Atomic Theory of Collective Excitations in Bose-Einstein Condensation and Spontaneously Broken Gauge Symmetry

S. J. Han

P.O. Box 4671, Los Alamos, NM 87544-4671

Abstract

A theory of collective excitations in Bose-Einstein condensation in a trap is developed based on the quantum Hamilton-Jacobi equation of Bohm and the phase coherence along with the idea of off-diagonal long range order of Penrose and Onsager. First, we show that a free surface behaves like a normal fluid - a breakdown of superfluidity. Second, inside the free surface it is shown that the spectrum of phonons is of the form $\omega = ck$ scaled with the external potential, where the speed of (first) sound, $c = [4\pi a \hbar^2]^{1/2}/M$ and $k$ is the wave number. Third, in the limit $a \to 0$, the hard spheres in the Bose-Einstein condensation collapse to a close-packed classical lattice with the zero-point vibrational motion about fixed points.

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The new Bose-Einstein condensation (BEC) in a trap is one of the remarkable phenomena exhibited by a macroscopic system of Bose gas at low temperature \[1\]. In his seminal paper on a theoretical model of a Bose gas \[2\], Bogoliubov has laid out the basis for much of our understanding of superfluidity of a system of weakly interacting Bose particles. The concept of BEC is generalized to strongly interacting systems by Penrose and Onsager based on the notion of off-diagonal long range order (ODLRO) \[3\] which gives rise to superfluidity in He II. Many properties of an interacting Bose system can be understood in terms of independent quasi-particle excitations (phonons and rotons) of the system. These excitations are essential for a complete understanding of certain properties of the interacting Bose system, in particular, the longitudinal excitation spectrum in the phonon regime is crucial for a quantitative experimental confirmation of BEC in a trap.

However, the Bogoliubov theory of collective excitations (first sound) differs so basically in its formulation from that of Feynman’s atomic theory of two-fluid model \[4\] and Landau’s two-fluid model \[5\] that the problem of the collective excitations in BEC represents the major stumbling block in our attempt to confirm the new BEC in terms of the Bogoliubov theory. For instance, the new Bose condensates in a trap are extremely small in size (\(i.e., a\) liquid droplet of tens of microns in radius), and unlike He II they are highly compressible and have a nonuniform density profile. In addition, the new BEC is dilute (mean particle density \(\bar{\rho}\) satisfies \(\bar{\rho}a^3 \ll 1\) \[7\]). It is therefore obvious that the formal extension of Bogoliubov’s theory to the new BEC is difficult on account of its inhomogeneous density profile. Furthermore, it is impossible in our study of the collective excitations to make use of the elegant quantum field theory technique developed in many-body theory which is valid only for a uniform system \[2, 6\].

The new BEC is, however, the simplest of all condensed many-body entities from the microscopic point of view. The atoms in a trap may be treated as a weakly interacting Bose gas which can be accurately described by a self-consistent Hartree equation with the well-known scattering length in the hard sphere approximation for a pair-interaction \[8\].

On the experimental side, the first systematic experimental study on the collective excitation was recently carried out in a long \(^{87}\text{Rb}\) Bose condensate in which the sound wave propagation can be treated approximately as a 1-D problem \[9\]. Yet the experimental data yield a qualitatively correct dispersion curve of the Bogoliubov spectrum \(\omega = ck\) with the estimated speed of first sound \(c_{eff} = 2.0 \pm 0.1 mm sec^{-1}\). Before the ideas of the new BEC
in a trap become tenable, it will be necessary to show experimentally that the collective excitation spectrum in the new BEC indeed agrees well with the Bogoliubov spectrum with the proper geometrical corrections \[2, 3\].

Theory, in the absence of credible experiments, could wander off down a blind alley \[7\]. And yet it is the theoretical insight that could unravel how Nature creates such a unique object with a peculiar property. Our analysis shows a remarkable new phenomenon - a breakdown of superfluidity at the free surface of BEC in a trap. This may be interpreted by broken gauge symmetry \[10, 12, 13\] and is precisely the reason why the speed of first sound was difficult to measure \[9\].

