Information Signal Design for Incentivizing Team Formation

Chamsi Hssaine, Siddhartha Banerjee
School of Operations Research and Information Engineering, Cornell University
{ch822,sbanerjee}@cornell.edu

Abstract. We study the use of Bayesian persuasion (i.e., strategic use of information disclosure/signaling) in endogenous team formation. This is an important consideration in settings such as crowdsourcing competitions, open science challenges and group-based assignments, where a large number of agents organize themselves into small teams which then compete against each other. A central tension here is between the strategic interests of agents who want to have the highest-performing team, and that of the principal who wants teams to be balanced. Moreover, although the principal cannot choose the teams or modify rewards, she often has additional knowledge of agents’ abilities, and can leverage this information asymmetry to provide signals that influence team formation. Our work uncovers the critical role of self-awareness (i.e., knowledge of one’s own abilities) for the design of such mechanisms. For settings with two-member teams and binary-valued agents partitioned into a constant number of prior classes, we provide signaling mechanisms which are asymptotically optimal when agents are agnostic of their own abilities. On the other hand, when agents are self-aware, then we show that there is no signaling mechanism that can do better than not releasing information, while satisfying agent participation constraints.

1 Introduction

Consider a course instructor who wants to pair up students into teams of two for completing assignments. Each student has an unknown ‘aptitude’ for the course, with past performance (academic background, GPA, etc.) providing a prior on these aptitudes; moreover, everyone wants a teammate with the highest possible aptitude. In the absence of any other information, the priors translate into a perceived ranking over students, and if left alone, students will pair up according to this ranking (i.e., the top two form one team, the next two form another, and so on). On the other hand, the instructor would prefer it if high-performing and low-performing students worked together, to encourage better learning outcomes. However, she does not want to decide the teams herself, or alter the grading to incentivize such teams to form. Is there anything she can do in such a setting?

The above problem takes on a much more meaningful form in the context of collaboration in open science challenges and crowdsourcing competitions [3]. As an example, consider the DREAM Challenges [8] – an online crowdsolving platform which leverages researchers from various backgrounds to solve problems in biology and medicine. A critical design feature of these challenges is that after an initial exploration round, competitors are required to pair up into teams – this is inspired by observations that the performance of individual contestants’ solutions perform much worse than ensembles of these solutions [18]. The team composition however is completely decided by the participants, with minimal involvement of the designers (and no change in reward structures). Thus, gaining insight into team formation and dynamics is key to the success of these platforms; this sentiment is echoed in a report from the National Academy of Sciences highlighting the need to find ways to foster team effectiveness [5].

Our work focuses on the use of Bayesian persuasion (sometimes referred to as strategic information revelation, or signaling) for incentivizing team formation. The main idea is that many
strategic settings have an inherent information asymmetry, where the principal has more information than the participating agents. By controlling the release of this information, the principal can influence agents' decisions. For example, for the formation of course assignment groups, suppose the instructor first administers a test to get a more accurate read on each student's true aptitude for the course. How she releases the results of this test will affect what teams are formed. If she chooses not to release any information, then the students will pair up as before according to their perceived aptitudes; if she releases the scores as is, the students will then pair up according to these true aptitudes, leading to even more imbalanced teams. The main idea we pursue in this work is to understand if there is any way of releasing the scores that can lead to more balanced teams; in other words, we want to know how the principal can influence endogenous team formation using strategic signaling.

1.1 Overview of Model and Results

We consider a setting with $n$ agents who endogenously form teams, leading to some utility for both the agents and the principal. For convenience, we focus here on teams of two agents (i.e., matchings) – the basic setting and questions (as well as many of our results) however are more general. The teams are chosen endogenously by the agents, in the form of a stable matching; the principal however can influence agents’ preferences via strategic release of information.

In more detail: each agent has an intrinsic (numerical) type, drawn from some publicly-known prior. Crucially, we assume the types are known to the principal, but unknown to the agents. Moreover, an agent’s utility is an increasing function of her and her teammates’ types, while the principal’s utility function depends on the set of resulting teams, and favors having more ‘balanced’ teams. For convenience, we focus on settings with a binary type-space \{0, 1\}, and a constant number (K) of prior distributions. Our insights however generalize to other settings, as we discuss below.

The main tool available to the principal is Bayesian persuasion, whereby she can leverage her information asymmetry by committing to a signaling scheme based on the realized types. This signaling scheme can be verified by the agents (for example, the principal can commit to using an open-source script that inputs the true types and generates the signal). Thus, the signal affects the agents’ posterior over the types, which then determines their choice of teams via a stable matching. The aim of the principal is to choose a signaling scheme which is individually rational (i.e., participation constraints, wherein all agents are weakly better off by agreeing to receive the signal), and for which the resulting stable matching maximizes the principal’s utility.

In the context of the above setting, our contributions are summarized as follows:

1. We characterize the optimal signaling schemes in the form of a linear program which implements a persuasive recommendation – a consistent posterior ranking of the agents. This LP however has $\Omega(n^n)$ variables, and hence is computationally intractable.
2. In settings where agents are self-agnostic (i.e., are uncertain even about their own type), we demonstrate a signaling mechanism which (i) under uniform priors (i.e., $K = 1$ clusters, wherein agents have i.i.d. types) achieves the optimal in hindsight, and (ii) for finite $K$, is asymptotically optimal (in $n$) compared to the best signaling scheme.
3. In settings where agents are self-aware (i.e., know their own type), we show that even for i.i.d. types ($K = 1$), no signaling scheme can do better than random matching (i.e., not releasing any information) while preserving participation constraints.

To the best of our knowledge, we are the first to formally characterize the effectiveness of signaling mechanisms for team formation. Moreover, despite the simplicity of our setting, our work provides...
important insights and techniques for the design of Bayesian persuasion schemes for general team formation settings. Our results indicate the importance of self-awareness in determining the success of signaling mechanisms, and provides a novel policy for self-agnostic settings based on inter-cluster pairing of type-profiles. Moreover, showing this strategy is asymptotically optimal requires a novel dual-certification argument, which may be useful in related settings.

1.2 Related work

Our work locates itself squarely within the framework of Bayesian persuasion, a topic which has garnered much recent attention. We briefly summarize some of the main ideas of this topic below; for a more detailed survey of this literature, refer [10] for a survey).

The basic idea originates from the seminal work of Kamenica and Gentzkow [14], which considers a principal who commits to a signaling policy which maps the true state of the world to a signal sent to a single agent, and derives conditions on the principal’s utility function under which she strictly benefits from persuasion. Going beyond, Kremer et al. [15] consider a dynamic setting in which agents arrive sequentially and choose an action with an unknown (but deterministic) reward. The goal of the principal is to find the optimal disclosure (or recommendation) policy of a planner who wants to maximize social welfare, whereas agents simply seek to maximize their own expected reward. The authors show that the optimal policy is a threshold policy which explores as much as possible, and then always recommends the best action, and obtain a similar near-optimal threshold policy with stochastic rewards; recent papers greatly generalize this line of work [17,2].

In the context of multi-agent settings with no externalities, Arieli and Babichenko [1] look at a model where the principal is trying to persuade individuals to adopt a product by sending private signals. They characterize the optimal policy for supermodular, submodular, and supermajority utility functions of the principal. More recent work has been devoted to proving hardness or inefficacy results for finding the optimal information disclosure policy in such settings [9,11].

In contrast to these papers, our work involves a multi-agent scenario with externalities – in other words, not only is the principal playing a game with multiple agents, but the agents are playing a game amongst themselves. Recent work has looked at related models in the context of routing and queueing games. In [4], the authors present hardness results on public signaling for Bayesian two-player zero-sum games and Bayesian network routing games; more recent works [7,19] consider practical variants of such policies in restricted settings. In the setting of strategic delay announcements for queueing, [16] show that when the principal is a revenue maximizer, a binary signaling mechanism with a threshold structure is optimal. Our work however is, to best of our knowledge, the first to consider the problem of finding the optimal signaling policy for team formation.

2 Preliminaries

2.1 Basic Setup

We consider a setting with a principal and $n$ agents (where $n$ is even), where the agents endogenously partition themselves into two-member teams. Each agent $i \in [n]$ has a random type $\theta_i \in \{0, 1\}$, which can be interpreted as her intrinsic ability to perform the task at hand. We use $\Theta \triangleq \{0, 1\}^n$ to denote the space of type-profiles of all $n$ agents. Additionally, we use $\mathcal{M}$ to denote the set of all matchings. Finally, for any given type-profile $\theta \in \Theta$, we define $h(\theta) = \sum_i \mathbb{1}_{\theta_i = 1}$ to be the number
of agents with type 1 (i.e., ‘high’ type) in $\theta$, and $\ell(\theta) = n - h(\theta)$ to be the number of agents with type 0 (i.e., ‘low’ type) in $\theta$.

Each agent's payoff depends only on her own type and that of her match. We denote the utility function of agent $i$ as $u_i : \{0,1\} \times \{0,1\} \to \mathbb{R}$, and assume they obey the following properties:

i. symmetric (i.e., $u(\theta_i,\theta_j) = u(\theta_j,\theta_i)$)
ii. non-decreasing in each type (i.e., $u(\theta,0) \leq u(\theta,1)$ and $u(0,\theta) \leq u(1,\theta)$)
iii. concave (i.e., $u(0,0) + u(1,1) \leq 2u(1,0)$)

The principal’s utility $f : \Theta \times \mathcal{M} \to \mathbb{R}$ is a function of the agents’ types, as well as the matching chosen by the agents. As with the agents’ utilities, we assume $f$ is symmetric in each type (i.e., anonymous), and hence, only depends on the number of matches between two type 1 agents $m_{11}$, two type 0 agents $m_{00}$, and one type 1 and one type 0 agent $m_{10}$. Note though that, for any given type profile $\theta$, the vector $(m_{00},m_{10},m_{11})$ (i.e., over all choices of teams) has only a single degree of freedom, and in particular, maximizing (resp. minimizing) $m_{10}$ is equivalent to minimizing (resp. maximizing) $m_{11}$ and $m_{00}$. Moreover, note that if $f$ is nondecreasing in $m_{11}$ or $m_{00}$, then the agents’ and the principal’s incentives are aligned, and thus the principal can reveal $\theta$ to realize the optimal matching. The tension arises because the principal wants teams which are more ‘balanced’ – to model this, we henceforth assume that $f$ is strictly increasing in $m_{10}$ (equivalently, strictly decreasing in $m_{00}$ and $m_{11}$).

In the spirit of the framework used in other work on optimal signaling [14,11], we assume that the type of each agent $i$ is drawn independently from a $Ber(p_i)$ distribution, and denote $\lambda$ to be the product distribution over $\Theta$. We assume that this distribution is known to all agents, and that each agent’s realized type is unknown to other agents. On the other hand, we assume that the principal has full knowledge of the realized types $\theta = (\theta_1, \ldots, \theta_n)$. Finally, with regard to what an agent knows about her own type, we consider two cases: one in which agents are self-aware, and one in which agents are self-agnostic.

**Definition 1.** A self-aware agent is an agent who knows her own type. A self-agnostic agent is an agent who does not know her own type.

For example, a self-agnostic student in a course is one who has no prior experience in the subject, and hence no idea of her aptitude for it; on the other hand, a self-aware student has some idea of her abilities, perhaps based on experience of similar courses or independent reading.

We assume the principal has the ability to commit to an information disclosure policy – also termed a signaling scheme – which is a mapping from the realized state of nature $\theta$ to a signal of some sort. For example, in crowdsourcing platforms, the principal can administer a test to each participant, and choose whether or not (and how) to reveal their scores.

We note that this signaling policy may be randomized. In this paper, we restrict the principal to public signaling schemes, i.e., the principal sends the same signal to all agents. Although private signaling schemes have been considered in other settings (for example, signaling in games without externalities [11]), it is unclear how an agent can reason about the beliefs and abilities of other agents when evaluating a match.

Formally, the sequence of events is as follows. First, the principal commits to a randomized signaling policy $\phi : \Theta \to \Sigma$, where $\Sigma$ is the set of all possible signals $\sigma$. Next, the state of nature $\theta \in \Theta$ is drawn from the prior distribution $\lambda$, and the types of all agents are revealed to the principal. The principal then draws a signal $\sigma$ from the distribution $\phi(\theta)$, and broadcasts it to the agents. The agents in turn compute a posterior on the state of nature $\theta$ given the signal $\sigma$. They compare their optimal match given $\sigma$ and their optimal match without $\sigma$, and choose the agent to
match with which maximizes their expected utility. As a result, a matching \( m \in M \) is induced. The optimization problem that we are interested in is finding the signaling scheme that maximizes the principal’s expected utility.

### 2.2 Solution Concept

We assume that agents are expected utility maximizers. Upon receiving the signal from the principal, self-agnostic agents form a posterior on the state of nature \( \theta \), given the common filtration \( \mathcal{F}_\sigma \) induced by the public signal \( \sigma \). A subgame is consequently induced, in which each agent maintains a rank ordered preference list over all other agents, and chooses to match with the agent who maximizes her expected utility, given her available information. Formally, each agent \( i \) computes \( \mathbb{E}[u(\theta_i, \theta_j) \mid \mathcal{F}_\sigma] \) for each agent \( j \neq i \), and maintains the preference list \( j_1 \succ j_2 \succ \cdots \succ j_{n-1} \) such that \( \mathbb{E}[u(\theta_i, \theta_{j_i}) \mid \mathcal{F}_\sigma] \geq \mathbb{E}[u(\theta_i, \theta_{j_{j_2}}) \mid \mathcal{F}_\sigma] \geq \cdots \geq \mathbb{E}[u(\theta_i, \theta_{j_{j_{n-1}}}) \mid \mathcal{F}_\sigma] \).

