Unifying structures in quantum integrable systems

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Abstract

Basic concepts of quantum integrable systems (QIS) are presented stressing on the unifying structures underlying such diverse models. Variety of ultralocal and nonultralocal models is shown to be described by a few basic relations defining novel algebraic entries. Such properties can generate and classify integrable models systematically and also help to solve exactly their eigenvalue problem in an almost model-independent way. The unifying thread stretches also beyond the QIS to establish its deep connections with statistical models, conformal field theory etc. as well as with abstract mathematical objects like quantum group, braided or quadratic algebra.

1 Introduction

A number of quantum models of diverse nature shows the important property of integrability in low dimensions, enabling us to solve them exactly. Examples of such models may extend from field models like sine-Gordon (SG), nonlinear Schrödinger equation (NLS), derivative NLS (DNLS) etc. to the discrete models like Toda chain (TC), relativistic TC, isotropic (XXX) or anisotropic spin chains (XXZ, XYZ) etc. Similarly they may have varied basic commutation relations ranging from ultralocal to nonultralocal models. However, the fascinating feature of such models, as revealed in recent years, is their common underlying algebraic structures defined through a basic relation called Yang-Baxter equation. The unifying spirit goes also beyond the domain of the quantum integrable systems (QIS) to reveal its deep relations with variety of other disciplines, which were thought to be completely unrelated in the recent past. Thus in one hand the QIS is deeply connected with abstract mathematical objects like quantum group, noncocommutative Hopf algebras,

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quadratic and braided algebra, universal $R$-matrix etc. and on the other hand it has intimate relation with stat-mech problems, conformal field theory (CFT), knot and braid theories etc.

The starting of QIS-age should possibly be counted from 1931, when Hans Bethe was the first to solve exactly an interacting many-body quantum problem, e.g. the isotropic Heisenberg spin chain, through his by now celebrated ansatz called the Bethe ansatz [1]. This was followed by many other applications and successes of the method in solving various physically interesting models [2] like $XXZ$-spin chain [2], many-body problem with attractive [3] or repulsive [4] $\delta$-function interactions, Hubbard model [5], Kondo problem [6] etc. Next came the landmark discovery of Rodney Baxter, in which he not only solved the completely anisotropic $XYZ$ spin chain but also established its deep link with a stat-mech problem, e.g. the eight-vertex model [2]. However, the genuine theory of QIS based on all the preceding achievements in quantum as well as classical integrable theories was started taking shape only in the late seventies [7][8]. Synthesising the concept of the Lax-operator, borrowed from the classical theory, along with the Bethe ansatz the quantum inverse scattering method (QISM) was developed mainly through the works of the Russian school, where the celebrated Yang-Baxter equation plays the central role and an algebraic formulation of the Bethe ansatz was invented for covering wider range of models including field theoretic models [1]. The connection of the integrable system with the conformal field theory (CFT) [12][13] as well as with the quantum group [14] and the braid and knot theory [15] was understood quite recently. Thanks to this ever-growing progress, today the theory of QIS is considered to be a major and important branch of theoretical and mathematical physics, with many research groups all over the globe engaged in active research in the field.

We consider the notion of integrability in the Liouville sense, i.e. call a system integrable, when the number of independent conserved quantities coincides with the degree of freedom of the system. In the quantum case they correspond to conserved operators with one being the Hamiltonian, commuting with them. Moreover, their independence demands that all of them must form a mutually commuting set of operators, which in field models are infinite in number. We will see that the quantum Yang-Baxter equation, which plays the role of the basic equation in QIS, ensures in fact such a property. Though in physics we are interested mainly in the Hamiltonian and its eigenvalue solutions, a completely integrable system actually guarantees much more. it should in principle give the whole hierarchy of conserved operators along with their exact eigenvalue solutions, which is one of the main aims of the QISM.

We present here the basic concepts of QISM with a major stress on the unifying scheme for both ultralocal and nonultralocal quantum models. We elaborate on the fascinating and novel algebraic structures revealed by the theory, which help to generate and solve such models in a systematic way. We also focus on the intriguing connections of the theory with the stat-mech and CFT models. The section headings explain the topics dealt in each of them.
2 Lax operator and examples of integrable systems

The central idea of classical ISM [16] is that, instead of dealing with the nonlinear equation in (1 + 1)-dimensions directly, it constructs the corresponding linear scattering problem
\[ T(x, \lambda) = L(u(x, t), p(x, t), \lambda) \mathcal{T}(x, \lambda), \]
where the Lax operator \( L(u, p, \lambda) \) depending on the fields \( u, p \) and the spectral parameter \( \lambda \) contains all information about the original nonlinear system and may serve therefore as the representative of a concrete model. The field \( u \) acts as the scattering potential. The aim of ISM is to solve the inverse problem by first finding a canonical mapping from the spectral data, which are like action-angle variables, to the scattering potential, i.e. the original field and using it to construct the exact solution for the given nonlinear equation. Soliton is a special solution, which corresponds to the reflection-less potential.

The QISM succeeds to generalise the notion of the Lax operator also to the quantum case. Since the canonical variables \( u, p \) or \( \psi, \psi^\dagger \) etc. now become operators acting on the Hilbert space, the quantum \( L(\lambda) \)-operators are unusual matrices with non-commuting matrix elements. This intriguing feature leads to nontrivial underlying algebraic structures in QIS. We present below a few concrete examples of the Lax operators associated with well known models to give an idea about the structure of this immensely important object in the integrable theory. The field models are represented by field equations and the corresponding continuum Lax operator \( \mathcal{L} \), while the lattice models are given through the Hamiltonian \( H \) and the discrete Lax operator \( L \).

