Can the Lepton Flavor Mixing Matrix Be Symmetric?

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Abstract

Current neutrino oscillation data indicate that the $3 \times 3$ lepton flavor mixing matrix $V$ is likely to be symmetric about its $V_{e3}$-$V_{\mu2}$-$V_{\tau1}$ axis. This off-diagonal symmetry corresponds to three pairs of congruent unitarity triangles in the complex plane. Terrestrial matter effects can substantially modify the genuine CP-violating parameter and off-diagonal asymmetries of $V$ in realistic long-baseline experiments of neutrino oscillations.

PACS number(s): 14.60.Pq, 13.10.+q, 25.30.Pt

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I. INTRODUCTION

The observed anomalies of atmospheric [1] and solar [2] neutrinos strongly suggest that neutrinos be massive and lepton flavors be mixed. In the framework of three charged leptons and three active neutrinos, the phenomena of flavor mixing and \( CP \) violation are described by a unitary matrix \( V \), which relates the neutrino mass eigenstates \((\nu_1, \nu_2, \nu_3)\) to the neutrino flavor eigenstates \((\nu_e, \nu_\mu, \nu_\tau)\):

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} =
\begin{pmatrix}
V_{e1} & V_{e2} & V_{e3} \\
V_{\mu1} & V_{\mu2} & V_{\mu3} \\
V_{\tau1} & V_{\tau2} & V_{\tau3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix} .
\] (1)

The unitarity of \( V \) represents two sets of normalization and orthogonality conditions:

\[
\sum_i (V_{\alpha i} V^*_{\beta i}) = \delta_{\alpha \beta} ,
\]

\[
\sum_{\alpha} (V_{\alpha i} V^*_{\alpha j}) = \delta_{ij} ,
\] (2)

where Greek and Latin subscripts run over \((e, \mu, \tau)\) and \((1, 2, 3)\), respectively. If neutrinos are Dirac particles, a full parametrization of \( V \) requires four independent parameters – three mixing angles and one \( CP \)-violating phase, for example. If neutrinos are Majorana particles, however, two additional \( CP \)-violating phases need be introduced for a complete parametrization of \( V \). In both cases, \( CP \) and \( T \) violation in normal neutrino oscillations depends only upon a single rephasing-invariant parameter \( J \) [3], defined through

\[
\text{Im} \left( V_{\alpha i} V^*_{\beta j} V^*_{\gamma i} V^*_{\beta j} \right) = J \sum_{\gamma, k} (\epsilon_{\alpha \beta \gamma} \epsilon_{ijk}) ,
\] (3)

where \((\alpha, \beta, \gamma)\) and \((i, j, k)\) run respectively over \((e, \mu, \tau)\) and \((1, 2, 3)\). A major goal of the future long-baseline neutrino oscillation experiments is to measure \( |V_{\alpha i}| \) and \( J \) as precisely as possible [4]. Once the matrix elements of \( V \) are determined to a good degree of accuracy, a stringent test of its unitarity will become available.

As a straightforward consequence of the unitarity of \( V \), two interesting relations can be derived from the normalization conditions in Eq. (2):

\[
|V_{e2}|^2 - |V_{\mu1}|^2 = |V_{\mu3}|^2 - |V_{\tau2}|^2 = |V_{\tau1}|^2 - |V_{e3}|^2 \equiv \Delta_L ,
\] (4)

and

\[
|V_{e2}|^2 - |V_{\mu3}|^2 = |V_{\mu1}|^2 - |V_{e2}|^2 = |V_{e3}|^2 - |V_{e1}|^2 \equiv \Delta_R .
\] (5)

The off-diagonal asymmetries \( \Delta_L \) and \( \Delta_R \) characterize the geometrical structure of \( V \) about its \( V_{e1}-V_{\mu2}-V_{\tau3} \) and \( V_{e3}-V_{\mu2}-V_{\tau1} \) axes, respectively. If \( \Delta_L = 0 \) held, \( V \) would be symmetric about the \( V_{e1}-V_{\mu2}-V_{\tau3} \) axis. Indeed the counterpart of \( \Delta_L \) in the quark sector is very small (of order \( 10^{-5} \) [5]); i.e., the \( 3 \times 3 \) quark mixing matrix is almost symmetric about its \( V_{ud}-V_{cs}-V_{tb} \).
axis. An exactly symmetric flavor mixing matrix may hint at an underlying flavor symmetry, from which some deeper understanding of the fermion mass texture can be achieved. In this sense, the tiny off-diagonal asymmetry of the quark flavor mixing matrix is likely to arise from a slight breakdown of certain flavor symmetries of quark mass matrices.

