COHOMOLOGY OF FINITE $p$-GROUPS OF FIXED NILPOTENCY CLASS

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Abstract. Let $p$ be a prime number and let $c, d$ be natural numbers. Then, the number of possible isomorphism types for the mod $p$ cohomology algebra of a $d$-generated finite $p$-group of nilpotency class $c$ is bounded by a function depending only on $p$, $c$ and $d$.

1. Introduction

The cohomology of a finite group $G$ was initially defined as the cohomology of the associated classifying space $BG$ [10]. The cohomology functor $H^*(BG; \mathbb{Z})$ translates the topological properties of $BG$ into the algebraic properties of its cohomology group $H^*(BG; \mathbb{Z})$, where the latter category is easier to work in. These graded abelian groups can be used to distinguish non homotopy equivalent topological spaces. The above definition of cohomology of groups, however, does not give much information about the algebraic properties of the given group.

The algebraic interpretations of low dimensional cohomology groups were the first evidence to which kind of algebraic properties such groups encoded. For instance, if $G$ acts trivially on $\mathbb{Z}$, then $H^1(G; \mathbb{Z})$ is the group of all the group homomorphisms $G \to \mathbb{Z}$ and $H^2(G; \mathbb{Z})$ classifies central extensions of $G$ by $\mathbb{Z}$ up to equivalence [1, Chapter IV]. Later, Quillen proved the following remarkable result: given a finite group $G$, the Krull dimension of $H^*(G; \mathbb{F}_p)$ is the $p$-rank of $G$ [12]. Here, $\mathbb{F}_p$ denotes the finite field of $p$ elements with trivial $G$-action.

We focus our study in cohomology rings $H^*(G; \mathbb{F}_p)$ of finite $p$-groups $G$. We would like to know whether there exists an analogous algebraic property to that of being homotopy equivalent in the category of topological spaces and that it is detected by the cohomology algebra. In fact, in the same way that cohomology cannot tell two homotopy equivalent spaces apart, if two $p$-groups have isomorphic cohomology algebras, then these groups should share a good or powerful group property. In this manner, given an infinite family $\{G_i\}_{i \in I}$ of finite $p$-groups having a ‘good’ common group property, then there should only be a finite number of isomorphism types of cohomology algebras in $\{H^*(G_i; \mathbb{F}_p)\}_{i \in I}$.

The most striking result of the last decade in cohomology algebras that puts in evidence the above study was proven by J. F. Carlson.

**Theorem 1.1.** [2, Theorem 5.1] Let $m$ be an integer. Then, there are only finitely many isomorphism types of cohomology algebras in $\{H^*(G_i; \mathbb{F}_2)\}$, where $\{G_i\}$ are 2-groups of coclass $m$.

The coclass of a finite $p$-group of size $p^n$ and nilpotency class $c$ is $m = n - c$. In the same paper, J. F. Carlson conjectures that the analogous result should hold for the $p$ odd case [2, Conjecture 6.1] and this conjecture is supported by the classification of finite $p$-groups.

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by their coclass by Leedham-Green [9]. There have been several partial proofs of the above conjecture: B. Eick and D. J. Green made the first steps by considering Quillen categories of groups [6] and later, the authors of this paper together with Antonio Díaz Ramos, made a considerable progress to proving the conjecture [4]. However, the conjecture has not been proved in its totality yet.

The dual problem of this conjecture was studied in [5], where it is proved that the number of isomorphism types of cohomology algebras of \( d \)-generated finite \( p \)-groups of nilpotency class smaller than \( p \) is bounded by a function that depends only on \( p \) and \( d \) (see [5, Theorem 5.1]). The proof of this result is based on three keystones: the Lazard Correspondence, the description of the cohomology ring of powerful \( p \)-central \( p \)-groups with the \( \Omega \)-extension property (see [13]) and some counting arguments in cohomology algebras using spectral sequences (see [2],[3]).

This paper is a continuation to [5] and we study the cohomology of \( d \)-generated \( p \)-groups of arbitrary (but fixed) nilpotency class \( c \).

**Main Theorem.** Let \( p \) be a prime number and let \( c \) and \( d \) be natural numbers. Then, the number of possible isomorphism types for the mod \( p \) cohomology algebra of a \( d \)-generated \( p \)-group of nilpotency class \( c \) is bounded by a function depending only on \( p \), \( c \) and \( d \).