The object of this paper is, first, to show how Bohm’s quantum theory \[14\] can be applied to the problem of collective excitations in BEC in such a way as to incorporate Feynman’s atomic theory of two-fluid model \[4\] and the concept of phase coherence in ODLRO, and, secondly, to present a demonstration that symmetry breaking is taking place at the free surface with the first-order symmetry breaking perturbations in Lagrangian coordinates \[17, 18, 19\]. It is shown here for the first time that the dispersion relation for the surface wave is a manifestation of the broken symmetry at the free surface. The symmetry breaking also accompanies massless phonons (Nambu-Goldstone bosons) \[13\], the spectrum of which is shown to be the same in form as that of the Bogoliubov dispersion relation \(\omega = ck\) scaled by the external trapping force with a small geometrical correction, where the speed of (first) sound, \(c = [4\pi a\rho \hbar^2]^{1/2}/M\) and \(k\) is the wave number \[2\].

Since the perturbation techniques of quantum field theory \[2, 6\] cannot be applied to a finite, spatially inhomogeneous system, we must find a new perturbation method that yields the Bogoliubov spectrum of collective excitations in the new BEC in a trap. Here we present straightforward, but mathematically precise, first-order perturbations by particle displacements in Lagrangian coordinates to an ensemble of particle trajectories defined by the condition of BEC that automatically include all the excited modes which tend to be otherwise omitted. In the present analysis, the problem of collective excitations reduces to a boundary-value problem for a BEC droplet in a trap bounded by a free surface. The present analysis also gives a good account of the possible formation of a close-packed lattice of a trapped Bose gas.

The essential point of ODLRO is that BEC (and superfluidity) be described as a state
in which the density matrix can be factorized as
\[
\rho(x, x') = \psi^\dagger(x)\psi(x') + \gamma(|x - x'|),
\]  
(1)
where \( \gamma \to 0 \) as \( |x - x'| \to \infty \). Here the single particle wave function \( \psi(x) \) (mean field) represents the Bose condensed state \[21\] and \( \int \psi^\dagger \psi dx = N \), where \( N \) is the number of particles in BEC. With the the hard sphere approximation of the inter-particle repulsive force for Bose particles \[8\], one can show that the mean field satisfies the nonlinear Schrödinger equation (Gross-Pitaevskii),
\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2M} \nabla^2 \psi + V(x)_{\text{ext}} + g|\psi|^2 \psi,
\]  
(2)
where \( g = 4\pi\hbar^2 a/M \) and \( a \) is the s-wave scattering length. Eq. (2) is a self-consistent Hartree equation for the Bose condensed wave function. This short-range interaction brings about the Bose condensed state also ensures that the system possess the longitudinal collective excitations (phonons) in BEC. It should be noted that the nonlinear term \(|\psi|^2\) is invariant under a \( U(1) \) group transformation. Hence we can apply Bohm’s interpretation of quantum theory to Eq. (2) with the understanding that it is a part of the potential defined in his quantum theory.

We write the equations for the ensemble average energy in the usual quantum theory \[14, 15, 16\]:
\[
\mathcal{H} = \int \psi^\dagger \left( -\frac{\hbar^2}{2M} \nabla^2 + V(x)_{\text{ext}} + \frac{g}{2}|\psi|^2 \right) \psi dx,
\]  
(3a)
\[
\mathcal{E}_{\text{ave}} = \int \left( \frac{\hbar^2}{2M} |\nabla \psi|^2 + V(x)_{\text{ext}} |\psi|^2 + \frac{g}{2} |\psi|^4 \right) dx, \int \psi^\dagger \psi dx = N.
\]  
(3b)

