The Hopf Skyrmion in QCD with Adjoint Quarks

S. Bolognesi\(^{(1)}\) and M. Shifman\(^{(2)}\)

\(^{(1)}\) The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
\(^{(2)}\) William I. Fine Theoretical Physics Institute, University of Minnesota, 116 Church St. S.E., Minneapolis, MN 55455, USA

Abstract

We consider a modification of QCD in which conventional fundamental quarks are replaced by Weyl fermions in the adjoint representation of the color SU\((N)\). In the case of two flavors the low-energy chiral Lagrangian is that of the Skyrme–Faddeev model. The latter supports topologically stable solitons with mass scaling as \(N^2\). Topological stability is due to the existence of a nontrivial Hopf invariant in the Skyrme–Faddeev model. Our task is to identify, at the level of the fundamental theory, adjoint QCD, an underlying reason responsible for the stability of the corresponding hadrons. We argue that all “normal” mesons and baryons, with mass \(O(N^0)\), are characterized by \((-1)^Q (-1)^F = 1\), where \(Q\) is a conserved charge corresponding to the unbroken U(1) surviving in the process of the chiral symmetry breaking \((\text{SU}(2) \rightarrow \text{U}(1)\) for two adjoint flavors). Moreover, \(F\) is the fermion number (defined mod 2 in the case at hand). We argue that there exist exotic hadrons with mass \(O(N^2)\) and \((-1)^Q (-1)^F = -1\). They are in one-to-one correspondence with the Hopf Skyrmions. The transition from nonexotic to exotic hadrons is due to a shift in \(F\), namely \(F \rightarrow F - \mathcal{H}\) where \(\mathcal{H}\) is the Hopf invariant. To detect this phenomenon we have to extend the Skyrme–Faddeev model by introducing fermions.
1 Introduction

In connection with the recent work on planar equivalence between supersymmetric and non-supersymmetric gauge theories, and orientifold large-$N$ expansion [1], there is a renewed interest in dynamics of Yang–Mills theories with massless quarks in representations other than fundamental. In particular, the orientifold large-$N$ expansion is based on the fact that the quarks in the two-index antisymmetric representation at $N = 3$ are identical to conventional fundamental quarks.

If the number of flavors $N_f > 1$, the theory under consideration has a chiral symmetry which is spontaneously broken. The pattern of the chiral symmetry breaking ($\chi$SB) depends on particular representation to which the massless fermions of the theory belong [2, 3, 4]. For $N_f$ Dirac fermions in the two-index antisymmetric representation the pattern of $\chi$SB is identical to that of QCD, namely,

$$\text{SU}(N_f)_L \times \text{SU}(N_f)_R \rightarrow \text{SU}(N_f)_V$$ (1)

As a result, the low-energy limit is described by the same chiral Lagrangian as in QCD, with the only distinction that in the case of two-index antisymmetric quarks the pion constant $F_\pi$ scales as

$$F_\pi^2 \sim N^2,$$

while in QCD it scales linearly (in QCD $F_\pi^2 \sim N$). As was noted in [5], this seemingly minor difference leads to a crucial consequence. Indeed, in both theories there exist Skyrmion solitons [6]. The issue of Skyrmions in QCD is well-studied. Witten showed [7, 8] that Skyrmions in QCD can be identified with baryons, their topological charge being in one-to-one correspondence with the baryon charge of the microscopic theory (for reviews see [9]). The Skyrmion mass is proportional to $F_\pi^2$ and grows linearly with the number of colors. This is fully compatible with our intuition since baryons in multicolor QCD are composed of $N$ quarks. At the same time the very same dependence of the Skyrmion mass on $F_\pi^2$ in the theory with the two-index antisymmetric quarks implies that the Skyrmion mass grows as $N^2$ rather than $N$.

At first sight this is totally counterintuitive since in this case, just as in QCD, one can construct a color-singlet baryon from $N$ antisymmetric quarks.

This puzzle formulated in [5] was successfully solved in [10] (for follow-up works see [11]). It turns out that if one considers $N$-quark colorless bound
states, not all quarks can be in the $S$-wave state, and, as a result, the mass of such objects scales as $N^{7/6}$ [10]. The minimal number of quarks of which one can build a particle with all quarks in the $S$-wave state is $N^2$. This $N^2$ quark particle is the stable state described by the Skyrmion. As for the $N$-quark particles they are unstable with respect to fusion of $N$ species into one $N^2$ quark state, with a huge release of energy in the form of pion emission.

It is clear that with massless quarks in higher representations of the gauge group, we deal with very interesting and intriguing dynamics calling for further investigation. In this work we will continue in this direction. We will consider $N_f$ massless Weyl (or, which is the same, Majorana) fermions in the adjoint representation of the SU($N$) gauge theory. These fermions will be referred to as “quarks” or “adjoint quarks.” To ensure a chiral symmetry in the fundamental Lagrangian on the one hand, and to keep the microscopic theory asymptotically free on the other, we must impose the following constraints on the value of $N_f$:

$$2 \leq N_f \leq 5.$$  \hspace{1cm} (2)

The pattern of the $\chi$SB in this case is also known [2, 3, 4],

$$SU(N_f) \times \mathbb{Z}_{2 N_f} \rightarrow SO(N_f) \times \mathbb{Z}_2,$$  \hspace{1cm} (3)

where the discrete factors are the remnants of the anomalous singlet axial U(1). Below we will focus on the case $N_f = 2$. A related discussion of the three-flavor case is presented in Ref. [12].

