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Linear $r$-matrix algebra for classical separable systems

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Abstract

We consider a hierarchy of the natural type Hamiltonian systems of $n$ degrees of freedom with polynomial potentials separable in general ellipsoidal and general paraboloidal coordinates. We give a Lax representation in terms of $2 \times 2$ matrices for the whole hierarchy and construct the associated linear $r$-matrix algebra with the $r$-matrix dependent on the dynamical variables. A Yang-Baxter equation of dynamical type is proposed. Using the method of variable separation we provide the integration of the systems in classical mechanics constructing the separation equations and, hence, the explicit form of action variables. The quantisation problem is discussed with the help of the separation variables.

Key words: separable systems, linear $r$-matrix algebras, classical Yang-Baxter equation, Lax representation, Liouville integrable systems

AMS classification: 70H20, 70H33, 58F07
1 Introduction

The method of separation of variables in the Hamilton-Jacobi equation,

\[ H(p_1, \ldots, p_n, x_1, \ldots, x_n) = E, \quad p_i = \frac{\partial W}{\partial x_i}, \quad i = 1, \ldots, n, \quad (1.1) \]

is one of the most powerful methods for the construction of action for the Liouville integrable systems of classical mechanics [3]. We consider below the systems of the natural form described by the Hamiltonian

\[ H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(x_1, \ldots, x_n), \quad p_i, x_i \in \mathbb{R}. \quad (1.2) \]

The separation of variables means the solution of the partial differential equation (1.1) for the action function \( W \) in the following additive form

\[ W = \sum_{i=1}^{n} W_i(\mu_i; H_1, \ldots, H_n), \quad H_n = H, \]

where \( \mu_i \) will be called separation variables. Notice that the partial functions \( W_i \) depend only on their separation variables \( \mu_i \), which define new set of variables instead of the old ones \( \{x_k\} \), and on the set of constants of motion, or integrals of motion, \( \{H_k\} \). In the sequel we shall speak about coordinate separation where the separation variables \( \{\mu_i\} \) are the functions of the coordinates \( \{x_k\} \) only. (The general change of variables may include also corresponding momenta \( \{p_k\} \)).

For a free particle (\( V = 0 \)), the complete classification of all orthogonal coordinate systems in which the Hamilton-Jacobi equation (1.1) admits the separation of variables is known, they are generalized \( n \)-dimensional ellipsoidal and paraboloidal coordinates [8, 9] (see also the references therein). It is also known that the Hamiltonian systems (1.2) admitting an orthogonal coordinate separation with \( V \neq 0 \) are separated only in the same coordinate systems.

The modern approach to finite-dimensional integrable systems uses the language of the representations of \( r \)-matrix algebras [10, 15, 17, 18]. The classical method of separation of variables can be formulated within this language dealing with the representations of linear and quadratic \( r \)-matrix algebras [11, 12, 17, 19]. For the \( 2 \times 2 \) \( L \)-operators the recipe consists of considering the zeros of one of the off-diagonal elements as the separation variables (see also generalization of this approach to the higher dimensions of \( L \)-matrix [19]). For \( V = 0 \) in [11, 12] \( 2 \times 2 \) \( L \)-operators were given satisfying the standard linear \( r \)-matrix algebra [10, 15],

\[ \{L_1(u), L_2(v)\} = [r(u - v), L_1(u) + L_2(v)], \quad r(u) = \frac{1}{u} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.3) \]

and the link with the separation of variables method was elucidated. Here in (1.3) we use the familiar notations for the tensor products of \( L(u) \) and \( 2 \times 2 \) unit matrix \( I \), \( L_1(u) = L(u) \otimes I \), \( L_2(v) = I \otimes L(v) \).
In the present article we construct $2 \times 2$ $L$-operators for the systems \([1,2]\) being separated in the generalized ellipsoidal and paraboloidal coordinates. In the case when the degree $N$ of the potential $V$ is equal to 1 or 2 the associated linear $r$-matrix algebra appears to be the standard one \([1,3]\). In the case $N > 2$ the algebra is of the form

$$\{L_1(u), L_2(v)\} = [r(u - v), L_1(u) + L_2(v)] + [s(u, v), L_1(u) - L_2(v)],$$

(1.4)

with $s(u, v) = \alpha_N(u, v) \sigma_\perp \otimes \sigma_\perp$, where $\sigma_\perp = \sigma_1 - i \sigma_2$ and $\sigma_i$ are the Pauli matrices, and $\alpha_N(u, v)$ is the function which equals 1 for $N = 3$ and depends on the dynamical variables $\{x_k, p_k\}_{k=1}^n$ for $N > 3$.

The study of completely integrable systems admitting a classical $r$-matrix Poisson structure with the $r$-matrix dependent on dynamical variables has attracted some attention \([3, 7]\). It is remarkable that the celebrated Calogero-Moser system, whose complete integrability was shown a number of years ago (c.f. \([14]\)), has been found only recently to possess a classical $r$-matrix of dynamical type \([4]\).