The purpose of this paper is to examine whether the lepton flavor mixing matrix \( V \) is really symmetric or not. In section II, we find that current neutrino oscillation data strongly favor \( \Delta_R = 0 \); i.e., \( V \) is possible to be symmetric about its \( V_{e3} - V_{\mu2} - V_{\tau1} \) axis. It remains too early to get any phenomenological constraints on \( \Delta_L \), unless very special assumptions are made. In section III, we point out that the off-diagonal symmetry \( \Delta_R = 0 \) corresponds to three pairs of congruent unitarity triangles in the complex plane. Taken realistic long-baseline experiments of neutrino oscillations into account, the terrestrial matter effects on \( J \), \( \Delta_L \) and \( \Delta_R \) are briefly discussed in section IV. Section V is devoted to some further discussions about possible implications of \( \Delta_R = 0 \) on specific textures of lepton mass matrices. Finally we summarize our main results in section VI.

II. OFF-DIAGONAL SYMMETRY

Current experimental data\(^1\) strongly favor the hypothesis that atmospheric and solar neutrino oscillations are dominated by \( \nu_\mu \to \nu_\tau \) and \( \nu_e \to \nu_\mu \) transitions, respectively. Thus their mixing factors \( \sin^2 2\theta_{\text{atm}} \) and \( \sin^2 2\theta_{\text{sun}} \) have rather simple relations with the elements of the lepton flavor mixing matrix \( V \). The mixing factor associated with the CHOOZ (or Palo Verde) reactor neutrino oscillation experiment\(^7\), denoted as \( \sin^2 2\theta_{\text{chz}} \), is also a simple function of \( |V_{ei}| \) in the same hypothesis. The explicit expressions of \( \sin^2 2\theta_{\text{sun}}, \sin^2 2\theta_{\text{atm}} \) and \( \sin^2 2\theta_{\text{chz}} \) read as follows:

\[
\begin{align*}
\sin^2 2\theta_{\text{sun}} & = 4|V_{e1}|^2|V_{e2}|^2, \\
\sin^2 2\theta_{\text{atm}} & = 4|V_{\mu3}|^2\left(1 - |V_{\mu3}|^2\right), \\
\sin^2 2\theta_{\text{chz}} & = 4|V_{e3}|^2\left(1 - |V_{e3}|^2\right).
\end{align*}
\]

An analysis of the Super-Kamiokande data on atmospheric neutrino oscillations\(^4\) yields \( 0.88 \leq \sin^2 2\theta_{\text{atm}} \leq 1.0 \) and \( 1.6 \times 10^{-3} \text{eV}^2 \leq \Delta m_{\text{atm}}^2 \leq 4.0 \times 10^{-3} \text{eV}^2 \) at the 90\% confidence level. Corresponding to \( \Delta m_{\text{chz}}^2 > 2.0 \times 10^{-3} \text{eV}^2 \), \( \sin^2 2\theta_{\text{chz}} < 0.18 \) can be drawn from the CHOOZ experiment\(^5\). We restrict ourselves to the large-angle Mikheyev-Smirnov-Wolfenstein (MSW) solution to the solar neutrino problem\(^6\), as it gives the best global fit of present data. At the 99\% confidence level, \( 0.56 \leq \sin^2 2\theta_{\text{sun}} \leq 0.99 \) and \( 2.0 \times 10^{-5} \text{eV}^2 \leq \Delta m_{\text{sun}}^2 \leq 5.0 \times 10^{-4} \text{eV}^2 \) have been obtained\(^6\).

