GENERIC EQUIVARIANT BIFURCATION FROM RELATIVE EQUILIBRIA

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ABSTRACT. We develop a new method to study equivariant bifurcation from relative equilibria in dynamical systems with symmetry. As an example of our approach we generalize the Equivariant Branching Lemma and the Equivariant Hopf Theorem to relative equilibria. The heart of our method is that, since orbit spaces are in general not smooth manifolds, given a proper action of a Lie group on a manifold one should consider the stack quotient instead of the classical orbit space construction. To carry out this program we categorify the space of equivariant bifurcation problems. This allows us to prove that generic equivariant bifurcation problems from a relative equilibrium on a proper action are equivalent to generic equivariant bifurcation problems from an equilibrium.

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1. Introduction

In this paper we develop a new method to study equivariant bifurcation from relative equilibria in one-parameter families of equivariant smooth dynamical systems. The systems we consider are equivariant with respect to the proper action of a (possibly noncompact) Lie group. We think of bifurcation problems as paths in the space of equivariant vector fields. Equivariant bifurcation theory is well-developed in the case of “honest” equilibria and compact Lie group actions. We note in particular the Equivariant Branching Lemma [4, 18], and the spatial and spatio-temporal Equivariant Hopf Theorems [8, 9]. In this paper we reduce the case of relative equilibria and proper actions to the case of equilibria on the slice representation of the compact isotropy group of the relative equilibrium. Reduction of such bifurcation problems via slices for the action is not a new idea (see [13]). What is new in this paper is our way of making sure that such a reduction preserves genericity.

The generic conditions for equivariant bifurcation in the case of equilibria involve the eigenvalues of the linearization at the given equilibrium. However, there are issues with linearizing vector fields at relative equilibria. Since relative equilibria descend to equilibria of the flow on the orbit space, one could try to linearize on the orbit space [14, 2]. The obstacle is that orbit spaces of group actions are generally not smooth. A brute force approach is to embed the orbit space in some Euclidean space $\mathbb{R}^n$. Instead, we prefer to think of the orbit space as a stack (see, for example, [15] for general background on stacks). While this may sound intimidating, thinking of quotients as stacks in practice amounts to replacing the vector space of equivariant vector fields by a 2-term chain complex of vector spaces, which we describe next.

Given an action of a group $G$ on a manifold $M$, we categorify the space $\mathcal{X}(M)^G$ of $G$-equivariant vector fields by introducing an action on $\mathcal{X}(M)^G$ of the vector space:

$$C^\infty(M, g)^G := \{ \psi : M \to g \mid \psi(g \cdot m) = \text{Ad}(g)\psi(m), \ g \in G, \ m \in M \},$$

where $g$ is the Lie algebra of $G$ (see (2.2) and Definition 2.7). The motivation comes from [11] where it was shown that the resulting action groupoid is equivalent to the category of vector fields on the stack quotient $[M/G]$. Equivalently, we consider the vector space $\mathcal{X}(M)^G$ as part of the 2-term chain complex of vector space:

$$C^\infty(M, g)^G \xrightarrow{\partial} \mathcal{X}(M)^G,$$

where the boundary map $\partial$ is induced by the infinitesimal action of the Lie algebra $g$ on the manifold $M$ (Definition 2.11).

In this paper we construct a 2-term chain complex $\text{Bif}(M)^G$ of equivariant bifurcation problems. This consists of certain paths in the 2-term chain complex (1.1) of

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1 We make no attempt at summarizing the literature in the area. A good overview can be found in the text [10], as well as the seminal volume [9] and the comprehensive references [3, 6].

2 A 2-term chain complex of vector spaces is a linear map thought of as a chain complex with all other terms being zero. The point of thinking of it as a 2-term chain complex is to see it as an object of the strict 2-category of 2-term chain complexes of vector spaces. In [11] it was shown that the strict 2-category of 2-term chain complexes is equivalent to the strict 2-category of 2-vector spaces; the latter being categories internal to the 1-category of vector spaces. Hepworth showed that vector fields on stacks form 2-vector spaces [11]. In light of the Baez-Crans equivalence in [1], we prefer to work with 2-term chain complexes.
equivariant vector fields (Definition 6.8 and Definition 6.19). By this we mean that the 2-term chain complex $\text{Bif}(M)^G_{\bullet}$ consists of certain smooth paths in each of the spaces $C^\infty(M, g)^G$ and $\mathfrak{X}(M)^G$ (Definition 2.13). The boundary map $\partial$ of $\text{Bif}(M)^G_{\bullet}$ is given by taking the boundary map of the chain complex (1.1) parameter-wise.

In order to talk about generic equivariant bifurcation problems, we topologize the 2-term chain complex $\text{Bif}(M)^G_{\bullet}$ of equivariant bifurcation problems. The 2-term chain complex $\text{Bif}(M)^G_{\bullet}$ is then a 2-term chain complex of topological abelian groups. With this we can define a category $\text{Gen}(\text{Bif}(M)^G_{\bullet})$ of generic equivariant bifurcation problems on the manifold $M$ (Definition 5.7). The category $\text{Gen}(\text{Bif}(M)^G_{\bullet})$ is a subcategory of the category of equivariant bifurcation problems corresponding to the 2-term chain complex $\text{Bif}(M)^G_{\bullet}$; or equivalently, it corresponds to a sub-chain complex of the 2-term chain complex $\text{Bif}(M)^G_{\bullet}$. The main result of this paper is the following (see Theorem 6.26 and Corollary 6.30):

**Theorem 1.1.** Let $G$ be a (possibly noncompact) Lie group acting properly on a smooth manifold $M$, let $m$ be a point in $M$, and let $V$ be the canonical slice representation of the isotropy group $K$ of the point $m$ (for simplicity, assume this slice is global). Under suitable irreducibility conditions on the slice representation, there is a homotopy equivalence of 2-term chain complexes of topological abelian groups:

$$\text{Bif}(M)^G_{\bullet} \simeq \text{Bif}(V)^K_{\bullet}$$

between the 2-term chain complexes of equivariant bifurcation problems on $M$ and $V$ respectively. In particular, there is an equivalence of categories:

$$\text{Gen}(\text{Bif}(M)^G_{\bullet}) \simeq \text{Gen}(\text{Bif}(V)^K_{\bullet})$$

between the categories of generic equivariant bifurcation problems on $M$ and $V$ respectively.

The equivariant bifurcation problems on a representation are necessarily bifurcation problems from an equilibrium (Remark 6.4). Thus, by Theorem 1.1 generic equivariant bifurcation problems from a relative equilibrium on a proper action are equivalent to generic equivariant bifurcation problems from an equilibrium.

Using the notation of Theorem 1.1 and assuming for simplicity that the slice representation $V$ is global, note that there is a canonical inclusion $\mathfrak{X}(V)^K \hookrightarrow \mathfrak{X}(M)^G$ of the equivariant vector fields on the canonical slice representation into the equivariant vector fields on the manifold $M$. The inclusion corresponds to equivariant extension of these vector fields. The image of this inclusion consists of vertical vector fields in the bundle $M \rightarrow G \cdot m$, and thus has infinite codimension in the space $\mathfrak{X}(M)^G$. Hence, no subcollection of these can be generic. A similar argument can be said of equivariant bifurcation problems. Corollary 6.32 shows how Theorem 1.1 implies that a subclass of generic bifurcating equivariant bifurcation problems yields a large subclass of generic bifurcating equivariant bifurcation problems on the given manifold.

In [13] Krupa provides a decomposition of the flows of equivariant vector fields which can be used to determine equivariant bif urcations from relative equilibria on the manifold $M$ from equivariant bifurcations on the slice representation $V$.

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3 In [13] Krupa studied equivariant bifurcations from relative equilibria in orthogonal representations on Euclidean space. Some of this work has been generalized to proper actions in [12] (see also [3] 7.8.4 and [2] Lemma 8.5.3 for discussion of Krupa’s work and its generalizations). Some aspects of our approach generalize Krupa’s methods. In particular, our projection map in
defines an equivariant bifurcation problem on the manifold $M$ to be generic if its reduction to the slice representation $V$ is generic. By contrast, in this paper we define an equivariant bifurcation problem to be generic if it is generic (in the usual sense) in the space of equivariant bifurcation problems on $M$ (Definition 6.29). Theorem 1.1 states how to recover the generic equivariant bifurcation problems on the given proper action from those on the slice representation; essentially by considering their equivariant extensions up to isomorphism.

Theorem 1.1 implies that any result about generic equivariant bifurcations from equilibria on representations yields a corresponding result about generic equivariant bifurcations from relative equilibria. As an illustration of this, we generalize the Equivariant Branching Lemma and the spatial version of the Equivariant Hopf Theorem to relative equilibria. In particular, we provide generic conditions for symmetry breaking bifurcations from relative equilibria to either relative equilibria, which we call Relative Equivariant Branching (Theorem 7.11), or to relative periodic trajectories, which we call Relative Spatial Equivariant Hopf (Theorem 7.14).

1.1. Organization of the paper. The paper is organized as follows.

- In section 2 the goal is to define isomorphisms of paths of equivariant vector fields (Definition 2.20). In particular, we define a 2-term chain complex of paths of equivariant vector fields in this section (Definition 2.11). The 2-term chain complexes of equivariant bifurcation problems defined in section 6 is a sub-chain complex of this one.
- In section 3 we discuss some preliminaries on bifurcating branches up to isomorphism.
- In section 4 we show that the 2-term chain complex of equivariant vector fields is a chain complex of topological abelian groups when topologized (Proposition ??).
- The main goal of section 5 is to show that the generic degree 0 elements in a 2-term chain complex of topological abelian groups form a category (Definition 5.7), and that homotopic 2-term chain complexes of topological abelian groups have equivalent categories of generic elements (Theorem 5.11).
- Section 6 has three parts:
  - The main goal of the first subsection is to define the 2-term chain complex of equivariant bifurcations problems in the case of representations of compact Lie groups (Definition 6.8).
  - The second subsection is concerned with defining the 2-term chain complex of equivariant bifurcation problems (from relative equilibria) in the case of proper actions (Definition 6.19). The main obstacle in doing this is overcoming the lack of a linearization by reducing to the case in the previous subsection.

Recall that there are two versions of the Equivariant Hopf Theorem each predicting different branches of periodic trajectories with different symmetry conditions; the spatial version which predicts periodic trajectories with certain spatial symmetries \[ Ch.XVI \text{ Theorem 2.2}, \] and the spatio-temporal version which predicts periodic trajectories with certain spatio-temporal symmetries \[ \text{(see also Ch.XVI Theorem 4.1 for the spatio-temporal case)}. \] Both can be generalized with our approach, but we stick to the spatial case for the sake of simplicity. The interested reader may want to adapt the proof presented here to the spatio-temporal case.
The final subsection contains the main theorem (Theorem 6.26), which is concerned with proving that the 2-term chain complex of equivariant bifurcation problems on a given proper action is equivalent to that on a representation. We also prove that the corresponding categories of generic equivariant bifurcation problems are equivalent (Corollary 6.30). Additionally, we prove that a generic subcollection of bifurcating equivariant bifurcation problems on the representation induces a generic subcollection on the proper action consisting of equivariant bifurcation problems that also bifurcate (Corollary 6.32).

Section 7 contains our generalizations of the Equivariant Branching Lemma and the (spatial) Equivariant Hopf Theorem; namely, the Relative Equivariant Branching Theorem (Theorem 7.11) and the Relative Spatial Equivariant Hopf Theorem (Theorem 7.14). For the sake of completeness, we prove the genericity of the classical eigenvalue crossing conditions for equivariant bifurcation problems on representations in Appendix A.

1.2. Acknowledgements. The author would like to thank Eugene Lerman for his enduring guidance, thoughtful discussions, and continued patience with my many questions throughout this project.

2. Isomorphisms of paths of equivariant vector fields

We review how to think of equivariant vector fields on a smooth manifold as objects of a category. We then describe how to view 1-parameter families of equivariant vector fields as paths in said category. First, recall:

**Definition 2.1.** An $H$-manifold $N$ is a smooth manifold $N$ with a smooth action of a Lie group $H$. A proper $H$-manifold is one where the action is proper.

**Definition 2.2.** An equivariant vector field on an $H$-manifold $N$ is a smooth vector field $X : N \to TN$ such that:

$$X(h \cdot n) = h \cdot X(n),$$

for all $h \in H$ and $n \in N$.

**Notation 2.3.** We will denote the vector space of equivariant vector fields on an $H$-manifold $N$ by $X(N)^H$.

Morphisms between equivariant vector fields will be built out of the following class of maps:

**Definition 2.4.** An infinitesimal gauge transformation on an $H$-manifold $N$ is an equivariant smooth map $\psi : N \to \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of $H$. That is,

$$\psi(h \cdot n) = \text{Ad}(h)\psi(n),$$

for all $h \in H$ and $n \in N$, where $\text{Ad}$ is the adjoint representation.

**Notation 2.5.** We will denote the space of infinitesimal gauge transformations by $C^\infty(N, \mathfrak{h})^H$. For the sake of brevity, we will sometimes refer to infinitesimal gauge transformations simply as gauge transformations.
A gauge transformation $\psi: N \to h$ on an $H$-manifold $N$ induces an equivariant vector field $\partial(\psi): N \to TN$. It is given by:

$$
\partial(\psi)(n) := \left. \frac{d}{d\tau} \right|_0 \exp(\tau \psi(n)) \cdot n,
$$

for any $\psi \in C^\infty(N, h)^H$ and any $n \in N$. The map $\partial : C^\infty(N, h)^H \to \mathfrak{X}(N)^H$ is linear. Consequently, the abelian group $C^\infty(N, h)^H$ acts on the space $\mathfrak{X}(N)^H$. The action is given by:

$$
\psi \cdot X := X + \partial(\psi),
$$

where $\psi$ is a gauge transformation, $X$ is an equivariant vector field, and the addition is the pointwise addition of vector fields.

**Remark 2.6.** Recall that the action of a group $G$ on a space $Y$ defines an action groupoid $G \times Y \rightrightarrows Y$ (see, for example, [17, Example 5.1 (5)]). The objects of the action groupoid of the action in 2.2 are the equivariant vector fields, while morphisms are pairs $(\psi, X)$ consisting of a gauge transformation $\psi$ and an equivariant vector field $X$. The source map is the projection onto the second factor, the target map is the action map, and the composition corresponds to addition of the first factors.

**Definition 2.7.** The groupoid of equivariant vector fields on an $H$-manifold $N$ is the action groupoid (see Remark 2.6) of the action of the space of gauge transformations $C^\infty(N, h)^H$ on the space of equivariant vector fields $\mathfrak{X}(N)^H$.

**Definition 2.8.** Two equivariant vector fields $X$ and $Y$ on an $H$-manifold $N$ are isomorphic if they are isomorphic as objects of the groupoid of equivariant vector fields. That is, they are isomorphic if there exists a gauge transformation $\psi : N \to h$ such that $Y = X + \partial(\psi)$.

Recall that the flow of an equivariant vector field on an $H$-manifold $N$ descends to give a continuous flow on the orbit space $N/H$. The following result has as corollary that isomorphic vector fields descend to the same continuous flow on the orbit space:

**Lemma 2.9** (Lerman). Let $X$ and $Y$ be two isomorphic equivariant vector fields on an $H$-manifold $N$ with flow $\phi^X$ and $\phi^Y$ respectively. Let $\mathcal{O} \subseteq \mathbb{R} \times N$ denote the domain of the flow $\phi^X$. Then $\mathcal{O}$ is also the domain of $\phi^Y$. Furthermore, there exists a smooth map $F: \mathcal{O} \to H$ such that $F(0, n)$ is the identity of $H$ for all $n \in N$, and:

$$
\phi^Y(\tau, n) = F(\tau, n) \cdot \phi^X(\tau, n)
$$

for all $(\tau, n) \in \mathcal{O}$.

**Proof.** See [16, Theorem 1.6].

**Remark 2.10.** In [11] Hepworth defined vector fields on stacks. According to his definition, vector fields on a stack $\mathcal{Y}$ are objects of a category $\text{Vect}(\mathcal{Y})$. In the case where the stack $\mathcal{Y}$ is the stack quotient $[N/H]$ of an $H$-manifold $N$, where $H$ is a compact Lie group, the category $\text{Vect}(\mathcal{Y})$ is equivalent to the corresponding groupoid of equivariant vector fields of Definition 2.7 [11, Proposition 6.1]. This groupoid is further explored in [16] and [12].
The groupoid of equivariant vector fields is in fact a 2-vector space in the sense of Baez and Crans [1]. That is, it is a small category internal to the category of vector spaces and linear maps. This means that the space of objects and the space of morphisms are vector spaces, and all the structure maps are linear. There is an equivalent point of view on 2-vector spaces: one can view a linear map as a chain complex with only two nonzero terms. Such a chain complex is called a 2-term chain complex. Thus, we have:

**Definition 2.11.** The 2-term chain complex $\mathfrak{X}(N)^H$ of equivariant vector fields on an $H$-manifold $N$ is the 2-term chain complex of vector spaces:

$$C^\infty(N,h)^H \xrightarrow{\partial} \mathfrak{X}(N)^H$$

where $\partial$ is the linear map defined by (2.1).

**Remark 2.12.** A 2-term chain complex of vector spaces $A_1 \xrightarrow{\partial} A_0$ defines an action of the abelian group $A_1$ on the space $A_0$. The action is given by $\psi \cdot x = x + \partial(\psi)$, for any $\psi \in A_1$ and $x \in A_0$. Conversely, given an action $\alpha : A_1 \times A_0 \to A_0$ of an abelian group $A_1$ on a vector space $A_0$, make the identification $A_1 \cong A_1 \times \{0\}$. Then the restriction $\alpha| : A_1 \to A_0$ gives a 2-term chain complex of vector spaces. In [1] Baez and Crans prove there is an equivalence of strict 2-categories:

$$2\text{TermVect} \simeq 2\text{Vect}$$

between the 2-category of 2-term chain complexes of vector spaces and the 2-category of 2-vector spaces. This equivalence means we can freely work with the 2-term chain complex of equivariant vector fields of Definition 2.11 in place of the groupoid of equivariant vector fields of Definition 2.7. We will refer to this equivalence as the Baez-Crans equivalence.

**Remark 2.13.** For our purposes, the scalar multiplication will not be important in the 2-term chain complexes of vector spaces that we consider. In fact, once we introduce topologies, we will need to “forget” the scalar multiplication and work with the underlying abelian groups. This is further discussed in section 5.

In this paper we think of 1-parameter families of equivariant vector fields on an $H$-manifold $N$ as “smooth” paths in the space of equivariant vector fields $\mathfrak{X}(N)^H$. Thus, we need to discuss what it means for such a path to be “smooth”. We must address the same question for paths in the space of gauge transformations $C^\infty(N,h)^H$. There are several ways to do it. For instance, we can turn the spaces $\mathfrak{X}(N)^H$ and $C^\infty(N,h)^H$ into Fréchet spaces. However, it is enough for our purposes to use the following simpler definition:

**Definition 2.14.** Let $N$ be an $H$-manifold. A map $X : \mathbb{R} \to \mathfrak{X}(N)^H$ is a smooth path of equivariant vector fields on $N$ if the associated map:

$$\tilde{X} : \mathbb{R} \times N \to TN, \quad \tilde{X}(\lambda, n) := X(\lambda)(n),$$

is smooth in the usual sense. An analogous definition gives smooth paths of infinitesimal gauge transformations $\psi : \mathbb{R} \to C^\infty(N,h)^H$.

**Notation 2.15.** The space of paths of equivariant vector fields on an $H$-manifold $N$ will be denoted by $C^\infty(\mathbb{R}, \mathfrak{X}(N)^H)$, and the space of paths of infinitesimal gauge transformations on an $H$-manifold $N$ will be denoted by $C^\infty(\mathbb{R}, C^\infty(N,h)^H)$. Given a path of equivariant vector fields $X$, a path of gauge transformations $\psi$, and...
a parameter value $\lambda \in \mathbb{R}$, we will denote the corresponding vector field and gauge transformation by $X_\lambda$ and $\psi_\lambda$ respectively.

**Remark 2.16.** Let $N$ be an $H$-manifold. Note that the path spaces $C^\infty (\mathbb{R}, \mathfrak{X}(N)^H)$ and $C^\infty (\mathbb{R}, C^\infty(N, \mathfrak{h})^H)$ are vector spaces.

**Definition 2.17.** The 2-term chain complex of paths of equivariant vector fields on an $H$-manifold $N$ is the 2-term chain complex of vector spaces:

$$C^\infty (\mathbb{R}, C^\infty(N, \mathfrak{h})^H) \xrightarrow{\partial} C^\infty (\mathbb{R}, \mathfrak{X}(N)^H)$$

where for any path of infinitesimal gauge transformations $\psi$ and any parameter value $\lambda \in \mathbb{R}$ we define:

$$\partial(\psi)(\lambda) := \partial(\psi_\lambda),$$

with the $\partial$ on the right-hand side being the map defined by (2.1).

**Notation 2.18.** We denote the 2-term chain complex of paths of equivariant vector fields on an $H$-manifold $N$ by $C^\infty (\mathbb{R}, \mathfrak{X}(N)^H)$.

**Definition 2.19.** The groupoid of paths of equivariant vector fields on an $H$-manifold $N$ is the 2-vector space corresponding to the 2-term chain complex of paths of equivariant vector fields of Definition 2.17 under the Baez-Crans equivalence (see Remark 2.12).

Thus, we have:

**Definition 2.20.** Two paths of equivariant vector fields $X$ and $Y$ on an $H$-manifold $N$ are *isomorphic* if they are isomorphic as objects of the groupoid of paths of equivariant vector fields (Definition 2.19). That is, they are isomorphic if there exists a path of infinitesimal gauge transformations $\psi : \mathbb{R} \to C^\infty(N, \mathfrak{h})^H$ such that $Y = X + \partial(\psi)$.

**Notation 2.21.** In the groupoid of paths of equivariant vector fields on an $H$-manifold $N$, an isomorphism $X \to Y$ is given by a pair $(\psi, X)$, where $\psi$ is an infinitesimal gauge transformation. We will sometimes refer to the path $\psi$ as an isomorphism between $X$ and $Y$ for the sake of simplicity.

Our position in this paper is that one should study bifurcating families of equivariant vector fields as objects of the groupoid of paths of equivariant vector fields. In practice, this perspective amounts to viewing bifurcating families up to isomorphism in the sense of Definition 2.20 or as degree 0 elements of the 2-term chain complex of Definition 2.17.

### 3. Bifurcating branches up to isomorphism

In this section we consider bifurcating paths of equivariant vector fields up to isomorphism. First, recall:

**Definition 3.1.** Let $X$ be an equivariant vector field on an $H$-manifold $N$. A point $n \in N$ is a relative equilibrium of the vector field $X$ if the vector $X(n)$ is tangent to the group orbit $H \cdot n$ of the point $n$. Equivalently, the point $n$ is a relative equilibrium if the integral curve of the vector field $X$ starting at the point $n$ projects to a constant path on the orbit space $N/H$. 
While it may happen that an equivariant vector field $X$ has an “honest” equilibrium at a point, it is more natural to consider relative equilibria in the presence of symmetries. The following result gives some justification for this perspective.

**Lemma 3.2.** Let $X$ and $Y$ be isomorphic vector fields on an $H$-manifold $N$, and let $n \in N$ be a relative equilibrium of the vector field $X$. Then the point $n$ is a relative equilibrium of the vector field $Y$.

