1 Introduction

The complex general orthogonal group \( GO(r) \) by definition consists of all \( r \times r \) complex matrices \( g \) such that \( t^* gg = cI \) where \( c \) is an invertible scalar. (For any integer \( r \geq 1 \), we will write \( GO(r, \mathbb{C}), GL(r, \mathbb{C}), O(r, \mathbb{C}) \) simply as \( GO(r), GL(r) \) and \( O(r) \), omitting the \( \mathbb{C} \).) A principal \( GO(r) \)-bundle on any base space \( X \) is then equivalent to a non-degenerate triple \( \mathcal{T} = (E, L, b) \) consisting of a rank \( r \) complex vector bundle \( E \), a complex line bundle \( L \) on \( X \) and a symmetric bilinear form \( b : E \otimes E \to L \) on \( E \) taking values in \( L \), such that \( b \) is non-degenerate on each fiber. The vector bundle \( E \) corresponds to the defining representation \( GO(r) \hookrightarrow GL(r) \) of \( GO(r) \), the line bundle corresponds to the character \( \sigma : GO(r) \to \mathbb{C}^* \) defined by the equation \( t^* gg = \sigma(g)I \), and \( b \) is induced by the standard bilinear form \( \sum x_i y_i \) on \( \mathbb{C}^r \). The reverse correspondence is by a Gram-Schmidt process applied locally (étale-locally in the algebraic category). We define a non-degenerate quadric bundle \( Q \) of rank \( r \) on a base \( X \) to be an equivalence class \( [\mathcal{T}] \) of non-degenerate rank \( r \) triples \( \mathcal{T} \), where for any line bundle \( K \) on \( X \), we regard the triples \( \mathcal{T} \) and \( \mathcal{T} \otimes K = (E \otimes K, L \otimes K^2, b \otimes 1_K) \) to be equivalent.

The central result (called the Main Theorem) is proved for cohomology with coefficients in an arbitrary ring. The computational results and examples involving
cohomological classes are formulated and proved for the case of $\mathbb{F}_2 = \mathbb{Z}/(2)$ coefficients. We always work with singular cohomology.

A cohomological **quadratic invariant** $\alpha$ of non-degenerate quadratic bundles of rank $r$ associates to any such quadratic bundle $Q$ on any base $X$ an element $\alpha(Q) \in H^*(X)$, with $f^*(\alpha(Q)) = \alpha(f^*Q) \in H^*(X')$ under any pullback $f : X' \to X$. Let $H^*(BGO(r))$ denote the singular cohomology ring with coefficients $\mathbb{F}_2$ of the classifying space $BGO(r)$ of $GO(r)$. It can be seen that the cohomological invariants of non-degenerate quadratic bundles of rank $r$ bijectively correspond to elements of the so-called **primitive subring** $PH^*(BGO(r)) \subset H^*(BGO(r))$, which consists of all elements $x \in H^*(BGO(r))$ such that $B(\mu)^*(x) = 1 \otimes x$. $B(\mu)^* : H^*(BGO(r)) \to H^*(BC^r) \otimes H^*(BGO(r))$ induced by the multiplication map $\mu : C^* \times GO(r) \to GO(r)$, which corresponds to the operation $T \mapsto T \otimes K$ on triples.

More generally, we consider triples $T = (E, L, b)$ which are not assumed to be non-degenerate, but for which for each $x \in X$, the restriction $b_x : E_x \otimes E_x \to L_x$ to the fiber over $x$ is non-zero. Again defining an equivalence relation $T \sim T \otimes K$, we call an equivalence class $Q = [T]$ as a **quadratic bundle** on $X$. We say that the triple $T$, or the corresponding quadratic bundle $Q$, is **minimally degenerate** if the rank of $b_x$ is constant equal to $r - 1$ for all $x \in X$, where $r = \text{rank}(E)$. Note that such a $Q$ naturally defines a non-degenerate quadratic bundle $\overline{Q}$ of rank $r - 1$ on $X$, where $r = \text{rank}(Q)$. If a triple $(E, L, b)$ represents $Q$, then $\overline{Q}$ is represented by the triple $(\overline{E}, L, \overline{b})$ where $\overline{E} = E/\ker(b)$ and $\overline{b}$ is induced by $b$.

Given a Hausdorff topological space $X$ of the homotopy type of a CW-complex, we say that a closed subspace $Y \subset X$ of the homotopy type of a CW-complex is a **topological divisor** if there exists an open neighbourhood $U$ of $Y$ in $X$, such that $U$ is homeomorphic to the total space of a real rank 2 vector bundle $N$ on $Y$, under a homeomorphism which is identity on $Y$ (where we regard $Y$ as the zero section of $N$). An important example of this is when $X$ is a complex algebraic variety, and $Y \subset X$ is a smooth Weil divisor, which does not meet the singular locus of $X$. Given such a pair $(X, Y)$, we will consider quadratic bundles $Q = [T]$ on $X$ which are non-degenerate on $X - Y$, and are minimally degenerate on $Y$. We will say that such a $Q$ is a **mildly degenerating** quadratic bundle on $(X, Y)$, of **generic rank** $r = \text{rank}(E)$.

This paper is devoted to the following three questions.

(1) Let the coefficients be $\mathbb{F}_2$. What is the cohomology ring $H^*(BGO(r))$, in terms of generators and relations? Under the canonical inclusions $O(r) \hookrightarrow GO(r)$ and $GO(r) \twoheadrightarrow GL(r)$, what are the induced ring homomorphisms $H^*(BGO(r)) \to H^*(BGL(r))$ and $H^*(BGO(r)) \to H^*(BO(r))$?

(2) Let the coefficients be $\mathbb{F}_2$. What are the invariants of non-degenerate quadratic bundles? Describe concretely the primitive subring $PH^*(BGO(r)) \subset H^*(BGO(r))$.

(3) Let the coefficients be in an arbitrary ring $R$. Consider a space $X$ and together with a topological divisor $Y \subset X$. For any mildly degenerating quadratic bundle $Q$ on $(X, Y)$ of generic rank $r$, how are the quadratic invariants of the restriction $Q_{X-Y}$ related to the quadratic invariants of the rank $r-1$ non-degenerate quadratic bundle
and let $Q_T$ be the corresponding primitive ring $\mathbb{Z}/(2)$, so $\ker(b) = \ker(E \to L \otimes E^*)$ is a line subbundle of $E$. Let $\overline{E}$ be the quotient $E/\ker(b)$, and let $\overline{b} : \overline{E} \otimes \overline{E} \to \overline{L}$ be induced by $b$. Note that $(\overline{E}, \overline{L}, \overline{b})$ is a non-degenerate rank $r - 1$ triple, where $r = \text{rank}(E)$, which defines $\overline{Q}$. We define the rank $r - 1$ triple $T^Q$ as

$$T^Q = (\overline{E}, \overline{L}, \overline{b}) \otimes (\ker(b))^{-1}$$

This is well-defined, independent of the initial choice of a representative $(E, L, b)$ for $Q$. In particular, if $Y \subset X$ is a topological divisor and $Q$ is a mildly degenerating quadric bundle on $(X, Y)$ of generic rank $r$, then we get a canonically defined principal $GO(r - 1)$-bundle on $Y$ corresponding to the triple $T^{Q_Y}$.

Next, observe that it follows from the definition of a topological divisor $Y \subset X$ that we have a Gysin boundary map

$$\partial : H^*(X - Y) \to H^{*-1}(Y)$$

which is an additive map, homogeneous of degree $-1$. Our main theorem says that for any quadric invariant $\alpha(Q_{X - Y}) \in H^*(X - Y)$ of the non-degenerate rank $r$
quadric bundle $Q_{X-Y}$, the Gysin boundary $\partial(\alpha(Q_{X-Y})) \in H^*(Y)$ is expressible in terms of the $GO(r-1)$-characteristic classes of the triple $T^{O_Y}$ as follows. First we need some preliminaries.

If $Y \subset X$ is a topological divisor, and $K$ is a line bundle on $X$ together with a section $s \in \Gamma(X-Y, K)$ which is everywhere non-vanishing on $X-Y$, we can define its absolute topological vanishing multiplicity $|\nu_Y(s)| \in H^0(Y; \mathbb{Z})$, and its topological vanishing parity $\overline{\nu}_Y(s) \in H^0(Y; \mathbb{Z})$ (see Section 6). In the algebraic category, this is the usual vanishing multiplicity $\nu_Y(s)$ and its parity, where even is 0 and odd is 1. Given a quadric bundle $Q = [E, L, b]$ on $X$ which is non-degenerate outside a topological divisor $Y \subset X$, the discriminant $\text{det}(b) \in \Gamma(X, L^{\text{rank}(E)} \otimes \text{det}(E)^{-2})$, and its topological vanishing multiplicity $\nu_Y(\text{det}(b))$ along $Y$, are independent of the choice of a triple $(E, L, b)$ that represents the given quadric bundle $Q$, which allows us to denote these simply as $\text{det}(Q) \in \Gamma(X, L^{r} \otimes \text{det}(E)^{-2})$ and as $\overline{\nu}_Y(\text{det}(Q))$.

Let $B(v)^*: H^*(BGO(r)) \to H^{r-1}(BO(r-1))$ be the ring homomorphism induced by the group homomorphism $v: O(r-1) \to GO(r)$ defined by $g \mapsto \begin{pmatrix} 1 \\ g \end{pmatrix}$.

Let $(E, L, b)$ be the universal triple over $BGO(r-1)$. The complement $L_0$ of the zero section of $L$ can be taken to be $BO(r-1)$ (see Section 2), so there is a Gysin boundary map $d: H^*(BO(r-1)) \to H^*(BGO(r-1))$. Finally, let $\delta: PH^*(BGO(r)) \to H^{r-1}(BGO(r-1))$ be the composite linear map

$$PH^*(BGO(r)) \hookrightarrow H^*(BGO(r)) \xrightarrow{B(v)^*} H^*(BO(r-1)) \xrightarrow{d} H^{r-1}(BGO(r-1)).$$

Here the cohomologies have coefficients in some fixed ring $R$. With the above notations, we can now state the following.

**Main Theorem** Let $Q$ be a quadric bundle on a space $X$, generically non-degenerate of rank $r \geq 2$, mildly degenerating over a topological divisor $Y \subset X$. Let $Q_{X-Y}$ be the resulting rank $r$ non-degenerate quadric bundle on $X-Y$, and let $T^{O_Y}$ be the resulting rank $r-1$ non-degenerate canonical triple on $Y$. Let $\alpha \in PH^*(BGO(r))$ be a universal quadric invariant, and let $\alpha(Q_{X-Y}) \in H^*(X-Y)$ be its value on $Q_{X-Y}$. Let $\delta: PH^*(BGO(r)) \to H^{r-1}(BGO(r-1))$ be the linear map defined above, and let $(\delta(\alpha))(T^{O_Y})$ be the value of the resulting $GO(r-1)$-characteristic class $\delta(\alpha)$ on $T^{O_Y}$. Let $\overline{\nu}_Y(\text{det}(Q)) \in H^0(Y, \mathbb{Z})$ be the topological vanishing parity along $Y$ of the discriminant $\text{det}(Q) \in \Gamma(X, L^{r} \otimes \text{det}(E)^{-2})$. Then under the Gysin boundary map $\partial: H^*(X-Y) \to H^{r-1}(Y)$, we have the equality

$$\partial(\alpha(Q_{X-Y})) = \overline{\nu}_Y(\text{det}(Q)) \cdot (\delta(\alpha))(T^{O_Y}).$$

**Sketch of the proof** If $X$ is an algebraic variety with an effective Cartier divisor $Y$, and $E$ is a vector bundle on $X$ with a given line subbundle $E'$ of the restriction $E|_Y$, then recall that the Hecke transform $F$ of $(E, E')$ along $Y$ is the vector bundle (locally free sheaf) on $X$ consisting of germs of all sections of $E$ which when restricted to $Y$ lie in $E'$. Doing what may be described as a topological analog of this procedure,
we first reduce to the case where the absolute topological vanishing multiplicity of \( \det(Q) \) along each component of \( Y \) is either 0 or 1, depending on whether the original absolute topological vanishing multiplicity was even or odd, respectively.

Without loss of generality, one can assume that \( Y \) is connected. When the absolute topological vanishing multiplicity of the discriminant is zero along the divisor \( Y \), we show that topologically \( Q \) admits a non-degenerate prolongation from \( X - Y \) to all of \( X \). In particular, all quadric invariants map to zero under the Gysin boundary map. This is necessarily the case when the real vector bundle \( N \) on \( Y \) (the ‘topological normal bundle’ of \( Y \) in \( X \)) is not orientable, as then the absolute topological vanishing multiplicity of the discriminant is always zero.

So now remains the case where \( |\nu|_Y(\det(Q)) = 1 \). (In the algebraic or complex analytic category, if the base \( X \) and the divisor \( Y \) are non-singular varieties, then the condition \( \nu_Y(\det(b)) = 1 \) on a mildly degenerating algebraic or holomorphic triple \( (E, L, b) \) is equivalent to demanding that the closed subvariety \( V \subset P(E) \) defined by \( b \) is non-singular.) To study the case \( |\nu|_Y(\det(Q)) = 1 \), we construct a particular such quadric bundle \( Q \) which is quasi-universal in the sense that for any \( Q \) on a base \( X \), degenerating over \( Y \), there is a tubular neighbourhood of \( Y \) in \( X \) over which the original quadric bundle may be replaced by a pullback of the quasi-universal bundle \( Q \), for the purpose of the main theorem. By this device, we reduce the problem to understanding the Gysin boundary map \( \partial \) in the case of the quasi-universal quadric bundle \( Q \).

The quasi-universal bundle \( Q \) is constructed as follows, for all \( r \geq 2 \). Let \( (E, L, b) \) be the universal triple over \( BGO(r - 1) \). The total space \( L \) of the line bundle \( \pi : L \to BGO(r - 1) \) serves as the base space for \( Q \), and the zero section \( BGO(r - 1) \subset L \) is the degeneration locus of \( Q \). Let \( \tau \in \Gamma(L, \pi^*(L)) \) be the tautological section of the pullback of \( L \) under \( \pi \), which vanishes with multiplicity 1 along the zero section \( BGO(r - 1) \subset L \). We may regard \( \tau \) as defining a bilinear form \( \tau : O_L \otimes O_L \to \pi^*(L) \) on the trivial line bundle \( O_L \) on \( L \). The direct sum triple \( (O_L, \pi^*(L), \tau) \oplus \pi^*(E, L, b) \) on \( L \) defines the quadric bundle

\[
Q = [O_L \oplus \pi^*(E), \pi^*(L), \tau \oplus \pi^*(b)]
\]

which is the desired quasi-universal mildly degenerating quadric bundle with generic rank \( r \). By the construction of the quasi-universal triple, the theorem follows.

The above theorem tells us how to compute explicitly \( \partial(\alpha(Q)) \), once we know how to compute the Gysin boundary map \( H^*(BO(r - 1)) \to H^{*-1}(BGO(r - 1)) \). We write this Gysin boundary map explicitly in terms of generators of the cohomologies. The answer naturally falls into two cases.

(1) **Odd rank degenerating to even rank** When \( Q \) is non-degenerate of rank \( 2n + 1 \) on \( X - Y \), degenerating to rank \( 2n \) on \( Y \), the required Gysin boundary map \( d : H^*(BO(2n)) \to H^{*-1}(BGO(2n)) \) is explicitly given in Section 4.

(2) **Even rank degenerating to odd rank** When \( Q \) is non-degenerate of rank \( 2n + 2 \) on \( X - Y \), degenerating to rank \( 2n + 1 \) on \( Y \), the required Gysin boundary map \( d : H^*(BO(2n + 1)) \to H^{*-1}(BGO(2n + 1)) \) is already known, as recalled in Section 3.
The “rank 3 degenerating to rank 2” case of the above theorem, where a conic bundle \( Q \) degenerates into a bundle whose fiber is a pair of distinct lines, was considered earlier in [N]. In this case, \( PH^*(BGO(3)) = \mathbb{P}_2[\hat{w}_2, \hat{w}_3] \), while \( H^*(BGO(2)) = \mathbb{P}_2[\lambda, a_1, b_1]/(\lambda a_1) \) (as proved in [H-N]). The invariant \( \hat{w}_2(Q) \in H^2(X - Y) \) is the Brauer class of the \( \mathbb{P}^1 \)-bundle \( Q_{X-Y} \) on \( X - Y \). The restriction \( Q_Y \) defines a 2-sheeted cover of \( Y \), and the quadric invariant \( a_1 \in H^1(Y) \) is the class of this cover.

It was proved in [N] that \( \partial(\hat{w}_2) = \nu_Y(\det(Q))a_1 \), but nothing was proved there about the behavior of the general invariant \( f(\hat{w}_2, \hat{w}_3) \) under \( \partial \). As a consequence of our Main Theorem, we now know \( \partial(f(\hat{w}_2, \hat{w}_3)) \) explicitly. For example, \( \hat{w}_3 \mapsto \nu_Y(\det(Q))a_1^2 \), and \( \hat{w}_2^3 \mapsto \nu_Y(\det(Q))(a_1^3 + a_1b_4) \). This last example is noteworthy, as \( a_1^3 + a_1b_4 \) is a characteristic class for \( GO(2) \) which is not a quadric invariant.

This paper is arranged as follows. The Sections 2 to 4 deal with non-degenerate quadric bundles. In Section 2, the cohomological invariants of non-degenerate quadric bundles in odd ranks (which is the easy case) are described, together with the Gysin boundary map \( d : H^*(BO(2n + 1)) \to H^{*-1}(BGO(2n + 1)) \). In Section 3, which treats even ranks \( r = 2n \), we first describe (following [H-N]) the characteristic classes for \( GO(2n) \) and the Gysin boundary map \( d : H^*(BO(2n)) \to H^{*-1}(BGO(2n)) \). Then we determine the the ring homomorphism \( H^*(BGL(2n)) \to H^*(BGO(2n)) \) induced by the inclusion \( GO(2n) \hookrightarrow GL(2n) \), which should be of independent interest. This in particular allows us to write the action of \( T \hookrightarrow T \otimes K \) on the \( GO(2n) \)-characteristic classes, which gives a computational procedure to decide whether a given characteristic class is quadric invariant. The ring of quadric invariants in ranks 4 and \( 4m + 2 \) is described in the Section 4, making use of the corresponding results of Toda [T] for the orthogonal groups. In Section 5, we recall the properties of the Gysin boundary map that we need, and prove a basic lemma about the Gysin boundary map. In Section 6, we consider the topological behavior of sections of complex line bundles on a space \( X \) which vanish on a topological divisor \( Y \). In Section 7, we establish some basic properties of mildly degenerating triples, and complete the proof our main theorem for Gysin boundary behavior of quadric invariants. In Section 8, we show how the main theorem leads to an algorithm for calculations, and as illustrations we give explicitly the behavior of some of the quadric invariants for the degenerations from ranks 3 to 2, 4 to 3, 5 to 4, and 6 to 5. The paper ends with an Appendix (Section 9) which recalls the required results of Toda.

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2 Quadric invariants in odd ranks

Some generalities for all ranks

All topological spaces will be assumed to be Hausdorff, and of the homotopy type of a CW complex. In this section we assume that all cohomologies will be singular cohomology with coefficients in the field $\mathbb{F}_2 = \mathbb{Z}/(2)$, unless otherwise indicated.

By a Gram-Schmidt argument, isomorphism classes of principal $O(r)$-bundles on a base $X$ are the same as isomorphism classes of pairs $(E, q)$ where $E$ is a vector bundle on $X$ of rank $r$, and $q : E \otimes E \to O_X$ is an everywhere non-degenerate symmetric bilinear form. More generally, isomorphism classes of principal $GO(r)$-bundles on a base $X$ are the same as isomorphism classes of triples $(E, L, b)$ where $E$ is a vector bundle on $X$ of rank $r$, $L$ is a line bundle on $X$, and $b : E \otimes E \to L$ is an everywhere non-degenerate symmetric bilinear form. It follows that isomorphisms $L \to O_X$ are the same as reductions of structure group from $GO(r)$ to $O(r)$. In particular, if $L$ denotes the total space of the line bundle $\pi : L \to X$, then the pullback of $(E, L, b)$ to $L_o = L - X$ (complement of the zero section of $L$) has a canonical reduction of structure group to $O(r)$.

