RENORMALIZATION OF RATIONAL FUNCTIONS OFTEN HAS SMALL MODULUS

ALEXANDER BLOKH, GENADI LEVIN, LEX OVERSTEEGEN, AND VLADLEN TIMORIN

ABSTRACT. The modulus of a polynomial-like (PL) map controls distortion of the corresponding straightening map and hence geometry of the corresponding PL Julia set. For this reason it is considered an important invariant. Lower bounds on the modulus, so called complex (a priori) bounds, are known in a great variety of contexts. We complement these by upper bounds under rather general geometric assumptions on renormalizable rational functions.

1. MAIN RESULTS

Since Sullivan’s work on Feigenbaum’s universality, complex a priori bounds for polynomial-like renormalizations play a key role in the polynomial dynamics. Such bounds, though usually difficult to obtain, have been established and/or crucially used for various classes of quadratic polynomials (see, e.g., [25, 11, 17, 5, 16, 8, 9, 10]). In particular, the bounds imply that the Julia set $J(f)$ is locally connected and, in a lot of cases, the MLC conjecture at the corresponding points. On the other hand, examples (by Douady and Hubbard) of infinitely (satellite) renormalizable quadratic polynomials with non-locally connected Julia sets show that the bounds do not hold in general (see [22, 24] for qualitative versions and [12, 13, 14] for quantitative ones). Until now, these have been the only known examples of infinitely renormalizable quadratic polynomials without complex bounds.

In the present paper we show that such bounds do not exist for many more explicit combinatorics. In fact, they cannot exist once a map admits infinitely many satellite renormalizations having relative periods tending to infinity. To this end we first make simple geometric assumptions concerning

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cycles of connected polynomial-like Julia sets of rational functions; under these assumptions we estimate the associated moduli of annuli, and deduce from them the above mentioned conclusions.

Let \( \mathbb{P}^1 \) be the Riemann sphere, i.e., the complex projective line over \( \mathbb{C} \). The space \( \mathbb{P}^1 \) comes with a conformal structure and a conformal Riemannian metric of constant positive curvature, the spherical metric. Normalize the spherical metric so that the total area of \( \mathbb{P}^1 \) is \( \pi \). With this normalization, \( \mathbb{P}^1 \) is isometric to a sphere in the Euclidean 3-space of radius \( \frac{1}{2} \). For two points \( z, w \in \mathbb{P}^1 \), we let \( \text{dist}(z, w) \) denote the spherical distance between \( z \) and \( w \). All lengths and areas are also with respect to the spherical metric.

For a compact subset \( X \subset \mathbb{C} \), let \( \text{diam}(X) \) be its (spherical) diameter.

We assume basic knowledge of complex dynamics (see Appendix on some background). Let a rational function \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be renormalizable in the sense of Douady and Hubbard, i.e., there is \( q > 0 \) and Jordan disks \( U \subset V \) such that \( f^q : U \to V \) is a degree \( d^* \) polynomial-like (PL) map with connected Julia set \( K^* \). The annulus \( A = V \setminus \overline{U} \) is called a fundamental annulus of \( K^* \), and the annulus \( U \setminus K^* \) is called the root annulus (of \( K^* \)) (associated with \( A \)). The set

\[
K = \{K^*, f(K^*), \ldots, f^{q-1}(K^*)\}
\]

is called a PL cycle, and each set \( f^i(K^*) \), where \( i \geq 0 \), is called an element of \( K \). Each element of a PL cycle is a connected filled Julia set of some PL map obtained as a suitable restriction of \( f^q \) (these sets are abbreviated as PL sets). In what follows, \( K \) always denotes a PL cycle of period \( q \) such that the first return map \( f^q \) has degree \( d^* \) on each element of \( K \). We always assume that every critical point of \( f \) belongs to at most one element of \( K \); this condition is standard, \([7, 19, 20]\). Given a collection \( \mathcal{H} \) of sets, let \( \mathcal{H}^+ \) be the union of elements of \( \mathcal{H} \).

**Definition 1.1 (Primitive and satellite).** Let \( f \) be a rational function.

1. If all elements of \( K \) are pairwise disjoint (in which case all \( f^q \)-fixed points in \( K^+ \) have exact period \( q \)) then \( K \) and all its elements are called primitive.

2. If there is a repelling \( f \)-cycle \( \alpha_0, \ldots, \alpha_{r-1} \) of period \( r < q \) (called the base cycle (of \( K \)) such that \( K^+ \) has \( r \) components \( C_0, \ldots, C_{r-1} \) and for each \( i \) the set \( C_i \) is the union of all elements of \( K^+ \) containing \( \alpha_i \) as a non-separating point and otherwise pairwise disjoint, then \( K \) and its elements are called satellite of base period \( r \). Setting \( q = rs \), the number \( s \) is called the relative period of \( K \).

By \([7, \text{Proposition 3.4}]\), a PL cycle is primitive or satellite except that points of the base cycle do not have to be non-separating points of the elements of \( K \). From now on we consider only primitive or satellite PL cycles.
Main Theorem. Let $K$ be a PL cycle for a rational function $f$ such that at least $t$ elements of $K$ have diameter at least $k$ and a root annulus of modulus at least $m$. Then $4tk^2 \leq e^{\pi/m}$ provided that $t$ is sufficiently large.

Theorem A is related to the Main Theorem. Elements of $K$ that contain critical values of $f$ are called critical value elements of $K$. Observe that the inequality in Theorem A does not depend on $r$.

Theorem A. Let $f$ be a rational function with a satellite PL cycle $K$ of relative period $s$. Let $D$ be the smallest diameter of a critical value element of $K$, and let $N \leq 2d - 2$ be the number of critical points of $f$ that belong to $K^+$. Then there is a number $s(D)$ such that for any $s > s(D)$, any element of $K$, and any root annulus $A$ of it, we have

$$\text{mod}(A) \leq \frac{2\pi d^*}{\ln(sD^2/N)}.$$ 

Consider an important case $f(z) = z^2 + c$ for some $c \in \mathbb{C}$.

Corollary 1.2. Suppose that a quadratic polynomial $f$ has a satellite PL cycle $K$ of sufficiently large period $s$ and base period $1$. Then the modulus of a root annulus of a PL set from $K$ is at most $4\pi/\ln(s)$.

Proof. Write $f$ as $f(z) = \lambda z(1 - \frac{z}{4})$ for a suitable affine coordinate $z$; then $z = 0$ is the repelling fixed point of $f$ of multiplier $\lambda$. The critical value $-2\lambda$ of $f$ has absolute value greater than 2 since $|\lambda| > 1$. Thus, the spherical diameter of the critical value element of $K$ is at least $\arctan(2) > 1$. Now the claim follows from Theorem A with $d^* = 2$ and $D = 1$. 

A compactness argument yields Corollary 1.3 proven in Section 3.

Corollary 1.3. Consider any compact set $N$ of degree $d$ rational functions. There exist numbers $D_N(r)$ and $s_N(r)$ depending only on $N$ and $r$ such that, for any $f \in N$, any satellite PL cycle $K$ for $f$ of relative period $s > s_N(r)$ and base period at most $r$, any element $K^*$ of $K$, and any root annulus $A^\text{root}$ of $K^*$, we have

$$\text{mod}(A^\text{root}) \leq \frac{2\pi d^*}{\ln(sD_N^2(r))}.$$ 

Some applications deal with the infinitely-renormalizable sets.

