Reeb components with complex leaves and their symmetries I:
The automorphism groups and Schröder’s equation on the half line

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Abstract
We review the standard Hopf construction of Reeb components with leafwise complex structure and determine the group of leafwise holomorphic smooth automorphisms for tame Reeb components in the case of complex leaf dimension one. For this, we solve the Schröder type functional equation on the half line for expanding diffeomorphism. As a result, we see that the automorphism group of one with trivial linear holonomy on the boundary contains an infinite dimensional vector space, while in the case of non-trivial linear holonomy the group is of finite dimensional.

0 Introduction

The aim of this article is to begin a study of Reeb components in foliation with complex leaves of codimension one, especially focused on the symmetry in the real 3-dimensional case.

Quite often we call them leafwise complex foliations.

Recall that a \((p+1)\)-dimensional Reeb component is a compact manifold \(R = D^p \times S^1\) with a (smooth) foliation of codimension one, whose leaves are graphs of smooth functions \(f : \text{int}D^p \to \mathbb{R}\) where \(\lim_{z \to \partial D^p} f(z) = +\infty\), and a compact leaf which is the boundary \(S^{p-1} \times S^1\). Here we identify \(R\) with \((D^p \times \mathbb{R})/\mathbb{Z}\). See also the figures in Section 2.

Foliations of codimension one with complex leaves are drawing attentions in several complex variables because it appears as the Levi foliations of Levi-flat real hypersurfaces in complex manifolds. A simple construction of Hopf manifolds admits a Levi-flat real hypersurface, whose Levi foliation consists of a pair of Reeb components. This construction is generalized in Nemirovskii’s examples \([Ne]\). They have non-trivial linear holonomy along toral leaves. In this paper, from rather topological points of view, we study Reeb components with all kinds of holonomy. Of course, the case where the holonomy is \textit{flat to the identity}, i.e., it is infinitely tangent to the identity at the origin, is included. Such Reeb components appear in

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turbulization of a codimension one foliation along a closed transversal, or in pasting constructions like Dehn surgery of 3-manifolds.

The present paper is organized as follows. In Part I we study Reeb components with complex leaves and their symmetries, as well as leafwise complex foliations in general. Part II is devoted to the study of Schröder’s functional equation on the half line, whose results are the core part of the study of the automorphism group of Reeb components.

In Section 1 we fix the basic notions on foliations with complex leaves. They are also called leafwise complex foliations. After these preparations, Reeb components and the turbulization are reviewed in the context of leafwise complex foliations in Section 2. Here we also review the notion of tameness of Reeb components, which we always assume in studying the symmetries.

Then, we study the symmetries of a Reeb component with complex leaves on a 3-manifold.

In section 3 we investigate the structure of the group of automorphisms of Reeb components with complex leaves of complex-dimension 1. Except for the case where the centralizer of the holonomy diffeomorphism in \( \text{Diff}^\infty [0, \infty) \) is not exactly known, we completely determine the structure of the automorphism group. This exception happens for some of diffeomorphisms which is flat to the identity. In anyway, we see that the automorphism group is of finite dimensional if the linear holonomy of the boundary leaf is non-trivial. On the other hand, if the linear holonomy is trivial, the automorphism group always contains an infinite dimensional vector space. (Theorem 3.9). Such a clear contrast results from the analysis of Schröder’s equation in Part II. Similar results are obtained for Reeb components of complex leaf dimension 2. They are explained by one of the authors in [Ho].

In Section 4 some direct corollaries to the results in Section 3 are stated. For example, the automorphism group of a Reeb foliation with complex leaves on the three sphere is understood.

The study of moduli space of tame Reeb component is studied by Meersseman and Verjovsky [MV]. The moduli exhibits to a certain degree a similar phenomena to those of compact complex manifolds, especially concerning the finite dimensionality. As to automorphism groups our result tells that only the Reeb components with non-trivial linear holonomy on the boundary shows such a similarity.

The second part of the paper is devoted to the study of Schröder’s equation on the half line. It is in a form which is looking for eigen solutions for a pull-back operator. Here the pull-back diffeomorphism is nothing but the holonomy of the Reeb component when it is applied to Part I. In fact the results are the main ingredients in describing the automorphism groups of Reeb components in Part I. In section 5, we describe the space of solutions to Schröder’s equation. We also extend the values of the equation to \( \mathbb{C}^2 \) or still higher dimensional case, which is used in [Ho].
For a diffeomorphism of the half line with non-trivial linear part, the computation is easy and in fact is well-known. For expanding diffeomorphisms with trivial linear part at the origin, the proofs are given in the subsequent sections. We present two different proofs.

In Section 6 a direct proof is given for the diffeomorphisms which are flat to the identity. This proof has a similar flavor to one by the center manifold theory, which is given in the final section.

In Section 7, a proof given for diffeomorphisms with non-trivial finite jets, i.e., those with the Taylor expansion at the origin different from the identity. This proof relies on Takens’ normal form [1a] and the classical Fourier series on smooth function on the circle.

Neither of the proofs in Section 7 nor 8 works in other cases.

Then in the final section, we give a unified proof which is applicable for both of the above cases. The main tool is the center manifold theory of partially hyperbolic dynamical systems. This proof was suggested by Masayuki Asaoka.

Throughout this article, we assume manifolds and foliations to be smooth unless otherwise stated.

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Part I : Reeb components with complex leaves

1 Basic definitions

Let $M$ be a $(2n + q)$-dimensional smooth manifold and $\mathcal{F}$ be a smooth foliation of codimension $q$ on $M$ and let $p = 2n$ be the dimension of leaves. In this section and in the next, $n$ and $q$ are reserved for the complex leaf dimension and the codimension. We refer general basics for foliation theory to [CC].

