Existence and uniqueness of time-fractional diffusion equation on a metric star graph

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Abstract

In this paper, we study the time-fractional diffusion equation on a metric star graph. The existence and uniqueness of the weak solution are investigated and the proof is based on eigenfunction expansions. Some priori estimates and regularity results of the solution are proved.

Keywords. Time-fractional diffusion equation; Caputo fractional derivative; Weak solution.

1 Introduction

We consider a graph \( G = (V, E) \) consisting of a finite set of vertices (nodes) \( V = \{ v_i : i = 0, 1, 2, \ldots, k \} \) and a set of edges \( E \) (such as heat conducting elements) connecting these nodes. The graph considered in this work is a metric graph [29]. Therefore, each edge \( e_i, i = 1, 2, \ldots, k \) is parametrised by an interval \((0, l_i)\). The study of partial differential equations (PDEs) on networks or metric graphs is not just the analysis of known mathematical objects on special domains, since in our context, graphs or networked domains are not manifolds. Thus, we investigate PDEs on single edges of graph (interpreted as continuous curves or manifolds) [38] along with certain transmission conditions such as continuity and Kirchoff condition at junction node. Hence, we define a coordinate system on each edge \( e_i \) by taking \( v_0 \) as the origin and \( x \in (0, l_i) \) as the coordinate. We consider a time-fractional diffusion equation on a metric star graph \( G \), which is a graph consisting of \( k \) edges incident to a common vertex \( v_0 \) (see Figure 1):

\[
C_D^\alpha_{0,t} y(x, t) = \frac{\partial^2 y(x, t)}{\partial x^2} + f(x, t), \quad x \in G, \ t \in (0, T), \ 0 < \alpha < 1.
\]

\[
y(x, 0) = y^0(x), \quad x \in G.
\]

More precisely, at each edge we have following fractional diffusion equation

\[
C_D^\alpha_{0,t} y_i(x, t) = \frac{\partial^2 y_i(x, t)}{\partial x^2} + f_i(x, t), \quad x \in (0, l_i), \ t \in (0, T), \ 0 < \alpha < 1
\]

\[
y_i(x, 0) = y^0_i(x), \quad x \in (0, l_i), \ i = 1, 2, \ldots, k,
\]

along with the continuity and Kirchoff conditions at junction node \( v_0 \) as

\[
y_i(0, t) = y_j(0, t), \quad i \neq j, \ i, j = 1, 2, \ldots, k, \ t \in (0, T),
\]

\[
\sum_{i=1}^{k} \frac{\partial y_i(0, t)}{\partial x} = 0,
\]

and Dirichlet boundary conditions at boundary nodes \( v_i \)

\[
y_i(l_i, t) = 0, \quad t \in (0, T).
\]

Here \( C_D^\alpha_{0,t} \) denotes the Caputo fractional derivative of order \( \alpha \) with respect to \( t \) defined as

\[
C_D^\alpha_{0,t} y(x, t) = \frac{1}{\Gamma(1 - \alpha)} \left( \int_0^t (t - \xi)^{-\alpha} \frac{\partial y(x, \xi)}{\partial \xi} d\xi \right), \quad 0 < \alpha < 1, \ t \in (0, T),
\]
where $\Gamma(.)$ denotes the Euler gamma function. In this paper we prove the existence and uniqueness of the weak solution of initial-value problem (IVP) \((1.1)-(1.2)\) whose restriction to the edge $e_i$ gives the weak solution of initial-boundary value problem (IBVP) \((1.3)-(1.7)\). When $\alpha$ approaches 1, the Caputo fractional derivative $C\mathcal{D}_0^\alpha u$ approaches the ordinary derivative $\frac{\partial u}{\partial t}$ and, thus, IBVP \((1.3)-(1.7)\) represents the standard diffusion equation on graphs for which existence and uniqueness was proved in [41]. Recently in [27], authors established the existence and uniqueness of nonlinear fractional boundary value problem on a star graph. Hence, this work could be seen as the extension of [27] for time dependent problem.

The origin of the study of differential equation on graphs can be traced back to 1980s with Lumer’s work [24] on ramification spaces. In [30], Nicaise investigated the propagation of nerve impulses. Since then, considerable work related to eigenvalue problems (Sturm-Liouville type problems) on networks, i.e., metric graphs has been done, for instance see the article by von Below [4] and [33, 8, 32]. Partial differential equations on graphs or multi-link structures plays important role in the field of science and engineering. For instance, the flows on the nets of gas pipeline [39], controlled vibrations of networks of strings (hyperbolic wave equations) [9], water wave propagation in open channel networks (Burgers type equation) [43] naturally lead to partial differential equation on graphs.

Evolutionary problems (such as parabolic equations) on metric graphs were considered in [40]. The dynamic networks of strings and beams along with their control properties were studied by Lagnese et al. in [18], see also e.g. [17, 16, 15, 14]. The progress of problems defined on metric graphs until 2006 has been presented in an excellent survey by Dager and Zuazua [6]. Since then, modeling, analysis and optimal control problems for linear and nonlinear partial differential equations on metric graphs has become an active area of research. Nonlinear Schrödinger equation on metric star graph were studied by Adami et al. in [1]. In [43], Yoshioka et al. considered the Burger type equation models on connected graph and discussed the existence and uniqueness of the model along with the energy estimates. Inverse problem on metric graphs were initiated by Nicaise in [31] for the wave equation. In [3], Avdonin and Nicaise considered the source identification problem for the wave equation on an interval and extended their approach to study the problem on trees (graphs which do not contain a cycle), while source identification problem for the heat equation on metric graphs was discussed in [2]. Recently, Grigor et al. [10] studied the Yamabe type equations on graphs and proved the existence of positive solution using the mountainpass theorem due to Ambrosetti-Rabinowich. On the other hand, fractional calculus find its importance in different fields of science and engineering [11, 7, 25, 5]. A strong motivation for the study and analysis of fractional diffusion equations comes from the fact that they efficiently describe the phenomenon of anomalous diffusion.

![Figure 1: A sketch of the star graph with $k$ edges](image)
tion [23]. Fractional diffusion equations on bounded domains has been studied by various authors. For instance in [21], Luchko gave the maximum principle for the time-fractional diffusion equation, while in [22] he established the existence and uniqueness results for time-fractional diffusion equation using eigenfunction expansion by taking source term \( f = 0 \). In [37], the existence results for fractional diffusion-wave equations were established by Sakamoto and Yamamoto, while IBVP for a coupled fractional diffusion system was discussed in [19]. For more results we refer [26, 20] and references therein.

To the best of our knowledge, there has not been any published work related to the existence and uniqueness results for the time-fractional diffusion equation on metric graphs so far. In this paper, we focus on proving the existence and uniqueness of IBVP (1.3)-(1.7) and study the regularity of solution given by the eigenfunction expansions.

The rest of the paper is divided into three sections. In Sec 2, we define the function spaces for star graph \( G \), state some propositions regarding Mittag-Leffler function (defined in section 2) and prove a Lemma by means of eigenfunction expansion which plays an important role in developing the detailed analysis of the problem. In Sec. 3, we prove the main results on existence and uniqueness of IBVP (1.3)-(1.7) under different regularity conditions on initial data \( y_0(x) \). In Sec. 4, we conclude the work done and provide a brief idea of the future direction.

