Topological Quantum Field Theory
for
Calabi-Yau threefolds and
$G_2$-manifolds

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1 Introduction

In the past two decades we have witnessed many fruitful interactions between mathematics and physics. One example is in the Donaldson-Floer theory for oriented four manifolds. Physical considerations lead to the discovery of the Seiberg-Witten theory which has profound impact to our understandings of four manifolds. Another example is in the mirror symmetry for Calabi-Yau manifolds. This duality transformation in the string theory leads to many surprising predictions in the enumerative geometry.
String theory in physics studies a ten dimensional space-time $X \times \mathbb{R}^{3,1}$. Here $X$ a six dimensional Riemannian manifold with its holonomy group inside $SU(3)$, the so-called Calabi-Yau threefold. Certain parts of the mirror symmetry conjecture, as studied by Vafa’s group, are specific for Calabi-Yau manifolds of complex dimension three. They include the Gopakumar-Vafa conjecture for the Gromov-Witten invariants of arbitrary genus, the Ooguri-Vafa conjecture on the relationships between knot invariants and enumerations of holomorphic disks and so on. The key reason is they belong to a duality theory for $G_2$-manifolds. $G_2$-manifolds can be naturally interpreted as special Octonion manifolds [23]. For any Calabi-Yau threefold $X$, the seven dimensional manifold $X \times S^1$ is automatically a $G_2$-manifold because of the natural inclusion $SU(3) \subset G_2$.

In recent years, there are many studies of $G_2$-manifolds in M-theory including works of Archaya, Atiyah, Gukov, Vafa, Witten, Yau, Zaslow and many others (e.g. [1], [5], [13], [2]).

In the studies of the symplectic geometry of a Calabi-Yau threefold $X$, we consider unitary flat bundles over three dimensional (special) Lagrangian submanifolds $L$ in $X$. The corresponding geometry for a $G_2$-manifold $M$ is called the special $\mathbb{H}$-Lagrangian geometry (or $C$-geometry in [19]). where we consider Anti-Self-Dual (abbrev. ASD) bundles over four dimensional coassociative submanifolds, or equivalently special $\mathbb{H}$-Lagrangian submanifolds of type II [23], (abbrev. $\mathbb{H}$-SLag) $C$ in $M$.

Counting ASD bundles over a fixed four manifold $C$ is the well-known theory of Donaldson differentiable invariants, $Don(C)$. Similarly, counting unitary flat bundles over a fixed three manifold $L$ is Floer’s Chern-Simons homology theory, $HF_{CS}(L)$. When $C$ is a connected sum $C_1 \#_L C_2$ along a homology three sphere, the relative Donaldson invariants $Don(C_i)$’s take values in $HF_{CS}(L)$ and $Don(C)$ can be recovered from individual pieces by a gluing theorem, $Don(C) = \langle Don(C_1), Don(C_2) \rangle_{HF_{CS}(L)}$ (see e.g. [7]). Similarly when $L$ has a handlebody decomposition $L = L_1 \#_\Sigma L_2$, each $L_i$ determines a Lagrangian subspace $\mathcal{L}_i$ in the moduli space $\mathcal{M}_{flat}(\Sigma)$ of unitary flat bundles over the Riemann surface $\Sigma$ and Atiyah conjectures that we can recover $HF_{CS}(L)$ from the Floer’s Lagrangian intersection homology group of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{M}_{flat}(\Sigma)$, $HF_{CS}(L) = HF_{flat}(\Sigma)(\mathcal{L}_1, \mathcal{L}_2)$. Such algebraic structures in the Donaldson-Floer theory can be formulated as a Topological Quantum Field Theory (abbrev. TQFT), as defined by Segal and Atiyah [3].

In this paper, we propose a construction of a TQFT by counting ASD bundles over four dimensional $\mathbb{H}$-SLag $C$ in any closed (almost) $G_2$-manifold.
We call these \( H\)-Slag cycles and they can be identified as zeros of a naturally defined closed one form on the configuration space of topological cycles. We expect to obtain a homology theory \( H_C(M) \) by applying the construction in the Witten’s Morse theory. When \( M \) is non-compact with an asymptotically cylindrical end, \( X \times [0, \infty) \), then the collection of boundary data of relative \( H\)-Slag cycles determines a Lagrangian submanifold \( \mathcal{L}_M \) in the moduli space \( M^{SLag}(X) \) of special Lagrangian cycles in the Calabi-Yau threefold \( X \).

