How Private Is Your Voting? A Framework for Comparing the Privacy of Voting Mechanisms

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Abstract. Voting privacy has received a lot of attention across several research communities. Traditionally, the cryptographic literature has focused on how to privately implement/emulate a voting mechanism. Yet, a number of recent works attempt to capture (and minimize) the amount of information one can infer from the output (rather than the implementation) of the voting mechanism. These works apply differential privacy, in short DP, techniques which noise the outcome to achieve privacy. This approach intrinsically compromises accuracy, rendering such a voting mechanism unsuitable for most realistic scenarios.

In this work we address the question of what is the inherent privacy that different voting rules achieve, without noising the result. To this end we utilize a well accepted notion of noiseless privacy introduced by Bassily et al. [FOCS 2013] called Distributional Differential Privacy, in short DDP. We argue that under standard assumptions in the voting literature about the distribution of votes, most natural mechanisms achieve a satisfactory level of DDP, indicating that noising—and its negative side-effects for voting—is unnecessary in most cases.

We then put forth a systematic study of noiseless privacy of commonly studied of voting rules, and compare these rules with respect to their achieved privacy. Note that both DP and DDP induce (possibly loose) upper bounds on the amount of information that can be inferred, which makes them insufficient for such a task. To circumvent this, we introduce an exact notion of privacy, which requires the bound to be exact (i.e. optimal) in a well defined manner. This allows us to order different voting rules with respect to their achieved privacy. Although motivated by voting, our definitions and techniques can be generically applied to address the optimality (with respect to privacy) of general mechanisms for privacy-preserving data release.
1 Introduction

Privacy-ensuring voting has been extensively studied in the cryptographic literature. The scope of these works is to enable voters to cast their votes (possibly online) in a manner that ensures that someone with access to their messages cannot link them to their votes. However, another, equally important question about voting privacy has received much less attention: How much can someone infer about your vote by observing information on the outcome released by different mechanisms? Suppose, for example, you cast a vote in a presidential election indicating your preference over the candidates. After the election, the winner is announced. What can an (adversarial) observer infer about your vote if a count of votes that indicate the same preference—i.e., the histogram of votes—is announced? How about if the number of votes for each candidate in your city is announced? How about in your building or in your family? Arguably, answering the above privacy-related questions is highly relevant in practice. For example, minimizing the amount of information leaked by the information announced by the voting mechanism can help protect against censorship and coercion and prevent vote buying.

The above questions are closely related to questions about releasing information on a dataset in a privacy-preserving manner. Indeed, a voting system can be cast as a two-step process: first the votes (data) are collected by a trusted authority into a database and then a voting rule is applied on the collected data to announce the outcome. Casting the problem in such a way indicates that standard methods for protecting the privacy of released data could be also applied to voting for addressing the above questions.

The most widely accepted such method is differential privacy (in short, DP) \[8\]. At a high level, DP bounds the amount of information that a function of a dataset reveals on any record of data. Informally, for any given database \(X\) and any given query \(f\) on \(X\), DP requires that no observer can distinguish the output \(f(X)\) from the output \(f(X')\) of \(f\) applied on any neighboring database \(X'\)—i.e., \(X'\) is derived from \(X\) by removing (or modifying) any given record. The main method for achieving DP for a given query is by means of a mechanism which noises the outcome of the query, so that one cannot decide whether or not any individual record was included in computing the query’s outcome—i.e., the response distribution should be almost the same with or without any given record.

The privacy level of a DP mechanism is indicated by a scalar \(\epsilon \geq 0\)—resulting in what is known as \(\epsilon\)-DP—which upper bounds the ability of any distinguisher to distinguish between the (noised) output applied on \(X\) and \(X'\). In order to allow for better DP mechanisms, the above traditional definition of DP is often extended by another parameter \(\delta \in [0, 1]\) which, informally, bounds the probability that any privacy guarantee is compromised. This yields the notion of \((\epsilon, \delta)\)-DP \[9\]. In both these notions, the smaller the \(\epsilon\) (and \(\delta\)) the better the corresponding privacy guarantee.

Given its wide adoption as a mechanism for making data release private a number of works has recently applied DP in voting privacy as a tool for limiting the inference from the output of voting mechanisms \[25,17,20,4\]. Alas, as natural as the above idea might seem at first, it has two critical issues.

First, noising the outcome inherently reduces the accuracy of the output, i.e., introduces a probability of error, which is unacceptable in many high-stakes situations. To see this, imagine a situation where candidate A wins by majority, but the mechanism outputs candidate B as the winner, arguing that this is to make the votes more private. And, of course, mechanisms that apply more noise are more private (i.e., achieve better DP), but they are also less accurate. Thus, these solutions result in voting mechanisms which might, with some probability that get higher with more privacy,
announce a wrong winner; this makes such solutions unusable for most applications. To resolve this problem, in this work we employ one \cite{1} of several distributional notions of differential privacy, that allow for noiseless mechanisms \cite{3,2,11,17,1}.

The second, and more delicate issue, is that differential privacy puts only an upper bound on the privacy leakage of a mechanism. In other words, given two \( \epsilon \)-DP mechanisms \( M_1 \) and \( M_2 \), one cannot directly answer which one is more private, as one is often not able to directly use the mechanisms’ DP parameters to compare voting rules—or any other type of mechanisms—with respect to their privacy. To rectify this, in this work we introduce the notion of exact (distributional) differential privacy. We note in passing that the question of comparing DP mechanisms with respect to their utility, has been extensively studied in the related literature \cite{23,5,16,2,12}. However, these works introduce utility as a function of a mechanism’s accuracy (according some proposed metric) thus making them insufficient to answer our original question: Given two \( \epsilon \)-DP mechanisms, which one is the most private?

1.1 Contributions

Motivated by the above voting-privacy questions, we propose a framework for comparing (voting) mechanisms according to the privacy they provide. To this end, we put forth the notion of exact privacy. Intuitively, a mechanism is exact private with respect to a set of privacy parameters, if there are no strictly better parameters that the mechanism can achieve. More concretely, we start by devising an exact version of \( \epsilon \)-differential privacy (\( \epsilon \)-DP) \cite{8} which we term \( \epsilon \)-exact Differential Privacy (in short, \( \epsilon \)-eDP). Our notion renders a mechanism \( \epsilon \)-eDP if and only if it satisfies the following conditions: (1) it is \( \epsilon \)-DP and (2) there exists no \( \epsilon' < \epsilon \) such that the mechanism is \( \epsilon' \)-DP. This definition of exactness is trivially extended to \((\epsilon, \delta)\)-eDP by requiring exactness, as in Condition 2, above, with respect to both \( \epsilon \) and \( \delta \).

Next, we use exact privacy to answer questions of the type: Given two \( \epsilon \)-DP mechanisms, which one is preferred? Exact privacy gives a way to answer to this question: prefer the mechanism which is inherently more private, i.e., achieves better (smaller) exact privacy parameters. The above natural question has been approached in the past by introducing explicit utility functions and using them as a measure of mechanism quality (eg. \cite{23,5,12,16,2}). In fact, exact privacy and utility turn out not to be completely unrelated quantities. Indeed, we show that a \( \epsilon \)-DP mechanism is \( \epsilon \)-eDP if it is optimal for some utility function that monotonically increases as privacy deteriorates (Theorem 4). We exemplify this relationship between exactness and utility by showing that the \( \alpha \)-truncated geometric mechanism—a provably optimal DP mechanism for a natural class of utility functions \cite{12}—is \( \ln(1/\alpha) \)-exact DP; to prove this we prove that one of the above class of utilities is monotonically increasing and apply our theorems. Given that our notion of monotonicity implies exactness, one might ask whether the converse direction is true: given an \( \epsilon \)-exact DP mechanism and a monotonically increasing utility function, is the mechanism optimal for this utility? We show that this is not the case, by analyzing the well known Laplace mechanism \cite{10} which, as we prove is exact DP, but it is not optimal for a monotonically increasing utility function.

Having defined exact privacy as a notion for comparing differentially private mechanisms, we next turn to using this notion in voting, and comparing the privacy of different voting rules. We first observe that, by its nature, noising the outcome of a voting mechanism renders the mechanism unsuitable for most voting applications. For this reason we resort to distributional differential privacy (DDP) \cite{1} and extend the notion of exactness to this definition as well, resulting to \((\epsilon, \delta)\)-eDDP.

Avoiding noising the output does not come for free, as DDP—and any other noiseless privacy notion—requires assumptions on (1) the distribution from which the data is generated and (2) on
the auxiliary information the observer/distinguisher has on it. We tackle the first issue by looking back into our problem, i.e., voting, and employ a standard assumption about the distribution of votes in the related literature, concretely, that that each row of the database (i.e., each vote) is independently and identically distributed (i.i.d.). In social choice, this is a natural assumption called impartial culture.

The second issue, i.e., auxiliary information, is more subtle as different auxiliary information trivially affects the privacy of different mechanisms. Hence, to order DDP mechanisms with respect to their privacy it is necessary to consider them both under the same auxiliary information. The most objective such information is naturally one which is not correlated with the input distribution or the output of the mechanism. For this reason, in this work we choose the empty (set) the auxiliary information.

We remark that, to our knowledge, no prior work has considered applying the idea of DDP on voting privacy. Therefore, as a sanity check we demonstrate that reasonable DDP parameters can be proven for (noiseless) voting rules under our assumptions above (Theorem 3). Concretely, we prove that when database rows are i.i.d., the histogram mechanism (eg. outputting the profile, which is a histogram of votes) is\(\left(\epsilon \geq 2 \ln(1 + \frac{1}{n}), \delta = \exp(-\Omega(n\min(2 \ln(2), \epsilon)^2))\right)\)-DDP\(^2\) for every database size \(n\). As DDP is immune to post-processing, voting rules which depend only on the profile also satisfy the same parameters as above.

Having demonstrated that DDP is a well-behaved privacy notion in the context of voting, we dive into exactness which will allow us to compare different voting rules. Here we are faced with a multi-parametric problem: Recall that DDP (hence also exact DDP) has two parameters \(\epsilon\) and \(\delta\) which means that the induced ordering if both parameters are left free is a partial order. For example, the following question has not natural answer: Is \((\epsilon = 0.1, \delta = 0.2)\) more, or less private than \((\epsilon = 0.2, \delta = 0.1)\)?

To avoid incomparable results, we fix one parameter to its optimal value (i.e., 0) and compare the other. We observe that fixing \(\delta = 0\) can yield a notion of (distributional) differential privacy which is in many cases cannot be instantiated\(^3\). Therefore we choose to fix \(\epsilon = 0\) and compare mechanisms with respect to how good a \(\delta\) they achieve. We remark that if we would choose \(\epsilon > 0\), then Theorem 3 give us a negligible \(\delta\) for outputting the histogram of votes—which means that any mechanism which depends on this histogram (like voting rules) have at most negligible \(\delta\). Hence, by choosing \(\epsilon = 0\), we get a much more interesting asymptotic separation of \(\delta\) values (for example, \(\Theta\left(\frac{1}{\sqrt{n}}\right)\) versus negligible in \(n\), as we see in Theorem 6). In addition, the choice \(\epsilon = 0\) yields a nice interpretation of exactness, since \(\delta\)—which is now the distinguishing advantage of the DDP experiment—becomes the absolute value of the difference between probability of the mechanism’s output and the simulator’s output, in other words, their statistical distance.

We use eDDP to order standard voting rules, namely, those in the class of Generalized Scoring Rules (GSR) \(^2\)8. We study the privacy achieved of these mechanisms for two or more \((m)\) candidates, as the number of votes \((n)\) increases.

We will first study the case when the number of candidates \(m = 2\). We consider the class of \(\alpha\)-majority rules (also called \(\alpha\)-biased plurality). For two candidates A and B, the \(\alpha\)-majority winner is A if \(\alpha\) fraction of the voters voted for A; otherwise, the winner is B. We note that \(\alpha\)-

\(^2\) When the database distribution and auxiliary information (which is the empty set for our results) are clear from context, we sometimes omit them.

\(^3\) For example, consider a simple mechanism \(M\) which counts the number of \(x\)’s in the database, and a database where one row is fixed to \(x\). Then, it is not possible for \(M\) to output a count of 0. When \(\delta\) is fixed to zero in the DDP definition, \(\epsilon^*\) is the ratio of probabilities of the the mechanisms and the simulators output, both probabilities conditioned on a database row being fixed to some value. In our example one of these probabilities is zero, so no finite \(\epsilon\) can satisfy this ratio. Thus, this simple mechanism does not satisfy \((\epsilon, \delta = 0)\)-DDP for any \(\epsilon\).
majority rules cover a wide range of commonly used voting rules for two candidates, including simple majority/plurality ($\alpha = 1/2$), supermajority ($\alpha > 1/2$) and submajority ($\alpha < 1/2$).

**Theorem 1 (Informal, see Theorem 6).** Let each vote be iid, and let $p$ be the probability a voter chooses the first candidate and $(1−p)$ be the probability a voter chooses the second. The mechanism outputting the $\alpha$-biased plurality winner is $\epsilon = 0, \delta = \Theta \left( \sqrt{\frac{1}{n} \left( \frac{1}{2} \right)^\alpha \left( \frac{1−\alpha}{2−\alpha} \right)^{1−\alpha}} \right)$-eDDP.

The above result highlights a separation of the privacy of the same mechanism, based on the database distribution. When the bias $\alpha$ is equal to the probability $p$, then $\delta = \Theta \left( \frac{1}{\sqrt{n}} \right)$. Otherwise, $\delta$ is negligible in the database size (number of votes) $n$.

The analysis of the above uses a new technical tool we introduce, termed the trails technique. The trails technique can be used to compute $(0, \delta)$-eDDP for mechanisms which depend only on the histogram (such as voting rules), and database distributions where rows are independent. To apply this technique we introduce an alternative definition of DDP which is equivalent to DDP in our setting, where database rows are independently distributed and the observer has no auxiliary information. A mechanism $M$ is $(\epsilon, \delta)$-DDP in the alternative definition if the distributions $M(X)|X_i = x$ and $M(X)|X_i = x'$ are $(\epsilon, \delta)$ close.

The idea of the trails techniques is as follows: Since $M$ depends only on the histogram, we can consider any subset of $M$’s range $S$ as a subset of histograms that are mapped to $S$. We can thus split up computing this distance over any $S$, into computing over disjoint trails. A trail is a set of histograms formed the following way: starting with histogram $t$, subtract one from bin $j$ and add one to bin $k$ from the previous histogram, and repeat $q$ times. In other words, a set of trails is of the form $\{ t - z \cdot bin_j + z \cdot bin_k : 0 \leq z \leq q \}$. The main observation is that when each database row is independently distributed, for any trail, computing the distance over it cancels all but the probability of the first and last histogram in the trail. Then, computing $(0, \delta)$-exact DDP is the same as computing the probabilities of the first and last histograms of a set of trails. We detail this technique in Section 5. We stress that the trail technique is not restricted to the two candidate case; in fact the technique is described for the multi-candidate setting and is used also in later sections. We believe that this technical tool is of independent interest, i.e, relevant for the analysis of exact privacy of also other mechanisms, beyond just voting.

Next, we consider the number of candidates $m \geq 2$. We begin by studying the exact privacy of outputting an entire histogram of votes, and the privacy of outputting a quantity related to plurality, the plurality score, which is a histogram of the top-ranked candidate in each vote. To do so, we show that outputting a histogram of any constant $b$ bins satisfies $(\epsilon = 0, \delta = \Theta \left( \frac{1}{\sqrt{n}} \right))$-eDDP (Theorem 7). By setting $b = m!$ for histogram of votes (where $m!$ is the number of linear orders on $m$ candidates and is constant when $m$ is constant) and $b = m$ for plurality score, we show that both these mechanisms satisfy $(\epsilon = 0, \delta = \Theta \left( \frac{1}{\sqrt{n}} \right))$-eDDP.

We proceed to present our main two results, which prove the exact privacy of (a large subset of) GSRs. The subset of GSRs we consider are the voting rules which satisfy the properties of monotonicity (we note this is not related to monotonically increasing utilities), canceling-out, and for which there exists a locally stable profile/histogram of votes. In the simple case of two candidates A and B, these are voting rules whose winner does not change even if one were to add both a vote for A and a vote for B, or if one were to raise the ranking of the winner in some votes. Moreover, a locally stable profile is one where, if we only replace a few of the votes, the winner does not change.

