Abstract. Since the seminal result of Karger, Motwani, and Sudan, algorithms for approximate 3-coloring have primarily centered around SDP-based rounding. However, it is likely that important combinatorial or algebraic insights are needed in order to break the \( n^{0.13} \) threshold. One way to develop new understanding in graph coloring is to study special subclasses of graphs. For instance, Blum studied the 3-coloring of random graphs, and Arora and Ge studied the 3-coloring of graphs with low threshold-rank.

In this work, we study graphs which arise from a tensor product, which appear to be novel instances of the 3-coloring problem. We consider graphs of the form \( H = (V, E) \) with \( V = V(K_3 \times G) \) and \( E = E(K_3 \times G) \setminus E', \) where \( E' \subseteq E(K_3 \times G) \) is any edge set such that no vertex has more than an \( \epsilon \) fraction of its edges in \( E' \). We show that one can construct \( \tilde{H} = K_3 \times \tilde{G} \) with \( V(\tilde{H}) = V(H) \) that is close to \( H \). For arbitrary \( G, \tilde{H} \) satisfies \( |E(H)|\Delta E(\tilde{H})| \leq O(\epsilon|E(H)|) \). Additionally when \( G \) is a mild expander, we provide a 3-coloring for \( H \) in polynomial time. These results partially generalize an exact tensor factorization algorithm of Imrich. On the other hand, without any assumptions on \( G \), we show that it is NP-hard to 3-color \( H \).

Key words. graph reconstruction, tensor factorization, 3-coloring, approximation algorithms

MSC codes. 05C70, 05C15, 05C85, 68Q25, 68R10, 68W25

1. Introduction. The 3-coloring problem is one of the most classical problems in theoretical computer science [36]. Although it is NP-hard, much effort has been made to understand the approximate 3-coloring problem: given a 3-colorable graph\(^1\) on \( n \) vertices as input, what is the fewest number of colors one can efficiently color the graph with? Initially, combinatorial algorithms were dominant in approximate 3-coloring, bringing us Wigderson’s famous \( O(\sqrt{n}) \)-approximation [52], as well as Blum’s \( O(n^{1/8}) \)-approximation and Blum’s 3-coloring algorithm for many random 3-colorable graphs [5]. Then, SDP-algorithms took center stage\(^2\). This began with the celebrated work of Karger, Motwani, and Sudan [35]. With a lot of extra work, clever observations were made to augment their algorithm or combine it with combinatorial algorithms and obtain better approximation results [1, 7, 38]. We give a more complete history of the 3-coloring problem in Section 1.2.

It is unclear how much further success can be obtained by combining SDP algorithms with fancier combinatorial techniques, and it is likely that completely new ideas are needed. One way to continue building insight on 3-coloring is to focus on interesting subclasses of 3-colorable graphs. Indeed, Blum did just this in targeting random 3-colorable graphs [5], and Arora and Ge generalized this result by studying low threshold rank 3-colorable graphs [2]. Additionally, the improvement from Kawarabayashi and Thorup comes from focusing on graphs with high degree [38]. Overall, we do not fully understand what properties make graphs easy or hard to color, and our work is a further exploration of this.

In this work, we propose a new class of 3-colorable graphs that are interesting for the 3-coloring problem: graphs that are close to the tensor graph \( K_3 \times G \). For undirected graphs \( F \) and \( G \), their tensor product\(^3\) \( F \times G \) is a graph on the vertex set \( V(F) \times V(G) \), where vertices \( (f, g) \) and \( (f', g') \) are incident if and only if \( (f, f') \in E(F) \) and \( (g, g') \in E(G) \). Observe that if \( G \) is connected, we can set \( F = K_3 \) and the graph \( P = K_3 \times G \) is easy to 3-color. In particular, we first locate (say by brute force) one triple of the form \( K_3 \times \{g\} \) for some \( g \in G \) in \( P \), which we call a core triple. We color the core triple with three distinct colors, and then observe the colors of the neighbors in the graph are forced. That is, for any \( g' \) in \( g \)'s neighborhood, the core triples \( K_3 \times \{g\} \) and \( K_3 \times \{g'\} \) have 6 edges between them such that if \( g \)'s copy of \( K_3 \) is colored with three distinct colors, there is only one valid coloring for \( g' \)'s copy. This coloring propagates through out \( P \). See Figure 1 for an illustration. On the other hand, suppose we delete edges from \( P \) to form a graph \( H \). If the number of deletions is large enough that a coloring is not immediately forced from fixing the colors on one core triple, then it is not obvious how to 3-color \( H \).

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\(^1\)Also known as the cardinal, direct, or Kronecker product, among other names.

\(^2\)The one exception to this trend is Kawarabayashi and Thorup’s \( O(n^{4/11}) \)-approximation [37].

\(^3\)In this paper, all graphs are undirected, simple (i.e., no double edges) and loopless (no edge from a vertex to itself).

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Another way to view this is via LP hierarchies. In particular, consider a $O(1)$-level Sherali-Adams lift on the basic 3-coloring LP. This simple coloring algorithm that was successful for 3-coloring $P$ is also successful for 3-coloring $H$ exactly when the lifted LP variables directly provide the core triples. In fact, we prove this algorithm is successful when $G$ is an expander for our deletion model in Theorem 1.2. Instead of arguing that a lifted LP/SDP rounding procedure could succeed through properties of the LP solution—we do not know whether such a proof is possible in our setting—we study its combinatorial (or topological!) analog. We believe this is a more promising avenue to complement the existing 3-coloring work, and overall lead to more progress on the problem.

However, this seemingly simple family of graphs does not seem to be properly captured by the current literature on coloring algorithms. In particular, to the best of our knowledge, there is no guarantee that previous analyses of coloring algorithms would work well on $H$. First, $H$ can look far from random, which prohibits a guaranteed success by Blum’s coloring tools [5]. Additionally, the threshold rank of $H$ is uncontrollable. If $G$ has high threshold rank, then the tensor $K_3 \times G$ has high threshold rank, and the deletion of edges to form $H$ has varied effects on the spectrum. Overall, this means we have no guarantee for a polynomial time $O(\log n)$-coloring by the algorithm from Arora and Ge [2]. To the best of our knowledge, current analyses of Gaussian cap rounding procedures would not color near tensors in constant colors. In response, we ask the following question:

**Suppose edges are deleted from $P = K_3 \times G$ to build $H$. When can one 3-color $H$ in polynomial time?**

If no edges are deleted from the tensor product, polynomial time factorization is possible. We will describe an algorithm presented by Imrich [32] in Section 1.3.1, which shows that if $F$ and $G$ are connected non-bipartite, and prime with respect to the tensor product one can reduce to a factoring problem for the Cartesian product. In particular, Imrich constructs a Cartesian product graph $F' \square G'$ (without knowing what $F'$ and $G'$ are), where $V(F') = V(F)$ and $V(G') = V(G)$. Then, a similarity metric by Winkler [53] (see also [31, 33, 20]) can be used on $F' \square G'$ to identify components, one of which will build the graph $F$ and the other $G$. Until then, one important thing to note about these procedures is that they are very brittle to any deviations from a tensor graph. In other words, these procedures cannot be run on graphs $H$ which are very close to a tensor, e.g. adding a small number of edges to $H$ would turn it into a tensor. Tensor product graphs (and graphs close to a tensor product) are in a wide range of applications, including image processing [14], network design [40], complex datasets [47], dynamic location theory [26], and chemical graph theory [13]. We note that graphs near tensor products are especially important in modeling social networks [47]. Therefore, an interesting combinatorial question is

**When can one approximately factor a graph that is close to a tensor product in polynomial time?**

We now present some models and results on these questions.

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4It is possible that there exists some reweighting of the edges of $H$ such that the reweighted graph has at most polylog($n$) threshold rank. This leaves open the possibility that a combination of combinatorial techniques together with the algorithm by Arora and Ge [2] could produce a quasi-polynomial time algorithm using $O(\log n)$ many colors.

5I.e. there is no such graphs $F_1, F_2$ on more than 1 vertex where $F = F_1 \times F_2$, and same for $G$.

6The Cartesian product of two graphs $F, G$ is a graph $F \square G$ on $V(F) \times V(G)$ such that $(u_1, v_1) \sim (u_2, v_2)$ in $F \square G$ if $(u_1, u_2) \in F$ and $v_1 = v_2$ or $u_1 = u_2$ and $(v_1, v_2) \in F$. Note that unless $F$ and $G$ contain loops, $F \times G$ and $F \square G$ have disjoint sets of edges.
1.1. Problem and theorem statements. Due to both the motivation from 3-coloring and our interest in robust tensor graph factoring algorithms, we study the following question. Let $K_3$ be the clique on three vertices, $G$ be a graph on $n$ vertices, and $P = K_3 \times G$ be their tensor product. We consider graphs of the form $H = (V(P), E(P) \setminus E')$, for $E' \subseteq E(P)$ an edge set such that no vertex $v$ in $H$ has more than an $\epsilon$ fraction of its incident edges in $E'$. We say that such an $H$ is $\epsilon$-near the triangle tensor product $P$. Our deletion model is very general, as the deleted edges could be adversarially chosen in such a way that nodes with substantially different looking neighborhoods in $P$ look the same in $H$. This model appears to be novel, as approximate graph products have mainly only been studied for the Cartesian product, not for the tensor product—we will discuss the related work more in Subsection 1.2.

Suppose we are given $H$ as above. Our primary reconstruction goal is the $\ell_1$ reconstruction goal,\(^7\) in which we aim to construct a graph $\tilde{H} = K_3 \times \tilde{G}$ with $V(\tilde{H}) = V(H)$ and

$$|E(H) \Delta E(\tilde{H})| \leq O(\epsilon|E(H)|).$$

We prove the following algorithmic results in Section 3. Our first theorem holds for any $G$.

**Theorem 1.1.** Assume $\epsilon = \Omega(\sqrt{|V(H)|}/|E(H)|)$. Let $H$ be $\epsilon$-near $K_3 \times G$. Then, there is an algorithm running in time $O(n^6)$ that constructs a tensor $\tilde{H} \cong K_3 \times \tilde{G}$ with $V(\tilde{H}) = V(H)$ achieving the $\ell_1$ reconstruction goal.

Our second theorem holds when $G$ is an expander. For any vertex $v \in V(G)$, let $\Gamma_G(v) = \{u : (u, v) \in E\}$ denote the neighbors of $v$. We will call $G$ an $\alpha$-edge-expander if for every $S \subseteq G$, the following holds

$$|E(G) \cap (S \times \tilde{S})| \geq \alpha \min \left(\sum_{v \in S} |\Gamma_G(v)|, \sum_{v \in S} |\Gamma_G(v)|\right).$$

**Theorem 1.2.** Fix $\epsilon < 1/40$. Let $H$ be $\epsilon$-near $K_3 \times G$, where graph $G$ is a $3\epsilon$-edge-expander. Then, there is an algorithm running in time $\text{poly}(n)$ that extracts a valid 3-coloring of $H$.

As we shall see, this theorem is possible as the expansion of $G$ allows for the partial three-coloring found in Theorem 1.1 to be converted into a globally consistent three-coloring. We can replace the condition in Theorem 1.2 that $G$ is a $3\epsilon$-edge-expander with a small-set expansion condition on $G$. See Section 3.6 for more details.

We complement our algorithmic results by showing that 3-coloring $H$ is hard for general $G$.

**Theorem 1.3.** Given as input a graph $H$ which is $\epsilon$-near $K_3 \times G$ for some $G$, it is NP-hard to find a 3-coloring of $\tilde{H}$.

Our algorithms are combinatorial, although they seem to draw on some topological properties of graph tensors (see the technical overview).

1.2. Related work. As our work bridges the rich theory of graph factoring algorithms with the world of approximate graph coloring, there are a number of prior works which relate to our investigation.

**Approximate coloring algorithms.** We refer the reader to the introduction in [38] for a detailed recap of 3-coloring progress over the past several decades. A simple algorithm by Wigderson [52] uses $O(\sqrt{n})$ colors and has remained an important subroutine in coloring graphs with high degree [35]. The barrier of $O(\sqrt{n})$ colors was broken by Berger and Rompel [4]. Blum introduced some intricate combinatorial techniques which inspired much future coloring work, in particular those of Kawarabayashi and Thorup [37, 38]. Since the seminal result of Karger, Motwani, and Sudan [35], algorithmic results in approximate graph coloring have focused on SDP based algorithms [1, 7, 2, 38]. Several integrality gap instances for the original 3-coloring SDP formulation presented by Karger, Motwani, and Sudan [35] have been found by Karger, Motwani, and Sudan [35]; Frankl et. al [22, 2]; and Feige et. al [18]. These graphs have valid solutions to the SDP, but have chromatic number at least $n^c$, for small constant $c > .01$ and $n$ the size of the vertex set. On the other hand, if a 3-colorable graph has threshold rank $D$, i.e. once its eigenvalues are scaled to be in $[-1, 1]$, at most $D$ of them are less than $-1/16$, then one can color the graph in $O(\log n)$ colors in time $n^{O(D)}$ [2].

\(^7\) As the name suggests, there are other natural reconstruction goals, see Section 5 for more details.
Hardness of approximate graph coloring. The hardness results for approximate graph coloring are quite far from the algorithmic results. For a 3-colorable graph it is known that it is NP-hard to color with 5 colors \([3]\), beating a long-standing record of 4 colors \([39, 24, 6]\). For \(k\)-colorable graphs, it is NP-hard to color with \(\binom{k}{k/2}\) colors \([54]\). Under a variety of conditional assumptions, it is hard to color a 3-colorable graph with superconstant colors (up to roughly \(\text{polylog} \, n\)), see \([11, 12, 54, 25]\).

Graph factoring algorithms. Given a graph product, one can efficiently factor a graph with respect to the product into prime graphs, i.e. graphs that cannot be further non-trivially factored according to the product. For the Cartesian product, any finite, simple, connected graph can be factored in polynomial time \([20, 53, 16, 33]\). Moreover, Imrich and Peterin \([33]\) present an algorithm that performs this factorization in time linear in the number of edges. Similarly for the strong product, whose graph unions the edges of the Cartesian product and the tensor product graphs, a polynomial time factorization was found by Feigenbaum and Schäffer \([21]\) for finite, simple, connected graphs. To factor a graph with respect to the tensor product, one can combine algorithms of Imrich and Winkler \([32, 53]\); we detail these algorithms—and how they can be combined to decompose a tensor product—at the end of Section 1.3.1. For both the Cartesian product and the strong product, prime factorizations (i.e., a decomposition into irreducible factors) are unique for finite, connected, simple graphs \([26]\). In the case of the tensor product, we are required to make the additional assumption that the graph is non-bipartite, as without it, a unique factorization with respect to the tensor product does not exist. With this additional assumption, the unique factorization can also be found in polynomial time \([32]\).