With the help of Eqs. (2) and (6), one may express \( |V_{e1}|^2, |V_{e2}|^2, |V_{e3}|^2 \) and \( |V_{\mu3}|^2 \) in terms of \( \theta_{\text{sun}}, \theta_{\text{atm}} \) and \( \theta_{\text{chz}} \):

\[
\begin{align*}
|V_{e1}|^2 & = \frac{1}{2} \left( \cos^2 \theta_{\text{chz}} \pm \sqrt{\cos^4 \theta_{\text{chz}} - \sin^2 2\theta_{\text{sun}}} \right), \\
|V_{e2}|^2 & = \frac{1}{2} \left( \cos^2 \theta_{\text{chz}} \mp \sqrt{\cos^4 \theta_{\text{chz}} - \sin^2 2\theta_{\text{sun}}} \right), \\
|V_{e3}|^2 & = \sin^2 \theta_{\text{chz}} + \cos^2 \theta_{\text{chz}}, \\
|V_{\mu3}|^2 & = \sin^2 \theta_{\text{atm}} - \cos^2 \theta_{\text{atm}}.
\end{align*}
\]
Without loss of generality, three mixing angles \((\theta_{\text{sun}}, \theta_{\text{atm}} \text{ and } \theta_{\text{chz}})\) can all be arranged to lie in the first quadrant. Then we need only adopt the solution \(|V_{e3}|^2 = \sin^2 \theta_{\text{chz}}\) \[^{[1]}\], in accord with \(\sin^2 2\theta_{\text{chz}} < 0.18\). We may also express \(|V_{\tau 3}|^2\) in terms of \(\theta_{\text{atm}}\) and \(\theta_{\text{chz}}\), once the normalization relation \(|V_{e3}|^2 + |V_{\mu 3}|^2 + |V_{\tau 3}|^2 = 1\) is taken into account. It turns out that useful experimental constraints are achievable for those matrix elements in the first row and in the third column of \(V\). However, it is impossible to get any constraints on the other four matrix elements of \(V\), unless some special assumptions are made \[^{[2]}\]. This observation means that it remains too early to get any instructive information on the off-diagonal asymmetry \(\Delta_L\) from current neutrino oscillation experiments, but it is already possible to examine whether \(\Delta_R = 0\) coincides with current data and what its implications can be on leptonic \(CP\) violation and unitarity triangles.

To see whether \(\Delta_R = 0\) is compatible with the present data of solar, atmospheric and reactor neutrino oscillations, we simply set \(|V_{e2}|^2 = |V_{\mu 3}|^2\) in Eq. (7) and then obtain

\[
\sin^2 2\theta_{\text{sun}}^{(\pm)} = \sin^2 2\theta_{\text{atm}} - \left(1 \pm \sqrt{1 - \sin^2 2\theta_{\text{atm}}} \right) \left(1 - \sqrt{1 - \sin^2 2\theta_{\text{chz}}} \right).
\]

As \(\sin^2 2\theta_{\text{chz}} < 0.18\) \[^{[3]}\], the second term on the right-hand side of Eq. (8) serves as a small correction to the leading term \(\sin^2 2\theta_{\text{atm}}\). The difference between \(\sin^2 2\theta_{\text{sun}}^{(+)}\) and \(\sin^2 2\theta_{\text{sun}}^{(-)}\) is therefore insignificant. Indeed \(\sin^2 2\theta_{\text{chz}} = 0\) leads definitely to \(\sin^2 2\theta_{\text{sun}}^{(+)} = \sin^2 2\theta_{\text{sun}}^{(-)} = \sin^2 2\theta_{\text{atm}}\), as a straightforward result of \(\Delta_R = 0\). Allowing \(\sin^2 2\theta_{\text{atm}}\) to vary in the experimental range \(0.88 \leq \sin^2 2\theta_{\text{atm}} \leq 1.0\), we plot the numerical dependence of \(\sin^2 2\theta_{\text{sun}}^{(\pm)}\) on \(\sin^2 2\theta_{\text{chz}}\) in Fig. 1. One can observe that the values of \(\sin^2 2\theta_{\text{sun}}^{(\pm)}\) predicted from Eq. (8) are consistent very well with current experimental data. Thus we conclude that a vanishing or tiny off-diagonal asymmetry of \(V\) about its \(V_{e3}-V_{\mu 2}-V_{\tau 1}\) axis is strongly favored.

III. UNITARITY TRIANGLES

Let us proceed to discuss possible implications of \(\Delta_R = 0\) on the leptonic unitarity triangles. It is known that six orthogonality relations of \(V\) in Eq. (2) correspond to six triangles in the complex plane \[^{[4]}\], as illustrated in Fig. 2. These six triangles totally have eighteen different sides and nine different inner angles. Unitarity requires that all six triangles have the same area amounting to \(J/2\), where \(J\) is just the rephasing-invariant measure of leptonic \(CP\) violation defined in Eq. (3).