**Theorem 2 (Informal, see Theorem 8).** For any Generalized Scoring Rule which satisfies the properties of monotonicity, canceling-out, and for which there exists a locally stable profile, when each vote is iid:
1. The mechanism that outputs its winner is \( (\epsilon = 0, \delta = \Theta \left( \frac{1}{\sqrt{n}} \right)) \)-exact DDP, when each vote is also uniform over the set of linear orders over candidates.

2. There exist database distributions whose Lebesgue measure can be arbitrarily close to 1, where the mechanism that outputs its winner is \( (\epsilon = 0, \delta = O(\exp(-\Theta(1)n))) \)-DDP.

The above leads to the somewhat surprising conclusion that, when each vote is uniformly distributed, common voting rules have asymptotically comparable exact privacy \( \delta(n) = \Theta \left( \frac{1}{\sqrt{n}} \right) \). As this is the privacy of the histogram mechanism, this means that the uniform distribution induces the worst case asymptotic privacy for voting rules. Moreover, it shows that for these GSR voting rules, the privacy differs asymptotically depending on the database distribution.

Finally, we apply our main result by showing a ranking of different voting rules based on concrete \( \delta \) values. For each voting rule, we compute the \( \delta \) values for the mechanism which outputs the winner, and the distribution where each vote is uniformly distributed. We use linear regression (on the inverse square of these values) to fit them to \( \delta = \frac{1}{\sqrt{an+b}} \) (where \( \delta = \Theta \left( \frac{1}{\sqrt{n}} \right) \) is our asymptotic theoretical bound). We find that fit is reasonable, with mean square error of 0.038 to 0.057, even for relatively few number of votes \( (n < 50) \). Based on the constant \( a \) in the fitted result (where the larger the \( a \), the smaller the \( \delta \) and the more private) the ranking from least to most private is: Borda, STV, Maximin, Plurality, and 2-approval. This ranking confirms our intuition that, for example, a voting rule which does not use the whole ranking in the vote (eg. Plurality, 2-approval) is more private than those which do. Yet it also informs us of the comparative information leakage between rules like Borda, STV, and Maximin, all of which use the entire ranking of the vote. We remark that STV is not in the subset of GSRs for which we have proven the asymptotic bound; however, its concrete \( \delta \)'s still fit well to \( \frac{1}{\sqrt{an+b}} \).

1.2 Organization

The remainder of the paper is organized as follows. In Section 3 we compute the DDP parameters for the histogram mechanism. In Sections 4 we describe our definition of exactness and its relation with utility based optimality. Then, in Sections 5 and 6 we prove our main asymptotic results about eDDP for the case of two and arbitrary number of candidates, respectively. Finally, Section 7 includes our empirical estimations of the exact privacy parameters for the various rules used in this work. For space reasons several details and proofs have been moved to a clearly marked appendix. We have already discussed most relevant literature; for completeness in Appendix ?? we include a more detailed review of these works.

2 Preliminaries and Notation

In this section we present our notation, and give an overview of Generalized Scoring Rules. We use standard Differential Privacy (DP) \[8\] and Distributional Differential Privacy (DDP) definitions \[1,14\]. For self-containment, we refer to Appendix ?? for these definitions.

We call the set of values a row in the database can take the universe, denoted by \( U \). The set of all databases of any size (i.e. number of rows), is denoted \( U^* \), and the set of all databases of size \( n \in \mathbb{N}^+ \) is \( U^n \). For a universe of finite \( b \in \mathbb{N}^+ \) values, and constant probabilities \((p_1, \cdots, p_b)\), we denote by \( X_{n,(p_1,\cdots,p_b)} \) the distribution on \( U^n \), where the support is only in \( U^n \), and each of the \( n \) rows is an independent and identically distributed \((i.i.d.)\) random variable which takes the value \( x_i \) with probability \( p_i \).
Let $C = \{x_1, \ldots, x_m\}$ denote a set of $m \geq 1$ alternatives (also called candidates). Let $L(C)$ denote the set of all linear orders over $C$, that is, all the set of antisymmetric, transitive, and total binary relations. Each vote is a linear order $V \in L(C)$. Let $n$ denote the number of votes. In general, the profile is the database of votes in the set $L(C)^*$, but for the voting rules we consider, the profile $P \in \mathbb{N}^{m!}$ is the histogram of all votes. We consider a voting rule $r$ to be a mapping that takes a profile as input and outputs a unique winner in $C$.

For example, a positional scoring rule is characterized by a scoring vector $s = (s_1, \ldots, s_m)$ with $s_1 \geq s_2 \geq \cdots \geq s_m$. For any candidate $c$ and any linear order $V \in L(C)$, we let $s(V, c) = s_j$, where $j$ is the rank of $c$ in $V$. Given a profile $P$, the positional scoring rule chooses a candidate $c$ that maximize $\sum_{V \in P} s(V, c)$ and break ties when multiple candidates have the highest score. Plurality, $k$-approval (for any $1 \leq k \leq m$), and Borda are positional scoring rules, with scoring vectors $(1, 0, \ldots, 0)$, $(1, \ldots, 10, \ldots, 0)$, $(m - 1, m - 2, \ldots, 0)$, respectively.

Another commonly used voting rules is the single transferable vote (STV) rule, which determines the winner in steps: in each step, the alternative ranked in the top positions least often is eliminated from the profile (and break ties when necessary), and the winner is the remaining candidate.

It turns out that many commonly studied voting rules belong to the class of Generalized Scoring Rules (GSRs).

**Definition 1 (Generalized Scoring Rules (GSR)).** A Generalized Scoring Rules (GSR) is defined by a number $K \in \mathbb{N}$ and two functions $f : L(C) \to \mathbb{R}^K$ and $g$, which maps any weak order over the $\{1, \ldots, K\}$ to $C$.

Given a vote $V \in L(C)$, $f(V)$ is the generalized score vector of $V$. Given a profile $P$, we let $f(P) = \sum_{V \in P} f = f(V)$, called the score. Then, then winning candidate is given by $g(\text{Ord}(f(P)))$, where $\text{Ord}$ is the function which outputs the weaker order of the $K$ components in $f(P)$.

We say that a rule is in GSR if it can be described by some $f, g$ as above. Examples of GSR rule can be found in Appendix [B.1]

### 3 Noiseless Privacy in Voting

In this section, we demonstrate that under standard assumptions in the voting literature, noising the outcome is not necessary for achieving a reasonable notion of privacy. Concretely, we show that the histogram mechanism $\text{Hist} : U^* \to \mathbb{N}^{[\ell]}$ satisfies Distributional Differential Privacy (DDP) with good parameters (Theorem 3).

**Theorem 3 (DDP of the histogram mechanism).** Let $U = \{x_1, \ldots, x_b\}$ and $p_{\min} = \min_{i \in [\ell]}(p_i)$.

For all $\epsilon(n) \geq 2\ln(1 + \frac{1}{p_{\min}n})$, there is a $\delta(n) = \exp(-\Omega(np_{\min}\min(2\ln(2), \epsilon(n))^2))$ such that for all $n \in \mathbb{N}$, the histogram mechanism $\text{Hist} : U^* \to \mathbb{N}^b$ is $(\epsilon(n), \delta(n), \Delta = \{(X_{n, (p_1, \ldots, p_b)}, \emptyset)\})$-DDP.

**Proof.** (sketch) Let $X = X_{n, (p_1, \ldots, p_b)}$ and $X_{-i}|X_i = x$ be the database distribution except for the $i$th row, which has been set to $x$. Since every row of $X$ is independent, $X_{-i}|X_i = x$ is the same distribution as $X_{-i}|X_i = x'$ for any $x'$. Moreover, since every row is identically distributed, the choice of row $i$ is not important. On input a database missing the $i$th row, we let the simulator guess the missing row as some $x'$, and apply the histogram mechanism to the resulting database. Since the distribution $X_{-i}$ is independent of the value of missing row, the distribution of the simulator’s output is $\text{Hist}(X)|X_i = x'$. The main idea here is that for any two sets $S, B$ and any value $x$, the probability $Pr(\text{Hist}(X) \in S|X_i = x)$, is less than or equal to $Pr(\text{Hist}(X) \in S \cap \overline{B}|X_i = x) + Pr(\text{Hist}(X) \in B|X_i = x)$. For a choice of $B$, we can show that for any value $x'$ (which the
simulator guesses to be the value of the missing row), the ratio \( \frac{\Pr(\text{Hist}(X) \in S \cap B | X_i = x)}{\Pr(\text{Hist}(X) \in S \cap B | X_i = x')} \leq e^\epsilon \). Then, we show by Chernoff bound that \( \Pr(\text{Hist}(X) \in B | X_i = x) \) is exponentially small in \( n \), and let this be \( \delta \).

Like differential privacy, DDP is immune to post-processing\(^4\) (This is formally argued in Lemma 3 in Appendix C). Thus, the result for histograms directly implies the same parameters for any mechanisms that only depend on the histogram of the database—for example, the voting rules we consider, which depend on the histogram of votes. This is formally stated (and proved) in Appendix C (see Corollary 3).

### 4 Exact Privacy for Comparing Mechanisms

In this section we introduce a natural extension to privacy definitions, which we call exact privacy. We first study this notion with respect to Differential Privacy (DP), by presenting its relationship to the notion of utility. Then, we apply exact privacy to distributional differential privacy (DDP) so that we can study noiseless mechanisms.

Intuitively, a mechanism has exact privacy with parameters \( \epsilon \) and \( \delta \) if the mechanism cannot satisfy the privacy definition with strictly better parameters.

**Definition 2 (Exact Distributional Differential Privacy (eDDP)).** A mechanism \( M \) is \((\epsilon, \delta, \Delta)-\)Exact Distributional Differential Privacy (eDDP) if it is \((\epsilon, \delta, \Delta)-\)DDP and there does not exist \((\epsilon' \leq \epsilon, \delta' < \delta)\) nor \((\epsilon' < \epsilon, \delta' \leq \delta)\) such that \( M \) is \((\epsilon', \delta', \Delta)-\)DDP.

The above definition can easily be altered to define \((\epsilon, \delta)-\)exact differential privacy (eDP), by replacing each instance of DDP in definition with DP.

In order to better understand the use of exact privacy for comparing mechanisms it is useful to investigate the relationship between utility and exact DP. Informally, Theorem 4 says that when the utility is monotonically increasing with \( \epsilon \), then any optimal \( \epsilon \)-DP mechanism is also \( \epsilon \)-exact DP. We show an example of an exact DP mechanism by applying this theorem (Corollary 3). However, the converse of the theorem is not true—there exists an \( \epsilon \)-exact DP mechanism that is not optimal for a monotonically increasing utility function (Lemma 5). We refer to Appendix D for definitions and proofs.

**Theorem 4 (Utility optimality implies Exact Privacy).** Let \( M \) be any set of mechanisms and \( u \) be any utility function monotonically increasing over \( M \). Then, an optimal (over \( M \)) \( \epsilon \)-differentially private mechanism \( M \) is \( \epsilon \)-exact differentially private.

### 5 Comparing Voting Mechanisms: The Two-Candidate Case

In this section, we completely characterize exact distributional differential privacy (exact DDP) for two candidates under any biased majority rule w.r.t. any i.i.d. distribution. To this end, we first introduce an alternative definition of DDP which is equivalent to DDP in our setting (independent database rows, no auxiliary information, and \( \epsilon = 0 \)), then a technique called “trails” to bound the exact DDP under the alternative definition.

**Definition 3 (Alternative Definition of DDP).** A mechanism \( M : U^* \rightarrow \mathcal{R} \) is \((\epsilon, \delta, \Delta = \{(X, \emptyset)\})-\)DDP if for all \( X \in \Delta, \) the following inequality is satisfied for any \( i, \) any \( x, x' \in \text{Supp}(X_i) \), and \( S \in \mathcal{R} \)

\[
\Pr(M(X) \in S | X_i = x) \leq e^\epsilon \Pr(M(X) \in S | X_i = x') + \delta
\]

\(^4\) Note that post-processing is not a property of all privacy definitions, such as exact privacy defined in the next section.
The next lemma shows that when there is no auxiliary information, and when database rows are independently distributed, Definition 3 above is equivalent to (the simulation-based) DDP \([1]\), see Definition 8) up to parameter changes. In particular, when \(\epsilon = 0\), the \(\delta\) of the two definitions differs by a constant factor of two, and since our results are asymptotic, our results for eDDP in the alternative definition also holds for the simulation-based one.

**Lemma 1.** Suppose a mechanism \(M : \mathcal{U}^* \rightarrow \mathcal{R}\) is \((\epsilon, \delta, \Delta = \{(X, \emptyset)\})\)-(simulation-based) DDP, then \(M\) is \((2\epsilon, (1 + e^\epsilon)\delta, \Delta)\)-DDP for Definition 3. Conversely, if \(M\) is \((\epsilon, \delta, \Delta = \{(X, \emptyset)\})\)-DDP for Definition 3 then \(M\) satisfies \((\epsilon, \delta, \Delta)\)-(simulation-based) DDP.

In light of Definition 3, the \((0, \delta, \Delta)\)-eDDP of a mechanism can be characterized by

\[
\delta = \max_{S, x, x'} \left[ \Pr(M(X) \in S | X_i = x) - \Pr(M(X) \in S | X_i = x') \right]
\]

However, the RHS of the equation is hard to bound. We address this challenge by breaking the RHS into a summation over multiple sets, each of which contains consecutive histograms and is called a trail. For any histogram \((t_1, \cdots, t_b) \in \mathbb{N}^b\), any \(z \in \mathbb{Z}\) and \(j \leq b\), we let \((t_1, \cdots, t_j + z, \cdots, t_b)\) denote the histogram \((t_1, \cdots, t_j, t_{j+1}, \cdots, t_b)\).

**Definition 4 (Trail).** Given a pair of data entries \((j, k)\) where \(j \neq k\), a histogram \(H\), and a length \(q\), we define the trail \(T_{H,x_j,x_k,q} = \{H - zx_j + zx_k : 0 \leq z \leq q\}\), where \((j, k)\) is called the direction of the trail, \(H\) is called the entry of the trail, also denoted by \(Entr(T_{H,x_j,x_k,q})\), and \(H - qx_j + qx_k\) is called the exit of the trail, denoted by \(Exit(T_{H,x_j,x_k,q})\).

Alternatively, a trail \(T\) can be defined by its direction, entry \(Entr(T)\), and exit \(Exit(T) = Entr(T) - qx_j + qx_k\).

**Example 1.** Figure 1 illustrates a trail \(T_{x_1,x_2,(6,1),4}\) for \(b = 2\), where the direction is \((1, 2)\), the entry is \((6, 1)\), the exit is \((2, 5)\), and the length is 4.

![Fig. 1. Example of trail \(T_{x_1,x_2,(6,1),4}\) for \(b = 2\).](image)

To bound the RHS of (1), we will divide the histograms which \(M\) maps to \(S\ \{X : M(X) \in S\}\) into trails, and use the following theorem about trails to simplify the RHS.

**Theorem 5 (Trails).** Let \(T\) be a trail with direction \((j, k)\), and let \(X\) be a database distribution where each row is independently distributed. For any \(i\), \(x_j, x_k \in \text{Supp}(X_i)\),

8
\[ \Pr(\text{Hist}(X) \in T | X_i = x_j) - \Pr(\text{Hist}(X) \in T | X_i = x_k) = \Pr(\text{Hist}(X) = \text{Exit}(T) | X_i = x_j) - \Pr(\text{Hist}(X) = \text{Enter}(T) | X_i = x_k) \]

We are now ready to characterize exact DDP for any majority rule for two candidates. Let \( \mathcal{U} = \mathcal{L}(\{x_1, x_2\}) \). For any \( \alpha \in [0, 1] \), let \( M_\alpha \) denote the biased majority rule that outputs \( x_1 \) when at least \( \alpha n \) entries have value \( x_1 \), and otherwise outputs \( x_2 \). For any \( p \in (0, 1) \), let \( X_n(p, 1-p) \) denote the distribution over the database with \( n \) entries, where each entry is \( x_1 \) with probability \( p \) independently.

