Light-Cone Gauge for 1+1 Strings

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ABSTRACT

Explicit construction of the light-cone gauge quantum theory of bosonic strings in 1+1 spacetime dimensions reveals unexpected structures. One is the existence of a gauge choice that gives a free action at the price of propagating ghosts and a nontrivial BRST charge. Fixing this gauge leaves a U(1) Kac-Moody algebra of residual symmetry, generated by a conformal tensor of rank two and a conformal scalar. Another is that the BRST charge made from these currents is nilpotent when the action includes a linear dilaton background, independent of the particular value of the dilaton gradient. Spacetime Lorentz invariance in this theory is still elusive, however, because of the linear dilaton background and the nature of the gauge symmetries.

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1 Introduction

The invention of matrix models[1] opened a whole new pathway to the understanding of quantum string theories and even string field theories. However, since the domain of matrix models is limited to dimensions less than one, the problem of inventing higher dimensional (presumably more physically realistic) string field theories is still approachable only in the continuum formulation. The hope of applying what has been learned in less than one dimension to problems stated in higher dimensions thus focuses attention on the correspondence between matrix models and continuum theories in the one case for which both are defined, strings in one dimension.

The one-dimensional noncritical theory of strings actually has two degrees of freedom and may be cast as a critical theory of strings in 1+1 dimensions in the presence of nontrivial background fields. Understanding of the discreet-continuum correspondence would benefit greatly from an explicit continuum formulation of first a quantum theory and then a field theory of strings in 1+1 dimensions. The construction of the quantum theory will be undertaken here.

One issue that must be addressed in quantization is the choice of gauge in which to work. One usually thinks of a choice between conformal and light-ccone gauges as a choice between manifest covariance at the price of nontrivial ghosts and a ghost-free spectrum at the price of hidden covariance. However, because one-dimensional string theory has no transverse dimensions, there is
another relevant consideration in this problem, the possibility of obtaining a free action for the dynamical fields which remain after gauge fixing. In string field theory, manifest covariance actually becomes an inconvenience, making it tricky to consistently define the string interaction vertices\(^2\), so the intended string field application of this work recommends a light-cone gauge, in which the string vertices are simply specified\(^3\). Further, the presence of a nontrivial dilaton background in the critical 1 + 1 theory makes it possible to choose an off-diagonal world-sheet/spacetime light-cone gauge which gives a free action. This choice, however, leaves a residual reparametrization symmetry which leads to propagating ghosts and a nontrivial BRST charge.

We will use this gauge, and sacrifice both manifest covariance and ghost-triviality in exchange for a free action and a simpler vertex in the future field theory. By checking the nilpotency of the BRST charge, we will ensure that the reparametrization symmetry we want is not violated by anomalies. The outline of this work is as follows.

Section 2 introduces the assumptions that define the theory. Quantization will be done in the language of functional integrals, in which anomalies are seen as violations by the measure of the gauge symmetries which have been built into the action. In this work, the choices of defining symmetry and the appropriate forms for the action and measure to implement them will be drawn from previous work done with the Liouville theory and other linear dilaton backgrounds.
Section 3 contains the treatment of the Faddeev-Popov gauge-fixing of the theory and the resulting well-defined functional integral. The BRST charge is constructed in section 4, and related to the generators of residual symmetry in the gauge-fixed functional integral. Finally section 5 points out some of the nonstandard features that arise in this construction.

2 String Theories in Two Dimensions

We wish to build a quantum theory of strings in two spacetime dimensions, where possible using forms that will make second quantization easier. We specify the theory by its desired symmetries, and express physical amplitudes as correlation functions of the form

\[ < \cdots > = \int \mathcal{D} fields \, e^{iS} \cdots \]  \hspace{1cm} (1)

where \( \cdots \) is some function of the fields invariant under the desired symmetry group.

