Recall that the solid of a subset \(A \subseteq K\) of a simplicial complex is \(|A| := \bigcup \{ \partial \sigma \mid \sigma \in A \}\). Let simplicial complexes \(K_i\), cds’s \(F_i\) and Morse decompositions \(M_i\) for \(i = 1, 2, \ldots n\) be such as in Section 3. Moreover, assume \(f_i : K_i \to K_{i+1}\) for \(i = 1, 2, \ldots n\) are simplicial maps. Let \(O^i\) denote the metric topology of the polytope \(|K_i|\). Denote by \(M^i_q := \{ |M| \mid M \in M_i \}\) the family of solids of Morse sets in \(M_i\). Consider the map \(\nu_i : \langle M^i_q \rangle \ni x \mapsto f_i(x) \in \langle M^i_{q+1} \rangle\). It is easily seen to be continuous with respect to topologies \(O_{M^i_q}\) and \(O_{M^i_{q+1}}\).

**Theorem 9** The persistence diagram of (1) is the same as the persistence diagram of the persistence module

\[
H(\langle M^i_1 \rangle, O^i_{M^i_1}) \xrightarrow{H(\nu_i)} H(\langle M^i_2 \rangle, O^2_{M^i_2}) \xrightarrow{H(\nu_2)} \cdots \xrightarrow{H(\nu_{n-1})} H(\langle M^i_n \rangle, O^n_{M^i_n}). \quad (5)
\]

**Proof:** By Proposition 7, it suffices to prove that the diagrams of (3) and (5) are isomorphic. By Theorem 2(iii), any two Morse sets in \(M_i\) are disconnected. Hence, it follows from [11 Proposition 1.2.4] (see Proposition 12 in Appendix A.1) that the nerve \(N(\langle M_i \rangle, T^i_{M_i})\) splits as the disjoint union \(\bigcup_{M \in M_i} N(M, T_M)\). In consequence, the whole diagram (4) splits as the direct sum of diagrams for individual Morse sets. Again by Theorem 2(iii), any two sets in \(M^i_q\) are \(O^i_{M^i_q}\)-disconnected. Therefore, diagram (5) also splits as the direct sum of diagrams for individual sets in \(M^i_q\). Thus, it suffices to prove that the respective diagrams for individual Morse sets are isomorphic. This follows easily from Proposition 10 below. \(\square\)

Note that, without loss of generality, given a simplicial complex \(K\), we may fix a geometric realization of \(K\) and take its barycentric subdivision as the geometric realization of \(N(K)\). Then, for any set of simplices \(M \subseteq K\) we have \(|N(M)| \subseteq |M|\).

**Proposition 10** The inclusion map \(\iota_M : |N(M)| \to |M|\) is a homotopy equivalence. Moreover, if \(f : K \to K'\) is a simplicial map, then the map \(\iota_M\) and the map \(\iota_{f(M)} : |N(f(M))| \to |M|\)
$|f(M)|$ commute with the restrictions $|N(f)||f(M)||M|$ and $|f||M|$, that is $\iota_{f(M)} \circ |N(f)||f(M)||M| = |f||M| \circ \iota_M$.

Proof: To prove that $\iota_M$ is a homotopy equivalence, it suffices to show that $|N(M)|$ is a deformation retract of $|M|$. To this end, order the simplices $\sigma_1, \ldots, \sigma_n$ in $\text{cl} M \setminus M$ so that if $\sigma_j \leq \sigma_k$, then $k \leq j$. Let $M_i = \text{cl} M \setminus \{\sigma_1, \ldots, \sigma_i\}$ and consider the sequence $\text{cl} M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_n = M$. We prove by induction on $n$ that $|M|$ deformation retracts to $|N(M)|$. Observe that the poset nerve $N(M_0) = N(\text{cl} M)$ coincides with the barycentric subdivision of $\text{cl} M$ and thus $|M_0| = |\text{cl} M| = |N(M_0)|$. Therefore, for $n = 0$, the claim is satisfied trivially.

Inductively assume that $|M_{i-1}|$ deformation retracts to $|N(M_{i-1})|$ for all $i \in [1, n]$. We observe the following:

1: In general $N(M_i) = N(M_{i-1}) \setminus C(\sigma_i)$ where $C(\sigma_i)$ denotes the set of all chains containing $\sigma_i$ in the poset $(N(M_{i-1}), \preceq)$. If $\sigma_i^*$ denotes the vertex corresponding to $\sigma_i$ in $N(M_{i-1})$, then $C(\sigma_i)$ is the star $\text{St} \sigma_i^*$ in $N(M_{i-1})$. Also, $|\sigma_i^*|$ is the barycenter $b(\sigma_i)$.

