Research Article

Eigenvalue Criteria for Existence of Positive Solutions to Fractional Boundary Value Problem

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The existence and multiplicity of positive solutions for the nonlinear fractional differential equation boundary value problem (BVP)

\[ C^\alpha_0 y(x) + f(x, y(x)) = 0, \quad 0 < x < 1, \]

\[ y(0) = y'(1) = y''(0) = 0, \]

where \(2 < \alpha \leq 3\), \(C^\alpha_0\) is the Caputo fractional derivative, and \(f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)\) is a continuous function. The conclusion relies on the fixed-point index theory and the Leray-Schauder degree theory. The growth conditions of the nonlinearity with respect to the first eigenvalue of the related linear operator is given to guarantee the existence and multiplicity.

1. Introduction

In this paper, we concentrate on the existence and multiplicity of positive solutions for the following problem:

\[ C^\alpha_0 y(x) + f(x, y(x)) = 0, \quad 0 < x < 1, \tag{1} \]

\[ y(0) = y'(1) = y''(0) = 0, \tag{2} \]

where \(2 < \alpha \leq 3\), \(C^\alpha_0\) is the Caputo fractional derivative, and \(f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)\) is a continuous function.

In the past twenty years, the fractional differential equation has aroused great consideration [1–21] not only in its application in mathematics but also in other applications in science and engineering, for example, fluid mechanics, viscoelastic mechanics, electroanalytical chemistry, and biological engineering. Bai and Qiu [22, 23] have investigated the existence and multiplicity of positive solutions of (1) and (2) by using the nonlinear alternative of the Leray-Schauder type and Krasnoselskii’s fixed-point theorem in a cone, but they did not consider its eigenvalue criteria.

The rest of the paper is organized as follows. In Section 2, we recall some concepts relative to fractional calculus and give some lemmas with respect to the corresponding Green function. In Section 3, with the use of the fixed-point theory, some existence and multiplicity results of positive solutions are obtained. At last, two examples are given.

2. Background Materials

For the convenience of the reader, we give some definitions and lemmas.

Definition 1 (see [23]). The Caputo’s fractional derivative of order \(\alpha > 0\) of a continuous function \(y : [0, +\infty) \rightarrow \mathbb{R}\) is given by

\[ C^\alpha_0 y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{y^n(t)}{(x-t)^{n-\alpha+1}} dt, \tag{3} \]

where \(n-1 < \alpha < n\) and \(n \in \mathbb{N}\), provided that the right-hand side is pointwise defined on \([0, +\infty)\).

Lemma 2 (see [15]). Given \(h \in C[0, 1]\), the unique solution of

\[ C^\alpha_0 y(x) + h(x) = 0, \quad 0 < x < 1, \]

\[ y(0) = y'(1) = y''(0) = 0, \tag{4} \]

is
Given $f \in C([0,1] \times [0,\infty) \cup [0,\infty))$. Let $T, A : K \to C [0,1]$ be the operators defined by

$$
(Tu)(x) = \int_0^1 G(x,t)f(t,u(t))dt, \quad u \in K, x \in [0,1],
$$

and

$$
(Au)(x) = \int_0^1 G(x,t)u(t)dt, \quad u \in K, x \in [0,1].
$$

It is well known that $T, A : K \to K$ are all completely continuous [23].

Denote

$$
M = \left( \int_0^1 \max G(x,t)dt \right)^{-1},
$$

$$
N = \left( \int_{(1/4)}^{(3/4)} G(\frac{1}{2}, t)dt \right)^{-1},
$$

where $M, N$ are positive constants.

**Lemma 7.** Suppose $A$ is defined by (11), then the spectral radius $r(A) > 0$ and $A$ has a positive eigenfunction $\varphi_1$ corresponding to its first eigenvalue $\lambda_1 = (r(A))^{-1}$.

