On the Existence of Time Delay for Rotating Beam with Proportional–Derivative Controller

Y. A. Amer¹, Taher A. Bahnasy²* and Ashraf M. Elmhlawy²

¹Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt.
²Department of Engineering Physics and Mathematics, Faculty of Engineering, Tanta University, Tanta, Egypt.

Authors’ contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2021/v17i130268
(1) Dr. Nikolaos D. Bagis, Aristotle University of Thessaloniki, Greece.
Editors:
(1) S. Maragatha Sundari, Kalasalingam University, India.
(2) Ramjeet Singh Yadav, Ashoka Institute of Technology and Management, Varanasi, India.
Complete Peer review History: [http://www.sdiarticle.com/review-history/66021](http://www.sdiarticle.com/review-history/66021)

Received: 06 January 2021
Accepted: 12 March 2021
Published: 22 March 2021

Abstract

A rotating beam at varying speed mathematical model is studied. Multiple time scales method is applied to the nonlinear system of differential equations and investigated the system behavior approximate solution in the instance of resonance case. We studied the system in case of applying the delayed control on the displacement and the velocity with Proportional–derivative (PD) controller. The consistency of the steady state solution in the near-resonance case is reviewed and analyzed using the Routh-Hurwitz approach. The factors on the steady state solution of the various parameters are recognized and discussed. Simulation effects are obtained using MATLAB software package. Different response curves are involved to show and compare controller effects at various system parameters.

Keywords: Non-linear dynamical system; multiple time scales method; active feedback controller; time delay.

*Corresponding author: E-mail: taher.bahnasy@gmail.com, Taher.bahnasy@f-eng.tanta.edu.eg;
1 Introduction

In dynamical and structural structures, disturbances and complex instability are always undesired phenomena. These systems face nonlinear vibrations for numerous purposes, such as materials' nonlinear properties, geometric nonlinearities, and nonlinear powers of excitation. Much time, money and efforts are spent on minimizing these systems' vibrations and oscillations for longer life and preventing them from failure or damage.

Many scholars and scientists have paid attention to and attempted to alleviate this topic that affects equipment, industry, and frugality. The high amplitude nonlinear vibration activity of a revolving cantilever beam is treated by Thomas et al. [1], with applications for turbo machinery and turbo-propeller blades. The effect of rotation speed on the nonlinear vibrations of the beam and particularly on the hardening/softening behavior of its resonances and the occurrence of high amplitude jump phenomena were investigated. A new dynamic model of a rotating flexible beam with a condensed mass positioned in an arbitrary location, based on the absolute nodal coordinate formulation, was investigated by Zhang et al. [2]. They found that both the magnitude and the direction of the condensed mass impact the normal frequencies and the mode shapes. Aeroelastic analysis of a spinning wind turbine blade was conducted by Rezaei et al. [3] by considering the effects of geometrical nonlinearities associated with large blade deflection created during the operation of the wind turbine. Through applying the concepts of quasi-steady and unsteady airfoil aerodynamics, they proposed an aerodynamic model based on the strip theory. The results showed that geometrical nonlinearity,
especially for larger structural deformations, had a significant impact. The effect of rotation velocity on nonlinear resonances is considered in [4], and the multi-scale perturbation approach is used and solved in the von Kármán [5] model. In order to simulate nonlinear resonances via a one-mode Galerkin expansion, nonlinear beam models such as axial inertia and nonlinear curvature are used. Nonlinear resonance curves are also computed, based on a Galerkin discretization with Legendre polynomials and a continuity process, with a completely numerical approach (harmonic balance coupled to an asymptotic numerical technique). For more detailed and effective dynamic analysis of a rotating cantilever beam with elastic deformation defined by partial integro-differential equations with non-Cartesian deformation variables, Kim and Chung [6] suggested a nonlinear model. They showed that the proposed model not only provided good numerical precision and efficiency, but also overcome the constraints expressed by Cartesian variables of a previous traditional nonlinear model. The dynamics of a structure consisting of a rotating rigid hub and a thin-walled composite beam with an embedded active part were introduced by Latalski [7].

Based on the device rotation velocity and laminae fiber orientation angle, they studied natural mode shapes and electrical field spatial distribution. A Proportional Derivative (PD) controller was applied by Kandil, H. El-Gohary [8] to research the effects of time delay on its output to decrease the oscillations of a spinning beam at different speeds. Although the vibrational modes of the dual system are linearly coupled, the controller is applied to only one mode and the other coupled mode tracks it. In the case of the worst resonance cases that were verified numerically, they regulated the device. Yao et al. [9,10] applied the theory and isotropic constitutive law of Hamilton in order to infer the beam's governing equations. Of supersonic gas flow and high temperature, they studied the dynamics at different speeds. Choi et al. [11,12] showed that an active damping effect can be obtained with polyvinylidene fluoride (PVDF) sensors and macrofiber composite (MFC) actuators through a nonlinear velocity feedback control algorithm. MFC is a composite form of piezoelectric material. Through the required arrangement and distribution scale of the sensor/actuator pair, ample vibration suppression efficiency would therefore be obtained.

Joy Mondal, and S. Chatterjee [13] proposed the efficacy of velocity feedback based nonlinear resonant controller to control the free and forced self-excited vibration of a nonlinear beam. The control force is determined using the nonlinear function of the derivative of the filter vector, which is fed through a second-order filter with the velocity signal from the sensor. Liang Li et al. [14] has developed a new hierarchical model for vibration studies of rotating versatile beams with improved active constrained layer damping (EACLD) treatment that is partially shielded. The mass effect of the two added edge components is included by modeling the EACLD patch's edge element as an analogous spring with attached point mass. The assumed mode approach and Lagrange's equations are used to obtain the discrete rigid-flexible coupled dynamic equations of hub-beam systems with EACLD treatment in the open-loop and closed-loop situations.

Boumediène, and Smaoui [15] believed that the beam is to be non-uniform and clamped at its left end to the disk's core, where torque control occurs, while a memory boundary control resides at the right end. The standard torque control is first proposed, followed by the boundary control, which is designed using a special type of memory phenomenon as well as the input's dynamic features. L.F. Lyu, W.D. Zhu [16] demonstrated a new operational modal analysis (OMA) method for a rotating structure based on a rigorous rotating beam vibration theory, an image processing method, and the lifting method of data processing. They developed a novel tracking continuously scanning laser Doppler vibrometer (TCSLDV) method to monitor and scan a rotating structure, and image processing was used to determine the rotating structure's real-time location, enabling the TCSLDV system to track a time-varying scan direction on the rotating structure.

In this article, the PID control with time delay control are applied to the system of rotating beam at varying speeds shown in Fig. 1a [8,9,10] subjected external and parametric force in order to reduce its oscillations and enhance its efficiency. The displacements of the blade cross section are measured by using MFC sensors that are distributed over the bottom surface of the blade, as shown in Fig. 1b. The measured signals will be sent back to the computer to analyze and compute the appropriate control signal as shown in Fig. 1c. Once the control signal is calculated, it is passed through conditioning circuit and then be applied to the embedded MFC actuators that are distributed over the top of the blade so that they can modify the blade position and reduce its vibration, a control loop feedback mechanism illustrated in Fig. 2 are continuously calculates an error value e(t) as the difference between a desired setpoint (SP) and a measured process variable (PV) and
applies a correction based on proportional, integral, and derivative terms (denoted P, I, and D respectively). The multiple time scales perturbation technique (MSPT) was applied to obtain an approximate solution and showing the response equation. The stability of the system at primary and principle parametric resonance case is investigated using both of phase plane and frequency response equation. The numerical solution and the effect of the different parameters for the response of the nonlinear dynamic system.

