Theoretical analysis of $^8\text{Li} + ^{208}\text{Pb}$ reaction and the critical angular momentum for complete fusion

B. Mukeru$^{a,b}$, M. L. Lekala$^a$, J. Lubian$^c$, Lauro Tomio$^b$

$^a$Department of Physics, University of South Africa, PO Box 392, Pretoria 0003, South Africa.
$^b$Instituto de Física Teórica, Universidade Estadual Paulista, 01140-070 São Paulo, SP, Brazil.
$^c$Instituto de Física, Universidade Federal Fluminense, Avenida Litoranea s/n, Gragoatá, Niterói, RJ, 24210-340, Brazil.

Abstract

In a theoretical approach, the complete and incomplete fusions are investigated by considering the $^8\text{Li} + ^{208}\text{Pb}$ reaction. By decreasing the projectile ground-state binding energy $\varepsilon_b$ from its known experimental value, the complete fusion is shown to have insignificant dependence on such variations, whereas the incomplete fusion strongly depends on that. The complete and incomplete fusion cross sections are calculated by using a combination of both continuum-discretized coupled-channel and sum-rule models. To this end, an incident-energy dependent cut-off angular momentum $L_c$ is first obtained by using the available complete fusion experimental data, within an approach which is extended to model results obtained for other incident-energies. An approximated fitted expression linking $L_c$ to the well-known critical value $L_{\text{crit}}$ derived by Wilczyński [Nucl. Phys. A 216 (1973) 386] suggests a generalization of the corresponding sum-rule model to energies around and below the Coulomb barrier.

Keywords: nuclear fusion reactions, cross sections, one-neutron halo nuclei, lithium-8, Pb-208, critical angular momentum

1. Introduction

The studies related to fusion reactions induced by loosely-bound projectiles, and their corresponding break up possibilities, are currently among the hottest subjects in Nuclear Physics [1–27]. Two kinds of fusions have emerged from these investigations, namely the complete fusion (CF), and the incomplete fusion (ICF), whose sum amounts to the total fusion ($\text{TF} = \text{CF} + \text{ICF}$). Given the low projectile binding energy, there is a high probability that the latter breaks up into two or more fragments before reaching the absorption region. In this case, all the fragments may be absorbed by the target, leading to the complete fusion. On the other hand, the target may absorb some but not all the fragments, while the scattered ones fly on the outgoing trajectory, leading to incomplete fusion. Many other phenomena such that transfer, delayed breakup may occur during such collisions, which are not discussed in this work. Nevertheless, one should point out that if the projectile breaks up when is moving apart of the target, this so-called delayed breakup will affect the reaction cross section, but not the fusion cross section [11]. Complete fusion is regarded as the absorption of the whole charge of the projectile, and not necessarily its whole mass, while incomplete fusion refers to the absorption of only a part of the projectile charge [11,8,17,26]. On the other hand, complete fusion is also interpreted as the absorption from bound states, with the incomplete fusion as the absorption from breakup states [2]. Yet, another model being used to study complete and incomplete fusions is the sum-rule model presented in Refs. [28–30], in which the idea of partial statistical equilibrium is combined with the generalized concept of critical angular momentum. According to this model, the CF process occurs at lower angular momenta ($L \leq L_{\text{crit}}$), where $L_{\text{crit}}$ is a critical angular momentum, that separates both complete and incomplete processes. In other words, the incomplete fusion process occurs at higher angular momenta ($L > L_{\text{crit}}$). Complete and incomplete fusion cross sections obtained

Email address: lauro.tomio@unesp.br (Lauro Tomio)
by using this model are reported in several works (see, for example, Refs. [31, 39]). However, this approach has being applied only for energies well above the Coulomb barrier, with $L_{\text{crit}}$ being independent of the incident energy. In view of that, an interesting study could be to verify whether such an approach can be extended to lower incident energies, around the Coulomb barrier, where in fact the incomplete fusion is important and both fusion processes are expected to have strong dependence on the incident energy.

