A gravity term from spontaneous symmetry breaking

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Abstract
In this model, the gravity term in the Lagrangean comes from spontaneous symmetry breaking of an additional scalar quadruplet field Υ. The resulting gravitational field is approximate to one of the models of coframe gravity with parameters \( \rho_1 + 4\rho_2 = 0, \rho_3 = 0 \). This article includes an exact solution of coframe gravity with model parameters \( \rho_1 \neq 0, \rho_2 \) any, \( \rho_3 = 0 \), which is Newtonian at infinity. An iteration process is given to construct a solution for a given matter-radiation stress-energy tensor.

1 Introduction
In this model, I use spontaneous symmetry breaking to add a gravity term to the Lagrangean. The resulting gravitational field is approximate to one of the models of coframe gravity with parameters \( \rho_1 + 4\rho_2 = 0, \rho_3 = 0 \).

The layout of the article is as follows: In section 2 I explain the notation I use. In section 3 I introduce the field which has a spontaneously broken symmetry. In section 4 I calculate the stress energy tensor of another field \( \phi \) to show that the sign of the stress-energy tensor is correct. In section 5 I calculate the gravity term added to the Lagrangean by spontaneous symmetry breaking. In section 6 an exact solution which is static and has spherical symmetry is calculated. In section 7 the field equation is linearized.

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2 Notation

I use a coframe 1-form $\theta : T(\mathcal{M}^4) \to \mathbb{R}^4$ and a dual vector field $\theta^i(v_k) = \delta_k^i$.

I use, $\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (1)

$\gamma^0 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\gamma^j = \begin{bmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{bmatrix}$ (2)

$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (3)

$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}$ (4)

$\Psi = \Psi^i \gamma^0$ (5)

$S^{ab} = \frac{1}{4} [\gamma^a, \gamma^b]$ (6)

$\epsilon^{0123} = \epsilon_{0123}$ (7)

Latin indices are Minkowskian and Greek indices are spacetime indices. Latin indices are raised or lowered using the Minkowski metric:

$A^a = \eta^{ab} A_b$ (8)

The coframe $\theta$ can be expressed using the coordinate 1-forms,

$\theta^a = \theta^a_{\mu} dx^\mu$ (9)

and the metric is,

$g_{\mu\nu} = \eta_{ab} \theta^a_{\mu} \theta^b_{\nu}$ (10)

(10) is 10 equations for 16 components, so for a given metric, $\theta$ has 6 degrees of freedom.

In differential geometry style (11), vectors are represented as derivatives, so $v_a \Psi$ is an equivalent notation to $d\Psi(v_a)$.

In this article, the Hodge dual is defined as,

$\star (\theta_i \wedge \ldots \wedge \theta_{i_{k-1}}) = \frac{1}{(n-k)!} \epsilon_{i_{0} \ldots i_{k-1} j_{0} \ldots j_{n-k-1}} \theta^{j_0} \wedge \ldots \wedge \theta^{j_{n-k-1}}$ (11)
This means that,

*1 = θ₀ ∧ θ₁ ∧ θ² ∧ θ³ (12)

**1 = *(θ₀ ∧ θ₁ ∧ θ² ∧ θ³) (13)

= *(η₀₀ η₁¹ η²² η³³ θ₀ ∧ θ₁ ∧ θ₂ ∧ θ₃) (14)

= − *(θ₀ ∧ θ₁ ∧ θ₂ ∧ θ₃) (15)

= −1 (16)

### 3 The Υ field

In addition to the Dirac field Ψ and the Glashow–Weinberg–Salam electroweak field φ, I add an additional field Υ (Greek letter Upsilon). Υ is a scalar complex quadruplet field. SL(2, ℂ) is a double cover of the group of Lorentz transformations SO(3, 1). Under a local SL(2, ℂ) gauge transform, the Dirac field Ψ and the field Υ transform as,

Ψ₂ = exp(\(\frac{k_{ab}}{2} S^{ab}\))Ψ₁ (17)

Υ₂ = exp(\(\frac{ck_{ab}}{2} S^{ab}\))Υ₁ (18)

with c a constant. \(k_{ab}\) can take different values at different events in space-time.

The Lagrangean is,

\[ L = i \frac{1}{2} (DΥ) \gamma^0 \wedge *DΥ − i c_1 ^2 *ΥΥ *1 + c_2 ^2 (ΥΥ)^2 *1 \] (19)

where D is the gauge-covariant exterior derivative:

\[ D = d − \frac{c}{2} S^{bc} ω_{bc} \] (20)

The constants \(c_1\) and \(c_2\) are used for spontaneous symmetry breaking. The Lagrangean can be written as,

\[ L = (K − V) *1 \] (21)

where \(K\) are the kinetic terms (terms containing a derivative) and \(V\) are the potential terms (terms without a derivative). If Υ is of the form,

\[ Υ = \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix} \] (22)
with $\xi$ representing two complex scalars, then the potential is,

$$V = -c_1^2|\xi|^2 + c_2^2|\xi|^4$$  \hfill (23)

The derivative of this potential is,

$$\frac{\partial V}{\partial |\xi|} = -2c_1^2|\xi| + 4c_2^2|\xi|^3$$  \hfill (24)

Solving $V' = 0$:

$$-2c_1^2|\xi| + 4c_2^2|\xi|^3 = 0$$  \hfill (25)

$$-2c_1^2 + 4c_2^2|\xi|^2 = 0$$  \hfill (26)

$$4c_2^2|\xi|^2 = 2c_1^2$$  \hfill (27)

$$|\xi| = \left|\frac{c_1}{\sqrt{2}c_2}\right|$$  \hfill (28)

This value of $|\xi|$ is a minimum of the potential $V$. It is the vacuum expectation value of $\Upsilon$, noted $\Upsilon_0$.

To calculate the field equation from this Lagrangean, I proceed in the usual manner, setting up a region of spacetime and a small variation of the field $\delta\Upsilon$ which vanishes on the region boundary. The variation of the Lagrangean is,

$$\delta\mathcal{L} = \delta\left( \frac{i}{2}(d\Upsilon - \frac{c}{2}S_{bc}\omega_{bc}\Upsilon)\gamma^0 \wedge *(d\Upsilon - \frac{c}{2}S_{bc}\omega_{bc}\Upsilon) - ic_1^2\Upsilon\gamma \right)$$

$$2 \delta\mathcal{L} = \Re\left( \delta(A^\dagger \cdot A) \right) = 2\Re(\delta A^\dagger \cdot A)$$  \hfill (32)