**Proof.** The operator $A : C[0,1] \to C[0,1]$ is a completely continuous linear operator and $A(K) \subset K$ (see [23]). Choose $\psi \in C[0,1]$ and $[x_1, x_2] \subset (0,1)$ such that $\psi(x) \geq 0$ for $x \in [0,1]$; $\psi(x) > 0$ for $x \in (x_1, x_2)$. By the use of Lemma 2, for $x \in [0,1]$, there holds

$$
(A\psi)(x) = \int_0^1 G(x,t)\psi(t)dt \geq \int_{x_1}^{x_2} G(x,t)\psi(t)dt > 0.
$$

So, we can choose $c \in \mathbb{R}$ so large that

$$
c(A\psi)(x) \geq \psi(x), \quad \text{for } x \in [0,1].
$$

By Lemma 6, we complete the proof.

**Theorem 8.** Suppose the following conditions hold:

$$
(I_0) \lim_{y \to 0^+} \frac{f(x,y)}{y} > \lambda_1,
$$

$$
(S_{\infty}) \lim_{y \to +\infty} \frac{f(x,y)}{y} < \lambda_1,
$$

where $\lambda_1$ is the first eigenvalue of the operator $A$ defined by (11). Then, BVP (1) and (2) have at least one positive solution.

**Proof.** By condition $(I_0)$, there exists $r_1 > 0$ small enough such that

$$
f(x,y) \geq \lambda_1 y, \quad \text{for } 0 \leq x \leq 1, 0 \leq y \leq r_1.
$$
Let \( \varphi^* \) be the positive eigenfunction of \( A \) corresponding to \( \lambda_1 \), thus \( \varphi^* = \lambda_1 A \varphi^* \).

For every \( \varphi \in K_1 \), for \( x \in [0, 1] \)

\[
(T \varphi)(x) = \int_0^1 G(x, t)f(t, \varphi(t))dt \\
\geq \lambda_1 \int_0^1 G(x, t)\varphi(t)dt = \lambda_1 (A \varphi)(x).
\]  

Suppose without loss of generality that \( T \) has no fixed point on \( \partial K_1 \) (otherwise, the proof is completed). We claim that

\[
\varphi - T \varphi \neq \mu \varphi^*, \quad \text{for all } \varphi \in \partial K_1, \mu > 0.
\]  

In fact, if there exist \( \varphi_1 \in \partial K_1 \) and \( \mu_0 > 0 \) such that \( \varphi_1 - T \varphi_1 = \mu_0 \varphi^* \), then

\[
\varphi_1 = T \varphi_1 + \mu_0 \varphi^* \geq \mu_0 \varphi^*.
\]  

Let

\[
\mu^* = \sup \{ \mu | \varphi_1 \geq \mu \varphi^* \}.
\]  

It is easy to see that \( +\infty > \mu^* \geq \mu_0 > 0 \) and \( \varphi_1 \geq \mu^* \varphi^* \). Taking into account that \( A \) is a linear positive operator, we have

\[
\lambda_1 A \varphi_1 \geq \mu^* \lambda_1 A \varphi^* = \mu^* \varphi^*.
\]  

Therefore, by (17),

\[
\varphi_1 = T \varphi_1 + \mu_0 \varphi^* \geq \lambda_1 A \varphi_1 + \mu_0 \varphi^* \geq (\mu^* + \mu_0) \varphi^*,
\]  

which contradicts the definition of \( \mu^* \). Hence (18) holds and we have from Lemma 3 that

\[
i(T, K_1, K) = 0.
\]  

On the other hand, by (S\(_\infty\)), there exist \( 0 < \sigma < 1 \) and \( r_2 > r_1 \) such that

\[
f(x, y) \leq \sigma \lambda_1 y, \quad \text{for all } y \geq r_2, x \in [0, 1].
\]  

Define \( A_1 : C[0, 1] \rightarrow C[0, 1] \) as \( A_1 \varphi = \sigma \lambda_1 A \varphi \), \( \varphi \in C[0, 1] \). Then \( A_1 \) is a bounded linear operator and \( A_1 (K) \subseteq K \).