Definition 1.4 (Infinitely renormalizable sets). Consider a sequence of PL cycles $K_n$ of $f$ such that elements of $K_{n+1}$ are proper subsets of elements of $K_n$. In this case, if $p_n$ is the period of $K_n$, then $p_{n+1}/p_n = q_n$, where $q_n \geq 2$ is the number of elements of $K_{n+1}$ in every element of $K_n$. Under these assumptions, the set $S = \bigcap_n \bigcup_{K \in K_n} K$ is called an infinitely-renormalizable set for $f$, while the sequence $(K_n)$ is said to generate $S$, or to be a generating sequence for $S$.
In the mentioned earlier examples by Douady and Hubbard, quadratic polynomials have infinitely many satellite PL cycles of relative periods $s_i$ and base periods $r_i$, where both sequences $\{s_i\}$, $\{r_i\}$ tend to infinity and the diameters of critical value elements stay away from zero uniformly in $i$. Then Theorem A implies an explicit uniform upper bound for the moduli of all such PL cycles. In Corollary 1.5 we use notation from Definition 1.4.

**Corollary 1.5.** Let $S$ be an infinitely renormalizable set for a rational function $f$ generated by a sequence of PL cycles $K_n$. Suppose that there is an infinite subsequence $K_{n_i}$ such that $K_{n_i+1}$ is a satellite PL cycle of relative period $s_i$ and base period $r_i$. 

1. There exists $D(r) > 0$ with the following property. If the moduli of some fundamental annuli for $K_{n_i}$ are at least $m > 0$ and $s_i D(r_i) \to \infty$, then the moduli of all root annuli for $K_{n_i+1}$ tend to zero.

2. Assume that every component of $S$ that contains a critical point of $P$ is non-degenerate. Then

$$\text{mod}(A_{n_i+1}^{\text{root}}) \leq \frac{C}{\ln s_i}$$

for some $C > 0$ and all root annuli $A_{n_i+1}^{\text{root}}$ of $K_{n_i+1}$.

Corollary 1.5 is proved in Section 3. Recall that $p_{n_i}$ is the period of $K_{n_i}$, and each element of $K_{n_i}$ contains $q_{n_i+1} = r_i s_i$ elements of $K_{n_i+1}$. In part (2), the constant $C$ depends on $S$ but not on $i$. Item (1) means that an infinitely renormalizable polynomial admitting infinitely many satellite renormalizations (i.e., such that $r_i = 1$) with relative periods $s_i$ tending to infinity cannot have complex bounds.

For a sequence of polynomials $f_i$ of degree $d$ with connected $K(f_i)$ and with satellite PL cycles of base period 1 and relative periods $s_i \to \infty$, the moduli of root annuli tend (uniformly) to zero. As we show in Theorem B, the latter conclusion holds not only in the satellite case.

**Theorem B.** Let $\{f_n\}$ be a sequence of degree $d \geq 2$ rational functions that converges to a rational function $f$ of degree $d$. Assume that for each $n$ there is a renormalization $f_n^{q_n} : U_n \to V_n$ of period $q_n$ with connected PL set $K_n$, and $K_n \to K$ in the Hausdorff metric where $K$ is non-degenerate. Assume that $q_n \to \infty$ and $\text{mod}(V_n \setminus K_n) \not\to 0$ as $n \to \infty$. Then $K$ is contained in a periodic parabolic domain of $f$. Thus, if (1) $K$ is non-degenerate, (2) $q_n \to \infty$, and (3) $K$ is not contained in a parabolic periodic domain of $f$ (e.g., if $f$ has no parabolic points), then $\lim_{n \to \infty} \text{mod}(V_n \setminus K_n) = 0$.

Note that, in contrast to Theorem A, it is only required in Theorem B that at least one element of the PL cycle of $f_n$ has large diameter. On the other hand, Theorem B provides no explicit upper bound for the modulus.
None of conditions (1) – (3) in Theorem B can be dropped, see Section 5.

**Definition 1.6.** Let $S$ be an infinitely-renormalizable set. Consider its generating sequence $K_n$ of PL cycles, and, for each $n$, take the supremum modulus $m_n$ of a root annulus of an element of $K_n$. If $m_n \to 0$ as $n \to \infty$, then $S$ is said to have vanishing annuli.

Corollary 1.7 can be deduced from Theorem B.

**Corollary 1.7.** If an infinitely renormalizable set $S$ for a rational function $f$ is not a Cantor set, then it has vanishing annuli.

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2. PACKING MODULUS

Recall that, in our normalization, the spherical metric is given by

$$ds = \frac{|dz|}{1 + |z|^2}$$

with respect to an affine coordinate $z$ given by the stereographic projection of the sphere. In terms of the $z$-coordinate, the sphere is identified with $\mathbb{C} \cup \{\infty\}$, where $\mathbb{C}$ is the $z$-plane, and $\infty$ is the only point on the sphere that is not represented by a finite value of $z$. In particular, a circle of (Euclidean) radius $\rho$ centered at 0 in the $z$-plane has radius

$$\int_0^\rho \frac{dx}{1 + x^2} = \arctan(\rho)$$

in spherical geometry.

A **pointed continuum** in $\mathbb{P}^1$ is defined as a pair $(z, Z)$ consisting of a continuum $Z \subset \mathbb{P}^1$ and a point $z \in Z$. Suppose now that we are given $t$ pointed continua $(z_i, Z_i)$, where $i = 1, \ldots, t$. The **packing modulus** of this collection of pointed continua is defined as the supremum of positive real numbers $m$ with the following property. There are Jordan domains $U_i \supset Z_i$, $i = 1, \ldots, t$ such that $\operatorname{mod}(U_i \setminus Z_i) \geq m$ and $z_i \notin U_j$ for $j \neq i$.

**Theorem 2.1.** The packing modulus $m$ of $t$ pointed continua of diameter at least $k$ satisfies the inequality $4tk^2 \leq e^{\pi/m}$ if $t$ is sufficiently large.
Recall the Teichmüller extremal problem: find the maximal possible value of $\text{mod}(U \setminus Z)$, where an open topological disk $U \subset \mathbb{C}$ and a continuum $Z$ are such that $0, -1 \in Z$ while $\mathbb{C} \setminus U$ contains a point $w$ with $|w| = \varepsilon$. A solution to the Teichmüller extremal problem is given, e.g., in \cite{1}, paragraphs 4.11 – 4.12. It reads that the optimal value $m$ of $\text{mod}(U \setminus Z)$ is given by

$$m = \frac{-\pi}{2 \ln(\varepsilon/8)}$$

and it is attained when $Z = [-1, 0]$ while $U = \mathbb{C} \setminus [\varepsilon, \infty)$.

**Lemma 2.2.** Let $U \subset \mathbb{P}^1$ be a topological disk, and $Z \subset U$ a continuum of spherical diameter $k$. Then, for every point $z \in Z$, the round disk of spherical radius $\rho = 4ke^{-\pi/2m}$ is contained in $U$ if $\rho$ is sufficiently small.