Although the complete fusion process have been extensively studied, the full understanding of its suppression due to the projectile breakup is yet to be fully understood. For example, while it is widely believed that the suppression of complete fusion has a stronger dependence on the projectile breakup threshold (see, for instance, Refs. [9, 40] and references therein), recent experimental results in $^7\text{Li} + ^{209}\text{Bi}$ reactions have suggested charge clustering rather than weak projectile binding (i.e., breakup prior to reaching the fusion barrier) as the crucial factor in complete fusion suppression [26, 27]. It is further asserted in these references that weak projectile binding energy leads to incomplete fusion enhancement, while strong charge clustering leads to complete fusion suppression. These fascinating results call for further investigation in the role of the projectile breakup on the complete fusion process. Among other approaches, this can be achieved by artificially varying the projectile ground-state binding energy. One of the advantages of such an approach is that, it keeps the projectile mass and charge unchanged, thus minimizing their effects. In fact, this procedure was adopted in Ref. [41] for $^8\text{Li}$ projectiles, providing numerical support to results reported in Ref. [26]. The artificial variation of the ground-state binding energy in that reference, revealed another interesting aspect: in the case of $^7\text{Li}$ projectile, for binding energies weaker than the experimental values, the complete fusion cross section displays an insignificant dependence on this energy variation, while the opposite was observed when considering the $^6\text{Li}$ projectile (see Fig. 3 of Ref. [41]). For energies weaker than the experimental breakup threshold, one is representing a clear case of projectile breakup prior to reaching the fusion barrier. Therefore, to some extend, these results clarify the fact that weak binding energy alone is not a sufficient condition to warrant the suppression of the complete fusion cross section, in accordance to Refs. [26, 27]. Since the complete fusion process can also be regarded as the absorption of the whole projectile charge, one would as well expect this process to exhibit an insignificant dependence on variations of the ground-state binding energy in the case of $^7\text{Li}$ ($^7\text{Li}$+$n$), regardless the target mass.

By considering the $^7\text{Li}$ results [41] together with the sum-rule model of Ref. [28], it can be argued that the total fusion cross section would exhibit an insignificant dependence on the variation of the ground-state binding energy in the case the angular momentum $L$ is smaller than a cutoff value $L_c$, such that for the angular momentum window $L \leq L_c$, the complete fusion is the most dominant process. For higher angular momenta where the incomplete fusion is more important than its complete counterpart, the total fusion would be expected to strongly depend on the variation of the binding energy. This assessment is further justified by the fact that complete fusion is derived from bound-state absorptions, whereas incomplete fusion derives from breakup state absorptions. Such study would serve not only as a further numerical proof of the conclusions drawn in Refs. [26, 27], but to some extend can be useful to establish a connection between different definitions for complete and incomplete fusion processes.

In this paper, we study total, complete and incomplete fusion processes in the breakup of $^8\text{Li}$ projectile on a lead target. We are particularly interested in investigating the dependence of the complete fusion cross section on the variation of the ground-state binding energy below the experimental value. As a first step, we calculate the total fusion cross section by means of the of the Continuum-Discretized Coupled-Channel (CDCC) method [42]. Next, the complete fusion cross section is obtained from the calculated total fusion cross section, by following the sum-rule model, as given in Ref. [28]. Without diving into the details about the model, the interested reader can also find some fundamentals and relevant discussion in section 4 of Ref. [30]. In this case, we sum up the angular momentum distribution total fusion cross sections from zero up to some upper limit $L_c$, identifying the remaining part of the sum as corresponding to incomplete fusion cross section. Therefore, the crucial ingredient in this process is the angular momentum cutoff $L_c$, which is first determined by using the available complete fusion experimental data, as it will be shown. Next, this procedure is extended to regions where experimental data are not available, by considering complete fusion model results. This parameter $L_c$ emerges naturally as dependent on the incident energy, as well as on the projectile binding energy. For the reaction under study, an attempt is made in establishing a relationship between the energy-dependent $L_c$ and the $L_{\text{crit}}$ of Ref. [28]. Once $L_c$ is obtained, our next step will be to analyze the dependence of the total fusion cross section on the variation of the ground-state binding energy below the experimental value, for $L \leq L_c$ and for $L > L_c$, leading to the study the dependence of the complete and incomplete fusion cross sections on such binding energy variation. Apart from the $^8\text{Li}$ experimental ground-state energy $\epsilon_0 \approx 2.033$ MeV [43], three more
values are arbitrarily considered below the experimental one: $\epsilon_b = 1.5, 1.0$ and 0.5 MeV, obtained by adjusting the depths of the central and spin-orbit coupling components of the Woods-Saxon potential used for the bound as well as the continuum wave functions.