Let the homomorphism $\sigma : GO(r) \to \mathbb{C}^*$ be defined by $t gg = \sigma(g)I$. This fits in the short exact sequence

$$1 \to O(r) \to GO(r) \xrightarrow{\sigma} \mathbb{C}^* \to 1$$

Hence the classifying space $BO(r)$ is the principal $\mathbb{C}^*$-bundle on $BGO(r)$ associated to $\sigma$. Let $(E, L, b)$ be the universal triple on the classifying space $BGO(r)$. Then $L$ is associated to $\sigma$, which (again) shows that the complement $L_o$ of the zero section of $L$ may be identified with $BO(r)$.

Let $\mathbb{Z}/(2) = \{ \pm 1 \}$, and let $\mu : \mathbb{Z}/(2) \times O(r) \to O(r)$ be the multiplication map, sending $(a, g) \mapsto ag$. Similarly, let $\mu : \mathbb{C}^* \times GO(r) \to GO(r)$ be the multiplication map, sending $(a, g) \mapsto ag$. As these maps are group homomorphisms, which commute with the inclusion homomorphisms $\mathbb{Z}/(2) \hookrightarrow \mathbb{C}^*$ and $O(r) \hookrightarrow GO(r)$, we have the following commutative diagram of classifying spaces, where the vertical maps are induced by the inclusions.

$$
\begin{array}{ccc}
B(\mathbb{Z}/(2)) \times BO(r) & \xrightarrow{B(\mu)} & BO(r) \\
\downarrow & & \downarrow \\
BC^* \times BGO(r) & \xrightarrow{B(\mu)} & BGO(r)
\end{array}
$$

Recall that $B(\mathbb{Z}/(2)) = \mathbb{R}P^\infty$, and its cohomology ring is the polynomial ring $\mathbb{F}_2[w]$ where $w \in H^1(B(\mathbb{Z}/(2)))$. Also, $BC^* = \mathbb{C}P^\infty$, and its cohomology ring is the polynomial ring $\mathbb{F}_2[t]$ where $t \in H^2(BC^*)$ is the image under $H^2(BC^*; \mathbb{Z}) \to H^2(BC^*)$ of the first Chern class of the line bundle $O(1)$ on $\mathbb{C}P^\infty$. The inclusion $\mathbb{Z}/(2) \hookrightarrow \mathbb{C}^*$ induces the ring homomorphism $H^*(BC^*) \to H^*(B(\mathbb{Z}/(2)))$ under which $c$ maps to $w^2$. Hence we get the following.

**Remark 2.1** The following diagram of ring homomorphisms is commutative where the first vertical map $\pi^*$ is induced by the projection $\pi : BO(r) \to BGO(r)$, and...
the second vertical map $\theta$ is induced by $\pi^*$ together with $w \mapsto t^2$.

$$
\begin{align*}
H^*(BGO(r)) \xleftarrow{B(\mu)^*} H^*(BGO(r))[t] \\
\pi^* \downarrow \downarrow \theta \\
H^*(BO(r)) \xleftarrow{B(\mu)^*} H^*(BO(r))[w]
\end{align*}
$$

In the rest of this section, we record various facts about non-degenerate triples and quadric bundles in odd ranks, for later use.

**The cohomology ring $H^*(BGO(2n+1))$**

The map $\mathbb{C}^* \times SO(2n+1) \to GO(2n+1)$ defined by $(a,h) \mapsto ah$ is an isomorphism. As $H^*(BC^*) = \mathbb{F}_2[c]$ where $c$ has degree 2, while $H^*(BSO(2n+1)) = \mathbb{F}_2[\hat{w}_2, \ldots, \hat{w}_{2n+1}]$ where $\hat{w}_i \in H^i(BSO(2n+1))$ are the universal special Stiefel-Whitney classes for $SO(2n+1)$, we get an identification

$$
H^*(BGO(2n+1)) = \mathbb{F}_2[c, \hat{w}_2, \ldots, \hat{w}_{2n+1}]
$$

where we write $c$ for $t \otimes 1$, and write $\hat{w}_i$ for $1 \otimes \hat{w}_i$.

**Quadric invariants in odd rank**

**Proposition 2.2** The ring of quadric invariants in any odd rank $2n+1$ is the subring $H^*(BSO(2n+1)) = \mathbb{F}_2[\hat{w}_2, \ldots, \hat{w}_{2n+1}]$ of $H^*(BGO(2n+1))$.

**Gysin sequence for** $BO(1) \to BC^*$

The character $2\chi : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto z^2$ has kernel $O(1) = \{\pm 1\} \subset \mathbb{C}^*$, hence the classifying space $BO(1)$ can be taken to be the complement of the zero section of the line bundle $O(2)$ on $BC^* = \mathbb{CP}^\infty$ defined by the character $2\chi$. The Euler class of $O(2)$ is $0 \in H^2(BC^*)$ as $2 = 0$ in $\mathbb{F}_2$. Hence the long exact Gysin sequence for the principal $\mathbb{C}^*$-bundle $\pi : BO(1) \to BC^*$ splits to give short exact sequences

$$
0 \to H^i(BC^*) \xrightarrow{\pi_*} H^i(BO(1)) \xrightarrow{d} H^{i-1}(BC^*) \to 0
$$

For all $i \geq 0$, $H^{2i}(BC^*) = \{0, c^i\}$, $H^{2i+1}(BC^*) = 0$, and $H^i(BO(1)) = \{0, w^i\}$, where we take $c^0 = 1$ and $w^0 = 1$. Hence the above short exact Gysin sequences implies that

$$
\pi^*(c^i) = w^{2i}, \quad d(w^{2i}) = 0, \quad \text{and} \quad d(w^{2i+1}) = c^i,
$$

for all $i \geq 0$.

**Gysin sequence for** $BO(2n+1) \to BGO(2n+1)$

The classifying space $BO(2n+1)$ has cohomology ring $\mathbb{F}_2[w_1, \ldots, w_{2n+1}]$, the polynomial ring in the Stiefel-Whitney classes $w_i$. With $O(1) = \{\pm 1\} \subset \mathbb{C}^*$, we have an isomorphism $O(1) \times SO(2n+1) \to O(2n+1) : (a,h) \mapsto ah$, which gives an isomorphism $BO(1) \times BSO(2n+1) \to BO(2n+1)$. In our previous notation,
$H^*(BO(1)) = \mathbb{F}_2[w]$ and $H^*(BSO(2n+1)) = \mathbb{F}_2[\hat{w}_2, \ldots, \hat{w}_{2n+1}]$, so taking tensor product,

\[ H^*(BO(2n+1)) = \mathbb{F}_2[w, \hat{w}_2, \ldots, \hat{w}_{2n+1}] \]

Note that if $s_1, \ldots, s_m$ are the elementary symmetric functions in the variables $x_1, \ldots, x_m$, and $u$ is another variable, then we have

\[ s_r(u + x_1, \ldots, u + x_m) = \sum_{0 \leq i \leq r} \binom{m - i}{r - i} u^{r-i} s_i(x_1, \ldots, x_m) \]

Hence by the splitting principle we get the following.

**Lemma 2.3** In $H^*(BO(2n+1)) = \mathbb{F}_2[w_1, \ldots, w_{2n+1}] = \mathbb{F}_2[w, \hat{w}_2, \ldots, \hat{w}_{2n+1}]$, we have the identities

\[
\begin{align*}
  w_1 &= w, \\
  w_r &= \binom{2n+1}{r} w^r + \sum_{2 \leq i \leq r} \binom{2n+1-i}{r-i} w^{r-i} \hat{w}_i \quad \text{for } 2 \leq r \leq 2n+1, \\
  \hat{w}_r &= \binom{2n}{r} w^r + \sum_{2 \leq i \leq r} \binom{2n+1-i}{r-i} w^{r-i} w_i \quad \text{for } 2 \leq r \leq 2n+1.
\end{align*}
\]

**Lemma 2.4** For the $\mathbb{C}^*$-fibration $\pi : BO(2n+1) \to BGO(2n+1)$, we have the following.

(i) The Euler class is zero.

(ii) The ring homomorphism $\pi^* : H^*(BGO(2n+1)) \to H^*(BO(2n+1))$ is given in terms of generators by

\[
\begin{align*}
  \pi^*(c) &= w_1^2, \\
  \pi^*(\hat{w}_r) &= \binom{2n}{r} w_1^r + \sum_{2 \leq i \leq r} \binom{2n+1-i}{r-i} w_1^{r-i} w_i \quad \text{for } 2 \leq r \leq 2n+1.
\end{align*}
\]

(iii) The Gysin boundary map $d : H^*(BO(2n+1)) \to H^{*-1}(BGO(2n+1))$ has the following expression. Using the identities given by Lemma 2.3, any element of $H^*(BO(2n+1)) = \mathbb{F}_2[w_1, \ldots, w_{2n+1}]$ can be uniquely expressed as a polynomial $\sum w^i f_i(\hat{w}_2, \ldots, \hat{w}_{2n+1})$, where the $f_i$ are polynomials in $2n$ variables. Then the additive group homomorphism $d$ is given by

\[
d(\sum_{i \geq 0} w^i f_i(\hat{w}_2, \ldots, \hat{w}_{2n+1})) = \sum_{j \geq 0} c^j f_{2j+1}(\hat{w}_2, \ldots, \hat{w}_{2n+1})
\]
Proof: The isomorphisms $O(1) \times SO(2n+1) \to O(2n+1)$ : $(a, h) \mapsto ah$ and $C^* \times SO(2n+1) \to GO(2n+1)$ : $(a, h) \mapsto ah$ fit in the commutative square

$$
\begin{array}{ccc}
O(1) \times SO(2n+1) & \to & O(2n+1) \\
\downarrow & & \downarrow \\
C^* \times SO(2n+1) & \to & GO(2n+1)
\end{array}
$$

Hence by the previous calculation of the Gysin sequence for $BO(1) \to BC^*$, the Gysin sequence for $BO(2n+1) \to BGO(2n+1)$ breaks into short exact sequences

$$0 \to H^i(BGO(2n+1)) \xrightarrow{\pi^*} H^i(BO(2n+1)) \xrightarrow{d} H^{i-1}(BGO(2n+1)) \to 0$$

with $\pi^*(c) = w^2$, $\pi^*(\hat{w}_r) = \hat{w}_r$ for all $r \geq 2$, $d(w^2j) = 0$ for all $j \geq 0$, $d(w^{2j+1}) = c^j$ for all $r \geq 0$, and $d(f(\hat{w}_2, \ldots, \hat{w}_{2n+1})) = 0$ for any polynomial $f$ in the variables $\hat{w}_2, \ldots, \hat{w}_{2n+1}$.

As $f(\hat{w}_2, \ldots, \hat{w}_{2n+1})$ is pulled back from $BGO(2n+1)$, it follows from Lemma 5.1 that $d(w^4f(\hat{w}_2, \ldots, \hat{w}_{2n+1})) = d(w^2)f(\hat{w}_2, \ldots, \hat{w}_{2n+1})$, which completes the proof. □

Example 2.5 In particular $d(w_1) = 1$, and for all $r \geq 1$ we have $d(w_{2r}) = 0$ and $d(w_{2r+1}) = (\frac{2n}{2r}) c^r + \sum_{1 \leq j \leq r} (\frac{2n-2j}{2r-2j}) c^{r-j} \hat{w}_{2j}$.

Remark 2.6 The ring $H^*(BGL(2n+1); \mathbb{Z})$ equals $\mathbb{Z}[c_1, \ldots, c_{2n+1}]$ where the $c_i$ are the Chern classes, therefore modulo 2 we have $H^*(BGL(2n+1)) = \mathbb{F}_2[c_1, \ldots, c_{2n+1}]$ with $\tau_i \in H^{2i}(BGL(2n+1))$. Under the map $H^*(BGL(2n+1)) \to H^*(BGO(2n+1))$, it can be seen that

$$
\tau_r \mapsto \left(\binom{2n+1}{r}\right) c^r + \sum_{2 \leq i \leq r} \binom{m-i}{r-i} c^{r-i} \hat{w}_i^2
$$

3 Characteristic classes for $GO(2n)$

Generators $\lambda$, $a_{2i-1}$, $b_{4j}$, $d_t$ of $H^*(BGO(2n))$, and relations.

The cohomology ring $H^*(BGO(2n))$ with coefficients $\mathbb{F}_2$ has been explicitly determined in terms of generators and relations in [H-N], which we recall. Let $(\mathbf{E}, \mathbf{L}, \mathbf{b})$ denote the universal triple on $BGO(2n)$. The characteristic class $\lambda \in H^2(BGO(2n))$ is by definition the Euler class of $\mathbf{L}$.

The character $\sigma : GO(2n) \to C^*$ defined by $\check{g}g = \sigma(g)I$ has kernel $O(2n)$, and $\mathbf{L}$ is associated to the character $\sigma$, hence $BO(2n)$ can be taken to be the complement $\mathbf{L}_o$ of the zero section of the line bundle $\mathbf{L}$. For the principal $C^*$-bundle $\pi : BO(2n) \to BGO(2n)$, consider the long exact Gysin sequence

$$\cdots \to H^i(BGO(2n)) \xrightarrow{\pi^*} H^i(BO(2n)) \xrightarrow{d} H^{i-1}(BGO(2n)) \xrightarrow{\lambda} H^{i+1}(BGO(2n)) \xrightarrow{\pi^*} \cdots$$

For each $1 \leq j \leq n$, we put

$$a_{2j-1} = dw_{2j} \in H^{2j-1}(BGO(2n)).$$
More generally, for any subset \( T = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\} \) of cardinality \( r \geq 2 \), let 
\[ v_T = w_{2i_1} \cdots w_{2i_r} \in H^{2 \deg(T)}(BO(2n)) \]
where \( \deg(T) = i_1 + \cdots + i_r \). We put
\[ d_T = d(v_T) \in H^{2 \deg(T) - 1}(BGO(2n)). \]

We recall from [H-N] that the composite \( \pi^*d : H^*(BO(2n)) \to H^{*-1}(BO(2n)) \) is the derivation \( s = \sum w_{2i_1} \frac{\partial}{\partial w_{2i_1}} \). Hence we have
\[ \pi^*(a_{2j-1}) = w_{2j-1} \quad \text{and} \quad \pi^*(d_T) = \sum_{i \in T} w_{2i-1} v_{T-\{i\}} \quad \text{in} \quad H^*(BO(2n)). \]

Finally, let \( c_i(E) \in H^{2i}(BGL(2n); \mathbb{Z}) \) denote the \( i \) th Chern class of the vector bundle \( E \), and let \( \overline{c}_i(E) \in H^{2i}(BGL(2n)) \) be its image under the coefficient map \( \mathbb{Z} \to \mathbb{F}_2 \). We put
\[ b_{ij} = \overline{c}_{2j}(E) \in H^{4j}(BGO(2n)) \quad \text{for each} \quad 1 \leq j \leq n. \]

These satisfy
\[ \pi^*(b_{ij}) = w_{2j}^2 \quad \text{for each} \quad 1 \leq j \leq n. \]

As shown in Theorem 3.3.5 of [H-N], these elements \( \lambda, a_{2i-1}, b_{ij}, \) and \( d_T \) generate the ring \( H^*(BGO(2n)) \), with relations given as follows.

**Theorem 3.1** For any \( n \geq 1 \), the cohomology ring of \( BGO(2n) \) with coefficients \( \mathbb{F}_2 \) is the quotient
\[ H^*(BGO(2n)) = \mathbb{F}_2[\lambda; (a_{2i-1})_i; (b_{4i})_i; (d_T)_T] / I \]
where \( I \) is the ideal generated by the elements \( \lambda a_{2i-1} \) for \( 1 \leq i \leq n \), the elements \( \lambda d_T \) where \( T \subset \{1, \ldots, n\} \) is a subset of cardinality \( |T| \geq 2 \), the elements \( \sum_{i \in T} a_{2i-1} d_T-\{i\} \) where \( T \subset \{1, \ldots, n\} \) with \( |T| \geq 3 \), the elements \( (d_{\{i,j\}})^2 + a_{2i-1} b_{ij} + a_{2j-1} b_{4i} \) where \( \{i, j\} \subset \{1, \ldots, n\} \) is a subset of cardinality 2, and the elements \( d_T d_U + \sum_{p \in T \cap U \subset \{1, \ldots, n\}} a_{2p-1} b_{4i} d_{\{T-(p)\} \Delta U} \) where \( T \neq U \) in case \( |T| = |U| = 2 \), and where \( \Delta \) denotes the symmetric difference of sets.

**The map** \( \pi^* : H^*(BGO(2n)) \to H^*(BO(2n)) \)

The pullback map \( \pi^* : H^*(BGO(2n)) \to H^*(BO(2n)) \) satisfies the following equations, as shown in [H-N].
\[ \pi^*(\lambda) = 0, \]
\[ \pi^*(a_{2i-1}) = w_{2i-1} \quad \text{for all} \quad 1 \leq i \leq n, \]
\[ \pi^*(b_{4i}) = w_{2i}^2 \quad \text{for all} \quad 1 \leq i \leq n, \]
\[ \pi^*(d_T) = \sum_{i \in T} w_{2i-1} v_{T-\{i\}} \quad \text{for all} \quad T \subset \{1, \ldots, n\} \text{ with} \quad |T| \geq 2, \]

where \( v_J = \prod_{j \in J} w_{2j} \) for any subset \( J \subset \{1, \ldots, n\} \).
Moreover, the map $\pi^*: H^{2i+1}(BGO(2n)) \to H^{2i+1}(BO(2n))$ is injective for each $i \geq 0$, by Remark 3.3.4 of [H-N]. Hence the Gysin sequence breaks into exact sequences

$$H^{2i}(BGO(2n)) \xrightarrow{\pi^*} H^{2i}(BO(2n)) \xrightarrow{d} H^{2i-1}(BGO(2n)) \to 0$$

and

$$0 \to H^{2i+1}(BGO(2n)) \xrightarrow{\pi^*} H^{2i+1}(BO(2n)) \xrightarrow{d} H^{2i}(BGO(2n))$$

**Gysin boundary map** $d : H^*(BO(2n)) \to H^{*-1}(BGO(2n))$

The Gysin boundary map $d : H^*(BO(2n)) \to H^{*-1}(BGO(2n))$ satisfies the following equations, as shown in [H-N].

$$d(w_{2i-1}) = 0 \text{ for all } 1 \leq i \leq n, \text{ as } w_{2i-1} = \pi^*(a_{2i-1}),$$

$$d(w_{2i}^2) = 0 \text{ for all } 1 \leq i \leq n, \text{ as } w_{2i}^2 = \pi^*(b_{4i}),$$

$$d(w_{2i}) = a_{2i-1} \text{ for all } 1 \leq i \leq n,$$

$$d(w_{2i_1} \cdots w_{2i_r}) = d_T \text{ for all } T = \{i_1 < \ldots < i_r\} \subset \{1, \ldots, n\}.$$

Moreover, by Lemma 5.1 proved later, it follows that for any $x \in H^*(BGO(2n))$ and $y \in H^*(BO(2n))$, we have $d(\pi^*(x)y) = \pi^*(x)dy$. Hence $d : H^*(BO(2n)) \to H^{*-1}(BGO(2n))$ is determined for any monomial in the $w_i$'s as follows. Any such monomial can be uniquely written as a product $f(w_{2i-1}) \cdot g(w_{2i}^2) \cdot w_{2i_1} \cdots w_{2i_r}$, where $f$ and $g$ are monomials in $n$ variables, and $1 \leq i_1 < i_2 < \ldots < i_r \leq n$. Then we have

$$d(f(w_{2i-1}) \cdot g(w_{2i}^2) \cdot w_{2i_1} \cdots w_{2i_r}) = \begin{cases} 0 & \text{if } r = 0 \\ f(a_{2i-1}) \cdot g(b_{4i}) \cdot a_{2i_1-1} & \text{if } r = 1 \\ f(a_{2i-1}) \cdot g(b_{4i}) \cdot d_T & \text{if } r \geq 2, \text{ where} \\ T = \{i_1 < \ldots < i_r\}. \end{cases}$$

This completes our description of the Gysin boundary map for $BO(2n) \to BGO(2n)$.

