Analytic theory of Hund’s metals: a renormalization group perspective

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We study the emergence of quasiparticles in Hund’s metals. We derive a precise class of Kondo impurity models carrying both spin and orbital degrees of freedom, and show that the coupling of the impurity spin to the conduction electrons can be ferromagnetic. We derive the corresponding weak-coupling renormalization group equations for arbitrary representations of SU(M) × SU(N).

Introduction to Hund’s Metals. There is a large class of materials where strong electronic correlations, in the form of large mass renormalizations or power laws in the optical conductivity, arise from the Hund’s coupling rather than from the Hubbard U term. Noticeable examples are the recently discovered iron pnictides and chalco-

genides high-temperature superconductors [1, 2], ruthenates [3, 4], as well as other 4d transition metal oxides [5]. LDA+DMFT studies have established that electrons in these Hund’s metals, form a Fermi liquid below a coherence temperature which is remarkably low [1]. The physical degrees of freedom at higher energies are well formed fluctuating moments [6] which are observed in XES measurements [7] and incoherent electronic excitations which are observed in their optical properties [8–10]. The DMFT mapping reduces the complexity of the many-body problem to an impurity problem in a self-consistent environment which, in the case of Hund’s metals, has a finite density of states at the Fermi level. Further steps have been taken towards an analytic theory of Hund’s metals. [11] derived a simplified low-energy Hamiltonian for this problem. The relevance of multi-channel Kondo physics for the intermediate asymptotics was conjectured and the operator responsible for the crossover to the Fermi liquid at low energies was identified [12]. Still, very basic questions such as what is the renormalization group flow that results from these Hamiltonians, what sets the low-energy scale of the Hund’s metals, and how the latter depends on the different material parameters such as the representation of the impurity, remain. This questions are quite pressing since both numerical and experimental studies indicate a strong dependence of this scale on doping, with the electron-doped iron pnictides such as Fe1−xCoxBa2As2 [13] having a much larger coherence temperature than hole-doped materials such as KFe2As2 [14].

In this Letter, we address all these questions by modeling such Hund’s metals by a class of Kondo impurity models that we derive from an Anderson impurity model, and by accessing their infrared properties by means of a renormalization group (RG) approach. Starting from an Anderson impurity model, we derive a class of SU(M) × SU(N) Kondo impurity models. Importantly, we find and discuss the possibility of a ferromagnetic coupling between the impurity spin and the conduction electrons. Then, we derive the associated weak-coupling renormalization group (RG) equations for arbitrary representations of SU(M) × SU(N). These general equations have a broad range of applicability since the spin-orbital Kondo effect can be realized in multiple settings such as, for example, bilayer graphene [15]. They provide a framework for understanding a very old puzzle dating back to Schrieffer’s early investigations of the dependence of the Kondo temperature of transition metal oxides who observed that the Kondo temperature decreases dramatically as the d shell filling increases towards half-filling [16, 17]. The models that were used to interpret this decrease involved only the spin but not the orbital degrees of freedom [18, 19] which we show are crucial for determining the coherence scale.

Model. Let us start our modelization of Hund’s metals with the following SU(M) × SU(N)-symmetric Anderson impurity Hamiltonian [20], $H_{\text{AIM}} = H_{\text{imp}} + H_{\text{hyb}} + H_{\text{bath}}$ with
with momentum \( k \) in a non-interacting medium with dispersion \( \epsilon_k \). In practice, we consider a flat density of states \( \rho \) (we later set \( \rho = 1 \) to simplify expressions) with a large bandwidth \( 2D_0 \). \( H_{\text{hyb}} \) is the bilinear hybridization between the impurity and the conduction electrons.

We refine our modelization of Hund’s metals by considering valences close but below \( 1/N \)-filling, i.e., \( d \equiv M - n_d \geq 1 \), and the hierarchy of energy scales \( U \gg D_0 \gg J_1 \gg V \). The charge degrees of freedom of the impurity are therefore frozen and the impurity is a combination of a fixed number \( n_d \) of elementary electrons which can be described by a total \( SU(N) \) spin \( S \) and an orbital \( SU(M) \) isospin \( T \) interacting together and with the local spin and isospin of the conduction electrons. Furthermore, since a strong Hund’s coupling tends to maximize the total spin, we achieve this by considering that \( S \) lives in the totally symmetric representation of \( n_d \) fundamental \( SU(N) \) spins. Consequently, we take \( T \) to live in the totally antisymmetric representation composed of \( n_d < M \) fundamental \( SU(M) \) isospins. These representations correspond to the Young’s tableaux in Fig. 1. Notice that at exactly \( 1/N \)-filling, i.e., \( n_d = M \), the orbital isospin is in a singlet state (scalar representation) and the model reduces to an \( M \)-channel Coqblin-Schrieffer model with a totally antisymmetric spin representation [21]. The precise Kondo-like Hamiltonian is obtained by performing a Schrieffer-Wolff transformation (second order perturbation theory) from \( H_{\text{AM}} \) to \( H_K = H_{\text{int}} + H_{\text{bath}} \), with (summing over repeated indices)

\[
H_{\text{int}} = J_p \psi_{\alpha \sigma}^\dagger \psi_{\alpha \sigma} + J_0 S^\alpha (\psi_{m \sigma}^\dagger \frac{\sigma_{\alpha \beta}}{2} \psi_{m \sigma' \beta}) + K_0 T^\alpha (\psi_{m \sigma}^\dagger \frac{\tau_{\alpha \beta}}{2} \psi_{m \sigma' \beta}) + I_0 S^\alpha T^\alpha (\psi_{m \sigma}^\dagger \frac{\sigma_{\alpha \beta}}{2} \frac{\tau_{\alpha \beta}}{2} \psi_{m \sigma'}).
\]

We introduced the local conduction electron \( \psi_{m \sigma} \). \( \sigma^\alpha (\alpha = 1 \ldots N^2 - 1) \) and \( \tau^\alpha (\alpha = 1 \ldots M^2 - 1) \) are the generators of \( SU(N) \) and \( SU(M) \) respectively in their fundamental representations. They obey the usual Lie algebra commutation relations and are normalized such that \( \text{Tr} [\sigma^\alpha \sigma^\beta] = 2\delta_{\alpha \beta} \) and \( \text{Tr} [\tau^\alpha \tau^\beta] = 2\delta_{\alpha \beta} \). For \( SU(2) \) and \( SU(3) \), they correspond to the well-known Pauli and Gell-Mann matrices respectively. \( J_p, J_0, K_0, \) and \( I_0 \) are respectively the bare potential, spin-spin Kondo, orbital-orbital Kondo, and spin-orbital Kondo coupling constants. We obtain [22]

\[
J_p = \frac{1}{MN} \left[ \frac{n_d}{\Delta E^--\frac{n_d+1}{\Delta E^+}} - \frac{N-n_d}{n_d+1} \frac{1}{\Delta E^+} \right] V^2, \tag{5}
\]