2 Preliminaries

First of all, we define the following function spaces on a star graph \( G \):

\[
L_2(G) = \prod_{i=1}^{k} L_2(0, l_i),
\]

\[
H_m(G) = \prod_{i=1}^{k} H_m(0, l_i),
\]

with the corresponding inner products

\[
\langle y, w \rangle_{L_2(G)} := \sum_{i=1}^{k} \langle y_i, w_i \rangle_{L_2(0, l_i)}
\]

and

\[
\langle y, w \rangle_{H_m(G)} := \sum_{i=1}^{k} \langle y_i, w_i \rangle_{H_m(0, l_i)},
\]

where \( L_2(0, l_i) \) and \( H_m(0, l_i) \) are standard Sobolev spaces. The spaces \( L_2(G) \) and \( H_m(G) \) are Hilbert spaces with inner products \( \langle \cdot, \cdot \rangle_{L_2(G)} \) and \( \langle \cdot, \cdot \rangle_{H_m(G)} \) respectively [28].

We define the following operator \( \mathcal{L} \) on the Hilbert space \( L_2(G) \):

\[
D(-\mathcal{L}) = \left\{ y \in L_2(G) : y_i \in H_2(0, l_i), \right. \]

\[
y_i(l_i) = 0, y_i(0) = y_j(0), \ i \neq j, \ i, j = 1, 2, \ldots, k \text{ and } \sum_{i=1}^{k} y_i'(0) = 0 \}
\]

\[
\forall y \in D(-\mathcal{L}) : \mathcal{L}y = \left( \frac{\partial^2 y_i}{\partial x^2} \right)_{i=1}^{k}.
\]

Remark 2.1. The operator \( -\mathcal{L} \) is a non-negative self-adjoint operator since it is the Friedrichs extension of the triple \( (L_2(G); V; a) \) defined by [13]

\[
V = \left\{ y \in \prod_{i=1}^{k} H_1(0, l_i) : y_i(l_i) = 0, y_i(0) = y_j(0), \ i \neq j, \ i, j = 1, 2, \ldots, k \right\},
\]
which is a Hilbert space with the inner product
\[ \langle u, w \rangle_V := \sum_{i=1}^{k} \langle y_i, w_i \rangle_{H_1(0, l_i)} = \sum_{i=1}^{k} \int_{0}^{l_i} y'_i w'_i \, dx, \]
and
\[ a(y, w) = \sum_{i=1}^{k} \int_{0}^{l_i} y'_i(x) w'_i(x) \, dx. \]

The spectrum of operator \(-L\) consists of eigenvalues, having the form
\[ 0 < \mu_1(\mathcal{G}) \leq \mu_2(\mathcal{G}) \leq \ldots \rightarrow \infty; \]
and the eigenfunction \(\Psi_n = (\psi_{n,1}, \psi_{n,2}, \ldots, \psi_{n,k})\) corresponding to eigenvalue \(\mu_n: -L \Psi_n = \mu_n \Psi_n\), \(n \in \mathbb{N}\). Then the sequence \(\{\Psi_n\}_{n \in \mathbb{N}}\) forms an orthonormal basis of \(L_2(\mathcal{G})\) (see [36, 39]). Hence \(\{\mu_n, \Psi_n\}_{n \in \mathbb{N}}\) is the eigensystem of following problem:
\[ \psi''_{n,i}(x) = -\mu_n \psi_{n,i}(x), \quad 0 < x < l_i, \quad (2.1) \]
\[ \psi_{n,i}(l_i) = 0, \quad i = 1, 2, \ldots, k, \quad (2.2) \]
\[ \psi_{n,i}(0) = \psi_{n,j}(0), \quad i, j = 1, 2, \ldots, k, \quad i \neq j, \quad (2.3) \]
\[ \sum_{i=1}^{k} \psi'_{n,i}(0) = 0. \quad (2.4) \]

Now, we define the fractional power \((-L)^{\gamma}\), \(\gamma \in \mathbb{R}\) using the spectral decomposition of operator \(L\).

For any \(y \in L_2(\mathcal{G})\), we have
\[ y = \sum_{n=1}^{\infty} \langle y, \Psi_n \rangle \Psi_n, \]
which gives
\[ y_i = \sum_{n=1}^{\infty} \langle y, \Psi_n \rangle \psi_{n,i}, \]
where \(y_i\) is the restriction of \(u\) to the edge \(e_i\).

Hence we define \((-L)^{\gamma} y = ((-M)^{\gamma} y_i)_i\), where \((M)^{\gamma} y_i = \sum_{n=1}^{\infty} \mu_n^{\gamma} \langle y, \Psi_n \rangle \psi_{n,i}\). Now
\[ \left\| (-L)^{\gamma} y \right\|_{L_2(\mathcal{G})}^2 = \sum_{i=1}^{k} \left\| (-M)^{\gamma} y_i \right\|_{L_2(0, l_i)}^2 = \sum_{n=1}^{\infty} \mu_n^{2\gamma} \left| \langle y, \Psi_n \rangle \right|^2. \]

Then we define
\[ D((-L)^{\gamma}) = \left\{ y \in L_2(\mathcal{G}) : \sum_{n=1}^{\infty} \mu_n^{2\gamma} \left| \langle y, \Psi_n \rangle \right|^2 < \infty \right\}. \]

It follows that \(D((-L)^{\gamma})\) is a Hilbert space with the norm
\[ \| y \|_{D((-L)^{\gamma})} = \left\| (-L)^{\gamma} y \right\|_{L_2(\mathcal{G})} = \left( \sum_{n=1}^{\infty} \mu_n^{2\gamma} \left| \langle y, \Psi_n \rangle \right|^2 \right)^{\frac{1}{2}}. \quad (2.5) \]
Remark 2.2. Using Parseval’s identity, we have
\[ \|u\|^2_{V} \sim \|u\|^2_{D(-L^{1/2})}, \]
while in general \( D((-L)^{\gamma}) \subset H_{2,\gamma}(\mathcal{G}) \) holds for \( \gamma > 0 \). Hence, in view of (2.5), the spaces \( V, L_{2}(\mathcal{G}) \) and \( H_{2}(\mathcal{G}) \) can be characterized as follows:

\[
V = \left\{ y = \sum_{n=1}^{\infty} \langle y, \Psi_{n} \rangle \Psi_{n} : \|y\|_{V}^{2} = \sum_{n=1}^{\infty} \mu_{n} |\langle y, \Psi_{n} \rangle|^{2} < \infty \right\},
\]

\[
L_{2}(\mathcal{G}) = \left\{ y = \sum_{n=1}^{\infty} \langle y, \Psi_{n} \rangle \Psi_{n} : \|y\|_{L_{2}(\mathcal{G})}^{2} = \sum_{n=1}^{\infty} |\langle y, \Psi_{n} \rangle|^{2} < \infty \right\}
\]

and

\[
H_{2}(\mathcal{G}) = \left\{ y = \sum_{n=1}^{\infty} \langle y, \Psi_{n} \rangle \Psi_{n} : \|y\|_{H_{2}(\mathcal{G})}^{2} = \sum_{n=1}^{\infty} \mu_{n}^{2} |\langle y, \Psi_{n} \rangle|^{2} < \infty \right\}.
\]

Now, we give the following definition and propositions regarding Mittag-Leffler function which will be used later.

