Extreme Compressive Sampling for Covariance Estimation

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Abstract

We consider the problem of estimating the covariance of a collection of vectors given extremely compressed measurements of each vector. We propose and study an estimator based on back-projections of these compressive samples. We show, via a distribution-free analysis, that by observing just a single compressive measurement of each vector one can consistently estimate the covariance matrix, in both infinity and spectral norm. Via information theoretic techniques, we also establish lower bounds showing that our estimator is minimax-optimal for both infinity and spectral norm estimation problems. Our results show that the effective sample complexity for this problem is scaled by a factor of \( m^2/d^2 \) where \( m \) is the compression dimension and \( d \) is the ambient dimension. We mention applications to subspace learning (Principal Components Analysis) and distributed sensor networks.

1 Introduction

Covariance matrices provide second-order information between a collection of random variables and play a fundamental role in statistics and signal processing. A concrete example is dimensionality reduction, where covariance information is a sufficient statistic for principal components analysis, one of the most widely used methods. An important statistical task is covariance estimation, where the goal is to recover the covariance matrix of a distribution, given samples from that distribution.

In this paper we study a variant of the covariance estimation problem, when the samples are observed only through low-dimensional random projections. This estimation problem has roots in compressed sensing, where random projections have been used to reduce measurement overhead associated with high-dimensional signals. It is also motivated by problems in learning over distributed sensor networks, where both power and communication constraints may limit the measurement capabilities of a single sensor.

In the first part of the paper, we propose and analyze a covariance estimator based on these low-dimensional compressed observations. We show that even when each vector is observed only via projection onto a one-dimensional subspace, one can consistently and accurately estimate the sample covariance matrix of the data vectors in both spectral and infinity norms. In our analysis, we make no distributional assumptions on the data vectors themselves, and attempt to recover the sample covariance, since no population

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covariance exists. We present specializations to known distributions as corollaries of our main theorems. We also consider applications to subspace learning and learning in distributed sensor networks.

In the second part of the paper, we consider the fundamental limits of this estimation problem. Using information-theoretic tools, we present a number of lower bounds for a variety of settings, including the distribution and distribution-free settings under which we analyze our estimator. This analysis reveals that our covariance estimator achieves the minimax-rate for this problem, i.e. modulo constants and logarithmic factors, our estimator is the best one can hope for. We also consider a slightly different measurement paradigm, where a fixed low-dimensional random projection is used for all data vectors, and show that this approach is not consistent for the covariance estimation problem.

Our work deviates from the majority of work on compressive covariance estimation in that we do not make structural assumptions on the estimand, in this case the target covariance. A number of papers assume that the target covariance is low rank, sparse, or that the inverse covariance is sparse [4, 3, 1]. The broad theme of this line of work is that when the target covariance has some low-dimensional structure, far fewer total measurements (via random project) are necessary to achieve the same error as direct observation in the unstructured case. However when the target covariance does not have low-dimensional structure, these methods can fail dramatically, as we show with our lower bounds.

In contrast, our work instead examines the statistical price one pays for compressing the data vectors when the covariance matrix does not exhibit any low dimensional structure. Instead of using fewer measurements than direct observation, in this setting, compressing the data requires that one use significantly more measurements to achieve the same level of accuracy as direct observation. We precisely quantify this increase in measurement, showing that the effective sample size shifts from $n$ to $nm^2/d^2$, where the projection dimension is $m$ and the ambient dimension is $d$. Since we must have $m \leq d$, this means that one needs more samples to achieve a specified accuracy under our measurement model, when compared with direct observation. This effective sample size is present in all of our upper and lower bounds, showing that there is a price to pay for compression without structural assumptions. Note that this quadratic growth in effective sample size also matches recent results on covariance estimation from missing data [11, 14].

Regarding proof techniques, our upper bounds are based on analysis of a carefully constructed unbiased estimator for the target covariance matrix. The natural estimator for this problem is biased, and by exploiting properties of the Beta distributions that arises from random projections, we can analytically de-bias this natural estimator. The challenge with obtaining sharp error bounds on this estimator is that the random variables have large range, so a straightforward application of the Matrix Bernstein inequality does not suffice. We therefore use a conditioning argument, first showing that the random variables have much smaller range with high probability, and then applying the Matrix Bernstein inequality conditioned on this event. For the lower bounds, we derive a strong data processing inequality which upper bounds the Kullback-Leibler divergence between two compressed gaussian distributions by a small (less than one) multiple of the KL-divergence before compression. This contraction in KL-divergence, in concert with Fano’s method, gives the lower bounds in this paper.

The remainder of this paper is organized as follows: We conclude this section with a formal specification of the covariance estimation problem and the observation model. In Section 2, we mention related results on covariance estimation and matrix approximation. In Section 3, we develop our covariance estimator, providing a theoretical analysis in Section 4. Section 5 also contains discussion of applications to subspace learning and learning in distributed sensor networks. We present all of our lower bounds in Section 5. All proofs of our theorems are in Section 6. Several technical lemmas are deferred to the appendices.
1.1 Setup

Let \( x_1, \ldots, x_n \) be a collection of vectors in \( \mathbb{R}^d \) and define the covariance \( \Sigma = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T \). Note that we make no distributional assumptions on the data sequence \( \{x_i\}_{i=1}^{n} \), and therefore aim to recover the sample covariance \( \Sigma \), since there is no well-defined population covariance. We will assume that the vectors have bounded \( \ell_2 \) and \( \ell_\infty \) norms, as these quantities will appear in our error bounds. Nevertheless, the sequence could be adversarially generated, subject to these boundedness conditions.

Independently for all \( t \), let \( A_t \in \mathbb{R}^{d \times m} \) be an orthonormal basis for a \( m \)-dimensional subspace drawn uniformly at random. We are interested in estimating \( \Sigma \) from the samples \( \{(A_t, A_t^T x_i)\}_{i=1}^{n} \), so that each vector is compressed from \( d \) dimensions down to \( m \) dimensions. Note that this sampling scheme is equivalent to drawing a \( m \)-dimensional orthogonal projection \( \Phi_t \in \mathbb{R}^{d \times d} \) uniformly at random, and independently for all \( t \), and observing \( \{(\Phi_t, \Phi_t x_i)\}_{i=1}^{n} \). In both cases the vectors \( x_t \) have been compressed down to \( m \) dimensions.

We use several standard matrix and vector norms throughout the paper. For a symmetric matrix \( M \in \mathbb{R}^{d \times d} \) let \( \|M\|_2 = \max_{x \in \mathbb{R}^d} \frac{x^T M x}{x^T x} \) denote the spectral or operator norm, let \( \|M\|_F = \sqrt{\sum_{i,j=1}^{d} M_{i,j}^2} \) denote the Frobenius norm and let \( \|M\|_\infty = \max_{i,j} |M_{i,j}| \) denote the element-wise infinity norm. We also use \( \|M\|_{p,q} \) to denote the \( \ell_q \) norm of the \( \ell_p \) norms of the matrix, e.g., \( \|M\|_{2,\infty} = \max_{j} \|m_j\|_2 \) for a matrix with columns \( m_j \). For a vector \( v \), \( \|v\|_2, \|v\|_\infty \) are the Euclidean and infinity norm, respectively. Lastly, we use the standard Big-O and Little-O notation for asymptotic characterizations.

2 Related Work

Estimating a covariance matrix from samples is a classical problem in statistics with applications across the spectrum of scientific disciplines. More recently work has focused on the high-dimensional setting, where the dimensionality of the data points is large relative to the number of samples. In this setting, a number of structural assumptions that lead to tractable estimators have been proposed and studied.

One approach to these estimation problems is through compressive sensing, where the data is observed through low dimensional random projections from which the estimand can be algorithmically recovered. Compressive sensing has been a flourishing area of research in the past decade, starting with the seminal works of Donoho [7] and Candés et al. [2]. The main insight behind compressed sensing is that one can exactly solve underdetermined linear systems, provided that the parameter vector exhibits some structure, which was classically sparsity. These ideas have been extended to the matrix setting, where a number of papers study low-rank matrix recovery and covariance estimation from compressive measurements. The vast majority of these results make structural assumptions about the parameter of interest (the vector or matrix). Our work deviates from these results precisely in this way; we make no structural assumptions about the underlying covariance matrix, and still aim for consistent recovery.

For example, Chen et al. [3] study the low-rank matrix recovery problem, where the matrix \( M \) is observed through noisy quadratic measurements of the form \( a_i^T M a_i + \epsilon \). Similarly, Cai and Zhang [11] have similar results but also consider the spiked covariance model. Both of these works make strong structural assumptions on the matrix, namely that it is low rank possibly plus a diagonal element, and can consequently get very low sample complexity. Our setting is highly unstructured, and consequently the existing results do not apply. On the other hand, due to the lack of structure, we cannot hope to match the related sample complexity bounds, and as we show our bounds are minimax optimal for our setting.

In the theoretical computer science and numerical linear algebra literature, there are a several works that use random projections for the purposes of fast approximation to the singular value decomposition of
a unstructured matrix \([17, 10, 13]\). In this line of work, the matrix is unstructured, but the goal is to only recover the principal components rather than our much more stringent goal of spectral or infinity-norm estimation. Moreover, this line uses compression as a technique for developing fast algorithms, but actually operates in a model where the data matrix is available to the algorithm. This is a crucial difference as the best algorithms for this problem make multiple compressive measurements of the target matrix, which we cannot do once compression is baked into the measurement paradigm. Thus, these techniques are not immediately applicable to our setting.

A representative example from this line of work is the paper by Halko, Martinsson and Tropp [10]. To approximate the principal components of a matrix \(M\), they propose to first right-multiply by a small random matrix \(R\) to obtain \(Y = MR\), which approximates the column space of the matrix \(M\). By then projecting the columns of \(M\) onto the subspace spanned by \(Y\), one obtains a highly accurate estimate of the principal components of \(M\). Notice that this algorithm both pre- and post-multiplies the target matrix \(M\), which amounts to obtaining compressive measurements of both the rows and columns of the data matrix \(M\). This is possible when compression is used as an algorithmic technique, but not in our setting, where the data is only available through compressive measurements of the columns.

Another closely related line of work focuses on matrix recovery from missing data. While the majority of the results in this line focus on low rank matrices or other structured settings, there have been recent results focusing explicitly on the covariance estimation problem in the unstructured setting. For example, Kolar and Xing [11] consider a setting where each coordinate of each data point is missing with probability \(1 - \alpha\), independently from other coordinates and points. They show that one can estimate the covariance matrix in \(\ell_\infty\) norm with \(\tilde{O}(n\alpha^2)\) samples and use this estimator to learn the structure of a Gaussian graphical model. Similarly, Gonen et al. [9] study the subspace learning problem in the missing data setting, and show that again the effective sample complexity is reduced by a factor related to the squared fraction of entries observed per column. While the sample model in our work is different, qualitatively the main result is similar; we show that a similar reduction in effective sample complexity when the data is observed via random projection.

### 3 The Covariance Estimator

In this section, we develop a covariance estimator from compressed observations. The estimator is based on an adjustment to the observed covariance, i.e. the covariance of the samples \(\Phi_t x_t\). This adjustment is motivated by a characterization of the bias of the observed covariance.

Specifically, let \(\hat{\Sigma}_1 = \frac{d^2}{nm^2} \sum_{i=1}^n (\Phi_i x_i)(\Phi_i x_i)^T\) be a rescaled version of the observed covariance. Our estimate for the sample covariance \(\Sigma\) is:

\[
\hat{\Sigma} = \frac{m}{d(dm + d - 2)} \left( (d + 2)(d - 1) \hat{\Sigma}_1 - (d - m) \text{tr}(\hat{\Sigma}_1) I_d \right)
\]  

This estimator is a de-biased version of \(\hat{\Sigma}_1\), which, as we show, is not centered around \(\Sigma\).

