AZUMAYA MONADS AND COMONADS

BACHUKI MESABLISHVILI, TBILISI
AND
ROBERT WISBAUER, DÜSSELDORF

Abstract. The definition of Azumaya algebras over commutative rings \( R \) require the tensor product of modules over \( R \) and the twist map for the tensor product of any two \( R \)-modules. Similar constructions are available in braided monoidal categories and Azumaya algebras were defined in these settings. Here we introduce Azumaya monads on any category \( A \) by considering a monad \( F \) on \( A \) endowed with a distributive law \( \lambda : FF \to FF \) satisfying the Yang-Baxter equation (BD-law). This allows to introduce an opposite monad \( F^\lambda \) and a monad structure on \( FF^\lambda \). For an Azumaya monad we impose the condition that the canonical comparison functor induces an equivalence between the category \( A \) and the category of \( FF^\lambda \)-modules. Properties and characterisations of these monads are studied, in particular for the case when \( F \) allows for a right adjoint functor. Dual to Azumaya monads we define Azumaya comonads and investigate the interplay between these notions.

In braided categories \((V, \otimes, I, \tau)\), for any \( V \)-algebra \( A \), the braiding induces a BD-law \( \tau_{A,A} : A \otimes A \to A \otimes A \) and \( A \) is called left (right) Azumaya, provided the monad \( A \otimes - \) (resp. \( - \otimes A \)) is Azumaya. If \( \tau \) is a symmetry, or if the category \( V \) admits equalisers and coequalisers, the notions of left and right Azumaya algebras coincide. The general theory provides the definition of coalgebras in \( V \). Given a cocommutative \( V \)-coalgebra \( D \), coalgebras \( C \) over \( D \) are defined as coalgebras in the monoidal category of \( D \)-comodules and we describe when these have the Azumaya property. In particular, over commutative rings \( R \), a coalgebra \( C \) is Azumaya if and only if the dual \( R \)-algebra \( C^* = \text{Hom}_R(C, R) \) is an Azumaya algebra.

Contents

Introduction 1
1. Preliminaries 3
2. Azumaya monads 10
3. Azumaya comonads 15
4. Azumaya algebras in braided monoidal categories 20
5. Azumaya coalgebras in braided monoidal categories 34
References 40

Introduction

Azumaya algebras \( A = (A, m, e) \) over a commutative ring \( R \) are characterised by the fact that the functor \( A \otimes_R - \) induces an equivalence between the category of \( R \)-modules and the category of \( (A, A) \)-bimodules. In this situation Azumaya algebras are

Key words and phrases. Azumaya algebras, category equivalences, monoidal categories, (co)monads.
separable algebras, that is, the multiplication $A \otimes_R A \to A$ splits as $(A, A)$-bimodule map.

Braided monoidal categories allow for similar constructions as module categories over commutative rings and so - with some care - Azumaya monoids (algebras) and Brauer groups can be defined for such categories. For finitely bicomplete categories this was worked out by J. Fisher-Palmquist in [8], for symmetric monoidal categories it was investigated by B. Pareigis in [21], and for braided monoidal categories the theory was outlined by F. van Oystaeyen and Y. Zhang in [29] and B. Femić in [7]. It follows from the observations in [21] that - even in symmetric monoidal categories - the category equivalence requested for an Azumaya monoid $A$ does not imply separability of $A$ (defined as for $R$-algebras).

In our approach to Azumaya (co)monads we focus on properties of monads and comonads on any category $A$ inducing equivalences between certain related categories. Our main tools are distributive laws between monads (and comonads) as used in the investigations of Hopf monads in general categories (see [17], [19]).

We begin by recalling basic facts about the related theory - including Galois functors - in Section 1. Then, in Section 2 we consider monads $F = (F, m, e)$ on any category $A$ endowed with a distributive law $\lambda : FF \to FF$ satisfying the Yang Baxter equation (BD-laws). The latter enables the definition of a monad $F^\lambda = (F^\lambda, m^\lambda, e^\lambda)$ where $F^\lambda = F$, $m^\lambda = m \cdot \lambda$, and $e^\lambda = e$. Furthermore, $\lambda$ can be considered as distributive law $\lambda : F^\lambda F \to FF^\lambda$ and this allows to define a monad structure on $FF^\lambda$. Then, for any object $A \in A$, $F(A)$ allows for an $FF^\lambda$-module structure, thus inducing a comparison functor $K : A \to A_{FF^\lambda}$. We call $F$ an Azumaya monad (in 2.3) if this functor is an equivalence of categories. Properties and characterisations of such monads are given, in particular for the case that they allow for a right adjoint functor (Theorem 2.10).

These notions lead to an intrinsic definition of Azumaya comonads as outlined in Section 3 where also the relationship between the Azumaya properties of a monad $F$ and a right adjoint comonad $R$ is investigated (Proposition 3.15). It turns out that for a Cauchy complete category $A$, $F$ is an Azumaya monad and $FF^\lambda$ is a separable monad if and only if $R$ is an Azumaya comonad and $G^nG$ is a separable comonad (Theorem 3.16).

In Section 4 our theory is applied to study Azumaya algebras in braided monoidal categories $(V, \otimes, I, \tau)$. Then, for any $V$-algebra $A$, the braiding induces a distributive law $\tau_{A,A} : A \otimes A \to A \otimes A$, and $A$ is called left (right) Azumaya if the monad $A \otimes - : V \to V$ (resp. $- \otimes A : V \to V$) is Azumaya. In [29], $V$-algebras which are both left and right Azumaya are used to define the Brauer group of $V$. We will get various characterisations for such algebras but will not pursue their role for the Brauer group. In braided monoidal categories with equalisers and coequalisers, the notions of left and right Azumaya algebras coincide (Theorem 4.19).

The results from Section 3 provide an extensive theory of Azumaya coalgebras in braided categories $V$ and the basics for this are described in Section 5. Besides the formal transfer of results known for algebras, we introduce coalgebras $C$ over cocommutative coalgebras $D$ and for this, Section 5 provides conditions which make them Azumaya. This extends the corresponding notions studied for coalgebras over cocommutative coalgebras in vector space categories by B. Torrecillas, F. van Oystaeyen and Y. Zhang in [28]. Over a commutative ring $R$, Azumaya coalgebras $C$ turn out to be
coseparable and are characterised by the fact that the dual algebra \( C^* = \text{Hom}(C, R) \) is an Azumaya \( R \)-algebra. Notice that coalgebras with the latter property were first studied by K. Sugano in [27].

1. Preliminaries

Throughout this section \( A \) will stand for any category.

1.1. Modules and comodules. For a monad \( T = (T, m, e) \) on \( A \), we write \( A_T \) for the Eilenberg-Moore category of \( T \)-modules and denote the corresponding forgetful-free adjunction by

\[ \eta_T, \varepsilon_T : \phi_T \dashv U_T : A_T \to A. \]

Dually, if \( G = (G, \delta, \varepsilon) \) is a comonad on \( A \), we write \( A_G \) for the Eilenberg-Moore category of \( G \)-comodules and denote the corresponding forgetful-cofree adjunction by

\[ \eta_G, \varepsilon_G : U_G \dashv \phi_G : A \to A_G. \]

For any monad \( T = (T, m, e) \) and an adjunction \( \eta, \varepsilon : T \dashv R \), there is a comonad \( R = (R, \delta, \varepsilon) \), where \( m \dashv \delta, \varepsilon \dashv e \) (mates) and there is an isomorphism of categories (e.g. [17])

\[ \Psi : A_T \to A_R, \quad (A, h) \mapsto (A, A \eta \to R_T(A) \xrightarrow{R(h)} R(A)). \]

Note that, for any \( (A, \theta) \in A_R \), \( \Psi^{-1}(A, \theta) = (A, T(A) \xrightarrow{F(\theta)} TR(A) \xrightarrow{\varepsilon_A} A) \).

1.2. Monad distributive laws. Given two monads \( T = (T, m, e) \) and \( S = (S, m', e') \) on \( A \), a natural transformation \( \lambda : TS \to ST \) is a (monad) distributive law of \( T \) over \( S \) if it induces commutativity of the diagrams

\[
\begin{array}{ccc}
S & \xrightarrow{eS} & TS \\
\downarrow e & & \downarrow \lambda \\
ST & \xrightarrow{S \varepsilon} & ST \\
\end{array}
\quad
\begin{array}{ccc}
TSS & \xrightarrow{\lambda S} & STS \\
\downarrow Tm' & & \downarrow S\lambda \\
STS & \xrightarrow{S\varepsilon} & SST \\
\end{array}
\quad
\begin{array}{ccc}
TTS & \xrightarrow{T\lambda} & TST \\
\downarrow mS & & \downarrow \lambda T \\
TSS & \xrightarrow{\lambda S} & STS \\
\end{array}
\quad
\begin{array}{ccc}
TTS & \xrightarrow{T\lambda} & TST \\
\downarrow mS & & \downarrow \lambda T \\
TSS & \xrightarrow{\lambda S} & STS \\
\end{array}
\]

Given a distributive law \( \lambda : TS \to ST \), the triple \( ST = (ST, m'm \cdot S\lambda T, e'e) \) is a monad on \( A \) (e.g. [1], [32]). Notice that the monad structure on \( ST \) depends on \( \lambda \) and if the choice of \( \lambda \) needs to be specified we write \( (ST)_\lambda \).

Furthermore, a distributive law \( \lambda \) corresponds to a monad \( \hat{S}_\lambda = (\hat{S}, \hat{m}, \hat{e}) \) on \( A_T \) that is lifting of \( S \) to \( A_T \) in the sense that

\[ U_T\hat{S} = SU_T, \quad U_T\hat{m} = m'U_T \quad \text{and} \quad U_T\hat{e} = e'U_T. \]

This defines the Eilenberg-Moore category \( (A_T)_{\hat{S}_\lambda} \) of \( \hat{S}_\lambda \)-modules whose objects are triples \( ((A, t), s) \), with \( (A, t) \in A_T \), \( (A, s) \in A_S \) with a commutative diagram

\[
\begin{array}{ccc}
TS(A) & \xrightarrow{\lambda A} & ST(A) \\
\downarrow T(s) & & \downarrow s(t) \\
T(A) & \xrightarrow{t} & A & \xleftarrow{s} & S(A).
\end{array}
\]
There is an isomorphism of categories \( P_\lambda : \mathcal{A}(ST)_\lambda \to (\mathcal{A}_T)_{S,\lambda} \) by the assignment

\[
(A, ST(A) \overset{\delta}{\to} A) \mapsto ((A, T(A) \overset{\epsilon_{T(A)}}{\to} ST(A) \overset{\delta}{\to} A), S(A) \overset{Se_A}{\to} ST(A) \overset{\delta}{\to} A),
\]

and for any \(((A, t), s) \in (\mathcal{A}_T)_{S,\lambda}^\lambda\),

\[
P_\lambda^{-1}((A, t), s) = (A, ST(A) \overset{S(t)}{\to} S(A) \overset{\delta}{\to} A).
\]

When no confusion can occur, we shall just write \( \hat{S} \) instead of \( \hat{S}_\lambda \).

1.3. Proposition. In the setting of 1.2, let \( \lambda : TS \to ST \) be an invertible monad distributive law.

(1) \( \lambda^{-1} : ST \to TS \) is again a monad distributive law;
(2) \( \lambda : TS \to ST \) can be seen as a monad isomorphism \((TS)_\lambda^{-1} \to (ST)_\lambda \) defining a category isomorphism

\[
\mathcal{A}_\lambda : \mathcal{A}(ST)_\lambda \to \mathcal{A}(TS)_{\lambda^{-1}}, \quad (A, ST(A) \overset{\delta}{\to} A) \mapsto (A, TS(A) \overset{\lambda}{\to} ST(A) \overset{\delta}{\to} A);
\]
(3) \( \lambda^{-1} \) induces a lifting \( \hat{T}_{\lambda^{-1}} : \mathcal{A}_S \to \mathcal{A}_S \) of \( T \) to \( \mathcal{A}_S \) and an isomorphism of categories

\[
\Phi : (\mathcal{A}_T)_{\hat{S}_\lambda} \to (\mathcal{A}_S)_{\hat{T}_{\lambda^{-1}}}, \quad ((A, t), s) \mapsto ((A, s), t),
\]

leading to the commutative diagram

\[
\begin{array}{ccc}
\mathcal{A}(ST)_\lambda & \xrightarrow{P_\lambda} & (\mathcal{A}_T)_{\hat{S}_\lambda} \\
\downarrow{\mathcal{A}_\lambda} & & \downarrow{\Phi} \\
\mathcal{A}(TS)_{\lambda^{-1}} & \xleftarrow{P_{\lambda^{-1}}} & (\mathcal{A}_S)_{\hat{T}_{\lambda^{-1}}}
\end{array}
\]

Proof. (1), (2) follow by \([11, \text{Lemma 4.2}]\), (3) is outlined in \([3, \text{Remark 3.4}]\). \( \Box \)

1.4. Comonad distributive laws. Given comonads \( \mathcal{G} = (G, \delta, \varepsilon) \) and \( \mathcal{H} = (H, \delta', \varepsilon') \) on \( \mathcal{A} \), a natural transformation \( \kappa : HG \to GH \) is a (comonad) distributive law of \( \mathcal{G} \) over \( \mathcal{H} \) if it induces commutativity of the diagrams

\[
\begin{array}{ccc}
HG & \xrightarrow{\kappa} & GH \\
\downarrow{\varepsilon G} & & \downarrow{G \varepsilon'} \\
G & \xrightarrow{\varepsilon H} & H \\
\end{array}
\quad
\begin{array}{ccc}
HGG & \xrightarrow{\kappa G} & GGH \\
\downarrow{\delta H} & & \downarrow{G \delta'} \\
GHG & \xrightarrow{\kappa H} & GHH
\end{array}
\]

Given this, the triple \((\mathcal{H}G)_\kappa = (HG, H\kappa G \cdot \delta' \delta, \varepsilon \varepsilon') \) is a comonad on \( \mathcal{A} \) (e.g. \([11, 32])\).

Also, the distributive law \( \kappa \) corresponds to a lifting of the comonad \( \mathcal{H} \) to a comonad \( \mathcal{H}_\kappa : \mathcal{A}_\mathcal{G} \to \mathcal{A}_\mathcal{G} \), leading to the Eilenberg-Moore category \((\mathcal{A}_\mathcal{G})_{\mathcal{H}_\kappa} \) of \( \mathcal{H}_\kappa \)-comodules.
whose objects are triples \(((A, g), h)\) with \((A, g) \in \mathcal{A}^G\) and \((A, h) \in \mathcal{A}^H\) with commutative diagram

\[
\begin{array}{ccc}
H(A) & \xrightarrow{h} & A \\
\downarrow{H(g)} & \quad & \quad & \downarrow{G(h)} \\
HG(A) & \xrightarrow{\kappa_A} & GH(A).
\end{array}
\]

There is an isomorphism of categories \(\mathcal{Q}_\kappa : \mathcal{A}^{(\mathcal{H}G)\kappa} \rightarrow (\mathcal{A}^G)\tilde{\kappa}\) given by

\[
(A, A \xrightarrow{\rho} HG(A)) \mapsto (A, A \xrightarrow{\rho} HG(A) \xrightarrow{\varepsilon_G(A)} G(A)), A \xrightarrow{\rho} HG(A) \xrightarrow{H(\varepsilon_A)} H(A),
\]

and for any \(((A, g), h) \in (\mathcal{A}^G)\tilde{\kappa}\),

\[
\mathcal{Q}_\kappa^{-1}((A, g), h) = (A, A \xrightarrow{h} H(A) \xrightarrow{H(g)} HG(A)).
\]

The following observations are dual to 1.3.

1.5. Proposition. In the setting of 1.4, let \(\kappa : HG \rightarrow GH\) be an invertible comonad distributive law.

1. Mixed distributive laws. Given a monad \(\mathcal{T} = (T, m, e)\) and a comonad \(\mathcal{G} = (G, \delta, \varepsilon)\) on \(\mathcal{A}\), a **mixed distributive law** (or **entwining**) from \(\mathcal{T}\) to \(\mathcal{G}\) is a natural transformation \(\omega : TG \rightarrow GT\) with commutative diagrams

\[
\begin{array}{ccc}
TG & \xrightarrow{T\omega} & TGT \\
\downarrow{T\varepsilon} & \quad & \quad & \downarrow{\varepsilon_T} \\
T & \xrightarrow{T\varepsilon} & GT
\end{array} \quad \begin{array}{ccc}
TTG & \xrightarrow{T\omega} & TGT \\
\downarrow{mG} & \quad & \quad & \downarrow{Gm} \\
TG & \xrightarrow{\omega T} & GT
\end{array}
\]

Given a mixed distributive law \(\omega : TG \rightarrow GT\) from the monad \(\mathcal{T}\) to the comonad \(\mathcal{G}\), we write \(\hat{\mathcal{G}}_\omega = (\hat{G}, \hat{\delta}, \hat{\varepsilon})\) for a comonad on \(\mathcal{A}_T\) lifting \(\mathcal{G}\) (e.g. [32, Section 5]).

It is well-known that for any object \((A, h)\) of \(\mathcal{A}_T\),
\[ \tilde{G}(A, h) = (G(A), G(h) \cdot \omega_A), \quad (\tilde{\delta})_{(A, h)} = \delta_A, \quad (\tilde{\varepsilon})_{(A, h)} = \varepsilon_A, \]

and the objects of \((\mathcal{A}_T)^\tilde{G}\) are triples \((A, h, \vartheta)\), where \((A, h) \in \mathcal{A}_T\) and \((A, \vartheta) \in \mathcal{A}_G\) with commuting diagram

\[
\begin{array}{ccc}
T(A) & \xrightarrow{h} & A \\
\downarrow{T(\vartheta)} & \nearrow{G(h)} \\
TG(A) & \xrightarrow{\omega_A} & GT(A).
\end{array}
\]

1.7. Distributive laws and adjoint functors. Let \(\lambda : TS \rightarrow ST\) be a distributive law of a monad \(T = (T, m, e)\) over a monad \(S = (S, m', e')\) on \(\mathcal{A}\). If \(T\) admits a right adjoint comonad \(R\) (with \(\overline{\eta}, \overline{\varepsilon} : T \dashv R\)), then the composite

\[ \lambda_\shuffle : SR \xrightarrow{\pi SR} RTSR \xrightarrow{R\lambda R} RSTR \xrightarrow{RSR} RS \]

is a mixed distributive law from \(S\) to \(R\) (e.g. [2], [17]) and the assignment

\[ (A, \nu : ST(A) \rightarrow A) \mapsto (A, h_\nu : S(A) \rightarrow A, \vartheta_\nu : A \rightarrow R(A)), \]

with

\[ h_\nu : S(A) \xrightarrow{S(e_A)} ST(A) \xrightarrow{\nu} A, \quad \vartheta_\nu : A \xrightarrow{\pi_A} RT(A) \xrightarrow{R(e'_T(A))} RST(A) \xrightarrow{R(\nu)} R(A), \]

yields an isomorphism of categories \(\mathcal{A}_{(ST)\lambda} \simeq (\mathcal{A}_S)^{\tilde{\lambda}_{\shuffle}}\).

1.8. Invertible distributive laws and adjoint functors. Let \(\lambda : TS \rightarrow ST\) be an invertible distributive law of a monad \(T = (T, m, e)\) over a monad \(S = (S, m', e')\) on \(\mathcal{A}\). Then \(\lambda^{-1} : ST \rightarrow TS\) is a distributive law of the monad \(S\) over the monad \(T\) \([1.3]\), and if \(S\) admits a right adjoint comonad \(H\) (with \(\overline{\eta}, \overline{\varepsilon} : S \dashv H\)), then the previous construction can be repeated with \(\lambda\) replaced by \(\lambda^{-1}\). Thus the composite

\[ (\lambda^{-1})_\shuffle : TH \xrightarrow{\pi TH} HSTH \xrightarrow{H\lambda^{-1}H} HTSH \xrightarrow{HT} HT \]

is a mixed distributive law from the monad \(S\) to the comonad \(H\). Moreover, there is an adjunction \(\alpha, \beta : \tilde{S}_\lambda \dashv \tilde{H}_{(\lambda^{-1})}\) : \(\mathcal{A}_T \rightarrow \mathcal{A}_T\), where \(\tilde{S}_\lambda\) is the lifting of \(S\) to \(\mathcal{A}_T\) considered in \([1.2]\) (e.g. [10], Theorem 4) and the canonical isomorphism \(\Psi\) from \([1.1]\) yields the commutative diagram

\[
\begin{array}{ccc}
(\mathcal{A}_T)^{\tilde{S}_\lambda} & \xrightarrow{\Psi} & (\mathcal{A}_T)^{\tilde{H}_{(\lambda^{-1})}} \\
\downarrow{U_{\tilde{S}_\lambda}} & \nearrow{U_{\tilde{H}_{(\lambda^{-1})}}} \\
\mathcal{A}_T & \xrightarrow{=} & \mathcal{A}_T.
\end{array}
\]

Note that \(U_T(\alpha) = \overline{\eta}\) and \(U_T(\beta) = \overline{\varepsilon}\).

1.9. Entwinings and adjoint functors. For a monad \(T = (T, m, e)\) and a comonad \(G = (G, \delta, \varepsilon)\), consider an entwining \(\omega : TG \rightarrow GT\). If \(T\) admits a right adjoint comonad \(R\) (with \(\overline{\eta}, \overline{\varepsilon} : T \dashv R\)), then the composite

\[ \omega^\shuffle : GR \xrightarrow{\pi GR} RTGR \xrightarrow{R\omega R} RGTR \xrightarrow{RG\varepsilon} RG \]

is a comonad distributive law of \(G\) over \(R\) (e.g. [2], [17]) inducing a lifting \(\tilde{G}_\omega\) of \(G\) to \(\mathcal{A}_R\) and thus an Eilenberg-Moore category \((\mathcal{A}_R)^{\tilde{G}_\omega}\) of \(\tilde{G}_\omega\)-comodules whose objects
are triples \(((A, d), g)\) with commutative diagram

\[
\begin{array}{ccc}
G(A) & \xleftarrow{\ g \ } & A \xrightarrow{\ d \ } R(A) \\
\downarrow & & \downarrow Rg \\
GR(A) & \xrightarrow{\ \omega^A_R \ } & RG(A).
\end{array}
\]

The following notions will be of use for our investigations.