Definition 1.1 (Leafwise complex foliation, cf. [MV]) A smooth foliation $\mathcal{F}$ on a smooth manifold $M$ is said to be a leafwise complex foliation or foliation with complex leaves if there exists a system of local smooth foliated coordinate charts $(U_\lambda, \varphi_\lambda)$ where $\varphi_\lambda : U_\lambda \to V_\lambda \subset \mathbb{C}^n \times \mathbb{R}^q = \{(z_1, \cdots, z_n, y_1, \cdots, y_q)\}$ is a smooth diffeomorphism onto an open set $V_\lambda$ such that the coordinate change $(w_1, \cdots, w_n, t_1, \cdots, t_q) = \gamma_{\mu\lambda}(z_1, \cdots, z_n, y_1, \cdots, y_q)$ is smooth, $t_j$’s depend only on $y_k$’s ($j, k = 1, \cdots, q$), and when $y_k$’s are fixed $w_l$’s are holomorphic in $z_m$’s, where $\gamma_{\mu\lambda} : \varphi_\lambda(V_\lambda \cap U_\mu) \to \varphi_\mu(V_\lambda \cap U_\mu)$. 


Remark 1.2  Instead of assuming local coordinate system as above, it is also natural to consider the following condition that the smooth foliation $\mathcal{F}$ admits a smooth almost complex structure $J$ acting on the tangent bundle $T\mathcal{F}$ to the foliation, which is integrable on each leaves, namely there exists local holomorphic coordinates on each leaves. We assume here $J$ is smooth on the ambient manifold $M$. (Of course $J$ becomes more than smooth in each leaf.)

This might appear slightly weaker than Definition 1.1, but eventually they are equivalent to each other. To prove from the weaker to the stronger is nothing but the parametric version of Newlander-Nirenberg’s theorem. The Newlander-Nirenberg theorem [NN] in the usual sense claims that an almost complex manifold $(L,J)$ admits a complex structure (a holomorphic local coordinate system) if the Nijenhuis tensor $N_J$ vanishes. In dead in [NN] Newlander and Nirenberg mentioned in the very last paragraph that the parametric version holds. For the case of $n = 1$, even the parametric version seems to be classically known, e.g., see [Mo].

It is also well known that if an almost complex structure $J$ is real analytic on $2n$-dimensional real analytic manifold $L$, the Newlander-Nirenberg theorem has a simple geometric proof. See, for example, Appendix A4 of [Hu]. Using this argument, if we can take such a smooth foliated chart $(u_1,u_2,\cdots,u_{2n},y_1,\cdots,y_q)$ that $J$ is real analytic on $(u_1,\cdots,u_{2n})$, we can show the existence of a local coordinate system in Definition 1.1.

It should be also remarked that the case of Levi-flat real hypersurface $M$ in an $(n+1)$-dimensional complex manifold $W$, the stronger one is easily satisfied. (If $M$ is of class $C^r$, then we can only assure that $TM$ is of class $C^{r-1}$, so that the resultant local coordinate system is assured to be of class $C^{r-1}$.)

Definition 1.3  A diffeomorphism between two foliated manifolds with complex leaves is said to be an isomorphism between leafwise complex foliations iff it preserves the foliations and gives rise to biholomorphisms between leaves. An automorphism is an isomorphisms between the same one.

In this paper we are mainly concerned with foliation of codimension one. In particular, our interest will be focused on Reeb components of real dimension 3, namely in the case of $n = 1$ and $q = 1$. As we see from the examples of Nemirovskii [Ne] even a real analytic Levi-flat hypersurface in a complex manifold can admit Reeb components in its Levi foliation. In such a case, the holonomy along the toral boundary leaf has a non-trivial linear part.

Apart from Levi-flat real hypersurfaces, for example, if we perform a turbulization we easily find various leafwise complex foliations admitting Reeb components with holonomy flat to the identity. See the next section for more detail.
2 Reeb components with complex leaves

In this section we review a particular construction of Reeb component with complex leaves and a process of turbulization which produces a new Reeb component in a leafwise complex foliation.

In order to make pasting construction easier, we introduce the following notions. Let \((R, F, J)\) (or simply \(R\) for short) be a Reeb component with leafwise complex structure of complex leaf dimension \(n\) and \((H, J_H)\) be its boundary leaf.

**Definition 2.1** The Reeb component \(R\) has a tame boundary (or ‘\(R\) is tame at boundary’ for short, or even shorter ‘\(R\) is tame’) with respect to a product coordinate \(H \times [0, \varepsilon)\) of a collar neighborhood of \(H\) if it gives rise to a smooth foliation with leafwise complex structure when it is pasted with the product foliation \((H \times (-\varepsilon, 0), \{H \times \{x\}| x \in (-\varepsilon, 0]\}, J_H)\) along their boundary. Here each leaf \(H \times \{x\}\) has the same complex structure as \(H\) when identified with the natural projection. Namely, the Reeb component is extended to the outside as a product foliation.

The notion of tameness was introduced in [MV].

**Remark 2.2** If we forget the leafwise complex structure and consider the same notion only as foliation of codimension one, it does not depend on the choice of product coordinate on the positive side and the tameness implies exactly that the holonomy is tangent to the identity to the infinite order. This is because the set of expanding diffeomorphisms of the half line \([0, \infty)\) which are infinitely tangent to the identity is an open convex cone and invariant under conjugation by any diffeomorphism. Also remark that the tameness depends only on the smooth projection of the collar neighborhood to the boundary, which the product coordinate defines. If two projections have the same infinite jets on the boundary, the tameness notion coincides for the two.

**Definition 2.3** The leafwise complex structure of a Reeb component \(R\) is simple around boundary (or \(R\) has a simple complex structures around boundary) if the boundary has a collar neighborhood \(U \cong H \times [0, \varepsilon)\) such that the restriction of the projection \(U = H \times [0, \varepsilon) \to H\) to each leaf in \(U\) is holomorphic.

This notion should also be understood relative to the projection from a collar neighborhood to the boundary.

The notions of tameness and simpleness apply not only to Reeb components but also to more general leafwise complex foliations of codimension one with a compact leaf or a boundary leaf.