**Proof.** Since the vector fields $X$ and $Y$ are isomorphic, there exists an infinitesimal gauge transformation $\psi : N \to \mathfrak{h}$ such that $Y = X + \partial(\psi)$. Note that the point $n$ is a relative equilibrium of the vector field $\partial(\psi)$ since the latter is defined as the derivative of a curve in the group orbit $H \cdot n$. Therefore, the vector $Y(n)$ is tangent to the group orbit $H \cdot n$ since the vector $Y(n)$ is the sum of the vectors $X(n)$ and $\partial(\psi)(n)$, which both lie in the tangent space $T_n(H \cdot n)$ of the group orbit $H \cdot n$. □

Now recall:

**Definition 3.3.** Let $X$ be an equivariant vector field on an $H$-manifold $N$ and let $\gamma : I \to N$ be an integral curve of $X$. The integral curve $\gamma$ is a relative periodic trajectory of the vector field $X$ if it projects to a periodic path on the orbit space $N/H$.

As with relative equilibria, it is more natural to consider relative periodic trajectories in the presence of symmetries, rather than “honest” periodic trajectories. The following lemma provides some justification for this assertion, and is analogous to Lemma 3.2.

**Lemma 3.4.** Let $X$ and $Y$ be isomorphic vector fields on an $H$-manifold $N$, and let $\gamma^X : I \to N$ be a relative periodic trajectory of the vector field $X$. Then there exists a smooth curve $h : I \to H$, where $I$ is an open interval containing 0, such that $h(0)$ is the identity element of $H$, and the curve:

$$\gamma^Y : I \to N, \quad \gamma^Y(\tau) := h(\tau) \cdot \gamma^X(\tau),$$

is a relative periodic trajectory of $Y$.

**Proof.** Let $F : O \to H$ be the smooth map of Lemma 2.9 where $O$ is the common domain of the flows of $X$ and $Y$. Then the curve:

$$h : I \to N, \quad h(\tau) := F(\tau, \gamma^X(0)),$$

where $I$ is the maximal interval in $O$ corresponding to the point $\gamma^X(0)$, is such that $\gamma^Y(\tau, \gamma^X(0)) = h(\tau) \cdot \gamma^X(\tau)$ for all $\tau \in I$. Thus, the curve:

$$\gamma^Y : I \to N, \quad \gamma^Y(\tau) := h(\tau) \cdot \gamma^X(\tau),$$

is the integral curve of the vector field $Y$ starting at the point $\gamma^X(0)$. Since the curve $\gamma^X$ is a relative periodic trajectory of $X$, its projection is a periodic trajectory. By definition, the curve $\gamma^Y$ projects to the same curve on the orbit space $N/H$ as the curve $\gamma^X$. Thus, the curve $\gamma^Y$ projects to a periodic trajectory on the orbit space, meaning it is also a relative periodic trajectory of $Y$. □

We now turn our attention to bifurcating branches. Recall:

**Definition 3.5.** Let $X$ be a path of equivariant vector fields on an $H$-manifold $N$, let $n \in N$ be a point, and let $\gamma : [0, \epsilon) \to N$ be a smooth curve with $\gamma(0) = n$. Furthermore, suppose points of $\gamma$ that are distinct from the starting point $n$ have group orbits distinct from $H \cdot n$. Then:
(1) The curve $\gamma$ is a branch of relative equilibria of $X$ if for all $\lambda \in (0, \epsilon)$ the point $\gamma(\lambda)$ is a relative equilibrium of the vector field $X_\lambda$.

(2) The curve $\gamma$ is a branch of relative periodic trajectories of $X$ if for all $\lambda \in (0, \epsilon)$ the integral curve of the vector field $X_\lambda$ starting at the point $\gamma(\lambda)$ is a relative periodic trajectory of $X_\lambda$.

A branch $\gamma$ is a trivial branch if it is a constant path.

One can analogously define bifurcating branches of “honest” equilibria and periodic trajectories. Again it is more natural to consider their relative counterparts in the presence of symmetries. This is partly justified by the following lemma.

**Lemma 3.6.** Let $X$ and $Y$ be isomorphic paths of equivariant vector fields on an $H$-manifold $N$. Suppose that $\gamma : [0, \epsilon) \to N$ is a bifurcating branch of relative equilibria or relative periodic trajectories of the path $X$. Then the curve $\gamma$ is also a bifurcating branch of relative equilibria or relative periodic trajectories, respectively, of the path $Y$.

**Proof.** The result for branches of relative equilibria follows immediately from Lemma 3.2, whereas the result for branches of relative periodic trajectories follows immediately from Lemma 3.4. \hfill \Box

## 4. Topologies on path spaces

In order to talk about generic bifurcations, we need to endow the space of paths of equivariant vector fields and the space of paths of gauge transformations with topologies.

Given an $H$-manifold $N$, the path space $C^\infty \left(\mathbb{R}, \mathfrak{X}(N)^H\right)$ can be identified with a subset of the mapping space $C^\infty \left(\mathbb{R} \times N, TN\right)$ (see Definition 2.14). Similarly, the path space $C^\infty \left(\mathbb{R}, C^\infty(N, \mathfrak{h})^H\right)$ can be identified with a subset of the space $C^\infty \left(\mathbb{R} \times N, \mathfrak{h}\right)$. Thus, we can topologize the path spaces by giving the corresponding mapping spaces some mapping space topology, and endowing the path spaces with the subspace topology. We will use the following mapping space topologies:

**Definition 4.1.** Let $U$ and $V$ be smooth manifolds.

- Given an integer $r \in \mathbb{Z}_{\geq 0}$, let $J^r(U, V)$ be the space of $r$-jets of mappings from $U$ to $V$. For a subset $O$ of $J^r(U, V)$ define the collection:
  \[ B^r(O) := \{ f \in C^\infty(U, V) \mid j^r f(U) \subseteq O \} \]
  The Whitney $C^r$-topology on $C^\infty(U, V)$ is the topology generated by the basis:
  \[ B^r := \{ B^r(O) \mid O \text{ is an open subset of } J^r(U, V) \} \]
  We will refer to the space $C^\infty(U, V)$ equipped with the Whitney $C^r$ topology as a **Whitney $C^r$ space**.

- The Whitney $C^\infty$-topology on $C^\infty(U, V)$ is the topology generated by the basis:
  \[ B^\infty := \bigcup_{r=0}^\infty B^r. \]
  We will refer to the space $C^\infty(U, V)$ equipped with the Whitney $C^\infty$ topology as a **Whitney $C^\infty$ space**.
Remark 4.2. Let $U$ and $V$ be smooth manifolds. Following Golubitsky and Guillemin [7, p. 43], we can get some intuition for the Whitney $C^r$-topology on $C^\infty(U,V)$ as follows. Pick a distance function $d$ on the space of $r$-jets $J^r(U,V)$, compatible with the topology on $J^r(U,V)$. Let $f$ be an arbitrary smooth map in $C^\infty(U,V)$, and let $\delta : U \to \mathbb{R}_+$ be a continuous function. Then the set:

$$B_\delta(f) := \{ g \in C^\infty(U,V) \mid d(j^rf(u), j^rg(u)) < \delta(u) \text{ for all } u \in U \}$$

is a neighborhood of $f$ in the Whitney $C^r$-topology. One can think of $B_\delta(f)$ as those maps in $C^\infty(U,V)$ that are, together with their first $r$ partial derivatives, $\delta$-close to the map $f$ and its first $r$ partial derivatives. In fact, the collection:

$$\{ B_\delta(f) \mid \delta : U \to \mathbb{R}_+ \text{ is a continuous function} \}$$

forms a neighborhood basis for the map $f$ in the Whitney $C^r$-topology.

Lemma 4.3. Let $U, V, W$, and $B$ be manifolds, and let $f : V \to W$, $g : V \to B$, and $h : W \to B$ be smooth maps.

1. The map:

$$f_* : C^\infty(U,V) \to C^\infty(U,W), \quad h \mapsto fh,$$

is continuous with respect to the Whitney topologies.

2. The canonical bijection of sets:

$$C^\infty(U,V \times W) \cong C^\infty(U,V) \times C^\infty(U,W)$$

is a homeomorphism with respect to the Whitney topologies.

3. The canonical bijection of sets:

$$C^\infty(U,V \times W)_{g,B,h} \cong C^\infty(U,V)_{g_*} \times C^\infty(U,W)_{h_*}$$

is a homeomorphism with respect to the Whitney topologies.

Proof. See [7, Proposition 3.5] for (1) and [7, Proposition 3.6] for (2). The continuity of the maps in the bijection of (3) follows by viewing the fiber products:

$$V \times W_{g,B,h} \quad C^\infty(U,V) \times C^\infty(U,W)_{g_*} \times C^\infty(U,W)_{h_*}$$

as subspaces of the products $V \times W$ and $C^\infty(U,V) \times C^\infty(U,W)$ respectively, and then applying parts (1) and (2) and the universal property of the subspace topology.

□

Lemma 4.4. Let $U, V, X$ and $Y$ be smooth manifolds. Then the map:

$$C^\infty(X,V) \times C^\infty(Y,W) \to C^\infty(X \times Y, U \times V), \quad (f, g) \mapsto f \times g,$$

where the map $f \times g : X \times Y \to U \times V$ is given by $(f \times g)(x, y) := (f(x), g(y))$, is a continuous map with respect to the Whitney topologies.

Proof. The proof of this fact is completely analogous to the proof of [7, Proposition 3.10].

□

Remark 4.5. Part (1) of Lemma 4.3 says that pushforwards by smooth maps are continuous with respect to the Whitney topologies. As discussed in the notes in [7, p. 49], pullbacks by smooth maps are in general not continuous with respect to the Whitney topologies. However, the following lemmas are two special cases of interest to us where the pullback is continuous.
Lemma 4.6. Let $U, V,$ and $W$ be manifolds, and let $f : V \to U$ be a smooth proper map. Then the pullback:

$$f^* : C^\infty(U, W) \to C^\infty(V, W), \quad h \mapsto hf,$$

is continuous with respect to the Whitney topologies.

*Proof.* See the second note in [7, p. 49]. □

Lemma 4.7. Let $K$ be a compact Lie group, let $P \xrightarrow{\pi} B$ be a principal $K$-bundle, and let $N$ be a manifold with a trivial action of $K$. Then the map:

$$\pi^* : C^\infty(B, N) \to C^\infty(P, N)^K, \quad f \mapsto f\pi,$$

is a homeomorphism. The inverse of $\pi^*$ is the map that takes an equivariant map $f : P \to N$ to the unique map $\tilde{f} : B \to N$ such that $f = f\pi$.

*Proof.* Since the group $K$ is compact, the bundle projection $\pi : P \to B$ is a proper map. Hence, the pullback $\pi^* : C^\infty(B, N) \to C^\infty(P, N)^K$ is continuous by Lemma 4.6. The remaining task is to show the continuity of the inverse map $(\pi^*)^{-1}$. The inverse map $(\pi^*)^{-1}$ sends a $K$-invariant map $f : P \to N$ to the unique map $\tilde{f} : B \to N$ such that the following diagram commutes:

$$
\begin{array}{ccc}
P & \xrightarrow{f} & N \\
\pi \downarrow & & \downarrow \pi \\
B & \xrightarrow{\tilde{f}} & N
\end{array}
$$

It suffices to show that the basis subsets given in Definition 4.1

$$\{B^r(O) \mid r \in \mathbb{Z}_{\geq 0}, O \subseteq J^r(B, N), O \text{ open}\}$$

have open preimages under the map $(\pi^*)^{-1}$ in the mapping space $C^\infty(P, N)^K$. We will show that the preimages $((\pi^*)^{-1})^{-1}(B^r(O)) = \pi^*(B^r(O))$ are themselves basis sets:

$$(4.1) \quad \pi^*(B^r(O)) = B^r(F_r^{-1}(O)),$$

where $F_r$ is a continuous map we define next.

Let $J^r(P, N)^K$ be the $r$-jets of $K$-invariant maps $P \to N$ and let $F_r$ be the map given by:

$$F_r : J^r(P, N)^K \to J^r(B, N), \quad j^r f(x_0) \mapsto j^r \tilde{f}(\pi(x_0)).$$

To verify that the map $F_r$ is continuous, pick local coordinates $U \subseteq \mathbb{R}^b$ for $B$, $V \subseteq \mathbb{R}^k$ for $K$, and $W \subseteq \mathbb{R}^n$ for $N$, and note that the $K$-invariant maps are represented by maps $f : U \times V \to W$ that are independent of the $V$ variables. Furthermore, given a map $f : U \times V \to W$, let $(u_0, v_0)$ be a point in $U \times V$, let $T_r f(u_0, v_0)$ be the coefficients of the $r^{th}$-order Taylor polynomial at the point $(u_0, v_0)$, let $T^U_r(u_0, v_0)$ denote the coefficients of the $r^{th}$-order Taylor polynomial at the point $(u_0, v_0)$ consisting only of those partial derivatives with respect to the $U$-variables only, and let $T^C_r(u_0, v_0)$ correspond to the rest of the coefficients in $T_r(u_0, v_0)$. Then note that the map $\tilde{f} = (\pi^*)^{-1}(f)$ has $r^{th}$-order Taylor polynomial such that $T_r \tilde{f}(u_0) = T^U_r f(u_0, v_0)$. Hence, the map $F_r$ is continuous since it is just the projection:

$$j^r f(u_0, v_0) = (u_0, v_0, T^U_r f(u_0, v_0), T^C_r f(u_0, v_0)) \mapsto (u_0, T^U_r f(u_0, v_0)).$$
for any map \( f : U \times V \to W \), independent of the \( V \) variables, and any point \((u_0, v_0) \in U \times V\).

We now proceed to verify \[\text{4.1}\]. Consider an arbitrary map \( f \in \pi^*(B^r(O))\). Then \( \tilde{f} = (\pi^*)^{-1}f \in B^r(O) \), so the image \((j^\ast \tilde{f})(B)\) is contained in the open set \( O \).

Taking the preimage of this inclusion under the map \( F_r \), we obtain that:

\[
j^\ast f(P) \subseteq F_r^{-1}(F_r(j^\ast f(P))) \subseteq F_r^{-1}\left((j^\ast \tilde{f})(B)\right) \subseteq F_r^{-1}(O),
\]

where we also use that \( F_r(j^\ast f(P)) = j^\ast \tilde{f}(B) \). Consequently the map \( f \) is an element of the basis set \( B^r(F_r^{-1}(O)) \) as desired.

For the converse inclusion, consider an arbitrary map \( f \in B^r(F_r^{-1}(O)) \). Then the image \((j^\ast f)(P)\) is contained in the preimage \( F_r^{-1}(O) \). Taking the image of this inclusion under the map \( F_r \) we obtain that:

\[
F_r(j^\ast f(P)) \subseteq F_r(F_r^{-1}(O)) \subseteq O,
\]

which in turn implies that:

\[
(j^\ast \tilde{f})(B) = F_r(j^\ast f(P)) \subseteq O.
\]

Consequently, the map \( \tilde{f} = (\pi^*)^{-1}(f) \) is in the basis set \( B^r(O) \), meaning that the map \( f \) is an element of the preimage \( \pi^*(B^r(O)) \). This proves the equality \[\text{4.1}\] and hence the continuity of the inverse map \((\pi^*)^{-1}\). Hence, the map \( \pi^* \) is a homeomorphism with respect to the Whitney \( C^\infty \) topology. \qed

**Corollary 4.8.** Let \( G \) be a Lie group, let \( K \) be a compact Lie subgroup of \( G \), let \( V \) be a finite-dimensional real representation of the compact Lie group \( K \), and let \( N \) be a \( G \)-manifold. Furthermore, let \( G \times^K V \) be the quotient of the action of the group \( K \) on the product \( G \times V \) given by:

\[
k \cdot (g,v) := (gk^{-1}, k \cdot v), \quad k \in K, (g,v) \in G \times V.
\]

Then the map:

\[
\epsilon : C^\infty(V,N)^K \to C^\infty(G \times^K V,N)^G, \quad f \mapsto \epsilon f,
\]

where the map \( \epsilon f : G \times^K V \to N \) is defined by \( \epsilon f([g,v]) := g \cdot f(v) \), is continuous with respect to the Whitney \( C^\infty \)-topologies.

**Remark 4.9.** We call the map \( \epsilon : C^\infty(V,N)^K \to C^\infty(G \times^K V,N)^G \) in the statement of Corollary \[\text{4.8}\] the *equivariant extension* of maps from the representation \( V \) to the associated bundle \( G \times^K V \).

**Proof.** First, note that the map \( \epsilon \) is well-defined for if \( f : V \to N \) is a \( K \)-equivariant map then, for \( k \in K \) and \((g,v) \in G \times V \) we have that:

\[
\epsilon f([gk^{-1}, k \cdot v]) = gk^{-1} \cdot f(k \cdot v) = gk^{-1} \cdot (k \cdot f(v)) = g \cdot f(v) = \epsilon f([g,v]).
\]

To prove that the map \( \epsilon \) is continuous, it suffices to prove that the map \( \iota \epsilon \) is continuous, where \( \iota \) is the inclusion of the subspace of equivariant maps \( C^\infty(G \times^K V,N)^G \) into the space of maps \( C^\infty(G \times^K V,N) \). We show that the composition \( \iota \epsilon \) factors as the composition of continuous maps. For this, note that the map:

\[
C^\infty(V,N)^K \hookrightarrow C^\infty(G,G) \times C^\infty(V,N), \quad f \mapsto (\text{id}_G, f),
\]

where \( \text{id}_G \) is the identity map of \( G \), is a continuous inclusion. By Lemma \[\text{4.4}\] the map:

\[
C^\infty(G,G) \times C^\infty(V,N) \hookrightarrow C^\infty(G \times V,G \times N), \quad (\varphi, f) \mapsto \varphi \times f,
\]
where $\varphi \times f$ is defined by $(\varphi \times f)(g, v) := (\varphi(g), f(v))$, is continuous. Furthermore, let the action of the group $G$ on the manifold $N$ be given by the map $ac : G \times N \to N$. The pushforward:

$$(4.4) \quad ac_* : C^\infty(G \times V, G \times N) \to C^\infty(G \times V, N), \quad \psi \mapsto ac \circ \psi,$$

is a continuous map by part (1) of Lemma 4.3. The composition of the maps in (4.4) ac factors as a composition of continuous maps. Thus, the pushforward:

$$(4.5) \quad C^\infty(V, N)^K \to C^\infty(G \times V, N), \quad f \mapsto \tilde{f},$$

where the map $\tilde{f} : G \times V \to N$ is defined by $\tilde{f}(g, v) := g \cdot f(v)$. Note that for any map $f \in C^\infty(V, N)^K$, the map $\tilde{f}$ is $K$-invariant. Hence, the map in (4.5) restricts to a continuous map:

$$(4.6) \quad \alpha : C^\infty(V, N)^K \to C^\infty(G \times V, N)^{K-\text{inv}}, \quad f \mapsto \tilde{f},$$

where $C^\infty(G \times V, N)^{K-\text{inv}}$ is the space of $K$-invariant maps $G \times V \to N$. On the other hand, let $\pi : G \times V \to G \times K V$ be the quotient map of the quotient space $G \times K V$. Then, since $\pi : G \times V \to G \times K V$ is a principal $K$-bundle, the pullback:

$$\pi^* : C^\infty(G \times K V, N) \to C^\infty(G \times V, N)^{K-\text{inv}}, \quad f \mapsto f \pi,$$

is a homeomorphism by Lemma 4.11. Finally, note that the following diagram commutes:

$$\begin{array}{ccc}
C^\infty(V, N)^K & \xrightarrow{\epsilon} & C^\infty(G \times K V, N)^G \\
\downarrow{\alpha} & & \downarrow{\iota} \\
C^\infty(G \times V, N)^{K-\text{inv}} & \xrightarrow{(\pi^*)^{-1}} & C^\infty(G \times K V, N)
\end{array}$$

and hence the composition $\iota \epsilon$ factors as a composition of continuous maps. Thus, the map $\epsilon$ is continuous as claimed. \hfill \Box

**Remark 4.10.** Let $N$ be an $H$-manifold. Note that the scalar multiplication in the path space $C^\infty(\mathbb{R}, \mathcal{X}(N)^H)$ need not be continuous (see the discussion after the proof of Proposition 3.5 in [7, pp. 46-47]). Here the path space has the subspace topology as a subspace of the mapping space $C^\infty(\mathbb{R} \times N, T N)$ with the Whitney topology. Hence, the path space $C^\infty(\mathbb{R}, \mathcal{X}(N)^H)$ is not a topological vector space. The same observation applies to the path space $C^\infty(\mathbb{R}, C^\infty(N, h)^H)$. However, as the following lemma shows, the addition and inversion maps are continuous. Therefore, these paths spaces are topological abelian groups.

**Lemma 4.11.** Let $N$ be an $H$-manifold. The space of paths of equivariant vector fields $C^\infty(\mathbb{R}, \mathcal{X}(N)^H)$ and the space of paths of gauge transformations $C^\infty(\mathbb{R}, C^\infty(N, \mathfrak{h})^H)$ are topological abelian groups.

**Proof.** We consider the case of the space $C^\infty(\mathbb{R}, \mathcal{X}(N)^H)$. The case of the space $C^\infty(\mathbb{R}, C^\infty(N, \mathfrak{h})^H)$ is analogous by thinking of the Lie algebra $\mathfrak{h}$ as a vector bundle over a point. It suffices to prove that the addition and additive inverse maps on the vector space $C^\infty(\mathbb{R}, \mathcal{X}(N)^H)$ are continuous. For the addition note that the
following diagram commutes:

\[
\begin{array}{c}
C^\infty (\mathbb{R}, \mathfrak{X}(N)^H) \times C^\infty (\mathbb{R}, \mathfrak{X}(N)^H) \xrightarrow{+} C^\infty (\mathbb{R}, \mathfrak{X}(N)^H) \\
\end{array}
\]

(4.7)

\[
\begin{array}{c}
C^\infty (\mathbb{R} \times N, TN) \times C^\infty (\mathbb{R} \times N, TN) \xrightarrow{+,*} C^\infty (\mathbb{R} \times N, TN) \\
\end{array}
\]

where the vertical maps are subspace inclusions, the top map \(+\) is the desired addition map, the pushforward \(+,*\) is the pushforward of the fiberwise addition
\(+ : TN \times_N TN \to TN\) with the domain of the pushforward identified with the fiber product via the canonical homeomorphism:

\[
\begin{array}{c}
C^\infty (\mathbb{R} \times N, TN) \times C^\infty (\mathbb{R} \times N, TN) \cong C^\infty (\mathbb{R} \times N, TN \times_N TN) \\
\end{array}
\]

(4.8)

of part (3) of Lemma 4.3. The continuity of the addition \(+\) now follows by the universal property of the subspace topology, the commutativity of diagram (4.7), and the fact that the composition along the left and bottom of this diagram is continuous.

For the additive inverse, note that the following diagram commutes:

\[
\begin{array}{c}
C^\infty (\mathbb{R}, \mathfrak{X}(N)^H) \xrightarrow{-} C^\infty (\mathbb{R}, \mathfrak{X}(N)^H) \\
\end{array}
\]

(4.9)

\[
\begin{array}{c}
C^\infty (\mathbb{R} \times N, TN) \xrightarrow{-,*} C^\infty (\mathbb{R} \times N, TN) \\
\end{array}
\]

where the vertical maps are inclusions, the top map is the desired additive inverse map, and the bottom map is the pushforward \(-,*\) of the fiberwise additive inverse map \(- : TN \to TN\). The continuity of the additive inverse \(-\) now follows by the universal property of the subspace topology, the commutativity of diagram (4.9), the continuity off the pushforward \(-,*\) by part (1) of Lemma 4.3, and the continuity of the inclusion on the left. \(\Box\)

**Remark 4.12.** A 2-term chain complex of topological abelian groups is a 2-term chain complex where the spaces are topological abelian groups and the boundary map is a continuous group homomorphism. Given an \(H\)-manifold \(N\), the spaces of the 2-term chain complex \(C^\infty (\mathbb{R}, \mathfrak{X}(N)^H)\) are topological abelian groups by Lemma 4.11. The following lemma proves that the boundary map of this chain complex is a continuous group homomorphism.