The novelty in the proof of the above theorem is that once the nilpotency class of a finite \( p \)-group is bigger than \( p \), there is no an analogue of the Lazard Correspondence. Instead, we prove that every \( d \)-generated \( p \)-group of nilpotency class \( c \) contains a powerful \( p \)-central subgroup with the \( \Omega \)-extension property (see Proposition 3.3 and Lemma 3.4). The cohomology of such \( p \)-groups is very well-understood (see Theorems 2.2 and 2.3) and in fact, the number of their isomorphism types is bounded in terms of \( p \) and \( d \). In the last part, using the counting argument result in Section 4, we prove the Main Theorem (compare Theorem 5.1).

**Notation.** Let \( G \) be a group, \( G^{pk} \) denotes the subgroup generated by the \( p^k \) powers of \( G \). The reduced mod \( p \) cohomology ring \( H^*(G;\mathbb{F}_p)_{\text{red}} \) is the quotient \( H^*(G;\mathbb{F}_p)/\text{nil}(H^*(G;\mathbb{F}_p)) \), where \( \text{nil}(H^*(G;\mathbb{F}_p)) \) is the ideal of all nilpotent elements in the mod \( p \) cohomology. We will use \([,]\) to denote the group commutator. If \( D \) is a group and \( I \) is a normal subgroup of \( D \), then we denote

\[
[I,cD] = [I,D,\ldots,D].
\]

We say that a function \( k = k(a,b,\ldots,A,B,\ldots) \) is \((a,b,\ldots)\)-bounded if there exist a function \( f \) depending on \((a,b,\ldots)\) such that \( k \leq f(a,b,\ldots) \). The rank of a \( p \)-group \( G \) is the sectional rank, that is,

\[
\text{rk}(G) = \max\{d(H) \mid \text{for all } H \leq G\},
\]

where \( d(H) \) is the minimal number of generators of \( H \).

2. **Powerful \( p \)-central \( p \)-groups**

Following [13], we define powerful \( p \)-central groups with the \( \Omega \)-extension property. We also recall a result about the cohomology algebra of such \( p \)-groups. Recall that for a \( p \)-group \( G \), we denote

\[
\Omega_1(G) = \langle g \in G \mid g^p = 1 \rangle.
\]

Similarly,

\[
\Omega_2(G) = \langle g \in G \mid g^{p^2} = 1 \rangle.
\]
Definition 2.1. Let $p$ be a prime number and let $G$ be a $p$-group. Then,

1. $G$ is powerful if $[G,G] \leq G^p$ for $p$ odd and, if $[G,G] \leq G^4$ for $p = 2$.
2. $G$ is $p$-central if $\Omega_1(G) \leq Z(G)$ for $p$ odd. For $p = 2$, we say that $G$ is $4$-central if $\Omega_2(G) \leq Z(G)$ and moreover, if $\Omega_2(G) = (\mathbb{Z}/4\mathbb{Z})^r$ for some $r \in \mathbb{N}$, we say that $G$ is $4^*$-central.
3. $G$ has the $\Omega$-extension property ($\Omega$EP for short) if there exists a $p$-central group $H$ such that $G = H/\Omega_1(H)$.

An easy computation shows that if $G$ has the $\Omega$EP, then $G$ is $p$-central. We finish this subsection by describing the mod $p$ cohomology algebra of a powerful $p$-central $p$-group with the $\Omega$EP and of a $4^*$-central 2-group.

Theorem 2.2. Let $p$ be an odd prime, let $G$ be a powerful $p$-central $p$-group with the $\Omega$EP and let $d$ denote the $\mathbb{F}_p$-rank of $\Omega_1(G)$. Then,

(a) $H^*(G;\mathbb{F}_p) \cong \Lambda(y_1, \ldots, y_d) \otimes \mathbb{F}_p[x_1, \ldots, x_d]$ with $|y_i| = 1$ and $|x_i| = 2$,
(b) the reduced restriction map $j_{\text{red}} : H^*(G;\mathbb{F}_p)_{\text{red}} \to H^*(\Omega_1(G);\mathbb{F}_p)_{\text{red}}$ is an isomorphism.

