A KRICHEVER–NOVIKOV FORMULATION OF CLASSICAL
W ALGEBRAS ON RIEMANN SURFACES

by

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Abstract

It is shown how the theory of classical $W$–algebras can be formulated on a higher genus Riemann surface in the spirit of Krichever and Novikov. An intriguing relation between the theory of $A_1$ embeddings into simple Lie algebras and the holomorphic geometry of Riemann surfaces is exhibited.
0. Introduction

Recently, a large body of literature has been devoted to the development of the theory of $W$--algebras (see refs. [1,2,3] for a comprehensive review). Such studies show that one can associate a $W$--algebra $W^g_t$ to any non trivial $A_1$ subalgebra $t$ of the Lie algebra $g$ of a complex simple Lie group $G$. The reduction of the adjoint representation of $g$ with respect to $t$ plays a fundamental role. The issue of the potentially non trivial relation between the algebraic structure of $W^g_t$ and the geometry and the topology of the base surface is not addressed since the latter is assumed to be merely a sphere with two punctures. A few years ago, I. M. Krichever and S. P. Novikov showed that there exist generalizations of the Heisenberg and Virasoro algebra on any compact connected oriented Riemann surface of genus $\ell \geq 2$ with two distinguished points $P_+$ and $P_-$ in general position [4,5] (see also ref. [6] for a pedagogical introduction). Their construction was subsequently generalized to super Virasoro [7,8] and Kac–Moody [9] algebras. In this communication, I shall outline a formulation à la Krichever–Novikov (KN) of classical $W$–algebras. An intriguing relation between the theory of the $A_1$ embeddings into simple Lie algebras and the holomorphic geometry of Riemann surfaces will emerge (see ref. [10] for an early attempt in this direction). The theory of reduction of Poisson manifolds is an essential ingredient.

1. The spaces $KN_j$ and $W_j$ and the Drinfeld–Sokolov holomorphic vector bundle

Let us briefly recall the basic notions of the KN theory $^1$. Let $\Sigma^\circ = \Sigma \setminus \{P_+, P_-\}$. The basic functional spaces of the KN theory are the KN spaces

$$KN_j = \Gamma(\Sigma^\circ, O(k^{\otimes j})), \quad (1.1)$$

where $j \in \mathbb{Z}/2$. There is a non singular pairing of $KN_j$ and $KN_{1-j}$, the KN pairing, defined by

$$\langle p, q \rangle = \oint_{C_\tau} \frac{dz}{2\pi i} pq, \quad p \in KN_j, \ q \in KN_{1-j}, \quad (1.2)$$

where $C_\tau$ is the curve in $\Sigma$ of KN time $\tau \in \mathbb{R}$. The pairing is actually independent from $\tau$, since the curves $C_\tau$ for varying $\tau$ are all homologous [4,5]. For each $j$, the space $KN_j$

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$^1$ The conventions adopted in this paper are the following. I denote by $O(K)$ the sheaf of germs of holomorphic sections of a 1 cocycle $K$ on $\Sigma$ and by $\Gamma(U, O(K))$ the space of sections of $O(K)$ on a open set $U$ of $\Sigma$. I denote by $k$ the holomorphic canonical line bundle of $\Sigma$. For any $(1,0)$ connection $C$ of $K$, $C_b = k_{ba} (\text{Ad} K_{ba} C_a + \partial_b K_{ba} K_{ba}^{-1})$, where $a, b, c, \ldots$ are patch indices. For any projective connection $R$, $R_b = k_{ba}^2 (R_a - \{z_b, z_a\})$. 
admits a standard basis, the KN basis, which is very useful in calculations (see ref. [6] for explicit expressions in terms of theta functions). The KN bases of $KN_j$ and $KN_{1-j}$ are dual with respect to the pairing (1.2).

The basic symmetry group of the KN theory is the group $Conf_0(\Sigma^0)$ of holomorphic diffeomorphisms $f$ of $\Sigma^0$ onto itself continuously connected to $id_\Sigma$. Its Lie algebra is $LieConf_0(\Sigma^0) = KN^{-1}$. The Lie brackets are given by
\[
[v, w] = v\partial w - w\partial v, \quad v, w \in LieConf_0(\Sigma^0). \tag{1.3}
\]
$Conf_0(\Sigma^0)$ acts on the KN spaces $KN_j$
\[
f^*p = (\partial f)^j p \circ f, \quad f \in Conf_0(\Sigma^0), \quad p \in KN_j. \tag{1.4}
\]
At the infinitesimal level, this relation reduces into
\[
\theta_v p = v\partial p + j(\partial v)p, \quad v \in LieConf_0(\Sigma^0), \quad p \in KN_j. \tag{1.5}
\]
The KN pairing (1.2) is invariant under $Conf_0(\Sigma^0)$.

