Temporal vortex morphology and time–varying angular momentum

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It is known that the fundamental mode of rotation in quantum fluids is represented by a quantum vortex. Its quantized value represents the orbital angular momentum (OAM) per particle, which is reduced in case of offset vortices. Here, we study analytically the dynamics of a displaced quantum vortex in a number of 2D systems, including infinite or harmonic confining potentials and a two–component condensate. The vortex core demonstrates peculiar motion in all such systems, however, the OAM content may vary in time or not, depending on the underlying physics. For the off–axis vortex in the radial potentials, the OAM remains invariant, despite the core moves on circular or elliptic orbits and even when secondary vortex–antivortex pairs proliferate due to self-interference of the wavepacket being reshaped. However, including harmonic time perturbations with a gain–dissipation term or adding a spatial anisotropy, confers time–varying OAM to the morphology reshaping. Considering a binary condensate, the Rabi dynamics can imply time–dependent angular momentum and core orbital motion in each of the components, as a natural result of their coherent coupling. The salient result across all the different cases is that a weird motion of the vortex core, and also its dynamical morphology, are distinct from and do not necessarily imply a time-varying OAM per particle, whereas the latter always imply some offset core and its morphology reshaping.

I. INTRODUCTION

Topology is indispensable to different areas of physics, in particular to condensed matter physics [1]. This involves invariant under deformation from ground state to a new configuration for which any path to the ground state is energetically too costly. As such, topological invariant powered by Homotopy groups provide subtle tool to detect holes in a given space; examples are topological defects in system with continuous broken symmetry. Vortex, a typical defect in system with global U(1) broken symmetry, is characterized by a null density region and an integer (topological) charge associated with the singularity in the gradient of the order parameter phase at the core position; indeed, the topological charge (TC) defines the intrinsic orbital angular momentum (OAM) available in the vortex state. Quantum vortices are common in many phases including superconductors [2], superfluids [3], atomic Bose Einstein condensates [4], and polaritons [5], among others.

Vortex beam (VB), confined vortex state in the form of beam with spiral wavefront, is exotic in a sense that it can carry angular momentum [6]; indeed one can distinguish two kinds of VBs, one with non–uniform phase–varying, which is called anisotropic VB, and the other with uniform phase variation, which is referred to as isotropic VB. In both cases, the fields, being associated to a displaced vortex, carry reduced–fixed amount of angular momentum and have modified morphology distributions. For static VB (not varying in time), either isotropic or anisotropic, it is known that the vortex helicity can be described based on the combination of some elementary vortices with integer topological charges which determines the overall morphology distribution and associated angular momentum across the beam [7].

Vortex beams can be dynamic, i.e., the angular content can be time dependent. First report regarding time–varying angular momentum was given in Ref. [8], where two pulses, one delayed with respect to the other, are imprinted on microcavity polariton (coupled photonic and excitonic fields) fluids. This leads to temporally and spatially structured components (photon and exciton), that is, the off–center core in one component (photon) is dynamically matched by an off-axis core in the counterpart field (exciton). The coherent transfer between the two components lets the two cores moving on the same orbit (with a reciprocal delay), resulting in the oscillating linear and angular momentums in the emitted light. Such oscillations happen in the linear regime of the dynamics. Quite recently a similar scheme, with two retarded pulses but now partially overlapping in time, has been used in the nonlinear process of high harmonic generation, to result in the vortex beam with continuously time–varying angular momentum [9].