**Theorem 6 (Exact DDP for Majority Rules).** For any \( \alpha \in [0, 1] \) and any \( p \in (0, 1) \), \( M_\alpha \) is \((0, \delta(n), \Delta = \{(X_n(p, 1-p), \emptyset)\})\)-DDP, where

\[ \delta = \Theta \left( \sqrt{\frac{1}{n}} \left[ \left( \frac{p}{\alpha} \right)^{\alpha} \left( \frac{1-p}{1-\alpha} \right)^{1-\alpha} \right]^n \right). \]

In particular, \( \delta = \Theta \left( \sqrt{\frac{1}{n}} \right) \) if and only if \( \alpha = p \); otherwise \( \delta \) is exponentially small.

**Proof.** Let \( X = X_n(p_1, p_2) \). Since there is no auxiliary information and \( \epsilon = 0 \), according to Definition 3, to bound \( \delta \) for \( M_\alpha \), we just need to bound the RHS of (1). We first gives an equivalent definition of \( M_\alpha(X) = x_i \) using trails. Let trail \( T_1 = \{X : X = (i, n-i), i \geq \alpha n\} \) and trail \( T_2 = \{X : X = (i, n-i), i < \alpha n\} \). It follows that \( \text{Hist}(X) \in T_i \) is equivalent to \( M_\alpha(X) = x_i \) for any \( i \in [2] \). Also, Definition 3 implies that \( S \neq \{x_1, x_2\} \) and \( S \neq \emptyset \). Therefore, continuing Equation (1), we have

\[ \delta = \max_{j \in [2]: x, x'} \left[ \Pr(\text{Hist}(X) \in T_j | X_i = x) - \Pr(\text{Hist}(X) \in T_j | X_i = x') \right] \]

\[ = \max_{j \in [2]: x, x'} \left[ \Pr(X = \text{Exit}(T_j) | X_i = x) - \Pr(X = \text{Enter}(T_j) | X_i = x') \right] \quad \text{(Theorem 5)} \]

\[ = \max_{j \in [2]} \left[ \max_x \Pr(X = \text{Exit}(T_j) | X_i = x) - \min_{x'} \Pr(X = \text{Enter}(T_j) | X_i = x') \right] \]

We first discuss the case that \( S = \{x_1\} \) \((j = 1)\), where trail \( T_1 \) defined above starts at \( \text{Enter}(T_1) = (n, 0) \) and exits at \( \text{Exit}(T_1) = (\lceil \alpha n \rceil, \lfloor (1-\alpha)n \rfloor) \). For the \( \text{Enter}(T_1) \) term in \( \delta \), we have,

\[ \min_{x'} \Pr(X = \text{Enter}(T_j) | X_i = x') = \Pr(X = (n, 0) | X_i = x_2) = 0 \]

For the \( \text{Exit}(T_1) \) term in \( \delta \), we have,

\[ \Pr(X = \text{Exit}(T_1) | X_i = x_1) \]

\[ = \Pr(X = (\lceil \alpha n \rceil, \lfloor (1-\alpha)n \rfloor) | X_i = x_1) = \Pr(X = (\lceil \alpha n \rceil - 1, \lfloor (1-\alpha)n \rfloor)) \]

\[ = p^{\lceil \alpha n \rceil - 1} (1-p)^{\lfloor (1-\alpha)n \rfloor} \frac{(n-1)!}{\lceil \alpha n - 1 \rceil! \cdot \lfloor (1-\alpha)n \rfloor!} \]

\[ = \Theta \left( \sqrt{\frac{1}{n}} \left[ \frac{pn}{\alpha} \right]^{\lceil \alpha n \rceil - 1} \cdot \left( \frac{1-p}{1-\alpha} \right)^{\lfloor (1-\alpha)n \rfloor} \right) \quad \text{(Stirling’s formula)} \]

\[ = \Theta \left( \sqrt{\frac{1}{n}} \left[ \left( \frac{p}{\alpha} \right) \left( \frac{1-p}{1-\alpha} \right)^{1-\alpha} \right]^n \right) \]
Similarly, we have \( \Pr(X = \text{Exit}(T_1)|X_i = x_2) = \Theta\left(\sqrt{\frac{1}{n}}\left[\frac{p}{\alpha}\left(\frac{1-p}{1-\alpha}\right)^{1-\alpha}\right]^n\right) \). It follows that

\[
\left[\max_x \Pr(X = \text{Exit}(T_1)|X_i = x)\right] - \left[\min_{x'} \Pr(X = \text{Enter}(T_1)|X_i = x')\right] = \Theta\left(\sqrt{\frac{1}{n}}\left[\frac{p}{\alpha}\left(\frac{1-p}{1-\alpha}\right)^{1-\alpha}\right]^n\right)
\]

The case for \( S = \{x_2\} \) is similar. We note that \( \left(\frac{p}{\alpha}\right)^\alpha \left(\frac{1-p}{1-\alpha}\right)^{1-\alpha} \leq 1 \) and the equation holds if and only if \( p = \alpha \).

\[ \square \]

6 Comparing Voting Mechanisms: Two Candidates or More

In this section we characterize \((0, \delta)\)-exact DDP for arbitrary number of candidates \( m \geq 2 \). We start with the general case for the histogram mechanism with any constant number of bins \( b \). This would immediately imply that the same results for the histogram mechanism in the voting setting by letting \( b = m! \) (the number of possible linear orders on \( m \) candidates).

**Theorem 7 (Exact DDP of Histogram Mechanism).** For any \( b \geq 2 \), let \( (p_1, \cdots, p_b) \) be a fixed nonzero probability distribution and let \( p_{\min} = \min_{i \neq j \in [b]} (p_i + p_j) \). There exists \( \delta(n) = \Theta\left(\sqrt{\frac{1}{n p_{\min}}}\right) \) such that for all \( n \in \mathbb{N} \), the histogram mechanism \( \text{Hist} \) is \((0, \delta(n), \Delta = \{(X_{n,(p_1,\cdots,p_b)}, \emptyset)\})\)-eDDP.

**Proof.** [Sketch] First we present the case for \( b = 2 \).

**Lemma 2 (Exact DDP for Histogram, when \( c = 2 \)).** Let \( \mathcal{U} = \{x_1, x_2\} \). The histogram mechanism \( \text{Hist} : \mathcal{U}^* \rightarrow \mathbb{N}^2 \) is \((0, \Theta(1/\sqrt{n})), \Delta = \{(X_2,(p_1,p_2), \emptyset)\})\)-eDDP.

**Proof.** Let \( X = X_{n,(p_1,p_2)} \). Without loss of generality, we can let \( x = x_1 \) and \( x' = x_2 \) (otherwise, rename them). Then, the maximizing set \( S \) in equation [1] is exactly the set of histograms such that \( \Pr(\text{Hist}(X) \in S|X_i = x_1) > \Pr(\text{Hist}(X) \in S|X_i = x_2) \). Since our database distribution has iid rows, with support on the set of size \( n \) databases, the histogram follows the binomial distribution (with \( n \) trials). Below we find that \( S \) is the set of histograms \((i, n - i)\) where \( i > pn \).

\[
\Pr(\text{Hist}(X) = (i, n - i)|X_i = x_1) > \Pr(\text{Hist}(X) = (i, n - i)|X_i = x_2)
\]

\[
p^{i-1}(1-p)^{n-i} \frac{(n-1)!}{(n-i)!(i-1)!} > p^i(1-p)^{n-i-1} \frac{(n-1)!}{(n-i-1)!i!}
\]

\[
i > pn
\]

The histograms \((i, n - i)\) where \( i > pn \) forms a trail \( T \) which starts from \((n,0)\) and exits at \((pm + 1, n - (pn + 1))\). Thus,

\[
\delta = \Pr(\text{Hist}(X) \in S|X_i = x) - \Pr(\text{Hist}(X) \in S|X_i = x')
\]

\[
= \Pr(\text{Hist}(X) = \text{Exit}(T)|X_i = x_1) - \Pr(\text{Hist}(X) = \text{Enter}(T)|X_i = x_2) \quad \text{(By Theorem 5)}
\]

\[
= \Pr(\text{Hist}(X) = (pn + 1, n - (pn + 1))|X_i = x_1) - \Pr(\text{Hist}(X) = (n,0)|X_i = x_2)
\]

\[
= p^{pn}(1-p)^{n-pn-1} \frac{(n-1)!}{(pn)!(n-pn-1)!}
\]

(When one row is fixed to \( x_2 \), the probability of histogram being \((n,0)\) is zero.)

\[
= \Theta(1/\sqrt{n}) \quad \text{(By applying Stirling’s formula)}
\]

\[ \square \]
We can generalize the result for \( b = 2 \), by using the trail technique, but for arbitrary number of bins. Again we assume WLOG that \( x = x_1 \) and \( x' = x_2 \). Let \( t_i \) denote the number of items in the bin for \( x_i \). We observe that, when rows are i.i.d, the bins \( t_3, \ldots, t_b \) are independent of \( t_1, t_2 \), conditioned on the sum \( s = t_1 + t_2 \). This means that we can compute \( \delta \) for general number of bins \( b \), as a sum

\[
\delta = \sum_{0 < s \leq n} \delta_s \Pr(\text{Bin}(n, p_1 + p_2) = s)
\]

Where \( \delta_s \) is the \( \delta \)-value for 2 bins, with database size \( s \). Using Chernoff bound we see that \( \text{Bin}(n, p_1 + p_2) \) is concentrated at its mean \( n(p_1 + p_2) \). Using the result for \( b = 2 \), we get \( \delta = \Theta\left(\frac{1}{\sqrt{n(p_1+p_2)}}\right) \).

We now define a set of properties for GSRs to characterize their eDDP.

**Definition 5 (Canceling-out, Monotonicity and Locally stability).** A voting rule satisfies canceling-out, if for any profile \( P \), adding a copy of every ranking does not change the winner. More precisely, \( r(P) = r(P \cup L(C)) \).

A voting rule satisfies monotonicity if it is not possible to prevent a candidate from winning by raising its ranking in a vote while keeping the order of other candidates the same.

A profile \( P^* \) is locally stable, if there exists an alternative \( a \), a ranking \( W \), and another ranking \( V \) that is obtained from \( W \) by raising the position of \( a \) without changing the order of other alternatives, such that for any \( P' \) in the \( \gamma \) neighborhood of \( P^* \) in terms of \( L_\infty \) norm, we have (1) \( r(P') \neq a \), and (2) the winner is \( a \) when all \( W \) votes in \( P' \) becomes \( V \) votes.

To present the result, we first introduce an equivalent definition of GSR that is similar to the ones used in [29,24].

**Definition 6 (The \((H, g_H)\) definition of GSR).** A GSR over \( m \) alternatives is defined by a set of hyperplanes \( H = \{h_1, \ldots, h_R\} \subseteq \mathbb{R}^m \) and a function \( g_H : \{+, 0, -\}^{|H|} \rightarrow C \). For any anonymous profile \( p \in \mathbb{R}^m \), we let \( H(p) = (\text{Sign}(h_1 \cdot p), \ldots, \text{Sign}(h_R \cdot p)) \), where \( \text{Sign}(x) \) is the sign \( (+, - \) or 0) of a number \( x \). We let the winner be \( g_H(H(p)) \).

That is, to determine the winner, we first use each hyperplane in \( H \) to classify the profile \( p \), to decide whether \( p \) is on the positive side \( (+) \), negative side \( (-) \), or is contained in the hyperplane \( (0) \). Then \( g_H \) is used to choose the winner from \( H(p) \). We refer to this definition the \((H, g_H)\) definition.

**Claim 1** The \((H, g_H)\) definition of GSR is equivalent to the \((f, g)\) definition of GSR in Definition 3.

We are now ready to present our theorem on GSRs that satisfy canceling-out, monotonicity, and local stability. We will characterize exact DDP under uniform distribution and give an exponential upper bound on DDP under some other distributions. For any pair of vectors \( D \) and \( h \), we let \( \text{Dist}(D, h) = \frac{D \cdot h}{|h|^2} \).

**Theorem 8 ((Exact) DDP for GSR).** Fix \( m \geq 2 \). Any mechanism \( M \) outputting the winner of a GSR rule that satisfies canceling-out, monotonicity, and local stability is \( (0, \delta(n) = \Theta\left(\frac{1}{\sqrt{n}}\right) \)-eDDP when each vote is i.i.d. and uniform over all linear orders on \( m \) candidates. Moreover, for any vector \( \pi \) of constant probabilities where component \( \pi[i] > 0 \) is the probability of the \( i \)th (type of) vote, then \( M \) is \( (0, \delta(n), \Delta = \{((X_n, \pi, \emptyset))\}) \)-DDP, where

\[
\delta(n) = O\left[\min\left\{\exp\left(-\frac{\min_{h \in H} \text{Dist}(\pi, h)^2}{3(m!)(\max_{i \in [m]} \pi[i])^2} \cdot n\right), \sqrt{n} \right\}\right]
\]
The exponential upper bound in Theorem 8 applies to any distribution characterized by \( \pi \) that is not on any hyperplane in GSR. Notice that the Lebesgue measure of distributions that is contained in any hyperplane is 0. It follows that the upper bound on \( \delta \) is exponentially small for any distribution in a closed set that does not intersect with any hyperplane in the GSR, whose Lebesgue measure can be arbitrarily close to 1.

**Proposition 1.** All positional scoring rules and all Condorcet consistent and monotonic rules satisfy the conditions described in Theorem 8.

**Corollary 1.** Plurality, veto, \( k \)-approval, Borda, maximin, Copeland, Bucklin, Ranked Pairs, Schulze are \((0, \Theta \left( \frac{1}{\sqrt{n}} \right))\)-eDDP when only the winner is announced, where each vote is i.i.d. and uniform over linear orders on a constant number of candidates.

STV is not one of them. However, empirical results (Section 7) suggest that STV is likely also \((0, \Theta \left( \frac{1}{\sqrt{n}} \right))\)-eDDP for this distribution.

### 7 Concrete Estimation of the Privacy Parameters

In this section we compute the concrete \((0, \delta)\)-exact DDP values for several voting rules in the class of Generalized Scoring Rules (GSR). Recall that all voting rules in GSR are characterized by functions \((f, g)\). On any profile (histogram of votes) \( P \), the vector \( f(P) \) is the score, and the output of \( g \) on the score is the winner. We refer to [28] for the \( f \) and \( g \) functions of different rules in GSR. The table below shows concrete \((\varepsilon = 0, \delta)\)-exact distributional differential privacy (exact DDP) values, using the database distribution where each row is iid and uniform over the set of linear orders on three candidates. We compute concrete \( \delta \) values for database sizes \( n < 50 \). Then, we fit them to \( \delta(n) = \frac{1}{\sqrt{an+b}} \) (our theoretical bound is \( \delta(n) = \Theta \left( \frac{1}{\sqrt{n}} \right) \)), using linear regression to find the specific \( a, b \) values for each voting rule. We rank from least to most private, by the value \( a \) for outputting the winner. The larger the \( a \), the smaller the \( \delta \) value and more private. The resulting ranking is: Borda, STV, Maximin, Plurality, and 2-approval.

| Rule \ Observable | Winner | Score |
|-------------------|--------|-------|
| Borda             | \( \delta(n) = \frac{1}{\sqrt{1.347n + 0.5263}} \) | \( \delta(n) = \frac{1}{\sqrt{0.7857n + 0.8742}} \) |
| STV               | \( \delta(n) = \frac{1}{\sqrt{1.495n + 0.02669}} \) | \( \delta(n) = \frac{1}{\sqrt{0.5226n + 0.8244}} \) |
| Maximin           | \( \delta(n) = \frac{1}{\sqrt{1.553n + 4.433}} \) | \( \delta(n) = \frac{1}{\sqrt{0.5226n + 0.8244}} \) |
| Plurality         | \( \delta(n) = \frac{1}{\sqrt{1.717n - 0.09225}} \) | \( \delta(n) = \frac{1}{\sqrt{1.047n + 1.173}} \) |
| 2-approval        | \( \delta(n) = \frac{1}{\sqrt{1.786n + 0.3536}} \) | \( \delta(n) = \frac{1}{\sqrt{1.047n + 1.173}} \) |

**Table 1.** Concrete \((0, \delta)\)-exact DDP values for outputting score and winner of various voting rules. Here we let each row of the database to be independently and uniformly distributed over linear orders on three candidates.

In Figure 2 below, we show the comparison between Plurality, Borda, and STV voting rules. As expected, outputting just the winner is more private than outputting the score (more easily seen...
Moreover, Plurality, a rule which only depends on the top-ranked candidate in each vote, is more private (leaks less information) than a rule like Borda, which uses the entire ranking. We remark that when we use the GSR winner to rank the privacy of voting rules, then STV is more private than Borda, but the opposite is true if we instead use the GSR score. We do not use the GSR score to compare the privacy of voting rules. This is because rules like STV and Maximin, the score has the same privacy as simply outputting the entire histogram of votes, and 2-approval would have the same privacy as plurality. Thus, mechanism outputting the winner allows for a better comparison.