Now that the off-diagonal asymmetries \(\Delta_L\) and \(\Delta_R\) describe the geometrical structure of \(V\), they must have direct relations with the unitarity triangles in the complex plane. Indeed it is easy to show that \(\Delta_L = 0\) or \(\Delta_R = 0\) corresponds to the congruence between two unitarity triangles; i.e.,

\[^{[1]}\]Allowing the Dirac-type \(CP\)-violating phase of \(V\) to vary between 0 and \(\pi\), Fukugita and Tanimoto \[^{[1]}\] have presented the numerical ranges of all nine \(|V_{\alpha i}|\) by use of current neutrino oscillation data. This rough construction of the lepton flavor mixing matrix is actually unable to shed light on its off-diagonal asymmetries and \(CP\)-violating features.
\[ \Delta_L = 0 \implies \Delta_e \cong \Delta_1, \]
\[ \Delta_\mu \cong \Delta_2, \]
\[ \Delta_\tau \cong \Delta_3; \quad (9) \]

and

\[ \Delta_R = 0 \implies \Delta_e \cong \Delta_3, \]
\[ \Delta_\mu \cong \Delta_2, \]
\[ \Delta_\tau \cong \Delta_1. \quad (10) \]

As \( \Delta_R = 0 \) is expected to be rather close to reality, we draw the conclusion that the unitarity triangles \( \Delta_e \) and \( \Delta_3 \) must be approximately congruent with each other. A similar conclusion can be drawn for the unitarity triangles \( \Delta_\mu \) and \( \Delta_2 \) as well as \( \Delta_\tau \) and \( \Delta_1 \). The long-baseline experiments of neutrino oscillations in the near future will tell whether an approximate congruence exists between \( \Delta_e \) and \( \Delta_1 \) or between \( \Delta_\tau \) and \( \Delta_3 \). A particularly interesting possibility would be \( \Delta_L \approx \Delta_R \approx 0 \); i.e., only two of the six unitarity triangles are essentially distinct.

Next we examine how large the area of each unitarity triangle (i.e., \( J/2 \)) can maximally be in the limit \( \Delta_R = 0 \), in which \( V \) is parametrized as follows:

\[
V = \begin{pmatrix}
  c_x c_z & s_x c_z & s_z \\
  -c_x s_z - c_z s_x e^{-i\delta} & -s_x^2 s_z + c_x^2 e^{-i\delta} & s_x c_z \\
  -c_z^2 s_x + s_x^2 e^{-i\delta} & -c_x s_x s_z - c_x^2 s_x e^{-i\delta} & c_x c_z
\end{pmatrix}
\begin{pmatrix}
  e^{i\rho} & 0 & 0 \\
  0 & e^{i\sigma} & 0 \\
  0 & 0 & 1
\end{pmatrix},
\]

(11)

where \( s_x \equiv \sin \theta_x, c_z \equiv \cos \theta_z \), and so on. The merit of this phase choice is that the Dirac-type \( CP \)-violating phase \( \delta \) does not appear in the effective mass term of the neutrinoless double beta decay [12], which depends only upon the Majorana phases \( \rho \) and \( \sigma \). Without loss of generality, one may arrange the mixing angles \( \theta_x \) and \( \theta_z \) to lie in the first quadrant. Three \( CP \)-violating phases \( (\delta, \rho, \sigma) \) can take arbitrary values from 0 to \( 2\pi \). Clearly \( J = c_x^2 s_x^2 c_z s_z \sin \delta \) holds. With the help of Eq. (6) or (7), we are able to figure out the relations between \( (\theta_x, \theta_z) \) and \( (\theta_{sun}, \theta_{chz}) \). The result is

\[
\sin^2 2\theta_x = \frac{4 \sin^2 2\theta_{sun}}{\left(1 + \sqrt{1 - \sin^2 2\theta_{chz}}\right)^2},
\]

\[
\sin^2 2\theta_z = \sin^2 2\theta_{chz}.
\]

(12)

