Hawking Like radiation from the Dynamic horizon in Lemaitre-Tolman-Bondi Model of the Universe : Quantum Prescription

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Abstract

Recent progress of black hole thermodynamics due to Hawking radiation has been applied to universal thermodynamics with universe as inhomogeneous LTB model. Hawking-like temperature with quantum corrections has been evaluated using HJ method and the semiclassical part of the temperature coincides with that formulated in radial null geodesic method.

Keywords : Hawking-like Temperature, LTB model, Tunnelling of Particles.

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1 Introduction

Hawking radiation has been extensively discussed since its discovery\cite{1} and different approaches\cite{2,3,4} have been formulated to study it. Recently there is a lot of attraction to the newly developed semi classical tunnelling method\cite{5,6,7}. The results agree with those of Hamilton-Jacobi (HJ) method\cite{8,9} at least at the semiclassical level. In recent past a fully quantum mechanical description has been formulated by Banerjee et al\cite{10} using HJ formulation. They have shown logarithmic correction to the entropy functions in leading order to quantum formulation. Usually, Hawking radiation is analysed\cite{11,12,13,14} on static background space-times where there is a global horizon—the event horizon. But dynamical space-time, it is not definite to have an event horizon. Recently, Hayward et.al.\cite{15} presented a locally defined Hawking temperature for dynamical black holes by using tunnelling method. Subsequently, this approach has been used to universal thermodynamics particularly to the FRW universe\cite{16,17,18,19} with apparent horizon as the boundary. Also inhomogeneous model of the universe in LTB space time has been discussed recently\cite{20}.

In the present work we study Hawking like radiation from the dynamic horizon in LTB model of the universe using both the approaches. In section 2 Hawking like temperature has been evaluated using the radial null geodesic method (popularly known as tunnelling method). Then HJ approach has been used to formulated quantum prescription of the Hawking like temperature in section 3. An expression for entropy has been presented in section 4. At the end there is a brief summary of the present work in the section 5.

2 Semiclassical Radial Null Geodesic Method : Hawking like Temperature

The inhomogeneous spherically symmetric model of the universe is described by the Lemaitre-Tolman-Bondi model with metric ansatz

\[ ds^2 = -dt^2 + \frac{R^2}{1 + f(r)}dr^2 + R^2 d\Omega^2 \]

(1)
Here \((t, r, \theta, \phi)\) is an orthogonal co moving coordinate with \('t'\) the co-moving time corresponding to a co-moving observer, \(R = R(r, t)\) is the area radius of the spherical surfaces and \(\dot{t} = \frac{\partial}{\partial r}\) and \(\dot{r} = \frac{\partial}{\partial t}\). The scalar function \(f(r)(> -1)\) (known as curvature scalar) identifies the space time as

(i) bounded if \(-1 < f(r) < 0\)
(ii) marginally bounded, if, \(f(r) = 0\)
and
(iii) unbounded if \(f(r) > 0\).

It is possible to decompose the above space time metric into metric ansatz on the surface of the 2-sphere and on the 2D hyper surface normal to the 2-sphere as

\[
ds^2 = h_{ab}dx^a dx^b + R^2 d\Omega_2^2
\]  

(2)

where \(h_{ab} = diag\left(-1, \frac{R^2}{1 + f(r)}\right)\) is the 2D metric normal to the 2-sphere. The trapping horizon, a hyper-surface foliated by marginal spheres is defined by \[21\]

\[
\partial^+ R = 0, \text{ i.e., } \dot{R} = \sqrt{1 + f(r)}
\]  

(3)

where \(\partial^\pm = -\sqrt{2} \left(\partial_t \mp \sqrt{1 + f(r)} \partial_r\right)\) are null vectors normal to the 2-sphere. Thus the trapping horizon coincides with apparent horizon as in FRW space time.

To study the radial null geodesics, we transform our space time coordinates to painleve-type coordinates by the transformation \(r \rightarrow R(r, t)\) and the metric (1) now becomes

\[
ds^2 = -\left(1 - \frac{\dot{R}^2}{1 + f(r)}\right)dt^2 - 2 \frac{\dot{R}}{1 + f(r)} dtdR + \frac{dR^2}{1 + f(r)} + R^2 d\Omega_2^2
\]  

(4)

In this coordinate system the Kodama vector has the form \[22\]

\[
K = \left(\sqrt{1 + f(r)}, 0, 0, 0\right)
\]  

(5)

and the associated conserved energy is given by

\[
\omega = -\sqrt{1 + f(r)} \frac{\partial S}{\partial t}
\]  

(6)

with \(S\), the classical action of a massless particle. So \(\frac{\omega}{\sqrt{1 + f(r)}}\) can be interpreted as the energy of the particle as measured by an observer with the Kodama vector.