**The map** $H^*(BGL(2n)) \to H^*(BGO(2n))$

The determination of the ring homomorphism $H^*(BGL(2n)) \to H^*(BGO(2n))$, which is carried out in the rest of this section, is needed in order to determine the quadric invariants.

The cohomology ring $H^*(BGL(2n))$ is the polynomial ring $\mathbb{F}_2[\overline{c}_1, \ldots, \overline{c}_{2n}]$ where the $2n$ variables $\overline{c}_r$ are the Chern classes mod 2. By the definition of the elements $b_{4r} \in H^{4r}(BGO(2n))$ in terms of the universal triple $(E, L, b)$, we have $\overline{c}_{2r}(E) = b_{4r}$ for $1 \leq i \leq n$. Hence under the ring homomorphism $H^*(BGL(2n)) \to H^*(BGO(2n))$ induced by the inclusion $GO(2n) \hookrightarrow GL(2n)$, we have $\overline{c}_{2r} \mapsto b_{4r}$. The following proposition gives the images of the odd Chern classes $\overline{c}_{2r-1}$ in $H^*(BGO(2n))$, which completes the determination of the ring homomorphism $H^*(BGL(2n)) \to H^*(BGO(2n))$.

**Proposition 3.2** Consider the lower triangular $n \times n$ matrix $A$ over $\mathbb{F}_2[\lambda^2]$, with all diagonal entries 1, and below-diagonal entries given by

$$A_{r,k} = \binom{n-k}{2r-2k} \lambda^{2r-2k} \text{ for } r > k.$$
Let $B = A^{-1}$ be its matrix inverse over the ring $\mathbb{F}_2[\lambda^2]$. For $1 \leq r \leq n$, let the polynomial $f_{n,r}(\lambda, b_4, \ldots, b_{4r-4})$ be defined by

$$f_{n,r} = \binom{n}{2r-1} \lambda^{2r-2} + \sum_{1 \leq k \leq r-1} \binom{n-k}{2r-1-2k} \lambda^{2r-2-2k} \left( \sum_{1 \leq j \leq k} B_{k,j}(b_{4j} - \binom{n}{2j} \lambda^{2j}) \right)$$

Then in the cohomology ring $H^*(BGO(2n))$, we have

$$\overline{c}_{2r-1}(E) = a_{2r-1}^2 + \lambda \cdot f_{n,r}(\lambda, b_4, \ldots, b_{4r-4})$$

**Proof** We divide the proof into two steps, (a) and (b).

(a) For each $1 \leq r \leq n$, there exists a unique polynomial $g_r(\lambda, b_4, \ldots, b_{4r-4})$ such that $\overline{c}_{2r-1}(E) = a_{2r-1}^2 + \lambda \cdot g_r(\lambda, b_4, \ldots, b_{4r-4})$.

(b) The polynomials $g_r(\lambda, b_4, \ldots, b_{4r-4})$ are the polynomials $f_{n,r}(\lambda, b_4, \ldots, b_{4r-4})$ defined in the statement of the proposition.

**Proof of (a)** The composite homomorphism $O(2n) \hookrightarrow GO(2n) \hookrightarrow GL(2n)$ induces $H^*(BGL(2n)) \to H^*(BGO(2n)) \to H^*(BO(2n))$ under which $\overline{c}_i \mapsto w_i^2$. As $a_{2i-1}^2 \mapsto w_{2i-1}^2$ under $H^*(BGO(2n)) \to H^*(BO(2n))$, we must have

$$\overline{c}_{2i-1}(E) = a_{2i-1}^2 + z_{4i-2}$$

where $z_{4i-2}$ lies in the kernel of $\pi^*: H^{4i-2}(BGO(2n)) \to H^{4i-2}(BO(2n))$. By exactness of Gysin, we have $z_{4i-2} = \lambda y_{4i-4}$ for some $y_{4i-4} \in H^{4i-4}(BGO(2n))$. Now, by the structure of the ring $H^*(BGO(2n))$, we know that $\lambda$ annihilates the $a_{2j-1}$’s and the $d_T$’s. Hence we can replace $y_{4i-4}$ by a polynomial $g_i(\lambda, b_{4j})$ in $\lambda$ and the $b_{4j}$’s. As the variables $\lambda$ and the $b_{4j}$ are algebraically independent, $g_i$ is unique. By degree considerations, the highest $b_{4j}$ that occurs in $g_i$ can be at most $b_{4i-4}$. This completes the proof of (a).

**Proof of (b)** We first determine the polynomial $g_i(\lambda, b_4, \ldots, b_{4i-4})$ in the following Example 3.3 of a triple $\mathcal{T}$.

**Example 3.3** Let $\mathcal{O}(1)$ be the universal line bundle on $BC^* = \mathbb{P}^\infty_C$. Let $X = BC^* \times \ldots \times BC^*$ be the product of $n + 1$-copies, and let $p_i: X \to BC^*$ be the projections for $0 \leq i \leq n$. Let $L = p_{\ast}^\ast(\mathcal{O}(1))$, and for $1 \leq i \leq n$ let $K_i = p_i^{\ast}(\mathcal{O}(1))$. On $X$, we get the non-degenerate triples $\mathcal{T}_i = ((L \otimes K_i) \oplus K_i^{-1}, L, b_i)$ where $b_i$ is induced by the canonical isomorphism $(L \otimes K_i) \otimes K_i^{-1} \sim L$. Let $\mathcal{T} = (E, L, b)$ be the direct sum triple $\oplus \mathcal{T}_i$ (see Definition 7.2). We now write down the characteristic classes of $\mathcal{T}$. Let $\lambda = \overline{c}_1(L)$. Then by definition, $\lambda(\mathcal{T}) = \lambda$. As the odd cohomologies of $X$ are zero, the classes $a_{2i-1}(\mathcal{T})$ and $d_T(\mathcal{T})$ are zero for all $1 \leq i \leq n$ and for all $T \subset \{1, \ldots, n\}$ with $|T| \geq 2$.

Let $x_i = \overline{c}_1(K_i)$ for $1 \leq i \leq n$, and let $s_k(\lambda x_1 + x_2^2)$ denote the $k$ th elementary symmetric polynomial in the variables $\lambda x_1 + x_2^2, \ldots, \lambda x_n + x_n^2$. As $E$ is the direct
As \( b_{4r}(T) = \overline{c}_{2r}(E) \), we have the following equations for \( 1 \leq r \leq n \).

\[
b_{4r} - \left( \begin{array}{c} n \\ 2r \end{array} \right) \lambda^{2r} = \sum_{1 \leq k \leq r} \left( \begin{array}{c} n - k \\ 2r - 2k \end{array} \right) \lambda^{2r - 2k}s_k(\lambda x_i + x_i^2)
\]

This is a system of linear equations with coefficients in \( \mathbb{F}_2[\lambda^2] \), for the \( b_{4r} - \left( \begin{array}{c} n \\ 2r \end{array} \right) \lambda^{2r} \) in terms of the \( s_k(\lambda x_i + x_i^2) \). It is given by the \( n \times n \) matrix \( A \) over \( \mathbb{F}_2[\lambda^2] \), with entries

\[
A_{r,k} = \left( \begin{array}{c} n - k \\ 2r - 2k \end{array} \right) \lambda^{2r - 2k}
\]

This is a lower triangular matrix, with all diagonal entries equal to 1, so is invertible over the ring \( \mathbb{F}_2[\lambda^2] \). Let \( B = A^{-1} \) be its matrix inverse. Hence we get

\[
s_r(\lambda x_i + x_i^2) = \sum_{1 \leq k \leq r} B_{r,k}(b_{4k} - \left( \begin{array}{c} n \\ 2k \end{array} \right) \lambda^{2k})
\]

Substituting this in the equation for \( \overline{c}_{2r-1}(E) \), we get equations

\[
\overline{c}_{2r-1}(E) = \lambda \cdot f_{n,r}(\lambda, b_4, \ldots, b_{4r-4})
\]

where

\[
f_{n,r} = \left( \begin{array}{c} n \\ 2r - 1 \end{array} \right) \lambda^{2r - 2} + \sum_{k=1}^{r-1} \lambda^{2r - 2k} \left( \sum_{1 \leq j \leq k} B_{k,j}(b_{4j} - \left( \begin{array}{c} n \\ 2j \end{array} \right) \lambda^{2j}) \right)
\]

**Proof of (b) continued** In the Example 3.3, we have the desired equality \( g_i = f_{n,i} \). Note that the cohomology ring \( H^*(X) \) is the polynomial ring \( \mathbb{F}_2[\lambda, x_1, \ldots, x_n] \), in which the \( n + 1 \) elements \( \lambda, b_4, \ldots, b_{4n} \) are algebraically independent, where \( b_{4r} = \left( \begin{array}{c} n \\ 2r \end{array} \right) \lambda^{2r} + \sum_{1 \leq k \leq r} \left( \begin{array}{c} n - k \\ 2r - 2k \end{array} \right) \lambda^{2r - 2k}s_k(\lambda x_i + x_i^2) \). Hence as \( g_i = f_{n,i} \) in this example, we get \( g_i = f_{n,i} \) universally.

This completes the proof of Proposition 3.2. \( \square \)

**Example 3.4** The above proposition in particular gives the following identities in \( H^*(BGO(2n)) \).

\[
\overline{c}_1 = a_4 + n \lambda \text{ for all } n \geq 1, \text{ and } \\
\overline{c}_3 = a_4^2 + \frac{n(n-1)(2n-1)}{6} \lambda^3 + (n-1)\lambda b_4 \text{ for all } n \geq 2.
\]
4 Quadric invariants in even ranks

The ring homomorphism $B(\mu)^* : H^*(BGO(2n)) \to H^*(BGO(2n))[t]$

Proposition 4.1 The ring homomorphism

$B(\mu)^* : H^*(BGO(2n)) \to H^*(B\mathbb{C}^* \otimes BGO(2n)) = H^*(BGO(2n))[t]$

induced by the group homomorphism $\mu : \mathbb{C}^* \times GO(2n) \to GO(2n)$ which sends $(a, g) \mapsto ag$, is given in terms of the generators of $H^*(BGO(2n))$ as follows.

$B(\mu)^* \lambda = \lambda$

$B(\mu)^* b_4 = b_4 + (a_1^2 + n\lambda)t + nt^2$

$B(\mu)^* b_{4r} = \sum_{i=1}^{r} \left( \frac{2n - 2i}{2r - 2i} \right) (b_{4i} + \lambda f_{n,i}t + a_{2i-1}^2 t) t^{2r-2i} + \left( \frac{2n}{2r} \right) t^{2r}$

where the elements $f_{n,i}$ are as in Proposition 3.2.

$B(\mu)^* a_1 = a_1$

$B(\mu)^* a_{2r-1} = \sum_{i=1}^{r} \left( \frac{2n + 1 - 2i}{2r - 2i} \right) a_{2i-1} t^{r-i}$

$B(\mu)^* d_{(1,2)} = d_{(1,2)} + na_3 t + \left( \frac{n}{2} \right) a_1 t^2$

$B(\mu)^* d_{(p,q)} = \sum_{i=1}^{p} \left( \frac{2n}{2q} \right) \sum_{i=1}^{q} \left( \frac{2n - 2i}{2p - 2i} \right) a_{2i-1} t^{p+q-i} + \sum_{j=1}^{q} \left( \frac{2n}{2p} \right) \sum_{j=1}^{q} \left( \frac{2n - 2j}{2q - 2j} \right) a_{2j-1} t^{p+q-j}$

$+ \sum_{i=1}^{p} \sum_{j=1}^{q} t^{p+q-i-j} d_{i,j}$ where by convention $d_{k,\ell} = 0$ for $k = \ell$.

(We do not give a closed formula for $B(\mu)^* d_T$ for a general $T$, but explain in the course of the proof how to explicitly compute $B(\mu)^* d_T$ for any given $T$.)

Proof If $T = (E, L, b)$ is a rank $2n$ triple, and $K$ any line bundle, then $c_1(L \otimes K^2) = c_1(L) + 2c_1(K)$ for the Chern class $c_1$. Hence $\overline{c}_1(L \otimes K^2) = \overline{c}_1(L)$ in $\mathbb{F}_2$ coefficients. This gives $B(\mu)^* \lambda = \lambda$.

Next we determine $B(\mu)^* b_{4r}$, using the expression $\overline{c}_{2i}(E) = b_{4i}$ and the expression for $\overline{c}_{2j-1}(E)$ given by the Proposition 3.2.

$\overline{c}_{2r}(E \otimes K)$

$= \sum_{p=0}^{2r} \left( \frac{2n - p}{2r - p} \right) \overline{c}_1(K)^{2r-p} \overline{c}_p(E)$

$= \sum_{i=0}^{r} \left( \frac{2n - 2i}{2r - 2i} \right) \overline{c}_1(K)^{2r-2i} \overline{c}_{2i}(E) + \sum_{j=1}^{r} \left( \frac{2n - 2j + 1}{2r - 2j + 1} \right) \overline{c}_1(K)^{2r-2j+1} \overline{c}_{2j-1}(E)$

$= \sum_{i=0}^{r} \left( \frac{2n - 2i}{2r - 2i} \right) \overline{c}_1(K)^{2r-2i} b_{4i} + \sum_{j=1}^{r} \left( \frac{2n - 2j + 1}{2r - 2j + 1} \right) \overline{c}_1(K)^{2r-2j+1} (a_{2j-1}^2 + \lambda f_{n,j})$
From this, the expression for \( B(\mu)^* b_{4r} \) follows, using standard binomial identities modulo 2.

Note that for any \( j \geq 0 \), any element of \( H^{2j-1}(BGO(2n)) \) is a sum of terms, each of which has an \( a_{2i-1} \) or a \( d_T \) as a factor, and so is killed by \( \lambda \), so the map \( \lambda : H^{2j-1}(BGO(2n)) \to H^{2j+1}(BGO(2n)) \) is identically zero, and hence \( \pi^* : H^{2j+1}(BGO(2n)) \to H^{2j+1}(BO(2n)) \) is injective. As \( t \) has even degree, it follows that the graded ring homomorphism \( \theta : H^*(BGO(2n))[t] \to H^*(BO(2n))[w] \) of Remark 2.1 is injective on each graded piece of \( H^*(BGO(2n))[t] \) of odd degree. It therefore follows from the commutative diagram in Remark 2.1 that one can calculate \( B(\mu)^* a_{2r-1} \) and \( B(\mu)^* d_T \) by determining the images of \( w_{2r-1} = \pi^*(a_{2r-1}) \) and of \( \sum_{i \in T} w_{2i-1} v_{T-i} = \pi^*(d_T) \) under \( B(\mu)^* : H^*(BO(2n)) \to H^*(BO(2n))[w] \) and then expressing them in terms of the generators (namely, \( \pi^*(a_{2i-1}), \pi^*(d_S), \pi^*(b_{4i}), \pi^*(b_{4i}) \), and \( \theta(t) = w^2 \) of the sub-algebra \( \theta(H^*(BGO(2n))[t]) \subset H^*(BO(2n))[w] \). This calculation is short in the case of the \( a_{2r-1} \), and in the case of \( d_T \) it can be done by a standard algorithm using Grobner bases. As an example, we have given the answer when \( T \) has cardinality 2.

\[ \square \]

The invariants \( \lambda \) and \( a_1 \)

The elements \( \lambda \) and \( a_1 \) of \( H^*(BGO(2n)) \) are in \( PH^*(BGO(2n)) \) for all even ranks \( 2n \). The element \( a_1 \) is commonly known as the ‘discriminant’, and is associated to the character \( \psi : GO(2n) \to \{ \pm 1 \} \) defined by \( g \mapsto \sigma(g)^n/\det(g) \) where recall that \( \sigma : GO(2n) \to \mathbb{C}^* \) was defined by the equality \( 'gg = \sigma(g)I \). It also has a well-known Gauss-Manin description in terms of the fibration \( V \to X \) where \( V \) is the subvariety of \( P(E) \) defined by the vanishing of \( b \). (So, \( V \) is a quadric bundle in the original sense).

The ring of quadric invariants in rank 2

In rank 2, we have \( B(\mu)^* \lambda = \lambda \), \( B(\mu)^* a_1 = a_1 \), and \( B(\mu)^* b_4 = b_4 + (a_1^2 + \lambda) t + t^2 \). Hence the ring of quadric invariants is

\[
PH^*(BGO(2)) = \mathbb{F}_2[\lambda, a_1, b_4] / (\lambda a_1) = H^*(BGO(2))
\]

The ring of quadric invariants in rank 4

Generators for the subring \( PH^*(BO(4)) \subset H^*(BO(4)) \) of orthogonal quadric invariants have been given by Toda in Proposition 3.12 of [T], which we recall (correcting a minor misprint).

**Proposition 4.2** (Toda [T] Proposition 3.12) The elements \( w_1, w_2^2 + w_1 w_3 \), and \( w_1 w_2 w_3 + w_2^2 w_4 + w_1^2 w_4 \) generate the subring \( PH^*(BO(4)) \subset H^*(BO(4)) \).

To give a set of generators for the ring of quadric invariants \( PH^*(BGO(4)) \subset H^*(BGO(4)) \), we combine the above and Remark 2.1, with the following table of the action of \( B(\mu)^* \) in rank 4 which follows from Proposition 3.1.

\[
B(\mu)^* \lambda = \lambda
\]
Proposition 4.3

The ring \( \text{PH}^*(BGO(4)) \subset H^*(BGO(4)) \) of quadric invariants in rank 4 is generated by the elements \( \lambda, a_1, a_1a_3 + b_4, \) and \( a_3^2 + a_1d_{1,2} \).

**Proof.** Let \( P' \) be the subring of \( H^*(BGO(4)) \) generated by the elements \( \lambda, a_1, a_1a_3 + b_4, \) and \( a_3^2 + a_1d_{1,2} \). We want to show that \( P' = \text{PH}^*(BGO(4)) \). From the above table of the effect of \( B(\mu)^* \), it follows that these elements are indeed quadric invariants, so \( P' \subset \text{PH}^*(BGO(4)) \). Further note that \( \pi^*(\lambda) = 0 \in H^*(BO(4)) \), while the images under \( \pi^* \) of \( x = a_1, y = a_1a_3 + b_4, \) and \( z = a_3^2 + a_1d_{1,2} \) are respectively the generators \( x' = w_1, y' = w_2^3 + w_1w_3, \) and \( z' = w_1w_2w_3 + w_3^2 + w_1^2w_4 \) of the subring \( \text{PH}^*(BO(4)) \subset H^*(BO(4)) \). Let \( g \in H^*(BGO(4)) \) be any quadric invariant. By the Remark [23], its image in \( H^*(BO(4)) \) is an orthogonal quadric invariant, so is expressible as a polynomial \( p(x', y', z') \) by Proposition 4.2. Hence \( h = g - p(x, y, z) \) lies in the kernel of \( \pi^*: H^*(BGO(4)) \rightarrow H^*(BO(4)) \).

We claim that

\[
\ker(\text{PH}^*(BGO(4)) \rightarrow \text{PH}^*(BO(4))) = \lambda k[\lambda, b_4]
\]

If the claim is true then to prove the proposition we have to show that for any polynomial \( p(\lambda, b_4) \), the element \( \lambda p(\lambda, b_4) \) lies in \( P' \). This holds for the monomials of the form \( \lambda^ib_4^j \) for any \( i > 0 \) and \( j \geq 0 \), in view of the equality \( \lambda^ib_4^j = \lambda^iy^l \), which holds as \( \lambda a_1 = 0 \). Hence the claim implies the proposition.

