Numerical Simulation of Thermal Noise in Heavy Ion Collisions

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Abstract.
Thermal noise is present in any viscous fluid, making the simulation of relativistic noise in heavy ion collisions a necessity. It is likely possible to use it to make an independent measurement of viscosity in heavy ion collisions. The size, energy densities, and time scales of the collisions determine the relative importance of thermal noise. This causes a non-trivial contribution to two-particle correlations as well as event-by-event fluctuations in observables.

1. Fluctuations, dissipation, and noise in heavy ion collisions
The production of mesons and baryons in heavy ion collisions is consistent with the creation of a nearly perfect, relativistic fluid [1, 2]. Like any viscous fluid at finite temperature, the fluid created in heavy ion collisions has thermal fluctuations: stochastic contributions to the energy-momentum tensor $T^{\mu \nu}$. These fluctuations are related to the statistical fluctuations that must exist after coarse-graining: for example, while it is possible to approximate the number
of particles in a box $N$ with an average $\bar{N}$, there are fluctuations away from this average with variance $\sim \sqrt{N}$. The energy-momentum tensor has a fluctuating part, but the total is conserved:

$$T^{\mu\nu} = T^{\mu\nu}_{\text{ave}} + \delta T^{\mu\nu}, \quad \partial_{\mu}(T^{\mu\nu}_{\text{ave}} + \delta T^{\mu\nu}) = 0; \quad (1)$$

in the later sections the fluctuating part is separated further into ideal, viscous, and noise terms $\delta T^{\mu\nu} = \delta T^{\mu\nu}_{i} + \delta W^{\mu\nu} + \Xi^{\mu\nu}$.

Thermal fluctuations can be understood from the very general methods used to derive the fluctuation-dissipation relation. As an example familiar from quantum mechanics, consider a bosonic operator $\hat{\phi}$. The damping of the retarded Green function $G_{R}$ becomes the imaginary part of $G_{R}(\omega)$ in Fourier space, and as discussed in [3]:

$$\text{Im} G_{R}(\omega) = -\frac{i}{2} \left[ -i \int dt \theta(t)e^{i\omega t} \langle [\hat{\phi}(t), \hat{\phi}(0)] \rangle - i \int dt \theta(t)e^{-i\omega t} \langle [\hat{\phi}(0), \hat{\phi}(t)] \rangle \right]$$

$$= -\frac{1}{2} \int dt \theta(t)e^{i\omega t} \langle [\hat{\phi}(t), \hat{\phi}(0)] \rangle + \int dt \theta(t)e^{-i\omega t} \langle [\hat{\phi}(0), \hat{\phi}(t)] \rangle$$

$$= -\frac{1}{2} \int dt e^{i\omega t} \langle [\hat{\phi}(t), \hat{\phi}(0)] \rangle = -\frac{1}{2} \left( 1 - e^{-\omega/T} \right) \int dt e^{i\omega t} \langle \hat{\phi}(t)\hat{\phi}(0) \rangle$$

$$= -\frac{i}{2} (1 - e^{-\omega/T}) G_{>}(\omega). \quad (2)$$

Functions that describe different physical phenomena (the retarded Green function $G_{R}$ and the expectation value of a transition $G_{\Delta}$) are related near thermal equilibrium. Similarly, the autocorrelation function $G_{S} = \frac{1}{2} \langle [\hat{\phi}(t), \hat{\phi}(t')] \rangle$, which describes random fluctuations in a system, is related to the response function after a few more steps:

$$G_{S}(\omega) = -(1 + 2n_{E}(\omega))\text{Im} G_{R}(\omega) \approx \frac{-2T}{\omega}\text{Im} G_{R}(\omega),$$

where the final approximate equality comes from assuming the system is nearly classical. In [4], the equal-time autocorrelation of variances in velocity $\delta v^{i}$ and in mass density $\rho$ were found in first-order non-relativistic hydrodynamics:

$$\langle \delta v^{i}(x)\delta v^{i}(x') \rangle = (T/\rho)\delta^{ij}\delta^{3}(x-x'),$$

$$\langle \delta \rho(x)\delta \rho(x') \rangle = \rho T(\frac{\partial p}{\partial T})_{T}\delta^{3}(x-x'), \quad (3)$$

showing how the strengths of these variances are related to temperature and other properties of the fluid.