Clearly if a Reeb component has simple complex structures around the boundary and the holonomy of the boundary leaf is infinitely tangent to
the identity, it is tame with respect to the appropriate projection. The tame-
ness condition prohibits unexpected wild behaviour around boundary. In
particular in the case of complex leaf dimension $= 1$, it induces a strong
consequence due to Meersseman and Verjovsky. See the following subsection.

2.1 Reeb component by Hopf construction

Let us recall the Hopf construction which is one of the standard ways to
construct Reeb components. This construction gives rise to a tame Reeb
component if the holonomy $\varphi$ is infinitely tangent to the identity at $x = 0$.

Construction 2.4 (Hopf construction) Let $\varphi \in \text{Diff}^\infty([0, \infty))$ a diffeomor-
phism of the half line $\mathbb{R}_{\geq 0} = [0, +\infty)$ satisfying $\varphi(x) - x > 0$ for $x > 0$,
namely the origin is an expanding unique fixed point. Also take a (lo-
cal) biholomorphic diffeomorphism $G \in \text{Diff}^{\text{hol}}(\mathbb{C}^n, O)$ which is expand-
ing. This implies that for some small neighborhood $D$ of the origin $O$ with
smooth boundary $G(\text{int}D) \supset D$ and $\cap_{k=1}^{\infty} G^{-k}(D) = \{O\}$. Now take
$U = \bigcup_{k=1}^{\infty} G^k(D) \subset \mathbb{C}^n.$

Then on $\tilde{R} = U \times [0, \infty) \setminus \{(O, 0)\} \subset \mathbb{C}^n \times \mathbb{R}$, take the restriction $\tilde{F}$ of
the product foliation $\{\mathbb{C}^n \times \{x\}\}$ together with the natural complex struc-
ture on leaves and a diffeomorphism $T = G \times \varphi$ on $U \times [0, \infty) \setminus \{(O, 0)\}$. Prac-
tically we take fairly simple diffeomorphisms such as linear maps as $G$ so that $U$ becomes the whole $\mathbb{C}^n$. Then on the quotient $R = \tilde{R}/T^\mathbb{Z}$, a
foliation $\mathcal{F}$ with complex leaves is naturally induced.

From the construction, it is simple around the boundary. If the holon-
omy is infinitely tangent to the identity it is also tame with respect to the
coordinate in the construction.

The boundary $U \setminus \{O\}/G^\mathbb{Z}$ is a complex manifold which is a so called
Hopf manifold. In the case $n = 1$ it is an elliptic curve and the construction
is equivalent to one with linear map as $G$.

Remark 2.5 It is well known as an elementary fact in complex dynamical
systems that assuming $G(\text{int}D) \supset \overline{D}$ for a bounded connected domain
$D \subset \mathbb{C}^n$ is enough to conclude that there is a unique linearly expanding
fixed point in $D$ and $D$ is included in the attracting basin of $G^{-1}$.

Theorem 2.6 (Meersseman-Verjovsky, [MV]) Any tame Reeb component
with complex leaves of complex dimension 1 is isomorphic to one of those
given by the Hopf construction.

We present a couple of extensions (variants) of the above construction.

Construction 2.7 Now, let us take the product not with the half line but
with the whole real line $\mathbb{R}$. Let $M$ and $\Phi \in \text{Diff}^\infty_+(\mathbb{R})$ be as follows.
\[ M = (U \times \mathbb{R} \setminus \{(0,0)\}) / T' \mathbb{Z}, \quad T' = G \times \Phi, \]
\[ x = 0 \] is an expanding unique fixed point of \( \Phi \).

\( M \) consists of two Reeb components and in exactly the same way as above a foliation with leafwise complex structure is induced on \( M \).

Note that in this construction and in the previous one, the holonomy of the toral leaf is given by \( \Phi \) and by \( \varphi \) respectively.

Construction 2.8 Next, we further extend the previous construction to obtain a Reeb component as a part of a Levi-flat hypersurface in a Hopf surface. We take \( \Phi \) to be a linear expansion in order to extend it a biholomorphic (in fact linear) expansion \( \tilde{\Phi} \) of \( \mathbb{C} \).

Note that in this construction and in the previous one, the holonomy of the toral leaf is given by \( \Phi \) and by \( \varphi \) respectively.

Construction 2.10 We introduce one more construction, which is a preparation for turbulization. Take \( \tilde{M} = (\mathbb{C}^n \times \mathbb{R}) \setminus \{(0)\} \times (-\infty,0] \) and restrict the product action \( \tilde{T} = G \times \Psi \) to \( \tilde{M} \), where \( \Psi \) is an orientation preserving diffeomorphism of \( \mathbb{R} \) which fixes 0, expanding on \([0,\infty)\), and contracting on the negative side \((-\infty,0]\), i.e., \( \Psi(x) > x \) for \( x < 0 \). On \( \tilde{M} \) we take (the restriction of) the horizontal foliation \( \tilde{F} \). Then take the quotient \((M, F, J_F) = (\tilde{M}, \tilde{F}, J_{\text{std}}) / \tilde{T} \mathbb{Z} \).

The non-negative part is nothing but the Reeb component constructed in [2.4] regarding \( \varphi = \Psi|_{[0,\infty)} \). The non-positive side \((N, \mathcal{G}) = (M, F)|_{x<0} \) remains non-compact and is in fact a foliated \((-\infty,0]\)-bundle with holonomy \( \psi = \Psi|_{(-\infty,0]} \).

Problem 2.9 Theorem 2.6 due to Meersseman and Verjovsky poses the following questions. We assume the complex leaf dimension to be one. Provided that two Reeb components with leafwise complex structures have the same boundary holonomy and their boundary leaves are biholomorphic to each other, are they isomorphic as leafwise complex foliations? Does there exist a Reeb component with complex leaves which is not isomorphic to a tame one but with holonomy infinitely tangent to the identity? Or does there exist one which is not isomorphic to any of those given by the Hopf construction? One more similar but subtle question is to ask whether if a tame Reeb component is always isomorphic to one given by the Hopf construction.