At this point it is possible to show that the ground state density is given in terms of an external potential and the chemical potential upon minimizing the energy functional Eq. (3a) for a fixed number of particles in a trap with the condition \( p = 0 \) (Penrose-Onsager criterion for BEC):
\[
\rho(x) = |\psi(x)|^2 = \frac{M}{4\pi\hbar^2 a}[\mu - V(x)_{\text{ext}}],
\]  
(4)
where \( \mu \) is a Lagrangian multiplier and is the chemical potential. It should be noted that the ground-state wave function Eq. (4) has a nodal surface at the boundary where the symmetry of the Bose system breaks down.
To briefly recapitulate Bohm’s interpretation of quantum theory [14], we write the wave function in the form
\[ \psi(r, t) = f(r, t) \exp\left[\frac{i}{\hbar} S(r, t)\right] = \rho^{1/2}(r, t) \exp\left[\frac{i}{\hbar} S(r, t)\right], \]
where \( S(r, t) \) is an action (or a phase) [15]. We then rewrite Eq. (2) to obtain,
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \frac{\nabla S}{M}) = 0 \quad (5a) \]
\[ \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2M} + V(x) - \frac{\hbar^2}{4M} \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} = 0, \quad (5b) \]
where \( V(x) = V(x)_{\text{ext}} + g\rho \). The last term \( U_{\text{eqmp}} \) of Eq. (5b) is the effective quantum-mechanical potential (EQMP) which plays an important role in an inhomogeneous condensate and fluctuates near the free surface as will be shown shortly.

From the point of view of macroscopic physics, the coordinates and momenta of an individual atoms are hidden variables, which in a macroscopic system manifest themselves only as statistical averages. In the following we show this interpretation by deriving the dynamical equations from Eqs. (5) with \( v(r_0, t) = \nabla S(r_0, t) / M \) for a single particle in condensation represented by the mean field in Eq. (2).

In a dilute Bose gas in a trap, the force derived from the potential \( V(x)_{\text{ext}} \) balances the repulsive force due to \( g\rho(x) \) to create BEC for which \( p(x) = 0 \). Hence in the limit \( \hbar \to 0 \) or with the less stringent condition \( \nabla \rho = 0 \) in a homogeneous medium for which \( U_{\text{eqmp}} = 0 \), \( S(x, t) \) is a solution of the Hamilton-Jacobi equation in Bohm’s theory [23]. Eq. (5a) expresses a probability density of a statistical ensemble for a system. Eq. (5b) implies, however, that the particle moves under the action of the force which is not entirely derivable from the potential \( V(x) \), but which also obtains contributions from \( U(x)_{\text{eqmp}} \). It is the essence of Bohm’s interpretation of quantum theory. In the classical description, the particle follows a classical orbit determined by the Hamilton-Jacobi equation. Eqs. (5) is a function of macroscopic dynamical variables and suggest us how we may choose the most stable particle trajectories (classical orbits) about which we may linearize the equations of motion.

The first step in the development of a theory of collective excitations in BEC is to associate with a single particle wave function in ODLRO with precisely definable, symmetry breaking perturbations to the first order. Since external perturbations are always in the classical domain, the problem of collective excitations is reduced to finding atomic displacements that are consistent with the atomic theory of the two-fluid model of He II by Feynman in
which the exact partition function is given as an integral over particle trajectories, using
his space-time approach to quantum mechanics \[4\]. We wish to stress that the collective
excitation spectrum is also a test of the perturbation method for calculating the longitudinal
excitations of (first) sound in BEC in a trap.

Our perturbation method is based on the following physical picture: the collective coor-
dinates are attached to the particles in motion as the phase coherent sound wave propagates
from the center to the surface of BEC, and that the sound wave must also satisfy the
prescribed initial-boundary conditions at the surface. The surface excitation involves the
breaking up of phase coherence over the entire surface and is accompanied by a dissipation
process of the sound wave. This gives rise a surface energy to conserve the total energy of
an isolated system.

To describe this, we introduce symmetry breaking perturbations to the particles in BEC in
Lagrangian coordinates \[17, 18, 19\], which can be defined as \(x = x_0 + \xi(x_0, t)\). Here \(\xi(x_0, t)\)
is a function of the unperturbed position of a particle in the condensation, and remains
attached to the particle in the condensate as it moves. This semiclassical perturbation
method is in fact a well-defined accurate quantum mechanical approximation scheme which
yields the Bogoliubov spectrum for the collective excitations \[2, 20\].

At this stage we find it more convenient to work with the familiar dynamical equations
instead of Eqs. \[3\]. We also note that the use of the Hamilton-Jacobi equation in solving
for the motion of a particle is only matter of convenience.