1.10. Monadic and comonadic functors. Let \(\eta, \varepsilon : F \dashv R : \mathbb{B} \to \mathbb{A}\) be an adjoint pair of functors. Then the composite \(RF\) allows for a monad structure \(RF\) on \(\mathbb{A}\) and the composite \(FR\) for a comonad structure \(FR\) on \(\mathbb{B}\). By definition, \(R\) is monadic and \(F\) is comonadic provided the respective comparison functors are equivalences,

\[
K_R : \mathbb{B} \to \mathbb{A}_{RF}, \quad B \mapsto (R(B), R(\varepsilon_B)),
\]

\[
K_F : \mathbb{A} \to \mathbb{B}_{FR}, \quad A \mapsto (F(A), F(\eta_A)).
\]

For an endofunctor we have, under some conditions on the category:

1.11. Lemma. Let \(F : \mathbb{A} \to \mathbb{A}\) be a functor that allows for a left and a right adjoint functor and assume \(\mathbb{A}\) to have equalisers and coequalisers. Then the following are equivalent:

(a) \(F\) is conservative;

(b) \(F\) is monadic;

(c) \(F\) is comonadic.

If \(\mathcal{F} = (F, m, e)\) is a monad, then the above are also equivalent to

(d) the free functor \(\phi_F : \mathbb{A} \to \mathbb{A}_F\) is comonadic.

Proof. Since \(F\) is a left as well as a right adjoint functor, it preserves equalisers and coequalisers. Moreover, since \(\mathbb{A}\) is assumed to have both equalisers and coequalisers, it follows from Beck’s monadicity theorem (see \([14]\)) and its dual that \(F\) is monadic or comonadic if and only if it is conservative.

(a)\(\Leftrightarrow\)(d) follows from \([16, \text{Corollary 3.12}]\). \(\Box\)

1.12. \(\mathcal{T}\)-module functors. Given a monad \(\mathcal{T} = (T, m, e)\) on \(\mathbb{A}\), a functor \(R : \mathbb{B} \to \mathbb{A}\) is said to be a (left) \(\mathcal{T}\)-module if there exists a natural transformation \(\alpha : TR \to R\) with \(\alpha \cdot eR = 1\) and \(\alpha \cdot mR = \alpha \cdot T\alpha\).

This structure of a left \(\mathcal{T}\)-module on \(R\) is equivalent to the existence of a functor \(\overline{R} : \mathbb{B} \to \mathbb{A}_T\) with commutative diagram (see \([6, \text{Proposition II.1.1}]\))

\[
\begin{array}{ccc}
\mathbb{B} & \xrightarrow{\ \overline{R} \ } & \mathbb{A}_T \\
\downarrow R & & \downarrow U_T \\
\mathbb{A} & \xrightarrow{\ T \ } & \mathbb{A}.
\end{array}
\]

If \(\overline{R}\) is such a functor, then \(\overline{R}(B) = (R(B), \alpha_B)\) for some morphism \(\alpha_B : TR(B) \to R(B)\) and the collection \(\{\alpha_B, B \in \mathbb{B}\}\) forms a natural transformation \(\alpha : TR \to R\) making \(R\) a \(\mathcal{T}\)-module. Conversely, if \((R, \alpha : TR \to R)\) is a \(\mathcal{T}\)-module, then \(\overline{R} : \mathbb{B} \to \mathbb{A}_T\) is defined by \(\overline{R}(B) = (R(B), \alpha_B)\).
For any $\mathcal{T}$-module $(R : \mathcal{B} \to \mathcal{A}, \alpha)$ admitting an adjunction $F \dashv R : \mathcal{B} \to \mathcal{A}$ with unit $\eta : 1 \to RF$, the composite

$$t_T : T \xrightarrow{\eta_T} TRF \xrightarrow{\alpha_F} RF$$

is a monad morphism from $\mathcal{T}$ to the monad $\mathcal{RF}$ on $\mathcal{A}$ generated by the adjunction $F \dashv R$ with unit $\eta : 1 \to RF$. If $t_T : T \to RF$ is an isomorphism (i.e. $A^{t_T}$ is an isomorphism), then $R$ is called a $\mathcal{T}$-Galois module functor. Since $R = A^{t_T} : K_R$ (see 1.10) we have (dual to [15] Theorem 4.4):

1.13. **Proposition.** The functor $\overline{R}$ is an equivalence of categories if and only if the functor $R$ is monadic and a $\mathcal{T}$-Galois module functor.

1.14. **\(G\)-comodule functors.** Given a comonad $\mathcal{G} = (G, \delta, \varepsilon)$ on a category $\mathcal{A}$, a functor $L : \mathcal{B} \to \mathcal{A}$ is a left $\mathcal{G}$-functor if there exists a natural transformation $\alpha : L \to GL$ with $\varepsilon L \cdot \alpha = 1$ and $\delta L \cdot \alpha = G\alpha \cdot \alpha$. This structure on $L$ is equivalent to the existence of a functor $L : \mathcal{B} \to \mathcal{AG}$ with commutative diagram (dual to 1.12)

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\mathcal{T}} & \mathcal{A}^\mathcal{G} \\
L \downarrow & & \downarrow U^\mathcal{G} \\
\mathcal{A} & \xrightarrow{i} & \mathcal{A}^\mathcal{G}
\end{array}$$

If a $\mathcal{G}$-functor $(L, \alpha)$ admits a right adjoint $S : \mathcal{A} \to \mathcal{B}$, with counit $\sigma : LS \to 1$, then (see Propositions II.1.1 and II.1.4 in [6]) the composite

$$t_L : LS \xrightarrow{\alpha_S} GLS \xrightarrow{G\sigma} G$$

is a comonad morphism from the comonad generated by the adjunction $L \dashv S$ to $\mathcal{G}$. $L : \mathcal{B} \to \mathcal{A}$ is said to be a $\mathcal{G}$-Galois comodule functor provided $t_L : LS \to G$ is an isomorphism.

Dual to Proposition 1.13 we have (see also [18], [19]):

1.15. **Proposition.** The functor $\overline{L}$ is an equivalence of categories if and only if the functor $L$ is comonadic and a $\mathcal{G}$-Galois comodule functor.

1.16. **Right adjoint for $L$.** If the category $\mathcal{B}$ has equalisers of coreflexive pairs and $L \dashv S$, the functor $\overline{L}$ (in 1.14) has a right adjoint $\overline{S}$, which can be described as follows (e.g. [6], [15]): With the composite

$$\gamma : S \xrightarrow{\eta_S} SLS \xrightarrow{S\tau} SG,$$

the value of $\overline{S}$ at $(A, \vartheta) \in \mathcal{A}^\mathcal{G}$ is given by the equaliser

$$\overline{S}(A, \vartheta) \xrightarrow{i(A, \vartheta)} S(A) \xrightarrow{S(\vartheta)} SG(A).$$

If $\overline{\sigma}$ denotes the counit of the adjunction $\overline{L} \dashv \overline{S}$, then for any $(A, \vartheta) \in \mathcal{A}^\mathcal{G}$,

$$U^\mathcal{G}(\overline{\sigma}(A, \vartheta)) = \sigma_A \cdot L(i(A, \vartheta)),$$

where $\sigma : LS \to 1$ is the counit of the adjunction $L \dashv S$. 


1.17. **Separable functors.** *(e.g. [23]*) A functor $F : \mathcal{A} \to \mathcal{B}$ between any categories is said to be *separable* if the natural transformation

$$F_{-,-} : \mathcal{A}(-,-) \to \mathcal{B}(U(-), U(-))$$

is a split monomorphism.

If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{D}$ are functors, then

(i) if $F$ and $G$ are separable, then $GF$ is also separable;

(ii) if $GF$ is separable, then $F$ is separable.

1.18. **Separable (co)monads.** *(2, 2.9)* Let $\mathcal{A}$ be any category.

(1) For a monad $\mathcal{F} = (F, m, e)$ on $\mathcal{A}$, the following are equivalent:

(a) $m$ has a natural section $\omega : F \to FF$ such that $Fm \cdot \omega F = \omega \cdot m = mF \cdot F\omega$;

(b) the forgetful functor $U\mathcal{F} : \mathcal{A}_F \to \mathcal{A}$ is separable.

(2) For a comonad $\mathcal{G} = (G, \delta, \varepsilon)$ on $\mathcal{A}$, the following are equivalent:

(a) $\delta$ has a natural retraction $\rho : GG \to G$ such that $\rho G \cdot G\delta = \delta \cdot \rho = G\rho \cdot \delta G$;

(b) the forgetful functor $U\mathcal{G} : \mathcal{A}_G \to \mathcal{A}$ is separable.

1.19. **Separability of adjoints.** *(2, 2.10)* Let $G : \mathcal{A} \to \mathcal{A}$ and $F : \mathcal{A} \to \mathcal{A}$ be an adjoint pair of functors with unit $\bar{\eta} : 1_\mathcal{A} \to FG$ and counit $\bar{\varepsilon} : GF \to 1_\mathcal{A}$.

(1) $F$ is separable if and only if $\bar{\eta} : 1_\mathcal{A} \to FG$ is a split monomorphism;

(2) $G$ is separable if and only if $\bar{\varepsilon} : GF \to 1_\mathcal{A}$ a split epimorphism.

Given a comonad structure $\mathcal{G}$ on $G$ with corresponding monad structure $\mathcal{F}$ on $F$ (see 1.1), there are pairs of adjoint functors

$\mathcal{A} \xrightarrow{\phi^F} \mathcal{A}_F$, $\mathcal{A}_F \xrightarrow{UF} \mathcal{A}$, $\mathcal{A} \xrightarrow{U^G} \mathcal{A}_G$, $\mathcal{A}_G \xrightarrow{\phi^G} \mathcal{A}$.

(1) $\phi^G$ is separable if and only if $\phi^F$ is separable.

(2) $U^G$ is separable if and only if $UF$ is separable and then any object of $\mathcal{A}_G$ is injective relative to $U^G$ and every object of $\mathcal{A}_F$ is projective relative to $UF$.

The following generalises criterions for separability given in [23 Theorem 1.2].

1.20. **Proposition.** Let $U : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$ be a pair of functors.

(i) If there exist natural transformations $1 \xrightarrow{\kappa} FU \xrightarrow{\kappa'} 1$ such that $\kappa' \cdot \kappa = 1$, then both $FU$ and $U$ are separable.

(ii) If there exist natural transformations $1 \xrightarrow{\eta} UF \xrightarrow{\eta'} 1$ such that $\eta' \cdot \eta = 1$, then both $UF$ and $F$ are separable.

**Proof.** (i) Inspection shows that

$$\mathcal{A}(-,-) \xrightarrow{(FU)_{-,-}} \mathcal{A}(FU(-), FU(-)) \xrightarrow{\mathcal{A}(\kappa, \kappa')} \mathcal{A}(-,-)$$

is the identify and hence $FU$ is separable. By 1.17, this implies that $U$ is also separable. (ii) is shown symmetrically. □
2. AZUMAYA MONADS

An algebra $A$ over a commutative ring $R$ is Azumaya provided $A$ induces an equivalence between $\mathcal{M}_R$ and the category $\mathcal{M}_A$ of $(A,A)$-bimodules. The construction uses properties of the monad $A \otimes_R -$ on $\mathcal{M}_R$ and the purpose of this section is to trace this notion back to the categorical essentials to allow the formulation of the basic properties for monads on any category. Throughout again $\mathcal{A}$ will denote any category.

2.1. Definitions. Given an endofunctor $F : \mathcal{A} \to \mathcal{A}$ on $\mathcal{A}$, a natural transformation $\lambda : FF \to FF$ is said to satisfy the Yang-Baxter equation provided it induces commutativity of the diagram

$$
\begin{array}{ccc}
FFF & F\lambda & FF \\
\lambda F & & \lambda F \\
\downarrow & & \downarrow \\
FFF & F\lambda & FF \\
\end{array}
$$

For a monad $\mathcal{F} = (F,m,e)$ on $\mathcal{A}$, a monad distributive law $\lambda : FF \to FF$ satisfying the Yang-Baxter equation is called a (monad) BD-law (see [11, Definition 2.2]). Here the interest in the YB-condition for distributive laws lies in the fact that it allows to define opposite monads and comonads.

2.2. Proposition. Let $\mathcal{F} = (F,m,e)$ be a monad on $\mathcal{A}$ and $\lambda : FF \to FF$ a BD-law.

1. $\mathcal{F}^\lambda = (F^\lambda, m^\lambda, e^\lambda)$ is a monad on $\mathcal{A}$, where $F^\lambda = F$, $m^\lambda = m \cdot \lambda$ and $e^\lambda = e$.

2. $\lambda$ defines a distributive law $\lambda : \mathcal{F}^\lambda \mathcal{F} \to \mathcal{F}^\lambda \mathcal{F}$ making $\mathcal{F}^\lambda \mathcal{F} = (FF, m, e)$ a monad where $m = mm^\lambda \cdot F\lambda F : FFFF \to FF$, $e := ee : 1 \to FF$.

3. The composite $FFF \xrightarrow{F\lambda} FFF \xrightarrow{Fm} FF \xrightarrow{m} F$ defines a left $\mathcal{F}^\lambda \mathcal{F}$-module structure on the functor $F : \mathcal{A} \to \mathcal{A}$.

4. There is a comparison functor $\overline{K}_F : \mathcal{A} \to \mathcal{A}_{F^\lambda}$ given by

$$
A \mapsto (F(A), FFF(A) \xrightarrow{F(m\lambda)} FFF(A) \xrightarrow{F(m\lambda)} FF(A) \xrightarrow{m\lambda} F(A)).
$$

Proof. (1) is easily verified (e.g. [3] Remark 3.4], [17] Section 6.9]).

2.3. Definition. A monad $\mathcal{F} = (F,m,e)$ on any category $\mathcal{A}$ is said to be Azumaya provided it allows for a BD-law $\lambda : FF \to FF$ such that the comparison functor $\overline{K}_F : \mathcal{A} \to \mathcal{A}_{F^\lambda}$ is an equivalence of categories.

2.4. Proposition. If $\mathcal{F}$ is an Azumaya monad on $\mathcal{A}$, then the functor $F$ admits a left adjoint.
Proof. With our previous notation we have the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\mathcal{K}_F} & \mathbb{A}_{F^{\lambda}} \\
F & \downarrow & \downarrow U_{F^{\lambda}} \\
\mathbb{A} & \xrightarrow{\mathcal{U}} & \mathbb{A}.
\end{array}$$

Since $U_{F^{\lambda}} : \mathbb{A}_{F^{\lambda}} \to \mathbb{A}$ always has a left adjoint, and since $\mathcal{K}_F$ is an equivalence of categories, the composite $F = U_{F^{\lambda}} \cdot \mathcal{K}_F$ has a left adjoint. \(\square\)

This observation allows for a first characterisation of Azumaya monads.

2.5. Theorem. Let $\mathcal{F} = (F, m, e)$ be a monad on $\mathbb{A}$, $\lambda : FF \to FF$ a BD-law. The following are equivalent:

(a) $F$ is an Azumaya monad;
(b) the functor $F : \mathbb{A} \to \mathbb{A}$ is monadic and the left $\mathcal{F} \mathcal{F}^{\lambda}$-module structure on $F$ defined in Proposition 2.2 is Galois;
(c) the functor $F : \mathbb{A} \to \mathbb{A}$ is monadic (with some adjunction $\eta, \varepsilon : L \dashv F$) and the composite (as in 1.12)

$$t_{\mathcal{F}} : FF \xrightarrow{F \eta} FFL \xrightarrow{F \lambda} FFL \xrightarrow{FmL} FFL \xrightarrow{mL} FL$$

is an isomorphism of monads $\mathcal{F} \mathcal{F}^{\lambda} \to \mathcal{T}$, where $\mathcal{T}$ is the monad on $\mathbb{A}$ generated by this adjunction $L \dashv F$.

Proof. That (a) and (b) are equivalent follows from Proposition 1.15.
(b)\(\leftrightarrow\)(c) In both cases, $F$ is monadic and thus $F$ allows for an adjunction, say $L \dashv F$ with unit $\eta : 1 \to FL$. Write $\mathcal{T}$ for the monad on $\mathbb{A}$ generated by this adjunction. Since the left $\mathcal{F} \mathcal{F}^{\lambda}$-module structure on the functor $F$ is the composite

$$FFF \xrightarrow{F \lambda} FFF \xrightarrow{Fm} FFF \xrightarrow{m} F,$$

it follows from 1.12 that the monad morphism $t_{\mathcal{F}} : \mathcal{F} \mathcal{F}^{\lambda} \to \mathcal{T}$ induced by the diagram

is the composite

$$t_{\mathcal{F}} : FF \xrightarrow{F \eta} FFL \xrightarrow{F \lambda} FFL \xrightarrow{FmL} FFL \xrightarrow{mL} FL.$$}

Thus $F$ is $\mathcal{F} \mathcal{F}^{\lambda}$-Galois if and only if $t_{\mathcal{F}}$ is an isomorphism. \(\square\)

2.6. The isomorphism $\mathbb{A}_{F^{\lambda}} \simeq (\mathbb{A}_{F^{\lambda}})$. According to 1.2, for any BD-law $\lambda : FF \to FF$, the assignment

$$(A, FF(A) \xrightarrow{\varrho} A) \mapsto ((A, F(A) \xrightarrow{e_{F(A)}} FF(A) \xrightarrow{\varrho} A), F(A) \xrightarrow{F \varrho} FF(A) \xrightarrow{\varrho} A)$$
yields an isomorphism of categories $\mathcal{P}_\lambda : \hat{A}_{F,F^\lambda} \rightarrow (\hat{A}_{F^\lambda})_{F^\lambda}$, where for any $((A, h), g) \in (\hat{A}_{F,F^\lambda})_{F}$,

$$\mathcal{P}_\lambda^{-1}((A, h), g) = (A, FF(A) \xrightarrow{Fh} F(A) \xrightarrow{g} A).$$

There is a comparison functor $K = K_F : \hat{A} \rightarrow (\hat{A}_{F,F^\lambda})_{F}$,

$$A \mapsto ((F(A), FF(A) \xrightarrow{\lambda A} FF(A) \xrightarrow{mA} F(A)), FF(A) \xrightarrow{mA} F(A)),$$

with $K_F = \mathcal{P}_\lambda^{-1}K_F$ and commutative diagram

\[
\begin{array}{ccc}
\hat{A} & \xrightarrow{K_F} & (\hat{A}_{F,F^\lambda})_{F} \\
\downarrow{\phi_F} & & \downarrow{\mathcal{P}_\lambda^{-1}} \\
\hat{A}_{F,F^\lambda} & \xrightarrow{U_{\hat{F}}} & \hat{A}_{F^\lambda} \\
\end{array}
\]

**Proof.** Direct calculation shows that

$$\mathcal{P}_\lambda K_F(A) = ((F(A), FF(A) \xrightarrow{\lambda A} FF(A) \xrightarrow{mA} F(A)), FF(A) \xrightarrow{mA} F(A)),$$

for all $A \in \hat{A}$. $\square$

It is obvious that $K_F : \hat{A} \rightarrow (\hat{A}_{F,F^\lambda})_{F}$ is an equivalence (i.e. $F$ is Azumaya) if and only if $K_F : \hat{A} \rightarrow (\hat{A}_{F,F^\lambda})_{F}$ is an equivalence. To apply Proposition [1, 13] to the functor $K_F$, we will need a functor left adjoint to $\phi_{F^\lambda}$ whose existence is not a consequence of the Azumaya condition. For this the invertibility of $\lambda$ plays a crucial part.

2.7. **Proposition.** Let $F = (F, m, e)$ be a monad on $A$ with an invertible BD-law $\lambda : FF \rightarrow FF$.

(1) $\lambda^{-1} : F^\lambda F \rightarrow F^\lambda F$ is a distributive law inducing a monad $(F^\lambda F)_{\lambda^{-1}} = (FF, m, e)$ where

$$m = m^\lambda m \cdot F\lambda^{-1}F : FFFF \rightarrow FF, \quad e = ee : 1 \rightarrow FF,$$

and $\lambda$ is an isomorphism of monads $(F^\lambda F)_{\lambda^{-1}} \rightarrow (FF^\lambda)_{\lambda}$.

(2) There is an isomorphism of categories

$$\Phi : (\hat{A}_{F,\lambda})_{\hat{F}_\lambda} \rightarrow (\hat{A}_{F^\lambda})_{(\hat{F}^\lambda)_{\lambda^{-1}}}, \quad ((A, h), g) \mapsto ((A, g), h).$$

(3) $\lambda^{-1}$ induces a comparison functor $K'_F : \hat{A} \rightarrow (\hat{A}_{F^\lambda})_{(\hat{F}^\lambda)_{\lambda^{-1}}} \simeq \hat{A}_{(F^\lambda F)_{\lambda^{-1}}}$,

$$A \mapsto ((F(A), FF(A) \xrightarrow{mA} F(A)), FF(A) \xrightarrow{\lambda A} FF(A) \xrightarrow{mA} F(A)),$$

with commutative diagrams

\[
\begin{array}{ccc}
\hat{A} & \xrightarrow{K'_F} & (\hat{A}_{F^\lambda})_{(\hat{F}^\lambda)_{\lambda^{-1}}} \\
\downarrow{\phi_{F^\lambda}} & & \downarrow{\Phi_{(\hat{F}^\lambda)_{\lambda^{-1}}}} \\
\hat{A}_{F^\lambda} & \xrightarrow{U_{(\hat{F}^\lambda)_{\lambda^{-1}}}} & \hat{A}_{(\hat{F}^\lambda)_{\lambda^{-1}}} \\
\end{array}
\]

\[
\begin{array}{ccc}
\hat{A} & \xrightarrow{K_F} & (\hat{A}_{F^\lambda})_{\lambda^{-1}} \\
\downarrow{\phi_F} & & \downarrow{\Phi_{(\hat{F}^\lambda)_{\lambda^{-1}}}} \\
\hat{A}_{F,F^\lambda} & \xrightarrow{U_{(\hat{F}^\lambda)_{\lambda^{-1}}}} & \hat{A}_{(\hat{F}^\lambda)_{\lambda^{-1}}} \\
\end{array}
\]
Proof. (1), (2) follow by Proposition \([1.3]\) (3) is shown similarly to \([2.6]\) \[\square\]

For \(\lambda\) invertible, it follows from the diagrams in the Sections \([2.6] [2.7]\) that \(F\) is an Azumaya monad if and only if the functor
\[
K'_{F\lambda} : \mathbb{A} \rightarrow (\mathbb{A}_{F\lambda})_{(\overline{\mathcal{F}}_{\lambda})_{\lambda^{-1}}}
\]
is an equivalence of categories.

