Balanced Allocations in Batches: Simplified and Generalized

Dimitrios Los∗ and Thomas Sauerwald†

Department of Computer Science & Technology, University of Cambridge

March 29, 2022

Abstract

We consider the allocation of $m$ balls (jobs) into $n$ bins (servers). In the TWO-CHOICE process, for each of $m$ sequentially arriving balls, two randomly chosen bins are sampled and the ball is placed in the least loaded bin. It is well-known that the maximum load is $m/n + \log_2 \log n + O(1)$ with high probability.

Berenbrink, Czumaj, Englert, Friedetzky and Nagel [7] introduced a parallel version of this process, where $m$ balls arrive in consecutive batches of size $b = n$ each. Balls within the same batch are allocated in parallel, using the load information of the bins at the beginning of the batch. They proved that the gap of this process is $O(\log n)$ with high probability.

In this work, we present a new analysis of this setting, which is based on exponential potential functions. This allows us to both simplify and generalize the analysis of [7] in different ways:

1. Our analysis covers a broad class of processes. This includes not only TWO-CHOICE, but also processes with fewer bin samples like $(1 + \beta)$, processes which can only receive one bit of information from each bin sample and graphical allocation, where bins correspond to vertices in a graph.

2. Balls may be of different weights, as long as their weights are independent samples from a distribution satisfying a technical condition on its moment generating function.

3. For arbitrary batch sizes $b \geq n$, we prove a gap of $O(b/n \cdot \log n)$. For any $b \in [n, n^3]$, we improve this to $O(b/n + \log n)$ and show that it is tight for a family of processes. This implies the unexpected result that for e.g. $(1 + \beta)$ with constant $\beta \in (0, 1]$, the gap is $\Theta(\log n)$ for all $b \in [n, n \log n]$.

We also conduct experiments which support our theoretical results, and even hint at a superiority of less powerful processes like $(1 + \beta)$ for large batch sizes.

∗dimitrios.los@cl.cam.ac.uk
†thomas.sauerwald@cl.cam.ac.uk
## Contents

1 Introduction 3

2 Notation 6
   2.1 Basic Notation and Specific Processes 6
   2.2 Classes of Processes 8
   2.3 Batched Model and Weights 10

3 Analysis of the Hyperbolic Cosine Potential 12

4 A Simple Upper Bound 20

5 An Improved Upper Bound for Batch Sizes $n \leq b \leq n^3$ 23
   5.1 Step 1: $\tilde{\Gamma}$ is linear w.h.p. 24
      5.1.1 Preliminaries 24
      5.1.2 Completing the Proof of Lemma 5.3 28
   5.2 Step 2: Completing the Proof of Theorem 5.1 31

6 Application to Graphical Allocations and $(1 + \beta)$-process 33

7 Lower Bounds 36

8 Experiments 40

9 Conclusions 41

A Tools 45
   A.1 Concentration inequalities 45
   A.2 Auxiliary Probabilistic Claims 46
   A.3 Auxiliary Non-Probabilistic Claims 47
1 Introduction

Motivation. We study the classical problem of allocating $m$ balls (jobs) into $n$ bins (servers). This framework also known as balls-into-bins or balanced allocations \cite{13} is a popular abstraction for various resource allocation and storage problems such as load balancing, scheduling or hashing (see surveys \cite{37,46}). Following a long line of previous works, we consider randomized allocation schemes where for each ball we take a certain number of bin samples and then allocate the ball into one of these samples.

For the simplest allocation scheme, called One-Choice, each of the $m$ balls is placed in a random bin chosen independently and uniformly. It is well-known that the maximum load is $\Theta(\log n / \log \log n)$ w.h.p.\footnote{In general, with high probability refers to probability of at least $1 - n^{-c}$ for some constant $c > 0$.} for $m = n$, and $m/n + \Theta(\sqrt{(m/n) \log n})$ w.h.p. for $m \gg n$. While this allocation scheme can be of course executed completely in parallel, it results in a significantly large gap between the maximum load and average load if $m$ gets large.

Azar, Broder, Karlin and Upfal \cite{13} (and implicitly Karp, Luby and Meyer auf der Heide \cite{19}) proved that if the $m$ balls are allocated sequentially, and each ball is placed in the lesser loaded of $d \geq 2$ randomly chosen bins, then the maximum load drops to $\log_d \log n + O(1)$ w.h.p., if $m = n$. This dramatic improvement from $d = 1$ (One-Choice) to $d = 2$ (Two-Choice) is known as “power of two choices”, and similar effects have been observed in other problems including routing, hashing and randomized rounding \cite{37}. Vöcking \cite{33} proved that further improvements on the gap bound (which are more significant for larger $d$) are possible if one employs an asymmetric tie-breaking rule.

Berenbrink, Czumaj, Steger and Vöcking \cite{8} extended the analysis of \cite{13} (and \cite{33}) to the so-called heavily loaded case, where $m \geq n$ can be arbitrarily large. In particular, for Two-Choice an upper bound on the gap (the difference between the maximum and average load) of $\log_2 \log n + O(1)$ w.h.p. was shown. A simpler but slightly weaker analysis was later given by Talwar and Wieder \cite{32}.

The above studies for $d$-Choice, as well as many other works in balanced allocations, usually make the following assumptions:

1.) All $m$ balls have to be allocated sequentially, with the load information being updated immediately.

2.) All $m$ balls are of the same weight.

3.) All $m$ balls need to take $d \geq 2$ independent and uniform bin samples.

While these assumptions are crucial in many of the mathematical analyses, they may be difficult to satisfy in practical applications. For example, in a highly parallel environment, the load information of a bin may not include the most recent allocations. Further, processing times of jobs (size of data items) may not be identical but rather follow a heterogeneous distribution, which leads to the so-called weighted balls-into-bins setting. Finally, Two-Choice (and $d$-Choice) assume that balls are able to sample two (or $d$) bins which are chosen independently and uniformly at random, in every round. It is natural to consider scenarios where the $d$ samples are correlated (e.g., through a network structure), or for some balls only one sample is available.

Related Work. In order to relax assumption 1.), Berenbrink, Czumaj, Englert, Friedetzky and Nagel \cite{7} proposed a model where balls are allocated in consecutive batches of size $b$. Each ball in a batch is allocated using Two-Choice, but based on the load values of the bins prior to the batch. This means, that the decisions among the balls within the same batch do not depend on each other in any way and can therefore be executed in parallel. In \cite{7}, it was shown that for $b = n$ the gap is $O(\log n)$ w.h.p.

A related setting with ball removals was studied in Berenbrink, Friedetzky, Kling, Mallmann-Trenn, Nagel and Wastell \cite{10}, where at each batch $\lambda \cdot n$ arriving balls are allocated in parallel,
and every non-empty bin removes one ball. The authors prove an upper bound of $O\left(\frac{\log n}{1-\lambda}\right)$ on the gap, when balls are allocated using Two-Choice. As mentioned in [10, Lemma 3.5], their analysis can be modified to re-derive the main result in [7] for the batched setting, using a simpler proof.

In [26], Mitzenmacher studied a similar model to the batched setting called bulletin board model with periodic updates. However, this model assumes stochastic arrivals and removals of balls, and the paper also does not provide any rigorous and quantitative bounds on the gap. On a high level, this work, as well as a study by Dahlin [12] raise the general question on how useful old load information is, and both works suggest that using a “less aggressive” strategy than $d$-Choice may lead to better performance in practice.

Other parallel allocation schemes which are typically based on a small number of communication rounds between bins and balls were studied in [11, 23, 22]. For example, Lenzen and Wattenhofer [23] proved that, for $m = n$, a maximum load of 2 is possible using only $\log^* n + O(1)$ communication rounds. Recently, [22] also extended this direction to the heavily loaded case $m \geq n$, and proved that a maximum load of $m/n + O(1)$ is possible using $O(\log \log (m/n) + \log^* n)$ rounds. While the gap bounds in [23, 22] are stronger than in our setting, they require more communication and coordination.

Assumption 2.) that balls are unweighted is made in the vast majority of theoretical works in balanced allocations. One exception is the work of Talwar and Wieder [31], who analyzed a wide class of weight distributions satisfying some mild conditions on its second and fourth moment. They proved that the gap remains independent of $m$, even though heavier and heavier weights may be encountered if $m$ gets large. For arbitrary weight distributions, results of [31] demonstrate that this setting is considerably harder than the unweighted setting, as many couplings and majorization results no longer hold.

Concerning assumption 3.) on how bins are sampled, many allocation schemes with fewer or correlated bin samples have been analyzed. One key example is the $(1 + \beta)$-process introduced by Peres, Wieder and Talwar [30], where each ball is allocated using One-Choice with probability $1 - \beta$ and otherwise is allocated using Two-Choice. The authors proved that for any $\beta \in (0, 1]$, the gap is only $O(\log n/\beta + \log(1/\beta)/\beta)$ for any $m \geq n$. Hence, only a “small” fraction of Two-Choice rounds are enough to inherit the property of Two-Choice that the gap is independent of $m$. This result also applies to weighted balls into bins for a large class of weight distributions.

A nice application of the $(1 + \beta)$-process is in the analysis of the so-called graphical balanced allocations [30]. In this model, first studied in [20] for $m = n$, we are given a graph where each bin corresponds to a vertex. For each ball, we pick a uniform edge in $G$ and place the ball in the lesser loaded endpoint of the edge. A reduction to the $(1 + \beta)$-process implies that, if $G$ is a regular expander graph, then for any $m \geq n$ the gap is $O(\log n)$ w.h.p. Very recently, Bansal and Feldheim [5] presented a more elaborate protocol based on multi-commodity-flows that achieves a poly-logarithmic gap for any bounded-degree regular graph. A natural extension of the graphical process was also studied for hypergraphs, see, e.g., [16, 17]. Other applications of the $(1 + \beta)$-process in parallel computing include population protocols [3] and distributed data structures [2, 29].

Related to $(1 + \beta)$ process is the two-THINNING process [15, 18, 13] with some load threshold $\ell$ which works in a two-stage procedure: First, take a uniform bin sample $i$. Secondly, if the load of bin $i$ is at most $\ell$ then allocate a ball into $i$, otherwise place a ball into another bin sample $j$ (without comparing its load with $i$). This process has received some attention recently, and several variations were studied in [15] for $m = n$ and [14, 24, 25] for $m \geq n$. [24] investigated a variant of THINNING called QUANTILE, which uses relative instead of absolute loads. This means the ball is allocated in the first sample if its load is among the $(1 - \delta) \cdot n$ lightest, for some quantile $\delta \in \{1/n, 2/n, \ldots, 1\}$, and otherwise the ball is allocated into a second bin sample. For both THINNING and QUANTILE, extensions exist which use more than two bin samples, and correspondingly, stronger gap bounds can be shown [14, 24].
An even stronger class of adaptive sampling schemes was analyzed by Czumaj and Steemann [11], where unlike Thinning or Quantile, the ball is always placed in the least loaded bin among all samples. However, their results hold only for $m = n$.

Another model relaxing the uniform sampling assumption was introduced in Wieder [34], where the minimum and maximum probability for sampling a bin may deviate from the uniform distribution by some factors $\alpha, \beta$. Wieder [34] proved some tight trade-off between $\alpha, \beta$ and $d$, so that d-Choice still achieves a small gap for any $m \geq n$. A related model with heterogeneous bins capacities was studied in Berenbrink, Brinkmann, Friedetzky and Nagel [6]. The authors proved that d-Choice achieves a gap bound of $\log_d n + O(1)$, matching the result in the classical setting.

**Our Results.** In this work we revisit the batched model from [7], which allocates balls in batches of size $b$, but here we allow any value of $b \geq n$. Additionally, we consider a wider range of allocation processes, including not only Two-Choice, but also $(1 + \beta)$ or Quantile. This relaxes the requirement of Two-Choice of always taking two uniform bin samples at each round and allocating into the less loaded of the two (given the available load information).

Our results hold for any process satisfying two natural conditions: (i) there is a suitable bias away from allocating into the heavily loaded bins and (ii) no single bin experiences are a “too large” bias. The second condition may seem a bit counter-intuitive at first, but it is crucial in the batched setting to prevent a lightly loaded bin from receiving too many allocations within the same batch. The precise definition of these conditions is given in Section 2.2.

Furthermore, our results are valid for the same class of weight distributions as considered in [30], which includes, for example, the geometric and the exponential distributions.

Our first result is that for any batch size $b \geq n$, after allocating any number of balls $m \geq n$,
a gap bound of \( O(b/n \cdot \log n) \) holds w.h.p. For \( b = n \), this matches the result of [7] for the Two-Choice process in the unweighted setting. Unlike the analysis in [7], which relies on some sophisticated Markov chain tools from [8] to prove a “short memory behaviour”, the derivation of this gap bound \( O(b/n \cdot \log n) \) is based on a hyperbolic cosine potential function (a version of two exponential potential functions), and thus we believe it to be more elementary and self-contained. On a high level, this analysis shares some of the ideas from [10] [30] which both uses similar versions of exponential functions, but it seems difficult to apply these existing approaches directly to the general setting with weighted balls and any \( b \geq n \).

We then proceed to a tighter bound and prove that for any \( n \log n \leq b \leq n^3 \) and any number of balls \( m \geq n \), the gap is \( O(b/n + \log n) \) w.h.p. This bound is derived through an interplay between different potential functions, in particular, we relate three hyperbolic cosine potential functions with different smoothing parameters.

Next we turn to proving asymptotically tight lower bounds. We prove that for any \( b \geq n \), there are processes falling into our framework that produce for certain values of \( m \) a gap of \( \Omega(b/n + \log n) \). These lower bounds are proven in the unweighted setting where all balls have weight one. For the Two-Choice process, the lower bounds are tight for \( b \geq n \log n \).

Combining our upper with lower bounds reveals an interesting behavior: For any \( b \in [n, O(n \log n)] \), the gap is \( \Theta(\log n) \) w.h.p., whereas for \( b \geq n \log n \), the gap is \( \Theta(b/n) \) w.h.p. In particular, the asymptotic gap bound does not change as \( b \) moves from \( n \) to \( n \log n \).

We further demonstrate the flexibility of our techniques by deriving results for the graphical allocation model from [30], where bins are arranged as a graph and at each round a pair of bins is sampled by picking a random edge from the graph. One open question in [30] Section 4 was to derive results for graphical allocation with weights. In this work, we make progress towards that question by proving gap bounds that hold not only for weighted balls but also in the batched setting. For example, if the graph is a bounded-degree expander, then we recover the gap bound of \( O(\log n) \) from [30] even if balls are weighted and are allocated in batches up to a size of \( n \log n \). Finally, another consequence of our approach is a tight \( O(\log n/\beta) \) upper bound for the \((1 + \beta)\) process for any \( \beta \leq 1/2 \).

Our results are summarized in Table 1.

**Organization.** In Section 2, we present some standard notation for balanced allocations and define the processes and models used. In Section 3, we generalize (and strengthen) the analysis of the hyperbolic cosine potential of [30]. In Section 4, we apply this analysis to obtain an \( O(\frac{b}{n} \log n) \) gap bound for a family of processes in the batched model with weighted balls. In Section 5, we improve this upper bound on the gap to \( O(\frac{b}{n} + \log n) \), for any \( n \leq b \leq n^3 \). In Section 6, we demonstrate applications of our analysis to graphical allocation and the \((1 + \beta)\) process. In Section 7, we show that our upper bound from Section 5 is asymptotically tight, by providing lower bounds for a large family of processes. In Section 8, we present some experimental results. Finally, in Section 9, we summarize the main results and point to some open problems.

## 2 Notation

### 2.1 Basic Notation and Specific Processes

We consider the allocation of \( m \) balls into \( n \) bins, which are labeled \([n] := \{1, 2, \ldots, n\}\). For the moment, the \( m \) balls are unweighted (or equivalently, all balls have weight \( 1 \)). For any round \( t \geq 0 \), \( x^t \) is the \( n \)-dimensional load vector, where \( x^t_i \) is the number of balls allocated into bin \( i \) in the first \( t \) allocations. In particular, \( x^0_i = 0 \) for every \( i \in [n] \). Finally, the gap is defined as

\[
\text{Gap}(t) = \max_{i \in [n]} x^t_i - \frac{t}{n}.
\]

It will be also convenient to keep the load vector \( x \) sorted. To this end, let \( \tilde{x}^t := x^t - \frac{t}{n} \). Then, relabel the bins such that \( y^1_i \) is a permutation of \( \tilde{x}^t \) and \( y^1_1 \geq y^1_2 \geq \cdots \geq y^1_n \). Note that
\[ \sum_{i \in [n]} y^t_i = 0 \] and \( \text{Gap}(t) = y^t_1 \). We will call a bin \( i \in [n] \) overloaded, if \( y_i \geq 0 \) and underloaded otherwise. Further, we say that a vector \( v = (v_1, v_2, \ldots, v_n) \) majorizes \( u = (u_1, u_2, \ldots, u_n) \) if for all \( 1 \leq k \leq n \), the prefix sums satisfy: \( \sum_{i=1}^{k} v_i \geq \sum_{i=1}^{k} u_i \).

Following [30], many allocation processes can be described by a time-invariant probability vector \( p_i, 1 \leq i \leq n \), such that at each step \( t \geq 0 \), \( p_i \) is the probability for allocating a ball into the \( i \)-th most heavily loaded bin (or equivalently, incrementing \( y^t_i \) by one).

By \( \mathcal{F}_t \) we denote the filtration of the process until step \( t \), which in particular reveals the load vector \( x^t \).

We continue with a formal description of the Two-Choice process.

**Two-Choice Process:**

**Iteration:** For each \( t \geq 0 \), sample two bins \( i_1 \) and \( i_2 \) with replacement, independently and uniformly at random. Let \( i \) be one bin with \( x^t_i = \min\{x^t_{i_1}, x^t_{i_2}\} \), breaking ties randomly. Then update:

\[
x^{t+1}_i = x^t_i + 1.
\]

It is immediate that the probability vector of Two-Choice is

\[
p_i = \frac{2(i - 1)}{n^2}, \quad \text{for all } i \in [n].
\]

Following [30], we recall the definition of \((1 + \beta)\) which is a process interpolating between One-Choice and Two-Choice:

**\((1 + \beta)\)-Process:**

**Parameter:** A mixing factor \( \beta \in (0, 1] \).

**Iteration:** For each \( t \geq 0 \), sample two bins \( i_1 \) and \( i_2 \) with replacement, independently and uniformly at random. Let \( i \) be one bin with \( x^t_i = \min\{x^t_{i_1}, x^t_{i_2}\} \), breaking ties randomly. Then update:

\[
\begin{cases}
  x^{t+1}_{i_2} = x^t_{i_2} + 1 & \text{if } i_1 \text{ is among the } \delta \cdot n \text{ most loaded bins}, \\
  x^{t+1}_{i_1} = x^t_{i_1} + 1 & \text{otherwise}.
\end{cases}
\]

In other words at each step, \((1 + \beta)\)-process allocates the ball following the Two-Choice rule with probability \( \beta \), and otherwise allocates the ball following the One-Choice rule. Therefore, the probability vector is given by [30]:

\[
p_i = (1 - \beta) \cdot \frac{1}{n} + \beta \cdot \frac{2(i - 1)}{n^2}, \quad \text{for all } i \in [n].
\]

The next process is another relaxation of Two-Choice.

**Quantile(\(\delta\)) Process:**

**Parameter:** A quantile \( \delta \in \{1/n, 2/n, \ldots, 1\} \).

**Iteration:** For each \( t \geq 0 \), sample two bins \( i_1 \) and \( i_2 \) with replacement, independently and uniformly at random, and update:

\[
\begin{cases}
  x^{t+1}_{i_2} = x^t_{i_2} + 1 & \text{if } i_1 \text{ is among the } \delta \cdot n \text{ most loaded bins}, \\
  x^{t+1}_{i_1} = x^t_{i_1} + 1 & \text{otherwise}.
\end{cases}
\]

Note that the Quantile(\(\delta\)) processes can be also implemented as a two-phase procedure: First probe the bin \( i_1 \) and place the ball there if \( i_1 \) is not among the \( \delta \cdot n \) heaviest bins. Otherwise,
Two-Choice \((1 + \beta)\), \(\beta = 0.4\), and \(\text{Quantile}(0.6)\), which is sandwiched between the other two processes.

Figure 1: Illustration of the probability vector \((p_1, p_2, \ldots, p_{10})\) and cumulative probability distribution of Two-Choice, \((1 + \beta)\) with \(\beta = 0.4\) and \(\text{Quantile}(0.6)\), which is sandwiched between the other two processes.

Another, equivalent description of \(\text{Quantile}(\delta)\) is that we perform Two-Choice, but only get to know whether a sampled bin’s rank is below or above \(\delta \cdot n\) and breaking ties randomly.

An example of the probability vectors of the three processes above can be found in Fig. 1.

Finally, we will also consider a graph-based version of balanced allocation, called \textit{graphical balanced allocation} \cite{30}. This process involves running Two-Choice on a graph, where only bin pairs can be sampled which are connected by an edge.

| Graphical(G) |
|---------------|
| **Parameter:** An undirected, connected, regular graph \(G\). |
| **Iteration:** For each \(t \geq 0\), sample an edge \(e = \{i_1, i_2\} \in E\) uniformly at random. Let \(i\) be one bin with \(x^t_i = \min\{x^t_{i_1}, x^t_{i_2}\}\), breaking ties randomly. Then update: |
| \[ x^{t+1}_i = x^t_i + 1 \] |

Note that unlike the other processes, the probability vector of Graphical will generally not be time-invariant.

2.2 Classes of Processes

We will now formulate general conditions based on the probability vector \(p\) to which our analysis will apply:

- **Condition \(\mathcal{D}_0\):** \((p_i)_{i \in [n]}\) is a non-decreasing probability vector in \(1 \leq i \leq n\).

- **Condition \(\mathcal{D}_1\):** There exist constant \(\delta \in (0, 1)\) and (not necessarily constant) \(\varepsilon \in (0, 1)\),

\[ p_{\delta n} \leq \frac{1 - \varepsilon}{n}. \]

- **Condition \(\mathcal{D}_2\):** For some constant \(C > 1\), \(\max_{i \in [n]} p_i \leq \frac{C}{n} \).
Note that condition $D_1$ only provides an upper bound for allocating into the heavier bins. However, due to $D_0$, this also implies a lower bound on the probability for allocating into the lighter bins (see Observation 2.1). Finally, we remark any $d$-Choice process satisfies $D_2$ for $C = d$.

For the application to graphical allocation, we will relax these conditions slightly and drop the assumption that $(p_i)_{i \in [n]}$ is non-increasing in exchange for a stronger version of condition $D_1$ that involves both prefix and suffix sums of $p$.