![Fig. 2. (0, δ)-exact DDP for GSR winner and score, as a function of the size of the database (the number of votes).](image)

![Fig. 3. Plurality score vs winner](image)

In Table 2, we show the mean square error between the concrete δ values for the mechanism outputting the winner, and the fitted curve in Table 1.
| Rule    | Mean Square Error ($n \in [50]$) |
|---------|---------------------------------|
| Borda   | 0.0566844201243                 |
| STV     | 0.0542992943035                 |
| Maximin | 0.0377631805983                 |
| Plurality | 0.0477175838906               |
| 2-approval | 0.0454223047191             |

**Table 2.** Mean square error between concrete $\delta$ privacy parameter for outputting the winner, and the fitted curve in Table 1.
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A Related Literature

The first works on DP described how one can create mechanisms for answering standard statistical queries on a database (e.g., number of records with some property or histograms) in a way that satisfies the DP definition. This ignited a vast and rapidly evolving line of research on extending the set of mechanisms and achieving different DP guarantees—we refer the reader to [11] for an (already outdated) survey—to a rich literature of relaxations to the definition, e.g., [3,22,7,1], that capture among others, noiseless versions of privacy, as well as works studying the trade-offs between privacy and utility of various mechanisms [23,5,16,2,12].

Generalized Scoring Rules (GSRs) is a class of voting rules that include many commonly studied voting rules, such as Plurality, Borda, Copeland, Maximin, and STV [28]. It has been shown that for any GSR the probability for a group of manipulators to be able to change the winner has a phase transition [28,24]. An axiomatic characterization of GSRs is given in [29]. The most robust GSR with respect to a large class of statistical models has been characterized [6]. Recently GSRs have been extended to an arbitrary decision space, for example to choose a set of winners or rankings over candidates [27].

Differential privacy and applications to voting Differential privacy [8] was recently used to add privacy to voting. Shang et al. [25] applied Gaussian noise to the histogram of linear orders, while Hay et al. [17] used Laplace and Exponential mechanisms applied to specific voting rules. Lee [20] also developed a method of random selection of votes to achieve differential privacy. [25,17] also address the accuracy/privacy tradeoff of their DP mechanisms. An aspect of noising the output that was observed in [11,20] is that it enables an approximate strategy-proofness; the idea here is that the added noise dilutes the effect of any individual deviation, thereby making strategies which would slightly perturb the outcome irrelevant. We remark that if one wishes to achieve DP for a large number of voting rules, well-known DP mechanisms (like adding Laplace noise [10]) can be applied to rules in GSR in a straightforward way, by adding noise to each component of the score vector and outputting the winner based on the noised score vector. However, here we focus on exact privacy of noiseless mechanisms.

Utility of mechanisms A number of works has explored the utility of mechanisms that achieve a particular level of differential privacy. In [23], the utility is an arbitrary user-defined function, used in the exponential mechanism. The works of [5,16,2] define utility in terms of error, where the closer (by some metric) the output of the mechanism is from the query’s, the higher the utility. The definition of [12] in addition allows the user to define a prior distribution on the query output.

Relaxations to Differential Privacy and Noiseless mechanisms Relaxations to differential privacy have been proposed to allow mechanisms with less to no noise to achieve a DP-style notion of privacy. Kasiviswanathan and Smith [19] formally proved that differential privacy holds in presence of arbitrary adversarial information, and formulated a Bayesian definition of differential privacy which makes adversarial information explicit. Hall et al. [15] suggested noising only certain values (such as low-count components in histograms) to achieve a relaxed notion of Random Differential Privacy with higher accuracy. Taking advantage of (assumed) inherent randomness in the database, several works have also put forward DP-style definitions which allow for noiseless mechanisms. Duan [7] showed that sums of databases with iid rows can be outputted without noise. Bhaskar et al. [3] introduced Noiseless Privacy for database distributions with iid rows, whose parameters depend on how far the query is from a function which only depends on a subset of the database. Motivated by Bayesian mechanism design, Leung and Lui [21] suggested noiseless sum queries and
introduced Bayesian differential privacy for database distributions with independent rows, where the auxiliary information is some number of revealed rows. This idea was generalized and extended by Bassily et al. who introduced distributional differential privacy (DDP) which captures both the randomness of the database and possible adversary auxiliary information. Informally, given a distribution \((X, Z)\) on both databases and the auxiliary information leaked, we say a mechanism \(M\) is \((\epsilon, \delta, (X, Z))\)-DDP if its output distribution \(M(X)|Z\) can be simulated by a simulator that is given the database missing one row. In these works, noiseless mechanisms which have been shown to satisfy DDP are exact sums, truncated histograms, and stable functions where with large probability, the output is the same given neighboring databases.

B Preliminaries (Cont’d)

Two databases are neighbors if they differ in exactly one row. Informally, a mechanism \(M\)'s differential privacy measures the maximal distance between distributions \(M(X)\) and \(M(X')\), for any two neighbors \(X\) and \(X'\). Intuitively, a row in the database represents one individual’s data, so differential privacy describes how much the output is perturbed, if one person has changed (or removed/added) his or her information.

**Definition 7 (Differential Privacy (DP) \[1\]).** A mechanism \(M: \mathcal{U}^* \rightarrow \mathcal{R}\) is \((\epsilon, \delta)\)-differentially private (DP) if for all neighboring databases \(X, X'\) and \(S \subseteq \text{Range}(M)\),

\[
\Pr(M(X) \in S) \leq e^\epsilon \Pr(M(X') \in S) + \delta
\]

Since the databases \(X\) and \(X'\) have no randomness, a mechanism \(M\) must be randomized to satisfy differential privacy for any non-trivial parameters. In contrast, relaxations to DP like Distributional Differential Privacy (DDP) below have allowed for noiseless mechanisms by letting the \(X\) be a random variable. In this definition, the two neighboring databases are replaced by two “neighboring distributions”, one with all the database rows \((X)\), and one where the \(i\)th row is removed \((X_{-i})\). Both these distributions are conditioned on the \(i\)th row \(X_i\) being fixed to some value \(x\). Intuitively, the goal of the simulator \(\text{Sim}\) is to take the database missing the \(i\)th row, and emulate what \(M\) would output on the database including its \(i\)th row.

**Definition 8 (Distributional Differential Privacy (DDP) \[1\]).** A mechanism \(M: \mathcal{U}^* \rightarrow \mathcal{R}\) is \((\epsilon, \delta, \Delta)\)-distributionally differentially private (DDP) if there is a simulator \(\text{Sim}\) such that for all \(D = (X, Z) \in \Delta\), for all \(i, (x, z) \in \text{Supp}(X_i, Z)\) (where \(\text{Supp}(.)\) denotes the support of a distribution), and all sets \(S \subseteq \mathcal{R}\),

\[
\Pr(M(X) \in S|X_i = x, Z = z) \leq e^\epsilon \Pr(\text{Sim}(X_{-i}) \in S|X_i = x, Z = z) + \delta
\]

and

\[
\Pr(\text{Sim}(X_{-i}) \in S|X_i = x, Z = z) \leq e^\epsilon \Pr(M(X) \in S|X_i = x, Z = z) + \delta
\]

As we will see in Lemma \[1\] in the case without auxiliary information, and when each row in the database is independently distributed, this definition is equivalent to the case where \(\text{Sim}\) guesses the missing row and applies the mechanism on the resulting database (up to difference of \((2\epsilon, (1 + e^\epsilon)\delta)\) in the privacy parameter, see Theorem \[1\].

\(5\) “Differ in one row” can mean two things: 1. (Unbounded differential privacy) Database \(X\) can be obtained from database \(X'\) by adding or removing one row, or 2. (Bounded differential privacy) \(X\) can be obtained from \(X'\) by changing value in exactly one row.
B.1 Examples of GSR

For example, any positional scoring rule with scoring vector \( s \) is a GSR, where \( K = m \) and the corresponding \( f_s \) and \( g_s \) are defined as follows. For any \( V \in \mathcal{L}(\mathcal{C}) \) and any \( i \leq m \), \([f_s(V)]_i \) is the score of candidate \( x_i \) in \( V \), that is, \( s(V,x_i) \). \( g_s \) selects the candidate that corresponds to the largest component in \( f_s(P) \) (and uses a tie-breaking mechanism when necessary).

STV is also a GSR with exponentially large \( K \). For every proper subset \( A \) of \( \mathcal{C} \) and every candidate \( c \) not in \( A \), there is a component in the generalized score vector that contains the number of times that \( c \) is ranked first if all alternatives in \( A \) are removed. Let \( K = \sum_{i=0}^{m-1} \binom{m}{i}(m-i) \); the coordinates are indexed by \((A, j)\), where \( A \) is a proper subset of \( \mathcal{C} \) and \( j \leq m, x_j \notin A \). Let \([f(V)]_{(A,j)} = 1\), if after removing \( A \) from \( V \), \( x_j \) is at the top of the modified \( V \); otherwise, \([f(V)]_{(A,j)} = 0\). Then, \( g \) mimics the process of STV to select a winner.

C Noiseless Privacy in Voting (Cont’d)

\textbf{Proof.} [Theorem 3] The histogram mechanism satisfies DDP

The proof is similar to Theorem 8 of \cite{21}.

Let \( X = X_{n,(p_1,\cdots,p_h)} \). Since database rows are i.i.d., we simplify \( \Pr(\text{Hist}(X_{-i}) \in S|X_i = x) \) as \( \Pr(\text{Hist}(x,X_{-1}) \in S) \).

We let the simulator \textbf{Sim} in the definition be the function which guesses the missing row in the database, and applies histogram mechanism on the resulting database. In other words, we need to show that for all \( x_i, x_j \in \{x_1, \cdots, x_b\} \), and all \( S \subseteq \mathbb{N}^b \):

\[ \Pr(\text{Hist}(x_i, X_{-1}) \in S) \leq e^{\epsilon} \Pr(\text{Hist}(x_j, X_{-1}) \in S) + \delta \]

We observe that for any set \( B \) and \( x \):

\[ \Pr(\text{Hist}(x,X_{-1}) \in S) = \Pr(\text{Hist}(x,X_{-1}) \in S \cap B) + \Pr(\text{Hist}(x,X_{-1}) \in S \cap \overline{B}) \]  \hspace{1cm} (3)
\[ \leq \Pr(\text{Hist}(x,X_{-1}) \in S \cap B) + \Pr(\text{Hist}(x,X_{-1}) \in B) \]  \hspace{1cm} (4)

Let \( B \) be the set of all histogram \( t \in \mathbb{N}^b \) where \( t_i > p_i(n-1)e^{\epsilon/2} \) and \( t_j < p_j(n-1)e^{-\epsilon/2} \). Fix a choice of \( \epsilon > 2\ln(1 + \frac{1}{p_{\text{min}}n}) \). We claim that for \( \delta = \exp(O(np_{\text{min}}(\min(2\ln(2), \epsilon))^2)) \), the following hold:

\textbf{Claim 1:} \( \Pr(\text{Hist}(x_i, X_{-1}) \in S \cap \overline{B}) \leq e^{\epsilon} \Pr(\text{Hist}(x_j, X_{-1}) \in S \cap \overline{B}) \)

\textbf{Claim 2:} \( \Pr(\text{Hist}(x_i, X_{-1}) \in B) \leq \delta \)

If both claims are true, then by inequality [4]

\[ \Pr(\text{Hist}(x_i,X_{-1}) \in S) \leq \Pr(\text{Hist}(x_i,X_{-1}) \in S \cap B) + \Pr(\text{Hist}(x_i,X_{-1}) \in B) \]
\[ \leq e^{\epsilon} \Pr(\text{Hist}(x_j,X_{-1}) \in S \cap B) + \delta \]
\[ \leq e^{\epsilon} \Pr(\text{Hist}(x_j,X_{-1}) \in S) + \delta \]

which proves the theorem. Below we will prove both claims.

\textbf{Claim 1 proof:}

Since all entries in random variable \( X_{-1} \) are i.i.d., the random variable
**Claim 2 proof:** Recall $\mathbf{B}$ is the set of all histogram $t \in \mathbb{N}^b$ where $t_i > p_i(n-1)e^{\epsilon/2}$ and $t_j < p_j(n-1)e^{-\epsilon/2}$. For any $i \in \{1, \cdots, b\}$ let $\mathbf{Hist}(x, X_{-1})_i$ denote $i$th component of the random variable $\mathbf{Hist}(x, X_{-1})$.

$$
\Pr(\mathbf{Hist}(x_i, X_{-1}) \in \mathbf{B}) \\
= \Pr \left( \mathbf{Hist}(x_i, X_{-1})_i > p_i(n-1)e^{\epsilon/2} \text{ or } \mathbf{Hist}(x_i, X_{-1})_j < p_j(n-1)e^{-\epsilon/2} \right) \\
\leq \Pr \left( \mathbf{Hist}(x_i, X_{-1})_i > p_i(n-1)e^{\epsilon/2} \right) + \Pr \left( \mathbf{Hist}(x_i, X_{-1})_j < p_j(n-1)e^{-\epsilon/2} \right) \\
\quad \text{(By union bound)} \\
= \Pr \left( 1 + \text{Bin}(n-1, p_i) > p_i(n-1)e^{\epsilon/2} \right) + \Pr \left( \text{Bin}(n-1, p_j) < p_j(n-1)e^{-\epsilon/2} \right) \\
\quad \text{(Where Bin}(n, p) \text{ denotes binomial r.v. with } n \text{ trials and success probability } p) \\
= \Pr(\text{Bin}(n-1, p_i) > p_i(n-1)(e^{\epsilon/2} - (p_i(n-1))^{-1})) \\
\quad + \Pr(\text{Bin}(n-1, p_j) < p_j(n-1)e^{-\epsilon/2})
$$
The random variable $\text{Bin}(n - 1, p_i)$ has mean $\mu = p_i(n - 1)$. When

$$2\ln(1 + \frac{1}{p_i(n - 1)}) < 2\ln(1 + \frac{1}{p_{\text{min}}(n - 1)}) < \epsilon \leq 2\ln(2) < 2\ln(2 + \frac{1}{p_i(n - 1)})$$

we have $0 < \beta = e^{\epsilon/2} - (p_i(n - 1))^{-1} - 1 < 1$. By Chernoff bound,

$$Pr(\text{Bin}(n - 1, p_i) > (1 + \beta)\mu \leq e^{-\mu\beta^2/3} = \exp(-\Omega(p_i(n - 1)(e^{\epsilon/2} - (p_i(n - 1))^{-1} - 1)^2))$$

$$= \exp(-\Omega(p_i p\epsilon^2))$$

The random variable $\text{Bin}(n - 1, p_j)$ has mean $\mu = p_j(n - 1)$. By Chernoff bound, for any $0 < \beta = 1 - e^{-\epsilon/2} < 1$ (ie. $\epsilon > 0$),

$$Pr(\text{Bin}(n - 1, p_j) < (1 - \beta)\mu \leq e^{-\mu\beta^2/2} = \exp(-\Omega(p_j(n - 1)(1 - e^{-\epsilon/2})^2))$$

$$= \exp(-\Omega(p_j p\epsilon^2))$$

So that:

$$Pr(\text{Hist}(x_i, X_{\cdot -1}) \in S) \leq Pr(\text{Bin}(n - 1, p_i) > p_i(n - 1)(e^{\epsilon/2} - (p_i(n - 1))^{-1}))$$

$$+ Pr(\text{Bin}(n - 1, p_j) < p_j(n - 1)e^{-\epsilon/2})$$

$$\leq \exp(-\Omega(p_i p\epsilon^2)) + \exp(-\Omega(p_j p\epsilon^2))$$

$$\leq \exp(-\Omega(p_{\text{min}} p\epsilon^2)) = \delta$$

for $2\ln(1 + \frac{1}{p_{\text{min}}(n - 1)}) < \epsilon \leq 2\ln(2)$. To get rid of the upper bound on $\epsilon$, notice when $\epsilon = 2\ln(2)$, $\delta = \exp(-\Omega(p_{\text{min}} n(2\ln(2))^2))$ suffices to satisfy the inequality

$$Pr(\text{Hist}(x_i, X_{\cdot -1}) \in S) \leq e^{\epsilon} Pr(\text{Hist}(x_j, X_{\cdot -1}) \in S) + \delta$$

Thus, when $\epsilon > 2\ln(2)$, the same $\delta = \exp(\Omega(np_{\text{min}} \min(2\ln(2), \epsilon)^2)) = \exp(-\Omega(p_{\text{min}} n(2\ln(2))^2))$ also suffices, as a larger $\epsilon$ only makes the right hand side of the inequality larger.