In the case where agents are self-agnostic, the public signal \( \sigma \) induces a common filtration \( \mathcal{F}_\sigma \) among all agents. Consequently, all agents will have the same preference profile. This resulting preference profile induces a matching \( m \). A natural solution concept for endogenous team formation is that of stability \([12]\):

**Definition 2.** A matching \( m \) is stable if it has no blocking pair, i.e., there exist no two agents who are not matched together but would prefer to be matched together.

Stability is a natural desiderata to impose on team formation. In addition, however, we also need to ensure that agents are willing to participate in any signaling scheme proposed by the principal. Note that in the absence of a signal, agents can still compute \( \mathbb{E}[u(\theta_i, \theta_j)] \), and form a preference profile and resulting matching \( m(\emptyset) \) (where \( \emptyset \) refers to the absence of signal). This suggests the following natural participation constraint:

**Definition 3.** A signaling scheme is individually rational if it ensures that each agent is weakly better off under any stable matching \( m \) induced by the signaling scheme as compared to her utility under the baseline stable matching \( m(\emptyset) \).

Thus, our aim is to design individually rational signaling schemes so as to maximize the principal’s utility under an induced stable matching. The following proposition helps further simplify this, by showing that every agent’s ranking over other agents induced by a signal \( \sigma \) is identical to the ranking induced by the conditional expectation over agent types.

**Proposition 1.** Given the filtration \( \mathcal{F}_\sigma \), the expected utility of agent \( i \) from matching with agent \( j \) is monotone increasing in the expected type of agent \( j \).

**Proof.** Define \( \mathbb{E}_\sigma[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_\sigma] \). Now consider \( \mathbb{E}_\sigma[u(\theta_i, \theta_j)] \), for arbitrary \( j \in [n] \). We have:

\[
\mathbb{E}_\sigma[u(\theta_i, \theta_j)] = u(0,0)\mathbb{P}_\sigma[\theta_i = 0, \theta_j = 0] + u(1,1)\mathbb{P}_\sigma[\theta_i = 1, \theta_j = 1] \\
+ u(1,0)\left(\mathbb{P}_\sigma[\theta_i = 1, \theta_j = 0] + \mathbb{P}_\sigma[\theta_i = 0, \theta_j = 1]\right) \\
= u(0,0)\mathbb{P}_\sigma[\theta_i = 0 \mid \theta_j = 0] \mathbb{P}_\sigma[\theta_j = 0] + u(1,1)\mathbb{P}_\sigma[\theta_i = 1 \mid \theta_j = 1] \mathbb{P}_\sigma[\theta_j = 1] \\
+ u(1,0)\left(\mathbb{P}_\sigma[\theta_i = 0 \mid \theta_j = 1] \mathbb{P}_\sigma[\theta_j = 1] + \mathbb{P}_\sigma[\theta_i = 1 \mid \theta_j = 0] \mathbb{P}_\sigma[\theta_j = 0]\right)
\]

\(^1\) The induced matching \( m \) may not be unique, since the ordering is not necessarily strict. We assume that agents break ties in favor of the principal.
Define \( q = \mathbb{P}_\sigma[\theta_j = 1] = \mathbb{E}_\sigma[\theta_j] \), and \((p_0, p_1)\) such that \( p_k = \mathbb{P}_\sigma[\theta_i = 1 | \theta_j = k] \). Additionally, let \( g(q, p_0, p_1) = \mathbb{E}_\sigma[u(\theta_i, \theta_j)] \). Substituting above, we get:

\[
g(q, p_0, p_1) = u(1, 1)p_0q + u(1, 0)((1 - p_0)q + p_1(1 - q)) + u(0, 0)(1 - p_1)(1 - q)
\]

Our goal is to show that \( g(q, p_0, p_1) \) is monotone increasing in \( q \). For this, we take the partial derivative with respect to \( q \):

\[
\frac{\partial g}{\partial q} = u(1, 1)p_1 + u(1, 0)(1 - p_0 - p_1) - u(0, 0)(1 - p_0)
\]

\[
= p_1(u(1, 1) - u(1, 0)) + (1 - p_0)(u(1, 0) - u(0, 0)) \geq 0,
\]

where the inequality follows from the fact that \( u(1, 1) \geq u(1, 0) \geq u(0, 0) \).

In the absence of signal, the strategy of a self-agnostic agent is obvious. Since \( \theta_j \sim \text{Ber}(p_j) \) for all \( j \in [n] \), we have that \( \mathbb{E}[\theta_j] = p_j \). Thus, each agent’s preference profile results by ordering agents according to \( p(1) \gg p(2) \gg \ldots \gg p(n-1) \), and the resulting stable matching is what we define above as \( m(\emptyset) \). Moreover, given a rank-order of agents \( a \gg b \gg c \gg d \gg \ldots \gg m \gg n \), the resulting teams (formed via stable matching) are \((a, b), (c, d), \ldots, (m, n)\). Thus, under any signal \( \sigma \), agents pair up sequentially according to the rank-order of their conditional expected types.

### 2.3 Finding the Optimal Policy

To summarize from above: the goal of the principal is to find a signaling policy that induces a stable matching and maximizes her expected utility. Additionally, the signaling policy must satisfy an individual rationality constraint, such that the agents are better off following the signal than ignoring it. Let \( m(\sigma) \) denote the matching induced when the principal sends signal \( \sigma \), and \( m^i(\sigma) \) denote the agent that agent \( i \) is matched with in matching \( m \). Similarly, \( m^i(\emptyset) \) is the agent that agent \( i \) is matched with in the absence of a signal. The principal’s optimization problem is given by

\[
\max_{\phi} \quad \mathbb{E}[f(\theta, m(\sigma))] \\
\text{s.t.} \quad m(\sigma) \text{ is stable} \quad \forall \sigma \in \Sigma \\
\mathbb{E}[\theta_{m^i(\sigma)} | \sigma] \geq \mathbb{E}[\theta_{m^i(\emptyset)}] \quad \forall \sigma \in \Sigma, \forall i \in [n]
\]  

(1)

where the maximization is over the set of all randomized maps \( \phi \) from type-profiles \( \theta \) to signals \( \sigma \), and the expectations are taken over the joint distribution of \((\sigma, \theta)\) induced by \( \phi \).

A standard revelation-principle style argument tells us that there exists an optimal signaling policy that is straightforward and persuasive \[3\]. A persuasive signaling policy is a policy for which the induced actions form a Bayesian Nash equilibrium. A straightforward signaling policy corresponds to space of signals equaling the agents’ action space. In our setting, a straightforward signaling scheme is one that signals a matching for a given realization \( \theta \). Additionally, by Proposition \[1\] matchings are induced by a rank ordered preference list of agents’ expected posterior types. Combining these three facts, we get that it is sufficient to restrict our attention to signals that are rank-orderings of agents according to expected posterior type.

We first need some additional notation. Let \( \Sigma \) now denote the set of all orderings (or permutations) over the \( n \) agents. We use the notation \( \sigma^i \) to denote the identity of the \( i \)-th-placed individual with respect to the ordering \( \sigma \). Additionally, \( \sigma^i(\theta) \) denotes the type of agent \( \sigma^i \) for the specific
realization \( \theta \). We abuse notation and use \( \sigma^{m(i)} \) to denote the agent that the \( i \)th agent in ordering \( \sigma \) is matched to. Finally, \( \sigma^{m(i)}(\theta) \) denotes the type of the match of agent \( i \) under ordering \( \sigma \) of a given realization \( \theta \). The following example helps clarify this notation.

Example 1. Suppose \( n = 4 \), with agents A, B, C and D. Consider realization \( \theta = (\theta_A, \theta_B, \theta_C, \theta_D) = (1,1,0,0) \), and \( \sigma = (A \succ C \succ B \succ D) \). Recall that for a given rank-order, the agents pair up sequentially; Thus, \( \sigma(\theta) = (1,0,1,0) \), \( \sigma^2 = C \), \( \sigma^2(\theta) = 0 \). Similarly, \( \sigma^{m(2)} = A \), and \( \sigma^{m(2)}(\theta) = 1 \).

Proposition 2 now gives us a tractable way of representing a given signal is persuasive, i.e., that the matching induced by the announced ordering is indeed stable.

Proposition 2. Suppose it is optimal for agents to follow the signal announced by the principal. The matching induced by the announced ordering is stable if and only if the following holds:

\[
E \left[ \sigma^{m(i)}(\theta) \mid F_\sigma \right] \geq E \left[ \sigma^j(\theta) \mid F_\sigma \right] \quad \forall i \in [n-1], \forall j \text{ s.t. } \sigma^j > \sigma^{m(i)}, \sigma^j \neq \sigma^i. \tag{2}
\]

Proof. ( \( \implies \) ) : Suppose constraint (2) is satisfied. For agent \( i = 1 \), \( \arg \max_j E[\theta_j \mid F_\sigma] = \sigma^2 \). Similarly, for agent \( i' = \sigma^2 \), \( \arg \max_j E[\theta_j \mid F_\sigma] = \sigma^1 \). Thus, agents \( \sigma^1 \) and \( \sigma^2 \) will match in \( m(\sigma) \).

Replicating this logic, agents will never want to match with agents who are ranked below their match which ordering \( \sigma \) induced. Though constraint (2) leaves open the possibility that an agent \( \sigma^i \) may want to match with \( \sigma^{i^*} \) such that \( \sigma^{i^*} < \sigma^{m(i)} \) (i.e. ranked above their match in the above ordering), such a match would never occur since \( \sigma^{i^*} \) has no incentive to deviate and match with someone ranked lower than her own partner. Thus, the resulting matching is stable.

( \( \impliedby \) ) : Suppose there exists an ordering \( \sigma \) and two agents \( \sigma^{(i)}, \sigma^{(j)} \) such that \( \sigma^{(i)} < \sigma^{(j)} \) but \( E[\sigma^{(i)}(\theta) \mid F_\sigma] < E[\sigma^{(j)}(\theta) \mid F_\sigma] \). Since \( E[\sigma^{(i)}(\theta) \mid F_\sigma] < E[\sigma^{(j)}(\theta) \mid F_\sigma] \), agent \( \sigma^{m(i)} \) would rather match with agent \( \sigma^j \). Additionally, since \( E[\sigma^{m(i)}(\theta) \mid F_\sigma] \geq E[\sigma^{m(j)}(\theta) \mid F_\sigma] \), agent \( \sigma^j \) would rather match with agent \( \sigma^{m(i)} \) than agent \( \sigma^{m(j)} \). Thus, \( (\sigma^{m(i)}, \sigma^j) \) constitute a blocking pair, and \( m(\sigma) \) is unstable.

Let \( m_{00}(\sigma(\theta)) \) denote the number of ‘0-0’ matches induced when ordering \( \sigma \) is announced for realization \( \theta \). We abuse notation and write the principal’s utility function \( f(m_{00}(\sigma(\theta))) \) since, for a given realization \( \theta \) and an ordering \( \sigma \), there is a one-to-one correspondence between \( m_{00}, m_{10}, \) and \( m_{11} \). Moreover, under our assumption that \( f \) is strictly decreasing in \( m_{00}(\sigma(\theta)) \), for all \( \theta \), it suffices to minimize the expected number of matches composed of two type 0 individuals, i.e., \( \max_\phi E[f(m_{00}(\sigma(\theta)))] \equiv \min_\phi E[m_{00}(\sigma(\theta))] \). Putting this all together, we obtain a concise representation of the principal’s optimization problem as follows:

\[
\min_\phi \quad E[m_{00}(\sigma(\theta))]
\]
\[
\text{s.t.} \quad E[\sigma^{m(i)}(\theta) \mid F_\sigma] \geq E[\sigma^{j}(\theta) \mid F_\sigma] \quad \forall \sigma \in \Sigma, \forall i \in [n-1], \forall j \text{ s.t. } \sigma^j > \sigma^{m(i)}, \sigma^j \neq \sigma^i \quad (P)
\]
\[
\quad \quad E[\sigma^{m(i)}(\theta) \mid F_\sigma] \geq E[\theta^{m(i)}(\theta)] 
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\]

We henceforth denote the optimal value of the principal’s optimization \( (P) \) as \( OPT \).
3 Main Results

We now present our main results, wherein we characterize optimal and near-optimal signaling policies in two different scenarios: with self-agnostic agents (i.e., those who do not know their own types), and with self-aware agents (i.e., those who know their own types). In the former scenario, we first consider a setting with symmetric agents who have i.i.d. types (drawn from a $\text{Ber}(p)$ distribution), wherein we provide an optimal signaling policy. We then generalize this to a setting with a constant number ($K$) of clusters of symmetric agents, with the types of agents in cluster $k \in [K]$ drawn from a $\text{Ber}(p_k)$ distribution – here, we provide a signaling scheme which is asymptotically optimal in $n$ (for fixed $K$). On the other hand, in settings with self-aware agents, we show that no individually rational signalling scheme can do better than having no signal.

