MUB-like structures and tomographic reconstruction for N-ququart systems

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Abstract. We construct an informationally complete set of mutually unbiased-like bases for N ququarts. These bases are used in an explicit tomographic protocol which performance is analyzed by estimating quadratic errors and compared to other reconstruction schemes.

Keywords: quantum tomography, mutually unbiased bases, error estimation

1. Introduction

Discrete quantum systems, i.e. systems with a finite number of energy levels subjected only to Clifford-type physical operations (those that preserve the generalized Pauli group), play a fundamental role in a wide range of quantum information protocols as teleportation, quantum key distribution, error correction codes \cite{1, 2, 3, 4, 5} and in problems related to state characterization and quantum tomography \cite{6, 7}. In most experiments related to information transmission, only two level systems (qubits) are involved. However, it has been shown (for a recent review see \cite{8} and references therein) that quantum information processing can be optimized by using systems of larger dimension (qudits) and (effective) non-unitary operations \cite{9}. In recent years special attention has been paid to ququarts (four-level systems), since they can be efficiently simulated in atomic systems \cite{10, 11}, in single photon setups \cite{12, 13, 14} and in nuclear spins \cite{15, 16}. Effective ququart systems have been experimentally applied in quantum computing without entanglement \cite{13, 16}, high-dimensional quantum key distribution \cite{17}, and four-dimensional entanglement distribution \cite{18, 19}. In general, manipulations with a multipartite quantum system of a given dimension become “cheaper”, i.e. it is required less non-local operations, when the dimensionality of its components increase. For instance, an elementary controlled-ququart gate costs no more that 8 CNOT gates, while 104 CNOT gates are needed for an implementation of a full four qubit logic \cite{14}.
One of the fundamental problems in quantum information processing is the reconstruction of quantum states with a certain fidelity. The lower intrinsic error generated in tomographic protocols depends both on the type of the system and on the operations available in a given experimental setup [20]. A variety of reconstruction schemes for \(N\)-qubit systems where proposed and experimentally implemented. The most theoretically advantageous tomographic method is based on the so-called Mutually Unbiased Bases (MUB), which can be constructed in multiple ways in a \(2^N\) dimensional Hilbert space. Most of the states composing the MUBs are highly entangled and require a large amount of non-local resources for their generation. Thus, the MUB tomography, while being optimal with respect to the minimization of statistical errors, turns out to be extremely “expensive”, for the experimental implementation, due to a high entanglement cost of states used in the respective projective measurements [21, 22, 23, 25]. Considering this, the reconstruction of a compound quantum system by using their (effective) higher dimensional constituents could be an attractive alternative.

On the other hand, the mathematical construction of MUBs for \(N\)-qubit (and any prime power) systems is heavily based on the underlying structure of finite fields [21, 25, 26]. That allows to simplify the inversion of high-rank matrices leading to explicit reconstruction expressions and a direct quantification of the estimation errors [27].

The simplest \(N\)-partite system that can not be treated in this simple way, i.e. using the standard language of finite fields, is a collection of four level systems, where \(SU(4)\) transformations can be locally implemented. The analysis of a system of \(N\) ququarts, and in particular, the problem of state reconstruction, requires the use of the Galois rings, whose structure is more involved than the one of the finite fields. This complexity is particularly reflected in the necessity of redundant measurements for a direct tomographic protocol [30], [31].

In this paper, we construct an informationally complete set of \(4^N + 2^N\) mutually unbiased MU-like bases for \(N\)-ququarts. We also obtain an explicit state reconstruction expression, which is used for the estimation of average square error of the Hilbert-Schmidt distance between the real and the estimated state. We compare the efficiency of our reconstruction scheme with the corresponding \(2N\) qubit MUB tomography.

The paper is organized as follows: in section 2 we briefly review \(N\)-qubit MUB construction and the corresponding tomographic scheme; in section 3 we construct MU-like bases for single and \(N\) ququarts and discuss their properties; in section 4 we obtain an explicit reconstruction expression for \(N\) ququarts, and analyze its performance. A summary of finite fields and Galois rings can be found in Appendices, for more details see [32].

2. Review for \(N\) qubit tomography

It is convenient to label the computational basis \(\{|k_1\rangle|k_2\rangle\cdots|k_N\rangle, k_i = 0, 1\}\) in the \(N\)-qubit Hilbert space \(\mathcal{H}_{2^N} = \mathcal{H}_2^{\otimes N}\) with elements of the finite field \(\mathbb{F}_{2^N}\), considered as
a linear space spanned by a basis \( \{ \theta_1, ..., \theta_N \} \), allowing to decompose any \( \kappa \in \mathbb{F}_2^N \) as,

\[
\kappa = \sum_{i=1}^{N} k_i \theta_i, \quad k_i \in \mathbb{Z}_2, \quad (1)
\]

so that,

\[
|k_1 \rangle |k_2 \rangle \cdots |k_N \rangle = |\kappa \rangle, \quad \langle \kappa | \kappa' \rangle = \delta_{\kappa, \kappa'}, \quad (2)
\]

see Appendix A. In \( \mathbb{F}_2^N \) there always exists the so-called self-dual basis, orthonormal with respect to the trace operation, \( \text{tr}(\theta_i \theta_j) = \delta_{ij}, \text{tr}(\kappa) = \kappa + \kappa^2 + ... + \kappa^{2^{N-1}} \in \mathbb{Z}_2 \). Thus, in Eq. (1), \( k_i \) are components of \( \kappa \) when it is written in a self-dual basis \( k_i = \text{tr}(\kappa \theta_i) \). This allows to establish a correspondence between qubits and elements of the basis in \( \mathbb{F}_2^N \): qubit \( i \leftrightarrow \theta_i \), through the trace operation.

The operators, elements of the generalized Pauli group \( \mathcal{P}^N = \mathcal{P}^1 \otimes ... \otimes \mathcal{P}^1 \) \[33], \[34], \[24] acting in \( \mathcal{H}_2^{\otimes N} \) are defined according to,

\[
Z_\gamma = \sum_{\kappa \in \mathbb{F}_2^N} (-1)^{\text{tr}(\gamma \kappa)} |\kappa \rangle \langle \kappa |, \quad X_\delta = \sum_{\kappa \in \mathbb{F}_2^N} |\kappa + \delta \rangle \langle \kappa |, \quad \gamma, \delta \in \mathbb{F}_2^N, \quad (3)
\]

satisfying the commutation relations,

\[
X_\delta Z_\gamma = (-1)^{\text{tr}(\delta \gamma)} Z_\gamma X_\delta, \quad (4)
\]

and are related through the finite Fourier transform operation, \( X_\delta = F_{2^N}^{-1} Z_\delta F_{2^N} \),

\[
F_{2^N} = 2^{-N/2} \sum_{\alpha,\beta \in \mathbb{F}_2^N} (-1)^{\text{tr}(\alpha \beta)} |\alpha \rangle \langle \beta |. \quad (5)
\]

The operators (3) are factorized into a direct product of single qubit operators,

\[
Z_\gamma = \sigma^g_x \otimes ... \otimes \sigma^g_z, \quad g_i = \text{tr}(\gamma \theta_i),
\]

\[
X_\delta = \sigma^d_x \otimes ... \otimes \sigma^d_z, \quad d_i = \text{tr}(\delta \theta_i), \quad (6)
\]

as well as Fourier operators

\[
F_{2^N} = F_2 \otimes ... \otimes F_2, \quad F_2 = 2^{-1/2} \sum_{\ell,\ell'=0}^{1} (-1)^{\ell \ell'} |\ell \rangle \langle \ell'|.
\]

where \( \sigma_{x,y,z} \) are Pauli operators. The set of \( 2^{2N} \) monomials \( \{Z_\gamma X_\delta, \gamma, \delta \in \mathbb{F}_2^N\} \) form an operational basis in \( \mathcal{H}_2^{\otimes N} \) and can be separated into \( 2^N + 1 \) subsets of \( 2^N \) commuting monomials,

\[
\{Z_\gamma X_\lambda, Z_\gamma X_\lambda', Z_\gamma' X_\lambda, Z_\gamma' X_\lambda'\} = 0, \quad \lambda, \gamma, \gamma' \in \mathbb{F}_2^N \} \cup \{X_\gamma, \gamma \in \mathbb{F}_2^N\}. \quad (7)
\]

The eigenstates of the commuting sets (7),

\[
Z_\gamma X_\lambda |\psi^\lambda_\kappa \rangle = (-1)^{\text{tr}(\gamma \kappa)} |\psi^\lambda_\kappa \rangle, \quad \text{fixed } \lambda,
\]

\[
X_\gamma |\bar{\kappa} \rangle = (-1)^{\text{tr}(\gamma \kappa)} |\bar{\kappa} \rangle, \quad |\bar{\kappa} \rangle = F_{2^N}^{-1} |\kappa \rangle, \quad (8,9)
\]

are mutually unbiased \[21], \[22], \[23], \[25], \[26],

\[
|\langle \psi^\lambda_\kappa | \psi^\lambda_\kappa' \rangle|^2 = \delta_{\lambda \lambda'} \delta_{\kappa \kappa'} + \frac{1 - \delta_{\lambda \lambda'}}{2^N}, \quad (10)
\]

\[
|\langle \bar{\kappa}' | \bar{\kappa} \rangle|^2 = \delta_{\kappa, \kappa'}, \quad |\langle \bar{\kappa}' | \psi^\lambda_\kappa \rangle|^2 = \frac{1}{2^N}. \quad (11)
\]
Explicitly, the eigenstates of a set \( \{ Z_\gamma X_\lambda \}, \) fixed \( \lambda \) are obtained from elements of the computational basis \( |\kappa\rangle \) through the following unitary transformation,

\[
|\psi_\kappa^\lambda\rangle = V_\lambda |\kappa\rangle, \quad V_\lambda = \frac{1}{2^N} \sum_{\alpha, \beta, \gamma \in \mathbb{F}_2} c_{\gamma,\lambda} (-1)^{\text{tr}(\gamma(\alpha-\beta))} |\alpha\rangle \langle \beta|,
\]

(12)

\[
V_\lambda Z_\gamma V_\lambda^\dagger = c_{\gamma,\lambda} Z_\gamma X_{\gamma\lambda},
\]

(13)

where the phases \( c_{\gamma,\lambda}, |c_{\gamma,\lambda}| = 1 \), satisfy the functional equation,

\[
c_{\alpha+\gamma,\lambda} = c_{\alpha,\lambda} c_{\gamma,\lambda} (-1)^{\text{tr}(\alpha\gamma\lambda)}.
\]

(14)

As discussed in Appendix B, a set of solutions of (14) can be easily obtained considering the indices of \( c_{\gamma,\lambda} \) as elements of the ring \( GR(4,N), \gamma, \lambda \in T_2 = GR(4,N)/(2) \subset GR(4,N) \) which are in one-to-one correspondence with \( \mathbb{F}_2 \).