\[
J_0 = \frac{2}{M} \left[ \frac{1}{\Delta E^+} - \frac{1}{n+1} \frac{1}{\Delta E^+} \right] V^2, \tag{6}
\]

\[
K_0 = \frac{2}{N} \left[ \frac{1}{\Delta E^+} + \frac{N-n_d}{n_d+1} \frac{1}{\Delta E^+} \right] V^2, \tag{7}
\]

\[
I_0 = 4 \left[ \frac{1}{n_d \Delta E^+} + \frac{1}{n_d+1} \frac{1}{\Delta E^+} \right] V^2, \tag{8}
\]

in which the virtual charge excitation energies to the \( n_d \pm 1 \) valence states, \( \Delta E^\pm \approx \epsilon_d + n_d U \) and \( \Delta E^\pm \approx -\epsilon_d - (n_d - 1)U \), are both positive if \( \epsilon_d = -(n_d - 1 + \alpha)/U \) with \( \alpha \in [0,1] \). In turn, the asymmetry between the factors of \( 1/\Delta E^+ \) and \( 1/\Delta E^- \) will explain the differences of coherence temperatures observed between hole-doped and electron-doped Hund’s metals. Notice also the minus sign in front of the second term of \( J_0 \) above which implies that \( J_0 \) can be anti-ferromagnetic or ferromagnetic depending on the value of \( \epsilon_d \). When \( \alpha < \alpha^* \equiv (n_d + 1)/(M + 1) \), the virtual transitions to states of valence \( n_d - 1 \) dominate, and the spin coupling is anti-ferromagnetic. On the contrary, for \( \alpha > \alpha^* \) virtual transitions to valence \( n_d + 1 \) dominate and \( J_0 \) is ferromagnetic. In the example of \( \text{Sr}_2\text{RuO}_4 \) \( (M = 3, N = 2, n_d = 2) \), one has \( J_0 = 8(\alpha - \alpha^*)/[9U\alpha(1 - \alpha)] \) with \( \alpha^* = 3/4 \).

The possibility of such a ferromagnetic spin coupling is a direct consequence of the large Hund’s coupling which is encoded in our choice of representations. Indeed, one can show [22] that setting \( J_1 = 0 \) yields positive Kondo couplings with \( J_0 = 2/M [1/\Delta E^+ + 1/\Delta E^-] V^2 \) and \( n_d I_0 = 2M J_0 = 2NK_0 \). In this case, the model reduces to a single-channel Coqblin-Schrieffer model with a spin living in the totally antisymmetric representation of \( SU(N) \), \( N \equiv MN \), composed of \( n_d \) impurity electrons.

We shall understand below how such a ferromagnetic spin coupling affects the properties of Hund’s metals at intermediate energies.

**RG equations.** To study the infrared properties of our Kondo model, we use a poor man’s scaling approach at zero temperature [23, 24]. This consists in reducing the bandwidth by perturbatively integrating over the degrees of freedom of those conduction electrons with an energy in the edge \( \delta D \) of the band and requiring that the physics remain invariant. The corresponding renormalization
of the couplings is given by the so-called $\beta$ functions, $\beta_i = dJ_i/d \ln D$ with $J_i = J, K, I$, together with the initial conditions set by the bare couplings, $J(D_0) = J_0$, $K(D_0) = K_0$ and $I(D_0) = I_0$. The expansion of the $\beta$ functions to any order in the couplings can be expressed in terms of $C_{n}^S$ and $C_{n}^T$, the eigenvalues of $n$-th order Casimir invariants of the representations of $S$ and $T$ respectively [28]. Up to the third order in the couplings, we obtain (see Fig. 2)

$$\beta_J = -\frac{N}{2} \left(1 - \frac{M}{2} J\right) \left(J^2 + \frac{C_T^2}{4M} J^4\right) + \ldots, \quad (9)$$

$$\beta_K = -\frac{M}{2} \left(1 - \frac{N}{2} K\right) \left(K^2 + \frac{C_S^2}{2N} K^4\right) + \ldots, \quad (10)$$

$$\beta_I = -\frac{MN}{4} \left[\left(\frac{4}{M} J + \frac{4}{N} K - J^2 - K^2\right) I + \left(C_T^2 / MC_2^2 + C_S^2 / NC_2^2\right) I^2 + \left(\frac{1}{4} C_T^2 / 2M + C_S^2 / 2N\right) I^3\right] + \ldots. \quad (11)$$

For sakes of generality, we gave the $\beta$ functions for $S$ and $T$ living in arbitrary faithful representations of $SU(N)$ and $SU(M)$. We shall later return to our particular model by specifying the Casimirs corresponding to our class of Hund’s metals. We discarded the $\beta$ function of the potential scattering since it does not renormalize the other couplings. We also discarded the $\beta$ functions corresponding to the two quadrupolar spin-orbital interaction terms which are generated by the pertubative expansion but are not initially present in $H_{\text{int}}$. For example, the term in $I^2 (S \cdot \sigma) (Q \cdot \tau)$ with $Q^c \equiv \{Q^o, T^o\} \text{Tr} \left[\tau^{(a,b)} c\right]$ was projected on $(S \cdot \sigma) (T \cdot \tau)$ [22] [29]. Notice that up to the third order in the couplings, the representation-dependent factors in the $\beta$ equations above vanish for $I = 0$, as it is expected for two uncoupled multi-channel Coqblin-Schrieffer models.

The fixed points of the RG Eqs. (9), (10) and (11) are easily identified as

$$I_0^* = \frac{C_S^2}{NC_2^2} + \frac{C_T^2}{MC_2^2} \pm \sqrt{8 \left(\frac{C_S^2}{N^2} + \frac{C_T^2}{M^2} - \frac{1}{2}\right) \left(\frac{1}{N^2} + \frac{1}{M^2}\right) + \left(\frac{C_S^2}{NC_2^2} + \frac{C_T^2}{MC_2^2}\right)^2} / \left(\frac{C_S^2}{N^2} + \frac{C_T^2}{M^2} - \frac{1}{2}\right). \quad (12)$$

(i) is the non-interacting fixed point. (ii) corresponds to the well-known intermediate-coupling fixed point of the $N$-channel $SU(M)$ Coqblin-Schrieffer model while (iii) corresponds to the one of the $M$-channel $SU(N)$ Coqblin-Schrieffer model. (i), (ii) and (iii) are unstable against $J_0 > 0$ or $K_0 > 0$ and, as long as $I_0 = 0$, the RG flows to the fixed point (iv) which corresponds to the fixed point of two uncoupled multi-channel Coqblin-Schrieffer models and the low-energy physics is dominated by the one with the smallest Kondo scale. As soon as $I_0 \neq 0$, the fixed points (i)-(iv) are all unstable and the RG eventually flows towards (v) or (vi) depending on the sign of $I_0$. Notice that, contrary to (i)-(iv), the locations of (v) and (vi) appear to depend on the specific representations of the spin $S$ and of the orbital isospin $T$ of the impurity.

