Optimal Exact Repair Strategy for the Parity Nodes of the \((k + 2, k)\) Zigzag Code

Jie Li and Xiaohu Tang, Member, IEEE

Abstract

In this paper, we reinterpret the \((k + 2, k)\) Zigzag code in coding matrix and then propose an optimal exact repair strategy for its parity nodes, whose repair disk I/O approaches a lower bound derived in this paper.

Index Terms

Distributed storage, MSR code, optimal repair, Zigzag code.

I. INTRODUCTION

Distributed storage systems built on huge numbers of storage nodes have wide applications in peer-to-peer storage systems such as OceanStore [12], Total Recall [1] and DHash++ [5]. Erasure code, which can provide both protection against node failures and efficient data storage, is very common in distributed storage systems [2], [3], [4], [8], [18], [19]. For instance, as a special class of erasure code, RAID-6 is a popular scheme for tolerating any two node failures [11].

Recently, Dimakis et al. [6] introduced a new class of erasure code for distributed storage systems named minimum storage regenerating (MSR) code. The distributed storage system deploys a \((k + r, k)\) MSR code to store a file of size \(M = kN\) symbols across \(n\) nodes, each node keeping \(N\) symbols. The \((k+r, k)\) MSR code has the optimal repair property that the repair bandwidth \(\gamma = \frac{d}{d-k+1}N\) is minimal, which is achieved by downloading \(\frac{N}{d-k+r}\) symbols from each of any \(k \leq d \leq k+r-1\) surviving nodes when repairing a failed node. In this paper, we only focus on the exact repair of high rate MSR codes. When \(r = 1\), the repair bandwidth is the highest, i.e., \(\gamma = M\). When \(r = 2\) and \(d = k + 1\), MSR code is very desirable since it can achieve the highest rate \(\frac{1}{k+2}\) for \(\gamma = (k+1)N/2 < M\). In addition, \((k + 2, k)\) MSR code can be alternative to RAID-6 schemes.

So far, several explicit constructions of \((k + 2, k)\) MSR codes have been presented [9], [10], [13], [15]. Among them, the \((k + 2, k)\) Zigzag code in [13], which is defined by a series of permutations, is of great interest because of:

(i) Optimal update disk I/O property (also known as optimal update property in [13]) that only itself and one symbol at each parity node need an update when a symbol in a systematic node is rewritten;

(ii) Optimal repair disk I/O property (also known as optimal rebuilding in [13]) for systematic nodes that the repair disk I/O of a systematic node is equal to the minimal repair bandwidth;

(iii) Small alphabet size of 3 so that it can be easily implemented;

(iv) The storage \(N = 2^{k-1}\) achieves the theoretic lower bound on the storage per node for \((k + 2, k)\) MSR codes with both optimal update disk I/O and optimal repair disk I/O for systematic nodes [13].

However, the parity nodes of the \((k + 2, k)\) Zigzag code was trivially repaired by downloading all the original data in [13], i.e., the download bandwidth reaches the maximal value \(\gamma = M\). In order to acquire the optimal repair property for both systematic nodes and parity nodes, a \((k, k-2)\) MSR code was presented in [15] based on a modification of the \((k + 2, k)\) Zigzag code, but at cost of sacrificing two systematic nodes while maintaining the same storage per node \(N = 2^{k-1}\). It should be noted that only the \((k + 2, k)\) Hadamard MSR code in [10] shares the optimally repair property of all the nodes in the all aforementioned codes.

In this paper, without changing the original structure of the \((k + 2, k)\) Zigzag code, we propose an optimal repair strategy for the two parity nodes, whose download bandwidth achieves the minimal value \(\gamma = (k+1)N/2\). A comparison of the properties of various known \((k + 2, k)\) MSR codes, such as the Zizag code employing our repair strategy, the original Zigzag
code \[^{[3]}\]\), the modified Zigzag code \[^{[16]}\]\), and Hadamard code \[^{[10]}\]\), is given in Table I. It is seen that the new repair strategy does not lose any good properties of the original Zigzag code, for example, the optimal update disk I/O property, the optimal repair disk I/O property for systematic nodes, small alphabet size of 3, and so on. In contrast to the modified Zigzag code and Hadamard code with the same optimal repair property of all nodes, the Zigzag code employing the new repair strategy shows a clear advantage over the storage per node. Although the repair disk I/O of the parity node is not optimal, which is \(kN + N - k\), larger than the minimal repair bandwidth \((k + 1)N/2\), it indeed approaches a lower bound on the disk I/O of Zigzag code given in this paper.

| Table I | Comparison of the Properties of Some \((k + 2, k)\) MSR Codes Where \(q\) and \(N\) Denote the Size of the Finite Field Required and the Storage Per Node, Respectively. |
|---------|--------------------------------------------------|
| Zigzag Code Employing New Repair Strategy | \(q\) | \(N\) | Optimal Update Disk I/O | Optimal Repair Disk I/O | Optimal Repair |
| Original Zigzag Code \[^{[13]}\] | 3 | \(2^{k-1}\) | Yes | Yes | No | Yes | Yes |
| Modified Zigzag Code \[^{[16]}\] | 3 | \(2^{k+1}\) | Yes | Yes | No | Yes | No |
| Hadamard Code \[^{[10]}\] | \(2k + 3\) | \(2^{k+1}\) | Yes | No | No | Yes | Yes |

The rest of this paper is organized as follows. Section II introduces the structure of a \((k + 2, k)\) MSR code and the necessary and sufficient conditions for optimal repair of parity nodes. Section III proposes the \((k + 2, k)\) Zigzag code and reinterprets it in coding matrix. In Section IV, a lower bound on disk I/O to optimally repair the parity nodes of the \((k + 2, k)\) Zigzag code is presented. The optimal repair strategy for the parity nodes of the \((k + 2, k)\) Zigzag code is given in Section V.

