ON THE FUNDAMENTAL GROUP AND TRIPLE MASSEY’S PRODUCT

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INTRODUCTION

We study the relations between the fundamental group and the homological operations on integer homology. For the “rational” fundamental group (Malcev completion) see [4, 5, 11, 16, 17].

This work is an attempt to understand the invariant of the fundamental group of the complement of a complex hyperplane arrangement that was used in [13]. Note that this invariant necessarily vanishes over $\mathbb{Q}$ (see [5]).

All homology and cohomology groups are with integer coefficients. By $|X|$ we denote the geometric realization of a simplicial set $X$.

1. PSEUDO-ISOMORPHISMS AND PSEUDO-HOMEOMORPHISMS

Theorem 1. Suppose an arcwise connected topological space $U$ has the homotopy type of a CW complex. Let $G = \pi_1(U, u)$, and let $Y = BG$ be the nerve of $G$. Then there is a continuous map $U \to BG$ inducing the natural isomorphism $\pi_1(U, u) = G \cong \pi_1(BG, (u))$ (hence, also an isomorphism $H_1(U) \to H_1(BG)$) and an epimorphism $H_2(U) \to H_2(BG)$.

Proof. Since $U$ has the homotopy type of a CW complex, $U$ is homotopy equivalent to $|S(U)|$, where $S(U)$ denotes the simplicial set of all singular simplices in $U$. Since $U$ is arcwise connected, $|S(U)|$ is homotopy equivalent to $|S_u(U)|$, where $u \in U$ is an arbitrary point and $S_u(U)$ denotes the simplicial set of all singular simplices in $U$ with all vertices equal to $u$. Therefore, it suffices to prove the theorem for $U = |X|$, where $X$ is a simplicial set with single vertex $x$, $u = x$.

In fact, we will construct a simplicial map $X \to BG$ inducing the natural isomorphism $\pi_1(|X|, x) = G \cong \pi_1(BG, (x))$ and an epimorphism $H_2(|X|) \to H_2(BG)$.

Let $X_k$ denote the set of non-degenerate $k$-simplices in $X$. Consider the free group $F = F(\hat{X}_1)$ and its nerve $BF$. It is well known that $H_0(BF) = \mathbb{Z}$, $H_1(BF) = \mathbb{Z}^{|X_1|}$, and $H_i(BF) = 0$ for $i > 1$. Denote by $X^1$ the minimal simplicial subset of $X$ containing $\hat{X}_1$. We identify $X^1$ with the corresponding simplicial subset of $BF$ and glue $|X|$ and $BF$ via this identification. Denote the resulting simplicial set by $Y$.

It is clear that $|X^1|$ is a union of 1-dimensional spheres, hence the inclusion $|X^1| \hookrightarrow BF$ induces isomorphism $H_i(|X^1|) \cong H_i(BF)$ for any $i$. Hence, from the long homological exact sequence of the pair $(BF, |X^1|)$ we see that $H_i(BF, |X^1|) = 0$ for any $i$. But $H_i(|Y|, |X|) = H_i(BF, |X^1|)$. From the long homological exact sequence of the pair $(|Y|, |X|)$ we see that the natural map $H_i(|X|) \to H_i(|Y|)$ is

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Research was supported in part by Grant M8H000/M8H300 from the International Science Foundation and Russian Government and by INTAS Grant 94-4720.
The fundamental group of $\sigma$ if it gives rise to an isomorphism $e_Z$ algebra variant of topological space w.r.t. pseudo-homeomorphisms is an invariant of its fundamental groups. Thus we can use $Y$ instead of $X$.

Let us define a simplicial map $\varphi : Y \to BG$. We have a natural homomorphism $F \to G$ that maps each element $a \in X$ to the corresponding element $g_a$ of the fundamental group $\pi_1(X, x) = G$. We put $\varphi((a_1, \ldots, a_n)) = (g_{a_1}, \ldots, g_{a_n})$ for each $(a_1, \ldots, a_n) \in B_n F$. Suppose $\sigma$ is a $k$-simplex in $X$. Let $a_k$ be 1-face of $\sigma$ corresponding to the inclusion $f_k : [0, 1] \to [0, m]$ given by $f_k(j) = j + k$. We put $\varphi(\sigma) = (g_{a_1}, \ldots, g_{a_k})$. It is obvious that $\varphi$ is a simplicial map. Besides, it is clear that the map $\varphi_* : \pi_1(\{Y\}, x) \to \pi_1(BG, (\))$ is an isomorphism.

Note that the map $\varphi_* : C(Y) \to C(BG)$ is surjective. Denote by $K$ the kernel of this map. We have $K_0 = 0$ and $K_1 = \langle (gn) - (g) \rangle_{n \in N}$, where $N$ is the kernel of the natural homomorphism $F \to G$. From the long exact sequence

$$\cdots \to H_2(Y) \to H_2(BG) \to H_1(K) \to \cdots$$

we see that it suffices to show that $H_1(K) = 0$.

It is clear that for any $g_1, g_2, h \in F$ such that $(g_1) - (g_2) \in \partial K_2$ we have $(h g_1) - (h g_2) \in \partial K_2$ and $(g_1 h) - (g_2 h) \in \partial K_2$. It follows that for any $g, h, f \in F$ such that $(g) - (1) \in \partial K_2$ we have $(h^{-1}g h) - (1) \in \partial K_2$ and for any $g, h, f \in F$ such that $(g) - (1) \in \partial K_2$ and $(h) - (1) \in \partial K_2$ we have $(g h) - (1) = (g h) - (h) + (h) - (1) \in \partial K_2$.

Let us consider arbitrary element $\sigma \in X_2$ and let $a = d_2 \sigma$, $b = d_0 \sigma$, and $c = d_1 \sigma$. Denote the corresponding elements of $F$ by $g_a, g_b$, and $g_c$. We have $(g_a g_b) - (g_c) = \partial((g_a, g_b) - \sigma) \in \partial K_2$. Hence $(g_a g_b g_c^{-1}) - (1) \in \partial K_2$. The elements of the form $g_a g_b g_c^{-1}$ and their conjugates generate the subgroup $N$. Therefore for any $n \in N$ we have $(n) - (1) \in \partial K_2$. It follows that $(g n) - (g) \in \partial K_2$ for any $n \in N, g \in F$, thus $K_1 = \partial K_2$. Hence $H_1(K) = 0$ and the map $\varphi_* : H_2(\{Y\}) \to H_2(BG)$ is an epimorphism.