When we decompose \( M = M_1 \#_X M_2 \) along an infinite asymptotically cylindrical neck, it is reasonable to expect to have a gluing formula,

\[ H_C(M) = HF_{\text{Lag}}^{M^{SLag}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2}). \]

The main technical difficulty in defining this TQFT rigorously is the compactness issue for the moduli space of \( H\)-Slag cycles in \( M \). We do not know how to resolve this problem and our homology groups are only defined in the formal sense (and physical sense?).

### 2 \( G_2\)-manifolds and \( H\)-Slag geometry

We first review some basic definitions and properties of \( G_2\)-geometry, see [19] for more details.

**Definition 1.** A seven dimensional Riemannian manifold \( M \) is called a \( G_2\)-manifold if the holonomy group of its Levi-Civita connection is inside \( G_2 \subset SO(7) \).

The simple Lie group \( G_2 \) can be identified as the subgroup of \( SO(7) \) consisting of isomorphism \( g : \mathbb{R}^7 \rightarrow \mathbb{R}^7 \) preserving the linear three form \( \Omega \),

\[
\Omega = f^1 f^2 f^3 - f^1 (e^1 e^0 + e^2 e^3) - f^2 (e^2 e^0 + e^3 e^1) - f^3 (e^3 e^0 + e^1 e^2),
\]

where \( e^0, e^1, e^2, e^3, f^1, f^2, f^3 \) is any given orthonormal frame of \( \mathbb{R}^7 \). Such a three form, or up to conjugation by elements in \( GL(7, \mathbb{R}) \), is called positive, and it determines a unique compatible inner product on \( \mathbb{R}^7 \) [6].

Gray [12] shows that \( G_2\)-holonomy of \( M \) can be characterized by the existence of a positive harmonic three form \( \Omega \).

**Definition 2.** A seven dimensional manifold \( M \) equipped with a positive closed three form \( \Omega \) is called an almost \( G_2\)-manifold.
Remark: The relationship between $G_2$-manifolds and almost $G_2$-manifolds is analogous to the relationship between Kahler manifolds and symplectic manifolds. Namely we replace a parallel non-degenerate form by a closed one.

For example, suppose that $X$ is a complex three dimensional Kähler manifold with a trivial canonical line bundle, i.e. there exists a nonvanishing holomorphic three form $Ω_X$. Yau’s celebrated theorem says that there is a Kähler form $ω_X$ on $X$ such that the corresponding Kahler metric has holonomy in $SU(3)$, i.e. a Calabi-Yau threefold. In particular both $Ω_X$ and $ω_X$ are parallel forms. Then the product $M = X × S^1$ is a $G_2$-manifold with

$$Ω = \text{Re} Ω_X + ω_X ∧ dθ.$$ 

Conversely, one can prove, using Bochner arguments, every $G_2$-metric on $X × S^1$ must be of this form. More generally, if $ω_X$ is a general Kähler form on $X$, then $(X × S^1, Ω)$ is an almost $G_2$-manifold and the converse is also true.

Next we quickly review the geometry of $\mathbb{H}$-SLag cycles in an almost $G_2$-manifold (see [19]).

**Definition 3.** An orientable four dimensional submanifold $C$ in an almost $G_2$-manifold $(M, Ω)$ is called a coassociative submanifold, or simply a $\mathbb{H}$-SLag, if the restriction of $Ω$ to $C$ is identically zero,

$$Ω|_C = 0.$$ 

If $M$ is a $G_2$-manifold, then any coassociative submanifold $C$ in $M$ is calibrated by $⋆Ω$ in the sense of Harvey and Lawson [14], in particular, it is an absolute minimal submanifold in $M$. The normal bundle of any $\mathbb{H}$-SLag $C$ can be naturally identified with the bundle of self-dual two forms on $C$. McLean [27] shows that infinitesimal deformations of any $\mathbb{H}$-SLag are unobstructed and they are parametrized by the space of harmonic self-dual two forms on $C$, i.e. $H^2_+(C, \mathbb{R})$.