The variation of the Lagrangean becomes,

$$\delta\mathcal{L} = \Re\left( i\delta(d\Upsilon - \frac{c}{2}S_{bc}\omega_{bc}\Upsilon)\gamma^0 \wedge *(d\Upsilon - \frac{c}{2}S_{bc}\omega_{bc}\Upsilon) \right)$$

$$- c_1^2\delta(i\overline{\Upsilon}\Upsilon) \gamma \cdot 1 + 2c_2^2(-i)(i\overline{\Upsilon}\Upsilon)\delta(i\overline{\Upsilon}\Upsilon) \gamma \cdot 1$$  \hfill (33)
For the second and third terms,

\[ \delta (i\overline{\gamma} \gamma) = i\delta \gamma^\dagger \gamma^0 \gamma + i\gamma^\dagger \gamma^0 \delta \gamma \] (34)

\[ (i\gamma^\dagger \gamma^0 \delta \gamma)^\dagger = -i\delta \gamma^\dagger (-\gamma^0) \gamma \] (36)

\[ = i\delta \gamma^\dagger \gamma^0 \gamma \] (37)

So the second term of (34) is the Hermitean conjugate of the first:

\[ \delta (i\overline{\gamma} \gamma) = 2\text{Re}(i\delta \gamma^\dagger \gamma^0 \gamma) \] (38)

The variation of the Lagrangean becomes,

\[ \delta \mathcal{L} = \text{Re}(i\delta (d\gamma - \frac{c}{2} S^{bc}_{\omega_{bc}}) \gamma^\dagger \gamma^0 \wedge \ast (d\gamma - \frac{c}{2} S^{bc}_{\omega_{bc}} \gamma)) \]

\[ - 2ic^2 \delta \gamma^\dagger \gamma^0 \gamma^* \gamma^0 + 4c^2 (\overline{\gamma} \gamma) \delta \gamma^\dagger \gamma^0 \gamma^* \gamma^0 \] (39)

\[ = \text{Re}(i(d\gamma - \frac{c}{2} S^{bc}_{\omega_{bc}} \gamma) \gamma^\dagger \gamma^0 \wedge \ast (d\gamma - \frac{c}{2} S^{bc}_{\omega_{bc}} \gamma)) \]

\[ - 2ic^2 \delta \gamma^\dagger \gamma^0 \gamma^* \gamma^0 + 4c^2 (\overline{\gamma} \gamma) \delta \gamma^\dagger \gamma^0 \gamma^* \gamma^0 \] (40)

\[ = \text{Re}(i(d\gamma^\dagger - \frac{c}{2} \delta \gamma^\dagger S^{bc}_{\omega_{bc}} \gamma^0 \wedge \ast (d\gamma - \frac{c}{2} S^{bc}_{\omega_{bc}} \gamma)) \]

\[ - 2ic^2 \delta \gamma^\dagger \gamma^0 \gamma^* \gamma^0 + 4c^2 (\overline{\gamma} \gamma) \delta \gamma^\dagger \gamma^0 \gamma^* \gamma^0 \] (41)

\[ = \text{Re}(id\delta \gamma^\dagger \gamma^0 \wedge \ast (d\gamma - \frac{c}{2} S^{bc}_{\omega_{bc}} \gamma)) \]

\[ - i\frac{c}{2} \delta \gamma^\dagger S^{bc}_{\omega_{bc}} \gamma^0 \wedge \ast (d\gamma - \frac{c}{2} S^{bc}_{\omega_{bc}} \gamma) \]

\[ - 2ic^2 \delta \gamma^\dagger \gamma^0 \gamma^* \gamma^0 + 4c^2 (\overline{\gamma} \gamma) \delta \gamma^\dagger \gamma^0 \gamma^* \gamma^0 \] (42)

I have,

\[ S^{bc}_{\gamma^0} = \frac{1}{4} \{ \gamma^b, \gamma^c \}^\dagger \gamma^0 \] (43)

\[ = \frac{1}{4} [\gamma^c, \gamma^b] \gamma^0 \] (44)

\[ = \frac{1}{4} \gamma^0 [\gamma^c, \gamma^b] \] (45)

\[ = -\gamma^0 S^{bc} \] (46)
So the variation of the Lagrangean is,

\[
\delta L = \text{Re}(i\delta \Upsilon^\dagger \gamma^0 \wedge *(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon)) \\
+ i\frac{c}{2} \delta \Upsilon^\dagger \gamma^0 S^{bc}_{\omega bc} \wedge *(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon) \\
- 2ic_1^2 \delta \Upsilon^\dagger \gamma^0 \Upsilon \star 1 + 4c_2^2 (\Upsilon \Upsilon) \delta \Upsilon^\dagger \gamma^0 \Upsilon \star 1)
\]

(47)

I have,

\[
d(\delta \Upsilon^\dagger \gamma^0 \wedge *(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon)) \\
= d\delta \Upsilon^\dagger \gamma^0 \wedge *(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon) + \delta \Upsilon^\dagger \gamma^0 d*(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon)
\]

(48)

so,

\[
d\delta \Upsilon^\dagger \gamma^0 \wedge *(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon) \\
= -\delta \Upsilon^\dagger \gamma^0 d*(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon) + d(\delta \Upsilon^\dagger \gamma^0 \wedge *(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon))
\]

(49)

Substituting this into the variation of the Lagrangean:

\[
\delta L = \text{Re}(-i\delta \Upsilon^\dagger \gamma^0 d*(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon)) \\
+ i\frac{c}{2} \delta \Upsilon^\dagger \gamma^0 S^{bc}_{\omega bc} \wedge *(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon) \\
- 2ic_1^2 \delta \Upsilon^\dagger \gamma^0 \Upsilon \star 1 + 4c_2^2 (\Upsilon \Upsilon) \delta \Upsilon^\dagger \gamma^0 \Upsilon \star 1) + d(\ldots)
\]

(50)

\[
= \text{Re}(i\delta \Upsilon^\dagger \gamma^0 ( - d*(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon)) \\
+ \frac{c}{2} S^{bc}_{\omega bc} \wedge *(d\Upsilon - \frac{c}{2} S^{bc}_{\omega bc} \Upsilon)) \\
- 2ic_1^2 \delta \Upsilon^\dagger \gamma^0 \Upsilon \star 1 + 4c_2^2 (\Upsilon \Upsilon) \delta \Upsilon^\dagger \gamma^0 \Upsilon \star 1) + d(\ldots)
\]

(51)

\[
= \text{Re}(i\delta \Upsilon^\dagger \gamma^0 (-D*\Upsilon - 2c_1^2 \Upsilon \star 1 - 4ic_2^2 (\Upsilon \Upsilon) \Upsilon \star 1)) + d(\ldots)
\]

(52)

The variation of the Lagrangean has to vanish up to a 4-divergence for any small variation of the field \(\delta \Upsilon^\dagger\), so the field equation is,

\[
-D*\Upsilon - 2c_1^2 \Upsilon \star 1 - 4ic_2^2 (\Upsilon \Upsilon) \Upsilon \star 1 = 0
\]