**Lemma 4.13.** Let \(N\) be an \(H\)-manifold. The boundary map:

\[
\partial : C^\infty (\mathbb{R}, C^\infty (N, b)^H) \to C^\infty (\mathbb{R}, \mathfrak{X}(N)^H)
\]

of the 2-term chain complex \(C^\infty (\mathbb{R}, \mathfrak{X}(N)^H)\) of paths of equivariant vector fields (Definition 2.17) is a continuous group homomorphism with respect to the Whitney topologies.
Proof. It suffices to prove that the boundary map is continuous. We prove that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{C}^\infty(\mathbb{R}, \mathcal{H}) & \xrightarrow{\partial} & \mathbb{C}^\infty(\mathbb{R}, \mathcal{X}(N)^H) \\
\downarrow & & \downarrow \\
\mathbb{C}^\infty(\mathbb{R} \times N, \mathbb{C}^\infty(\mathbb{N}, h)) & \xrightarrow{\alpha_*} & \mathbb{C}^\infty(\mathbb{R} \times N, \mathbb{T}N)
\end{array}
\]  

(4.10)

where the top map is the boundary map, the left-hand map is the inclusion defined by \( \psi \mapsto (\psi, \text{pr}_2) \) with \( \text{pr}_2 : \mathbb{R} \times N \to \mathbb{N} \) being the projection onto the second factor, the right-hand map is the obvious inclusion, and the bottom map is the pushforward of the map:

\[
a : \mathbb{R} \times N \to \mathbb{T}N, \quad (\xi, n) \mapsto \left. \frac{d}{d\tau} \right|_0 \exp(\tau \xi) \cdot n,
\]

(4.11)

where we have also used part (3) of Lemma 4.3 to write the domain as a product.

Note that the map \( a \) is smooth since it is obtained by differentiating the action \( H \times N \to N \), with respect to the \( H \)-variables only, at the identity of \( H \). Hence, the pushforward \( \alpha_* \) is continuous by part (1) of Lemma 4.3. On the other hand, the inclusion on the left-hand side of diagram (4.10) is continuous since it is the product of the inclusion \( \mathbb{C}^\infty(\mathbb{R}, \mathcal{H}) \hookrightarrow \mathbb{C}^\infty(\mathbb{R} \times N, h) \) and the constant map:

\[
\mathbb{C}^\infty(\mathbb{R}, \mathcal{H}) \to \mathbb{C}^\infty(\mathbb{R} \times N, \mathbb{N}), \quad \psi \mapsto \text{pr}_2.
\]

The continuity of the map \( \partial \) now follows by the universal property of the subspace topology, the commutativity of diagram (4.10), and the fact that the composition along the left and bottom of this diagram is continuous. \( \square \)

5. Genericity

In keeping with the philosophy of this paper, we want to study the genericity of families of equivariant vector fields up to isomorphism. Motivated by Remarks 4.10 and 4.12, we show that generic points in a 2-term chain complex of topological abelian groups form a category, and that the homotopic 2-term chain complexes of topological abelian groups have equivalent categories of generic objects. Recall the following definition:

**Definition 5.1.** A subset \( \mathcal{U} \) of a topological space \( \mathcal{Y} \) is residual if it is the countable intersection of open and dense subsets of \( \mathcal{Y} \). A point \( x \in \mathcal{Y} \) is generic if it is an element of a residual subset of \( \mathcal{Y} \).

Thus, we introduce the following:

**Definition 5.2.** Let \( A_\bullet = \{ A_1 \xrightarrow{\partial} A_0 \} \) be a 2-term chain complex of topological abelian groups. A point \( x \in A_0 \) is generic in \( A_\bullet \) if it is generic in the space \( A_0 \) in the sense of Definition 5.1.

**Remark 5.3.** A 2-term chain complex of topological abelian groups \( A_\bullet \) determines a continuous action of the abelian group \( A_1 \) on the space \( A_0 \). The action is given by:

\[
\psi \cdot x := x + \partial(\psi), \quad \psi \in A_1, \ x \in A_0,
\]
where $\partial : A_1 \to A_0$ is the boundary map. As with all continuous group actions, the quotient map $A_0 \to A_0/A_1$ of this action is an open map. Let $x \in A_0$ be a point. We will denote the orbit of $x$, with respect to the action of $A_1$, by $A_1 \cdot x$ when viewing it as a subset of the space $A_0$, and denote it by $[x]$ when viewing it as a point in the orbit space $A_0/A_1$.

**Proposition 5.4.** Given a 2-term chain complex of topological abelian groups $A_\bullet$, a point $x \in A_0$ is generic in $A_\bullet$ if and only if its orbit $[x]$ is a generic point in the orbit space $A_0/A_1$.

**Proof.** Let $\pi : A_0 \to A_0/A_1$ be the quotient map of the 2-term chain complex. First, suppose the point $x$ is generic in the space $A_0$. We need to show that its orbit $[x]$ is contained in a residual subset of the orbit space. Let $\{U_\alpha\}_{\alpha=1}^\infty$ be a countable family of open dense subsets of the space $A_0$ whose intersection contains the point $x$. Consider the countable family $\{\pi(U_\alpha)\}_{\alpha=1}^\infty$ of subsets of the orbit space. The orbit $[x]$ is contained in the intersection of these sets, and each of them is open since the map $\pi$ is open. Finally, note that:

\[
A_0/A_1 = \pi(A_0) \quad \text{by the surjectivity of } \pi
\]

\[
= \pi \left( \bigcap_{\alpha} U_\alpha \right) \quad \text{since the } U_\alpha \text{ are dense in } A_0
\]

\[
\subseteq \pi \left( \bigcap_{\alpha} U_\alpha \right) \quad \text{by the continuity of } \pi.
\]

This shows that the sets $\pi(U_\alpha)$ are dense in the orbit space $A_0/A_1$. Hence, the orbit $[x]$ is a generic point in the orbit space $A_0/A_1$.

Conversely, suppose that the orbit $[x]$ is a generic point in the orbit space. We need to show that the point $x$ is contained in a residual subset of the space $A_0$. Let $\{V_\alpha\}_{\alpha=1}^\infty$ be a countable family open dense subsets of the orbit space whose intersection contains the orbit $[x]$. Consider the countable family $\{\pi^{-1}(V_\alpha)\}_{\alpha=1}^\infty$ of subsets of the space $A_0$. The point $x$ is contained in the intersection of these sets, and each of them is open since the map $\pi$ is continuous. Now observe that, for all $\alpha \in \mathbb{Z}_{\geq 1}$, we have:

\[
\pi^{-1}(V_\alpha) = \pi^{-1} \left( V_\alpha \right) \quad \text{since } \pi \text{ is an open map}
\]

\[
= \pi^{-1} \left( A_0/A_1 \right) \quad \text{since the } V_\alpha \text{ are dense in the orbit space}
\]

\[
= A_0.
\]

This shows that the sets $\pi^{-1}(V_\alpha)$ are dense in the space $A_0$. Hence, the point $x$ is generic in the orbit space $A_0/A_1$. \qed

**Corollary 5.5.** Let $x, y \in A_0$ be two isomorphic points in a 2-term chain complex of topological abelian groups. If the point $x$ is generic then so is the point $y$.

**Proof.** By definition isomorphic points in the space $A_0$ have the same orbit in the orbit space. Thus, the result follows immediately from Proposition 5.4. \qed

**Remark 5.6.** Given a 2-term chain complex of topological abelian groups $A_\bullet$, the action of the topological abelian group $A_1$ on the space $A_0$ determines an action groupoid $A_0 \times A_1 \rightrightarrows A_0$, as with the case of vector spaces. The structure maps of this action groupoid are continuous group homomorphisms, so the action groupoid is in fact a topological abelian 2-group. That is, $A_0 \times A_1 \rightrightarrows A_0$ is a small category internal to the category of topological abelian groups. The Baez-Crans equivalence
mentioned in Remark 2.12 extends to this context. That is, there is an equivalence of strict 2-categories:

\[ \text{2TermTopAb} \simeq \text{2TopAb} \]

between the strict 2-category of 2-term chain complexes of topological abelian groups and the strict 2-category of topological abelian 2-groups. The equivalence is the same as that of the case of 2-vector spaces, since all the required maps turn out to be continuous. In particular, note that a homotopic 2-term chain complexes of topological abelian groups have equivalent topological abelian 2-groups.

Thanks to Corollary 5.5 the following category is well-defined:

**Definition 5.7.** Let \( A_\bullet \) be a 2-term chain complex of topological abelian groups. The category \( \text{Gen}(A_\bullet) \) of generic elements of the 2-term chain complex \( A_\bullet \) is the full subcategory of the action groupoid of \( A_1 \times A_0 \rightrightarrows A_0 \) with objects the generic elements of \( A_\bullet \).

The point of the category of Definition 5.7 is to relate the generic points of homotopic chain complexes. For this, we need be precise about what we mean by homotopy chain complexes:

**Definition 5.8.** A map \( F_\bullet : A_\bullet \to B_\bullet \) between 2-term chain complexes of topological abelian groups is a map of chain complexes that is a continuous group homomorphisms in each degree.

**Definition 5.9.** A homotopy \( h : F_\bullet \cong G_\bullet \) between maps of 2-term chain complexes of topological abelian groups \( F_\bullet : A_\bullet \to B_\bullet \) and \( G_\bullet : A_\bullet \to B_\bullet \) is a homotopy of chain complex maps such that the corresponding map \( h : A_0 \to B_1 \) is a continuous group homomorphism.

**Definition 5.10.** Two 2-term chain complexes of topological abelian groups \( A_\bullet \) and \( B_\bullet \) are homotopic or equivalent, which we write \( A_\bullet \simeq B_\bullet \), if there exist maps \( F_\bullet : A_\bullet \to B_\bullet \) and \( G_\bullet : B_\bullet \to A_\bullet \), and homotopies \( \mu : G_\bullet F_\bullet \cong 1_{A_\bullet} \) and \( \eta : F_\bullet G_\bullet \cong 1_{B_\bullet} \). We say the maps \( F_\bullet \) and \( G_\bullet \) are homotopy inverses of each other as maps of 2-term chain complexes of topological abelian groups.

The category of generic points respects the principle of equivalence in the following sense:

**Theorem 5.11.** Let \( A_\bullet \) and \( B_\bullet \) be 2-term chain complexes of topological abelian groups. If the chain complexes \( A_\bullet \) and \( B_\bullet \) are homotopic, then the categories of generic elements \( \text{Gen}(A_\bullet) \) and \( \text{Gen}(B_\bullet) \) are equivalent categories.

**Proof.** Denote the maps that are part of the equivalence by \( E_\bullet : A_\bullet \to B_\bullet \) and \( P_\bullet : B_\bullet \to A_\bullet \) and the homotopies by \( \mu : P_\bullet E_\bullet \cong 1_{A_\bullet} \) and \( \eta : E_\bullet P_\bullet \cong 1_{B_\bullet} \). By the Baez-Crans equivalence between 2-term chain complexes of topological abelian groups and topological abelian 2-groups, this equivalence induces an equivalence between the topological abelian 2-groups of \( A_\bullet \) and \( B_\bullet \) (see Remark 5.6). Specifically, the map \( E_\bullet \) determines the functor:

\[
\left( E_1 \times E_0, E_0 \right) : (A_1 \times A_0 \rightrightarrows A_0) \to (B_1 \times B_0 \rightrightarrows B_0),
\]

and analogously for the map \( P_\bullet \). Furthermore, the homotopy \( \mu : P_\bullet E_\bullet \cong 1_{A_\bullet} \) determines the natural isomorphism \( \mu \times P_0 E_0 : P_\bullet E_\bullet \cong 1 \) given by:

\[
\mu \times P_0 E_0 : A_0 \to B_1 \times B_0, \quad x \mapsto \left( \mu(x), P_0 E_0(x) \right),
\]
and analogously for the homotopy \( \eta : E_\bullet \sim 1_{B_\bullet} \). We will show that these functors and natural isomorphisms restrict to give the desired equivalence \( \text{Gen}(A_\bullet) \simeq \text{Gen}(B_\bullet) \).

Since the categories of generic elements are full subcategories of the respective 2-groups, it suffices to check that the functors restrict on objects. Note that the natural isomorphisms will restrict to natural isomorphisms of the restricted functors, again since the categories of generic objects are full subcategories. That is, it suffices to verify that the maps \( E_0 \) and \( P_0 \) take generic points to generic points.

Since the compositions \( E_\bullet P_\bullet \) and \( P_\bullet E_\bullet \) are homotopic to the identities, the maps \( E_0 \) and \( P_0 \) induce inverse maps \( [E] \) and \( [P] \) such that the following diagrams commute:

\[
\begin{array}{ccc}
A_0 & \xrightarrow{E_0} & B_0 \\
|E| & \downarrow & \downarrow \\
A_0/A_1 & \xrightarrow{\sim} & B_0/B_1 \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
B_0 & \xrightarrow{P_0} & A_0 \\
|P| & \downarrow & \downarrow \\
B_0/B_1 & \xrightarrow{\sim} & A_0/A_1 \\
\end{array}
\]

(5.1)

where the vertical maps are the quotient maps. Since the maps \( E_0 \) and \( P_0 \) are continuous, the maps \( [E] \) and \( [P] \) are both continuous. Since they are homeomorphisms, the maps \( [E] \) and \( [P] \) take generic points to generic points.

Now let \( x \) be a generic point in the space \( A_0 \). Hence, the orbit \( [x] \) is a generic point in the orbit space \( A_0/A_1 \) by Proposition 5.4. Since the map \( [E] \) takes generic points to generic points, the orbit \( [E]( [x] ) = [E_0(x)] \) is generic, where we also use the left hand side diagram in (5.1). Hence, the point \( E_0(x) \) is generic in the space \( B_0 \), again by Proposition 5.4. A similar argument holds for the map \( P_\bullet \). Thus, we obtain an equivalence \( \text{Gen}(A_\bullet) \simeq \text{Gen}(B_\bullet) \). \( \square \)

The following will be useful for relating collections of generic points between homotopic chain complexes:

**Corollary 5.12.** Let \( A_\bullet \) and \( B_\bullet \) be 2-term chain complexes of topological abelian group, and let \( E_\bullet : A_\bullet \to B_\bullet \) and \( P_\bullet : B_\bullet \to A_\bullet \) be (part of) an equivalence of chain complexes. If \( U \subseteq A_0 \) is an open, dense, and invariant subset of the space \( A_0 \) with respect to the action of the group \( A_1 \), then:

\[ P_0^{-1}(U) = B_1 \cdot E_0(U). \]

In particular, the preimage \( P_0^{-1}(U) \) is an open, dense, and invariant subset of \( B_0 \) with respect to the action of the group \( B_1 \).

**Proof.** Let \( \mu : P_\bullet E_\bullet \simeq 1_{A_\bullet} \) and \( \eta : E_\bullet P_\bullet \simeq 1_{B_\bullet} \) be the homotopies in the equivalence \( A_\bullet \simeq B_\bullet \). We first prove the equality of sets. Let \( y \in P_0^{-1}(U) \), then note that \( P_0(y) \in U \) and \( \eta(P_0(y)) \in B_1 \) are such that:

\[ y = E_0 P_0(y) + \partial\left(\eta(P_0(y))\right), \]

since \( \eta \) is a homotopy \( E_\bullet P_\bullet \simeq 1_{B_\bullet} \). Hence, \( y \in B_1 \cdot E_0(U) \). For the converse inclusion, let \( y \in B_0 \) be such that \( y = E_0(x) + \partial(\psi) \) for some \( x \in U \) and some
\[ \psi \in B_1. \] Then:

\[ P_0(y) = P_0 E_0(x) + P_0(\partial(\psi)) \quad \text{since } P_0 \text{ is a homomorphism} \]
\[ = x + \partial(\mu(\psi)) + P_0(\partial(\psi)) \quad \text{since } \mu \text{ is a homotopy } PE \cong 1_A \]
\[ = x + \partial(\mu(x)) + \partial(P_1(\psi)) \quad \text{since } P_* \text{ is a map of chain complexes} \]
\[ = x + \partial(\mu(x) + P_1(\psi)) \quad \text{since } \partial \text{ is a homomorphism.} \]

This shows that the point \( P_0(y) \) is in the \( A_1 \)-orbit of the point \( x \), which is in \( U \). Thus, the point \( P_0(y) \) is in the set \( U \) since \( U \) is invariant with respect to the action of the group \( A_1 \). This proves the desired equality.

It now suffices to show that the set \( B_1 \cdot E_0(U) \) is open, dense, and invariant. It is immediate that it is invariant. Let \( \pi_A : A_0 \to A_0/A_1 \) and \( \pi_B : B_0 \to B_0/B_1 \) be the quotient maps of the orbit spaces. As in the proof of Theorem 5.11, the map \( E_* \) induces a homeomorphism \( [E] : A_0/A_1 \xrightarrow{\cong} B_0/B_1 \) such that \( \pi_B E_0 = [E] \pi_A \).

Since the quotient map is open and the map \( [E] \) is a homeomorphism, the set \( \pi_B E_0(U) = [E] \pi_A(U) \) is open. Hence, since:

\[ B_1 \cdot E_0(U) = \pi_B^{-1}(\pi_B(E_0(U))), \]

the set \( B_1 \cdot E_0(U) \) is open. To see that the set \( B_1 \cdot E_0(U) \) is dense, note that:

\[
B_1 \cdot E_0(U) = \pi_B^{-1}(\pi_B(E_0(U)))
\]
\[
= \pi_B^{-1}(\pi_B(E_0(U))) \quad \text{since } \pi_B \text{ is an open map}
\]
\[
= \pi_B^{-1}(E(\pi_A(U))) \quad \text{since } \pi_B E_0 = [E] \pi_A
\]
\[
= \pi_B^{-1}(E(\pi_A(U))) \quad \text{since } [E] \text{ is a homeomorphism}
\]
\[
\supseteq \pi_B^{-1}(E(\pi_A(U))) \quad \text{by the continuity of the map } \pi_A
\]
\[
= \pi_B^{-1}(E(\pi_A(A_0))) \quad \text{since } U \text{ is dense in } A_0
\]
\[
= \pi_B^{-1}(E(\pi_A(A_0))) \quad \text{by the surjectivity of } \pi_A
\]
\[
= \pi_B^{-1}(E(\pi_A(A_0))) \quad \text{since } [E] \text{ is a homeomorphism}
\]
\[
= B_0.
\]

Since the converse inclusion is trivial, the set is dense as claimed. \( \square \)

6. Relative bifurcations up to isomorphism

Given a proper action of a Lie group \( G \) on a manifold \( M \), we define a 2-term chain complex of equivariant bifurcation problems on \( M \) (Definition 6.8 and Definition 6.10). We first consider the case of representations of compact Lie groups (section 6.1), and then generalize this to proper actions via slices (section 6.2).

Suppose the canonical slice representation through a point in \( M \) is global, we prove that the 2-term chain complex of equivariant bifurcation problems on the manifold \( M \) is equivalent to the 2-term chain complex of equivariant bifurcation problems on the slice for the action through the point (Theorem 6.20). As a corollary, we prove that generic equivariant bifurcation problems on the given proper action are equivalent to generic equivariant bifurcation problems on a global slice for the action (Corollary 6.30).
6.1. **Equivariant bifurcation problems on representations.** In this subsection we define a 2-term chain complex of equivariant bifurcation problems on a finite-dimensional real representation $V$ of a compact Lie group $K$ (Definition 6.8).

**Remark 6.1.** Let $V$ be a representation of a compact Lie group $K$. In what follows we will often suppose the representation is irreducible or the sum $W \oplus W$, where $W$ is an absolutely irreducible representation of $K$. The motivation for this is the following. It is standard to study bifurcations on a representation $V$ of a compact Lie group $K$ by considering the dynamics on the center manifold since this is the part of the dynamics that exhibits bifurcations. In the case of representations of compact Lie groups, the representation on the center subspace of a generic equivariant bifurcation problem is irreducible or the sum $W \oplus W$, where $W$ is an absolutely irreducible representation of $K$ (see [9, Ch. XIII Proposition 3.2] and [9, Ch. XVI Proposition 1.4]).

**Remark 6.2.** Recall that a representation is irreducible if the only invariant linear subspaces are the zero subspace or the whole vector space. Also recall that the ring of equivariant linear endomorphisms $\text{End}(V)^K$ of an irreducible real representation $V$ of a compact Lie group $K$ is isomorphic to one of the division algebras by Schur’s Lemma (see, for example, [9, Ch. XII §3]). We say that such a representation is of *real, complex, or quaternionic type* when the ring $\text{End}(V)^K$ is isomorphic to the respective division algebra. It is standard to call the representation $V$ of the compact Lie group $K$ *absolutely irreducible* if it is irreducible of real type.

**Remark 6.3.** Let $V$ be a representation of a compact Lie group $K$. The linearization of a vector field $Y$ with an equilibrium at the origin $0 \in V$ is a linear endomorphism $DY(0) : T_0V \to T_0V$. At times it may be more convenient to use the canonical identification $T_0V \cong V$ and think of the linearization as a linear endomorphism $DY(0) : V \to V$.

**Remark 6.4.** Let $X$ be a path of equivariant vector fields on a representation $V$ of the compact Lie group $K$ with a trivial branch of relative equilibria at the origin in the sense of Definition 3.5. Since the action is linear, the orbit of the origin is trivial. Hence the path $X$ has an “honest” equilibrium at the origin. Equivalently, it has a trivial branch of equilibria at the origin.

Recall the following necessary condition for bifurcations from an equilibrium to occur:

**Definition 6.5** (critical equilibrium). Let $V$ be a representation of a compact Lie group $K$. Let $X$ be a path of equivariant vector fields on $V$ with a relative equilibrium at the origin. The origin $0 \in V$ is a **critical (relative) equilibrium** of $X$ if the linearization $DX_0(0)$ has eigenvalues with zero real part. Such eigenvalues are called **critical eigenvalues** of $X$.

We introduce the following definition:

**Definition 6.6** (equivariant bifurcation problem on a representation). Let $V$ be a representation of a compact Lie group $K$. An **equivariant bifurcation problem** on $V$ is a path of equivariant vector fields $X$ on $V$ with a critical (relative) equilibrium at the origin $0 \in V$.

**Definition 6.7.** Let $V$ be a representation of a compact Lie group $K$. Two equivariant bifurcations problems on $V$ are **isomorphic equivariant bifurcation problems**
if they are isomorphic as paths of equivariant vector fields in the sense of Definition 2.20.

The following definition is the main point of this section:

**Definition 6.8** (2-term chain complex of equivariant bifurcation problems on a representation). Let $V$ be a representation of the compact Lie group $K$ satisfying the assumptions in Remark 6.1. The 2-term chain complex $\text{Bif}(V)^K_0$ of equivariant bifurcation problems on $V$ is the 2-term chain complex of topological abelian groups:

$$\text{Bif}(V)^K_1 \xrightarrow{\partial} \text{Bif}(V)^K_0,$$

where $\text{Bif}(V)^K_1$ is the space $C^\infty(\mathbb{R}, C^\infty(V, \mathfrak{k})^K)$ of paths of infinitesimal gauge transformations, $\text{Bif}(V)^K_0$ is the space of equivariant bifurcation problems of Definition 6.6, and the boundary map $\partial$ is given by restricting the boundary map of the 2-term chain complex of paths of equivariant vector fields on $V$ (Definition 2.17).

To verify that Definition 6.8 is well-defined we verify that the space of equivariant bifurcation problems $\text{Bif}(V)^K_0$ is a subgroup of the space $C^\infty(\mathbb{R}, \mathfrak{X}(V)^K)$ of paths of equivariant vector fields on $V$ (Lemma 6.10), and that the image of the boundary map $\partial$ in Definition 2.17 has image contained in the space $\text{Bif}(V)^K_0$ (Lemma 6.13).