Equation (3) can be elucidated as follows. If we denote the adjoint quark field as $\lambda^a_{\alpha_f}$ (here $a, \alpha, f$ are the color, Lorentz-spinorial, and flavor indices, respectively; we use the Weyl representation), the Lorentz-scalar bilinear $\lambda^a_{\alpha_f} \lambda^{\alpha_a}$ is expected to condense,

$$\langle \lambda^a_{\alpha_f} \lambda^{\alpha_a} \rangle \sim \Lambda^3.$$  \hspace{1cm} (4)

The above bilinear is obviously symmetric with respect to $f, g$; it is a vector in the flavor space. It is convenient to choose the condensate in such a way that the third component of the vector has a nonvanishing vacuum expectation value,

$$\langle \lambda^a_{\alpha} (i \tau_3 \tau_2) \lambda^{\alpha_a} \rangle = 2 \Lambda^3,$$  \hspace{1cm} (5)

where $\tau_i$ ($i = 1, 2, 3$) stand for the Pauli matrices acting in the flavor space (the same matrices acting in the spinor space will be denoted by $\sigma_i$). Equation
\[ N_f = 2 \quad N_f = 3 \quad N_f = 4 \quad N_f = 5 \]
\[
\begin{array}{cccc}
\pi_3 & Z & Z_4 & Z_2 \\
\end{array}
\]

Table 1: The third homotopy group for sigma models emerging in Yang–Mills with two, three, four and five adjoint flavors.

Equation (5) implies that
\[ \langle \lambda_{a_1}^{\alpha} \lambda_{a_2}^{\alpha} \rangle = \Lambda^3, \]  
with other flavor components vanishing.

The above order parameter stays intact under those transformations from SU(2) which generate rotations around the third axis in the flavor space. There is one such transformation,
\[ \lambda \rightarrow e^{i\alpha \tau_3} \lambda. \]  

Thus, the low-energy pion Lagrangian is a nonlinear sigma model with the target space \( \mathcal{M} \) given by the coset space
\[ \mathcal{M}_2 = SU(2)/U(1) = S^2. \]  

This is the famous O(3) sigma model which was treated in two dimensions in \[13\]. In four dimensions, with quartic in derivatives terms included (for stabilization), this is the model \[14\] which is sometimes referred to as the Skyrme–Faddeev and sometimes as the Faddeev–Hopf model. Soliton’s topological stability is due to the existence of the Hopf invariant. In fact, the Faddeev–Niemi solitons are knots \[14\]. For generic values of \( N_f \) the sigma model is defined on the coset space
\[ \mathcal{M}_{N_f} = SU(N_f)/SO(N_f). \]  

The starting point of the present work is the observation that the third homotopy group for \( (9) \) is nontrivial in all four cases \( (2) \), as shown in Table \( \text{II} \).\footnote{This is also true for SO(\( N \)) Yang–Mills theories with vectorial quarks in which the structure of the chiral Lagrangians and solitons is similar. In this case the Skyrmion was identified with the baryon \[8\]. Its \( \mathbb{Z}_2 \) stability is due to the fact that the \( \epsilon \) tensor cannot be written as the sum of \( \delta \)'s.}
Hence, topologically stable solitons (whose mass scales as $N^2$) exist much in the same way as Skyrmions in QCD. Unlike QCD, where the relation between the Skyrmions and microscopic theory is well established, in our case it is far from being clear. How can one interpret the topologically stable solitons of the low-energy effective Lagrangian from the standpoint of the fundamental theory with the adjoint quarks? With regards to stability, is it an artifact of the low-energy approximation? If no, what prevents these particle whose mass scales as $N^2$ from decaying into “light” color-singlet mesons and baryons with mass $O(N^0)$? What plays the role of the QCD baryon charge?

Below we will answer these and similar questions. We will argue that the Hopf Skyrmion stability is not a low-energy artifact; rather it can be understood in the fundamental theory – Yang–Mills with two adjoint quarks. This is due to the fact that all conventional mesons and baryons with $m = O(N^0)$ in the theory at hand have

\[( -1)^Q \cdot ( -1)^F = 1, \tag{10} \]

while for the Hopf Skyrmion

\[( -1)^Q \cdot ( -1)^F = 1. \tag{11} \]

Here $Q$ is the conserved global $U(1)$ charge which survives after the spontaneous breaking, and $F$ is the fermion number. Note that the fermion number per se is not conserved in the microscopic theory under consideration, but $( -1)^F$ is.

Organization of the paper is as follows. Section 2 summarizes known facts about QCD with adjoint quarks as well as our findings regarding the existence of exotic stable hadrons with mass $O(N^2)$ in the two-flavor model. In Sect. 3 we discuss the low-energy limit of two-flavor adjoint QCD. We present the corresponding chiral Lagrangian and discuss its features. In Sect. 4 we extend the model by introducing appropriate fermion fields. In Sect. 5 we calculate the induced fermion charge.

2 General considerations

The $SU(2)$ flavor group is broken down to the $U(1)$ subgroup generated by the Pauli matrix $\tau_3$. The unbroken $U(1)$ is the exact global symmetry of the
microscopic theory acting on the adjoint quark field as indicated in Eq. (7). Correspondingly, the U(1) charge (to be referred to as $Q$) of the quark of the first flavor is $+1$ while that of the second flavor is $-1$. It is obvious that the condensate (6) is neutral.