Now to prove the claim we consider any polynomial \( g = g(\lambda, b_4, b_8) \in k[\lambda, b_4, b_8] \). We have to show that if it lies in \( \ker(\text{PH}^*(BGO(4)) \rightarrow \text{PH}^*(BO(4))) \) then it lies in \( \lambda k[\lambda, b_4] \). We write the polynomial as \( g = \sum_{i=0}^{r} p_i(\lambda, b_4)b_8^i \), where \( p_r(\lambda, b_4) \neq 0 \).

Consider the equalities

\[
B(\mu)^*(\lambda g) = B(\mu)^* \left( \sum_{i=0}^{r} (\lambda p_i(\lambda, b_4))b_8^i \right)
\]

\[
= \sum_{i=0}^{r} B(\mu)^*(\lambda p_i(\lambda, b_4))B(\mu)^*(b_8^i)
\]

\[
= \sum_{i=0}^{r} (\lambda p_i(\lambda, b_4))(B(\mu)^*(b_8))^i.
\]

Using the fact that \( B(\mu)^*(b_8) = b_8 + a_1^2t^3 + b_4t^2 + (a_3^2 + \lambda^3 + \lambda b_4)t^3 + t^4 \), we find that the coefficient of \( t^{4r} \) in the expression for \( B(\mu)^*(\lambda g) \) is exactly equal to \( \lambda p_r(\lambda, b_4) \),
which is a contradiction unless \( r = 0 \). Hence \( g = p_0(\lambda, b_4) \). This completes the proof of the claim, hence the proposition. 

\[ \square \]

**The ring \( PH^*(BGO(4m + 2)) \) of quadric invariants in rank \( 4m + 2 \)**

We first recall the following short exact sequence which is used in \([H-N]\) to compute the cohomology ring of \( BGO(2n) \) (in the present case, \( BGO(4m + 2) \))

\[
0 \to \lambda \mathbb{F}_2[b_1, \ldots, b_{8m+4}] \to H^*(BGO(4m + 2)) \to C \to 0
\]

where \( C \) is the subring of \( H^*(BO(4m+2)) \) which is the image of \( \pi^* : H^*(BGO(4m+2)) \to H^*(BO(4m+2)) \). We recall from \([H-N]\) that \( C \) equals the kernel of the derivation \( d_1 = s = \sum w_{2i-1} \frac{\partial}{\partial w_{2i}} \) on \( H^*(BO(4m+2)) = \mathbb{F}_2[w_1, \ldots, w_{4m+2}] \).

Recall that by Remark \( \Sigma1 \) for \( A = H^*(BO(4m + 2)) \) and \( A' = H^*(BGO(4m + 2)) \), the map \( \pi^* : A' \to A \) maps \( PA' \to PA \). Also recall from \([H-N]\) that the map \( \pi^* : A' \to A \) is injective in odd ranks.

**Generators \( \alpha'_{2i-1}, \delta'_T \) and \( \beta'_{4i} \) for the primitive ring \( PH^*(BGO(4m + 2)) \)**

Consider the maps \( d_j : A \to A \) for \( j \geq 1 \), defined by \( d_j(w_r) = (\binom{m+2}{j}) w_{r-j} \). By definition of the \( d_j \), we have \( PA \subset \ker(d_j) \) for \( j \geq 1 \), in particular, the elements \( \alpha'_{2i-1} \) and \( \delta'_T \in PH^*(BO(4m + 2)) \) are in the kernel of the map \( d_j \) for \( j \geq 1 \). Hence these elements lie in \( C = \ker(d_1) = \text{im}(\pi^*) \). As the map \( \pi^* \) is injective in odd ranks, there exist unique elements

\[
\alpha'_{2i-1} \in H^{2i-1}(BGO(4m + 2)) \text{ for } 1 \leq i \leq 2m + 1
\]

and

\[
\delta'_T \in H^{2\sum p_i-1}(BGO(4m + 2)) \text{ for } T = \{p_1, \ldots, p_r\} \subset \{2, \ldots, 2m + 1\}; \ r \geq 2
\]

such that they map to \( \alpha_{2i-1} \) and \( \delta_T \) respectively. By the injectivity of \( \pi^* \) in odd dimensions, these elements actually lie in \( PH^*(BGO(4m + 2)) \).

Recall the definition of the elements \( \beta_{4i} \in PA \) by

\[
\beta_{4i} = \hat{w}_{2i}^2 + \hat{w}_{2i-1}(\hat{w}_{2i} w_1 + \hat{w}_{2i-1} w_2) \text{ for } 2 \leq i \leq 2m + 1
\]

As \( \hat{w}_{2i} w_1 + \hat{w}_{2i-1} w_2 = d_1(\hat{w}_{2i} w_2) \) and \( d_1 \circ d_1 = 0 \), \( \hat{w}_{2i} w_1 + \hat{w}_{2i-1} w_2 \in \ker(d_1) = \text{im}(\pi^*) \).

As \( \hat{w}_{2i} w_1 + \hat{w}_{2i-1} w_2 \in H^{2i+1}(BO(4m + 2)) \) is of odd rank \( 2i + 1 \), there exist a unique element

\[
g_{2i+1} \in H^*(BGO(4m + 2)) \text{ with } \pi^*(g_{2i+1}) = \hat{w}_{2i} w_1 + \hat{w}_{2i-1} w_2
\]

Recall from the Appendix the definition of elements \( \widehat{c}_i \in H^*(BGL(4m + 2)) \) for \( i > 2 \), following Toda \([T]\). We now define elements \( \widehat{b}_{4i} \) by

\[
\widehat{b}_{4i} = p^*(\widehat{c}_{2i}) \in H^{4i}(BGO(4m + 2)) \text{ for } 2 \leq i \leq 2m + 1
\]

where \( p^* : H^*(BGL(4m + 2)) \to H^*(BGO(4m + 2)) \) is the map induced by the inclusion \( GO(4m + 2) \hookrightarrow GL(4m + 2) \). Then we observe that \( \pi^*(\widehat{b}_{4i}) = \hat{w}_{2i}^2 \), as
\( \hat{c}_i \mapsto \hat{w}_i^2 \) as already seen. We define the element \( \beta'_{4i} \in H^{4i}(BGO(4m + 2)) \) by the equality

\[
\beta'_{4i} = \hat{b}_{4i} + \alpha'_{2i-1}g_{2i+1}
\]

so that we have

\[
\pi^*(\beta'_{4i}) = \beta_{4i}
\]

**Lemma 4.4** The elements \( \beta'_{4i} \) lie in \( PH^*(BGO(4m + 2)) \subset H^*(BGO(4m + 2)) \).

**Proof** By definition of \( g_{2i+1} \), we have \( \pi^*(g_{2i+1}) = \hat{w}_{2i}w_1 + \hat{w}_{2i-1}w_2 \). Hence,

\[
B(\mu)^*(\pi^*(g_{2i+1})) = B(\mu)^*(\hat{w}_{2i}w_1 + \hat{w}_{2i-1}w_2) = (\hat{w}_{2i}w_1 + \hat{w}_{2i-1}w_2) + w^2\hat{w}_{2i-1}
\]

Therefore by the commutativity of the square

\[
\begin{array}{ccc}
H^*(BGO(2n)) & \xrightarrow{B(\mu)^*} & H^*(BGO(2n))[t] \\
\downarrow & & \downarrow \\
H^*(BO(2n)) & \xrightarrow{B(\mu)^*} & H^*(BO(2n))[w]
\end{array}
\]

we see that \( B(\mu)^*(g_{2i+1}) = g_{2i+1} + ta'_{2i-1} + t\lambda f \) for some \( f \in H^{2i-3}(BGO(2n))[t] \). As \( \lambda a'_{2i-1} = 0 \), \( B(\mu)^*(\alpha'_{2i-1}g_{2i+1}) = \alpha'_{2i-1}g_{2i+1} + ta^2_{2i-1} \). This equality is used in the third line of the following chain of equalities:

\[
B(\mu)^*(\beta'_{4i}) = B(\mu)^*(\hat{b}_{4i} + \alpha'_{2i-1}g_{2i+1}) \quad \text{by definition of } \beta'_{4i},
\]

\[
= B(\mu)^*(p^*(\hat{c}_{2i})) + B(\mu)^*(\alpha'_{2i-1}g_{2i+1}) \quad \text{by definition of } \hat{b}_{4i},
\]

\[
= p^*(B(\mu)^*(\hat{c}_{2i})) + \alpha'_{2i-1}g_{2i+1} + ta^2_{2i-1} \quad \text{as explained above},
\]

\[
= \hat{b}_{4i} + \alpha'_{2i-1}g_{2i+1} + (p^*(\hat{c}_{2i-1}) + \alpha^2_{2i-1})
\]

The last line above follows from the fact that \( B(\mu)^*(\hat{c}_{2i}) = \hat{c}_{2i} + t\hat{c}_{2i-1} \) and that \( p^* \circ B(\mu)^* = B(\mu)^* \circ p^* \).

Hence to complete the proof of the Lemma 4.4, it is sufficient to prove the following.

**Lemma 4.5** With the above notations, we have \( p^*(\hat{c}_{2i-1}) = \alpha^2_{2i-1} \) for \( 2 \leq i \leq 2m + 1 \).

**Proof of Lemma 4.5** We first observe that \( B(\mu)^*(p^*(\hat{c}_{2i-1})) = B(\mu)^*(\alpha^2_{2i-1}) = \hat{w}_{2i-1}^2 \). Hence \( p^*(\hat{c}_{2i-1}) = \alpha^2_{2i-1} + \lambda h \), where \( h = h(\lambda, b_1, \ldots, b_{8m+4}) \). We just have to show that \( h = 0 \).

For that we consider the group homomorphisms \( \tau : O(2) \times O(2m + 1) \to O(4m + 2) \), \( \tau' : GO(2) \times O(2m + 1) \to GO(4m + 2) \) and \( \tau'' : GL(2) \times GL(2m + 1) \to GL(4m + 2) \), induced by the tensor product \( \mathbb{C}^2 \otimes \mathbb{C}^{2m+1} \to \mathbb{C}^{4m+2} \). These fit in the following
commutative diagram of group homomorphisms, in which the vertical maps are natural inclusions.

\[
\begin{array}{ccc}
O(2) \times O(2m + 1) & \xrightarrow{\tau} & O(4m + 2) \\
\downarrow & & \downarrow \\
GO(2) \times O(2m + 1) & \xrightarrow{\tau'} & GO(4m + 2) \\
\downarrow & & \downarrow \\
GL(2) \times GL(2m + 1) & \xrightarrow{\tau''} & GL(4m + 2)
\end{array}
\]

Hence we get the following commutative diagram of ring homomorphisms.

\[
\begin{array}{ccc}
H^*(BGL(4m + 2)) & \xrightarrow{B(\tau'')^*} & H^*(BGL(2)) \otimes H^*(BGL(2m + 1)) \\
p^* \downarrow & & p^* \otimes (p\tau)^* \\
H^*(BGO(4m + 2)) & \xrightarrow{B(\tau')^*} & H^*(BGO(2)) \otimes H^*(BO(2m + 1)) \\
\pi^* \downarrow & & \pi^* \otimes \text{id} \\
H^*(BO(4m + 2)) & \xrightarrow{B(\tau)^*} & H^*(BO(2)) \otimes H^*(BO(2m + 1))
\end{array}
\]

By Propositions 9.9 and 9.8 we have \(B(\tau'')^*(\widehat{c}_{2i-1}) = 0\) and \(B(\tau')^*(\alpha'_{2i-1}) = 0\). These facts imply that \(B(\tau'')^*(\lambda h) = 0\). From this, we can deduce that \(\lambda h = 0\) by means of the following lemma (statement 4.6.(3) below), completing the proof of the Lemma 4.5.

**Lemma 4.6** With the above notations we have

(i) \(B(\tau')^*(\lambda) = \lambda \otimes 1\), where we also denote by \(\lambda\) the class \(\tau_1(L) \in H^*(BGO(2))\),

(ii) The map \(B(\tau')^*\) is injective on \(F_2[\lambda, b_1, \ldots, b_{8m+1}]\).

**Proof** of Lemma 4.6: To prove the first part, we use the commutativity of the square

\[
\begin{array}{ccc}
H^*(BGO(4m + 2)) & \xrightarrow{B(\tau')^*} & H^*(BGO(2)) \otimes H^*(BO(2m + 1)) \\
\pi^* \downarrow & & \pi^* \otimes \text{id} \\
H^*(BO(4m + 2)) & \xrightarrow{B(\tau)^*} & H^*(BO(2)) \otimes H^*(BO(2m + 1))
\end{array}
\]

and the fact that the map

\[
\bigoplus_{i=0}^1(H^i(BGO(2)) \otimes H^{2-i}(BO(2m + 1))) \rightarrow \bigoplus_{i=0}^1(H^i(BO(2)) \otimes H^{2-i}(BO(2m + 1)))
\]

is injective. This implies that the image of \(\lambda\) under the map \(B(\tau')^*\) is contained in \(H^2(BGO(2)) \otimes H^0(BO(2m + 1))\). But the Künneth projection

\[
H^*(BGO(2)) \otimes H^*(BO(2m + 1)) \rightarrow H^*(BGO(2))
\]

composed with \(B(\tau')^*\) is a map \(H^*(BGO(4m + 2)) \rightarrow H^*(BGO(2))\) which is induced by the inclusion \(GO(2) \subset GO(4m + 2)\) defined by

\[
A \mapsto \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}
\]

20
Hence the map \( H^*(BGO(4m + 2)) \rightarrow H^*(BGO(2)) \) takes \( \lambda \mapsto \lambda \). This implies that \( B(\tau^r)(\lambda) = \lambda \otimes 1 \). This completes the proof of the first assertion.

For proving statement (ii), first consider the commutative square

\[
\begin{array}{ccc}
H^*(BGO(4m + 2)) & \xrightarrow{B(\tau^r)^*} & H^*(BGO(2)) \otimes H^*(BO(2m + 1)) \\
\pi^* \downarrow & & \downarrow B(u)^* \\
H^*(BO(4m + 2)) & \xrightarrow{B(i)^*} & H^*(BO(2m + 1))
\end{array}
\]

where \( B(i)^* \) is induced by the the inclusion

\[
i : O(2m + 1) \hookrightarrow O(4m + 2) : A \mapsto \begin{pmatrix} A \\ A \end{pmatrix}
\]

and \( u : O(2m + 1) \rightarrow GO(2) \times O(2m + 1) \) is the homomorphism defined by \( g \mapsto (1, g) \). It follows from the definition of \( B(i)^* \) that \( B(i)^*(w_{2j}) = w_{2j}^2 \). Hence for any polynomial \( p(w_2, \ldots, w_{4m + 2}) \) we have \( B(i)^*(p) = p(w_1^2, \ldots, w_{2m + 1}^2) \). By the commutativity of the above square it follows that for any polynomial \( p = p(b_4, \ldots, b_{8m + 4}) \in H^*(BGO(4m + 2)) \) we have \( B(\tau^r)(p) = 1 \otimes p(w_4^2, \ldots, w_{2m + 1}^2) + q \) where \( q \in \oplus_{j=1}^r H^j(BGO(2)) \otimes H^{2r-j}(BO(2m + 1)) \).

Now consider a polynomial \( p = p(\lambda, b_4, \ldots, b_{8m + 4}) \in H^{2r}(BGO(4m + 2)) \). Writing it as \( p = \sum_{i=0}^r \lambda^i p_i(b_4, \ldots, b_{8m + 4}) \), let \( \ell \) be the smallest integer such that \( p_\ell(b_4, \ldots, b_{8m + 4}) \neq 0 \). Then we see that \( B(\tau^r)(p) = \lambda^\ell \otimes p_\ell(w_4^2, \ldots, w_{2m + 1}^2) + q \), where \( q \in \oplus_{j=\ell+1}^{2r} H^j(BGO(2)) \otimes H^{2r-j}(BO(2m + 1)) \). Hence \( B(\tau^r)(p) \neq 0 \). This proves Lemma 4.6 (ii), and hence completes the proofs of the Lemmas 4.5 and 4.4. \( \square \)

**Proposition 4.7** The map \( PH^*(BGO(4m + 2)) \rightarrow PH^*(BO(4m + 2)) \) is surjective.

**Proof** We know that \( PH^*(BO(4m + 2)) \) is generated by elements of the form \( \alpha_{2i-1}, \beta_{4i} \) and \( \delta_T \). These are the images of \( \alpha_{2i-1}^{\prime}, \beta_{4i}^{\prime} \) and \( \delta_T^{\prime} \) in \( PH^*(BGO(4m + 2)) \), so the proposition follows. \( \square \)

**Theorem 4.8** The primitive subring \( PH^*(BGO(4m + 2)) \subset H^*(BGO(4m + 2)) \) is generated by the finite set consisting of the following elements.

(i) The element \( \lambda \),
(ii) the element \( \alpha_1^{\prime} = a_1 \),
(iii) the elements \( \alpha_{2i-1} \) for \( 2 \leq i \leq 2m + 1 \),
(iv) the elements \( \beta_{4j}^{\prime} \) for \( 2 \leq j \leq 2m + 1 \), and
(v) the elements \( \delta_T^{\prime} \), for \( T \subset \{2, \ldots, 2m + 1\} \) with cardinality \( |T| \geq 2 \).

**Proof** Let \( P' \) be the subring of \( PH^*(BGO(4m + 2)) \) generated by the elements listed in the statement of the theorem. We want to show that \( P' = PH^*(BGO(4m + 2)) \). By Proposition 4.4 the map \( PB(\mu)^* : PH^*(BGO(4m + 2)) \rightarrow PH^*(BO(4m + 2)) \) is surjective. \( \square \)
(2)) is surjective. As its kernel is \( \lambda \mathbb{F}_2[\lambda, b_4, \ldots, b_{8m+4}] \cap PH^*(BGO(4m+2)) \), it is enough to show that we have the following inclusion of sets:

\[
\lambda \mathbb{F}_2[\lambda, b_4, \ldots, b_{8m+4}] \cap PH^*(BGO(4m+2)) \subset P'
\]

Consider any polynomial \( p = p(\lambda, b_4, \ldots, b_{8m+4}) \). We can write \( b_{4i} \) as \( \hat{b}_{4i} + z \) where \( z \) is a polynomial in elements of \( H^j(BGO(4m+2)) \) for \( j < 4i \) (this holds because the elements \( b_{4i} \) and \( \hat{b}_{4i} \) come from the elements \( \tau_{2i} \) and \( \hat{\tau}_{2i} \) of \( H^i(BGL(4m+2)) \), and the corresponding statement holds in \( H^*(BGL(4m+2)) \)).

As \( \lambda \) annihilates odd degree elements of \( H^*(BGO(2n)) \), we can assume that \( z \) is a polynomial in the elements \( \lambda, b_4, \ldots, b_{4i-4} \). We substitute this in \( p \) to conclude, by iteration, that the polynomial \( \lambda p \) is an element of \( \lambda \mathbb{F}_2[\lambda, b_4, \hat{b}_8, \ldots, \hat{b}_{8m+4}] \).

It follows from the definition of \( \beta'_{4i} \) that \( \lambda \beta'_{4i} = \lambda \hat{b}_{4i} \). This implies that for any polynomial \( p = p(\lambda, b_4, \ldots, b_{8m+4}) \), we have \( \lambda p \in \lambda \mathbb{F}_2[\lambda, b_4, \beta'_8, \ldots, \beta'_{8m+4}] \).