Let us consider only arcwise connected topological spaces that have the homotopy type of CW complex. Any continuous map inducing isomorphism of $H_1$ and epimorphism of $H_2$ will be called a pseudo-homeomorphism. We see that any invariant of topological space w.r.t. pseudo-homeomorphisms is an invariant of its fundamental group.

Now we want to know what information about the fundamental group can be contained in such invariants.

**Definition 1.** Let $G$ be a group. Denote by $I$ the augmentation ideal of the group algebra $\mathbb{Z}G$ (that is, $I$ is generated by all elements of the form $g - e$, where $g \in G$ and $e$ is the identity element of $G$). We put $D^{(k)}(G) = \mathbb{Z}G/I^k$. Denote by $D(G)$ the projective system of $\mathbb{Z}$-algebras

$$\to D^{(k+1)}(G) \to D^{(k)}(G) \to D^{(k-1)}(G) \to \cdots \to D^{(1)}(G) = \mathbb{Z}.$$

Suppose $\varphi : G_1 \to G_2$ is a group homomorphism. We say that it is a pseudo-isomorphism if it gives rise to an isomorphism $D(G_1) \to D(G_2)$.

Let $X = (X_n)_{n \in \mathbb{N}}$ be a simplicial set with single vertex (that is, $X_0 = \{x\}$). The fundamental group of $|X|$ can be described as the group with generators $g_a$ ($a \in X_1$) and relations $g_d \sigma g_{d \sigma} = g_{d_1 \sigma}$ for any $\sigma \in X_2$, where $d_1 \sigma$ means $i$-th face of $\sigma$ (with $i$-th vertex missing).
Let \( C = C(X) \) be the chain complex of \( X \) over \( \mathbb{Z} \). It has the standard structure of coalgebra: for any \( n \)-simplex \( \sigma \) we have
\[
\Delta \sigma = \sum_{i+j=n} (i) \sigma \otimes (j),
\]
where \( (i) \sigma \) is the front \( i \)-dimensional face and \( (j) \sigma \) is the back \( j \)-dimensional face of \( \sigma \). Let \( \overline{\mathcal{C}} = C/C_0 \); since \( C_0 = \mathbb{Z}x \) is subcoalgebra of \( C \), we obtain comultiplication \( \Delta : \overline{\mathcal{C}} \to \overline{\mathcal{C}} \otimes \overline{\mathcal{C}} \). Denote by \( \mathcal{F}(\mathcal{C}) \) the tensor algebra \( T(s^{-1}\overline{\mathcal{C}}) \) (cobar construction \[ \square \]). Note that \( \mathcal{F}(\mathcal{C})_0 \) is a free associative algebra generated by \( X_1 \). We write \( [c_1,c_2,\ldots,c_k] \) instead of \( s^{-1}c_1 \otimes s^{-1}c_2 \otimes \cdots \otimes s^{-1}c_k \). The differential in \( \mathcal{F}(\mathcal{C}) \) is a derivation of the tensor algebra defined on generators as
\[
\partial[c] = [-\delta c] + \sum (-1)^{\deg a_i} [a_i] b_i,
\]
where \( \overline{\Delta} = \sum a_i \otimes b_i \) and \( \delta \) is the differential in \( \overline{\mathcal{C}} \).

Let \( \mathcal{F}^k(\mathcal{C}) = \mathcal{F}(\mathcal{C})/(T^k(s^{-1}\overline{\mathcal{C}})) \). It is clear that the ideal \( (T^k(s^{-1}\overline{\mathcal{C}})) = \bigoplus_{r \geq k} T^r(s^{-1}\overline{\mathcal{C}}) \) is a subcomplex of \( \mathcal{F}(\mathcal{C}) \); hence \( \mathcal{F}^k(\mathcal{C}) \) is a complex. Denote \( H_0(\mathcal{F}^k(\mathcal{C})) \) by \( A^k = A^k(X) \).

**Proposition 2.** Let \( G = \pi_1([X],v) \). Then there is an isomorphism \( D^k(G) \to A^k(X) \) such that for each \( a \in X_1 \) the image of \( g_a \) in \( D^k(G) \) corresponds to the image of \( 1 + [a] \) in \( A^k(X) \).

**Proof.** Denote \( 1 + [a] \in \mathcal{F}(\mathcal{C})_0 \) by \( \tilde{g}_a \). Let \( \sigma \) be a 2-simplex and let \( a = d_2 \sigma, b = d_0 \sigma, \) and \( c = d_1 \sigma \). Then \( \tilde{g}_a \tilde{g}_b - \tilde{g}_c = [a] + [b] + [a][b] - [c] = -\partial[\sigma] \), thus the corresponding element in \( A^k(X) \) is zero. Hence there is a homomorphism \( \mathbb{Z}G \to A^k \) sending each generator \( g_a \) to the image of \( \tilde{g}_a \) in \( A^k(X) \). From the definitions it is clear that its kernel is \( P^k \). \( \square \)

**Theorem 3.** Let \( U \) and \( V \) be arcwise connected topological spaces having the homotopy types of CW complexes. Suppose \( f : U \to V \) is a pseudo-homeomorphism. Then
\[
f_* : \pi_1(U,u) \to \pi_1(V,f(u))
\]
is a pseudo-isomorphism.

**Proof.** As in the proof of Theorem \[ \square \] we assume that \( U = [X] \) and \( V = [Y] \), where \( X \) and \( Y \) are simplicial spaces with single vertices \( x \) and \( y \) respectively. Besides, we assume that \( f = |F| \), where \( F : X \to Y \) is a simplicial map, \( u = x \) and, thus, \( f(u) = y \).