The density matrix has a simple expansion on the projectors on the MUBs

\[
\rho = \sum_{\lambda, \kappa \in \mathbb{F}_{2N}} p_\lambda^\kappa |\psi_\kappa^\lambda\rangle \langle \psi_\kappa^\lambda| + \sum_{\kappa \in \mathbb{F}_{2N}} \tilde{p}_\kappa |\tilde{\kappa}\rangle \langle \tilde{\kappa}| - \mathbb{I},
\]

(15)

where the measured probabilities \( p_\lambda^\kappa = \langle \psi_\kappa^\lambda| \rho |\psi_\kappa^\lambda\rangle \) and \( \tilde{p}_\kappa = \langle \tilde{\kappa}| \rho |\tilde{\kappa}\rangle \) satisfy the normalization conditions,

\[
\sum_{\kappa \in \mathbb{F}_{2N}} p_\lambda^\kappa = \sum_{\kappa \in \mathbb{F}_{2N}} \tilde{p}_\kappa = 1.
\]

(16)

The above reconstruction scheme is optimal in the sense of minimisation of the statistical error associated with the \( 2^N + 1 \) measurement setups.

Unfortunately, among all MUBs there are only 3 completely factorized bases, eigenstates of \( \{ |\psi_\kappa^{\lambda=0}\rangle \equiv |\kappa\rangle \}, |\tilde{\kappa}\rangle \) which are eigenstates of the sets \( \{ Z_\gamma \}, \{ X_\gamma \} \) and \( \{ Z_\gamma X_\gamma \} \) correspondingly. All the other MUBs of the type (12) are entangled, which makes the experimental realization of MUB tomography very “expensive” [28, 29].

3. N ququart MU-like bases

3.1. Single ququart

Let us consider a single particle with four energy levels (ququart). Although the dimension of the Hilbert space for a ququart is the same as for 2 qubits \( \mathcal{H}_4 = \mathcal{H}_2^\otimes 2 \), there is a crucial difference in the type of operations available in both systems: non-local transformations in two-qubit systems correspond to local \( SU(4) \) operations in ququart systems. Here we focus only on discrete transformations generated by unitary operators

\[
Z = \sum_{k=0}^3 i^k |k\rangle \langle k|, \quad X = \sum_{k=0}^3 |k+1\rangle \langle k|,
\]

(17)

where the algebraic operations are mod 4. The operators (17) are cyclic, \( Z^4 = X^4 = \mathbb{I} \), and satisfy the commutation relation,

\[
Z^a X^b = i^{ab} X^b Z^a, \quad a, b \in \mathbb{Z}_4.
\]

(18)
Table 1. Table of commuting monomials for one ququart. Each row corresponds to a commuting set. The first four rows corresponds to the sets \( Z^a X^l a \), with \( l \in \mathbb{Z}_4 \) and the last two rows correspond to the sets \( Z^m b X^b \), with \( m \in (2) = \{0, 2\} \).

| \( Z \)  | \( Z^2 \)  | \( Z^3 \) | \( l = 0 \)  | \( Z^a X^{l=0} a \) |
|-------|-------|-------|---------|----------------|
| \( ZX^2 \)  | \( Z^2 \)  | \( Z^3 X^2 \) | \( l = 2 \)  | \( Z^a X^{l=2} a \) |
| \( ZX \)  | \( Z^2 X^2 \)  | \( Z^3 X^3 \) | \( l = 1 \)  | \( Z^a X^{l=1} a \) |
| \( ZX^3 \)  | \( Z^2 X^2 \)  | \( Z^3 X \) | \( l = 3 \)  | \( Z^a X^{l=3} a \) |
| \( X \)  | \( X^2 \)  | \( X^3 \) | \( m = 0 \)  | \( Z^{m=0} b X^b \) |
| \( Z^2 X \)  | \( X^2 \)  | \( Z^2 X^3 \) | \( m = 2 \)  | \( Z^{m=2} b X^b \) |

The full set of 16 monomials \{\( Z^a X^b, a, b \in \mathbb{Z}_4 \)\} form an operational basis in \( \mathcal{H}_4 \), and can be grouped into 6 commuting sets, which in contrast to qubit systems are not disjoint. There are two types of commuting sets:

i) four sets of the form,
\[
\{ Z^a X^l a, a \in \mathbb{Z}_4 \}, \quad l = 0, 1, 2, 3;
\]

ii) two sets of the form,
\[
\{ Z^m b X^b, b \in \mathbb{Z}_4 \}, \quad m = 0, 2;
\]

see Table I. The sets (19) and (20) are not equivalent since \( m = 0, 2 \) are zero divisors, i.e. have no multiplicative inverse in \( \mathbb{Z}_4 \). One can observe that only 3 mutually disjoint sets can be chosen out of 6.

The bases corresponding to commuting sets \{\( Z^a X^l a \)\} are labelled as \(| \psi^l_k \rangle, k \in \mathbb{Z}_4 \rangle\}, being the computational basis \(|k \rangle = |\psi^l_{=0}\rangle, k \in \mathbb{Z}_4 \rangle\}; the eigenstates of the sets \{\( Z^m b X^b \)\} are \{| \tilde{\psi}^m_k \rangle, k \in \mathbb{Z}_4 \rangle\}. According to the general approach, the bases corresponding to disjoint sets are unbiased [26]. The overlap relations between eigenstates of sets that share operators are more involved, but possess certain symmetries, that can be used for constructing tomographic protocols. The explicit form of all 6 bases is given in [Appendix C].

The overlap relations between the elements of the bases can be represented in a compact form by introducing the bar map [32]: \( \overline{\cdot} : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \) which is defined as \( \overline{a} = a \mod 2, \forall a \in \mathbb{Z}_4 \).

(i) The eigenbases of \{\( Z^a X^l a \)\} are unbiased to the eigenstates of \{\( Z^m b X^b \)\} for every \( l = 0, \ldots, 3 \), and \( m = 0, 2 \),
\[
|\langle \psi^l_k | \tilde{\psi}^m_{k'} \rangle|^2 = \frac{1}{4}, \quad \forall k, k' \in \mathbb{Z}_4, \quad \forall l, m = 0, 2.
\]

(ii) The eigenstates of \{\( Z^a X^{l=a} \)\} and \{\( Z^a X^{l'=a} \)\} are mutually unbiased if and only if \( \overline{l} \neq \overline{l'} \); if \( \overline{l} = \overline{l'} \), the overlap is either 0 or 1/2,
\[
|\langle \psi^l_k | \psi^{l'}_{k'} \rangle|^2 = \delta_{kk'} \delta_{ll'} + (1 - \delta_{ll'}) \left( \frac{\delta_{kk'} \delta_{ll'}}{2} + \frac{1 - \delta_{ll'}}{4} \right), \quad k, k' \in \mathbb{Z}_4.
\]
(iii) The eigenbases of \( \{X^b\} \) and \( \{Z^{2b}X^b\} \) are not unbiased and satisfy the relation,
\[
|\langle \tilde{\psi}_k^0 | \tilde{\psi}_{k'}^2 \rangle|^2 = \frac{1}{2} \delta_{k,k'}, \quad k, k' \in \mathbb{Z}_4.
\]

Thus, there are only 3 MUBs among 6 bases, according to the number of disjoint sets.

3.2. \( N \) ququarts

In case of \( N \) ququarts it is convenient to label both states and operators acting in the Hilbert space \( \mathcal{H}^{4N} = \mathcal{H}_4^\otimes N \), with elements of the Galois Ring \( GR(4, N) \), that is considered as a linear space spanned by a basis \( \{\theta_i,i = 1, ..., N\} \),
\[
\alpha = a_1\theta_1 + a_2\theta_2 + ... + a_N\theta_N, \quad \alpha \in GR(4, N), \quad a_i \in \mathbb{Z}_4.
\]

The components in the above decomposition are related to the ring element through the trace operation \( T_4 : GR(4, N) \rightarrow \mathbb{Z}_4 \),
\[
a_i = T_4(\alpha \theta_i^*),
\]
being \( \{\theta_i^*\} \) the basis dual to \( \{\theta_i\} \), i.e. \( T_4(\theta_i\theta_j^*) = \delta_{ij} \), see Appendix B.1 and each element of the basis can be put in correspondence with a ququart. Elements of the ring are separated into units, having a multiplicative inverse, and zero divisors.

