Minimax estimation of Functional Principal Components from noisy discretized functional data

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Abstract

Functional Principal Component Analysis is a reference method for dimension reduction of curve data. Its theoretical properties are now well understood in the simplified case where the sample curves are fully observed without noise. However, functional data are noisy and necessarily observed on a finite discretization grid. Common practice consists in smoothing the data and then to compute the functional estimates, but the impact of this denoising step on the procedure’s statistical performance are rarely considered. Here we prove new convergence rates for functional principal component estimators. We introduce a double asymptotic framework: one corresponding to the sampling size and a second to the size of the grid. We prove that estimates based on projection onto histograms show optimal rates in a minimax sense. Theoretical results are illustrated on simulated data and the method is applied to the visualization of genomic data.

Keywords— Functional data analysis, Principal Components Analysis, minimax rates.

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1 Introduction

Functional data analysis is a dedicated statistical framework considering that observed curve data are the realizations of random functions [Ferraty and Vieu 2006, Ramsay and Silverman 2010, Ferraty and Romain 2011]. Hence this framework is intrinsically infinite-dimensional. Functional Principal Components Analysis is a common method to reduce dimensionality of data with the associated eigenvalues, assumed to be distinct, $\mu^* = \{\mu^*_d, d \in \mathbb{N}^*; \mu^*_1 > \mu^*_2 > \ldots\}$ of the operator $\Gamma$, we obtain the Karhunen-Loève representation of $Z$ [Bosq 2000]:

$$Z = \sum_{d \in \mathbb{N}^*} \zeta^*_d \mu^{*1/2}_d \eta^*_d,$$

(1)

where $\zeta^*_d, d \in \mathbb{N}^*$ is a sequence of non-correlated centered random variables of variance $1$, usually called the principal components scores. Then, for any integer $D$, the best $D$-dimensional approximation of process $Z$ is spanned by the $D$ first eigenfunctions. Our aim is to provide estimators of these eigenelements based on noisy discretized data, and to assess their statistical performance.

In the ideal case, when the $Z_i(t)$'s are observed for all $t \in [0, 1]$ without noise, the estimator of $\eta^*_d$ is the eigenfunction $\hat{\eta}_d$ associated to the $d$-th largest eigenvalue of the empirical covariance operator

$$\hat{\Gamma}(f)(\cdot) = \frac{1}{n} \sum_{i=1}^n \langle f, Z_i \rangle Z_i(\cdot), \quad f \in \mathcal{L}^2.$$

However, in practice, functional data are observed on a discretization grid (fixed or random) and are usually corrupted by random noise. Hence we consider the following statistical model

$$Y_i(t_h) = Z_i(t_h) + \epsilon_i,h, \quad i = 1, \ldots, n,$$

(2)

with the $\epsilon_i,h$'s being independent Gaussian errors with variance $\sigma^2$. The errors are assumed to be independent of the $Z_i$'s. Here, we assume that the grid is fixed and regular with $p$ points $\{t_h = h/(p - 1); h = 0, \ldots, p - 1\}$ but the approach could be generalized to the case where $|t_h - h/p| \leq 1/p$. The random design case must be treated with different approaches, for instance with kernel methods [Hall et al. 2006, Li and Hsing 2010].

In practice, when process $Z$ is observed on a grid, the covariance operator $\Gamma$ must be approximated. In this case, the problem is restated as a covariance matrix estimation problem. Bunea and Xiao [2013] compared the rescaled eigenelements of the covariance matrix of the discretized data with the corresponding eigenelements of the discretized kernel. However, this approach makes it more difficult to account for the functional regularity of the underlying random process $Z$. A common alternative method consists in denoising the observations by projecting the observed curves $Z_i(t)$ on a smoothing basis denoted by $\phi = (\phi_\lambda; \lambda \in \Lambda)$, that can be typically splines [Ramsay and Silverman 2010, Lin and Carroll 2000], but we will consider histograms and the Haar wavelet basis here. This preliminary step produces a continuous estimate of the covariance kernel $K$ defined as

$$K(s, t) = E\{Z(s)Z(t)\}, \quad (s, t) \in [0, 1]^2.$$

(3)
From a theoretical perspective, the main challenge is to assess the rates of convergence of the obtained estimators, in a very specific framework since functional principal component analysis combines two very different convergence settings: a first one associated with the sampling of \( n \) independent processes with same distribution of \( Z \), and a non-parametric setting since data are functions, here observed at \( p \) points. Early works focused on the continuous non-noisy case, corresponding to \( p = +\infty \) and \( \sigma = 0 \), for which the empirical estimator achieves an optimal \( n^{-1} \) parametric rate (up to a logarithmic factor) for the risk associated to the \( L^2 \)-error [Bosq 2000] or to the operator norm of the projector [Mas and Ruymgaart 2015]. But the challenge is now to determine how discretization and how the noise impact the estimation of the eigenelements of the covariance operator. In particular, the statistical implications of the smoothing step are rarely debated, whereas it raises some concern, mainly related to the level of regularity of the underlying process versus the choice of the smoothing basis, and the capacity of distinguishing noise from signal through this method. In particular, the regularity of the functional data has also deep impact on the mathematical developments required to account for the projection step on the smoothing basis \( \phi \). Indeed, the operator \( \Gamma \) being trace-class,

\[
\text{tr}(\Gamma) = \sum_{\lambda \in \Lambda} \text{Var}(\langle Z, \phi_\lambda \rangle) = \sum_{\lambda \in \Lambda} \langle \Gamma(\phi_\lambda), \phi_\lambda \rangle
\]

is finite. Consequently, we cannot resume the problem to a setting where functional data \( Z \) would be like an infinite vector of coefficients \( \langle Z, \phi_\lambda \rangle, \lambda \in \Lambda \) of equal variance. This distinction is crucial and comes from the fact that the dimensionality of functional data does not arise in same terms as in the vector case since the variance of most coefficients vanishes at infinity.

Another positive consequence is that functional principal component analysis may avoid the inconsistency problems raised by [Johnstone and Lu 2009]. Hence, the asymptotic properties of functional principal component when \( n \) and \( p \) both tend to infinity differ from those of standard principal component analysis for high-dimensional data.

To fully understand the statistical complexity of functional principal component analysis, it is necessary to compute the minimax rates of estimation and to compare it with the parametric bounds obtained by [Mas and Ruymgaart 2015]. Upper bounds in the case of noisy discretized data have also been proposed [Bunea and Xiao 2015, Hall et al. 2006, Hall et al. 2006] established optimal rates of convergence of order \( n^{-2r/(2r+1)} \) in the case where the number of per-individual observations is bounded and the signal has \( r \) bounded derivatives. [Descary and Panaretos 2019] studied a generalization to heterogeneous noise with possible time dependency at the price of two strong assumptions: analyticity of the eigenfunctions and finite rank of the covariance operator of the signal; the achieved rate is then \( n^{-1} + p^{-2} \).

Here we study non-asymptotic convergence rates for the estimation of the eigenelements of the covariance operator \( \Gamma \) under a mild regularity assumption on the process \( Z \). Denoting by \( \alpha \) this regularity, our assumption is equivalent to assuming that the kernel \( K \) is a bivariate \( \alpha \)-Hölder continuous function. Under a moment assumption for process \( Z \), we obtain rates of the form \( n^{-1} + p^{-2\alpha} \).

These rates, which are new, are, moreover, optimal in the minimax sense for the estimation of the first eigenfunction (we prove a lower bound). We illustrate these rates in practice on simulations. These rates tell us a lot about the behavior of the estimated eigenfunctions under the double asymptotic in \( p \) and \( n \). When \( p \) is large compared to \( n^{1/(2\alpha)} \), we find the optimal parametric rate \( n^{-1} \) obtained by [Dauxois et al. 1982, Mas and Ruymgaart 2015] when the curves \( Z_i \) are fully observed and without noise. Moreover, even though the problem is intrinsically non-parametric, and in the presence of noisy observations, the simple estimator obtained by projection on the
Since this estimator cannot be calculated directly from the data, we introduce an orthonormal system of $\mathcal{L}^2$ curves on the entire interval and we define, for $i \leq n$ and $\lambda \in \Delta_D$, an approximation of $(Y_i, \phi_\lambda)$.

2 Estimation of the eigenfunctions and the eigenvalues

Following the usual approach of functional data analysis [Dauxois et al., 1982, Bosq, 2000, Ramsay and Silverman, 2010, Hall et al., 2006], the estimation procedure for functional principal component analysis consists in estimating the covariance operator $\Gamma$ from the data:

$$\Gamma(f)(\cdot) = \int_0^1 K(s, \cdot) f(s) ds, \quad f \in \mathcal{L}^2;$$

that is well defined provided $E(||Z||^2) < +\infty$, which is assumed in the following. Then, a natural approach consists in estimating $K$ in a first step using the empirical covariance kernel:

$$\hat{K}(s, t) = \frac{1}{n} \sum_{i=1}^n Z_i(t)Z_i(s), \quad (s, t) \in [0, 1]^2.$$ 

Since this estimator cannot be calculated directly from the data, we introduce $(\phi_\lambda)_{\lambda \in \Delta_D}$, an orthonormal system of $\mathcal{L}^2$ with $\Delta_D$ a finite set of size $D$. In the following, we will consider histograms and Haar wavelets. Then, in the setting of Model (2), we first reconstruct the observed curves on the entire interval and we define, for $i = 1, \ldots, n$,

$$\tilde{Y}_i(t) = \sum_{\lambda \in \Delta_D} \tilde{y}_{i, \lambda} \phi_\lambda(t), \quad \tilde{y}_{i, \lambda} = \frac{1}{p} \sum_{h=0}^{p-1} Y_i(t_h) \phi_\lambda(t_h), \quad t \in [0, 1],$$

with $\tilde{y}_{i, \lambda}$ an approximation of $(Y_i, \phi_\lambda)$. Similarly, we define $\tilde{Z}_i(t), \tilde{Z}_{i, \lambda}, \tilde{E}_i(t), \tilde{E}_{i, \lambda}$ by replacing $Y_i(t_h)$ in the previous expressions by $Z_i(t_h)$, and $\varepsilon_{i,h}$. A natural estimator of the covariance kernel $K$ is then

$$\hat{K}(s, t) = \frac{1}{n} \sum_{i=1}^n \tilde{Y}_i(t)\tilde{Y}_i(s), \quad (s, t) \in [0, 1]^2,$$
and the covariance operator $\Gamma$ can be estimated by
\[
\hat{\Gamma}_\phi(f)(\cdot) = \int_0^1 \hat{K}_\phi(s,\cdot) f(s) ds, \quad f \in L^2.
\]
Since $\hat{K}_\phi$ is symmetric, the operator $\hat{\Gamma}_\phi$ is self-adjoint and is also finite-rank since $\text{Im}(\hat{\Gamma}_\phi) \subset \text{span}(\hat{Y}_1,\ldots,\hat{Y}_n)$. Therefore, $\hat{\Gamma}_\phi$ is a compact operator. From the spectral theorem, we know that there exists a $L^2$-basis of eigenfunctions of $\hat{\Gamma}_\phi$, denoted by $\hat{\eta}_\phi = \{\hat{\eta}_{\phi,d}, d \in \mathbb{N}^*\}$, with associated eigenvalues $\hat{\nu}_\phi = \{\hat{\mu}_{\phi,d}, d \in \mathbb{N}^*; \hat{\nu}_{\phi,1} \geq \hat{\mu}_{\phi,2} \geq \ldots\}$. We then obtain estimates of the principal components that are analyzed in the minimax setting.

3 Minimax rates of the eigenfunction estimator

3.1 Smoothness class for the functional curve $Z$

Minimax rates of convergence depend on the underlying smoothness of the process of interest. In the sequel, for any $\alpha \in (0,1]$ and $L > 0$ we consider the regularity class
\[
\mathcal{R}_\alpha(L) = \{P, \text{ probability measure on } C^0 \text{ such that } \int_0^1 |z(t) - z(s)|^2 dP(z) \leq L|t - s|^{2\alpha}, \quad (s,t) \in [0,1]^2\}.
\]
The use of these regularity sets is natural. Indeed, we can for instance remark that $P_Z$, the distribution of $Z$, satisfies
\[
P_Z \in \mathcal{R}_\alpha(L) \Leftrightarrow E_Z[(Z(t) - Z(s))^2] \leq L|t - s|^{2\alpha}, \quad (s,t) \in [0,1]^2.
\]
Our regularity condition implies that kernel $K$ is bounded
\[
\|K\|_\infty = \sup_{(s,t) \in [0,1]^2} |K(s,t)| < \infty,
\]
and is an $\alpha$-Hölder continuous function. More precisely, for any $(s,s',t,t') \in [0,1]^4$,
\[
P_Z \in \mathcal{R}_\alpha(L) \Rightarrow |K(s,t) - K(s',t')| \leq (\|K\|_\infty L)^{1/2} (|s - s'|^\alpha + |t - t'|^\alpha).
\]
Conversely, if $K$ is a bivariate $\alpha$-Hölder continuous function, we know that there exists $L' > 0$ such that
\[
|K(s,t) - K(s',t')| \leq L' (|s - s'|^2 + |t - t'|^2)^{\alpha/2}.
\]
Then
\[
E_Z[(Z(t) - Z(s))^2] = K(s,s) - 2K(s,t) + K(t,t) \leq 2L'|s - t|^{\alpha},
\]
and $P_Z \in \mathcal{R}_\alpha(2L')$.

Classical Gaussian processes belong to $\mathcal{R}_\alpha(L)$ for $\alpha$ and $L$ well chosen. For instance, if $Z$ is a standard Brownian motion or a Brownian bridge then $P_Z \in \mathcal{R}_{1/2}(1)$. More generally, fractional Brownian motions with Hurst exponent $\alpha$ and Hurst index $C_\alpha$ belong to $\mathcal{R}_\alpha(C_\alpha)$. If $Z$ is an Ornstein-Uhlenbeck process, its covariance function is $K(s,t) = \exp(-|t - s|/2)$, then it verifies
\[
E_Z[(Z(t) - Z(s))^2] = 2(1 - e^{-|t - s|/2}) \leq |t - s|, \quad (s,t) \in \mathbb{R}^2,
\]
which implies $Z \in \mathcal{R}_{1/2}(1)$. We refer to [Lifshits, 1995] for the precise definitions and properties of these processes.
3.2 Lower bound

The lower bound of the risk for estimating eigenfunctions can be viewed as a benchmark to achieve. We focus on the first eigenfunction but a similar result, though more technical, could be obtained for the other eigenfunctions.

