Smooth interpolation of lattice gauge fields by signal processing methods

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We digitally filter the Fourier modes of the link angles of an abelian lattice gauge field which produces the Fourier modes of a continuum \( A_\mu(x) \) that exactly reproduces the lattice links through their definition as phases of finite parallel transport. The constructed interpolation is smooth \( (C^\infty) \), free from transition functions, and gauge equivariant. After discussing some properties of this interpolation, we discuss the non-abelian generalization of the method, arriving for SU(2), at a Cayley parametrization of the links in terms of the Fourier modes of \( A_\mu^c(x) \). We then discuss the use of a maximum entropy type method to address gauge invariance in the non-abelian case.

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1. Lattices and signal filters

The explicit relationship between an array of links \( U_\mu \) and the continuum field \( A_\mu(x) \) to which it corresponds is an important ingredient in comparing lattice results with perturbation theory. While the mapping of continuum interpolations to lattice configurations is (very) many-to-one, there are various simple algorithms which give unique continuum fields that serve our purposes to lesser or greater degree. Probably the simplest of these is the piecewise constant interpolation where links are \( U_\mu = e^{i \theta_\mu} \), and \( A_\mu(x) \) is constant along links. Despite its simplistic properties, this interpolation is a mainstay of lattice gauge theory.

The purpose of this contribution is to explore better interpolations of a lattice configuration using some ideas borrowed from signal processing. The starting point is that the interpolating field be smooth \( (C^\infty) \), and that it reproduces the lattice links exactly through the definition of parallel transport between \( x \) and \( x + a \)

\[
U_\mu(x, x + a) = P e^{i \int_x^{x+a} dt \cdot A_\mu(x^1, x^2, ..., x^D)} 
\]

(1)

In light of developments in improved actions over the past few years, we know that a better representation of the underlying continuum field is extremely powerful. Knowing \( A_\mu(x) \) allows us to compute (classically) "perfect" operators directly in the continuum such as \( F_{\mu\nu}^2, F_{\mu\nu} F^{\mu\nu}, \partial_\mu A_\mu, \) and \( \text{Det}[\partial (A_\mu)] \), providing a greatly improved action, topological charge density, gauge fixing functional, and fermion effective action. It also provides a method of implementing chiral fermions del Desperados [1,2].

The interpolation of an unknown function from a discrete set of measurements is the bread and butter of digital signal processing [3] and I find it useful to think in terms of the following analogy between the lattice and a digital filter:

\[
A_\mu(x) \rightarrow \text{lattice} \rightarrow U_\mu \\
\text{input} \rightarrow \text{filter} \rightarrow \text{output}.
\]

Thus, a lattice is nothing but a band limited digital filter of continuum gauge fields. Usually a signal filter is represented by its transfer function \( \lambda_k \) in \( k \)-space:

\[
\hat{f}_\text{in}(k) \rightarrow \lambda_k \hat{f}_\text{in}(k) \rightarrow \hat{f}_\text{out}(k).
\]

Similarly the effect of a lattice on \( A_\mu(x) \) has a simple representation, as follows.

2. An abelian extrapolation

An abelian link \( U_\mu(x, x + a) = e^{i \theta} \) is defined by an angle \( -\pi \leq \theta_\mu(x + a) < \pi \), assigned to the point \( (x^1, x^2, ..., x^\mu, x_{\mu+1} + a, ..., x^D) \). The lattice transfer function \( \lambda_k \) then follows immediately. We first Fourier transform the angle field

\[
\theta_\mu(x + a) =
\]
This filter is so common in signal analysis that it has a special name, sinc($k$); it arises when the data are integrals of the underlying signal, rather than samples. A familiar example is a CCD camera where pixels represent the integrated count of photons over some length of time.