This proves Claim 2. \(\square\)

The definition of distributional differential privacy, like differential privacy, is immune to post-processing. This means that if $M$ is a $(\epsilon, \delta, \Delta)$ private mechanism, and $f$ is a function on the output of $M$, then $f \circ M$ (their composition) is also a $(\epsilon, \delta, \Delta)$ private mechanism. Note that post-processing immunity is not a property of exact privacy.

**Lemma 3 (Immunity to Post-processing).** Suppose mechanism $M : U^* \rightarrow R$ is $(\epsilon, \delta, \Delta)$-DDP. Let $f : R \rightarrow R'$ be a deterministic function. Then $f \circ M : U^* \rightarrow R'$ is also $(\epsilon, \delta, \Delta)$-DDP.

**Proof.** For any $X \in \Delta$, $x_i, x'_i \in \text{Supp}(X_i)$ and $S \subseteq R'$, let $W = \{w : f(w) \in S\} \subseteq U^*$. Then

$$Pr(f(M(X))) \in S \mid X_i = x_i$$

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Corollary 2 (DDP of any voting rule). Let \( \mathcal{U} = \mathcal{L}(\{x_1, \cdots, x_m\}) \) be the set of votes on \( m \) candidates, let \( f : \mathbb{N}^m! \rightarrow \mathcal{R} \) be any function on the set of histograms of votes (ie. profiles) \( \mathbb{N}^m! \), and \( \pi_{\text{min}} = \min_{i \in [m]}(\pi[i]) \), where \( \pi[i] \) is the probability of the \( i \)th type of vote. For all \( \epsilon(n) \geq 2 \ln(1 + \frac{1}{\text{quant}}) \), there is a \( \delta(n) \in \exp(-\Omega(n\pi_{\text{min}}[\min(2\ln(2), \epsilon(n))]^2)) \) such that the mechanism \( f \circ \text{Hist} : \mathcal{U}^* \rightarrow \mathcal{R} \) is \((\epsilon(n), \delta(n), \Delta = \{X_n, \pi\})\)-DDP.

D Exact Privacy for Comparing Mechanisms (Cont’d)

Below we present the definitions of a (monotonically increasing) utility function, and what it means for an \( \epsilon \)-DP mechanism to be optimal for a utility function.

**Definition 9 (Utility function).** A utility function \( u : (\mathcal{U}^* \rightarrow \mathcal{R}) \rightarrow \mathbb{R} \) is any function taking as input a mechanism \( M : \mathcal{U}^* \rightarrow \mathcal{R} \) and outputting real number.

A property we are interested in is when the best possible utility achievable among \( \epsilon \)-DP mechanisms is **monotonically increasing** with \( \epsilon \). Intuitively, this means when we allow for worse privacy, we can always achieve better utility.

**Definition 10 (Monotonically increasing).** We say a utility function \( u \) is monotonically increasing over a set of mechanisms \( \mathcal{M} \) if its supremum increases monotonically with \( \epsilon \), ie.

\[
\forall \epsilon > \epsilon' \geq 0 \quad \sup_{\text{\( \epsilon \)}-DP \ mechanism } M \ u(M) > \sup_{\text{\( \epsilon' \)}-DP \ mechanism } M \ u(M)
\]

For any utility function, the optimal \( \epsilon \)-DP mechanism is the one which maximizes the utility among all \( \epsilon \)-DP mechanisms.

**Definition 11 (Optimal mechanism).** For any utility function \( u \), an optimal \( \epsilon \)-differentially private mechanism (DP) over set \( \mathcal{M} \) is one that maximizes \( u \) among all \( \epsilon \)-DP mechanisms in \( \mathcal{M} \), ie.

\[
u(M) = \max_{\text{\( \epsilon \)}-DP \ mechanism M} u(M)
\]

Here, we show the proof for Theorem 4.

**Proof.** [Theorem 4] Let \( u \) be an utility function whose supremum increases monotonically with \( \epsilon \). Suppose \( M \) is \( \epsilon \)-DP and is optimal over \( \mathcal{M} \). In other words, for \( \epsilon \), the supremum is in fact the maximum. To show \( M \) is \( \epsilon \)-exact DP, it must satisfy:

1. \( M \) is \( \epsilon \)-DP: This is satisfied by definition.
2. There is no \( \epsilon' < \epsilon \) such that \( M \) is \( \epsilon' \)-DP. For contradiction assume there is such \( \epsilon' \). Then,

\[
u(M) = \max_{\text{\( \epsilon \)}-DP \ mechanism M} u(M) = \sup_{\text{\( \epsilon \)}-DP \ mechanism M} u(M) > \sup_{\text{\( \epsilon' \)}-DP \ mechanism M} u(M) \geq u(M)
\]

The last inequality is by assumption that \( M \) is \( \epsilon' \)-DP. This is a contradiction (we get \( u(M) > u(M) \)). Thus, there must not exist such \( \epsilon' \) and \( M \) is \( \epsilon \)-exact DP.

□
Example: Truncated Geometric mechanism \[12\] Here we show that the $\alpha$-truncated geometric mechanism is $\epsilon = \ln(1/\alpha)$-exact DP (Corollary 3). By \[12\], the truncated geometric mechanism is the unique optimal mechanism for a monotonically increasing utility function (see Lemma 4). By Theorem 4, we see that this mechanism is $\epsilon$-exact private.

Definition 12 (Loss function \[12\]). A loss function $l(i,r)$ denotes an user’s loss when the query result is $i$ and the mechanism’s (perturbed) output is $r$. We allow $l$ which is nonnegative and nondecreasing in $|i - r|$ for each fixed $i$.

Definition 13 (Oblivous mechanism). For a query $f$, A mechanism $M$ is oblivious if its output only depends on the output of the query on the database, $f(X)$.

Definition 14 (Utility given loss function and priors). \[12\] Let $M$ with range $R$ be an oblivious mechanism. Let $f: U^n \rightarrow \mathcal{N} = \{1, \cdots, n\}$ be a count query. Let $x_{i,r}$ denote the probability that $M$ will output $r$ given query output $i = f(X)$. A user with prior $\{p_i = \Pr(f(D) = i)\}_{i \in \mathcal{N}}$, and loss function $l$ has utility

$$u(M) = -\sum_{i \in \mathcal{N}} p_i \sum_{r \in R} x_{i,r} \cdot l(i,r)$$

We remark that \[12\] defines disutility/expected loss \[6\] instead of utility, where the disutility is $-u$. However, maximizing utility is the same as minimizing disutility.

Definition 15 (Geometric Mechanism). For a query $f$, and $\alpha \in [0,1]$, the $\alpha$-geometric mechanism is an oblivious mechanism, where for $\alpha \in (0,1)$, $M(D) = f(D) + Y$, 

$$\Pr(Y = z) = \frac{1 - \alpha}{1 + \alpha} \alpha^z$$

When $\alpha = 1$, the geometric mechanism outputs $f(D)$. When $\alpha = 0$, the geometric mechanism outputs a value drawn from uniform distribution.

Definition 16 (Truncated Geometric Mechanism). The truncated $\alpha$-geometric mechanism, denoted $M^\alpha$, truncates the output of the $\alpha$-geometric mechanisms by remapping every value greater than $n$ to $n$, and every value less than $0$ to $0$.

Lemma 4. Let the loss function $l(i,r) = 1 + \gamma |i - r|$ for a small $\gamma$ when $i \neq r$ and 0 otherwise. For this loss function and uniform prior $\{p_i = \frac{1}{|\mathcal{N}|}\}_{i \in \mathcal{N}}$ the utility is monotonically increasing over the set of oblivious mechanisms.

Proof. By Lemma 5.12 in \[12\], the truncated $\alpha$-geometric mechanism $M^\alpha$ is the (unique) optimal mechanism over the set of oblivious $\epsilon$-DP mechanisms $\mathcal{M}$. In other words

$$u(M^{\alpha=\epsilon'}) = \sup_{\substack{\epsilon\text{-DP mechanism } M \in \mathcal{M} \epsilon=\epsilon'}} u(M)$$

To show that the utility is monotonically increasing, we show that for $\epsilon > \epsilon'$

$$u(M^{\alpha=\epsilon'}) > u(M^{\alpha'=\epsilon'})$$

They also allow for an optimal remapping function which post-processes the output of a mechanism. But for our example we do not need this remapping function.
Now we compute $u(M^α)$. For a fixed $i$, let $\text{Geometric}(α)$ denote the probability distribution on the distance between the query output and the output of $M^α$ (essentially, the absolute value of the (truncated geometric) noise).

$$u(M^α) = - \sum_{i \in N} p_i \sum_{r \in R} \Pr(M^α(i) = r)l(i, r)$$

$$= - \frac{1}{|N|} \sum_{i \in N} \sum_{r \in R} \Pr(M^α(i) = r)l(i, r) \quad (p_i = \frac{1}{N})$$

$$= - \frac{1}{|N|} \sum_{i \in N} \sum_{r \in R} \Pr(\text{Geometric}(α) = |i - r|)l(i, r)$$

(For a fixed $i$, the (truncated) geometric noise only depends on the distance between $i$ and $r$)

$$= - \frac{1}{|N|} \sum_{i \in N} \sum_{r \in N, r \neq 0} \Pr(\text{Geometric}(α) = |i - r|)(1 + γ|i - r|)$$

(The loss function equals zero when $i = r$ and $1 + γ|i - r|$ otherwise.)

$$= - \frac{1}{|N|} \left( \sum_{i \in N} \sum_{r \in N, r \neq 0} \Pr(\text{Geometric}(α) = |i - r|) + γ \sum_{i \in N} \sum_{r \in N, r \neq 0} \Pr(\text{Geometric}(α) = |i - r|)|i - r| \right)$$

For each $i$,

$$\sum_{r \in N, r \neq 0} \Pr(\text{Geometric}(α) = |i - r|) = \sum_{r \in N} \Pr(\text{Geometric}(α) = |i - r|) - \Pr(\text{Geometric}(α) = 0)$$

$$= 1 - \Pr(\text{Geometric}(α) = 0)$$

With smaller $α$, the distribution $\text{Geometric}(α)$ is more concentrated around zero. Thus, $\Pr(\text{Geometric}(α) = 0)$ is larger, and $1 - \Pr(\text{Geometric}(α) = 0)$ is smaller. Moreover, for each $i$,

$$\sum_{r \in N, r \neq 0} \Pr(\text{Geometric}(α) = |i - r|)|i - r| = \sum_{r \in N} \Pr(\text{Geometric}(α) = |i - r|)|i - r|$$

(When $i = r$, $|i - r| = 0$.)

$$= \mathbb{E}\text{Geometric}(α)$$

With smaller $α$, the distribution $\text{Geometric}(α)$ is more concentrated around zero. Thus, the expected value is smaller. Now, both values decrease monotonically with decreasing $α$. Thus, the utility (the negative of these values) increases. In other words, with larger $ε$, the greater the utility $u(M^{α=ε})$.

Then, for all $ε > ε' \geq 0$, we have

$$\sup_{ε-DP \text{ mechanism } M \in \mathcal{M}} u(M) = u(M^{α=ε}) > u(M^{α'=ε'}) = \sup_{ε-DP \text{ mechanism } M' \in \mathcal{M}} u(M)$$

Then, since the truncated geometric mechanism is optimal for the monotonically increasing utility function in Lemma 4, it is $ε$-exact differentially private.
Corollary 3. For \(0 \leq \alpha \leq 1\), and \(\epsilon = \ln(1/\alpha)\), the \(\alpha\)-truncated geometric mechanism is \(\epsilon\)-exact differentially private.

The other direction of Theorem 4, however, is not true. By Lemma 5.12 in [12], the Laplace mechanism [10] does not optimize a monotonically increasing utility function and yet it satisfies exact privacy.

Lemma 5 (Exactness and Monotonically Increasing Does Not Imply Optimality). There is a utility function \(u\) that is monotonically increasing over some set \(M\), and a mechanism \(M \in M\) that satisfies \(\epsilon\)-exact differential privacy, such that \(M\) does is not optimal for \(u\).

Proof. [Lemma 5] Here we choose our monotonically increasing utility function \(u\) to be the one used in Lemma 12, our set of mechanisms \(M\) to be the set of oblivious mechanisms, and our \(\epsilon\)-exact DP oblivious mechanism to be the Laplace mechanism. By Lemma 5.12 in [12], the truncated \(\alpha\)-geometric mechanism is the unique mechanism that optimizes \(u\) and so Laplace mechanism is not optimal for \(u\). Thus, to prove this lemma we need to show Laplace mechanism is \(\epsilon\)-exact DP.

Fix \(\epsilon\). For a function \(f : U^* \to \mathbb{R}^k\), the sensitivity of the function is

\[
\Delta f = \max_{X,X' \text{neighbouring databases}} ||f(X) - f(X')||_1
\]

The Laplace mechanism, denoted \(M : U^* \to \mathbb{R}^k\) outputs \(M(X) = f(X) + (Y_1, \cdots, Y_k)\) where \(Y_i\) is iid drawn from \(\text{Lap}(\Delta f/\epsilon)\). Let \(p_X\) denote the probability density function of \(M(X)\) and \(p_{X'}\) denote the probability density function of \(M(X')\). Then,

\[
\max_{z, \text{neighbouring } X,X'} \frac{p_{X}(z)}{p_{X'}(z)} = \max_{z, \text{neighbouring } X,X'} \prod_{i=1}^{k} \exp(-\epsilon |f(X)_i - z_i|/\Delta f) \exp(-\epsilon |f(X')_i - z_i|/\Delta f) = \max_{z, \text{neighbouring } X,X'} \prod_{i=1}^{k} \exp \left( \epsilon |f(X')_i - f(X)_i|/\Delta f \right) \quad \text{(When } z_i \leq f(X) \text{ and } z_i \leq f(X'))
\]

\[
= \max_{\text{neighbouring } X,X'} \exp \left( \epsilon ||f(X') - f(X)||_1/\Delta f \right) \quad \text{(When } ||f(X) - f(X')||_1 = \Delta f) \]

\[
= \exp(\epsilon)
\]

□

E Comparing Voting Mechanisms: The Two-Candidate Case \((m = 2)\) (Cont’d)

E.1 Proof for Lemma 1

Proof. [Lemma 1] We prove the first statement, that is, if \(M\) is \((\epsilon, \delta, \Delta = \{(X, \emptyset)\})\)-(simulation-based) DDP [1], then \(M\) is \((2\epsilon, (1 + e^\epsilon)\delta, \Delta)\)-DDP of Definition 3.
By the definition of \(M\) being \((\epsilon, \delta, \Delta)\)-simulation-based DDP, the simulator \(\text{Sim}\) has to satisfy the below inequalities for any \((X, Z = \emptyset)\) ∈ \(\Delta\), any \(i\), and \(x \in \text{Supp}(X_i)\). With no auxiliary information (ie. \(Z = \emptyset\)), we can write the inequalities in the DDP definition without \(Z\) as

\[
\Pr(M(X) \mid X_i = x) \leq e^\epsilon \Pr(\text{Sim}(X_{-i}) \mid X_i = x) + \delta
\]

\[
\Pr(\text{Sim}(X_{-i}) \in S \mid X_i = x) \leq e^\epsilon \Pr(M(X) \in S \mid X_i = x) + \delta
\]  

(5)

Now consider any \(x' \in \text{Supp}(X_i)\), possibly different from the \(x\) above. By the definition of DDP, the inequalities should also hold for \(x'\), ie.

\[
\Pr(M(X) \mid X_i = x') \leq e^\epsilon \Pr(\text{Sim}(X_{-i}) \mid X_i = x') + \delta
\]

Since the simulator is given the database without the \(i\)th row, its output does not depend on the value of the \(i\)th row. Moreover, if database rows are independent, the distributions \(X_{-i}\mid X_i = x' = X_{-i}\mid X_i = x\). Thus \(\Pr(\text{Sim}(X_{-i}) \mid X_i = x') = \Pr(\text{Sim}(X_{-i}) \mid X_i = x)\). So,

\[
\Pr(M(X) \mid X_i = x') \leq e^\epsilon \Pr(\text{Sim}(X_{-i}) \mid X_i = x) + \delta
\]

\[
\Pr(M(X) \mid X_i = x') \leq e^\epsilon (e^\epsilon \Pr(M(X) \in S \mid X_i = x) + \delta) + \delta \quad \text{(By Equation 5 above.)}
\]

\[
\Pr(M(X) \mid X_i = x') \leq e^{2\epsilon} \Pr(M(X) \mid X_i = x) + e^\epsilon \delta + \delta
\]

Thus, we have shown that for all \(x, x' \in \text{Supp}(X_i)\) (and all \(i\)),

\[
\Pr(M(X) \mid X_i = x') \leq e^{2\epsilon} \Pr(M(X) \mid X_i = x) + (e^\epsilon + 1)\delta
\]

So, \(M\) is \((2\epsilon, (1 + e^\epsilon)\delta, \Delta)\)-DDP, proving the first statement.