By Proposition \[ \square \] it suffices to show that the natural map \( A^k(X) \to A^k(Y) \) is an isomorphism for any \( k \in \mathbb{N} \). Consider the filtration of the complex \( \mathcal{F}^k(\mathcal{C}) \) for both \( C = C(X) \) and \( C = C(Y) \)
\[
\mathcal{F}^k(C) = \mathcal{F}^k_0(C) \supset \mathcal{F}^k_{-1}(C) \supset \cdots \supset \mathcal{F}^k_{-k+1}(C) \supset \mathcal{F}^k_{-k}(C) = \{0\}
\]
where \( \mathcal{F}^k_p(C) = (T^{-p}(s^{-1}\overline{\mathcal{C}})/T^k(s^{-1}\overline{\mathcal{C}})) \). We have a natural map of the corresponding spectral sequences \( E_{p,q}(X) \to E_{p,q}(Y) \).

Note that \( E_{p,q}(X) = H_{q-p}(s^{-1}\overline{\mathcal{C}})^{\otimes r} \).

**Lemma 3.1.** Let \( K, L, \) and \( M \) be chain complexes of free \( \mathbb{Z} \)-modules and let \( f : K \to L \) be a map of complexes such that \( f_* : H_i(K) \to H_i(L) \) is an isomorphism for \( i = 1,\ldots,k \) and an epimorphism for \( i = k+1 \). Then \( (f \otimes \text{id}_M)_* : H_i(K \otimes M) \to H_i(L \otimes M) \) is an isomorphism for \( i = 1,\ldots,k \) and an epimorphism for \( i = k+1 \).
Proof. The condition of the Lemma is equivalent to the fact that Cone(\( f \)) is acyclic in dimensions 0, 1, \ldots, \( k \). But Cone(\( f \otimes \text{id}_M \)) \( \simeq \) Cone(\( f \)) \( \otimes M \), therefore it is also acyclic in dimensions 0, 1, \ldots, \( k \).

Let us continue the proof of the Theorem. Repeatedly applying the Lemma we see that the natural map \( E^r_{p,q}(X) \to E^r_{p,q}(Y) \) is an isomorphism for \( p + q = 0 \) and an epimorphism for \( p + q = 1 \). It follows that it is also true for \( E^\infty_{r,p,q} \) for any \( r > 0 \), therefore it is true for \( E^\infty_{r,p,q} \) and hence for \( H_1(\mathcal{F}^k(C)) \).

2. **Triplet Massey’s product**

From theorem it follows that the invariants of a topological space w.r.t. pseudo-homeomorphisms distinguish fundamental groups (at least) up to a pseudo-isomorphism.

Clearly, in our study of such invariants it suffices to consider simplicial sets with single vertex. A simplicial map \( f : X \to Y \) of simplicial sets with single vertex will be called a simplicial pseudo-homeomorphism, if its geometric realization \( |f| : |X| \to |Y| \) is a pseudo-homeomorphism.

**Proposition 4.** Let \( f : X \to Y \) be a simplicial pseudo-homeomorphism. Then the map \( f^* : H^i(Y) \to H^i(X) \) is an isomorphism for \( i = 1 \) and a monomorphism for \( i = 2 \).

**Proof.** Let us consider the map \( \varphi : C(X) \to C(Y) \) of chain complexes. We put \( C = C(X), K_0 = C(Y) \). Thus \( C_0 \simeq \mathbb{Z}, K_0 \simeq \mathbb{Z}, \partial_{C_1} = 0, \partial_{K_1} = 0 \). We have

\[
\begin{align*}
K_1 &= \varphi C_1 + \partial K_2, \\
\varphi^{-1}(\partial K_2) &= \partial C_2, \\
\text{Ker} \partial K_2 &= \varphi \text{Ker} \partial C_2 + \partial K_3.
\end{align*}
\]

Let \( C = \text{Hom}(C, \mathbb{Z}) \) and \( K^\ast = \text{Hom}(K, \mathbb{Z}) \). From the first equation in \( (\text{I}) \) we get \( \text{Ker} \varphi^* \cap \text{Ker} \partial_{K_1} = 0 \), combining the first and the second equations in \( (\text{I}) \) we get \( \text{Ker} \partial_{C_1} = \varphi^* \text{Ker} \partial_{K_1}, \) and with the help of all \( (\text{I}) \) we get \( \varphi^* \partial_{C_2} \cap \text{Ker} \partial_{K_2} = \partial K_3 \). This is just what we need.

Let \( X \) be a simplicial set. We write \( H^i \) instead of \( H^i(X) \) and use standard notations \( C^i, Z^i, \) and \( B^i \) for \( \mathbb{Z} \)-modules of \( i \)-cochains, \( i \)-cocycles, and \( i \)-coboundaries respectively. Let us denote by \( \mu \) the map of cup-product \( C^i \otimes C^j \to C^{i+j} \) (the map \( \mu : C^i \otimes C^j \to C^{i+j} \)) is the adjoint operator to comultiplication \( \Delta : C^1 \to C^1 \otimes C^1 \). The cup-product in cohomology \( H^i \otimes H^j \to H^{i+j} \) is denoted by \( \bar{\mu} \).

Now we will construct an invariant of pseudo-homeomorphisms that is closely related to the triple Massey product. Let us recall its definition. Suppose \( \theta_1, \theta_2, \theta_3 \in H^1 \) satisfy the conditions \( \theta_1 \cdot \theta_2 = 0 \) and \( \theta_2 \cdot \theta_3 = 0 \). Choose \( \omega_i \in Z^1 \) such that \( \theta_i = [\omega_i] (i = 1, 2, 3); \) then there exist \( \omega_{12}, \omega_{23} \in C^1 \) such that \( \omega_1 \cdot \omega_2 = d \omega_{12} \) and \( \omega_2 \cdot \omega_3 = d \omega_{23} \). Consider the cochain \( \omega_{12} \cdot \omega_3 + \omega_{23} \cdot \omega_1; \) clearly, it is a cocycle; its class in \( H^2 \) is denoted by \( \langle \theta_1, \theta_2, \theta_3 \rangle \) and is called triple Massey product of \( \theta_1, \theta_2, \theta_3 \). This product is not defined uniquely: only the set \( \langle \theta_1, \theta_2, \theta_3 \rangle + H^1 \cdot \theta_3 + \theta_1 \cdot H^1 \) has an invariant sense.

