ERGODICITY OF MAPPING CLASS GROUP ACTIONS ON SU(2)-CHARACTER VARIETIES

WILLIAM M. GOLDMAN AND EUGENE Z. XIA

To Bob Zimmer, on his sixtieth birthday

Abstract. Let Σ be a compact orientable surface with genus $g$ and $n$ boundary components $∂_1, \ldots, ∂_n$. Let $b = (b_1, \ldots, b_n) \in [-2, 2]^n$. Then the mapping class group $\text{Mod}(Σ)$ acts on the relative $SU(2)$-character variety $X_b := \text{Hom}_b(\pi, SU(2))/SU(2)$, comprising conjugacy classes of representations $\rho$ with $\text{tr}(\rho(∂_i)) = b_i$. This action preserves a symplectic structure on the open dense smooth submanifold of $\text{Hom}_b(\pi, SU(2))/SU(2)$ corresponding to irreducible representations. This subset has full measure and is connected. In this note we use the symplectic geometry of this space to give a new proof that this action is ergodic.

1. Introduction

Let $Σ = Σ_{g,n}$ be a compact oriented surface of genus $g$ with $n$ boundary components $∂_1(Σ), \ldots, ∂_n(Σ)$. Choose basepoints $p_0 \in Σ$ and $p_i \in ∂_i(Σ)$. Let $π = π_1(Σ, p_0)$ denote the fundamental group of $Σ$. Choosing arcs from $p_0$ to each $p_i$ identifies each fundamental group $π_1(∂_i(Σ), p_i)$ with a subgroup $π_1(∂_i) \hookrightarrow π$. The orientation on $Σ$ induces orientations on each $∂_i(Σ)$. For each $i$, denote the positively oriented generator of $π_1(∂_i Σ)$ also by $∂_i$.

The mapping class group $\text{Mod}(Σ)$ consists of isotopy classes of orientation-preserving homeomorphisms of $Σ$ which pointwise fix each $∂_i$. The Dehn-Nielsen Theorem (see for example Farb-Margalit [1] or Morita [19]), identifies $\text{Mod}(Σ)$ with a subgroup of $\text{Out}(π) := \text{Aut}(π)/\text{Inn}(π)$.

Consider a connected compact semisimple Lie group $G$. Its complexification $G^C$ is the group of complex points of a semisimple linear algebraic group defined over $\mathbb{R}$. Fix a conjugacy class $B_i \subset G$ for each

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boundary component \( \partial_i \). Then the relative representation variety is

\[
\Hom_B(\pi, G) := \{ \rho \in \Hom(\pi, G) \mid \rho(\partial_j) \in B_j, \text{ for } 1 \leq j \leq n \}.
\]

The action of the automorphism group \( \text{Aut}(\pi) \) on \( \pi \) induces an action on \( \Hom_B(\pi, G^C) \) by composition. Furthermore this action descends to an action of \( \text{Mod}(\Sigma) \subset \text{Out}(\pi) \) on the categorical quotient or the relative character variety

\[
\mathfrak{X}_B^C(G) := \Hom_B(\pi, G^C)//G^C.
\]

The moduli space \( \mathfrak{X}_B^C(G) \) has an invariant dense open subset which is a smooth complex submanifold. This subset has an invariant complex symplectic structure \( \omega^C \), which is algebraic with respect to the structure of \( \mathfrak{X}_B^C(G) \) as an affine algebraic set. The pullback \( \omega \) of the real part of this complex symplectic structure under

\[
\mathfrak{X}_B(G) := \Hom_B(\pi, G)/G \longrightarrow \mathfrak{X}_B^C(G)
\]

defines a symplectic structure on a dense open subset, which is a smooth submanifold. The smooth measure defined by the symplectic structure is finite \([3, 14, 11]\) and \( \text{Mod}(\Sigma) \)-invariant. The main result of Goldman \([6]\) (when \( G \) has \( \text{SU}(2) \) and \( \text{U}(1) \)-factors) and Pickrell-Xia \([24]\) (when \( g > 1 \)) is:

**Theorem.** The action of \( \text{Mod}(\Sigma) \) on each component of \( \mathfrak{X}_B(G) \) is ergodic with respect to the measure induced by \( \omega \).

The goal of this note is to give a short proof in the case that \( G = \text{SU}(2) \).

Recently F. Palesi \([23]\) proved ergodicity of \( \text{Mod}(\Sigma) \) on \( \mathfrak{X}_B(\text{SU}(2)) \) when \( \Sigma \) is a compact connected nonorientable surface with \( \chi(\Sigma) \leq -2 \). When \( \Sigma \) is nonorientable, the character variety fails to possess a symplectic structure (in fact its dimension may be odd) and it would be interesting to adapt the proof given here to the nonorientable case.

