An Isoperimetric Problem With Density and the Hardy Sobolev Inequality in $\mathbb{R}^2$.

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Abstract
We prove the following generalization of the isoperimetric inequality: if $p \in \mathbb{R}$, $\Omega \subset \mathbb{R}^2$ then the inequality

$$\left( \frac{|\Omega|}{\pi} \right)^{\frac{p+1}{2}} \leq \frac{1}{2\pi} \int_{\partial \Omega} |x|^p d\sigma(x).$$

holds true under appropriate assumptions on $\Omega$ and $p$. We prove the equivalence with a Hardy-Sobolev inequality, giving the best constant, and generalize thereby the equivalence between the classical isoperimetric inequality and the Sobolev Inequality. We give a further application arising in the context of the Moser-Trudinger embedding.

1 Introduction
Let $\Omega \subset \mathbb{R}^2$ be a bounded open simply connected set with a piecewise $C^1$ boundary $\partial \Omega$. Suppose that $0 \in \Omega$, and let $p \geq -1$. The following inequality is our main result:

$$\left( \frac{|\Omega|}{\pi} \right)^{\frac{p+1}{2}} \leq \frac{1}{2\pi} \int_{\partial \Omega} |x|^p d\sigma(x),$$

(1)

where $|\Omega|$ denotes the 2 dimensional volume of $\Omega$ and $d\sigma(x)$ is the 1 dimensional Hausdorff measure. All other results will be deduced from this inequality. Setting $p = 0$ we obtain the classical isoperimetric inequality. We discuss the inequality for different values of $p$ more precisely and also the case of equality (see Theorem 1). The proof is very simple, self-contained and uses only basic properties of holomorphic functions and the Riemann mapping theorem. It

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seems that this is a new approach for tackling an isoperimetric problem with density.

Isoperimetric problems with densities, or also called weights, are being investigated with increasing interest in the recent years. We give here the overview in only two dimensions, although several of the below mentioned results deal with arbitrary dimension $n \geq 2$. The general question is the following: Given two positive functions $f, g$ and defining the weighted volume $V_f$ and weighted perimeter $P_g$ as

$$V_f(\Omega) = \int_{\Omega} f(x) dx, \quad P_g(\Omega) = \int_{\partial \Omega} g(x) d\sigma(x),$$

one studies the existence of minimizers of

$$I(C) = \inf \{ P_g(\Omega) : \Omega \subset \mathbb{R}^2 \text{ and } V_f(\Omega) = C \}.$$  \hspace{1cm} (2)

Most of the recent results are for the case $f = g$. We refer to the papers of Fusco-Maggi-Pratelli [19], Morgan-Pratelli [29], Rosales-Canete-Bayle-Morgan [3] and in particular to Figalli-Maggi [15] which has an extensive list of references. The case where $f = 1$ and $g \neq f$ has received less attention, but is also studied. An example for such a problem has been investigated by Figalli-Maggi [16]. Examples of weights where $g = 1$ and $f \neq g$ can be found in Carroll-Jacob-Quinn-Walters [7], Section 3. Motivated by two important applications in the theory of partial differential equations it turns out to be natural to investigate the following special case: let $p > -1$ and $q > -2$ and consider

$$I(C) = \inf \left\{ \int_{\partial \Omega} |x|^p d\sigma : \Omega \subset \mathbb{R}^2 \text{ and } \left( \int_{\Omega} |x|^q dx \right)^{\frac{p+1}{p+q}} = C \right\}.$$  \hspace{1cm} (3)

In contrast to (2) one of the main differences is that this problem is scaling invariant in the sense that if $\Omega$ is a minimizer for $I(C)$, then $\lambda \Omega$ is a minimizer for $I(\lambda^{p+1}C)$. Hence, for the study of optimal shapes the value of $C$ becomes irrelevant and can be fixed. In the case $p = q$, the works of Carroll-Jacob-Quinn-Walters [7], Cañete-Miranda-Wittone [5] and finally Dahlberg-Dubbs-Newkirk-Tran [12] have led to the solution, which states that balls whose boundary contains the origin are the minimizers. Díaz-Harman-Howe-Thompson [13] are able to obtain more general results. They solve the problem for the two cases (see [13] Proposition 4.21)

(i) $p \in (-1,0)$ and $q \in (-2,2p)$ \hspace{1cm} (ii) $p \geq 0$ and $q \in (-2,p-1] \cup [p,2p]$.  \hspace{1cm} (4)

Note that in both cases there is no open interval for neither $p$ nor $q$ such that the classical isoperimetric inequality appears as a boundary case of that interval. This is precisely one of the remaining open problems, see [13] Open Questions 1.5 (4), i.e. $q = 0$ and $p \in (0,1)$. We prove in the present paper that balls centered at the origin are minimizers. Diaz-Harman-Howe-Thompson also conjecture many more results for more general values of $p$ and $q$, see [13] Conjecture 4.22. In the present paper we precisely deal mainly with the case $q = 0$ and $p \geq -1$.  

2
Another closely related work is by Betta-Brock-Mercaldo-Posteraro [2], whose result implies the case $q = 0$ and $p \geq 1$. We will state and use this result (see Theorem 3) to simplify our proof for the case $p \geq 1$ and to give a complete picture. The result in [2] gives an inequality for more general radial weights $g(x) = a(|x|)$ (and $f = 1$), which satisfy a monotonicity condition and are of the form

$$a(|x|) = C + \frac{f(|x|)}{|x|^2},$$

where $C$ is a constant and $f$ is a convex function. In the proof one adds the classical isoperimetric inequality, arising from $C$, to rather generous estimates arising from the term $f(|x|)/|x|^2$. We point out that the relevant applications in this paper are for the range $p \in [-1, 1]$, where convexity fails and which will be precisely those including the classical isoperimetric inequality as a special case.

It is well known since the paper of Talenti [31] that the Sobolev embedding, with the best constant,

$$||u||_{L^2} \leq \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}^2} |\nabla u(x)| dx \quad \text{for all } u \in C^\infty_c (\mathbb{R}^2)$$

is equivalent to the isoperimetric inequality. This equivalence is essentially due to Federer-Fleming [14] and Fleming-Rishel [17], by means of the coarea formula. The concept of relating functional inequalities with isoperimetric problems has also been investigated recently by Maggi-Villani [25], [26] for the Brézis-Lieb trace Sobolev inequality. However, it seems that the connection has not yet been observed for the Hardy-Sobolev inequality. More precisely it turns out that for values

$$-2 < p - 1 \leq q \leq 2p$$

the problem is equivalent with determining the best constant in the following case of the Caffarelli-Kohn-Nirenberg [4] inequality:

$$||u |x|^\gamma||_{L^r} \leq C||Du |x|^\alpha||_{L^1}, \quad \text{for all } u \in C^\infty_c (\mathbb{R}^2),$$

where $\alpha, \gamma$ and $r$ satisfy the conditions

$$r > 0, \quad 0 \leq \alpha - \gamma \leq 1, \quad 0 < \frac{1}{r} + \frac{\gamma}{2} = \frac{\alpha + 1}{2}.$$ 

If $q = 0$, the range $p \in (-1, 1]$ is the best possible to establish such an equivalence with this method. For $p \in [0, 1]$ we obtain precisely [14], whereas for $p \in (-1, 0)$ the set of admissible functions $C^\infty_c (\mathbb{R}^2)$ has to be restricted appropriately. For $p \in [0, 1]$ our result states that

$$||u||_{L^r(\mathbb{R}^2)} \leq \frac{\pi^{1/r}}{2\pi} \int_{\mathbb{R}^2} |\nabla u(x)| |x|^p dx, \quad \forall u \in C^\infty_c (\mathbb{R}^2), \quad r = \frac{2}{p + 1}.$$ 

Note that if $p \notin [0, 1]$ there exists no constant $C > 0$ such that $||u||_{L^r(\mathbb{R}^2)} \leq ||\nabla u(x)|x|^p||_{L^1}$ for all $u \in C^\infty_c (\mathbb{R}^2)$, see [4]. Best constants and extremal
functions have been investigated for all kinds of cases for the Sobolev and Caffarelli-Kohn-Nirenberg inequality and the literature is very large. We refer to Cordero-Villani-Nazaret [10], Catrina-Wang [8], Chou-Chu [9], Horiuchi [22] and Lieb [24], and the references therein. However, to the knowledge of the author, the best constant has not yet been established in the present case. We also discuss the existence of extremal functions.