The specification of this estimator is motivated by the following proposition, which gives an exact characterization of the bias of \(\hat{\Sigma}_1\) based on properties of the Beta distribution (See Fact 9):

**Proposition 1.** Let \(\Sigma = \frac{1}{n} \sum_{i=1}^n x_i x_i^T\) and \(\hat{\Sigma}_1 = \frac{d^2}{m} \frac{1}{n} \sum_{i=1}^n (\Phi_i x_i)(\Phi_i x_i)^T\). Then:

\[
E\hat{\Sigma}_1 = \frac{d}{m} \frac{(dm + d - 2)}{(d + 2)(d - 1)} \Sigma + \frac{d}{m} \frac{d - m}{(d + 2)(d - 1)} \text{tr}(\Sigma) I_d.
\]  

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See Section 6.2 for a proof.

With this expansion of the bias, it is also easy to see that \( \text{tr}(E\hat{\Sigma}_1) = \frac{d}{m} \text{tr}(\Sigma) \). Substituting in for \( \text{tr}(\Sigma) \) and re-arranging, we see that:

\[
\Sigma = \frac{m}{d(m + d - 2)} \left( (d + 2)(d - 1)E\hat{\Sigma}_1 - (d - m) \text{tr}(E\hat{\Sigma}_1)I_d \right)
\]

Since trace is a linear operator, we immediately see that our estimator is unbiased for \( \Sigma \).

### 4 Upper Bounds and Consequences

We now turn to our analysis of the estimator \( \hat{\Sigma} \). In this section we upper bound the error in both spectral and \( \ell_\infty \) norms for \( \hat{\Sigma} \) under our distribution-free setting. We also specialize these results to the problem of estimating the population covariance of a collection of gaussian vectors. Lastly, we discuss applications to subspace learning and learning in distributed sensor networks.

**Theorem 2.** Let \( d \geq 6 \) and \( \delta \in (0, 1) \) such that \( \delta \leq nd^2/e \) and \( \delta \geq d^2 \exp(-n/5) \). There exist universal constants \( \kappa_1, \kappa_2 > 0 \) such that with probability at least \( 1 - \delta \):

\[
\|\hat{\Sigma} - \Sigma\|_\infty \leq \kappa_1 \sqrt{\frac{d^2 \log \left( \frac{nd}{\delta} \right)}{nm^2}} \left( \|X\|_\infty^2 + \frac{\|X\|_{2,\infty}^2 d}{n} \right) + \kappa_2 \frac{d^2 \log \left( \frac{nd}{\delta} \right)}{nm^2} \left( \|X\|_\infty + \sqrt{\|X\|_{2,\infty}^2} \right)
\]

**Theorem 3.** There exist universal constants \( \kappa_1, \kappa_2 > 0 \) such that for any \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \):

\[
\|\hat{\Sigma} - \Sigma\|_2 \leq \kappa_1 \sqrt{\frac{d^2 \|\Sigma\|_2 \|X\|_{2,\infty}^2 \log \left( \frac{2d}{\delta} \right)}{nm^2}} + \kappa_2 \frac{d^2 \|X\|_{2,\infty}^2 \log \left( \frac{2d}{\delta} \right)}{nm^2}
\]

We defer the proofs of both theorems to Section 6. Note that both theorems are distribution-free; we make no assumptions on the data apart from the boundedness of \( \|X\|_\infty \) and \( \|X\|_{2,\infty} \) which both appear in our deviation bounds. In particular, the data sequence can be adversarially generated. However, to compare with existing work, it is best to specialize to the case when the data vectors come from a gaussian distribution. Using gaussian tail bounds on \( \|X\|_{2,\infty} \) and \( \|X\|_\infty \) yields the following corollary:

**Corollary 4.** Let \( X_1, \ldots, X_n \sim \mathcal{N}(0, \Sigma) \) and construct \( \hat{\Sigma} \) as in Equation 7 using compressive measurements. Then for any \( \delta \in (0, 1) \), there exist universal constants \( c, \kappa_1, \kappa_2, \kappa_3 > 0 \) such that, with probability at least \( 1 - c\delta \):

\[
\|\hat{\Sigma} - \Sigma\|_\infty \leq \kappa_1 \|\Sigma\|_\infty \left( \sqrt{\frac{d^2 \log^3 (nd/\delta)}{nm^2}} + \frac{\log(d/\delta)}{n} \right) + \kappa_2 \sqrt{\|\Sigma\|_\infty \left( \frac{d^2 \log^3 (nd/\delta)}{nm^2} \right)} \quad (5)
\]

\[
\|\hat{\Sigma} - \Sigma\|_2 \leq \kappa_2 \|\Sigma\|_2 \left( \sqrt{\frac{d^3 \log^2 (nd/\delta)}{nm^2}} + \frac{d^3 \log^2 (nd/\delta)}{nm^2} + \frac{\log(2d/\delta)}{n} \right) \quad (6)
\]

Here we make several remarks:
1. Ignoring logarithmic factors, the rates of convergence for our estimator are $\tilde{O}(\sqrt{\frac{d}{nm^2}})$ in $\ell_\infty$ norm and $\tilde{O}(\sqrt{\frac{d}{nm^2}})$ in spectral norm. In comparison, to estimate the population covariance of a Gaussian distribution in the fully observed setting, it is well known that the sample covariance achieves rates $\tilde{O}(\sqrt{\frac{d}{n}})$ and $\tilde{O}(\sqrt{\frac{d}{n}})$ in infinity and spectral norm respectively. Thus, the effective sample size shrinks from $n$ to $nm^2/d^2$ in the compressed setting.

2. Apart from our previous work [12], this is the first estimator with such strong guarantees in the compressed setting. Existing work focuses on recovery under strong structural assumptions of the population covariance, for example low rank [3, 1] or spiked covariance [1]. The other related line of work is on using random projections for matrix approximation, but the majority of the methods are not applicable here as they require observing the target matrix, which is not available in our setting.

3. In comparison with our previous work [12], the results here are significantly more refined. First, the bounds here hold even when the compression dimension $m$ is one, which previously we were unable to handle as we used a data-splitting technique to avoid the bias demonstrated in Proposition 1. Secondly, in terms of rates, the results here are actually sharper. In Krishnamurthy et al. [12], the rate of convergence in spectral norm for the Gaussian case is $\tilde{O}(\sqrt{\frac{d}{nm^2}} + \frac{d^2}{nm^2})$ which is polynomially worse than our $\tilde{O}(\sqrt{\frac{d^3}{nm^2}})$ bound here.

The proof of these results are based on showing that $\hat{\Sigma}_1$ concentrates sharply around its mean. For the spectral norm bound in Theorem 3 one can use the Matrix Bernstein inequality, and use properties of Beta random variables to upper bound the variance term. For the infinity norm bound, such a crude analysis does not suffice, as the random variables used have large range, although the tails decay quite quickly. Instead we use a conditioning argument where we first provide a probabilistic bound on the range of the random variables, and then apply the Matrix Bernstein inequality conditioned on this event. The corollaries are based on using well-known gaussian deviation inequalities to bound the quantities that depend on $X$ in the theorems.

4.1 Guarantees for Subspace Learning

In the subspace learning problem, the goal is just to estimate the principal components of the data, which amounts to the leading eigenvectors of the covariance matrix. Specifically, if the covariance matrix $\Sigma$ has eigendecomposition $\sum_{i=1}^d \lambda_i v_i v_i^T$ with $\lambda_1 \geq \ldots \geq \lambda_d$, then our goal is to estimate the leading $k$ eigenvectors $v_1, \ldots, v_k$. Let $V_k \in \mathbb{R}^{d \times k}$ be a matrix whose columns are the leading $k$ eigenvectors and let $\Pi_k = V_k V_k^T$ be the orthogonal projection onto the span of these vectors. The goal of subspace learning is to recover a projection matrix $\hat{\Pi} \in \mathbb{R}^{d \times d}$ that is close to $\Pi_k$ in spectral norm.

The usual approach to subspace learning is to first build an estimate of the covariance matrix $\hat{\Sigma}$ and then use the leading eigenvectors of the covariance estimate as the subspace estimate. In our compressive setting, we estimate the covariance with $\hat{\Sigma}$ from Equation 1 and compute the eigendecomposition $\hat{\Sigma} = \sum_{i=1}^d \hat{\lambda}_i \hat{v}_i \hat{v}_i^T$ and let $\hat{\Pi}$ be the projection onto the leading $k$ eigenvectors of $\hat{\Sigma}$.

To obtain theoretical guarantees for this problem, we introduce the standard notion of signal strength for the subspace learning problem. The signal strength in this problem is $\gamma_k = \lambda_k - \lambda_{k+1}$ which is commonly referred to as the eigengap. If $\gamma_k$ is large, then the principal subspace is well separated from the remaining directions, whereas if $\gamma_k$ is zero, then the principal subspace is actually unidentifiable.
Incorporating this signal strength into our estimation error bounds immediately results in the following result on subspace learning from compressive measurements. For clarity we present this result in the Gaussian setting.

**Corollary 5.** Let $X_1, \ldots, X_n \sim \mathcal{N}(0, \Sigma)$ and consider the compressive sampling model. For any $\delta \in (0, 1)$, with probability at least $1 - \delta$:

$$||\hat{\Pi} - \Pi_k||_2 \leq \frac{\kappa_2 ||\Sigma||_2}{\gamma_k} \left( \frac{d^3 \log^2(nd/\delta)}{nm^2} + \frac{d^3 \log^2(nd/\delta)}{nm^2} + \sqrt{\frac{\log(2d/\delta)}{n}} \right)$$

This corollary is a consequence of Corollary 4 followed by the celebrated Davis-Kahan theorem [6] characterizing how perturbing a matrix affects the eigenvectors. The only other result for this specific problem is our previous work [12], which as we mentioned uses a guarantee for covariance estimation, and hence is weaker than this result. Other results for subspace learning in other measurement settings are similar in spirit to the one here [9].

### 4.2 Consequences for Distributed Compressive Sensing

In distributed sensor networks, one is often tasked with performing statistical analysis under both measurement and communication constraints. In the distributed covariance estimation problem, the data vectors $x_1, \ldots, x_n$ are observed at $n$ sensors $s_1, \ldots, s_n$ (i.e. sensor $s_t$ observes sample $x_t$), and we would like to estimate the covariance structure of the $x_t$s using limited measurement and communication overhead. Typically communication is with a fusion center that aggregates the measurements from all of the sensors and performs any additional computation. Our approach provides a low-cost solution to these problems.

In our approach, each sensor $s_t$ makes $m$ compressive measurements of the signal $x_t$, computes the back-projection $\Phi_t x_t$ and sends this to the fusion center. This approach has measurement cost $O(nm)$ and communication cost $O(nd)$ as $d$-dimensional vectors must be transmitted to the fusion center. If the orthonormal bases $A_t$ used for sensing are synchronized with the fusion center before data acquisition, then the sensors can instead transmit $A_t^T x_t$, which would result in $O(nm)$ communication cost. As we saw the error depends on the effective sample size $nm^2/d^2$ so a practitioner can adjust $m$ to tradeoff between measurement overhead and statistical accuracy. One extreme of this tradeoff does not compress the signals at all during measurement; this naïve approach has $O(nd)$ measurement and communication cost.

The other natural approach synchronizes a single measurement matrix $A \in \mathbb{R}^{d \times m}$ with all sensors prior to data acquisition and obtains the samples $A^T x_t$. As we will see in Proposition 8, this approach is not consistent for the covariance estimation problem unless $m = d$ in which case it is equivalent to the naïve approach. Lastly, one could imagine a strategy based on the matrix approximation algorithm of Halko, Martinsson, and Tropp [10] or any of the related matrix approximation algorithms. However, these approaches all take compressive measurements on both the rows and the columns of the matrix $X$ with columns $x_t$, and it is not clear how the compressive measurements of the rows can be done in a distributed setting. So these methods do not appear to be applicable.

