Genuine-multipartite entanglement criteria based on positive maps

Fabien Clivaz,1 Marcus Huber,2 Ludovico Lami,3 and Gláucia Murta4,5
1Group of Applied Physics, University of Geneva, 1211 Geneva 4, Switzerland
2Institute for Quantum Optics and Quantum Information (IQOQI),
Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria
3Física Teórica: Informació i Fenòmens Quàntics,
Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain
4QuTech, Delft University of Technology,
Lorentzweg 1, 2628 CJ Delft, the Netherlands
5Departamento de Física, Universidade Federal de Minas Gerais,
Caixa Postal 702, 30123-970, Belo Horizonte, MG, Brazil

Positive maps applied to a subsystem of a bipartite quantum state constitute a central
tool in characterising entanglement. In the multipartite case, however, the direct application
of a positive but not completely positive map cannot distinguish if a state is genuinely multi-
partite entangled or just entangled across some bipartition. We thus generalise this bipartite
concept to the multipartite setting by introducing non-positive maps that are positive on the
subset of bi-separable states, but can map to a non-positive element if applied to a genuine
multipartite entangled state. We explicitly construct examples of multipartite non-positive
maps, obtained from positive maps via a lifting procedure, that in this fashion can reveal
genuine multipartite entanglement in a robust way.

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I. INTRODUCTION

Due to the importance of entanglement as a resource for quantum information processing,
the task of determining whether a quantum state is entangled or not plays a crucial role for the
theoretical and practical developments of the field. While the usefulness of bipartite entanglement
is well established, the development of applications using entanglement in multipartite systems
is still in its early stages, however, it has already been shown to play a fundamental role for
universal quantum computation in the measurement-based quantum computation paradigm [1]
and for cryptographic tasks such as secret sharing [2].

Concerning the characterisation of entanglement, one of the biggest challenges to start with is
the fact that it is NP-hard in the Hilbert space dimension to decide whether a given quantum state
is entangled [3]. Since the dimension grows exponentially in the number of involved parties, the
exact answer to that question will probably remain elusive for many-body systems.

Nonetheless there exists an abundance of necessary separability criteria capable of certifying en-
taglement in a practically satisfying way. For bipartite entanglement, the two most paradigmatic
techniques are positive maps which are not completely positive [1] and entanglement witnesses [5].
These two concepts are intimately related, as every positive map directly leads to a multitude of
entanglement witnesses and every witness can be associated to a positive map [4]. Both concepts
are sufficient for the characterisation of entanglement, in the sense that, for every entangled state,
there exists both a positive map and an entanglement witness certifying its entanglement [4, 5].

Turning to multipartite systems, the potentially underlying separability structure adds a layer of complexity to the detection of entanglement. As opposed to just entanglement in multipartite systems [6], mixed multipartite states pose another fundamental challenge. Counterintuitively, multipartite systems can exhibit entanglement across every partition, yet still not feature any genuine multipartite entanglement (GME). To certify GME, an abundance of entanglement witnesses were derived (see e.g. reviews [7, 8]) and a characterisation in terms of semi-definite programs (SDP) was developed [9, 10]. There is however no direct correspondence to the concept of positive map based criteria, not even for the undoubtedly most used criterion of positivity under partial transposition (PPT). Indeed, the naive application of a positive map to a subsystem in correspondence to the bipartite case must inevitably fail, as all it can reveal is entanglement across that bipartition, which, as we just mentioned, is never enough to infer GME.

In this manuscript we expand the notion of positive map based criteria from the bipartite to the multipartite case, thus filling a gap in the set of available tools. Our main idea is to derive maps which are positive on all biseparable states, whilst not positive on the set of all states. We first give a general description of the approach and then showcase some exemplary maps. The maps we derive are based on convex combinations of positive maps that are used in the bipartite case, thus generalising the PPT criterion and others to the multipartite case.

The manuscript is organised as follows: In Section II we set the stage by recalling some important bipartite and multipartite entanglement definitions and giving the motivation of this work. In Section III we introduce our framework and general construction method based on positive maps. We also make some general existence considerations. In Section IV we give some explicit examples of our method. In particular, a remarkably simple partial transposed based criteria is worked out and a Choi map construction is shown to robustly detect noisy n-qudit GHZ-like states as well as PPT states. Finally in Section V we discuss possible extensions and applications of our method.

II. PRELIMINARIES

To set the stage and notation we first remind the reader of the concepts of separability, entanglement witnesses and positive maps in the bipartite case. A finite dimensional state is called separable iff it can be decomposed into a convex combinations of pure product states, i.e.

\[ \rho_{\text{sep}} := \sum_i p_i |\phi_A^i \rangle \langle \phi_A^i| \otimes |\phi_B^i \rangle \langle \phi_B^i|, \]  

(1)

where \( p_i \) is a probability distribution.

The states which can be written in the form of Eq. (1) form a closed convex set and therefore, as a consequence of the Hahn-Banach theorem (see [4]), the set of separable states can be separated from any point in its complement by a hyperplane that can be written as Tr(\( \rho W \)) = 0 for a self-adjoint operator (observable) \( W = W^\dagger \). Now if (by convention) Tr(\( \rho_{\text{sep}} W \)) ≥ 0 for all \( \rho_{\text{sep}} \) and there exits at least one (entangled) state \( \rho \) such that Tr(\( \rho W \)) < 0, the operator \( W \) is referred to as an entanglement witness. The above thus states that any entangled state can be detected by an entanglement witness \( W \).
From another perspective a linear positive map $\Lambda$, $\Lambda[\alpha \rho + \beta \sigma] = \alpha \Lambda[\rho] + \beta \Lambda[\sigma]$, $\Lambda[\rho] \geq 0 \forall \rho \geq 0$, can be used to detect entanglement if it is not completely positive, since the application of a non-completely positive map to an entangled state $\rho_{AB} \geq 0$ can result in a non-positive operator, $\Lambda_A \otimes I_B[\rho_{AB}] \nless 0$. The fact that the extension of a positive map remains positive on separable states can be easily seen by applying it to the state $\rho_{sep}$ in Eq. (1). The equivalence of the two approaches was established in Ref. [4], where the authors had proven that for every entangled state $\rho_{AB}$ there exists a positive but non-completely positive map, $\Lambda_A$, whose extension applied to $\rho_{AB}$ maps it into an operator with negative eigenvalues.

It is straightforward to generate a witness out of a positive map: if $\Lambda_A \otimes I_B[\rho_{AB}] \nless 0$, it implies that there exists a pure state $|\psi\rangle$ such that $\text{Tr}(|\psi\rangle\langle\psi|\Lambda_A \otimes I_B[\rho_{AB}]) < 0$. Now, by invoking the concept of the dual of a map $\Lambda^*$, uniquely defined by $\text{Tr}(\rho \Lambda[\rho']) = \text{Tr}(\Lambda^*\rho \rho')$, $\forall \rho, \rho'$, it can immediately be seen that $\Lambda_A^* \otimes I_B[|\psi\rangle\langle\psi|]$ is an entanglement witness detecting $\rho_{AB}$. The converse construction goes as follows. Every witness $W$ is a block-positive operator, i.e. it is positive on product vectors: $\langle \alpha\beta | W | \alpha\beta \rangle \geq 0$. This implies the positivity of the map $\Lambda$ whose Choi matrix is $W$, defined in the computational basis (up to a constant) by $\Lambda(|i\rangle\langle j|) = \sum_{k,l} (|i\rangle |W|kj\rangle) |l\rangle |m\rangle$. If $W$ detects $\rho$, then $0 > \text{Tr} W \rho = \text{Tr}(\Lambda \otimes I)(|\varepsilon\rangle\langle\varepsilon|) \rho = \langle\varepsilon| (\Lambda^* \otimes I)(\rho) |\varepsilon\rangle$, where $|\varepsilon\rangle = \sum_i |ii\rangle$ is the (unnormalised) maximally entangled state on the bipartite system $AB$. Thus, we see that the positive map $\Lambda^*$ reveals the entanglement of $\rho$. For details, we refer the reader to [4].

To appreciate the challenges in the multipartite case, we first review the different levels of separability a system can exhibit. The strongest notion of separability is the complete absence of any type of entanglement, i.e. fully separability. An $n$-partite finite dimensional quantum state $\rho_{sep} \in \mathcal{P}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ is called fully-separable, and will be denoted by $\rho_{sep}$, iff it can be decomposed as

$$
\rho_{sep} = \sum_i p_i \rho_1^i \otimes \cdots \otimes \rho_n^i,
$$

where as before $p_i$ is a probability distribution. If a quantum state $\rho$ cannot be decomposed into the form (2) there must be some entanglement in the system. This concept of separability can also be revealed in terms of general linear maps [6]. However, as we have mentioned before, for multipartite systems, many levels of separability can exist, according to how many subsystems share entanglement. We now present the weakest notion of separability, usually referred to as biseparability: Let $\rho_{2-sep} \in \mathcal{P}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ be an $n$-partite quantum state and let $A \subset \{1, \ldots, n\}$ denote a proper subset of the parties. A state $\rho_{2-sep}$ is biseparable iff it can be decomposed as

$$
\rho_{2-sep} = \sum_A \sum_i p_A^i \rho_A^i \otimes \rho_A^i \otimes \rho_A^i, \quad p_A^i \geq 0, \quad \sum_A \sum_i p_A^i = 1,
$$

where $\rho_A$ denotes a quantum state for the subsystem defined by the subset $A$ and $\sum_A$ stands for the sum over all bipartitions $A \bar{A}$.

An $n$-partite state which cannot be decomposed as (3) is called genuine $n$-partite entangled. If the number of parties is clear from the context, we call those states genuine multipartite entangled (GME). Note that since the biseparable states form a convex closed set, the Hahn-Banach theorem ensures that given any GME-state there exists a GME-witness detecting it. Just as in the bipartite
FIG. 1: An illustration of the connections between entanglement witnesses and non-positive maps. The bottom of the figure illustrates the bipartite situation, where the non-positive maps $\Phi$ are obtained via tensoring a positive map with the identity. The top represents the multipartite situation, where the non-positive maps $\Phi_{\text{GME}}$, so called GME-maps, can in some cases be constructed from positive maps via a lifting procedure. $W$ and $W_{\text{GME}}$ denote the witnesses of the respective cases. The black arrows denote previously known connections, the red arrows indicate our contribution with this paper.