**Proof.** By a suitable rotation of the sphere, we may arrange that $U$ lies in an affine chart $\mathbb{C}$, and the diameter of $Z$ is realized as the spherical distance $k$ between points $0 \in Z$ and $-\tan(k) \in Z$. Clearly, $U$ contains the set $\{|z| < \varepsilon\}$ for every $\varepsilon > 0$ such that $m = \text{mod}(U \setminus Z)$ is equal to or exceeds the solution of the Teichmüller problem for $\varepsilon/\tan(k)$. In other words, $\varepsilon \geq 8 \tan(k) \exp(-\pi/2m) \geq 8k \exp(-\pi/2m)$. Note also that $\{|z| < \varepsilon\}$ has spherical radius $\arctan \varepsilon$. For small $\varepsilon$, this radius is bigger than $\varepsilon/2$, and the desired claim follows. \hfill \Box

We are now ready to prove Theorem 2.1

**Proof of Theorem 2.1.** Consider a collection of $t$ pointed continua $(z_i, Z_i)$ with $\text{diam}(Z_i) \geq k$. Suppose that $m$ is less than the packing modulus of these continua. Then there are topological disks $U_i \supset Z_i$ such that $z_i \notin U_j$ for $i \neq j$. Let $B_i$ be the disk of spherical radius $2ke^{-\pi/2m}$ centered at $z_i$. Then, by Lemma 2.2, for sufficiently small $\rho = 4ke^{-\pi/2m}$, the disks $B_i$ are pairwise disjoint. The spherical area of each of these disks is less than the Euclidean area of a Euclidean disk of the same radius, that is, $\text{area}(B_i) \leq 4\pi k^2 e^{-\pi/m}$. On the other hand, $t$ such pairwise disjoint disks fit into the sphere $\mathbb{P}^1$ of total area $\pi$. It follows that

$$4t \pi k^2 e^{-\pi/m} \leq \pi \Rightarrow 4tk^2 \leq e^{\pi/m}.$$ 

If $t$ is large, then the radii of $B_i$ are necessarily small. Therefore, the above inequality holds for every $m$ less than the packing modulus. Passing to the limit, we see that it also holds for the packing modulus. \hfill \Box

By “period” we always mean “minimal period”.

**Definition 2.3.** Let $\mathcal{K}$ be a period $q$ PL cycle for $f$, and let $K^*$ be an element of $\mathcal{K}$. Then any point $z \in K^*$ of period $q$ is called a proper.

The proof of the next lemma is left to the reader.
Lemma 2.4. Proper points exist.

We can now complete the proof of the Main Theorem.

Proof of the Main Theorem. Let \( \mathcal{K} \) be a PL cycle satisfying the assumptions of the Main Theorem. Consider elements \( Z_1, \ldots, Z_t \) of this cycle that have diameter at least \( k \) and root annuli \( U_i \setminus Z_i \) of modulus at least \( m \). Choose points \( z_i \in Z_i \) as proper points; they exist by Lemma 2.4. Thus, we obtain a collection of \( t \) pointed continua in \( \mathbb{P}^1 \). If \( m_+ \) is the packing modulus of these pointed continua, then, by Theorem 2.1, we have \( 4tK^2 \leq e^{\pi/m_+} \). Note that \( z_i \) cannot belong to \( U_j \) for \( i \neq j \). Indeed, all points of \( U_j \) escape under \( f^q \), hence all \( f^q \)-fixed points in \( U_j \) must lie in \( Z_j \). It follows that \( m \leq m_+ \), which implies the Main Theorem. \( \square \)

3. The satellite case

Assume that \( f \) is a degree \( d \) rational function with PL cycle \( \mathcal{K} \) of period \( q \). Fix an element \( K^* \) of a PL cycle \( \mathcal{K} \). Consider the sets \( K^*_i \), \( f(K^*_i) \), \( \ldots \), \( f^{q-1}(K^*) \). Let \( V \setminus U \) be a fundamental annulus of \( K^* \). Define \( U_i \), \( V_i \) as appropriate iterated pullbacks of \( U \) and \( V \) containing \( K^*_i = f^i(K^*) \). Namely, \( U_i \) is the component of \( f^{i-q}(U) \) containing \( K^*_i \). Similarly, \( V_i \) is the component of \( f^{i-q}(V) \) containing \( K^*_i \). Clearly, \( V_0 = U \). Observe that \( U_i \setminus K^*_i \) is a root annulus of \( K^* \). Also, the map \( f^q : U_i \to V_i \) is a PL map with PL set \( K^*_i \), where \( i = 0, \ldots, q-1 \). Thus, the sets \( K^*_i \) are also PL sets of \( f \). Let \( d^* \) be the degree of the PL map \( f^* = f^q : U \to V \).

Lemma 3.1. Let \( K^* \) be an \( q \)-periodic PL set for a degree \( d \) rational function \( f \). Then \( d^* \leq 2^{2d-2} \), and for each \( i = 0, \ldots, q-1 \),

\[
\frac{\text{mod}(U \setminus K^*)}{d^*} \leq \text{mod}(U_i \setminus K^*_i) \leq \text{mod}(U \setminus K^*).
\]

Proof. Indeed, \( d^* \) is the product of \( d_i^* = \deg(f^i(K^*)) \rightarrow f^{i+1}(K^*) \) over \( i = 0, \ldots, q-1 \). The function \( f \) has \( 2d - 2 \) critical points, counting multiplicities. Positive integers \( d_i^* \) satisfy the inequality

\[
(d_0^* - 1) + \cdots + (d_{s-1}^* - 1) \leq 2d - 2,
\]

and the maximal possible value of the product of \( d_i^* \) subject to this inequality is attained when \( 2d-2 \) different numbers \( d_i^* \) are equal to 2, and all remaining \( d_j^* \) are equal to 1. In this case, the product is \( 2^{2d-2} \), as claimed. \( \square \)

Call \( i \) (and \( f^i(K^*) = K^*_i \)) critical if \( K^*_i \) contains a critical point of \( f \). The image of a critical element of \( \mathcal{K} \) is called a critical value element. Observe that if \( i \) is not critical and \( A^* \) is a root annulus of \( K^*_i \), then \( U_i = A^* \cup K^*_i \), contains no critical points of \( f \). Also, let \( B(\alpha, \varepsilon) \) be a round ball of (spherical) radius \( \varepsilon \) centered at \( \alpha \).
Let \( \tilde{K} \) be a critical value element of \( \mathcal{K} \). If for some \( j \geq 0 \) the map \( f^j|_{\tilde{K}} \) is injective, then we call a collection \( \mathcal{I} = (\tilde{K}, f(\tilde{K}), \ldots, f^j(\tilde{K})) \) an injective stretch (of length \( j + 1 \)); the set \( \tilde{K} \) is called the initial element of \( \mathcal{I} \).

**Lemma 3.2.** Assume that \( \mathcal{K} \) is a PL cycle of base period 1. Let \( \{\alpha\} \) be the base cycle of \( \mathcal{K} \). Suppose that injective stretches of lengths \( s_1, \ldots, s_j \) are given so that their initial elements \( \tilde{K}_i \) have points \( \Delta \)-distant from \( \alpha \) for some \( \Delta > 0 \). If \( \bar{s} = s_1 + \cdots + s_j \) is sufficiently large, then \( m^* \leq 2\pi d^*/\ln(4\bar{s}\Delta^2) \), where \( m^* \) is the modulus of any root annulus of any element of \( \mathcal{K} \).

Thus, if \( D \) is the least diameter of the initial elements of the given injective stretches, then \( m^* \leq 2\pi d^*/\ln(\bar{s}D^2) \) for sufficiently large \( \bar{s} \).

**Proof.** Fix some \( K^* \in \mathcal{K} \), and set \( m^* = \text{mod}(U \setminus K^*) \), for some \( U \) such that \( f^q : U \to V \) is a PL map with PL set \( K^* \). Set \( m = \text{mod}(U_0 \setminus K^*) = m^*/d^* \). Then any element \( K_j^* \) of \( \mathcal{K} \) has root annulus \( U_j \setminus K_j^* \) of modulus at least \( m \).

