Quantum Cosmology and Dark Energy Model of Born-Infeld Type Scalar Field

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Abstract

In this paper, we consider a quantum model of gravitation interacting with a Born-Infeld(B-I) type scalar field $\varphi$. The corresponding Wheeler-Dewitt equation can be solved analytically for both very large and small cosmological scale factor. In the condition that small cosmological scale factor tend to limit, the wave function of the universe can be obtained by applying the methods developed by Vilenkin, Hartle and Hawking. Both Vilenkin’s and Hartle-Hawking’s wave function predicts nonzero cosmological constant. The Vilenkin’s wave function predicts a universe with a cosmological constant as large as possible, while the Hartle-Hawking’s wave function predicts a universe with positive cosmological constant, which equals to $\frac{1}{\lambda}$. It is different from Coleman’s result that cosmological constant is zero, and also different from Hawking’s prediction of zero cosmological constant in quantum cosmology with linear scalar field. We suggest that dark energy in the universe might result from the B-I type scalar field with potential and the universe can undergo a phase of accelerating expansion. The equation of state parameter lies in the range of $-1 < w < -\frac{1}{3}$. When the potential $V(\varphi) = \frac{1}{\lambda}$, our Lagrangian describes the Chaplygin gas. In order to give an explanation to the observational results of state parameter $w < -1$, we also investigate the phantom model that posses negative kinetic energy. We find that weak and strong energy conditions are violated for phantom B-I type scalar field. At last, we study a specific potential with the form $V_0(1 + \frac{\varphi}{\varphi_0})e^{-(\frac{\varphi}{\varphi_0})}$ in phantom B-I scalar field in detail.

The attractor property of the system is shown by numerical analysis.

Keywords: Born-Infeld type scalar field; Quantum cosmology; Dark energy; Phantom field

PACS: 98.80.Cq, 04.50.+h
I. INTRODUCTION

Today, one of the central questions in cosmology is the origin of the dark energy. Many candidates for dark energy have been proposed. Among these models, the important ones are cosmological constant, ”quintessence” [7], ”K-essence” [8] and ”tachyon field” [9]. On the other hand, nonlinear Born-Infeld theory has been considered in string theory and cosmology.

In this paper, we consider the quantum cosmology and dark energy model of nonlinear Born-Infeld type scalar field. The corresponding Lagrangian has been first proposed by Heisenberg [1] to describe the process of meson multiple production connected with strong field regime, as a generalization of the B-I electromagnetic field Lagrangian $L_{B-I} = b^2 [1 - \sqrt{1 + (\frac{1}{2\pi}) F_{\mu\nu} F^{\mu\nu}}]$. The Lagrangian of B-I type scalar field is

$$L_s = \frac{1}{\lambda} (1 - \sqrt{1 - \lambda \varphi,\mu \varphi,\nu g_{\mu\nu}}) - V(\varphi)$$  \hspace{1cm} (1)

Eq.(1) possesses some interesting characteristics [3] that nonsingular scalar field solution can be generated, and shock waves don’t develop under smooth and continuous initial conditions. Second when $g^{\mu\nu} \varphi,\mu \varphi,\nu \ll \frac{1}{\lambda}$, by Taylor expansion, Eq.(1) approximates to

$$L_s = \frac{1}{2} g^{\mu\nu} \varphi,\mu \varphi,\nu - V(\varphi)$$  \hspace{1cm} (2)

the linear theory is recovered. Third, when potential $V = \frac{1}{\lambda}$, our Lagrangian represents a universe filled with Chaplygin gas which maybe unify dark energy and dark matter [5].

We consider quantum creation of universe based on the Wheeler-Dewitt(WD) equation $\hat{H}\psi = 0$ in the superspace. This quantum approach applying to cosmology may help us avoid the cosmology singularity problem and understand what determined the initial state of the universe. In section II, we consider quantum cosmology with B-I type scalar field and constant potential $V(\varphi)$ which describes cosmological constant effectively. We also find the wave function of universe by applying the methods developed by Vilenkin and Hartle-Hawking [4]. Both Vilenkin’s and Hartle-Hawking’s wave function predicts nonzero positive cosmological constant. It is different from zero cosmological constant that has been predicted by Coleman [10] and Hawking [4]. In fact the H-H’s wave function with B-I type scalar field predicts a universe filled with Chaplygin gas. In section III, we investigate the cosmology with B-I type scalar field as dark energy. Section IV is conclusion.