Here, we study analytically some simple physical systems that can sustain a dynamical vortex, in the sense that the vortex morphology is temporally being stretched or folded up, while the angular momentum content may vary with time. Indeed, one may expect a time–varying morphology also results in a time–varying angular momentum. We rather show that the time–varying morphology can be associated to but does not necessarily imply time–varying angular momentum. To this end, we consider an off–axis vortex in some elementary physical examples: infinite (circular) quantum well potential, harmonic potential perturbed by spatially anharmonic component and time–harmonic term, and two coupled condensates. We consider in Sec. [1] the case of displaced
vortex in quantum well with hard boundary, for which we introduce the mechanism of pair excitation induced exclusively by quantum effects associated with interference. While the vortex morphology is peculiarly varied in time, the angular momentum content remains steady. Section [III] accounts for the off-center vortex in the subtle example of harmonic potential, where we extend the study to consider the effect of the spatial anisotropic anharmonicity (deviation from harmonic potential) and time–harmonic perturbation (oscillatory in time) with a gain/dissipative term. We show that while the harmonic potential alone sustains an angular momentum which is constant, on the other hand the time harmonic perturbation induces periodically varying angular momentum. Interestingly, anharmonicity modifies greatly the morphology of the vortex alongside with the vortex core motion, as removing the degeneracy of the excited states of harmonic potential mediates oscillatory time–varying angular momentum. The other possibility to observe the time–varying angular momentum is through binary condensates, namely, Rabi coupled fields, which is studied in Sec. [IV] Such behavior was shown recently for the case of large imbalance between masses of coupled fields, corresponding to the microcavity polaritons. Finally, Sec. [V] provides conclusion remarks. It is worth noting that all examples considered here are in the linear regime, where the self–interaction between particles are negligible.

II. QUANTUM WELL

It might be one of the simplest case to consider an off–center vortex in an infinite quantum well. We consider the Hamiltonian of a circular quantum well of radius \( R \):

\[
H_{qw} = -\frac{\hbar^2}{2m} \nabla^2 + V(r),
\]

where

\[
V(r, \varphi) = \begin{cases} 
0 & r < R \\
\infty & r > R 
\end{cases},
\]

where \( r = \sqrt{x^2 + y^2} \) and \( \varphi = \arg(x + iy) \). Solving equation \( H_{qw}\chi_{n,l} = E_{n,l}\chi_{n,l} \), one can find states and energies through:

\[
\chi_{n,l} = \frac{N_{n,l}}{\sqrt{2\pi}} e^{i\varphi} J_{|l|}(k_{n,l}r),
\]

\[
E_{n,l} = \frac{\hbar^2 k_{n,l}^2}{2m} = \frac{\hbar^2}{2mR^2} \beta_{n,l}^2,
\]

where \( J_{|l|} \) is the Bessel function of integer order, \( N_{n,l} \) is the normalization constant, \( l = 0, \pm 1, \pm 2, \ldots \), and \( \beta_{n,l} = k_{n,l}R \) is the \( l \)th zero of the \(|l|\)th Bessel function for which \( J_{|l|}(\beta_{n,l}) = 0 \); this is a condition imposed by wavefunction on the boundary, namely \( \chi_{n,l}(R, \varphi) = 0 \).

In addition to stationary solutions \( \chi_{n,l} \), one can solve Schrödinger equation \( i\hbar \partial_t \psi = H\psi \) for a given arbitrary initial state. To end this, we note that:

\[
\psi(r, \varphi, t) = \sum_{n,l} \alpha_{n,l} e^{-iE_{n,l}t/\hbar} \chi_{n,l}(r, \varphi),
\]

\[
\alpha_{n,l} = \int d\varphi \int rdr \psi^*_{n,l}(r, \varphi) \chi_{n,l}(r, \varphi),
\]

where \( \psi_0(r, \varphi) \equiv \psi(r, \varphi, 0) \). Here, initial condition is a superposition of two Gaussian wavepackets one with topological charge \( TC = 1 \) and another with \( TC = 0 \); it reads:

\[
\psi_0 = \frac{e^{-r^2/2w^2}}{w \sqrt{\pi (w^2 + r_c^2)} (re^{i\varphi} - r_c e^{i\varphi})},
\]

where \( w \) is the spot size of the wavepacket and \( \{x_c \equiv r_c \cos \varphi_c, y_c \equiv r_c \sin \varphi_c \} \) gives the position of the core. A similar initial state could be implemented in an experiment with two delayed resonant pulses [8] in a polariton fluid.