The computational basis in \( \mathcal{H}^{4N} \) is formed by the states
\[
|k_1\rangle \otimes ... \otimes |k_N\rangle = |\kappa\rangle, \quad k_i = T_4(\kappa \theta_i^*) \in \mathbb{Z}_4, \quad \kappa \in GR(4, N).
\]
The vectors (23) are eigenstates of the set of \( 4^N \) commuting operators,
\[
Z_{\gamma} = \sum_{\kappa \in GR(4, N)} i^{T_4(\gamma \kappa)}|\kappa\rangle \langle \kappa|, \quad Z_{\gamma} = Z^{g_1} \otimes ... \otimes Z^{g_N},
\]
where \( \gamma = g_1\theta_1^* + \cdots + g_N\theta_N^* \in GR(4, N) \), \( g_j = T_4(\gamma \theta_j) \in \mathbb{Z}_4 \), and a single ququart operator \( Z \) is defined by (17). From now on, all the sums run over \( GR(4, N) \), unless otherwise specified. The shift operators,
\[
X_\delta = X^{d_1} \otimes ... \otimes X^{d_N}, \quad X_\delta = \sum_{\kappa} |\kappa + \delta\rangle \langle \kappa|,
\]
where \( \delta = d_1\theta_1 + \cdots + d_N\theta_N \in GR(4, N) \), \( d_j = T_4(\delta \theta_j^*) \in \mathbb{Z}_4 \), and the operator \( X \) defined in (17), form a complementary set to (24),
\[
X_\delta Z_{\gamma} = i^{T_4(\delta \gamma)}Z_{\gamma}X_\delta.
\]
Observe that if \( \delta = \gamma \), the powers \( \{d_j, j = 1, ..., N\} \) in (25) are related to \( \{g_j, j = 1, ..., N\} \) in (24) according to \( d_j = \sum_{i=1}^N g_i T_4(\theta_i^* \theta_j^*) \). Thus, if \( \delta = \gamma \), \( d_j = g_j \) only in the case of decomposition on the self-dual basis. This, in particular, entails that the Fourier operator
\[
F_{4N} = \frac{1}{2N} \sum_{\alpha, \beta} i^{T_4(\alpha \beta)}|\alpha\rangle \langle \beta|, \quad X_\delta = F_{4N}^{-1}Z_\delta F_{4N},
\]
is factorized only in a self-dual basis. A self-dual basis exists in \( GR(4, N) \) only for odd \( N \) values.
Table 2. Table of the sets of commuting operators for two ququarts. Each entry corresponds to one set. In every row, the sets are not disjoint. First four rows correspond to the sets \( \{Z\gamma X_{\lambda \gamma}\} \), the last row corresponds to the sets \( \{Z_{\mu \delta} X_{\delta}\} \). The right column contains the operators shared between the sets of each row. \( \xi \) is a root of the irreducible polynomial on \( \mathbb{Z}_2 \): \( x^2 + x + 1 = 0 \).

| \(Z\gamma\) | \(Z\gamma X_{2\gamma}\) | \(Z\gamma X_{2\xi \gamma}\) | \(Z\gamma X_{2\xi^2 \gamma}\) | \(\lambda = 0\) | \(Z_2, Z_{2\xi}, Z_{2\xi^2}\) |
|---|---|---|---|---|---|
| \(Z\gamma X_{\gamma}\) | \(Z\gamma X_{3\gamma}\) | \(Z\gamma X_{(1+2\xi)\gamma}\) | \(Z\gamma X_{(1+2\xi^2)\gamma}\) | \(\lambda = 1\) | \(Z_2 X_2, Z_{2\xi} X_{2\xi}, Z_{2\xi^2} X_{2\xi^2}\) |
| \(Z\gamma X_{\xi \gamma}\) | \(Z\gamma X_{(\xi+2)\gamma}\) | \(Z\gamma X_{3\gamma}\) | \(Z\gamma X_{(\xi+2\xi^2)\gamma}\) | \(\lambda = \xi\) | \(Z_2 X_{2\xi}, Z_{2\xi} X_{2\gamma}, Z_{2\xi^2} X_{2\xi}\) |
| \(Z\gamma X_{\xi^2 \gamma}\) | \(Z\gamma X_{(\xi^2+2)\gamma}\) | \(Z\gamma X_{(\xi^2+2\xi^2)\gamma}\) | \(Z\gamma X_{3\xi^2 \gamma}\) | \(\lambda = \xi^2\) | \(Z_2 X_{2\xi^2}, Z_{2\xi^2} X_{2\xi}, Z_{2\xi} X_{2\xi}\) |
| \(X_{\delta}\) | \(Z_{2\delta} X_{\delta}\) | \(Z_{2\xi \delta} X_{\delta}\) | \(Z_{2\xi^2 \delta} X_{\delta}\) | \(\mu = 0\) | \(X_2, X_{2\xi}, X_{2\xi^2}\) |

The monomials \(Z\gamma X_{\delta}\) form an operational basis in \(\mathcal{H}_4\) and can be separated into two types of commuting sets,

\[
\{Z\gamma X_{\lambda \gamma} | \gamma, \lambda \in GR(4, N)\}, \tag{28}
\]
\[
\{Z_{\mu \delta} X_{\delta} | \delta \in GR(4, N), \mu \in (2)\}, \tag{29}
\]
labelled with the indices \(\lambda \in GR(4, N)\) and zero-divisors \(\mu\) contained in the principal ideal (2) of \(GR(4, N)\). Since there are \(2^N\) zero divisors in \(GR(4, N)\), there exist in total \(4^N + 2^N\) commuting sets.

Not all the sets (28)–(29) are disjoint. It immediately follows from (26) that,

i) the sets (28) are disjoint only if \(\bar{\lambda} \neq \bar{\lambda}\), where the bar map, \(GR(4, N) \rightarrow \mathbb{F}_{2N}\), is defined as \(\bar{\lambda} = \lambda \mod 2\); Thus, the sets \(\{Z\gamma X_{\lambda \gamma}\}\) and \(\{Z\gamma X_{\lambda' \gamma}\}\) with \(\bar{\lambda} = \bar{\lambda'}\) share the elements \(\{Z\gamma X_{\lambda \gamma} | \gamma \in (2)\}\);

ii) any two sets \(\{Z_{\mu \delta} X_{\delta}\}\) and \(\{Z_{\mu' \delta} X_{\delta}\}\) are not disjoint since \(\mu = 0\) if \(\mu \in (2)\), and share the elements \(\{X_{\delta} | \delta \in (2)\}\);

iii) the sets (28) and (29) are mutually disjoint.

Thus, there are \(2^N + 1\) groups each containing \(2^N\) non disjoint sets of commuting monomials, where the sets from different groups are mutually disjoint. In a sense, this is a generalization of the \(N\)-qubit case, where each group contains only one commuting set. In Table II the structure of commuting sets (28)–(29) for two ququarts is presented.

The above structure can be nicely represented in a non-Euclidian finite plane, where the axis \((\gamma, \delta)\) are labeled by the indexes of \(Z\gamma\) and \(X_{\delta}\) operators. In the geometrical representation the commuting monomials (28)–(29) are put in correspondence with two inequivalent bundles of rays \(\delta = \lambda \gamma, \lambda \in GR(4, N)\) and \(\lambda \delta = \gamma, \lambda \in (2)\), respectively. However, in contrast to prime power dimensions [22], the rays with the same \(\bar{\lambda}\) are intersected in sublines \(\delta = \bar{\lambda} \gamma\), when \(\lambda \in GR(4, N)\) and \(\gamma = 0\) if \(\lambda \in (2)\) [30, 31].

In practice, it is frequently convenient to use \(2\)-adic notation (see Appendix B.2) for labelling the operators \(Z\alpha (X\alpha)\): \(\alpha \in GR(4, N)\) is uniquely represented in terms of the Teichmuller set, \(\mathcal{T}_2 = \{0, \xi, \xi^2, \ldots, \xi^{2^N-2}, \xi^{2^N-1} = 1\}\), as \(\alpha = a + 2b\), for \(a, b \in \mathcal{T}_2\).
3.3. MU-like structure

The eigenstates of commuting monomials form peculiar bases, that resemble the properties of MUBs (8)-(11) in the N-qubit case. The eigenstates of the commuting sets \{Z_\gamma X_\lambda\}, \lambda \in GR(4, N) are obtained as unitary transformations of the computational basis (23),

\[ |\psi_\kappa^\lambda\rangle = V_\lambda |\kappa\rangle, \quad V_\lambda = \frac{1}{4^N} \sum_{\alpha,\alpha',\beta} c_{\beta,\lambda} i^{T_4(\beta(\alpha - \alpha'))} |\alpha\rangle \langle \alpha'|, \]

where \( c_{\gamma,\lambda} |\gamma\rangle = 1 \) satisfy the following functional equation,

\[ c_{\alpha + \gamma,\lambda} c_{\gamma,\lambda}^* = c_{\alpha,\lambda} i^{3T_4(\alpha\gamma\lambda)}. \]

According to the general method (see Appendix B.4), a solution of (31) can be found considering the indices of \( c_{\kappa,\lambda} \) as elements of \( T_3 = GR(8, N)/(4) \): \( c_{\gamma,\lambda} = \omega^{T_8(\lambda\gamma^2)} \), where \( \omega = (1 + i)/\sqrt{2} \) is the eighth root of unity and \( T_8 \) is the trace operation defined in \( GR(8, N) \) over \( Z_8 \), see Appendix B.1. From now on, we will use indistinctly \( |\kappa\rangle \) or \( |\psi_\kappa^0\rangle \) for the computational basis.

The eigenstates \{\( |\tilde{\kappa}\rangle, \kappa \in GR(4, N) \}\} of the set \{\( X_\delta \}\} are obtained as the Fourier transform of the computational basis (23),

\[ |\tilde{\kappa}\rangle = F_{4^N}^{-1} |\kappa\rangle. \]

The eigenstates \{\( |\tilde{\psi}_\kappa^\mu\rangle \}\} of the sets \{\( Z_{\mu\delta}X_\delta, \mu \in (2) \}\} are unitary equivalent to \{\( |\tilde{\kappa}\rangle = |\tilde{\psi}_\kappa^0\rangle \}\} according to,

\[ |\tilde{\psi}_\kappa^\mu\rangle = \tilde{V}_\mu |\tilde{\kappa}\rangle, \quad \tilde{V}_\lambda = F_{4^N}^{-1} V_\lambda F_{4^N}. \]

Thus, the spectral decomposition of monomials \( Z_\gamma X_\lambda \) and \( Z_{\mu\delta}X_\delta \) has the form,

\[ Z_\gamma X_\lambda = c_{\gamma,\lambda}^* \sum_\eta i^{T_4(\eta\gamma)} |\psi_\eta^\lambda\rangle \langle \psi_\eta^\lambda|, \quad Z_{\mu\delta}X_\delta = c_{\delta,\mu}^* \sum_\eta i^{T_4(\delta\eta)} |\tilde{\psi}_\eta^\mu\rangle \langle \tilde{\psi}_\eta^\mu|. \]

The overlap relations between the bases (30) and (33) can be easily found:

- The bases \{\( |\psi_\kappa^\lambda\rangle \}\} and \{\( |\psi_\eta^\lambda\rangle \}\}, where \( \lambda, \lambda' \in GR(4, N) \), are unbiased if \( \tilde{\lambda} \neq \tilde{\lambda}' \). If \( \tilde{\lambda} = \tilde{\lambda}' \), the squared overlap is either 0 or \( 2^{-N} \) depending on whether the bar maps of \( \kappa \) and \( \eta \) are the same or not,

\[ |\langle \psi_\eta^\lambda | \psi_\kappa^\lambda \rangle|^2 = \delta_{\kappa,\eta} \delta_{\lambda,\lambda'} + (1 - \delta_{\lambda,\lambda'}) \left( \frac{\delta_{\kappa,\eta} \delta_{\lambda,\lambda'}}{2^N} + \frac{1 - \delta_{\lambda,\lambda'}}{4^N} \right). \]

- The bases \{\( |\psi_\kappa^\lambda\rangle, \lambda \in GR(4, N) \}\} and \{\( |\tilde{\psi}_\kappa^\mu\rangle, \mu \in (2) \}\} are unbiased,

\[ |\langle \psi_\kappa^\lambda | \tilde{\psi}_\kappa^\mu \rangle|^2 = \frac{1}{4^N}. \]

- The bases \{\( |\tilde{\psi}_\kappa^\mu\rangle \}\} and \{\( |\tilde{\psi}_\eta^\mu\rangle \}\}, where \( \mu, \mu' \in (2) \), are not unbiased. The square overlap is either 0 or \( 2^{-N} \) depending on whether the bar maps of \( \kappa \) and \( \eta \) are the same or not,

\[ |\langle \tilde{\psi}_\eta^\mu | \tilde{\psi}_\kappa^\mu \rangle|^2 = \delta_{\kappa,\eta} \delta_{\mu,\mu'} + (1 - \delta_{\mu,\mu'}) \frac{\delta_{\kappa,\eta}}{2^N}. \]
There are $2^N + 1$ MUBs among $4^N + 2^N$ bases, with $2^N(2^N + 1)$ possibilities of choosing them.

In other words, the eigenstates of disjoint sets are mutually unbiased, while sets sharing elements have either orthogonal eigenstates or their overlap is a constant (up to a phase). As a consequence, the projectors on the bases (30) and (33) satisfy the following redundancy conditions

$$\sum_{\gamma \in (2)} |\psi_{\lambda+\gamma}^{\lambda+\delta}\rangle\langle\psi_{\lambda+\gamma}^{\lambda+\delta}| = \sum_{\gamma \in (2)} |\psi_{\lambda+\gamma}^{\lambda}\rangle\langle\psi_{\lambda+\gamma}^{\lambda}|,$$

$$\sum_{\gamma \in (2)} |\tilde{\psi}_{\lambda+\gamma}^{\mu}\rangle\langle\psi_{\lambda+\gamma}^{\mu}| = \sum_{\gamma \in (2)} |\tilde{\psi}_{\lambda+\gamma}^{0}\rangle\langle\tilde{\psi}_{\lambda+\gamma}^{0}|,$$

where $\delta, \mu \in (2)$, which also differentiate them from the mutually unbiased bases in the $N$-qubit case. In composite dimensions a construction similar to (35)-(37) is called weak MUBs [30], [31].

The factorization structure of the bases (30) and (33) is similar to the qubits case. There are four factorized bases of the type (30), and two factorized bases of the type (33). All the other bases are non-factorized. This can be seen straightforward to see for an odd number of qubits, when there exists a self-dual basis $\{\theta_i\}, i = 1, \ldots, N$ in $GF(4, N)$, so that the monomials (28) for $\lambda = l = 0, 1, 2, 3$ and (29) for $\mu = m = 0, 2$ are factorized according to,

$$Z_{\gamma} X_{\gamma} = Z^{g_1} X^{l_{g_1}} \otimes \cdots \otimes Z^{g_N} X^{l_{g_N}}, \quad \gamma = \sum g_i \theta_i, \quad g_i \in \mathbb{Z}_4,$$

$$Z_{m\delta} X_{\delta} = Z^{d_1} X^{d_1} \otimes \cdots \otimes Z^{d_N} X^{d_N}, \quad \delta = \sum d_i \theta_i, \quad d_i \in \mathbb{Z}_4.$$

The monomials in each of the sets above commute by particle, which guarantees that their eigenstates, $\{|\psi_{\lambda}^{\lambda}\rangle, \lambda = 0, 1, 2, 3\}$ and $\{|\tilde{\psi}_{\mu}^{\mu}\rangle, \mu = 0, 2\}$ are factorized.

In case of even number of ququarts the elements $GR(4, N)$, that label the factorized bases, depend on the choice of the basis in the ring. For instance, for two ququarts, the sets (28) with $\lambda = 0, 2, \xi + 3\xi^2, 3\xi + \xi^2$, where $\xi$ is a root of the irreducible polynomial $x^2 + x + 1$, are factorized when the particles are associated to the elements of the basis $\{\theta_1 = \xi, \theta_2 = \xi^2\}$. It is worth noting that in this case the rotation operator (30) is proportional to the $CNOT_4$ operator,

$$V_\lambda \sim CNOT_4^{l_1 + l_2}, \quad \lambda = l_1 \theta_1 + l_2 \theta_2,$$

where,

$$CNOT_4 = \sum_{k_1=0}^{3} |\tilde{k}_1\rangle_{11} \langle\tilde{k}_1| \otimes X_2^{k_1}.$$  

4. MU-like ququart tomography

4.1. General reconstruction formula

In the $N$-qubit case $2^N - 1$ independent measurements in each of $2^N + 1$ (unbiased) bases determine $2^{2N} - 1$ entries of the density matrix, in such a way that every measured
probability defines one matrix element according to the reconstruction equation (15). The mutually unbiased-like structure (35)-(37) allows to construct a tomographic scheme for $N$-ququarts. However, $4^N - 1$ measured probabilities in each of $4^N + 2^N$ bases lead to an informationally overcomplete reconstruction of $4^{2N} - 1$ independent parameters, that determine the $N$-ququart density matrix.

The density matrix $\rho$ is expanded in the monomial operational basis (28)-(29) as follows,

$$\rho = \sum_{\mu \in (2)} \sum_{\kappa \in GR(4,N)/\{0\}} \tilde{A}_\mu^\kappa Z_{\mu\kappa} X_\kappa + \sum_{\lambda \in GR(4,N)/\{0\}} \sum_{\kappa \in (2)\{0\}} A_\kappa^\lambda Z_{\kappa\lambda} X_\kappa + \frac{I}{4^N}$$

$$- \frac{2^N - 1}{2^N} \left( \sum_{\mu \in (2)} \sum_{\kappa \in (2)\{0\}} \tilde{A}_\mu^\kappa Z_{\mu\kappa} X_\kappa + \sum_{\lambda \in (2)\{0\}} \sum_{\kappa \in (2)\{0\}} A_\kappa^\lambda Z_{\kappa\lambda} X_\kappa \right). \quad (41)$$

The last term in (41) takes into account the repetitions present in the first two terms, i.e. every monomial appears only one time in the above expression. The expansion coefficients $A_\kappa^\lambda$ and $A_\kappa^\mu$ are easily computed by considering the spectral representations (34),

$$A_\kappa^\lambda = \frac{1}{4^N} Tr\{\rho (Z_{\kappa\lambda} X_\kappa)\} = \frac{1}{4^N} c_{\kappa,\lambda} \sum_\eta i^{3T_4(\kappa \eta)} \tilde{p}_\eta^\lambda, \quad (42)$$

$$\tilde{A}_\mu^\kappa = \frac{1}{4^N} Tr\{\rho (Z_{\mu\kappa} X_\kappa)\} = \frac{1}{4^N} c_{\mu,\kappa} \sum_\eta i^{3T_4(\kappa \eta)} \tilde{\tilde{p}}_\eta^\mu, \quad (43)$$

where $p_\kappa^\lambda$ and $\tilde{p}_\mu^\kappa$ are the measured probabilities $p_\kappa^\lambda = \langle \psi_\kappa^\lambda | \rho | \psi_\kappa^\lambda \rangle$, $\tilde{p}_\mu^\kappa = \langle \tilde{\psi}_\mu^\kappa | \rho | \tilde{\psi}_\mu^\kappa \rangle$.

Substituting (43) and (43) into (41) and taking into account the summation rules on the ring,

$$\sum_{\alpha \in GR(4,N)} i^{T_4(\alpha \kappa)} = 4^N \delta_{\kappa,0}, \quad \sum_{\alpha \in (2)} i^{T_4(\alpha \kappa)} = 2^N \delta_{\kappa,0},$$

we arrive at the following tomographic expression for the $N$-ququart density matrix,

$$\rho = \sum_{\mu \in (2)} \sum_{\kappa \in GR(4,N)} \tilde{C}_\mu^\kappa |\tilde{\psi}_\mu^\kappa\rangle \langle \tilde{\psi}_\mu^\kappa| + \sum_{\lambda,\kappa \in GR(4,N)} C_\kappa^\lambda |\psi_\kappa^\lambda\rangle \langle \psi_\kappa^\lambda| - \frac{I}{2^N}, \quad (44)$$

where,

$$\tilde{C}_\mu^\kappa = \tilde{p}_\mu^\kappa - \frac{2^N - 1}{4^N} \sum_{\gamma \in (2)} \tilde{p}_\mu^\gamma, \quad (45)$$

$$C_\kappa^\lambda = p_\kappa^\lambda - \frac{2^N - 1}{4^N} \sum_{\gamma \in (2)} p_\kappa^\gamma. \quad (46)$$

The reconstruction equation (44) is similar to the corresponding one for the qubit MUB tomography (15), except that the coefficients are linear combinations of the measured probabilities. The probabilities $p_\kappa^\lambda$ and $\tilde{p}_\mu^\kappa$ are not independent and apart from the normalization conditions similar to (16), they satisfy the following relations directly.
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followed from (38) - (39),
\[
\sum_{\gamma \in (2)} p_{K+\gamma}^{\lambda+\delta} = \sum_{\gamma \in (2)} p_{K+\gamma}^\lambda, \quad \sum_{\gamma \in (2)} \tilde{p}_{K+\gamma}^\mu = \sum_{\gamma \in (2)} \tilde{p}_{K+\gamma}^0,
\]
where \(\delta, \mu \in (2)\), explicitly reflecting the overcompleteness of the present measurement scheme.