II. QUANTUM COSMOLOGY WITH B-I TYPE SCALAR FIELD
In order to find the solution of the WD equation, we shall apply the minisuperspace model—a closed Robertson-Walker (R-W) metric. In the minisuperspace there are only two degrees of freedom: the scale factor $a(t)$ and scalar field $\varphi(t)$. Using Eq.(1) and by integrating with respect to space-components, the action

$$S = \int \frac{3\pi}{4G} (1 - \dot{a}^2) dt + \int 2\pi^2 a^3 \left[ \frac{1}{\lambda} (1 - \sqrt{1 - \lambda \dot{\varphi}^2}) - V(\varphi) \right] dt = \int L_g dt + \int L_s dt$$

(3)

where the upper-dot means the derivative with respect the time $t$. From the Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L_s}{\partial \dot{\varphi}} \right) - \frac{\partial L_s}{\partial \varphi} = 0$$

(4)

we can obtain

$$\dot{\varphi} = \frac{c}{\sqrt{a^6 + \lambda c^2}}$$

(5)

where $c$ is integral constant. From the above equation we know that cosmological scale factor $a$ is very large or small when $\dot{\varphi}$ is very small or large respectively. The critical kinetic energy $\frac{1}{2}(\dot{\varphi})^2_{max}$ is $\frac{1}{2}\lambda$.

To quantize the model, we first find out the canonical momenta $P_a = \frac{\partial L_g}{\partial \dot{a}} = -(3/2G)a \dot{a}$, $P_\varphi = \frac{\partial L_\varphi}{\partial \dot{\varphi}} = 2\pi^2 a^3 \dot{\varphi} / \sqrt{1 - \lambda \dot{\varphi}^2}$ and the Hamiltonian $H = P_a \dot{a} + P_\varphi \dot{\varphi} - L_g - L_s$. $H$ can be written as the follows

$$H = -\frac{G}{3\pi a} P_a^2 - \frac{3\pi}{4G} a \left[ 1 - \frac{8\pi G}{3} a^2 V(\varphi) \right] - \frac{2\pi^2 a^3}{\lambda} \left[ 1 - \sqrt{1 + \frac{\lambda P_\varphi^2}{4\pi^4 a^6}} \right]$$

(6)

For $\dot{\varphi}^2 \ll \frac{1}{\lambda}$, the Hamiltonian Eq.(6) can be simplified by using the Taylor expansion, and the terms smaller than $\ddot{\varphi}^6$ can be ignored, so the Hamiltonian becomes

$$H = -\frac{G}{3\pi a} P_a^2 - \frac{3\pi}{4G} a \left[ 1 - \frac{8\pi G}{3} a^2 V(\varphi) \right] + \frac{P_\varphi^2}{4\pi^4 a^3} - \frac{\lambda P_\varphi^4}{64\pi^6 a^9}$$

(7)

If $\dot{\varphi}(\lambda \dot{\varphi}^2 \sim 1)$ is very large, Eq.(6) becomes

$$H = -\frac{G}{3\pi a} P_a^2 - \frac{3\pi}{4G} a \left[ 1 - \frac{8\pi G}{3} a^2 V(\varphi) - \frac{1}{\lambda} \right]$$

(8)

The WD equation is obtained from $\hat{H} \psi = 0$, Eqs.(7) and (8) by replacing $P_a \rightarrow -i(\partial / \partial a)$ and $P_\varphi \rightarrow -i(\partial / \partial \varphi)$. Then we obtain

$$\left[ \frac{\partial^2}{\partial a^2} + \frac{p}{a} \frac{\partial}{\partial a} - \frac{1}{a^2} \frac{\partial^2}{\partial \Phi^2} - \frac{\lambda}{16\pi^4 a^8} \frac{\partial^4}{\partial \Phi^4} - U(a, \Phi) \right] \psi = 0$$

(9)

and

$$\left[ \frac{\partial^2}{\partial a^2} + \frac{p}{a} \frac{\partial}{\partial a} - u(a, \Phi) \right] \psi = 0$$

(10)
where $\tilde{\Phi}^2 = 4\pi G \varphi^2 / 3$ and the parameter $p$ represent the ambiguity in the ordering of factor $a$ and $\partial / \partial a$ in the first term of Eqs.(7) and (8). We have also denoted

\[
U(a, \Phi) = \left( \frac{3\pi}{2G} \right)^2 a^2 \left[ 1 - \frac{8\pi G}{3} a^2 V(\Phi) \right]
\]

\[
u(a, \Phi) = \left( \frac{3\pi}{2G} \right)^2 a^2 \left[ 1 - \frac{8\pi G}{3} a^2 [V(\Phi) - \frac{1}{\lambda}] \right]
\]

Eqs.(9) and (10) are the WD equations corresponding to the action (3) in the case of small and large $\dot{\varphi}$ respectively.