Thus, we find that our approach precisely quantifies the tradeoff between measurement and communication overhead on one hand and statistical accuracy on the other hand.
5 Lower Bounds

We now turn to establishing lower bounds for the compressive covariance estimation problem. These lower bounds show that, modulo logarithmic factors, our estimator is rate-optimal. This means that our estimator achieves the best performance one could hope for in terms of the problem parameters $n, m, d$.

The quantity of study in this section is the minimax risk, which is the worst case error of the best estimator. Specifically, it is the infimum over all measurable estimators $\hat{\Sigma}$ of the supremum over all covariance matrices $\Sigma$ of the expected error (in infinity or spectral norm) of the estimator when the data is generated according to a distribution with covariance $\Sigma$. Formally, we are interested in lower bounding:

$$\inf_{\hat{\Sigma}} \sup_{\Sigma \in \Theta} \mathbb{E}_{X^n_\ell \sim \mathcal{P}_\Phi} \left[ \|\hat{\Sigma}(\{(\Phi_i, \Phi_iX_i)\}_{i=1}^n) - \Sigma\|_{\ell} \right]$$

where the norm $\| \cdot \|$ is either the infinity or the spectral norm and the class $\Theta$ is some subset of the semidefinite cone in $d$ dimensions. Here we are parametrizing the estimator $\Sigma$ by the projection operators $\Phi_i$ and the observations $\Phi_iX_i$, and we are implicitly assuming that the projections are $m$-dimensional and drawn uniformly at random. Note that Theorems 6 and 7 give upper bounds on this minimax risk in both norms of interest for appropriate classes $\Theta$. Here we are interested in lower bounds.

Let $\Theta(\ell_\infty, \eta, d)$ denote the set of $d$-dimensional covariance matrices with $\ell_\infty$ norm bounded by $\eta$. Our first theorem lower bounds the minimax $\ell_\infty$ error when the data is generated according to a zero-mean gaussian with covariance matrix $\Sigma \in \Theta(\ell_\infty, \eta, d)$.

**Theorem 6.** If $\sqrt{\frac{1}{6} \frac{d^2}{nm^2}} \leq 1$, and $d \geq 4$, then we have:

$$\inf_{\Sigma} \sup_{\Sigma \in \Theta(\ell_\infty, \eta, d)} \mathbb{P}_{X^n_\ell \sim \mathcal{N}(0, \Sigma)} \left[ \|\hat{\Sigma}(\{(\Phi_i, \Phi_iX_i)\}_{i=1}^n) - \Sigma\|_{\ell_\infty} \right] \geq \eta \sqrt{\frac{d^2 \log d}{6nm^2}} \geq \frac{1}{3} - o(1) \quad (7)$$

For the spectral norm lower bound, let $\Theta(\ell_2, \eta, d)$ denote the set of $d$-dimensional covariance matrices with spectral norm bounded by $\eta$. The following theorem lower bounds the spectral norm error when the data is generated according to a zero-mean gaussian with covariance matrix $\Sigma \in \Theta(\ell_2, \eta, d)$.

**Theorem 7.** If $\sqrt{\frac{1}{192} \frac{d^3}{nm^2}} \leq 1$ and $d \geq 6$, then we have:

$$\inf_{\Sigma} \sup_{\Sigma \in \Theta(\ell_2, \eta, d)} \mathbb{P}_{X^n_\ell \sim \mathcal{N}(0, \Sigma)} \left[ \|\hat{\Sigma}(\{(\Phi_i, \Phi_iX_i)\}_{i=1}^n) - \Sigma\|_2 \right] \geq \frac{\eta}{60} \sqrt{\frac{d^3}{nm^2}} \geq \frac{1}{4}$$

These two bounds hold in the gaussian setting and should be compared with the bounds in Corollary 4.

While the constants and logarithmic factors disagree slightly, we see that the leading terms in the rates match in their dependence on $n, m, d$. This shows that our estimator achieves the minimax rate, modulo logarithmic factors. Note that in the lower bounds, one should set $\eta = \|\Sigma\|_{\ell_\infty}$ or $\|\Sigma\|_2$ respectively, so that our upper bounds also agree in the dependence on the boundedness of the covariance parameter.

Lastly, we consider a different compression scheme which has been popular in the literature 11, 4.

Here, rather than drawing an independent random projection for each data vector, we use the same random projection on every sample. While this approach has appealing properties in the distributed sensing setting, we show that one cannot consistently estimate the covariance matrix unless $m \rightarrow d$. Intuitively, the challenge is that one simply does not see $d - m$ directions of the covariance matrix, even though these directions are chosen at random, so one cannot hope to estimate the energy in these directions. This intuition is formalized in the following:
Proposition 8 (Fixed compression operator lower bound). As long as \( m < d \),
\[
\inf_T \sup_{\Sigma} \mathbb{E}_{\Pi} \|T(\Pi, \Pi\Sigma) - \Sigma\|_2 \geq \sqrt{1/2} \left( 1 - \frac{m}{d} \right)^{1/4}
\]
so that consistent estimation of \( \Sigma \) is impossible with fixed \( m \)-dimensional projection operator.

Notice that in this theorem, the estimator \( T \) actually has access to a compressed version of the population covariance \( \Sigma \). The expectation is over the randomness in the projection, and holds in the population sense, with no dependence on the number of samples \( n \). Consequently, we see that consistent recovery of the covariance matrix \( \Sigma \) is not possible unless \( m = d \), in which case it is trivial. This shows that this fixed-compression sampling scheme is not suitable for the unstructured covariance estimation problem. Note however that a fixed compression operator is suitable for problems where the target covariance is low rank or sparse, as shown in recent work [1, 4].

6 Proofs

In this section we provide proofs of our main theorems and corollaries. We begin by introducing some tools that we will use in the proofs, turn next to the upper bounds, and close this section with proofs of the lower bounds. To maintain readability, the proofs of all lemmas stated in this section are deferred to the appendices.

6.1 Preliminary Tools

We will make extensive use of the properties of Beta distribution. To aid readability, we collect several facts here. A random variable \( \omega \) supported on \([0, 1]\) is said to be Beta distributed with shape parameters \( \alpha, \beta > 0 \) if it has pdf
\[
p(\omega) = \omega^{\alpha-1}(1-\omega)^{\beta-1} \frac{1}{B(\alpha, \beta)}
\]
where \( B(\cdot, \cdot) \) is the Beta function.

Fact 9 (Properties of Beta Random Variables). The following facts involving random projections and the Beta hold:

1. Let \( a \sim \chi_m^2, b \sim \chi_{d-m}^2 \) be random variables. Then
\[
\frac{a}{a+b} \sim \text{Beta}\left( \frac{m}{2}, \frac{d-m}{2} \right).
\]

2. Let \( \omega \sim \text{Beta}\left( \frac{m}{2}, \frac{d-m}{2} \right) \). Then:
\[
\mathbb{E}[\omega^i] = \frac{m+2(j-1)}{d+2(j-1)}
\]
\[
\mathbb{E}[\omega^2] = \frac{m(d-m)}{d(d+2)}
\]

3. If \( x \in \mathbb{R}^d \) and \( \Phi \in \mathbb{R}^{d \times d} \) is a uniformly distributed random rank \( m \) orthogonal projection operator, then:
\[
\Phi x \overset{d}{=} \omega x + \sqrt{\omega - \omega^2} \|x\| W \alpha
\]
where \( \omega \sim \text{Beta}\left( \frac{m}{2}, \frac{d-m}{2} \right) \), \( \alpha \in \mathbb{R}^{d-1} \) distributed uniformly on the sphere and \( W \in \mathbb{R}^{d \times (d-1)} \) is an orthonormal basis for the subspace orthogonal to \( x \), i.e. \( x^T W = 0 \).
To establish our lower bounds on spectral norm error, we will need a packing of the $\ell_2$ sphere in an appropriate metric. This metric is the spectral norm between the corresponding rank one matrices, i.e. $d(u, v) = \|uu^T - vv^T\|_2$, which is a metric provided that $u$ is identified with $-u$. Asymptotic results of this form exist in the approximation theory literature, where one classical result is the Chabauty-Shannon-Wyner theorem [8]. For our purposes the following non-asymptotic statement, which is weaker than the results in the literature, will suffice: This lemma is proved in the appendix using the probabilistic method.

**Lemma 10** (Packing number lower bound). For any $\tau \in [0.5, 1]$, if $d(1 - \tau^2) \geq 4$, then there exists a set of unit vectors $\{v_j\}_{j=1}^M$ in $\mathbb{R}^d$ of size $M \geq \exp(-1/8)\tau^{-d/8}$ such that $\|v_i v_i^T - v_j v_j^T\|_2 \geq \tau$ for all $i \neq j$.

The next tool we require is the celebrated Davis-Kahan sine theorem, which is a standard result from matrix perturbation theory.

**Theorem 11** (Davis-Kahan Theorem [6]). Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix with eigenvectors $v_1, \ldots, v_n$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Let $A \in \mathbb{R}^{n \times n}$ also be a symmetric matrix with eigenvectors $u_1, \ldots, u_n$ and eigenvalues $\mu_1 \geq \ldots \geq \mu_n$. Then:

$$\sin \angle v_i, u_i \leq \frac{\|M - A\|_2}{\delta_i}$$

where $\delta_i = \min_{j \neq i} |\lambda_i - \lambda_j|$.

Finally we will use two standard concentration inequalities controlling the deviation between the sample covariance of a collection of gaussian vectors and the population covariance.

**Proposition 12.** Let $X_1, \ldots, X_n \sim \mathcal{N}(0, \Sigma)$ in $\mathbb{R}^d$ where $d \geq 4$. Then there exists a universal constant $c > 0$ such that for any $\delta \in (0, 1)$, the following tail bounds on the matrix $X \in \mathbb{R}^{n \times d}$ hold:

$$\mathbb{P}\left(\|X\|_\infty \leq \sqrt{2\|\Sigma\|_\infty \log(2d/\delta)}\right) \geq 1 - \delta$$

$$\mathbb{P}\left(\|X\|_{2,\infty} \leq \sqrt{2 \text{tr}(\Sigma) \log(2d/\delta)}\right) \geq 1 - \delta$$

$$\mathbb{P}\left(\left\|\frac{1}{n} \sum_{t=1}^n X_t X_t^T - \Sigma\right\|_2 \leq \|\Sigma\|_2 \sqrt{\frac{c \log(2d/\delta)}{n}}\right) \geq 1 - \delta$$

$$\mathbb{P}\left(\left\|\frac{1}{n} \sum_{t=1}^n X_t X_t^T - \Sigma\right\|_\infty \leq \|\Sigma\|_\infty \sqrt{\frac{\log(2d/\delta)}{n}}\right) \geq 1 - \delta$$

All four of these are standard results. The first two follow from gaussian tail bounds and a union bound over the $n$ vectors. The third result is based on random matrix theory, and shows that the usual sample covariance matrix is a good estimator for the population in spectral norm [20]. The last bound uses $\chi^2$ tails to give $\ell_\infty$ norm bounds on the error of sample covariance matrix.

### 6.2 Proof of Proposition 1

It suffices to consider $n = 1$ as the result will follow by linearity of expectation. Let $x = x_1$ and $\Phi = \Phi_1$.

By Fact [9] we know that $\Phi x \overset{d}{=} \omega x + \sqrt{\omega - \omega^2}||x||_W \alpha$. Since we now know the squared norm of $\Phi x$, we know that the angle between $x$ and $\Phi$ is $\cos^{-1}(\sqrt{\omega})$, which means that the magnitude of $\Phi x$ in the direction of $x$ is $||\Phi x||_2 \cos(\sqrt{\omega}) = \omega ||x||_2$. By the Pythagorean Theorem, the magnitude of $\Phi x$ in the orthogonal
direction must therefore be \(|x|_2 \sqrt{\omega - \omega^2}\), and the direction is chosen uniformly at random, subject to being orthogonal to \(x\). This gives the identity \(\Phi x = \omega x + \sqrt{\omega - \omega^2} \|x\| W\alpha\).