There are two Nambu–Goldstone bosons, $\pi^{++}$ and $\pi^{--}$. Roughly, $\pi^{++} \sim \lambda_1 \lambda_1$, $\bar{\lambda}_2 \bar{\lambda}_2$ and $\pi^{--} \sim \lambda_2 \lambda_2$, $\bar{\lambda}_1 \bar{\lambda}_1$. (Note that complex conjugation raises the flavor index). Two of four possible linear combinations produce massive scalar mesons $\sigma$ while two others produce massless pseudoscalar pions. Note that, although we use the Weyl formalism for the fermions, parity is conserved in the microscopic theory under consideration at any $N_f$.

The pion U(1) charges are

$$Q(\pi^{\pm\pm}) = \pm 2. \quad (12)$$

If we rely on a constituent quark model, as we do in QCD, we could say that for a generic hadronic state

$$Q = 2 \, J \pmod{2}, \quad (13)$$

where $J$ is the spin of the given hadron. Alternatively this relation can be represented as

$$(-1)^Q = (-1)^{2J}. \quad (14)$$

The current corresponding to the conserved U(1) charge in our conventions can be written as

$$J_{\mu}^{U(1)} \equiv \frac{1}{2} \langle \sigma_\mu \rangle_{\alpha \dot{\alpha}} \, J^{U(1)}_{\alpha \dot{\alpha}}, \quad (15)$$

The second linear combination,

$$J_{\mu}^{F, \alpha \dot{\alpha}} = \bar{\chi}^a \, \dot{\alpha} \, 1 \lambda_1^a \alpha - \bar{\chi}^a \, \dot{\alpha} \, 2 \lambda_2^a \alpha. \quad (16)$$

can be called the fermion current. Classically it is conserved too. However, at the quantum level, due to the chiral anomaly,

$$\partial_\mu \, J^{F}_{\mu} = \frac{NN_f}{16\pi^2} \, F_{\mu\nu}^{a} \, \tilde{F}^{a}_{\mu\nu}. \quad (17)$$
For two adjoint flavors \( N_f = 2 \) in Eq. (17). Thus, the fermion number \( F \) is not conserved. A discrete subgroup \( \mathbb{Z}_{2N_f} \) survives from (17). The nonvanishing condensate (5) breaks \( \mathbb{Z}_{2N_f} \) down to \( \mathbb{Z}_2 \). This means that nonconservation of \( F \) is quantized, \( |\Delta F| = 2 \). In other words, \( (-1)^F \) is conserved.

For all “ordinary” hadrons which can be produced from the vacuum by local currents, say, (16) or (19), determination of \( (-1)^F \) is straightforward. This is not the case, however, for hadrons build of \( \sim N^2 \) constituents, see below.

Now, we can decompose the Hilbert space of hadronic excitations in the direct sum of two spaces

\[
\mathcal{H}^{\text{(hadronic)}} = \mathcal{H}^{(+1,+1)} \oplus \mathcal{H}^{(-1,-1)}
\]

containing, respectively, the composite states with the even and odd \( U(1) \) charges. We have denoted the charges as \( ((-1)^Q, (-1)^F) \). From the point of view of the hadronic Hilbert space (18) this would appear a redundant notation. It will soon be clear that this is not.

Relying on the same constituent quark model for orientation, we would say that \( \mathcal{H}^{(+1,+1)} \) contains hadronic excitation of the boson type while \( \mathcal{H}^{(-1,-1)} \) of the fermion type. In particular, \( \mathcal{H}^{(+1,+1)} \) contains the massless Nambu–Goldstone bosons \( \pi^{\pm \pm} \) and, hence, there is no mass gap here. On the contrary, \( \mathcal{H}^{(-1,-1)} \) has a mass gap \( m \), the mass of the lightest composite fermion of the type

\[
\psi_{\beta f} = C \text{Tr} \left( \lambda_1^{\alpha} F_{\alpha \beta} \right) \equiv C \text{Tr} \left( \lambda_0^{\alpha} \sigma^{\mu \nu} F_{\mu \nu} \right),
\]

where \( F_{\alpha \beta} \) is the (anti)self-dual gluon field strength tensor (in the spinorial notation), and \( C \) is a normalizing factor,

\[
C \sim (N \Lambda^2)^{-1}.
\]

Two \( U(1) \)-charge \( \pm 1 \) composite fermions are

\[
\psi_\pm = C \text{Tr} \left( \lambda_1^\alpha F_{\alpha \beta} \right),
\]

(plus their antiparticles, of course). Note that \( \psi_- \) is not \( \psi_+ \)'s antiparticle. Moreover, we can combine \( \psi_+ \) and \( \psi_- \) in a single Dirac spinor \( \Psi_D \),

\[
\Psi_D = \left\{ \psi_1, -i \sigma_2 (\psi_2)^* \right\}.
\]
Here $\sigma_2$ lowers the spinorial dotted index. This construction is useful for comparison of our results with calculations one can find in the literature, see e.g. Ref. [15]. Although this work is intended for description of somewhat different (albeit related) physics, its mathematical aspect overlaps with that of our analysis\(^2\).

Below we will argue that in fact Eq. (18) is incomplete. An extra sector can and must be added,

$$\mathcal{H} = \mathcal{H}^{(\text{hadronic})} \oplus \mathcal{H}^{(\text{exotic})},$$

where $\mathcal{H}^{(\text{hadronic})}$ is given by (18) and the new sector is given by

$$\mathcal{H}^{(\text{exotic})} = \mathcal{H}^{(+1,-1)} \oplus \mathcal{H}^{(-1,+1)}.$$

$\mathcal{H}^{(\text{exotic})}$ includes hadrons with even $Q$ and odd $F$ and vice versa, odd $Q$ and even $F$. To build such a hadron one needs $\sim N^2$ constituents. In adjoint QCD they play the role of baryons of conventional QCD. Their existence is reflected in the Hopf-Skyrmions.