The second form of question seems less difficult and negative. Anyway, those questions are asking what should be the complete invariants to determine Reeb components without assuming the tameness.
If we remove the boundary compact leaf \( \{ x = 0 \} \) from the non-positive side \((N, \mathcal{G})\), it is isomorphic to \((C^n \setminus \{ O \}) \times S^1\). For a better description of turbulization process, let us be more precise about this identification. This is done by embedding \( \tilde{T} \) in a 1-parameter family. Take a smooth curve \( G_t \) in \( GL(2; \mathbb{C}) \) and also a smooth curve \( \psi_t \) in \( Diff^\infty(-\infty,0) \) satisfying the following conditions.

\[
\psi_k = \psi^k (k \in \mathbb{Z}), \quad \psi_{t+1} = \psi \circ \psi_t (t \in \mathbb{R}), \quad \frac{\partial \psi_t(x)}{\partial t} > 0 \quad (\forall x, t),
\]

\[
G_k = G^k (k \in \mathbb{Z}), \quad G_{t+1} = G \circ G_t (t \in \mathbb{R}).
\]

Then, fixing (any) \( x_0 < 0 \), \( x = \psi_t(x_0) \) gives a diffeomorphism between \((-\infty,0) (\ni x) \) and \( \mathbb{R}(\ni t) \). Then the identification of \((z,x) \in (C^n \setminus \{ O \}) \times (-\infty,0) \) with \((w,t) \in (C^n \setminus \{ O \}) \times \mathbb{R} \) by \((z = G_t(w), x = \psi_t(x_0)) \) conjugates \( \tilde{T}_{|x<0} \) into \((w,t) \mapsto (w, t+1) \).

Of course an easy way to choose such a 1-parameter family \( \{ \psi_t \} \) is to take a 1-parameter subgroup. Take a smooth vector field \( \rho(x) \frac{dx}{dt} \) on \((-\infty,0] \) with \( \rho(x) > 0 \) for \( x < 0 \) and \( \rho(0) = 0 \). Then putting \( \psi_t = \exp(tX) \) we obtain such \( \psi_t \) with \( \psi = \psi_1 \). If we choose \( \rho(x) \) to be flat at \( x = 0 \), \( \psi_t \) is infinitely tangent to the identity at \( x = 0 \). Even if \( \psi \) does not ly

It is worth remarking that this identification gives rise to a partial compactification of horizontally foliated manifold \((C^n \setminus \{ O \}) \times S^1\), \((C^n \setminus \{ O \}) \times \{ t \} \) by a Hopf manifold \( N \) as a boundary leaf so as to obtain \((N, \mathcal{G})\). If we take diffeormorphisms \( \psi_t \) infinitely tangent to the identity at the origin, we obtain a tame structure. Also on the non-negative side, by taking simlar family \( \phi_t \) for \( \varphi = \varphi_1 \), we also obtain a tame structure on the non-negative side. Once we obtain tame ones on both side with the same complex structure on the boundaries, we can paste them to obtain a smooth structure.

### 2.2 Turbulization in \( L \times S^1 \)

Here we review the turbulization, which is classically well-known modification of a foliation of codimension one to yield a new Reeb component. We start from a standard situation.

**Construction 2.11** Let \((M, \mathcal{F})\) be a leafwise complex foliation of codimension one and assume that there is an embedded solid torus \( U = intD^{2n} \times S^1 \) on which the the induced foliation is \( \{ intD^{2n} \times \{ * \} \} \) and the induced complex structure is also the canonical ones on each \( intD^{2n} \times \{ * \} \) \( \cong intD^{2n} \subset C^n \). Let \((w,t)\) denote the natural coordinate of \( U = intD^{2n} \times S^1 \) where \( S^1 \) is regarded as \( \mathbb{R}/\mathbb{Z} \). Then we remove \( \{ O \} \times S^1 \) from \( U \) and let \( U^* \) denote the result. Using the coordinate \((w,t)\) \( U^* \) is identified with an open subset of the negative side of Construction 2.10 together with leafwise complex foliations. Therefore we can compactify this end with the
Hopf manifold as in Construction 2.10 and also if we add positive side of Construction 2.10 we obtain a leafwise complex foliated manifold without boundary but with a new Reeb component. For this construction we can choose any of $G_t$, $\psi_t$, and $\Phi$ as in Construction 2.10. The above process including adding the positive side is the leafwise complex version of the turbulization. See also Figure 1 below.

2.3 General case

It is easy to find a closed transversal to a foliation of codimension one, namely, an embedded circle which is transverse to the foliation, unless the manifold is open and the foliation is too simple. Like in the case of a smooth foliation without leafwise complex structure, it is always possible to perform the turbulization in a tubular neighborhood of any closed transversal regarding leafwise complex structure. This fact also belongs a kind of folklore, while below it is reviewed.

**Theorem 2.12** Let $(M^{2n+1}, \mathcal{F}, J)$ be a smooth leafwise complex foliation of codimension one and $K \subset M$ is a closed transversal, namely there exists a smooth embedding $f : S^1 \to M$ which is transverse to the foliation $\mathcal{F}$ with its image $f(S^1) = K$.

Then, there exists a tubular neighborhood $U \cong K \times \text{int} \ D^{2n}$ such that the restricted foliation $(U, \mathcal{F}_U, J|_{\mathcal{F}_U})$ is isomorphic to the standard one $(S^1 \times \text{int} \ D^{2n}, \mathcal{F}_0 = \{1\} \times \text{int} \ D^{2n}, J_0)$ and through this isomorphism $K$ is identified with $S^1 \times \{O\}$.

In particular, we can perform the standard turbulization 2.11 in $U$.

This theorem is a direct corollary to the following lemma.

**Lemma 2.13** The group $\text{Diff}^{\text{hol}}(\mathbb{C}^n, O)$ of germs of holomorphic diffeomorphisms of $(\mathbb{C}^n, O)$ which fix the origin is pathwise connected.