Note that if \(\lambda : \mathcal{F}\mathcal{F} \rightarrow \mathcal{F}\mathcal{F}\) is a BD-law, then \(\lambda\) can be seen as a BD-law \(\lambda : \mathcal{F}^\lambda \mathcal{F}^\lambda \rightarrow \mathcal{F}^\lambda \mathcal{F}^\lambda\), and it is not hard to see that the corresponding comparison functor
\[
K_{F\lambda} : \mathbb{A} \rightarrow (\mathbb{A}_{(F\lambda)})(\overline{\mathcal{F}}_{\lambda})_{\lambda}
\]
takes \(A \in \mathbb{A}\) to
\[
(F(A), FFF(A) \xrightarrow{F(\lambda)_A} FFF(A) \xrightarrow{F((m^\lambda)_{A})} FF(A) \xrightarrow{(m^\lambda)_A} F(A)).
\]

Now, if \(\lambda^2 = 1\), then \(\lambda = \lambda^{-1}\) and \((\mathcal{F}^\lambda)^\lambda = \mathcal{F}\). Thus, the category \((\mathbb{A}_{(F\lambda)})(\overline{\mathcal{F}}_{\lambda})_{\lambda}\) can be identified with the category \((\mathbb{A}_{\mathcal{F}\lambda})_{(\overline{\mathcal{F}}_{\lambda})_{\lambda^{-1}}}\). Modulo this identification, the functor \(K'_{F\lambda}\) corresponds to the functor \(K_{F\lambda}\). It now follows from the preceding remark:

2.8. Proposition. Let \(\mathcal{F} = (F, m, e)\) be a monad on \(\mathbb{A}\) with a BD-law \(\lambda : FF \rightarrow FF\). If \(\lambda^2 = 1\), then the monad \(\mathcal{F}\) is Azumaya if and only if the monad \(\mathcal{F}^\lambda\) is so.

2.9. Azumaya monads with right adjoints. Let \(\mathcal{F} = (F, m, e)\) be a monad with an invertible BD-law \(\lambda : FF \rightarrow FF\). Assume \(F\) to admit a right adjoint functor \(R\), with \(\eta : F \dashv R\), inducing a comonad \(\mathcal{R} = (R, \delta, \varepsilon)\) (see \([1.3]\)). Since \(\lambda : \mathcal{F}^\lambda \mathcal{F} \rightarrow \mathcal{F} \mathcal{F}^\lambda\) is an invertible distributive law, there is a comonad \(\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(\lambda^{-1})\) on \(\mathbb{A}_{\mathcal{F}\lambda}\) lifting the comonad \(\mathcal{R}\) and is right adjoint to the monad \(\widehat{\mathcal{F}}\) (see \([1.7]\)) yielding a category isomorphism
\[
\Psi_{\mathcal{F}\lambda} : (\mathbb{A}_{\mathcal{F}\lambda})_{\overline{\mathcal{F}}_{\lambda}} \rightarrow (\mathbb{A}_{\mathcal{F}\lambda})_{\widehat{\mathcal{R}}},
\]
where for any \(((A, h), g) \in (\mathbb{A}_{\mathcal{F}\lambda})_{\overline{\mathcal{F}}_{\lambda}}\),
\[
\Psi_{\mathcal{F}\lambda}((A, h), g) = ((A, h), \overline{g})\quad\text{with}\quad \overline{g} : A \xrightarrow{\overline{\eta}_{F(A)}} RF(A) \xrightarrow{R(g)} R(A),
\]
and a commutative diagram (see \([1.4]\))
\[
(2.2)
\begin{array}{ccc}
A & \xrightarrow{K} & (\mathbb{A}_{\mathcal{F}\lambda})_{\overline{\mathcal{F}}_{\lambda}} & \xrightarrow{\Psi_{\mathcal{F}\lambda}} & (\mathbb{A}_{\mathcal{F}\lambda})_{\widehat{\mathcal{R}}} \\
& & \phi_{\mathcal{F}\lambda} & \downarrow U_{\overline{\mathcal{F}}_{\lambda}} & \downarrow U_{\widehat{\mathcal{R}}} \\
\mathbb{A}_{\mathcal{F}\lambda} & = & \mathbb{A}_{\mathcal{F}\lambda} & \xrightarrow{U_{\mathcal{F}\lambda}} & \mathbb{A}_{\mathcal{F}\lambda}.
\end{array}
\]

Putting \(K := A \xrightarrow{K} (\mathbb{A}_{\mathcal{F}\lambda})_{\overline{\mathcal{F}}_{\lambda}} \xrightarrow{\Psi_{\mathcal{F}\lambda}} (\mathbb{A}_{\mathcal{F}\lambda})_{\widehat{\mathcal{R}}}\), one has for any \(A \in \mathbb{A}\),
\[
K(A) = ((F(A), m_A \cdot \lambda_A), R(m_A) \cdot \overline{\eta}_{F(A)}).
\]
So the \(A\)-component \(\alpha_A\) of the induced \(\widehat{\mathcal{R}}\)-comodule structure \(\alpha : \phi_{\mathcal{F}\lambda} \rightarrow \widehat{\mathcal{R}} \phi_{\mathcal{F}\lambda}\) on the functor \(\phi_{\mathcal{F}\lambda}\) induced by the commutative diagram \((2.2)\) (see \([1.4]\)) is the composite
\[
\alpha_A : F(A) \xrightarrow{\overline{\eta}_{F(A)}} RFF(A) \xrightarrow{R(m_A)} RF(A).
\]
It then follows that for any \((A, h) \in \mathbb{A}_F\), the \((A, h)\)-component \(t_{(A,h)}\) of the corresponding comonad morphism \(t : \phi_F U_F \to \hat{\mathcal{R}}\) (see 1.14) is the composite
\[
t_{(A,h)} : F(A) \xrightarrow{\eta_{F(A)}} RFF(A) \xrightarrow{R(m_A)} RF(A) \xrightarrow{R(h)} R(A).
\]

These observations lead to the following characterisations of Azumaya monads.

2.10. **Theorem.** Let \(\mathcal{F} = (F, m, e)\) be a monad on \(\mathbb{A}\), \(\lambda : FF \to FF\) an invertible BD-law, and \(\mathcal{R}\) a comonad right adjoint to \(F\) (with \(\eta, \varepsilon : F \dashv R\)). Then the following are equivalent:

(a) \(F\) is an Azumaya monad;
(b) (i) \(\phi_F \lambda\) is comonadic and
(ii) \(\phi_F \lambda\) is \(\hat{\mathcal{R}}\)-Galois, that is,
\[t_{(A,h)}\ in \ (2.3) \ is \ an \ isomorphism \ for \ any \ (A, h) \in \mathbb{A}_F, \ or \ the \ composite\]
\[\chi : FF \xrightarrow{\pi_{FF}} RFFF \xrightarrow{RmF} RFF \xrightarrow{R\lambda} RFF \xrightarrow{Rm} RF\ is \ an \ isomorphism.\]

**Proof.** Recall first that the monad \(F^\lambda\) is of effective descent type means that \(\phi_F \lambda\) is comonadic. By Proposition 1.15, the functor \(K\) making the triangle (2.2) commute is an equivalence of categories (i.e., the monad \(F\) is Azumaya) if and only if the monad \(F^\lambda\) is of effective descent type and the comonad morphism \(t : \phi_F \lambda U_F \to \hat{\mathcal{R}}\) is an isomorphism. Moreover, according to [19, Theorem 2.12], \(t\) is an isomorphism if and only if for any object \(A \in \mathbb{A}\), the \(\phi_F \lambda(A)\)-component \(t_{\phi_F \lambda(A)} : F(\phi_F \lambda(A)) \to R\phi_F \lambda(A)\) is an isomorphism. Using now that \(\phi_F \lambda(A) = (F(A), m_A^\lambda = m_A \cdot \lambda_A)\), it is easy to see that \(\chi_A = t_{\phi_F \lambda(A)}\) for all \(A \in \mathbb{A}\). This completes the proof.

The existence of a right adjoint of the comparison functor \(K\) can be guaranteed by conditions on the base category.

2.11. **Right adjoint for \(K\).** With the data given above, assume \(\mathbb{A}\) to have equalisers of coreflexive pairs. Then

(1) the functor \(K : \mathbb{A} \to (\mathbb{A}_F)^{\mathcal{R}}\) (see 2.9) admits a right adjoint \(R : (\mathbb{A}_F)^{\mathcal{R}} \to \mathbb{A}\) whose value at \(((A, h), \vartheta) \in (\mathbb{A}_F)^{\mathcal{R}}\) is the equaliser
\[
\begin{array}{ccc}
R((A,h),\vartheta) & \xrightarrow{i_{((A,h),\vartheta)}} & A \\
\downarrow & \searrow & \downarrow \vartheta \\
RF(A) & \underset{\pi_A}{\rightarrow} & R(A) \\
\end{array}
\]

(2) for any \(A \in \mathbb{A}\), \(RK(A)\) is the equaliser
\[
\begin{array}{ccc}
RK(A) & \xrightarrow{i_{\pi(A)}} & F(A) \\
\downarrow & \nearrow & \downarrow \pi_F(A) \\
RFF(A) & \underset{R(m_A), \pi_F(A)}{\rightarrow} & RF(A) \\
\end{array}
\]

\[R(m_A) \cdot \pi_F(A) \to RF(A)\]
Proof. (1) According to 1.16, $R((A, h), \vartheta)$ is the object part of the equaliser of

\[
\begin{array}{ccc}
A & \xrightarrow{\vartheta} & R(A),
\end{array}
\]

where $\gamma$ is the composite $U_F \xrightarrow{U_F \phi \lambda} U_F \phi \lambda U_F = U_F \xrightarrow{U_F \lambda} U_F \hat{R}$. It follows from the description of $t$ that $\gamma((A, h))$ is the composite

\[
\begin{array}{ccc}
A & \xrightarrow{e_A} & F(A) & \xrightarrow{\eta F(A)} & RF(A) & \xrightarrow{R(m_A)} & RF(A) & \xrightarrow{R(h)} & R(A)
\end{array}
\]

which is just the composite $R(h) \cdot \eta_A$ since

- $\eta F(A) \cdot e_A = RF(e_A) \cdot \eta_A$ by naturality of $\eta$,
- $m_A \cdot F(e_A) = 1$ because $e$ is the unit for $F$.

(2) For any $A \in \mathbb{A}$, $K(A)$ fits into the diagram (2.2). □

2.12. Definition. Write $F_F$ for the subfunctor of the functor $F$ determined by the equaliser of the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{Rm \cdot \eta F} & RF & \xrightarrow{Rm} & RF F
\end{array}
\]

We call the monad $F$ central if $F_F$ is (isomorphic to) the identity functor.

Since $R$ is right adjoint to the functor $K$, $K$ is fully faithful if and only if $RK \simeq 1$.

2.13. Theorem. Assume $\mathbb{A}$ to admit equalisers of coreflexive pairs. Let $F = (F, m, e)$ be a monad on $\mathbb{A}$, $\lambda : FF \to FF$ an invertible BD-law, and $R$ a comonad right adjoint to $F$. Then the comparison functor $K : \mathbb{A} \to (\mathbb{A}_F)^{\hat{R}}$ is

(i) full and faithful if and only if the monad $F$ is central;
(ii) an equivalence of categories if and only if the monad $F$ is central and the functor $R$ is conservative.

Proof. (i) follows from the preceding proposition.

(ii) Since $F$ is central, the unit $\eta : 1 \to RK$ of the adjunction $K \dashv R$ is an isomorphism by (i). If $\varepsilon$ is the counit of the adjunction, then it follows from the triangular identity $R \varepsilon \cdot \eta R = 1$ that $R \varepsilon$ is an isomorphism. Since $R$ is assumed to be conservative (reflects isomorphisms), this implies that $\varepsilon$ is an isomorphism, too. Thus $K$ is an equivalence of categories. □

3. Azumaya comonads

Following the pattern for monads we introduce the corresponding definitions for comonads. Again $\mathbb{A}$ denotes any category. The following results and definitions are dual to those in the preceding section.

3.1. Definition. For a comonad $G = (G, \delta, \varepsilon)$ on $\mathbb{A}$, a comonad distributive law $\kappa : GG \to GG$ (see 1.4) satisfying the Yang-Baxter equation is called a comonad BD-law or just a BD-law if the context is clear.
3.2. **Proposition.** Let $\mathcal{G} = (G, \delta, \varepsilon)$ be a comonad on $\mathbb{A}$ with BD-law $\kappa : GG \rightarrow GG$.

1. $\mathcal{G}^{\kappa} = (G^{\kappa}, \delta^{\kappa}, \varepsilon^{\kappa})$ is a comonad on $\mathbb{A}$, where $G^{\kappa} = G$, $\delta^{\kappa} = \kappa \cdot \delta$ and $\varepsilon^{\kappa} = \varepsilon$.

2. $\kappa$ defines a comonad distributive law $\kappa : \mathcal{G}G^{\kappa} \rightarrow G^{\kappa}\mathcal{G}$ making the triple $\mathcal{G}G^{\kappa} = (GG, \delta, \varepsilon)$ a comonad with

   \[ \delta : GG \xrightarrow{\delta_{\kappa}} GGGG \xrightarrow{G_{\kappa}G} GGGG, \quad \varepsilon : GG \xrightarrow{\varepsilon_{\kappa}} 1. \]

3. The composite $G \xrightarrow{\delta} GG \xrightarrow{G\delta} GGG \xrightarrow{G_{\kappa}} GGG$ defines a left $\mathcal{G}G^{\kappa}$-comodule structure on the functor $G : \mathbb{A} \rightarrow \mathbb{A}$.

4. There is a comparison functor $\mathcal{K}_{\kappa} : \mathbb{A} \rightarrow \mathbb{A}^{\mathcal{G}^{\kappa}}$ given by

   \[ A \mapsto (G(A), G(A) \xrightarrow{\delta_{A}} GG(A) \xrightarrow{G_{\kappa}A} GGGG(A)). \]

Comonad BD-laws are obtained from monad BD-laws by adjunctions (see [17, 7.4]):

3.3. **Proposition.** Let $\mathcal{F} = (F, m, e)$ be a monad on $\mathbb{A}$ and $\lambda : FF \rightarrow FF$ a BD-law. If $F$ has a right adjoint $R$, then there is a comonad $R = (R, \delta, \varepsilon)$ (where $m \dashv \delta$, $e \dashv \varepsilon$) with a comonad YB-distributive law $\kappa : RR \rightarrow RR$. Moreover, $\lambda$ is invertible if and only if $\kappa$ is so.

3.4. **Definition.** A comonad $\mathcal{G} = (G, \delta, \varepsilon)$ on a category $\mathbb{A}$ is said to be **Azumaya** provided it allows for a (comonad) BD-law $\kappa : GG \rightarrow GG$ such that the comparison functor $\mathcal{K}_{\kappa} : \mathbb{A} \rightarrow \mathbb{A}^{\mathcal{G}^{\kappa}}$ is an equivalence.

3.5. **Proposition.** If $\mathcal{G}$ is an Azumaya comonad on $\mathbb{A}$, then the functor $G$ admits a right adjoint.

This leads to a first characterisation of Azumaya comonads.

3.6. **Theorem.** Consider a comonad $\mathcal{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$ with a comonad BD-law $\kappa : GG \rightarrow GG$. The following are equivalent:

(a) $\mathcal{G}$ is an Azumaya comonad;

(b) the functor $G : \mathbb{A} \rightarrow \mathbb{A}$ is comonadic and the left $\mathcal{G}G^{\kappa}$-comodule structure on $G$ defined in Proposition 3.2 is Galois;

(c) the functor $G : \mathbb{A} \rightarrow \mathbb{A}$ is comonadic (for some adjunction $G \dashv R$ with counit $\sigma : GR \rightarrow 1$) and the composite

   \[ GGR \xrightarrow{G_{\kappa}R} GGGR \xrightarrow{G_{\kappa}G} GGGG \xrightarrow{G_{\kappa}A} GGGG \xrightarrow{\varepsilon_{\kappa}} GG, \]

   is an isomorphism of comonads $\mathcal{H} \rightarrow \mathcal{G}^{\kappa}$, where $\mathcal{H}$ is the comonad on $\mathbb{A}$ generated by the adjunction $G \dashv R$.

3.7. **The isomorphism** $\mathbb{A}^{\mathcal{G}^{\kappa}} \simeq (\mathbb{A}^{\mathcal{G}^{\kappa}})^{\tilde{G}}$. Write $\tilde{\mathcal{G}}$ for the lifting of the comonad $\mathcal{G}$ to $\mathbb{A}^{\mathcal{G}^{\kappa}}$ corresponding to the distributive law $\kappa : \mathcal{G}G^{\kappa} \rightarrow G^{\kappa}\mathcal{G}$. Then (see 1.4), the assignment

   \[ (A, A \xrightarrow{\rho} GG(A)) \mapsto ((A, A \xrightarrow{\rho} GG(A) \xrightarrow{\varepsilon_{G(A)}} G(A)), A \xrightarrow{\rho} GG(A) \xrightarrow{G_{\varepsilon(A)}} G(A)) \]

yields an isomorphism of categories

   \[ \mathcal{Q}_{\kappa} : \mathbb{A}^{\mathcal{G}^{\kappa}} \rightarrow (\mathbb{A}^{\mathcal{G}^{\kappa}})^{\tilde{G}}, \]
where for any \(((A, \theta), \vartheta) \in (\mathbb{A}^{G^*})^\theta\),

\[
Q^{-1}_\kappa((A, \theta), \vartheta) = (A, A \xrightarrow{\theta} G(A) \xrightarrow{G(\theta)} GG(A)).
\]

There is a comparison functor \(K_\kappa : \mathbb{A} \to (\mathbb{A}^{G^*})^\theta\),

\[
A \mapsto ((G(A), G(A) \xrightarrow{\delta_A} GG(A)), G(A) \xrightarrow{\delta_A} GG(A) \xrightarrow{\kappa A} GG(A)),
\]
with \(K_\kappa = Q^{-1}_\kappa K_\kappa\) and commutative diagram

\[
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{K_\kappa} & (\mathbb{A}^{G^*})^\theta \\
\downarrow{G^\kappa} & \simeq & \downarrow{U^\theta} \\
\mathbb{A}^{G^\kappa} & \xrightarrow{U^G} & \mathbb{A}.
\end{array}
\]

### 3.8. Proposition

Let \(\mathcal{G} = (G, \delta, \varepsilon)\) be a comonad on \(\mathbb{A}\) and \(\kappa : GG \to GG\) an invertible BD-law.

1. \(\kappa^{-1} : \mathcal{G}^* \mathcal{G} \to \mathcal{G} G^\kappa\) is a comonad distributive law and hence induces a comonad \((\mathcal{G}^* \mathcal{G})_{\kappa^{-1}} = (G^* G, \delta, \varepsilon)\) where

\[
\delta : GG \xrightarrow{\delta \delta} GGGG \xrightarrow{G^{\kappa^{-1}G}} GGGG, \quad \varepsilon : GG \xrightarrow{\varepsilon \varepsilon} 1,
\]

and \(\kappa : \mathcal{G} G^\kappa \to (\mathcal{G}^* \mathcal{G})_{\kappa^{-1}}\) is a comonad isomorphism.