This identity means that:

\[
\Phi xx^T \Phi = \left( \omega x + \sqrt{\omega - \omega^2} \|x\| W\alpha \right) \left( \omega x + \sqrt{\omega - \omega^2} \|x\| W\alpha \right)^T
\]

\[
= \omega^2 xx^T + (\omega - \omega^2) \|x\|^2 W\alpha \alpha^T W^T + \omega \sqrt{\omega - \omega^2} \|x\| \left( x\alpha^T W^T + W\alpha x^T \right)
\]

By linearity of expectation we can analyze each term individually. By Fact 9 and the distribution of \(\Phi xx\), we know that \(E\omega = \frac{m}{d}\), \(E\omega^2 = \frac{m(m+2)}{d(d+2)}\), \(E\alpha = 0\) and \(E\alpha \alpha^T = \frac{1}{d-1} I_{d-1}\) since \(\alpha\) is distributed uniformly on the \(d - 1\) dimensional sphere. This means that:

\[
E\Phi xx^T \Phi = \frac{m(m+2)}{d(d+2)} xx^T + \frac{m(d-m)}{d(d+2)} \|x\|^2 \left( I - \frac{xx^T}{\|x\|^2} \right)
\]

\[
= \frac{m(m+2)}{d(d+2)} xx^T + \frac{m(m-m)}{d(d+2)(d-1)} \|x\|^2 \left( I - \frac{xx^T}{\|x\|^2} \right)
\]

Note that \(\|x\|^2 = \text{tr}(xx^T)\). Proposition 9 then follows by linearity of expectation, and using the same expansion for all of the \(n\) samples and rescaling by \(d^2/m^2\).

### 6.3 Upper Bounds

Recall that \(\hat{\Sigma}_1 = \frac{d^2}{nm^2} \sum_{t=1}^n \Phi_t x_t x_t^T \Phi_t\) is the observed covariance. Define:

\[
\Sigma = \frac{d(d + d - 2)}{m(m + 2)(d - 1)} \Sigma + \frac{d(d - m)}{m(d + 2)(d - 1)} \text{tr}(\Sigma) I
\]

which is the expectation of \(\hat{\Sigma}_1\) from Proposition 1. The proofs of both infinity and spectral norm bounds follow by arguing that \(\Sigma_1\) is close to \(\Sigma\) and then using this fact to relate our estimator \(\hat{\Sigma}\) to the estimand of interest \(\Sigma\).

#### 6.3.1 \(\ell_\infty\)-norm Bound

We begin with the \(\ell_\infty\) norm bound. For this result, we use an intermediary deviation bound on quadratic forms.

**Lemma 13** (Quadratic-Form Deviation Bound). Let \(d \geq 6\). For any unit vector \(\mathbf{u} \in \mathbb{R}^d\) and for any \(\delta \in (0, 1)\) with \(\delta \leq n/e \) and \(\log(1/\delta) \leq \frac{n}{4(d-2)(d+5)^2(d+1)}\), with probability \(\geq 1 - 4\delta\) we have:

\[
| \mathbf{u}^T \hat{\Sigma}_1 \mathbf{u} - \mathbf{u}^T \Sigma \mathbf{u} | \leq \sqrt{\frac{d^2 \log(2/\delta)}{nm^2}} \left[ \sqrt{18 \log(n/\delta)} \left( b^2 + \frac{8c^2}{d} \right) + \frac{8c^2}{d} \right] + \frac{d^2 \log(2/\delta)}{nm^2} \left( b + 2 \sqrt{c^2 \log(n/\delta) \over d} \right)
\]
where \( b = \max_{i \in [n]} x_i^T u \) and \( c = \max_{i \in [n]} \sqrt{\|x_i\|^2_2 - (x_i^T u)^2} \).

**Proof.** The proof involves a careful concentration of measure argument and is the crux of our analysis for the \( \ell_\infty \) norm bound. First we use the distributional characterization of the projection operator to expand the expression \( u^T \hat{\Sigma}_1 u \) in terms of several Beta random variables. Then we perform a two-step analysis; we first control the randomness in one set of these random variables, leaving dependence on the other set. This gives a large-deviation bound involving the remaining random variables in several places. We next control all of these terms, which involves several more deviation bounds.

Let \( b_t = x_t^T u \) and let \( c_t = \sqrt{\|x_t\|^2_2 - (x_t^T u)^2} \). Then:

\[
\begin{align*}
(u^T \hat{\Sigma}_1 u) &= \frac{d(dm + d - 2)}{m(d + 2)(d - 1)} u^T \Sigma u + \frac{d(d - m)}{m(d + 2)(d - 1)} \text{tr}(\Sigma) \\
&= \frac{1}{n} \sum_{t=1}^n \frac{d(m + 2)}{m(d + 2)} (x_t^T u)^2 + \frac{d(d - m)}{m(d + 2)(d - 1)} (\|x_t\|^2_2 - (x_t^T u)^2) \\
&= \frac{1}{n} \sum_{t=1}^n \frac{d(m + 2)}{m(d + 2)} b_t^2 + \frac{d(d - m)}{m(d + 2)(d - 1)} c_t^2.
\end{align*}
\]

We now expand the term involving \( \hat{\Sigma}_1 \). Write \( \Phi_t u = \omega_t u + \sqrt{\omega_t - \omega_t^2} W_t \alpha_t \) where \( \omega_t \sim \text{Beta}(\frac{d}{2}, \frac{d - m}{2}) \), \( \alpha_t \in \mathbb{R}^{d-1} \) is distributed uniformly on the unit sphere, and \( W_t \) is an orthonormal basis for the subspace orthogonal to \( u \). Then:

\[
\begin{align*}
(u^T \hat{\Sigma}_1 u) &= \frac{d}{nm^2} \sum_{t=1}^n \left( \omega_t x_t^T u + \sqrt{\omega_t - \omega_t^2} x_t^T W \alpha_t \right)^2 \\
&= \frac{d}{nm^2} \sum_{t=1}^n \left( \omega_t^2 (x_t^T u)^2 + (\omega_t - \omega_t^2) (x_t^T W \alpha_t)^2 + 2 \omega_t x_t^T u \sqrt{\omega_t - \omega_t^2} x_t^T W \alpha_t \right) \\
&= \frac{d}{nm^2} \sum_{t=1}^n \left( \omega_t^2 b_t^2 + (\omega_t - \omega_t^2) \|x_t^T W\|_2^2 c_t \nu_t + 2 \sigma_t \omega_t \sqrt{\omega_t - \omega_t^2} \|x_t^T W\|_2 \sqrt{\nu_t} \right).
\end{align*}
\]

Here \( \nu_t \sim \text{Beta}(\frac{1}{2}, \frac{d-2}{2}) \) while \( \sigma_t \) is a Rademacher random variable, i.e. it takes value \(-1\) with probability \( \frac{1}{2} \) and value \( 1 \) with probability \( \frac{1}{2} \).

The first equivalence follows from writing \( \hat{\Sigma}_1 = \frac{d}{nm^2} \sum_{t=1}^n \Phi_t x_t x_t^T \Phi_t \) and grouping the projections instead with the \( u \) vectors. The second equivalence is just an expansion of the squared term. For the third equivalence, notice that \( x_t^T W \in \mathbb{R}^{d-1} \) while \( \alpha_t \in \mathbb{R}^{d-1} \) is distributed uniformly on the unit sphere. We can think of \( \alpha_t \) as a one-dimensional projection operator, and by the geometric argument from before, we know that the squared norm of the projection is distributed as a Beta(\( \frac{1}{2}, \frac{d-2}{2} \)) random variable, scaled by the squared-length of the original vector \( x_t^T W \). We use the same argument for the third term, except we introduce the Rademacher random variable because the \( x_t^T W \alpha_t \) is symmetric about zero. The fourth equivalence follows from the fact that \( WW^T = I - uu^T \) and therefore \( \|x_t^T W\|_2^2 = x_t^T WW^T x_t = \|x_t\|^2_2 - (x_t^T u)^2 \).

Now consider all of the \( \nu_t \) random variables fixed, and we will develop a deviation bound for the remaining randomness. We will apply Bernstein’s inequality, so we need to bound the variance and the range. By
Fact 9 we know that:

\[
\text{Var}(\frac{d^2}{m^2}\omega_t^2b_t^2) = \frac{d^4b_t^4}{m^4}(\mathbb{E}\omega_t^2 - (\mathbb{E}\omega_t^2)^2) = \frac{d^4b_t^4}{m^4}\left(\frac{m(m+2)(m+4)(m+6)}{(d+2)(d+4)(d+6)} - \left(\frac{m(m+2)}{d(d+2)}\right)^2\right) \leq V_{1t}
\]

\[
\text{Var}(\frac{d^2}{m^2}\omega_t^2(\omega_t - \omega_t^2)\nu_t) = \frac{d^4c_t^4}{m^4}\nu_t^2\left(\frac{m(m+2)}{d(d+2)}\left(1 + \frac{(m+4)(m+6)}{(d+4)(d+6)} - 2\left(\frac{m(m+4)}{(d+4)}\right) - \left(\frac{m(m-m)}{d(d+2)}\right)\right)^2 \leq V_{2t}
\]

\[
\text{Var}\left(\frac{2d^2}{m^2}\sigma_t b_t c_t \omega_t \sqrt{\omega_t - \omega_t^2} \nu_t\right) = 4\frac{d^4c_t^2}{m^4}\nu_t^2\frac{m(m+2)(m+4)}{d(d+2)(d+6)}\left(1 - \frac{m+6}{d+6}\right) \leq V_{3t}
\]

Therefore,

\[
\text{Var}\left(\frac{d^2}{m^2}\left(\omega_t^2b_t^2 + (\omega_t - \omega_t^2)c_t^2\nu_t + 2\sigma_t b_t c_t \omega_t \sqrt{\omega_t - \omega_t^2} \nu_t\right)\right) \leq 3(V_{1t} + V_{2t} + V_{3t})
\]

As for the range, by straightforward calculation, we have:

\[
\left|\frac{d^2}{m^2}\left(\omega_t^2b_t^2 + (\omega_t - \omega_t^2)c_t^2\nu_t + 2\sigma_t b_t c_t \omega_t \sqrt{\omega_t - \omega_t^2} \nu_t\right)\right| \leq \frac{d^2}{m^2}\left(b_t^2 + \frac{1}{4}c_t^2\nu_t + \frac{2}{3}b_t c_t \sqrt{\nu_t}\right)
\]

The last term is actually maximized when \(\omega_t = 3/4\) and takes value \(3\sqrt{3}/8 \leq 2/3\). Bernstein’s inequality now reveals that with probability at least \(1 - \delta\):

\[
\left|u^T \hat{S}_1 u - \mathbb{E}_{\omega_t, \sigma_t} u^T \hat{S}_1 u\right| \leq \sqrt{\frac{6\log(2/\delta)}{n}} \sqrt{\frac{1}{n} \sum_{t=1}^n \left(V_{1t} + V_{2t} + V_{3t}\right) + \frac{2d^2\log(2/\delta)}{3nm^2} \max_{t \in [n]} \left(b_t^2 + \frac{1}{4}c_t^2\nu_t + \frac{2}{3}b_t c_t \sqrt{\nu_t}\right)}
\]

The expectation here is:

\[
\mathbb{E}_{\omega_t, \sigma_t} u^T \hat{S}_1 u = \frac{1}{n} \sum_{t=1}^n \frac{d(m+2)}{m(d+2)} b_t^2 + \frac{d(d-m)}{m(d+2)} c_t^2 \nu_t
\]