The wavepacket is given by:

\[
\psi(r, \varphi, t) = \sum_n \left[ \alpha_{n,1} e^{-iE_{n,1}t/\hbar} \chi_{n,1} + \alpha_{n,0} e^{-iE_{n,0}t/\hbar} \chi_{n,0} \right].
\]

The dynamics is intriguingly highlighted. Examples are shown in Fig. [I]. We can detect different stages in the dynamics; at first we see an increment of the wavepacket size inside the well, which brings the density null and crest in larger and larger separation; however, the wavepacket should be zero on the boundary and there is a time instant when diffusion stops and a second crest starts to form, in half–moon shape, which is thinner than the first one but larger in radius. At the same time the vortex core rotates and the distortion of the phase excites further vortices, and precisely a vortex–antivortex pair, that move until they again recombine. Very shortly after the outer density reforms to an inner density, followed by rotation of the core and once again by creation and annihilation of the vortex pairs. Such processes are repeated during the time evolution of the dynamics.

Despite the complex motion of the core, the angular momentum content remains steady, one can show that:

\[
\langle L_z \rangle = -i\hbar \int d\varphi \int rdr \psi^* \partial_\varphi \psi = \frac{\hbar R^2}{2} \sum_n |\alpha_{n,1}|^2 N_{n,1}^2 [J_2(\beta_{n,1})]^2.
\]

The desired quantity is the angular momentum per particle, which in the current physics is less than one (as expected for off–center vortex [III]), namely, \( l \equiv \langle L_z \rangle/N < 1 \), where \( N \) is the total number of the particles and is given by:

\[
N = \frac{R^2}{2} \left[ \sum_n |\alpha_{n,1}|^2 N_{n,1}^2 [J_2(\beta_{n,1})]^2 + \sum_m |\alpha_{m,0}|^2 N_{m,0}^2 [J_1(\beta_{m,0})]^2 \right].
\]
The fact that \(l < 1\) comes from the initial field \([5]\), where we introduce two kinds of particles with a fraction only of them carrying (integer and equal to one) angular momentum. Here the infinite external potential is a purely confining one, its shape is radially symmetric and azimuthally homogeneous. Considering the potential gradient and its symmetry, its boundaries act both as a local and net force on the fluid at any moment, however there is no net torque acting on the fluid. Basically, while the center of mass of the fluid and its net linear momentum keep changing due to the continuous bouncing of the fluid against the well boundaries, its net angular momentum remains constant. Yet, some peculiar morphology reshaping, involving also the topological charges, happens during the evolution.

Once the expanding cloud reaches the boundary and is partially reflected, a circular ripple of lower density with larger phase gradient (visible starting from Fig. 1f,g on) is formed. It can be understood in terms of interference between the outward diffusing and the inward reflected waves. The secondary vortex-antivortex pair is nucleated starting from this loop of local low density. The pair proliferation is understandable in the realm of wave interference as well. Consider the interference of two waves, both with a Laguerre Gaussian type envelope of same amplitude: one propagating in the \(k_1\) direction, \(A_1 = e^{-ik_1 \cdot r} e^{-\gamma t}/(2\pi)\), and the other propagating in the different direction \(k_2\) but with a topological charge \(TC \leq 1\), \(A_2 = e^{-ik_2 \cdot r} e^{-\gamma t}/(2\pi)\left(r e^{i\varphi} - r e^{i\varphi_c}\right)\). If, for simplicity, \(A_1\) travels along the \(x\) direction, for which \(k_1 = k_1\), and for \(A_2\) we assume \(k_2 = 0\), then the two interfering waves can yield a total density \(I \propto |e^{ik_1 \cdot r} + re^{i\varphi} - re^{i\hat{\varphi}c}|^2\). Depending on wavenumber \(k\), such a density can have one root, corresponding to the initial topological charge \(TC \leq 1\), or can sustain three roots corresponding to a vortex and a vortex-antivortex pair. Example of this behavior of the phase is shown in Fig. 2 where we take \(x_c = 1 \mu m\) and \(y_c = 0\) (that is obtained by polar parameters \(r_c = 1 \mu m\) and \(\varphi_c = 0\)). Upon increasing \(k\), the phase gets distorted, resulting in vortex pair nucleation. This condition also depends on the relative amplitude of the two waves, which we didn’t considered in order to highlight the importance of the momentum. Coming back to the displaced vortex in the circular quantum well, the excitement of the pair comes to play when a reflected
wave from the hard boundary interferes with the field inside the well, as described partially in the above example, where a vortex–antivortex pair is created and annihilated repeatedly. There could be higher energetic initial conditions for which a larger number of vortex–antivortex pairs are cyclically observed. All these dynamics and morphology reshaping happen keeping a constant angular momentum.