(53)

\[
- *D*\Upsilon + 2c_1^2 \Upsilon + 4ic_2^2 (\Upsilon \Upsilon) \Upsilon = 0
\]

(54)
If the metric is Minkowskian and the coframe are the coordinate differentials, then,

\[ \ast D \ast D \Upsilon = \ast d \ast d \Upsilon \]  
\[ = - \Box \Upsilon \]  

(55)  
(56)

To show that the minus sign is correct, I assume that \( \Upsilon \) depends only on time. Then,

\[ \ast d \ast d \Upsilon = \ast d (\partial_0 \Upsilon) dx^0 \]  
\[ = - \ast d (\partial_0 \Upsilon) dx_0 \]  
\[ = - \ast d (\partial_0 \Upsilon) dx^1 \wedge dx^2 \wedge dx^3 \]  
\[ = - \ast (\partial_0 \partial_0 \Upsilon) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \]  
\[ = \partial_0 \partial_0 \Upsilon \]  
\[ = - \Box \Upsilon \]  

(57)  
(58)  
(59)  
(60)  
(61)  
(62)  
(63)

And the field equation is,

\[ \Box \Upsilon + 2 c_1^2 \Upsilon + 4 i c_2^2 (\Upsilon \Upsilon) \Upsilon = 0 \]  

(64)

If \( \Upsilon \) is of the form,

\[ \Upsilon = \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix} \]  

(65)

then the field equation has a solution with non-zero constant \( \Upsilon \):

\[ 4 i c_2^2 (\Upsilon \Upsilon) = -2 c_1^2 \]  
\[ -4 c_2^2 |\xi|^2 = -2 c_1^2 \]  
\[ |\xi| = \left| \frac{c_1}{\sqrt{2} c_2} \right| \]  

(66)  
(67)  
(68)

It is a minimum of the potential.

If the metric is Minkowskian and the coframe are the coordinate differentials and \( \Upsilon \) is of the form,

\[ \Upsilon = \begin{bmatrix} u(t) \\ 0 \\ 0 \end{bmatrix} \]  

(69)
with \( u(t) \) representing two complex scalars, then the kinetic term of the Lagrangean is,

\[
K * 1 = \frac{i}{2} dY^\dagger \gamma^0 \wedge * dY
\]

\[
= \frac{i}{2} \partial_0 Y^\dagger dx^0 \gamma^0 \wedge * \partial_0 Y dx^0
\]

\[
= \frac{i}{2} \partial_0 Y^\dagger dx^0 \gamma^0 \wedge - * \partial_0 Y dx_0
\]

\[
= \frac{i}{2} \partial_0 Y^\dagger dx^0 \gamma^0 \wedge - \partial_0 Y dx^1 \wedge dx^2 \wedge dx^3
\]

\[
= - \frac{i}{2} \partial_0 Y^\dagger \gamma^0 \partial_0 Y * 1
\]

Using that \( Y \) is an eigenvector of \( \gamma^0 \) with eigenvalue \( i \),

\[
K * 1 = \frac{1}{2} \partial_0 u^\dagger \partial_0 u * 1
\]

\[
= \frac{1}{2} |\partial_0 u|^2 * 1
\]

The cost to the kinetic term of the Lagrangean of random big changes with respect to time of the wave function is positive, therefore the system shall gravitate toward a minimum of the potential \( V \).

### 4 The stress-energy tensor

I shall not calculate the stress-energy tensor of the \( \Upsilon \) field in this article. Instead, I shall calculate the stress-energy tensor of a scalar complex doublet field \( \phi \) with spontaneous symmetry breaking. This is to show that the stress-energy tensor has the correct sign.

The Lagrangean of \( \phi \) is,

\[
\mathcal{L} = -\frac{1}{2} d\phi^\dagger \wedge * d\phi + c_1^2 |\phi|^2 * 1 - c_2^2 |\phi|^4 * 1
\]

The variation of the first term of the Lagrangean is,

\[
\delta \left( -\frac{1}{2} (v_\alpha \phi^\dagger) (v^\alpha \phi) * 1 \right) = -\frac{1}{2} \left( \delta ((v_\alpha \phi^\dagger) * 1) (v^\alpha \phi) + (v_\alpha \phi^\dagger) \delta ((v^\alpha \phi) * 1) \right)
\]

\[
- (v_\alpha \phi^\dagger) (v^\alpha \phi) \delta * 1
\]
The variation of the second part is,

\[
\delta(c_1^2 \phi^\dagger \phi * 1) = c_1^2 \phi^\dagger \phi \delta * 1
\]

\[
= c_1^2 \phi^\dagger \phi \delta \theta^m \wedge * \theta_m
\]

\[
= c_1^2 \phi^\dagger \phi \delta \theta_m \wedge * \theta^m
\]

\[
= c_1^2 \phi^\dagger \phi \delta \theta_m \wedge * \eta^{mn} \wedge \theta_n
\]

(87)

(88)

(89)

(90)

The variation of the third part is,

\[
\delta(-c_2^2 |\phi|^4 * 1) = -c_2^2 |\phi|^4 \delta * 1
\]

\[
= -c_2^2 |\phi|^4 \delta \theta^m \wedge * \theta_m
\]

\[
= -c_2^2 |\phi|^4 \delta \theta_m \wedge * \theta^m
\]

\[
= -c_2^2 |\phi|^4 \delta \theta_m \wedge * \eta^{mn} \wedge \theta_n
\]

(91)

(92)

(93)

(94)

The variation of the Lagrangean can be expressed as,

\[
\delta \mathcal{L} = \delta \theta_m \wedge T^m
\]

(95)
with $T$ the stress-energy tensor. The latter can also be expressed as,

$$T^m = T^{mn} * \theta_n$$  \hspace{1cm} (96)$$

So $T$ is,

$$T^{mn} = \frac{1}{6} (v^m \phi^\dagger) v^n \phi + \frac{1}{6} (v^n \phi^\dagger) v^m \phi + \frac{1}{6} \eta^{mn} (v^a \phi^\dagger) v_a \phi + c_1^2 |\phi|^2 \eta^{mn} - c_2^2 |\phi|^4 \eta^{mn}$$  \hspace{1cm} (97)$$

5 A gravity term

In this theory, the gravity term comes from spontaneous symmetry breaking of the $\Upsilon$ field. The vacuum expectation of $\Upsilon$ is,

$$\Upsilon_0 = \begin{bmatrix} \xi_0 \\ 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (98)$$

$$|\xi| = \frac{c_1}{\sqrt{2c_2}}$$  \hspace{1cm} (99)$$

where $\xi$ represents two complex components.