Definition 2.1. The Mittag-Leffler function is defined as follows
\[
E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^{j}}{\Gamma(\alpha j + \beta)}, \quad z \in \mathbb{C},
\]
where \( \alpha > 0 \) and \( \beta \in \mathbb{R} \) are arbitrary constants.

Proposition 2.1 (see [34]). Let \( 0 < \alpha < 2, \beta \in \mathbb{R} \) be arbitrary and \( \mu \) be such that \( \pi \alpha / 2 < \mu < \min\{\pi, \pi \alpha\} \), then there exists a constant \( C = C(\alpha, \beta, \mu) > 0 \) such that
\[
|E_{\alpha,\beta}(z)| \leq \frac{C}{|\Gamma(z)|}, \quad \mu \leq |\arg(z)| \leq \pi.
\]

Proposition 2.2. Let \( 0 < \alpha < 1 \) and \( \eta > 0 \), then we have \( 0 < E_{\alpha,\alpha}(-\eta) < \frac{1}{\Gamma(\alpha)} \). Furthermore, \( E_{\alpha,1}(-\eta) \) is a monotonically decreasing function with \( \eta > 0 \).

Proposition 2.3 (see [35]). Let \( 0 < \alpha < 1 \) and \( t > 0 \), then we have \( 0 < E_{\alpha,1}(-t) < 1 \). Furthermore, \( E_{\alpha,1}(-t) \) is completely monotonic that is
\[
(-1)^{n} \frac{d^{n}}{dt^{n}} E_{\alpha,1}(-t) \geq 0, \quad n \in \mathbb{N}.
\]

Proposition 2.4 (see [37]). Let \( \alpha > 0, \lambda > 0 \) and \( m \in \mathbb{N} \), then we have
\[
\frac{d^{m}}{dt^{m}} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^{\alpha}), \quad t > 0.
\]

Proposition 2.5 (see [13]). Let \( \alpha > 0 \) and \( \lambda > 0 \), then we have
\[
cD_{t}^{\alpha} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda E_{\alpha,1}(-\lambda t^{\alpha}), \quad t > 0.
\]

Next, we prove a lemma that plays an important role in further analysis.

Lemma 2.1. Let \( f(\cdot,t) \in L_{2}(\mathcal{G}) \), \( y^{0} \in L_{2}(\mathcal{G}) \), then the solution \( y_{i}(x,t) \) of IBVP (1.3)–(1.7) has the form
\[
y_{i}(x,t) = \sum_{n=1}^{\infty} \langle y^{0}, \Psi_{n} \rangle E_{\alpha,1}(-\mu_{n} t^{\alpha}) \psi_{n,i}(x)
\]
\[
+ \sum_{n=1}^{\infty} \left( \int_{0}^{t} \langle f(x,\xi), \Psi_{n} \rangle (t-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{n} (t-\xi)^{\alpha}) d\xi \right) \psi_{n,i}(x),
\]
where \( \{\mu_{n}, \Psi_{n}\}_{n \in \mathbb{N}} \) is the eigensystem of (2.1)–(2.4) and \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L_{2}(\mathcal{G}) \).
Proof. We will use the method of eigenfunction expansions for the solution of (1.3), that is we write the solution in terms of Fourier series whose coefficients are vary with time. Hence, we try to find a solution of the equation (1.1) in the form
\[ y(x,t) = \sum_{n=1}^{\infty} T_n(t) \Psi_n(x), \]
which gives
\[ y_i(x,t) = \sum_{n=1}^{\infty} T_n(t) \psi_{n,i}(x), \quad i = 1, 2, \ldots, k. \] (2.7)

Now
\[ \frac{\partial^2 y_i(x,t)}{\partial x^2} = \sum_{n=1}^{\infty} T_n(t) \psi''_{n,i}(x), \]
\[ cD_{0,t}^{\alpha} y_i(x,t) = \sum_{n=1}^{\infty} \left( cD_{0,t}^{\alpha} T_n(t) \right) \psi_{n,i}(x). \]

After substituting the value of above expressions in equation (1.3), we get
\[ \sum_{n=1}^{\infty} \left[ \left( cD_{0,t}^{\alpha} T_n(t) \right) \psi_{n,i}(x) - T_n(t) \psi''_{n,i}(x) \right] = f_i(x,t) \]
and
\[ \sum_{n=1}^{\infty} \left[ cD_{0,t}^{\alpha} T_n(t) + \mu_n T_n(t) \right] \psi_{n,i}(x) = f_i(x,t), \] (2.8)
where we used the fact that \( \psi''_{n,i}(x) = -\mu_n \psi_{n,i}(x) \). Now we expand \( y^0(x) \) and \( f(x,t) \) in terms of Fourier series,
\[ f(x,t) = \sum_{n=1}^{\infty} f_n(t) \Psi_n(x) \] (2.9)
and
\[ y^0(x) = \sum_{n=1}^{\infty} a_n \Psi_n(x), \] (2.10)
which gives
\[ f_i(x,t) = \sum_{n=1}^{\infty} f_n(t) \psi_{n,i}(x) \quad \text{and} \quad y^0_i(x) = \sum_{n=1}^{\infty} a_n \psi_{n,i}(x), \] (2.11)
where
\[ f_n(t) = \langle f(x,t), \Psi_n(x) \rangle \quad \text{and} \quad a_n = \langle y^0_i(x), \Psi_n(x) \rangle. \]

Hence, from equations (2.8) and (2.9), we obtain
\[ \sum_{n=1}^{\infty} \left[ cD_{0,t}^{\alpha} T_n(t) + \mu_n T_n(t) \right] \Psi_n(x) = \sum_{n=1}^{\infty} f_n(t) \Psi_n(x). \]

Using the uniqueness of Fourier series we get the family of fractional ODE’s
\[ cD_{0,t}^{\alpha} T_n(t) + \mu_n T_n(t) = f_n(t) \] (2.12)
and
\[ y_i(x,0) = \sum_{n=1}^{\infty} T_n(0) \psi_{n,i}(x) = y^0_i(x) = \sum_{n=1}^{\infty} a_n \psi_{n,i}(x), \]
so that

\[ T_n(0) = a_n \quad n \geq 1. \]  \tag{2.13}

The solution of fractional differential equation (2.12) subject to initial condition (2.13) is given by

\[ T_n(t) = a_n E_{\alpha,1}(-\mu_n t^\alpha) + \int_0^t (t-\xi)^{\alpha-1} E_{\alpha,\alpha} (-\mu_n (t-\xi)^\alpha) f_n(\xi) d\xi. \]

Hence, from equation (2.7), we have

\[ y_i(x, t) = \sum_{n=1}^{\infty} a_n E_{\alpha,1}(-\mu_n t^\alpha) \psi_{n,i}(x) \]

\[ + \sum_{n=1}^{\infty} \left( \int_0^t (t-\xi)^{\alpha-1} E_{\alpha,\alpha} (-\mu_n (t-\xi)^\alpha) f_n(\xi) d\xi \right) \psi_{n,i}(x). \]  \tag{2.14}

After substituting the value of \( a_n \) and \( f_n(t) \) in equation (2.14), we get the desired result. \( \square \)

### 3 Existence and uniqueness results of a weak solution

In this section, the existence and uniqueness of weak solutions will be proved. So let us first define the weak solution as follows.

