Proposal for implementing the three-qubit refined Deutsch-Jozsa quantum algorithm

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We propose a way to implement a three-qubit refined Deutsch-Jozsa (DJ) algorithm. The present proposal is based on the construction of the 35 \(f\)-controlled phase gates, which uses single-qubit \(\sigma_z\) gates and two-qubit standard controlled-phase (CP) gates only. This proposal is implementable because a single-qubit \(\sigma_z\) gate can be easily realized by applying a single classical pulse and a two-qubit CP gate has been experimentally demonstrated in various physical systems. Finally, it is noted that this proposal is quite general, and can be applied to implement a three-qubit refined DJ algorithm in a cavity-based or noncavity-based physical system.

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I. INTRODUCTION

The interest in quantum computation is stimulated by the discovery of quantum algorithms which can solve problems of significance much more efficiently than their classical counterparts. Quantum algorithms could be implemented using various types of qubits [1] or quantum simulators [2]. Among important quantum algorithms, there exist the Deutsch algorithm [3], the Deutsch-Jozsa algorithm [4], the Shor algorithm [5], the Simon algorithm [6], the quantum Fourier transform algorithm, and the Grover search algorithm [7]. In addition, proposals for implementing other quantum algorithms have been also presented (e.g., see [8]).

As is well known, the Deutsch algorithm and the Deutsch-Jozsa algorithm were the first two that make use of the features of quantum mechanics for quantum computation. Compared with other quantum algorithms, these two algorithms are easy to be implemented and thus have been considered as the natural candidates for demonstrating power of quantum computation.

In this article, we restrict ourselves to a refined Deutsch-Jozsa (DJ) algorithm. We will propose a way to implement a three-qubit refined DJ algorithm. As shown below, this proposal is based on the construction of the 35 \(f\)-controlled phase gates, which employs single-qubit \(\sigma_z\) gates and two-qubit standard controlled-phase (CP) gates. To the best of our knowledge, our work is the first to consider how to construct the 35 \(f\)-controlled phase gates using these two kinds of basic gates. The construction of the 35 \(f\)-controlled phase gates here is general, and can be applied to implement the three-qubit refined DJ algorithm in a cavity-based or noncavity-based physical system.

II. MOTIVATIONS

There are several motivations for this work:

(i) In Ref. [9], the authors showed that implementing a one-qubit or two-qubit refined DJ algorithm does not require entanglement between the input query qubits, and thus test of the one-qubit or two-qubit refined DJ algorithm is not meaningful because it can be solved in a classical way. However, they showed that entanglement exists between the input query qubits during performing a \(n\)-qubit refined DJ algorithm with \(n \geq 3\), and thus the meaningful test of the refined DJ algorithm occurs only and only if \(n \geq 3\). Hence, as far as the refined DJ algorithm is concerned, a three-qubit refined DJ algorithm (i.e., the case for \(n = 3\)) is the smallest one that needs to be tested or implemented, in order to demonstrate the full power of quantum computation as applied to the Deutsch problem. For a detailed discussion, see [9].

(ii) A second motivation of this work is as follows. A single-qubit \(\sigma_z\) gate can be easily realized by applying a single classical pulse. In addition, it is well known that a two-qubit CP gate (i.e., the key element in the construction of

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the 35 f-controlled phase gates, see Table II below) has been experimentally demonstrated in many physical systems such as quantum dots, trapped ions, atoms in cavity QED, and superconducting qubits/qutrits coupled to a single circuit cavity. Thus, it is straightforward to implement a three-qubit refined DJ algorithm by applying the present proposal.

(iii) Quantum information processing using superconducting qubits coupled to a cavity has attracted considerable interest during the past ten years. Based on cavity QED, many theoretical proposals have been presented for realizing two-qubit gates [10-14] and multiple qubit gates [15,16] with superconducting qubits. Moreover, experimental demonstration of two-qubit gates [17-20] and three-qubit gates [21-23] with superconducting qubits in cavity QED has been also reported. However, after a deep investigation, we noted that how to implement a three-qubit (original and refined) DJ algorithm with superconducting qubits or qutrits in cavity QED has not been reported in both theoretical and experimental aspects. As is known, the experimental realization of the three-qubit DJ algorithm with a cavity-superconducting-device system is important because it would be an important step toward more complex quantum computation in circuit cavity QED.