The gravity term comes from evaluating the Lagrangean at the vacuum expectation value of $\Upsilon$:

$$L|_{\Upsilon=\Upsilon_0} = i \left( -\frac{c}{2} S^{bc} \omega_{bc} \gamma_0^\dagger \wedge - * \frac{c}{2} S^{de} \omega_{de} \gamma_0 + \frac{c^4}{4c_2^2} * 1 \right)$$  \hspace{1cm} (100)$$

The last term is a contribution to the cosmological constant from spontaneous symmetry breaking. I shall neglect it for now because the situation is similar for all theories with spontaneous symmetry breaking. The first term on the right of (100) is,

$$-\frac{i c}{2} \Upsilon_0^{\dagger} S^{bc} \omega_{bc} \gamma_0^\dagger \wedge - * \frac{c}{2} S^{de} \omega_{de} \gamma_0$$

$$= -\frac{i c}{2} \Upsilon_0^{\dagger} (-\gamma_0^0 S^{bc}) \omega_{bc} \wedge - * \frac{c}{2} S^{de} \omega_{de} \gamma_0$$  \hspace{1cm} (101)$$

$$= \frac{c^2}{8} \Upsilon_0^{\dagger} S^{bc} \omega_{bc} \wedge * S^{de} \omega_{de} \gamma_0$$  \hspace{1cm} (102)$$

$$= \frac{c^2}{8} \Upsilon_0^{\dagger} S^{bc} S^{de} \gamma_0 \omega_{bc} \wedge * \omega_{de}$$  \hspace{1cm} (103)$$
\[= \frac{c^2}{32} (|\xi|^2 \eta^{bd} \eta^{ce}) \omega_{bc} \wedge \star \omega_{de} \quad (104)\]

\[= - \frac{c^2}{32} |\xi|^2 \eta^{bd} \eta^{ce} \omega_{bc} \wedge \star \omega_{de} \quad (105)\]

\[= - \frac{c^2}{32} |\xi|^2 \omega_{bc} \wedge \star \omega_{bc} \quad (106)\]

\[= - \frac{c^2}{32} |\xi|^2 \omega_{bc}(v_a) \omega^{bc}(v^a) * 1 \quad (107)\]

\[= - \frac{c^2}{32} |\xi|^2 \left( d\theta_a(v_b, v_c) + d\theta_b(v_a, v_c) - d\theta_c(v_a, v_b) \right) \times \left( d\theta^a(v^b, v^c) + d\theta^b(v^a, v^c) - d\theta^c(v^a, v^b) \right) * 1 \quad (108)\]

\[= - \frac{c^2}{32} |\xi|^2 \left( 3d\theta_a(v_b, v_c) d\theta^a(v^b, v^c) + d\theta_a(v_b, v_c) \left( d\theta^b(v^a, v^c) - d\theta^c(v^a, v^b) \right) + d\theta_b(v_a, v_c) \left( d\theta^a(v^b, v^c) - d\theta^c(v^a, v^b) \right) - d\theta_c(v_a, v_b) \left( d\theta^a(v^b, v^c) + d\theta^b(v^a, v^c) \right) \right) * 1 \quad (109)\]

For the term:

\[-d\theta_a(v_b, v_c)d\theta^c(v^a, v^b) = -d\theta_a(v_c, v_b)d\theta^b(v^a, v^c) \quad (111)\]

\[= d\theta_a(v_b, v_c)d\theta^b(v^a, v^c) \quad (112)\]

For the term:

\[d\theta_b(v_a, v_c)d\theta^a(v^b, v^c) = d\theta^a(v^b, v^c)d\theta_b(v_a, v_c) \quad (113)\]

\[= d\theta_a(v_b, v_c)d\theta^b(v^a, v^c) \quad (114)\]

For the term:

\[-d\theta_b(v_a, v_c)d\theta^c(v^a, v^b) = d\theta_b(v_a, v_c)d\theta^c(v^a, v^b) \quad (115)\]

\[= -d\theta_b(v_c, v_a)d\theta^c(v^b, v^a) \quad (116)\]

\[= -d\theta_a(v_b, v_c)d\theta^b(v^a, v^c) \quad (117)\]

For the term:

\[-d\theta_c(v_a, v_b)d\theta^a(v^b, v^c) = -d\theta^a(v^b, v^c)d\theta_c(v_a, v_b) \quad (118)\]

\[= -d\theta^a(v^c, v^b)d\theta_b(v_a, v_c) \quad (119)\]

\[= d\theta^a(v^b, v^c)d\theta_b(v_a, v_c) \quad (120)\]

\[= d\theta_a(v_b, v_c)d\theta^b(v^a, v^c) \quad (121)\]
For the term:

\[-d\theta_c(v_a, v_b)d\theta^b(v^a, v^c) = -d\theta_a(v_c, v_b)d\theta^b(v^c, v^a) = -d\theta_a(v_b, v_c)d\theta^b(v^a, v^c)\] (122)

The addition to the Lagrangean from spontaneous symmetry breaking is,

\[\Delta L = -\frac{c^2|\xi|^2}{32} \left(3d\theta_a(v_b, v_c)d\theta^a(v^b, v^c) + 2d\theta_a(v_b, v_c)d\theta^b(v^a, v^c)\right) * 1\] (124)

The term,

\[d\theta_a \wedge \theta^a \wedge *(d\theta_b \wedge \theta^b)\] (125)

is equal to,

\[d\theta_a \wedge \theta^a \wedge *(d\theta_b \wedge \theta^b) = d\theta_a \wedge \theta^a \wedge *(d\theta^b \wedge \theta_b) = d\theta_a(v_c, v_d)\frac{1}{2}\theta^c \wedge \theta^d \wedge \theta^a \wedge *(d\theta^b(v^f, v^g)\frac{1}{2}\theta_f \wedge \theta_g \wedge \theta_b) = d\theta_a(v_c, v_d)\frac{1}{2}\theta^c \wedge \theta^d \wedge \theta^a \wedge d\theta^b(v^f, v^g)\frac{1}{2}\epsilon_{fghb}\theta^h = \frac{1}{4}d\theta_a(v_c, v_d)\epsilon^{cdab}d\theta^b(v^f, v^g)\epsilon_{fghb} * 1 = \left(\frac{1}{2}d\theta_a(v_c, v_d)d\theta^a(v^c, v^d) - d\theta_a(v_c, v_d)d\theta^c(v^a, v^d)\right) * 1\] (130)

The term,

\[d\theta_a \wedge *d\theta_a\] (131)

is equal to,

\[\frac{1}{2}d\theta_a(v_b, v_c)d\theta^a(v^b, v^c) * 1\] (132)

So the addition to the Lagrangean from spontaneous symmetry breaking is,

\[\Delta L = -\frac{c^2|\xi|^2}{32} \left(8d\theta_a \wedge *d\theta^a - 2d\theta_a \wedge \theta^a \wedge *(d\theta_b \wedge \theta^b)\right)\] (133)

The coframe gravity Lagrangean is,

\[L = \frac{1}{2} \sum_{i=1}^{3} \rho_i \langle i \rangle L\] (134)
with,

\begin{align}
(1) L &= \text{d}\theta_a \wedge *\text{d}\theta^a \\
(2) L &= \text{d}\theta_a \wedge \theta^a \wedge *(\text{d}\theta_b \wedge \theta^b) \\
(3) L &= \text{d}\theta_a \wedge \theta^b \wedge *(\text{d}\theta_b \wedge \theta^a)
\end{align}

This model is approximately a coframe gravity model with \( \rho_1 = -\frac{1}{2}c^2|\xi|^2 \), \( \rho_2 = \frac{1}{8}c^2|\xi|^2 \), \( \rho_3 = 0 \), except that it is not exactly a coframe gravity model because there is an extra field \( \Upsilon \).