In the next, we organize the presentation of this paper as follows: the CDCC formalism is briefly described in Section 2. Projectile-target basic CDCC formalism

2. Projectile-target basic CDCC formalism

Following Ref. [42] for a system having a target with a weakly-bound core-plus-neutron projectile, the corresponding three-body Schrödinger equation is transformed into the CDCC differential equation after an expansion of the three-body wave function on a complete basis of bound and continuum states of the projectile [44]. With $R$ the vector position of the target in relation to the projectile center-of-mass, the corresponding radial coupled differential equations for the wave-function components $\chi_{\alpha}^{LJ}(R)$ are given by

$$\left[-\frac{\hbar^2}{2\mu_{pt}} \left( \frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} \right) + U_{\alpha\alpha}^{LJ}(R) \right] \chi_{\alpha}^{LJ}(R) + \sum_{a \neq \alpha} U_{\alpha a}^{LJ}(R) \chi_{a}^{LJ}(R) = (E - \epsilon_\alpha) \chi_{\alpha}^{LJ}(R), \quad (1)$$

where $\mu_{pt}$ is the projectile-target reduced mass, with the quantum numbers $L$ and $J$ being identified, respectively, with the orbital and total angular momentum (which, in the following, will be given in units of $\hbar$). The other relevant quantum numbers to describe the states of the projectile are represented by $a \equiv [i, l, s, j]$ ($i = 0, 1, 2, \ldots, N_b$, with $N_b =$ number of bins), with $\epsilon_\alpha$ being the projectile bin energies. By having $j \equiv l + s$, $l$ is the relative angular momentum between $^7$Li and the neutron, with $s$ being the spin of the neutron, considering that the interaction does not depend on the spin of the core. $U_{\alpha a}^{LJ}$ are the potential matrix elements, which are defined by

$$U_{\alpha a}^{LJ}(R) = \langle \mathcal{F}_{\alpha l}(\hat{r}, \hat{R})|V_{pt}(R, r)|\mathcal{F}_{a l'}(\hat{r}, \hat{R}) \rangle, \quad (2)$$

where

$$\mathcal{F}_{\alpha l}(\hat{r}, \hat{R}) = [i^L \Phi_{\alpha}(r) \otimes Y_{l}^{\alpha}(\hat{R})]_{LM}, \quad (3)$$

with the function $\Phi_{\alpha}(r)$ containing the bound and discretized bin wave functions of the projectile. The potential $V_{pt}(R, r)$ in Eq. (3) is a sum of core($^7$Li)-target and neutron-target potentials, such that

$$V_{pt}(R, r) = V_{ct}(R_c) + V_{nt}(R_{nt}), \quad (4)$$

It contains both Coulomb and nuclear components to account for the total breakup. The imaginary part of the projectile-target nuclear potential, $W_{nt}(R, r) = W_{ct}(R_c) + W_{nt}(R_{nt})$ is responsible for the absorption of the projectile by the target. Therefore, the coupling matrix elements corresponding to the absorption is given by

$$W_{nt}(R) = \langle \mathcal{F}_{\alpha l}(\hat{r}, \hat{R})|W_{nt}(R_c) + W_{nt}(R_{nt})|\mathcal{F}_{a l'}(\hat{r}, \hat{R}) \rangle. \quad (5)$$