Now let \( p = \sum_{i=0}^{r} b_{4i} p_i(\beta'_8, \ldots, \beta'_{8m+4}) \) \( \in H^*(BGO(4m+2)) \) be such that \( \lambda p \in \lambda \mathbb{F}_2[\lambda, b_4, \ldots, b_{8m+4}] \cap PH^*(BGO(4m+2)) \). Suppose \( r \geq 1 \). As \( B(\mu)^*(p_i) = p_i \) and \( B(\mu)^*(b_4) = b_4 + a_4^2 t + t^2 \), the coefficient of \( t^{2r} \) in the expansion of \( B(\mu)^*(\sum_{i=0}^{r} \lambda p_i b_{4i}^4) \) is equal to \( \lambda p_r(\beta'_8, \ldots, \beta'_{8m+4}) \). Hence if \( \lambda p \) is in \( PH^*(BGO(4m+2)) \), we must have \( \lambda p_r(\beta'_8, \ldots, \beta'_{8m+4}) = 0 \). Continuing this way, we get \( \lambda p = \lambda \cdot p_0(\beta'_8, \ldots, \beta'_{8m+4}) \). This lies in \( P' \), which shows the desired inclusion and completes the proof of the theorem.

\[ \square \]

**Remark 4.9** In fact the method of the proof gives us the short exact sequence

\[
0 \to \mathbb{F}_2[\lambda, \beta'_8, \ldots, \beta'_{8m+4}] \to PH^*(BGO(4m+2)) \to PH^*(BO(4m+2)) \to 0
\]

which can be used to write down the relations in the ring \( PH^*(BGO(4m+2)) \). Similarly, we can write the relations between our generators of \( PH^*(BO(4)) \). We do not include the description of relations in this paper, as it does not contribute here to our main theorems on degenerating quadric bundles.

## 5 The Gysin boundary map

**Lemma 5.1** Let \( Y \) be a Hausdorff topological space, and let \( \pi : N \to Y \) be a real vector bundle bundle on \( Y \), of real rank \( r \). Let \( N_0 \subset N \) denote the complement of the zero section \( i : Y \to N \) of \( N \to Y \). With the above notation, the following diagram is commutative

\[
\begin{array}{ccc}
H^p(N) & \times & H^q(N_0) \\
\downarrow r & & \downarrow \partial \\
H^p(Y) & \times & H^{q-r+1}(Y)
\end{array} \xrightarrow{\cup} \begin{array}{c} H^{p+q}(N_0) \\
\downarrow \partial \end{array} \xrightarrow{\cup} H^{p+q-r+1}(Y)
\]

In other words, given \( a \in H^p(N) \) and \( b \in H^q(N_0) \), we have \( \partial(a \cup b) = i^*(a) \cup \partial(b) \in H^{p+q-r+1}(Y) \).
Proof Let $\tau \in H^r(N, N_o)$ be the Thom class, so that for each $p$ we have the Thom isomorphism $T$, which is the composite

$$T : H^p(Y) \xrightarrow{\pi^*} H^p(N) \xrightarrow{-\cup \tau} H^{p+r}(N, N_o)$$

where both $\pi^*$ and $-\cup \tau$ are isomorphisms. The Gysin boundary map $\partial : H^p(N_o) \to H^{p-r+1}(Y)$ is the composite

$$\partial : H^p(N_o) \xrightarrow{\delta} H^{p+1}(N, N_o) \xrightarrow{T^{-1}} H^{p-r+1}(Y)$$

of the connecting homomorphism $\delta$ for the pair $(N, N_o)$ with the inverse of the Thom isomorphism $T$. If $\alpha \in H^p(N)$ and $\beta \in H^{q-r+1}(N)$, then by associativity of cup product, $(\alpha \cup \beta) \cup \tau = \alpha \cup (\beta \cup \tau)$, in other words, the following diagram is commutative.

$$
\begin{array}{ccc}
H^p(N) & \times & H^{q-r+1}(N) \\
\downarrow \text{id} & & \downarrow \cup \tau \\
H^p(N) & \times & H^{q+1}(N, N_o)
\end{array}
\quad
\begin{array}{ccc}
& & H^{p+q-r+1}(N) \\
\downarrow \cup \tau & & \downarrow \cup \tau \\
& & H^{p+q+1}(N, N_o)
\end{array}
$$

By composing with the isomorphism $\pi^* : H^*(Y) \to H^*(N)$, this gives the following commutative diagram, where $T$ is the Thom isomorphism.

$$
\begin{array}{ccc}
H^p(Y) & \times & H^{q-r+1}(Y) \\
\downarrow \pi^* & & \downarrow T \\
H^p(N) & \times & H^{q+1}(N, N_o)
\end{array}
\quad
\begin{array}{ccc}
& & H^{p+q-r+1}(Y) \\
\downarrow T & & \downarrow T \\
& & H^{p+q+1}(N, N_o)
\end{array}
$$

Hence as $i^*$ is the inverse of $\pi^*$, the following diagram commutes.

$$
\begin{array}{ccc}
H^p(N) & \times & H^{q+1}(N, N_o) \\
\downarrow i^* & & \downarrow T^{-1} \\
H^p(Y) & \times & H^{q-r+1}(Y)
\end{array}
\quad
\begin{array}{ccc}
& & H^{p+q+1}(N, N_o) \\
\downarrow T^{-1} & & \downarrow T^{-1} \\
& & H^{p+q-r+1}(Y)
\end{array}
$$

For any integer $p$, let $\delta : H^p(N_o) \to H^{p+1}(N, N_o)$ denote the connecting homomorphism for the singular cohomology of the pair $(N, N_o)$. For any integers $p$ and $q$, consider the cup product maps $H^p(N) \otimes H^q(N_o) \to H^{p+q}(N_o)$ and $H^p(N) \otimes H^{q+1}(N, N_o) \to H^{p+q+1}(N, N_o)$. It is known that these fit in a commutative diagram

$$
\begin{array}{ccc}
H^p(N) & \times & H^q(N_o) \\
\downarrow \text{id} & & \downarrow \delta \\
H^p(N) & \times & H^{q+1}(N, N_o)
\end{array}
\quad
\begin{array}{ccc}
& & H^{p+q}(N_o) \\
\downarrow \delta & & \downarrow \delta \\
& & H^{p+q+1}(N, N_o)
\end{array}
$$

Now the lemma follows by juxtaposing the above two commutative diagrams. \qed

Remark 5.2 The above lemma is for coefficients $\mathbb{F}_2$. A version with arbitrary coefficient ring $R$ is possible, where we have to assume that $N$ is orientable with respect to coefficients $R$, and the conclusion is that the the diagram in statement 5.1 is graded-commutative, that is, $\partial(a \cup b) = (-1)^{\deg(a)}i^*(a) \cup \partial(b)$.  

23
Remark 5.3 We will apply the above lemma in the following situation. Let $X$ be a topological space and $i : Y \rightarrow X$ a closed subspace which is a topological divisor in $X$, that is, there exists a rank 2 real vector bundle $\pi : N \rightarrow Y$ over $Y$, together with a continuous map $\varphi : N \rightarrow X$ which is a homeomorphism of $N$ with an open neighbourhood $U$ of $Y$ in $X$, such that $\varphi$ restricted to the zero section $Y \subset N$ is id$_Y$. Composing the restriction $H^p(X - Y) \rightarrow H^p(U - Y)$ with the map $H^p(U - Y) \rightarrow H^p(N_o)$ and the Gysin boundary map $\partial : H^p(N_o) \rightarrow H^{p-1}(Y)$, we get a map

$$
\partial : H^p(X - Y) \rightarrow H^{p-1}(Y)
$$

which is by definition the Gysin boundary map for the pair $(X,Y)$. It can be shown that the map $\partial$ is independent of the choice of $\pi : N \rightarrow Y$ and $\varphi$. The Lemma [b.1] implies that given $a \in H^p(X)$ and $b \in H^q(X - Y)$, we have

$$
\partial(a \cup b) = i^*(a) \cup \partial(b) \in H^{p+q-r+1}(Y)
$$

Remark 5.4 We now summarize the properties of the Gysin boundary map that we will use. Let $L \rightarrow Y$ be a complex line bundle. Let $L_o \subset L$ be the complement of the zero section, and let $\pi : L_o \rightarrow Y$ be the projection. Let $\partial : H^*(L_o) \rightarrow H^{*-1}(Y)$ be the Gysin boundary, and $\pi^* : H^*(Y) \rightarrow H^*(L_o)$ the pullback under $\pi : L_o \rightarrow Y$. Let $s = \pi^* \circ \partial : H^*(L_o) \rightarrow H^{*-1}(L_o)$. Let $\mu : \mathbb{C}^* \times L_o \rightarrow L_o$ be the scalar multiplication, and let $p : \mathbb{C}^* \times L_o \rightarrow L_o$ be the projection. Let $\eta \in H^1(\mathbb{C}^*)$ be the generator. Let $Sq^i$ denote the $i$ th Steenrod operation. Then for any $x, x' \in H^*(L_o)$ and $y \in H^*(Y)$, we have the following basic equalities.

$$
\partial(x \cup \pi^* y) = \partial(x) \cup y \quad (1) \\
(\mu^* - p^*)(x) = \eta \otimes s(x) \quad (2) \\
\partial Sq^i x = Sq^i \partial x \quad (3) \\
\partial(x^2) = 0 \quad (4) \\
s(x \cup x') = s(x) \cup x' + x \cup s(x') \quad (5)
$$

The property (1) is given by Lemma [b.1]. The properties (2)-(5) are proved in Section 2 of [H-N]. Note that property (5) can be expressed by saying that the map $s$ is a derivation on the cohomology ring $H^*(L_o)$. The above equalities are used only in the explicit computations of the Gysin boundary images of the quadric invariants (see Section 8).

6 Topological vanishing multiplicity

Definition of vanishing multiplicities

Let $N$ be a 2-dimensional real vector space, together with an orientation, and let $L$ be a 1-dimensional complex vector space. Let $f : N - \{0\} \rightarrow L - \{0\}$ be a map. We denote by $\nu_0(\sigma) \in \mathbb{Z}$ the winding number of this map. Its sign depends on
the orientation on $N$, but its absolute value $|\nu_0|(f) \in \mathbb{Z}_{\geq 0}$ and its parity $\overline{\nu}_Y(f) \in \{0, 1\} \subset \mathbb{Z}_{\geq 0}$ are independent of the orientation chosen. Here the parity $\overline{\nu}_Y(f)$ is defined to be 0 or 1 depending on whether $|\nu_0|(f)$ is even or odd.

Next, we globalize the above. Let $Y$ be a Hausdorff space, of the homotopy type of a CW complex, and let $\pi : N \to Y$ be a real rank 2 vector bundle on $Y$. Let $L$ be a complex line bundle on the total space of $N$. Let $N - Y$ be the complement of the zero section $Y \subset N$, and let $\sigma \in \Gamma(N - Y, L)$ be a section, which is nowhere vanishing. Choose an isomorphism $\theta : L \to \pi^*(L_Y)$ of complex line bundles, that is identity over $Y$ (which exists by Remark 6.3). For any $y \in Y$, $\sigma$ gives a map $f_y : N_y - \{0\} \to L_y - \{0\}$. The numbers $|\nu_0|(f_y) \in \mathbb{Z}_{\geq 0}$ and $\overline{\nu}_0(f_y) \in \{0, 1\} \subset \mathbb{Z}$ are well-defined, independent of the choice of the isomorphism $\theta : L \to \pi^*(L_Y)$ as follows from Remark 6.3. Hence we regard them as functions on $Y$. As such, these are constant on path components of $Y$, so give functions

$$|\nu_0|(\sigma) : \pi_0(Y) \to \mathbb{Z}_{\geq 0} \text{ and } \overline{\nu}_Y(\sigma) : \pi_0(Y) \to \{0, 1\} \subset \mathbb{Z}$$

If $N$ is orientable, then a choice of orientation gives rise to a function

$$\nu_0(\sigma) : \pi_0(Y) \to \mathbb{Z}$$

similarly defined.

**Definition 6.1** By definition of 0th singular cohomology, we thus have an element $|\nu_0|(\sigma) \in H^0(Y; \mathbb{Z})$ which we call the **absolute topological vanishing multiplicity**, an element $\overline{\nu}_Y(\sigma) \in H^0(Y; \mathbb{Z})$ which we call the **topological vanishing parity**, and when $N$ is given an orientation, an element $\nu_0(\sigma) \in H^0(Y; \mathbb{Z})$ which we call the **oriented topological vanishing multiplicity** of the non-vanishing section $\sigma \in \Gamma(N - Y, L)$.

**Remark 6.2** More generally, if $X$ is a paracompact, Hausdorff topological space, $Y \subset X$ is a topological divisor, $L$ a complex line bundle on $X$, and $\sigma \in \Gamma(X - Y, L)$ is a nowhere vanishing section, we can define the elements $|\nu_0|(\sigma) \in H^0(Y; \mathbb{Z})$, $\overline{\nu}_Y(\sigma) \in H^0(Y; \mathbb{Z})$ by choosing a pair $(N, \varphi)$ where $N$ is a rank 2 real vector bundle on $Y$ and $\varphi : N \to X$ is an open embedding which maps the zero section of $N$ identically to $Y$, and then applying the above definition to the pullback under $\varphi$. The resulting topological vanishing multiplicity $|\nu|$ and its parity $\overline{\nu}$ are well-defined, independent of the choice of the tubular neighbourhood $(N, \varphi)$. Moreover, if the bundle $N$ is orientable, then the choice of an orientation gives $\nu_0(\sigma) \in H^0(Y; \mathbb{Z})$.

For a definition of $|\nu_0|(\sigma)$ in terms of local cohomology, see [N].

**Remark 6.3** As the zero section $Y \subset N$ is a strong deformation retract of $N$, for any complex line bundle $L$ on the total space $N$ there exists an isomorphism of complex line bundles $\theta : L \to \pi^*(L_Y)$ which is identity on the zero section $Y \subset N$, where $L_Y = L|_Y$, and any two such isomorphisms $\theta_1, \theta_2 : L \to \pi^*(L_Y)$ are related by $\theta_1 = f \theta_2$ where $f : N \to \mathbb{C}^*$ is a function such that $f$ is homotopic to 1 relative to $Y$ (in symbols, $f \sim 1$ rel $Y$).
Remark 6.4 Let \( \pi : N \rightarrow Y \) will be a real vector bundle of rank 2 on a space \( Y \), equipped with a fiber-wise Riemannian metric (that is, a continuous positive definite symmetric real bilinear form on \( N \)). Let \( \sigma \in \Gamma(N - Y, L) \) be a nowhere vanishing section with \(|\nu_Y| (\sigma) \neq 0 \). Then \( N \) is orientable, and has a unique orientation such that \( \nu_Y (\sigma) \) is positive. This orientation, together with the given metric, defines a \( \mathbb{C} \)-linear structure on \( N \). Using this, we can regard \( N \) as a complex line bundle.

Lemma 6.5 Let \( N \rightarrow Y \) be a rank 2 real vector bundle on \( Y \), let \( D \subset N \) be the disk bundle over \( Y \) of radius 1 with respect to a metric on the vector bundle \( N \), and let \( \partial D \) denote the boundary of \( D \). Let \( g : \partial D \rightarrow \mathbb{C}^* \) be a continuous function such that the restriction of \( g \) to any fiber of \( p : \partial D \rightarrow Y \) is homotopic to a constant function. Then \( g : \partial D \rightarrow \mathbb{C}^* \) admits a prolongation \( \overline{g} : D \rightarrow \mathbb{C}^* \).

Proof By working on one component of \( Y \) at a time, we can assume that \( Y \) is connected, that is, \( \pi_0 (Y) = 1 \). Consider the homotopy exact sequence

\[
\pi_1 (S^1) \xrightarrow{i} \pi_1 (\partial D) \xrightarrow{p} \pi_1 (Y) \rightarrow 1
\]

for the fibration \( \partial D \rightarrow Y \), where \( i : S^1 \hookrightarrow \partial D \) denotes one of the fibers. The hypothesis of the lemma shows that the homomorphism \( g_* : \pi_1 (\partial D) \rightarrow \pi_1 (\mathbb{C}^*) \) is trivial on the image of \( \pi_1 (S^1) \rightarrow \pi_1 (\partial D) \), so factors through \( \pi_1 (\partial D) \xrightarrow{p} \pi_1 (Y) \) giving \( \theta : \pi_1 (Y) \rightarrow \pi_1 (\mathbb{C}^*) \). By identifying \( \pi_1 (D) \) with \( \pi_1 (Y) \) via the projection \( D \rightarrow Y \), we have a group homomorphism \( \theta : \pi_1 (D) \rightarrow \pi_1 (\mathbb{C}^*) \). As \( \mathbb{C}^* \) is a \( K(\pi, 1) \)-space, every group homomorphism \( \pi_1 (X) \rightarrow \pi_1 (\mathbb{C}^*) \) for a CW complex \( X \) is of the form \( f_* \) for some continuous function \( f : X \rightarrow \mathbb{C}^* \) which is unique up to homotopy. We can therefore choose a function \( f : D \rightarrow \mathbb{C}^* \) with \( f_* = \theta : \pi_1 (D) \rightarrow \pi_1 (\mathbb{C}^*) \). On \( \partial D \), we have \( g_* = (f|\partial D)_* \), so by the above there exists a homotopy \( H : \partial D \times I \rightarrow \mathbb{C}^* \) from \( g \) to \( f|\partial D \). Now let \( D_{1/2} \subset D \) denote the disk bundle of radius half, and let \( D_o \) denote its interior. Note that we have a canonical homeomorphism \( \psi : \partial D \times I \rightarrow \partial D - D_{1/2}^o \), defined by the formula \( \psi(v,t) = (1-t/2)v \). Now define \( \overline{g} : D \rightarrow \mathbb{C}^* \) by putting

\[
\overline{g}(v) = \begin{cases} 
H \circ \psi^{-1}(v) & \text{for } |v| \geq 1/2, \\
f & \text{for } |v| \leq 1/2.
\end{cases}
\]

This proves the lemma. \( \square \)

Lemma 6.6 (Non-Vanishing prolongation) Let \( N \rightarrow Y \) be a rank 2 real vector bundle on \( Y \) equipped with a metric, let \( L \) be a complex line bundle on \( N \), and let \( \sigma \in \Gamma(N, L) \) be a global section, which is non-vanishing outside \( Y \), with \(|\nu_Y| (\sigma) = 0 \). Then the line bundle \( L \) is trivial on \( N \), and moreover there exists a non-vanishing global section \( \sigma' \in \Gamma(N, L) \) such that \( \sigma' \) and \( \sigma \) coincide on the open set \( N - D \), where \( D \subset N \) be the disk bundle over \( Y \) of radius 1.

Proof We first prove that \( L \) is trivial on \( N \), equivalently, that the restriction \( M = L|_Y \) is trivial on \( Y \). We can assume by Remark 6.3 that \( L = \pi^* (M) \) for the
projection $\pi : N \to Y$. Let $N^0 = N - Y$ and $M^0 = M - Y$, and let $\nu : N^0 \to Y$ and $\mu : M^0 \to Y$ denote the projections. We have the commutative diagram of homotopy exact sequences

$$
\begin{array}{ccc}
\pi_2(Y) & \to & \pi_1(N^0) \\
\downarrow & & \downarrow \\
\pi_2(Y) & \to & \pi_1(M^0)
\end{array}
\quad
\begin{array}{ccc}
\psi & \to & \pi_1(N^0) \\
\downarrow & & \downarrow \\
\psi & \to & \pi_1(M^0)
\end{array}
\quad
\begin{array}{ccc}
\mu & \to & \pi_1(Y) \\
\downarrow & & \downarrow \\
\mu & \to & \pi_1(Y)
\end{array}
\to 1
$$

where the middle two vertical maps are induced by $\sigma$. As $|\nu\gamma|(\sigma) = 0$, the map $\pi_1(N^0) \to \pi_1(M^0)$ is zero. Hence from the above diagram there exists a group homomorphism $\psi : \pi_1(Y) \to \pi_1(M^0)$ such that $\mu_s \circ \psi = \text{id}\pi_1(Y)$.