III. HARMONIC POTENTIAL

Next, we consider the displaced vortex in a harmonic potential. The main equation is $i\hbar \partial_t \psi = H_{so} \psi$ with:

$$H_{so} = -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m \omega_{so}^2 r^2,$$

where $\omega_{so}$ stands for the natural frequency of the oscillations. Ring potentials in general have been shown to sustain rotating patterns [11, 12], based on combinations of rotational modes of different eigenenergies. The harmonic potential provides the staple model for many vibrating systems. Here, we show interesting features of the vortex fluid inside such potential and add a further degree, that is, temporally periodic angular momentum. To this end, we start our analysis with a spatially equipotential initial condition, namely, where the width parameter $w = \beta \equiv \sqrt{\hbar/(m\omega_{so})}$ is assumed in the initial configuration expressed by Eq. (5).

The wavepacket can be obtained in a close form:

$$\psi = \frac{e^{-r^2/(2\beta^2)} e^{-i\omega_{so} t}}{\beta \sqrt{\pi (\beta^2 + r^2)}} \left[ r e^{i(\varphi - \omega_{so} t)} - r_c(0) e^{i(\varphi_c(0))} \right].$$

This gives the orbit where the core moves on, which is a circle of radius $r_c(0)$; such a motion happens with a constant speed $v_c = r_c(0) \omega_{so}$. Although the core oscillates in time, the angular momentum content remains steady:

$$\langle L_z \rangle = \frac{\hbar \beta^2}{\beta^2 + r_c^2(0)}.$$  

(11)

This is consistent with a fractional value due to the offset core, which remains constant due to the fixed radius of its orbit. Then, the rotational content depends on the initial radial distance of the core from the origin, ranging from $\langle L_z \rangle = 1$ for $r_c(0) = 0$ to some very tiny value for large $r_c(0)$.

Interesting dynamics happens when in the initial condition we set $w \neq \beta$, corresponding to a non-equipotential isotropic case. Depending on the ratio of $w/\beta$, approximate solution can be provided through:

$$\psi = \sum_{nm} \alpha_{n,m}(t) \phi_{n,m}(x,y),$$

(12)

where $H_{so} \phi_{n,m} = E_{n,m} \phi_{n,m}$, and $\alpha_{n,m}(t) = \int e^{-iE_{n,m}/\hbar} \psi \phi_{n,m} dx dy$ with $n,m = 0,1,2,\cdots$. The evolution of the density is shown in Fig. 3 for $w = 1.2\beta$ and for the core located initially in $(-1,0)$. First, the wavepacket shrinks in size and at the same time the core also moves, but this time it moves on an ellipse rather than a circle: $(\beta x_c(t)/w)^2 + (\beta y_c(t)/\beta)^2 = (\beta/w)^2$. Then, the wavepacket shapes to its initial size, and the core completes one period of its motion. Such oscillations are repeated during the time evolution of the dynamics. Apart from the peculiar motion of the vortex core, that is being temporally sped up or down, and also the time-varying morphology, the quantum average of angular momentum is yet constant. In general and for any initial condition, one can show that for a simple harmonic potential the expectation value of angular momentum reads as:

$$\langle L_z \rangle_{so} = i\hbar \sum_{n,m} \alpha_{n,m}(0) \left[ \alpha_{n-1,m+1}(0) \sqrt{n(m+1)} - \alpha_{n+1,m-1}(0) \sqrt{n(m+1)} \right],$$

(13)

which is clearly time-independent. This is once again (as in the case of the circular quantum well) in agreement with the symmetry of the radial force field (energy gradient) of such potentials, which do have a zero net torque. In their interaction with fluid, they cannot exert any change of its overall angular momentum, which

![FIG. 3. Phase map profiles of displaced vortex in non-equipotential case ($w = 1.2\beta$). Panels (a) to (d) corresponds to time $[0, 0.75, 1.5, 3]$ ps. The trajectory of the core is an ellipse shown in cyan. For the matter of comparison, trajectory for the equipotential ($w = \beta$) case also is shown. The vortex morphology, $|\psi| = \text{constant}$, is shown with red contours. The enlarged view in panel (c) helps to visualize the highly compact shape at the perihelion core position along the orbit.](image)
keeps constant.