The total fusion cross section (absorption cross section) is then obtained as the expectation given by

$$\sigma_{TF} = \sum_{L=0}^{L_{max}} \sigma_{TF}^{(L)} = \sum_{L=0}^{L_{max}} \frac{2\mu_{pt}}{\hbar^2 K_0} (2L + 1) \sum_{a \neq \alpha} \langle \chi_{\alpha}^{LJ}(R)|W_{nt}(R)|\chi_{a}^{LJ}(R) \rangle, \quad (6)$$

where $\chi_{\alpha}^{LJ}(R)$ is the full-radial wave function, and and $K_0$ is the projectile-target relative wave number in the incident channel. The complete fusion cross section can be directly obtained from Eq. (6), according to the sum-rule model as follows

$$\sigma_{CF} = \sum_{L=0}^{L_{max}} \sigma_{CF}^{(L)} \quad (7)$$
where $L_c$ is the angular momentum cutoff. The incomplete fusion is then given by

$$\sigma_{ICF} = \sigma_{TF} - \sigma_{CF} = \sum_{L > L_c} \sigma_{TF}^{(L)}.$$  \hfill (8)

### 3. Calculation details

In the CDCC formalism, the projectile energies and wave functions are crucial inputs in the coupling matrix elements. In the present work, the projectile being considered is the $^6\text{Li}$, which is known to be a $n$-$^{7}\text{Li}$ bound system, having a ground-state binding energy $\varepsilon_b = 2.033$ MeV, with $j^p = 2^+$. In a single-particle (or shell-model) picture, this ground state can be interpreted as a valence neutron in a $(\ell = 1, n = 1)$ single-particle configuration. This nucleus exhibits also a $j^p = 1^+$ first excited state with energy $\varepsilon_{ex} = 0.98$ MeV \cite{43}. The ground and excited states, as well as the continuum wave functions, are first obtained by using the parametrization presented in Ref. \cite{46} for the usual Woods-Saxon plus spin-orbit (SO) nuclear potential, as given in Refs. \cite{47,48}, with $V_{SO} = 4.89$ MeV·fm$^2$, $r_{SO} = r_0 = 1.25$ fm and $a_{SO} = a_0 = 0.52$ fm. The depth of the central potential, $V_0$, which is adjusted to take into account the ground, excited states and continuum wave functions, is also used as a parameter for tuning the ground-state binding energies being considered in the present study. The global parametrization of Akyuz-Winther \cite{59} is used to obtain the parameters of real parts of the $^7\text{Li}$-target and $n$-target optical potentials. As for the imaginary parts, we follow Ref. \cite{2} and consider short-ranged imaginary potentials with parameters $W = 50$ MeV, $r_w = 1.0$ fm and $a_w = 0.10$ fm. These parameters are used for both $^7\text{Li}$-target and $n$-target imaginary potentials. The short-ranged nature of these potentials implies that they quickly vanish beyond $r_w (A_{1/3}^{1/3} + A_t^{1/3})$ for the core-target system, and $r_w A_t^{1/3}$ for the $n$-target system, where $A_c = 7$ and $A_t = 208$ are the core and target atomic mass numbers, respectively. We also verify that, as long as this potential is well inside the Coulomb barrier, the results have a small dependence on such parameters. For the numerical calculations, we consider a maximum bin energy $\varepsilon_{max} = 8$ MeV. The $[0, \varepsilon_{max}]$ interval is discretized into bins of width $\Delta \varepsilon = 0.5$ MeV for $s$- and $p$-waves, $\Delta \varepsilon = 1.5$ MeV for $d$-waves, and $\Delta \varepsilon = 2.0$ MeV for higher partial waves. The angular momentum between $^7\text{Li}$ and the neutron is truncated at $\ell_{max} = 4$, with the maximum matching radius for the bin integration over the coordinate being $r_{max} = 70$ fm. The $^7\text{Li}$-target and $n$-target potentials are expanded into potential multipoles of maximum $\lambda_{max} = 4$. For the integration of the coupled differential equations over relative centre-of-mass distances, the matching radius is $R_{max} = 600$ fm, with the angular momentum of the relative centre-of-mass motion truncated at $L_{max} = 500$. These parameters are selected in accordance with the convergence requirements, for the considered $^6\text{Li}$ binding energies. This convergence is exemplified in Fig. \ref{fig:1} for results obtained for the total fusion cross section as a function of the incident center-of-mass (c.m.) energy $E_{cm}$, assuming different maximum bin energies for $\varepsilon_b = 2.03$ MeV, which are indicating that $\varepsilon_{max} = 8$ MeV is quite enough to insure converged results in all the cases.