Finally, let us mention the problem which originally motivated the weighted isoperimetric inequality, in particular the range $p \in [-1, 0]$. The existence of an extremal function for the Moser-Trudinger embedding when $\Omega = B_1$ (unit ball) has been shown by Carleson-Chang [6]; see also Malchiodi-Martinazzi [27] who recently complemented and improved this result for the ball in dimension 2. Flucher [18] developed a method how to deduce from the existence of extremal functions on balls the same result in general domains $\Omega \subset \mathbb{R}^2$, by proving a so-called functional isoperimetric inequality. As the name suggests, Flucher’s proof uses the isoperimetric inequality in a crucial way. The weighted isoperimetric inequality will allow us to generalize Flucher’s method to more general functionals involving weights, such as (cf. Adimurthi-Sandeep [1])

$$u \in W^{1,2}_0(\Omega) \to \int_\Omega e^{\alpha u^2} |x|^\beta \, dx.$$  

where $\Omega \subset \mathbb{R}^2$ is a domain containing the origin and $\alpha/(4\pi) + \beta/2 \leq 1$. We shall establish this in a forthcoming paper [11] and restrict ourselves here to generalizing only that crucial inequality where the isoperimetric inequality is used: If $G_{\Omega,x}$ is the Green’s function with singularity at $x \in \Omega$, then Flucher [18] has shown that $|\Omega|$ can be estimated by

$$|\Omega| \leq \frac{1}{4\pi} \int_{\partial \Omega} \frac{1}{|\nabla G_{\Omega,x}(y)|} d\sigma(y).$$

In Theorem 12 we will prove a generalization of this estimate.

## 2 A Weighted Isoperimetric Inequality

The following is the main theorem of this section.

**Theorem 1** Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with piecewise $C^1$ boundary $\partial \Omega$. Regarding the inequality

$$\left( \frac{|\Omega|}{\pi} \right)^{\frac{1}{p+1}} \leq \frac{1}{2\pi} \int_{\partial \Omega} |x|^p d\sigma,$$  

the following statements hold true:

(i) If $p \geq -1$, then [14] holds if $\Omega$ is connected and $0 \in \Omega$.

(ii) If $p \geq 0$, then [14] holds for all $\Omega$. 


(iii) If \( p \geq -1 \), \( \Omega \) satisfies the conditions of (i), \( \partial \Omega \) is \( C^2 \) and there is equality in (5), then \( \Omega \) is a ball. If \( p \neq 0 \), then this ball must be centered at the origin.

(iv) If \( p > 0 \), \( 0 \notin \partial \Omega \), \( \partial \Omega \) is \( C^2 \), and there is equality in (5), then \( \Omega \) is a ball centered at the origin. If in addition \( p \geq 1 \), then the same conclusion holds without the assumption \( 0 \notin \partial \Omega \).

Remark 2 (a) We will sometimes use the following equivalent formulation: let \( B_R = B_R(0) \) be the ball of radius \( R \) and center at the origin such that \( |\Omega| = |B_R| = \pi R^2 \). Then we have

\[
\int_{\partial \Omega} |x|^p \, d\sigma \geq \int_{\partial B_R} |x|^p \, d\sigma = 2\pi R^{p+1}.
\]

Or in other words: Among all admissible sets with given fixed volume the set \( B_R(0) \) is a (if \( p \neq 0 \)) minimizer of the weighted perimeter.

(b) If \(-1 \leq p < 0\), neither the assumption that \( \Omega \) is connected, nor that \( 0 \in \Omega \) can be dropped. Thus the hypothesis in (i) is optimal. For \( \Omega \) not connected, take the following example: fix \( r > 0 \) and \( \Omega \) is the union of two disconnected balls \( \Omega = B_r(0) \cup B_r(y) \). Obviously \( 0 \in \Omega \). For \( |y| \) large enough we get a contradiction. For the case \( 0 \notin \Omega \), simply take \( \Omega = B_r(y) \) and let again \( |y| \to \infty \). If \(-1 < p < 0\), actually something slightly stronger holds true: there exists no constant \( C > 0 \) such that

\[
\left( \frac{|\Omega|}{\pi} \right)^{\frac{p+1}{p}} \leq C \int_{\partial \Omega} |x|^p \, d\sigma
\]

for all bounded sets containing 0. This can be seen by defining for all \( m \in \mathbb{N} \) the sets \( \Omega_m = B_{1/m}(0) \cup B_1(m+1) \). We easily calculate that

\[
\lim_{m \to \infty} |\Omega_m| = 1, \quad \text{but} \quad \int_{\partial \Omega_m} |x|^p \, d\sigma \leq 2\pi \left( \frac{1}{m^{p+1}} + m^p \right) \to 0,
\]

for \( m \to \infty \), since \( p \in (-1,0) \).

(c) The condition \( 0 \notin \partial \Omega \) in Part (iv) if \( p \in (0,1) \) is for technical reasons, due probably to the method of proof. The hypothesis \( 0 \notin \partial \Omega \) can clearly be removed if \( p \geq 1 \), in view of Theorem 4.3 in [2].

The case \( p \geq 1 \) is already known since it follows from Betta-Brock-Mercaldo-Posteraro [2]. We state here their result in dimension 2, however it is also valid in higher dimensions, cf. Theorem 2.1 in [2].

Theorem 3 (Betta-Brock-Mercaldo-Posteraro) Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set with Lipschitz boundary. Let \( R > 0 \), be such that \( |\Omega| = \pi R^2 \). Let \( a \) be a function \( a : [0, \infty) \to [0, \infty) \) with the following properties

\[
a \text{ is nondecreasing}
\]

\[
z \mapsto \left( a \left( \sqrt{z} \right) - a(0) \right) \sqrt{z} \quad \text{is convex.}
\]

\[
15
\]
Then
\[ \int_{\partial \Omega} a(|x|) d\sigma(x) \geq \int_{\partial B_R(0)} a(|x|) d\sigma(x) = 2\pi R a(R). \]

We illustrate the idea of the proof of this theorem on the example \(a(z) = C + z^p, C \in \mathbb{R}, p \geq 1\) and for a domain starshaped with respect to the origin. Under these assumptions \(\partial \Omega\) can be parametrized in the form \(\theta \in [0, 2\pi] \mapsto r(\theta)(\cos \theta, \sin \theta)\) for some \(2\pi\)-periodic function \(r > 0\).

