Some free-by-cyclic groups

Ian J. Leary∗  Graham A. Niblo †  Daniel T. Wise ‡

May 3, 2014

A group is said to be locally free if every finitely generated subgroup of it is free. One example is the additive group of the rationals. We exhibit a finitely generated group $G$ that is free-by-cyclic and contains a non-free, locally free subgroup. The smallest such example that we have found is of the form $G \cong F_n \rtimes \mathbb{Z}$ for $n = 3$. We also construct word-hyperbolic examples for larger values of $n$, and show that the groups are not subgroup separable.

We used Bestvina and Brady’s ‘Morse theory for cube complexes’ in the construction of these groups. The authors thank Jim Anderson, who posed a question concerning 3-manifolds that led to these examples, and the referee, whose comments were very helpful. This work was started at a conference at Southampton, immediately before Groups St. Andrews, which was funded by EPSRC visitor grants GR/L06928 and GR/L31135, and by a grant from the LMS.

Throughout this note, $F_n$ denotes a free group of rank $n$, $\bar{x}$ denotes $x^{-1}$, and $x^y = yxy$.

**Proposition 1.** The group $G$ given by the presentation

$$G = \langle a, b, t : a^t = b, b^t = ab\bar{a} \rangle$$

contains a non-free, locally free subgroup, and is isomorphic to a split extension $F_3 \rtimes \mathbb{Z}$.

**Proof** The given presentation expresses $G$ as an ascending HNN extension, with base group freely generated by $a$ and $b$ and with stable letter $t$. Define $\phi : G \to \mathbb{Z}$ by $\phi(t) = 1$, $\phi(a) = \phi(b) = 0$, and let $K$ be the kernel of $\phi$. Then $K$ is a strictly ascending union of 2-generator free groups

$$\langle a, b \rangle \subseteq \langle a, b \rangle^t \subseteq \langle a, b \rangle^{t^2} \subseteq \cdots \subseteq K.$$ 

∗Partially supported by EPSRC grant no. GR/L69398
†Partially supported by EPSRC grant no. GR/K25618
‡Supported by NSF grant no. DMS-9627506
Any such group is locally free and not free, since it is not finitely generated and the rank of its abelianization is at most two. (In fact, the abelianization of $K$ is infinite cyclic.)

It remains to show that $G$ is free-by-cyclic. Define $\psi: G \to \mathbb{Z}$ by $\psi(t) = \psi(a) = \psi(b) = 1$. It will be shown that the kernel of $\psi$ is free of rank 3.

A presentation 2-complex $Y$ for $G$ may be constructed by attaching two 2-cells to a rose with edges $a$, $b$, and $t$ according to the maps given in figure 1. Since $Y$ is obtained from the presentation of $G$ as an HNN extension with free base group, it follows that $Y$ is an Eilenberg-Mac Lane space for $G$ (see proposition 3.6 of [3]). Represent the three 1-cells of $Y$ as unit intervals, and represent the two 2-cells of $Y$ as a unit square and a $2 \times 1$ rectangle, as indicated in figure 1. This makes $Y$ into an affine cell complex in the sense of Bestvina and Brady ([1], Def. 2.1).

![Figure 1](image-url)