For an element \( K' \) of a given injective stretch, let \( \mathcal{A}^{root}(K') = U(K') \setminus K' \) be a root annulus of \( K' \) of modulus at least \( m \). Set \( \varepsilon = \min(\Delta, \Delta \cdot 4e^{-\pi/2m}) \).

Let \( \tilde{K}_i \) be the initial element of a given injective stretch. Then the boundary of \( B(\alpha, \varepsilon) \) intersects \( \tilde{K}_i \) while \( B(\alpha, \varepsilon) \) by Lemma 2.2, and the map \( f^{s_i-1} \) is univalent on \( B(\alpha, \varepsilon) \). Moreover, since \( \alpha \) is repelling and by the Koebe \( \frac{1}{4} \)-theorem, the boundary of \( B(\alpha, \frac{\varepsilon}{4}) \) contains points of each of the sets \( \tilde{K}_i, f(\tilde{K}_i), \ldots, f^{s_i-1}(\tilde{K}_i) \). The diameter of these \( \bar{s}_i \) sets is at least \( \frac{\varepsilon}{4} = \tilde{k} \). Apply the Main Theorem to elements of all given injective stretches (the role of \( t \) in the Main Theorem is now played by \( \bar{s} = s_1 + \cdots + s_j \), and the role of \( k \) by \( \tilde{k} \)). Suppose that \( \bar{s} \) is sufficiently large so that the Main Theorem applies and \( \bar{s}^2 \leq 4e^\pi/m \). We have \( 4e^{-\pi/m} \leq 1 \) as if \( 4e^{-\pi/m} > 1 \) then \( \varepsilon = \Delta \), and, by the Main Theorem, \( 16 < s\Delta^2 \leq 4e^\pi/m \), a contradiction with \( 4e^{-\pi/m} > 1 \). So, \( 4e^{-\pi/m} \leq 1 \) and \( \varepsilon = \Delta \cdot 4e^{-\pi/m} \). It now follows from the Main Theorem that \( 4\bar{s}\Delta^2 \leq e^{2\pi/m} \). Since \( \bar{s} \) is large, this implies that \( m^* \leq 2\pi d^*/\ln(4\bar{s}\Delta^2) \). It remains to notice that we can set \( \Delta = D/2 \).

Corollary 3.3 follows from Lemma 3.2 and is left to the reader.

**Corollary 3.3.** Assume that \( \mathcal{K} \) is a PL cycle of relative period \( s \) and base period 1. If \( D \) is the least diameter of a critical value element of \( \mathcal{K} \) then \( m^* \leq 2\pi d^*/\ln(sD^2) \), where \( m^* \) is the modulus of any root annulus of any element of \( \mathcal{K} \) and \( s \) is sufficiently large.

We are ready to prove Theorem A.

**Proof of Theorem A.** Let \( A = \{\alpha_0, \ldots, \alpha_{r-1}\} \) be the base cycle of \( \mathcal{K} \). Set \( \tilde{f} = f^r \). The set \( \mathcal{K}^+ \) consists of \( r \) components. Each component \( C \) of \( \mathcal{K}^+ \) is the union of elements of a certain PL cycle for \( \tilde{f} \), of base \( \tilde{f} \)-period 1 and relative period \( s \); moreover, there exists a unique point \( \alpha_i \in A \) in \( C \).
Assume first that \( N = 1 \). Then Theorem A immediately follows from Corollary \ref{cor:1.3}. Let \( N > 1 \). Then we claim that, for \( \tilde{s} \leq s/N \), there exists a critical value element \( \tilde{K} \) of \( \mathcal{K} \) such that \( \tilde{f}^{\tilde{s}-1} \) is injective on \( \tilde{K} \). Indeed,

\[
\tilde{s} \leq \frac{s}{N} \Rightarrow (\tilde{s}-1)r \leq \frac{sr-Nr}{N} \leq \frac{sr-N}{N} \Rightarrow (\tilde{s}-1)r \leq \frac{sr}{N} - 1;
\]
on the other hand, if \( j \) is the largest integer such that \( f^j|_\tilde{K} \) is injective for a critical value element \( \tilde{K} \) of \( \mathcal{K} \), then \( j \geq \frac{s}{N} - 1 \), and so \( (\tilde{s}-1)r \leq j \), as desired. Now the claim of Theorem A follows from Lemma \ref{lem:3.2} applied to the maximal injective stretch of \( \tilde{f} \) with initial element \( \tilde{K} \).

We can now prove Corollary \ref{cor:1.3}.

\textbf{Proof of Corollary \ref{cor:1.3}.} Let \( R(f) \) be the minimum of \( \text{dist}(z, w) \) taken over all non-attracting \( f \)-periodic points \( z \) of period at most \( r \), and points \( w \in f^{-1}(f(z)), w \neq z \). Clearly, \( R(f) > 0 \) is a continuous function of \( f \in \mathcal{N} \). Since \( \mathcal{N} \) is compact, the smallest value \( \delta_N(r) > 0 \) of \( R \) is positive; in particular, the diameter of any critical element of \( \mathcal{K} \) is greater than or equal to \( \delta_N(r) \). Since \( f \) belongs to a compact family \( \mathcal{N} \) of rational functions, then there exists a number \( \tilde{D}_N(r) > 0 \) depending only on \( \mathcal{N} \) and \( \delta(r) \) such that the diameter of the \( f \)-image of any continuum of diameter at least \( \delta_N(r) \) has diameter at least \( \tilde{D}_N(r) \). Thus, any critical value element of a PL cycle with base period at most \( r \) of \( f \in \mathcal{N} \) has diameter at least \( \tilde{D}_N(r) \). Now Theorem A yields the desired, where \( D_N(r) = \tilde{D}_N^2(r)/(2d - 2) \) (note that \( N \) from Theorem A is at most \( 2d - 2 \) for a degree \( d \) rational function).

Let us now prove the last corollary of Theorem A, namely Corollary \ref{cor:1.5}.

\textbf{Proof of Corollary \ref{cor:1.5}.} Let us first prove part (1). Assume that there is \( m > 0 \) such that, for each \( i \), the following holds:

1. there is a PL map \( f^{\tilde{p}_i} : U_{n_i} \to V_{n_i} \) of degree \( d^*_{n_i} \) with \( \text{mod}(V_{n_i} \setminus U_{n_i}) \geq m \) and a connected PL set \( K_{n_i} \subset \mathcal{K}_{n_i} \);
2. for some \( K_{n_i+1} \subset \mathcal{K}_{n_i+1} \), \( K_{n_i+1} \subset K_{n_i} \), the PL map \( f^{\tilde{p}_ir_is_i} : U_{n_{i+1}} \to V_{n_{i+1}} \) of degree \( d^*_{n_{i+1}} \) has root annulus \( U_{n_{i+1}} \setminus K_{n_{i+1}} \) of modulus \( \geq m \).

Here, \( \tilde{p}_i := p_{n_i} \) is the period of \( K_{n_i} \), and \( \tilde{q}_i := p_{n_{i+1}} = \tilde{p}_ir_is_i \) is the period of \( K_{n_{i+1}} \).

As \( d^*_{n_{i+1}} \leq 2^{2d-2} \) passing to a further subsequence one can assume that \( d^*_{n_{i+1}} = d^* \geq 2 \) for all \( i \). Moreover, replacing \( U_{n_{i+1}} \) and \( V_{n_{i+1}} \) by their \( l \)-preimages, with \( l \) large enough, under the PL map \( f^{\tilde{q}_i} : U_{n_{i+1}} \to V_{n_{i+1}} \) one can also assume that \( m \leq \text{mod}(V_{n_{i+1}} \setminus K_{n_{i+1}}) \leq M = m d^* \).