**Remark 6.9.** Let $V$ be a real representation of a compact Lie group $K$. Recall that if the representation $V$ of $K$ is absolutely irreducible, then all equivariant linear endomorphisms are real multiples of the identity map. If the representation $V$ of $K$ is of complex type there is an equivariant linear endomorphism $J$ such that:

$$\text{End}(V)^K = \{\alpha I + \beta J | \alpha, \beta \in \mathbb{R}\}.$$

Furthermore, the eigenvalues of an equivariant linear endomorphism $\alpha I + \beta J$ are $\alpha \pm \beta i$. If the representation $V$ of the compact Lie group $K$ is of quaternionic type there exist three equivariant endomorphisms $I, J, K$, playing the role of the basis quaternions, such that:

$$\text{End}(V)^K = \{\alpha I + \beta I + \gamma J + \delta K | \alpha, \beta, \gamma, \delta \in \mathbb{R}\}.$$

In this case, the eigenvalues of an arbitrary equivariant linear endomorphism $\alpha I + \beta I + \gamma J + \delta K$ are $\alpha \pm i \sqrt{\beta^2 + \gamma^2 + \delta^2}$. Finally, if the representation $V$ of $K$ is the sum $W \oplus W$ of an absolutely irreducible representation $W$ of the compact Lie group $K$, the equivariant linear endomorphisms can be written in block form so that:

$$\text{End}(V)^K = \left\{ \begin{pmatrix} \alpha I & b I \\ c I & d I \end{pmatrix} | a, b, c, d \in \mathbb{R} \right\},$$

where $I$ is the identity map of $W$. In this case the eigenvalues of an arbitrary equivariant linear endomorphism are $2 \pm \sqrt{4a^2 - 4(ad - bc)}$.

**Lemma 6.10.** Let $V$ be a representation of the compact Lie group $K$ satisfying the assumptions in Remark 6.1. Then the space of equivariant bifurcation problems $\text{Bif}(V)^K_0$ is a topological abelian subgroup of the space of paths of equivariant vector fields $C^\infty(\mathbb{R}, \mathfrak{X}(V)^K)$.

**Proof.** We prove that the space $\text{Bif}(V)^K_0$ is a subgroup. Given equivariant bifurcation problems $X$ and $Y$ on $V$, we show that the path $X - Y$ is an equivariant bifurcation problem on $V$. The proof consists of showing that the linearization
$D(X_0 - Y_0)(0)$ has critical eigenvalues. Note that the path $X - Y$ has a trivial branch of equilibria at the origin since both $X$ and $Y$ do, so the linearization $D(X_0 - Y_0)(0)$ is well-defined.

First, suppose the representation $V$ of the compact Lie group $K$ is absolutely irreducible. Since the linearizations $DX_0(0)$ and $DY_0(0)$ are equivariant linear endomorphisms and have critical eigenvalues the linearizations are both 0 by Remark 6.9. Thus:

$$D(X_0 - Y_0)(0) = DX_0(0) - DY_0(0) = 0,$$

meaning that the path $X - Y$ is also an equivariant bifurcation problem.

Now suppose that the representation $V$ of the compact Lie group $K$ is of complex type. Since the linearizations $DX_0(0)$ and $DY_0(0)$ are equivariant linear endomorphisms and have critical eigenvalues there exist real numbers $\beta_X$ and $\beta_Y$ such that $DX_0(0) = \beta_X I$ and $DY_0(0) = \beta_Y I$ by Remark 6.9. Thus:

$$D(X_0 - Y_0)(0) = DX_0(0) - DY_0(0) = (\beta_X - \beta_Y)I.$$

Hence, the linearization $D(X_0 - Y_0)(0)$ has pure imaginary eigenvalues, meaning the path $X - Y$ is also an equivariant bifurcation problem.

Now suppose that the representation $V$ of the compact Lie group $K$ is of quaternionic type. Since the linearizations $DX_0(0)$ and $DY_0(0)$ are equivariant linear endomorphisms and have critical eigenvalues, by Remark 6.9 there exist real numbers $\beta_X, \gamma_X, \delta_X, \beta_Y, \gamma_Y,$ and $\delta_Y$ such that:

$$DX_0(0) = \beta_X I + \gamma_X J + \delta_X K, \quad \text{and} \quad DY_0(0) = \beta_Y I + \gamma_Y J + \delta_Y K.$$

Thus:

$$D(X_0 - Y_0)(0) = DX_0(0) - DY_0(0) = (\beta_X - \beta_Y)I + (\gamma_X - \gamma_Y)J + (\delta_X - \delta_Y)K.$$

Hence, the linearization $D(X_0 - Y_0)(0)$ has pure imaginary eigenvalues, meaning that the path $X - Y$ is an equivariant bifurcation problem.

Finally, suppose that the representation $V$ of the compact Lie group $K$ is the sum $W \oplus W$ of an absolutely irreducible representation $W$ of the compact Lie group $K$. Hence, since the linearizations $DX_0(0)$ and $DY_0(0)$ are equivariant linear endomorphisms and have critical eigenvalues, by Remark 6.9 there exist real numbers $a_X, b_X, c_X, d_X, a_Y, b_Y, c_Y,$ and $d_Y$ such that $a_X + d_X = 0, a_Y + d_Y = 0,$ and:

$$DX_0(0) = \begin{pmatrix} a_X I \\ b_X I \end{pmatrix}, \quad \text{and} \quad DY_0(0) = \begin{pmatrix} a_Y I \\ b_Y I \end{pmatrix}.$$

Thus:

$$D(X_0 - Y_0)(0) = DX_0(0) - DY_0(0) = \begin{pmatrix} (a_X - a_Y)I \\ (b_X - b_Y)I \\ (c_X - c_Y)I \\ (d_X - d_Y)I \end{pmatrix}.$$

with $a_X - a_Y + d_X - d_Y = (a_X + d_X) - (a_Y + d_Y) = 0$. Hence, by Remark 6.9 the linearization $D(X_0 - Y_0)(0)$ has pure imaginary eigenvalues, meaning that the path $X - Y$ is an equivariant bifurcation problem.

By all of the above, the space $\text{Bif}(V)_0^K$ is a subgroup of the space of paths of equivariant vector fields. The continuity of the addition and inversion maps follows immediately by the universal property of the subspace topology. Hence, the space $\text{Bif}(V)_0^K$ is a topological abelian subgroup of the space of paths of equivariant vector fields.
Remark 6.11. Lemma 6.10 need not be true without the irreducibility assumptions. To see this, consider the finite group generated by the rotation of the plane by $\pi$, and its standard representation on the plane $\mathbb{R}^2$. The constant paths of linear vector fields on $\mathbb{R}^2$:

$$X_{\lambda} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_{\lambda} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

are both equivariant with respect to this representation. Both paths of equivariant vector fields have critical eigenvalues of 0, but their sum does not have critical eigenvalues. However, by Remark 6.1 and Lemma 6.10 the space of equivariant bifurcation problems on a fixed problem’s center subspace generically is a group. This will be enough for our purposes.

Note the following theorem:

**Theorem 6.12** (Theorem 1.9 in [16]). Let $V$ be a representation of a compact Lie group $K$. Let $\psi : V \to \mathfrak{k}$ be an infinitesimal gauge transformation on $V$ (Definition 2.4). The induced vector field $\partial(\psi)$, in the sense of (2.1), has a relative equilibrium at the origin. Furthermore, the linearization at the origin is given by the equality:

$$D\partial(\psi)(0) = \delta \rho(\psi(0)),$$

where $\rho : K \to \text{GL}(V)$ is the representation and $\delta \rho := (T\rho)_1 : \mathfrak{k} \to \mathfrak{gl}(V)$ is the infinitesimal representation. In particular, the linearization has purely imaginary eigenvalues.

With this theorem, we can complete the verification that Definition 6.8 is well-defined:

**Lemma 6.13.** Let $V$ be a representation of the compact Lie group $K$ satisfying the assumptions in Remark 6.1. Then the boundary map:

$$\partial : C^\infty(\mathbb{R}, C^\infty(V, \mathfrak{k}))^K \to C^\infty(\mathbb{R}, \mathfrak{X}(V)^K)$$

of the 2-term chain complex of paths of equivariant vector fields $C^\infty(\mathbb{R}, \mathfrak{X}(V)^K)$ (Definition 2.14) has image contained in the space $\text{Bif}(V)_0^K$ of equivariant bifurcation problems on $V$.

**Proof.** By Theorem 6.12 the induced vector field $\partial(\psi)_0$ has critical eigenvalues, and so the origin is a critical equilibrium. \qed

6.2. **Equivariant bifurcation problems on proper actions.** The main goal of this subsection is to define a 2-term chain complex of equivariant bifurcations problems as in Definition 6.8 but for the more general case of proper actions.

**Remark 6.14.** For the case of proper actions, it suffices to consider an invariant neighborhood of the group orbit of the relative equilibrium. Let $K$ be a compact Lie subgroup of a Lie group $G$, and let $V$ be a representation of the Lie subgroup $K$. Recall that the associated bundle $G \times^K V$ is the quotient $(G \times V)/K$ of the action of the group $K$ on the product $G \times V$ given by:

$$k \cdot (g, v) := (gk^{-1}, k \cdot v), \quad k \in K, (g, v) \in G \times V.$$

The bundle $G \times^K V$ is a $G$-manifold with the action of the group $G$ given by:

$$g' \cdot [g, v] := [g'g, v], \quad g' \in G, [g, v] \in G \times^K V.$$
The stabilizer of the point \([1, 0] \in G \times^K V\), with respect to the action of the group \(G\), is the subgroup \(K\). The vector space \(V\), with equivariant embedding \(j : V \hookrightarrow G \times^K V\) defined by \(j(v) := [1, v]\), is a global slice for the action of the group \(G\) at the point \([1, 0]\). Note that the group orbit of the point \([1, 0]\) is isomorphic to the quotient group \(G/K\), and that the map:

\[
G \times^K V \to G/K, \quad [g, v] \mapsto gK,
\]

is the bundle projection of a smooth vector bundle with typical fiber \(V\).

We seek to define what it means to have an equivariant bifurcation problem on a proper \(G\)-manifold \(M\). This requires we define critical relative equilibria, which in turn means we must address the fact that the linearization at a relative equilibrium is not well-defined.

**Remark 6.15.** Let \(M\) be a proper \(G\)-manifold. Given an equilibrium \(m \in M\) of a vector field \(X\) on \(M\), the linearization is the linear endomorphism \(DX(m) : T_m M \to T_m M\) that makes the following diagram commute:

\[
\begin{array}{ccc}
T_m M & \xrightarrow{TX(m)} & T_{(m, 0)} TM \\
& \searrow & \downarrow \text{pr}_2 \\
& & T_m M
\end{array}
\]

where the projection \(\text{pr}_2\) is the projection onto the second factor of the canonical splitting \(T_{(m, 0)} TM \cong T_0(T_m M) \oplus T_0(T_m M)\), and where we identify the second factor with the tangent space \(T_m M\). However, when the point \(m\) is a relative equilibrium of the vector field \(X\), the vector \(X(m)\) is not necessarily zero, and hence there is no canonical splitting of the tangent space \(T_{(m, X(m))} TM\). Thus, the usual linearization is not well-defined at a relative equilibrium.

Consider an associated bundle \(G \times^K V\) as in Remark 6.14 and let the map \(j : V \hookrightarrow G \times^K V\) be the embedding of the slice \(V\). Note that there is a canonical inclusion of the vector space \(\mathfrak{X}(V)^K\) of \(K\)-equivariant vector fields on the representation \(V\) into the vector space \(\mathfrak{X}(G \times^K V)^G\) of \(G\)-equivariant vector fields on the \(G\)-manifold \(G \times^K V\). The inclusion is given by the linear map:

\[
E : \mathfrak{X}(V)^K \hookrightarrow \mathfrak{X}(G \times^K V)^G, \quad X \mapsto EX,
\]

where the vector field \(EX : G \times^K V \to T(G \times^K V)\) is defined by:

\[
EX([g, v]) := g \cdot (Tj)X(v).
\]

The map \(E\) extends in a straightforward way to a map of paths of equivariant vector fields by using the map in (6.2) parameterwise.

Observe that there is no canonical map in the other direction. Pick an equivariant connection \(\Phi \in \Omega^1(G \times^K V; \mathcal{V}(G \times^K V))^G\) on the vector bundle \(G \times^K V \to G/K\), where the bundle map is as in (6.1) and \(\mathcal{V}(G \times^K V)\) is the corresponding vertical bundle. Given such a connection, we can define the linear map:

\[
P : \mathfrak{X}(G \times^K V)^G \to \mathfrak{X}(V)^K, \quad X \mapsto PX,
\]

where the vector field \(PX : V \to TV\) is defined by:

\[
PX(v) := j^*(\Phi \circ X)(v).
\]
That is, the vector field $PX$ is the pullback by the embedding $j : V \hookrightarrow G \times K V$ of the vector field $\Phi \circ X$. Using the connection $\Phi$, note that the vector field $PX$ is the component of $X$ that is tangent to $V \cong j(V)$. As with the map $E$, the map $P$ extends to a map of paths of equivariant vector fields by using the map in \((6.3)\) parameterwise.

Given an equivariant vector field $X$ on the proper $G$-manifold $G \times K V$ with a relative equilibrium at the point $[1, 0] \in G \times K V$, its projection $PX$ has an equilibrium at the origin $0 \in V$. Thus, we can linearize the projection $PX$ at the origin $0 \in V$ to obtain a linear map $D(PX)(0) : V \to V$. Hence, we have the following definition:

**Definition 6.16.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, let $X$ be a path of equivariant vector fields on the associated bundle $G \times K V$, and let $P : \mathfrak{X}(G \times K V)^G \to \mathfrak{X}(V)^K$ be a projection map as in \((6.3)\). The point $[1, 0] \in G \times K V$ is a critical relative equilibrium of the path of vector fields $X$ if the linearization $D(PX)_0(0)$ has eigenvalues with zero real part. Such eigenvalues are called critical eigenvalues of $X$ (with respect to the projection $P$).

**Remark 6.17.** It was noted in \([6]\) Lemma 8.5.2] that the real parts of the eigenvalues of a linearization $D(PX_0)(0)$ as in Definition 6.16 are independent of the choice of slice and projection (though the construction of the projection in \([6]\) is somewhat different to ours, it is straightforward to check that our projection matches the slice vector field constructed in \([6]\) and hence their result applies). Thus, while the critical eigenvalues of a projection may vary with respect to the choice of projection, the existence of critical eigenvalues is independent of the choice. Hence, Definition 6.16 is well-defined.

With this, we can define:

**Definition 6.18.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. An equivariant bifurcation problem on the bundle $G \times K V$ is a path of equivariant vector fields $X : \mathbb{R} \to \mathfrak{X}(G \times K V)^G$ with a critical relative equilibrium at the point $[1, 0] \in G \times K V$ in the sense of Definition 6.16.

We can now give the main definition of this subsection:

**Definition 6.19.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. The 2-term chain complex $\text{Bif}(G \times K V)^G_\bullet$ of equivariant bifurcation problems on the bundle $G \times K V$ is the 2-term chain complex of topological abelian groups:

$$\text{Bif}(G \times K V)^G_0 \xrightarrow{\partial} \text{Bif}(G \times K V)^G_1$$

where $\text{Bif}(G \times K V)^G_0$ is the space $C^\infty(\mathbb{R}, C^\infty(G \times K V, G)^G)$ of paths of infinitesimal gauge transformations, $\text{Bif}(G \times K V)^G_0$ is the space of equivariant bifurcation problems of Definition 6.18, and the boundary map $\partial$ is given by restricting the boundary map of the 2-term chain complex of paths of equivariant vector fields on $G \times K V$ (Definition 2.17).

To verify that Definition 6.19 is well-defined we will prove that the space $\text{Bif}(G \times K V)^G_0$ is a subgroup of the space $C^\infty(\mathbb{R}, \mathfrak{X}(G \times K V)^G)$ of paths of equivariant vector fields on $G \times K V$ (Lemma 6.20), and that the image of the boundary map $\partial$ in Definition 2.17 has image contained in the space $\text{Bif}(G \times K V)^G_0$ (Lemma 6.21).
Lemma 6.20. Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. The space $\text{Bif}(G \times K V)^G$ of equivariant bifurcation problems on the bundle $G \times K V$ (Definition 6.19) is a topological abelian subgroup of the group of paths of equivariant vector fields $C^\infty(\mathbb{R}, \mathfrak{X}(G \times K V)^G)$ (Definition 2.17).

Proof. Let $X$ and $Y$ be two equivariant bifurcation problems on the $G$-manifold $G \times K V$. We show that the path of vector fields $X - Y$ is also an equivariant bifurcation problem. Let the map $P : \mathfrak{X}(G \times K V)^G \to \mathfrak{X}(V)^K$ be any projection as in (6.3). Note that the projections $PX$ and $PY$ are paths of equivariant vector fields on the vector space $V$. Since the paths $X$ and $Y$ are equivariant bifurcation problems on $G \times K V$, the linearizations of the vector fields $PX_0$ and $PY_0$ at the origin $0 \in V$ have critical eigenvalues. Consequently, the projected paths $PX$ and $PY$ are equivariant bifurcation problems on the representation $V$ of the Lie subgroup $K$.

Since the representation $V$ satisfies the assumptions in Remark 6.1 the space $\text{Bif}(V)_0^K$ of equivariant bifurcation problems on the representation $V$ is a group (Lemma 6.21). Hence, the path of vector fields $PX - PY$ is an equivariant bifurcation problem in the space $\text{Bif}(V)_0^K$, meaning that the linearization:

$$DP(X_0 - Y_0)(0) = D(PX_0 - PY_0)(0)$$

has critical eigenvalues. Thus, the path of vector fields $X - Y$ is an equivariant bifurcation problem on $G \times K V$. This proves that the space $\text{Bif}(G \times K V)^G$ is a subgroup of the group of paths $C^\infty(\mathbb{R}, \mathfrak{X}(G \times K V)^G)$. It is straightforward to verify that it is a topological abelian subgroup when equipped with the subspace topology.

We complete the verification that Definition 6.19 is well-defined.

Lemma 6.21. Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. Then the boundary map:

$$\partial : C^\infty(\mathbb{R}, C^\infty(G \times K V, \mathfrak{g})^G) \to C^\infty(\mathbb{R}, \mathfrak{X}(G \times K V)^G)$$

of the 2-term chain complex of paths of equivariant vector fields (Definition 2.17) has image contained in the space $\text{Bif}(G \times K V)^G$ of equivariant bifurcation problems on the associated bundle $G \times K V$.

Proof. We can reduce to the considering vector fields. For this, let $\partial : C^\infty(G \times K V, \mathfrak{g})^G \to \mathfrak{X}(G \times K V)^G$ be the boundary map of the 2-term chain complex of equivariant vector fields $\mathfrak{X}(G \times K V)^G$ (Definition 2.11). It suffices to make a choice of projection map $P : \mathfrak{X}(G \times K V)^G \to \mathfrak{X}(V)^K$ as in (6.3) and prove that, for any infinitesimal gauge transformation $\psi \in C^\infty(G \times K V, \mathfrak{g})^G$, the origin $0 \in V$ is a critical equilibrium of the vector field $P\partial(\psi)$. To make the choice of projection, pick a $K$-invariant complement $\mathfrak{q}$ to the Lie algebra $\mathfrak{k}$ of $K$ in the Lie algebra $\mathfrak{g}$ of $G$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ is a $K$-equivariant splitting. This defines a principal connection $\Phi \in \Omega^1(G; \mathcal{V}(G))$ on the principal $K$-bundle $G \to G/K$. Explicitly, the connection is given by:

$$\Phi(q, \xi) := (g, \mathbb{P}(\xi)), \quad (g, \xi) \in G \times \mathfrak{g} \cong TG,$$

where $\mathbb{P} : \mathfrak{g} \to \mathfrak{k}$ is the $K$-equivariant projection corresponding to the splitting. This defines a $G$-equivariant connection $\Phi \in \Omega^1(G \times K V; \mathcal{V}(G \times K V))^G$ on the associated
bundle $G \times^K V \to G/K$ such that the following diagram commutes

$$
\begin{array}{ccc}
TG \times TV & \xrightarrow{\Phi \times \text{id}} & TG \times TV \\
\downarrow & & \downarrow \\
T(G \times^K V) & \xrightarrow{\psi} & T(G \times^K V)
\end{array}
$$

where $\varpi$ is the quotient map of the associated bundle $T(G \times^K V) \cong TG \times^T K TV$. We will consider the projection map $P : \mathfrak{X}(G \times^K V)^G \to \mathfrak{X}(V)^K$ as in \eqref{eq:6.3} corresponding to this connection.

To prove the origin $0 \in V$ is a critical equilibrium of the vector field $P \partial(\psi)$, consider the gauge transformation:

$$
\varphi : V \to \mathfrak{k}, \quad \varphi := P \circ \psi \circ j,
$$

where $j : V \hookrightarrow G \times^K V$ is the embedding given by $j(v) := [1, v]$. It suffices to prove that the vector fields $\Phi \circ \partial(\psi) \in \mathfrak{X}(G \times^K V)^G$ and $\partial(\varphi) \in \mathfrak{X}(V)^K$ are $j$-related.

For this, let $v \in V$ be a point on the slice, and note that the evaluation map $\text{ev}_{(1, v)} : G \to G \times V$ at the point $(1, v) \in G \times V$ of the $G$-action on the product $G \times V$ is such that:

$$
\text{ev}_{(1, v)} : g \mapsto g \times V, \quad \text{ev}_{(1, v)}(\xi) = (\xi, v).
$$

Also, if $\text{ev}_{j(v)} : G \times G \times^K V$ is the evaluation map at the point $j(v) \in G \times^K V$ of the $G$-action on $G \times^K V$, then the two evaluation maps make the following diagram commute:

$$
\begin{array}{ccc}
\emptyset & \xrightarrow{\text{ev}_{(1, v)}} & \mathfrak{g} \times V \\
& \downarrow \text{ev}_{j(v)} & \\
T_{j(v)}(G \times^K V) & \xrightarrow{\varpi} & (\xi, v)
\end{array}
$$

Now note that:

$$
\Phi_{j(v)} \partial(\psi)(j(v)) = T\text{ev}_{j(v)}(j(v)) = \Phi_{j(v)} \varpi \text{ev}_{(1, v)}(\psi(j(v))) = \varpi(P \times \text{id}_V) \text{ev}_{(1, v)}(\psi(j(v))) = \varpi(\varphi(v), v) = \varpi \text{ev}_{(1, v)}(\varphi(v)) = (Tj) \text{ev}_v(\varphi(v)) = (Tj) \partial(\varphi)(v).
$$

Thus, the vector fields $\Phi \circ \partial(\psi)$ and $\partial(\varphi)$ are $j$-related, which completes the proof. \qed
6.3. Equivalence of equivariant bifurcation problems on proper actions and representations. In this subsection we state and prove the main theorem of this paper (Theorem 6.20). Without loss of generality, we restrict ourselves to the case of associated bundles as in the previous subsection (see Remark 6.14). We will need the following definition:

**Definition 6.22.** An inclusion of 2-term chain complexes of topological abelian groups is a map $\iota : A_\bullet \hookrightarrow B_\bullet$ of 2-term chain complexes of topological abelian groups (Definition 6.22) that is also a subspace inclusion in each degree.