We stress that the pertubative expansion of the $\beta$ functions can only be reliable around the non-interacting fixed point (i) and one must be careful before giving a physical meaning to the previous fixed points. In the special case in which both sectors of the impurity, $S$ and $T$, are in their fundamental representation, $C_S^2 = (N^2 - 1)/2N$ and $C_T^2 = (N^2 - 1)(N^2 - 4)/4N^2$ (and similar expressions for $C_T^2$ and $C_T^4$), one recovers the $\beta$ equations derived in [25]. For $SU(2) \times SU(2)$, it is well known that the fixed points (v) and (vi) with $I_0^* = \pm 4$ are artefacts of the third order expansion, and the correct fixed point is a strong-coupling fixed point at $I, J, K \to \infty$. For arbitrary $M$ and $N$, the fixed point (v) with $I_0^* = -4N^2M^2 + M^2 - M_0^2$ is well defined at large $N$ and $M$ and it was conjectured to be stable for all $N$ and $M$ except in the case $N = M = 2$ [25]. On the other hand, the fixed point (vi) with $I_0^* = 4$ lies out of the scope of the pertubative analysis. Kuramoto argued that, similarly to the $SU(2) \times SU(2)$ case, it should be replaced by a strong-coupling fixed point. This is particularly clear at the special values of the couplings $2MJ = 2NK = I$ for which the model reduces to the $SU(M \times N)$ Coqblin-Schrieffer model which has only a non-interacting and a strong-coupling fixed point.

**Application to Hund’s metals.** We now turn to the study of the intermediate to low-energy properties our class of Hund’s metals by working with the totally symmetric and totally antisymmetric representations intro-
duced before (see Fig. 1). The corresponding Casimirs read $C_{3}^{S} = (N - 1) n_{d} (N + n_{d})/2N$, $C_{3}^{D} = (N - 2) (N - 1) n_{d} (N + n_{d}) (N + 2 n_{d})/4 N^{2}$, $C_{4}^{S} = (M + 1) n_{d} (M - n_{d})/2M$, and $C_{4}^{D} = (M + 2) (M + 1) n_{d} (M - n_{d}) (M - 2 n_{d})/4 M^{2}$ [2]. Henceforth, we work in the large-$N$ large-$M$ limit while keeping both the ratio $q = M/N$ and the distance to $1/N$-filling, $d = M - n_{d} \geq 1$, finite. In this limit, the fixed points $(v)$ and $(vi)$ are located at

$$I_{+}^{*} \propto -\frac{4}{N M}, \quad \text{and} \quad I_{+}^{*} \propto \frac{4}{M}. \quad (13)$$

However, a rapid inspection shows that the higher order terms in the expansion of $\beta_{I}$ scale as $N^{2n-2} I^{n}$ for $n \geq 4$ (see e.g. the fourth order term in Fig. 2). This indicates that the perturbative expansion converges if $|I| \ll 1/N^{2}$. Noteworthy, both fixed points $(v)$ and $(vi)$ lie out this convergence domain. Based on the numerical findings of [3, 11], we conjecture that the flow towards the fixed point $(vi)$ at $(J^{*}, K^{*}, I_{+}^{*})$ should be understood as a flow to a strong-coupling fixed point yielding a Fermi-liquid behavior.

Below, we show that when considering a ferromagnetic spin coupling, $J_{0} < 0$, and the hierarchy of couplings $0 < I_{0} \ll K_{0} \ll |J_{0}| \ll J^{*}$, the flow to strong coupling proceeds in two stages: first the renormalization of orbital coupling $K$ and the slow approach to the fixed point $(ii)$ with a quenched ferromagnetic spin coupling, the sudden renormalization of $I$, and then the renormalization of $J$ to an anti-ferromagnetic value and the approach to the strong-coupling fixed point $(vi)$. We shall see that this scenario is radically different from starting with an anti-ferromagnetic ($J_{0} > 0$) or in the absence of Hund’s coupling ($J_{H} = 0$).

At high energies, the terms involving $I$ in the RG Eqs. (9) and (10) can initially be neglected. In this regime, the spin and orbital degrees of freedom are decoupled and the high energy features of our model are those of two uncoupled multi-channel Coqblin-Schrieffer models. One corresponds to an anti-ferromagnetic totally antisymmetric $SU(M)$ spin [21] with an approach to its non-Fermi-liquid fixed point $(ii)$ controlled by the Kondo scale

$$T_{K}^{i} \approx \exp \left(-\frac{2}{MK_{0}}\right) D_{0}, \quad (14)$$

and the scaling exponent $\Delta_{K} \equiv d \beta_{K}/d K \approx q$. The other one corresponds to a ferromagnetic totally symmetric $SU(N)$ spin [27] flowing much slowly to weak coupling with an exponent $\Delta_{J} \equiv d \beta_{J}/d J \approx 0$. Importantly, as long as $K \ll |J_{0}|$, the renormalization of $I$ is controlled by $J$: $\beta_{J} \approx -N J_{0} I > 0$. Therefore, $I$ first slowly renormalizes to weak coupling, $I(D) \approx I_{0} (D/D_{0})^{W[J_{0}]}$, down to energy scales on the order of $T_{K}^{i}$.

At energy scales on the order of $T_{K}^{i}$, the spin coupling has little renormalized, $J(T_{K}^{i}) \sim J_{0}$, while the orbital coupling reaches its fixed point, $K(T_{K}^{i}) \approx K^{*}$. It is now the latter that controls $\beta_{K} \approx -NK^{*} I$ and the spin-orbital coupling therefore renormalizes suddenly to strong coupling, $I(T_{K}^{K}) \approx I_{+}^{*}$.

In the first stage described above, as $I$ was flowing to weak coupling, the $I^{2}$ term could be neglected in $\beta_{J}$ given in Eq. (9). However, after $I$ has renormalized to strong coupling below $T_{K}^{K}$, it is eventually this $I^{2}$ term which drives the growth of $J$ and in particular the crossing of the $J = 0$ axis. We are left with the RG equation for the renormalized spin coupling which reads in the large-$N$ limit

$$\beta_{J} \approx -\frac{N}{2} \left(1 - \frac{M}{2} J \right) \left(J^{2} + \frac{1}{4} d I_{+}^{2} \right). \quad (15)$$

This equation can be easily integrated. In particular, the crossing of $J = 0$ occurs at an intermediate energy scale on the order of

$$\tilde{D} \approx \exp \left(-\frac{1}{2} q^{2} N \right) T_{K}^{K}. \quad (16)$$

In this regime, the renormalization rate of $J$ is rather constant and given by $\tilde{\Delta}_{J} \equiv -\beta_{J}(J \approx 0) \approx 2d/q^{2} N$. The integration of Eq. (15) also provides an estimate of the typical energy scale $T_{K}^{i}$ at which the spin coupling renormalizes to $J^{*}$, i.e. at which the strong-coupling regime establishes,

$$T_{K}^{i} \approx \exp \left(-\frac{1}{2} q^{2} \right) \tilde{D}. \quad (17)$$

Notice that this emergent energy scale depends on $d$, the distance to $1/N$-filling. In agreement with the experimental and numerical evidence, $T_{K}^{i}$ is found to be smaller as one gets closer to $1/N$-filling. At a more formal level, this also means that $T_{K}^{i}$ depends explicitly on the representations of the spin and the orbital isospin. This is unlike the typical Kondo scales emerging in Kondo models without spin-orbital coupling which can only depend implicitly on the representations through the values of the bare couplings (see for instance the expression of $T_{K}^{i}$ above).