2: Let $Y \subseteq \text{St} \sigma_i^*$ be any set of simplices in $N(M_{i-1})$ including $\sigma_i^*$. Then $|N(M_{i-1})| \setminus |Y|$ deformation retracts to $|N(M_i)|$. This follows from the fact that $\text{St} \sigma_i^* \setminus |\sigma_i^*| = \text{St} \sigma_i^* \setminus b(\sigma_i)$ retracts to the link of $b(\sigma_i)$ along the segments that connect $b(\sigma_i)$ to the points in the link and the restriction of this retraction to points in $\text{St} \sigma_i^* \setminus |Y|$ provides the necessary deformation retraction.

For induction, observe that $|N(M_{i-1})|$ contains a subdivision of $|\sigma_i| = \sigma_i$ because $M_{i-1}$ contains $\sigma_i$ and all its faces by definition of $M_i$s. Let $Y$ denote the set of simplices that subdivide $\sigma_i$. Then, according to (2), $|N(M_{i-1})| \setminus \sigma_i$ deformation retracts to $|N(M_i)|$. We construct a deformation retraction of $|M_i|$ to $|N(M_i)|$ by first retracting $|M_{i-1}|$ to $|N(M_{i-1})|$ by the inductive hypothesis and then retracting $|N(M_{i-1})| \setminus \sigma_i$ to $|N(M_i)|$. The remaining part of the lemma is an immediate consequence of Proposition 6. □

5 Examples

Example 1. Kuznetsov map. Let us consider the following planar map analyzed by Kuznetsov in the context of the Neimark-Sacker bifurcation [20, Subsection 4.6].

$$N\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(1 + \alpha \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_1^2 + x_2^2 \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (6)$$

For parameters $\theta = \pi/17$, $\alpha = 0.5$, $a = -1$ and $b = 0.5$ the system restricted to cube $[-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2$ admits a Morse decomposition consisting of an unstable fixed point
and an attracting invariant circle. (see Figure 3 upper left). We want to detect this Morse decomposition just from a finite sample of the map and in the presence of Gaussian noise. The setup is similar to the toy example in Section 2 (see Appendix A.2 for more details). Results are presented in Figure 3. The persistence diagram indicates the presence of two 0-dimensional and one 1-dimensional homology generators with large persistence, as expected. Bottom row of Figure 3 shows Morse sets of selected filtration steps.

**Example 2. Lotka Volterra model.** Consider the Lotka Volterra (LV) model:

\[
\left( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right) = (x(1 - \frac{x}{k}) - \frac{a_1 xy}{b + x}, \frac{a_2 xy}{b + x} - gy) \tag{7}
\]
where $k = 3.5$, $b = 1$, $g = 0.5$, $a_1 = (1 - \frac{1}{k})(b + 1)$, $a_2 = g(b + 1)$ (see [6, Chapter 2, Eq. 2.13 and 2.14]). The system has a Morse decomposition consisting of a repelling stationary point and an attracting periodic orbit. We want to observe this Morse decomposition in a cds constructed from a finite sample of the vector field. In Appendix A.3 we present an algorithm for constructing a cmf from a sampled vector field. The algorithm requires an angle parameter $\alpha$. The constructed cmf and hence its cds depend on this parameter. We execute
the algorithm for varying $\alpha$ and construct the zigzag filtration \cite{2}. Since the supporting simplicial complex (mesh) remains fixed, we obtain zigzag persistence under inclusion maps. Experiments with varying mesh, utilizing non-inclusion maps, are in progress. The outcome for the LV model is presented in Figure \ref{fig:4}. We note that the trivial Morse sets that is Morse sets consisting of just one multivector $V$ such that $H(\text{cl}V, \text{cl}V \setminus V) = 0$ are excluded from the presentation of Morse decompositions and from the barcode, because such Morse sets are considered spurious due to the triviality of their Conley index (see \cite{23}).

References

[1] Z. Alexander, E. Bradley, J.D. Meiss, and N. Sanderson. Simplicial Multi-valued Maps and the Witness Complex for Dynamical Analysis of Time Series, *SIAM Journal on applied dynamical systems* 14 (2015), 1278–1307.

[2] P.S. Alexandrov. Diskrete Räume, *Mathematicskii Sbornik (N.S.)* 2 (1937), 501–518.