**Definition 3.1.** We define \( y \) as a weak solution of \([1.1] - [1.2]\) if \([1.1]\) holds in \( L_2(\mathcal{G}) \) and \( y(\cdot, t) \in V \) for almost all \( t \in (0, T) \) and satisfy

\[ \lim_{t \to 0} \left\| y(\cdot, t) - y^0 \right\|_{L_2(\mathcal{G})} = 0. \]

Now we state our first main result as follows.

**Theorem 3.1.** Let \( y^0 \in L_2(\mathcal{G}) \) and \( f(x, t) \in L_\infty(0, T; L_2(\mathcal{G})) \). Then there exists a unique weak solution \( y \in C([0, T]; L_2(\mathcal{G})) \cap C((0, T]; D(-\mathcal{L})) \) such that \( C D_{\alpha,1}^\alpha u \in L_\infty(0, T; L_2(\mathcal{G})) \). Furthermore, there exists a positive constant \( C_1 \) such that

\[ \|y\|_{C([0, T]; L_2(\mathcal{G}))} \leq C_1 \left( \left\| y^0 \right\|_{L_2(\mathcal{G})} + \|f\|_{L_\infty(0, T; L_2(\mathcal{G}))} \right), \]  \tag{3.1}

\[ \|y(\cdot, t)\|_{\Pi_{i=1}^{k} H_2(0, l_i)} \leq C_1 \left( \left\| y^0 \right\|_{L_2(\mathcal{G})} l^{-\alpha} + \|f\|_{L_\infty(0, T; L_2(\mathcal{G}))} \right). \]  \tag{3.2}

**Proof.** We will show that \( y(x, t) = (y_i(x, t))_{i=1}^k \), where \( y_i(x, t) \) is given by equation (2.6), certainly gives the weak solution to \( [1.1] - [1.2] \). We assume \( C > 0 \) to be a generic constant in the following proof. Hence, using equation (2.6) and the fact that

\[ \sum_{i=1}^{k} \langle \psi_{n,i}, \psi_{m,i} \rangle_{L_2(0, l_i)} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{for } m \neq n \end{cases}, \]

we have

\[ \sum_{i=1}^{k} \left\| y_i(\cdot, t) \right\|_{L_2(0, l_i)}^2 = \sum_{n=1}^{\infty} \left( \left\| y^0, \Psi_n \right\|_{E_{\alpha,1}(-\mu_n t^\alpha)} \right)^2 + \sum_{n=1}^{\infty} \int_0^t \left( f(\cdot, \xi) \Psi_n(t-\xi)^{\alpha-1} E_{\alpha,\alpha} (-\mu_n (t-\xi)^\alpha) \right) d\xi. \]
Using Propositions 2.2 and 2.3 we get
\[
\sum_{i=1}^{k} \|y_i(\cdot, t)\|_{L_2(0, l_i)}^2 \leq \sum_{n=1}^{\infty} \left| \langle y^0_n, \Psi_n \rangle \right|^2 + \sum_{n=1}^{\infty} \left| \int_0^t \langle f(\cdot, \xi), \Psi_n \rangle \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} d\xi \right|^2 \\
\leq \|y^0\|_{L_2(\mathcal{G})}^2 + \sup_{0 \leq t \leq T} \left| \langle f(\cdot, t), \Psi_n \rangle \right|^2 \left( \frac{t^\alpha}{\Gamma(\alpha+1)} \right)^2 \\
\leq \|y^0\|_{L_2(\mathcal{G})}^2 + \|f\|_{L_\infty(0, T; L_2(\mathcal{G}))}^2 \frac{T^{2\alpha}}{(\Gamma(\alpha+1))^2}.
\]

Hence,
\[
\|y(\cdot, t)\|_{L_2(\mathcal{G})} \leq C_1 \left( \|y^0\|_{L_2(\mathcal{G})} + \|f\|_{L_\infty(0, T; L_2(\mathcal{G}))} \right), \quad t \in [0, T],
\]
where \(C_1 = \max \left\{ 1, \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right\} \).

Now, given \(t, t+h \in [0, T]\), we have
\[
y_i(x, t+h) - y_i(x, t) = \sum_{n=1}^{\infty} \langle y^0_n, \Psi_n \rangle \left( E_{\alpha,1}(-\mu_n(t+h)^\alpha) - E_{\alpha,1}(-\mu_n t^\alpha) \right) \psi_{n,i}(x) \\
+ \sum_{n=1}^{\infty} (u_n(t+h) - u_n(t)) \psi_{n,i}(x),
\]
where
\[
u_n(t) = \int_0^t \langle f(\cdot, \xi), \Psi_n \rangle (t-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\mu_n(t-\xi)^\alpha) d\xi.
\]

Again, using Propositions 2.2 and 2.3 we get
\[
\sum_{i=1}^{k} \|y_i(\cdot, t+h) - y_i(\cdot, t)\|_{L_2(\mathcal{G})}^2 = \sum_{n=1}^{\infty} \left| \langle y^0_n, \Psi_n \rangle \left( E_{\alpha,1}(-\mu_n(t+h)^\alpha) - E_{\alpha,1}(-\mu_n t^\alpha) \right) \right|^2 \\
+ \sum_{n=1}^{\infty} \left| (u_n(t+h) - u_n(t)) \right|^2.
\]

Since \(\lim_{h \to 0} \left| E_{\alpha,1}(-\mu_n(t+h)^\alpha) - E_{\alpha,1}(-\mu_n t^\alpha) \right| = 0\), \(\lim_{h \to 0} |u_n(t+h) - u_n(t)| = 0\) (see Lemma 2.14 (42)). Hence, using Lebesgue dominated convergence theorem, we get
\[
\lim_{h \to 0} \|y(\cdot, t+h) - y(\cdot, t)\|_{L_2(\mathcal{G})}^2 = \lim_{h \to 0} \left( \sum_{i=1}^{k} \|y_i(\cdot, t+h) - y_i(\cdot, t)\|_{L_2(0, l_i)}^2 \right) = 0.
\]

Therefore, \(y \in C((0, T]; L_2(\mathcal{G})).\)

Now it will be shown that \(y \in C((0, T]; D(-\mathcal{L}))\) and \(C\cdot D_{0,t}^\alpha y \in L_\infty(0, T; L_2(\mathcal{G})).\) We have
\[
(-\mathcal{M})y_i(x, t) = \sum_{n=1}^{\infty} \mu_n \langle y^0_n, \Psi_n \rangle E_{\alpha,1}(-\mu_n t^\alpha) \psi_{n,i}(x) \\
+ \sum_{n=1}^{\infty} \mu_n \left( \int_0^t \langle f(\cdot, \xi), \Psi_n \rangle (t-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\mu_n(t-\xi)^\alpha) d\xi \right) \psi_{n,i}(x).
\]
Now,

\[
\left\| (-\mathcal{L})y(\cdot,t) \right\|_{L^2(\mathcal{G})}^2 = \sum_{i=1}^{k} \left\| (-\mathcal{M})y_i(\cdot,t) \right\|_{L^2(0,t_i)}^2
= \sum_{n=1}^{\infty} \mu_n^2 \left| \langle y^0, \Psi_n \rangle E_{\alpha,1}(-\mu_n t^n) \right|^2
+ \sum_{n=1}^{\infty} \mu_n^2 \int_0^t \left| \langle f(\cdot,\xi), \Psi_n \rangle (t-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\mu_n(t-\xi)^\alpha) \right| d\xi.
\]