4.2. Error estimation

Projectors on the elements of the bases (30) and (33) can be considered as output channels corresponding to specific setups, specified by \(\lambda \in GR(4, N)\) and \(\mu \in (2)\). Performing a finite number of measurements, \(M\), in each setup one counts the frequencies
\[
f_{\lambda, \kappa} = \frac{m_{\lambda, \kappa}}{M}, \quad \tilde{f}_{\mu, \kappa} = \frac{\tilde{m}_{\mu, \kappa}}{M},
\]
where \(m_{\lambda, \kappa}\) and \(\tilde{m}_{\mu, \kappa}\) are the number of times the state is projected into \(|\psi_{\lambda, \kappa}\rangle\) and \(|\tilde{\psi}_{\mu, \kappa}\rangle\), obeying the multinomial statistics \(\propto \Pi_{\kappa} (p_{\lambda, \kappa}^\lambda)^{m_{\lambda, \kappa}}\). (\(\propto \Pi_{\kappa} (\tilde{p}_{\mu, \kappa}^\mu)^{\tilde{m}_{\mu, \kappa}}\)).

The accuracy of the reconstruction scheme (44) can be measured by the statistical average of the Hilbert-Schmidt distance between the real \((\rho)\) and estimated \((\rho_{est})\) density matrices [37],
\[
\langle E^2 \rangle = \langle \text{Tr}(\rho - \rho_{est})^2 \rangle.
\]

In terms of independent probabilities (see Appendix D) the square error [48] is represented as,
\[
\langle E^2 \rangle = \Delta p^T Q \Delta p,
\]
where \(\Delta p = p - p_{est}\) is the vector of the difference between real and estimated (independent) probabilities and the matrix \(Q\) is expressed in terms of scalar products between the projectors appearing in (44). It is worth noting that for \(N\)-qubits the \(Q\) matrix has a trivial form,
\[
Q = \bigoplus_{\lambda \in \mathbb{F}_{22N}} Q^{(\lambda)} q_{\kappa, \eta}^{\lambda} = 1 + \delta_{\kappa, \eta}, \quad \kappa, \eta, \lambda \in \mathbb{F}_{22N}.
\]

This is not the case for \(N\)-ququarts, since probabilities labelled with the same \(\bar{\lambda}\) are related according to (47) (see the explicit expressions for the \(Q\)-matrix in Appendix D).

The minimum square error [48] is fixed by the Cramér-Rao bound [27],
\[
\langle E^2 \rangle \geq \text{Tr} \left( Q \mathcal{F}^{-1} \right),
\]
where \(\mathcal{F}\) is the Fisher matrix. The algebraic structure of the bases (30)-(33) allows to compute explicitly both \(Q\) and Fisher matrices, and thus, analytically estimate the minimum mean square error, see Appendix D.

It is instructive to compare the performance of tomographic protocols for i) \(N\)-ququarts (44), ii) 2\(N\) qubits (15), and iii) informationally complete symmetric POVMs (SIC-POVMs) [36] in the Hilbert space \(\mathcal{H}\) of dimension \(d = 4^N\).

• For a system of 2\(N\) qubits the lower bound on the MSE is [37],
\[
\langle E^2_{MUB} \rangle_{\text{min}} = 2^{2N} + 1 \left[ \sum_{\kappa \in \mathbb{F}_{22N}} \left( \sum_{\lambda \in \mathbb{F}_{22N}} (p_{\kappa}^{\lambda})^2 + (\tilde{p}_{\kappa}^{0})^2 \right) \right],
\]
Table 3. Minimum square error $\sqrt{\langle E^2 \rangle_{\min}}$ averaged over $10^3$ pure states, randomly generated using the Fubini-Study metric (second column) and separately over $10^3$ mixed states randomly generated using the Hilbert-Schmidt metric (third column) for different tomographic schemes.

| Tomographic scheme       | $\sqrt{\langle E^2 \rangle_{\min}}$ (pure) | $\sqrt{\langle E^2 \rangle_{\min}}$ (mixed) |
|--------------------------|--------------------------------------------|--------------------------------------------|
| single ququart MUB-like  | 1.72                                       | 1.84                                       |
| MUB 2 qubits             | 1.88                                       | 1.95                                       |
| $d=4$ SIC-POVM           | 4.24                                       | 4.44                                       |
| 2 ququarts MUB-like      | 3.16                                       | 3.54                                       |
| MUB 4 qubits             | 3.87                                       | 3.98                                       |
| $d=16$ SIC-POVM          | 16.43                                      | 16.49                                      |

where $p_{\lambda} = \langle \psi_{\lambda} | \rho | \psi_{\lambda} \rangle$ and $\tilde{p}_{\kappa} = \langle \tilde{\kappa} | \rho | \tilde{\kappa} \rangle$ are the probabilities of projecting on the unbiased bases (9) and (12).

- The lower bound of the MSE for SIC-POVMS in dimension $d = 4^N$ is, [39, 40]
  $$\langle E^2_{\text{SIC}} \rangle_{\min} = 4^{2N} + 4^N - 1 - \text{Tr}(\rho^2),$$  
  (51)

where $p_k$, $k = 1, ..., 4^N$, represents the probability of obtaining the outcome associated with $\Pi_k$, which are the first-rank projectors $\{\Pi_k, k = 1, ..., 4^N\}$, such that, $\text{Tr}(\Pi_k \Pi_l) = 1/(4^N + 1), k \neq l$.

In table 3 we show $\sqrt{\langle E^2 \rangle_{\min}}$ averaged over $10^3$ pure states, randomly generated using the Fubini-Study metric (second column) and separately over $10^3$ mixed states randomly generated using the Hilbert-Schmidt metric (third column) for i) one and two ququarts; ii) two and four qubits and iii) dim $\mathcal{H} = 4$ and 16. One can observe that the ququart tomography exhibits a better performance that MUB qubit and SIC POVM reconstructions in the Hilbert spaces of the same dimension. This can be attributed to the redundancy in the acquired data needed for the reconstruction protocol (44). In addition, it is worth noting that the amount of non-local gates required for the generation of $N$-ququart bases (30) and (33) is significantly lower than the one for the generation of 2$N$-qubit MUBs (12). For instance, for 2 ququarts one 14 $CNOT_1$ gates (40) are needed against 40 $CNOT_2$ gates required in the four-qubit case [29, 38].

5. Conclusions

The standard algebraic approach, based on the finite field structure, to construct a complete set of mutually unbiased bases can be applied only to systems of power-prime dimensions [21]. Such bases are employed in the optimal tomographic scheme, characterized by a non-redundant set of measurements. In the simplest multipartite case of $N$-ququarts such an approach is not feasible, since the underlying algebraic structure Galois rings does not allow to construct genuine set of MUBs. However,
the algebraic structure of the bases (30)-(33) is sufficiently simple so that an explicit reconstruction protocol can be found. In contrast to the qubit case, it requires redundant measurements. The extra information contained in these measurements leads to a reduction of the statistical error in comparison with the qubit MUB and SIC POVM schemes in a Hilbert space of the same dimension.

Appendix A. Finite fields

A set \( \mathcal{R} \) is a commutative ring if two commutative and associative binary operations: addition (+) and multiplication (\( \cdot \)) are defined. \((\mathcal{R}, +)\) forms a group, i.e. for any \( a \in \mathcal{R} \) there exist \(-a \in \mathcal{R}\) such that \( a + (-a) = 0 \). The set \( \mathbb{Z}_N = \{0, 1, \ldots, N - 1\} \) forms a ring, where all algebraic operations are mod \( N \).

A field \( \mathbb{F} \) is a commutative ring with division, i.e. for any \( a \in \mathbb{F} \) there exists \( a^{-1} \in \mathbb{F} \) so that \( a^{-1}a = aa^{-1} = I \) (excluding the zero element). Elements of a field form groups with respect to addition in \( \mathbb{F} \) and multiplication in \( \mathbb{F}^* = \mathbb{F} - \{0\} \). The set \( \mathbb{Z}_2 = \{0, 1\} \) is a field, while the set \( \mathbb{Z}_4 = \{0, 1, 2, 3\} \) does not have a finite field structure since 2 has not inverse in \( \mathbb{Z}_4 \).

A finite field is a field with a finite number of elements, its characteristic is the smallest integer \( p \), so that,

\[
p \cdot 1 = 1 + 1 + \ldots + 1 = 0,
\]

and it is always a prime number.

Any finite field contains a prime subfield \( \mathbb{Z}_p \) and has \( p^N \) elements, where \( N \) is a natural number. The finite field containing \( p^N \) elements is unique and is usually referred as Galois field, \( \mathbb{F}_{p^N} \). \( \mathbb{F}_{p^N} \) is an extension of degree \( N \) of \( \mathbb{Z}_p \), i.e. elements of \( \mathbb{F}_{p^N} \) can be obtained with \( \mathbb{Z}_p \) and all the roots of an irreducible (i.e. non-factorable in \( \mathbb{Z}_p \)) polynomial of degree \( N \) with coefficients in \( \mathbb{Z}_p \).

The multiplicative group of \( \mathbb{F}_{p^N} \) \( : \mathbb{F}_{p^N}^* = \mathbb{F}_{p^N} - \{0\} \) is cyclic \( \theta^{p^N} = \theta, \theta \in \mathbb{F}_{p^N} \). The generators of this group are called primitive elements of the field. A primitive element of \( \mathbb{F}_{p^N} \) is a root of an irreducible polynomial of degree \( N \) over \( \mathbb{Z}_p \). This polynomial is called a primitive polynomial \( h(x) \).