For example, if $S$ is a complex surface in a Calabi-Yau threefold $X$, then $S × \{t\}$ is a $\mathbb{H}$-SLag in $M = X × S^1$ for any $t ∈ S^1$. Notice that $H^2_+(S, \mathbb{R})$ is spanned by the Kahler form and the real and imaginary parts of holomorphic two forms on $S$, and the latter can be identified holomorphic normal vector fields along $S$ because of the adjunction formula and the Calabi-Yau condition on $X$. Thus all deformations of $S × \{t\}$ in $M$ as $\mathbb{H}$-SLag submanifolds are of the same form. Similarly, if $L$ is a three dimensional special
Lagrangian submanifold in \( X \) with phase \( \pi/2 \), i.e.
\[ \omega|_L = \text{Re} \Omega_X|_L = 0, \]
then \( L \times S^1 \) is also a \( \mathbb{H} \text{-SLag} \) in \( M = X \times S^1 \). Furthermore, all deformations of \( L \times S^1 \) in \( M \) as \( \mathbb{H} \text{-SLag} \) submanifolds are of the same form because
\[ H^2_+(L \times S^1) \cong H^1(L), \]
which parametrizes infinitesimal deformations of special Lagrangian submanifolds in \( X \).

**Definition 4.** A \( \mathbb{H} \text{-SLag} \) cycle in an almost \( G_2 \)-manifold \((M, \Omega)\) is a pair \((C, D_E)\) with \( C \) a \( \mathbb{H} \text{-SLag} \) in \( M \) and \( D_E \) an ASD connection over \( C \).

**Remark:** \( \mathbb{H} \text{-SLag} \) cycles are supersymmetric cycles in physics as studied in [26]. Their moduli space admits a natural three form and a cubic tensor [19], which play the roles of the correlation function and the Yukawa coupling in physics.

We assume that the ASD connection \( D_E \) over \( C \) has rank one, i.e. a \( U(1) \) connection. This avoids the occurrence of reducible connections, thus the moduli space \( \mathcal{M}^{\mathbb{H} \text{-SLag}}(M) \) of \( \mathbb{H} \text{-SLag} \) cycles in \( M \) is a smooth manifold. It has a natural orientation and its expected dimension equals \( b^1(C) \), the first Betti number of \( C \). This is because the moduli space of \( \mathbb{H} \text{-SLag} \)s has dimension equals \( b^1(C) \) [27] and the existence of an ASD \( U(1) \)-connection over \( C \) is equivalent to \( H^2(C, \mathbb{R}) \cap H^2(C, \mathbb{Z}) \neq \phi \). The number \( b^1(C) \) is responsible for twisting by a flat \( U(1) \)-connection.

For simplicity, we assume that \( b^1(C) = 0 \), otherwise, one can cut down the dimension of \( \mathcal{M}^{\mathbb{H} \text{-SLag}}(M) \) to zero by requiring the ASD connections over \( C \) to have trivial holonomy around loops \( \gamma_1, \ldots, \gamma_{b^1(C)} \) in \( C \) representing an integral basis of \( H_1(C, \mathbb{Z}) \). We plan to count the algebraic number of points in this moduli space \( \# \mathcal{M}^{\mathbb{H} \text{-SLag}}(M) \).

This number, in the case of \( X \times S^1 \), can be identified with a proposed invariant of Joyce [17] defined by counting rigid special Lagrangian submanifolds in any Calabi-Yau threefold. To explain this, we need the following proposition on the strong rigidity of product \( \mathbb{H} \text{-SLag} \)s.

**Proposition 5.** If \( L \times S^1 \) is a \( \mathbb{H} \text{-SLag} \) in \( M = X \times S^1 \) with \( X \) a Calabi-Yau threefold, then any \( \mathbb{H} \text{-SLag} \) representing the same homology class must also be a product.

**Proof:** For simplicity we assume that the volume of the \( S^1 \) factor is unity, \( \text{Vol}(S^1) = 1 \). If \( L \times S^1 \) is a \( \mathbb{H} \text{-SLag} \) in \( M \) then \( L \) is special Lagrangian submanifold in \( X \) with phase \( \pi/2 \), i.e. \( \text{Re} \Omega_X|_L = \omega|_L = 0 \). Suppose \( C \) is another \( \mathbb{H} \text{-SLag} \) in \( M \) representing the same homology class, we have \( \text{Vol}(C) = \text{Vol}(L) \). If we write \( C_\theta = C \cap (X \times \{\theta\}) \) for any \( \theta \in S^1 \), then
$\text{Vol} (C_\theta) \geq \text{Vol} (L)$, as $L$ is a calibrated submanifold in $X$. Furthermore the equality sign holds only if $C_\theta$ is also calibrated. In general we have

$$\text{Vol} (C) \geq \int_{S^1} \text{Vol} (C_\theta) \, d\theta,$$

with the equality sign holds if and only if $C$ is a product with $S^1$. Combining these, we have

$$\text{Vol} (L) = \text{Vol} (C) \geq \int_{S^1} \text{Vol} (C_\theta) \, d\theta \geq \int_{S^1} \text{Vol} (L) \, d\theta = \text{Vol} (L).$$