**Theorem 1** Suppose that \( Z \) is a Gaussian process and \( p \geq 3 \). Let \( \alpha \in (0, 1] \) and \( L > 0 \). There exists \( n_0 \) only depending on \( L \) and \( \alpha \) such that, for all \( n \geq n_0 \),

\[
\inf_{\tilde{\eta}} \sup_{P \in \mathcal{R}_n(L)} E(\|\tilde{\eta} - \eta\|^2) \geq c(\sigma)(p^{-2\alpha} + n^{-1}),
\]

where \( c(\sigma) \) is a constant only depending on \( \sigma \) and the infimum is taken over all estimators \( \hat{\eta} \) of \( \eta \) with first eigenfunctions distant of \( 1 \). A generalization to non-Gaussian processes could be done with an assumption on the Kullback-Leibler divergence [Tsybakov, 2009]. The lower bound of the minimax risk by \( p^{-2\alpha} \) is true whatever the distribution of \( Z \). Indeed we can construct two processes \( Z_0, Z_1 \) with first eigenfunctions distant of \( p^{-2\alpha} \) from each other and such that \( Z_0(t_h) = Z_1(t_h) \) almost surely for all \( h = 0, \ldots, p-1 \) (see Appendix B).

3.3 General upper bounds

We now derive upper bounds for estimates \( \hat{\eta}_{\phi,d} \). For this purpose, we set

\[
b_1 = 8(\mu_1^* - \mu_2^*)^{-2}
\]

and for \( d = 2, \ldots, D \),

\[
b_d = 8/\min(\mu_d^* - \mu_{d+1}^*, \mu_{d-1}^* - \mu_d^*)^2.
\]

Since we supposed that all the true eigenvalues \( \mu_d^* \)'s are distinct, the quantities \( b_d \)'s are well defined and finite.

The eigenfunction \( \eta_d \) being defined up to a sign change \( \pm \eta_d \) is also an eigenfunction associated to the eigenvalue \( \mu_d^* \), we cannot assess our procedure by using the classical risk \( E(\|\tilde{\eta} - \eta_d\|^2) \). Following [Bosq, 2000], we evaluate the risk of \( \eta_{\phi,d} = \text{sign}(\langle \tilde{\eta}_{\phi,d}, \eta_d \rangle) \times \eta_d \).

We consider the following mild assumptions on the 4th moment of the vector \( Z = \{Z(t_0), \ldots, Z(t_{p-1})\}^T \).

**Assumption 1** We assume that there exists \( C_1 > 0 \) such that

\[
E(\langle v^T Z \rangle^4) \leq C_1 [E(\langle v^T Z \rangle^2)]^2, \quad v \in \mathbb{R}^p.
\]

Assumption 1 ensures a control of the fourth moment of \( \tilde{z}_{1,\lambda} \). It is satisfied with \( C_1 = 3 \) if \( Z \) is Gaussian. Then we obtain the following result:

**Theorem 2** Let \( d \) be fixed. Under Assumption 1 we have

\[
E(\|\tilde{\eta}_{\phi,d} - \eta_{\phi,d}^*\|^2) \leq 5b_d \left[ \|\Pi_D \Gamma_{\Pi_D} - \Gamma\|_\infty + \max(C_1 + 3; 6) \right] \left( \sum_{\lambda \in \Lambda_D} \left( \sigma_\lambda^2 + s_\lambda^2 \right) \right)^2 + A_1^{(K)}(\phi, D) + A_1^{(\alpha)}(\phi, D) + \frac{\sigma^4}{p^2},
\]

where...
where $\Pi_D$ is the orthogonal projection onto $S_D = \text{span}(\phi_{\lambda}, \lambda \in \Lambda_D)$.

$$
\sigma^2_{\lambda} = \text{Var}(\tilde{z}_{1,\lambda}) = \frac{\sigma^2}{p^2} \sum_{h=0}^{p-1} \phi^2_h(t_h), \quad s^2_{\lambda} = \text{Var}(\tilde{z}_{1,\lambda}) = \frac{1}{p^2} \sum_{h,h'=0}^{p-1} K(t_h, t_{h'}) \phi_{\lambda}(t_h) \phi_{\lambda}(t_{h'})
$$

and

$$
A^{(K)}_p(\phi, D) = \sum_{\lambda, \lambda' \in \Lambda_D} \left\{ \frac{1}{p^2} \sum_{h,h'=0}^{p-1} K(t_h, t_{h'}) \phi_{\lambda}(t_h) \phi_{\lambda'}(t_{h'}) - \int_0^1 \int_0^1 K(s, t) \phi_{\lambda}(s) \phi_{\lambda'}(t) ds dt \right\}^2,
$$

$$
A^{(\varnothing)}_p(\phi, D) = \frac{\sigma^2}{p^2} \sum_{\lambda, \lambda' \in \Lambda_D} \left\{ \frac{1}{p} \sum_{h=0}^{p-1} \phi_{\lambda}(t_h) \phi_{\lambda'}(t_h) - 1_{(\lambda=\lambda')} \right\}^2.
$$

The first term of the upper bound is a bias term corresponding to the projection step, that decreases with $D$, the dimension of the approximation space. The second term is a variance term that increases with $D$ but contrary to what happens generally in non-parametric statistics, it is bounded by $n^{-1}$ up to a constant under mild assumptions on the orthonormal system (details in Section 3.4). Indeed, heuristically, when $p$ grows, the term $\sigma^2_{\lambda}$ is of order $\sigma^2/p$ (the variance due to the noise is tempered by the repetition of the observations) and the term $s^2_{\lambda}$ is of order $\int \int K(s, t) \phi_{\lambda}(s) \phi_{\lambda'}(t) ds dt = E((Z, \phi_{\lambda})^2)$ so $\sum_{\lambda \in \Lambda_D} s^2_{\lambda}$ is bounded by a constant (independent of $D$) of order $E(\|Z\|^2) < +\infty$. By taking $D = \text{card}(\Lambda_D) \leq p$, the second term is of order $n^{-1}$. The third and fourth terms are linked to the discretization and are usually negligible with respect to both the bias and variance terms (see Section 3.4). The term $\sigma^2/p^2$ is also negligible.

We can refine the previous result and obtain similar upper bounds in probability. To state them, we first recall the definition of sub-Gaussian variables. We refer to [Vershynin, 2012] for more details.

**Definition 1** We say that a random variable $W$ is sub-Gaussian if

$$
\|W\|_{\psi_2} := \sup_{q \geq 1} \left\{ q^{-1/2} \left\{ E(\|W\|^q) \right\}^{1/q} \right\} < \infty.
$$

In this case, $\|W\|_{\psi_2}$ is called the sub-Gaussian norm of $W$.

Assumption [1] is extended to all moments of order $q$ as follows.

**Assumption 2** We assume that there exists $C_2 > 0$ such that

$$
\|v^T Z\|_{\psi_2}^2 \leq C_2 E(v^T Z)^2, \quad v \in \mathbb{R}^p.
$$

Assumption [2] of Theorem [3] is stronger than Assumption [1] of Theorem [2] but it allows us to obtain an inequality in probability, which is stronger than in expectation; the price to pay is the logarithmic factor in the variance term as show in the following. Using quantities introduced in Theorem [2] we then obtain the following result.

**Theorem 3** Let $d$ be fixed. Then, under Assumption [2] for all $\gamma > 0$, with probability larger than $1 - 2 \exp(-1/64 \min(\gamma^2, 16\gamma^2/\pi))$,

$$
\|\tilde{\eta}_{\phi,d} - \eta^*_{\phi,d}\|^2 \leq 5b_d \left[ \|\Pi_D \Gamma \Pi_D - \Gamma\|^2_{\infty} + \frac{(e^{1/2} + \gamma)^2 C_2^2 (C_2 + 1)^2}{\pi} \sum_{\lambda \in \Lambda_D} \left( \sigma^2_{\lambda} + s^2_{\lambda} \right) \right]^2 \left[ A^{(K)}_p(\phi, D) + A^{(\varnothing)}_p(\phi, D) + \frac{\sigma^2}{p^2} \right],
$$

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where $\overline{C}$ is an absolute constant.

Observe that if we take $\gamma = 8(\beta \log n)^{1/2}$, then for $n$ large enough, the upper bound holds with probability larger than $1 - 2n^{-\beta}$. In this case, the order of the variance term is the same as for Theorem 2 up to the log $n$-factor.

Theorem 3 is based on Assumption 2, namely a control of the sub-Gaussian norm of $v^T Z$ for all vectors $v$. Such controls are standard to obtain concentration inequalities which are at the core of the proof of Theorem 3 see for instance Vershynin [2012], Kolchinskii and Lounici [2017] or Section 2.3 of Boucheron et al. [2013]. This assumption enables us to apply large deviation bounds for martingale differences established by Juditsky and Nemirovski [2008] to specific covariance matrices. See Proposition 5 in the Appendix for more details. Observe that Assumption 2 is satisfied if $Z$ is Gaussian.

Remark. The classical functional assumption (see e.g. Mas and Ruymgaart 2015) amounts to bound the moments of the variable $\zeta_\star^d$ appearing in the Karhunen-Loève decomposition (1) of $Z$ as follows:

$$\sup_{q \geq 1} \sup_{d \in \mathbb{N}^\star} E\left( |\zeta_\star^d|^2q \right) \leq q! b^{q-1}$$

for $b > 0$ a constant. This type of assumption is not well adapted to the case where the data are discretized but shows strong similarities with ours since $\zeta_\star^d = \langle Z, \eta_\star^d \rangle \times V(\langle Z, \eta_\star^d \rangle)^{-1/2}$.

Remark. The first step of the proof of our results consists in applying Bosq inequalities to bound $\|\hat{\eta} \phi, d - \eta_\star \phi, d\|$. Similar bounds also hold for $|\hat{\mu} d - \mu_\star d|$. See Section C of the Appendix for more details. Therefore, bounds of the previous theorems also hold for $|\hat{\mu} d - \mu_\star d|^2$. Obtaining lower bounds for the estimation of the eigenvalues remains an open interesting question.

3.4 Upper bound for histograms

In this paragraph, we specify our results for the case of the following piecewise orthonormal system.

Assumption 3. There exists an integer $D$ such that $D$ divides $p$ and for any $\lambda \in \Lambda_D = \{0, \ldots, D - 1\}$,

$$\phi_\lambda(t) = D^{1/2} \times 1_{I_\lambda}(t), \quad t \in [0, 1],$$

with $I_\lambda = (\lambda/D, (\lambda + 1)/D)$. In this framework, all terms appearing in upper bounds of Theorems 2 and 3 can be easily controlled.

Proposition 1. Under Assumption 3 if $P_Z \in \mathcal{R}_\alpha(L)$, we have

$$\|\Pi_D \Gamma_D - \Gamma\|_\infty \leq \frac{16L \|K\|_\infty}{(\alpha + 1)^2} D^{-2\alpha},$$

$$A_p^K(\phi, D) \leq \frac{16\|K\|_\infty L}{(\alpha + 1)^2} p^{-2\alpha}, \quad A_p^K(\phi, D) = 0$$

and

$$\sum_{\lambda \in \Lambda_D} (\sigma_\lambda^2 + \bar{s}_\lambda^2) \leq \|K\|_\infty + \frac{\sigma^2 D}{p}.$$

Combining Proposition 1 with Theorems 2 and 3 we finally deduce the following corollary.

Corollary 1. Let $d$ be fixed. Assume that $P_Z \in \mathcal{R}_\alpha(L)$ and Assumption 3 holds with $D = p$.

Under Assumption 2,

$$E\left( \|\hat{\eta}_d - \eta_\star^d\|^2 \right) \leq b_d \left\{ \frac{B(L, K, \alpha)}{p^{2\alpha}} + \frac{\sigma^4}{p^2} + \frac{V_1(K, \sigma, C_1)}{n} \right\}$$
and under Assumption \[2\] for any $\beta > 0$, for $n$ large enough, with probability larger than $1 - 2n^{-\beta}$,

$$\|\hat{\eta}_{0,d} - \eta^*_d, d\|^2 \leq b_4 \left\{ \frac{B(L, K, \alpha)}{p^{2\alpha}} + \frac{\sigma^4}{p^2} + \frac{V_2(K, \sigma, C_2, \beta)}{n} \right\},$$

where $B(L, K, \alpha)$ depends on $L$, $\|K\|$ and $\alpha$ and $V_1(K, \sigma, C_1)$ (resp. $V_2(K, \sigma, C_2, \beta)$) depends on $\|K\|_\infty$, $\sigma$ and $C_1$ (resp. $\|K\|_\infty$, $\sigma$, $C_2$ and $\beta$) (the constants $B(L, K, \alpha)$, $V_1(K, \sigma, C_1)$ and $V_2(K, \sigma, C_2, \beta)$ are deterministic).

Since $\alpha \leq 1$, the term $\sigma^4/p^2$ is not larger than the first term $B(L, K, \alpha)/p^{2\alpha}$ (up to a constant).

In particular, under Assumption \[1\]

$$\sup_{P_c \in \mathcal{R}_\alpha(L)} E \left( \|\hat{\eta}_{0,d} - \eta^*_d, d\|^2 \right) \leq C \left( p^{-2\alpha} + n^{-1} \right),$$

for $C$ a constant. This upper bound and the lower bound of Theorem \[1\] match, meaning that our estimation procedure is optimal in our setting. Observe that Assumption \[1\] is very mild. If we replace it with the stronger Assumption \[2\] we obtain a control in probability, coming from exponential bounds on probabilities of large deviations (for the Frobenius norm) for specific matrices. The price to pay is a logarithmic term in the variance term.

As expected, parameters $p$ and $n$ have a strong influence on rates. In our framework with two asymptotics very different in nature, we note that if $p$ is large enough (depending on $n$ and $\alpha$), then our procedure achieves the parametric rate $n^{-1}$ already obtained by Dauxois et al. [1982] and  [Mas and Ruymgaart 2015] when the curves $(Z_1, \ldots, Z_n)$ are fully observed and without noise. It means that discretization has no influence on theoretical performances. Conversely, if $p$ is not very large with respect to $n$, discretization has a deep impact and rates depend strongly on the underlying smoothness of the curves observed in a noisy setting. The obtained rate $p^{-2\alpha} + n^{-1}$ describes very precisely the competition between the number of discretization points and the number of observations in functional principal component analysis. To the best of our knowledge, these rates are new.