Then we obtain

\[
J(\pm) = \frac{\sqrt{2}}{4} \sin^2 2\theta_{sun} \sin \delta \sqrt{1 - \sqrt{1 - \sin^2 2\theta_{chz}}} \left(1 + \sqrt{1 - \sin^2 2\theta_{chz}}\right),
\]

(13)

where \( \sin^2 2\theta_{sun}^{(\pm)} \) has been given in Eq. (8). Again the difference between \( J(+) \) and \( J(-) \) is insignificant. If \( \delta = 0 \) or \( \pi \) held, we would arrive at \( J(\pm) = 0 \). In general, however, \( CP \) symmetry is expected to break down in the lepton sector. For illustration, we plot the
numerical dependence of $\mathcal{J}(\pm) / \sin \delta$ on $\sin^2 2\theta_{\text{chz}}$ in Fig. 3, where the experimentally allowed values of $\sin^2 2\theta_{\text{atm}}$ are used. It is obvious that the upper bound of $\mathcal{J}(\pm)$ (when $\delta = \pi/2$) can be as large as a few percent, only if $\sin^2 2\theta_{\text{chz}} \geq 0.01$. This result implies that leptonic $CP$ and $T$ violation might be observable in the future long-baseline neutrino oscillation experiments.

IV. MATTER EFFECTS

In realistic long-baseline experiments of neutrino oscillations, the terrestrial matter effects must be taken into account [13]. The pattern of neutrino oscillations in matter can be expressed in the same form as that in vacuum, however, if we define the effective neutrino masses $\tilde{m}_i$ and the effective lepton flavor mixing matrix $\tilde{V}$ in which the terrestrial matter effects are already included [14]. Note that $\tilde{m}_i$ are remarkably large, if $\sin^2 2\theta_{\text{chz}}$ is expressed in the same form as that in vacuum, however, if we define the matter parameter and its magnitude depends upon the neutrino beam energy $E$ and the background density of electrons $N_e$. The analytically exact relations between $(\tilde{m}_i, \tilde{V}_{ai})$ and $(m_j, V_{ai})$ can be found in Ref. [14], if $N_e$ is assumed to be a constant. In analogy to the definitions of $\mathcal{J}$, $\Delta_L$ and $\Delta_R$, the effective $CP$-violating parameter $\tilde{\mathcal{J}}$ and off-diagonal asymmetries $\tilde{\Delta}_L$ and $\tilde{\Delta}_R$ can be defined as follows:

$$\text{Im} \left( \tilde{V}_{ai} \tilde{V}_{\beta j} \tilde{V}_{\alpha j}^* \tilde{V}_{\beta i}^* \right) = \tilde{\mathcal{J}} \sum_{\gamma, k} (\epsilon_{\alpha\beta\gamma} \epsilon_{ijk}) ,$$

(14)

where $(\alpha, \beta, \gamma)$ and $(i, j, k)$ run respectively over $(e, \mu, \tau)$ and $(1, 2, 3)$;

$$|\tilde{V}_{e2}|^2 - |\tilde{V}_{\mu1}|^2 = |\tilde{V}_{\mu3}|^2 - |\tilde{V}_{\tau2}|^2 = |\tilde{V}_{\tau1}|^2 - |\tilde{V}_{e3}|^2 \equiv \tilde{\Delta}_L ,$$

(15)

and

$$|\tilde{V}_{e2}|^2 - |\tilde{V}_{\mu3}|^2 = |\tilde{V}_{\mu1}|^2 - |\tilde{V}_{\tau2}|^2 = |\tilde{V}_{\tau3}|^2 - |\tilde{V}_{e1}|^2 \equiv \tilde{\Delta}_R .$$

(16)

It is then interesting to examine possible departures of $(\tilde{\mathcal{J}}, \tilde{\Delta}_L, \tilde{\Delta}_R)$ from $(\mathcal{J}, \Delta_L, \Delta_R)$ in a concrete experimental scenario.