In this Section we present a method of developing GME criteria based on positive but not completely positive maps. Before we introduce our method, we first note that it is always possible to find a GME-map, as defined in Eq. (4), detecting a given GME-state. Indeed let $\rho_{\text{GME}}$ be a setting, the witnesses of genuine multipartite entanglement (GME-witnesses) are defined by hermitian operators $W_{\text{GME}}$ that, for all $\rho_{2-\text{sep}}$, fulfil $\text{Tr}(\rho_{2-\text{sep}}W_{\text{GME}}) \geq 0$ and for which there exists a multipartite state $\rho$ such that $\text{Tr}(\rho W_{\text{GME}}) < 0$. However, as mentioned in the introduction, positive maps fail to capture the concept of genuine multipartite entanglement for a simple reason: applying a positive map to any marginal subsystem $A$ of a biseparable state does not necessarily result in a positive operator, as it only guarantees positivity for the portion of the state which is separable with respect to the partition $A|\bar{A}$.

Let us remark that, for bipartite systems, the crucial point of entanglement criteria based on positive maps $\Lambda$ is the following: If we consider their extension $\Phi := \Lambda_A \otimes I_B$ they become non-positive maps which are nonetheless positive on all separable states, see Fig. 1. This is the point of view that allows for a straightforward multipartite generalisation. The objects we are thus looking for are non-positive maps $\Phi_{\text{GME}}$ such that

$$
\Phi_{\text{GME}}[\rho_{2-\text{sep}}] \geq 0, \quad \forall \rho_{2-\text{sep}}.
$$

We call those maps GME-maps. In the following Sections we explore the possibility of constructing such maps from the lifting of positive maps (see Fig. 1 for an illustration of our concept).

### III. GME CRITERIA BASED ON POSITIVE MAPS

In this Section we present a method of developing GME criteria based on positive but not completely positive maps. Before we introduce our method, we first note that it is always possible to find a GME-map, as defined in Eq. (4), detecting a given GME-state. Indeed let $\rho_{\text{GME}}$ be a
GME-state. Then we know from \cite{10} that there exists a GME-witness detecting \( \rho_{\text{GME}} \) of the form

\[
W_{\text{GME}} = \sum_A \Lambda_A^* \otimes I_A (|\psi_A\rangle \langle \psi_A|) + M_{\{\Lambda_A,|\psi_A\rangle\}_A},
\]

(5)

where for each partition \( A \), \( \Lambda_A \) is a positive map, \( |\psi_A\rangle \) is chosen such that \( \langle \psi_A| \Lambda_A \otimes I_A (\rho_{\text{GME}}) |\psi_A\rangle < 0 \), and \( M_{\{\Lambda_A,|\psi_A\rangle\}_A} \) is a positive matrix depending on the choice of \( \Lambda_A \) and \( |\psi_A\rangle \). In Appendix \( A \) it is shown how one obtains \( W_{\text{GME}} \) in (5) from \cite{10} and in particular how \( M_{\{\Lambda_A,|\psi_A\rangle\}_A} \) is constructed. Then the map

\[
\Phi_{\text{GME}}[\rho] = \text{Tr}(W_{\text{GME}} \cdot \rho) I
\]

(6)

is a GME-map detecting \( \rho_{\text{GME}} \). We should also note that given any GME-map one can associate a GME-witness to each state detected by the map in the same way as done in the bipartite setting. Indeed given \( \Phi_{\text{GME}} \) and \( \rho_{\text{GME}} \) such that \( \Phi_{\text{GME}}(\rho_{\text{GME}}) \not\geq 0 \), there exists a pure state \( |\psi\rangle \) such that

\[
\Phi_{\text{GME}}^*[|\psi\rangle \langle \psi|] \text{ is a GME-witness.}
\]

This looks all good but upon inspecting the GME-map defined by Eq.(6), one remarks that it is only gained from positive maps in an indirect way. Indeed from the family of maps \( \{\Lambda_A \otimes I_A\}_A \) (bottom left of Fig. \( 1 \)), a family of bipartite witnesses \( \{\Lambda_A^* \otimes I_A (|\psi_A\rangle \langle \psi_A|)\}_A \) was first associated (bottom right of Fig. \( 1 \)), from which a GME-witness was extracted (top right of Fig. \( 1 \)). This GME-witness finally defined the non-positive map \( \text{Tr}(W_{\text{GME}} \cdot \rho) I \) of Eq.(6) (top left of Fig. \( 1 \)).

Now, our goal is to explore a direct lifting method, illustrated by the red arrow from \( \Phi \) to \( \Phi_{\text{GME}} \) in Fig. \( 1 \) from positive maps to GME maps, without passing through the witnesses. The reason for pursuing this is that GME-maps of the form of Eq.(6) are quite trivial in the sense that the boundary of the set \( \{\rho \mid \text{Tr}(W_{\text{GME}} \cdot \rho) I \geq 0\} \) is a hyperplane rather than the boundary of a more complex convex set, which could possibly explore more subtleties of genuine multipartite entanglement.

Inspiring ourselves from the structure of \( W_{\text{GME}} \) in Eq. (5), our goal is to seek for maps of the form

\[
\Phi_{\text{GME}} := \sum_A \Lambda_A \otimes I_A \circ U^{(A)} + M,
\]

(7)

where \( M \) is a positive map, \( U^{(A)}[\rho] := \sum_i P_i^{(A)} U_i^{(A)} \rho (U_i^{(A)})^\dagger \) is a family of convex combinations of local unitaries, and \( \Phi_{\text{GME}}[\rho_{\text{sep}}] \geq 0 \forall \rho_{\text{sep}} \). In Appendix \( B \) we derive some general conditions as to when one can ensure that for a given GME-state a GME-map of the form of Eq. (7) detecting this GME-state exists. Unfortunately, it is not clear from those conditions whether the construction of \( \Phi_{\text{GME}} \) via lifting of positive maps is general enough, that is, whether for every GME-state \( \rho_{\text{GME}} \), there exists a GME-map of the form of Eq. (7) detecting \( \rho_{\text{GME}} \).

After those general considerations, we want to give in the remaining of the manuscript concrete examples of our method by exhibiting GME-maps constructed via the lift of positive but non-completely positive maps.
IV. SOME DIRECT LIFTING EXAMPLES

A. Transposition-based GME criteria

Our first example is possibly the most naive attempt, where we set $M = cI \cdot \text{Tr}$, $U^{(A)} = I$ and $\Lambda = T$, the transposition, in a tripartite setting, and find the value of $c$ for which $\Phi_{\text{GME}}$ defined by Eq. 7 is a GME-map. That is, we consider the map

$$\Phi_T[\cdot] = (T_A \otimes I_B \otimes I_C + I_A \otimes T_B \otimes I_C + I_A \otimes I_B \otimes T_C + c I \cdot \text{Tr})[\cdot].$$

(8)

**Theorem 1.** For $c = 1$ it holds that for all tripartite biseparable states $\rho_{2-sep}$

$$\Phi_T[\rho_{2-sep}] \geq 0,$$

(9)

and this value of $c$ is optimal, i.e. it is the least compatible with the above constraint.

The proof of Theorem 1 consists in determining the minimal eigenvalue of the operators $T_A \otimes I_{\overline{A}}$, which turns out to be $-\frac{1}{2}$ for any partition $A$ (see [11]). The detailed proof of Theorem 1 is presented in Appendix C.

The intuition behind this construction rests upon the prerequisite that the negative eigenstates under application of partial transposition for every cut have at least some nonzero overlap. As we will see in the following examples, some states will exhibit this property and the map works in a straightforward way. The additional unitary transformation in eq. (7) is supposed to systematically map the corresponding negative eigenstates into the same space, opening up the possibility to detect many more states. We will see that such unitary corrections can be readily constructed, but their success depends also on the symmetries exhibited by the state.

To start, let us consider an example where the map directly works. If we consider the state

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$

(10)

we find $\Phi_T[|W\rangle\langle W|] \neq 0$ with negative eigenvalue $1 - \frac{2}{\sqrt{3}} \approx -0.15$ and therefore $\Phi_T$ provides a GME criterion. The map $\Phi_T$ can detect the noisy $W$ state, $p \ |W\rangle\langle W| + (1-p) \ I_8$, for $p > \frac{11\sqrt{3}}{16+3\sqrt{3}} \approx 0.90$.

The map $\Phi_T$ can be modified to detect the GHZ state, by choosing $U^{(A)} = \tilde{\sigma}_x^A$, where $\tilde{\sigma}_x^A[\rho] = \prod_{i \in A} \sigma_{x_i} \otimes I_{\overline{i}} \rho \prod_{i \in A} \sigma_{x_i} \otimes I_{\overline{i}}$. Let us consider

$$\Phi_{Tx}[\cdot] = (\tilde{\sigma}_x \circ T_A \otimes I_B \otimes I_C + I_A \otimes \tilde{\sigma}_x \circ T_B \otimes I_C + I_A \otimes I_B \otimes \tilde{\sigma}_x \circ T_C + I \cdot \text{Tr})[\cdot]$$

(11)

where the positive map $\tilde{\sigma}_x \circ T$ denotes the transposition followed by the application of the unitary Pauli operator $\sigma_x$. Since we only added a local unitary operation, the map $\Phi_{Tx}$ remains positive on all biseparable states. One calculates $\Phi_{Tx}[|GHZ\rangle\langle GHZ|] \neq 0$ with negative eigenvalue $-\frac{1}{2}$ and $p > \frac{11}{15} \approx 0.73$. Therefore, $\Phi_{Tx}$ is also a GME-map and it detects the noisy GHZ state more robustly than $\Phi_T$ detects the $W$ state.

The reason why the map $\Phi_{Tx}$ successfully detects the GHZ state, as opposed to $\Phi_T$, is that by composing the transposition with the unitary operation $\sigma_x$ we are exploring the symmetries of the
state: note that the off-diagonal elements of the GHZ state remain invariant under the application of partial transposition followed by the Pauli matrix $\sigma_x$. Further considerations of symmetry will be made in section [IV C].

