Hybrid black-hole binary initial data

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Abstract

Traditional black-hole (BH) binary puncture initial data is conformally flat. This unphysical assumption is coupled with a lack of radiation signature from the binary’s past life. As a result, waveforms extracted from evolutions of this data display an abrupt jump. In Kelly \textit{et al} (2010, \textit{Class. Quantum Grav.} \textbf{27} 114005), a new binary BH initial data with radiation content derived from post-Newtonian (PN) theory was adapted to puncture evolutions in numerical relativity. This data satisfies the constraint equations to the 2.5PN order, and contains a transverse-traceless ‘wavy’ metric contribution, violating the standard assumption of conformal flatness. Although the evolution contained less spurious radiation, there were undesirable features: unphysical horizon mass loss and large initial orbital eccentricity. Introducing a hybrid approach to the initial data evaluation, we significantly reduce these undesired features.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The field of numerical relativity (NR) has progressed at a remarkable pace since the breakthroughs of 2005 \cite{2-4}, with the first successful fully nonlinear dynamical numerical simulation of the inspiral, merger and ringdown of an orbiting black-hole binary (BHB) system. In particular, the ‘moving-punctures’ approach, developed independently by the NR groups at NASA/GSFC and at RIT, has now become the most widely used method in the field and has been successfully applied to evolve generic BHBs. This approach
regularizes a singular term in the spacetime metric and allows the black holes (BHs) to move across the computational domain. Since this breakthrough, BHB physics has rapidly matured into a critical tool for gravitational wave (GW) data analysis and astrophysics. Recent developments include: studies of the orbital dynamics of spinning BHBs [5–11], calculations of recoil velocities from the merger of unequal-mass BHBs [12–14], the surprising discovery that very large recoils can be acquired by the remnant of the merger of two spinning BHs [8, 15–30], empirical models relating the final mass and spin of the remnant with the spins of the individual BHs [31–38], and comparisons of waveforms and orbital dynamics of BHB inspirals with post-Newtonian (PN) predictions [39–47].

One of the important applications of NR is the generation of waveforms to assist GW astronomers in their search for, and analysis of, waveforms in the data collected by ground-based interferometers, such as LIGO [48] and VIRGO [49], and future missions, such as LCGT [50], LISA [51], ET [52] and DECIGO [53]. BHBs are particularly promising sources, with the final merger event producing a strong burst of GWs at a luminosity of $L_{GW} \sim 10^{22} L_\odot$, greater than the combined luminosity of all stars in the observable universe. The central goal of the field has been to develop the theoretical techniques, and perform numerical simulations, needed to explore the highly dynamical regions and thus generate GW signals from a representative sample of the full BHB parameter space. Accurate waveforms are important for extracting physical information about the binary system, such as the masses of the component BHs, as well as the magnitude and orientation of their spins.

With this in mind, we note a drawback shared by most present-day comparable-mass BHB simulations in three spatial dimensions: they are performed using conformally flat initial data given by the Bowen–York (BY) [54] prescription as applied by Brandt and Brügmann [55]. This prescription, while numerically convenient, lacks physical realism. The unique stationary vacuum BH solution to Einstein’s equations is the Kerr solution, which is not conformally flat for non-zero spin. We cannot approximate this with single-puncture spinning BY data without also including unphysical radiation.

Conversely, approximating an inspiralling BHB system with two-puncture BY data will leave out the gravitational radiation expected in physical situations—radiation inextricably linked to the past history that produced the inspiral. This is demonstrated in plots of extracted radiation from current simulations, where observers at a distance $R_{ext}$ from the binary only see a flat radiation profile for the first $t \approx R_{ext}$ of evolution time.

The PN initial data developed in this and preceding papers [1, 56, 57] is an attempt to address this shortcoming in initial data by incorporating to leading PN accuracy the GW content of a physical binary inspiral. The previous paper in this series, [57], demonstrated the evolution behavior of the data evolved with no numerical conditioning of the constraints. In summary, we noted that the extracted radiation agreed with physical expectations from the very start of the simulation (though the burst of junk radiation associated with puncture evolutions was not completely removed). We also encountered several related weaknesses in the new evolved data, including:

- very high eccentricity ($\sim 10\%$) in puncture trajectories, until around $100M$ before merger;
- (possibly related) an extremely slow stabilization of pre-merger horizon masses during evolution;
- large constraint violations when all the GW terms are included.