We then have our extrapolation: the Fourier modes of the continuum field corresponding to the lattice with angles $\lambda_k$ are

$$\lambda(k)\hat{A}_\mu(k) = \frac{2\sin(ak\mu/2)}{k}$$

(2)

2.1. Properties

- **Gauge equivariance**

  It is very easy to show that the above construction is gauge invariant when the continuum gauge section $G(x)$ is extracted likewise from its lattice counterpart by sampling (ie. $\lambda_k = 1$). This gives us a continuum $\omega(x)$ whose exponential $G(x_i) = e^{i\omega(x_i)}$ at lattice sites $x_i$ is the gauge transformation applied to the lattice field. Then the following diagram holds:

  \[
  \begin{array}{ccc}
  \text{Lattice} & \xrightarrow{U_\mu(x_i + \frac{a}{2})} & \text{Continuum} \\
  G(x) & \rightarrow & e^{i\omega(x)} \\
  \downarrow & & \downarrow \\
  U_\mu'(x_i + \frac{a}{2}) & \rightarrow & A_\mu'(x) = A_\mu(x) - \partial_\mu\omega(x) \\
  \end{array}
  \]

  ie. the extrapolation of the gauge transformed lattice field $U_\mu'$, is identical to the continuum gauge transformation $A_\mu'$ (using the extrapolated section $\omega(x)$) of the original extrapolation $A_\mu(x)$. This is true at arbitrary $x$.

- **Topological charge**

  Since the extrapolated gauge field is $C^\infty$ and periodic, on a manifold without boundaries it must have topological charge $Q = 0$, a simple fact maintaining Stokes law. While some might be unhappy that an extrapolation has $Q = 0$ when other lattice definitions give nonzero charge, I view this as a feature and not a bug. We should remember that nontrivial topological charge stems from nontrivial transition functions; if our lattices are periodic, then our gauge fields should have $Q = 0$. Any non-trivial topological charge arises from a misrepresentation of the geometric relationship between $A_\mu$ and $F_{\mu\nu}$, and means that we have allowed singular gauge sections.

  The topological charge density on the other hand, will not be zero on the extrapolation, and should be a perfectly good measure of the true topological fluctuations and character of the field. In particular, $\hat{F} \hat{F}$ computed from the extrapolated $A_\mu(x)$ can be integrated over many regions small compared to the volume of the lattice, whose average will then give an accurate measure of $Q/\text{Vol}$.

- **Large gauge transformations**

  In the diagram (4), extrapolated gauge transformations $e^{i\omega(x)}$ are restricted to those with zero winding if we use a strictly periodic Fourier basis. However two configurations $U_\mu$ and $U_\mu'(x)$ which are related by a large gauge transformation will only differ in their zero momentum modes, and the extrapolation to $A_\mu(x)$ and $A_\mu'(x)$ will reproduce this difference correctly, despite the fact that it cannot produce the continuum gauge transformation between them.

3. Nonabelian extrapolation

There are two essential difficulties in applying the idea of the last section to nonabelian gauge fields.

3.1. Nonabelian parallel transport

...is very nonlinear. The relationship between link and gauge field is given by the path ordered exponential

$$U_\mu(x, x + a) = \mathcal{P} e^{i \int_{x}^{x+a} dt A_\mu}$$
\[
\begin{align*}
&= 1 + i \int_{x}^{x+a} dt A_\mu(t) \\
&\quad - \frac{1}{2i} \int_{x}^{x+a} dt_2 \int_{x}^{x+a} dt_1 \left\{ \Theta(t_2 - t_1) A_\mu(t_2) A_\mu(t_1) \\
&\quad \quad \quad \quad \quad \quad \quad \quad + \Theta(t_1 - t_2) A_\mu(t_1) A_\mu(t_2) \right\} \\
&\quad + \cdots
\end{align*}
\]

where \( \Theta(t) = (x > 0)? 1 : 0 \) is a step function, and the permutations over ordering are explicitly displayed. We must evaluate the Fourier transform of each term, then do the sum. The main part of the computation is evaluating integrals of the form

\[
\int_{x}^{x+a} dt_n \cdots \int_{x}^{x+a} dt_2 \int_{x}^{x+a} dt_1 \Theta(t_n - t_{n-1}) \cdots \Theta(t_2 - t_1) \\
\times e^{ik_n t_n + ik_{n-1} t_{n-1} + \cdots + ik_1 t_1}.
\]