- **Condition $C_1$:** There exist constant $\delta \in (0, 1)$ and (not necessarily constant) $\varepsilon \in (0, 1)$, such that for any $1 \leq k \leq \delta \cdot n$,
  \[
  \sum_{i=1}^{k} p_i \leq (1 - \varepsilon) \cdot \frac{k}{n},
  \]
  and similarly for any $\delta \cdot n + 1 \leq k \leq n$,
  \[
  \sum_{i=k}^{n} p_i \geq \left(1 + \varepsilon \cdot \frac{\delta}{1 - \delta}\right) \cdot \frac{n - k + 1}{n}.
  \]

- **Condition $C_2$:** For some constant $C > 1$, $\max_{i \in [n]} p_i \leq \frac{C}{n}$.

**Observation 2.1.** Conditions $D_0$ and $D_1$ with $\delta$ and $\varepsilon$ imply condition $C_1$ with the same $\delta$ and $\varepsilon$.

*Proof.* Since $p_{\delta n} \leq \frac{1 - \varepsilon}{n}$ and $p_i$ is non-decreasing, it follows that $p_j \leq \frac{1 - \varepsilon}{n}$ for all $1 \leq j \leq \delta n$, and thus the prefix sum condition of $C_1$ holds with equality. We can also conclude that
  \[
  \sum_{i=1}^{\delta n} p_i \leq (1 - \varepsilon) \cdot \delta,
  \]
  and hence
  \[
  \sum_{i=\delta n+1}^{n} p_i \geq 1 - (1 - \varepsilon) \cdot \delta.
  \]
  Since $p_i$ is non-decreasing, it follows for any $\delta n + 1 \leq k \leq n$,
  \[
  \sum_{i=k}^{n} p_i \geq \frac{n - k + 1}{(1 - \delta)n} \cdot (1 - (1 - \varepsilon) \cdot \delta) = \frac{n - k + 1}{n} \cdot \left(1 + \varepsilon \cdot \frac{\delta}{1 - \delta}\right).
  \]

Using this observation, it is easy to verify that Two-Choice, $(1 + \beta)$ and Quantile satisfy the two conditions $C_1$ and $C_2$.

**Proposition 2.2.** For any $\beta \in (0, 1]$, the $(1 + \beta)$-process satisfies condition $C_1$ with $\delta = \frac{1}{4}$ and $\varepsilon = \frac{\beta}{2}$ and condition $C_2$ with $C = 2$. Further, for any constant $\delta \in (0, 1)$, the Quantile($\delta$) process satisfies condition $C_1$ with $\delta$ and $\varepsilon = 1 - \delta$, and condition $C_2$ with $C = 2$.

*Proof.* For any $i \in [n]$,
  \[
  p_i = (1 - \beta) \cdot \frac{1}{n} + \beta \cdot \frac{2(i - 1)}{n^2}.
  \]
  This shows that $p_i$ is increasing in $i \in [n]$ (condition $D_0$), and thus also $\max_{i \in [n]} p_i \leq \frac{2}{n}$ (condition $C_2$). Further, for $\delta = 1/4$,
  \[
  p_{\delta n} \leq (1 - \beta) \cdot \frac{1}{n} + \beta \cdot \frac{1}{2n} = \left(1 - \beta \cdot \frac{1}{2}\right) \cdot \frac{1}{n},
  \]
proving that $D_1$ holds with $\varepsilon = \beta/2$. By Observation 2.1, $C_1$ holds with the same $\varepsilon$ and $\delta$.

For $\text{Quantile}(\delta)$, it is obvious that condition $D_0$ holds, as well as condition $C_2$ with $C = 2$. Further, for any $i \leq \delta \cdot n$, we have $p_i \leq \frac{\delta}{n}$, which means $D_1$ holds with $\varepsilon = 1 - \delta$. \hfill $\square$

Note that for $\beta = 1$, the $(1 + \beta)$-process equals Two-Choice, so the above statement also applies to Two-Choice. Finally, since Two-Choice satisfies $C_1$, by majorisation also $d$-Choice for any $d > 2$ satisfies $C_1$ with the same $\delta$ and $\varepsilon$. Further, $d$-Choice satisfies $C_2$ with $C = d$ and thus:

**Proposition 2.3.** For any $d \geq 2$, $d$-Choice satisfies condition $C_1$ with $\delta = \frac{1}{4}$, $\varepsilon = \frac{1}{2}$ and condition $C_2$ with $C = d$.

### 2.3 Batched Model and Weights

We will now extend the definitions of Section 2.1 and Section 2.2 to weighted balls into bins. To this end, let $w^t \geq 0$ be the weight of the $t$-th ball to be allocated ($t \geq 1$). By $W^t$ we denote the the total weights of all balls allocated after the first $t \geq 0$ allocations, so $W^t := \sum_{i=1}^{t} x_i = \sum_{i=1}^{t} w^i$. The normalized loads are $\tilde{x}_i := x_i - \frac{W^t}{n}$, and with $y_i$ being again the decreasingly sorted, normalized load vector, we have $\text{Gap}(t) = y_i$.

The weight of each ball will be drawn independently from a fixed distribution $W$ over $[0, \infty)$. Following [30], we assume that the distribution $W$ satisfies:

- $\mathbf{E}[W] = 1$.
- $\mathbf{E}[e^{\lambda W}] < \infty$ for some constant $\lambda > 0$.

It is clear that when $\mathbf{E}[W] = \Theta(1)$, by scaling $W$, we can always achieve $\mathbf{E}[W] = 1$. Specific examples of distributions satisfying above conditions (after scaling) are the geometric, exponential, binomial and Poisson distributions.

Similar to the arguments in [30], the above two assumptions can be used to prove that:

**Lemma 2.4.** There exists $S := S(\lambda) \geq \max(1, 1/\lambda)$, such that for any $\alpha \in (0, \min(\lambda/2, 1))$ and any $\kappa \in [-1, 1]$,

$$\mathbf{E}[e^{\alpha \kappa - \lambda W}] \leq 1 + \alpha \cdot \kappa + S\alpha^2 \cdot \kappa^2.$$  

**Proof.** This proof closely follows the argument in [30 Lemma 2.1]. Let $M(z) = \mathbf{E}[e^{z W}]$, then using Taylor’s Theorem (mean value form remainder), for any $z \in [-\alpha, \alpha]$ there exists $\xi \in [-\alpha, \alpha]$ such that

$$M(z) = M(0) + M'(0) \cdot z + M''(\xi) \cdot \frac{1}{2} \cdot z^2 = 1 + z + M''(\xi) \cdot \frac{1}{2} \cdot z^2.$$  

By the assumptions on $\alpha$ and $\lambda$,

$$M''(\xi) = \mathbf{E}[W^2 e^{\xi W}] \leq \sqrt{\mathbf{E}[W^4] \cdot \mathbf{E}[e^{2\xi W}]} \leq \frac{1}{2} \cdot \left( \mathbf{E}[W^4] + \mathbf{E}[e^{2\xi W}] \right) \leq \frac{1}{2} \cdot \left( \frac{8}{\lambda} \cdot \log \left( \frac{8}{\lambda} \right) \right)^4 + \mathbf{E}[e^{\lambda W}] + \mathbf{E}[e^{\lambda W}],$$  

where (a) uses the Cauchy-Schwartz inequality $|\mathbf{E}[X \cdot Y]| \leq \sqrt{\mathbf{E}[X^2] \mathbf{E}[Y^2]}$ for random variables $X$ and $Y$, (b) uses a mean inequality, and (c) uses Lemma A.5. Now defining

$$S := 2 \cdot \max \left\{ \left( \frac{8}{\lambda} \cdot \log \left( \frac{8}{\lambda} \right) \right)^4, 2 \cdot \mathbf{E}[e^{\lambda W}], 1/2 \right\},$$  

and choosing $z := \kappa \cdot \alpha$, the lemma follows. \hfill $\square$
We will now describe the allocation of weighted balls into bins using a batch size of \( b \geq n \). For the sake of concreteness, let us first describe the batched model if the allocation is done using Two-Choice. For a given batch size consisting of \( b \) consecutive balls, each ball of the batch performs the following. First, it samples two bins \( i_1 \) and \( i_2 \) and compares the load the two bins had at the beginning of the batch (let us denote the bin which has less load by \( i \)). Secondly, a weight is sampled from the distribution \( W \). Then a weighted ball is added to bin \( i \). Recall that since the load information is only updated at the beginning of the batch, all allocations of the \( b \) balls within the same batch can be performed in parallel.

In the following, we will use a more general framework, where the process of sampling (one or more) bins and then deciding where to allocate the ball to is described by a probability vector \( p \) over the \( n \) bins (Section 2.1 and Section 2.2). Also for the analysis, it will be convenient to focus on the normalized and sorted load vector \( y \), which is why the definition below is based on \( y \) rather than the actual load vector \( x \).

Batched Allocation with Weights

**Parameters:** Batch size \( b \geq n \), probability vector \( p \), weight distribution \( W \).

**Iteration:** For each \( t = 0 \cdot b, 1 \cdot b, 2 \cdot b, \ldots \):

1. Sample \( b \) bins \( i_1, i_2, \ldots, i_b \) from \([n]\) following \( p \).
2. Sample \( b \) weights \( w^{t+1}, w^{t+2}, \ldots, w^{t+b} \) from \( W \).
3. Update for each \( i \in [n] \),
   \[
   z_i^{t+b} = y_i^t + \sum_{j=1}^{b} w^{t+j} \cdot 1_{i_j = i} - \frac{1}{n} \sum_{j=1}^{b} w^{t+j},
   \]
4. Let \( y^{t+b} \) be the vector \( z^{t+b} \), sorted decreasingly.

We also look at the version of the processes that performs random tie-breaking between bins of the same load. For \( b = 1 \), this makes no observable difference to the process, but for multiple steps, this effectively averages out the probability over (possibly) multiple bins that have the same load. This would, for instance, correspond to Two-Choice, randomly deciding between the two bins if they have the same load. In particular, if \( p \) is the original probability vector, then the one with random tie-breaking is \( \tilde{p}(y^t) \) (for \( t \) being the beginning of the batch), where

\[
\tilde{p}_i(y^t) := \frac{1}{|\{J \in [n] : y_J^t = y_i^t\}|} \sum_{j \in [n] : y_j^t = y_i^t} p_j, \quad \text{for } i \in [n].
\] (2.1)

Batched Allocation with Weights and Random Tie-Breaking

**Parameters:** Batch size \( b \geq n \), probability vector \( p \), weight distribution \( W \).

**Iteration:** For each \( t = 0 \cdot b, 1 \cdot b, 2 \cdot b, \ldots \):

1. Let \( \tilde{p} := \tilde{p}(y^t) \) be the probability vector accounting for random tie-breaking.
2. Sample \( b \) bins \( i_1, i_2, \ldots, i_b \) from \([n]\) following \( \tilde{p} \).
3. Sample \( b \) weights \( w^{t+1}, w^{t+2}, \ldots, w^{t+b} \) from \( W \).
4. Update for each \( i \in [n] \),
   \[
   z_i^{t+b} = y_i^t + \sum_{j=1}^{b} w^{t+j} \cdot 1_{i_j = i} - \frac{1}{n} \sum_{j=1}^{b} w^{t+j},
   \]
5. Let \( y^{t+b} \) be the vector \( z^{t+b} \), sorted decreasingly.
Following [7], our goal will be to bound the gap at the end of a batch, i.e., $m$ will be a multiple of $b$.

Next we prove the following simple lemma, that high probability gap bounds at the end of batches imply high probability gap bounds at all steps in between, for batch sizes $b = \text{poly}(n)$. Thus, Theorem 4.2 and Theorem 5.1 only prove gap bounds at the end of batches.

**Lemma 2.5 (Smoothing argument).** Consider any allocation process in the weighted batched setting with $b \geq n$ and a weight distribution satisfying Lemma 2.4 for some constants $\lambda > 0$ and $S := S(\lambda) \geq 1$. If for some $m$ being a multiple of $b$, some (not necessarily constant) $c > 0$ and constant $\kappa > 0$,

\[
\Pr[ y_1^m \leq c ] \geq 1 - n^{-\kappa},
\]

then for any $t \in [m - b, m]$,

\[
\Pr[ y_1^t \leq c + \frac{2 \ln(S)}{\lambda} \cdot \frac{b}{n} ] \geq 1 - 2n^{-\kappa}.
\]

Similarly, if for some (not necessarily constant) $c > 0$,

\[
\Pr[ -y_n^m \leq c ] \geq 1 - n^{-\kappa},
\]

then there exists a constant $\kappa := \kappa(S) > 0$, such that for any $t \in [m, m + b]$,

\[
\Pr[ -y_n^t \leq c + \frac{2 \ln(S)}{\lambda} \cdot \frac{b}{n} ] \geq 1 - 2n^{-\kappa}.
\]

**Proof.** Applying Lemma A.1 for the $k := b \geq n$ weights $(w^{m-b+j})_{j=1}^b$,

\[
\Pr\left[ \sum_{j=1}^b w^{m-b+j} \leq \frac{2 \ln(S)}{\lambda} \cdot b \right] \geq 1 - e^{-\Omega(b)}.
\]

Hence, w.h.p. the mean load does not increase by more than $\frac{2 \ln(S)}{\lambda} \cdot \frac{b}{n}$. Therefore by the union bound, for any $t \in [m - b, m]$, any bin load can increase by at most that amount,

\[
\Pr[ y_1^t \leq c + \frac{2 \ln(S)}{\lambda} \cdot \frac{b}{n} ] \geq 1 - e^{-\Omega(b)} - n^{-\kappa} \geq 1 - 2n^{-\kappa},
\]

Similarly, by the union bound, for any $t \in [m, m + b]$, any bin load can decrease by at most $\frac{2 \ln(S)}{\lambda} \cdot \frac{b}{n}$,

\[
\Pr[ -y_n^t \leq c + \frac{2 \ln(S)}{\lambda} \cdot \frac{b}{n} ] \geq 1 - e^{-\Omega(b)} - n^{-\kappa} \geq 1 - 2n^{-\kappa}.
\]

which concludes the claim.

\[\square\]

## 3 Analysis of the Hyperbolic Cosine Potential

In this section we generalize [30, Theorem 2.10]. This generalization allows us to apply it to multi-step changes in Section 4 handle general quantile conditions (arbitrary constant $\delta > 0$ instead of $\delta = 1/3$) and obtain tighter bounds of $O(n)$ on the expectation of the potential, which we make use of in Section 5. Further, using this generalization, we obtain bounds on graphical allocation with weights and batches and a tighter upper bound for the $(1 + \beta)$ process for very small $\beta$ (Section 6). The **hyperbolic cosine potential** is defined as

\[
\Gamma^t := \Phi^t + \Psi^t := \sum_{i=1}^n e^{\alpha y_i^t} + \sum_{i=1}^n e^{-\alpha y_i^t},
\]  

(3.1)
for $\alpha > 0$. We also decompose $\Gamma_i^t$ across bins as follows, and define for any bin $i \in [n]$:

$$\Gamma_i^t := \Phi_i^t + \Psi_i^t = e^{\alpha y_i^t} + e^{-\alpha y_i^t}.$$

Further, we use the following shorthands to denote the changes in the potentials $\Delta \Phi_i^t := \Phi_i^{t+1} - \Phi_i^t$, $\Delta \Psi_i^t := \Psi_i^{t+1} - \Psi_i^t$ and $\Delta \Gamma_i^t := \Gamma_i^{t+1} - \Gamma_i^t$.

The next result holds for any probability vector $p$ satisfying condition $C_1$ and any load vector $x$. As we show in Corollary 3.2, this implies upper bounds on the expected change of the $\Gamma$ potential, under certain conditions.

**Theorem 3.1.** Consider any probability vector $p$ satisfying condition $C_1$ for constant $\delta \in (0,1)$ and $\varepsilon > 0$, and any load vector $x$ with $\Phi := \Phi(x)$, $\Psi := \Psi(x)$ and $\Gamma := \Gamma(x)$. Further for some $K > 0$ define,

$$\Delta \Phi := \sum_{i=1}^{n} \Delta \Phi_i = \sum_{i=1}^{n} \Phi_i \cdot \left( p_i - \frac{1}{n} \right) \cdot \alpha + K \cdot \frac{\alpha^2}{n},$$

and

$$\Delta \Psi := \sum_{i=1}^{n} \Delta \Psi_i = \sum_{i=1}^{n} \Psi_i \cdot \left( \frac{1}{n} - p_i \right) \cdot \alpha + K \cdot \frac{\alpha^2}{n}.$$

Then, there exists a constant $c := c(\delta) > 0$, such that for any $0 < \alpha < \min(1, \frac{\varepsilon \delta}{\frac{8}{n} \Gamma})$,

$$\Delta \Gamma := \Delta \Phi + \Delta \Psi \leq -\frac{\varepsilon \delta}{8} \cdot \frac{\alpha}{n} \cdot \Gamma + c \cdot \varepsilon \cdot \alpha.$$

Before presenting the proof, we begin with an outline of the key observations in the proof. Let $\Delta \Gamma_i := \Delta \Phi_i + \Delta \Psi_i$.

1. It suffices to analyze the potential for a process with probability vector,

$$q_i := \begin{cases} \frac{1-\varepsilon}{n} & \text{if } i \leq \delta n, \\ \frac{1+\varepsilon}{n} & \text{otherwise}, \end{cases} \quad (3.2)$$

where $\bar{\varepsilon} := \varepsilon \cdot \frac{\delta}{1-\delta}$, as this maximizes the terms $\Delta \Phi$ and $\Delta \Psi$.

2. For any bin $i \in [n]$, there is one dominant term in $\Gamma_i$: for overloaded bins it is $\Phi_i$ (and $\Psi_i \leq 1$) and for underloaded bins it is $\Psi_i$ (and $\Phi_i \leq 1$). The change of the smaller term is absorbed by the change of the dominant and the additive term, i.e., $c \cdot \varepsilon \cdot \alpha$.

3. It suffices to show that

$$\sum_{i=1}^{n} \left( \Phi_i \cdot \left( p_i - \frac{1}{n} \right) \cdot \alpha + \Psi_i \cdot \left( \frac{1}{n} - p_i \right) \cdot \alpha \right) \leq -\frac{\varepsilon \delta}{4} \cdot \frac{\alpha}{n} \cdot \Gamma + c \cdot \varepsilon \cdot \alpha,$$

as half of the decrease term, i.e., $-\frac{\varepsilon \delta}{n} \cdot \Gamma$ will counteract the increase term $K \cdot \alpha^2 \cdot \Gamma$ for sufficiently small $\alpha$. So, the main focus is on the coefficients of $\alpha$.

4. Any overloaded bin $i \in [n]$ with $i \leq \delta n$, satisfies $p_i = \frac{1-\varepsilon}{n}$ and so $\Delta \Phi_i \leq -\Phi_i \cdot \frac{\alpha}{n} + \mathcal{O}(\alpha^2)$. We call these the set $\mathcal{G}_+$ of good overloaded bins. The rest of the overloaded bins are the bad overloaded bins $\mathcal{B}_+$, and these still satisfy $\Delta \Phi_i \leq +\Phi_i \cdot \frac{\alpha}{n} + \mathcal{O}(\alpha^2)$.

Similarly, good underloaded bins $\mathcal{G}_-$ with $i > \delta n$, satisfy $\Delta \Psi_i \leq -\Psi_i \cdot \frac{\alpha}{n} + \mathcal{O}(\alpha^2)$ and bad underloaded bins $\mathcal{B}_-$ satisfy $\Delta \Psi_i \leq +\Psi_i \cdot \frac{\alpha}{n} + \mathcal{O}(\alpha^2)$.
5. We can either have $B_+ \neq \emptyset$ or $B_- \neq \emptyset$ (Fig. 2).

**Figure 2:** The two cases of bad bins in a configuration and their dominating terms in $\Delta \Gamma$ for each of the set of bins.

The handling of one case is symmetric to the other due to the symmetric nature of $\Delta \Phi$ and $\Delta \Psi$ (with $\delta$ being replaced by $1 - \delta$). So, from here on we only consider cases with $B_+ \neq \emptyset$ (and $B_- = \emptyset$).

6. **Case A:** When the number of bad overloaded bins is small (i.e., $|B_+| \leq \frac{n}{2} \cdot (1 - \delta)$), the positive contribution of the bins in $B_+$ is counteracted by the negative contribution of the bins in $G_+$ as in Case A. This is shown by analyzing the worst-case, where all bad bins are equal to $y_{\delta n}$. All underloaded bins are good and so on aggregate we get a decrease.

**Figure 3:** Case A: The positive dominant term in the contribution of bins in $B_+$ is counteracted by a fraction of the negative contribution term of the good bins $G_+$.

7. **Case B:** Consider the case when $|B_+| > \frac{n}{2} \cdot (1 - \delta)$. The positive contribution of the first $\frac{n}{2} \cdot (1 - \delta)$ of the bins $B_+$, call them $B_1$, is counteracted by the negative contribution of the bins in $G_+$ as in Case A. The positive contribution of the remaining bad bins $B_2$ is
counteracted by a fraction of the negative contribution of the bins in \( G_- \). This is because the number of holes in the bins of \( G_- \) are significantly more than the number of bins in \( B_2 \). Hence, again on aggregate we get a decrease (Fig. 4).

\[
y(n) \cdot \delta_n + \frac{1}{4} + \frac{1}{4} \cdot \alpha \cdot \frac{1}{n} \leq \frac{1}{4} \cdot \frac{1}{4} \cdot z_1 \cdot z_2 \cdot \delta_n + \frac{1}{4} \cdot \alpha \cdot \frac{1}{n} \leq \frac{1}{4} \cdot \frac{1}{4} \cdot \alpha \cdot \frac{1}{n} \leq \frac{1}{4} \cdot \frac{1}{4} \cdot \alpha \cdot \frac{1}{n} \leq \frac{1}{4} \cdot \frac{1}{4} \cdot \alpha \cdot \frac{1}{n}
\]

\( \text{Figure 4: Case B: The dominant change of the bins in } B_1 \text{ is counteracted by a fraction of the decrease of the bins in } G_+ \text{ as in Case A. The dominant change of the bins in } B_2 \text{ is counteracted by a fraction of the decrease of the bins in } G_- \text{, when } z_2 \text{ is sufficiently large.} \)

\textbf{Proof.} Fix a labeling of the bins so that they are sorted non-increasingly according to their load in \( x \). Let \( p \) be the probability vector satisfying condition \( C_1 \) for some \( \varepsilon \in (0, 1) \) and \( \delta \in (0, 1) \). Then define another probability vector, \[ q_i := \begin{cases} 
\frac{1}{n} - \frac{\varepsilon}{n} & \text{if } i \leq \delta n, \\
\frac{1}{n} - \frac{\varepsilon}{n} + \frac{\frac{\varepsilon}{n}}{\delta} & \text{otherwise}, 
\end{cases} \]

where \( \bar{\varepsilon} := \varepsilon \cdot \frac{\delta}{\delta} \). Thanks to the definition of \( \bar{\varepsilon} \), it is clear that this is a probability vector.