2. There is an isomorphism of categories

\[
\Phi' : (\mathbb{A}^{G^*})^\theta \simeq (\mathbb{A}^G)^{(G^e)^\kappa}, \quad ((A, \theta), \vartheta) \mapsto ((A, \theta), \theta).
\]

3. \(\kappa^{-1}\) induces a comparison functor

\[
K'_\kappa : \mathbb{A} \to (\mathbb{A}^G)^{(G^e)^\kappa}, \quad A \mapsto ((G(A), G(A) \xrightarrow{\delta_A} GG(A)), G(A) \xrightarrow{\delta_A} GG(A)),
\]

with commutative diagrams (with \(K_\kappa\) from 3.7)

\[
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{K'_\kappa} & (\mathbb{A}^G)^{(G^e)^\kappa} \\
\downarrow{G^\kappa} & \simeq & \downarrow{U^{G^e}} \\
\mathbb{A}^{G^\kappa} & \xrightarrow{U^G} & \mathbb{A} \\
\end{array}
\quad
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{K_\kappa} & (\mathbb{A}^{G^*})^\theta \\
\downarrow{G^\kappa} & \simeq & \downarrow{U^G} \\
\mathbb{A}^{G^\kappa} & \xrightarrow{K'_\kappa} & (\mathbb{A}^G)^{(G^e)^\kappa} \\
\end{array}
\quad
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\Phi'} & (\mathbb{A}^{G^*})_{\kappa^{-1}} \\
\downarrow{G^\kappa} & \simeq & \downarrow{U^G} \\
\mathbb{A}^{G^\kappa} & \xrightarrow{K'_\kappa} & (\mathbb{A}^{G*})_{\kappa^{-1}}. \\
\end{array}
\]

Note that, for \(\kappa\) invertible, it follows from the diagrams in the Sections 3.7, 3.8 that \(G\) is an Azumaya comonad if and only if the functor

\[
K'_\kappa : \mathbb{A} \to (\mathbb{A}^G)^{(G^e)^\kappa}
\]

is an equivalence of categories. Dualising Proposition 2.8 gives:

### 3.9. Proposition

Let \(\mathcal{G} = (G, \delta, \varepsilon)\) be a comonad on \(\mathbb{A}\) with an invertible BD-law \(\kappa : GG \to GG\) and assume \(\kappa^2 = 1\). Then the comonad \(\mathcal{G}\) is Azumaya if and only if the comonad \(G^\kappa\) is so.
3.10. **Azumaya comonads with left adjoints.** Again let $G = (G, \delta, \varepsilon)$ be a comonad on $A$ with an invertible BD-law $\kappa : GG \to GG$. Assume now that the functor $G$ admits a left adjoint functor $L$, with $\eta, \varepsilon : L \dashv G$, inducing a monad $L = (L, m, e)$ on $A$ (see §1.1). Since $\kappa$ is invertible, $\kappa^{-1}$ can be seen as a distributive law $G \kappa G \to GG \kappa$. It then follows from the dual of §1.7 that the composite 

$$\omega : LG \xrightarrow{LG\eta} LGGL \xrightarrow{L\kappa^{-1}L} LGGL \xrightarrow{\pi_A} GL$$

is a mixed distributive law from the monad $L$ to the comonad $G \kappa$ leading to an isomorphism of categories

$$(A^{G^\kappa})^{\tilde{L}^{\kappa}} \simeq (A^{G^\kappa})_{\bar{\kappa}}, \quad ((A, \theta), \vartheta) \mapsto ((A, \vartheta), L(A) \xrightarrow{L(\theta)} LG(A) \xrightarrow{\pi_{G(A)}} A),$$

where $\tilde{L}$ is the lifting of $L$ to $A^{G^\kappa}$ (corresponding to $\omega$). Then the composite

$$K_{\kappa} : A \xrightarrow{K_{\kappa}} (A^{G^\kappa})^{\tilde{L}^{\kappa}} \xrightarrow{\sim} (A^{G^\kappa})_{\bar{\kappa}}$$

takes an arbitrary $A \in A$ to

$$((G(A), \delta^A), LG(A) \xrightarrow{L(\delta_A)} LGG(A) \xrightarrow{\pi_{G(A)}} A),$$

thus inducing commutativity of the diagram

$$\begin{array}{ccc}
A & \xrightarrow{K_{\kappa}} & (A^{G^\kappa})_{\bar{\kappa}} \\
\downarrow{\phi^{G^\kappa}} & & \downarrow{U_{\bar{\kappa}}} \\
A^{G^\kappa} & \xrightarrow{U_{\bar{\kappa}}} & A^{G^\kappa}
\end{array}$$

3.11. **Theorem.** Let $G = (G, \delta, \varepsilon)$ be a comonad on $A$ with an invertible comonad BD-law $\kappa : GG \to GG$ and $L$ a monad left adjoint to $G$ (with $\eta, \varepsilon : L \dashv G$). Then the following are equivalent:

(a) $G$ is an Azumaya comonad;
(b) (i) the functor $\phi^{G^\kappa} : A \to A^{G^\kappa}$ is monadic and
(ii) $\phi^{G^\kappa}$ is $\tilde{L}$-Galois, that is,

$$t_{(A, \theta)} : L(A) \xrightarrow{L(\theta)} LG(A) \xrightarrow{L(\delta_A)} LGG(A) \xrightarrow{\pi_{G(A)}} G(A), \text{ is an isomorphism for any } (A, \theta) \in A^{G^\kappa} \text{ or}$$

$$\chi : LG \xrightarrow{L\delta} LGG \xrightarrow{L\delta_G} LGGG \xrightarrow{\pi_{G(A)}} GG \text{ is an isomorphism.}$$

**Proof.** This follows by applying the dual of Theorem [2.10] to the last diagram. $\Box$

3.12. **Proposition.** If $A$ has coequalisers of reflexive pairs, then $K_{\kappa} : A \to (A^{G^\kappa})_{\bar{\kappa}}$ admits a left adjoint functor $L : (A^{G^\kappa})_{\bar{\kappa}} \to A$ whose value at $((A, \vartheta), h) \in (A^{G^\kappa})_{\bar{\kappa}}$ is given as the coequaliser

$$\begin{array}{ccc}
L(A) & \xrightarrow{h} & A \\
\downarrow{L(\vartheta)} & & \downarrow{\pi_A} \\
LG(A) & \xrightarrow{q_{((A, \vartheta), h)}} & L((A, \vartheta), h)
\end{array}$$
3.13. **Definition.** Write $G^G$ for the quotient functor of the functor $G$ determined by the coequaliser of the diagram

![Diagram](https://example.com/diagram.png)

We call the comonad $G$ **cocentral** if $G^G$ is (isomorphic to) the identity functor.

3.14. **Theorem.** Assume $\mathbb{A}$ to admit coequalisers of reflexive pairs. Let $G = (G, \delta, \varepsilon)$ be a comonad on $\mathbb{A}$, $\kappa : GG \to GG$ an invertible comonad BD-law, and $\mathcal{L}$ a monad left adjoint to $G$. Then the comparison functor $K_\kappa : \mathbb{A} \to (\mathbb{A}^G)^\mathcal{L}$ is

(i) full and faithful if and only if the comonad $G$ is cocentral;

(ii) an equivalence of categories if and only if the comonad $G$ is cocentral and the functor $\mathcal{L}$ is conservative.

The next observation shows the transfer of the Galois property to an adjoint functor.

3.15. **Proposition.** Assume $\mathcal{F} = (F, m, e)$ to be a monad on $\mathbb{A}$ with invertible BD-law $\lambda : FF \to FF$, and $\eta, \varepsilon : F \dashv R$ an adjunction inducing a comonad $\mathcal{R} = (R, \delta, \varepsilon)$ with invertible BD-law $\kappa : RR \to RR$ (see Proposition 3.3). Then the functor $\phi_{\mathcal{F} \lambda}$ is $\mathcal{R}$-Galois if and only if the functor $\phi_{\mathcal{R}^\mathcal{F}}$ is $\mathcal{F}$-Galois.

**Proof.** By Theorems 2.10 and 3.11, we have to show that, for any $(A, h) \in \mathbb{A}_{\mathcal{F} \lambda}$, the composite

$$t_{(A, h)} : F(A) \xrightarrow{\eta_F(A)} RFF(A) \xrightarrow{R(m_A)} RF(A) \xrightarrow{R(h)} R(A)$$

is an isomorphism if and only if, for any $(A, \theta) \in \mathbb{A}_{\mathcal{R}^\mathcal{F}}$, this is so for the composite

$$t_{(A, \theta)} : F(A) \xrightarrow{F(\theta)} FR(A) \xrightarrow{F(\delta_A)} FRR(A) \xrightarrow{\varepsilon_R(A)} R(A).$$

By symmetry, it suffices to prove one implication. So suppose that the functor $\phi_{\mathcal{F} \lambda}$ is $\mathcal{R}$-Galois. Since $m \dashv \delta$, $\delta$ is the composite

$$R \xrightarrow{\pi_R} RFR \xrightarrow{\pi_R \pi_F} RRF \xrightarrow{Rm_R} RR \xrightarrow{RR} R.$$

Considering the diagram

![Diagram](https://example.com/diagram.png)

in which the top left triangle commutes by one of the triangular identities for $F \dashv R$ and the other partial diagrams commute by naturality, one sees that $t_{(A, \theta)}$ is the...
composite
\[ F(A) \xrightarrow{\pi_F(A)} RFF(A) \xrightarrow{Rm_A} RF(A) \xrightarrow{RF(\theta)} RFR(A) \xrightarrow{R\pi_A} R(A). \]

Since \((A, \theta) \in \mathbb{A}^{R_e}\), the pair \((A, F(A)) \xrightarrow{F(\theta)} FR(A) \xrightarrow{\pi_A} A\) – being \(\Psi^{-1}(A, \theta)\) (see [1.1]) – is an object of the category \(\mathbb{A}_{\mathcal{F}^\Delta}\). It then follows that \(t_{(A, \theta)} = t_{(A, \pi_A, F(\theta))}\). Since the functor \(\phi_{\mathcal{F}^\Delta}\) is assumed to be \(\tilde{\mathcal{R}}\)-Galois, the morphism \(t_{(A, \pi_A, F(\theta))}\), and hence also \(t_{(A, \theta)}\), is an isomorphism, as desired. \(\square\)

In view of the properties of separable functors (see [1.19] and Definition [2.3] for an Azumaya monad \(\mathcal{F}\), \(\mathcal{F}\mathcal{F}^\lambda\) is a separable monad if and only if \(F\) is a separable functor. In this case \(\phi_{\mathcal{F}^\Delta}\) is also a separable functor, that is, the unit \(e : 1 \to F\) splits. Dually, for an Azumaya comonad \(\mathcal{R}\), \(\mathcal{R}\mathcal{R}^\kappa\) is separable if and only if the functor \(R\) is separable. Thus we have:

3.16. **Theorem.** Under the conditions of Proposition [3.13], suppose further that \(\mathbb{A}\) is a Cauchy complete category. Then the following are equivalent:

(a) \((\mathcal{F}, \lambda)\) is an Azumaya monad and \(\mathcal{F}\mathcal{F}^\lambda\) is a separable monad;

(b) \((\mathcal{F}, \lambda)\) is an Azumaya monad and the unit \(e : 1 \to F\) is a split monomorphism;

(c) \(\phi_{\mathcal{F}^\Delta}\) is \(\tilde{\mathcal{R}}\)-Galois and \(e : 1 \to F\) is a split monomorphism;

(d) \((\mathcal{R}, \kappa)\) is an Azumaya comonad and the counit \(\varepsilon : R \to 1\) is a split epimorphism;

(e) \(\phi_{\mathcal{R}^\kappa}\) is \(\tilde{\mathcal{F}}\)-Galois and \(\varepsilon : R \to 1\) is a split epimorphism;

(f) \(\phi_{\mathcal{R}^\kappa}\) is \(\tilde{\mathcal{F}}\)-Galois and \(\mathcal{R}\mathcal{R}^\kappa\) is a separable comonad.

**Proof.** (a)⇒(b)⇒(c) follow by the preceding remarks.

(c)⇒(a) Since \(\mathbb{A}\) is assumed to be Cauchy complete, by [10] Corollary 3.17, the splitting of \(e\) implies that the functor \(\phi_{\mathcal{F}^\Delta}\) is comonadic. Now the assertion follows by Theorem [2.10].

Since \(\varepsilon\) is the mate of \(e\), \(\varepsilon\) is a split epimorphism if and only if \(e\) is a split monomorphism (e.g. [17, 7.4]) and the splitting of \(\varepsilon\) implies that the functor \(\phi_{\mathcal{R}^\kappa}\) is monadic.

Applying now Theorems [2.10] [3.11] and Proposition [3.15] gives the desired result. \(\square\)

### 4. Azumaya Algebras in Braided Monoidal Categories

#### 4.1. Algebras and modules in monoidal categories.

Let \((\mathcal{V}, \otimes, I, \tau)\) be a strict monoidal category ([14]). An algebra \(\mathcal{A} = (A, m, e)\) in \(\mathcal{V}\) (or \(\mathcal{V}\)-algebra) consists of an object \(A\) of \(\mathcal{V}\) endowed with multiplication \(m : A \otimes A \to A\) and unit morphism \(e : I \to A\) subject to the usual identity and associative conditions.

For a \(\mathcal{V}\)-algebra \(\mathcal{A}\), a left \(\mathcal{A}\)-module is a pair \((V, \rho_V)\), where \(V\) is an object of \(\mathcal{V}\) and \(\rho_V : A \otimes V \to V\) is a morphism in \(\mathcal{V}\), called the left action (or \(\mathcal{A}\)-left action) on \(V\), such that \(\rho_V(m \otimes V) = \rho_V(A \otimes \rho_V)\) and \(\rho_V(e \otimes V) = 1\).

Left \(\mathcal{A}\)-modules are objects of a category \(\mathcal{A}\mathcal{V}\) whose morphisms between objects \(f : (V, \rho_V) \to (W, \rho_W)\) are morphism \(f : V \to W\) in \(\mathcal{V}\) such that \(\rho_W(A \otimes f) = f \rho_V\). Similarly, one has the category \(\mathcal{V}_{\mathcal{A}}\) of right \(\mathcal{A}\)-modules.

The forgetful functor \(\mathcal{A}\mathcal{U} : \mathcal{A}\mathcal{V} \to \mathcal{V}\), taking a left \(\mathcal{A}\)-module \((V, \rho_V)\) to the object \(V\), has a left adjoint, the free \(\mathcal{A}\)-module functor

\[ \phi_A : \mathcal{V} \to \mathcal{A}\mathcal{V}, \quad V \mapsto (A \otimes V, m_A \otimes V). \]
There is another way of representing the category of left $\mathcal{A}$-modules involving modules over the monad associated to the $\mathcal{V}$-algebra $\mathcal{A}$.

Any $\mathcal{V}$-algebra $\mathcal{A} = (A, m, e)$ defines a monad $\mathcal{A}_l = (T, \eta, \mu)$ on $\mathcal{V}$ by putting

- $T(V) = A \otimes V$,
- $\eta_V = e \otimes V : V \to A \otimes V$,
- $\mu_V = m \otimes V : A \otimes A \otimes V \to A \otimes V$.

The corresponding Eilenberg-Moore category $\mathcal{V}_{\mathcal{A}_l}$ of $\mathcal{A}_l$-modules is exactly the category $\mathcal{A}\mathcal{V}$ of left $\mathcal{A}$-modules, and $\mathcal{A}U \dashv F$ is the familiar forgetful-free adjunction between $\mathcal{V}_{\mathcal{A}_l}$ and $\mathcal{V}$. This gives in particular that the forgetful functor $\mathcal{A}U : \mathcal{A}\mathcal{V} \to \mathcal{V}$ is monadic. Hence the functor $\mathcal{A}U$ creates those limits that exist in $\mathcal{V}$.

Symmetrically, writing $\mathcal{A}_r$ for the monad on $\mathcal{V}$ whose functor part is $- \otimes A$, the category $\mathcal{V}_{\mathcal{A}_r}$ is isomorphic to the Eilenberg-Moore category $\mathcal{V}_{\mathcal{A}_r}$ of $\mathcal{A}_r$-modules, and the forgetful functor $U_\mathcal{A} : \mathcal{V}_{\mathcal{A}_r} \to \mathcal{V}$ is monadic and creates those limits that exist in $\mathcal{V}$.

If $\mathcal{V}$ admits coequalisers, $\mathcal{A}$ is a $\mathcal{V}$-algebra, $(V, \rho_V) \in \mathcal{V}_{\mathcal{A}}$ a right $\mathcal{A}$-module, and $(W, \rho_W) \in \mathcal{V}_A$ a left $\mathcal{A}$-module, then their tensor product (over $\mathcal{A}$) is the object part of the coequaliser

$$V \otimes A \otimes W \xrightarrow{\rho_V \otimes W} V \otimes W \xrightarrow{\rho_W} V \otimes_A W.$$ 

### 4.2. Bimodules

If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-algebras, an object $V$ in $\mathcal{V}$ is called an $(\mathcal{A}, \mathcal{B})$-bimodule if there are morphisms $\rho_V : A \otimes V \to V$ and $\varrho_V : V \otimes B \to V$ in $\mathcal{V}$ such that $(V, \rho_V) \in \mathcal{A}\mathcal{V}$, $(V, \varrho_V) \in \mathcal{V}_B$ and $\varrho_V(\rho_V \otimes B) = \rho_V(A \otimes \varrho_V)$. A morphism of $(\mathcal{A}, \mathcal{B})$-bimodules is a morphism in $\mathcal{V}$ which is a morphism of left $\mathcal{A}$-modules as well as of right $\mathcal{B}$-modules. Write $\mathcal{A}\mathcal{V}_B$ for the corresponding category.

Let $\mathcal{I}$ be the trivial $\mathcal{V}$-algebra $(I, 1_I : I = I \otimes I \to I, 1_I : I \to I)$. Then, $\mathcal{I}\mathcal{V} = \mathcal{V}_I = \mathcal{V}$, and for any $\mathcal{V}$-algebra $\mathcal{A}$, the category $\mathcal{A}\mathcal{V}_I$ is (isomorphic to) the category of left $\mathcal{A}$-modules $\mathcal{A}\mathcal{V}$, while the category $\mathcal{I}\mathcal{V}_\mathcal{A}$ is (isomorphic to) the category of right $\mathcal{A}$-modules $\mathcal{V}_A$. In particular, $\mathcal{I}\mathcal{V}_\mathcal{I} = \mathcal{V}$.

### 4.3. The monoidal category of bimodules

Suppose now that $\mathcal{V}$ admits coequalisers and that the tensor product preserves these coequaliser in both variables (i.e. all functors $\otimes : \mathcal{V} \to \mathcal{V}$ as well as $- \otimes V : \mathcal{V} \to \mathcal{V}$ for $V \in \mathcal{V}$ preserve coequalisers). The last condition guarantees that if $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{V}$-algebras and if $M \in \mathcal{A}\mathcal{V}_B$ and $N \in \mathcal{B}\mathcal{V}_C$, then

- $M \otimes_B N \in \mathcal{A}\mathcal{V}_C$;
- if $\mathcal{D}$ is another $\mathcal{V}$-algebra and $P \in \mathcal{C}\mathcal{V}_D$, then the canonical morphism

$$(M \otimes_B N) \otimes_C P \to M \otimes_B (N \otimes_C P)$$

induced by the associativity of the tensor product, is an isomorphism in $\mathcal{A}\mathcal{V}_D$;

- $(\mathcal{A}\mathcal{V}_A, - \otimes_A -, \mathcal{A})$ is a monoidal category.

Note that (co)algebras in this monoidal category are called $\mathcal{A}$-(co)rings.

### 4.4. Coalgebras and comodules in monoidal categories

Associated to any monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$, there are three monoidal categories $\mathcal{V}^{op}$, $\mathcal{V}^{r}$ and $(\mathcal{V}^{op})^{r}$ obtained from $\mathcal{V}$ by reversing, respectively, the morphisms, the tensor product and both the morphisms and tensor product, i.e., $\mathcal{V}^{op} = (\mathcal{V}^{op}, \otimes, I)$, $\mathcal{V}^{r} = (\mathcal{V}, \otimes^{r}, I)$,
where $V \otimes^r W := W \otimes V$, and $(V^{op})^r = (V^{op}, \otimes^r, I)$ (see, for example, [24]). Note that $(V^{op})^r = (V^r)^{op}$.

Coalgebras and comodules in a monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ are respectively algebras and modules in $\mathcal{V}^{op} = (\mathcal{V}^{op}, \otimes, I)$. If $\mathcal{C} = (\mathcal{C}, \delta, \varepsilon)$ is a $\mathcal{V}$-coalgebra, we write $\mathcal{V}^\mathcal{C}$ (resp. $\mathcal{C}^\mathcal{V}$) for the category of right (resp. left) $\mathcal{C}$-comodules. Thus, $\mathcal{V}^\mathcal{C} = (\mathcal{V}^{op})_{\mathcal{C}}$ and $\mathcal{C}^\mathcal{V} = \mathcal{C}(\mathcal{V}^{op})$. Moreover, if $\mathcal{C}'$ is another $\mathcal{V}$-coalgebra, then the category $\mathcal{C}^\mathcal{V}$ of $(\mathcal{C}, \mathcal{C}')$-bicomodules is $\mathcal{C}(\mathcal{V}^{op})_{\mathcal{C}'}$. Writing $\mathcal{C}_t$ (resp. $\mathcal{C}_r$) for the comonad on $\mathcal{V}$ with functor-part $\mathcal{C} \otimes -$ (resp. $- \otimes \mathcal{C}$), one has that $\mathcal{V}^\mathcal{C}$ (resp. $\mathcal{C}^\mathcal{V}$) is just the category of $\mathcal{C}_t$-comodules (resp. $\mathcal{C}_r$-comodules).

4.5. **Duality in monoidal categories.** One says that an object $V$ of $\mathcal{V}$ admits a left dual, or a left adjoint, if there exist an object $V^*$ and morphisms $db : I \to V \otimes V^*$ and $ev : V^* \otimes V \to I$ such that the composites

$$V \xrightarrow{db \otimes V} V \otimes V^* \otimes V \xrightarrow{V \otimes ev} V, \quad V^* \otimes db : V^* \otimes V \otimes V^* \xrightarrow{ev \otimes V^*} V^*,$$

yield the identity morphisms. $db$ is called the unit and $ev$ the counit of the adjunction. We use the notation $(db, ev : V^* \dashv V)$ to indicate that $V^*$ is left adjoint to $V$ with unit $db$ and counit $ev$. This terminology is justified by the fact that such an adjunction induces an adjunction of functors

$$db \otimes - , ev \otimes - : V^* \otimes - \dashv V \otimes - : \mathcal{V} \to \mathcal{V},$$

as well as an adjunction of functors

$$- \otimes db , - \otimes ev : - \otimes V \dashv - \otimes V^* : \mathcal{V} \to \mathcal{V},$$

i.e., for any $X, Y \in \mathcal{V}$, there are bijections

$$\mathcal{V}(V^* \otimes X, Y) \simeq \mathcal{V}(X, V \otimes Y) \quad \text{and} \quad \mathcal{V}(X \otimes V, Y) \simeq \mathcal{V}(X, Y \otimes V^*).$$

Any adjunction $(db, ev : V^* \dashv V)$, induces a $\mathcal{V}$-algebra and a $\mathcal{V}$-coalgebra,

$$\mathcal{S}_{V^*,V} = (V \otimes V^*, V \otimes V^* \otimes V \otimes V^* \xrightarrow{V^* \otimes ev \otimes V} V \otimes V^*, \text{db} : I \to V \otimes V^*), \quad \mathcal{C}_{V^*,V} = (V \otimes V^*, V \otimes V^* \xrightarrow{V^* \otimes db \otimes V} V \otimes V^* \otimes V \otimes V^*, \text{ev} : V^* \otimes V \to I).$$

Dually, one says that an object $V$ of $\mathcal{V}$ admits a right adjoint if there exist an object $V^\sharp$ and morphisms $db' : I \to V^\sharp \otimes V$ and $ev' : V^\sharp \otimes V \to I$ such that the composites

$$V^\sharp \xrightarrow{db' \otimes V^\sharp} V^\sharp \otimes V \otimes V^\sharp \xrightarrow{V^\sharp \otimes ev} V^\sharp, \quad V \xrightarrow{V \otimes db} V \otimes V^\sharp \otimes V \xrightarrow{ev \otimes V} V,$$

yield the identity morphisms.