3.1 Self-Agnostic Agents

When agents do not have access to their own types, they share a common posterior induced by the announced ordering $\sigma$. Thus, the set of ‘persuasiveness’ constraints in Problem (P) reduces to:

$$
\mathbb{E} [\sigma^i(\theta) | \sigma] \geq \mathbb{E} [\sigma^{i+1}(\theta) | \sigma] \quad \forall \sigma \in \Sigma, \forall i \in [n - 1].
$$

Note that the above constraints require that, for two agents matched together, in expectation the agent who comes first in the ranking must indeed have a higher type than the agent who comes second. Adding this constraint is without loss of generality due to the assumption that agents’ utilities are anonymous.

Let $x_{\theta,\sigma}$ denote the probability of announcing ordering $\sigma$ when the realization is $\theta$. We consider the relaxed problem which ignores the individual rationality constraints of agents, and consequently is an upper bound on (P):

$$
\min_{\{x_{\theta,\sigma}\}} \sum_{\theta \in \Theta} \sum_{\sigma \in \Sigma} \lambda(\theta) x_{\theta,\sigma} m_{00}(\sigma(\theta))
$$

s.t. $\sum_{\theta \in \Theta} \sum_{\sigma \in \Sigma} \lambda(\theta) x_{\theta,\sigma} \sigma^i(\theta) \geq \sum_{\theta \in \Theta} \lambda(\theta) x_{\theta,\sigma} \sigma^{i+1}(\theta) \quad \forall \sigma \in \Sigma, \forall i \in [n - 1]$

$$
\sum_{\sigma \in \Sigma} x_{\theta,\sigma} = 1 \quad \forall \theta \in \Theta
$$

$$
x_{\theta,\sigma} \geq 0 \quad \forall \theta \in \Theta, \forall \sigma \in \Sigma
$$

Let $\tilde{\text{OPT}}$ denote the optimal value of (P). This LP is clearly computationally intractable, even for small values of $n$, since it is exponentially large in both the number of variables (of which there are $n!2^n$) and the number of constraints (of which there are $O(n!2^n)$).

**Single cluster** Despite the complexity of the above LP, we now show that for the special case of i.i.d. agent types (i.e., where all agents form a single cluster with the same prior on their types), we can actually demonstrate an optimal signaling scheme. Moreover, the resulting utility obtained by the principal matches the first-best solution, i.e., the utility of the optimal matching ignoring the individual rationality and persuasiveness constraints.
**Theorem 1.** In the single cluster setting where all agents have i.i.d. types, for any realized type-profile $\theta$, the principal can achieve $m_{00}(\sigma^*(\theta)) = \min_{\sigma} m_{00}(\sigma(\theta))$. Thus,

$$OPT = \mathbb{E} \left[ \min_{\sigma} m_{00}(\sigma(\theta)) \right].$$

**Proof outline.** To prove the theorem, we first solve the relaxed problem $[\tilde{P}]$, and show that, for all realizations $\theta$, the principal can construct a scheme such that the number of matches between two type 0 agents is as small as possible. We henceforth refer to this scheme as First Best (denoted $FB$) since this is indeed the best achievable utility without strategic considerations. We then show that this signaling scheme satisfies the individual rationality constraint for all agents, and hence is feasible for $[P]$. We provide a brief outline of the scheme here, but owing to space constraints, defer the complete proof to the appendix.

The main intuition behind designing our persuasive signaling scheme to achieve the first-best outcome is to pair up type-profiles for a given signal $\sigma$ such that together they satisfy the rank-order in the signal. In particular, since agents are self-agnostic, we can leverage this by pairing up type-profiles such that for each type-profile $\theta$ under which two agents of different types are matched, there exists a profile $\bar{\theta}$, realized with equal probability, in which the same agents are matched, but have their types flipped. This ensures that the incentive constraints in $[\tilde{P}]$ are satisfied, as we are essentially incentivizing a strong agent to accept being matched with weak agent via a promise of matching with a strong agent when they themselves are weak.

Finally, the fact that the above scheme satisfies the participation constraints depends crucially on the concavity of the agents' utility function. The intuition behind this is that under a concave $u(\cdot)$, the additional value of a ‘1-1’ match compared to a ‘1-0’ match is less than the value of being in a ‘1-0’ match than being in a ‘0-0’ match. Consequently, having an assurance of being in a 0-0 team as rarely as possible dominates the match under $m(\emptyset)$.

To summarize, Theorem 1 states that when agents have i.i.d. types and are agnostic of their own types, the principal has enough freedom to match as many type 1 agents with type 0 agents as possible, and to do so such that (i) the declared rank-ordering is ‘persuasive’ and (ii) agents are better off under this mechanism than under their myopic strategy. In the next section, we use the $FB$ signaling scheme as a critical primitive for the multi-cluster setting.

**Multiple clusters** A natural next question to ask would be how well the principal can do if there are multiple clusters of agents, with agents within each cluster drawing i.i.d. types. Formally, suppose there are $K \geq 2$ groups of agents, with the types of the $n_k$ agents in each group $k$ drawn independently from a $Ber(p_k)$ distribution. We assume that $p_k \neq p_{k'}$ for all $k \neq k'$. Additionally, we denote $\ell_k(\theta), h_k(\theta)$ the number of agents with a true type of 0 and 1, respectively, in cluster $k$.

Consider the scheme that only matches agents within their own clusters, and does so using the $FB$ signaling scheme from above; we refer to this scheme as ‘Cluster First Best’, denoted as $FB_C$. Given Theorem 1, this scheme is feasible, and thus gives us an upper bound on $OPT$. However, the scheme is sub-optimal for $K \geq 2$ clusters, since it misses opportunities to match excess type 1 and type 0 agents across clusters, an event which occurs a non-trivial number of times.

Despite this, as the number of agents $n$ scales, while keeping the number of clusters constant, then we show that the Cluster First Best policy gets arbitrarily close to the optimal solution for $K = 2$ clusters.
Theorem 2. For \( K = 2 \) clusters and \( n_1, n_2 \) such that \( \sqrt{\frac{\ln n_1}{n_1}} \leq |p_1 - 1/2|, \sqrt{\frac{\ln n_2}{n_2}} \leq |p_2 - 1/2| \), then Cluster First Best achieves \( o(1) \) regret (with respect to \( n \)).

As noted above, Cluster First Best is suboptimal since it misses occasions where it could induce agents of different types to match across different clusters. The first important realization is that the only realizations for which Cluster First Best is suboptimal are realizations for which there is an excess of type 1 agents in one cluster, and an excess of type 0 agents in the other cluster. For all other realizations, Cluster First Best is in fact optimal.

For the settings where there are potential gains from matching agents across clusters, we can no longer use the first-best solution as a benchmark. To show that the \( FB_C \) policy is asymptotically optimal in these cases is much more challenging. We provide a brief sketch of this proof below, and defer the formal proof to the appendix.

Proof outline. The proof of the theorem considers three regions of the \((p_1, p_2)\) space:

(i) \( p_1 > p_2 > 1/2 \) , (ii) \( p_2 < p_1 < 1/2 \) , (iii) \( p_2 < 1/2 < p_1 \)

Moreover, via standard measure concentration arguments, we can restrict ourselves to considering realizations that lie in a 'typical set' (following standard information theoretic definitions; cf. [6]):

\[ A_{(n_1, n_2)} = \{ \theta : |h_1(\theta) - n_1p_1| \leq \epsilon_1n_1, |h_2(\theta) - n_2p_2| \leq \epsilon_2n_2 \}. \]

Note that for sufficiently large \( n \), the realized type-profiles in \( A_{(n_1, n_2)} \) are such that \( h_1(\theta), h_2(\theta) \) are in the same orthant as \( n_1p_1, n_2p_2 \), respectively.

Our proof for cases (i) and (ii) relies on the key observation that in these regions, the induced matching under Cluster First Best is exactly First Best. In Case (i), for example, all realizations in \( A_{(n_1, n_2)} \) will have an excess of type 1 agents in each cluster. Thus, if we simply match optimally within each cluster, only type 1 agents remain to be matched, and we could not have done better. A similar argument holds for Case (ii). We finish our argument by using concentration inequalities for large enough \( n_1, n_2 \).

The main technical challenge in the proof is in dealing with Case (iii). Here, we exhibit a dual certificate solution that gives a lower bound on the optimal solution for the restricted space \( A_{(n_1, n_2)} \). In particular, we construct a feasible solution for the dual LP of \((\mathcal{P})\) whose value is exactly that of Cluster First Best. The construction relies on an intricate inductive argument, the details of which we defer to the appendix.

Finally, again using standard concentration inequalities, we show that, in going from the typical set \( A_{(n_1, n_2)} \) to the entire space \( \Theta \), the lower bound decreases at most by \( o(1) \) for large enough \( n_1, n_2 \).

Moreover, Theorem\(^2\) can be extended to provide a similar result for \( K > 2 \) clusters.

Corollary 1. For \( K > 2 \) clusters with \( n_1, \ldots, n_K \) agents, such that \( \sqrt{\frac{\ln n_1}{n_1}}, \ldots, \sqrt{\frac{\ln n_K}{n_K}} \leq |p_K - 1/2|, \), Cluster First Best achieves \( o(1) \) regret.

The proof of the corollary requires an additional induction on the number of clusters. The high-level idea remains: for large clusters, the number of type 1 agents in each cluster is in the \((n_1p_1, \ldots, n_Kp_K)\) orthant with high probability. Thus, there do not exist enough realizations in

\(^2\) Restricting to \( p_1 > p_2 \) is without loss of generality since we can relabel the clusters.
The other orthants that would ‘fix’ the excesses in this high probability orthant. We defer the formal proof to the appendix.

Theorem 2 and Corollary 1 tell us that, though sophisticated information disclosure policies can induce agents to pair with agents in other clusters, for a large enough number of agents, the fraction of type 1 agents in the ‘stronger’ clusters concentrates so much so that the principal has no leverage to induce them to match with an agent in a weaker cluster. Thus, the principal does not gain a significant amount from more sophisticated signaling schemes, and Cluster First Best is close to optimal.

3.2 Self-Aware Agents

We now turn our attention to settings where agents are aware of their own types. We use the same solution concepts as for self-agnostic agents: individual rationality and stability. In this case, however, the filtration induced by the signal is no longer common to all agents, since they have the additional knowledge of their own type. Consequently, stability is a much more difficult goal to achieve in this scenario.

Following a similar argument as in Section 3.1, the principal’s optimization problem in the setting with self-aware agents can be written as:

\[
\begin{align*}
\min_{\{x_{\theta,\sigma}\}} & \sum_{\theta \in \Theta} \sum_{\sigma \in \Sigma} \lambda(\theta) x_{\theta,\sigma} m_{00}(\sigma(\theta)) \\
\text{s.t.} & \sum_{\theta,\sigma^{t}(\theta)=1} \lambda(\theta) x_{\theta,\sigma} \sigma^{m(i)}(\theta) \geq \sum_{\theta,\sigma^{t}(\theta)=1} \lambda(\theta) x_{\theta,\sigma} \sigma^{j}(\theta) \quad \forall \sigma \in \Sigma, \forall i \in [n-1], \forall j > m(i) \\
& \sum_{\theta,\sigma^{t}(\theta)=0} \lambda(\theta) x_{\theta,\sigma} \sigma^{m(i)}(\theta) \geq \sum_{\theta,\sigma^{t}(\theta)=0} \lambda(\theta) x_{\theta,\sigma} \sigma^{j}(\theta) \quad \forall \sigma \in \Sigma, \forall i \in [n-1], \forall j > m(i) \\
& \mathbb{E} \left[ \sigma^{m(i)}(\theta) \mid \theta_i, \sigma \right] \geq \mathbb{E} \left[ \sigma_{m(\theta)}(\emptyset) \mid \theta_i \right] \quad \forall \sigma \in \Sigma, \forall i \in [n], \theta_i \in \{0,1\} \\
& \sum_{\sigma \in \Sigma} x_{\theta,\sigma} = 1 \quad \forall \theta \in \Theta \\
& x_{\theta,\sigma} \geq 0 \quad \forall \theta \in \Theta, \forall \sigma \in \Sigma
\end{align*}
\]

Intuitively, the principal has less leverage when agents know their own types than in the scenario with self-agnostic agents, since she cannot compensate for the times where a type 1 agent was matched with a type 0 agent by the times where this same agent was type 0 and was matched with a type 1 agent instead. Additionally, from a computational point of view, the optimization problem appears more difficult to solve since the information disclosure policy now has to take into account the different priors that agents have over one another.

This point however turns out to be moot, as we show below that this setting suffers from a strong impossibility result:

**Theorem 3.** Consider a single cluster of agents with i.i.d. types. If the agents are self-aware, then no individually rational signaling policy can perform better than random.

**Proof.** Suppose that each agent has i.i.d. type drawn from a Ber(p) distribution, and is aware of her own type. Now consider any signaling policy \( S \). The individual rationality constraint imposes
that the expected utility of each agent \(i\) under \(S\) must be at least her expected utility when matched with a random agent.