The differential equation for the radial null geodesic (i.e., \(ds^2 = 0 = d\Omega_2^2\)) can be written as

\[
\frac{dR}{dt} = \dot{R} \pm \sqrt{1 + f(r)}
\]  

(7)

where as usual \(+/-\) sign stands for outgoing/incoming null geodesic. As for the universal model we consider tunnelling from the outside to the inside of the horizon so we consider only the incoming geodesics. According to Parikh and Wilczek \[\text{[5]}\], the tunnelling probability is characterized by the imaginary part of the action corresponding to tunnelling of particles through a barrier (i.e., the classically forbidden region) and we have

\[
Im S = Im \int_{R_{in}}^{R_{out}} p_R dR = Im \int_{R_{in}}^{R_{out}} dR \int_{0}^{\dot{p}_R} dp_R = Im \int_{R_{in}}^{R_{out}} dR \int_{0}^{E} \frac{dH'}{R}
\]  

(8)

where to obtain the last equality we have used the Hamiltonian equation namely,

\[
\dot{R} = \frac{\partial H}{\partial p_R} = \frac{dH}{dp_R} |_R
\]

(9)

In the above equalities we denote by \(p_R\) the radial momentum of the tunnelling particle, \(R_{in}\) and \(R_{out}\) are positions very close to the horizon with \(R_{in}\) the initial position and \(R_{out}\), a classical turning point. Now substituting the value of \(\dot{R}\) from the equation (7) into (9) we have

\[
Im S = Im \int_{R_{in}}^{R_{out}} dR \frac{\omega}{R \sqrt{1 + f(r)}}
\]
Thus the tunnelling probability $\Gamma \sim \exp \left\{ -\frac{2}{\hbar} Im S \right\}$ can be compared with the Boltzmann factor $\exp \left\{ -\frac{\omega}{T} \right\}$ to obtain the temperature of the trapping (apparent) horizon as

$$T = \frac{\hbar}{2} \left[ Im \int_{R_{in}}^{R_{out}} \frac{dR}{\sqrt{1 + f(r)} \left( R - \sqrt{1 + f(r)} \right)} \right]^{-1}.$$  

(10)

This is the semiclassical Hawking like temperature (except for a factor of $\frac{1}{2}$) of the inhomogeneous LTB universe for tunnelling of massless particles across the trapping (apparent) horizon. This problem of 'two' factor has been pointed out in [10, 20] and it is mainly of the following two facts: Firstly, $Im \int_{R_{in}}^{R_{out}} p_{R} dR$ is not canonically invariant (i.e., not a paper observable) and secondly, the canonically invariant quantity $Im \int_{R_{in}}^{R_{out}} p_{R} dR$ is normally equivalent to $2 Im \int_{R_{in}}^{R_{out}} p_{R} dR$, because for ordinary tunnelling the amplitude of tunnelling is same for left $\leftrightarrow$ right, but in our case the trapping horizon is an one way membrane and the amplitude for left $\leftrightarrow$ right are not equivalent. However, in the following section, the HJ method is free from this discrepancy we shall have correct Hawking-like temperature.

3 Hamiltonian-Jacobi approach: quantum corrections

Due to non-static nature of the LTB model there is no time like killing vector, but the Kodama vector [22]

$$K^\mu = \left( \sqrt{1 + f(r)}, - \frac{\dot{R}}{R} \sqrt{1 + f(r)}, 0, 0 \right)$$

which is time like inside the horizon, takes the role of the time like Killing vector as in stationary black hole space time. Accordingly, there is a conserved quantity of a particle moving in this space time as

$$\omega = -\sqrt{1 + f(r)} \frac{\partial S_0}{\partial t} + \frac{\dot{R}}{R} \sqrt{1 + f(r)} \frac{\partial S_0}{\partial t}$$