2 System Model and Mathematical Analysis

The equations of motion for the rotating beam shown in Fig. 1 is introduced by Bekhoucha [5] and Yao et al. [9, 10] by applying the Hamilton’s principle as:

$$\int_0^T (\delta K - \delta U + \delta W) dt ,$$

(1)
where $K$ denote the kinetic energy, $U$ the strain energy, and $W$ is the virtual work of external forces, $t$ denotes time, and $\delta$ is the variation operator. By calculating the variation in kinetic, strain energy, and the virtual work of non-conservative external forces (given in Appendix), and substituting Equation (1), then the governing equations of the nonlinear vibration system for the rotating beam are as the following:

$$\begin{align*}
\ddot{u} - \Omega^2 [u + R(z)u'' + R' (z)u'] + \alpha \Delta T u'' &= a_i(z) v'' - a_3(z) u'' - \delta (R_o + z) + p_x, \\
\ddot{v} - \Omega^2 [R(z)v'' + R'(z)v'] + \alpha \Delta T v'' &= a_i(z) u'' - a_3(z) v'' + p_y,
\end{align*}$$

(2)

where $u, v$ are the translations along the $x, y$ axes, $p_x, p_y$ are the external forces per unit axial length in the $x$ and $y$ direction. The values of $p_x, p_y$ and the variables $a_i(z), i=1,2,3$ are given in Appendix.

The dots and primes, respectively, represent partial differentiation with respect to $t$ and $z$, $R(X,Y,Z)$ is the vector function of a point $M(X,Y,Z)$ of the deformed thin wall beam, and given by

$$R(X,Y,Z) = (X + u(i) + (Y + v(j) + Z k) + R_z.$$ Applying Galerkin’s approach [17] on system (2), the horizontal and vertical displacements $u,v$ have been approximated to the modes $X_1, X_2$ respectively to have the dimensionless two degree of freedom non-linear rotating beam system in the form:

$$\begin{align*}
\ddot{X}_1 + 2\mu_1 \dot{X}_1 + \omega^2 X_1 + \beta_1 \dot{X}_2 + \beta_2 X_2 + \beta_3 X_1^2 + \beta_4 X_1^3 &= 2f_o x \beta_{21} X_1 \cos(\Omega t) \\
+ f^2 \beta_{11} \dot{X}_1 \cos^2(\Omega t) + f \beta_{16} \Omega \sin(\Omega t) - k_1 \dot{X}_1(t - \tau) - k_2 \dot{X}_2(t - \tau), \\
\ddot{X}_2 + 2\mu_2 \dot{X}_2 + \omega^2 X_2 + \beta_2 \dot{X}_1 + \beta_3 X_2 + \beta_4 X_2^2 + \beta_5 X_2^3 &= 2f_o x \beta_{22} X_2 \cos(\Omega t) \\
+ f^2 \beta_{22} \dot{X}_2 \cos^2(\Omega t),
\end{align*}$$

(3a)

(3b)

where all system parameters are defined before.

Scaling the previous parameters as:

$$\begin{align*}
\hat{\beta}_1 &= \epsilon \hat{\beta}_{11}, \beta_{13} = \epsilon \hat{\beta}_{13}, \beta_{14} = \epsilon \hat{\beta}_{14}, \beta_{16} = \epsilon \hat{\beta}_{16}, \beta_{21} = \epsilon \hat{\beta}_{21}, \beta_{22} = \epsilon \hat{\beta}_{22}, \beta_{24} = \epsilon \hat{\beta}_{24}, \beta_5 = \epsilon \hat{\beta}_5, \\
k_1 = \epsilon \hat{k}_1, k_2 = \epsilon \hat{k}_2, \mu_1 = \epsilon \hat{\mu}_1, \mu_2 = \epsilon \hat{\mu}_2.
\end{align*}$$

(4)

Applying multiple time scales method [18], an asymptotic expansion is sought as:

$$\begin{align*}
X_1(T_o, T_1, \epsilon) &= X_{10}(T_o, T_1) + \epsilon X_{11}(T_o, T_1) + O(\epsilon^2), \\
X_2(T_o, T_1, \epsilon) &= X_{20}(T_o, T_1) + \epsilon X_{21}(T_o, T_1) + O(\epsilon^2), \\
X_3(T_o - \tau, T_1 - \epsilon \tau, \epsilon) &= X_{10}(T_0, T_1) + \epsilon X_{11}(T_o, T_1) + O(\epsilon^2),
\end{align*}$$

(5)

where the time derivative will takes the values:

$$\begin{align*}
\frac{d}{dt} &= D_o + \epsilon D_1 + O(\epsilon^2), \\
\frac{d^2}{dt^2} &= D_o^2 + 2\epsilon D_o D_1 + O(\epsilon^2)
\end{align*}$$

(6)
Applying Eqs. (4) - (6) into Eq. (3), then equating same powers of \( \varepsilon \) coefficients to obtain the following:

\[
O(\varepsilon^0) : (D_o^2 + \omega^2) X_{10} = 0
\]

(7a)

\[
(D_o^2 + \omega^2) X_{20} = 0
\]

(7b)

\[
O(\varepsilon) : \begin{align*}
(D_o^2 + \omega^2) X_{11} &= \hat{\beta}_1 \omega X_{10} \cos(\Omega T_0) + f \omega \cos(\Omega T_0) + f \hat{\beta}_i \omega \Omega \sin(\Omega T_0) \\
-2D_oD_{10}X_{10} - 2\hat{\mu}_cX_{10} - \hat{\beta}_1D_oX_{20} - \hat{\beta}_iD_{10}X_{20} - \hat{\beta}_sX_{10}^2 - \hat{\beta}_sX_{10}^3 \\
-k_1X_{10} - k_2D_oX_{10} - k_1X_{10} + k_2D_oX_{10}.
\end{align*}
\]

(8a)

\[
(D_o^2 + \omega^2) X_{21} = \hat{\beta}_22\omega X_{20} \cos(\Omega T_0) + f \omega \cos(\Omega T_0) - 2D_oD_{10}X_{20} - 2\hat{\mu}_cX_{20} - \hat{\beta}_sX_{10}^2 - \hat{\beta}_sX_{10}^3.
\]

(8b)

It is well known that solutions of (7a), (7b) are

\[
X_{10} = A(T_1)e^{i\omega T_0} + cc.,
\]

\[
X_{20} = B(T_1)e^{i\omega T_0} + cc.,
\]

\[
X_{10r} = A_r(T_1)e^{i\omega(T_0 - \tau)} + cc.,
\]

(9)

Using Taylor expansion, then the value of \( A_r(T_1) \) is given by [19]:

\[
A_r(T_1) = A(T_1 - \varepsilon \tau) = A(T_1) - \varepsilon \tau \dot{A}(T_1) + \cdots
\]

As approximation, we keep only the first term of this expansion, then, \( X_{10r} = A(T_1)e^{i\omega(T_0 - \tau)} + cc., \)

where \( cc. \) represents the complex conjugates of the preceding terms and \( A, B \) are complex functions of \( T_1 \).