Below we recap briefly how to get the $2 \times 2$ $L$-operators for the separable systems \([1,2]\) without potential $V$ \([1]\). Let us consider the direct sum of the Lie algebras each of rank 1: $\mathcal{A} = \bigoplus_{k=1}^n \mathfrak{so}(2,1)$. Generators $\vec{s}_k \in \mathbb{R}^3$, $k = 1, \ldots, n$ of the $\mathcal{A}$ algebra satisfy the following Poisson brackets:

$$\{s^i_k, s^j_m\} = \delta_{km} \varepsilon_{ijl} g_{ll} s^j_k, \quad g = \text{diag}(1, -1, -1).$$

(1.5)

Everywhere below, we will imply the $g$ metric to calculate the norm and scalar product of the vectors $\vec{s}_i$:

$$\vec{s}_i^2 \equiv (\vec{s}_i, \vec{s}_i) = (s^{11}_i)^2 - (s^{22}_i)^2 - (s^{33}_i)^2,$$

$$\vec{s}_i \vec{s}_k = s^1_is^1_k - s^2_is^2_k - s^3_is^3_k.$$

Let us fix the values of the Casimir elements of the $\mathcal{A}$ algebra: $\vec{s}_i^2 = c^2_i$, then variables $\vec{s}_i$ lie on the direct product of $n$ hyperboloids in $\mathbb{R}^3$. Let $c_i \in \mathbb{R}$ and choose the upper sheets of these double-sheeted hyperboloids. Denote the obtained manifold as $K_n^+$. We will denote by hyperbolic Gaudin magnet \([7]\) integrable Hamiltonian system on $K_n^+$ given by $n$ integrals of motion $H_i$ which are in involution with respect to the bracket \([1, 3]\),

$$H_i = 2 \sum_{k=1}^n (\vec{s}_i, \vec{s}_k) e_i - e_k, \quad \{H_i, H_k\} = 0, \quad e_i \neq e_k \quad \text{if} \quad i \neq k.$$

(1.6)

To be more exact one has to call this model an $n$-site $so(2,1)$-$XXX$ Gaudin magnet. Notice that all the $H_i$ are quadratic functions on generators of the $\mathcal{A}$ algebra and the following equalities are valid

$$\sum_{i=1}^n H_i = 0, \quad \sum_{i=1}^n e_i H_i = \vec{J}^2 - \sum_{i=1}^n c^2_i;$$

where the new variable $\vec{J} = \sum_{i=1}^n \vec{s}_i$ is introduced which is the total sum of the hyperbolic momenta $\vec{s}_i$. The components of the vector $\vec{J}$ obey $so(2,1)$ Lie algebra with respect to the bracket \([1, 3]\) and are in involution with all the $H_i$. The complete set of involutive integrals of motion is provided by the following choice: $H_i, \vec{J}^2$ and, for
example, \((J^3)^2\). The integrals \((1.6)\) are generated by the \(2 \times 2\) \(L\)-operator (as well as the additional integrals \(\vec{J}_L(u)\))

\[
L(u) = \sum_{j=1}^{n} \frac{1}{u - e_j} \begin{pmatrix}
    is_j^3 & -(s_j^1 - s_j^2) \\
    -(s_j^1 + s_j^2) & -is_j^3
\end{pmatrix},
\]

\[
\det L(u) = -\sum_{j=1}^{n} \left( \frac{H_j}{u - e_j} + \frac{c_j^2}{(u - e_j)^2} \right),
\]

(1.7)
satisfying the standard linear \(r\)-matrix algebra \((1.3)\). Let \(c_i = 0, i = 1, \ldots, n\), then the hyperboloids \(s_i^2 = c_i^2\) turn into cones. The manifold \(K_n^+\) admits in this case the following parameterization \((p_i, x_i \in \mathbb{R})\):

\[
s_1^i = \frac{p_i^2 + x_i^2}{4}, \quad s_2^i = \frac{p_i^2 - x_i^2}{4}, \quad s_3^i = \frac{p_i x_i}{2},
\]

(1.8)

where the variables \(p_i\) and \(x_i\) are canonically conjugated. Using the isomorphism \((1.8)\) the complete classification of the separable orthogonal coordinate systems was provided in \([11, 12]\) by means of the corresponding \(L\)-operators satisfying the standard linear \(r\)-matrix algebra \((1.3)\). The starting point for our investigation are these \(L\)-operators written for the cases of free motion on a sphere and in the Euclidean space.