Thermal hydrodynamical fluctuations are often neglected in physical applications; for example, for the average variance of $\delta v^{i}$ over a volume of size $\Delta V$

$$\left\langle \frac{\int d^{3}x \delta v^{i}(x) \int d^{3}x' \delta v^{i}(x')}{\Delta V} \right\rangle = \frac{T}{\rho \Delta V} \delta^{ij} \quad (4)$$

is very small whenever $T \ll \rho \Delta V$. Thermal fluctuations are usually unimportant at everyday length scales, temperatures, and densities. However, in heavy ion collisions, the equation of state is ultrarelativistic while the fluid created is small, typically about 10 fm across. Therefore, a heavy ion collision should be relatively noisy. Thanks to the fluctuation-dissipation theorem, this noise might be used to provide an independent measurement of transport coefficients.
Heavy ion collisions produce a fluid expanding at relativistic speeds, and thermal fluctuations need to be determined in this setting. Recently, Kapusta, Müller, and Stephanov determined the correlation functions for thermal noise in first-order relativistic hydrodynamics [5]. The thermal noise was calculated for both the Landau-Lifshitz frame, where the velocity $u^\mu$ represents the flow of energy density in the fluid, and the Eckart frame, where $u^\mu$ is the flow of a particle number density $n$. In first-order hydrodynamics and assuming a Bjorken expansion, analytic solutions are possible, and in [5] the authors found a contribution of thermal noise to two-particle correlation functions at rapidity gaps as large as 3, shown in Fig. 1.

2. Thermal fluctuations in Israel-Stewart hydrodynamics

In first order viscous hydrodynamics, shear modes propagate faster than light. This fact is problematic in heavy-ion collisions where the flow velocities approach the speed of light. The Israel-Stewart equations describe second order hydrodynamics with a system of equations:

\[
\partial_\nu (T^{\mu\nu}_{\text{id}} + W^{\mu\nu}) = 0, \\
\Delta^\alpha_\mu \Delta_\nu^\beta (u \cdot \partial) W^{\alpha\beta} = \frac{1}{\tau_\pi} (W^{\mu\nu} - S^{\mu\nu}) - \frac{4}{3} (\partial \cdot u) W^{\mu\nu},
\]

where $S^{\mu\nu} = \eta(\Delta^{\mu}u^{\nu} + \Delta^{\nu}u^{\mu}) + (\zeta - 2\eta/3) \Delta^{\mu\nu}(\partial \cdot u)$ is the first order viscous correction to the energy-momentum tensor, $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}$, and $\Delta^{\mu} = \Delta^{\mu\nu} \partial_\nu$. The equations are closed by an equation of state relating energy density and pressure $e(p)$ and the requirement that the viscous correction $W^{\mu\nu}$ is transverse: $u_\mu W^{\mu\nu} = 0$. With an appropriate choice of relaxation time $\tau_\pi$, this system does not have modes propagating faster than light; one can imagine the second equation as slowing the medium’s response to a shear stress. Heavy-ion collisions are often described with second order hydrodynamics, and thermal fluctuations need to be determined again for this system of equations.