Finally, let us emphasize the simplicity of our optimal estimation procedure. It is based on the most classical ideas: projection by using piecewise constant bases and empirical mean estimation. In particular, regularization is not necessary and the knowledge of $\alpha$ is not required. The use of such standard tools may be surprising in view of results obtained by Johnstone [2001] and Baik and Silverstein [2006]. But, as already mentioned, functional principal component analysis is a very specific setting. The rates we obtain are very similar to those of Descary and Panaretos [2019] for which strong assumptions on the covariance operator (finite rank and analyticity of the eigenvalues $\eta_d^\alpha$) are required and the noise may exhibit local correlations. With the same assumption on noise as ours, Hall et al. [2006] obtained an $L^2$ convergence rate $n^{-(2r+1)/(2r+1)}$ for kernel estimators when $\eta_d^\alpha$ has a $r$-th order derivative and the number of observations per curve is bounded by a constant. Bunea and Xiao [2015] obtained a rate of convergence that is difficult to compare with ours, because their assumptions, concerning the rate of decay of the eigenvalues, differ significantly from our regularity assumption on the process. In the case of Brownian motion the rate of convergence of the $L^2$- risk of reconstruction of the estimator of Bunea and Xiao [2015] is of order $\log^2(n)n^{-1} + p^{-1/3}$ which is suboptimal compared to the minimax rate of $n^{-1/2} + p^{-1}$ that we have proven.

## 4 Simulation results

We assess the statistical performance of functional principal components estimators with simulations. We consider two eigenfunctions such that $\eta_1^\alpha(\cdot) = \sqrt{2} \sin(2\pi \cdot)$ and $\eta_2^\alpha(\cdot) = \sqrt{2} \cos(2\pi \cdot)$,
with eigenvalues $\mu_1 = 1.1$ and $\mu_2 = 0.1$. Simulated functional data are sampled on regularly spaced discretization points $t_h = h/(p - 1)$ with $h = 0, \ldots, p - 1$, and we compute the covariance matrix $\Sigma$ such that:

$$
\Sigma_{h,h'} = \sum_{d=1}^{2} \mu_d^2 \eta_d^2(t_h) \eta_d^2(t_{h'}) + \sigma^2 \mathbb{1}_{h = h'}
$$

from which we sample $n$ random functions $Y_1, \ldots, Y_n \sim \mathcal{N}(0, \Sigma)$, following Model (2). Then we consider different values for the number of observations $n \in \{256, 512, 1024, 2048, 4096\}$ to study the asymptotic performance of our estimators, and we will also consider different values of $p \in \{16, 32, 64, 128, 256\}$ to study the impact of discretization. The noise level $\sigma$ is chosen to match a given signal to noise ratio defined by $\sigma^{-2} \sum_{d=1}^{2} \mu_d^2$ (the variance of the signal divided by the variance of the noise), that takes value in $\{0.25, 1, 4\}$. We consider two smoothing systems, histograms and the Haar wavelets, as detailed in the Appendix. We report average the performance on $nb_{test} = 25$ independent simulations. Even if our theoretical results do not include regularized estimators, we also consider a hard thresholded version of these estimators to improve reconstruction (as detailed in the Appendix).

### 4.1 Reconstruction Errors

To assess the empirical performance of our approach we study the behavior of the mean reconstruction error

$$
E \left( \| \eta_{\Sigma,d} - \hat{\eta}_{\phi,d} \|^2 \right)
$$

according to the number of observations $n$, the size of the discretization grid $p$, and the signal to noise ratio. More precisely, we introduce a second finer grid $s_h = h/p'$, with $h = 0, \ldots, p' - 1$, such that $p' \gg p$ ($p' = 8192$ in practice) and use the approximation

$$
E \left( \| \eta_{\Sigma,d}^{*} - \hat{\eta}_{\phi,d}^{*} \|^2 \right) \approx \frac{1}{nb_{test}} \sum_{j=1}^{nb_{test}} \frac{1}{p'} \sum_{h=1}^{p'} \left\{ \eta_{\Sigma,d}^{*}(s_h) - \hat{\eta}_{\phi,d}^{*}(s_h) \right\}^2.
$$

To deduce the values of our estimator outside of the original grid, we use the piecewise constant property of the Haar and the histogram systems. In the following we also compute the estimation error on eigenvalues and assess $E \left( (\mu_{d}^{*} - \hat{\mu}_{\phi,d})^2 \right)$.

### 4.2 Results

The empirical error rates on eigenfunctions match the theoretical ones (Fig. 1), with orders $(\mu_1^2 - \mu_2^2)^{-2}(n^{-1} + p^{-2\alpha})$ ($\alpha = 1$ in our case) for the first eigenfunction estimator and $(\mu_2^2 - \mu_1^2)^{-2}(n^{-1} + p^{-2\alpha})$ for the second (Fig. 4 and 5). $\mu_3^2 = 0$ in our setting. Computed errors exhibit a double asymptotic behavior in $n$ and $p$. The rates in $n$ are slower than those in $p$, and exactly match $n^{-1}$ and $p^{-2}$. Also, the difference in terms of mean square errors between the first and the second eigenfunctions is due to the gap, since $\mu_1 - \mu_2 = 10(\mu_2^2 - \mu_3^2)$, which means that the estimation of the second eigenfunction is 10 times harder in terms of speed of convergence. The estimation error on eigenvalues also matches the theoretical upper bound (Fig. 2).

We also show that regularization does not necessarily improve the rate of convergence of eigen-elements (Fig. 6 and 7). We showed that projection-based functional principal component should attain minimax rates without any regularization. Consequently, at best, regularization should induce a better variance but comparable rates. Since one could operate only on the observed part of the function, at best, one could improve results on the grid without going beyond $n^{-1}$, which is already attained by the non-smoothed estimator.
Figure 1: Mean square error for the first eigenfunction $\eta^*_1$ according to the number of discretization points $p$ (left), and the number of samples $n$ (right). Left: the number of samples is $n \in \{256, 1024, 4096\}$ (light gray, gray, black respectively). Right: the number of discretization points is $p \in \{16, 32, 256\}$ (light gray, gray, black respectively). The signal to noise ratio is 0.25.

Figure 2: Mean square error for the first eigenvalue $\mu^*_1$ according to the number of discretization points $p$ (left), and the number of samples $n$ (right). Left: the number of samples is $n \in \{256, 1024, 4096\}$ (light gray, gray, black respectively). Right: the number of discretization points is $p \in \{16, 32, 256\}$ (light gray, gray, black respectively). The signal to noise ratio is 0.25.
5 Application

Genomics offers an original application of functional principal component analysis, to reduce the
dimensionality of data that are structured in one dimension along the genome. As an illustration,
we consider the fine mapping of replication origins in the human genome, that constitute the
starting points of chromosomes duplication. Replication origins are under a very strong spatio-
temporal control, and are part of the integrity maintenance of genomes. The investigation of
their spatial organization has become central to better understand genomes architecture and reg-
ulation, which remains challenging due to a complex interplay between genetic and epigenetic
regulations. Part of the genetic component of their regulation involve particular sequence motifs,
called G-quadruplexes, that have the property to form complex four-stranded structures whose
role in replication remains unsolved. A crucial aspect to better understand their implication is
to determine if these sequence motifs have a preferential positioning upstream replication ori-
gins. To investigate this matter, we considered the ∼130,000 replication origins of the human
genome [Picard et al., 2014], and we defined by \( Y_i(t) \) the process that equals 1 if there is a
G-quadruplex at position \( t \) in replication origin \( i \), taking motifs coordinates from [Zheng et al.]
[2020]. By convention, \( t = 0 \) corresponds to the peak of replication, and we consider positions
500 bases upstream this peak (in negative coordinates). The continuous aspect of the model is
not mandatory since positions along the genome are discrete. However, the functional setting
allows us to consider the spatial dependencies between the occurrences of these motifs, which
is very informative. Given the discrete nature of the data, we smoothed the data using the his-
togram system, with bin size of 25 bases (corresponding to the average size of G-quadruplexes).
Then we performed functional principal component analysis, and we used functional principal
components to perform a downstream clustering. We projected every observed curve on prin-
cipal components to obtain a new representation of the functional data based on general terms
\( \langle \tilde{Y}_i, \hat{\eta}_d \rangle \), and performed a \( k \)-means clustering to regroup replication origins that share the same
spatial distribution of these G-quadruplex motifs. We considered 6 principal components along
with 6 clusters, and we considered the spatial distribution of G-quadruplexes within clusters as a
result (Figure 3). Functional principal component analysis appears to catch the spatial structure
that makes the clusters, as different clusters of replication origins are characterized by specific
patterns of G-quadruplexes accumulation upstream the replication peak. Interestingly, the ob-
served periodicity can be related to a biophysical property of chromatin fibers. Indeed, the DNA
molecule is in the form of chromatin fibers in the nucleus, wrapped around the so-called nucle-
osomes with a periodicity of 144 base pairs. The formation of stable G-quadruplexes has been
shown to take place in nucleosome-free regions [Prorok et al., 2019], hence, the periodicity of
their accumulation upstream replication origins indicates that their positioning is directly linked
to the epigenetic context of replication initiation. These new biological results are currently
under further investigation.

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Figure 3: Density of G-quadruplexes accumulation in human replication origins clusters, determined by functional principal component analysis combined with $k$-means clustering. Each color correspond to a particular cluster.

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A Simulation study

We consider two smoothing systems, the histogram system, and the Haar wavelet system. In the case of histograms, we denote by \( D \) the number of bins (such that \( D \) divides \( p \) in practice), then

\[
\Lambda_D = \{ 0, \ldots, D - 1 \}
\]

and

\[
\phi_\lambda(t) = D^{1/2} \times 1_{\{\lambda/D, (\lambda+1)/D\}}(t), \quad t \in [0,1], \lambda \in \Lambda_D.
\]

Then in the case of the Haar system, we consider

\[
\varphi_{0,0}(t) = 1_{[0,1]}(t),
\]

\[
\psi_{j,k}(x) = 2^{j/2} \times 1_{[\frac{2k-2}{2^{j+1}}, \frac{2k-1}{2^{j+1}}]}(t) - 2^{j/2} \times 1_{[\frac{2k-2}{2^{j+1}}, \frac{2k-1}{2^{j+1}}]}(t), \quad t \in [0,1].
\]

We introduce a cross-validation procedure to regularize the eigenfunctions estimators. For each fold \( r \in \{1, \ldots, n_{\text{folds}}\} \) we split the observations \( Y = (Y_1, \ldots, Y_n) \) into two training and test sets \( Y_{\text{train}}^r, Y_{\text{test}}^r \) such that \( \lvert \text{train}_r \cup \text{test}_r \rvert = n \). Then we introduce \( \zeta \), a thresholding parameter, and we set

\[
\hat{Y}_{i,\zeta}(t) = \tilde{y}_{i,0,0}(t)\varphi_{0,0}(t) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \tilde{y}_{i,j,k} 1_{|\tilde{y}_{i,j,k}|>\zeta} \psi_{j,k}(t), \quad i = 1, \ldots, n, \quad t \in [0,1],
\]

with

\[
\tilde{y}_{i,0,0} = \frac{1}{p} \sum_{h=0}^{p-1} Y_i(t_h) \varphi_{0,0}(t_h),
\]

\[
\tilde{y}_{i,j,k} = \frac{1}{p} \sum_{h=0}^{p-1} Y_i(t_h) \psi_{j,k}(t_h).
\]

Then we compute the \( \tilde{f}_d^{\zeta} \)'s on \( Y_{\text{train}}^r \) for each fold such that

\[
(\tilde{f}_d^{\zeta})_d \in \arg \min_{(f_d,f_d')} \sum_{i \in \text{train}_r} \| \hat{Y}_{i,\zeta} - \sum_{d=1}^{2} (\tilde{Y}_{i,\zeta}, f_d) f_d \|_2^2.
\]

We select \( \hat{\zeta} \), the minimizer of the cross validated errors:

\[
\hat{\zeta} = \min_{\zeta} \frac{1}{n_{\text{folds}}} \sum_{r=1}^{n_{\text{folds}}} \sum_{i \in \text{test}_r} \| Y_i - \sum_{d=1}^{2} (Y_i, \tilde{f}_d^{\zeta}) \tilde{f}_d^{\zeta} \|_2^2.
\]
Figure 4: Mean square error for the second eigenfunction \( \eta_2^* \) according to the number of discretization points \( p \) (left), and the number of samples \( n \) (right). Left: the number of samples is \( n \in \{256, 1024, 4096\} \) (light gray, gray, black respectively). Right: the number of discretization points is \( p \in \{16, 32, 256\} \) (light gray, gray, black respectively). The signal to noise ratio is 0.25.

Once \( \widehat{\zeta} \) is chosen we compute the final estimator \( \widehat{\eta}_d, \widehat{\zeta} \) as :

\[
(\widehat{\eta}_d, \widehat{\zeta}) \in \arg \min_{\eta_d \in \mathcal{H}} \sum_{i=1}^n \| \widehat{Y}_i - \sum_{d=1}^2 \langle \widehat{Y}_i, f_d \rangle f_d \|_2^2.
\]

We use the same score function to select the number of bins for histograms.

B Proof of Theorem 1

To establish Theorem 1 we prove following Propositions 2 and 3.

**Proposition 2** Under Assumptions of Theorem 1, there exists \( n_0 \) depending only on \( L \) and \( \alpha \) such that, for all \( n \geq n_0 \),

\[
\inf_{\eta_1} \sup_{P \in \mathcal{R}_\alpha(L)} \mathbb{E}[\|\widehat{\eta}_1 - \eta_1^*\|_2^2] \geq c_1 n^{-1},
\]

where \( c_1 > 0 \) is a constant depending on \( \sigma \).

**Proposition 3** There exists a universal constant \( c_2 > 0 \) such that

\[
\inf_{\eta_1} \sup_{P \in \mathcal{R}_\alpha(L)} \mathbb{E}[\|\widehat{\eta}_1 - \eta_1^*\|_2^2] \geq c_2 p^{-2\alpha}.
\]

The result of Theorem 1 is deduced from Propositions 2 and 3 by taking

\[
c(\sigma) = \frac{1}{2} \min(c_1; c_2) > 0.
\]

**B.1 Proof of Proposition 2**

The proof of Proposition 2 follows the general scheme described in Tsybakov [2009] Section 2.4.2]. Let

\[
\phi(t) = e^{-\frac{t}{1-\sigma^2}} 1_{(-1,1)}(t).
\]
Figure 5: Mean square error for the second eigenvalue $\mu_2^*$ according to the number of discretization points $p$ (left), and the number of samples $n$ (right). Left: the number of samples is $n \in \{256, 1024, 4096\}$ (light gray, gray, black respectively). Right: the number of discretization points is $p \in \{16, 32, 256\}$ (light gray, gray, black respectively). The signal to noise ratio is 0.25.