To highlight that this Kondo scale is particularly small, we compare it to the case in which the bare spin coupling is anti-ferromagnetic, $J_{0} > 0$. In this scenario, $\beta_{I}$ is controlled by $J$ in the UV regime, $\beta_{I} \approx -NJ/J \ll 0$, and therefore the spin-orbital coupling rapidly grows to $I_{+}^{*}$. This drives the concomitant growths of $J$ and $K$ which quickly reach strong coupling. Our result can also be compared to the case of the absence of Hund’s coupling, $J_{H} = 0$, for which the model reduces to the $SU(N)$ Coqblin-Schrieffer model referring el described before ($N = M N$). There, it is well known that a ferromagnetic coupling $J_{0}$ renormalizes to strong coupling at energies on the order of $\exp(-2/N J_{0}) D_{0} \gg T_{K}^{i}$.

**Conclusion.** By a precise modeling and analytical treatment of the infrared physics, we proposed a precise
mechanism that successfully reproduces the low coherence energy scale, its dependence on the distance to half-filling, its asymmetry against hole or electron doping, as well as the two-stage renormalization of the couplings, that have been observed in Hund’s metals.

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[28] $C_\alpha S_\beta \equiv \text{Tr}[\frac{\sigma^{\alpha_1}}{2} \ldots \frac{\sigma^{\alpha_n}}{2}] S^{\alpha_1} \ldots S^{\alpha_n}$ where $I_S$ is the identity in the representation of $S$ and $\{a_1 \ldots a_n\}$ stands for the sum over all permutations weighted by $1/n!$. In the fundamental representations, quadrupolar terms are simply absent since one necessarily has $Q \propto S$ owing to the fact that $\{I, S^a; a = 1 \ldots N^2 - 1\}$ is a complete basis of the $SU(N)$ operators.
SUPPLEMENTARY MATERIAL
Camille Aron, Gabriel Kotliar

In this note, we give some details on the derivation of our $SU(M) \times SU(N)$ Kondo model starting from an $SU(M) \times SU(N)$ Anderson impurity model, both in the presence of a strong Hund’s coupling as well as in the absence of Hund’s coupling ($J_H = 0$). We also detail some group-theoretic aspects of the derivation of the weak-coupling RG equations.

To simplify, we replace the notation for the impurity valence $n_d \mapsto n$ throughout this note.

**Schrieffer-Wolff transformation**

We recall the definition of the Anderson Hamiltonian introduced in Eqs. (1)-(3) of the Letter:

$$H_{\text{AIM}} = H_{\text{imp}} + H_{\text{hyb}} + H_{\text{bath}},$$

with

$$H_{\text{imp}} \equiv \epsilon_d n + \frac{1}{2} U n(n - 1) + \frac{1}{2N} J_H \sum_{m, \sigma, \sigma'} d^\dagger_{m \sigma} d_{n \sigma'} d_{m \sigma'} d_{n \sigma},$$

$$H_{\text{hyb}} \equiv V \sum_{k, m, \sigma} \psi^\dagger_{km \sigma} d_\sigma + \text{h.c.},$$

$$H_{\text{bath}} \equiv \sum_{k, m, \sigma} \epsilon_k \psi^\dagger_{km \sigma} \psi_{km \sigma}.$$

where $n = \sum_{m, \sigma} d^\dagger_{m \sigma} d_{m \sigma}$ and $s_m = \sum_{\sigma, \sigma'} d^\dagger_{m \sigma} \sigma \sigma' d_{m \sigma}$ are the $SU(N)$-spins in their fundamental representation and $s^2_m = \frac{N^2 - 1}{2N}$. For a given valence $n$, the lowest energy level of $H_{\text{imp}}$ corresponds to the situation in which the $n$ $SU(N)$-spins are aligned and

$$E_n = n \epsilon_d + \frac{1}{2} n(n - 1)(U - J_H).$$

The goal is to re-write $H_{\text{AIM}}$ as the Kondo model $H_K = H_{\text{int}} + H_{\text{bath}}$ with

$$H_{\text{int}} = J_p (I_T \otimes I_S) \otimes (\psi^\dagger_{m \sigma} \psi_{m \sigma}) + J_0 (I_T \otimes S) \otimes (\psi_{m \sigma} \frac{\sigma \sigma'}{2} \psi_{m \sigma'}) + K_0 (T \otimes I_S) \otimes (\psi_{m \sigma} \frac{\tau_{mn}}{2} \psi_{n \sigma}) + I_0 (T \otimes S) \otimes (\psi_{m \sigma} \frac{\sigma \sigma'}{2} \tau_{mn} \psi_{n \sigma'}).$$

The Schrieffer-Wolff transformation (second order perturbation theory in the hybridization) is given by

$$H_{\text{eff}} = -P_n H_{\text{hyb}} \left[ \frac{P_{n+1}}{\Delta E^+} + \frac{P_{n-1}}{\Delta E^-} \right] H_{\text{hyb}} P_n,$$
where $P_{n\pm 1}$ are the projectors on the Hilbert space of valence $n \pm 1$ and
\begin{align}
\Delta E^+ &\equiv E_{n+1} - E_n = \epsilon_d + n(U - J_H) \simeq \epsilon_d + nU , \\
\Delta E^- &\equiv E_{n-1} - E_n = -\epsilon_d - (n-1)(U - J_H) \simeq -\epsilon_d - (n-1)U ,
\end{align}
where we used the fact that $U \gg J_H$ to simplify the expressions. Notice that $\Delta E^+ + \Delta E^- = U > 0$. The conditions $\Delta E^+ > 0$ and $\Delta E^- > 0$ are simultaneously fulfilled for
\[ \epsilon_d = -(n-1+\alpha)U \text{ with } \alpha \in ]0,1[ . \]

$\alpha \to 1$ favors the virtual processes to the states with valence $n + 1$ whereas $\alpha \to 0$ favors the virtual processes to the states with valence $n - 1$.