Approximate graph products. The theory of approximate graph products has also been studied before, but in different settings and with different goals from us. For a graph \(H\) that is close to a product graph \(H'\), i.e., it takes a small number of edge deletions or insertions to transform \(H\) to the product graph \(H'\), previous works study how to find \(H'\) \([57, 58, 28, 29, 30]\). Feigenbaum and Haddad \([19]\) showed that for the Cartesian product, obtaining such an \(H'\) with the fewest possible edge insertions or the fewest possible edge deletions is NP-hard. Overall, the Cartesian product is the most well studied graph product, and both approximate Cartesian product and strong product graphs have connections to theoretical biology, as they model evolutionary relationships of observable characteristics \([28, 29]\). To the best of our knowledge, prior to our work, approximate graph products have only been studied for the Cartesian product and the strong product, and not with respect to the tensor product. The closest problem is the Nearest Kronecker Product problem, which given \(A \in \mathbb{R}^{m \times n}\) seeks to find \(B \in \mathbb{R}^{m_1 \times n_1}\) and \(C \in \mathbb{R}^{n_2 \times n_2}\) such that \(\|A - B \times C\|_F^2\) is minimized, for \(B \times C\) the Kronecker product of \(B\) and \(C\) and \(m = m_1 \cdot n_2\), \(n = n_1 \cdot n_2\) \([50, 49]\). Note that the relation between this problem and the approximate tensor product problem lies in the fact that the adjacency matrix of the tensor product of two graphs is the Kronecker product of the underlying adjacency matrices. Another similar problem is the closest separable state problem in Quantum systems, which approximates the entanglement of a system by measuring is how far it is from a composite of separable states \([43, 51]\). More on product graphs, their factorizations, and approximate graph products can be found in the book by Hammack, Imrich, and Klavžar \([26]\).

Learning theory. A related line of work to ours is tensor decomposition in the learning theory community. Tensors (not necessarily tensor graphs, just tensors) represent higher order information from data. A common goal is uncover the latent (hidden) variables underlying some data in order to understand it in a lower dimensional form \([44]\). One popular way to achieve this is the CP (CANDECOMP/ PARAFAC) decomposition, which writes a tensor as a sum of rank 1 tensors \([46, 34]\). Other related decompositions are PCA, Tensor Robust PCA, and the Tucker decomposition \([46, 34, 42]\). A similar research topic is reconstructing a partially observed tensor (e.g., \([55]\)).

1.3. Technical overview. We now present overviews for the proofs of Theorems 1.1, 1.2, and 1.3. The full proofs for Theorems 1.1 and 1.2 are in Section 3, and the full proof for Theorem 1.3 is in Section 4.

1.3.1. Overview of reconstructing a tensor graph. To give intuition for our results, we first summarize Imrich’s algorithm for efficiently factoring a tensor \(P = F \times G\). The first goal is to find a graph \(S\) on the same vertex set as \(P\) that is isomorphic to a Cartesian product \(S = F' \square G'\). The procedure Imrich uses to construct \(S\) was inspired by an algorithm of Feigenbaum and Schäffer \([21]\) for the strong product.

The key to constructing \(S\) is the following observation on intersections of neighborhoods in tensor graphs. For any vertices \(u, v\) in \(P\), let \(I_P(u, v) = \Gamma_P(u) \cap \Gamma_P(v)\). Since \(P\) is a tensor graph, one can show that \(I_P(u, v) = I_F(u_f, v_f) \times I_G(u_g, v_g)\), where \(u_{f}\) is the projection of \(u\) onto \(V(F)\), etc. In particular, if \(I_P(u, v)\) is
maximal (as a set) among \( v \neq u \), then it must be that either \( u_f = v_f \) or \( u_g = v_g \). In particular, adding all such maximal edges \((u, v)\) to \( S \) will keep \( S \) consistent with a Cartesian product \( F' \square G' \), where \( F' \) is on the same vertices as \( F \) and \( G' \) is on the same vertices as \( G \). If \( S \) is not a connected graph, we repeat this procedure, where the maximal \( I_P(u, v)'s \) are found for \( v's \) which are not in the same connected component as \( u \).

Once \( S \) is constructed, factoring the Cartesian product can be done with a variety of algorithms, including the one of Winkler [53]. Winkler’s algorithm is rather elegant. Let \( d_S(u, v) \), the length of the shortest path between \( u \) and \( v \) in \( S \). Define \((u_1, v_1), (u_2, v_2) \in S \) to be similar if \( d_S(u_1, v_2) + d_S(v_1, u_2) = d_S(u_1, v_1) + d_S(v_2, u_2) \). While this similarity relation may not be transitive, we can perform a depth first search to find all components that are transitively similar. The components found from Winkler’s algorithm either correspond precisely to the Cartesian product factors \( F' \) and \( G' \) of \( S \), with \( V(F') = V(F) \) and \( V(G') = V(G) \), or \( S \) (and thus \( P \)) can be decomposed into more than two factors. From this factorization, we can extract \( F \) and \( G \) from \( P \) with the following projection trick. Given \( u_f, v_f \in V(F') \) and \( u_g, v_g \in V(G') \) with \( ((u_f, v_g), (v_f, v_g)) \in E(P) \), we add \((u_f, v_f)\) to \( E(F) \) and \((u_g, v_g)\) to \( E(G) \). See Figure 2 for an illustration. This completes the factoring algorithm.

**1.3.2. Reconstruction Algorithm.** To approximately factor a graph \( H \), we need to approximate for each vertex of \( v \) the \( t \in K_3 \) and \( g \in G \) corresponding to \( v \), which we call the “color class” and “\( G \) class” of \( v \) respectively. Since \( G \) is unknown, we first try to group the vertices of \( H \) into triangles, which estimate \( K_3 \times \{ g \} \) for some \( g \in G \), then use the edges between these triangles to estimate the edges of \( G \).

**Candidate edge graph.** For Imrich, it sufficed to connect pairs of nodes in the surrogate Cartesian product graph whose neighborhoods’ intersection in \( P \) satisfied some maximality criteria. To make this maximality criteria more robust, we define what is known as the candidate edge graph \( C \) on \( V(H) \). Informally, two vertices \( u, v \in V(H) \) form an edge of \( C \) if the intersection of their neighborhoods has size approximately half their degrees (see section 2.2). Note that the edges of \( C \) and the edges of \( H \) are qualitatively quite different (they can even be disjoint). However, a key property of \( C \) is that for every vertex \( g \in G \), the three vertices of \( H \) corresponding to \( K_3 \times \{ g \} \) form a triangle in \( C \). We call this a core triangle.

We show the triangles of \( C \), which we call \( T(C) \), have a very particular form. Such a triple is one of two types: it is either (1) “close” to some core triangle (which we call quasi-core) or (2) contains vertices whose color classes are all the same and whose \( G \) classes have very structured pairwise intersection (which we call monochrome, see Lemma 3.4). Because the color classes of \( H \) are hidden information, we cannot directly determine if any triangle of \( C \) is quasi-core or monochrome.

**Triangle components.** Instead, we separate these two types of triangles topologically. We say that two triangles of \( T(C) \) are compatible if the subgraph of \( H \) on the six vertices of these triangles is isomorphic to \( K_3 \times K_2 \). In other words, it is consistent that these two triangles correspond to \( K_3 \times \{ g \} \) and \( K_3 \times \{ g' \} \) for some \( (g, g') \in E(H) \). This compatible relation divides \( T(C) \) into connected components; specifically, one can build a graph with vertex set \( T(C) \), and where two triangles in \( T(C) \) are connected exactly when they are compatible. Perhaps the most crucial (although easy to prove) technical lemma of this paper is that each component of \( T(C) \) consists only of quasi-core triangles (which we call a core component) or only of monochrome triangles (see Lemma 3.5).

Fig. 2. An illustration of Imrich’s algorithm. The algorithm is given an unknown connected, nonbipartite tensor \( P \). Imrich’s algorithm builds \( S \cong F' \square G' \), with \( V(G) = V(G') \) and \( V(F) = V(F') \). Then, the algorithm factors \( F' \) and \( G' \), and projects \( P \) onto \( F' \) and \( G' \) to find \( F \) and \( G \).

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There are other considerations which Imrich carefully handles, such as if there is a third vertex \( w \) with \( I_P(u, v) = I_P(u, w) \) and \( \Gamma_P(w) \subseteq \Gamma_P(v) \).
Coloring algorithm. Assume we pick an arbitrary component $Y_j$ of $T(C)$. Let $U_j$ be the vertices of $H$ covered by $Y_j$. By guessing the colors of one of the triangles of $Y_j$ and then performing a depth-first search, we can efficiently color all the vertices of $H[U_j]$ (i.e., the subgraph of $H$ induced by $U_j$); or, if it fails, we can deduce that $Y_j$ is not a core component (see Proposition 3.9). If $G$ is a sufficiently good expander, then we can show that $H$ has enough edges to force there to be only a single core component (although there can be a large number of monochrome components). Thus, by looping over all possible $Y_j$’s (of which there are clearly at most $O(n^3)$), our coloring algorithm will succeed on one of them, proving Theorem 1.2.

Matching algorithm to build triangles. By finding the valid 3-coloring of $H[U_j]$, we can augment this by finding an approximate tensor factorization of $H[U_j]$. This is the heart of Lemma 3.10. The key idea is we take the three color classes of $H[U_j]$, which we call $A, B, C$, and perform a tripartite matching algorithm on them. More formally, one can build a weighted tripartite graph on $(A \cup B \cup C, E)$, where the bipartite subgraphs between $A$ and $B$, as well as $B$ and $C$, are complete, with each edge having weight corresponding to their pairwise intersection, normalized by the degrees of the vertices. By finding a max-min matching, that is a matching which maximizes the weight of the minimum weight edge (which can be done in polynomial time), on this tripartite graph, we find a chosen set of triples on the vertices in $Y_j$. In particular, assuming $Y_j$ is a core component, each of the triples found will be approximately quasi-core triangles. We can then build a tensor graph corresponding to $H[U_j]$ by having each triple found by the matching correspond to a vertex of the reconstructed graph $G$, and have each edge of $G$ correspond to any pair of triples sharing at least one edge in $H$. Analyzing this factorization accurately requires showing that the every error in the reconstruction can be “charged” to a mismatch in the neighborhoods of the matched triples. The max-min guarantee implies that the number of such mismatches is of $O(\epsilon)$ edge density, allowing us to achieve the $\ell_1$ reconstruction goal for this subgraph.

Finishing the factorization. To factor the whole graph, we recursively apply Lemma 3.10. In particular, we loop through the triangle components of $T(C)$. If the lemma successfully factors the subgraph of $H$ induced by that component and the cut between that subgraph and the rest of $H$ has sufficiently few edges, we recurse on the remainder of $H$. Our tensor graph is then the disjoint union of the tensors produced for each subgraph. By combining the reconstruction guarantees for each connected component, we have the factorization achieves the $\ell_1$ reconstruction goal, proving Theorem 1.1.

Note that we cannot easily achieve a 3-coloring through such a recursive algorithm, as even though each component is correctly 3-colored, the sparse edges between the components make finding a globally consistent coloring intractable. We formalize this hardness in Theorem 1.3.

1.3.3. Hardness. The proof of Theorem 1.3 is ultimately a reduction from 3-coloring, albeit in a round-about manner. Given an instance $G$ of the 3-coloring problem, we replace each vertex of $u \in G$ with 27 copies $(u, x)$ for $x \in [3]^3$ corresponding to a copy of $K_3 \times K_3 \times K_3$. By a result of Greenwell and Lovasz [23], there are three types of 3-colorings of $K_3 \times K_3 \times K_3$, arising from the three different triangles in the tensor product. For each edge $\{u, v\} \in E(G)$ in the base graph, we add edges between their copies such that any valid 3-colorings of these copies must have different types. One can show that $G$ is 3-colorable if and only if the graph produced by the gadget reduction is. Many similar gadget reductions have been performed in the hardness of approximation literature, such as in [11].

One can show that if $G$ is 3-colorable, then the graph resulting from this reduction is a subset of $K_3 \times G'$ from some graph $G'$. However, the reduction may not be $\epsilon$-near to this tensor. To circumvent this, instead of reducing from 3-coloring directly, we reduce from “3-coloring with equality”\(^{10}\), where each vertex now has many copies which are forced to be equal by equality constraints. There are so many copies that less than an $\epsilon$ fraction of the edges from each vertex correspond to 3-coloring constraints. If we repeat the aforementioned reduction, we then obtain a graph which is $\epsilon$-near to a tensor of the form $K_3 \times G'$.

One can show that if $G$ is 3-colorable, then this graph is $\epsilon$-near a $K_3$ tensor. However, if $G$ is not 3-colorable, then the resulting graph isn’t even 3-colorable. This is enough to establish Theorem 1.3, for if we had a polynomial-time algorithm for 3-coloring these graphs $\epsilon$-near a $K_3$ tensor, then we could solve the general 3-coloring problem.

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\(^{9}\)Also known as a bottleneck matching.

\(^{10}\)This name was coined by a reviewer.
1.4. Paper outline. In Section 2, we define the necessary terms and concepts needed to prove our algorithmic results. Then, in Section 3 we prove the main algorithmic results: Theorem 1.1 and Theorem 1.2. Section 4 contains the proof of our main hardness theorem: Theorem 1.3. We conclude in Section 5 with directions for future work.

2. Algorithm Preliminaries. In this section, we give the key definitions and other concepts needed to prove the algorithmic results.

2.1. Important definitions and propositions. We let \( P = K_3 \times G \) denote the tensor product. As in the problem statement, the graph \( H \) is \( \epsilon \)-near a triangle tensor, as the following definition details.

**Definition 2.1.** A graph \( H \) is \( \epsilon \)-near a triangle tensor if there exists some product \( P \) with \( H \cong (V(P), E(P) \setminus E') \), for \( E' \) a set of edges where no vertex in \( P \) has more than an \( \epsilon \) fraction of its incident edges in \( E' \).

Let \( X \) be an arbitrary graph. We denote the neighborhood of a vertex \( v \) of \( X \) by \( \Gamma_X(v) \), where we specify the graph in which we take the neighborhood in the subscript. Additionally, for every subset \( S \subseteq V(X) \), define

\[
\text{vol}_X(S) = \sum_{v \in S} |\Gamma_X(v)|.
\]

Recall that Theorem 1.2 considers graphs whose tensor component \( G \) is an \( \alpha \)-edge-expander. That is, for every \( S \subseteq V(G) \), the following holds\(^\text{11}\)

\[
|E(G) \cap (S \times \bar{S})| \geq \min \left( \text{vol}_G(S), \text{vol}_G(\bar{S}) \right).
\]

Additionally, in our arguments, we need to argue about vertices of \( H \) and \( G \) with similar degrees. We choose the following definition

**Definition 2.2.** For any graph \( X \), vertices \( u, v \in X \) with \( |\Gamma_X(u)| \geq |\Gamma_X(v)| \) have \( \epsilon \)-similar degree when \( |\Gamma_X(u)| - |\Gamma_X(v)| \leq 2\epsilon|\Gamma_X(u)| \).

Frequently, we refer to the intersection of two vertices. Let the set of vertices in the intersection of \( u \) and \( v \) in graph \( X \) be denoted by \( I_X(u, v) = \Gamma_X(u) \cap \Gamma_X(v) \), and similarly let the intersection of \( u, v, w \) in graph \( X \) be \( I_X(u, v, w) = \Gamma_X(u) \cap \Gamma_X(v) \cap \Gamma_X(w) \).

We denote the vertices of \( K_3 \) as \( \{a, b, c\} \), and often refer to these as *colors*. To refer to an arbitrary color in \( K_3 \), we use the variable \( t \in \{a, b, c\} \). For every vertex \( v \) in \( H \), there is a hidden pair of labels associated with \( v \) from \( P \). Formally, \( v \) corresponds to a node from the tensor \( P \), whose vertices are tuples \( (t, g) \in V(K_3) \times V(G) \). We call \( t \) the unknown color class of \( v \) and \( g \) the unknown \( G \)-class of \( v \). We use the same notation as in our informal problem statement, and say that the function \( \psi_{K_3} : V(H) \to V(K_3) = \{a, b, c\} \) extracts the unknown color class of \( v \), and the function \( \psi_G : V(H) \to V(G) \) extracts the unknown \( G \)-class of \( v \). Then, we can define the complete hidden label function for \( v \in V(H) \) as \( \psi(v) = (\psi_{K_3}(v), \psi_G(v)) \).

