Geometry of the Gromov product: Geometry at infinity of Teichmüller space

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Abstract. This paper is devoted to studying transformations on metric spaces. It is done in an effort to produce qualitative version of quasi-isometries which takes into account the asymptotic behavior of the Gromov product in hyperbolic spaces. We characterize a quotient semigroup of such transformations on Teichmüller space by use of simplicial automorphisms of the complex of curves, and we will see that such transformation is recognized as a “coarsification” of isometries on Teichmüller space which is rigid at infinity. We also show a hyperbolic characteristic that any finite dimensional Teichmüller space does not admit (quasi)-invertible rough-homothety.

1. Introduction.

1.1. Backgrounds.

Let \((X, d_X)\) be a metric space. The Gromov product with reference point \(x_0 \in X\) is defined by

\[
\langle x_1 | x_2 \rangle_{x_0} = \langle x_1 | x_2 \rangle_{x_0} = \frac{1}{2}(d_X(x_0, x_1) + d_X(x_0, x_2) - d_X(x_1, x_2)).
\]  

We define the Gromov product of two sequences \(x = \{x_n\}_{n \in \mathbb{N}}, y = \{y_n\}_{n \in \mathbb{N}}\) in \(X\) by

\[
\langle x | y \rangle_{x_0} = \lim \inf_{n, m \to \infty} \langle x_n | y_m \rangle_{x_0}
\]  

Convention 1.1. When the metric space and the reference point in the discussion are clear in the context, we omit to specify them in denoting the Gromov product. We always assume in this paper that any metric space is of infinite diameter.

Let \(USq(X) \subset X^\mathbb{N}\) is the set of unbounded sequences in \(X\). We call a sequence \(x \in USq(X)\) convergent at infinity if

\[
\langle x | x \rangle = \infty
\]  

(cf. Section 8 in [14]). Any sequence satisfying (3) is contained in USq(X). The definition (3) is independent of the choice of the reference point. Let

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Table 1. Comparison with the coarse geometry for general metric spaces $X$. For details, see Section 1.2 and Section 3.

| Coarse geometry                  | Geometry on the Gromov product |
|----------------------------------|--------------------------------|
| Quasi-isometries (qi)            | Asymptotically conservative (ac) mappings |
| Coarsely Lipschitz               | weakly ac |
| Coarsely co-Lipschitz            | $\omega(z) \in \text{Vis}(\omega(x)) \Rightarrow z \in \text{Vis}(x)$ ($\forall x, z \in \text{USq}(X)$) |
| Cobounded                        | Asymptotically surjective |
| Quasi-inverse                    | Invertible ac |
| Parallelism                      | Close at infinity |
| $\Omega I(X)$ (cf. (21))         | $\mathfrak{A}C(X)$ (Section 1.3) |

Table 2. Comparison with the coarse geometry for metric spaces which are WBGP. For details, see Section 4.

| Coarse geometry                  | Geometry on the Gromov product |
|----------------------------------|--------------------------------|
| Quasi-isometries                  | Asymptotically conservative mappings |
| Coarsely Lipschitz                | $\langle x | z \rangle = \infty \Rightarrow \langle \omega(x) | \omega(z) \rangle = \infty$ ($\forall x, z \in \text{Sq}^\infty(X)$) |
| Coarsely co-Lipschitz            | $\langle \omega(x) | \omega(z) \rangle = \infty \Rightarrow \langle x | z \rangle = \infty$ ($\forall x, z \in \text{Sq}^\infty(X)$) |

$$\text{Sq}^\infty(X) = \{ x \in \text{USq}(X) \mid \langle x | x \rangle = \infty \}.$$  

We say that a sequence $y \in X^N$ is visually indistinguishable from $x \in X^N$ if

$$\langle x | y \rangle = \infty.$$  

(4)

For a sequence $x \in X^N$, we define

$$\text{Vis}(x) = \{ y \in \text{Sq}^\infty(X) \mid \langle y | x \rangle = \infty \}.$$  

Notice that $\text{Vis}(x) = \emptyset$ when a sequence $x$ is bounded. Two sequences $x^1, x^2 \in \text{USq}(X)$ are said to be asymptotic if $\text{Vis}(x^1) = \text{Vis}(x^2)$. When $X$ is a Gromov hyperbolic space, $\text{Vis}(x)$ defines a point in the Gromov boundary (cf. [6]).

In this paper, aiming for developing the coarse geometry and the asymptotic geometry on Teichmüller space, we investigate the theory of mappings on metric spaces with respecting for asymptotic behavior of sequences converging at infinity (cf. Tables 1 and 2). Namely, we (pretend to) recognize that two unbounded sequences $x^1, x^2 \in \text{USq}(X)$ determine the same ideal point at infinity if two sequences $x^1, x^2$ converging at infinity are asymptotic. Intuitively, asymptotically conservative mappings given in this paper are mappings keeping the divergence conditions of the Gromov products of two sequences converging at infinity.

**1.2. Definitions.**

Let $X$ and $Y$ be metric spaces. A mapping $\omega \in Y^X$ is said to be asymptotically conservative with the Gromov product (asymptotically conservative for short) if for any
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sequence \( x \in \text{USq}(X) \), the following two conditions hold;

1. \( \omega(\text{Vis}(x)) \subset \text{Vis}(\omega(x)) \).
2. For any \( z \in \text{USq}(X) \), if \( \omega(z) \in \text{Vis}(\omega(x)) \), then \( z \in \text{Vis}(x) \).

We will call a map \( \omega \in Y \times \text{X} \) with the condition (1) above \textit{weakly asymptotically conservative} (cf. Section 3.2).

Here, for a sequence \( x = \{x_n\}_{n \in \mathbb{N}} \subseteq X \subseteq X \) and a mapping \( \omega \in Y \times \text{X} \), we define usually \( \omega(x) = \{\omega(x_n)\}_{n \in \mathbb{N}} \subseteq Y \subseteq Y \) and \( \omega(E) = \{\omega(x) | x \in E\} \).

Two mappings \( \omega_1, \omega_2 \in Y \times \text{X} \) are said to be \textit{close at infinity}, if for any \( x^1, x^2 \in \text{Sq}^\infty(X), \text{Vis}(\omega_1(x^1)) = \text{Vis}(\omega_2(x^2)) \) holds whenever \( \text{Vis}(x^1) = \text{Vis}(x^2) \). An asymptotically conservative mapping \( \omega \in Y \times \text{X} \) is said to be \textit{invertible} if there is an asymptotically conservative mapping \( \omega' \in \text{X} \times Y \) such that \( \omega' \circ \omega \) and \( \omega \circ \omega' \) are close to the identity mappings on \( X \) and \( Y \), respectively. We call such \( \omega' \) an \textit{asymptotic quasi-inverse} of \( \omega \). Let \( \text{AC}_{\text{inv}}(X) \) be the set of invertible asymptotically conservative mappings on \( X \) to itself. For instance, any isometric isomorphism between metric spaces is invertible asymptotically conservative. The notions of mappings given above are stable under parallelism (cf. Proposition 3.1). In Section 3, we will give more discussion.

1.3. Results.

We first observe the following theorem (cf. Section 2.7).

\textbf{Theorem A} (The group \( \text{AC}(X) \)). \textit{Let} \( X \) \textit{be a metric space. The set} \( \text{AC}_{\text{inv}}(X) \) \textit{admits a monoid structure with respect to the composition of mappings. Furthermore, the relation “closeness at infinity” is a semigroup congruence on} \( \text{AC}_{\text{inv}}(X) \) \textit{and the quotient semigroup} \( \text{AC}(X) \) \textit{is a group.}

\textbf{Large scale geometry of Teichmüller space.} \textbf{Our main interest is to clarify the large scale geometry of Teichmüller space} \( \mathcal{T} \) \textbf{in respecting for asymptotic behaviors of sequences converging at infinity.}

\textbf{Rigidity theorem.} \textit{Let} \( S \) \textit{be a compact orientable surface. We denote the complexity of} \( S \) \textit{by}

\[ \text{cx}(S) = 3 \text{genus}(S) - 3 + \#\{\text{components of } \partial S\}. \]

\textit{The Euler characteristic of} \( S \) \textit{is denoted by} \( \chi(S) \). \textit{Throughout this paper, we always assume that} \( \chi(S) < 0 \). \textit{S} \textit{is said to be in the sporadic case if} \( \text{cx}(S) \leq 1 \).

\textit{Let} \( \mathcal{T} \) \textit{be the Teichmüller space of} \( S \) \textit{endowed with the Teichmüller distance. The extended mapping class group} \( \text{MCG}^*(S) \) \textit{of} \( S \) \textit{acts isometrically on} \( \mathcal{T} \) \textit{and we have a group homomorphism}

\[ I_0: \text{MCG}^*(S) \to \text{Isom}(\mathcal{T}). \]

\textit{We also have a monoid homomorphism}
\[ \mathcal{I}: \text{Isom}(\mathcal{T}) \to \text{AC}_{\text{inv}}(\mathcal{T}) \]
defined by the inclusion (see Section 9.3.1). We will prove the following rigidity theorem (cf. Theorem 9.2).

**Theorem B (Rigidity).** Suppose \( \text{cx}(S) \geq 2 \). Let \( \mathcal{X}(S) \) be the complex of curves on \( S \). Then, there is a monoid epimorphism
\[ \Xi: \text{AC}_{\text{inv}}(\mathcal{T}) \to \text{Aut}(\mathcal{X}(S)) \]
which descends to an isomorphism
\[ \mathfrak{A}\mathcal{C}(\mathcal{T}) \to \text{Aut}(\mathcal{X}(S)) \]
satisfying the following commutative diagram
\[
\begin{array}{ccc}
\text{MCG}^*(S) & \xrightarrow{\phi} & \text{Isom}(\mathcal{T}) \\
\downarrow & & \downarrow \Xi \\
\mathfrak{A}\mathcal{C}(\mathcal{T}) & \xrightarrow{\text{group iso}} & \text{Aut}(\mathcal{X}(S)),
\end{array}
\]
where \( \text{Aut}(\mathcal{X}(S)) \) is the group of simplicial automorphisms of \( \mathcal{X}(S) \).

**Relation to the coarse geometry.** Recently, A. Eskin, H. Masur and K. Rafi [9] and B. Bowditch [5] independently observed a remarkable result that any cobounded quasi-isometry of \( \mathcal{T} \) is parallel to an isometry, and the inclusion \( \text{Isom}(\mathcal{T}) \hookrightarrow \text{QI}(\mathcal{T}) \) induces an isomorphism
\[ \text{Isom}(\mathcal{T}) \cong \Omega I(\mathcal{T}) = \text{QI}(\mathcal{T})/(\text{parallelism}) \]
when \( S \) is in the non-sporadic case. Especially, any self quasi-isometry on \( \mathcal{T} \) is weakly asymptotically conservative (cf. Proposition 3.1). Hence, we have the following sequence of monoid homomorphisms
\[ \text{Isom}(\mathcal{T}) \hookrightarrow \text{QI}(\mathcal{T}) \hookrightarrow \text{AC}_{\text{inv}}(\mathcal{T}) \]  \( (5) \)
by inclusions (see Corollary 3.1). Theorem B implies the following.

**Corollary 1.1 (Relation to the coarse geometry on \( \mathcal{T} \)).** For non-sporadic cases, a quotient set \( \text{QI}(\mathcal{T})/(\text{close at infinity}) \) admits a group structure equipped with the operation defined by composition, and the sequence \( (5) \) descends to the following sequence of isomorphisms
\[ \text{Isom}(\mathcal{T}) \cong \Omega I(\mathcal{T}) \cong \text{QI}(\mathcal{T})/(\text{close at infinity}) \cong \mathfrak{A}\mathcal{C}(\mathcal{T}). \]

**Corollary 1.2 (Criterion for parallelism).** Let \( \omega, \psi: \mathcal{T} \to \mathcal{T} \) be cobounded quasi-isometries. The following are equivalent.
1. **For any** \( x, y \in \text{Sq}_\infty(T) \) **with** \( \langle x \mid y \rangle = \infty \), **it holds** \( \langle \omega(x) \mid \psi(y) \rangle = \infty \).

2. \( \psi \) **is parallel to** \( \omega \).

Corollary 1.2 gives a hyperbolic nature and Teichmüller space. Indeed, the same conclusion holds when we consider the hyperbolic space \( \mathbb{H}^n \) \((n \geq 2)\) instead of the Teichmüller space \( T \).

**No-rough homothety.** By applying the discussion in the proof of Theorem B, we also obtain a hyperbolic characteristic of Teichmüller space. In fact, we will give a proof of the following folklore result in Section 9.5.

**Theorem C** (No rough-homothety with \( K \neq 1 \)). **There is no** \((K,D)\)-**rough homothety with asymptotic quasi-inverse on the Teichmüller space of** \( S \) **unless** \( K = 1 \).

Here, a mapping \( \omega: (X, d_X) \to (Y, d_Y) \) between metric spaces is said to be a \((K,D)\)-**rough homothety** if

\[
|d_Y(\omega(x_1), \omega(x_2)) - K d_X(x_1, x_2)| \leq D
\]

(6)

for \( x_1, x_2 \in X \) (cf. Chapter 7 of [7]). Any rough-homothety is asymptotically conservative. Theorem C implies that there is no non-trivial similarity in Teichmüller space, like in hyperbolic spaces. Since rough homotheties are quasi-isometries, if \( \omega \) in Theorem C is cobounded, the rigidity in the theorem follows from Eskin–Masur–Rafi–Bowditch’s quasi-isometry rigidity theorem. However, the author does not know whether rough homotheties in the statement are always cobounded or not.

Recently, enormous hyperbolic characteristics are observed in Teichmüller space (e.g. [4], [5], [9]), though Teichmüller space is not Gromov hyperbolic (cf. [30]). See also Section 6.4). Some of these hyperbolic nature might naturally imply that a measurable \((K,D)\)-homothety on Teichmüller space does not exist unless \( K \neq 1 \).

### 1.4. Plan of this paper.

This paper is organized as follows: In Section 2, we will introduce asymptotically conservative mappings on metric spaces. We first start with the basics for the Gromov product, and we next develop the properties of asymptotically conservative mappings. We will prove Theorem A in Section 2.7. In Section 3 and Section 4, we will discuss a relation between our geometry and the coarse geometry.

From Section 5 to Section 7, we devote to preparing for the proofs of Theorems B and C. In Section 5, we give basic notions of Teichmüller theory including the definitions of Teichmüller space, measured foliations and extremal length. In Section 6, we recall our unification theorem for extremal length geometry on Teichmüller space via intersection number. One of the key for proving our rigidity theorem is to characterize the null sets for points in the GM-cone (cf. Theorem 7.1) The characterization is also applied to proving a rigidity theorem of holomorphic disks in the Teichmüller space and to studying the null-set reductions of several compactifications of Teichmüller space (cf. [3] and [37]). In Section 8, We define the reduced Gardner–Masur compactification and study the action of asymptotically conservative mappings on the reduced Gardiner–Masur compactification. In Section 9, we will prove Theorems B and C.
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2. Asymptotically conservative with the Gromov product.

2.1. Basics of the Gromov product.

Let \((X, d_X)\) be a metric space. The following is known for \(x_1, x_2, x_3, z_1, w_1 \in X\):

\[
\langle x_1 | x_2 \rangle_{z_1} \geq 0, \tag{7}
\]

\[
\langle x_1 | x_2 \rangle_{z_1} \leq \min\{d_X(z_1, x_1), d_X(z_1, x_2)\}, \tag{8}
\]

\[
\langle x_1 | x_1 \rangle_{z_1} = d_X(z_1, x_1), \tag{9}
\]

\[
|\langle x_1 | x_2 \rangle_{z_1} - \langle x_1 | x_2 \rangle_{w_1}| \leq d_X(z_1, w_1), \tag{10}
\]

\[
|\langle x_1 | x_2 \rangle_{z_1} - \langle x_1 | x_3 \rangle_{z_1}| \leq d_X(x_2, x_3). \tag{11}
\]

2.2. Sequences converging at infinity.

We notice the following.

**Remark 2.1 (Basic properties).** Let \(X\) be a metric space. The following hold:

1. The relation “visually indistinguishable” is reflexive on \(\text{Sq}^\infty(X)\): \(x \in \text{Sq}^\infty(X)\) if and only if \(x \in \text{Vis}(x)\).

2. The relation “visually indistinguishable” is symmetric on \(\text{Sq}^\infty(X)\): If \(z \in \text{Vis}(x)\), then \(x \in \text{Vis}(z)\).

3. The relation “visually indistinguishable” is not transitive in general. Namely, it is possible that \(\text{Vis}(z) \neq \text{Vis}(x)\) for some unbounded sequences \(x, z\) with \(\text{Vis}(x) \cap \text{Vis}(z) \neq \emptyset\).

4. For \(x \in X^\mathbb{N}\), any subsequence \(z'\) of \(z \in \text{Vis}(x)\) is also in \(\text{Vis}(x)\).

5. Any subsequence \(x'\) of \(x \in X^\mathbb{N}\) satisfies \(\text{Vis}(x) \subset \text{Vis}(x')\).

Indeed, (1) and (2), follow from the definitions. Notice that for any subsequences \(x' \subset x\) and \(y' \subset y\), it holds

\[
\langle x | y \rangle \leq \langle x' | y' \rangle. \tag{12}
\]

In particular, any subsequence of a sequence converging at infinity also converges at infinity. (4) and (5) follow from (12). For (3), we will see that on Teichmüller space \(\mathcal{T}\) equipped with the Teichmüller distance, the relation “visually indistinguishable” does not define an equivalence relation on \(\text{Sq}^\infty(\mathcal{T})\), when the base surface is neither a torus with one hole nor a sphere with four holes (cf. Section 6.4).
2.3. Asymptotically conservative.

For metric spaces $X$ and $Y$, we define

$$AC(X, Y) = \{ \omega \in Y^X \mid \omega \text{ is asymptotically conservative} \}$$

(for the definition, see Section 1.2). Set $AC(X) = AC(X, X)$.

**Proposition 2.1.** Let $\omega \in AC(X, Y)$. For a sequence $x \in USq(X)$, $x \in Sq^\infty(X)$ if and only if $\omega(x) \in Sq^\infty(Y)$.

**Proof.** Let $x \in USq(X)$. Suppose first that $x \in Sq^\infty(X)$. From (1) of Remark 2.1, $x \in Vis(x)$. Since $\omega$ is asymptotically conservative, $\omega(x) \in \omega(Vis(x)) \subset Vis(\omega(x))$ and $\omega(x) \in Sq^\infty(Y)$.

Conversely, assume that $\omega(x) \in Sq^\infty(Y)$. Since $\omega(x) \in Vis(\omega(x))$, from the definition of asymptotically conservative mappings, we have $x \in Vis(x)$ and hence $x \in Sq^\infty(X)$. \qed

**Proposition 2.2 (Composition in AC).** Let $X$, $Y$ and $Z$ be metric spaces. For $\omega_1 \in AC(Y, Z)$ and $\omega_2 \in AC(X, Y)$, $\omega_1 \circ \omega_2 \in AC(X, Z)$.

**Proof.** Let $x \in USq(X)$. Then,

$$\omega_1 \circ \omega_2(Vis(x)) \subset \omega_1(Vis(\omega_2(x))) \subset Vis(\omega_1 \circ \omega_2(x)).$$

Let $z \in USq(X)$ with $\omega_1 \circ \omega_2(z) \in Vis(\omega_1 \circ \omega_2(x))$. Since $\omega_1$ is asymptotically conservative, $\omega_2(z) \in Vis(\omega_2(x))$. Since $\omega_2$ is also asymptotically conservative again, we have $z \in Vis(x)$.

\qed

2.4. Remark on closeness.

Recall that two mappings $\omega_1, \omega_2 \in Y^X$ are close at infinity if for any $x^1, x^2 \in Sq^\infty(X)$, it holds $Vis(\omega_1(x^1)) = Vis(\omega_2(x^2))$ whenever $Vis(x^1) = Vis(x^2)$. In particular, such $\omega_1$ and $\omega_2$ satisfy

$$Vis(\omega_1(x)) = Vis(\omega_2(x))$$

for all $x \in Sq^\infty(X)$.

**Proposition 2.3 (Composition and closeness).** Let $X$, $Y$ and $Z$ be metric spaces. Let $\omega_1, \omega'_1 \in Z^Y$ and $\omega_2, \omega'_2 \in Y^X$. If $\omega_i$ and $\omega'_i$ are close at infinity for $i = 1, 2$, $\omega_1 \circ \omega_2$ is close to $\omega'_1 \circ \omega'_2$ at infinity.

**Proof.** Let $x^1, x^2 \in Sq^\infty(X)$ with $Vis(x^1) = Vis(x^2)$. By definition, $Vis(\omega_2(x^1)) = Vis(\omega'_2(x^2))$, and hence $Vis(\omega_1 \circ \omega_2(x^1)) = Vis(\omega'_1 \circ \omega'_2(x^2))$. \qed

2.5. Asymptotic surjectivity and closeness at infinity.

A mapping $\omega \in Y^X$ is said to be asymptotically surjective if for any $y \in Sq^\infty(Y)$, there is $x \in Sq^\infty(X)$ with $Vis(y) = Vis(\omega(x))$. Let

$$AC_{as}(X, Y) = \{ \omega \in AC(X, Y) \mid \omega \text{ is asymptotically surjective} \}.$$
**Proposition 2.4.** Let $\omega \in \text{AC}_{\text{as}}(X,Y)$. For $x^1, x^2 \in \text{Sq}^\infty(X)$, if $\text{Vis}(x^2) \subset \text{Vis}(x^1)$, then $\text{Vis}(\omega(x^2)) \subset \text{Vis}(\omega(x^1))$. In particular, if $x^1$ and $x^2$ are asymptotic, so are $\omega(x^2)$ and $\omega(x^1)$.

**Proof.** Let $y \in \text{Vis}(\omega(x^2))$. Since $\omega$ is asymptotically surjective, there is $z \in \text{Sq}^\infty(X)$ such that $\text{Vis}(y) = \text{Vis}(\omega(z))$. Since $\omega$ is asymptotically conservative and $\omega(x^2) \in \text{Vis}(y) = \text{Vis}(\omega(z))$, we have $x^2 \in \text{Vis}(z)$ and $z \in \text{Vis}(x^2) \subset \text{Vis}(x^1)$.

Therefore, $x^1 \in \text{Vis}(z)$ (cf. (2) of Remark 2.1). Hence we deduce

$$\omega(x^1) \in \omega(\text{Vis}(z)) \subset \text{Vis}(\omega(z)) = \text{Vis}(y),$$

and $y \in \text{Vis}(\omega(x^1))$. □

**Proposition 2.5 (Composition of mappings in AC$_{\text{as}}$).** For $\omega_1 \in \text{AC}_{\text{as}}(Y,Z)$ and $\omega_2 \in \text{AC}_{\text{as}}(X,Y)$, we have $\omega_1 \circ \omega_2 \in \text{AC}_{\text{as}}(X,Z)$.