Next, we use the following orthonormal basis of $SU(1) \times SU(N)$
\begin{equation}
I^\alpha \equiv \begin{cases}
\frac{1}{\sqrt{M}}|I_T\rangle \otimes \frac{1}{\sqrt{N}}|I_S\rangle \\
\frac{1}{\sqrt{M}}|I_T\rangle \otimes \frac{1}{\sqrt{2}}|\sigma\rangle \\
\frac{1}{\sqrt{2}}|I_T\rangle \otimes \frac{1}{\sqrt{N}}|I_S\rangle \\
\frac{1}{\sqrt{2}}|I_T\rangle \otimes \frac{1}{\sqrt{2}}|\sigma\rangle 
\end{cases}
\end{equation}
and the completeness relation
\[ \delta_{ij} \delta_{jk} = \sum_\alpha I^\alpha_{ik} I^\alpha_{kl} , \]
where we gathered spin and orbital indices into $i \equiv (m, \sigma)$ and we sum over indices repeated twice, to re-write
\begin{align}
H_{\text{eff}} &= -\left\{ \frac{V^2}{\Delta E^+} \left[ I^\alpha_{kl} P_n d_l P_{n+1} d_k P_n \right] \left[ \psi^\dagger I^\alpha_{ij} \psi_j \right] \\
&\quad - \frac{V^2}{\Delta E^-} \left[ I^\alpha_{kl} P_n d_i P_{n+1} d_k P_n \right] \left[ \psi^\dagger I^\alpha_{ij} \psi_j \right] \right\} .
\end{align}
The invariance of the Hamiltonian with respect to rotations of the spin and the orbital isospin implies that we can simply identify $J_p$, $J_0$, $K_0$ and $I_0$ by computing single matrix elements:
\begin{align}
-\frac{1}{MN} \left[ \frac{V^2}{\Delta E^+} \langle \Phi|d_{m\sigma} P_{n+1} d_{m\sigma}^\dagger |\Phi\rangle - \frac{V^2}{\Delta E^-} \langle \Phi|d_{m\sigma} P_{n-1} d_{m\sigma}^\dagger |\Phi\rangle \right] &= J_p \langle \Phi|I_T \otimes I_S |\Phi\rangle , \\
-\frac{1}{M} \sigma_{\sigma'}^\alpha \sigma_0 \left[ \frac{V^2}{\Delta E^+} \langle \Phi|d_{m\sigma'} P_{n+1} d_{m\sigma}^\dagger |\Phi\rangle - \frac{V^2}{\Delta E^-} \langle \Phi|d_{m\sigma} P_{n-1} d_{m\sigma'}^\dagger |\Phi\rangle \right] &= J_0 \langle \Phi|I_T \otimes S^{\alpha_0} |\Phi\rangle , \\
-\frac{1}{N} \tau_{\tau'}^\alpha_{\mu_0} \left[ \frac{V^2}{\Delta E^+} \langle \Phi|d_{n\sigma'} P_{n+1} d_{n\sigma}^\dagger |\Phi\rangle - \frac{V^2}{\Delta E^-} \langle \Phi|d_{n\sigma} P_{n-1} d_{n\sigma'}^\dagger |\Phi\rangle \right] &= K_0 \langle \Phi|T^{\alpha_0} \otimes I_S |\Phi\rangle , \\
-\sigma_{\sigma'}^\alpha_{\mu_0} \tau_{\tau'}^\alpha_{\mu_0} \left[ \frac{V^2}{\Delta E^+} \langle \Phi|d_{n\sigma'} P_{n+1} d_{n\sigma}^\dagger |\Phi\rangle - \frac{V^2}{\Delta E^-} \langle \Phi|d_{n\sigma} P_{n-1} d_{n\sigma'}^\dagger |\Phi\rangle \right] &= I_0 \langle \Phi|T^{\alpha_0} \otimes S^{\alpha_0} |\Phi\rangle ,
\end{align}
where for each identification, we are free to choose a convenient state $|\Phi\rangle$ in the Hilbert space of valence $n$, as well as the components $\alpha_0 \in \{1, \ldots, N^2 - 1\}$ and $\alpha_0 \in \{1, \ldots, M^2 - 1\}$.

**Computation of the Kondo couplings.** Henceforth, we denote the weights of the fundamental representation of $SU(N)$ from the highest to the lowest by $1, 2, \ldots, N$. The elements of the Cartan sub-algebra of $SU(N)$ are labelled correspondingly: $\sigma^1, \ldots, \sigma^{N-1}$, with $\text{Tr}[\sigma^\alpha \sigma^\beta] = 2\delta_{\alpha\beta}$ where $\alpha, \beta = 1, \ldots, N - 1$.

Typically, we choose the state
\begin{equation}
|\Phi\rangle = \left( \underbrace{|1\ldots10\ldots0\rangle}_n \right) .
\end{equation}
"1" labels the highest weight of the fundamental representation of $SU(N)$ [$1 = \uparrow$ along $\sigma^z$ in $SU(2)$] and "0" labels a vacant orbital. $|\Phi\rangle$ is normalized $\langle\Phi|\Phi\rangle = 1$, automatically anti-symmetrized in the orbital indices and it is symmetric by permutations of the spins since they are all "1". We also choose $\alpha_0$ and $\alpha_0$ so that $\sigma^{\alpha_0}$ and $\tau^{\alpha_0}$ are elements of the Cartan sub-algebras of $SU(N)$ and $SU(M)$. This simplifies greatly the analysis because those elements can be represented by $N \times N$ (or $M \times M$) diagonal matrices.

We now detail the computation of $J_0$, sketch the computation of $K_0$, and leave the computation of $J_p$ and $I_0$ to the reader since they go along the same lines as for $J_0$.

**Computation of $J_0$**

To compute $J_0$, we need to compute both sides of

$$-\frac{1}{M} \sigma^{\alpha_0} \sigma^{\alpha_0} \left[ \frac{V^2}{\Delta E^+} \langle\Phi|d_{m\sigma}^* P_{n+1} d_{m\sigma}^l|\Phi\rangle - \frac{V^2}{\Delta E^-} \langle\Phi|d_{m\sigma}^l P_{n-1} d_{m\sigma}|\Phi\rangle \right] = J_0 \langle\Phi|\Sigma_{\tau} \otimes S^{\alpha_0}|\Phi\rangle. \tag{36}$$

We choose $\alpha_0 = 1$ and work with the 'first' element of the Cartan sub-algebra of $SU(N)$, which is the diagonal $N \times N$ matrix reading

$$\frac{\sigma^1}{2} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{37}$$

[for $SU(2)$, it is $\frac{\sigma^y}{2}$] and we choose the state

$$|\Phi\rangle = \left[ \underbrace{1 \cdots 1}_{M} 0 \cdots 0 \right]. \tag{38}$$

It is normalized $\langle\Phi|\Phi\rangle = 1$, automatically anti-symmetrized in the orbital indices and it is symmetric by permutations of the spins since they are all 1.