The above framework can be extended as follows. Let $t_{-1}, t_0, t_{+1}$ be the standard generators of a $A_1$ subalgebra $t$ of the complex simple Lie algebra $g$. They satisfy the relations
\[
[t_{+1}, t_{-1}] = 2t_0, \quad [t_0, t_{\pm 1}] = \pm t_{\pm 1}. \tag{1.6}
\]
The adjoint representation of $g$ is completely reducible with respect to the subalgebra $t$. Let us denote by $\Pi$ the set of the representations of $A_1$ appearing in the reduction, each counted with its multiplicity, by $j_\eta$ the spin of a representation $\eta \in \Pi$ and by $I_\eta$ the set \{\(m|m \in \mathbb{Z}/2, |m| \leq j_\eta, m - j_\eta \in \mathbb{Z}\). To a representation $\eta \in \Pi$, there is associated a distinguished set of generators $t_{\eta,m}, m \in I_\eta$ of $g$ such that
\[
[t_d, t_{\eta,m}] = C^d_{j_\eta,m} t_{\eta,m+d}, \quad d = -1, 0, +1, \tag{1.7a}
\]
\[
C^{\pm 1}_{j_\eta,m} = [j(j+1) - m(m \pm 1)]^{\frac{1}{2}}, \quad C^0_{j_\eta,m} = m. \tag{1.7b}
\]
$t_{-1}, t_0, t_{+1}$ themselves span a representation $o \in \Pi$ with $j_o = 1$ and $t_{o,\pm 1} = \mp 2^{-\frac{1}{2}} t_{\pm 1}$ and $t_{o,0} = t_0$. The non degeneracy of the Cartan form of $g$ implies that each representation $\eta \in \Pi$ admits a conjugate representation $\bar{\eta}$. $\bar{\eta} = \eta, \bar{j_\eta} = j_{\bar{\eta}}$ and $\bar{\eta} = \eta$ if and only if $j_\eta \in \mathbb{Z}$. For any representation $D$ of $g$,
\[
\text{tr}_D(t_{\eta,m} t_{\xi,-n}) = N_D \eta (-1)^{j_o-m} \delta_{\bar{\eta},\xi} \delta_{m,n}, \tag{1.8}
\]
where \( N_{D\eta} \) is a normalization constant such that \( N_{D\eta} = (-1)^{2j_0} N_{D\eta} \). In what follows, I shall assume tacitly that all Lie algebra and group elements are taken in the fundamental defining representation of \( \mathfrak{g} \).

The Drinfeld–Sokolov vector bundle is defined by

\[
L_{ab} = k_{ab}^{-t_0} \exp(\partial_a k_{ab}^{-1} t_{-1})
\]  

[11]. \( L \) possesses a distinguished \((1,0)\) holomorphic connection, the Drinfeld–Sokolov connection, given by

\[
A = (1/2)t_{+1} - Rt_{-1},
\]

where \( R \) is a reference holomorphic projective connection [11]. The structure of this connection justifies the name adopted for \( L \) and \( A \) [12]. The existence of \( A \) shows the flatness of \( L \). The flat structures of \( L \) are parametrized by the \((1,0)\) holomorphic connections \( A_\zeta \) of the form

\[
A(\zeta) = A + \sum_{\eta \in \Pi} \zeta_\eta t_{\eta, -j_\eta}, \text{ where } \zeta_\eta \in \Gamma(\Sigma, \mathcal{O}(k^{\otimes j_\eta + 1})) \]  

[13]. \( L \) is unstable, since \( \sum_{\eta \in \Pi} \zeta_\eta t_{\eta, -j_\eta} \) is a non trivial element of \( \Gamma(\Sigma, \mathcal{O}(\text{Ad}L)) \) [13].

In the present context, besides the KN spaces, one needs the spaces

\[
W_j = \Gamma(\Sigma^\circ, \mathcal{O}(k^{\otimes j} \otimes \text{Ad}L)),
\]

[11]. There is a non singular pairing of \( W_j \) and \( W_{1-j} \) given by

\[
\langle X, W \rangle = \oint_{C_\tau} \frac{dz}{2\pi i} \text{tr}(XW), \quad X \in W_j, \ W \in W_{1-j}.
\]

The spaces \( W_j \) admit standard bases. However, since here one aims at classical \( W \)-algebras, such bases are not necessary and the KN basis of the KN spaces \( \text{KN}_j \) suffice for calculations.