### 4. Results and Discussion

The ground-state radial wave functions for $n$-$^7\text{Li}$ system, given by $r\phi(r)$, are displayed in Fig. \ref{fig:2} by considering different binding energies, which are obtained by varying the strength of the corresponding nuclear potential. As one can notice from this figure, as the binding energy decreases, it appears that the magnitude of the wave functions also decrease in the inner part ($r \leq 4$ fm), while the corresponding densities are extended to larger distances, due to the longer tails of the ground-state wave functions. Therefore, it is of interest to verify how this behavior will affect the complete and incomplete fusion cross sections. As anticipated in the introduction, the crucial step is the determination of a cutoff angular momentum $L_c$. This is done by calculating the complete fusion cross section within the sum-rule model, i.e., by using Eq. \ref{eq:7}. We proceed as follows: the partial total fusion cross sections $\sigma_{TF}^{(L)}$ are summed from $L = 0$ up to some angular momentum, such that the obtained sum fairly agrees with the experimental complete fusion data from Ref. \cite{50}. Then such angular momentum that corresponds to this sum is taken as the cutoff angular momentum $L_c$ for a specific incident energy $E_{cm}$. Within this procedure, we should allow a possible deviation in the determination of $L_c$, with $\Delta L_c \sim \pm0.5$. In Fig. \ref{fig:3} we notice that the calculated complete fusion cross section is in excellent agreement with the experimental data. The corresponding numerical values of $L_c$ are summarized in Table \ref{tab:1}. Recalling that the angular momentum of the relative center-of-mass motion was truncated at 500, these numerical values serve to
Further confirm that indeed the complete fusion is a process that occurs at lower angular momenta, as long asserted in Ref. [28]. For example, by considering $E_{\text{cm}} = 37.5$ MeV, one can deduce that $L_c$ represents only about 3% of $L_{\text{max}}$. Another relevant aspect to be noticed in this table is the observed dependence of $L_c$ on the incident energy, whereas $L_{\text{crit}}$ of Ref. [28] is fixed for a specific reaction, as given for energies well above the Coulomb barrier.

Having obtained the cutoff angular momentum $L_c$, considering available experimental data, let us now study the dependence of the total fusion cross section on the variation of the projectile ground-state binding energy below the experimental value for $L \leq L_c$. To this end, the $L$-distribution total fusion cross sections, given by $\sigma_L \equiv \sigma_{\text{TF}}^{(L)}$, are depicted in Fig.4. The blue arrows in the panels (a)-(g) of this figure represent the values of $L_c$ corresponding to each incident energy as listed in Table 1. Looking carefully at this figure, it is interesting to observe that the total fusion cross section has a rather insignificant dependence on the binding energy for $L \leq L_c$, whereas it strongly depends on this energy for $L > L_c$, in agreement with our assessment in the introduction. Consequently, the complete fusion cross section, which is defined in the $L \leq L_c$ window, will exhibit a weaker dependence on the projectile ground-state binding energy. On the other hand, the incomplete fusion, which is defined in the $L > L_c$ window, will have a stronger dependence on this energy. These results show that $L_c$ can also be regarded as the maximum angular momentum such that, for $L \leq L_c$, the total fusion cross sections exhibit an insignificant dependence on variations of the projectile ground-state binding energy. In fact, we have used this assessment to empirically determine $L_c$ for incident energies.
Figure 3: Complete fusion cross sections (solid line) calculated by using Eq.(7), with $L_c$ as given in Table 1. The experimental data (symbols) are from Table 1 of Ref.[50], with the corresponding error bars (absolute values $\leq 13$ mb) being approximately within the size of the symbols.