I. Trigonometric Class:
1. Sine-Gordon (SG) model
\[ u(x, t)_{tt} - u(x, t)_{xx} = \frac{m^2}{\eta} \sin(\eta u(x, t)), \quad \mathcal{L}_{SG} = \left( \begin{array}{cc} ip, & m \sin(\lambda - \eta u) \\ m \sin(\lambda + \eta u), & -ip \end{array} \right), \quad p = \dot{u} \] (2.2)

2. Liouville model (LM)
\[ u(x, t)_{tt} - u(x, t)_{xx} = \frac{1}{2} e^{2\eta u(x, t)}, \quad \mathcal{L}_{LM} = i \left( \begin{array}{cc} p, & \xi e^{\eta u} \\ \frac{1}{\xi} e^{\eta u}, & -p \end{array} \right). \] (2.3)

3. Anisotropic XXZ spin chain
\[ H = \sum_n (\sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \cos \eta \sigma_n^3 \sigma_{n+1}^3), \]
\[ L_n(\xi) = i \left( \begin{array}{cc} \sin(\lambda + \frac{\eta}{2} \sigma_n^3), & \sin \eta \sigma_n^3 \\ \sin \eta \sigma_n^3, & \sin(\lambda - \frac{\eta}{2} \sigma_n^3) \end{array} \right). \] (2.4)

II. Rational Class:
1. Isotropic XXX spin chain

\[ H = \sum_n \sum_a \left( \sigma^a_n \sigma^a_{n+1} \right) \]  
\[ L_n(\xi) = i \left( \begin{array}{cc} \lambda + \frac{1}{2} \sigma^3_n & \sigma^-_n \\ \sigma^+_n & \lambda - \sigma^3_n \end{array} \right) \]  
(2.5)

2. Nonlinear Schrödinger equation (NLS)

\[ i\psi_t(x,t) + \psi_{xx} + \eta(\psi^\dagger(x,t)\psi(x,t))\psi(x,t) = 0, \quad \mathcal{L}_{NLS}(\lambda) = \left( \begin{array}{cc} \lambda & \eta \psi^\dagger \\ \eta^\dagger \psi, & -\lambda \end{array} \right). \]  
(2.6)

3. Toda chain (TC)

\[ H = \sum_i \left( \frac{1}{2} p_i^2 + e^{(u_i - u_{i+1})} \right), \quad L_n(\lambda) = \left( \begin{array}{cc} p_n - \lambda & e^{u_n} \\ -e^{-u_n} & 0 \end{array} \right). \]  
(2.7)

Notice that we have put the models under different classes and apparently diverse looking models into the same class. The meaning of this will be understood as we go further, though it might have already sent a signal of the fascinating unifying feature in the integrable systems. Observing carefully we also see that the off-diagonal elements (as \( \psi, \psi^\dagger \) in (2.6) and \( \sigma^-, \sigma^+ \) in (2.4)) involve creation and annihilation operators while the diagonal terms are the number like operators. It is crucial to note that under matrix multiplication this property is preserved, which as we will see below, has important consequences in their algebraic Bethe ansatz solution.

3 Yang-Baxter equation, \( R \)-matrix and notion of quantum integrability

As we see, the Lax operators are local functions of \( x \), or if we discretise the space of every lattice site \( i \). However, since the integrability is defined through the conserved quantities, which are global objects, in proving integrability we evidently need some global entries constructed from the local Lax operators. Such an object can be formed by matrix multiplication of the Lax operators at all lattice sites as

\[ T(\lambda) = \prod_{i=1}^N L_i(\lambda) = \left( \begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array} \right). \]  
(3.1)

As we have indicated above, the off-diagonal global operators \( B(\lambda), C(\lambda) \) are like creation/annihilation operators and related to the angle-variables, while \( A(\lambda), D(\lambda) \) correspond to action variables. For ensuring integrability one must show that \( \tau(\lambda) = trT(\lambda) = A(\lambda) + D(\lambda) \) generates the conserved operators : \( \ln \tau(\lambda) = \sum_j C_j \lambda^j \) with \( [H, C_m] = 0 \) and mutual commutation: \( [C_n, C_m] = 0 \). This is in fact achieved by a key condition on the quantum Lax operators (for ultralocal models) given by

\[ R_{12}(\lambda - \mu) L_{1i}(\lambda) L_{2i}(\mu) = L_{2i}(\mu) L_{1i}(\lambda) R_{12}(\lambda - \mu). \]  
(3.2)
with the notations \( L_{1i} = L_i \otimes I, \ L_{2i} = I \otimes L_i \). This matrix relation is known as the Quantum Yang–Baxter equation (QYBE), where apart from \( L \)-operator \( R_{12}(\lambda, \mu) \)-matrix with \( c \)-number functions of spectral parameters and acting nontrivially only in the first two spaces \( V \otimes V \otimes I \) appears, which in turn satisfies the YBE

\[
R_{12}(\lambda - \mu) \ R_{13}(\lambda - \gamma) \ R_{23}(\mu - \gamma) = \ R_{23}(\mu - \gamma) \ R_{13}(\lambda - \gamma) \ R_{12}(\lambda - \mu).
\]

(3.3)

Due to an additional ultralocal property of the \( L \)-operators: \([L_{1i}(\lambda), L_{2j}(\mu)] = 0, i \neq j\) of a large class of models, one can treat these 'specially separated' operators almost as classical objects. Therefore multiplying them for all sites and repeatedly using (3.2) one may arrive at the same QYBE

\[
R_{12}(\lambda - \mu) \ T_1(\lambda) \ T_2(\mu) = \ T_2(\mu) \ T_1(\lambda) \ R_{12}(\lambda - \mu),
\]

(3.4)

but for the global object \( T(\lambda) \) defined in (3.1). This invariance relation for the tensorial product reflects a deep algebraic property related to the Hopf algebra, to which we will return later. Taking now the trace of relation (3.4), (since under the trace \( R \)-matrices can rotate cyclically and thus cancel out) one gets \([\tau(\lambda), \tau(\mu)] = 0\), proving the commutativity of \( C_n \) for different \( n \)'s. Thus starting from the local QYBE (3.2) and following several logical steps we finally establish the complete integrability of the quantum system. Therefore the validity of (3.2) may be considered to be sufficient for the quantum integrability of the ultralocal systems and the local QYBE as the basic equation in the QIS. We will show below that even for nonultralocal models one can formulate the local QYBE in some generalised form and prove the integrability, though the procedure is much more involved. The \( 4 \times 4 \) \( R(\lambda) \)-matrix solution of YBE (3.3) is rather easy to find, which in the simplest form may be given as

\[
R(\lambda) = \begin{pmatrix}
    1 & f_1(\lambda) \\
    f_1(\lambda) & 1 \\
    f(\lambda) & 1 \\
    f(\lambda) & f(\lambda)
\end{pmatrix}.
\]

(3.5)

There are usually only two types of solutions (we shall not speak here of more general elliptic solutions), namely trigonometric with

\[
f(\lambda) = \frac{\sin(\lambda + \eta)}{\sin \lambda}, \quad f_1(\lambda) = \frac{\sin \eta}{\sin \lambda}
\]

and the rational with

\[
f(\lambda) = \frac{\lambda + \eta}{\lambda}, \quad f_1(\lambda) = \frac{\eta}{\lambda}.
\]

(3.7)

In the examples of integrable systems the trigonometric and rational classes mentioned are associated with the respective \( R \)-matrices presented here.