The map \( \sigma(\alpha) = \alpha^p \) on \( \mathbb{F}_{p^N} \) (over \( \mathbb{Z}_p \)) is a linear automorphism of \( \mathbb{F}_{p^N} \), called Frobenius automorphism. Frobenius automorphisms leave the prime subfield \( \mathbb{Z}_p \) invariant. The trace operation \( \text{tr} : \mathbb{F}_{p^N} \rightarrow \mathbb{Z}_p \) is defined as,

\[
\text{tr}(\alpha) = \alpha + \sigma(\alpha) + \ldots + \sigma^{N-1}(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \ldots + \alpha^{p^{N-1}}, \quad \alpha \in \mathbb{F}_{p^N}.
\]

Any \( \alpha \in \mathbb{F}_{p^N} \) can be written as,

\[
\alpha = a_1 \theta_1 + \cdots + a_N \theta_N,
\]

where \( a_i \in \mathbb{Z}_p \) and \( \{\theta_1, \ldots, \theta_N\} \) is a basis for \( \mathbb{F}_{p^N} \). Two bases \( \{\theta_1, \ldots, \theta_N\} \) and \( \{\theta_1^*, \ldots, \theta_N^*\} \) are dual if \( \text{tr}(\theta_i^* \theta_j) = \delta_{ij} \). A basis which is dual to itself is called self-dual basis, \( \text{tr} (\theta_i \theta_j) = \delta_{ij} \). For finite fields of characteristic 2, i.e. \( \mathbb{F}_{2^N} \), there always exists a self-dual basis.
Example: For $\mathbb{F}_{2^2}$, the primitive polynomial is $x^2 + x + 1 = 0$, has the roots $\{\xi, \xi^2\}$. The basis $\{\xi, \xi^2\}$ is self-dual,

$$
\text{tr}(\xi \xi) = 1, \quad \text{tr}(\xi \xi^2) = 0, \quad \text{(A.1)}
$$

$$
\text{tr}(\xi^2 \xi) = 0, \quad \text{tr}(\xi^2 \xi^2) = 1. \quad \text{(A.2)}
$$

Appendix B. Galois Rings

We focus on commutative rings with an identity element, which main difference from a field is the existence of elements without a multiplicative inverse. An element $a \neq 0 \in \mathcal{R}$ is called zero divisor if there exists $b \neq 0 \in \mathcal{R}$, such that $ab = 0 \in \mathcal{R}$. For instance, in $\mathbb{Z}_4$ the set of zero divisors is $\{2\}$, while for $\mathbb{Z}_8$ is $\{2, 4, 6\}$.

An non empty set $\mathcal{I} \subset \mathcal{R}$ is called an ideal of $\mathcal{R}$, if for any $a, b \in \mathcal{I}$ and $r \in \mathcal{R}$, the sum $a + b \in \mathcal{I}$ and $ra \in \mathcal{I}$. The ideal $Ra = \{ra : r \in \mathcal{R}\} \equiv (a)$, for some $a \in \mathcal{R}$ is called principal ideal (generated by $a$). The set $(2) = \{0, 2\} \subset \mathbb{Z}_4$ is the principal ideal, while in $\mathbb{Z}_8$ the principal ideals are the sets $(2) = \{0, 2, 4, 6\}$, and $(4) = \{0, 4\}$. The maximal ideal is a principal ideal, such that any other principal ideal is proper. The maximal ideal of $\mathbb{Z}_8$ is $(2)$. In this paper we will refer indistinctly as zero divisors to the set of zero divisors and zero, i.e. the maximal ideal $(2)$.

A Galois ring is a finite ring with an identity element, such that the set of its zero divisors added with zero form a maximal principal ideal ($p$), where $p$ is always a prime number. $\mathbb{Z}_{p^s}$ is a Galois ring with $p^s$ elements, for any $p$ prime and $s$ positive integer; its principal ideals are $(1), (p), (p^2), \ldots (p^{s-1})$, and the unique maximal ideal is $(p)$.

Any element of $\mathbb{Z}_{p^s}$ can be written in the form,

$$
c_1 + c_2p + c_3p^2 + \cdots + c_sp^{s-1},
$$

where $c_i \in \mathbb{Z}_p$. The characteristic of a Galois ring is the smallest integer such that,

$$
p^s \cdot 1 = 1 + 1 + \ldots + 1 = 0, \quad \text{p^s times}
$$

and it is always a power of a prime number.

The bar map is a homomorphism, $\overline{\cdot} : \mathbb{Z}_{p^s} \to \mathbb{Z}_p = \mathbb{F}_p$, given by,

$$
\overline{a} = a \mod p, \quad a \in \mathbb{Z}_{p^s},
$$

or equivalently,

$$
\overline{c_1 + c_2p + c_3p^2 + \cdots + c_sp^{s-1}} \to c_1.
$$

A Galois ring of characteristic $p^s$ and the cardinality $p^{sN}$ is denoted as $GR(p^s, N)$, $\mathbb{Z}_{p^s} \subset G(p^s, N)$. The bar map extended to the polynomial ring $\mathbb{Z}_{p^s}[x]$ over $\mathbb{Z}_{p^s}$ gives the polynomial ring $\mathbb{F}_p[x]$ over $\mathbb{F}_p$;

$$
a_1 + a_2x + \cdots + a_Nx^{N-1} \to \overline{a_1} + \overline{a_2}x + \cdots + \overline{a_N}x^{N-1}.
$$

The Galois ring $GR(p^s, N)$ is an extension of degree $N$ of $\mathbb{Z}_{p^s}$, i.e. elements of $GR(p^s, N)$ are obtained adding to $\mathbb{Z}_{p^s}$ all the roots of a basic monic irreducible (non-factorizable
in $\mathbb{Z}_{p^s}$, whose bar map is monic irreducible in $\mathbb{F}_{p^N}$ polynomial $f_s(x)$ of degree $N$ that divides $x^{p^N} - 1$ with coefficients in $\mathbb{Z}_{p^s}$. There are $p^N$ zero divisors in $GR(p^s, N)$.

Any element $\alpha \in GR(p^s, N)$ can be uniquely written in the additive representation:

$$\alpha = a_1 + a_2\xi + a_3\xi^2 + \cdots + a_N\xi^{N-1},$$

being $\xi$ a root of $f_s(x)$, and $a_i \in \mathbb{Z}_{p^s}$.

**Appendix B.1. The trace map and the self-dual basis**

The generalized trace map from $GR(p^s, N)$ to $\mathbb{Z}_{p^s}$, which is an additive operation given by

$$T_{p^s}(\alpha) = \sum_{i=0}^{N-1} \phi^i(\alpha) = \alpha + \phi(\alpha) + \cdots + \phi^{N-1}(\alpha),$$

where $\alpha = a_0 + a_1\xi + \cdots + a_{N-1}\xi^{N-1}$ and $\phi(\alpha)$ is the generalized Frobenius automorphism,

$$\phi^i(\alpha) = a_0 + a_1\xi^i + \cdots + a_{N-1}\xi^{i(N-1)},$$

where $\phi^0(\alpha) = \alpha$.

As well as in the case of the field $\mathbb{F}_{p^N}$, the ring $GR(p^s, N)$ can be considered as a linear space spanned by a basis $\{\theta_1, \theta_2, \ldots, \theta_N\}$, which is self-dual if $T_{p^s}(\theta_i\theta_j) = \delta_{ij}$. Self-dual bases do not always exist. For instance, there is no self-dual basis in $GR(4, 2)$.

**Appendix B.2. 2-adic Representation**

Let us consider the Galois ring $GR(2^s, N)$ and $\xi$ is a root of the basic primitive polynomial $f_s(x)$ of degree $N$ over $\mathbb{Z}_{2^s}$. The subset (the Teichmuller set) $\mathcal{T}_s = \{0, \xi, \xi^2, \ldots, \xi^{2N-1} = 1\} \subset GR(2^s, N)$ allows to represent any element $\alpha \in GR(2^s, N)$ as,

$$\alpha = a_1 + 2a_2 + 4a_3 + \cdots + 2^{s-1}a_s, \quad a_i \in \mathcal{T}_s,$$

which is 2-adic representation of the ring $GR(2^s, N)$, and $\overline{a_i} \in \mathbb{F}_{2^N}$, for all $a_i \in \mathcal{T}_s$. In this representation, invertible elements of the ring (that are called units) are such that $a_1 \neq 0$ and the zero divisors correspond to $a_1 = 0$. The application of the bar map becomes trivial in this representation: $\overline{\alpha} = \overline{a_1}$.

**Appendix B.3. Hensel Lift**

A basic monic irreducible polynomial $f_s(x)$ on $\mathbb{Z}_{2^s}[x]$ can be considered as the Hensel Lift of a monic irreducible polynomial $f_1(x) = \overline{f_s(x)}$ on $\mathbb{Z}_2[x]$. Such a lifting is unique. The lifting can be divided in steps: there is a unique Hensel lift from $f_2(x)(x^{2N} - 1)$ in $\mathbb{Z}_{2^s}[x]$ to $f_{s+1}(x)(x^{2^N} - 1)$ in $\mathbb{Z}_{2^{s+1}}[x]$, so that $f_{s+1}(x) \bmod 2^s = f_s(x)$. The roots of $f_s$ and $f_{s+1}(x)$ define $GR(2^s, N)$ and $GR(2^{s+1}, N)$ respectively \[5\]. The corresponding trace operations are related as,

$$T_{2^{s+1}}(\alpha) \bmod 2^s = T_{2^s}(\alpha \bmod 2^s),$$

(B.2)
where \( \alpha \in GR(2^{s+1}, N) \) and \( \alpha \mod 2^s \in GR(2^s, N) \).

For instance, i) the Hensel lift \( Z_2[x] \rightarrow Z_{2^s}[x] \) of the irreducible polynomial \( x^2 + x + 1 \) that divides \( x^4 - 1 \) is \( x^2 + x + 1 \), and, ii) the Hensel lift \( Z_{2^s}[x] \rightarrow Z_{2^{s+1}}[x] \) of the irreducible polynomial in \( x^2 + x + 1 \) that divides \( x^4 - 1 \) is \( x^2 + x + 1 \).