Table 1: For each incident energy, given in the first line, the second line provides the approximate values for the corresponding cutoff angular momenta $L_c$ (in units of $\hbar$), consistent with experimental data for complete fusion cross sections, as given in Fig.1.

| $E_{c.m.}$ (MeV) | 32.10 | 33.00 | 33.90 | 34.80 | 35.70 | 36.60 | 37.50 |
|-----------------|-------|-------|-------|-------|-------|-------|-------|
| $L_c$           | 9     | 11    | 12    | 13    | 14    | 15    | 16    |

where experimental data are not available. The computed results, considering several values for $E_{c.m.}$ from 45 to 75 MeV, are obtained by analyzing the corresponding behaviors of $\sigma_L$, for different projectile binding energies, which will indicate $L_c$ as the approximate upper limit for $L$, such that the results become almost independent on the binding energies. The panels (h)-(l) of Fig.4 are exemplifying how these approximate results given in Table 2 are obtained, where the arrows (red) indicate the $L_c$ positions. In order to search for a possible relationship between $L_c$ and $L_{crit}$, we look for an expression to fit the values presented in the Tables 1 and 2. As shown in Fig.5 for colliding energies above the Coulomb barrier $V_B$, they can be well represented by the empirical expression

$$L_c \equiv [1 - \exp \{-1.7\ (E_{c.m.}/V_B - 1)\}] L_{crit}, \quad (9)$$

where the factor 1.7 in the exponential is an adjustable parameter. In our approach, we assume $V_B = 28.65$ MeV, from the São Paulo Potential [53], with $L_{crit}$ being the well-known critical limit for the complete fusion according to the sum-rule model, which is obtained from the equilibrium condition of the Coulomb, nuclear and centrifugal forces [28,40]:

$$\left( L_{crit} + \frac{1}{2} \right)^2 = \frac{\mu (R_p + R_t)^3}{\hbar^2} \left[ 2\pi (\gamma_p + \gamma_t) \frac{R_p R_t}{R_p + R_t} - \frac{Z_p Z_t e^2}{(R_p + R_t)^2} \right], \quad (10)$$

where $\gamma_i = 0.95 \left[ 1 - 1.78(1 - 2Z_i/A_i)^2 \right]$ MeV fm$^{-2}$ ($i \equiv p, t$) are the surface tension coefficients, $R_i$ are the half-density nuclear radii, $\mu$ is the reduced mass, and $A_i$ the mass numbers. For the present $^8$Li + $^{208}$Pb reaction, by assuming $(R_p, R_t)$ from Ref. [51], with $R_p = R_{8Li} = 2.339$ fm and $R_t = R_{208Pb} = 5.501$ fm, we obtain $L_{crit} \approx 41$. The expression (9) is indicating that a more general relation may be found for the energy-dependent $L_c$ in terms of $L_{crit}$, which requires further investigation to be well established.