4 Universality in underlying algebraic structure

In sect. 2 we have given the Lax operators for a number of models and grouped some like SG, LM and XXZ in the same trigonometric class, while equally varied models like NLS,
TC, XXX etc. are in the same rational class. Here we address to the intriguing question that why a wide range of diverse models share the same $R(\lambda)$ and construct also their Lax operators exploring the universality feature underlying such systems.

To generate models of the trigonometric class we start with the $R_{\text{trig}}$-matrix solution (3.6) and look for the the Lax operator of the corresponding ancestor model in a generalised form

$$L_t(\lambda) = i\left( \frac{\sin(\lambda + \eta s^3)(c_1^+ + c_1^-) - i\cos(\lambda + \eta s^3)(c_1^+ - c_1^-),}{(2\sin\eta)S^+}, \quad \frac{(2\sin\eta)S^-}{\sin(\lambda - \eta s^3)}(c_2^+ + c_2^-) - i\cos(\lambda + \eta s^3)(c_2^+ - c_2^-) \right), \quad (4.1)$$

The abstract operators $s^3, S^\pm$ describe an algebra given by the following relations

$$[s^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = -\frac{1}{2} \left( M^+[2s^3]_q + \frac{M^-}{\sin\eta} \cos(2\eta s^3) \right). \quad (4.2)$$

where $[x]_q = \frac{q^x-q^{-x}}{q-q^{-1}} = \frac{\sin(\eta x)}{\sin\eta}$, $q = e^{i\eta}$ and $M^\pm$ are related to the central elements of the algebra: $c_1^\pm, i = 1, 2$ as $M^\pm = c_1^c c_2^\pm + c_1^c c_2^\pm$. Note that (4.2) is a new type of algebra underlying such QIS. This is in fact a quadratic algebra and unlike the known Lie algebra or its deformations there are arbitrary multiplicative central elements $M^\pm$ in the RHS of (4.2), which can take trivial values. For different values of these elements one gets different kinds of the algebra along with different types of the Lax operators. Therefore starting from this universal structure of the ancestor model (4.1) and considering various representation of (4.2) with concrete choices for $c_i^\pm$, one can generate systematically the Lax operators of diverse models belonging to this class.

Remarkably, for the choice $M^- = 0, M^+ = -2$ one gets the well known quantum algebra (QA) $U_q(su(2))$ with the defining relations

$$[s^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = [2s^3]_q \quad (4.3)$$

Let us consider first the models related directly to this QA with the Lax operators obtained from (4.1) for the choice $c_1^+ = c_2^+ = c_1^- = c_2^- = 1$. The fundamental representation

$$S^\pm = \frac{1}{2} \sigma^\pm, \quad s^3 = \frac{1}{2} \sigma^3 \quad (4.4)$$

constructs clearly the Lax operator of the spin-$\frac{1}{2}$ XXZ-chain (2.4). Another nontrivial example obtained through the bosonic representation ( with $[u_n, p_m] = \frac{i\Lambda}{2}\delta_{nm}$) of QA:

$$s_n^3 = u_n, \quad S_n^- = g(u_n)e^{i\Delta p_n}, \quad S_n^+ = (S_n^-)^\dagger, \quad (4.5)$$

where $g(u_n) = [1 + \frac{i}{2} m^2 \Delta^2 \cos 2\eta (u_n + \frac{1}{2})]^{\frac{1}{2}}$ yields an exact lattice version of the sine-Gordon model. The Lax operator of the model can be derived readily from (4.1) using (4.5). At $\Delta \to 0$ one gets the SG field model with the Lax operator obtained as $\sigma^1 L_n = I + \Delta L_{SG}(x) + O(\Delta)$ reproducing (2.2).

We now go beyond QA and consider more general structure (1.2). Note that the choice $c_1^+ = c_2 = \Delta, \ c_1^- = c_2^+ = 0$ giving $M^+ = M^- = \Delta^2$ corresponds to an intriguing algebra with $[S^+, S^-] = i\Delta q_3^{2s^3}. \ A$ bosonic realisation of this algebra as

$$s_n^3 = u_n, \quad S_n^+ = f(u_n)e^{i\Delta p_n}, \quad S_n^- = (S_n^-)^\dagger, \quad (4.6)$$
with \( f(u) = \left(1 + \Delta^2 e^{i\eta(2u+i\hbar)}\right)^{\frac{1}{2}} \) constructs again from the same \((4.1)\) the Lax operator of an exact lattice Liouville model and at \( \Delta \to 0 \) the Liouville field model \((2.3)\). We have witnessed here an important consequence of the the underlying algebraic structure, from which through simple realisations we could generate exact lattice versions of SG and LM, which were originally found in an intuitive and much involved way \([38]\) \([37]\).

We have shown thus that the Lax operators of the models, at least for the trigonometric class presented in sect. 2, can be constructed systematically and the set of diverse models put under the same class are in fact descendants of the same ancestor model \((4.1)\). We continue this procedure to generate other members of this class like the quantum Derivative NLS (DNLS), Ablowitz-Ladik model, relativistic Toda chain etc. as different realisations of the unifying algebra \((4.2)\). For example, at

\[
c_1^+ = c_2^+ = 1, \quad c_1^- = i\frac{\Delta}{4} \bar{q}, \quad c_2^- = i\frac{\Delta}{4} q
\]

(a) corresponds to

\[
M_+ = \Delta^2 \sin \eta, \quad M_- = i\frac{\Delta^2}{2} \cos \eta
\]

(b) after the substitution

\[
S^+ = \kappa A, \quad S^- = \kappa A^\dagger, \quad s^3 = N, \quad \kappa = -i(\cot \eta)^{\frac{1}{2}}
\]

leads directly to the well known \(q\)-oscillator algebra

\[
[A, A^\dagger] = \Delta \cos \eta \cos(\eta(2N + 1)), \quad [A, N] = A, \quad [A^\dagger, N] = -A^\dagger
\]

and therefore to a novel integrable \(q\)-oscillator model with explicit \(L\)-operator and the \(R\) \(trig\) matrix \((3.6)\). A realisation of the \(q\)-oscillators in bosonic operators: \([\psi_n, \psi^\dagger_m]\) \(\bar{\hbar}\Delta \delta_{nm}\) in the form

\[
A_n = (\frac{\Delta}{\hbar})^\frac{1}{2} \psi_n \sqrt{\frac{[N_n]_{\eta}}{N_n}}, \quad N_n = \frac{\Delta}{\hbar} \psi^\dagger_n \psi_n
\]

(4.8)

derives an interesting exact lattice version of the quantum DNLS model, which at the \(\lim_{\Delta \to 0}\) yields the Lax operator

\[
L^{dnls}(\xi) = i \left(-\frac{1}{4} \xi^2 \sigma^3 + \psi^\dagger(x)\psi(x)\kappa + \xi(\psi^\dagger(x)\sigma^+ + \psi(x)\sigma^-)\right), \quad \kappa = diag(\kappa_-, -\kappa_+), \quad (4.9)
\]

of the DNLS field model, which is exactly solvable through the Bethe ansatz \([20]\). A fusion of such DNLS model in turn can generate the well known massive Thirring model \([35]\).