**Definition 2.3.** For \( v \in H \), the functions \( \psi_{K_3} : V(H) \to V(K_3) \) and \( \psi_G : V(H) \to V(G) \) map \( v \) to its unknown, underlying color class and \( G \)-class of \( v \), respectively. Together, they give the hidden label \( \psi(v) = (\psi_{K_3}(v), \psi_G(v)) \).

We refer to the hidden label function’s inverse, \( \psi^{-1} \), when discussing the unknown location of \( (t, g) \in K_3 \times G = P \) in the graph \( H \). Instead of writing out \( \{\psi(v_1), \psi(v_2), \ldots\} \) for a set \( S = \{v_1, v_2, \ldots\} \), we use the shorthand \( \psi(S) \), with similar notation for \( \psi^{-1} \). Note that \( \psi \) is a bijection.

Next, we define a set of triples that will be important for our reconstruction algorithm.

**Definition 2.4.** In the graph \( H \), the core triangles\(^\text{12}\) are the triples of the form

\[
\psi^{-1}(K_3 \times \{g\}) \subseteq \{\psi^{-1}(a, g), \psi^{-1}(b, g), \psi^{-1}(c, g)\}.
\]

One might wish to find the core triangles of all vertices in \( G \) in order to prove our reconstruction goal, but such a task is impossible. In particular, nodes in \( G \) may have such similar neighborhoods that it is impossible to distinguish vertices in \( H \) with their \( G \)-classes. This motivates our next definition.

\(^\text{11}\)See [8] for the spectral implications of this definition of an expander.

\(^\text{12}\)Note that the use of the word “core” is unrelated to cores as defined by Hell and Nešetřil [27]. Also note the core triangles are not copies of \( K_3 \) in \( H \).
DEFINITION 2.5. Vertices $g, g' \in G$ are $\epsilon$-confusable when $|I_G(g, g')| > (1 - 9\epsilon) \max \{|\Gamma_G(g)|, |\Gamma_G(g')|\}$.

Since the $\epsilon$-confusable vertices of $G$ make finding the core triangles impossible, we are content to reconstruct quasi-core triangles.

DEFINITION 2.6. A set of 3 vertices $\{v_1, v_2, v_3\} \subseteq H$ is a quasi-core triangle if the color classes of all three vertices are different and the $G$ classes of all three vertices are all $\epsilon$-confusable with each other, i.e. for $\psi_G(v_i) = g_i$, for $i \in [3]$, $g_1, g_2, g_3$ are all $\epsilon$-confusable with each other.

Note that core triangles are quasi-core. Finding a set of disjoint quasi-core triangles that cover all nodes in $V(H)$ is challenging because of triangles that are not quasi-core, but may look similar to quasi-core triangles (see Proposition 3.2). We call such triangles monochrome–as their nodes have the same color class–in order to distinguish them from quasi-core triangles.

DEFINITION 2.7. A set of 3 vertices $\{v_1, v_2, v_3\} = \psi^{-1}(t \times \{g_1, g_2, g_3\}) \subseteq H$ with $|\Gamma_H(v_1)| \geq |\Gamma_H(v_2)|$, $|\Gamma_H(v_3)|$ is a monochrome triangle if

\[(1 - 8\epsilon) \cdot \frac{|\Gamma_H(v_1)|}{4} \leq |I_G(g_i, g_j)| \leq (1 + 9\epsilon) \cdot \frac{|\Gamma_H(v_1)|}{4},\]

for all $i \neq j$.

In fact, the only triples of vertices from $H$ that will be relevant to us are the quasi-core (which includes core) and monochrome triangles (see Lemma 3.4).

The following two propositions will be used very frequently in proving Theorem 1.1. The first, Proposition 2.8, implies that $|I_H(u, v)|$ is very close to $|I_{K_3}(t_1, t_2)| \cdot |I_G(g_1, g_2)|$, and it follows from the definition of the tensor structure, plus the constraint that no node has more than an $\epsilon$ fraction of its edges removed to form $H$ from $P$. Similarly, the second, Proposition 2.9, says that for $u \in H$ with $\psi_G(u) = g$, $|\Gamma_G(g)|$ is very close to $|\Gamma_H(u)|/2$.

**Proposition 2.8.** For $u, v \in H$ with $|\Gamma_H(u)| \geq |\Gamma_H(v)|$, let $\psi(u) = (t_1, g_1)$ and $\psi(v) = (t_2, g_2)$. Then for $\epsilon \leq 1/3$,

\[|I_{K_3}(t_1, t_2)| \cdot |I_G(g_1, g_2)| - 3\epsilon |\Gamma_H(u)| \leq |I_H(u, v)| \leq |I_{K_3}(t_1, t_2)| \cdot |I_G(g_1, g_2)|.\]

**Proof.** From the definition of the tensor structure, we have that

\[|I_{K_3}(t_1, t_2)| \cdot |I_G(g_1, g_2)| = |I_P(u, v)|.\]

Further, deleting edges can never increase the size of an intersection in the graph, so $|I_H(u, v)| \leq |I_P(u, v)|$. The upper bound follows by combining the two equations. Since no node has more than an $\epsilon$ fraction of the edges that are deleted from $P$ to form $H$, we see that

\[|I_H(u, v)| \geq |I_P(u, v)| - \epsilon (|\Gamma_P(u)| + |\Gamma_P(v)|) \geq |I_P(u, v)| - \frac{\epsilon}{1 - \epsilon} (|\Gamma_H(u)| + |\Gamma_H(v)|)\]

The lower bound follows since $|\Gamma_H(u)| \geq |\Gamma_H(v)|$, and $2 \cdot \frac{\epsilon}{1 - \epsilon} \leq 3\epsilon$ for $\epsilon \leq \frac{1}{3}$. □

**Proposition 2.9.** For any vertex $u$ in $H$ with $\psi_G(u) = g$,

\[|\Gamma_H(u)| \leq |\Gamma_G(g)| \leq \frac{1}{1 - \epsilon} \cdot \frac{|\Gamma_H(u)|}{2}.\]

**Proof.** The lower bound on $|\Gamma_G(g)|$ follows from the observation that $|\Gamma_H(u)| \leq |\Gamma_P(u)| = 2|\Gamma_G(g)|$. On the other hand, since the number of deletions from each vertex is bounded,

\[|\Gamma_H(u)| \geq (1 - \epsilon)|\Gamma_P(u)| = 2(1 - \epsilon)|\Gamma_G(g)|.\]

□

2.2. **Important graphs.** We define several graphs here that will be used throughout the proof section.
The candidate edge graph, $C$. First, we construct a graph on $V(H)$, whose edges contain those from the core triangles (see Proposition 3.1). Since our eventual goal is to find quasi-core triangles and the edges in our graph are candidates for edges in the core triangles, we will call this graph the candidate edge graph, $C$. We define the graph $C$ on $V(H)$ such that distinct $u, v \in V(H)$ with $|\Gamma_H(u)| \geq |\Gamma_H(v)|$ form an edge $(u, v) \in E(C)$ if and only if

(i) $u$ and $v$ have $\epsilon$-similar degree in $H$, and

(ii)

\[
(1 - 6\epsilon) \cdot \frac{|\Gamma_H(u)|}{2} \leq |I_H(u, v)| \leq \frac{1}{1 - \epsilon} \cdot \frac{|\Gamma_H(u)|}{2}.
\]

In what follows, $C_i$ will denote a component of $C$.

The triangle graph, $T(C)$. Next, we construct a graph $T(C)$, whose vertices are all sets of triples from $V(C)$ such that for $v_1, v_2, v_3 \in V(C)$, $(v_i, v_j) \in E(C)$ for all $i \neq j$ with $i, j \in \{1, 2, 3\}$. See Figure 3 for an illustration. Note that by definition, since there are no self-loops in $C$, $v_1, v_2, v_3$ are all distinct. We will refer to the vertices of $T(C)$ as triangles. An important property of $T(C)$ is that it includes all the core triangles (see Proposition 3.1). We say the triangles $T_1, T_2 \in T(C)$ are adjacent if and only if they are compatible in the following sense.

**Definition 2.10.** Triangles $T_1, T_2 \in T(C)$ are compatible exactly when

(i) $T_1$ and $T_2$ are on disjoint sets of vertices of $C$, and

(ii) there is an indexing $(u_1, u_2, u_3)$ of $T_1$ and $(v_1, v_2, v_3)$ of $T_2$ such that $(u_i, v_j) \in E(H)$ if and only if $i \neq j$.

The compatibility condition is defined so that triangles $T_1$ and $T_2$ corresponding to core triangles of $g$ and $g'$ in $T$, i.e. $T_1 = \psi^{-1}(K_3 \times \{g\})$ and $T_2 = \psi^{-1}(K_3 \times \{g'\})$, are adjacent in $T$ if $(g, g') \in E(G)$ and none of the edges between $T_1$ and $T_2$ are deleted in building $H$ from $P$.

We say that two triangles of $T(C)$ are connected if there is a path of compatible triangles from one to the other. Let $Y_j$ denote a connected component of $T(C)$. Note that a component $Y_j$ of $T(C)$ might intersect many components of $C$, because the triangles of $Y_j$ are connected via edges of $H$, which may cross different components of $C$. We will show in Lemma 3.7 that for any $Y_j$ and any $C_i, Y_j$ either contains every vertex in $C_i$ or none of them.

3. Algorithmic Results. In this section, we provide formal algorithm statements and proofs for our algorithmic results. Through out, we take $\epsilon_0 = 1/40$.

3.1. Properties of $C$, the Graph of Candidate Edges. We use the graph $C$ to identify the core triangles, or triangles that are close to the core triangles. The following proposition shows that the edges within core triangles will be kept in $C$.

**Proposition 3.1.** Fix $\epsilon < \epsilon_0$. For all $g \in G$, all pairs of distinct vertices in $\psi^{-1}(K_3 \times \{g\})$ contain an edge in $C$.

**Proof.** Fix distinct $u, v \in C$ with $\psi(u) = (a, g)$ and $\psi(v) = (b, g)$. By rewriting the equation in Proposition 2.9, both $u$ and $v$ have neighborhood sizes satisfying

\[
(1 - \epsilon) \cdot 2|\Gamma_G(g)| \leq |\Gamma_H(u)|, |\Gamma_H(v)| \leq 2|\Gamma_G(g)|.
\]

Therefore, $u$ and $v$ have $\epsilon$-similar degree in $H$.
Now, we show \( |I_H(u, v)| \) falls in the specified range for \((u, v)\) to be an edge in \( C \). Using Proposition 2.8, set \( t_1 = a, t_2 = b, \) and \( g_1 = g_2 = g \) to see that
\[
|I_{K_3}(a, b)| \cdot |I_G(g, g)| - 3\epsilon |\Gamma_H(u)| \leq |I_H(u, v)| \leq |I_{K_3}(a, b)| \cdot |I_G(g, g)|
\]
Since \( |I_G(g, g)| = |I_G(g)| \) and \( |I_{K_3}(a, b)| = 1 \), the above simplifies to
\[
|\Gamma_G(g)| - 3\epsilon |\Gamma_H(u)| \leq |I_H(u, v)| \leq |\Gamma_G(g)|.
\]
We can further lower bound \( |I_H(u, v)| \) by using the fact that \( |\Gamma_H(u)| \leq 2|\Gamma_G(g)| \) from Proposition 2.9, so
\[
(1 - 6\epsilon) \cdot |\Gamma_H(u)| / 2 \leq |I_H(u, v)|.
\] (3.1)
On the other hand, \( |\Gamma_H(u)| \geq 2(1 - \epsilon)|\Gamma_G(g)| \) from the lower bound in Proposition 2.9. Using this as an upper bound on \( |\Gamma_G(g)| \), we see that \( |I_H(u, v)| \leq |\Gamma_H(u)|/(2(1 - \epsilon)) \). Comparing with Equation 2.2, it follows that \((u, v)\) is an edge in \( C \).

Next, Proposition 3.2 together with Proposition 2.8 proves that edges in \( C \) either come from (1) vertices whose unknown color classes are the same and whose \( G \) classes have intersection size roughly half that of their neighborhoods in \( G \) or (2) vertices whose unknown color classes are different and whose \( G \) classes have almost identical intersection. The edges from core triangles are of type (2).

**Proposition 3.2.** Fix \( \epsilon < \epsilon_0 \). For \((u, v) \in C\), let \( \psi(u) = (t_1, g_1) \) and \( \psi(v) = (t_2, g_2) \), where \( |\Gamma_H(u)| \geq |\Gamma_H(v)| \).

Either (1) \( t_1 = t_2 \) and
\[
(1 - 6\epsilon) \cdot \frac{|\Gamma_H(u)|}{4} \leq |I_G(g_1, g_2)| \leq (1 + 8\epsilon) \cdot \frac{|\Gamma_H(u)|}{4}.
\]
or (2) \( t_1 \neq t_2 \) and
\[
(1 - 6\epsilon) \cdot \frac{|\Gamma_H(u)|}{2} \leq |I_G(g_1, g_2)| \leq (1 + 8\epsilon) \cdot \frac{|\Gamma_H(u)|}{2}.
\]

**Proof.** If \( t_1 = t_2 \), then by Proposition 2.8
\[
2 \cdot |I_G(g_1, g_2)| - 3\epsilon |\Gamma_H(u)| \leq |I_H(u, v)| \leq 2 \cdot |I_G(g_1, g_2)|,
\]
and recall from Equation 2.2 that
\[
(1 - 6\epsilon) \cdot \frac{|\Gamma_H(u)|}{2} \leq |I_H(u, v)| \leq \frac{1}{1 - \epsilon} \cdot \frac{|\Gamma_H(u)|}{2}.
\]
Crossing the upper and lower bounds from the inequalities, we see that
\[
(1 - 6\epsilon) \cdot \frac{|\Gamma_H(u)|}{4} \leq |I_G(g_1, g_2)| \leq \left( \frac{1}{1 - \epsilon} + 6\epsilon \right) \cdot \frac{|\Gamma_H(u)|}{4}.
\]
The same argument holds when \( t_1 \neq t_2 \), but when applying Proposition 2.8, the factor \( |I_{K_3}(t_1, t_2)| \) is now a 1 instead of a 2, so we obtain
\[
(1 - 6\epsilon) \cdot \frac{|\Gamma_H(u)|}{2} \leq |I_G(g_1, g_2)| \leq \left( \frac{1}{1 - \epsilon} + 6\epsilon \right) \cdot \frac{|\Gamma_H(u)|}{2}.
\]
Then we see that the statement holds, since for \( \epsilon < \epsilon_0 \)
\[
\frac{1}{1 - \epsilon} + 6\epsilon \leq 1 + 8\epsilon.
\]

Next, we consider certain triples in \( C \) with the graph \( T(C) \). Recall that the triples in \( T(C) \) have an identifiable structure that is found in the core triangles.
3.2. Properties of $T(C)$, the Graph of Triangles of $C$. Recall we let $C_i$ denote a component of $C$. We will show that one can use the partitioning induced by the components of $C$ to partition the core triangles. Further, if $\psi^{-1}(t, g)$ and $\psi^{-1}(t', g')$ are in the same component of $C$, then they remain in the same component of $T(C)$, i.e., $\psi^{-1}(K_3 \times \{g\})$ and $\psi^{-1}(K_3 \times \{g'\})$ are in the same component in $T(C)$, as shown in the following lemma.