The lemma immediately follows from the two facts that $GL(n; \mathbb{C})$ is pathwise connected and that such a germ with identical linear part can be joined by a straight segment to the identity.

2.4 Dehn surgery in $\dim = 3$ vs. higher dimensional turbulization

In order to close the section, this subsection provides with some remarks concerning the possibility of pasting the Reeb component in a different way in a turbulization. In the rest of this section, we assume the holonomy $\Psi$, and eventually $\phi$ and $\psi$, to be infinitely tangent to the identity at the origin, so that it is easier to past two pieces along their boundary.
Remark 2.14  If we forget the leafwise complex structure and treat foliations only as smooth objects, basically there are two ways to perform the turbulization. The one has been already described above and is indicated in Figure 1. For the other one we can reverse the top and bottom of the Reeb component (Figure 2). This is because the cyclic (universal for \( n \geq 2 \)) covering of the boundary leaf is \( \mathbb{R}^{2n} \setminus \{O\} \cong S^{2n} \setminus \{N,S\} \) and two ends are exchangeable by a diffeomorphism. However, as a complex manifold, \( C^n \setminus \{O\} \) has one convex end and the another concave. For the case of complex leaf dimension \( n \) greater than one, these two ends are not exchangeable. In particular, for \( n \geq 2 \), the turbulization for leafwise complex foliations does not change the homotopy class of the tangent bundle.

![Figure 1](image1.png)  ![Figure 2](image2.png)

The green lines indicate the boundary leaves of Reeb components. The axes of the rotational symmetries of the Reeb components, which are not drawn in the figure, correspond to \( \{O\} \times \mathbb{R}_+ \).

Remark 2.15  In the case of complex leaf dimension one, the ‘upside-down’ construction always works. Namely, in Construction 2.4, \( z \leftrightarrow z^{-1} \) always induces an biholomorphism on the boundary elliptic curve. Therefore we can regard it as a turbulization indicated in Figure 2.

If the boundary elliptic curve admits a complex multiplication, namely finite but discrete symmetries of order 2, 3 or 4, removing the Reeb component and pasting it back with one of those symmetries is a special kind of Dehn surgeries.

More generally, in the process of turbulization, after removing a tubular neighborhood of the closed transversal and compactify the new boundary (namely after the process of Construction 2.10), instead of filling up
with a Reeb component as explained above, we fill up the boundary as follows. We prepare another Reeb component with a different complex structure. If their boundaries match up through some diffeomorphism, we can fill up the boundary with that Reeb component. In this way, a Dehn surgery corresponding to any element of the mapping class group $\mathcal{M}_1(\cong SL(2;\mathbb{Z}))$ of a 2-dimensional torus $T^2$ is realized for a closed transversal in a leafwise complex codimension one foliation of $n = 1$.

3 Symmetries of 3-dimensional Reeb components

In this section we compute the group of automorphisms of a Reeb component of dimension 3 with complex leaves, which is given by the Hopf construction. In order to fix notations, we present our objects again. Let $\tilde{R}$ be $\mathbb{C} \times [0, \infty) \setminus \{(0, 0)\}$, take $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and a diffeomorphism $\varphi \in \text{Diff}^{\infty}([0, \infty))$ which is expanding, namely, satisfying $\varphi(x) - x > 0$ for $x > 0$. Let $G$ denote the linear expansion of $\mathbb{C}$ defined as the multiplication by $\lambda$ and $T : \tilde{R} \to \tilde{R}$ be $T = G \times \varphi$. Then we obtain a Reeb component $(\tilde{R}, \tilde{F}, J) = (\tilde{R}, \tilde{F}, J_{\text{std}}) / T\mathbb{Z}$ as the quotient, as well as the boundary elliptic curve $H = \mathbb{C} \setminus \{0\} / G\mathbb{Z}$. Here, on the upstairs the leaves of the foliation $\tilde{F} = \{\mathbb{C} \times \{x\} | x > 0\} \sqcup \{\mathbb{C}^* \times \{0\}\}$ are equipped with the natural complex structure $J_{\text{std}}$ which is inherited by those of $F$.

We classify the diffeomorphisms $\varphi$ into the following three cases according to the nature of its jet at $x = 0$. Here, $\varphi^{(i)}$ denotes the $i$-th derivative of $\varphi(x)$ and $j^i \varphi(0)$ denotes the $i$-th jet at $x = 0$.

Case (1) : $\varphi'(0) = \mu > 1$, 

Case (2) : for some $n \geq 2$, $j^{n-1} \varphi(0) = j^{n-1} \text{id}(0)$ and $\varphi^{(n)}(0) > 0$, 

Case (3) : $j^\infty \varphi(0) = j^\infty \text{id}(0)$.

The discussions in Subsection 3.1 does not depend on the above classification. As reviewed in Section 3.2, the structure of the centralizer of $\varphi$ is very subtle for a certain class in Case (3). For the rest of Case (3) and for Case (1) and (2), the centralizer is fairly simple.

The main result of this paper is the computation of the automorphisms which fixes the boundary and the transverse space. This is in deed the main results of Part II of this paper. Concerning this part, for Case (2) and for Case (3) the results are the same. For Case (1), such automorphisms are very few.

The internal structure of the automorphism group for Case (2) and (3) rather depends only on the nature of the centralizer (see Subsection 3.3). The centralizer itself is included in the automorphism group. Except for this part, the extendability to the outside is basically the same for any $\varphi$ in Case (2) and (3).
3.1 Lift to $\tilde{R}$ and restriction to $H$

Let us consider the group $Aut(R, F, J)$, which is also denoted by $Aut R$ for short, of all foliation preserving diffeomorphisms of $R$ whose restriction to each leaf is holomorphic. Also we consider the group of holomorphic diffeomorphisms $Aut H$ of the boundary elliptic curve $H$ as well as its identity component $Aut_0 H$ which is isomorphic to $T^2$ and can be identified with $H$ itself.