Coming back to the dependence of total, complete and incomplete fusion cross sections on the projectile ground-state binding energy, these cross sections are plotted in Fig.6 for the different four ground-state binding energies. Indeed, the inspection of this figure shows that, the complete fusion cross section [Fig.6(b)] has an insignificant dependence on the variation of the ground-state binding energy, in clear agreement with the results obtained for $^7$Li.
in Ref. [41]. while the incomplete fusion cross section [Fig. 6(c)] is strongly dependent on this energy. It follows that the $\text{^7Li}$ and $\text{^8Li}$ complete fusion cross sections are both insignificantly dependent on the variation of ground-state energy below the respective experimental values, although the former is modeled as a cluster of alpha and triton nuclei. These results further indicate that breakups of these isotopes prior to reaching the fusion barrier alone is not a sufficient condition to explain the suppression of complete fusion, a conclusion reported in Refs. [26, 27]. For completeness, Fig. 7 displays the competition between the complete and incomplete fusion cross sections for the different binding energies considered. As expected, it is seen that, the incomplete fusion cross section is dominant over its complete counterpart at low incident energies, whereas the complete fusion cross section prevails for larger incident energies [see Fig. 7(a)-(c)]. However, looking carefully at this figure, it appears that as the binding energy decreases, the incomplete fusion cross section becomes dominant for all incident energies within the interval studied here (around
Figure 5: Cutoff angular momenta $L_c$ (in units of $\hbar$), are shown as a function of $E_{c.m.}/V_B$, where the solid curve is given by Eq. (9). The data points are those given in Tables 1 (diamond symbols) and 2 (square symbols), from which one should allow a possible error in the extracted $L_c$ of about ±0.5. The arrow is indicating the corresponding sum-rule critical position for the $^8$Li $+^{208}$Pb reaction, $L_{\text{crit}} \approx 41$.

Table 2: For each incident energy, given in the first line, in the second line we present the corresponding approximate cutoff angular momenta $L_c$ (in units of $\hbar$), within uncertainty of about $\pm \hbar$ given inside the parenthesis. For $L \leq L_c$, the total fusion cross section should exhibit an insignificant dependence on variations of $\epsilon_b$. The panels (h) to (l) of Fig. 4 are exemplifying how the $L_c$ positions are obtained.

| $E_{c.m.}$ (MeV) | 45   | 50   | 55   | 60   | 65   | 70   | 75   |
|------------------|------|------|------|------|------|------|------|
| $L_c$            | (24-25) | (28-29) | (32-33) | (34-35) | (36-37) | (37-38) | (38-39) |

Figure 6: Total, complete and incomplete fusion cross sections, given as functions of the $E_{c.m.}$, for different ground-state energies $\epsilon_b$, as indicated inside the frame (a). The triangles in panel (b) refer to data results given in Table 1, obtained for $\epsilon_b = 2.03$MeV.

the Coulomb barrier) [see Fig. 7(d)]. Another interesting aspect in this figure that better depicts the results in Fig. 5 (b) is that all the different considered binding energies provide an excellent fit of the experimental data.

Since the complete fusion is verified to be independent on variations of the binding energy below the experimental value, it is important to investigate how is it suppressed owing to the variation of the binding energy. It has been shown that the complete fusion in the $^8$Li $+^{208}$Pb, $^{209}$Bi reactions, is suppressed by about 30% [26, 50], although the $^8$Li projectile breaks up into one charged and one uncharged fragments. As the complete fusion suppression can be analyzed through its contribution to the total fusion cross section, we first define the complete ($\Delta_{\text{CF}}$) and incomplete
The corresponding results for complete and incomplete fusions are plotted in the panels (a) and (b) of Fig. 8 respectively. In these two panels, the results are plotted as functions of binding energies $\varepsilon_b$. By inspecting this figure, one observes that the complete fusion is well-suppressed owing to the decrease of the binding energy [Fig. 8(a)], while the incomplete is enhanced [Fig. 8(b)]. In particular, a careful look at this figure shows that for $E_{cm} = 37.5$ MeV the complete fusion accounts for over 60% of the total fusion cross section for $\varepsilon_b = 2.03$ MeV, which represents a suppression of about 35% which fairly agrees with suppression factor reported in Refs. [26, 50]. As observed, this contribution drops significantly as $\varepsilon_b$ decreases, and it is about 25% (which represents a suppression of about 75%) for $\varepsilon_b = 0.50$ MeV. It is clear from Eq. (11), that the suppression of the complete fusion owing to the decrease of the projectile ground-state binding energy, is due to a stronger dependence of the incomplete fusion on this energy.