The paper is organized as follows. In Section 2 we describe the classical Poisson structure associated with the hierarchy of natural type Hamiltonians separable in the three coordinate systems—spherical (for motion on a sphere), and general ellipsoidal and paraboloidal (for \(n\)-dimensional Euclidean motion) coordinates. This structure is given in terms of the linear \(r\)-matrix formalism, providing new example of the dynamical dependence of the \(r\)-matrices. We also introduce analogue of Yang-Baxter equation for our dynamical \(r\)-matrices. In the Section 3 we derive the Lax representation for all the hierarchy as a consequence of the \(r\)-matrix representation given in the Section 2. Section 3 deals also with the variable separation material. The question of quantisation of the considered systems is briefly discussed.

### 2 Classical Poisson structure

Let us consider the following ansatz for the \(2 \times 2\) \(L\)-operator

\[
L_N(u) = \begin{pmatrix}
    A(u) & B(u) \\
    C_N(u) & -A(u)
\end{pmatrix}
\]

(2.1)

where

\[
B(u) = \varepsilon - \sum_{i=1}^{n} \frac{x_i^2}{u - e_i}, \quad \varepsilon = 0, 1, \text{ or } 4(u - x_{n+1} + B),
\]

(2.2)

\[
A(u) = \dot{\varepsilon} + \frac{1}{2} \sum_{i=1}^{n} \frac{x_i p_i}{u - e_i},
\]

(2.3)

\[
C_N(u) = \sum_{i=1}^{n} \frac{p_i^2}{u - e_i} - V_N(u), \quad V_N(u) = \sum_{k=0}^{N} V_k u^{N-k}.
\]

(2.4)
Here the \( x_i, p_j \) are canonically conjugated variables (\( \{ p_i, x_j \} = \delta_{ij} \)), \( V_k \) are indeterminate functions of the \( x \)-variables; \( B \) and \( e_i \) are non-coincident real constants. Note that \( \dot{\varepsilon} \) means differentiation by time, and for natural Hamiltonian (1.2) one has \( \dot{x}_{n+1} = p_{n+1} \).

**Theorem 1** Let the curve \( \det(L(u) - \lambda I) = 0 \) for the \( L \)-operator (2.1) have the form

\[
\lambda^2 - A(u)^2 - B(u)C_N(u) = \lambda^2 + \varepsilon u^N - \sum_{i=1}^{n} \frac{H_i}{u - e_i} = 0,
\]

for \( \varepsilon = 0, 1 \), and

\[
\lambda^2 - A(u)^2 - B(u)C_N(u) = \lambda^2 + 16u^{N-2}(u + B)^2 + 8H - \sum_{i=1}^{n} \frac{H_i}{u - e_i} = 0,
\]

for \( \varepsilon = 4(u - x_{n+1} + B) \), (2.5)

with some integrals of motion \( H_i \) and \( H_1, H \) in the case of (2.6). Then the following recurrence relations for \( V_k \) are valid

\[
V_k = \sum_{i=1}^{n} x_i^2 \sum_{j=0}^{k-1} V_{k-1-j} e_i^j, \quad V_0 = 1,
\]

for \( \varepsilon = 0, 1 \); \( \varepsilon = 0, 1 \); \( \varepsilon = 4(u - x_{n+1} + B) \); (2.7)

\[
V_k = (x_{n+1} + B)V_{k-1} + \frac{1}{2} \sum_{i=1}^{n} x_i^2 \sum_{j=1}^{k-1} V_{k-1-j} e_i^j, \quad V_0 = 0,
\]

for \( \varepsilon = 4(u - x_{n+1} + B) \). (2.8)

The explicit formulae for the integrals \( H_i \) have the form

\[
H_i = -\sum_{j=1}^{n} \frac{M_{ij}^2}{e_i - e_j} + \varepsilon \cdot p_i^2 + x_i^2 \sum_{k=0}^{N} V_k e_i^{N-k}, \quad \text{for} \quad \varepsilon = 0, 1,
\]

\[
H_i = 2x_i^2 \sum_{j=1}^{N-1} (-1)^{j-1} e_i^j V_{N-j} + 4p_{n+1}x_i - p_i^2(e_i + 4x_{n+1} - 4B)
\]

\[+ \sum_{j \neq i}^{n} \frac{M_{ij}^2}{e_i - e_j}, \quad \text{for} \quad \varepsilon = 4(u - x_{n+1} + B),
\]

(2.9)

where \( M_{ij} = x_i p_j - x_j p_i \). The Hamiltonians \( H \) are given by

\[
H \equiv \sum_{i=1}^{n} H_i = \varepsilon \cdot \sum_{i=1}^{n} p_i^2 + V_{N+1}, \quad \text{for} \quad \varepsilon = 0, 1,
\]

\[
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V_N, \quad \text{for} \quad \varepsilon = 4(u - x_{n+1} + B).
\]

(2.11)

(2.12)