A different set of choice for the elements \(c_i^\pm\) likewise results from \((4.2)\) other algebras and related integrable models. The simplest choice \(c_1^- = 0, c_1^+ = 1\) leads to a light-cone SG model, while \(c_2^+ = 0, c_1^- = -c_1^+ = 1\) giving \(M^\pm = 0\) with realisation

\[
s^3 = p, \quad S^- = \frac{\eta}{\sin \eta} e^u, \quad S^+ = \frac{\eta}{\sin \eta} e^{-u}
\]

(4.10)
gives from the ancestor model \((4.2)\) the Lax operator of a quantum relativistic Toda chain \([34]\). It should also be noted that under a twisting transformation:

\[
R^{\theta}_{trig}(\lambda) = F(\theta)R_{trig}(\lambda)F(\theta), \quad \text{with} \quad F(\theta) = e^{i\theta(\sigma^3 \otimes 1 - 1 \otimes \sigma^3)}
\]

(4.11)

one gets a twisted trigonometric \(R\)-matrix and a corresponding \(\theta\)-parameter extension of the algebra \((4.3)\). Proceeding as above and choosing properly \(\theta\) one gets yet another set of
integrable models, a prominent example of which is the Ablowitz-Ladik model [33][35]. This
demonstrates clearly the unifying feature of the algebraic structure in generating a wide
range of descendant models from the Lax operator \((4.1)\) all sharing the same trigonometric
\(R\)-matrix inherited from the ancestor model.

Let us consider now the \(q \to 1\) limit when \(R_{\text{trig}} \to R_{\text{rat}}\) giving the rational \(R_{\text{rat}}\)-matrix
\((3.7)\), which can be expressed also as \(R_{\text{rat}}(\lambda) = \lambda + \hbar P\). The ancestor model reduces to the
corresponding rational form

\[
L(\lambda)_{\text{rat}} = \begin{pmatrix} K_1^0 + i\lambda c_1^0 & K_{21} \\ K_{12} & K_2^0 + i\lambda c_2^0 \end{pmatrix}, \tag{4.12}
\]

As a consequence of the QYBE \((3.2)\) the \(K\) operators satisfy the algebra given through the
relations

\[
[K_{12}, K_{21}] = (\epsilon_1 K_1^0 - c_1^0 K_2^0), \quad [K_1^0, K_2^0] = 0
\]

\[
[K_1^0, K_{12}] = \epsilon_1 K_{12} c_1^0, \quad [K_1^0, K_{21}] = -\epsilon_1 K_{21} c_1^0,
\tag{4.13}
\]

with \(\epsilon_1 = 1, \epsilon_2 = -1\) and \(c_1^0, c_2^0\) are the central elements. Note that though it is much
similar to a Lie algebra, it is in fact again a quadratic algebra with arbitrary multiplicative
central elements. Since \(L(\lambda)\) operator given in the general form \((4.12)\) satisfies QYBE
and associated with the rational \(R_{\text{rat}}\)-matrix, it may be taken as the ancestor model for
generating integrable models belonging to the rational class.

A nontrivial choice for the central elements \(c_1^0 = c_2^0 = 1\) and the notation

\[
K_1^0 = -K_2^0 = s^3, \quad K_{12} = s^+, \quad K_{21} = s^-
\tag{4.14}
\]

reverses the \(su(2)\) spin algebra

\[
[s^3, s^\pm] = \pm s^\pm, \quad [s^+, s^-] = 2s^3,
\tag{4.15}
\]

which is a Lie algebra. The integrable models like the XXX spin chain and the NLS model,
our acquaintances from sect. 2, can be generated from the case \((4.14)\). In fact the simplest
spin-\(\frac{1}{2}\) representation through Pauli matrices \(s^a = \frac{1}{2} \sigma^a\) reduces \((4.12)\) to the Lax operator
of the XXX chain \((4.5)\). On the other hand bosonic representation of the spin operators
given by the Holstein-Primakov transformation

\[
s^3 = s - \Delta \psi^\dagger \psi, \quad s^- = \Delta \frac{1}{2} (2s - \Delta \psi^\dagger \psi) \frac{1}{2} \psi^\dagger, \quad s^+ = (s^-)^\dagger
\tag{4.16}
\]

leads \((4.12)\) to the quantum integrable Lattice NLS model. The continuum limit takes it
to the NLS field model \((2.6)\).

The general algebra \((4.13)\) permitting trivial values for \(c_1^0\) however allows more freedom
for constructing models. Choosing \(c_2^0 = 0, \quad c_1^0 = i\) for example one can set \(K_2^0 = 0\),
reducing the algebra to

\[
[K_1^0, K_{12}] = iK_{12}, \quad [K_1^0, K_{21}] = -iK_{21} \quad [K_{12}, K_{21}] = 0.
\tag{4.17}
\]
A bosonic representation of the generators:

\[ K_1^0 = p, \ K_{12} = e^{-u}, \ K_{21} = e^u, \quad (4.18) \]

as is evident from (4.12) yields the Lax operator of the Toda chain (2.7). Another simple lattice NLS model can be obtained taking again the values as \( c_0^3 = 0, \ c_0^1 = -\Delta \kappa \) with the representation

\[ K_1^0 = \kappa \Delta^2 \phi \psi + 1, \ K_2^0 = 1, \ K_{12} = i\Delta \sqrt{\kappa} \psi, \ K_{21} = -i\Delta \sqrt{\kappa} \phi \quad (4.19) \]

where the operators \( \psi, \phi \) are canonical. At the continuum limit the same standard NLS field model (2.6) is recovered.