Figure 6: Mean square error for the first eigenfunction $\eta_1^*$ according to the number of discretization points $p$ and the smoothing system (Haar, left panel, histograms, right panel). The number of samples is $n \in \{256, 1024, 4096\}$ (light gray, gray, black respectively). Plain line: regularized estimator based on cross validation, dashed line: non regularized estimator. The signal to noise ratio is 0.25.
Figure 7: Mean square error for the first eigenfunction \( \eta_1^* \) according to the number samples \( n \) and the smoothing system (Haar, left panel, histograms, right panel). The number of discretization points is \( p \in \{16, 32, 256\} \) (light gray, gray, black respectively). Plain line: regularized estimator based on cross validation, dashed line: non regularized estimator. The signal to noise ratio is 0.25.

We then define
\[
\varphi(t) = \begin{cases} 
\phi(4t - 3) & \text{if } t \in [1/2, 1), \\
-\phi(4t - 1) & \text{if } t \in (0, 1/2), \\
0 & \text{if } t \not\in (0, 1).
\end{cases}
\]

Both functions \( \phi \) and \( \varphi \) are \( C^\infty \) on \( \mathbb{R} \) with bounded support, then are \( \alpha \)-Hölder continuous, for all \( \alpha > 0 \). Moreover \( \int_0^1 \varphi(t)dt = 0 \). We note \( L_{\alpha} > 0 \) such that, for all \( t, u \in \mathbb{R} \),
\[
|\varphi(t) - \varphi(u)| \leq L_{\alpha}|t - u|^\alpha.
\]

Let us now define two test eigenfunctions
\[
\eta_{1,0}^*(t) = 1_{[0,1]}(t),
\]
and, with
\[
\varphi_{a,s}(t) = a\varphi(st),
\]
for \( a > 0 \) and \( s \geq 1 \) specified later,
\[
\eta_{1,1}^*(t) = C \left( \eta_{1,0}^*(t) + \frac{1}{\sqrt{n}}\varphi_{a,s}(t) \right), \quad t \in [0, 1],
\]
with \( C \) such that \( ||\eta_{1,1}^*|| = 1 \). We first calculate \( C \):
\[
||\eta_{1,1}^*||^2 = C^2 \left( ||\eta_{1,0}^*||^2 + \frac{2}{\sqrt{n}} \int_0^1 \varphi_{a,s}(t)dt + \frac{1}{n}||\varphi_{a,s}||^2 \right) = C^2 \left( 1 + \frac{1}{n}||\varphi_{a,s}||^2 \right).
\]

Now, since \( s \geq 1 \),
\[
||\varphi_{a,s}||^2 = a^2 \int_0^1 \varphi^2(st)dt = \frac{a^2}{s} \int_0^s \varphi^2(t)dt = \frac{a^2}{s} \int_0^1 \varphi^2(t)dt = \frac{a^2}{s} ||\varphi||^2.
\]

Then, we set
\[
C := \left( 1 + \frac{a^2}{sn}||\varphi||^2 \right)^{-1/2} < 1. \tag{6}
\]
Now, for $\xi \sim N(0, 1)$ and $\mu^*_1 > 0$, we introduce

$$Z^j(t) = \sqrt{\mu^*_1} \xi \eta^*_j(t), \quad j = 0, 1$$

and we consider Model 2 such that $Z^1_0, \ldots, Z^0_n$ (resp. $Z^1_0, \ldots, Z^1_n$) are i.i.d copies of $Z^0$ (resp. $Z^1$). Let, for $j = 0, 1$, $P^j_z$ the distribution of $Z^j$. We have for any $(t, u) \in [0, 1]^2$,

$$\int_{c_0} (z(t) - z(u))^2 dP^j_z(z) = \mathbb{E}[(Z^j(t) - Z^j(u))^2]$$

$$= \mu^*_1 \mathbb{E}[\xi^2] (\eta^*_j(t) - \eta^*_j(u))^2.$$  

We have easily that $P^2_0 \in \mathcal{R}_\alpha(L)$ since $\eta^*_1,0$ is constant on $[0, 1]$, implying

$$\int_{c_0} (z(t) - z(u))^2 dP^2_0(z) = 0.$$

We have

$$\int_{c_0} (z(t) - z(u))^2 dP^2_z(z) = \mu^*_1 \mathbb{E}[\xi^2] (\eta^*_1,1(t) - \eta^*_1,1(u))^2$$

$$= \frac{C^2 \mu^*_1}{n} (\varphi_{a,s}(t) - \varphi_{a,s}(u))^2 = \frac{C^2 a^2 \mu^*_1}{n} (\varphi(st) - \varphi(su))^2$$

$$\leq \frac{C^2 L^2 a^2 \mu^*_1 s^{2\alpha}}{n} |t - u|^{2\alpha},$$

and, since $C \leq 1$, $P^2_1 \in \mathcal{R}_\alpha(L)$ if

$$\frac{L^2 a^2 \mu^*_1 s^{2\alpha}}{n} \leq L.$$  

This allows to deduce that

$$\inf_{\hat{\eta}_1} \sup_{P \in \mathcal{R}_\alpha(L)} \mathbb{E}[\|\hat{\eta}_1 - \eta^*_1\|^2] \geq \inf_{\hat{\eta}_1} \sup_{j = 0, 1} \mathbb{E}[\|\hat{\eta}_1 - \eta^*_1, j\|^2],$$

and the aim of what follows is to prove a lower bound for $\mathbb{E}[\|\hat{\eta}_1 - \eta^*_1, j\|^2]$.

Let $\hat{\eta}_1$ an estimator and $\psi$ the minimum distance test defined by

$$\hat{\psi} = \arg\min_{j = 0, 1} \|\hat{\eta}_1 - \eta^*_1, j\|^2,$$

we have for $j = 0, 1$,

$$\|\hat{\eta}_1 - \eta^*_1, j\| \geq \frac{1}{2} \|\eta^*_1, \hat{\psi} - \eta^*_1, j\|.$$

Now, since $\int_0^T \eta_0(t) \varphi_{a,s}(t) dt = 0$, if

$$\frac{a^2}{n} \leq 1,$$

we have $C \geq (1 + \|\varphi\|^2)^{-1/2}$, and

$$\|\eta^*_1, \hat{\psi} - \eta^*_1, j\|^2 = 1_{(\hat{\psi} \neq j)} \|\eta^*_1,0 - \eta^*_1, 1\|^2 = 1_{(\hat{\psi} \neq j)} \|1 - C\| \eta^*_1,0 - \frac{C}{\sqrt{n}} \varphi_{a,s}\|^2$$

$$= 1_{(\hat{\psi} \neq j)} \left( (1 - C)^2 + \frac{C^2}{n} \|\varphi_{a,s}\|\right)$$

$$\geq 1_{(\hat{\psi} \neq j)} \frac{C^2 a^2}{sn} \|\varphi\|^2 \geq 1_{(\hat{\psi} \neq j)} \frac{a^2}{sn} \|\varphi\|^2.$$

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Then,
\[
\inf_{\eta_1} \sup_{p \in \mathbb{R}^n(L)} \mathbb{E}[(\|\eta_1 - \eta_1^*\|^2)] \geq \frac{\|\varphi\|^2}{4(\|\varphi\|^2 + 1)} \frac{a^2}{sn} \times \inf_{\psi} \max_{j=0,1} \mathbb{P}(\hat{\psi} \neq j). \tag{9}
\]
We now prove that the quantity \(\inf_{\psi} \max_{j=0,1} \mathbb{P}(\hat{\psi} \neq j)\) can be bounded from below by an absolute positive constant. For this purpose, we control the Hellinger distance between the data generated by the two models. More precisely, we have to prove that for some constant \(H_{\text{max}}^2 < 2\), we have
\[
\mathcal{H}^2((\ell_0^{\text{obs}})^{\otimes n}, (P_1^{\text{obs}})^{\otimes n}) \leq H_{\text{max}}^2
\]
where \(P_j^{\text{obs}}\) is the law of the random vector \(Y_j^{\text{obs}} := (Y_j(t_0), \ldots, Y_j(t_{p-1}))\) such that
\[
Y_j(t_k) = Z^j(t_k) + \varepsilon^j_k
\]
with \(\varepsilon^0_{p-1}, \ldots, \varepsilon^0_1, \varepsilon^1_{p-1}, \ldots, \varepsilon^1_{p-1} \sim \text{i.i.d.} \mathcal{N}(0, \sigma^2)\). Indeed, in this case, Theorem 2.2 of Tsybakov [2009] shows that
\[
\inf_{\psi} \max_{j=0,1} \mathbb{P}(\hat{\psi} \neq j) \geq \frac{1}{2} \left(1 - \sqrt{H_{\text{max}}^2(1 - H_{\text{max}}^2/4)} \right) > 0
\]
and Inequality (9) provides the desired lower bound. First remark that
\[
\mathbf{Y}_{j, \text{obs}} \sim \mathcal{N}(0, \mathbf{G}_j),
\]
where \(G_j = ([G_j]_{\ell,k})_{0 \leq \ell, k \leq p-1}\)
\[
[G_j]_{\ell,k} = \mathbb{E}[Y_j(t_{\ell}) Y_j(t_k)] = \mu_1 \eta_{1,j}(t_{\ell}) \eta_{1,j}(t_k) + \sigma^2 \mathbf{1}_{(\ell=k)}.
\]
We have, for \(j = 0\),
\[
\mathbf{G}_0 = \mu_1 \mathbf{1}_{p \times p} + \sigma^2 \mathbf{I}_p,
\]
where \(\mathbf{1}_{p \times p}\) is the \(p \times p\) matrix whose coefficients are all equal to 1 and, for \(j = 1\),
\[
[G_1]_{\ell,k} = \mu_1 C^2 \left(1 + \frac{1}{\sqrt{n}} \varphi_{a,s}(t_{\ell})\right) \left(1 + \frac{1}{\sqrt{n}} \varphi_{a,s}(t_k)\right) + \sigma^2 \mathbf{1}_{(\ell=k)},
\]
hence
\[
G_1 = \mu_1 C^2 \left(1 + \frac{1}{\sqrt{n}} \right) \mathbf{1}_{p \times p} + \frac{1}{n} \mathbf{1}_{p \times p} \varphi_{a,s}^2 \mathbf{1}_p + \mathbf{1}_{n \times p} \varphi_{a,s} \varphi_{a,s}^T + \sigma^2 \mathbf{I}_p,
\]
where \(\mathbf{1}_p = (1, \ldots, 1) \in \mathbb{R}^p\), \(\varphi_{a,s} = (\varphi_{a,s}(t_0), \ldots, \varphi_{a,s}(t_{p-1}))\). Considering \(A(P_0^{\text{obs}}, P_1^{\text{obs}})\) the Hellinger affinity of \((P_0^{\text{obs}}, P_1^{\text{obs}})\), we get
\[
\mathcal{H}^2((P_0^{\text{obs}})^{\otimes n}, (P_1^{\text{obs}})^{\otimes n}) = 2 - 2A(P_0^{\text{obs}}, P_1^{\text{obs}})^n.
\]
In our case where the variables are Gaussian with equal mean vectors, the Hellinger affinity writes (see e.g. Pardo [2006] pp. 45, 46 and 51),
\[
A(P_0^{\text{obs}}, P_1^{\text{obs}}) = \frac{\det((G_0 G_1)^{1/4})}{\det((G_0^1 + G_1)/2)^{1/2}}. \tag{10}
\]
Matrices \(G_0\) and \(G_1\) can be analyzed in terms of eigenvalues and eigenfunctions. Indeed, assuming that \(p \geq 3\), we take \(s \geq 1\) such that
\[
s = \begin{cases} 
 1 & \text{if } (p-1)/2 \\
 p-1 &= \text{is an integer} \\
 p/2 & \text{if } p/2 \text{ is an integer} \quad \text{and} \quad q := \frac{p-1}{2s} 
\end{cases}
\]
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is an integer such that \( q \leq (p - 1)/2 \leq p - 1 \). In this case,

\[
1_p^t \varphi_{a,s} = \varphi_{a,s} 1_p = \sum_{k=0}^{p-1} \varphi_{a,s}(t_k) = a \sum_{k=0}^{p-1} \varphi \left( \frac{sk}{p-1} \right) = a \left( \sum_{k=0}^{p-1} \varphi \left( \frac{4sk}{p-1} - 3 \right) 1_{\{k \in [1/2, 1]\}} - \sum_{k=0}^{p-1} \varphi \left( \frac{4sk}{p-1} - 1 \right) 1_{\{k \in (0, (p-1)/2s)\}} \right) = a \left( \sum_{k=0}^{p-1} \varphi \left( \frac{4sk}{p-1} - 3 \right) 1_{\{k \in [0, (p-1)/2s)\}} - \sum_{k=0}^{p-1} \varphi \left( \frac{4sk}{p-1} - 1 \right) 1_{\{k \in (0, (p-1)/2s)\}} \right) \]

replacing the variable \( k \) in the first sum by \( \ell = k - q \) and observing that \( q \leq p - 1 - q \). We also have

\[
1_{p \times p}^2 = p 1_{p \times p}, \quad 1_{p \times p} 1_p^t = 1_p 1_p^t.
\]

We set

\[
v_1 := \frac{1}{\sqrt{p}} 1_p, \quad v_2 := \| \varphi_{a,s} \|_{\ell_2}^{-1} \varphi_{a,s} = a^{-1} \left( \sum_{k=0}^{p-1} \varphi^2(s_k) \right)^{-1/2} \varphi_{a,s},
\]

so that \( \| v_1 \|_{\ell_2} = \| v_2 \|_{\ell_2} = 1 \) and \( v_3, \ldots, v_p \) an orthonormal basis of span\( \{ v_1, v_2 \}^\perp \), and \( V \) the orthogonal matrix

\[
V := [v_1; v_2; \cdots; v_p].
\]

We have

\[
G_0 v_1 = (p \mu^*_1 + \sigma^2) v_1, \quad G_0 v_2 = \sigma^2 v_2
\]

\[
G_1 v_1 = (p \mu^*_1 C^2 + \sigma^2) v_1 + \mu^*_1 C^2 \sqrt{\frac{p}{n}} \| \varphi_{a,s} \|_{\ell_2} v_2,
\]

\[
G_1 v_2 = \left( \frac{\mu^*_1 C^2 \| \varphi_{a,s} \|_{\ell_2}^2 + \sigma^2}{n} \right) v_2 + \mu^*_2 C^2 \sqrt{\frac{p}{n}} \| \varphi_{a,s} \|_{\ell_2} v_1
\]

and

\[
G_0 = V \begin{pmatrix}
p \mu^*_1 + \sigma^2 & 0 & \cdots & 0 \\
0 & \sigma^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma^2
\end{pmatrix} V^T,
\]

\[
G_1 = V \begin{pmatrix}
p \mu^*_1 C^2 + \sigma^2 & \mu^*_1 C^2 \sqrt{\frac{p}{n}} \| \varphi_{a,s} \|_{\ell_2} & 0 & \cdots & 0 \\
0 & \mu^*_1 C^2 \sqrt{\frac{p}{n}} \| \varphi_{a,s} \|_{\ell_2} & \sigma^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \sigma^2
\end{pmatrix} V^T.
\]
In particular, we have:

\[
\frac{G_0 + G_1}{2} = V \begin{pmatrix}
\frac{\mu_1^* C^2}{2} \sqrt{\frac{n}{\pi}} \|\varphi_{a,s}\|_{\ell_2} & \frac{\mu_1^* C^2}{2n} \sqrt{\frac{\pi}{n}} \|\varphi_{a,s}\|_{\ell_2} & 0 & \cdots & 0 \\
\frac{\mu_1^* C^2}{2} \sqrt{\frac{n}{\pi}} \|\varphi_{a,s}\|_{\ell_2} & \frac{\mu_1^* C^2}{2n} \sqrt{\frac{\pi}{n}} \|\varphi_{a,s}\|_{\ell_2} + \sigma^2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \sigma^2 \\
\end{pmatrix} V^T.
\]