In the course of the proof below, all constants \( \varepsilon, m', L, C \) etc will depend merely on \( d^* \) and \( m \).
By Lemma 2.2, there exists \( \varepsilon = \varepsilon(m) \) that depends only of \( m \) such that 
\( \varepsilon \text{diam}(K_{n_i}) \)-neighborhood of \( K_{n_i} \) is contained in \( U_{n_i} \), and \( \varepsilon \text{diam}(K_{n_i+1}) \)-neighborhood \( W_{n_i+1} \) of \( K_{n_i+1} \) is contained in \( U_{n_i+1} \), for all \( i \). As \( K_{n_i+1} \subset K_{n_i} \), it follows that \( W_{n_i+1} \subset U_{n_i} \).

Now fix \( i \) and consider a PL map \( P : U \to V \) of degree \( d^* \) where \( P := f^i \), \( U := U_{n_i+1} \), \( V := V_{n_i+1} \) and \( m \leq \text{mod}(V \setminus K) \leq M \). Then \( V \) contains the \( \varepsilon \text{diam}(K) \)-neighborhood \( W := W_{n_i+1} \) of the filled (connected) Julia set \( K := K_{n_i+1} \) of \( P : U \to V \).

This allows us to apply [20, Proposition 4.10], according to which there is a restriction \( P : U' \to V' \) of \( P : U \to V \) which is also a polynomial-like of the same degree \( d^* \) such that \( M \geq \text{mod}(V' \setminus U') \geq m' > 0 \), domains \( U' \), \( V' \) are \( L \)-quasidisks and \( \text{diam}(V') \leq C \text{diam}(K) \). By our choice of \( \varepsilon \) and by [20, Proposition 4.10], \( m', L \) and \( C \) here depend only on \( d^* \) and \( m \). By Lemma 2.2 there exists \( \varepsilon' = \varepsilon'(m') \) that depends only of \( m' \) such that the \( \varepsilon' \text{diam}(K) \)-neighborhood of \( K \) is contained in \( U' \).

**Claim.** There is an integer \( k_0 \geq 0 \) which depends only on \( d^*, L, C, \varepsilon, M \) and \( m' \) (that is, after all, only on \( d^* \), \( m \) and \( M \)), such that \( P^{-k_0}(V') \) is contained in the \( \varepsilon' \text{diam}(K) \)-neighborhood \( W \) of \( K \).

Indeed, assume the contrary. Without loss of generality, one can also always assume that \( \text{diam}(K) = 1 \) and \( 0 \in K \). Then we find a sequence of PL maps \( P_j : U'_j \to V'_j \) of degree \( d^* \) and connected filled Julia set \( 0 \in K_j \) of diameter 1 with the properties listed in the Claim and the minimal \( k(j) \to \infty \) such that \( P_j^{-k(j)}(V'_j) \) is contained in the \( \varepsilon' \text{-neighborhood} \) \( W_j \) of \( K_j \). Passing to a subsequence if necessary one can further assume that the sequence \( P_j : (U'_j, 0) \to (V'_j, 0) \) converges in the Carathéodory topology [19] to a PL map \( P_* : (U_*, 0) \to (V_*, 0) \) of degree \( d^* \) with connected filled Julia set \( K_* \). As all \( U'_j, V'_j \) are \( L \)-quasidisks with the same \( L \), domains \( U_*, V_* \) are in fact Hausdorff limits of sequences \( (U'_j), (V'_j) \) (pieces of \( U'_j, V'_j \) cannot “pinch off” and disappear in the limit because their boundaries are \( L \)-quasicircles) and \( K_* \) is the Hausdorff limit of the sequence \( (K_j) \). It follows that the \( \varepsilon' / 2 \)-neighborhood \( W_* \) of \( K_* \) is contained in \( V_* \). As \( \cap_{k \geq 0} P_*^{-k}(U_*) = K_* \), there is some positive integer \( k_* \) such that \( P_*^{-k_*}(U_*) \) is compactly contained in \( W_* \). By continuity, then \( P_j^{-k_*}(V'_j) \) is in the \( \varepsilon' \text{-neighborhood} \) \( W_j \) of \( K_j \), for all big \( j \), a contradiction that proves the Claim.

By the Claim, if \( \bar{V} = P^{-k_0}(V') \), \( \bar{U} = P^{-k_0}(U') \), then \( P : \bar{U} \to \bar{V} \) is a PL restriction such that \( \bar{V} \) is a subset of the \( \varepsilon' \text{diam}(K) \)-neighborhood of \( K \) while \( \text{mod}(\bar{V} \setminus K) \geq m' / (d^*)^{k_0} \). Going back to the original notation, we see that \( K = K_{n_i+1} \subset K_{n_i} \), hence \( \bar{V} \) is contained in the \( \varepsilon' \text{diam}(K_{n_i}) \)-neighborhood of \( K_{n_i} \) that in turn is contained in \( U_{n_i} \) by Lemma 2.2. Thus reducing \( V_{n_i+1} \) if necessary to \( \bar{V} = P^{-k_0}(V') \) (while keeping property
(2) with \( m/(d^*)^{k_0} \) instead of \( m \), one can assume from the beginning that \( V_{n+1} \subset U_n \) (recall that \( k_0 \) is independent of \( i \)).

By the theory of polynomial-like mappings and relying upon property (1), find \( c > 0 \) such that each \( \tilde{f}_{p_i} : U_n \to V_n \) is conjugate, by a \( c \)-quasiconformal map \( h_i \), to a monic centered polynomial of degree \( d^* n_i \leq 2^{2d-2} \). Set \( D(r) = \max_{d_+} D_{N_{d_+}}(r) \) (the function from Corollary [1.3], where \( N_{d_+} \) is the space of all monic centered degree \( d_+ \)-polynomials with connected Julia sets (this space of polynomials is compact), and \( d_+ \) ranges from 2 to \( 2^{2d-2} \). As \( s_i D(r_i) \to \infty \), by applying Corollary [1.3] we obtain that \( \text{mod}(h_i(U_{n+1}) \setminus h_i(K_{n+1})) \to 0 \) as \( i \to \infty \). Therefore,

\[
\text{mod}(U_{n+1} \setminus K_{n+1}) \leq c \text{mod}(h_i(U_{n+1}) \setminus h_i(K_{n+1})) \to 0,
\]
a contradiction. Part (1) of Corollary [1.5] is proved.

Turning to part (2), observe that by Theorem A and since the degree of \( f \) is \( d \), we see that

\[
\text{mod}(A_{n+1}^{\text{root}}) \leq \frac{2\pi \cdot 2^{2d-2}}{\ln(s_i D^2/(2d - 2))}
\]

where \( D \) is a constant that depends only on \( S \). Observe that several initial terms of the sequence \( s_i \) may not be sufficiently large in the sense of Theorem A. This is easily compensated by the choice of \( C \), and it follows that choosing large \( C \) we guarantee that (1) \( \text{mod}(A_{n+1}^{\text{root}}) \leq \frac{C}{\ln s_i} \) for several initial \( i \)'s to which Theorem A does not apply, and (2) we always have

\[
\frac{2\pi \cdot 2^{2d-2}}{\ln(s_i D^2/(2d - 2))} < \frac{C}{\ln s_i}
\]

so that the desired inequality from Corollary [1.5] always hold. \( \square \)

4. THEOREM B AND RELATED COMMENTS

In this section, we prove Theorem B.