In this paper, we present hybrid BY–PN data with partial constraint conditioning, aimed at resolving these issues.

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4 This luminosity estimate is independent of the binary mass and takes into account that 3−10% of the total mass $M$ of the binary is radiated over a time interval of $\sim 100M$ [38].
In section 2, we summarize the standard conformally flat BY puncture initial data. In section 3, we review the theoretical work that led to the PN data, and discuss the encoding of the past history of the binary in section 4. In section 5, we present the new hybrid initial data, and demonstrate its improved numerical behavior. The numerical evolutions and comparison for various initial data are discussed in section 6. We conclude in section 7 with a discussion of future development and application of this data.

2. Bowen–York initial data

Initial data \{\gamma_{ij}, K_{ij}\} for a vacuum numerical evolution must satisfy the Hamiltonian and momentum constraints:

\[ R + K^2 - K^{ij}K_{ij} = 0, \]  
\[ \nabla_j(K^{ij} - \gamma^{ij}K) = 0, \]  

where \(\nabla_j\) is the covariant derivative compatible with \(\gamma_{ij}\), \(R\) is the three-Ricci scalar, and \(K \equiv \gamma^{ij}K_{ij}\) is the trace of the extrinsic curvature. Most NR groups use the puncture prescription of Brandt and Brügmann [55] with the BY extrinsic curvature [54]. In this prescription, the three-metric \(\gamma_{ij}\) is conformally flat:

\[ \gamma_{ij} = \psi^4 \delta_{ij}, \]  

and the Hamiltonian constraint, equation (1), becomes an elliptic equation for the conformal factor \(\psi\)

\[ \Delta \psi + \frac{1}{4} \psi K_{ij} \psi^{-7} = 0, \]  

where the conformal BY extrinsic curvature \(\hat{K}_{ij}\) already satisfies the momentum constraint, equation (2), for holes with arbitrary momentum and spin. Brandt and Brügmann’s insight to find the solution to equation (4) was to factor out the divergent parts of \(\psi\), leaving a well-behaved, simply-connected sheet on which to solve their modified constraint:

\[ \psi_{BY} = 1 + \sum_{a=1}^{2} \frac{m_a}{2r_a} + u, \]  

where BY refers hereafter to BY puncture data, \(r_a = |x - x_a|\), and \(m_a\) and \(x_a\) denote the ‘bare’ mass and location of each hole, respectively. In this prescription, only a single elliptic equation for \(u\) has to be solved, and \(u\) will be regular everywhere on the grid. Despite the great success in simplifying the form of the constraint equations, this prescription does not make use of any information to accurately describe the past evolution of the binary BHs. Conformal flatness, for example, prevents the astrophysically expected gravitational radiation from being included in the initial data.

3. Post-Newtonian initial data in ADM-TT gauge

PN techniques are considered to accurately represent an astrophysical system in the limit of slow motion/widely separated BHs. See Blanchet [58] and Schäfer [59] for a review of the PN approach.

In the canonical Hamiltonian formulation with point-like sources in general relativity, Ohta et al [60–62] investigated a class of coordinate systems in which the metric tensor becomes Minkowskian at spatial infinity, and developed the 2PN-order calculation. In particular, they found a compact way of expressing a compatible coordinate condition with the help of the
ADM formalism. By applying a transverse-traceless decomposition of the three-metric and its
conjugate momentum, they were able to encode the full information about the dynamics of the
canonical fields and particle variables in a reduced Hamiltonian, which is obtained by solving
the constraint equations only. To solve the constraints, we follow the steps of Schäfer [63].
Historically, the 2PN-order calculation was completed by Damour and Schäfer [64]. Shortly
afterward, Schäfer [63, 65] included the 2.5PN radiation–reaction terms in this Hamiltonian
approach.