Although the angular momentum is steady in the above cases, the time-varying angular momentum could be induced by means of some perturbations; here we consider two examples. The first is an external time harmonic (and spatially homogeneous) potential $V(t) = V_0 \cos(\omega_p t) + i V_1 \sin(\omega_p t)$ which could be induced by an external oscillating field such as an incident laser. Here, we also add a complex term to take into account decay and/or gain. The coefficient $\alpha_{n,m}$ in Eq. (12) is given by:

$$\alpha_{n,m}(t) = e^{-i \frac{V_0 \sin(\omega_p t)}{\hbar \omega_p} E_n/m} e^{-i \frac{V_1 \cos(\omega_p t)}{\hbar \omega_p}} \times \int_0^\infty \psi_0 \phi_{n,m} \, dx \, dy.$$  

Corresponding average of the angular momentum reads as:

$$\langle L_z \rangle = e^{\frac{2V_1 \cos(\omega_p t)}{\hbar \omega_p}} \langle L_z \rangle_{so},$$

which is periodically time dependent. The amplitude of oscillations depend strongly on ratio of $V_1/\hbar \omega_p$, and may go beyond $\hbar$, for example, $\langle L_z \rangle > 2 \hbar$. The morphology of the vortex now attains an extra prefactor weight of $e^{-V_1 \cos(\omega_p t)/\hbar \omega_p}$ which results in oscillatory angular momentum.

The second example accounts for the effect of spatial anharmonicity, namely, a quartic contribution which is added to the harmonic oscillator potential. Indeed, a fluid in power law trap demonstrated some rich vortex anharmonicity, namely, a quartic contribution which is:

$$\text{oscillations depend strongly on ratio of } \rho \text{, for example, } \langle L_z \rangle > 2 \hbar.$$
any of the two fields. For the initial condition we assume:

\[ C \text{mass in the} \quad m \]

Here, \( \Omega \) is the Rabi frequency, and \( m_{C}(m_{X}) \) shows the mass in the \( C(X) \) field. We consider no decay term in any of the two fields. For the initial condition we assume:

\[ (\psi_{C,X})|_{t=0} = \frac{e^{-t^2/2w^2}}{\sqrt{\pi}(w^2 + r_{c,x}^2)}(r e^{i\varphi} - r_{c,x} e^{i\varphi_{c,x}}), \quad (22) \]

where the vortex cores are located in different points in real space, defined by \( (r_{c,x}, \varphi_{c,x}) \) in polar coordinates. We are not considering here the degree represented by a different amplitude factor between the two fields. One can find a close form solution in reciprocal space \( (k = \sqrt{k_x^2 + k_y^2} \text{ and } \theta_k = \arg[k_x + ik_y]) \) by applying the Fourier transform \( \mathcal{F} \) on each field, and obtain a general expression for the amplitudes of two coupled fields [15]:

\[
\tilde{\psi}_{C} = \frac{1}{k_{\Omega}} e^{-i \frac{k^2 M_{+}}{2h} t} \times \left[ -i \sin \left( \frac{k_{\Omega}}{2} \right) \left( 2 \hbar \Omega \tilde{\psi}_{X} \right) |_{t=0} + (\tilde{\psi}_{C}) |_{t=0} (\hbar \delta + k^2 M_{-}) \right] \] \quad (23a)

\[
\tilde{\psi}_{X} = \frac{1}{k_{\Omega}} e^{-i \frac{k^2 M_{-}}{2h} t} \times \left[ -i \sin \left( \frac{k_{\Omega}}{2} \right) \left( 2 \hbar \Omega \tilde{\psi}_{C} \right) |_{t=0} - (\tilde{\psi}_{X}) |_{t=0} (\hbar \delta + k^2 M_{-}) \right] \] \quad (23b)