**Proof.** Let $z \in \text{Sq}^\infty(Z)$. By definition, there are $y \in \text{Sq}^\infty(Y)$ and $x \in \text{Sq}^\infty(X)$ such that $\text{Vis}(z) = \text{Vis}(\omega_1(y))$ and $\text{Vis}(y) = \text{Vis}(\omega_2(x))$. Since $\omega_1$ is asymptotically surjective again, from Proposition 2.4, we conclude

$$\text{Vis}(z) = \text{Vis}(\omega_1(y)) = \text{Vis}(\omega_1(\omega_2(x))) = \text{Vis}(\omega_1 \circ \omega_2(x))$$

and hence $\omega_1 \circ \omega_2$ is asymptotically surjective. □

**Proposition 2.6 (Closeness is an equivalence relation on AC$_{\text{as}}$).** Let $X$ and $Y$ be metric spaces. The relation “closeness at infinity” is an equivalence relation on AC$_{\text{as}}(X,Y)$.

**Proof.** Let $x^1, x^2 \in \text{Sq}^\infty(X)$. Suppose $x^1$ and $x^2$ are asymptotic.

**(Reflexive law)** This follows from Proposition 2.4.

**(Symmetric law)** Take two mappings $\omega_1, \omega_2 \in \text{AC}_{\text{as}}(X,Y)$. Since $\omega_1$ is close to $\omega_2$ at infinity, $\text{Vis}(\omega_1(x^1)) = \text{Vis}(\omega_2(x^2))$. By interchanging the roles of $x^1$ and $x^2$, $\text{Vis}(\omega_2(x^1)) = \text{Vis}(\omega_1(x^2))$. This means that $\omega_2$ is close to $\omega_1$ at infinity.

**(Transitive law)** Take three mappings $\omega_1, \omega_2, \omega_3 \in \text{AC}_{\text{as}}(X,Y)$. Suppose that $\omega_i$ is close to $\omega_{i+1}$ at infinity ($i = 1, 2$). Then, from (13),

$$\text{Vis}(\omega_1(x^1)) = \text{Vis}(\omega_2(x^1)) = \text{Vis}(\omega_3(x^2))$$

and hence, $\omega_1$ is close to $\omega_3$ at infinity. □

### 2.6. Invertibility and asymptotic quasi-inverse.

Define

$$\text{AC}^{\text{inv}}(X,Y) = \{\omega \in \text{AC}(X,Y) \mid \omega \text{ is invertible}\}$$
Since \( \omega \) (for the definition, see Section 1.2). Set \( AC_{\text{inv}}(X) = AC_{\text{inv}}(X,X) \) as the introduction. Notice that any \( \omega \in AC_{\text{inv}}(X,Y) \) admits an asymptotic quasi-inverse \( \omega' \in AC_{\text{inv}}(Y,X) \), and \( \omega \) is also an asymptotic quasi-inverse of \( \omega' \).

**Proposition 2.7** (Invertibility implies asymptotic-surjectivity). For any metric spaces \( X \) and \( Y \), \( AC_{\text{inv}}(X,Y) \subset AC_{\text{as}}(X,Y) \).

**Proof.** Let \( \omega \in AC_{\text{inv}}(X,Y) \). Let \( \omega' \) be an asymptotic quasi-inverse of \( \omega \). Let \( y \in Sq^\infty(Y) \). Set \( x = \omega'(y) \). Since \( \omega' \) is asymptotically conservative, \( x \in Sq^\infty(X) \). Since \( \omega \circ \omega' \) is close to the identity mapping on \( Y \) at infinity, from (13),

\[
Vis(\omega(x)) = Vis(\omega \circ \omega'(y)) = Vis(y).
\]

Therefore, we conclude \( \omega \in AC_{\text{as}}(X,Y) \). \( \square \)

**Proposition 2.8** (Composition of mappings in \( AC_{\text{inv}} \)). For \( \omega_1 \in AC_{\text{inv}}(Y,Z) \) and \( \omega_2 \in AC_{\text{inv}}(X,Y) \), we have \( \omega_1 \circ \omega_2 \in AC_{\text{inv}}(X,Z) \).

**Proof.** Let \( \omega_1' \) be an asymptotic quasi-inverse of \( \omega_1 \) for \( i = 1, 2 \). Suppose \( z^1, z^2 \in Sq^\infty(Z) \) are asymptotic. From Propositions 2.1 and 2.2, \( x^i = \omega_2 \circ \omega_1'(z^1) \in Sq^\infty(X) \) for \( i = 1, 2 \). Since \( \omega_2 \circ \omega_2' \) is close to the identity mapping on \( Y \),

\[
Vis(\omega_2(z^1)) = Vis(\omega_2(\omega_2' \circ \omega_1'(z^1))) = Vis(\omega_2 \circ \omega_2'(\omega_1'(z^1))) = Vis(\omega_1'(z^1)).
\]

From Proposition 2.7, \( \omega_1' \) is asymptotically surjective, and from Proposition 2.4, \( \omega_1'(z^1) \) and \( \omega_1'(z^2) \) are asymptotic. Therefore, \( Vis(\omega_2(z^1)) = Vis(\omega_1'(z^2)) \). Since \( \omega_1 \) is also asymptotically surjective, by applying Proposition 2.4 again, we have

\[
Vis(\omega_1 \circ \omega_2(z^1)) = Vis(\omega_1(\omega_2(z^1)))
\]

\[
= Vis(\omega_1(\omega_1'(z^2))) = Vis(\omega_1 \circ \omega_1'(z^2)) = Vis(z^2)
\]

since \( \omega_1 \circ \omega_1' \) is asymptotically close to the identity mapping on \( Z \) at infinity. Therefore,

\[
Vis((\omega_1 \circ \omega_2) \circ (\omega_2 \circ \omega_1')(z^1)) = Vis(\omega_1 \circ \omega_2(z^1)) = Vis(z^2),
\]

which means that \( (\omega_1 \circ \omega_2) \circ (\omega_2 \circ \omega_1') \) is close to the identity mapping on \( Z \) at infinity. By the same argument, we can see that \( (\omega_1 \circ \omega_1') \circ (\omega_1 \circ \omega_2) \) is close to the identity mapping on \( X \). Therefore, \( \omega_2' \circ \omega_1' \) is an asymptotic quasi-inverse of \( \omega_1 \circ \omega_2 \) and \( \omega_1 \circ \omega_2 \in AC_{\text{inv}}(X,Z) \). \( \square \)

**Proposition 2.9** (Stability of \( AC_{\text{inv}} \) in \( AC_{\text{as}} \)). Let \( \omega_1, \omega_2 \in AC_{\text{as}}(X,Y) \). Suppose that \( \omega_1 \) and \( \omega_2 \) are close at infinity. If \( \omega_1 \in AC_{\text{inv}}(X,Y) \), so is \( \omega_2 \). In addition, any asymptotic quasi-inverse of \( \omega_1 \) is also that of \( \omega_2 \).

**Proof.** Let \( \omega_1' \) be an asymptotic quasi-inverse of \( \omega_1 \). Suppose \( x^1, x^2 \in Sq^\infty(X) \) are asymptotic. Since \( \omega_1 \) and \( \omega_2 \) are close at infinity,

\[
Vis(\omega_1(x^1)) = Vis(\omega_2(x^2)).
\]
Since \( \omega'_1 \) is asymptotically surjective, by Proposition 2.4, we have

\[
\text{Vis}(\omega'_1 \circ \omega_2(x^2)) = \text{Vis}(\omega'_1 \circ \omega_1(x^1)) = \text{Vis}(x^1).
\]  
(14)

Suppose that \( y^1, y^2 \in \text{USq}(Y) \) are asymptotic. Since \( \omega'_1 \) is asymptotically surjective again,

\[
\text{Vis}(\omega'_1(y^1)) = \text{Vis}(\omega'_1(y^2)).
\]

Since \( \omega_1 \) and \( \omega_2 \) are close at infinity, we deduce

\[
\text{Vis}(\omega_2 \circ \omega'_1(y^1)) = \text{Vis}(\omega_1 \circ \omega'_1(y^2)) = \text{Vis}(y^2).
\]  
(15)

From (14) and (15), we conclude that \( \omega'_1 \) is an asymptotic quasi-inverse of \( \omega_2 \), and hence \( \omega_2 \in \text{AC}_{\text{inv}}(X,Y) \). \( \square \)

### 2.7. Monoids and Semigroup congruence.

We have defined three kinds of classes of mappings between metric spaces. By definition and Proposition 2.7, the relation of the classes is given as

\[
\text{AC}_{\text{inv}}(X,Y) \subset \text{AC}_{\text{as}}(X,Y) \subset \text{AC}(X,Y)(\subset Y^X)
\]  
(16)

for metric spaces \( X \) and \( Y \).

The following theorem follows from Propositions 2.2, 2.5, and 2.8.

**Theorem 2.1.** \( \text{AC}(X) \) admits a canonical monoid structure with respect to the composition of mappings. The identity element of \( \text{AC}(X) \) is the identity mapping on \( X \). In addition, \( \text{AC}_{\text{as}}(X) \) and \( \text{AC}_{\text{inv}}(X) \) are submonoids of \( \text{AC}(X) \).

Let \( G \) be a semigroup. A *semigroup congruence* is an equivalence relation \( \sim \) on \( G \) with the property that for \( x, y, z, w \in G \), \( x \sim y \) and \( z \sim w \) imply \( xz \sim yw \). Then, the congruence classes

\[
G/\sim = \{ [g] \mid g \in G \}
\]

is also a semigroup with the product \( [g_1][g_2] = [g_1g_2] \). We call \( G/\sim \) the *quotient semigroup* of \( G \) with the semigroup congruence \( \sim \).

We define a relation on \( \text{AC}_{\text{as}}(X) \) by using the closeness at infinity. Namely, for two \( \omega_1 \) and \( \omega_2 \in \text{AC}_{\text{as}}(X) \), \( \omega_1 \) is *equivalent* to \( \omega_2 \) if \( \omega_1 \) is close to \( \omega_2 \) at infinity. From Propositions 2.3 and 2.6, this relation is a subgroup congruence on \( \text{AC}_{\text{as}}(X) \). We define the quotient monoid by

\[
\mathfrak{AC}_{\text{as}}(X) = \text{AC}_{\text{as}}(X)/(\text{close at infinity}).
\]

We also define the semigroup congruence on \( \text{AC}_{\text{inv}}(X) \) in the same procedure, and obtain the quotient semigroup by

\[
\mathfrak{AC}(X) = \text{AC}_{\text{inv}}(X)/(\text{close at infinity}).
As a result, we summarize as follows.

**Theorem 2.2 (Group $\mathcal{AC}(X)$).** Let $X$ be a metric space. Then, the quotient semigroup $\mathcal{AC}(X)$ is a group. The identity element of $\mathcal{AC}(X)$ is the congruence class of the identity mapping, and the inverse of the congruence class $[\omega]$ of $\omega \in \mathcal{AC}_{\text{inv}}(X)$ is the congruence class of an asymptotic quasi-inverse of $\omega$.

**Corollary 2.1.** Let $X$ and $Y$ be metric spaces. Let $\omega \in \mathcal{AC}_{\text{inv}}(X,Y)$ and $\omega'$ an asymptotic quasi-inverse of $\omega$. Then, the mapping

$$AC_{\text{as}}(X) \ni f \mapsto \omega \circ f \circ \omega' \in AC_{\text{as}}(Y),$$

$$AC_{\text{inv}}(X) \ni f \mapsto \omega \circ f \circ \omega' \in AC_{\text{inv}}(Y)$$

induces isomorphisms

$$\mathcal{AC}_{\text{as}}(X) \ni [f] \mapsto [\omega \circ f \circ \omega'] \in \mathcal{AC}_{\text{as}}(Y),$$

$$\mathcal{AC}(X) \ni [f] \mapsto [\omega \circ f \circ \omega'] \in \mathcal{AC}(Y).$$

Notice from Proposition 2.9 that any equivalence class in $\mathcal{AC}(X)$ consists of elements in $\mathcal{AC}_{\text{inv}}(X)$. Hence, we conclude the following.

**Theorem 2.3.** The inclusion $\mathcal{AC}_{\text{inv}}(X) \hookrightarrow \mathcal{AC}_{\text{as}}(X)$ induces a monoid monomorphism

$$\mathcal{AC}(X) \hookrightarrow \mathcal{AC}_{\text{as}}(X).$$

In other words, $\mathcal{AC}(X)$ is a subgroup of $\mathcal{AC}_{\text{as}}(X)$.

The monomorphism (17) could be an isomorphism. The author does not know whether it is true or not in general.

### 3. Comparison with the coarse geometry.

#### 3.1. Backgrounds from the coarse geometry.

##### 3.1.1. Parallelism.

Two mappings $\omega, \xi \in Y^X$ between metric spaces are said to be *parallel* if and only if

$$\sup_{x \in X} d_Y(\omega(x), \xi(x)) < \infty$$

(cf. Section 1.1 in [15]). The “parallelism” defines an equivalence relation on any subclass in $Y^X$. If two mappings $\omega, \xi \in Y^X$ are parallel,

$$\sup_{x, z \in X} |\langle \omega(x) | \omega(z) \rangle_{y_0} - \langle \xi(x) | \xi(z) \rangle_{y_0}| < \infty,$$

$$\sup_{x \in X, y \in Y} |\langle \omega(x) | y \rangle_{y_0} - \langle \xi(x) | y \rangle_{y_0}| < \infty.$$  

From (19), for any sequence $\mathbf{x} \in \text{USq}(X)$, it holds
Vis(ω(x)) = Vis(ξ(x)).

3.1.2. Quasi-isometries.
A mapping ω ∈ Y^X satisfies
\[ d_Y(ω(x_1), ω(x_2)) \leq K d_X(x_1, x_2) + D \]
for all x_1, x_2 ∈ X, we call ω a coarsely (K, D)-Lipschitz. If ω ∈ Y^X satisfies
\[ \frac{1}{K} d_X(x_1, x_2) - D \leq d_Y(ω(x_1), ω(x_2)) \]
for all x_1, x_2 ∈ X, we call ω a coarsely co-(K, D)-Lipschitz. A mapping ω ∈ Y^X is said to be \((K, D)\)-quasi-isometry if ω is both coarsely (K, D)-Lipschitz and coarsely co-(K, D)-Lipschitz. A mapping ω ∈ Y^X is D-cobounded if the D-neighborhood of the image of X under ω coincides with Y. A quasi-inverse of a mapping ω ∈ Y^X is a mapping ω′ ∈ X^Y such that ω′ ∘ ω and ω ∘ ω′ are parallel to the identity mappings. Usually, quasi-inverses are assumed to be quasi-isometry. However, we do not assume so for our purpose. Any quasi-inverse of a quasi-isometry is automatically a quasi-isometry. Any cobounded quasi-isometry admits a quasi-inverse.

Let QI(X, Y) be the set of cobounded quasi-isometries from X to Y. Then, QI(X) = QI(X, X) admits a monoid structure defined by composition. One can easily check that the parallelism is a subgroup congruence on QI(X). Hence we have a quotient group defined by
\[ QI(X) = QI(X)/(\text{parallelism}). \]

The group QI(X) is a central object in the coarse geometry (cf. Section I.8 in [6]).

3.2. Asymptotically conservative mappings in the coarse geometry.
Recall that a mapping ω ∈ Y^X is called weakly asymptotically conservative if ω(Vis(x)) ⊂ Vis(ω_1(x)) for all sequence x ∈ USq(X). Let
\[ AC^w(X, Y) = \{ ω ∈ Y^X | ω \text{ is weakly asymptotically conservative} \}. \]

Set AC^w(X) = AC^w(X, X). From (16), we have
\[ AC_{\text{inv}}(X, Y) ⊂ AC_{as}(X, Y) ⊂ AC(X, Y) ⊂ AC^w(X, Y)(⊂ Y^X) \]
for metric spaces X and Y.

A class M of Y^X is said to be stable under parallelism if a mapping ω ∈ Y^X is parallel to some ω ∈ M, then ω ∈ M.

**Proposition 3.1 (Stability under parallelism).** All classes AC^w(X, Y), AC(X, Y), AC_{as}(X, Y) and AC_{inv}(X, Y) are stable under the parallelism.

**Proof.** Suppose that ω, ξ ∈ Y^X are parallel.

(i) Suppose ξ ∈ AC^w(X, Y). Let x ∈ USq(X). Take z ∈ Vis(x). Since ξ is weakly
asymptotically conservative, \( \xi(z) \in \xi(\text{Vis}(x)) \subset \text{Vis}(\xi(x)) \). From (18), we have \( \omega(z) \in \text{Vis}(\omega(x)) \) and \( \omega(\text{Vis}(x)) \subset \text{Vis}(\omega(x)) \). Hence \( \omega \in \text{AC}^w(X,Y) \).

(ii) Suppose \( \xi \in \text{AC}(X,Y) \). From the argument above, \( \omega \in \text{AC}^w(X,Y) \). Suppose a sequence \( z \) in \( X \) satisfies \( \omega(z) \in \text{Vis}(\omega(x)) \). From (18) again, \( \xi(z) \in \text{Vis}(\xi(x)) \). Since \( \xi \) is asymptotically conservative, \( z \in \text{Vis}(x) \) and \( \omega \in \text{AC}(X,Y) \).

(iii) Suppose \( \xi \in \text{AC}_{as}(X,Y) \). Let \( y \in \text{Sq}^{\infty}(Y) \). Take \( x \in \text{Sq}^{\infty}(X) \) such that \( \text{Vis}(y) = \text{Vis}(\xi(x)) \). From (20), we have

\[
\text{Vis}(y) = \text{Vis}(\xi(x)) = \text{Vis}(\omega(x)),
\]

which implies \( \omega \in \text{AC}_{as}(X,Y) \).

(iv) Suppose \( \xi \in \text{AC}_{inv}(X,Y) \). Let \( \xi' \in \text{AC}_{inv}(Y,X) \) be an asymptotic quasi-inverse of \( \xi \). Let \( x^1, x^2 \in \text{Sq}^{\infty}(X) \) with \( \text{Vis}(x^1) = \text{Vis}(x^2) \). Since \( \xi' \) is asymptotically surjective, by Proposition 2.4 and (20), we have

\[
\text{Vis}(x^1) = \text{Vis}(\xi' \circ \xi(x^2)) = \text{Vis}(\xi' \circ \omega(x^2)),
\]

and hence \( \xi' \circ \omega \) is close to the identity mapping on \( X \).

To prove the converse, we notice from the above that \( \omega \) is asymptotically surjective and asymptotically conservative from Proposition 2.7. Then, by Propositions 2.2 and 2.5, \( \omega \circ \xi' \) is also asymptotically surjective and asymptotically conservative.

Let \( y^1, y^2 \in \text{Sq}^{\infty}(Y) \) with \( \text{Vis}(y^1) = \text{Vis}(y^2) \). Then, since \( \xi'(y^i) \in \text{Sq}^{\infty}(X) \) for \( i = 1, 2 \), by Proposition 2.4 and (20) again, we have

\[
\text{Vis}(\xi'(y^1)) = \text{Vis}(\xi'(y^2))
\]

and

\[
\text{Vis}(\omega \circ \xi'(y^1)) = \text{Vis}(\xi \circ \xi'(y^2)) = \text{Vis}(y^2),
\]

and \( \omega \circ \xi' \) is close to the identity mapping on \( Y \). Therefore, we conclude that \( \omega \in \text{AC}_{inv}(X,Y) \).

The following proposition gives comparisons between items in rows of the correspondence table in the introduction (cf. Table 1).

**Proposition 3.2 (Comparison).**

1. A cobounded asymptotically conservative mapping is asymptotically surjective.

2. If two asymptotically surjective, asymptotically conservative mappings are parallel, they are close at infinity.

3. If \( \omega \in Y^X \) admits a quasi-inverse \( \omega' \in X^Y \) in the sense of the coarse geometry, \( \omega' \circ \omega \) and \( \omega \circ \omega' \) are close to the identity mappings on \( X \) and \( Y \) at infinity, respectively.

**Proof.** (1) Let \( y = \{y_n\}_{n \in \mathbb{N}} \in \text{Sq}^{\infty}(Y) \). Take \( x_n \in X \) such that \( d_Y(\omega(x_n), y_n) \leq D_0 \) where \( D_0 > 0 \) is independent of \( n \). Since
for \( y_0 \in Y \), we have \( \omega(x) \in \text{Sq}^\infty(Y) \) and \( x \in \text{Sq}^\infty(X) \) from Proposition 2.1. Since
\[
|\langle \omega(x) | \omega(x) \rangle^Y_{y_0} - \langle y | y \rangle^Y_{y_0}| \leq 2D_0
\]
for every sequence \( z = \{z_n\}_{n \in \mathbb{N}} \in \text{USq}(Y) \), we obtain \( \text{Vis}(y) = \text{Vis}(\omega(x)) \). Thus, \( \omega \) is asymptotically surjective.

(2) Let \( \omega_1, \omega_2 \in \text{AC}_{as}(X,Y) \). Suppose that \( \omega_1 \) is parallel to \( \omega_2 \). Take asymptotic sequences \( x^1, x^2 \in \text{Sq}^\infty(X) \). Since \( \omega_2 \) is asymptotically surjective, from Proposition 2.4 and (20), we conclude that
\[
\text{Vis}(\omega_1(x^1)) = \text{Vis}(\omega_2(x^1)) = \text{Vis}(\omega_2(x^2))
\]
and \( \omega_1 \) and \( \omega_2 \) are close at infinity.

(3) Since the identity mapping is asymptotically surjective and asymptotically conservative, from Proposition 3.1, \( \omega' \circ \omega \) and \( \omega \circ \omega' \) are also asymptotically surjective and asymptotically conservative. Hence, from above (2), we conclude what we wanted. \( \square \)

### 3.3. Criterion for subclasses to be compatible in \( \mathcal{AC} \).

Let \( M \) be a subclass of \( X^X \). Consider the following conditions.

(S1) \( M \) is a monoid with the operation defined by composition, and the parallelism is a semigroup congruence on \( M \).

(S2) Any element in \( M \) is cobounded.

(S3) Any element in \( M \) admits a quasi-inverse in \( M \) in the coarse geometry.

Notice that the condition (S3) implies (S2). Under the condition (S1), the quotient set \( M = M/\text{parallelism} \) has a canonical monoid structure, and if \( M \) satisfies all conditions, \( M \) has a canonical group structure. For instance, the monoid \( \text{QI}(X) \) of cobounded self quasi-isometries on \( X \) satisfies all conditions above.

**Proposition 3.3 (Criterion).** Let \( M \) be a subclass of \( X^X \) satisfying the condition (S1) posed above.

1. Suppose in addition that \( M \) satisfies the condition (S2) posed above. When \( M \subset \text{AC}(X) \), then \( M \subset \text{AC}_{as}(X) \). The inclusion \( M \hookrightarrow \text{AC}_{as}(X) \) induces maps (as sets)
\[
\mathcal{M} = M/\text{parallelism} \rightarrow M/\text{close at infinity} \rightarrow \mathcal{AC}_{as}(X)
\]
such that the composition of the maps \( M \hookrightarrow \mathcal{AC}_{as}(X) \) is a monoid homomorphism.

2. Suppose that \( M \) satisfies the condition (S3) posed above. When \( M \subset \text{AC}^w(X) \), then \( M \subset \text{AC}_{inv}(X) \). The inclusion \( M \hookrightarrow \text{AC}_{inv}(X) \) induces maps (as sets)
\[
\mathcal{M} \rightarrow M/\text{close at infinity} \rightarrow \mathcal{AC}(X)
\]
such that the composition of the maps \( M \hookrightarrow \mathcal{AC}(X) \) is a group homomorphism.
Proof. (1) From (1) of Proposition 3.2, $M \subset AC_{as}(X)$, and the “closeness at infinity” is an equivalence relation on $M$ by Proposition 2.6. Therefore, from (2) of Proposition 3.2, we have well-defined mappings between quotient sets

$$M \to M/(\text{close at infinity})$$
$$\to AC_{as}(X)/(\text{close at infinity}) = \mathfrak{AC}_{as}(X).$$

From (2) of Proposition 3.2 again, the composition

$$M \to \mathfrak{AC}_{as}(X)$$

induces a monoid homomorphism.