The map $\Phi_{Tx}$ can be generalised for an arbitrary number of parties. Let $A \subset \{1, \ldots, n\}$ be a proper subset of the parties, we define the map

$$\Phi_{Tx, n}[\cdot] = \left(\sum_{A} \tilde{\sigma}^A_X \circ T_A \otimes \mathbb{I}_{\bar{A}} + (2^{n-1} - 2)\frac{1}{2} \mathbb{I} \cdot \text{Tr}\right)[\cdot],$$  

(12)

**Theorem 2.** For every $n$-partite biseparable state $\rho_{2-\text{sep}}$ it holds that

$$\Phi_{Tx, n}[\rho_{2-\text{sep}}] \geq 0.$$  

(13)

The proof of Theorem 2 follows in the same line as the proof of Theorem 1. Moreover, $\Phi_{Tx, n}[|GHZ_n\rangle\langle GHZ_n|] \not\geq 0$ for all $n$, with negative eigenvalue $-\frac{1}{2}$ which implies $p = \frac{2^{2n-2} - 2^{n-1}}{2^{2n-2} - 2^{n-1} - 1} \to 1$ as $n \to \infty$, where $GHZ_n$ is the $n$-partite GHZ state. The map $\Phi_T$ can also be similarly generalised for $n$ parties, however, already for $n = 4$, the $n$-partite $W$ state is no more detected by it. We therefore see that this construction although generating GME-maps, does not provide a noise resistant detection. This might have two causes, namely the choice of the positive map or the naive choice of $M$. See sections [IV B] or [IV C] for less naive constructions.

**B. Optimized transposition criteria**

In this section we will modify the map $\Phi_{Tx, n}$ of section [IV A] by looking at a slightly better choice for the correction map $M$. This will enable us to detect the GHZ state with an optimal noise resistance. We will choose $M = (2^{n-1} - 2)\text{Diag} \circ \phi$, where here $\phi = \sum_{A} \tilde{\sigma}^A_X \circ T_A \otimes \mathbb{I}_{\bar{A}}$ and $\text{Diag}[\rho]$ maps $\rho$ to a diagonal matrix with the same elements as $\rho$. Hence we look at the map

$$\eta := \phi + (2^{n-1} - 2)\text{Diag} \circ \phi.$$  

(14)

We also want to project our $n$-qubit state onto a subspace before applying $\eta$ to it. The subspace we want to project onto is

$$\text{span}\{|i\rangle \langle j| \mid |i\rangle \langle j| \text{GHZ}_{n, \text{cyclic}} |j\rangle \neq 0\},$$  

(15)

where for $X^C := \mathbb{I} \otimes X^{C_2} \otimes \cdots \otimes X^{C_n}$, with $X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $C := (C_2, \ldots, C_n) \in \{0, 1\}^{n-1}$, we have

$$\text{GHZ}_{n, \text{cyclic}} := \sum_{C \in \{0, 1\}^{n-1}} X^C |\text{GHZ}_n\rangle \langle \text{GHZ}_n| (X^C)^\dagger.$$  

(16)

The class of states invariant under the projection onto subspace (16) is also known as X-states [16], and genuine multipartite entanglement in this class of states was fully characterised in Ref. [17]. The notation is more extensively explained in section [IV C]. As we are going to see in Proposition 5
in section IV.C, it turns out that one can project onto this subspace via a mixture of local unitaries (the assertion holds for any \(d \geq 2\) but for now we are interested in the case \(d = 2\)), thus ensuring that no entanglement is created in the process. We denote this projection by \(\mathcal{X}_n\). Our map of interest is thus \(\eta \circ \mathcal{X}_n\).

**Theorem 3.** \(\eta \circ \mathcal{X}_n[\rho_{2\text{-sep}}] \geq 0\), for all \(n\)-qubit biseparable states \(\rho_{2\text{-sep}}\).

**Proof.** The proof relies on the following fact:

**Lemma 4.**

\[
OD \circ \phi[\mathcal{X}_n[\rho]] = (2^{n-1} - 1)OD[\tilde{\sigma}^A \circ T_A \otimes I_A[\mathcal{X}_n[\rho]], \quad \forall \rho, \forall A,
\]

where \(OD[X] := X - \text{Diag}[X] \) and \(\phi = \sum_A \tilde{\sigma}^A \circ T_A \otimes I_A\).

The intuition behind this fact is that for X-states, the off-diagonal elements are equally permuted by partial transposition in subsystem \(A\) and \(\sigma_x\) flips in the same subsystem, i.e. all their off-diagonal elements are invariant under application of \(\tilde{\sigma}^A \circ T_A \otimes I_A\).

Now let us consider an \(n\)-partite state biseparable with respect to partition \(A|\bar{A}\): \(\rho = \rho_A \otimes \rho_{\bar{A}}\). By projecting \(\rho\) onto the above subspace the resultant state \(\hat{\rho} := \mathcal{X}_n(\rho)\) is still biseparable with respect to partition \(A|\bar{A}\) since the projection \(\mathcal{X}_n\) is a separable operation. Analysing the map \(\eta\) applied to \(\hat{\rho}\) gives:

\[
\eta[\hat{\rho}] = \text{Diag}[\phi[\hat{\rho}]] + OD[\phi[\hat{\rho}]] + (2^{n-1} - 2)\text{Diag}[\phi[\hat{\rho}]] = \text{Diag}[\tilde{\sigma}^A \circ T_A \otimes I_A[\hat{\rho}]] + \sum_{B \neq A} \text{Diag}[\tilde{\sigma}^B \circ T_B \otimes I_B[\hat{\rho}]] \\
+ (2^{n-1} - 1)OD[\tilde{\sigma}^A \circ T_A \otimes I_A[\hat{\rho}]] \\
+ (2^{n-1} - 2)\text{Diag}[\tilde{\sigma}^A \circ T_A \otimes I_A[\hat{\rho}]] \\
+ (2^{n-1} - 2)\sum_{B \neq A} \text{Diag}[\tilde{\sigma}^B \circ T_B \otimes I_B[\hat{\rho}]] \\
\geq 0,
\]

where in the second step we have used Lemma 4.

The proof of Theorem 3 for an arbitrary biseparable state \(\rho\) follows from the linearity of the map \(\eta \circ \mathcal{X}_n\).

The map \(\eta \circ \mathcal{X}_n\) furthermore detects the \(n\)-qubit GHZ state but now with a improved noise resistance. Indeed for all \(n \geq 2\),

\[
\eta \circ \mathcal{X}_n[p \text{ GHZ}_n + (1 - p) \frac{I}{2^n}] \not\geq 0, \quad \forall 1 \geq p > \frac{2^{n-1} - 1}{2^n - 1}.
\]
For \( n = 3 \) for example this means \( 1 \geq p > \frac{3}{7} \), meaning \( \eta \circ X_3 \) optimally detects the 3-qubit GHZ state. In fact, it reproduces the necessary and sufficient conditions first presented in Ref. [15] for all GHZ-diagonal states and any number of qubits \( n \). This has the added benefit that the map is simple to apply and invariant under many local unitary operations, detecting a multitude of states using the same criterion.

C. Choi-based GME criteria

The transposition-based GME criteria of section [IV A] illustrates the general idea of our method and provides a GME criteria for any number of parties. However the application of Theorem [1] to noisy states does not provide a robust criterion. Modifying the correction term \( M \) as in section [IV B] delivered an improved criteria for \( n \)-qubit GHZ states that was even revealed to be optimal for any \( n \). We now want to optimise the criterion for \( n \)-qubit GHZ states by defining a GME-map based on the non-completely positive Choi-map [12]. An important step in the construction of our GME-map is to explore the symmetries of the Choi-map in order to add a correction term reflecting them. As a result we obtain a map which is very robust to detect GHZ-like states for any dimension and number of parties.

The construction of the Choi-based GME criteria consists, as in section [IV B] of two steps. The first step is a projection onto a subspace of GHZ-like states by a mixture of local unitary operations, and the second step is the application of a non-positive map based on the Choi-map.

First of all, let us introduce some notation. A suitable generalisation of the GHZ state to \( n \)-parties and \( d \)-dimensions can be defined as

\[
|GHZ^d_n\rangle = \frac{1}{\sqrt{d}}(|00\ldots0\rangle + |11\ldots1\rangle + \ldots + |(d-1)(d-1)\ldots(d-1)\rangle). \tag{19}
\]

And we denote by GHZ-like state an \( n \)-partite \( d \)-dimensional state that differs from (19) by local unitary operations. We will be interested in the subspace generated by cyclic permutations of \( |GHZ^d_n\rangle \), which we can think of as the subspace that leaves invariant the family of states that generalises X-states to higher dimensions. In order to describe our subspace we are going to use the shift operator, the generator of cyclic permutations:

\[
X_d := \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0 & 1 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}, \tag{20}
\]

and consider the matrix

\[
GHZ_{n,cyclic}^d := \sum_{C \in \{0,1,\ldots,d-1\}^{n-1}} X_d^C |GHZ_n^d\rangle \langle GHZ_n^d| (X_d^C)^\dagger, \tag{21}
\]
where for $C := (C_2, \ldots, C_n) \in \{0, 1\}^{n-1}$, $X^C := \mathbb{I} \otimes X^{C_2} \otimes \cdots \otimes X^{C_n}$. The span of $\{|i\langle j| \mid \langle i|GHZ_{n,cyclic}^d |j\rangle \neq 0\}$ defines a subspace that we denote by $\{GHZ_{n,cyclic}^d\}$. Note that $\{GHZ_{n,cyclic}^2\}$ is the subspace considered in section IV.B.

We now want to show that we can project any $n$-partite state $\rho \in \mathcal{P}(C^d \otimes \cdots \otimes C^d)$ onto $\{GHZ_{n,cyclic}^d\}$. In order to prove it we construct our projection by making use of the clock-matrices:

$$Z_k = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & e^{2\pi ik/d} & \cdots & 0 \\
0 & 0 & e^{2\pi ik2/d} & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & e^{2\pi i(d-1)k/d}
\end{pmatrix}, \quad (22)$$

where $k \in \{0, 1, \ldots, d-1\}$.