As usual, we introduce a world sheet with metric, on which the spacetime embedding coordinates appear as scalar fields. Bosonic string theory in other than 26 dimensions can be cast in two forms, either as a noncritical theory in which world sheet diffeomorphism is a symmetry but variation of the world sheet geometry is not, or as a critical theory with one more dimension, in which both diffeomorphism and geometric variation of the world sheet are symmetries. The form of the action in the critical version of the theory
is\[4, 5, 6\]

\[
S_C = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left( g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \alpha' n \cdot X R^{(2)} \right)
\]

(2)

where \(\alpha'\) is the inverse string tension, \(g_{ab}\) is the world sheet metric, and \(X^\mu\) are the fields that embed the string in spacetime. \(\eta_{\mu\nu}\) are the components of the flat spacetime metric, \(n\) is a constant spacetime vector and the term linear in \(X\) is the dilaton background.

World sheet diffeomorphism invariance of (2) is automatic, and the pseudo-Weyl variation

\[
g_{ab} \rightarrow (1 + \epsilon) g_{ab}; \quad X^\mu \rightarrow X^\mu - \frac{\alpha' n^\mu}{2} \epsilon.
\]

(3)

will also be a symmetry of the quantum theory if the term linear in \(R^{(2)}\) in the resulting variation of the action is scaled correctly to cancel the variation of the measure\[7\]. The scaling expected from the Liouville theory is the same as that proposed by Myers in a slightly different context\[8\]. Considering general linear dilaton backgrounds, he has shown that if the generators of Lorentz transformations of the embedding fields are supplemented with generators which transform \(n\), the algebra of the Lorentz group can be recovered if the norm \(n^2 = \frac{26 - D}{6\alpha'}\), where \(D\) is the dimension of the embedding spacetime. Thus \(n\) is spacelike when \(D < 26\) and when \(D = 26\) the dilaton gradient vanishes and one obtains the standard simplest critical bosonic string action.

We will work in \(D = 2\) and include the nonzero dilaton gradient, but will allow it to take some general spacelike norm.
We will find that the gauge choice that makes the action (2) free results in a nontrivial Faddeev-Popov determinant and so leads to propagating ghosts. The naive version of this gauge gives dynamics to both reparametrization and Weyl ghosts in an unnecessarily complicated way. This can be avoided if we work with the Weyl-invariant fields

\[ \hat{g}^{ab} \equiv \sqrt{g} g^{ab}; \quad \hat{X}^\mu \equiv X^\mu + \frac{\alpha' n^\mu}{2} \log \sqrt{g} \]  

in terms of which the action becomes

\[ S_C = \frac{-1}{4\pi\alpha'} \int d^2 \sigma \left[ \hat{g}^{ab} \partial_a \hat{X}^\mu \partial_b \hat{X}^\nu \eta_{\mu\nu} + \alpha' n \cdot \nabla \hat{R}^{(2)} ight. 
\left. - \frac{\alpha'^2 n^2}{2} \log \sqrt{g} \left( \hat{R}^{(2)} - \frac{1}{2} \Box \log \sqrt{g} \right) \right] \]  

where \( \hat{R}^{(2)} \) and \( \Box \) are respectively the scalar curvature and the covariant Laplacian in the “metric” \( \hat{g} \). The natural light-cone gauge to make the action (3) free gives a simpler ghost action, from which the Weyl ghost can be eliminated by an equation of constraint, leaving all residual symmetry in reparametrizations. It would be nice to find a gauge which eliminates propagating ghosts altogether and still gives a free action, but that has not been accomplished here. The action (3) will be the starting point in our construction of the quantum theory.

3 Fixing Light-Cone Gauge

The world sheet time and space coordinates are labeled \((\tau, \sigma)\) and the spacetime embedding fields are \(X^+, X^-\), in terms of which the spacetime metric
components are $\eta_{++} = \eta_{--} = -1$, $\eta_{+-} = \eta_{-+} = 0$. In a light-cone gauge one of the world sheet coordinates is specified in terms of one of the spacetime embedding fields, such as $X^+ = \tau$. In addition, two other degrees of freedom of the world sheet metric must be constrained in order to fix the gauge freedom. To do this it is convenient to introduce world sheet coordinates $\sigma^\pm \equiv \frac{1}{\sqrt{2}}(\tau \pm \sigma)$.