**Remark 6.23.** Let $N$ be an $H$-manifold. Consider the $H$-manifold $\mathbb{R} \times N$ where the group $H$ acts trivially on the $\mathbb{R}$ factor. There is a subspace inclusion and continuous group homomorphism:

$$C^\infty (\mathbb{R}, \mathcal{X}(N)^H) \hookrightarrow \mathcal{X}(\mathbb{R} \times N)^H, \quad X \mapsto 0_{\mathbb{R} \times \mathbb{R}} \times X,$$

where the vector field $0_{\mathbb{R} \times \mathbb{R}} \times X$ on $\mathbb{R} \times N$ is defined by:

$$(0_{\mathbb{R} \times \mathbb{R}} \times X)(\lambda, n) := ((\lambda, 0), X_\lambda(n)), \quad (\lambda, n) \in \mathbb{R} \times N.$$

Similarly, there is a homeomorphism and group isomorphism:

$$C^\infty (\mathbb{R}, C^\infty (N, h)^H) \rightarrow C^\infty (\mathbb{R} \times N, h)^H, \quad \psi \mapsto \tilde{\psi},$$

where $\tilde{\psi}$ is the associated map of $\psi$ (Definition 2.14). These two maps define an inclusion of 2-term chain complexes of topological abelian groups (Definition 6.22):

$$(6.8) \quad C^\infty (\mathbb{R}, \mathcal{X}(N)^H) \hookrightarrow \mathcal{X}(\mathbb{R} \times N)^H$$

of the 2-term chain complex of paths of equivariant vector fields on $N$ (Definition 2.11) into the 2-term chain complex of equivariant vector fields on $\mathbb{R} \times N$ (Definition 2.17). Now suppose that the $N$-manifold $H$ is either a representation of a compact Lie group $H$ or is an associated bundle, so that we can define a 2-term chain complex of equivariant bifurcation problems on $N$ (Definition 6.8 or Definition 6.19). Then the inclusion (6.8) restricts to an inclusion of 2-term chain complexes of topological abelian groups (Definition 6.22):

$$(6.9) \quad \text{Bif}(N)^H_\bullet \hookrightarrow \mathcal{X}(\mathbb{R} \times N)^H$$

of the 2-term chain complex of equivariant bifurcation problems on $N$ into the 2-term chain complex of equivariant vector fields on $\mathbb{R} \times N$.

**Remark 6.24.** Consider an associated bundle $G \times^K V \rightarrow G/K$ as in Remark 6.14. Motivated by the inclusion $\text{Bif}(G \times^K V)^G_\bullet \hookrightarrow \mathcal{X}(\mathbb{R} \times G \times^K V)^G$ of (6.9) we consider the associated bundle $\mathbb{R} \times (G \times^K V) \rightarrow G/K$. The total space $\mathbb{R} \times (G \times^K V)$ is the quotient of the action of the compact Lie group $K$ on the product $\mathbb{R} \times (G \times V)$ given by:

$$k \cdot (\lambda, g, v) := (\lambda, g k^{-1}, k \cdot v), \quad k \in K, \ (\lambda, g, v) \in \mathbb{R} \times G \times V.$$

The bundle $\mathbb{R} \times (G \times^K V)$ has typical fiber $\mathbb{R} \times V$. If $j : V \hookrightarrow G \times^K V$ is the slice embedding defined by $j[v] = [1, v]$, then the map $\tilde{j} := 1_{\mathbb{R} \times \mathbb{R}} \times j : \mathbb{R} \times V \rightarrow \mathbb{R} \times (G \times^K V)$ is also a slice embedding. Let $\mathfrak{k} \oplus \mathfrak{q}$ be a choice of $K$-equivariant splitting of the Lie algebra $\mathfrak{g}$, where $\mathfrak{k}$ is the Lie algebra of the Lie subgroup $K$. This induces an isomorphism of the tangent bundle:

$$T(\mathbb{R} \times (G \times^K V)) \cong T\mathbb{R} \times (G \times \mathfrak{q}) \times^K (V \times V).$$
The splitting also gives us an isomorphism of the vertical bundle:

$$\mathcal{V}(\mathbb{R} \times (G \times^K V)) \cong T\mathbb{R} \times G \times^K (V \times V).$$

If $\Phi : T(G \times^K V) \to \mathcal{V}(G \times^K V)$ is the connection on the bundle $G \times^K V \to G/K$ induced by the splitting $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{q}$, then:

$$1_{T\mathbb{R}} \times \Phi : T(\mathbb{R} \times (G \times^K V)) \to \mathcal{V}(\mathbb{R} \times (G \times^K V))$$

is the corresponding connection induced on the bundle $\mathbb{R} \times (G \times^K V) \to G/K$. The splitting gives an isomorphism of the horizontal bundle as well:

$$\mathcal{H}(\mathbb{R} \times (G \times^K V)) \cong \mathcal{Z} \times (G \times \mathfrak{q}) \times^K V,$$

where $\mathcal{Z}$ is the zero section of the bundle $T\mathbb{R}$.

The following proposition extends the inclusion of topological abelian groups in (6.2) to an inclusion of 2-term chain complexes of topological abelian groups:

**Proposition 6.25.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. There is a canonical inclusion:

$$E_* : \text{Bif}(V)_* \to \text{Bif}(G \times^K V)_*^G$$

of 2-term chain complexes of topological abelian groups given by equivariant extension in each degree.

**Proof.** We define the inclusion by degrees. In degree 1, the inclusion $E_1$ is defined as the composition filling in the dashed arrow in the following diagram:

$$\begin{array}{ccc}
C^\infty(\mathbb{R} \times V, \mathfrak{g})^K & \xrightarrow{\epsilon} & C^\infty(\mathbb{R} \times G \times^K V, \mathfrak{g})^G \\
\cong & & \cong \\
\text{Bif}(V)^K & \xrightarrow{E_1} & \text{Bif}(G \times^K V)^G
\end{array}$$

where $\iota_*$ is the pushforward of the inclusion $\iota : \mathfrak{t} \to \mathfrak{g}$, the map $\epsilon$ is the equivariant extension map (Remark 6.1). Explicitly, for any path $\psi \in \text{Bif}(V)^K$, the map is given by:

$$(6.11) \quad E_1 \psi_\lambda([g, v]) := \text{Ad}(g)_* \iota_* (\psi_\lambda(v)), \quad \lambda \in \mathbb{R}, \ [g, v] \in G \times^K V,$$

The map $E_1$ is a continuous injective group homomorphism since the maps $\iota_*$ are $\epsilon$ are thus (see part (1) of Lemma 4.34 and Corollary 4.3). We define the inclusion in degree 0 as the unique map filling in the dashed arrow in the following diagram:

$$\begin{array}{ccc}
C^\infty(\mathbb{R} \times V, T\mathbb{R} \times G \times^K V)^K & \xrightarrow{\epsilon} & C^\infty(\mathbb{R} \times G \times^K V, T\mathbb{R} \times G \times^K V)^G \\
\cong & & \cong \\
\text{Bif}(V)^0 & \xrightarrow{E_0} & \text{Bif}(G \times^K V)^G
\end{array}$$
Here the vertical maps are subspace inclusions as in Remark 6.24. Such a map $E_0$ is a continuous injective group homomorphism since the map $(T_j)_*$, the map $\epsilon$, and the vertical maps are thus. Explicitly, for any bifurcation problem $X \in \text{Bif}(V)_0^K$, this map is given by:

$$(6.12) \quad E_0X_\lambda([g,v]) := g \cdot (T_j)X_\lambda(v), \quad \lambda \in \mathbb{R}, \ [g,v] \in G \times^K V,$$

for a given bifurcation problem $X \in \text{Bif}(V)_0^K$. In particular, note that for any projection $P : \mathfrak{X}(G \times^K V) \to \mathfrak{X}(V)_K$ as in (6.3) we have that:

$$PE_0X_\lambda = X_\lambda, \quad \lambda \in \mathbb{R}.$$ 

Consequently, since the vector field $X_\lambda$ has a critical equilibrium at the origin $0 \in V$, the vector field $E_0X_\lambda$ has a critical relative equilibrium at the point $[1,0] \in G \times^K V$ (see Definition 6.16). Thus, the map $E_0$ is well-defined.

It remains to verify that the following diagram commutes:

$$
\begin{array}{ccc}
\text{Bif}(V)^K & \xrightarrow{\partial} & \text{Bif}(V)_0^K \\
E_1 & \downarrow & \downarrow E_0 \\
\text{Bif}(G \times^K V)^G & \xrightarrow{\partial} & \text{Bif}(G \times^K V)_0^G 
\end{array}
$$

For this, observe that for a path of gauge transformations $\psi$ on $V$, a point $[g,v] \in G \times^K V$, and a parameter $\lambda \in \mathbb{R}$ we have:

$$E_0\partial(\psi)_\lambda([g,v]) = g \cdot (T_j)(\partial\psi)_\lambda(v)$$
$$= g \cdot (T_j)Te_{\psi}(\psi_\lambda(v))$$
$$= g \cdot Te_{[1,v]}(\psi_\lambda(v)) \quad \text{by the chain rule}$$
$$= Te_{g,v} \text{Ad}(g)\psi_\lambda(v) \quad \text{since } g \cdot Te_{[1,v]} = Te_{[g,v]} \text{Ad}(g)$$
$$= Te_{g,v}(E_1\psi_\lambda([g,v]))$$
$$= \partial(E_1\psi)_\lambda([g,v]).$$

Hence, the map $E_\bullet$ is a map of chain complexes as claimed. \qed

We can now state the main theorem:

**Theorem 6.26.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. The map:

$$E : \text{Bif}(V)_\bullet^K \hookrightarrow \text{Bif}(G \times^K V)_G^\bullet$$

of Proposition 6.25 is an equivalence of 2-term chain complexes of topological abelian groups. That is, this map has a homotopy inverse as a map of 2-term chain complexes of topological abelian groups (Definition 5.10).

**Remark 6.27.** Lerman extended the canonical inclusion (6.2) to an equivalence of 2-term chain complexes of vector spaces $E : \mathfrak{X}(V)_\bullet^K \to \mathfrak{X}(G \times^K V)_G^G$ between the 2-term chain complexes of equivariant vector fields [16, Theorem 4.3]. This can be extended to paths of equivariant vector fields. Theorem 6.26 is saying that this restricts to equivariant bifurcation problems, and can be shown to be an equivalence of 2-term chain complexes of topological abelian groups.

We state two corollaries of Theorem 6.26 before proceeding with its proof.
**Remark 6.28.** Recall that if $A_\bullet$ is a 2-term chain complex of topological abelian groups, then there is a corresponding category $\text{Gen}(A_\bullet)$ with objects the generic elements of $A_\bullet$ (Definition 5.7). Thus, we have the following definition, which we need to state the first corollary of Theorem 6.26.

**Definition 6.29.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. Then, using Definition 5.7:

1. The category of generic equivariant bifurcation problems on the representation $V$ is the category $\text{Gen}(\text{Bif}(V)^K)$. 
2. The category of generic equivariant bifurcation problems on the associated bundle $G \times^K V$ is the category $\text{Gen}\left(\text{Bif}\left(G \times^K V\right)^G\right)$.

The first corollary of Theorem 6.26:

**Corollary 6.30.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. Then there is an equivalence:

$$\text{Gen}\left(\text{Bif}\left(G \times^K V\right)^G\right) \simeq \text{Gen}(\text{Bif}(V)^K)$$

between the categories in Definition 6.29.

**Proof.** The result follow from Theorem 5.11 and Theorem 6.26.

**Remark 6.31.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. Recall there is an action of the group $\text{Bif}(G \times^K V)^G$ on the space $\text{Bif}(G \times^K V)^G_0$ given by:

$$\psi \cdot X = X + \partial(\psi),$$

for $X \in \text{Bif}(G \times^K V)^G_0$ and $\psi \in \text{Bif}(G \times^K V)^G_1$.

The second corollary of Theorem 6.26 states how to obtain *generic* bifurcations from a point on an associated bundle from *generic* bifurcations on the slice representation at the point:

**Corollary 6.32.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. Furthermore,

1. Let the map $E_\bullet : \text{Bif}(V)^K \to \text{Bif}(G \times^K V)^G$ be the inclusion map of Proposition 6.29.
2. Let $\mathcal{E}$ be a subcollection of the space of bifurcation problems $\text{Bif}(V)^K_0$ that is open, dense, and invariant with respect to the action in Remark 6.31, consisting of bifurcation problems exhibiting bifurcations to relative equilibria.

Then the subcollection:

$$\tilde{\mathcal{E}} := \{ E_0 X + \partial(\psi) \mid X \in \mathcal{E}, \psi \in \text{Bif}(G \times^K V)^G_1 \}$$

is an open and dense subcollection of the space $\text{Bif}(G \times^K V)^G_0$, and the bifurcation problems in the subcollection $\tilde{\mathcal{E}}$ bifurcate to relative equilibria. The analogous statement for relative periodic trajectories is also true.
Proof. First, with respect to the action in Remark 6.31 note that:

$$\tilde{E} = \text{Bif}(G \times K V)^{G} \cdot E_{0}(E).$$

Hence, the subcollection $\tilde{E}$ is open and dense in the space $\text{Bif}(G \times K V)^{G}$ by Corollary 5.12. The remaining part of the corollary consists of two cases.

For the first case, let $X$ be a bifurcation problem in $E$ bifurcating to relative equilibria and let $\psi \in \text{Bif}(G \times K V)^{G}$. We must show that the path $E_{0}X + \partial(\psi)$ also bifurcates to relative equilibria. It suffices to prove that, for any $\lambda \in \mathbb{R}$, if the vector field $X_{\lambda}$ has a relative equilibrium at a point $v \in V$, the vector field $E_{0}X_{\lambda} + \partial(\psi)_{\lambda}$ has a relative equilibrium at the point $[1, v] \in G \times K V$. Then, since $X_{\lambda}(v) \in T_{v}(K \cdot v)$, we have that:

$$E_{0}X_{\lambda}([1, v]) = 1 \cdot (Tj)X_{\lambda}(v) \in (Tj)T_{v}(K \cdot v) \subseteq T_{[1, v]}(G \cdot [1, v]),$$

where $j : V \hookrightarrow G \times K V$ is the $K$-equivariant slice embedding. This means that the vector field $E_{0}X_{\lambda}$ has a relative equilibrium at the point $v \in V$. Hence, so does the vector field $E_{0}X_{\lambda} + \partial(\psi)_{\lambda}$ since it is isomorphic to it. This proves the first case.

For the second case, let $X$ be a bifurcation problem in $E$ bifurcating to relative periodic trajectories and let $\psi \in \text{Bif}(G \times K V)^{G}$. We must show that the path $E_{0}X + \partial(\psi)$ bifurcates to relative periodic trajectories at the point $[1, 0]$. It suffices to prove that, for any $\lambda \in \mathbb{R}$, if the vector field $X_{\lambda}$ has a relative periodic trajectory $\gamma$ starting at a point $v \in V$, the vector field $E_{0}X_{\lambda} + \partial(\psi)_{\lambda}$ has a relative periodic trajectory $\eta$ starting at the point $[1, v] \in G \times K V$. For this, let $\theta$ be the integral curve of the vector field $E_{0}X_{\lambda}$ starting at the point $[1, v]$. Denote by $[\gamma]$ and $[\theta]$ the curves on the orbit spaces $V/K$ and $(G \times K V)/G$ induced by the curves $\gamma$ and $\theta$ respectively. Let $[j] : V/K \hookrightarrow (G \times K V)/G$ be the subspace inclusion induced by the slice embedding $j : V \hookrightarrow G \times K V$. Then note that the following diagram commutes:

$$\begin{array}{c}
\mathbb{R} \\
\downarrow \gamma \\
V \\
\downarrow j \\
V/K \\
\downarrow [j] \\
(G \times K V)/G \\
\downarrow [\theta] \\
G \times K V \\
\downarrow \theta \\
\mathbb{R}
\end{array}$$

(6.14)

since the top square commutes because the vector fields $X_{\lambda}$ and $E_{0}X_{\lambda}$ are $j$-related by definition, and the left triangle, the right triangle, and the middle square commute by the definition of the maps $[\gamma]$, $[\theta]$, and $[j]$ respectively. Consequently, the outer edge of diagram (6.14) commutes and gives the equality:

$$[\theta] = [j][\gamma].$$

(6.15)

Since the curve $\gamma$ is a relative periodic trajectory of the vector field $X_{\lambda}$ starting at the point $v \in V$, the curve $[\gamma]$ is a periodic curve on the orbit space $V/K$. Hence, by (6.15), the curve $[\theta]$ is a periodic curve on the orbit space $(G \times K V)/G$, and thus the curve $\theta$ is a relative periodic trajectory of the vector field $E_{0}X_{\lambda}$. Thus, the integral curve $\eta$ of the vector field $E_{0}X_{\lambda} + \partial(\psi)_{\lambda}$ starting at the point $[1, v]$ is
a relative periodic trajectory since the vector fields $E_0 X_\lambda$ and $E_0 X_\lambda + \partial(\psi)\lambda$ are isomorphic (Lemma 3.4). This proves the second case. \qed

We proceed with the proof of Theorem 6.26. Recall that a choice of $G$-equivariant connection on an associated bundle $G \times^K V \to G/K$ induces a choice of projection map $P : \mathfrak{X}(G \times^K V)^G \to \mathfrak{X}(V)^K$ as in (6.3). In the next proposition, we show that this choice also yields a map $P_\ast$ of 2-term chain complexes of equivariant bifurcation problems, which will be the homotopy inverse of the map $E_\ast$ in Proposition 6.25.

**Proposition 6.33.** Let $V$ be a representation of a compact Lie subgroup $K$ of a Lie group $G$, and suppose $V$ satisfies the assumptions in Remark 6.1. Furthermore, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be a $K$-equivariant splitting of the Lie algebra $\mathfrak{g}$ of the group $G$, where $\mathfrak{k}$ is the Lie algebra of the subgroup $K$. Then there is a map:

$$P_\ast : \text{Bif}(G \times^K V)^G \to \text{Bif}(V)^K$$

of 2-term chain complexes of topological abelian groups corresponding, in each degree, to an equivariant projection induced by the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$.

**Proof.** We define the projection by degrees. For this, let $P : \mathfrak{g} \to \mathfrak{k}$ be the $K$-equivariant projection corresponding to the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, and let $\Phi \in \Omega^1(G \times^K V, V(G \times^K V))^G$ be the $G$-equivariant connection on the bundle $G \times^K V \to G/K$ corresponding to the projection $P$ as in (6.5). In degree 1, the projection $P_1$ is defined as the composition filling in the dashed arrow in the following diagram:

$$\begin{array}{ccc}
C^\infty(R \times G \times V, \mathfrak{g})^K & \xrightarrow{\pi^*} & C^\infty(R \times G \times^K V, \mathfrak{g})^G \\
\pi | & & \downarrow | \\
C^\infty(R \times G \times^K V, \mathfrak{g}) & \xrightarrow{\mathfrak{j}^*} & C^\infty(R \times V, \mathfrak{g})^K \\
\downarrow | & & \downarrow | \\
\text{Bif}(G \times^K V)^G_1 & \xrightarrow{- - - - - - - - - - - - - - - -} & \text{Bif}(V)^K_1 \\
\end{array}$$

where the map $I$ is the obvious inclusion, the map $\pi^*$ is the pullback via the principal bundle projection $\pi : R \times G \times V \to R \times G \times^K V$, the map $\mathfrak{j}^*$ is the pullback via the slice embedding $\mathfrak{j} : R \times V \hookrightarrow R \times G \times^K V$, and the map $P_\ast$ is the pushforward of the projection $P : \mathfrak{g} \to \mathfrak{k}$. Explicitly, for any path $\psi \in \text{Bif}(G \times^K V)^G_1$, the path $P_1 \psi$ is given by:

$$P_1 \psi_\lambda([g, v]) := P(\psi_\lambda(\mathfrak{j}(v))), \quad \lambda \in \mathbb{R}, [g, v] \in G \times^K V.$$  

This map is a continuous group homomorphism since each of the maps in diagram (6.16) is such. In particular, the pullback $\pi^*$ is continuous by Lemma 4.7, the pullback $\mathfrak{j}^*$ is continuous by Lemma 4.6, and the pushforward $P_\ast$ is continuous by part (1) of Lemma 4.3.
In degree 0, the projection \( P_0 \) is defined as the map filling in the dashed arrow in the following diagram:

\[
\begin{array}{c}
\Gamma(V(\mathbb{R} \times G \times K V)) \\
\downarrow (1_{TR} \times \Phi)_* \\
\mathfrak{X}(\mathbb{R} \times G \times K V) \\
\downarrow I_0 \\
\text{Bif}(G \times K V)^G_0 \end{array}
\begin{array}{c}
\downarrow (T\tilde{j})^{-1} \circ (-) \circ \tilde{j} \\
\mathfrak{X}(\mathbb{R} \times K V) \\
\downarrow I_1 \\
\text{Bif}(V)^K_0
\end{array}
\]

(6.18)

where the maps \( I_0 \) and \( I_1 \) are subspace inclusions following Remark 6.24, the map \((1_{TR} \times \Phi)_*\) is the pushforward of the connection:

\[
1_{TR} \times \Phi : T(\mathbb{R} \times G \times K V) \to V(\mathbb{R} \times G \times K V)
\]

on the associated bundle \( \mathbb{R} \times G \times K V \to G/K \), and the map \((T\tilde{j})^{-1} \circ (-) \circ \tilde{j}\) consists of pullback by the slice embedding \( \tilde{j} : \mathbb{R} \times V \hookrightarrow \mathbb{R} \times G \times K V \) and pushforward by the inverse of the restriction of the tangent map \( T\tilde{j} \) to a bundle map \( T(\mathbb{R} \times V) \to V(\mathbb{R} \times G \times K V) \| \mathbb{R} \times V \) (here \( V(\mathbb{R} \times G \times K V) \| \mathbb{R} \times V \) denotes the restriction of the vertical bundle of \( \mathbb{R} \times G \times K V \to G/K \) to the slice \( \mathbb{R} \times V \)). Explicitly, for any bifurcation problem \( X \) on \( G \times K V \), the bifurcation problem \( P_0 X \) is defined by:

\[
P_0 X_\lambda(v) := j^*(\Phi(X_\lambda(v))), \quad \lambda \in \mathbb{R}, v \in V.
\]

The composition \( I_1 \circ P_0 \) is a continuous group homomorphism since it factors as the composition of continuous group homomorphisms by diagram (6.18). In particular, the pushforward \((1_{TR} \times \Phi)_*\) is continuous by part (1) of Lemma 4.3, the pushforward \((T\tilde{j})^{-1} \circ (-) \circ \tilde{j}\) is continuous by part (1) of Lemma 4.3, and the pullback \( \tilde{j}^* \) is continuous by Lemma 4.6 since the map \( \tilde{j} \) is a closed embedding (hence proper). Consequently, the map \( P_0 \) is a continuous group homomorphism.