As complex line bundles $M$ are classified by their first Chern class $c_1(M) \in H^2(Y; \mathbb{Z})$, we have to show that $c_1(M) = 0$. Given any non-zero element $c$ of $H^2(Y; \mathbb{Z})$, there exists a compact 2-manifold $K$ and a continuous map $f : K \to Y$, such that $f^*(c) \neq 0$ in $H^2(K; \mathbb{Z})$. The section $f^*(\sigma) \in \Gamma(f^*N, f^*L)$ has the same vanishing multiplicity as $\sigma$. Hence we can base change to $K$, and therefore we can assume that $Y$ is a compact 2-manifold for the sake of proving that $L$ is trivial.

Unless such a $Y$ is homeomorphic to $S^2$ or $\mathbb{P}^2\mathbb{R}$ (we treat these two cases separately later), we know that $Y$ is a $K(\pi, 1)$ space. Hence it follows from the homotopy exact sequence for $\mu : M^0 \to Y$ that $M^0$ is also a $K(\pi, 1)$ space. Hence the group homomorphism $\psi : \pi_1(Y) \to \pi_1(M^0)$ is induced by a map $s : Y \to M^0$. As $\mu_s \circ s = \text{id}\pi_1(Y)$, and as $Y$ is a $K(\pi, 1)$ space, the map $\mu \circ s : Y \to Y$ is homotopic to $\text{id}Y$. Hence by the homotopy lifting property of $\mu : M^0 \to Y$, the map $\text{id}Y : Y \to Y$ lifts to give a section of $\mu$, proving triviality of $M$.

Next, suppose $Y = S^2$. Then $M = O(d)$ for $d \in \mathbb{Z}$, with $c_1(M) = d \in H^2(Y; \mathbb{Z}) = \mathbb{Z}$. It can be seen that $\pi_1(M^0) = \mathbb{Z}/(d)$. Now as the map $\pi_1(N^0) \to \pi_1(M^0)$ is zero, from the above commutative diagram comparing the long exact homotopy sequences we see that the map $\pi_2(Y) \to \pi_1(M^0)\pi_1(N^0)$ is zero, hence the map $\pi_1(M^0) \to \pi_1(M^0)$ is an isomorphism, showing $\pi_1(M^0) = \mathbb{Z}$, therefore $M$ is trivial when $Y = S^2$.

Finally, suppose $Y = \mathbb{P}^2\mathbb{R}$. Then as $H^2(Y; \mathbb{Z}) = \mathbb{Z}/(2)$, $Y$ has exactly one non-trivial complex line bundle $M$, up to isomorphism. It can be seen that for this $M$, we have $\pi_1(M^0) = \mathbb{Z}$, so there cannot exist a section to $\pi_1(M^0) \to \pi_1(Y) = \mathbb{Z}/(2)$. Hence $M$ must be trivial, even when $Y = \mathbb{P}^2\mathbb{R}$.

Now choose a non-vanishing global section $e$ of $L$, and let $g : \partial D \to \mathbb{C}^*$ be defined by $\sigma = ge$ on $\partial D$. The hypothesis $\nu\gamma(\sigma) = 0$ implies that the restriction of $g$ to any fiber of $p : \partial D \to Y$ is homotopic to the constant function 1. By Lemma 6.3 above, $g$ admits a prolongation $\overline{g} : D \to \mathbb{C}^*$. Put $\sigma' = \overline{g}e$ on $D$, and $\sigma' = \sigma$ on $N - D$. This proves the lemma.

**Lemma 6.7** Let $Y$ be a connected Hausdorff space of homotopy type of a CW complex, and let $\pi : N \to Y$ be a real vector bundle on $Y$ of rank 2, with a metric. Let $M \to Y$ be a complex line bundle on $Y$, let $L = \pi^*(M)$, and let $u \in \Gamma(N, L)$ such that $u$ is non-vanishing on $N - Y$ and $\nu\gamma(u) = m \geq 1$. Let $N$ be given the $\mathbb{C}$-linear structure induced by the orientation defined by $u$ (see Remark 6.4), making...
it a complex line bundle on \( Y \). Let \( N^m \) be the \( m \) th tensor power of the complex line bundle \( N \).

Then we have the following.

(i) The complex line bundle \( M \) is isomorphic to \( N^m \).

(ii) In fact, there exists an isomorphism \( \psi : N^m \to M \) of complex line bundles on \( Y \) with the following property. Let \( \tau \in \Gamma(N, \pi^*(N)) \) be the tautological section (defined by \( \text{id} : N \to N \)), and let \( \tau^m \in \Gamma(N, \pi^*(N^m)) \) be its \( m \) th power. Consider the section \( \psi(\tau^m) \in \Gamma(N, L) \), and let \( f : N - Y \to \mathbb{C}^* \) be defined by \( u|_{N-Y} = f \cdot \psi(\tau^m)|_{N-Y} \). Then \( f \) is homotopic to the constant function 1.

**Proof** As \( \nu_Y(u) \neq 0 \), by Remark 6.4 \( u \) defines an orientation on the real vector bundle \( N \). Together with the chosen metric on \( N \), this gives a \( \mathbb{C} \)-linear structure on \( N \), so we regard it as a complex line bundle. Let \( K \) be the complex line bundle on \( Y \) defined by \( K = M \otimes N^{-m} \). Let \( \pi^*(K) \) denote the pullback of \( K \) to the total space of \( N \). Let the continuous map \( v : N \to M \) be the composite of the section \( u : N \to N \times_Y M \) with the projection \( N \times_Y M \to M \). (Note that \( v \) is over \( Y \), but may not be linear on fibers.) Given any \( x \in N - Y \) over \( y \in Y \), let

\[
\sigma_x : (N_y)^\otimes m \to M_y
\]

be the unique \( \mathbb{C} \)-linear map under which \( x^\otimes m \mapsto v(x) \). We can regard \( \sigma_x \) as an element of \( K_y \), and as \( x \) varies we get a section \( \sigma \in \Gamma(N - Y, \pi^*(K)) \). From the hypothesis that \( \nu_Y(u) = m \) it follows that \( \nu_Y(\sigma) = 0 \). Hence by Lemma 6.6, the complex line bundle \( K \) is trivial. This proves (i).

Now suppose \( \psi_1 : N \to M \) is some isomorphism of complex bundles, which exists by (i). Consider the resulting section \( \psi_1(\tau^m) \in \Gamma(N, M_N) \), and let \( f_1 : N - Y \to \mathbb{C}^* \) be defined by \( u = f_1\psi_1(\tau^m) \). As both \( u \) and \( \psi_1(\tau^m) \) have oriented topological vanishing multiplicity 1 along \( Y \), it follows that \( f_1 \) has topological vanishing multiplicity 0 along \( Y \). By Lemma 6.3, there exists a function \( f_2 : N \to \mathbb{C}^* \) such that \( f_2|_{N-D} = f_1|_{N-Y} \) where \( D \subset N \) is the unit disk bundle in \( N \). Now put \( \psi = f_2\psi_1(\tau^m) \). Then \( u|_{N-Y} = f \cdot \psi|_{N-Y} \), where \( f = f_1/f_2 : N - Y \to \mathbb{C}^* \). As \( f|_{N-D} \) is the constant function 1, and as \( N - D \leftrightarrow N - Y \) is a homotopy equivalence, \( f|_{N-Y} \) is homotopic to the constant function 1. This proves (ii) and completes the proof of the lemma. \( \square \)

## 7 Proof of the Main Theorem

In this section we allow the singular cohomologies to have coefficients in an arbitrary ring \( R \).

**Vanishing multiplicity of discriminant**

Recall that for any triple \( (E, L, b) \) on \( X \), the form \( b \) is given on a trivializing open cover of \( X \) by \( r \times r \) matrices of functions (where \( r = \text{rank}(E) \)), whose determinants fit together to define a global section \( \det(b) \in \Gamma(X, L^r \otimes \det(E)^{-2}) \) called the **discriminant** of \( (E, L, b) \). It is clear that if \( K \) is any line bundle on \( X \), then for the
triple \((E \otimes K, L \otimes K^\otimes 2, b \otimes 1_K)\) we have \(\det(b \otimes 1_K) = \det(b) \in \Gamma(X, L^r \otimes \det(E)^{-2})\). Hence for a quadric bundle \(Q = [E, L, b]\), the section \(\det(Q) = \det(b)\) is well defined. The non-degeneracy condition on \((E, L, b)\) just means that \(\det(b)\) is nowhere zero. More generally, if \(Z \subset X\) is the vanishing locus of \(\det(b)\) then the restriction \((E, L, b)|_{X-Z}\) is non-degenerate.

Now consider a pair \((X,Y)\) where \(X\) is a topological space, and \(Y\) a topological divisor in \(X\). Given a quadric bundle \(Q = [E, L, b]\) on \(X\) which is non-degenerate on \(X - Y\), the non-negative integer valued functions \(\nu_Y|\det(Q)\) and \(\nu_Y|\det(Q)\) on \(Y\) are the absolute vanishing multiplicity of the discriminant \(\det(b)\) along \(Y\), and its parity, as in Definition 6.1 and Remark 6.2 above.

Minimally degenerate quadric bundles and canonical triples

Recall from the introduction that a triple \(T = (E, L, b)\), or the corresponding quadric bundle \(Q = [E, L, b]\) on a base \(Y\) is minimally degenerate if \(\text{rank}(b_y) = \text{rank}(E) - 1\) at all points \(y \in Y\). This means that the linear map \(b : E \to L \otimes E^*\) is of constant rank, with kernel a line subbundle \(K \subset E\). Let \(\overline{E} = E/K\) be the quotient bundle, on which we get an induced bilinear form \(\overline{b} : \overline{E} \otimes \overline{E} \to L\) which is non-degenerate. Hence we get a non-degenerate triple \(\overline{T} = (\overline{E}, L, \overline{b})\) of rank \(r - 1\) where \(r = \text{rank}(E)\).

Recall that we have defined the canonical triple \(\overline{T}^Q\) associated to such a minimally degenerate quadric bundle \(Q\) to be the non-degenerate rank \(r - 1\) triple

\[
\overline{T}^Q = \overline{T} \otimes \ker(b)^{-1} = (\overline{E} \otimes K^{-1}, L \otimes K^{-2}, \overline{b} \otimes 1_{K^{-1}})
\]

This is well-defined, independent of the choice of the representative \(T\) for \(Q\). It represents the non-degenerate quadric bundle \(\overline{Q} = [\overline{T}]\) of rank \(r - 1\).

Mild degeneration

Let \(Y \subset X\) be a topological divisor. Recall from the introduction that we call \(T = (E, L, b)\) a mildly degenerating triple on \((X,Y)\) if \(T_{X-Y}\) is non-degenerate of rank \(r\) say, and \(T_Y\) is of constant rank \(r - 1\). We say that the corresponding \(Q = [T]\) is a mildly degenerating quadric bundle on \((X,Y)\) of generic rank \(r\).

In the algebraic category, we have the following.

**Proposition 7.1** Let \(X\) be a non-singular complex algebraic variety, and let \(Y \subset X\) be a smooth divisor. Let \(Q \to X\) be mildly degenerating quadric bundle over \((X,Y)\) in the algebraic category. Then the following are equivalent.

(i) The total space of \(Q\) is non-singular.

(ii) The algebraic vanishing multiplicity \(\nu_Y(\det(Q))\) (which is the same as the oriented topological vanishing multiplicity with respect to the natural orientation) is 1 over all components of \(Y\).

**Proof** This is a simple generalization of the Proposition 3 of [N], and follows by the Jacobian criterion. \(\square\)

Orthogonal decomposition
Definition 7.2 (Direct sums of triples) Let \((E_1, L, b_1)\) and \((E_2, L, b_2)\) be two triples on a space \(X\), where the line bundle \(L\) is common to the two triples. We define the \textbf{direct sum} of \((E_1, L, b_1) \oplus (E_2, L, b_2)\) to be the triple \((E_1 \oplus E_2, L, b_1 \oplus b_2)\), where \(b_1 \oplus b_2\) has the matrix form \(
abla \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \). More generally, if \((E_1, L_1, b_1)\) and 
\((E_2, L_2, b_2)\) are two triples, and we are given an isomorphism \(\psi : L_1 \to L_2\), Then the triple \((E_1, L_1, b_1) \oplus \psi (E_2, L_2, b_2)\), which we call the \textbf{direct sum via} \(\psi\), is defined to be \((E_1 \oplus E_2, L_2, \psi b_1 \oplus b_2)\), where \(\psi b_1 \oplus b_2\) has the matrix form \(
abla \begin{pmatrix} \psi b_1 \\ b_2 \end{pmatrix} \).

Lemma 7.3 Let \(\pi : N \to Y\) be a rank 2 real vector bundle, and let \((E, L, b)\) be a mildly degenerating triple on \((N,Y)\), of generic rank \(r\). Let the line subbundle \(K \subset E_Y\) be the kernel of the restriction of \(b\) to \(Y\), and let \((E_Y/K, L_Y, \overline{b_Y})\) be the non-degenerate triple of rank \(r - 1\) on \(Y\), where \(\overline{b_Y}\) is induced by \(b\). Then there exists a bilinear form \(b' : \pi^*(K) \otimes \pi^*(K) \to \pi^*(L_Y)\) such that the triple \((E, L, b)\) on \(N\) is isomorphic to the direct sum

\[(E, L, b) \cong (\pi^*(K), \pi^*(L_Y), b') \oplus \pi^*(E_Y/K, L_Y, \overline{b_Y})\]

Proof As \(Y \subset N\) is a strong deformation retract, we assume without loss of generality that \(E\) is of the form \(\pi^*(E_Y)\) and \(L\) is of the form \(\pi^*(L_Y)\). We choose a Hermitian metric \(h_Y\) on the vector bundle \(E_Y\). Let \(E''_Y\) be the orthogonal complement of \(K\) in \(E_Y\) with respect to \(h_Y\). Then the quotient map \(E_Y \to E_Y/K\) gives an isomorphism \(E''_Y \to E_Y/K\).

Let \(E'' = \pi^*(E''_Y)\) which is a rank \(n - 1\) subbundle of \(E = \pi^*(E_Y)\). Let \(b'' : E'' \otimes E'' \to L\) be induced by \(b : E \otimes E \to L\). Note that \(b''|_{Y} = \overline{b_Y}\), so the discriminant \(\det(b'')\) is non-zero on \(Y\). Since \(Y\) is paracompact we can find a positive continuous function \(f\) on \(Y\) such that for any \(x \in N\) we have \(\det(b'')_x \neq 0\) whenever \(0 \leq \|x\| \leq f(\pi(x))\). By scaling the metric using \(f\), we assume that \(\det(b'')_x \neq 0\) whenever \(0 \leq \|x\| \leq 1\).

Consider the continuous map \(\gamma : N \to N\) defined by

\[
\gamma(x) = \begin{cases} 
x & \text{if } \|x\| \leq 1, \\
 x/\|x\| & \text{if } \|x\| \geq 1.
\end{cases}
\]

The triple \((E, L, b)\) on \(N\) is then isomorphic to the pullback \(\gamma^*(E, L, b)\), which allows us to assume without loss of generality that the restriction \(b'' = b|_{E''}\) is everywhere non-degenerate on \(N\).

Let \(E' \subset E\) be the orthogonal complement of \(E''\) with respect to the form \(b : E \otimes E \to L\). As \(b\) is non-degenerate on \(E''\), it follows that \(E'\) is a line subbundle of \(E\). From its definition, we have the following equality of subbundles of \(E_Y\)

\[E'|_{Y} = K \subset E_Y\]

Let \(b' = b|_{E'}\). By its construction, the triple \((E', L, b')\) on \(N\) is non-degenerate of rank \(1\) on \(N - Y\), and degenerates on \(Y\), and we have the direct sum decomposition

\[(E, L, b) = (E', L, b') \oplus (E'', L, b'')\]
Note that as \((E'', L, b'')\) is non-degenerate, it is equivalent to a principal \(GO(n-1)\)-bundle \(P\) on \(N\). As \(Y \subset N\) is a strong deformation retract with inverse \(\pi : N \to Y\), it follows that \(P\) is isomorphic to \(\pi^*(P|_Y)\). Hence \((E'', L, b'')\) is isomorphic to \(\pi^*(E_Y/K, L_Y, \overline{b_Y})\). Also note that for the same reason, the line bundle \(E'\) is isomorphic to \(\pi^*\overline{K}^*\). This completes the proof of the lemma. \(\square\)

**Non-degenerate prolongation when \(|\nu_Y|(\det(b)) = 0\)**

**Lemma 7.4** If the absolute vanishing multiplicity \(|\nu_Y|(\det(b)) = 0\), then there exists an everywhere non-degenerate symmetric bilinear form \(b_* : E \otimes E \to L\) such that \(b_*\) coincides with \(b\) on \(N^\geq 1 = \{ v \in N \mid \|v\| \geq 1 \}\). In particular, all quadric invariants \(\alpha \in H^*(X - Y)\) of \([E, L, b]_{X - Y}\) prolong to \(H^*(X)\) as the corresponding invariants of \([E, L, b_*]\), hence map to 0 under the Gysin boundary map \(H^*(X - Y) \to H^{* -1}(Y)\).

**Proof** By Lemma 7.3 we can assume that

\[(E, L, b) = (\pi^*(K), \pi^*(L_Y), b') \oplus \pi^*(E_Y/K, L_Y, \overline{b_Y})\]

As \(\overline{b_Y}\) is non-degenerate, it follows that \(|\nu_Y|(\det(b')) = |\nu_Y|(\det(b)) = 0\). We now apply Lemma 7.6 with \(L = \pi^*(L_Y \otimes K^{-2})\) and \(\sigma = \det(b')\). Hence there exists a non-vanishing section \(b'_* \in \Gamma(N, \pi^*(L_Y \otimes K^{-2}))\) such that \(b'_*\) coincides with \(b'\) on \(N - D\), where \(D \subset N\) is the unit disk subbundle. Consider the resulting triple \((\pi^*(K), \pi^*(L_Y), b'_*)\). Then the direct sum \((\pi^*(K), \pi^*(L_Y), b'_*) \oplus \pi^*(E_Y/K, L_Y, \overline{b_Y})\) satisfies the lemma. \(\square\)

**Standard models of mildly degenerating triples**

In view of the above lemma, to prove the main theorem we only need to consider the situation where the absolute vanishing multiplicity \(|\nu_Y|(\det(Q)) \geq 1\). Therefore, \(N\) has a unique orientation such that the oriented vanishing multiplicity \(\nu_Y(\det(Q)) \geq 1\). Together with the metric on \(N\), this defines a \(\mathbb{C}\)-linear structure on \(N\). Hence in what follows, we assume that \(N\) is a complex line bundle, and \(\nu_Y(\det(Q)) = m \geq 1\).

Now let \(\tau \in \Gamma(N, \pi^*(N))\) be the tautological section defined by the identity homomorphism \(N \to N\). Note that its \(m\) th tensor power \(\tau^m \in \Gamma(N, \pi^*(N^m))\), where \(N^m\) is the \(m\) th tensor power of the line bundle \(N\), is non-vanishing outside \(Y\) and has oriented topological vanishing multiplicity \(m\) along \(Y\). On the total space of \(N\), consider the triple \((\mathcal{O}_N, \pi^*(N^m), \tau^m)\) where the bilinear form \(\tau^m : \mathcal{O}_N \otimes \mathcal{O}_N \to \pi^*(N^m)\) sends \(1 \otimes 1 \mapsto \tau^m \in \Gamma(N, \pi^*(N^m))\). Note that this triple is non-degenerate of rank \(m\) outside \(Y\) and minimal degenerate on \(Y\), with \(\nu_Y(\det(\tau^m)) = m\).

Now let \((F, N^m, q)\) be a non-degenerate triple of rank \(r - 1\) on \(Y\), where \(r \geq 2\). Then on \(N\) we have the direct sum triple

\[\mathcal{M} = (\mathcal{O}_N, \pi^*(N^m), \tau^m) \oplus (F, N^m, q)\]

From its construction, the above triple is mildly degenerating on \((N, Y)\), with \(\nu_Y(\det(\mathcal{M})) = m\).