Proof. See [13, Theorem 2.1 and Corollary 4.2] □

Theorem 2.3. Let $G$ be a $4^*$-central 2-group and let $d$ denote the $\mathbb{F}_2$-rank of $\Omega_2(G)$. Then, the following are equivalent:

(a) $H^*(G;\mathbb{F}_2)_{\text{red}} \cong \mathbb{F}_2[x_1, \ldots, x_d]$, where $|x_i| = 2$ for all $i = 1, \ldots, d$.
(b) $G$ has the $\Omega$EP.

Proof. See [13, Theorem 2.1]. □

3. $p$-GROUPS OF FIXED NILPOTENCY CLASS

The results in this section give a preliminary setting for the proof of the main theorem in Section 4. We start by proving an easy lemma about $p$-groups of fixed nilpotency class $c$ and rank $r$. We will show that such $p$-groups have a descending normal series of bounded length $h(r,c)$. Then, the proof of the Main theorem will proceed by induction on such a series.

Lemma 3.1. Let $p$ be a prime number and let $c, r$ be positive integers. Let $G$ be a finite $p$-group of nilpotency class $c$ and rank $r$. Then, there exists a series of normal subgroups $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_h = \{1\}$, such that

(a) $G_i \triangleleft G$, $G_i/G_{i+1}$ is a cyclic $p$-group,
(b) if $G/G_{i+1}$ has nilpotency class $l$, then $G_i \leq \gamma_l(G)$ and, in particular, $G_i/G_{i+1}$ is central in $G/G_{i+1}$,
(c) $h = h(c,r) \leq cr$ is a number that depends only on $c$ and $r$.

Proof. Let $G$ be a finite $p$-group of nilpotency class $c$. That is, the lower central series $\{\gamma_i(G)\}_{i \geq 1}$ satisfies that $\gamma_{i+1}(G) = \{1\}$ and $\gamma_i(G) \neq 0$ for all $i \leq c$. Note that each factor group $A_i := \gamma_i(G)/\gamma_{i+1}(G)$ is an abelian $p$-group of rank at most $r$ and it is central in $G/\gamma_{i+1}(G)$. Moreover, each $A_i$ has a decomposition with cyclic factor groups, say,

$$A_i = B_0^i \supseteq B_1^i \supseteq \cdots \supseteq B_{l_i}^i = \{1\}.$$
where \( B^i_j / B^i_{j+1} \) is cyclic and the values \( l_i \) depend on the rank of \( A_i \) (which in turn depends on the rank of \( G \)). Thus, we can form a normal series

\[
G = \gamma_1(G) \supseteq B^1_1 \supseteq \cdots \supseteq B^1_{l_1} = \gamma_2(G) \supseteq B^2_1 \supseteq \cdots \supseteq \gamma_{c+1}(G) = \{1\},
\]

where the length of the series depends on \( c \) and \( r \). For simplicity, denote the above series by \( \{G_i\}^h_{i=0} \). Finally, it is clear by construction that if \( G / G_{i+1} \) has nilpotency class \( l \), then \( G_{i+1} \leq \gamma_i(G) \) and in particular, \( G_i / G_{i+1} \) is central in \( G / G_{i+1} \). Moreover, since the rank of \( G \) is \( r \), each factor \( \gamma_i(G) / \gamma_{i+1}(G) \) has rank at most \( r \) and the bound on \( h(c, r) \) is attained. \( \square \)

We continue by proving a result in commutator subgroups of finitely generated pro-\( p \) groups and subgroups in the category of (finitely generated) pro-\( p \) groups are closed unless otherwise stated.

**Lemma 3.2.** Let \( c \) and \( k \) be non-negative integers. Let \( G \) be a finitely generated pro-\( p \) group of nilpotency class \( c \) and let \( K = G^{p^k} \).

1. Suppose that \( p \) is odd and that \( k - 1 \geq \log_p(c + 1) \), then
   \[
   [N, K] \leq N^p,
   \]
   for all normal subgroups \( N \) of \( G \) and in particular, \( K \) is powerful.

2. Suppose that \( p = 2 \) and that \( k - 2 \geq \log_2(c + 1) \), then
   \[
   [N, K] \leq N^4,
   \]
   for all normal subgroups \( N \) of \( G \) and in particular, \( K \) is powerful.