Suppose agent \(i\) is a type 1 agent. Under her myopic strategy, agent \(i\)’s expected utility is:

\[
E[u_i \mid \theta_i = 1] = pu(1,1) + (1-p)u(1,0)
= u(1,0) + p(u(1,1) - u(1,0)).
\]

Let \(p_S\) denote the probability that agent \(i\) is matched with another type 1 agent under signaling policy \(S\). Under \(S\), agent \(i\)’s expected utility is:

\[
E[u_i \mid \theta_i = 1, S] = p_Su(1,1) + (1-p_S)u(1,0)
= u(1,0) + p_S(u(1,1) - u(1,0)).
\]

To satisfy the agent rationality constraint, \(\mathcal{M}\) must satisfy:

\[
\begin{align*}
&u(1,0) + p_S(u(1,1) - u(1,0)) \\
&\geq u(1,0) + p(u(1,1) - u(1,0)).
\end{align*}
\]

Equivalently, since \(u(1,1) > u(1,0)\), \(\mathcal{S}\) must satisfy \(p_S \geq p\).

Recall the objective function from Problem \([P]\):

\[
\sum_{\theta \in \Theta} \sum_{\sigma \in \Sigma} \lambda(\theta)x_{\theta,\sigma}m_{00}(\sigma(\theta)),
\]

where \(m_{00}(\sigma(\theta))\) denotes the number of matches of two type 0 agents when signal \(\sigma\) is announced. Moreover, since \((m_{00}, m_{10}, m_{11})\) is one-dimensional, it is enough for us to reason about the number of matches between two type 1 agents. Now note that we can expand \(m_{11}(\sigma(\theta))\) as:

\[
m_{11}(\sigma(\theta)) = \frac{1}{2} \sum_{i=1}^{n} \mathbb{1}\{\sigma^i(\theta) = 1, \sigma^{m(i)}(\theta) = 1\}.
\]

Plugging this into our objective of interest, we obtain:

\[
\begin{align*}
\sum_{\theta \in \Theta} \sum_{\sigma \in \Sigma} \lambda(\theta)x_{\theta,\sigma} \sum_{i=1}^{n} \frac{1}{2} \mathbb{1}\{\sigma^i(\theta) = 1, \sigma^m(\theta) = 1\}
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{\theta, \sigma} \lambda(\theta)x_{\theta,\sigma} \sigma^m(\theta) \\
&= \frac{1}{2} \sum_{i=1}^{n} p_S \\
&\geq \frac{1}{2} \sum_{i=1}^{n} p \\
&= \frac{np}{2},
\end{align*}
\]

where inequality \([4]\) follows from the individual rationality constraint. This lower bound is achieved by matching each type 1 agent with another type 1 agent with probability \(p\), which is equivalent to having agents match randomly.

\[\square\]

Moreover, given Theorem \([3]\) the following corollary is immediate, since the \(K = 1\) cluster case is one where the principal has the most leverage.

**Corollary 2.** In a setting with self-aware agents belonging to \(K \geq 1\) clusters, any individually rational signaling policy cannot achieve a utility that is better than that achieved with no signalling (i.e., wherein agents pair randomly within clusters).
4 Extensions

Although our results consider the setting in which agents’ types are binary, the methods easily extend to more general discrete distributions (e.g., multinomial) by carefully defining analogs for the principal’s and agents’ utility functions. For example, in the self-agnostic case with one cluster, with linear and symmetric agent utilities, and principal’s utility that is increasing in the difference between match types, a nearly identical construction gives us that First Best is achievable by the principal. The setting with many clusters requires additional assumptions on the distributions from which each cluster is drawn. Namely, for binary types, our results relied heavily on the fact that $p_1 > p_2$. Intuitively, first-order stochastic dominance should be sufficient for similar results to hold (i.e., Cluster First Best achieves sublinear regret).

Another important setting for which our techniques should extend is that of bipartite matching. One motivating example is that of matching riders and drivers on ridesharing platforms. On these platforms, both riders and drivers have ratings. Companies such as Lyft and Uber have an incentive to want diverse matches, since matches of poorly rated riders to poorly rated drivers result in negative experiences for all users, and are likely to create self-reinforcing cycles of poor ratings. If there was a way for the platform to obtain private information on each side (say, selectively display some of the ratings), then using our techniques, there is hope for more diverse matches to be induced.

In the bipartite matching problem, the one-cluster analog would be each side of the platform drawn from a $Ber(p_{left}), Ber(p_{right})$ distribution. Again, an analogous construction using the ‘complement’ of a realized type profile imply similar results. Further, the case with multiple clusters on each side lends itself to the intuitive result that the platform cannot benefit from having, for example, the highest ranked cluster on the left match with any other cluster than the highest ranked cluster on the right. We leave the derivation of this very natural and important extension as future work.

References

1. Arieli, I., Babichenko, Y.: Private bayesian persuasion. SSRN preprint [http://dx.doi.org/10.2139/ssrn.2721307](http://dx.doi.org/10.2139/ssrn.2721307) (2016)
2. Bahar, G., Smorodinsky, R., Tennenholtz, M.: Economic recommendation systems. arXiv preprint [arXiv:1507.07191](https://arxiv.org/abs/1507.07191) (2015)
3. Bender, E., et al.: Crowdsourced solutions. Nature 533(3) (2016)
4. Bhaskar, U., Cheng, Y., Ko, Y.K., Swamy, C.: Hardness results for signaling in bayesian zero-sum and network routing games. In: Proceedings of the 2016 ACM Conference on Economics and Computation. pp. 479–496. ACM (2016)
5. Council, N.R., et al.: Enhancing the effectiveness of team science. National Academies Press (2015)
6. Cover, T.M., Thomas, J.A.: Elements of information theory. John Wiley & Sons (2012)
7. Das, S., Kamenica, E., Mirka, R.: Reducing congestion through information design. In: Communication, Control, and Computing (Allerton), 2017 55th Annual Allerton Conference on. pp. 1279–1284. IEEE (2017)
8. DREAM Challenge: [dreamchallenges.org](https://dreamchallenges.org) (Accessed July 27, 2018)
9. Dughmi, S.: On the hardness of signaling. In: Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on. pp. 354–363. IEEE (2014)
10. Dughmi, S.: Algorithmic information structure design: a survey 15(2) (2017)
11. Dughmi, S., Xu, H.: Algorithmic persuasion with no externalities. In: Proceedings of the 2017 ACM Conference on Economics and Computation. pp. 351–368. ACM (2017)
12. Gale, D., Shapley, L.S.: College admissions and the stability of marriage. The American Mathematical Monthly 69(1), 9–15 (1962)
13. Hoeffding, W.: Probability inequalities for sums of bounded random variables. Journal of the American statistical association 58(301), 13–30 (1963)
14. Kamenica, E., Gentzkow, M.: Bayesian persuasion. American Economic Review 101(6), 2590–2615 (2011)
15. Kremer, I., Mansour, Y., Perry, M.: Implementing the “wisdom of the crowd”. Journal of Political Economy 122(5), 988–1012 (2014)
16. Lingenbrink, D., Iyer, K.: Optimal signaling mechanisms in unobservable queues with strategic customers (2017)
17. Mansour, Y., Slivkins, A., Syrgkanis, V.: Bayesian incentive-compatible bandit exploration. In: Proceedings of the Sixteenth ACM Conference on Economics and Computation. pp. 565–582. ACM (2015)
18. Marbach, D., Mattiussi, C., Floreano, D.: Combining multiple results of a reverse-engineering algorithm: Application to the dream five-gene network challenge. Annals of the New York Academy of Sciences 1158(1), 102–113 (2009)
19. Tavafoghi, H., Teneketzis, D.: Informational incentives for congestion games. In: Communication, Control, and Computing (Allerton), 2017 55th Annual Allerton Conference on. pp. 1285–1292. IEEE (2017)
A Omitted Proofs

Proof of Theorem 1

We first show that solving the relaxed LP \( \tilde{P} \) allows the principal to achieve the minimum number of matches between two type 0 agents, for each realization \( \theta \).

Lemma 1.

\[
\text{OPT} = \mathbb{E} \left[ \min_{\sigma} m_{00}(\sigma(\theta)) \right]
\]

Proof. The proof is constructive. Let \( \ell(\theta), h(\theta) \) denote the number of agents of type 0 and type 1, respectively. Clearly, we have that

\[
\ell(\theta) = n - h(\theta).
\]

Case 1: \( h(\theta) \in \{0, n\} \)

When \( h(\theta) = 0 \), for all \( \sigma \in \Sigma \), \( m_{00}(\sigma(\theta)) = n/2 \). Thus, the principal is indifferent among orderings, and can choose an arbitrary ordering \( \tilde{\sigma} \) and announce it with probability 1 (\( x_{\tilde{\theta}, \tilde{\sigma}} = 1 \)).

Similarly, when \( h(\theta) = n \), \( m_{00}(\sigma(\theta)) = 0 \) for all \( \sigma \in \Sigma \). As in the case above, the principal can choose an arbitrary ordering to announce with probability 1.

Case 2: \( h(\theta) \in \{1, n - 1\} \)

When \( h(\theta) = 1 \), \( m_{00}(\sigma(\theta)) = n/2 - 1 \) for all \( \sigma \). Additionally, when \( h(\theta) = n - 1 \), \( m_{00}(\sigma(\theta)) = 0 \) for all \( \sigma \). As in Case 1, the principal is indifferent between orderings.

Let \( \sigma_{\text{truth}} \) denote any ordering that correctly orders agent for a realization \( \theta \). Pick an arbitrary \( \sigma_{\text{truth}} \) and set \( x_{\tilde{\theta}, \sigma_{\text{truth}}} = 1 \).

Case 3: \( 1 < h(\theta) < n - 1 \)

We first illustrate our construction via an example.

Example 2. Consider \( \theta = (\theta_A, \theta_B, \theta_C, \theta_D, \theta_E, \theta_F) = (1, 1, 1, 1, 0, 0) \). Then, \( \sigma^* = (A \succ B \succ C \succ E \succ D \succ F) \) achieves zero ‘0-0’ matches. Consider \( \tilde{\theta} = (1, 1, 0, 0, 1, 1) \). Announcing the same signal \( \sigma^* \) for this realization also achieves zero ‘0-0’ matches. Additionally, note that since \( h(\theta) = h(\tilde{\theta}) \) and agents have i.i.d. types, \( \lambda(\theta) = \lambda(\tilde{\theta}) \). Suppose that, given either of these realizations (and for no other realization), the principal announces \( \sigma^* \) with probability 1. Then, when agents receive the signal \( \sigma^* \), the posterior probability of being in realization \( \theta \) is equal to the posterior probability of being in \( \tilde{\theta} \). Thus, the expected posterior types of agents \( A \) and \( B \) is 1; the expected posterior type of all other agents is 0.5, and the persuasiveness constraints are thus respected.

In the same spirit as the above example, our goal will be to find a realization \( \tilde{\theta} \) with which to pair \( \theta \) such that \( \sigma^*(\tilde{\theta}) \in \arg\min_{\sigma} m_{00}(\sigma(\theta)) \), and \( \sigma^*(\tilde{\theta}) \in \arg\min_{\sigma} m_{00}(\sigma(\tilde{\theta})) \).