Thus both inequalities are indeed equal. Hence $C = L' \times S^1$ for some special Lagrangian submanifold $L'$ in $X$. ■

Suppose $M = X \times S^1$ is a product $G_2$-manifold and we consider product $\mathbb{H}$-SLag cycles $C = L \times S^1$ in $M$. From the above proposition, every $\mathbb{H}$-SLag representing $[C]$ must also be a product. Since $b^2_+ (C) = b^1 (L)$, the rigidity of the $\mathbb{H}$-SLag $C$ in $M$ is equivalent to the rigidity of the special Lagrangian submanifold $L$ in $X$. When this happens, i.e. $L$ is a rational homology three sphere, we have $b^2 (C) = 0$ and

$$\text{No. of ASD U}(1)\text{-bdl}/C = \# H^2 (C, \mathbb{Z}) = \# H^2 (L, \mathbb{Z}) = \# H_1 (L, \mathbb{Z}).$$

Here we have used the fact that the first cohomology group is always torsion free. Thus the number of such $\mathbb{H}$-SLag cycles in $X \times S^1$ equals the number of special Lagrangian rational homology three spheres in a Calabi-Yau threefold $X$, weighted by $\# H_1 (L, \mathbb{Z})$. Joyce [17] shows that with this particular weight, the numbers of special Lagrangians in any Calabi-Yau threefold behave well under various surgeries on $X$, and expects them to be invariants. Thus in this case, we have

$$\# \mathcal{M}^{\mathbb{H} - \text{SLag}} (X \times S^1) = \text{Joyce's proposed invariant for } \# \text{SLag. in } X.$$

In the next section, we will propose a homology theory, whose Euler characteristic gives $\# \mathcal{M}^{\mathbb{H} - \text{SLag}} (M)$.

### 3 Witten’s Morse theory for $\mathbb{H}$-SLag cycles

We are going to use the parametrized version of $\mathbb{H}$-SLag cycles in any almost $G_2$-manifold $M$. We fix an oriented smooth four dimensional manifold $C$ and
a rank $r$ Hermitian vector bundle $E$ over $C$. We consider the configuration space

$$
\mathcal{C} = \text{Map}(C, M) \times \mathcal{A}(E),
$$

where $\mathcal{A}(E)$ is the space of Hermitian connections on $E$.

**Definition 6.** An element $(f, D_E)$ in $\mathcal{C}$ is called a parametrized $\mathbb{H}$-SLag cycles in $M$ if

$$
f^*\Omega = F_E^+ = 0,
$$

where the self-duality is defined using the pullback metric from $M$.

Instead of $\text{Aut}(E)$, the symmetry group $\mathcal{G}$ in our situation consists of gauge transformations of $E$ which cover arbitrary diffeomorphisms on $M$,

$$
\begin{array}{ccc}
E & \xrightarrow{g} & E \\
\downarrow & & \downarrow \\
M & \xrightarrow{g_M} & M.
\end{array}
$$

It fits into the following exact sequence,

$$
1 \to \text{Aut}(E) \to \mathcal{G} \to \text{Diff}(C) \to 1.
$$

The natural action of $\mathcal{G}$ on $\mathcal{C}$ is given by

$$
g \cdot (f, D_E) = (f \circ g_M, g^*D_E),
$$

for any $(f, D_E) \in \mathcal{C} = \text{Map}(C, M) \times \mathcal{A}(E)$. Notice that $\mathcal{G}$ preserves the set of parametrized $\mathbb{H}$-SLag cycles in $M$.

The configuration space $\mathcal{C}$ has a natural one form $\Phi_0$: At any $(f, D_E) \in \mathcal{C}$ we can identify the tangent space of $\mathcal{C}$ as

$$
T_{(f,D_E)}\mathcal{C} = \Gamma(C, f^*T_M) \times \Omega^1(C, \text{ad}(E)).
$$

We define

$$
\Phi_0(f, D_E)(v, B) = \int_C \text{Tr} \left[ f^*(\iota_v \Omega) \wedge F_E + f^*\Omega \wedge B \right],
$$

for any $(v, B) \in T_{(f,D_E)}\mathcal{C}$.