Denote by \( \text{Gau}_0(\Sigma^\circ, L) \) the group of \( G \)-valued gauge transformations \( \gamma \) of \( L \) holomorphic off \( P_+ \) and \( P_- \) and continuously connected to the identity. Its Lie algebra \( \text{LieGau}_0(\Sigma^\circ, L) \) is \( W_0 \) with Lie brackets

\[
[\xi, \eta] = [e(\xi), e(\eta)], \quad \xi, \eta \in \text{LieGau}_0(\Sigma^\circ, L),
\]

[13] where in the right hand side \( e \) is the evaluation map at a given point of \( \Sigma^\circ \) and the Lie brackets are those of \( \mathfrak{g} \). \( \text{Gau}_0(\Sigma^\circ, L) \) acts on \( W_j \) through the adjoint representation

\[
\gamma W = \text{Ad}\gamma W, \quad \gamma \in \text{Gau}_0(\Sigma^\circ, L), \ W \in W_j.
\]

At the infinitesimal level, this relation becomes

\[
\delta_\xi W = [\xi, W], \quad \xi \in \text{LieGau}_0(\Sigma^\circ, L), \ W \in W_j.
\]
The pairing (1.12) is invariant under Gau$_0$(Σ$^\circ$, L).

The action of Conf$_0$(Σ$^\circ$) extends naturally to the spaces $W_j$:

$$f^*W = (\partial f)^j \text{Ad} L_f W \circ f, \quad f \in \text{Conf}_0(\Sigma^\circ), \; W \in W_j, \quad (1.16a)$$

$$L_f = (\partial f)^{-t_0} \exp(\partial(\partial f)^{-1} t_{-1}). \quad (1.16b)$$

At the infinitesimal level, this relation reduces into

$$\theta_v W = v \partial A W + j(v)W + [L_v,W], \quad v \in \text{LieConf}_0(\Sigma^\circ), \; W \in W_j, \quad (1.17a)$$

$$L_v = D_v, \quad D = (1/2)t_{+1} - \partial t_0 - (\partial^2 + R)t_{-1}, \quad (1.17b)$$

where $\partial A = \partial - \text{ad} A$ is the covariant derivative of $A$. One can show that $L_v \in W_0$ and that $L_v$ satisfies the important equation

$$\partial A L_v = -D_1 v t_{-1}, \quad D_1 = \partial^3 + 2R\partial + (\partial R), \quad (1.18)$$

where $D_1$ is a Bol operator [14]. The pairing (1.12) is invariant under Conf$_0(\Sigma^\circ)$.

2. **The Poisson manifold** (W$_1$, {·, ·}$\kappa$)

W$_1$ can be endowed with a Poisson structure depending on a parameter $\kappa \in \mathbb{C}$. The Poisson structure is completely defined by giving the Poisson brackets of the linear functionals on W$_1$. The Poisson brackets of general functions on W$_1$ are obtained by enforcing the Leibniz rule. Since the pairing (1.12) is non singular, every linear functional on W$_1$ is of the form

$$\lambda_X(W) = \langle X, W \rangle, \quad W \in W_1, \quad (2.1)$$

for some $X \in W_0$. One sets

$$\{\lambda_X + a, \lambda_Y + b\}_\kappa = \lambda_{[X,Y]} + \kappa \lambda(X,Y), \quad X, Y \in W_0, \; a, b \in \mathbb{C} \quad (2.2a)$$

$$\lambda(X,Y) = \langle X, \partial_A Y \rangle. \quad (2.2b)$$

It is straightforward to verify that the Poisson brackets {·, ·}$\kappa$ are bilinear, antisymmetric and satisfy the Jacobi identity as the should. The above Poisson structure clearly resembles that of Kac–Moody phase space. Namely, the level and the Kac–Moody current would be $-\kappa$ and $\kappa A + W$, respectively. However, the geometrical interpretation is completely different as the relevance of the Drinfeld–Sokolov vector bundle shows.

For any $\gamma \in \text{Gau}_0(\Sigma^\circ, L)$, one has that $\chi(\gamma X, \gamma Y) = \chi(X,Y) - \langle [X,Y], \gamma^{-1} \partial_A \gamma \rangle$. So, the ordinary action of Gau$_0(\Sigma^\circ, L)$ on W$_1$, defined by (1.14), is not Poisson: it does not
leave the Poisson brackets invariant. However, there exists a deformation of the action enjoying such property, namely

\[(\gamma W)_{\kappa} = \gamma W + \kappa \partial_A \gamma^{-1}, \quad W \in W_1.\] 

(2.3)

The deformation induces an action of $\text{Gau}_0(\Sigma^0, L)$ on the functionals $\lambda_X + a, \ X \in W_0, \ a \in \mathbb{C}$:

\[(\gamma(\lambda_X + a))_{\kappa}(W) = \lambda_X((\gamma^{-1}W)_{\kappa}) + a = \lambda_{\gamma X}(W) + a - \kappa(X, \gamma^{-1}\partial_A \gamma), \quad W \in W_1. \] 

(2.4)

By combining (2.2) and (2.4), one verifies that the deformed action thus defined is Poisson. At the infinitesimal level, (2.3) and (2.4) become

\[(\delta_\xi W)_{\kappa} = \kappa \partial_A \xi + \delta_\xi W, \] 