Finally we can consider the twisted rational \( R_{\text{rat}}^\theta (\lambda) \)-matrix through transformation (4.11) with the same twisting operator \( F(\theta) \). As a result one gets a \( \theta \)-deformation of the algebra (4.13) in which \( c_0^1 \)'s no longer remain central elements. We can construct again in a similar way the integrable models belonging to this class. Thus one can obtain the lattice \( \theta \)-deformed NLS, \( \theta \)-deformed Toda chain, Tamm-Dancoff \( q \)-bosonic model or Dzyaloshinsky-Moriya spin chain, which is nothing other than the \( \theta \)-deformed XXX chain [35] [31].

Thus the algebraic structure in the QIS clearly plays a crucial role in unifying diverse models of the same class as descendants from the same ancestor model and at the same time realisations like (4.5) gives a criterion for defining integrable nonlinearity as different nonlinear realisations of the underlying quantum algebras (4.1) or (4.13). We see below that this universality feature is also present curiously in their Bethe ansatz solution, which facilitates their exact eigenvalue solution in an almost model-independent way.

5 Universality in Bethe ansatz solutions

Coordinate formulation of the Bethe ansatz, though more effective for finding the energy spectrum \( H \mid m \rangle = E_m \mid m \rangle \) of concrete models, dependents heavily on the structure of the Hamiltonian and consequently lacks the uniform approach of its algebraic formulation. We therefore focus briefly only on the algebraic Bethe ansatz method for solving the general eigenvalue problem

\[ \tau(\lambda) \mid m \rangle = \Lambda_m(\lambda) \mid m \rangle \quad (5.1) \]

and highlight its universal feature.

Apart from the integrability condition, the QYBE (3.4) represents also a set of commutation relations between action and angle variables given in the matrix form. They can be obtained by inserting matrix \( T \) in the form (3.1) and the \( R(\lambda) \)-matrix solution as (3.5) in the equation (3.4). Such generalised commutation relations dictated by the QYBE are of the form

\[ A(\lambda)B(\mu) = f(\mu - \lambda)B(\mu)A(\lambda) - f_1(\mu - \lambda)B(\lambda)A(\mu), \quad (5.2) \]

\[ D(\lambda)B(\mu) = f(\lambda - \mu)B(\mu)D(\lambda) - f_1(\mu - \lambda)B(\lambda)D(\mu), \quad (5.3) \]
together with the trivial commutations for \([A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [D(\lambda), D(\mu)] = [A(\lambda), D(\mu)] = 0\) etc. It is now important to note that the off-diagonal element \(B(\lambda)\) acts like a creation operator (induced by the local creation operators of \(L(\lambda)\) as argued above). Therefore the \(m\)-particle state \(| m >\) may be considered to be created by \(B(\lambda_i)\) acting \(m\) times on the pseudovacuum \(| 0 >\):

\[
| m > = B(\lambda_1)B(\lambda_2) \cdots B(\lambda_m) | 0 >
\] (5.4)

If one can solve the general problem (5.1), the eigenvalue problem for all

\[
C_n = \frac{1}{n!} \frac{\partial}{\partial \lambda} \tau_N(\lambda) |_{\lambda=0}
\] (5.5)

can be obtained simultaneously by simply expanding \(\Lambda(\lambda)\) as

\[
C_1 | m > = \Lambda'_m(0)\Lambda^{-1}_m(0) | m >, \quad C_2 | m > = (\Lambda'_m(0)\Lambda^{-1}_m(0))' | m >
\] (5.6)

e etc., where say the Hamiltonian \(H = C_1\).

Evidently for solving (5.1) through the Bethe ansatz we have to drag \(\tau(\lambda) = A(\lambda) + D(\lambda)\) through the string of \(B\)’s without spoiling their structures (and thereby preserving the eigenvector) and hit finally the pseudovacuum giving \(A(\lambda) | 0 > = \alpha(\lambda) | 0 >\) and \(D(\lambda) | 0 > = \beta(\lambda) | 0 >\).

Notice that for this purpose (5.2,5.3) coming from the QYBE are indeed the right kind of commutation relations but for the second terms in both the RHS, where the argument of \(B\) has changed: \(\tilde{B} \to B\) spoiling the structure of the eigenvector. However, if we put the sum of all such unwanted terms \(= 0\), we should be able to achieve our goal. In field models such unwanted terms are however absent, while in lattice models they may be removed by the Bethe equations induced by the periodic boundary condition giving

\[
\left( \frac{\alpha(\lambda_k)}{\beta(\lambda_k)} \right)^N = \prod_{l \neq k} \frac{f(\lambda_k - \lambda_l)}{f(\lambda_l - \lambda_k)} = \prod_{l \neq k} \frac{-a(\lambda_k - \lambda_l)}{a(\lambda_l - \lambda_k)}, \quad k = 1, 2, \ldots, m.
\] (5.7)

This in turn serves as the determining equations for parameters \(\lambda_j\). Ignoring therefore the second terms for the time being and making use of the first terms only, as argued above we finally solve the eigenvalue problem to yield

\[
\Lambda_m(\lambda) = \prod_{j=1}^m f(\lambda_j - \lambda)\alpha(\lambda) + \prod_{j=1}^m f(\lambda - \lambda_j)\beta(\lambda).
\] (5.8)

The structure of the eigenvalue \(\Lambda_m(\lambda)\) reveals the curious fact about the Bethe ansatz result, that apart from the \(\alpha(\alpha), \beta(\alpha)\) factors the eigenvalue (5.8) and the Bethe equation (5.7) depend mainly on the nature of the function \(f(\lambda - \lambda_j)\), which are universal within a class of models and determined only by the choice of \(R\)-matrix as (3.6) or (3.7). The coefficients \(\alpha(\lambda), \beta(\lambda)\) determined by the concrete form of the Lax operators and the definition of the pseudovacuum are the only model-dependent part. Therefore models like the DNLS, SG, Liouville and the \(XXZ\) chain belonging to the trigonometric class share
similar type of eigenvalue relations with individual differences only in the form of \(\alpha(\lambda)\) and \(\beta(\lambda)\) coefficients. Thus this deep rooted universality in integrable systems helps to solve the eigenvalue problem for the whole class of models and for the full hierarchy of their conserved currents in a systematic way. For example, for the \(XXZ\) model using the Lax operator structure \((2.4)\) and \(|0\rangle\) defined as the all spins up state, one easily finds \(\alpha(\lambda) = \sin^N(\lambda + \eta), \ \beta(\lambda) = \sin^N \lambda\). Using subsequently the \(R_{\text{trig}}\)-matrix information \((3.4)\) one derives from \((5.7)\) (with a shift \(\lambda \rightarrow \lambda + \frac{\eta}{2}\)) the Bethe ansatz result