It remains to show that the following diagram commutes:

\[
\begin{array}{c}
\text{Bif}(G \times K V)^G_1 \\
\downarrow P_1 \\
\text{Bif}(V)^K_0
\end{array}
\begin{array}{c}
\downarrow \partial \\
\text{Bif}(G \times K V)^G_0 \end{array}
\]

(6.19)

Given a path of gauge transformations \( \psi \) on the associated bundle \( G \times K V \), it suffices to prove, for each \( \lambda \in \mathbb{R} \), that the vector field \( \partial(P_1 \psi)_\lambda \) on the representation \( V \) is the unique vector field on \( V \) that is \( j \)-related to the vector field \( \Phi \circ \partial(\psi)_\lambda \), where \( \Phi \) is the given connection on the associated bundle \( G \times K V \to G/K \). For this, let \( v \in V \) be a point, and let \( \varpi : TG \times TV \to TG \times T^K TV \) be the quotient map of the
tangent bundle \( T(G \times^K V) \). Then note that:
\[
\Phi_{j(v)} \partial(\psi)_{\lambda}(j(v)) = \Phi_{j(v)} T ev_{j(v)}(\psi_{\lambda} j(v)) \\
= \Phi_{j(v)} \varpi T ev_{(1,v)}(\psi_{\lambda} j(v)) \quad \text{by (6.7)} \\
= \Phi_{j(v)} \varpi (\psi_{\lambda} j(v)), v) \quad \text{by (6.6)} \\
= \varpi (F \times \text{id})(\psi_{\lambda} j(v)), v) \quad \text{by (6.5)} \\
= \varpi (F \psi_{\lambda} j(v)), v) \\
= \varpi T ev_{(1,v)}(F \psi_{\lambda} j(v)) \quad \text{by (6.6)} \\
= \varpi T ev_{(1,v)}(P_1 \psi_{\lambda} (v)) \quad \text{by (6.17)} \\
= T ev_{j(v)}(P_1 \psi_{\lambda} (v)) \quad \text{by (6.7)} \\
= (T j) T ev_{\nu}(P_1 \psi_{\lambda} (v)) \quad \text{by the chain rule} \\
= (T j) \partial(P_1 \psi)_{\lambda} (v).
\]
Hence, the vector fields \( \Phi \circ \partial(\psi)_{\lambda} \) and \( \partial(P_1 \psi)_{\lambda} \) are \( j \)-related for all \( \lambda \in \mathbb{R} \), meaning that \( P_0 \partial(\psi) = \partial P_1 (\psi) \) as required. \( \square \)

We will need the following technical lemma to prove that the map in Proposition 6.33 is the homotopy inverse of the map in Proposition 6.25.

**Lemma 6.34.** Let \( V \) be a representation of a compact Lie subgroup \( K \) of a Lie group \( G \), and suppose \( V \) satisfies the assumptions in Remark 6.1. Furthermore,

1. Let \( g = \mathfrak{k} \oplus q \) be a \( K \)-equivariant splitting of the Lie algebra \( g \) of the Lie group \( G \), where \( \mathfrak{k} \) is the Lie algebra of the Lie subgroup \( K \).
2. Let \( \mathcal{H} \to G \times^K V \) be the horizontal bundle of the connection induced by the splitting \( g = \mathfrak{k} \oplus q \) on the associated bundle \( G \times^K V \to G/K \).
3. Let \( C^\infty(\mathbb{R}, \Gamma(\mathcal{H})^G) \) be the space of paths of horizontal equivariant vector fields on the associated bundle \( G \times^K V \).

Then the map:
\[
C^\infty(\mathbb{R}, C^\infty(V,q)^K) \to C^\infty(\mathbb{R}, \Gamma(\mathcal{H})^G), \quad \psi \mapsto X^\psi,
\]
where the path \( X^\psi \) is defined by:
\[
X^\psi_{\lambda}([g,v]) := T ev_{[g,v]}(\text{Ad}(g)\psi_{\lambda}(v)), \quad \lambda \in \mathbb{R}, \ [g,v] \in G \times^K V,
\]
is a well-defined homeomorphism and group isomorphism.

**Proof.** It is straightforward to verify that the map in the statement is a group homomorphism. Hence, we restrict ourselves to proving that the map in the statement of the lemma is a well-defined homeomorphism \( C^\infty(\mathbb{R}, C^\infty(V,q)^K) \to C^\infty(\mathbb{R}, \Gamma(\mathcal{H})^G) \). We do this by constructing a homeomorphism filling in the dashed arrow in the following diagram:
\[
\begin{array}{ccc}
C^\infty(\mathbb{R}, C^\infty(V,q)^K) & \cong & C^\infty(\mathbb{R}, \Gamma(\mathcal{H})^G) \\
\downarrow \cong & & \downarrow \cong \\
C^\infty(\mathbb{R} \times V,q)^K & \to & \Gamma(Z \times \mathcal{H})^G
\end{array}
\]
where \( Z \) is the zero section of the tangent bundle \( T\mathbb{R} \) and the vertical maps are the topological identifications as in section 4.
Consider the trivial bundle \( \mathbb{R} \times V \times q \xrightarrow{pr_{\mathbb{R} \times V}} \mathbb{R} \times V \) with the action of the group \( K \) on the total space given by:

\[
k : (\lambda, v, \xi) := (\lambda, k \cdot v, \text{Ad}(k)\xi), \quad k \in K, \quad (\lambda, v, \xi) \in \mathbb{R} \times V \times q,
\]

and the action on the base given by:

\[
k \cdot (\lambda, v) := (\lambda, k \cdot v), \quad k \in K, \quad (\lambda, v) \in \mathbb{R} \times V.
\]

We build the dashed map in diagram (6.20) as the composition filling in the dashed arrow in the following diagram:

\[
\begin{array}{ccc}
C^\infty(\mathbb{R} \times V, q)^K & \to & \Gamma(\mathbb{R} \times V)^G \\
1_{\mathbb{R} \times V} \times (-) & \downarrow & \epsilon \\
\Gamma(\mathbb{R} \times V \times q)^K & \xrightarrow{\beta_*} & \Gamma(\mathbb{Z} \times H \mid \mathbb{R} \times V)^K
\end{array}
\]

where the map \( 1_{\mathbb{R} \times V} \times (-) \) is the map defined by \( \psi \mapsto 1_{\mathbb{R} \times V} \times \psi \), the map \( \beta_* \) is the pushforward of a diffeomorphism \( \beta : \mathbb{R} \times V \times q \to \mathbb{Z} \times H \mid \mathbb{R} \times V \) we define below, and the map \( \epsilon \) is the equivariant extension map obtained via restriction of the corresponding equivariant extension map in Remark 4.9.

To define the diffeomorphism \( \beta \) referenced in diagram (6.21), note that the restricted bundle \( \mathbb{Z} \times H \mid \mathbb{R} \times V \to \mathbb{R} \times V \) is trivializable:

\[
\mathbb{Z} \times H \cong \mathbb{Z} \times (k \times q) \times K V \cong \mathbb{R} \times V \times K V.
\]

Explicitly, the isomorphism is given by:

\[
\mathbb{R} \times V \times q \to \mathbb{Z} \times H \mid \mathbb{R} \times V, \quad (\lambda, v, 0) \mapsto \left( 0, \text{Te}_v(\lambda) \right),
\]

where \( 0 \) is the zero vector in the tangent space \( T_0 \mathbb{R} \). Hence, the pushforward \( \beta_* \) of this map is a homeomorphism by part (1) of Lemma 4.3.

The map \( 1_{\mathbb{R} \times V} \times (-) \) is also a homeomorphism since its inverse is given by the pushforward of the bundle projection \( pr_{\mathbb{R} \times V} : \mathbb{R} \times V \times q \to \mathbb{R} \times V \), which is continuous by part (1) of Lemma 4.3. On the other hand, the inverse of the map \( \epsilon \) in diagram (6.21) is the pullback by the slice embedding \( \tilde{j} : \mathbb{R} \times V \hookrightarrow \mathbb{R} \times G \times K V \) of the associated bundle \( \mathbb{R} \times G \times K V \to G/K \). This pullback is continuous by Lemma 4.1. Hence, the map \( \epsilon \) is also a homeomorphism. Consequently, the dashed arrow in diagram (6.21) is a homeomorphism. It is straightforward to verify that this map makes diagram (6.20) commute, concluding the proof. \( \square \)

The following proposition completes the proof of the main theorem:

**Proposition 6.35.** Let \( V \) be a representation of a compact Lie subgroup \( K \) of a Lie group \( G \), and suppose \( V \) satisfies the assumptions in Remark 6.1. Let \( E_* : \text{Bif}(V) \to \text{Bif}(G \times K V) \) be the map of Proposition 6.25 and let \( \hat{P}_* : \text{Bif}(G \times K V) \to \text{Bif}(V) \) be a choice of projection map as in Proposition 6.33. These maps are homotopy inverses of each other as maps of 2-term chain complexes of topological abelian groups (Definition 5.10).

**Proof.** It suffices to prove that \( \hat{P}_* E_* = \text{id}_{\text{Bif}(V)} \) and construct a homotopy \( h : E_* \hat{P}_* \cong \text{id}_{\text{Bif}(G \times K V)} \). We begin with the first claim. Let \( j : V \to G \times K V \) be the slice embedding of the representation \( V \) into the bundle \( G \times K V \to G/K \), and let
\[ P : \mathfrak{g} \to \mathfrak{t} \] be the equivariant projection corresponding to the map \( P \). Note that for any path \( \psi \in \text{Bif}(V)_1^K \), any parameter \( \lambda \in \mathbb{R} \), and any point \( v \in V \), we have that:

\[ PE\psi_\lambda(v) = P\psi_\lambda_j(v) = P(\text{Ad}(1)\psi_\lambda(v)) = P(\psi_\lambda(v)) = \psi_\lambda(v), \]

where the last equality follows since \( \psi_\lambda(v) \in \mathfrak{t} \). This proves the equality in degree 1. Consider an arbitrary bifurcation problem \( X \in \text{Bif}(V)_0^K \). For each \( \lambda \in \mathbb{R} \), the vector field \( E_0X_\lambda \) is vertical in the bundle \( G \times V \to G/K \), so \( P_0E_0X_\lambda \) is the unique vector field that is \( j \)-related to the vector field \( E_0X_\lambda \) by definition of the map \( P_0 \) (see Proposition 6.33). On the other hand,

\[ E_0X_\lambda j(v) = 1 \cdot (T_jX)_\lambda(v) = (T_jX)_\lambda(v), \quad \lambda \in \mathbb{R}, \ v \in V. \]

Hence, for each \( \lambda \in \mathbb{R} \), the vector field \( X_\lambda \) is \( j \)-related to the vector field \( E_0X_\lambda \). Thus, the vector fields \( P_0E_0X_\lambda \) and \( X_\lambda \) are equal. This proves the equality in degree 0, and completes the proof of the first claim.

We now construct the homotopy \( h : E_0P_0 \cong \text{id}_{\text{Bif}(G \times K V)_0^G} \). For this, let \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{q} \) be the \( K \)-equivariant splitting corresponding to the map \( P_0 \), and let \( \mathcal{H} \to G \times K V \) be the corresponding horizontal bundle of the bundle \( G \times K V \to G/K \). We define the homotopy \( h \) as the composition filling in the dashed arrow in the following diagram:

\[
\begin{array}{ccc}
C^\infty(\mathbb{R}, C^\infty(V, \mathfrak{q})^K) & \xrightarrow{\beta} & C^\infty(\mathbb{R}, C^\infty(V, \mathfrak{g})_1^G) \\
& \downarrow{\iota_*} & \\
C^\infty(\mathbb{R}, \Gamma(\mathcal{H})^G) & \xrightarrow{\epsilon} & C^\infty(\mathbb{R}, C^\infty(V, \mathfrak{g})_1^G) \\
& \uparrow{\alpha} & \\
\text{Bif}(G \times K V)_0^G & \xrightarrow{-h} & \text{Bif}(G \times K V)_1^G
\end{array}
\]

where the maps in this diagram are defined as follows. The map \( \alpha \) is the map defined by \( X \mapsto E_0P_0(X) - X \). This is well-defined as a map into horizontal paths since, for all \( \lambda \in \mathbb{R} \), the vector field \( E_0P_0X_\lambda - X_\lambda \) is horizontal in the bundle \( G \times K V \to G/K \). To see this, observe that the space \( C^\infty(\mathbb{R}, \Gamma(\mathcal{H})^G) \) is the kernel of the map \( P_0 \). The map \( \beta \) is the continuous inverse of the map in the statement of Lemma 6.34. The map \( \iota_* \) is the pushforward \( \iota_* : C^\infty(\mathbb{R} \times V, \mathfrak{q})^K \to C^\infty(\mathbb{R} \times V, \mathfrak{g})^K \) of the canonical inclusion \( \iota : \mathfrak{q} \hookrightarrow \mathfrak{g} \), where we have used the usual identifications. Finally, the map \( \epsilon \) is the equivariant extension map \( \epsilon : C^\infty(\mathbb{R} \times V, \mathfrak{g})^K \to C^\infty(\mathbb{R} \times G \times K V, \mathfrak{g})^G \) as in Remark 4.9, where we have used the usual identifications again.

The map \( \alpha \) is a continuous group homomorphism since the addition in the group \( \text{Bif}(G \times K V)_0^G \) is continuous and the maps \( E_0 \) and \( P_0 \) are continuous group homomorphisms. The map \( \beta \) is a continuous group homomorphism by Lemma 6.34. The map \( \iota_* \) is continuous by part (1) of Lemma 4.3. The map \( \epsilon \) is continuous by Corollary 4.8. It is straightforward to verify that both maps \( \iota_* \) and \( \epsilon \) are group homomorphisms. Hence, the map \( h \) is a continuous group homomorphism.

It remains to show that for all equivariant bifurcation problems \( X \) on the associated bundle \( G \times K V \), the path of gauge transformations \( hX \) is such that \( E_0P_0X = X + \partial(hX) \). For this, observe that, by definition of the maps involved,
the following diagram commutes:

\[
\begin{array}{ccc}
C^\infty (\mathbb{R}, C^\infty (V, q)^K) & \xrightarrow{\beta^{-1}} & C^\infty (\mathbb{R}, \Gamma (H)^G) \\
\epsilon_* & & \epsilon_* \\
\text{Bif}(G \times K V)^G_1 & \xrightarrow{\partial} & \text{Bif}(G \times K V)^G_0
\end{array}
\]

where the right vertical map is the inclusion. Therefore, for all equivariant bifurcation problems \(X\) on the associated bundle \(G \times K V\) we have that:

\[
\partial(hX) = \partial \epsilon_* \beta \alpha(X) = \beta^{-1} \beta \alpha(X) = \alpha(X) = E_0 P_0(X) - X,
\]

which is what we needed. Hence, the map \(h\) is the desired homotopy. \(\square\)

7. Relative symmetry breaking bifurcations

In this section we apply Theorem 6.26 and its corollaries to obtain relative symmetry breaking theorems from relative equilibria. In particular, we obtain a Relative Equivariant Branching Lemma (Theorem 7.11) giving generic conditions for branching to relative equilibria from a relative equilibrium, as well as a Relative Equivariant Hopf Theorem (Theorem 7.14) giving generic conditions for branching to relative periodic trajectories from a relative equilibrium.

First, we review some basic definitions about symmetry breaking bifurcations.

Remark 7.1. Let \(N\) be an \(H\)-manifold. Throughout this subsection we refer to isotropy subgroups of the symmetry group \(H\) acting on the manifold \(N\). These are subgroups of the symmetry group \(H\) that are the isotropy subgroup of some point in the manifold \(N\).

Definition 7.2. Let \(X\) be a path of equivariant vector fields on an \(H\)-manifold \(N\) with a relative equilibrium at a point \(n \in N\) with isotropy subgroup \(K\), and let \(\Sigma\) be a proper Lie subgroup of \(K\). We say that \(\Sigma\) is (spatially) symmetry breaking for \(X\) at the point \(n\) if there exists a bifurcating branch \(\gamma : [0, \epsilon) \to N\) starting at \(n\) such that the group \(\Sigma\) is the isotropy subgroup of the point \(\gamma(\lambda)\) for all \(\lambda \in (0, \epsilon)\).

Notation 7.3. In Definition 7.2 we will often specify whether the points on the branch are relative equilibria, relative periodic trajectories, or their “honest” counterparts, by saying that the group \(\Sigma\) is symmetry breaking to relative equilibria, relative periodic trajectories, equilibria, or periodic trajectories respectively.

Definition 7.4 is invariant in the following sense:

Lemma 7.4. Let \(X\) and \(Y\) be isomorphic paths of equivariant vector fields on an \(H\)-manifold \(N\) in the sense of Definition 2.20, let \(n \in N\) be a point with isotropy \(K\), and let \(\Sigma\) be a proper Lie subgroup of \(K\). Then if \(\Sigma\) is symmetry breaking for \(X\) at \(n\), it is also symmetry breaking for \(Y\) at \(n\). Furthermore, the nontrivial branches of relative equilibria or relative periodic trajectories of the path \(X\) are also nontrivial branches of relative equilibria or relative periodic trajectories, respectively, of the path \(Y\).

Proof. The result follows immediately from Lemma 3.6. \(\square\)
We now recall the eigenvalue crossing conditions used to predict strict symmetry breaking bifurcation in the literature.

**Remark 7.5.** Let $V$ be a real representation of a compact Lie group $K$ satisfying the irreducibility assumptions in Remark 6.1. Recall that an equivariant linear endomorphism of such a representation has a unique complex conjugate pair of eigenvalues (Remark 6.9). Thus, if $X$ is an equivariant bifurcation problem on $V$ there exist curves $\sigma_X : \mathbb{R} \to \mathbb{R}$ and $\mu_X : \mathbb{R} \to \mathbb{R}$, smooth near $\lambda = 0$, such that the complex conjugate pair $\sigma_X(\lambda) \pm i\mu_X(\lambda)$ correspond to the eigenvalues of the linearizations $DX_\lambda(0)$ for each $\lambda \in \mathbb{R}$.

It will be helpful to define the following:

**Definition 7.6.** Let $V$ be a real representation of a compact Lie group $K$ satisfying the irreducibility assumptions in Remark 6.1. Let $X$ be an equivariant bifurcation problem on $V$ and let $\sigma_X(\lambda) \pm i\mu_X(\lambda)$ be the eigenvalues of the linearizations $DX_\lambda(0)$ (see Remark 7.5). Then:

1. The equivariant bifurcation problem $X$ satisfies the **eigenvalue crossing condition** if $\sigma'_X(0) \neq 0$.
2. The equivariant bifurcation problem $X$ satisfies the **eigenvalue crossing condition with nonzero period** if $\sigma'_X(0) \neq 0$ and $\mu_X(0) \neq 0$.

**Notation 7.7.** We will denote the collection of equivariant bifurcation problems on a representation satisfying the eigenvalue crossing condition by $E$, and the collection of equivariant bifurcation problems on a representation satisfying the eigenvalue crossing condition with nonzero period by $E'$.

**Remark 7.8.** The equivariant bifurcation problems satisfying the eigenvalue crossing conditions of Definition 7.6 are significant since they exhibit symmetry breaking bifurcations to equilibria or periodic trajectories by the classic Equivariant Branching Lemma [4, 18] and the Equivariant Hopf Theorems [8]. Additionally, as the next proposition states, they form a generic class of equivariant bifurcation problems.

**Proposition 7.9.** Let $V$ be a real representation of a compact Lie group $K$ satisfying the irreducibility assumptions in Remark 6.1. Then the collections of equivariant bifurcation problems $E$ and $E'$ (Definition 7.6) are open and dense subsets of the space of equivariant bifurcation problems $\text{Bif}(V, 0)_K^G$ (Definition 6.8).

**Proof.** See Appendix A. □

The first eigenvalue crossing condition is invariant under isomorphisms in the following sense:

**Lemma 7.10.** Let $V$ be an absolutely irreducible representation of a compact Lie group $K$ satisfying the assumptions in Remark 6.1. Then the collection of bifurcation problems $E$ (Definition 7.6) is invariant with respect to the action of the group $\text{Bif}(V)_K^G$ on the space $\text{Bif}(G \times K V)_G^G$ (Definition 6.8).

**Proof.** We show that if $X$ and $Y$ are two equivariant bifurcation problems on $V$ such that $Y = X + \partial(\psi)$ for some path $\psi \in \text{Bif}(V)_K^G$, then, for any $\lambda \in \mathbb{R}$, the linearizations $DX_\lambda(0)$ and $DY_\lambda(0)$ are equal. Consequently, if $\sigma_X$ and $\sigma_Y$ are the respective eigenvalue curves then $\sigma_X = \sigma_Y$ (Remark 7.5). In particular, if the bifurcation problem $X$ satisfies the eigenvalue crossing condition, then so does the isomorphic bifurcation problem $Y$. 

For this, recall that the eigenvalues of the linearization \( D\partial(\psi)_\lambda(0) \) are a pair of purely imaginary eigenvalues \( \pm \mu i \) (Remark 6.9). On the other hand, since the path \( \partial(\psi) \) is an equivariant bifurcation problem (Remark 6.13) and the representation \( V \) of \( K \) is absolutely irreducible, the linearizations \( D(\partial(\psi)_\lambda)(0) \) are real multiples of the identity. Consequently, the eigenvalues \( \pm \mu i \) must be zero and the linearizations \( D\partial(\psi)_\lambda(0) \) are zero. Now note that:

\[
DY_\lambda(0) = D(X_\lambda + \partial(\psi)_\lambda)(0) = DX_\lambda(0) + D\partial(\psi)_\lambda(0) = DX_\lambda(0),
\]

for all \( \lambda \in \mathbb{R} \). That is, the linearizations are equal as claimed. \( \square \)

The following theorem generalizes the Equivariant Branching Lemma to relative equilibria. It is the first of the two main theorems of this section:

**Theorem 7.11** (Relative Equivariant Branching). Let \( V \) be an absolutely irreducible representation of a compact Lie subgroup \( K \) of a Lie group \( G \). Furthermore:

1. Let \( \Sigma \) be an isotropy subgroup for the representation \( V \) of the group \( K \) such that \( \dim(\text{Fix}(\Sigma)) = 1 \), where \( \text{Fix}(\Sigma) \) is the fixed-subspace of \( \Sigma \) in the representation.
2. Let \( E \) be the collection of equivariant bifurcation problems on the representation \( V \) satisfying the eigenvalue crossing condition (Definition 7.6).
3. Let \( \tilde{E} := \{ EY + \partial(\psi) | Y \in E, \psi \in \text{Bif}(G \times^K V)_1 \} \)

Then the subcollection of bifurcation problems:

\[
\tilde{E} := \{ EY + \partial(\psi) | Y \in E, \psi \in \text{Bif}(G \times^K V)_1 \}
\]

is open and dense in the space \( \text{Bif}(G \times^K V)_0 \) of equivariant bifurcation problems on the associated bundle \( G \times^K V \) (Definition 6.19), and the subgroup \( \Sigma \) is symmetry breaking to relative equilibria for all equivariant bifurcation problems in the subcollection \( \tilde{E} \) (Definition 7.2).

**Proof.** By the classical Equivariant Branching Lemma [4, 18], the subgroup \( \Sigma \) is symmetry breaking to equilibria for all equivariant bifurcation problems in the subcollection \( E \). By Proposition 7.9, the subcollection \( E \) is open and dense in the space of equivariant bifurcation problems on the representation \( V \). By Lemma 7.10 the subcollection \( E \) is invariant with respect to the action of the group \( \text{Bif}(V)_1 \) on the space of equivariant bifurcation problems \( \text{Bif}(V)_0 \). The result now follows by Corollary 6.32. \( \square \)

We introduce the following definition:

**Definition 7.12.** A representation \( V \) of a Lie group \( K \) is **gauge-free at a point** \( v \in V \) if \( \psi(v) = 0 \) for all gauge transformations \( \psi \in C^\infty(V, \mathfrak{g})^K \).

With this definition we can obtain the following lemma:

**Lemma 7.13.** Let \( V \) be a representation of a compact Lie group \( K \). Suppose it is irreducible of complex or quaternionic type, or is the sum \( V = W \oplus W \) of an absolutely irreducible representation \( W \) of \( K \) (see Remark 6.1). Then:

1. If the representation is gauge-free at the origin, then the subcollection \( E' \) of the space of equivariant bifurcation problems \( \text{Bif}(V)_0^K \) (Definition 7.6) is invariant with respect to the action of the group \( \text{Bif}(V)_1^K \) on the space \( \text{Bif}(V)_0^K \).
(2) If the representation is not gauge-free at the origin, then the subcollection \( \mathcal{E} \) of the space of equivariant bifurcation problems \( \text{Bif}(V)^K_0 \) (Definition 7.6) is invariant with respect to the action of the group \( \text{Bif}(V)^K_1 \) on the space \( \text{Bif}(V)^K_0 \).