Denote

\[
B = \left( \max_{0 \leq x \leq 1} G(x, t) \right) \sup_{y \in B_2} \int_0^1 f(t, y(t))dt.
\]  

It is clear that \( B < +\infty \). Let

\[
W = \{ \varphi \in K | \varphi = \mu T \varphi, 0 \leq \mu \leq 1 \}.
\]  

In the following, we firstly prove that the set \( W \) is bounded.

For any \( \varphi \in W \), set \( \varphi(x) = \min \{ \varphi(x), r_2 \} \) for \( x \in [0, 1] \) and denote \( E(\varphi) = \{ x \in [0, 1] | \varphi(x) > r_2 \} \), then for \( x \in [0, 1] \),

\[
\varphi(x) = \mu(T \varphi)(x) \leq (T \varphi)(x) = \int_0^1 G(x, t)f(t, \varphi(t))dt \\
= \int_{E(\varphi)} G(x, t)f(t, \varphi(t))dt + \int_{[0,1] \setminus E(\varphi)} G(x, t)f(t, \varphi(t))dt \\
\leq \int_0^1 G(x, t)\sigma \lambda_1 \varphi(t)dt + \int_0^1 G(x, t)f(t, \varphi(t))dt \\
\leq (A_1 \varphi)(x) + B.
\]  

Thus \( (I - A_1) \varphi(x) \leq B, x \in [0, 1] \). Since \( \lambda_1 \) is the first eigenvalue of \( A \) and \( 0 < \sigma < 1 \), the first eigenvalue of \( A_1 \), \( \sigma \lambda_1 \lambda_1 \), exists and

\[
(I - A_1)^{-1} = I + A_1 + A_1^2 + \cdots = A_1 + A_2 + \cdots.
\]  

It follows from \( A_1 (K) \subseteq K \) that \( (I - A_1)^{-1} (K) \subseteq K \). So we have \( \varphi(x) \leq (I - A_1)^{-1} B, x \in [0, 1] \) and the set \( W \) is bounded.

Choose \( r_3 > \max \{ r_2, \| (I - A_1)^{-1} B \| \} \). Then by Lemma 4, we have

\[
i(T, K_{r_3}, K) = 1.
\]  

By (23) and (29), one has

\[
i\left( T, \frac{K_{r_3}}{K_1}, K \right) = i(T, K_{r_3}, K) - i(T, K_1, K) = 1.
\]  

Then, \( T \) has at least one fixed point on \( K_{r_3} \setminus K_1 \). This means that problem (1) and (2) have at least one positive solution. The proof is complete.

**Theorem 9.** Suppose the following conditions are met:

\[
(I_\infty) \lim_{y \to +\infty} \frac{f(x, y)}{y} < \lambda_1,
\]  

\[
(S_0) \lim_{y \to 0^+} \frac{f(x, y)}{y} < \lambda_1,
\]  

where \( \lambda_1 \) is the first eigenvalue of the operator \( A \) defined by (11). Then BVP (1) and (2) have at least one positive solution. The proof is similar to Theorem 8.

**Theorem 10.** Suppose there exist two numbers \( b > a > 0 \) such that the following conditions are met:

\[
(C_1) f(x, y) \leq Ma, \quad \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq a,
\]  

\[
(C_2) f(x, y) \geq Nb, \quad \text{for } \frac{1}{4} \leq x \leq \frac{3}{4} \text{ and } \frac{1}{4} \leq y \leq b.
\]  

Then, BVP (1) and (2) have at least one positive solution.
\textbf{Proof.} If \( C_1 \) and \( C_2 \) hold, similar to Lemma 3 [6], we have
\begin{equation}
\begin{aligned}
i(A, K_a, K) &= 1, \\
i(A, K_b, K) &= 0.
\end{aligned}
\end{equation}
(33)

Consequently, the additivity of the fixed-point index implies
\begin{equation}
i(A, K_b, K) = i(A, K_b, K) - i(A, K_a, K) = -1.
\end{equation}
(34)

Consequently, \( A \) has a fixed point \( y(x) \) in \( K_b \setminus \bar{K}_a \).