To find the field equation for gravity in this model, I proceed in the usual manner, with a small variation of the fields \( \delta \theta \) which vanishes on the region boundary. The variation of the Lagrangean is,

\[ \delta^\text{(1)}L = \text{d}\delta\theta_a \wedge *\text{d}\theta^a + \text{d}\theta_a \wedge *\text{d}\delta\theta^a \]

If \( A \) and \( B \) are 2-forms, then

\begin{align}
A \wedge *B &= A \wedge *B(v^a, v^b)\frac{1}{2}\theta_a \wedge \theta_b
\end{align}

\begin{align}
&= A \wedge B(v^a, v^b)\frac{1}{4}\epsilon_{abcd}\theta^c \wedge \theta^d
\end{align}

\begin{align}
&= A(v_e, v_f)\frac{1}{2}\theta^e \wedge \theta^f \wedge B(v^a, v^b)\frac{1}{4}\epsilon_{abcd}\theta^c \wedge \theta^d
\end{align}

\begin{align}
&= \frac{1}{8}A(v_e, v_f)B(v^a, v^b)\epsilon_{abcd}\epsilon^{efcd} * 1
\end{align}

\begin{align}
&= \frac{1}{2}A(v_a, v_b)B(v^a, v^b) * 1
\end{align}

\begin{align}
&= \frac{1}{2}A(v_a, v_b)\eta^{ac}\eta^{bd}B(v_e, v_d) * 1
\end{align}

\begin{align}
&= B \wedge *
\end{align}

The variation of the Lagrangean is of the form,

\[ \delta^\text{(1)}L = \delta A \wedge *A + A \wedge \delta A \]

however,

\[ A \wedge *\delta A = \delta A \wedge *A \]

so the variation of the Lagrangean has the form,

\[ \delta^\text{(1)}L = 2\delta A \wedge *A \]
The variation of the Lagrangean becomes,

\[ \delta^{(1)}L = 2(\delta(d\theta^a)) \land \ast d\theta^a \]  
\[ = 2d\delta\theta^a \land \ast d\theta^a \]  
\[ d(2\delta\theta^a \land \ast d\theta^a) = 2d\delta\theta^a \land \ast d\theta^a - 2\delta\theta^a \land d\ast d\theta^a \]  
so,

\[ 2d\delta\theta^a \land \ast d\theta^a = 2\delta\theta^a \land d\ast d\theta^a + d(2\delta\theta^a \land \ast d\theta^a) \]  

Substituting this, the variation of the Lagrangean becomes,

\[ \delta^{(1)}L = 2\delta\theta^a \land d\ast d\theta^a + d(2\delta\theta^a \land \ast d\theta^a) \]  

For the second part of the Lagrangean, the variation is,

\[ \delta^{(2)}L = 2\left(\delta(d\theta^a \land \theta^a)\right) \land \ast(d\theta^b \land \theta^b) \]  
\[ = 2\left(d\delta\theta^a \land \theta^a + d\theta^a \land \delta\theta^a\right) \land \ast(d\theta^b \land \theta^b) \]  

I have,

\[ d\left(\delta\theta^a \land \theta^a \land \ast(d\theta^b \land \theta^b)\right) = d\delta\theta^a \land \theta^a \land \ast(d\theta^b \land \theta^b) - \delta\theta^a \land d\theta^a \land \ast(d\theta^b \land \theta^b) \]  
\[ + \delta\theta^a \land \theta^a \land d\ast(d\theta^b \land \theta^b) \]  
so,

\[ d\delta\theta^a \land \theta^a \land \ast(d\theta^b \land \theta^b) = \delta\theta^a \land d\theta^a \land \ast(d\theta^b \land \theta^b) - \delta\theta^a \land \theta^a \land d\ast(d\theta^b \land \theta^b) \]  
\[ + d(\delta\theta^a \land \theta^a \land \ast(d\theta^b \land \theta^b)) \]  

Substituting this, the variation of the Lagrangean becomes,

\[ \delta^{(2)}L = 2\delta\theta^a \land d\theta^a \land \ast(d\theta^b \land \theta^b) - 2\delta\theta^a \land \theta^a \land d\ast(d\theta^b \land \theta^b) \]  
\[ + 2d\delta\theta^a \land \delta\theta^a \land \ast(d\theta^b \land \theta^b) + d(2\delta\theta^a \land \theta^a \land \ast(d\theta^b \land \theta^b)) \]  
\[ = \delta\theta^a \land (4d\theta^a \land \ast(d\theta^b \land \theta^b) - 2\theta^a \land d\ast(d\theta^b \land \theta^b)) + d(\ldots) \]  

If \( \eta \) is any p-form, then,

\[ \ast v^a J\eta = (-1)^{p-1}\theta^a \land \ast\eta \]  

14
with $\mathbf{J}$ being the inner product between a vector and a p-form. If $X_1, X_2, \ldots, X_{p-1}$ are vectors, then,

$$(v^a \mathbf{J}\eta)(X_1, X_2, \ldots, X_{p-1}) = \eta(v^a, X_1, X_2, \ldots, X_{p-1})$$  \hspace{1cm} (161)

I have,

$$*v^a \mathbf{J}(d\theta^b \wedge \theta^b) = \theta^a \wedge *(d\theta^b \wedge \theta^b)$$  \hspace{1cm} (162)

$$d*v^a \mathbf{J}(d\theta^b \wedge \theta^b) = d \left( \theta^a \wedge *(d\theta^b \wedge \theta^b) \right)$$  \hspace{1cm} (163)

$$= d\theta^a \wedge *(d\theta^b \wedge \theta^b) - \theta^a \wedge d*(d\theta^b \wedge \theta^b)$$  \hspace{1cm} (164)

so,

$$\theta^a \wedge d*(d\theta^b \wedge \theta^b) = -d*v^a \mathbf{J}(d\theta^b \wedge \theta^b) + d\theta^a \wedge *(d\theta^b \wedge \theta^b)$$  \hspace{1cm} (165)