Further, for any \( 1 \leq k \leq \delta n \),

\[ \sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i, \]

and any \( \delta \cdot n + 1 \leq k \leq n \),

\[ \sum_{i=k}^{n} p_i \geq \sum_{i=k}^{n} q_i. \]

This implies that \( p \) is majorized by \( q \). Since \( \Phi_i \) and \( \Psi_i \) are non-increasing in \( i \in [n] \), using Lemma A.7 the terms

\[ \Delta \Phi = \sum_{i=1}^{n} \Phi_i \cdot \left( (p_i - \frac{1}{n}) \cdot \alpha + K \cdot \frac{\alpha^2}{n} \right), \]

and

\[ \Delta \Psi = \sum_{i=1}^{n} \Psi_i \cdot \left( (\frac{1}{n} - p_i) \cdot \alpha + K \cdot \frac{\alpha^2}{n} \right) \]

are larger for \( q \) than for \( p \). Hence, from now on, we will be working with \( p_i = q_i \) for all \( i \in [n] \).

Recall, that we partition overloaded bins \( i \) with \( y_i \geq 0 \) into \textit{good overloaded bins} \( G_+ \) with \( p_i = \frac{1 - \varepsilon}{n} \) and into \textit{bad overloaded bins} \( B_+ \) with \( p_i = \frac{1 + \varepsilon}{n} \). These are called good bins, because any bin \( i \in G_+ \) satisfies \( \Delta \Phi_i \leq \frac{\alpha \cdot \frac{1 - \varepsilon}{n} + K \cdot \frac{\alpha^2}{n}}{n} \) and since \( \Psi_i \leq 1 \) for overloaded bins, this will imply the drop condition for \( \Gamma_i \).

\textbf{Case A} \([1 \leq |B_+| \leq \frac{n}{2} \cdot (1 - \delta)]\): Intuitively, in this case the contribution of the bad bins is counteracted by the contribution of the good overloaded bins (Fig. 3). To formalize this, let
Hence, combining Eq. (3.4) and Eq. (3.5), the contribution for $\Phi$ of overloaded bins is given by

$$
\sum_{i \in B_+} \Delta \Phi_i \leq \sum_{i \in B_+} \Phi_i \cdot \left( \frac{\alpha \varepsilon}{n} + K \cdot \frac{\alpha^2}{n} \right)
\leq \sum_{i \in B_+} e^{\alpha z_i} \cdot \frac{\alpha \varepsilon}{n} + \sum_{i \in B_+} \Phi_i \cdot K \cdot \frac{\alpha^2}{n}
\leq \frac{n}{2} \cdot (1 - \delta) \cdot e^{\alpha z_i} \cdot \frac{\alpha \varepsilon}{n} + \sum_{i \in B_+} \Phi_i \cdot K \cdot \frac{\alpha^2}{n}
= e^{\alpha z_i} \cdot \frac{\alpha \varepsilon \delta}{2} + \sum_{i \in B_+} \Phi_i \cdot K \cdot \frac{\alpha^2}{n},
$$

(3.4)

where we have used in the last inequality that $|B_+| \geq 1$ implies $|G_+| = \delta \cdot n$. For bins in $G_+$,

$$
\sum_{i \in G_+} \Delta \Phi_i \leq \sum_{i \in G_+} \Phi_i \cdot \left( -\frac{\alpha \varepsilon}{n} + K \cdot \frac{\alpha^2}{n} \right)
= -\sum_{i \in G_+} \Phi_i \cdot \frac{\alpha \varepsilon}{4n} - \sum_{i \in G_+} \frac{3 \alpha \varepsilon}{4n} + \sum_{i \in G_+} \Phi_i \cdot K \cdot \frac{\alpha^2}{n}
\leq -\sum_{i \in G_+} \Phi_i \cdot \frac{\alpha \varepsilon}{4n} - \sum_{i \in G_+} e^{\alpha z_i} \cdot \frac{3 \alpha \varepsilon}{4n} + \sum_{i \in G_+} \Phi_i \cdot K \cdot \frac{\alpha^2}{n}
= -\sum_{i \in G_+} \Phi_i \cdot \frac{\alpha \varepsilon}{4n} - e^{\alpha z_i} \cdot \frac{3 \alpha \varepsilon \delta}{4} + \sum_{i \in G_+} \Phi_i \cdot K \cdot \frac{\alpha^2}{n}.
$$

(3.5)

Hence, combining Eq. (3.4) and Eq. (3.5), the contribution for $\Phi$ of overloaded bins is given by

$$
\sum_{i : y_i \geq 0} \Delta \Phi_i \leq -\sum_{i : y_i \geq 0} \Phi_i \cdot \frac{\alpha \varepsilon}{4n} - e^{\alpha z_i} \cdot \frac{\alpha \varepsilon \delta}{4} + \sum_{i : y_i \geq 0} \Phi_i \cdot K \cdot \frac{\alpha^2}{n}
\leq -\sum_{i : y_i \geq 0} \Phi_i \cdot \frac{\alpha \varepsilon}{4n} - \sum_{i : y_i \geq 0} \frac{\alpha \varepsilon \delta}{2(1 - \delta)n} + \sum_{i : y_i \geq 0} \Phi_i \cdot K \cdot \frac{\alpha^2}{n}
\leq \sum_{i : y_i \geq 0} \Phi_i \cdot \left( -\frac{\alpha \varepsilon \delta}{4n} + K \cdot \frac{\alpha^2}{n} \right).
$$

So, using that $\Psi_i \leq 1$ for overloaded bins,

$$
\sum_{i : y_i \geq 0} \Delta \tilde{\Phi}_i = \sum_{i : y_i \geq 0} \Delta \Phi_i + \sum_{i : y_i \geq 0} \Delta \Psi_i
\leq \sum_{i : y_i \geq 0} \Phi_i \cdot \left( -\frac{\alpha \varepsilon \delta}{4n} + K \cdot \frac{\alpha^2}{n} \right) + \sum_{i : y_i \geq 0} \Psi_i \cdot \left( \frac{\alpha \varepsilon}{n} + K \cdot \frac{\alpha^2}{n} \right)
\leq \sum_{i : y_i \geq 0} \Gamma_i \cdot \left( -\frac{\alpha \varepsilon \delta}{4n} + K \cdot \frac{\alpha^2}{n} \right) + \sum_{i : y_i \geq 0} \frac{2\alpha}{n} \cdot \max \left( \varepsilon, \varepsilon \delta \right).
$$

(3.6)

Since in this case all underloaded bins are good, i.e., for any $i \in [n]$ with $y_i < 0$, we have $p_i = \frac{1 + \varepsilon}{n}$, we have

$$
\sum_{i : y_i < 0} \Delta \Psi_i \leq \sum_{i : y_i < 0} \Psi_i \cdot \left( -\frac{\alpha \varepsilon}{n} + K \cdot \frac{\alpha^2}{n} \right).
$$

(3.7)
Combining the contribution across all underloaded bins,

\[
\sum_{i : y_i < 0} \Delta \Gamma_i = \sum_{i : y_i < 0} \Delta \Phi_i + \sum_{i : y_i < 0} \Delta \Psi_i \\
\leq \sum_{i : y_i < 0} \Phi_i \cdot \left( \frac{\alpha \tilde{\varepsilon}}{n} + K \cdot \frac{\alpha^2}{n} \right) + \sum_{i : y_i < 0} \Psi_i \cdot \left( -\frac{\alpha \tilde{\varepsilon}}{n} + K \cdot \frac{\alpha^2}{n} \right) \\
\leq \sum_{i : y_i < 0} \Gamma_i \cdot \left( -\frac{\alpha \tilde{\varepsilon}}{n} + K \cdot \frac{\alpha^2}{n} \right) + \sum_{i : y_i < 0} 2\alpha \tilde{\varepsilon},
\]

(3.8)

where in the first inequality we used Eq. (3.7) and the precondition of the theorem, while in the last inequality we used that \( \Phi_i \leq 1 \) for underloaded bins.

Combining Eq. (3.6) and Eq. (3.8),

\[
\Delta \Gamma = \sum_{i : y_i > 0} \Delta \Gamma_i + \sum_{i : y_i < 0} \Delta \Gamma_i \\
\leq \sum_{i : y_i > 0} \Gamma_i \cdot \left( -\frac{\alpha \varepsilon \delta}{4n} + K \cdot \frac{\alpha^2}{n} \right) + \sum_{i : y_i > 0} 2\alpha \cdot \max \left( \tilde{\varepsilon}, \frac{\varepsilon \delta}{4} \right) + \sum_{i : y_i < 0} \Gamma_i \cdot \left( -\frac{\alpha \tilde{\varepsilon}}{n} + K \cdot \frac{\alpha^2}{n} \right) + \sum_{i : y_i < 0} 2\alpha \tilde{\varepsilon} \\
\leq \sum_{i=1}^{n} \Gamma_i \cdot \left( -\frac{\alpha \varepsilon \delta}{4n} + K \cdot \frac{\alpha^2}{n} \right) + \sum_{i=1}^{n} 2\alpha \cdot \max \left( \tilde{\varepsilon}, \varepsilon \delta \cdot \frac{\alpha}{4} \right) \\
\leq -\Gamma \cdot \frac{\alpha \varepsilon \delta}{8n} + 2\alpha \cdot \max \left( \tilde{\varepsilon}, \frac{\varepsilon \delta}{4} \right),
\]

(3.9)

using in the last line that \( \alpha \leq \frac{\varepsilon \delta}{4n} \).

**Case B** \([|B_+| > \frac{n}{2} \cdot (1 - \delta)]\): We partition \( B_+ \) into \( B_1 := B_+ \cap \{ i \in [n] : i \leq \frac{n}{2} \cdot (1 + \delta) \} \) and \( B_2 := B_+ \setminus B_1 \). The positive contribution \( \Delta \Phi_i \) for bins \( i \in B_1 \) will be counteracted by that of the bins in \( G_+ \) as in Case A. For that of bins in \( B_2 \) we consider two cases based on \( z_2 := y_{\frac{n}{2} \cdot (1 + \delta)} > 0 \), the load of the first bin in \( B_2 \). Similarly to Eq. (3.4),

\[
\sum_{i \in B_2} \Delta \Phi_i \leq e^{\alpha z_2} \cdot \frac{\alpha \varepsilon \delta}{2} + \sum_{i \in B_2} \Phi_i \cdot K \cdot \frac{\alpha^2}{n}.
\]

(3.10)

**Case B.1** \([z_2 \leq \frac{1}{n} \cdot \frac{1 - \delta}{2} \cdot \ln(8/3)]\): In this case, we will show that the contribution of the bad bins can be absorbed by the additive term. In particular, the contribution of the bins in \( B_2 \) is

\[
\sum_{i \in B_2} \Delta \Gamma_i \leq \sum_{i \in B_2} 2 \cdot e^{\alpha z_2} \cdot \left( \frac{\alpha \tilde{\varepsilon}}{n} + K \cdot \frac{\alpha^2}{n} \right) \\
\leq \sum_{i \in B_2} 4 \cdot e^{\frac{1 - \delta}{2} \ln(8/3)} \cdot \frac{\alpha \tilde{\varepsilon}}{n} \\
< 4 \cdot e^{\frac{1 - \delta}{2} \ln(8/3)} \cdot \alpha \tilde{\varepsilon}.
\]

Hence, counteracting the positive contribution of the bins in \( B_1 \) using that of the bins in \( G_+ \) as in Case A (since \(|B_1| \leq \frac{n}{2} \cdot (1 - \delta)\)) as in Eq. (3.9) we have

\[
\Delta \Gamma \leq -\Gamma \cdot \frac{\alpha \varepsilon \delta}{8n} + \max \left( \frac{\varepsilon \delta}{4} \cdot \tilde{\varepsilon}, 2 \cdot e^{\frac{1 - \delta}{2} \ln(8/3)} \cdot \tilde{\varepsilon} \right) \cdot 2\alpha.
\]

17
Case B.2 \([z_2 > \frac{1}{\alpha} \cdot \frac{1-\delta}{2\alpha} \cdot \ln(8/3)]\): In this case, it means that there are substantially more holes in the underloaded bins than balls in the overloaded bins of \(B_1\). Hence, as we will prove below, the negative contribution \(\Delta \Psi\) for bins in \(G_-\) will counteract the positive contribution of \(\Delta \Phi\) for \(B_1\) [Fig. 4],

\[
\sum_{i \in G_-} \Delta \Psi_i \leq \sum_{i \in G_-} \Psi_i \cdot \left( -\frac{\alpha \varepsilon}{n} + K \cdot \frac{\alpha^2}{n} \right) \\
\leq \sum_{i \in G_-} \Psi_i \cdot \left( -\frac{\alpha \varepsilon}{4n} + K \cdot \frac{\alpha^2}{n} \right) - \sum_{i \in G_-} \Psi_i \cdot \frac{3\alpha \varepsilon \delta}{4 \cdot (1-\delta) \cdot n}. \tag{3.11}
\]

The term \(\sum_{i \in G_-} \Psi_i\) is minimized when all underloaded bins are equal to the same load \(-z_3 < 0\), i.e. \(\sum_{i \in G_-} \Psi_i \geq |G_-| \cdot e^{\alpha z_3}\). Note that \(z_3 \geq \frac{z_2 \cdot (|B_1| + |G_+|)}{|G_-|} \geq \frac{z_2 \cdot \frac{n}{2} \cdot (1+\delta)}{|G_-|}\) and that the function \(f(z) = z \cdot e^{k/z}\) is decreasing for \(0 \leq z \leq k\) [Lemma A.8]. Hence, for \(k = z_2 \cdot \frac{n}{2} \cdot (1+\delta)\), the maximum size \(|G_-| = \frac{n}{2} \cdot (1-\delta) \leq k\), minimizes the term \(\sum_{i \in G_-} \Psi_i\). We lower bound \(z_3\) as follows,

\[
z_3 \geq \frac{z_2 \cdot \frac{n}{2} \cdot (1+\delta)}{\frac{n}{2} \cdot (1-\delta)} = z_2 + z_2 \cdot \frac{2\delta}{1-\delta} \geq z_2 + \frac{1}{\alpha} \cdot \ln(8/3),
\]

using the lower bound on \(z_2\). Hence,

\[
\sum_{i \in G_-} \Psi_i \geq \frac{n}{2} \cdot (1-\delta) \cdot e^{\alpha \cdot (z_2 + \frac{1}{\alpha} \cdot \ln(8/3))} \cdot \frac{3\alpha \varepsilon \delta}{4 \cdot (1-\delta) \cdot n} = e^{\alpha z_2} \cdot (\alpha \varepsilon \delta).
\]

Applying this to \textbf{Eq. (3.11)}

\[
\sum_{i \in G_-} \Delta \Psi_i \leq \sum_{i \in G_-} \Psi_i \cdot \left( -\frac{\alpha \varepsilon}{4n} + K \cdot \frac{\alpha^2}{n} \right) - e^{\alpha z_2} \cdot (\alpha \varepsilon \delta). \tag{3.12}
\]

Aggregating \textbf{Eq. (3.10)} and \textbf{Eq. (3.12)} the contribution of underloaded bins to \(\Delta \Phi\) is

\[
\sum_{i \in G_-} \Delta \Psi_i + \sum_{i \in B_2} \Delta \Phi_i \leq \sum_{i \in G_-} \Psi_i \cdot \left( -\frac{\alpha \varepsilon}{4n} + K \cdot \frac{\alpha^2}{n} \right) - e^{\alpha z_2} \cdot (\alpha \varepsilon \delta) \\
+ e^{\alpha z_2} \cdot \frac{\alpha \varepsilon \delta}{2} + \sum_{i \in B_2} \Phi_i \cdot K \cdot \frac{\alpha^2}{n} \\
= \sum_{i \in G_-} \Psi_i \cdot \left( -\frac{\alpha \varepsilon}{4n} + K \cdot \frac{\alpha^2}{n} \right) - e^{\alpha z_2} \cdot \frac{\alpha \varepsilon \delta}{2} + \sum_{i \in B_2} \Phi_i \cdot K \cdot \frac{\alpha^2}{n} \\
\leq \sum_{i \in G_-} \Psi_i \cdot \left( -\frac{\alpha \varepsilon}{4n} + K \cdot \frac{\alpha^2}{n} \right) - \sum_{i \in B_2} \Phi_i \cdot \frac{\alpha \varepsilon \delta}{(1-\delta) \cdot n} + \sum_{i \in B_2} \Phi_i \cdot K \cdot \frac{\alpha^2}{n}.
\]

Hence,

\[
\sum_{i \in G_-} \Delta \Gamma_i + \sum_{i \in B_2} \Delta \Gamma_i \\
\leq - \sum_{i \in G_-} \Psi_i \cdot \frac{\alpha \varepsilon}{4n} + \sum_{i \in G_-} \Phi_i \cdot \frac{\alpha \varepsilon}{n} - \sum_{i \in B_2} \Phi_i \cdot \frac{\alpha \varepsilon \delta}{(1-\delta) \cdot n} + \sum_{i \in B_2} \Psi_i \cdot \frac{\alpha \varepsilon}{n} + \sum_{i \in G_- \cup B_2} \Gamma_i \cdot K \cdot \frac{\alpha^2}{n} \\
\leq \sum_{i \in G_- \cup B_2} \Gamma_i \cdot \left( -\frac{\alpha \varepsilon}{4n} + K \cdot \frac{\alpha^2}{n} \right) + \sum_{i \in G_- \cup B_2} 2\alpha \varepsilon \delta \cdot \frac{\varepsilon \delta}{4 \cdot \varepsilon}. \tag{3.13}
\]
Aggregating similarly, to Case A, for $G_+$ and $B_1$, 

$$
\sum_{i \in G_+} \Delta \Gamma_i + \sum_{i \in B_1} \Delta \Gamma_i \\
\leq \sum_{i \in G_+ \cup B_1} \Gamma_i \cdot \left(-\frac{\alpha \varepsilon \delta}{4n} + K \cdot \frac{\alpha^2}{n}\right) + \sum_{i \in G_+ \cup B_1} \frac{2\alpha}{n} \cdot \max \left(\bar{\varepsilon}, \frac{\varepsilon \delta}{4}\right).
$$

(3.14)

Hence, combining Eq. (3.13) and Eq. (3.14),

$$
\Delta \Gamma \leq -\Gamma \cdot \frac{\alpha \varepsilon \delta}{8n} + \max \left(\frac{\bar{\varepsilon}}{4}, \bar{\varepsilon}\right) \cdot 2\alpha.
$$

Case C, D: These are symmetric to Case A and Case B, but interchanging $\Phi$ with $\Psi$, $\delta$ with $1 - \delta$ and negating the normalized load vector and flipping the load vector.

Combining the four cases, we get that

$$
\Delta \Gamma \leq -\Gamma \cdot \frac{\alpha \varepsilon \delta}{8n} + c \cdot \varepsilon \cdot \alpha,
$$

where $c := 2 \cdot \max \left(\frac{\delta}{4}, \frac{\bar{\varepsilon}}{4}, 2 \cdot e^{\frac{\varepsilon \delta}{4n}} \cdot \frac{\delta}{4}, 2 \cdot e^{\frac{\delta}{4n}} \cdot \ln(8/3)\right)$, using that $\bar{\varepsilon} := \varepsilon \cdot \frac{\delta}{4}$. \qed

By scaling the quantities $\Delta \Phi$ and $\Delta \Psi$ in Theorem 3.1 by some $\kappa > 0$, we obtain:

**Corollary 3.2.** Consider any allocation process with probability vector $p$ satisfying conditions $C_1$ for constant $\delta \in (0, 1)$ and $\varepsilon > 0$. Further assume that it satisfies for some $K > 0$ and some $\kappa > 0$, for any $t \geq 0$,

$$
\sum_{i=1}^{n} \mathbb{E} \left[ \Delta \Phi_i^{t+1} \mid \tilde{\delta}^t \right] \leq \sum_{i=1}^{n} \Phi_i^{t} \cdot \left(\left(p_i - \frac{1}{n}\right) \cdot \kappa \cdot \alpha + K \cdot \kappa \cdot \frac{\alpha^2}{n}\right),
$$

and

$$
\sum_{i=1}^{n} \mathbb{E} \left[ \Delta \Psi_i^{t+1} \mid \tilde{\delta}^t \right] \leq \sum_{i=1}^{n} \Psi_i^{t} \cdot \left(\left(\frac{1}{n} - p_i\right) \cdot \kappa \cdot \alpha + K \cdot \kappa \cdot \frac{\alpha^2}{n}\right).
$$

Then, there exists a constant $c := c(\delta) > 0$, such that for $0 < \alpha < \min(1, \frac{\varepsilon \delta}{8K})$

$$
\mathbb{E} \left[ \Delta \Gamma_{t+1} \mid \tilde{\delta}^t \right] \leq -\frac{\varepsilon \delta}{8} \cdot \kappa \cdot \frac{\alpha}{n} \cdot \Gamma^t + c \cdot \kappa \cdot \varepsilon \cdot \alpha,
$$

and

$$
\mathbb{E} \left[ \Gamma^t \right] \leq \frac{8c}{\delta} \cdot n.
$$

**Proof.** Applying Theorem 3.1 for the current load vector $x^t$ and for

$$
\Delta \Phi := \sum_{i=1}^{n} \Phi_i^{t} \cdot \left(\left(p_i - \frac{1}{n}\right) \cdot \alpha + K \cdot \frac{\alpha^2}{n}\right) \quad \text{and} \quad \Delta \Psi := \sum_{i=1}^{n} \Psi_i^{t} \cdot \left(\left(\frac{1}{n} - p_i\right) \cdot \alpha + K \cdot \frac{\alpha^2}{n}\right),
$$

we get

$$
\Delta \Phi + \Delta \Psi \leq -\frac{\varepsilon \delta}{8} \cdot \frac{\alpha}{n} \cdot \Gamma^t + c \cdot \varepsilon \cdot \alpha.
$$

(3.15)

By the assumptions,

$$
\mathbb{E} \left[ \Delta \Gamma_{t+1} \mid \tilde{\delta}^t \right] = \mathbb{E} \left[ \Delta \Phi_{t+1} \mid \tilde{\delta}^t \right] + \mathbb{E} \left[ \Delta \Psi_{t+1} \mid \tilde{\delta}^t \right] \leq \kappa \cdot (\Delta \Phi + \Delta \Psi).
$$

(3.16)
Hence, combining Eq. (3.15) and Eq. (3.16) we get
\[
\mathbb{E} \left[ \Delta \Gamma^{t+1} | \tilde{\mathcal{F}}^t \right] \leq -\frac{\epsilon \delta}{8} \cdot \kappa \cdot \frac{\alpha}{n} \cdot \Gamma^t + c \cdot \kappa \cdot \epsilon \cdot \alpha.
\]

Now, we will show by induction that for any \( t \geq 0 \), \( \mathbb{E} \left[ \Gamma^t \right] \leq \frac{8c}{\delta} \cdot n \). Assume true for \( t \), then
\[
\mathbb{E} \left[ \Gamma^{t+1} \right] = \mathbb{E} \left[ \mathbb{E} \left[ \Gamma^{t+1} | \tilde{\mathcal{F}}^t \right] \right] \\
\leq \mathbb{E} \left[ \Gamma^t \cdot \left( 1 - \frac{\epsilon \delta}{8} \cdot \kappa \cdot \frac{\alpha}{n} \right) \right] + c \cdot \kappa \cdot \epsilon \cdot \alpha \\
\leq \frac{8c}{\delta} \cdot n \cdot \left( 1 - \frac{\epsilon \delta}{8} \cdot \kappa \cdot \frac{\alpha}{n} \right) + c \cdot \kappa \cdot \epsilon \cdot \alpha \\
= \frac{8c}{\delta} \cdot n - c \cdot \kappa \cdot \epsilon \cdot \alpha + c \cdot \kappa \cdot \epsilon \cdot \alpha = \frac{8c}{\delta} \cdot n. \quad \Box
\]

### 4 A Simple Upper Bound

In this section we derive an upper bound of \( \mathcal{O}(b/n \cdot \log n) \) for the weighted batched setting. This upper bound is tight for \( b = \Theta(n) \), as shown in Section 7. We will make use of the hyperbolic cosine potential as defined in Eq. (3.1). This will also serve as the base case for the tighter analysis in Section 5.