Up to linear order in response, the equations for thermal fluctuations in a system described
with the Israel-Stewart equations were determined in [8]:

\[
\partial_\mu (\delta T^{\mu\nu}_{id} + \delta W^{\mu\nu}) = 0, \\
(u \cdot \partial) \delta W^{\mu\nu} = -\frac{1}{\tau_\pi} (\delta W^{\mu\nu} - \delta S^{\mu\nu} - \xi^{\mu\nu}) - \frac{4}{3} (\partial \cdot \delta u) W^{\mu\nu} - \frac{4}{3} (\partial \cdot u) \delta W^{\mu\nu} - (\delta u \cdot \partial) W^{\mu\nu}
\]

\[
-\delta u^\mu ((u \cdot \partial) u_\alpha) W^{\alpha\nu} - u^\mu ((\delta u \cdot \partial) u_\alpha) W^{\alpha\nu} + (u \cdot \partial) \delta u_\alpha W^{\alpha\nu} + (u \cdot \partial) u_\alpha \delta W^{\alpha\nu}
\]

\[
-\delta u^\nu ((u \cdot \partial) u_\alpha) W^{\alpha\mu} - u^\nu ((\delta u \cdot \partial) u_\alpha) W^{\alpha\mu} + (u \cdot \partial) \delta u_\alpha W^{\alpha\mu} + (u \cdot \partial) u_\alpha \delta W^{\alpha\mu},
\]

\[
\langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(x') \rangle = \left[ 2\eta T(\Delta^{\alpha\mu} \Delta^{\nu\beta} + \Delta^{\mu\beta} \Delta^{\nu\alpha}) + 2(\zeta - 2\eta/3) T \Delta^{\mu\nu} \Delta^{\alpha\beta} \right] \delta^4(x - x').
\]

Here, \(\delta T^{\mu\nu}_{id} = -\delta p^{\mu\nu} + \delta p \left( 1 + (\partial e/\partial p)_{n/s} \right) u_0^\mu u_0^\nu + (e_0 + p_0)(u_0^\mu \delta u^\nu + u_0^\nu \delta u^\mu)\) is the fluctuation in the ideal part of the energy-momentum tensor, \(\delta S^{\mu\nu}\) represents the change in \(S^{\mu\nu}\) up to linear order in fluctuations, \(\delta W^{\mu\nu} = \delta W^{\mu\nu} + \Xi^{\mu\nu}\) is the sum of the thermal fluctuation of the viscous energy-momentum tensor and the noise \(\Xi^{\mu\nu}\) is a stochastic source term driving these fluctuations to thermal expectation values. The noisy part of the energy-momentum tensor \(\Xi^{\mu\nu}\) now has a non-trivial autocorrelation in time:

\[
\langle \Xi^{\mu\nu}(x) \Xi^{\alpha\beta}(x') \rangle \propto \exp(-|t - t'|/\tau_s).
\]

Eqs. 5 are stochastic, due to the presence of the random variables \(\xi^{\mu\nu}\). Even for a single initial condition, the solution to these equations is not a single event but more properly an ensemble of events, which in practice is approximated with a large set sampled with Monte Carlo event generation. In the following section, each sampling corresponds to a single heavy ion collision while a set of samplings is similar to a centrality class.

3. Results at the Large Hadron Collider

Where can effects from thermal fluctuations be expected in heavy ion collisions? One can determine this by considering the thermal expectation values of fluctuations. First of all, \(\langle \delta T^{\mu\nu} \rangle = 0\): while each event has thermal fluctuations, the average of the fluctuations across many events is zero. As a result, thermal fluctuations should have a vanishingly small effect on one-particle observables averaged over an event class, such as average yields of pions as a function of \(p_T\) and \(\eta\). However, \(\langle \delta T^{\mu\nu} \delta T^{\alpha\beta} \rangle \neq 0\), and thermal fluctuations contribute to two-particle correlations, as discussed in [5]. Finally, in a given event, \(\delta T^{\mu\nu} \neq 0\): thermal fluctuations are a source for event-by-event variations in all observables, even if the average effect for a given observable should be zero.

The solution of the thermally fluctuating Israel-Stewart equations describing heavy-ion collisions is discussed in [8]: the hydrodynamical code MUSIC has been extended to include linearized fluctuations. MUSIC [9, 10] solves relativistic hydrodynamics in \(\tau-\eta\) coordinates, using initial conditions determined with the Glauber model for heavy-ion collisions and an equation of state determined with lattice QCD calculations [12]. Equations [5] are solved using the MacCormack method [11]. For the remainder of this work, we will discuss simulations using this extension of MUSIC meant to describe lead-lead collisions at \(\sqrt{s}/A = 2.76\) TeV created and
measured at the Large Hadron Collider (LHC), in both the 10-20% centrality class and for the “ultra-central” event class examined by CMS [13].