It was emphasized by Tzani\[9\] that the world sheet metric cannot be fixed in the conformal gauge once light-cone coordinates are chosen because that would overly constrain the system. Therefore, in this gauge one of the metric components must remain dynamical. Polyakov\[10\] has pointed out that an off-diagonal gauge, such as $g^{--} = 0$, $\sqrt{g} = 1$ produces a simple measure for the remaining degree of freedom of the metric, which can be regulated by modifying the propagator without introducing vertices. In this theory it will also produce the desired free action for the remaining fields. Therefore we will make the gauge choice

\[
\begin{align*}
\dot{X}^+ &= \tau \\
\dot{g}^{--} &= 0 \\
\sqrt{g} &= 1,
\end{align*}
\]

It will be convenient for later steps to define a scale $\frac{1}{m} \equiv \frac{\alpha'\eta^+}{\sqrt{2}}$.

General diffeomorphisms and Weyl transformations are parametrized by vector field components $\epsilon^+$, $\epsilon^-$ and a Weyl parameter $\epsilon$. The infinitessimally
shifted fields are given by

\[ \hat{X}^+ = \hat{X}^+ - \epsilon^a \hat{X}^+_a - \frac{1}{\sqrt{2m}} \epsilon^a, \]
\[ \hat{g}^- = \hat{g}^- + (\epsilon^a \hat{g}^-)_a + 2\epsilon^- \hat{g}^- - 2\epsilon^- \hat{g}^+ \]  \tag{7} \]
\[ \sqrt{\hat{g}} \varepsilon = \sqrt{\hat{g}} - (\epsilon^a \sqrt{\hat{g}})_a + \epsilon \sqrt{\hat{g}} \]

In the Faddeev-Popov gauge-fixing procedure\[11\] a well defined and gauge-fixed functional integral is given by

\[ Z = \int \mathcal{D} \left( \hat{g}^a, \sqrt{\hat{g}}, \hat{X}^\mu \right) \delta \left[ \sqrt{2} \left( \hat{X}^+ - \tau \right) \right] \delta \left[ \hat{g}^- \right] \delta \left[ \sqrt{\hat{g}} - 1 \right] \Delta_{FP} e^{iS_C} \]  \tag{8} \]

where the Faddeev-Popov determinant is given in terms of the variation of the gauge conditions with respect to the generators of the gauge symmetry:

\[ \Delta_{FP} \equiv \det \left[ \frac{\partial \left( \sqrt{2} \hat{X}^+ + \hat{g}^-, \sqrt{\hat{g}} \right)}{\partial (\epsilon^+, \epsilon^-, \epsilon)} \right] \bigg|_{\epsilon^+, \epsilon^-, \epsilon=0} \]  \tag{9} \]

Adding sources coupled to additional terms in the action in (8) produces the desired generator of the correlation functions (11). Physically meaningful correlation functions are generated by sources that couple to field quantities invariant under diffeomorphisms and the transformation (3).

4 Residual Symmetry and the BRST Charge

One way to expose the final group of invariances of the integral (8) is to study its BRST symmetry\[12\]. By introducing auxiliary fields \( B_f, B, \tilde{B}_{-}, \) antighosts \( b_f, b, \tilde{b}_{-} \) and ghosts \( c^+, c^-, c \), one can write the same integral as

\[ Z = \int \mathcal{D} \left( \hat{g}^a, \sqrt{\hat{g}}, \hat{X}^\mu, B_f, B, \tilde{B}_{-}, b_f, b, \tilde{b}_{-}, c^+, c^-, c \right) e^{i(S_C + S_{GF} + S_{GH})}, \]  \tag{10} \]
where

\[ S_{G.F.} = \int d^2\sigma \left[ B_f \sqrt{2} \left( \dot{X}^+ - \tau \right) + \dot{B}_{--} \hat{g}^{--} + B \left( \sqrt{g} - 1 \right) \right] \]  