Using polar coordinates we get
\[ \pi R^2 = |\Omega| = \int_0^{2\pi} \int_0^{r(\theta)} r^2 \, ds \, d\theta = \frac{1}{2} \int_0^{2\pi} r^2(\theta) \, d\theta. \]

Let us first assume that \(C = 0\). Using that \(p \geq 1\) implies that the map \(g : (0, \infty) \to \mathbb{R}\) given by \(g(s) = s^{\frac{p+1}{2}}\) is convex. We therefore obtain from Jensen inequality that
\[ 2\pi R^{p+1} = 2\pi \left( \frac{1}{2\pi} \int_0^{2\pi} r^2(\theta) \, d\theta \right)^{\frac{p+1}{2}} \leq \int_0^{2\pi} r^{p+1}(\theta) \, d\theta \leq \int_0^{2\pi} r^p(\theta) \sqrt{r'^2(\theta) + r^2(\theta)} \, d\theta = \int_{\partial \Omega} |z|^p d\sigma(x). \]

If \(C \neq 0\), then one adds the classical isoperimetric inequality to (8) and the same proof works. The disadvantage of this result is that all our relevant applications are for \(p \in [-1, 1]\), in particular the classical isoperimetric inequality.

We now prove Part (i) of Theorem 1. This is the central part of the proof and the most difficult case. If we set \(p = 0\) everywhere in the following proof we recover the proof sketched by Mateljevic-Pavlovic \[28\] for the classical isoperimetric inequality. Part (ii) will be deduced from (i) by fairly obvious geometric arguments and elementary inequalities, ideas which already appear in the same or similar form in \[7\] (Proposition 4.5) and \[29\] (Proposition 6.10). In what follows we can always assume, by approximation, that \(\partial \Omega\) is \(C^2\). If \(U \subset \mathbb{C}\), then \(\text{Hol}(U)\) will denote the set of holomorphic functions in \(U\).

**Proof (Theorem 1 Part (i)).** Step 1. Let us first assume that \(\Omega\) is simply connected and let \(\overline{\Omega} = \mathbb{C} \setminus \Omega\) denote the complement of \(\Omega\). Consider the set \(D \subset \mathbb{C}\) defined by
\[ D = \left\{ \frac{1}{z} : z \in \overline{\Omega} \right\} \cup \{0\}. \]

Note that the map \(\eta(z) = 1/z\) is one-to-one and \(\partial \Omega\) is a simple closed curve. Therefore \(\eta\) maps \(\partial \Omega\) onto a simple closed curve \(\Gamma\). By the Jordan curve theorem the interior of \(\Gamma\) is a bounded simply connected set. It must coincide with \(D\) since \(\eta\) maps \(\infty\) to 0. Thus \(D\) is a bounded open simply connected set with \(C^2\) boundary such that \(0 \in D\). By the Riemann mapping theorem there exists a conformal map \(h : B_1 \to D\) (which extends to a \(C^1\) diffeomorphism on \(\overline{B_1}\), cf. for instance Theorem 5.2.4 page 121 in Krantz \[23\]) such that
\[ h(0) = 0 \quad \text{and} \quad h \in \text{Hol}(B_1) \cap C^1(\overline{B_1}; D). \]
Define now \( g \in \text{Hol}(B_1 \setminus \{0\}) \cap C^1(\overline{B_1} \setminus \{0\}) \) by

\[
g = \frac{1}{h} : \overline{B_1} \setminus \{0\} \to \Omega^c.
\]

Since \( h(0) = 0 \), there exists \( G \in \text{Hol}(B_1) \) such that \( h(z) = zG(z) \) for all \( z \in B_1 \). Because \( h'(0) \neq 0 \), we must have that \( G(0) \neq 0 \). Moreover, using again that \( h(0) = 0 \) and that \( h \) is one-to-one, we get that \( G(z) \neq 0 \) for all \( z \in B_1 \). Moreover \( h(z) \neq 0 \) for all \( z \in \partial B_1 \) and therefore

\[
G(z) \neq 0 \quad \text{for all} \quad z \in \overline{B_1} \quad \text{and} \quad G \in \text{Hol}(B_1) \cap C^1(\overline{B_1}).
\]

Finally we define \( Q \in \text{Hol}(B_1) \cap C^1(\overline{B_1}) \) and \( \lambda \in \mathbb{C} \) by

\[
Q(z) = \frac{1}{G(z)} \quad \text{and} \quad \lambda = Q(0).
\]

Note that \( \lambda \neq 0 \). In view of the above definitions, there exists a holomorphic function \( P = Q - \lambda \in \text{Hol}(B_1) \) such that

\[
g(z) = \frac{1}{h(z)} = \frac{Q(z)}{z} = \frac{\lambda}{z} + P(z) \quad \text{for all} \quad z \in B_1.
\]

**Step 2.** We will prove in this step that

\[
|\Omega| \leq \pi |\lambda|^2.
\]

**Step 2.1.** The map \( h : \partial B_1 \to \partial D \) is one-to-one and onto. The same holds for the map \( \eta(z) = 1/z \), which maps \( \partial D \) to \( \partial \Omega \). Therefore the curve \( \gamma(t) = g(e^{-it}) \), \( t \in [0, 2\pi] \), is a parametrization of \( \partial \Omega \). Since \( \eta \) inverses the orientation of the curve, we have taken \( e^{-it} \) (instead of \( e^{it} \)) and therefore \( |\Omega| \) computes as

\[
|\Omega| = \frac{1}{2} \int_{\Omega} \text{div}(x_1, x_2) dx = \frac{1}{2} \int_0^{2\pi} \left( \gamma_1(t)\gamma_2'(t) - \gamma_2(t)\gamma_1'(t) \right) dt,
\]

where \( \gamma_1(t) = \text{Re}(\gamma(t)) \) and \( \gamma_2(t) = \text{Im}(\gamma(t)) \). Since \( \gamma \) is not contained in the open set \( B_1 \) we do an approximation and define \( \gamma_r(t) = g(re^{-it}) \), where \( 0 < r < 1 \), and define

\[
A_r = \frac{1}{2} \int_0^{2\pi} \left( \left( \gamma_r \right)_1(t)\left( \gamma_r \right)_2'(t) - \left( \gamma_r \right)_2(t)\left( \gamma_r \right)_1'(t) \right) dt.
\]

Since \( g \in C^1(\overline{B_1} \setminus \{0\}) \), we have that

\[
\lim_{r \to 1} A_r = |\Omega|.
\]