**Lemma 3.3.** Fix $\epsilon < \epsilon_0$. If $\psi(v) = (t, g)$ and $\psi(v') = (t', g')$ are in the same component of $C$, then $\psi^{-1}(K_3 \times \{g\})$ and $\psi^{-1}(K_3 \times \{g'\})$ are in the same component in $T(C)$.

**Proof.** Suppose for now that $v$ and $v'$ are in the same component of $C_i$. We want to show that there is some $g'' \in I_G(g, g')$ where $\psi^{-1}(K_3 \times \{g''\})$ is compatible with both $\psi^{-1}(K_3 \times \{g\})$ and $\psi^{-1}(K_3 \times \{g'\})$. Such a $g''$ guarantees that triangle $\psi^{-1}(K_3 \times \{g''\})$ is incident to both triangles $\psi^{-1}(K_3 \times \{g'\})$ and $\psi^{-1}(K_3 \times \{g\})$ in $T(C)$, proving they are in the same component of $T(C)$. We have a lower bound on $|I_G(g, g')|$ from Proposition 3.2, so it remains to argue that not too many compatibility relations are destroyed in the process of deleting edges from $P = K_3 \times G$ to form $H$.

Recall that for any node $u$ in $\psi^{-1}(K_3 \times \{g\})$, $|\Gamma_H(u)| \leq 2|\Gamma_G(g)|$. Using this and the fact that at most an $\epsilon$ fraction of edges are deleted from any vertex, there are at most $6\epsilon \cdot |\Gamma_G(g)|$ edges deleted from vertices $

\psi^{-1}(K_3 \times \{g\})$ that would connect to core triangles. That implies that out of the $|I_G(g, g')|$ possible triangles $\psi^{-1}(K_3 \times \{g''\})$ for $g'' \in I_G(g, g')$, at least $|I_G(g, g')| - 6\epsilon|\Gamma_G(g)|$ of them still maintain all 6 edges between $\psi^{-1}(K_3 \times \{g''\})$ and $\psi^{-1}(K_3 \times \{g\})$ that are necessary for the triangles to be compatible. This similarly holds for $\psi^{-1}(K_3 \times \{g''\})$, at least $|I_G(g, g')| - 6\epsilon|\Gamma_G(g')|$ of them still maintain all 6 edges between $\psi^{-1}(K_3 \times \{g''\})$ and $\psi^{-1}(K_3 \times \{g'\})$.

Then from Proposition 3.2, we have that

$$(1 - 6\epsilon) \cdot \max(|\Gamma_H(v)|, |\Gamma_H(v')|)/4 \leq |I_G(g, g')|.$$ 

Since for $u \in \{v, v', |\Gamma_H(u)| \geq 2|\Gamma_G(\psi(u))|(1 - \epsilon)$ by Proposition 2.9, it follows that

$$(1 - 6\epsilon) \cdot 2 \max(|\Gamma_G(g)|, |\Gamma_G(g')|) \cdot (1 - \epsilon)/4 \leq |I_G(g, g')|. $$

Therefore, for $g'' \in I_G(g, g')$, the number of triangles $\psi^{-1}(K_3 \times \{g''\})$ in $T(C)$ that have the 12 required edges to be compatible with both $\psi^{-1}(K_3 \times \{g\})$ and $\psi^{-1}(K_3 \times \{g'\})$ is at least

$$(1 - 6\epsilon)(1 - \epsilon)/2 - 12\epsilon \max(|\Gamma_G(g)|, |\Gamma_G(g')|),$$

which is positive since $(1 - 6\epsilon)(1 - \epsilon)/2 - 12\epsilon > 0$ for $\epsilon < \epsilon_0$.

Now suppose that $v$ and $v'$ are not necessarily incident in $C_i$. There is some path of nodes in $C_i$ connecting them, and every pair of incident vertices in that path is in the same component of $T(C)$. Transitivity, it must be that $v$ and $v'$ are also in the same component of $T(C)$.

Next, we discuss the structure of $T(C)$ in more depth. In particular, we will look at which triples of nodes from $C$ become vertices in $T(C)$, and we will study the edge relations between the triangles of $T(C)$. Recall that the triples we find may be quasi-core triangles and not necessarily core triangles (see Definition 2.6). Let $\psi(v_1) = (a, g_1), \psi(v_2) = (b, g_2)$, and $\psi(v_3) = (c, g_3)$. Overall, the edges could be deleted in such a way that makes it impossible to distinguish triangles $\psi^{-1}(K_3 \times \{g\})$ and $\psi^{-1}(a, g_1), (b, g_2), (c, g_3)$, if the vertices $g, g_1, g_2, g_3$ are $\epsilon$-confusable (see Definition 2.5), i.e. almost all of their neighbors in $G$ are in common. Thus the need to allow quasi-core triangles in the reconstruction. For example, consider the case when $G$ is the $\epsilon$-noisy hypercube. This is the graph on vertices $\{0, 1\}^n$ for $n = 2^\ell$, where nodes are adjacent exactly when their hamming distance is $\ell$. Here, every vertex $u \in G$ has neighbors whose neighborhoods are all only an $\epsilon$ fraction different from $u$'s, so in $H$ it might be that $u$ has the same neighborhood as one–or many–of its neighbors.

We have seen from Proposition 3.2 that edges in $C$ can also be from nodes whose color classes are the same and whose $G$ classes have intersection size roughly $|\Gamma_H(v_1)|/4$, for $|\Gamma_H(v_1)| \geq |\Gamma_H(v_2)|, |\Gamma_H(v_3)|$. Such triangles $\{v_1, v_2, v_3\}$ will be monochrome (see Definition 2.7). Now we can exactly describe all of the types of triangles in $T(C)$.

**Lemma 3.4.** Fix $\epsilon < \epsilon_0$. The vertices of $T(C)$, which are all triangles of $C$, are all quasi-core or monochrome triangles.
Proof. Fix a triangle \( T = \{v_1, v_2, v_3\} \) in \( T(C) \), where \( \psi(v_i) = (t_i, g_i) \), for \( i \in [3] \). Without loss of generality that \( |\Gamma_H(v_1)| \geq |\Gamma_H(v_2)| \geq |\Gamma_H(v_3)| \).

First, suppose that \( t_1 = t_2 \). For sake of deriving a contradiction, suppose also that \( t_3 \neq t_1, t_2 \). By Proposition 3.2,
\[
|I_G(g_1, g_2)| \leq (1 + 8\epsilon) \cdot \frac{|\Gamma_H(v_1)|}{4}.
\]
We will show that the above upper bound cannot hold when \( t_3 \neq t_1, t_2 \) by lower bounding \( |I_G(g_1, g_2)| \) with \( |I_G(g_1, g_2, g_3)| \), where \( I_G(g_1, g_2, g_3) \) the intersection of \( \cap_{i \in [3]} \Gamma_G(g_i) \).

Again by Proposition 3.2, for there to be an edge between \( v_1 \) and \( v_3 \), as well as between \( v_2 \) and \( v_3 \), it must be that
\[
|I_G(g_1, g_3)| \geq (1 - 6\epsilon)|\Gamma_H(v_1)|/2 \quad \text{and} \quad |I_G(g_2, g_3)| \geq (1 - 6\epsilon)|\Gamma_H(v_2)|/2.
\]
Since \( v_1, v_2, v_3 \) all have \( \epsilon \)-similar degree, \( |\Gamma_H(v_2)|, |\Gamma_H(v_3)| \geq (1 - 2\epsilon)|\Gamma_H(v_1)| \). Using the lower bounds on \( |I_G(g_2, g_3)| \) and \( |I_G(g_1, g_3)| \), the \( \epsilon \)-similar degrees property, and Proposition 2.9, we have that
\[
|I_G(g_1, g_2, g_3)| \geq |\Gamma_G(g_3)| - |\Gamma_G(g_3) \setminus \Gamma_G(g_1)| - |\Gamma_G(g_3) \setminus \Gamma_G(g_2)|
\geq |\Gamma_G(g_3)| - (|\Gamma_G(g_3)| - (1 - 6\epsilon)|\Gamma_H(v_1)|/2) - (|\Gamma_G(g_3)| - (1 - 6\epsilon)|\Gamma_H(v_2)|/2)
\geq \frac{1 - 6\epsilon}{2} \cdot (|\Gamma_H(v_1)| + |\Gamma_H(v_2)|) - |\Gamma_G(g_3)|
\geq \frac{(1 - 6\epsilon)(2 - 2\epsilon)}{2} \cdot |\Gamma_H(v_1)| - |\Gamma_H(v_1)|/(2(1 - \epsilon)).
\]
Putting it all together,
\[
(1 + 8\epsilon) \cdot \frac{|\Gamma_H(v_1)|}{4} \geq |I_G(g_1, g_2)| \geq |I_G(g_1, g_2, g_3)| \geq \left( \frac{1 - 6\epsilon)(2 - 2\epsilon)}{2} - \frac{1}{2(1 - \epsilon)} \right) \cdot |\Gamma_H(v_1)|
\]
is false for all \( \epsilon < \epsilon_0 \), so we reach a contradiction.\(^{13}\) Therefore, if two vertices of \( C \) in a triangle of \( T(C) \) have the same color class, then the third vertex also has that same color class.

It follows that all triangles consist of vertices in \( C \) that either have the same color class, or all different color classes. For a triangle with vertices in \( C \) having all the same color class, by Proposition 3.2 and the fact that the degrees are \( \epsilon \)-similar, for all distinct \( i, j \in [3] \),
\[
(1 - 6\epsilon)(1 - 2\epsilon) : \frac{|\Gamma_H(v_i)|}{4} \leq |I_G(g_i, g_j)| \leq (1 + 8\epsilon) : \frac{|\Gamma_H(v_1)|}{4}.
\]
Given that \( (1 - 6\epsilon)(1 - 2\epsilon) \geq (1 - 8\epsilon) \) and \( 1 + 8\epsilon \leq 1 + 9\epsilon \) for all \( \epsilon < \epsilon_0 \), we can compare this inequality on \( |I_G(g_i, g_j)| \) with that from the definition of monochrome (Definition 2.7) to see that such a triangle is monochrome.

For a triangle with vertices having all different color classes, using Proposition 3.2, the fact that \( g_1, g_2, g_3 \) are \( \epsilon \)-similar, and Proposition 2.9, for all distinct \( i, j \in [3] \)
\[
(1 - 6\epsilon)(1 - 2\epsilon)(1 - \epsilon) : |\Gamma_G(g_i)| \leq |I_G(g_i, g_j)| \leq (1 + 8\epsilon)|\Gamma_G(g_1)|.
\]
Note that the reasoning for the above bound is exactly the same as when the vertices of the triangle all had the same color class, but the corresponding inequality in Proposition 3.2 has an additional factor of 2 here, and the other additional factor of 2 (and \( 1 - \epsilon \) in the lower bound) come from using \( \Gamma_G \) to bound instead of \( \Gamma_H \). It follows that such a triangle is quasi-core since in comparing with the definitions of quasi-core and \( \epsilon \)-confusable (Definitions 2.6, 2.5)
\[
(1 - 6\epsilon)(1 - 2\epsilon)(1 - \epsilon) \geq (1 - 9\epsilon).
\]

The graph \( T(C) \) will have several components. The next lemma shows that core triangles are only compatible with core or quasi-core triangles.

\(^{13}\) The choice of \( \epsilon_0 = 1/40 \) comes from \( \epsilon_0 \) being the largest reasonable looking fraction that this holds for.
Lemma 3.5. Fix $\epsilon < \epsilon_0$. For $T \in \mathcal{T}(C)$ quasi-core, we have that $T$ is only compatible with other quasi-core triangles. Consequently, any component of $\mathcal{T}(C)$ consists of only monochrome triangles or of only quasi-core triangles.

Proof. Suppose that $T', T \in \mathcal{T}(C)$ are compatible triangles where $T = \{\psi^{-1}(a, g_1), \psi^{-1}(b, g_2), \psi^{-1}(c, g_3)\}$ is quasi-core and $T' = \{\psi^{-1}(t_1, g'_1), \psi^{-1}(t_2, g'_2), \psi^{-1}(t_3, g'_3)\}$. Assume the vertices of $T$ and $T'$ are labeled so that Condition 2 from Definition 2.10 holds; so $t_1 \neq b, c$ and $t_2 \neq a, c$, and $t_3 \neq a, b$. That is, $t_1 = a, t_2 = b, t_3 = c$. By Lemma 3.4, all triangles are either quasi-core or monochrome, so $T'$ is quasi-core, and every component consists of only monochrome triangles or only quasi-core triangles.

Given that all components of $\mathcal{T}(C)$ consist of either monochrome triangles or quasi-core triangles, we will refer to the former type of component as monochrome and the latter as core. In particular, our choice to refer to the latter components as core and not quasi-core is purposeful, as the next lemma shows that any component of $\mathcal{T}(C)$ containing a triangle with the node $u$ also contains the core triangle that $u$ is a part of.

Lemma 3.6. Fix $\epsilon < \epsilon_0$. Fix $T \in \mathcal{T}(C)$ in core component $Y_j$ of $\mathcal{T}(C)$. For all $g = \psi_C(u)$ for some $u \in T$, the core triangle $\psi^{-1}(T' \times \{g\})$ is in component $Y_j$ of $\mathcal{T}(C)$.

Proof. As in the statement of the lemma, fix $T \in \mathcal{T}(C)$ in component $Y_j$ of $\mathcal{T}(C)$. For any $u \in T$, take $g = \psi_C(u)$. We seek to show that $\psi^{-1}(T' \times \{g\}) \in Y_j$.

First off, if $T'$ is core, there is nothing more to prove, so suppose that $T$ is quasi-core, but not core. Denote the vertices of $T$ as $u, v, w$, where without loss of generality $\psi(u) = (a, g), \psi(v) = (b, g')$, and $\psi(w) = (c, g'')$. Overall, we will show that there is a core triangle $\psi^{-1}(T' \times \{g''\})$ that is disjoint from $\psi^{-1}(T' \times \{g\})$ and $T'$, but is also compatible with both triangles.

We give a few sentences describing our high-level strategy. Note that we can always consider the compatibility relation in Definition 2.10 and $\psi^{-1}$ on the graph $P = K_3 \times G$, as $H$ is formed from just removing edges in $P$. As such, observe that in $P$, the triangle $\psi^{-1}(K_3 \times \{g''\})$ is compatible with (as in Definition 2.10 but with respect to $P$) and vertex disjoint from the copy of $T$ in $P$ and $\psi^{-1}(K_3 \times \{g\})$, when $g'' \in I_C(g, g', g'')$. When edges are removed to form $H$ from $P$, some triangles which were compatible are no longer. It suffices to show that $|I_C(g, g', g'')|$ is large, i.e., bigger than $(1 - O(\epsilon)) \max \{\Gamma_C(g), \Gamma_C(g'), \Gamma_C(g'')\}$, and that there exists at least one $\psi^{-1}(K_3 \times \{g''\})$ whose edges required for the compatibility relation remain intact in $H$.

This proof follows very similarly to that of Lemma 3.3.

