The study of phase transitions and critical phenomena is one of the cornerstones of modern theoretical physics. Away from the critical point, correlations between individual units decay exponentially with separation. Here the correlation length defines a characteristic scale. At the critical point, where the correlation length is infinite and two or more macroscopic phases become indistinguishable, there is long-range power-law decay – the hallmark of criticality. Critical points are ‘scale free’, there being no characteristic scale associated with a simple power law.

Our general understanding of phase transitions and critical phenomena has been greatly enhanced by the study of exactly solved lattice models in statistical mechanics. Here I give a brief account of some recent developments. These involve (i) the most general network of self-avoiding polymer chains attached to a boundary and (ii) a model in the same universality class as the two-dimensional Ising model in a magnetic field. The common feature in both studies is an exact solution by means of the Bethe Ansatz.

**Polymer Networks**

Progress in polymer physics has been significantly advanced by the relation with critical phenomena. Long polymer chains in a solvent can be described by self-avoiding walks (SAWs) on a lattice. Of particular utility is the correspondence between SAWs and the $O(n)$ model in the limit $n \to 0$. It transpires that this latter model can be solved exactly on the honeycomb lattice at criticality.

The investigation of the configurational properties of SAWs in the vicinity of a boundary already dates back some twenty years. The SAW is restricted to say the upper half plane and originates on the boundary. The surface interaction is represented by an energy, $\epsilon$, associated with each contact between the polymer and the surface. The Boltzmann weight for a configuration of the polymer is given by $\kappa^m = e^{-\epsilon_{\text{contact}}} / kT$, where $T$ is the temperature of the solvent and $m$ is the number of contacts with the surface. At some critical temperature $T_c$ the polymer becomes adsorbed onto the surface. For $T > T_c$ the polymer is in a desorbed phase where it extends a large distance into the solvent above the surface. For $T < T_c$ the polymer is in an adsorbed phase. The configurational properties are phase dependent.

The $O(n)$ model has been solved for a number of different boundary conditions. In the terminology of surface critical phenomena these boundary conditions correspond to the ‘ordinary’, ‘special’ and ‘extraordinary’ transitions. The critical adsorption temperature, $T_a$, for SAWs corresponds to the special transition, whilst the ordinary transition corresponds to the presence of an effectively repulsive surface. The extraordinary transition has also been discussed.

This model for polymer adsorption has been studied on a number of lattices via different techniques. The advantage of the present approach is that the relevant quantities have been calculated exactly on the honeycomb lattice. These results can then be combined with general arguments from scaling and conformal invariance to yield the configurational exponents.

The critical adsorption temperature depends on the particular orientation of the honeycomb lattice. For the boundary orientation of Fig. 1(a) the critical adsorption temperature is known to be given by

$$\exp\left(\frac{\epsilon}{kT_a}\right) = 1 + \sqrt{2} = 2.414 \ldots$$

For the orientation of Fig. 1(b) we find

$$\exp\left(\frac{\epsilon}{kT_a}\right) = \sqrt{\frac{2 + \sqrt{2}}{1 + \sqrt{2} - \sqrt{2 + \sqrt{2}}} = 2.455 \ldots}$$

Both results follow from the boundary Boltzmann weights of the exactly solved $O(n)$ model at $n = 0$ and have been confirmed via series expansion techniques. However, being specific to the honeycomb lattice, the adsorption temperatures are non-universal quantities. Of most interest are the universal configurational exponents.

**FIG. 1.** The two orientations of the honeycomb lattice.

Our chief result concerns the number of configurations $Z_G$ of the monodisperse network of $N$ self-avoiding polymer chains of length $S$ depicted in Fig. 2. The network is attached to the boundary.
The boundary conditions differ on each side of the origin marked m. To the left is the ordinary boundary condition and to the right is the special boundary condition. \( Z_G \) scales as
\[
Z_G \sim \mu^{NS} S^\gamma_{o}^{1} \quad \text{as} \quad S \to \infty, \tag{3}
\]
where \( \gamma_G \) is a universal exponent. Here \( \mu \) is the lattice-dependent connective constant for SAWs.

\[\begin{align*}
\text{FIG. 2. A network of } N = 11 \text{ chains attached to the mixed o-s boundary.}
\end{align*}\]

The general scaling and conformal invariance arguments can be extended to the mixed boundary. The key ingredients are the scaling dimensions for each of the bulk, ordinary and special boundary conditions, all of which have been derived from the exact Bethe Ansatz solution of the honeycomb lattice model. In this way the universal exponent
\[
\gamma_G = \frac{1}{4} + \frac{1}{64} \sum_L n_L (2 - L)(9L + 50) + \frac{1}{16} \sum_L n_L^2 (9L^2 + 22L - 24) + \frac{1}{32} \sum_L n_L^3 (9L^2 + 10L - 24) - \frac{1}{32} \sum_L n_L^3 (9L^2 + 2L - 16) \quad \text{(4)}
\]
governing the asymptotic number of configurations of the most general planar network is obtained.