\[
\left(\frac{\sin(\lambda_k + \frac{\eta}{2})}{\sin(\lambda_k - \frac{\eta}{2})}\right)^N = \prod_{j \neq k} \frac{\sin(\lambda_k - \lambda_j + \eta)}{\sin(\lambda_k - \lambda_j - \eta)},
\]

for \(j = 1, 2, \ldots, m\). Similarly \((5.8)\) gives the eigenvalue

\[
\Lambda_{XXZ}(\lambda) = \sin^N(\lambda + \eta) \prod_{j=1}^m \frac{\sin(\lambda_j - \lambda + \frac{\eta}{2})}{\sin(\lambda_j - \lambda - \frac{\eta}{2})} + \sin^N \lambda \prod_{j=1}^m \frac{\sin(\lambda - \lambda_j + \frac{3\eta}{2})}{\sin(\lambda - \lambda_j + \frac{\eta}{2})} + \sin^N \frac{\eta}{2},
\]

yielding for \(H_{xxz} = C_1\), the energy spectrum

\[
E^{(m)}_{xxz} = \Lambda(\lambda)\Lambda^{-1}(\lambda) \big|_{\lambda=0} = \sin \eta \prod_{j=1}^m \frac{1}{\sin(\lambda_j - \frac{\eta}{2})\sin(\lambda_j + \frac{\eta}{2})} + N \cot \eta.
\]

At the limit \(\eta \rightarrow 0, \sin \lambda \rightarrow \lambda\), when the \(R\)-matrix along with its associated models reduce to the rational class, one can derive the Bethe ansatz result of the isotropic \(XXX\) chain directly by taking the limits of the above expressions for \(XXZ\) model. The Bethe ansatz result for the NLS model of the same rational class also have very similar structure.

It should be remarked that unlike the coordinate formulation, the algebraic Bethe ansatz does not require explicit form of the Hamiltonian. The information about the Lax operator and the \(R\)-matrix is more important for this method. However using the definitions \((5.3)\) and \((8.1)\) we can construct Hamiltonian of the model from the Lax operator. For example, the \(XXZ\) chain with Lax operator \((2.4)\) (with a shift \(\lambda \rightarrow \lambda + \frac{\eta}{2}\)) satisfies the condition \(L_{ai}(0) = P_{ai} = I + \sigma \cdot \sigma\), where \(P_{ai}\) is the permutation operator with the property \(P^2 = 1\) and \(P_{ai}L'_{ai+1}(0) = L'_{ai+1}(0)P_{ai}\). Using this property we derive from \((3.1)\)

\[
\tau(0) = \tau_a(P_{aj}P_{aj+1} \cdots P_{aj-1}) = (P_{jj+1} \cdots P_{jj-1})\tau_a(P_{aj})
\]

for any \(j\), applying freedom of cyclic rotation of matrices under the trace. Taking derivative with respect to \(\lambda\) in \((8.1)\) we similarly get

\[
\tau'(0) = \tau_a \sum_{j=1}^N \left(P_{aj}L'_{aj+1}(0) \cdots P_{aj-1}\right) = \sum_{j=1}^N (L'_{jj+1}(0) \cdots P_{jj-1})\tau_a(P_{aj}),
\]

where we have assumed the periodic boundary condition: \(L_{aN+j} = L_{aj}\). Defining now from \((5.3)\) \(H = c C_1 = c \frac{d}{d\lambda} \ln \tau(\lambda) |_{\lambda=0} = c \tau'(0)\tau^{-1}(0), \ \ c = \text{const}.\) we get the Hamiltonian from \((5.12)\) and \((5.13)\) as

\[
H = c \sum_{j=1}^N (L'_{jj+1}(0)P_{jj+1})
\]
with only 2-neighbor interactions, due to cancelation of all other nonlocal factors and the property $P^{-1} = P$. Calculating $L'$ from the Lax operator (2.4) and inserting in (5.14) we finally get the explicit form of $H$ as in (2.4) for the $XXZ$ chain.

6 Algebraic structures in ultralocal and nonultralocal integrable systems

We have seen that the algebraic property in integrable systems is dictated by the QYBE. However since the QYBE for the nonultralocal systems is different from (3.2), the related algebraic structure can not be described by the above procedure. As a result progress in this field is not much satisfactory compared to the above described ultralocal theory, though many important models like Nonabelian Toda chain, quantum KdV and modified KdV model, nonlinear $\sigma$-model etc. belong to this class. Nevertheless, it is possible now to describe such models to considerable extent by some extensions of the QYBE [22] [23].

However for better understanding of the algebraic structure of the nonultralocal models let us take a closer look at that of the ultralocal Lax operators themselves. The underlying quadratic quantum algebra (4.2), as mentioned before, exhibits Hopf algebra property. The most prominent characteristic of it is the coproduct structure given by

$$\Delta(s^3) = s^3 \otimes I + I \otimes s^3, \quad \Delta(S^+) = S^+ \otimes c^+_1 q^{-s^3} + c^+_1 q^{s^3} \otimes S^+ , \quad \Delta(S^-) = S^- \otimes c^-_1 q^{-s^3} + c^-_2 q^{s^3} \otimes S^-$$

Note that the $c^+_i = 1$ case recovers the well known result for the standard quantum algebra. This means that if $S^+_i = S^+ \otimes I$ and $S^+_2 = I \otimes S^+$ satisfy the quantum algebra separately, then their tensor product $\Delta(S^+)$ given by (6.1) should also satisfy the same algebra. This Hopf algebraic property induces the crucial transition from the local QYBE (3.2) to its tensor product given by the global equation (3.4) as discussed above. Note that thanks to the property of the generators of the algebra: $[S^a_i, S^b_j] = 0$, $i \neq j$ the ultralocality of the Lax operators is achieved. However for nonultralocal models with the underlying braided algebra having a different multiplication rule though a similar Hopf algebra property [21], we get $[L_{1i}, L_{2j}] \neq 0$ requiring generalisation of the QYBE.