Also from Propositions 2.3 and 2.4

\[
\int_0^t |\xi^{\alpha-1} E_{\alpha,\alpha}(-\mu_n \xi)\| d\xi = \int_0^t \xi^{\alpha-1} E_{\alpha,\alpha}(-\mu_n \xi)\| d\xi = -\frac{1}{\mu_n} \int_0^t \frac{d\xi}{\xi^{\alpha}} E_{\alpha,\alpha}(-\mu_n \xi)\| d\xi = \frac{1}{\mu_n} (1 - E_{\alpha,\alpha}(-\mu_n t^n)) \leq \frac{1}{\mu_n}.
\]

Now, using equation (3.3), Proposition 2.1 and Young inequality for the convolution, we get

\[
\left\| (-\mathcal{L})y \right\|_{L^2(\mathcal{G})}^2 \leq \sum_{n=1}^{\infty} \mu_n^2 \left| \langle y^0, \Psi_n \rangle \right|^2 \left( \frac{C_1}{1 + \mu_n t^n} \right)^2
+ \sum_{n=1}^{\infty} \mu_n^2 \sup_{0 \leq t \leq T} \left| \langle f(\cdot,t), \Psi_n \rangle \right|^2 \left| \int_0^T t^{\alpha-1} E_{\alpha,\alpha}(-\mu_n t^n) dt \right|^2.
\]

Hence, we obtain

\[
\left\| (-\mathcal{L})y \right\|_{L^2(\mathcal{G})}^2 \leq \left\| y^0 \right\|_{L^2(\mathcal{G})}^2 t^{-2\alpha} + \left\| f \right\|_{L^\infty(0,T;L^2(\mathcal{G}))}^2.
\]

Since \(-\mathcal{L}y\) is convergent in \(L^2(\mathcal{G})\) uniformly on \(t \in (t_0, T]\) with any given \(t_0 > 0\), we deduce that \(-\mathcal{L}y \in C((0,T]; L^2(\mathcal{G}))\), that is \(-\mathcal{M}y_i \in C((0,T]; L^2(0,t_i))\), \(i = 1, 2, \ldots, k\) and hence \(y \in C((0,T]; D(-\mathcal{L}))\). Moreover, we obtain the following estimate from equation (3.4)

\[
\left\| y(\cdot,t) \right\|_{H(z(0,t),i) = 1} L^2(0,t_i) = \sum_{i=1}^{k} \left\| y_i(\cdot,t) \right\|_{H^2(0,t_i)}^2
\leq C' \sum_{i=1}^{k} \left\| (-\mathcal{L})y_i(\cdot,t) \right\|_{L^2(0,t_i)}^2
= C' \left\| (-\mathcal{L})y_i(\cdot,t) \right\|_{L^2(\mathcal{G})} \leq C \left( \left\| y^0 \right\|_{L^2(\mathcal{G})} t^{-\alpha} + \left\| f \right\|_{L^\infty(0,T;L^2(\mathcal{G}))} \right).
\]

By (1.1), we see that \(CD_{0,t}^\alpha y \in L^\infty(0,T; L^2(\mathcal{G}))\) and (1.1) holds in \(L^2(\mathcal{G})\) for \(t \in (0,T]\).

Next it will be shown that \(\lim_{t \to 0} \left\| y(\cdot,t) - y^0 \right\|_{L^2(\mathcal{G})} = 0\). From equations (2.6) and (2.11), we have

\[
y_i(x,t) - y_i^0(x) = \sum_{n=1}^{\infty} \left( y^0, \Psi_n \right) \left( E_{\alpha,1}(-\mu_n t^n) - 1 \right) \psi_n(x)
+ \sum_{n=1}^{\infty} \left( \int_0^t \langle f(x,\xi), \Psi_n \rangle (t-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\mu_n(t-\xi)^\alpha) d\xi \right) \psi_n(x).
\]
Hence,  
\[ \sum_{i=1}^{k} \left\| y_i(\cdot, t) - y_i^0(\cdot) \right\|_{L^2(0, t_i)}^2 \leq \sum_{n=1}^{\infty} \left| \langle y^0, \Psi_n \rangle \left( E_{\alpha, 1}(-\mu_n t^\alpha) - 1 \right) \right|^2 \]
\[ + \sum_{n=1}^{\infty} \left\| \int_0^t \langle f(x, \xi), \Psi_n \rangle (t - \xi)^{\alpha-1} E_{\alpha, \alpha} \left(-\mu_n (t - \xi)^\alpha \right) d\xi \right\|^2 \]
\[ =: V_1(t) + V_2(t). \]

Clearly, \( \lim_{t \to 0} V_2(t) = 0 \), using Proposition 2.3
\[ V_1(t) = \sum_{n=1}^{\infty} \left| \langle y^0, \Psi_n \rangle \left( E_{\alpha, 1}(-\mu_n t^\alpha) - 1 \right) \right|^2 \leq C \| y^0 \|_{L^2(G)}^2 \]

and \( \lim_{t \to 0} \left( E_{\alpha, 1}(-\mu_n t^\alpha) - 1 \right) = 0 \). Hence, by using Lebesgue dominated convergence theorem, we have \( \lim_{t \to 0} V_1(t) = 0 \). Hence \( \lim_{t \to 0} \sum_{i=1}^{k} \left\| y_i(\cdot, t) - y_i^0(\cdot) \right\|_{L^2(0, t_i)} = 0 \), which shows that

\[ \lim_{t \to 0} \left\| y(\cdot, t) - y^0 \right\|_{L^2(G)} = 0. \]

Finally, we show the uniqueness of the weak solution to initial-value problem (1.1)-(1.2).