(2) We first check that $M \subset AC(X)$. Let $\omega \in M$ and $\omega' \in M$ a quasi-inverse of $\omega$. Let $z$ be an unbounded sequence in $X$ with $\omega(z) \in Vis(\omega(x))$. Since $\omega'$ is weakly asymptotically conservative, we have

$$\omega' \circ \omega(z) \in \omega'(Vis(\omega(x))) \subset Vis(\omega' \circ \omega(x)) = Vis(x)$$

and

$$|\langle x | z \rangle_{x_0}^{X} - \langle \omega' \circ \omega(x) | \omega' \circ \omega(z) \rangle_{x_0}^{X}| = O(1)$$

for all $x \in x$ and $z \in z$, since is parallel to the identity mapping on $X$ and infinity from Proposition 2.4, Proposition 3.1 and (2) of Proposition 3.2. Therefore, $z \in Vis(x)$, and $\omega$ is asymptotically conservative.

Then, by applying the same argument as above, we have a mappings

$$M \to M/(\text{close at infinity}) \to \mathfrak{AC}(X)$$

such that the composition

$$M \to \mathfrak{AC}(X)$$

is a monoid homomorphism. From (3) of Proposition 3.2, any quasi-inverse of $\omega \in M$ corresponds to a asymptotic quasi-inverse of $\omega$ in $AC_{inv}(X)$ under the inclusion $M \subset AC_{inv}(X)$. Hence (24) is a group homomorphism. □

Corollary 3.1 (Criterion for quasi-isometries). Let $X$ be a metric space. If $QI(X) \subset AC^{w}(X)$, then the inclusion $QI(X) \hookrightarrow AC_{inv}(X)$ induces a group homomorphism

$$QI(X) \to \mathfrak{AC}(X).$$

3.4. Remarks.

The notions of quasi-isometries and the asymptotically conservation are independent in general:

1. An asymptotically conservative mapping need not be a quasi-isometry. Indeed, let
\[ X = [0, \infty) \text{ with } d_X(x_1, x_2) = |x_1 - x_2|. \] Let \( x_0 = 0 \) be the reference point. Then, any increasing function \( f : X \to X \) with \( f(0) = 0 \) is asymptotically conservative.

2. Meanwhile, little is known as to when quasi-isometries become asymptotically conservative. For instance, any rough homothety is asymptotically conservative (cf. (6)). Actually, it follows from the following fact that any rough homothety \( \omega \) satisfies
\[
|K \langle x_1 | x_2 \rangle^X_{x_0} - \langle \omega(x_1) | \omega(x_2) \rangle^{Y}_{y_0}| \leq D' \quad (x_1, x_2 \in X)
\]
for some \( K, D' > 0 \).

3. In general, the homomorphism (23) is not injective: Take an increasing sequence \( \{a_n\}_{n=0}^{\infty} \subset \mathbb{Z} \) such that \( a_0 = 0 \) and \( a_{n+1} - a_n \to \infty \). Consider a graph in \( \mathbb{R}^2 \) defined by
\[
X = [0, \infty) \times \{-1, 1\} \cup \bigcup_{n=0}^{\infty} \{a_n\} \times [-1, 1] \subset \mathbb{R}^2.
\]
Then, \( X \) is a metric space equipped with the graph metric such that the length of the edges are measured by the Euclidean metric. Let \( x_0 = (0, 0) \in X \) as a base point. In this case, \( \text{Sq}^\infty(X) = \text{USq}(X) \) and \( \text{Vis}(x) = \text{Sq}^\infty(X) \) for all \( x \in \text{USq}(X) \). Hence, \( \mathfrak{AC}(X) \) is the trivial group. Furthermore, \( X \) is WBGP in the sense of Section 4, and any quasi-isometry on \( X \) is weakly asymptotically conservative (cf. Proposition 4.2). Thus, we have a homomorphism (23) in this case. Define an isometry \( r \) on \( X \) by \( r(x, y) = (x, -y) \). Then, \( r \) is not parallel to the identity mapping \( \text{id}_X \), and \( \text{id}_X \) and \( r \) are contained in the different classes in \( \mathfrak{QI}(X) \).

4. In general, the homomorphism (23) is not surjective: When \( X = Y = \mathbb{D} \) equipped with the Poincaré distance, \( \mathfrak{AC}(\mathbb{D}) \) is canonically isomorphic to the group of homeomorphism on \( \partial \mathbb{D} \) via extensions. Hence, we can find an invertible asymptotically conservative mapping on \( \mathbb{D} \) which is not parallel to any cobounded quasi-isometry.

4. Relaxation of the definition.

4.1. Metric spaces which are WBGP.
Let \( X \) be a metric space. For \( x \in \text{USq}(X) \), we define
\[
\text{sub}^\infty(x) = \{x' \mid \text{subsequences of } x \text{ with } x' \in \text{Sq}^\infty(X)\}.
\]
A metric space \( X \) is called \textit{well-behaved at infinity with respect to the Gromov product (WBGP)} if \( \text{sub}^\infty(x) \neq \emptyset \) for all \( x \in \text{USq}(X) \).

**Examples.** The following are metric spaces which are WBGP:

1. Proper geodesic spaces that are Gromov-hyperbolic (of infinite diameter).
2. Teichmüller space equipped with the Teichmüller distance.
3. The Cayley graphs for pairs \((G, S)\) of finitely generated infinite group \( G \) and a finite system \( S \) of symmetric generators.
Indeed, (1) follows from the compactness of the Gromov’s bordification (compactification) (e.g. Proposition 2.14 in [20]). (2) is proven at Proposition 6.1.

We check (3). Let $\Sigma(G,S)$ be the Cayley graph. Let $F_S$ be the free group generated by $S$. There is a canonical surjection $\pi: \Sigma(F_S,S) \to \Sigma(G,S)$ induced by the quotient map $F_S \to G$. Let $x = \{x_n\}_{n \in \mathbb{N}}$ be an unbounded sequence in $\Sigma(G,S)$. Take $y_n \in \Sigma(F_S,S)$ such that $\pi(y_n) = x_n$ and $d_G(id,x_n) = d_{F_S}(id,y_n)$. Then $y = \{y_n\}_{n \in \mathbb{N}}$ is an unbounded sequence in $G(F_S,S)$, and hence we can find a subsequence $y' = \{y_{n_j}\}_j$ of $y$ such that $(y' | y') = \infty$ from (1) above. Since the projection $\pi$ is 1-Lipschitz, $d(x_{n_j}, x_{n_k}) \leq d(y_{n_j}, y_{n_k})$ and we have

$$\langle \pi(y') | \pi(y') \rangle_{id} \geq \langle y' | y' \rangle_{id} = \infty.$$ 

Thus $\pi(y')$ is a desired subsequence of $x$.

4.2. Properties.

We shall give a couple of properties of metric spaces which are WBGP.

**Proposition 4.1.** Suppose a metric space $X$ is WBGP. For any $x \in USq(X)$, we have

$$\text{Vis}(x) = \bigcap_{x' \in \text{sub}^\infty(x)} \text{Vis}(x').$$

**Proof.** From (5) of Remark 2.1, we have

$$\text{Vis}(x) \subseteq \bigcap_{x' \in \text{sub}^\infty(x)} \text{Vis}(x').$$

Let $z \in USq(X) - \text{Vis}(x)$. Then $\langle z | x \rangle < \infty$. Since $X$ is WBGP, we can take subsequences $z' \subset z$ and $x' \in \text{sub}^\infty(x)$ such that $\langle z' | x' \rangle < \infty$. Hence $z' \notin \text{Vis}(x')$ and $z \notin \text{Vis}(x')$ from (4) of Remark 2.1. Therefore, $z \notin \bigcap_{x' \in \text{sub}^\infty(x)} \text{Vis}(x').$ 

**Proposition 4.2** (Relaxation of the definition). Let $X$ and $Y$ be metric spaces which are WBGP.

1. A mapping $\omega \in Y^X$ is in $\text{AC}^w(X,Y)$ if and only if for any $x, z \in \text{Sq}^\infty(X)$, $\langle \omega(x) | \omega(z) \rangle = \infty$ whenever $\langle x | z \rangle = \infty$.

2. A mapping $\omega \in Y^X$ is in $\text{AC}(X,Y)$ if and only if for any $x, z \in \text{Sq}^\infty(X)$, $\langle \omega(x) | \omega(z) \rangle = \infty$ implies $\langle x | z \rangle = \infty$, and vice versa.

**Proof.** (1) The condition is paraphrased that $\omega(\text{Vis}(x)) \subseteq \text{Vis}(\omega(x))$ for all $x \in \text{Sq}^\infty(X)$. Hence, the “only if” part follows from the definition. We show the “if” part. Let $x \in USq(X)$. Suppose to the contrary that there is $z \in \text{Vis}(x)$ such that $\omega(z) \notin \text{Vis}(\omega(x))$. From Proposition 4.1, there is $y \in \text{sub}^\infty(\omega(x))$ such that $\omega(z) \notin \text{Vis}(y)$. By taking subsequences $z' \subset \text{sub}^\infty(z)$ and $x' \in \text{sub}^\infty(x)$ respectively, we may assume that $\omega(x') \in \text{sub}^\infty(y) \subset \text{sub}^\infty(\omega(x))$ and $\omega(z') \notin \text{Vis}(\omega(x'))$.

On the other hand, since $x' \in \text{sub}^\infty(x) \subset \text{Sq}^\infty(X)$, from the condition (1), we have $\omega(\text{Vis}(x')) \subseteq \text{Vis}(\omega(x'))$. Hence $z' \notin \text{Vis}(x')$, which implies $z' \notin \text{Vis}(x)$ because
Vis(x) ⊂ Vis(x′) from (5) of Remark 2.1. This contradicts to (4) of Remark 2.1. Thus, we conclude that ω(Vis(x)) ⊂ Vis(ω(x)).

(2) We only show the “if” part. From (1) above, ω ∈ ACw(X, Y). Let z ∈ USq(X) with ω(z) ∈ Vis(ω(x)). Suppose z ∉ Vis(x). From the argument in Proposition 4.1, there is a subsequence z′ ∈ sub∞(z) with z′ ∉ Vis(x). This means that ω(z′) ∉ Vis(ω(x)) from the assumption, and hence ω(z) ∉ Vis(ω(x)) from (4) of Remark 2.1. This is a contradiction. □

From Proposition 3.3, we conclude the following.

COROLLARY 4.1. Let X is a metric space which is WBGP. Let M be a subclass of X satisfying (S1) and (S3) in Section 3.3. Suppose that any ω ∈ M satisfies the condition that ⟨ω(x)|ω(z)⟩ = ∞ whenever ⟨x|z⟩ = ∞ for all x, z ∈ Sq∞(X). Then, M ⊂ ACinv(X) and the inclusion M → ACinv(X) induces a group homomorphism

\[ M/(\text{parallelism}) \to \mathfrak{A}(X). \]

5. Teichmüller theory.

In this section, we recall basics in the Teichmüller theory. For details, the readers can refer to [1], [10], [17] and [18].

5.1. Teichmüller space.

The Teichmüller space \( \mathcal{T} = \mathcal{T}(S) \) of \( S \) is the set of equivalence classes of marked Riemann surfaces \((Y, f)\) where \( Y \) is a Riemann surface of analytically finite type and \( f: \text{Int}(S) \to Y \) is an orientation preserving homeomorphism. Two marked Riemann surfaces \((Y_1, f_1)\) and \((Y_2, f_2)\) are Teichmüller equivalent if there is a conformal mapping \( h: Y_1 \to Y_2 \) which is homotopic to \( f_2 \circ f_1^{-1} \).

Teichmüller space \( \mathcal{T} \) is topologized with a canonical complete distance, called the Teichmüller distance \( d_\mathcal{T} \) (cf. (32)). It is known that the Teichmüller space \( \mathcal{T} = \mathcal{T}(S) \) of \( S \) is homeomorphic to \( \mathbb{R}^{2\times \text{ex}(S)} \).

CONVENTION 5.1. Throughout this paper, we fix a conformal structure \( X \) on \( S \) and consider \( x_0 = (X, \text{id}) \) as the base point of the Teichmüller space \( \mathcal{T} \) of \( S \).

5.2. Measured foliations.

Let \( S \) be the set of homotopy classes of non-trivial and non-peripheral simple closed curves on \( S \). Consider the set of weighted simple closed curves \( \mathcal{WS} = \{ t\alpha \mid t \geq 0, \alpha \in \mathcal{S} \} \), where \( t\alpha \) is the formal product between \( t \geq 0 \) and \( \alpha \in \mathcal{S} \). We embed \( \mathcal{WS} \) into the space \( \mathbb{R}_+^\mathcal{S} \) of non-negative functions on \( S \) by

\[ \mathcal{WS} \ni t\alpha \mapsto [S \ni \beta \mapsto t \cdot i(\alpha, \beta)] \in \mathbb{R}_+^\mathcal{S} \]  \hspace{1cm} (26)

where \( i(\cdot, \cdot) \) is the geometric intersection number on \( S \). The closure \( \mathcal{MF} \) of the image of the mapping (26) is called the space of measured foliations on \( S \). The space \( \mathbb{R}_+^\mathcal{S} \) admits a canonical action of \( \mathbb{R}_{>0} \) by multiplication. The quotient space
\[ \mathcal{PMF} = (\mathcal{MF} - \{0\})/\mathbb{R}_{>0} \subset \mathcal{PR}^S_+ = (\mathbb{R}^S_+ - \{0\})/\mathbb{R}_{>0} \]

is said to be the space of projective measured foliations. By definition, \( \mathcal{MF} \) contains \( \mathcal{WS} \) as a dense subset. The intersection number function on \( \mathcal{WS} \) defined by

\[ \mathcal{WS} \times \mathcal{WS} \ni (t\alpha, s\beta) \mapsto ts i(\alpha, \beta) \]

extends continuously on \( \mathcal{MF} \times \mathcal{MF} \). It is known that \( \mathcal{MF} \) and \( \mathcal{PMF} \) are homeomorphic to \( \mathbb{R}^{2cx(S)} \) and \( S^{2cx(S)-1} \), respectively.

**Normal forms.** Any \( G \in \mathcal{MF} \) is represented by a pair \( (\mathcal{F}_G, \mu_G) \) of a singular foliation \( \mathcal{F}_G \) and a transverse measure \( \mu_G \) to \( \mathcal{F}_G \). The intersection number \( i(G, \alpha) \) with \( \alpha \in S \) is obtained as

\[ i(G, \alpha) = \inf_{\alpha' \sim \alpha} \int_{\alpha'} d\mu_G. \]

The support \( \text{Supp}(G) \) of a measured foliation \( G \) is, by definition, the minimal essential subsurface containing the underlying foliation. A measured foliation is said to be minimal if it intersects any curves in \( S \) in its support.

According to the structure of the underlying foliation, any \( G \in \mathcal{MF} \) has the normal form: Any measured foliation \( G \in \mathcal{MF} \) is decomposed as

\[ G = G_1 + G_2 + \cdots + G_{m_1} + \beta_1 + \cdots + \beta_{m_2} + \gamma_1 + \cdots + \gamma_{m_3} \quad (27) \]

where \( G_i \) is a minimal foliation in its support \( X_i = \text{Supp}(G_i) \), \( \beta_j \) and \( \gamma_k \) are simple closed curves such that each \( \beta_j \) cannot be deformed into any \( X_i \) and \( \gamma_k \) is homotopic to a component of \( \partial X_i \) for some \( i \) (cf. Section 2.4 of [18]). In this paper, we call \( G_i, \beta_j \) and \( \gamma_k \) a minimal component, an essential curve, and a peripheral curve of \( G \) respectively.

**5.3. Null sets of measured foliations.**

For a measured foliation \( G \), we define the null set of \( G \) by

\[ \mathcal{N}_{MF}(G) = \{ F \in \mathcal{MF} \mid i(F, G) = 0 \}. \quad (28) \]

We denote by \( G^\circ \) the measured foliation defined from \( G \) by deleting the foliated annuli associated to the peripheral curves in \( G \). We here call \( G^\circ \) the distinguished part of \( G \) on nullity. Notice that \( (G^\circ)^\circ = G^\circ \). The following might be well-known. However, we give a proof of the proposition for completeness.

**PROPOSITION 5.1 (Null sets and topologically equivalence).** Let \( G, H \in \mathcal{MF} \). Then, the following are equivalent.

1. \( \mathcal{N}_{MF}(G) = \mathcal{N}_{MF}(H) \).
2. \( G^\circ \) is topologically equivalent to \( H^\circ \).

In particular, \( \mathcal{N}_{MF}(G) = \mathcal{N}_{MF}(G^\circ) \).

We say that two measured foliations \( F_1 \) and \( F_2 \) are topologically equivalent if the underlying foliations of \( F_1 \) and \( F_2 \) are modified by Whitehead operations to foliations with
trivalent singularities such that the resulting foliations (without transversal measures) are isotopic (cf. Section 3.1 of [19]).

**Proof.** Suppose (1) holds. We decompose $G$ as (27):

$$G = \sum_{i=1}^{m_1} G_i + \sum_{i=1}^{m_2} \beta_i + \sum_{i=1}^{m_3} \gamma_i.$$  

Since $i(G, H) = 0$, the decomposition of $H$ is represented as

$$H = \sum_{i=1}^{m_1} H_i + \sum_{i=1}^{m_2} a_i \beta_i + \sum_{i=1}^{m_1} \sum_{\gamma \subset \partial X_i} b_\gamma \gamma + H_0 \quad (29)$$

where $H_i$ is either topologically equivalent to $G_i$ or is 0, $a_i, b_\gamma \geq 0$ and Supp($H_0$) $\subset X - \text{Supp}(G)$. In the summation $\sum_{\gamma \subset \partial X_i}$ in (29), $\gamma$ runs over all component of $\partial X_i$. See Proposition 3.2 of Ivanov [18] or Lemma 3.1 of Papadopoulos [38]. Indeed, Ivanov in [18] works under the assumption that each $G_i$ is a stable lamination for some pseudo-Anosov mapping on $X_i$. However, the discussion of his proof can be applied to our case.

If $H_0 \neq 0$, there is an $\alpha \subset S$ with $i(G, \alpha) = 0$ but $i(H_0, \alpha) \neq 0$. Since $\alpha \in N_{MF}(G) = N_{MF}(H)$ from the assumption, this is a contradiction. Hence $H_0 = 0$.

Suppose $a_i = 0$ for some $i$. Since $\beta_i$ is essential, we can find an $\alpha \subset S$ such that $i(G, \alpha) = i(\beta_i, \alpha) = 0$. Such an $\alpha$ satisfies $i(H, \alpha) = a_i i(\beta_i, \alpha) = 0$, which is a contradiction. With the same argument, we can see that $H_i \neq 0$. Thus,

$$G^0 = \sum_{i=1}^{m_1} G_i + \sum_{i=1}^{m_2} \beta_i,$$

$$H^0 = \sum_{i=1}^{m_1} H_i + \sum_{i=1}^{m_2} a_i \beta_i$$

are topologically equivalent.

Suppose (2) holds. Let $F \in N_{MF}(G)$. Consider the decomposition (29) for $F$ instead of $H$, one can easily deduce that $F \in N_{MF}(H)$.

5.4. Extremal length.

Let $X$ be a Riemann surface and let $A$ be a doubly connected domain on $X$. If $A$ is conformally equivalent to a round annulus $\{1 < |z| < R\}$, we define the modulus of $A$ by

$$\text{Mod}(A) = \frac{1}{2\pi} \log R.$$  

**Extremal length** of a simple closed curve $\alpha$ on $X$ is defined by

$$\text{Ext}_X(\alpha) = \inf \left\{ \frac{1}{\text{Mod}(A)} \left| \text{the core curve of } A \subset X \text{ is homotopic to } \alpha \right. \right\}. \quad (30)$$

In [23], Kerckhoff showed that if we define the extremal length of $t\alpha \in WS$ by
\[ \text{Ext}_X(t\alpha) = t^2 \text{Ext}_X(\alpha), \]

then the extremal length function \( \text{Ext}_X \) on \( W\Sigma \) extends continuously to \( \mathcal{MF} \). For \( y = (Y, f) \in \mathcal{T} \) and \( G \in \mathcal{MF} \), we define

\[ \text{Ext}_y(G) = \text{Ext}_Y(f(G)). \]

We define the unit sphere in \( \mathcal{MF} \) by

\[ \mathcal{MF}_1 = \{ F \in \mathcal{MF} \mid \text{Ext}_{x_0}(F) = 1 \}. \]

The projection \( \mathcal{MF} - \{0\} \to \mathcal{PMF} \) induces a homeomorphism \( \mathcal{MF}_1 \to \mathcal{PMF} \).

Kerckhoff’s formula. The Teichmüller distance \( d_T \) is expressed by extremal length, which we call Kerckhoff’s formula:

\[ d_T(y_1, y_2) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{\text{Ext}_{y_2}(\alpha)}{\text{Ext}_{y_1}(\alpha)} \]  \hspace{1cm} (32)

(see [23]).

Minsky’s inequality. Minsky [31] observed the following inequality, which we recently call Minsky’s inequality:

\[ i(F, G)^2 \leq \text{Ext}_y(F) \text{Ext}_y(G) \]  \hspace{1cm} (33)

for \( y \in \mathcal{T} \) and \( F, G \in \mathcal{MF} \). Minsky’s inequality is sharp in the sense that for any \( y \in \mathcal{T} \) and \( F \in \mathcal{MF} \), there is a unique \( G \in \mathcal{MF} \) up to multiplication by a positive constant such that \( i(F, G)^2 = \text{Ext}_y(F) \text{Ext}_y(G) \) (cf. [13]).
5.5. Teichmüller rays.

Let \( x = (X, f) \in \mathcal{T} \) and \([G] \in \mathcal{PMF}\). By the Ahlfors–Bers theorem, we can define an isometric embedding

\[ [0, \infty) \ni t \mapsto R_{G,x}(t) \in \mathcal{T} \]

with respect to the Teichmüller distance by assigning the solution of the Beltrami equation defined by the Teichmüller Beltrami differential

\[ \tanh(t) \frac{|J_{G,x}|}{J_{G,x}} \]

for \( t \geq 0 \). We call \( R_{G,x} \) the Teichmüller (geodesic) ray associated to \([G] \in \mathcal{PMF}\). Notice that the differential (34) depends only on the projective class of \( G \). The exponential map

\[ \mathcal{PMF} \times [0, \infty) / (\mathcal{PMF} \times \{0\}) \ni ([G], t) \mapsto R_{G,t}(t) \in \mathcal{T} \]

which is a homeomorphism (see also [17]).

6. Thurston theory with extremal length.

In this section, we recall the unification of extremal length geometry via intersection number developed in [36].