4.6. **Proposition.** Let $V \in \mathcal{V}$ be an object with a left dual $(V^*, db, ev)$.

(i) For a $\mathcal{V}$-algebra $A$ and a left $A$-module structure $\rho_V : A \otimes V \to V$ on $V$, the morphism

$$t_{(V,\rho_V)} : A \xrightarrow{A \otimes db} A \otimes V \otimes V^* \xrightarrow{\rho_V \otimes V^*} V \otimes V^*$$

(mate of $\rho_V$ under $\mathcal{V}(A \otimes V, V) \simeq \mathcal{V}(A, V \otimes V^*)$) is a morphism from the $\mathcal{V}$-algebra $A$ to the $\mathcal{V}$-algebra $\mathcal{S}_{V^*,V}$.  

(ii) For a $\mathcal{V}$-coalgebra $\mathcal{C}$ and a right $\mathcal{C}$-comodule structure $\varrho_V : V \to V \otimes C$, the morphism

$$t'_{(V,\varrho_V)} : V^* \otimes V \xrightarrow{V^* \otimes \varrho_V} V^* \otimes V \otimes C \xrightarrow{ev \otimes C} C$$
4.7. Definition. Let $V \in \mathcal{V}$ be an object with a left dual $(V^*, \text{db}, \text{ev})$.

(i) For a $\mathcal{V}$-algebra $A$, a left $A$-module $(V, \rho_V)$ is called Galois if the morphism $t_{(V, \rho_V)} : A \to V \otimes V^*$ is an isomorphism between the $\mathcal{V}$-algebras $A$ and $S_{V,V^*}$, and faithfully Galois if, in addition, the functor $V \otimes - : \mathcal{V} \to \mathcal{V}$ is conservative.

(ii) For a $\mathcal{V}$-coalgebra $C$, a right $C$-comodule $(V, \varrho_V)$ is called Galois if the morphism $t_{(V, \varrho_V)}^* : V^* \otimes V \to C^*$ is an isomorphism between the $\mathcal{V}$-coalgebras $C_{V,V^*}$ and $C$, and faithfully Galois if, in addition, the functor $V \otimes - : \mathcal{V} \to \mathcal{V}$ is conservative.

4.8. Braided monoidal categories. A braided monoidal category is a quadruple $(\mathcal{V}, \otimes, I, \tau)$, where $(\mathcal{V}, \otimes, I)$ is a monoidal category and $\tau$, called the braiding, is a collection of natural isomorphisms

$$\tau_{V,W} : V \otimes W \to W \otimes V, \quad V, W \in \mathcal{V},$$

subject to two hexagon coherence identities (e.g. [14]). A symmetric monoidal category is a monoidal category with a braiding $\tau$ such that $\tau_{V,W} \cdot \tau_{W,V} = 1$ for all $V, W \in \mathcal{V}$. If $\mathcal{V}$ is a braided category with braiding $\tau$, then the monoidal category $\mathcal{V}^r$ becomes a braided category with braiding given by $\tau_{V,W} := \tau_{W,V}$. Furthermore, given $\mathcal{V}$-algebras $A$ and $B$, the triple

$$A \otimes B = (A \otimes B, (m_A \otimes m_B) \cdot (A \otimes \tau_{B,A} \otimes B), e_A \otimes e_B)$$

is also a $\mathcal{V}$-algebra, called the braided tensor product of $A$ and $B$.

The braiding also has the following consequence (e.g. [26]):

If an object $V$ in $\mathcal{V}$ admits a left dual $(V^*, \text{db}, \text{ev})$ such that $\tau_{V,W} \cdot \tau_{W,V} = 1$, then $(V^*, \text{db}', \text{ev}')$ is right adjoint to $V$ with unit and counit

$$\text{db}' : I \xrightarrow{\text{db}} V \otimes V^* \xrightarrow{\tau_{V,W}^{-1}} V^* \otimes V, \quad \text{ev}' : V \otimes V^* \xrightarrow{\tau_{V,W}} V^* \otimes V \xrightarrow{\text{ev}} I.$$

Thus there are isomorphisms $(V^*)^r \simeq V$ and $(V^r)^* \simeq V$, and we have the following

4.9. Definition. An object $V$ of a braided monoidal category $\mathcal{V}$ is said to be finite if $V$ admits a left (and hence also a right) dual.

For the rest of this section, $\mathcal{V} = (\mathcal{V}, \otimes, I, \tau)$ will denote a braided monoidal category.

Finite objects in a braided monoidal category have the following relationship between the related functors to be (co)monadic or conservative. Recall that a morphism $f : V \to W$ in $\mathcal{V}$ called a copure epimorphism (monomorphism) if for any $X \in \mathcal{V}$, the morphism $f \otimes X : V \otimes X \to W \otimes X$ (and hence also the morphism $X \otimes f : X \otimes V \to X \otimes W$) is a regular epimorphism (monomorphism).

4.10. Proposition. Let $\mathcal{V}$ be a braided monoidal category admitting equalisers and coequalisers. For a finite object $V \in \mathcal{V}$ with left dual $(V^*, \text{db}, \text{ev})$, the following are equivalent:

(a) $V \otimes - : \mathcal{V} \to \mathcal{V}$ is conservative (monadic, comonadic);
(b) $\text{ev} : V^* \otimes V \to I$ is a copure epimorphism,
(c) $- \otimes V : \mathcal{V} \to \mathcal{V}$ is conservative (monadic, comonadic);
(d) $\text{db} : I \to V \otimes V^*$ is a pure monomorphism.
Proof. First observe that since \( V \) is assumed to admit a left dual, it admits also a right dual (see [12]). Hence the equivalence of the properties listed in (a) (and in (c)) follows from [12]. It only remains to show the equivalence of (a) and (b), since the equivalence of (c) and (d) will then follow by duality.

(a)\(\Rightarrow\)(b) If \( V \otimes - : \mathcal{V} \to \mathcal{V} \) is monadic, then it follows from [12, Theorem 2.4] that each component of the counit of the adjunction \( V^* \otimes - \to V \otimes - \), which is the natural transformation \( ev \otimes - \), is a regular epimorphism. Thus, \( ev : V^* \otimes V \to I \) is a copure epimorphism.

(b)\(\Rightarrow\)(a) To say that \( ev : V^* \otimes V \to I \) is a copure epimorphism is to say that each component of the counit \( ev \otimes - \) of the adjunction \( V^* \otimes - \to V \otimes - \) is a regular epimorphism, which implies (see, for example, [12]) that \( V \otimes - : \mathcal{V} \to \mathcal{V} \) is conservative.

\[ \square \]

4.11. Remark. In Proposition 4.10, if the tensor product preserves regular epimorphisms, then (b) is equivalent to require \( ev : V^* \otimes V \to I \) to be a regular epimorphism.

If the tensor product in \( \mathcal{V} \) preserves regular monomorphisms, then (d) is equivalent to require \( db : I \to V \otimes V^* \) to be a regular monomorphism.

4.12. Opposite algebras. For a \( \mathcal{V} \)-algebra \( A = (A, m, e) \), define the opposite algebra \( A^r = (A, m^r, e^r) \) in \( \mathcal{V} \) with multiplication \( m^r = m \cdot \tau_{A,A} \) and unit \( e^r = e \). Denote by \( A^e = A \otimes A^r \) and by \( \mathcal{A} = A^r \otimes A \) the braided tensor products.

Then \( A \) is a left \( A^e \)-module as well as a right \( \mathcal{A} \)-module by the structure maps

\[
\begin{align*}
A \otimes A^r \otimes A & \xrightarrow{A \otimes \tau_{A,A}} A \otimes A \otimes A \xrightarrow{A \otimes m} A \otimes A \xrightarrow{m} A, \\
A \otimes A^r \otimes A & \xrightarrow{\tau_{A,A} \otimes A} A \otimes A \otimes A \xrightarrow{m \otimes A} A \otimes A \xrightarrow{m} A.
\end{align*}
\]

By properties of the braiding, the morphism \( \tau_{A,A} : A \otimes A \to A \otimes A \) induces a distributive law from the monad \( (A^r)_l \) to the monad \( \mathcal{A}_l \) satisfying the Yang-Baxter equation and the monad \( \mathcal{A}_r \) is just the monad \( (A^r)_r \). Thus the category of \( \mathcal{A}_l(\mathcal{A}^r)_l \)-modules is the category \( \mathcal{A}_r \mathcal{V} \) of left \( \mathcal{A}^e \)-modules. Symmetrically, the category of \( \mathcal{A}_r(\mathcal{A}^r)_r \)-modules is the category \( \mathcal{V}_{\mathcal{A}} \) of right \( \mathcal{A} \)-modules.

4.13. Azumaya algebras. Given a \( \mathcal{V} \)-algebra \( A = (A, m, e) \), by Proposition 2.2, there are two comparison functors

\[
\begin{align*}
\overline{K}_l : \mathcal{V} & \to \mathcal{V}_{\mathcal{A}_l(\mathcal{A}^r)_l} = \mathcal{A}_r\mathcal{V}, \\
\overline{K}_r : \mathcal{V} & \to \mathcal{V}_{\mathcal{A}_r(\mathcal{A}^r)_r} = \mathcal{V}_{\mathcal{A}},
\end{align*}
\]

given by the assignments

\[
\begin{align*}
\overline{K}_l : V & \mapsto (A \otimes V, A \otimes A \otimes A \otimes V \xrightarrow{A \otimes m^r \otimes V} A \otimes A \otimes V \xrightarrow{m \otimes V} A \otimes V), \\
\overline{K}_r : V & \mapsto (V \otimes A, V \otimes A \otimes A \otimes A \xrightarrow{V \otimes m^r \otimes A} V \otimes A \otimes A \xrightarrow{V \otimes m} V \otimes A)
\end{align*}
\]

with commutative diagrams

\[
\begin{align*}
\mathcal{V} & \xrightarrow{\overline{K}_l} \mathcal{A}_r\mathcal{V} & \mathcal{V} & \xrightarrow{\overline{K}_r} \mathcal{V}_{\mathcal{A}} \\
A \otimes - & \xrightarrow{\mathcal{A}_r U} \mathcal{A}_r\mathcal{V} & - \otimes A & \xrightarrow{U_{\mathcal{A}}} \mathcal{V}_{\mathcal{A}}.
\end{align*}
\]

The \( \mathcal{V} \)-algebra \( A \) is called left (right) Azumaya provided \( \mathcal{A}_l (\mathcal{A}_r) \) is an Azumaya monad.
4.14. **Remark.** It follows from Remark 2.8 that if \( \tau^2_{A,A} = 1 \), the monad \( \mathcal{A}_I \) (resp. \( \mathcal{A}_I^r \)) is Azumaya if and only if \( (\mathcal{A}^r)_{I} \) (resp. \( (\mathcal{A}^r)_I \)) is. Thus, in a symmetric monoidal category, a \( \mathcal{V} \)-algebra is left (right) Azumaya if and only if its opposite is so.

A basic property of these algebras is the following.

4.15. **Proposition.** Let \( \mathcal{V} \) be a braided monoidal category and \( \mathcal{A} = (A, m, e) \) a \( \mathcal{V} \)-algebra. If \( \mathcal{A} \) is left Azumaya, then \( A \) is finite in \( \mathcal{V} \).

**Proof.** It is easy to see that when \( \mathcal{V} \) and \( \mathcal{A}^c \mathcal{V} \) are viewed as right \( \mathcal{V} \)-categories (in the sense of [22]), \( \mathcal{K}_I \) is a \( \mathcal{V} \)-functor. Hence, when \( \mathcal{K}_I \) is an equivalence of categories (that is, when \( \mathcal{A} \) is left Azumaya), its inverse equivalence \( \mathcal{K} \) is also a \( \mathcal{V} \)-functor. Moreover, since \( \mathcal{R} \) is left adjoint to \( \mathcal{K}_I \), it preserves all colimits that exist in \( \mathcal{A}^c \mathcal{V} \). Obviously, the functor \( \phi(A^c)_I : \mathcal{V} \to \mathcal{A}^c \mathcal{V} \) is also a \( \mathcal{V} \)-functor and, moreover, being a left adjoint, it preserves all colimits that exist in \( \mathcal{V} \). Consequently, the composite \( \mathcal{R} \cdot \phi(A^c)_I : \mathcal{V} \to \mathcal{V} \) is a \( \mathcal{V} \)-functor and preserves all colimits that exist in \( \mathcal{V} \). It then follows from [22, Theorem 4.2] that there exists an object \( A^* \) such that \( \mathcal{R} \cdot \phi(A^c)_I \simeq A^* \otimes - \). Using now that \( \mathcal{K}_I \cdot \phi(A^c)_I \) is left adjoint to the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \), it is not hard to see that \( A^* \) is a left dual to \( A \). \( \square \)

4.16. **Left Azumaya algebras.** Let \( (\mathcal{V}, \otimes, I, \tau) \) be a braided monoidal category and \( \mathcal{A} = (A, m, e) \) a \( \mathcal{V} \)-algebra. The following are equivalent:

(a) \( \mathcal{A} \) is a left Azumaya algebra;

(b) the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \) is monadic and the left \( (\mathcal{A}^c)_I \)-module structure on it induced by the left diagram in \([4.1]\), is Galois.

(c) (i) \( A \) is finite with left dual \((A^*, db : I \to A \otimes A^*, ev : A^* \otimes A \to I)\), the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \) is monadic and

(ii) the composite \( \mathcal{X}_0 \):

\[
A \otimes A \xrightarrow{\Delta \otimes A \otimes db} A \otimes A \otimes A \otimes A^* \xrightarrow{A \otimes \tau_A \otimes A^*} A \otimes A \otimes A \otimes A^* \xrightarrow{m \otimes A \otimes A^*} A \otimes A \otimes A^* \xrightarrow{m \otimes A^*} A \otimes A^*
\]

is an isomorphism (between the \( \mathcal{V} \)-algebras \( \mathcal{A}^c \) and \( \mathcal{S}_{A,A^*} \));

(d) (i) \( A \) is finite with right dual \((A^\sharp, db' : I \to A^\sharp \otimes A, ev' : A \otimes A^\sharp \to I)\), the functor \( \phi(A^\sharp)_I : \mathcal{V} \to \mathcal{V}(A^\sharp)_I = A^\sharp \mathcal{V} \) is comonadic and

(ii) the composite \( \mathcal{X} \):

\[
A \otimes A \xrightarrow{db' \otimes A \otimes A} A^\sharp \otimes A \otimes A \otimes A \xrightarrow{A^\sharp \otimes m \otimes A} A^\sharp \otimes A \otimes A \xrightarrow{A^\sharp \otimes \Delta \otimes A} A^\sharp \otimes A \otimes A \xrightarrow{A^\sharp \otimes m} A^\sharp \otimes A
\]

is an isomorphism.

**Proof.** (a) \( \iff \) (b) This follows by Proposition 1.13.

(a) \( \iff \) (c) If \( \mathcal{A} \) is a left Azumaya algebra, then \( A \) has a left dual by Proposition 4.15. Thus, in both cases, \( A \) is finite, i.e. there is an adjunction \((db, ev : A^* \dashv A)\). Then the functor \( A^* \otimes - : \mathcal{V} \to \mathcal{V} \) is left adjoint to the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \), and the monad on \( \mathcal{V} \) generated by this adjunction is \((\mathcal{S}_{A,A^*})_I\). It is then easy to see that the monad morphism \( \mathcal{X}_0 : (A^\sharp)_I \to (\mathcal{S}_{A,A^*})_I \) corresponding to the left commutative diagram in \([11]\), is just \( \mathcal{X}_0 \otimes - \). Thus, \( \mathcal{X}_0 \) is an isomorphism if and only if \( \mathcal{X}_0 \) is so. It now follows from Theorem 2.5 that (a) and (c) are equivalent.
(a) $\iff$ (d) Any left Azumaya algebra has a left (and hence also a right) dual by Proposition 4.15. Moreover, if $A$ has a right dual $A^\sharp$, then the functor $A^\sharp \otimes -$ is right adjoint to the functor $A \otimes -$. The desired equivalence now follows by applying Theorem 2.10 to the monad $\mathcal{A}_l$ and using that the natural transformation $\chi$ is just $\chi \otimes -$.

Each statement about a general braided monoidal category $\mathcal{V}$ has a counterpart statement obtained by interpreting it in $\mathcal{V}^\ast$. We do this for Theorem 4.16.

4.17. Right Azumaya algebras. Let $(\mathcal{V}, \otimes, I, \tau)$ be a braided monoidal category and $\mathcal{A} = (A, m, e)$ a $\mathcal{V}$-algebra. The following are equivalent:

(a) $\mathcal{A}$ is right Azumaya;

(b) the functor $- \otimes A : \mathcal{V} \to \mathcal{V}$ is monadic and the right $(\mathcal{A})_r$-module structure on it induced by the right diagram in (4.7), is Galois;

(c) (i) $A$ is finite with right dual $(A^\sharp, db : I \to A^\sharp \otimes A, ev : A \otimes A^\sharp \to I)$, the functor $- \otimes A : \mathcal{V} \to \mathcal{V}$ is monadic and

(ii) the composite $\phi_{A^\sharp} : \mathcal{V} \to \mathcal{V} (A^\sharp)_r = \mathcal{V} A^\sharp$ is comonadic and

(ii) the composite $\phi_{A^\sharp} : A \otimes A \to A \otimes A \otimes A^\ast$ is an isomorphism. 

(d) (i) $A$ is finite with left dual $(A^\ast, db : I \to A \otimes A^\ast, ev : A^\ast \otimes A \to I)$, the functor $\phi_{A^\ast} : \mathcal{V} \to \mathcal{V} (A^\ast)_l = \mathcal{V} A^\ast$ is comonadic and

(ii) the composite $\phi_{A^\ast} : A \otimes A \to A \otimes A \otimes A^\ast$ is an isomorphism.

4.18. Proposition. In any braided monoidal category, an algebra is left (right) Azumaya if and only if its opposite algebra is right (left) Azumaya.

Proof. We just note that if $(\mathcal{V}, \otimes, I, \tau)$ is a braided monoidal category and $\mathcal{A}$ is a $\mathcal{V}$-algebra, then $(\tau_{-A})^{-1} : A \otimes - \to - \otimes A^\ast$ is an isomorphism of monads $\mathcal{A}_l \to (\mathcal{A}^\ast)_r$, while $(\tau_{A^{-1}})^{-1} : - \otimes A \to A^\ast \otimes -$ is an isomorphism of monads $\mathcal{A}_r \to (\mathcal{A}^\ast)_l$.

Under some conditions on $\mathcal{V}$, left Azumaya algebras are also right Azumaya and vice versa:

4.19. Theorem. Let $\mathcal{A} = (A, m, e)$ be a $\mathcal{V}$-algebra in a braided monoidal category $(\mathcal{V}, \otimes, I, \tau)$ with equalisers and coequalisers. Then the following are equivalent:

(a) $\mathcal{A}$ is a left Azumaya algebra;

(b) the left $\mathcal{A}^e$-module $(A, m \cdot (A \otimes m^r))$ is faithfully Galois;

(c) $A$ is finite with right dual $(A^\sharp, db : I \to A^\sharp \otimes A, ev : A \otimes A^\sharp \to I)$, the functor $\phi_{A^\sharp} : \mathcal{V} \to \mathcal{V} (A^\sharp)_r = \mathcal{V} A^\sharp$ is comonadic, and the composite $\chi$ in (4.17) (d) is an isomorphism;

(d) $A$ is finite with right dual $(A^\sharp, db : I \to A^\sharp \otimes A, ev : A \otimes A^\sharp \to I)$, the functor $- \otimes A : \mathcal{V} \to \mathcal{V}$ is monadic, and the composite $\chi$ in (4.17) (c) is an isomorphism;

(e) the right $\mathcal{A}^e$-module $(A, m \cdot (m^r \otimes A))$ is faithfully Galois;
(f) \( \mathcal{A} \) is a right Azumaya algebra.

**Proof.** In view of Proposition 4.10 and Remark 4.11, (a), (b), and (c) are equivalent by 4.10 and (d), (e), and (f) are equivalent by 4.17.