**Proposition 7.** The one form $\Phi_0$ on $\mathcal{C}$ is closed and invariant under the action by $\mathcal{G}$.
Proof: Recall that there is a universal connection $D_E$ over $C \times \mathcal{A}(E)$ whose curvature $F_E$ at a point $(x, D_E)$ equals,
\[
F_E|_{(x, D_E)} = \left( F_{E, 0}^2, F_{E, 1}^1, F_{E}^0 \right)
\in \Omega^2(C) \otimes \Omega^0(A) + \Omega^1(C) \otimes \Omega^1(A) + \Omega^0(C) \otimes \Omega^2(A)
\]
with
\[
F_{E, 0}^2 = F_E, \quad F_{E, 1}^1(v, B) = B(v), \quad F_{E}^0 = 0,
\]
where $v \in T_x C$ and $B \in \Omega^1(C, \text{ad}(E)) = T_{D_E} A(E)$ (see e.g. [20]). The Bianchi identity implies that $\text{Tr} F_E$ is a closed form on $C \times \mathcal{A}(E)$. We also consider the evaluation map,
\[
ev : C \times \text{Map}(C, M) \to M
\]
\[
ev(x, f) = f(x).
\]
It is not difficult to see that the pushforward of the differential form $\ev^* (\Omega) \wedge \text{Tr} F_E$ on $C \times \text{Map}(C, M) \times \mathcal{A}(E)$ to $\text{Map}(C, M) \times \mathcal{A}(E)$ equals $\Phi_0$, i.e.
\[
\Phi_0 = \int_C \ev^* (\Omega) \wedge \text{Tr} F_E.
\]
Therefore the closedness of $\Phi_0$ follows from the closedness of $\Omega$. It is also clear from this description of $\Phi_0$ that it is $G$-invariant. ■

From this proposition, we know that $\Phi_0 = d\Psi_0$ locally for some function $\Psi_0$ on $C$. As in the Chern-Simons theory, this function $\Psi_0$ can be obtained explicitly by integrating the closed one form $\Phi_0$ along any path joining to a fixed element in $C$. When $M = X \times S^1$ and $C = L \times S^1$, this is essentially the functional used by Thomas in [30].

From now on, we assume that $E$ is a rank one bundle.

**Lemma 8.** The zeros of $\Phi_0$ are the same as parametrized $\mathbb{H}$-SLag cycles in $M$.

Proof: Suppose $(f, D_E)$ is a zero of $\Phi_0$. By evaluating it on various $(0, B)$, we have $f^* \Omega = 0$, i.e. $f : C \to M$ is a parametrized $\mathbb{H}$-SLag. This implies that the map
\[
\delta \Omega : T_{f(x)} M \to \Lambda^2 T_x^* C
\]
has image equals $\Lambda^2 T_x^* C$, for any $x \in C$. By evaluating $\Phi_0$ on various $(v, 0)$, we have $F_{E}^0 = 0$, i.e. $(f, D_E)$ is a parametrized $\mathbb{H}$-SLag cycle in $M$. The converse is obvious. ■
From above results, \( \Phi_0 \) descends to a closed one form on \( \mathcal{C}/\mathcal{G} \), called \( \Phi \). Locally we can write \( \Phi = dF \) for some function \( F \) whose critical points are precisely (unparametrized) \( \mathbb{H} \)-SLag cycles in \( M \). Using the gradient flow lines of \( F \), we could formally define a Witten’s Morse homology group, as in the famous Floer’s theory. Roughly speaking one defines a complex \((C_*, \partial)\), where \( C_* \) is the free Abelian group generated by critical points of \( F \) and \( \partial \) is defined by counting the number of gradient flow lines between two critical points of relative index one.

Remark: The equations for the gradient flow are given by

\[
\frac{\partial f}{\partial t} = * (f^* \xi \wedge F_E), \quad \frac{\partial D_E}{\partial t} = * (f^* \Omega),
\]

where \( \xi \in \Omega^2 (M, T_M) \) is defined by \( \langle \xi (u, v), w \rangle = \Omega (u, v, w) \).

The equation

\[
\partial^2 = 0
\]

requires a good compactification of the moduli space of \( \mathbb{H} \)-SLag cycles in \( M \), which we are lacking at this moment (see [31] however). We denote this proposed homology group as \( H_C (M) \), or \( H_C (M, \alpha) \) when \( f_* [C] = \alpha \in H_4 (M, \mathbb{Z}) \).

This homology group should be invariant under deformations of the almost \( G_2 \)-metric on \( M \) and its Euler characteristic equals,

\[
\chi (H_C (M)) = \# \mathcal{M}_{\mathbb{H}\text{-SLag}} (M).
\]

Like Floer homology groups, they measure the middle dimensional topology of the configuration space \( \mathcal{C} \) divided by \( \mathcal{G} \).