**Proof** is straightforward and based on direct computations.

We remark, that the above recurrence formulae for the potentials can be written in differential form. In particular, for the paraboloidal coordinates we have

\[
\frac{\partial V_N}{\partial x_i} = \frac{1}{2} \frac{\partial V_{N-1}}{\partial x_{n+1}} x_i - A_i \frac{\partial V_{N-1}}{\partial x_i}, \quad i = 1, \ldots, n,
\]

\[= V_{N-1} + \frac{1}{2} \sum_{i=1}^{n} x_i \frac{\partial V_{N-1}}{\partial x_i} + (x_{n+1} - B) \frac{\partial V_{N-1}}{\partial x_{n+1}}.
\]

(2.13)

(2.14)
Notice that the case of $\varepsilon = 0$ is connected with the ellipsoidal coordinates on a sphere and two other cases $\varepsilon = 1, 4(u - x_{n+1} + B)$ describe the ellipsoidal and paraboloidal coordinates in the Euclidean space, respectively (see Section 3.2 and [11, 12] for more details). Recall that we study now the motion of a particle on these manifolds under the external field with the potential $V$ that could be any linear combination of the homogeneous ones $\mathcal{V}_k$.

Now we are ready to describe the linear algebra for the $L$-operator (2.1).

**Theorem 2** Let the $L$-matrix be of the form (2.1) and satisfy the conditions of the Theorem 1, then the following algebra is valid for its entries

\[
\{B(u), B(v)\} = \{A(u), A(v)\} = 0, \quad (2.15)
\]

\[
\{C_N(u), C_N(v)\} = -4 \alpha_N(u, v) (A(u) - A(v)), \quad (2.16)
\]

\[
\{B(u), A(v)\} = \frac{2}{u - v} (B(u) - B(v)), \quad (2.17)
\]

\[
\{C_N(u), A(v)\} = -\frac{2}{u - v} (C_N(u) - C_N(v)) - 2 \alpha_N(u, v) B(v), \quad (2.18)
\]

\[
\{B(u), C_N(v)\} = -\frac{4}{u - v} (A(u) - A(v)), \quad (2.19)
\]

where the function $\alpha_N(u, v)$ has the form

\[
\alpha_N(u, v) = \frac{Q_N(u) - Q_N(v)}{u - v} = \sum_{k=1}^{N} Q_k \frac{u^k - v^k}{u - v}, \quad (2.20)
\]

\[
Q_N(u) = \sum_{k=0}^{N} Q_k u^k, \quad Q_k = \sum_{m=0}^{k} \mathcal{V}_m \mathcal{V}_{k-m}.
\]

**Proof** is based on the recurrence relations (2.7), (2.8).

We remark that for the paraboloidal coordinates the following formula is valid

\[
Q(u) = u^{N-2} - \frac{1}{2} \sum_{k=0}^{N-3} \frac{\partial \mathcal{V}_{N-k-1}}{\partial x_{n+1}}, \quad (2.21)
\]

therefore in this case we have

\[
Q(u) = \frac{1}{4} \frac{\partial C(u)}{\partial x_{n+1}}. \quad (2.22)
\]

The algebra (2.15)-(2.19) can be rewritten in the matrix form as linear $r$-matrix algebra

\[
\{L_1(u), L_2(v)\} = [r(u - v), L_1(u) + L_2(v)] + [s_N(u, v), L_1(u) - L_2(v)], \quad (2.23)
\]

using $4 \times 4$ notations $L_1(u) = L(u) \otimes I, L_2(v) = I \otimes L(v)$; the matrices $r(u - v)$ and $s_N(u, v)$ are given by

\[
r(u - v) = \frac{2}{u - v} P, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.24)
\]

\[
s_N(u, v) = 2 \alpha_N(u, v) \sigma_- \otimes \sigma_-, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
The algebra \((2.15)-(2.19)\) or \((2.23)-(2.24)\) contains all the information about the system under consideration. From it there follows the involutivity of the integrals of motion. Indeed, the determinant \(d(u) \equiv \det L(u)\) is the generating function for the integrals of motion and it is simply to show that

\[
\{d(u), d(v)\} = 0. \tag{2.25}
\]

In particular, the integrals \(H_i \ (2.9), \ (2.10)\) are the residues of the function \(d(u)\):

\[
H_i = \text{res}_{u=e_i} d(u), \quad i = 1, \ldots, n.
\]

The Hamiltonians \(H \ (2.11), (2.12)\) appear to be a residue at infinity. Let us rewrite the relation \((2.23)\) in the form

\[
\{L_1(u), L_2(v)\} = [d_{12}(u,v), L_1(u)] - [d_{21}(u,v)L_2(v)], \tag{2.26}
\]

with \(d_{ij} = r_{ij} + s_{ij}, \ d_{ji} = s_{ij} - r_{ij} \) at \(i < j\).