If we write \(v(x_0, t) = \nabla S(x_0, t)/M\) - (a solution of the Hamilton-Jacobi equation), we
then obtain the following three dynamical equations from Eqs. \[3\] and \[5\]:

- the equation of motion for a single particle,
  \[
  M \frac{\partial}{\partial t} v + v \cdot \nabla v = -\nabla \mu,
  \]  
  \[\text{(6)}\]

- the equation of continuity,
  \[
  \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho v) = 0,
  \]  
  \[\text{(7)}\]

where \(\rho\) is henceforth interpreted as the number density,

and the equation of state,

\[
\mu(r, t) = \mu_{\text{loc}}[\rho(r, t)] + V_{\text{ext}},
\]  
  \[\text{(8)}\]

where \(\mu_{\text{loc}}[\rho(r, t)]\) is the chemical potential in the local density approximation \[22\].
Next we expand $S$, $v$, and $\rho$ to the first-order,

$$S(r, t) = S(r_0, t) + \xi \cdot \nabla_0 S(r_0, t) \quad (9a)$$

$$v(x, t) = \frac{\partial}{\partial t} \xi \quad (9b)$$

$$\rho(x, t) = \rho(x_0) - \nabla_0 \cdot [\rho(x_0) \xi] = \rho(x_0) + \delta \rho, \quad (9c)$$

where $p(x_0) = 0$ in BEC and $\nabla_0$ denotes the partial derivative with respect to $x_0$ with $\nabla \to \nabla_0 - \nabla_0 \xi \cdot \nabla_0$. Eq. (9c) was derived by substituting Eq. (9b) to Eq. (7) with $\nabla S(x, t) = p(x, t)$ without taking into account of EQMP in Eq. (5b), then integrating over time.

Now it is easy to express from Eqs. (9) the boundary conditions in terms of $\xi$ by the usual definition of incompressibility and irrotational motion of a fluid at the free surface in BEC [25, 26]; they are given mathematically by $\nabla \cdot \xi = 0$ and $\nabla \times \xi = 0$.

The phase coherence is essential to relate the Bose condensed wave function (mean field) in ODLRO to a many-body ground-state wave function, $\xi \cdot \nabla S(x_0, t) = \sum_i \xi_i \cdot \nabla S_0,i(x_0,i, t)$ in our analysis, where $S_0,i(x_0,i, t)$ is the phase of a single Bose particle in the system, and is a necessary and sufficient condition for phase coherence. Hence the collective excitations we study in this paper are phase coherent waves.

The basis of this paper is the set of linear equations, Eqs. (6), (7), (8), and (9) that describe the collective excitations in BEC. It should be remarked that $\psi$ can support the collective excitations (phonons) only if it obeys the nonlinear Schrödinger equation for which we take $\nabla S(x_0, t) = p(x_0, t)$ as a solution of the Hamilton-Jacobi just as the nonlinear Maxwell equations describe radiation processes (photons).

By virtue of Eq. (9c), the chemical potential may be expanded as

$$\mu(x, t) = \mu_{loc}(x_0) + V_{\text{ext}} - \frac{\partial}{\partial \rho} \mu_{loc}[\nabla_0 \cdot (\rho_0 \xi)]. \quad (10)$$

In BEC, $p(x_0) = 0$, and the equation of motion gives

$$\mu_{loc}(\rho_0(x_0)) + V_{\text{ext}} = \text{constant} = \mu_{loc}(\rho_0(0)) = \mu_0. \quad (11)$$

If we now set $V_{\text{ext}} = \omega_0^2 r^2/2$, the density profile describes a spherically symmetric, nonuniform Bose-Einstein condensation (SSNBEc). This is the problem from which our investigation started. For the special example of collective excitations in BEC in trap, we limit our analysis to the problem of SSNBEc.
In order to describe the collective excitations, we linearize Eqs. (6) along with Eq. (9b), Eq. (9c), Eq. (10), and Eq. (11). It is straightforward algebra, though somewhat tedious, to arrive at the first-order equation of motion in $\xi$. The result is

$$\frac{\partial^2}{\partial t^2} \xi = \frac{\mu_0}{M} \nabla \sigma - \omega_0^2 [(\xi \cdot \nabla) r + (r \cdot \nabla \xi)] - \omega_0^2 [\sigma r + \frac{1}{2} r^2 \nabla \sigma].$$

(12)

Here we have taken $V_{\text{ext}}(r) = M \omega_0^2 r^2/2$, $\partial \mu_{\text{loc}}/\partial \rho = 4 \pi \hbar^2 a/M$, $\sigma = \nabla \cdot \xi$, and have also dropped the subscript in $r_0$.