\section{4 TQFT of \( \mathbb{H} \)-SLag cycles}

In this section we study complete almost \( G_2 \)-manifold \( M_i \) with asymptotically cylindrical ends and the behavior of \( H_C (M) \) when a closed almost \( G_2 \)-manifold \( M \) decomposes into connected sum of two pieces, each with an asymptotically cylindrical end,

\[
M = M_1 \#_X M_2.
\]

Nontrivial examples of compact \( G_2 \)-manifolds are constructed by Kovalev [18] using such connected sum approach. The boundary manifold \( X \) is necessary a Calabi-Yau threefold. We plan to discuss analytic aspects of \( M_i \)’s in a future paper [24].
Each $M_i$'s will define a Lagrangian subspace $\mathcal{L}_{M_i}$ in the moduli space of special Lagrangian cycles in $X$. Furthermore we expect to have a gluing formula expressing the above homology group for $M$ in terms of the Floer Lagrangian intersection homology group for the two Lagrangian subspaces $\mathcal{L}_{M_1}$ and $\mathcal{L}_{M_2}$,

$$H_C(M) = HF_{\text{Lag}}^{\mathcal{M}_{\text{SLag}}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2}).$$

These properties can be reformulated to give us a topological quantum field theory. To begin we have the following definition.

**Definition 9.** An almost $G_2$-manifold $M$ is called cylindrical if $M = X \times \mathbb{R}^1$ and its positive three form respect such product structure, i.e.

$$\Omega_0 = \text{Re} \Omega_X + \omega_X \wedge dt.$$

A complete almost $G_2$-manifold $M$ with one end $X \times [0, \infty)$ is called asymptotically cylindrical if the restriction of its positive three form equals to the above one for large $t$, up to a possible error of order $O(e^{-t})$. More precisely the positive three form $\Omega$ of $M$ restricted to its end equals,

$$\Omega = \Omega_0 + d\zeta$$

for some two form $\zeta$ satisfying $|\zeta| + |\nabla \zeta| + |\nabla^2 \zeta| + |\nabla^3 \zeta| \leq C e^{-t}$.

Remark: If $M$ is an almost $G_2$-manifold with an asymptotically cylindrical end $X \times [0, \infty)$, then $(X, \omega_X, \Omega_X)$ is a complex threefold with a trivial canonical line bundle, but the Kähler form $\omega_X$ might not be Einstein. This is so, i.e. a Calabi-Yau threefold, provided that $M$ is a $G_2$-manifold. We will simply write $\partial M = X$.

We consider $\mathbb{H}$-SLags $C$ in $M$ which satisfy a Neumann condition at infinity. That is, away from some compact set in $M$, the immersion $f : C \to M$ can be written as

$$f : L \times [0, \infty) \to X \times [0, \infty)$$

with $\partial f / \partial t$ vanishes at infinite [24]. A relative $\mathbb{H}$-SLag itself has asymptotically cylindrical end $L \times [0, \infty)$ with $L$ a special Lagrangian submanifold in $X$. A relative $\mathbb{H}$-SLag cycle in $M$ is a pair $(C, D_E)$ with $C$ a relative $\mathbb{H}$-SLag in $M$ and $D_E$ a unitary connection over $C$ with finite energy,

$$\int_C |F_E|^2 dv < \infty.$$ 

Any finite energy connection $D_E$ on $C$ induces a unitary flat connection $D_E^\prime$ on $L$ [7].
Such a pair \((L, D_{E'})\) of a unitary flat connection \(D_{E'}\) over a special Lagrangian submanifold \(L\) in a Calabi-Yau threefold \(X\) is called a special Lagrangian cycle in \(X\). Their moduli space \(\mathcal{M}^{SLag}(X)\) plays an important role in the Strominger-Yau-Zaslow Mirror Conjecture [29] or [22]. The tangent space to \(\mathcal{M}^{SLag}(X)\) is naturally identified with \(H^2(L, \mathbb{R}) \times H^1(L, \text{ad}(E'))\). For line bundles over \(L\), the cup product \(\cup: H^2(L, \mathbb{R}) \times H^1(L, \mathbb{R}) \to \mathbb{R}\), induces a symplectic structure on \(\mathcal{M}^{SLag}(X)\) [15]. Using analytic results from [24] about asymptotically cylindrical manifolds, we can prove the following theorem.

Claim 10. Suppose \(M\) is an asymptotically cylindrical (almost) \(G_2\)-manifold with \(\partial M = X\). Let \(\mathcal{M}^{\mathbb{H}-SLag}(M)\) be the moduli space of rank one relative \(\mathbb{H}\)-SLag cycles in \(M\). Then the map defined by the boundary values,

\[b: \mathcal{M}^{\mathbb{H}-SLag}(M) \to \mathcal{M}^{SLag}(X),\]

is a Lagrangian immersion.