Now we will study the system worst operating modes due to resonance cases.

### 2.1 Case 1 Primary resonance

The primary resonance occur when the value of \( \Omega \) is equal to \( \omega \) so we study the behavior of the system near this case i.e.

\[
\Omega = \omega + \sigma_1 = \omega + \varepsilon \hat{\sigma}_1,
\]

(10)

Combining Eq. (9) and (10) into (8), we get:
(D_o^2 + \omega^2)X_{11} = -2D_o D_1 (A e^{i\alpha T} + \bar{A} e^{-i\alpha T}) - 2\mu_1 D_o (B e^{i\alpha T} + \bar{B} e^{-i\alpha T})$

$- \beta_{13} D_o (B e^{i\alpha T} + \bar{B} e^{-i\alpha T}) - \beta_{11} (B e^{i\alpha T} + \bar{B} e^{-i\alpha T})$

$- \beta_5 (A e^{i\alpha T} + \bar{A} e^{-i\alpha T}) (B e^{i\alpha T} + \bar{B} e^{-i\alpha T})^2 - \beta_5 (A e^{i\alpha T} + \bar{A} e^{-i\alpha T})^3$

+$f_o \beta_{14} (A e^{i\alpha T} + \bar{A} e^{-i\alpha T}) (e^{i(\alpha + \sigma)} T) - e^{-i(\alpha + \sigma)} T) + k_1 (A e^{i\alpha(T_{t-r})} + \bar{A} e^{-i\alpha(T_{t-r})})$

$- k_2 D_o (A e^{i\alpha(T_{t-r})} + \bar{A} e^{-i\alpha(T_{t-r})}),$

$(D_o^2 + \omega^2)X_{21} = -2D_o D_1 (B e^{i\alpha T} + \bar{B} e^{-i\alpha T}) - 2\mu_2 D_o (B e^{i\alpha T} + \bar{B} e^{-i\alpha T})$

$- \beta_{22} D_o (A e^{i\alpha T} + \bar{A} e^{-i\alpha T}) - \beta_{21} (A e^{i\alpha T} + \bar{A} e^{-i\alpha T})$

$- \beta_b (B e^{i\alpha T} + \bar{B} e^{-i\alpha T}) (A e^{i\alpha T} + \bar{A} e^{-i\alpha T})^2 - \beta_b (B e^{i\alpha T} + \bar{B} e^{-i\alpha T})^3$

+$f_o \beta_{23} (B e^{i\alpha T} + \bar{B} e^{-i\alpha T}) (e^{i(\alpha + \sigma)} T) + e^{-i(\alpha + \sigma)} T) + k_1 (A e^{i\alpha(T_{t-r})} + \bar{A} e^{-i\alpha(T_{t-r})})$

$- k_2 D_o (A e^{i\alpha(T_{t-r})} + \bar{A} e^{-i\alpha(T_{t-r})}).$

Eliminating all secular terms in Eqs. (11) and (12), we obtain:

$$\frac{\beta_{14} f_1}{2} (A + \bar{A} e^{2i\sigma}) - 2i \alpha \bar{A} - 2i \mu_1 \omega \bar{A} - \beta_{13} B - i \omega \beta_{12} B - 2\beta_{2} A \bar{B} - 3\beta_A \bar{A}$$

$$- \beta_5 B \bar{A} - 0.5i \beta_{16} Q_f e^{i\sigma} - k_1 A e^{-i\alpha \tau} - i k_2 \omega A e^{-i\alpha \tau} = 0,$$

$$\frac{\beta_{23} f_2}{2} (B + \bar{B} e^{2i\sigma}) - 2i \alpha \bar{B} - 2i \mu_2 \omega B - \beta_{23} A - i \omega \beta_{22} A - 2\beta_{2} A \bar{B} - 3\beta_B \bar{B} - \beta_B \bar{B} = 0$$

Converting $A, B$ to the polar form then we have:

$$A = \frac{a_1}{2} e^{i\beta_1},$$

$$B = \frac{a_2}{2} e^{i\beta_1},$$

where $a_j, \beta_j, (j = 1, 2)$ are the system amplitude and phase respectively.

Introducing Eq. (15) in Eqn. (13) and (14) and equating the real and imaginary parts we get:
\[
\dot{a}_1 = -\mu a_1 - \frac{\beta_{11} a_1}{2\omega} \sin(\varphi_2) - \frac{\beta_{13} a_1^2}{2} \cos(\varphi_2) + \frac{\beta_{16} a_1^2}{8\omega} \sin(2\varphi_2) + \frac{\beta_{14} a_1 f^2}{8\omega} \sin(2\varphi_2) - \frac{\beta_{16} f \Omega}{2\omega} \cos(\varphi_1) + \frac{k_1 a_1}{2\omega} \sin(\omega \tau) - \frac{k_2 a_1}{2} \cos(\omega \tau),
\]
(16a)

\[
\phi_1 = \sigma_1 - \frac{\beta_{11} a_2}{2\omega a_1} \cos(\varphi_2) + \frac{\beta_{13} a_2}{2 a_1} \sin(\varphi_2) - \frac{\beta_{16} a_2^2}{8\omega} \cos(2\varphi_2) - \frac{\beta_{14} a_2^2}{4\omega},
\]
(16b)

\[
\dot{a}_2 = -\mu a_2 + \frac{\beta_{21} a_1}{2\omega} \sin(\varphi_2) - \frac{\beta_{23} a_1^2}{2} \cos(\varphi_2) + \frac{\beta_{26} a_1^2}{8\omega} \sin(2\varphi_2) - \frac{\beta_{24} a_1 f^2}{8\omega} \sin(2\varphi_2),
\]
(16c)

\[
\phi_2 = \sigma_2 + \frac{\beta_{21} a_1}{2\omega a_2} \sin(\varphi_2) - \frac{\beta_{23} a_1^2}{2 a_2} \cos(\varphi_2) - \frac{\beta_{26} a_1^2}{8\omega} \cos(2\varphi_2) + \frac{\beta_{24} a_1^2}{4\omega} \cos(2\varphi_2) + \frac{f^2}{4\omega} (\beta_{14} - \beta_{24}) a_1 + \frac{\beta_{15} a_2}{2 a_1} \sin(\varphi_2).
\]
(16d)

where

\[
\varphi_1 = \sigma_1 t - \beta_1,
\]
\[
\varphi_2 = \beta_2 - \beta_1.
\]
(17)