**Uniqueness:** Under the conditions \( y^0 = 0 \) and \( f = 0 \), we need to show that system (1.3)-(1.7) has only the trivial solution. On taking the inner product of (1.1) with \( \Psi_n(x) \), applying Green’s formula and setting \( y^\alpha(t) = (y(\cdot, t), \Psi_n) \), we obtain
\[ CD_{0,t}^\alpha y^\alpha(t) = \int_G \frac{\partial^2 y(x, t)}{\partial x^2} \Psi_n(x) dx \]
\[ = \sum_{i=1}^{k} \int_0^{t_i} \frac{\partial^2 y_i(x, t)}{\partial x^2} \psi_{n,i}(x) dx \]
\[ = - \sum_{i=1}^{k} \int_0^{t_i} \frac{\partial y_i(x, t)}{\partial x} \psi_{n,i}'(x) dx + \sum_{i=1}^{k} \frac{\partial y_i(x, t)}{\partial x} \psi_{n,i}(x) \bigg|_0^{t_i}. \]

Using equations (1.6) and (2.3), we get
\[ \sum_{i=1}^{k} \frac{\partial y_i(x, t)}{\partial x} \psi_{n,i}(x) \bigg|_0^{t_i} = \sum_{i=1}^{k} \frac{\partial y_i(l_i, t)}{\partial x} \psi_{n,i}(l_i) - \sum_{i=1}^{k} \frac{\partial y_i(0, t)}{\partial x} \psi_{n,i}(0) \]
\[ = - \sum_{i=1}^{k} \frac{\partial y_i(0, t)}{\partial x} \psi_{n,i}(0) = -\phi_n(0) \sum_{i=1}^{k} \frac{\partial y_i(0, t)}{\partial x} = 0, \]

where \( \psi_{n,i}(0) = \psi_{n,j}(0) = \phi_n(0), i \neq j, \ i, j = 1, 2, \ldots, k \). Hence we get
\[ CD_{0,t}^\alpha y^\alpha(t) = - \sum_{i=1}^{k} \int_0^{t_i} \frac{\partial y_i(x, t)}{\partial x} \psi_{n,i}'(x) dx \]
\[ = \sum_{i=1}^{k} \int_0^{t_i} y_i(x, t) \psi_{n,i}''(x) dx - \sum_{i=1}^{k} y_i(x, t) \psi_{n,i}'(x) \bigg|_0^{t_i}. \]

Again using equations (1.5) and (2.4) and a similar approach as above, we get
\[ \sum_{i=1}^{k} y_i(x, t) \psi_{n,i}'(x) \bigg|_0^{t_i} = 0. \]
Therefore,
\[
C D_{0,t}^\alpha y^n(t) = \sum_{i=1}^{k} \int_0^{t_i} y_i(x,t) \psi''_{n,i}(x) \, dx
\]
\[
= - \mu_n \sum_{i=1}^{k} \int_0^{t_i} y_i(x,t) \psi_{n,i}(x) \, dx = - \mu_n \langle u(\cdot,t), \Psi_n \rangle.
\]

Hence, we obtain the following initial value fractional differential equation
\[
\begin{cases}
C D_{0,t}^\alpha y^n(t) = - \mu_n y^n(t), & t \in (0,T), \\
y^n(0) = 0.
\end{cases}
\]

Due to the existence and uniqueness of the above fractional differential equation, we get that \(y^n(t) = 0, \; n = 1, 2, \ldots\). Since \(\Psi_n\) is a complete orthonormal basis in \(L_2(\mathcal{G})\), we have \(y = 0\) in \(\mathcal{G} \times (0,T]\).

**Theorem 3.2.** Let \(y^0 \in V, f(x,t) \in L_\infty(0,T;L_2(\mathcal{G}))\). Then there exists a unique weak solution \(y \in L_2((0,T];D(-\mathcal{L}))\) such that \(C D_{0,t}^\alpha y \in L_2(\mathcal{G} \times (0,T))\) and the following inequality holds:

\[
\|y\|_{L_2((0,T];\Pi_{n=1}^\infty H_2(0,l_n))} + \|C D_{0,t}^\alpha y\|_{L_2(\mathcal{G} \times (0,T))} \leq C \left( \|y^0\|_V + \|f\|_{L_\infty(0,T;L_2(\mathcal{G}))} \right). \tag{3.5}
\]

**Proof.**
\[
\|(-\mathcal{L})y(\cdot,t)\|_{L_2(\mathcal{G})}^2 = \sum_{n=1}^{\infty} \|(-\mathcal{M})y_n(\cdot,t)\|_{L_2(0,l_n)}^2
\]
\[
= \sum_{n=1}^{\infty} \mu_n \langle y^0, \Psi_n \rangle E_{\alpha,1}(-\mu_n t^\alpha) + \mu_n \int_0^T \|f(\cdot,\xi), \Psi_n\|_{t=\xi}^2 E_{\alpha,\alpha}(-\mu_n(\xi - t)^\alpha) \, d\xi.
\]

Now, using Proposition 2.1 and Young inequality for the convolution, we get
\[
\|(-\mathcal{L})y(\cdot,t)\|_{L_2(\mathcal{G})}^2 \leq \sum_{n=1}^{\infty} \mu_n \|\langle y^0, \Psi_n \rangle\|^2 \left( \frac{C_1 \sqrt{\mu_n}}{1 + \mu_n t^\alpha} \right)^2
\]
\[
+ \sum_{n=1}^{\infty} \mu_n^2 \sup_{0 \leq \xi \leq T} \|f(\cdot,\xi), \Psi_n\|^2 \left( \int_0^T t^{\alpha-1} E_{\alpha,\alpha}(-\mu_n t^\alpha) \, dt \right)^2
\]
\[
= \sum_{n=1}^{\infty} \mu_n \|\langle y^0, \Psi_n \rangle\|^2 \left( \frac{C_1 \sqrt{\mu_n}}{1 + \mu_n t^\alpha} \right)^2 \left( \int_0^T t^{\alpha-1} E_{\alpha,\alpha}(-\mu_n t^\alpha) \, dt \right)^2
\]
\[
+ \sum_{n=1}^{\infty} \mu_n^2 \sup_{0 \leq \xi \leq T} \|f(\cdot,\xi), \Psi_n\|^2 \left( \int_0^T t^{\alpha-1} E_{\alpha,\alpha}(-\mu_n t^\alpha) \, dt \right)^2
\]
\[
\leq C \|y^0\|_V^2 t^{-\alpha} + \|f\|_{L_\infty(0,T;L_2(\mathcal{G}))}^2,
\]
where we have used equation (3.3).
Theorem 3.3. Let \( y^0 \in D(-\mathcal{L}) \), \( f(x, t) \in L_\infty(0, T; L_2(\mathcal{G})) \). Then there exists a unique weak solution \( y \in C([0, T]; L_2(\mathcal{G})) \cap C((0, T); D(-\mathcal{L})) \) such that \( c D^\alpha_{0,t} y \in L_2(\mathcal{G} \times (0, T)) \). Moreover
there exists a constant $C_1 > 0$ such that
\[
\|y\|_{C([0,T];\Pi_{i=1}^k H_2(0,t_i))} + \left\| cD_0^\alpha y \right\|_{L_2(\mathcal{G} \times (0,T))} \leq C_1 \left( \|y^0\|_{\Pi_{i=1}^k H_2(0,t_i)} + \|f\|_{L_\infty(0,T;L_2(\mathcal{G}))} \right). \tag{3.6}
\]