(11)

\[ S_{GH} = \int d^2\sigma \left[ -b_f \left( \sqrt{2} \dot{X}^+ a e^a + \frac{1}{m} e_{-a} \right) \right. 
+ \left. \hat{b}_{--} \left( e^-, -\hat{g}^{--} - c^- \hat{g}^{--}, - + (c^+ \hat{g}^{--}), + + 2c^-, + \hat{g}^{++} \right) \right. 
- \left. b \sqrt{g} \left( e^a, a + e^a \partial_a \log \sqrt{g} - c \right) \right] \]  

(12)

In the Faddeev-Popov procedure, one usually discovers a global nilpotent symmetry of the classical action (for both matter and ghosts) in (10), called the BRST symmetry. Requiring that this transformation preserve its classical properties in the full quantum theory is then the check that anomalies do not spoil the gauge invariance of the theory assumed at the classical level. This condition constrains the central charge of the theory, which may appear as a restriction on the number of embedding dimensions or in some other way. In this string theory, the gauge transformation (3) is not a symmetry of the action at the classical level (the term from variation of the measure is required to cancel the resulting term from variation of the action), so the distinction between classical and quantum seems artificial. However, we can still guess what should be the form of the BRST transformation on fields at the classical level from the algebra of the gauge constraints, as one would do in the Kugo-Uehara procedure[13] for gauge fixing the operator Hamiltonian of a constrained system.

The BRST symmetry is conveniently introduced using an anticommuting
parameter $\theta$ to produce the following field variations:

$$
\begin{align*}
\delta b_f &= -\theta B_f & \delta B_f &= 0 \\
\delta b_- &= -\theta B_- & \delta B_- &= 0 \\
\delta b &= -\theta B & \delta B &= 0 \\
\delta c^a &= -\left(\theta c^b\right)_{a,b} & \delta c &= -\left(\theta c^b\right)_{c,b} \\
\delta \hat{X}^+ &= -\left(\theta c^a\right) \hat{X}^+_{a} - \frac{1}{\sqrt{2m}} \left(\theta c^a\right)_{a} \\
\delta \hat{g}^{ab} &= \left(\theta c^a\right) \hat{g}^{cb} + \left(\theta c^b\right) \hat{g}^{ac} - \left(\theta c^c\hat{g}^{ab}\right)_{c} \\
\delta \sqrt{\hat{g}} &= -\left(\theta c^a\sqrt{\hat{g}}\right)_{a} + \left(\theta c\right) \sqrt{\hat{g}} \\
\end{align*}
$$

When $\theta$ is a constant $\delta$ should be a global symmetry of the theory. Here the term $\theta$ has been written as it would appear in general variations, such as one would make to find the associated Noether current. As well as being an expansion parameter, $\theta$ is a sign-tracking parameter, and when it is constant it is possible to use it without listing it explicitly to define the unparametrized global transformation $\Delta$ by setting $\delta \equiv \theta \Delta$. It is straightforward to check that $\Delta^2 = 0$ as a transformation of fields at the classical level.

The action of the BRST symmetry on the canonical variables is precisely the action of the gauge symmetry, with the ghosts substituted as the parameters in the gauge transformations, and further it may be seen that the gauge-fixing and ghost actions themselves come from a BRST variation,

$$
S_{G.F.} + S_{GH} = \Delta \int d^2 \sigma \left[ -b_f \sqrt{2} \left( \hat{X}^+ - \tau \right) - \hat{b}_- \hat{g}^{--} - b (\sqrt{\hat{g}} - 1) \right]
$$

Thus if the action and measure for the canonical fields respect the gauge symmetries used to define $\Delta$, nilpotency ensures that it will be a global symmetry of the gauge fixing and ghost actions as well. Therefore, it will have an associated conserved charge, called the BRST charge (usually labeled
\( Q \), which as an operator in the quantum theory is the generator of the BRST symmetry.