**Step 2.2.** In this Step we will estimate \( A_r \). We have by definition of \( \gamma_r(t) \)

\[
\gamma_r(t) = g \left( re^{-it} \right) \quad \text{and} \quad \gamma_r'(t) = -ig'(re^{-it}) \cdot r^{-it}.
\]
Therefore if we set
\[ z = g(re^{-it}) = \frac{\lambda}{r} e^{it} + P(re^{-it}) \]
\[ w = -g'(re^{-it}) r^{-it} = \frac{\lambda}{r} e^{it} - re^{-it} P'(re^{-it}) \]
and use the formula: \( \text{Re}(z) \text{Im}(iw) - \text{Im}(z) \text{Re}(iw) = \text{Re}(zw) \), we get that
\[ A_r = \frac{1}{2} \int_0^{2\pi} \text{Re}(zw) dt = \frac{1}{2} \text{Re} \int_0^{2\pi} z\overline{w} dt. \quad (12) \]
Since \( P \) is holomorphic in \( B_1 \), we can write \( P \) as \( P(z) = \sum_{n=0}^{\infty} a_n z^n \), where the series converges absolutely and uniformly on every \( B_r \), and thus in particular on \( \partial B_r \). If we define \( a_{-1} = \lambda \), then \( z \) and \( \overline{w} \) can be written as
\[ z = \frac{\lambda}{r} e^{it} + \sum_{n=0}^{\infty} a_n r^n e^{-int} = \sum_{n=-1}^{\infty} a_n r^n e^{-int} \]
\[ \overline{w} = \frac{\lambda}{r} e^{-it} - \sum_{m=1}^{\infty} m e^{imt} = - \sum_{m=1}^{\infty} m e^{imt}. \]
We therefore obtain that
\[ \int_0^{2\pi} (zw) dt = -\int_0^{2\pi} \sum_{n,m=-1}^{\infty} m a_n \overline{a}_m r^{n+m} e^{i(m-n)t} dt \]
\[ = -\sum_{n,m=-1}^{\infty} m a_n \overline{a}_m r^{n+m} \int_0^{2\pi} e^{i(m-n)t} dt = -2\pi \sum_{n=-1}^{\infty} n |a_n|^2 r^{2n}. \]
We recall that \( a_{-1} = \lambda \) and set the previous equation into (12) to get
\[ A_r = -\pi \sum_{n=-1}^{\infty} n |a_n|^2 r^{2n} = \pi \left( \frac{|\lambda|^2}{r^2} - \sum_{n=1}^{\infty} n |a_n|^2 r^{2n} \right) \leq \frac{\pi |\lambda|^2}{r^2}. \quad (13) \]
Hence this inequality with (11) and letting \( r \to 1 \) proves the claim of Step 2.

**Step 3.** We will prove in this step that
\[ |\lambda|^{p+1} \leq \frac{1}{2\pi} \int_{\partial\Omega} |x|^p d\sigma(x). \]
As in Step 2, we get that \( t \in [0, 2\pi] \to \alpha(t) = g(e^{it}) \) is a parametrization of \( \partial\Omega \). Using that \( |\alpha'(t)| = |g'(e^{it})| e^{it} | \) gives
\[ \int_{\partial\Omega} |x|^p d\sigma(x) = \int_0^{2\pi} |g'(e^{it})| |g(e^{it})|^p dt. \]

8
As in Step 2, we set \( \alpha_r(t) = g(re^{it}) \) for \( 0 < r < 1 \). Observe that, since \( g \in C^1(\overline{B_1}\setminus\{0\}) \), we get that

\[
\int_{\partial \Omega} |x|^p d\sigma(x) = \lim_{r \to 1} S_r, \tag{14}
\]

where

\[
S_r = \int_0^{2\pi} |\alpha'_r(t)| |\alpha_r(t)|^p dt = r \int_0^{2\pi} |g'(re^{it})| |g(re^{it})|^p dt.
\]

Let us define \( u \), using (9), by

\[
u(z) = z^2 g'(z) = zQ'(z) - Q(z).
\]

Note that \( u \in \text{Hol}(B_1) \) and \( u(0) = \lambda \). Since \( B_1 \) is simply connected and \( Q \) does not vanish on \( B_1 \), there exists a holomorphic logarithm \( \varphi \) of \( Q \), that is

\[
\varphi \in \text{Hol}(B_1) \quad \text{and} \quad Q(z) = \exp(\varphi(z)) \quad \text{for all} \quad z \in B_1.
\]

In particular the function \( \tau \) defined by \( \tau(z) = u(z) \exp(p\varphi(z)) \) is a holomorphic function in \( B_1 \). We can therefore apply the Cauchy mean value integral formula

\[
\tau(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it}) \exp(p\varphi(re^{it})) dt. \tag{15}
\]

Note that for any \( z \in \mathbb{C} \) and \( p \in \mathbb{R} \) the identity \( |\exp(pz)| = |\exp(z)|^p \) holds true.

The previous equality leads to the estimate

\[
|\tau(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |u(re^{it})| |Q(re^{it})|^p dt. \tag{16}
\]

From the definitions of \( Q, u \) and \( \tau \) we get the following three identities

\[
|\tau(0)| = |u(0)| |Q(0)|^p = |\lambda|^{p+1} \quad \text{and} \quad |u(re^{it})| = r^2 |g'(re^{it})| \quad |Q(re^{it})| = |re^{it}| |g(re^{it})| = r |g(re^{it})|
\]

Setting these identities into (16) yields

\[
|\lambda|^{p+1} = |\tau(0)| \leq \frac{p+2}{2\pi} \int_0^{2\pi} |g'(re^{it})| |g(re^{it})|^p dt = \frac{p+1}{2\pi} S_r.
\]

Finally, letting \( r \to 1 \) and using (14) proves the claim of Step 3.

**Step 4.** Sinc \( p \geq -1 \), Steps 2 and 3 imply that

\[
\left( \frac{1}{|\lambda|^{p+1}} \right)^{\frac{p+1}{p+2}} \leq \frac{1}{2\pi} \int_{\partial \Omega} |x|^p d\sigma(x), \tag{17}
\]

which proves the Part (i) the theorem in the case that \( \Omega \) is simply connected. If \( \Omega \) is not simply connected, then there exists an integer \( m \) and simply connected
open bounded sets $\Omega_i$, $i = 0, \ldots, m$, such that for $i = 1, \ldots, m$, the sets $\overline{\Omega}_i$ are disjoint and $\overline{\Omega}_i \subset \Omega_0$; $0 \in \Omega_0$ and

$$
\Omega = \Omega_0 \setminus \left( \bigcup_{i=1}^m \overline{\Omega}_i \right).
$$

Let $|\Omega_0| = \pi R_0^2$ and $|\Omega| = \pi R^2$. Using Part (i) and that $p + 1 \geq 0$ we obtain that

$$
\int_{\partial \Omega} |x|^p d\sigma(x) \geq \int_{\partial \Omega_0} |x|^p d\sigma(x) \geq 2\pi R_0^{p+1} \geq 2\pi R^{p+1}.
$$

This proves Part (i) of the theorem. ■

We now turn to the proof of Part (ii). For the case $0 \not\in \Omega$ the idea is very simple and consists roughly speaking of the following: If $p \geq 0$, then $|x|^p$ decreases for all $x \in \partial \Omega$, if $\Omega$ is shifted closer to the origin in an appropriate way. We introduce the following notations: Let $a, b \in \mathbb{R}^2$ and

$$
[a, b] = \{ x \in \mathbb{R}^2; x = \lambda a + (1 - \lambda)b, \ 0 \leq \lambda \leq 1 \} \quad \overline{ab} = \{ x \in \mathbb{R}^2; x = \lambda a + (1 - \lambda)b, \ \lambda \in \mathbb{R} \}.
$$

We will use the following simple lemma.