6.1. Gardiner–Masur closure.

Consider a mapping

\[ \tilde{\Phi}_{GM} : \mathcal{T} \ni y \mapsto [S \ni \alpha \mapsto \text{Ext}_y(\alpha)^{1/2}] \in \mathbb{R}_+^S, \]
\[ \Psi_{GM} : \mathcal{T} \ni y \mapsto [S \ni \alpha \mapsto e^{-d_T(x_0, y) \text{Ext}_y(\alpha)^{1/2}}] \in \mathbb{R}_+^S. \]

Let \( \text{proj} : \mathbb{R}_+^S - \{0\} \rightarrow \mathbb{P}\mathbb{R}_+^S \) be the quotient mapping of the action. In [13], Gardiner and Masur showed that the mapping

\[ \Phi_{GM} = \text{proj} \circ \Psi_{GM} = \text{proj} \circ \tilde{\Phi}_{GM} : \mathcal{T} \rightarrow \mathbb{P}\mathbb{R}_+^S \]

is an embedding with compact closure. The closure \( \text{cl}_{GM}(\mathcal{T}) \) of the image is called the Gardiner–Masur closure or the Gardiner–Masur compactification, and the complement \( \partial_{GM} \mathcal{T} = \text{cl}_{GM}(\mathcal{T}) - \Phi_{GM}(\mathcal{T}) \) is called the Gardiner–Masur boundary. They also observed that the space \( \mathcal{PMF} \) of projective measured foliaitons is contained in \( \partial_{GM} \mathcal{T} \).

6.2. Cones, the intersection number and the Gromov product.

We define

\[ C_{GM} = \text{proj}^{-1}(\text{cl}_{GM}(\mathcal{T})) \cup \{0\} \subset \mathbb{R}_+^S, \]
\[ \mathcal{I}_{GM} = \text{proj}^{-1}(\Phi_{GM}(\mathcal{T})) \subset \mathbb{R}_+^S, \]
\[ \tilde{\partial}_{GM} = \text{proj}^{-1}(\partial_{GM} \mathcal{T}) \cup \{0\} \subset \mathbb{R}_+^S. \]
Since $\mathcal{PM}\subset \partial_{GM}T$, $\mathcal{MF}\subset \tilde{\partial}_{GM}\subset C_{GM}$. From Proposition 1 of [36], $\Psi_{GM}: T \to C_{GM}$ extends to an injective continuous mapping on $\text{cl}_{GM}(T)$.

**CONVENTION 6.1.** We denote by $[a] \in \text{cl}_{GM}(T)$ the projective class of $a \in C_{GM}$. Unless otherwise stated, we always identify $y \in T$ with the projective class $\Phi_{GM}(y) = [\tilde{\Phi}_{GM}(y)] = [\Psi_{GM}(y)]$.

In [36], the author established the following *unification* of extremal length geometry via the intersection number.

**Theorem 6.1 (Theorem 1.1 in [36]).** Let $x_0 \in T$ be the base point taken as above. There is a unique continuous function $i(\cdot, \cdot): C_{GM} \times C_{GM} \to \mathbb{R}$ with the following properties.

(i) $i(\tilde{\Phi}_{GM}(y), F) = \text{Ext}_y(F)^{1/2}$ for any $y \in T$ and $F \in \mathcal{MF}$.

(ii) For $a, b \in C_{GM}$, $i(a, b) = i(b, a)$.

(iii) For $a, b \in C_{GM}$ and $t, s \geq 0$, $i(ta, sb) = ts i(a, b)$.

(iv) For any $y, z \in T$,

$$i(\tilde{\Phi}_{GM}(y), \Phi_{GM}(z)) = \exp(d_T(y, z)),$$

$$i(\Psi_{GM}(y), \Phi_{GM}(z)) = \exp(-2\langle y | z \rangle_{x_0}).$$

(v) For $F, G \in \mathcal{MF} \subset C_{GM}$, the value $i(F, G)$ is equal to the geometric intersection number $I(F, G)$ between $F$ and $G$.

We define the *extremal length* of $a \in C_{GM}$ on $y \in T$ by

$$\text{Ext}_y(a) = \sup_{F \in \mathcal{MF} - \{0\}} \frac{i(a, F)^2}{\text{Ext}_y(F)}$$

(cf. Corollary 4 in [36]). One see that

$$e^{-2d_T(x, y)}\text{Ext}_x(a) \leq \text{Ext}_y(a) \leq e^{2d_T(x, y)}\text{Ext}_x(a)$$

from (32) (see also (5.6) in [36]). From (33) and Gardiner–Masur’s work in [13], (36) coincides with the original extremal length when $a \in \mathcal{MF}$. The extremal length $\text{Ext}_y$ is continuous on $C_{GM}$ and satisfies

$$e^{-d_T(x_0, y)}\text{Ext}_y(\Psi_{GM}(z))^{1/2} = \exp(-2\langle y | z \rangle_{x_0}) = i(\Psi_{GM}(y), \Phi_{GM}(z)),$$

$$e^{-d_T(x_0, y)}\text{Ext}_y(a)^{1/2} = i(\Psi_{GM}(y), a)$$

for $y, z \in T$ and $a \in C_{GM}$ (cf. Theorem 4 and Proposition 7 in [36]). The extremal length (36) also satisfies the following *generalized Minsky inequality*:
for all $y \in \mathcal{T}$ and $a, b \in \mathcal{C}_{GM}$ (cf. Corollary 3 in [36]).

### 6.3. Intersection number with base point

We define the intersection number with base point $x_0$ by

$$i_{x_0}(p, q) = i(\Psi_{GM}(p), \Psi_{GM}(q))$$

(41)

for $p, q \in \text{cl}_{GM}(\mathcal{T})$ (cf. Section 8.2 in [36]). Since the intersection number is continuous, so is $i_{x_0}$ on the product $\text{cl}_{GM}(\mathcal{T}) \times \text{cl}_{GM}(\mathcal{T})$. From Theorem 6.1, the Gromov product

$$\langle y | z \rangle_{x_0} = -\frac{1}{2} \log i_{x_0}(y, z)$$

(42)

extends continuously to $\text{cl}_{GM}(\mathcal{T}) \times \text{cl}_{GM}(\mathcal{T})$ with values in the closed interval $[0, \infty]$ (cf. Corollary 1 in [36]).

**Proposition 6.1.** Teichmüller space $(\mathcal{T}, d_{\mathcal{T}})$ is WBGP.

**Proof.** Let $x = \{x_n\}_{n \in \mathbb{N}} \in \text{USq}(\mathcal{T})$. Since the Gardiner–Masur closure is compact, we find a subsequence $x' = \{x_n(k)\}_{k \in \mathbb{N}}$ and $p \in \partial_{GM} \mathcal{T}$ such that $x_n(k) \to p$ as $k \to \infty$. Since

$$i_{x_0}(x_n(k), x_n(l)) \to i_{x_0}(p, p) = 0 \quad (k, l \to \infty),$$

we have $x' \in \text{Sq}^\infty(\mathcal{T})$ from (42).

**Proposition 6.2** (Intersection number with base point). For any $[a], [b] \in \text{cl}_{GM}(\mathcal{T})$, it holds

$$i_{x_0}([a], [b]) = \frac{i(a, b)}{\text{Ext}_{x_0}(a)^{1/2} \text{Ext}_{x_0}(b)^{1/2}}.$$  

(43)

Notice that the intersection number in the right-hand side of (43) is the original intersection number on $\mathcal{M}\mathcal{F}$ (cf. (v) of Theorem 6.1).

**Proof of Proposition 6.2.** Let $y \in \mathcal{T}$. Notice that

$$\text{Ext}_{x_0}(\Psi_{GM}(y)) = \exp(-2\langle x_0 \mid y \rangle_{x_0}) = 1.$$  

Since $\text{Ext}_{x_0}$ is continuous on $\mathcal{C}_{GM}$, we have

$$\Psi_{GM}([a]) = \frac{a}{\text{Ext}_{x_0}(a)^{1/2}}.$$  

(44)

Therefore,

$$i_{x_0}([a], [b]) = i(\Psi_{GM}([a]), \Psi_{GM}([b])) = \frac{i(a, b)}{\text{Ext}_{x_0}(a)^{1/2} \text{Ext}_{x_0}(b)^{1/2}}.$$
for $a, b \in C_{GM}$. 

6.4. A short proof for non-Gromov hyperbolicity.

We check that the relation “visually indistinguishable” is not an equivalence relation on $\text{Sq}^\infty(T)$ when $\text{cx}(S) \geq 2$. This also implies that Teichmüller space $(T, d_T)$ is not Gromov hyperbolic.

Indeed, let $\alpha, \beta, \gamma \in S$ with $i(\alpha, \beta) = i(\alpha, \gamma) = 0$, but $i(\beta, \gamma) \neq 0$. Consider sequences $x = \{x_n\}_{n \in \mathbb{N}}$, $y = \{y_n\}_{n \in \mathbb{N}}$ and $z = \{z_n\}_{n \in \mathbb{N}}$ in $T$ with $x_n \to [\alpha]$, $y_n \to [\beta]$ and $z_n \to [\gamma]$ in $c_{GM}(T)$, where the projective classes $[\alpha]$, $[\beta]$ and $[\gamma]$ are recognized as points in $\partial_{GM} T$. Then,

$$i_{x_0}(x_n, y_n) \to i_{x_0}([\alpha], [\beta]) = 0,$$

$$i_{x_0}(x_n, z_n) \to i_{x_0}([\alpha], [\gamma]) = 0,$$

but

$$i_{x_0}(y_n, z_n) \to i_{x_0}([\beta], [\gamma]) \neq 0.$$

From (42), these observations imply that $y, z \in \text{Vis}(x)$ but $y \notin \text{Vis}(z)$.

6.5. Subadditivity of the intersection number.

The intersection number has the following subadditive property.

Lemma 6.1 (Subadditivity). Let $F, G \in MF \subset C_{GM}$ with $i(F, G) = 0$. Then, for any $a \in C_{GM}$ we have

$$(i(a, F)^2 + i(a, G)^2)^{1/2} \leq i(a, F + G) \leq i(a, F) + i(a, G).$$

Proof. Let $y \in T$. Then, we have

$$i(\tilde{\Phi}_{GM}(y), F + G) = \text{Ext}_y(F + G)^{1/2}$$

$$= \sup_{H \in \mathcal{MF} - \{0\}} \frac{i(H, F + G)}{\text{Ext}_y(H)^{1/2}} = \sup_{H \in \mathcal{MF} - \{0\}} \frac{i(H, F) + i(H, G)}{\text{Ext}_y(H)^{1/2}}$$

$$\leq \sup_{H \in \mathcal{MF} - \{0\}} \frac{i(H, F)}{\text{Ext}_y(H)^{1/2}} + \sup_{H \in \mathcal{MF} - \{0\}} \frac{i(H, G)}{\text{Ext}_y(H)^{1/2}}$$

$$= \text{Ext}_y(F)^{1/2} + \text{Ext}_y(G)^{1/2} = i(\tilde{\Phi}_{GM}(y), F) + i(\tilde{\Phi}_{GM}(y), G).$$

Hence, the right-hand side of (45) follows from the density of $\mathcal{T}_{GM}$ in $C_{GM}$.

We prove the left-hand side of (45). We first show the case where $F$ and $G$ are rational. Let $F = \sum_{i=1}^{N_1} t_i \alpha_i + \sum_{j=1}^{N_2} u_j \gamma_j$ and $G = \sum_{i=1}^{N_3} s_i \beta_i + \sum_{j=1}^{N_4} v_j \gamma_j$ where $\alpha_i, \beta_i, \gamma_j$ are mutually disjoint and distinct simple closed curves and $t_i, s_i > 0$ and $u_j, v_j \geq 0$. Let $y \in T$ and $A_{\alpha_i}, A_{\beta_i}, A_{\gamma_j}$ be the characteristic annuli for $\alpha_i$, $\beta_i$ and $\gamma_j$ of $J_{F+G,y}$ (cf. [43]). From (30) and Theorem 20.5 in [43], we have

$$i(\tilde{\Phi}_{GM}(y), F + G)^2 = \text{Ext}_y(F + G) = \|J_{F+G,y}\|$$
\[ \begin{align*}
&= \sum_{i=1}^{N_1} \frac{t_i^2}{\text{Mod}(A_{\alpha_i})} + \sum_{i=1}^{N_2} \frac{s_i^2}{\text{Mod}(A_{\beta_i})} + \sum_{j=1}^{N_3} \frac{(u_j + v_j)^2}{\text{Mod}(A_{\gamma_j})} \\
&\geq \left( \sum_{i=1}^{N_1} \frac{t_i^2}{\text{Mod}(A_{\alpha_i})} + \sum_{j=1}^{N_3} \frac{u_j}{\text{Mod}(A_{\gamma_j})} \right) \\
&\quad + \left( \sum_{i=1}^{N_2} \frac{s_i^2}{\text{Mod}(A_{\beta_i})} + \sum_{j=1}^{N_3} \frac{v_j^2}{\text{Mod}(A_{\gamma_j})} \right) \\
&\geq \|J_{F,y}\| + \|J_{G,y}\| = \text{Ext}_y(F) + \text{Ext}_y(G) \\
&= i(\tilde{\Phi}_{GM}(y),F)^2 + i(\tilde{\Phi}_{GM}(y),G)^2. 
\end{align*} \]

Since \( T_{GM} \) is dense in \( C_{GM} \), the above calculation implies

\[ i(a,F)^2 + i(a,G)^2 \leq i(a,F + G)^2 \]

for all \( a \in C_{GM} \). Hence, the left-hand side of (45) also follows by approximating arrational components by weighted multicurves (cf. Theorem C of [24]). \[ \Box \]

7. Structure of the null sets.

We define the null set for \( a \in C_{GM} \) by

\[ \mathcal{N}(a) = \{ b \in C_{GM} \mid i(a,b) = 0 \}. \]

This section is devoted to showing the following theorem.

**Theorem 7.1 (Structure of the null set).** For any \( a \in \partial_{GM} - \{0\} \), any associated foliation \([G] \in \mathcal{PMF}\) for \( a \) satisfies \( \mathcal{N}(a) = \mathcal{N}(G) = \mathcal{N}(G^t) \).

The associated foliation for \( a \) in Theorem 7.1 is defined in Section 7.1. We will see that the associated foliations for \( a \) are essentially uniquely determined from \( a \) (cf. Theorem 7.2). The following is known (cf. Proposition 9.1 in [36]).

**Proposition 7.1.** For \( a \in C_{GM} - \{0\} \), \( \mathcal{N}(a) \neq \{0\} \) if and only if \( a \in \partial_{GM} \). In any case, we have \( \mathcal{N}(a) \subset \partial_{GM} \), and \( a \in \mathcal{N}(a) \) if \( a \in \partial_{GM} \).

7.1. Associated foliations.

Let \([a] \in \partial_{GM} T \) and \( a \in \partial_{GM} - \{0\} \). A projective measured foliation \([G] \in \mathcal{PMF}\) is said to be an associated foliation for \([a] \in \partial_{GM} T \) if there exist \( x \in T \), a sequence \([G_n] \in \mathcal{PMF}\) and \( t_n > 0 \) such that \( R_{G_n,x}(t_n) \rightarrow [a] \) and \([G_n] \rightarrow [G] \) as \( n \rightarrow \infty \). We call the point \( x \) the base point for the associated foliation \([G]\). We denote by \( \mathcal{AF}([a]) \) the set of associated foliations for \([a]\).

In this section, we prove the following.

**Proposition 7.2 (Uniqueness of vanishing curves).** Let \( a \in \partial_{GM} - \{0\} \). For any \([G] \in \mathcal{AF}([a])\), we have
Geometry of the Gromov product

Figure 1. Associated foliation \([G]\): In the figure, we set \(x_n = R_{G_n,x}(t_n)\). \(x\) is the base point for \([G]\).

\[
\mathcal{N}(G) \cap \mathcal{S} = \mathcal{N}(a) \cap \mathcal{S}.
\]

7.1.1. Lemmas.

Let

\[
\mathcal{N}_{MF}(a) = \mathcal{N}(a) \cap \mathcal{MF}.
\]  

(47)

When \(a \in \mathcal{MF}\), the set (47) coincides with the set defined as (28).

**Lemma 7.1.** The following hold:

1. \(\{G \in \mathcal{MF} \mid [G] \in \mathcal{AF}(a)\} \subset \mathcal{N}_{MF}(a)\).

2. For \([G] \in \mathcal{AF}(a)\), we have \(\mathcal{N}(a) \subset \mathcal{N}(G)\) and \(\mathcal{N}_{MF}(a) \subset \mathcal{N}_{MF}(G)\).

In particular \(i(G_1, G_2) = 0\) for \([G_1], [G_2] \in \mathcal{AF}(a)\).

**Proof.**

1. Let \([G] \in \mathcal{AF}(a)\). Take \(x \in \mathcal{T}\), \(\{[G_n]\}_{n \in \mathbb{N}} \subset \mathcal{PMF}\), and \(t_n > 0\) such that \(R_{G_n,x}(t_n) \to [a]\) and \(G_n \to G\) as \(n \to \infty\). From (44), \(\Psi_{GM} \circ R_{G_n,x}(t_n) = e^{-t_n} \Phi_{GM} \circ R_{G_n,x}(t_n)\) converges to \(a' \in C_{GM} - \{0\}\) which is projectively equivalent to \(a\). Therefore

\[
i(a', G) = \lim_{n \to \infty} i(e^{-t_n} \Phi_{GM} \circ R_{G_n,x}(t_n), G_n)
= \lim_{n \to \infty} e^{-t_n} \text{Ext}_{R_{G_n,x}(t_n)}(G_n)^{1/2} = \lim_{n \to \infty} e^{-2t_n} \text{Ext}_x(G_n)^{1/2} = 0
\]

and \(G \in \mathcal{N}_{MF}(a)\).

2. Let \(b \in \mathcal{N}(a)\). Take \(x \in \mathcal{T}\), \(\{[G_n]\}_{n \in \mathbb{N}} \subset \mathcal{PMF}\), \(t_n > 0\), and \(a'\) as above. From (39) and (40), we have

\[
i(G, b) = \lim_{n \to \infty} i(G_n, b)
\leq \lim_{n \to \infty} \text{Ext}_{R_{G_n,x}(t_n)}(G_n)^{1/2} \text{Ext}_{R_{G_n,x}(t_n)}(b)^{1/2}
= \lim_{n \to \infty} e^{-t_n} \text{Ext}_x(G_n)^{1/2} \text{Ext}_{R_{G_n,x}(t_n)}(b)^{1/2}
= \lim_{n \to \infty} \text{Ext}_x(G_n)^{1/2} i(e^{-t_n} \Phi_{GM} \circ R_{G_n,x}(t_n), b)
\]
and $b \in \mathcal{N}(G)$. From the definition,

$$\mathcal{N}_{MF}(a) = \mathcal{N}(a) \cap \mathcal{M} \subset \mathcal{N}(G) \cap \mathcal{M} = \mathcal{N}_{MF}(G).$$

And we are done. □

For $a \in \partial_{GM} - \{0\}$, we define

$$\mathcal{A}V(a) = \bigcup_{[G] \in \mathcal{A}F([a])} \mathcal{N}_{MF}(G) \subset \mathcal{M}.$$  

**Lemma 7.2.** $\mathcal{A}V(a) \cap S \subset \mathcal{N}(a) \cap S$ for all $a \in C_{GM} - \{0\}$.

**Proof.** Let $\alpha \in \mathcal{A}V(a) \cap S$. Let $[G] \in \mathcal{A}F([a])$ with $i(G, \alpha) = 0$. Then, there are $x \in T$, a sequence $\{[G_n]\}_{n \in \mathbb{N}}$ converging to $[G]$ and $t_n \rightarrow 0$ such that $R_{G_n,x}(t_n)$ tends to $[a]$ as $n \rightarrow \infty$. Let $y_t = (Y_t, f_t) = R_{G,x}(t)$.

We refer to the argument in Section 5.3 of [34] (see also [19] and [29]). Let $\Gamma_G$ be the critical vertical graph of the holomorphic quadratic differential of $J_{G,x}$. We add mutually disjoint critical vertical segments to $\Gamma_G$ emanating from critical points to get a graph $\Gamma_G^0$ whose edges are all vertical. The degree of a vertex $\Gamma_G^0$ is one-prong if it is one of endpoints of an added vertical segment. Take $\epsilon > 0$ sufficiently small such that the $\epsilon$-neighborhood $C(\epsilon)$ (with respect to the $|J_{G,x}|$-metric) is embedded in $X$. Then, as the argument in the proof of Theorem 3.1 in [19], by shrinking with a factor $e^{-t}$, we get a canonical conformal embedding $g_t : C(\epsilon) \rightarrow Y_t$ such that $g_t(\Gamma_G) = f_t(\Gamma_G)$. Since $i(\alpha, G) = 0$, $\alpha$ can be deformed into $C(\epsilon)$. Hence, by from the geometric definition (30) of extremal length, the conformal embedding $g_t : C(\epsilon) \rightarrow Y_t$ induces

$$\text{Ext}_{y_t}(\alpha) \leq \text{Ext}_{C(\epsilon)}(\alpha) =: c_0$$

for some $c_0 > 0$ independent of $t$.

Let $\epsilon > 0$. Take $T > 0$ such that $2c_0e^{-T} < \epsilon$. Since $[G_n] \rightarrow [G]$, by (35), there exists an $n_0 > 0$ such that $d(R_{G,x}(T), R_{G_n,x}(T)) \leq (\log 2)/2$ and $t_n \geq T$ for $n \geq n_0$. It has shown from Lemma 1 of [34] that a function

$$[0, \infty) \ni t \mapsto e^{-t} \text{Ext}_{y_t}(F)^{1/2}$$

is a non-increasing function for any $F \in \mathcal{M}$. Hence, from (48), we have

$$i(e^{-t_n} (\Phi_{GM} \circ R_{G_n,x}(t_n), \alpha) = e^{-t_n} \text{Ext}_{R_{G_n,x}(t_n)}(\alpha)^{1/2} \leq e^{-T} \text{Ext}_{R_{G_n,x}(T)}(\alpha)^{1/2} \leq 2e^{-T} \text{Ext}_{y_T}(\alpha)^{1/2} \leq 2c_0 e^{-T} < \epsilon.$$
converges to $a' \in C_{GM} - \{0\}$ with $[a'] = [a]$. Therefore, we get
\[ i(a', \alpha) = \lim_{n \to \infty} i(e^{-t_n} \Phi_{GM} \circ R_{G_n,x}(t_n), \alpha) = 0 \]
and $\alpha \in \mathcal{N}(a) \cap \mathcal{S}$.

7.1.2. Proof of Proposition 7.2.
Let $[G] \in \mathcal{AF}([a])$. From (2) of Lemma 7.1 and Lemma 7.2, we have
\[ \mathcal{N}(a) \cap \mathcal{S} \subset \mathcal{N}(G) \cap \mathcal{S} = \mathcal{N}_{MF}(G) \cap \mathcal{S} \subset \bigcup_{[G] \in \mathcal{AF}([a])} \mathcal{N}_{MF}(G) \cap \mathcal{S} = \mathcal{N}(a) \cap \mathcal{S} \subset \mathcal{N}(a) \cap \mathcal{S}. \]

7.2. Vanishing surface.
The aim of this section is to define the vanishing surface for $a$, which is used for proving Theorem 7.2 stated in the next section.