For obtaining the steady state solution for amplitude and phase, putting \(\dot{a}_1 = \phi_1 = \dot{a}_2 = \phi_2 = 0\) into Eq. (16), the resultant formulas can be solved numerically. To discuss the stability behavior of these solutions, linearizing these equations according to Lyapunov first (indirect) method [20] to give the following system:

\[
\begin{bmatrix}
\frac{\partial \dot{a}_1}{\partial a_1} & \frac{\partial \dot{a}_1}{\partial \phi_1} & \frac{\partial \dot{a}_1}{\partial a_2} & \frac{\partial \dot{a}_1}{\partial \phi_2} \\
\frac{\partial \phi_1}{\partial a_1} & \frac{\partial \phi_1}{\partial \phi_1} & \frac{\partial \phi_1}{\partial a_2} & \frac{\partial \phi_1}{\partial \phi_2} \\
\frac{\partial \dot{a}_2}{\partial a_1} & \frac{\partial \dot{a}_2}{\partial \phi_1} & \frac{\partial \dot{a}_2}{\partial a_2} & \frac{\partial \dot{a}_2}{\partial \phi_2} \\
\frac{\partial \phi_2}{\partial a_1} & \frac{\partial \phi_2}{\partial \phi_1} & \frac{\partial \phi_2}{\partial a_2} & \frac{\partial \phi_2}{\partial \phi_2}
\end{bmatrix}
\begin{bmatrix}
\dot{a}_1 \\
\phi_1 \\
\dot{a}_2 \\
\phi_2
\end{bmatrix}
= \begin{bmatrix}
\gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & a_1 \\
\gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \phi_1 \\
\gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} & a_2 \\
\gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} & \phi_2
\end{bmatrix}
= \mathbf{J}
\begin{bmatrix}
\dot{a}_1 \\
\phi_1 \\
\dot{a}_2 \\
\phi_2
\end{bmatrix}
\]
(18)
where the values of \( \gamma_{mn} \), \( (m, n = 1, 2, 3, 4) \) are included in “Appendix”. Stability of a particular fixed point with respect to perturbation proportional to \( \exp(\lambda T) \) is determined by zeros of characteristic equation of the jacobian determinate \( |J - \lambda I| \) which gives:

\[
\lambda^4 + \Pi_1 \lambda^3 + \Pi_2 \lambda^2 + \Pi_3 \lambda + \Pi_4 = 0,
\]

(19)

where \( \Pi_{mn}, m, n = 1:4 \) are given in appendix. According to Routh-Hurwitz criteria [20,21], the necessary and sufficient condition for all characteristic roots of the characteristic equation (19) to have negative real parts if and only if the determinate \( D \) and all its principle minors are positive, where

\[
D = \begin{vmatrix}
\Pi_1 & 1 & 0 & 0 \\
\Pi_3 & \Pi_2 & \Pi_1 & 1 \\
0 & \Pi_4 & \Pi_3 & \Pi_2 \\
0 & 0 & 0 & \Pi_4 \\
\end{vmatrix},
\]

then the stability conditions will be

\[
\Pi_1 > 0, \quad \Pi_4 > 0, \quad (\Pi_1 \Pi_2 - \Pi_3) > 0, \quad [\Pi_3(\Pi_1 \Pi_2 - \Pi_3) - \Pi_1^2 \Pi_4] > 0,
\]

(20)

2.2 Case 2 Principal Parametric resonance

Assume that the detuning parameter \( \sigma_2 \) is to be used to depict the principal parametric resonance as shown in the following relation:

\[
\Omega = 2\omega + \sigma_2 = 2\omega + \sigma \hat{\sigma}_2
\]

(21)

Similarly as in case 1 combining Eq. (9) and (21) into (8) and eliminating all secular terms from the resulting equations to have:

\[
\frac{\beta_{14} f}{2} A + \beta_{44} f f \bar{A} e^{i \sigma t} - 2i \omega \bar{A} - 2i \omega A - \beta_1 B - i \omega \beta_3 B
\]

\[
-2\beta_5 A B \bar{B} - 3\beta_5 A^2 \bar{A} - \beta_5 B^2 \bar{A} - k_1 A e^{-i \sigma t} - i k_2 \omega A e^{-i \sigma t} = 0,
\]

\[
\beta_{24} (\frac{B f}{2} + f f \bar{B} e^{i \sigma t}) - 2i \omega \bar{B} - 2i \omega B - \beta_2 A - i \omega \beta_{22} A
\]

\[
-2\beta_6 A B \bar{A} - 3\beta_6 B^2 \bar{B} - \beta_6 A^2 \bar{B} = 0
\]

(22a)

(22b)

Using Eq. (15) into (22) and equating the real and imaginary parts to obtain the following system of ordinary differential equations:

\[
\dot{a}_1 = -\mu a_1 - \frac{\beta_{11} a_1}{2 \omega} \sin(\varphi_2) - \frac{\beta_{13} a_2}{2} \cos(\varphi_2)
\]

\[
- \frac{\beta_3 a_2^2}{4 \omega} \sin(2\varphi_2) + \frac{\beta_{14} f f \bar{A} e^{i \sigma t}}{2 \omega} \sin(\varphi_2) - \frac{k_1 a_1}{2 \omega} \sin(\omega \tau) - \frac{k_2 a_1}{2} \cos(\omega \tau)
\]

(23a)
\[ \phi_1 = \sigma_2 - \frac{\beta_{12} a_2}{\omega a_i} \cos(\phi_2) + \frac{\beta_{13} a_2}{a_i} \sin(\phi_2) - \frac{\beta_2 a_2^2}{2 \omega} \cos(2\phi_2) \]
\[ - \frac{\beta_3 a_2^2}{2 \omega} - \frac{3 \beta_3 a_2^2}{4 \omega} + \frac{\beta_3 f_o}{\omega} \left( \frac{f}{2} + f_o \cos(\phi_1) \right) - \frac{k_1}{\omega} \cos(\omega \tau) - k_2 \sin(\omega \tau), \]

(23b)

\[ \dot{a}_2 = -\mu_2 a_2 + \frac{\beta_{21} a_1 \sin(\phi_2)}{2 \omega} - \frac{\beta_{22} a_2}{2} \cos(\phi_2) + \frac{\beta_3 a_2 a_1^2}{8 \omega} \sin(2\phi_2) \]
\[ + \frac{\beta_{23} a f_o f}{2 \omega} \sin(\phi_1 - 2\phi_2), \]

(23c)

\[ \phi_2 = \frac{\beta_{22} a_2}{2 a_2} \sin(\phi_2) + \frac{\beta_{23} a_2}{2 \omega a_2} \cos(\phi_2) + \frac{\beta_3 a_2^2}{4 \omega} (1 + \cos(2\phi_2)) + \frac{3 \beta_3 a_2^2}{8 \omega} \]
\[ - \frac{\beta_{24} f_o f}{2 \omega} \cos(\phi_1 - 2\phi_2) + \frac{f^2}{4 \omega} \beta_{24} - \frac{\beta_{11} a_2}{2 \omega a_1} \cos(\phi_2) + \frac{\beta_{13} a_2}{2 a_1} \sin(\phi_2) \]
\[ - \frac{\beta_3 a_2^2}{4 \omega} \cos(2\phi_2) + \frac{3 \beta_3 a_2^2}{8 \omega} - \frac{\beta_{3} a_2^2}{2 \omega} - \frac{\beta_{3} a_2^2}{8 \omega} \]
\[ + \frac{\beta_{3} a_2^2}{2 \omega} \left( \frac{f}{2} + f_o \cos(\phi_1) \right) - \frac{k_1}{2 \omega}, \]

(23d)

where

\[ \phi_1 = \sigma_2 - 2\beta_1, \]
\[ \phi_2 = \beta_2 - \beta_1. \]

(24)

Similarly, for obtaining the steady state solution for amplitude and phase putting \(\dot{a}_1 = \dot{\phi}_1 = \dot{a}_2 = \dot{\phi}_2 = 0\) into Eq. (23), the resultant formulas can be solved numerically using MATLAB software.