Now we take the ambiguity of the ordering factor $p = -1$ and set the transformation \((a/a_0)^2 = \sigma\), with $a_0$ being the Plank’s length. Taking the Plank constant $h = 1$ and the speed of light $c = 1$, $a_0 \sim \sqrt{4G/3\pi}$, we obtain from Eq.(9)

\[
\frac{\partial^2 \psi}{\partial \sigma^2} - \frac{1}{\sigma^2} \frac{\partial^2 \psi}{\partial \Phi^2} - \frac{\lambda}{16\pi^4 a_0^6 \sigma^5} \frac{\partial^4 \psi}{\partial \Phi^4} - \tilde{U} \psi = 0
\]

where $\tilde{U} = (3\pi/4G)^2 a_0^4 (1 - \alpha a_0^2 \sigma)$ and $\alpha = (8\pi G/3)V$. Denoting $m = (\lambda / 16\pi^4 a_0^6)$ and assuming $\psi(\sigma, \Phi) \sim Q(\sigma)e^{-K\Phi}$ with $K$ being an arbitrary constant, we can take the Eq.(13) as follows

\[
\frac{d^2 Q}{d\sigma^2} - \left( \frac{K^2}{\sigma^2} + \frac{mK^4}{\sigma^5} + \tilde{U} \right) Q = 0
\]

If $a(\text{or } \sigma)$ is large, $\alpha a^2 = (8\pi GV(\varphi)/3)a_0^2 \sigma \gg 1$ and Eq.(14) approximates to

\[
\frac{d^2 Q}{d\sigma^2} + \hat{\mu} a_0^2 \sigma Q = 0
\]

where $\hat{\mu} = (3\pi/4G)\alpha$. The general solution of Eq.(15) can be expressed in terms of Bessel function and is given by

\[
Q(\sigma) = \sqrt{\sigma} Z_\frac{1}{3} \left( \frac{2\hat{\mu} a_0^2}{3} \right)^{3/2} \sigma^{-3/2}
\]

$Q(\sigma)$ is an oscillatory function.

In the next step, we consider the solution of the WD Eq.(10) with $\dot{\varphi}$ being large. We still keep the ambiguity of the ordering of factor $p = -1$. By the same transformation \((a/a_0)^2 = \sigma\), Eq.(10) becomes

\[
\frac{d^2 \psi}{d\sigma^2} - (3\pi/4G)a_0^4 (1 - H^2 a_0^2 \sigma) \psi = 0
\]

where $H^2 = (8\pi G/3)[V(\varphi) - \frac{1}{\lambda}]$. When $H^2 a^2 \ll 1$, Eq.(17) approximates to

\[
\frac{d^2 \psi}{d\sigma^2} - (3\pi/4G)^2 a_0^4 \psi = 0
\]

Solving Eq.(18), we get

\[
\psi = Ne^{-\frac{3\pi}{4G} a_0^2 \sigma} = Ne^{-\frac{3\pi}{4G} a^2}
\]
where $N$ is a constant.

We can see that solution (19) is consistent with Vilenkin’s tunnelling wave function Eq.(29). When $a$ is larger, $H^2 a^2 = H^2 a_0^2 \sigma \gg 1$, and Eq.(17) approximates to

$$\frac{d^2 \psi}{d\sigma^2} + \left(\frac{3\pi}{4G}\right)^2 a_0^6 H^2 \sigma \psi = 0$$

(20)

Its solution is

$$\psi = \sqrt{\sigma} Z_3 \left(\frac{2 \tilde{H}^2 a_0^3}{3} \sigma^3\right)$$

(21)

where $\tilde{H} = \left(\frac{3\pi}{4G}\right)H$. The wave function $\psi$ given by Eq.(21) is an oscillatory function.

Next we will consider the cosmology in case of very large $\dot{\varphi}$ (correspondingly very small $a(t)$) by Vilenkin’s quantum tunnelling approach. Eq.(10) has the form of a one-dimensional Schrödinger equation for a “particle” described by a coordinate $a(t)$, which is zero energy and moves in a potential $u$. The classically allowed region is $u \leq 0$ or $a \geq H^{-1}$, with $H$ being defined in Eq.(17). In this region, disregarding the pre-exponential factor, the WKB solutions of Eq.(10) are

$$\psi_{\pm}^{(1)}(a) = \exp\{\pm i \int_{H^{-1}}^{a} P(a') da'\} \mp \frac{i\pi}{4}$$

(22)

The under-barrier ($a < H^{-1}$, classically forbidden or Euclidean region) solutions are

$$\psi_{\pm}^{(2)}(a) = \exp\{\pm \int_{a}^{H^{-1}} |P(a')| da'\}$$

(23)

where $P(a) \equiv \sqrt{1 - u(a)}$.

The classical momentum conjugate to $a$ is $P_a = -a \ddot{a}$. For $a > H^{-1}$, we have

$$(-i \frac{d}{da})\psi_{\pm}^{(1)}(a) = \pm P(a)\psi_{\pm}^{(1)}(a)$$

(24)

and thus $\psi_{-}^{(1)}(a)$ and $\psi_{+}^{(1)}(a)$ describe the expanding and contracting universe respectively. The tunneling boundary condition requires that only the expanding component should be present at large $a$,

$$\psi_T(a > H^{-1}) = \psi_{-}^{(1)}(a)$$

(25)

The under-barrier wave function is found from WKB connection formula

$$\psi_T(a < H^{-1}) = \psi_{+}^{(2)}(a) - \frac{i}{2} \psi_{-}^{(2)}(a)$$

(26)