Tichy et al [56] adapted the 2.5PN ADM-TT results to puncture initial data for numerical
relativity. We start then from the PN expression for the spatial metric, which differs from
conformal flatness by a radiative term $h_{ij}^{TT}$:

$$\gamma_{ij}^{\text{PN}} = \psi_{\text{PN}}^{4} \delta_{ij} + h_{ij}^{TT},$$

where the PN conformal factor is given by

$$\psi_{\text{PN}} = 1 + \sum_{a=1}^{2} \frac{E_{a}}{2r_{a}} + O(\epsilon^{6}),$$

where $\epsilon \equiv 1/c$ is the PN order parameter and

$$E_{a} = \epsilon^{2} m_{a} + \epsilon^{4} \left( \frac{p_{a}^{2}}{2m_{a}} - \frac{m_{1} m_{2}}{2r_{12}} \right).$$

Here, $p_{a}$ represents the linear momentum of each hole, and $r_{12}$ is the separation between
the holes.

The accompanying extrinsic curvature $K_{ij}^{\text{PN}}$ is related to the three-metric’s (trace-free)
conjugate momentum $\pi_{ij}^{\text{PN}}$:

$$\pi_{ij}^{\text{PN}} = \frac{1}{\psi_{\text{PN}}^{4}} \left( \epsilon^{3} \pi_{ij}^{(3)} + \epsilon^{5} \pi_{ij}^{(5)} \right) + O(\epsilon^{6}).$$

(See the complete equation (17) in [56].) Explicit expressions for $\pi_{ij}^{(3)}$ and $\pi_{ij}^{(5)}$ can be found
in [66]; the leading-PN-order term, $\pi_{ij}^{(3)}$, coincides with the standard BY linear-momentum
contribution for a nonspinning binary.

Near each particle, the spatial metric can be approximated by

$$\gamma_{ij}^{\text{PN}} \sim \left( 1 + \frac{E_{a}}{2r_{a}} \right)^{4} \delta_{ij} + O(1/r_{a}^{4}),$$

which is just the Schwarzschild metric in isotropic coordinates. For $r_{a} \to 0$, the
coordinate singularity is approached. This represents the inner asymptotically flat end of
the Schwarzschild metric in isotropic coordinates, or the puncture representation of the
Schwarzschild solution. This shows that if the metric form is kept as in equation (6), then
there is a BH centred on each particle.

The transverse-traceless part of the spatial three-metric, on the other hand, can be
constructed by imposing an outgoing-wave condition. We can rewrite the evolution equation
for $h_{ij}^{TT}$ as

$$h_{ij}^{TT} = -\square_{\text{ret}}^{-1} \delta_{ij}^{TT} \left[ \sum_{a=1}^{N} \frac{p_{ak} \phi_{ai}}{m_{a}} \delta(x - x_{a}) + \frac{1}{4} \phi_{k}^{(2)} \delta_{ij}^{(2)} \right],$$

where $\delta_{ij}^{TT}$ is the TT-projection operator as defined in [63]. Schäfer [63] suggested a ‘near
zone’ (NZ) approximation for $h_{ij}^{TT}$, by replacing the retarded inverse d’Alembertian in equation
with an inverse Laplacian:
\[ h_{ij}^{TT} = -\left[ \Delta^{-1} + (\Box_{\text{ret}}^{-1} - \Delta^{-1}) \right] h_{ij}^{TT[\cdots]} \]
\[ = h_{ij}^{TT(\text{NZ})} + h_{ij}^{TT(\text{remainder})} + O(\epsilon^5). \]  

The explicit form of this NZ approximation was given by Jaranowski and Schäfer [66]. A few years later, Kelly et al. [57] completed the picture for nonspinning BHs by determining the ‘remainder’ TT term, \( h_{ij}^{TT(\text{remainder})} \), to 2PN order. The structure of the remainder term divides into three segments, according to time of evaluation:

\[ h_{ij}^{TT(\text{remainder})} = h_{ij}^{TT(\text{present})} + h_{ij}^{TT(\text{retarded})} + h_{ij}^{TT(\text{interval})}. \]

For each field point where \( h_{ij}^{TT(\text{remainder})} \) is to be evaluated, the ‘present’ term is evaluated using the particle positions and momenta at \( t = 0 \). The ‘retarded’ term is evaluated using positions and momenta at the retarded time of each source particle relative to the field point. The ‘interval’ term is an integral over the particles’ paths from the retarded time to the present.

The present-time piece almost completely cancels the NZ solution. The kinetic terms (i.e. those involving particle momenta) cancel exactly, while potential terms (involving the particles’ separation \( r_{12} \)) are strongly suppressed. The retarded-time piece reduces to the quadrupole solution for a nonspinning binary as \( r_{12}/r \to 0 \) where \( r \) denotes the field distance. The interval piece is too difficult to do in generality: we must integrate numerically.