II. Optimal Repair for Parity Nodes of \((k + 2, k)\) MSR Codes

Let \(q\) be a prime power and \(F_q\) be the finite field with \(q\) elements. Assume that a file of size \(M = kN\) is equally partitioned into \(k\) parts, respectively denoted by \(f_0, f_1, \ldots, f_{k-1}\), where \(f_j\) is a column vector of length \(N\) for \(0 \leq j < k\). The file is encoded to a \((k + 2, k)\) MSR code and then stored across \(k\) systematic and two parity storage nodes, each node having storage \(N\). The first \(k\) nodes are systematic nodes, which store the file parts \(f_0, f_1, \ldots, f_{k-1}\) in an uncoded form respectively. Without loss of generality, assume that the two parity nodes, nodes \(k\) and \(k + 1\), respectively store \(f_k = f_0 + f_1 + \cdots + f_{k-1}\) and \(f_{k+1} = A_0 f_0 + A_1 f_1 + \cdots + A_{k-1} f_{k-1}\) for some \(N \times N\) matrices \(A_0, \ldots, A_{k-1}\) over \(F_q\), where the matrix \(A_j\) is called the coding matrix for systematic node \(j\), \(0 \leq j < k\). To guarantee the MDS property, it is required that \[^{[10]}, \[^{[14]}\]

\[
\text{rank}(A_i) = \text{rank}(A_i - A_j) = N, 0 \leq i \neq j < k. \tag{1}
\]

Table I illustrates the structure of a \((k + 2, k)\) MSR code.

| Table II | Structure of a \((k + 2, k)\) MSR Code |
|---------|-------------------------------------|
| \(f_0\) | \(f_1\) | \(\cdots\) | \(f_{k-1}\) | \(f_k = \sum_{i=0}^{k-1} f_i\) | \(f_{k+1} = \sum_{i=0}^{k-1} A_i f_i\) |

When repairing a failed node \(j\), the optimal repair property demands to download half data from each surviving node \(l\), \(0 \leq l \neq j < k + 2\), by multiplying its original data \(f_l\) with an \(N/2 \times N\) matrix of rank \(N/2\), called repair matrix. In what follows, we review the requirement on repair matrices for the optimal repair of parity nodes of a \((k + 2, k)\) MSR code \[^{[10]}, \[^{[14]}\]

Upon failure of the first parity node (node \(k\)), respectively downloading \(S_a f_j\) and \(\tilde{S}_a f_{k+1}\), \(0 \leq j < k\), where \(S_a\) and \(\tilde{S}_a\) are two \(N/2 \times N\) repair matrices of rank \(N/2\), eventually one gets the following system of linear equations

\[
\left(\begin{array}{c}
S_a f_0 \\
\tilde{S}_a f_{k+1}
\end{array}\right) = \left(\begin{array}{c}
S_a \\
\tilde{S}_a A_0
\end{array}\right) f_k. \tag{2}
\]

To cancel all the interference terms and then recover the target data \(f_k\), the optimal repair requires \[^{[10]}, \[^{[14]}\]

\[
\text{rank} \left(\begin{array}{c}
S_a \\
\tilde{S}_a A_0
\end{array}\right) = N
\]

The modified Zigzag code given in this paper.

Upon failure of the first parity node (node \(k\)), respectively downloading \(S_a f_j\) and \(\tilde{S}_a f_{k+1}\), \(0 \leq j < k\), where \(S_a\) and \(\tilde{S}_a\) are two \(N/2 \times N\) repair matrices of rank \(N/2\), eventually one gets the following system of linear equations

\[
\left(\begin{array}{c}
S_a f_0 \\
\tilde{S}_a f_{k+1}
\end{array}\right) = \left(\begin{array}{c}
S_a \\
\tilde{S}_a A_0
\end{array}\right) f_k. \tag{2}
\]

To cancel all the interference terms and then recover the target data \(f_k\), the optimal repair requires \[^{[10]}, \[^{[14]}\]

\[
\text{rank} \left(\begin{array}{c}
S_a \\
\tilde{S}_a A_0
\end{array}\right) = N
\]
and
\[
\text{rank} \left( \begin{pmatrix} S_a & S_a A_{0} - A_{i} \end{pmatrix} \right) = \frac{N}{2}, \quad 1 \leq l < k. \tag{3}
\]

Clearly, the disk I/O to optimally repair the first parity node is \(kN_1 + N_2\) where \(N_1\) and \(N_2\) denote the nonzero columns of \(S_a\) and \(S_a\) respectively.

To repair the second parity node (node \(k + 1\)), downloading \((S_b A_{1}) f_j\) and \(\tilde{S}_b f_k, \ 0 \leq j < k\), where \(S_b\) and \(\tilde{S}_b\) are two \(N/2 \times N\) matrices of rank \(N/2\), one obtains the following system of linear equations
\[
\begin{pmatrix} S_b A_{0} f_0 \\ \tilde{S}_b A_{1}^{-1} f_k \end{pmatrix} = \begin{pmatrix} S_b \\ \tilde{S}_b A_{0}^{-1} \end{pmatrix} f_{k+1} - \sum_{l=1}^{k-1} \begin{pmatrix} S_b \\ \tilde{S}_b (A_{0}^{-1} - A_{l}^{-1}) \end{pmatrix} A_{l} f_{l},
\]

Similarly, optimal repair demands \([10], [14]\)
\[
\text{rank} \left( \begin{pmatrix} S_b & S_b A_{0}^{-1} \end{pmatrix} \right) = N
\tag{4}
\]
and
\[
\text{rank} \left( \begin{pmatrix} S_b & S_b (A_{0}^{-1} - A_{l}^{-1}) \end{pmatrix} \right) = \frac{N}{2}, \quad 1 \leq l < k.
\tag{5}
\]

Accordingly, the disk I/O to optimally repair the second parity node is the total number of nonzero columns of \(\tilde{S}_b\) and \(S_b A_{1}\), \(0 \leq i < k\).