**Proof.** We start with the odd case. By Theorem 2.4 in [8], we have that for all normal subgroups \( N \) of \( G \),

\[
[N, K] = [N, G^{p^k}] \equiv [N, G]^{p^k} (\text{mod}[N, G]^{p^{k-1}} [N, p^2 G]^{p^{k-2}} \cdots [N, p^k G])
\]

\[
\leq N^{p^k} (\text{mod}[N, G]^{p^{k-1}} [N, p^2 G]^{p^{k-2}} \cdots [N, p^k G]),
\]

and since \( [N, p^i G]^{p^{k-i}} \leq N^{p^{k-i}} \) for \( i = 0, \ldots, k - 1 \) and \( [N, p^k G] \leq \gamma_{c+1}(G) = 1 \), the statement follows.

If \( p = 2 \), again by Theorem 2.4 in [8], we have that

\[
[N, K] = [N, G^{2^k}] \equiv [N, G]^{2^k} (\text{mod}[N, G]^{2^{k-1}} [N, 2 G]^{2^{k-2}} \cdots [N, 2^k G])
\]

\[
\leq N^{2^k} (\text{mod}[N, G]^{2^{k-1}} [N, 2 G]^{2^{k-2}} \cdots [N, 2^k G]).
\]

Now, since \( [N, 2 G]^{2^{k-i}} \leq N^{2^{k-i}} \) for \( i = 0, \ldots, k - 2 \) and both \( [N, 2^k G] \) and \( [N, 2^{k-1} G]^2 \) are contained in \( \gamma_{c+1}(G) = 1 \), the statement follows. \( \square \)

**Notation.** For a prime number \( p \), set \( p^* = p \) if \( p \) is odd and \( p^* = 4 \) if \( p = 2 \).

Using the above result, we prove that a finite \( p \)-group \( G \) of fixed nilpotency class \( c \) contains a powerful \( p^* \)-central \( p \)-subgroup \( B \) with the \( \Phi \)EP. The existence of this subgroup implies that the restriction map from \( H^*(G; \mathbb{F}_p) \) to \( H^*(B; \mathbb{F}_p) \) is non-trivial (see Theorems 2.2 and 2.3) and this is crucial to apply Theorem 4.1 in the proof of the Main result.

**Proposition 3.3.** Let \( c \) and \( k \) be positive integers and let \( p \) be a prime number. Let \( G \) be a finite \( p \)-group of nilpotency class \( c \), let \( C = \langle y \rangle \) be a cyclic subgroup of \( G \) such that \( C \leq \gamma_c(G) \). Set \( K := G^{p^k} \).

1. If \( p \) is odd, suppose that \( k - 1 \geq \log_p(c + 1) \), then there exists a \( p \)-group \( H \) such that
(a1) $H$ is powerful $p$-central.

(a2) $K \cdot C \cong H/\Omega_1(H)$, that is, $K \cdot C$ is a powerful $p$-central $p$-group with the $\Omega$EP.

(b) If $p = 2$, suppose that $k - 2 \geq \log_2(c + 1)$, then there exists a 2-group $H$ such that

(b1) $H$ is powerful 4-central.

(b2) $K \cdot C \cong H/\Omega_1(H)$, that is, $K \cdot C$ is a powerful 4-central 2-group with the $\Omega$EP.

Proof. Let $X$ be a minimal system of generators of $G$ and let $F$ denote the free pro-$p$ group on $X$ of nilpotency class $c$. Let $\pi : F \to G$ denote the natural projection map, let $N$ denote its kernel and let $x$ be a preimage of $y$ contained in $\gamma_c(F)$. From now on, the proof of the proposition deals with the $p$ odd case and the $p = 2$ case, separately.