The main goal is to derive the preconditions of Corollary 3.2 and apply it for \( \kappa := b \) over the batches (not individual time steps).

**Lemma 4.1.** Consider the weighted batched setting with batch size \( b \geq n \), for a process with probability vector \( p \) satisfying condition \( C_2 \) for some \( C > 1 \) and the weight distribution satisfying Lemma 2.4 for some \( S \geq 1 \). Then for any \( 0 < \alpha \leq \frac{n}{2C^2 S^2 b} \), for any \( t \geq 0 \) being a multiple of \( b \),
\[
\sum_{i=1}^{n} \mathbb{E} \left[ \Phi_{i}^{t+b} | \tilde{\mathcal{F}}^t \right] \leq \sum_{i=1}^{n} \Phi_{i}^{t} \cdot \left( 1 + \left( \frac{1}{n} - p_i \right) \cdot \alpha \cdot b + 5 \cdot C^2 \cdot S^2 \cdot \frac{b}{n} \cdot \left( \frac{\alpha^2}{n} \cdot b \right) \right), \quad (4.1)
\]
and
\[
\sum_{i=1}^{n} \mathbb{E} \left[ \Psi_{i}^{t+b} | \tilde{\mathcal{F}}^t \right] \leq \sum_{i=1}^{n} \Phi_{i}^{t} \cdot \left( 1 + \left( \frac{1}{n} - p_i \right) \cdot \alpha \cdot b + 5 \cdot C^2 \cdot S^2 \cdot \frac{b}{n} \cdot \left( \frac{\alpha^2}{n} \cdot b \right) \right). \quad (4.2)
\]

**Proof.** Consider an arbitrary bin \( i \in [n] \). Let \( Z \in \{0,1\}^b \) be the indicator vector, where \( Z_j \) indicates whether the \( j \)-th ball was allocated to bin \( i \). The expected change for the overload potential \( \Phi_{i}^{t} \), is given by
\[
\mathbb{E} \left[ \Phi_{i}^{t+b} | \tilde{\mathcal{F}}^t \right] = \Phi_{i}^{t} \cdot \sum_{z \in \{0,1\}^b} \mathbb{P} \left[ Z = z \right] \cdot \mathbb{E} \left[ e^{\alpha \sum_{j=1}^{n} (z_j w_{t+j} - w_{t+j}^{(j)})} \bigg| \tilde{\mathcal{F}}^t, Z = z \right].
\]
In the following, let us upper bound the factor of $\Phi^t_i$:

$$
\sum_{z \in \{0,1\}^b} \Pr[Z = z] \cdot E\left[ e^{\alpha \sum_{j=1}^{n} (z_j w_t^{i,j} - 2^{i,j})} \mid \mathcal{F}^t, Z = z \right]
$$

\begin{align*}
&= \sum_{z \in \{0,1\}^b} \left( \prod_{j=1}^{b} (p_i)^{z_j} (1 - p_i)^{1-z_j} \left( E[e^{\alpha W(1 - \frac{1}{n})}] \right)^{z_j} \left( E[e^{-\alpha \frac{w}{n}}] \right)^{1-z_j} \right) \\
&\leq \sum_{z \in \{0,1\}^b} \left( \prod_{j=1}^{b} \left( p_i \cdot \left( 1 + \alpha \cdot \left( 1 - \frac{1}{n} \right) + S \alpha^2 \right)^{z_j} \cdot \left( 1 - p_i \cdot \left( 1 - \frac{\alpha}{n} + S \cdot \frac{\alpha^2}{n^2} \right)^{1-z_j} \right) \right) \\
&\leq \left( 1 + \left( p_i - \frac{1}{n} \right) \cdot \alpha + 2 \cdot p_i \cdot S \alpha^2 \right)^b, \\
&\text{using in (a) that the weights are independent given } \mathcal{F}^t, \text{ in (b) the Lemma 2.4 and in (c) the binomial theorem. Let us define} \\
y := \left( p_i - \frac{1}{n} \right) \cdot \alpha + 2 \cdot p_i \cdot S \alpha^2.
\end{align*}

We first claim that $y \cdot b \leq 1$, which holds indeed since

$$
y \cdot b = \left( p_i - \frac{1}{n} \right) \cdot \alpha \cdot b + 2 \cdot p_i \cdot S \alpha^2 \cdot b \\
\leq \frac{C}{n} \cdot \alpha \cdot b + 2 \cdot \frac{C}{n} \cdot S \alpha^2 \cdot b \leq 2CS \cdot \alpha \cdot b \leq 1,
$$

where in the last inequality we used $\alpha \leq \frac{n}{2CS}$.

Then,

$$
E[\Phi^{t+b}_i \mid \mathcal{F}^t] \leq \Phi^t_i \cdot e^{y\cdot b}
$$

\begin{align*}
&\leq \Phi^t_i \cdot \left( 1 + y \cdot b + y^2 \cdot b^2 \right) \\
&= \Phi^t_i \cdot \left( 1 + \left( p_i - \frac{1}{n} \right) \cdot \alpha \cdot b + 2 \cdot p_i \cdot S \alpha^2 \cdot b + \left( p_i - \frac{1}{n} \right) \cdot \alpha \cdot b + 2 \cdot p_i \cdot S \alpha^2 \cdot b \right)^2,
\end{align*}

using in (a) that $1 + y \leq e^y$ for any $y$, and in (b) that $e^y \leq 1 + y + y^2$ for $y \leq 1.75$. Since $p_i \leq \frac{C}{n}$ for all $i \in [n]$, we conclude

$$
E[\Phi^{t+b}_i \mid \mathcal{F}^t] \leq \Phi^t_i \cdot \left( 1 + \left( p_i - \frac{1}{n} \right) \cdot \alpha \cdot b + 2 \cdot \frac{CS}{n} \cdot \alpha^2 \cdot b + \left( \frac{2CS}{n} \cdot \alpha \cdot b \right)^2 \right) \\
\leq \Phi^t_i \cdot \left( 1 + \left( p_i - \frac{1}{n} \right) \cdot \alpha \cdot b + 5 \cdot \frac{C^2 \cdot S^2}{n} \cdot \frac{\alpha^2}{n} \cdot b \right).
$$

Similarly, for the underloaded potential $\Psi^t_i$, for any bin $i \in [n],$

$$
E[\Psi^{t+b}_i \mid \mathcal{F}^t] = \Psi^t_i \cdot \sum_{z \in \{0,1\}^b} \Pr[Z = z] \cdot E\left[ e^{\alpha \sum_{j=1}^{n} (z_j w_t^{i,j} - 2^{i,j})} \mid \mathcal{F}^t, Z = z \right].
$$
As before, we will upper bound the factor of $\Psi^t_i$:

$$
\sum_{z \in \{0,1\}^b} \Pr[Z = z] \cdot E \left[ e^{-\alpha \sum_{j=1}^{n} (z_j w^{t+j} - w^{t+j})} \mid \mathfrak{A}^t, Z = z \right] = \sum_{z \in \{0,1\}^b} \prod_{j=1}^{b} (p_i)^{z_j} (1 - p_i)^{1-z_j} (E[e^{-\alpha W^{t+j} - \frac{1}{n}}])^{z_j} (E[e^{\alpha \frac{W}{n}}])^{1-z_j} \leq \sum_{z \in \{0,1\}^b} \prod_{j=1}^{b} (1 - p_i) \cdot \left( 1 - \frac{1}{n} + S \alpha^2 \right)^{z_j} \cdot \left( 1 - p_i \cdot \left( 1 + \frac{\alpha}{n} + S \cdot \frac{\alpha^2}{n^2} \right) \right)^{1-z_j}
$$

$$
\leq \left( p_i \cdot \left( 1 - \frac{1}{n} + S \alpha^2 \right) + (1 - p_i) \cdot \left( 1 + \frac{\alpha}{n} + S \cdot \frac{\alpha^2}{n^2} \right) \right)^b
$$

$$
(1 + \frac{1}{n}) \cdot \alpha + p_i \cdot \alpha \cdot S \alpha^2 + (1 - p_i) \cdot \alpha \cdot S \alpha^2 = \left( 1 + \frac{1}{n} - p_i \right) \cdot \alpha + 2 \cdot p_i \cdot \alpha \cdot S \alpha^2 ,
$$

using in (a) that the weights $W$ are independent given $\mathfrak{A}^t$, in (b) Lemma 2.4 and in (c) the binomial theorem. So,

$$
E[\Psi^{i+b} | \mathfrak{A}^t] = \Psi^t_i \cdot e^{(\frac{1}{n} - p_i) \cdot \alpha \cdot b + 2 \cdot p_i \cdot \alpha \cdot S \alpha^2 \cdot b}
$$

$$
\leq \Psi^t_i \cdot \left( 1 + \frac{1}{n} - p_i \right) \cdot \alpha \cdot b + \frac{CS}{n} \cdot \alpha \cdot b + \left( \frac{2CS}{n} \cdot \alpha \cdot b \right)^2
$$

$$
\leq \Psi^t_i \cdot \left( 1 + \frac{1}{n} - p_i \right) \cdot \alpha \cdot b + \frac{CS}{n} \cdot \alpha \cdot b + \frac{2CS}{n} \cdot \alpha \cdot b ,
$$

using in (a) that $1 + y \leq e^y$ for any $y$, in (b) that $e^y \leq 1 + y + y^2$ for $y \leq 1.75$ and that $(\frac{1}{n} - p_i) \cdot \alpha \cdot b + \frac{CS}{n} \cdot \alpha \cdot b \leq \frac{\alpha}{n} \cdot \alpha \cdot b + \frac{2CS}{n} \cdot \alpha \cdot b \leq 2CS \cdot \alpha \cdot b \leq 1$, since $\alpha \leq \frac{n}{12CS} \cdot b$.

We are now ready to apply Corollary 3.2 for $\kappa = b$.

**Theorem 4.2.** Consider any process $p$ satisfying conditions $C_1$ for constant $\delta \in (0, 1)$ and (not necessarily constant) $\epsilon \in (0, 1)$ as well as condition $C_2$ for some constant $C > 1$. Further, consider the batched setting with any $b \geq n$ and a weight distribution satisfying Lemma 2.4 with constant $S \geq 1$. Then there exists a constant $k := k(\delta, C, S) > 0$, such that for any $m \geq 0$ being a multiple of $b$,

$$
\Pr \left[ \max_{i \in [n]} |y^{m}_i| < k \cdot \frac{1}{\epsilon} \cdot \frac{b}{n} \cdot \log n \right] \geq 1 - n^{-2}.
$$

**Remark 4.3.** The same upper bound as in Theorem 4.2 holds also for processes with a time-dependent probability vector $p^t$, as long as for all $t$ being multiples of $b$, the probability vector $p^t$ satisfies $C_1$ and $C_2$ for the same $\epsilon, \delta$ and $C$.

**Remark 4.4.** The same upper bound as in Theorem 4.2 holds also for processes with random tie-breaking and a probability vector $p$ satisfying the preconditions of Lemma 4.1. The reason for this is that (i) averaging probabilities in Eq. (2.1) can only reduce the maximum entry, i.e. max$_{i \in [n]} \hat{p}_i(x^t) \leq \max_{i \in [n]} p_i$, so it satisfies $C_1$ and (ii) moving probability between bins $i, j$ with $x_i = x_j$ (and thus $\Psi^t_i = \Psi^t_j$ and $\Psi^t_j = \Psi^t_j$), implies that the aggregate upper bounds in (4.1) and (4.2) remain the same.
Proof of [Theorem 4.2]. By [Lemma 4.1] the preconditions of [Corollary 3.2] are satisfied for \( K := 5 \cdot C^2 \cdot S^2 \cdot \frac{b}{n} \), \( \kappa := b \) and \( \alpha := \frac{\varepsilon \delta}{8} \). Hence, there exists a constant \( c := c(\delta) > 0 \) such that for any step \( m \geq 0 \) which is a multiple of \( b \),

\[
E[\Gamma^m] \leq \frac{8c}{\delta} \cdot n.
\]

Hence, by Markov’s inequality

\[
\Pr \left[ \Gamma^m \leq \frac{8c}{\delta} \cdot n^3 \right] \geq 1 - n^{-2}.
\]

To prove the claim, note that when \( \{\Gamma^m \leq \frac{8c}{\delta} \cdot n^3\} \) holds, then also,

\[
\max_{i \in [n]} |y_i^m| \leq \frac{1}{\alpha} \cdot \left( \log \left( \frac{8c}{\delta} \right) + 3 \cdot \log n \right) \leq 4 \cdot \frac{8 \cdot 5 \cdot C^2 \cdot S^2 \cdot b}{\varepsilon \delta} \cdot \frac{n \cdot \log n}{n}.
\]

5 An Improved Upper Bound for Batch Sizes \( n \leq b \leq n^3 \)

In this section, we will prove an improved (and tight) upper bound of \( O(\log n) \) on the gap for the weighted batched setting with batch size \( n \leq b \leq n^3 \). We will be assuming processes satisfying conditions \( C_1 \) for constant \( \delta > 0 \) and constant \( \varepsilon > 0 \) and \( C_2 \) with constant \( C > 1 \).

**Theorem 5.1.** Consider any process \( p \) satisfying conditions \( C_1 \) for constant \( \delta \in (0, 1) \) and constant \( \varepsilon \in (0, 1) \) as well as \( C_2 \) for some constant \( C > 1 \). Further, consider the batched setting with any \( n \leq b \leq n^3 \) and a weight distribution satisfying [Lemma 2.4] with constant \( S \geq \max(1, 1/\lambda) \). Then, there is a constant \( \kappa := \kappa(\delta, \varepsilon, C, S) > 0 \), such that for any \( m \geq 0 \) being a multiple of \( b \),

\[
\Pr \left[ \bigcap_{j \in [0, \log n]} \{\tilde{\Gamma}^t + j \cdot b \leq \tilde{c} \cdot n\} \right] \geq 1 - n^{-3}.
\]

**Remark 5.2.** The same gap bound holds also for processes with a time-dependent probability vector \( p^t \), as long as for all \( t \) being a multiple of \( b \), the probability vector \( p^t \) satisfies \( C_1 \) and \( C_2 \) for the same \( \varepsilon, \delta \) and \( C \).

There are two key steps in the proof:

**Step 1:** Similar to the analysis in [24, Theorem 5.3], we will be using two instances of the \( \Gamma \) potential defined in Section 4 for \( \alpha := \frac{\varepsilon \delta}{40 \cdot C^2 \cdot S^2} \cdot \min(\frac{1}{\log n}, \frac{n}{2}) \). The second instance \( \tilde{\Gamma} \) has a smaller smoothing factor \( \tilde{\alpha} := \frac{\varepsilon \delta}{80} \).

\[
\tilde{\Gamma}^t := \sum_{i=1}^{n} \left( e^{\tilde{\alpha} y_i^t} + e^{-\tilde{\alpha} y_i^t} \right).
\]

So, in particular \( \tilde{\Gamma}^t \leq \Gamma^t \) holds. Note that by varying \( b \in [n, n \log n] \), both smoothing factors do not change, but this will not affect the upper bound, as we shall see below.

We will show that w.h.p. \( \tilde{\Gamma} = O(n) \) for \( \log^3 n \) batches.

**Lemma 5.3.** Let \( \tilde{c} := 2 \cdot \frac{8c}{\delta} \) where \( c := c(\delta) > 0 \) is the constant from [Corollary 3.2]. Then, for any \( t \geq 0 \) being a multiple of \( b \),

\[
\Pr \left[ \bigcap_{j \in [0, \log^3 n]} \{\tilde{\Gamma}^t + j \cdot b \leq \tilde{c} \cdot n\} \right] \geq 1 - n^{-3}.
\]
We prove this by conditioning on $\Gamma^t = \text{poly}(n)$ which implies that $\Delta \tilde{\Gamma}^{t+1} = O\left(\frac{n}{b} \cdot n^{1/4}\right)$ (Lemma 5.5 (ii)). This in turn allows us to apply a bounded difference inequality (Theorem A.4) to prove concentration for $\Gamma$. The complete proof is given in Section 5.1.

**Step 2:** We start by exploiting that conditioning on $\{\Gamma^{t+j:b} \leq \tilde{c} \cdot n\}$, the number of bins with load at least $k := \frac{1}{n} \cdot \log(c/\delta) = \Theta(\max(b/n, \log n))$ is at most $\delta n$. We define the following potential function which only takes bins into account that are overloaded by at least $k$ balls:

$$\Lambda^t := \sum_{i: y_i^t \geq k} \Lambda^t_i \cdot e^{\gamma(y_i^t - k)},$$

where $\gamma := \min\left(\frac{1}{2}, \frac{n \log n}{k}\right)$. This means that when $\{\Gamma^{t+1:b} \leq \tilde{c} \cdot n\}$ holds, the probability of allocating to one of these bins is $p_i \leq \frac{1 - \epsilon}{n}$, because of condition $C_2$. Hence, the potential drops in expectation (Lemma 5.9) and this means that w.h.p. $\Lambda_n = \text{poly}(n)$, implying an $O(k + \gamma^{-1} \cdot \log n) = O(b/n + \log n)$ gap.

### 5.1 Step 1: $\tilde{\Gamma}$ is linear w.h.p.

In this subsection, we will prove Lemma 5.3. In Section 5.1.1, we prove some properties of the $\Gamma$ and $\tilde{\Gamma}$ potential and in Section 5.1.2 we combine these to show that w.h.p. $\tilde{\Gamma}^t = O(n)$ for $\log^3 n$ batches.

#### 5.1.1 Preliminaries

For constant $\lambda > 0$ as defined in Section 2.3, we define the following event, for any round $t \geq 0$

$$H^t := \left\{ w^t \leq \frac{15}{\lambda} \cdot \log n \right\},$$

which means that the weight of the ball sampled in round $t$ is $O(\log n)$.

**Lemma 5.4.** For any $b \leq n^3$ and for any $t \geq 0$,

$$\Pr\left[ \bigcap_{s \in [t, t+2b \log^3 n]} H^s \right] \geq 1 - n^{-10}$$

**Proof.** Since $w^t$ is sampled according to $W$ with $\mathbb{E}[e^{\lambda W}] < \infty$, by Lemma A.1

$$\Pr\left[ w^t \geq \frac{15}{\lambda} \cdot \log n \right] \leq n^{-14}.$$  

By taking the union bound over the interval $[t, t + 2b \log^3 n]$ and since $b \leq n^3$ we get the conclusion. \qed

We will now show that when $\Gamma^t = \text{poly}(n)$ and $H^t$ holds, then $\Delta \tilde{\Gamma}^{t+1}$ is small.

**Lemma 5.5.** Let $\tilde{c} := \tilde{c}(\delta) > 0$ be the constant defined in Lemma 5.3. For any $t \geq 0$, where $\Gamma^t \leq 2\tilde{c} \cdot n^{26}$ and $H^t$ holds, then (i) $\Gamma^t \leq n^{5/4}$ and (ii) $|\Gamma^{t+1} - \Gamma^t| \leq \frac{c}{n} \cdot n^{1/4}$. Further, let $\tilde{x}^t$ be the load vector obtained by moving the $t$-th ball of the load vector $x^t$ to some other bin, then (iii) when $H^t$ holds holds, $\Gamma^t(\tilde{x}^t) \leq 2 \cdot \Gamma^t(x^t)$.

**Proof.** For any bin $i \in [n],

$$\Gamma^t \leq 2\tilde{c} \cdot n^{26} \Rightarrow e^{\alpha y_i^t} + e^{-\alpha y_i^t} \leq \tilde{c} \cdot n^{26} \Rightarrow y_i^t \leq \frac{27}{\alpha} \log n \land -y_i^t \leq \frac{27}{\alpha} \log n,$$
where in the second implication we used $\log(2\tilde{c}) + \frac{2\tilde{c}}{\alpha} \log n \leq \frac{27}{\alpha} \log n$, for sufficiently large $n$.

This implies that
\[
\tilde{\Gamma}_i^t \leq e^{\tilde{y}_i t} + e^{-\tilde{y}_i t} \leq 2 \cdot e^{\tilde{\alpha} \cdot \frac{27}{\alpha} \log n} \leq 2 \cdot n^{1/8},
\]
using that $\tilde{\alpha} := \frac{n}{8 \cdot 30}$. Hence, by aggregating, we get the first claim $\Gamma^t = \sum_{i=1}^n \tilde{\Gamma}_i^t \leq 2 \cdot n \cdot n^{1/8} \leq n^{5/4}$.