To discuss the stability behavior of these solutions, linearizing these equations according to Lyapunov first (indirect) method to give the following system:

\[
\begin{bmatrix}
\dot{a}_1 \\
\dot{\phi}_1 \\
\dot{a}_2 \\
\dot{\phi}_2
\end{bmatrix} =
\begin{bmatrix}
\zeta_{11} & \zeta_{12} & \zeta_{13} & \zeta_{14} \\
\zeta_{21} & \zeta_{22} & \zeta_{23} & \zeta_{24} \\
\zeta_{31} & \zeta_{32} & \zeta_{33} & \zeta_{34} \\
\zeta_{41} & \zeta_{42} & \zeta_{43} & \zeta_{44}
\end{bmatrix} \begin{bmatrix}
a_1 \\
\phi_1 \\
a_2 \\
\phi_2
\end{bmatrix}
\]

(25)

where the values of \(\zeta_{mn}, (m, n = 1, 2, 3, 4)\) are given in “Appendix”. Numerically, primary resonance is the worst resonance case that is taken into account in the discussions.

3 Results and Discussion

In this section, system behaviors of the amplitude and phase at various resonance cases are illustrated. A comparison between active and time delay control and the effect of some system parameters on its amplitude are shown.
3.1 Time history

Fig. 3(a, b) shows the time response for the amplitude $X_1, X_2$, where Fig. 3(c) illustrates the system phase plane. Without resonance case and without applying any control system (i.e. $k_1 = k_2 = 0$) at the following parameter variables:

$$\begin{align*}
\omega &= 65, \quad \Omega = 100, \quad \mu_1 = \mu_2 = 0.5, \quad \beta_{11} = -0.003, \quad \beta_{13} = -0.82, \quad \beta_{14} = 0.55, \quad \beta_{16} = 6.55, \\
\beta_5 &= 0.9, \quad \beta_{22} = -0.82, \quad \beta_{21} = -0.001, \quad \beta_{24} = 0.5, \quad f_\omega = 7, \quad f = 2, \quad \tau = 0.
\end{align*}$$

![Fig. 3(a). the time response for the amplitude $x_1$](image_url)

![Fig. 3(b). The time response for the amplitude $x_1$](image_url)

![Fig. 3 (c). System phase plane](image_url)
We can see that the steady state amplitudes are stable in the case of non-resonance operating mode. Fig. 4 clarifies the time history without control and with primary resonance at the same previous parameters except that $\Omega = \omega = 99$, we observe that the amplitudes have been increased due to the resonance operating point.

Fig. 4(a). The time response for the amplitude $x_1$

Fig. 4(b). The time response for the amplitude $x_2$

Fig. 4(c). System phase plane

Now applying active and time delay control for the system with primary resonance and comparing the amplitudes. Fig. 5, 6 shows the effect of active and time delay control on both $X_1, X_2$. We observe that the
effective of active control is about 105%, and Time delay controller is about 125% so the time delay controller is more efficient than active velocity feed-back controller for this system.

![Graph of active control](image1)

**Fig. 5 (a,b). Effect of active control on $x_1$, $x_2$ respectively at primary resonance case**

![Graph of time delay control](image2)

**Fig. 6 (a,b). Effect of time delay control on $x_1$, $x_2$ respectively at primary resonance case, $\tau = 0.0015$**

### 3.2 Comparisons with numerical method

In this sub-section we compared the approximate solution induced by (MTSM) and numerical solution using Rung-Kutta Method (RKM). Figure 7, and 8 show good agreement between the approximate solution (blue curves) and the numerical results (red curves) in case of $\tau = 0$, $0.0015$ respectively.

![Graph of MTSM and RKM comparison](image3)

**Fig. 7(a). Time history for the amplitude $x_1$ using MTSM (blue curve) and RKM (red curve)**
Fig. 7(b). Time history for the amplitude $x_2$ using RKM (blue curve) and MTSM (red curve)

Fig. 8(a). Time history for the amplitude $x_1$ using MTSM (blue curve) and RKM (red curve) for $\tau = 0.0015$

Fig. 8 (b). Time history for the amplitude $x_2$ using MTSM (blue curve) and RKM (red curve) for $\tau = 0.0015$

3.3 Frequency response

Now the following figures show the system amplitude against the detuning parameter $\sigma_1$ with change in specified values for system parameters. In Fig. 9 the parameters $a_1, a_2$ with $\sigma_1$ in case of primary resonance case with:

$$\omega = \Omega = 100, \mu_1 = 0.9, \mu_2 = 0.7, \beta_{11} = -0.003, \beta_{15} = -0.82, \beta_{14} = 0.55, \beta_{16} = 6.55, \beta_3 = 0.9, \beta_{23} = -0.82, \beta_{21} = -0.001, \beta_{24} = 0.5, f_o = 7, f = 3, \tau = 0.0015, \varepsilon = 0.001, k_1 = 1000, k_2 = 0.7, 1, 1.5.$$
We observe that the amplitude decreases with the increase of the gain $k_2$, then the delayed velocity feedback control is more efficient than the delay on the displacement. Figs. 10, and 11 illustrate the effect of $\sigma_1$ on the amplitude with various values of the damping parameter $\mu_1, \mu_2$ as given in these figures respectively. The same system parameters values as given for Fig. 9 are used and $k_2 = 1$. We observe in Fig. 10 that the values of $a_1, a_2$ are proportional inversely with the damping parameter $\mu_1$, but in Fig. 11 the value of $a_1$ is approximately constant with $\mu_2$ as it is effect on the velocity $\dot{x}_2$ of the system second mode with two peaks.

3.4 Amplitude vs. certain system parameters

Let us consider the parameters given in sub-section 3.3 unless otherwise specified. In this sub-section we shows the change of amplitude range with varying of the constant and variable rotating forces $f_x, f_x$ as shown in Fig. 12(a,b) respectively $\omega = 15, k_2 = 100, \Omega = 90$. The steady state amplitude of the main system is a monotonic increasing function of the excitation amplitude up to maximum amplitude at saturation. The saturation value may lead to an unstable or damaged system due to its large value. Figure 13 (a, b) describe the behavior of the amplitude with damping parameters $\mu_1, \mu_2$ respectively at $\Omega = \omega = 10$. We observe in Fig. 13 that the suitable range for $\mu_2 \leq 0.003$, and $\mu_1 \geq 0.2$, it is useful for the system to choose a large value for $\mu_1$, but an expensive material should be used, so we use suitable materials with appropriate cost and adding a specified controller for reducing the amplitude for minimum values in the instance of resonance cases.