The proof given here arose from our investigation \([10]\) of ergodic properties of subgroups of \( \text{Mod}(\Sigma) \) on character varieties. The closed curves on \( \Sigma \) play a central role. Namely, every closed curve defines a conjugacy class of elements in \( \pi \), and hence a regular function

\[
\Hom(\pi, G^C) \xrightarrow{f_\alpha} \mathbb{C}
\]

\[
\rho \mapsto \text{tr}(\rho(\alpha)).
\]

for some representation \( G^C \longrightarrow \text{GL}(N, \mathbb{C}) \). These trace functions \( f_\alpha \) are \( G^C \)-conjugate invariant and results of Procesi \([25]\) imply that such functions generate the coordinate ring \( \mathbb{C}[\mathfrak{X}_B(\text{SL}(2, \mathbb{C}))] \) of \( \mathfrak{X}_B(\text{SL}(2, \mathbb{C})) \).
Simple closed curves $\alpha$ determine elements of $\text{Mod}(\Sigma)$, namely the Dehn twists $\tau_{\alpha}$. Let $S$ be a set of simple closed curves on $\Sigma$. Our methods apply to the subgroup $\Gamma_S \subset \text{Mod}(\Sigma)$ generated by $\tau_{\alpha}$, where $\alpha \in S$. Our proof may be summarized: if the trace functions $f_{\alpha}$ generate $\mathbb{C}[\mathcal{X}_B(\text{SL}(2, \mathbb{C}))]$, then the action of $\Gamma_S$ on each component of $\mathcal{X}_B(\text{SU}(2))$ is ergodic.

The original proof [6] decomposes $\Sigma$ along a set $\mathcal{P}$ of $3g-3+2n$ disjoint curves into

$$2g-2+n = -\chi(\Sigma)$$

3-holed spheres (a pants decomposition.) The subgroup $\Gamma_{\mathcal{P}}$ of $\text{Mod}(\Sigma)$ stabilizing $\mathcal{P}$ is generated by Dehn twists along curves in $\mathcal{P}$. The corresponding trace functions define a map

$$\mathcal{X}_b \xrightarrow{f_{\mathcal{P}}} [-2, 2]^{\mathcal{P}}$$

which is an ergodic decomposition for the action of $\Gamma_{\mathcal{P}}$. Thus any measurable function invariant under $\Gamma_{\mathcal{P}}$ must factor through $f_{\mathcal{P}}$. Changing $\mathcal{P}$ by elementary moves on 4-holed spheres, and a detailed analysis in the case of $\Sigma_{0,4}$ and $\Sigma_{1,1}$ implies ergodicity under all of $\text{Mod}(\Sigma)$. The present proof uses the commutative algebra of the character ring (in particular the work of Horowitz [13], Magnus [17] and Procesi [25], and the identification of the twist flows with the Hamiltonians of trace functions [4]). Although it is not used in [6], the map $f_{\mathcal{P}}$ is the moment map for the $\mathbb{R}^{\mathcal{P}}$-action by twist flows, as well as the ergodic decomposition for $\Gamma_{\mathcal{P}}$. Finding sets $S$ of simple curves whose trace functions generate the character ring promises to be useful to prove ergodicity of the subgroup of $\text{Mod}(\Sigma)$ generated by Dehn twists along elements of $S$ (Goldman-Xia [10].)

In a similar direction, Sean Lawton has pointed out that this method of proof, (combined with Lawton [Lawton1,Lawton2]) implies in at least some cases ergodicity of $\text{Mod}(\Sigma)$ on the relative $\text{SU}(3)$-character varieties (except when $\Sigma \approx \Sigma_{0,3}$, where it is not true).

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With great pleasure we dedicate this paper to Bob Zimmer. Goldman first presented this result in Zimmer’s graduate course at Harvard University in Fall 1985. Goldman would like to express his warm gratitude to Zimmer for the friendship, support and mathematical inspiration he has given over many years.
2. Simple generators for the character ring

In the paper we restrict to the case \( G = \text{SU}(2) \) and \( G^c = \text{SL}(2, \mathbb{C}) \). Conjugacy classes in \( G = \text{SU}(2) \) are the level sets of the trace function \( \text{SU}(2) \to [-2, 2] \). Thus a collection \( B = (B_1, \ldots, B_n) \) of conjugacy classes in \( \text{SU}(2) \) is precisely given by an \( n \)-tuple \( b = (b_1, \ldots, b_n) \in [-2, 2]^n \).

We denote the relative representation variety by:

\[
\text{Hom}_b(\pi, \text{SU}(2)) := \{ \rho \in \text{Hom}(\pi, \text{SU}(2)) \mid \text{tr}(\rho(\partial)) = b_i \}
\]

and its quotient, the relative character variety by:

\[
\mathcal{X}_b := \text{Hom}_b(\pi, \text{SU}(2))/\text{SU}(2).
\]

**Theorem 2.1.** There exists a finite subset \( S \subset \pi \) corresponding to simple closed curves on \( \Sigma \) such that \( \{ f_\gamma : \gamma \in S \} \) generates the coordinate ring \( \mathbb{C}[\mathcal{X}_b] \).