(a) Suppose that $p$ is an odd prime and let $k$ be an integer such that $k - 1 \geq \log_p(c + 1)$. Put $M := N \cap \langle x, F^{p^k} \rangle$ and let $H := \langle x, F^{p^k}/M^p \rangle$ be a finite $p$-group. Then, we have that

$$K \cdot C = G^{p^k} \cdot \langle y \rangle \cong \left( \frac{F}{N} \right)^{p^k} \cdot \frac{\langle x, N \rangle}{N} \cong \frac{\langle x, F^{p^k} \rangle}{N \cap \langle x, F^{p^k} \rangle} = \frac{F^{p^k}}{M}.$$  

Take $N = F^{p^k}$ in the previous lemma, to obtain that $[\langle x, F^{p^k} \rangle, \langle x, F^{p^k} \rangle] = [F^{p^k}, F^{p^k}] \leq (F^{p^k})^p$. Then, $\langle x, F^{p^k} \rangle$ is a powerful pro-$p$ group and whence, $K \cdot C = G^{p^k} \cdot C$ is also a powerful $p$-group.

Note that $M$ is a normal subgroup of $F$ that is contained in $F^{p^k}$. Then,

$$[M, M] \leq [M, F^{p^k}] \leq M^p,$$

where in the first inequality we used the fact that $x$ is central and in second inequality we used the previous lemma with $N = M$. This shows that $M$ is a finitely generated powerful pro-$p$ group. Moreover, as $F$ and $M$ are torsion-free, we have that $M$ is a uniform pro-$p$ group [1, Theorem 4.5]. Thus, $M^p = \{m^p \mid m \in M\}$ (see [1, Theorem 3.6]). In particular,

$$\Omega_1(H) = \Omega_1(\langle x, F^{p^k}/M^p \rangle) = M/M^p,$$

and

$$[\Omega_1(H), H] = [M/M^p, \langle x, F^{p^k}/M^p \rangle] = [M, F^{p^k}]M^p/M^p = 1.$$  

Therefore, $H$ is a powerful $p$-central $p$-group and

$$H/\Omega_1(H) = \frac{\langle x, F^{p^k}/M^p \rangle}{M/M^p} \cong \frac{\langle x, F^{p^k} \rangle}{M} \cong \langle y, G^{p^k} \rangle = K \cdot C.$$  

The last equality shows that $K \cdot C$ has the $\Omega$EP and by [11 Corollary 2.4], we conclude that $G^{p^k} \cdot C$ is a powerful $p$-central $p$-group with the $\Omega$EP.

(b) Suppose that $p = 2$ and let $k$ be an integer such that $k - 2 \geq \log_2(c + 1)$. Construct $M$ and $H$ as in (a) by taking $p = 2$. Also, take $N = F^{2^k}$ in the previous lemma to obtain that $[\langle x, F^{2^k} \rangle, \langle x, F^{2^k} \rangle] = [F^{2^k}, F^{2^k}] \leq (F^{2^k})^4$. Then, $\langle x, F^{2^k} \rangle$ is a powerful pro-2 group and whence, $K \cdot C = G^{2^k} \cdot C$ is also a powerful 2-group.

Note that again $M$ is a normal subgroup of $F$ that is contained in $F^{2^k}$. Then,

$$[M, M] \leq [M, F^{2^k}] \leq M^4,$$

where in the first inequality we used the fact that $x$ is central and in second inequality we used the previous lemma with $N = M$. This shows that $M$ is a finitely generated powerful
pro-2 group. Moreover, as $F$ and $M$ are torsion-free, we have that $M$ is a uniform pro-2 group and thus, $M^4 = \{m^4 \mid m \in M\}$ (see [3] Theorems 4.5, 3.6). In particular, 
$$\Omega_2(H) = \Omega_2((x, F^{2k})/M^4) = M/M^4,$$

and 
$$[\Omega_2(H), H] = [M/M^4, (x, F^{2k})/M^4] = [M, F^{2k}]M^4/M^4 = 1.$$

Therefore, $H$ is a powerful 4-central 2-group. Notice that by [11, Corollary 2.4] the group $H/\Omega_1(H)$ is also a powerful 4-central 2-group. Furthermore 
$$\frac{H/\Omega_1(H)}{\Omega_1(H/\Omega_1(H))} \cong H/\Omega_2(H) = \frac{(x, F^{2k})/M^4}{M/M^4} \cong \frac{\langle x, F^{2k} \rangle}{M} \cong \langle y, G^{2k} \rangle = K.C.$$ 

The last equality shows that $K$ has the ΩEP and by [11, Corollary 2.4], we conclude that $G^{2k}$ is a powerful 4-central 2-group with the ΩEP. The last property easily follows by construction.