(c) \( \text{\Leftrightarrow} \) (d) The composite \( \chi \) is the upper path and \( \chi_1 \) is the lower path in the diagram

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\mathbf{d}b'.A.A} & A^z \otimes A \otimes A \otimes A \otimes A \xrightarrow{\mathbf{A}^z.A-\mathbf{m}A} A^z \otimes A \otimes A \xrightarrow{\mathbf{A}^z.A-\tau} A^z \otimes A \\
\downarrow & & \downarrow \mathbf{A}^z.A-\tau \\
A \otimes A & \xrightarrow{\mathbf{d}b'.A.A} & A^z \otimes A \otimes A \otimes A \xrightarrow{\mathbf{A}^z.A-\mathbf{m}A} A^z \otimes A \otimes A \xrightarrow{\mathbf{A}^z.A-\mathbf{m}A} A^z \otimes A
\end{array}
\]

where \( \tau = \tau_{A,A} \) and \( \cdot = \otimes \). The left square is commutative by naturality, the pentagon is commutative since \( \tau \) is a braiding, and the parallelogram commutes by the associativity of \( m \). So the diagram is commutative and hence \( \chi = \chi_1 \cdot \tau_{A,A} \), that is, \( \chi \) is an isomorphism if and only if \( \chi_1 \) is so. Thus, in order to show that (c) and (d) are equivalent, it is enough to show that the functor \( \phi_{(\mathcal{A}^r)_i} : \mathcal{V} \to \mathcal{A}^r \mathcal{V} \) is comonadic if and only if the functor \( \mathcal{V} \to \mathcal{A}^r \mathcal{V} \) is monadic. Since \( \mathcal{V} \) is assumed to have equalisers and coequalisers, this follows from Lemma 1.11 and Proposition 4.10. \( \square \)

4.20. **Remark.** A closer examination of the proof of the previous theorem shows that if a braided monoidal category \( \mathcal{V} \) admits

- coequalisers, then any left Azumaya \( \mathcal{V} \)-algebra is right Azumaya,
- equalisers, then any right Azumaya \( \mathcal{V} \)-algebra is left Azumaya.

In the setting of 4.12 by Proposition 2.2 the assignment

\[
\mathcal{V} \mapsto ((A \otimes \mathcal{V}, A \otimes A \otimes \mathcal{V} \xrightarrow{\mathbf{m}^z \otimes \mathcal{V}}, A \otimes A \otimes \mathcal{V} \xrightarrow{\mathbf{m} \otimes \mathcal{V}}, A \otimes A \otimes \mathcal{V} \xrightarrow{\mathbf{m} \otimes \mathcal{V}}, A \otimes A \otimes \mathcal{V})
\]

yields the comparison functor \( K : \mathcal{V} \to (\mathcal{V}_{(\mathcal{A}^r)_i})_{\mathcal{V}_i} = (\mathcal{A}^r \mathcal{V})_{\mathcal{V}_i} \).

Now assume \( A \otimes - : \mathcal{V} \to \mathcal{V} \) to have a right adjoint functor \( [A, -] : \mathcal{V} \to \mathcal{V} \) with unit \( \eta^A : 1 \to [A, A \otimes -] \). Then there is a unique comonad structure \( [A, -] \) on \( [A, -] \) (right adjoint to \( \mathcal{A}_i \), see 1.1) leading to the commutative diagram

\[
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{K} & (\mathcal{A}^r \mathcal{V})_{\mathcal{V}_i} \\
\phi_{(\mathcal{A}^r)_i} \downarrow & & \downarrow \Psi \phi_{(\mathcal{A}^r)_i} \\
\mathcal{A}^r \mathcal{V} & \xrightarrow{U \phi_{(\mathcal{A}^r)_i}} & (\mathcal{A}^r \mathcal{V})_{[A,-]}
\end{array}
\]

where \( \Psi = \Psi_{(\mathcal{A}^r)_i} \). This is just the diagram (2.2) and Theorem 2.10 provides characterisations of left Azumaya algebras.

4.21. **Theorem.** Let \( \mathcal{A} = (A, m, e) \) be an algebra in a braided monoidal category \( (\mathcal{V}, \otimes, I, \tau) \) and assume \( A \otimes - \) to have a right adjoint \( [A, -] \) (see above). Then the following are equivalent:

(a) \( \mathcal{A} \) is left Azumaya;
(b) the functor \( \phi_{(\mathcal{A}^r)_i} : \mathcal{V} \to \mathcal{A}^r \mathcal{V} \) is comonadic and, for any \( \mathcal{V} \in \mathcal{V} \), the composite:

\[
\chi_{\mathcal{V}} : A \otimes A \otimes \mathcal{V} \xrightarrow{(\eta^A)_{\otimes A \otimes \mathcal{V}}} [A, A \otimes A \otimes A \otimes A] \xrightarrow{(A, m \otimes A \otimes A)} [A, A \otimes A \otimes A] \xrightarrow{(A, \tau_{A, A} \otimes \mathcal{V})} [A, A \otimes A \otimes \mathcal{V}] \xrightarrow{(A, m \otimes \mathcal{V})} [A, A \otimes \mathcal{V}]
\]
rewriting the morphism $\bar{\phi}$

Theorem.

is an isomorphism;

(c) $A$ is finite, the functor $\phi_{(A^r)_r} : V \to A^r$ is comonadic, and the composite

\[ \chi_A : A \otimes A \xrightarrow{(\eta^A)_A \otimes \eta^A} [A, A] \xrightarrow{[A, m \otimes A]} [A, A] \xrightarrow{[A, m^r]} [A, A] \]

is an isomorphism;

Proof. (a)\(\Rightarrow\)(b) follows by Theorem 2.10.

(a)\(\Leftrightarrow\)(c) Since $A$ turns out to be finite, there is a right dual $(A^r, db', ev')$ of $A$. Then $A^r \otimes - : V \to V$ and $[A, -] : V \to V$ are both right adjoint to $A \otimes - : V \to V$, and thus there is an isomorphism of functors $t : [A, -] \to A^r \otimes -$ inducing the commutative diagram

\[
\begin{diagram}
V & \xrightarrow{(\eta^A)_V} & [A, A \otimes V] \\
\downarrow{db' \otimes V} & & \downarrow{t_{A \otimes V}} \\
A^r \otimes A \otimes V. & & \\
\end{diagram}
\]

Rewriting the morphism $\chi$ from (4.16(d)) yields the morphism $\chi_A$ in (c).

Considering the symmetric situation we get:

4.22. Theorem. Let $A = (A, m, e)$ be an algebra in a braided monoidal category $V$ and assume $- \otimes A$ to have a right adjoint $\{A, -\}$ with unit $\eta_A : 1 \to \{A, - \otimes A\}$. Then the following are equivalent:

(a) $A$ is right Azumaya;

(b) the functor $\phi_{(A^r)_r} : V \to V_{A^r}$ is comonadic and for any $V \in V$, the composite

\[ \chi_V : V \otimes A \otimes A \xrightarrow{V \otimes (\eta_A)_A \otimes A} [A, V \otimes A \otimes A] \xrightarrow{[A, V \otimes m \otimes A]} [A, A \otimes A] \xrightarrow{[A, V \otimes m]} [A, V \otimes A] \]

is an isomorphism;

(c) $A$ is finite, the functor $\phi_{(A^r)_r} : V \to V_{A^r}$ is comonadic, and the composite

\[ \chi_A : A \otimes A \xrightarrow{(\eta_A)_A \otimes \eta^A} [A, A \otimes A] \xrightarrow{[A, m \otimes A]} [A, A] \xrightarrow{A, m^r]} [A, A] \]

is an isomorphism.

4.23. Remark. In [29], F. van Oystaeyen and Y. Zhang defined Azumaya algebras $A = (A, m, e)$ in $V$ by requiring $A$ to be left and right Azumaya in our sense (see 4.13). The preceding theorems 4.21 and 4.22 correspond to the characterisation of these algebras in [29 Theorem 3.1]. As shown in Theorem 4.19 if $V$ admits equalisers and coequalisers it is sufficient to require the Azumaya property on one side.

Given an adjunction $(db, ev : V^* \to V)$ in $V$, we know from 4.5 that $S_{V, V^*} = V \otimes V^*$ is a $V$-algebra. Moreover, it is easy to see that the morphism $V^* \otimes V \otimes V^* \xrightarrow{ev \otimes V^*} V^*$ defines a left $S_{V, V^*}$-module structure on $V^*$, while the composite $V \otimes V^* \otimes V \xrightarrow{V \otimes ev} V$ defines a right $S_{V, V^*}$-module structure on $V$.

Recall from [29] that an object $V \in V$ with a left dual $(V^*, db, ev)$ is right faithfully projective if the morphism $\bar{ev} : V^* \otimes S_{V, V^*} = V \to I$ induced by $ev : V^* \otimes V \to I$ is an isomorphism. Dually, an object $V \in V$ with a right dual $(V^*, db', ev')$ is left faithfully...
projective if the morphism \( \overrightarrow{ev} : V \otimes_{S_{V, V}} V^\sharp \to I \) induced by \( ev' : V \otimes V^\sharp \to I \) is an isomorphism.

Since in a braided monoidal category an object is left faithfully projective if and only if it is right faithfully projective (e.g. \cite[Proposition 4.14.]{}), we do not have to distinguish between left and right faithfully projective objects and we shall call them just faithfully projective.

4.24. **Theorem.** Let \( (V, \otimes, I, \tau) \) be a braided closed monoidal category with equalisers and coequalisers. Let \( \mathcal{A} = (A, m, e) \) be a \( V \)-algebra such that the functor \( A \otimes - \) admits a right adjoint \( [A, -] \) (and hence the functor \( - \otimes A \) also admits a right adjoint \( \{A, -\} \)). Then the following are equivalent:

(a) \( A \) is left Azumaya;
(b) \( A \) is right Azumaya;
(c) \( A \) is faithfully projective and the composite

\[
A \otimes A \xrightarrow{(\eta^A)_{A \otimes A}} [A, A \otimes A \otimes A] \xrightarrow{[A, m \otimes A]} [A, A \otimes A] \xrightarrow{[A, m^r]} [A, A],
\]

where \( \eta^A \) is the unit of the adjunction \( A \otimes - \dashv [A, -] \), is an isomorphism.
(d) \( A \) is faithfully projective and the composite

\[
A \otimes A \xrightarrow{(\eta_A)_{A \otimes A}} \{A, A \otimes A \otimes A\} \xrightarrow{[A, m \otimes A]} \{A, A \otimes A\} \xrightarrow{[A, m^r]} \{A, A\},
\]

where \( \eta_A \) is the unit of the adjunction \( - \otimes A \dashv \{A, -\} \), is an isomorphism.

**Proof.** That (a) and (b) are equivalent follows from Theorem 4.19.

(a) \( \Leftrightarrow \) (c) Since in both cases \( A \) is finite and thus the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \) has both left and right adjoints, in view of Proposition 4.10, we get from Lemma 1.11 that the functor \( \phi_{(A^\vee)^\vee} : \mathcal{V} \to \mathcal{A} \mathcal{V} \) is comonadic if and only if the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \) is conservative. According to \cite[2.5.1, 2.5.2]{}, \( A \) is faithfully projective if and only if \( A \) is finite and the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \) is conservative and hence the equivalence of (a) and (c) follows by Theorem 4.21.

Similarly, one proves that (b) and (d) are equivalent. \( \square \)

4.25. **Braided closed monoidal categories.** A braided monoidal category \( \mathcal{V} \) is said to be **left closed** if each functor \( V \otimes - : \mathcal{V} \to \mathcal{V} \) has a right adjoint \( [V, -] : \mathcal{V} \to \mathcal{V} \), we write \( \eta^V, ev^V : V \otimes - \dashv [V, -] \). \( \mathcal{V} \) is called **right closed** if each functor \( - \otimes V : \mathcal{V} \to \mathcal{V} \) has a right adjoint \( \{V, -\} : \mathcal{V} \to \mathcal{V} \), we write \( \eta_V, ev_V : - \otimes V \dashv \{V, -\} \). \( \mathcal{V} \) being braided left closed implies that \( \mathcal{V} \) is also right closed. So assume \( \mathcal{V} \) to be closed.

If \( \mathcal{A} \) is a \( \mathcal{V} \)-algebra, and \( (V, \rho_V) \in \mathcal{A} \mathcal{V} \), then for any \( X \in \mathcal{V} \),

\[
(V \otimes X, A \otimes V \otimes X \xrightarrow{\rho_V \otimes X} V \otimes X) \in \mathcal{A} \mathcal{V},
\]

and the assignment \( X \to (V \otimes X, \rho_V \otimes X) \) defines a functor \( V \otimes - : \mathcal{V} \to \mathcal{A} \mathcal{V} \). When \( \mathcal{V} \) admits equalisers, this functor has a right adjoint \( \mathcal{A}[V, -] : \mathcal{A} \mathcal{V} \to \mathcal{V} \), where for any \( (W, \rho_W) \in \mathcal{A} \mathcal{V} \), \( \mathcal{A}[V, W] \) is defined to be the equalizer in \( \mathcal{V} \) of

\[
[V, W] \xrightarrow{[\mathcal{A}[V, -]\rho_V]} [A \otimes V, W],
\]

where one of the morphisms is \( [\rho_V, W] \), and the other one is the composition

\[
[V, W] \xrightarrow{(A \otimes -)_{V, W}} [A \otimes V, A \otimes W] \xrightarrow{[A \otimes V, \rho_W]} [A \otimes V, W].
\]
Symmetrically, for $V, W \in \mathcal{V}_A$, one defines $\{V, W\}_A$.

The functor $\overline{K} = \Psi K : \mathcal{V} \to (\mathcal{A}^\mathcal{V})^\mathcal{A}_\mathcal{A}$ (in diagram (4.2)) has as right adjoint $\overline{R} : (\mathcal{A}^\mathcal{V})^\mathcal{A}_\mathcal{A} \to \mathcal{V}$ (see 1.16), and since $\Psi$ is an isomorphism of categories, the composition $\overline{R} \Psi$ is right adjoint to the functor $K : \mathcal{V} \to (\mathcal{A}^\mathcal{V})^\mathcal{A}_\mathcal{A}$. Using now that $\mathcal{P}$ (see 2.6) is an isomorphism of categories, we conclude that $\overline{R} \Psi \mathcal{P}$ is right adjoint to the functor $\mathcal{P}^{-1}K : \mathcal{V} \to \mathcal{A}^\mathcal{V}$. For any $(V, h) \in \mathcal{A}^\mathcal{V}$, we put $\mathcal{A}^\mathcal{V} := \overline{R} \Psi \mathcal{P}(V, h)$.

Taking into account the description of the functors $\mathcal{P}$, $\Psi$ and $\overline{R}$, one gets that $\mathcal{A}^\mathcal{V}$ can be obtained as the equaliser of the diagram

![Diagram](https://example.com/diagram)

The functor $\mathcal{P}^{-1}K : \mathcal{V} \to \mathcal{A}^\mathcal{V}$ is just the functor $A \otimes - : \mathcal{V} \to \mathcal{A}^\mathcal{V}$ and admits as a right adjoint the functor $\mathcal{A}^\mathcal{V} \times \mathcal{A}^\mathcal{V} : \mathcal{A}^\mathcal{V} \to \mathcal{V}$ (see 4.25). As right adjoints are unique up to isomorphism, we get an alternative proof for B. Femić's [7, Proposition 3.3]:

4.26. Proposition. Let $\mathcal{V}$ be a braided closed monoidal category with equalisers. For any $\mathcal{V}$-algebra $A$, the functors $A(-), \mathcal{A}^\mathcal{V}[A, -] : \mathcal{A}^\mathcal{V} \to \mathcal{V}$ are isomorphic.

This isomorphism allows for further characterisations of Azumaya algebras.

4.27. Theorem. Let $\mathcal{V}$ be a braided closed monoidal category with equalisers. Then a $\mathcal{V}$-algebra $A = (A, m, e)$ is left Azumaya if and only if

(i) the morphism $e : I \to A$ is a pure monomorphism, and

(ii) for any $(V, h) \in \mathcal{A}^\mathcal{V}$, with the canonical inclusion $i_V : \mathcal{A}^\mathcal{V} \to \mathcal{V}$, the composite

$$A \otimes \mathcal{A}^\mathcal{V} \xrightarrow{A \otimes i_V} A \otimes \mathcal{A}^\mathcal{V} \xrightarrow{h \otimes A \otimes \mathcal{A}^\mathcal{V}} \mathcal{A}^\mathcal{V}$$

is an isomorphism.

Proof. The $\mathcal{V}$-algebra $A$ is left Azumaya provided the functor $\overline{K}_l : \mathcal{V} \to \mathcal{A}^\mathcal{V}$ is an equivalence of categories. It follows from equation (1.6) that the composite

$$h \cdot (A \otimes e \otimes V) \cdot (A \otimes i_V) : A \otimes \mathcal{A}^\mathcal{V} \to \mathcal{V}$$

is just the $\mathcal{P} \mathcal{P}(V, h)$-component of the counit of $\overline{K}_l \Rightarrow \overline{R}$ and hence is an isomorphism. Moreover, by Proposition 1.15, the functor $\phi_{(\mathcal{A}^\mathcal{V})_l} : \mathcal{V} \to \mathcal{A}^\mathcal{V}$ is comonadic, whence the morphism $e : I \to A$ is a pure monomorphism (e.g. [16, Theorem 2.1(2.(i))]). This proves one direction.

For the other direction we note that, under the conditions (i) and (ii), the counit of the adjunction $\mathcal{P}^{-1}\overline{K}_l \Rightarrow \overline{R} \mathcal{P} \mathcal{P}$ (and hence also of the adjunction $\overline{K}_l = \Psi K \Rightarrow \overline{R}$) is an isomorphism and the functor $\phi_{(\mathcal{A}^\mathcal{V})_l}$ (and hence also $\overline{K}_l$) is conservative (again [16, Theorem 2.1(2.(i))]), implying (as in the proof of Theorem 2.13 (ii)) that $\overline{K}_l$ is an equivalence of categories.

\(\square\)
Symmetrically, for any \((V, h) \in \mathcal{V}_V\) defining \(V^A\) as the equaliser of the diagram

\[
\begin{array}{c}
V \xrightarrow{(\eta_A)_V} \{A, V \otimes A\} \xrightarrow{\{A, V \otimes A \otimes e\}} \{A, V \otimes A \otimes A\} \xrightarrow{\{A, h\}} \{A, V\},
\end{array}
\]

one has an isomorphism of functors \((-)^A, \{A, -\}_A : \mathcal{V}_A \to \mathcal{V}\).

Dualising the previous theorem gives:

4.28. \textbf{Theorem.} Let \(\mathcal{V}\) be a braided closed monoidal category with equalisers. Then a \(\mathcal{V}\)-algebra \(A = (A, m, e)\) is right Azumaya if and only if

(i) the morphism \(e : I \to A\) is a pure monomorphism, and

(ii) for any \((V, h) \in \mathcal{V}_V\), with the canonical inclusion \(i_V : V^A \to V\), the composite

\[
V^A \otimes V \xrightarrow{i_V \otimes A} V \otimes A \xrightarrow{V \otimes e \otimes A} V \otimes A \otimes A \xrightarrow{h} V
\]

is an isomorphism.

4.29. \textbf{Definition.} A \(\mathcal{V}\)-algebra \(A\) is called left (resp. right) central if there is an isomorphism \(I \simeq \mathcal{A}^e[A, -]\) (resp. \(I \simeq \{A, -\}_A\)). \(A\) is called central if it is both left and right central.

4.30. \textbf{Proposition.} Let \(\mathcal{V}\) be a braided closed monoidal category with equalisers. Then

(i) any left (resp. right) Azumaya algebra is left (resp. right) central;

(ii) if, in addition, \(\mathcal{V}\) admits also coequalisers, then any \(\mathcal{V}\) algebra that is Azumaya on either side is central.

\textbf{Proof.} (i) follows by the Theorems 4.27 and 4.28 while (ii) follows from (i) and Theorem 4.19.

Recall that for any \(\mathcal{V}\) algebra \(A\), an \(A^e\)-module \(M\) is \(U_{A^e}\)-projective provided for morphisms \(g : N \to L\) and \(f : M \to L\) in \(A^e\mathcal{V}\) with \(U_{A^e}(g)\) a split epimorphism, there exists an \(h : M \to N\) in \(A^e\mathcal{V}\) with \(gh = f\). This is the case if and only if \(M\) is a retract of a (free) \(A^e\)-module \(A^e \otimes X\) with some \(X \in \mathcal{V}\) (e.g. [25]). This is applied in the characterisation of separable algebras.

4.31. \textbf{Proposition.} The following are equivalent for a \(\mathcal{V}\)-algebra \(A = (A, m, e)\):

(a) \(A\) is a separable algebra;

(b) \(m : A \otimes A \to A\) has a section \(\xi : A \to A \otimes A\) in \(\mathcal{V}\) such that

\[
(A \otimes m) \cdot (\xi \otimes A) = \xi \cdot m = (m \otimes A) \cdot (A \otimes \xi);
\]

(c) the left \(A^e\)-module \((A, m \cdot (A \otimes m^*))\) is \(A^eU\)-projective;

(d) the functor \(A^eU : A^e\mathcal{V} \to \mathcal{V}\) is separable.

4.32. \textbf{Proposition.} Consider \(\mathcal{V}\)-algebras \(A\) and \(B\) such that the unit \(e : I \to B\) of \(B\) is a split monomorphism. If \(A \otimes B\) is separable in \(\mathcal{V}\), then \(A\) is also separable in \(\mathcal{V}\).

\textbf{Proof.} Since \(I\) is a retract of \(B\) in \(\mathcal{V}\), \(A\) is a retract of \(A \otimes B\) in \(A^e\mathcal{V}\). Since \(A \otimes B\) is assumed to be separable in \(\mathcal{V}\), \(A \otimes B\) is a retract of \((A \otimes B)^e\) in \((A \otimes B)^e\mathcal{V}\), and hence also in \(A^e\mathcal{V}\). Thus \(A\) is a retract of \(A^e \otimes B^e \simeq (A \otimes B)^e\) in \(A^e\mathcal{V}\). Since \(A^e \otimes B^e = \phi_{A^e}(B^e)\), it follows that \(A^e \otimes B^e\) is \(A^eU\)-projective, and since retracts of \(A^eU\)-projectives are \(A^eU\)-projective, \(A\) is \(A^eU\)-projective and \(A\) is separable by Proposition 4.31. \qed
Following [21], a finite object \( V \) in \( \mathcal{V} \) is said to be a progenerator if the counit morphism \( \text{ev} : V^* \otimes V \to I \) is a split epimorphism. The following list describes some of its properties.