Denote by \( \eta_2 \) the natural projection \( H^1 \otimes H^1 \to \Lambda^2 H^1; \) choose a homomorphism of \( \mathbb{Z} \)-modules \( \chi_2 : \Lambda^2 H^1 \to H^1 \otimes H^1 \) such that \( \eta_2 \circ \chi_2 = \text{id} \) (in other words, \( \chi_2 \) is right inverse to \( \eta_2 \)). Since \( H^1 \) is a free Abelian group, we can fix a homomorphism \( \chi : H^1 \to Z^1 \) right inverse to the canonical projection.

**Proposition 5.** There is a natural (non-linear) map \( \zeta : Z^1 \to C^1 \) such that \( \omega \cdot \omega = d \zeta(\omega) \) for any \( \omega \in Z^1 \).
Proof. The map $\zeta$ may be defined as follows: $\zeta(\omega)(a) = (\omega(a) - \omega(a^2))/2$ for any 1-simplex $a$. It is clear that this map is natural; the equality $\omega \sim \omega = d\zeta(\omega)$ is easy to check.

From Proposition 3 it follows that the map $\bar{\mu} : H^1 \otimes H^1 \to H^2$ may be factored through $\Lambda^2 H^1$; it means that there is a map $\bar{\mu} : \Lambda^2 H^1 \to H^2$ such that $\bar{\mu} \circ \eta_2 = \bar{\mu}$. Let $R^2 = \text{Ker} \, \bar{\mu}$, $\bar{R}^2 = \text{Ker} \, \bar{\mu}$. Clearly, $S(2) H^1 \subset R^2$, where $S(2) H^1 \subset H^1 \otimes H^1$ is the set of symmetric tensors.

Denote $R^2 \otimes H^1 \cap H^1 \otimes R^2 \subset H^1 \otimes H^1 \otimes H^1$ by $Q^3$. Let us choose a homomorphism of Abelian groups $\nu : R^2 \to C^1$ such that $\mu \circ (\sigma \otimes \sigma)(r) = d\nu(r)$ for any $r \in R^2$.

Now define $\lambda : Q^3 \to C^2$ as $\lambda = \mu \circ (\nu \otimes \sigma + \sigma \otimes \nu)$. We have $d \circ \lambda = \mu \circ ((d \circ \nu) \otimes \sigma - \sigma \otimes (d \circ \nu)) = \mu \circ (d \circ (\nu \otimes \sigma) \otimes \sigma - \sigma \otimes (d \circ \sigma \otimes \sigma)) = 0$ by associativity of the cup-product. For any $q \in Q^3$ we denote the image of $\lambda(q)$ in $H^2$ by $\lambda(q)$.

Let $\theta_1$, $\theta_2$, and $\theta_3$ be as in the definition of the triple Massey product. Then $q = \theta_1 \otimes \theta_2 \otimes \theta_3 \in Q^3$ and $(\theta_1, \theta_2, \theta_3) = \lambda(q)$. Thus, the homomorphism $\bar{\lambda} : Q^3 \to H^2$ can be viewed as a form of the triple Massey product.

Clearly, $\bar{\lambda}$ depends on the choices of $\sigma$ and $\nu$. If we change $\sigma$ to $\sigma' = \sigma + d \circ \varepsilon$, where $\varepsilon : H^1 \to C^0$ is an arbitrary homomorphism, then $\nu' = \nu + \mu \circ (\sigma' \otimes \varepsilon) + \mu \circ (\varepsilon \otimes \sigma')$ satisfies the condition $\mu \circ (\sigma' \otimes \sigma') = d \circ \nu'(r)$ and gives rise to the same $\lambda$. On the other hand, if we change $\nu$ to $\nu' = \nu + \rho$, where $\rho : R^2 \to Z^1$ is an arbitrary homomorphism, then $\bar{\lambda}$ changes to $\bar{\lambda} + \bar{\mu} \circ (\text{id} \otimes \bar{\rho} + \bar{\rho} \otimes \text{id})$, where $\bar{\rho}$ is the composition of $\rho$ with the canonical projection $Z^1 \to H^1$. Denote by $\delta$ the map from $\text{Hom}(R^2, H^1)$ to $\text{Hom}(Q^3, H^2)$ sending $f \in \text{Hom}(R^2, H^1)$ to $\bar{\mu} \circ (\text{id} \otimes f + f \otimes \text{id})$.

We see that the class of $\bar{\lambda}$ in $\text{Hom}(Q^3, H^2)/\delta \text{Hom}(R^2, H^1)$ is well-defined. Clearly, it is an invariant of pseudo-homeomorphisms.

We can further reduce $\lambda$ with the help of Proposition 3.

Let us fix a basis $(\xi_1, \ldots, \xi_n)$ of $H^1$. We choose $\nu$ so that $\nu(\xi_i \otimes \xi_i) = \zeta(\sigma(\xi_i))$ for $i = 1, \ldots, n$ and $\nu(\xi_i \otimes \xi_j + \xi_j \otimes \xi_i) = \zeta(\sigma(\xi_i + \xi_j)) - \zeta(\sigma(\xi_j))$ for $i \neq j$.

Let $\delta$ be the map from $\text{Hom}(R^2, H^1)$ to $\text{Hom}(Q^3, H^2)$ sending $f \in \text{Hom}(R^2, H^1)$ to $\delta(f \circ \eta_2)$. Clearly, the class of $\bar{\lambda}$ in $\text{Hom}(Q^3, H^2)/\delta \text{Hom}(R^2, H^1)$ is a well-defined invariant of pseudo-homeomorphisms.