The ADM-TT PN data introduced here has several attractive properties, which we briefly restate here. The three-metric and extrinsic curvature expressions are easily found. Unlike harmonic coordinates, ADM-TT has no logarithmic divergences. For a single BH, the data reduces to the Schwarzschild metric in isotropic coordinates. Up to 1.5PN order, the data coincides with the (unsolved) puncture approach—conformally flat, with the BY extrinsic curvature. The trace of the extrinsic curvature vanishes up to 3PN order. The Hamiltonian constraint decouples from the momentum constraints.

4. Encoding the past history of the binary

To evaluate this initial data for a given separation at \( t = 0 \), we must know not only the particle momenta at \( t = 0 \), but also the position and momentum of each particle over the past history of the binary, at least from the retarded time relative to the most distant grid point in the numerical domain. That is, the larger the numerical domain, the further back in time we must look for the information needed to fill in the \( h_{ij}^{TT(\text{remainder})} \) fields at these spatially distant points.

To supply this position and momentum information, we evolve the Hamiltonian equations of motion, using a standard Taylor PN Hamiltonian and flux function evaluated at 3PN and 3.5PN order, respectively. The set of equations that we use in this paper can be found in [67]. This evolution is started at a suitably large separation (\( r_{12} \sim 40M \) is easily sufficient for current purposes), and the separation and orbital phase are saved for later interpolation. We then apply a shift in time and orbital frequency to match the initial conditions of the fully numerical evolution.

This shifted trajectory data is used directly to evaluate the present time fields. We also use it to evaluate the two retarded times (one for each particle) at each field point using a nonlinear Newton-search algorithm. Then we can evaluate the retarded time fields, and numerically integrate the interval terms.
5. Hybrid initial data

The similarity between the BY puncture data and the leading-PN-order terms of the ADM-TT data may be the key to further constraining the initial data. We use the rationale that the largest Hamiltonian violation contributions for field points closer to the punctures are due to the PN conformal factor, $\psi_{\text{PN}}$, since $h_{ij}^{\text{TT}}$ is relatively much smaller. We propose then a hybrid initial data prescription in which we first solve the Hamiltonian constraint for the traditional BY puncture initial data and then rescale and superpose the higher order PN terms to this solution. More specifically, the BY extrinsic curvature is used as usual to source the Hamiltonian constraint for a conformally flat spatial metric:

$$\Delta \psi_{\text{BY}} + \frac{1}{8} K_{ij}^{\text{BY}} K_{ij}^{\text{BY}} \psi_{\text{BY}}^{-7} = 0.$$  \hspace{1cm} (14)

The solution then follows from the puncture trick and is given by

$$\psi_{\text{BY}} = 1 + \frac{1}{2} \left( \frac{m_1}{r_1} + \frac{m_2}{r_2} \right) + u,$$  \hspace{1cm} (15)

where $u$, the regular part of the solution, is determined by the numerical solution of the resulting Poisson-like elliptic equation. The functional similarity of this solution to the one provided by the PN expansion, equation (7), is quite striking. The singular part, or the Brill–Lindquist part [68] if you wish, has the same functional form in each case, differing only in how the bare masses of particles and BHs are treated, and the missing regular contribution $u$ in the PN conformal factor. In this new approach the ADM canonical quantities are then rescaled by the BY solution, $\psi_{\text{BY}}$, instead of the PN conformal factor $\psi_{\text{PN}}$:

$$\gamma_{ij} = \psi_{\text{BY}}^4 \delta_{ij} + h_{ij}^{\text{TT}},$$  \hspace{1cm} (16)

$$\pi^{ij} = \frac{1}{\psi_{\text{BY}}^4} \left( \epsilon^3 \pi^{ij}_{(3)} + \epsilon^5 \pi^{ij}_{(5)} \right),$$  \hspace{1cm} (17)

where we note that $\pi^{ij}_{(3)}$ corresponds exactly to the spinless BY contribution. The transverse-traceless part of the metric, $h_{ij}^{\text{TT}}$, and the PN contributions to the conjugate momentum, $\pi^{ij}_{(3)}$ and $\pi^{ij}_{(5)}$, are just those from the standard PN data.

