A Three Species Prey-Predator Holling Type-II Non-Autonomous Discrete Model

B. S. N. Murthy¹, M. A. S. Srinivas²* and Y. Narasimhulu³

¹Department of Mathematics, Aditya College of Engineering and Technology, Kakinada-533001, Andhra Pradesh, India; bsn3213@gmail.com
²Department of Mathematics, Jawaharlal Nehru Technological University, Hyderabad-500085, Andhra Pradesh, India; massrinivas@gmail.com
³Rayalaseema University, Kurnool-518002, Andhra Pradesh, India; ynarasimhulu@rediffmail.com

Abstract

Objectives: This study involves a three species fishery system having two prey and two predators (one species act as both prey and predator simultaneously). A discrete non-autonomous system of difference model for this three species prey-predator model is proposed and analyzed. Methods/Statistical Analysis: The zooplankton (predator) grows based on the phytoplankton (prey) population. For the fish population the food sources are the phytoplankton (prey) population and zooplankton (prey) populations. The interesting part is that the toxin released by the Phytoplankton causes death to Zooplankton and controls the growth of Zooplankton population and also the death rate of Zooplankton is relative to the Phytoplankton population. The functional responses for Phytoplankton, Zooplankton and Fish population are of Holling type-II. Findings: A discrete non autonomous system of difference equations which governs P-Z-F model are analyzed under certain conditions on the parameters the existence of lower and a upper bounds of the densities of all the three species are identified for established the permanence of the P-Z-F system and existence of a unique globally stable periodica solutiona. It is also established under certaina conditions athe prey species will tend to global solution as the predator species become null. Application /Improvements: In this discrete non autonomous model, introducing density restriction terms for zooplankton and fish populations. The results indicate that, death rate of predator is very high or the conservative of Fish for Phytoplankton and Zooplankton are very small, then the Fish population will be driven to extinction.

Keywords: Fish, Global Stability, Periodic Solution, Permanence, Phytoplankton, Semi-Saturation Constant, Zooplankton

1. Introduction

One of the important interdisciplinary activities which cover the study of limited aspects of diverse disciplines is Mathematical modeling. This area has reached to its apex in the recent years and has become a part and parcel of our life in all spheres of activity. The lively relationship in between the prey and predator has existed since a long time and there is a chance for it to be continued in the nearest future, as it is a dominant theme in the ecology. The fact here is that the model what we have proposed i.e., predator method has been used by many scholars for their research. Adding to this; a major issue has been recognized in the ecology. The hiding nature of the prey on the dynamics of predator prey interactions has become the most highlighted issue to study the after math results of their hiding which studied the permanence and global a attractively of the discrete predator-prey system a with a Holling type-Ia functional response.

Plankton is one of its kinds of micro organisms which have a feature of floating freely on the surface of water. These planktons are madea with a combination of tiny animals and tiny plants. Here, tiny plants are phytoplankton which can be considered as a primary producer as it

*Author for correspondence
prepares carbohydrates using energy from sunlight. Now, tiny animals which are considered to be zooplankton consume the above mentioned food as it is the most favorable food for fish and other aquatic species. Normally, phytoplankton area presents with in fresha wasser and marine, buta have the chance to increase quickly to form “blooms”. Researchers have continued their research on interaction with phytoplankton and zooplankton in the prey-predator models and proved that among several thousanda species of phytoplankton, only a few species such as Alexandra spa. and Amphidinium carterae reproduce a toxin which affects the growth of a zooplankton population and has an impact on phytoplankton and zooplankton population.

Fish, generally is considered to be a good source of high-quality protein and contains vitamins & minerals. Due to this fact, fish is consumed as a food by many species including humans all through the world. Therefore, the study of the survival of fish species has become a very important aspect in the nature. The existence of fish population as it’s a known fact, naturally depends on phytoplankton and zooplankton. In the recent times, clear evidences both biological and physiological in many situations have been identified, that when predators are in search for food, a more relatively genera predator-prey should be totally dependent on the ratio-dependent theory, which can be a stated as, the per-capitaa predator growth rate should be a function of the ratioa of prey to predator abundance and therefore it is finalized as the so-calleda ratio-dependenta functional result. Many laboratory experiments and observations strongly supported this statement.

Few scholars studied that the non autonomous case is more reliable and real, because number of biological or environmental parameters does really involve in fluctuation method with the time, and therefore more critical and complex equations are to be introduced. Many intellectuals studied the dynamic behaviors of the above mentioned non-autonomous system which incorporates prey refuge. Adding to this aspect, on the other end, a few discrete time models which are governed by difference equations are founda, which are termed to be more suitable, than the original ones, specially when the population have non-overlapping generations. Finally, according to the researcher’s idea and argument, it has beena proved that the dynamic behaviors of the discretea system are termed to be crucial and complex with more richa vibrant than the continuous ones.

### 2. Phytoplankton-Zooplankton-Fish Model (P-Z-F MODEL)

A new prey-predator system for three species such as Phytoplankton, Zooplankton and Fish has been considered. Here Fish (predator) grows by eating Phytoplankton (prey) and Zooplankton (prey) and again Zooplankton (predator) grows by eating Phytoplankton (prey) so, there are two predators (Fish and Zooplankton) and two preys (Zooplankton and Phytoplankton). Therefore, the specialty in this paper is that Zooplankton acts as a predator in one side as well as a prey in another side and Phytoplankton is purely a prey and Fish is purely predator. Also the Phytoplankton releases toxic substance which kills the Zooplankton and controls the growth of the Zooplankton population. It is assumed that the death rate of a Zooplankton is proportional to the consumption rate of the Phytoplankton.

Suppose \( x(t), y(t), z(t) \) denotes the densities of P-Z-F model at time ‘\( t \)’.

The functional response for Phytoplankton by Zooplankton is \( \frac{\beta_{xy}}{\alpha + x} \), \( \beta \) being the maximum uptake rate for Zooplankton species. The functional response for Zooplankton by Phytoplankton \( \frac{1}{\alpha + x} \), \( \beta \) being the ratio of biomass protection for Zooplankton species. The functional response for Phytoplankton and Zooplankton by Fish is \( \frac{s_{1}yz}{\alpha + x} \), \( s_{1} \) and \( s \) being the ratio of biomass protection of Fish population for Phytoplankton and Zooplankton respectively. The death rate of Zooplankton is \( \frac{\rho_{xy}}{\alpha + x} \) which is proportional to the Phytoplankton population, \( \rho \) being the rate of releasing the toxic substances produced by per unit biomass of Phytoplankton. Here \( \alpha \) is semi saturationa constanta for Holling type-II functional food.

The mathematical form of this P-Z-F model is

\[
\frac{dx}{dt} = rx \left( 1 - \frac{x}{k} \right) - \frac{\beta_{xy}}{\alpha + x} - \frac{\gamma_{xz}}{\alpha + x} 
\]

\[
\frac{dy}{dt} = \frac{\beta_{xy}}{\alpha + x} - dy - \frac{\rho_{xy}}{\alpha + x} - \frac{\gamma_{yz}}{\alpha + x} 
\]
\[
\frac{dz}{dt} = \frac{sxz}{\alpha + x} + \frac{s_1yz}{\alpha + x} - \delta z
\]  
(3)

In [19] studied the above autonomous model by treating \(x(t), y(t), z(t)\) as continuous on \(t > 0\).

In the literature majority of the dynamic behavior of inhabitants models are based on autonomous incessant models governed by a set of differential equations. Whenever the populations have non overlapping generations and the size of the population is very small, the dynamic behavior of the population model can be better analyzed by discrete models than continuous models. In fact the discrete models are more intricate and affluent dynamite to analyze. The discrete time models can also provide proficient computational models of incessant models for numerical simulations.