**Lemma 4** Let $a, b \in \mathbb{R}^2$ be such that $0 \in \overline{ab}$ and assume that $p \geq 0$. Then

$$
\min \left\{ \int_{\gamma} |x|^p d\sigma; \gamma \in C^1 \left( [0, 1], \mathbb{R}^2 \right), \gamma(0) = a, \gamma(1) = b \right\} = \int_{[a, b]} |x|^p d\sigma.
$$

**Proof** Let $\gamma \in C^1 \left( [0, 1], \mathbb{R}^2 \right)$ be any curve such that $\gamma(0) = a$ and $\gamma(1) = b$. Using a rotation, we can assume without loss of generality that $\overrightarrow{ab} = \{(t, 0); t \in \mathbb{R}\}$. Let us call $\tilde{\gamma} = (\gamma_1, 0)$ the projection of $\gamma = (\gamma_1, \gamma_2)$ onto $\overrightarrow{ab}$. We immediately get the two inequalities $|\gamma'(t)| \geq |\tilde{\gamma}'(t)|$ and $|\gamma(t)|^p \geq |\tilde{\gamma}(t)|^p$. We therefore obtain that

$$
\int_{\gamma} |x|^p d\sigma = \int_0^1 |\gamma(t)|^p |\gamma'(t)| dt \geq \int_0^1 |\tilde{\gamma}(t)|^p |\tilde{\gamma}'(t)| dt \geq \int_{[a, b]} |x|^p d\sigma.
$$

From this inequality the lemma follows. ■

**Proof (Theorem 1 Part (ii)).** Step 1. Note that if $p \geq 0$, the map $x \mapsto |x|^p$ is continuous. We thus obtain from Part (i) and by a continuity argument that \(\mathbb{M}\) holds for all $\Omega$ such that $0 \in \overline{\Omega}$. Moreover, as in the proof of Part (i) Step 4, we can assume that $\Omega$ is simply connected.

Step 2. We show in this step that we can drop the assumption $0 \in \Omega$. So suppose that $0 \not\in \overline{\Omega}$, but we still assume that $\Omega$ is connected. Then we can also assume, as in the proof of Part (i) Step 4, that $\Omega$ is simply connected. Define $E = \text{conv}(\overline{\Omega})$ as the convex hull of $\overline{\Omega}$. We now distinguish two cases.

Case 1: $0 \not\in E$. By the definition of $E$ we can assert the existence of $x_0 \in \partial E$, $y_1, y_2 \in \partial \Omega$ and $0 \leq \lambda \leq 1$ with the following properties
(i) \(|x_0| = \min_{y \in E} |y|\) and \(x_0 = \lambda y_1 + (1 - \lambda)y_2\).

(ii) \(\overline{y_1y_2}\) is a separating hyperplane for \(\{0\}\) and \(E\), i.e. \(\langle x_0; x - x_0 \rangle \geq 0\) for all \(x \in E\).

We have assumed \(y_1 \neq y_2\), otherwise we can take \(x_0 \in \partial \Omega\) and the following proof simplifies considerably, as can be easily verified. It follows from property (ii), that \(|x - x_0| \leq |x|\) for all \(x \in E\) and therefore

\[|x - x_0|^p \leq |x|^p\quad \text{for all } x \in E.\]  

Since \(\Omega\) is simply connected, \(\partial \Omega\) is the image of an oriented simple closed curve \(\gamma\). The points \(y_1, y_2\) split \(\gamma\) into two curves \(\gamma_1\) and \(\gamma_2\) which do not intersect (except at \(y_1, y_2\)), such that \(\gamma = \gamma_1 + \gamma_2\) and have the properties (reparametrizing, if necessary)

\[\gamma_1(0) = y_1, \quad \gamma_1(1) = y_2, \quad \gamma_2(0) = y_2, \quad \gamma_2(1) = y_1.\]

By abuse of notation, let us consider \([y_1, y_2]\) as an oriented curve. Then the two closed simple curves \(\gamma_1 - [y_1, y_2]\) and \(\gamma_2 + [y_1, y_2]\) both bound a simply connected bounded set, let’s say \(\Omega_1\) and \(\Omega_2\). Since \(\overline{y_1y_2}\) is a separating hyperplane and \(\gamma_1\) and \(\gamma_2\) do not intersect, it follows that \(\Omega\) must be contained in one of the \(\Omega_1\) or \(\Omega_2\). (It is enough to note that \([y_1, y_2]\) is a separating hyperplane for \(\{0\}\)) Let us assume that \(\Omega \subset \Omega_2\), and in particular \(|\Omega| \leq |\Omega_2|\). Let us define \(\Omega'_2 = \Omega_2 - x_0\) and \(\Omega' = \Omega - x_0\). It now follows from Step 1 applied to \(\Omega'_2\), from Lemma \(\Box\) and from \(\Box\) that

\[\left(\frac{|\Omega|}{\pi}\right)^{\frac{1}{n+1}} \leq \left(\frac{|\Omega'_2|}{\pi}\right)^{\frac{1}{n+1}} \leq \int_{\partial \Omega'_2} |x|^p d\sigma \leq \int_{\partial \Omega} |x|^p d\sigma \leq \int_{\partial \Omega} |x|^{n+1} d\sigma.\]

This proves the claim in the present case.

Case 2: \(0 \in E\). The argument is very similar to that of Case 1 and we omit the details. Since \(0 \in E\), there exists \(y_1, y_2 \in \partial \Omega\) such that \(0 \in [y_1, y_2]\) and \((y_1, y_2) \cap \overline{\Omega} = \emptyset\) (where \((y_1, y_2)\) is the open line segment). We define \(\gamma_1, \gamma_2\), respectively \(\Omega_1, \Omega_2\) as in Case 1. Using that neither \(\gamma_1\) nor \(\gamma_2\) intersect \((y_1, y_2)\), that \(\gamma_1\) and \(\gamma_2\) do not intersect and that \(0 \notin \overline{\Omega}\), one easily deduces that \(\Omega\) must be contained in \(\Omega_1\) or \(\Omega_2\). We then argue exactly as in Case 1, whereby we set \(x_0 = 0\), i.e. there is no need to shift the domain to origin.

Step 3. We now show that we can also drop the assumption that \(\Omega\) is connected. In view of Theorem \(\Box\) we can assume that \(0 \leq p \leq 1\). Let \(\Omega_i, i = 1, \ldots, l\) denote the connected components of \(\Omega\), and \(R, R_i > 0\) be such that \(\pi R^2_i = |\Omega_i|\), respectively \(\pi R^2 = |\Omega|\). We know from Step 2 that for each \(i = 1, \ldots, l\)

\[2\pi R_i^{p+1} \leq \int_{\partial \Omega_i} |x|^p d\sigma.\]  

\[11\]
It follows from $|\Omega| = \sum_{i=1}^{l} |\Omega_i|$ that $R_{p+1} = \left( R_1^2 + \cdots + R_l^2 \right)^{\frac{p+1}{2}}$. The result now follows from (19) and the inequality

\[
(R_1^2 + \cdots + R_l^2)^{\frac{p+1}{2}} \leq \left( R_1^{p+1} + \cdots + R_l^{p+1} \right)^{\frac{1}{2}},
\]

see for instance [21] Theorem 19 page 28, where we have used that $0 \leq p \leq 1$.

**Remark 5** We have used in the proof of Theorem 1 Part (ii) the result of Betta-Brock-Mercaldo-Posteraro [2], i.e. Theorem 3, for the case $p > 1$. Note however that only Step 3 of the proof requires that $p > 1$. Actually Step 3 can be proven without the use of [2] by a similar argument as Step 2: One shifts together two disjoint components of $\Omega$ appropriately until they touch and then argue by induction on the number of disjoint components. In this way one can obtain an independent proof.