Furthermore, let $\rho \in \mathcal{P}(C^d \otimes \cdots \otimes C^d)$ be an $n$-qudit state. For $i \in \{1, \ldots, n\}$ we define

$$h_{n,i}^d(\rho) := \frac{1}{d} d_{i}^{1} \sum_{k=0}^{d-1} Z_k^i (Z_k^i)^\dagger \rho (Z_k^i)^\dagger Z_k^i, \quad (23)$$

where $Z_k^i := \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes Z_k \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}$.

And we finally construct the operator:

$$\Lambda_n^d[\cdot] := \sum_{j=2}^{n} h_{n,j}^d[\cdot]. \quad (24)$$

The following proposition states the equivalence of the the map (24) with the projection on the subspace $\{GHZ_{n,cyclic}^d\}$.

**Proposition 5.** For all $n \in \mathbb{N}$, $d \geq 2$, $\Lambda_n^d[\rho]$ projects any $n$-partite $d$-dimensional state $\rho$ into the subspace $\{GHZ_{n,cyclic}^d\}$.

The proof of Proposition 5 can be found in Appendix D.

Now that we have a way to project any $n$-qudit state into the subspace spanned by GHZ-like cyclically permuted states, we can continue to discuss the positive maps employed.

**Definition 6.** The Choi-map $\Lambda$ of dimension $d$ is defined as follows

$$\Lambda[\rho] = 2 \text{Diag}[\rho] + \sum_{j=1}^{d-2} X_j^d \text{Diag}[\rho] X_j^d \dagger - \rho, \quad (25)$$

where $X_d$ is the shift operator defined in Eq. (20).

We next want to look at

$$\phi = \sum_A \Lambda_A \otimes I_A, \quad (26)$$
where $A \subset \{1, \ldots, n\}$ and $\Lambda_A$ denote the Choi-map applied to the subsystems contained in $A$. With the above we can consider the map

$$
\mu[\rho] = \phi[\rho] + (2^{n-1} - 2) \left[ \text{Diag}[\phi[\rho]] - \sum_A \text{Diag}[\rho] \right].
$$

(27)

Now we are ready to state the main result of this section.

**Theorem 7.** $\mu \circ X_n^d[\rho_{2-sep}] \geq 0$.

Note that in order to express $\mu \circ X_n^d$ in the form of $\Phi_{GME}$ of equation (7) in section III, we should set $U(A) = X_n^d$ for every $A$ and $M = (2^{n-1} - 2) \left[ \text{Diag}(\phi[X_n^d(\rho)]) - \sum_A \text{Diag}(X_n^d(\rho)) \right]$. Theorem 7 gives us a sufficient condition for a state to be GME: if upon projecting an $n$-partite state $\rho$ into the GHZ-like cyclically permuted subspace and applying the map $\mu$ results into a negative eigenvalue, one can assure that the state $\rho$ is genuinely $n$-partite entangled.

**Proof.** The proof is analogous to the one of Theorem 3 of section IV B. Indeed, we make use of the following Lemma that will be proven in Appendix E.

**Lemma 8.**

$$
OD \circ \phi[X_n^d[\rho]] = (2^{n-1} - 1) OD[\Lambda_A \otimes I_{\bar{A}}[X_n^d[\rho]]] \forall \rho, \forall A,
$$

(28)

where $OD[X] = X - \text{Diag}[X]$;

Now consider an $n$-partite state biseparable with respect to partition $A$, $\rho = \rho_A \otimes \rho_{\bar{A}}$. We note that $\tilde{\rho} := X_n^d(\rho)$ is still biseparable with respect to the partition $A|\bar{A}$ and hence, like in section IV B

$$
\mu[\tilde{\rho}] = \text{Diag}[\phi[\tilde{\rho}]] + \text{OD}[\phi[\tilde{\rho}]] + (2^{n-1} - 2) \left( \text{Diag}[\phi[\tilde{\rho}]] - \sum_A \text{Diag}[\tilde{\rho}] \right)
$$

$$
= \text{Diag}[\Lambda_A \otimes I_{\bar{A}}[\tilde{\rho}]] + \sum_{B \neq A} \text{Diag}[\Lambda_B \otimes I_B[\tilde{\rho}]]
$$

$$
= \text{Diag}[\Lambda_A \otimes I_{\bar{A}}[\tilde{\rho}]] + (2^{n-1} - 1) \text{Diag}[\Lambda_A \otimes I_{\bar{A}}[\tilde{\rho}]]
$$

$$
+ (2^{n-1} - 2) \text{Diag}[\Lambda_A \otimes I_{\bar{A}}[\tilde{\rho}]]
$$

$$
+ (2^{n-1} - 2) \sum_{B \neq A} \text{Diag}[\Lambda_B \otimes I_B[\tilde{\rho}]]
$$

$$
- (2^{n-1} - 1) \sum_{B \neq A} \text{Diag}[\tilde{\rho}]
$$

$$
\geq 0,
$$

(29)
where in the second step we have used Lemma 8.

The proof of Theorem 7 for an arbitrary biseparable state $\rho$ follows again from the linearity of the map $\mu \circ \mathcal{X}_n^d$.

We now want to look at some states detected by $\mu \circ \mathcal{X}_n^d$. Our first example is the noisy $n$-partite $d$-dimensional GHZ state:

$$\rho_{\text{GHZ}}^{n,d} = \alpha \left| \text{GHZ}_{d,n} \right\rangle \left\langle \text{GHZ}_{d,n} \right| + (1 - \alpha) \frac{I}{d^n}. \quad (30)$$

Applying our map to this state gives us the critical value $\alpha_c$ for which genuine multipartite entanglement is detected by it:

$$\alpha_c = \frac{(d - 2)(2^n - 1) + 1}{(d - 2)(2^{n-1} - 1) + 1 + (d - 2)d^{n-1}}, \quad (31)$$

which for $d > 2$ fixed goes to zero exponentially with the number of parties, as indeed $\alpha_c = o((\frac{2}{d})^n)$ ($n \rightarrow \infty$). This means that for $d > 2$ and $n$ big enough, our map detects the $n$-qudit GHZ state with up to almost 100% white noise.

Furthermore, since our map is based on the non-decomposable Choi-map it can also detect genuine multipartite entanglement in systems which are positive under partial transposition with respect to any bipartition. To illustrate that we can consider the family of 3-qutrit states $\{\rho(\lambda)\}_{\lambda \in \mathbb{R}^+}$ introduced in Ref. [13]. These states are obtained by setting $\lambda_i = \lambda$, $\forall i \in \alpha$ in the definition of Eq. (13) of Ref. [13]. More explicitly, for $\lambda \in \mathbb{R}^+$

$$\rho(\lambda) := \sum_{i=1}^{10} |\psi_i\rangle \langle \psi_i| + |\text{GHZ}\rangle \langle \text{GHZ}|, \quad (32)$$

where the $|\psi_i\rangle$, $i = 1, \ldots, 10$ are defined in Eq. (19) of Ref. [10]. In any case, the states $\rho(\lambda)$ have the property to be invariant under partial transposition, therefore partial transposition based criteria cannot even detect bipartite entanglement. By applying our map we recover the results obtained in Ref. [13], showing that the states $\rho(\lambda)$ are GME for $0 \leq \lambda < \frac{1}{3}$, which to our knowledge is still the best known detection range. Setting $\lambda = \frac{1}{9}$, the noisy state

$$\rho_{\text{noise}}(p) = p \frac{I}{27} + (1 - p) \rho \left( \frac{1}{9} \right) \quad (33)$$

is detected with our criteria with white noise up to $\frac{9}{179} \approx 5\%$. Apart from Refs. [10] [13] and Ref. [14] we are not aware of any GME detection technique powerful enough to achieve this feature.

We have just seen that our Choi based criteria is able to detect the $n$-qudit GHZ state with incredible white noise resistance for big enough $n$. It can also detect sates that are PPT with respect to every cut with the best known white noise resistance. It is thus fair to say that our scheme enabled us to derive a strong criterion. The key for obtaining such a criterion was to explore the symmetries of the Choi-map, namely that it acts the same on every off-diagonal element of a state, and to be able to project in a subspace, $\{\text{GHZ}_{n,\text{cyclic}}^d\}$, exhibiting this symmetry without creating any entanglement.
We believe that this strategy can be applied to other maps, such as the Breuer-Hall map to cite only one. One then is left with the task of identifying the symmetry that characterizes the map, and more difficultly to find a way of making a projection into a subspace exhibiting this symmetry without creating any entanglement. In order to shed some light to the possibility of accomplishing the second step of this process, we include an alternative proof of Proposition 5 in Appendix D. Indeed, since the alternative proof is based on group theory, we believe it will be easier to adapt it to a map having another symmetry.

V. DISCUSSION

In this manuscript we investigate the generalisation of the bipartite concept of positive maps to GME-maps in order to detect genuine multipartite entanglement, therefore closing a gap in the set of available tools for revealing GME. We introduce a general method of constructing GME-maps based on positive but non-completely positive maps. We have furthermore illustrated our construction method by generalising the paradigmatic PPT criterion to the multipartite case. As a first application, we showed that our approach provides a novel solution to the problem of determining the bi-separability threshold of GHZ mixtures. Moreover, by exploring the symmetries of the Choi-map we were able to design a very robust criterion (for GHZ-like states), which can recover the results of some of the strongest known criteria and even reveal entanglement in states that are PPT with respect to every partition. We are therefore strongly convinced that this construction could be a first step on a path to a better understanding of GME, even if only for the sole purpose to derive new GME-witnesses by lifting known positive maps to GME-maps.

There are, however, still many open questions concerning our construction. In the bipartite case, one of the upsides of the approach based on positive maps is that in some cases it is possible to achieve local unitary invariance of the resulting criteria (as with PPT). This also directly leads to the fact that every entangled pure state can easily be detected by this one simple map. Now the question that arises here is analogous: can there be a single multipartite map that reveals GME for all locally unitarily related (or even all) GME pure states? While our maps are performing very strongly in terms of noise resistance and can even detect states which are PPT with respect to every bipartition, they are not local unitary invariant and will probably fail to reveal GME of all GME pure states. The main challenge appears due to the fact that the negative eigenstates under every partition need to mutually overlap, in order for the construction to work. The lack of a unique Schmidt decomposition for multipartite systems and the hardness of computing the tensor rank make this a tough challenge, that we were only able to overcome in scenarios with a high degree of symmetry.

