ON A THEOREM OF GRIGOR’YAN, HU AND LAU

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Abstract. We refine a result of Grigor’yan, Hu and Lau to give a moment condition on a heat kernel which characterizes the critical exponent at which a family of Besov spaces associated to the Dirichlet energy becomes trivial.

Following [1], consider a metric measure space \((M, d, \mu)\), with \(M\) nonempty and \(\mu\) Borel, that admits a heat kernel \((p_t)_{t>0}\). The latter is a collection of symmetric, non-negative, measurable functions on \(M \times M\) all of which have unit integral, satisfy the semigroup property \(p_{t+\tau}(x, y) = \int p_t(x, z)p_\tau(z, y)d\mu(z)\) for all \(s, t > 0\), and approximate the identity in the sense that if \(f \in L^2\) then \(\int p_t(x, y)f(y)d\mu(y) \to f\) in \(L^2\) as \(t \downarrow 0\). This hypothesis has many consequences, among which we will need that setting

\[
T_t u(x) = \int_M p_t(x, y)u(y) \, d\mu(y), \quad E_t(u) = t^{-1} \langle u - T_t u, u \rangle,
\]

where \(\langle , \rangle\) is the \(L^2\) inner product, we find \(E_t(u)\) is decreasing in \(t > 0\) so \(E(u) = \lim_{t \downarrow 0} E_t(u)\) exists, though it may be infinite. Moreover setting \(\mathcal{D}(E) = \{u \in L^2 : E(u) < \infty\}\) we have that \(E\) is a Dirichlet form with domain \(\mathcal{D}(E)\).

In [1] the authors further assume the heat kernel has a two-sided estimate of the form

\[
\frac{1}{r^{\alpha/\beta}} \Phi_1\left(\frac{d(x, y)}{r^{1/\beta}}\right) \leq p_t(x, y) \leq \frac{1}{r^{\alpha/\beta}} \Phi_2\left(\frac{d(x, y)}{r^{1/\beta}}\right)
\]

for \(\mu\)-a.e. \(x, y \in M\) and all \(t > 0\), where \(\alpha\) and \(\beta\) are positive constants and \(\Phi_1\) and \(\Phi_2\) are non-negative monotone decreasing functions on \([0, \infty)\). They then show that \(\alpha\) and \(\beta\) are determined by \((M, d, \mu)\) provided that \(\Phi_1(1) > 0\) and \(\Phi_2\) satisfies a moment condition of the form

\[(H_\gamma) \quad \int_0^\infty s^\gamma \Phi_2(s) \, \frac{ds}{s} < \infty \]

for some suitable value of \(\gamma\). Some of these results are stated in terms of a Besov space which they denote \(W^{\sigma, 2}\) but which is sometimes called \(\text{Lip}(\sigma, 2, \infty)\). To define this space let

\[
W^\sigma(u) = \sup_{0<r<1} r^{-2\sigma} \int_M \int_{B(x, r)} |u(y) - u(x)|^2 \, d\mu(y) \, d\mu(x)
\]

where \(\int_B = \mu(B)^{-1} \int_B\) is the average and \(B(x, r)\) is the ball of radius \(r\) with center \(x\). Then let \(W^{\sigma, 2} = \{u \in L^2 : W^\sigma(u) < \infty\}\). This is a Banach space with norm \(\|u\|_{L^2} + W^\sigma_{1/2}\). Also let \(\beta' = 2 \sup(\sigma : \dim W^{\sigma, 2} = \infty)\).

Among the main results in [1] are the following:

**Theorem 1** ([1] Theorems 3.2, 4.2, 4.6).

Suppose \((M, d, \mu)\) has a heat kernel satisfying ([1]) and that \(\Phi_1(1) > 0\).

1. If \((H_\gamma)\) holds for \(\gamma = \alpha\) then \(\mu\) is Ahlfors regular with exponent \(\alpha\).
(2) If $\{H_2\}$ holds for $\gamma = \alpha + \beta$ then $D(E) = W^{\beta/2, 2}$ and $E(u) \approx W_{\beta/2}(u)$.