**Proof (Theorem 1 Part (iii) and (iv)).** Step 1 (Proof of Part (iii)). Step 4 in the proof of Part (i) shows that we cannot have equality if $\Omega$ is not simply connected. Thus we assume that $\Omega$ is simply connected. If there is equality in the theorem, then we must have equality in both of the inequalities in (17). We obtain from the first one, equations (11) and (13) that

\[
\lim_{r \to 1} \sum_{n=1}^{\infty} n|a_n|^2 r^{2n} = 0.
\]

Therefore $a_n = 0$ for all $n \geq 1$, $Q(z) = \lambda + a_0 z$ and

\[
g(z) = \frac{\lambda + a_0 z}{z} : \partial B_1 \to \partial \Omega.
\]

It is easy to check that $g$ is a Möbius transformation which sends the unit circle to a circle with center $a_0$ and radius $|\lambda|$. This proves that $\Omega$ must be a ball. Note that $|\lambda| > |a_0|$, since by assumption $0 \in \Omega$. It remains to show that $a_0 = 0$, if $p \neq 0$.

We now use that also the second inequality in (17) must be an equality. Since $Q(z) = \lambda + a_0 z$ and $u(z) = -\lambda$ are holomorphic in $C$ we have that (16) holds now also for $r = 1$, and we must have equality. We therefore obtain from (15) and (16) that the following equalities must hold

\[
|\lambda|^{p+1} = |\tau(0)| = |\lambda| \cdot \left( \int_0^{2\pi} \exp(p\varphi(e^{it}))dt \right) = |\lambda| \cdot \int_0^{2\pi} \left| \exp(p\varphi(e^{it})) \right| dt.
\]

Defining $f(t) = \exp(p\varphi(e^{it}))$, the previous equality implies that

\[
\left| \int_0^{2\pi} f(t) dt \right| = \int_0^{2\pi} |f(t)| dt.
\]

12
This is only possible if there is a function $\phi : [0, 2\pi] \to \mathbb{R}$ and two constants $c_1, c_2 \in \mathbb{R}$, such that $\text{Re}(f(t)) = c_1 \phi(t)$ and $\text{Im}(f(t)) = c_2 \phi(t)$. (cf. for instance [21], Theorem 201 page 148 applied to $f_1 = \text{Re}(f)$ and $f_2 = \text{Im}(f)$). We thus get that $\varphi$ has to satisfy the two equations, defining $c = c_1 + ic_2$:

$$
\exp(p\varphi(e^{it})) = c\varphi(t) \quad \text{and} \quad \exp(\varphi(e^{it})) = \lambda + a_0 e^{it}.
$$

We multiply the second equation by $p$ and obtain by deriving the two equations

$$
\frac{\varphi'(t)}{\varphi(t)} = p\varphi'(e^{it})ie^{it} = \frac{a_0 e^{it}}{\lambda + a_0 e^{it}}.
$$

Thus if $p \neq 0$ we must have that

$$
\frac{a_0 e^{it}}{\lambda + a_0 e^{it}} = \in i\mathbb{R} \quad \text{for all } t \in [0, 2\pi].
$$

This is only possible if $a_0 = 0$. Recall that the image of $\partial B_1$ under the M"obius transformation $g$ is a circle and the image of that circle under the map $z \to 1/z$ cannot be line.

Step 2 (Proof of Part (iv)). We can again assume that $p < 1$, by inferring to [2] Theorem 4.3, where it is proven for the case $p \geq 1$ that $\Omega$ has to be a ball centered at the origin. Note first that if $0 \leq p < 1$, then we have equality in (20) if and only if $l = 1$ (see e.g. [21] Theorem 19). Thus $\Omega$ has to be connected. Moreover, as in (ii), we must have that $\Omega$ must be simply connected. We now distinguish two cases: If $0 \in \Omega$, then the result follows from Part (iii). If $0 \notin \Omega$, then Step 2 in the proof Theorem [1] Part (ii), shows that there exist a domain $\Omega' \neq \Omega$, which has piecewise $C^1$ boundary and the properties

$$
|\Omega| < |\Omega'| \quad \text{and} \quad \int_{\partial \Omega'} |x|^p d\sigma \leq \int_{\partial \Omega} |x|^p d\sigma.
$$

We therefore cannot have equality in (45) for $\Omega$. $
$

3 Best Constants for Hardy-Sobolev Inequalities

In what follows $C_c^\infty(\mathbb{R}^2)$ shall denote the space of smooth functions with compact support. It is known, see for instance Caffarelli-Kohn-Nirenberg [4] that there exists a constant $C > 0$ such that

$$
\|u |x|^\gamma\|_{L^r} \leq C\|Du |x|^\alpha\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^2),
$$

if and only if $\alpha, \gamma$ and $r$ satisfy the conditions

$$
r > 0, \quad 0 \leq \alpha - \gamma \leq 1, \quad 0 < \frac{1}{r} + \frac{\gamma}{2} = \frac{\alpha + 1}{2}.
$$
If $\Omega \subset \mathbb{R}^2$ is some smooth set we can take $u_i$ as a smooth approximation of the characteristic function $\chi_\Omega$ and obtain by a limiting process (or alternatively generalize \[21\] to functions of bounded variation) that

$$\left( \int_{\Omega} |x|^{r} dx \right)^{\frac{1}{r}} \leq C \int_{\partial \Omega} |x|^\alpha d\sigma(x).$$

We set $p = \alpha$ and $q = \gamma r$, which satisfy, in view of \[22\],

$$0 < 2 + q = r(p + 1), \quad 0 \leq pr - q \leq r.$$

From the first condition we immediately get that $q > -2$ and $p > -1$. Moreover solving the first condition for $r$ and setting into the second one gives that

$$p - 1 \leq q \leq 2.$$

Thus we have obtained that any $\Omega$ satisfies

$$\left( \int_{\Omega} |x|^q dx \right)^{\frac{p+1}{p+q}} \leq C \int_{\partial \Omega} |x|^p d\sigma(x) \quad (23)$$

if

$$-2 < p - 1 \leq q \leq 2p. \quad (24)$$

Note that the constant $C$ is the same as in \[21\] and also that the conditions $p > -1$ and $q > -2$ imply that the integrals are finite for any domain $\Omega \subset \mathbb{R}^2$. The next proposition shows that the converse is also true if $r \geq 1$.

**Proposition 6** Let $p, q \in \mathbb{R}$ be such that $p > -1$, $q > -2$ and

$$\frac{q + 2}{p + 1} = r \geq 1.$$

Suppose that there exists a constant $C$ such that \[20\] holds for all bounded open smooth sets $\Omega \subset \mathbb{R}^2$. Then the following inequality holds

$$\|u \, |x|^{\gamma}\|_{L^r} \leq C\|Du \, |x|^\alpha\|_{L^1}, \quad \text{for all } u \in C_c^\infty(\mathbb{R}^2),$$

where $\gamma = q/r$, $\alpha = p$ and the constant $C$ is the same as in \[23\].

In the proof we follow Struwe \[30\] page 43, i.e. in the case $p = q = 0$. Note that if $q = 0$, the Minkowsky inequality in the below proof is presicely valid for $p \in (-1, 1]$. We use the notation: if $u \in C_c^\infty(\mathbb{R}^2)$, then $\Omega(t) = \Omega_u(t) = \{x \in \mathbb{R}^2; |u(x)| > t\}$ shall denote the level sets of $u$.