(2.5)

and

\[(\delta_\xi(\lambda_X + a))_{\kappa}(W) = -\lambda_X(\delta_\xi W)_{\kappa}) = \lambda_{[\xi, X]}(W) + \kappa\langle \xi, \partial_A X \rangle, \] 

(2.6)

where $\xi \in \text{Lie Gau}_0(\Sigma^0, L)$ (cf. eq. (1.15)). From (2.2) and (2.6), one has

\[(\delta_\xi(\lambda_X + a))_{\kappa} = \{J_\xi, \lambda_X + a\}_\kappa, \] 

(2.7a)

\[J_\xi(W) = \langle \xi, W \rangle, \quad W \in W_1. \] 

(2.7b)

From here, it appears that the deformed action of $\text{Gau}_0(\Sigma^0, L)$ on $W_1$ is Hamiltonian with respect to the Poisson structure (2.2), the Hamiltonian functions being the $J_\xi$. $J_\xi$ can trivially be written as

\[J_\xi(W) = \langle \xi, J(W) \rangle, \quad W \in W_1. \] 

(2.8a)

\[J(W) = W. \] 

(2.8b)

So, the map $W \in W_1 \rightarrow J(W) \in W_1$ can be identified with the moment map of the Hamiltonian action.

Next consider $\text{Conf}_0(\Sigma^0)$. For any $f \in \text{Conf}_0(\Sigma^0)$, one has that $\chi(f^*X, f^*Y) = \chi(X, Y) - \kappa([X, Y], A - f^{-1}*A)$, where for $f \in \text{Conf}_0(\Sigma^0)$, $f^*A = \partial L_f L_f^{-1} + \partial f \text{Ad} L_f A \circ f$. Because of the non invariance of $\chi$, the action of $\text{Conf}_0(\Sigma^0)$ on $W_1$, defined by (1.16), is not Poisson. However, in this case too, there is a deformation of the action enjoying this property. Set

\[(f^*W)_{\kappa} = \kappa(f^*A - A) + f^*W, \quad W \in W_1. \] 

(2.9)

The deformation induces an action of $\text{Conf}_0(\Sigma^0)$ on the functionals $\lambda_X + a, \ X \in W_0, \ a \in \mathbb{C}$ given by

\[(f^*(\lambda_X + a))_{\kappa}(W) = \lambda_X((f^{-1}*W)_{\kappa}) + a = \lambda_{f^*X}(W) + a + \kappa(X, f^{-1}*A - A), \quad W \in W_1. \] 

(2.10)
From (2.2) and (2.10), it follows that the action (2.9) is Poisson. At the infinitesimal level, (2.9) and (2.10) become

\[ (\theta_v W)_\kappa = \kappa \partial A L_v + \theta_v W, \]  
\[ (\theta_v (\lambda X + a))_\kappa (W) = -\lambda X ((\theta_v W)_\kappa) = \lambda_{\theta_v X} (W) - \kappa \langle X, \partial A L_v \rangle, \]  

where \( v \in \text{LieConf}_0(\Sigma^\circ) \) (cf. eq. (1.17)). Now, it can be verified that

\[ (\theta_v (\lambda X + a))_\kappa = \{ T_v, \lambda X + a \}_\kappa, \]  
\[ T_v (W) = (1/2\kappa) \langle vW, W \rangle + \langle L_v, W \rangle, \quad W \in \mathcal{W}_1. \]  

This shows that the action (2.11) is Hamiltonian, the Hamiltonian functions being the \( T_v \). \( T_v \) can be written as

\[ T_v (W) = \langle v, T(W) \rangle, \quad W \in \mathcal{W}_1 \]  
\[ T(W) = \text{tr} (\mathcal{D}^t W + (1/2\kappa) W^2), \quad \mathcal{D}^t = (1/2)t_{+1} + \partial t_0 - (\partial^2 + R) t_{-1}. \]  

So the map \( W \in \mathcal{W}_1 \to T(W) \in \text{Kn}_2 \) is the moment map of the Hamiltonian action.

From (2.2), (2.7b) and (2.13b), one gets

\[ \{ J_\xi, J_\eta \}_\kappa = J_{[\xi, \eta]} + \kappa \chi (\xi, \eta), \quad \xi, \eta \in \text{LieGau}_0(\Sigma^\circ, L), \]  
\[ \{ T_v, T_w \}_\kappa = T_{[v, w]} + 12 \kappa \text{tr} (t_0^2) \sigma (v, w), \quad v, w \in \text{LieConf}_0(\Sigma^\circ), \]  
\[ \{ T_v, J_\xi \}_\kappa = J_{\theta_v \xi} + \kappa \chi (L_v, \xi), \quad \xi \in \text{LieGau}_0(\Sigma^\circ, L), \quad v \in \text{LieConf}_0(\Sigma^\circ), \]

where \( \sigma (v, w) = -(1/12) \langle v, D_1 w \rangle \) is the \( \text{Kn} \) 1–cocycle and \( D_1 \) is given in (1.18).