I have evaluated terms up to third order in \( A_\mu \) using

\[
\Theta(t) = \frac{1}{2\pi} \left[ \sum_{n\neq 0} \frac{1}{in} e^{it} + t + \pi \right],
\]

on a gauge field with no zero momentum mode. There are interesting cancellations which give hope that the entire sum can be done analytically. For example in the second order term

\[
I_2 = \int_{x}^{x+a} dt_2 \int_{x}^{x+a} dt_1 \Theta(t_2 - t_1) e^{ik_2 t_2 + ik_1 t_1} \\
= \frac{1}{2\pi i} \left\{ \sum_{n\neq 0} \frac{1}{n} \lambda_k \lambda_{k+n} \right. \\
&\quad + \left( \frac{\partial}{\partial k_2} - \frac{\partial}{\partial k_1} \right) \lambda_k \lambda_{k_1} \\
&\quad + \pi i \lambda_k \lambda_{k_1} \left\} e^{(k_2 + k_1)(x + \frac{a}{2})},
\]

the sum over \( n \neq 0 \) (done via the \( \zeta \) function contour) exactly cancels the partial derivative term, leaving

\[
I_2 = \frac{1}{2} \lambda_k \lambda_{k_1} e^{(k_2 + k_1)(x + \frac{a}{2})}
\]

which is nicely factorized in \( k_1 \) and \( k_2 \), linear in each \( \lambda_k \), and centered at \( x + \frac{a}{2} \).

The third order term with two nested theta functions \( \Theta(t_3 - t_2) \Theta(t_2 - t_1) \) is tedious but straightforward. While many terms cancel, certain derivatives of \( \lambda_k \) remain in an individual integral of type (4), yielding

\[
I_3 = \frac{1}{4\pi^2} \left\{ \lambda_k \lambda_{k_1} \lambda_{k_2} \lambda_{k_1} - \lambda_{k_3} \lambda_{k_2} \lambda_{k_1} \\
+ \pi^2 \lambda_{k_3} \lambda_{k_2} \lambda_{k_1} \right\} e^{(k_3 + k_2 + k_1)(x + \frac{a}{2})},
\]

However the derivative terms do cancel when summed over the 3! permutations of the ordering of \( A_\mu(k_n) \) leaving only the last term in (10): again factorized in \( k_n \), linear in each \( \lambda_k \), and centered at \( x + \frac{a}{2} \).

For SU(2) fields, I have summed the series in (11) based on the above results and using explicit formulae for the permutations of products of \( \sigma_i \); the result is interesting but incomplete. Assuming that permutations cancel the derivative terms at each order, the coefficient of the last term \( \lambda_{k_n} \lambda_{k_{n-1}} \cdots \lambda_{k_1} \) is easily obtained and produces a series which evaluates to the unitary matrix

\[
U_\mu(x, x + a) = \frac{1 + i \sum_k \lambda_k \hat{A}_\mu(k)}{1 - i \sum_k \lambda_k \hat{A}_\mu(k)}.
\]

This is a Cayley representation of \( U_\mu \), similar to that recently espoused by Periwal (4). This shows that there must be more terms appearing at higher orders, since the correct result must faithfully represent

\[
U_\mu(a, b) U_\mu(b, c) = U_\mu(a, c),
\]

which (11) unfortunately does not do. Nonetheless, it seems an interesting result.

### 3.2. Gauge invariance and \( k_c \)

In the abelian case we can content ourselves with a frequency cutoff in \( \hat{A}_\mu(k) \) and \( \hat{\omega}(k) \), and due to the linearity of gauge rotations, the entire construction is equivariant. In the non-abelian case gauge transformations, which involve products of fields \( G^\dagger(x) A_\mu(x) G(x) \cdots \), mix Fourier modes of \( \hat{G} \) and \( \hat{A}_\mu \) beyond the cutoff momentum \( k_c \).