The increasing exponential $\psi_{+}^{(2)}(a)$ and the decreasing exponential $\psi_{+}^{(2)}(a)$ have comparable amplitudes at the nucleation point $a = H^{-1}$, but away from that point the decreasing
The "tunneling amplitude" (probability distribution for the initial values of \( V \) in nucleating universe) is

\[
\frac{\psi_T(H^{-1})}{\psi_T(0)} = e^{-\frac{\pi^2}{2G\mathcal{H}^2}}
\]

(28)

From Eq.(28) we obtain the result that the tunneling wave function predicts a nucleating universe with the vacuum energy (i.e. cosmological constant) as large as possible and critical kinetic energy as small as possible (defining the kinetic energy when \( a = 0 \) as critical kinetic energy, which equals \( \frac{1}{2\lambda} \) from Eq.(5)). If \( H^2a^2 \ll 1 \), by Taylor expansion Eq.(27) becomes

\[
\psi_T(a) \approx \exp\left[\frac{\pi^2}{2G\mathcal{H}^2} - \frac{3\pi a^2}{4G}\right]
\]

(29)

Comparing Eqs.(19) and (29), we find that the only difference is just an unimportant pre-exponential factor.

The Hartle-Hawking (H-H) no boundary wave function is given by the path integral

\[
\psi_{HH} = \int [dg][d\phi]e^{-S_E(g,\phi)}
\]

(30)

In order to determine \( \psi_{HH} \), we assume that the dominant contribution to the path integral is given by the stationary points of the action (the wormhole instantons) and evaluates \( \psi_{HH} \) simply as \( \psi_{HH} \propto e^{-S_E|_{\text{saddle point}}} \). When \( (\dot{\phi})^2 \sim \frac{1}{\lambda} \), from action (3) we can obtain

\[
S = \int \frac{3\pi}{4G}[1 - \dot{a}^2]dt - \int 2\pi^2 a^3 H^2 dt
\]

(31)

where \( H^2 = \frac{8\pi}{3}(v - \frac{1}{\lambda}) \). The corresponding Euclidean action \( S_E = -i(S)_{\text{continue}} \) is

\[
S_E = \int -\frac{3\pi}{4G}[1 + \left(\frac{da}{d\tau}\right)^2]d\tau + \int 2\pi^2 a^3 H^2 d\tau
\]

(32)

where \( \tau = it \). From action (31), we can obtain that the \( a(t) \) satisfies the following classical equation of motion

\[
-\left(\frac{da}{dt}\right)^2 - 1 + H^2 a^2 = 0
\]

(33)

The solution of Eq.(33) is the de sitter space with \( a(t) = H^{-1}cosh(Ht) \). The corresponding Euclidean version (replacing \( t \to -i\tau \)) of Eq.(33) is

\[
\left(\frac{da}{d\tau}\right)^2 - 1 + H^2 a^2 = 0
\]

(34)
The solution of Eq.(34) is
\[ a(\tau) = H^{-1} \sin(H\tau) \] (35)

Using Eqs.(32)(35), we obtain
\[ \psi_{HH}(a) \propto \exp\left[-\frac{\pi}{2G\eta^2}(1 - H^2a^2)^{\frac{3}{2}}\right] \] (36)

The only difference between the H-H’s wave function(36) and Vilenkin’s wave function(27) is the sign of the exponential factor. The H-H’s wave function(36) gives the probability distribution
\[ P_{HH} \propto e^{\frac{\pi}{2G\eta^2}} \] (37)

The H-H’s distribution(37) is the same as Vilenkin’s one(28), except a sign of the exponential factor. The distribution(37) is peaked at \( V - (1/\lambda) = 0 \) and it predicts a universe with a positive cosmological constant \( \frac{1}{\lambda} \). This is a very significative result, it is different from zero cosmological constant predicted by Coleman and Hawking.

When \( V = \frac{1}{\lambda} \) for our lagrangian, it represents a universe filled with Chaplygin gas[5]. In fact H-H’s wave function predicts a universe filled with Chaplygin gas. Next, we will discuss dark energy model of B-I type scalar field and the relationship between our Lagrangian and Chaplygin gas.

### III. Dark Energy Model of B-I Type Scalar Field

1. The Model With Lagrangian \( \frac{1}{\lambda}[1 - \sqrt{1 - \lambda g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}}] - V \)

In the spatially flat R-W metric, Einstein equation \( G_{\mu\nu} = KT_{\mu\nu} \) can be written as
\[ \left(\frac{\dot{a}}{a}\right)^2 = \frac{K}{3} T_0^0 = \frac{K}{3} \rho \]
(38)
\[ 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = KT_1^1 = KT_2^2 = KT_3^3 = -K\rho \]
(39)