Proposition 3.1 The image of the restriction map $r_H : Aut R \to Aut H$ is exactly $Aut_0 H$.

Proof. If we regard $Aut H / Aut_0 H$ as a subgroup of $SL(2; \mathbb{Z})$, in most cases it is just $\{ \pm E \}$ where $E$ denotes the identity matrix. In a few cases where the elliptic curve $H$ admits complex multiplications, they are of order 3, 4, or of 6 and a kind of ‘rotations’ on the universal covering, i.e., elliptic matrices in $SL(2; \mathbb{Z})$. In any of those cases, no element in $Aut H \setminus Aut_0 H$ preserves the direction of holonomy and thus none extends to $R$ as a foliation preserving diffeomorphism.

On the other hand, any element in $Aut_0 H$ is obtained as the quotient of the scalar multiplication $m_a : C^* \to C^*$ by some nonzero complex number $a$. The automorphism $m_a \times \text{id}_{[0, \infty)}$ of $\tilde{R}$ clearly descends to $R$ and defines an element in $Aut R$. \hfill $\square$

By this proposition, the study of the structure of $Aut R$ breaks into two parts, that of the kernel $Aut(R, H)$ and the study of the restriction map $r_H$.

Now it is easier to look at the lifts of automorphisms on $\tilde{R}$. Any element $f \in Aut R$ has a unique lift $\tilde{f} \in Aut(\tilde{R}, \tilde{F}, I_{\text{std}}) (= Aut \tilde{R})$ which takes the form

$$\tilde{f}(z, x) = (\xi(z, x), \eta(x))$$

in $C \times [0, \infty)$-coordinate. A lift $\tilde{f}$ should commutes with the covering transformation $T$, because, $T \circ \tilde{f} = \tilde{f} \circ T^k$ for some $k \in \mathbb{Z}$ but it is easy to see that $k = 1$ when it is restricted to the boundary. Therefore an element in $Aut \tilde{R}$ is a lift of some element in $Aut R$ if and only if it commutes with $T$. Let $Aut(\tilde{R}; T)$ denote the centralizer of $T$ in $Aut \tilde{R}$, namely, the group of all such lifts. It contains an abelian subgroup $\{ m_a \times \text{id}_{[0, \infty)} | a \in C^* \} \cong C^*$. This subgroup injectively descends to a subgroup of $Aut R$ which restricts exactly to $Aut_0 H \cong C^* / \lambda \mathbb{Z}$. It is important to remark that whether $Aut_0 H$ admits a homomorphic section is not a trivial question. Postponing this question until the end of this section, we go on an easier way.

Let us introduce one more subgroup $Aut(\tilde{R}, \tilde{H}; T)$ of $Aut(\tilde{R}; T)$ which consists of all elements which act trivially on the boundary $\tilde{H}$. Any element $f \in Aut(R, H)$ has a unique lift to an element $\tilde{f} \in Aut(\tilde{R}, \tilde{H}; T)$ Namely,
Corollary 3.2  \( \text{Aut}(R, H) \) is isomorphic to \( \text{Aut}(\tilde{R}, \tilde{H}; T) \).

Again, let \( g \in \text{Aut}\tilde{R} \) be presented in the form \( g(z, x) = (\xi(z, x), \eta(x)) \).

Lemma 3.3  The element \( g \) in \( \text{Aut}\tilde{R} \) belongs to \( \text{Aut}(\tilde{R}; T) \) if and only if it satisfies the following conditions.

(a) \( \xi(z, x) = az + b(x), b(0) = 0 \) for some \( b \in C^\infty([0, \infty), \mathbb{C}) \) and \( a \in \mathbb{C}^* \).

(b) \( b(\varphi(x)) = \lambda b(x) \).

(c) \( \varphi \circ \eta = \eta \circ \varphi \), namely, \( \eta \in Z_\varphi = \) the centralizer of \( \varphi \) in \( \text{Diff}^\infty([0, \infty)) \).

Further more, \( g \) belongs to \( \text{Aut}(\tilde{R}, \tilde{H}; T) \) if and only if the above conditions are satisfied with \( a = 1 \).

Proof.  Let us first show the only if direction, then the if direction will become almost trivial.

Assume \( g \in \text{Aut}(\tilde{R}; T) \). \( \xi(z, x) \) is smooth and holomorphic in \( z \). If \( x \) is fixed, \( \xi(\cdot, x) : \mathbb{C} \to \mathbb{C} \) is a holomorphic automorphism even in the case where \( x = 0 \) because the origin is a removable singularity; it is a linear map with nontrivial linear term. Therefore it is written in the following form:

\[
\xi(z, x) = a(x)z + b(x) \quad \text{where} \quad a(x), b(x) \in C^\infty([0, \infty), \mathbb{C}) \quad \text{with} \quad a(x) \neq 0 \quad \text{and} \quad b(0) = 0.
\]

These also apply to elements in \( \text{Aut}\tilde{R} \).

Now look at the commutation relation \( g \circ T = T \circ g \). This implies

\[
(a(\varphi(x))\lambda z + b(\varphi(x)), \eta(\varphi(x)) = (\lambda a(x)z + \lambda b(x), \varphi(\eta(x))).
\]

Thus we obtain (b) and (c). This also tells us that \( a(\varphi(x)) = a(x) \), so that for any \( x \geq 0 \) we have \( a(x) = \lim_{n \to \infty} a(\varphi^{-n}(x)) = a(0) \) and (a) is concluded.

For \( g \in \text{Aut}(\tilde{R}, \tilde{H}; T) \) we just need to confirm that \( a = 1 \). \( \square \)

Remark 3.4  The condition (b) appears as Equation (I) in Part II. Solving this Schröder type functional equation on the half line \([0, \infty)\) for given \( \lambda \) and \( \varphi \) is the main theme in Part II.