[3] Z. Arai, W. Kalies, H. Kokubu, K. Mischaikow, H. Oka and P. Pilarczyk. A Database Schema for the Analysis of Global Dynamics of Multiparameter Systems, *SIAM J. Applied Dyn. Syst.* 8 (2009), 757–789.

[4] J.A. Barmak. Algebraic Topology of Finite Topological Spaces and Applications, Lecture Notes in Mathematics 2032, Springer-Verlag, 2011.

[5] U. Bauer, H. Edelsbrunner, G. Jabłoński, M. Mrozek. Persistence in sampled dynamical systems faster, *arXiv:1709.04068* (2017).

[6] N. Boccara. *Modeling Complex Systems*, Springer, New York, NY 2004, ISBN 978-0-387-40462-2, https://doi.org/10.1007/b97378.

[7] M.C. Bortolan, T. Caraballo, A.N. Carvalho, J.A. Langa. Skew product semiflows and Morse decomposition, *J. Differential Equations* 255 (2013), 2436–2462.

[8] J. Bush, M. Gameiro, S. Harker, H. Kokubu, K. Mischaikow, I. Obayashi and P. Pilarczyk. Combinatorial-topological framework for the analysis of global dynamics, *Chaos* 22 (2012), 047508.

[9] C. Conley. *Isolated Invariant Sets and the Morse Index*, American Mathematical Society, Providence, RI, 1978.
[10] H.B. da Costa, J. Valero. Morse Decompositions with Infinite Components for Multivalued Semiflows, *Set-Valued Var. Anal* 25(2017), 25–41.

[11] T. K. Dey, F. Fan, Y. Wang. Computing topological persistence for simplicial maps, *Proc. 30th Annu. Sympos. Comput. Geom.* (2014).

[12] H. Edelsbrunner, G. Jabłoński, M. Mrozek. The Persistent Homology of a Self-map, *Foundations of Computational Mathematics*, 15(2015), 1213–1244. DOI: 10.1007/s10208-014-9223-y.

[13] H. Edelsbrunner, D. Letscher, A. Zomorodian. Topological Persistence and Simplification, *Discrete and Computational Geometry* 28(2002), 511–533.

[14] R. Forman. Morse theory for cell complexes, *Advances in Mathematics*, 134 (1998), 90–145.

[15] R. Forman. Combinatorial vector fields and dynamical systems, *Mathematische Zeitschrift* 228 (1998), 629–681.

[16] J. Garland, E. Bradley, J.D. Meiss. Exploring the topology of dynamical reconstructions, *Physica D* 334(2016), 49–59.

[17] G. Guerrero, J.A. Langa and A. Suárez. Architecture of attractor determines dynamics on mutualistic complex networks, Nonlinear Analysis: Real World Applications 34(2017), 17–40.

[18] D.H. Knipl, P. Pilarczyk and G.Röst. Rich Bifurcation Structure in a Two-Patch Vaccination Model, *SIAM J. Applied Dynamical Systems* 14(2015), 980–1017.

[19] J. Kubica. M. Lipiński, M. Mrozek. Conley-Morse-Forman theory for generalized combinatorial multivector fields, in preparation.

[20] Y.A. Kuznetsov. *Elements of Applied Bifurcation Theory*, Springer-Verlag, 1995.

[21] M.C. McCord. Singular homology and homotopy groups of finite spaces, *Duke Math. J.* 33(1966), 465–474.

[22] K. Mischaikow, M. Mrozek. Chaos in Lorenz equations: a computer assisted proof, *Bull. AMS (N.S.)*, 33(1995), 66-72.

[23] K. Mischaikow, M. Mrozek, J. Reiss, A. Szymczak. Construction of Symbolic Dynamics from Experimental Time Series *Physical Review Letters* 82(1999), p. 1144–1147.
Appendix.

A.1 Finite topological spaces

A topology $\mathcal{T}$ on $X$ is a family of subsets of $X$ satisfying the conditions:

(i) $\emptyset$ and $X$ are in $\mathcal{T}$,

(ii) $\mathcal{T}$ is closed under union and finite intersection.

The sets in $\mathcal{T}$ are called open. The interior of $A$ is the union of open subsets of $A$. A set $A \subseteq X$ is closed if $X \setminus A$ is open. The closure of $A$, denoted $\text{cl} \ A$, is the intersection of all closed supersets of $A$.