So in expectation over \(\omega_t, \sigma_t\) we have the deviation:

\[
\left|\mathbb{E}_{\omega_t, \sigma_t} u^T \hat{S}_1 u^T - u^T \hat{S}_1 u\right| = \left|\frac{1}{n} \sum_{t=1}^n \frac{d(d-m)}{m(d+2)} c_t^2 \nu_t - \frac{d(d-m)}{m(d+2)(d+1)} c_t^2\right|
\]

(9)

We are left to control all of the terms involving the \(\nu_t\) random variables. By Proposition 21 we have that for any \(\delta_1 > 0\), provided that \(\log(1/\delta_1) \leq \frac{n}{4} \frac{(d-2)(d+5)^2}{(d-1)^2(d+1)}\), then:

\[
\mathbb{P}\left(\frac{1}{n} \sum_{t=1}^n c_t^2 \nu_t - \frac{1}{n} \sum_{t=1}^n \frac{1}{d-1} c_t^2 \nu_t \right) > \sqrt{\frac{12 \log(1/\delta_1)}{n} \frac{2(d-2)c_t^4}{(d-1)^2(d+1)}} \leq \delta_1
\]

where \(c = \max_{t \in [n]} c_t\). By Proposition 22 we have that for any \(\delta > 0\),

\[
\mathbb{P}\left(\frac{1}{n} \sum_{t=1}^n \frac{1}{d-1} c_t^2 - \frac{1}{n} \sum_{t=1}^n c_t^2 \nu_t \right) > \sqrt{\frac{6c^4 \log(1/\delta)}{(d^2-1)n}} \leq \delta.
\]

13
These two bounds control the deviation in the right hand side of Equation 9. Finally, by Proposition 25 we have that for any \( \delta > 0 \),
\[
P \left( \max_{t \in [n]} \nu_t > \frac{2}{d-3} \log(n/\delta) \right) \leq \delta
\]
This last bound allows us to control the \( V_2t, V_3t \) terms. Specifically, we have:
\[
V_{1t} \leq \frac{d^4b^4}{m^4} \left( \frac{m(m+2)(m+4)(m+6)}{d(d+2)(d+4)(d+6)} - \left( \frac{m(m+2)}{d(d+2)} \right)^2 \right) \triangleq V_1'
\]
\[
V_{2t} \leq \frac{d^4c^4}{m^4} \frac{4 \log^2(n/\delta)}{(d-3)^2} \left( \frac{m(m+2)}{d(d+2)} \left( 1 + \frac{(m+4)(m+6)}{(d+4)(d+6)} - \frac{2(m+4)}{(d+4)} - \left( \frac{m(d-m)}{d(d+2)} \right)^2 \right) \right) \triangleq V_2'
\]
\[
V_{3t} \leq \frac{4d^4b^2c^2}{m^4} \frac{2 \log(n/\delta)}{(d-3)m(d+2)} \left( \frac{m(m+2)(m+4)}{d(d+2)(d+4)} \left( 1 - \frac{m+6}{d+6} \right) \right) \triangleq V_3'
\]

where \( b = \max_{t \in [n]} b_t \). It also controls the terms \( c_t^2 \nu_t \) in the range term of our application of Bernstein’s inequality. Combining all of the bounds gives:
\[
\left| u^T \hat{\Sigma} u - u^T \bar{\Sigma} u \right| \leq \sqrt{\frac{6 \log^2(2/\delta)}{n}} \sqrt{V_1' + V_2' + V_3'} +
\]
\[
+ \frac{2d^2 \log(2/\delta)}{3nm^2} \left( b^2 + \frac{1}{4}c^2 \frac{2}{d-3} \log(n/\delta) + \frac{2}{3}bc \sqrt{\frac{2}{d-3} \log(n/\delta)} \right) +
\]
\[
+ \frac{d(d-m)}{m(d+2)} \left( \sqrt{\frac{12 \log(1/\delta)}{n}} \frac{2(d-2)c^4}{(d-1)^2(d+1)} + \sqrt{\frac{6c^4 \log(1/\delta)}{n(d^2-1)}} \right)
\]
The second term on the right hand side can be upper bounded by:
\[
\frac{2d^2 \log(2/\delta)}{3nm^2} \left( b + 2c \sqrt{\frac{\log(n/\delta)}{d}} \right)^2
\]
While the third term, by setting \( \delta = \delta_1 \) can be upper bounded by:
\[
\frac{8d(d-m)}{m(d+2)} \sqrt{\frac{c^4 \log(1/\delta)}{n(d^2-1)}} \leq \frac{8d}{m} \sqrt{\frac{c^4 \log(1/\delta)}{nd^2}}
\]
We have bounds for \( V_1', V_2' \) and \( V_3' \):
\[
V_1' \leq \frac{b^4d^4(m+2)}{m^4(d+2)}
\]
\[
V_2' \leq \frac{8c^4d^4 \log(n/\delta) \ m(m+2)}{m^4(d-3)^2 \ d(d+2)}
\]
\[
V_3' \leq \frac{8d^2b^2c^2 \log(n/\delta) \ m(m+2)}{m^4(d-3) \ d(d+2)}
\]
Here we use the fact that \( m \leq d \) so terms of the form \( \frac{m+x}{d+x} \leq 1 \). This means that for \( \delta \leq n/e \) we have:

\[
\sqrt{V_1^2 + V_2^2 + V_3^2} \leq \frac{d^2}{m^2} \sqrt{\frac{m(m+2)}{d(d+2)} \log(n/\delta)} \left( b^2 + \frac{4c^2}{d} \right) \leq \sqrt{\frac{3d^2}{m^2} \log(n/\delta)} \left( b^2 + \frac{8c^2}{d} \right)
\]

This last inequality holds provided that \( d \geq 6 \). Putting everything together proves the claim.

We are now able to prove Theorem 2. Taking a union bound over all vectors \( u = \frac{x_i + x_j}{\sqrt{2}} \) for \( i, j \in [d] \) we obtain a bound on \( \| \Sigma_1 - \bar{\Sigma} \|_\infty \). Specifically, with probability \( 1 - \delta \) we have:

\[
\| \hat{\Sigma}_1 - \bar{\Sigma} \|_\infty \leq \kappa_1 \frac{d^2 \log(d/\delta)}{nm^2} \left( \sqrt{\log(nd/\delta)}(b^2 + c^2/d) + c^2/d \right) + \kappa_2 \frac{d^2 \log(d/\delta)}{nm^2} \left( b + \sqrt{c^2 \log(nd/\delta)} \right)
\]

for universal constants \( \kappa_1, \kappa_2 > 0 \) and with \( b = \| X \|_\infty \) and \( c = \| X \|_{2,\infty} \).

Using the relationship between \( \Sigma \) and \( \hat{\Sigma}_1 \) and the equivalent relationship between \( \Sigma \) and \( \bar{\Sigma} \), we then have:

\[
\| \hat{\Sigma} - \Sigma \|_\infty \leq \frac{m}{d(m + d - 2)} \left( (d + 2)(d - 1) \| \hat{\Sigma}_1 - \bar{\Sigma} \|_\infty + (d - m)d \| \hat{\Sigma}_1 - \bar{\Sigma} \|_\infty \right)
\]

which holds provided that \( d \geq 4 \). This proves the theorem.

### 6.3.2 Spectral-norm Bound

The proof for the spectral norm lower bound is much more straightforward as it follows from a direct application of the Matrix Bernstein Inequality (Theorem 18). We apply the Matrix Bernstein Inequality to the unnormalized sum \( \frac{m}{d^2} \hat{\Sigma}_1 \). Using Fact 9, we have:

\[
\frac{m^2}{d^2} \hat{\Sigma}_1 = \frac{1}{n} \sum_{i=1}^{n} \left( \omega_i x_i + \sqrt{\omega_i - \omega_i^2} \| x_i \|_2 W_i \alpha_t \right) \left( \omega_i x_i + \sqrt{\omega_i - \omega_i^2} \| x_i \|_2 W_i \alpha_t \right)^T
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \omega_i^2 x_i x_i^T + (\omega_i - \omega_i^2) \| x_i \|_2^2 W_i \alpha_t \alpha_t^T W_i^T \right) + \omega_i \sqrt{\omega_i - \omega_i^2} \| x_i \|_2 \left( x_i \alpha_t^T W_i^T + W_i \alpha_t x_i^T \right)
\]

\[
\triangleq \frac{1}{n} \sum_{i=1}^{n} Z(x_i, \omega_i, \alpha_t)
\]

It is easy to see that, almost surely:

\[
\max_{t \in [n]} \| Z(x_i, \omega_i, \alpha_t) \|_2 \leq 3 \max_{t \in [n]} \| x_i \|_2^2
\]

so that, almost surely:

\[
\max_{t \in [n]} \| Z(x_i, \omega_i, \alpha_t) - \mathbb{E} Z(x_i, \omega_i, \alpha_t) \|_2 \leq 6 \max_{t \in [n]} \| x_i \|_2^2
\]
It remains to bound the variance term. We use the following properties of \( \alpha_t, x_t \) and \( W_t \): (a) \( \mathbb{E}\alpha_t = 0 \), (b) \( \|\alpha_t\| = 1 \), (c) \( \mathbb{E}\alpha_t\alpha_t^T = \frac{1}{d-1}I_{d-1} \), (d) \( x_t^TW_t = 0 \), (e) \( W_t^T W_t = I_d \) and (f) \( W_t W_t^T = I_d - x_t x_t^T \).

These identities give us:

\[
\mathbb{E}Z(x_t, \omega_t, \alpha_t)^2 = (\mathbb{E}\omega_t^4)\|x_t\|^2 x_t^T + (\mathbb{E}(\omega_t^2 - \omega_t^3)^2)\|x_t\|^2 \mathbb{E}W_t \alpha_t \alpha_t^T W_t^T + \\
+ (\mathbb{E}\omega_t^2 (\omega_t - \omega_t^3))\|x_t\|^2 (x_t^T + \|x_t\|^2 \mathbb{E}W_t \alpha_t \alpha_t^T W_t^T) \\
= \left( \mathbb{E}\omega_t^3 - \mathbb{E}(\omega_t^2 - \omega_t^3) \right) \frac{1}{d-1} \|x_t\|^2 x_t^T + \mathbb{E}(\omega_t^2 - \omega_t^3) \frac{1}{d-1} \|x_t\|^2 Id.
\]

And this means that the variance term is bounded as:

\[
\left\| \frac{1}{n} \sum_{t=1}^n \mathbb{E}Z(x_t, \omega_t, \alpha_t)^2 - (\mathbb{E}Z(x_t, \omega_t, \alpha_t))^2 \right\|_2 \leq \left\| \frac{1}{n} \sum_{t=1}^n \mathbb{E}Z(x_t, \omega_t, \alpha_t^T)^2 \right\|_2 \\
\leq \left( \mathbb{E}\omega_t^3 \right) \|\Sigma\|_2 \|X\|_2^2 + \mathbb{E}(\omega_t^2 - \omega_t^3) \frac{1}{d-1} \left( \frac{1}{n} \sum_{t=1}^n \|x_t\|_2^4 \right) \\
\leq 15 \frac{m^3}{d^2} \|\Sigma\|_2 \|X\|_2^2 + 3 \frac{m^2}{d} \left( \frac{1}{n} \sum_{t=1}^n \|x_t\|_2^4 \right) \\
\leq 15 \frac{m^3}{d^2} \|\Sigma\|_2 \|X\|_2^2 + 3 \frac{m^2}{d} \|\Sigma\|_2 \|X\|_2^2 \\
\leq 18 \frac{m^2}{d} \|\Sigma\|_2 \|X\|_2^2
\]

Plugging these two bounds into Theorem 18 and adjusting for the normalization reveals that:

\[
\|\hat{\Sigma}_1 - \Sigma\|_2 \leq \frac{d^2}{m^2} \left( \sqrt{\frac{30 m^2}{n} \|\Sigma\|_2 \|X\|_2^2 \log \frac{2d}{\delta}} + 4 \frac{\|X\|_2^2 \log \frac{2d}{\delta}}{n} \right)
\]

As in the proof of the \( \ell_\infty \)-norm bound, we know that as long as \( d \geq 4 \), we have \( \|\hat{\Sigma} - \Sigma\|_2 \leq 3 \|\hat{\Sigma}_1 - \Sigma\|_2 \).