(iv) Over the past decade, there has been much interest in quantum information processing with atoms in cavity QED. Based on cavity QED technique, many theoretical methods have been proposed for implementing a two-qubit CP or controlled-NOT gate and multiple qubit controlled gates with atoms [24-28]. Moreover, a two-qubit CP gate between a cavity mode and an atom has been experimentally demonstrated [29]. However, after a thorough search, we found that only several proposals [30-32] were proposed for implementing the original or refined n-qubit DJ algorithm for n ≥ 3, using atoms in cavity QED. As discussed below, these proposals have some drawbacks. For instances, the one in [30] is incomplete, and the ones in [31,32] are difficult to implement in experiments when compared with our current proposal.

III. DEUTSCH-JOZSA ALGORITHM

The DJ algorithm is aimed at distinguishing the constant function from the balanced functions on 2^n inputs. The function f(x) takes either 0 or 1. For the constant function, the function values are constant (0 or 1) for all 2^n inputs. In contrast, for the balanced function, the function values are equal to 1 for half of all the possible inputs, and 0 for the other half. Using the DJ algorithm, whether the function is constant or balanced can be determined by only one query. However, a classical algorithm would require 2^{n-1} + 1 queries to answer the same problem, which grows exponentially with input size.

The original DJ algorithm involves the n input query qubits each initially prepared in the state \(|0\rangle\) and an auxiliary working qubit initially prepared in the state \(\langle - \rangle \rangle = ((0) - |1\rangle) / \sqrt{2}\). It operates by: (i) applying a Hadamard gate on each input query qubit, which results in \(|0\rangle \rightarrow ((0) + |1\rangle) / \sqrt{2}\) and \(|1\rangle \rightarrow ((0) - |1\rangle) / \sqrt{2}\), (ii) performing a f-controlled-NOT gate \(\hat{U}_f\), and (iii) applying a Hadamard gate again on each input query qubit followed by a measurement on each input query qubit along the single-qubit z base formed by the two states \{|0\rangle, |1\rangle\}. Here, \(\hat{U}_f\) is defined as \(\hat{U}_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle\), where \(|x\rangle\) is the state of the n input query qubits with x being an n-qubit binary decomposition, \(|y\rangle\) is the state of the working qubit with \(y \in \{0,1\}\), and \(\oplus\) means modulo 2. Note that the operation \(\hat{U}_f\) results in the following transformation [2]

\[
|x\rangle \langle - \rangle \rightarrow (-1)^{f(x)}|x\rangle \langle - \rangle ,
\]  

which shows that there is no entanglement between the query qubits and the working qubit in the output of the \(\hat{U}_f\). Thus there is no need for any coupling between the query qubits and the working qubit during the entire process. Hence, in the original DJ algorithm the working qubit is completely redundant. 

FIG. 1: Quantum circuit for the refined n-qubit Deutsch-Jozsa algorithm. Here, \(H\) represents a single-qubit Hadamard gate on an input query qubit, which results in \(|0\rangle \rightarrow ((0) + |1\rangle) / \sqrt{2}\) and \(|1\rangle \rightarrow ((0) - |1\rangle) / \sqrt{2}\). In addition, \(U_f\) indicates a f-controlled phase gate on the n input query qubits, described by Eq. (2).
| $U_1: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{10}: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ |
| $U_2: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{11}: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ |
| $U_3: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{12}: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ |
| $U_4: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{13}: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ |
| $U_5: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{14}: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ |
| $U_6: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{15}: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ |
| $U_7: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{16}: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ |
| $U_8: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{17}: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ |
| $U_9: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{18}: f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ |

TABLE I: List of the 35 balanced functions for a three-qubit refined Deutsch-Jozsa algorithm. Here, $x_1 = 000$, $x_2 = 001$, $x_3 = 010$, $x_4 = 011$, $x_5 = 100$, $x_6 = 101$, $x_7 = 110$, and $x_8 = 111$. For simplicity, we only list the function values, which are “0”, for four inputs corresponding to each balanced function. Note that for each balanced function, the function values for the other four inputs (not listed) take a value “1”. For instance, for the balanced function corresponding to $U_{f_1}$, the function values for the other four inputs (not listed) are $f(x_5) = f(x_6) = f(x_7) = f(x_8) = 1$.