**Proof.** Under the assumption $y^0 \in D(-\mathcal{L})$, using Proposition 2.4, Proposition 2.3 and Young inequality for the convolution, we have
\[
\|(-\mathcal{L})y(\cdot,t)\|_{L_2(\mathcal{G})}^2 = \sum_{i=1}^k \|(-\mathcal{M})y_i(\cdot,t)\|_{L_2(0,t_i)}^2 \\
= \sum_{n=1}^\infty \mu_n^2 \left| \langle y^0, \Psi_n \rangle E_{\alpha,1}(-\mu_n t^\alpha) \right|^2 \\
+ \sum_{n=1}^\infty \mu_n^2 \left| \int_0^t \langle f(\cdot,\xi), \Psi_n \rangle(t-\xi)^{\alpha-1}E_{\alpha,\alpha}(-\mu_n(t-\xi)^\alpha) \, d\xi \right|^2 \\
\leq \sum_{n=1}^\infty \mu_n^2 \left| \langle y^0, \Psi_n \rangle \right|^2 \\
+ \sum_{n=1}^\infty \mu_n^2 \sup_{0 \leq \xi \leq T} \left| \langle f(\cdot,\xi), \Psi_n \rangle \right|^2 \left| \int_0^t t^{\alpha-1}E_{\alpha,\alpha}(-\mu_n t^\alpha) \, dt \right|^2 \\
\leq \|y^0\|_{\Pi_{i=1}^k H_2(0,t_i)}^2 + \|f\|_{L_\infty(0,T;L_2(\mathcal{G}))}^2.
\]

Hence, we obtain
\[
\|y(\cdot,t)\|_{\Pi_{i=1}^k H_2(0,t_i)} \leq C\|(-\mathcal{L})y(\cdot,t)\|_{L_2(\mathcal{G})} \\
\leq C \left( \|y^0\|_{\Pi_{i=1}^k H_2(0,t_i)} + \|f\|_{L_\infty(0,T;L_2(\mathcal{G}))} \right).
\]

Now,
\[
\sum_{i=1}^k \left\| cD_0^\alpha y_i(\cdot,t) \right\|_{L_2(0,t_i)}^2 \leq \sum_{n=1}^\infty \mu_n^2 \left| \langle y^0, \Psi_n \rangle E_{\alpha,1}(-\mu_n t^\alpha) \right|^2 + \sum_{n=1}^\infty \left| \langle f(x,t), \Psi_n \rangle \right|^2 \\
+ \sum_{n=1}^\infty \mu_n^2 \left| \int_0^t \langle f(x,\xi), \Psi_n \rangle(t-\xi)^{\alpha-1}E_{\alpha,\alpha}(-\mu_n(t-\xi)^\alpha) \, d\xi \right|^2 \\
\leq \sum_{n=1}^\infty \mu_n^2 \left| \langle y^0, \Psi_n \rangle \right|^2 + \sum_{n=1}^\infty \left| \langle f(x,t), \Psi_n \rangle \right|^2 \\
+ \sum_{n=1}^\infty \mu_n^2 \sup_{0 \leq \xi \leq T} \left| \langle f(\cdot,\xi), \Psi_n \rangle \right|^2 \left| \int_0^t t^{\alpha-1}E_{\alpha,\alpha}(-\mu_n t^\alpha) \, dt \right|^2.
\]

Using equation (3.3), we obtain
\[
\left\| cD_0^\alpha y(\cdot,t) \right\|_{L_2(\mathcal{G})}^2 = \sum_{i=1}^k \left\| cD_0^\alpha y_i(\cdot,t) \right\|_{L_2(0,t_i)}^2 \\
\leq \|y^0\|_{\Pi_{i=1}^k H_2(0,t_i)}^2 + \|f(\cdot,t)\|_{L_2(\mathcal{G})}^2 + \|f\|_{L_\infty(0,T;L_2(\mathcal{G}))}^2 \\
\leq \|y^0\|_{\Pi_{i=1}^k H_2(0,t_i)}^2 + C\|f\|_{L_\infty(0,T;L_2(\mathcal{G}))}^2.
\]
Hence,
\[
\left\| c D_{0,t}^\alpha y \right\|_{L^2(\mathcal{G} \times (0,T))}^2 = \int_0^T \left\| c D_{0,t}^\alpha y(\cdot,t) \right\|_{L^2(\mathcal{G})}^2 dt \\
\leq T \left( \|y^0\|_{\Pi_{i=1}^k H_2(0,l_i)}^2 + C \|f\|_{L^\infty(0,T;L^2(\mathcal{G}))}^2 \right) \\
\leq C_1 \left( \|y^0\|_{\Pi_{i=1}^k H_2(0,l_i)}^2 + \|f\|_{L^\infty(0,T;L^2(\mathcal{G}))}^2 \right).
\]

\[\square\]

**Corollary 3.3.1.** Let \( y^0 \in L^2(\mathcal{G}) \) and \( f = 0 \). Then we obtain the following estimate for the unique weak solution \( y \in C([0,T]; L^2(\mathcal{G})) \cap C((0,T]; D(-\mathcal{L})) : \)

\[
\|y(\cdot,t)\|_{L^2(\mathcal{G})} \leq \frac{C}{1 + \mu_1 t^\alpha} \|y^0\|_{L^2(\mathcal{G})}, \quad t \in (0,T).
\]

**Proof.** Substituting \( f = 0 \) in equation (2.6) and using Proposition 2.1 we obtain

\[
\sum_{i=1}^k \|y_i(\cdot,t)\|_{L^2(0,l_i)}^2 = \sum_{n=1}^\infty \left| \langle y^0, \Psi_n \rangle \right| E_{\alpha,1}(\mu_n t^\alpha) \leq \sum_{n=1}^\infty \left| \langle y^0, \Psi_n \rangle \right| \left( \frac{C}{1 + \mu_n t^\alpha} \right)^2 \leq \left( \frac{C}{1 + \mu_1 t^\alpha} \right)^2 \|y^0\|_{L^2(\mathcal{G})}, \quad t \in (0,T).
\]

\[\square\]

**Corollary 3.3.2.** Let \( y^0 \in D(-\mathcal{L}) \) and \( f = 0 \). Then there exists a constant \( C_1 > 0 \) such that

\[
\|y(\cdot,t)\|_{\Pi_{i=1}^k H_2(0,l_i)} + \left\| c D_{0,t}^\alpha y(\cdot,t) \right\|_{L^2(\mathcal{G})} \leq \frac{C_1}{1 + \mu_1 t^\alpha} \|y^0\|_{\Pi_{i=1}^k H_2(0,l_i)}, \quad t \in (0,T).
\]

**Proof.**

\[
\|(-\mathcal{L})y(\cdot,t)\|_{L^2(\mathcal{G})}^2 = \sum_{i=1}^k \left\| (-\mathcal{M})y_i(\cdot,t) \right\|_{L^2(0,l_i)}^2 \\
= \sum_{n=1}^\infty \mu_n^2 \left| \langle y^0, \Psi_n \rangle \right| E_{\alpha,1}(\mu_n t^\alpha) \leq \left( \frac{C_1}{1 + \mu_1 t^\alpha} \right)^2 \sum_{n=1}^\infty \mu_n^2 \left| \langle y^0, \Psi_n \rangle \right|^2 \leq \left( \frac{C_1}{1 + \mu_1 t^\alpha} \right)^2 \|y^0\|_{\Pi_{i=1}^k H_2(0,l_i)}^2, \quad t \in (0,T).
\]

Since \( f = 0 \), equation (1.1) implies that \( c D_{0,t}^\alpha y(\cdot,t) = \mathcal{L} y(\cdot,t) \) and hence estimate (3.8) follows.

\[\square\]

### 4 Conclusion and future work

In this paper, the existence and uniqueness of time-fractional diffusion equation on a star graph is established. By using the method of eigenfunction expansion the existence and uniqueness of the weak solution and the regularity of the solution is derived. In future we will consider fractional diffusion equation on more general graphs (i.e. graphs containing cycles) and investigate the existence and uniqueness of solution. We will also consider optimal control problems for fractional diffusion equation on metric graphs.
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References

[1] R. Adami, C. Caaciapuoti, D. Finco, and D. Noja. Variational properties and orbital stability of standing waves for nls equation on a star graph. Journal of Differential equations, 257:3738–3777, 2014.