### III. REINTERPRETATION OF \((k + 2, k)\) ZIGZAG CODE IN CODING MATRIX

Throughout this paper, let \(k \geq 2\) and \(N = 2^{k-1}\). Given an integer \(0 \leq i < N\), let \((i_1, \ldots, i_{k-1})\) be its binary expansion, i.e., \(i = \sum_{j=1}^{k-1} 2^{k-1-j} i_j\). For simplicity, we do not distinguish a nonnegative integer \(i\) and its binary expansion if the context is clear.

Let \(\{e_j\}_{j=1}^{k-1}\) be the standard vector basis over \(F_2\) of dimension \(k - 1\), i.e.,
\[
e_j = (0, \ldots, 0, 1, 0, \ldots, 0), \quad 1 \leq j < k
\]
with only the \(j\)th entry being nonzero. By convenience, set \(e_0\) to be the all-zero vector.

In [13], the \((k + 2, k)\) Zigzag code is characterized by the following permutation \(P_j : [0, N - 1] \to [0, N - 1]\)
\[
P_j(x) = x \oplus e_j = \begin{cases} (x_1, \ldots, x_{k-1}), & j = 0 \\ (x_1, \ldots, x_{j-1}, x_j \oplus 1, x_{j+1}, \ldots, x_{k-1}), & 0 < j < k \end{cases}
\]

where \(\oplus\) denotes the addition in \(F_2\). Obviously,
\[
P_j^{-1}(x) = x \oplus e_j = P_j(x), \quad 0 \leq j < k.
\tag{6}
\]

For any integer \(0 \leq l < N\), define \(Z_l\) as \(Z_l = \{(i, j)|i = P_j^{-1}(l), 0 \leq j < k\}\), i.e.,
\[
Z_l = \{(i, j)|i = l \oplus e_j, 0 \leq j < k\}
\]
by (6). The structure of the \((k + 2, k)\) Zigzag code is depicted in Table II, where the first parity node stores \(f_{i,k} = \sum_{j=0}^{k-1} f_{i,j}\)
and the second parity node stores \(f_{i,k+1} = \sum_{(i,j) \in Z_l} \beta_{i,j} f_{i,j}, \ 0 \leq i < N\) and \(0 \leq j < k\), \(\beta_{i,j} = (-1)^{l+\sum_{l=0}^{i} e_i}\), i.e.,
\[
\beta_{i,j} = \begin{cases} 1, & \text{if } j = 0 \\ (-1)^{l+i_1+\cdots+i_j}, & \text{otherwise} \end{cases}
\tag{7}
\]

In the following, we reinterpret the data stored at the second parity node of the \((k + 2, k)\) Zigzag code in the form of coding matrix so that we can use Equations (3)-(5) to check the optimality of our new repair matrices in the next section.

Given an integer \(k \geq 2\), recursively define \(k\) matrices \(A_{0}^{(k)}, \ldots, A_{k-1}^{(k)}\) of order \(N\) over \(F_3\) as
\[
A_{0}^{(k)} = I_{2^{k-1}}, \quad A_{1}^{(k)} = \begin{pmatrix} I_{2^{k-2}} & -I_{2^{k-2}} \end{pmatrix}, \quad A_{j}^{(k)} = \begin{pmatrix} A_{j-1}^{(k)} & -A_{j-1}^{(k)} \end{pmatrix} \quad \text{for } 2 \leq j < k
\tag{8}
\]
TABLE III
STRUCTURE OF THE \((k + 2, k)\) ZIGZAG CODE

| Node 0 | \cdots | Node \(k\) | Node \(k + 1\) |
|--------|--------|-------------|---------------|
| \(f_{0,0}\) | \cdots | \(f_{0,k-1}\) | \(f_{0,k} = \sum_{j=0}^{k-1} f_{0,j}\) |
| \(f_{0,k+1} = \sum_{(i,j) \in Z_0} \beta_{i,j} f_{i,j}\) |
| \(f_{1,0}\) | \cdots | \(f_{1,k-1}\) | \(f_{1,k} = \sum_{j=0}^{k-1} f_{1,j}\) |
| \(f_{1,k+1} = \sum_{(i,j) \in Z_1} \beta_{i,j} f_{i,j}\) |
| \(\vdots\) | \cdots | \(\vdots\) | \(\vdots\) |
| \(f_{N-1,0}\) | \cdots | \(f_{N-1,k-1}\) | \(f_{N-1,k} = \sum_{j=0}^{k-1} f_{N-1,j}\) |
| \(f_{N-1,k+1} = \sum_{(i,j) \in Z_{N-1}} \beta_{i,j} f_{i,j}\) |

where

\[ A^{(2)}_0 = I_2, \quad A^{(2)}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

First of all, the following properties of the matrices in (8) are obvious.

**Property 1.** For any \(k \geq 2\), the matrix \(A^{(k)}_j\) in (8) with \(1 \leq j < k\) satisfies

(i) \((A^{(k)}_j)^2 = -I_{2^{k-1}}\);

(ii) Both each row and each column of \(A^{(k)}_j\) have only one nonzero entry.

Next, we show that the matrix \(A^{(k)}_j\) in (8) is just the coding matrix for systematic node \(j\) of the \((k + 2, k)\) Zigzag code for all \(0 \leq j < k\).

**Theorem 1.** The coding matrices of the \((k + 2, k)\) Zigzag code are \(A^{(k)}_0, \ldots, A^{(k)}_{k-1}\), i.e.,

\[ f_{k+1} = A^{(k)}_0 f_0 + \cdots + A^{(k)}_{k-1} f_{k-1} \]

where \(f_j = (f_{0,j}, \ldots, f_{N-1,j})^T\).