(3) If $\{H_2\}$ holds for $\gamma > \alpha + \beta$ then for $\sigma > \beta/2$ the space $W^{\sigma, 2} = \{0\}$ and $\beta = \beta^*$.

The purpose of this note is to show that the third of the above implications may be improved as follows:

**Theorem 2.** Suppose $(M, d, \mu)$ has a heat kernel satisfying $(1)$, that $\Phi(1) > 0$ and $\{H_2\}$ holds for $\gamma = \alpha + \beta$. Then $W^{\sigma, 2} = \{0\}$ and $\beta = \beta^*$.

**Proof.** We follow the proof of Theorem 4.6 of [1]. They decompose $E_\delta(u) = A(t) + B(t)$ where for $t = \delta$

$$B(t) = \frac{1}{2t} \int M \int E_B(x, 1) (u(x) - u(y))^2 p_t(x, y) d\mu(y) d\mu(x)$$

$$= \frac{1}{2t} \sum_{k=1}^{\infty} \int M \int E_{B(x, 2^{-k-1})} (u(x) - u(y))^2 p_t(x, y) d\mu(y) d\mu(x)$$

$$\leq \frac{\sigma^{\beta + \epsilon}}{2} \sum_{k=1}^{\infty} 2^{-k \beta} \Phi_2 \left( \frac{2^{-k}}{\beta} \right) \int M \int E_{B(x, 2^{-k-1})} (u(x) - u(y))^2 d\mu(y) d\mu(x)$$

$$\leq CW_{\gamma}(u)^{\sigma^{\beta + \epsilon}} \int_0^{1/\beta} s^{\sigma^{\beta + \epsilon}} \Phi_2(s) \frac{ds}{s}$$

in which the first inequality is from the upper bound in (1) and the second is from the definition of $W_{\gamma}(u)$ and the fact that part (1) of Theorem 1 implies $\mu(B(x, 2^{-k-1})) \approx 2^k$.

The above is essentially shown in the proof of Theorem 4.6 in (1); they then assume $\{H_2\}$ for $\gamma = \alpha + \beta + \epsilon$ to establish that the integral is bounded independent of $t$ and conclude $\lim_{t \downarrow 0} B(t) = 0$. However this also follows from $\{H_2\}$ for $\gamma = \alpha + \beta$. This is actually a standard exercise: given $\delta > 0$ use $\{H_2\}$ for $\gamma = \alpha + \beta$ to obtain $T$ so small that

$$\int_{T-1/\beta}^{\infty} s^{\sigma^{\beta + \epsilon}} \Phi_2(s) \frac{ds}{s} < \delta$$

from which

$$B(t) \leq CW_{\gamma}(u)^{\sigma^{\beta + \epsilon}} \int_0^{T^{-1/\beta}} s^{\sigma^{\beta + \epsilon}} \Phi_2(s) \frac{ds}{s} + CW_{\gamma}(u)^{\sigma^{\beta + \epsilon}} \int_{T^{-1/\beta}}^{\infty} s^{\sigma^{\beta + \epsilon}} \Phi_2(s) \frac{ds}{s}$$

$$\leq CW_{\gamma}(u) \left( \frac{1}{T} \right) \int_0^{T^{-1/\beta}} s^{\sigma^{\beta + \epsilon}} \Phi_2(s) \frac{ds}{s} + CW_{\gamma}(u) \int_{T^{-1/\beta}}^{\infty} s^{\sigma^{\beta + \epsilon}} \Phi_2(s) \frac{ds}{s}$$

and $\lim_{t \downarrow 0} B(t)$ follows. Since it is established in equation (4.17) of (1) that $\lim_{t \downarrow 0} A(t) = 0$ we conclude

$$E(u) = \lim_{t \downarrow 0} E_\delta(u) = \lim_{t \downarrow 0} A(t) + B(t) = 0.$$

**Remark.** A similar argument is used for a slightly different purpose in (3), and a slightly less general result with the same proof is in [2]. Nonetheless this specific result does not seem to be known, and the weaker result in part (3) of Theorem 1 is frequently cited.
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