Eq. (12) is a typical second-order (inhomogeneous) partial differential equation subject to the initial-boundary conditions and has two particular solutions, one corresponding to the surface waves and the other to longitudinal sound waves. The general solution to Eq. (12) is a combination of the two solutions.

We begin with the surface phenomena, and shall study later the more complex problem of the longitudinal sound waves in SSNBE. We limit the surface waves by imposing the boundary conditions $\nabla \cdot \xi = 0$ and $\nabla \times \xi = 0$, which simplifies the algebra considerably. This implies that we may solve for the potential flow as $\nabla^2 \chi = 0$ and $\xi_s = -\nabla \chi$. Here the subscript $s$ stands for the surface waves. The solution for $\chi$ is given by

$$\chi(r, t) = \sum_{\ell,m} [\chi_{\ell}^+(t) r^\ell + \chi_{\ell}^-(t) r^{-(\ell+1)}] Y_{\ell,m}(\theta, \phi),$$

(13)

where we may set $\chi_{\ell}^+(t) = 0$ for a quantum liquid droplet. We may now expand $\xi_s$ in terms of three orthogonal vector spherical harmonics [24],

$$\xi_s = \sum_{\ell,m} [\xi_{1s}^{\ell,m}(r, t) a_1 + \xi_{2s}^{\ell,m}(r, t) a_2 + \xi_{3s}^{\ell,m}(r, t) a_3].$$

(14)

Here we have defined the three vector spherical harmonics as

$$a_1 = e_r Y_{\ell,m}(\theta, \phi), \quad a_2 = r \nabla Y_{\ell,m}(\theta, \phi), \quad a_3 = r \times \nabla Y_{\ell,m}(\theta, \phi).$$

It is now only a matter of elementary algebra, by substituting Eq. (14) into Eq. (12) to obtain a dispersion relation,

$$\omega_{\text{surf}}^2 = \ell \omega_0^2.$$  

(15)

Here we have taken $\chi_+(t) = e^{i \omega t}$. It is at once apparent that the dispersion relation for the surface waves is independent of the internal dynamics of the trapped Bose gas (e.g., the
pair interaction potential). Another way of looking at the property of the free surface is to note that the surface waves are driven by the action of external perturbations over the surface of a spherical droplet in equilibrium, similar to that of gravity waves on the surface of a fluid in equilibrium in a gravitational field. More importantly, the dispersion relation is independent of $\bar{\hbar}$ which implies that the free surface behaves like a classical fluid. It is, therefore, evident that Eq. (15) is the first definite proof that the free surface of a superfluid behaves like a normal fluid; it is a peculiarly universal property of the free surface of a superfluid under an external field. Furthermore, Eq. (15) is also an excellent example of broken symmetry in Nature. The breakdown of superfluidity has been already observed at the vortex core in He II. The breakdown of superfluidity at the free surface in rotating He II also resolves a long standing puzzle in Landau’s two fluid model in connection with the question of why a superfluid component rotates with the normal fluid at the free surface.

Since in practice the free surface is never confined to a strictly mathematical surface, the thin surface layer which also contains phonons and particles driven by EQMP,

$$U_{eqmp} = -\frac{\hbar^2}{4M} \left[ \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right] = -\frac{\hbar^2}{M} \frac{\nabla^2 f}{f}. \quad (16)$$

Thus the particles experience a force from $U_{eqmp}$ which fluctuates with the momentum $p = \nabla S$ and energy of a particle near the surface with the degree of divergence $M\omega_0^2 \mu_0 / D^2$ as $D = [\mu_0 - (1/2)M\omega_0^2 r^2] \to 0$ at the free surface as emphasized by Bohm. $U_{eqmp}$, therefore, breaks up phase coherence in the collective excitations in the surface layer.