To illustrate, we compute $\tilde{\mathcal{J}}$, $\tilde{\Delta}_L$ and $\tilde{\Delta}_R$ using the typical inputs $\theta_x = 40^\circ$, $\theta_z = 5^\circ$ and $\delta = 90^\circ$, which yield $\mathcal{J} = 0.021$, $\Delta_L = 0.166$ and $\Delta_R = 0$ in vacuum. The neutrino mass-squared differences are taken to be $\Delta m_{21}^2 = \Delta m_{\text{sun}}^2 = 5 \times 10^{-5} \text{ eV}^2$ and $\Delta m_{31}^2 = \Delta m_{\text{atm}}^2 = 3 \times 10^{-3} \text{ eV}^2$. The explicit formulas relevant to our calculation have been given in Ref. [14]. We plot the numerical dependence of $\tilde{\mathcal{J}} / \mathcal{J}$, $\tilde{\Delta}_L$ and $\tilde{\Delta}_R$ on the matter parameter $A$ in Fig. 4, where both the cases of neutrinos $(+A)$ and antineutrinos $(-A)$ are taken into account. One can see that the off-diagonal symmetry $\Delta_R = 0$ in vacuum can substantially be spoiled by terrestrial matter effects. The deviation of $\tilde{\Delta}_L$ from $\Delta_L$ and that of $\tilde{\mathcal{J}}$ from $\mathcal{J}$ are remarkably large, if $A > 10^{-5} \text{ eV}^2$.

As emphasized in Ref. [13], there exists the simple reversibility between the fundamental neutrino mixing parameters in vacuum and their effective counterparts in matter. The
former can therefore be expressed in terms of the latter, allowing more straightforward extraction of the genuine lepton mixing quantities (including $\mathcal{J}$, $\Delta_L$ and $\Delta_R$) from a variety of long-baseline neutrino oscillation experiments. If $|V_{\alpha i}|$ are determined to a very high degree of accuracy, it will be possible to test the unitarity of $V$ and establish leptonic $CP$ violation through the non-zero area of six unitarity triangles even in the absence of a direct measurement of $\mathcal{J}$.

V. FURTHER DISCUSSIONS

If $\Delta_R = 0$ really holds, one may wonder whether this off-diagonal symmetry of $V$ hints at very special textures of the neutrino mass matrix $M_{\nu}$ and (or) the charged lepton mass matrix $M_l$. In the following, we take two simple but instructive examples to illustrate possible implications of $\Delta_R = 0$ on $M_l$ and $M_{\nu}$.

Example A

In no conflict with current data on atmospheric, solar and reactor neutrino oscillations, a remarkably simplified form of $V$ with $\Delta_R = 0$ and $\delta = \rho = \sigma = 0$ is

$$V = \begin{pmatrix}
c & c & s \\
\frac{1 + s}{\sqrt{2}} & \frac{1 - s}{\sqrt{2}} & c \\
\frac{1 - s}{2} & \frac{1 + s}{2} & c
\end{pmatrix}, \quad (17)$$

where $s \equiv \sin \theta \ll 1$ and $c \equiv \cos \theta \approx 1$ [17]. In the flavor basis where $M_l$ is diagonal, $M_{\nu}$ can be given as

$$M_{\nu} = V \begin{pmatrix} m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3
\end{pmatrix} V^T = m_1 N_1 + m_2 N_2 + m_3 N_3, \quad (18)$$

where symmetric matrices $N_1$, $N_2$ and $N_3$ read

$$N_1 = \begin{pmatrix}
\frac{c^2}{2} & \frac{c(1 + s)}{2\sqrt{2}} & \frac{c(1 - s)}{2\sqrt{2}} \\
\frac{(1 + s)^2}{4} & -\frac{1 - s^2}{4} & \frac{(1 - s)^2}{4}
\end{pmatrix},$$

$$N_2 = \begin{pmatrix}
\frac{c^2}{2} & \frac{c(1 - s)}{2\sqrt{2}} & -\frac{c(1 + s)}{2\sqrt{2}} \\
\frac{(1 - s)^2}{4} & -\frac{1 - s^2}{4} & \frac{(1 + s)^2}{4}
\end{pmatrix},$$

$$N_3 = \begin{pmatrix}
\frac{c^2}{2} & \frac{c(1 - s)}{2\sqrt{2}} & \frac{c(1 + s)}{2\sqrt{2}} \\
\frac{(1 - s)^2}{4} & -\frac{1 - s^2}{4} & \frac{(1 + s)^2}{4}
\end{pmatrix}.$$
\[
N_3 = \begin{pmatrix}
  s^2 & \frac{cs}{\sqrt{2}} & \frac{cs}{\sqrt{2}} \\
  \frac{c^2}{2} & \frac{c^2}{2} & \frac{c^2}{2}
\end{pmatrix}
\]  

(19)

The texture of \( M_\nu \) is rather complicated, hence it is difficult to observe any hidden flavor symmetry associated with lepton mass matrices.