### 3 The Skyrme–Faddeev model

Now, let us briefly review the effective low-energy pion Lagrangian corresponding to the given pattern of the $\chi$SB, see Eq. (3) with $N_f = 2$. We describe the pion dynamics by the $O(3)$ nonlinear sigma model (in four dimensions)

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{2} \partial_\mu \vec{n} \cdot \partial^\mu \vec{n} + \text{higher derivatives}$$

where the three-component field $\vec{n}$ is a vector in the flavor space subject to the condition

$$\vec{n}^2 = 1,$$

and a higher derivative term is needed for stabilization. The “plain” vacuum corresponds to a constant value of $\vec{n}$ which we are free to choose as $\langle n_3 \rangle = 1$. Due to (25) the target space of the sigma model at hand is $S^2$.

Usually the higher derivative term is chosen as follows (for a review see [17]):

$$\delta \mathcal{L}_{\text{eff}} = -\frac{\lambda}{4} (\partial_\mu \vec{n} \times \partial_\nu \vec{n}) \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n}).$$

\(^2\) For a supersymmetric follow up see [16].
Equations (24) and (26) constitute the Skyrme–Faddeev (or the Faddeev–Hopf) model. Note that the Wess–Zumino–Novikov–Witten (WZNW) term [18] does not exist in this model.

To have a finite soliton energy the vector $\vec{n}$ for the soliton solution must tend to its vacuum value at the spatial infinity,

$$\vec{n} \to \{0, 0, 1\} \text{ at } |\vec{x}| \to \infty .$$

(27)

Two elementary excitations near the vacuum $n_3 = 1$,

$$\frac{1}{\sqrt{2}} (n_1 \pm i n_2),$$

can be identified with the pions. The boundary condition (27) compactifies the space to $S^3$. Since $\pi_3(S^2) = \mathbb{Z}$, see Table 1, solitons present topologically nontrivial maps of $S^3 \to S^2$. As was noted in [14], there is an associated integer topological charge $N$, the Hopf invariant, which presents the soliton number. This charge cannot be the degree of the mapping $S^3 \to S^2$ because dimensions of $S^3$ and $S^2$ are different.

Figure 1: The simplest Hopf soliton, in the adiabatic limit, correspond to a Belavin-Polyakov soliton closed into a donut after a $2\pi$ twist of the internal phase.

The solitons in the the Skyrme–Faddeev model are of the knot type. The simplest of them is toroidal, it looks as a “donut,” see e.g. [17]. Qualitatively it is rather easy to understand, in the limit when the ratio of the periods is a large number, that the Hopf topological number combines the instanton number in two dimensions, with twist in the perpendicular third dimension.

Let us slice the “donut” soliton by a perpendicular plane $AB$. In the vicinity of this plane the soliton can be viewed as a cylinder, so that the
problem becomes effectively two-dimensional. In two dimensions the O(3) sigma model has the Polyakov–Belavin instantons \[13\] whose topological stability is ensured by the existence of the corresponding topological charge. The Polyakov–Belavin instanton has an orientational collective coordinate describing its rotation in the unbroken U(1) subgroup (for a review see \[19\]). In two dimensions for each given instanton this collective coordinate is a fixed number. In the Hopf soliton of the type shown in Fig. 1 as we move the plane \(AB\) in the direction indicated by the arrow, this collective coordinate changes (adiabatically), so that the \(2\pi\) rotation of the plane \(AB\) in the direction of the arrow corresponds to the \(2\pi\) rotation of the orientational modulus of the Polyakov–Belavin instanton. This is the twist necessary to make the Hopf soliton topologically stable.

Another way to understand the geometry of the Hopf map is given in Fig. 2. Let us visualize the space \(R^3\) as a “book.” Every “page” of this book is a semi-infinite plane attached to the axial line. The axial line plus the points at infinity are mapped onto the north pole of the target space \(S^2\). The preimage of the south pole is a circle linked with the axial line. Every semi-infinite plane is wrapped around the target space. The U(1) phase, which in this picture is the rotation of \(S^2\) that keeps fixed the north and south poles, is twisted while the semi-infinite plane is rotated around the axial line.

The Hopf charge cannot be written as an integral of any density which is local in the field \(\vec{n}\). However, if one uses a U(1) gauged formulation of the CP(1) sigma model in terms of the doublet fields \(z^\alpha, \bar{z}_\alpha (\alpha = 1, 2)\) and

\[
\bar{z}_\alpha z^\alpha = 1, \quad \vec{n} = -\bar{z} \vec{\tau} z, \tag{28}
\]
(for a review see \[20\]), in this formulation the Hopf invariant reduces to the Chern–Simons term for the above gauge field \[14\],

$$\mathcal{H} = \frac{1}{4\pi^2} \int d^3x \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho),$$

(29)

$$A_\mu = -\frac{i}{2} \bar{z} \partial_\mu z$$

(30)

(see Sect. 4). Spherically symmetric field configurations automatically have vanishing Hopf charge.