A standard result\(^{[12, 14, 15]}\) is that maintaining nilpotency of the BRST variation in the quantum theory and identifying physical states with the elements of the BRST cohomology class is equivalent to preserving the physical gauge equivalence relation in the quantum correlation functions. Thus the requirement that the classical nilpotency of the BRST transformation be preserved as the operator statement \( Q^2 \equiv 0 \) is our check that quantization preserves the gauge symmetry we have assumed to define the theory.

While it is possible to calculate \( Q \) directly as a Noether charge and compute its square using an operator product expansion, the algebra of the residual symmetry left after gauge fixing and its relation to the original gauge invariance imposed on the theory can be more easily seen if we express \( Q \) in terms of the components of the energy-momentum tensor which generate the residual symmetry transformations in the gauge-fixed functional integral (10).

To find these we first perform the integrations over the auxiliary fields and the antighost \( b \) that couples to \( \sqrt{g} \). The auxiliary field integrations simply restore the original gauge-fixing \( \delta \)-functionals, and because of the simple form of its ghost term the \( b \) integration produces a \( \delta \)-functional which constrains the value of the Weyl ghost \( c \). Next integrating over \( \hat{g}^- \), \( \sqrt{\hat{g}} \), \( \hat{X}^+ \) and \( c \) places these fields on the constraint surface. The resulting functional integral
is

\[ Z = \int D \left( \hat{g}^{++}, \hat{X}^{-}, b_f, \hat{b}_{--}, c^+, c^- \right) e^{i(S_C + S_{GH})}, \tag{15} \]

in which the remaining actions are reduced to

\[ S_C = \frac{\sqrt{2}}{4\pi\alpha'} \int d^2\sigma \left( 1 + \frac{\partial^+}{m} \right) \hat{g}^{++} \hat{X}^{-, +} \tag{16} \]

\[ S_{GH} = \int d^2\sigma \left[ -b_f \left( 1 + \frac{\partial_a}{m} \right) c^a + 2\hat{b}_{--} c^{-, +} \right] \tag{17} \]

As desired the Weyl ghost is present neither explicitly nor through possible BRST variations of the remaining dynamical quantities.

By construction the ghost equations of motion are precisely the equations satisfied by gauge parameters that preserve the gauge slice. Thus the EOM for the reparametrization ghosts tell us what part of the diffeomorphism invariance was left unfixed by the gauge condition. Making the general decomposition

\[ c^+ = -(1 + \frac{\partial}{m})\tilde{u} + e^{-\sqrt{2m}\tau} \tilde{v} \quad b_f = -me^{-\sqrt{2m}\tau} \tilde{z} \]

\[ c^- = \tilde{u} \quad \hat{b}_{--} = \frac{\tilde{v}}{2} - \frac{1}{2m}e^{\sqrt{2m}\tau} \tilde{z}, \tag{18} \]

the ghost action becomes

\[ S_{GH} = \int d^2\sigma \left( \tilde{w}_+ + \tilde{z}_{++} \right) \tag{19} \]

and we find that the residual symmetry unfixed by the gauge condition is the collection of reparametrizations generated by the vector field components

\[ \epsilon^+ = -(1 + \frac{\partial}{m})u^\epsilon + e^{-\sqrt{2m}\tau} v^\epsilon \quad u^\epsilon_{++} = 0 \]

\[ \epsilon^- = u^\epsilon \quad v^\epsilon _{++} = 0 \tag{20} \]
One may check directly that the algebra of these classical transformations is a $U(1)$ Kac-Moody algebra, which we will see again as the algebra of their associated Noether currents.