Moreover, from a more technical point of view, in order to develop new examples of the maps we propose it would be crucial to have a general method of deriving a valid, nontrivial, compensation map $M$, without having to first project into a subspace of the multipartite system (as in this way we expect to miss many important features and focus only on a small niche of the richness of GME correlations).
Finally, an important open point is whether every GME state can be detected by a GME-map lifted from positive maps. For the GME-witnesses a similar construction method was indeed recently shown to suffice to reveal the entanglement of any GME-state \cite{10}. In our case it is unfortunately not clear if the restrictions exposed in Proposition \cite{10} of Appendix \cite{12} still enable to detect every GME-state. An affirmative answer would show that among all the GME-maps, one would only need to consider the ones lifted from positive maps to fully characterize genuine multipartite entanglement, thus greatly simplifying the search for such maps.

VI. ACKNOWLEDGEMENT

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\[1\] R. Raussendorf, H. J. Briegel, A One-Way Quantum Computer, Phys. Rev. Lett., 86, 22, 5188, (2001).
\[2\] R. Cleve, D. Gottesman, H.-K. Lo, How to Share a Quantum Secret, Phys. Rev. Lett., 83, 3, 648, (1999).
\[3\] L. Gurvits, Classical deterministic complexity of Edmonds’ problem and quantum entanglement, in Proceedings of the thirty-fifth annual ACM symposium on Theory of computing, 10 (2003).
\[4\] M. Horodecki, P. Horodecki and R. Horodecki. Separability of mixed states: necessary and sufficient conditions. Physics Letters A 223, 1-2, 1-8, (1996).
\[5\] D. Bruss, Characterizing Entanglement, J. Math. Phys. 43, 4237 (2002).
\[6\] M. Horodecki, P. Horodecki, R. Horodecki, Separability of n-particle mixed states: necessary and sufficient conditions in terms of linear maps, Physics Letters A 283, 1-2, 1-7, (2001).
\[7\] O. Gühne, G. Toth, Entanglement detection, Physics Reports 474, 1 (2009).
\[8\] C. Eltschka, J. Siewert, Quantifying entanglement resources, J. Phys. A: Math.Theor. 47, 424005 (2014).
\[9\] B. Jungnitsch, T. Moroder, O. Gühne, Taming multiparticle entanglement, Phys. Rev. Lett. 106, 190502 (2011).
\[10\] C. Lancien, O. Gühne, R. Sengupta, M. Huber, Relaxations of separability in multipartite systems: Semidefinite programs, witnesses and volumes, J. Phys. A: Math. Theor. 48 505302 (2015).
\[11\] S.Rana, Negative eigenvalues of partial transposition of arbitrary bipartite states, Phys, Rev. A 87, 054301 (2013).
\[12\] M. D. Choi, T. Y. Lam, Extremal positive semidefinite forms, Math. Ann. 231, 1 (1977/78).
\[13\] M. Huber, R. Sengupta, Witnessing genuine multipartite entanglement with positive maps, Phys. Rev. Lett. 113, 100501 (2014).
\[14\] M. Piani, C. Mora, Class of PPT bound entangled states associated to almost any set of pure entangled states, Phys. Rev. A 75, 012305 (2007).
Appendix A: Deriving $W_{\text{GME}}$

In this section we want to derive the form of $W_{\text{GME}}$ as presented in (5). We know from Theorem III.1. of [10] that for any GME-state $\rho_{\text{GME}}$ there exists a set of weakly optimal bipartite witnesses $\{W_A\}_A$ of the form

$$W_A := Q + \sum_n [T_A]_+$$

is a multipartite witness detecting $\rho_{\text{GME}}$. Furthermore by Corollary III.2 of [10], for each A, the $W_A$ can be assumed to be of the form $W_A = \Lambda_A^* \otimes I_A [\psi_A \langle \psi_A |]$, where for each A, $\Lambda_A$ is a positive map detecting $\rho_{\text{GME}}$ and $|\psi_A\rangle$ is chosen such that $\langle \psi_A | \Lambda_A \otimes I_A (\rho_{\text{GME}}) | \psi_A \rangle < 0$. Then by defining

$$M_{\{\Lambda_A, |\psi_A\rangle\}_A} := \sum_n [(2^n - 1)[T_A]_+ - T_A] \geq 0; \quad (A2)$$

we have that

$$W_{\text{GME}} := \sum_A \Lambda_A^* \otimes I_A [\psi_A \langle \psi_A |] + M_{\{\Lambda_A, |\psi_A\rangle\}_A}$$

$$= \sum_A (Q + T_A) + \sum_A ((2^n - 1)[T_A]_+ - T_A)$$

$$= (2^n - 1)W;$$

proving that there exists a GME-witness detecting $\rho_{\text{GME}}$ of the desired form, namely $W_{\text{GME}}$.

Appendix B: Considerations on GME-maps

In this section we want to derive some general conditions about the existence of a GME-map of the form

$$\Phi_{\text{GME}} = \sum_A \Lambda_A \otimes I_A \circ U^{(A)} + M \quad (B1)$$

detecting a given GME-state $\rho_{\text{GME}}$. Quite trivially, we first note that

**Proposition 9.** For every family of positive but non-completely positive maps $\{\Lambda_A\}_A$, the map

$$\Phi_{\text{GME}} := \sum_A \Lambda_A \otimes I_A \circ U^{(A)} + M, \quad (B2)$$

for \( \{U^{(A)}\}_A \) a family of convex combination of local unitaries and \( M \) positive satisfying

\[
M[\rho_{2-sep}] \geq -\sum_A \Lambda_A \otimes I_A \circ U^{(A)}[\rho_{2-sep}] \quad \forall \rho_{2-sep},
\]

is such that \( \Phi_{GME}[\rho_{2-sep}] \geq 0 \quad \forall \rho_{2-sep} \).

The proof of Proposition 9 follows directly from the condition (B3) imposed to the positive map \( M \). If for the particular family of maps \( \{\Lambda_A\}_A \) and \( \Phi_{GME} \) as in Proposition 9 there exists a GME state \( \rho_{GME} \) such that \( M[\rho_{GME}] \neq -\sum_A \Lambda_A \otimes I_A \circ U^{(A)}[\rho_{GME}] \) it follows that \( \Phi_{GME} \) is a GME-map constructed from the lifting of positive but non-completely positive operators. Of course, given a GME-state, it is not clear from Proposition 9 if a family \( \{\Lambda_A, U^{(A)}\}_A \) and a map \( M \) exist such that \( \Phi_{GME} \) as in Eq. (B1) is a GME-map. Conditions for the existence of such a map are given by the following

**Proposition 10.** Given a GME-state \( \rho_{GME} \) and a GME-witness \( W_{GME} \) detecting \( \rho_{GME} \) of the form

\[
W_{GME} = M_{\{\Lambda_A, \psi_A\}_A} + \sum_A \Lambda_A^* \otimes I_A [\psi_A \langle \psi_A \rangle],
\]

there exists a GME-map \( \Phi_{GME} \) detecting \( \rho_{GME} \) of the form

\[
\Phi_{GME} = \sum_A \Lambda_A \otimes I_A \circ U^{(A)} + M,
\]

with \( M(\rho) > 0 \), \( \forall \rho > 0 \) and \( \{U^{(A)}\}_A \) a family of local unitaries if

i) there exists a family of local unitaries \( \{V^{(A)}\}_A \) such that \( V^{(A)}|\psi_A \rangle \) are equal for all \( A \),

ii) \( \text{Tr}(\rho M_{\{\Lambda_A, \xi_A\}_A}) V^{(A)}|\xi_A \rangle \langle \xi_A \rangle V^{(A)}^\dagger \) is independent of \( \{\xi_A\}_A \) for any chosen family \( \{\xi_A\}_A \).

**Proof.** First of all recall that the states \( \{|\psi_A\rangle\}_A \) were chosen such that \( \langle \psi_A | \Lambda_A \otimes I_A | \rho_{GME} \rangle |\psi_A \rangle < 0 \) and are fixed. We then pick a family of local unitaries as in i) that we will refer to as \( \{V^{(A)}\}_A \). In particular this means that \( V^{(A)}|\psi_A \rangle \) is equal for all \( A \). For ease of notation we denote \( V^{(A)}|\psi_A \rangle \) by \( |\phi \rangle \), that is

\[
|\phi \rangle := V^{(A)}|\psi_A \rangle.
\]

We then have for all \( \rho \)

\[
\text{Tr}(W_{GME} : \rho) = \text{Tr}(|\phi \rangle \langle \phi | \sum_A \Lambda_A \otimes I_A \circ U^{(A)}[\rho]) + \text{Tr}(M_{\{\Lambda_A, \phi\}_A} : \rho),
\]

where \( U^{(A)}[\rho] := U^{(A)} \cdot \rho \cdot (U^{(A)})^\dagger \) and \( U^{(A)} := ((V^{(A)})^{-1})^\dagger \). Since by ii) \( \text{Tr}(\rho M_{\{\Lambda_A, \phi\}_A}) |\phi \rangle \langle \phi | \) is \( |\phi \rangle \) independent, we can define

\[
M[\rho] := \text{Tr}(\rho M_{\{\Lambda_A, \phi\}_A}) |\phi \rangle \langle \phi |,
\]
where we did not explicitly write the dependence of $M$ on the choice of $\{\Lambda_A\}_A$. Note that $M[\rho] \geq 0, \forall \rho$ and $\text{Tr}(\phi \langle \phi | M[\rho]) = \text{Tr}(M_{\{\Lambda_A\}_A}\phi \cdot \rho)$. We then have

$$\text{Tr}(W_{\text{GME}} \cdot \rho) = \langle \phi | \left( \sum_A \Lambda_A \otimes \mathbb{I}_A \circ \mathcal{U}^{(A)}[\rho] + M[\rho] \right) | \phi \rangle.$$  \hfill (B9)

Since by construction $W_{\text{GME}}$ detects $\rho_{\text{GME}}$, equation (B9) tells us that

$$\sum_A \Lambda_A \otimes \mathbb{I}_A \circ \mathcal{U}^{(A)}(\rho_{\text{GME}}) + M(\rho_{\text{GME}}) \not\geq 0.$$  \hfill (B10)