(12)

where $S_0$ is the classical action of a massless particle. So $\omega \frac{\partial}{\partial \sqrt{1 + f(r)}}$ is the energy of the particle as measured by an observer with the Kodama vector. We start with the Klein-Gordon(KG) equation

$$- \frac{\hbar^2}{\sqrt{-g}} \partial_{\mu} \left( g^{\mu\nu} \sqrt{-g} \partial_{\nu} \right) \psi = 0$$

(13)

where the scalar field $\psi$ describes a massless scalar particle. As we concentrate our attention to radial geodesics only and further due to spherical symmetric nature of the space-time so it is sufficient to consider the above KG equation in the 2D hyperplane $(t, r)$ for which the metric $h_{ab}$ is given in equation (2). Thus the explicit form of the above wave equation is

$$\frac{\partial^2 \psi}{\partial t^2} - \left( 1 + f(r) \right) \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{2} \frac{\partial}{\partial r} \left( 1 + f(r) \right) \frac{\partial \psi}{\partial r} - \frac{1}{2} \frac{R^2}{1 + f(r)} \frac{\partial}{\partial t} \left( 1 + f(r) \right) \frac{\partial \psi}{\partial t} = 0$$

(14)

Now writing

$$\psi(r, t) = \exp \left\{ \frac{i}{\hbar} S(r, t) \right\}$$

(15)

we obtain the differential equation for $S$ as

$$\frac{\partial S}{\partial t} \left( \frac{\partial S}{\partial t} \right) - \left( 1 + f(r) \right) \frac{\partial S}{\partial r} \left( \frac{\partial S}{\partial r} \right) \frac{R^2}{2(1 + f(r))} \frac{\partial}{\partial t} \left( 1 + f(r) \right) \frac{\partial S}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r} \left( 1 + f(r) \right) \frac{\partial S}{\partial r} = 0$$

(16)
we now try to solve this complicated partial differential equation (p.d.e.) in perturbative approach with Planck’s constant as the perturbation parameter, unperturbed term as semiclassical approximation and terms in different powers of \( \hbar \) correspond to different order quantum corrections. So we write

\[
S(r, t) = S_0(r, t) + \Sigma_k \hbar^k S_k(r, t)
\]

with \( k \), a positive integer. If we plug this action ansatz into p.d.e. \( (16) \) and equate equal powers of \( \hbar \) on both sides then we have the following set of p.d.e.s :

\[
h^0 = (\frac{\partial S_0}{\partial t})^2 - \left( \frac{1 + f(r)}{R^2} \right) \left( \frac{\partial S_0}{\partial r} \right)^2 = 0
\]

\[
h^1 = 2 \frac{\partial S_0}{\partial t} \frac{\partial S_1}{\partial t} - \left( \frac{1 + f(r)}{R^2} \right) \frac{\partial S_0}{\partial r} \frac{\partial S_1}{\partial r}
+ i \left[ \frac{\partial^2 S_0}{\partial t^2} - \left( \frac{1 + f(r)}{R^2} \right) \frac{\partial^2 S_0}{\partial r^2} - \frac{R^2}{2(1 + f(r))} \frac{\partial}{\partial t} \left( \frac{1 + f(r)}{R^2} \right) \frac{\partial S_0}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r} \left( \frac{1 + f(r)}{R^2} \right) \frac{\partial S_0}{\partial r} \right] = 0
\]

\[
h^2 = (\frac{\partial S_1}{\partial t})^2 + 2 \frac{\partial S_0}{\partial t} \frac{\partial S_2}{\partial t} - \left( \frac{1 + f(r)}{R^2} \right) \left( \frac{\partial S_1}{\partial r} \right)^2 + 2 \frac{\partial S_0}{\partial r} \frac{\partial S_2}{\partial r}
+ i \left[ \frac{\partial^2 S_1}{\partial t^2} - \left( \frac{1 + f(r)}{R^2} \right) \frac{\partial^2 S_1}{\partial r^2} - \frac{R^2}{2(1 + f(r))} \frac{\partial}{\partial t} \left( \frac{1 + f(r)}{R^2} \right) \frac{\partial S_1}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r} \left( \frac{1 + f(r)}{R^2} \right) \frac{\partial S_1}{\partial r} \right] = 0
\]

and so on.