where \( \tilde{\psi}_{C,X} = \mathcal{F}[\psi_{C,X}] \) and we introduce \( M_{\pm} = \frac{\hbar \Omega}{2} (1 \pm (m_{C}/m_{X})) \) and \( k_{\Omega} = \sqrt{(2\hbar \Omega)^2 + (\hbar \delta + k^2 M_{-})^2} \) with \( \hbar \delta = E_{C} - E_{X} \) as the energy detuning. In general it is not possible to find a close form solution in real space, due to polaritonic factor [15], however, with respect to the initial conditions [22], we can find some approximate solutions in the regime of \( w \gg 1 \) which is the case for the experimental observations [8]. In this regime, one notes that the dynamics is limited to a very small in-
moves along a circle orbit, shown in cyan, and similarly
through resonant pulse pumping and their offset tuned
\( \psi \) (Fig. 5 for \( \rho \) pled fields [17, 18]. Example of the dynamics is shown in
dynamics, which is here Rabi oscillations between cou-
the center). In fact, average quantities, such as angu-
ular momentum oscillates regularly.

where \( \omega_R \equiv \sqrt{(2\Omega)^2 + (\delta)^2}/2 \) is introduced as the fre-
quency of oscillations, and \( \Omega' \equiv \Omega(1 + m_r) \) with \( m_r \equiv m_{C}/m_{X} \). It is noted that if initially vortex cores are po-
sitioned in the same point of the real space \((r_{c},e^{i\varphi_{c}})\), they do not move at all; however, detuning \( (\delta \neq 0) \) can induce dephased density oscillations [16–18],
while the morphology remains constant, otherwise mor-
phology undergoes a complex orbiting reshaping. The
mean angular momentum content of each field is then
given as:

\[
\langle L_z \rangle_{C,X} = \frac{\hbar w^2}{4} \left[ \frac{4 \cos^2 \omega_{R} t}{w^2 + r_{c}^2} + \sin^2 \omega_{R} t \left( \frac{(2\Omega)^2}{w^2 + r_{c}^2} - \frac{4\delta}{\sqrt{(w^2 + r_{c}^2)(w^2 + r_{x}^2)}} \right) \right] + \frac{\hbar w^2}{4} \left[ \frac{4 \cos^2 \omega_{R} t}{w^2 + r_{x}^2} + \sin^2 \omega_{R} t \left( \frac{(2\Omega)^2}{w^2 + r_{x}^2} - \frac{4\delta}{\sqrt{(w^2 + r_{c}^2)(w^2 + r_{x}^2)}} \right) \right].
\]

which is clearly independent of \( m_{r} \), however, oscillatory
in time. Again, for zero detuning it can be observed that
for co-aligned vortices, the momentum stays constant. If
however \( \delta \neq 0 \), even though the cores do not move, there
are some spatial density waves and angular momentum
oscillations induced by detuning. Most interestingly, in
the general case, the two vortex cores keep orbiting each
other. Then the angular momentum, alongside density,
is coherently transferred and oscillating between the
coupled fields, while the total angular momentum in the sum
of the two fields is conserved. In other terms, due to the
exchange of particles with different position and moment-
num, also the average angular momentum per particle in
each of the two fields is periodically time-varying. This
is well represented by the cores moving in the off-axis
orbit (which is, periodically changing their distance from
the center). In fact, average quantities, such as angular
momentum, depend eventually on the nature of the dyna-
mics, which is here Rabi oscillations between cou-
pied fields [17–18]. Example of the dynamics is shown in
Fig. 3 for \( m_{c} \ll 1 \), corresponding to the strong coupling
regime of cavity photons and quantum well excitons [19].
Here we show some frames of the phase map for the pho-
ton (\( \psi_{C} \)) field. Initial vortex states can be introduced
through resonant pulse pumping and their offset tuned
by means of two-pulses coherent control [8]. The core
moves along a circle orbit, shown in cyan, and similarly
the morphologies of the vortex, shown by red isodensity
contours, illustrates nontrivial oscillatory shapes. While the
core and the vortex itself evolve oddly, in a sense that
the core is faster in the outer part of the beam [8–16],
the angular momentum oscillates regularly.