**Proof of Theorem B.** By way of contradiction, assume that \( \text{mod}(V_n \setminus K_n) \) stay away from 0 for an infinite subsequence of numbers \( n \). Pulling \( V_n \) and \( U_n \) back under \( f_n^{s_n} \), we may assume that there are also Jordan disks \( W_n \) with \( f_n^{s_n} : V_n \to W_n \) being PL-maps. Then, by Lemma [2.2] a neighborhood of \( K \) is contained in all sets \( U_n \). Passing to a subsequence, assume that there is a domain \( U \) such that for all \( n \) we have \( K_n \subset U \subset U_n \), and \( \text{mod}(U \setminus K_n) \geq m \) for some \( m > 0 \).

**Lemma 4.1.** The set \( U \) is contained in the Fatou set of \( f \).

**Proof.** Recall that the exceptional set of \( f \) is defined as the maximal finite subset \( E \subset \mathbb{P}^1 \) with the property \( f^{-1}(E) \subset E \). By [21, Lemma 4.9], the exceptional set \( E_f \) of \( f \) consists of one or two points. Moreover, if \( E_f \) is
nonempty, then $f^2$ is Möbius conjugate to a polynomial. Assume, by way of contradiction, that $U \cap J(f) \neq \emptyset$. Then, by [21] (see Corollary 14.2 and the following remark), the complement of $f^j(U)$ is a subset of an arbitrarily small neighborhood of $E_f$, for all sufficiently large $j$. Since $s_n \to \infty$, we may assume that $s_1$ is already sufficiently large, so that $f^{s_1}(U) \cup O(E_f) = \mathbb{P}^1$ for a small neighborhood $O(E_f)$ of $E_f$ (if $E_f$ is empty, then $f^{s_1}(U) = \mathbb{P}^1$). We may also assume that $s_1 < s_2 < \cdots < s_n < \ldots$.

Suppose first that $E_f = \emptyset$. Then $f^{s_1}(U) = \mathbb{P}^1$. It follows that $f^{s_2}(U) = \mathbb{P}^1$. Indeed, since $f$ is an open map, for any point $w \in \mathbb{P}^1$, there is a point $z \in \mathbb{P}^1$, a positive real number $\varepsilon$, and a Jordan neighborhood $O(z)$ of $z$ such that $w = f^{s_1}(z)$, and $f^{s_1}(O(z))$ includes the $\varepsilon$-neighborhood of $w$. Then $f^{s_1}(O(z))$ includes the $(\varepsilon/2)$-neighborhood of $w$, for large $n$. By compactness, one can arrange that $\varepsilon$ and $n$ are independent of $w$. Thus we have $f^{s_2}(U) = \mathbb{P}^1$ for large $n$. It follows that $f^{s_n}(U) = \mathbb{P}^1$, a contradiction with $f^{s_n}(U) \subset V_n$.

Suppose now that $E_f \neq \emptyset$. Then it suffices to assume that $f$ is a polynomial, possibly replacing $f$ with $f^2$ and $f_n$ with $f_n^2$. In this case $f^{s_1}(U)$ is the whole of $\mathbb{C}$ except possibly for a small neighborhood of $\infty$, and $V_n \supset f^{s_1}(U) \supset J(f_n)$ for large $n$. This is a contradiction with Lemma 2.4 applied to $K_n$ and $V_n$.

It follows that $U \subset \Omega$ where $\Omega$ is a Fatou component of $f$. The set $\Omega$ cannot be a component of the basin of an attracting cycle of $f$ as otherwise, for large $n$, the PL set $K_n \subset U$ would be in the basin of an attracting cycle of $f_n$, a contradiction. By way of contradiction assume that $\Omega$ is not in the basin of a parabolic cycle of $f$. Thus $\Omega$ is eventually mapped to a periodic rotation domain (Siegel disk or Herman ring) under $f$. It is safe to assume that $\Omega$ is itself a periodic rotation domain of period $p$.

As $\text{mod}(U \setminus K_n) \geq m$, there is $\delta > 0$ such that $\text{dist}(\partial U, K_n) > \delta$ for all $n$. Now, since $f^p : \Omega \to \Omega$ is conjugate to an irrational rotation, one can fix $s > 0$ such that, for each $z \in \overline{U}$ we have $\text{dist}(z, f^{sp}(z)) < \delta/2$. As $f_n \to f$, it follows that for every $n$ large enough, $f^{sp}(K_n) \subset U \subset U_n$ and $f^{sp}(K_n) \neq K_n$. The sets $K_n$ and $K'_n = f^{sp}(K_n)$ are in the same PL cycle. Hence the proper point of $K'_n$ that exists by Lemma 2.4 cannot lie in $U_n$. This contradiction concludes the proof.

Observe that the last part of the proof of Theorem B can be deduced from the Main Theorem. Indeed, the fact that the limit set $K$ of the sequence $\{K_n\}$ is contained in a rotation domain implies that for some $k > 0$ the number of components of the $f_n$-orbit of $K_n$ of diameter greater than $k$ goes to infinity with $n$, which contradicts the Main Theorem.

Corollary 1.7 easily follows from Theorem B because no point of an infinitely-renormalizable set can belong to a parabolic domain.
5. An example

All three conditions (1) – (3) are essential for the conclusion of the second part of Theorem B. Namely, it is clear that the conclusion breaks down without condition (2). As for condition (1), counterexamples can be found in the real unimodal family \( f_c(z) = z^d + c \), with \( c \in \mathbb{R} \), for every even positive \( d \). Indeed, in that case it is known that there is a universal complex bound for a specific choice of domains for any renormalization of \( f_c \) of period \( s \) whenever \( f_c^{2^s} \) has no attracting or neutral fixed point, see Theorem A of [11] (complemented by the following paragraph), see also [6, 17]. Thus it is enough to find corresponding examples of real \( f_c \) with periodic intervals. It follows that the conclusion breaks down without condition (1).

As for (3), the following counterexample shows that this assumption is also necessary.

**Example 5.1.** We claim that there exists a real sequence \((a_n)\) such that:

(i) the sequence \(a_n\) is decreasing, and \(a_n \searrow a\);

(ii) the map \(f_a\) has a parabolic 3-cycle of multiplier 1;

(iii) there is a symmetric \(f_{a_n}\)-periodic interval \(L_n \supseteq 0\) of period \(q_n\);

(iv) maps \(f_{a_n}^{2q_n}\) have no attracting/parabolic fixed points;

(v) periods \(q_n\) tend to infinity;

(vi) diameters of \(L_n\) stay away from 0.

Here, symmetric mean invariant under \(x \mapsto -x\). As mentioned, by applying the above universal complex bounds, the conclusion of Theorem B breaks down, for a special choice of pairs \(V_n, U_n\). On the other hand, conditions (1) – (2) still hold.