**Proof:** Let \(A(l, i)\) denote the entry at row \(l\) and column \(i\) of matrix \(A\). By Property 1-(ii), equations (6) and (7), it suffices to prove \(A^{(k)}_j(l, P^{-1}_j(l)) = \beta_{P^{-1}_j(l),j}\), i.e.,

\[ A^{(k)}_0(l, l) = A^{(k)}_0(l, l \oplus e_0) = \beta_{l,0} = 1, \quad 0 \leq l < N \quad \text{(9)} \]

and

\[ A^{(k)}_j(l, l \oplus e_j) = \beta_{l \oplus e_j, j} = (-1)^{l_1 + \cdots + l_j + 1}, \quad 1 \leq j < k, 0 \leq l < N. \quad \text{(10)} \]

Obviously, (9) holds since \(A^{(k)}_0\) is the identity matrix and (10) holds for \(j = 1\), i.e., \(A^{(k)}_1(l, l \oplus e_1) = (-1)^{l_1 + 1}, 0 \leq l < N\), by the definition in (8).

Hereafter, we prove (10) for \(j \geq 2\) by the induction. Suppose that (10) holds for \(k \geq 2\) and \(1 \leq j < k\). Then,

\[ A^{(k+1)}_j(l, l \oplus e_j) = A^{(k+1)}_j((l_1, \ldots, l_k), (l_1, \ldots, l_{j-1}, l_j \oplus 1, l_{j+1}, \ldots, l_k)) = (-1)^{l_j} A^{(k)}_{j-1}((l_2, \ldots, l_k), (l_2, \ldots, l_{j-1}, l_j \oplus 1, l_{j+1}, \ldots, l_k)) = (-1)^{l_j} A^{(k)}_{j-1} + 1 \]

for \(2 \leq j < k + 1\) and \(0 \leq l < 2^k\), where the last two equalities respectively follow from (8) and the assumption. \hfill \blacksquare

IV. BOUNDS ON DISK I/O TO OPTIMALLY REPAIR THE PARITY NODES OF THE ZIGZAG CODE

For a general \((k + 2, k)\) MSR code over \(F_q\) defined in Table I, Wang et al. [17] proved that the minimal disk I/O to repair the first and second parity nodes are respectively at least \((k + 1)N/2\) and \(kN\) if \(q = 2\). In fact, the assertion can be proved for \(q > 2\) by almost the same proof in [17].

Specifically for the Zigzag code, in this section we give a more tight bound on the minimal disk I/O for the optimal repair of the parity nodes.

Firstly, we state a connection between the optimal repair strategies for the two parity nodes of the Zigzag code.

**Lemma 1.** If \(S^{(k)}_1\) and \(\tilde{S}^{(k)}_j\) are the repair matrices for the first parity node of the \((k + 2, k)\) Zigzag code, then \(\tilde{S}^{(k)}_j A^{(k)}_j, 0 \leq j < k, \) and \(S^{(k)}_j\) are the repair matrices for the second parity node, and vice versa.
Proof: Note from (1) and (8) that \( A_0^{(k)} - A_1^{(k)} = I_N - A_1^{(k)} \) is nonsingular for \( 1 \leq l < k \). Then,

\[
\text{rank}\left( \begin{pmatrix} S^{(k)}(A_0^{(k)}) \end{pmatrix}^{-1} - (A_l^{(k)})^{-1} \right) = \text{rank}\left( \begin{pmatrix} S^{(k)} \end{pmatrix} (I_N + A_1^{(k)}) \right) = \text{rank}\left( \begin{pmatrix} S^{(k)} \end{pmatrix} (I_N - A_1^{(k)}) \right)
\]

where in the first and fourth identities we use Property 1-(i), i.e., \( (A_1^{(k)})^2 = -I_N \) and then \( (A_l^{(k)})^{-1} = -A_1^{(k)} \). In addition,

\[
\text{rank}\left( \begin{pmatrix} S^{(k)}(A_0^{(k)}) \end{pmatrix}^{-1} \right) = \text{rank}\left( \begin{pmatrix} S^{(k)} \end{pmatrix} \right) = \text{rank}\left( \begin{pmatrix} S^{(k)} \end{pmatrix} (A_0^{(k)} - A_1^{(k)}) \right).
\]

Therefore, the result can be obtained from (2), (3), (4) and (5).

\[\Box\]

**Theorem 2.** The disk I/O to optimally repair the first or second parity node of the \((k + 2, k)\) Zigzag code is at least \( kN + \frac{k-3}{2(k-1)}N \).

Proof: Suppose that \( S_a^{(k)} \) and \( \tilde{S}_a^{(k)} \) are two repair matrices for the first parity node of \((k + 2, k)\) Zigzag code. According to the definition of repair disk I/O, we need to prove \( kN_1 + N_2 \geq kN + \frac{k-3}{2(k-1)}N \), where \( N_1 \) and \( N_2 \) respectively denote the number of nonzero columns of the matrices \( S_a^{(k)} \) and \( \tilde{S}_a^{(k)} \).

By (2) and (3), we have

\[
\text{rank}\left( \begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} \end{pmatrix} \right) = \text{rank}\left( \begin{pmatrix} S_a^{(k)}(A_0^{(k)}) \\ \tilde{S}_a^{(k)}(A_0^{(k)}) \end{pmatrix} \right) = N \tag{12}
\]

and

\[
\text{rank}\left( \begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} \end{pmatrix} (A_0^{(k)} - A_1^{(k)}) \right) = \text{rank}\left( \begin{pmatrix} S_a^{(k)} \end{pmatrix} (I_N - A_1^{(k)}) \right) = \frac{N}{2}, \quad 1 \leq l < k. \tag{13}
\]