5.1. Constraint violations

Looking at the violation of the constraint equations, figure 1, it becomes clear that the PN metric becomes inaccurate close to the punctures, at least when compared to the traditional BY data. The long-dash-dotted brown curve represents the constraint violation for the PN metric in the ADM-TT gauge, labeled in the figure and in all plots hereafter as ‘ADMTT PN’. The (purple) vertical dotted lines indicate the apparent horizon boundary around one of the punctures, here the one located on the $y$-axis at $y = -4M$. Focusing our attention on the plot of the left panel, the Hamiltonian constraint violation, we can observe a negative violation outside of the horizon for the PN data. This can be interpreted as, and has the effect of, an unphysical negative mass gravitating around each BH. Unfortunately, the hybrid approach introduced in the last section did not reduce this violation considerably, as the green dashed line in the plot shows. We decided then to investigate a way to improve the data approximation closer to the apparent horizons, leading to the second crucial element of the approach proposed here: the use of attenuation functions.
Figure 1. Violation of the Hamiltonian constraint, equation (1), (left panel) and momentum constraint, equation (2), (right panel) around a puncture located on the y-axis at \( y = -4M \). The (purple) dotted vertical lines indicate the location of the apparent horizon. The (red) solid line represents the violation for the traditional BY data. The long-dash-dotted (brown) line describes the violation for the ADM-TT PN data, while the (green) dashed and the (blue) long-dashed lines correspond to the Hybrid and attenuated Hybrid function, respectively. We keep the line styles consistently the same in all figures hereafter to facilitate comparison.

5.2. Attenuation function

Since the PN approximation of the metric and extrinsic curvature is inaccurate for the inner or internal zone, i.e. the region around each BH in binary systems where \( r_1 \ll r_{12} \) and \( r_2 \ll r_{12} \), we need to consider a different approach to construct the initial data around that region. In a sequence of papers, Yunes et al [69, 70] and Johnson-McDaniel et al [71] have added tidal deformations to the inner zone metric, and asymptotically matched the PN metric of the NZ metric to construct the data for the whole initial time slice.

Here, we treat the inner-zone metric with a simpler approach. We discard the PN contribution around each BH by applying an attenuation function \( F \) defined as follows (see figure 2 for its functional form along an interval on the y-axis):

\[
F = \frac{1}{4} \left[ \tanh \left( \ln \left( \frac{r_1}{a_1 m_1} \right) \right) + 1 \right] \left[ \tanh \left( \ln \left( \frac{r_2}{a_2 m_2} \right) \right) + 1 \right]
\]

\[
= \left( 1 + \frac{a_1^2 m_1^2}{r_1^2} \right)^{-1} \left( 1 + \frac{a_2^2 m_2^2}{r_2^2} \right)^{-1},
\]

(18)

where \( a_i \) is a constant parameter. Our hybrid initial data approach with the use of an attenuation function then becomes

\[
\gamma_{ij} = \psi^4_{BY} \delta_{ij} + F h_{ij}^{TT},
\]

(20)

Note that when we apply this attenuation function to \( h_{ij}^{TT} \), the correction at large distances is at higher PN order than we treat in this paper. At large distances, a Taylor expansion with respect to \( m_i / r_i \) (i.e. a PN expansion) yields

\[
F h_{ij}^{TT} = h_{ij}^{TT} \left( 1 - \frac{a_1^2 m_1^2}{r_1^2} - \frac{a_2^2 m_2^2}{r_2^2} + \cdots \right).
\]

Therefore, the corrections due to the attenuation function \( F \) are at 2PN higher order than the leading order of \( h_{ij}^{TT} \).
Figure 2. The attenuation function \( F \), equation (18), along the \( y \)-axis for a binary with \( m_1 = m_2 = M/2 \) at \( y_1 = -y_2 = 4M \), and \( a_t = 1 \).

\[
\pi^{ij} = \frac{1}{\psi_{BY}} \left( \epsilon^{3} \pi^{ij}_{(3)} + F \epsilon^{5} \pi^{ij}_{(5)} \right).
\]  

(21)

This means that we ignore both the tidal deformations of each BH and the higher-order PN terms in order to consider only the BY initial data around each hole. This prescription is similar to the zeroth-order asymptotic matching between the inner and NZ metrics described in Yunes et al [69]. The effect of this procedure is a considerable reduction of the Hamiltonian constraint violation, as the blue long-dashed curve indicates on the left panel of figure 1.