The sequence \((c_n)\) can be defined using the Lavaurs maps. The construction is motivated by [15]. Start with a map \(f_a\) satisfying (ii). Let \(F = f_a^3\) and \(x_0, x_0 < 0\), be the point of the parabolic 3-cycle of \(f\) that contains 0 in its immediate basin of attraction. Locally,

\[
F(z) = z + A(z - x_0)^2 + B(z - x_0)^3 + \ldots
\]

where \(A < 0\). Let \([c_{-2}, 0]\) be the maximal interval containing \(x_0\) on which \(F\) is increasing. Here \(c_{-2} < 0\), \(f_a^2(c_{-2}) = 0\) and \(F(c_{-2}) = c_1 := f_a(0)\) where \(c_1 < c_{-2}\). The interval \((c_1, c_{-2})\) contains a point \(c_{-1}\) such that \(f_a(c_{-1}) = 0\) so that \(f_a^2\) is decreasing on \([c_1, c_{-1}]\) and increasing on \([c_{-1}, 0]\). Consider the fundamental interval \(I_- = [F(0), 0]\) in the immediate attracting basin of \(x_0\). The interval \(I_-\) being a fundamental interval means that the sets \(F^n(I_-)\) for \(n \geq 0\) have disjoint interiors, and their union covers the interval \((x_0, 0]\) all of whose points converge to \(x_0\) under forward iterations of \(F\). Also, consider the interval \(I_+ = [c_1, c_{-2}]\). This is a fundamental interval for the backward iteration of \(F^{-1} : [c_1, x_0] \rightarrow [c_{-2}, x_0]\) which, from now on, denote the inverse branch of strictly increasing \(F : [c_{-2}, x_0] \rightarrow [c_1, x_0]\).
Now, consider attracting \( \varphi_- \) and repelling \( \varphi_+ \) Fatou coordinates of \( F \) near \( x_0 \). Let us very briefly recall necessary definitions, see e.g., \([21, 23]\) for details. Let \( \delta > 0 \) be very small and \( D_- \), \( D_+ \) be the disks centered at \( x_0 + \delta, x_0 - \delta \) respectively with radius \( \delta \). The map \( F \) sends \( D_- \) into itself, while \( D_+ \subset F(D_+) \). This defines, after identification of \( z \) with \( F(z) \) at the boundary, two cylinders \( U_- = D \setminus F(D_-) \) and \( U_+ = F(D_+) \setminus D_+ \). By the Riemann mapping theorem these two cylinders may be uniformized by “straight” cylinders. In other words, there exists \( \varphi_\pm \) mapping the cylinders \( U_\pm \) conformally to vertical strips of width 1 conjugating \( F \) to the translation by 1. That is,

\[
\varphi_-(F(z)) = T_1(\varphi_-(z))
\]

for \( z \in U_- \) where \( T_\sigma : z \mapsto z + \sigma \) denotes the translation by \( \sigma \). Correspondingly, \( \varphi_+ \) satisfied the same equation:

\[
\varphi_+(F(z)) = T_1(\varphi_+(z))
\]

for \( z \in U_+ \). By symmetry, we may assume that \( \varphi_\pm(\bar{z}) = \overline{\varphi_\pm(z)} \).

Let us concentrate at the moment to their restrictions to the real axis. Then \( \varphi_- \) extends by \((5.1)\) to an orientation reversing homeomorphism \( \varphi_- : (x_0, 0] \to [\varphi_-(0), +\infty) \) and \( \varphi_+ \) extends by \((5.2)\) to an orientation reversing homeomorphism \( \varphi_+ : [c_1, x_0) \to (-\infty, \varphi_+(c_1)] \). In particular, \( \varphi_-(I_-) = [\varphi_-(0), \varphi_-(F(0))] \) where \( \varphi_-(F(0)) = \varphi_-(0) + 1 \) and \( \varphi_+(I_+) = [\varphi_+(c_2), \varphi_+(c_1)] \) where \( \varphi_+(c_1)) = \varphi_+(c_2) + 1 \). Notice that the Fatou coordinates \( \varphi_\pm \) are unique up to post-composition by a real translation. This allows us to fix the choice of \( \varphi_\pm \) in such a way that

\[
X := \varphi_-(0) = \varphi_+(c_2)
\]

which means that \( \varphi_-(x_0, 0]) = [X, +\infty) \) while \( \varphi_+([c_1, x_0)) = (-\infty, X + 1] \). Hence, \( \varphi_-(I_-) = \varphi_+(I_+) = [X, X + 1] \) and the following map is a well-defined orientation preserving homeomorphism:

\[
g_0 := \varphi_+^{-1} \circ \varphi_- : I_- \to I_+.
\]

More generally, let

\[
g_\sigma = \varphi_+^{-1} \circ T_\sigma \circ \varphi_-
\]

be the Lavaurs map. If \( \sigma \leq 0 \) then \( T_\sigma \circ \varphi_- (I_-) = [X + \sigma, X + 1 + \sigma] \subset \varphi_+([c_1, x_0)) \), i.e., for each \( \sigma \leq 0 \), the map \( g_\sigma : I_- \to [c_1, x_0) \) is a well-defined orientation preserving homeomorphism on its image \([g_\sigma(F(0))], g_\sigma(0)] \) where \( g_\sigma(F(0)) \) is the image \( g_\sigma(I_-) \) move monotonically to the right. There is a unique \( \sigma_0 < 0 \) such that \( g_{\sigma_0}(F(0)) = c_1 \). For every \( \sigma \in [\sigma_0, 0] \), there exists a unique solution \( g_\sigma \in I_- \) of the equation
\[g_\sigma(x) = c_{-1},\] so that \(g_\sigma([q_\sigma, 0]) = [c_{-1}, g_\sigma(0)]\). Note that \(q_\sigma\) monotonically increases from \(q_0\) to \(q_\sigma_0\) as \(\sigma\) monotonically decreases from 0 to \(\sigma_0\).

Let \(G_\sigma = f_\sigma^2 \circ g_\sigma\). As \(f_\sigma^2\) increases on \([c_{-1}, 0]\), for every \(\sigma \in [\sigma_0, 0]\), \(G_\sigma : [q_\sigma, 0] \to [c_1, f_\sigma^2(g_\sigma(0))]\) is an orientation preserving homeomorphism. The map \(g_\sigma\) extends immediately by symmetry to an even map on \([F(0), -F(0)]\) (which is again denoted by \(g_\sigma\)). Therefore, we get a unimodal map \(G_\sigma\) on \([q_\sigma, -q_\sigma]\), for each \(\sigma \in [\sigma_0, 0]\). The following relations hold:

1. \(G_\sigma\) is increasing on \([q_\sigma, 0]\) and is an even function on \([q_\sigma, -q_\sigma]\), for \(\sigma_0 \leq \sigma \leq 0\),
2. \(G_\sigma(q_\sigma) = c_1 < q_\sigma < 0\) for \(\sigma_0 \leq \sigma \leq 0\),
3. \(G_\sigma(0) = 0\) and \(G_\sigma(0) = -c_{-2} > -F(0) > -q_\sigma_0 > 0\),
4. \(G_\sigma(0)\) decreases from \(G_\sigma(0) > 0\) to \(G_\sigma(0) = 0\) as \(\sigma\) increases from \(\sigma_0\) to 0.

Indeed, (1)-(2) is by the construction and \(G_\sigma(0) = f_\sigma^2(c_{-2}) = 0\). Now, \(G_\sigma(0) = f_\sigma^2(F^{-1}(c_{-1}))\) is a point of \(f_\sigma^{-1}(\{c_{-1}\})\) where \(c_{-1} < 0\) and \(f_\sigma(c_{-1}) = 0\). On the other hand, \(c_1 < c_{-1} < c_{-2} < F^{-1}(c_{-1}) < 0\) and \(f_\sigma^2\) increases on \([c_{-1}, 0]\), hence, \(0 < G_\sigma(0)\). There is just one positive point of \(f_\sigma^{-1}(\{c_{-1}\})\), which is \(-c_{-2}\). To finish with (3), it remains to note that \(c_{-2} < x_0 < F(0) < q_\sigma_0\). Finally, (4) follows from (3).