For \( 0 \leq i < N \), denote by \( S_a^{(k)}[i] \) and \( \tilde{S}_a^{(k)}[i] \) the column \( i \) of \( S_a^{(k)} \) and \( \tilde{S}_a^{(k)} \). Assume that columns \( i_1, i_2, \ldots, i_{N-N_1} \) of \( S_a^{(k)} \) are zero columns. Note that in (13), \( \text{rank}(S_a^{(k)}) = \text{rank}(S_a^{(k)}(I_N - A_1^{(k)})) = N/2 \). Then, we have that \( \tilde{S}_a^{(k)}(I_N - A_1^{(k)})[i_s] = \tilde{S}_a^{(k)}[i_s] - (\tilde{S}_a^{(k)}A_1^{(k)})[i_s] \) is also a zero column, i.e.,

\[
(\tilde{S}_a^{(k)}A_1^{(k)})[i_s] = \tilde{S}_a^{(k)}[i_s] \text{ for } 1 \leq l < k \text{ and } 1 \leq s \leq N - N_1.
\]

Further, it follows from Property 1-(ii) and (10) that only the \((i \oplus e_l)\)th entry in \( A_1^{(k)}[i] \) is \pm 1, which implies \( (\tilde{S}_a^{(k)}A_1^{(k)})[i_s] = \pm \tilde{S}_a^{(k)}[i_s] \). Thus,

\[
\tilde{S}_a^{(k)}[i_s \oplus e_l] = \pm \tilde{S}_a^{(k)}[i_s] \text{ for } 1 \leq l < k \text{ and } 1 \leq s \leq N - N_1. \tag{14}
\]

On the other hand, it is seen from (12) that all the columns \( i_1, i_2, \ldots, i_{N-N_1} \) of \( \tilde{S}_a^{(k)} \) are linearly independent, which indicates that

\[
\{i_u \oplus e_l : 1 \leq l < k\} \cap \{i_v \oplus e_l : 1 \leq l < k\} = \emptyset \text{ for } 1 \leq u \neq v \leq N - N_1. \tag{15}
\]

Therefore, applying (14) and (15) to \( \text{rank}(\tilde{S}_a^{(k)}) = N/2 \), we obtain \( N/2 \leq N - (k-1)(N - N_1) \), i.e., \( N_1 \geq N - \frac{N}{2(k-1)} \).

By means of (11), we can prove \( N_2 \leq N - \frac{N}{2(k-1)} \) in the same fashion. Hence,

\[
kN_1 + N_2 \geq (k+1)(N - \frac{N}{2(k-1)}) = kN + N - \frac{N(k+1)}{2(k-1)} = kN + \frac{k-3}{2(k-1)}N.
\]

That is, the assertion is valid for the first parity node.

For the second parity node of the \((k + 2, k)\) Zigzag code, assume that \( S_b^{(k)}A_j^{(k)} \), \( 0 \leq j < k \), and \( \tilde{S}_b^{(k)} \) are the repair matrices. According to the definition, the repair disk I/O is the total number of nonzero columns of the matrices \( S_b^{(k)}A_j^{(k)} \).
and \( \tilde{S}^{(k)}_{b}, 0 \leq j < k \), which is \( kN_1 + N_2 \) by Property (ii), where \( N_1 \) and \( N_2 \) respectively denote the number of nonzero columns of the matrices \( S^{(k)}_{b} \) and \( \tilde{S}^{(k)}_{b} \). By Lemma II it is known that \( \tilde{S}^{(k)}_{b} \) and \( S^{(k)}_{b} \) are two repair matrices for the first parity node. Therefore, by the analysis for the first parity node we have

**Proposition 2.**

Thus, the proof is finished by the above induction.  

**V. REPAIR MATRICES FOR THE PARITY NODES OF THE ZIGZAG CODE**

In this section, we give the repair matrices for the parity nodes of the \((k + 2, k)\) Zigzag code and verify that they satisfy (2), (3), (4) and (5).

Recursively define the \( 2^{k-2} \times 2^{k-1} \) matrices \( E^{(k)} \) and \( F^{(k)} \) over \( F_3 \) as

\[
E^{(k)} = \begin{pmatrix} E^{(k-1)} & F^{(k-1)} \end{pmatrix}, \quad F^{(k)} = \begin{pmatrix} F^{(k-1)} & E^{(k-1)} \end{pmatrix}, \quad k \geq 3
\]

where

\[
E^{(2)} = \begin{pmatrix} 0 & -1 \end{pmatrix}, \quad F^{(2)} = \begin{pmatrix} -1 & 0 \end{pmatrix}.
\]

Next recursively define the \( 2^{k-2} \times 2^{k-1} \) matrices \( S^{(k)}_a \) and \( \tilde{S}^{(k)}_a \) over \( F_3 \) as

\[
S^{(k)}_a = \begin{pmatrix} S^{(k-1)}_a & E^{(k-1)} \\ \tilde{S}^{(k-1)}_a & F^{(k-1)} \end{pmatrix}, \quad \tilde{S}^{(k)}_a = \begin{pmatrix} \tilde{S}^{(k-1)}_a & -F^{(k-1)} \\ -S^{(k-1)}_a & S^{(k-1)}_a \end{pmatrix}, \quad k \geq 3
\]

where

\[
S^{(2)}_a = \begin{pmatrix} 0 & 1 \\ \tilde{S}^{(2)}_a = \begin{pmatrix} 1 & 1 \end{pmatrix}.
\]

**Proposition 1.** For \( k \geq 2 \), \n
\[
\text{rank}\left( \begin{pmatrix} S^{(k)}_a \\ \tilde{S}^{(k)}_a \end{pmatrix} \left( A_0^{(k)} \right) \right) = N.
\]

**Proof:** When \( k = 2 \), the statement is easily checked. For any given \( k \geq 2 \), suppose that the statement is true. According to recursive definition in (18), we have

\[
\text{rank}\left( \begin{pmatrix} S^{(k+1)}_a \\ \tilde{S}^{(k+1)}_a \end{pmatrix} \left( A_0^{(k+1)} \right) \right) = \text{rank}\left( \begin{pmatrix} S^{(k+1)}_a \\ \tilde{S}^{(k+1)}_a \end{pmatrix} \left( A_0^{(k)} \right) \right) = \text{rank}\left( \begin{pmatrix} S^{(k)}_a & E^{(k)} \\ \tilde{S}^{(k)}_a & -F^{(k)} \end{pmatrix} \right) = \text{rank}\left( \begin{pmatrix} S^{(k)}_a & E^{(k)} \\ \tilde{S}^{(k)}_a & -F^{(k)} \end{pmatrix} \right) = 2N
\]

since \( \begin{pmatrix} S^{(k)}_a \\ \tilde{S}^{(k)}_a \end{pmatrix} = \begin{pmatrix} S^{(k)}_a \\ \tilde{S}^{(k)}_a \end{pmatrix} \) is an \( N \times N \) matrix of full rank.