We obtain

\[
\det(G_0) = (p\mu_1^* + \sigma^2)\sigma^{2(p-1)} = \sigma^{2p}(1 + \sigma^{-2}p\mu_1^*),
\]

\[
\det(G_1) = \sigma^{2(p-2)} \left( (p\mu_1^* C^2 + \sigma^2) \left( \frac{\mu_1^* C^2}{n} \|\varphi_{a,s}\|_{\ell_2}^2 + \sigma^2 \right) - \frac{\mu_1^* C^2}{n} \|\varphi_{a,s}\|_{\ell_2}^2 \right)
\]

\[
\quad = \sigma^{2(p-2)} \left( \sigma^2 + p\mu_1^* C^2 \sigma^2 + \frac{\mu_1^* C^2 \sigma^2}{n} \|\varphi_{a,s}\|_{\ell_2}^2 \right)
\]

\[
\quad = \sigma^{2p} \left( 1 + p\mu_1^* C^2 \sigma^2 + \frac{\mu_1^* C^2 \sigma^2}{n} \|\varphi_{a,s}\|_{\ell_2}^2 \right)
\]

and

\[
\det((G_0 + G_1)/2) = \sigma^{2(p-2)} \left( \frac{p \mu_1^*}{2} (C^2 + 1) + \sigma^2 \right) \left( \frac{\mu_1^* C^2}{2n} \|\varphi_{a,s}\|_{\ell_2}^2 + \sigma^2 \right) - \frac{p \mu_1^* C^2}{2n} \|\varphi_{a,s}\|_{\ell_2}^2
\]

\[
\quad = \sigma^{2(p-2)} \left( \sigma^2 + \frac{p \mu_1^*}{2} \sigma^2 + \left( \sigma^2 + \frac{p \mu_1^*}{2} \right) \frac{\mu_1^* C^2}{2n} \|\varphi_{a,s}\|_{\ell_2}^2 + \frac{p \mu_1^* C^2 \sigma^2}{2} \right)
\]

\[
\quad = \sigma^{2p} \left( 1 + \frac{p \mu_1^* \sigma^{-2}}{2} \right) + \left( \sigma^{-2} + \frac{p \mu_1^*}{2} \sigma^{-4} \right) \frac{\mu_1^* C^2}{2n} \|\varphi_{a,s}\|_{\ell_2}^2 + \frac{p \mu_1^* C^2 \sigma^{-2}}{2}
\]

Now, we take

\[
\mu_1^* = \frac{1}{p}, \quad a = \frac{\sqrt{a}}{\|\varphi\|}
\]

and we remark that since \( p \geq 3 \), we have \( s \leq 1 \) so that \( 7 \) and \( 8 \) are satisfied as soon as

\[
n \geq \frac{8L_d^2}{3\|\varphi\|L} \geq \frac{L_d^2 2^{1+2\alpha}}{p\|\varphi\|L} \quad \text{and} \quad n \geq \|\varphi\|^{-2}.
\]

Moreover we observe that

\[
\frac{1}{p} \|\varphi_{a,s}\|_{\ell_2}^2 = \frac{1}{p} \sum_{k=0}^{p-1} a^2 \varphi^2(st_k) = \frac{s}{p\|\varphi\|} \sum_{k=0}^{p-1} \varphi^2 \left( \frac{sk}{p-1} \right) \to 1,
\]

when \( p \to +\infty \), so

\[
u_p : = \frac{1}{p} \|\varphi_{a,s}\|_{\ell_2}^2
\]

is bounded from below and above uniformly in \( p \) (and \( n \)). Furthermore,

\[
\det(G_0) = \sigma^{2p}(1 + \sigma^{-2})
\]

and using \( 6 \), we have

\[
C^2 = \left( 1 + \frac{1}{n} \right)^{-1} = 1 - \frac{1}{n} + O\left( \frac{1}{n^2} \right),
\]
which implies
\[
\det(G_1) = \sigma^{2p} \left( 1 + C^2 \sigma^{-2} + \frac{C^2 \sigma^{-2}}{n} u_p \right)
\]
\[
= \sigma^{2p} \left( 1 + \sigma^{-2} + \frac{\sigma^{-2}}{n} (u_p - 1) + O \left( \frac{1}{n^2} \right) \right)
\]
and
\[
\det((G_0 + G_1)/2) = \sigma^{2p} \left( 1 + \frac{\sigma^{-2}}{2} + \left( \sigma^{-2} + \frac{\sigma^{-2}}{2} \right) C^2 \frac{u_p}{2n} + \frac{C^2 \sigma^{-2}}{2} \right)
\]
\[
= \sigma^{2p} \left( 1 + \sigma^{-2} + \frac{\sigma^{-2}}{2n} (u_p - 1) + \frac{\sigma^{-4} u_p}{4n} + O \left( \frac{1}{n^2} \right) \right).
\]
Now let \( \epsilon > 0 \). For \( p \) large enough, \( |u_p - 1| \leq \epsilon \) and using [10],
\[
A(P_0^{obs}, P_1^{obs}) = \frac{\det(G_0 G_1)^{1/4}}{\det((G_0 + G_1)/2)^{1/2}} \geq \frac{\left( 1 - \sigma^{-2} (1 + \sigma^{-2})^{-1} \frac{n}{\sigma} + O \left( \frac{1}{n^2} \right) \right)^{1/4}}{\left( 1 + (1 + \sigma^{-2})^{-1} \frac{\sigma^{-2}}{2n} \frac{n}{\sigma} + \sigma^{-4} (1 + \sigma^{-2})^{-1} \frac{1 + n}{4n^2} + O \left( \frac{1}{n^2} \right) \right)^{1/2}}
\]
implying
\[
A(P_0^{obs}, P_1^{obs})^n \geq \frac{\left( 1 - \sigma^{-2} (1 + \sigma^{-2})^{-1} \frac{n}{\sigma} + O \left( \frac{1}{n^2} \right) \right)^{n/4}}{\left( 1 + (1 + \sigma^{-2})^{-1} \frac{\sigma^{-2}}{2n} \frac{n}{\sigma} + \sigma^{-4} (1 + \sigma^{-2})^{-1} \frac{1 + n}{4n^2} + O \left( \frac{1}{n^2} \right) \right)^{n/2}}
\]
so
\[
\liminf_{n \to +\infty} A(P_0^{obs}, P_1^{obs})^n \geq \exp \left( -0.5 \sigma^{-2} (1 + \sigma^{-2})^{-1} \frac{n}{\sigma} - 0.125 \sigma^{-4} (1 + \sigma^{-2})^{-1} (1 + \sigma) \right)
\]
and the last quantity is positive for any \( \epsilon > 0 \). This implies that
\[
\limsup_{n \to +\infty} H^2((P_0^{obs})^\otimes n, (P_1^{obs})^\otimes n) < 2.
\]

### B.2 Proof of Proposition [3]

The proof is based on Assouad’s Lemma and follows the general scheme described in [Tsybakov 2009 Sections 2.6 and 2.7]. Let
\[
\phi(t) = e^{-\frac{|t|}{1 + |t|}} 1_{(-1,1)}(t).
\]
We then define
\[
\varphi(t) = \begin{cases} 
\phi(4t - 1) & \text{if } t \in [0, 1/2), \\
-\phi(4t + 1) & \text{if } t \in (-1/2, 0], \\
0 & \text{if } t \notin (-1/2, 1).
\end{cases}
\]
Both functions \( \phi \) and \( \varphi \) are \( C^\infty \) on \( \mathbb{R} \) with bounded support, then are \( \alpha \)-Hölder continuous, for all \( \alpha > 0 \). The function \( \varphi \) has its support included in \( (-1/2, 1/2) \) and verifies \( \int_{-1/2}^{1/2} \varphi(t) dt = 0 \). We note \( L_\alpha \) such that, for all \( t, u \in \mathbb{R} \),
\[
|\varphi(t) - \varphi(u)| \leq L_\alpha |t - u|^{\alpha}.
\]
Let us now define test eigenfunctions. For $\omega = (w_0, \ldots, w_{p-1}) \in \{0, 1\}^p$, we set

$$\eta^*_1, \omega(t) = C_\omega \left( \gamma + \sum_{k=0}^{p-1} \omega_k \left( p^{-\alpha} \varphi\left( p(t - t_k) - 1/2 \right) \right) \right),$$

with $C_\omega$ and $\gamma > 0$ two positive constants to be specified later. To be an eigenfunction, $\eta^*_1, \omega$ has to be of norm 1, which writes

$$\|\eta^*_1, \omega\|^2 = C^2_\omega \int_0^1 \left( \gamma + \sum_{k=0}^{p-1} \omega_k \left( p^{-\alpha} \varphi\left( p(t - t_k) - 1/2 \right) \right) \right)^2 dt$$

$$= C^2_\omega \left( \gamma^2 + 2\gamma \sum_{k=0}^{p-1} \omega_k \left( p^{-\alpha} \int_0^1 \varphi\left( p(t - t_k) - 1/2 \right) dt \right) \right.$$  

$$+ \int_0^1 \left( \sum_{k=0}^{p-1} \omega_k \left( p^{-\alpha} \varphi\left( p(t - t_k) - 1/2 \right) \right) \right)^2 dt \bigg).$$

Using successively that the support of $\varphi$ is in $(-1/2, 1/2)$ and that $\int_{-1/2}^{1/2} \varphi(t) dt = 0$, we have

$$\int_0^1 \varphi\left( p(t - t_k) - 1/2 \right) dt = \int_{t_k}^{t_{k+1}} \varphi\left( p(t - t_k) - 1/2 \right) dt = p^{-1} \int_{-1/2}^{1/2} \varphi(t) dt = 0,$$

and

$$\int_0^1 \left( \sum_{k=0}^{p-1} \omega_k \varphi\left( p(t - t_k) - 1/2 \right) \right)^2 dt = \sum_{k=0}^{p-1} \omega_k \int_0^1 \varphi^2\left( p(t - t_k) - 1/2 \right) dt = p^{-1} \sum_{k=0}^{p-1} \omega_k \|\varphi\|^2.$$

This implies that

$$\|\eta^*_1, \omega\|^2 = C^2_\omega \left( \gamma^2 + p^{-2\alpha - 1} \|\varphi\|^2 \sum_{k=0}^{p-1} \omega_k \right).$$

We then fix the quantity

$$C_\omega = \left( \gamma^2 + p^{-2\alpha - 1} \|\varphi\|^2 \sum_{k=0}^{p-1} \omega_k \right)^{-1/2},$$

so that $\|\eta^*_1, \omega\| = 1$ and observe that $C_\omega$ verifies

$$(\gamma^2 + \|\varphi\|^2)^{-1/2} \leq (\gamma^2 + p^{-2\alpha} \|\varphi\|^2)^{-1/2} \leq C_\omega \leq \gamma^{-1}.$$

We now define the associated distribution of our observations: for $\xi \sim \mathcal{N}(0, 1)$ and $\mu^*_1, \omega = \frac{L_1}{\sqrt{2p \|\varphi\|^2}}$, we set

$$Z_\omega(t) = \sqrt{\mu^*_1, \omega} \xi \eta^*_1, \omega(t).$$

Let $P^2_\omega$ be the distribution of $Z_\omega$. We have that $P^2_\omega \in \mathcal{R}_\omega(L)$ since

$$\int_{C((0,1))} (z(t) - z(s))^2 dP^2_\omega(z) = \mathbb{E}[(Z_\omega(t) - Z_\omega(s))^2] = \mu^*_1, \omega(\eta_1, \omega(t) - \eta_1, \omega(s))^2 \mathbb{E}[\xi^2]$$

$$= \mu^*_1, \omega(\eta_1, \omega(t) - \eta_1, \omega(s))^2$$

$$= \mu^*_1, \omega C^2_\omega \left( \sum_{k=0}^{p-1} \omega_k p^{-\alpha} \varphi\left( p(t - t_k) - 1/2 \right) - \varphi\left( p(s - t_k) - 1/2 \right) \right)^2.$$
Then, using the properties of $\varphi$, we have two cases:

- If $s, t \in [t_\ell, t_{\ell+1}]$ for some $\ell \in \{0, \ldots, p-1\}$,

$$
\left( \sum_{k=0}^p \omega_k p^{-\alpha} (\varphi(p(t-t_k) - 1/2) - \varphi(p(s-t_k) - 1/2)) \right)^2
\leq \omega^2 \sum_{k=0}^p \omega_k p^{-2\alpha} (\varphi(p(t-t_k) - 1/2) - \varphi(p(s-t_k) - 1/2))^2,
$$

- If $s \in [t_\ell, t_{\ell+1}]$ and $t \in [t_{\ell'}, t_{\ell'+1}]$ with $\ell \neq \ell'$,

$$
\left( \sum_{k=0}^p \omega_k p^{-\alpha} (\varphi(p(t-t_k) - 1/2) - \varphi(p(s-t_k) - 1/2)) \right)^2
\leq 2L^2_{\alpha} |t - s|^{2\alpha}.
$$

Finally,

$$
\int_{C([0,1])} (z(t) - z(s))^2 dP_\omega(z) \leq 2\mu^1_\omega C^2_{\alpha} L^2_{\alpha} |t - s|^{2\alpha} = L |t - s|^{2\alpha}.
$$

This allows to deduce that

$$
\inf_{\tilde{\eta}_1} \sup_{P_\omega \in R_\alpha(L)} \mathbb{E}[\|\tilde{\eta}_1 - \eta^*_1\|^2] \geq \inf_{\tilde{\eta}_1} \sup_{\omega \in \{0,1\}^p} \mathbb{E}[\|\tilde{\eta}_1 - \eta^*_1, \omega\|^2],
$$

and the aim of what follows is to prove a lower bound for $\mathbb{E}[\|\tilde{\eta}_1 - \eta^*_1, \omega\|^2]$.