In the literature many biological parameters were considered as constants and they do take fixed values for all \(t\). But in reality these biological parameters fluctuate with time. Hence non autonomous systems are more realistic than autonomous systems. It is the property of the environment that the density of any species will be restricted by several reasons. To have much more realistic model we introduce the density restriction terms in a P-Z-F model (1-3).

Now we propose a discrete non autonomous Phytoplankton-Zooplankton-Fish system where Zooplankton and Fish population densities are restricted by the environment. The difference equations for the discrete non autonomous system which is counter part of the model (1-3) by introducing density restriction terms in Zooplankton and Fish.

The differences are:

\[
x(k+1) = x(k) \exp \left( a_1(k)x(k) - b_1(k)x(k) - \frac{c_{12}(k)y(k)}{\alpha(k) + x(k)} - \frac{c_{13}(k)z(k)}{\alpha(k) + x(k)} \right)
\]  
(4)

\[
y(k+1) = y(k) \exp \left( \frac{c_{21}(k)x(k)}{\alpha(k) + x(k)} - b_2(k)y(k) - \frac{c_{23}(k)z(k)}{\alpha(k) + x(k)} - d_1(k) \right)
\]  
(5)

\[
z(k+1) = z(k) \exp \left( \frac{c_{31}(k)x(k)}{\alpha(k) + x(k)} - b_3(k)z(k) + \frac{c_{32}(k) y(k)}{\alpha(k) + x(k)} - d_3(k) \right)
\]  
(6)

Where \(x(k), y(k), z(k)\) are respectively the inhabitant sizes of the Phytoplankton, Zooplankton and Fish at \(k^{th}\) generation respectively.

The parameters in the system (4-6) represent ecologically the following:

- \(a_1(k)\) is intrinsic growth rate of Phytoplankton population;
- \(b_1(k), b_2(k), b_3(k)\) represents the density restriction terms of Phytoplankton, Zooplankton and Fish;
- \(c_{12}(k), c_{13}(k), c_{23}(k)\) are respectively the maximum uptake rate of Phytoplankton by Zooplankton, Phytoplankton by Fish and Zooplankton by Fish;
- \(c_{21}(k), c_{31}(k), c_{32}(k)\) are respectively the ratio of biomass conservation of Zooplankton for Phytoplankton, Fish for Phytoplankton and Fish for Zooplankton;
- \(\gamma(k)\) is the releasing rate of the toxic substances a per unit biomass a of Phytoplankton.

Here all \(a_1(k), b_1(k), c_{ij}(k), d_1(k)\) and \(\alpha(k)(i,j=1,2,3)\) are all nonnegative sequences.

The \(k^{th}\) generation of the Phytoplankton-Zooplankton-Fish model is given by

\[
x(k) = x(0) \exp \left( \sum_{i=1}^{3} \frac{a_i(L) - b_i(L)x(k) - \frac{c_{i2}(L)y(L) - \gamma(L)z(L)}{\alpha(L) + x(L)} - \frac{c_{i3}(L)z(L)}{\alpha(L) + x(L)} - d_i(L)}{\alpha(L) + x(L)} \right)
\]  
(7)

\[
y(k) = y(0) \exp \left( \sum_{i=1}^{3} \frac{c_{i2}(L)x(L)}{\alpha(L) + x(L)} - b_i(L)y(k) - \frac{c_{i3}(L)z(L)}{\alpha(L) + x(L)} - d_i(L) \right)
\]  
(8)

\[
z(k) = z(0) \exp \left( \sum_{i=1}^{3} \frac{c_{i1}(L)x(L)}{\alpha(L) + x(L)} - b_3(k)z(k) + \frac{c_{i2}(L)y(L)}{\alpha(L) + x(L)} - d_3(L) \right)
\]  
(9)

\(x(0) > 0, y(0) > 0, z(0) > 0\) are the initial population of the system (7-9).

3. Permanence

**Definition:** System (4-6) is said to be permanents if there exista positive constants \(N_i, M_i, i = 1, 2, 3\), such that

- \(N_1 \leq \liminf_{n \to \infty} x(k) \leq \limsup_{n \to \infty} x(k) \leq M_1\).
- \(N_2 \leq \liminf_{n \to \infty} y(k) \leq \limsup_{n \to \infty} y(k) \leq M_2\).
- \(N_3 \leq \liminf_{n \to \infty} z(k) \leq \limsup_{n \to \infty} z(k) \leq M_3\).

In this paper \(p^w \) denotes sup \(p(k)\) and \(p^l \) denotes inf \(p(k)\).
Theorem 3.1

\[ \limsup_{k \to \infty} x(k) \leq M_1 \text{ Where } M_1 = \frac{1}{b'_1} \exp\left(a''_1 - 1\right) \] (10)

Proof. Since \( x(0) > 0, y(0) > 0, z(0) > 0 \) from the equation (7-9) it is clear that
\( x(k) > 0, y(k) > 0, z(k) > 0 \) for all \( k \geq 0 \).

Case 1. First assume that \( x(k+1) \geq x(k) \)

Then
\[ a_i(k) - b_i(k)x(k) + \frac{c_{12}(k)y(k)}{\alpha(k) + x(k)} - \frac{c_{13}(k)z(k)}{\alpha(k) + x(k)} \geq 0 \]

\[ \Rightarrow a_i(k) - b_i(k)x(k) \geq \frac{c_{12}(k)y(k)}{\alpha(k) + x(k)} + \frac{c_{13}(k)z(k)}{\alpha(k) + x(k)} \geq 0 \]

\[ \Rightarrow x(k) \leq \frac{a_i(k)}{b_i(k)} = \frac{a''_1}{b'_1} \]

From the elementary calculus, it is known that
\[ \max_{x \in R} \left( x \exp\left(a - bx\right) \right) = \frac{\exp\left(a - bx\right)}{b} \]

\[ x(k) \exp\left( a - bx(k) \right) \leq \frac{\exp\left(a'' - 1\right)}{b'} \] (11)

\[ x(k+1) = x(k) \exp\left( a_i(k) - b_i(k)x(k) + \frac{c_{12}(k)y(k)}{\alpha(k) + x(k)} - \frac{c_{13}(k)z(k)}{\alpha(k) + x(k)} \right) \]

\[ \leq x(k) \exp\left( a''_1 - b'_1x(k) \right) \]

\[ \leq \frac{1}{b'_1} \exp\left( a''_1 - 1\right) \text{ (in view of 11)} = M_1 \] (12)

Thus it is established that \( x(k+1) \geq x(k) \) then
\[ x(k) \leq \frac{a''_1}{b'_1} \text{ and } x(k+1) \leq M_1 \] (13)

Claim. \( x(k) \leq M_1 \forall k \in N \).

We prove this by the method of contradiction.

Suppose there exist some \( u \in N \) such that \( x(u) > M_1 \).

Let \( u^* \) be the smallest positive integer such that
\[ x(u^*) > M_1 \] (14)

Since \( u^* \) is smallest, such that \( x(u^*) > M_1 \), we have
\[ x(u^* - 1) \leq M_1 \]

Therefore \( x(u^*) > M_1 \geq x(u^* - 1) \) that implies
\[ x(u^*) \geq x(u^* - 1) \]

In view of (13), we have \( x(u^*) \leq M_1 \) and
\[ x(u^* - 1) \leq \frac{a''_1}{b'_1} \]

This is a contradiction to (14).