**Theorem 3** The following equations (dynamical Yang-Baxter equations) are valid for the algebra \((2.24)\)

\[
[d_{12}(u,v), d_{13}(u,w)] + [d_{12}(u,v), d_{23}(v,w)] + [d_{32}(w,v), d_{13}(u,w)] +
+\{L_2(v), d_{13}(u,w)\} - \{L_3(w), d_{12}(u,v)\} +
+[c(u,v,w), L_2(v) - L_3(w)] = 0, \tag{2.27}
\]

where \(c(u,v,w)\) is some matrix dependent on dynamical variables. Other two equations are obtained from \((2.27)\) by cyclic permutations.

**Proof** Let us write the Jacobi identity as

\[
\{L_1(u), \{L_2(v), L_3(w)\}\} + \{L_3(w), \{L_1(u), L_2(v)\}\}
+\{L_2(v), \{L_3(w), L_1(u)\}\} = 0 \tag{2.28}
\]

with \(L_1(u) = L(u) \otimes I \otimes I, \ L_2(v) = I \otimes L(v) \otimes I, \ L_3(w) = I \otimes I \otimes L(w)\). The extended form of the \((2.28)\) reads \[13\],

\[
[L_1(u), [d_{12}(u,v), d_{13}(u,w)]] + [d_{12}(u,v), d_{23}(v,w)] +
+[d_{32}(w,v), d_{13}(u,w)] +
+[L_1(u), [L_2(v), d_{13}(u,w)]] - \{L_3(w), d_{12}(u,v)\} +
+\text{cyclic permutations} = 0. \tag{2.29}
\]

Further we restrict ourselves proving the \((2.27)\) only in the paraboloidal case (other cases can be handled in a similar way). To complete the derivation of \((2.27)\) we shall prove the following equality for all the members of the hierarchy

\[
\{L_2(v), s_{13}(u,w)\} - \{L_3(w), s_{12}(u,v)\} = 2\beta_N(u,v,w)[P_{23}, s_{13} + s_{12}]
- \frac{\partial \beta_N(u,v,w)}{\partial x_n+1}[s, L_2(v) - L_3(w)] \tag{2.30}
\]
(with cyclic permutations). In (2.30) the matrix \( s = \sigma_- \otimes \sigma_- \otimes \sigma_- \) and
\[
\beta_N(u, v, w) = \frac{Q_N(u)(v - w) + Q_N(v)(w - u) + Q_N(u)(u - v)}{(u - v)(v - w)(w - u)}.
\]

In the extended form (2.30) can be rewritten as
\[
\{Q(u), Q(v)\} = \{B(u), Q(v)\} = 0,
\]
\[
\{A(u), Q(v)\} = 4 \alpha_N(u, v) - \frac{1}{2} \frac{\partial \alpha_N(u, v)}{\partial x_{n+1}} B(u),
\]
\[
\{A(w), \alpha_N(u, v)\} = \frac{4}{u - v} (\alpha_N(w, u) - \alpha_N(w, v))
- \frac{1}{2} \frac{B(w)}{u - v} \left( \frac{\partial \alpha_N(w, v)}{\partial x_{n+1}} - \frac{\partial \alpha_N(w, u)}{\partial x_{n+1}} \right),
\]
\[
\{Q(u), C(v)\} + \{C(u), Q(v)\} = \frac{\partial \alpha_N(u, v)}{\partial x_{n+1}} (A(u) - A(v)).
\]

The equality (2.32) is trivial and equation (2.33) is derived by differentiating (2.18). Equation (2.34) follows from the definition of \( Q(u) \) and (2.33). To prove (2.34) we write it using the explicit form of \( C_N(u) \) and \( A(u) \) as follows
\[
\sum_{i=1}^{n} \frac{p_i}{(v - e_i)(u - e_i)} \left( (u - e_i) \frac{\partial \alpha_N(w, u)}{\partial x_{i}} - (v - e_i) \frac{\partial \alpha_N(w, v)}{\partial x_{i}} \right)
- \frac{1}{2} \sum_{i=1}^{n} \frac{p_i x_i}{(u - e_i)(v - e_i)} \left( \frac{\partial \alpha_N(w, u)}{\partial x_{n+1}} - \frac{\partial \alpha_N(w, v)}{\partial x_{n+1}} \right).
\]

Using the identity
\[
\frac{w^k - u^k}{w - u} - \frac{w^k - v^k}{w - v} = \frac{w^{k+1} - u^{k+1}}{w - u} - \frac{w^{k+1} - v^{k+1}}{w - v},
\]
and the recurrence relation (2.13), we find that the equality (2.36) is valid. Therefore the equations (2.27) follow with the matrix \( c(u, v, w) = \partial \beta(u, v, w) / \partial x_{n+1} \sigma_- \otimes \sigma_- \otimes \sigma_- \).

The proof is completed.