![Fig. 9(a). System amplitude $a_1$ against detuning parameter $\sigma_1$ at $k_2 = 0.7, 1, 1.5$](image1.png)

![Fig. 9(b). System amplitude $a_2$ against detuning parameter $\sigma_1$ at $k_2 = 0.7, 1, 1.5$](image2.png)
Fig. 10(a). System amplitude $a_1$ against detuning parameter $\sigma_1$ at $\mu_1 = 0.4, 0.5, 0.7$

Fig. 10(b). System amplitude $a_2$ against detuning parameter $\sigma_1$ at $\mu_1 = 0.4, 0.5, 0.7$

Fig. 11 (a) System amplitude $a_1$ against detuning parameter $\sigma_1$ at $\mu_1 = 0.4, 0.5, 0.6, 0.9$

Fig. 11 (b). System amplitude $a_1$ against detuning parameter $\sigma_1$ at $\mu_1 = 0.4, 0.5, 0.6, 0.9$
Fig. 12(a). System amplitude against constant rotating forces $f_o$

Fig. 12(b). System amplitude against variable rotating forces $f$

Fig. 13(a). System amplitude against damping parameters $\mu_i$
4 Conclusion

In this research, a system of nonlinear ordinary differential equations that describing a rotating beam is analyzed approximately via multiple time scales method. We studied the effect of existence and nonexistence of the time delay on the velocity and the displacement feedback the system amplitude. The study are in case of the worst resonance cases that are primary and principal parametric resonance. We concluded that the time delay controller is more efficient than active feed-back controller on the velocity for this system, as the effective of active control is about 105\%, and Time delay controller is about 125\%, so the Time delay control is recommended to use in this system. The Lyapunov first method and Routh–Hurwiz criteria are adopted to achieve the stability analysis. In addition, approximate solution induced by (MSPT) is compared with numerical approximation solution using Rung-Kutta of fourth order method. The distinction offered a good agreement between approximately and numerical approaches. The effects of system parameters on the amplitude are discussed for choosing appropriate values for these parameters that attaining the system stability.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Thomas O, Sénéchal A, Deü JF. Hardening / softening behavior and reduced ordermodeling of nonlinear vibrations of rotating cantilever beams. Nonlinear Dyn. 2016;86:1293–1318.

[2] Zhang X, Zhang D, Chen S, Hong J. Modal characteristics of a rotating flexible beam with a concentrated mass based on the absolute nodal coordinate formulation. Nonlinear Dyn. 2017;88:61–77.

[3] Rezaei MM, Behzad M, Haddadpour H, Moradi H. Aeroelastic analysis of a rotating wind turbine blade using geometrically exact formulation. Nonlinear Dyn. 2017;89:2367–2392.

[4] Arvin H, Bakhtiari-Nejad F. Non-linear modal analysis of a rotating beam. Int. J. Non-Linear Mech. 2011;46(6):877–897.

[5] Bekhoucha F, Rechak S, Duigou L, Cadou JM. Nonlinear forced vibrations of rotating anisotropic beams. Nonlinear Dyn. 2013;74(4):1281–1296.
[6] Kim H, Chung J. Nonlinear modeling for dynamic analysis of a rotating cantilever beam, Nonlinear Dyn. 2016;86:1981–2002.

[7] Latalski J. A coupled-field model of a rotating composite beam with an integrated nonlinear piezoelectric activeshield. Nonlinear Dyn. 2017;90:2145–2162.

[8] Ali Kandil, Hany El-Gohary. Investigating the performance of a time delayed proportional derivative controller for rotating bladervibrations, Nonlinear Dyn. 2018;91:2631–2649.

[9] Yao MH, Chen YP, Zhang W. Nonlinear vibrations of blade with varying rotating speed, Nonlinear Dyn. 2012;68:487–504.

[10] Yao MH, Zhang W, Chen YP. Analysis on nonlinear oscillations and resonant responses of a compressor blade, Acta Mech. 2014;225:3483–3510.

[11] Choi SC, Park JS, Kim JH. Active damping of rotating composite thin-walled beams using MFC actuators and PVDF sensors, Compos. Struct. 2006;76:362–374.

[12] Choi SC, Park JS, Kim JH. Vibration control of pre-twisted rotating composite thin-walled beams with piezoelectricfiber composites, J. Sound Vib. 2007;300:176–196.

[13] Joy Mondal, S Chatterjee. Controlling self-excited vibration of a nonlinear beam by nonlinear resonant velocity feedback with time-delay, International Journal of Non-Linear Mechanics 2021;131:103684.

[14] Liang Li, Wei-Hsin Liao, Dingguo Zhang, Yongbin Guo. Dynamic modeling and analysis of rotating beams with partially covered enhanced active constrained layer damping treatment, Journal of Sound and Vibration. 2019;455:107368.

[15] Boumediène Chentouf, Nejib Smaoui, Exponential stabilization of a non-uniform rotating disk-beam system via a torque control and a finite memory type dynamic boundary control, Journal of the Franklin Institute. 2019;356:11318–11344.

[16] Lyu LF, Zhu WD. Operational modal analysis of a rotating structure under ambient excitation using a tracking continuously scanning laser Doppler vibrometer system, Mechanical Systems and Signal Processing. 2021;152:107367.

[17] Andrea Bonito, Ricardo H Nochetto, Dimitris Ntogkas, Discontinuous Galerkin approach to large bending deformation of a bilayer plate with isometry constraint, Journal of Computational Physics. 2020;423:109785.

[18] Nayfeh AH, Mook DT. Nonlinear Oscillations. Wiley, New York; 1995.

[19] YA Amer, AT El-Sayed, Darwesh FO. Active and time delay controls on vibrations of the Micro Electro-Mechanical System (MEMS) resonator, Asian Research Journal of Mathematics. 2019;12(4):1-17. Article no.ARIOM.46389 ISSN: 2456-477X.

[20] Robert L Borrelli, Courtney S Colman. Differential Equation, John Wiley and Sons, INC., New York; 1998.

[21] Gradshteyn IS, Ryzhik LM. 6th Ed. San Diego, Ca: Academic Press. 2000;1076.
Appendix

\[ p_x = C_p \rho \frac{a v}{2} \sin(2 \beta + 2 \gamma) - a u \sin^2(\beta + \gamma) - a u P \sin^3(\beta + \gamma) \]
\[ - a u P \beta \sin^2(\beta + \gamma) \cos(\beta + \gamma) + a v \sin^2(\beta + \gamma) \cos(\beta + \gamma) \]
\[ - a v \beta \sin^3(\beta + \gamma) - b u \cos^2(\beta + \gamma) - \frac{b v}{2} \sin(2 \beta + 2 \gamma) - b u' P \cos^3(\beta + \gamma) \]
\[ + b u P \beta \cos^2(\beta + \gamma) \sin(\beta + \gamma) - b v' P \cos^2(\beta + \gamma) \sin(\beta + \gamma) - b u P \frac{\beta}{L} \cos^3(\beta + \gamma) \]

\[ p_y = C_p \rho \frac{a v}{2} \sin(2 \beta + 2 \gamma) - a v' \cos^2(\beta + \gamma) + a u' P \sin^2(\beta + \gamma) \cos(\beta + \gamma) \]
\[ + a u P \frac{\beta}{L} \cos^2(\beta + \gamma) \sin(\beta + \gamma) - a v' P \sin(\beta + \gamma) \cos^2(\beta + \gamma) \]
\[ + a v P \frac{\beta}{L} \sin^2(\beta + \gamma) - \frac{b u}{2} \sin(2 \beta + 2 \gamma) \]
\[ - b v' \sin^2(\beta + \gamma) - b u' P \cos^2(\beta + \gamma) \sin(\beta + \gamma) \]
\[ + b u P \frac{\beta}{L} \cos(\beta + \gamma) \sin^2(\beta + \gamma) \]
\[ - b v' P \cos(\beta + \gamma) \sin^2(\beta + \gamma) - b v P \frac{\beta}{L} \cos^3(\beta + \gamma) \sin(\beta + \gamma) \]