Substituting Eq.(38) into Eq.(39), we get
\[ \frac{\ddot{a}}{a} = -\frac{K}{6}(T_0^0 - 3T_1^1) \]
(40)

where
\[ T_{\mu}^{\nu} = \frac{g^{\mu\rho}\varphi_{,\nu}\varphi_{,\rho}}{\sqrt{1 - \eta g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}}} - \delta_{\mu}^{\nu} L_s \]
(41)
The energy density $\rho_s = T_0^0$ and pressure $p_s = -T_1^1 = -T_2^2 = -T_3^3$ are defined as following

$$\rho_s = V - \frac{1}{\lambda} + \frac{1}{\lambda \sqrt{1 - \lambda \dot{\varphi}^2}} = B + \sqrt{1 + \lambda c^2 a^{-6}}$$  \hspace{1cm} (42)$$

$$p_s = \frac{1}{\lambda} [1 - \sqrt{1 - \lambda \dot{\varphi}^2}] - V$$  \hspace{1cm} (43)$$

where the upper index "." denotes the derivative with respect to $t$, and $V(\varphi)$ is taken $V_0$ as a constant. We define $B = \lambda V_0 - 1$. When $a(t) = 0$ the kinetic energy $\frac{\dot{a}^2}{2} = \frac{1}{2\lambda}$ is critical value from Eq.(5). According to Eqs.(38) and (42) we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{K}{3\lambda} \left[\sqrt{1 + \lambda c^2 a^{-6}} + B\right]$$  \hspace{1cm} (44)$$

When $a(t)$ is very small ($a(t) \to 0$), Eq.(44) approximates to

$$\dot{a} = \frac{\sqrt{Kc}}{3\lambda^{3/2} a}$$  \hspace{1cm} (45)$$

$$a \propto t^{3/2}$$  \hspace{1cm} (46)$$

When $a(t)$ increases little by little until $a \gg (\lambda c^2)^{1/6}$, Einstein equation(44) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{K}{3} \left[1 + \frac{1}{2} e^2 a^{-6}\right]$$  \hspace{1cm} (47)$$

From Eq.(47) we obtain

$$\ddot{a} = \frac{KV_0}{3} a - \frac{Kc^2}{3} a^{-5}$$  \hspace{1cm} (48)$$

From the above equation we find that when $a > (\frac{2}{Kc})^{1/6}$, the universe is undergoing a accelerated phase. Integrating Eq.(47) we obtain

$$a^3 \propto \sqrt{\frac{c^2}{2V_0} \left(e^{\sqrt{12KV_0}t} - 1\right)}$$  \hspace{1cm} (49)$$

When $a \to \infty$, Eq.(44) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{K}{3} V_0$$  \hspace{1cm} (50)$$

From the above equation, we see that the universe is undergoing a inflation phase. However, the dark energy model interpolates between a dust dominated phase in the past and a de-sitter phase at the late time.

Especially when $V_0 = \frac{1}{\lambda}$, our Lagrangian describes the Chaplygin gas that was proposed as a model for both dark energy and dark matter in the present universe[5]. Correspondingly, the density and pressure are

$$\rho_c = \frac{\sqrt{1 + \lambda c^2 a^{-6}}}{\lambda}$$  \hspace{1cm} (51)$$
Substituting Eq.(51) into Eq.(38), we obtain the solution of Einstein equation(38)

\[ t = \frac{\lambda^\frac{1}{2}}{6} \left[ \ln \left( \frac{1}{\lambda^2} + \frac{c^2}{\lambda^2}a^{-6} \right)^{\frac{1}{2}} + \left( \frac{1}{\lambda^2} \right)^{\frac{1}{2}} - 2 \arctan(\sqrt{1 + c^2 \lambda a^{-6}}) \right] \] (53)

When \( a(t) \) is very small\((a(t) \to 0)\), the density is approximated by

\[ \rho_c \sim \frac{c}{a^3 \lambda^\frac{1}{2}} \] (54)

that corresponds to a universe dominated by dust-like matter. For large values of the cosmological scalar factor \( a(t) \) it follows that

\[ \rho_c \sim \frac{1}{\lambda} \] (55)
\[ p_c \sim -\frac{1}{\lambda} \] (56)

which in turn corresponds to a universe with a cosmological constant \( \frac{1}{\lambda} \)(i.e., a de-Sitter universe). At early time, i.e., the cosmological scale factor \( a(t) \) is small, \( \rho_c \sim \frac{c}{a^3 \lambda^\frac{1}{2}} \), which corresponds to a dust like dominated universe. At late time, i.e., the cosmological scale factor is large, \( \rho_c \sim \frac{1}{\lambda} \), which corresponds to a cosmological constant like dominated universe. Therefore the Chaplygin gas interpolates between a dust dominated phase in the past and a de-Sitter phase at late time.