**Proof.** Let \( X \) be an equivariant bifurcation problem on \( V \), let \( \psi \) be a path of gauge transformations on \( V \), and set \( Y := X + \partial(\psi) \). Suppose that we are in the first case and that \( X \in \mathcal{E}' \). That is, suppose that the eigenvalues \( \sigma_X(\lambda) \pm i\mu_X(\lambda) \) of the linearizations \( DX_\lambda(0) \) satisfy \( \sigma'_X(0) \neq 0 \) and \( \mu_X(0) \neq 0 \). We must show that the eigenvalues \( \sigma_Y(\lambda) \pm i\mu_Y(\lambda) \) of the linearizations \( DY_\lambda(0) \) are such that \( \sigma'_Y(0) \neq 0 \) and \( \mu_Y(0) \neq 0 \). For this, let \( \rho : K \to \text{GL}(V) \) be the representation and note that:

\[
\begin{align*}
DY_0(0) &= DX_0(0) + D\partial(\psi)_0(0) \\
&= DX_0(0) + \delta\rho(\psi_0(0)) & \text{by Theorem 6.12} \\
&= DX_0(0) + \delta\rho(0) & \text{since the representation is gauge-free at the origin} \\
&= DX_0(0) & \text{since } \delta\rho \text{ is a linear map.}
\end{align*}
\]

Hence, the real parts satisfy \( \sigma'_Y(0) = \sigma'_X(0) \neq 0 \) and the imaginary parts satisfy \( \mu_Y(0) = \mu_X(0) \neq 0 \). Thus, the bifurcation problem \( Y \) is in the subcollection \( \mathcal{E}' \).

Now suppose we are in the second case and that \( X \in \mathcal{E} \). That is, suppose that the eigenvalues \( \sigma_X(\lambda) \pm i\mu_X(\lambda) \) of the linearizations \( DX_\lambda(0) \) satisfy \( \sigma'_X(0) \neq 0 \). We must show that the eigenvalues \( \sigma_Y(\lambda) \pm i\mu_Y(\lambda) \) of the linearizations \( DY_\lambda(0) \) are such that \( \sigma'_Y(0) \neq 0 \). Note that the eigenvalues of the linearizations \( DY_0(0) = DX_0(0) + \delta\partial(\psi)_\lambda(0) \) are the sums of the real parts of the eigenvalues of the linearizations \( DX_\lambda(0) \) and \( D\partial(\psi)_\lambda(0) \) (see Remark 6.9). Since the linearizations \( D\partial(\psi)_\lambda(0) = \delta\rho(\psi_\lambda)(0) \) have pure imaginary eigenvalues, this means that the real parts of the eigenvalues of the linearizations \( DY_\lambda(0) \) are equal to the real parts \( \sigma_X(\lambda) \). Hence, in particular, they satisfy \( \sigma'_Y(0) = \sigma'_X(0) \neq 0 \). Consequently, the bifurcation problem \( Y \) is in the subcollection \( \mathcal{E} \). \( \square \)

The following theorem is our generalization of the (spatial) Equivariant Hopf Theorem. It is the second of the two main theorems of this section.

**Theorem 7.14** (Relative Spatial Equivariant Hopf Theorem). Let \( V \) be a representation of a compact Lie group \( K \). Suppose that the representation is irreducible of complex or quaternionic type, or is the sum \( V = W \oplus W \) of an absolutely irreducible representation \( W \) of \( K \). Furthermore:

1. Let \( \Sigma \) be an isotropy subgroup for the representation \( V \) of the group \( K \) such that \( \dim(\text{Fix}(\Sigma)) = 2 \), where \( \text{Fix}(\Sigma) \) is the fixed-subspace of \( \Sigma \) in the representation.
2. Let \( E_* : \text{Bif}(V)^K_* \to \text{Bif}(G \times K V)^\Sigma_* \) be the canonical inclusion of Proposition 6.25.
3. Let \( \mathcal{E} \) be the collection of equivariant bifurcation problems on the representation \( V \) satisfying the eigenvalue crossing condition and let \( \mathcal{E}' \) be the collection of those satisfying the eigenvalue crossing condition with nonzero period (Definition 7.6).

Then:

1. If the representation is gauge-free at the origin, then the subgroup \( \Sigma \) is symmetry breaking to relative periodic trajectories for all bifurcation problems...
in the open and dense subcollection:

$$\tilde{E}':=\{ E\psi + \partial(\psi) \mid Y \in \mathcal{E}', \psi \in \text{Bif}(G \times K V)^G_1 \}$$

of the space of bifurcation problems $\text{Bif}(G \times K V)^G_1$.

(2) If the representation is not gauge-free at the origin, then the subgroup $\Sigma$ is symmetry breaking to relative periodic trajectories for all bifurcation problems in the open and dense subcollection:

$$\tilde{E}:=\{ E\psi + \partial(\psi) \mid Y \in \mathcal{E}, \psi \in \text{Bif}(G \times K V)^G_1 \}$$

of the space of bifurcation problems $\text{Bif}(G \times K V)^G_1$.

**Proof.** There are two cases. First, suppose the representation $V$ of the group $K$ is gauge-free at the origin. By Proposition 7.9, the subcollection $\mathcal{E}'$ is open and dense in the space $\text{Bif}(V)^K_0$. By Lemma 7.13, the subcollection $\mathcal{E}'$ is invariant with respect to the action of the group $\text{Bif}(V)^K_1$ on the space $\text{Bif}(V)^K_0$. By the spatial Equivariant Hopf Theorem [9, Ch. XVI, Theorem 2.2], the subgroup $\Sigma$ is symmetry breaking to periodic trajectories for all equivariant bifurcation problems in the subcollection $\mathcal{E}'$. The result now follows by Corollary 6.32.

Now suppose that the representation $V$ is not gauge-free at the origin. By Proposition 7.9, the subcollection $\mathcal{E}$ is open and dense in the space of equivariant bifurcation problems on the representation $V$. By Lemma 7.11, the subcollection $\mathcal{E}$ is invariant with respect to the action of the group $\text{Bif}(V)^K_1$ on the space $\text{Bif}(V)^K_0$. We now prove that the symmetry group $\Sigma$ is symmetry breaking to relative periodic trajectories for all equivariant bifurcation problems in the subcollection $\mathcal{E}$. The result will then follow by Corollary 6.32.

Let $X$ be an arbitrary equivariant bifurcation problem in the subcollection $\mathcal{E}$. Note that the subcollection $\mathcal{E}'$ is contained in the subcollection $\mathcal{E}$. If $X \in \mathcal{E}'$, then the subgroup $\Sigma$ is symmetry breaking to relative periodic trajectories for the path $X$ by the spatial Equivariant Hopf Theorem [9, Ch. XVI, Theorem 2.2]. Thus, suppose that $X \notin \mathcal{E}'$. We will construct an isomorphic equivariant bifurcation problem $Y$ in $\mathcal{E}'$. Since $Y \in \mathcal{E}'$, the subgroup $\Sigma$ will be symmetry breaking to relative periodic trajectories for the path $Y$. This will imply, by Lemma 7.4, that the subgroup $\Sigma$ is symmetry breaking to relative periodic trajectories for the path $X$, concluding the proof.

We proceed with the construction of the bifurcation problem $Y$. Let $\sigma(\lambda) \pm i\mu(\lambda)$ be the eigenvalues of the linearizations $DX_\lambda(0)$; noting that $\sigma'(0) \neq 0$ and $\mu(0) = 0$. Since the representation is not gauge-free at the origin, let $\psi \in \text{Bif}(V)^K_1$ be a path such that $\psi_0(0) \neq 0$. By Lemma 7.13, the linearizations $D\partial(\psi_\lambda)(0)$ equal the linear maps $\delta \rho(\psi_\lambda(0))$, where $\delta \rho$ is the infinitesimal representation. Hence the linearization $D\partial(\psi_0)(0)$ is nonzero since $\psi_0(0) \neq 0$. By the same lemma, the linearizations $D\partial(\psi_\lambda)(0)$ have pure imaginary eigenvalues. Note that the linearizations $D\partial(\psi_\lambda(0))$ have a single pair of complex imaginary numbers $\pm i\nu(\lambda)$ (see Remark 6.9). Since we also know that the linearization $D\partial(\psi_0(0))$ is nonzero, the eigenvalues $\pm i\nu(0)$ must be nonzero.

Consider the bifurcation problem $Y = X + \partial(\psi)$. Note that the linearization:

$$DY_\lambda(0) = DX_\lambda(0) + D\partial(\psi_\lambda)(0), \quad \lambda \in \mathbb{R},$$

has eigenvalues $\mu(\lambda) \pm i\nu(\lambda)$ (see Remark 6.9). Thus, we have that $Y \in \mathcal{E}'$ since $\mu'(0) \neq 0$ and $\nu(0) \neq 0$, meaning that $Y$ is the desired bifurcation problem and concluding the proof. \qed
A. Genericity of the eigenvalue crossing conditions

The goal of this appendix is to prove the genericity of the eigenvalue crossing conditions of Definition 7.6. This is a well-known known fact in the equivariant dynamics community. We include it here for the sake of completeness since we could not find a written proof.

Remark A.1. Throughout this subsection we will consider a real representation $V$ of a compact Lie group $K$ satisfying the assumptions of Remark 6.1. This covers the cases considered in the symmetry breaking theorems of section 7 (Theorems 7.11 and 7.14). We will also consider the following subcollections of equivariant bifurcation problems on $V$:

1. The subcollection $\mathcal{E}$ of equivariant bifurcation problems on $V$ satisfying the eigenvalue crossing condition.
2. The subcollection $\mathcal{E}'$ of equivariant bifurcation problems on $V$ satisfying the eigenvalue crossing condition with nonzero period.

Both $\mathcal{E}$ and $\mathcal{E}'$ are subcollections of the space $\text{Bif}(V)^K_0$ of equivariant bifurcation problems on $V$ (Definition 7.6).

The main theorem of this appendix is the following:

Theorem A.2. Let $V$ be a representation of a compact Lie group $K$ satisfying the assumptions in Remark 6.1. Let $\mathcal{E}$ and $\mathcal{E}'$ be the subcollections of equivariant bifurcation problems on $V$ described in Remark A.1. Then:

1. The subcollection $\mathcal{E}$ is open and dense in the space of bifurcation problems $\text{Bif}(V)^K_0$.
2. Suppose additionally that the representation is not absolutely irreducible, then the subcollection $\mathcal{E}'$ is open and dense in the space of bifurcation problems $\text{Bif}(V)^K_0$.

Remark A.3. Note that if the representation $V$ of the compact Lie group $K$ is absolutely irreducible, then the subcollection $\mathcal{E}'$ is empty. This explains the cases considered in Theorem A.2.

Remark A.4. Recall that a path of equivariant vector fields $X : \mathbb{R} \to \mathfrak{X}(V)^K$ is smooth if and only if the associated map $X : V \times \mathbb{R} \to TV$ is smooth (Definition 2.14). Equivalently, since we are considering a vector space in this appendix, the path is smooth if and only if the associated map $X : V \times \mathbb{R} \to V$ is smooth. Throughout this appendix we freely identify the path $\mathbb{R} \to \mathfrak{X}(V)^K$ with the associated map $V \times \mathbb{R} \to V$.

We prove the openness and density statements of Theorem A.2 separately, beginning with openness. We will need the following technical lemma:

Lemma A.5. Let $V$ be a representation of a compact Lie group $K$ satisfying the assumptions in Remark 6.1. For any equivariant bifurcation problem $X \in \text{Bif}(V)^K_0$, let $\sigma_X, \rho_X : \mathbb{R} \to \mathbb{R}$ be the eigenvalue functions of the family $X$ (Remark 7.5). Then there exist continuous functions:

$$
\begin{align*}
\sigma : J^1_{(0,0,0)}(V \times \mathbb{R}, V) &\to \mathbb{R}, \\
\rho : J^1_{(0,0,0)}(V \times \mathbb{R}, V) &\to \mathbb{R}, \\
\bar{\sigma} : J^2_{(0,0,0)}(V \times \mathbb{R}, V) &\to \mathbb{R},
\end{align*}
$$
such that:

\[
\begin{align*}
\sigma_X(0) &= \sigma \left( j^1 X(0,0) \right) \\
\rho_X(0) &= \rho \left( j^1 X(0,0) \right) \\
\sigma'_X(0) &= \tilde{\sigma} \left( j^2 X(0,0) \right).
\end{align*}
\]

(A.1)

**Proof.** First, we describe global coordinates for the jet spaces \(J^1_{(0,0,0)}(V \times \mathbb{R}, V)\) and \(J^2_{(0,0,0)}(V \times \mathbb{R}, V)\). Fix a basis \(v^1, \ldots, v^{\dim(V)}\) on \(V\). Then, we have that:

\[
\begin{align*}
J^1_{(0,0,0)}(V \times \mathbb{R}, V) &= \left\{ j^1 X(0,0) = \left( 0, 0, 0, \partial_1 X(0,0), \ldots, \partial_m X(0,0) \right) \mid X \in C^\infty(V \times \mathbb{R}, V) \right\},
\end{align*}
\]

where \(\partial_j X(0,\lambda),\) for \(j = 1, \ldots, m,\) are the first-order partial derivatives of \(X\) with respect to the basis \(v^1, \ldots, v^{\dim(V)}\) evaluated at the point \((0,0) \in V \times \mathbb{R}\). Similarly, we have that:

\[
\begin{align*}
J^2_{(0,0,0)}(V \times \mathbb{R}, V) &= \left\{ j^2 X(0,0) = \left( 0, 0, 0, \partial_1 X(0,0), \ldots, \partial_n X(0,0) \right) \mid X \in C^\infty(V \times \mathbb{R}, V) \right\},
\end{align*}
\]

where \(\partial_j X(0,\lambda),\) for \(j = 1, \ldots, n,\) are the partial derivatives of \(X\) of first and second order with respect to the basis \(v^1, \ldots, v^{\dim(V)}\) evaluated at the point \((0,0) \in V \times \mathbb{R}\). Note that there is a canonical projection:

\[
\text{pr}_{2,1} : J^2_{(0,0,0)}(V \times \mathbb{R}, V) \to J^1_{(0,0,0)}(V \times \mathbb{R}, V), \quad j^2 X(0,0) \to j^1 X(0,0).
\]

Given a map \(X : V \times \mathbb{R} \to V,\) with \(X(0,0) = 0,\) note that the linearization \(DX_0(0)\) has a matrix representation, in the basis \(v^1, \ldots, v^{\dim(V)}\), with entries consisting of the partial derivatives:

\[
DX_0(0)_{i,j} = \frac{\partial X_i}{\partial v^j}_0(0,\lambda), \quad i, j = 1, \ldots, \dim(V).
\]

Hence, using the coordinates on the jet spaces we introduced, one can see there is also a projection:

\[
\text{pr}_{\text{lin}} : J^1_{(0,0,0)}(V \times \mathbb{R}, V) \to \text{End}(V), \quad j^1 X(0,0) \to DX_0(0).
\]

We will use these projections to build the desired maps.

Furthermore, by the eigenvalue formulas given in Remark 6.9 there exist continuous functions:

\[
\mu : \text{End}(V) \to \mathbb{R}, \quad \nu : \text{End}(V) \to \mathbb{R},
\]

such that the eigenvalues of any equivariant linear endomorphism \(L \in \text{End}(V)^K\) are given by \(\mu(L) \pm i\nu(L).\) We will use these maps as well.

The desired map \(\sigma\) is defined as the composition making the following diagram commute:

\[
\begin{array}{ccc}
J^1_{(0,0,0)}(V \times \mathbb{R}, V) & \xrightarrow{\text{pr}_{\text{lin}}} & \text{End}(V) \\
\downarrow \sigma & & \downarrow \mu \\
\mathbb{R} & & \mathbb{R}
\end{array}
\]
The desired map $\rho$ is defined as the composition making the following diagram commute:

$$
\begin{array}{ccc}
J^1_{(0,0,0)}(V \times \mathbb{R}, V) & \xrightarrow{pr_{lin}} & \text{End}(V) \\
\downarrow \rho & & \downarrow \nu \\
\mathbb{R} & & \mathbb{R}
\end{array}
$$

Finally, to define the map $\tilde{\sigma}$ let $y^{ij}$, for $i,j = 1, \ldots, \dim(V)$, be the coordinates induced by the basis of $V$ on the space $\text{End}(V)$. Then define the map $\tilde{\sigma}$ by:

$$
\tilde{\sigma}(j^2X(0,0)) := \sum_{i,j=1,\ldots,\dim(V)} \frac{\partial \mu}{\partial y^{i,j}} (DX_0(0)) \frac{\partial^2 X_i}{\partial \lambda \partial v^j}(0,0),
$$

for any 2-jet $j^2X(0,0) \in J^2_{(0,0,0)}(V \times \mathbb{R}, V)$. In the above formula, we have implicitly used the projection $pr_{lin}$. The map $\tilde{\sigma}$ is well-defined since it involves the partial derivatives of $X$ at the point $(0,0) \in V \times \mathbb{R}$ only up to second order.

Note that the maps $\sigma$ and $\rho$ are continuous since they are the composition of continuous maps. The map $\tilde{\sigma}$ is continuous since the projection $pr_{lin}$ and the map $\mu$ are continuous. It remains to show that the functions $\sigma$, $\rho$, and $\tilde{\sigma}$ satisfy the desired relationships in (A.1). For that, let $X \in \text{Bif}(V)^K_0$ be an arbitrary equivariant bifurcation problem on $V$. Let $\sigma_X : \mathbb{R} \to \mathbb{R}$ and $\rho_X : \mathbb{R} \to \mathbb{R}$ be the eigenvalue functions of the bifurcation problem $X$. Thus, by the definition of $\sigma$ and $\rho$, we have that:

$$
\begin{align*}
\sigma_X(\lambda) &= \mu(DX_\lambda(0)) = \sigma(j^1X(0,0)) \\
\rho_X(\lambda) &= \nu(DX_\lambda(0)) = \rho(j^1X(0,0)),
\end{align*}
$$

for all $\lambda \in \mathbb{R}$. Finally, we also have that:

$$
\begin{align*}
\sigma'_X(0) &= \left. \frac{d}{d\lambda} \mu(DX_\lambda(0)) \right|_{\lambda=0} \quad \text{by (A.2)} \\
&= \sum_{i,j=1,\ldots,\dim(V)} \frac{\partial \mu}{\partial y^{i,j}} (DX_0(0)) \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} \frac{\partial X^i}{\partial v^j}(0,0) \quad \text{by the chain rule} \\
&= \sum_{i,j=1,\ldots,\dim(V)} \frac{\partial \mu}{\partial y^{i,j}} (DX_0(0)) \frac{\partial^2 X^i}{\partial \lambda \partial v^j}(0,0) \\
&= \tilde{\sigma}(j^2X(0,0)).
\end{align*}
$$

Hence, the functions $\sigma$, $\rho$, and $\tilde{\sigma}$ satisfy the equalities in (A.1) for all families $X \in \text{Bif}(V)^K$. Therefore, the functions $\sigma$, $\rho$, and $\tilde{\sigma}$ are the desired functions. \hfill \square

We provide a direct proof that the subcollections $\mathcal{E}$ and $\mathcal{E}'$ are open by constructing open neighborhoods satisfying the eigenvalue crossing conditions.

Proof of the openness claim in Theorem A.2. Let $X$ be an equivariant bifurcation problem on the representation $V$ of the compact Lie group $K$. First, we describe an open neighborhood basis of the bifurcation problem $X$ with respect to the Whitney $C^\infty$ topology. Recall that for every nonnegative integer $r$ there exists a nonnegative
integer $n_r$ such that for every point $(v, \lambda, w) \in V \times \mathbb{R} \times V$ we may assume that the $r$-jet space at $(v, \lambda, w)$ is given by the space:

$$J^r_{(v, \lambda, w)}(V \times \mathbb{R}, V) = \left\{ j^r X(v, \lambda) = \left( v, \lambda, w, \partial_1 X(v, \lambda), \ldots, \partial_n X(v, \lambda) \right) \mid X \in C^\infty(V \times \mathbb{R}, V) \right\},$$

where $\partial_1 X(v, \lambda), \ldots, \partial_n X(v, \lambda)$ are the partial derivatives of $X$ up to order $r$ with respect to the basis of $V$. Furthermore, the $r$-jet bundle is then given by:

$$J^r(V \times \mathbb{R}, V) = \bigsqcup_{(v, \lambda, w) \in V \times \mathbb{R} \times V} J^r_{(v, \lambda, w)}(V \times \mathbb{R}, V)$$

There exists a nonnegative integer $N_r$, such that the $r$-jet bundle $J^r(V \times \mathbb{R}, V)$ is homeomorphic to $\mathbb{R}^{N_r}$ with respect to the usual topology on $\mathbb{R}^{N_r}$. In particular, we can take $d$ and $|| \cdot ||$ to be the canonical distance function and norm, respectively, on the space $\mathbb{R}^{N_r}$, and think of them as a distance function and norm on the space $J^r(V \times \mathbb{R}, V)$. With this distance function, recall that a neighborhood basis of the given bifurcation problem $X$, with respect to the Whitney $C^\infty$ topology on the space $\text{Bif}(V)^K_0$, is given by the collection of sets:

$$B_{r, \delta}(X) := \left\{ Y \in \text{Bif}(V)^K_0 \mid d\left( j^r X(v, \lambda), j^r Y(v, \lambda) \right) < \delta(v, \lambda) \right\},$$

where $r$ ranges over all nonnegative integers, and $\delta : V \times \mathbb{R} \rightarrow \mathbb{R}_+$ is any continuous function (see Remark [A.2]). We will also need the functions:

$$\sigma : J^1_{(0,0,0)}(V \times \mathbb{R}, V) \rightarrow \mathbb{R},$$
$$\rho : J^1_{(0,0,0)}(V \times \mathbb{R}, V) \rightarrow \mathbb{R},$$
$$\tilde{\sigma} : J^2_{(0,0,0)}(V \times \mathbb{R}, V) \rightarrow \mathbb{R},$$

of Lemma [A.3].

There are two cases to consider. First, we will suppose the bifurcation problem $X$ is in the subcollection $\mathcal{E}$. We will show that if we choose a continuous function $\delta : V \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that $\delta(0,0) > 0$ is small enough, then the neighborhood $B_{2,\delta}(X)$, as in [A.3], is contained in the set $\mathcal{E}$. Let $\sigma_X : \mathbb{R} \rightarrow \mathbb{R}$ be the function giving the real part of the eigenvalues of the given bifurcation problem $X$, and consider the real number:

$$\epsilon := \frac{\left| \sigma_X(0) \right|}{2} > 0,$$

which is positive since the bifurcation problem $X$ is in the set $\mathcal{E}$ and therefore satisfies the eigenvalue crossing condition. Since the function $\tilde{\sigma}$ in [A.4] is continuous, let $\delta_0 > 0$ be a real number such that:

$$\left| \tilde{\sigma} \left( j^2 X(0,0) \right) - \tilde{\sigma} \left( j^2 Y(0,0) \right) \right| < \epsilon,$$

for all jets $j^2 Y(0,0) \in J^2_{(0,0)}(V \times \mathbb{R}, V)$ such that $d\left( j^2 X(0,0), j^2 Y(0,0) \right) < \delta_0$. Now let $\delta : V \times \mathbb{R} \rightarrow \mathbb{R}_+$ be any continuous function such that $\delta(0,0) = \delta_0 > 0$. Let $Y$ be an arbitrary bifurcation problem in the neighborhood $B_{2,\delta}(X)$. Then we have
that \( d \left( j^2X(0,0), j^2Y(0,0) \right) < \delta(0,0) = \delta_0 \). Now let \( \sigma_Y : \mathbb{R} \to \mathbb{R} \) be the eigenvalue function of \( Y \), then:

\[
|\sigma'_X(0) - \sigma'_Y(0)| = \left| \frac{\sigma' (j^2X(0,0)) - \sigma' (j^2Y(0,0))}{\delta} \right| \quad \text{by the definition of } \sigma
\]

\[< \frac{|\sigma'_X(0)|}{2} \quad \text{by } \text{(A.5)}. \]

Hence, by the reverse triangle inequality we have that:

\[|\sigma'_Y(0)| > \frac{|\sigma'_X(0)|}{2} > 0.\]

Consequently, the bifurcation problem \( Y \) satisfies the eigenvalue crossing condition. Therefore, the neighborhood \( B_{2,\delta}(X) \) is contained in the set \( \mathcal{E} \). This shows that the set \( \mathcal{E} \) is open in the space \( \text{Bif}(V)^K_\delta \) with respect to the Whitney \( C^\infty \) topology.