Now the form of the action (16) is very simple, but the measure in (15) is not. The presence of the Weyl anomaly tells us that it is metric-dependent, and in this gauge the metric has a dynamical component. Therefore, as in the work of David[4], Distler and Kawai[5], we make the ansatz that the measure in (15) may be written as a field-independent measure times a counterterm, as

$$D\left(\hat{g}^{++}, \hat{X}^-, b_f, \hat{b}_{-\cdot}, c^+, c^-\right) = \tilde{D}\left(\hat{g}^{++}, \hat{X}^-, b_f, \hat{b}_{-\cdot}, c^+, c^-\right) e^{i(S_M)}. \quad (21)$$

We can choose the counterterm by requiring that the reparametrizations generated by (20), and thus the BRST transformation, be classical symmetries of the resulting improved canonical action $S_C + S_M$. This leads to the conclusion that on this gauge surface of constraint

$$S_M = \frac{1}{4\pi\alpha'} \int d^2\sigma \hat{g}^{++}. \quad (22)$$

It then remains to show that the measure $\tilde{D}$ is indeed free from anomalies by demonstrating nilpotency of the BRST charge in the quantum theory.

Associated with the symmetries generated by $u^\epsilon(\sigma^-)$ and $v^\epsilon(\sigma^-)$ are Noether currents $J_u$ and $J_v$, obtained by varying with general $u^\epsilon(\sigma^-, \sigma^+)$ and $v^\epsilon(\sigma^-, \sigma^+)$, which obey $J_{u^{++}} = 0$; $J_{v^{++}} = 0$ when the dynamical fields

\[1\]This simplification was proposed by J. Polchinski
are on shell. The equations of motion from $S_C + S_M$ are satisfied by functions of the form

$$
\hat{g}^{++} = \tilde{f} + e^{-\sqrt{2m}\tau} \tilde{g} \quad (23)
$$

$$
\hat{X}^- = \tilde{h} + e^{\sqrt{2m}\tau} \tilde{j} - \tau \quad (24)
$$

with

$$
\tilde{f}_+ = 0; \quad \tilde{g}_+ = 0 \\
\tilde{h}_+ = 0; \quad \tilde{j}_+ = 0 \quad (25)
$$

(If we were working in suitably defined complex world sheet coordinates the fields $\tilde{f}$, $\tilde{g}$, $\tilde{h}$, $\tilde{j}$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{z}$ would be analytic functions).

It is straightforward to show that with the proper transformation properties for the ghosts, the diffeomorphisms generated by (20) are also symmetries of $S_{GH}$. In order that the ghost action (12) be a scalar under diffeomorphism it is necessary that $b_f$ transform as a scalar density and $\hat{b}_{--}$ as a component of a rank-2 contravariant tensor ($\hat{g}_{--}$ is already the component of a covariant tensor density). Similarly $c^+$, $c^-$ are components of a covariant vector.

Using these reparametrization properties, one may find the currents for canonical fields and ghosts separately in terms of analytic variables. Their algebra is simplest if they are written using the rescaled or shifted fields

$$
\begin{align*}
  f &= \tilde{f} + 2 \\
  g &= \tilde{g} \\
  h &= \frac{\sqrt{2}}{4\alpha'} \tilde{h} \\
  j &= -\frac{\sqrt{2}}{4\alpha'} \tilde{j}
\end{align*}
$$

$$
\begin{align*}
  u &= \tilde{u} \\
  v &= \tilde{v} \\
  w &= \pi \tilde{w} \\
  z &= \pi \tilde{z}
\end{align*} \quad (26)
$$

The expression for the canonically normalized currents in terms of these fields
is

\[
\begin{align*}
J_{uC} &= \left(-fh, -g, -2gj, -\frac{2m}{\alpha}h, + \frac{1}{2\alpha}\right) \\
J_{vC} &= \left(mfj + 2j\right) \\
J_{uGH} &= \left(-vz, -w, -2wu\right) \\
J_{vGH} &= \left(-z, -u\right)
\end{align*}
\]

The one algebraically nonstandard feature of these currents is the constant term in \(J_{uC}\) that arises from the explicit time dependence in the decomposition of the field \(X^-\) in Eq. (24). This is avoided if we consider instead the algebraic relations involving the (also conserved) current \(\tilde{J}_{uC} \equiv J_{uC} - \frac{1}{2\alpha}\).