Now consider the map $l : H^1 \otimes R^2 \to \Lambda^3 H^1$ arising from the wedge product in $\Lambda^* H^1$. We set $Q^3 = \text{Ker} \, l$.

**Proposition 6.** The image of the map $(\text{id}_{H^1} \otimes \eta_2) \circ (\text{id} - s_{(123)}) : Q^3 \to H^1 \otimes R^2$ is $Q^3$.

**Proof.** Clearly, the image of this map belongs to $Q^3$. Let us construct a map $p : Q^3 \to Q^3$ right inverse to the map under consideration.

Let $t = \sum_j \xi_j \otimes r_j \in Q^3$, $r_j = \sum_{i < k} \alpha_{ijk} \xi_j \wedge \xi_k$. We put
\[
pt = \sum_{i < j < k} (\alpha_{ijk}(\xi_i \otimes \xi_j \otimes \xi_k + \xi_j \otimes \xi_i \otimes \xi_k) + \alpha_{kj}(\xi_i \otimes \xi_k \otimes \xi_j + \xi_k \otimes \xi_i \otimes \xi_j)) \\
+ \sum_{i < j} (\alpha_{ij}(\xi_i \otimes \xi_j \otimes \xi_j - \alpha_{iij}(\xi_j \otimes \xi_j \otimes \xi_j)).
\]

Since $\alpha_{ijk} - \alpha_{ikj} + \alpha_{kij} = 0$ for $i < j < k$, we have $pt \in H^1 \otimes R^2 \cap S(2) H^1 \otimes H^1$ and $(\text{id}_{H^1} \otimes \eta_2) \circ (\text{id} - s_{(123)})pt = t$. \qed
Let us fix the map $p$ constructed above.
Let now $t \in H^1 \otimes \bar{R}^2$, $t = \sum \xi_i \otimes r_i$, $r_i = \sum_{j<k} \alpha_{ijk} \xi_j \wedge \xi_k$. We put
\[
qt = \sum_i \sum_{j<k} \alpha_{ijk} (\xi_i \otimes \xi_j \otimes \xi_k + \xi_j \otimes \xi_i \otimes \xi_k + \xi_j \otimes \xi_k \otimes \xi_i).
\]
Clearly, $q$ is the map $H^1 \otimes \bar{R}^2 \to Q^3$ satisfying the conditions $(\text{id}_{H^1} \otimes \eta_2) \circ q = \text{id}$ and $(\text{id}_{H^1} \otimes \eta_2) \circ s_{(123)} \circ q = \text{id}.

**Proposition 7.** We have $Q^3 = p\bar{Q}^3 \oplus q(H^1 \otimes \bar{R}^2) \oplus S^{(3)}H^1$.

**Proof.** Clear.

Now let $\bar{\lambda} : Q^3 \to H^2$ be as above.

**Proposition 8.** We have $\bar{\lambda}(t) = 0$ for any $t \in S^{(3)}H^1$ and $\bar{\lambda}(qt) = \sum_{i<j} (\alpha_{iij} + \alpha_{ijj}) \xi_i \wedge \xi_j$ for $t = \sum_i \xi_i \otimes \sum_{j<k} \alpha_{ijk} \xi_j \wedge \xi_k \in H^1 \otimes \bar{R}^2$.

**Proof.** Let $h \in H^1$, $\tilde{h} = \varepsilon h$. We define $f \in C^1$ by the formula $f(a) = \tilde{h}(a)(\tilde{h}(a) - 1)(\tilde{h}(a) - 2)/6$ for all $a \in X_1$. It is easy to check that $\lambda(h \otimes h \otimes h) = df$. Therefore, $\bar{\lambda}$ vanishes on $S^{(3)}H^1$.

To prove the second assertion of the proposition we recall that there is a natural map $\mu_1 : C_n \otimes C_n \to C_{n+1}$ of degree $-1$ such that for any $f \in C_n \otimes C_n$ one has
\[
\mu(f - sf) = d\mu_1 f + \mu_1 df,
\]
where $s(a \otimes b) = (-1)^{\deg a \deg b} a \otimes b$ for homogeneous $a, b \in C^*_n$. (This is a part of the structure of $E_\infty$-algebra on $C_*$, see [13].) The map $\mu_1$ is adjoint to the map $\Delta_1 : C_n \to C_n \otimes C_n$ of degree 1. For the standard simplices of dimensions 1 and 2 the map $\Delta_1$ is given by the formulas $\Delta_1[01] = [01] \otimes [01]$ and $\Delta_1[012] = [012] \otimes [02] + ([01] + [12]) \otimes [012]$.

For $t = \sum \xi_i \otimes \sum_{j<k} \alpha_{ijk} \xi_j \wedge \xi_k \in H^1 \otimes \bar{R}^2$ we have
\[
\lambda(qt) = \mu_1 \circ (\mu \otimes \varepsilon - \varepsilon \circ (\mu \otimes \varepsilon) + (\varpi \otimes \mu) \circ (\text{id} - s_{(123)}))(qt)
= d \circ \mu_1 \circ (\mu \otimes \varepsilon)(qt) + \mu_1 \circ (\mu \otimes \text{id})(\varepsilon \otimes \varepsilon)(qt) + (\varpi \otimes \mu) \circ (\text{id} - s_{(123)})(qt).
\]
Note that $(\text{id} - s_{(123)})$ is symmetric w.r.t. the last two indices. Now it is easy to check that
\[
\mu_1 \circ (\mu \otimes \text{id}) \circ (\varpi \otimes \varepsilon)(qt) + (\varpi \otimes \mu) \circ (\text{id} - s_{(123)})(qt) = -d \sum \sum \alpha_{ijk} \xi_i \xi_j \xi_k + \sum \sum \alpha_{ijij} \xi_i \xi_j + \alpha_{ijji} \xi_i \xi_j,
\]
where $\tilde{h} = \varepsilon h$ and $\tilde{\xi}_i \xi_j \xi_k$ is a cochain in $C^1$ given by
\[
\tilde{\xi}_i \xi_j \xi_k(a) = \tilde{\xi}_i(a) \xi_j(a) \xi_k(a)
\]
for all $a \in X_1$.