Combining this inequality with above proves the theorem.

### 6.3.3 Proof of Corollary 4

The corollary follows from Theorems 2 and 3 along with the Gaussian tail bounds in Proposition 12. Equipped with these bounds, we can plug in for \( \|X\|_\infty \) and \( \|X\|_2,\infty \) in Theorems 2 and 3. Then by the triangle inequality, we can bound \( \|\hat{\Sigma}_1 - \Sigma\| \leq \|\hat{\Sigma} - \frac{1}{n} \sum_{t=1}^n x_t x_t^T\| + \|\frac{1}{n} \sum_{t=1}^n X_t X_t^T - \Sigma\| \) and use the latter two bounds to complete the proof.

### 6.4 Lower Bounds

Our lower bounds involve a standard argument based on Fano’s inequality. In particular, we will apply the following theorem (See [18]).

**Theorem 14.** Assume that \( M \geq 2 \) and suppose that a parameter space \( \Theta \) contains elements \( \theta_0, \theta_1, \ldots, \theta_M \) associated with probability measures \( \mathbb{P}_0, \ldots, \mathbb{P}_M \) such that:
1. $d(\theta_i, \theta_j) \geq 2s > 0$ for all $0 \leq j < k \leq M$.

2. $P_j \ll P_0$ for all $j \in [M]$ and:

$$\frac{1}{M} \sum_{j=1}^{M} KL(P_j, P_0) \leq \alpha \log M$$

with $0 < \alpha < 1/8$.

Then:

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} P_{\theta} \left[ d(\hat{\theta}, \theta) \geq s \right] \geq \frac{\sqrt{M}}{1 + \sqrt{M}} \left( 1 - 2\alpha - \sqrt{\frac{2\alpha}{\log M}} \right) > 0 \quad (10)$$

To apply the theorem we will use our packing construction in Lemma[10] but will also need to control the Kullback-Leibler divergence between these distributions. The following lemma provides precisely this KL-divergence bound.

**Lemma 15 (KL-divergence bound).** Let $P_0$ be a distribution on $(z, U)$ where $U$ is an orthonormal basis for a uniform-at-random $m$-dimensional subspace, $x \sim \mathcal{N}(0, \eta I)$ and $z = U^T x$. Let $P_1$ be the same distribution but where $x \sim \mathcal{N}(0, \eta I + \gamma vv^T)$. We have:

$$KL(P_1^{n} || P_0^{n}) \leq 3 \gamma^2 \frac{nm^2}{2 \eta^2} d^2 \quad (11)$$

The lemma demonstrates that our compression model results in a contraction in the Kullback-Leibler divergence between these two Gaussian distributions. As the KL-divergence between the two Gaussians is $\Theta(\frac{\gamma^2}{\eta^2} n)$, this contraction is over order $m^2/d^2$. This is known in the literature as a form of **strong data-processing inequality** and it allows us to easily adapt existing lower bound constructions to our setting.

**Proof of Theorem 6** The goal of this proof is to apply Theorem[14] for a set of $d + 1$ distributions. The first distribution $P_0$ has the data vectors drawn from $\mathcal{N}(0, \eta I)$, while the distribution $P_j$ has the vectors drawn from $\mathcal{N}(0, \eta I - \gamma e_j e_j^T)$. Clearly the first condition of Theorem[14] is satisfied with $s = \gamma/2$. Secondly, the infinity norm bound on all covariance matrices is $\eta$ and to ensure positive semi-definiteness we require $\gamma \leq \eta$. Lastly, by Lemma[15] we have:

$$\frac{1}{d} \sum_{j=1}^{M} KL(P_j || P_0) \leq 3 \frac{\gamma^2}{2 \eta^2} \frac{nm^2}{d^2} \cdot$$

which means that we can set $\gamma \leq \eta \sqrt{\frac{2\alpha d^2 \log d}{3 \eta^2 nm^2}}$. The PSD constraint means that we require:

$$\sqrt{\frac{2\alpha d^2 \log d}{3 \eta^2 nm^2}} \leq 1,$$

which happens for $n$ large enough. Theorem[14] now states that:

$$\inf_{\Sigma} \sup_{\Sigma} \mathbb{P}_{\Sigma} \left[ \|\Sigma - \Sigma\|_{\infty} \geq \eta \sqrt{\frac{2\alpha d^2 \log d}{3 \eta^2 nm^2}} \right] \geq \frac{\sqrt{d}}{1 + \sqrt{d}} \left( 1 - 2\alpha - \sqrt{\frac{2\alpha}{\log d}} \right) \geq \nu^2 \left( 1 - 2\alpha - \sqrt{\frac{2\alpha}{\log d}} \right) \geq \nu^2 > 0 \quad (11)$$
We set $\alpha = 1/4$ and provided that $d \geq 4$, the bound becomes:

$$\inf \sup_{\Sigma} \left\| \hat{\Sigma} - \Sigma \right\|_\infty \geq \eta \sqrt{\frac{1}{6} \frac{d^2 \log d}{nm^2}} \geq \frac{1}{3} - o(1)$$

**Proof of Theorem 7** As before, the proof is based on an application of Theorem 14, but we will use exponentially many distributions. The first distribution $P_0$ has the data vectors drawn from $\mathcal{N}(0, \eta I)$. For the remaining distribution, let $\{v_j\}_{j=1}^M$ be the $1/2$-packing in the projection metric from Lemma 10. We know that $M \geq \exp(-1/8)d^2/8$. For each $j$, let $P_j$ be the distribution where the data vectors are drawn from $\mathcal{N}(0, \eta I - \gamma v_j v_j^T)$.

As before, the first condition of Theorem 14 is satisfied with $s = \gamma / 2$. Moreover the spectral norm of all covariance matrices is bounded by $\eta$ and to ensure positive semi-definiteness we require $\gamma \leq \eta$. Lastly, by Lemma 15, we have that the average KL is at most $\frac{3}{2} \frac{\eta^2 nm^2}{d^2}$. Plugging in for $M$, we require that:

$$\frac{3}{2} \frac{\gamma^2 nm^2}{d^2} \leq \alpha \left( \frac{d}{8} \log 2 - \frac{1}{8} \right)$$

which is satisfied if we set $\gamma^2 = \frac{\alpha d^3}{48 nm^2}$, provided that $d \geq 4$. Setting $\alpha = 1/16$ the right hand side of Theorem 14 is lower bounded by $1/4$ while the separation is $\left( \frac{d}{8} \log 2 - \frac{1}{8} \right)$. The condition on involving $n, m$ and $d$ is based on requiring that $\gamma \leq \eta$ for positive semidefiniteness, while the condition that $d \geq 6$ ensures that we may apply the Lemma 10. The constant $\frac{1}{60}$ is a lower bound on $\frac{1}{2} \sqrt{\frac{1}{768}}$.

**6.5 Proof of Proposition 8**

To prove Proposition 8, we will need one intermediate result. The following lemma lower bounds the minimax risk (in squared $\ell_2$ norm) in estimating a vector when it is observed via a low-dimensional random projection.

**Lemma 16** (Lower bound for compressed vector estimation). Let $U$ be the uniform distribution over unit-vectors in $\mathbb{R}^d$ and let $\mathcal{P}$ be the uniform distribution over $m$-dimensional projection matrices over $\mathbb{R}^d$ Then:

$$\inf_{T} \mathbb{E}_{x \sim U} \mathbb{E}_{\Pi \sim \mathcal{P}} \| T(\Pi, \Pi x) - x \|^2 \geq 1 - \frac{m}{d}$$

We will now prove the theorem. We can lower bound the minimax risk by:

$$\inf_T \sup_{\Sigma} \| T(\Pi, \Pi \Sigma \Pi) - \Sigma \|_2 \geq \inf_T \mathbb{E}_{x \sim U} \mathbb{E}_{\Pi \sim \mathcal{P}} \| T(\Pi, \Pi xx^T \Pi) - xx^T \|_2$$

$$\geq \inf_T \mathbb{E}_{x \sim U} \mathbb{E}_{\Pi \sim \mathcal{P}} \sin \angle(v_{\max}(T(\Pi, \Pi xx^T \Pi)), x)$$

where $v_{\max}(M)$ is the eigenvector corresponding to the largest eigenvalue of $M$. Here we are applying Theorem 11 and using the eigengap for the matrix $xx^T$, which is 1. We can now eliminate the dependence on $T$ and instead take infimum over estimators $v(\Pi, \Pi x)$ for the vector $x$. We may replace $\Pi xx^T \Pi$ with $\Pi x$ since they contain precisely the same information.
For unit normed vectors $x$ and $v$, some algebra reveals that:

$$\sin \angle(v, x) \geq \sqrt{\frac{\|x - v\|_2}{2}}$$

so that we can lower bound the minimax risk by:

$$\inf_{\mathcal{P}} \sup_{\Sigma} \mathbb{E}_{\Pi \sim \mathcal{P}} \|T(\Pi, \Pi \Sigma \Pi) - \Sigma\|_2 \geq \inf_v \mathbb{E}_{x \sim \mathcal{U}} \mathbb{E}_{\Pi \sim \mathcal{P}} \sqrt{\frac{\|x - v(\Pi, \Pi x)\|_2}{2}}.$$  

Proposition 8 now follows by applying Lemma 16 to the right hand side of this inequality.

7 Conclusions

In this paper we studied the problem of estimating a covariance matrix from highly compressive measurements. We proposed an estimator based on projecting the observations back into the high-dimensional space, and bounded the infinity- and spectral-norm error of this estimator. We complement these bounds with minimax lower bounds for this problem, showing that our estimator is rate-optimal. We mention one concrete application to subspace learning. Note that many other consequences, for example to the task of learning the structure of a gaussian graphical model are immediate following the results of Ravikumar et al. [16].

The main insight of our work is that by leveraging independent random projection operators for each data point, we can build consistent covariance estimators from compressive measurements even in an unstructured setting. However, due to the absence of structure in this problem, the effective sample size shrinks from $n$ in the classical setting to $nm^2/d^2$ in this setting. This gives a precise characterization of the effects of data compression in the covariance estimation problem.