Let $\tilde{\eta}_1$ an estimator and

$$
\tilde{\omega} \in \arg \min_{\omega \in \{0,1\}^p} \|\tilde{\eta}_1 - \eta^*_1, \omega\|^2,
$$

we have

$$
\|\tilde{\eta}_1 - \eta^*_1, \tilde{\omega}\| \geq \frac{1}{2} \|\tilde{\eta}_1 - \eta^*_1, \omega\|.
$$

Now, still from the support properties of $\varphi$,

$$
\|\tilde{\eta}^*_1, \omega - \eta^*_1, \omega\|^2
\geq \sum_{k=0}^{p-1} \int_{t_k}^{t_{k+1}} \left( C_\omega (\gamma + \tilde{\omega}_k p^{-\alpha} \varphi(p(t - t_k) - 1/2)) - C_\omega (\gamma + \omega_k p^{-\alpha} \varphi(p(t - t_k) - 1/2)) \right)^2 dt
\geq \sum_{k=0}^{p-1} \int_{t_k}^{t_{k+1}} \left( C_\omega (\gamma + \tilde{\omega}_k p^{-\alpha} \varphi(u)) - C_\omega (\gamma + \omega_k p^{-\alpha} \varphi(u)) \right)^2 du
\geq (C_\omega - C_\omega)^2 \gamma^2 + \|\varphi\|^2 p^{-2\alpha-1} \sum_{k=0}^{p-1} (C_\omega \tilde{\omega}_k - C_\omega \omega_k)^2
\geq \|\varphi\|^2 p^{-2\alpha-1} \sum_{k=0}^{p-1} (C_\omega \tilde{\omega}_k - C_\omega \omega_k)^2
\geq \|\varphi\|^2 p^{-2\alpha-1} \sum_{k=0}^{p-1} 1(\tilde{\omega}_k \neq \omega_k)
\geq (\gamma^2 + \|\varphi\|^2)^{-1} \|\varphi\|^2 p^{-2\alpha-1} \rho(\tilde{\omega}, \omega).
$$

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where \( \rho(\omega, \omega') = \sum_{k=0}^{p-1} \omega_k \neq \omega'_k \) is the Hamming distance on \( \{0, 1\}^p \).

Combining all the inequalities above, we have the existence of a constant \( \tilde{c} = \|\varphi\|^2 / (4(\gamma^2 + \|\varphi\|^2)) \) such that

\[
\inf \sup \mathbb{E}[\|\widehat{\eta}_1 - \eta_{\omega}^*\|^2] \geq \tilde{c} \rho_{\omega}^{2n-1} \max_{\omega \in \{0, 1\}^p} \mathbb{E}[\rho(\tilde{\omega}, \omega)].
\]

By Assouad’s lemma (see e.g. Tsybakov, 2009, Theorem 2.12), there exists a constant \( c > 0 \) such that

\[
\inf \max_{\omega \in \{0, 1\}^p} \mathbb{E}[\rho(\tilde{\omega}, \omega)] \geq c \rho_{\omega}^{2n},
\]

provided we are able to prove that for some constant \( K_{\max} \geq 0 \),

\[ KL((P_\omega^{\text{obs}})^{\otimes n}, (P_0^{\text{obs}})^{\otimes n}) \leq K_{\max}, \text{ for all } \omega \in \{0, 1\}^p, \]

where \( P_\omega^{\text{obs}} \) is the law of the random vector

\[ Y_\omega^{\text{obs}} := (Y_\omega(t_0), \ldots, Y_\omega(t_{p-1})) \]

such that

\[ Y_\omega(t_j) = Z_\omega(t_j) + \varepsilon_j \]

with \( \varepsilon_0, \ldots, \varepsilon_{p-1} \sim_{i.i.d.} \mathcal{N}(0, \sigma^2) \) and \( KL(P, Q) \) is the Kullback-Leibler divergence between two measures \( P \) and \( Q \). In (11), the constant \( c \) only depends on \( K_{\max} \). We observe that, for all \( \omega \in \{0, 1\}^p \), for all \( j = 0, \ldots, p-1 \),

\[ Y_\omega(t_j) = Z_\omega(t_j) + \varepsilon_j = \sqrt{\mu_1^+ \omega \xi_{\eta_j} \varepsilon_j + \varepsilon_j}.
\]

Now

\[ \eta_j^* \omega(t_j) = C_\omega \left( \gamma + \sum_{k=0}^{p-1} \omega_k \varphi(p(t_j - t_k) - 1/2) \right) = C_\omega \gamma,
\]

since \( \varphi((p(t_j - t_k) - 1/2) = \varphi(-1/2) = 0 \) if \( j = k \) and \( \varphi((p(t_j - t_k) - 1/2) = 0 \) if \( j \neq k \) by the support properties of \( \varphi \) and the fact that

\[ p(t_j - t_k) - 1/2 = \frac{P}{p-1} (j - k) - 1/2 \geq \frac{P}{p-1} - 1/2 \geq 1/2
\]

if \( j > k \) and \( p(t_j - t_k) - 1/2 \leq 1/2 \) if \( j < k \). Hence

\[ Y_\omega(t_j) = \sqrt{\mu_1^+ \omega \xi_{C_\omega \gamma} + \varepsilon_j} = \frac{\gamma \sqrt{L}}{L_n \sqrt{2}} \xi + \varepsilon_j \]

and the distribution of \( Y_\omega^{\text{obs}} \) does not depend on \( \omega \). Therefore,

\[ KL((P_\omega^{\text{obs}})^{\otimes n}, (P_0^{\text{obs}})^{\otimes n}) = nKL(P_\omega^{\text{obs}}, P_0^{\text{obs}}) = 0.
\]

### C Proof of Theorems 2 and 3

#### C.1 Preliminary result

The proof of Theorems 2 and 3 is based on Bosq inequalities stated in the following theorem.
\textbf{Theorem 4 \cite{Bosq2000}.} Let $\Gamma$ and $\widehat{\Gamma}$ be two linear compact operators on a separable Hilbert space $(\mathcal{H}, \| \cdot \|, \langle \cdot , \cdot \rangle)$. We denote by
\[
\Gamma = \sum_{d=1}^{\infty} \mu_d^* \eta_d \otimes \eta_d \quad \text{and} \quad \widehat{\Gamma} = \sum_{d=1}^{\infty} \mu_d \widehat{\eta}_d \otimes \widehat{\eta}_d
\]
their spectral decomposition with the eigenvalues $(\mu_d^*)_{d \geq 1}$ and $(\mu_d)_{d \geq 1}$ sorted in decreasing order. Then
\[
\| \mu_d - \mu_d^* \| \leq \| \widehat{\Gamma} - \Gamma \|_\infty , \tag{12}
\]
where $\| \cdot \|_\infty$ is the operator norm associated to $\| \cdot \|$ defined by $\| T \|_\infty = \sup_{\| f \|_\infty = 1} \| Tf \|$ for all continuous operator $T \in \mathcal{L}(\mathcal{H})$. Suppose moreover that, for $d \geq 1$, the eigenspace associated to the eigenfunction $\eta_d^*$ is one-dimensional and denote, to avoid sign confusion, $\eta_{\pm,d}^* = \text{sign}((\eta_{0,d}, \eta_d^*)) \times \eta_d^*$. Then, we have
\[
\| \eta_d - \eta_{\pm,d}^* \| \leq b_d \| \widehat{\Gamma} - \Gamma \|_\infty , \tag{13}
\]
where
\[
b_d = 8 \frac{(\mu_1^* - \mu_2^*)^{-2}}{\min(\mu_d^* - \mu_{d+1}^*, \mu_d^*-\mu_d^*)^2}.
\]

The proof of Theorem 4 comes directly from \cite{Bosq2000} Theorem 4.2, p. 103 for the upper bound \eqref{12} on the eigenvalues and \cite{Bosq2000} Lemma 4.3, p. 104 for the upper bound \eqref{13} on the eigenfunctions. We use the previous result to establish the following proposition.

\textbf{Proposition 4} Setting $K_\phi = E[\widehat{K}_\phi]$, we have
\[
\| \eta_{0,d} - \eta_{\pm,d}^* \|^2 \leq 5b_d \left[ \| \widehat{K}_\phi - \Gamma \phi \|_\infty^2 + \| \Pi_D \Gamma \Pi_D - \Gamma \|^2_\infty + \frac{\sigma^4}{p^2} + A_p^{(K)}(\phi, D) + A_p^{(\phi)}(\phi, D) \right]. \tag{14}
\]

\textbf{Proof (of Proposition 4).} In the sequel, we denote $\Gamma_\phi = E[\widehat{\Gamma}_\phi]$. We have
\[
K_\phi(s,t) = \sum_{\lambda, \lambda' \in \Lambda_D} \frac{1}{p^2} \sum_{h,h'=0}^{p-1} K(t_h, t_{h'}) \phi_{\lambda}(t_h) \phi_{\lambda'}(t_{h'}) \phi_{\lambda'}(s) \phi_{\lambda'}(t)
\]
\[
+ \frac{\sigma^2}{p} \sum_{\lambda, \lambda' \in \Lambda_D} \sum_{h=0}^{p-1} \phi_{\lambda}(t_h) \phi_{\lambda'}(t_h) \phi_{\lambda}(s) \phi_{\lambda'}(t)
\]
\[
= \Pi_{S_D^2} K(s,t) + \frac{\sigma^2}{p} \sum_{\lambda \in \Lambda_D} \phi_{\lambda}(s) \phi_{\lambda}(t) + R^{(K)}(s,t) + R^{(\phi)}(s,t), \tag{15}
\]
where $\Pi_{S_D^2}$ is the orthogonal projection onto $S_D^2 = \text{span}\{(s,t) \twoheadrightarrow \phi_{\lambda}(s) \phi_{\lambda'}(t), \lambda, \lambda' \in \Lambda_D\}$.

\[
R^{(K)}(s,t) = \sum_{\lambda, \lambda' \in \Lambda_D} \frac{1}{p^2} \sum_{h,h'=0}^{p-1} K(t_h, t_{h'}) \phi_{\lambda}(t_h) \phi_{\lambda'}(t_{h'}) \phi_{\lambda}(s) \phi_{\lambda'}(t) - \Pi_{S_D^2} K(s,t)
\]
\[
= \sum_{\lambda, \lambda' \in \Lambda_D} \left( \frac{1}{p^2} \sum_{h,h'=0}^{p-1} K(t_h, t_{h'}) \phi_{\lambda}(t_h) \phi_{\lambda'}(t_{h'}) - \int_0^1 \int_0^1 K(s,t) \phi_{\lambda}(s) \phi_{\lambda'}(t) ds dt \right) \phi_{\lambda}(s) \phi_{\lambda'}(t)
\]

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and

\[ R^{(a)}(s,t) = \frac{\sigma^2}{p} \sum_{\lambda, \lambda' \in \Lambda_D} \sum_{k=0}^{p-1} \phi_{\lambda}(t_h)\phi_{\lambda'}(t_h)\phi_{\lambda}(s)\phi_{\lambda'}(t) - \frac{\sigma^2}{p} \sum_{\lambda \in \Lambda_D} \phi_{\lambda}(s)\phi_{\lambda}(t) \]

\[ = \frac{\sigma^2}{p} \sum_{\lambda, \lambda' \in \Lambda_D} \left( \frac{1}{p} \sum_{k=0}^{p-1} \phi_{\lambda}(t_h)\phi_{\lambda'}(t_h) - 1_{(\lambda=\lambda')} \right) \phi_{\lambda}(s)\phi_{\lambda'}(t). \]

Then, from the decomposition of the kernel \( K_\phi \) given in Equation (13), we have for any function \( f \) and any \( t \in [0,1] \),

\[ \Gamma_\phi(f)(t) = \int_0^1 K_\phi(s,t)f(s)ds \]

\[ = \int_0^1 \Pi_{S_D} K(s,t)f(s)ds + \frac{\sigma^2}{p} \sum_{\lambda \in \Lambda_D} \int_0^1 \phi_{\lambda}(s)f(s)ds \phi_{\lambda}(t) + T^{(K)}(f)(t) + T^{(\sigma)}(f)(t) \]

\[ = \int_0^1 \Pi_{S_D} K(s,t)f(s)ds + \frac{\sigma^2}{p} \Pi_D(f)(t) + T^{(K)}(f)(t) + T^{(\sigma)}(f)(t), \]

where \( \Pi_D \) is the orthogonal projection onto \( S_D = \text{span}\{\phi_\lambda, \lambda \in \Lambda_D\} \) and \( T^{(K)} \) (resp. \( T^{(\sigma)} \)) is the integral operator associated to the kernel \( R^{(K)} \) (resp. \( R^{(\sigma)} \)):

\[ T^{(K)}(t) := \int_0^1 R^{(K)}(s,t)f(s)ds, \quad T^{(\sigma)}(f)(t) := \int_0^1 R^{(\sigma)}(s,t)f(s)ds. \]

Now,

\[ \int_0^1 \Pi_{S_D} K(s,t)f(s)ds = \sum_{\lambda, \lambda' \in \Lambda_D} \int_0^1 \int_0^1 K(u,v)\phi_{\lambda}(u)\phi_{\lambda'}(v)du dv \phi_{\lambda}(s)\phi_{\lambda'}(t)f(s)ds \]

\[ = \sum_{\lambda, \lambda' \in \Lambda_D} \langle \phi_{\lambda}, f \rangle \int_0^1 \int_0^1 K(u,v)\phi_{\lambda}(u)\phi_{\lambda'}(v)du dv \phi_{\lambda'}(t) \]

\[ = \sum_{\lambda, \lambda' \in \Lambda_D} \langle \phi_{\lambda}, f \rangle \langle \Gamma(\phi_{\lambda}), \phi_{\lambda'} \rangle \phi_{\lambda'}(t) \]

\[ = \sum_{\lambda, \lambda' \in \Lambda_D} \langle \Gamma(\sum_{\lambda' \in \Lambda_D} \langle \phi_{\lambda}, f \rangle \phi_{\lambda'}), \phi_{\lambda'} \rangle \phi_{\lambda'}(t) \]

\[ = \Pi_D(\Gamma(\Pi_D(f)))(t). \quad (16) \]

Hence, we obtain:

\[ \Gamma_\phi = \Pi_D \Gamma \Pi_D + \frac{\sigma^2}{p} \Pi_D + T^{(K)} + T^{(\sigma)}. \]