**Example B**

The result of \( M_\nu \) in Example A will become simpler, if \( s = 0 \) is further taken (i.e., \( V \) is of the bi-maximal mixing form \([18]\)). In this special case, however, simple textures of \( M_l \) and \( M_\nu \) can be written out in a more general flavor basis. It is easy to show that \( M_l \) and \( M_\nu \) of the following textures lead to \( V \) with \( s = 0 \):

\[
M_l = \frac{C_l}{2} \left[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \delta_l & 0 & 0 \\ 0 & \varepsilon_l & 0 \\ 0 & \varepsilon_l & 0 \end{pmatrix} \right],
\]

\[
M_\nu = \frac{C_\nu}{2} \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon_\nu & 0 \\ \varepsilon_\nu & 0 & 0 \\ 0 & 0 & \delta_\nu \end{pmatrix} \right],
\]

(20)

where \( \delta_{l,\nu} \) and \( \varepsilon_{l,\nu} \) are small perturbative parameters \([19]\). In the limit \( \delta_{l,\nu} = \varepsilon_{l,\nu} = 0 \), \( M_l \) has the \( S(2)_L \times S(2)_R \) symmetry and \( M_\nu \) displays the \( S(3) \) symmetry \([20]\). The perturbative corrections in \( M_l \) allow electron and muon to acquire their masses:

\[
\{m_e, m_\mu, m_\tau\} = \frac{C_l}{2} \{ |\delta_l|, |\varepsilon_l|, 2 + \varepsilon_l \}.
\]

(21)

We then arrive at \( C_l = m_\mu + m_\tau \approx 1.88 \) GeV, \( |\varepsilon_l| = 2m_\mu/(m_\mu + m_\tau) \approx 0.11 \) and \( |\delta_l| = 2m_e/(m_\mu + m_\tau) \approx 5.4 \times 10^{-4} \). The perturbative corrections in \( M_\nu \) make three neutrino masses non-degenerate:

\[
\{m_1, m_2, m_3\} = C_\nu \{1 + \varepsilon_\nu, 1 - \varepsilon_\nu, 1 + \delta_\nu \}.
\]

(22)

As a result, \( |\varepsilon_\nu|/|\delta_\nu| \approx \Delta m^2_{\text{sun}}/(2\Delta m^2_{\text{atm}}) \sim 10^{-2} \) for the large-angle MSW solution to the solar neutrino problem.

Examples A and B illustrate how the off-diagonal symmetry of \( V (\Delta_R = 0) \) can be reproduced from specific textures of lepton mass matrices.

**VI. SUMMARY**

In view of current experimental data on solar and atmospheric neutrino oscillations, we have discussed the geometrical structure of the \( 3 \times 3 \) lepton flavor mixing matrix \( V \). We find
that the present data strongly favor the off-diagonal symmetry of $V$ about its $V_{e3}-V_{\mu2}-V_{\tau1}$ axis. This symmetry, if really exists, will correspond to three pairs of congruent unitarity triangles in the complex plane. It remains too early to tell whether $V$ is symmetric or not about its $V_{e1}-V_{\mu2}-V_{\tau3}$ axis. A brief analysis of terrestrial matter effects on the universal $CP$-violating parameter and off-diagonal asymmetries of $V$ has also been made. We expect that future long-baseline experiments of neutrino oscillations can help establish the texture of the lepton flavor mixing matrix, from which one could get some insights into the underlying flavor symmetries responsible for the charged lepton and neutrino mass matrices.