While the Hamiltonian constraint violation is improved—and we are going to discuss its effect on the data’s evolution later on—the momentum constraint violation seems to be unaffected by these two key ingredients of our procedure. Both the hybrid and the attenuated hybrid data seem to affect the momentum constraint residuals only negligibly when compared to the PN data. This is possibly because the metric considered is very close to conformal flatness in the inner zone, so the tidal deformations induced by the other hole are not considered; and also because the non-conformal piece of the data satisfies the constraints only up to 2.5PN order. Would a higher-order PN approximation improve the violation observed? While the answers to these issues are out of this paper’s scope, we are also going to discuss later how the lack of these ingredients may be affecting the data evolution.

Finally, we should note that the way the ADM-TT PN and BY data are combined in our hybrid procedure above is not unique. We were guided, though, by its simplicity and the similarities between the data, particularly in the inner zone, where the PN approximation is not accurate. Before discussing any further properties of this data, let us say a few words on how the numerical experiments were prepared and performed. This is the topic of the next section.

6. Numerical evolution

The BHB studied in these experiments had a mass ratio of 1, was nonspinning, with punctures initially located on the \( y \)-axis at \( \pm 4M \). In this framework we use the puncture approach [55] along with the \textsc{twopunctures} thorn [72] to calculate the BY initial data. We evolved these BHB data-sets using the \textsc{lahz} [73] implementation of the moving puncture formalism [3, 4] with the conformal factor \( W = \sqrt{\lambda} = \exp(-2\phi) \) suggested by [10] as a dynamical variable. For the runs presented here we use centered, eighth-order finite differencing in space [74] and an RK4 time integrator (note that we do not upwind the advection terms).

We obtain accurate, convergent waveforms and horizon parameters by evolving this system in conjunction with a modified 1 + log lapse and a modified Gamma-driver shift condition...
Figure 3. Phase accuracy. This figure shows the difference in phase for the (2, 2) mode of the radiative Weyl scalar \( \psi_4 \) at different resolutions. We compare the difference of the medium and low resolutions with that of the high and medium ones. We see that the phase error is below the threshold of 0.05 at \( M \Omega = 0.2 \) (vertical dotted green line) suggested by the NRAR collaboration [83].

[3, 75, 76], and an initial lapse \( \alpha(t = 0) = 2/(1 + \psi_{4L}^4) \). The lapse \( \alpha \) and shift \( \beta^a \) are evolved with

\[
(\partial_t - \beta^i \partial_i) \alpha = -2 \alpha K, \\
\partial_t \beta^a = \frac{3}{4} \Gamma^a - \eta \beta^a ,
\]

(22)

where \( \eta \) is constant.

We use the Cactus/Einstein Toolkit code [77, 78] to provide the parallel infrastructure and the Carpet [79] mesh refinement driver to provide a ‘moving boxes’ style mesh refinement. In this approach refined grids of fixed size are arranged about the coordinate centers of both holes. The Carpet code then moves these fine grids through the computational domain by following the trajectories of the two punctures.

We use the AHFINDERDIRECT thorn [80] to locate apparent horizons. We measure the magnitude of the horizon spin \( S \) using the Isolated Horizon algorithm detailed in [81]. Once we have the horizon spin, we can calculate the horizon mass \( M_{\text{AH}} \) via the Christodoulou formula [82]:

\[
M_{\text{AH}} = \sqrt{M_{\text{irr}}^2 + \frac{S^2}{4M_{\text{irr}}^2}} ,
\]

(23)

where \( M_{\text{irr}} = \sqrt{A/(16\pi)} \) is the irreducible mass and \( A \) is the surface area of the horizon.