We now proceed to the second statement. Consider the change for the bin $j \in [n]$ where the ball was allocated. Since $\tilde{\alpha} < \frac{1}{40 \cdot S \cdot \log n}$ and $S > \frac{1}{\lambda}$, we have $\tilde{\alpha} \cdot \frac{15}{\lambda} \cdot \log n \leq 1$ and so by a Taylor estimate, $e^{\alpha \cdot \frac{15}{\lambda} \cdot \log n} \leq 1 + 2 \cdot \tilde{\alpha} \cdot \frac{15}{\lambda} \cdot \log n$. If $j \in [n]$ is an overloaded bin, then
\[
|\Delta \tilde{\Gamma}_j^t| \leq \tilde{\Gamma}_j^t \cdot e^{\tilde{\alpha} \cdot \frac{15}{\lambda} \cdot \log n} - \tilde{\Gamma}_j^t \leq \tilde{\Gamma}_j^t \cdot \left(1 + \tilde{\alpha} \cdot \frac{30}{\lambda} \cdot \log n\right) - \tilde{\Gamma}_j^t
= \tilde{\Gamma}_j^t \cdot \tilde{\alpha} \cdot \frac{30}{\lambda} \cdot \log n \leq \frac{n}{b} \cdot n^{1/8} \cdot \log n,
\]
using Eq. (5.1) and $\tilde{\alpha} \leq \frac{\epsilon \delta}{40 \cdot c^2 \cdot S \cdot \tau \cdot n^2}$. Similarly, if $j$ is underloaded, then
\[
|\Delta \tilde{\Gamma}_j^t| \leq \tilde{\Gamma}_j^t - \tilde{\Gamma}_j^t \cdot e^{-\tilde{\alpha} \cdot \frac{15}{\lambda} \cdot \log n} \leq \tilde{\Gamma}_j^t - \tilde{\Gamma}_j^t \cdot \left(1 - \tilde{\alpha} \cdot \frac{30}{\lambda} \cdot \log n\right)
= \tilde{\Gamma}_j^t \cdot \tilde{\alpha} \cdot \frac{30}{\lambda} \cdot \log n \leq \frac{n}{b} \cdot n^{1/8} \cdot \log n.
\]

The rest of the bins’ contributions change due to the change in the average load. In particular, for any overloaded bin $i \in [n] \setminus \{j\}$,
\[
|\Delta \tilde{\Gamma}_i^t| \leq \tilde{\Gamma}_i^t \cdot e^{\tilde{\alpha} \cdot \frac{15}{\lambda} \cdot \log n} - \tilde{\Gamma}_i^t \leq \tilde{\Gamma}_i^t \cdot \left(1 + 2 \cdot \tilde{\alpha} \cdot \frac{15}{\lambda} \cdot \log n\right) - \tilde{\Gamma}_i^t
= \tilde{\Gamma}_i^t \cdot \tilde{\alpha} \cdot \frac{30}{\lambda} \cdot \log n \leq \frac{1}{b} \cdot \log n \cdot n^{1/8}.
\]
Similarly, for an underloaded bin $i \in [n] \setminus \{j\}$,
\[
|\Delta \tilde{\Gamma}_i^t| \leq \tilde{\Gamma}_i^t - \tilde{\Gamma}_i^t \cdot e^{-\tilde{\alpha} \cdot \frac{15}{\lambda} \cdot \log n} \leq \tilde{\Gamma}_i^t - \tilde{\Gamma}_i^t \cdot \left(1 - 2 \cdot \tilde{\alpha} \cdot \frac{15}{\lambda} \cdot \log n\right)
= \tilde{\Gamma}_i^t \cdot \tilde{\alpha} \cdot \frac{30}{\lambda} \cdot \log n \leq \frac{1}{b} \cdot \log n \cdot n^{1/8}.
\]
Hence, aggregating over all bins
\[
|\Delta \tilde{\Gamma}^t+1| \leq |\Delta \Gamma_j^t+1| + \sum_{i \in [n] \setminus \{j\}} |\Delta \Gamma_i^t+1| \leq 2 \cdot \frac{n}{b} \cdot n^{1/8} \cdot \log n + \frac{1}{b} \cdot \log n \cdot n^{1/8} \leq \frac{n}{b} \cdot n^{1/4},
\]
for sufficiently large $n$.

For statement $(iii)$, let $i, j \in [n]$ be the differing bins between $x^t$ and $\tilde{x}^t$. Then since $\mathcal{H}^t$ holds, $w^t \leq \frac{15}{\lambda} \cdot \log n$, so
\[
\Gamma_i(\tilde{x}^t) \leq e^{\alpha w^t} \cdot \Gamma_i^t(x^t) \leq 2 \cdot \Gamma_i^t(x^t),
\]
since $\alpha < \frac{1}{40 \cdot S \cdot \log n}$ and $S > 1/\lambda$. Similarly, for $j$,
\[
\Gamma_j(\tilde{x}^t) \leq e^{\alpha w^t} \cdot \Gamma_j^t(x^t) \leq 2 \cdot \Gamma_j^t(x^t),
\]
Hence,
\[
\Gamma^t(\tilde{x}^t) = \sum_{k=1}^n \Gamma_k^t(\tilde{x}^t) \leq \sum_{k=1}^n 2 \cdot \Gamma_k^t(x^t) = 2 \cdot \Gamma^t(x^t).
\]
\[\square\]
Next, we will show that $\mathbb{E}[\hat{\Gamma}] = \mathcal{O}(n)$ and that when $\hat{\Gamma}$ is sufficiently large, it drops in expectation over the next batch.

**Lemma 5.6.** Let $\tilde{c} := 2 \cdot \frac{8c}{\delta}$ where $c := c(\delta) > 0$ is the constant from Corollary 3.2. Then, for any step $t \geq 0$ being a multiple of $b$,

$$(i) \quad \mathbb{E}[\hat{\Gamma}^t] \leq \frac{\tilde{c}}{2} \cdot n,$$

and

$$(ii) \quad \mathbb{E}[\Gamma^t] \leq \frac{\tilde{c}}{2} \cdot n.$$ 

Further, there exists a constant $\tilde{c}_1 := \tilde{c}_1(\varepsilon, \delta) > 0$ such that

$$(iii) \quad \mathbb{E}\left[\hat{\Gamma}^{t+b} \mid \hat{\Gamma}^t \geq \tilde{c} \cdot n\right] \leq \left(1 - \frac{\tilde{c}_1}{\log n}\right) \cdot \hat{\Gamma}^t,$$

and

$$(iv) \quad \mathbb{E}\left[\hat{\Gamma}^{t+b} \mid \hat{\Gamma}^t \leq \tilde{c} \cdot n\right] \leq \tilde{c} \cdot n - \frac{n}{\log^2 n}.$$ 

**Proof.** The first two statements follow immediately by Lemma 4.1 and Corollary 3.2 by setting $\tilde{c} := 16c/\delta$, since $c := c(\delta) > 0$.

Also, using Lemma 4.1 and Corollary 3.2 for $\tilde{\alpha}$, we get that for any $t \geq 0$,

$$\mathbb{E}\left[\hat{\Gamma}^{t+b} \mid \hat{\Gamma}^t \right] \leq \hat{\Gamma}^t \cdot \left(1 - \frac{\varepsilon \delta}{8} \cdot \frac{b}{n} \cdot \tilde{\alpha}\right) + c \cdot b \cdot \varepsilon \cdot \tilde{\alpha}. \quad (5.2)$$

Let $\tilde{c}_3 := \frac{1}{2} \cdot \frac{\varepsilon \delta}{8} \cdot \frac{b}{n} \cdot \tilde{\alpha} \geq \tilde{c}_1 / \log n$, for some constant $\tilde{c}_1 > 0$ since $\tilde{\alpha} = \Theta(\min(n/b, 1/\log n))$ and $\varepsilon$ is constant. When $\hat{\Gamma}^t \geq \tilde{c} \cdot n$, then Eq. (5.2) yields,

$$\mathbb{E}\left[\hat{\Gamma}^{t+b} \mid \hat{\Gamma}^t \geq \tilde{c} \cdot n\right] \leq \hat{\Gamma}^t \cdot \left(1 - 2 \cdot \tilde{c}_3 \right) + c \cdot b \cdot \varepsilon \cdot \tilde{\alpha}$$

$$\leq \hat{\Gamma}^t - \tilde{c}_3 \cdot \hat{\Gamma}^t + \left(c \cdot b \cdot \varepsilon \cdot \tilde{\alpha} - \tilde{c}_3 \cdot \hat{\Gamma}^t\right)$$

$$\leq \hat{\Gamma}^t - \tilde{c}_3 \cdot \hat{\Gamma}^t + \left(c \cdot b \cdot \varepsilon \cdot \tilde{\alpha} - \frac{1}{2} \cdot \frac{\varepsilon \delta}{8} \cdot \frac{b}{n} \cdot \tilde{\alpha} \cdot \frac{16c}{\delta} \cdot n\right)$$

$$\leq \left(1 - \frac{\tilde{c}_1}{\log n}\right) \cdot \hat{\Gamma}^t.$$

Similarly, when $\Gamma^t < \tilde{c} \cdot n$, Eq. (5.2) yields,

$$\mathbb{E}\left[\hat{\Gamma}^{t+b} \mid \hat{\Gamma}^t < \tilde{c} \cdot n\right] \leq \tilde{c} \cdot n \cdot \left(1 - 2 \cdot \tilde{c}_3 \right) + c \cdot b \cdot \varepsilon \cdot \tilde{\alpha}$$

$$= \tilde{c} \cdot n - c \cdot \tilde{c}_3 \cdot n + \left(c \cdot b \cdot \varepsilon \cdot \tilde{\alpha} - \tilde{c}_3 \cdot n\right)$$

$$\leq \tilde{c} \cdot n - \frac{c \cdot \tilde{c}_1}{\log n} \leq \tilde{c} \cdot n - \frac{n}{\log^2 n}. \quad \square$$

In the next lemma, we show that w.h.p. $\Gamma$ is poly($n$) for every step in an interval of length $2b \log^3 n$.

**Lemma 5.7.** Let $\tilde{c} := 2 \cdot \frac{8c}{\delta}$ be the constant defined in Lemma 5.6. For any $n \leq b \leq n^3$ and for any $t \geq 0$ being a multiple of $b$,

$$\Pr \left[ \bigcap_{s \in [t, t + 2b \log^3 n]} \{\Gamma^s \leq \tilde{c} \cdot n^20\} \right] \geq 1 - n^{-10}.$$ 

26
Proof. Using [Lemma 5.6 (i)], Markov’s inequality and the union bound, we have for any $t \geq 0$,
\[
\Pr \left[ \bigcap_{s \in [0, 2 \log^3 n]} \left\{ r^{t+s-b} \leq \tilde{c} \cdot n^{12} \right\} \right] \geq 1 - \frac{2 \log^3 n}{n^{11}}. \tag{5.3}
\]
Given that $\Gamma^{t+s-b} \leq \tilde{c} \cdot n^{12}$, we will upper bound $\Gamma^{t+s-b+r}$ for any $r \in [0, b]$. To this end, we will upper bound for each bin $i \in [n]$ the terms $\Phi_i^{t+s-b+r}$ and $\Psi_i^{t+s-b+r}$ separately. Proceeding using Eq. (4.3) in Lemma 4.1
\[
\Pr \left[ \Phi_i^{t+s-b} \right] \leq \Phi_i^{t+s-b} \cdot \left( 1 + \left( \frac{1}{n} - p_i \right) \cdot \alpha + 2 \cdot p_i \cdot S \cdot \alpha^2 \right)^r
\]
\[
\leq \Phi_i^{t+s-b} \cdot \left( 1 + \frac{C \alpha}{n} + 2 \cdot \frac{C}{\alpha} \cdot S \cdot \alpha^2 \right)^r
\]
\[
(a) \leq \Phi_i^{t+s-b} \cdot \left( 1 + \frac{2C \alpha}{n} \right)^r
\]
\[
\leq \Phi_i^{t+s-b} \cdot e^{2\alpha C \cdot \frac{r}{n}} \leq \Phi_i^{t+s-b} \cdot e^{2\alpha C \cdot \frac{b}{n}} \leq 2 \cdot \Phi_i^{t+s-b},
\]
using in (a) that $\alpha \leq \frac{\epsilon_0}{40 C^{\alpha^2} S^2} \leq \frac{2}{5}$ and in (b) that $\alpha \leq \frac{\epsilon_0}{40 C^{\alpha^2} S^2} \cdot \frac{n}{\alpha} \leq \frac{1}{40 C} \cdot \frac{n}{\alpha}$. Similarly, using Eq. (4.5) in Lemma 4.1
\[
\Pr \left[ \Psi_i^{t+s-b} \right] \leq \Psi_i^{t+s-b} \cdot \left( 1 + \left( \frac{1}{n} - p_i \right) \cdot \alpha + 2 \cdot p_i \cdot S \cdot \alpha^2 \right)^r
\]
\[
\leq \Psi_i^{t+s-b} \cdot \left( 1 + \frac{C \alpha}{n} + 2 \cdot \frac{C}{\alpha} \cdot S \cdot \alpha^2 \right)^r
\]
\[
(a) \leq \Psi_i^{t+s-b} \cdot \left( 1 + \frac{2C \alpha}{n} \right)^r
\]
\[
\leq \Psi_i^{t+s-b} \cdot e^{2\alpha C \cdot \frac{r}{n}} \leq \Psi_i^{t+s-b} \cdot e^{2\alpha C \cdot \frac{b}{n}} \leq 2 \cdot \Psi_i^{t+s-b},
\]
using in (a) that $\alpha \leq \frac{\epsilon_0}{40 C^{\alpha^2} S^2} \leq \frac{2}{5}$ and in (b) that $\alpha \leq \frac{\epsilon_0}{40 C^{\alpha^2} S^2} \cdot \frac{n}{\alpha} \leq \frac{1}{40 C} \cdot \frac{n}{\alpha}$. Hence, aggregating over the bins,
\[
\Pr \left[ \Gamma^{t+s-b+r} \right] \leq 2 \cdot \Gamma^{t+s-b}.
\]
Applying Markov’s inequality, for any $r \in [0, b)$,
\[
\Pr \left[ \Gamma^{t+s-b+r} \leq n^{14} \cdot \Gamma^{t+s-b} \right] \geq 1 - 2 \cdot n^{-14}.
\]
Hence, by a union bound over the $2b \cdot \log^3 n \leq 2 \cdot n^3 \cdot \log^3 n$ possible rounds for $s \in [0, 2 \log^3 n]$ and $r \in [0, b]$,
\[
\Pr \left[ \bigcap_{s \in [0, 2 \log^3 n]} \bigcap_{r \in [0, b]} \left\{ \Gamma^{t+s-b+r} \leq n^{14} \cdot \Gamma^{t+s-b} \right\} \right] \geq 1 - 2 \cdot n^{-14} \cdot 2b \log^3 n \geq 1 - \frac{1}{2} \cdot n^{-10}. \tag{5.4}
\]
Finally, taking the union bound of Eq. (5.3) and Eq. (5.4) we conclude
\[
\Pr \left[ \bigcap_{s \in [t, t+2b \log^3 n]} \left\{ \Gamma^s \leq \tilde{c} \cdot n^{26} \right\} \right] \geq \Pr \left[ \bigcap_{r \in [0, b]} \bigcap_{s \in [0, 2 \log^3 n]} \left\{ \Gamma^{t+s-b+r} \leq n^{14} \cdot \Gamma^{t+s-b} \right\} \cap \bigcap_{s \in [0, 2 \log^3 n]} \left\{ \Gamma^{t+s-b} \leq \tilde{c} \cdot n^{12} \right\} \right] \geq 1 - \frac{1}{2} \cdot n^{-10} - \frac{2 \log^3 n}{n^{11}} \geq 1 - n^{-10}. \square
We will now show that w.h.p. there is a step where the exponential potential $\tilde{\Gamma}$ becomes $O(n)$.

**Lemma 5.8.** Let $\tilde{c} := 2 \cdot \frac{8c}{9}$ be the constant defined in Lemma 5.6. For any $t \geq 0$ being a multiple of $b$,

$$\Pr \left[ \bigcup_{s \in [0,b \log^3 n]} \{ \tilde{\Gamma}^{t+s} \leq \tilde{c} \cdot n \} \right] \geq 1 - 2 \cdot n^{-8}.$$  

**Proof.** By Lemma 5.6 (ii), using Markov’s inequality at time $t$ being a multiple of $b$, we have

$$\Pr \left[ \tilde{\Gamma}^t \leq \tilde{c} \cdot n^9 \right] \geq 1 - n^{-8}. \quad (5.5)$$

Assuming $\tilde{\Gamma}^t \leq \tilde{c} \cdot n^9$ and by Lemma 5.6 (iii) if at some step $\tilde{\Gamma}^r > \tilde{c} \cdot n$, then

$$E \left[ \tilde{\Gamma}^{r+1} \mid \tilde{\Gamma}^r, \tilde{\Gamma}^r > \tilde{c} \cdot n \right] \leq \left( 1 - \frac{\tilde{c}_1}{\log n} \right) \cdot \tilde{\Gamma}^r,$$

where $\tilde{c}_1 > 0$ is some constant. For any $r \in [0, \log^3 n]$, we define the “killed” potential function,

$$\hat{\Gamma}^{t+r-b} := \tilde{\Gamma}^{t+r-b} \cdot 1_{\{r \in [0,r) \cap (\tilde{\Gamma}^{t+r} > \tilde{c} \cdot n)\}}.$$

This potential satisfies the drop inequality of Lemma 5.6 without any condition on the value of $\tilde{\Gamma}^r$, that is,

$$E \left[ \hat{\Gamma}^{t+(r+1)-b} \mid \hat{\Gamma}^{t+r-b} \right] \leq \left( 1 - \frac{\tilde{c}_1}{\log n} \right) \cdot \hat{\Gamma}^{t+r-b}.$$

Inductively applying this for $\log^3 n$ batches, and since $\tilde{c}_1 := \tilde{c}(\delta, \delta) > 0$ is a constant,

$$E \left[ \hat{\Gamma}^{t+(\log^3 n)-b} \mid \hat{\Gamma}^t \right] \leq \left( 1 - \frac{\tilde{c}_1}{\log n} \right)^{\log^3 n} \cdot \hat{\Gamma}^t \leq e^{-\tilde{c}_1 \cdot \log^3 n} \cdot \tilde{c} \cdot n^9 < n^{-7}.$$

So by Markov’s inequality,

$$\Pr \left[ \hat{\Gamma}^{t+(\log^3 n)-b} \geq n \mid \hat{\Gamma}^t \leq \tilde{c} \cdot n^9 \right] \leq n^{-8}$$

By union bound with Eq. (5.5)

$$\Pr \left[ \hat{\Gamma}^{t+(\log^3 n)-b} \geq n \right] = \Pr \left[ \hat{\Gamma}^{t+(\log^3 n)-b} \geq n \mid \hat{\Gamma}^t \leq \tilde{c} \cdot n^9 \right] \cdot \Pr \left[ \hat{\Gamma}^t \leq \tilde{c} \cdot n^9 \right] + \Pr \left[ \hat{\Gamma}^t > \tilde{c} \cdot n^9 \right]$$

$$< n^{-8} + n^{-8} = 2 \cdot n^{-8}.$$

Due to the definition of $\tilde{\Gamma}$, at any step $t \geq 0$, deterministically $\tilde{\Gamma}^t \geq 2n$. So, we conclude that w.p. at least $1 - 2 \cdot n^{-8}$, there must be at least one time step $r \in [0, \log^3 n]$, with $\tilde{\Gamma}^{t+r-b} = 0$ and so $\tilde{\Gamma}^{t+s+b} \leq \tilde{c} \cdot n$ for some $s \in [0, \log^3 n]$. \hfill \square

### 5.1.2 Completing the Proof of Lemma 5.3

We are now ready to prove Lemma 5.3 using a method of bounded differences with a bad event Theorem A.4 ([21, Theorem 3.3]).

**Lemma 5.3.** Let $\tilde{c} := 2 \cdot \frac{8c}{9}$ where $c := c(\delta) > 0$ is the constant from Corollary 3.2. Then, for any $t \geq 0$ being a multiple of $b$,

$$\Pr \left[ \bigcap_{j \in [0, \log^3 n]} \{ \tilde{\Gamma}^{t+j} \leq \tilde{c} \cdot n \} \right] \geq 1 - n^{-3}.$$
Proof. Our starting point is to apply Lemma 5.8 which proves that there is at least one time step \( t + \rho \cdot b \in [\ell - b \log^3 n, t] \) with \( \rho \in [-\log^2 n, 0] \) such that the potential \( \hat{\Gamma} \) is small,

\[
\Pr \left[ \bigcup_{\rho \in [-\log^2 n, 0]} \left\{ \hat{\Gamma}^{t+\rho b} \leq \tilde{c} \cdot n \right\} \right] \geq 1 - 2 \cdot n^{-8}. \tag{5.6}
\]

Note that if \( t < b \log^3 n \), then deterministically \( \hat{\Gamma}^0 = 2n \leq \tilde{c} \cdot n \) (which corresponds to \( \rho = -t/b \)).

We are now going to apply the concentration inequality Theorem A.4 to each of the batches starting at \( t + \rho \cdot b, \ldots, t + (\log^3 n) \cdot b \) and show that the potential remains \( \leq \tilde{c} \cdot n \) at the end of each batch. In particular, we will show that for any \( \tilde{r} \in [\rho, \log^3 n] \), for \( r = t + b \cdot \tilde{r} \),

\[
\Pr \left[ \hat{\Gamma}^{r+b} > \tilde{c} \cdot n \mid \hat{\Gamma}^r \leq \tilde{c} \cdot n \right] \leq 3 \cdot n^{-4}.
\]

We will show this by applying Theorem A.4 for all steps of the batch \( [r, r+b] \). We define the good event

\[
\mathcal{G}_r := \mathcal{G}_r^{r+b} := \bigcap_{s \in [r,r+b]} \left( \{ \Gamma^s \leq \tilde{c} \cdot n^{26} \} \cap \mathcal{H}^s \right),
\]

and \( \mathcal{B}_r := (\mathcal{G}_r)^c \) the bad event. Using a union bound over Lemma 5.4 and Lemma 5.7

\[
\Pr \left[ \bigcap_{s \in [t-b \log^3 n,t+b \log^3 n]} \left( \{ \Gamma^s \leq \tilde{c} \cdot n^{26} \} \cap \mathcal{H}^s \right) \right] \geq 1 - 2n^{-10}. \tag{5.7}
\]

Consider any \( u \in [r, r+b] \). Further, we define the slightly weaker good event, \( \tilde{\mathcal{G}}_r^u := \bigcap_{s \in [r,u]} \left( \{ \Gamma^s \leq 2\tilde{c} \cdot n^{26} \} \cap \mathcal{H}^s \right) \) and the “killed” potential,

\[
\hat{\Gamma}_r^u := \mathbf{1}_{\tilde{\mathcal{G}}_r^u} \cdot \hat{\Gamma}_r^u.
\]

We will show that the sequence \( \hat{\Gamma}_r^r, \ldots, \hat{\Gamma}_r^{r+b} \) is strongly difference-bounded by \( (n^{5/4}, n^{5/4}, 2, n^{-10}) \) (Definition A.3).