Therefore \( x(k) \leq M_1 \) for all \( k \in N \)

Hence the claim is proved.

Case 2. Now assume that \( x(k+1) < x(k) \)

Let \( x^* = \lim_{k \to \infty} x(k) \)

Claim. \( x^* \leq \frac{a''_1}{b'_1} \)

By contradiction if possible \( x^* > \frac{a''_1}{b'_1} \) this implies
\[ a''_1 - b'_1x^* < 0 \]

Taking limit as \( k \to \infty \) to the first equation of system (4-6)
\[ x^* = x^* \exp\left( a_i(k) - b_i(k)x^* + \frac{c_{12}(k)y(k)}{\alpha(k) + x(k)} - \frac{c_{13}(k)z(k)}{\alpha(k) + x(k)} \right) \]

\[ \Rightarrow a_i(k) - b_i(k)x^* \geq \frac{c_{12}(k)y(k)}{\alpha(k) + x(k)} + \frac{c_{13}(k)z(k)}{\alpha(k) + x(k)} = 0 \]

\[ \Rightarrow \frac{c_{12}(k)y(k)}{\alpha(k) + x(k)} + \frac{c_{13}(k)z(k)}{\alpha(k) + x(k)} = a_i(k) - b_i(k)x^* \]

\[ \leq a''_1 - b'_1x^* \text{ (in view of 11)} = M_1 \]

This is absurd as left hand side is positive.

Hence the claim is proved.

In Case 2 \( x(k) \leq \frac{a''_1}{b'_1} \), and in Case 1 \( x(k) \leq M_1 \)

Therefore in any case \( x(k) \leq M_1 \)

\[ \left\{ \begin{array}{l}
\text{since } M_1 = \frac{\exp\left(a''_1 - 1\right)}{b'_1} \geq \frac{a''_1}{b'_1} \\
\text{Hence the theorem is proved.}
\end{array} \right. \]
Theorem 3.2
When the inequalities
\[
\frac{c_{31}^u M_1}{\alpha^l} - d_2 > 0 \quad \text{and} \quad \frac{c_{31}^u M_1 + c_{32}^u M_2}{\alpha^l} - d_1 > 0
\]
holds (15),
\[
\lim_{k \to \infty} \sup y(k) \leq M_2 \quad \text{and} \quad \lim_{k \to \infty} \sup z(k) \leq M_3
\]
Where
\[
M_2 = \frac{1}{b_2} \exp \left( \frac{c_{31}^u M_1}{\alpha^l} - d_2^l - 1 \right),
\]
\[
M_3 = \frac{1}{l} \exp \left( \frac{c_{31}^u M_1 - c_{32}^u M_2 - d_3}{l} - 1 \right)
\]

Proof. Case 1. First assume that \( y(k+1) \geq y(k) \)
Then
\[
\frac{c_{31}(k)x(k)}{\alpha(k) + x(k)} - b_2(k)y(k) - \frac{y(k)x(k)}{\alpha(k) + x(k)} \geq 0
\]
\[
\Rightarrow y(k) \leq \frac{1}{b_2} \left( \frac{c_{31}^u M_1}{\alpha^l} - d_2^l \right)
\]
\[
y(k+1) = y(k) + \frac{c_{31}(k)x(k)}{\alpha(k) + x(k)} - b_2(k)y(k) + \frac{y(k)x(k)}{\alpha(k) + x(k)} - d_2(k) \leq \frac{1}{b_2} \exp \left( \frac{c_{31}^u M_1}{\alpha^l} - d_2^l - 1 \right) \quad \text{(In view of (11))} = M_2 \quad (16)
\]
Thus it is established that if \( y(k) \leq \frac{1}{b_2} \left( \frac{c_{31}^u M_1}{\alpha^l} - d_2^l \right) \) then \( y(k+1) \leq M_2 \)

Claim. \( y(k) \leq M_2 \quad \forall \ k \in \mathbb{N} \)

We prove this by the method of contradiction. Suppose there exist some \( v \in \mathbb{N} \) such that \( y(v) > M_2 \). Let \( v \) be such smallest positive integer such that \( y(v') > M_2 \) (18)

Since \( v' \) is smallest, such that \( y(v') > M_2 \), we have \( y(v' - 1) \leq M_2 \).

Therefore \( y(v') > M_1 \geq y(v' - 1) \Rightarrow v' \geq y(v' - 1) \), In view of (16), we have
\[
y(v') \leq M_2 \quad \text{and} \quad y(v' - 1) \leq \frac{1}{b_2^l} \left( \frac{c_{31}^u M_1}{\alpha^l} - d_2^l \right).
\]

This is a contradiction to (18). Therefore \( y(k) \leq M_2 \)
for all \( k \in \mathbb{N} \)
Hence the claim is proved.

Case 2. Now assume that \( y(k+1) < y(k) \)

Suppose \( y^* = \lim_{k \to \infty} y(k) \)

Claim. \( y^* \leq \frac{1}{b_2^l} \left( \frac{c_{31}^u M_1}{\alpha^l} - d_2^l \right) \)

If possible \( y^* > \frac{1}{b_2^l} \left( \frac{c_{31}^u M_1}{\alpha^l} - d_2^l \right) \) this implies
\[
\frac{c_{31}^u M_1}{\alpha^l} - d_2^l - b_2^l y^*(k) < 0
\]

Taking limit as \( k \to \infty \) to the second equation of system (4-6), i.e.,
\[
y' = y' \exp \left( \frac{c_{31}(k)x(k)}{\alpha(k) + x(k)} - b_2(k)y(k) - \frac{y(k)x(k)}{\alpha(k) + x(k)} \right) - \frac{c_{32}(k)z(k)}{\alpha(k) + x(k)} - d_2(k)
\]
\[
\Rightarrow y(k+1) = y(k) + \frac{c_{31}(k)x(k)}{\alpha(k) + x(k)} - b_2(k)y(k) + \frac{y(k)x(k)}{\alpha(k) + x(k)} - \frac{c_{32}(k)z(k)}{\alpha(k) + x(k)} = 0
\]
\[
\Rightarrow y(k) = \frac{c_{31}(k)x(k)}{\alpha(k) + x(k)} + \frac{c_{32}(k)z(k)}{\alpha(k) + x(k)} = \frac{c_{31}(k)x(k)}{\alpha(k) + x(k)} - b_2(k)y(k) - d_2(k)
\]
\[
\leq \frac{c_{31}^u M_1}{\alpha^l} - b_2^l y^*(k) - d_2^l < 0
\]

This is absurd as left hand side is positive. Therefore
\[
y^* \leq \frac{1}{b_2^l} \left( \frac{c_{31}^u M_1}{\alpha^l} - d_2^l \right)
\]

Hence the claim is proved.

From case 2 \( y(k) \leq \frac{1}{b_2^l} \left( \frac{c_{31}^u M_1}{\alpha^l} - d_2^l \right) \) and from case 1 \( y(k) \leq M_2 \)

Therefore in any case \( y(k) \leq M_2 \)

Hence the theorem is proved.
The proof of $\limsup_{k \to \infty} z(k) \leq M_3$ is similar to above analysis and we omit the detail here.