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A Deviation Bounds

**Theorem 17** (Bernstein Inequality). If \( U_1, \ldots, U_n \) are independent zero-mean random variables with \( |U| \leq B \), a.s., and \( \frac{1}{n} \sum_{i=1}^n \text{Var}(U_i) \leq \sigma^2 \), then for any \( \delta \in (0, 1) \):

\[
P \left( \left| \frac{1}{n} \sum_{t=1}^n U_t \right| \leq \sqrt{\frac{2\sigma^2 \log(2/\delta)}{n} + \frac{2B \log(2/\delta)}{3n}} \right) \geq 1 - \delta.
\]

**Theorem 18** (Matrix Bernstein Inequality). Let \( X_1, \ldots, X_n \) be independent, random, self-adjoint matrices with dimension \( d \) satisfying:

\[ E X_k = 0 \quad \text{and} \quad \|X_k\|_2 \leq R \text{ almost surely.} \]

Then, for all \( t \geq 0 \),

\[
P \left( \left\| \sum_{k=1}^n X_k \right\| \geq t \right) \leq d \exp \left( \frac{-t^2/2}{\sigma^2 + Rt/3} \right) \quad \text{where} \quad \sigma^2 = \left\| \sum_{k=1}^n E X_k^2 \right\|
\]

**Theorem 19** ([15]). Let \( X_1, \ldots, X_n \) be independent random variables with \( E X_t^2 < \infty \) and \( X_t \geq 0 \), a.s., for all \( t \in [n] \). Then for any \( \epsilon > 0 \):

\[
P \left( E \left( \sum_{t=1}^n X_t \right) - \sum_{t=1}^n X_t \geq \epsilon \right) \leq \exp \left( \frac{-\epsilon^2}{2 \sum_{t=1}^n E X_t^2} \right)
\]

**Theorem 20** ([19]). Let \( X_1, \ldots, X_n \) be independent random variables with \( E X_t = 0 \) and \( E X_t^2 = b_t \) and:

\[ E|X_t|^k \leq \frac{b_t}{2} k^k \alpha^{k-2}, \]

for all \( t \in [n] \), \( k \geq 3 \) and for some constant \( \alpha > 0 \). Then:

\[
E \exp \left( \lambda \sum_{t=1}^n X_t \right) \leq \exp \left( \frac{\lambda^2 \sum_{t=1}^n b_t}{2(1-s)} \right)
\]

for any \( s \in (0, 1) \) and \( \lambda > 0 \) provided that \( \lambda \alpha \leq s \).

**Proposition 21.** For any \( t \in [n] \) and \( d \geq 3 \), let \( c_t \geq 0 \) and \( \nu_t \sim \text{Beta}(\frac{1}{2}, \frac{d-2}{2}) \). Define \( c = \max_{t \in [n]} c_t \) and:

\[ B = \frac{2(d-2)nc^4}{(d-1)^2(d+1)} \]

For any \( s \in (0, 1) \) and \( \delta > 0 \), if \( \log(1/\delta) \leq \frac{(d+5)^2 Bs^2}{12c(1-s)} \), then:

\[
P \left( \frac{1}{n} \sum_{t=1}^n c_t^2 \nu_t - \frac{1}{n} \sum_{t=1}^n \frac{1}{d-1} c_t^2 > \sqrt{\frac{2B \log(1/\delta)}{n^2 (1-s)}} \right) \leq \delta
\]
Proof. For each $t \in [n]$, we have $c_t^2 \nu_t \geq 0$, $\mathbb{E} c_t^2 \nu_t = \frac{1}{d-1} c_t^2$, and $\mathbb{E}(c_t^2 \nu_t)^2 = c_t^4 \frac{3}{(d-1)}$. So by Theorem 19 for any $\epsilon > 0$, we have:

$$
\Pr\left( \frac{1}{n} \sum_{t=1}^{n} \frac{1}{d-1} c_t^2 - \frac{1}{n} \sum_{t=1}^{n} c_t^2 \nu_t \geq \epsilon \right) \leq \exp \left( -\frac{\epsilon^2}{2n \left( \frac{3c^4}{d^2-1}\right)} \right)
$$

and the result follows by inverting the inequality.

Proposition 22. For any $t \in [n]$ and $d \geq 3$, let $c_t \geq 0$ and $\nu_t \sim \text{Beta}\left(\frac{1}{2}, \frac{d-2}{2}\right)$. Define $c = \max_{t \in [n]} c_t$. Then for any $\delta > 0$:

$$
\Pr\left( \frac{1}{n} \sum_{t=1}^{n} \frac{1}{d-1} c_t^2 - \frac{1}{n} \sum_{t=1}^{n} c_t^2 \nu_t > \sqrt{\frac{6c^4 \log(1/\delta)}{n(d^2-1)}} \right) \leq \delta
$$

Proof. For $t \in [n]$, define $X_t = c_t^2 \left( \nu_t - \frac{1}{d-1} \right)$ and note that $\mathbb{E} X_t = 0$. Let $b_t = \mathbb{E} X_t^2 = c_t^4 \frac{2(d-2)}{(d-1)^2(d+1)}$.

Then for any $k \geq 3$ we have, using Minkowski’s Inequality:

$$
\mathbb{E}|X_t|^k = c_t^{2k} \mathbb{E}\left| \nu_t - \frac{1}{d-1} \right|^k 
\leq c_t^{2k} \left( \mathbb{E}(\nu_t^k)^{1/k} + \frac{1}{d-1} \right)^k 
= c_t^{2k} \left( \left( \prod_{r=0}^{k-1} \frac{1/2 + r}{(d-1)/2 + r} \right)^{1/k} + \frac{1}{d-1} \right)^k 
$$

We now leverage two claims:

Claim 23. For $k \geq 3$:

$$
\max \left( \frac{1/2 + k}{(d-1)/2 + k} \times \frac{1}{k+1} \frac{d-1}{d-2} \frac{10}{(d+3)} \right) \leq \frac{2}{d+5}
$$

Proof. The proof is elementary.

Claim 24. For any $\zeta \geq \frac{2}{d+5}$ and $k \geq 3$:

$$
\frac{d-1}{d-2} \frac{60}{d+3} \prod_{r=3}^{k-1} \frac{1/2 + r}{(d-1)/2 + r} \leq k! \zeta^{k-2}
$$

Proof. We proceed by induction. For $k = 3$, the expression simplifies to the second term in the previous claim:

$$
\frac{d-1}{d-2} \frac{60}{d+3} \leq 6\zeta
$$
For the inductive step, assume the claim holds for \( k \geq 3 \), then for \( k + 1 \) we have:

\[
\frac{d - 1}{d - 2} \frac{60}{d + 3} \prod_{r=3}^{k-1} \frac{1/2 + r}{(d - 1)/2 + r} \leq \frac{1/2 + k}{(d - 1)/2 + k} k! \zeta^{k-2} \leq (k + 1)! \zeta^{k-1}
\]

where the first step is the inductive hypothesis and the second is from the first part of the previous claim. \( \square \)

Armed with these two claims we can proceed with the proof of the proposition. For any \( a \geq \frac{3c^2}{d+5} \) we have:

\[
\mathbb{E}|X_t|^k \leq c_t^{2k} \left( \prod_{r=0}^{k-1} \frac{1/2 + r}{(d - 1)/2 + r} \right)^{1/k} + \frac{1}{d - 1} \right)^k \leq 2^k c_t^{2k} \prod_{r=0}^{k-1} \frac{1/2 + r}{(d - 1)/2 + r} = 2^k c_t^{2k} \prod_{r=0}^{k-1} \frac{1/2 + r}{(d - 1)/2 + r} \leq \frac{c_t^4}{(d - 1)(d + 1)} (2c_t^2)^{k-2} \frac{60}{d + 3} \prod_{r=3}^{k-1} \frac{1/2 + r}{(d - 1)/2 + r} \leq \frac{c_t^4}{(d - 1)(d + 1)} (2c_t^2)^{k-2} \frac{d - 2}{d - 1} k! \frac{2}{d + 5} \]

Here the first inequality is from the application of Minkowski’s inequality above, the second uses the fact that \( \frac{1}{d - 1} \leq \frac{1/2+r}{(d-1)/2+r} \) for any \( r \). In the third line we use pull out terms from the product, using the notation that \( \prod_{r=3}^{k-1} (\cdot) = 1 \). Then we apply the claim from before with \( \zeta = \frac{2}{d + 5} \) and finally substitute in for \( a \) and \( b_t \).

Now setting \( a = \frac{4c^2}{d+5} \), we have that the moment bound above holds for all \( X_t \). Let \( s \in (0, 1) \) and \( \lambda > 0 \) such that \( \lambda a \leq s \). Since \( B \triangleq \frac{2(d-2)n c^4}{(d-1)^2 (d+1)} \geq \sum_{t=1}^{n} b_t \), by Theorem 20 we have:

\[
\mathbb{E} \exp \left( \lambda \sum_{t=1}^{n} X_t \right) \leq \exp \left( \frac{B\lambda^2}{2(1-s)} \right)
\]

We may now apply the Chernoff trick, so that for any \( \epsilon > 0 \):

\[
\mathbb{P} \left( \sum_{t=1}^{n} X_t > \epsilon \right) \leq \exp \left( -\lambda \epsilon + \frac{B\lambda^2}{2(1-s)} \right) = \exp \left( \lambda \left( \frac{B\lambda}{2(1-s)} - \epsilon \right) \right)
\]

Set \( \lambda = \frac{1-s}{B} \epsilon \), so that if \( \frac{1-s}{B} \leq \frac{s}{a} \) we have:

\[
\mathbb{P} \left( \sum_{t=1}^{n} X_t > \epsilon \right) \leq \exp \left( \frac{-\epsilon^2(1-s)}{2B} \right)
\]
Inverting this inequality and the condition above proves the result. In particular, we require that for the $\delta > 0$
that we choose, $\frac{1 - s}{B} \sqrt{\frac{2B \log(1/\delta)}{1 - s}} \leq \frac{s}{a}$. If this is the case, we have:

$$P\left( \sum_{t=1}^{n} X_t \geq \sqrt{\frac{2B \log(1/\delta)}{1 - s}} \right) \leq \delta$$

**Proposition 25.** If $\nu \sim \text{Beta}(\frac{1}{2}, \frac{d-2}{2})$, for $d \geq 4$ then $P(\nu > \frac{2}{d-3} \log(1/\delta)) \leq \delta$ for any $\delta > 0$.

**Proof.** Let $\alpha \in \mathbb{R}^{d-1}$ be uniformly distributed on the unit sphere and let $\zeta \sim \text{Beta}(1, \frac{d-3}{2})$. Then $\alpha(1)^2 = \nu$ and $\alpha(1)^2 + \alpha(2)^2 = \zeta$. So for any $\epsilon \in (0, 1)$:

$$P(\nu \leq \epsilon) = P(\alpha(1)^2 \leq \epsilon) \geq P(\alpha(1)^2 + \alpha(2)^2 \leq \epsilon) = P(\zeta \leq \epsilon) = P(- \log(1 - \zeta) \leq - \log(1 - \epsilon)).$$

It is well known that $- \log(1 - \zeta)$ is exponentially distributed with rate $\frac{d-3}{2}$, and so:

$$P(- \log(1 - \zeta) \leq - \log(1 - \epsilon)) \geq P(- \log(1 - \zeta) \leq \epsilon) = 1 - \exp\left( -\epsilon \frac{d-3}{2} \right).$$

**B Proofs of Technical Lemmas**

**B.1 Proof of Lemma 10**

The proof is based on the probabilistic lemma. We will first show that for any fixed unit vector $x$, if we draw
another vector $v$ uniformly at random, then:

$$P[\|xx^T - vv^T\|_2 \leq \tau] \leq \exp\left( -\frac{d \log(1/\tau) + 1}{4} \right)$$

Armed with this deviation bound, if we draw $M$ points uniformly at random from the unit sphere in $d$
dimensions, then the probability that no two points are within $\tau$ of each other is (via a union bound):

$$P[\forall i \neq j : \|v_i v_i^T - v_j v_j^T\|_2 \geq \tau] = 1 - P\left( \bigcup_{i \neq j} \|v_i v_i^T - v_j v_j^T\| \leq \tau \right) \geq 1 - \left( \frac{M}{2} \right) \exp\left( -\frac{d \log(1/\tau) + 1}{4} \right)$$