Figure 2 shows thermal fluctuations $\delta\epsilon/e$ at different proper times of the evolution of a lead-lead collision typical at the LHC. The magnitude of $\delta\epsilon/e$ increases with dropping temperature. At this point, these simulations should be tested in meaningful ways. One important test is to make sure that the average of a large set of thermally fluctuating events does indeed give the same results for particle spectra as does a single noiseless hydrodynamical simulation, as is discussed in the first paragraph of this section. Figure 3 demonstrates this: the distributions for $dN/dp_T$ of $\pi^+$ mesons are compared for both a single simulation without noise and 200 simulations with thermal noise, and agreement between the two is found, as one would expect from the relation $\langle \delta T^{\mu\nu} \rangle = 0$. This is non-trivial: individual noisy events do not yield $dN/dp_T$ distributions identical to the results of the noiseless simulation.

We also examine the momentum eccentricity $\epsilon_p$, which is the addition in quadrature of $\langle T^{xx} - T^{yy} \rangle/\langle T^{xx} + T^{yy} \rangle$ and $\langle 2T^{xy} \rangle/\langle T^{xx} + T^{yy} \rangle$. These two quantities are, respectively, the expectation values $\langle \cos(2\phi) \rangle$ and $\langle \sin(2\phi) \rangle$ for the direction of the total transverse momentum in cells, weighted by the total energy of each cell. In [14], $\epsilon_p$ was used as a proxy for $v_2$ of hadrons after freeze-out. Figure 4 shows the distributions of $\epsilon_p$ for two different ensembles of events, one representing the 10-20% centrality class at the LHC and the other the ultra-central event class.
Figure 3. (Color online) Transverse momentum distribution for pions showing that the single particle distribution is unaffected by noise, from [8].

(in practice, the latter set of simulations just has the impact parameter set to zero).

Thermal noise affects both the average values of $\epsilon_p$ as well as the width of the distributions $dN/d\epsilon_p$ in a given centrality class. To see this, consider an event with zero impact parameter: in a simulation without thermal fluctuations, all azimuthal coefficients $v_n = 0$. However for a given fluctuating event, $v_n \neq 0$, and after defining the reaction plane, all $v_n$ are positive. This makes the expectation values $\langle v_n \rangle$ also non-zero and positive. Thermal noise tends to increase average values of $\epsilon_p$ while contributing to the variance of this observable across events.

For the 10-20% centrality class, the variance in $\epsilon_p$ is subleading compared with the variance caused by fluctuating initial conditions [15], and the increase in the average of $\epsilon_p$ is modest. However the ensemble of events with zero impact parameter has an average value of $\epsilon_p$ comparable to the $v_2$ in the 0-0.2% centrality class examined by the CMS collaboration [13]. This suggests that a significant contribution to the azimuthal anisotropy in this event class may come from thermal fluctuations.

4. Conclusions

Thermal fluctuations are demonstrably present in viscous fluids, and the autocorrelation of the thermal noise is proportional to the shear viscosity. This suggests a complementary way of measuring the transport coefficients of the fluid created in heavy ion collisions. Previous work has examined the contribution of thermal noise to the two particle correlation function in a viscous Bjorken expansion. We have determined the equations for thermal fluctuations in second order viscous hydrodynamics and implemented a simulation of this noise that works well for heavy ion collisions at the Relativistic Heavy Ion Collider and the Large Hadron Collider.

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Figure 4. (Color online) Top panel ($b = 6$ fm): Inclusion of noise increases the average $\epsilon_p$ and broadens its distribution. Bottom panel ($b = 0$ fm): Even for exactly central collisions there is a nonzero average $\epsilon_p$ due to noise. From [8].

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