Again let’s turn to engineering for a solution. In situations where one needs to know the power spectrum at a frequency other than at discrete
values of the measured FFT, the method of Maximum Entropy says that the FFT is actually equal to a Laurent series in the complex plane, which can be approximated by a Padé polynomial

\[ \hat{f}(k) = \sum_x f(x) z_k^x = \frac{1}{\sum_p \alpha_p z_k^p} \]  

where \( z_k = e^{2 \pi i a k} \). We can use the Padé approximation for \( \hat{f}(k) \) at any \( k : |k| \leq k_c \).

Another ingredient, known as aliasing, is that when doing an FFT, the “measured” values of \( \hat{f}_k \) include contributions from each multiple of \( k_c \). Thus,

\[ \hat{F}(k) = \sum_{n=-\infty}^{\infty} \hat{f}(k + nk_c) \]  

where \( \hat{F}(k) \) is the measured FFT, and \( \hat{f}(k) \) is the actual spectrum. Can we use the aliasing to get information about the spectrum at all \( k \)?

Consider the Padé approximation to \( \hat{f}(k) \)

\[ \tilde{f}(k) = \frac{A(k)}{B(k)} \]  

where \( A \) and \( B \) are polynomials. Including the effect of aliasing this becomes (here we use lattice units \( a = 1 \), so that \( k_c = L \) and the fourier frequencies \(-\frac{L}{2} \leq k \leq \frac{L}{2} - 1 \) are integer).

\[ \tilde{F}(k) = \sum_n \frac{A(k - nL)}{B(k - nL)} = -\sum_{\text{Res}} \frac{\pi \cot(\pi z)}{B(k - zL)} \]  

where the residues are computed at the zeros of \( B(k - zL) \). If we assume a simple form for \( B \) such as \( B(k) = L^{2L} - k^{2L} \)

then the poles and residues are easy to find, and we have a linear system of equations for the coefficients of \( \tilde{A}(k) \). Since \( \tilde{A}_c^x(x) \) is real, take \( \tilde{A}(k) \) real and even for the \( \text{Re} \tilde{f} \), and imaginary and odd for \( \text{Im} \tilde{f} \). \( A(k) \) is of order \( k^{2L - 2} \) ensuring finite power in \( \tilde{f}(k) \). Then, for example

\[ \text{Re} \tilde{f}(k) = \sum_{p=0}^{L} \left\{ \sum_{n=0}^{L} \frac{\pi \cot(\pi (k/L - \phi_n))}{\prod_{m \neq n} (\phi_n - \phi_m)^2} \right\} A_{2p} \]  

where \( \phi_n = e^{i\pi (n+1/2)/L} \), and \( A_{2p} \) is the coefficient of \( k^{2p} \) in \( A(k) \), with a similar formula in \( 2p - 1 \) for \( \text{Im} \tilde{f}(k) \).

4. Outlook

I have attempted to outline the issues surrounding a better interpolation of lattice gauge fields. In the nonabelian case there is a large technical difficulty in unraveling the nonlinearity of parallel transport, which however bears some promise of analytic tractability. On the other issue of gauge equivariance, it would seem we can at best make a good approximation to the continuous Fourier spectrum of the continuum field \( A_\mu(x) \). Notice though that this problem arises due to fact that we’ve used the Fourier basis.

A major development in signal processing are wavelets, which unlike Fourier modes are localized both in frequency and space. Perhaps in an appropriate wavelet basis, the convolution problem might not be so bad. In fact the simple piecewise linear extrapolation is an expansion in the Haar wavelet, one of the simplest examples of a wavelet basis. It is gauge invariant precisely because of its convolution properties. I hope to explore this approach in the future.

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