Let us know examine the the above results, interpret them and compare them with the known literature. (2.15) is a Poisson bracket algebra closely resembling a Kac–Moody algebra of level \( \kappa \). The moment map \( J(W) \), eq. (2.8b), plays here the role of the Kac–Moody current. Similarly, (2.16) is a Poisson bracket Virasoro algebra of central charge \( 12 \kappa \text{tr} (t_0^2) \). This is the well-known value of the classical central charge encountered in the theory of classical \( W \)–algebras [1,2,3]. The moment map \( T(W) \), eq. (2.14b), is the energy-momentum tensor. In the usual approach [1,3], the central charge originates from an improvement term added to the Sugawara energy-momentum tensor of Kac–Moody theory in order to maintain conformal invariance upon carrying out the Hamiltonian reduction of the Kac–Moody phase space. The first and second contributions in expression (2.13b) of \( T_v \) correspond more or less to such terms in the present formulation. Here, however, the improvement term is yielded \textit{ab initio} by the nature of the Drinfeld–Sokolov vector bundle and the action of the conformal group of \( \Sigma^\circ \). The second derivative term appearing in expression (2.14b) of \( T(W) \) has a counterpart in the usual approach where it is added \textit{ad }
hoc after the reduction of the phase space [1,3]. Here, it is present from the beginning and it strictly necessary to ensure the correct transformation properties of $T(W)$ under coordinate changes. From (2.17), the current $J(W)$ transforms as a primary field of conformal weight 1 under Poisson bracketting, except for the component corresponding to the generator $t_{+1}$ of $\mathfrak{g}$ (see eqs. (1.18) and (2.2b)). This also is familiar in the theory of classical $W$–algebras [1,3].

3. The reduction of the Poisson manifold $(W_1, \{\cdot, \cdot\}_\kappa)$

To obtain the classical $W$–algebras in the above framework, one has to impose a suitable set of constraints on the Poisson manifold $(W_1, \{\cdot, \cdot\}_\kappa)$ to reduce it. The constraints have the form

$$J_\xi \approx 0, \quad \xi \in \mathcal{X}, \quad (3.1)$$

where $\mathcal{X}$ is some subset of LieGau$_0(\Sigma^\circ, L)$. Such constraints are essentially of the same form as those used in [1] once one recalls that in the present formulation the counterpart of the Kac–Moody current is $A + J(W)$. To implement the reduction of $(W_1, \{\cdot, \cdot\}_\kappa)$, one demands that the constraints are first class. From (2.15), this yields the condition

$$[\xi, \eta] \in \mathcal{X} \quad \text{and} \quad \chi(\xi, \eta) = 0, \quad \xi, \eta \in \mathcal{X}. \quad (3.2)$$

One also requires that the constraint manifold is invariant under the action of Conf$_0(\Sigma^\circ)$. From (2.17), this yields the condition

$$\theta_v \xi \in \mathcal{X} \quad \text{and} \quad \chi(L_v, \xi) = 0, \quad v \in \text{LieConf}_0(\Sigma^\circ), \xi \in \mathcal{X}. \quad (3.3)$$

A maximal subspace $\mathcal{X}$ of LieGau$_0(\Sigma^\circ, L)$ satisfying (3.2) – (3.3) is obtained as follows. The treatment given here follows very closely that of [1]. Consider the 2-form $\omega \in \wedge^2 \mathfrak{g}^\vee$ defined by $\omega(x, y) = \text{tr}(t_{+1}[x, y]), \ x, y \in \mathfrak{g}$. The restriction of such form to $\mathfrak{g}_{-\frac{1}{2}}$ is non singular $^2$. By Darboux theorem, there is a direct sum decomposition $\mathfrak{g}_{-\frac{1}{2}} = \mathfrak{p}_{-\frac{1}{2}} \oplus \mathfrak{q}_{-\frac{1}{2}}$ into subspaces of $\mathfrak{g}_{-\frac{1}{2}}$ which are maximally isotropic and dual to each other with respect to $\omega$. Set

$$\mathfrak{r} = \mathfrak{g}_{\leq -1} \oplus \mathfrak{p}_{-\frac{1}{2}}, \quad (3.4)$$

which is a nilpotent subalgebra of $\mathfrak{g}$. Then, one can show that

$$\mathcal{X} = \{\xi | \xi \in \text{LieGau}_0(\Sigma^\circ, L), \ \xi \text{ valued in } \mathfrak{r}\}. \quad (3.5)$$