One can ask whether the Hopf Skyrmion represents a hadron with integer or semi-integer spin. Quantization of the Hopf solitons was considered in the literature previously \[21, 22\], with the conclusion that it can be quantized both as a boson and as a fermion. The possibility of the fermionic quantization is due to the Finkelstein–Rubinstein mechanism \[23\]. In both cases Eq. (13) is valid.

Since we would like to understand the relation between $Q$ and $F$ we need to introduce (composite) fermions in the Skyrme–Faddeev model. The low-energy description in terms of the $\vec{n}$ field knows nothing about baryons of the type (19). It is clear that it is impossible to decide this issue without including such fermions in the low-energy chiral model. This is done in Sect. 4.

4 Introducing massive fermions

The Hopf soliton is topologically stable due to nontriviality of the homotopy group $\pi_3 (S^3) = \mathbb{Z}$. However, the Skyrme–Faddeev model neglects excitations belonging to the odd sector of the Hilbert space, baryons of the type (19), whose mass is $O(\Lambda)$ and does not scale with $N$. Let us now switch them on. One can take baryons into account through the SU(2) invariant coupling $\psi^{\alpha_f} \bar{\vec{n}} \cdot (\vec{\tau})^g_f \psi_{\alpha_g} + \text{h.c.}$ We can write an effective Lagrangian that includes both the pions and the baryons,

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{2} \partial_\mu \bar{\vec{n}} \cdot \partial^\mu \vec{n} + \bar{\psi}_f^\dagger i \partial^\alpha \psi_{\alpha_f} - \frac{g}{2} \left\{ \psi^{\alpha_f} \bar{\vec{n}} \cdot (\vec{\tau})^g_f \psi_{\alpha_g} + \text{h.c.} \right\} + \ldots$$

(31)

where $g$ is a coupling constant of dimension of mass (which will be assumed to be positive). Similar (but not identical) couplings were considered in the
literature previously, see e.g. Ref. [24]. The Lagrangian (31) is obviously SU(2) invariant. The \( \psi \) baryon mass is generated in the process of \( \chi \)SB. Namely, if in the vacuum \( n_3 = 1 \), the baryon term becomes

\[
L_{\text{ferm}} = \bar{\psi}_f i \partial^{\alpha} \psi_{\alpha f} - g \{\psi_{\alpha 1}^2 \psi_{\alpha 2} + \text{h.c.}\} ,
\]

(cf. Eq. (5)) and the mass \( M_\psi^2 = g^2 \). The bilinear pion terms are

\[
\mathcal{L} = (\partial_\mu \pi^{++}) (\partial_\mu \pi^{--}) .
\]

Expanding near \( n_3 = 1 \) we get interactions terms. For what follows it is convenient to transfer the \( \psi \psi n \) interaction from the potential to the kinetic term. To this end let us introduce an SU(2) flavor matrix \( U \) such that

\[
U^\dagger (\vec{n} \cdot \vec{\tau}) U = \tau_3 ,
\]

where \( U \) is a function of \( \vec{n} \). It is not uniquely defined, of course. One can always choose it in the form

\[
U = \exp (i \vec{\nu} \cdot \vec{\tau}_\perp) , \quad \nu_i \propto \varepsilon_{ij} n_\perp j .
\]

Here \( \vec{\nu} \) and \( \vec{n}_\perp \) are two-dimensional vectors,

\[
\vec{\nu} = \{\nu_1, \nu_2\}, \quad \vec{n}_\perp = \{n_1, n_2\}
\]

while \( \vec{\tau}_\perp = \{\tau_1, \tau_2\} \). Furthermore, if

\[
\psi_f \equiv U^g f \chi_g , \quad \psi^f \equiv \chi^g (U^\dagger)^f_g
\]

the fermion part of the Lagrangian takes the form

\[
\mathcal{L}_{\text{ferm}} = \bar{\chi}_a \left( i \partial^{\alpha} + A^{\alpha a} \right) \chi_{\alpha} - g \{\chi_{\alpha 1}^2 \chi_{\alpha 2} + \text{h.c.}\} ,
\]

\[
A_\mu \equiv i U^\dagger \partial_\mu U \equiv A^{(i)}_\mu \tau_i .
\]

Written in terms of \( \chi \)'s the interaction terms have a generic form \( \bar{\chi} n \ldots \partial n \ldots n \chi \).

The fermion current we are interested in is

\[
J^{F0}_{\alpha \hat{\alpha}} = \bar{\chi}_a \chi_{\alpha f} , \quad F = \int d^3 x J^0
\]
The meaning of the superscript 0 will become clear shortly. The current (39) is flavor-singlet since the summation over \( f \) is implied. Naively, one would say that states containing odd number of \( \chi \)'s have \( F = 1 \), mod 2, while those with even number of \( \chi \)'s (in particular, no \( \chi \)'s at all) have \( F = 0 \), mod 2. The mass term in Eq. (37) satisfies the constraint \(|\Delta F| = 2|\). 

The question we ask is whether a state build from \( n \)'s can have \( F = 1 \), mod 2, and vice versa, a state containing a \( \chi \) quantum \( F = 0 \), mod 2. Such states cannot be produced from the vacuum by local sources constructed from (a finite number of) \( \chi \) and \( n \) fields. We want to show, however, that they do exist in the sector with nontrivial values of the Hopf invariant.