Corollary 3.5  \( \text{Aut}R \) is naturally isomorphic to \( \text{Aut}(\tilde{R}; T)/T^Z \).

3.2 Centralizer of \( \varphi \) in \( \text{Diff}^\infty([0, \infty)) \)

For an expanding diffeomorphism \( \varphi \in \text{Diff}^\infty([0, \infty)) \), concerning its centralizer in \( \text{Diff}^\infty([0, \infty)) \) and the embeddability in a (smooth) 1-parameter subgroup, the followings are known.

Theorem 3.6  1) For Case (1) thanks to Sternberg’s linearization [St] and for Case (2) thank to Takens’ normal form [13], there exists a smooth vector field \( X = \rho(x)\frac{d}{dx} \) on \([0, \infty)\) such that \( \varphi = \exp X \) and the centralizer \( Z_\varphi \) exactly coincides with the 1-parameter subgroup \( \exp(tX) ; t \in \mathbb{R} \).
2) In Case (3), if there exists a smooth vector field \( X = \rho(x) \frac{d}{dx} \) on \([0, \infty)\) with \( \varphi = \exp X \), then its centralizer \( Z_\varphi \) in coincides with the 1-parameter subgroup \( \exp(tX); t \in \mathbb{R} \). In general, the centralizer \( Z_\varphi \) of \( \varphi \) which is infinitely tangent to the identity at \( x = 0 \) is known to be fairy wild (see [Ey]). For an expanding \( \varphi \), it is known that there exists a unique \( C^1 \)-vector field \( X_\varphi = \rho(x) \frac{d}{dx} \) on the half line \([0, \infty)\), which is of class \( C^\infty \) on \((0, +\infty)\), in such a way that the exponential map \( \exp X_\varphi \) coincides with \( \varphi \) (see [Sz] and [Na]). This vector field is often called the Szekerez vector field of \( \varphi \). If the Szekerez vector field \( X_\varphi \) is of class \( C^\infty \) on \([0, \infty)\), namely \( \rho(x) \) is smooth and flat at \( x = 0 \), then the centralizer \( Z_\varphi \) coincides with the 1-parameter family \( \{ \exp(tX_\varphi) = \varphi^t; t \in \mathbb{R} \} \) generated by \( X_\varphi \).

In general case, using the order of real numbers, the centralizer \( Z_\varphi \) turns out to be a totally ordered abelian group which contains \( \{ \varphi^Z \} \cong \mathbb{Z} \). Therefore it is uniquely identified with a certain subgroup of the additive group \( \mathbb{R} \) under the identification \( \{ \varphi^Z \} \cong \mathbb{Z} \). Depending on \( \varphi \), \( Z_\varphi \) can be far beyond the normal expectation, e.g., it can be \( \mathbb{Z}, \mathbb{Q}, \) or \( \mathbb{Z} \oplus \mathbb{Z} \alpha \) where \( \alpha \) is a Liouville number [Ey], or far more complicated. The topology on \( Z_\varphi \) through this identification with natural topology of \( \mathbb{R} \) coincides with the one induced from the \( C^0 \)-topology on \( \text{Diff}^\infty([0, \infty)) \).

We should also remark that for \( \varphi \) which is infinitely tangent to the identity at \( x = 0 \), so is any element of \( Z_\varphi \).

At present it is not known whether \( Z_\varphi \cong \mathbb{R} \) implies the smoothness of \( X_\varphi \) at \( x = 0 \). This is a subtle point in Hilbert’s problem No. 5 when it is stated in the context of a continuous homomorphism from a Lie group to a group of diffeomorphisms. The difficulty occurs when the orbit is not compact.

### 3.3 Structure of \( AutR \)

Upon all the previous preparations we are able to describe the structure of \( AutR \) as follows.

**Proposition 3.7** Let \( R \) be a Reeb component of real dimension 3 which is given by the Hopf construction.

1) The group \( AutR \) of automorphisms of the Reeb component \( R \) admits a following sequence of extensions by abelian groups \( Aut_0H, Z_\varphi, \) and \( K_{\lambda, \varphi}, \)

\[
0 \to Aut(R, H) \to AutR \to Aut_0H \to 0
\]

\[
0 \to K_{\lambda, \varphi} \to Aut(R, H) \to Z_\varphi \to 0
\]

where \( Aut_0H \cong \mathbb{C}^* / \lambda \mathbb{Z} \) is the multiplication by the constant linear part \( a \mod \lambda \mathbb{Z} \) as described in Lemma 3.3 and \( Z_\varphi \) is the centralizer of \( \varphi \) which is
explained in the previous subsection, and $K_{\lambda, \varphi}$ is identified with the space of solutions to the equation (b) in Lemma 3.3. As explained in Part II, $K_{\lambda, \varphi}$ is isomorphic to an infinite dimensional vector space $Z_{\lambda, \varphi}$ in Case (2) and (3), while in Case (1) it is a complex vector space of dimension 1 or 0 according to the resonance condition $\lambda = (\varphi'(0))^n$ for some $n \in \mathbb{N}$ or not.

The following is an important consequence to Theorem 5.2 in Part II on $K_{\lambda, \varphi} \cong Z_{\lambda, \varphi}$ in Case (2) and (3).

**Corollary 3.8** If we paste $H \times (-\varepsilon, 0]$ to $R$ along the boundary $H$, in Case (3) any element of $\text{Aut}(R, H)$ extends to the other side, being the identity on $H \times (-\varepsilon, 0]$, as a diﬀeomorphisms of class $C^\infty$. In Case (2) the same applies to $K_{\lambda, \varphi}$.

**Proof of Proposition.** The first step of the extensions is obtained by looking at the action on the boundary, and once we assume that the action on the boundary is trivial, the second extension is obtained by looking at the action on the vertical line $\{0\} \times [0, \infty)$. We can interpret it as an action on the leaf space. \qed

In the above, the first extension does not yield a non-abelian group. Using the identification $\text{Aut}R \cong \text{Aut}(\tilde{R}; T)/T^Z$ in Corollary 3.5, we obtain a better description not only from the above point of view but also from that of the question whether the restriction map $r_H : \text{Aut}R \to \text{Aut}_0 H$ admits a homomorphic section. Note that $Z_{\varphi}$ admits a section to $\text{Aut}(R, H) \subset \text{Aut}R$.