**RHS of Eq. (36).** $|\Phi\rangle$ is the state with highest weight of our totally symmetric spin representation and the corresponding value of $S^z$ is simply

$$\langle\Phi|S^1|\Phi\rangle = n \times \frac{1}{2}, \tag{39}$$

so that the rhs of Eq. (36) is

$$J_0 \langle\Phi|\Sigma_{\tau} \otimes S^{\alpha_0}|\Phi\rangle = \frac{n}{2} J_0. \tag{40}$$

**LHS of Eq. (36).** Let us start with the term in $1/\Delta E^+$. The sum over $\sigma$ and $\sigma'$ is simplified because the matrix $\sigma^1$ is diagonal, and the only non-vanishing elements are $\sigma = \sigma' = 1, 2$. We have

$$-\frac{1}{M} \sigma^{\alpha_0} \sigma^{\alpha_0} \left[ \frac{V^2}{\Delta E^+} \langle\Phi|d_{m\sigma}^* P_{n+1} d_{m\sigma}^l|\Phi\rangle - \frac{V^2}{\Delta E^-} \langle\Phi|d_{m\sigma}^l P_{n-1} d_{m\sigma}|\Phi\rangle \right] \tag{41}$$

$$= -\frac{1}{M} \left[ \sum_{m=n+1}^M \left( \underbrace{1 \cdots 1}_{n} 0 \cdots 0 |P_{n+1} \right) \underbrace{1 \cdots 1}_{0 \cdots 0} \overline{0 \cdots 0} \left| P_{n+1} \right) \right] \tag{42}$$
The relevant states contributing to the projector on the Hilbert space of valence \( n + 1 \), \( P_{n+1} \), are

\[
|\Psi_m\rangle \equiv \frac{1}{\sqrt{M}} \sum_{m=n+1}^{M} \sum_{i \in \{1\ldots n,m\}} |\Psi_n \rangle \langle \Psi_n|, \quad m = n + 1 \ldots M .
\]

(43)

These states are totally antisymmetric in the orbital sector and totally symmetric in the spin sector and we were careful to normalize them properly. Performing the projection in Eq. (42) with

\[
P_{n+1} = \sum_{m=n+1}^{M} |\Psi_m\rangle\langle \Psi_m| + |\Psi'_m\rangle\langle \Psi'_m| + \ldots
\]

we obtain

\[
\frac{1}{M} \sum_{\sigma\sigma'} V^2 |\Psi_m\rangle \langle \Psi_m| + |\Psi'_m\rangle\langle \Psi'_m| = -\frac{1}{M} \sum_{\sigma\sigma'} V^2 \left[ (M-n) - (M-n) \frac{1}{n+1} \right] .
\]

(44)

Let us now cope with the term in \( 1/\Delta E^- \) in Eq. (36). Here also, the sum over \( \sigma \) and \( \sigma' \) is simplified because the matrix \( \sigma \) is diagonal, and only the element \( \sigma = \sigma' = 1 \) contribute to the sum (the element \( \sigma = \sigma' = 2 \) is irrelevant since there is no spin 2 to annihilate in \( |\Phi\rangle \)). We have

\[
\frac{1}{M} \sum_{\sigma\sigma'} V^2 |\Psi_m\rangle \langle \Psi_m| = \frac{1}{M} \sum_{\sigma\sigma'} V^2 \left[ (M-n) - (M-n) \frac{1}{n+1} \right] .
\]

(45)

Altogether, summing the terms in \( 1/\Delta E^+ \) and \( 1/\Delta E^- \), we obtain

\[
\frac{1}{M} \sum_{\sigma\sigma'} V^2 |\Psi_m\rangle \langle \Psi_m| = \frac{1}{M} \sum_{\sigma\sigma'} V^2 \left[ (M-n) - (M-n) \frac{1}{n+1} \right] .
\]

(46)

Performing the projection in Eq. (48) with

\[
P_{n-1} = \sum_{m=1}^{n} |\Psi_m\rangle\langle \Psi_m| + |\Psi'_m\rangle\langle \Psi'_m| + \ldots
\]

we obtain

\[
\frac{1}{M} \sum_{\sigma\sigma'} V^2 |\Psi_m\rangle \langle \Psi_m| = \frac{1}{M} \sum_{\sigma\sigma'} V^2 \left[ (M-n) - (M-n) \frac{1}{n+1} \right] .
\]

(47)

The relevant states contributing to the projector to the Hilbert space of valence \( n - 1 \), \( P_{n-1} \), are

\[
|\Psi_m\rangle \equiv \frac{1}{\sqrt{M}} \sum_{m=1}^{n} \sum_{i \in \{1\ldots n,m\}} |\Psi_n \rangle \langle \Psi_n|, \quad m = 1 \ldots n .
\]

(48)

Performing the projection in Eq. (48) with

\[
P_{n-1} = \sum_{m=1}^{n} |\Psi_m\rangle\langle \Psi_m| + \ldots
\]

we obtain

\[
\frac{1}{M} \sum_{\sigma\sigma'} V^2 |\Psi_m\rangle \langle \Psi_m| = \frac{1}{M} \sum_{\sigma\sigma'} V^2 \left[ (M-n) - (M-n) \frac{1}{n+1} \right] .
\]

(49)

Altogether, summing the terms in \( 1/\Delta E^+ \) and \( 1/\Delta E^- \), we obtain

\[
\frac{1}{M} \left[ \sum_{\sigma\sigma'} V^2 \left[ (M-n) - (M-n) \frac{1}{n+1} \right] \right] .
\]

(50)

and finally,

\[
J_0 = \frac{1}{2M} \left[ \frac{1}{\Delta E^-} - \frac{1}{\Delta E^+} \right] .
\]

(51)
Computation of $K_0$

To compute $K_0$, we need to compute both sides of

$$
-\frac{1}{N} e^{a_0}_{mn} \left[ \frac{V^2}{\Delta E^+} \langle \Phi | d_{n\sigma} P_{n+1} d_{n\sigma}^\dagger | \Phi \rangle - \frac{V^2}{\Delta E^-} \langle \Phi | d_{n\sigma}^\dagger P_{n-1} d_{n\sigma} | \Phi \rangle \right] = K_0 \langle \Phi | T^{a_0} \otimes I_S | \Phi \rangle ,
$$

(54)

We choose $a_0 = n$ and work with the $n^{th}$ element of the Cartan sub-algebra which is the diagonal $N \times N$ matrix reading

$$
\sigma^n = \frac{1}{\sqrt{2n(n+1)}} \begin{pmatrix}
1 & \ldots & 1 \\
\vdots & \ddots & \vdots \\
-1 & \ldots & 0 \\
0 & \ldots & 0
\end{pmatrix},
$$

(55)

in which the element of value $-n$ is located at the $n + 1^{th}$ position. and the state

$$
|\Phi\rangle = \begin{pmatrix}
M \\
\vdots \\
n \\
\vdots \\
M
\end{pmatrix}0 \ldots 0 .
$$

(56)

It is normalized $\langle \Phi | \Phi \rangle = 1$, automatically anti-symmetrized in the orbital indices and it is symmetric by permutations of the spins since they are all 1.