Now take $S^1 = \mathbb{R}/\mathbb{Z}$, viewed as a cell complex with one vertex and one edge of length 1, as an Eilenberg-Mac Lane space for the integers. A cellular map $g: Y \to S^1$ may be defined that induces the homomorphism $\psi: G \to \mathbb{Z}$ on fundamental groups and is affine on each cell. In figure 1 this map is represented by ‘height modulo one’, where the length of each edge is chosen so that its height is one. The inverse image of the vertex $v$ of $S^1$ is a rose consisting of one vertex and three 1-cells (the dotted lines on figure 1). Now let $X$ be the cover of $Y$ corresponding to the subgroup $H = \ker(\psi)$. The map $g$ lifts to a map $f: X \to \mathbb{R}$. $X$ is an affine cell complex, and $f$ is a Morse function in the sense of [1], Def. 2.2. By construction, $X$ is an Eilenberg-Mac Lane space for $H$, and for any integer $t$, $X_t = f^{-1}(t)$ consists of a disjoint union of copies of a 3-petalled rose. (Clearly, $X_t$ is a disjoint union of connected covers of $g^{-1}(v)$, but since any loop in $g^{-1}(v)$ represents an element of $G = \pi_1(Y)$ in the kernel of $\psi$, every lift in $X_t$ of such a loop is itself a loop, and hence $X_t$ is a disjoint union of 1-fold covers.)
Bestvina and Brady’s Morse theory allows one to compare $X$ and $X_t$: by lemma 2.5 of [1], a space homotopy equivalent to $X$ may be obtained from $X_t$ by coning off a subspace homeomorphic to a copy of the descending link (resp. ascending link) at $v$ for each vertex $v$ of $X$ such that $f(v) > t$ (resp. $f(v) < t$). (Ascending and descending links are defined in section 2 of [1].) All vertices of $X$ have isomorphic links, since $Y$ has only one vertex, and the ascending and descending links at each vertex are as shown in figure 2.

![The descending link](image1)

![The ascending link](image2)

Figure 2.

Both the ascending and descending link are contractible. Since coning off a contractible subspace does not change the homotopy type of a space, it follows that $X$ is homotopy equivalent to $X_t$. But it is already known that $X$ is an Eilenberg-Mac Lane space for $H$, and that $X_t$ is a disjoint union of 3-petalled roses. It follows that $X_t$ is connected, and that $H$ is free of rank three.

With the benefit of hindsight, a shorter proof that $G$ as above is free-by-cyclic may be given—see Proposition 2 below. Such a proof gives no indication as to how $G$ was discovered however. Moreover, the techniques of Proposition 1 generalize easily to more complicated presentations such as those given in Proposition 3.

**Proposition 2.** Let $H'$ be freely generated by $x$, $y$ and $z$, and define an automorphism $\theta$ of $H'$ by

$$\theta(x) = y, \quad \theta(y) = z, \quad \theta(z) = y^2x.$$ 

The group $G$ of Proposition 1 is isomorphic to $H' \rtimes \langle t \rangle$, where the conjugation action of $t$ on $H'$ is given by $\theta$.

**Proof** First, check that the endomorphism $\theta$ is an automorphism of $H'$ by exhibiting an inverse:

$$\theta^{-1}(z) = y, \quad \theta^{-1}(y) = x, \quad \theta^{-1}(x) = \bar{z}x^2.$$ 

Now, eliminate $b$ from the given presentation for $G$ to obtain

$$G = \langle a, t : \bar{t}\bar{a}a\bar{t} \bar{a} \rangle.$$ 

Substitute $a = xt$, and eliminate $a$, obtaining

$$G = \langle x, t : \bar{t}(xt)tt(xt)\bar{t}(\bar{t}x)t(\bar{t}x) \rangle = \langle x, t : t^2xt^3x\bar{t}x^2 \rangle.$$
Add new generators $y = x^t$ and $z = x^{t^2}$, obtaining
\[
G = \langle x, y, z, t : x^t = y, y^t = z, ztxy^2 = t \rangle = \langle x, y, z, t : x^t = y, y^t = z, z^t = y^2x \rangle.
\]
Thus $G$ is seen to be isomorphic to $H' \rtimes \langle t \rangle$ as claimed.

Next, we show how to construct a word-hyperbolic group having similar properties to the group $G$.

**Proposition 3.** For $s \geq 3$ define a word $W(x, y)$ by
\[
W(x, y) = xy^4xy^5x \cdots xy^{4+s}x.
\]
The group $G_s$ with presentation
\[
G_s = \langle a, b, t : a^t = b, b^t = W(b, a)b(W(a, b))^{-1} \rangle
\]
is free-by-cyclic and contains a non-free, locally free subgroup. For $s$ sufficiently large, $G_s$ is word-hyperbolic.