We prove this theorem in §2.1 and §2.2.

2.1. Magnus-Horowitz-Procesi generators. The following well known proposition is a direct consequence of the work of Horowitz [13] and Procesi [25]. Compare also Magnus [17], Newstead [22] and Goldman [8].

**Proposition 2.2.** Let \( F_N \) be the free group freely generated by \( A_1, \ldots, A_N \), and let

\[
\mathcal{X}(N) := \text{Hom}(F_N, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})
\]

be its \( \text{SL}(2, \mathbb{C}) \)-character variety. Denote by \( \mathcal{I}_N \) the collection of all

\[
I = (i_1, \ldots, i_k) \in \mathbb{Z}^k
\]

where

\[
1 \leq i_1 < \cdots < i_k \leq N
\]

and \( k \leq 3 \). For \( I \in \mathcal{I}_N \), define

\[
A_I := A_{i_1} \cdots A_{i_k}
\]

and let

\[
\mathcal{X}(N) \xrightarrow{f_I} \mathbb{C}
\]

\[
[\rho] \mapsto \text{tr}(\rho(A_I))
\]

the corresponding trace functions. Then the collection

\[
\{ f_I \mid I \in \mathcal{I}_N \}
\]

generates the coordinate ring \( \mathbb{C}[\mathcal{X}(N)] \).
We shall refer to the coordinate ring \( \mathbb{C}[X(N)] \) as the *character ring*. Recall that by definition it is the subring of the ring of regular functions
\[
\text{SL}(2, \mathbb{C})^N \longrightarrow \mathbb{C}
\]
consisting of \( \text{Inn}(\text{SL}(2, \mathbb{C})) \)-invariant functions.

2.2. Constructing simple loops. Suppose that \( \Sigma \) has genus \( g \geq 0 \) and \( n > 0 \) boundary components. (We postpone the case when \( \Sigma \) is closed, that is \( n = 0 \), to the end of this section.) We suppose that \( \chi(\Sigma) = 2 - 2g - n < 0 \). Then \( \pi_1(\Sigma) \) is free of rank \( N = 2g + n - 1 \). We describe a presentation of \( \pi_1(\Sigma) \) such that the above elements \( A_I \), for \( I \in \mathcal{I}_N \) can be represented by simple closed curves on \( \Sigma \). We also identify \( I \) with the subset
\[
\{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}.
\]
The fundamental group \( \pi_1(\Sigma) \) admits a presentation with generators
\[
A_1, \ldots, A_{2g}, A_{2g+1}, \ldots, A_{2g+n}
\]
subject to the relation
\[
A_1A_2A_1^{-1}A_2^{-1} \ldots A_{2g-1}A_{2g}A_{2g-1}^{-1}A_{2g}^{-1} \ldots A_{2g+1} \ldots A_{2g+n} = 1.
\]
Then
\[
\pi = \pi_1(\Sigma) \cong \mathbb{F}_{2g+n-1},
\]
freely generated by the set \( \{A_1, \ldots, A_{2g+n-1}\} \).

To represent the elements \( A_I \in \pi_1(\Sigma) \) explicitly as simple loops, we realize \( \Sigma \) as the union of a planar surface \( P \) and \( g \) handles \( H_1, \ldots, H_g \). In the notation of [8], \( P \approx \Sigma_{0,g+n} \) has \( g + n \) boundary components
\[
\alpha_1, \ldots, \alpha_g, \alpha_{g+1}, \ldots, \alpha_{g+n}
\]
and each handle \( H_j \approx \Sigma_{1,1} \) is a one-holed torus. The original surface \( \Sigma \) is obtained by attaching \( H_j \) to \( P \) along \( \alpha_j \) for \( j = 1, \ldots, g \).

We construct the curves \( A_i \), for \( i = 1, \ldots, 2g + n \) as follows. Choose a pair of basepoints \( p^+_j, p^-_j \) on each \( \alpha_j \) for \( j = 1, \ldots, g + n \). Let \( \alpha_j^- \) be the oriented subarc of \( \alpha_j \) from \( p^-_j \) to \( p^+_j \), and \( \alpha_j^+ \) the corresponding subarc from \( p^+_j \) to \( p^-_j \). Thus \( \alpha_j \simeq \alpha_j^- * \alpha_j^+ \) is a boundary component of \( P \).

Choose a system of disjoint arcs \( \beta_j \) from \( p^+_j \) to \( p^-_{j+1} \), where \( \beta_{g+n} \) runs from \( p^+_{g+n} \) to \( p^-_1 \) in the cyclic indexing of \( \{1, 2, \ldots, g + n\} \). Compare Figure [5].