The numbers \( n_L \) are topological indices describing any given network. For example, \( n_L \) is the number of \( L \)-leg vertices in the bulk, \( n_L^2 \) is the number of \( L \)-leg vertices attached to the ordinary surface, \( n_L^3 \) the number at the origin, etc. In each case there can be \( L \geq 1 \) vertices. The network depicted in Fig. 2 has \( n_1 = 3, n_2 = 2, n_4 = 1, n_6 = 1, \), and \( n^2_2 = 1 \). For mixed boundaries there is only one \( L \)-leg vertex emanating from the origin, thus \( n^3_L = 1 \).

All previously known examples follow from this most general formula. The exponents for a pure ordinary surface are recovered with \( n^2_L = n^3_L = 0 \). For the pure special surface we take \( n^2_L = n^3_L = 0 \). For a single walk in the half-plane, \( \gamma_T^{1} = \frac{43}{16} \) and \( \gamma_T^{2} = \frac{93}{16} \). If the walk also terminates on the boundary, \( \gamma_T^{11} = -\frac{43}{16} \) and \( \gamma_T^{11} = \frac{93}{16} \) (using the standard notation). These exponents satisfy Barber’s scaling law,
\[
2 \gamma_{1} - \gamma_{11} = \gamma + \nu L^{1/14} \quad \text{where} \quad \gamma = \frac{43}{32} \quad \text{and} \quad \nu = \frac{3}{4}.
\]

The network can be tied in a wedge of arbitrary angle \( \alpha \). Obtaining the wedge network exponents \( \gamma_G(\alpha) \) involves a conformal map of the wedge to the half-plane. As an example, consider an \( L \)-leg star polymer confined to a wedge with o-o, s-s or o-s surfaces (see Fig. 3). The \( \alpha \)-dependent exponents are given by
\[
\begin{align*}
\gamma_L^{o}(\alpha) &= 1 + \frac{27L - 3\pi L(3L + 2)}{32\alpha}, \\
\gamma_L^{s}(\alpha) &= 1 + \frac{27L - 9\pi L(3L - 2) + 8\pi}{32\alpha}, \\
\gamma_L^{ms}(\alpha) &= 1 + \frac{27L - 3\pi L(3L - 2)}{32\alpha}. \quad \text{(5)}
\end{align*}
\]

The o-o result is that obtained earlier.

\[\begin{align*}
\text{FIG. 3. Star polymer in a wedge of angle } \alpha \text{ with general boundaries A and B.}
\end{align*}\]

As particular examples, consider a single SAW confined to a quarter of the plane and emanating from the \( 90^\circ \) corner. In this case the above formulae reduce to \( \gamma_1^{o}(\frac{\pi}{2}) = \frac{93}{16}, \gamma_1^{s}(\frac{\pi}{2}) = \frac{95}{16}, \gamma_1^{ms}(\frac{\pi}{2}) = \frac{79}{16}. \) The exponents differ if the walk terminates on either boundary. In that case we have \( \gamma_1^{o}(\frac{\pi}{4}) = -\frac{43}{16} \) for the o-o corner and \( \gamma_1^{s}(\frac{\pi}{4}) = \frac{43}{64} \) for the s-s corner. For the o-s corner the walk can terminate on either the o side, with \( \gamma_1^{ms}(\frac{\pi}{4}) = \frac{19}{32} \), or on the s side, with \( \gamma_1^{ms}(\frac{\pi}{4}) = \frac{19}{32}. \)

A comprehensive study of a linear chain has been undertaken by exact enumeration on the honeycomb lattice, with various attachments of the walk’s ends to the surface, in wedges of angles \( \pi/2 \) and \( \pi \), with general mixed boundary conditions. A comparison between the numerical estimates of the \( 90^\circ \) wedge exponents discussed above and the exact values are given below. The exponent estimates are clearly seen to be in excellent agreement with the predicted values. The verification of the special exponents for the \( 90^\circ \) wedge involved the implicit verification of both the adsorption temperatures, (1) and (3).
Magnetic Ising Model

The two-dimensional Ising model in a magnetic field continues to defy an exact solution. However, a solvable two-dimensional lattice model—the dilute $A_3$ model—in the same universality class as the Ising model in a magnetic field has been found. This model is also known as the $E_8$ lattice realisation of the Ising model in a magnetic field at $T = T_c$. The model is a three-state interaction-round-a-face model. The Boltzmann weights, which are too complicated to reproduce here, are given in terms of elliptic theta functions. The elliptic nome plays the role of a variable magnetic field. The free energy has been derived, the singular part of which behaves as

$$f \sim h^{1+1/\delta} \quad \text{as} \quad h \to 0,$$

where the magnetic exponent $\delta = 15$. This was the first time that this exponent had been obtained without the use of scaling relations between critical exponents. The latest experimental estimate of the Ising magnetic exponent, on an atomic layer of ferromagnetic iron deposited on a substrate, is $\delta = 14 \pm 2$. The excess Ising magnetic surface exponent $\delta_m$ has been derived in a similar way from the excess surface free energy. This result is also in agreement with that predicted by proposed scaling relations between bulk and surface critical exponents.