For $p = 2$, we would also like to conclude that the restriction map $H^*(G; \mathbb{F}_p) \to H^*(B; \mathbb{F}_p)$ is non-trivial. To that aim, $B$ should also be a $4^s$-central subgroup (see Theorem [2.3]).

**Lemma 3.4.** Let $G$ be a d-generated 4-central powerful 2-group with the ΩEP. Then there exists a normal subgroup $N$ of $G$ such that $|G : N| \leq 2^d$ and $N$ is a $4^s$-central powerful 2-group with the ΩEP. Furthermore, if $C$ is a cyclic central subgroup of $G$ of size bigger than 2, then $N \cdot C$ is also a $4^s$-central subgroup with the ΩEP.

**Proof.** Let $\pi : G/G^2 \to G^2/G^4$ be the homomorphism defined by $\pi(xG^2) = x^2G^4$. Let $V$ be the kernel of $\pi$ and let $W$ be a complement of $V$ in $G/G^2$. Since $\pi$ is surjective, the restriction $\pi_W : W \to G^2/G^4$ is an isomorphism. Let $x_1G^2, \ldots, x_lG^2$ be generators of $W$ and put $N = \langle x_1, \ldots, x_l \rangle$. By construction, $N^2 = G^2$ and in particular, $N$ is a normal subgroup of $G$. Furthermore, $[N, N] \leq [G, G] \leq G^4 \cong N^4$ and thus, $N$ is a powerful subgroup of $G$.

Let us check that $N$ is $4^s$-central. Note that $\Omega_1(N^2) \subseteq \Omega_1(N)$ and we claim that $\Omega_1(N) = \Omega_1(N^2)$. Indeed, if $g \in N$ and $g^2 = 1$, then $gG^2 \in V$. As $W$ is a complement of the kernel $V$ of $\pi$, this can only happen if $gG^2 = 1G^2$. That is, $g$ must be contained in $G^2 = N^2$. Hence, $\Omega_1(N) = \Omega_1(G^2) \leq \Omega_1(G) \cong (\mathbb{Z}/2\mathbb{Z})^s$ for some $s \geq 1$, where the exponent of $\Omega_1(N)$ is strictly bigger than two. Thus, $N$ is a powerful $4^s$-central subgroup and so is $N \cdot C$. Finally, note that the ΩEP property is closed under taking subgroups and this finishes the proof.

### 4. A COUNTING ARGUMENT USING SPECTRAL SEQUENCES

In this section, we shall consider a central extension $C \to G \to Q$ of finite $p$-groups with $C$ a cyclic $p$-group and show that under mild assumptions, the cohomology algebra $H^*(G; \mathbb{F}_p)$ is determined up to a finite number of possibilities by the cohomology algebra $H^*(Q; \mathbb{F}_p)$. Henceforth, by a spectral sequence associated to a central extension, we mean the Lyndon-Hochschild-Serre spectral sequence (LHS for short) [1, VII.6]. The following result can be considered as a generalization of Theorem 4.2 in [5] and it is crucial in the proof of the main theorem.

**Theorem 4.1.** Let $p$ be a prime number, let $k, c, r, f$ be positive integers and suppose that

\begin{equation}
1 \to C \to G \to Q \to 1
\end{equation}

\begin{enumerate}
\item $k + c + r + f = 0$
\end{enumerate}
Lemma 3.4 or \( C \) is a central extension of finite \( p \)-group of size \( p^k \).

Proof. Let \( C = C_{p^k} \) denote a cyclic \( p \)-group of size \( p^k \). We start the proof by treating separately the \( p = 2 \) and \( k = 1 \) case.

Suppose that \( k \geq 1 \) if \( p \) is odd or \( k \geq 2 \) if \( p = 2 \). By Proposition 3.3 and Lemma 3.4 there exists a powerful \( p \)-central or \( 4 \)-central subgroup \( B \) of \( G \) with the \( \Omega EP \) for \( p \) odd or \( p = 2 \), respectively, that contains \( C \). The index of \( B \) in \( G \) is bounded in terms of \( p, r \) and \( f \). By Theorems 2.2(b) and 2.3 there exists a class \( \eta \in H^2(B; \mathbb{F}_p) \) such that \( \text{res}^B_C(\eta) \) is non-zero.