It has been shown that the incomplete fusion contribution, which is often referred to as incomplete fusion probability, decreases linearly as the projectile ground-state binding energy increases, see for example Ref. [52]. Generally, this analysis is done by considering different projectiles with different binding energies, atomic masses, as well as ground-state configurations. Since that $^4$Li is primarily formed by $^4$He+$^3$H+n (as studied in Ref. [52]), it should also be stressed that after the breakup the ICF process may involve fusion with charged clusters $^4$He+$^3$H. Therefore, in order to better elucidate the ICF for the reaction under study, a four-body CDCC calculation would be helpful, with the $^3$Li being treated within a three-body configuration. Or, within a sequential breakup, in which the $^4$Li first breaks into n+$^7$Li, followed by a second step with the $^7$Li split into $^4$He+$^3$H. Hence, the results obtained may also depend on different factors other than the binding energy. However, it is interesting to observe in Fig. 8(b) a perfect linear
dependence of the ICF contribution on the ground-state binding energy for all the incident energies; a linearity which is also displayed in Fig. 8(a) for the corresponding complementary complete fusion contribution.

5. Conclusions

We have investigated in some detail the dependence of the total, complete and incomplete fusion cross sections on variations of the projectile ground-state binding energy below the experimental value for the $^6$Li + $^{208}$Pb reaction. By adopting the sum-rule model of Refs. [28, 29], and by using the complete fusion experimental data, an angular momentum cutoff $L_c$ is determined, such that the complete fusion cross section occurs at angular momenta ($L$) lower than $L_c$ ($L \leq L_c$). By fitting the $L_c$ numerical values, we have obtained an expression linking both $L_c$ and the well-known critical value $L_{crit}$, which was derived in Ref. [28]. The expression for $L_c$ is a function of the incident energy $E_{cm}$ above the Coulomb barrier, whereas $L_{crit}$ is an energy independent parameter. Therefore, the results obtained hint to a possibility of extending the sum-rule model to energies around the Coulomb barrier. On the other hand, our findings indicate that a combination of the CDCC and sum-rule models can provide a better description of complete fusion processes, which is pointing out to an interesting extension of this study by considering incident energies below the Coulomb barrier.

The study of the dependence of the total fusion cross section on variations of the binding energy $\varepsilon_b$ shows that the total fusion cross section has an insignificant dependence on such variations for angular momenta in the limit for $L \leq L_c$, while it strongly depends on these variations for $L > L_c$. Consequently, the complete fusion, which is defined in the $L \leq L_c$ angular momentum window, is found to exhibit an insignificant dependence on $\varepsilon_b$ variations, whereas the incomplete fusion, defined in the $L > L_c$ window, is strongly dependent on $\varepsilon_b$. This is one of the main outcomes of this study. It follows that weak binding energy (or breakup prior to reaching the fusion barrier) alone, is not a sufficient ingredient to explain the suppression of complete fusion, as also it was concluded in Refs. [26, 27]. Therefore, the results in the present work can be regarded as a further theoretical support to the conclusions drawn in these references. Nevertheless, the complete fusion cross section is found to be stronger suppressed owing to the decrease of the binding energy. This comes rather from a strong enhancement of the incomplete fusion cross section due to the decreasing of the binding energy. We have also verified that the complementary complete and incomplete fusion contributions present approximate linear dependences on variations of the ground-state binding energy, when considering the available incident-energy experimental results.

Acknowledgements

B.M. is grateful to the South American Institute of Fundamental Research (ICTP-SAIFR) for local facilities. We also thank the following agencies for partial support: Conselho Nacional de Desenvolvimento Científico e Tecnológico
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