Suppose \( h(\theta) > \ell(\theta) \), and let \( e(\theta) = h(\theta) - \ell(\theta) \) denote the excess of type 1 agents over type 0 agents. (An entirely analogous construction can be shown to be feasible for \( h(\theta) \leq \ell(\theta) \).) Let ordering \( \sigma^* \) be any permutation of \( \theta \) such that type 1 and type 0 individuals are flipped as many...
times as possible. That is, for such \( \sigma^*, \sigma^{{1}}(\theta) = 1, \sigma^{{2}}(\theta) = 1, \ldots, \sigma^{e(\theta)}(\theta) = 1, \) and \( \sigma^{e(\theta)+2}(\theta) = 1, \sigma^{e(\theta)+2}(\theta) = 0, \ldots, \sigma^{n-1}(\theta) = 1, \sigma^{n}(\theta) = 0. \) Clearly, \( \sigma^* \in \arg \min_{\sigma} m_{00}(\sigma(\theta)). \)

Let \( \bar{\theta} \) be the realization with \( h(\bar{\theta}) = h(\theta), \) and such that \( \sigma^{e(\theta)}(\bar{\theta}) = 1, \sigma^{e(\theta)+2}(\bar{\theta}) = 1, \ldots, \sigma^{n}(\bar{\theta}) = 1. \) This construction also makes it clear that \( \sigma^* \in \arg \min_{\sigma} m_{00}(\sigma(\bar{\theta})). \)

Let \( x_{\bar{\theta}, \sigma^*} = x_{\bar{\theta}, \sigma^*} = 1. \) We show that this construction satisfies the set of ‘persuasive’ constraints of \( \{P\}, \) namely:

\[
\sum_{\theta \in \Theta} \lambda(\theta) x_{\theta, \sigma^*} \sigma^{x_i}(\theta) \geq \sum_{\theta \in \Theta} \lambda(\theta) x_{\theta, \sigma^*} \sigma^{x_i+1}(\theta) \quad \forall i \in [n-1]
\]

\[
\Leftrightarrow \sum_{\theta \in \Theta} \lambda(\theta) x_{\theta, \sigma^*} \bar{v}(\sigma^{x_i}(\theta)) \geq 0 \quad \forall i \in [n-1],
\]

where \( \bar{v}(\sigma^{x_i}(\theta)) \triangleq \sigma^{x_i}(\theta) - \sigma^{x_i+1}(\theta). \)

For \( 1 \leq i \leq e(\theta), \) all entries of \( \sigma^*(\theta) \) and \( \sigma^*(\bar{\theta}) \) are 1. Thus, for \( 1 \leq i \leq e(\theta) - 1 \)

\[
\bar{v}(\sigma^{x_i}(\theta)) = \bar{v}(\sigma^{x_i}(\bar{\theta})) = 0 \implies \lambda(\theta) x_{\theta, \sigma^*} \bar{v}(\sigma^{x_i}(\theta)) + \lambda(\theta) x_{\bar{\theta}, \sigma^*} \bar{v}(\sigma^{x_i}(\bar{\theta})) = 0.
\]

As noted in the example, since \( h(\bar{\theta}) = h(\theta), \) and all agent types are drawn independently from the same \( \text{Ber}(p) \) distribution, we have \( \lambda(\bar{\theta}) = \lambda(\theta). \) Additionally, for \( i \geq e(\theta) + 1, \) our construction is such that \( \bar{v}(\sigma^{x_i}(\theta)) = -\bar{v}(\sigma^{x_i}(\bar{\theta})). \) These two facts together give us that, for \( i \geq e(\theta) + 1 \)

\[
\lambda(\theta) x_{\theta, \sigma^*} \bar{v}(\sigma^{x_i}(\theta)) + \lambda(\bar{\theta}) x_{\bar{\theta}, \sigma^*} \bar{v}(\sigma^{x_i}(\bar{\theta})) = \lambda(\theta)(\bar{v}(\sigma^{x_i}(\theta)) - \bar{v}(\sigma^{x_i}(\bar{\theta}))) = 0.
\]

It remains for us to show that the constraint is satisfied for \( i = e(\theta). \) By construction, \( \bar{v}(\sigma^{e(\theta)}(\theta)) = 0, \) and \( \bar{v}(\sigma^{e(\theta)}(\bar{\theta})) = 1. \) Thus:

\[
\lambda(\theta) x_{\theta, \sigma^*} \bar{v}(\sigma^{e(\theta)}(\theta)) + \lambda(\bar{\theta}) x_{\bar{\theta}, \sigma^*} \bar{v}(\sigma^{e(\theta)}(\bar{\theta})) = \lambda(\theta) > 0.
\]

\[
\square
\]

We have shown that we can achieve First Best in our relaxed problem. Although the construction that we have presented is not necessarily symmetric (i.e., not all type 1 agents have the same probability of being matched with a type 0 agent), we can randomize this scheme over realizations, since agents themselves are symmetric. This allows the First Best Scheme to be both symmetric and feasible in \( \{P\}. \) Thus, henceforth we use the term ‘First Best’ to refer to the symmetric version of the above signaling scheme.

It remains for us to show that First Best is also individually rational. That is, each agent’s expected utility under the signaling mechanism is at least as large as her expected utility if she were to act myopically (which, in the one-cluster case, would be simply to pair up with someone randomly).

**Lemma 2.** The First Best signaling scheme is individually rational. Thus, it is optimal for all agents to follow the principal’s signal.
Proof. We first derive the expected utility of agent \( i \) under her myopic strategy, in the symmetric case.

\[
\mathbb{E}[u_i] = \mathbb{E}[u \mid \theta_i = 1] \mathbb{P}[	heta_i = 1] + \mathbb{E}[u \mid \theta_i = 0] \mathbb{P}[	heta_i = 0]
\]

\[
= p\left(pu(1,1) + (1-p)u(1,0)\right) + (1-p)\left(pu(1,0) + (1-p)u(0,0)\right)
\]

\[
= p\left(u(1,1) - (1-p)(u(1,1) - u(1,0))\right) + (1-p)\left(u(0,0) + p(u(1,0) - u(0,0))\right)
\]

\[
= pu(1,1) + (1-p)u(0,0) + p(1-p)\left(2u(1,0) - u(1,1) - u(0,0)\right).
\]

Under the First Best scheme \( \mathcal{M}^* \), the expected utility of agent \( i \) is given by:

\[
\mathbb{E}[u_i \mid \mathcal{M}^*] = \mathbb{E}\left[\theta_i \left(\min\left\{1, \frac{\ell(\theta)}{h(\theta)}\right\} u(1,0) + \left(1 - \min\left\{1, \frac{\ell(\theta)}{h(\theta)}\right\}\right) u(1,1)\right)\right]
\]

\[
+ (1-\theta_i) \left(\min\left\{1, \frac{h(\theta)}{\ell(\theta)}\right\} u(1,0) + \left(1 - \min\left\{1, \frac{h(\theta)}{\ell(\theta)}\right\}\right) u(0,0)\right)
\]

\[
= \mathbb{E}\left[\theta_i \left(\min\left\{1, \frac{\ell(\theta)}{h(\theta)}\right\} (u(1,0) - u(1,1)) + u(1,1)\right)\right]
\]

\[
+ (1-\theta_i) \left(\min\left\{1, \frac{h(\theta)}{\ell(\theta)}\right\} (u(1,0) - u(0,0)) + u(0,0)\right)
\]

\[
= pu(1,1) + (1-p)u(0,0) + (u(1,0) - u(0,0)) \mathbb{E}\left[\theta_i \min\left\{1, \frac{\ell(\theta)}{h(\theta)}\right\}\right]
\]

\[
+ (u(1,0) - u(0,0)) \mathbb{E}\left[(1-\theta_i) \min\left\{1, \frac{h(\theta)}{\ell(\theta)}\right\}\right]
\]

\[
= pu(1,1) + (1-p)u(0,0) + \frac{1}{n} \left(2u(1,0) - u(1,1) - u(0,0)\right) \mathbb{E}\left[\min\{i(\theta, j(\theta))\}\right],
\]

where the last equality results from the easy-to-show fact that in the symmetric setting

\[
\mathbb{E}\left[\theta_i \min\left\{1, \frac{\ell(\theta)}{h(\theta)}\right\}\right] = \mathbb{E}\left[(1-\theta_i) \min\left\{1, \frac{h(\theta)}{\ell(\theta)}\right\}\right] = \frac{1}{n} \mathbb{E}\left[\min\{\ell(\theta), h(\theta)\}\right].
\]

Since \( u_i \) is concave, it suffices for us to show that

\[
\frac{1}{n} \mathbb{E}\left[\min\{\ell(\theta), h(\theta)\}\right] \geq p(1-p) \quad \forall \, n \geq 2, p \in [0,1].
\]
We have:

\[
\frac{1}{n} \mathbb{E} \left[ \min \{\ell(\theta), h(\theta)\} \right] = \frac{1}{n} \sum_{k=1}^{n} k \binom{n}{k} \left[ p^k (1-p)^{n-k} + (1-p)^k p^{n-k} \right] \\
= p(1-p) \sum_{k=1}^{n} \binom{n-1}{k-1} \left[ p^{k-1} (1-p)^{n-k-1} + (1-p)^{k-1} p^{n-k-1} \right] \\
= p(1-p) \sum_{k=0}^{n-1} \binom{n-1}{k} \left[ p^{k} (1-p)^{n-k-2} + (1-p)^k p^{n-k-2} \right] \\
\geq p(1-p) \sum_{k=0}^{n-1} \binom{n-1}{k} \left[ p^{k} (1-p)^{n-k-1} + (1-p)^k p^{n-k-1} \right] \\
= p(1-p) \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} + \sum_{k=n}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k-1} \\
= p(1-p).
\]

\[\square\]

Theorem 1 follows immediately from Lemmas 1 and 2, since we have shown that the optimal scheme that achieves a lower bound on \(P\) is also feasible. \(\square\)

**Proof of Theorem 2:**

Let \(A_{\epsilon_1, \epsilon_2}^{(n_1, n_2)}\) be the set of all realizations \(\theta \in \Theta\) such that the number of type 1 individuals in each cluster is within an \(\epsilon_1 n_1, \epsilon_2 n_2\) neighborhood of \(n_1 p_1, n_2 p_2\), respectively.

\[A_{\epsilon_1, \epsilon_2}^{(n_1, n_2)} = \{ \theta : |h_1(\theta) - n_1 p_1| \leq \epsilon_1 n_1, |h_2(\theta) - n_2 p_2| \leq \epsilon_2 n_2 \}.
\]

We will require \(\epsilon_1, \epsilon_2\) to be such that \(A_{\epsilon_1, \epsilon_2}^{(n_1, n_2)}\) is entirely contained in the same orthant as \((n_1 p_1, n_2 p_2)\). That is, \(\epsilon_1 \leq |p_1 - 1/2|, \epsilon_2 \leq |p_2 - 1/2|\). Additionally, we define \(\overline{A}_{\epsilon_1, \epsilon_2}^{(n_1, n_2)}\) to be the complement of \(A_{\epsilon_1, \epsilon_2}^{(n_1, n_2)}\).

We begin by finding an upper bound on \(OPT\). Recall that our objective is with respect to \(m_{00}\), the number of ‘0-0’ matches. We abuse notation use \(FB_C(\theta)\) to denote the value achieved by the Cluster First Best scheme (optimally matching agents in each cluster), for a fixed realization \(\theta\).

**Lemma 3.**

\[OPT \leq \sum_{\theta \in \overline{A}_{\epsilon_1, \epsilon_2}^{(n_1, n_2)}} \lambda(\theta) FB_C(\theta) + o_{n_1+n_2}(1). \quad (UB)\]
Proof. We use the Cluster First Best scheme, which we know to be feasible, for our upper bound.

\[ OPT \leq \sum_{\theta \in \Theta} \lambda(\theta) F_{\text{BC}}(\theta) \]

\[ = \sum_{\theta \in \Theta_{1,2}^{n_1,n_2}} \lambda(\theta) F_{\text{BC}}(\theta) + \sum_{\theta \in \Theta_{1,2}^{n_1,n_2}} \lambda(\theta) F_{\text{BC}}(\theta) \]

\[ \leq \sum_{\theta \in \Theta_{1,2}^{n_1,n_2}} \lambda(\theta) F_{\text{BC}}(\theta) + n \sum_{\theta \in \Theta_{1,2}^{n_1,n_2}} \lambda(\theta) \]

\[ \leq \sum_{\theta \in \Theta_{1,2}^{n_1,n_2}} \lambda(\theta) F_{\text{BC}}(\theta) + n \left( e^{-2\epsilon_1^2 n_1} + e^{-2\epsilon_2^2 n_2} \right) \]

where the second inequality uses the crude upper bound of \( F_{\text{BC}}(\theta) \leq n \), and the final inequality follows from Hoeffding’s inequality \[13\].

Let \( \epsilon_1 = \sqrt{\frac{\ln n_1}{n_1}} \), \( \epsilon_2 = \sqrt{\frac{\ln n_2}{n_2}} \). Since \( \sqrt{\frac{\ln n}{n}} \) is monotonically decreasing for \( n \geq 2 \), there exists \( n_0 \) such that for all \( n_1 \geq n_0, n_2 \geq n_0, \epsilon_1 \leq |p_1 - 1/2|, \epsilon_2 \leq |p_2 - 1/2| \), which satisfies our requirement of staying within the same region as \( np_1, np_2 \). Thus, we obtain the bound:

\[ OPT \leq \sum_{\theta \in \Theta_{1,2}^{n_1,n_2}} \lambda(\theta) F_{\text{BC}}(\theta) + 2n \cdot \frac{1}{n_1^2 + n_2^2} \]

\[ = \sum_{\theta \in \Theta_{1,2}^{n_1,n_2}} \lambda(\theta) F_{\text{BC}}(\theta) + \frac{2(n_1 + n_2)}{n_1^2 + n_2^2} \]

\[ = \sum_{\theta \in \Theta_{1,2}^{n_1,n_2}} \lambda(\theta) F_{\text{BC}}(\theta) + o_{n_1 + n_2}(1). \]

\[ \square \]

We treat the analysis of the lower bound separately, depending on the orthant in which \((p_1, p_2)\) resides: 1. \( p_1 > p_2 > 1/2 \) , 2. \( p_2 < p_1 < 1/2 \) , 3. \( p_2 < 1/2 < p_1 \).

The fact that \( p_1 > p_2 \) is without loss of generality, since we can simply relabel the clusters otherwise. Additionally, recall that \( p_1 = p_2 \) is the scenario in which we only have one cluster, for which we have shown that we can achieve First Best.

Let \( F_B(\theta) = \min_\sigma m_{00}(\sigma(\theta)) \). That is, \( F_B(\theta) \) is the minimum number of \('0-0'\) matches the principal can achieve without any individual rationality or stability constraints.

**Proposition 3.** For all \( \theta \in \Theta_{1,2}^{n_1,n_2} \), and settings where \( p_1 > p_2 \geq 1/2 \) or \( 1/2 \leq p_2 < p_1 \), \( F_{\text{BC}}(\theta) = F_B(\theta) \).