$^2$ Here, $\mathfrak{g}_m = \{x | x \in \mathfrak{g}, \ \text{ad} t_0 x = mx\}, \ \mathfrak{g}_{\leq m} = \bigoplus_{k \leq m} \mathfrak{g}_k$, etc.. The orthogonal complement of a subspace $\mathfrak{v}$ of $\mathfrak{g}$ with respect to the invariant bilinear form defined by the trace $\text{tr}$ will be denoted by $\mathfrak{v}^\perp$. 
It can be proven that the condition of valuedness in $\mathfrak{r}$ is compatible with changes of trivializations of $L$ [13]. Such condition involves no restriction on the KN content of $\mathcal{X}$. In fact, it can be shown that

$$\mathcal{X} \simeq \text{KN}_{\frac{1}{2}} \oplus \cdots \oplus \text{KN}_{\frac{1}{2}} \oplus \bigoplus_{\eta \in \Pi, m \in I, m \geq 1} \text{KN}_m,$$

(3.6)

where there are $\dim \mathfrak{p}_{-\frac{1}{2}} \text{KN}_{\frac{1}{2}}$ spaces in the right hand side. The explicit form of the isomorphism will be given elsewhere [13].

The constraint manifold $\mathcal{W}_{\text{constr}}$ is given in terms of the orthogonal complement $\mathfrak{r}^\perp$ of $\mathfrak{r}$

$$\mathfrak{r}^\perp = \mathfrak{g}_{\leq 0} \oplus \text{ad} t_1 \mathfrak{p}_{-\frac{1}{2}}$$

(3.7)

and is explicitly given by

$$\mathcal{W}_{\text{constr}} = \{ W | W \in \mathcal{W}_1, W \text{ valued in } \mathfrak{r}^\perp \}.$$

(3.8)

Here too, one can show that the condition of valuedness in $\mathfrak{r}^\perp$ is compatible with changes of trivializations of $L$ and that in fact no restriction on the KN content of $\mathcal{W}_{\text{constr}}$ results in the sense that

$$\mathcal{W}_{\text{constr}} \simeq \text{KN}_{-\frac{1}{2}} \oplus \cdots \oplus \text{KN}_{-\frac{1}{2}} \oplus \bigoplus_{\eta \in \Pi, m \in I, m \geq 0} \text{KN}_m,$$

(3.9)

where the right hand side contains $\dim \mathfrak{p}_{-\frac{1}{2}} \text{KN}_{-\frac{1}{2}}$ spaces.

From (1.15) and (2.5), it follows that, for $\xi \in \mathcal{X}$ and $W \in \mathcal{W}_{\text{constr}}$, $(\delta \xi W)_\kappa \in \mathcal{W}_{\text{constr}}$. Similarly, from (1.17), (1.18) and (2.11), it follows that for $\nu \in \text{LieConf}_0(\Sigma^o)$ and $W \in \mathcal{W}_{\text{constr}}$, $(\theta_v W)_\kappa \in \mathcal{W}_{\text{constr}}$. The gauge symmetry, associated to the first class constraints (3.1), must be fixed. It can be shown that for any $W \in \mathcal{W}_{\text{constr}}$, there exists a unique element $\zeta_W \in \mathcal{X}$ depending polynomially on $W$, $R$ and their derivatives such that

$$(\exp \zeta_W W)_\kappa = W_c,$$

(3.10)

where $W_c$ is an element of $\mathcal{W}_{\text{constr}}$ such that

$$\text{ad} t_{-1} W_c = 0.$$

(3.11)

By the nilpotency of $\mathfrak{r}$, $W_c$ depends polynomially on $W$, $R$ and their derivatives as well. The uniqueness of $\zeta_W$ ensures further that the map $W \rightarrow W_c$ is gauge invariant, i.e.

$$\exp(\xi W)_{\kappa c} = W_c, \quad \xi \in \mathcal{X}, \; W \in \mathcal{W}_{\text{constr}}.$$

(3.12)
Here, it is important to realize that the standard proof of the existence and uniqueness of \( \zeta_W \) given in refs. [1,3] cannot be straightforwardly generalized to the present framework, for due account of global issues is not taken. In spite of this, the result remains true.

The above suggests the following gauge fixing condition

\[
W = W_c, \quad W \in \mathcal{W}_{\text{red}},
\]  

(3.13)

defining the reduced manifold \( \mathcal{W}_{\text{red}} \). \( \mathcal{W}_{\text{red}} \) can be characterized in terms of a set of second class constraints. Let

\[
\mathcal{X}' = \{ \xi | \xi \in \text{LieGau}_0(\Sigma^\circ, L), \xi \text{ valued in (} \ker \text{ad} \mathfrak{t}_{-1})^\perp \}. 
\]  

(3.14)

The KN content of \( \mathcal{X}' \) is expressed by the isomorphism

\[
\mathcal{X}' \simeq \bigoplus_{\eta \in \Pi, m \in I_\eta, m \geq -j_\eta + 1} \text{KN}_m. 
\]  

(3.15)