Furthermore we have from the definition of $W_{\text{GME}}$ that for all biseparable state $\rho_{2-sep}$, $\text{Tr}(W_{\text{GME}} \cdot \rho_{2-sep}) \geq 0$. In fact, one can plug any family of states $\{||\tilde{\psi}\rangle\}_A$ in the defining formula of $W_{\text{GME}}$ instead of $\{|\psi\rangle\}_A$ and $\text{Tr}(W_{\text{GME}} \cdot \rho) \geq 0$ still holds for any biseparable $\rho$; $\tilde{W}_{\text{GME}}$ denoting the newly obtained operator. In particular, the states can be of the form $||\tilde{\psi}\rangle = (V^{(A)})^{-1} |\tilde{\phi}\rangle$, for an arbitrarily chosen $|\tilde{\phi}\rangle$. As before, we obtain

$$\text{Tr}(\tilde{W}_{\text{GME}} \rho) = \langle \tilde{\phi} | \sum_A \Lambda_A \otimes \mathbb{I}_A \circ \mathcal{U}^{(A)}[\rho] + M[\rho] | \tilde{\phi} \rangle \geq 0, \forall \text{ biseparable } \rho.$$  \hfill (B11)

Since $|\tilde{\phi}\rangle$ is arbitrary, this means that $\sum_A \Lambda_A \otimes \mathbb{I}_A \circ \mathcal{U}^{(A)}[\rho] + M[\rho] \geq 0$ for all biseparable $\rho$, which means that

$$\Phi_{\text{GME}} := \sum_A \Lambda_A \otimes \mathbb{I}_A \circ \mathcal{U}^{(A)} + M$$  \hfill (B12)

is a GME-map detecting $\rho_{\text{GME}}$, as claimed. \hfill \Box

Note that in Proposition 10 although $W_{\text{GME}}$ was formally needed to construct $\Phi_{\text{GME}}$, the latter is actually independent of the former making $\Phi_{\text{GME}}$ be a direct lifting. Also note that in Proposition 10 we use a family on local unitaries rather than a family of convex combination of local unitaries. This is needed since we make use of the inverse of those local unitaries, which is not guaranteed to exist for convex combinations of local unitaries. The question that remains open though, is whether, given a GME-state, one can find a GME-map detecting that state and that fulfills the conditions required by Proposition 10, or more generally, if any GME-state can be detected by a map of the form of Eq. (B1).

**Appendix C: Transposition-based GME criteria**

Let $\rho \in \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$ be a tripartite quantum state and let $\rho^{T_A} := T_A \otimes \mathbb{I}_{\bar{A}}(\rho)$ denote the partial transposition in system $A$ acting on $\rho \geq 0$. We first want to prove that $T_A \otimes \mathbb{I}_{\bar{A}}$ has $-\frac{1}{2}$ as minimal eigenvalue. For an alternative way of proving this result see [11].

**Proposition 11.** $\chi^{T_A} := \min_{\rho} EV_{\text{min}}(\rho^{T_A}) = -\frac{1}{2}, \forall A \subset \{1, 2, 3\}$
Proof. W.l.o.g $A = \{1\}$ (the proof for the other cases is achieved by basis transformation and/or global transposition, both of which leave $\lambda^{T_A}$ invariant). Let $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ and $\rho \in \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$. First note that $\lambda^{T_A} = \min_{|\psi\rangle,\rho} \langle \psi | \rho^{T_A} | \psi \rangle$. We then evaluate $\min_{|\psi\rangle,\rho} \langle \psi | \rho^{T_A} | \psi \rangle$ and find:

$$
\min_{\rho, |\psi\rangle} \langle \psi | \rho^{T_A} | \psi \rangle = \min_{\rho, |\psi\rangle} \sum_i \langle i | \psi \rangle \langle \psi | \rho^{T_A} | i \rangle \\
= \min_{\rho, |\psi\rangle} \text{Tr}( \langle \psi | \rho^{T_A} \rangle )
$$

where we used that $T_A$ is trace invariant and $\text{Tr}((AB)^{T_A}) = \text{Tr}(A^{T_A}B^{T_A})$. Now $|\psi\rangle \langle \psi|^{T_A}$ may not be positive definite but it is hermitian. To see this, one expands $|\psi\rangle \langle \psi|$ in an hermitian product basis (an explicit one being the generalised Gell-Mann matrices) and use the fact that tensor products and linear combinations of hermitian matrices are hermitian. This in particular means that we can diagonalise $|\psi\rangle \langle \psi|^{T_A}$ and choose the order of the diagonal basis such that we have $|\psi\rangle \langle \psi|^{T_A} = \sum_{i=1}^{d^3} \lambda_i |\phi_i\rangle \langle \phi_i|$, $\lambda_i \geq \lambda_{i+1}$. Writing $|\psi\rangle \langle \psi|^{T_A}$ as such in Eq. (C1) and recalling that $\rho$ is positive semidefinite and normalised, we have that the minimum is achieved for $\rho = |\phi_{d^3}\rangle \langle \phi_{d^3}|$.

We have hence just proven the following:

**Claim 12.** $\lambda^{T_A} = \min_{|\psi\rangle} EV_{\min}(\langle \psi | \psi|^{T_A})$.

We now want to find the eigenvalues of $|\psi\rangle \langle \psi|^{T_A}$. Using the Schmidt decomposition on $|\psi\rangle \in \mathbb{C}^d \otimes (\mathbb{C}^d \otimes \mathbb{C}^d)$, that is viewing $\mathbb{C}^d$ as the first system and $\mathbb{C}^d \otimes \mathbb{C}^d$ as the second, we find $|\psi\rangle = \sum_{i=1}^{d} c_i |ii\rangle$, where the $c_i$'s are real and positive, $c_1^2 + \cdots + c_d^2 = 1$ and the $|ii\rangle$'s are orthonormal; such that

$$
|\psi\rangle \langle \psi| = \sum_{i,j=1}^{d} c_i c_j |ii\rangle \langle jj|
$$

$$
= \sum_{i=1}^{d} c_i^2 |ii\rangle \langle ii| + \sum_{i,j=1}^{d} c_i c_j |ii\rangle \langle jj|
$$

and hence

$$
|\psi\rangle \langle \psi|^{T_A} = \sum_{i=1}^{d} c_i^2 |ii\rangle \langle ii| + \sum_{i,j=1}^{d} c_i c_j |ji\rangle \langle ij|
$$

We next complete $(|ii\rangle)_{i=1,...,d}$ to an orthonormal basis of $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ choosing the ordering

as

$|11\rangle, |22\rangle, \ldots, |dd\rangle, |12\rangle, |21\rangle, |13\rangle, |31\rangle, \ldots, |1d\rangle, |d1\rangle, |23\rangle, |32\rangle, |24\rangle, \ldots, |2d\rangle, |d2\rangle, \ldots, |(d-1)d\rangle, |d(d-1)\rangle$, + completion.
In this basis, $|\psi\rangle\langle\psi|^T_A$ has the following block diagonal form

$$|\psi\rangle\langle\psi|^T_A = \begin{pmatrix} D & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $D = \text{diag}(c_1^2, \ldots, c_d^2)$ and $A = \text{diag}(A_{12}, \ldots, A_{1d}, A_{23}, \ldots, A_{2d}, \ldots, A_{(d-1)d})$, where

$$A_{ij} = \begin{pmatrix} 0 & c_j c_i \\ c_i c_j & 0 \end{pmatrix}, \quad i, j = 1, \ldots, d; i \neq j.$$

The eigenvalues of $|\psi\rangle\langle\psi|^T_A$ are hence

$$c_1^2, \ldots, c_d^2$$

$$\pm c_i c_j \quad ; i, j = 1, \ldots, d, i \neq j$$

$$0.$$

Recalling that $c_i \geq 0, \forall i = 1, \ldots, d$ and $c_1^2 + \cdots + c_d^2 = 1$, it is clear that the minimum eigenvalue of $|\psi\rangle\langle\psi|^T_A$ is obtained by $-c_i c_j$ for any $i, j = 1, \ldots, d, i \neq j$. Furthermore $-c_i c_j$ is minimal when $c_i = c_j = \frac{1}{\sqrt{2}}$, and hence $c_k = 0, \forall k \neq i, j$. Therefore with Claim 12 $\min_{|\psi\rangle, \rho} \langle\psi|\rho^T_A|\psi\rangle = -c_i c_j = -\frac{1}{2}$.

**Corollary 13.** Proposition 11 generalises to any $n$-qudit system for any partition $A \subset \{1, \ldots, n\}$.

**Proof.** In the proof of Proposition 11 only dim$(A) \leq \text{dim}(\bar{A})$ is used (not $\text{dim}(A)^2 = \text{dim}(\bar{A})$). So for $\text{dim}(A) \leq \text{dim}(\bar{A})$ the generalisation is straightforward, else note that

$$\lambda_{T_A} = \min_{\rho} \text{EV}_{\text{min}}(\rho^T_A)$$

$$= \min_{\rho} \text{EV}_{\text{min}}((\rho^T_A)^T)$$

$$= \min_{\rho} \text{EV}_{\text{min}}(\rho^T_A) = \lambda_{T_A}.$$

We now want to prove Theorem 1 of the main text.
Proof of Theorem 1. We first point out that

\[
\text{EV}_{\min} (\Phi^T_{\rho_{2-sep}}) = \text{EV}_{\min} \left( \sum_A T_A \otimes I_A \left[ \sum_{A'} \sum_i p^{i}_{A'} \cdot \rho^{i}_{A'} \otimes \rho^{i}_{A'} \right] + I \right)
\]

\[
= \text{EV}_{\min} \left( \sum_{A'} \sum_i \left( \sum_{A=A'} T_{A'} \otimes I_{A'} \left[ \rho^{i}_{A'} \otimes \rho^{i}_{A'} \right] + \sum_{A \neq A'} T_{A} \otimes I_A \left[ \rho^{i}_{A'} \otimes \rho^{i}_{A'} \right] \right) + I \right)
\]

\[
\geq \sum_{A'} \sum_i \left[ \sum_{A=A'} p^{i}_{A'} \text{EV}_{\min} \left( T_{A'} \otimes I_{A'} \left[ \rho^{i}_{A'} \otimes \rho^{i}_{A'} \right] \right) + \sum_{A \neq A'} p^{i}_{A'} \text{EV}_{\min} \left( T_{A} \otimes I_A \left[ \rho^{i}_{A'} \otimes \rho^{i}_{A'} \right] \right) \right]
\]

\[
\geq 0;
\]

since \( \sum_{A \neq A'} \) has two terms for 3 parties and \( \sum_{A'} \sum_i p^{i}_{A'} = 1 \). This proves the first claim. To see that \( c = 1 \) is optimal, it is enough to apply \( \Phi^T \) to the bi-separable state on \( ABC \) formed by a tensor product of a maximally entangled state of \( AB \) and an arbitrary pure state of \( C \).