Proof. Since the Cluster First Best scheme matches as many type 1 agents to type 0 agents as possible in each cluster, the number of \('0-0'\) matches in each cluster is simply half of the remaining
(or excess) type 0 agents. Thus, we have:

\[
FB_C(\theta) = \left( \frac{\ell_1(\theta) - h_1(\theta)}{2} \right)^+ + \left( \frac{\ell_2(\theta) - h_2(\theta)}{2} \right)^+
\]

\[
= \begin{cases} 
0 & \text{if } p_1 > p_2 \geq 1/2 \\
\frac{\ell_1(\theta) - h_1(\theta)}{2} + \frac{\ell_2(\theta) - h_2(\theta)}{2} & \text{if } p_2 < p_1 < 1/2 \\
\left( \frac{(\ell_1(\theta) + \ell_2(\theta)) - (h_1(\theta) + h_2(\theta))}{2} \right)^+ & \text{if } p_1 \geq 1/2 \\
= FB(\theta).
\end{cases}
\]

We can re-write the upper bound (UB) as follows:

\[
OPT \leq \sum_{\theta \in A_{n_1,n_2}} \lambda(\theta)FB(\theta) + o_{n_1+n_2}(1). \quad (UB1)
\]

The proof of the lower bound for Cases 1 and 2 is straightforward. We use First Best as a crude lower bound on \(OPT\).

\[
OPT \geq \sum_{\theta \in \Theta} \lambda(\theta)FB(\theta)
\geq \sum_{\theta \in A_{n_1,n_2}} \lambda(\theta)FB(\theta)
= \sum_{\theta \in A_{n_1,n_2}} \lambda(\theta)FB_C(\theta). \quad (LB1)
\]

Putting the upper and lower bounds (UB1) and (LB1) together, we get that Cluster First Best achieves \(o_{n_1+n_2}(1)\) regret for Cases 1 and 2.

**Case 3:** We consider the case where \(p_1 > 1/2 > p_2\).

We obtain a lower bound on \(OPT\) via the dual, given by:

\[
\max_{z_\theta, y_{\sigma_1} \geq 0} \sum_{\theta} z_\theta \\
\text{subject to } z_\theta + \sum_{i=1}^{n-1} \lambda(\theta)\bar{v}(\sigma^i(\theta))y_{\sigma_1} \leq \lambda(\theta)m_{00}(\sigma(\theta)) \quad \forall \theta \in \Theta, \sigma \in \Sigma
\]

Consider the restricted space of realizations \(A_{\epsilon_1,\epsilon_2}^{(n_1,n_2)}\). In this orthant, \(A_{\epsilon_1,\epsilon_2}^{(n_1,n_2)} = \{ \theta : h_1(\theta) > n_1/2, h_2(\theta) < n_2/2 \}\). This set is of interest because it corresponds exactly to the realizations for which Cluster First Best fails, and some more sophisticated policies would succeed. Note that, for \(\theta \in A_{\epsilon_1,\epsilon_2}^{(n_1,n_2)}\), \(FB_C(\theta) = \frac{\ell_2(\theta) - h_2(\theta)}{2}\). If we modify the dual to be over this restricted space \(A_{\epsilon_1,\epsilon_2}^{(n_1,n_2)}\), we have the following result.
Lemma 4. For all $\sigma \in \Sigma$, $i \in [n-1]$, there exists $y_{\sigma i}$ such that $z_\theta = \lambda(\theta) \cdot \left( \frac{\ell_2(\theta) - h_2(\theta)}{2} \right) = \lambda(\theta) F B C_i(\theta)$, and $(z_\theta, y_{\sigma i})$ form a feasible set of solutions in the restricted space of realizations $A_{k_1, k_2}^{(n_1, n_2)}$.

To show this, we need to show that there exists $y_{\sigma i} \geq 0$, $\forall \sigma, i$, such that:

$$\sum_{i=1}^{n-1} \lambda(\theta) \bar{v}(\sigma^i(\theta)) y_{\sigma i} \leq \lambda(\theta) m_{00}(\sigma(\theta)) - \frac{\ell_2(\theta) - h_2(\theta)}{2} \quad \forall \theta \in A_{k_1, k_2}^{(n_1, n_2)}$$

$$\iff \sum_{i=1}^{n-1} \bar{v}(\sigma^i(\theta)) y_{\sigma i} \leq m_{00}(\sigma(\theta)) - \frac{\ell_2(\theta) - h_2(\theta)}{2} \quad \forall \theta \in A_{k_1, k_2}^{(n_1, n_2)}.$$  (6)

We will actually prove something stronger – namely, we will find a set of binary $y_{\sigma i}$ such that Equation (6) holds. This problem, then, reduces to finding an index set $I_\sigma$ for each permutation $\sigma$ such that

$$m_{00}(\sigma(\theta)) - \sum_{i \in I_\sigma} \bar{v}(\sigma^i(\theta)) \geq \frac{\ell_2(\theta) - h_2(\theta)}{2} \quad \forall \theta \in A_{k_1, k_2}^{(n_1, n_2)}.$$  (7)

We approach finding this set $I_\sigma$ through the following key observation: any permutation $\sigma$ can be produced through a sequence of swaps of two agents. Let the ‘identity’ permutation be the ordering $\sigma_0 = (1, 2, \ldots, n_1, n_1 + 1, \ldots, n_1 + n_2)$. Without loss of generality, we can restrict to reasoning about swaps of two agents with respect to the identity permutation, since agents within clusters can be arbitrarily re-labeled.

At a high level, we seek to show that the principal does not benefit from swapping individual across clusters. (We ignore the effect of swapping individuals within clusters, since we already know that Cluster First Best does this in the optimal way, and again, agents within clusters can simply be relabeled.) Thus, reasoning about the swaps that have created inter-cluster matches will be key.

Let $N$ be the number of swaps that create inter-cluster matches, and $\sigma_N$ be the permutation which created the $N$th inter-cluster match. We denote the identities of the swapped individuals under $\sigma_N$ to be $\sigma_N^{k_1^{(1)}}, \sigma_N^{k_1^{(2)}}$, respectively, where the superscript represents the cluster that each agent finds herself in after the swap. Additionally, let $\sigma_N^{m(k_1^{(1)})}, \sigma_N^{m(k_1^{(2)})}$ denote the respective identities of the partners of the two swapped agents from the $N$th swap. Finally, without loss of generality we assume that $k_1^{(1)} < k_1^{(2)} < \ldots < k_N^{(1)}$ and $k_1^{(1)} > k_1^{(2)} > \ldots > k_N^{(2)}$. This is simply done for ease of notation, as the individuals are in each cluster are symmetric and can be relabeled. Let $N_{\text{max}}$ denote the maximum number of swaps that can create inter-cluster matches with respect to the identity permutation.

Example 3. Suppose $n = 4$ with agents A, B, C, and D. Additionally, suppose A and B are in Cluster 1, and C and D are in Cluster 2, and we’ve performed $N = 1$ inter-cluster swaps, with $\sigma_1 = (A \succ C \succeq D) \succ B$ and $\theta = (1, 1, 0, 0)$. Then, $\sigma_1(\theta) = (1, 0, 1, 0)$. Further, $\sigma_1^{k_1^{(1)}} = C \sigma_1^{k_1^{(2)}} = B$; and $\sigma_1^{k_1^{(1)}}(\theta) = 0$ and $\sigma_1^{k_1^{(2)}}(\theta) = 1$; $\sigma_1^{m(k_1^{(1)})} = A, \sigma_1^{m(k_1^{(2)})} = D$; and $\sigma_1^{m(k_1^{(1)})}(\theta) = 1, \sigma_1^{m(k_1^{(2)})}(\theta) = 0$.

Recall that for fixed $\sigma, \theta, i \in [n-1]$, we define $\bar{v}(\sigma^i(\theta))$ to be the difference between two consecutive agent types under ordering $\sigma$. That is, $\bar{v}(\sigma^i(\theta)) = \sigma^i(\theta) - \sigma^{i+1}(\theta)$. Finally, we define $I_{\sigma_N}$
Lemma 5. For the above choice of $\mathcal{I}_{\sigma_{N}}$, and for all $1 \leq N \leq N_{\max}$,

$$m_{00}(\sigma(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{N}}} \bar{v}(\sigma_{N}^{i}(\theta)) \geq \frac{\ell_{2}(\theta) - h_{2}(\theta)}{2} \ \forall \, \theta \in A_{k_{1},k_{2}}^{(1,2)}.$$ 

Proof of Lemma 5. To show the claim, we will do an induction on $N$, the number of swaps that have created inter-cluster matches.

For notational convenience we simply refer to $\ell_{2}, h_{2}$ instead of $\ell_{2}(\theta), h_{2}(\theta)$. From context it should be clear that we are referring to the number of type 0 and type 1 individuals for that specific realization.

Base cases (even and odd):

1. $N = 0$:

$N = 0$ simply corresponds to the identity permutation, $\sigma_{0}$. Consider $\theta^{*} \in \arg\min_{\theta} m_{00}(\sigma(\theta))$. This realization corresponds to the maximal pairing of 1s and 0s, in which we case we obtain $m_{00}(\sigma_{0}(\theta^{*})) = \frac{\ell_{2} - h_{2}}{2}$. Thus, when $N = 0$, for all $\theta$:

$$m_{00}(\sigma_{0}(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{0}}} \bar{v}(\sigma_{0}^{i}(\theta)) = m_{00}(\sigma_{0}(\theta))$$

$$\geq \frac{\ell_{2} - h_{2}}{2}$$

2. $N = 1$:

For $N = 1$, one individual from cluster 1 was placed in cluster 2, and one individual from cluster 2 was placed in cluster 1. Let $\mathcal{I}_{\sigma_{1}}$ be such that

$$\sum_{i \in \mathcal{I}_{\sigma_{1}}} \bar{v}(\sigma_{1}^{i}(\theta)) = \sum_{i \in \mathcal{I}_{\sigma_{0}}} \bar{v}(\sigma_{0}^{i}(\theta)) + \sigma_{1}^{k_{1}}(\theta) - \sigma_{1}^{k_{2}}(\theta)$$

$$= \sigma_{1}^{k_{1}}(\theta) - \sigma_{1}^{k_{2}}(\theta)$$

Thus, for all $\theta$, we have:

$$m_{00}(\sigma_{1}(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{1}}} \bar{v}(\sigma_{1}^{i}(\theta)) = m_{00}(\sigma_{1}(\theta)) - \sigma_{1}^{k_{1}}(\theta) + \sigma_{1}^{k_{2}}(\theta) \ \forall \theta.$$ 

For $\theta$ such that $\sigma_{1}^{k_{1}}(\theta) = \sigma_{1}^{k_{2}}(\theta)$, $m_{00}(\sigma_{1}(\theta)) = m_{00}(\sigma_{0}(\theta)) \geq \frac{\ell_{2} - h_{2}}{2}$, which we showed above. We now consider $\theta$ such that $\sigma_{1}^{k_{1}}(\theta) \neq \sigma_{1}^{k_{2}}(\theta)$. There are two possibilities:
(a) Post-swap $\sigma_1^{k(1)}(\theta) = 0, \sigma_1^{k(2)}(\theta) = 1 \iff$ Pre-swap $\sigma_0^{k(1)}(\theta) = 1, \sigma_0^{k(2)}(\theta) = 0:

In this case, we have:

$$m_{00}(\sigma_1(\theta)) - \sigma_1^{k(1)}(\theta) + \sigma_1^{k(2)}(\theta) = m_{00}(\sigma_1(\theta)) + 1$$

We note that, in the worst case, $m_{00}(\sigma_0(\theta))$ could only have decreased by 1 (i.e. there was a ‘0-0’ match which became a ‘0-1’ or ‘1-0’ match, with no additional ‘0-0’ match created). Thus, we have:

$$m_{00}(\sigma_1(\theta)) \geq m_{00}(\sigma_0(\theta)) - 1$$

$$\implies m_{00}(\sigma_1(\theta)) + 1 \geq m_{00}(\sigma_0(\theta))$$

$$\geq \frac{\ell_2 - h_2}{2}$$

where the last inequality follows from $N = 0$.

(b) Post-swap $\sigma_1^{k(1)}(\theta) = 1, \sigma_1^{k(2)}(\theta) = 0 \iff$ Pre-swap $\sigma_0^{k(1)}(\theta) = 0, \sigma_0^{k(2)}(\theta) = 1:

In this case, we have:

$$m_{00}(\sigma_1(\theta)) - \sigma_1^{k(1)}(\theta) + \sigma_1^{k(2)}(\theta) = m_{00}(\sigma_1(\theta)) - 1 \quad (8)$$

We consider four cases, representing the possible types of the partners of the flipped agents. Throughout these four cases, we will use the fact that $\sigma_1^{k(1)}(\theta) = 1, \sigma_1^{k(2)}(\theta) = 0 \iff \sigma_0^{k(1)}(\theta) = 0, \sigma_0^{k(2)}(\theta) = 1$. Additionally, we use the important fact that, between two swaps, the only change is the types of the swapped agents. The types of the partners of the swapped agents remain the same.

i. $\sigma_1^{m(k(1))}(\theta) = \sigma_1^{m(k(2))}(\theta) = 1$:

Since the partners of both agents have the same type, flipping a 0 and 1 has no effect on the number of ‘0-0’ matches. Thus, we have $m_{00}(\sigma_1(\theta)) = m_{00}(\sigma_0(\theta))$.