\( \mathcal{W}_{\text{red}} \) is defined by the second class constraints

\[
J\xi \approx 0, \quad \xi \in \mathcal{X}'.
\]  

(3.16)

So, \( \mathcal{W}_{\text{red}} \) is given by

\[
\mathcal{W}_{\text{red}} = \{ W | W \in \mathcal{W}_1, \ W \text{ valued in } \ker \text{ad} \mathfrak{t}_{-1} \}. 
\]  

(3.17)

The KN content of \( \mathcal{W}_{\text{red}} \) is expressed by the isomorphism

\[
\mathcal{W}_{\text{red}} \simeq \bigoplus_{\eta \in \Pi} \text{KN}_{j_\eta + 1}. 
\]  

(3.18)

a relation which could be deduced also from (3.11) and (3.13). It is readily verified that (3.3) holds with \( \mathcal{X} \) replaced by \( \mathcal{X}' \), showing that the reduced manifold is invariant under \( \text{Conf}_0(\Sigma^\circ) \). \( \mathcal{W}_{\text{red}} \) equipped with the Dirac brackets \( \{\cdot,\cdot\}_\kappa^* \) defines the reduced Poisson manifold \( (\mathcal{W}_{\text{red}}, \{\cdot,\cdot\}_\kappa^*) \), whose properties are the topic of the next section.

4. **The Poisson manifold \( (\mathcal{W}_{\text{red}}, \{\cdot,\cdot\}_\kappa^*) \)**

The task now facing one is the computation of the Dirac brackets \( \{\cdot,\cdot\}_\kappa^* \). Consider the dual space \( \mathcal{W}_{\text{red}}^\vee \) of \( \mathcal{W}_{\text{red}} \). This can be characterized as

\[
\mathcal{W}_{\text{red}}^\vee \simeq \bigoplus_{\eta \in \Pi} \text{KN}_{-j_\eta}. 
\]  

(4.1)
The dual pairing of $W_{\text{red}}$ and $W_{\text{red}}^\vee$ is defined as follows. Let $\nu = (\nu_{\eta})_{\eta \in \Pi}$, $\nu_{\eta} \in \mathbb{K}N_{-j_{\eta}}$ be an element $W_{\text{red}}^\vee$ and $\omega = (\omega_{\eta})_{\eta \in \Pi}$, $\omega_{\eta} \in \mathbb{K}N_{j_{\eta}+1}$ be one of $W_{\text{red}}$. Then

$$\langle \nu, \omega \rangle = \sum_{\eta \in \Pi} N_{\eta} \langle \nu_{\eta}, \omega_{\bar{\eta}} \rangle$$  \hspace{1cm} (4.2)

(cf. eq. (1.8)). The Dirac brackets $\{\cdot, \cdot\}_\kappa^*$ are completely defined by those of the linear functionals

$$l_{\nu}(\omega) = \langle \nu, \omega \rangle, \hspace{1cm} \omega \in W_{\text{red}},$$  \hspace{1cm} (4.3)

for $\nu \in W_{\text{red}}^\vee$. The calculation of the Dirac brackets of the $l_{\nu}$ involves the choice of a basis of $\mathcal{X}'$. Luckily, the explicit expression of the basis elements is not necessary to carry out the calculation. The result obtained is

$$\{l_{\mu}, l_{\nu}\}_\kappa^*(\omega) = \langle [E_{\mu,\kappa-1}\omega, E_{\nu,0}], Q_\omega \rangle + \kappa \chi(E_{\mu,0}, E_{\nu,0}), \hspace{1cm} \mu, \nu \in W_{\text{red}}^\vee, \hspace{0.5cm} \omega \in W_{\text{red}},$$  \hspace{1cm} (4.4a)

$$E_{\nu,\omega} = \left[1 + \text{Ad} t_{-1}(\partial_A - \text{ad} Q_\omega)\right]^{2j_{\max}+1} P_{\nu}, \hspace{1cm} P_{\nu} = \sum_{\eta \in \Pi} \nu_{\eta} t_{\eta,j_{\eta}}, \hspace{1cm} Q_\omega = \sum_{\eta \in \Pi} \omega_{\eta} t_{\eta,-j_{\eta}},$$  \hspace{1cm} (4.4b)

where $L$ is the formal inverse of $\frac{1}{2}\text{Ad} t_{-1} \text{Ad} t_{+1}$ extended by 0 on $\ker \text{ad} t_{+1}$ and $j_{\max} = \max\{j_{\eta}|\eta \in \Pi\}$. It can be shown that $E_{\nu,\omega} \in W_0$ and $Q_\omega \in W_1$, so that the above expression of the Dirac brackets is globally defined [13]. The first term in the right hand side of (4.4a) is a differential polynomial in $\mu$, $\nu$ and $\omega$ and is computable in principle using (4.4b). The second term, proportional to $\kappa$, is the anomaly. It can be calculated explicitly. The result is