**Proof** As in Proposition 1, $G_s$ is a strictly ascending HNN-extension with base group freely generated by $a$ and $b$, so contains a non-free, locally free subgroup. As in Proposition 1, an Eilenberg-Mac Lane space for $G_s$ with an affine cell structure can be made by attaching a unit square and an $m \times 1$ rectangle to a rose with three petals of length 1. (Here $m$ is one more than the length of the word $W$, as shown in figure 3.) The argument given in Proposition 1 shows that $G_s$ is expressible as $F_n \rtimes \mathbb{Z}$, where $n$ is the total area of the two 2-cells in figure 3, i.e., $n = m + 1 = s + 4 + (8 + s)(s + 1)/2$. 

![Figure 3](image-url)
It remains to show that $G_s$ is word-hyperbolic for $s$ sufficiently large. For this, it suffices to show that some presentation for $G_s$ satisfies the $C'(1/7)$ small cancellation condition (see [4]). Eliminate $b$ from the presentation for $G_s$. This leaves a 1-relator group, with relator

$$\tilde{t}a^2a(\tilde{t}a^4ta)(\tilde{t}a^5ta)\cdots(\tilde{t}a^4sta\bar{a}^2\bar{a}ta^4s\bar{a})(\tilde{t}a^3sta\bar{a})\cdots(\tilde{t}a^5\bar{a}\tilde{t})(\tilde{t}a^4\bar{a}).$$

The total length of this relator as a cyclic word in $a$ and $t$ may be seen to be $8 + (14 + s)(s + 1)$. (The bracketing of the word is intended to facilitate this check.) For any $4 \leq r \leq 4 + s$, the four words $(\tilde{t}a^r t)^{\pm 1}$ and $(ta^r \bar{t})^{\pm 1}$ occur exactly once each as a subword of the relator or its inverse, and any subword of the relator or its inverse of length at least $2s + 15$ contains a subword of this form. (The worst case is the subword $a^{4+s}ta\bar{a}^2\bar{a}ta^{4+s}$, of length $2s + 14$.) Hence any subword of the relator or its inverse of length $2s + 15$ occurs in a unique place. It follows that $G_s$ is word-hyperbolic whenever $2s + 15 \leq 1/7 (8 + (14 + s)(s + 1))$. This inequality is satisfied for all sufficiently large $s$. (In fact, $s \geq 9$ suffices.)

P. Scott asked if free-by-cyclic groups are necessarily subgroup separable. An example due to Burns, Karass and Solitar showed that this is not the case (see [2]). The groups constructed above give another, simpler, argument to show this.

**Proposition 4.** The groups $G$ and $G_s$, constructed in Propositions 1 and 3, are not subgroup separable.

**Proof** In each case, let $L_1$ be the subgroup generated by $a$ and $b$, and let $L_2 = tL_1\bar{t}$. Then $L_1$ and $L_2$ are free of rank two and are conjugate in $G$ (resp. in $G_s$). Moreover, $L_1$ is a proper subgroup of $L_2$. It follows that $L_1$ cannot be closed, since it cannot be separated from any element of $L_2 \setminus L_1$: in any finite quotient, the images of $L_1$ and $L_2$ have the same order, since they are conjugate, and so must be equal since the image of $L_1$ is a subgroup of the image of $L_2$. 

5
References

[1] M. Bestvina and N. Brady, Morse theory and finiteness properties of groups, to appear in *Inventiones Math.*

[2] R. G. Burns, A. Karrass, and D. Solitar, A note on groups with separable finitely generated subgroups, *Bull. Austral. Math. Soc.* 36 (1987), no. 1, 153–160.

[3] P. Scott and C. T. C. Wall, Topological methods in group theory, in *Homological Group Theory*, London Math. Soc. Lecture Notes 36 (ed. by C. T. C. Wall), Cambridge Univ. Press, Cambridge 1979.

[4] R. Strebel, Small cancellation groups, Appendix to: *Sur les groupes hyperboliques d’après Mikhael Gromov*, 227–273, Progr. Math., 83, Birkhäuser Boston, Boston, MA, 1990.

Authors’ addresses:
Faculty of Mathematical Studies,
University of Southampton,
Southampton,
SO17 1BJ.

Department of Mathematics,
Cornell University,
Ithaca, NY
14853