**Appendix B.4. Solution of equation**

Consider the following equation,

\[
c_{\alpha+\gamma,\lambda} + c_{\gamma,\lambda} = c_{\alpha,\lambda} \omega_s^{-T_{2^s}(\alpha\gamma\lambda)}, \quad |c_{\gamma,\lambda}| = 1, \quad c_{0,\lambda} = 1,
\]

where \( \alpha, \gamma, \lambda \in GR(2^s, N) \) and \( \omega_s \) is \( 2^s \)-th root of unity. The functional Eq. (B.3) admits a simple solution if the subindices of \( c_{\gamma,\lambda} \) are considered as elements of the set,

\[
\mathcal{T}_{s+1} = GR(2^{s+1}, N) / (2^s),
\]

which has the same cardinality as \( GR(2^s, N) \). Therefore, the elements of \( \mathcal{T}_{s+1} \) are in one-to-one correspondence with \( GR(2^s, N) \), and,

\[
\mathcal{T}_{s+1} \mod 2^s = GR(2^s, N).
\]

Then, Eq. (B.3) can be rewritten according to (B.2) as follows,

\[
c_{\alpha+\gamma,\lambda} = c_{\alpha,\lambda} c_{\gamma,\lambda} \omega_s^{-T_{2^s}(\alpha\gamma\lambda)}, \quad \alpha, \gamma, \lambda \in \mathcal{T}_{s+1} \subset GR(2^{s+1}, N),
\]

such that \( \alpha, \gamma, \lambda \mod 2^s \in GR(2^s, N) \). Imposing the condition \( c_{0,\lambda} = c_{2^s\gamma,\lambda} = 1 \), we immediately arrive at the following solution,

\[
c_{\gamma,\lambda} = \omega_s^{-T_{2^s+1}(\lambda\gamma^2)}.
\]

For instance, in the qubit case, corresponding to \( s = 1 \), \( c_{\gamma,\lambda} = i^{3T_2(\lambda\gamma^2)} \) is obtained, where \( \gamma, \lambda \in \mathcal{T}_2 = GR(4, N) / (2) \subset GR(4, N) \) and \( \mathcal{T}_2 \mod 2 = GR(2, N) = \mathbb{F}_{2N} \).

**Appendix B.5. GR(4,2)**

The basic monic irreducible polynomial \( h(x) = x^2 + x + 1 \) in \( \mathbb{Z}_4[x] \) uniquely defines the elements of \( GR(4, 2) \) as the residue class \( \mathbb{Z}_4[x] / (h(x)) \), where \( \xi \) is a root of \( h(x) \). In the 2-adic representation, the \( 4^2 \) elements are split into two sets,

(i) Units: \( \alpha = 1 + 2b, \xi + 2b, \xi^2 + 2b \)

(ii) Zero divisors: \( \alpha = 2b \)

where \( b \in \mathcal{T}_2 = \{0, 1 = \xi^3, \xi, \xi^2\} \). A suitable representation of \( GR(4, 2) \) as elements of \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) is \( \kappa = k_0 \xi + k_1 \xi^2 \), where \( k_0, k_1 \in GR(4, N) \) and \( \{\xi, \xi^2\} \) form a (non self-dual) basis, i.e.,

\[
T_4(\xi^i \xi^j) = \delta_{i,j} + 2,
\]

where,

\[
T_4(\xi^i \xi^j \xi^k) = \begin{cases} 
2 & \text{if } i = j = k; \\
3 & \text{otherwise}.
\end{cases}
\]

In table B3 the expansion in the basis \( \{\xi, \xi^2\} \) and 2-adic representation, for the irreducible polynomial \( \xi^2 + \xi + 1 = 0 \) are present.
Table B1. Expansion of elements of $GR(4, 2)$ in the basis $\{\xi, \xi^2\}$ and the corresponding 2-adic representation.

| 2-adic | Basis expansion | 2-adic | Basis expansion |
|--------|-----------------|--------|-----------------|
| 0      | 0               | $\xi$  | $\xi$           |
| 2      | $2\xi + 2\xi^2$| $\xi + 2$| $3\xi + 2\xi^2$|
| $2\xi$ | $2\xi$          | $3\xi$ | $3\xi$          |
| $2\xi^2$ | $2\xi^2$    | $\xi + 2\xi^2$| $\xi + 2\xi^2$|
| 1      | $3\xi + 3\xi^2$| $\xi^2$ | $\xi^2$         |
| 3      | $\xi + \xi^2$  | $\xi^2 + 2$| $2\xi + 3\xi^2$|
| $1 + 2\xi$ | $\xi + 3\xi^2$| $\xi^2 + 2\xi$| $\xi^2 + 2\xi$|
| $1 + 2\xi^2$ | $\xi^2 + 3\xi$| $3\xi^2$ | $3\xi^2$       |

Appendix B.6. $GR(4, 3)$

The ring $GR(4, 3)$ is uniquely defined by a root $\xi$ of the monic basic irreducible polynomial $h(x) = x^3 + 3x^2 + 2x + 3$ over $\mathbb{Z}_4[x]$. In the 2-adic representation the elements of $GR(4, 3)$ have the form $a_0 + 2a_1$, where,

$$a_0, a_1 \in \mathcal{T}_2 = \{0, 1 = \xi^7, \xi, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6\}.$$

There is a self-dual basis in $GR(4, 3)$,

$$\{\theta_1 = \xi + 2\xi^2, \theta_2 = \xi^2 + 2\xi^4, \theta_3 = \xi^4 + 2\xi\}.$$

Appendix C. MU-like bases for one ququart

Here, six MU-like bases for a single ququart are explicitly constructed.

i) Eigenstates of the set $\{Z^k, k = 0, 1, 2, 3\}$ form the computational basis $\{|\psi_k^0\rangle = |k\rangle, k = 0, 1, 2, 3\}$.

ii) The eigenstates of the set $\{Z^kX^k, k = 0, 1, 2, 3\}$ are

$$|\psi_0^1\rangle = \frac{1}{2} \left( \frac{1+i}{\sqrt{2}} |0\rangle + \frac{1-i}{\sqrt{2}} |2\rangle + |3\rangle \right),$$

$$|\psi_1^1\rangle = \frac{1}{2} \left( |0\rangle + \frac{1+i}{\sqrt{2}} |1\rangle + |2\rangle - \frac{1-i}{\sqrt{2}} |3\rangle \right),$$

$$|\psi_2^1\rangle = \frac{1}{2} \left( \frac{1+i}{\sqrt{2}} |0\rangle + |1\rangle + \frac{1-i}{\sqrt{2}} |2\rangle + |3\rangle \right),$$

$$|\psi_3^1\rangle = \frac{1}{2} \left( |0\rangle - \frac{1+i}{\sqrt{2}} |1\rangle + |2\rangle + \frac{1-i}{\sqrt{2}} |3\rangle \right);$$

iii) The eigenstates of the set $\{Z^kX^{2k}, k = 0, 1, 2, 3\}$ are

$$|\psi_0^2\rangle = \frac{1}{2} ((1+i) |0\rangle + (1-i) |2\rangle), \quad |\psi_1^2\rangle = \frac{1}{2} ((1+i) |1\rangle + (1-i) |3\rangle),$$

$$|\psi_2^2\rangle = \frac{1}{2} ((1-i) |0\rangle + (1+i) |2\rangle), \quad |\psi_3^2\rangle = \frac{1}{2} ((1-i) |1\rangle + (1+i) |3\rangle);$$
iv) The eigenstates of the set \( \{ Z^k X^k, k = 0, 1, 2, 3 \} \) are
\[
\begin{align*}
| \psi_0^3 \rangle &= \frac{1}{2} \left( -\frac{1-i}{\sqrt{2}} |0 \rangle + |1 \rangle + \frac{1-i}{\sqrt{2}} |2 \rangle + |3 \rangle \right), \\
| \psi_1^3 \rangle &= \frac{1}{2} \left( |0 \rangle - \frac{1-i}{\sqrt{2}} |1 \rangle + |2 \rangle + \frac{1-i}{\sqrt{2}} |3 \rangle \right), \\
| \psi_2^3 \rangle &= \frac{1}{2} \left( \frac{1-i}{\sqrt{2}} |0 \rangle + |1 \rangle - \frac{1-i}{\sqrt{2}} |2 \rangle + |3 \rangle \right), \\
| \psi_3^3 \rangle &= \frac{1}{2} \left( |0 \rangle + \frac{1-i}{\sqrt{2}} |1 \rangle + |2 \rangle - \frac{1-i}{\sqrt{2}} |3 \rangle \right);
\end{align*}
\]

v) Eigenstates of the set \( \{ Z^k X^k, k = 0, 1, 2, 3 \} \) are
\[
\begin{align*}
| \tilde{\psi}_0^0 \rangle &= \frac{1}{2} (|0 \rangle + |1 \rangle + |2 \rangle + |3 \rangle), \\
| \tilde{\psi}_1^0 \rangle &= \frac{1}{2} (|0 \rangle + i |1 \rangle - |2 \rangle - i |3 \rangle), \\
| \tilde{\psi}_2^0 \rangle &= \frac{1}{2} (|0 \rangle - |1 \rangle + |2 \rangle - |3 \rangle), \\
| \tilde{\psi}_3^0 \rangle &= \frac{1}{2} (|0 \rangle - i |1 \rangle - |2 \rangle + i |3 \rangle);
\end{align*}
\]

vi) The eigenstates of the set \( \{ X^k, k = 0, 1, 2, 3 \} \) are
\[
\begin{align*}
| \tilde{\psi}_0^2 \rangle &= \frac{1}{2} (|0 \rangle + i |1 \rangle + |2 \rangle + i |3 \rangle), \\
| \tilde{\psi}_1^2 \rangle &= \frac{1}{2} (|0 \rangle - |1 \rangle - |2 \rangle + |3 \rangle), \\
| \tilde{\psi}_2^2 \rangle &= \frac{1}{2} (|0 \rangle - i |1 \rangle + |2 \rangle - i |3 \rangle), \\
| \tilde{\psi}_3^2 \rangle &= \frac{1}{2} (|0 \rangle + |1 \rangle - |2 \rangle - |3 \rangle).
\end{align*}
\]

The bases i) and iii), ii) and iv), v) and vi) are not mutually unbiased by pairs.

**Appendix D. MSE for N ququarts**

In this Appendix, an analytical expression for the Cramer-Rao lower bound [27],
\[
\langle \mathcal{E}^2 \rangle_{\min} = \text{Tr}(Q \mathcal{F}^{-1}),
\]
(D.1)
of the mean square error (MSE) proper to the reconstruction protocol (44) is obtained.