Substituting, I get,

$$\delta^{(2)}L = \delta\theta^a \wedge \left( 2d*v^a \mathbf{J}(d\theta^b \wedge \theta^b) + 2d\theta^a \wedge *(d\theta^b \wedge \theta^b) \right) + d(\ldots)$$  \hspace{1cm} (166)

For the third part of the Lagrangean, the variation is,

$$\delta^{(3)}L = \left( \delta(d\theta_a \wedge \theta^b) \right) \wedge *(d\theta^b \wedge \theta^a) + d\theta_a \wedge \theta^b \wedge *(\delta(d\theta^b \wedge \theta^a))$$  \hspace{1cm} (167)

$$= \left( \delta(d\theta_a \wedge \theta^b) \right) \wedge *(d\theta^b \wedge \theta^a) + \left( \delta(d\theta^b \wedge \theta^a) \right) \wedge *(d\theta^b \wedge \theta^a)$$  \hspace{1cm} (168)

$$= \left( \delta(d\theta_a \wedge \theta^b) \right) \wedge *(d\theta^b \wedge \theta^a) + \left( \delta(d\theta^b \wedge \theta^a) \right) \wedge *(d\theta^b \wedge \theta^a)$$  \hspace{1cm} (169)

$$= 2 \left( \delta(d\theta_a \wedge \theta^b) \right) \wedge *(d\theta^b \wedge \theta^a)$$  \hspace{1cm} (170)

$$= 2 \left( d\delta\theta_a \wedge \theta^b + d\theta_a \wedge d\theta^b \right) \wedge *(d\theta^b \wedge \theta^a)$$  \hspace{1cm} (171)

I have,

$$d \left( \delta\theta_a \wedge \theta^b \wedge *(d\theta^b \wedge \theta^a) \right) = d\delta\theta_a \wedge \theta^b \wedge *(d\theta^b \wedge \theta^a) - \delta\theta_a \wedge d\theta^b \wedge *(d\theta^b \wedge \theta^a)$$  \hspace{1cm} (172)

$$+ \delta\theta_a \wedge \theta^b \wedge d*(d\theta^b \wedge \theta^a)$$

so,

$$d\delta\theta_a \wedge \theta^b \wedge *(d\theta^b \wedge \theta^a) = \delta\theta_a \wedge \theta^b \wedge *(d\theta^b \wedge \theta^a) - \delta\theta_a \wedge \theta^b \wedge d*(d\theta^b \wedge \theta^a)$$  \hspace{1cm} (173)
Substituting this, the variation of the third part of the Lagrangean becomes,

\[ \delta^3 L = 2 \delta \theta_a \wedge d \theta^b \wedge \ast (d \theta_b \wedge \theta^a) = 2 \delta \theta_a \wedge \theta^b \wedge d \ast (d \theta_b \wedge \theta^a) \]

\[ + 2 d \theta_a \wedge \delta \theta^b \wedge \ast (d \theta_b \wedge \theta^a) + d (2 \delta \theta_a \wedge \theta^b \wedge \ast (d \theta_b \wedge \theta^a)) \]

\[ = \delta \theta_a \wedge 2 \delta \theta^b \wedge \ast (d \theta_b \wedge \theta^a) + \delta \theta^b \wedge 2 d \theta_a \wedge \ast (d \theta_b \wedge \theta^a) \]

\[ - \delta \theta_a \wedge 2 \theta^b \wedge d \ast (d \theta_b \wedge \theta^a) + d (\ldots) \]

\[ = \delta \theta_a \wedge 2 \delta \theta^b \wedge \ast (d \theta_b \wedge \theta^a) \]

\[ + 2 d \theta^b \wedge \ast (d \theta^a \wedge \theta_b) - 2 \theta^b \wedge d \ast (d \theta_b \wedge \theta^a) \] + d (\ldots) \hspace{1cm} (177)

I have,

\[ \ast v^b \hookrightarrow (d \theta_b \wedge \theta^a) = \theta^b \wedge \ast (d \theta_b \wedge \theta^a) \]

\[ d \ast v^b \hookrightarrow (d \theta_b \wedge \theta^a) = d \left( \theta^b \wedge \ast (d \theta_b \wedge \theta^a) \right) \]

\[ = d \theta^b \wedge \ast (d \theta_b \wedge \theta^a) \]

\[ - \theta^b \wedge d \ast (d \theta_b \wedge \theta^a) \]

so,

\[ \theta^b \wedge d \ast (d \theta_b \wedge \theta^a) = -d \ast v^b \hookrightarrow (d \theta_b \wedge \theta^a) + d \theta^b \wedge \ast (d \theta_b \wedge \theta^a) \hspace{1cm} (182) \]

Substituting, I get,

\[ \delta^3 L = \delta \theta_a \wedge \left( 2 d \ast v^b \hookrightarrow (d \theta_b \wedge \theta^a) + 2 d \theta^b \wedge \ast (d \theta^a \wedge \theta_b) \right) + d (\ldots) \hspace{1cm} (183) \]

The variation of the Lagrangean can be written as,

\[ \delta L = \delta \theta_a \wedge \left( d \ast F^a + T_{\text{grav}}^a + T_{\text{mat}}^a \right) + d (\ldots) \hspace{1cm} (184) \]

The reason for writing the stress-energy tensor with a plus sign is because if a field has spontaneous symmetry breaking, the only way to satisfy both these requirements:

- The Lagrangean can be written as \( \mathcal{L} = (K - V) \ast 1 \)
- The energy of particles is positive.
is to write the stress-energy tensor with a plus sign.
The symbols are,

\[ F^a = \rho_1 \mathrm{d}\theta^a + \rho_2 v^a \mathbf{J}(\mathrm{d}\theta_b \wedge \theta^b) + \rho_3 v^b \mathbf{J}(\mathrm{d}\theta_b \wedge \theta^a) \]  
\[ T_{\text{grav}}^a = \rho_2 \mathrm{d}\theta^a \wedge *(\mathrm{d}\theta_b \wedge \theta^b) + \rho_3 \mathrm{d}\theta^b \wedge *(\mathrm{d}\theta^a \wedge \theta_b) \]  

(185)  
(186)

So the coframe gravity approximation field equation is,

\[ \mathrm{d} \ast F^a + T_{\text{grav}}^a + T_{\text{mat}}^a = 0 \]  

(187)

6 An exact solution

The field equation is,

\[ \mathrm{d} \ast F^a + T_{\text{grav}}^a + T_{\text{mat}}^a = 0 \]  

(188)

with,

\[ F^a = \rho_1 \mathrm{d}\theta^a + \rho_2 v^a \mathbf{J}(\mathrm{d}\theta_b \wedge \theta^b) + \rho_3 v^b \mathbf{J}(\mathrm{d}\theta_b \wedge \theta^a) \]  
\[ T_{\text{grav}}^a = \rho_2 \mathrm{d}\theta^a \wedge *(\mathrm{d}\theta_b \wedge \theta^b) + \rho_3 \mathrm{d}\theta^b \wedge *(\mathrm{d}\theta^a \wedge \theta_b) \]  