**RHS of Eq. (54).** The weight of the state $|\Phi\rangle$ on the direction $T^n$ can be computed as

$$
\langle \Phi | T^n | \Phi \rangle = \frac{n}{\sqrt{2n(n+1)}} ,
$$

(57)

so that the rhs of Eq. (54) is

$$
K_0 \langle \Phi | T^{a_0} \otimes I_S | \Phi \rangle = \frac{n}{\sqrt{2n(n+1)}} K_0 .
$$

(58)

**LHS of Eq. (54).** We leave this exercise to the reader since the computation goes along the same lines as for $J_0$.

Repeating the exercise for $J_p$ and $I_0$, we identify

$$
J_p = \frac{1}{MN} \left[ \frac{n}{\Delta E^-} - \frac{M - n N + n}{n + 1} \Delta E^+ \right] V^2 ,
$$

(59)

$$
J_0 = \frac{2}{M} \left[ \frac{1}{\Delta E^-} - \frac{M - n}{n + 1} \Delta E^+ \right] V^2 ,
$$

(60)

$$
K_0 = \frac{2}{N} \left[ \frac{1}{\Delta E^-} + \frac{N + n}{n + 1} \Delta E^+ \right] V^2 ,
$$

(61)

$$
I_0 = 4 \left[ \frac{1}{n \Delta E^-} + \frac{1}{n + 1} \Delta E^+ \right] V^2 .
$$

(62)

**Case without Hund’s coupling, $J_H = 0$**

In the absence of Hund’s coupling, since all states at a given valence are degenerate, the projectors $P_{n+1}$ and $P_{n-1}$ in the expressions of Eqs. (31)-(34) play no role and this simplifies greatly the analysis. Contrary to the case of a strong Hund’s coupling, all physical configurations are now allowed. We briefly detail the case of $J_0$ and leave the computation of the other couplings as an exercise.
Without the projectors $P_{n \pm 1}$, Eq. (32) reduces to
\[
- \frac{1}{M} \sigma^{\alpha \sigma} \left[ \frac{V^2}{\Delta E^+} \langle \Phi | d_{m \sigma} d_{m \sigma}^\dagger | \Phi \rangle - \frac{V^2}{\Delta E^-} \langle \Phi | d_{m \sigma}^\dagger d_{m \sigma} | \Phi \rangle \right] = J_0 \langle \Phi | I_T \otimes S^{\alpha \sigma} | \Phi \rangle .
\]
(63)

Similarly to what we did in the Hund’s case above, we choose $\alpha_0 = 1$ (i.e. working with $\sigma^1$, the first element of the Cartan sub-algebra) and the state $| \Phi \rangle = | 1 \ldots 1 0 \ldots 0 \rangle$. Using $\sum_{m=1}^M \langle \Phi | d_{m1} d_{m1}^\dagger | \Phi \rangle = M - n$, $\sum_{m=2}^M \langle \Phi | d_{m1}^\dagger d_{m2} | \Phi \rangle = M - n$, and $\sum_{m=1}^M \langle \Phi | d_{m2}^\dagger d_{m2} | \Phi \rangle = 0$, we get
\[
- \frac{1}{M} \left[ -n \frac{1}{\Delta E^+} - n \frac{1}{\Delta E^-} \right] V^2 = \frac{J_0}{2} n ,
\]
(64)
and therefore
\[
J_0 = \frac{2}{M} \left[ \frac{1}{\Delta E^+} + \frac{1}{\Delta E^-} \right] V^2 .
\]
(65)

Similarly, we find
\[
J_p = - \frac{1}{MN} \left[ \frac{M N - n}{\Delta E^+} - \frac{n}{\Delta E^-} \right] V^2 ,
\]
(66)
\[
K_0 = \frac{2}{N} \left[ \frac{1}{\Delta E^+} + \frac{1}{\Delta E^-} \right] V^2 ,
\]
(67)
\[
I_0 = \frac{4}{n} \left[ \frac{1}{\Delta E^+} + \frac{1}{\Delta E^-} \right] V^2 ,
\]
(68)
i.e. $n_d J_0 = 2MJ_0 = 2NK_0$. The model reduces to an $SU(M \times N)$ Coqblin-Schrieffer impurity model, with a single Kondo coupling, in which the spin lives in the totally anti-symmetric representation of $SU(M \times N)$, composed of $n_d$ impurity electrons (single column Young tableau).

**RG: Projection of terms $I^2 (S \cdot \sigma) (Q \cdot \tau)$ on $(S \cdot \sigma) (T \cdot \tau)$**

In this section, we provide the computational details needed to perform the projection of terms such as $I^2 (S \cdot \sigma) (Q \cdot \tau)$, with $Q^a \equiv \{ T^a, T^b \} \text{Tr} [\tau^{(a \beta \gamma)} \tau^b]$, on the original model Hamiltonian, namely on a term $b (S \cdot \sigma) (T \cdot \tau)$ where $b$ is a constant to determine which depends *a priori* on the representation of the orbital isospin $T$.

**$SU(N)$ conventions.** $S^\alpha$ ($\alpha = 1 \ldots N^2 - 1$) and $T^a$ ($a = 1 \ldots M^2 - 1$) are the respective generators of the $SU(N)$ and $SU(M)$ group in their respective generic representations $\{ S \}$ and $\{ T \}$. They obey the Lie algebra defining commutation relations
\[
[S^\alpha; S^\beta] = i f_{\alpha \beta \gamma} S^\gamma , \quad [T^a; T^b] = i f_{abc} T^c .
\]
(69)

$\sigma^\alpha$ and $\tau^a$ are the respective generators of $SU(N)$ and $SU(M)$ in their fundamental representations that we denote $\{ \sigma \}$ and $\{ \tau \}$. They also obey the Lie algebra defining commutation relations
\[
\left[ \frac{\sigma^\alpha}{2}; \frac{\sigma^\beta}{2} \right] = i f_{\alpha \beta \gamma} \frac{\sigma^\gamma}{2} , \quad \left[ \frac{\tau^a}{2}; \frac{\tau^b}{2} \right] = i f_{abc} \frac{\tau^c}{2} ,
\]
(70)
where $f_{abc}$ and $f_{\alpha \beta \gamma}$ are completely antisymmetric tensors. We choose them to be normalized such that
\[
\text{Tr} [\sigma^\alpha \sigma^\beta] = 2 \delta_{\alpha \beta} \quad \text{and} \quad \text{Tr} [\tau^a \tau^b] = 2 \delta_{ab} .
\]
(71)