Now, since the eigenvalues \( (\mu_\phi^a)_{a \geq 1} \) are all distincts, the eigenspace associated to the eigenvalue \( \mu_\phi^a \) is one-dimensional and we can apply Theorem 1 to the operators \( \Gamma \) and \( \Gamma_\phi \), which yields

\[ \| \eta_{\phi,d} - \eta_{\phi,d}^* \| \leq b_d^{1/2} \| \bar{\Gamma}_\phi - \Gamma \|_\infty \]

\[ \leq b_d^{1/2} \left( \| \bar{\Gamma}_\phi - \Gamma \|_\infty + \| \Pi_D \Gamma \Pi_D - \Gamma \|_\infty + \frac{\sigma^2}{p} + \| T^{(K)} \|_\infty + \| T^{(\sigma)} \|_\infty \right). \]

In the previous inequality, we have used that \( \| \Pi_D \|_\infty = 1 \). We now control each term of the previous inequality. For this purpose, introducing \( \| \cdot \|_{HS} \), the Hilbert-Schmidt norm of an
operator defined by \( \|T\|_{HS}^2 = \sum_{\lambda \in A} \|Te_\lambda\|^2 \) where \( (e_\lambda)_{\lambda \in A} \) is an orthonormal basis of \( L^2 \) (recall that the Hilbert-Schmidt norm is independent of the choice of the basis), we have, for all operator \( T: L^2 \to L^2 \), \( \|T\|_\infty \leq \|T\|_{HS} \) since

\[
\|T\|_\infty^2 = \sup_{f \in L^2, f \neq 0} \frac{\|Tf\|^2}{\|f\|^2}
\]

and, by Cauchy-Schwarz’s Inequality,

\[
\|Tf\|^2 = \sum_{\lambda \in A} (Tf, e_\lambda)^2 \leq \sum_{\lambda \in A} \left( \sum_{\lambda' \in A} (f, e_{\lambda'}) (Te_{\lambda'}, e_\lambda) \right)^2 \leq \sum_{\lambda \in A} \left( \sum_{\lambda' \in A} (f, e_{\lambda'})^2 \right) \left( \sum_{\lambda' \in A} \|Te_{\lambda'}\|^2 \right) = \|f\|^2 \sum_{\lambda' \in A} \|Te_{\lambda'}\|^2 = \|f\|^2 \|T\|_{HS}^2.
\]

Moreover, we also remark that if \( T \) is a kernel operator associated to a kernel \( R \),

\[
\|T\|_{HS}^2 = \sum_{\lambda \in A} \|Te_\lambda\|^2 = \sum_{\lambda \in A} \left\| \int_0^1 R(s, \cdot)e_\lambda(s)ds \right\|^2 = \sum_{\lambda \in A} \int_0^1 \left( \int_0^1 R(s, t)e_\lambda(s)ds \right)^2 dt = \int_0^1 \int_0^1 R^2(s, t)dsdt = \|R\|^2.
\]

In addition, if the kernel \( R \in S^2_0 \), i.e. if there exists a matrix \( G = (G_{\lambda,\lambda'})_{\lambda,\lambda' \in \Lambda_D} \) such that

\[
R(s, t) = \sum_{\lambda,\lambda' \in \Lambda_D} G_{\lambda,\lambda'} \phi_\lambda(s)\phi_{\lambda'}(t),
\]

we have \( \|R\|_{L^2} = \|G\|_F \), where, for a matrix \( G \),

\[
\|G\|_F = \sqrt{Tr(G^T G)} = \left( \sum_{\lambda,\lambda' \in \Lambda_D} G_{\lambda,\lambda'}^2 \right)^{1/2}
\]

is the Frobenius norm of the matrix \( G \). The fourth and fifth terms of Equation (17) are then bounded by the squared Frobenius norm of the associated matrices and we obtain

\[
\|\tilde{\eta}_{\phi, d} - \eta_{\phi, d}^*\|^2 \leq 5\delta_d \left[ \|\tilde{\Gamma}_{\phi} - \Gamma_{\phi}\|^2_{\infty} + \|\Pi_D \Gamma_{\Pi_D} - \Gamma\|^2_{\infty} + \frac{\sigma^4}{\rho^2} + A_{\phi}^{(K)}(\phi, D) + A_{\phi}^{(S)}(\phi, D) \right].
\]

Proposition 4 is proved.

To end the proof of Theorems 2 and 3, it remains to deal with the stochastic term \( \|\tilde{\Gamma}_{\phi} - \Gamma_{\phi}\|^2_{\infty} \), still bounded by using the Frobenius norm:

\[
\|\tilde{\Gamma}_{\phi} - \Gamma_{\phi}\|^2_{\infty} \leq \|\tilde{G}_{\phi} - G_{\phi}\|_{F},
\]

where

\[
\tilde{G}_{\phi} := \left( \frac{1}{n} \sum_{i=1}^n \tilde{y}_{i,\lambda\lambda'} \right)_{\lambda,\lambda' \in \Lambda_D}
\]

and \( G_{\phi} = E[\tilde{G}_{\phi}] \). The upper bound of \( E[\|\tilde{G}_{\phi} - G_{\phi}\|^2_{F}] \) gives Theorem 2 whereas Theorem 3 is deduced from the control in probability of \( \|\tilde{G}_{\phi} - G_{\phi}\|_{F} \) provided by Proposition 5 below.
C.2 End of the proof of Theorem 2

Lemma 1 Under Assumption 1, we have:

\[ E[\|\hat{G}_\phi - G_\phi\|^2_F] \leq \max(C_1 + 3; 6) \left( \sum_{\lambda \in \Lambda_D} \left[ \sigma_\lambda^2 + s_\lambda^2 \right] \right)^2. \]

Proof (of Lemma 1) We have

\[ E[\|\hat{G}_\phi - G_\phi\|^2_F] = \sum_{\lambda, \lambda' \in \Lambda_D} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^n \left[ \tilde{y}_{i, \lambda} \tilde{y}_{i, \lambda'} - \mathbb{E}[\tilde{y}_{i, \lambda} \tilde{y}_{i, \lambda'}] \right] \right)^2 \right] \]

\[ = \sum_{\lambda, \lambda' \in \Lambda_D} \text{Var} \left( \frac{1}{n} \sum_{i=1}^n \tilde{y}_{i, \lambda} \tilde{y}_{i, \lambda'} \right) \]

\[ \leq \frac{1}{n} \sum_{\lambda, \lambda' \in \Lambda_D} \mathbb{E}[\tilde{y}_{i, \lambda}^2 \tilde{y}_{i, \lambda'}^2] \]

\[ \leq \frac{1}{n} \left( \sum_{\lambda \in \Lambda_D} (E[\tilde{y}_{i, \lambda}]^4)^{1/2} \right)^2. \]

Now, since \( \tilde{z}_{1, \lambda} \sim \mathcal{N}(0, \sigma_\lambda^2) \), and \( \tilde{z}_{1, \lambda} = \frac{1}{p} \sum_{h=0}^{p-1} Z_1(t_h) \phi_\lambda(t_h) \), we have

\[ E[\tilde{y}_{i, \lambda}] = E[(\tilde{z}_{1, \lambda} + \tilde{e}_{1, \lambda})^4] \]

\[ = E[\tilde{z}_{i, \lambda}^4] + 6E[\tilde{z}_{i, \lambda}^2]E[\tilde{e}_{i, \lambda}^2] + E[\tilde{e}_{i, \lambda}^4] \]

\[ \leq C_1(E[\tilde{z}_{i, \lambda}^4] + 6E[\tilde{z}_{i, \lambda}^2]E[\tilde{e}_{i, \lambda}^2] + 3\sigma_\lambda^4) \]

\[ \leq (C_1 + 3)s_\lambda^4 + 6\sigma_\lambda^4 \]

and

\[ E[\|\hat{G}_\phi - G_\phi\|^2_F] \leq \frac{1}{n} \left( \sum_{\lambda \in \Lambda_D} ((C_1 + 3)s_\lambda^4 + 6\sigma_\lambda^4)^{1/2} \right)^2 \]

\[ \leq \max(C_1 + 3; 6) \left( \sum_{\lambda \in \Lambda_D} \left[ \sigma_\lambda^2 + s_\lambda^2 \right] \right)^2. \]

This ends the proof of Lemma 1.

Combining the upper bound of the previous lemma with (14) provides the stated result in Theorem 2.

C.3 End of the proof of Theorem 3

To complete the proof of Theorem 3, we need some technical lemmas. Before stating them, we recall that for all \( i = 1, \ldots, n \), we have set

\[ \tilde{y}_{i, \lambda} = \frac{1}{p} \sum_{h=0}^{p-1} Y_i(t_h) \phi_\lambda(t_h), \quad \tilde{e}_{i, \lambda} = \frac{1}{p} \sum_{h=0}^{p-1} Z_i(t_h) \phi_\lambda(t_h), \quad \tilde{e}_{i, \lambda} = \frac{1}{p} \sum_{h=0}^{p-1} e_{i, h} \phi_\lambda(t_h) \]
and

\[ s_\lambda^2 = \text{Var}(\tilde{z}_\lambda), \quad \sigma_\lambda^2 = \text{Var}(\varepsilon_\lambda). \]

In the sequel, we consider \( \tilde{y}_\lambda = (\tilde{y}_{\lambda,i})_{\lambda \in \Lambda_D}, \tilde{z}_\lambda = (\tilde{z}_{\lambda,i})_{\lambda \in \Lambda_D} \) and \( \varepsilon_\lambda = (\varepsilon_{\lambda,i})_{\lambda \in \Lambda_D} \).

**Lemma 2** Under Assumption 2, for any \( u \in [\Lambda_D]^p \),

\[ \|u^T \tilde{z}_1\|_2^2 \leq C_2 E[(u^T \varepsilon_1)^2]. \] \( \tag{18} \)

If we consider \( \varepsilon_1 \) instead of \( \tilde{z}_1 \), Inequality (18) holds with an absolute constant instead of \( C_2 \). Furthermore,

\[ \text{Tr}(E[\varepsilon_1 \varepsilon_1^T]) = \sum_{\lambda \in \Lambda_D} s_\lambda^2, \quad \text{Tr}(E[\tilde{z}_1 \tilde{z}_1^T]) = \sum_{\lambda \in \Lambda_D} \sigma_\lambda^2. \] \( \tag{19} \)

**Proof (of Lemma 2)** Since \( Z_1 := \{Z_1(t_0), \ldots, Z_1(t_{p-1})\}^T \) is a zero-mean sub-Gaussian vector, the vector \( \tilde{z}_1 \) is also a zero-mean sub-Gaussian vector. We have, for any \( u \in [\Lambda_D]^p \),

\[ \|u^T \tilde{z}_1\|_2^2 = \left\| \sum_{\lambda \in \Lambda_D} u_\lambda \tilde{z}_{1,\lambda} \right\|_2^2 = \left\| \sum_{\lambda \in \Lambda_D} u_\lambda \times \frac{1}{p} \sum_{h=0}^{p-1} Z_1(t_h) \phi_\lambda(t_h) \right\|_2^2 = \left\| u^T Z_1 \right\|_2^2, \]

with \( v = (v_h)_{h=0, \ldots, p-1} \) and \( v_h := \frac{1}{p} \sum_{h' \in \Lambda_D} u_\lambda \phi_\lambda(t_{h'}). \) Therefore,

\[ \|u^T \tilde{z}_1\|_2^2 \leq C_2 E[(u^T Z_1)^2] \leq C_2 E \left[ \sum_{h, h'=0}^{p-1} v_h Z_1(t_h) Z_1(t_{h'}) v_{h'} \right] \leq C_2 \sum_{\lambda, \lambda' \in \Lambda_D} u_\lambda u_{\lambda'} \frac{1}{p^2} E \left[ \sum_{h, h'=0}^{p-1} \phi_\lambda(t_h) \phi_{\lambda'}(t_{h'}) Z_1(t_h) Z_1(t_{h'}) \right] \leq C_2 \sum_{\lambda \in \Lambda_D} u_\lambda u_\lambda \left\| (\varepsilon_1)_{\lambda} \right\|_{2}^2 \leq C_2 \sum_{\lambda \in \Lambda_D} u_\lambda u_\lambda \|v\|_{2}^2 \leq C_2 E[(u^T \varepsilon_1)^2]. \]

Now, if we consider \( \varepsilon_1 \) instead of \( \tilde{z}_1 \), setting \( \varepsilon_1 := (\varepsilon_{1,0}, \ldots, \varepsilon_{1,p-1})^T \), and using Section 5.2.3 and Lemma 5.24 of Vershynin (2012),

\[ \left\| u^T \varepsilon_1 \right\|_2^2 \leq C \sigma^2 \|v\|_2^2 = C E[(u^T \varepsilon_1)^2], \]

with \( C \) an absolute constant, and

\[ \|u^T \tilde{z}_1\|_2^2 \leq C E[(u^T \varepsilon_1)^2]. \]

The equalities (19) are obvious.