Sketch of the proof ([24]): For any closed Calabi-Yau threefold \(X\) (resp. \(G_2\)-manifold \(M\)), the moduli space of rank one special Lagrangian submanifolds \(L\) (resp. \(\mathbb{H}\)-SLags \(C\)) is smooth [27] and has dimension \(b_2^2(L)\) (resp. \(b_2^2(C)\)). The same holds true for complete manifold \(M\) with a asymptotically cylindrical end \(X \times [0, \infty)\), where \(b_2^2(C)_{L^2}\) denote the dimension of \(L^2\)-harmonic self-dual two forms on a relative \(\mathbb{H}\)-SLag \(C\) in \(M\).

The linearization of the boundary value map \(\mathcal{M}^{\mathbb{H}-SLag}(M) \to \mathcal{M}^{SLag}(X)\) is given by \(H^2_+\ (C)_{L^2} \xrightarrow{\alpha} H^2(L)\). Similar for the connection part, where the boundary value map is given by \(H^1(C)_{L^2} \xrightarrow{\beta} H^1(L)\). We consider the following diagram where each row is a long exact sequence of \(L^2\)-cohomology groups for the pair \((C, L)\) and each column in a perfect pairing.

\[
\begin{array}{cccccccc}
0 & \to & H^2_+ (C, L) & \to & H^2_+ (C) & \xrightarrow{\alpha} & H^2 (L) & \to & H^3 (C, L) & \to & \cdots \\
& & \otimes & \otimes & \otimes & & \otimes & & \otimes & & \\
0 & \leftarrow & H^2_+ (C) & \leftarrow & H^2_+ (C, L) & \leftarrow & H^1 (L) & \xleftarrow{\beta} & H^1 (C) & \leftarrow & \cdots \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
& & \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} & & \mathbb{R} & & \\
\end{array}
\]

Notice that \(H^2_+ (C, L)\), \(H^2_+ (C)\) and \(H^2 (L)\) parametrize infinitesimal deformation of \(C\) with fixed \(\partial C\), deformation of \(C\) alone and deformation of \(L\) respectively.
By simple homological algebra, it is not difficult to see that $\text{Im} \alpha \oplus \text{Im} \beta$ is a Lagrangian subspace of $H^2(L) \oplus H^1(L)$ with the canonical symplectic structure. Hence the result.

Remark: The deformation theory of conical special Lagrangian submanifolds is developed by Pacini in [28].

We denote the immersed Lagrangian submanifold $b (\mathcal{M}^H_{-SLag}(M))$ in $\mathcal{M}^{SLag}(X)$ by $\mathcal{L}_M$. When $M$ decompose as a connected sum $M_1 \#_X M_2$ along a long neck, as in Atiyah’s conjecture on Floer Chern-Simons homology group [3], we expect to have an isomorphism,

$$H_C(M) \cong HF^*_{\text{Lag}}(M_1, M_2).$$

More precisely, suppose $\Omega_t$ with $t \in [0, \infty)$, is a family of $G_2$-structure on $M_t = M$ such that as $t$ goes to infinite, $M$ decomposes into two components $M_1$ and $M_2$, each has an asymptotically cylindrical end $X \times [0, \infty)$. Then we expect that $\lim_{t \to \infty} H_C(M_t) \cong HF^*_{\text{Lag}}(M_1, M_2)$. We summarize these structures in the following table:

| Manifold: | (almost) $G_2$ -manifold, $M^7$ | (almost) CY threefold, $X^6$ |
|-----------|---------------------------------|---------------------------|
| SUSY Cycles: | $\mathbb{H}$-SLag. submfds. + ASD bdl | SLag submfds. + flat bdl |
| Invariant: | Homology group, $H_C(M)$ | Fukaya category, $Fuk (\mathcal{M}^{SLag}(X))$. |

These associations can be formalized to form a TQFT [4]. Namely we associate an additive category $F(X) = Fuk (\mathcal{M}^{SLag}(X))$ to a closed almost Calabi-Yau threefold $X$, a functor $F(M) : F(X_0) \to F(X_1)$ to an almost $G_2$-manifold $M$ with asymptotically cylindrical ends $X_1 - X_0 = X_1 \cup X_0$.