When potential is taken to zero, the density and pressure are

\[ \rho_s = \frac{\dot{\varphi}^2}{\sqrt{1 - \lambda \dot{\varphi}^2}} - \frac{1}{\lambda} \left[ 1 - \sqrt{1 - \lambda \dot{\varphi}^2} \right] \] (57)
\[ p_s = \frac{1}{\lambda} \left[ 1 - \sqrt{1 - \lambda \dot{\varphi}^2} \right] \] (58)

Substituting Eq.(5) to the above two expressions, we have

\[ w = \frac{p_s}{\rho_s} = \frac{a^3}{\sqrt{a^6 + \lambda c^2}} \] (59)

and can see

\[ 0 \leq w < 1 \] (60)

So, there is no accelerated expansion in the universe of B-I type scalar field without potential. For B-I type scalar field with potential, we have

\[ \rho + 3p = \frac{2}{\lambda} + \frac{3\lambda \dot{\varphi}^2 - 2}{\lambda \sqrt{1 - \lambda \dot{\varphi}^2}} - 2V(\varphi) \] (61)
When potential is greater than $\frac{1}{\lambda}$ and the kinetic energy of $\varphi$ field evolves to region of $\dot{\varphi}^2 < \frac{2}{3\lambda}$, $\rho + 3p < 0$. The universe undergoes a phase of accelerating expansion.

We also get

$$\rho + p = \frac{\dot{\varphi}^2}{\sqrt{1 - \lambda \dot{\varphi}^2}} > 0$$

and

$$w = \frac{p}{\rho} > -1$$

However, some analysis to the observation data hold that the range of state parameter lies in $-1.32 < w < -0.82$[6]. From Eq.(63) we know that parameter of state equation is larger than -1. In order to give a favor explanation to the observation results, we investigate the phantom field model that possess negative kinetic energy and can realize $w < -1$ in their evolution. Next we discuss phantom field model.

2. The Model With Lagrangian $\frac{1}{\lambda}[1 - \sqrt{1 + \lambda g_{\mu\nu} \varphi,\mu \varphi,\nu}] - V(\varphi)$

We consider the case that the kinetic energy terms is negative. The energy-momentum tensor is

$$T^\mu_\nu = -\frac{g^{\mu_0} \varphi,_{\nu_0} \varphi,_{\mu} \varphi,_{\nu}}{\sqrt{1 + \lambda g^{\mu\nu} \varphi,_{\mu} \varphi,_{\nu}}} - \delta^\mu_\nu L$$

From Eq.(64), we have

$$\rho = T^0_0 = \frac{1}{\lambda \sqrt{1 + \lambda \dot{\varphi}^2}} - \frac{1}{\lambda} + V$$

$$p = -T^1_1 - T^2_2 - T^3_3 = \frac{1}{\lambda} - \frac{\sqrt{1 + \lambda \dot{\varphi}^2}}{\lambda} - V$$

Based on Eq.(65) and (66), we can obtain

$$\rho + p = -\frac{\dot{\varphi}^2}{\sqrt{1 + \lambda \dot{\varphi}^2}}$$

It is clear that the static equation $w < -1$ is completely decided by Eq.(67). We can also get

$$\rho + 3p = \frac{2}{\lambda} - \frac{2}{\lambda \sqrt{1 + \lambda \dot{\varphi}^2}} - \frac{\dot{\varphi}^2}{1 + \lambda \dot{\varphi}^2} - 2V$$

It is obvious that $\rho + 3p < 0$. Eq.(68) shows that the universe is undergoing a phase of accelerated expansion. The model of phantom B-I type scalar field without potential $V(\varphi)$ is hard to understand. In this model we can always find $\rho = \frac{1}{\lambda \sqrt{1 + \lambda \dot{\varphi}^2}} - \frac{1}{\lambda} < 0$ and $(\frac{\dot{\varphi}}{\lambda})^2 < 0$. It is unreasonable apparently. In the model of phantom B-I type scalar field with potential
$V(\varphi)$, if $V(\varphi) > \frac{1}{\lambda} - \frac{1}{\lambda\sqrt{1+\lambda\dot{\varphi}^2}}$, $\rho$ is greater than zero.

First, we consider the case of a specific simple example $V = u_0 = \text{const}$ and $u_0 - \frac{1}{\lambda} = \frac{A}{\lambda}(A > 0)$. So, Eq.(63) becomes

$$\rho = \frac{1}{\lambda\sqrt{1+\lambda\dot{\varphi}^2}} + \frac{A}{\lambda} \quad (69)$$

Substituting Eq.(69) into Eq.(38), we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{K}{3} \left[\frac{1}{\lambda\sqrt{1+\lambda\dot{\varphi}^2}} + \frac{A}{\lambda}\right] \quad (70)$$

We can obtain from the Euler-Lagrange equation (4)

$$\dot{\varphi} = \frac{c}{\sqrt{a^6 - \lambda c^2}} \quad (71)$$

where $c$ is integrate constant. Substituting Eq.(71) into Eq.(70), we get

$$\dot{a} = \sqrt{\frac{K a^2}{3\lambda} \left[\sqrt{1 - \lambda c^2 a^{-6}} + A\right]} \quad (72)$$

From the above equation, we know the universe is nonsingular because the minimum $a_{\text{min}}$ of scale factor is $(\lambda c^2)^{\frac{1}{2}}$. When the universe scale factor approximates to $a_{\text{min}}$, Eq.(72) becomes

$$\dot{a} = \sqrt{\frac{K A}{3\lambda}} a \quad (73)$$

$$a = e^{\sqrt{\frac{KA}{3\lambda}} t} \quad (74)$$

When $a \to \infty$, Eq.(72) becomes

$$\dot{a} = \sqrt{\frac{K(A+1)}{3\lambda}} a \quad (75)$$

$$a = e^{\sqrt{\frac{K(A+1)}{3\lambda}} t} \quad (76)$$

In phantom model with constant potential, the universe is always undergoing a phase of inflation at very small and larger scale factor.