7.2.1. Minimal vanishing surfaces.
Let $a \in \hat{\partial}_{GM} - \{0\}$. Let $Z^0_a$ be the minimal essential subsurface of $X$ which contains all simple closed curve in $\mathcal{N}(a) \cap \mathcal{S}$. We call $Z^0_a$ the minimal vanishing surface for $a$. By definition, any component $Z_i$ of $Z^0_a$ contains a collection of curves in $\mathcal{N}(a) \cap \mathcal{S}$ which fills up $Z_i$. It is possible that either $\mathcal{N}(a) \cap \mathcal{S}$ or $Z^0_a$ is empty.

7.2.2. Properties of minimal vanishing surfaces.
From Lemma 10.1 in Appendix, if $\alpha \in \mathcal{S}$ can be deformed into $Z^0_a$, then $i(a, \alpha) = 0$ (see also Theorem 6.1 of [13]).

**Proposition 7.3.** Let $[G] \in \mathcal{AF}([a])$. For $\alpha \in \mathcal{S}$, the following are equivalent.

1. $\alpha$ is homotopic to a component of $\partial Z^0_a$.
2. $\alpha$ is homotopic to either an essential curve or a peripheral curve of $G$.

**Proof.** (1) $\Rightarrow$ (2). Since $i(a, \alpha) = 0$, from Proposition 7.2, we have $i(\alpha, G) = 0$. Suppose that $\alpha$ is non-peripheral in a component $W$ of $X - \text{Supp}(G)$. Then, there is an $\alpha' \in \mathcal{S}$ which is non-peripheral in $W$ satisfying $i(\alpha, \alpha') \neq 0$. Since $i(\alpha', G) = 0$, $i(\alpha', a) = 0$ by Proposition 7.2. This means that $\alpha$ cannot be homotopic to a component of $\partial Z^0_a$ because $Z^0_a$ contains a regular neighborhood of $\alpha \cup \alpha'$. This contradicts our assumption.

(2) $\Rightarrow$ (1). Since $i(\alpha, G) = 0$, by Proposition 7.2, $\alpha$ can be deformed into the vanishing surface $Z^0_a$. Suppose to the contrary that $\alpha$ is non-peripheral in $Z^0_a$. Then, there is a non-peripheral curve $\delta$ in a component of $Z^0_a$ with $i(\alpha, \delta) \neq 0$. Since $\delta \in \mathcal{N}(a) \cap \mathcal{S}$, we have $i(\delta, G) = 0$ by Proposition 7.2 again.

If $\delta$ is a component of some $\partial X_i$, $i(\alpha, G) \geq i(\alpha, G_i) \neq 0$ by Lemma 2.14 of [18]. This contradicts that $\alpha \subset Z^0_a$. If $\delta$ is non-peripheral in a component of $X - \text{Supp}(G)$, so is $\alpha$ since $i(\alpha, \delta) \neq 0$. This contradicts to the assumption. \qed
Proposition 7.4. For $\alpha \in \partial GM - \{0\}$, none of components of $Z_0^\alpha$ are pairs of pants.

Proof. Let $Z$ be a component of $Z_0^\alpha$. Suppose to the contrary that $Z$ is a pair of pants. Since any simple closed curve in $Z$ is homotopic to a component of $\partial Z$, $(Z_0^\alpha - Z) \cup N(\partial Z)$ contains all curves in $N(\alpha) \cap S$, where $N(\partial Z)$ is the regular neighborhood of $\partial Z$. This contradicts the minimality of $Z_0^\alpha$. \hfill \Box

7.2.3. Vanishing surface.
We define a subsurface $Z_\alpha$ of $X$ as follows:

1. Remove annular components of $Z_0^\alpha$ whose core is homotopic to a component of $\partial W$, where $W$ runs components of $X - Z_0^\alpha$ which are pairs of pants.

2. To the resulting surface, add components of $X - Z_0^\alpha$ which are pairs of pants.

See Figure 2. We call $Z_\alpha$ the vanishing surface for $\alpha$. Notice from definition that $i(\partial Z, \alpha) = 0$ for every component $Z$ of $Z_\alpha$, and none of the components of $X - Z_\alpha$

![Diagram](image.png)

Figure 2. Case of $G = F + \sum_{i=1}^{4} \alpha_i$. In this case, $G^0 = G$. $Z_0^\alpha$ has three annular components. Two have the core curves which are homotopic to a peripheral curve. The other comes from an essential curve $\alpha_4$ of $G$. The complement $X - Z_0^\alpha$ has two components which are pairs of pants.
are pairs of pants. Recall that $G^\circ$ denotes the distinguished part of $G \in MF$ on nullity (cf. Section 5.3).

### 7.3. Uniqueness of the underlying foliations.

The following uniqueness theorem implies that the underlying foliations of associated foliations for $a$ is essentially determined from $a$.

**Theorem 7.2 (Uniqueness of the underlying foliations).** For any $[G_1], [G_2] \in AF([a])$, $G_1^\circ$ and $G_2^\circ$ are topologically equivalent.

The above uniqueness theorem follows from Proposition 7.5 below.

**Proposition 7.5 (Decomposition associated to $a$).** Let $a \in \tilde{\partial}_{GM} - \{0\}$ and $Z_a$ the vanishing surface for $a$. Then, the reference surface $X$ is decomposed into a union of compact essential surfaces with mutually disjoint interiors as

$$Z_a \cup X_1 \cup \cdots \cup X_{m_1} \cup B_1 \cup \cdots \cup B_{m_2}$$

such that for all $[G] \in AF([a])$, the following properties hold.

1. The family $\{X_i\}_{i=1}^{m_1}$ consists of all components of $X - Z_a$ whose complexities are at least 1. The support of any minimal component of $G$ is some $X_i$. For any for $i = 1, \ldots, m_1$, $X_i$ contains an irrational foliation $F_i$ such that the minimal component of $G$ whose support is $X_i$ is topologically equivalent to $F_i$. Conversely, for any $i$, $G$ contains an irrational component whose support is isotopic to $X_i$.

2. The family $\{B_i\}_{i=1}^{m_2}$ consists of all annular components of $X - Z_a$. Any essential curve of $G$ is homotopic to the core curve of some $B_i$. Conversely, the core of any $B_i$ is homotopic to some essential curve of $G$.

3. Any curve $\alpha \in S$ deformed into $Z_a$ satisfies $i(\alpha, a) = i(\alpha, G) = 0$.

**Proof of Proposition 7.5.** Proposition 7.5 follows from the combination of the following four lemmas given below.

**Lemma 7.3 (Non annular components of $Z_a$).** Let $a \in \tilde{\partial}_{GM} - \{0\}$ and $[G] \in AF([a])$. Every non-annular component of $Z_a$ is isotopic to a non-annular component of $X - \text{Supp}(G^\circ)$, and vice versa.

**Proof.** Let $Z$ be a non-annular component of $Z_a$. Suppose first that $Z$ is not a pair of pants. Then, $Z$ is also a component of $Z_a^\circ$ and $Z$ contains a finite family of curves in $\mathcal{N}(a)$ which fills up. By Proposition 7.2, there is a component $W$ of the component of $X - \text{Supp}(G)$ such that $Z \subset W$ in homotopy sense. Since $\text{cx}(W) \geq 1$, $W$ is also a component of $X - \text{Supp}(G^\circ)$ in homotopy sense. From Proposition 7.3, all components of $\partial Z$ are peripheral curves in $W$. Hence, $Z$ is isotopic to $W$.

Suppose $Z$ is a pair of pants. By definition, $i(\partial Z, a) = 0$ and $i(\partial Z, G) = 0$. Since $Z$ does not contain any minimal component of $G$, $Z$ is contained in a component $W$ of $X - \text{Supp}(G^\circ)$. By the same argument as above, we obtain that $Z$ is isotopic to $W$.
The converse follows from the same argument. However, let us give a sketch for the completeness. Let \( W \) be a non-annular component of \( X - \text{Supp}(G^\circ) \). If \( \text{cx}(W) \geq 1 \), by Proposition 7.2 again, \( W \) is contained in \( Z_a^0 \) in homotopy sense. From Proposition 7.3 again, \( W \) is isotopic to the component \( Z \) of \( Z_a^0 \) containing \( W \). Since \( \text{cx}(W) \geq 1 \), \( Z \) is also a component of \( Z_a \) in homotopy sense. Since \( i(\partial W, G) = 0, i(\partial W, a) = 0 \) and hence \( \overline{Z} \) is isotopic to \( \overline{W} \). If \( W \) is a pair of pants, since \( i(\partial W, a) = 0 \) again, we also conclude that \( \overline{Z} \) is isotopic to \( \overline{W} \).

**Lemma 7.4** (Non annular components of \( X - Z_a \)). Let \( a \in \partial_{GM} \setminus \{0\} \).

1. Let \( [G] \in \mathcal{AF}([a]) \). Let \( W \) be a component of \( X - Z_a \) with \( \text{cx}(W) \geq 1 \). There is a minimal component \( G_i \) of \( G \) such that \( \overline{W} = \text{Supp}(G_i) \) in homotopy sense. Conversely, the support of any arrational component of \( G \) is isotopic to the closure of a component \( W \) of \( X - Z_a \) with \( \text{cx}(W) \geq 1 \).

2. For \( [G_1], [G_2] \in \mathcal{AF}([a]) \), any arrational component of \( G_1 \) is topologically equivalent to that of \( G_2 \).

**Proof.** (1) Let \( W \) be a component of \( X - Z_a \) with \( \text{cx}(W) \geq 1 \). By definition, \( W \) is also a component of \( X - Z_a^0 \). From Proposition 7.2, we have \( i(\alpha, G) \neq 0 \) for every curve \( \alpha \) which is non-peripheral in \( W \). From Proposition 7.3 essential curves and peripheral curves of \( G \) are deformed into \( Z_a^0 \). Hence \( \alpha \) intersects some minimal component \( G_i \) of \( G \).

We check that \( \overline{W} = \text{Supp}(G_i) \) in homotopy sense. We first check \( \text{Supp}(G_i) \subset W \). Otherwise, there is a component \( \gamma \) of \( \partial W \subset \partial Z_a^0 \) which intersects non-trivially to \( \text{Supp}(G_i) \). This means that \( i(\gamma, G) \geq i(\gamma, G_i) \neq 0 \) and hence \( i(\gamma, a) \neq 0 \) from Proposition 7.2, which is a contradiction. If a component \( \gamma \) of \( \partial \text{Supp}(G_i) \) is non-peripheral in \( W \), \( \gamma \) cannot be deformed into \( Z_a^0 \) and hence \( i(\gamma, a) \neq 0 \). Therefore, \( i(\gamma, G) \neq 0 \), as we checked in the previous paragraph. Thus, we conclude that \( \text{Supp}(G_i) \leftrightarrow \overline{W} \) is a deformation retract.

Let \( G_i \) be a minimal component of \( G \). Since any simple closed curve which is non-peripheral in \( \text{Supp}(G_i) \) satisfies \( i(\alpha, G) = i(\alpha, G_i) \neq 0 \), we have \( i(\alpha, a) \neq 0 \). Therefore, \( \text{Supp}(G_i) \) is disjoint from \( Z_a^0 \) (in homotopy sense). Let \( W \) be a component of \( Z - Z_a^0 \) with \( \text{Supp}(G_i) \subset W \) in homotopy sense. Since \( i(\partial \text{Supp}(G_i), G) = 0 \), from Proposition 7.2, we can deduce that \( \text{Supp}(G_i) \) is isotopic to \( \overline{W} \).

(2) Let \( H_i \) be a minimal component of \( G_1 \). From (1) above, there is a minimal component \( H_2 \) of \( G_2 \) such that \( \text{Supp}(H_2) = \text{Supp}(H_1) \). Since \( i(H_1, H_2) \leq i(G_1, G_2) = 0 \) from Lemma 7.1. Hence \( H_1 \) is topologically equivalent to \( H_2 \) (e.g. Theorem 1.1 in [42]).

**Lemma 7.5** (Annular components of \( X - Z_a \)). Let \( a \in \partial_{GM} \setminus \{0\} \) and \( [G] \in \mathcal{AF}([a]) \). The core curve of any annular component of \( X - Z_a \) is homotopic to an essential curve of \( G \), and vice versa.

**Proof.** Let \( W \) be an annular component of \( X - Z_a \). Let \( Z_1 \) and \( Z_2 \) be components of \( Z_a \) adjacent to \( W \). Possibly \( Z_1 = Z_2 \). Suppose some \( Z_i \) is an annulus. Then, \( Z_i \) is also a component of \( Z_a \). Since \( W \) is also an annulus, \( Z_i \) is absorbed into the regular neighborhood of \( \partial Z_j \) where \( \{i, j\} = \{1, 2\} \). This contradicts to the minimality of \( Z_a^0 \),
because each component of $\partial Z_j$ is in $\mathcal{N}(a) \cap S$ and the regular neighborhood of $\partial Z_j$ is contained in $Z_a^0$. Hence, the core curve $\delta$ of $W$ is not a peripheral curve of $G$ from Lemma 7.3.

Since the core $\delta$ of $W$ is non-peripheral in $Z_1 \cup W \cup Z_2$, we can take a curve $\beta \in S$ such that $\beta \subset Z_1 \cup W \cup Z_2$ and $i(\delta, \beta) \neq 0$. If $\delta$ is not an essential curve of $G$, $i(\beta, G) = 0$ and hence $i(\beta, a) = 0$. Therefore, $Z_1 \cup W \cup Z_2$ is a non-annular component of $X - \text{Supp}(G^0)$, since each component of $X - \text{Supp}(G^0)$ is incompressible. This is a contradiction because $W$ can be deformed into the outside of $Z_a$ (cf. Lemma 7.3).

Conversely, let $\delta$ be a essential curve of $G$. Let $W_1$ and $W_2$ be components of $X - \text{Supp}(G^0)$ which are adjacent to the annular component $N_{\delta}$ of $\text{Supp}(G^0)$ whose core is $\delta$. Since neither $W_1$ nor $W_2$ is not annulus, from Lemma 7.3, each $W_i$ is a component of $Z_a$. Therefore, $N_{\delta}$ is a component of $X - Z_a$.

**Lemma 7.6 (Annular component of $Z_a$).** Let $a \in \partial_G M - \{0\}$ and $[G] \in AF([a])$. The core curve of any annular component of $Z_a$ is homotopic to a component of the boundary of the support of a minimal component of $G$.

**Proof.** Let $Z$ be an annular component of $Z_a$. Then, $Z$ is also a component of $Z_a^0$. Hence the core curve $\delta$ of $Z$ is not peripheral in $X$. Let $\partial Z = \gamma_1 \cup \gamma_2$ Let $W_1$ and $W_2$ be the closures of components of $X - Z_a$ such that $\gamma_i \subset \partial W_i$ ($i = 1, 2$). Possibly $W_1 = W_2$. Since each $W_i$ is not a pair of pants, if some $W_i$ is a non-annulus, $Z$ is absorbed in the component of $Z_a$ which is on the opposite side of $W_i$ to $Z$. This contradicts to the minimality of $Z_a^0$. Hence, each $W_i$ satisfies $c x(W_i) \geq 1$, from Lemma 7.4, we conclude that $\delta$ is homotopic to a component of the boundary of some minimal component of $G$.

**7.4. Intersection number lemma.**

The following intersection number lemma encodes the intersection number for two points in $\partial_G M T$ to that of those associated foliations up to multiple by positive constant.

**Lemma 7.7 (Intersection number lemma).** Let $a, b \in \partial_G M - \{0\}$ and $[G] \in AF([a])$ and $[H] \in AF([b])$. Then, there is an $[F_\infty] \in AF([a])$ in $PMF$ such that

$$D_0 i_{x_0}([G], [H]) \leq i_{x_0}([a], [b]) \leq i_{x_0}([F_\infty], [b])$$

where $D_0 = e^{-d_T(x_0, x) - d_T(x_0, y)}$ and $x$ and $y$ are base points for the associated foliations $[G]$ and $[H]$ respectively.

**Proof.** By definition, there are $\{[G_n]\}_{n \in \mathbb{N}}$, $\{[H_n]\}_{n \in \mathbb{N}} \subset PMF$ and $t_n, s_n > 0$ such that

- $R_{G_n, x}(t_n) \to [a]$ and $R_{H_n, y}(s_n) \to [b]$ as $n \to \infty$, and
- $G_n \to G$ and $H_n \to H$ as $n \to \infty$.

For simplicity, let $x_n = R_{G_n, x}(t_n)$ and $y_n = R_{H_n, y}(s_n)$.

Since $d_T(x_0, x_n) \leq t_n + d_T(x_0, x)$ and $d_T(x_0, y_n) \leq s_n + d_T(x_0, y)$, from Proposition 6.2, we deduce
\[ i_{x_0}(x_n, y_n) = \exp(-2\langle x_n \mid y_n \rangle x_0) \]
\[ = \exp(d_T(x_n, y_n) - d_T(x_0, x_n) - d_T(x_0, y_n)) \]
\[ \geq D_0 \exp(d_T(x_n, y_n))e^{-t_n}e^{-s_n} \]
\[ = D_0 \exp(d_T(x_n, y_n))\frac{\Ext_{x_0}(G_n)^{1/2}}{\Ext_{x_0}(G_n)^{1/2}} \frac{\Ext_{y_0}(H_n)^{1/2}}{\Ext_{y_0}(H_n)^{1/2}} \]
\[ \geq D_0 i(x_0, y_0)\frac{\Ext_{x_0}(G_n)^{1/2}}{\Ext_{x_0}(G_n)^{1/2}} \frac{\Ext_{y_0}(H_n)^{1/2}}{\Ext_{y_0}(H_n)^{1/2}} \]
\[ \geq D_0 i(x_0, y_0)\frac{\Ext_{x_0}(G_n)^{1/2}}{\Ext_{x_0}(H_n)^{1/2}} = D_0 i(x_0, [G_n], [H_n]). \]

By letting \( n \to \infty \), we obtain the left-hand side of (50).

Fix \( n \in \mathbb{N} \). Let \( F_{m,n} \in \mathcal{MF}_1 \) with \( x_m = R_{F_{m,n}}y_n(u_{m,n}) \), where \( u_{m,n} = d_T(x_n, y_m) \).

Notice that
\[
\Ext_{x_m}(F_{m,n}) = e^{-2u_{m,n}}\Ext_{y_n}(F_{m,n}). \tag{51}
\]

By taking a subsequence (or by the diagonal argument), we may assume that \( F_{m,n} \to F_{\infty,n} \in \mathcal{MF}_1 \) as \( m \to \infty \) for each \( n \), and \( F_{\infty,n} \) converges to \( F_{\infty} \in \mathcal{MF}_1 \). Since \( x_m \to [a] \), \( [F_{\infty,n}] \) is an associated foliation for \( a \) with base point \( y_n \). Therefore, the limit \( [F_{\infty}] \) is contained in the closure of \( \mathcal{AF}([a]) \) in \( \mathcal{PMF} \).

Since \( F_{m,n} \in \mathcal{MF}_1 \), from Theorem 6.1, (39), (44) and (51), we deduce
\[
i_{x_0}(y_n, x_m) = \exp(-2\langle y_n \mid x_m \rangle x_0) \]
\[ = \exp(u_{m,n} - d_T(x_0, x_m) - d_T(x_0, y_n)) \]
\[ = \exp(-d_T(x_0, y_n))\frac{\Ext_{y_n}(F_{m,n})^{1/2}}{\exp(d_T(x_0, x_m))\Ext_{x_m}(F_{m,n})^{1/2}} \]
\[ \leq \exp(-d_T(x_0, y_n))\Ext_{y_n}(F_{m,n})^{1/2} \]
\[ = i(\Psi_{GM}(y_n), F_{m,n}) = i(\Psi_{GM}(y_n), \Psi_{GM}(F_{m,n})) \]
\[ = i_{x_0}(y_n, [F_{m,n}]). \]

Letting \( m \to \infty \), we conclude
\[ i_{x_0}(y_n, [a]) \leq i_{x_0}(y_n, [F_{\infty,n}]). \tag{52} \]

Thus, if \( n \to \infty \) in (52), we obtain what we wanted. \( \square \)

7.5. Proof of structure theorem.

We first check the following.

**Lemma 7.8.** Let \([G] \in \mathcal{AF}([a])\). Let \( F \in \mathcal{MF} \) be a measured foliation which is topologically equivalent to a minimal component of \( G \). Then, \( i(F, a) = 0 \).
PROOF. Take \( x \in T, [G_n] \in \mathcal{PMF} \), and \( t_n > 0 \) such that \( R_{G_n,x}(t_n) \to [a] \) and \( G_n \to G \) as \( n \to \infty \). Let \( y_n = R_{G_n,x}(t_n) \). Let \( L_{F,y_n} \) be the geodesic current associated to the singular flat structure defined as \( Q_n := J_{F,y_n}/\|J_{F,y_n}\| \) given by Duchin, Leininger and Rafi in [8].

Suppose on the contrary that \( i(a,F) \neq 0 \). Then, by Proposition 4 in [34], \( \{Q_n\}_{n \in \mathbb{N}} \) is a stable sequence in the sense that the set of accumulation points of \( \{e^{-t_n}L_{F,y_n}\}_{n \in \mathbb{N}} \) in the space of geodesic currents is contained in \( \mathcal{MF} - \{0\} \) (as geodesic currents). In addition, any accumulation point \( L_\infty \in \mathcal{MF} - \{0\} \) satisfies

\[
i(L_\infty,F) = t_0i(a,F) \neq 0, \quad (53)
\]

\[
i(L_\infty,H) \leq t_0i(a,H) \quad (54)
\]

for some \( t_0 > 0 \) and any \( H \in \mathcal{MF} \) (see Proposition 5 in [34]).

Let \( G_0 \) be a minimal component of \( G \) which is topologically equivalent to \( F \) and \( X_0 \) be the support of \( G_0 \). From (54), \( i(L_\infty,G) = 0 \). Hence, if \( L_\infty \) has a component \( L_0 \) whose support intersects \( X_0 \), then \( L_0 \) is topologically equivalent to \( G_0 \) (cf. [18]). This means that \( i(L_\infty,F) = 0 \), which contradicts to (53).

PROOF OF THEOREM 7.1. We are ready to prove Theorem 7.1. Let \( [G] \in \mathcal{AF}([a]) \).

From (2) of Lemma 7.1, we need to show the converse \( \mathcal{N}(a) \supset \mathcal{N}(G) \).

We first claim that \( \mathcal{N}_{\mathcal{MF}}(a) = \mathcal{N}_{\mathcal{MF}}(G) \) for \([G] \in \mathcal{AF}([a]) \). We decompose \( G \) as (27):

\[
G = G_1' + G_2' + \cdots + G_m' + \beta_1 + \cdots + \beta_{m_2} + \gamma_1 + \cdots + \gamma_{m_3}.
\]

Let \( H \in \mathcal{N}_{\mathcal{MF}}(G) \). Then, \( H \) can be decomposed as

\[
H = \sum_{i=1}^{m_1} H_i + \sum_{i=1}^{m_2} a_i \beta_i + \sum_{i=1}^{m_1} \sum_{\gamma \subset \partial X_i} b_{i\gamma} + F_0 \quad (55)
\]

where \( a_i, b_{i\gamma} \geq 0 \), \( H_i \) is a measured foliation topologically equivalent to \( G_i' \) (possibly \( H_i = 0 \)), and \( F_0 \) is a measured foliation whose support is contained in the complement of Supp(\( G \)) (cf. [18]).