The refined DJ algorithm was proposed by Collins et al. in 2001 [9], which is illustrated in Fig. 1. This refined DJ algorithm follows a similar pattern of operations as the original DJ algorithm, which is described below:

(i) Each input query qubit is prepared in the initial state $|0\rangle$.

(ii) Perform a Hadamard transformation $H$ on each qubit. As a result, the $n$-qubit initial state $|00\ldots0\rangle$ changes to the state $\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle$ (denoted as $|\psi_1\rangle$).

(iii) Apply the $f$-controlled phase shift $U_f$, described by

$$|x\rangle \xrightarrow{U_f} (-1)^{f(x)} |x\rangle,$$  \hspace{1cm} (2)

which leads the state $|\psi_1\rangle$ to the state $\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle$ (denoted as $|\psi_2\rangle$).

(iv) Perform another Hadamard transformation $H$ on each qubit. As a result, the state $|\psi_2\rangle$ becomes $\frac{1}{\sqrt{F}} \sum_{z=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{z+f(x)} |z\rangle$.

(v) Measure the final state of the $n$ qubits. If the $n$ qubits are measured in the state $|00\ldots0\rangle$, then $f(x)$ is constant. However, if they are measured in other $n$-qubit computational basis states, then $f(x)$ is balanced. This is because the amplitude $a_{00\ldots0}$ of the state $|00\ldots0\rangle$ is given by $a_{00\ldots0} = \frac{1}{\sqrt{F}} \sum_{x=0}^{2^n-1} (-1)^{f(x)}$, which is $\pm 1$ for a constant $f(x)$ while 0 for a balanced $f(x)$.

One can see that when compared with the original DJ algorithm, this refined DJ algorithm does not need a working qubit. Hence, it requires one qubit fewer than the original DJ algorithm. Consequently, its physical implementation requires one fewer two-state system.

IV. CONSTRUCTION OF THE $f$-CONTROLLED PHASE GATES

For both of the original and refined DJ algorithms, there are a total of $C_{2^n}^{2^n-1}$ balanced functions, among which only $C_{2^n}^{2^n-1}/2$ balanced functions are nontrivial if the symmetry is taken into account. For the three-qubit DJ algorithm,
TABLE II: The construction of 35 $f$-controlled phase gates for a three-qubit refined Deutsch-Jozsa algorithm. Here, $\sigma_z$ represents a single-qubit $\sigma_z$ gate on qubit $j$ ($j=1,2,3$); while $C_{jk}$ indicates a two-qubit standard controlled-phase gate on qubits $j$ and $k$ ($j,k=1,2,3$), described by Eq. (3).

As discussed previously, a single-qubit $\sigma_z$ gate here can be easily realized by applying a single classical pulse, and a two-qubit CP gate (i.e., the central element in our gate construction) has been experimentally demonstrated in various physical systems. Hence, the gate construction given in Table II is implementable.

V. COMPARING WITH PREVIOUS PROPOSALS USING ATOMS IN CAVITY QED

After a thorough search, we found that only several proposals [30-32] were proposed for implementing the original or refined DJ algorithm for $n \geq 3$, using atoms in cavity QED. In the following we will briefly introduce these previous proposals and then give a comparison of them with our current proposal.

(i) The authors in [30] proposed an incomplete scheme for realizing the $n$-qubit original DJ algorithm using atoms coupled to a cavity, because they only discussed how to implement one balanced function, which completes the state

\begin{equation}
\begin{aligned}
|00\rangle_{jk} &\rightarrow |00\rangle_{jk}, \\
|10\rangle_{jk} &\rightarrow |10\rangle_{jk}, \\
|01\rangle_{jk} &\rightarrow |01\rangle_{jk}, \\
|11\rangle_{jk} &\rightarrow |11\rangle_{jk},
\end{aligned}
\end{equation}

which implies that if and only if the control qubit $j$ (the first qubit) is in the state $|1\rangle$, a phase flip happens to the state $|1\rangle$ of the target qubit $k$ (the second qubit), but nothing happens otherwise.