As long as this probability is non-zero, then we know that there exists such a packing set. In particular, if:

$$\frac{M(M-1)}{2} \exp\left( -\frac{d \log(1/\tau) + 1}{4} \right) \leq 1/2,$$

then we would show existence of a packing set of size $M$. This inequality is satisfied when $M \geq \exp(-1/8)\tau^{-d/8}$. 24
To prove the deviation bound above, note that since $x, v$ are unit vectors, the spectral norm difference is just the sine of the principal angle between their subspaces, which is just the sine of the angle between the two vectors. It is well known that (see [5]):

$$
\mathbb{P} \left[ (x^T v)^2 \geq \beta / d \right] \leq \exp \left( \frac{1}{2} (1 - \beta + \log \beta) \right)
$$

$$
\mathbb{P} \left[ \cos^2(\angle(x, v)) \geq (1 + \epsilon) / d \right] \leq \exp \left( \frac{1}{2} (-\epsilon + \log(1 + \epsilon)) \right)
$$

Therefore:

$$
\mathbb{P} \left[ \sin \angle(x, v) \leq \sqrt{1 - (1 + \epsilon) / d} \right] = \mathbb{P} \left[ \cos^2(\angle(x, v)) \geq (1 + \epsilon) / d \right] \leq \exp \left( \frac{1}{2} (-\epsilon + \log(1 + \epsilon)) \right)
$$

If $\epsilon \geq 3$, then $\epsilon - \log(1 + \epsilon) \geq \epsilon / 2$, so we can upper bound the probability by $\exp(-3\epsilon / 4)$. Setting $\epsilon = d(1 - \tau^2) - 1$ gives the inequality:

$$
\mathbb{P} \left[ \sin \angle(x, v) \leq \sqrt{1 - (1 + \epsilon) / d} \right] \leq \exp \left( \frac{-d(1 - \tau^2)^2 + 1}{4} \right)
$$

We now proceed to lower bound $(1 - \tau^2)$ by $\log(1 / \tau)$. This is possible for $\tau \in [0.5, 1]$ as both functions are monotonically decreasing in $\tau$ but $(1 - \tau^2)$ is concave while $\log(1 / \tau)$ is convex. At $\tau = 1/2$ the first is larger than the second, and they are both equal at $\tau = 1$. The condition on $\tau$ and this lower bound establishes the inequality used above.

**B.2 Proof of Lemma [15]**

We will prove Lemma [15] for the distributions based on $N(0, \eta I)$ and $N(0, \eta I + \gamma e_1 e_1^T)$. By rotational invariance, the bound holds if we replace $e_1$ with any unit vector $v$. The KL-divergence for a single sample is:

$$
KL(P_1, P_0) = \int N(0, U^T \Sigma_1 U) \text{Unif}(U) \log \left( \frac{N(0, U^T \Sigma_1 U) \text{Unif}(U)}{N(0, U^T \Sigma_0 U) \text{Unif}(U)} \right)
$$

$$
= E_{U \sim \text{Unif}} KL(N(0, U^T \Sigma_1 U), N(0, U^T \Sigma_0 U))
$$

$$
= E_{U \sim \text{Unif}} \frac{1}{2} \left( \frac{1}{\eta} \text{tr}(\eta I_m + \gamma U^T e_1 e_1^T U) - m - \log \frac{\det(\eta I_m + \gamma U^T e_1 e_1^T U)}{\eta^m} \right)
$$

Here Unif is the uniform distribution over orthonormal bases for $m$-dimensional subspaces of $\mathbb{R}^d$. To analyze the quantity inside the expectation, let $\lambda_1, \ldots, \lambda_m$ denote the eigenvalues of $\eta I_m + \gamma U^T e_1 e_1^T U$
and write:

\[
\frac{1}{2} \left( \frac{1}{\eta} \text{tr}(\eta I_m + \gamma U^T e_1 e_1^TU) - m - \log \frac{\det(\eta I_m + \gamma U^T e_1 e_1^TU)}{\eta^m} \right)
\]

\[
= \frac{1}{2} \left( \sum_{i=1}^{m} \frac{\lambda_i}{\eta} - \log(\lambda_i/\eta) - 1 \right) \leq \frac{1}{2} \left( \sum_{i=1}^{m} \left( \frac{\lambda_i}{\eta} \right)^2 - \frac{1}{(\lambda_i/\eta)^2 + 1} - 1 \right)
\]

\[
= \frac{1}{2} \sum_{i=1}^{m} \left( \frac{\lambda_i}{\eta} \right)^2 + 1 \left( (\lambda_i/\eta) - 1 \right)^2 \leq \frac{1}{2} \sum_{i=1}^{m} (\lambda_i/\eta - 1)^2 = \frac{1}{2\eta^2} \|U^T(\Sigma_0 - \Sigma_1)U\|^2_F
\]

\[
= \frac{\gamma^2}{2\eta^2} \|U^T e_1 e_1^TU\|^2_F.
\]

The first inequality above uses the inequality \(\log(x) \geq x^2 - 1/(x^2 + 1)\) which holds for \(x \geq 1\). The second inequality is that \(\frac{x^2}{x^2 + 1} \leq 1\) since \(x^2 - x + 1\) is convex and minimized at \(x = 1/2\) in which case it takes value \(3/4\). So we have show than:

\[
KL(\mathbb{P}_1, \mathbb{P}_0) \leq \frac{\gamma^2}{2\eta^2} \mathbb{E}_{U \sim \text{Unif}} \|U^T e_1 e_1^TU\|^2_F.
\]

We will no upper bound this expectation.

\[
\mathbb{E}_{U \sim \text{Unif}} \|U^T e_1 e_1^TU\|^2_F = \sum_{i,j=1}^{m} \mathbb{E}_{U \sim \text{Unif}} U_{i1}^2 U_{j1}^2
\]

This is the squared-Frobenius norm of the outer product of the first row (in \(\mathbb{R}^m\)) with itself. Marginally, each entry of \(U\), after squaring, is distributed as \(\text{Beta}(\frac{1}{2}, \frac{d-1}{2})\) so the diagonal terms of this matrix (the terms where \(i = j\) above) are just the second (non-central) moment of the Beta distribution. These are \(\frac{3}{d(d+2)} \leq \frac{3}{d^2}\).

For the off-diagonal terms, note that by spherical symmetry each row of \(U\) has a direction that is chosen uniform at random (in \(m\) dimensions) while the squared-norm of each row is distributed as \(\text{Beta}(\frac{m}{2}, \frac{d-m}{2})\). This second fact holds because \(UU^T e_1 = \omega e_1 + \sqrt{\omega(1-\omega)} z\) where \(z \perp e_1\) and \(\omega \sim \text{Beta}(\frac{m}{2}, \frac{d-m}{2})\), so that \(e_1^T UU^T e_1 = \|U^T e_1\|^2_F = \omega\). If we let \(v \in \mathbb{R}^m\) denote a uniform at random unit vector, the off-diagonal terms can be written as:

\[
\mathbb{E}_v \omega^2 v_i^2 v_j^2 = \mathbb{E}_v^2 v_i^2 v_j^2 \mathbb{E}_\omega^2 \leq \sqrt{\mathbb{E} v_i^4} \sqrt{\mathbb{E} v_j^4} \frac{m/2}{d/2} \frac{m/2 + 1}{d/2 + 1} = \frac{3}{m(m+2)} \frac{m}{d} \frac{m+2}{d+2} \leq \frac{3}{d^2}
\]

Here we use the Cauchy-Schwarz inequality, the fact that \(\omega \sim \text{Beta}(\frac{m}{2}, \frac{d-m}{2})\) and that marginally each \(v_i^2 \sim \text{Beta}(\frac{1}{2}, \frac{m-1}{2})\) since \(v\) is a uniform random vector in \(m\)-dimensions. So every term in the sum is bounded by \(3/d^2\). There are \(m^2\) terms producing the bound:

\[
\mathbb{E}_{U \sim \text{Unif}} \|U^T e_1 e_1^TU\|^2_F \leq \frac{3m^2}{d^2}
\]

Plugging this into our KL bound above and using additivity of KL-divergence for product measures, we arrive at the bound in the Lemma.
B.3 Proof of Lemma 16

Recall that the quantity we are interested in lower bounding is:

$$\inf_T E_{x,\Pi,T} \|T(\Pi, \Pi x) - x\|_2^2$$

The expectation over $T$ allows for randomized estimators, $x$ is drawn uniformly at random from the unit sphere, and $\Pi$ is a uniformly drawn $m$-dimensional projection operator. Instead of draw an $m$-dimensional projection matrix $\Pi$ uniformly at random, it is equivalent to draw an orthonormal basis $U \in \mathbb{R}^{d \times m}$ uniformly at random. The observation is then $(U, U^T x)$ which is clearly equivalent to observing $(\Pi, \Pi x)$ since one can be constructed from the other and vice-versa. So we will instead lower bound:

$$\inf_T E_{x,\Pi,T} \|T(U, U^T x) - x\|_2^2 \geq \inf_T E_{x,\Pi,T} \|T(U, U^T x)\|_2^2 - 2T(U, U^T x)^T x + 1$$

Before proceeding, we need to clarify one definition. We will use $\angle V, y = \angle(P_V y, y)$ to denote the angle between the subspace $V$ and the vector $y$. We can evaluate the integrals by first choosing a subspace $U$, choosing a vector $y \in U$, and finally choosing the vector $x$ so that $P_U x = y$.

$$\inf_T E_{x,\Pi,T} \|T(\Pi, \Pi x) - x\|_2^2 \geq \inf_T \int_U \int_y \int_x \left[ E_T \|T(U, U^T x)\|_2^2 - 2T(U, U^T x)^T x + 1 \right] dP(x; y, U) dQ(y) dR(U)$$

where $P$ is the conditional distribution of $x$ given that it projects to $y$ with subspace $U$, $Q$ is the distribution over projection vectors $y$ and $R$ is the uniform distribution on the $m$-dimensional Grassmanian manifold. Now we will push the $\inf_T$ inside of the first two integrals. Notice that all of the information the estimator has is $U, y$ since $U^T x = U^T y$, so for each $(U, y)$ pair, the estimator is just a distribution over vectors. Call this distribution $T(\cdot; y, U)$:

$$\inf_T E_{x,\Pi,T} \|T(\Pi, \Pi x) - x\|_2^2 \geq \int_U \int_y \inf_{T(\cdot; y, U)} \int_x \|v\|_2^2 - 2v^T x + 1 dP(x; y, U) dT(v; y, U) dQ(y) dR(U)$$

The only term depending on $x$ is the $v^T x$ term, for which:

$$\int_x 2v^T x dP(x; y, U) = 2v^T E_{x \sim P(x; y, U)} x = 2v^T y$$

which follows by spherical symmetry, since we draw $x$ uniformly from the unit sphere, which is symmetric about the subspace $U$. Therefore we have:

$$\inf_{T(\cdot; y, U)} \int_x \left( \|v\|_2^2 - 2v^T x + 1 \right) dP(x; y, U) dT(v; y, U) = \inf_T \int_y \left( \|v\|_2^2 - 2v^T y + 1 \right) dT(v; y, U) = 1 - \|y\|_2^2$$

So we can lower bound by:

$$\inf_T E_{x,\Pi,T} \|T(\Pi, \Pi x) - x\|_2^2 \geq 1 - \int_U \int_y \|y\|_2^2 dQ(y) dR(U)$$
and are left to control the expected norm of $y = P_U x$. We have the identity $\|y\| = \cos \angle (P_U x, x)$. By spherical symmetry, we can let $U$ to be the span of the first $m$ standard basis vectors, in which case:

$$\cos \angle P_U x, x = \frac{\sum_{i=1}^{m} x_i^2}{\sqrt{\sum_{i=1}^{m} x_i^2}}$$

Therefore $\|y\|_2^2 = Z = \frac{\sum_{i=1}^{m} x_i^2}{\sum_{i=1}^{d} x_i^2}$ where $x_1, \ldots, x_d \sim \mathcal{N}(0, 1)$. This gives the lower bound:

$$\inf_{T} \mathbb{E}_{x, \Pi, x} \|T(\Pi, \Pi x) - x\|_2^2 \geq 1 - \mathbb{E}[Z] = 1 - \frac{m}{d}$$