From Proposition 7.5, the support of \( F_0 \) is contained in the vanishing surface \( Z_a \). Therefore, \( i(F_0,a) = 0 \). Since any component of \( \partial X_i \) is deformed into \( Z'_{a} \), from Lemma 6.1 and Lemma 7.8, we have

\[
i(H,a) \leq \sum_{i=1}^{m_1} i(H_i,a) + \sum_{i=1}^{m_2} a_i i(\beta_i,a) + \sum_{i=1}^{m_1} \sum_{\gamma \subset \partial X_i} b_{i\gamma} i(\gamma,a) + i(F_0,a) = 0 \quad (56)
\]

and hence \( \mathcal{N}_{\mathcal{MF}}(G) \subset \mathcal{N}_{\mathcal{MF}}(a) \).

Let \( b \in \mathcal{N}(G) \) and take \( \{y_n\}_{n=1}^\infty \) such that \( y_n \to [b] \) as \( n \to \infty \). Let \( H_n \in \mathcal{MF}_1, s_n > 0 \) such that \( y_n = R_{H_n,x_0}(s_n) \). By taking a subsequence, we may assume that \( H_n \to H_\infty \). Then, \( [H_\infty] \in \mathcal{AF}([b]) \). Let \( [F_\infty] \in \mathcal{AF}([a]) \) as Lemma 7.7 for \([G], [H_\infty], a \) and \( b \). To show that \( b \in \mathcal{N}(a) \), it suffices to show that \( i(b,F_\infty) = 0 \) from Lemma 7.7.

Since \( b \in \mathcal{N}(G) \) and \( \mathcal{N}_{\mathcal{MF}}(H_\infty) = \mathcal{N}_{\mathcal{MF}}(b) \), we have \( i(G,H_\infty) = 0 \). Therefore, \( H_\infty \) is decomposed as
where $H'_i$ is topologically equivalent to $G'_i$, $a_i, b_\gamma \geq 0$, $H_0$ is a measured foliation whose support is contained in the complement of $\text{Supp}(G)$. Since $[F_\infty] \in \mathcal{AF}([a])$, from Theorem 7.2, $F_\infty$ is decomposed as

$$F_\infty = \sum_{i=1}^{m_1} F'_i + \sum_{i=1}^{m_2} a_i \beta_i + \sum_{i=1}^{m_1} \sum_{\gamma \subset \partial X} b_\gamma \gamma + H_0$$  \hspace{1cm} (57)$$

where $F'_\infty$ is topologically equivalent to $G'_i$ (possibly $F'_i = 0$) and $a_i, b_\gamma \geq 0$. From (57) and (58), we have $i(F_\infty, H_\infty) = 0$. Since $N_{MF}(H_\infty) = N_{MF}(b)$ again, we conclude that $i(b, F_\infty) = 0$ as desired. \hfill \Box

7.6. Topological equivalence revisited.

Before closing this section, we notice the following expected property.

**Corollary 7.1 (Topological equivalence and null sets).** For $G, H \in MF$, the following are equivalent:

1. $G^\circ$ and $H^\circ$ are topologically equivalent;
2. $N_{MF}(G) = N_{MF}(H)$;
3. $N(G) = N(H)$.

In particular, $N(G) = N(G^\circ)$ for any $G \in MF$.

**Proof.** From Proposition 5.1, the conditions (1) and (2) are equivalent. Since $N_{MF}(G) = N(G) \cap MF$, (2) follows from (3). Hence, we need to show that (1) implies (3). From the symmetry of the topological equivalence, it suffices to show that $N(G) \subset N(H)$.

Let $a \in N(G)$ and $[F] \in \mathcal{AF}([a])$. Then, $i(G, F) = 0$ from Theorem 7.1. Since $H^\circ$ is topologically equivalent to $G^\circ$, by Proposition 5.1, we have $i(H, F) = 0$. Hence, by applying Theorem 7.1 again, we have $i(H, a) = 0$ and $a \in N(H)$.

8. Action on the Reduced boundary.

Let $S$ and $S'$ be compact orientable surfaces of non-sporadic type. In this section, we study maps in $AC_{inv}(T(S), T(S'))$.

8.1. Null sets and accumulation sets.

For $p \in \cl GM(T(S))$, we define the null set for $p$ by

$$\mathcal{N}_S(p) = \{ q \in \cl GM(T(S)) \mid i_{x_0}(p, q) = 0 \}.$$  \hspace{1cm} (59)

For $x \in Sq^\infty(T(S))$, we define

$$\mathcal{ACM}_S(x) = \bigcup \{ z \cap \partial GM T(S) \mid z \in \text{Vis}(x) \},$$  \hspace{1cm} (59)
where \( \mathcal{F} \) is the closure of \( z \) in \( \text{cl}_{GM}(T(S)) \). The following proposition follows from (42).

**Proposition 8.1.** Let \( p, p^1, p^2 \in \partial_{GM}T(S) \) and \( x, x^1, x^2 \in USq(T(S)) \).

1. If \( x \) converges to \( p \), \( \mathcal{N}_S(p) = ACM_S(x) \).

2. Suppose each \( x^i \) converges to \( p^i \) for \( i = 1, 2 \). Then, \( \mathcal{N}_S(p^2) \subset \mathcal{N}_S(p^1) \) if and only if \( \text{Vis}(x^2) \subset \text{Vis}(x^1) \).

**Proposition 8.2 (Structure of accumulation points).** Let \( x \in \text{Sq}^\infty(T(S)) \). Then, there is \( G \in \mathcal{M} \mathcal{F} \) such that

\[
ACM_S(x) = \mathcal{N}_S([G]).
\]

Furthermore, the following are equivalent for \( q \in \partial_{GM}T(S) \):

1. \( q \in \mathcal{N}_S([G]) \);

2. for any \( p \in \mathcal{F} \cap \partial_{GM}T(S) \) and \( [G_p] \in \mathcal{A} \mathcal{F}(p) \), \( q \in \mathcal{N}_S([G_p]) \);

3. for any \( p \in \mathcal{F} \cap \partial_{GM}T(S) \) \( q \in \mathcal{N}_S(p) \).

**Proof.** For \( p \in \mathcal{F} \cap \partial_{GM}T(S) \), fix \( [G_p] \in \mathcal{A} \mathcal{F}(p) \). From (42) and Theorem 7.1,

\[
i(G_{p^1}, G_{p^2}) = 0 \text{ for } p^1, p^2 \in \mathcal{F} \cap \partial_{GM}T(S).
\]

Hence, we can find \( G \in \mathcal{M} \mathcal{F} \) such that

1. for any \( p \in \mathcal{F} \cap \partial_{GM}T(S) \), \( G \cap p \) is topologically equivalent to a subfoliation of \( G \), and

2. any component of \( G^\circ \) is topologically equivalent to a component of some \( G_p, p \in \mathcal{F} \cap \partial_{GM}T(S) \).

We check that \( G \) satisfies the desired property. Let \( q \in ACM_S(x) \) be an accumulation point of \( z \in \text{Vis}(x) \). Let \( [H] \in \mathcal{A} \mathcal{H}(q) \). Since \( i(H, G_p) = 0 \) for all \( p \in x \cap \partial_{GM}T(S) \), from the condition (2) of \( G \), we have \( i(G, H) = 0 \) and hence \( \Psi_{GM}(q) \in \mathcal{N}(G) \) by Theorem 7.1. This means that \( q \in \mathcal{N}_S([G]) \) and \( ACM_S(x) \subset \mathcal{N}_S([G]) \).

Conversely, let \( q \in \mathcal{N}_S([G]) \). Take a sequence \( z \) in \( X \) converging to \( q \). By the condition (1) of \( G \) above, \( \mathcal{N}_S([G]) \subset \mathcal{N}_S([G_p]) \) for all \( p \in \mathcal{F} \cap \partial_{GM}T(S) \). In other words, any subsequence of \( x \) contains a subsequence which is visually indistinguishable from \( z \). Therefore, we have \( x \in \text{Vis}(z) \) and hence \( z \in \text{Vis}(x) \).

The last statement follows from the construction of \( G \) and Theorem 7.1. \( \square \)

**Proposition 8.3.** Let \( x^1, x^2 \in \text{Sq}^\infty(T(S)) \). The following are equivalent:

1. \( ACM_S(x^1) \subset ACM_S(x^2) \);

2. \( \text{Vis}(x^1) \subset \text{Vis}(x^2) \).

**Proof.** From the definition (59), the condition (2) implies (1).

Suppose the condition (1). Assume to the contrary that there is \( z \in \text{Vis}(x^1) \setminus \text{Vis}(x^2) \). Take subsequences \( z^1 = \{z_n^1\}_{n \in \mathbb{N}} \) of \( z \) and \( x^2 = \{x_n^2\}_{n \in \mathbb{N}} \) of \( x^2 \) such that

\[
\langle x_n^1, z_n^1 \rangle_{x_0} < M_1
\]
for all \( n \in \mathbb{N} \). Then, any \( q' \in \overline{T} \cap \partial_{GM} \mathcal{T}(S) \subset \overline{\mathbb{P}} \cap \partial_{GM} \mathcal{T}(S) \) and \( p' \in \overline{x^2} \cap \partial_{GM} \mathcal{T}(S) \subset \overline{\mathbb{P}} \cap \partial_{GM} \mathcal{T}(S) \) satisfy \( i_{x_0}(p', q') \neq 0 \) (cf. (42)). By Proposition 8.2, \( q' \notin \mathcal{AC}_S(x^2) \). Since \( \mathcal{AC}_S(x^1) \subset \mathcal{AC}_S(x^2) \) from the assumption, \( q' \notin \mathcal{AC}_S(x^1) \). On the other hand, since \( z \in \text{Vis}(x^1) \), \( q' \in \overline{x} \cap \partial_{GM} \mathcal{T}(S) \subset \mathcal{AC}_S(x^1) \). This is a contradiction. \( \Box \)

8.2. Accumulation sets.
Let \( \omega \in \mathcal{AC}(\mathcal{T}(S), \mathcal{T}(S')) \), for \( p \in \text{cl}_{GM}(\mathcal{T}(S)) \), we define the accumulation set by

\[
\mathcal{A}(\omega; p) = \{ q \in \text{cl}_{GM}(\mathcal{T}(S')) \mid \exists \{ y_n \}_{n \in \mathbb{N}} \in \mathbb{S}_\infty(\mathcal{T}(S)) \text{ s.t. } y_n \to p \text{ and } \omega(y_n) \to q \).
\]

The following lemma will be applied for defining the extension to the reduced Gardiner–Masur closure in Section 8.4.

**Lemma 8.1** (Null sets and accumulation points). Let \( \omega \in \mathcal{AC}_{as}(\mathcal{T}(S), \mathcal{T}(S')) \). Let \( p^1, p^2 \in \partial_{GM} \mathcal{T}(S) \) and \( q^i \in \mathcal{A}(\omega; p_i) \) for \( i = 1, 2 \). If \( \mathcal{R}_S(p^2) \subset \mathcal{R}_S(p^1) \), then \( \mathcal{R}_{S'}(q^2) \subset \mathcal{R}_{S'}(q^1) \). Especially, \( \mathcal{R}_{S'}(q^2) = \mathcal{R}_{S'}(q^1) \) for \( p \in \partial_{GM} \mathcal{T}(S) \) and \( q^1, q^2 \in \mathcal{A}(\omega; p) \).

**Proof.** For \( i = 1, 2 \), let \( x^i \) be a sequence converging to \( p_i \) such that \( \omega(x^i) \) converges to \( q_i \). From Proposition 8.1, the assumption \( \mathcal{R}_S(p_2) \subset \mathcal{R}_S(p_1) \) implies \( \text{Vis}(x^2) \subset \text{Vis}(x^1) \). By Propositions 2.4, we have \( \text{Vis}(\omega(x^2)) \subset \text{Vis}(\omega(x^1)) \). Therefore, by applying Proposition 8.1 again, we obtain \( \mathcal{R}_{S'}(q_2) \subset \mathcal{R}_{S'}(q_1) \). \( \Box \)

8.3. Reduced Gardiner–Masur closure and boundary.
We say two points \( p, q \in \text{cl}_{GM}(\mathcal{T}(S)) \) are equivalent if one of the following holds:

1. \( p, q \in \mathcal{T}(S) \) and \( p = q \);
2. \( p, q \in \partial_{GM} \mathcal{T}(S) \) and \( \mathcal{R}_S(p) = \mathcal{R}_S(q) \).

We denote by \([p]\) the equivalence class of \( p \in \text{cl}_{GM}(\mathcal{T}(S)) \). We abbreviate the equivalence class \([[G]]\) of the projective class \([G] \in \mathcal{PMF} \subset \partial_{GM} \mathcal{T}(S)\) as \([[G]]\). We denote by \( \text{cl}_{GM}^{\text{red}}(\mathcal{T}(S)) \) the quotient of \( \text{cl}_{GM}(\mathcal{T}(S)) \) under this equivalence relation. Let \( \pi_{GM}: \text{cl}_{GM}(\mathcal{T}(S)) \to \text{cl}_{GM}^{\text{red}}(\mathcal{T}(S)) \) be the quotient map. We always identify \( \pi_{GM}(\mathcal{T}(S)) \) with \( \mathcal{T}(S) \). We call \( \text{cl}_{GM}^{\text{red}}(\mathcal{T}(S)) \) the reduced Gardiner–Masur closure of \( \mathcal{T}(S) \). From the definition, the space \( \text{cl}_{GM}^{\text{red}}(\mathcal{T}(S)) \) contains \( \mathcal{T}(S) \) canonically. We call the complement

\[
\partial_{GM}^{\text{red}} \mathcal{T}(S) = \text{cl}_{GM}^{\text{red}}(\mathcal{T}(S)) - \mathcal{T}(S)
\]

the reduced Gardiner–Masur boundary of \( \mathcal{T}(S) \).

The reduced Gardiner–Masur closure is a variation of the reduced compactifications of Teichmüller space. See [40].

8.4. Boundary extension.
For \( \omega \in \mathcal{AC}_{as}(\mathcal{T}(S), \mathcal{T}(S')) \), we define the boundary extension \( \partial_{\infty}(\omega): \text{cl}_{GM}^{\text{red}}(\mathcal{T}(S)) \to \text{cl}_{GM}^{\text{red}}(\mathcal{T}(S')) \) by

\[
\partial_{\infty}(\omega)([[p]]) = \begin{cases} [[\omega(p)]] & (p \in \mathcal{T}(S)), \\ [[q]] & (q \in \mathcal{A}(\omega; p) \text{ if } p \in \partial_{GM} \mathcal{T}(S)). \end{cases}
\]

(60)
From Lemma 8.1, the extension $\partial_\infty(\omega)$ is well-defined.

**Lemma 8.2 (Composition).** For $\omega_1, \omega_2 \in \text{AC}_{\text{as}}(\mathcal{T}(S), \mathcal{T}(S'))$, the extensions satisfy

$$
\partial_\infty(\omega_1 \circ \omega_2) = \partial_\infty(\omega_1) \circ \partial_\infty(\omega_2)
$$
on $\partial_{\text{red}}^{\text{GM}} \mathcal{T}(S)$.

**Proof.** Let $[[p]] \in \partial_{\text{red}}^{\text{GM}} \mathcal{T}(S)$. Take $x = \{x_n\}_{n \in \mathbb{N}} \subset \mathcal{T}(S)$ such that $x_n \to p$ and $\omega_1 \circ \omega_2(x_n) \to p' \in \partial_{\text{GM}} \mathcal{T}(S')$. By definition,

$$
\partial_\infty(\omega_1 \circ \omega_2)([[p]]) = [[p']].
$$

On the other hand, from Proposition 8.1, we may assume that $\omega_2(x)$ converges to $q \in A(\omega_2; p)$. From the definition, we have $\partial_\infty(\omega_2)([[p]]) = [[q]]$. Since $\omega_1 \circ \omega_2(x) = \omega_1(\omega_2(x))$, $p' \in A(\omega_1; q)$ and hence

$$
[[p']] = \partial_\infty(\omega_1)([[q]]) = \partial_\infty(\omega_1) \circ \partial_\infty(\omega_2)([[p]]).
$$

**Lemma 8.3 (Close at infinity).** Let $\omega_1, \omega_2 \in \text{AC}_{\text{as}}(\mathcal{T}(S), \mathcal{T}(S'))$. If $\omega_1$ is close to $\omega_2$ at infinity, $\partial_\infty(\omega_1) = \partial_\infty(\omega_2)$ on $\partial_{\text{red}}^{\text{GM}} \mathcal{T}(S)$.

**Proof.** Let $p \in \partial_{\text{GM}} \mathcal{T}(S)$. Take $x = \{x_n\}_{n \in \mathbb{N}} \in \text{USq}(\mathcal{T}(S))$ with $x_n \to p$ as $n \to \infty$ such that $\omega_i(x_n) \to q^i \in A(\omega_i; p)$ for $i = 1, 2$. Since $\text{Vis}(\omega_1(x)) = \text{Vis}(\omega_2(x))$, by Proposition 8.1,

$$
\mathcal{M}_{S'}(q^1) = \text{ACM}_{S'}(\omega_1(x)) = \text{ACM}_{S'}(\omega_2(x)) = \mathcal{M}_{S'}(q^2).
$$

Hence

$$
\partial_\infty(\omega_1)([[p]]) = [[q^1]] = [[q^2]] = \partial_\infty(\omega_2)([[p]])
$$

and $\partial_\infty(\omega_1) = \partial_\infty(\omega_2)$ on $\partial_{\text{red}}^{\text{GM}} \mathcal{T}(S)$.

**Corollary 8.1 (Inverse).** Let $\omega \in \text{AC}_{\text{inv}}(\mathcal{T}(S), \mathcal{T}(S'))$ and $\omega'$ be an asymptotic quasi-inverse of $\omega$. Then, $\partial_\infty(\omega') \circ \partial_\infty(\omega)$ and $\partial_\infty(\omega) \circ \partial_\infty(\omega')$ are identity mappings on $\partial_{\text{red}}^{\text{GM}} \mathcal{T}(S)$ and $\partial_{\text{red}}^{\text{GM}} \mathcal{T}(S')$, respectively.

9. **Rigidity of asymptotically conservative mappings.**

9.1. **Heights of reduced boundary points.**

An ordered sequence $\{[[p_k]]\}_{k=1}^m$ in $\partial_{\text{red}}^{\text{GM}} \mathcal{T}(S)$ is said to be an adherence tower starting at $[[p_1]]$ if

$$
\mathcal{M}_S(p_1) \supsetneq \mathcal{M}_S(p_2) \supsetneq \cdots \supsetneq \mathcal{M}_S(p_m).
$$

The adherence tower is named with referring Ohshika’s paper [39]. See also Papadopoulos’s paper [38]. We call the number $m$ the length of the adherence tower. Let $[[p]] \in \partial_{\text{red}}^{\text{GM}} \mathcal{T}(S)$. We define the height $\text{ht}([[p]])$ of $[[p]]$ by
\( \text{ht}([p]) = \sup \{ \text{lengths of adherence towers starting } [p] \} \).

For a measured foliation \( G \), we set \( \{ X_i \}_{i=1}^{m_1} \) be the supports of the minimal components of \( G \). We define the complexity of \( G \) by

\[
\xi_0(G) = \left( -\sum_{i=1}^{m_1} \text{cx}(X_i), \# \{ \text{essential curves in } G \} \right) \in \mathbb{Z} \times \mathbb{Z}.
\]

(61)

(cf. Theorem 1 in [39]).

**Lemma 9.1 (Heights of boundary points).** The height of any \([p] \in \partial_{\text{red}} \mathcal{T}(S)\) is at most \( \text{cx}(S) \). The equality \( \text{ht}([p]) = \text{cx}(S) \) holds if and only if the support of any \( [G] \in \mathcal{AF}(p) \) is a simple closed curve.

**Proof.** We first discuss the associated foliations of points in an adherence tower of length two. Let \([p_1], [p_2] \in \partial_{\text{red}} \mathcal{T}(S)\). Let \([G_i] \in \mathcal{AF}(p_i) \) for \( i = 1, 2 \). Suppose that \([p_1], [p_2] \) is an adherence tower. From the definition, \( \mathcal{N}(\Psi_{GM}(p_1)) \supseteq \mathcal{N}(\Psi_{GM}(p_2)) \). From Corollary 7.1, we see

\[
\mathcal{N}_{MF}(G_1^\circ) = \mathcal{N}_{MF}(\Psi_{GM}(p_1)) \supseteq \mathcal{N}_{MF}(\Psi_{GM}(p_2)) = \mathcal{N}_{MF}(G_2^\circ).
\]

We decompose \( G_1^\circ \) as in (27):

\[
G_1^\circ = \sum_{i=1}^{m_1} G_i' + \sum_{i=1}^{m_2} \beta_i.
\]

(62)

where \( G_i' \) is a minimal component, and \( \beta_i \) is a (weighted) essential curve of \( G_1 \). Since \( G_2 \in \mathcal{N}_{MF}(G_1) \), the decomposition of \( G_2^\circ \) is represented as

\[
G_2^\circ = \sum_{i=1}^{m_1} H_i' + \sum_{i=1}^{m_2} a_i \beta_i + G_3.
\]

(63)

where \( H_i' \) is topologically equivalent to \( G_i' \), \( a_i \geq 0 \) and the support of \( G_3 \) is contained in the complement of the support of \( G_1^\circ \) (at this moment, \( G_2^\circ \) may contain curves homotopic to boundary components of arrational components of \( G_1 \) as essential curves). Since \( \mathcal{N}_{MF}(G_2) \subset \mathcal{N}_{MF}(G_1) \), \( H_i' \neq 0 \) and \( a_i \neq 0 \). Moreover, from the assumption \( \mathcal{N}_{MF}(G_2) \neq \mathcal{N}_{MF}(G_1) \) implies that \( G_3 \neq 0 \). Therefore, from (62) and (63), we have

\[
\xi_0(G_1) < \xi_0(G_2)
\]

in the lexicographical order in \( \mathbb{Z} \times \mathbb{Z} \), since \( G_3 \) in (63) contains either a minimal component or an essential curve of \( G_2 \).

Let us return to the proof of the lemma. Let \( \{ [p_i] \}_{i=1}^{m} \) be an adherence tower of length \( m \). Let \([G_i] \in \mathcal{AF}(p_i) \). From the above argument, we have

\[
\xi_0(G_1) < \xi_0(G_2) < \cdots < \xi_0(G_m).
\]

(64)
Since the number of essential curves is at most $\text{cx}(S)$ and the sum of the first and second coordinates of $\xi_{0}(G)$ is at most $\text{cx}(S)$ minus the number of boundary components of minimal foliations of $G$ which are non-peripheral in $S$, we have $m \leq \text{cx}(S)$. In addition, if $m = \text{cx}(S)$, each $G_{i}$ consists of essential curves. Hence, in this case, the adherence tower starts with a simple closed curve.

9.2. Induced isomorphism.

Let $X_{0}(S)$ be the 0-skeleton of $\mathcal{X}(S)$. We identify each vertex of $X_{0}(S)$ with its projective class in $\partial_{GM} \mathcal{T}(S)$.

**Theorem 9.1 (Induced isomorphism).** Let $S$ and $S'$ be compact orientable surfaces of non-sporadic type. For $\omega \in \text{AC}_{\text{inv}}(\mathcal{T}(S), \mathcal{T}(S'))$, there is a simplicial isomorphism $h_{\omega}: \mathcal{X}(S) \rightarrow \mathcal{X}(S)$ such that for any $\alpha \in X_{0}(S)$, and any sequence $\{x_{n}\}_{n} \subset \mathcal{T}(S)$ with $x_{n} \rightarrow [\alpha]$, we have $\omega(x_{n}) \rightarrow [h_{\omega}(\alpha)]$. Furthermore, When $\omega$ and $\omega'$ are close at infinity, $h_{\omega} = h_{\omega'}$.