Such a peculiar dynamics can be perceived readily by
the following simplified version of the coupled field solu-
tions in matrix representation (refer to appendix A for
more details):

\[
\begin{pmatrix}
\psi_{C} \\
\psi_{X}
\end{pmatrix}
= \mathcal{J}
\begin{pmatrix}
\psi_{C} \\
\psi_{X}
\end{pmatrix}
|_{t=0},
\]

with

\[
\mathcal{J} \equiv \begin{pmatrix}
\cos \omega_{R} t - \frac{i\delta}{2w_{R}^{c}} \sin \omega_{R} t & -\frac{\delta}{2w_{R}^{c}} \sin \omega_{R} t \\
-\frac{\delta}{2w_{R}^{x}} \sin \omega_{R} t & \cos \omega_{R} t + \frac{i\delta}{2w_{R}^{x}} \sin \omega_{R} t
\end{pmatrix}.
\]

Once more, it is noted that the solutions [26–27] are valid
in the regime of \( w \gg 1 \) and not correct in general cases,
for example, when the dispersion effect plays a central
rule [20].

One may have déjà vu that the aforementioned represen-
tation is reminiscent of rotation matrix in linear alge-
bra, however, there is stark contrast due to the complex
representation is reminiscent of rotation matrix in linear alge-
bra, however, there is stark contrast due to the complex
dynamics of binary fields in linear regime [18], and
especially is different than merely a rotation matrix as
happens in real space. Yet, one can be hopeful for an
interpretation similar to rotation. To this end, let in-
trude the complex number defining the polariton state
(in the basis of the coupled fields) \( Z = \psi_{C}/\psi_{X} \) so that
the matrix representation in Eqs. (26–27) reappears as a bilinear transformation \( Z_0 \rightarrow Z = M(Z_0) \), where \( Z_0 \) is
the initial state and we introduce:

\[
M \equiv \begin{pmatrix}
\cos \omega_{R} t & -\frac{i\delta}{2w_{R}^{c}} \sin \omega_{R} t \\
\frac{\delta}{2w_{R}^{x}} \sin \omega_{R} t & \cos \omega_{R} t + \frac{i\delta}{2w_{R}^{x}} \sin \omega_{R} t
\end{pmatrix}.
\]

Assuming \( \delta = 0 \) hereafter for simplicity, the transfor-
mation [28] holds two fixed states \( Z_0 = \pm 1 \), which also
represent the vortex cores in the dressed states (normal
modes) [21]. Rewriting the transformation \( M \) in the nor-
FIG. 5. Profiles of the phase map for different times: \{0, 0.15, 0.25, 0.39, 0.65, 1.55\}, for panel (a) to (f), respectively. Time is in ps. The core is initially located at \((-w, 0)\). Here, the dynamics is shown for the limit of \(m_r = 0.001 \ll 1\) for \(\psi_c\) in Eq. 24. The position of the core is shown by a white point, and trajectory of its motion is illustrated in cyan. The red contours show the shape (morphology) of the vortex.

one finds that any \(Z\) moves in the complex plane mapping the polariton state around Apollonius circles that are symmetric with respect to fixed points, and then \(M\) maps each circle to itself. In advanced complex analysis, the specific transform \(M\) is an example of elliptic transform \([22]\). Furthermore, one can find an explicit representation of the \(Z\) point in terms of its stereographic projection on the Riemann sphere (here also coinciding with the Bloch sphere of polariton states). Denoting \((X, Y, Z)\) as a point from the Riemann sphere, corresponding stereographic expressions read as:

\[
X + iY = \frac{2Z}{1 + |Z|^2}, \quad \text{(30a)}
\]

\[
Z = \frac{|Z|^2 - 1}{|Z|^2 + 1}. \quad \text{(30b)}
\]

The representation on the sphere helps to visualize the smooth motion of the dynamics. On the sphere indeed the evolution of every point is a rotation around the axis passing along the two poles which represent the normal modes. Such axis is here set horizontal with respect to the complex plane. The previous considerations only concern the polariton states evolution, independently from any mapping or distribution they could assume in real space. Very interestingly, it is the specific and special initial condition with the two offset vortices of the said form \([22]\) that creates a homeomorphism which links the Riemann sphere to the real space too \([8]\) and mirror the complex plane into it. The vortex core moving along a finite size circle orbit in Fig. 5 is such an example, obtained upon a given set of parameters. A further example of straight line orbit (circle at infinite) is considered in appendix B.