Results of the previous lemma are useful for the following result.
Lemma 3  We denote \( X = (X_{\lambda, \lambda'})_{\lambda, \lambda' \in \Lambda_D} \) the matrix whose entries are
\[
X_{\lambda, \lambda'} = \tilde{y}_{1, \lambda} \tilde{y}_{1, \lambda'} - E[\tilde{y}_{1, \lambda} \tilde{y}_{1, \lambda'}].
\]
Setting
\[
M_D := \sum_{\lambda \in \Lambda_D} s_\lambda^2 + \sum_{\lambda \in \Lambda_D} \sigma_\lambda^2 = \sum_{\lambda \in \Lambda_D} \left( \frac{1}{p^2} \sum_{h, h'=0}^{p-1} K(t_h, t_{h'}) \phi_\lambda(t_h) \phi_{\lambda'}(t_{h'}) + \frac{\sigma_\lambda^2}{p^2} \sum_{h=0}^{p-1} \phi_\lambda(t_h) \right),
\]
under Assumption 2 there exists an absolute constant \( C \) such that for any \( t \geq C(C_2 + 1)M_D, \)
\[
E \left[ \exp(t^{-1} \| X \|_F) \right] \leq \exp(1).
\]
Proof (of Lemma 3) We have
\[
\| X \|_F = \| \tilde{y}_{1, \lambda} \tilde{y}_{1, \lambda'}^T - E[\tilde{y}_{1, \lambda} \tilde{y}_{1, \lambda'}^T] \|_F
\leq \| (\tilde{z}_1 + \tilde{\varepsilon}_1) (\tilde{z}_1 + \tilde{\varepsilon}_1)^T \|_F + \| E[(\tilde{z}_1 + \tilde{\varepsilon}_1)(\tilde{z}_1 + \tilde{\varepsilon}_1)^T] \|_F
\leq \| \tilde{z}_1 \tilde{z}_1^T \|_F + \| \tilde{\varepsilon}_1 \tilde{\varepsilon}_1^T \|_F + 2\| \tilde{z}_1 \tilde{\varepsilon}_1^T \|_F + \| E[\tilde{\varepsilon}_1 \tilde{\varepsilon}_1^T] \|_F + \| E[\tilde{\varepsilon}_1 \tilde{\varepsilon}_1^T] \|_F + \| E[\tilde{\varepsilon}_1 \tilde{\varepsilon}_1^T] \|_F
\leq \| \tilde{z}_1 \|_{\ell_2}^2 + \| \tilde{\varepsilon}_1 \|_{\ell_2}^2 + 2\| \tilde{z}_1 \|_{\ell_2} \| \tilde{\varepsilon}_1 \|_{\ell_2} + \| E[\tilde{\varepsilon}_1 \tilde{\varepsilon}_1^T] \|_F + \| E[\tilde{\varepsilon}_1 \tilde{\varepsilon}_1^T] \|_F + \| E[\tilde{\varepsilon}_1 \tilde{\varepsilon}_1^T] \|_F.
\]
We also have
\[
\| E[\tilde{z}_1 \tilde{z}_1^T] \|_F = \sum_{\lambda, \lambda' \in \Lambda_D} \left( E[\tilde{z}_1, \tilde{z}_1, \lambda, \lambda'] \right)^2 \leq \sum_{\lambda \in \Lambda_D} \sum_{\lambda' \in \Lambda_D} E[\tilde{z}_1, \lambda] E[\tilde{z}_1, \lambda'] \leq \left( \sum_{\lambda \in \Lambda_D} s_\lambda^2 \right)^2.
\]
Therefore
\[
\| E[\tilde{z}_1 \tilde{z}_1^T] \|_F \leq \sum_{\lambda \in \Lambda_D} s_\lambda^2
\]
and similarly,
\[
\| E[\tilde{\varepsilon}_1 \tilde{\varepsilon}_1^T] \|_F \leq \sum_{\lambda \in \Lambda_D} \sigma_\lambda^2.
\]
We finally obtain
\[
\| X \|_F \leq 2\| \tilde{z}_1 \|_{\ell_2}^2 + 2\| \tilde{\varepsilon}_1 \|_{\ell_2}^2 + M_D
\]
and we have
\[
E \left[ \exp(t^{-1} \| X \|_F) \right] \leq E \left[ \exp(2t^{-1} \| \tilde{z}_1 \|_{\ell_2}^2) \right] \times E \left[ \exp(2t^{-1} \| \tilde{\varepsilon}_1 \|_{\ell_2}^2) \right] \times \exp(t^{-1}M_D).
\]
Then, using Lemma 2 and Proposition A.1. of Bunea and Xiao (2015), we obtain for \( C_* \) and \( c_* \) two absolute positive constants, if \( t > c_*(4C_2 + 1) \sum_{\lambda \in \Lambda_D} s_\lambda \),
\[
E \left[ \exp(2t^{-1} \| \tilde{z}_1 \|_{\ell_2}^2) \right] \leq \exp \left( \exp \left( 2t^{-1} \left( \| \tilde{z}_1 \|_{\ell_2}^2 - \sum_{\lambda \in \Lambda_D} s_\lambda^2 \right) \right) \right) \times \exp \left( 2t^{-1} \sum_{\lambda \in \Lambda_D} s_\lambda^2 \right) \leq \exp \left( C_* \left( \frac{4C_2 + 1}{t} \sum_{\lambda \in \Lambda_D} s_\lambda \right)^2 + 2t^{-1} \sum_{\lambda \in \Lambda_D} s_\lambda^2 \right).
\]
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Similarly, for $t$ larger than $\sum_{\lambda \in \Lambda_D} \sigma_\lambda^2$ up to a multiplicative absolute constant,

$$E \left[ \exp(2t^{-1} \| \tilde{\varepsilon}_1 \|_2^2) \right] \leq \exp \left( C^{**} \left( \frac{\sum_{\lambda \in \Lambda_D} \sigma_\lambda^2}{t} \right)^2 + 2t^{-1} \sum_{\lambda \in \Lambda_D} \sigma_\lambda^2 \right),$$

where $C^{**}$ is an absolute constant. This ends the proof of the lemma.

The following proposition controls the term $\| \hat{G}_\phi - G_\phi \|_F$ as required to complete the proof of Theorem 3.

**Proposition 5** We assume that Assumption 2 is satisfied. For $\gamma > 0$, with probability larger than $1 - 2 \exp(-1/64 \min(\gamma^2, 16\gamma \sqrt{n}))$,

$$\| \hat{G}_\phi - G_\phi \|_F \leq \frac{\bar{C}(e^{1/2} + \gamma)(C_2 + 1)}{\sqrt{n}} \sum_{\lambda \in \Lambda_D} \left[ \sigma_\lambda^2 + s_\lambda^2 \right],$$

where $\bar{C}$ is an absolute constant.

**Proof (of Proposition 5)** We apply Theorem 4.1 of Juditsky and Nemirovski (2008) with $\alpha = 1$, since $(\mathbb{R}^{|\Lambda_D|^2}, \| \cdot \|_F)$ is 1-smooth (see Definition 2.1 of Juditsky and Nemirovski (2008)). Since Lemma 3 gives for $t \geq \bar{C}(C_2 + 1) \sum_{\lambda \in \Lambda_D} \left[ \sigma_\lambda^2 + s_\lambda^2 \right]$,

$$E \left[ \exp(t^{-1} \| X \|_F) \right] \leq \exp(1),$$

for $\gamma > 0$, with probability larger than $1 - 2 \exp(-1/64 \min(\gamma^2, 16\gamma \sqrt{n}))$,

$$\| \hat{G}_\phi - G_\phi \|_F \leq \frac{\bar{C}(e^{1/2} + \gamma)(C_2 + 1)}{\sqrt{n}} \sum_{\lambda \in \Lambda_D} \left[ \sigma_\lambda^2 + s_\lambda^2 \right].$$

**Proposition 5** is proved.

Plugging the upper bound of Proposition 5 in (14) provides the stated result of Theorem 3.

**C.4 Proof of Proposition 1**

We control each deterministic term of the bound obtained in Theorems 2 and 3. Using (16), we first have for any $f \in L_2$,

$$\| \Pi_D (\Pi_D f) - \Gamma(f) \|^2 = \int_0^1 \left( \Pi_D (\Pi_D f)(t) - \Gamma(f)(t) \right)^2 dt$$

$$= \int_0^1 \left( \int_0^1 \Pi_{S_D^2} K(s, t)f(s)ds - \int_0^1 K(s, t)f(s)ds \right)^2 dt$$

$$= \int_0^1 \left( \int_0^1 (\Pi_{S_D^2} K(s, t) - K(s, t))f(s)ds \right)^2 dt$$

$$\leq \int_0^1 \left[ \int_0^1 (\Pi_{S_D^2} K(s, t) - K(s, t))^2 ds \int_0^1 f^2(s)ds \right] dt$$

$$\leq \| \Pi_{S_D^2} K - K \|^2 \| f \|^2$$

and then

$$\| \Pi_D \Gamma \Pi_D - \Gamma \|_\infty^2 \leq \| \Pi_{S_D^2} K - K \|^2.$$
Now, we take \((s, t) \in \mathbb{I}²\). Then there exists a unique couple \((\lambda, \lambda') \in \mathbb{D}²\) such that \(s \in I_\lambda\) and \(t \in I_{\lambda'}\). Therefore, \(\phi_{\lambda'}(s) = 0\) for \(\lambda' \neq \lambda\) and \(\phi_{\lambda'}(t) = 0\) for \(\lambda'' \neq \lambda'\) and then,

\[
\Pi_{\mathcal{S}_D}K(s, t) - K(s, t) = \sum_{\lambda'' \in \mathbb{D}} \int_0^1 \int_0^1 K(s', t') \phi_{\lambda''}(s') \phi_{\lambda''}(t') ds' dt' \phi_{\lambda''}(s) \phi_{\lambda''}(t) - K(s, t)
\]

\[
= \int_0^1 \int_0^1 K(s', t') \phi_{\lambda'}(s') \phi_{\lambda'}(t') ds' dt' \phi_{\lambda'}(s) \phi_{\lambda'}(t) - K(s, t)
\]

\[
= D^2 \int_{I_\lambda} \int_{I_{\lambda'}} (K(s', t') - K(s, t)) ds' dt'.
\]

Then, Eq. \((\text{4})\) gives

\[
\left| \Pi_{\mathcal{S}_D}K(s, t) - K(s, t) \right| \leq D^2 \sqrt{L} \|K\|_{\infty} \int_{I_\lambda} \int_{I_{\lambda'}} \left[ |s' - s|^\alpha + |t - t'|^\alpha \right] ds' dt'
\]

\[
\leq \frac{4 \sqrt{L} \|K\|_{\infty}}{\alpha + 1} D^{-\alpha},
\]

meaning that

\[
\|\Pi_D \Gamma \Pi_D - \Gamma\|_2 \leq \frac{16 \sqrt{L} \|K\|_{\infty}}{(\alpha + 1)^2} D^{-2\alpha}.
\]

For studying the terms \(A_p^{(K)}(\phi, D)\) and \(A_p^{(s)}(\phi, D)\), we set for any \(h = 0, \ldots, p - 1\), \(b_h = h/p\). Observe that \(t_h = h/(p - 1) \in [b_h, b_{h+1}]\). We also set for any \(\lambda = 0, \ldots, D - 1\),

\[
J_\lambda = \{h = 0, \ldots, p - 1 : \text{Leb}[b_h, b_{h+1} \cap I_\lambda] \neq 0\}.
\]

Remember that \(m := p/D\) is an integer, so that, \(J_\lambda = \{m \lambda, \ldots, m \lambda + m - 1\}\) and

\[
I_\lambda = \left[ \frac{m \lambda}{p}, \frac{m \lambda + m}{p} \right] = \bigcup_{h \in J_\lambda} [b_h, b_{h+1}].
\]

Then, since \(\phi_\lambda(x) = \sqrt{D} \chi_{I_\lambda}(x)\) and \(\text{card}(J_\lambda) = m = p/D\), for any \(\lambda, \lambda' = 0, \ldots, D - 1\),

\[
G_{\lambda, \lambda'}^{(K)} := \frac{1}{p^2} \sum_{h, h' = 0}^{p-1} K(t_h, t_{h'}) \phi_\lambda(t_h) \phi_{\lambda'}(t_{h'}) - \int_0^1 \int_0^1 K(s, t) \phi_\lambda(s) \phi_{\lambda'}(t) ds dt
\]

\[
= \sum_{h, h' = 0}^{p-1} \int_{b_h}^{b_{h+1}} \int_{b_{h'}}^{b_{h'+1}} \left[ K(t_h, t_{h'}) - K(s, t) \right] \phi_\lambda(s) \phi_{\lambda'}(t) ds dt
\]

\[
+ \sum_{h, h' = 0}^{p-1} \int_{b_h}^{b_{h+1}} \int_{b_{h'}}^{b_{h'+1}} K(t_h, t_{h'}) \left[ \phi_\lambda(t_h) \phi_{\lambda'}(t_{h'}) - \phi_\lambda(s) \phi_{\lambda'}(t) \right] ds dt
\]

\[
= D \sum_{h \in J_\lambda} \sum_{h' \in J_{\lambda'}} \int_{b_h}^{b_{h+1}} \int_{b_{h'}}^{b_{h'+1}} \left[ K(t_h, t_{h'}) - K(s, t) \right] ds dt.
\]
Therefore,

\[
|G^{(K)}_{\lambda, \lambda'}| \leq D \sum_{h \in J_{\lambda}} \sum_{h' \in J_{\lambda'}} \int_{b_h}^{b_{h'+1}} \int_{b_h'}^{b_{h'+1}} \sqrt{\|K\|_\infty L \left( |s - t_h|^\alpha + |t - t_{h'}|^\alpha \right)} ds dt \\
\leq 2 \sqrt{\|K\|_\infty L} \times D p^{-1} \text{card}(J_{\lambda'}) \sum_{h \in J_{\lambda}} |s - t_h|^\alpha ds \\
\leq 2 \sqrt{\|K\|_\infty L} \times D p^{-1} \text{card}(J_{\lambda'}) \times \frac{2}{\alpha + 1} p^{-\alpha - 1} \\
\leq \frac{4 \sqrt{\|K\|_\infty L}}{\alpha + 1} D^{-1} p^{-\alpha}.
\]

Finally,

\[
A_p^{(K)}(\phi, D) = \|G^{(K)}\|_F^2 = \sum_{\lambda, \lambda' \in \Lambda_D} \left( G^{(K)}_{\lambda, \lambda'} \right)^2 \leq \frac{16 \|K\|_\infty L}{(\alpha + 1)^2} p^{-2\alpha}.
\]

Similarly, for any \( \lambda, \lambda' = 0, \ldots, D - 1 \), observing that for \( \lambda \neq \lambda' \), \( J_{\lambda} \cap J_{\lambda'} = \emptyset \),

\[
G^{(\sigma)}_{\lambda, \lambda'} := \frac{\sigma^2}{p} \left( \frac{1}{p} \sum_{h=0}^{p-1} \phi_{\lambda}(t_h) \phi_{\lambda'}(t_{h'}) - \langle \phi_{\lambda}, \phi_{\lambda'} \rangle \right) \\
= \frac{\sigma^2}{p} \left( \frac{1}{p} \sum_{h \in J_{\lambda} \cap J_{\lambda'}} D - 1_{(\lambda = \lambda')} \right) = 0
\]

and

\[
A_p^{(\sigma)}(\phi, D) = \|G^{(\sigma)}\|_F^2 = \sum_{\lambda, \lambda' \in \Lambda_D} \left( G^{(\sigma)}_{\lambda, \lambda'} \right)^2 = 0.
\]

Finally, for any \( \lambda = 0, \ldots, D - 1 \),

\[
\sigma^2 + s^2_{\lambda} = \frac{\sigma^2}{p^2} \sum_{h=0}^{p-1} \phi_{\lambda}^2(t_h) + \frac{1}{p^2} \sum_{h, h' = 0}^{p-1} K(t_h, t_{h'}) \phi_{\lambda}(t_h) \phi_{\lambda'}(t_{h'}) \\
\leq \frac{D \sigma^2}{p^2} \text{card}(J_{\lambda}) + \frac{\|K\|_\infty D}{p^2} (\text{card}(J_{\lambda}))^2 \\
\leq \frac{\sigma^2}{p} + \frac{\|K\|_\infty}{D}
\]

and

\[
\sum_{\lambda \in \Lambda_D} \left( \sigma^2 + s^2_{\lambda} \right) \leq \|K\|_\infty + \frac{\sigma^2 D}{p}.
\]

This ends the proof of Proposition 1.