For the runs in this paper, we used a grid hierarchy spanning nine levels of refinement. We ran three different sets of grid-hierarchy resolutions in order to determine the waveform accuracy, shown in figure 3. The low-, medium- and high-resolution hierarchies have mesh spacings of \( h_c = [5M, 4M, 3.2M] \) for the coarsest grid and \( h_f = [M/51.2, M/64, M/80] \) for the finest ones, respectively. The mesh spacings in the wave-extraction region are \( h_{\text{ext}} = [1.25M, 1M, 0.8M] \). We apply reflection-boundary symmetry at the \( z = 0 \) plane and ‘\( \pi \)-symmetry’ (that is, particle-exchange symmetry) at the \( x = 0 \) plane. The outer boundary is located at 400M.
6.1. Apparent horizon mass

The first result from the evolution of the data introduced in the previous section that we would like to discuss concerns the apparent-horizon mass. It was reported earlier [1] for the PN data that the mass was not being conserved with time—some sort of mass ‘leak’ was plaguing the data evolution. We confirm this problem in figure 4, as we can observe the long time it takes for the PN irreducible mass to relax toward a constant value.

The first step we took when trying to tackle this issue was to introduce the attenuation function $F$, equation (18), for the PN data. As the (purple, with a short time evolution) dotted curve in the figure reveals, this approach alone was not successful. Only when we introduced the hybrid approach to the BHB initial data did we see an improvement in this unwanted feature. Now the mass conservation seems much more accurate, leaving only the presence of some bumps in the curve.

We believe that the presence of these bumps in the data is related to unphysical negative mass, resulting from the initial Hamiltonian constraint violation. Since these bumps appear approximately every half-orbital period, or roughly when the punctures cross the y-axis, it may be reasonable to assume that the initial Hamiltonian violations persist in the region around the initial puncture locations or in the holes’ ‘wakes’, occasionally being absorbed by the passing hole. The attenuation function seems to reduce this effect slightly, however only when associated with the hybrid data.

6.2. Eccentricity

Several methods have been proposed to estimate the eccentricity of numerically evolved binary inspirals; for recent examples, see [39, 84, 85]. We estimate the eccentricity of all four different
data sets by first $\chi^2$-fitting a polynomial function, $r_{\text{fit}}(t)$, to the binary coordinate separation, $r_{\text{NR}}(t)$, as a function of time. We then subtract the numerical data from the fitted function and normalize the difference with the same fitted function:

$$e_r(t) = \frac{r_{\text{NR}}(t) - r_{\text{fit}}(t)}{r_{\text{fit}}(t)} \simeq e_r \cos(\Omega_1 r t + \phi_0).$$  \hspace{1cm} (24)

The interpolating polynomial must be monotonically decreasing with no extrema in the fitting interval. We use the highest possible polynomial order satisfying these constraints. The assumption behind this procedure is a secular quasi-adiabatic shrinking of the orbital separation. The amplitude of the sinusoidal part extracted out of this orbital decay then provides an estimate of the orbital eccentricity.

Specifically for these runs, where the initial separation is $r_0 = 8M$, we fit a second-order polynomial curve to the numerical data. We were careful to choose a time to stop fitting much earlier than the coalescence time. Usually we would eliminate the initial gauge effects from the data, and start fitting somewhere around $t = 40M$ or later, as was the case for both instances of hybrid data presented in figure 5. However, this procedure turned out to be very sensitive to the initial fit separation for the BY and PN data, so we decided to include this initial transient in order to establish only an upper bound estimate for the eccentricity. We believe that this will not be necessary for binaries initially farther apart.

Amplitude readings from the eccentricity estimator plot, as well as from a bare inspection of the coordinate separation as a function of time, figure 5, indicate that the BY data has an eccentricity of the order of $10^{-2}$. Clearly the PN eccentricity is around five times this value. The hybrid data alone did not have a considerable impact on the eccentricity, possibly due to the fact that the momentum constraint violations were not improved as well. We could reduce the high-eccentricity effects somewhat by introducing the attenuation function, but this problem remains open.

6.3. Waveforms

We illustrate in figure 6 the presence of the realistic wave content in the hybrid initial data. The figure shows the $z = 0$ slice of the data for the real part of the Weyl scalar $\psi_4$ multiplied by the areal radius $R_{\text{areal}}$. We show the traditional BY data for comparison in the right panel of the same figure.
Figure 6. Comparison between the wavy pattern at $t = 0$ present in $R_{\text{real}} \Re (\psi_4)$ for the hybrid initial data (left) and the traditional Bowen–York data (right).

Figure 7. Real and imaginary parts of the $(2, 2)$ mode of the Weyl scalar $\psi_4$ resulting from the evolution of Bowen–York, ADM-TT PN, hybrid and attenuated hybrid initial data. The extraction radius is $R_{\text{ext}} = 90M$.