We now prove the second statement. That is, if \(M\) is \((\epsilon, \delta, \Delta = \{(X, \emptyset)\})\)-DDP of Definition 3 then \(M\) is \((\epsilon, \delta, \Delta)\)-(simulation-based) DDP. To do so, we define the simulator \(\text{Sim}\) to be the algorithm which inserts any \(x' \in \text{Supp}(X_i)\) to the missing \(i\)th row of the database, and apply \(M\) to the result. By independence of rows, \(\Pr(\text{Sim}(X_{-i}) \mid X_i = x) = \Pr(\text{Sim}(X_{-i}) \mid X_i = x')\) by our definition of \(\text{Sim}\), equal to \(\Pr(M(X) \mid X_i = x')\). Then, for any \(X \in \Delta, i, \) and \(x, x' \in \text{Supp}(X_i)\),

\[
\Pr(\text{Sim}(X_{-i}) \mid X_i = x) = \Pr(M(X) \mid X_i = x') \leq e^\epsilon \Pr(M(X) \mid X_i = x) + \delta
\]

by inequality of Definition 3. This proves the second statement. \(\square\)

E.2 Proof for Theorem 5

Proof. [Proof for Theorem 5] This equality comes from the simple observation that when the database rows are independently distributed, for any histogram \(t \in \mathbb{N}^b\) and any \(j \in [b]\)

\[
\Pr(\text{Hist}(X) = t \mid X_i = x_j) = \Pr(\text{Hist}(X_{-i}) = t - x_j)
\]

Where \(X_{-i}\) is distribution on the database without the \(i\)th row (well-defined as database rows are independent). Let \(q\) be the length of the trail. For any 0 ≤ \(z < q\), let \(t_z = \text{Enter}(T) - zx_j + zx_k\). Then,

\[
\Pr(\text{Hist}(X) = t_z \mid X_i = x_j) = \Pr(\text{Hist}(X_{-i}) = t_z - x_j)
\]

\[
= \Pr(\text{Hist}(X) = t_z - x_j + x_k \mid X_i = x_k) = \Pr(\text{Hist}(X) = t_{z+1} \mid X_i = x_k)
\]

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In other words,

\[
\Pr(\text{Hist}(X) \in T|X_i = x_j) - \Pr(\text{Hist}(X) \in T|X_i = x_k)
= \Pr(\text{Hist}(X) = t_q|X_i = x_j) - \Pr(\text{Hist}(X) = t_0)
+ \sum_{0 \leq z < q} \Pr(\text{Hist}(X) = t_z|X_i = x_j) - \Pr(\text{Hist}(X) = t_{z+1}|X_i = x_k)
= \Pr(\text{Hist}(X) = t_q|X_i = x_j) - \Pr(\text{Hist}(X) = t_0|X_i = x_k) \quad \text{(Every term in the sum equals 0.)}
= \Pr(\text{Hist}(X) = \text{Exit}(T)|X_i = x_j) - \Pr(\text{Hist}(X) = \text{Enter}(T)|X_i = x_k)
\]

\[\square\]

### F Comparing Voting Mechanisms: The \(m > 2\) Candidate Case (Cont’d)

#### F.1 Proof for Theorem 7

**Lemma 6 (Conditional independence).** Let \(U = \{x_1, \ldots, x_b\}\) and \(D\) be distribution on databases \(U^n\) where each row is iid. Let \(#x_i\) denote the r.v. of the number of occurrences of the value \(x_i\) in \(D\). Then, for all \(0 \leq l \leq n\), the random variables \((#x_1, #x_2)\) and \((#x_3, \ldots, #x_b)\) are independent conditioned on \(#x_1 + #x_2 = l\). In other words, for any \((t_1, \ldots, t_b)\) such that \(\sum_i t_i = n\), we have

\[
\Pr((#x_1, \ldots, #x_b) = (t_1, \ldots, t_b) | #x_1 + #x_2 = l)
= \Pr((#x_1, #x_2) = (t_1, t_2) | #x_1 + #x_2 = l) \times \Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | #x_1 + #x_2 = l)
\]

**Proof.** [Proof for Lemma 6] We equivalently show that

\[
\Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | #x_1 + #x_2 = l)
= \Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | #x_1 + #x_2 = l \land (#x_1, #x_2) = (t_1, t_2))
\]

Let \(D_1 > D_2 > \cdots > D_l\) denote the random variable of row numbers (i.e., locations in the database) with value \(x_1\) or \(x_2\), in ascending order, conditioned on that there are exactly \(l\) such locations. By total probability, the left hand side of Equation 6 is:

\[
\Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | #x_1 + #x_2 = l)
= \sum_{d_1 > d_2 > \cdots > d_l} \Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | #x_1 + #x_2 = l \land (D_1, \ldots, D_l) = (d_1, \ldots, d_l))
\times \Pr((D_1, \ldots, D_l) = (d_1, \ldots, d_l) | #x_1 + #x_2 = l)
\]

We already assume there are exactly \(l\) entries with values \(x_1\) or \(x_2\), so

\[
\Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | #x_1 + #x_2 = l)
= \sum_{d_1 > d_2 > \cdots > d_l} \Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | (D_1, \ldots, D_l) = (d_1, \ldots, d_l))
\times \Pr((D_1, \ldots, D_l) = (d_1, \ldots, d_l))
\]

The right hand side of Equation 6 is:

\[
\Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | #x_1 + #x_2 = l \land (#x_1, #x_2) = (t_1, t_2))
\]
Now given database locations \((d_1, \ldots, d_t)\), \((#x_3, \ldots, #x_b)\) only depends on the rows in database \(D\setminus\{d_1, \ldots, d_t\}\). Since rows are independent, \((#x_3, \ldots, #x_b)\) is independent of \((#x_1, #x_2)\) (which are values in locations \((d_1, \ldots, d_t)\)). Moreover, the locations \((D_1, \ldots, D_t)\) are independent of \((#x_1, #x_2)\). As database rows are iid, no matter the locations \((d_1, \ldots, d_t)\), the distributions of \((#x_1, #x_2)\) remain the same. Thus,

\[
\Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | #x_1 + #x_2 = l \land (#x_1, #x_2) = (t_1, t_2)) = \sum_{d_1 > d_2 > \cdots > d_t} \Pr((#x_3, \ldots, #x_b) = (t_3, \ldots, t_b) | (D_1, \ldots, D_t) = (d_1, \ldots, d_t)) \times \Pr((D_1, \ldots, D_t) = (d_1, \ldots, d_t))
\]

(By total probability.)

This concludes that the left hand side and right hand side probabilities of Equation \([6]\) are equal. The random variables \((#x_1, #x_2)\) are independent conditioned on \((#x_1, #x_2)\).

Below we present the proof of Theorem \([7]\) using Lemma \([2]\) which showed the case for \(b = 2\).

**Proof.** [Proof of Theorem \([7]\) Exact DDP of Histogram Mechanism]

Let \(X = X_n({p_1, \ldots, p_b})\). Since there is no auxiliary information and \(\epsilon = 0\), we will use an equivalent definition of DDP (Definition \([3]\)), setting \(\epsilon = 0\). The histogram mechanism \(\text{Hist} : \{x_1, \ldots, x_b\}^n \to \mathbb{N}^b\) is \((0, \delta, \Delta = \{(X, \emptyset)\})\)-DDP if for any \(x, x' \in \text{Supp}(X_i)\), and \(S \in \mathcal{R}\)

\[
\delta \geq \Pr(\text{Hist}(X) \in S | X_i = x) - \Pr(\text{Hist}(X) \in S | X_i = x')
\]

For exact DDP, it is Equation \([1]\)

\[
\delta = \max_{x, x', S} \Pr(\text{Hist}(X) \in S | X_i = x) - \Pr(\text{Hist}(X) \in S | X_i = x')
\]

Like in the \(b = 2\) case, without loss of generality, we can let \(x = x_1\) and \(x' = x_2\) (otherwise, rename). Then, the maximizing set \(S\) is exactly the set of histograms such that \(\Pr(\text{Hist}(X) \in S | X_i = x_1) > \Pr(\text{Hist}(X) \in S | X_i = x_2)\). The set \(S\) from the \(b = 2\) case generalizes to general \(b\). Since our database distribution has iid rows, with support on size \(n\) databases, the histogram follows the multinomial distribution (with \(n\) trials). For any \(0 < s \leq n\) and \((t_3, \ldots, t_b)\) which sum to \(n - s\).

\[
\Pr(\text{Hist}(X) = (i, s - i, t_3, \ldots, t_b) | X_i = x_1) > \Pr(\text{Hist}(X) = (i, s - i, t_3, \ldots, t_b) | X_i = x_2)
\]

\[
p_1^{i-1}p_2^{n-i}p_3^{t_3} \cdots p_b^{t_b} \frac{(n-1)!}{(i-1)!(s-i)!t_3! \cdots t_b!} > p_1^{i}p_2^{n-i-1}p_3^{t_3} \cdots p_b^{t_b} \frac{(n-1)!}{(s-i-1)!t_3! \cdots t_b!}
\]

\[
\frac{p_2}{s-i} > \frac{p_1}{i}
\]

\[
i > \left( \frac{p_1}{p_1 + p_2} \right) s
\]
Let $p = \frac{p_1}{p_1 + p_2}$. For any $0 < s \leq n$ and $(t_3, \cdots, t_b)$ which sum to $n - s$, let $T_{s,(t_3,\cdots, t_b)}$ be the trail starting from $(s, 0, t_3, \cdots, t_b)$ and exiting at $(ps + 1, s - (ps + 1), t_3, \cdots, t_b)$. The set $\mathcal{S}$ then can be partitioned into such trails. Thus,

$$\delta = \Pr(\text{Hist}(X) \in \mathcal{S}|X_i = x_1) - \Pr(\text{Hist}(X) \in \mathcal{S}|X_i = x_2)$$

$$= \sum_{T_{s,(t_3,\cdots, t_b)}} \Pr(\text{Hist}(X) \in T_{s,(t_3,\cdots, t_b)}|X_i = x_1) - \Pr(\text{Hist}(X) \in T_{s,(t_3,\cdots, t_b)}|X_i = x_2)$$

$$= \sum_{T_{s,(t_3,\cdots, t_b)}} \Pr(\text{Hist}(X) = \text{Exit}(T_{s,(t_3,\cdots, t_b)}|X_i = x_1) - \Pr(\text{Hist}(X) = \text{Enter}(T_{s,(t_3,\cdots, t_b)}|X_i = x_2)$$

(By Theorem 5)

$$= \sum_{0 < s \leq n} \sum_{(t_3,\cdots, t_b) \quad t_3 + \cdots + t_b = n-s} \Pr(\text{Hist}(X) = (ps + 1, s - (ps + 1), t_3, \cdots, t_b)|X_i = x_1)$$

$$- \Pr(\text{Hist}(X) = (s, 0, t_3, \cdots, t_b)|X_i = x_2)$$

Now let us consider the value of these two probabilities. Consider the distribution $X_{-i}$, which is $X$ but without the $i$th row. Let the random variables that are individual components of $\text{Hist}(X_{-i})$ be $(a_1, \cdots, a_b)$. Since rows are distributed independently, for $(t_1, \cdots, t_b)$,

$$\Pr(\text{Hist}(X) = (t_1, \cdots, t_b)|X_i = x_1) = \Pr(\text{Hist}(X_{-i}) = (t_1 - 1, t_2, t_3, \cdots, t_b))$$

(Recall these a’s are components of $\text{Hist}(X_{-i})$)

$$= \Pr((a_1, \cdots, a_b) = (t_1 - 1, t_2, t_3, \cdots, t_b)|a_1 + a_2 = s) \times \Pr(a_1 + a_2 = s)$$

(By Lemma 6) $(a_1, a_2)$ and $(a_3, \cdots, a_b)$ are independent conditioned on $a_1 + a_2 = s$)

Similar to the $b = 2$ case, $\Pr(\text{Hist}(X) = (s, 0, t_3, \cdots, t_b)|X_i = x_2) = 0$. This is because when one row in the database is fixed to $x_2$, it is impossible to have zero in the second component in the histogram (the number of occurrences of $x_2$). Thus,

$$\delta = \sum_{0 < s \leq n} \sum_{(t_3,\cdots, t_b) \quad t_3 + \cdots + t_b = n-s} \Pr((a_1, a_2) = (ps, s - (ps + 1))|a_1 + a_2 = s)$$

$$\times \Pr((a_3, \cdots, a_b) = (t_3, \cdots, t_b)|a_1 + a_2 = s) \times \Pr(a_1 + a_2 = s)$$

$$= \sum_{0 < s \leq n} \Pr((a_1, a_2) = (ps, s - (ps + 1))|a_1 + a_2 = s) \times \Pr(a_1 + a_2 = s)$$

$$\times \sum_{(t_3,\cdots, t_b) \quad t_3 + \cdots + t_b = n-s} \Pr((a_3, \cdots, a_b) = (t_3, \cdots, t_b)|a_1 + a_2 = s)$$

$$= \sum_{0 < s \leq n} \Pr((a_1, a_2) = (ps, s - (ps + 1))|a_1 + a_2 = s) \times \Pr(a_1 + a_2 = s)$$

(For any $s$, the second sum equals one.)

Where $\Pr((a_1, a_2) = (ps, s - (ps + 1))|a_1 + a_2 = s)$ is the $\delta$ value for the histogram mechanism, when $b = 2$ and the distribution is $X_{s,(p,1-p)}$ (recall $p = \frac{p_1}{p_1 + p_2}$). We denote this by $\delta_s$. Moreover,

$$\Pr(a_1 + a_2 = s) = \Pr(\text{Bin}(n, p_1 + p_2 = p) = s)$$

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Where \( \text{Bin}(n, p) \) is the binomial distribution with \( n \) trials and probability \( p \). Then

\[
\delta = \sum_{0 < s \leq n} \delta_s \Pr(\text{Bin}(n, p) = s) \\
= \sum_{s \geq (1-\sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s) \times \delta_s + \sum_{s < (1-\sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s) \times \delta_s + \sum_{s > (1+\sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s) \times \delta_s
\]

Lower bound of \( \delta \):

\[
\delta \geq \sum_{s \geq (1-\sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s) \times \delta_s
\]

Since \( \delta_s \) decreases with larger \( s \), \( \delta_{(1+\sqrt{\frac{3}{4}})np} \) is the minimum.

\[
\geq \delta_{(1+\sqrt{\frac{3}{4}})np} \times \sum_{s \geq (1-\sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s)
\]

\[
= \delta_{(1+\sqrt{\frac{3}{4}})np} \times \left[ 1 - \Pr(\text{Bin}(n, p) > (1 + \sqrt{\frac{3}{4}})np) - \Pr(\text{Bin}(n, p) < (1 - \sqrt{\frac{3}{4}})np) \right]
\]

By Chernoff bound for binomial distribution, for any \( 0 < \beta < 1 \), we have:

\[
\Pr(\text{Bin}(n, p) > (1 + \beta) \mu) \leq e^{-\beta^2 \mu}
\]

\[
\Pr(\text{Bin}(n, p) < (1 - \beta) \mu) \leq e^{-\beta^2 \mu}
\]

Where \( \mu = np \) is the mean of \( \text{Bin}(n, np) \). Now let \( \beta = \sqrt{\frac{3}{4}} \), which is between 0 and 1. Then,

\[
1 \geq \left[ 1 - \Pr(\text{Bin}(n, p) > (1 + \sqrt{\frac{3}{4}})np) - \Pr(\text{Bin}(n, p) < (1 - \sqrt{\frac{3}{4}})np) \right]
\]

\[
\geq 1 - e^{-\frac{3}{4} \mu} - e^{-\frac{3}{4} \mu}
\]

\[
= 1 - e^{-\frac{np}{2}} - e^{-\frac{3np}{2}}
\]

(For large enough \( n, np \geq 1 \), so \( e^{-\frac{np}{2}} \leq e^{-1/2} \) and \( e^{-\frac{3np}{2}} \leq e^{-3/2} \))

\[
\geq 1 - e^{-1/2} - e^{-3/2} \geq 1/10
\]

Which means

\[
1 - \Pr(\text{Bin}(n, p) > (1 + \sqrt{\frac{3}{4}})np) - \Pr(\text{Bin}(n, p) < (1 - \sqrt{\frac{3}{4}})np) = \Theta(1).
\]

By Stirling formula, we have

\[
\delta_{(1+\sqrt{\frac{3}{4}})np} = \Theta\left(\frac{1}{\sqrt{(1 + \sqrt{\frac{3}{4}})np}}\right)
\]
\[
= \Theta \left( \frac{1}{np} \right)
\]
(Recall we assumed the maximizing \(x, x'\) are \(x_1, x_2\), up to renaming the \(x_i\)'s, and that \(p = p_1 + p_2\))

\[
= \Theta \left( \frac{1}{np_{\min}} \right)
\]
(In general, \(p_{\min} = \min_{i \neq j \in [b]} (p_i + p_j)\).)