The generalization of the above map to \( n \) parties follows straightforwardly.

Corollary 14. For \( \Phi^T = \sum_A T_A \otimes I_A + \frac{2^{n-1}-2}{2} \cdot \text{tr} \) and for any \( n \)-partite biseparable states \( \rho_{2-sep} \) we have

\[
\Phi^T_{\rho_{2-sep}} \geq 0.
\]

Proof. As before we obtain

\[
\text{EV}_{\min} (\Phi^T_{\rho_{2-sep}})
\]

\[
\geq \sum_{A'} \sum_i \left[ \sum_{A=A'} p^{i}_{A'} \text{EV}_{\min} \left( T_{A'} \otimes I_{A'} \left[ \rho^{i}_{A'} \otimes \rho^{i}_{A'} \right] \right) + \sum_{A \neq A'} p^{i}_{A'} \text{EV}_{\min} \left( T_{A} \otimes I_A \left[ \rho^{i}_{A'} \otimes \rho^{i}_{A'} \right] \right) \right]
\]

\[
\geq 0;
\]

since this time \( \sum_{A \neq A'} \) has \( 2^{n-1} - 2 \) terms.

Theorem 2 of the main text follows from Corollary 14 since \( \text{EV}_{\min} \) is invariant under unitary transformations. Note however that already for 4 parties \( \Phi^T \) fails to detect the \( W \)-state, whereas \( \Phi_{T_x} \) detects the GHZ-state for any number of parties.

Appendix D: Proof of Proposition 5

In the following we prove Proposition 5 of the main text. To do this, we first prove the following
Proposition 15. For all \( n \in \mathbb{N}, \ n \geq 2 \) we have for the mixture of local unitaries \( X_n^d \) that

\[
X_n^d ((\langle + | (\langle + |)^\otimes n) = GHZ_{n,\text{cyclic}}^d
\]

where \( |+\rangle := |0\rangle + |1\rangle + \cdots + |d-1\rangle \) and \((|+\rangle \langle +|)^\otimes n\) denotes the matrix whose elements are all equal to one.

Proof. For \( n = 2 \) we have:

\[
X_2^d = h_2^d((\langle + | (\langle + |)^\otimes 2)
\]

\[
= \frac{1}{d} \sum_{k=0}^{d-1} Z_k^d (Z_k^d)^\dagger (\langle + | (\langle + |)^\otimes 2 (Z_k^d)^\dagger Z_k^d
\]

\[
= (|00\rangle + |11\rangle + \cdots + |(d-1)(d-1)\rangle)\langle 00 | + (|11\rangle + \cdots + \langle (d-1)(d-1) |)
\]

\[
+ (|01\rangle + |12\rangle + \cdots + |(d-1)0\rangle)\langle 01 | + |12\rangle + \cdots + \langle (d-1)0 |)
\]

\[
+ \ldots
\]

\[
+ (|0(d-1)\rangle + |10\rangle + \cdots + |(d-1)(d-2)\rangle)\langle 0(d-1) | + |10\rangle + \cdots + \langle (d-1)(d-2) |)
\]

\[
= GHZ_{2,\text{cyclic}}^d.
\]

To see that \( \ast \) holds one can, representing \( Z_k \) by \( \text{diag}(1, (k), (2k), \ldots, ((d-1)k)) \), where for \( l \in \{0, \ldots, (d-1)\} \), \((lk) := e^{2\pi ilk/d}, \text{ view } Z_k \otimes (Z_k)^\dagger (\langle + | (\langle + |)^\otimes n (Z_k)^\dagger \otimes Z_k \text{ as}

\[
\begin{array}{cccccccc}
1 & (k) & (2k) & \cdots & ((d-1)k) & (-k) & 1 & \cdots & ((d-2)k) & \cdots & ((-d+1)k) & \cdots & 1 \\
1 & -k & \cdots & (d-1)k & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & \cdots & 1 \\
-2k & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
((d-1)k) & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
((d-2)k) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
((d-1)k) & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
((d-2)k) & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

The leftmost column (left of the vertical bar) represents \( Z_k \otimes (Z_k)^\dagger \) as the latter multiplies each row of \((|+\rangle \langle +|)^\otimes n\) by the displayed factor. Similarly, \((Z_k)^\dagger \otimes Z_k \) is represented by the topmost row since it multiplies each column of \((|+\rangle \langle +|)^\otimes n\) by the displayed factor.
Now, we see that the first row (|00\rangle) has ones exactly at \langle 00 |, \langle 11 |, \ldots, \langle (d - 1)(d - 1) | such that the first row of this matrix can be represented by |00 \rangle \langle \text{GHZ}|. The second row where a 1 appears on the leftmost column is the |11 \rangle row, and again this row can be represented by |11 \rangle \langle \text{GHZ}|. Hence we see that we can represent the matrix where all the ones of the topmost row and leftmost column meet by |\text{GHZ} \rangle \langle \text{GHZ}|. Similarly, the matrix representing where all the (k) of the topmost column and (-k) of the leftmost row meet can be written as \( X_d^{(l)} \langle \text{GHZ}| \langle \text{GHZ}| X_d^{(l)} \rangle \). In general, for \( l \in \{0, 1, \ldots, d - 1\} \), the matrix representing where all the \((lk)\) of the topmost column and \((-lk)\) of the leftmost row can be written as \( X_d^{(l)} |\text{GHZ}\rangle \langle \text{GHZ}| X_d^{(l)} \rangle \). Those matrices represent all the entries where \( Z_k \otimes Z_k \rangle \langle \rangle \otimes |n\rangle \langle n| Z_k \) has ones. The other entries of this matrix have a phase \((lk)\) for some \( l \in \{0, 1, \ldots, d - 1\} \). When summing over all the \( k = 0, \ldots, d - 1 \) we have

\[
\sum_{k=0}^{d-1} (lk) = \sum_{k=0}^{d-1} e^{2\pi i lk/d} = \frac{1 - e^{2\pi id/d}}{1 - e^{2\pi i/d}} = 0;
\]

such that we have

\[
\sum_{k=0}^{d-1} Z_k \otimes (Z_k)^\dagger (|+) \langle +|)^\otimes n (Z_k)^\dagger \otimes Z_k = \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} X_d^{(l)} |\text{GHZ}\rangle \langle \text{GHZ}| X_d^{(l)} \rangle = d \cdot \text{GHZ}_{2,cyclic}^d.
\]

This proves the Claim for \( n = 2 \). For \( n > 2 \) notice that

\[
h_{n,2}^d (|+) \langle +|)^\otimes n = \frac{1}{d} \sum_{k=0}^{d-1} Z_k (Z_k)^\dagger (|+) \langle +|)^\otimes n (Z_k)^\dagger Z_k = \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} X_d^{(l)} |\text{GHZ}\rangle \langle \text{GHZ}| X_d^{(l)} \rangle \sum_{i_3 \ldots i_n=0}^{d-1} |i_3 \ldots i_n\rangle \langle j_3 \ldots j_n|\].
\]

Assuming that we are working in the following basis of \( \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d: \langle e_1^1 \otimes \cdots \otimes e_n^1, e_1^1 \otimes \cdots \otimes e_n^2, \ldots, e_1^1 \otimes \cdots \otimes e_n^{d-1} \rangle \) we define the basis transformation \( D_{2k} \) by

\[
D_{2k}(e_1^{l_1} \otimes e_2^{l_2} \otimes \cdots \otimes e_n^{l_n}) := e_1^{l_1} \otimes e_2^{l_2} \otimes \cdots \otimes e_n^{l_n},
\]

\( l_1, \ldots, l_n \in \{1, \ldots, d\} \). This has the effect of exchanging the subspaces 2 and k. Indeed we have

\[
h_{n,k}^d = D_{2k} h_{n,2}^d D_{2k}^{-1},
\]

\[
X_d^{(C_2 \ldots C_k \ldots C_n)} = D_{2k} X_d^{(C_k \ldots C_2 \ldots C_n)} D_{2k}^{-1};
\]
and hence
\[
\begin{split}
h_{n,k}^d((|+\rangle \langle +|)^{\otimes n}) &= D_{2k} h_{n,2}^d D_{2k}^{-1}((|+\rangle \langle +|)^{\otimes n}) \\
&= D_{2k} h_{n,2}^d((|+\rangle \langle +|)^{\otimes n}) D_{2k}^{-1} \\
&= \sum_{i_3,\ldots,i_n=0}^{d-1} \sum_{j_3,\ldots,j_n=0}^{d-1} X_d^{0,\ldots,C_2,\ldots,0} |\text{GHZ}_{1k}\rangle \langle i_k, i_3, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n| (X_d^{0,\ldots,C_2,\ldots,0})^\dagger,
\end{split}
\]
where $|\text{GHZ}_{1k}\rangle \langle i_k, i_3, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n| := \sum_{l=0}^{d-1} |l, i_k, i_3, \ldots, i_{k-1}, l, i_{k+1}, \ldots, i_n\rangle$. Therefore, since $X_d^{C_2,0,\ldots,0} \cdots X_d^{0,0,C_n} = X_d^C$ we find
\[
X_d^d((|+\rangle \langle +|)^{\otimes n}) = h_{n,2}^d \circ h_{n,3}^d \circ \cdots \circ h_{n,n}^d((|+\rangle \langle +|)^{\otimes n})
\]
\[
= \sum_{C \in \{0,\ldots,d\}^{n-1}} X_d^C \langle \text{GHZ} | (X_d^C)^\dagger
\]
\[
= \text{GHZ}_n^{d,cyclic}.
\]

With this we can now easily prove Proposition 5 of the main text.