[2] S. Avdonin, G. Murzabekova, and K. Nurtazina. Source identification for the heat equation with memory. 2015.

[3] S. Avdonin and S. Nicaise. Source identification for the wave equation on graphs. C.R. Acad. Sci. Paris, Ser.I, 352:907–912.

[4] J. Von Below. A characteristic equation associated to an eigenvalue problem on $c^\infty$-net. Linear. Alg. Appl., 71:309–325, 1985.

[5] G. W. Bohannan. Analog fractional order controller in temperature and motor control applications. Journal of Vibration and Control, 14:1487–1498, 2008.

[6] René Dáger and Enrique Zuazua. Wave propagation, observation and control in 1-d flexible multi-structures, volume 50 of Mathématiques & Applications (Berlin) [Mathematics & Applications]. Springer-Verlag, Berlin, 2006.

[7] Chr. Friedrich. Rheological material functions for associating comb-shaped or h-shaped polymers a fractional calculus approach. Philosophical magazine letters, 66:287–292, 1992.

[8] N. I. Gerasimenko and B. S. Pavlov. The scattering problem on noncompact graphs. Teor. Mat. Fiz., 74:345–359, 1988.

[9] G. Leugering. On the semi-discretization of optimal control problems for networks of elastic strings:global optimality systems and domain decomposition. J. Comput. Appl. Math., 120:133–157, 2000.

[10] A. Grigor’yan, Y. Lin, and Y. Yang. Yamabe type equations on graph. Journal of Differential Equations, 261:4924–4943, 2016.

[11] R. Hilfer. Applications of fractional calculus in physics. World Scientific, Singapore, 2000.

[12] T. Kato. Perturbation theory for linear operators. Springer, Berlin, 1980.

[13] A. A. Kilbas, H.M. Srivastava, and J.J. Trujillo. Theory and Applications of Fractional Differential Equations. Elsevier, 2006.

[14] J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. Modelling and controllability of networks of thin beams. In System modelling and optimization (Zurich, 1991), volume 180 of Lect. Notes Control Inf. Sci., pages 467–480. Springer, Berlin, 1992.

[15] J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. Control of planar networks of Timoshenko beams. SIAM J. Control Optim., 31(3):780–811, 1993.

[16] J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. Modelling of dynamic networks of thin thermoelastic beams. Math. Methods Appl. Sci., 16(5):327–358, 1993.
[17] J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. On the analysis and control of hyperbolic systems associated with vibrating networks. *Proc. Roy. Soc. Edinburgh Sect. A*, 124(1):77–104, 1994.

[18] J. E. Lagnese, Günter Leugering, and E. J. P. G. Schmidt. *Modeling, analysis and control of dynamic elastic multi-link structures*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[19] Lang Li, Lingyu Jin, and Shaomei Fang. Existence and uniqueness of the solution to a coupled fractional diffusion system. *Advances in Difference Equations*, 2015(1):370, 2015.

[20] Y. S. Li and T. Wei. An inverse time-dependent source problem for a time-space fractional diffusion equation. *Applied Mathematics and Computation*, 336:257–271, 2018.

[21] Y. Luchko. Maximum principle for the generalized time-fractional diffusion equation. *J. Math. Anal. Appl.*, 351:218–223, 2009.

[22] Y. Luchko. Some uniqueness and existence results for the initial-boundary value problems for the generalized time-fractional diffusion equations. *Comput. Math. Appl.*, 59:1766–1772, 2010.

[23] Y. Luchko. Anomalous diffusion: models, their analysis, and interpretation. In *Advances in Applied Analysis*, pages 115–145. Springer, 2012.

[24] G. Lumer. Connecting of local operators and evolution equations on a network. *Lect. Notes Math.*, 787:219–234, 1980.

[25] R. L. Magin and M. Ovadia. Modeling the cardiac tissue electrode interface using fractional calculus. *Journal of Vibration and Control*, 14:1431–1442, 2008.

[26] F. Mainardi. The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett.*, 9:23–28, 1996.

[27] Vaibhav Mehandiratta, Mani Mehra, and Günter Leugering. Existence and uniqueness results for a nonlinear Caputo fractional boundary value problem on a star graph. *Journal of Mathematical Analysis and Applications*, 477:1243–1264, 2019.

[28] D. Mugnolo. Gaussian estimates for a heat equation on a network. *Networks and Heterogeneous Media*, 2:55–79, 2007.

[29] D. Mugnolo. *Semigroup Methods for Evolution Equations on Networks*. Springer, 2014.

[30] S. Nicaise. Some results on spectral theory over networks, applied to nerve impulses transmission. *Lect. Notes Math.*, 1771:532–541, 1985.

[31] S. Nicaise and O. Zair. Identifiability, stability and reconstruction results of point sources by boundary measurements in heterogeneous trees. *Revista Matematica Complutense*, 16, 2003.

[32] B. S. Pavlov and M. D. Faddeev. Model of free electrons and the scattering problem. *Teor. Mat. Fiz.*, 55:257–269, 1983.

[33] O. M. Penkin, Yu. V. Pokornyi, and E. N. Provotorova. On one vector boundary-value problem. *Boundary -Value Problems*.

[34] I. Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[35] H. Pollard. The completely monotonic character of Mittag-Leffler function $E_{\alpha(-x)}$. *Bulletin of the American Mathematical Society*, 54:2233–2244, 1948.

[36] V.V. Provotorov. Eigenfunctions of the Sturm-Liouville problem on a star graph. *Sbornik:Mathematics*, 199:1528–1545, 2008.
[37] K. Sakamoto and M. Yamamoto. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.*, 382:426–447, 2011.

[38] A. Shukla, M. Mehra, and G. Leugering. A fast adaptive spectral graph wavelet method for the viscous burgers’ equation on a star-shaped connected graph. *Mathematical Methods in the Applied Sciences*, 2019.

[39] M. C. Steinbach. On pde solution in transient optimization of gas networks. *J. Comput. Appl. Math.*, 203:345–361, 2007.

[40] Joachim von Below. Classical solvability of linear parabolic equations on networks. *J. Differential Equations*, 72(2):316–337, 1988.

[41] Maximilian Walther. *Simulation-Based Model Reduction for Partial Differential Equations on Networks*. PhD thesis, FAU Studies Mathematics and Physics, Erlangen, 2018.

[42] T. Wei, X. L. Li, and Y. S. Li. An inverse time dependent source problem for a time-fractional diffusion equation. *Inverse Probl.*, 32, 2016.

[43] H. Yoshioka, K. Unami, and M. Fujihara. Burgers type equation models on connected graphs and their application to open channel hydraulics. 2014. [http://hdl.handle.net/2433/195771](http://hdl.handle.net/2433/195771).