We say that $x$ is separated from $y$ in $\mathcal{T}$ if there exists a $U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$. We say that $\mathcal{T}$ is $T_0$ topology if for any $x, y \in X$ such that $x \neq y$ either $x$ is separated from $y$ or $y$ is separated from $x$. We say that $\mathcal{T}$ is $T_1$ topology if for any $x, y \in X$ where $x \neq y$, both $x$ is separated from $y$ and $y$ is separated from $x$. We say that $\mathcal{T}$ is $T_2$ topology if for any $x, y \in X$ where $x \neq y$, there exist disjoint sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.
A topological space is a pair \((X, \mathcal{T})\) where \(\mathcal{T}\) is a topology in \(X\). It is a finite topological space if \(X\) is finite. For two topological spaces \((X, \mathcal{T})\) and \((X', \mathcal{T}')\) we say that a map \(f : (X, \mathcal{T}) \to (X', \mathcal{T}')\) is continuous if \(U \in \mathcal{T}'\) implies \(f^{-1}(U) \in \mathcal{T}\).

Recall that the space \(X\) is \(\mathcal{T}\)-disconnected if there exist disjoint, non-empty sets \(U, V \in \mathcal{T}\) such that \(X = U \cup V\). The space \(X\) is \(\mathcal{T}\)-connected if it is not \(\mathcal{T}\)-disconnected. A subset \(A \subseteq X\) is \(\mathcal{T}\)-connected if it is connected as a space with induced topology \(\mathcal{T}_A\).

The connected component of \(x \in X\), denoted \([x]_{\mathcal{T}}\), is the union of all connected subsets of \(X\) containing \(x\). Note, that \([x]_{\mathcal{T}}\) is a connected set and \(\{ [x]_{\mathcal{T}} \mid x \in X \}\) is a partition of \(X\).

**Theorem 11** (P. Alexandrov, [2]) For a partial order \(\leq\) on a finite set \(X\), there is a \(T_0\) topology \(\mathcal{T}_{\leq}\) on \(X\) whose open sets are upper sets with respect to \(\leq\). For a \(T_0\) topology \(\mathcal{T}\) on \(X\), there is a partial order \(\leq_{\mathcal{T}}\) where \(x \leq_{\mathcal{T}} y\) if and only if \(x\) is in the closure of \(y\) with respect to \(\mathcal{T}\). The correspondences \(\mathcal{T} \mapsto \leq_{\mathcal{T}}\) and \(\leq_{\mathcal{T}} \mapsto \mathcal{T}_{\leq}\) are mutually inverse. They transform continuous maps into an order-preserving maps and vice versa.

A fence in a poset \(X\) is a sequence \(x_0, x_1, \ldots, x_n\) of points in \(X\) such that any two consecutive points are comparable. \(X\) is order-connected if for any two points \(x, y \in X\) there exists a fence starting in \(x\) and ending in \(y\).

**Proposition 12** ([4, Proposition 1.2.4]) Let \((X, \mathcal{T})\) be a finite \(T_0\) topological space. Then, the following conditions are equivalent:

(i) \(X\) is a connected topological space.

(ii) \(X\) is an order-connected preorder.

(iii) \(X\) is a path-connected topological space.

Consider the map \(\mu_{(X, \mathcal{T})} : |N(X, \mathcal{T})| \ni x \mapsto \min \sigma_x \in X\), where \(\sigma_x\) denotes the unique simplex \(\sigma \in N(X, \mathcal{T})\) such that \(x \in \sigma\) and the minimum is taken with respect to the partial order \(\leq_{\mathcal{T}}\).

**Theorem 13** (M. C. McCord, [21]) The map \(\mu_{(X, \mathcal{T})}\) is continuous and a weak homotopy equivalence. Moreover, if \(f : (X, \mathcal{T}) \to (X', \mathcal{T}')\) is a continuous map of two finite \(T_0\) topological spaces, then the following diagrams commute.

\[
\begin{array}{ccc}
|N(X, \mathcal{T})| & \xrightarrow{|N(\mu)|} & |N(X', \mathcal{T}')| \\
\mu_{(X, \mathcal{T})} & & \mu_{(X', \mathcal{T}')}
\end{array}
\]

\[
\begin{array}{ccc}
H_k(|N(X, \mathcal{T})|) & \xrightarrow{H(|N(\mu)|)} & H_k(|N(X', \mathcal{T}')|) \\
H_k(X, \mathcal{T}) & \xrightarrow{H(f)} & H_k(X', \mathcal{T}')
\end{array}
\]
A.2 Sampled map