(189)  
(190)

I shall look for a diagonal solution. If the coframe is diagonal, then,

\[ \mathrm{d}\theta_a \wedge \theta^a = 0 \]  

(191)

If \( \rho_3 = 0 \), then the field equation becomes,

\[ \rho_1 \mathrm{d} \ast \mathrm{d}\theta^a + T_{\text{mat}}^a = 0 \]  

(192)

Using,

\[ T^{ab} = - \ast (\theta^b \wedge T^a) \]  

(193)

the field equation becomes,

\[ -\rho_1 \ast (\theta^b \wedge \mathrm{d} \ast \mathrm{d}\theta^a) + T_{\text{mat}}^{ab} = 0 \]  

(194)

\[ -\rho_1 \ast (\theta^b \wedge \mathrm{d} \ast \mathrm{d}\theta^a) = -T_{\text{mat}}^{ab} \]  

(195)
I shall do some calculations to bring this into a form with which it is easy to calculate with. The field equation is,

\[ -\rho_1 \ast (\theta^b \wedge d \ast d\theta^a) = -T_{\text{mat}}^{ab} \]  

(196)

\[ -\rho_1 \ast (\theta^b \wedge d \ast d\theta^a (v^d, v^e) \frac{1}{2}\theta_d \wedge \theta_e) = -T_{\text{mat}}^{ab} \]  

(197)

\[ -\rho_1 \ast \left( \theta_b \wedge d \left( d\theta_a (v^d, v^e) \frac{1}{4}\epsilon_{defg}\theta^f \wedge \theta^g \right) \right) = -T_{\text{mat}}^{ab} \]  

(198)

\[ \rho_1 \left( \left( v_h d\theta_a (v^d, v^e) \right) \frac{1}{4}\epsilon_{defg}\theta^h \wedge \theta^i \wedge \theta^g \right) \]  

(200)

I shall look for a solution of the form,

\[ \theta^a_{\mu} = \begin{bmatrix} \exp(f(x)) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  

(204)

Then the field equation becomes the equation,

\[ -\rho_1 v^\pi \partial_\pi (\theta^0_{\mu,\nu} v^{0\nu} v^{h\nu}) = -T_{\text{mat}}^{00} \]  

(205)

\[ \rho_1 v^\pi \partial_\pi (\theta^0_{\mu,\nu} v^0_{\mu} v^{h\nu}) = -T_{\text{mat}}^{00} \]  

(206)
\( v \) is equal to,

\[
v_a^\mu = \begin{bmatrix}
\exp(-f(x)) & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]  

(207)

The equation becomes,

\[
\rho_1 \delta_h \pi \partial_\pi (e^{f(x)}(\partial_\nu f)e^{-f(x)}\delta^{h\nu}) = -T_{\text{mat}}^{00}
\]

(208)

\[
-\rho_1 \Delta f = T_{\text{mat}}^{00}
\]

(209)

An exact solution for all static spacetimes in which the only component of \( T_{\text{mat}} \) which is non-zero is \( T_{\text{mat}}^{00} \), is,

\[
f(x) = \frac{1}{-8\pi \rho_1} \int d^3y \frac{T_{\text{mat}}^{00}(y)}{|x - y|}
\]

(210)

An exact homogeneous solution which is static and has spherical symmetry is,

\[
f(x) = -\frac{GM}{rC^2}
\]

(211)

Corresponding to the coframe,

\[
\theta^0_0 = \exp \left( -\frac{GM}{rC^2} \right)
\]

(212)

\[
\theta^a_\mu = \delta^a_\mu \text{ for } a, \mu \text{ not both } 0
\]

(213)

These solutions are common to a one-parameter set of models with \( \rho_1 \neq 0, \rho_2 \text{ any}, \rho_3 = 0 \).

7 The linearized field equation

I linearize the field equation using,

\[
\theta_{a\mu} = \eta_{a\mu} + h_{a\mu}
\]

(214)

and using,

\[
T_{ab} = -*(\theta_b \wedge T_a)
\]

(215)
In the following calculation, I shall ignore non-linear terms. Taking \( \rho_3 = 0 \), the linearized field equation becomes,

\[
- \ast \left( \theta_b \wedge d \ast F_a \right) = - T_{\text{mat} \, ab} \tag{216}
\]

\[
= - \ast \left( \theta_b \wedge d \ast \left( \rho_1 d \theta_a + \rho_2 v_a J(d \theta_c \wedge \theta^e) \right) \right) = - T_{\text{mat} \, ab} \tag{217}
\]

\[
- \rho_1 \ast \left( \theta_b \wedge d \ast d \theta_a \right) - \rho_2 \ast \left( \theta_b \wedge d \ast v_a J(d \theta_c \wedge \theta^e) \right) = - T_{\text{mat} \, ab} \tag{218}
\]

\[
- \rho_1 \ast \left( \theta_b \wedge d \ast d \theta_a(v^d, v^e) \frac{1}{2} \theta_d \wedge \theta_c \right)
\]

\[
- \rho_2 \ast \left( \theta_b \wedge d \ast v_a J(d \theta^c(v^d, v^e) \frac{1}{2} \theta_d \wedge \theta_c) \right) = - T_{\text{mat} \, ab} \tag{219}
\]

\[
- \rho_1 \ast \left( \theta_b \wedge d \left( d \theta_a(v^d, v^e) \frac{1}{4} \epsilon_{defg} \theta^f \wedge \theta^g \right) \right)
\]

\[
- \rho_2 \ast \left( \theta_b \wedge d \left( \theta_a \wedge \ast \left( d \theta^c(v^d, v^e) \frac{1}{2} \theta_d \wedge \theta_c \right) \right) \right) = - T_{\text{mat} \, ab} \tag{220}
\]

\[
- \rho_1 \ast \left( \theta_b \wedge \left( v_h d \theta_a(v^d, v^e) \frac{1}{4} \epsilon_{defg} \theta^h \wedge \theta^f \wedge \theta^g \right) \right)
\]

\[
- \rho_2 \ast \left( \theta_b \wedge v_g d \theta^c(v^d, v^e) \frac{1}{2} \epsilon_{def} \theta_b \wedge \theta^g \wedge \theta_a \wedge \theta^f \right) = - T_{\text{mat} \, ab} \tag{222}
\]

\[
\rho_1 \left( v_h d \theta_a(v^d, v^e) \frac{1}{4} \eta_{ba} \epsilon_{defg} \epsilon^{hfg} \right)
\]