For \( I \in \mathcal{I}_N \), the curve \( A_I \) will be the concatenation \( E_1^I \ldots E_{g+n}^I \) of simple arcs \( E_j^I \) running from \( p^-_j \) to \( p^-_{j+1} \). Define
\[
E_j^0 := \alpha_j^- * \beta_j,
\]
so that

\[ A^0 := E_1^0 * \cdots * E_N^0 \]

is a contractible loop.

Suppose first that \( i > 2g \). Then the curve \( A_i \) will be freely homotopic to the oriented loop \( \alpha_i^{-1} \), corresponding to a component of \( \partial \Sigma \). The arc

\[ E_i^+ := (\alpha_i^+)^{-1} * \beta_i \]

goes from \( p_i^- \) to \( p_{i+1}^- \) (cyclically). Then \( A_i \) corresponds to the arc

\[ A_i := E_1^0 * \cdots * E_{2g}^0 * E_{2g+1}^0 * \cdots * E_i^+ * \cdots E_{2g+n-1}^0 \]

For \( i \leq 2g \), the curves \( A_i \) will lie on the handles \( H_j \). The curves \( A_{2j-1} \) and \( A_{2j} \) define a basis for the relative homology of \( H_j \) and the relative homology class of the curve

\[ A_{2j-1,2j} := A_{2j-1}A_{2j} \]

is their sum. Compare Figures 6,7.

As above we define three simple arcs \( \gamma_j, \delta_j, \eta_j \) running from \( p_j^- \) to \( p_j^+ \) to build these three curves respectively.

The boundary \( \partial H_j \) identifies with \( \alpha_j \) for \( j = 1, \ldots, g \). The two points on \( \partial H_j \) which identify to

\[ p_j^\pm \in \alpha_j \subset \partial P \]

divide \( \partial H_j \) into two arcs. Without danger of confusion, denote these arcs by \( \alpha_j^\pm \) as well. On the handle \( H_j \), choose disjoint simple arcs \( \gamma_j, \delta_j \) and \( \eta_j \) running from \( p_i^+ \) to \( p_i^- \) such that the

\[ H_j \setminus (\gamma_j \cup \delta_j) \]

is an hexagon. Two of its edges correspond to the arcs \( \alpha_j^\pm \). Its other four edges are the two pairs obtained by splitting \( \gamma_j \) and \( \delta_j \). (Compare Figure 6.) Let \( \eta_j \) to be a simple arc homotopic to \( \gamma_j * (\alpha_j^+)^{-1} * \delta_j \), where * denotes concatenation. For each \( j \leq g \), the arcs

\[ E_j^\gamma = \gamma_j * \beta_j \]
\[ E_j^\delta = \delta_j * \beta_j \]
\[ E_j^\eta = \eta_j * \beta_j \]
run from $p_j^-$ to $p_j^+$ and define:

\[
\begin{align*}
A_{2j-1} &= E_1^\emptyset \ast \cdots \ast E_j^\emptyset \ast \cdots \ast E_g^\emptyset \ast \cdots \ast E_{g+n}^\emptyset \\
A_{2j} &= E_1^\emptyset \ast \cdots \ast E_j^\emptyset \ast \cdots \ast E_g^\emptyset \ast \cdots \ast E_{g+n}^\emptyset \\
A_{2j-1,2j} &= E_1^\emptyset \ast \cdots \ast E_j^n \ast \cdots \ast E_g^\emptyset \ast \cdots \ast E_{g+n}^\emptyset .
\end{align*}
\]

In general, suppose that $I \in \mathcal{I}_N$. Define

\[
A_I := E_{1}^I \ast \cdots \ast E_{g+n}^I
\]

where

\[
E_j^I = \begin{cases} 
E_j^\emptyset & \text{if } j \notin I \\
E_j^+ & \text{if } j \in I
\end{cases}
\]

if $j > g$ and

\[
E_j^I = \begin{cases} 
E_j^\emptyset & \text{if } 2j - 1, 2j \notin I \\
E_j^\gamma & \text{if } 2j - 1 \in I, 2j \notin I \\
E_j^\delta & \text{if } 2j - 1 \notin I, 2j \in I \\
E_j^n & \text{if } 2j - 1, 2j \in I.
\end{cases}
\]

if $j \leq g$.

Now each $A_I$ is simple: Each of the oriented arcs $\alpha_i^\pm, \beta_i, \gamma_i, \delta_i, \eta_i$ are embedded and intersect only along $p_i^\pm$. In particular each of the above oriented arcs begins at some $p_i^\pm$ and ends at some $p_i^\pm$. Thus each

\[
E_j^\emptyset, E_j^+, E_j^\gamma, E_j^\delta, E_j^n
\]

is a simple arc running from $p_j^-$ to $p_{j+1}^-$, cyclically. The loop $A_I$ concatenates these arcs, which only intersect along the $p_i^\pm$. Each of these endpoints occurs exactly twice, once as the initial endpoint and once as the terminal endpoint. Therefore the loop $A_I$ is simple.