We remark that validity of the equations (2.27) with an arbitrary matrix \( c(u, v, w) \) is sufficient for the validity of (2.28) and, therefore, (2.27) can be interpreted as some dynamical classical Yang-Baxter equation, i.e. the associativity condition for the linear \( r \)-matrix algebra. These equations have an extra term \( [c, L_i - L_j] \) in comparison with the extended Yang-Baxter equations in [13].

We would like to emphasize that all statements of the Section can be generalized to the following form of the potential term \( V_N(u) \) in (2.4)
\[
V_{MN}(u) = \sum_{k=-M}^{N} f_k \mathcal{V}_k u^{N-k}, \quad f_k \in \mathbb{C}
\]
that corresponds to the linear combinations of homogeneous terms \( \mathcal{V}_k \) as potential \( V \) and also includes the negative degrees to separable potential. See the end of Section 3.1 for more details.
3 Consequences of the $r$-matrix representation

3.1 Lax representation

Following the article [5] we can consider the Poisson structure (2.26) for the powers of the $L$-operator

$$\{(L_1(u))^k, (L_2(v))^l\} = [d_{ij}^{(k,l)}(u,v), L_1(u)] - [d_{ij}^{(k,l)}(u,v)L_2(v)], \quad (3.1)$$

with

$$d_{ij}^{(k,l)}(u,v) = \sum_{p=0}^{k-1} \sum_{q=0}^{l-1} (L_1(u))^{k-p-1}(L_2(v))^{k-1}d_{ij}(u,v)(L_1(u))^p(L_2(v))^q. \quad (3.2)$$

As an immediate consequence of (3.1)-(3.2) we obtain that the conserved quantities $H, H_i$ are in involution. Indeed, we have

$$\{\text{Tr}(L_1(u))^2, \text{Tr}(L_2(v))^2\} = \text{Tr}\{(L_1(u))^2, (L_2(v))^2\}, \quad (3.3)$$

and after applying the equality (3.1) at $k = l = 2$ to this equation and taking the trace, we obtain the desired involutivity. Further, let us define differentiation by time as follows:

$$\dot{L}(u) = \frac{d}{dt} L(u) = \text{Tr}_2\{L_1(u), (L_2(u))^2\}, \quad (3.4)$$

where the trace is taken over the second space. Applying the equation (3.1) at $k = 1, l = 2$ to (3.4), we obtain the Lax representation in the form $\dot{L}(u) = [M(u), L(u)]$ with the matrix $M(u)$ given as

$$M(u) = 2 \lim_{v \to u} \text{Tr}_2 L_1(v)(r(u-v) - s(u,v)). \quad (3.5)$$

After the calculation in which we take into account the asymptotic behaviour of the $L$-operator (2.1), we obtain the following explicit Lax representation:

$$\dot{L}(u) = [M(u), L(u)], \quad L(u) = \begin{pmatrix} A(u) & B(u) \\ C_N(u) & -A(u) \end{pmatrix}, \quad M(u) = \begin{pmatrix} 0 & 1 \\ Q_N(u) & 0 \end{pmatrix}, \quad (3.6)$$

where $Q_N(u)$ was defined by the equations (2.20). Lax representations for the higher flows can be obtained in a similar way.

It follows from the equation (3.6) that

$$A(u) = -\frac{1}{2} \dot{B}(u), \quad C_N(u) = -\frac{1}{2} \dot{B}(u) - B(u)Q_N(u),$$

so our $L$-matrix can be given in the form

$$L(u) = \begin{pmatrix} -\frac{1}{2} \dot{B}(u) & B(u) \\ -\frac{1}{2} \dot{B}(u) - B(u)Q_N(u) & \frac{1}{2} \dot{B}(u) \end{pmatrix}. \quad (3.7)$$
The equations of motion, which follow from (3.6) with the $L$-matrix from (3.7), have the form

$$B_1[Q_N] \cdot B(u) = 0,$$

where

$$B_1[Q_N] \equiv \frac{1}{4} \partial^2 + \frac{1}{2} \{ \partial, Q_N \}, \quad \partial \equiv \frac{d}{dt},$$

with curly brackets standing for the anticommutator. Operator $B_1$ is the Hamiltonian operator of the first Hamiltonian structure for the coupled KdV equation [1, 2]. Equation (3.8), considered as one for the unknown function $B(u)$, was solved in the three cases (2.2), $B(u) = \varepsilon - \sum_{i=1}^{n} \frac{x_i^2}{u - e_i}$, $\varepsilon = 0, 1, 4(u - x_{n+1} + B)$, in [1] and [2]. General solution of this equation as one for the unknown function $Q(u)$ has the form

$$Q_{MN}(u) = \sum_{k=-M}^{N} f_k Q_k u^k, \quad f_k \in \mathbb{C},$$

where the coefficients $Q_k$ are defined from the generating function $\tilde{Q}(u)$

$$\tilde{Q}(u) \equiv B^{-2}(u) = \sum_{k=-\infty}^{+\infty} Q_k u^k.$$

Recall that we can write the element $C_N(u)$ of the $L$-matrix (2.1) in two different forms (using $Q$ or $V$ functions)