\[ a_1(z) = \left[ \int K_{32} \frac{dx}{ds} \frac{dy}{ds} ds - K_{11} x y ds \right], \]
\[ a_2(z) = \left[ \int K_{11} y^2 ds - K_{32} \left( \frac{dx}{ds} \right)^2 ds \right], \]
\[ a_3(z) = \left[ \int K_{11} x^2 + K_{32} \left( \frac{dx}{ds} \right)^2 ds \right], \]
\[ K_{11} = A_{22} - \frac{A_{12}^2}{A_{11}}, \quad K_{32} = D_{22}, \quad A_{11} = \int_{-h/2}^{h/2} Q_y ds, \]
\[ D_y = \int_{-h/2}^{h/2} Q_y n^2 ds, \quad Q_{11} = Q_{22} = \frac{E}{1 - \nu^2}, \]
\[ Q_{12} = Q_{21} = \frac{E \nu}{1 - \nu^2}. \]
\[ \delta K = \int_0^l \left\{ b_1 \left[ \dot{u} + \dot{\Omega} (R_u + z) - u \Omega^2 \right] + b_1 \delta u + b_1 \delta v \right\} dz, \]
\[ b_i = \int_{-h/2}^{h/2} \rho d n d s, \]
\[ W = \int_0^l \left( p_p u + p_p v \right) dz, \]
\[ U = U_1 + U_2, \]
\[ U_1 = \int_0^l \rho A \Omega^2 \left[ R_u (L - z) + 0.5 (L^2 - z^2) \right] \left[ (u')^2 + (v')^2 \right] ds, \]
\[ U_2 = \frac{1}{2} \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \sigma_{zz} \varepsilon_{zz} d n d s d z, \]
\[ \varepsilon_{zz} = 0.5 \left[ (u')^2 + (v')^2 \right] - y v^* - A u^* + n \left[ v^* \frac{dx}{ds} - u^* \frac{dy}{ds} \right] + \varepsilon_{zz}, \]
\[ \sigma_{zz} = Q_{zz} e_{zz} + \alpha \Delta T, \]
\[ \gamma_{11} = -\mu - \beta_2 a_s^2 \sin(2\phi) + \frac{\beta_4 f^2 \sin(2\phi)}{8 \omega} - \frac{k_z}{2}, \]
\[ \gamma_{12} = \frac{\beta_4 f^2 a_i \sin(2\phi)}{4 \omega} + \frac{\beta_6 \Omega f \sin(\phi)}{2 \omega}, \]
\[ \gamma_{13} = -\frac{\beta_3 \cos(\phi)}{2} - \frac{\beta_1 \sin(\phi)}{2 \omega} - \frac{\beta_5 a_i a_s \sin(2\phi)}{4 \omega}, \]
\[ \gamma_{14} = \frac{\beta_3 a_s \sin(\phi)}{2} - \frac{\beta_1 a_i \cos(\phi)}{2 \omega} - \frac{\beta_5 a_i a_s \cos(2\phi)}{4 \omega}, \]
\[ \gamma_{21} = -\frac{\beta_1 a_s \sin(\phi)}{2 a_i^2} + \frac{\beta_5 a_i \cos(\phi)}{2 \omega a_i^2} - \frac{3 \beta_2 a_s}{2 \omega a_i^2} - \frac{\beta_6 \Omega f \sin(\phi)}{4 \omega}, \]
\[ \gamma_{22} = -\frac{\beta_1 f^2 \sin(2\phi)}{4 \omega} + \frac{\beta_6 \Omega f \cos(\phi)}{2 \omega a_i}, \]
\[ \gamma_{23} = \frac{\beta_3 \sin(\phi)}{2 a_i} - \frac{\beta_1 \cos(\phi)}{2 \omega a_i} - \frac{\beta_5 a_s}{2 \omega} - \frac{\beta_6 a_s \cos(2\phi)}{4 \omega}, \]
\[ \gamma_{24} = \frac{\beta_1 a_s \cos(\phi)}{2 a_i} + \frac{\beta_5 a_s \sin(\phi)}{2 \omega a_i} + \frac{\beta_5 a_s^2 \sin(2\phi)}{4 \omega}, \]
\[
\gamma_{31} = \frac{-\beta_{22} \cos(\phi_2) + \beta_{22} \sin(\phi_2)}{2} + \frac{\beta_5 a_1 a_2 \sin(2\phi_2)}{4\omega},
\]
\[
\gamma_{32} = \frac{\beta_{24} f^2 a_2 \cos(2\phi_2 - 2\phi_1)}{4\omega},
\]
\[
\gamma_{33} = -\frac{\mu_2 + \beta_5 a_2^2 \sin(2\phi_2) - \beta_{24} f^2 \sin(2\phi_2 - 2\phi_1)}{8\omega},
\]
\[
\gamma_{34} = \frac{\beta_{22} a_1 \sin(\phi_2) + 2\omega a_2^2}{2\omega} + \frac{\beta_5 a_1 a_2 \cos(2\phi_2) - \beta_{24} f^2 a_2 \cos(2\phi_2 - 2\phi_1)}{4\omega} + \frac{\beta_5 a_1 a_2 \cos(2\phi_2) - 2\omega a_2^2}{2\omega a_1^2} - \frac{\beta_{16} \Omega f \sin(\phi_1)}{2\omega a_1^2},
\]
\[
\gamma_{41} = \frac{\beta_{22} \sin(\phi_2)}{2a_1} + \frac{\beta_{22} \cos(\phi_2)}{2\omega a_2} + \frac{\beta_4 a_1^2 a_2}{2\omega} + \frac{\beta_5 a_1 a_2 \cos(2\phi_2)}{2\omega a_2} + \frac{\beta_{24} f^2 a_2 \cos(2\phi_2 - 2\phi_1)}{4\omega} + \frac{1}{2} - \frac{\beta_{16} \Omega f \cos(\phi_1)}{2\omega a_1},
\]
\[
\gamma_{42} = \frac{\beta_{24} f^2 \sin(2\phi_1 - 2\phi_2)}{4\omega} - \frac{\beta_{24} f^2 \cos(2\phi_1)}{4\omega} + \frac{\beta_{16} \Omega f \cos(\phi_1)}{2\omega a_1},
\]
\[
\gamma_{43} = -\frac{\beta_{22} a_1 \sin(\phi_2)}{2a_1^2} - \frac{\beta_{22} a_1 \cos(\phi_2)}{2\omega a_2^2} + \frac{\beta_5 a_1 a_2 \cos(2\phi_2)}{2\omega a_2^2} + \frac{1}{2} + \frac{\beta_{13} \sin(\phi_1) - \beta_{13} \cos(\phi_1)}{2\omega a_2},
\]
\[
\gamma_{44} = \frac{\beta_{22} a_1 \cos(\phi_2)}{2a_2} + \frac{\beta_{22} a_1 \sin(\phi_2)}{2\omega a_2} + \frac{\beta_5 \sin(2\phi_2)}{4\omega} - \frac{\beta_{24} f^2 \sin(2\phi_1 - 2\phi_2)}{4\omega} + \frac{\beta_5 a_1 a_2 \cos(2\phi_2)}{2\omega a_1^2} + \frac{\beta_{13} a_2 \cos(\phi_2)}{2\omega a_1^2} + \frac{\beta_{13} a_2 \sin(\phi_2)}{2\omega a_1},
\]
\[
\zeta_{11} = -\frac{\beta_5 a_2^2 \sin(2\phi_2)}{8\omega} + \frac{\beta_{14} a_1 f \sin(\phi_1)}{2\omega} - \frac{k_2}{2},
\]
\[
\zeta_{12} = \frac{\beta_{14} a_1 f \sin(\phi_1)}{2\omega},
\]
\[
\zeta_{13} = \frac{-\beta_{13} \cos(\phi_2) - \beta_{13} \sin(\phi_2) - \beta_5 a_2 a_2 \sin(2\phi_2)}{4\omega},
\]
\[
\zeta_{14} = \frac{-\beta_{13} a_2 \sin(\phi_2) - \beta_1 a_2 \cos(\phi_2) - \beta_5 a_1 a_2^2 \cos(2\phi_2)}{4\omega},
\]