**Proof** Let $\chi_{\Omega(t)} : \mathbb{R}^2 \to \{0, 1\}$ denote the characteristic function of $\Omega(t)$. Let $M = \max |u|$. Then we can write for all $x \in \mathbb{R}^2$

$$|u(x)| = \int_0^M \chi_{\Omega(t)}(x) dt.$$

Thus we obtain that

$$\|u \, |x|^{\gamma}\|_{L^r} = \left( \int_{\mathbb{R}^2} \left( \int_0^M \chi_{\Omega(t)}(x)|x|^\gamma dt \right)^r dx \right)^{\frac{1}{r}}.$$
Using that \( r \geq 1 \) we can apply the Minkowsky inequality (see for instance Hardy Littlewood and Pólya [21, Theorem 202, page 148]), and we get

\[
\|u| x| \gamma \|_{L^r} \leq \int_0^M \left( \int_{\mathbb{R}^2} \chi_{\Omega(t)}(x)|x|^{\gamma r} dx \right)^{\frac{1}{r}} dt = \int_0^M \left( \int_{\Omega(t)} |x|^q dx \right)^{\frac{1}{q}} dt.
\]

(25)

Note that by Sard’s theorem \(|\nabla u| \neq 0\) on \(|u| = t\) for almost every \( t \in [0, M]\), hence the sets \( \Omega(t) \) are smooth for almost every \( t \). We therefore obtain from the hypothesis, namely inequality (23), that

\[
\left( \int_{\Omega(t)} |x|^q dx \right)^{\frac{1}{q} + 1} \leq C \int_{\partial \Omega(t)} |x|^\alpha d\sigma \left( x \right) dt.
\]

(26)

We substitute this inequality into the previous one

\[
\|u| x| \gamma \|_{L^r} \leq C \int_0^M \left( \int_{\{ |u| = t \}} |x|^\alpha d\sigma \right) dt = C \int_0^M \left( \int_{\{ |u| = t \}} |x|^\alpha d\sigma \right) dt.
\]

We now apply the coarea formula to get

\[
\|u| x| \gamma \|_{L^r} \leq C \int_{\mathbb{R}^2} |\nabla u(x)| |x|\alpha dx = C \int_{\mathbb{R}^2} |\nabla u(x)| |x|\alpha dx,
\]

which proves the proposition. \( \blacksquare \)

We now apply Theorem [21] and the previous proposition to obtain the following imbedding with best constant.

**Theorem 7** Let \( p \in (-1, 1) \) and define \( r \) as

\[
\frac{2}{p + 1}.
\]

Consider the inequality

\[
\|u\|_{L^r(\mathbb{R}^2)} \leq \frac{\pi^{1/r}}{2\pi} \int_{\mathbb{R}^2} |\nabla u(x)| |x|^\alpha dx.
\]

(27)

Then we have the following statements:

(i) If \( p \in [0, 1] \) the inequality (27) holds for all \( u \in C^\infty_c(\mathbb{R}^2) \).

(ii) If \( p \in (-1, 0) \) inequality (27) holds for all \( u \in C^\infty_c(\mathbb{R}^2) \) with the property that for all \( t \in (0, \max |u|) \) the sets \( \Omega(t) \) are connected and \( 0 \in \Omega(t) \).

(iii) In both cases the constant \( \pi^{1/r}/2\pi \) is optimal and it is not attained.
Proof Step 1. Part (i) follows directly from Theorem 1 Part (ii) and Proposition 6. For Part (ii) it is sufficient to note that the proof of Proposition 6 carries through without modification, since by hypothesis $\Omega(t)$ satisfies the hypothesis of Part (i) in Theorem 1 for every $t$.

Step 2. Let us now show that the constant in the imbedding is optimal. Note that we have equality in (25) if and only if $\chi_{\Omega(t)}$ can be factorized as $\chi_{\Omega(t)} = \Phi(x)\Psi(t)$ (see [21], Theorem 202). And in (26) we have equality if $\Omega(t)$ is a ball centered at the origin (unless $0 \in \partial\Omega(t)$). This suggests that one should take $u$ as the characteristic function of a ball. We proceed thus by approximating $\chi_{B_R(0)}$ for some $R > 0$ and define for each $m \in \mathbb{N}$

$$f_m(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq R \\ -mr + mR + 1 & \text{if } R \leq r \leq R + \frac{1}{m} \\ 0 & \text{if } r \geq R + \frac{1}{m}. \end{cases}$$

And finally we define $u_m(x) = f_m(|x|)$. The function $u_m$ is not smooth but it is clear that (27) must also hold for it. One easily verifies by a direct calculation that

$$\lim_{m \to \infty} \|u_m\|_L^r \int_{\mathbb{R}^2} |\nabla u_m(x)| |x|^p dx = \frac{\pi^{\frac{p+1}{2}}}{2\pi} = \frac{\pi^{\frac{1}{2}}}{2\pi},$$

which shows that the constant is optimal. This also shows that there is no extremal function attaining inequality in the embedding, since the characteristic function of any set is not smooth.

In the embedding (ii) of Theorem 7 neither of the hypothesis $0 \in \Omega(t)$ nor $\Omega(t)$ is connected can be relaxed. More precisely we have the following Proposition.

Proposition 8 Let $p \in (-1,0)$ and $r = \frac{2}{(p+1)}$. Then there exists no constant $C > 0$ such that

$$\|u\|_{L^r(\mathbb{R}^2)} \leq C \int_{\mathbb{R}^2} |\nabla u(x)| |x|^p dx \quad \text{for all } u \in X,$$

where either $X = C_c^\infty(\mathbb{R}^2)$ or

$$X = \{ u \in C_c^\infty(\mathbb{R}^2); 0 \in \Omega_u(t) \text{ for all } t \in (0, \max |u|) \}.$$ 

Proof The case $X = C_c^\infty(\mathbb{R}^2)$ is standard, cf. [4] (choose any $v \in C_c^\infty(B_1)$ and define $u_m = v(m - x)$ for all $m \in \mathbb{N}$ and let $m \to \infty$). So let us consider the second definition of $X$. Choose $v \in C_c^\infty(B_1)$ such that $0 \leq v \leq 1$ and $v(0) = 1$, extend it by zero in $\mathbb{R}^2 \setminus B_1$, and define for $m \in \mathbb{N}$

$$u_m(x) = v(mx) + v((m,0) - x).$$

We have that for all $m$ and all $t \in (0,1) = (0, \max |u_m|)$

$$0 \in \Omega_{u_m}(t) \quad \text{but} \quad \Omega_{u_m}(t) \text{ is not connected}.$$
One easily verifies by a direct calculation, using that $|x|^p \leq |m-1|^p$ for all $x \in B_1((m,0))$, that

$$\int_{\mathbb{R}^2} |\nabla u_m| |x|^p \leq \frac{1}{m^{p+1}} C_2 + (m-1)^p C_3,$$

where $C_2 = \int_{B_1} |\nabla v| |x|^p dx > 0$ and $C_3 = \int_{B_1} |\nabla v| dx > 0$, Since $p \in (-1,0)$, we therefore obtain that if $m \to \infty$

$$\int_{\mathbb{R}^2} |\nabla u_m| |x|^p dx \to 0 \quad \text{but} \quad \|u_m\|_{L^r} \to \|v\|_{L^r},$$

for some constant $C \in \mathbb{R}$. ■

If $q = p-1$, then the weighted isoperimetric inequality and the best constant in the Hardy-Sobolev inequality can be obtained in an almost trivial way. This is in striking contrast to the case $q = p$, where the isoperimetric problem is much more difficult and has been solved by Dahlberg-Dubbs-Newkirk-Tran [12].