### 4.33. Proposition

Assume \( \mathcal{V} \) to admit equalisers and coequalisers. For an algebra \( \mathcal{A} = (A, m, e) \) in \( \mathcal{V} \) with \( A \) admitting a left adjoint \( (V^*, \text{db}, \text{ev}) \), consider the following statements:

1. \( A \) is a progenerator;
2. the morphism \( \text{db} : I \to A \otimes A^* \) is a split monomorphism;
3. the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \) is separable;
4. the unit morphism \( e : I \to A \) is a split monomorphism;
5. the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \) is conservative (monadic, comonadic);
6. \( A \otimes A^* \) is a separable \( \mathcal{V} \)-algebra.

One always has \((1) \iff (2) \iff (3) \iff (4) \Rightarrow (5) \) and \( (1) \Rightarrow (6) \).

If \( I \) is projective (w.r.t. regular epimorphisms) in \( \mathcal{V} \), then \( (5) \Rightarrow (1) \).

**Proof.**

Since \( A \) is assumed to be admit a left adjoint \( (V^*, \text{db}, \text{ev}) \), the functor \( A^* \otimes - : \mathcal{V} \to \mathcal{V} \) is left as well as right adjoint to the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \). For any \( V \in \mathcal{V} \), the composite

\[
V \xrightarrow{\text{db} \otimes V} A \otimes A^* \otimes V \xrightarrow{\tau_{A^*, A \otimes A}^{-1} \otimes \text{ev}} A^* \otimes A \otimes V
\]

is the \( V \)-component of the unit of the adjunction \( A \otimes - \dashv A^* \otimes - : \mathcal{V} \to \mathcal{V} \), while the morphism \( A^* \otimes A \otimes V \xrightarrow{\text{ev} \otimes V} V \) is the \( V \)-component of the counit of the adjunction \( A^* \otimes - \dashv A \otimes - : \mathcal{V} \to \mathcal{V} \). To say that \( \text{db} : I \to A \otimes A^* \) (resp. \( \text{ev} : A^* \otimes A \to I \)) is a split monomorphism (resp. epimorphism) is to say that the unit (resp. counit) of the adjunction \( A \otimes - \dashv A^* \otimes - \) (resp. \( A^* \otimes - \dashv A \otimes - \)) is a split monomorphism (resp. epimorphism). From the observations in [1.17] one gets \((1) \iff (2) \iff (3) \).

By Proposition 4.10, the properties listed in (5) are equivalent.

Since \( \mathcal{V} \) admits equalisers, it is Cauchy complete, and the implication \((3) \Rightarrow (5) \) follows from [16, Proposition 3.16].

If \( e : I \to A \) is a split monomorphism, then the natural transformation

\[
e \otimes - : 1_{\mathcal{V}} \to A \otimes -
\]

is a split monomorphism and applying Proposition I.20 to the pair of functors \((A \otimes -, 1_{\mathcal{V}})\) gives that the functor \( A \otimes - : \mathcal{V} \to \mathcal{V} \) is separable, proving \((4) \Rightarrow (3) \).

If \( A \) is a progenerator, then \( \text{ev} : A^* \otimes A \to I \) has a splitting \( \zeta : I \to A^* \otimes A \). Consider the composite

\[
\phi : A \xrightarrow{\zeta \otimes A} A^* \otimes A \otimes A \xrightarrow{A^* \otimes m} A^* \otimes A \xrightarrow{\text{ev}} I.
\]

We claim that \( \phi \cdot e = 1 \). Indeed, we have

\[
\text{ev} \cdot A^* \otimes m \cdot \zeta \otimes A \cdot e = \text{ev} \cdot A^* \otimes m \cdot A^* \otimes A \otimes e \cdot \zeta = \text{ev} \cdot \zeta = 1.
\]

The first equality holds by naturality, the second one since \( e \) is the unit for the \( \mathcal{V} \)-algebra \( \mathcal{A} \), and the third one since \( \zeta \) is a splitting for \( \mathcal{V} \)-algebra \( \mathcal{A} \). Thus (2) implies (4).
Now, if $A$ is again a progenerator, then the morphism $ev : A^* \otimes A \to I$ has a splitting $\zeta : I \to A^* \otimes A$, and direct inspection shows that the morphism

$$\xi = A \otimes \zeta \otimes A^* : A \otimes A^* \to A \otimes A^* \otimes A \otimes A^*$$

is a splitting for the multiplication $A \otimes ev \otimes A^*$ of the $V$-algebra $A \otimes A^*$ satisfying condition (b) of Proposition 4.31. Thus $A \otimes A^*$ is a separable $V$-algebra, proving the implication $(5) \Rightarrow (6)$.

Finally, suppose that $I$ is projective (w.r.t. regular epimorphisms) in $V$ and that the functor $A \otimes - : V \to V$ is monadic. Then, by [12, Theorem 2.4], each component of the counit of the adjunction $A^* \otimes - \to A \otimes -$ is a regular epimorphism. Since $ev : A^* \otimes A \to I$ is the $I$-component of the counit, ev is a regular epimorphism, and hence splits, since $I$ is assumed to be projective w.r.t. regular epimorphisms. Thus $A$ is a progenerator. This proves the implication $(5) \Rightarrow (1)$. $\square$

4.34. Theorem. Let $V$ be a braided monoidal category with equalisers and coequalisers. For an algebra $A = (A, m, e)$ in $V$, the following are equivalent:

(a) $A$ is a separable left Azumaya $V$-algebra;

(b) $A$ is a progenerator and the morphism $\overline{\chi}_0 : A \otimes A \to A \otimes A^*$ in 4.10(c) is an isomorphism between the $V$-algebras $A^v$ and $S_{A,A^v}$;

(c) $e : I \to A$ is a split monomorphism and $(A, m \cdot (A \otimes m^\tau)) \in A^V V$ is a Galois module.

Proof. (a)\(\Rightarrow\)(c) In view of Proposition 4.31, this is a special case of 3.16. (b)\(\Rightarrow\)(c) is an easy consequence of Proposition 4.33 and Theorem 4.16. $\square$

To bring back our general theory to the starting point, let $R$ be a commutative ring with identity and $M_R$ the category of $R$-modules. Then for any $M, N \in M_R$, there is the canonical twist map $\tau_{M,N} : M \otimes_R N \to N \otimes_R M$. Putting $[M,N] := \text{Hom}_R(M,N)$, then $([M,N], - \otimes - , R, [-,-], \tau)$ is a symmetric monoidal closed category. We have the canonical adjunction $\eta^M, \varepsilon^M : M \otimes_R - \to [M,-]$.

4.35. Algebras in $M_R$. For any $R$-algebra $\mathcal{A} = (A, m, e)$, $\tau_{A,A} : A \otimes_R A \to A \otimes_R A$ is an invertible (involutive) BD-law allowing for the definition of the (opposite) algebra $A^\tau = (A, m \cdot \tau, e)$. The monad $A \otimes_R -$ is Azumaya provided the functor

$$K : M_R \to A^\tau M,$$

$$M \mapsto ((A \otimes_R M, A \otimes_R A \otimes_R M) \xrightarrow{A \otimes_R m^\tau \otimes_R M} A \otimes_R A \otimes_R M \xrightarrow{m \otimes_R M} A \otimes_R M),$$

is an equivalence of categories. Obviously this holds if and only if $A$ is an Azumaya $R$-algebra in the usual sense. We have the commutative diagram

$$(4.4)$$

where $(e \otimes_R A^\tau)^*$ is the restriction of scalars functor induced by the ring morphism $e \otimes_R A^\tau : A^\tau \to A \otimes_R A^\tau$. 
It is not hard to see that, for any \((M,h) \in A^r \mathbb{M}_r\), the \((M,h)\)-component \(t_{(M,h)} : A \otimes_R M \to [A,M]\) of the comonad morphism \(t : \phi(A^r)_h U(A^r)_h \to [A,A]\) corresponding to the functor \(K = \Psi K\), takes any element \(a \otimes_R m\) to the map \(b \mapsto h((ba) \otimes_R m)\). Thus, writing \(a \cdot m\) for \(h(a \otimes_R m)\), one has for \(a, b \in A\) and \(m \in M\),
\[ t_{(M,h)}(a \otimes_R m) = (b \mapsto (ba) \cdot m). \]
In particular, for any \(N \in \mathbb{M}_R\), \(t_{\phi(A^r)_h(N)}(a \otimes_R b \otimes_R n) = (c \mapsto (bca) \cdot n)\).

Since the canonical morphism \(i : R \to A\) factorises through the center of \(A\), it follows from \([16, \text{Theorem } 8.11]\) that the functor \(A \otimes_R - : \mathbb{M}_R \to A\mathbb{M}\) (and hence also \(A^r \otimes_R - : \mathbb{M}_R \to A^r \mathbb{M}\)) is comonadic if and only if \(i\) is a pure morphism of \(R\)-modules. Applying Theorem \([4.21]\) and using that \(K = \Psi K\) is so, we get several characterisations of Azumaya \(R\)-algebra.

**4.36. Theorem.** An \(R\)-algebra \(A\) is an Azumaya \(R\)-algebra if and only if the canonical morphism \(i : R \to A\) is a pure morphism of \(R\)-modules, and one of the following holds:

(a) for any \(M \in A^r \mathbb{M}\), there is an isomorphism
\[ A \otimes_R M \to [A,M], \quad a \otimes_R m \mapsto [b \mapsto (ba) \cdot m]; \]
(b) for any \(N \in \mathbb{M}_R\), there is an isomorphism
\[ A \otimes_R A \otimes_R N \to [A,A \otimes_R N], \quad a \otimes_R b \otimes_R n \mapsto [c \mapsto bca \otimes_R n]; \]
(c) \(A\) is finitely generated projective and there is an isomorphism
\[ A \otimes_R A \to [A,A], \quad a \otimes_R b \mapsto [c \mapsto bca]; \]
(d) for any \((A,A)\)-bimodule \(M\), the evaluation map is an isomorphism
\[ A \otimes_R M^A \to M, \quad a \otimes_R m \mapsto a \cdot m. \]

**Proof.** (a) follows by Theorem \([2.10]\); (b) and (c) are derived from Theorem \([4.21]\).
(c) An \(R\)-module is finite in the monoidal category \(\mathbb{M}_R\) if and only if it is finitely generated and projective over \(R\) and Theorem \([4.15]\) applies.
(d) is a translation of Theorem \([4.27]\) into the present context. \(\square\)

For a (von Neumann) regular ring \(R\), \(i : R \to A\) is always a pure \(R\)-module morphism, and hence over such rings the (equivalent) properties (a) to (d) are sufficient to characterise Azumaya algebras.

### 5. Azumaya coalgebras in braided monoidal categories

Throughout \((\mathcal{V}, \otimes, I, \tau)\) will denote a strict monoidal braided category. The definition of coalgebras \(\mathcal{C} = (C, \Delta, \epsilon)\) in \(\mathcal{V}\) was recalled in \([4.4]\).

#### 5.1. The coalgebra \(C^c\).
Let \(C\) be a \(\mathcal{V}\)-coalgebra. The braiding \(\tau_{C,C} : C \otimes C \to C \otimes C\) provides a BD-law allowing for the definition of the opposite coalgebra \(C^\tau = (C^\tau, \Delta^\tau = \tau_{C,C} \cdot \Delta, \epsilon^\tau = \epsilon)\) and a coalgebra
\[ C^c := (C \otimes C^\tau, (C \otimes \tau \otimes C^\tau)(\Delta \otimes \Delta^\tau), \epsilon \otimes \epsilon). \]

Writing \(\tau : C_l(C^\tau)_l \to (C^\tau)_l C_l\) for the induced distributive law of the comonad \(C_l\) over the comonad \((C^\tau)_l\), we have an isomorphism of categories \(\mathcal{V}(C^\tau)_l C_l \simeq \mathcal{V}(C_c)_l = C^c \mathcal{V}\).
5.2. **Definition.** (see [3.4]) A \( \mathcal{V} \)-coalgebra \( C \) is said to be left Azumaya provided the comonad \( \mathcal{C}_r = C \otimes - : \mathcal{V} \to \mathcal{V} \) is Azumaya, i.e. the comparison functor
\[
\mathcal{K}_r : \mathcal{V} \to \mathcal{C} V, \quad V \rightsquigarrow (C \otimes V, C \otimes V \xrightarrow{\Delta \otimes V} C \otimes C \otimes V \xrightarrow{C \otimes \Delta \otimes V} C \otimes C \otimes C \otimes V),
\]
is an equivalence of categories. It fits into the commutative diagram
\[
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\mathcal{K}_r} & \mathcal{C} V \\
\downarrow C \otimes - & & \downarrow C \otimes - \\
\mathcal{C} C & = \mathcal{V}(\mathcal{C} C) & \mathcal{C} C \end{array}
\]
\( \mathcal{C} \) is said to be right Azumaya if the corresponding conditions for \( \mathcal{C}_r = - \otimes C \) are satisfied. Similar to [4.15] we have:

5.3. **Proposition.** Let \( C = (C, \Delta, \varepsilon) \) be a coalgebra in a braided monoidal category \( \mathcal{V} \). If \( \mathcal{C} \) is left Azumaya, then \( C \) is finite in \( \mathcal{V} \).

**Proof.** Suppose that a \( \mathcal{V} \)-coalgebra \( C \) is left Azumaya. Then the functor \( C \otimes - : \mathcal{V} \to \mathcal{V} \) admits a right adjoint \( \mathcal{C}, - : \mathcal{V} \to \mathcal{V} \) by Proposition [3.5]. Write \( \vartheta \) for the composite \( (C \otimes \Delta^e) \otimes \Delta : C \to C \otimes C \otimes C \). Then for any \( V \in \mathcal{V} \), \( \mathcal{K}_r(V) = (C \otimes V, \vartheta \otimes V) \) and thus the \( V \)-component of the left \( \mathcal{C}^e \)-comodule structure on the functor \( C \otimes - : \mathcal{V} \to \mathcal{V} \), induced by the commutative diagram (5.1), is the morphism \( \vartheta \otimes V : C \otimes V \to C \otimes C \otimes C \otimes V \). From [1.14] we then see that the \( V \)-component \( t_V \) of the comonad morphism induced by the above diagram is the composite
\[
C \otimes [C, V] \xrightarrow{\vartheta \otimes [C, V]} C \otimes C \otimes C \otimes [C, V] \xrightarrow{C \otimes \sigma_C \otimes (\mathcal{V} C)} C \otimes C \otimes V,
\]
where \( \mathcal{V} C \) is the counit of the adjunction \( C \otimes - \dashv \mathcal{V} C \).

Next, let \( \sigma_V : [C, I] \otimes V \to [C, V] \) be the transpose of the morphism \( (\mathcal{V} C)_I \otimes V : C \otimes [C, I] \otimes V \to V \) and consider the following diagram
\[
\begin{array}{ccc}
C \otimes [C, I] \otimes V & \xrightarrow{\vartheta \otimes [C, I] \otimes V} & C \otimes C \otimes C \otimes [C, I] \otimes V \\
C \otimes [C, V] & \xrightarrow{\vartheta \otimes [C, V]} & C \otimes C \otimes C \otimes [C, V] \\
C \otimes [C, V] & \xrightarrow{\vartheta \otimes [C, V]} & C \otimes C \otimes C \otimes [C, V] \xrightarrow{C \otimes \sigma_C \otimes (\mathcal{V} C)} C \otimes C \otimes V.
\end{array}
\]
In this diagram the rectangle is commutative by naturality of composition. Since \( \sigma_V \) is the transpose of the morphism \( (\mathcal{V} C)_I \otimes V \), the transpose of \( \sigma_V \) – which is the composite \( C \otimes [C, I] \otimes V \xrightarrow{C \otimes \sigma_V} C \otimes [C, V] \xrightarrow{(\mathcal{V} C)_V} V \) – is \( (\mathcal{V} C)_I \otimes V \). Hence the triangle in the diagram is also commutative. Now, since
\[
(C \otimes C \otimes (\mathcal{V} C)_I \otimes V) \cdot (\vartheta \otimes [C, I] \otimes V) = t_I \otimes V,
\]
it follows from commutativity of the diagram that \( t_I \otimes V = t_V \cdot (C \otimes \sigma_V) \), and since \( \mathcal{C} \) is assumed to be left Azumaya, both \( t_I \) and \( t_V \) are isomorphisms, one concludes that \( C \otimes \sigma_V \) is an isomorphism. Moreover, the functor \( C \otimes - : \mathcal{V} \to \mathcal{V} \) is comonadic, hence conservative. It follows that \( \sigma_V : [C, I] \otimes V \to [C, V] \) is an isomorphism for all \( V \in \mathcal{V} \). Thus the functor \( [C, I] \otimes - : \mathcal{V} \to \mathcal{V} \) is also right adjoint to the functor \( C \otimes - : \mathcal{V} \to \mathcal{V} \). It is now easy to see that \( [C, I] \) is right adjoint to \( C \).

Theorem [3.6] provides a first characterisation of left Azumaya coalgebras.
5.4. **Theorem.** For a \( \mathcal{V} \)-coalgebra \( C = (C, \Delta, \varepsilon) \), the following are equivalent:

(a) \( C \) is a left Azumaya \( \mathcal{V} \)-coalgebra;

(b) the functor \( C \otimes - : \mathcal{V} \to \mathcal{V} \) is comonadic and the left \( (C^e)_I \)-comodule structure on it, induced by the commutative diagram \( [5.7] \), is Galois;

(c) (i) \( C \) is finite with right dual \( (C^r, \text{db} : I \to C^r \otimes C, \text{ev} : C^r \otimes C \to I) \), the functor \( C \otimes - : \mathcal{V} \to \mathcal{V} \) is comonadic and

(ii) the composite \( \overline{\chi}_0 : \)

\[
\begin{array}{cccc}
C \otimes C^r & \overset{\Delta \otimes C}{\longrightarrow} & C \otimes C^r \otimes C^r & \overset{C \otimes \Delta \otimes C}{\longrightarrow} & C \otimes C \otimes C \otimes C^r & \overset{C \otimes \tau \otimes C}{\longrightarrow} & C \otimes C \otimes C \otimes C^r & \overset{C \otimes \Delta \otimes \text{ev}'}{\longrightarrow} & C \otimes C \\
\end{array}
\]

is an isomorphism (between the \( \mathcal{V} \)-coalgebras \( \mathcal{S}_{C,C^r} \) and \( C^e \));

(d) (i) \( C \) is finite with left dual \( (C^l, \phi : I \to C \otimes C^*, \text{ev} : C^* \otimes C \to I) \), the functor \( \phi_{(C^r)_I} : \mathcal{V} \to \mathcal{V}^{(C^r)_I} = C^r \mathcal{V} \) is monadic and

(ii) the composite \( \overline{\chi} : \)

\[
\begin{array}{cccc}
C^r \otimes C & \overset{C^r \otimes \Delta}{\longrightarrow} & C^r \otimes C \otimes C^r & \overset{C^r \otimes \tau}{\longrightarrow} & C^r \otimes C \otimes C \otimes C^r & \overset{C^r \otimes \Delta}{\longrightarrow} & C^r \otimes C \otimes C \otimes C^r & \overset{\text{ev} \otimes C \otimes C}{\longrightarrow} & C \otimes C \\
\end{array}
\]

is an isomorphism.

**Proof.** (a) and (b) are equivalent by Theorem 3.6.

The equivalences (a) \( \Leftrightarrow \) (c) and (a) \( \Leftrightarrow \) (d) follow from Proposition 5.3 by dualising the proofs of the corresponding equivalences in Theorem 4.16 \( \square \)

Similarly, writing out Theorem 3.11 and the dual form of Theorem 4.17 yields conditions for right Azumaya coalgebras \( C \), that is, making \( C_r = - \otimes C \) an Azumaya comonad. Dualising Theorem 4.19 gives:

5.5. **Theorem.** Let \( C = (C, \Delta, \varepsilon) \) be a \( \mathcal{V} \)-coalgebra in a braided monoidal category \( \mathcal{V} \) with equalisers and coequalisers. Then the following are equivalent:

(a) \( C \) is a left Azumaya coalgebra;

(b) the left \( C^e \)-comodule \( (C, (C \otimes \Delta^r) \cdot \Delta) \) is cofaithfully Galois;

(c) there is an adjunction \( \text{db}', \text{ev}' : C \dashv C^r \), the functor \( - \otimes C : \mathcal{V} \to \mathcal{V} \) is comonadic, and the composite \( \overline{\chi} \) in (c) is an isomorphism;

(d) the right \( ^1C \)-comodule \( (C, (\Delta^r \otimes C) \cdot \Delta) \) is cofaithfully Galois;

(e) \( C \) is a right Azumaya coalgebra.

Under suitable assumptions, the base category \( \mathcal{V} \) may be replaced by a comodule category over a cocommutative coalgebra. For this we consider the

5.6. **Cotensor product.** Suppose now that \( \mathcal{V} = (\mathcal{V}, \otimes, I, \tau) \) is a braided monoidal category with equalisers and \( \mathcal{D} = (D, \Delta_D, \varepsilon_D) \) is a coalgebra in \( \mathcal{V} \). If \((V, \rho^V) \in \mathcal{D} \mathcal{V}\) and \((W, \varphi^W) \in \mathcal{V}^\mathcal{D}\), then their cotensor product (over \( \mathcal{D} \)) is the object part of the equaliser

\[
V \otimes D W \overset{\text{iv} \otimes W}{\longrightarrow} V \otimes W \overset{\varphi \otimes W}{\longrightarrow} V \otimes D \otimes W
\]

Suppose in addition that either

- for any \( V \in \mathcal{V} \), \( V \otimes - : \mathcal{V} \to \mathcal{V} \) and \( - \otimes V : \mathcal{V} \to \mathcal{V} \) preserve equalisers, or
- \( \mathcal{V} \) is Cauchy complete and \( \mathcal{D} \) is coseparable.
Each of these conditions guarantee that for $V, W, X \in \mathcal{P}\mathcal{V}^D$,

- $V \otimes^D W \in \mathcal{P}\mathcal{V}^D$;
- the canonical morphism (induced by the associativity of the tensor product)
  $$\left( V \otimes^D W \right) \otimes^D V \to V \otimes^D \left( W \otimes^D X \right)$$
  is an isomorphism in $\mathcal{P}\mathcal{V}^D$;
- $(\mathcal{P}\mathcal{V}^D, - \otimes^D -, D, \bar{\tau})$, where $\bar{\tau}$ is the restriction of $\tau$, is a braided monoidal category.