Let \( \omega \in [n]^b \) be an allocation vector encoding the allocations made in \( [r, r+b] \). Let \( \omega' \) be an allocating vector resulting from \( \omega \) by changing one arbitrary allocation. It follows that,

\[
|\hat{\Gamma}_r^{r+b}(\omega) - \hat{\Gamma}_r^{r+b}(\omega')| \leq \max_\omega \hat{\Gamma}_r^{r+b}(\tilde{\omega}) - \min_\tilde{\omega} \hat{\Gamma}_r^{r+b}(\tilde{\omega}) \\
\leq \max_{\tilde{\omega} \in \tilde{\mathcal{G}}_r^{r+b}} \hat{\Gamma}_r^{r+b}(\tilde{\omega}) - 0 \leq n^{5/4},
\]

where in the last inequality we used Lemma 5.5 (i) that for any \( \tilde{\omega} \in \tilde{\mathcal{G}}_r^{r+b} \), we have \( \hat{\Gamma}_r^{r+b}(\tilde{\omega}) \leq \hat{\Gamma}_r^{r+b}(\omega) \leq n^{5/4} \).

We will now derive a refined bound by additionally assuming that \( \omega \in \mathcal{G}_r \). Then, for any \( u \in [r, r+b] \),

\[
\Gamma^{r+u}(\omega') \leq 2 \cdot \Gamma^{r+u}(\omega) \leq 2\tilde{c} \cdot n^{26},
\]

where the first inequality is by Lemma 5.5 (iii). Hence \( \omega' \in \tilde{\mathcal{G}}_r^{r+b} \), so \( \mathbf{1}_{\tilde{\mathcal{G}}_r^{r+b}}(\omega') = 1 \) and \( \hat{\Gamma}_r^{r+b}(\omega') = \hat{\Gamma}_r^{r+b}(\omega) \). Similarly, for \( \omega \in \mathcal{G}_r \subseteq \tilde{\mathcal{G}}_r^{r+b} \), we have \( \hat{\Gamma}_r^{r+b}(\omega) = \hat{\Gamma}_r^{r+b}(\omega) \) and by Lemma 5.5 (ii),

\[
|\hat{\Gamma}_r^{r+b}(\omega) - \hat{\Gamma}_r^{r+b}(\omega')| = |\hat{\Gamma}_r^{r+b}(\omega) - \hat{\Gamma}_r^{r+b}(\omega')| \leq \frac{n}{b} \cdot n^{1/4}.
\]

29
Within a single batch all allocations are independent, so we apply \( \text{Theorem A.4} \) choosing \( \gamma_k := \frac{1}{b} \) and \( N := b \), which states that for any \( \lambda > 0 \) and \( \mu := E[\hat{\gamma}^{r+b}] = \mu + \lambda \mid \hat{\gamma}^r, \Gamma^r \leq \hat{\gamma} \cdot n \),

\[
\Pr \left[ \hat{\gamma}^{r+b} > \mu + \lambda \mid \hat{\gamma}^r, \Gamma^r \leq \hat{\gamma} \cdot n \right] \leq \exp \left( -\frac{\lambda^2}{2 \cdot \sum_{k=1}^{b} (\frac{n}{b} \cdot \frac{1}{n^{1/4}} + \frac{n^{5/4}}{b} \cdot \frac{1}{b} \cdot \frac{1}{2})^2} \right) + 2 \cdot n^{-10} \cdot b.
\]

By \( \text{Lemma 5.6 (iv)} \), we have \( \mu \leq E[\hat{\gamma}^{r+b}] \mid \hat{\gamma}^r < \hat{\gamma} \cdot n \] \( \leq \exp \left( -\frac{n^2/\log^4 n}{2 \cdot b \cdot (2 \cdot \frac{b}{n^{1/4}})^2} \right) + 2n^{-10} \cdot b^2 \leq \exp \left( -\frac{b}{8 \cdot \log^4 n \cdot n^{1/2}} \right) + 2n^{-10} \cdot n^6 \leq 3 \cdot n^{-4}.

Hence, for \( \lambda := n/\log^2 n \), since \( n \leq b \leq n^3 \), we have

\[
\Pr \left[ \hat{\gamma}^{r+b} > \hat{\gamma} \cdot n \mid \hat{\gamma}^r, \Gamma^r \leq \hat{\gamma} \cdot n \right] \leq \exp \left( -\frac{n^2/\log^4 n}{2 \cdot b \cdot (2 \cdot \frac{n}{n^{1/4}})^2} \right) + 2n^{-10} \cdot b^2 \leq \exp \left( -\frac{b}{8 \cdot \log^4 n \cdot n^{1/2}} \right) + 2n^{-10} \cdot n^6 \leq 3 \cdot n^{-4}.
\]

(5.8)

By union bound of \( \text{Eq. (5.6)} \) and \( \text{Eq. (5.7)} \),

\[
\Pr \left[ \bigcup_{\rho \in [-\log^3 n]} \mathcal{K}_{\rho}^{log^3 n} \right] \geq \Pr \left[ \mathcal{G}^{log^3 n} \cap \bigcup_{\rho \in [-\log^3 n, 0]} \left\{ \hat{\gamma}^{t+\rho b} \leq \hat{\gamma} \cdot n \right\} \right] 
\geq 1 - 2 \cdot n^{-8} - 2 \cdot n^{-10} \geq 1 - 3 \cdot n^{-8}.
\]

(5.9)

Let \( A := \bigcap_{\rho \in [0,\log^3 n]} \left\{ \hat{\gamma}^{t+\rho b} \leq \hat{\gamma} \cdot n \right\} \) and \( A_{\rho} := \bigcap_{\rho \in [0,\log^3 n]} \left\{ \hat{\gamma}^{t+\rho b} \cdot 1_{\mathcal{K}_{\rho}^{\log^3 n}} \leq \hat{\gamma} \cdot n \right\} \). Then,

\[
\Pr \left[ A_{\rho} \mid \hat{\gamma}^{t+\rho b} \leq \hat{\gamma} \cdot n \right] \geq \prod_{\rho \in [0,\log^3 n-1]} \Pr \left[ \bigcap_{\rho \in [0,\log^3 n-1]} \left\{ 1_{\mathcal{K}_{\rho}^{\log^3 n}} \cdot \hat{\gamma}^{t+\rho b} \leq \hat{\gamma} \cdot n \right\} \right] 
\geq \prod_{\rho \in [0,\log^3 n-1]} \left( 1 - 3n^{-4} \right)^2 \log^3 n \geq 1 - 6 \cdot n^{-4} \cdot \log^3 n
\]

where in the last inequality we have used \( \text{Eq. (5.8)} \) and the fact \( \rho \geq -\log^3 n \). So,

\[
\Pr \left[ A_{\rho} \right] = \Pr \left[ A_{\rho} \mid \hat{\gamma}^{t+\rho b} \leq \hat{\gamma} \cdot n \right] \cdot \Pr \left[ \hat{\gamma}^{t+\rho b} \leq \hat{\gamma} \cdot n \right] + 1 \cdot \Pr \left[ -\left\{ \hat{\gamma}^{t+\rho b} \leq \hat{\gamma} \cdot n \right\} \right]
\geq 1 - 6 \cdot n^{-4} \cdot \log^3 n
\]

(5.10)

Note that for any \( \rho \in [-\log^3 n, 0] \), we have that \( A_{\rho} \cap \mathcal{K}_{\rho}^{log^3 n} \subseteq A \). Hence we conclude by the union bound of \( \text{Eq. (5.9)} \) and \( \text{Eq. (5.10)} \) that

\[
\Pr \left[ A \right] \geq \Pr \left[ \bigcup_{\rho \in [-\log^3 n, 0]} \mathcal{K}_{\rho}^{log^3 n} \cap \bigcap_{\rho \in [-\log^3 n, 0]} A_{\rho} \right] \geq 1 - 3 \cdot n^{-8} - 6 \cdot n^{-4} \cdot \log^6 n \geq 1 - n^{-3}.
\]
5.2 Step 2: Completing the Proof of Theorem 5.1

Recall the definition of the $\Lambda$ potential function,

$$\Lambda^t := \sum_{i:y_i^t \geq k} \Lambda_i^t \cdot e^{\gamma(y_i^t - k)},$$

where $\gamma := \min \left( \frac{c}{\log n}, \frac{n \log n}{2} \right)$ and $k := \frac{1}{n} \cdot \log (\tilde{c}/\delta) = \Theta(\max(b/n, \log n))$.

We will now show that when $\tilde{\Gamma}^t = \mathcal{O}(n)$, the stronger potential function $\Lambda^t$ drops in expectation. This will allow us to prove that $\Lambda^m = \text{poly}(n)$ and deduce that w.h.p $\text{Gap}(m) = \mathcal{O}(b/n + \log n)$.

**Lemma 5.9.** Let $\tilde{c} := 2 \cdot \frac{3c}{n}$ be the constant defined in **Lemma 5.6**. For any $t \geq 0$ being a multiple of $b$,

$$\mathbf{E} \left[ \Lambda^{t+b} | \tilde{S}^t, \tilde{\Gamma}^t \leq \tilde{c} \cdot n \right] \leq \Lambda_i^t \cdot \left( 1 + \left( p_i - \frac{1}{n} \right) \cdot \gamma + 2 \cdot p_i \cdot S \gamma^2 \right)^b.$$

*Proof.* When $\{\tilde{\Gamma}^t \leq \tilde{c} \cdot n\}$ holds, the number of bins with load $y_i^t \geq k$ is at most

$$\tilde{c} \cdot n \cdot e^{-\tilde{c}k} = \tilde{c} \cdot n \cdot e^{-\log(\tilde{c}/\delta)} = \delta \cdot n.$$

For any bin $i$ with $y_i^t \geq k$, we get as in **Eq. (4.3)**

$$\mathbf{E} \left[ \Lambda_i^{t+b} | \tilde{S}^t, \tilde{\Gamma}^t \leq \tilde{c} \cdot n \right] \leq \Lambda_i^t \cdot \left( 1 - \frac{\gamma}{n} + 2 \cdot C \cdot S \gamma^2 \right)^b \leq \Lambda_i^t \cdot \left( 1 - \frac{\gamma}{2n} \right)^b \leq \Lambda_i^t \cdot e^{-\frac{\gamma}{2n} \cdot b},$$

using in (a) that $p_i \leq C/n$, in (b) that $\gamma \leq \frac{2c}{\log n}$ and in (c) that $1 + z \leq e^z$ for any $z$. For the rest of the bins with $i > \delta n$,

$$\mathbf{E} \left[ \Lambda_i^{t+b} | \tilde{S}^t \right] \leq \Lambda_i^t \cdot \left( 1 + \left( p_i - \frac{1}{n} \right) \cdot \gamma + 2 \cdot p_i \cdot S \gamma^2 \right)^b \leq \Lambda_i^t \cdot \left( 1 + \frac{C}{n} \gamma - \frac{1}{n} \cdot \gamma + 2 \cdot \frac{C}{n} \cdot S \gamma^2 \right)^b \leq \Lambda_i^t \cdot \left( 1 + \frac{C}{n} \gamma \right)^b \leq \left( 1 + \frac{C}{n} \right)^b \leq e^{\frac{C}{n} \cdot b},$$

using in (a) that $p_i \leq C/n$, in (b) that $\gamma \leq \frac{2c}{\log n}$, in (c) that $\Lambda_i^t \leq 1$ and in (d) that $1 + z \leq e^z$ for any $z$.

Aggregating the contributions of all bins,

$$\mathbf{E} \left[ \Lambda^{t+b} | \tilde{S}^t, \tilde{\Gamma}^t \leq \tilde{c} \cdot n \right] \leq \sum_{i:y_i^t \geq k} \Lambda_i^t \cdot e^{-\frac{\gamma}{2n} \cdot b} + \sum_{i:y_i^t < k} e^{\frac{C}{n} \cdot b} \leq \Lambda^t \cdot e^{-\frac{\gamma}{2n} \cdot b} + n \cdot e^{\frac{C}{n} \cdot b}.$$
Theorem 5.1. Consider any process \( p \) satisfying conditions \( C_1 \) for constant \( \delta \in (0,1) \) and constant \( \varepsilon \in (0,1) \) as well as \( C_2 \) for some constant \( C > 1 \). Further, consider the batched setting with any \( n \leq b \leq n^3 \) and a weight distribution satisfying Lemma 2.4 with constant \( S \geq \max(1,1/\lambda) \). Then, there is a constant \( \kappa := \kappa(\delta, \varepsilon, C, S) > 0 \), such that for any \( m \geq 0 \) being a multiple of \( b \),

\[
\Pr \left[ \gamma^m_1 \leq \kappa \cdot \left( \frac{b}{n} + \log n \right) \right] \geq 1 - n^{-2}.
\]

Proof. Consider first the case when \( m \geq b \cdot \log^3 n \). Let \( t_0 = m - b \cdot \log^3 n \). Let \( \mathcal{E}' := \{ \tilde{\gamma}^t \leq \tilde{c} \cdot n \} \). Then using Lemma 5.3,

\[
\Pr \left[ \bigcap_{j \in [0, \log^3 n]} \mathcal{E}^{t_0 + j \cdot b} \right] \geq 1 - n^{-3}. \tag{5.11}
\]

We define the killed potential \( \tilde{\Lambda} \), with \( \tilde{\Lambda}^{t_0} := \Lambda^{t_0} \) and for \( j > 0 \),

\[
\tilde{\Lambda}^{t_0 + j \cdot b} := \mathbb{1}_{\tilde{\gamma}^{t_0 + j \cdot b} \in \mathcal{E}^{t_0 + j \cdot b}} \cdot \tilde{\Lambda}^{t_0 + j \cdot b}.
\]

By Lemma 5.9 we have

\[
\mathbb{E} \left[ \tilde{\Lambda}^{t_0 + (j+1) \cdot b} \mid \mathcal{F}^{t_0 + j \cdot b} \right] \leq \tilde{\Lambda}^{t_0 + j \cdot b} \cdot e^{-\frac{\varepsilon}{\log^3 n} b} + n \cdot e^{\frac{C_2}{\log^3 n} \cdot b} \cdot \tilde{\Lambda}^{t_0 + (j+1) \cdot b}. \tag{5.12}
\]

Assuming \( \mathcal{E}^{t_0} \) holds, we have

\[
y_0^{t_0} \leq \frac{1}{\alpha} \cdot (\log \tilde{c} + \log n) \leq \frac{2}{\alpha} \cdot \log n,
\]

for sufficiently large \( n \). Hence for some constant \( \kappa_1 > 0 \),

\[
\tilde{\Lambda}^{t_0} \leq n \cdot e^{\gamma y_1^{t_0}} \leq e^{\kappa_1 \log^2 n}.
\]

Applying Lemma A.6 to Eq. (5.12) with \( a := e^{-\frac{\varepsilon}{\log^3 n} \cdot b} \) and \( b := n \cdot e^{\frac{C_2}{\log^3 n} \cdot b} \) for \( \log^3 n \) steps,

\[
\mathbb{E} \left[ \tilde{\Lambda}^{m} \mid \mathcal{F}^{t_0}, \tilde{\Lambda}^{t_0} \leq e^{\kappa_1 \log^2 n} \right] \leq e^{\kappa_1 \log^2 n} \cdot a^{\log^3 n} \cdot \frac{b}{1 - a}
\]

\[
\leq 1 + 1.5 \cdot b \leq 2 \cdot n \cdot e^{\frac{C_2}{\log^3 n} \cdot b} \leq 2 \cdot n^{1 + \kappa_2}. \tag{5.13}
\]

using in (a) that \( \frac{\varepsilon}{2n} \cdot b = \Omega(1) \) and \( a \) a constant \( < 1 \) and in (b) that \( e^{\frac{C_2}{n} \cdot b} \leq \kappa_2 \cdot \log n \) for some constant \( \kappa_2 > 0 \), since \( \gamma = \min \left( \frac{\varepsilon}{2n}, \frac{n \cdot \log n}{b} \right) \).

By Markov’s inequality, we have

\[
\Pr \left[ \tilde{\Lambda}^{m} \leq 2 \cdot n^{4 + \kappa_2} \mid \mathcal{F}^{t_0}, \tilde{\Lambda}^{t_0} \leq e^{\kappa_1 \log^2 n} \right] \geq 1 - n^{-3}.
\]

Hence, by Eq. (5.11)

\[
\Pr \left[ \tilde{\Lambda}^{m} \leq 2 \cdot n^{4 + \kappa_2} \right] = \Pr \left[ \tilde{\Lambda}^{m} \leq 2 \cdot n^{4 + \kappa_2} \mid \mathcal{E}^{t_0} \right] \cdot \Pr \left[ \mathcal{E}^{t_0} \right] \geq (1 - n^{-3}) \cdot (1 - n^{-3}). \tag{5.14}
\]

Combining Eq. (5.11) and Eq. (5.14), we have

\[
\Pr \left[ \tilde{\Lambda}^{m} \leq 2 \cdot n^{4 + \kappa_2} \right] \geq \Pr \left[ \tilde{\Lambda}^{m} \leq 2 \cdot n^{4 + \kappa_2} \right] \cap \bigcap_{j \in [0, \log^3 n]} \mathcal{E}^{t_0 + j \cdot b}
\]

\[
\geq (1 - n^{-3}) \cdot (1 - n^{-3}) - n^{-3} \geq 1 - n^{-2}.
\]
Finally, \( \{ \Lambda^m \leq 2 \cdot n^{4+\kappa_2} \} \) implies
\[
y_1^m \leq k + \frac{\log 2}{\gamma} + \frac{1}{\gamma} \cdot (4 + \kappa_2) \cdot \log n = O(b/n + \log n),
\]
since \( \gamma = \min \left( \frac{s}{|\mathcal{CS}|}, \frac{n \log n}{b} \right) \) and \( \Theta(\max(b/n, \log n)) \), so the claim follows.

For the case when \( m < b \cdot \log^3 n \), note that \( \Lambda_{t_0} = 2n \) deterministically, which is a stronger starting point in Eq. (5.13) to prove that \( \mathbb{E}[\Lambda^m] \leq 2 \cdot n^{1+\kappa_2} \), which in turn implies the gap bound.

\[ \square \]

6 Application to Graphical Allocations and \((1 + \beta)\)-process

In [30], the authors proved several bounds on the gap for the \((1 + \beta)\) process (in the setting without batches) where balls are sampled from a weight distribution with constant \( \lambda > 0 \) as defined in Section 2.3. In the second part of [30], the authors used a majorization argument to deduce gap bounds for graphical balanced allocation. However, due to the involved majorization argument not working for weights, all results for graphical allocation in [30] assume balls are unweighted. This lack of results for weighted graphical allocations is summarized as Open Question 1 in [30]. By leveraging the results in previous sections, we are able to fill this “gap”.

For a \( d \)-regular (and connected) graph, let us define the conductance as:
\[
\Phi(G) := \min_{S \subseteq V: 1 \leq |S| \leq n/2} \frac{|E(S, V \setminus S)|}{|S| \cdot d}.
\]
We will call a family of graphs an \textit{expander}, if \( \Phi \) is at least a constant bounded below from 0 (as \( n \to \infty \)).

\textbf{Lemma 6.1.} Consider \textsc{Graphical} on a \( d \)-regular graph with conductance \( \Phi \) with batch size \( b = 1 \). Then for any \( 1 \leq k \leq n/2 \),
\[
\sum_{i=1}^{k} p_i^t \leq (1 - \Phi) \cdot \frac{k}{n},
\]
and similarly, for any \( n/2 + 1 \leq k \leq n \),
\[
\sum_{i=k}^{n} p_i^t \geq (1 + \Phi) \cdot \frac{n - k + 1}{n}.
\]
Further, \( \max_{i \in [n]} p_i^t \leq \frac{d}{n} \). Thus, the vector \( p^t \) satisfies condition \( C_1 \) with \( \delta = 1/2 \), \( \varepsilon = \Phi \) and condition \( C_2 \) with \( C = d \).

The proof of this lemma closely follows [30, Proof of Theorem 3.2].

\textbf{Proof.} Fix any load vector \( x^t \) in round \( t \). Consider any \( 1 \leq k \leq n/2 \). Let \( S_k \) be the \( k \) “heaviest” bins with the largest load. Hence in order to allocate a ball into \( S_k \), both endpoints of the sampled edge must be in \( S_k \), and hence
\[
\begin{align*}
\sum_{i=1}^{k} p_i^t &= \frac{2 \cdot |E(S_k, S_k)|}{2 \cdot |E|} \\
&= \frac{|E(S_k, V)| - |E(S_k, V \setminus S_k)|}{2 \cdot |E|} \\
&\leq \frac{|S_k| \cdot d - |S_k| \cdot \Phi \cdot d}{nd} = (1 - \Phi) \cdot \frac{k}{n}.
\end{align*}
\]
where the inequality used the definition of conductance $\Phi$. Hence $p_t$ satisfies condition $C_1$ with $\varepsilon = \Phi$. Now, we will consider the suffix sums for $n/2 + 1 \leq k \leq n$. We start by upper bounding the prefix sum up to $k - 1$,
\[
\sum_{i=1}^{k-1} p_t^i \leq \frac{|S_k| \cdot d - |V \setminus S_k| \cdot \Phi \cdot d}{nd} \\
\leq \frac{(k - 1) \cdot d - (n - k + 1) \cdot \Phi \cdot d}{nd} \\
= \frac{(k - 1) - (n - k + 1) \cdot \Phi}{n},
\]
where the inequality used our assumption that $G$ has expansion $\Phi > 0$. Hence the suffix sum is
\[
\sum_{i=k}^{n} p_t^i = 1 - \sum_{i=1}^{k-1} p_t^i \geq 1 - \frac{(k - 1) - (n - k + 1) \cdot \Phi}{n} = (1 + \Phi) \cdot \frac{n - k + 1}{n}.
\]
Finally, we also know that for any bin $i \in [n]$, $p_t^i \leq \frac{d}{n}$, since in the worst-case we allocate a ball into bin $i$ whenever an edge incident to $i$ is chosen.