Suppose that \( p = 2 \) and \( k = 1 \), then \( C = C_2 \) is either contained in \( N \) constructed in Lemma 3.4 or \( C \) is not contained in \( N \). In the former case, considering \( B = N \), we can proceed as in the previous paragraph and there exists a class \( \eta \in H^2(B; \mathbb{F}_p) \) such that \( \text{res}^B_C(\eta) \) is non-zero. In the latter case, \( C \cap N = \{1\} \) and \( B = N \cdot C \cong N \times C_2 \). In this case there also exists a class \( \eta \in H^2(B; \mathbb{F}_p) \) such that \( \text{res}^B_C(\eta) \) is non-zero.

Now, following the arguments of the proof of [2, Lemma 3.2], we shall show that the spectral sequence \( E \) arising from

\[
1 \to C \to G \to Q \to 1,
\]

stops at most at page \( 2|G : B| + 1 \). As the extension \( C \) is central, the second page \( E_2 \) of the spectral sequence is isomorphic to the tensor product (as bigraded algebras),

\[
E_2^{s, t} \cong H^*(Q; \mathbb{F}_p) \otimes H^*(C; \mathbb{F}_p).
\]

Let \( \zeta = \text{Norm}^G_B(\eta) \in H^{2|G : B|}(G; \mathbb{F}_p) \), where \( \text{Norm}(\cdot) \) is the Evens norm map. Write \( |G : B| = p^n \) for short. From the usual Mackey formula for the double coset decomposition \( G = \bigcup_{x \in D} CxB \) with \( D = C \backslash G / B \),

\[
\text{res}^G_C(\text{Norm}^G_B(\eta)) = \text{res}^C_C(\zeta) = \prod_{x \in D} \text{Norm}^{C \cap x Bx^{-1}}_C(\text{res}^C_{C \cap x Bx^{-1}}(\eta)),
\]

we have that \( \text{res}^C_C(\zeta) \neq 0 \). The restriction map on cohomology from \( G \) to \( C \) is the edge homomorphism on the spectral sequence and thus, the image of the restriction map

\[
\text{res}^G_C : H^{2p^n}(G; \mathbb{F}_p) \to H^{2p^n}(C; \mathbb{F}_p)
\]

is isomorphic to \( E_\infty^{0, 2p^n} \). Let \( \zeta' \in E_\infty^{0, 2p^n} \) be an element representing \( \text{res}^G_C(\zeta) \). Suppose that \( t \geq 2p^n + 1 \) and let \( \mu \in E_t^{s, 0} \) with \( s = a(2p^n) + b \) where \( b < 2p^n \). We may write \( \mu = (\zeta')^a \mu' \) for some \( \mu' \in E_t^{s, b} \). Then,

\[
d_t(\mu) = d_t((\zeta')^a \mu') = d_t((\zeta')^a) \mu' + (\zeta')^a d_t(\mu') = (\zeta')^a d_t(\mu') = 0
\]

as \( d_t(\mu') \in E_t^{s + t, b + 1 - 2p^n} \) with \( b + 1 - 2p^n < 0 \). Then, \( \zeta' \) is a regular element on \( E_t^{s, 0} \), in turn, \( \zeta' \) is regular on \( H^*(G; \mathbb{F}_p) \) and \( d_t = 0 \) for all \( t \geq 2p^n + 1 \). Thus, the spectral sequence collapses at most at page \( 2p^n + 1 \). We shall finish the proof by following the proof of [2, Proposition 3.1].

Let \( \gamma_1, \ldots, \gamma_n \) be elements in \( H^*(C; \mathbb{F}_p) \) generating a polynomial subalgebra over which \( H^*(C; \mathbb{F}_p) \) is a finitely generated module. Note that if an element \( \gamma \otimes 1 \in H^*(C; \mathbb{F}_p) \otimes 1 \subset
$E_{n}^{0,*}$ survives to the $n^\text{th}$ page then, as $d_n$ is a derivation,
\[ d_n(\gamma^p) = p^\gamma p^\gamma d_n(\gamma) = 0. \]