**Theorem 3.3**

When the inequalities $a'_i - \left( \frac{c_{i2}^i M_2 + c_{i3}^i M_3}{\alpha'} \right) > 0$,

$$\frac{c_{21} N_1}{\alpha'' + M_1} - d''_u - \left( \frac{\gamma'^* M_1 + c_{23}^u M_3}{\alpha''} \right) > 0$$

and $\frac{c_{i1} N_1 + c_{i2} N_i}{\alpha'' + M_i} - d'_i > 0$ holds, $\liminf_{k \to \infty} x(k) \geq N_1$, $\inf y(k) \geq i$.

$$\liminf_{k \to \infty} z(k) \geq N_3$$  \hspace{1cm} (19)

Where $M_1, M_2$, and $M_3$ are the same as in Theorem (3.1a), Theorem (3.2)a and

$$N_i = \frac{1}{b_3} \left( a'_i - \frac{c_{i2}^i M_2 + c_{i3}^i M_3}{\alpha'} \right) \exp \left( a'_i - b_i^u M_i - c_{i2}^i M_2 - c_{i3}^i M_3 \right)$$

$$N_i = \frac{1}{b_3} \left( c_{i1} N_1 + c_{i2} N_i - d'_i \right) \exp \left( c_{i1} N_1 + c_{i2} N_i - b_i^u M_i \right)$$

$$N_i = \frac{1}{b_3} \left( c_{i1} N_1 + c_{i2} N_i - d'_i \right) \exp \left( c_{i1} N_1 + c_{i2} N_i - b_i^u M_i \right)$$

**Proof.**

We first establish $\liminf_{k \to \infty} z(k) \geq N_3$  \hspace{1cm} (20)

**Case 1.** First, assume that $z(k + 1) \leq z(k)$.

Then $\frac{c_{31}^i(x(k) - b_j^u z(k)) + c_{32}^i(y(k) - d_3^i k)}{\alpha(k) + x(k)} \leq 0$.

$\Rightarrow z(k) \geq \frac{1}{b_3} \left( c_{31}^i x(k) + c_{32}^i y(k) \right) \frac{1}{\alpha(k) + x(k)} - d_3^i(k) \geq 0$

$\Rightarrow \frac{1}{b_3} \left( c_{31} N_1 + c_{32} N_2 - d'_3 \right)$

$\Rightarrow \liminf_{k \to \infty} z(k) \geq N_3$  \hspace{1cm} (21)

**Claim.** $z(k) \geq N_3 \forall k \in N$

We prove this by the method of contradiction. Suppose there exist some $w \in N$ such that $z(w) < N_3$.

Let $w^*$ be such smallest positive integer such that $z(w^*) < N_3$  \hspace{1cm} (23)

Since $w^*$ is smallest such that, $z(w^*) < N_3$ and we have $z(w^* - 1) \geq N_3$.

Therefore $z(w^*) < N_3 \leq z(w^* - 1)$ that implies $z(w^*) \leq z(w^* - 1)$.

In view of (22), we have $z(w^*) \geq N_3$ and

$$z(w^* - 1) \geq \frac{1}{b_3} \left( c_{31}^i N_1 + c_{32}^i N_2 - d''_u \right)$$

This is a contradiction to (23). Therefore $z(k) \geq N_3 \forall k \in N$. Hence the claim is proved.

**Case 2.** Now assume that $z(k + 1) > z(k)$.

Let $z^* = \lim_{k \to \infty} z(k)$

**Claim.** $z^* \geq \frac{1}{b_3} \left( c_{31}^i N_1 + c_{32}^i N_2 - d''_u \right)$

This implies $c_{31}^i N_1 + c_{32}^i N_2 - d''_u - b_3^i z^* > 0$

Taking limit as $k \to \infty$ to the third equation of system (4-6)

$$z^* = z^* \exp \left( \frac{c_{31}^i x(k) - b_j^u z^*(k) + c_{32}^i y(k)}{\alpha(k) + x(k)} - d_3^i(k) \right)$$

$$\Rightarrow \frac{c_{31}^i x(k) - b_j^u z^*(k) + c_{32}^i y(k)}{\alpha(k) + x(k)} - d_3^i(k) = 0$$

$$\Rightarrow \frac{c_{31}^i x(k) + c_{32}^i y(k)}{\alpha(k) + x(k)} = b_j^u z^*(k) + d_3^i(k)$$

and $z(k + 1) \geq N_3$  \hspace{1cm} (22)
Indian Journal of Science and Technology

Vol 10 (22) | June 2017 | www.indjst.org

B. S. N. Murthy, M. A. S. Srinivas and Y. Narasimhulu

\[
\leq b_3^u z^*(k) + d_3^u < \frac{c_{31}^l N_1 + c_{32}^l N_2}{\alpha^* + M_1} \Rightarrow
\]

This is a contradiction since \(c_{31}(k)x(k) > c_{31}^l N_1\) and \(c_{32}(k)y(k) > c_{32}^l N_2\).

Therefore \(z^* \geq \frac{1}{b_3^u} \left( \frac{c_{31}^l N_1 + c_{32}^l N_2 - d_3^u}{\alpha^* + M_1} \right)\) in any case \(z(k) \geq N_3\). Hence the theorem is proved.

The proof of the other two inequalities is similarly to above analysis and we omit the detail here.

**Theorem 3.4**

When conditions (15) and (19) holds, the system (4-6) possess permanence.

**Proof.** Theorem (3.1), Theorem (3.2) and Theorem (3.3) give the lower and upper bounds of \(x(n), y(n), z(n)\). Hence the system (4-6) is a permanent.

The parameters of the system (4-6) may be assumed to be cyclic or periodic by the effect of environment such as weather, temperature, food supply and other environmental quantities. Here we consider in the system (4-6)

\[
\alpha(k), a_i(k), b_i(k), c_i(k), d_i(k) (i, j = 1, 2, 3),
\]

are being periodic with a common period \(\theta\). Figure 1 represents the dynamic behavior of the system (67-69) with the initial conditions \((x(0), y(0), z(0)) = (0.5, 0.8, 0.3), (0.05, 0.1, 0.2) \text{ and } (0.4, 0.2, 0.6)\) respectively.

**4. Survival and Stability of a Periodic Solution**

By adopting some extra conditions, global stability of the system (4-6) is established.

**Theorem 5.1**

Assume that (19) and (24) hold, and

\[
\pi_i = \max \left\{ -b_i^r M_i, \left| -b_i^l N_i \right|, \frac{c_i^u M_i + c_i^l M_i}{\alpha^* + N_i}, \frac{c_i^u M_i + c_i^l M_i}{\alpha^* + N_i} \right< 1,
\]

then for every \(\theta\) periodic solution \((\overline{x}(k), \overline{y}(k), \overline{z}(k))\) of system a (4-6), we have

\[
\left| \overline{x}(k), \overline{y}(k), \overline{z}(k) \right| < 1.
\]

**Figure 1.** The dynamic behavior of the system (67-69) with the initial conditions \((x(0), y(0), z(0)) = (0.5, 0.8, 0.3), (0.05, 0.1, 0.2) \text{ and } (0.4, 0.2, 0.6)\) respectively.
\[ \lim_{k \to \infty} x(k) - \overline{x}(k) = 0 \quad \lim_{k \to \infty} y(k) - \overline{y}(k) = 0 \]
\[ \lim_{k \to \infty} z(k) - \overline{z}(k) = 0 \]  

(28)

**Proof:** Let 
\[ x(k) = \tau(k) \exp(\delta(k)), y(k) = \tau(k) \exp(\beta(k)), z(k) = \tau(k) \exp(\chi(k)) \]
then system (4-6) is equivalent to 
\[ \delta(k+1) = \delta(k) + \beta(k) \tau(k)(1 - \exp(\delta(k))) + \frac{c_{1}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\beta(k))) \]
\[ + \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ - \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\chi(k))) \]
\[ \beta(k+1) = \beta(k) + \beta(k) \tau(k)(1 - \exp(\beta(k))) - \frac{c_{1}(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ + \frac{\gamma(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{1}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ - \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ \gamma(k+1) = \gamma(k) + \beta(k) \tau(k)(1 - \exp(\beta(k))) - \frac{c_{1}(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ + \frac{\gamma(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{1}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ - \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]