\[ \zeta_{21} = -\frac{\beta_{13} a_1 \sin(\varphi_2)}{a_i^2} + \frac{\beta_{11} a_3 \cos(\varphi_2)}{\omega a_i^2} - \frac{3 \beta_{11} a_1}{2 \omega}, \]

\[ \zeta_{22} = -\frac{\beta_{13} \sin(\varphi_2)}{\omega}, \]

\[ \zeta_{23} = \frac{\beta_{13} \sin(\varphi_2)}{a_i} - \frac{\beta_{11} \cos(\varphi_2)}{\omega a_i} - \frac{\beta_{2} a_2}{\omega} - \frac{\beta_{3} a_3 \cos(2\varphi_2)}{2 \omega}, \]

\[ \zeta_{24} = \frac{\beta_{13} a_2 \cos(\varphi_2)}{a_i} + \frac{\beta_{11} a_3 \sin(\varphi_2)}{\omega a_i} + \frac{\beta_{5} a_3^2 \sin(2\varphi_2)}{2 \omega}, \]

\[ \zeta_{31} = \frac{\beta_{22} \cos(\varphi_2)}{2} + \frac{\beta_{23} \sin(\varphi_2)}{2 \omega} + \frac{\beta_{5} a_3 \sin(2\varphi_2)}{4 \omega}, \]

\[ \zeta_{32} = \frac{\beta_{24} f_0 f a_2 \cos(2\varphi_2 - \varphi_1)}{2 \omega}, \]

\[ \zeta_{33} = -\mu_2 + \frac{\beta_{4} a_3^2 \sin(2\varphi_2)}{8 \omega} + \frac{\beta_{24} f_0 f \sin(2\varphi_2 - \varphi_1)}{2 \omega}, \]

\[ \zeta_{34} = \frac{\beta_{22} a_1 \sin(\varphi_2)}{2} + \frac{\beta_{21} a_4 \cos(\varphi_2)}{2 \omega} + \frac{\beta_{5} a_3^2 a_4 \cos(2\varphi_2)}{4 \omega} - \frac{\beta_{24} f_0 f a_2 \cos(2\varphi_2 - \varphi_1)}{2 \omega}, \]

\[ \zeta_{41} = \frac{\beta_{22} \sin(\varphi_2)}{2 a_2} + \frac{\beta_{21} \cos(\varphi_2)}{2 \omega a_2} + \frac{\beta_{5} a_3 \sin(2\varphi_2)}{2 a_2} + \frac{1}{2} - \frac{\beta_{13} a_3 \sin(\varphi_2)}{2 a_i^2} + \frac{\beta_{11} a_3 \cos(\varphi_2)}{2 \omega a_i^2}, \]

\[ \zeta_{42} = -\frac{\beta_{24} f_0 f \sin(2\varphi_2 - \varphi_1)}{2 \omega} - \frac{\beta_{4} f_0 f \sin(\varphi_1)}{2 \omega}, \]

\[ \zeta_{43} = -\frac{\beta_{22} a_1 \sin(\varphi_2)}{2 a_i^2} - \frac{\beta_{21} a_4 \cos(\varphi_2)}{2 \omega a_i^2} + \frac{\beta_{5} a_3 \sin(2\varphi_2)}{2 \omega a_i^2} + \frac{\beta_{13} \sin(\varphi_2)}{2 a_i} - \frac{\beta_{11} \cos(\varphi_2)}{2 \omega a_i}, \]

\[ \zeta_{44} = \frac{\beta_{22} a_2 \cos(\varphi_2)}{2 a_i} - \frac{\beta_{24} f_0 f \sin(2\varphi_2 - \varphi_1)}{4 \omega} + \frac{\beta_{4} f_0 f \sin(2\varphi_2 - \varphi_1)}{4 \omega} + \frac{\beta_{13} a_3 \sin(\varphi_2)}{2 \omega a_i} + \frac{\beta_{11} a_3 \sin(\varphi_2)}{2 \omega a_i}. \]
\[
\Pi_1 = -(y_{11} + y_{22} + y_{33} + y_{44}),
\]
\[
\Pi_2 = y_{11} y_{22} + y_{33} y_{44} + (y_{11} + y_{22}) (y_{33} + y_{44}) - [y_{12} y_{21} + y_{13} y_{31} + y_{14} y_{41} + y_{23} y_{32} + y_{24} y_{42} + y_{34} y_{43}],
\]
\[
\Pi_3 = -y_{11} [y_{22} (y_{33} + y_{44}) - y_{43} + y_{33} y_{44} - y_{23} y_{32} - y_{24} y_{42}) - y_{22} (y_{33} y_{44} - y_{34} y_{43})
\]
\[
- y_{23} y_{34} y_{24} - y_{24} (y_{32} y_{43} - y_{22} y_{33}) + y_{12} (y_{21} (y_{33} + y_{44}) - y_{23} y_{31} + y_{24} y_{41})
\]
\[
+ y_{13} [y_{31} (y_{22} + y_{44}) - y_{32} y_{21} - y_{34} y_{41} + y_{14} (y_{41} (y_{22} + y_{33}) - y_{42} y_{21} - y_{43} y_{31})],
\]
\[
\Pi_4 = -\Pi_1 + y_{11} [y_{33} y_{44} - y_{34} y_{43})
\]
\[
+ y_{21} y_{12} (y_{34} y_{43} - y_{33} y_{44}) + y_{12} y_{23} (y_{31} y_{44} - y_{34} y_{41}) - y_{12} y_{24} (y_{31} y_{43} - y_{33} y_{41})
\]
\[
+ y_{13} (y_{23} y_{44} - y_{34} y_{42}) + y_{22} (y_{34} y_{41} - y_{31} y_{44}) + y_{24} (y_{31} y_{42} - y_{32} y_{41})
\]
\[
+ y_{14} [y_{21} (y_{33} y_{42} - y_{32} y_{43}) + y_{22} (y_{31} y_{43} - y_{41} y_{33}) + y_{14} (y_{41} y_{32} - y_{42} y_{31})].
\]

© 2021 Amer et al.; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

**Peer-review history:**
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/66021