This collection $A_I$, for $I \in \mathcal{I}_N$, of simple curves determines a collection of regular functions $f_I$ on $\mathcal{X}_C$ which generate the character ring. Since the inclusion

\[
\mathcal{X}_b^C \hookrightarrow \mathcal{X}_C
\]

is a morphism of algebraic sets, the restrictions of $f_I$ to $\mathcal{X}_b^C$ generate the coordinate ring of $\mathcal{X}_b^C$.

The case $n = 0$ remains. To this end, the character variety of $\Sigma_{g,0}$ appears as the relative character variety of $\Sigma_{g,1}$ with boundary condition $b_1 = 2$. As above, the restrictions of the $f_I$ from $\Sigma_{g,1}$ to the character variety of $\Sigma_{g,0}$ generate its coordinate ring. The proof of Theorem 2.1 is complete.
Figure 1. Simple loops on $\Sigma_{1,2}$ corresponding to words $A_1, A_2, A_3, A_1A_3$ and $A_1^{-1} = A_1A_2A_1^{-1}A_2^{-1}A_3$ in free generators $\{A_1, A_2, A_3\}$.

Figure 2. Simple loop corresponding to $A_2A_3$.

Figure 3. Simple loop corresponding to $A_1A_2$.

Figure 4. Simple loop corresponding to $A_1A_2A_3$. 
Figure 5. A planar surface $P \approx \Sigma_{0,4}$

Figure 6. A handle $H_j \approx \Sigma_{1,1}$

Figure 7. A $(1, 1)$-curve $\eta_j$ on the handle $H_j$
3. Infinitesimal Transitivity

The application of Theorem 2.1 involves several lemmas to deduce that the flows of the Hamiltonian vector fields $\text{Ham}(f_\gamma)$, where $\gamma \in \mathcal{S}$, generate a transitive action on $\mathcal{X}_b$.

**Lemma 3.1.** Let $X$ be an affine variety over a field $k$. Suppose that $\mathcal{F} \subset k[X]$ generates the coordinate ring $k[X]$. Let $x \in X$. Then the differentials $df(x)$, for $f \in \mathcal{F}$, span the cotangent space $T^*_x(X)$.

**Proof.** Let $\mathcal{M}_x \subset k[X]$ be the maximal ideal corresponding to $x$. Then the functions $f - f(x)1$, where $f \in \mathcal{F}$, span $\mathcal{M}_x$. The correspondence

$$\mathcal{M}_x \rightarrow T^*_x(X)$$

$$f \mapsto df(x)$$

induces an isomorphism $\mathcal{M}_x/\mathcal{M}_x^2 \cong T^*_x(X)$. In particular it is onto. Therefore the covectors $df(x)$ span $T^*_x(X)$ as claimed. □

**Lemma 3.2.** Let $X$ be a connected symplectic manifold and $\mathcal{F}$ be a set of functions on $X$ such that at every point $x \in X$, the differentials $df(x)$, for $f \in \mathcal{F}$, span the cotangent space $T^*_x(X)$. Then the group $\mathcal{G}$ generated by the Hamiltonian flows of the vector fields $\text{Ham}(f)$, for $f \in \mathcal{F}$, acts transitively on $X$.

**Proof.** The nondegeneracy of the symplectic structure implies that the vector fields $\text{Ham}(f)(x)$ span the tangent space $T_xX$ for every $x \in X$. By the inverse function theorem, the $\mathcal{G}$-orbit $\mathcal{G} \cdot x$ of $x$ is open. Since the orbits partition $X$ and $X$ is connected, $\mathcal{G} \cdot x = X$ as claimed. □

**Proposition 3.3.** Let $b = (b_1, \ldots, b_m) \in [-2,2]^n$. Then $\mathcal{X}_b$ is either empty or connected.

The proof follows from Newstead [21] and Goldman [5]. Alternatively, apply Mehta-Seshadri [18] to identify $\mathcal{X}_b$ with a moduli space of semistable parabolic bundles, and apply their result that the corresponding moduli space is irreducible.

**Corollary 3.4.** Let $\mathcal{G}$ be the group generated by the flows of the Hamiltonian vector fields $\text{Ham}(f_\gamma)$, where $\gamma \in \mathcal{S}$. Then $\mathcal{G}$ acts transitively on $\mathcal{X}_b$.

**Proof.** By Theorem 2.1,

$$\{f_\gamma \mid \gamma \in \mathcal{S}\}$$

generates $\mathbb{C}[\mathcal{X}_b]$. Lemma 3.1 implies that at every point $x \in \mathcal{X}_b$ the differentials $df_\gamma(x)$ span $T^*_x(\mathcal{X}_b)$. Proposition 3.3 implies that $\mathcal{X}_b$ is connected. Now apply Lemma 3.2. □
4. Hamiltonian twist flows

We briefly review results of Goldman [4], describing the flows generated by the Hamiltonian vector fields $\text{Ham}(f_\alpha)$, when $\alpha$ represents a simple closed curve. In that case the local flow of this vector field on $\mathfrak{X}(G)$ lifts to a flow $\xi_t$ on the representation variety $\text{Hom}_B(\pi, G)$. Furthermore this flow admits a simple description [4] as follows.