(29)

(30)

(31)

Following the mean value theorem, it follows that 
\[ \delta(k+1) = \delta(k) + \beta(k) \tau(k)(1 - \exp(\beta(k))) - \frac{c_{1}(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ + \frac{\gamma(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{1}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ - \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ \beta(k+1) = \beta(k) + \beta(k) \tau(k)(1 - \exp(\beta(k))) - \frac{c_{1}(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ + \frac{\gamma(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{1}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ - \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ \gamma(k+1) = \gamma(k) + \beta(k) \tau(k)(1 - \exp(\beta(k))) - \frac{c_{1}(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ + \frac{\gamma(k) \alpha(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{1}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]
\[ - \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) + \frac{c_{13}(k) \tau(k)}{\alpha(k) + \chi(k)}(1 - \exp(\delta(k))) \]

(32)

(33)

(34)

For the completion of the proof, it is enough to prove that 
\[ \lim_{k \to \infty} \delta(k) = \lim_{k \to \infty} \beta(k) = \lim_{k \to \infty} \chi(k) = 0 \]  

(35)

From (25-27), we can choose \( \varepsilon > 0 \) such that 
\[ \pi_{\varepsilon}^{*} = \max \left[ \left| -b_{1}^{*}(M_{1}+\varepsilon) \right|, \left| -b_{1}^{*}(N_{1}+\varepsilon) \right| + \frac{c_{13}^{*}(M_{1}+\varepsilon) + c_{13}^{*}(M_{1}+\varepsilon)}{\alpha^{*} + (N_{1}+\varepsilon)} \right] < 1, \]
\[ (36) \]

\[ \pi_{\varepsilon}^{*} = \max \left[ \left| -b_{1}^{*}(M_{1}+\varepsilon) \right|, \left| -b_{1}^{*}(N_{1}+\varepsilon) \right| + \frac{c_{13}^{*}(M_{1}+\varepsilon) + c_{13}^{*}(M_{1}+\varepsilon)}{\alpha^{*} + (N_{1}+\varepsilon)} \right] < 1, \]
\[ (37) \]

\[ \pi_{\varepsilon}^{*} = \max \left[ \left| -b_{1}^{*}(M_{1}+\varepsilon) \right|, \left| -b_{1}^{*}(N_{1}+\varepsilon) \right| + \frac{c_{13}^{*}(M_{1}+\varepsilon) + c_{13}^{*}(M_{1}+\varepsilon)}{\alpha^{*} + (N_{1}+\varepsilon)} \right] < 1. \]
\[ (38) \]

From the Theorem (3.1)a, Theorem (3.2)a and Theorem (3.3)a there exists \( k_{0} \in \mathbb{N} \) such that 
\[ N_{1} \leq x(k) \leq M_{1}+\varepsilon, N_{1} \leq \overline{x}(k) \leq M_{1}+\varepsilon, N_{1} \leq y(k) \leq M_{1}+\varepsilon, N_{1} \leq \overline{y}(k) \leq M_{1}+\varepsilon, N_{1} \leq z(k) \leq M_{1}+\varepsilon, \]
\[ \overline{x}(k) \leq N_{1} \geq k_{0} \]

Note that \( \xi_{1}(k) \in [0,1] \Rightarrow \overline{x}(k) \exp \{ \xi_{1}(k) \delta(k) \} \]
lies between \( x(k) \) and \( \overline{x}(k) \)

Similarly, \( \overline{y}(n) \exp \{ \xi_{2}(n) \beta(n) \} \) lies between \( y(n) \) and \( \overline{y}(n) \exp \{ \xi_{2}(n) \beta(n) \} \) lies between \( \overline{x}(k) \) and \( \overline{z}(k) \).

From (32-38), we get 
\[ |\delta(k+1) - |\delta(k)| < \left| \max \left[ \left| -b_{1}^{*}(M_{1}+\varepsilon) \right|, \left| -b_{1}^{*}(N_{1}+\varepsilon) \right| + \frac{c_{13}^{*}(M_{1}+\varepsilon) + c_{13}^{*}(M_{1}+\varepsilon)}{\alpha^{*} + (N_{1}+\varepsilon)} \right] \right| \]
\[ \beta(k) \left| \frac{c_{13}^{*}(M_{1}+\varepsilon) + c_{13}^{*}(M_{1}+\varepsilon)}{\alpha^{*} + (N_{1}+\varepsilon)} \right| \]
\[ \chi(k) \left| \frac{c_{13}^{*}(M_{1}+\varepsilon) + c_{13}^{*}(M_{1}+\varepsilon)}{\alpha^{*} + (N_{1}+\varepsilon)} \right| \]

(39)
\[ |\beta(k+1)| \leq |\beta(k)| \max \left| b_i^c (M_i + \varepsilon) - b_i^c (N_i - \varepsilon) \right| + |\varepsilon(k)| \left( \frac{c_{32} (M_i + \varepsilon)}{\alpha' + (N_i - \varepsilon)} \right)^2 \]

\[ + |\delta(k)| \left( \frac{c_{32} M_i + \varepsilon}{\alpha' + (N_i - \varepsilon)} \right)^2 \]  \hspace{1cm} (40)

\[ |\varepsilon(k+1)| \leq |\varepsilon(k)| \max \left| b_i^c (M_i + \varepsilon) - b_i^c (N_i - \varepsilon) \right| + |\beta(k)| \left( \frac{c_{32} (M_i + \varepsilon)}{\alpha' + (N_i - \varepsilon)} \right)^2 \]

\[ + |\delta(k)| \left( \frac{c_{32} M_i + \varepsilon}{\alpha' + (N_i - \varepsilon)} \right)^2 \]

for \( k \geq k_0 \)  \hspace{1cm} (41)

Let \( \pi = \max \{ \pi_1, \pi_2, \pi_3 \} \), then \( \pi < 1 \), from (39-41) we get

\[ \max \{ |\beta(k+1)|, |\beta(k)|, |\varepsilon(k+1)| \} \leq \pi \max \{ |\beta(k)|, |\varepsilon(k)| \}, k \geq k_0. \]

This implies

\[ \max \{ |\beta(k)|, |\varepsilon(k)| \} \leq \pi^{k-k_0} \max \{ |\beta(k)|, |\varepsilon(k)| \}, k \geq k_0. \]

Therefore (39-41) holds and the complete the proof.

6. Extinction of Predator (Fish) and Stability of Prey Species

System (4-6) with \( \alpha(k), a_i(k), b_i(k), c_{ij}(k), d_i(k) \) being a periodic with a common period \( \theta > 0 \). Following a few assumptions in this particular case, the predator will be automatically driven to extinction, while prey will be world wide attractive to a certain solution of a logistic equation. We consider a discrete logistic equation

\[ T(k+1) = T(k) \exp \left( a_i(k) - b_i(k) T(k) \right), k \in N \]  \hspace{1cm} (42)

**Theorem 6.1**

The positive solution \( T^* \) of (42), we have

\[ m \leq \liminf_{k \to +\infty} T^* \leq \limsup_{k \to +\infty} T^* \leq M_1 \]

Where \( m = \frac{a_i}{b_i} \exp \left( a_i^* - b_i^* M_1 \right) \) and \( M_1 \) is defined by Theorem (3.1)a. Furthermore, there exist a \( \theta \)-periodic solution for equation (42).