**Proof.** Let $\omega \in \text{AC}_{\text{inv}}(\mathcal{T}(S), \mathcal{T}(S'))$ and $\alpha \in X_{0}(S)$. From Lemma 9.1, there is an adherence tower $\{([p_{i}])/_{\text{cx}(S)}\}_{i=1}^{\text{cx}(S)}$ with $([p_{1}]) = ([\alpha])$. From Lemma 8.1, $\partial_{\omega}(\alpha)([[p_{i}]])_{\text{cx}(S)}$ is also an adherence tower starting $\partial_{\omega}(\alpha)([[p_{1}]])) = \partial_{\omega}(\alpha)([\alpha]))$. Applying the above argument for asymptotic quasi-inverse of $\omega$, we see that the adherence tower $\{\partial_{\omega}(\alpha)([[p_{i}]])_{\text{cx}(S)}\}_{i=1}$ has this maximal height. From Lemma 9.1 and Corollary 8.1, we obtain a bijection $h_{\omega}: X_{0}(S) \rightarrow X_{0}(S')$ such that

$$ \partial_{\omega}(\alpha)([[\alpha]]) = ([h_{\omega}(\alpha)]). \tag{65} $$

Let $\alpha, \beta \in X_{0}(S)$ with $i(\alpha, \beta) = 0$. Then, $G = \alpha + \beta \in \mathcal{M}F$ and $N(\alpha) \cap N(\beta) \subset N(G)$. Therefore, $\{([[\alpha]], [[G]])\}$ and $\{([[\beta]], [[G]])\}$ are adherence towers. From Lemma 8.1, $\{\partial_{\omega}(\alpha)([[\alpha]]), \partial_{\omega}(\alpha)([[G]])\}$ and $\{\partial_{\omega}(\alpha)([[\beta]]), \partial_{\omega}(\alpha)([[G]])\}$ are also adherence towers. From Theorem 7.1, there is an $H \in \mathcal{M}F$ such that $\partial_{\omega}(\alpha)([[G]]) = ([H])$. Since $h_{\omega}$ is bijective, $h_{\omega}(\alpha)$ and $h_{\omega}(\beta)$ represent different components of $H$. Therefore, $i(h_{\omega}(\alpha), h_{\omega}(\beta)) = 0$. This means that $h_{\omega}$ extends a simplicial isomorphism from $\mathcal{X}(S)$ to $\mathcal{X}(S')$. From Lemma 8.3, one can easily see that $h_{\omega'} = h_{\omega}$ when $\omega'$ is close to $\omega$ at infinity.

Let $x = \{x_{n}\}_{n}$ be a sequence in $\mathcal{T}(S)$ converging to a simple closed curve $[\alpha] \in \text{cl}_{GM}(\mathcal{T}(S))$. By (60) and (65), any accumulation point $q \in \partial_{GM} \mathcal{T}(S')$ of a sequence $\omega(x)$ satisfies $\mathfrak{n}_{G}(q) = \mathfrak{n}_{G}(\omega([\alpha]))$ from Lemma 8.1. Hence $q$ satisfies $i_{\omega(x)}(F, q) = 0$ for all $F \in \mathcal{N}_{\mathcal{M}F}(h_{\omega}(\alpha)) (\subset \mathcal{M}F(S'))$. From Theorem 3 in [34], we conclude that $q = [h_{\omega}(\alpha)]$ in $\partial_{GM} \mathcal{T}(S')$. This means that $\omega(x)$ converges to $[h_{\omega}(\alpha)]$ in $\text{cl}_{GM}(\mathcal{T}(S'))$.

9.3. Rigidity theorem.

9.3.1. Actions of extended mapping class group.

The extended mapping class group $\text{MCG}^{\ast}(S)$ of $S$ is the group of all isotopy classes of homeomorphisms on $S$. The extended mapping class group $\text{MCG}^{\ast}(S)$ acts on $\mathcal{T}(S)$ isometrically by
\[ \mathcal{T}(S) \ni y = (Y, f) \mapsto [h]_*(y) = (Y, f \circ h^{-1}) \in \mathcal{T}(S) \]

for \([h] \in \text{MCG}^*(S)\). Hence, we have a group homomorphism
\[ I_0 : \text{MCG}^*(S) \ni [h] \to [h]_* \in \text{Isom}(\mathcal{T}(S)), \tag{66} \]
where \(\text{Isom}(\mathcal{T}(S))\) is the group of all isometries of \(\mathcal{T}(S)\).

Let \(\mathcal{X}(S)\) be the complex of curves of \(S\) and \(\text{Aut}(\mathcal{X}(S))\) be the simplicial automorphisms on \(\mathcal{X}(S)\). Since \(\text{MCG}^*(S)\) acts on \(\mathcal{X}(S)\) canonically, we have a (group) homomorphism
\[ J : \text{MCG}^*(S) \to \text{Aut}(\mathcal{X}(S)). \tag{67} \]

It is known that \(J\) is an isomorphism if \(S\) is neither a torus with two holes nor a closed surface of genus 2, and an epimorphism if \(S\) is not a torus with two holes (cf. Ivanov [19], Korkmaz [22] and Luo [26]).

The action of any isometry on \(\mathcal{T}(S)\) extends homeomorphically to the Gardiner–Masur boundary (cf. [25]). We can observe that the extension of the action leaves \(S \subset \partial_{GM} \mathcal{T}(S)\) invariant, and it induces a canonical homomorphism \(J_1 : \text{Isom}(\mathcal{T}(S)) \to \text{Aut}(\mathcal{X}(S))\) such that the diagram
\[ \begin{array}{ccc}
\text{MCG}^*(S) & \xrightarrow{I_0} & \text{Isom}(\mathcal{T}(S)) \\
\downarrow{J} & & \downarrow{J_1} \\
\text{Aut}(\mathcal{X}(S)) & & \end{array} \]

is commutative (cf. [36]). The homomorphism \(J_1\) is an isomorphism for any \(S\) with \(\text{cx}(S) \geq 2\) (cf. [19]). The reason why (67) is not surjective when \(S\) is a torus with two holes is that there is no homeomorphism on \(S\) which sends a non-null-homologous curve to a null-homologous curve, while each curve on \(S'\) is null-homologous. Thus, in any case, the homomorphism \(J_1\) is surjective (cf. [26]).

**9.3.2. Rigidity theorem.**

Recall that any isometry is an invertible asymptotically conservative mapping. Hence, we have a monoid monomorphism
\[ I : \text{Isom}(\mathcal{T}(S)) \hookrightarrow \text{AC}_{\text{inv}}(\mathcal{T}(S)). \]

Our rigidity theorem is given as follows.

**Theorem 9.2 (Rigidity theorem).** There is a monoid epimorphism
\[ \Xi : \text{AC}_{\text{inv}}(\mathcal{T}(S)) \to \text{Aut}(\mathcal{X}(S)) \]

with the following properties:

(1) If \(\omega' \in \text{AC}_{\text{inv}}(\mathcal{T}(S))\) is an asymptotic quasi-inverse of \(\omega \in \text{AC}_{\text{inv}}(\mathcal{T}(S))\), \(\Xi(\omega') = \Xi(\omega)^{-1}\);
(2) $J_t = \Xi \circ I$ as monoid homomorphisms.

In addition, $\Xi$ descends to a group isomorphism

$$\mathfrak{N}(\mathcal{T}(S)) \rightarrow \text{Aut}(\mathbb{X}(S)).$$

which satisfies the following commutative diagram:

\[
\begin{array}{ccc}
\text{MCG}^*(S) & \xrightarrow{T_0} & \text{Isom}(\mathcal{T}(S)) & \xrightarrow{T} & \mathfrak{N}_{\text{inv}}(\mathcal{T}(S))
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\text{group iso}}
\end{array}
\]

\[
\begin{array}{ccc}
\mathfrak{N}(\mathcal{T}(S)) & \xrightarrow{\text{group iso}} & \text{Aut}(\mathbb{X}(S))
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\Xi}
\end{array}
\]

PROOF. When $S$ is a torus with two holes, the quotient map $S \rightarrow S'$ by the hyper-elliptic action induces an isometry between the Teichmüller spaces of $S$ and $S'$ and an isomorphism between $\mathbb{X}(S)$ and $\mathbb{X}(S')$, where $S'$ is a sphere with five holes (cf. [11] and [26]). Hence, we may assume that $S$ is not a torus with two holes. For $\omega \in \mathfrak{N}_{\text{inv}}(\mathcal{T}(S))$, we take $h_\omega \in \text{Aut}(\mathbb{X}(S))$ as Theorem 9.1. Define a homomorphism $\Xi$ by $\Xi(\omega) = h_\omega$.

Theorem 9.1 asserts that $\Xi$ satisfies the condition (1) in the statement and descends to a homomorphism

$$\mathfrak{N}(\mathcal{T}(S)) \ni [\omega] \mapsto \Xi(\omega) = h_\omega \in \text{Aut}(\mathbb{X}(S)).$$

We next check the condition (2) in the statement. Since $\omega \in \text{Isom}(\mathcal{T}(S))$ preserves $S$ in $\mathcal{PMF} \subset \partial_{GM} \mathcal{T}(S)$, from the definition of $h_\omega$, for any $\alpha \in S$, $\Xi(\omega)(\alpha)$ coincides with $\omega(\alpha)$ (cf. Section 9 in [36]). This means that $\mathcal{J}(\omega) = \Xi \circ I(\omega)$.

We here check (68) is an epimorphism. Since $S$ is not a torus with two holes, $\mathcal{J}$ is an epimorphism, and so are $\Xi$ and (69). The injectivity of (68) (or (69)) is proven in the next section. □

9.4. Injectivity of homomorphism.

In this section, we shall show that the epimorphism (68) is an isomorphism. We first check the following.

PROPOSITION 9.1. Suppose that $S$ is not a torus with two holes. For $\omega \in \mathfrak{N}_{\text{inv}}(\mathcal{T}(S))$, there is a homeomorphism $f_\omega$ of $S$ with the following property: For any $p \in \partial_{GM} \mathcal{T}(S)$, $[G] \in \mathcal{AF}(p)$ and $q \in \mathcal{A}(\omega: p)$, we have $\mathfrak{N}_S(q) = \mathfrak{N}_S([f_\omega(G)])$.

PROOF. From the assumption and Theorem 9.2, there is a homeomorphism $f_\omega$ of $S$ such that $h_\omega(\alpha) = f_\omega(\alpha)$. From Theorem 7.1, if we take $[H] \in \mathcal{AF}(q)$, then

$$\mathfrak{N}_S(q) = \mathfrak{N}_S([H]).$$

From Theorem 9.1, for any $\alpha \in S$, $i(G, \alpha) = 0$ if and only if $[f_\omega(\alpha)] = [h_\omega(\alpha)] \in \mathfrak{N}_S(q)$. Hence, we deduce that

$$\mathfrak{N}_S([H]) \cap S = \mathfrak{N}_S(q) \cap S = \mathfrak{N}_S([f_\omega(G)]) \cap S$$

(70)
where $\mathcal{S}$ stands for a subset of $\partial_G \mathcal{T}(S)$ in (70). Therefore, the support of $H^\circ$ coincides with the support of $f_\omega(G)^\circ$. In particular, any essential curve of $H$ is also that of $f_\omega(G)$, and vice versa. As (27), we decompose $G$ as

$$G = G_1 + G_2 + \cdots + G_{m_1} + \beta_1 + \cdots + \beta_{m_2} + \gamma_1 + \cdots + \gamma_{m_3}.$$ 

Let $X_i$ be the support of a minimal component $G_i$ of $G$.

It is known that $\text{cl}_{GM}(\mathcal{T}(S))$ is metrizable. For instance

$$d_\infty(p^1, p^2) = \sup_{p \in \partial_{GM}(\mathcal{T}(S))} |i_{x_0}(p^1, p) - i_{x_0}(p^2, p)|$$

(71)

is a metric on $\text{cl}_{GM}(\mathcal{T}(S))$ since $\mathcal{S} \subset \partial_{GM}(\mathcal{T}(S))$ (cf. Theorem 1.2 in [33]).

Fix $i = 1, \ldots, k$. Take a sequence $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ such that $\alpha_n \subset X_i$ and

$$d_\infty([\alpha_n], [G_i]) < 1/n.$$ 

Since $f_\omega$ is a homeomorphism, $[f_\omega(\alpha_n)]$ tends to $[f_\omega(G_i)]$ in $\mathcal{P}\mathcal{M}\mathcal{F}$ (and hence in $\text{cl}_{GM}(\mathcal{T}(S))$) as $n \to \infty$. By taking a subsequence, we may assume that

$$d_\infty([f_\omega(\alpha_n)], [f_\omega(G_i)]) < 1/n$$

for all $n \in \mathbb{N}$.

Let $x^n = \{x^m_n\}_{m \in \mathbb{N}}$ be a sequence in $\mathcal{T}(S)$ converging to $[\alpha_n]$ in $\text{cl}_{GM}(\mathcal{T}(S))$. Since $\omega \in \text{AC}_{\text{inv}}(\mathcal{T}(S))$, by Theorem 9.1, $\omega(x^n)$ converges to $[f_\omega(\alpha_n)]$ in $\text{cl}_{GM}(\mathcal{T}(S))$ for all $n$. By applying the diagonal argument and taking a subsequence if necessary, we can take $m(n) \in \mathbb{N}$ such that if we put $z_n = x^m_n$ and $z = \{z_n\}_{n \in \mathbb{N}}$, then

$$\max\{d_\infty(z_n, [G_i]), d_\infty(\omega(z_n), [f_\omega(\alpha_n)])\} < 2/n$$

(72)

in $\text{cl}_{GM}(\mathcal{T}(S))$. Since $f_\omega$ is a homeomorphism of $S$, $[f_\omega(\alpha_n)]$ tends to $[f_\omega(G_i)]$ in $\mathcal{P}\mathcal{M}\mathcal{F}$ and hence in $\text{cl}_{GM}(\mathcal{T}(S))$. From (72), we have

$$d_\infty(\omega(z_n), f_\omega([G_i])) \leq d_\infty(\omega(z_n), f_\omega([\alpha_n])) + d_\infty(f_\omega([\alpha_n]), [f_\omega(G_i)]) \to 0$$

as $n \to \infty$. Therefore $\omega(z)$ converges to $[f_\omega(G_i)]$. This means that $[f_\omega(G_i)] \in \mathcal{A}(\omega; [G_i])$.

Since $G_i$ is a minimal component of $G$, $\mathcal{N}_S(p) = \mathcal{N}_S([G_i]) \subset \mathcal{N}_S([G_i])$. Therefore, by Lemma 8.1, we conclude

$$\mathcal{N}_S([H]) = \mathcal{N}_S(q) \subset \mathcal{N}_S([f_\omega(G_i)])$$

since $q \in \mathcal{A}(\omega; p)$. Therefore, $H$ contains a minimal component $H_i$ which is topologically equivalent to $f_\omega(G_i)$. Since the support of $H^\circ$ coincides with that of $G^\circ$, minimal components of $H$ are contained in $\bigcup f_\omega(X_i)$. Hence, the normal form of $H$ should be

$$H = \sum_{i=1}^{m_1} H_i + \sum_{i=1}^{m_2} a_i f_\omega(\beta_i) + \sum_{i=1}^{m_1} \sum_{\gamma \subset \partial f_\omega(X_i)} b_i \gamma$$
where \( a_i > 0 \) and \( b_i \geq 0 \). Thus, \( H^0 \) is topologically equivalent to \( f_\omega(G)^0 \). Hence by Corollary 7.1, we deduce

\[
\mathfrak{N}_S([f_\omega(G)]) = \mathfrak{N}_S([H]) = \mathfrak{N}_S(q),
\]

which is what we desired. \( \square \)

**Proposition 9.2 (Induced isometry).** For any \( \omega \in \text{AC}_{\text{inv}}(T(S)) \), there is a unique isometry \( \xi_\omega \) on \( T(S) \) which is close to \( \omega \) at infinity.

**Proof.** The case where \( S \) is a torus with two holes follows from the fact that the Teichmüller space of \( S \) is isometric to the Teichmüller space of a sphere with five holes. Hence, we may suppose that \( S \) is not a torus with two holes.

Take \( f_\omega \) as in Proposition 9.1. Since \( f_\omega \) is a homeomorphism of \( S \), \( f_\omega \) induces an isometry \( \xi_\omega \) on \( T(S) \). When \( S \) is a closed surface of genus 2, there is an ambiguity of the choice of \( f_\omega \), which is caused by the hyperelliptic involution. However, the isometry \( \xi_\omega \) is independent of the choice.

Let \( x^1, x^2 \in \text{Sq}^\infty(T(S)) \) satisfying \( \text{Vis}(x^1) = \text{Vis}(x^2) \). From Proposition 8.2, there are \( G, H_1, H_2 \in \mathcal{MF} \) such that

\[
\mathfrak{N}_S([G]) = \mathcal{AM}_S(x^1) = \mathcal{AM}_S(x^2),
\]

\[
\mathfrak{N}_S([H_1]) = \mathcal{AM}_S(\omega(x^1)),
\]

\[
\mathfrak{N}_S([H_2]) = \mathcal{AM}_S(\xi_\omega(x^2)).
\]

Hence, our assertion follows from Proposition 8.3 and the following lemma. \( \square \)

**Lemma 9.2.** It holds

\[
\mathfrak{N}_S([H_1]) = \mathfrak{N}_S([f_\omega(G)]) = \mathfrak{N}_S([H_2]).
\]

**Proof.** Let \( \overline{w} \in \text{Vis}(\omega(x^1)) \) and \( q \in \overline{w} \cap \partial_{GM} T(S) \). Let \( p \in \overline{x^1} \cap \partial_{GM} T(S) \) and fix \( [G_p] \in \mathcal{AF}(p) \). From Proposition 9.1, \( i_{x_0}(q, [f_\omega(G_p)]) = 0. \) Since \( p \) is taken arbitrarily in \( \overline{x^1} \cap \partial_{GM} T(S) \), from the proof of Proposition 8.2, we have \( i_{x_0}(q, [f_\omega(G)]) = 0. \) Hence

\[
\mathfrak{N}_S([H_1]) \subset \mathfrak{N}_S([f_\omega(G)]). \tag{73}
\]

Let \( q' \in \mathfrak{N}_S([f_\omega(G)]) \). For \( q \in \overline{\omega(x^1)} \cap \partial_{GM} T(S) \), we take a subsequence \( z \) of \( x^1 \) such that \( \omega(z) \) converges to \( q \) and \( z \) converges to some \( p \in \overline{x^1} \cap \partial_{GM} T(S) \). Fix \( [G_p] \in \mathcal{AF}(p) \). From Proposition 9.1, we have \( \mathfrak{N}_S(q) = \mathfrak{N}_S([f_\omega(G_p)]) \). From the construction of \( G, G_p^0 \) is topologically equivalent to a subfoliation of \( G \). Hence, \( q' \in \mathfrak{N}_S([f_\omega(G_p)]) = \mathfrak{N}_S(q) \). Since \( q \) is taken arbitrarily from \( \overline{\omega(x^1)} \cap \partial_{GM} T(S) \), from (73), we deduce that \( q' \in \mathfrak{N}_S([H_1]) \) and

\[
\mathfrak{N}_S([H_1]) = \mathfrak{N}_S([f_\omega(G)]). \tag{74}
\]

Since \( \xi_\omega \) is an isometry,
Since $\xi_\omega$ extends to $\text{cl}_{GM}(T(S))$ homeomorphically and coincides with the action of $f_\omega$ on $\mathcal{PMF} \subset \partial_{GM}T(S)$, we have

$$\mathcal{N}_S([H_2]) \cap \mathcal{PMF} = AC\mathcal{MS}(\xi_\omega(x^2)) \cap \mathcal{PMF} = \xi_\omega(AC\mathcal{MS}(x^2)) \cap \mathcal{PMF} = \xi_\omega(\mathcal{N}_S([G])) \cap \mathcal{PMF} = \mathcal{N}_S([f_\omega(G)]) \cap \mathcal{PMF}.$$ 

This equality means that $\mathcal{N}_{MF}(H_2) = \mathcal{N}_{MF}(f_\omega(G))$. By Corollary 7.1, we have $\mathcal{N}(H_2) = \mathcal{N}(f_\omega(G))$ and $\mathcal{N}_S([H_2]) = \mathcal{N}_S([f_\omega(G)])$. \hfill \(\square\)

**Proof of the injectivity of the homomorphism (68).** Let $\omega \in AC_{\text{as}}(T(S))$ be in the kernel of $\Xi$. From Proposition 9.2, there is an isometry $\xi_\omega$ which is close to $\omega$. Since $\Xi(\xi_\omega) = \Xi(\omega) = id$, $\xi_\omega$ is the identity mapping on $T(S)$, and hence $\omega$ is close to the identity. \hfill \(\square\)

### 9.5. Rough homotheties on Teichmüller space.

In this section, we shall prove Theorem C.

Suppose first that $\text{cx}(S) \geq 2$. We may assume that $S$ is not a torus with two holes. Suppose to the contrary that there is a $(K, D)$-rough homothety $\omega$ with asymptotic quasi-inverse for some $K \neq 1$. Notice that $\omega \in AC_{\text{inv}}(T(S))$. Take a homeomorphism $f_\omega$ on $S$ as Proposition 9.1.

Let $\alpha, \beta \in S$. Consider the projective classes $[\alpha]$ and $[\beta]$ as points in $\partial_{GM}T(S)$. Then, from (25), Theorem 9.1 and Proposition 9.1, we have

$$e^{-D_0}i_{x_0}([\alpha], [\beta])^K \leq i_{x_0}([f_\omega(\alpha)], [f_\omega(\beta)]) \leq e^{D_0}i_{x_0}([\alpha], [\beta])^K \quad (75)$$

where $D_0$ is a constant depending only on $D$ and $d_T(x_0, \omega(x_0))$. On the other hand, let $K_0 = e^{2d_T(x_0, \xi_\omega(x_0))}$, where $\xi_\omega$ is an isometry associated to $\omega$ taken as Proposition 9.2. From the definition, $\text{Ext}_{x_0}(f_\omega(G)) = \text{Ext}_{\xi_\omega^{-1}(x_0)}(G)$ for $G \in \mathcal{MF}$. By the quasiconformal invariance of extremal length, we obtain

$$K_0^{-1}i_{x_0}([\alpha], [\beta]) \leq i_{x_0}([f_\omega(\alpha)], [f_\omega(\beta)]) \leq K_0i_{x_0}([\alpha], [\beta])$$

since $f_\omega$ is a homeomorphism on $S$ and $i(f_\omega(\alpha), f_\omega(\beta)) = i(\alpha, \beta)$. Therefore, we deduce

$$i_{x_0}([\alpha], [\beta])^{1-K} \leq K_0e^{D_0}, \quad (76)$$

$$i_{x_0}([\alpha], [\beta])^{K-1} \leq K_0e^{D_0}, \quad (77)$$

for any $\alpha, \beta \in S$ with $i_{x_0}([\alpha], [\beta]) \neq 0$. Since the left-hand sides of (76) and (77) are projectively invariant, when the projective classes $[\alpha], [\beta]$ tend together to some projective measured foliation $[G] \in \mathcal{PMF}$ with keeping satisfying $i(\alpha, \beta) \neq 0$, the left-hand side in (76) diverges if $K > 1$, otherwise the left-hand side in (77) diverges. In any case, we get a contradiction.