V. CONCLUSIONS

Understanding the quantum fluid systems with time-varying angular content matures a further degree of freedom in the capacity of structured wavepacket to exert torque to different objects, ranging from some small size of nanometer to the size as large as milliliter. We studied analytically some possible systems, either based on confining potentials or on coupled condensates, for which the output vortex beams can carry dynamical orbital angular momentum. One can find a peculiar motion of the vortex core that is nested into a nontrivial morphology of the packet, which could be considered as a kind of temporal–anisotropic vortex structure, namely, its shape is being modified in the time course. In general, one can find the vortex core in motion with some local angular momentum distributed in space, however the expectation value of the angular content of the beam, which is done over the whole of wavepacket as a global quantity, may keep constant or be time-varying, depending on the physical system and the specific settings. The torque applied onto an object immersed into the beam would hence depend on both the beam and object morphologies, and can be made, for example periodic such as in a hybrid optomechanical torsion pendulum. This further degree of time-varying vortex and orbital control could benefit applications of OAM signal encoding and transmission, or precision metrology, such as gyroscopes and even Casimir torque measurements.
Appendix A: Simplified—coupled fields expression

It is possible to find a simpler version of the solutions, which is given in Eqs. (26–27). Let start from the exact expression for the two coupled fields in the reciprocal space [23], and ignore all terms with $k$ dependency which means, indeed, the dynamics remains in the real space domain size of the initial packet. Then, there are neither associated diffusive effects nor dispersive ones in the dynamics of the solutions. It is worth noting that real space solutions of (24) include diffusion, which mainly comes from the exponential term $e^{-\frac{t^2}{2\sigma^2}}$ in Eqs. (23). Yet, by ignoring dispersive and diffusive effects, one can find the desired solutions:

$$\tilde{\psi}_C = \left[ \cos(\omega_c t) - \frac{i\delta}{2\omega_c} \sin(\omega_c t) \right] (\tilde{\psi}_C)_{t=0} \quad - \frac{i\Omega}{\omega_c} \sin(\omega_c t) (\tilde{\psi}_C)_{t=0}, \quad (A1a)$$

$$\tilde{\psi}_X = - \frac{i\Omega}{\omega_c} \sin(\omega_c t) (\tilde{\psi}_C)_{t=0} + \left[ \cos(\omega_c t) + \frac{i\delta}{2\omega_c} \sin(\omega_c t) \right] (\tilde{\psi}_X)_{t=0}, \quad (A1b)$$

that after taking inverse Fourier transform, they finally give the simplified real-space form (26).

Appendix B: Vortex core motion on the Riemann sphere

Here we describe an example of the dynamics of the $Z$ point on the Riemann sphere. To keep the example interesting let consider vortex cores in photon and exciton fields where they are located, respectively, at $(-w,0)$ and $(0,0)$ in real space. We assume zero energy detuning. Corresponding $Z$ points are then given as $Z_C(t) = i\tan\Omega t$ and $Z_X(t) = -i\cot\Omega t$, which move along the imaginary axis of the complex plane. In $t = 0$, $Z_X$ is a point at infinity, in $t = \pi/(2\Omega)$ it reaches the origin of the complex plane and then in $t = \pi/\Omega$ it is again at infinity. Contrarily, $Z_C$ is initially positioned at origin, then goes to infinity, and finally in $t = \pi/\Omega$ it reappears at origin. Clearly, the motion of the points are not smooth, that is, $\frac{d}{dt}Z_{C,X}$ is not a constant. However, points on the Riemann sphere show smooth dynamics. To consider this, we note that the corresponding points on the sphere are $(0, -\sin 2\Omega t, \cos 2\Omega t)$ for $Z_C$, and $(0, \sin 2\Omega t, -\cos 2\Omega t)$ for $Z_X$, which can be obtained directly form Eqs. (30). These provide a circle on the sphere, and the points on the sphere move apparently with constant angular speed of $2\Omega$. The described situation can be realized starting from a specific initial condition of the said form (22). However the very same picture can be extended, upon using different parameters in the same form, to otherwise points of real space, where the relevant $Z$ point moves on an associated Apollonius circle [8].

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