Which gives us the lower bound \(\delta \geq \Theta \left( \frac{1}{np_{\min}} \right)\).

Upper bound of \(\delta\):

\[
\delta = \sum_{s \geq (1 - \sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s) \times \delta_s \\
+ \sum_{s < (1 - \sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s) \times \delta_s \\
+ \sum_{s > (1 + \sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s) \times \delta_s
\]

Since \(\delta_s \leq 1\) for all \(s\) and \(\sum_{s \geq (1 - \sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s) \leq 1\)

\[
\leq \max_{(1 - \sqrt{\frac{3}{4}})np \leq s \leq (1 + \sqrt{\frac{3}{4}})np} (\delta_s) + \sum_{s < (1 - \sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s) \\
+ \sum_{s > (1 + \sqrt{\frac{3}{4}})np} \Pr(\text{Bin}(n, p) = s)
\]

\[
= \delta (1 - \sqrt{\frac{3}{4}})np + \Pr(\text{Bin}(n, p) < \left(1 - \frac{\sqrt{3}}{4}\right)np) + \Pr(\text{Bin}(n, p) > \left(1 + \frac{\sqrt{3}}{4}\right)np)
\]

\[
\leq \delta (1 - \sqrt{\frac{3}{4}})np + e^{-\frac{np}{2}} + e^{\frac{3np}{2}}
\]

(By Chernoff bound for binomial)

\[
\leq \delta (1 - \sqrt{\frac{3}{4}})np + 2\sqrt{\frac{1}{np}}
\]

(Since \(np \geq 0\), both \(e^{-\frac{np}{2}}, e^{\frac{3np}{2}} \leq \sqrt{\frac{1}{np}}\))

By Stirling’s formula,

\[
\delta (1 - \sqrt{\frac{3}{4}})np = \Theta \left( \frac{1}{\sqrt{(1 - \sqrt{\frac{3}{4}})np}} \right)
\]

\[
= \Theta \left( \sqrt{\frac{1}{np}} \right)
\]

As is with the lower bound, in general (without assuming \((x, x') = (x_1, x_2)\)), we have \(p = p_{\min} = \min_{i \neq j \in [b]} (p_i + p_j)\). Since both lower and upper bounds of \(\delta\) are \(\Theta \left( \sqrt{\frac{1}{np_{\min}}} \right)\), \(\delta = \Theta \left( \sqrt{\frac{1}{np_{\min}}} \right)\). □

### F.2 Proof for Claim 1, Theorem 8 and Proposition 1

**Proof.** [Proof for Claim 1] We first show that any \((H, g_H)\) GSR can be represented by a \((f, g)\) GSR in the following way: for each ranking \(V\), we let \(f(V) = (h_1 \cdot e_V, h_2 \cdot e_V, \ldots, h_R \cdot e_V, 0)\). Then, the \(g\) function mimics \(g_H\) by only focusing on orderings between the \(k\)th component of \(f(P)\) and the last
component, which is always 0, for all \( k \leq R \). More precisely, ordering between the \( k \)th component of \( f(P) \) and 0 uniquely determines \( \text{Sign}(h_k \cdot p) \).

We now prove that any \( f - g \) GSR can be represented by an \( H - g_H \) GSR. For any pair of distinct component \( k_1, k_2 \leq K \), we introduce a hyperplane \( h_{k_1,k_2} = ([f(V)]_{k_1} - [f(V)]_{k_2})_{V \in L(C)} \). Therefore, for any profile \( p \), \( h_{k_1,k_2} \cdot p = [f(p)]_{k_1} - [f(p)]_{k_2} \). The sign of \( h_{k_1,k_2} \cdot p \) corresponds to the order between \([f(p)]_{k_1}\) and \([f(p)]_{k_2}\). Then, \( g_H \) mimics \( g \).

**Proof.** [Theorem 8 (Exact) DDP for GSR]

We first shows that w.l.o.g. we can assume that all hyperplanes in \( H \) passes 1.

**Lemma 7.** A GSR satisfies canceling-out, if and only if there exists another equivalent GSR \( r = (H, g_H) \), where all hyperplanes passes 1.

**Proof.** The “if” direction is straightforward. To prove the “only if” part, it suffices to prove that \( g_H \) does not depend on outcomes of hyperplanes in \( H \) that does not pass 1. W.l.o.g. let \( h_1 \in H \) denote the hyperplane that does not pass 1, that is, \( h \cdot 1 \neq 0 \). We will prove that for any \( u_{-1} \in \{-1,0,1\}^{L-1} \) and any \( u_1, u'_1 \in \{-1,0,1\} \), such that there exist profiles \( P, Q \) with \( H(P) = (u_1, u_{-1}) \) and \( H(Q) = (u'_1, u_{-1}) \), we have \( g_H(u_1, u_{-1}) = g_H(u'_1, u_{-1}) \).

For the sake of contradiction, suppose this does not hold and let \( P, Q \) be the profiles such that \( H(P) \) and \( H(Q) \) differ on the first coordinate, and \( r(P) \neq r(Q) \). Then, for sufficiently large \( n \) we have that \( H(P+nL(C)) = H(Q+nL(C)) \). This is because for any \( h \in H \) that passes 1, we have \( h \cdot (P+nL(C)) = h \cdot P = h \cdot (Q+nL(C)) \). For any \( h \in H \) that does not pass 1, we have \( h \cdot (P+nL(C)) = h \cdot P + nh \cdot 1 \), and when \( n \) is sufficiently large, the sign of \( h \cdot (P+nL(C)) \) is the same as the sign of \( nh \cdot 1 \), which is the sign of \( h \cdot (Q+nL(C)) \). This means that \( \text{Sign}(h \cdot P) = \text{Sign}(h \cdot (P+nL(C))) = \text{Sign}(h \cdot (Q+nL(C))) = \text{Sign}(h \cdot Q) \), which is a contradiction. □

Let \( r \) be a GSR, \( P^* \) be the locally stable profile and \( a \) be the alternative, \( V, W \) be the rankings as in the statement of Definition 5. W.l.o.g. suppose \( V \) is the first type ranking and \( W \) is the second type ranking. In other words, \( V \), respectively, \( W \) is the first (respectively, second) coordinate in the \( m \)-profiles space. We will show that the exact DDP bound is achieved when \( S \) is the set of all profiles where the winner is \( a \).

We recall that for any profile \( P \), a pair of different votes \( V, W \), and a length \( q \in \mathbb{N} \), \( T_{P,V,W,q} \) is the trail starting at \( P \), going along the \( V - W \) direction, and contains \( q \) profiles. We let \( T_{P,V,W,\infty} = \max_q T_{P,V,W,q} \) denote the longest \( V - W \) trail starting at \( P \). For a GSR \( r \), we define \( \text{End}(a) = \{ \text{Exit}(T_{P,V,W,\infty}) : \forall V, W \in \mathcal{U}, r(P) = a \} \). In other words, there is no \( W \) votes in \( \text{End}(a) \).

Because \( r \) satisfies monotonicity, for any profile \( P \) such that \( r(P) = a \), we must have that \( a \) is the winner under all profiles in the \( V \)-\( W \) trail starting at \( P \). Therefore, \( S \) can be partitioned into multiple non-overlapping trails, each of which starts at a different profile, where \( a \) is the winner, and \( a \) is no longer the winner if we go one step into the \( W \)-\( V \) direction. Formally, we let \( \text{End}(a) \) (shown in Figure 4) denote all \( n \)-profiles \( P \) such that (1) \( r(P) = a \) and (2) \( r(P + W - V) \neq a \).

Then, we define a partition \( S_a \) as follows.

\[
S_a = \{ P : r(P) = a \} = \bigcup_{P \in \text{End}(a)} T_{P,V,W,\infty}
\]

It follows from Theorem 5 that

\[
\Pr(P \in S_a | X_1 = V) - \Pr(P \in S_a | X_1 = W) = \sum_{P \in \text{End}(a) : P(V) > 0} \Pr(P - V). \]

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We will define a subset of $n$-profile, $\mathcal{R}_n$ and prove the lower bound on it. For a locally stable profile $P^*$ (with constant $\gamma$ in the statement of Definition 5), let $p_0 = P^* - 1 \cdot \frac{|P^*|}{m!}$. That is, $p_0$ be obtained from $P^*$ by subtracting a constant in each component, such that $p_0 \cdot 1 = 0$. For any $n$, we define $\mathcal{R}_n$ to be the set of $n$-profiles that are in the $\gamma\sqrt{n}$ neighborhood of $\frac{n}{m!} \cdot 1 + p_0 \cdot \sqrt{n}$ w.r.t. $L_\infty$ norm for last $m! - 2$ dimensions. That is,

$$\mathcal{R}_n = \left\{ P : P[V] = 0 \text{ and } \forall j \geq 3, \left| P[j] - \left( \frac{n}{m!} + p_0[j] \cdot \sqrt{n} \right) \right| \leq \gamma \sqrt{n} \right\}$$

Throughout the proof in Theorem 8, we will use $\pi$ to denote the database distribution $D$, and $\pi[j]$ denote the probability of $j$-th kind of ranking. Here $P[V]$ is the number of $V$ votes in $P$ and $P[j]$ is the number of $j$-th type of vote in $P$. For any $P \in \mathcal{R}_n$, we let $\text{Piv}(P) = \text{End}(a) \cap \mathcal{T}_{P,V,W,\infty}$ denote the intersection of End$(a)$ and the $V$-$W$ trail starting at $P$. That is, $\text{Piv}(P) = P + l(V - W)$ for some $l \in \mathbb{Z}$, $r(\text{Piv}(P)) = a$, and $r(\text{Piv}(P) - V + W) \neq a$.

We next prove that the number of $V$ votes in $\text{Piv}(P)$ and the number of $W$ votes in $\text{Piv}(P)$ are close—the difference is $O(\sqrt{n})$.

**Claim 2** For any $P \in \mathcal{R}_n$, we have $|\text{Piv}(P)[V] - \text{Piv}(P)[W]| = O(\sqrt{n})$.

**Proof.** Let $Q^+ = \text{Piv}(P)$ and $Q^- = \text{Piv}(P) - V + W$. We note that $\text{Piv}(P)$ is at the boundary of $S$, which means that $r(Q^+) \neq r(Q^-)$. Therefore, because $r$ is a GSR, the line segment between $Q^+$ and $Q^-$ must contain the intersection of $\mathcal{T}_{P,V,W,\infty}$ and a hyperplane $h \in H$. Therefore, it suffices to show that the difference in number of $V$ votes and number of $W$ votes at the intersection of $\mathcal{T}_{P,V,W,\infty}$ and any hyperplane $h$ is $O(\sqrt{n})$.

We recall that by Lemma 7 all hyperplanes for $r$ pass 1. For any $h \in H$, we recall that we assumed that $V$ and $W$ corresponds to the first and second coordinate, respectively. Because $h \cdot (P + l(V - W)) = 0$, we have $(h_2 - h_1)l = h \cdot P = h \cdot (P - 1 \cdot \frac{n}{m!}) = O(\sqrt{n})$. This means that $|l| = |\text{Piv}(P)[V] - \text{Piv}(P)[W]| = O(\sqrt{n})$. $\square$

**Claim 3** For any $P \in \mathcal{R}_n$, there is a $V$-$W$ trail passing $P$.

**Proof.** According to the cancelling our property of $r$, we can construct profile $P' = P \frac{n}{m!} |P^*| \sqrt{n}$, which is equivalent to $P$. For any profile $P \in \mathcal{R}_n$, we have $|P[j] - (\frac{n}{m!} + p_0[j] \cdot \sqrt{n})| \leq \gamma \sqrt{n}$, which
is equivalent with \(|P'[j] - P^*[j]| \leq \sqrt{n}P^*| \leq \gamma\sqrt{n}\), which means \(\frac{P^*}{\sqrt{n}}\) is in the \(\gamma\) neighborhood of profile \(P^*\) in terms of the 3-rd to \(m\)!-th dimensions. According to the \((H, g_H)\) definition of GSR, we know \(r(P^*) = r(P')\) and the claim follows by local stability of \(P^*\).

We will show that the probability of a subset of \(\text{End}(a)\)—the pivotal profiles on trails starting at profiles in \(\mathcal{R}_n\)—is \(\Theta(1/\sqrt{n})\) for the condition that \(\pi\) is uniform over \(\mathcal{U}\). Let \(\mathcal{R}_n^- \subseteq \mathbb{R}^{m!-2}\) and for any \(p_- \in \mathcal{R}_n^-\), we define \(\text{Piv}(p_-) = \text{Piv}(P)\), where \(P \in \mathcal{R}_n\) and \(P[3, \ldots, m!] = p_-\).

\[
\sum_{P \in \text{End}(a)} \Pr(P - V) \geq \sum_{P \in \mathcal{R}_n} \Pr(\text{Piv}(P) - V)
\]

\[
= \sum_{p_- \in \mathcal{R}_n^- |P| = n - 1} \Pr(P[3, \ldots, m!] = p_-) \left( \Pr(P[1] = \text{Piv}(p_-)[1] - 1) + \Pr(P[2] = \text{Piv}(p_-)[2]|P[3, \ldots, m!] = p_-) \right)
\]

\[
= \sum_{p_- \in \mathcal{R}_n^- |P| = n - 1} A(p_-)B(p_-)
\]

where \(A(p_-) = \Pr(P[3, \ldots, m!] = p_-)\) and

\(B(p_-) = \Pr(P[1] = \text{Piv}(p_-)[1] - 1)\). It follows that \(B(p_-)\) is equivalent to probability of flipping a coin \((\frac{\pi[W]}{\pi[V] + \pi[W]}\) probability for head) for \(\text{Piv}(p_-)[1] + \text{Piv}(p_-)[2] - 1\) times, with \(\text{Piv}(p_-)[1] - 1\) heads and \(\text{Piv}(p_-)[2]\) tails. The next lemma gives a lower bound to \(\sum_{p_- \in \mathcal{R}_n^- |P| = n - 1} A(p_-)B(p_-)\) when \(\pi\) is a uniform distribution.

**Lemma 8.** \(\sum_{p_- \in \mathcal{R}_n^- |P| = n - 1} A(p_-)B(p_-) = \Omega\left(\frac{1}{\sqrt{n}}\right)\) if \(\pi\) is uniform over \(\mathcal{U}\).

**Proof.** We first give a claim saying the total number of \(V\) and \(W\) votes in \(P \in \mathcal{R}_n\) is \(\Theta(n)\).