To see that this is the case let us examine the divergence of \( J_{\mu}^{F0} \). Naively one would say that \( \partial^\mu J_{\mu}^{F0} \) is given by the classical mass term in Eq. (37). A closer inspection reveals an anomaly. Indeed, the U(1) symmetry corresponding to rotations around the third axis in the flavor space is a strictly conserved global symmetry in the model at hand. Maintaining conservation of the corresponding current produces an anomalous contribution in \( \partial^\mu J_{\mu}^{F0} \), as it usually happens with the triangle graphs. Calculation of the anomaly is a straightforward task, given that we keep only \( A_{\mu}^{(3)} \) in Eq. (37) treating \( A_{\mu}^{(3)} \) as an external field,

\[
\partial^\mu J_{\mu}^{F0} = \frac{1}{8\pi^2} F_{\mu\nu}^{(3)} \tilde{F}_{\mu\nu}^{(3)} + \text{classical term}, \tag{40}
\]

where

\[
F_{\mu\nu}^{(3)} = \partial_\mu A_{\nu}^{(3)} - \partial_\nu A_{\mu}^{(3)}, \tag{41}
\]

and the extra \( 1/2 \) in the coefficient \( (8\pi^2 \text{ in the denominator instead of } 4\pi^2) \) reflects the fact that \( \chi \) is the Weyl rather than Dirac spinor.

Now let us take into account the fact that

\[
F_{\mu\nu}^{(3)} \tilde{F}_{\mu\nu}^{(3)} = 2 \partial_\mu K_\mu, \quad K_\mu = \epsilon^{\mu\nu\alpha\beta} A_{\nu}^{(3)} \partial_\alpha A_{\beta}^{(3)}. \tag{42}
\]

Here \( K_\mu \) is the Chern-Simons current. Thus, we see that the genuine fermion current is

\[
J_{\mu}^{F} = J_{\mu}^{F0} - \frac{1}{4\pi^2} K_\mu \tag{43}
\]

rather than \( J_{\mu}^{F0} \). The fermion charge \( F \) is shifted accordingly.
5 The impact of $K_\mu$

Consider first the Skyrme–Faddeev theory in 2+1 dimensions. The conserved topological current of the Belavin–Polyakov (BP) instanton \cite{13} is

$$j^\mu_{\text{BP}} = \frac{1}{8\pi} \varepsilon^{\mu\nu\rho} \varepsilon^{abc} n^a \partial_\nu n^b \partial_\rho n^c ,$$  \hspace{1cm} (44)

normalized so that $\int d^2 x j^0_{\text{BP}} = 1$ on the instanton of topological charge 1 (see e.g. \cite{19}, Sect. 6).

![Adiabatic twist](image)

Figure 3: A 2+1 dimensional (cylindrical) field configuration with an adiabatically twisted BP instanton corresponding to the unit Hopf invariant.

The conservation of $j^\mu_{\text{BP}}$ is due to the condition $\vec{n}^2 = 1$. Since $\partial_\mu j^\mu_{\text{BP}} = 0$, we can introduce an auxiliary field $a_\mu$ defined as follows:

$$j^\mu_{\text{BP}} = \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho .$$  \hspace{1cm} (45)

Then, the Hopf invariant can be expressed as \cite{25}

$$\mathcal{H} = \int d^3 x \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho .$$  \hspace{1cm} (46)

It correspond to a Chern-Simon term\footnote{If a term $\theta \mathcal{H}$ is added to the action, the BP particle acquires a spin equal to $\theta/2\pi$} for the auxiliary gauge field $a_\mu$. The Hopf invariant is normalized in such a way that it is 1 on the BP field.
configuration of topological charge 1 which adiabatically rotates around the third axis by the overall angle $2\pi$ as we move from $-\infty$ to $+\infty$ in the perpendicular direction (Fig. 3). This can be shown by explicitly evaluating the integral (46) for the adiabatic rotation and then by using the fact that the Hopf term is topological and depends only on the homotopy class of the field configuration [25].

Furthermore, it is crucial that $a_\mu$ in Eq. (46) coincides up to a normalization factor with $A_\mu$ introduced in Eq. (30) which in turn exactly coincides with $A_\mu^{(3)} = (i/2) \text{Tr}(\tau_3 U^\dagger \partial_\mu U)$, see Eq. (38). The latter statement follows from a straightforward calculation which we present in Appendix. As for the former, let us note that the $z, \bar{z}$ fields realize the SU(2) target space. Division by U(1) is implemented through gauging of U(1), with no kinetic term for the $A_\mu$ field (see e.g. [20]). The following identification

$$a_\mu = \frac{1}{2\pi} A_\mu$$  (47)

ensues [20, 21].

Thus, we conclude that the Hopf invariant is

$$\mathcal{H} = \frac{1}{4\pi^2} \int d^3 x \epsilon^{\mu\nu\rho} A^{(3)}_\mu \partial_\nu A^{(3)}_\rho = \frac{1}{4\pi^2} \int d^3 x K_0.$$  (48)

The same conclusion was achieved in [24] where the author used the formalism of the Dirac spinors (cf. Eq. (21)) in a slowly-varying background field configurations which are relevant for sufficiently wide solitons, and the Goldstone–Wilczek diagrammatic technique [28].

Comparing (43) with (48) we see that the induced fermion number $F$ for a minimal Hopf Skyrmion is 1. In general,

$$F = F_0 - \mathcal{H}.$$  (49)

In a related situation the shift of the fermion charge by the Hopf invariant in the background of the knot soliton was demonstrated in [15]. Although the model of Ref. [15] deals with a single Dirac fermion, from the purely mathematical standpoint core calculations run in parallel.