The next result is for Graphical in the unbatched setting.

**Theorem 6.2.** Consider Graphical on a $d$-regular graph with conductance $\Phi > 0$. Further, consider the non-batched setting, i.e., $b = 1$ and assume that balls are sampled from a weight distribution with constant $\lambda > 0$. Then there is a constant $k := k(\lambda) > 0$ such that for any $m \geq 0$,
\[
\Pr \left[ \max_{i \in [n]} |y_m^i| \leq k \cdot \frac{d}{\Phi} \cdot \log n \right] \geq 1 - n^{-2}.
\]

**Proof.** Using [30, Lemma 2.1], we have for $\alpha = \frac{\Phi}{16dS}$ and some constant $S := S(\lambda) > 1$,
\[
E \left[ \Delta \Phi_{x'}^{i+1} \mid x' \right] \leq \Phi^i \cdot \left( \left( p_i - \frac{1}{n} \right) \cdot \alpha + dS \cdot \frac{\alpha^2}{n} \right),
\]
and using [30, Lemma 2.3],
\[
E \left[ \Delta \Phi_{x'}^{i+1} \mid x' \right] \leq \Phi^i \cdot \left( \left( \frac{1}{n} - p_i \right) \cdot \alpha + dS \cdot \frac{\alpha^2}{n} \right).
\]
Hence, applying Corollary 3.2 for $\varepsilon := \Phi$, $\delta := 1/2$, we get for any $m \geq 0$,
\[
E [\Gamma_m] \leq \frac{8c}{\delta} \cdot n,
\]
for some constant $c := c(\delta) > 0$. Hence, by Markov’s inequality
\[
\Pr \left[ \Gamma_m \leq \frac{8c}{\delta} \cdot n^3 \right] \geq 1 - n^{-2}.
\]
The event $\{\Gamma_m \leq \frac{8c}{\delta} \cdot n^3\}$ implies that
\[
\max_{i \in [n]} |y_m^i| \leq \log \left( \frac{8c}{\delta} \right) + 3 \cdot \frac{16 \cdot d \cdot S}{\Phi} \cdot \log n \leq k \cdot \frac{d}{\Phi} \cdot \log n,
\]
for some constant $k := k(\lambda) > 0$. □

The next result is the batched version of Theorem 6.2.
Theorem 6.3. Consider Graphical on a $d$-regular graph with conductance $\Phi > 0$. Further, consider the batched setting with $b \geq n$ and assume that balls are sampled from a weight distribution with constant $\lambda > 0$. Then there is a constant $k := k(\lambda) > 0$ such that it holds for any $m \geq 0$ being a multiple of $b$, 
\[
\Pr \left[ \max_{i \in [n]} |y_i^m| \leq k \cdot \frac{d^2}{\Phi} \cdot \frac{b}{n} \cdot \log n \right] \geq 1 - n^{-2}.
\]
Further, if the conductance $\Phi$ is lower bounded by a constant $\Phi > 0$ (i.e., $G$ is an expander), $d > 0$ is constant and $n \leq b \leq n^3$, then there is a constant $k := k(\lambda, d) > 0$ such that for any $m \geq 0$ being a multiple of $b$, 
\[
\Pr \left[ y_1^m \leq k \cdot \left( \frac{b}{n} + \log n \right) \right] \geq 1 - n^{-2}.
\]

Note that our first gap bound for constant $d > 0$, generalizes [30, Theorem 3.2], which is a gap bound of $O(\log \frac{n}{\Phi})$ in the setting without batches and weights. Similarly, our second result extends the $O(\log \frac{n}{\Phi})$ bound from [30] for expanders, and proves that the same gap bound applies in the weighted batched setting with any $b = O(n \log n)$.

Proof. The first result follows directly from Lemma 6.1 and Theorem 4.2. For the second result, $\varepsilon = \Phi$ is a constant $> 0$, and we can apply the refined gap bound from Theorem 5.1. \hfill $\square$

Next we improve the upper bound on the gap for $(1 + \beta)$ for very small $\beta$. In [30, Corollary 2.1], it was shown that this gap is $O(\log \frac{n}{\beta} + \log(1/\beta)/\beta)$. For $1/\beta = n^{o(1)}$, the second term dominates. We improve this gap bound to $O(\log \frac{n}{\beta})$. This is tight up to multiplicative constants for $\beta \leq 1/2$, due to a lower bound of $\Omega(\log \frac{n}{\beta})$ as shown in [30, Section 4].

Theorem 6.4. Consider the $(1 + \beta)$ process for any $\beta \in (0, 1]$. Then there exists a constant $k > 0$, such for any $m \geq 1$, 
\[
\Pr \left[ \text{Gap}(m) \leq k \cdot \frac{\log n}{\beta} \right] \geq 1 - n^{-2}.
\]

Proof. Using [30, Lemma 2.1], we have for $\alpha = \frac{\beta}{4}$ and some constant $S > 1$, 
\[
E \left[ \Delta \Phi_t^{t+1} \mid x^t \right] \leq \Phi_t^t \cdot \left( \left(p_t - \frac{1}{n}\right) \cdot \alpha + 2S \cdot \frac{\alpha^2}{n} \right),
\]
and using [30, Lemma 2.3],
\[
E \left[ \Delta \Psi_t^{t+1} \mid x^t \right] \leq \Psi_t^t \cdot \left( \left(\frac{1}{n} - p_t\right) \cdot \alpha + 2S \cdot \frac{\alpha^2}{n} \right).
\]
By Proposition 2.2 the $1 + \beta$ process satisfies the $C_1$ condition for $\varepsilon = \frac{\beta}{4}$ and $\delta = \frac{1}{4}$. By Corollary 3.2 there exists $c := c(\delta) > 0$ such that for any $m \geq 1$
\[
E [\Gamma^m] \leq \frac{8c}{\delta} \cdot n.
\]
Hence, using Markov’s inequality
\[
\Pr \left[ \Gamma^m \leq n^4 \right] \geq 1 - n^{-2}.
\]
Note that when $\{\Gamma^m \leq n^4\}$ holds, we have
\[
\text{Gap}(m) \leq \frac{4}{\alpha} \cdot \log n = 4 \cdot \frac{8 \cdot (2S)}{\beta \delta} \cdot \log n = O\left( \frac{\log n}{\beta} \right).
\]
7 Lower Bounds

For the lower bounds we always assume that balls are unweighted (or equivalently, have unit weight). We recall the following result which assumes no batching, i.e., balls are allocated sequentially using perfect knowledge about the bin loads.

Lemma 7.1 ([25, Theorem 10.4]). Consider any allocation process in the unweighted setting with probability vector \( q \) with \( \min_{i \in [n]} q_i \geq C/n \) for some constant \( C > 0 \). Then there exists a constant \( k > 0 \), such that for \( m = \Theta(n \log n) \),

\[
\Pr \left[ \text{Gap}(m) \geq k \cdot \log n \right] \geq 1 - n^{-2}.
\]

We use the following majorization result from [25] (see also [30, Section 3]).

Lemma 7.2 (Lemma 4.13 in [25]). Consider two allocation processes \( Q \) and \( P \). The allocation process \( Q \) uses at each round a fixed allocation distribution \( q \). The allocation process \( P \) uses a time-dependent allocation distribution \( p^t \), which may depend on \( F_t \) but majorizes \( q \) at each round \( t \geq 0 \). Let \( y^t(Q) \) and \( y^t(P) \) be the two normalized load vectors, sorted decreasingly. Then there is a coupling such that for all rounds \( t \geq 0 \), \( y^t(P) \) majorizes \( y^t(Q) \).

Combining the two lemmas above, we can now prove a lower bound which holds for any batch size:

Proposition 7.3. Consider any allocation process with probability vector \( p \) with \( \min_{i \in [n]} p_i \geq C/n \) for some constant \( C > 0 \), in the unweighted batched setting for any \( b \geq 1 \). Then there exists a constant \( k > 0 \) such that for \( m = \Theta(n \log n) \),

\[
\Pr \left[ \text{Gap}(m) \geq k \cdot \log n \right] \geq 1 - n^{-2}.
\]

Note that this statement applies to the \((1+\beta)\)-process for constant \( \beta \in (0, 1) \) and \textsc{Quantile}(\( \delta \)) and constant \( \delta \in (0, 1) \), but it does not apply to \textsc{Two-Choice}.

Proof. For the purpose of this lower bound derivation, we assume that the batched setting allocates all \( m \) balls sequentially in rounds \( t = 1, 2, \ldots, m \). As the load information does not get updated within each batch of size \( b \), this means that the allocation made in each round is described by an allocation vector \( p^t \), which depends on \( F_t \) but also on the history of the process, i.e., \( F^t \).

Let \( q \) be the vector \( p \) sorted in non-decreasing order. Then, at each round \( p^t \) majorizes \( q \), since the outdated information implies that \( p^t \) is a permutation of \( q \).

We apply Lemma 7.2 with \( p^t \) and \( q \) as defined above. Hence for \( t = m \), there is a coupling such that the load vector \( y^m(P) \) majorizes \( y^m(Q) \), in particular,

\[
y^m_1(P) \geq y^m_1(Q),
\]

which is equivalent to \( \text{Gap}(P, m) \geq \text{Gap}(Q, m) \). Hence the statement of the lemma follows by Lemma 7.1.

Proposition 7.4. Consider any allocation process with probability vector \( p \) with \( \max_{i \in [n]} p_i \geq C/n \) for some \( C > 1 \), in the unweighted batched setting with \( b \geq n \log n \). Then, for \( \gamma := \min(C - 1, 0.5) \), any bin \( j = \arg\max_{i \in [n]} p_i \) satisfies

\[
\Pr \left[ y^b_j \geq \frac{\gamma}{4} \cdot \frac{b}{n} \right] \geq 1 - n^{-\gamma^2/8}.
\]
Picking \( \lambda \)

Hence it follows that \( k \) smallest of the \( X_i \) with probability \( C/n \) independently. Let \( X := \sum_{j=1}^{b} X_j \), where the \( X_j \)'s are independent Bernoulli random variables with \( \mathbb{E}[X_j] \geq \frac{1+\gamma}{n} \). Hence \( \mathbb{E}[X] \geq b \cdot \frac{1+\gamma}{n} \). Using the following Chernoff bound, which states that for any \( \lambda > 0 \),

\[
\Pr[X \leq (1-\lambda) \cdot \mathbb{E}[X]] \leq \exp\left(-\lambda^2/2 \cdot \mathbb{E}[X]\right).
\]

Picking \( \lambda = \gamma/2 \) implies

\[
\Pr\left[x_i^b \leq (1 - \gamma/2) \cdot (1 + \gamma) \cdot \frac{b}{n}\right] \leq \exp\left(-\frac{\gamma^2}{8} \cdot \frac{b}{n}\right) \leq n^{-\gamma/8},
\]

where the last inequality used our assumption that \( b \geq n \log n \). If \( x_i^b \geq (1 - \gamma/2) \cdot (1 + \gamma) \cdot \frac{b}{n} \), then this implies for the normalised load,

\[
y_j^b = x_i^b - \frac{b}{n} \geq \frac{\gamma}{2} \cdot b - \frac{\gamma^2}{2} \cdot \frac{b}{n} \geq \frac{\gamma}{4} \cdot \frac{b}{n},
\]

where the last inequality used \( \gamma \leq 1/2 \).

The lemma above can be applied to any process satisfying condition \( C_2 \), so unlike Proposition 7.3, it applies to Two-Choice.

Note that for \( \text{d-CHOICE} \), \( \max_{i \in [n]} p_i \approx \frac{d}{n} \). Hence this lower bound shows that, in sharp contrast to the classical setting without batches, that large values of \( d \) lead to a worse performance. This is explained by the higher bias towards underloaded bins, which when given no (or outdated) information about the bins, will lead to a larger gap.

Let us remark that in the proof above, we assumed that the allocation process uses the same probability vector and bin labeling in all rounds of the same batch. In particular, this analysis does not apply to Two-Choice with random tie-breaking. However, Two-Choice with random tie-breaking will allocate all balls in the first batch following One-Choice. Exploiting this, we can then prove that by the end of the batch, there is a unique bin which attains the minimum load if \( b = \Omega(n \log n) \), which means for the second batch we can apply Proposition 7.3 and conclude that a lower bound of \( \Omega\left(\frac{b}{n}\right) \) holds with constant probability \( > 0 \).

First, we will make use of the following property of \( n \) independent Poisson random variables:

**Lemma 7.5.** Consider any \( n \geq 2 \) and \( \lambda \geq 16 \cdot \log n \). Let \( X_1, \ldots, X_n \) be independent Poisson random variables with \( X_i \sim \text{Pois}(\lambda) \), and denote by \( Y_{(n)}, Y_{(n-1)} \) the smallest and second smallest of the \( X_i \)'s. Then there exist constants \( \kappa_1, \kappa_2 > 0 \) such that

\[
\Pr\left[Y_{(n-1)} - Y_{(n)} \geq \kappa_1 \cdot \sqrt{\lambda / \log n}\right] \geq \kappa_2.
\]

**Proof.** Let \( X \sim \text{Pois}(\lambda) \), where \( \lambda := m/n \geq 16 \cdot \log n \). Let \( k \geq 0 \) be the minimal integer such that

\[
\Pr[\text{Pois}(\lambda) \leq k] \geq n^{-1}.
\]

By Lemma A.2 for \( \delta := \sqrt{4 \cdot \lambda^{-1} \cdot \log n} \), we have

\[
\Pr\left[X \leq \lambda - \sqrt{4 \cdot \lambda \cdot \log n}\right] \leq e^{-\lambda \delta^2/2} = e^{-2 \log n} = n^{-2}.
\]

Hence it follows that \( k \geq \lambda - 2 \cdot \sqrt{\lambda \cdot \log n} \). Next note that

\[
\frac{\Pr[\text{Pois}(\lambda) = k + 1]}{\Pr[\text{Pois}(\lambda) = k]} = \frac{\lambda}{k + 1},
\]

(7.1)
which, since \( k \geq \frac{1}{2} \lambda \) (as \( \lambda \geq 16 \log n \)), also implies that
\[
\Pr[\text{Pois}(\lambda) \leq k] \leq 2 \cdot n^{-1}.
\]

Our next claim is that
\[
\Pr[\text{Pois}(\lambda) = k] \leq 2 \cdot n^{-1} \cdot 1/\sqrt{\lambda/\log n}.
\]

We will now derive this claim. We have
\[
2n^{-1} \geq \Pr[\text{Pois}(\lambda) \leq k]
= \sum_{j=0}^{k} \Pr[\text{Pois}(\lambda) = j]
\geq \sqrt{\lambda/\log n} \cdot \Pr[\text{Pois}(\lambda) = k] \cdot \left( \frac{k - \sqrt{\lambda}}{\lambda} \right)^{\sqrt{\lambda/\log n}}
\geq \sqrt{\lambda/\log n} \cdot \Pr[\text{Pois}(\lambda) = k] \cdot \left( \frac{\lambda - 3\sqrt{\lambda \log n}}{\lambda} \right)^{\sqrt{\lambda/\log n}}
= \sqrt{\lambda/\log n} \cdot \Pr[\text{Pois}(\lambda) = k] \cdot \left( 1 - \frac{3}{\sqrt{\lambda/\log n}} \right)^{\sqrt{\lambda/\log n}}
\geq \sqrt{\lambda/\log n} \cdot \Pr[\text{Pois}(\lambda) = k] \cdot c_1,
\]
for some constant \( c_1 > 0 \), where in (a) we used Eq. (7.1) and in (b) we used that \( k \geq \lambda - 2\sqrt{\lambda \log n} \), and in (c) that \( \lambda \geq 16 \log n \).

Next we wish to upper bound
\[
\Pr[\text{Pois}(\lambda) \leq k + c_2 \cdot \sqrt{\lambda/\log n}]
\]
for some constant \( c_2 > 0 \). Note that
\[
\Pr[\text{Pois}(\lambda) \leq k + c_2 \cdot \sqrt{\lambda/\log n}]
\leq \Pr[\text{Pois}(\lambda) \leq k] + \sum_{i=1}^{c_2 \sqrt{\lambda/\log n}} \frac{\lambda^i}{k \cdot (k+1) \cdot \ldots \cdot (k+i-1)} \cdot \Pr[\text{Pois}(\lambda) = k]
\leq 2n^{-1} + c_2 \cdot \sqrt{\lambda/\log n} \cdot \frac{\lambda^{c_2 \sqrt{\lambda/\log n}}}{k^{c_2 \sqrt{\lambda/\log n}}} \cdot 2 \cdot n^{-1} \cdot 1/\sqrt{\lambda/\log n}
= 2n^{-1} + 2c_2 \cdot \left( 1 - \frac{1}{c\sqrt{\lambda \cdot \log n}} \right)^{-c_2 \sqrt{\lambda/\log n}} \cdot n^{-1}
\leq c_3 \cdot n^{-1},
\]
for another constant \( c_3 > 0 \), where (a) is due to Eq. (7.1).

We now use the principle of deferred decisions when exposing the \( n \) independent Poisson variables with mean \( \lambda \) denoted by \( X_1, X_2, \ldots, X_n \) one by one. Let \( \tau := \min\{ j : X_j \leq k \} \). With probability \( 1 - (1 - 1/n)^n \geq 1 - 1/e \), we have \( \tau < n \). Conditional on that, \( X_{\tau+1}, \ldots, X_n \) are
still $n - \tau$ independent Poisson variables with mean $\lambda$. Due to Eq. (7.2), the probability that all of the following Poisson random variables are larger than $k + c_2 \cdot \sqrt{\lambda} / \log n$ is at least
\[(1 - c_3 \cdot n^{-1})^r \geq (1 - c_3 \cdot n^{-1})^n \geq c_4,
\]
where $c_4 > 0$ is another constant.

Hence with probability at least $(1 - 1/e) \cdot c_4$, we have a gap of at least $c_2 \cdot \sqrt{\lambda} / \log n$ between $Y_{(n-1)}$ and $Y_{(n)}$.

We can now derive the lower bound for allocation processes with random tie-breaking.

**Lemma 7.6.** Consider an allocation process with probability vector $p$ and random tie-breaking, such that $p_n \geq \frac{C}{n}$ for some constant $C \in (1, 1.5]$ in the unweighted batched setting with $b \geq \frac{384}{(C-1)^2} n \log n$. Then, there exist constants $\kappa_1 := \kappa_1(C), \kappa_2 := \kappa_2(C) > 0$, such that
\[
\Pr \left[ \text{Gap}(2b) \geq \frac{C - 1}{8} \cdot \frac{b}{n} \right] \geq \kappa_2.
\]

**Proof.** Initially, all bins have load 0, so the first $b$ balls will be allocated using One-Choice. In order to use the Poisson Approximation Method [28, Theorem 5.6], let $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ be $n$ independent Poisson distributed random variables with rate $\lambda = (b - 4 \cdot \sqrt{b}) / n$. By Lemma 7.4, the sum $S_n := \sum_{i=1}^{n} \tilde{X}_i$ is in the range $[b - 8\sqrt{b}, b]$, with probability at least $1 - o(1)$. By Lemma 7.5, we have that with at least constant $\kappa_2 > 0$ probability, the difference between the smallest and second smallest bin is at least $\kappa_1 \cdot \sqrt{\lambda} / \log n$, for some constant $\kappa_1 > 0$.

Consider now the allocation of the remaining $b - S_n \leq 8\sqrt{b}$ balls. The average load of a bin through these balls is $8\sqrt{b} / n$. Using Markov’s inequality, the smallest bin does not receive more than $16\sqrt{b} / n$ additional balls with probability at least $1/2$.

Since $\kappa_1 \cdot \sqrt{\lambda} / \log n \geq \kappa_1 \cdot \sqrt{0.5 \cdot b / n / 1 / \log n} \geq 16\sqrt{b} / n$ we can conclude that there is still a unique minimally loaded bin after the allocation of all $b$ balls. Further, by using a Chernoff bound for One-Choice, it follows that
\[
\Pr \left[ y_n^b \leq b / n - \sqrt{6 \cdot b / n \log n} \right] \geq 1 - n^{-2}.
\]

Taking the union bound, we conclude that at the end of the first batch, the following holds:
\[
\Pr \left[ y_n^b \in \left[ -\sqrt{6 \cdot b / n \log n}, y_n^{b-1} - 1 \right] \right] \geq \kappa_1 \cdot \frac{1}{2} - o(1) - n^{-2}. \tag{7.3}
\]

Conditioning on $y_n^b \leq y_n^{b-1} - 1$, we have $\tilde{p}_n(x^b) \geq p_n \geq \frac{C}{n}$. For simplicity, let us fix label $n$ to be the index of the bin with smallest load at time $b$. Applying Proposition 7.4 to the allocations made in the second batch to bin $n$, we conclude that there is a constant $\gamma > 0$ such that
\[
\Pr \left[ x_n^{2b} - x_n^b \geq \left( 1 + \frac{\gamma}{4} \right) \cdot \frac{b}{n} \mid y_n^b \in \left[ -\sqrt{6 \cdot b / n \log n}, y_n^{b-1} - 1 \right] \right] \geq 1 - n^{-\gamma^2/8}. \tag{7.4}
\]

Both events in Eq. (7.3) and Eq. (7.4) hold with probability at least $\kappa_1 \cdot \frac{1}{8}$, and in this case,
\[
x_n^{2b} = x_n^b + x_n^{2b} - x_n^b \geq \frac{b}{n} - \sqrt{6 \cdot b / n \log n} + \left( 1 + \frac{C - 1}{4} \right) \cdot \frac{b}{n} \geq 2b - 6 \cdot b / n \log n + \frac{C - 1}{4} \cdot \frac{b}{n} \geq \frac{2b}{n} + \frac{C - 1}{8} \cdot \frac{b}{n},
\]
where we have used in (a) that if $b \geq \frac{384}{(C-1)^2} n \log n$ then,
\[
\frac{C - 1}{4} \cdot \frac{b}{n} \geq 2 \cdot \sqrt{6 \cdot b / n \log n} \iff b \geq \frac{384}{(C-1)^2} \cdot n \log n.
\]
Hence $\text{Gap}(2b) \geq \frac{C - 1}{8} \cdot \frac{b}{n}$. 

\[\square\]
8 Experiments

In this section, we complement our analysis with some experiments (Fig. 5, Fig. 6, Fig. 7 and Fig. 8).