In our model of an imperfect Bose gas, between each pair of particles there is a hard sphere repulsion of range $a$ and no long-range interaction. This pair-interaction ensures that the system possesses the longitudinal collective excitations (phonons) in BEC. Thus the problem of collective excitations in SSNBEC should be considered as a spherical longitudinal, phase-coherent sound wave propagation. In the phonon regime, the dispersion relation for phonons must be of the form $\omega = ck$ scaled with the external trapping force, where the speed of first sound, $c = [4\pi a \rho \hbar^2]^{1/2} / M$, and $k$ is the wave number. Because of our approximation for the pair-interactions of Bose particles in the system, the presence of the s-wave scattering length $a$ in the speed of (first) sound wave $c$ is essential. Also this approximation should be valid so long as the condition $\bar{\rho}a^3 \ll 1$ is met, a condition which restricts the discussion to the region of temperature near absolute zero, which explains why there cannot be roton.
excitations in BEC in a trap.

Now we are ready to define the compressibility of the fluid as \( \sigma = \nabla \cdot \xi \) from Eq. (9c), and then its spectrum can be obtained from Eq. (12) by taking the divergence on both sides. After straightforward algebra with the help of the vector identity \( \nabla \cdot (r \cdot \nabla \xi) = r \cdot \nabla \sigma + \sigma \) and the condition of superfluidity, \( \nabla \times \xi = 0 \), we obtain

\[
\frac{\partial^2}{\partial t^2} \sigma(r, t) = \frac{1}{2} \omega_0^2 (\alpha^2 - r^2) \nabla^2 \sigma - 3 \omega_0^2 r \frac{\partial}{\partial r} \sigma - 5 \omega_0^2 \sigma, \tag{17}
\]

where \( \alpha^2 = 8 \pi \hbar^2 a \rho_0(0)/(M \omega_0)^2 \).

The solution of the equation is not entirely trivial. Writing \( \sigma(r, t) = S(t) W(r) Y_{\ell,m}(\theta, \phi) \), we obtain the variable separated equations:

\[
\frac{d^2}{dt^2} S(t) + \lambda_n S(t) = 0 \tag{18}
\]

and

\[
(\alpha^2 - r^2)[\frac{1}{r} \frac{d^2}{dr^2}(r W_n(r)) - \frac{\ell(\ell + 1)}{r^2} W_n(r)] - 6r \frac{d}{dr} W_n(r) + (2 \lambda_n/\omega_0^2 - 10) W_n(r) = 0, \tag{19}
\]

where \( \lambda_n \) is a constant of separation. The eigenvalues are determined by Eq. (19), and by the boundary conditions on \( \sigma(r, t) \), i.e., \( \sigma(r, t) = 0 \) at the free surface and the origin. Here we tacitly assume an infinitely small point source at the origin to drive an outgoing spherical sound wave. The speed of first sound \( c = \sqrt{[4 \pi a \rho(r) \hbar^2]^{1/2}/M} \) is the maximum at the origin and becomes zero at the free surface. To see this is indeed the case we recall that the surface layer is no longer a superfluid but is a normal fluid. Hence the phonons can interact with the normal fluid and, upon the dissipation process, give rise to a surface energy.

The solution of Eq. (18) then gives the dispersion relation for the collective excitations with the eigenvalues that are a function of the s-wave scattering length, the radial trap frequency, the peak density at the center of the trap, i.e., the speed of first sound \( c_{ctr} = \sqrt{[4 \pi a \rho(0) \hbar^2]^{1/2}/M} \), the trapping frequency, and the wave number with \( k_\theta \simeq \ell/r \). To show this explicitly, we transform Eq. (19) to the Sturm-Liouville problem and obtain the eigenvalues in terms of a complete set of functions with the orthogonal properties:

\[
\lambda_n = \int_0^b r^2 (\alpha^2 - r^2)^3 \left[ \left( \frac{d}{dr} W_n(r) \right)^2 + \frac{\ell(\ell + 1)}{r^2} W_n^2(r) \right] dr \tag{20}
\]
and with
\[
\int_0^b r^2(\alpha^2 - r^2)^2W_m(r)W_n(r)dr = \delta_{m,n},
\]
(21)

Here \(\lambda_n = (2\lambda_n/\omega_0^2 - 10)\) and \(b = \alpha = [8\pi a h^2 \rho(0)]^{1/2}/(M\omega_0)\) is the radius of the condensate.

This procedure, though somewhat a digression, is the only way we can explicitly show the functional relation between the eigenvalues and the speed of first sound.