Since $T$ is quasi-core, the vertices $g, g'$, and $g''$ are all $\epsilon$-confusable, i.e. they all have large pairwise intersection. If $|\Gamma_C(g)| \geq |\Gamma_C(g')| \geq |\Gamma_C(g'')|$, then since $g, g'$, and $g''$ are $\epsilon$-confusable, $|\Gamma_C(g) \setminus \Gamma_C(g')| \leq 9\epsilon|\Gamma_C(g)|$ and $|\Gamma_C(g') \setminus \Gamma_C(g'')| \leq 9\epsilon|\Gamma_C(g')|$. Using these bounds, we can lower bound $|I_C(g, g', g'')|$ with

$$|I_C(g, g', g'')| \geq |\Gamma_C(g)| - |\Gamma_C(g) \setminus \Gamma_C(g')| - |\Gamma_C(g') \setminus \Gamma_C(g'')|$$

$$|I_C(g, g', g'')| \geq |\Gamma_C(g)| - 18\epsilon|\Gamma_C(g)| = (1 - 18\epsilon)|\Gamma_C(g)|.$$

The choice of $|\Gamma_C(g)|$ as the largest neighborhood was arbitrary and only chosen to show that more generally, the following bound holds:

$$|I_C(g, g', g'')| \geq (1 - 18\epsilon) \max(|\Gamma_C(g)|, |\Gamma_C(g')|, |\Gamma_C(g'')|).$$

Now, we will show that not enough edges are deleted to destroy the necessary compatibility relations. Since the number of edges in $P$ incident to $\psi^{-1}(K_3 \times \{g\})$ is $6|\Gamma_C(g)|$, there are at most $6\epsilon|\Gamma_C(g)|$ edges deleted between $\psi^{-1}(K_3 \times \{g\})$ and another core triangle. Thus, there are at least

$$(1 - 18\epsilon) \max(|\Gamma_C(g)|, |\Gamma_C(g')|, |\Gamma_C(g'')|) - 6\epsilon|\Gamma_C(g)|$$

core triangles $\psi^{-1}(K_3 \times \{g''\})$ that have $g'' \in I_C(g, g', g'')$ and the 6 edges necessary for the compatibility relation to be preserved between $\psi^{-1}(K_3 \times \{g''\})$ and $\psi^{-1}(K_3 \times \{g\})$. Let $J_g$ be the set of such $g''$. It suffices to prove that at least one $g'' \in J_g$ has that $\psi^{-1}(K_3 \times \{g\})$ is compatible with $T$. Note that since $g'' \in I_C(g, g', g'')$, every $g'' \in J_g$ is compatible with $T$ according to the edges of $P$. Thus, in order for $\psi^{-1}(K_3 \times \{g\})$ to not be compatible with $T$ at least one of the 6 edges between them in $P$ must be deleted. However, at most $6\epsilon \max(|\Gamma_C(g)|, |\Gamma_C(g')|, |\Gamma_C(g'')|)$ edges incident to $T$ were deleted from $P$ to get $H$.

Therefore, the number of triangles of the form $\psi^{-1}(K_3 \times \{g''\})$ that are compatible with both $T$ and $\psi^{-1}(K_3 \times \{g\})$ is at least

$$(1 - 30\epsilon) \max(|\Gamma_C(g)|, |\Gamma_C(g')|, |\Gamma_C(g'')|).$$
We just require one such triangle to prove the statement, so it suffices that \( 1 - 30\epsilon > 0 \) to prove the claim, and this is true for all \( \epsilon < \epsilon_0 \).

Now, putting together several of the previous lemmas, we will see that a core component of \( T(C) \) must either not intersect a component \( C_i \) of \( C \), or it must contain all the core triangles of \( C_i \).

**Lemma 3.7.** Fix \( \epsilon < \epsilon_0 \). For any component \( C_i \) of \( C \) and any core component \( Y_j \) of \( T(C) \), \( Y_j \) either contains every core triangle of \( C_i \) or does not contain any node from \( C_i \).

**Proof.** Fix a component \( C_i \) of \( C \), and suppose \( Y_j \) is a core component of \( T(C) \) that contains a triangle with some node \( v \in C_i \). By Lemma 3.6, the core triangle \( T \) with \( v \in T \) must also be in \( Y_j \). To finish, by Lemma 3.3, \( Y_j \) contains every node of \( C_i \) (and thus every triangle of \( C_i \)).

### 3.3. Structure of core components: Proof of Theorem 1.2.

We say that \( X \subseteq V(H) \) is atomic if for every component \( C_i \) of \( C \), we have that \( C_i \cap X = \emptyset \) or \( C_i \subseteq X \). In other words, a set of vertices is atomic if it is a union of components. For each component \( Y_j \) of \( T(C) \), let \( U_j \) be the vertices of \( C \) (and thus \( H \)) which are covered by at least one triangle in \( Y_j \). Note that \( U_j \) is atomic by Lemma 3.6. Let \( H[U_j] \) be the subgraph of \( H \) induced by the vertices \( U_j \). First, we show that if \( Y_j \) is a core component, then the cut between \( U_j \) and \( H \setminus U_j \) is sparse.

**Proposition 3.8.** Let \( Y_j \) be a core component of \( T(C) \). Let \( X \subseteq U_j \) and \( Z \subseteq H \setminus U_j \) be atomic. Then,

\[
|E(H) \cap (X \times Z)| \leq \frac{5\epsilon}{1-\epsilon} \min(\text{vol}_H(X), \text{vol}_H(Z)).
\]

**Proof.** Recall that we denote the full product graph from which \( H \) is constructed by \( P = K_3 \times G \). By Proposition 3.1, if \( X \) is atomic, then the vertices of \( X \) can be expressed as a disjoint union of core triangles. That is, there exists \( \psi_G(X) \subseteq G \) such that \( \psi(X) = K_3 \times \psi_G(X) \). Define \( \psi_G(Z) \) likewise, and note that \( \psi_G(X) \) and \( \psi_G(Z) \) must be disjoint. Since \( Y_j \) is core, each core triangle of \( X \) must not be compatible with each core triangle of \( Z \). In particular, since \( X \) and \( Z \) are atomic, for any edge \((g, g')\) between \( \psi_G(X) \) and \( \psi_G(Z) \), at least one of the six edges between \((\psi^{-1}(t, g), \psi^{-1}(t', g'))\) must be in \( E' = E(P) \setminus E(H) \). In particular, this implies that

\[
|E(H) \cap (X \times Z)| \leq 5|E' \cap (X \times Z)|
\]

\[
= 5 \min \left( \sum_{u \in X} |\Gamma_{E'}(u) \cap Z|, \sum_{u \in Z} |\Gamma_{E'}(u) \cap X| \right)
\]

\[
\leq 5\epsilon \min \left( \sum_{u \in X} |\Gamma_{P}(u)|, \sum_{u \in Z} |\Gamma_{P}(u)| \right)
\]

\[
\leq \frac{5\epsilon}{1-\epsilon} \min \left( \sum_{u \in X} |\Gamma_{H}(u)|, \sum_{u \in Z} |\Gamma_{H}(u)| \right)
\]

\[
\leq \frac{5\epsilon}{1-\epsilon} \min(\text{vol}_H(X), \text{vol}_H(Z)),
\]

as desired.

We further show that if \( Y_j \) is a core component, then \( H[U_j] \) can be colored by a simple algorithm.

**Proposition 3.9.** Given a core component \( Y_j \) of \( T(C) \), one can in \( O(|Y_j|^2) \) time find subsets \( A_j, B_j, C_j \) of \( U_j \) such that \( A_j, B_j, \) and \( C_j \) equal \( \psi_{K_3}(u) \cap U_j, \psi_{K_3}(b) \cap U_j, \) and \( \psi_{K_3}(c) \cap U_j \), up to a permutation of the three sets.

**Proof.** Pick an arbitrary triangle \( T_0 = \{u, v, w\} \subseteq Y_j \). We choose to initialize \( A = \{u\}, B = \{v\}, C = \{w\} \). Initialize \( S = \{T_0\} \). We now iterate the following procedure.

Consider a triangle \( T' \in Y_j \setminus S \) which is compatible with some \( T \in S \). Assume that \( T = \{u_a, v_b, w_c\} \), with \( u_a \in A, v_b \in B \) and \( w_c \in C \). Since \( T \) and \( T' \) are compatible, there is a unique vertex \( u' \in T' \) which is non-adjacent to \( u_a \) in \( H \). We place this vertex in \( A \). Likewise, we place the unique \( v' \in T' \) non-adjacent to \( v_b \) in \( B \) and the unique \( w' \in T' \) non-adjacent to \( w_c \) in \( C \). Finally, we add \( T' \) to \( S \). This procedure will iterate until \( S = Y_j \) because \( Y_j \) is connected.
It suffices to show that if \( Y_j \) is core, then \( A, B, C \) will respect the color classes induced by \( \psi_{K_3}^{-1} \). To see why, each triangle \( T \) of \( Y_j \) is quasi-core by Lemma 3.5, so its three vertices land in the different color classes induced by \( \psi_{K_3}^{-1} \). Further, for any pair \( T, T' \) of compatible triangles the non-adjacent pairs of vertices must lie in the same color class. Thus, at every step of the algorithm, the sets \( A, B, C \) are consistent with \( \psi_{K_3}^{-1} \). Therefore, the final \( A, B, C \) must precisely be \( \psi_{K_3}^{-1}(a) \cap U_j, \psi_{K_3}^{-1}(b) \cap U_j, \) and \( \psi_{K_3}^{-1}(c) \cap U_j \), up to a permutation of the three sets.

We now have the tools to prove Theorem 1.2.

Proof of Theorem 1.2. We claim that if \( G \) is a 3\( \epsilon \)-edge-expander, then there is a single core component \( Y_j \in \mathcal{T}(C) \). Therefore, we can loop over all the \( Y_j \)'s (which can be done efficiently) and output one that gives a full 3-coloring of \( H \). There exists a core component by Lemma 3.5 and we can consider one such component as input to Proposition 3.9. As we loop over the \( Y_j \)'s, we will encounter components that are not core, but the procedure in the proof of Proposition 3.9 can still be applied on each component, where if a 3-coloring is not produced on \( H \) as a result, then that component is definitively not core.

Consider a core component \( Y_j \). Let \( G_j \) be the vertices of \( G \) which correspond to vertices of \( U_j \). Since \( G \) is a 3\( \epsilon \)-edge-expander, we know that

\[
|E(G) \cap (G_j \times (G \setminus G_j))| \geq 3\epsilon \min(\text{vol}_G(G_j), \text{vol}_G(G \setminus G_j)),
\]

Thus, since each edge of \( G \) becomes 6 edges in \( P \), we have that

\[
|E(P) \cap (U_j \times (V \setminus U_j))| = 6|E(G) \cap (G_j \times (G \setminus G_j))| \geq 18\epsilon \min(\text{vol}_G(G_j), \text{vol}_G(G \setminus G_j)) = 9\epsilon \min(\text{vol}_P(U_j), \text{vol}_P(V \setminus U_j)) \geq 9\epsilon \min(\text{vol}_H(U_j), \text{vol}_H(V \setminus U_j)).
\]

Also, note that by the definition of \( E' \) (the edges deleted from \( P \)),

\[
|E' \cap (U_j \times (P \setminus U_j))| \leq \epsilon \min(\text{vol}_P(U_j), \text{vol}_P(V \setminus U_j)) \leq \frac{\epsilon}{1-\epsilon} \min(\text{vol}_H(U_j), \text{vol}_H(V \setminus U_j)).
\]

Thus,

\[
|E(H) \cap (U_j \times (V \setminus U_j))| \geq \left(9 - \frac{1}{1-\epsilon}\right) \epsilon \min(\text{vol}_H(U_j), \text{vol}_H(V \setminus U_j))
\]

Since \( U_j \) is atomic by Lemma 3.6, we can use Proposition 3.8 to see that

\[
|E(H) \cap (U_j \times (H \setminus U_j))| \leq \frac{5\epsilon}{1-\epsilon} \min(\text{vol}_H(U_j), \text{vol}_H(H \setminus U_j)).
\]

Thus, since \( 9 - \frac{1}{1-\epsilon} > \frac{5\epsilon}{1-\epsilon} \) for \( \epsilon < \epsilon_0 \), we must have that \( \min(\text{vol}_H(U_j), \text{vol}_H(H \setminus U_j)) = 0 \). Every vertex of \( H \) has at least one edge coming from it, since in constructing \( H \) at least a \( (1-\epsilon) \) fraction of edges were preserved from the underlying tensor, and therefore \( Y_j \) does indeed cover all of \( H \), as desired.

3.4. Core Factoring Algorithm. Leading up to the proof of Theorem 1.1, we now use Proposition 3.9 to prove our Core Factoring Lemma (Lemma 3.10). Algorithm 3.1 outlines our approach to Lemma 3.10. The proof of Lemma 3.10 also relies on Propositions 3.11, 3.12, and 3.13, which are stated within the proof of Lemma 3.10.

For the algorithm, we define a matching on a weighted bipartite graph as the max-min matching (also known as a bottleneck matching) to be the matching that maximizes the minimum weight of any edge in the matching (e.g., [45] and references therein). One efficient algorithm for max-min matching is to sort the edges by weight and iteratively delete the smallest one until no matching remains. The last edge deleted is the objective value. This can be made more efficient with a binary search (c.f., [15]).

Recall for the next lemma that \( U_j \) are the vertices of \( H \) that are in some triangle in \( Y_j \).

Lemma 3.10 (Core Factoring Lemma). Fix \( \epsilon < \epsilon_0 \). Fix \( Y_j \) a component of \( \mathcal{T}(C) \). Let \( U \) be an atomic subset of \( U_j \). In \( O(|Y_j|^{1/3}) \) time, Algorithm 3.1 outputs either FAIL or a graph \( \tilde{H}_U \) on \( U \) with a factorization \( K_3 \times \tilde{G}_U \), as given by maps \( \psi_{K_3}^U : U \rightarrow K_3 \) and \( \psi_{\tilde{G}_U}^U : U \rightarrow \tilde{G}_U \) with the following properties:
Algorithm 3.1 Core Factoring Algorithm

Input: A component \( Y_j \) of \( T(C) \) and \( U \subseteq U_j \) atomic.
Output: FAIL or a factorization \( K_3 \times \tilde{G}_U \) of \( H_U \) on vertex set \( U \) with maps \( \psi_{K_3}^U : U \rightarrow K_3 \) and \( \psi_G^U : U \rightarrow \tilde{G}_U \)

Compute \( A_j, B_j, C_j \subseteq H[U_j] \) according to Proposition 3.9.
Let \( A = A_j \cap U, B = B_j \cap U, C = C_j \cap U \)
if \( A, B, C \) have unequal sizes or is not a 3-coloring of \( H[U] \) then return FAIL

// Max-min matching phase
Construct graph \( (A \cup B \cup C, E) \), where \( E = (A \times B) \cup (B \times C) \).
for each edge \( (u, v) \) of \( E \) do assign weight \( 2|I_H(u, v)| \max\{|\Gamma_H(u)|, |\Gamma_H(v)|\} \).
Find a max-min matching \( M_1 \) between \( A \) and \( B \)
Find a max-min matching \( M_2 \) between \( B \) and \( C \)
if objective value of either \( M_1 \) or \( M_2 \) is less than \( 1 - 6\epsilon \) then return FAIL

// Building the tensor decomposition
Initialize \( V(\tilde{G}_U) = \{g_v : v \in B\} \) and \( V(\tilde{H}_U) = U \).
for each \( v \) in \( B \) do
   Let \( u \) be unique \( (u, v) \in M_1, w \in C \) be unique \( (v, w) \in M_1 \)
   Set \( \psi_{K_3}^U(u) = a, \psi_{K_3}^U(v) = b, \psi_{K_3}^U(w) = c \).
   Set \( \psi_G^U(u) = \psi_G^U(v) = \psi_G^U(w) = g_v \).
   for each \( v' \) in \( B \) do
      Let \( u' \) be unique \( (u', v') \in M_1, w' \) be unique \( (v', w') \in M_2 \).
      if any of \( (u, v'), (u, w'), (v, u'), (v, w'), (w, v'), (w, u'), (w, w') \in H \) then
         Add \((g_v, g_{v'})\) to \( E(\tilde{G}_U) \)
         Add \((u, v'), (u, w'), (v, u'), (v, w'), (w, v'), (w, w') \) to \( E(\tilde{H}_U) \)
      if \( |E(H[U]) \Delta E(\tilde{H}_U)| \leq 260|E(H[U])| + |U| \) then return \( \tilde{G}_U, \tilde{H}_U, \psi_{K_3}^U, \psi_G^U \)
else return FAIL

(a) \( \psi_{K_3}^U \) is a 3-coloring of \( H[U] \).
(b) \( |E(H[U]) \Delta E(\tilde{H}_U)| \leq 260 \text{vol}(H[U]) + |U| \).
(c) If \( Y_j \) is a core component, the algorithm will never output FAIL.