In order to find the \( Q \)-matrix Eq. (44) is substituted into (48) and only the independent probabilities are kept by:

i) removing the \( 2^N + 1 \) dependent probabilities \( \{ p_0^\lambda, \forall \lambda \in T_2; \tilde{p}_0^0 \} \) as a consequence of the normalization conditions,
\[
\Delta p^\lambda_0 = - \sum_{\kappa \in GR(4,N)/\{0\}} \Delta p^\lambda_\kappa, \quad \Delta \tilde{p}_0^0 = - \sum_{\kappa \in GR(4,N)/\{0\}} \Delta \tilde{p}_\kappa^0;
\]
(D.2)
ii) removing the $4^N(2^N - 1)$ probabilities $\{p^\lambda_\kappa, \forall \lambda, \kappa \in T_2 \land \forall \delta \in (2), \delta \neq 0\}$ due to relation (47)

$$
\Delta p^{\lambda + \delta}_\kappa = - \sum_{\gamma \in (2)/\{0\}} \Delta p^{\lambda + \delta}_{\kappa + \gamma} + \sum_{\gamma \in (2)} \Delta p^\lambda_{\kappa + \gamma}, \ \ \delta \in (2), \delta \neq 0; \tag{D.3}
$$

iii) removing the $2^N(2^N - 1)$ probabilities $\{\tilde{p}_\mu^\kappa, \forall \kappa \in T_2 \land \forall \mu \in (2), \mu \neq 0\}$ due to (47),

$$
\Delta \tilde{p}_\mu^\kappa = - \sum_{\gamma \in (2)/\{0\}} \Delta \tilde{p}^\mu_{\kappa + \gamma} + \sum_{\gamma \in (2)} \Delta \tilde{p}^0_{\kappa + \gamma}, \ \ \mu \in (2), \mu \neq 0. \tag{D.4}
$$

Then, the MSE takes the form,

$$
\langle \mathcal{E}^2 \rangle = \sum_{\lambda \in T_2} \sum_{\kappa, \eta \in GR(4,N)/\{0\}} (Q_\lambda)^{\overline{\lambda},\lambda}_{\overline{\kappa},\kappa,\eta,\eta} \langle \Delta p^\lambda_\kappa \Delta p^{\overline{\lambda}}_\eta \rangle \\
+ \sum_{\lambda, \eta \in T_2} \sum_{\kappa \in GR(4,N)/\{0\}} \sum_{\delta, \gamma \in (2)/\{0\}} (Q_\lambda)^{\overline{\lambda},\lambda + \delta,\lambda + \delta}_{\overline{\kappa},\kappa,\eta,\eta + \gamma} \langle \Delta p^\lambda_\kappa \Delta p^{\overline{\lambda} + \delta}_\eta \rangle \\
+ \sum_{\lambda, \eta, \gamma \in T_2} \sum_{\delta, \delta' \in (2)/\{0\}} \sum_{\gamma', \gamma' \in (2)/\{0\}} (Q_\lambda)^{\overline{\lambda},\lambda + \delta,\lambda + \delta'}_{\overline{\kappa},\kappa,\eta,\eta + \gamma} \langle \Delta p^{\overline{\lambda} + \delta}_\kappa \Delta p^{\overline{\lambda} + \delta'}_\eta \rangle \\
+ \sum_{\kappa, \eta \in GR(4,N)/\{0\}} (\tilde{Q}_0)^{0,0,0}_{\kappa,\eta,\eta} \langle \Delta \tilde{p}^0_\kappa \Delta \tilde{p}^0_\eta \rangle \\
+ \sum_{\kappa \in GR(4,N)/\{0\}} \sum_{\eta \in T_2/\{0\}} \sum_{\gamma \in (2)/\{0\}} (\tilde{Q}_0)^{0,\mu,0}_{\kappa,\eta,\eta + \gamma} \langle \Delta \tilde{p}^0_\kappa \Delta \tilde{p}^\mu_\eta \rangle \\
+ \sum_{\mu, \nu \in (2)/\{0\}} \sum_{\kappa, \eta \in T_2} \sum_{\gamma, \gamma' \in (2)/\{0\}} (\tilde{Q}_0)^{\mu,\nu,0}_{\kappa,\eta,\eta + \gamma} \langle \Delta \tilde{p}^\mu_\kappa \Delta \tilde{p}^\nu_\eta \rangle, \tag{D.5}
$$

and the matrix $Q$ has a block diagonal structure,

$$
Q = \bigoplus_{\lambda} Q_{\lambda} \bigoplus \tilde{Q}_0.
$$

The non-zero matrix elements of the blocks labelled by $\overline{\lambda} \in T_2$ are,

$$
(Q_\lambda)^{\overline{\lambda},\overline{\lambda}}_{\kappa + \gamma, \eta + \gamma'} = \frac{1}{2^N} [(4^N - 2^N + 1)(\delta_{\kappa,\eta} + 1) \\
+ (2^N - 1) [\delta_{\kappa,\eta} (1 - \delta_{\gamma,\gamma'}) - 2\delta_{\kappa,\eta} (\delta_{\kappa,0} + 1)]] \tag{D.6}
$$

$$
(Q_\lambda)^{\lambda + \delta,\lambda + \delta}_{\kappa,\kappa,\eta,\eta} = (1 - \delta_{\kappa,0}) (\delta_{\kappa,\eta} + \delta_{\kappa,\eta}), \tag{D.7}
$$

$$
(Q_\lambda)^{\lambda,\lambda + \delta}_{\kappa,\kappa,\eta,\eta} = -2 (1 - \delta_{\kappa,0} - \delta_{\eta,0}), \tag{D.8}
$$

where,

$$
\delta_{\kappa,\eta} = \sum_{\gamma \in (2)} \delta_{\kappa,\eta + \gamma}.
$$

The non-zero elements of the block $\tilde{Q}_0$ have the same structure as [D.6]-[D.8], with $\overline{\kappa} = \overline{\lambda} = 0$.

The $\left[(4^N)^2 - 1\right] \times \left[(4^N)^2 - 1\right]$ dimensional Fisher matrix is formed by $2^N + 1$ diagonal blocks of dimension $\left[(4^N - 1) + 2^N (2^N - 1)^2\right] \times \left[(4^N - 1) + 2^N (2^N - 1)^2\right]$,

$$
\mathcal{F} = \bigoplus_{\lambda} \mathcal{F}_{\lambda} \bigoplus \tilde{\mathcal{F}}_0. \tag{D.9}
$$
where each value of $\lambda$ defines one diagonal block whose elements are,

$$
(F_{\lambda})_{\kappa,\eta}^{\lambda,\sigma} = \frac{1}{M} \left( \frac{\partial \log \mathcal{L}(n_{\lambda} | p_{\lambda})}{\partial p_{\kappa}^{\lambda}} \frac{\partial \log \mathcal{L}(n_{\lambda} | p_{\lambda})}{\partial p_{\eta}^{\sigma}} \right),
$$

$$
(F_{0})_{\kappa,\eta}^{\mu,\nu} = \frac{1}{M} \left( \frac{\partial \log \mathcal{L}(n_{0} | \tilde{p}_{0})}{\partial p_{\kappa}^{\mu}} \frac{\partial \log \mathcal{L}(n_{0} | \tilde{p}_{0})}{\partial p_{\eta}^{\nu}} \right),
$$

also,

$$
\mathcal{L}(n_{\lambda} | p_{\lambda}) \propto \Pi_{\kappa}(p_{\kappa}^{\lambda})^{-\lambda}, \quad \mathcal{L}(n_{0} | \tilde{p}_{0}) \propto \Pi_{\kappa}(p_{\kappa}^{\mu})^{-\mu},
$$

is the likelihood. The non-zero matrix elements for any of the $2^N$ blocks labelled by $\lambda \in \mathcal{T}_2$ are,

$$
(F_{\lambda})_{\kappa,\eta}^{\lambda,\lambda} = \frac{\delta_{\kappa,\eta}}{p_{\kappa}^{\lambda}} + \frac{1}{p_{0}^{\lambda}} + (1 - \delta_{\kappa,0})(1 - \delta_{\eta,0}) \sum_{\delta \in \{2\}} \left( \frac{\delta_{\delta,0}}{p_{\kappa}^{\lambda+\delta}} + \frac{1}{p_{0}^{\lambda+\delta}} \right), \quad \kappa, \eta \neq 0,
$$

$$
(F_{\lambda})_{\kappa+\gamma,\theta+\gamma'}^{\lambda+\delta,\lambda+\delta} = \frac{\delta_{\gamma,\gamma'}}{p_{\kappa+\gamma}^{\lambda+\delta}} + \frac{1}{p_{0}^{\lambda+\delta}}, \quad (F_{\lambda})_{\kappa+\gamma,\theta+\gamma'}^{\lambda,\lambda+\delta} = -(1 - \delta_{\kappa,0}) \frac{1}{p_{\kappa}^{\lambda+\delta}}, \quad \delta, \gamma, \gamma' \neq 0;
$$

the non-zero elements of the single matrix block labelled by $\bar{\mu} = 0$ are,

$$
(F_{0})_{\kappa,\eta}^{0,0} = \frac{\delta_{\kappa,\eta}}{p_{0}^{0}} + \frac{1}{p_{0}^{0}} + (1 - \delta_{\kappa,0})(1 - \delta_{\eta,0}) \sum_{\nu \in \{2\}} \left( \frac{\delta_{\kappa,\nu}}{p_{\kappa}^{0}} + \frac{1}{p_{\nu}^{0}} \right), \quad \kappa, \eta \neq 0,
$$

$$
(F_{0})_{\kappa+\gamma,\theta+\gamma'}^{\mu,\mu} = \frac{\delta_{\gamma,\gamma'}}{p_{\kappa+\gamma}^{0}} + \frac{1}{p_{\eta}^{0}}, \quad (F_{0})_{\kappa+\gamma,\theta+\gamma'}^{\mu,0} = -(1 - \delta_{\kappa,0}) \frac{1}{p_{\kappa}^{0}}, \mu, \gamma, \gamma' \neq 0.
$$

Therefore, the Crâmer-Rao lower bound can be written as,

$$
\langle \mathcal{E}^2 \rangle_{\min} = \text{Tr}(\hat{Q}_0 \hat{F}_0^{-1}) + \sum_{\lambda \in \mathcal{T}_2} \text{Tr}(Q_\lambda \mathcal{F}_\lambda^{-1}).
$$

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