Next we study the cosmological evolution by numerical analysis in the phantom model with a specific potential[11]

$$V(\varphi) = V_0 (1 + \frac{\varphi}{\varphi_0}) e^{(-\frac{\varphi}{\varphi_0})} \quad (77)$$
As we consider the phantom field becomes dominant, we can neglect the nonrelativistic and relativistic components (matter and radiation) in the universe, then from Eqs.(5,38), we have

$$\ddot{\phi} + \dot{\phi}(1 + \lambda \dot{\phi}^2) \sqrt{3K \left[ \frac{1}{\lambda \sqrt{1 + \lambda \dot{\phi}^2}} - \frac{1}{\lambda} + V(\varphi) \right]} - V'(\varphi)(1 + \lambda \dot{\phi}^2)^{\frac{3}{2}} = 0$$

(78)

where the overdot denotes the differentiation with respect to $t$ and the prime denotes the differentiation with respect to $\varphi$.

To study an numerical computation, it is convenient to introduce two independent variables

$$\begin{cases}
X = \varphi \\
Y = \dot{\varphi}
\end{cases}$$

(79)

then Eq.(78) can be written

$$\begin{cases}
\frac{dX}{dt} = Y \\
\frac{dY}{dt} = V'(X)(1 + \lambda Y^2)^{\frac{3}{2}} - Y(1 + \lambda Y^2) \sqrt{3K \left[ \frac{1}{\lambda \sqrt{1 + \lambda Y^2}} - \frac{1}{\lambda} + V(X) \right]}
\end{cases}$$

(80)

We can obtain this system’s critical point from

$$\begin{cases}
\frac{dX}{dt} = 0 \\
\frac{dY}{dt} = 0
\end{cases}$$

(81)

then its critical point is $(X_c,0)$, where the critical value $X_c$ is determined by $V'(X_c) = 0$.

Linearizing Eq.(80) around the critical point, we have

$$\begin{cases}
\frac{dX}{dt} = Y \\
\frac{dY}{dt} = V''(X_c)(X - X_c) - \sqrt{3KV(X_c)}Y
\end{cases}$$

(82)

the types of the critical point are determined by the eigenequation of system

$$\varepsilon^2 + \alpha \varepsilon + \beta = 0$$

(83)

where $\alpha = \sqrt{3KV(X_c)}$, $\beta = -V''(X_c)$, the two eigenvalues are $\varepsilon_1 = \frac{-\sqrt{3KV(X_c)} + \sqrt{3KV(X_c)^2 + 4V''(X_c)}}{2}$, $\varepsilon_2 = \frac{-\sqrt{3KV(X_c)} - \sqrt{3KV(X_c)^2 + 4V''(X_c)}}{2}$. For a positive potentials, if $V''(X_c) < 0$, then the critical point $(X_c,0)$ is a stable node, which implies that the dynamical system admits attractor solutions. We can also conclude that if a potential possesses the general properties: $V(X_c) > 0, V'(X_c) = 0$ and $V''(X_c) < 0$, then our phantom model with this potential will
have a attractor solution and predict a late time de-sitter like behavior ($w_\varphi = -1$). Substituting Eq.(77) into Eq.(80), we obtain

$$\begin{align*}
\frac{dX}{dt} &= Y \\
\frac{dY}{dt} &= -\frac{V_0}{\varphi_0} X e^{(-\frac{X}{\varphi_0})} (1 + \lambda Y^2)^{\frac{3}{2}} - Y(1 + \lambda Y^2) \sqrt{3K[\frac{1}{1+y^2} - \frac{1}{\lambda} + V_0(1 + \frac{X}{\varphi_0})e^{(-\frac{X}{\varphi_0})}]} \tag{84}
\end{align*}$$

From Eqs(81,84) we obtain the critical $X_c = 0$ and $V''(X_c) = -\frac{V_0}{\varphi_0} < 0$ from Eq(77). Therefore this model has an attractor solution which corresponds to its attractor regime, the equation of state $w \leq -1$. To solve this equations system via the numerical approach, we re-scale the quantities as $x = \frac{X}{\varphi_0}$, $s = (\varphi_0^2 \lambda)^{-\frac{1}{2}} t$, $y = \sqrt{\lambda} Y$. Then Eq.(84) becomes

$$\begin{align*}
\frac{dx}{ds} &= y \\
\frac{dy}{ds} &= -\gamma xe^{(-x)}(1 + y^2)^{\frac{3}{2}} - \phi_0 y(1 + y^2) \sqrt{3K[\frac{1}{1+y^2} - 1 + \gamma(1 + x)e^{(-x)}]} \tag{85}
\end{align*}$$

where we set $K = 1$ and $\gamma = V_0 \lambda$ is parameter. The numerical results with different initial condition are plotted in Figs.1 – 3 and the parameters $\varphi_0 = \sqrt{0.1}$, $\gamma = 3$.