Proof. The reader may use Algorithm 3.1 as an outline for this proof. Using Proposition 3.9, we can find \( A_j, B_j, C_j \subseteq U_j \) which cover every triangle of \( Y_j \). Note that if \( Y_j \) is core, then \( |A_j| = |B_j| = |C_j| \) by Lemma 3.6, as \( U_j \) can be partitioned by the core triangles of \( Y_j \). Further, if \( Y_j \) is core, then \( A_j, B_j, C_j \) induce a 3-coloring of \( H[U_j] \).
Let \( A = A_j \cap U, B = B_j \cap U, \) and \( C = C_j \cap U \).
Since \( U \) is atomic, \( A_j, B_j, C_j \) each intersect each core triangle of \( U \) in exactly one vertex. Thus, the partition from \( A, B, C \) induces a 3-coloring of \( H[U] \) and each set has equal size. We can safely report FAIL if one of these conditions fails to hold.

Max-min matching phase. Consider the tripartite graph on \( U \) partitioned into \( A, B, \) and \( C \), with edge set \( (A \times B) \cup (B \times C) \).
For every edge \( (u, v) \) in this tripartite graph, assign a weight to it of \( 2|I_H(u, v)| \max\{|\Gamma_H(u)|, |\Gamma_H(v)|\} \).
We find a max-min matching between sets \( A, B, M_1 \), and between sets \( B, C, M_2 \). Define the quality of the matchings on \( (A, B, C) \) to be the minimum objective of the max-min matchings \( M_1 \) and \( M_2 \).

We now show that if \( Y_j \) is core, the quality of the matchings is at least \( 1 - 6\epsilon \).
If node \( u \) with \( \psi(u) = (t, g) \) is in \( A \), then, by Lemma 3.6, for distinct \( t', t'' \neq t \), one of \( \psi^{-1}(t', g), \psi^{-1}(t'', g) \) is in \( B \) and the other is in \( C \).
Letting \( v \) be such that \( \psi(v) \in \{(t', g), (t'', g)\} \), recall Equation 3.1 states that
\[
(1 - 6\epsilon) \cdot \max\{|\Gamma_H(u)|, |\Gamma_H(v)|\}/2 \leq |I_H(u, v)|.
\]
This implies that the quality of the matchings is at least \( 1 - 6\epsilon \), since that value is achievable when the matching edges are all a subset of the core triangle edges. Thus, if the quality is less than \( 1 - 6\epsilon \), we may safely output FAIL, as \( Y_j \) cannot be core.

Building the tensor decomposition. It remains to build \( \tilde{G}_U \) and \( \tilde{H}_U \). For each \( v \in B \), let \( g_v \) be a vertex of \( V(\tilde{G}_U) \). We let the vertices of \( \tilde{H}_U \) be \( U \). For each \( v \in B \), let \( T_v \) be the triangle \( \{u, v, w\} \), where \( u \in A \) is such that \( (u, v) \in M_1 \) and \( w \in C \) is such that \( (v, w) \in M_2 \).
By definition of \( M_1 \) and \( M_2 \), the \( T_v \)'s partition \( U \).
For every pair \( v, v' \in B \), we add \((g_v, g_{v'})\) to \( E(\tilde{G}_U) \) if \( T_v \) and \( T_{v'} \) share at least one edge in \( H \). Likewise,
if \( T_v \) and \( T_{v'} \) share at least one edge in \( H \), we add \((x, y)\) to \( E(\tilde{H}_U) \) for all \( x \in T_v \) and \( y \in T_{v'} \) with \( x \) and \( y \) elements of distinct \( A, B, C \). For each \( u \in A \), we define \( \psi_{K_3}^U(u) = a \). Likewise, for each \( v \in B \), \( \psi_{K_3}^U(v) = b \) and each \( w \in C \), \( \psi_{K_3}^U(w) = c \). For each \( v \in B \), we define \( \psi_{G}^U(x) = g_v \) for all \( x \in T_v \). Note by definition \( \tilde{H}_U = K_3 \times \tilde{G}_U \) as induced by \( \psi_{K_3}^U \) and \( \psi_{G}^U \).

**Bounding reconstruction error.** It is clear at this point that if the output does not FAIL, then conditions (a), (b), (c) of the lemma hold. Note in particular that condition (b) holds due to the line preceding the else statement. It suffices to prove that the algorithm does not return FAIL when \( Y_j \) is a core component. In particular, it suffices to check condition (b) when \( Y_j \) is core. Observe that

\[
|E(\tilde{H}_U) \Delta E(H_U)| = |E(\tilde{H}_U) \setminus E(H_U)| + |E(H_U) \setminus E(\tilde{H}_U)|.
\]

Note that \((x, y) \in E(H_U) \setminus E(\tilde{H}_U)\) only if \( x, y \in T_v \) for some \( v \in B \), as any edge between different \( T_v \)'s is accounted for in \( \tilde{H}_U \). Thus, \(|E(H_U) \setminus E(\tilde{H}_U)| \leq 3|B| = |U|\).

Now, we bound \(|E(\tilde{H}_U) \setminus E(H_U)|\). For simplicity of the remainder of the argument, we assume without loss of generality that \( A = \psi_{K_3}^{-1}(a) \cap U, B = \psi_{K_3}^{-1}(b) \cap U, \) and \( C = \psi_{K_3}^{-1}(c) \cap U \) (recall by Proposition 3.9 that this is always true up to permutation of \( A, B, C \)). We observe the following structural property about the matched triangles.

**Proposition 3.11.** Assume \( Y_j \) is a core component, with \( A, B, C \) its color classes when restricted to \( U \). Fix \( T_v = \{ u \in A, v \in B, w \in C \} \) found by the matching algorithm. Then,

\[
(1 - 14\epsilon) \max(|\Gamma_H(u)|, |\Gamma_H(v)|, |\Gamma_H(w)|) \leq \min(|\Gamma_H(u)|, |\Gamma_H(v)|, |\Gamma_H(w)|).
\]

**Proof of Proposition 3.11.** Note that by the defining property of \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), we have that

\[
\begin{align*}
(3.2) & \quad (1 - 6\epsilon) \max(|\Gamma_H(u)|, |\Gamma_H(v)|)/2 \leq |I_H(u, v)| \\
(3.3) & \quad (1 - 6\epsilon) \max(|\Gamma_H(v)|, |\Gamma_H(w)|)/2 \leq |I_H(v, w)|.
\end{align*}
\]

Further, since \( \psi_{K_3}(u) = a, \psi_{K_3}(v) = b, \psi_{K_3}(w) = c \), we have that any vertex \( x \in I_H(u, v) \) must have \( \psi_{K_3}(x) = c \). Thus,

\[
|I_H(u, v)| \leq \min(|\Gamma_H(u) \cap \psi_{K_3}^{-1}(c)|, |\Gamma_H(v) \cap \psi_{K_3}^{-1}(c)|)
\]
\[
\leq \min(|\Gamma_P(u) \cap \psi_{K_3}^{-1}(c)|, |\Gamma_P(v) \cap \psi_{K_3}^{-1}(c)|)
\]
\[
= \frac{1}{2} \min(|\Gamma_P(u)|, |\Gamma_P(v)|)
\]
\[
\leq \frac{1}{2(1 - \epsilon)} \min(|\Gamma_H(u)|, |\Gamma_H(v)|).
\]

We can further show that \( 2(1 - \epsilon)|I_H(v, w)| \leq \min(|\Gamma_H(v)|, |\Gamma_H(w)|) \). Thus, when combined with (3.2) and (3.3), we get that

\[
\begin{align*}
(1 - 7\epsilon) \max(|\Gamma_H(u)|, |\Gamma_H(v)|) & \leq \min(|\Gamma_H(u)|, |\Gamma_H(v)|), \\
(1 - 7\epsilon) \max(|\Gamma_H(v)|, |\Gamma_H(w)|) & \leq \min(|\Gamma_H(v)|, |\Gamma_H(w)|).
\end{align*}
\]

By suitably combining these two inequalities, we can further deduce that

\[
(1 - 14\epsilon) \max(|\Gamma_H(u)|, |\Gamma_H(v)|) \leq \min(|\Gamma_H(u)|, |\Gamma_H(v)|).
\]

To see why the result follows, assume that \( x \in \{u, v, w\} \) has the highest degree and \( y \in \{u, v, w\} \) has the lowest. If \( x = y \), the result is obvious. Otherwise, we pick the inequality of the previous three featuring \( x \) and \( y \).

Given vertices \( g_1, g_2, g_3 \in G \), define their **disjunction** \( \Delta(g_1, g_2, g_3) \) to be

\[
\Delta(g_1, g_2, g_3) = |(\Gamma_G(g_1) \cup \Gamma_G(g_2) \cup \Gamma_G(g_3)) \setminus I_G(g_1, g_2, g_3)|
\]

We give a bound on the disjunction of the triangles we matched.
Proposition 3.12. Assume $Y_j$ is a core component, with $A, B, C$ its color classes when restricted to $U$. Fix $T_v = \{u \in A, v \in B, w \in C\}$. Let $g_u, g_v, g_w \in G$ be such that $u = \psi^{-1}(a, g_u), v = \psi^{-1}(b, g_v), w = \psi^{-1}(c, g_w)$. Then,

$$|\Delta(g_u, g_v, g_w)| \leq 50\epsilon \min(|\Gamma_H(u)|, |\Gamma_H(v)|, |\Gamma_H(w)|).$$

Proof of Proposition 3.12. Note that

$$|\Delta(g_u, g_v, g_w)| \leq |\Gamma_G(g_u)| + |\Gamma_G(g_v)| + |\Gamma_G(g_w)| - 3|I_G(g_u, g_v, g_w)|.

Further,

$$|I_G(g_u, g_v, g_w)| \geq |\Gamma_G(g_v)| - |\Gamma_G(g_v) \setminus I_G(g_u, g_v)| - |\Gamma_G(g_v) \setminus I_G(g_v, g_w)|.$$

See that via (3.2)

$$|\Gamma_G(g_v) \setminus I_G(g_u, g_v)| = \frac{1}{2} |\Gamma_P(v)| - |I_P(u, v)|$$

$$\leq \frac{1}{2(1 - \epsilon)} |\Gamma_H(v)| - |I_H(u, v)|$$

$$\leq \left(1 - \frac{2}{2 - 2\epsilon} - \frac{1}{2} - 6\epsilon\right) |\Gamma_H(v)|$$

$$\leq 4\epsilon |\Gamma_H(v)|,$$

where the last line follows for $\epsilon < \epsilon_0$. Thus, working out the symmetric bound for $|\Gamma_G(g_u) \setminus I_G(g_v, g_w)|$, we have that

$$|\Delta(g_u, g_v, g_w)| \leq |\Gamma_G(g_u)| + |\Gamma_G(g_v)| - 2|\Gamma_G(g_v)| + 24\epsilon |\Gamma_H(v)|$$

$$\leq \frac{1}{2(1 - \epsilon)} |\Gamma_H(u)| + \frac{1}{2(1 - \epsilon)} |\Gamma_H(w)| - (1 - 24\epsilon) |\Gamma_H(v)|.$$

Further using Proposition 3.11, we get that

$$|\Delta(g_u, g_v, g_w)| \leq \frac{1}{(1 - \epsilon)(1 - 14\epsilon) |\Gamma_H(u)|, |\Gamma_H(v)|, |\Gamma_H(w)|}$$

$$- (1 - 24\epsilon) \min|\Gamma_H(u)|, |\Gamma_H(v)|, |\Gamma_H(w)|$$

$$\leq 50\epsilon \min|\Gamma_H(u)|, |\Gamma_H(v)|, |\Gamma_H(w)|,$$

for $\epsilon < \epsilon_0$, as desired.

Recall from Definition 2.10 that triangles $T_v = \{u, v, w\}$ and $T_{v'} = \{u', v', w'\}$ (with $u, u' \in A, v, v' \in B, w, w' \in C$) are compatible if

$$(u, v', (u, w), (v, u'), (v, w'), (w, u'), (w, v') \in E(H).$$

We will call triangles $T_v$ and $T_{v'}$ weakly linked if at least one but not all of those six pairs is an edge of $H$. The edges in $E(\overline{H}_U) \setminus E(H_U)$ come from weakly linked triangles. Thus, we can upper bound $|E(\overline{H}_U) \setminus E(H_U)|$ with 5 times the number of weakly linked triangles, as each pair of weakly linked triangles contributes at most 5 edges to $E(\overline{H}_U) \setminus E(H_U)$.

Proposition 3.13. Assume $Y_j$ is a core component, with $A, B, C$ its color classes when restricted to $U$. There are at most $52\epsilon \text{vol}_H(U)$ weakly linked pairs of triangles.

Proof. Assume that $T_v = \{u, v, w\}$ and $T_{v'} = \{u', v', w'\}$ are weakly linked. They can be weakly linked for two reasons. The first is that $T_v$ and $T_{v'}$ are compatible in $P[U]$, but at least one edge was deleted going from $P$ to $H$. The second case is that $T_v$ and $T_{v'}$ are also weakly linked in $P$. For the first type, since at most $2\epsilon \text{vol}_H(U)$ edges were deleted in $P[U]$, this is also an upper bound on the number of weakly linked pairs of triangles of this type. Thus, the remainder of the proof is bounding the number of pairs of the second type.

Let $\psi_G(u) = g_u$, and so forth. Since $T_v$ and $T_{v'}$ are weakly linked, we know that at least one (but not all) of $(g_u, g_{v'}), (g_u, g_w), (g_v, g_{v'}), (g_v, g_w), (g_w, g_{v'}), (g_w, g_{w'})$ is an edge of $G$. From this, we can deduce that either
one of $g_{uv}, g_{uw}$ is contained in $\Delta(g_{uv}, g_{uv}, g_{uw})$ or one of $g_{uv}, g_{vw}, g_{uw}$ is contained in $\Delta(g_{uv}, g_{uv}, g_{uw})$. Further, a single $g \in \Delta(g_{uv}, g_{uv}, g_{uw})$ can appear in at most 3 other triangles—namely those that the vertices $\psi^{-1}(a, g), \psi^{-1}(b, g), \psi^{-1}(c, g)$ appear in. Therefore, we can upper bound the number of weakly linked triangles of this type by

$$\sum_{v \in E \cap \{u, v, w\} = T_v} 3|\Delta(g_{uv}, g_{uv}, g_{uw})|$$

Using Proposition 3.12, we can upper bound this quantity by

$$\sum_{v \in E \cap \{u, v, w\} = T_v} 50(\Gamma_H(u) + \Gamma_H(v) + \Gamma_H(w)) = 50\epsilon \text{vol}_H(U),$$

where we use the fact that $A \cup B \cup C$ is a partition of $U$. Therefore, there are at most $52\epsilon \text{vol}_H(U)$ weakly linked pairs of triangles.