Direct calculation of the BRST charge is slightly more involved, because one must first substitute the fields shifted by Eq. (13) into the integral Eq. (10), which gives modified equations of constraint from the \(B_f, \hat{B}_-, B, b\) integrations. Imposed on the remaining (also shifted) actions by integration of \(\hat{X}^+, \hat{g}^-, \sqrt{g}, c\), these yield the so-called “on shell BRST variation”. The resulting Noether charge from this variation is related to the currents \(J_u\) and \(J_v\) by

\[
Q = \int d\sigma \left[uJ_{uC} + vJ_{vC} + \frac{1}{2} (uJ_{uGH} + vJ_{vGH})\right]
\]

We may now check the value of \(Q^2\) and its relation to the operator products of the currents that generate the residual symmetry. Reading from the ghost action and the form of the propagator for \(\hat{g}^{++}\) with \(\hat{X}^-\) it is possible to show that the analytic fields have the propagators

\[
\begin{align*}
\langle fh' \rangle &= \langle gj' \rangle = -\frac{1}{(\sigma - \sigma')} \\
\langle vz' \rangle &= \langle wu' \rangle = \frac{1}{(\sigma - \sigma')}
\end{align*}
\]

\[
(29)
\]
The terms in the operator product expansions for the currents \(^{27}\) singular at short distance are

\[
\begin{align*}
\tilde{J}_{uC} \tilde{J}_{uC}' & \sim \frac{-28}{2(\sigma^{-}-\sigma^{-'})^4} + \frac{2 \tilde{J}_{uC}}{(\sigma^{-}-\sigma^{-'})^2} + \frac{\tilde{J}_{uC}'}{(\sigma^{-}-\sigma^{-'})} \\
J_{vC} J'_{vC} & \sim 0 \\
\tilde{J}_{uC} J'_{vC} & \sim \frac{J'_{vC}}{(\sigma^{-}-\sigma^{-'})} \\
J_{uGH} J'_{uGH} & \sim \frac{-28}{2(\sigma^{-}-\sigma^{-'})^4} + \frac{2 J'_{uGH}}{(\sigma^{-}-\sigma^{-'})^2} + \frac{J'_{uGH}'}{(\sigma^{-}-\sigma^{-'})} \\
J_{vGH} J'_{vGH} & \sim 0 \\
J_{uGH} J'_{vGH} & \sim \frac{J'_{vGH}}{(\sigma^{-}-\sigma^{-'})}.
\end{align*}
\] (30)

The functional integral (15) described a decoupled product of canonical and ghost theories, which are now seen to possess the same algebra of residual symmetry. The currents separately generate \(U(1)\) Kac-Moody algebras, with \(\tilde{J}_{uC}\) a conformal tensor of rank 2 and central charge 28, \(J_{uGH}\) a conformal tensor of rank 2 and central charge \(-28\), and \(J_{vC}\) and \(J_{vGH}\) conformal scalars.

Using the relation (28), the operator product \(Q^2\) is a double integral over the positions of two currents on a spatial slice of the worldsheet, in which only single pole terms from the OPEs of the currents survive[9]. A direct calculation of this product reveals that the pole is proportional to the total central charge of the theory,

\[
Q^2 \propto C_{TOT} = C_C + C_{GH} = 0
\] (31)

Therefore \(Q^2 = 0\). This completes the proof of gauge invariance of the full quantum theory.
Several things about the previous construction deserve mention, and a few puzzles remain. The first notable point is the number of degrees of freedom of the residual symmetry in the gauge-fixed functional integral. In the light-cone theory with transverse dimensions and no dilaton\cite{9} the only residual degree of freedom was in one set of analytic reparametrizations. The associated ghost current had central charge $-26$ and the current from the canonical sector had central charge $D$. Thus, the condition $D = 26$ was necessary for the nilpotency of the BRST charge. In this theory without transverse dimensions we have replaced the standard Weyl transformation with (3), and so admitted another degree of freedom between the light cone and $\hat{g}^{--}$ ghost equations of motion. This changes the ghost central charge from $-26$ to $-28$.