When $\mathcal{D}$ is cocommutative (i.e. $\tau_{\mathcal{D}, \mathcal{D}} \cdot \Delta = \Delta$), then for any $(V, \rho^V) \in \mathcal{P}\mathcal{V}$, the composite $\rho_1^V = \tau_{\mathcal{D}, V}^{-1} \cdot \rho^V : V \to V \otimes D$, defines a right $\mathcal{D}$-comodule structure on $V$. Conversely, if $(W, \varrho^W) \in \mathcal{P}\mathcal{D}$, then $\varrho_1^W = \tau_{W, \mathcal{D}} \cdot \varrho^W : W \to D \otimes W$ defines a left $\mathcal{D}$-comodule structure on $W$. These two constructions establish an isomorphism between $\mathcal{P}\mathcal{D}$ and $\mathcal{V}^\mathcal{D}$, and thus we do not have to distinguish between left and right $\mathcal{D}$-comodules. In this case, the tensor product of two $\mathcal{D}$-comodules is another $\mathcal{D}$-comodule, and cotensoring over $\mathcal{D}$ makes $\mathcal{P}\mathcal{V}$ (as well as $\mathcal{V}^\mathcal{D}$) a braided monoidal category with unit $\mathcal{D}$.

5.7. $\mathcal{D}$-coalgebras. Consider $\mathcal{V}$-coalgebras $\mathcal{C} = (C, \Delta_C, \varepsilon_C)$ and $\mathcal{D} = (D, \Delta_D, \varepsilon_D)$ with $\mathcal{D}$ cocommutative. A coalgebra morphism $\gamma : \mathcal{C} \to \mathcal{D}$ is called cocentral provided the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta_C} & C \otimes C \\
\downarrow{\Delta_C} & & \downarrow{\tau_{C,D}} \\
C \otimes C & \xrightarrow{\gamma \otimes C} & D \otimes C
\end{array}
$$

is commutative. When this is the case, $(\mathcal{C}, \gamma)$ is called a $\mathcal{D}$-coalgebra.

To specify a $\mathcal{P}\mathcal{V}$-coalgebra structure on an object $C \in \mathcal{V}$ is to give $C$ a $\mathcal{D}$-coalgebra structure $(\mathcal{C} = (C, \Delta_C, \varepsilon_C), \gamma)$. Indeed, if $\gamma : \mathcal{C} \to \mathcal{D}$ is a cocentral morphism, $\mathcal{C}$ can be viewed as an object of $\mathcal{P}\mathcal{V}$ (and $\mathcal{V}^\mathcal{D}$) via

$$
C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{\gamma \otimes C} D \otimes C,
$$

and $\Delta_C$ factors through the $i_{C,C} : C \otimes_D C \to C \otimes C$ by some (unique) morphism $\Delta_C : C \to C \otimes_D C$, that is $\Delta_C = i_{C,C} \cdot \Delta_C$.

The triple $\mathcal{C}_D = (C, \Delta'_C, \gamma)$ is a coalgebra in the braided monoidal category $\mathcal{P}\mathcal{V}$.

Conversely, any $\mathcal{P}\mathcal{V}$-coalgebra, $(\mathcal{C}, \Delta'_C : C \to C \otimes^D C, \varepsilon_C : C \to D)$ induces a $\mathcal{V}$-coalgebra

$$
\mathcal{C} = (C, C \xrightarrow{\Delta'_C} C \otimes^D C \xrightarrow{i_{C,C}} C \otimes C, C \xrightarrow{\varepsilon_C} D \xrightarrow{\varepsilon_D} 1),
$$

and the pair $(\mathcal{C}, \varepsilon_C)$ is a $\mathcal{D}$-coalgebra.

Related to any $\mathcal{V}$-coalgebra morphisms $\gamma : \mathcal{C} \to \mathcal{D}$, there is the corestriction functor

$$(-)_{\gamma} : \mathcal{C} \to \mathcal{P}\mathcal{V}, \quad (V, \varrho^V) \mapsto (V, (V \otimes \gamma)^{-1} \cdot \varrho^D),$$

and usually one writes $(V)_{\gamma} = V$. If the category $\mathcal{C}\mathcal{V}$ admits equalisers, then one has the coinduction functor

$$C \otimes^D - : \mathcal{P}\mathcal{V} \to \mathcal{C}\mathcal{V}, \quad W \mapsto (C \otimes^D W, \Delta_C \otimes^D W),$$

defining an adjunction

$$(-)_{\gamma} \dashv C \otimes^D - : \mathcal{P}\mathcal{V} \to \mathcal{C}\mathcal{V}.$$
Considering \( \mathcal{C} \) as a \((\mathcal{D}, \mathcal{C})\)-bicomodule by \( C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{\gamma \otimes_C} D \otimes_R C \), the corestriction functor is isomorphic to \( C \otimes^C - : \mathcal{C} \mathcal{V} \to \mathcal{D} \mathcal{V} \).

If \((\mathcal{C}, \gamma)\) is a \(\mathcal{D}\)-coalgebra, then the category \( C \mathcal{D}(\mathcal{D} \mathcal{V}) \) can be identified with the category \( C \mathcal{V} \) and, modulo this identification, the functor
\[
C \mathcal{D} \otimes^D - : \mathcal{D} \mathcal{V} \to C \mathcal{V}
\]
corresponds to the coinduction functor \( C \otimes^D - : \mathcal{D} \mathcal{V} \to C \mathcal{V} \).

5.8. \textbf{Azumaya} \(\mathcal{D}\)-\textbf{coalgebras.} Let \( \mathcal{D} \) be a cocommutative \( \mathcal{V}\)-coalgebra. Then a \(\mathcal{D}\)-coalgebra \( \mathcal{C} = (C, \Delta, \varepsilon) \) is said to be \textit{left Azumaya} provided the comonad
\[
\mathcal{C}l = C \otimes^D - : \mathcal{D} \mathcal{V} \to \mathcal{D} \mathcal{V}
\]
is Azumaya, i.e. (see [3.4], the comparison functor \( \mathcal{T}_\mathcal{D} : \mathcal{D} \mathcal{V} \to C \otimes^D C \mathcal{D} \mathcal{V} \) defined by
\[
V \mapsto (C \otimes^D V, C \otimes^D V \xrightarrow{\Delta \otimes^D \varepsilon} C \otimes^D C \otimes^D V \xrightarrow{\Delta \otimes^D \Delta \otimes^D \varepsilon} C \otimes^D C \otimes^D V)
\]
is an equivalence of categories.

In this setting, the results from Section 3 and also specializing Theorem 5.4 yield various characterisations of Azumaya \(\mathcal{D}\)-coalgebras.

Now let \( R \) be again a commutative ring with identity and \( \mathbb{M}_R \) the category of \( R \)-modules. As an additional notion of interest the dual algebra of a coalgebra comes in.

5.9. \textbf{Coalgebras in} \(\mathbb{M}_R\). An \( R \)-coalgebra \( \mathcal{C} = (C, \Delta, \varepsilon) \) consists of an \( R \)-module \( C \) with the \( R \)-linear maps multiplication \( \Delta : C \to C \otimes_R C \) and counit \( \varepsilon : C \to R \) subject to coassociativity and counitality conditions. \( C \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R \) is a comonad and it is customary to write \( C \mathbb{M} := \mathbb{M}_R^{C \otimes -} \) for the category of left \( \mathcal{C} \)-comodules. We write \( \text{Hom}^C(M, N) \) for the comodule morphisms between \( M, N \in C \mathbb{M} \). In general, \( C \mathbb{M} \) need not be a Grothendieck category unless \( C_R \) is a flat \( R \)-module (e.g. [1, 3.14]).

The dual module \( C^* = \text{Hom}_R(C, R) \) has an \( R \)-algebra structure by defining for \( f, g \in C^* \), \( f \cdot g = (g \otimes f) \cdot \Delta \) (definition opposite to [1, 1.3]) and there is a faithful functor
\[
\Phi : C \mathbb{M} \to C^* \mathbb{M}, \quad (M, \varrho) \mapsto C^* \otimes_R M \xrightarrow{\delta \otimes \varrho} C^* \otimes_R C \otimes M \xrightarrow{ev \otimes M} M,
\]
where \( ev \) denotes the evaluation map. The functor \( \Phi \) is full if and only if for any \( N \in \mathbb{M}_R \),
\[
\alpha_N : C \otimes_R N \to \text{Hom}_R(C^*, N), \quad c \otimes n \mapsto [f \mapsto f(c)n],
\]
is injective and this is equivalent to \( C_R \) being locally projective (\( \alpha \)-condition, e.g. [1, 4.2]). In this case \( C \mathbb{M} \) can be identified with the full subcategory \( \sigma_{[C^* C]} \subset C^* \mathbb{M} \) subgenerated by \( C \) as \( C^* \)-module.

The \( R \)-module structure of \( C \) is of considerable relevance for the related constructions and for convenience we recall:

5.10. \textbf{Remark.} For \( C_R \) the following are equivalent:
(a) \( C_R \) is finitely generated and projective;
(b) \( C \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R \) has a left adjoint;
(c) \( \text{Hom}_R(C, -) : \mathbb{M}_R \to \mathbb{M}_R \) has a right adjoint;
(d) $C^* \otimes_R - \to \operatorname{Hom}_R(C, -)$, $f \otimes_R - \mapsto (c \mapsto f(c) \cdot -)$, is a (monad) isomorphism;
(e) $C \otimes_R - \to \operatorname{Hom}_R(C^*, -)$, $c \otimes_R - \mapsto (f \mapsto f(c) \cdot -)$, is a (comonad) isomorphism;
(f) $\Phi : \mathcal{CM} \to \mathcal{CM}$ is a category isomorphism.

If this holds, there is an algebra anti-isomorphism $\operatorname{End}_R(C) \simeq \operatorname{End}_R(C^*)$ and we denote the canonical adjunction by $\eta^C, \varepsilon^C : C \otimes_R - \dashv C^* \otimes_R -$.

**5.11. The coalgebra** $C^e$. As in [5.11] the twist map $\tau_{C,C} : C \otimes_R C \to C \otimes_R C$ provides an (involutive) BD-law allowing for the definition of the opposite coalgebra $C^\tau = (C^\tau, \Delta^\tau, \varepsilon^\tau)$ and a coalgebra

$$
C^e := (C \otimes_R C^\tau, (C \otimes_R \tau \otimes_R C^\tau)(\Delta \otimes_R \Delta^\tau), \varepsilon \otimes_R \varepsilon).
$$

The category $C^e\mathcal{M}$ of left $C^e$-comodules is just the category of $(C, C)$-bicomodules (e.g. [13, 4.3.26]). A direct verification shows that the endomorphism algebra of $C$ as $C^e$-comodule is just the center of $C^e$, that is,

$$Z(C^e) = \operatorname{Hom}^{C^e}(C, C) \subset \mathcal{C} \operatorname{Hom}(C, C) \simeq C^*.$$

If $C_R$ is locally projective, an easy argument shows that $C \otimes_R C$ is also locally projective as $R$-module and then $C^e\mathcal{M}$ is a full subcategory of $(C^e)_*\mathcal{M}$.

**5.12. Definition.** An $R$-coalgebra $C$ is said to be an *Azumaya coalgebra* provided the comonad $\mathcal{G} = C \otimes_R - : \mathcal{M}_R \to \mathcal{M}_R$ is Azumaya, i.e. (see [3.4], the comparison functor $K : \mathcal{M}_R \to C^e\mathcal{M}$ defined by

$$M \mapsto (C \otimes_R M, C \otimes_R M \xrightarrow{\Delta \otimes_R M} C \otimes_R C \otimes_R M \xrightarrow{C \otimes \Delta^e \otimes_R M} C \otimes C \otimes_R C \otimes_R M)$$

is an equivalence of categories. We have the commutative diagram

$$
\begin{array}{cccc}
R\mathcal{M} & \xrightarrow{K} & C^e\mathcal{M} \\
C \otimes_R - & \searrow & & \downarrow C^eU \\
& & R\mathcal{M} & 
\end{array}
$$

By Proposition [1.15] the functor $K$ is an equivalence provided

(i) the functor $C \otimes_R - : R\mathcal{M} \to R\mathcal{M}$ is comonadic, and

(ii) the induced comonad morphism $C \otimes_R \operatorname{Hom}_R(C, -) \to C^e \otimes_R -$ is an isomorphism.

If $R \simeq \operatorname{End}^{C^e}(C) \simeq Z(C^*)$, the morphism in (ii) characterises $C$ as a $C^e$-*Galois comodule* as defined in [31, 4.1] and if $C_R$ is finitely generated and projective, the condition reduces to an $R$-coalgebra isomorphism $C \otimes_R C^* \simeq C^e$.

In module categories, separable coalgebras are well studied and we recall some of their characterisations (e.g. Section [1.19, 13, 9, 4.3.29], [2, 2.10]).

**5.13. Coseparable coalgebras.** An $R$-coalgebra $C = (C, \Delta, \varepsilon)$ is called *coseparable* if any of the following equivalent conditions is satisfied:

(a) $C \otimes_R - : \mathcal{M}_R \to \mathcal{M}_R$ is a separable comonad;
(b) $\Delta : C \to C \otimes_R C$ splits in $C^e\mathcal{M}$;
(c) $C$ is $(C^e, R)$-injective;
(d) the forgetful functor $C^e\mathcal{M} \to \mathcal{M}_R$ is separable;
(e) the forgetful functor $\mathcal{C}^e \mathbb{M} \to \mathbb{M}_R$ is separable;
(f) $\text{Hom}_R(C, -) : \mathbb{M}_R \to \mathbb{M}_R$ is a separable monad.

For any coseparable coalgebra $C$, $Z(C^*)$ is a direct summand of $C^*$.

**Proof.** Let $\omega : C \otimes_R C \to C$ denote the splitting morphism for $\Delta$. Then we obtain the splitting sequence of $Z(C^*)$-modules

$$\mathcal{C}^* \simeq \text{Hom}^{\mathbb{C}^e}(C, C \otimes_R C) \xrightarrow{\text{Hom}^{\mathbb{C}^e}(C, \omega)} \text{Hom}^{\mathbb{C}^e}(C, C) \simeq Z(C^*) \tag*{\Box}$$

For an Azumaya coalgebra $C$, the free functor $\phi: M_R \to C^e \mathbb{M}$ is monadic (see Theorem 3.11), and hence, in particular, it is conservative. It then follows that, for each $X \in \mathbb{M}_R$, the morphism $\varepsilon \otimes_R X : C \otimes_R X \to X$ is surjective. For $X = R$ this yields that $\varepsilon : C \to R$ is surjective (hence splitting). By Theorem 3.16 this means that $C$ is also a coseparable coalgebra.

It follows from the general Hom-tensor relations that the functor $K : \mathbb{M}_R \to C^e \mathbb{M}$ has a right adjoint $C^e \text{Hom}(C, -) : C^e \mathbb{M} \to \mathbb{M}_R$ (e.g. [4, 3.9]) and we denote the unit and counit of this adjunction by $\eta$ and $\varepsilon$, respectively.

Besides the characterisations derived from Theorem 5.4 we have:

**5.14. Characterisation of Azumaya coalgebras.** For an $R$-coalgebra $C$ the following are equivalent:

(a) $C$ is an Azumaya coalgebra;
(b) (i) $\varepsilon_X : C \otimes_R C^e \text{Hom}(C, X) \to X$ is an isomorphism for any $X \in \mathbb{M}_R$,
    (ii) $\eta_M : M \mapsto C^e \text{Hom}(C, C \otimes_R M)$ is an isomorphism for any $M \in \mathbb{M}_R$.
(c) $C$ is a $C^e$-Galois comodule, $C^*$ is a central $R$-algebra, and the functor $C \otimes_R -$ : $R\mathbb{M} \to R\mathbb{M}$ is comonadic;
(d) $C^*$ is an Azumaya algebra.

**Proof.** This is essentially Theorem 3.16. \tag*{\Box}

As shown in Proposition 5.3, an Azumaya coalgebra $C$ is finite in $\mathbb{M}_R$, that is, $C_R$ is finitely generated and projective (see Remark 5.10).

Coalgebras $C$ with $C_R$ finitely generated and projective for which $C^*$ is an Azumaya $R$-algebra were investigated by K. Sugano in [27]. As an easy consequence he also observed that an $R$-algebra $A$ with $A_R$ finitely generated and projective is Azumaya if and only if $A^*$ is an Azumaya coalgebra.

For vector space categories, Azumaya $D$-coalgebras $C$ over a cocommutative coalgebra $D$ (over a field) were defined and characterised in [28, Theorem 3.14].

**Acknowledgments.** This research was partially supported by Volkswagen Foundation (Ref.: I/85989). The first author also gratefully acknowledges the support of the Shota Rustaveli National Science Foundation Grant DI/12/5-103/11.

**References**

[1] Beck, J., *Distributive laws*, Lecture Notes in Math., 80, 119–140 (1969).
[2] Böhm, G., Brzeziński, T. and Wisbauer, R., *Monads and comonads in module categories*, J. Algebra 322, 1719–1747 (2009).
[3] Böhm, G. and Stefan, D., *Examples of para-cocyclic objects induced by BD-laws*, Algebra Represent. Theory 12(2-5), 153-180 (2009).
[4] Brzeziński, T. and Wisbauer, R., *Corings and Comodules*, London Math. Soc. LNS 309, Cambridge University press (2003).
[5] Cuadra, J. and Femic, B., *A Sequence to Compute the Brauer Group of Certain Quasi-Triangular Hopf Algebras*, Appl. Categor. Struct. 20, 433–512 (2012).
[6] Dubuc, E., *Kan extensions in enriched category theory*, Lecture Notes in Math. 145, Berlin-Heidelberg-New York, Springer-Verlag (1970).
[7] Femic, B., *Some remarks on Morita theory, Azumaya algebras and center of an algebra in braided monoidal categories*, Rev. Un. Mat. Argentina 51, 27–50 (2010).
[8] Fisher-Palmquist, J., *The Brauer group of a closed category*, Proc. Amer. Math. Soc. 50, 61–67 (1975).
[9] Guzmán, F., *Cointegrations, relative cohomology for comodules, and coseparable corings*, J. Algebra 126(1), 211–224 (1989).
[10] Johnstone, P., *Adjoint lifting theorems for categories of algebras*, Bull. London Math. Soc. 7, 294–297 (1975).
[11] Kasangian, S., Lack, S. and Vitale, E.M., *Coalgebras, braidings, and distributive laws*, Theory Appl. Categ. 13(2), 129–146 (2004).
[12] Kelly, G.M. and Power, A.J., *Adjunctions whose counits are coequalizers, and presentations of finitary enriched monoids*, J. Pure Appl. Algebra 89, 163–179 (1993).
[13] Larson, R.G., *Coseparable Hopf algebras*, J. Pure Appl. Algebra 3, 261–267 (1973).
[14] Mac Lane, S., *Categories for the Working Mathematician*, 2nd edn, Springer-Verlag, New York (1998).
[15] Mesablishvili, B., *Entwining Structures in Monoidal Categories*, J. Algebra 319(6), 2496–2517 (2008).
[16] Mesablishvili, B., *Monads of effective descent type and comonadicity*, Theory Appl. Categ. 16, 1–45 (2006).
[17] Mesablishvili, B. and Wisbauer, R., *Bimonads and Hopf monads on categories*, J. K-Theory 7(2), 349–388 (2011).
[18] Mesablishvili, B. and Wisbauer, R., *Galois functors and entwining structures*, J. Algebra 324, 464–506 (2010).
[19] Mesablishvili, B. and Wisbauer, R., *Notes on bimonads and Hopf monads*, Theory Appl. Categ. 26, 281–303 (2012).
[20] Mesablishvili, B. and Wisbauer, R., *QF functors and (co)monads*, J. Algebra 376, 101–122 (2013).
[21] Pareigis, B., *Non-additive ring and module theory IV. The Brauer group of a symmetric monoidal category*, Lecture Notes in Math., 549, 112–133 (1976).
[22] Pareigis, B., *Non-additive ring and module theory II. C-categories, C-functors and C- morphisms*, Algebra Berichte, 24, 351–361 (1977).
[23] Rafael, M.D., *Separable functors revisited*, Comm. Algebra, 18, 1445–1459 (1990).
[24] Saavendra, R., *Categories Tannakiennes*, Lecture Notes in Math., 265, Berlin-Heidelberg-New York, Springer-Verlag (1972).
[25] Sobral, M., *Restricting the comparison functor of an adjunction to projective objects*, Quaest. Math., 6, 303–312 (1983).
[26] Street, R., *Quantum Groups*, Cambridge University Press (2007).
[27] Sugano, Kozo, *A characterization of Azumaya coalgebras over a commutative ring*, J. Math. Soc. Japan 34(4), 719–726 (1982).
[28] Torrecillas, B., Van Oystaeyen, F. and Zhang, Y. H., *The Brauer group of a cocommutative coalgebra*, J. Algebra 177(2), 536–568 (1995).
[29] Van Oystaeyen, F. and Zhang, Yinhuo, *The Brauer group of a braided monoidal category*, J. Algebra 202(1), 96–128 (1998).
[30] Wisbauer, R., *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia (1991).
[31] Wisbauer, R., *On Galois comodules*, Commun. Algebra 34, 2683–2711 (2006).
[32] Wisbauer, R., *Algebra Versus Coalgebras*, Appl. Categor. Struct. 16, 255–295 (2008).

Addresses:
A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6, Tamarashvili Str., and Tbilisi Centre for Mathematical Sciences, Chavchavadze Ave. 75, 3/35, Tbilisi 0177, Georgia.

bachi@rmi.ge

Department of Mathematics of HHU, 40225 Düsseldorf, Germany,
wisbauer@math.uni-duesseldorf.de