There is a very intimate relation between exactly solved lattice models and integrable quantum field theory. In a remarkable paper, Zamolodchikov considered the magnetic perturbation of the $c = \frac{1}{2}$ conformal field theory and showed that there are a number of nontrivial local integrals of motion and thus an integrable field theory. He then conjectured the $S$-matrix and mass spectrum of this field theory. The masses coincide with the components of the Perron-Frobenius vector of the Cartan matrix of the Lie algebra $E_8$. Their ratios are

$$
\begin{align*}
\gamma_1^\nu(\pi/2) & = 0.4843(9) \quad \left(\frac{\pi}{4}\right) = 0.483475 \\
\gamma_1^\nu(\pi/2) & = -0.655(3) \quad \left(\frac{-\pi}{4}\right) = -0.65625 \\
\gamma_1^\nu(\pi/2) & = 1.482(8) \quad \left(\frac{\pi}{2}\right) = 1.484375 \\
\gamma_1^\nu(\pi/2) & = 0.851(1) \quad \left(\frac{\pi}{4}\right) = 0.84375 \\
\gamma_1^\nu(\pi/2) & = 1.233(6) \quad \left(\frac{3\pi}{4}\right) = 1.234375 \\
\gamma_1^{m\nu}(\pi/2) & = 0.091(1) \quad \left(\frac{-\pi}{4}\right) = 0.09375 \\
\gamma_1^{m\nu}(\pi/2) & = 0.596(7) \quad \left(\frac{\pi}{4}\right) = 0.59375 \\
\end{align*}
$$

at $T = T_c$. The conjectured $S$-matrix and the mass spectrum were later confirmed by a thermodynamic Bethe Ansatz calculation on the Hamiltonian formulation of the lattice model in the same universality class.

Our approach has been to derive the correlation lengths $\xi_j$ of the lattice model using an exact perturbative approach based on the Bethe Ansatz solution. The input to our calculations are the various root distributions revealed by numerical investigations. The calculations are long and tedious, involving perturbation in an exact manner from the high magnetic field limit. Our final result is that the elementary masses appearing in the eigen-spectrum are of the form

$$m_j = \xi_j^{-1} = 2 \sum a \log_4 \frac{\vartheta_4(a \pi h + \pi)}{\vartheta_4(\pi a h - 4 \pi, h^{8/15})},$$

where $\vartheta_4$ is a standard elliptic theta function. Here $h$ is the elliptic nome with $h = 0$ at the critical point and $h = 1$ in the high field limit. The numbers $a$ as tabulated below have already appeared for the related Hamiltonian.

| $j$ | $a$ |
|-----|-----|
| 1 | 1, 11 |
| 2 | 7, 13 |
| 3 | 2, 10, 12 |
| 4 | 6, 10, 14 |
| 5 | 3, 9, 11, 13 |
| 6 | 8, 12, 14 |
| 7 | 4, 8, 10, 12, 14 |
| 8 | 5, 7, 9, 11, 13, 15 |

In particular, as $h \to 0$,

$$m_j \sim 8 h^{8/15} \sum a \sin \frac{a\pi}{4},$$

This is the formula obtained previously. The $E_8$ masses in $\xi$ are recovered from $E$ by virtue of trig identities. The mass ratios show remarkably little variation as a function of $h$. In the high field limit the mass ratios are given exactly by $\frac{1}{3}, 2, 2\frac{1}{3}, 3, 4, 8, 5$, which are to be compared with the $h = 0$ results $E_8$.

Each of the masses tend to zero at criticality. The related correlation lengths diverge with magnetic correlation length exponent $\nu_m = \frac{8}{15}$. This exponent satisfies the general scaling relation $2\nu_m = 1 + 1/\delta$, which follows from the thermodynamic relation

$$f \xi^2 \sim \text{constant},$$

which is expected to hold near criticality. Here the constant can easily be evaluated. The singular part of the free energy scales like
\[ f \sim 4\sqrt{3} \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{15}} h^{16/15} \quad \text{as} \quad h \to 0. \]  \hspace{1cm} (13)

On the other hand, from (10)

\[ \xi_1 \sim \frac{1}{8\sqrt{3}} \frac{1}{\sin \frac{\pi}{5}} h^{-8/15} \quad \text{as} \quad h \to 0. \]  \hspace{1cm} (14)

Combining these results gives

\[ f \xi_1^2 = \frac{1}{16\sqrt{3}} \frac{1}{\sin \frac{\pi}{5}} \cos \frac{\pi}{15} = 0.061\ 728\ 589\ldots \]  \hspace{1cm} (15)

This is the universal magnetic Ising amplitude. It has been predicted earlier by other means. Namely by thermodynamic Bethe Ansatz calculations based on the \( E_8 \) scattering theory (see also Ref. 30). Here it is obtained for an explicit lattice model.

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