They satisfy

(i) $F(\phi) =$ the additive tensor category of vector spaces $(\mathbb{V}ec)$,

(ii) $F(X_1 \amalg X_2) = F(X_1) \otimes F(X_2)$.

For example, when $M$ is a closed $G_2$-manifold, that is a cobordism between empty manifolds, then we have $F(M) : ((\mathbb{V}ec)) \to ((\mathbb{V}ec))$ and the image of the trivial bundle is our homology group $H_C(M)$. 
5 More TQFTs

Notice that all TQFTs we propose in this paper are formal mathematical constructions. Besides the lack of compactness for the moduli spaces, the obstruction issue is also a big problem if we try to make these theories rigorous. This problem is explained to the author by a referee.

There are other TQFTs naturally associated to Calabi-Yau threefolds and $G_2$-manifolds but (1) they do not involve nontrivial coupling between submanifolds and bundles and (2) new difficulties arise because of corresponding moduli spaces for Calabi-Yau threefolds have virtual dimension zero and could be singular. They are essentially in the paper by Donaldson and Thomas [9].

TQFT of associative cycles

We assume that $M$ is a $G_2$-manifold, i.e. $\Omega$ is parallel rather than closed. Three dimensional submanifolds $A$ in $M$ calibrated by $\Omega$ is called associative submanifolds and they can be characterized by $\chi|_A = 0$ ([14]) where $\chi \in \Omega^3 (M, T_M)$ is defined by $\langle w, \chi (x, y, z) \rangle = \ast \Omega (w, x, y, z)$. We define a parametrized $A$-cycle to be a pair $(f, D_E) \in C_A = \text{Map} (A, M) \times A (E)$, with $f : A \to M$ a parametrized $A$-submanifold and $D_E$ is a unitary flat connection on a Hermitian vector bundle $E$ over $A$. There is also a natural $G$-invariant closed one form $\Phi_A$ on $C_A$ given by

$$\Phi_A (f, D_E) (v, B) = \int_A Tr F_E \wedge B + \langle f^* \chi, v \rangle_{T_M},$$

for any $(v, B) \in \Gamma (A, f^* T_M) \times \Omega^1 (A, ad (E)) = T_{(f, D_E)} C_A$. Its zero set is the moduli space of $A$-cycles in $M$. As before, we could formally apply arguments in Witten’s Morse theory to $\Phi_A$ and define a homology group $H_A (M)$.

The corresponding category associated to a Calabi-Yau threefold $X$ would be the Fukaya-Floer category of the moduli space of unitary flat bundles over holomorphic curves in $X$, denote $\mathcal{M}^{\text{curve}} (X)$. We summarize these in the following table:

| Manifold:     | $G_2$ -manifold, $M^7$ | CY threefold, $X^6$ |
|---------------|-------------------------|---------------------|
| SUSY Cycles:  | A-submfd.+ flat bundles | Holomorphic curves+ flat bundles |
| Invariant:    | Homology group, $H_A (M)$ | Fukaya category, $\text{Fuk} (\mathcal{M}^{\text{curve}} (X))$. |
TQFT of Donaldson-Thomas bundles

We assume that $M$ is a seven manifold with a $G_2$-structure such that its positive three form $\Omega$ is co-closed, rather than closed, i.e. $d\Theta = 0$ with $\Theta = *\Omega$. In [9] Donaldson and Thomas introduce a first order Yang-Mills equation for $G_2$-manifolds,

$$F_E \wedge \Theta = 0.$$ 

Their solutions are the zeros of the following gauge invariant one form $\Phi_{DT}$ on $A(E)$,

$$\Phi_{DT} (D_E)(B) = \int_M Tr [F_E \wedge B] \wedge \Theta,$$

for any $B \in \Omega^1 (M, ad (E)) = T_{D_E} A(E)$. This one form $\Phi_{DT}$ is closed because of $d\Theta = 0$. As before, we can formally define a homology group $H_{DT} (M)$. The corresponding category associated to a Calabi-Yau threefold $X$ should be the Fukaya-Floer category of the moduli space of Hermitian Yang-Mills connections over $X$, denote $\mathcal{M}^{HYM}(X)$. Again we summarize these in a table:

| Manifold: | $G_2$-manifold, $M^7$ | CY threefold, $X^6$ |
|-----------|---------------------|---------------------|
| SUSY Cycles: | DT-bundles | Hermitian YM-bundles |
| Invariant: | Homology group, $H_{DT} (M)$ | Fukaya category, $Fuk (\mathcal{M}^{HYM}(X))$. |

It is an interesting problem to understand the transformations of these TQFTs under dualities in M-theory.

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