4.1. Invariant functions and centralizing one-parameter subgroups. Let $Ad$ be the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$. We suppose that $Ad$ preserves a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. In the case $G = \text{SU}(2)$, this will be $\langle X, Y \rangle := \text{tr}(XY)$.

Let $G \xrightarrow{f} \mathbb{R}$ be a function invariant under the inner automorphisms $\text{Inn}(G)$. Following [4], we describe how $f$ determines a way to associate to every element $x \in G$ a one-parameter subgroup

$$\zeta^t(x) = \exp(tF(x))$$

centralizing $x$. Given $f$, define its variation function $G \xrightarrow{F} \mathfrak{g}$ by:

$$\langle F(x), v \rangle = \frac{d}{dt} \bigg|_{t=0} f(x \exp(tv))$$

for all $v \in \mathfrak{g}$. Invariance of $f$ under $Ad(G)$ implies that $F$ is $G$-equivariant:

$$F(gxg^{-1}) = Ad(g)F(x).$$

Taking $g = x$ implies that the one-parameter subgroup

$$(4.1) \quad \zeta^t(x) := \exp(tF(x))$$

lies in the centralizer of $x \in G$.

Intrinsically, $F(x) \in \mathfrak{g}$ is dual (by $\langle \cdot, \cdot \rangle$) to the element of $\mathfrak{g}^*$ corresponding to the left-invariant 1-form on $G$ extending the covector $df(x) \in T^*_x(G)$.

4.2. Nonseparating loops. There are two cases, depending on whether $\alpha$ is nonseparating or separating. Let $\Sigma|\alpha$ denote the surface-with-boundary obtained by splitting $\Sigma$ along $\alpha$. The boundary of $\Sigma|\alpha$ has two components, denoted by $\alpha_{\pm}$, corresponding to $\alpha$. The original surface $\Sigma$ may be reconstructed as a quotient space under the identification of $\alpha_-$ with $\alpha_+$. 
If $\alpha$ is nonseparating, then $\pi = \pi_1(\Sigma)$ can be reconstructed from the fundamental group $\pi_1(\Sigma|\alpha)$ as an HNN-extension:

$$\pi \cong \left( \pi_1(\Sigma|\alpha) \amalg \langle \beta \rangle \right) / \left( \beta \alpha_\beta \beta^{-1} = \alpha_+ \right).$$

A representation $\rho$ of $\pi$ is determined by:

- the restriction $\rho'$ of $\rho$ to the subgroup $\pi_1(\Sigma|\alpha) \subset \pi$, and

- the value $\beta' = \rho(\beta)$

which satisfies:

$$\beta' \rho'(\alpha_-) \beta'^{-1} = \rho'(\alpha_+).$$

Furthermore any pair $(\rho', \beta')$ where $\rho'$ is a representation of $\pi_1(\Sigma|\alpha)$ and $\beta' \in G$ satisfies (4.3) determines a representation $\rho$ of $\pi$.

The twist flow $\xi^t_{\alpha}$, for $t \in \mathbb{R}$ on $\text{Hom}(\pi, \text{SU}(2))$, is then defined as follows:

$$\xi^t_{\alpha}(\rho) : \gamma \mapsto \begin{cases} 
\rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\
\rho'(\beta) \zeta^t(\rho(\alpha_-)) & \text{if } \gamma = \beta.
\end{cases}$$

where $\zeta^t$ is defined in (4.1). This flow covers the flow generated by $\text{Ham}(f_{\alpha})$ on $\mathcal{X}_b$ (See [4]).

4.3. Separating loops. If $\alpha$ separates, then $\pi = \pi_1(\Sigma)$ can be reconstructed from the fundamental groups $\pi_1(\Sigma_i)$ of the two components $\Sigma_1$, $\Sigma_2$ of $\Sigma|\alpha$, as an amalgam

$$\pi \cong \pi_1(\Sigma_1) \amalg (\pi_1(\Sigma_2)).$$

A representation $\rho$ of $\pi$ is determined by its restrictions $\rho_i$ to $\pi_1(\Sigma_i)$. Furthermore any two representations $\rho_i$ of $\pi$ satisfying $\rho_1(\alpha) = \rho_2(\alpha)$ determines a representation of $\pi$.

The twist flow is defined by:

$$\xi^t_{\alpha}(\rho) : \gamma \mapsto \begin{cases} 
\rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\
\zeta^t(\rho(\alpha)) \rho(\gamma) \zeta^{-t}(\rho(\alpha)) & \text{if } \gamma \in \pi_1(\Sigma_2)
\end{cases}$$

where $\zeta^t$ is defined in (4.1).