We now consider the case where $\text{cx}(S) = 1$. This case is indeed a prototype of our study. In this case, there is an isometry $T(S) \rightarrow \mathbb{D}$ sending $x_0$ to the origin 0. Furthermore, the Gromov product $\langle x_1 | x_2 \rangle_0$ for $x_1, x_2 \in \mathbb{D}$ satisfies
\[ |\langle x_1 \mid x_2 \rangle_0 - d_D(0, [x_1, x_2])| \leq D_1 \] (78)

for some universal constant \( D_1 > 0 \), where \([x_1, x_2]\) is the geodesic connecting between \( x_1 \) and \( x_2 \) (cf. Section 2.33 in [44]).

Suppose on the contrary that there is a \((K, D)\)-rough homothety \( \omega \) with \( K \neq 1 \). Notice from the definition that any \( \omega \in AC_{inv}(\mathbb{D}) \) extends to a bijective mapping on \( \partial \mathbb{D} \). We can easily see that the extension is continuous, and hence, \( \omega \) extends to a self-homeomorphism on \( \partial \mathbb{D} \). We may assume that \( \omega(0) = 0 \).

From (78), for \( x_1, x_2 \in \mathbb{D} \),
\[ |d_D(0, [\omega(x_1), \omega(x_2)]) - K d_D(0, [x_1, x_2])| \leq D_2 \]
for some constant \( D_2 > 0 \). Therefore, for any \( p_1, p_2 \in \partial \mathbb{D} \), we have
\[ C_1 |p_1 - p_2|^K \leq |\omega(p_1) - \omega(p_2)| \leq C_2 |p_1 - p_2|^K \] (79)
with positive constants \( C_1, C_2 \). If \( K > 1 \), \( \omega \) is differentiable and the derivative is zero at any \( \partial \mathbb{D} \). Hence, \( \omega \) should be a constant on \( \partial \mathbb{D} \), which is a contradiction. Suppose \( K < 1 \). Since the lift of a self-homeomorphism on \( \partial \mathbb{D} \) to \( \mathbb{R} \) is a monotone function, the extension of \( \omega \) to \( \partial \mathbb{D} \) is differentiable almost everywhere on \( \partial \mathbb{D} \). However, from (79), \( \omega \) is not differentiable any point on \( \partial \mathbb{D} \). This is also a contradiction.

10. Appendix.

The main result of this section is Lemma 10.1. The estimates in the lemma look similar to those in Theorem 6.1 of [13]. However, our advantage here is that we treat the extremal lengths of all non-trivial (possibly peripheral) curves of subsurfaces and give a constant \( C_\gamma \) concretely (cf. (81) and (85)).

10.1. Measured foliations and intersection numbers.

Let \( Q \) be a holomorphic quadratic differential on \( X \). The differential \( |Re \sqrt{Q}| \) defines a measured foliation on \( X \). We say that such a measured foliation the vertical foliation of \( Q \). The vertical foliation of \( -Q \) is called the horizontal foliation of \( Q \).

By a step curve, we mean a geodesic polygon in \( X \) the sides of which are horizontal and vertical arcs of \( Q \) (cf. Figure 3). For the intersection number functions defined by the vertical foliations of holomorphic quadratic differentials, it is known the following.

\[ \beta_1 \]
\[ \alpha_1 \]
\[ \beta_2 \]
\[ \gamma_0 \]

Figure 3. A step curve with the property stated in Proposition 10.1.
Proposition 10.1 (Theorem 24.1 of [43]). Let $Q$ be a quadratic differential and $F$ the vertical foliation of $Q$. Let $\gamma_0$ be a simple closed step curve with the additional property that for any vertical side $\alpha_i$ of $\gamma_0$ the two neighboring horizontal sides $\beta_1$ and $\beta_2$ are on different sides of $\alpha_i$ (there are no zeros of $Q$ on $\gamma_0$). Then,

$$i(\gamma, F) = \int_{\gamma_0} |\Re \sqrt{Q}|,$$

where $\gamma$ is the homotopy class containing $\gamma_0$.

It can be also observed that a step curve with the property stated in Theorem 10.1 is quasi-transversal. For instance, see the proof of Proposition II.6 or the curve (4) of Figure 10 of Exposé 5 in [10].

10.2. Filling curves and Extremal length.

Let $X_0$ be an essential subsurface of $X$. Denote by $S(X_0)$ a subset of $S$ consisting of curves which are non-peripheral in $X_0$. Let $S_0(X_0)$ be a subset of $S$ consisting of curves which can be deformed into $X_0$.

Lemma 10.1. Let $X_0$ be a connected, compact and essential subsurface of $X$ with negative Euler characteristic. Let $\{\alpha_i\}_{i=1}^m \subset S(X_0)$ be a system of curves which fills up $X_0$. Then, for $\gamma \in S_0(X_0)$, we have

$$\text{Ext}_X(\gamma) \leq C_\gamma \max_{1 \leq i \leq m} \text{Ext}_X(\alpha_i),$$

where

$$C_\gamma = C(g, n, m) \left( \sum_{i=1}^m i(\alpha_i, \gamma) \right)^2 + 4(6g - 6 + n)^2$$

and $C(g, n, m)$ depends only on the topological type $(g, n)$ of $X$ and the number $m$ of the system $\{\alpha_i\}_{i=1}^m$. In particular, we have

$$\text{Ext}_X(F) \leq C(g, n, m) \left( \sum_{i=1}^m i(\alpha_i, F) \right)^2 \max_{1 \leq i \leq m} \text{Ext}_X(\alpha_i),$$

for all $F \in M\mathcal{F}(X_0) \subset M\mathcal{F}$.

Proof. Let $\gamma \in S_0(X_0)$. We divide the proof into two cases.

Case 1: $\gamma$ is peripheral in $X_0$. Suppose first that $\gamma$ is represented by a component of $\partial X_0$. When $\gamma$ is homotopic to a puncture of $X$, $\text{Ext}_X(\gamma) = 0$ since $X$ contains an arbitrary wide annulus whose core is homotopic to $\gamma$. Hence we have nothing to do (in fact, we can set $C_\gamma = 0$).

Suppose that $\gamma$ is not peripheral in $X$. Let $J_\gamma$ be a Jenkins–Strebel differential for $\gamma$ on $X$. Let $A_\gamma$ be the characteristic annulus of $J_\gamma$. We consider a “compactification” $\overline{A_\gamma}$ by attaching two copies of circles as its boundaries. The induced flat structure on $A_\gamma$ from $J_\gamma$ canonically extends to the compactification $\overline{A_\gamma}$ and components of the boundary $\partial \overline{A_\gamma}$.
are closed regular trajectories under this flat structure. There is a canonical surjection \( I_\gamma : \overline{A_\gamma} \to \overline{X} \) (the completion of \( X \) at the punctures). Namely, \( \overline{X} \) is reconstructed by identifying disjoint vertical straight arcs in \( \partial \overline{A_\gamma} \) along vertical saddle connections of \( J_\gamma \). (In this sense, \( I_\gamma \) is a quotient map). Without any confusion, we may recognize the characteristic annulus \( A_\gamma \) itself as a subset of \( \overline{X} \).

Let \( \gamma^* \) and \( \alpha_i^* \) be the core trajectory in \( A_\gamma \) and the geodesic representative of \( \alpha_i \) with respect to \( J_\gamma \) respectively. Since \( \gamma \) is parallel to \( \partial X_0 \), by taking an isotopy, we may assume that \( \gamma^* \) is a component of \( \partial X_0 \). Furthermore, since \( \alpha_i \in S(X_0) \), \( \gamma \) does not intersect any \( \alpha_i \) for all \( i \). Hence, each \( \alpha_i^* \) consists of vertical saddle connections. In other words, \( \alpha_i^* \) is contained in the critical graph \( \Sigma_\gamma = I_\gamma(\partial \overline{A_\gamma}) \) of \( J_\gamma \) in \( X \), which consists of vertical saddle connections of \( J_\gamma \).

Let \( \gamma_1 \) and \( \gamma_2 \) be components of \( \partial \overline{A_\gamma} \). Each \( \gamma_i^* := I_\gamma(\gamma_i) \) is canonically recognized as a path in \( \Sigma_\gamma \) consisting of vertical saddle connections. We claim:

**Claim 1.** One of \( \gamma_i^* \), say \( \gamma_1^* \), is contained in the union \( \bigcup_{i=1}^m \alpha_i^* \).

**Proof of Claim 1.** Suppose \( \gamma_1^* \cap \alpha_i^* \neq \emptyset \) for some \( i \) and \( \gamma_1^* \) contains a vertical saddle connection \( s_0 \) such that \( s_0 \not\subset \alpha_i^* \) for all \( i \). Then, \( s_0 \) intersects all \( \alpha_i^* \) at most at endpoints (critical points of \( J_\gamma \)). Let \( \text{Int}(s_0) = s_0 \setminus \partial s_0 \). Let \( h_1 \) be a horizontal arc in \( A_\gamma \) starting at \( p_1 \in \gamma^* \) and terminating at a point of \( \text{Int}(s_0) \). Since the both side of \( s_0 \) is in \( A_\gamma \), after \( h_1 \) passes through \( s_0 \), \( h_1 \) terminates at a point \( p_2 \in \gamma^* \). Let \( \gamma_0^* \) be a segment of \( \gamma^* \) connecting \( p_1 \) and \( p_2 \) (cf. Figure 4). Set \( \beta = \gamma_0^* \cup h_1 \). By definition, \( \beta \) does not intersect any \( \alpha_i^* \) and hence \( i(\beta, \alpha_i) = 0 \) for all \( i \).

Suppose first that \( h_1 \) arrived at \( p_2 \) from the different side from that where \( h_1 \) departed at \( p_1 \) (cf. (1) of Figure 4). Then, we have

\[
i(\gamma, \beta) = \int_{\beta} |\text{Re} \sqrt{J_\gamma}| = 1,
\]
since the width of \( A_\gamma \) is one and \( \beta \) is a step curve with the property stated in Proposition 10.1. Hence \( \beta \) is non-trivial and non-peripheral simple closed curve in \( X \). However, this contradicts that \( \{ \alpha_i \}_{i=1}^m \) fills up \( X_0 \), since such a \( \beta \cap X_0 \) contains homotopically non-trivial arc connecting \( \partial X_0 \) because \( \gamma \) is parallel to a component of \( \partial X_0 \).

![Figure 4. Trajectories in \( A_\gamma \).](image)
Suppose $h_1$ arrived at $p_2$ from the same side as that where $h_1$ departed (cf. (2) of Figure 4). We may also assume that $h_1$ departs from $p_1$ into $X_0$. Indeed, suppose we cannot assume so. Then, the component of $\partial A_\gamma$ that lies on the same side as that of $X_0$ (near $\gamma^*$) is covered by $\{\alpha_i^*\}_{i=1}^m$, which contradicts what we assumed first.

Then, there is an open rectangle $R_0$ in $A_\gamma$ such that $\beta$ and a segment in $\gamma_1$ surround $R_0$ in $A_\gamma$. From the assumption, we may assume that the closure of $I_\gamma(R_0)$, say $X_1$, intersects some $\alpha_i^*$. Suppose that $\beta$ is trivial. Then, $X_1$ is a disk in $X$ surrounded by $\beta$, since $\gamma^*$ can be homotopic to the outside of $X_1$. This means that $\alpha_i^*$ is contained in a disk $X_1$ because $\alpha_i^*$ does not intersect $\beta$, which is a contradiction. By the same argument, we can see that $\beta$ is non-peripheral (otherwise, $\alpha_i^*$ were peripheral). Since $h_1$ departs into $X_0$ at $p_1$ and returns to $\gamma^*$ on the side where $X_0$ lies, after taking an isotopy if necessary, we can see that $h_1$ contains a subsegment which is nontrivial in $X_0$ and connecting $\partial X_0$, which contradicts again that $\{\alpha_i\}_{i=1}^m$ fills $X_0$ up. □

Let us continue to prove Lemma 10.1 for peripheral $\gamma \in S_0(X_0)$. We take $\gamma_i^*$ as in Claim 1. Since both sides of every vertical saddle connection face $A_\gamma$, $\gamma_i^*$ visits each vertical saddle connection at most twice. Notice that the number of vertical saddle connections is at most $6g - 6 + n$. Since each vertical saddle connection in $\gamma_i^*$ is contained in some $\alpha_i^*$, we have

$$\ell_{J_\gamma}(\gamma) = \ell_{J_\gamma}(\gamma_i^*) \leq 2(6g - 6 + n) \max\{\ell_{J_\gamma}(\alpha_i^*) \mid i = 1, \ldots, n\}$$

$$= 2(6g - 6 + n) \max\{\ell_{J_\gamma}(\alpha_i) \mid i = 1, \ldots, n\},$$

since $\alpha_i^*$ is the geodesic representative of $\alpha_i$. Since the width of $A_\gamma$ is one, from (31), we conclude

$$\text{Ext}_X(\gamma) = \ell_{J_\gamma}(\gamma)^2/\|J_\gamma\|$$

$$\leq 4(6g - 6 + n)^2 \max\{\ell_{J_\gamma}(\alpha_i)^2/\|J_\gamma\|\}$$

$$\leq 4(6g - 6 + n)^2 \max_{1 \leq i \leq m} \text{Ext}_X(\alpha_i). \quad (83)$$

**Case 2 : $\gamma \in S(X_0)$**. We next assume that $\gamma$ is not parallel to any component of $\partial X_0$. Let $\{\beta_i\}_{i=1}^s$ be components of $\partial X_0$ each of which is non-peripheral in $X$. Let $\epsilon > 0$ and set

$$F_\epsilon = \gamma + \epsilon \sum_{i=1}^s \beta_i \quad (84)$$

(cf. [18]). It is possible that two curves $\beta_{i_1}$ and $\beta_{i_2}$ are homotopic in $X$. In this case, we recognize $\beta_{i_1} + \beta_{i_2} = \beta_{i_1}$ in (84). However, for the simplicity of the discussion, we shall assume that any two of $\{\beta_i\}_{i=1}^s$ are not isotopic. The general case can be treated in a similar way.

Let $J_\gamma^\epsilon$ be the holomorphic quadratic differential on $X$ whose vertical foliation is $F_\epsilon$. Since $F_\epsilon \to \gamma$ in $\mathcal{MF}$, $J_\gamma^\epsilon$ tends to $J_\gamma$ in $Q_X$ (cf. [16]. See also Theorem 21.3 in [43]). Let $A_\gamma^\epsilon$ and $A_i^\epsilon$ denote the characteristic annuli of $J_\gamma^\epsilon$ for $\gamma$ and $\beta_i$, respectively.
Set $\gamma^*_{\epsilon,*}$ and $\beta^*_{\epsilon,*}$ to be closed trajectories in homotopic to $\gamma$ and $\beta_i$, respectively. Let $Y_0^\gamma$ be the closure of the component of $\epsilon/4$-neighborhood of the cores $\beta^*_{\epsilon,*}$, containing $A^\gamma_0$. By definition, we may identify $Y_0^\gamma$ with $X_0$. Let $\alpha^*_{\epsilon,*}$ be the geodesic representation of $\alpha_i$ with respect to $J_\gamma^\epsilon$.

We fix an orientation on $\gamma^*_{\epsilon,*}$. Let $\xi$ be a component of $\gamma^*_{\epsilon,*} \cup \bigcup_{i=1}^m \alpha^*_{\epsilon,*}$. Let $I_0(\xi)$ be the set of points $p \in \xi$ such that the horizontal ray $r_p$ departing at $p$ from the right of $\xi$ terminates at a curve in $\{\alpha^*_{\epsilon,*}, \beta^*_{\epsilon,*}\}_{i,j}$ before intersecting $\xi$ twice. Let $C_0(\xi)$ be the set of $p \in \xi$ such that $r_p$ terminates at a critical point of $J_\gamma^\epsilon$. Then, we claim

**Claim 2.** $\xi \setminus I_0(\xi) \subset C_0(\xi)$, and $I_0(\xi) \setminus C_0(\xi)$ is open in $\xi$.

**Proof of Claim 2.** Let $p \in \xi \setminus I_0(\xi)$. Suppose $p \notin C_0(\xi)$. Since the completion $\overline{X}$ with respect to the punctures is closed, $r_p$ is recurrent (cf. Section 10 of Chapter IV in [43]). By the definition of $I_0(\xi)$ and $p \notin I_0(\xi)$, $r_p$ intersects $\xi$ at least twice before intersecting curves in $\{\alpha^*_{\epsilon,*}, \beta^*_{\epsilon,*}\}_{i,j}$. Hence, $r_p$ contains a consecutive horizontal segments $h_1$ and $h_2$ such that each $h_i$ intersects $\xi$ only at its endpoints, and does not intersect any curves in $\{\alpha^*_{\epsilon,*}, \beta^*_{\epsilon,*}\}_{i,j}$.

When one of the segments, say $h_1$, connects both sides of $\xi$, $\xi$ contains a vertical segment $v_1$ connecting endpoints of $h_i$, and two trajectories $h_1$ and $v_1$ make a closed curve $\delta$ on $X$. Since the two ends of $h_i$ terminate at $\xi$ from different sides, the intersection number satisfies

$$i(F_\epsilon, \delta) = \int_{\delta} \text{Re} \sqrt{J_\gamma^\epsilon}$$

and is greater than or equal to the width of $A^\gamma_0$, by Proposition 10.1. Therefore $\delta$ is non-trivial and non-peripheral in $X$. Since $h_i$ does not intersect $\beta^*_{\epsilon,*}$, $\delta$ is contained in $Y_0^\gamma$, where we have identified with $X_0$. Furthermore, $\delta$ is not peripheral in $X_0$ because $\delta$ has non-trivial intersection with $\gamma$. By definition, $\delta$ does not intersect all $\alpha_i$, which is a contradiction because $\{\alpha_i\}_{i=1}^m$ fills $X_0$ up.

We assume that two ends of each $h_i$ terminate at $\xi$ from the same side. In this case, we can also construct a simple closed step curve $\delta$ with the property stated in Proposition 10.1 from $h_1$, $h_2$ and a subsegment of $\xi$ (cf. Figure 5). This is a contradiction as above. Thus we conclude that $\xi \setminus I_0(\xi) \subset C_0(\xi)$.

We show that $I_0(\xi) \setminus C_0(\xi)$ is open in $\xi$. Let $p \in I_0(\xi) \setminus C_0(\xi)$ such that the horizontal ray $r_p$ defined above does not terminate at critical points of $J_\gamma^\epsilon$. By definition, the horizontal ray $r_p$ terminate the interior of a straight arc contained in either $\alpha^*_{\epsilon,*}$ or $\beta^*_{\epsilon,*}$, respectively. Hence, when $p' \in \xi$ is in some small neighborhood of $p$, $r_{p'}$ also terminates at such a straight arc, and hence $p' \in I_0(\xi)$ for any point $p'$ in a small neighborhood of $p$.  

Let us return to the proof of Case 2 of the lemma. Let $\xi$ be a component of $\gamma^* \cup \bigcup_{i=1}^m \alpha^*_{\epsilon,*}$. By definition, for $p \in I_0(\xi)$, $r_p$ terminates at $\xi$ at most once before intersecting curves in $\{\alpha^*_{\epsilon,*}, \beta^*_{\epsilon,*}\}_{i,j}$. Since any horizontal ray $r_p$ with $p \notin C_0(\xi)$ can terminate at a curve in $\{\alpha^*_{\epsilon,*}, \beta^*_{\epsilon,*}\}_{i,j}$ from at most two sides. Hence for almost all point $q$ in a curve in $\{\alpha^*_{\epsilon,*}, \beta^*_{\epsilon,*}\}_{i,j}$, there are at most 4 points in $I_0(\xi)$ such that the horizontal rays emanating there land at $q$. From Claim 2, we get
Figure 5. How to get a closed curve $\delta$: There are two cases. In the case (1), the initial point of $h_1$ and the terminal point of $h_2$ are separated by the terminal point of $h_1$. The case (2) describes the other case.

$$|\xi| = |I_0(\xi) \setminus C_0(\xi)| \leq 4 \left( \sum_{i=1}^{m} \ell_{J_{\gamma_i}}(\alpha_i) + \sum_{i=1}^{s} \ell_{J_{\gamma_i}}(\beta_i) \right),$$

where $| \cdot |$ means linear measure. Since $\alpha_i^{\epsilon,*}$ is the geodesic representative of $\alpha_i$, the number of components of $\gamma^{\epsilon,*} \setminus \bigcup_{i=1}^{m} \alpha_i^{\epsilon,*}$ is

$$\sum_{i=1}^{m} i(\alpha_i, \gamma).$$

Therefore, we obtain

$$\ell_{J_{\gamma}}(\gamma) \leq 4 \left( \sum_{i=1}^{m} i(\alpha_i, \gamma) \right) \left( \sum_{i=1}^{m} \ell_{J_{\gamma_i}}(\alpha_i) + \sum_{i=1}^{s} \ell_{J_{\gamma_i}}(\beta_i) \right).$$

Since $J_{\gamma}^{\epsilon}$ tends to $J_{\gamma}$ as $\epsilon \to 0$, for all $\eta > 0$ we find an $\epsilon > 0$ such that

$$\text{Ext}_X(\gamma) = \frac{\ell_{J_{\gamma}}(\gamma)^2}{\|J_{\gamma}\|} \leq \frac{\ell_{J_{\gamma}}(\gamma)^2}{\|J_{\gamma}^{\epsilon}\|} + \eta$$

$$\leq 16 \left( \sum_{i=1}^{m} i(\alpha_i, \gamma) \right)^2 (m + s)^2 \left( \sum_{i=1}^{m} \frac{\ell_{J_{\gamma_i}}(\alpha_i)^2}{\|J_{\gamma_i}\|} + \sum_{i=1}^{s} \frac{\ell_{J_{\gamma_i}}(\beta_i)^2}{\|J_{\gamma_i}\|} \right) + \eta$$

$$\leq C'_{\gamma, \text{max}} \text{Ext}_X(\alpha_i) + \eta$$

by Corollary 21.2 in [43] and the Cauchy–Schwartz inequality, where

$$C'_{\gamma} = 16(m + s)^2(m + 4s(6g - 6 + n)^2) \left( \sum_{i=1}^{m} i(\alpha_i, \gamma) \right)^2$$

from (83).

Notice that the number $s$ of components of $\partial X_0$ satisfies $s \leq 2g + n$. Indeed, we fix a hyperbolic metric on $X$ and realize $X_0$ as a convex hyperbolic subsurface of $X$. Let
$g'$ and $n'$ be the genus and the number of punctures in $X_0$. Since $X_0 \subset X$ and $X_0$ is essential, by comparing to the hyperbolic area, we have

$$2\pi(s - 2) \leq 2\pi(2g' - 2 + s + n') = \text{Area}(X_0) \leq \text{Area}(X) = 2\pi(2g - 2 + n),$$

and hence $s \leq 2g + n$.

Thus, by (83), we conclude that (80) holds with

$$C(g, n, m) := 16(m + 2g + n)^2(m + 4(2g + n)(6g - 6 + n)^2),$$

which implies what we wanted. □

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