Thus, the proof is finished by the above induction.  

**Proposition 2.** For \( k \geq 2 \), \n
\[
\text{rank}\left( \begin{pmatrix} S^{(k)}_a \\ \tilde{S}^{(k)}_a \end{pmatrix} \left( A_0^{(k)} - A_1^{(k)} \right) \right) = N/2.
\]

**Proof:** When \( k = 2 \), the statement is easily checked. When \( k > 2 \), by the recursive definitions in (8) and (18), we have

\[
\begin{align*}
\left[ S^{(k)}_a \left( A_0^{(k)} - A_1^{(k)} \right) \right] &= \left[ S^{(k)}_a \left( I_N - A_1^{(k)} \right) \right] \\
&= \left[ \tilde{S}^{(k+1)}_a \right] \\
&= \left[ \tilde{S}^{(k+1)}_a \left( I_N - F^{(k-1)} \right) \right] \\
&= \left[ \tilde{S}^{(k+1)}_a \right] \\
&= \left[ \tilde{S}^{(k+1)}_a \left( I_N - F^{(k-1)} \right) \right] \\
&= \left[ \tilde{S}^{(k+1)}_a \right] \\
&= \left[ \tilde{S}^{(k+1)}_a \left( I_N - F^{(k-1)} \right) \right]
\end{align*}
\]
Therefore,

\[
\text{rank}\left( \begin{pmatrix}
S_a^{(k)} & \tilde{S}_a^{(k)}(A_0^{(k)} - A_1^{(k)}) \\
S_a^{(k-1)} & E^{(k-1)} \\
\tilde{S}_a^{(k-1)} + F^{(k-1)} & \tilde{S}_a^{(k-1)} - F^{(k-1)} \\
-\tilde{S}_a^{(k-1)} & \tilde{S}_a^{(k-1)}
\end{pmatrix} \right)
\]

\[
= \text{rank}\left( \begin{pmatrix}
P & \cdot \\
\tilde{S}_a^{(k-1)} + F^{(k-1)} & \tilde{S}_a^{(k-1)} - F^{(k-1)} \\
\tilde{S}_a^{(k-1)} & \tilde{S}_a^{(k-1)}
\end{pmatrix} \cdot Q \right)
\]

\[
= \text{rank}\left( \begin{pmatrix}
S_a^{(k-1)} + E^{(k-1)} \\
\tilde{S}_a^{(k-1)}
\end{pmatrix} \right) + \text{rank}\left( \begin{pmatrix}
\tilde{S}_a^{(k-1)} + F^{(k-1)} \\
\tilde{S}_a^{(k-1)}
\end{pmatrix} \right)
\]

(20)

where the two matrices \(P, Q\) are respectively defined by

\[
P = \begin{pmatrix}
I_{N/4} & I_{N/4} & I_{N/4} \\
-I_{N/4} & -I_{N/4} & I_{N/4}
\end{pmatrix}, \quad Q = \begin{pmatrix}
I_{N/2} & -I_{N/2}
\end{pmatrix}.
\]

Next, we prove

\[
\text{rank}\left( \begin{pmatrix}
S_a^{(k)} + E^{(k)} \\
\tilde{S}_a^{(k)}
\end{pmatrix} \right) = \text{rank}\left( \begin{pmatrix}
\tilde{S}_a^{(k)} + F^{(k)} \\
\tilde{S}_a^{(k)}
\end{pmatrix} \right) = N/2
\]

for any \(k \geq 2\) by the induction.

When \(k = 2\), the statement is easily verified. For any \(k \geq 2\), suppose that it is true. By the definition of \(S_a^{(k+1)}\) and \(\tilde{S}_a^{(k+1)}\) in (18), we then have

\[
\text{rank}\left( \begin{pmatrix}
S_a^{(k+1)} + E^{(k+1)} \\
\tilde{S}_a^{(k+1)}
\end{pmatrix} \right) = \text{rank}\left( \begin{pmatrix}
\tilde{S}_a^{(k)} + F^{(k)} \\
\tilde{S}_a^{(k)}
\end{pmatrix} \right) = N/2
\]
Thus the disk I/O to repair the first parity node is in each of the systematic nodes and all the by the recursive definitions which satisfy the proof after substituted into (20).

**Proposition 3.** Given \( k \geq 3 \), \( \text{rank} \left( \begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} (A_0^{(k)} - A_i^{(k)}) \end{pmatrix} \right) = N/2 \) for all \( 2 \leq i < k \).