$$\chi(E_{\mu,0}, E_{\nu,0}) = \sum_{\eta \in \Pi} N_{\eta} \left[ \prod_{m \in \{\eta, m \geq -j_{\eta}+1\}} \frac{2}{C-1_{j_{\eta},m}} \right] \langle \mu_{\eta}, D_{j_{\eta}} \nu_{\bar{\eta}} \rangle,$$  \hspace{1cm} (4.5a)

$$D_0 = \partial,$$
$$D_{\frac{1}{2}} = \partial^2 + (1/2)R,$$
$$D_1 = \partial^3 + 2R\partial + (\partial R),$$
$$D_{\frac{3}{2}} = \partial^4 + 5R\partial^2 + 5(\partial R)\partial + (3/2)(\partial^2 R + (3/2)R^2),$$
$$D_2 = \partial^5 + 10R\partial^3 + 15(\partial R)\partial^2 + [9(\partial^2 R) + 16R^2]\partial + 2[(\partial^3 R) + 8R(\partial R)],$$
$$\text{etc}..$$  \hspace{1cm} (4.5b)

The $D_j$ are the well-known Bol operators [14].
There are other relevant Dirac brackets. Consider the energy-momentum tensor $T$. For any $v \in \text{LieConf}_0(\Sigma^o)$, the restriction of $T_v$ to $W_{\text{red}}$ is given by

$$T_v(\omega) = 2^{-\frac{1}{2}} \text{tr}(t_0^2)(v, \omega_o) + (1/2\kappa) \sum_{\eta \in \Pi, j \eta = 0} N_\eta \langle v, \omega_\eta \rangle^2$$  \hspace{1cm} (4.6)

(see below eq. (1.7b)). From (4.4a) and (4.6), one has

$$\{T_v, T_w\}_\kappa = T_{[v,w]} + 12\kappa \text{tr}(t_0^2)\sigma(v, w), \hspace{0.5cm} v, w \in \text{LieConf}_0(\Sigma^o),$$  \hspace{1cm} (4.7)

which is to be compared with (2.16). One further finds that

$$\{T_v, l_\mu\}_\kappa = l_\theta_v \mu + \kappa \chi(L_v, E_{\mu,0}), \hspace{0.5cm} v \in \text{LieConf}_0(\Sigma^o), \hspace{0.2cm} \mu \in W_{\text{red}}^\vee,$$  \hspace{1cm} (4.8)

where $\theta_v \mu_\eta$ is given by (1.5) with $p = \mu_\eta$ and $j = -j_\eta$.

Let us discuss briefly the results just obtained. (4.4) defines a Dirac bracket $W$–algebra in the so-called lowest weight gauge. In fact, analogous expression have been worked out in the literature following analogous techniques (see f. i. ref. [3] for a review). The $W$–algebra proper is obtained by letting $\mu_\eta$ and $\nu_\eta$ in (4.4a) be elements of the KN basis of $\mathfrak{KN}_{-j_\eta}$. The form of the anomaly was first found in [6] in a different approach where however the deep relation with the theory of $A_1$ embeddings into simple Lie algebras was not apparent. From (4.6) and (4.7), it follows that the $T_v$ form a Dirac bracket Virasoro algebra of classical central charge $12\kappa \text{tr}(t_0^2)$. From (4.8), it also appears that the functions $l_\mu$ with $\mu_o = 0$ are primary with respect to the Virasoro algebra. All the above properties have a counterpart in the standard algebraic formulation to $W$–algebras [1,3].

The present paper provides a synthesis of the algebraic and geometrical approaches to $W$–algebras and shows them in a completely new light. As a possible application, one may consider a special class of flat forms of the Drinfeld–Sokolov vector bundle $L$, the ones corresponding to equivalence classes of elements $W \in \mathcal{W}_{\text{constr}}$ modulo $G_f$ gauge transformations, where $G_f$ is the subgroup of $G$ corresponding to the Lie subalgebra $\mathfrak{g}$. Such flat forms may be viewed as functions on $W_{\text{red}}$. It would be interesting to compute the Dirac brackets $\{\iota(l_1), \iota(l_2)\}_\kappa^*$ where $\iota$ is the characteristic homomorphism of $\pi_1(\Sigma^o)$ associated to a flat form of $L$ and, for fixed $l \in \pi_1(\Sigma^o)$, $\iota(l)$ is viewed as a function on $W_{\text{red}}$. This would probe world sheet topological effects of $W$–algebras. It also would be interesting to develop a BRST formalism and study quantization. Lastly, it is also important to analyze the relation between the above approach to $W$–algebras and the formulation of Toda theory on Riemann surfaces of ref. [15], where the basic holomorphic vector bundle can be shown to be holomorphically equivalent on $\Sigma^o$ to the Drinfeld–Sokolov bundle $L$.  

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