**Proof:** The proof of above claim follows that of the proof of the Theorem (3.1)a, Theorem (3.2) a and Theorem (3.3) a with slight modifications and wea omit the detail here.

**Theorem 6.2**

When the inequality \( \frac{b_i^* \exp \left( a_i^* - 1 \right)}{b_i} < 2 \)  \hspace{1cm} (43)

holds. Let \( \bar{T}(k) \) be a periodic solution of (42), then for every positive solution \( T(n) \) of (42), we have \( \lim_{k \to +\infty} T(k) - \bar{T}(k) = 0 \).

**Proof:** Let \( T(k) = \bar{T}(k) \exp \left( F(k) \right) \) then system (42) is equal to \( F(k+1) = F(k) + b_i(k)(\bar{T}(k) - T(k)) \)

\[ = F(k) - b_i(k) \bar{T}(k)(\exp F(k) - 1) \]

Following the mean value theorem, the following are;

\[ F(k+1) = F(k)(1 - b_i(k) \bar{T}(k) \exp \{ \xi_4(k) F(k) \}) \]

where \( \xi_4(k) \in [0,1] \).  \hspace{1cm} (44)

To validate the proof, it is enough to prove that

\[ \lim_{n \to +\infty} F(n) = 0 \]  \hspace{1cm} (45)

We first assume that

\[ \pi^* = \max \left\{ \left| 1 - b_i^* M \right|, \left| 1 - b_i^* m \right| \right\} < 1 \]  \hspace{1cm} (46)

then a positive constant can be chosen \( \varepsilon > 0 \) small enough such that

\[ \pi_e^* = \max \left\{ \left| 1 - b_i^* (M_i + \varepsilon) \right|, \left| 1 - b_i^* (m - \varepsilon) \right| \right\} < 1 \]  \hspace{1cm} (47)

From the Theorem (6.2), there exists \( k^* \in N \) such that

\[ m - \varepsilon \leq T(k), \bar{T}(k) \leq M_i + \varepsilon, k \geq k^* \]

note that \( \xi_4(k) \in [0,1] \) implies that

\[ \bar{T}(k) \exp \{ \xi_4(k) F(k) \} \] exists \( \bar{T}(k) \) and \( T(k) \).

From (44), we get

\[ |F(n+1)| \leq |F(n)| \max \left\{ \left| 1 - b_i^* (M_i + \varepsilon) \right|, \left| 1 - b_i^* (m - \varepsilon) \right| \right\} \]

\[ \Rightarrow |F(k)| \leq |F(k^*)| \left| \pi_e^* \right|^{k-k^*}, k \geq k^* \]  \hspace{1cm} (48)
Since \( \pi^* < 1 \) and \( \varepsilon \) is an arbitrary small, we obtain
\[
\lim_{k \to \infty} F(k) = 0,
\]
and it means that (45) holds when
\[
\pi^* < 1.
\]
Note that
\[
\pi^* < 1
\]
is equivalent to
\[
1 - \frac{b_i}{b'_1} M_1 > -1.
\]
Or
\[
b_i M_1 = \frac{b_u}{b'_1} \exp\left(a_i - 1\right) < 2.
\]

Now we can conclude that (45) is satisfied as (43) holds, and so
\[
\lim_{k \to \infty} T(k) - T(k) = 0
\]

**Theorem 6.3**

When the inequality
\[
\frac{c_{i1}^w M_1 + c_{i2}^w M_2}{\alpha' + (N_1 - \varepsilon)} - \frac{b'_1}{b'_1} 0
\]
holds. Where \( M_1, M_2 \) and \( N_1 \) are defined by Theorem (3.1), Theorem (3.2) and Theorem (3.3)

Let \( (x(k), y(k), z(k)) \) be any positive solution of system (4-6), then \( z(k) \to 0 \) as \( k \to \infty \).

**Proof.** From (49) we can choose a constant \( \varepsilon > 0 \) small enough such that inequality
\[
\frac{c_{i1}^w (M_1 + \varepsilon) + c_{i2}^w (M_2 + \varepsilon)}{\alpha' + (N_1 - \varepsilon)} - \frac{b'_1}{b'_1} < 0
\]
holds. Therefore
\[
c_{i1}^w (M_1 + \varepsilon) + c_{i2}^w (M_2 + \varepsilon) - b'_1 < -\sigma < 0
\]

Let \( (x(n), y(n), z(n)) \) be any positive solution of system (4-6) for any \( l \in N \), according to the equation a of system (4-6), we obtain
\[
\ln \frac{z(l+1)}{z(l)} = \frac{c_{i1}(l)x(l)}{\alpha(l)x(l)} - b(l)z(l) + \frac{c_{i2}(l)y(l)}{\alpha(l)x(l)} + \frac{d_i(l)}{d_i(l)}
\]
\[
\leq \frac{c_{i1}(l)x(l)}{\alpha(l)x(l)} + \frac{c_{i2}(l)y(l)}{\alpha(l)x(l)} - d_i(l)
\]
\[
\leq -d_i(l) + \frac{c_{i1}^w (M_1 + \varepsilon)}{\alpha' + (N_1 - \varepsilon)} + \frac{c_{i2}^w (M_2 + \varepsilon)}{\alpha' + (N_1 - \varepsilon)} \leq -\sigma < 0
\]

Summating both sides of the above equations from 0 to \( k-1 \), we obtain
\[
\ln \frac{z(k)}{z(0)} < -\sigma k
\]
then
\[
z(n) < z(0) \exp(-\sigma k).
\]
The above inequality shows that \( z(k) \to 0 \) exponentially as \( k \to \infty \).

This completes the proof of Theorem 6.3.

**Theorem 6.4**

When (43), (50) and \( \frac{c_{i1}^w M_1 + c_{i2}^w M_2}{\alpha' + M_1} - d_i l < 2 \) holds, also
\[
\frac{c_{i1}^w M_1 + c_{i2}^w M_2}{\alpha' + d_i M_1} - d_i l > 0
\]
and
\[
\frac{c_{i1}^w M_1 + c_{i2}^w M_2}{\alpha' + \sigma + M_1} - d_i l > 0
\]
then for any positive solution \( (x(k), y(k), z(k)) \) of system (4-6), we have
\[
\lim_{k \to \infty} x(k) - x^*(k) = 0, \quad \lim_{k \to \infty} y(k) - y^*(k) = 0
\]
\( x^*(k) \) is any positive solution of system (42) and \( y^*(k) \) is any positive solution of the second equation of system (4-6).