**Proof of Proposition 5.** By noting that for any $n$-qudit state $\rho$ we have $X_d^d(\rho) = X_d^d(|+\rangle \langle +|)^{\otimes n}) \circ_s \rho$, where $\circ_s$ denotes the Schur product, Proposition 15 means exactly that $X_n^d$ projects any $n$-qudit state into the subspace spanned by the entries of the GHZ cyclically permuted states, which is the statement of Proposition 5 of the main text.

Although this way of proving Proposition 5 gives great insights into how the projection is concretely performed, we would like to present an alternative way of proving it that, using a group theoretical approach, puts more emphasis on the symmetries of the GHZ cyclically permuted states that are made used of here, i.e. the symmetry of the Choi map.

**Alternative Proof of Proposition 5.** Let $\mathcal{G}$ be a compact group whose associated Haar integral we denote by $\int_{\mathcal{G}}$. Given a unitary representation $\zeta : \mathcal{G} \to \mathcal{L}(\mathcal{H})$ of $\mathcal{G}$ on a finite–dimensional complex Hilbert space, an interesting super–operator is given by the expression
\[
\mathcal{P} \equiv \int_{\mathcal{G}} dg \text{Ad}_\zeta(g) : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H}) ,
\]
where $\text{Ad}_\zeta(g)$ acts on linear operators $X \in \mathcal{L}(\mathcal{H})$ as $(\text{Ad}_\zeta(g))(X) \equiv \zeta(g)X\zeta(g)^\dagger$. The superoperator $\mathcal{P}$ given by (D2) is an orthogonal projector with respect to the Hilbert–Schmidt product on $\mathcal{L}(\mathcal{H})$. Moreover, let the irreducible decomposition of $\mathcal{L}(\mathcal{H})$ under the action of $\zeta$ be given by
\[
\mathcal{H} = \bigoplus_\alpha V_\alpha^\otimes n_\alpha = \bigoplus_\alpha V_\alpha \otimes \mathbb{C}^{n_\alpha} ,
\]
\[
\zeta = \sum_\alpha n_\alpha \zeta_\alpha ,
\]
where $\zeta_\alpha$ is irreducible for all $\alpha$ and $\zeta_\alpha$, $\zeta_\alpha'$ inequivalent if $\alpha \neq \alpha'$. Then $\mathcal{P}$ acts as

$$\mathcal{P}(\cdot) = \bigoplus_\alpha \frac{1}{d_\alpha} \otimes \text{Tr}_{V_\alpha}[\Pi_\alpha(\cdot) \Pi_\alpha], \quad (D5)$$

where $d_\alpha \equiv \dim V_\alpha$, $\Pi_\alpha$ is the projector onto the $\alpha$–th block of the direct sum in $\{D3\}$, and the partial trace is over the first component of the bipartite Hilbert space $V_\alpha \otimes \mathbb{C}^{n_\alpha}$.

Despite the complicated appearance, the above observation is just a slightly more sophisticated application of Schur’s Lemma.

In our case, we choose $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$ and $\mathcal{G} = \mathbb{Z}_d^{n-1}$, where $\mathbb{Z}_d$ is the group of integers with the operation of sum modulo $d$. The Haar integral over $\mathcal{G}$ is simply the normalized sum, i.e. $f_\mathcal{G} = \frac{1}{d^n-1} \sum_{g \in \mathcal{G}}$. For $g = (k_2, \ldots, k_n) \in \mathcal{G}$, we define

$$\zeta(g) \equiv Z_{k_2+\ldots+k_n} \otimes Z_{k_2}^\dagger \otimes \ldots \otimes Z_{k_n}^\dagger. \quad (D6)$$

Then, it is not difficult to see that $\chi_n^d = \mathcal{P}$ as defined by Eq. 24 of section IV C of the main text and (D2) here. Now, all we have to do is to decompose $\zeta$ in irreducible representations (irreps), all one-dimensional because $\mathcal{G}$ is abelian. For all $0 \leq r, h_2, \ldots, h_n \leq d$, we observe that $|r, r + h_2, \ldots, r + h_n\rangle$ satisfies

$$\zeta(k_2, \ldots, k_n) |r, r + h_2, \ldots, r + h_n\rangle = \omega^{-h_2 k_2 - \ldots - h_n k_n} |r, r + h_2, \ldots, r + h_n\rangle, \quad (D7)$$

i.e. it is an irreducible subspace for $\zeta$. It turns out that the irreps of $\mathcal{G}$ are all present in $\zeta$ and indexed by $h_2, \ldots, h_n$, and each of them has multiplicity $d$ (internal index $r$). This can be easily seen by verifying that the characters of $|r, r + h_2, \ldots, r + h_n\rangle$ and $|r', r' + h'_2, \ldots, r' + h'_n\rangle$ are orthogonal if $(h_2, \ldots, h_n) \neq (h'_2, \ldots, h'_n)$, and by remembering that the number of non-isomorphic irreps of an abelian group coincides with its cardinality ($d^{n-1}$ in this case). Finally, the action of $\mathcal{P}$ as specified in $\{D5\}$ gives exactly the claim of Proposition 5.

Appendix E: Choi Map

We want here to prove Lemma 4 and Lemma 8 of the main text. We begin by proving Lemma 8 that we state again for ease of read.

**Lemma 16 (Lemma 8).**

$$\text{OD}(\phi[X_n^d(\rho)]) = (2^{n-1} - 1) \text{OD}[\Lambda_A \otimes I_A[X_n^d(\rho)]] \quad \forall A, \quad (E1)$$

where $\text{OD}[X] = X - \text{Diag}[X]$.

**Proof.** First of all note that

$$\text{GHZ}_{n,\text{cyclic}}^d = \sum_{C \in \{0,1,\ldots,d-1\}^{n-1}} X_C^{\dagger} \left[\text{GHZ}_{n}^d\right] \left(\text{GHZ}_{n}^d\right)^\dagger \left(X_C^{\dagger}\right)$$

$$= \sum_{C \in \{0,1,\ldots,d-1\}^{n-1}} \frac{1}{d} \sum_{i,j=0}^{d-1} X_C^{\dagger} |i i \ldots i\rangle \langle j j \ldots j| \left(X_C^{\dagger}\right)^\dagger$$

$$= \sum_{C \in \{0,1,\ldots,d-1\}^{n-1}} \frac{1}{d} \sum_{i,j=0}^{d-1} |i i + C_2 \ldots i + C_n\rangle \langle j j + C_2 \ldots j + C_n|,$$
where the sums $k + C_l$, $k = i, j$, $l = 2, \ldots, n$ are to be understood as modulo $d$. For any state $ho = (\rho_{k_1, k_2}) \geq 0$, with $k_1, k_2 \in \{0, \ldots, d - 1\}^n$ we hence have
\[
X_{n}^d(\rho) = X_{n}^d[|+\rangle\langle +|] \circ \rho
\]
\[
= \text{GHZ}_{n,\text{cyclic}} \circ \rho
\]
\[
= \sum_{C \in \{0, 1, \ldots, d-1\}} \frac{1}{d} \sum_{i,j=0}^{d-1} \rho_{i+(0,C),j+(0,C)} |i\ i + C_2 \ldots i + C_n\rangle \langle j\ j + C_2 \ldots j + C_n|.
\]

Note that trivially $i \neq j \iff i + C_k \neq j + C_k$, $\forall k \in \{2, \ldots, n\}$ such that, the main diagonal excepted, all non-vanishing elements of $X_{n}^d(\rho)$ are off-diagonal with respect to any partition $(A \mid \bar{A})$. In particular, this means that for any partition $(A \mid \bar{A})$
\[
\Lambda_A \otimes I_{\bar{A}}[\text{OD}[X_{n}^d(\rho)]] = -\text{OD}[X_{n}^d(\rho)];
\] (E2)
that is the diagonal part of $\Lambda_A$ acts trivially. In other words, the above means that $\text{Diag}_A$ only affects the diagonal elements of $X_{n}^d[\rho]$. Furthermore, since the latter becomes diagonal upon applying $\text{Diag}_A$, it follows
\[
\Lambda_A \otimes I_{\bar{A}}[\text{OD}[X_{n}^d(\rho)]] = \text{OD}[\Lambda_A \otimes I_{\bar{A}}[X_{n}^d(\rho)]],
\] (E3)
such that
\[
\text{OD}[\phi[X_{n}^d(\rho)]] = \text{OD}\left[\sum_{B \mid \bar{B}} \Lambda_B \otimes I_{\bar{B}}\right]
\]
\[
= \sum_{B \mid \bar{B}} \text{OD}[\Lambda_B \otimes I_{\bar{B}}[X_{n}^d(\rho)]]
\]
\[
\leq \sum_{B \mid \bar{B}} \Lambda_B \otimes I_{\bar{B}}[\text{OD}[X_{n}^d(\rho)]]
\]
\[
\leq \sum_{B \mid \bar{B}} (-\text{OD}[X_{n}^d(\rho)])
\]
\[
= (2^{n-1} - 1)(-\text{OD}[X_{n}^d(\rho)])
\]
\[
\leq (2^{n-1} - 1)\Lambda_A \otimes I_{\bar{A}}[\text{OD}[X_{n}^d(\rho)]], \forall A.
\]

Lemma [4] is the same assertion as Lemma [8] but for $\tilde{\sigma}_x^A \circ T_A$ instead of $\Lambda_A$. The proof is also identical and can be carried through by systematically replacing $\Lambda_A$ by $\tilde{\sigma}_x^A \circ T_A$, setting $d = 2$ and noting that $X_{n}^2 \equiv X_n$ in the above proof. Only equation [E2] has to be adapted to
\[
\tilde{\sigma}_x^A \circ T_A \otimes I_{\bar{A}}[\text{OD}[X_n(\rho)]] = \text{OD}[X_n(\rho)]
\] (E4)
since $\tilde{\sigma}_x^A \circ T_A \otimes I_{\bar{A}}$ acts as the identity on $\text{OD}[X_n(\rho)]$, not as minus the identity.