ACKNOWLEDGMENTS

This work was supported in part by the National Natural Science Foundation of China.
REFERENCES

[1] Super-Kamiokande Collaboration, Y. Fukuda et al., Phys. Lett. B 467, 185 (1999); S. Fukuda et al., Phys. Rev. Lett. 85, 3999 (2000).
[2] Super-Kamiokande Collaboration, S. Fukuda et al., Phys. Rev. Lett. 86, 5651 (2001); Phys. Rev. Lett. 86, 5656 (2001); SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett 87, 071301 (2001).
[3] C. Jarlskog, Phys. Rev. Lett. 55, 1839 (1985).
[4] For a review with extensive references, see: P. Fisher, B. Kayser, and K.S. McFarland, Annu. Rev. Nucl. Part. Sci. 49, 481 (1999); M. Aoki et al., hep-ph/0112338.
[5] Z.Z. Xing, Phys. Rev. D 51, 3958 (1995); Nuovo Cim. A 109, 115 (1996); J. Phys. G 23, 717 (1997).
[6] H. Fritzsch and Z.Z. Xing, Prog. Part. Nucl. Phys. 45, 1 (2000).
[7] CHOOZ Collaboration, M. Apollonio et al., Phys. Lett. B 420, 397 (1998); Palo Verde Collaboration, F. Boehm et al., Phys. Rev. Lett. 84, 3764 (2000).
[8] S.P. Mikheyev and A.Yu. Smirnov, Yad. Fiz. (Sov. J. Nucl. Phys.) 42, 1441 (1985); L. Wolfenstein, Phys. Rev. D 17, 2369 (1978).
[9] G.L. Fogli, E. Lisi, D. Montanino, and A. Palazzo, Phys. Rev. D 64, 093007 (2001); J.N. Bahcall, M.C. Gonzalez-Garcia, and C. Pena-Garay, JHEP 0108, 014 (2001); P.I. Krastev and A.Yu. Smirnov, hep-ph/0108177; P. Aliani, V. Antonelli, M. Picariello, and E. Torrente-Lujan, hep-ph/0111418.
[10] Z.Z. Xing, hep-ph/0202034; to appear in Phys. Rev. D.
[11] M. Fukugita and M. Tanimoto, Phys. Lett. B 515, 30 (2001).
[12] H. Fritzsch and Z.Z. Xing, Phys. Lett. B 517, 363 (2001).
[13] V. Barger, K. Whisnant, S. Pakvasa, and R.J. Phillips, Phys. Rev. D 22, 2718 (1980); H.W. Zaglauer and K.H. Schwarzer, Z. Phys. C 40, 273 (1988); T.K. Kuo and J. Panteleone, Rev. Mod. Phys. 61, 937 (1989).
[14] Z.Z. Xing, Phys. Lett. B 487, 327 (2000); Phys. Rev. D 63, 073012 (2001).
[15] Z.Z. Xing, Phys. Rev. D 64, 073014 (2001).
[16] Y. Farzan and A.Yu. Smirnov, hep-ph/0201103.
[17] Z.Z. Xing, Phys. Rev. D 61, 057301 (2000); Phys. Rev. D 63, 057301 (2001).
[18] See, e.g., V. Barger, S. Pakvasa, T.J. Weiler, and K. Whisnant, Phys. Lett. B 437, 107 (1998); and references therein.
[19] H. Fritzsch and Z.Z. Xing, Phys. Lett. B 372, 265 (1996); Phys. Lett. B 440, 313 (1998); Phys. Rev. D 61, 073016 (2000).
[20] M. Fukugita, M. Tanimoto, and T. Yanagida, Phys. Rev. D 57, 4429 (1998); K. Kang and S.K. Kang, hep-ph/9802328.
FIG. 1. Dependence of $\sin^2 2\theta^{(+)}_{\text{sun}}$ on $\sin^2 2\theta_{\text{chz}}$, where $0.88 \leq \sin^2 2\theta_{\text{atm}} \leq 1.0$ has been input.
FIG. 2. Unitarity triangles of the lepton flavor mixing matrix in the complex plane. Each triangle is named by the Greek or Latin subscript that does not manifest in its three sides.
FIG. 3. Dependence of $\mathcal{J}_{(\pm)}/\sin\delta$ on $\sin^22\theta_{\text{chz}}$, where $0.88 \leq \sin^22\theta_{\text{atm}} \leq 1.0$ has been input.
FIG. 4. Illustrative plots for terrestrial matter effects on $\mathcal{F}$, $\Delta_L$ and $\Delta_R$ associated with neutrinos (+$A$) and antineutrinos (−$A$), where $\Delta m_{21}^2 = 5 \times 10^{-5} \text{ eV}^2$, $\Delta m_{31}^2 = 3 \times 10^{-3} \text{ eV}^2$, $\theta_x = 40^\circ$, $\theta_z = 5^\circ$ and $\delta = 90^\circ$ have typically been input.