**Proposition 9** Let $p > -1$ and suppose that $\Omega \subset \mathbb{R}^2$ is a bounded open set with piecewise $C^1$ boundary. Then

$$\int_{\Omega} |x|^{p-1} dx \leq \frac{1}{p+1} \int_{\partial \Omega} |x|^p d\sigma.$$

Moreover, if there is equality, then $\Omega$ has to be a ball centered at the origin.

**Proof** Note that

$$\text{div}(|x|^{p-1} x) = (p+1)|x|^{p-1}.$$

Let $\nu$ be the unit outward normal to $\partial \Omega$. We therefore get from the divergence theorem and using the Cauchy-Schwarz inequality $|(x, \nu)| \leq |x|$, that

$$\int_{\Omega} |x|^{p-1} dx = \frac{1}{p+1} \int_{\partial \Omega} (x, \nu)|x|^{p-1} d\sigma \leq \frac{1}{p+1} \int_{\partial \Omega} |x|^p d\sigma.$$

Note that if we have equality, then $x$ must be parallel to $\nu$ for every $x \in \partial \Omega$. This is only possible if $\partial \Omega$ is a circle centered at the origin. ■

**Corollary 10** Let $p > -1$. Then the following inequality holds

$$\|u|^{p-1}\|_{L^1} \leq \frac{1}{p+1} \int_{\mathbb{R}^2} |\nabla u(x)||x|^p dx \quad \text{for all} \ u \in C^\infty_c(\mathbb{R}^2),$$

where $1/(p+1)$ is the best constant for the embedding.

**Proof** This follows from Proposition 9 and Proposition 10. Taking $u_m$ as in the proof of Theorem 7 shows that this is the best constant. ■

It is the main result of Dahlberg-Dubbs-Newkirk-Tran [12] Theorem 3.16 that if $p > 0$, then balls of radius $R$ and centered at a distance $R$ from the origin are minimizers of the isoperimetric problem

$$I(k) = \inf \left\{ \int_{\partial \Omega} |x|^p d\sigma; \ \Omega \subset \mathbb{R}^2 \text{ and } \int_{\Omega} |x|^p dx = k \right\}.$$
For convenience we will abbreviate $B_R((R,0)) = B_R(R)$. Thus we obtain that any $\Omega$ with the property
\[
\int_{\Omega} |x|^p dx = \int_{B_R(R)} |x|^p dx,
\]
has to satisfy the inequality
\[
\int_{\partial \Omega} |x|^p d\sigma \geq \int_{\partial B_R(R)} |x|^p d\sigma = \frac{1}{C \left( \int_{\Omega} |x|^p dx \right)^{\frac{p+2}{p+2}}},
\]
where, by scaling invariance,
\[
C = \frac{\left( \int_{B_R(R)} |x|^p dx \right)^{\frac{p+2}{p+2}}}{\int_{\partial B_R(R)} |x|^p d\sigma} = \frac{\left( \int_{B_1(1)} |x|^p dx \right)^{\frac{p+2}{p+2}}}{\int_{\partial B_1(1)} |x|^p d\sigma}.
\]
\[(28)\]
We therefore obtain that every bounded open smooth set $\Omega \subset \mathbb{R}^2$ has to satisfy
\[
\left( \int_{\Omega} |x|^p dx \right)^{\frac{p+2}{p+2}} \leq C \int_{\partial \Omega} |x|^p d\sigma.
\]
As before, we obtain from Proposition 6 the best constant in a Hardy-Sobolev inequality.

**Proposition 11** Let $p > 0$ and
\[
r = \frac{p + 2}{p + 1}, \quad \gamma = \frac{p(p + 1)}{p + 2}.
\]
Then
\[
\|u |x|^\gamma\|_{L^1} \leq C \|Du |x|^p\|_{L^1} \quad \text{for all } u \in C_0^\infty(\mathbb{R}^2),
\]
where $C$ is the best constant and is given by (28).

4 **Flucher’s Estimate for $|\Omega|$ in Terms of Green’s Function**

Another application is a generalization of an estimate by Flucher [18] for $|\Omega|$ in terms of the Green’s function (see Theorem 17 in [18]). The Green’s function for $\Omega$ with singularity at $x \in \Omega$ will be denoted by $G_{\Omega,x}$.

**Theorem 12** Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with smooth boundary $\partial \Omega$, and $x \in \Omega$. Then the inequality
\[
|\Omega|^{1 - \frac{p}{2}} \leq \frac{1}{4\pi^{1 + \frac{p}{2}}} \int_{\partial \Omega} \frac{1}{|y|^{p + 2}} \frac{1}{|\nabla G_{\Omega,x}(y)|} d\sigma(y).
\]
holds true
(i) for all $0 \leq \beta \leq 2$ if $\Omega$ is connected and $0 \in \Omega$.
(ii) for all $\beta \leq 0$ without restriction on $\Omega$. 

18
In the proof will use the following simple lemma.

**Lemma 13** \( G_{\Omega,x} \) satisfies for every \( t \in [0, \infty) \) the two equations:

\[
\int_{\{G_{\Omega,x} < t\}} |\nabla G_{\Omega,x}(y)|^2 \, dy = t \quad \text{and} \quad \int_{\{G_{\Omega,x} = t\}} |\nabla G_{\Omega,x}(y)| \, d\sigma(y) = 1.
\]

**Proof** By the definition of Green’s function \( \int_{\Omega} \nabla G_{\Omega,x}(y) \nabla f(y) \, dy = f(x) \) for all \( f \in W_0^{1,2}(\Omega) \). In particular, choosing \( f(y) = \inf\{G_{\Omega,x}(y), t\} \) gives the first equality. From the coarea formula we have

\[
\int_{\{G_{\Omega,x} < t\}} |\nabla G_{\Omega,x}(y)|^2 \, dy = \int_0^t \left( \int_{\{G_{\Omega,x} = s\}} |\nabla G_{\Omega,x}(y)| \, d\sigma(y) \right) \, ds.
\]

The second equality therefore follows from the first one by derivation. \( \square \)

**Proof (Theorem 12).** Set \( p = -\beta/2 \) and let \( R > 0 \) be such that \( |\Omega| = \pi R^2 \). Then we get from Theorem [1]

\[
2\pi R^{1+p} \leq \int_{\partial\Omega} |y|^p \, d\sigma(y) = \int_{\partial\Omega} \frac{\sqrt{\nabla G_{\Omega,x}(y)}}{\sqrt{|y|^2 |\nabla G_{\Omega,x}(y)|}} \, d\sigma(y).
\]

We apply Hölder inequality and the second equality in Lemma 13 with \( t = 0 \) to get that

\[
2\pi R^{1+p} \leq \left( \int_{\partial\Omega} |\nabla G_{\Omega,x}(y)| \, d\sigma(y) \right)^{\frac{1}{2}} \left( \int_{\partial\Omega} \frac{1}{|y|^2 |\nabla G_{\Omega,x}(y)|} \, d\sigma(y) \right)^{\frac{1}{2}}
\]

\[
= \left( \int_{\partial\Omega} \frac{1}{|y|^2 |\nabla G_{\Omega,x}(y)|} \, d\sigma(y) \right)^{\frac{1}{2}}.
\]

From this the theorem follows immediately. \( \square \)

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