At a glance it seems that different order p.d.e.s are very complicated, but surprisingly there is lot of simplifications by using previous equations of the set. Finally, we obtain exactly same p.d.e for \( S_0 \) as well as for \( S_k \), \( k = 0, 1, 2, \ldots \) i.e.,

\[
(\frac{\partial S_0}{\partial t})^2 - \left( \frac{1 + f(r)}{R^2} \right) \left( \frac{\partial S_0}{\partial r} \right)^2 = 0
\]

Thus different order quantum corrections are not independent, they are proportional to the semiclassical action \( S_0 \). The proportionality constants can be determined from the dimensional analysis as follows : In the choice of units \( G = c = K_p = 1 \), the plank constant \( \hbar \) has the dimension \( M_p^2 \) (\( M_p \) = Planck mass), so \( S_k \) has the dimension \( \hbar^{-K} \equiv M^{-2k} \), where \( M \) is identified as the mass of the universe. So we write

\[
S_k = \alpha_k M^{-2k} S_0.
\]

where \( \alpha_k \)s are dimensionless constant parameters.

As a result the series \( (17) \) now becomes

\[
S(r, t) = S_0(r, t) \left[ 1 + \Sigma_k \alpha_k \left( \frac{\hbar}{M^2} \right)^k \right]
\]

Thus solution of \( S_0 \) from equation \( (18) \) gives the complete solution for the action \( S \). Using \( (12) \) and \( (18) \) the solution for \( S_0 \) can be written in the integral form as

\[
S_0(r, t) = - \int_{\sqrt{1+f} - \bar{R}}^{\sqrt{1+f} + \bar{R}} \frac{\omega dt}{\sqrt{1+f} - \bar{R}} + \int_{\sqrt{1+f} - \bar{R}}^{\sqrt{1+f} + \bar{R}} \frac{R' dr}{\sqrt{1+f} \sqrt{1+f} - \bar{R}}
\]

where \(-/+\) sign corresponds to ingoing/outgoing scalar particle and the corresponding wave functions are given by (using \( (13) \))

\[
\psi_{in} = \exp \left[ \frac{i}{\hbar} \left( 1 + \Sigma_k \alpha_k \left( \frac{\hbar}{M^2} \right)^k \right) \left\{ \int_{\sqrt{1+f} - \bar{R}}^{\sqrt{1+f} + \bar{R}} \frac{\omega dt}{\sqrt{1+f} - \bar{R}} + \int_{\sqrt{1+f} - \bar{R}}^{\sqrt{1+f} + \bar{R}} \frac{R' dr}{\sqrt{1+f} \sqrt{1+f} - \bar{R}} \right\} \right]
\]
At this point it is to be noted that in crossing the horizon the matrix \( h_{ab} \) of the metric corresponding to 2D hyperplane \(- (r, t)\) becomes \(- h_{ab}\). i.e., the metric coefficient \( g_{tt} \) and \( g_{rr} \) change their signs, and consequently, the above time integration may have imaginary part and hence contribute to the probabilities for the incoming and outgoing particles. So the probabilities are given by

\[
P_{\text{in}} = |\psi_{\text{in}}|^2 = \exp \left[ \frac{i}{\hbar} \left\{ 1 + \Sigma_k \alpha_k \left( \frac{\hbar}{M^2} \right)^k \right\} \left\{ \int \frac{\omega dt}{\sqrt{1 + f - \dot{R}}} - \omega \int \frac{R' dr}{\sqrt{1 + f \{ \sqrt{1 + f - \dot{R}} \}} \} \right\} \right] \quad (26)
\]

and

\[
P_{\text{out}} = |\psi_{\text{out}}|^2 = \exp \left[ \frac{2}{\hbar} \left\{ 1 + \Sigma_k \alpha_k \left( \frac{\hbar}{M^2} \right)^k \right\} \left\{ \int \frac{\omega dt}{\sqrt{1 + f - \dot{R}}} + \omega \int \frac{R' dr}{\sqrt{1 + f \{ \sqrt{1 + f - \dot{R}} \}} \} \right\} \right]
\] \quad (27)