\[
- \rho_2 \ast \left( v_g d \theta^c(v^d, v^e) \frac{1}{2} \epsilon_{def} \theta_b \wedge \theta^g \wedge \theta_a \wedge \theta^f \right) = - T_{\text{mat} \, ab} \tag{223}
\]

\[
\rho_1 \left( v_h d \theta_a(v^d, v^e) \frac{1}{4} \eta_{ba} \epsilon_{defg} \epsilon^{hfg} \right)
\]

\[
+ \rho_2 \frac{1}{2} \left( v_g d \theta^c(v^d, v^e) \epsilon_{def} \eta_{ah} \eta_{ba} \epsilon^{jhg} \right) = - T_{\text{mat} \, ab} \tag{224}
\]

\[
\rho_1 \frac{1}{2} \left( v_h d \theta_a(v_b, v^h) - v_h d \theta_a(v^h, v_b) \right)
\]

\[
+ \rho_2 \frac{1}{2} \left( v^g d \theta_a(v_b, v_g) - v^g d \theta_a(v_g, v_b) - v^g d \theta_g(v_b, v_a) \right)
\]

\[
+ v^g d \theta_g(v_a, v_b) + v^g d \theta_b(v_g, v_a) - v^g d \theta_b(v_a, v_g) \right) = - T_{\text{mat} \, ab} \tag{225}
\]

\[
\rho_1 \left( v_h d \theta_a(v_b, v_h) \right)
\]

\[
+ \rho_2 \left( v^g d \theta_a(v_b, v_g) - v^g d \theta_a(v_g, v_b) + v^g d \theta_b(v_g, v_a) \right) = - T_{\text{mat} \, ab} \tag{226}
\]
\[
\rho_1(h_{am,b}^{m} - h_{ab,m}^{m}) + \rho_2(h_{am,b}^{m} - h_{ab,m}^{m} - h_{ma,b}^{m}) + h_{mb,a}^{m} + h_{ba,m}^{m} - h_{bm,a}^{m} = -T_{\text{mat}ab}
\]

(227)

\[
-\rho_1(\square h_{ab} - h_{am,b}^{m}) - \rho_2 h_{[ab,m]}^{m} = -T_{\text{mat}ab}
\]

(228)

At all orders, the field equation can be written as,

\[
-\rho_1(\square h_{ab} - h_{am,b}^{m}) - \rho_2 h_{[ab,m]}^{m} = (\text{non-linear terms})_{ab} - T_{\text{mat}ab}
\]

(229)

\[
-\rho_1(\square h_{ab} - h_{am,b}^{m}) = \rho_2 h_{[ab,m]}^{m} + (\text{non-linear terms})_{ab} - T_{\text{mat}ab}
\]

(230)

The left side has a vanishing divergence, so if the field equation is consistent, the right side must also have a vanishing divergence.

I set as an Ansatz,

\[
-\rho_1(\square h_{ab} = (\text{right side})_{ab}
\]

(231)

with (right side) the right side of (230).

The following function follows the homogeneous 3+1-dimensional Laplace equation. In the following calculation, the index \(i\) runs from 1 to 3:

\[
\square r \cos(\omega(x^0 - r)) = \omega^2 r \cos(\omega(x^0 - r)) + \partial_i \partial_i r \cos(\omega(x^0 - r))
\]

(232)

\[
= \omega^2 r \cos(\omega(x^0 - r)) + \partial_i \left( \frac{\omega \sin(\omega(x^0 - r)) x^i}{r^2} - \frac{\cos(\omega(x^0 - r)) x^i}{r^3} \right)
\]

(233)

\[
= \omega^2 r \cos(\omega(x^0 - r))
\]

\[
- \frac{\omega^2 \cos(\omega(x^0 - r))(x^i)^2}{r^2} + \delta_i^j \frac{\omega \sin(\omega(x^0 - r)) x^j}{r^4} - \frac{2 \omega \sin(\omega(x^0 - r))(x^i)^2}{r^4}
\]

\[
- \frac{\omega \sin(\omega(x^0 - r))(x^i)^2}{r^4} - \delta_i^j \frac{\cos(\omega(x^0 - r)) x^j}{r^3} + 3 \frac{\cos(\omega(x^0 - r))(x^i)^2}{r^5}
\]

(234)

\[
= 0
\]

(235)

This is used to construct a solution:

\[
h_{ab}(x) = \frac{1}{-8\pi \rho_1} \int d^4y \frac{(\text{right side})_{ab}(y) \delta(x^0 - y^0 - |x - y|)}{|x - y|}
\]

(236)
For $i$ a space index,

$$\partial_i h_{ab}(x) = \lim_{n \to 0} \frac{h_{ab}(x + n\hat{x}^i) - h_{ab}(x)}{n} \quad (237)$$

In mathematics, we cannot always interchange differentiation and integration. But in physics, we can.

$$= \lim_{n \to 0} \frac{1}{n} \left( \frac{1}{-8\pi \rho_1} \int d^4y \frac{(\text{right side})_{ab}(y)\delta(x^0 - y^0 - |x + n\hat{x}^i - y|)}{|x + n\hat{x}^i - y|} \right) \quad (238)$$

Changing the variable of integration to $z = y - n\hat{x}^i$:

$$= \lim_{n \to 0} \frac{1}{n} \left( \frac{1}{-8\pi \rho_1} \int d^4z \frac{(\text{right side})_{ab}(z + n\hat{x}^i)\delta(x^0 - z^0 - |x - z|)}{|x - z|} \right) \quad (239)$$

$$= \frac{1}{-8\pi \rho_1} \int d^4y \frac{(\text{right side})_{ab,i}(y)\delta(x^0 - y^0 - |x - y|)}{|x - y|} \quad (240)$$

Similarly,

$$\partial_0 h_{ab} = \frac{1}{-8\pi \rho_1} \int d^4y \frac{(\text{right side})_{ab,0}(y)\delta(x^0 - y^0 - |x - y|)}{|x - y|} \quad (241)$$

Therefore,

$$h_{am,b}^m = \frac{1}{-8\pi \rho_1} \int d^4y \frac{(\text{right side})_{am,b}^m(y)\delta(x^0 - y^0 - |x - y|)}{|x - y|} \quad (242)$$

$$= 0 \quad (243)$$

This proves the Ansatz to be a solution.

The field equation at all orders is,

$$- \rho_1 \Box h_{ab} = \rho_2 h_{[ab,m]}^m + (\text{non-linear terms})_{ab} - T_{mat ab} \quad (244)$$

This field equation can be solved iteratively by starting with the coframe $h_{ab} = 0$. At each step of the iteration, the old value of $h$ is substituted in the right side of the equation. And the new value of $h$ is solved for in the left side of the equation.
Reference

[1] R. W. Sharpe. *Differential Geometry: Cartan’s Generalization of Klein’s Erlangen Program*. Springer.