**Figure 5:** Comparison between \((1 + \beta)\) for \(\beta = 0.5\) and \(\beta = 0.7\), Two-Choice and Three-Choice without random tie-breaking, for the unweighted batched setting with \(b \in \{n, 2n, \ldots, 50n\}\) for \(n = 10^3\) and \(m = n^2\) (100 runs). For small \(b\), Two-Choice and Three-Choice outperform the \((1 + \beta)\) processes, which is caused by the smaller probabilities for the heavily loaded bins. Conversely, for larger \(b\), the larger probabilities for the lightly loaded bins are responsible for creating larger gaps for Two-Choice and Three-Choice, as suggested by Proposition 7.4. For \(d\)-Choice the largest term is approximately \(d/n\) and for \((1 + \beta)\) this is \((1 + \beta)/n\), which corresponds to the observed performance: \(3/n \geq 2/n \geq 1.7/n \geq 1.5/n\). Similar observations were made in the queuing setting in [26].

**Figure 6:** Empirical results for the unweighted batched setting showing that the \(\text{Quantile}(\log n)^{-1}\) process and \((1 + \beta)\) with \(\beta = (\log n)^{-1}\) achieve better gaps than \((1 + \beta)\) for \(\beta = 0.5\) for large values of \(b \geq 150n\), \(n = 10^3\) and \(m = n^2\) (100 runs). This is probably due to the smaller maximum entry in the probability vector, which for the first two processes is \((1 + (\log n)^{-1})/n\), while for the third process it is \(1.5/n\).
Three-Choice
Two-Choice
(1 + β) for β = 0.7
(1 + β) for β = 0.5.

Figure 7: Empirical results for the weighted batched setting, where weights are sampled from an exponential distribution with mean 1. Further, n = 1000 and m = n² (100 runs). Overall, we seem to have a similar ordering among the four processes as in Figure 5, but for small values of b the weights of the balls create larger gaps (in comparison to the unweighted setting). This makes sense, as some of the balls will be of weight Ω(log n).

Two-Choice without random tie-breaking
Two-Choice with random tie-breaking

Figure 8: Empirical results for the unweighted batched setting, for the Two-Choice process with and without random tie-breaking, for b = 25 · n, n = 10³ and 100 runs. When using random tie-breaking the maximum load is slightly smaller, especially for the first batch where it allocates using One-Choice (so max i∈[n] p_i = 1/n instead of 2/n). However, from the second batch onwards these are asymptotically Ω(b/n log n) for b ≥ n log n (see Lemma 7.6).

9 Conclusions

In this work, we studied balanced allocations in a batched setting, following the model proposed in [7]. As our main result, we proved that for any batch size n ≤ b ≤ n³ and m ≥ n, a gap bound of O(b/n + log n) holds with high probability. This analysis covered both weighted balls and a number of allocation processes satisfying two mild conditions on their probability vector, thereby demonstrating that many of the sequential allocation processes perform well in the batched setting, and can thus be “parallelized”. We also proved lower bounds which match our upper bound up to multiplicative constants for a family of processes.

Our results also imply a slight improvement on the gap for the (1 + β)-process with very small β. Further, we proved the first gap bounds for graphical allocation with weights, thereby addressing Open Question 1 in [30].

A natural open problem is to investigate other batch sizes, e.g., b < n or b = Ω(n³), or consider a dynamic setting where the batch sizes may vary over time. Our new bounds for graphical allocation crucially depend on how much the maximum probability deviates from 1/n, and thus on the maximum degree of the graph. Improving this dependence may lead to stronger bounds for dense graphs.

The experimental results exhibit an interesting trade-off between the probability vector and
the achieved gap; having small probabilities for the heavily loaded bins is not significant for large $b$, but more important is to avoid large probabilities for the lightly loaded bins, which is achieved by processes like $\text{QUANTILE}((\log n)^{-1})$ or $(1 + \beta)$ for $\beta = (\log n)^{-1}$. In other words, following a more powerful process with more choices like $d$-CHOICE leads to worse performance than a more “indifferent” allocation scheme like $(1 + \beta)$ with $\beta = (\log n)^{-1}$.

Recall that our lower bound in Proposition 7.4 sheds some light onto this phenomenon, but further bounds are needed (in particular, refined upper bounds based on $\max_{i\in[n]} p_i$) so that we can rigorously compare the performance of these processes.

References

[1] Micah Adler, Soumen Chakrabarti, Michael Mitzenmacher, and Lars Rasmussen. Parallel randomized load balancing. *Random Structures Algorithms*, 13(2):159–188, 1998.

[2] Dan Alistarh, Trevor Brown, Justin Kopinsky, Jerry Zheng Li, and Giorgi Nadiradze. Distributionally linearizable data structures. In *Proceedings of 30th on Symposium on Parallelism in Algorithms and Architectures (SPAA’18)*, pages 133–142, New York, NY, USA, 2018. Association for Computing Machinery.

[3] Dan Alistarh, Rati Gelashvili, and Joel Rybicki. Brief announcement: Fast graphical population protocols. In Seth Gilbert, editor, *35th International Symposium on Distributed Computing, DISC 2021, October 4-8, 2021, Freiburg, Germany (Virtual Conference)*, volume 209 of *LIPIcs*, pages 43:1–43:4. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. Full version at https://arxiv.org/abs/2102.08808.

[4] Yossi Azar, Andrei Z. Broder, Anna R. Karlin, and Eli Upfal. Balanced allocations. *SIAM J. Comput.*, 29(1):180–200, 1999.

[5] Nikhil Bansal and Ohad N. Feldheim. Well-balanced allocation on general graphs. *CoRR*, abs/2106.06051, 2021.

[6] Petra Berenbrink, André Brinkmann, Tom Friedetzky, and Lars Nagel. Balls into non-uniform bins. *J. Parallel Distributed Comput.*, 74(2):2065–2076, 2014.

[7] Petra Berenbrink, Artur Czumaj, Matthias Englert, Tom Friedetzky, and Lars Nagel. Multiple-choice balanced allocation in (almost) parallel. In *Proceedings of 16th International Workshop on Approximation, Randomization, and Combinatorial Optimization (RANDOM’12)*, pages 411–422, Berlin Heidelberg, 2012. Springer-Verlag.

[8] Petra Berenbrink, Artur Czumaj, Angelika Steger, and Berthold Vöcking. Balanced allocations: the heavily loaded case. *SIAM J. Comput.*, 35(6):1350–1385, 2006.

[9] Petra Berenbrink, Tom Friedetzky, Zengjian Hu, and Russell A. Martin. On weighted balls-into-bins games. *Theor. Comput. Sci.*, 409(3):511–520, 2008.

[10] Petra Berenbrink, Tom Friedetzky, Peter Kling, Frederik Mallmann-Trenn, Lars Nagel, and Chris Wastell. Self-stabilizing balls and bins in batches: the power of leaky bins. *Algorithmica*, 80(12):3673–3703, 2018.

[11] Artur Czumaj and Volker Stemann. Randomized allocation processes. *Random Structures Algorithms*, 18(4):297–331, 2001.

[12] Michael Dahlin. Interpreting stale load information. *IEEE Trans. Parallel Distributed Syst.*, 11(10):1033–1047, 2000.
[13] Derek L. Eager, Ed D. Lazowska, and John Zahorjan. Adaptive load sharing in homogeneous distributed systems. *IEEE Transactions on Software Engineering, SE-12(5):662–675, 1986.*

[14] Ohad N. Feldheim, Ori Gurel-Gurevich, and Jiange Li. Long-term balanced allocation via thinning, 2021.

[15] Ohad Noy Feldheim and Jiange Li. Load balancing under d-thinning. *Electronic Communications in Probability, 25:Paper No. 1, 13, 2020.*

[16] P. Brighten Godfrey. Balls and bins with structure: balanced allocations on hypergraphs. In *Proceedings of 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA’08),* pages 511–517, 2008.

[17] Catherine Greenhill, Bernard Mans, and Ali Pourmiri. Balanced Allocation on Dynamic Hypergraphs. In Jarosław Byrka and Raghu Meka, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2020),* volume 176 of *Leibniz International Proceedings in Informatics (LIPIcs),* pages 11:1–11:22, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

[18] Kazuo Iwama and Akinori Kawachi. Approximated two choices in randomized load balancing. In *Proceedings of 15th International Symposium on Algorithms and Computation (ISAAC’04),* volume 3341, pages 545–557. Springer-Verlag, 2004.

[19] Richard M. Karp, Michael Luby, and Friedhelm Meyer auf der Heide. Efficient PRAM simulation on a distributed memory machine. *Algorithmica, 16(4-5):517–542, 1996.*

[20] Krishnaram Kenthapadi and Rina Panigrahy. Balanced allocation on graphs. In *Proceedings of 17th ACM-SIAM Symposium on Discrete Algorithms (SODA’06),* pages 434–443, USA, 2006. Society for Industrial and Applied Mathematics.

[21] Samuel Kutin. Extensions to McDiarmid’s inequality when differences are bounded with high probability. Technical report, University of Chicago, 2002.

[22] Christoph Lenzen, Merav Parter, and Eylon Yogev. Parallel balanced allocations: The heavily loaded case. In Christian Scheideler and Petra Berenbrink, editors, *The 31st ACM on Symposium on Parallelism in Algorithms and Architectures, SPAA 2019, Phoenix, AZ, USA, June 22-24, 2019,* pages 313–322, New York, NY, USA, 2019. ACM.

[23] Christoph Lenzen and Roger Wattenhofer. Tight bounds for parallel randomized load balancing: Extended abstract. In *Proceedings of the Forty-Third Annual ACM Symposium on Theory of Computing, STOC ’11,* page 11–20, New York, NY, USA, 2011. Association for Computing Machinery.

[24] Dimitrios Los and Thomas Sauerwald. Balanced Allocations with Incomplete Information: The Power of Two Queries. In Mark Braverman, editor, *13th Innovations in Theoretical Computer Science Conference (ITCS 2022),* volume 215 of *Leibniz International Proceedings in Informatics (LIPIcs),* pages 103:1–103:23, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[25] Dimitrios Los, Thomas Sauerwald, and John Sylvester. Balanced Allocations: Caching and Packing, Twinning and Thinning. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA),* pages 1847–1874, Alexandria, Virginia, 2022. SIAM.

[26] Michael Mitzenmacher. How useful is old information? *IEEE Trans. Parallel Distributed Syst., 11(1):6–20, 2000.*
[27] Michael Mitzenmacher, Andrée W. Richa, and Ramesh Sitaraman. The power of two random choices: a survey of techniques and results. In Handbook of randomized computing, Vol. I, II, volume 9 of Comb. Optim., pages 255–312. Kluwer Acad. Publ., Dordrecht, Netherlands, 2001.

[28] Michael Mitzenmacher and Eli Upfal. Probability and computing. Cambridge University Press, Cambridge, second edition, 2017. Randomization and probabilistic techniques in algorithms and data analysis.

[29] Giorgi Nadiradze. On Achieving Scalability through Relaxation. PhD thesis, IST Austria, 2021.

[30] Yuval Peres, Kunal Talwar, and Udi Wieder. Graphical balanced allocations and the $(1 + \beta)$-choice process. Random Structures Algorithms, 47(4):760–775, 2015.

[31] Kunal Talwar and Udi Wieder. Balanced allocations: the weighted case. In Proceedings of 39th ACM Symposium on Theory of Computing (STOC’07), pages 256–265, 2007.

[32] Kunal Talwar and Udi Wieder. Balanced allocations: a simple proof for the heavily loaded case. In Automata, languages, and programming. Part I, volume 8572 of Lecture Notes in Comput. Sci., pages 979–990. Springer, Heidelberg, Berlin, Heidelberg, 2014.

[33] Berthold Vöcking. How asymmetry helps load balancing. J. ACM, 50(4):568–589, 2003.

[34] Udi Wieder. Balanced allocations with heterogenous bins. In Proceedings of the Nineteenth Annual ACM Symposium on Parallel Algorithms and Architectures, SPAA ’07, page 188–193, New York, NY, USA, 2007. Association for Computing Machinery.

[35] Udi Wieder. Hashing, load balancing and multiple choice. Found. Trends Theor. Comput. Sci., 12(3-4):275–379, 2017.
A Tools

A.1 Concentration inequalities

The first lemma is a standard Chernoff bound for sum of independent random variables whose moment generating function is bounded.

**Lemma A.1.** Assume $Z_1, Z_2, \ldots, Z_k$ are independent samples from a distribution $W$, for which there is a constant $\lambda > 0$ such that $E[W] = 1$ and $E[e^{\lambda W}] \leq S$. Then for $Z := \sum_{i=1}^{k} Z_i$, it holds for that

$$
\Pr[Z \geq 2 \ln(S)/\lambda \cdot k] \leq \exp(-\ln(S) \cdot k).
$$

Furthermore, for the special case $k = 1$, we have for any $c > 0$,

$$
\Pr[Z_1 \geq 1/\lambda \cdot (c \cdot \ln(n) + \ln(S))] \leq n^{-c}.
$$

**Proof.** Let $t \in (0, \lambda]$ to be specified later. Then,

$$
\Pr[Z \geq 2 \ln(S)/\lambda \cdot k] = \Pr[e^{tZ} \geq e^{t2\ln(S)/\lambda \cdot k}]
\leq E[e^{tZ}] \cdot \exp(-t \cdot 2 \log(S)/\lambda \cdot k)
= (E[e^{tZ_1}])^k \cdot \exp(-t \cdot 2 \log(S)/\lambda \cdot k)
\leq (E[e^{\lambda Z_1}])^{k/t\lambda} \cdot \exp(-t \cdot 2 \log(S)/\lambda \cdot k)
\leq S^{k/t\lambda} \cdot \exp(-t \cdot 2 \log(S)/\lambda \cdot k)
= \exp(k \cdot (\ln(S) \cdot t/\lambda - t \cdot 2 \ln(S)/\lambda))
,$$

where the second inequality is due to Jensen’s inequality. Choosing $t = \lambda$ yields the claim.

For the second statement, for any $c > 0$,

$$
\Pr[Z_1 \geq 1/\lambda \cdot (c \cdot \ln(n) + \ln(S))] \leq \Pr[e^{\lambda Z_1} \geq e^{-c \cdot \ln(n) - \ln(S)}]
\leq E[e^{\lambda W}] \cdot e^{-c \cdot \ln(n) - \ln(S)}
\leq S \cdot n^{-c} \cdot \frac{1}{S} = n^{-c}.
$$

Next we state a Chernoff bound for Poisson random variables.

**Lemma A.2 (Theorem 5.4 from [28]).** Let $X \sim \text{Pois}(\lambda)$, then for any $0 < \delta < 1$,

$$
\Pr[X \leq (1 - \delta) \cdot \lambda] \leq e^{-\lambda \delta^2/2},
$$

and

$$
\Pr[X \geq (1 + \delta) \cdot \lambda] \leq e^{-\lambda \delta^2/3}.
$$

Following [21], we will now give the definition for strongly difference-bounded and then give the statement for a bounded differences inequality with bad events.

**Definition A.3 (Strongly difference-bounded – Definition 1.6 in [21].** Let $\Omega_1, \ldots, \Omega_N$ be probability spaces. Let $\Omega = \prod_{k=1}^{N} \Omega_k$ and let $X$ be a random variable on $\Omega$. We say that $X$ is strongly difference-bounded by $(\eta_1, \eta_2, \xi)$ if the following holds: there is a “bad” subset $B \subseteq \Omega$, where $\xi = \Pr[\omega \in B]$. If $\omega, \omega' \in \Omega$ differ only in the $k$-th coordinate, and $\omega \notin B$, then

$$
|X(\omega) - X(\omega')| \leq \eta_2.
$$

Furthermore, for any $\omega$ and $\omega'$ differing only in the $k$-th coordinate,

$$
|X(\omega) - X(\omega')| \leq \eta_1.
$$
Theorem A.4 (Theorem 3.3 in [21]). Let \( \Omega_1, \ldots, \Omega_N \) be probability spaces. Let \( \Omega = \prod_{k=1}^N \Omega_k \), and let \( X \) be a random variable on \( \Omega \) which is strongly difference-bounded by \( (\eta_1, \eta_2, \xi) \). Let \( \mu = E[X] \). Then for any \( \lambda > 0 \) and any \( \gamma_1, \ldots, \gamma_N > 0 \),
\[
\Pr[X \geq \mu + \lambda] \leq \exp\left(-\frac{\lambda^2}{2\cdot \sum_{k \in [N]} (\eta_2 + \eta_1 \gamma_k)^2}\right) + \xi \cdot \sum_{k \in [N]} \frac{1}{\gamma_k}.
\]

A.2 Auxiliary Probabilistic Claims

We give a proof for the well-known fact that when \( E[e^{\lambda W}] < \infty \) then \( E[W^4] \) is also bounded.

Lemma A.5. Consider a random variable \( W \) with \( E[e^{\lambda W}] < \infty \) for some \( \lambda > 0 \). then
\[
E[W^4] \leq \left( \frac{8}{\lambda} \right)^4 + E[e^{\lambda W}].
\]

Proof. Let \( \kappa := (8/\lambda) \cdot \log(8/\lambda) \). Consider \( x \geq \max(0, \kappa) =: \kappa^* \). Then
\[
e^{\lambda x/4} = e^{\lambda x/8} \cdot e^{\lambda x/8} \geq e^{\log(8/\lambda)} \cdot e^{\lambda x/8} \geq \frac{8}{\lambda} \cdot \frac{\lambda x}{8} = x,
\]
using that \( e^z \geq z \) for any \( z \). Hence,
\[
e^{\lambda x} = (e^{\lambda x/4})^4 \geq x^4.
\]
Hence, if \( p_x \) is the pdf of \( W \), then
\[
E[W^4] = \int_{x=0}^{\infty} x^4 \cdot p_x \, dx = \int_{x=0}^{\kappa^*} x^4 \cdot p_x \, dx + \int_{x=\kappa^*}^{\infty} x^4 \cdot p_x \, dx
\leq \int_{x=0}^{\kappa^*} \kappa^4 \cdot p_x \, dx + \int_{x=\kappa^*}^{\infty} e^{\lambda x} \cdot p_x \, dx
\leq \kappa^4 \cdot \int_{x=0}^{\infty} p_x \, dx + \int_{x=0}^{\infty} e^{\lambda x} \cdot p_x \, dx = \kappa^4 + E[e^{\lambda W}].
\]

Next we state an inequality for a sequence of random variables, related through a recurrence inequality.

Lemma A.6. Consider a sequence of random variables \( (Z_i)_{i \in \mathbb{N}} \) such that there are \( 0 < a < 1 \) and \( b > 0 \) such that every \( i \geq 1 \),
\[
E[Z_i | Z_{i-1}] \leq Z_{i-1} \cdot a + b.
\]
Then for every \( i \geq 1 \),
\[
E[Z_i | Z_0] \leq Z_0 \cdot a^i + \frac{b}{1 - a}.
\]

Proof. We will prove by induction that for every \( i \in \mathbb{N} \),
\[
E[Z_i | Z_0] \leq Z_0 \cdot a^i + b \cdot \sum_{j=0}^{i-1} a^j.
\]
For \( i = 0 \), \( E[Z_0 | Z_0] \leq Z_0 \). Assuming the induction hypothesis holds for some \( i \geq 0 \), then since \( a > 0 \),
\[
E[Z_{i+1} | Z_0] = E[E[Z_{i+1} | Z_i] | Z_0] \leq E[Z_i | Z_0] \cdot a + b
\leq (Z_0 \cdot a^i + b \cdot \sum_{j=0}^{i-1} a^j) \cdot a + b
= Z_0 \cdot a^{i+1} + b \cdot \sum_{j=0}^{i} a^j.
\]
The claims follows using that for \( a \in (0,1) \), \( \sum_{j=0}^{\infty} a^j = \frac{1}{1-a} \). \( \square \)
A.3 Auxiliary Non-Probabilistic Claims

For the next lemma, we define for two $n$-dimensional vectors $x, y$, $\langle x, y \rangle := \sum_{i=1}^{n} x_i \cdot y_i$.

**Lemma A.7.** Let $(p_k)_{k=1}^{n}$, $(q_k)_{k=1}^{n}$ be two probability vectors and $(c_k)_{k=1}^{n}$ be non-negative and non-increasing. Then if $p$ majorizes $q$, i.e., for all $1 \leq k \leq n$, $\sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} q_i$ holds, then

$$\langle p, c \rangle \geq \langle q, c \rangle.$$  

**Proof.** We will consider a sequence of moves between $p$ and $q$, which gradually moves probability mass from lower to higher coordinates. Specifically, we define the following sequence:

$$r^1 = (p_1, p_2, p_3, p_4, \ldots, p_n) = p$$
$$r^2 = (q_1, p_2 + (p_1 - q_1), p_3, p_4, \ldots, p_n)$$
$$r^3 = (q_1, q_2, p_3 + (p_1 + p_2 - q_1 - q_2), p_4, \ldots, p_n)$$
$$r^4 = (q_1, q_2, q_3, p_4 + (p_1 + p_2 + p_3 - q_1 - q_2 - q_3), \ldots, p_n)$$
$$\vdots$$
$$r^n = (q_1, q_2, q_3, \ldots, q_{n-1}, p_n + \sum_{i=1}^{n-1} (p_i - q_i)) = q,$$

where in the last equation we used $p_n + \sum_{i=1}^{n-1} (p_i - q_i) = p_n - p_n + q_n = q_n$.

For any $1 \leq k < n$, since $r^k$ and $r^{k+1}$ differ only in the $k$-th and $(k+1)$-st coordinate, and $\sum_{i=1}^{k} (p_i - q_i) \geq 0$, we conclude it follows that

$$\langle r^k, c \rangle - \langle r^{k+1}, c \rangle \geq r^k c_k + r^{k+1} c_{k+1} - r_k c_k + r^{k+1} c_{k+1}$$
$$= c_k \cdot \left( \sum_{i=1}^{k-1} (p_i - q_i) - q_k \right) + c_{k+1} \cdot \left( p_{k+1} - \sum_{i=1}^{k} (p_i - q_i) \right)$$
$$= (c_k - c_{k+1}) \cdot \sum_{i=1}^{k} (p_i - q_i)$$
$$\geq 0.$$

Hence $\langle p, c \rangle = \langle r^1, c \rangle \geq \langle r^2, c \rangle \geq \cdots \geq \langle r^n, c \rangle = \langle q, c \rangle$.  

**Lemma A.8.** The function $f(z) = z \cdot e^{k/z}$ for $k > 0$, is decreasing for $z \in (0, k]$.

**Proof.** By differentiating,

$$f'(z) = e^{k/z} - z \cdot e^{k/z} \cdot \frac{k}{z^2} = e^{k/z} \cdot \left( 1 - \frac{k}{z} \right).$$

For $z \in (0, k]$, $f'(z) \leq 0$, so $f$ is decreasing.