$$C_N(u) = -\frac{1}{2} \tilde{B}(u) - B(u)Q_N(u) = -\sum_{i=1}^{n} \frac{p_i^2}{u - e_i} - V(u)$$

where function $V(u) = \sum_{k=0}^{N} \nu_k u^{N-k}$ was defined in (2.4). The general form of the function $V(u)$ is as follows:

$$V_{MN}(u) = \sum_{k=-M}^{N} f_k \nu_k u^k, \quad f_k \in \mathbb{C},$$

where coefficients $\nu_k$ are defined by the generating function $\tilde{V}(u)$

$$\tilde{V}(u) \equiv B^{-1}(u) = \sum_{k=-\infty}^{+\infty} \nu_k u^k.$$

Potentials $\nu_k$ are connected with coefficients $Q_k$. Indeed, using generating functions (3.10) and (3.12), we have

$$\tilde{Q} = B^{-2}(u) = \tilde{V}(u) \cdot \tilde{V}(u) = \left( \sum_{k=-\infty}^{+\infty} \nu_k u^k \right) \left( \sum_{k=-\infty}^{+\infty} \nu_k u^k \right) = \sum_{k=-\infty}^{+\infty} u^k \sum_{j=0}^{k} \nu_j \nu_{k-j},$$

and, therefore, $Q_k = \sum_{j=0}^{k} \nu_j \nu_{k-j}$. Thus we have recovered the formula (2.20) for the $s$-matrix.
3.2 Separation of Variables

Let $K$ denote the number of degrees of freedom: $K = n - 1$ for ellipsoidal coordinates on a sphere, $K = n$ for ellipsoidal coordinates in the Euclidean space, and $K = n + 1$ for paraboloidal coordinates in the Euclidean space. The separation of variables (c.f. [12, 13]) is understood in the context of the given hierarchy of Hamiltonian systems as the construction of $K$ pairs of canonical variables $\pi_i, \mu_i, i = 1, \ldots, K$,

\[
\{\mu_i, \mu_k\} = \{\pi_i, \pi_k\} = 0, \quad \{\pi_i, \mu_k\} = \delta_{ik}, \tag{3.13}\]

and $K$ functions $\Phi_j$ such that

\[
\Phi_j (\mu_j, \pi_j, H_N^{(1)}, \ldots, H_N^{(K)}) = 0, \quad j = 1, 2, \ldots, K, \tag{3.14}\]

where $H_N^{(i)}$ are the integrals of motion in involution. The equations (3.14) are the separation equations. The integrable systems considered admit the Lax representation in the form of $2 \times 2$ matrices (3.6) and we will introduce the separation variables $\pi_i, \mu_i$ as

\[
B(\mu_i) = 0, \quad \pi_i = A(\mu_i), \quad i = 1, \ldots, K. \tag{3.15}\]

Below we write explicitly these formulae for our systems. The set of zeros $\mu_j, j = 1, \ldots, K$ of the function $B(u)$ defines the spherical ($\varepsilon = 0$), general ellipsoidal ($\varepsilon = 1$) and general paraboloidal ($\varepsilon = 4(u - x_{n+1} + B)$) coordinates given by the formulae [8, 9, 12]

\[
x_m^2 = c \frac{\prod_{j=1}^{n-1} (\mu_j - e_m)}{\prod_{k \neq m} (e_m - e_k)}, \quad m = 1, \ldots, n, \tag{3.16}\]

where \( c = \sum_{k=1}^{n} x_k^2 \), for $\varepsilon = 0$;

\[
x_m^2 = -4 \frac{\prod_{j=1}^{n} (\mu_j - e_m)}{\prod_{k \neq m} (e_m - e_k)}, \quad m = 1, \ldots, n, \tag{3.17}\]

for $\varepsilon = 1$;

\[
x_{n+1} = -\sum_{i=1}^{n} e_i + B + \sum_{i=1}^{n+1} \mu_i, \tag{3.18}\]

\[
x_m^2 = -4 \frac{\prod_{j=1}^{n+1} (\mu_j - e_m)}{\prod_{k \neq m} (e_m - e_k)}, \quad m = 1, \ldots, n, \tag{3.18}\]

for $\varepsilon = 4(u - x_{n+1} + B)$.