**Proof:** since (43) holds, it follows from Theorem (6.3) that
\[
\lim_{k \to \infty} z(k) < 0
\]
To prove
\[
\lim_{k \to \infty} x(k) - x^*(k) = 0
\]
Let \( x(k) = x^*(k) \exp(\delta(k)) \)
then from the first equation of system (4-6) and (54),
\[
\delta(k+1) - \delta(k) = x^*(k) \exp(\delta(k)) \exp(\delta(k-1)) - \frac{c_{i1}(k)x(k)}{\alpha(k)x(k)} - \frac{c_{i1}(k)z(k)}{\alpha(k)x(k)}
\]
Using the mean value theorem, one has
\[
\exp(\delta(k+1) - \delta(k)) = \exp(\delta(0), \delta(k)) \exp(\delta(k))
\]
\( \delta(k+1) - \delta(k) = 0 \)
Then the first equation of system (4-6) is equivalent to
\[
\delta(k+1) - \delta(k) = \frac{c_{i1}(k)x(k)}{\alpha(k)x(k)} - \frac{c_{i1}(k)z(k)}{\alpha(k)x(k)}
\]
To complete the proof, it is sufficient to prove that
\[
\lim_{k \to \infty} \delta(k) = 0
\]
We first assume that $\pi = \max \{ ||1 - b^u_i M_i ||, ||1 - b^l_i N_i || \} < 1$, and a positive constant can be chosen $\varepsilon > 0$ small enough such that

$$\pi^e = \max \{ ||1 - b^u_i (M_i + \varepsilon) ||, ||1 - b^l_i (N_i - \varepsilon) || \} < 1$$

(57)

For above $\varepsilon$, according to Theorem (3.1)a, Theorem (3.2)a, Theorem (3.3)a and Theorem (6.3), there exists an integer $k_i \in \mathbb{N}$ such that

$$N_i - \varepsilon \leq x(k) \leq M_i + \varepsilon, \quad m_i - \varepsilon \leq x^*(k) \leq M_i + \varepsilon,$$

(58)

$$z(k) \leq \varepsilon, \quad k \geq k_i.$$

Noting that $n \geq N_i$ , then

$$N_i - \varepsilon \leq x(k) \leq M_i + \varepsilon, \quad m_i - \varepsilon \leq x^*(k) \leq M_i + \varepsilon, \quad z(k) \leq \varepsilon, \quad k \geq k_i$$

(59)

It follows from (58) that

$$\frac{c_{12}(k)}{\alpha(k) + x(k)} \leq \frac{c_{12}^e}{\alpha^e} = p_x$$

and

$$\frac{c_{13}(k)}{\alpha(k) + x(k)} \leq \frac{c_{13}^e}{\alpha^e} = p_{x^*}$$

(60)

Noting that $\varphi(k) \in (0.1)$, it follows that

$$x^*(k) \exp (\varphi(k) \delta(k))$$

lies between $x^*(k)$ and $x(k)$.

From (55), (57)–(58), we get

$$|\varphi(k) - \varphi(n)| \leq |\varphi(k) - \varphi(n) - \delta(k)| + |\delta(k) + \delta(n)| \leq c_{12}^e (N_i + \varepsilon) + \alpha^e = c_{12}^e$$

(61)

This implies that

$$\left| \pi(k) \right| \leq \pi_{k+1} \left| \pi(k) \right| \left( \frac{1 - \pi_{k+1}^{n_i}}{1 - \pi_{k+1}^n} \right) (p_x (N_i + \varepsilon) + p_{x^*} \varepsilon) \quad k \geq k_i$$

(62)

Since $\pi_{k+1} < 1$ and $\varepsilon$ is an arbitrary small, we obtain

$$\lim_{k \to \infty} \delta(k) = 0$$

(63)

Note that $1 - b^u_i M_i \leq 1 - b^l_i N_i < 1$.

Thus, $\pi < 1$ is equivalent to $1 - b^u_i M_i > -1$

$$b^u_i M_i = \frac{b^u_i}{b^l_i} \exp \left[ a^u_i \left( \frac{1 - \pi}{1 - \pi_{k+1}} \right) \right] < 2.$$
We can conclude that \( \lim_{k \to \infty} x(k) - x^*(k) = 0 \), \( \lim_{k \to \infty} y(k) - y^*(k) = 0 \) \( k \geq k_1 \).

This completes the proof of Theorem (6.4).

7. Numerical Simulation

Here, an example is given to illustrate the viability of the main results.

**Example 1.**

Consider the following Phytoplankton-Zooplankton-Fish system

\[
\begin{align*}
x(k+1) &= x(k) \exp \left( (0.5 + 0.2 \cos k) - x(k) \right) - \frac{0.02 y(k)}{1 + x(k)} - \frac{0.001 z(k)}{1 + x(k)} \\
y(k+1) &= y(k) \exp \left( (0.2 - 0.02 \sin k) \frac{x(k)}{1 + x(k)} - y(k) \right) + \frac{0.001 y(k)}{1 + x(k)} - \frac{0.001 z(k)}{1 + x(k)} \frac{0.001 y(k)}{1 + x(k)} - 0.001 \\
z(k+1) &= z(k) \exp \left( (0.2 + 0.1 \cos k) \frac{x(k)}{1 + x(k)} - z(k) \right) + \frac{0.01 y(k)}{1 + x(k)} - \frac{0.001 y(k)}{1 + x(k)} - 0.001
\end{align*}
\]

Clearly we can easily see that \( \frac{c_{21}^u M_1 + c_{23}^u M_2}{\alpha^l} - d_1^l > 0 \),

\[
\frac{c_{31}^y M_1 + c_{32}^y M_2}{\alpha^l} - d_1^l > 0, \quad \alpha^l \left( \frac{c_{21}^u M_1 + c_{23}^u M_2}{\alpha^l} - d_1^l > 0 \right)
\]

\[
\frac{c_{21}^l N_1 + c_{23}^l M_1}{\alpha^l} - d_2^l > 0, \quad \alpha^l \left( \frac{c_{21}^l N_1 + c_{23}^l M_1}{\alpha^l} - d_2^l > 0 \right)
\]

The condition (19) is satisfied. From Theorem (3.1)a, Theorem (3.2)a and Theorem (3.3)a the system (67-69) permanent.

**Example 2**

\[
\begin{align*}
x(k+1) &= x(k) \exp \left( (0.5 - 0.1 \cos k) - x(k) \right) - \frac{0.02 y(k)}{1 + x(k)} - \frac{0.001 z(k)}{1 + x(k)} \\
y(k+1) &= y(k) \exp \left( (0.9 - 0.1 \sin k) \frac{x(k)}{1 + x(k)} - y(k) \right) + \frac{0.001 y(k)}{1 + x(k)} - \frac{0.001 z(k)}{1 + x(k)} - 0.001 \\
z(k+1) &= z(k) \exp \left( (0.2 x(k)) - z(k) \right) + \frac{0.01 y(k)}{1 + x(k)} - 0.5
\end{align*}
\]

We can easily see that \( \frac{c_{21}^y M_1 + c_{23}^y M_2}{\alpha^l} - d_1^l > 0 \), \( c_{21}^l N_1 + c_{23}^l M_1 > 0 \) and \( c_{21}^w M_1 + c_{23}^w M_2 - d_3^l < 0 \).

\[
\frac{b_1^l \exp \left( a_1^w - 1 \right)}{b_1^w} < 2, \quad \frac{c_{31}^w M_1 + c_{32}^w M_2}{\alpha^l + N_1} - d_3^w < 0
\]

\[
\text{and } M_3 b_2^u < 2.
\]

Clearly, Theorem (6.2)a, Theorem (6.3)a and Theorem (6.4)a are satisfied. So

\[
\lim_{k \to \infty} x(k) - x^*(k) = 0, \quad \lim_{k \to \infty} y(k) - y^*(k) = 0
\]

and \( \lim z(k) = 0 \). Figure 2 represents the dynamic behavior of the system (70-72) with the initial conditions \( (x(0), y(0), z(0)) = (0.5, 0.8, 1.3), (0.05, 0.1, 0.2) \) and \( (0.4, 0.2, 0.6) \) respectively.

**Figure 2:** The dynamic behavior of the system (70-72) with the initial conditions \( (x(0), y(0), z(0)) = (0.5, 0.8, 1.3), (0.05, 0.1, 0.2) \) and \( (0.4, 0.2, 0.6) \) respectively.