Additionally, since $\sigma_0^{k(2)}(\theta) = 1$, we have two type 1 individuals paired together in the 2nd cluster with respect to the identity permutation. Thus, in the worst case, $m_{00}(\sigma_0(\theta)) \geq \frac{\ell_2 - h_2 - 2}{2} = \frac{\ell_2 - h_2}{2} + 1$, since the minimum number of ‘0-0’ matches is achieved by a maximal pairing of 1s and 0s.

Thus, from Equation (8), we obtain:

$$m_{00}(\sigma_1(\theta)) - 1 = m_{00}(\sigma_0(\theta)) - 1$$

$$\geq \left(\frac{\ell_2 - h_2}{2} + 1\right) - 1$$

$$= \frac{\ell_2 - h_2}{2}.$$ 

ii. $\sigma_1^{m(k(1))}(\theta) = \sigma_1^{m(k(2))}(\theta) = 0$:

For the same reason as the case above, we have $m_{00}(\sigma_1(\theta)) = m_{00}(\sigma_0(\theta))$.

Additionally, note that in this case we have $\sigma_0^{k(1)}(\theta) = 0, \sigma_0^{m(k(1))}(\theta) = 0$. This implies that, pre-swap (when agents were in the identity permutation), there was at least 1 ‘0-0’ match in the first cluster. Thus, we can conclude that $m_{00}(\sigma_0(\theta)) \geq \frac{\ell_2 - h_2}{2} + 1$, where the first summand corresponds to the number of ‘0-0’ matches in the second cluster, and the second summand corresponds to the ‘0-0’ match in the first cluster.
From Equation (8):

\[ m_{00}(\sigma_1(\theta)) - 1 = m_{00}(\sigma_0(\theta)) - 1 \]
\[ \geq \left( \frac{\ell_2 - h_2}{2} + 1 \right) - 1 \]
\[ = \frac{\ell_2 - h_2}{2} \]

iii. \( \sigma_1^{m(k^{(1)})}(\theta) = 1, \sigma_1^{m(k^{(2)})}(\theta) = 0: \)

Since \( \sigma_1^{m(k^{(2)})}(\theta) = 0 \) and \( \sigma_1^{k^{(2)}}(\theta) = 0 \), flipping the two agents created a ‘0-0’ match. Thus \( m_{00}(\sigma_1(\theta)) = m_{00}(\sigma_0(\theta)) + 1 \), and Equation (8) becomes:

\[ m_{00}(\sigma_1(\theta)) - 1 = (m_{00}(\sigma_0(\theta)) + 1) - 1 \]
\[ = m_{00}(\sigma_0(\theta)) \]
\[ \geq \frac{\ell_2 - h_2}{2}. \]

iv. \( \sigma_1^{m(k^{(1)})}(\theta) = 0, \sigma_1^{m(k^{(2)})}(\theta) = 1: \)

We first note that, since \( \sigma_1^{m(k^{(1)})}(\theta) = 0 \) and \( \sigma_0^{k^{(1)}}(\theta) = 0 \), the swap destroyed one ‘0-0’ match in the first cluster. Additionally, since \( \sigma_1^{m(k^{(2)})}(\theta) = 1 \), we know that no ‘0-0’ match could have been created in the second cluster. Thus, \( m_{00}(\sigma_1(\theta)) = m_{00}(\sigma_0(\theta)) - 1 \).

Since \( \sigma_0^{k^{(2)}}(\theta) = \sigma_0^{m(k^{(2)})}(\theta) = 1 \), under \( \sigma_0 \) there are two fewer type 1 agents available to match with type 0 agents. Thus, the minimum number of ‘0-0’ matches in the second cluster pre-swap (under \( \sigma_0 \)) is \( \frac{\ell_2 - (h_2 - 2)}{2} \). Additionally, we just noted that there was one ‘0-0’ match in the first cluster before the swap. Thus \( m_{00}(\sigma_0(\theta)) \geq 1 + \frac{\ell_2 - (h_2 - 2)}{2} = \frac{\ell_2 - h_2 + 2}{2} \).

Finally, Equation (8) becomes:

\[ m_{00}(\sigma_1(\theta)) - 1 = (m_{00}(\sigma_0(\theta)) - 1) - 1 \]
\[ \geq \left( \frac{\ell_2 - h_2}{2} + 2 \right) - 2 \]
\[ = \frac{\ell_2 - h_2}{2}. \]

**Inductive step (even):** Let \( 1 \leq N^* \leq N_{\text{max}} \) be an arbitrary odd integer. Suppose that \( I_{\sigma_{N^*}} \) satisfies the inequality of interest. We will show that the set \( I_{\sigma_{N^*+1}} \) satisfying \( \sum_{i \in I_{\sigma_{N^*+1}}} \tilde{v}(\sigma_{N^*+1}(\theta)) = \sum_{i \in I_{\sigma_{N^*}}} \tilde{v}(\sigma_{N^*}(\theta)) - m_{N^*+1}^{(1)} + m_{N^*+1}^{(2)} \) also satisfies the inequality of interest.

As before, we consider four scenarios:

1. \( m_{N^*+1}^{(1)} = 1, m_{N^*+1}^{(2)} = 1: \)

\[ m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in I_{\sigma_{N^*}}} \tilde{v}(\sigma_{N^*}(\theta)) + \sigma_{N^*+1}^{(1)}(\theta) - \sigma_{N^*+1}^{(2)}(\theta) \]
\[ = m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in I_{\sigma_{N^*}}} \tilde{v}(\sigma_{N^*}(\theta)) \]

24
Since \( m(k^{(1)}_{N+1}) = m(k^{(2)}_{N+1}) = 1 \), swapping two individuals will neither create not destroy a ‘0-0’ pair. Thus, \( m_0(\sigma_{N+1}^\ast(\theta)) = m_0(\sigma_N^\ast(\theta)) \). So, we have:

\[
m_0(\sigma_{N+1}^\ast(\theta)) - \sum_{i \in I_{\sigma_{N+1}^\ast}} \bar{v}(\sigma_N^\ast_i(\theta)) = m_0(\sigma_N^\ast(\theta)) - \sum_{i \in I_{\sigma_{N+1}^\ast}} \bar{v}(\sigma_N^\ast_i(\theta))
\]

\[
\geq \frac{\ell_2 - h_2}{2}
\]

by the inductive hypothesis.

2. \( \sigma_{N+1}^\ast(\theta) = 0, \sigma_{N+1}^\ast(\theta) = 0 \):

Since \( \sigma_{N+1}^\ast(\theta) = \sigma_{N+1}^\ast(\theta) = 0 \), swapping two individuals has no effect on the number of ‘0-0’ matches – either the two swapped agents had the same type, in which case we have established that \( m_0(\sigma_{N+1}^\ast(\theta)) = m_0(\sigma_N^\ast(\theta)) \), or one is a type 0 individual and one is a type 1 individual, in which case swapping them will simultaneously destroy and create a ‘0-0’ match. Thus:

\[
m_0(\sigma_{N+1}^\ast(\theta)) = \sigma_{N+1}^\ast(\theta) + \sum_{i \in I_{\sigma_{N+1}^\ast}} \bar{v}(\sigma_N^\ast_i(\theta))
\]

\[
\geq \frac{\ell_2 - h_2}{2},
\]

where the inequality follows from the inductive hypothesis.

3. \( \sigma_{N+1}^\ast(\theta) = 1, \sigma_{N+1}^\ast(\theta) = 0 \):

\[
m_0(\sigma_{N+1}^\ast(\theta)) - \sum_{i \in I_{\sigma_{N+1}^\ast}} \bar{v}(\sigma_N^\ast_i(\theta)) + \sigma_{N+1}^\ast(\theta) - \sigma_{N+1}^\ast(\theta)
\]

\[
= m_0(\sigma_N^\ast(\theta)) - \sum_{i \in I_{\sigma_{N+1}^\ast}} \bar{v}(\sigma_N^\ast_i(\theta)) + 1
\]

In the worst case \( m_0(\sigma_N^\ast(\theta)) \) can only decrease by 1, since only one pair of individuals was swapped. Thus:

\[
m_0(\sigma_{N+1}^\ast(\theta)) - \sum_{i \in I_{\sigma_{N+1}^\ast}} \bar{v}(\sigma_N^\ast_i(\theta)) + 1 \geq (m_0(\sigma_N^\ast(\theta)) - 1) - \sum_{i \in I_{\sigma_{N+1}^\ast}} \bar{v}(\sigma_N^\ast_i(\theta)) + 1
\]

\[
= m_0(\sigma_N^\ast(\theta)) - \sum_{i \in I_{\sigma_{N+1}^\ast}} \bar{v}(\sigma_N^\ast_i(\theta)) + 1
\]

\[
\geq \frac{\ell_2 - h_2}{2}
\]

by the inductive hypothesis.
4. \( m_{k_{N^*+1}^{(1)}}(\theta) = 0, m_{k_{N^*+1}^{(2)}}(\theta) = 1: \)

\[
\begin{align*}
m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{N^*}}} \bar{v}(\sigma_i^{N^*}(\theta)) &= m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{N^*}}} \bar{v}(\sigma_i^{N^*}(\theta)) - 1 \\
&= m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{N^*}}} \bar{v}(\sigma_i^{N^*}(\theta)) - 1 \\
&\geq \frac{\ell_2 - (h_2 - 2)}{2} - 1 \\
&= \frac{\ell_2 - h_2}{2}
\end{align*}
\]

Within this case, we consider four cases, based on the types of the swapped individuals for the second swap:

i. Post-swap \( \sigma_{N^*+1}^{(1)}(\theta) = 1, \sigma_{N^*+1}^{(2)}(\theta) = 1 \iff \) Pre-swap \( \sigma_{N^*}^{(1)}(\theta) = 1, \sigma_{N^*}^{(2)} = 1: \)

Recall that swapping two individuals with the same type has no effect on the number of ‘0-0’ matches. Thus, \( m_{00}(\sigma_{N^*+1}(\theta)) = m_{00}(\sigma_{N^*}(\theta)). \)

Additionally, since \( \sigma_{N^*+1}^{(2)}(\theta) = \sigma_{N^*}^{m_{k_{N^*+1}^{(2)}}}(\theta) = 1, \) we can view the permutation \( \sigma_{N^*} \) as one on \( n_1 \) individuals in cluster 1, \( n_2 - 2 \) individuals in cluster 2, and \( h_2 - 2 \) type 1 individuals in the second cluster. Thus, by the inductive hypothesis, we have:

\[
m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{N^*}}} \bar{v}(\sigma_i^{N^*}(\theta)) - 1 = m_{00}(\sigma_{N^*}(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{N^*}}} \bar{v}(\sigma_i^{N^*}(\theta)) - 1 \\
\geq \frac{\ell_2 - (h_2 - 2)}{2} - 1 \\
= \frac{\ell_2 - h_2}{2}
\]

ii. Post-swap \( \sigma_{N^*+1}^{(1)}(\theta) = 0, \sigma_{N^*+1}^{(2)}(\theta) = 0 \iff \) Pre-swap \( \sigma_{N^*}^{(1)}(\theta) = 0, \sigma_{N^*}^{(2)}(\theta) = 0: \)

As above, we have \( m_{00}(\sigma_{N^*+1}(\theta)) = m_{00}(\sigma_{N^*}(\theta)). \) Additionally, since \( \sigma_{N^*+1}^{(2)}(\theta) = m_{k_{N^*+1}^{(2)}}(\theta) = 0, \) we know that there is a ‘0-0’ match in the first cluster. Thus, we can view the permutation \( \sigma_{N^*} \) as one on \( n_1 - 2 \) individuals in cluster 1 with \( \ell_1 - 2 \) type 0 individuals, and \( n_2 \) individuals in cluster 2, plus an extra ‘0-0’ match. By the inductive hypothesis:

\[
m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{N^*}}} \bar{v}(\sigma_i^{N^*}(\theta)) - 1 = m_{00}(\sigma_{N^*}(\theta)) - \sum_{i \in \mathcal{I}_{\sigma_{N^*}}} \bar{v}(\sigma_i^{N^*}(\theta)) - 1 \\
\geq \left( \frac{\ell_2 - h_2}{2} + 1 \right) - 1 \\
= \frac{\ell_2 - h_2}{2}
\]

iii. Post-swap \( \sigma_{N^*+1}^{(1)}(\theta) = 0, \sigma_{N^*+1}^{(2)}(\theta) = 1 \iff \) Pre-swap \( \sigma_{N^*+1}^{(1)}(\theta) = 1, \sigma_{N^*+1}^{(2)}(\theta) = 0: \)
Since \( \sigma_{N^*+1}^{(1)}(\theta) = \sigma_{N^*+1}^{m(1)}(\theta) = 0 \), the swap created an additional ‘0-0’ match. Thus, 
\( m_{00}(\sigma_{N^*+1}(\theta)) = m_{00}(\sigma_{N^*}(\theta)) + 1 \). We have:

\[
m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in I_{\sigma_{N^*}}^*} \bar{v}(i_{\sigma_{N^*}}(\theta)) - 1 = (m_{00}(\sigma_{N^*}(\theta)) + 1) - \sum_{i \in I_{\sigma_{N^*}}^*} \bar{v}(i_{\sigma_{N^*}}(\theta)) - 1
\]
\[
= m_{00}(\sigma_{N^*}(\theta)) - \sum_{i \in I_{\sigma_{N^*}}^*} \bar{v}(i_{\sigma_{N^*}}(\theta))
\]
\[
\geq \frac{\ell_2 - h_2}{2}
\]

by the inductive hypothesis.

iv. Post-swap \( \sigma_{N^*+1}^{(1)}(\theta) = 1, \sigma_{N^*+1}^{m(1)}(\theta) = 0 \iff \) Pre-swap \( \sigma_{N^*}^{(1)}(\theta) = 0, \sigma_{N^*}^{m(1)}(\theta) = 1 \):

Note that, in this case, a ‘0-0’ match in the first cluster was destroyed, and none was created. Thus, 
\( m_{00}(\sigma_{N^*+1}(\theta)) = m_{00}(\sigma_{N^*}(\theta)) - 1 \).