**Theorem 4** The coordinates $\mu_i, \pi_i$ given by (3.15) are canonically conjugated.

**Proof** Let us list the commutation relations between $B(v)$ and $A(u)$,

\[
\{B(u), B(v)\} = \{A(u), A(v)\} = 0, \tag{3.19}\]

\[
\{A(u), B(v)\} = \frac{2}{v - u} (B(u) - B(v)). \tag{3.20}\]
The equalities \( \{ \mu_i, \mu_j \} = 0 \) follow from (3.19). To derive the equality \( \{ \mu_i, \pi_j \} = -\delta_{ij} \) we substitute \( u = \mu_j \) in (3.20), obtaining thus
\[
\{ \pi_j, B(v) \} = -\frac{2}{v - \mu_j} B(v),
\]
which together with the equation
\[
0 = \{ \pi_j, B(\mu_i) \} = \{ \pi_j, B(v) \} \big|_{v=\mu_i} + B'(\mu_i) \{ \pi_j, \mu_i \}
\]
gives
\[
\{ \pi_j, \mu_i \} = -\frac{1}{B'(\mu_i)} \{ \pi_j, B(v) \} \big|_{v=\mu_i} = \delta_{ij}.
\]
Equalities \( \{ \pi_i, \pi_j \} = 0 \) can be verified in the similar way:
\[
\{ \pi_i, \pi_j \} = \{ A(\mu_i), A(\mu_j) \}
= \{ A(\mu_j), A(v) \} \big|_{v=\mu_i} + A'(\mu_i) \{ \mu_i, A(\mu_j) \}
= A'(\mu_j) \{ A(\mu_i), \mu_j \} + A'(\mu_i) \{ \mu_i, A(\mu_j) \} = 0.
\]
The separation equations have the form
\[
\pi_i^2 = d(\mu_i), \quad (3.21)
\]
where the function \( d(u) \) is the determinant of the \( L \)-operator (2.25).

### 3.3 Quantization

The separation of variables has a direct quantum counterpart [11, 20]. To pass to quantum mechanics we change the variables \( \pi_i, \mu_i \) to operators and the Poisson brackets (3.13) to the commutators
\[
[\mu_j, \mu_k] = [\pi_j, \pi_k] = 0, \quad [\pi_j, \mu_k] = -i\delta_{jk}. \quad (3.22)
\]
Suppose that the common spectrum of \( \mu_i \) is simple and the momenta \( \pi_i \) are realized as the derivatives \( \pi_j = -i \frac{\partial}{\partial \mu_j} \). The separation equations (3.21) become the operator equations, where the noncommuting operators are assumed to be ordered precisely in the order as those listed in (3.14), that is \( \pi_i, \mu_i, H_N^{(1)}, \ldots, H_N^{(K)} \). Let \( \Psi(\mu_1, \ldots, \mu_K) \) be a common eigenfunction of the quantum integrals of motion:
\[
H_N^{(i)} \Psi = \lambda_i \Psi, \quad i = 1, \ldots, K. \quad (3.23)
\]
Then the operator separation equations lead to the set of differential equations
\[
\Phi_j(-i \frac{\partial}{\partial \mu_j}, \mu_j, H_N^{(1)}, \ldots, H_N^{(K)}) \Psi(\mu_1, \ldots, \mu_K) = 0, \quad j = 1, \ldots, K, \quad (3.24)
\]
which allows the separation of variables
\[
\Psi(\mu_1, \ldots, \mu_K) = \prod_{j=1}^{K} \psi_j(\mu_j). \quad (3.25)
\]
The original multidimensional spectral problem is therefore reduced to the set of one-dimensional multiparametric spectral problems which have the following form in the context of the problems under consideration

\[
\left( \frac{d^2}{du^2} + \varepsilon u^N + \sum_{i=1}^{n} \frac{\lambda_i}{u - e_i} \right) \psi_j(u; \lambda_1, \ldots, \lambda_n) = 0,
\]

for \( \varepsilon = 0, 1, \) \hspace{1cm} (3.26)

\[
\left( \frac{d^2}{du^2} + 16u^{N-2}(u + B)^2 + 8\lambda_{n+1} + \sum_{i=1}^{n} \frac{\lambda_i}{u - e_i} \right) \psi_j(u; \lambda_1, \ldots, \lambda_{n+1}) = 0,
\]

for \( \varepsilon = 4(u - x_{n+1} + B), \) \hspace{1cm} (3.27)

with the spectral parameters \( \lambda_1, \ldots, \lambda_{n+1}. \) The problems (3.26), (3.27) must be solved on the different intervals—“permitted zones”—for the variable \( u. \)

4 Conclusion

We remark that all systems considered yield the algebra which has general properties being independent on the type of the system. Therefore it would be interesting to consider its Lie-algebraic origin within the general approach to the classical \( r \)-matrices \([16]\).

There exists an interesting link of the algebra studied here with the restricted flow formalism for the stationary flows of the coupled KdV (cKdV) equations \([1]\). The Lax pairs which have been derived in the paper from the algebraic point of view were recently found in \([2]\) by consideration of the bi-Hamiltonian structure of cKdV.

It seems to be interesting to study the same questions for the generalized hierarchy of differential operators of Gelfand-Dickey for which the corresponding \( L \)-operators have to be the \( n \times n \) matrices.

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