We now consider the case when the representation is not absolutely irreducible and the bifurcation problem is in the subcollection \( \mathcal{E}' \). The proof is analogous to the previous case, with some minor modifications we now describe. First, recall that there is a canonical projection:

\[ \text{pr}_{2,1} : J^2(V \times \mathbb{R}, V) \to J^1(V \times \mathbb{R}, V), \quad j^2Y(v, \lambda) \mapsto j^1Y(v, \lambda). \]

This implies, in particular, that for any smooth map \( Y \in C^\infty(V \times \mathbb{R}, V) \), with \( Y(0,0) = 0 \), we have that:

\[ \text{(A.6) } \quad d \left( j^1X(0,0), j^1Y(0,0) \right) \leq d \left( j^2X(0,0), j^2Y(0,0) \right) \]

Let \( \sigma_X : \mathbb{R} \to \mathbb{R} \) and \( \rho_X : \mathbb{R} \to \mathbb{R} \) be the eigenvalue functions of the bifurcation problem \( X \). Consider the real number:

\[ \epsilon := \min \left\{ \frac{|\sigma'_X(0)|}{2}, \frac{|\rho'_X(0)|}{2} \right\} > 0, \]

which we know is nonzero since the bifurcation problem \( X \) is in the set \( \mathcal{E}' \) and therefore satisfies the eigenvalue crossing condition with nonzero period. Using \( \text{(A.6)} \) and the continuity of the functions \( \rho : J^1_{(0,0,0)}(V \times \mathbb{R}, V) \to \mathbb{R} \) and \( \sigma : J^2_{(0,0,0)}(V \times \mathbb{R}, V) \to \mathbb{R} \), choose a real number \( \delta_0 > 0 \) such that:

\[ \text{(A.7) } \quad \left\{ \begin{array}{l}
|\rho \left( j^1X(0,0) \right) - \rho \left( j^1Y(0,0) \right) | < \epsilon, \\
|\sigma \left( j^2X(0,0) \right) - \sigma \left( j^2Y(0,0) \right) | < \epsilon,
\end{array} \right. \]

for all jets \( j^1Y(0,0) \) and \( j^2Y(0,0) \) such that \( d \left( j^2X(0,0), j^2Y(0,0) \right) < \delta_0 \). Now let \( \delta : V \times \mathbb{R} \to \mathbb{R}_+ \) be any continuous function such that \( \delta(0,0) = \delta_0 > 0 \). Let \( Y \) be an arbitrary bifurcation problem in the neighborhood \( B_{2,\delta}(X) \). Then we have that \( d \left( j^2X(0,0), j^2Y(0,0) \right) < \delta(0,0) = \delta_0 \). Let \( \sigma_Y : \mathbb{R} \to \mathbb{R} \) and \( \rho_Y : \mathbb{R} \to \mathbb{R} \) be the eigenvalue functions of \( Y \). Then, exactly as in the case of the subcollection \( \mathcal{E} \), we get that \( |\sigma'_Y(0)| > \frac{|\sigma'_X(0)|}{2} > 0 \) by \( \text{(A.7)} \). Similarly, we also have that:

\[ |\rho_X(0) - \rho_Y(0)| = \left| \rho \left( j^1X(0,0) \right) - \rho \left( j^1Y(0,0) \right) \right| \quad \text{by the definition of } \rho
\]

\[< \epsilon \quad \text{by } \text{(A.7)}. \]

\[\leq \frac{|\rho_X(0)|}{2} \quad \text{by the choice of } \epsilon.\]
Hence, by the reverse triangle inequality we get that:

$$|\rho_Y(0)| > \frac{|\rho_X(0)|}{2} > 0.$$  

Consequently, the bifurcation problem $Y$ satisfies the eigenvalue crossing condition with nonzero period. Therefore, the neighborhood $B_{2,\delta}(X)$ is contained in the set $E'$. This shows that the set $E'$ is open in the space $\text{Bif}(V)_K$ with respect to the Whitney $C^\infty$ topology. □

Before proving the density of the eigenvalue crossing conditions, we prove the following technical lemma:

**Lemma A.6.** Let $V$ be a representation of a compact Lie group $K$ satisfying the assumptions in Remark 6.1. Let $L : V \to V$ be a $K$-equivariant linear endomorphism of $V$, and let $\alpha, \beta \in \mathbb{R}$ be real numbers. Then:

1. If the representation $V$ is absolutely irreducible, let $l$ be the real eigenvalue of $L$. Then the eigenvalue of the equivariant linear map:

$$\tilde{L} : V \to V, \quad \tilde{L} := L + \alpha I,$$

where $I$ is the identity map of $V$, is the number $l + \alpha$.

2. If the representation $V$ is not absolutely irreducible, let $l$ be the real part of the eigenvalues of $L$. Then there exist an equivariant linear map $J : V \to V$ such that the eigenvalues of the equivariant linear map:

$$\tilde{L} : V \to V, \quad \tilde{L} := L + \alpha I + \beta J,$$

where $I$ is the identity map of $V$, are the number $l + \alpha$. Furthermore, the imaginary parts of the eigenvalues of $\tilde{L}$ are nonzero if $|\beta|$ is small enough.

**Proof.** In the case where the representation is irreducible of complex or quaternionic type, let $J$ be the map in Remark 6.9. In the case where the representation is the sum of two copies of an absolutely irreducible representation, let $J$ be the map given in block form by:

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where $I$ is the identity map on the absolutely irreducible representation of $K$. Now let $L : V \to V$ be an arbitrary equivariant linear map on $V$, let $\alpha$ and $\beta$ be real numbers, and form the equivariant linear endomorphism $\tilde{L} := L + \alpha I + \beta J$.

In the case where the representation $V$ of the group $K$ is absolutely irreducible, note that $L = UI$ (Remark 6.9). Thus, the map $\tilde{L}$ is the map $\tilde{L} = (l + \alpha)I$. Hence, the eigenvalues are as desired.

In the case where the representation $V$ of the group $K$ is irreducible of complex type, we may assume that the map $L$ is of the form $L = UI + bJ$. Thus, the map $\tilde{L}$ is the map $\tilde{L} = (l + \alpha)I + (b + \beta)J$. Hence, the eigenvalues are $(l + \alpha) \pm i(b + \beta)$ by the formulas in Remark 6.9 and so are as desired.

In the case where the representation $V$ of the group $K$ is irreducible of quaternionic type, we may assume that the map $L$ is of the form $L = UI + bI + cJ + dK$, where the maps $I$ and $K$ are as in Remark 6.9. Thus, the map $\tilde{L}$ is the map $\tilde{L} = (l + \alpha)I + bI + cJ + dK$. Hence, the eigenvalues are $(l + \alpha) \pm i(\sqrt{b^2 + c^2} + d)$ by the formulas in Remark 6.9 and so are as desired.
In the case where the representation \( V \) of the group \( K \) is the sum of two copies of an absolutely irreducible representation of the group \( K \), we may assume that the map \( L \) is of the form:

\[
L = \begin{pmatrix} 2\epsilon I & bI \\ cI & 2(l - \epsilon)I \end{pmatrix}.
\]

Thus, the map \( \tilde{L} \) is the map:

\[
\tilde{L} = \begin{pmatrix} (2\epsilon + \alpha)I & (b - \beta)I \\ (c + \beta)I & (2l - 2\epsilon + \alpha)I \end{pmatrix}.
\]

Hence, the eigenvalues are:

\[
l + \alpha \pm \sqrt{4l^2 - 4(4\epsilon(l - \epsilon) - bc) + b\beta - c\beta - \beta^2}.
\]

by the formulas in Remark 6.9 and so are as desired.

We prove the density claim in Theorem A.2 by constructing equivariant perturbations satisfying the eigenvalue crossing conditions.

Remark A.7. The eigenvalue crossing conditions in Definition 7.6 are essentially transversality conditions in the space of jets. Thus, one could prove density of the subcollections \( E \) and \( E' \) via Thom’s transversality theorem [7, Theorem 4.9]. We instead favor an explicit construction of equivariant perturbations. These perturbations themselves satisfy the eigenvalue crossing conditions and are arbitrarily close, in the Whitney \( C^\infty \) topology, to the given bifurcation problem.

Proof of the density claim in Theorem A.2. Fix an arbitrary bifurcation problem \( X \in \text{Bif}(V)_0^K \). Let \( r \) be a nonnegative integer, let \( \delta : V \times \mathbb{R} \to \mathbb{R}_+ \) be a continuous function, and consider the set:

\[
B_{r,\delta}(X) := \left\{ Y \in \text{Bif}(V)_0^K \mid \begin{array}{l}
d\left(j^*X(v,\lambda),j^*Y(v,\lambda)\right) < \delta(v,\lambda) \\
\text{for all } (v,\lambda) \in V \times \mathbb{R}
\end{array} \right\}.
\]

Since the integer \( r \) and the function \( \delta : V \times \mathbb{R} \to \mathbb{R}_+ \) are arbitrary, recall that the set \( B_{r,\delta}(X) \) is an arbitrary neighborhood of the bifurcation problem \( X \). The collection of such neighborhoods, ranging over nonnegative integer \( r \) and continuous functions \( \delta \), gives a neighborhood basis of \( X \) in the Whitney \( C^\infty \) topology. Hence, it suffices to show that every such neighborhood contains a \( K \)-equivariant perturbation \( Y \) of \( X \) that is arbitrarily close and satisfies the corresponding eigenvalue crossing condition.

For this, let \( B \) be a \( K \)-invariant open ball around the origin \((0,0) \in V \times \mathbb{R} \), let \( \hat{B} \) be a \( K \)-invariant closed ball containing the ball \( B \), and consider a \( K \)-invariant smooth bump function \( \mu : V \times \mathbb{R} \to \mathbb{R} \). That is, the function \( \mu \) is such that:

\[
\mu(v,\lambda) = \begin{cases} 1 & \text{if } (v,\lambda) \in B \\ 0 & \text{if } (v,\lambda) \in \hat{B} - B \\ 1 - \mu(v,\lambda) & \text{if } (v,\lambda) \in (V \times \mathbb{R}) - \hat{B} \end{cases}
\]

and \( \mu(k \cdot v,\lambda) = \mu(v,\lambda) \) for all \( k \in K \) and \((v,\lambda) \in V \times \mathbb{R} \). To construct such a bump function, just take any smooth bump function satisfying (A.8), then average that bump function with respect to the action of \( K \) to obtain the desired \( K \)-invariant
smooth bump function, which is possible since $K$ is a compact Lie group and the sets $B$ and $\hat{B}$ are $K$-invariant.

Now consider the case when the bifurcation problem $X$ is in the subcollection $\mathcal{E}$; that is, that it satisfies the eigenvalue crossing condition. Let $\sigma_X: \mathbb{R} \to \mathbb{R}$ be the function giving the real parts of the eigenvalues of the bifurcation problem $X$. That is, for each $\lambda \in \mathbb{R}$, the value $\sigma_X(\lambda)$ is the real part of the eigenvalues of the linearization of the vector field $X_\lambda$ at the origin. Define the family of smooth vector fields:

$$Z: V \times \mathbb{R} \to V, \quad Z(v, \lambda) := \lambda \mu(v, \lambda) v.$$ 

Note that $Z$ is the associated map of a path of equivariant vector fields on $V$ since the bump function $\mu$ is $K$-invariant (Definition 2.14). Without loss of generality, we will think of $Z$ as a path $Z: \mathbb{R} \to \mathfrak{X}(V)^K$ of equivariant vector fields on $V$. Furthermore, observe that $Z$ has a trivial branch of equilibria at the origin $0 \in V$.

Let $|| \cdot ||$ be the norm on the jet bundle $J^r(V \times \mathbb{R}, V)$, and define the continuous function:

$$H: V \times \mathbb{R} \to \mathbb{R}_+, \quad H(v, \lambda) := ||j^r Z(v, \lambda)||.$$ 

Since the function $H$ is continuous and the closed ball $\hat{B}$ is compact, there exists a real number $M \geq 0$, such that:

(A.9) \quad $H(v, \lambda) \leq M$, for all $(v, \lambda) \in \hat{B}$. 

On the other hand, since the function $\delta: V \times \mathbb{R} \to \mathbb{R}_+$ is a nonnegative continuous function and the closed ball $\hat{B}$ is compact, there exists a real number $m > 0$, such that:

(A.10) \quad $m \leq \delta(v, \lambda)$, for all $(v, \lambda) \in \hat{B}$. 

Pick a real number $\epsilon > 0$ small enough so that:

(A.11) \quad $\sigma'_X(0) + \epsilon \neq 0$, and $\epsilon M \leq m$, 

where $\sigma_X: \mathbb{R} \to \mathbb{R}$ is the eigenvalue function of the family $X$.

We will define the desired $K$-equivariant perturbation $Y: \mathbb{R} \to \mathfrak{X}(V)^K$ via the associated map (Definition 2.14) given by:

$$Y: V \times \mathbb{R} \to V, \quad Y := X + \epsilon Z,$$

where are also identifying the bifurcation problem $X$ with its associated map $V \times \mathbb{R} \to V$. Note that the path $Y$ has a trivial branch of equilibria at the origin $0 \in V$, since both $X$ and $Z$ have one. Now observe that for $(v, \lambda) \in B$, the path $Y$ is of the form:

$$Y(v, \lambda) = X(v, \lambda) + \epsilon \lambda v,$$

since $\mu(v, \lambda) = 1$ for $(v, \lambda) \in B$. Therefore, for parameters $\lambda \in \mathbb{R}$ near $0$, the linearization $DY_\lambda(0): V \to V$ of the vector field $Y_\lambda$ is given by:

$$DY_\lambda(0) = DX_\lambda(0) + \epsilon I,$$

where $I$ is the identity map of $V$. Let $\sigma_Y: \mathbb{R} \to \mathbb{R}$ be the function giving the real parts of the eigenvalues of the linearizations of the path $Y$ at the origin. For parameters $\lambda \in \mathbb{R}$ near $0$, Lemma A.6 implies that:

$$\sigma_Y(\lambda) := \sigma_X(\lambda) + \epsilon \lambda.$$ 

Consequently, we have that:

$$\sigma_Y(0) = \sigma_X(0) + \epsilon 0 = \sigma_X(0) = 0,$$
where the last equality follows since the origin $0 \in V$ is a critical equilibrium of the vector field $X_0$. Thus, the origin $0 \in V$ is a critical equilibrium of the vector field $Y_0$. Hence, the path $Y$ is an equivariant bifurcation problem in the space $\text{Bif}(V)^K_0$. On the other hand, note that:

$$\sigma_Y'(0) = \sigma_Y'(0) + \epsilon \neq 0,$$

where we use (A.11). Hence, the bifurcation problem $Y$ satisfies the eigenvalue crossing condition, so it is in the subcollection $\mathcal{E}$.

It remains to show that the path $Y$ is in the neighborhood $B_{r,\delta}(X)$. Fix a basis of the vector space $V$ and let:

$$\partial_j X(v, \lambda), \partial_j Y(v, \lambda), \text{ and } \partial_j Z(v, \lambda), \text{ for } j = 1, \ldots, n_r,$$

be the partial derivatives (with respect to the basis) at the point $(v, \lambda) \in V \times \mathbb{R}$ up to order $r$, of the maps $X$, $Y$, and $Z$ respectively. Now observe that for any $(v, \lambda) \in V \times \mathbb{R}$ we may compute the distance between the jets $j^r X(v, \lambda)$ and $j^r Y(v, \lambda)$ to be:

$$d\left(j^r X(v, \lambda), j^r Y(v, \lambda)\right)$$

$$= \left\| \left( v, \lambda, Y(v, \lambda), \partial_1 Y(v, \lambda), \ldots, \partial_n Y(v, \lambda) \right) \right\|$$

$$= \left\| \left( v, \lambda, X(v, \lambda), \partial_1 X(v, \lambda), \ldots, \partial_n X(v, \lambda) \right) \right\|$$

$$= \epsilon \left\| \left( 0, 0, Z(v, \lambda), \partial_1 Z(v, \lambda), \ldots, \partial_n Z(v, \lambda) \right) \right\|.$$

Now for any point $(v, \lambda) \in \hat{B}$ we have that:

$$d\left(j^r X(v, \lambda), j^r Y(v, \lambda)\right)$$

$$= \epsilon \left\| \left( 0, 0, Z(v, \lambda), \partial_1 Z(v, \lambda), \ldots, \partial_n Z(v, \lambda) \right) \right\|$$

$$\leq \epsilon \|j^r Z(v, \lambda)\|$$

$$= \epsilon H(v, \lambda)$$

$$\leq \epsilon M \quad \text{by (A.9)}$$

$$< m \quad \text{by (A.10)}$$

Now observe that for $(v, \lambda) \in (V \times \mathbb{R}) - \hat{B}$ we have that:

$$Z(v, \lambda) = \lambda \mu(v, \lambda) v = 0,$$

by (A.8). Hence, the map $Z$ is identically 0 on the open set $(V \times \mathbb{R}) - \hat{B}$. Consequently, for any point $(v, \lambda) \in (V \times \mathbb{R}) - \hat{B}$ we have that:

$$\partial_j Z(v, \lambda) = 0, \quad j = 1, \ldots, n_r.$$
This implies that:

\[ d\left(j^*X(v, \lambda), j^*Y(v, \lambda)\right) = \epsilon \left\| \left(0,0, Z(v, \lambda), \partial_1 Z(v, \lambda), \ldots, \partial_n Z(v, \lambda)\right) \right\| \quad \text{by} \quad (A.12) \]

\[ = \epsilon \left\| (0, \ldots, 0) \right\| \quad \text{by} \quad (A.14) \text{ and } (A.15) \]

\[ = 0 < \delta(v, \lambda) \quad \text{since} \quad \delta \text{ is nonnegative.} \quad (A.16) \]

Therefore, by \((A.13)\) and \((A.16)\), the bifurcation problem \(Y\) is in the neighborhood \(B_{r, \delta}(X)\) as desired.

We now define the perturbation for the case when the representation is not absolutely irreducible and the bifurcation problem \(X\) is in the subcollection \(\mathcal{E}'\). The strategy is similar, but we must define a different perturbation since we also want to affect the imaginary part of the eigenvalues. Let \(\sigma_X : \mathbb{R} \to \mathbb{R}\) and \(\rho_X : \mathbb{R} \to \mathbb{R}\) be the eigenvalue functions of \(X\). Let \(\mathbb{J} : V \to V\) be the equivariant linear endomorphism of \(V\) constructed in Corollary \(A.6\). Then define the path of smooth vector fields:

\[ Z : V \times \mathbb{R} \to V, \quad Z(v, \lambda) := \lambda \mu(v, \lambda)v + \mu(v, \lambda)\mathbb{J}v. \]

Note that \(Z\) is a path of equivariant vector fields since the bump function \(\mu\) is \(K\)-invariant and the transformation \(\mathbb{J}\) is \(K\)-equivariant. Furthermore, the path \(Z\) has a trivial branch of equilibria at the origin \(0 \in V\). As before, define the continuous map:

\[ H : V \times \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad H(v, \lambda) := \left\| j^*Z(v, \lambda) \right\|, \]

and choose real numbers \(M > 0\) and \(m > 0\) so that:

\[ (A.17) \quad H(v, \lambda) \leq M, \quad m < \delta(v, \lambda), \quad \text{for all } (v, \lambda) \in \hat{B}, \]

using that the closed ball \(\hat{B}\) is compact. Now choose a real number \(\epsilon > 0\) small enough so that:

\[ (A.18) \quad \epsilon M < m, \quad \sigma_X'(0) + \epsilon \neq 0, \]

and, by Lemma \(A.6\) such that the imaginary part of the equivariant linear endomorphism:

\[ DX_0(0) + \epsilon\mathbb{J} \]

has nonzero imaginary part. Then define the path of vector fields:

\[ Y : V \times \mathbb{R} \to V, \quad Y := X + \epsilon Z, \]

and observe that this is a path of \(K\)-equivariant vector fields since both \(X\) and \(Z\) are thus. Furthermore, as before, the path \(Y\) has a trivial branch of equilibria at the origin \(0 \in V\). Now observe that for points \((v, \lambda) \in B\) the path \(Y\) is of the form:

\[ Y(v, \lambda) = X(v, \lambda) + \epsilon \mu v + \epsilon\mathbb{J} v, \]

since the \(\mu(v, \lambda) = 1\) for \((v, \lambda) \in B\). Therefore, for parameters \(\lambda \in \mathbb{R}\) near \(0\), the linearization \(DY_\lambda(0) : V \to V\) is given by:

\[ (A.20) \quad DY_\lambda(0) = DX_\lambda(0) + \epsilon \mu + \epsilon\mathbb{J}. \]
Let $\sigma_Y : \mathbb{R} \to \mathbb{R}$ be the function giving the real parts of the eigenvalues of the linearizations of the path $Y$ at the origin. Then, for parameters $\lambda \in \mathbb{R}$ near 0, Lemma A.6 implies that:

$$\sigma_Y(\lambda) = \sigma_X(\lambda) + \epsilon \lambda.$$ 

Consequently, we have that:

$$\sigma_Y(0) = \sigma_X(0) + \epsilon(0) = \sigma_X(0) = 0,$$

where the last equality follows since the origin $0 \in V$ is a critical equilibrium of the vector field $X_0$. Hence, the path $Y$ is an equivariant bifurcation problem on the representation $V$ of the group $K$. On the other hand, as in the case of the subcollection $\mathcal{E}$, note that:

$$\sigma_Y'(0) = \sigma_X'(0) + \epsilon \neq 0,$$

where we use (A.18). Now we need to check that the imaginary part $\rho_Y(0)$ of the eigenvalues of $DY_0(0)$ is nonzero. By (A.20) we have that:

$$DY_0(0) = DX_0(0) + \epsilon J.$$

Hence, by the choice of $\epsilon$ and (A.19), the imaginary part $\rho_Y(0)$ of the eigenvalues of the linearization $DY_0(0)$ is nonzero. Since $\sigma_Y'(0) \neq 0$ and $\rho_Y(0) \neq 0$, the bifurcation problem $Y$ satisfies the eigenvalue crossing condition with nonzero period, so it is in the set $\mathcal{E}'$. The proof that the path $Y$ is in the neighborhood $B_{r,\delta}(X)$ is completely analogous to the case of the subcollection $\mathcal{E}$.

Therefore, in both cases we have constructed an equivariant bifurcation problem $Y$ in the neighborhood $B_{r,\delta}(X)$ and distinct from $X$, such that $Y$ satisfies the corresponding eigenvalue crossing condition. Since the neighborhood $B_{r,\delta}(X)$ is an arbitrary neighborhood in the neighborhood basis of $X$, this holds for all neighborhoods of $X$ in the Whitney $C^\infty$ topology. Since $X$ is an arbitrary bifurcation problem in the space $\text{Bif}(V)^K_0$, this shows that the subcollections $\mathcal{E}$ and $\mathcal{E}'$ are dense in the space $\text{Bif}(V)^K_0$ in the respective cases. □

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