We call this triple \(\mathcal{M}\) as the **standard model** of a mildly degenerating triple on \((N, Y)\), corresponding to the data \((F, N^m, q), m\). This name is justified by the Remark 7.3 which follows the lemma below.
Lemma 7.5 Let \( N \to Y \) be a complex line bundle, and let \((E,L,b)\) be a mildly degenerating triple on \((N,Y)\) of generic rank \( r \), with oriented vanishing multiplicity \( \nu_Y(\det(b)) = m \geq 1 \). Let \( K \subset E_Y \) be the kernel of \( b_Y : E_Y \to L_Y \otimes E_Y^* \), and let \((E_Y/K,L_Y,\overline{b_Y})\) be the induced non-degenerate triple of rank \( r-1 \) on \( Y \). Then there exists an isomorphism \( \psi : N^m \otimes K^2 \to L_Y \) of complex line bundles, such that on \( N - Y \), we have an isomorphism of non-degenerate triples

\[
(E,L,b)_{N-Y} \cong (O_{N-Y}, \pi^*(N^m), \tau^m) \otimes \pi^*(E_Y/K, L_Y, \overline{b_Y})
\]

where \( N^m \otimes K^2 \) is identified with \( L_Y \) via \( \psi \) for defining the direct sum of triples on the right hand side as in Definition 7.2, and \( \pi : N - Y \to Y \) is the projection.

Proof By Lemma 7.3 we can assume that

\[
(E,L,b) = (\pi^*(K), \pi^*(L_Y), b') \oplus \pi^*(E_Y/K, L_Y, \overline{b_Y})
\]

As \( \overline{b_Y} \) is non-degenerate, it follows that \( \nu_Y(\det(b')) = \nu_Y(\det(b)) = m \geq 1 \). We now apply Lemma 6.7 with \( M = L_Y \otimes K^{-2} \) and \( u = \det(b') \). Let \( \psi : N^m \to L_Y \otimes K^{-2} \) be an isomorphism of complex line bundles as given by Lemma 6.7(ii), so that

\[
\det(b')|_{N-Y} = f \cdot \psi(\tau^m)|_{N-Y}
\]

where \( f : N - Y \to \mathbb{C}^* \) is homotopic to the constant map \( 1 : N - Y \to \mathbb{C}^* \). Hence \( f \) admits a continuous square-root \( g : N - Y \to \mathbb{C}^* \), with \( g^2 = f \). It follows that the scalar multiplication \( g : K \to K \) gives an isomorphism of triples

\[
(g, \text{id}_L) : (\pi^*(K), \pi^*(L_Y), b') \to (\pi^*(K), \pi^*(L_Y), \psi(\tau^m))
\]

Substituting this in the direct sum decomposition of \((E,L,b)\) at the beginning of the proof, the lemma follows. \( \square \)

Remark 7.6 With notation as in the lemma above, let \( F \) be the vector bundle \((E_Y/K) \otimes K^{-1} \) on \( Y \). Then under the isomorphism \( \psi : N^m \otimes K^2 \to L_Y \), the triple \((E_Y/K,L_Y,\overline{b_Y})\) takes the form \((F,N^m,q) \otimes K\). Hence the above lemma says that given any triple \( T = (E,L,b) \) which minimally degenerates on \( Y \), there exists a standard model triple \( \mathcal{M} = (O_N, \pi^*(N^m), \tau^m) \oplus \pi^*(F,N^m,q) \) with the following properties.

1. The triples \( \mathcal{M} \otimes K \) and \( T \) have isomorphic restriction on \( Y \).
2. The triples \( \mathcal{M} \otimes K \) and \( T \) have isomorphic restriction on \( N - Y \).
3. The oriented topological vanishing multiplicity \( \nu_Y(\det(\mathcal{M})) = \nu_Y(\det(\mathcal{M} \otimes K)) \) of \( \mathcal{M} \) equals the oriented topological vanishing multiplicity \( \nu_Y(\det(T)) \) of \( T \).

Hence for the purpose of studying the topology of degeneration where \( \nu_Y(\det(b)) \geq 1 \), we can confine ourselves to quadric bundles \([\mathcal{M}]\) defined by standard models \( \mathcal{M} \).
Proof of the Main Theorem for $|\nu|_Y(\det(Q))$ even

Let $N \to Y$ be a complex line bundle. Consider a standard model triple

$$\mathcal{M} = (\mathcal{O}_N, \pi^*(N^m), \tau^m) \oplus \pi^*(F, N^m, q)$$

on $N$, where $m = 2k > 0$ is even. We define a new triple $\mathcal{N}$ by

$$\mathcal{N} = (\pi^*(N^k), \pi^*(N^m), t) \oplus \pi^*(F, N^m, q)$$

where $t : \pi^*(N^k) \otimes \pi^*(N^k) \to \pi^*(N^{2k}) = \pi^*(N^m)$ is the tensor product. We have a morphism of triples $f = (\tau^k, \text{id}) : (\mathcal{O}_N, \pi^*(N^m), \tau^m) \to (\pi^*(N^k), \pi^*(N^m), t)$, which together with identity on $\pi^*(F, N^m, q)$ gives a morphism $g : \mathcal{M} \to \mathcal{N}$. From its definition, it is clear that $g|_{N-Y}$ is an isomorphism of triples on $N-Y$, and also $g|_Y$ is an isomorphism of triples on $Y$.

We have thus shown that for any mildly degenerating triple $\mathcal{T}$ on $(N, Y)$ where $N$ is a rank 2 real vector bundle on $Y$, such that $|\nu|_Y(\det(\mathcal{T}))$ is even, there exists an everywhere non-degenerate triple $\mathcal{T}'$ on $N$ such that $\mathcal{T}$ is isomorphic to $\mathcal{T}'$ on $N-Y$. In particular, all quadric invariants $\alpha[\mathcal{T}_{N-Y}] \in H^*(N-Y)$ of $[\mathcal{T}]_{N-Y}$ prolong to $H^*(N)$ as the corresponding invariants of $[\mathcal{T}]$, hence map to 0 under the Gysin boundary map $H^*(N-Y) \to H^*+1(Y)$. This completes the proof of the main theorem when the topological vanishing parity $\nu_Y(\det(Q))$ is zero. \hfill \Box

Proof of the Main Theorem for $|\nu|_Y(\det(Q))$ odd

Let $N \to Y$ be a complex line bundle. Consider a standard model triple

$$\mathcal{M} = (\mathcal{O}_N, \pi^*(N^m), \tau^m) \oplus \pi^*(F, N^m, q)$$

on $N$, where $m = 2k + 1 > 0$ is odd. We define a new triple $\mathcal{N}$ by

$$\mathcal{N} = (\pi^*(N^k), \pi^*(N^m), \tau \circ t) \oplus \pi^*(F, N^m, q)$$

where $t : \pi^*(N^k) \otimes \pi^*(N^k) \to \pi^*(N^{2k+1}) = \pi^*(N^m)$ is induced by $\tau \in \Gamma(N, \pi^*(N))$. Again, we have a morphism of triples $f = (\tau^k, \text{id}) : (\mathcal{O}_N, \pi^*(N^m), \tau^m) \to (\pi^*(N^k), \pi^*(N^m), t)$, which together with identity on $\pi^*(F, N^m, q)$ gives a morphism $g : \mathcal{M} \to \mathcal{N}$ such that $g|_{N-Y}$ is an isomorphism of triples on $N-Y$, and also $g|_Y$ is an isomorphism of triples on $Y$. Moreover, note that $\nu_Y(\det(b_\mathcal{N})) = 1$. The triple $\mathcal{N} \otimes \pi^* (N^{-k})$ is the standard model triple

$$\mathcal{M} = (\mathcal{O}_N, \pi^*(N), \tau) \oplus \pi^*(F \otimes N^{-k}, N, q \otimes 1_{N^{-k}})$$

with $\nu_Y(\det(\mathcal{M})) = 1$. Hence we have the following.

Lemma 7.7 Let $\pi : N \to Y$ be a complex line bundle. Given any standard model triple $\mathcal{M}$ on $N$ with oriented topological vanishing multiplicity $\nu_Y(\det(\mathcal{M})) = 2k + 1 > 0$, there exists a standard model triple $\mathcal{M}$ on $N$ with the following three properties: (i) the triples $\mathcal{M} \otimes \pi^*(N^{-k})$ and $\mathcal{M}$ have isomorphic restriction on $Y$, (ii) the triples $\mathcal{M} \otimes \pi^*(N^{-k})$ and $\mathcal{M}$ have isomorphic restriction on $N-Y$, and (iii) The oriented topological vanishing multiplicity of $\mathcal{M}$ is 1.

33
Hence to prove the main theorem, it is enough to restrict ourselves to the case of quadric bundles \([M]\) on \(N\), defined by triples of the form
\[
M = (\mathcal{O}_N, \pi^*(N), \tau) \oplus \pi^*(V, N, b)
\]
where \((V, N, b)\) is a non-degenerate triple on \(Y\). Note that for such an \([M]\), the induced canonical triple \(T^M\) on \(Y\) is just \((V, N, b)\).

**Pairs of classifying maps**

Let \(T = (V, N, b)\) be a rank \(m\) non-degenerate triple on \(Y\). On \(N_o = N - Y\), the restriction of the tautological section \(\tau : \mathcal{O}_N \to \pi^*(N)\) admits an inverse \(t_{N_o}\). We get a non-degenerate pair \(t_{N_o}(T) = (\pi^*(V), t_{N_o} \circ \pi^*(b))\) of rank \(m\) on \(N - Y\) (which means \(t_{N_o} \circ \pi^*(b)\) is an \(\mathcal{O}_{N-Y}\)-valued non-degenerate symmetric bilinear form of rank \(m\) on \(\pi^*(V)\)). Let \(BGO(m)\) be a classifying space of \(GO(m)\), where \(m \geq 1\), and let \(U = (E, L, b)\) denote the universal triple on \(BGO(m)\). Consider the non-degenerate pair \(t_L(U)\) on \(L_o = L - BGO(m)\), which gives an identification of \(L_o\) with the classifying space \(BO(m)\) of \(O(m)\). Let \(\eta : Y \to BGO(m)\) be a classifying map for \((V, N, b)\). By the defining property of a classifying map, there exists an isomorphism of triples \((\alpha, \beta) : T \to \eta^*U\) where \(\alpha : V \to \eta^*E\) and \(\beta : N \to \eta^*L\) are vector bundle isomorphisms which take \(b\) to \(\eta^*b\). Let \(\eta' : \eta^*L \to L\) be the projection map, and consider the composite map \(\eta' \circ \beta : N \to L\). Then from its construction, its restriction \(\theta = (\eta' \circ \beta)|_{N_o} : N_o \to L_o = BO(m)\) is a classifying map for the non-degenerate quadratic pair \(t_{N_o}(T) = (\pi^*(V), t_{N_o} \circ \pi^*(b))\) on \(N_o\). Moreover, we have a pullback diagram (Cartesian square) of vector bundles on \(Y\) as follows
\[
\begin{array}{ccc}
N & \xrightarrow{\eta' \circ \beta} & L \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\eta} & BGO(m)
\end{array}
\]
As the Gysin boundary maps \(\partial\) commute with pullback of bundles, the above proves the following crucial lemma, by taking \(M = (\mathcal{O}_N, \pi^*(N), \tau) \oplus \pi^*(V, N, b)\), and choosing \(\theta\) as above.

**Lemma 7.8** Let \(\pi : N \to Y\) be a complex line bundle, let \(T = (V, N, b)\) be a non-degenerate triple of rank \(n - 1\) on \(Y\), where \(n \geq 2\), and let \(M\) be the mildly degenerating triple
\[
M = (\mathcal{O}_N, \pi^*(N), \tau) \oplus \pi^*(V, N, b)
\]
on \((N,Y)\). Let \(\eta : Y \to BGO(n-1)\) be a classifying map for the non-degenerate triple \(T^M = T\), and let \(\theta : N - Y \to BO(n-1)\) be a classifying map for the principal \(O(n-1)\)-bundle \(t_{N_o}(T)\). Then the following diagram is commutative for any integer \(i\).
\[
\begin{array}{ccc}
H^i(BO(n-1)) & \xrightarrow{\theta^*} & H^i(N - Y) \\
\downarrow \partial & & \downarrow \partial \\
H^{i-1}(BGO(n-1)) & \xrightarrow{\eta^*} & H^{i-1}(Y)
\end{array}
\]

34
The above lemma shows that Gysin boundary acts on invariants $\alpha[M_{N-Y}]$ of a mildly degenerating quadric $M$ on $(N,Y)$ defined by a triple $(O_N, \pi^*(N), \tau) \oplus \pi^*(V,N,b)$ exactly as given by the main theorem. As we have already reduced the proof of the main theorem to triples of the above form, this completes the proof of the main theorem. \hfill \Box

8 Sample calculations

The main theorem, combined with our earlier determination of the Gysin boundary for the fibrations $BO(2n+1) \to BGO(2n+1)$ and $BO(2n) \to BGO(2n)$ gives the following corollaries which tell us how to explicitly calculate the Gysin boundary image of any given quadric invariant.

**Corollary 8.1** (Odd rank degenerating to even rank) Let $Q$ be a mildly degenerating quadric bundle on $(X,Y)$, of generic rank $2n+1$, where $Y$ is connected. Suppose that the degeneration multiplicity $|\nu|_Y(\det(Q))$ is odd. Then under the Gysin boundary map $\partial : H^*(X - Y) \to H^{*-1}(Y)$, quadric invariants $h(\tilde{w}_2, \ldots, \tilde{w}_{2n+1})$ of $Q_{X-Y}$ map to characteristic classes $g(\lambda, a_{2i-1}, b_{ij}, d_T)$ of the triple $T_{QY}$ on $Y$ as follows. Under the substitutions $\tilde{w}_r \mapsto \left( \begin{array}{c} 2n \\ r \end{array} \right) w_1^r + \sum_{2 \leq i \leq r} \left( \begin{array}{c} 2n+1-i \\ r-i \end{array} \right) w_i^{r-i} w_i$ for $2 \leq r \leq 2n+1$ followed by putting $w_{2n+1} = 0$, let the polynomial $h(\tilde{w}_2, \ldots, \tilde{w}_{2n+1})$ map to the sum $\sum_T P_T$ indexed by subsets $T \subset \{1, \ldots, n\}$, where for each $T = \{i_1 < \ldots < i_r\}$, $P_T$ has the form

$$P_T = f_T(w_{2i-1}) \cdot g_T(w_{2i}) \cdot w_{2i_1} \cdots w_{2i_r}$$

where $f_T$ and $g_T$ are polynomials in $n$ variables each. Then $\partial(h) = \sum_T \partial(P_T)$ where

$$\partial(P_T) = \begin{cases} 0 & \text{if } r = 0 \\ f(a_{2i-1}) \cdot g(b_i) \cdot a_{2i-1} & \text{if } r = 1 \\ f(a_{2i-1}) \cdot g(b_i) \cdot d_T & \text{if } r \geq 2, \text{ where } \\ T = \{i_1 < \ldots < i_r\} \end{cases}$$

If the degeneration multiplicity $|\nu|_Y(\det(Q))$ is even, then $\partial(h) = 0$.

**Corollary 8.2** (Even rank degenerating to odd rank) Let $Q$ be a mildly degenerating quadric bundle on $(X,Y)$, of generic rank $2n+2$, where $Y$ is connected. If the degeneration multiplicity $|\nu|_Y(\det(Q))$ is odd, then under the Gysin boundary map $\partial : H^*(X - Y) \to H^{*-1}(Y)$, a quadric invariant of the form $h(\lambda, a_{2i-1}, b_{ij}, d_T)$ of $Q_{X-Y}$ maps to the characteristic class $g(c, \tilde{w}_2, \ldots, \tilde{w}_{2n+1})$ of the triple $T_{QY}$ as follows. Under the substitution

$$\lambda \mapsto 0, \ a_{2i-1} \mapsto w_{2i-1}, \ b_{ij} \mapsto w_{2j}^2 \text{ and } d_T \mapsto \sum_{i \in T} w_{2i-1}v_{T-\{i\}}$$

where $w_{2n+2} = 0$, let the polynomial $h(\lambda, a_{2i-1}, b_{ij}, d_T)$ map to the polynomial $h'(w_1, \ldots, w_{2n+1})$. Under the change of variables from $w_1, \ldots, w_{2n+1}$ to $w, \tilde{w}_2, \ldots,$
In particular, the triple

\[ \omega_{2n+1} \]

given by Lemma 2.3, let \( h' \) map to the sum \( \sum w^i f_i(\hat{w}_2, \ldots, \hat{w}_{2n+1}) \) where \( f_i \) are polynomials in \( 2n \) variables. We have

\[ \partial(h) = \sum_{j \geq 0} c_j f_{2j+1}(\hat{w}_2, \ldots, \hat{w}_{2n+1}) \]

If the degeneration multiplicity \( |\nu|_Y(\det(Q)) \) is even, then \( \partial(h) = 0 \).

**Remark 8.3** Even to odd degeneration over \((N,Y)\) Let \( \pi : N \rightarrow Y \) be a rank 2 real vector bundle, and \( \mathcal{T} = (E,L,b) \) a mildly degenerating triple on \((N,Y)\) of generic rank \( 2n \). Consider the resulting principal \( GO(2n-1) \)-bundle \( \mathcal{T}_Y \) on \( Y \). The character \( \kappa : GO(2n-1) \rightarrow \mathbb{C}^* \) defined by \( g \mapsto \sigma(g)^n / \det(g) \) is a square-root of the defining character \( \sigma \) of \( L_Y \), hence \( \kappa \) defines a line bundle \( K \) on \( Y \) together with an isomorphism \( \varphi : K^2 \rightarrow L_Y \). Hence the triple \( \mathcal{T} \otimes \pi^*(K^{-1}) \) takes the form \((V,\mathcal{O}_N,q)\). In particular, the triple \( \mathcal{T}_{N-Y} \otimes \pi^*(K^{-1}) \) on \( N-Y \) has its structure group reduced from \( GO(2n) \) to \( O(2n) \). Thus in the special case where \( X \) is the total space of a rank 2 real vector bundle on \( Y \) with \( Y \subset X \) the zero section, when studying mild degenerations in which the generic rank is even, it is enough to consider quadric invariants only in \( PH^*(BO(2n)) \).

**Examples**

In the following explicit examples, the multiplicity \( |\nu|_Y(\det(Q)) \) is assumed to be odd, as otherwise \( \partial \) maps each invariant to \( 0 \) as shown.

**Example 8.4** Rank 3 degenerating to rank 2 In this case, the ring of quadric invariant is \( PH^*(BGO(3)) = \mathbb{F}_2[\hat{w}_2, \hat{w}_3] \), while \( H^*(BGO(2)) = \mathbb{F}_2[\lambda, a_1, b_1]/(\lambda a_1) \).

Using the Corollary 8.1 we explicitly determine the following.

\[
\begin{align*}
\partial((\hat{w}_2^3 \hat{w}_3^2))(Q_{X-Y}) &= 0 \\
\partial((\hat{w}_2^2 \hat{w}_3^3))(Q_{X-Y}) &= (a_1^1 + b_1^1) a_1^3 b_1^2 (\mathcal{T}^Q_Y) \\
\partial((\hat{w}_2^2 \hat{w}_3^4))(Q_{X-Y}) &= (a_1^1 + b_1^1) a_1^2 b_1^4 (\mathcal{T}^Q_Y) \\
\partial((\hat{w}_2^2 \hat{w}_3^5))(Q_{X-Y}) &= (a_1^1 + b_1^1) a_1^2 b_1^4 (\mathcal{T}^Q_Y)
\end{align*}
\]

**Example 8.5** Rank 4 degenerating to rank 3 In this case the quadric invariants of \( Q_{X-Y} \), by Proposition 4.3, are the polynomials in \( \lambda(Q_{X-Y}), a_1(Q_{X-Y}), (a_1 a_3 + b_1)(Q_{X-Y}) \) and \((a_1^2 + a_1 d_{1,2})(Q_{X-Y}) \).