Since the spectral sequence stops at page $2p^n + 1$, we must have that for all $i$, $\tau_i = \gamma_p^{2p^n+1}$ is a universal cycle. Analogously, let $\tau_{u+1}, \ldots, \tau_v$ be homogeneous parameters for $H^*(Q; F_p) \otimes 1 \subset E_n^{*,0}$. Then, $E_2$ is a finitely generated module over the polynomial subalgebra $W$ generated by $\tau_1, \ldots, \tau_u$. Moreover, $d_j$ is a $W$-module homomorphism because $d_j(\tau_i) = 0$ and thus, $E_{j+1}^{*,*}$ is finitely generated $W$-module. As Carlson says, such generators $\alpha_1, \ldots, \alpha_q$ of $E_{j+1}^{*,*}$ can be chosen to be homogeneous and the key point is that then $d_j$ is determined by $d_j(\alpha_1), \ldots, d_j(\alpha_v)$. For each $i$, there is only a finite number of choices for the images $d_j(\alpha_i)$. So, for all $j \geq 2$, there are finitely many choices for $d_j$ and for the $W$-modules and algebra structures of $E_{j+1}$. Since the spectral sequence stops after finitely many steps, there are finitely many $W$-module and algebra structures for $E_{\infty}^{*,*}$. Now, the result holds from [2] Theorem 2.1.

\[ \Box \]

5. Proof of the Main Result and Further Work

The aim of this section is to prove the Main theorem of this paper using the results in Sections 3 and 4.

**Theorem 5.1.** Let $p$ be a prime number and let $c$ and $d$ be natural numbers. Then, the number of possible isomorphism types for the mod $p$ cohomology algebra of a $d$-generated $p$-group of nilpotency class $c$ is bounded by a function depending only on $p$, $c$ and $d$.

**Proof.** Let $G$ be a $d$-generated $p$-group of nilpotency class $c$. Let $r$ denote the sectional rank of $G$ that depends on $d$ and $c$ (see Notation in Section 1). Then, by Lemma 3.1, there is a central series,
\[ G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_h = \{1\}, \]
where $h = h(c,r) \leq cr$ is a finite number (depending only on $c$ and $r$) and the quotients $G_i/G_{i+1}$ are cyclic. Then, for each $i = 1, \ldots, h-1$, consider the following central extension
\[ G_i/G_{i+1} \to G/G_{i+1} \to G/G_i, \]
where $G_i/G_{i+1} \cong C_{p^{k_i}}$ is a cyclic group. Note that the rank of $G_i/G_{i+1}$ is at most $r$ and that $G_i/G_i$ is a $p$-group of nilpotency class smaller than $c$. Then, the theorem easily follows by induction on $i$ and Theorem 4.1.

We finish this paper with some comments and further questions. Finite $p$-groups of fixed nilpotency class and finite $p$-groups of fixed coclass share the following property: they all have bounded rank and therefore, they contain a powerful $p$-central subgroup of bounded index. Using the correspondence with Lie algebras in [14], one can easily show that these $p$-groups contain a powerful $p$-central $p$-subgroup with the $\Omega EP$ for $p \geq 5$. It is natural to expect that this also holds for the primes $p = 2$ and $p = 3$.

**Conjecture 5.1.** Let $G$ be a finite $p$-group of fixed rank $r$. Then, there exists a subgroup $K$ of bounded index by a function depending on $p$ and $r$ such that it is powerful $p^\gamma$-central with the $\Omega EP$.

For $p$ odd, the group $K$ above has a particular type of mod $p$ cohomology algebra and powerful $4^\gamma$-central $2$-groups with the $\Omega EP$ should have the same cohomological structure.
Conjecture 5.2. Let $G$ be a powerful 4*-central 2-group with the ΩEP property and let $d = \text{rk}(\Omega_1(G))$. Then,

$$H^*(G; \mathbb{F}_2) \cong \Lambda(y_1, \ldots, y_d) \otimes \mathbb{F}_2[x_1, \ldots, x_d],$$

where the generators $y_i$ have degree one and the generators $x_i$ have degree two.

In [5] the authors conjectured that for a given integer $r$, there only finitely many possibilities for the mod $p$ cohomology algebra of a finite $p$-group of rank $r$. We expect that the previous two conjectures and proving that the Lyndon-Hochschild-Serre spectral sequence arising from the extension of $p$-groups of rank at most $r$,

$$1 \to N \to G \to C_p \to 1$$

collapses in a bounded number of steps, would prove Conjecture 5.2 in [5].

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