Passing to analytic properties of the Lavaurs map \(g_\sigma\) (and aiming the map \(G_\sigma\), it is easy to see that \(5.1\) allows us to extend \(\varphi_-\) from \(U_-\) to an analytic function to the basin of attraction \(\Delta\) of the parabolic 3-cycle of \(f_\sigma\) while the inverse map \(\varphi_+^{-1}\) extends by \(5.2\) from \(\varphi_+(U_+)\) to an entire function. Therefore, \(g_\sigma\) extends to an analytic function to \(\Delta\). In fact, the main purpose of introducing \(g_\sigma\) (as well as our use of it) is the following theorem due to Douady and Lavaurs \([3]\) (stated in the particular case of 3-cycle) as follows: for every \(\sigma\) there exists a sequence \(a_n \searrow a\) and an increasing sequence of positive integers \(N_n\) such that \(g_\sigma(z) = \lim_{n \to \infty} f_\sigma^{3N_n}(z)\) uniformly on compact subsets of \(\Delta\).

As \([F(0), -F(0)] \subset \Delta\) and \(f_\sigma\) has a negative Schwarzian derivative \([18]\) \(Sf_\sigma < 0\) on \(\mathbb{R}\) this theorem implies, in particular, that \(Sg_\sigma \leq 0\) on \([F(0), -F(0)]\). (This fact follows also directly from the following: \(g_\sigma^{-1}\) extends from the real interval to a univalent function of the upper half plane into itself, see \([15]\).) As \(G_\sigma = f_\sigma^2 \circ g_\sigma\) and \(Sf_\sigma^2 < 0\) on \(\mathbb{R}\), then \(SG_\sigma < 0\) on \([F(0), -F(0)]\). Now, \(G_\sigma\) has on \([F(0), -F(0)]\) precisely 3 critical points: 0 and two symmetric critical points at \(\pm q_\sigma\). Along with relations \(G_\sigma(q_\sigma) < q_\sigma < 0 < G_\sigma(0)\) for \(\sigma_0 \leq \sigma < 0\), \(G_\sigma(0) = 0\), and general properties of maps with the negative Schwarzian \([18]\) (basically, that \(G'_\sigma\) cannot have a positive local minimum on \((q_\sigma, 0)\), we get that, for each \(\sigma \in [\sigma_0, 0]\) the map \(G_\sigma\) has a unique (orientation preserving) fixed point \(\beta_\sigma \in (q_\sigma, 0)\). Moreover, it is repelling. So we have a unimodal restriction of \(G_\sigma\) to \(L_\sigma := [\beta_\sigma, -\beta_\sigma]\).
Now, as \( \beta_0 < 0 = G_0(0) < \beta_0 \) while \(-\beta_0 < G_0(0)\), by continuity, there is a Chebyshev parameter \( \sigma_{\text{CH}} \in (\sigma_0, 0) \), i.e., such that \( G_{\sigma_{\text{CH}}}(0) = \beta_{\text{CH}} \).

Let us fix any \( \sigma_* \in (\sigma_{\text{CH}}, 0) \) such that \( G_* := G_{\sigma_*} : L_* \to L_* \) where \( L_* = [\beta_{\sigma_*}, -\beta_{\sigma_*}] \) has no attracting or neutral fixed point or 2-cycle. For example, any \( \sigma_* \) close enough \( \sigma_{\text{CH}} \) would work. Notice that \( L_* \subset [F(0), -F(0)] \).

Now, by the stated earlier theorem of Douady and Lavaurs, there exists a sequence \( a_n \searrow a \) and an increasing sequence of positive integers \( N_n \) such that \( g_{\sigma_*}(z) = \lim_{n \to \infty} f_{a_n}^{3N_n}(z) \) uniformly in some complex neighborhood of \([F(0), -F(0)]\). Let \( q_n = 3N_n + 2 \). Then, for each large enough \( n \), the map \( f_{a_n} \) has a symmetric periodic interval \( L_n \ni 0 \) of period \( q_n \) and \( L_n \to L_* \) as \( n \to \infty \). Besides, \( f_{2q_n}^2 \) has no attracting or neutral fixed point on \( L_n \). It follows that the sequence \((a_n)\) is as required.

Notice that in Example 5.1 the number of components of the orbit of \( K_n \) of size at least \( k \) is \( O(k^{-1/2}) \) as \( k \to \infty \).

APPENDIX: BACKGROUND

In this section, we recall some standard facts about polynomial-like (PL) maps; they go back to [2]. We also recall the definition of the modulus of an annulus.

Let \( U \) and \( V \) be Jordan disks such that \( U \subseteq V \) (i.e., \( \overline{U} \subseteq V \)). A proper holomorphic map \( f : U \to V \) is said to be polynomial-like (PL). The filled Julia set \( K(f) \) of \( f \) is defined as the set of points in \( U \), whose forward \( f \)-orbits stay in \( U \). Similarly to polynomials, the set \( K(f) \) is connected if and only if all critical points of \( f \) are in \( K(f) \).

**Straightening Theorem.** Let \( f_1 : U_1 \to V_1 \) and \( f_2 : U_2 \to V_2 \) be two PL maps. Consider Jordan neighborhoods \( W_1 \) of \( K(f_1) \) and \( W_2 \) of \( K(f_2) \). A quasiconformal homeomorphism \( \phi : W_1 \to W_2 \) is called a hybrid equivalence between \( f_1 \) and \( f_2 \) if \( f_2 \circ \phi = \phi \circ f_1 \) whenever both parts are defined, and \( \overline{\partial} \phi = 0 \) on \( K(f_1) \). By the Straightening Theorem of [2], a PL map \( f : U \to V \) is hybrid equivalent to a polynomial of the same degree restricted on a Jordan neighborhood of its filled Julia set.

**Moduli of annuli.** Let \( A \) be a Riemann surface homeomorphic to \( \mathbb{C}/\mathbb{R} \). Then, by the Uniformization Theorem, there is a conformal isomorphism between \( A \) and a Euclidean cylinder of height \( \mu \) and circumference \( 1 \), where \( \mu \in \mathbb{R}_{>0} \cup \{+\infty\} \). In this case, \( \mu \) is called the modulus of \( A \) and is denoted by \( \text{mod}(A) \). This is a conformal invariant. It is a straightforward computation using the complex logarithm function that the modulus of the round annulus \( A = \{ z \in \mathbb{C} \mid r_1 < |z| < r_2 \} \) is given by \( \text{mod}(A) = \log(r_2/r_1)/(2\pi) \). If \( U \subset \mathbb{P}^1 \) is an open topological disk, and
$X \subset U$ is a continuum (that is, a compact connected subset), then $U \setminus X$ is homeomorphic to a round annulus, hence $\text{mod}(U \setminus X)$ is defined.

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(Alexander Blokh and Lex Oversteegen) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, AL 35294

(Genadi Levin) INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, GIVAT RAM 91904, JERUSALEM, ISRAEL. TEMPORARY ADDRESS: INSTITUTE OF MATHEMATICS OF PAN, SNIADECKICH 8, WARSAW 00-656, POLAND

(Vladlen Timorin) FACULTY OF MATHEMATICS, HSE UNIVERSITY, 6 USACHEVA STR., MOSCOW, RUSSIA, 119048

*Email address*, Alexander Blokh: ablokh@uab.edu
*Email address*, Genadi Levin: genady.levin@mail.huji.ac.il
*Email address*, Lex Oversteegen: overstee@uab.edu
*Email address*, Vladlen Timorin: vtimorin@hse.ru