The characteristic classes of the triple \( \mathcal{T}^Q_Y \) are the polynomials in \( c(\mathcal{T}^Q_Y), \hat{w}_2(\mathcal{T}^Q_Y), \) and \( \hat{w}_3(\mathcal{T}^Q_Y) \).

Using the Corollary 8.2 we can explicitly determine the following.

\[
\begin{align*}
\partial(\lambda(Q_{X-Y})) &= 0 \\
\partial(a_1(Q_{X-Y})) &= 1 \\
\partial((a_1 a_3 + b_1)(Q_{X-Y})) &= \hat{w}_3(\mathcal{T}^Q_Y) \\
\partial((a_1^2 + a_1 d_{1,2})(Q_{X-Y})) &= (c\hat{w}_3 + \hat{w}_2 \hat{w}_3)(\mathcal{T}^Q_Y)
\end{align*}
\]

36
Example 8.6  **Rank 5 degenerating to rank 4** In this case, the quadric invariants of $Q_{X-Y}$ are polynomials in $\hat{w}_2(Q_{X-Y}), \ldots, \hat{w}_5(Q_{X-Y})$. The boundary behavior of the $\hat{w}_i(Q_{X-Y})$, following Corollary 8.1, is given in Example 8.8 below. As another example, $\partial((\hat{w}_2\hat{w}_3)(Q_{X-Y})) = (a_1^2b_3)(\mathcal{T}^{Q_Y})$.

Example 8.7  **Rank 6 degenerating to rank 5** In this case the quadric invariants of $Q_{X-Y}$, by Theorem 4.8, are the polynomials in the cohomology classes

$$
\lambda(Q_{X-Y}), \alpha'_1(Q_{X-Y}), \alpha'_3(Q_{X-Y}), \alpha'_5(Q_{X-Y}), \beta'_8(Q_{X-Y}), \beta'_{12}(Q_{X-Y}), \delta'_{(2,3)}(Q_{X-Y}).
$$

The characteristic classes of the triple $\mathcal{T}^{Q_Y}$ are the polynomials in $c(\mathcal{T}^{Q_Y}), \hat{w}_2(\mathcal{T}^{Q_Y}), \hat{w}_3(\mathcal{T}^{Q_Y}), \hat{w}_4(\mathcal{T}^{Q_Y}), \hat{w}_5(\mathcal{T}^{Q_Y})$. Using the Corollary 8.2 we get the following.

$$
\begin{align*}
\partial(\lambda(Q_{X-Y})) &= 0 \\
\partial(\alpha'_1(Q_{X-Y})) &= 1 \\
\partial(\alpha'_3(Q_{X-Y})) &= (c + \hat{w}_2)(\mathcal{T}^{Q_Y}) \\
\partial(\alpha'_5(Q_{X-Y})) &= (c \hat{w}_2 + \hat{w}_3^2)(\mathcal{T}^{Q_Y}) \\
\partial(\beta'_8(Q_{X-Y})) &= (\hat{w}_3 \hat{w}_4 + \hat{w}_3^2\hat{w}_2)(\mathcal{T}^{Q_Y}) \\
\partial(\beta'_{12}(Q_{X-Y})) &= (\hat{w}_2^2 \hat{w}_5 + \hat{w}_4\hat{w}_5 + c \hat{w}_2^3 \hat{w}_3 + \hat{w}_3^2 \hat{w}_3 \hat{w}_4)(\mathcal{T}^{Q_Y}) \\
\partial(\delta'_{(2,3)}(Q_{X-Y})) &= (c^2 \hat{w}_2 + \hat{w}_3^3)(\mathcal{T}^{Q_Y}) \\
\end{align*}
$$

Example 8.8  We consider the general case of odd degenerating to even, where the quadric invariants of $Q_{X-Y}$ are all the polynomials in $\hat{w}_2(Q_{X-Y}), \ldots, \hat{w}_{2n+1}(Q_{X-Y})$. The Corollary 8.1 in particular gives

$$
\begin{align*}
\partial(\hat{w}_{2r}(Q_{X-Y})) &= \sum_{i=1}^{r} \binom{2n + 1 - 2i}{2r - 2i} (a_1^{2r-2i}a_{2i-1})(\mathcal{T}^{Q_Y}) \\
\partial(\hat{w}_{2r+1}(Q_{X-Y})) &= \sum_{i=1}^{r} \binom{2n + 1 - 2i}{2r - 2i} (a_1^{2r-2i+1}a_{2i-1})(\mathcal{T}^{Q_Y})
\end{align*}
$$

Note that $\partial(\hat{w}_{2n+1}(Q_{X-Y})) = a_1(\mathcal{T}^{Q_Y})\partial(\hat{w}_{2r}(Q_{X-Y}))$, which reflects the well-known relation $\text{Sq}^1 w_{2i} = w_{2i+1} + w_1 w_{2i}$ (Wu’s formula).

9  **Appendix : Material from Toda [T]**

For any integer $N \geq 1$, let $x_1, \ldots, x_N$ and $y$ be independent variables over the field $\mathbb{F}_2$. Let $A$ be the polynomial ring $A = \mathbb{F}_2[x_1, \ldots, x_N]$, and let $\phi : A \rightarrow \mathbb{F}_2[t] \otimes A$ be the ring homomorphism defined by $\phi(x_i) = \sum_{i=0}^{N-1} \binom{N-1}{r} t^r \otimes x_i$ where $x_0 = 1$. This makes $A$ into a Hopf algebra comodule over $\mathbb{F}_2[t]$, as follows from the topological interpretation of $\phi$ as the map $H^*(BG) \rightarrow H^*(BG) \otimes H^*(BG)$ induced by the multiplication map $\Gamma \times G \rightarrow G$ where $(G, \Gamma)$ is the pair $(GL(N, \mathbb{C}), \mathbb{C}^\times)$ or the pair
The additive isomorphism of abelian groups gives a finite set of generators and relations for the ring \( PA \) when \( N = 4m + 2 \) for some \( m \geq 0 \), and also when \( N = 4 \), which is recalled below.

Consider the maps \( d_i : A \to A \), for \( i \geq 0 \), defined by \( d_i(x_r) = (N-r+i)x_{r-i} \), so that \( \phi(x) = \sum_{i \geq 0} x_i \otimes d_i(x) \). In particular, the map \( d_1 : A \to A \) is the derivation \( s = \sum_{1 \leq i \leq N}(N-i+1)x_{i-1}\frac{\partial}{\partial x_i} \) which played an important role in [H-N], with \( d_1 \circ d_1 = 0 \). The following proposition is an important step in Toda’s determination of the ring of primitive elements.

**Proposition 9.1** (Lemma 3.6 of Toda [T]) Let \( N = qh \) where \( q \) is a power of 2 and \( h \) is odd. Consider the \( \mathbb{F}_2 \)-vector subspace (that is, additive subgroup) \( B \) of \( A \), defined by \( B = \{ a \in A \mid d_i(a) = 0 \text{ for all } i \geq q \} \). Then the multiplication map \( \mathbb{F}_2[x_q] \times B \to A \) which sends \( (f(x_q), b) \mapsto f(x_q)b \) induces an \( \mathbb{F}_2[x_q] \)-linear bijection \( \mathbb{F}_2[x_q] \otimes_{\mathbb{F}_2} B \to A \). In particular, it induces an additive isomorphism \( \psi : B \to A/x_qA \).

**Remark 9.2** \( B \) is not a subring of \( A \), that is, the inclusion \( B \hookrightarrow A \) does not preserve multiplication. However, the primitive ring \( PA \) is a subring of both \( B \) and \( A \). The importance of the ring \( B \) is that it helps us get to \( PA \).

**Remark 9.3** The proof in [T] of the surjectivity of the map \( \mathbb{F}_2[x_q] \otimes B \to A \) is constructive, that is, given any element in \( A \), one can recursively calculate an explicit preimage in \( \mathbb{F}_2[x_q] \otimes B \). In particular, given any \( a \in A/x_qA \), we can determine explicitly, in a recursive manner, its preimage \( \psi^{-1}a \in B \) under \( \psi : B \to A/x_qA \).

It can be seen that \( d_i(x_q) = x_{q-i} \) for all \( 1 \leq i \leq q-1 \). Consider the elements \( \hat{x}_i \in A \) defined by

\[
\hat{x}_i = x_i \text{ for } 1 \leq i \leq q,
\]

\[
\hat{x}_{kq} = \psi^{-1}(x_{kq}) \text{ for } k \geq 2, \text{ and}
\]

\[
\hat{x}_{kq-i} = d_i(\hat{x}_{kq}) \text{ for } k \geq 2, \text{ and } 1 \leq i \leq q-1.
\]

The additive isomorphism of abelian groups \( B \to A/x_qA \) is used by Toda to define a ring structure (multiplication operation \( * \)) on the additive group \( B \), pulling back the ring structure on the quotient ring \( A/x_qA = \mathbb{F}_2[x_1, \ldots, x_{q-1}, x_{q+1}, \ldots, x_N] \). Under this, \( B \) becomes the polynomial ring \( B = \mathbb{F}_2[\hat{x}_1, \ldots, \hat{x}_{q-1}, \hat{x}_{q+1}, \ldots, \hat{x}_N] \) in the \( N-1 \) variables \( \hat{x}_i (i \neq q) \) defined above (see Proposition 3.7 of [T]).

**Example 9.4** When \( N = 2n + 1 \) is odd, the elements \( \hat{x}_r \) are given by \( \hat{x}_r = \sum_{i=0}^{r} \binom{2n+1-i}{r-i} x_{1}^{-i} x_i \). The primitive ring \( PA \), when \( N = 2n + 1 \) is odd, is the subring \( \mathbb{F}_2[\hat{x}_2, \ldots, \hat{x}_{2n+1}] \subset A \).
The above case of odd \( N \) is the case where \( q = 1 \). Next we consider the case where \( q = 2 \), that is, \( N = 2n = 4m + 2 \) is congruent to 2 modulo 4. The simplest such \( N \) is \( N = 2 \), where it can be seen that \( \hat{x}_1 = x_1 \) and \( PA = B = \mathbb{F}_2[x_1] \).

**Example 9.5** When \( N = 4m + 2 \) for some \( m \geq 0 \), we have \( \hat{x}_1 = x_1 \), and the elements \( \hat{x}_3, \ldots, \hat{x}_N \in B \) can be calculated as follows. To begin with, we define \( s_0 = 1 \) and \( t_0 = 0 \). Now for each \( i \geq 0 \) we define \( s_i \) and \( t_i \) recursively by \( s_i = x_2 t_{i-1} \) and \( t_i = s_{i-1} + x_1 t_{i-1} \). For example, \( s_1 = 0 \) and \( t_1 = 1 \), and \( s_2 = x_2 \) and \( t_2 = x_1 \). The elements \( \hat{x}_1, \ldots, \hat{x}_N \) are given by

\[
\hat{x}_{2r} = \sum_{i=0}^{2r} \binom{4m + 2 - 2r + i}{i} x_{2r-i} s_i \quad \text{and} \quad \hat{x}_{2r-1} = \sum_{i=0}^{2r} \binom{4m + 2 - 2r + i}{i} x_{2r-i} t_i
\]

**The ring \( PA \) when \( N = 2n = 4m + 2 \)**

In this case the action can be written down as follows (recall that \( \hat{x}_1 = x_1 \) and \( \hat{x}_2 = x_2 \) in this case).

\[
\phi(\hat{x}_{2k-1}) = 1 \otimes \hat{x}_{2k-1} \quad \text{for all } 1 \leq k \leq 2m + 1, \\
\phi(\hat{x}_2) = 1 \otimes \hat{x}_2 + t \otimes \hat{x}_1 + t^2 \otimes 1, \quad \text{and} \\
\phi(\hat{x}_{2k}) = 1 \otimes \hat{x}_{2k} + t \otimes \hat{x}_{2k-1} \quad \text{for all } 2 \leq k \leq 2m + 1.
\]

We now write Toda’s system of generators \( \alpha_{2k-1} \), \( \delta_T \), and \( \beta_{4k} \) for the ring \( PA \).

For \( 1 \leq k \leq 2m + 1 \), let

\[
\alpha_{2k-1} = \hat{x}_{2k-1} = d_1(\hat{x}_{2k})
\]

The ring multiplication \( * : B \times B \to B \) can be extended as a binary operation \( * : A \times A \to A \) by putting

\[
b \ast c = bc + d_1(b) d_1(c) x_2 \quad \text{for all } b, c \in A
\]

Using this, more generally for any subset \( T = \{p_1, \ldots, p_r\} \subset \{2, \ldots, 2m + 1\} \) of cardinality \( r > 1 \), we define

\[
\delta_T = d_1(\hat{x}_{2p_1} \ast \cdots \ast \hat{x}_{2p_r})
\]

Next, we define for \( 2 \leq k \leq 2m + 1 \),

\[
\beta_{4k} = \hat{x}_{2k} \ast \hat{x}_{2k} + x_1 \hat{x}_{2k-1} \hat{x}_{2k}
\]

**Proposition 9.6** (Proposition 3.11 of Toda [T]) Let \( N = 4m + 2 \) for \( m \geq 0 \). Then the primitive ring \( PA \) is generated by the elements \( \alpha_{2i-1}, \beta_{4i}, \delta_T \) defined above.

(Toda also gives the relations, for which see the above reference).

**Primitive ring when \( N = 4 \)**

In the case \( N = 4 \), Toda proves the following.

**Proposition** (Toda [T] Proposition 3.12) For \( N = 4 \), the ring \( PA \) is the polynomial ring \( \mathbb{F}_2[x_1, d_4, d_6] \), where \( d_4 = x_2^2 + x_1 x_3 \) and \( d_6 = x_3^2 + x_1^2 x_4 + x_1 x_2 x_3 \).
The primitive rings for $BO(N)$ and $BGL(N)$

The above algebraic material applies to the following topological cases. Consider a pair $(G, \Gamma)$ consisting of a Lie group $G$ and a central subgroup $\Gamma \subset G$. Let $\Gamma \times G \to G$ be the multiplication map, sending $(\gamma, g) \mapsto \gamma g$. This is a group homomorphism as $\Gamma$ is central, so induces a map $B(\Gamma \times G) = B\Gamma \times BG \to BG$ on the classifying spaces. Let $\phi : H^*(BG) \to H^*(B\Gamma) \otimes H^*(BG)$ be the induced ring homomorphism on cohomology. Now we consider two cases, as follows.

**Orthogonal group** When $(G, \gamma) = (O(N), \{\pm 1\})$, $H^*(BG) = \mathbb{F}[w_1, \ldots, w_N]$ is a polynomial ring on the Stiefel-Whitney classes $w_i$, and $H^*(B\Gamma) = \mathbb{F}[w]$. The map $\phi$ is given by $w_r \mapsto \sum_{0 \leq i \leq r} \binom{N-i}{r-i} w^{r-i} w_r$.

**General linear group** When $(G, \gamma) = (GL(N), \mathbb{C}^*)$, $H^*(BG) = \mathbb{F}[^2, \ldots, \mathbb{C}^N]$ is a polynomial ring on the mod 2 Chern classes classes $\mathbb{C}^i$, and $H^*(B\Gamma) = \mathbb{F}[t]$. The map $\phi$ is given by $t_r \mapsto \sum_{0 \leq i \leq r} \binom{N-i}{r-i} t^{r-i} \mathbb{C}^i$.

Hence with appropriate change of notation, the material above gives the primitive rings for the orthogonal groups or general linear groups in the cases when $N$ is odd, or when $N = 4$ or when $N = 4m + 2$ for some $m \geq 0$.

**Notation for generators of $PH^*(BO(N))$ and $PH^*(BGL(N))$**

For $N = 4m + 2$, we denote the generators for $PH^*(BO(N))$ by $\alpha_{2k-1} = \hat{w}_{2k-1}$, $\delta_T = d_1(\hat{w}_{2p_1} \cdots \hat{w}_{2p_r})$, and $\beta_{4k} = \hat{w}_{2k} \ast \hat{w}_{2k} + w_1 \hat{w}_{2k-1} \hat{w}_{2k}$. On the other hand, for $N = 4m + 2$ we will denote the generators for $PH^*(BGL(N))$ by $\alpha''_{2k-1} = \hat{c}_{2k-1}$, $\delta''_T = d_1(\hat{c}_{2p_1} \cdots \hat{c}_{2p_r})$, and $\beta''_{4k} = \hat{c}_{2k} \ast \hat{c}_{2k} + \mathbb{C} \hat{c}_{2k-1} \hat{c}_{2k}$.

**Remark 9.7** The ring homomorphism $\theta^* : H^*(BGL(N)) \to H^*(BO(N))$ induced by the inclusion $\theta : O(N) \to GL(N)$ maps $\mathbb{C}^i \mapsto w_i^2$ for each $i$. Hence under $\theta^* : H^*(BGL(4m + 2)) \to H^*(BO(4m + 2))$ the invariants $\alpha''_{2i-1}$, $\beta''_{4i}$ and $\delta''_T$ map to $\alpha''_{2i-1}$, $\beta''_{4i}$ and $\delta''_T$ respectively.

**The tensor product** $\mathbb{C}^2 \otimes \mathbb{C}^{2m+1} \to \mathbb{C}^{4m+2}$

Consider the homomorphism $\tau'' : GL(2m + 1) \times GL(2) \to GL(4m + 2)$ defined by the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^{2m+1} \to \mathbb{C}^{4m+2}$. This induces a ring homomorphism

$$B(\tau'')^* : H^*(BGL(4m + 2)) \to H^*(BGL(2)) \otimes H^*(BGL(2m + 1))$$

The following proposition is a part of the Proposition 4.2 in Toda [T].

**Proposition 9.8** With the above notations we have $B(\tau'')^*(\alpha''_{2i-1}) = 0$ for all $i > 1$, and $B(\tau'')^*(\delta''_T) = 0$ for every $T = \{i_1, \ldots, i_k\} \subset \{2, \ldots, 2m + 1\}$.

This has the following analog for $O(4m + 2)$.

**Proposition 9.9** Let the group homomorphism $\tau : O(2) \times O(2m + 1) \to O(4m + 2)$ be induced by the tensor product. Then we have $B(\tau)^*(\alpha_{2i-1}) = 0$ for all $i > 1$, and $B(\tau)^*(\delta_T) = 0$ for every $T = \{i_1, \ldots, i_k\} \subset \{2, \ldots, 2m + 1\}$.  

40
**Proof** Observe that the following diagram commutes

\[
\begin{array}{ccc}
H^*(BGL(4m+2)) & \xrightarrow{B(\tau'')^*} & H^*(BGL(2)) \otimes H^*(BGL(2m+1)) \\
\theta^* \downarrow & & \downarrow \theta^* \otimes \theta^* \\
H^*(BO(4m+2)) & \xrightarrow{B(\tau)^*} & H^*(BO(2)) \otimes H^*(BO(2m+1))
\end{array}
\]

where \(\theta^* : H^*(BGL(4m+2)) \to H^*(BO(4m+2))\) is the natural map which takes the element \(c_i \mapsto w_{2i}^2\). Hence this map also takes the elements \(\hat{c}_{2i-1}\) to \(\hat{w}_{2i-1}^2\) and \(\delta''_T\) to \(\delta_T^2\). Now for any \(a \in H^*(BGL(4m+2))\) such that \(B(\tau'')^*(a) = 0\), we have \(B(\tau)^*\theta^*(a) = 0\). This implies that \(B(\tau)^*(\hat{w}_{2i-1}^2) = 0\) and \(B(\tau)^*(\delta_T^2) = 0\). Now the proposition follows from the fact that \(H^*(BO(2)) \otimes H^*(BO(2m+1))\) has no zero divisors, being a polynomial ring.

\(\square\)

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