4.4. Dehn twists. Let $\alpha \subset \Sigma$ be a simple closed curve. The Dehn twist along $\alpha$ is the mapping class $\tau_\alpha \in \text{Mod}(\Sigma)$ represented by a homeomorphism $\Sigma \to \Sigma$ supported in a tubular neighborhood $N(\alpha)$ of $\alpha$ defined as follows. In terms of a homeomorphism $S^1 \times [0, 1] \xrightarrow{h} N(\alpha)$ which takes $\alpha$ to $S^1 \times \{0\}$, the Dehn twist is

$$\tau_\alpha \circ h(\zeta, t) = h(e^{2\pi i \zeta}, t).$$
If $\alpha$ is essential, then $\tau_\alpha$ induces a nontrivial element of $\text{Out}(\pi)$ on $\pi = \pi_1(\Sigma)$.

If $\alpha$ is nonseparating, then $\pi = \pi_1(\Sigma)$ can be reconstructed from the fundamental group $\pi_1(\Sigma|\alpha)$ as an HNN-extension as in (4.2). The Dehn twist $\tau_\alpha$ induces the automorphism $(\tau_\alpha)_* \in \text{Aut}(\pi)$ defined by:

$$(\tau_\alpha)_* : \gamma \mapsto \begin{cases} 
\gamma & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\
\gamma \alpha & \text{if } \gamma = \beta.
\end{cases}$$

The induced map $(\tau_\alpha)^*$ on $\text{Hom}(\pi,G)$ maps $\rho$ to:

$$(4.7) \quad (\tau_\alpha)^*(\rho) : \gamma \mapsto \begin{cases} 
\rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\
\rho(\gamma)\rho(\alpha)^{-1} & \text{if } \gamma = \beta.
\end{cases}$$

If $\alpha$ separates, then $\pi = \pi_1(\Sigma)$ can be reconstructed from the fundamental groups $\pi_1(\Sigma_i)$ as an amalgam as in (4.5). The Dehn twist $\tau_\alpha$ induces the automorphism $(\tau_\alpha)_* \in \text{Aut}(\pi)$ defined by:

$$(\tau_\alpha)_* : \gamma \mapsto \begin{cases} 
\gamma & \text{if } \gamma \in \pi_1(\Sigma_1) \\
\alpha \gamma \alpha^{-1} & \text{if } \gamma \in \pi_1(\Sigma_2).
\end{cases}$$

The induced map $(\tau_\alpha)^*$ on $\text{Hom}(\pi,G)$ maps $\rho$ to:

$$(4.8) \quad (\tau_\alpha)^*(\rho) : \gamma \mapsto \begin{cases} 
\rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\
\rho(\alpha)^{-1}\rho(\gamma)\rho(\alpha) & \text{if } \gamma \in \pi_1(\Sigma_2).
\end{cases}$$

5. The case $G = \text{SU}(2)$

Now we specialize the preceding theory to the case $G = \text{SU}(2)$. Its Lie algebra $\mathfrak{su}(2)$ consists of $2 \times 2$ traceless skew-Hermitian matrices over $\mathbb{C}$.

5.1. One-parameter subgroups. The trace function

$$\text{SU}(2) \xrightarrow{f} [-2,2]$$
$$x \mapsto \text{tr}(x)$$

induces the variation function

$$\text{SU}(2) \xrightarrow{F} \mathfrak{su}(2)$$
$$x \mapsto x - \frac{\text{tr}(x)}{2} \mathbb{I},$$
the projection of $x \in \text{SU}(2) \subset \mathbb{M}_2(\mathbb{C})$ to $\text{su}(2)$. Explicitly, if $x \in \text{SU}(2)$, there exists $g \in \text{SU}(2)$ such that

$$x = g \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} g^{-1}.$$ 

Then $f(x) = 2 \cos(\theta)$,

$$F(x) = g \begin{bmatrix} 2i \sin(\theta) & 0 \\ 0 & -2i \sin(\theta) \end{bmatrix} g^{-1} \in \text{su}(2)$$

and the corresponding one-parameter subgroup is

$$\zeta^t(x) = g \begin{bmatrix} e^{2i \sin(\theta)t} & 0 \\ 0 & e^{-2i \sin(\theta)t} \end{bmatrix} g^{-1} \in \text{SU}(2)$$

Except in two exceptional cases this one-parameter subgroup is isomorphic to $S^1$. Namely, $f(x) = \pm 2$, then $x = \pm I$. These comprise the center of $\text{SU}(2)$. In all other cases, $-2 < f(x) < 2$ and $\zeta^t(x)$ is a circle subgroup. Notice that this circle subgroup contains $x$:

$$x = \zeta^{s(x)}(x)$$

where

$$s(x) := \frac{2}{\sqrt{4 - f(x)^2}} \cos^{-1}\left( \frac{f(x)}{2} \right).$$

Furthermore

$$\zeta^t(x) = I$$

if and only if

$$t \in \frac{4\pi}{\sqrt{4 - f(x)^2}} \mathbb{Z}.$$ 

(Compare Goldman [6].)