At the same time, the current corresponding to the U(1) charge $Q$,

$$J^{U(1)}_{\alpha\bar{\alpha}} = \bar{\chi}_{\dot{\alpha}} \tau_3 \chi_\alpha$$  (50)
is anomaly-free. This implies that there is no induced $Q$ charge.

Now let us return to the issue of the stability of the Hopf Skyrmion and the hadrons it represents. The theory under consideration contains three kinds of hadrons whose quantum number assignments are summarized in Table 2. The Hopf Skyrmion can have charges $Q$ and $F$ respectively 0 and 1, if there is no fermion zero mode crossing in the process of evolution from the topologically trivial background to that of the Hopf Skyrmion, or 1 and 0 if there is a fermion zero mode crossing.

In both cases the lightest exotic hadrons represented by the Hopf Skyrmion are stable. They can not decay in any number of pions and/or pions plus “ordinary” baryons with mass $O(N^0)$. Note that this is a $Z_2$ stability. Two Hopf Skyrmions can annihilate and decay into an array of $\pi$’s and $\psi$’s. For nonexotic hadron excitations which can be seen in a constituent model and have mass $O(N^0)$ the combination $Q + F$ is always even while for exotic hadrons with mass $O(N^2)$ the sum $Q + F$ is odd.

### 6 Conclusions

We considered the low-energy limit of QCD with two adjoint quarks, summarized by a chiral Lagrangian corresponding to $\chi$SB of the type SU(2) → U(1). This Lagrangian supports topologically stable solitons described previously by Faddeev and Niemi. The stability of these solitons is due to the existence of the Hopf invariant.

We addressed the question of whether one can understand stability of the corresponding hadrons within the framework of the underlying microscopic

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3 A relevant discussion of the fermion zero mode crossings in 2+1 dimensions can be found in [29].
theory. There is a singly strictly conserved quantum number $Q$ in this theory. We observe, however, that although the fermion number is not conserved, $(-1)^F$ is. We argued that the hadrons represented by the Hopf Skyrmions are exotic in the sense that they have an induced fermion number coinciding with the Hopf number, so that unlike all “ordinary” hadrons they are characterized by negative $(-1)^{Q+F}$. This is the underlying reason explaining their stability.

Unlike the “pion” part of the Lagrangian (31) which is unambiguously fixed by symmetries, the “baryon” part (32) does not seem to be unique. Indeed, the theory under consideration has infinitely many fermionic interpolating operators, besides the one presented in (20). For example, any operator of the type $\text{Tr}(\lambda \ldots \lambda F \ldots)$ with an odd number of $\lambda$’s represents a baryon. One could pose the question of completeness. We would like to claim that the Lagrangian (31) and (32) is complete in the sense that it captures all relevant dynamics — to answer the question we pose there is no need to include in it additional baryon operators. The point is that any other baryons with mass $O(N^0)$ (which are necessary unstable particles, resonances) have a projection on the operator (20). The inclusion of additional baryon terms would have no impact on our result since they would not change the anomaly (40).

There are two obvious questions calling for future investigations:

(i) if one introduces a mass term for the adjoint quarks, how do the Hopf Skyrmions evolve as the mass term increases?

(ii) when we quantize the Hopf Skyrmions and consider excitations over the lowest-lying soliton, what is the quantum number assignments for the tower of the excitations?

Acknowledgments

We are grateful to Roberto Auzzi for useful discussions.

This work was supported in part by DOE grant DE-FG02-94ER408. M.S. would like to thank the Niels Bohr Institute in Copenhagen, where this work began, for kind hospitality extended to him in July 2006. S.B. was funded by the Marie Curie grant MEXT-CT-2004-013510 and in part by FTPI, University of Minnesota. S.B. wants to thank FTPI for the hospitality in the fall of 2006, when this work was completed.
Appendix

Let $U(x)$ be a matrix which rotates the vector $\vec{n}(x)$ to align it along the third direction at every given point $x$,

$$U^\dagger (\vec{n} \cdot \vec{\tau}) U = \tau_3.$$  \hfill (A.1)

Assume that we choose some point $x_0$ and the corresponding matrix $U(x_0)$. Consider $x_0$ as given and fixed.

In the vicinity of $x_0$,

$$x = x_0 + \delta x,$$

the matrix $U(x)$ can be decomposed in two factors

$$U(x) = U(x_0) \delta U(\delta x),$$  \hfill (A.2)

where the space-time dependence is now enclosed only in $\delta U(\delta x)$. Since $\delta U(0) = 1$, we can perform the expansion $\delta U(\delta x) = \exp \{ i (\nu_1 \tau_1 + \nu_2 \tau_2) \}$ where $\nu_{1,2}$ depend on $\delta x$. We will expand the exponent keeping the leading relevant terms. Then

$$n_1 = -2\nu_2, \quad n_2 = 2\nu_1,$$

$$n_3 = 1 - 2(\nu_1^2 + \nu_2^2).$$  \hfill (A.3)

The two-component $z$ appearing in the gauged definition of the sigma model is

$$z = \begin{pmatrix} \varepsilon \\ 1 - \frac{1}{2} |\varepsilon|^2 \end{pmatrix}, \quad \varepsilon = \nu_2 + i\nu_1.$$  \hfill (A.4)

The vector $A_\mu$ defined in Eq. (30) takes the form

$$A_\mu = -\nu_1 \leftrightarrow_\mu \nu_2.$$  \hfill (A.5)