Equations (20)-(21) exhibit the main feature of our method: that the eigenvalues are, indeed, a function of the speed of (first) sound, \(b = \sqrt{2/\omega_0} c_{ctr}\), where the sound speed \(c_{ctr} = [4\pi a \rho(0) h^2]^{1/2}/M\), and the trapping frequency \(\omega_0\), for all values of the angular momentum \(\ell\). This method of presentation also shows why the excitation spectrum has not shown the dependence, or lack thereof, the speed of (first) sound in previous studies [7]. The functional relation of the speed of (first) sound and the wave number in the excitation spectrum is also clear in this integral representation. More importantly, it is apparent that the perturbation method by Lagrangian displacement vectors [17, 18, 19] is in fact a well-defined quantum mechanical approximation scheme.

Another important point is that the eigenvalue is a function of the speed of first sound not at the free surface but at the center of SSNBEC. This has a simple physical interpretation: an outgoing spherical sound wave can travel with little energy loss toward the free surface where the sound wave (phonons) interacts only with the normal fluid of the surface layer and completely dissipates giving rise to a surface energy. It is precisely the nature of a superfluid that it cannot interact with phonons, but the phonons will interact with a normal fluid as Mott emphasized [31]. Therefore, the phenomenon of the fluctuation of particles due to \(U_{eqmp}\) and the dissipation of sound waves at the surface layer can be understood in terms of Kubo’s fluctuation-dissipation theory [32] and satisfies the conservation of energy for an isolated system. We thus see that the particles in the surface layer obey the Boltzman statistics just like rotons because of the elevated particle energy. Hence the surface layer is a normal fluid in every respect [5].

The analytical expression Eq. (20), in principle, gives a complete solution to the eigenvalue problem. A moment’s thought shows, however, that the expression for \(W_n\) is too complicated to evaluate the eigenvalues by this approach. Here we follow a different, but equivalent, approach to the eigenvalue problem; that is, we shall solve Eq. (19) as an eigenvalue equation.
With the substitution $W_n(r) = r^{\pm(\ell+1/2)-1/2}Z_n(r)$ together with $x = r^2/\alpha^2$, we obtain
\[ x(1 - x)\frac{d^2}{dx^2}Z_n + [c - (a + b + 1)x]\frac{d}{dx}Z_n - abZ_n = 0 \quad (22) \]
This is just the Gauss differential equation \[33\] with $c = \pm(\ell+1/2)+1$, $a+b = \pm(\ell+1/2)+3$, and $ab = (1/4) \hat{\lambda}_n - 6 [\pm(\ell + 1/2) - 1/2]$.

Eq. (22) has regular singular points at $x = 0$, $x = 1$ and $x = \infty$. Its solution is the hypergeometric function, which is analytic in the complex plane with a cut from 1 to $\infty$ along the real axis \[33\].

Since we are interested in the low-lying excited states (phonons), it is only necessary to find the smallest eigenvalues in the differential equation. This is consistent with Feynman’s picture of phonons - a sound (longitudinal) wave without nodes \[4\]. A simple, but accurate, numerical method \[34\] has been employed to evaluate the eigenvalues in the domain $[0, 1]$ in which the solutions are analytic.

Now returning to Eq. (18) and taking $S(t) = e^{i\omega t}$, we obtain the dispersion relation as
\[ \omega_{ph} = \pm[\hat{\lambda}_s/2]^{1/2}\omega_0, \quad (23) \]
where $\hat{\lambda}_s$ is the smallest eigenvalues from Eq. (22).

The principal result of this paper is Eq. (23). It is given in Fig.1, where the ratio $\omega_{ph}/\omega_0$ with respect to the angular momentum $\ell$ is plotted. A close examination of the dispersion curve shows that in the phonon regime the energy spectrum of longitudinal collective excitations is nearly linear with respect to $\ell$ (i.e., the wave number $k_\theta \simeq \ell/r$).