This would have been a disaster for the canonical sector without the linear dilaton, because there are no adjustable parameters in the $D = 2$ theory analogous to the number of transverse embedding dimensions. The linear dilaton comes to the rescue in a strange way, though. Because in the work of Myers\cite{8} its gradient was constrained in terms of the number of transverse dimensions, and in the work of Tzani\cite{9} they are linked to the nilpotency of the BRST symmetry, one may have expected the dilaton gradient to appear in the central charge of this theory. However, rather than generate a term in the central charge proportional to $\frac{1}{m^2}$ which could be tuned to make the total
charge 28, it introduces another whole degree of freedom into the solutions of the canonical equations of motion very similar to that in the ghost sector. In both sectors there are two conjugate analytic pairs, the elements of one pair multiplied by a constant \((f, h \text{ and } u, w)\) and the elements of the other multiplied by exponentials of \(\sqrt{2m\tau} \ (g, j \text{ and } v, z)\).

In the canonical sector the constant pair, which looks like the usual zero-dilaton solution, contributes \(c = 2\) and the exponential pair contributes \(c = 26\). In this way the addition of the dilaton mimics the substitution of the transverse dimensions (which had \(c = 24\)) with a conformal field theory with \(c = 26\). In the ghost sector the constant pair, which look like the usual reparametrization ghosts, contribute \(c = -26\) while the exponential pair account for the remaining \(c = -2\). In all of these central terms the particular value of \(\frac{1}{m}\) never appears, though it is still present in the BRST charge and can presumably be expected to effect the cohomology and hence the physical spectrum. The important conclusion is that the inclusion of the linear dilaton at all, together with this choice for the defining symmetries of the theory, is enough to guarantee the nilpotency of the BRST symmetry in the quantum theory.

Actually, the fact that the dilaton gradient is not constrained by anomaly cancellation is not really a surprise, because in the total absence of transverse dimensions the previous considerations do not apply. After finding the appropriately modified Lorentz generators, Myers\textsuperscript{8} was forced to fix the value of...
n^2 to ensure vanishing of an anomaly in the commutator of the unfixed light-like generator with the transverse generators. This theory has no transverse directions, so the potentially anomalous term never appears. This relates to one of the puzzles. In that work, the entire physical spectrum was given by excitations of the transverse oscillators. Therefore the absence of transverse embedding dimensions in this theory could be expected to lead to a trivial physical spectrum. However, we have found a nontrivial BRST charge which may well lead to a nontrivial spectrum\(^2\). More generally, the relation of the continuum theory, in which physical excitations are expected to be purely transverse, with the matrix models, which have a nontrivial physical spectrum even with central charge one, is very confusing. That relation and its implications for string field theory are currently under investigation.

It may yet be that the value of \( \frac{1}{m} \) will be constrained by something like Lorentz invariance in another way, but the whole issue of spacetime Lorentz invariance in this theory is still unclear. In addition to explicitly breaking it by introducing the constant vector \( n \), we have further singled out the same direction by mixing it with the metric when we defined the shifted variables (4). The preservation of Lorentz invariance when it is not manifest is usually regarded as the central problem in light cone gauge theories, so defining a useful notion of it here remains an important and central issue, which I hope also to address in future work.

\(^2\)In our discussions, the whole relation of the results obtained here to those reported by Myers has been a source of confusion to the author and to J. Polchinski.
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