We see that $\bar{\lambda}$ is determined by $\bar{\lambda} = \lambda \circ p : \bar{Q}^3 \to H^2$. Denote by $\bar{\delta}$ the map $\text{Hom}(\bar{R}^2, H^1) \to \text{Hom}(\bar{Q}^3, H_2)$ sending $f \in \text{Hom}(\bar{R}^2, H^1)$ to $\bar{\delta}(f) \circ p$. Clearly, the class $[\lambda] \in \text{Hom}(\bar{Q}^3, H^2)/\bar{\delta} \text{Hom}(\bar{R}^2, H^1)$ of $\bar{\lambda}$ is a well-defined invariant of pseudo-homeomorphisms. More precisely, we have

**Theorem 9.** Let $X$ and $Y$ be simplicial sets and let $\varphi : X \to Y$ be simplicial pseudo-homeomorphism. Suppose that the map $\lambda^{(X)} : \Lambda^2 H^1(X) \to H^1(X) \otimes H^1(X)$ and the basis of $H^1(X)$ are chosen as above. Let us transfer these structures to
3. \[Q\]

Proof. This is obvious since we have used only natural constructions to define \(\overline{F}\), \(\bar{Q}\), and \(\lambda\).

3. THE INVARIANT OF FUNDAMENTAL GROUP

Our next goal is to interpret the invariant \(\lambda\) in terms of the fundamental group. It is possible due to theorem \([\text{10}]\), but we want to produce an explicit construction.

Let \(G = \pi_1(U, u)\), where \(U\) is an arcwise connected topological space having the homotopy type of a CW-complex. Suppose the following conditions hold:

1. \(G \to G'\) is a free Abelian group of rank \(n\) (that is, \(H_1(U) \cong \mathbb{Z}^n\));
2. \(G\) is generated with \(n\) generators, thus \(G \cong F/R\), where \(F = F(w_1, \ldots, w_n)\) is a free group with generators \(w_1, \ldots, w_n\) and \(R\) is a normal subgroup of \(F\);
3. the comultiplication in \(C(U)\) gives rise to an injective homomorphism \(H_2(U) \to H_1(U) \otimes H_1(U)\).

Remark. These conditions are satisfied for the complement of a complex hyperplane arrangement (see \([10]\)).

By Theorem \([\text{10}]\), there is a pseudo-homeomorphism \(U \to BG\). Hence, by Theorem \([\text{10}]\) the invariants \(\overline{R}\), \(\overline{Q}\), and \(\lambda\) may be computed for the simplicial set \(X = BG\) with single vertex \(x = (\cdot)\).

Denote by \(g_1, \ldots, g_n\) the generators of \(G\) (i. e., the images of \(w_1, \ldots, w_n\)). Let \(h_1, \ldots, h_n\) be the corresponding elements in \(H_1 = H_1(X) \cong G/G' \cong F/F'\). Clearly, \((h_1, \ldots, h_n)\) is a basis of \(H_1\). Since \(H_2 = H_2(X)\) is imbedded into \(H_1 \otimes H_1\), it is also a free \(\mathbb{Z}\)-module. Hence \(H_1 \cong \text{Hom}_\mathbb{Z}(H^1, \mathbb{Z})\) and \(H_2 \cong \text{Hom}_\mathbb{Z}(H^2, \mathbb{Z})\). Denote by \((\xi_1, \ldots, \xi_n)\) the basis of \(H^1\) dual to \((h_1, \ldots, h_n)\).

As usual, denote by \(\gamma_k G\) the \(k\)-th term of the lower central series of \(G\) (that is, \(\gamma_1 G = G\) and \(\gamma_{k+1} G = \langle G, \gamma_k G \rangle = \langle g^{-1} f^{-1} g f \mid g \in G, f \in \gamma_k G \rangle\)). It is well known that \(\mathfrak{g} = \bigoplus_{k=1}^\infty \mathfrak{g}_k\), where \(\mathfrak{g}_k = \gamma_k G/\gamma_{k+1} G\), has the structure of the graded Lie algebra with the Lie commutator \([\cdot, \cdot] : \mathfrak{g}_k \times \mathfrak{g}_m \to \mathfrak{g}_{k+m}\) in \(\mathfrak{g}\) corresponding to the group commutator \((\cdot, \cdot) : \gamma_k G \times \gamma_m G \to \gamma_{k+m} G\). Note also that for the free group \(F\) the corresponding Lie algebra \(\mathfrak{f}\) is the free Lie algebra with generators \(x_i = w_i \gamma_i F \in \mathfrak{f}_1\) (see \([\text{13}]\)).

By the Magnus Theorem \([\text{13}]\) the subgroup \(\gamma_k F\) is the set of all \(w \in F\) such that \(w - 1\) belongs to the \(k\)-th power of the augmentation ideal \(I \in \mathbb{Z}F\). This is not generally true for an arbitrary group (see \([\text{14}]\)). But in our case we have the following

Proposition 10. For \(k = 1, 2, 3, 4\) the subgroup \(\gamma_k G\) is the set of all \(g \in G\) such that \(g - 1\) belongs to the \(k\)-th power of the augmentation ideal \(I \in \mathbb{Z}G\).
Proof. For \( k = 1 \) there is nothing to prove. The case \( k = 2 \) is not much harder.

Since \( G = F/R \), we have \( \gamma_k G = \gamma_k F/R \cap \gamma_k F \); thus, \( g = f/\tau \), where \( \tau = \bigoplus \tau_k \), \( \tau_k = R \cap \gamma_k F/R \cap \gamma_{k+1} F \).

Consider the graded algebra \( A = \sum_{k=0}^{\infty} I^k/I^{k+1} \). Clearly, it is a free associative algebra with the generators \( x_1, \ldots, x_n \), where \( x_i = (w_i - 1) + J_2 \); \( A \) is isomorphic to the universal enveloping algebra of \( f \). Let us compare the algebra \( A/(\tau) \) (it is isomorphic to the universal enveloping algebra of \( g \)) and the algebra \( B = \sum_{k=0}^{\infty} I^k/I^{k+1} \).