Here we present the details of Example 1 in Section 5. Let $x \in \mathbb{R}^2$ and let $X$ and $Y$ be random vectors in $\mathbb{R}^2$ with normal distribution centered at zero and standard deviation $\sigma_X$ and $\sigma_Y$ respectively. Let $\tilde{N}(x) := N(x + X) + Y$ be a noisy version of the map (6). Consider a triangulation $K$ of the square $Q := [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2$ obtained by splitting $Q$ into a $48 \times 48$ uniform grid of squares and dividing every square into two triangles. Then, $K_{\text{top}}$ consists of 4608 triangles. We take $\sigma_X, \sigma_Y$ proportional to the grid size $\tau$, that is $\tau = 1/24, \sigma_X = \tau/4 = 1/96$ and $\sigma_Y = \tau$. We generate a noisy sample $\{(x_i, y_i)\}$ of the map $N$ by taking a uniformly distributed sequence of points $x_i$ in $Q$ and taking $y_i := \tilde{N}(x_i)$. We reject a pair $(x_i, y_i)$ for which $y_i \not\in Q$. We define $n_{\tau, \tilde{\tau}}$ and $\text{cds} F_\mu$ the same way as in Section 2, that is

$n_{\tau, \tilde{\tau}} := \# \{(x_i, y_i) \mid x_i \in \tau \text{ and } y_i \in \tilde{\tau}\}$

$F_\mu(\sigma) := \co \bigcup_{\tau \in K_{\text{top}}, \sigma \preceq \tilde{\tau}} \{\tilde{\tau} \in K_{\text{top}} \mid \frac{n_{\tau, \tilde{\tau}}}{n_{\max}} \geq \mu\}$.

The family of Morse sets $\mathcal{M}(F_\mu)$ at level $\mu$ consists of all strongly connected components. The sequence of considered weights $\mu_n > \mu_{n-1} > \ldots > \mu_0 = 0$ leads to the filtration where $\mathcal{M}(F_{\mu_n}) \sqsubseteq \mathcal{M}(F_{\mu_{n-1}})$.

Algorithm 1 Construct cmf from a sampled vector field.

1: procedure VFCMF($K, V, \alpha$)
2: $m \leftarrow$ an identity map $K \rightarrow K$.
3: for all $\sigma \in K$ do
4: \hspace{1em} $m[\sigma] \leftarrow$ any toplex in the star of $\sigma$ pointed by mean of $\{\vec{v} \in V \mid s_v \in \sigma\}$
5: for all $\vec{v} = (s_v, t_v) \in V$ do \quad \triangleright Aligns vectors
6: \hspace{1em} $S \leftarrow \{(\dim \sigma, \angle(\sigma, \vec{v}), \sigma) \mid s_v \in \sigma \text{ and } \angle(\sigma, \vec{v}) \leq \alpha\}$
7: \hspace{1em} $S' \leftarrow$ sort $S$ using lexicographical order on first two positions \quad \triangleright (dim, \angle, \_)
8: \hspace{1em} $(\_., \sigma) \leftarrow$ first element of $S'$
9: \hspace{1em} $m[s_v] \leftarrow \sigma$
10: for all $\sigma \in K$ in descending dimension do \quad \triangleright Remove convexity conflicts
11: \hspace{1em} while exists $\tau \preceq \sigma$ s.t. $m[\tau]$ and $m[\sigma]$ belong to one toplex and $m[\tau] \neq m[\sigma]$ do
12: \hspace{2em} $m[\tau] \leftarrow \sigma$ and $m[\sigma] \leftarrow \sigma$
13: $\mathcal{V} \leftarrow$ build a partition of $K$ using nonempty pre-images of $m$
14: return $\mathcal{V}$.
A.3 Sampled vector field

We use Algorithm 1 to construct a cmf from a sampled vector field. The input consists of a simplicial mesh $K$, a cloud of vectors $V := \{ \vec{v} = (s_v, t_v) \mid s_v, t_v \in \mathbb{R}^d \}$ sampled at the vertices $s_v$ of $K$, and an input parameter $\alpha$. For a vector $\vec{v} = (s_v, t_v)$ and a simplex $\sigma$ such that $s_v \in \sigma$, we measure the angle between $\vec{v}$ and the flat supporting $\sigma$. We assume the angle is zero when a vector has length zero. For a toplex $\sigma$, we assume the angle to be zero when $\sigma$ is pointed by the vector and $\infty$ otherwise. When the angle is smaller than $\alpha$, we project $\vec{v}$ onto $\sigma$. Intuitively, it aligns the vectors to the simplices. After this alignment, a multivector field is constructed by removing the convexity conflicts. Obviously, the output depends on the parameter $\alpha$. We measure changes in the multivector field $V$ via persistence of its Morse decomposition. To compute such persistence we use Dionysus software [24].