**Proof:** If \( k = 3 \), the statement is obvious. For any \( k \geq 3 \), assume that it is true for all \( 2 \leq j < k \). When \( j \geq 2 \), according to the definitions of \( A_j^{(k+1)} \) in (8) and \( S_a^{(k+1)}, \tilde{S}_a^{(k+1)} \) in (18),

\[
\text{rank} \left( \begin{pmatrix} S_a^{(k+1)} \\ \tilde{S}_a^{(k+1)} (A_0^{(k+1)} - A_j^{(k+1)}) \end{pmatrix} \right) = \text{rank} \left( \begin{pmatrix} S_a^{(k+1)} \\ \tilde{S}_a^{(k+1)} (I_{2N} - A_j^{(k+1)}) \end{pmatrix} \right) = \text{rank} \left( \begin{pmatrix} U_j^{(k)} \\ W_j^{(k)} \end{pmatrix} \right)
\]

for three \( N \times N \) matrices

\[
U_j^{(k)} = \begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} (I_N - A_{j-1}^{(k)}) \end{pmatrix}, \quad V_j^{(k)} = \begin{pmatrix} \tilde{S}_a^{(k)} \\ S_a^{(k)} (I_N + A_{j-1}^{(k)}) \end{pmatrix}, \quad W_j^{(k)} = \begin{pmatrix} E^{(k)} \\ -F^{(k)} (I_N + A_{j-1}^{(k)}) \end{pmatrix},
\]

by the recursive definitions which satisfy

\[
W_j^{(k)} = \begin{cases} -U_j^{(k)} + R^{(k)} V_j^{(k)}, & \text{if } j = 2 \\ U_j^{(k)} Q^{(k)} - P^{(k)} V_j^{(k)}, & \text{if } j > 2 \end{cases}
\]

where

\[
R^{(k)} = \begin{pmatrix} 0_{N/4} & I_{N/4} & I_{N/4} & 0_{N/4} \\ -I_{N/4} & 0_{N/4} & 0_{N/4} & I_{N/4} \\ 0_{N/4} & 0_{N/4} & 0_{N/4} & I_{N/4} \\ 0_{N/4} & 0_{N/4} & -I_{N/4} & 0_{N/4} \end{pmatrix}, \quad P^{(k)} = \begin{pmatrix} 0_{N/4} & 0_{N/4} & 0_{N/4} & 0_{N/4} \\ I_{N/4} & 0_{N/4} & 0_{N/4} & 0_{N/4} \\ 0_{N/4} & 0_{N/4} & 0_{N/4} & 0_{N/4} \\ 0_{N/4} & 0_{N/4} & I_{N/4} & 0_{N/4} \end{pmatrix}, \quad Q^{(k)} = \begin{pmatrix} 0_{N/2} & 0_{N/2} \\ I_{N/2} & 0_{N/2} \end{pmatrix},
\]

and \( 0_N \) denotes the zero matrix of order \( N \).

Hence,

\[
\text{rank} \left( \begin{pmatrix} \tilde{S}_a^{(k)} \\ S_a^{(k+1)} (A_0^{(k+1)} - A_j^{(k+1)}) \end{pmatrix} \right) = \text{rank} \left( \begin{pmatrix} \tilde{S}_a^{(k)} \\ S_a^{(k)} (I_N - A_{j-1}^{(k)}) \end{pmatrix} \right) + \text{rank} \left( \begin{pmatrix} \tilde{S}_a^{(k)} \\ S_a^{(k)} (I_N + A_{j-1}^{(k)}) \end{pmatrix} \right)
\]

for \( j \geq 2 \).

Further, note from (11) that \( A_0^{(k)} - A_{j-1}^{(k)} = I_N - A_{j-1}^{(k)} \) is nonsingular if \( j \geq 2 \). Then,

\[
\text{rank} \left( \begin{pmatrix} \tilde{S}_a^{(k)} \\ S_a^{(k)} (I_N + A_{j-1}^{(k)}) \end{pmatrix} \right) = \text{rank} \left( \begin{pmatrix} \tilde{S}_a^{(k)} \\ S_a^{(k)} (I_N - A_{j-1}^{(k)}) \end{pmatrix} \right)
\]

\[
= N/2
\]

where in the first identity we use (11) and the in last identity we use the assumption if \( j \geq 3 \) and Proposition 2 if \( j = 2 \). This completes the proof after substituted into (21).

The following main result is immediate.

**Theorem 3.** \( S_a^{(k)} \) and \( \tilde{S}_a^{(k)} \) that defined by (16), (17), (18) and (19) are the repair matrices for the first parity node of the \( (k + 2, k) \) Zigzag code, whose repair disk I/O is \( kN + N - k \).

**Proof:** The optimal repair property of repair matrices \( S_a^{(k)} \) and \( \tilde{S}_a^{(k)} \) is obvious from Propositions 1, 2 and 3.

Note that there is only one zero column in \( S_a^{(k)} \) and no zero columns in \( \tilde{S}_a^{(k)} \), which means \( N - 1 \) elements should be read in each of the systematic nodes and all the \( N \) elements should be read in the second parity node to repair the first parity node. Hence, the disk I/O to repair the first parity node is \( kN + N - k \).

By Lemma 1 the second parity node of the \( (k + 2, k) \) Zigzag code can also be optimally repaired. However, if we use \( S_b^{(k)} A_i^{(k)} \), \( 0 \leq i < k \) and \( \tilde{S}_b^{(k)} \) as the repair matrices, where \( S_b^{(k)} = \bar{S}_a^{(k)} \) and \( \tilde{S}_b^{(k)} = S_a^{(k)} \) are defined by (16), (17), (18) and
Theorem 4. Let \( S_a^{(k)} \) and \( \tilde{S}_a^{(k)} \) be defined by (22), (16) and (18), then \( S_b^{(k)} A_i^{(k)} \), \( 0 \leq i < k \) and \( \tilde{S}_b^{(k)} \) are the repair matrices for the second parity node of the \((k + 2, k)\) Zigzag code where \( S_b^{(k)} = S_a^{(k)} \) and \( \tilde{S}_b^{(k)} = \tilde{S}_a^{(k)} \). Moreover, the disk I/O to optimally repair the second parity node is \( kN + N - k \).