Fig1. This plot shows the evolution of the scalar field in difference initial condition, solid line is for $\varphi_{in} = 0.7\varphi_0$, dotted line is for $\varphi_{in} = \varphi_0$, dashed line and dot-dashed line for $\varphi_{in} = 1.3\varphi_0, 1.7\varphi_0$ respectively, they are all plotted for a fixed value of $y_{in} = 0.1$. 

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Fig2. The evolution of $w$ with respect to $s$, the initial condition is the same as fig1.

Fig3. The attractor property of the system in the phase plane, the initial condition is the same as fig1.

As we know, when $\lambda \to 0$, our model turns to be phantom quintessence model. In order to see the nonlinear effect, we plot the phantom quintessence model with our model in fig4 and fig5.
Fig 4. The evolution of scalar field with respect to $s$, solid line is phantom quintessence model, dotted line and dashed line are nonlinear B-I phantom field, dotted line is for $\lambda = 2/3$, dashed line is for $\lambda = 1/3$.

Fig 5. The evolution of $w$ with respect to $s$, solid line is phantom quintessence model, dotted line and dashed line are nonlinear B-I phantom field, dotted line is for $\lambda = 2/3$, dashed line is for $\lambda = 1/3$.

From fig 1, fig 2 and fig 3, we can easily find the system admits an attractor solution. The equation of state parameter $w$ starts from regime of near -1, then quickly evolves to the regime of smaller than -1, last, turns back to execute the damped oscillation, and reaches to -1 eventually for ever. Due to the unusual physical properties in phantom model, the phantom field, releases at the distance from the origin with a small kinetic energy, and moves
towards the top of the potential and crosses over to the other side, then turns back to execute the damped oscillation around the critical point. After a certain time the motion ceases and the phantom field settles on the top of the potential permanently to mimic the de Sitter-like behavior \( w_\phi = -1 \). Fig4 and fig5 indeed shows that the nonlinear scalar phantom field will reduce to phantom quintessence model when \( \lambda \) decreases to zero. The nonlinear effect does not affect the global attractor behavior but change the evolution of \( w \) and scalar field \( \phi \) in details.

IV. CONCLUSION

It shows that the tachyon can be described by a B-I type Lagrangian resulting from string theory. It is clear that our Lagrangian \( \frac{1}{\lambda} [1 - \sqrt{1 - \lambda g^{\mu\nu}\Phi,_{\mu}\Phi,_{\nu}}] - V(\varphi) \) is equivalent formally to the tachyon type Lagrangian \( -\frac{1}{\lambda} \sqrt{1 - g^{\mu\nu}\Phi,_{\mu}\Phi,_{\nu}} \), with a potential \( \frac{1}{\lambda} - V(\varphi) \) where we re-scale the scalar field as \( \Phi = \lambda^{\frac{1}{2}} \varphi \). The WD equation are solved analytically for both very large and small \( a(t) \). For the very small cosmological scale factor, the Vilenkin’s tunnelling wave function Eq.(27) predicts a nucleating universe with the biggest possible vacuum energy \( V(\text{cosmological constant}) \) and the smallest possible critical kinetic energy \( \frac{1}{\lambda} \). The only difference between the H-H’s wave function Eq.(36) and Vilenkin’s wave function Eq.(27) is the sign of the exponential factor. Then the H-H’s probability distribution in Eq.(37) reaches to peak at \( V - \frac{1}{\lambda} = 0 \). It predicts a universe with a positive cosmological constant \( \frac{1}{\lambda} \). It is different from zero cosmological constant that has been predicted by Coleman and Hawking.

The parameter \( w \) of state equation lies in the range of \( 0 \leq w < 1 \) for B-I type scalar field without potential. The universe of B-I type scalar field with potential can undergo a phase of accelerated expansion. The corresponding parameter of state equation lies in the range of \( -1 < w < -\frac{1}{3} \). This model admits a late time attractor solution that leads to an equation of state \( w = -1 \).

In order to give a favor explanation to the observational results that the range of state parameter lies in \( -1.32 < w < -0.82 \), we investigate the phantom field model that possess negative kinetic energy. The weak and strong energy condition are violated for phantom B-I type scalar field. The parameter \( w \) of state equation lies in the range of \( w < -1 \). When the potential \( V(\varphi) = u_0 \), the universe is nonsingular and stays in de-sitter phase always. When we choose a potential as Eq.(77), the evolution behavior of the state parameter \( w \) is shown in Fig.2. By numerical analysis, we learn that there is an attractor solution in Fig.3.
ACKNOWLEDGEMENTS

This work is partly supported by NNSFC under Grant No.10573012 and No.10575068 and by Shanghai Municipal Science and Technology Commission No.04dz05905.

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