Furthermore, they have the property (special to the fundamental representations)
\[
\left\{ \frac{\sigma^\alpha}{2}; \frac{\sigma^\beta}{2} \right\} = d_{\alpha \beta \gamma} \frac{\sigma^\gamma}{2} + \frac{1}{N} \delta_{\alpha \beta} \mathbb{I}_\sigma , \quad \left\{ \frac{\tau^a}{2}; \frac{\tau^b}{2} \right\} = d_{abc} \frac{\tau^c}{2} + \frac{1}{M} \delta_{ab} \mathbb{I}_\tau ,
\]
(72)
where $d_{abc}$ and $d_{a\beta\gamma}$ are totally symmetric tensors (so-called the structure tensors of the Lie algebras). These two sets of properties lead to the following multiplication laws

$$\frac{\sigma^a \sigma^b}{2} = \frac{1}{2N} \delta_{ab} + \frac{1}{2} (d_{abc} + i f_{abc}) \frac{\sigma^c}{2}$$  \hspace{1cm} (73)$$

$$\frac{\tau^a \tau^b}{2} = \frac{1}{2M} \delta_{ab} + \frac{1}{2} (d_{a\beta\gamma} + i f_{a\beta\gamma}) \frac{\tau^\gamma}{2}$$  \hspace{1cm} (74)$$

that used iteratively can decomposed any product of generators $\sigma^{a_1} \sigma^{a_2} \ldots \sigma^{a_n}$ onto the complete basis formed by $I_\tau$ and the $\sigma^{a_i}$s.

The Casimir invariants are defined as

$$C_n^S \equiv \text{Tr} \left[ \frac{\sigma^{a_1}}{2} \ldots \frac{\sigma^{a_n}}{2} \right] S^{a_1} \ldots S^{a_n},$$

where $I_S$ is the identity in the representation of $S$ and $\{a_1 \ldots a_n\}$ stands for the sum over all permutations weighted by $1/n!$.

**Identification of $b$.** Let us work with the matrix $Q^c \equiv \{T^a; T^b\} d_{abc}$. It is an Hermitian matrix, therefore $\text{Tr} \left[ (Q^c)^2 \right] > 0$. A sub-basis of the vector space of such Hermitian matrices is given by $\{I, T^a (a = 1 \ldots M^2 - 1)\}$. The following scalar product can be given to that sub-vector space: $\langle Q^c | Q^d \rangle \equiv \text{Tr} [Q^c Q^d]$. Indeed one has $\text{Tr} [T^a T^b] \propto \delta_{ab}$. The basis can be completed to the full vector space by a set of traceless matrices $U^k$ with $\text{Tr} [T^a U^k] = 0$, $\text{Tr} [U^k U^k] \propto \delta_{kk}$. In full generality, $Q^c$ can be decomposed in this complete basis as

$$Q^c = a^c I_\tau + b^c T^a + c^c U^k.$$  \hspace{1cm} (76)$$

Now, in order for the expression $Q^c \tau^c$ to be a scalar (i.e. a quantity that is invariant under the simultaneous rotations of the isospin of the impurity and the one of the conduction electrons) we must have the coefficients $a^c = 0$. Below, we check that indeed $a^c = 0$ and we identify $b^c$. In order to extract the coefficient $a^c$, we trace the above expression:

$$\text{Tr} \left[ \{T^a; T^b\} \right] d_{abc} = a^c d_{\{T\}},$$

where $d_{\{T\}}$ is the dimension of the representation $\{T\}$. Using the relation

$$\text{Tr} [T^a T^b] = \frac{1}{2} \frac{d_{\{T\}} C_2(\{T\})}{d_{\{\tau\}} C_2(\{\tau\})} \delta_{ab},$$

where $d_{\{\tau\}} = M$ is the dimension of the fundamental representation of $SU(M)$, and $C_2(\{T\})$ is the eigen-value of the second order Casimir in the representation $\{T\}$ $[C_2(\{T\}) = \frac{M^2 - 1}{2M}]$ is the eigen-value of the second order Casimir in the fundamental representation], we obtain

$$\frac{d_{\{\tau\}} C_2(\{T\})}{d_{\{\tau\}} C_2(\{\tau\})} d_{abc} = a^c d_{\{T\}}.$$  \hspace{1cm} (79)$$

The LHS is zero because the tensor $d_{abc}$ is traceless, therefore we get $a^c = 0$.

In order to extract the coefficient $b^c$, we multiply Eq. (76) by $T^c$ and trace:

$$\text{Tr} [Q^c T^c] = b^c \text{Tr} [T^a T^c].$$  \hspace{1cm} (80)$$

Using the relation (78) we obtain

$$\text{Tr} \left[ \{T^a; T^b\} T^c \right] d_{abc} = \frac{1}{2} b^c \frac{d_{\{T\}} C_2(\{T\})}{d_{\{\tau\}} C_2(\{\tau\})}.$$  \hspace{1cm} (81)$$

One can also compute the LHS of Eq. (80) as

$$\text{Tr} \left[ \{T^a; T^b\} T^c \right] d_{abc} = 2 s \text{Tr} [T^a, T^b, T^c] d_{abc} \equiv 2 g^{(T)} d_{abc}.$$  \hspace{1cm} (82)$$

with the symmetrized trace defined as

$$s \text{Tr} [T^{a_1}, \ldots, T^{a_n}] = \frac{1}{n!} \sum_\pi \text{Tr} [T^{a_{\pi(1)}} \ldots T^{a_{\pi(n)}}].$$  \hspace{1cm} (83)$$
where the sum is performed over all permutations of the indices. Using the following property (c.f. Eq. (114) in [1])

\[
d_{abc}d_{abc} = \frac{1}{d_{\text{Adj}}}d_{ab\delta}d_{ab\delta}\delta_{ce}
\]  

(84)

where \(d_{\text{Adj}} = M^2 - 1\) is the dimension of the adjoint representation of \(SU(M)\), together with the contraction identity (c.f. Eq. (A21) in [2])

\[
d_{ab\delta}d_{ab\delta} = d_{(T)}C_3\{T\},
\]

(85)

[where \(C_3\{T\}\) is the eigen value of the third order Casimir – in the fundamental representation, \(C_3\{\tau\} = (M^2 - 1)(M^2 - 4)/4M^2\) we get

\[
2\frac{d_{(T)}}{d_{\text{Adj}}}C_3\{T\}\delta_{ce} = \frac{1}{2}b^c d_{\{\tau\}}C_2\{\tau\}C_3\{T\},
\]

(86)

implying that \(b^c = b\delta_{ce}\) where \(b\) depends on the representation and is given by

\[
b\{T\} = 4\frac{d_{\{\tau\}}C_2\{\tau\}}{d_{\text{Adj}}C_2\{T\}}\cdot C_3\{T\} = 2\frac{C_3\{T\}}{C_2\{T\}}.
\]

(87)

Notice that in the fundamental representation, \(C_3\{\tau\} = \frac{(M^2 - 1)(M^2 - 4)}{4M^2}\) so that \(b\{\tau\} = (M^2 - 4)/M\).

[1] T. van Ritbergen A.N. Schellekens, J.A.M Vermaseren, arXiv:hep-ph/9802376 (1998).
[2] S. Okubo, Phys. Rev. D 16, 3528 (1977).