Like the wavy ADM-TT PN data, the hybrid data significantly reduces the amplitude of the spurious radiation when compared to the traditional BY data, as is demonstrated in figure 7 for the real and imaginary parts of the $(2, 2)$ mode of the Weyl scalar $\psi_4$. The burst amplitude reduction is quite similar between the three data sets: ADM-TT PN, hybrid, and attenuated hybrid. This strongly suggests that the higher-order conformally curved PN terms have only a small effect on the initial burst of spurious radiation. We believe then that the neglected tidal effects may play a crucial role in the elimination of this spurious initial data radiation.

In figure 8, we compare the amplitude (left panel) and phase (right panel) of the $(2, 2)$ mode. We can observe a clear reduction in the burst amplitude and a dramatic improvement
in the phase for all cases, compared to BY data. Note that there is a decrease in the amplitude of the hybrid data, suggesting that the binary is cast into a larger orbital separation. However, the use of an attenuation function greatly reduces this effect. For the $\psi_4(4,4)$ mode we only achieve a visible reduction of the junk radiation in the amplitude with the attenuated hybrid data, figure 9, which suggests that we need to improve the analytical approximation of the ADM canonical quantities in the inner zone if we want to accurately describe BH tidal effects. The disturbances in phase still seem greatly reduced.

6.4. Final state

Finally, we would like to comment on another advantage of the hybrid data over the PN data. As figure 10 shows, the final state (mass and spin) of the hybrid data is much closer to the BY data than the PN one. Additionally, the post-merger horizon mass shows a significantly larger drift over time for the PN data than for the hybrid data.
7. Conclusion and discussion

Motivated by the benefits that the PN initial data could bring in bridging the fields of numerical and analytical relativity, we have introduced a hybrid approach to the initial-data evaluation. We were able to significantly reduce the unphysical horizon mass loss, as well as reduce the eccentricity present in the PN data in ADM-TT coordinates, both undesired features previously reported in [1]. The method consists of solving the traditional BY puncture data conformal factor and using this conformal factor to rescale the ADM canonical quantities as they appear in the PN data. The high eccentricity still present in the data was reduced by the use of an attenuation function in the BH inner zone, effectively decreasing the contributions of the higher-order PN terms and the transverse-traceless part of the metric to the canonical conjugate momentum and three-metric, respectively.

One possible objection to the data-construction procedure presented here is that its hybrid construction makes it difficult to unambiguously isolate and correct sources of error: an observed problem may be due to shortcomings in the PN terms, in the hybridization procedure, or in the attenuation functions. We argue, however, that such issues are already present in the more standard BY puncture data used by many groups: initial momentum choices derived from PN theory do not produce generically quasicircular numerical orbits, and the initial relaxation of BY curvature, with gauge pulses and uncertainty in the horizon masses, make these problems difficult to diagnose analytically. It does not seem that our more complicated data approach has added any qualitatively new problems; rather, it was designed to ameliorate one existing problem—poor GW content of the initial data.

We plan to continue our studies with two lines of research. First, as a short-term project, we would like to reduce the eccentricity of the hybrid data even further by exploring different attenuation functions and using the iterative eccentricity reduction procedure described in Pfeiffer et al [86]. We plan to use larger orbital separations to facilitate this modeling and, consequently, the eccentricity reduction.

The second plan of research, a long-term one, is two-fold. On the analytic side, we would like to explore alternative analytic approximations to the metric and extrinsic curvature in the BH inner zones, where the point-particle approximation breaks down. Our plan is to match the PN metric to a perturbed, boosted Schwarzschild BH, in the manner of Yunes et al [69, 70], with a trumpet topology in quasi-isotropic coordinates [87–89]; this should provide
an analytic solution with smaller constraint violations and less spurious radiation. On the numerical side, we will implement a generic constraint solver for conformally curved data, and use the ADM-TT PN data, and possibly other analytic approximations, as free data for this solver. In this direction, Lovelace [90] was able to reduce the junk radiation by using superposed-boosted-Schwarzschild initial data. In the same token, Lovelace et al [91, 92] showed that a superposed-Kerr-Schild initial data could provide BHBs with nearly-extremal spins. We hope to encode the past history of the binary better and provide initial data free of junk radiation by adopting a constraint-satisfying conformally curved metric.

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