In the classical domain the horizon is no longer a barrier (absorber in this case) and everything can go out so we must have \( \lim_{\hbar \to 0} P_{\text{out}} = 1 \), which gives

\[
\int \frac{\omega dt}{\sqrt{1 + f - \dot{R}}} = \omega \int \frac{R' dr}{\sqrt{1 + f \{ \sqrt{1 + f - \dot{R}} \}} \} \quad (29)
\]

and as a result \( P_{\text{in}} \) simplifies to

\[
P_{\text{in}} = \exp \left[ \frac{4 \omega}{\hbar} \left\{ 1 + \Sigma_k \alpha_k \left( \frac{\hbar}{M^2} \right)^k \right\} \int \frac{R' dr}{\sqrt{1 + f \{ \sqrt{1 + f - \dot{R}} \}} \} \right]
\] \quad (30)

so from the principle of 'detailed balance' we have,

\[
P_{\text{out}} = \exp \left\{ - \frac{\omega}{T_h} \right\} P_{\text{in}}
\] \quad (31)

and comparing with (30) we obtain the Hawking like temperature with quantum corrections as

\[
T_h = \frac{\hbar}{4} \left\{ 1 + \Sigma_k \alpha_k \left( \frac{\hbar}{M^2} \right)^k \right\}^{-1} \left\{ \int \frac{R' dr}{\sqrt{1 + f \{ \sqrt{1 + f - \dot{R}} \}} \} \right\}^{-1} = \left\{ 1 + \Sigma_k \alpha_k \left( \frac{\hbar}{M^2} \right)^k \right\}^{-1} T_c
\] \quad (32)

where

\[
T_c = \frac{\hbar}{4} \left\{ \int \frac{R' dr}{\sqrt{(1 + f(r))\{ \dot{R} - \sqrt{1 + f(r)} \}} \} \right\}
\]

is the semi-classical Hawking-like temperature at the trapping horizon of the LTB model.

Thus the quantum corrections appear as a multiplicative factor to the semiclassical temperature. Different order quantum corrections appear additively as an infinite series. The parameters \( \alpha_k \) are chosen in such a manner that the infinite series converges.

### 4 Entropy of the horizon with quantum corrections

We shall now express the entropy of the trapping horizon from the thermodynamical law namely,

\[
dM = T_h dS_h
\]

\[\boxed{5}\]
i.e. 

$$S_h = \int \frac{dM}{T_h} \quad (33)$$

where $M$ is identified as the Mishner-Sharp mass \[20\]

$$M = \frac{R}{2}.$$ 

Now using (32) in (33) we have the quantum correction entropy of the LTB universe bounded by the trapping horizon as

$$S_h = \frac{2}{\hbar} \int dR \left\{ 1 + \sum_k \alpha'_k \left( \frac{\hbar}{R^2} \right)^k \right\} \left\{ \text{Im} \int \frac{R'dr}{\sqrt{1 + f(r)(R - \sqrt{1 + f(r)})}} \right\} \quad (34)$$

Due to inhomogeneity of the space-time, one cannot evaluate the integrations, so it is not possible to derive the semiclassical Bekenstein-Hawking entropy formula from the above expression. Also we cannot predict whether the leading order quantum correction gives logarithmic contribution or not. However, the above integrand can be simplified to some extent by choosing $\alpha'_k = \alpha^k$ then the summation term can be written in compact form as $(1 - \frac{\alpha h}{R^2})^{-1}$ and hence the form of $S_h$ simplifies to

$$S_h = \frac{2}{\hbar} \int \frac{dR}{1 - \frac{\alpha h}{R^2}} \left\{ \text{Im} \int \frac{R'dr}{\sqrt{1 + f(r)(R - \sqrt{1 + f(r)})}} \right\} \quad (35)$$

It should be mentioned here that such choice of the proportionality parameter $\alpha'_k$ was used by Banerjee et al \[10\] in deriving one loop back reaction effects in the space-time for static black holes.

## 5 Summary

In the present work we have formulated a quantum prescription of the Hawking like radiation from the trapping horizon of the inhomogeneous LTB model of the universe. Due to non-static nature of the space-time we have used Kodama observer and the energy of the particle is associated with the corresponding invariant energy. The outgoing and incoming probabilities are matched using the principle of detailed balance. The semiclassical part of the quantum corrected temperature using HJ approach is compared with that obtained from tunnelling approach where there is a discrepancy of a factor “two”. Finally, an expression for the horizon entropy has been presented and it is not possible to deduce the standard entropy-area formula at the semi classical level due to its complicated form. For future work, we shall attempt to resolve the factor two discrepancy for non-static metric and try to find a physical interpretation to the different order quantum correction terms.

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