**Claim 4** \(\text{Piv}(p_-)[1] + \text{Piv}(p_-)[2] - 1 = \Theta(n)\) for all \(p_- \in \mathcal{R}_n^-\).

**Proof.**

\[
\left| \text{Piv}(p_-)[1] + \text{Piv}(p_-)[2] - \frac{2n}{m!} \right| = \sum_{j=3}^{m!} \left| P[j] - \frac{n}{m!} \right| \leq \sum_{j=3}^{m!} \left( \gamma \sqrt{n} + |p_0[j]| \sqrt{n} \right) \leq (\gamma + 1)n! \sqrt{n}
\]

According to Claim 2 & 4, we know that \(B(p_-)\) is equivalent to probability of flipping a fair coin for \(\frac{2n}{m!} + c_1 \sqrt{n}\) times and get \(\frac{n}{m!} + c_2 \sqrt{n}\), where \(c_1\) and \(c_2\) are bounded constants. In the next claim, we will give a tight bound to \(B(p_-)\) for uniform distributed entries.

**Claim 5** \(B(p_-) = \Theta\left(\frac{1}{\sqrt{n}}\right)\) for any \(p_- \in \mathcal{R}_n^-\).

**Proof.** Letting \(n' = \frac{2n}{m!} + c_1 \sqrt{n}\), \(c' = c_2 - \frac{c_1}{2}\) and assuming \(n'\) is a even number, for the lower bound, we have,

\[
B(p_-) = \left(\frac{1}{2}\right)^{\frac{2n}{m!} + c_1 \sqrt{n}} \left(\frac{2n}{m!} + c_1 \sqrt{n}\right) = \left(\frac{1}{2}\right)^{n'} \left(\frac{n'}{2} + c' \sqrt{n} + 1\right) \times \cdots \times \left(\frac{n'}{2} + c' \sqrt{n} - 1\right)
\]

\[
= \left(\frac{1}{2}\right)^{n'} \cdot \left(\frac{n'}{2}\right) \cdot \left(\frac{n'}{2} + c' \sqrt{n} - 1\right) \times \cdots \times \left(\frac{n'}{2}\right)
\]

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\[
\frac{1}{2n'} \left( \frac{n'}{n'/2} \right) \cdot \left( \frac{n'/2 - c' \sqrt{n'}}{n'/2} \right) ^{c' \sqrt{n'}} = \Omega \left( \frac{1}{\sqrt{n}} \right) \text{ (applying Stirling’s Formula)}
\]

Upper bound can be obtained using similar technique as lower bound. □

The next claim gives a lower bound on \( \sum_{p \in \mathcal{R}_m^-} A(p^-) \), using the main technique of Lindeberg-Levy Central Limit Theorem [13].

**Claim 6** \( \sum_{p \in \mathcal{R}_m^-} A(p^-) = \Omega (1) \).

**Proof.** We first define a set of \( m! - 2 \) dimensions random variables that \( Y_i = (Y_i[1], \cdots, Y_i[m! - 2]) \), where \( Y_i[j] = 1 \) if ranking \( j \) happens to \( i \)-th row and \( Y_i[j] = 0 \) otherwise. According to the definition of profile, we have \( P[j + 2] = \sum_{j=1}^n Y_i[j] \text{ and } \mathbb{E}(P[j]) = \frac{n}{m} \) for uniform case. We further define a \( m! - 2 \) dimensional random vector \( u \) such that \( u[j] = (P[j + 2] - \frac{n}{m}) / \sqrt{n} \), which is the scaled average of \( Y_1, \cdots, Y_n \). According to Lindeberg-Levy Central Limit Theorem [13], we know that the distribution of \( u \) converges in probability to multivariate normal distribution \( \mathcal{N}(0, \Sigma) \), where

\[
\Sigma = \begin{bmatrix}
\frac{m!-1}{(m!)^2} - \frac{1}{(m!)^2} & \cdots & -\frac{1}{(m!)^2} \\
\frac{1}{(m!)^2} & \frac{m!-1}{(m!)^2} & \cdots & -\frac{1}{(m!)^2} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{(m!)^2} & -\frac{1}{(m!)^2} & \cdots & \frac{m!-1}{(m!)^2} 
\end{bmatrix}.
\]

Since each diagonal element in \( \Sigma \) is strictly larger than the sum of the absolute value of all other elements in the same row, we know that \( \Sigma \) is non-singular according to Levy-Desplanques Theorem [18]. According to Varah et al. [26], we obtain a bound on \( \Sigma^{-1} \)'s \( L_\infty \) norm as,

\[
||\Sigma^{-1}||_\infty \leq \frac{1}{\min_i \left( |\Sigma_{ii} - \sum_{j \neq i} |\Sigma_{ij}|| \right)} \leq \frac{(m!)^2}{2}.
\]

For any \( m! - 2 \) dimensional random vector \( u \) constructed from a profile \( P \) using the procedure that \( u[j] = (P[j + 2] - \frac{n}{m}) / \sqrt{n} \), we have,

\[
P \in \mathcal{R}_m^- \text{ if and only if } u \in \mathbb{U} = \{ u : |u[j] - p_0[j]| \leq \gamma, \forall j \in [m! - 2] \}.
\]
Thus, for all \( u \in \mathbb{U} \) we know about its Probability Density Function (PDF) that,

\[
\text{PDF}(u) = \frac{1}{\sqrt{(2\pi)^{m!}} |\Sigma|} \exp \left( -\frac{1}{2} u^T \Sigma^{-1} u \right)
\]

\[
= \frac{1}{\sqrt{(2\pi)^{m!}} |\Sigma|} \exp \left( -\frac{1}{2} u^T \Sigma^{-1} u \right)
\]

\[
\geq \frac{1}{\sqrt{(2\pi)^{m!}} |\Sigma|} \exp \left( -\frac{1}{2} ||u^T \Sigma^{-1}||_\infty \cdot ||u||_1 \right) \quad \text{(Holder’s Inequality)}
\]

\[
\geq \frac{1}{\sqrt{(2\pi)^{m!}} |\Sigma|} \exp \left( -\frac{1}{2} ||u^T \Sigma^{-1}||_\infty \cdot ||u||_1 \right)
\]

\[
= \Omega(1).
\]

Thus, letting \( \text{Vol}(\cdot) \) be the volume function,

\[
\sum_{p - \in \mathcal{R}_n} A(p -) \geq \text{Vol}(\mathbb{U}) \cdot \min_{u \in \mathbb{U}} \text{PDF}(u) \geq \gamma^{m! - 2} \cdot \Omega(1) = \Omega(1).
\]

Lemma 8 follows be combining Claim 6 and Claim 5. □

Recalling Theorem 5, for the case that \( \pi \) is uniform over all ranking, we have,

\[
\delta = \max_{x,x',S} \text{Pr}(M(X) \in S|X_1 = x) - \text{Pr}(M(X) \in S|X_1 = x')
\]

\[
\leq \text{Pr}(M(X) \in S_a|X_1 = W) - \text{Pr}(M(X) \in S_a|X_1 = V)
\]

\[
= \sum_{P \in \text{End}(a)} \text{Pr}(P - V) = \Omega \left( \frac{1}{\sqrt{n}} \right).
\]

Then, we derive an upper bound of \( \delta \) using the similar technique of lower bound (\( \pi \) can be non-uniform for this bound). We first define \( \mathcal{R}'_n \), a subset of \( n \)-profile space, where event \( P \in \mathcal{R}'_n \) will be proved to happen with high probability.

\[
\mathcal{R}'_n = \left\{ P : P[V] = 0 \quad \text{and} \quad \forall j \geq 3, |P[j] - (n \cdot \pi[j])| \leq n^{3/4} \right\}.
\]

Then, we recall Theorem 5 for the case that \( \pi \) such that \( \min_i \pi[i] > 0 \), we have,

\[
\delta = \max_{V,W,S} \text{Pr}(P \in S|X_1 = V) - \text{Pr}(P \in S|X_1 = W)
\]

\[
\leq \max_{V,W} \sum_{i=1}^{m} \text{Pr}(P \in S_i|X_1 = V) - \text{Pr}(P \in S_i|X_1 = W) = \sum_{i=1}^{m} \sum_{P \in \text{End}(x_i)} \text{Pr}(P - V).
\]

where \( S_i = \{ X : r(X) = x_i \} = \bigcup_{P \in \text{End}(x_i)} T_{P,V,W,\infty} \). The next claim gives an upper bound on the number of pivotal profiles sharing one End.

**Claim 7** For any profile \( P \) in \( \mathcal{R}'_n \), there are at most \( |H| \) pivotal profiles following \( V-W \) direction.
Proof. We know from the \((H, g_H)\) definition of GSR that r’s output only changes while passing at least one hyperplane. Considering a trial \(T_{P_0}\) enter at \((P_0[1] + P_0[2], 0, P_0[3], \ldots, P_0[m!])\) and exit at \((0, P_0[1] + P_0[2], P_0[3], \ldots, P_0[m!])\) (\(P_0\) is an arbitrary n-profile). Thus, there are at most \(|H|\) pivotal profiles sharing the same end point because \(T_{P_0}\) passes hyperplanes at most \(|H|\) times. \(\square\) Using the partition of \(R_n'\) and arbitrarily selected candidate \(a\), we have,

\[
\sum_{P \in \text{End}(x_i)} \Pr(P - V) \leq |H| \left( \sum_{P \in R_n'} \Pr(\text{Piv}(P) - V) + \sum_{P \in \text{End}(x_i) \setminus R_n'} \Pr(\text{Piv}(P) - V) \right) \\
\leq |H| \left( \sum_{p_\in R_n', |P|=n-1} A(p_-) B(p_-) + \sum_{p_\not\in R_n', |P|=n-1} A(p_-) B(p_-) \right) \\
\leq |H| \left( \max_{p_\in R_n'} B(p_-) \cdot \sum_{p_\in R_n'} A(p_-) + \max_{p_\not\in R_n'} B(p_-) \cdot \sum_{p_\not\in R_n'} A(p_-) \right) \\
= O \left( \frac{1}{\sqrt{n}} \right) \cdot O(1) + O(1) \cdot O \left( \frac{1}{\sqrt{n}} \right) \quad \text{(by applying Claim 9)} \\
= O \left( \frac{1}{\sqrt{n}} \right)
\]

The next claim gives an upper bound to \(\sum_{p_\not\in R_n} A(p_-)\).

Claim 8 \(\sum_{p_\not\in R_n'} A(p_-) = O \left( \frac{1}{\sqrt{n}} \right)\).

Proof.

Let \(Y_j^{(i)} = \"the i-th agent gives vote of type j\"\). It is easy to find that \(P[j] = \sum_{i=1}^{n} Y_j^{(i)}, \mathbb{E}(P[j]) = n\pi[j]\) and \(\text{Var}(P[j]) = n\pi[j](1 - \pi[j])\). Thus,

\[
\sum_{p_\not\in R_n} A(p_-) = \Pr \left[ \bigcup_{j=3}^{m!} \{ |P[j] - n \cdot \pi[j]| \leq n^{3/4} \} \right] \\
\leq \sum_{j=3}^{m!} \Pr \left[ \{ |P[j] - \mathbb{E}(P[j])| \leq n^{3/4} \} \right] \\
\leq \sum_{j=3}^{m!} \frac{n\pi[j](1 - \pi[j])}{n^{3/2}} \quad \text{(by Chebyshev’s Inequality)} \\
= O \left( \frac{1}{\sqrt{n}} \right) \quad \text{\(\square\)}
\]

Then, all we need is a upper bound on \(B(p_-)\), and we first prove that the length of \(V - W\) sequence is \(\Theta(n)\) for all \(P \in R_n'\).

Claim 9 \(\text{Piv}(p_-)[1] + \text{Piv}(p_-)[2] - 1 = \Theta(n)\) for all \(P \in R_n'\).

Proof.

\[
|\text{Piv}(p_-)[1] + \text{Piv}(p_-)[2] - n(\pi[W] + \pi[V])| = \sum_{j=3}^{m!} |P[j] - n \cdot \pi[j]| \leq \sum_{j=3}^{m!} n^{3/4} \leq m! \cdot n^{3/4}
\]

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Then, using the same technique of Claim 5, we know that,

\[ B(p-) = \Theta \left( \sqrt{\frac{1}{n}} \right) \quad \text{for all} \quad p- \in \mathcal{R}_n^r \]

Thus, combining all results above, we have,

\[ \delta \leq \sum_{i=1}^{m} \sum_{P \in \text{End}(x_i)} \Pr(P - V) = \sum_{i=1}^{m} \sum_{P \in \text{End}(x_i)} \Pr(P - V) = O \left( \frac{1}{\sqrt{n}} \right) \]

Next, we will give an exponential (tighter) upper bound on \( \delta \) when \( \pi \) does not belong to any hyperplanes.

We first give a generalized definition of pivotal profile.

**Definition 17 (Generalized Pivotal Profile).** Profile \( P \) is a (generalized) pivotal profile if there exist a pair of votes \( V \) and \( W \) such that \( r(P) \neq r(P - V + W) \).

In the next lemma we will show that generalized pivotal profiles only stays close to hyperplanes. We first define the set of all generalized pivotal profiles \( P_{\text{Piv}} \). For any \( P \in P_{\text{Piv}} \), we know that there exist a hyperplane \( h \in H \) such that \( \text{Dist}(h, P) \leq \sqrt{2} \). According to triangular inequality, we have

\[ \text{Dist}(n\pi, P) \geq \text{Dist}(n\pi, h) - \text{Dist}(h, P) \geq n\text{Dist}(\pi, h) - \sqrt{2} \]

Thus, there must exist one hyperplane \( h \in H \) such that \( \text{Dist}(h, P) \leq \sqrt{2} \).

**Lemma 9.** Let \( D \) be the distribution on profiles (databases of votes), where each entry is iid according to distribution \( \pi \) over linear orders on \( m \) candidates. GSR \( r(H, h_H) \) is \( (0, \delta, D) \) DDP when only the winner is announced, where

\[ \delta = O \left[ \exp \left( -\frac{\min_{h \in H} \text{Dist}(\pi, h)^2}{3(m!)^2 \left( \max_{i \in [m]} \pi[i]\right)} \cdot n \right) \right] = O \left[ e^{-\Theta(n)} \right] \]

**Proof.** We first define the set of all generalized pivotal profiles \( \mathbb{P}_{\text{Piv}} \). For any \( P \in \mathbb{P}_{\text{Piv}} \), we know that there exist a hyperplane \( h \in H \) such that \( \text{Dist}(h, P) \leq \sqrt{2} \). According to triangular inequality, we have \( \text{Dist}(n\pi, P) \geq \text{Dist}(n\pi, h) - \text{Dist}(h, P) \geq n\text{Dist}(\pi, h) - \sqrt{2} \). Thus, there must exist one
dimension $j$ that $|P[j] - n\pi[j]| \geq \frac{n\text{Dist}(\pi, h) - \sqrt{2}}{m!}$. Then, we bound $\delta$ as,

$$\delta = \max_{V, W, S} [\Pr(P \in S | X_1 = V) - \Pr(P \in S | X_1 = W)]$$

$$\leq \sum_{P \in \mathbb{P}_{V|W}} \left[ \max_{V} \Pr(P \in \mathbb{P}_{V|W} | X_1 = V) \right]$$

$$\leq \max_{V, h, j} \Pr \left( |P[j] - n\pi[j]| \geq \frac{n\text{Dist}(\pi, h) - \sqrt{2}}{m!} | X_1 = V \right)$$

$$\leq \max_{h, j} \Pr \left( |P[j] - n\pi[j]| \geq \frac{n\text{Dist}(\pi, h) - \sqrt{2}}{m!} - 1 \right)$$

$$= O \left[ \exp \left( -\frac{[\min_{h \in H} \text{Dist}(\pi, h)]^2}{3(m!)^2 (\max_{i \in [m]} \pi[i])} \cdot n \right) \right]$$

by applying Chernoff bound.

Theorem 8 follows by combining all three bounds derived above.

Proof. Suppose $s_1 = \cdots = s_{l} > s_{l+1}$. We let $V = [a \succ c_1 \succ c_{l-1} \succ b \succ \text{others}]$ and $W = [c_1 \succ c_{l-1} \succ b \succ a \succ \text{others}]$. Let $M$ be the permutation $c_1 \rightarrow c_2 \rightarrow \ldots \rightarrow c_{m-2} \rightarrow c_1$. Let $V_1 = [a \succ b \succ \text{others}]$ and $V_2 = [b \succ a \succ \text{others}]$. Let $P^* = 2P' \cup \{V, W\}$. It follows that $a$ and $b$ are the only two alternatives tied in the first place in $P^*$. Therefore, there exists $\epsilon$ to satisfy the condition.

The same profile can be used to prove the local stability of all Condorcet consistent and monotonic rules.