The construction for each of the 35 $f$-controlled phase gates is listed in Table II. One can see from Table II that the 35 $f$-controlled phase gates $U_{f1}, U_{f2}, ..., U_{f35}$ are classified into the following four types: (i) Type 1 includes seven $f$-controlled phase gates each constructed with single-qubit $\sigma_z$ gates only; (ii) Type 2 contains twelve $f$-controlled phase gates each constructed with single-qubit $\sigma_z$ gates and one two-qubit CP gate; (iii) Type 3 has twelve $f$-controlled phase gates each constructed by using single-qubit $\sigma_z$ gates and two two-qubit CP gates; and (iv) Type 4 involves four $f$-controlled phase gates each implemented with two single-qubit $\sigma_z$ gates and three two-qubit CP gates.

As discussed previously, a single-qubit $\sigma_z$ gate here can be easily realized by applying a single classical pulse, and a two-qubit CP gate (i.e., the central element in our gate construction) has been experimentally demonstrated in various physical systems. Hence, the gate construction given in Table II is implementable.
transformation from the input state $2^{-n/2} (|0\rangle + |1\rangle)^\otimes n$ to the output state $2^{-n/2} (|0\rangle - |1\rangle)^\otimes n$, as shown in [30].

Furthermore, this state transformation can be reached in a classical way, since it can be alternatively achieved by simply performing a single-qubit $\sigma_z$ gate on each of individual qubits of the register.

(ii) In Ref. [31], a scheme was proposed for the implementation of the $n$-qubit original DJ algorithm via atoms interacting with a cavity. As discussed there, to implement each of $C^\otimes n - 1$ different $n$-qubit controlled-NOT gates, this scheme requires $2^n - 1$ different $n$-qubit controlled-NOT gates each with $n - 1$ qubits simultaneously controlling a target qubit. For a three-qubit DJ algorithm, $n = 3$. Thus, by applying this scheme to realize each balanced function for a three-qubit DJ algorithm, four different three-qubit controlled-NOT gates (each with 2 qubits simultaneously controlling a target qubit) would be required.

(iii) Ref. [32] presented a way for implementing a 3-qubit refined DJ algorithm with three atoms in cavity QED. This proposal is based on four three-qubit controlled phase gates (each with 2 qubits simultaneously controlling a target qubit). In addition, as discussed in [32], it is required to send three atoms through four cavities at the same time, or simultaneously sending the three atoms through a common cavity four times.

From the description given above, one can see that:

(i) The present proposal is different from the previous one in [30], since the latter is incomplete, which only considered realizing one balanced function that can be achieved in a classical way. In contrast to [30], we have presented a complete protocol for implementing all 35 $f$-controlled phase gates for a three-qubit refined DJ algorithm.

(ii) To implement a three-qubit DJ algorithm, the proposal in [31] requires using four different three-qubit gates, and the one in [32] requires sending three atoms through four cavities or needs the use of four cavities. In contrast, it can be seen from Table II that our proposal needs three two-qubit CP gates at most, which can be implemented using a single cavity or resonator only, as discussed below. Hence, when compared with the previous proposals in [31,32], it is simple and easy to implement a three-qubit refined DJ algorithm based on cavity QED, by using the present proposal.

We point out that it is not our intention to cast aspersions on existing approaches [30-32] to the DJ algorithm implementation; rather we simply wish to present a protocol to implement a three-qubit refined DJ algorithm.

As relevant to the cavity QED-based implementation of the present proposal, some points may need to be addressed here. First, the present proposal can be implemented with three superconducting qubits/qutrits placed in a single cavity or resonator. This is because during performing each two-qubit CP gate one can have the irrelevant qubit/qutrit to be decoupled from the cavity by adjusting its level spacings (e.g., see [15,16] and discussion therein). Second, it can be realized using three atoms plus a single cavity. During performing each two-qubit CP gate on atoms, the irrelevant atom can be made to be decoupled from the cavity, for instance, by moving it out of the cavity via translating optical lattices (e.g., see [16,28]).

VI. CONCLUSION

We have proposed a way to implement the three-qubit refined Deutsch-Jozsa quantum algorithm. The construction of the $f$-controlled phase gates can be applied to implement a three-qubit refined Deutsch-Jozsa algorithm in a cavity-based or noncavity-based physical system. This work is of interest because it provides a simple and general protocol for implementing a three-qubit refined Deutsch-Jozsa algorithm, which is an important step toward more complex quantum computation.

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