**Theorem 11.** The problem in (1) and (2) has at least two positive solutions if conditions \((I_0), (I_{\infty}), \) and \( C_1 \) hold, where \( \lambda_1 \) is the first eigenvalue of the operator \( A \) defined by (11).

**Proof.** Because \((I_0)\) and \((I_{\infty})\) hold, there exist \( 0 < r < a < R \) such that
\begin{equation}
\begin{aligned}
i(A, K_r, K) &= 0, \\
i(A, K_R, K) &= 0.
\end{aligned}
\end{equation}
(35)

On the other hand, \( C_1 \) implies \( i(A, K_a, K) = 1 \). So we have
\begin{equation}
\begin{aligned}
i(A, K_R, K) &= i(A, K_R, K) - i(A, K_a, K) = 1, \\
i(A, K_b, K) &= i(A, K_b, K) - i(A, K_a, K) = -1,
\end{aligned}
\end{equation}
(36)

therefore, \( A \) has two fixed points, \( y_1 \in K_a \setminus \bar{K}_r, \ y_2 \in K_R \setminus \bar{K}_a \).

**Theorem 12.** The problem in (1) and (2) has at least two positive solutions if conditions \((S_0), (S_{\infty}), \) and \( C_2 \) hold, where \( \lambda_1 \) is the first eigenvalue of the operator \( A \) defined by (10).

**Proof.** Because \((S_0)\) and \((S_{\infty})\) hold, there exist \( 0 < r < b < R \) such that
\begin{equation}
\begin{aligned}
i(A, K_r, K) &= 1, \\
i(A, K_R, K) &= 1.
\end{aligned}
\end{equation}
(37)

On the other hand \( C_2 \) implies \( i(A, K_b, K) = 0 \). So we have
\begin{equation}
\begin{aligned}
i(A, K_R, K) &= i(A, K_R, K) - i(A, K_b, K) = 1, \\
i(A, K_b, K) &= i(A, K_b, K) - i(A, K_b, K) = -1,
\end{aligned}
\end{equation}
(38)

Therefore, \( A \) has two fixed points, \( y_1 \in K_b \setminus \bar{K}_r, \ y_2 \in K_R \setminus \bar{K}_b \).

**4. Example**

To illustrate the main points, we give two examples.

**Example 13.** Let
\[ f(x, y) = y^{(1/2)} + \frac{2 + \sin x}{4}. \]
(39)

Consider the BVP
\[ C^D_y, y(x) + (y(x))^{(1/2)} + \frac{2 + \sin x}{4} = 0, \quad 0 < x < 1, \]
(40)
\[ y(0) = y'(1) = y''(0) = 0. \]
(41)

It is not difficult to see that
\[ \liminf_{y \to 0} \frac{f(x, y)}{y} = +\infty > \lambda_1, \]
(43)
\[ \limsup_{y \to +\infty} \frac{f(x, y)}{y} = 0 < \lambda_1, \]
where \( \lambda_1 \) is the first eigenvalue of the operator \( A \) defined by (11). By Theorem 11, BVP (40) and (41) have at least one positive solution.

**Example 14.** Let
\[ f(x, y) = y^3 + y^2 \sin x. \]
(44)

Consider the BVP
\[ C^D_y, y(x) + |y(x)|^3 + |y(x)|^2 \sin x = 0, \quad 0 < x < 1, \]
(45)
\[ y(0) = y'(1) = y''(0) = 0. \]
(46)

It is not difficult to see that
\[ \liminf_{y \to 0} \frac{f(x, y)}{y} = +\infty > \lambda_1, \]
(48)
\[ \limsup_{y \to +\infty} \frac{f(x, y)}{y} = 0 < \lambda_1, \]
where \( \lambda_1 \) is the first eigenvalue of the operator \( A \) defined by (11). By Theorem 12, BVP (45) and (46) have at least one positive solution.

**Data Availability**

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.


Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors’ Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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