Such generalised QYBE for nonultralocal systems with the inclusion of braiding matrices $Z$ (nearest neighbour braiding) and $\tilde{Z}$ (nonnearest neighbour braiding ) may be given by

$$R_{12}(u-v)Z_{21}^{-1}(u,v)L_{1j}(u)\tilde{Z}_{21}(u,v)L_{2j}(v) = Z_{12}^{-1}(v,u)L_{2j}(v)\tilde{Z}_{12}(v,u)L_{1j}(u)R_{12}(u-v).$$

In addition, this must be complemented by the braiding relations

$$L_{2j+1}(v)Z_{21}^{-1}(u,v)L_{1j}(u) = \tilde{Z}_{21}^{-1}(u,v)L_{1j}(u)\tilde{Z}_{21}(u,v)L_{2j+1}(v)\tilde{Z}_{21}^{-1}(u,v)$$

at nearest neighbour points and

$$L_{2k}(v)\tilde{Z}_{21}^{-1}(u,v)L_{1j}(u) = \tilde{Z}_{21}^{-1}(u,v)L_{1j}(u)\tilde{Z}_{21}(u,v)L_{2k}(v)\tilde{Z}_{21}^{-1}(u,v)$$

(6.2)
with \( k > j + 1 \) answering for the nonnearest neighbours. Note that along with the usual quantum \( R_{12}(u - v) \)-matrix like (3.3) additional \( \tilde{Z}_{12}, Z_{12} \) matrices appear, which can be (in-)dependent of the spectral parameters and satisfy a system of Yang-Baxter type relations [23].

7 Generation of nonultralocal models from the braided QYBE

Unlike ultralocal models due to the appearance of \( Z \) matrices in the braided QYBE relations one faces initial difficulty in trace factorisation. Nevertheless, in nonultralocal cases one can mostly bypass this problem by introducing a \( K(u) \) matrix and defining \( t(u) = tr(K(u)T(u)) \) as commuting matrices [30, 23] for establishing the quantum integrability for nonultralocal models. Though a well-framed theory for such systems is yet to be achieved, one can derive the basic equations for a series of nonultralocal models in a rather systematic way from the general relations (6.2-6.4) by particular explicit choices of \( Z, \tilde{Z}, R \)-matrices [23, 29]. The models which can be covered through this scheme are

I. Nonultralocal models with rational \( R \)-matrix

1. Nonabelian Toda chain [24]
   \[ \tilde{Z} = 1, Z = I + ih(e_{22} \otimes e_{12}) \otimes \pi. \]

2. Nonultralocal quantum mapping [25]
   \[ \tilde{Z} = 1 \text{ and } Z_{12}(u_2) = 1 + \frac{h}{2} \sum_{\alpha} N_{\alpha} e_{\alpha N} \otimes e_{\alpha N}. \]

3. Supersymmetric models
   \[ Z = \tilde{Z} = \sum_{\alpha, \beta} \eta_{\alpha \beta} g_{\alpha \beta}, \text{ where } \eta_{\alpha \beta} = e_{\alpha \alpha} \otimes e_{\beta \beta} \text{ and } g = (-1)^{\hat{\alpha} \hat{\beta}} \text{ with supersymmetric grading } \hat{\alpha}. \]

4. Anyonic type SUSY model
   \[ Z = \tilde{Z} = \sum_{\alpha, \beta} \eta_{\alpha \beta} \tilde{g}_{\alpha \beta}, \text{ with } \tilde{g}_{\alpha \beta} = e^{i\theta \hat{\alpha} \hat{\beta}}. \]

5. Kundu-Eckhaus equation [36]
   Classically integrable NLS equation with 5th power nonlinearity
   \[ i\psi_1 + \psi_{xx} + \kappa(\psi\psi^\dagger)\psi + \theta^2(\psi\psi^\dagger)^2\psi + 2i\theta(\psi\psi^\dagger)_x \psi = 0, \quad (7.1) \]
   as a quantum model involves anyonic type fields: \( \psi_n \psi_m = e^{i\theta} \psi_m \psi_n, \ n > m; \ [\psi_m, \psi_n^\dagger] = 1. \)

The choice \( \tilde{Z} = 1, Z = \text{diag}(e^{i\theta}, 1, 1, e^{i\theta}) \) constructs the braided QYBE, The trace factorisation problem has not been solved.

II. Nonultralocal models with trigonometric \( R \)-matrix

1. Current algebra in WZWN model [19]
   \[ \tilde{Z} = 1 \text{ and } R_{12} = Z_{12} = R_{q12}^-; \text{ where } R_{q}^{\pm} \text{ is the } \lambda \rightarrow \pm \infty \text{ limit of the trigonometric } R(\lambda)-\text{matrix} \]

2. Coulomb gas picture of CFT [26]
   \[ \tilde{Z} = 1, Z_{12} = q^{-\sum} H_i \otimes H_i; \text{ and } R_{12} = R_{q12}^+. \]

3. Integrable model on moduli space [27]
\[ \tilde{Z} = Z_{12} = R^+_q \] and \( \lambda \)-dependent \( R(\lambda) \)-matrix.

4. Quantum mKdV model \[28\]
\[ \tilde{Z} = 1, \ Z_{12} = Z_{21} = q^{-\frac{1}{2}} \sigma^3 \otimes \sigma^3, \] and the \( \lambda \)-dependent trigonometric \( R(\lambda) \)-matrix.

Other models of nonultralocal class are the well known Calogero-Sutherland (CS) and Haldane-Shastry (HS) models with interesting long-range interactions. Though spin CS model exhibits many fascinating features \[18\] and its integrability formulation through braided QYBE for both HS and CS models has not yet been achieved.

8 Connections with Stat Mech and CFT

As it was mentioned above the unifying feature of the QIS is present in its relationships with various other branches of physics and mathematics, where the knowledge and techniques of the QIS often proved to be extremely helpful. Among these various connections those with the Stat Mech problems and the CFT models seem to be the most interesting.

The \((1+1)\) dimensional quantum systems are intimately linked with 2 dimensional classical statistical models. Moreover the notions of integrability are equivalent in both these cases and are through the Yang-Baxter relations. In integrable statistical systems both QYBE \[3.2\] and the YBE \[3.3\] become the same and leads similarly to the commuting transfer matrices \( \tau(\lambda) \). A classical statistical vertex model may be given by \( N \times M \) lattice points connected by the bonds assigned with +ve (-ve) signs or equivalently, with right, up (left, down) arrows in a random way (see Fig. 1).