8. Conclusion

A three species prey-predator P-Z-F model has two preys and two predators (one species is prey as well as predator simultaneously) is considered. One of the prey (Phytoplankton) releases toxic substances which affect the predator (Zooplankton). The interesting part is Phytoplankton is food source for Zooplankton and Phytoplankton opposed to be preyed on them by releasing a toxic substance which controls the growth of Zooplankton simultaneously; the Zooplankton is a food source for another predator (Fish) population.

A discrete non autonomous system of difference equations which governs P-Z-F model are analyzed under certain conditions on the parameters the existence of lower and upper bounds of the densities of all the three species are identified for established the permanence of the P-Z-F system. In section (4) the parameters of the
P-Z-F system are assumed to be periodic and establish that the solution of the P-Z-F system has periodic solution. Sufficient conditions about the global stability of the periodic solutions of P-Z-F system are obtained. It is very much necessary to protect some species from extinction. Fish is one of such species which is to be protected. Hence in section (6) we establish the conditions under which the Fish species become extinct. Our results indicate that, death rate of predator is very high or the conservative of Fish for Phytoplankton and Zooplankton are very small, then the Fish population will be driven to extinction. The conditions (6.8) explained.

9. Acknowledgement

This research is supported by University Grants Commission, India. (Grant no 42-18/2013(SR)).

10. References

1. Agarwal RP. Difference Equations and Inequalities. Theory Methods and Applications. 2nd ed. Monographs and Textbooks in Pure and Applied Mathematics. New York: 2000.
2. Berryman AA. The origins and evolution of predator-prey theory. Ecology. 1992; 73(5):1530–5. Crossref
3. Chen LJ, Chen FD, Wang YQ. Influence of predator mutual interference and prey refuge a on Lotka-Volterra predator-prey dynamica. Communications in Nonlinear Science and Numerical a Simulation. 2013; 183:174–80.
4. Chen FaD, Maa ZZ, Zhanga HY. Global asymptotical stability of the Positive Equilibrium of Lotka-Volterra prey-predator model incorporating a constant number of prey refuge. Nonlineara Analysis of Real World Applications. 2012; 13:2790–3. Crossref
5. Krikorian N. The volterra model for three species predator-prey systems boundedness and Stability. Journal of Mathematical Biology. 1979; 7:117–32. Crossref
6. Lianga ZQ, Chena LS. Stabilitaet periodica solutioana for a discretea Leslie predator-prey System Acta Mathematica Scientia. 2006; 26(4):634–40. Crossref
7. Maa ZZ, Chena FD, Wua CQ, Chen WL. Dynamica behaviors of a Lotka-Volterra Predator prey model incorporating a prey refuge and predator mutual interference. Applied Mathematics and Computation. 2013; 219:7945–53. Crossref
8. Wang L, Li WT. Periodic a solutions a and a permanence a for a delayed a non autonomous Ratio dependent a predator-prey model with Holling type a functional response. Journal of a Computational and Applied Mathematics. 2004; 162:341–57. Crossref
9. Zhang a SW, Tan DJ. Study for three species model with Holling II a functional response and periodic coefficients. Journal of Biomathematics. 2000; 15:353–7.
10. Chen LJ, Chen FD. Global stability and bifurcation of a ratio-dependent predator-prey model with prey refuge. Acta a Math. Sinica. 2014; 57(2):301–10.
11. Chen LJ. Permanence of a discrete periodic a volterra model with mutual a Interference. Discrete Dynamics in Nature and Society. 2009; 9.
12. Hugoa A, Massawe ES, Makindea OD. An eco epidemiological mathematical model with treatment and disease infection in both prey and predator population. Journal aof Ecology and the natural environment. 2012; 4(10):266–79.
13. Kara TK, Borty CK. Effort dynamica in a prey– predator model with harvesting. International journal of information and systems sciences. 2010; 6(3):318–32.
14. Wu, a Li. Permanence and globala attract citya of the discrete predator-prey system With Hassell-Varley-Holling-III type functional response. Discrete Dynamics in Nature and society. 2013; 13–9.
15. Duinker J, Wefer G. Gas CO2 and Die Rolle Des Ozeans. Naturwissensc a haften. 1994; 81:237–42. Crossref
16. Chattopadhyay J. Effect of toxic a substances on a two species competitive system. Ecol. Model. 1996; 84:287–9. Crossref
17. Gopalsamy K. Global asymptotic stability in a periodic Lotka-Volterra system. Journal of the Australian Mathematical Society. Series B. Applied Mathematics. 1985; 27:66–72.
18. Smith JM. Models in ecology. Cambridge: Cambridgea University Press; 1974. p. 1363. PMid:4437142
19. Pradhana T, Chaudhuria KS. A dynamical reaction model of two species fishery with Taxation as a control instrument a capital theoretic analysis. Ecological Model. 1996; 121:1–16. Crossref
20. Panjaa P, Mandal a SK. Stability analysis of coexistence of three species prey–predator model. Nonlinear Dynamics. 2015; 81:373–82.
21. Sarkara RR, Chattopadhyaya J. The role of environmental stochasticity in a toxic phytoplankton-non-toxic phytoplankton – zooplankton system. Environ metrics. 2003; 14(8):775–92. Crossref
22. Sahaa T, Bandypadhyaya M. Dynamical analysis of toxin producing phytoplankton–zooplankton interactions: a Nonlinear Analysis. Real World Applications. 2009; 10:314–32. Crossref
23. Gakkhar S, Negi K. A Mathematical model for viral infection in toxin producing Phytoplankton and zooplankton system. Applied Mathematica Computation. 2006; 179:301–13. Crossref
24. Makinde OD. Solving a ratio-dependent predator–prey system with constant effort harvesting using Adomain decomposition method. Applied Mathematical Computation. 2007; 187:17–22. Crossref
25. Wang Y, Wang J. Influence of prey refuge on a predator-prey dynamics. Nonlinear Dynamics. 2012; 67(1):191–201. Crossref
26. Wang YS, Wu H, Sun S. Persistence of pollination mutualisms in plant pollinator-robbing systems. Theoretical Population Biology. 2014; 81:243–50. Crossref
27. Xiao D, Li W. Dynamics in a ratio dependent predator-prey model with predator harvesting. Journal of mathematical analysis and applications. 2006; 324(1):4–29.
28. Apanasov B, Xie X. Discrete actions on nilpotent Lie groups and negatively curved spaces. Differential Geometry and Its Applications. 2004; 20(1):11–29. Crossref
29. Chen YM, Zhou Z. Stable periodic solutions of a discrete periodic Lotka-Volterra Competition system. Journal of Mathematical Analysis and Applicators. 2003; 277(1):358– 66.
30. Chakrabortya Ka, Jana S, Kara TK. Effort dynamics of a delay-induced prey-predator system with reserve. Nonlinear Dynamics. 2012; 70:1805–29a.
31. Panjaa P, Mondala SK. A mathematical study on the spread of Cholera. South Asian Journal of Mathematics. 2004; 4(2):69–84.
32. Panga P, Wang M. Strategy and stationary pattern in a three-species predator-prey model. Journal of Differential Equations. 2004; 200:245–73. Crossref
33. Wu YM, Chen FD. Dynamic behaviors of a non-autonomous discrete predator-prey system incorporating a prey refuge and Holling type II functional response. Discrete Dynamics in Nature and Society. 2012; 14.
34. Yunfei L, Yongzhena P, Shujing G, Changguoa L. Harvesting of a phytoplankton–zooplankton model: Nonlinear Analysis. Real world applications. 2010; 11:3608–19. Crossref
35. Zhang a ZQ, Wu Ja, Wang ZC. Periodic solutions of non-autonomous stage-structured cooperative system. Computers and Mathematics with Applications. 2004; 47:699–706. Crossref