At the same time, the vector $A$ defined in Eq. (38) is

$$A_\mu = i U^\dagger \partial_\mu U = -\tau_3 \left( \nu_1 \leftrightarrow_\mu \nu_2 \right) + \text{terms with } \tau_{1,2}.$$  \hfill (A.6)

Thus we have $A_\mu = A_\mu^{(3)}$ that, in the original frame, means

$$A_\mu = A_\mu^{(i)} n^{(i)}.$$  \hfill (A.7)
References

[1] A. Armoni, M. Shifman and G. Veneziano, From super-Yang-Mills theory to QCD: Planar equivalence and its implications, in From Fields to Strings: Circumnavigating Theoretical Physics, Eds. M. Shifman, A. Vainshtein, and J. Wheater, (World Scientific, Singapore, 2005), Vol. 1, p. 353 [hep-th/0403071]; Phys. Rev. D 71, 045015 (2005) [hep-th/0412203].

[2] S. Dimopoulos, Nucl. Phys. B 168, 69 (1980); M. E. Peskin, Nucl. Phys. B 175, 197 (1980).

[3] Y. I. Kogan, M. A. Shifman and M. I. Vysotsky, Yad. Fiz. 42, 504 (1985) [Sov. J. Nucl. Phys. 42, 318 (1985)].

[4] J. J. Verbaarschot, Phys. Rev. Lett. 72, 2531 (1994) [hep-th/9401059]; A. Smilga and J. J. Verbaarschot, Phys. Rev. D 51, 829 (1995) [hep-th/9404031]; M. A. Halasz and J. J. Verbaarschot, Phys. Rev. D 52, 2563 (1995) [hep-th/9502096].

[5] A. Armoni and M. Shifman, Nucl. Phys. B 670, 148 (2003) [hep-th/0303109].

[6] T. H. R. Skyrme, Proc. Roy. Soc. Lond. A 262, 237 (1961); Nucl. Phys. 31, 556 (1962). See also Selected Papers of Skyrme, Ed. G. Brown (World Scientific, Singapore, 1994); for a review see Chiral Solitons, Ed. Keh-Fei Liu (World Scientific, Singapore, 1987).

[7] E. Witten, Nucl. Phys. B 223, 422 (1983).

[8] E. Witten, Nucl. Phys. B 223, 433 (1983) [reprinted in S. Treiman et al., Current Algebra and Anomalies, (Princeton University Press, 1985), p. 515.]

[9] A.P. Balachandran, Skyrmions, in High Energy Physics 1985, Eds. M. J. Bowick and F. Gürsey, (World Scientific, Singapore, 1985), Vol. 1, p. 1.

[10] S. Bolognesi, Baryons and Skyrmions in QCD with Quarks in Higher Representations, [hep-th/0605065]

[11] A. Cherman and T. D. Cohen, The Skyrmion Strikes Back: Baryons and a New Large N∞ Limit, [hep-th/0607028 Phys. Lett. B 641, 401 (2006) hep-th/0607110].
[12] R. Auzzi and M. Shifman, *Low-Energy Limit of Yang–Mills with Massless Adjoint Quarks: Chiral Lagrangian and Skyrmions*, hep-th/0612211.

[13] A. M. Polyakov and A. A. Belavin, JETP Lett. 22, 245 (1975) [Pisma Zh. Eksp. Teor. Fiz. 22, 503 (1975)].

[14] L. D. Faddeev and A. J. Niemi, Nature 387, 58 (1997) hep-th/9610193.

[15] L. Freyhult and A. J. Niemi, Phys. Lett. B 557, 121 (2003) hep-th/0212053.

[16] L. Freyhult, Nucl. Phys. B 681, 65 (2004) hep-th/0310261.

[17] N. Manton and P. Sutcliffe, *Topological Solitons*, (Cambridge University Press, 2004), Sect. 9.11.

[18] J. Wess and B. Zumino, Phys. Lett. B 37, 95 (1971); S. P. Novikov, *Multi-valued Functions and Functionals. An Analogue of the Morse Theory*, Dokl. Akad. Nauk SSSR 260, 31 (1981) [Soviet Math. Doklady, 24, 222 (1981)]; E. Witten, Commun. Math. Phys. 92, 455 (1984).

[19] For a review see e.g. V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Rept. 116, 103 (1984), Sect. 5.2.

[20] E. Witten, Nucl. Phys. B 149, 285 (1979).

[21] J. Gladikowski and M. Hellmund, Phys. Rev. D 56, 5194 (1997) hep-th/9609035.

[22] S. Krusch and J. M. Speight, Commun. Math. Phys. 264, 391 (2006) hep-th/0503067.

[23] D. Finkelstein and J. Rubinstein, J. Math. Phys. 9 (1968) 1762.

[24] T. Jaroszewicz, Phys. Lett. B 146, 337 (1984); Phys. Lett. B 159, 299 (1985).

[25] F. Wilczek and A. Zee, Phys. Rev. Lett. 51, 2250 (1983).

[26] Y. S. Wu and A. Zee, Phys. Lett. B 147, 325 (1984).

[27] Z. Hloušek, D. Senechal and S. H. H. Tye, Phys. Rev. D 41, 3773 (1990).

[28] J. Goldstone and F. Wilczek, Phys. Rev. Lett. 47, 986 (1981).
[29] M. Carena, S. Chaudhuri and C. E. M. Wagner, Phys. Rev. D 42, 2120 (1990); M. Carena, T. ter Veldhuis, C. E. M. Wagner and S. Chaudhuri, Phys. Lett. B 252, 101 (1990).