Thus, by Proposition 3.13, we have that $|E(\tilde{H}_U) \setminus E(H_U)| \leq 5 \cdot 52\epsilon \text{vol}_H(U)$. Therefore, $|E(\tilde{H}_U)\Delta E(H_U)| \leq 260\epsilon \text{vol}_H(U) + |U|$ as desired.

For the runtime, note that each phase of the algorithm can be done in $O(|Y_j|^3)$ time, including finding the max-min matching by performing a binary search on the bottleneck edge weight and then using a bipartite matching algorithm [9].

### Algorithm 3.2 Main Algorithm

**Input:** The graph $H$

**Output:** Graph $\tilde{H} = K_3 \times \tilde{G}$ on $V(H)$ with $|E(H)\Delta E(\tilde{H})| \leq O(\epsilon|E(H)|)$, and $\tilde{G}$

Construct graphs $C, T(C)$. Compute connected components of $T(C), Y_1, \ldots, Y_{\ell}$

Set $S = V(H)$ and $\mathcal{F} = \{\}$

for each $Y_j$ in $Y_1, \ldots, Y_{\ell}$ do

- Let $U_j$ be vertices induced by $Y_j$.
  - if two triangles of $Y_j$ share a common vertex then go to $Y_{j+1}$ in for loop
  - Run Algorithm 3.1 on $Y_j$ with $U = U_j \cap S$.
    - if output is FAIL then go to $Y_{j+1}$ in for loop
    - if $|E(H) \cap (U \times (S \setminus U))| \geq 5\epsilon(1 - \epsilon) \min(\text{vol}_H(U), \text{vol}_H(S \setminus U))$ then go to $Y_{j+1}$ in for loop
    - Add factorization of $H[U]$ to $\mathcal{F}$.

Set $S \leftarrow S \setminus U$.

return disjoint union of $\mathcal{F}$.

### 3.5. Proof of Theorem 1.1

We are now ready to prove Theorems 1.1.

**Proof of Theorem 1.1.** We set $S = V(H)$ and loop over the components $Y_j$ of $T(C)$. We shall maintain the invariant that $S$ is atomic. For each component, we run Algorithm 3.1 with $U = U_j \cap S$ (assuming $U \neq \emptyset$), which by Lemma 3.10 must either output FAIL or a factoring of $K_3 \times \tilde{G}_U$ of $H[U]$ with error bounded by $260\epsilon \text{vol}_H(U) + |U|$. If FAIL is output, we know that $Y_j$ cannot be core, so we can continue.

By Proposition 3.8, we know that if $Y_j$ is core then $|E(H) \cap (U \times (S \setminus U))| \leq \frac{5\epsilon}{1 - \epsilon} \min(\text{vol}_H(U), \text{vol}_H(S \setminus U))$. Thus, we can throw out any $Y_j$ which does not have this property.

Since each vertex of $S$ is covered by a core component, we shall accept at least one $Y_j$, we can replace $S$ with $S \setminus U$ and continue in the loop. Note that by the time we loop over all the $Y_j$’s, we must have $S = \emptyset$, because each vertex of $S$ is included in some core component. Further, at each step $U_j$ is atomic so $U_j \cap S$ is atomic and $S \setminus U_j$ is also atomic. Thus, the invariants are maintained throughout the algorithm.

Our final factorization is the disjoint union of all the factorizations found throughout the algorithm. Let $\tilde{H}$ be this graph, and let $U^{(1)}, \ldots, U^{(\ell)}$ be the vertices which induce the components of $\tilde{H}$. The total error is
then (by Proposition 3.8 and Lemma 3.10)

\[
|E(\tilde{H}) \Delta E(H)| \leq \sum_{i=1}^{\ell} \left( |E(\tilde{H}[U^{(i)}]) \Delta E(H[U^{(i)}])| + |E(H) \cap (U^{(i)} \times (U^{(i+1)}) \cup \ldots \cup U^{(\ell)})| \right)
\]

\[
\leq \sum_{i=1}^{\ell} \left( 260\epsilon \text{vol}_H(U^{(i)}) + |U^{(i)}| + \frac{5\epsilon}{1-\epsilon} \text{vol}_H(U^{(i)}) \right)
\]

\[
\leq \sum_{i=1}^{\ell} \left( 520\epsilon + \frac{10\epsilon}{1-\epsilon} \right) |E(H[U^{(i)}])| + |U^{(i)}| = \left( 520\epsilon + \frac{10\epsilon}{1-\epsilon} \right) |E(H)| + |V(H)|
\]

\[
\leq 550\epsilon |E(H)|,
\]

in the last line, we use that the sum of the degrees of the vertices of \( H \) is twice the number of edges and the fact that the \(|V(H)|\) term can be absorbed into the constant for \( \epsilon = \Omega(|V(H)|/|E(H)|) \).

For the runtime, computing the connected components of \( T(C) \) takes \( O(n^6) \) time. Consider iteration \( j \) of the loop. Note that checking if two triangles of \( Y_j \) overlap takes \( O(n) \) time at most. If \(|Y_j| > n\), then the loop iteration takes \( O(|Y_j|) = O(n) \) time, as the unique coverage of the vertices of \( U \) must fail. Otherwise, if \(|Y_j| \leq n\), the loop iteration takes \( O(|Y_j|^3 + n^2) = O(n^3) \) time as each \(|Y_j| = O(n)\). We know that \( \ell = O(n^3) \), so the runtime is \( O(n^6) \).

We remark that the runtime of our algorithm does not depend on \( \epsilon \). If one does not know \( \epsilon \) before running the algorithm and is only given a graph \( H \), one can binary search on its possible values to find a value for \( \epsilon \) close to the smallest one for which \( H \) is \( \epsilon \)-near some triangle tensor. This only increases the runtime by a factor of \( \log(1/\epsilon) \).

### 3.6. Extensions.

There are a number of settings in which we believe Theorem 1.1 and Theorem 1.2 can be extended. One such setting is allowing \( G \) to be a small-set expander, and we discuss this below in detail.

We propose a few other settings, which require more formal study in order to obtain a reconstruction goal, in Section 5.

**G is a small-set expander.** Recall that for Theorem 1.2, we prove that \( H \) can be efficiently 3-colored, if \( G \) is a 3\( \epsilon \)-edge-expander by showing that there can be at most one core triangle component, which we can efficiently 3-color by Lemma 3.10. If instead a graph \( H \) has the property that there are at most \( k \) core components, we can in \( n^{O(k)} \) time brute force guess which \( k \) are the core ones, determine their colorings and combine them to color \( H \).

We can prove such a property for \( H \) with a broader class of graphs \( G \), which are known as small set expanders. We say that \( G \) is a \((\delta, \alpha)\)-small set expander if for every set \( S \) of size at most \( \delta \cdot n \), we have that

\[
|E(G) \cap S \times (V(G) \setminus S)| \geq \alpha \text{vol}_G(S).
\]

Using methods similar to the proof of Theorem 1.2, we can show that if \( G \) is a \((1/k, \Omega(\epsilon))\)-small set expander, then \( H \) has at most \( k \) core components, and thus can be 3-colored in \( n^{O(k)} \) time.

One interesting example here is the \( \beta \)-noisy hypercube, the graph on \( \{0,1\}\)\(\ell\), where nodes are adjacent if and only if their hamming distance is \( \beta \ell \). This graph is a \((1/\log(n), 1/2)\)-small set expander (see for instance, [56]), and we can obtain a 3-coloring in quasi-polynomial time with our algorithm.

### 4. Hardness.

In this section, we will show that certain natural tensor reconstruction criteria are in fact NP-hard to achieve and prove Theorem 1.3. As an immediate corollary of Theorem 1.3, this rules out a monotone reconstruction for tensor graphs.

**Corollary 4.1.** Given as input an \( H \) which is \( \epsilon \)-near \( K_3 \times G \) for some \( G \), it is NP-hard to find a graph \( \tilde{H} = K_3 \times \tilde{G} \) on the same vertex set as \( H \) with \( \tilde{H} \supseteq H \).

**Proof.** Assume we found such an \( \tilde{H} \). Then by applying Imrich’s algorithm, we can factor a copy of \( K_3 \) out of \( \tilde{H} \) and thus determine a 3-coloring of \( \tilde{H} \). Since \( \tilde{H} \supseteq H \), this is also a 3-coloring of \( H \). Thus, by Theorem 1.3, finding \( \tilde{H} \) is NP-hard.

\[\text{[56]}\]
4.1. 3-Coloring with Equality. We show that Theorem 1.3 follows from the hardness of a problem we call “3-coloring with equality.”\footnote{In the hardness of approximation community, this problem can be viewed as a special case of a problem known as Label Cover.} An instance of 3-coloring with equality consists of a set of vertices with two edge sets: \((V, E_\#), E_=\). We call the edges \(E_\#\) coloring constraints and the edges \(E_=\) equality constraints. An assignment is a map \(\psi: V \rightarrow [3]\). We say this assignment satisfies the instance if for all \(\{u, v\} \in E_\#\), we have that \(\psi(u) \neq \psi(v)\) and for all \(\{u, v\} \in E_=\) we have that \(\psi(u) = \psi(v)\).

We say that a vertex \(v \in V\) is \(\epsilon\)-loose if at most an \(\epsilon\) fraction of its neighboring constraints are color constraints. We say an instance of 3-coloring with equality is \(\epsilon\)-loose if every vertex is \(\epsilon\)-loose. We now show that 3-coloring with equality is NP-hard even when the instance is \(\epsilon\)-loose for small \(\epsilon\).

**Lemma 4.2.** Assume that \(\epsilon \in (0, 1)\) is constant. Given an instance \((V, E_\#, E_=)\) of 3-coloring with equality which is \(\epsilon\)-loose, it is NP-hard to determine if the instance is satisfiable.

**Proof.** We reduce from the NP-hardness of 3-coloring. Let \(G := (V_3, E_3)\) be an instance of 3-coloring. Let \(d_v\) be the degree of each \(v \in V_3\). We now define our instance \(H := (V, E_\#, E_=)\) of 3-coloring with equality.

For each \(v \in V_3\), create \(n_v := \lceil d_v / \epsilon \rceil\) new vertices \(v_1, \ldots, v_{n_v}\) which we call \(v\)’s cloud vertices. Let \(C_v\) be this set of cloud vertices.

We then define
\[
V = V_3 \cup \bigcup_{v \in V_3} C_v
\]
\[
E_\# = E_3
\]
\[
E_= = \bigcup_{v \in V_3} \bigcup_{v' \in C_v} \{v, v'\}.
\]

First, note that \(H = (V, E_\#, E_=)\) is \(\epsilon\)-loose, as each vertex corresponding to a vertex of \(G\) is \(\epsilon\)-loose. Further, each of the cloud vertices has a single equality constraint and thus is \(\epsilon\)-loose.

Now, if \(G\) is 3-colorable, then \(H\) can be satisfied by copying the coloring for the copies of the vertices in \(G\), and then assigning the cloud vertices to have the same color as the vertex they are adjacent to. Likewise, if \(H\) is satisfied, the cloud vertices can be deleted to yield a 3-coloring of \(G\). Thus, we have completed the reduction.

4.2. The Tensor Reduction. We give an efficient reduction from an instance of 3-coloring with equality to a tensor reconstruction problem (such as in \([11, 6]\)). Let \((V, E_\#, E_=)\) be our instance of 3-coloring with equality, and let \((V', E')\) be the graph for our tensor reconstruction problem.

Our vertex set is \(V' = V \times [3]^3\), that is there are 27 copies of each vertex. For each \(v \in V\) and \(x, y \in [3]^3\), we have that \(\{(v, x), (v, y)\} \in E'\) if and only if \(x_i \neq y_j\) for all \(i \in [3]\). In other words, the subgraph induced by \(\{v\} \times [3]^3\) is isomorphic to the graph product of 3 copies of \(K_3\).

For each \(\{u, v\} \in E_\#\) and \(x, y \in [3]^3\), we have an edge between \(\{(u, x), (v, y)\} \in E'\) if and only if \(x_i \neq y_j\) for all \((i, j) \in K_3 := ([3], \neq)\). For each \(\{u, v\} \in E_=\) and \(x, y \in [3]^3\), we have \(\{(u, x), (v, y)\} \in E'\) if and only if \(x_i \neq y_i\) for all \(i \in [3]\).

Theorem 1.3 follows from the following two observations (which are proved in the following subsections).

**Lemma 4.3** (Completeness). For \(c_{\text{loose}} < 1/3\), given a \(c_{\text{loose}}\)-loose instance \((V, E_\#, E_=)\) of 3-coloring with equality which is satisfiable, the tensor reduction produces a graph \(H\) which is \(\epsilon\)-near \(K_3 \times G\) for some \(G\).

**Lemma 4.4** (Soundness). Given an instance \((V, E_\#, E_=)\) of 3-coloring with equality with no satisfying assignment, the tensor reduction produces a graph \(H\) which is not 3-colorable.

**Proof of Theorem 1.3.** Let \(A\) be an algorithm for 3-coloring graphs \(\epsilon\)-near \(K_3 \times G\) for some \(G\). By Lemma 4.2, it suffices to show that a polynomial-time modification of \(A\) can solve \(c_{\text{loose}}\)-loose instances of 3-coloring with equality.

Our modified algorithm \(A'\) is to take an instance \((V, E_\#, E_=)\) of 3-coloring with equality, apply the tensor reduction (which is polynomial-time) to produce a graph \(\tilde{H}\) and then apply \(A\) to \(\tilde{H}\). We then check that the output of \(A\) is indeed a 3-coloring of \(\tilde{H}\). If so, we output “satisfiable.” Otherwise, we output “unsatisfiable.”
If the 3-coloring with equality instance is satisfiable, by Lemma 4.3, we have that $\bar{H}$ is $\epsilon$-near a tensor with $K_3$. Thus, $A$ will output a valid 3-coloring, and thus we shall output “satisfiable” for this 3-coloring with equality instance.

On the other hand, if the 3-coloring with equality instance is not satisfiable, then $\bar{H}$ is not even 3-colorable by Lemma 4.4. Thus, even if $A$ outputs a coloring, it is not a valid 3-coloring, and thus we shall output “unsatisfiable.”

Thus, finding a 3-coloring of a graph $\epsilon$-near $K_3 \times G$ is NP-hard. □

4.3. Completeness: Proof of Lemma 4.3. Let $\psi : V \rightarrow [3]$ be a satisfying assignment to our $c_{\text{loose}}$-loose instance $(V, E_\neq, E_\approx)$ of 3-coloring with equality. Let $H$ be the graph produced by the tensor reduction. We show how to approximately factor this graph as a $K_3$ times another graph $G$. We first build the graph $G$ and then construct the approximate isomorphism between $K_3 \times G$ and $H$.

The vertices of $G$ are $V \times [3]^2$. For any $v \in V$ and $x, y \in [3]^2$, we have that $\{(v, x), (v, y)\} \in E(G)$ if and only if $x_1 \neq y_1$ and $x_2 \neq y_2$. Likewise, for any $\{(u, v), (w, z)\} \in E(G)$ if and only if $u_1 \neq w_1$ and $x_2 \neq w_2$. For each $\{(u, v)\} \in E_\neq$ and $x, y \in [3]^2$, we always have that $\{(u, v)\} \in E(G)$.