In an inhomogeneous medium, the Bogoliubov dispersion relation $\omega_{ph} = ck$ with $c = [4\pi a\rho(r)h^2]^{1/2}/M$ must be studied approximately with $k_\theta \simeq \ell/r$ at a given radius in SSNBEC. In particular $\omega_{ph}/\omega_0 = 1.5972$ for $\ell = 0$ is a unique value in a finite space problem, which corresponds to a uniform radial perturbation. Finally, since the eigenvalues are positive for all values of $\ell$, the Landau critical velocity $v_c = \varepsilon_{ph}/p$ is also finite. Hence the imperfect Bose gas is a superfluid, whereas an ideal Bose gas, for which $c$ is independent of the scattering length, is not, because $c$ and $v_c$ vanish identically. Moreover, the dispersion relation, Fig.1, also shows how the presence of the repulsive pair-interaction determines the excitation spectrum of phonons, which is nearly linear with respect to $k_\theta \[2, 35\]$.

As we have shown above, the symmetry of the ground state wave function breaks down at the free surface, i.e., a nodal surface. It is therefore natural to identify the underlying
basic mechanism for the symmetry breaking as a spontaneously broken gauge symmetry at the free surface which accompanies phonons as Nambu-Goldstone bosons \[12, 13\].

Next we discuss a possible formation of a close-packed lattice as \(a \to 0\) in BEC. Our ground state wave function describes a perfect Bose gas modified as little as possible by the presence of the repulsive pair-interactions (a hard sphere approximation). Hence the hard spheres in BEC could collapse to a close-packed lattice in the limit, \((i.e., a \to 0)\) in which each particle vibrates about a fixed center. Since the number of particles \(N\) in the system must be conserved in the limit, the density must satisfy \((4/3)\pi d_{cl}^3 \rho_{cl} = N; \rho_{cl}\) is the mean density of the collapsed spheres \[8, 36\]. It is unlikely that the closed-packed lattice will be realized in a trap, because the radius of the condensate \(b \to d_{cl} \ll 1\) as \(a \to 0\) [Eq. (20)], which implies a far stronger external trapping force to overcome the inter-particle repulsive force. However our analysis may provide a qualitative mechanism by which the close-packed lattice can be formed from BEC by symmetry breaking.

The separation of the zero-point vibration is not always possible, but in the limit \(a \to 0\), \(b = \alpha = [8\pi a^2 \rho(0)]^{1/2}/(M\omega_0) \to d_{cl}, \dot{X}_n \simeq 0\), we have the following zero-point vibrational motion about fixed points. Since \(\lambda_n = 5.0\omega_0^2\), we have a dispersion relation for the closed-packed lattice from Eq. (18),

\[
\omega_{\text{latt}} = (5)^{1/2}\omega_0,
\]

which is again independent of \(\hbar\). This implies that if there is actually a transition from BEC to a crystalline ground state, it is a classical lattice. It is gratifying to note that the condition under which the close-packed lattice could be realized in BEC is essentially the same as that of a crystalline lattice observed under an external high pressure applied to He II \[37, 38\]. The physical mechanism by which the close-packed lattice can be realized is again the spontaneously broken gauge symmetry which accompanies phonons as Nambu-Goldstone bosons \[12, 13\].

In summary, we have shown the perturbation method with the Lagrangian displacement vectors on particle trajectories in the many-system yields the results that are consistent with those of quantum field theory technique of Bogoliubov. The compressible perturbations give the dispersion relation for a spherical (longitudinal) sound wave which is of the form of the Bogoliubov spectrum for phonons. The incompressible perturbations drive the free surface to behave like a normal fluid. The dispersion relations Eqs. (15) and (23) are interpreted as the
result of the spontaneously broken gauge symmetry at the free surface which accompanies phonons (Nambu-Goldstone bosons). The present approach gives a unified picture of the collective excitations in BEC in a trap and also suggests a possible mechanism by which BEC reduces to a close-packed lattice under external pressure similar to that of the helium crystals at constant pressure.

Note added in proof. After this paper was written, I learned about 't Hooft’s conjecture that is essentially equivalent to our picture of fluctuation-dissipation of the longitudinal sound waves at the free surface, which is driven by the effective quantum potential and is due to the broken symmetry [Massimo Blasone, Petr Jizba, Giuseppe Vitiello, Phys. Lett. A 287 (2001) 205].

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FIG. 1: The ratio $\omega_{ph}/\omega_0 = [2\lambda_s]^{1/2}$ is plotted against the angular momentum $\ell$. It shows how the energy spectrum of phonons varies with the angular momentum.