**Proposition 5.1.** Let $\alpha \in \pi$ be represented by a simple closed curve, and $\xi^t_\alpha$ be the corresponding twist flow on $\text{Hom}(\pi, G)$ as defined in (4.4) and (4.6). Let $\rho \in \text{Hom}(\pi, G)$. Then

$$(\tau_\alpha)^*(\rho) = \xi^s_\alpha(\rho(\alpha))$$

where $s$ is defined in (5.2).

**Proof.** Combine (5.1) with (4.4) when $\alpha$ is nonseparating case and (4.6) when $\alpha$ separates.
The basic dynamical ingredient of our proof, (like the original proof in [6]) is the ergodicity of an irrational rotation of $S^1$. There is a unique translation-invariant probability measure on $S^1$ (Haar measure). Furthermore this measure is ergodic under the action of any infinite cyclic subgroup. Recall that an action of group $\Gamma$ of measure-preserving transformations of a measure space $(X, \mathcal{B}, \mu)$ is ergodic if and only if every invariant measurable set has either measure zero or has full measure (its complement has measure zero).

**Lemma 5.2.** If $\cos^{-1}(f(x)/2)/\pi$ is irrational, then the cyclic group $\langle x \rangle$ is a dense subgroup of the one-parameter subgroup $\langle \zeta^t(x) | t \in \mathbb{R} \rangle \cong S^1$ and acts ergodically on $S^1$ with respect to Lebesgue measure.

For these basic facts see Furstenberg [2], Haselblatt-Katok [12], Morris [20] or Zimmer [26].

**Corollary 5.3.** Let $\alpha, \xi_\alpha$ and $\tau_\alpha$ be as in Proposition 5.1. Then for almost every $b \in [-2, 2]$, $(\tau_\alpha)^*$ acts ergodically on the orbit $
abla_t \{\xi_\alpha((\rho))\}_{t \in \mathbb{R}},$

when $f_\alpha(\rho) = b$.

**Proof.** Combine Proposition 5.1 with Lemma 5.2. □

**Proposition 5.4.** Let $\alpha \in S$ be a simple closed curve, with twist vector field $\xi_\alpha$ and Dehn twist $\tau_\alpha$. Let $X_b \xrightarrow{\psi} \mathbb{R}$ be a measurable function invariant under the cyclic group $\langle (\tau_\alpha)^* \rangle$. Then there exists a nullset $\mathcal{N}$ of $X_b$ such that the restriction of $\psi$ to the complement of $\mathcal{N}$ is constant on each orbit of the twist flow $\xi_\alpha$.

**Proof.** Disintegrate the symplectic measure on $X_b$ over the quotient map $X_b \twoheadrightarrow X_b/\xi_\alpha$
as in Furstenberg [2] or Morris [20], 3.3.3, 3.3.4. By (5.3) almost all fibers of this map are circles.

The subset $\mathcal{N} := f_\alpha^{-1}(2 \cos(Q\pi)) \subset X_b$

has measure zero. By Corollary 5.3, the action of $(\tau_\alpha)^*$ is ergodic on each circle in the complement of $\mathcal{N}$. In particular, $\psi$ factors through the quotient map, as desired. □
Conclusion of proof of Main Theorem. Suppose that $\mathcal{X}_b \to \mathbb{R}$ is a measurable function invariant under $\text{Mod}(\Sigma)$; we show that $\psi$ is almost everywhere constant.

To this end let $\mathcal{S}$ be the collection of simple closed curves in Theorem 2.1. Then, for each $\alpha \in \mathcal{S}$, the function $\psi$ is invariant under the mapping $(\tau_\alpha)^*$ induced by the Dehn twist along $\alpha \in \mathcal{S}$. By Proposition 5.4, $\psi$ is constant along almost every orbit of the Hamiltonian flow $\xi_\alpha$ of $\text{Ham}(f_\alpha)$. Thus, up to a nullset, $\psi$ is constant along the orbits of the group $\mathcal{G}$ generated by these flows. By Corollary 3.4, $\mathcal{G}$ acts transitively on $\mathcal{X}_b$. Therefore $\psi$ is almost everywhere constant, as claimed. The proof is complete. □

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**Department of Mathematics, University of Maryland, College Park, MD 20742, wmg@math.umd.edu (Goldman)**

**Division of Mathematics, National Center for Theoretical Science (South), Department of Mathematics, National Cheng-kung University, Tainan 701, Taiwan, ezxia@ncku.edu.tw (Xia)**