We now construct a map $\pi : [3] \times V(G) \rightarrow V(H)$. For each $(v, x) \in V(G)$ and $y \in [3]$, we map the pair $(y, (v, x))$ based on the value of $\psi(v)$:

$$
\pi(y, (v, x)) = \begin{cases} 
(v, (y, x_1, x_2)) & \psi(v) = 1 \\
(v, (x_1, y, x_2)) & \psi(v) = 2 \\
(v, (x_1, x_2, y)) & \psi(v) = 3.
\end{cases}
$$

We now show that $E(H) \subseteq E(\pi(K_3 \times G))$, and that the lack of equality is only due to edges added to $G$ via edges of $E_\neq$.

Observe that for any edge $\{(y, y')\} \in K_3$ and $\{(v, x), (v, x')\} \in V(G)$, we have that the edge arising from the tensor product $\{(\pi(y, (u, x)), \pi(y', (v, x'))\}$ is an edge of $H$ as $y \neq y', x_1 \neq x_1'$ and $x_2 \neq x_2'$. Further, this is a bijection between the subgraph of $H$ induced by $\{(v)\} \times [3]^2$ and the tensor product of $K_3$ with the subgraph of $G$ induced by $\{v\} \times [3]^2$.

Likewise, if $\{(u, v)\} \in E_\neq$ and $\{(y, y')\} \in K_3$ and $\{(u, x), (v, x')\} \in V(G)$, we have that the corresponding product $\{(\pi(y, (u, x)), \pi(y', (v, x'))\}$ is an edge of $H$. Further, the edges of $H$ arising from $\{(u, v)\} \in E_\neq$ are in bijection with the tensor product of $K_3$ with the subgraph of $G$ with the edges added due to $\{(u, v)\}$.

If $\{(u, v)\} \in E_\neq$, we no longer have such a bijection between $K_3$ times the edges added to $G$ and the edges added to $H$. However, we can prove that every such edge of $H$ corresponds to an edge in $K_3 \times G$. Assume without loss of generality that $\psi(u) = 1$ and $\psi(v) = 2$. Consider $x, y \in [3]^2$ with $x_1 \neq y_j$ for all $(i, j) \in K_3$. Then, note that

$$
\pi^{-1}(u, x) = (x_1, (u, (x_2, x_3)))
$$

$$
\pi^{-1}(v, y) = (y_2, (v, (y_1, y_3)))
$$

As stated $x_1 \neq y_2$, so $\{(x_1, y_2)\} \in K_3$. Further, $\{(u, v)\} \in E_\neq$ so $\{(u, (x_2, x_3)), (v, (y_1, y_3))\} \in E(G)$.

Therefore, $E(H) \subseteq E(\pi(K_3 \times G))$, and that the lack of equality is only due to edges added to $G$ via edges of $E_\neq$.

To finish, for every $(u, v) \in V(H)$, we need to bound the number of incident edges in $E(\pi(K_3 \times G)) \setminus E(H)$. For each $\{(u, v)\} \in E_\neq$, there are 18 edges between $(u, x)$ and vertices of the form $(v, y) \in \pi(K_3 \times G)$. Further, for each $(u, v) \in E_\approx$, there are 8 edges between $(u, x)$ and vertices of the form $(v, y) \in V(H)$. Finally, there are 8 edges between $(u, x)$ and vertices of the form $(u, y) \in V(H)$.

Let $a_u = |\{v \in V : \{(u, v)\} \in E_\neq\}$ and $b_u = |\{v \in V : \{(u, v)\} \in E_\approx\}|$. Since our 3-coloring with equality instance is $c_{\text{loose}}$-loose, we have that $b_u \leq c_{\text{loose}}(a_u + b_u)$. Therefore, the fraction of edges of $\pi(K_3 \times G)$ incident to $(u, x)$ which are in $E(\pi(K_3 \times G)) \setminus E(H)$ is bounded by

$$
\frac{18b_u}{8 + 8a_u + 18b_u} \leq \frac{18b_u}{8a_u + 8b_u} = \frac{18}{8} < 1/3,
$$

as $c_{\text{loose}} < 1/3$.

Therefore, the graph $H$ produced by the tensor reduction is $\epsilon$-near $K_3 \times G$, as desired.
4.4. Soundness: Proof of Lemma 4.4. We prove the soundness via the contrapositive. Assume there exists a 3-coloring of the graph formed via the tensor reduction. That is, there exists a map \( \psi: V \times [3]^3 \rightarrow [3] \) forming a 3-coloring. A key result, taken from the literature, is that for each \( v \in V \), the induced map \( \psi(v, -): [3]^2 \rightarrow [3] \) can be unambiguously decoded to a single color \( \phi(v) \in [3] \).

Claim 1 ([23]). Given a graph \( T \) which is the tensor of \( L \geq 1 \) copies of \( K_3 \) and a 3-coloring \( c: [3]^4 \rightarrow [3] \) of \( T \), there exists an edge between \( x, y \) if and only if for any \( x, y \in [3]^3 \) with \( x_1 = x, y_2 = y \) and \( x_1 \neq y_1 \) for some \( i \), \( 1 \leq i \leq 3 \).

Proof of Lemma 4.4. For each \( v \in V \), let \( i_v \) and \( \eta_v \) be such that for all \( x \in [3]^4 \), \( \psi(v, x) = \eta_v(x_{i_v}) \). For all \( v \in V \), define \( \phi(v) = i_v \). We claim that \( \phi \) is a satisfying assignment to the instance \((V, E_{\neq}, E_{=})\) of 3-coloring with equality.

First, consider any equality constraint \((u, v) \in E_{=}\). We seek to show that \( \phi(u) = \phi(v) \). Assume for sake of contradiction that \( \phi(u) \neq \phi(v) \). Without loss of generality, we can assume that \( \phi(u) = 1 \) and \( \phi(v) = 2 \). This is NP-hard by a result of Dailey [10].

Second, consider any coloring constraint \((u, v) \in E_{\neq}\). We seek to show that \( \phi(u) = \phi(v) \). Assume for sake of contradiction that \( \phi(u) = \phi(v) \). Without loss of generality, we can assume that \( \phi(u) = \phi(v) = 1 \).

Thus, for any \( x, y \in [3]^3 \), we have that \( \psi(u, x) = \eta_v(x_{\phi(u)}) = \eta_v(x_1) \) and \( \psi(v, y) = \eta_v(y_2) \).

Note that for any choice of \( x_1, y_2 \in [3] \), there exists \( x, y \in [3]^3 \) with \( x_1 = x_1, y_2 = y_2 \) and \( x_1 \neq y_1 \) for some \( i \), \( 1 \leq i \leq 3 \). Thus, \( \eta_v(x_1) \neq \eta_v(y_1) \) for all \( x_1, y_1 \in [3] \), a contradiction. Therefore, \( \phi(u) \neq \phi(v) \). Therefore \( \phi \) is a satisfying assignment of the 3-coloring with equality instance.

This completes the proof of Theorem 1.3. We now prove the following corollary.

**Corollary 4.5.** For any \( \delta \in (0, 1) \), given as input a graph \( \tilde{H} \) on \( N \) vertices which is \( \epsilon = N^{-\delta} \)-near \( K_3 \times G \) for some \( G \), it is NP-hard to find a 3-coloring of \( \tilde{H} \).

**Proof.** We perform the same reductions as Theorem 1.3 from 3-coloring to 3-coloring with equality to tensor reconstruction. However, we need to verify that each step of the reduction accommodates non-constant \( \epsilon \). We start with an instance \((V, E)\) of 3-coloring on \( n \) vertices such that every vertex has degree 4. This is NP-hard by a result of Dailey [10], but \( \epsilon = n^{-\eta} \) for some \( \eta > 0 \) to be specified later. In particular, each vertex \( v \in V \) becomes a cloud of \([4n^\eta]\) vertices. Thus, the instance \((V', E_{\neq}', E_{=}')\) of 3-coloring with equality has \( |V'| \leq n(1 + [4n^\eta]) \leq 6n^{1+\eta} \) vertices and is \( n^{-\eta}\)-loose.

The tensor reduction then produces an instance \( \tilde{H} \) of tensor reconstruction on \( N := 27(6n^{1+\eta}) \) vertices. By Lemma 4.3 and Lemma 4.4, it is NP-hard to distinguish whether \( \tilde{H} \) is \( \epsilon' = n^{-\eta} \)-near a \( K_3 \)-tensor or is not 3-colorable. In terms of \( N \), we have that \( \epsilon' = O(N^{(1+\eta)/n}) \). If we pick \( \eta \) sufficiently large such that \( \frac{n}{1+\eta} \) is strictly greater than \( \delta \in (0, 1) \), then \( \epsilon' < N^{-\delta} \), as desired.

5. Conclusions and Open Questions. Inspired by finding novel tractable instances of the 3-coloring problem, in this paper we have studied the efficient 3-colorability of graphs that are approximately of the form \( K_3 \times G \). In particular, if \( G \) is a mild expander, then 3-coloring is indeed possible. However, it is NP-hard for general \( G \), although weaker tensor reconstruction criteria hold, such as the \( \ell_1 \) reconstruction goal achieved in Theorem 1.1. There are a number of directions which could be pursued to extend the results in this paper. We list a few of the most promising ones.

First, we believe one can extend our results to more general settings. We discuss a few such settings that seem like very tractable future work, as we think here, one can make use of our techniques.

**Allowing deletions and insertions.** One can change the model from strictly deletions to allowing deletions and insertions, so long as the insertions respect the tensor structure. In other words, the adversary can insert an edge between \( u \) and \( v \) only if \( \psi_{K_3}(u) \neq \psi_{K_3}(v) \) and \( \psi_G(u) \neq \psi_G(v) \). We observe that this model can be reduced to the deletion model that we have studied.\(^{16}\) Let \( G' \supset G \) be the graph where \( g' \) and \( g' \) are connected if there is an edge added between and \( u \) and \( v \) with \( \psi_G(u) = g \) and \( \psi_G(v) = g' \). Observe that \( H \), the graph

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\(^{16}\)A notable analogy in coding theory is that any error-correcting code for a deletion channel is also an error-correcting code for the insertion-deletion channel (cf. [41]).
the adversary produced, is a subset of $P' := K_3 \times G'$. Further, for any vertex $u$ of $H$, the number of incident edges in $P' \setminus H$ is at most $6\epsilon |\Gamma_{P'}(u)| \leq \frac{6\epsilon}{1-2\epsilon} |\Gamma_{P}(u)|$. Thus, we believe one can analyze $H$ as the result of deleting $O(\epsilon)$ fraction of the edges from each vertex of $P'$, and perhaps obtain similar guarantees to those in Theorems 1.1 and 1.2.

Further, one can allow insertions of edges $(u, v)$ for which $\psi_C(u) = \psi_C(v)$. At most two such edges can be added per vertex and so $\epsilon$ can be increased very slightly in each of the inequalities to accommodate these edges. Thus, we conjecture that the most general insertion model our algorithm can withstand is one in which the original color classes are respected.

Extending past $K_3$. We believe that one can use analogous arguments to the ones presented in this section to replace $K_3$ with any fixed size core graph, which is a graph such that every homomorphism on it is an isomorphism [27]. Some examples of core graphs are cliques and an odd-length cycles. Call this fixed graph $F$, and let $f$ denote the number of vertices in $F$. In these settings, there are an analogue of core triangles. First, one can build an analog of the graph $C$ that connects nodes that are candidates to be from the same component—that is, the sizes of the pairwise neighborhoods are proportional to the sizes of the pairwise neighborhoods of $F$. Then, one could form the equivalent of the graph $T(C)$ by considering tuples of size $f$ that form copies of $F$ in $C$, and such tuples are compatible with each other if they have the edges between them in $H$ that we would expect from the tensor. We conjecture that one can then prove that these components are either “core” or “monochrome,” though it is likely that the constants and runtime will get worse as $f$ increases.

**Question 1.** Let $F$ be a fixed size clique or an odd-length cycle and let $G$ be a $O(\epsilon)$-expander. Let $H$ be $\epsilon$-near $F \times G$. Given $H$, can one efficiently find a tensor graph $\bar{H}$ which is $\epsilon$-near $F \times G$?

A spectral variant of Imrich’s algorithm. The algorithms we present in this paper are highly combinatorial. Yet, as evidenced by Theorem 1.2, the spectral properties of $G$ are essential for learning some structure of the underlying graph. Thus, a natural goal would be to find a truly spectral variant of Imrich’s algorithm (e.g., via semidefinite programming) which allows for robust reconstruction. In particular, the following result could potentially result from such an investigation.

**Question 2.** Let $F$ and $G$ be $O(\epsilon)$-expanders. Let $H$ be $\epsilon$-near $F \times G$. Given $H$, can one efficiently find a tensor graph $\bar{H}$ which is $\epsilon$-near $F \times G$?

Perhaps related to this question is the work of Trevisan [48] which relates high-quality approximations of MAX CUT to spectral properties of the input graph.

Alternative reconstruction goals. We show that when $H$ is $\epsilon$-near $K_3 \times G$, we can find a tensor $\bar{H} = K_3 \times \bar{G}$ with $|E(\bar{H}) \Delta E(H)| \leq O(\epsilon|E(H)|)$. Since deletions are bounded on each vertex, another seemingly reasonable reconstruction goal is to try to find $\bar{H}$ that is close to $H$ on every vertex, e.g. for all $v \in H$, $|\Gamma_H(v) \Delta \Gamma_{\bar{H}}(v)| \leq O(\epsilon|\Gamma_H(v)|)$. This is just one other example reconstruction goal, and there may be other natural, tractable reconstruction goals for robust tensor factoring that are worth defining and studying.

Extensions to other deletion models. The deletion model we assumed in this paper is that $\epsilon$ fraction of the edges incident to each vertex were deleted. Another natural model is that $\epsilon$ fraction of the total edges are deleted. Analyzing this model appears more difficult as many vertices could have a large fraction of their neighbors deleted, which cause our heuristics to fail.

Another natural model to consider is random deletions with i.i.d. deletion probability $\epsilon$. With high probability, the results of the main theorems apply, but it may be possible to 3-color random deletions of $K_3 \times G$ for an adversarial $G$. “Semi-random” graph coloring problems such as this have been of interest in the literature (c.f., [17]).

Interesting subclasses of 3-colorable graphs. We might be able to better understand hardness of the 3-coloring problem by studying interesting special cases. Indeed, this was the motivation behind our work, as well as that of Blum in coloring random 3-colorable graphs [5], Arora and Ge [2] in studying 3-colorable graphs with low threshold rank, and Kawarabayashi and Thorup [38] in focusing on graphs with high degree. Based on these previous works, a natural extension of these ideas is the following question:

**Question 3.** What other structural properties of a 3-colorable graph imply one can color it with few (at most $O(\log n)$) colors efficiently?
Deep connections to coloring. This study was inspired by trying to better understand how to color 3-colorable graphs with a small number of colors. An important direction is to better understand the implications our results have for the approximate coloring problem. For instance, if a graph can be efficiently partitioned into approximate tensors, can we use our reconstruction algorithm on the approximate tensor pieces and some other meta algorithm on the full graph. More generally, we ask the following:

**Question 4.** Can we bootstrap our algorithm to give approximate colorings of general graphs?

Stronger hardness results. One may seek to strengthen Theorem 1.3 by imposing a weaker reconstruction criteria than finding a 3-coloring of $\tilde{H}$. By adapting the methods of [6], it should be possible to show that it is NP-hard to find even a 4-coloring of $\tilde{H}$. However, the approach we take is useless even for six colors, per observations of [6, 3], unless one makes conditional assumptions like in [11].

One may also seek to instead show that it is NP-hard to reconstruct $\tilde{H}$ with respect to some reconstruction metric. We are currently unaware of a method to obtain such results.

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