Proof: Firstly, it can be easily verified that the results in Propositions 1, 2 and 3 are also hold for \( S_a^{(k)} \) and \( \tilde{S}_a^{(k)} \) defined from the initial values \( E^{(2)} \), \( F^{(2)} \), \( S_a^{(2)} \) and \( \tilde{S}_a^{(2)} \) in (22). Secondly, it follows from Lemma 1 that \( S_a^{(k)} A_i^{(k)} \), \( 0 \leq i < k \) and \( S_a^{(k)} \) are the repair matrices for the second parity node of the \((k + 2, k)\) Zigzag code.

From Theorems 3 and 4, it is seen that the disk I/O to optimally repair the parity nodes of the Zigzag code is very close to the lower bound given in Lemma 2.

Finally, we give some examples of the repair matrices for the parity nodes of the \((k + 2, k)\) Zigzag code.

Example 1. The first parity node of the \((5, 3)\) Zigzag code, \((6, 4)\) Zigzag code, and \((7, 5)\) Zigzag code, can be respectively optimally repaired by the following matrices

\[
S_a^{(3)} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \tilde{S}_a^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
S_a^{(4)} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}, \quad \tilde{S}_a^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}
\]

\[
S_a^{(5)} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{S}_a^{(5)} = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
S_b^{(3)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \tilde{S}_b^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}
\]

\[
S_b^{(4)} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{S}_b^{(4)} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

The second parity node of the \((5, 3)\) Zigzag code, \((6, 4)\) Zigzag code, and \((7, 5)\) Zigzag code, can be respectively optimally repaired by the following matrices
\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

REFERENCES

[1] R. Bhagwan, K. Tati, Y.-C. Cheng, S. Savage, and G.M. Voelker, “Total recall: System support for automated availability management,” presented at the Symp. Networked Systems Design and Implementation (NSDI), 2004.

[2] M. Blaum, J. Brady, J. Bruck, and J. Menon, “EVENODD: An efficient scheme for tolerating double disk failures in RAID architectures,” IEEE Trans. Comput., vol. 44, no. 2, pp. 192-202, Feb. 1995.

[3] M. Blaum, J. Bruck, and E. Vardy, “MDS array codes with independent parity symbols,” IEEE Trans. Inform. Theory, vol. 42, no. 2, pp. 529-542, Mar. 1996.

[4] P. Corbett, B. English, A. Goel, T. Grcanac, S. Kleiman, J. Leong, and S. Sankar, “Row-diagonal parity for double disk failure correction,” in Proc. 3rd USENIX Symp. File Storage Technol., 2004.

[5] F. Dabek, J. Li, E. Sit, J. Robertson, M. Kaashoek, and R. Morris, “Designing a DHT for low latency and high throughput,” presented at the Symp. Networked Systems Design and Implementation (NSDI), 2004.

[6] A.G. Dimakis, P. Godfrey, Y. Wu, M. Wainwright, and K. Ramchandran, “Network coding for distributed storage systems,” IEEE Trans. Inform. Theory, vol. 56, no. 9, pp. 4539-4551, Sep. 2010.

[7] A.G. Dimakis, K. Ramchandran, Y. Wu, and C. Suh, “A survey on network codes for distributed storage,” IEEE Trans. Inform. Theory, vol. 57, no. 5, pp. 2548-2568, May 2011.

[8] C. Huang and L. Xu, “STAR: an efficient coding scheme for correcting triple storage node failures,” IEEE Trans. Comput., vol. 57, no. 7, pp. 889-901, Jul. 2008.

[9] J. Li, X.H. Tang, and U. Parampalli, “A framework of constructions of minimal storage regenerating codes with the optimal access/update property,” IEEE Trans. Inform. Theory, vol. 61, no. 4, pp. 1920-1932, Apr. 2015.

[10] D.S. Papailiopoulos, A.G. Dimakis, and V.R. Cadambe, “Repair optimal erasure codes through hadamard designs,” IEEE Trans. Inform. Theory, vol. 59, no. 5, pp. 3021-3037, May 2013.

[11] J.S. Plank, “The RAID-6 Liberation Code,” Int. J. High Perform. Comput. Appl., vol. 23, no. 3, pp. 242-251, Aug. 2009.

[12] S. Rhea, C. Wells, P. Eaton, D. Geels, B. Zhao, H. Weatherspoon, and J. Kubiatowicz, “Maintenance-free global data storage,” IEEE Internet Comput., pp. 40-49, Sep. 2001.

[13] T. Tamo, Z. Wang, and J. Bruck, “Zigzag codes: MDS array codes with optimal rebuilding,” IEEE Trans. Inform. Theory, vol. 59, no. 3, pp. 1597-1616, Mar. 2013.

[14] X.H. Tang, B. Yang, and J. Li, “New repair strategy of hadamard minimum storage regenerating code for distributed storage system,” [Online]. Available: arXiv: 1312.3537v1 [cs.IT].

[15] Z. Wang, I. Tamo, and J. Bruck, “Long MDS codes for optimal repair bandwidth,” in Proc. IEEE Int. Symp. Inform. Theory, Jul. 2012, pp. 1182-1186.

[16] Z. Wang, I. Tamo, and J. Bruck, “On codes for optimal rebuilding access,” in Proc. 49th Annu. Allerton Conf. Commun., Control, Comput., Sep. 2011, pp. 1374-1381.

[17] Y. Wang, X. Yin, and X. Wang, “MDR codes: A new class of RAID-6 codes with optimal rebuilding and encoding,” IEEE J. Sel. Areas Commun., vol. 32, no. 5, pp. 1008-1018, May 2014.

[18] L. Xu, V. Bohossian, J. Bruck, and D. Wagner, “Low-density MDS codes and factors of complete graphs,” IEEE Trans. Inform. Theory, vol. 45, no. 6, pp. 1817-1826, Sep. 1999.

[19] L. Xu and J. Bruck, “X-code: MDS array codes with optimal encoding,” IEEE Trans. Inform. Theory, vol. 45, no. 1, pp. 272-276, Jan. 1999.