Since \( R \in \gamma_2 F \), it is readily seen that \( B_2 = A_2/(\tau)_2 \) and \( B_3 = A_3/(\tau)_3 \). By the Poincaré-Birkhoff-Witt theorem (which is valid over \( \mathbb{Z} \), see [1]), \( g \) is imbedded in \( A/(\tau) \). Therefore, \( g_2 \) is imbedded in \( B_2 \) and \( g_3 \) is imbedded in \( B_3 \). The proposition for \( k = 3 \) and 4 follows.

Consider the algebra \( B = \sum_{k=0}^{\infty} I^k/I^{k+1} \). Note that \( B_0 = \mathbb{Z} \) and \( B_1 = H_1 \). Using Proposition 2, it is easily shown that \( B_2 = H_1 \otimes H_1/\Delta(H_2) \) and \( B_3 = H_1 \otimes H_1 \otimes H_1/(\Delta(H_2) \otimes H_1 + H_1 \otimes \Delta(H_2)) \). Denote by \( P_2 \) (resp. \( P_3 \)) the image of \([H_1, H_1] \subset H_1 \otimes H_1 \) (resp. \([H_1, [H_1, H_1]] \subset H_1 \otimes H_1 \otimes H_1 \)) under the canonical projection \( H_1 \otimes H_1 \to H_1 \otimes H_1/\Delta(H_2) \) (resp. \( H_1 \otimes H_1 \otimes H_1 \to H_1 \otimes H_1 \otimes H_1/(\Delta(H_2) \otimes H_1 + H_1 \otimes \Delta(H_2)) \)). By Proposition 3, the natural homomorphisms \( \gamma_2 G/\gamma_3 G \to B_2 \) and \( \gamma_3 G/\gamma_4 G \to B_2 \) are injective. Therefore, \( \gamma_2 G/\gamma_3 G \) is isomorphic to \( P_2 \), and \( \gamma_3 G/\gamma_4 G \) is isomorphic to \( P_3 \).

Let \( w \) be an element of \( H_2 \). We have \( \Delta(w) = \sum_{i \neq j} \alpha_{ij}[h_i, h_j] \). Let

\[
\tau(w) = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (g_i g_j g_i^{-1} g_j^{-1})^{\alpha_{ij}} \in G.
\]

Clearly, \( \tau(w) \in \gamma_3 G \). Denote by \( \tau(w) \) the corresponding element of \( \gamma_3 G/\gamma_4 G \). Thus, we get a homomorphism of Abelian groups \( \tilde{\tau} : H_2 \to P_3 \).

Note that

\[
R^2 \simeq \text{Hom}_\mathbb{Z}(H_1 \otimes H_1/\Delta(H_2), \mathbb{Z})
\]

and

\[
Q_3 = \text{Hom}_\mathbb{Z}(H_1 \otimes H_1 \otimes H_1/\Delta(H_2) \otimes H_1 + H_1 \otimes \Delta(H_2), \mathbb{Z}).
\]

We have \( P_2 \simeq [H_1, H_1]/\Delta(H_2) \). Let \( j : \Lambda^3 H_1 \to P_2 \otimes H_1 \) be given by \( j(x \wedge y \wedge z) = [x, y] \otimes z + [y, z] \otimes x + [z, x] \otimes y \). We have \( P_3 = (P_2 \otimes H_1)/j(\Lambda^3 H_1) \). Clearly, \( R^2 = \text{Hom}_\mathbb{Z}(P_2, \mathbb{Z}) \) and \( Q_3 = \text{Hom}_\mathbb{Z}(P_3, \mathbb{Z}) \).

Since \( X = B G \) is a simplicial set with single vertex, we see that the map \( x : H^1 \to Z^1 \) right inverse to the canonical projection is unique. It is readily seen that there is a unique map \( i : C_1 \to H_1 \otimes H_1/\Delta(H_2) \) such that \( i(g_i) = 0 \) for \( i = 1, \ldots, n \) and the diagram

\[
\begin{array}{ccc}
C_2 & \xrightarrow{\Delta} & C_1 \otimes C_1 \\
\downarrow i & & \downarrow \\
C_1 & \xrightarrow{i} & H_1 \otimes H_1/\Delta(H_2)
\end{array}
\]

is commutative. We set \( \nu : R^2 \to C^1 \) to be the conjugate of \( i \).

Note that \( \nu(\xi_i \otimes \xi_i) = \zeta(\omega(\xi_i)) \) for \( i = 1, \ldots, n \) and \( \nu(\xi_i \otimes \xi_j + \xi_j \otimes \xi_i) = \zeta(\omega(\xi_i + \xi_j)) - \zeta(\omega(\xi_i)) - \zeta(\omega(\xi_j)) \) for \( i \neq j \) (where \( \zeta(\omega)(a) = (\omega(a) - \omega(a)^2)/2 \) for any 1-simplex \( a \)). Indeed, for any \( r \in R^2 \), the condition \( d\nu(r) = \mu \circ (\omega \otimes \omega)(r) \) means that \( \nu(r) \) is determined uniquely by its values on 1-cycles \( (g_1), \ldots, (g_n) \).
By the definition of \( \nu \), these values are zero for any \( r \in \mathbb{R}^2 \). But \( \zeta(\nu(g_i))(g_j) = (\nu(g_i))(g_j) - \nu(g_i)(g_j)^2)/2 = 0 \) for any \( i, j \), since
\[
\nu(g_i) = \begin{cases} 
1, & \text{for } i = j, \\
0, & \text{for } i \neq j.
\end{cases}
\]
Similarly, \( \zeta(\nu(g_i) + \nu(g_j))(g_k) = 0 \) for any \( i, j, k, i \neq j \).

Thus, we can construct the map \( \tilde{\lambda} : \tilde{Q}^3 \rightarrow \tilde{H}^2 \) as above.

**Proposition 11.** The map \( \tilde{\lambda} : \tilde{Q}^3 \rightarrow \tilde{H}^2 \) is conjugate to the map \( \tilde{\tau} : \tilde{H}_2 \rightarrow \tilde{P}_3 \).