![Figure 1](image)

**Figure 1** Classical 2-dimensional vertex model is related with \((1+1)\)-quantum system. The row-to row transfer matrix \( \tau \) depends only on vertical space indices \( \alpha, \beta \) and made out of multiplying \( R^{(i)} \)-matrices with elements \( R^{(i)}_{kl} \), which represent the Boltzmann weights at vertex point \( i \).
Setting the corresponding Boltzmann weights for possible arrangements as matrix elements of a \(4 \times 4\)-matrix, we get the \(R_{12}^{(i)}\)-matrix with crucial dependence on spectral parameter \(\lambda\). The YBE (3.3) restricts the solution of the \(R\)-matrix to integrable models. Imposing extra symmetries on the \(R\)-matrix like the charge conserving: \(R_{kl}^{ij} \neq 0\), only when \(k + l = i + j\) and reversing: \(R_{kl}^{ij} = R_{-k,-l}^{i,-j}\) symmetry we get the Boltzmann weights of the 6-vertex model constituting the elements of the \(R\)-matrix:

\[
R_{++}^{++} = R_{--}^{--} = a(\lambda), \quad R_{+-}^{+-} = R_{-+}^{-+} = b(\lambda), \quad R_{++}^{-+} = R_{--}^{-+} = c(\lambda),
\]

where \(\frac{a}{b} = f, \frac{c}{b} = f_1\). Using an overall normalisation one immediately recognises this as our familiar \(R\)-matrix (3.3), which represents also the Lax operator of the anisotropic quantum spin-\(\frac{1}{2}\) chain (2.4).

The configuration probability for a string of \(N\)-lattice sites in a row may be given by the transfer matrix \(\tau_N(\vec{\alpha}, \vec{\beta}) = tr(\prod_1^N R^{(i)})\) in a similar way as in QIS. The free parameter \(\lambda\) may be linked with the temperature \(T = \frac{k_B}{\lambda}\) of the statistical model as \(\lambda = \frac{\text{const}}{M} \beta\). The direction along \(N\) in statistical system may be considered as the space direction of the quantum model, while the \(M\) direction is like the time direction in the quantum system (see Fig. 1), which again may be replaced in statistical systems by the temperature. Therefore, the finite-temperature behaviour can be studied by taking suitably the limit of \(M \to \infty\). Recalling that the Hamiltonian of the quantum model may be given by \(H = C_1 = -\tau_N(0)^{-1} \frac{\partial}{\partial \lambda} \tau_N(\lambda) \mid_{\lambda=0}\), we can establish the important connection \(\exp(-\beta H) = \lim_{M \to \infty} \left(\tau_D^{DTD}\right)_M^{\lambda}\), where \(\tau_D^{DTD} = \tau_N(0)^{-1} \tau_N(\vec{\beta})\) is the diagonal to diagonal transfer matrix. On the other hand the partition function \(Z_{N,M}(\lambda) = tr(\tau_D^{DTD}(\lambda))^M\). Interchanging \(N,M\) amounts to the rotation by 90\(^\circ\) and effected by changing the spectral parameter \(\lambda \to \eta - \lambda\) due to the crossing symmetry of the model. This gives therefore \(Z_{M,N}(\lambda) = tr(\tau_D^{DTD}(\eta - \lambda))^N\) yielding the free energy \(f = -\frac{1}{\beta} \lim_{M \to \infty} \Lambda_M(\lambda)\), where \(\Lambda_M(\lambda)\) is the largest eigenvalue of \(\tau_D^{DTD}(\eta - \lambda)\). Therefore using the eigenvalue expressions (5.8) and expanding it in powers of \(\beta\) one can find the finite-temperature corrections to the thermodynamical quantities just as in the case of expansion in large \(N\) for finding the finite-size corrections as shown above.

Similar deep interrelation exists also between the QIST and the CFT models, revealed first perhaps by Zamolodchikov (12) by showing that, if CFT is perturbed through relevant perturbation and the system goes away from criticality, it might generate hierarchies of integrable systems. For example \(c = \frac{1}{2}\) CFT perturbed by the field \(\sigma = \phi_{(1,2)}\) as \(H = H_\frac{1}{2} + h\sigma \int \sigma(x) d^2x\), represents in fact the Ising model at \(T = T_c\) with nonvanishing magnetic field \(h\). Similarly the WZWN model perturbed by the operator \(\phi_{(1,3)}\) generates integrable restricted sine-Gordon (RSG) model. Under such perturbations the trace of the stress tensor, unlike pure CFT becomes nonvanishing and generates infinite series of integrals of motion associated with the integrable systems.

Another practical application of this relationship is to extract important information about the underlying CFT in the scaling limit of the integrable lattice models. Interestingly,
from the finite size correction of the Bethe ansatz solutions, one can determine the CFT characteristics like the central charge and the conformal dimensions \[c\]. For example, one may analyse the finite size effect of the Bethe ansatz solutions of the 6-vertex model (with a seam given by \(\kappa\)). Considering the coupling parameter \(q = e^{i\pi \nu}/\nu\), one obtains from the Bethe solution at the large \(N\) limit the expression

\[E_0 = N f_\infty - \frac{1}{N} \frac{\pi}{6} c + O\left(\frac{1}{N^2}\right)\]

for the ground state energy and

\[E_m - E_0 = \frac{2\pi}{N}(\Delta + \tilde{\Delta}) + O\left(\frac{1}{N^2}\right) \quad P_m - P_0 = \frac{2\pi}{N}(\Delta - \tilde{\Delta}) + O\left(\frac{1}{N^2}\right)\]

for the excited states. Here \(\Delta, \tilde{\Delta}\) are conformal weights of unitary minimal models and \(c = 1 - \frac{6\kappa^2}{\nu(\nu + 1)}\), \(\nu = 2, 3, \ldots\) is the central charge of the corresponding conformal field theory.

9 Concluding remarks

The variety of models within the quantum integrable systems are linked together by an unifying algebraic structure induced by the quantum Yang-Baxter equation. Such structures not only systematically generate different classes of integrable models but also gives an unifying approach in their Bethe ansatz solutions. The theory of nonultralocal models, though not so successful allows to derive the models from few basic relations, which are braided extensions of the Yang-Baxter equations. The relationship of QIS with stat mech and CFT gives practical results in finding finite-temperature and finite size corrections of the quantum model as well as its underlying conformal properties.

Recently found link of the QIS with diverse subjects like link and knot polynomials [15], reaction-diffusion processes [39], Seiberg-Witten model [40] etc. are also becoming more and more important.

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