Additionally, since \( \sigma_{N^*}^{m(k^{(2)}_{N^*+1})} = 1 \) and \( \sigma_{N^*}^{k^{(1)}_{N^*+1}} = 0 \), we can view the permutation \( \sigma_{N^*} \) as one on \( n_1 - 2 \) individuals in cluster 1, plus a ‘0-0’ match in cluster 1, and \( n_2 - 2 \) individuals in cluster 2, with \( h_2 - 2 \) type 1 individuals in the second cluster. Thus, by the inductive hypothesis:

\[
m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in I_{\sigma_{N^*}}^*} \bar{v}(i_{\sigma_{N^*}}(\theta)) - 1 = (m_{00}(\sigma_{N^*}(\theta)) - 1) - \sum_{i \in I_{\sigma_{N^*}}^*} \bar{v}(i_{\sigma_{N^*}}(\theta)) - 1
\]
\[
= m_{00}(\sigma_{N^*+1}(\theta)) - \sum_{i \in I_{\sigma_{N^*}}^*} \bar{v}(i_{\sigma_{N^*}}(\theta)) - 2
\]
\[
\geq \left( \frac{\ell_2 - (h_2 - 2)}{2} + 1 \right) - 2
\]
\[
= \frac{\ell_2 - h_2}{2}.
\]

**Inductive step (odd):** Let \( 2 \leq N^* \leq N_{\text{max}} \) be an arbitrary even integer. Suppose that \( I_{\sigma_{N^*}} \) satisfies the inequality of interest. We will show that the set \( I_{\sigma_{N^*+1}} \) satisfying \( \sum_{i \in I_{\sigma_{N^*+1}}} \bar{v}(i_{\sigma_{N^*+1}}(\theta)) = \sum_{i \in I_{\sigma_{N^*}}^*} \bar{v}(i_{\sigma_{N^*}}(\theta)) - \sigma_{N^*+1}^{(1)}(\theta) + \sigma_{N^*+1}^{m(1)}(\theta) \) also satisfies the inequality of interest.

We omit this proof, since it is entirely analogous to the case where \( N = 1 \), and the case-by-case work is identical to what was done for the even inductive step.

Thus, we have shown that this choice of \( I_{\sigma_N} \) satisfies Equation (7), and Lemma 4 immediately follows.

Using Lemma 4, we now have the lower bound for the case where \( p_1 > 1/2 > p_2 \), in our restricted space of realizations.

We construct a feasible solution to the dual using our \( \{y_{\sigma i}\} \) construction. We already showed that, for all \( \theta \in A_{\ell_{2}, \epsilon_{2}}^{(n_1, n_2)} \) (i.e. such that \( h_1(\theta) > n_1/2, h_2(\theta) < n_2/2 \), \( z_{\theta} = \lambda(\theta) \cdot \frac{\ell_2 - h_2}{2} \) is feasible. Thus, it remains to consider \( \theta \in A_{\ell_{2}, \epsilon_{2}}^{(n_1, n_2)} \). For such \( \theta \), and the set of \( y_{\sigma i} \) that we constructed, we
Lemma 6. Proposition 3.

As before, we will require that

\[ m_{00}(\sigma(\theta)) \geq 0, \quad \text{and} \quad \sum_{i \in I_{\sigma}} \bar{v}(\sigma^i(\theta)) \leq n. \]

By weak duality:

\[
\text{OPT} \geq \sum_{\theta \in A_{n_1,n_2}^{(n_1,n_2)}} z_{\theta} - n \sum_{\theta \in A_{n_1,n_2}^{(n_1,n_2)^{<}}} \lambda(\theta)
\]

\[
= \sum_{\theta \in A_{n_1,n_2}^{(n_1,n_2)}} z_{\theta} - \frac{2(n_1 + n_2)}{n_1 + n_2}
\]

\[
= \sum_{\theta \in A_{n_1,n_2}^{(n_1,n_2)}} \lambda(\theta) \cdot \left( \frac{\ell_2(\theta) - h_2(\theta)}{2} - o_{n_1+n_2}(1) \right)
\]

\[
= \sum_{\theta \in A_{n_1,n_2}^{(n_1,n_2)}} \lambda(\theta)FB_{C}(\theta) - o_{n_1+n_2}(1). \tag{LB2}
\]

where the second inequality follows from Hoeffding’s inequality, as was done to derive the upper bound.

Putting the upper and lower bounds \( (UB) \) and \( (LB2) \) together, we obtain our final result.

\[ \square \]

Proof of Corollary 1. Given \( p_1, p_2, \ldots, p_K \), we define \( A_{\ell_1,\ell_2,\ldots,\ell_K}^{(n_1,n_2,\ldots,n_K)} \) analogously to the \( K = 2 \) cluster case. That is:

\[
A_{\ell_1,\ell_2,\ldots,\ell_K}^{(n_1,n_2,\ldots,n_K)} = \left\{ \theta : |h_1(\theta) - n_1p_1| \leq \epsilon_1n_1, |h_2(\theta) - n_2p_2| \leq \epsilon_2n_2, \ldots, |h_k(\theta) - n_Kp_K| \leq \epsilon_Kn_K \right\}. \tag{9}
\]

As before, we will require that \( A_{\ell_1,\ell_2,\ldots,\ell_K}^{(n_1,n_2,\ldots,n_K)} \) be contained in the same orthant as \( (n_1p_1, \ldots, n_Kp_K) \), i.e. \( \epsilon_i \leq |p_i - 1/2| \forall i \in [K] \).

We will focus only on the interesting cases, in which there exists at least one cluster \( i \) such that \( p_i < 1/2 \), and one cluster \( j \) such that \( p_j > 1/2 \). Otherwise, the proof is identical to that of Proposition 3.

Lemma 6. For all \( \theta \in A_{\ell_1,\ell_2}^{(n_1,n_2)} \), there exists \( y_{\sigma_1} \) such that

\[
z_{\theta} = \min_{\sigma} \lambda(\theta) \left[ \left( \frac{\ell_1(\theta) - h_1(\theta)}{2} \right)^+ + \ldots + \left( \frac{\ell_K(\theta) - h_K(\theta)}{2} \right)^+ \right] = \lambda(\theta)FB_{C}(\theta) \text{ is a feasible dual solution.}
\]

Proof. We prove this by induction.

Base case: We already showed this for \( K = 2 \).

Inductive step: Let \( K^* > 2 \) be an arbitrary integer. Suppose the statement is true for \( K^* \).

The goal is to show that it must be true for \( K^* + 1 \). We prove this by contrapositive.
Suppose the statement is false for $K^* + 1$ clusters. This means that, for all $y_{σi} \geq 0$, there exists a pair $(θ, σ)$ such that

$$m_{00}(σ(θ)) - \sum_i \bar{v}(σ^i(θ))y_{σi} < \left(\frac{ℓ_1 - h_1}{2}\right)^+ + \cdots + \left(\frac{ℓ_{K^*} - h_{K^*}}{2}\right)^+ + \left(\frac{ℓ_{K^*+1} - h_{K^*+1}}{2}\right)^+ \quad (10)$$

We consider two cases:

1. $p_{K^*} < 1/2, p_{K^*+1} < 1/2$:

   In this case, from inequality (10) we obtain:

   $$m_{00}(σ(θ)) - \sum_i \bar{v}(σ^i(θ))y_{σi} < \left(\frac{ℓ_1 - h_1}{2}\right)^+ + \cdots + \left(\frac{ℓ_{K^*} - h_{K^*}}{2}\right)^+ + \left(\frac{ℓ_{K^*+1} - h_{K^*+1}}{2}\right)^+$$

   $$\quad = \left(\frac{ℓ_1 - h_1}{2}\right)^+ + \cdots + \left(\frac{ℓ_{K^*} - h_{K^*}}{2}\right)^+ + \left(\frac{ℓ_{K^*+1} - h_{K^*+1}}{2}\right)^+ \quad (11)$$

   Consider the right-hand side of inequality (11). Note that this corresponds exactly to the scenario with $K^*$ clusters with $δ_{K^*}$ agents, where $δ_{K^*} = n_{K^*} + n_{K^*+1}$. Let $ℓ_{K^*}, h_{K^*}$ be the number of type 0 and type 1 agents, respectively, in the $K^*$th cluster. Let $θ$ be such that $ℓ_{K^*} = ℓ_{K^*} + ℓ_{K^*+1}, h_{K^*} = h_{K^*} + h_{K^*+1}$. Additionally, since $θ ∈ A_{ε_1, ε_2, ..., ε_{K^*+1}}(n_1, n_2, ..., n_{K^*+1})$, we have:

   $$ℓ_{K^*} > n_{K^*/2}, ℓ_{K^*+1} > n_{K^*+1}/2 \implies ℓ_{K^*} > \frac{n_{K^*} + n_{K^*+1}}{2}$$

   $$\implies \tilde{θ} ∈ A_{ε_1, ε_2, ..., ε_{K^*}}(n_1, n_2, ..., n_{K^*+1})$$

   Thus, by inequality (11), there exists $σ ∈ Σ$ such that, for any $y_{σi} \geq 0$, for this specific realization $θ$:

   $$m_{00}(σ(θ)) - \sum_i \bar{v}(σ^i(θ))y_{σi} < \left(\frac{ℓ_1 - h_1}{2}\right)^+ + \cdots + \left(\frac{\tilde{ℓ}_{K^*} - \tilde{h}_{K^*}}{2}\right)^+,$$

   which implies that $z_{θ} = λ(θ) \left[\sum_{k=1}^{K^*} \left(\frac{ℓ_k(θ) - h_k(θ)}{2}\right)^+\right]$ is not feasible in the $K^*$-cluster case.

2. $p_{K^*} > 1/2, p_{K^*+1} < 1/2$:

   In this case, by inequality (10):

   $$m_{00}(σ(θ)) - \sum_i \bar{v}(σ^i(θ))y_{σi} < \frac{ℓ_{K^*+1} - h_{K^*+1}}{2}$$

   $$\quad = \left(\frac{ℓ_1 - h_1}{2}\right)^+ + \left(\frac{ℓ_{K^*} - h_{K^*}}{2}\right)^+ + \left(\frac{ℓ_{K^*+1} - h_{K^*+1}}{2}\right)^+$$

   $$\quad = \left(\frac{ℓ_1 + ℓ_{K^*} - (h_1 + h_{K^*})}{2}\right)^+ + \left(\frac{ℓ_{K^*+1} - h_{K^*+1}}{2}\right)^+$$

   where the last equality follows from the fact that $ℓ_1 < n_1/2, ℓ_{K^*} < n_{K^*}/2 \implies ℓ_1 + ℓ_{K^*} < h_1 + h_{K^*}$.
Applying a similar argument as the one in Case 1, we find that, if we are in the $K^*$-cluster case and fix $y_{σi} ≥ 0$, for $θ$ such that $\tilde{ℓ}_{K^*} = ℓ_1 + ℓ_{K^*}, \tilde{h}_{K^*} = h_1 + h^*_{K^*}, z_θ = λ(θ) \left[ \sum_{k=1}^{K^*} \left( \frac{ℓ_k - h_k}{2} \right)^+ \right]$ is not feasible.

This inductive argument gives us many properties about the $K > 2$ setting for free, namely that, for this set of $z_θ, θ ∈ A_{n_1,n_2}$, one can construct a $y_{σi} ∈ \{0,1\}, \forall σ ∈ Σ, i ∈ [n − 1]$, such that $(z_θ, y_{σi})$ is dual feasible. The argument for this is immediate, once one realizes that the $K > 2$ case can be reduced to the $K = 2$ case, by grouping all clusters such that $p_k > 1/2$, and $p_k < 1/2$, respectively.

We derive the upper and lower bounds in an identical fashion to the $K = 2$ case, and obtain our main result:

$$\sum_{θ ∈ A_{n_1,n_2,...,n_K}} λ(θ)FB_C(θ) - o_{n_1+...+n_K}(1) ≤ OPT ≤ \sum_{θ ∈ A_{n_1,n_2,...,n_K}} λ(θ)FB_C(θ) + o_{n_1+...+n_K}(1)$$