**Proof.** Let \( w \in \tilde{H}_2 \); \( \tilde{\Delta}(w) = \sum_{i<j} \alpha_{ij}[h_i, h_j] \). We want to describe \( \tilde{\tau}(w) \in \tilde{P}_3 \subset \tilde{B}_3 \) in terms of comultiplication in the chain complex of \( X \).

Note that \( \tilde{B}_3 \) is isomorphic to the kernel of the projection \( D^{(4)}(G) \rightarrow D^{(3)}(G) \). We identify \( D^{(4)}(G) \) with \( A^{(4)}(X) \) (by Proposition 2). Denote the image of \( (g_i) \in X_1 \) in \( A^{(4)}(G) \) by \( a_i \). Then the image of \( \tau(w) \) in \( A^{(4)}(G) \) is equal to
\[
\prod_{i<j}(1 + a_i)(1 + a_j)(1 - a_i + a_i^2 - a_i^3)(1 - a_j + a_j^2 - a_j^3) = 1 + \sum_{i<j} \alpha_{ij}(a_i a_j - a_j a_i)(1 - a_i - a_j).
\]

Let us consider \( \tilde{\tau} \) as the map \( H_2 \rightarrow \tilde{B}_3 \). We see that it is the sum of two maps—\( \tilde{\tau}_1 \) and \( \tilde{\tau}_2 \)—where
\[
\tilde{\tau}_1(w) = -\sum_{i<j} \alpha_{ij}(h_i \otimes h_j - h_j \otimes h_i) \otimes (h_i + h_j) + (\tilde{\Delta}(H_2) \otimes H_1 + H_1 \otimes \tilde{\Delta}(H_2))
\]
and \( \tilde{\tau}_2(w) \) is the preimage of \( \sum_{i<j} \alpha_{ij}(a_i a_j - a_j a_i) \in A^{(4)}(G) \) under the injection \( \tilde{B}_3 \rightarrow A^{(4)}(G) \).

The element \( \tilde{\tau}_2(w) \) can be described as follows. Let \( c \in \mathbb{Z}_2 \) be a representative of the class \( w \). The image of \( \Delta c \in C_1 \otimes C_1 \) in \( A^{(4)}(X) \) equals zero. Therefore, in the definition of \( \tilde{\tau}_2(w) \) we can use the image of the element \( t = -\Delta c + \sum_{i<j} \alpha_{ij}(g_i) \otimes (g_j) - (g_j) \otimes (g_i) \in C_1 \otimes C_1 \) in \( A^{(4)}(X) \) instead of \( \sum_{i<j} \alpha_{ij}(a_i a_j - a_j a_i) \) (which is the image of \( \sum_{i<j} \alpha_{ij}(g_i) \otimes (g_j) - (g_j) \otimes (g_i) \in C_1 \otimes C_1 \)). Clearly, \( t \) is the projection of \( -\Delta c \) to \( C_1 \otimes B_1 + B_1 \otimes C_1 \) along the linear span of \( (g_i) \otimes (g_j) \). Hence, there is an element \( \tilde{t} \in C_2 \otimes C_1 \otimes C_1 \) such that \( t = (\partial \otimes \text{id} \oplus \text{id} \otimes \partial)\tilde{t} \). From the definition of \( A^{(4)}(X) \) it follows that \( \tilde{\tau}_2(w) \) is the image of \( -\Delta \otimes \text{id} \oplus \text{id} \otimes \Delta)\tilde{t} \). Hence, \( \tilde{\tau}_2(w) = \varphi(\tilde{\nu} + pr \otimes \nu)\Delta c \), where \( pr \) is the canonical projection \( C_1 \rightarrow H_1 \) and \( \varphi \) is the natural map \( (H_1 \otimes H_1/\Delta(H_2)) \otimes H_1 \oplus H_1 \otimes (H_1 \otimes H_1/\Delta(H_2)) \rightarrow B_3 \).

Thus, we see that \( \tilde{\lambda} \) is the conjugate of \( \tilde{\tau}_2 \). Note that \( \tilde{\tau}_1(w) \) is orthogonal to \( p\tilde{Q}^3 \).

On the other hand, the injections \( p : \tilde{Q}^3 \rightarrow \tilde{Q}^3 \) and \( p_3 \rightarrow \tilde{P}_3 \) preserve the natural pairing. Since the image of \( \tilde{\tau} = \tilde{\tau}_1 + \tilde{\tau}_2 \) belongs to \( \tilde{P}_3 \), the theorem follows. \( \square \)

**Theorem 12.** Suppose that the the groups \( G_a = \pi_1(U_a, u_a) \) and \( G_b = \pi_1(U_b, u_b) \) satisfy conditions \((1)–(3)\). We identify \( H_1(U_a) \) with \( G_a/\gamma_2 G_a \) and \( H_1(U_b) \) with \( G_b/\gamma_2 G_b \). Let \( f : H_1(U_a) \rightarrow H_1(U_b) \) be any isomorphism and let \( \varphi : H^1(U_b) \rightarrow H^1(U_a) \) be its conjugate. Then we claim the following.
1. The isomorphism $f$ can be extended to an isomorphism

$$G_a/\gamma_3 G_a \to G_b/\gamma_3 G_b$$

if and only if $\Lambda^2 \varphi(\overline{R}_a^2) = \overline{R}_a^2$.

2. Suppose that the previous condition hold and $P_3^{(a)} \cong P_3^{(b)}$ is a free Abelian group. Then the isomorphism $f$ can be extended to an isomorphism

$$G_a/\gamma_4 G_a \to G_b/\gamma_4 G_b$$

if and only if $[\lambda_a]$ corresponds to $[\lambda_b]$, i.e., there is a commutative diagram

![Diagram](https://example.com/diagram.png)

Proof. Clear.

Remark. It is readily seen that $[\lambda]$ is just the invariant used in [13] to distinguish fundamental groups of combinatorially equivalent complex hyperplane arrangements.

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