An element $f \in \text{Aut}(\tilde{R}; T)/T^Z$ admits a presentation $f(z, x) = (az + b(x), \eta(x))$ up to $T^Z$ where $T(z, x) = (\lambda z, \varphi(x))$. Therefore ignoring $b(x)$ from this presentation and assigning $f \mapsto (a, \eta) \bmod (\lambda, \varphi)^Z$, we obtain a surjective homomorphism $\text{Aut}(\tilde{R}; T)/T^Z \twoheadrightarrow (\mathbb{C}^* \times Z_{\varphi})/(\lambda, \varphi)^Z$ to an abelian group. Also, by setting $b(x) = 0$, we see this abelian group can be realized as a subgroup of $\text{Aut}(\tilde{R}; T)/T^Z$. This enables us to describe the structure of $\text{Aut}R$ as follows.

**Theorem 3.9** The automorphism group $\text{Aut}R \cong \text{Aut}(\tilde{R}; T)/T^Z$ is isomorphic to the semi-direct product

$$K_{\lambda, \varphi} \rtimes \left\{ (\mathbb{C}^* \times Z_{\varphi})/(\lambda, \varphi)^Z \right\}$$

where $a \in \mathbb{C}^*$ acts on $b(x) \in K_{\lambda, \varphi}$ by multiplication $b(x) \mapsto a^{-1}b(x)$, i.e., the conjugation in the affine transformations of each leaf, and $\eta \in Z_{\varphi}$ acts by $b(x) \mapsto b(\eta(x))$.

**proof.** Let us only verify the action of $a$. The conjugation by the multiplication by $a$ is $[z \mapsto z + b(x)] \mapsto [z \mapsto a^{-1}(az + b(x))] = z + a^{-1}b(x)$. \qed
To close this section, consider the liftability of $\text{Aut}_0 H$ to $\text{Aut} R$. This is nothing but the liftability of the surjective homomorphism

$\left( \mathbb{C}^* \times \mathbb{Z}_\varphi \right) / (\lambda, \varphi)^{\mathbb{Z}} \rightarrow \mathbb{C}^*/\lambda^{\mathbb{Z}}$.

Here we assume the continuity of splitting, otherwise the question should include thinking about non-continuous homomorphism $\mathbb{R} \rightarrow \mathbb{R}$ with $1 \mapsto 1$. If the centralizer $Z_\varphi$ is the total of $\mathbb{R}$, it implies $Z_\varphi$ is a $C^0$-family of 1-parameter subgroup $\{ \eta_t ; t \in \mathbb{R} \}$ in $\text{Diff}^\infty([0, \infty))$ with $\varphi = \eta_1$. Then we obtain easily a lift defined as

$$a \left( \text{mod} \lambda^{\mathbb{Z}} \right) \mapsto \left( a, \eta_t(a) \right) \left( \text{mod} (\lambda, \varphi)^{\mathbb{Z}} \right), \quad t(a) = \frac{\log |a|}{\log |\lambda|}.$$  

The converse is almost the same. If we have a continuous lift to $\text{Aut}(\tilde{\mathbb{R}}; T)/T^{\mathbb{Z}}$, choose a value of $\log \lambda$ and look at the lift of a circle subgroup $e^{t \log \lambda}$ ($0 \leq t \leq 1$) to a continuous path in $\text{Aut}(\tilde{\mathbb{R}}; T)$ starting from the identity. Then its projection to $Z_\varphi$ gives rise to a 1-parameter family in $Z_\varphi$. If this curve is smooth, it implies that the Szekeres vector field $X_\varphi$ of $\varphi$ is smooth. Thus we obtain the following.

**Theorem 3.10** The restriction map $r_H : \text{Aut} R \rightarrow \text{Aut}_0 H$ admits a continuous [resp. smooth] homomorphic section if and only if the centralizer $Z_\varphi$ is isomorphic to $\mathbb{R}$ as an ordered abelian group [resp. the Szekeres vector field $X_\varphi$ is smooth]. In such cases, $\text{Aut} R$ admits not only a structure of twice semi-direct products

$$\text{Aut} R \cong (\mathcal{K}_{\lambda, \varphi} \times Z_\varphi) \times \text{Aut}_0 H \cong (\mathcal{K}_{\lambda, \varphi} \times \mathbb{R}) \times \mathbb{R}^2/\mathbb{Z}^2,$$

but also a structure of simple semi-direct product

$$\text{Aut} R \cong \mathcal{K}_{\lambda, \varphi} \rtimes (\text{Aut}_0 H \times Z_\varphi) \cong \mathcal{K}_{\lambda, \varphi} \rtimes (\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}).$$

of two abelian groups, where the action of the right group on the left is continuous with respect to the smooth topology on $\mathcal{K}_{\lambda, \varphi} \cong Z_{\lambda, \varphi} \subset C^\infty([0, \infty); \mathbb{C})$ in Case (2) and (3) and $\mathcal{K}_{\lambda, \varphi} \cong \mathbb{C}$ or $\{0\}$ in Case (1) [resp. smooth in a usual sense].

**Remark 3.11** We saw that in Case (2) and (3) the automorphism groups are of infinite dimension, while in Case (1) it is of finite dimension and shows a similarity to compact complex manifold.

In fact, in Case (3), not only any Reeb component is realized in a Hopf surface in Construction [2.8] but also any of its automorphisms extends to the ambient Hopf surface. Undr the setting $n = 1$, the Hopf surface $W$ is obtained as $W = (\mathbb{C}^2 \setminus \{O\})/T''$ where $T''(z, w) = (\lambda \cdot z, \mu \cdot w)$. Then a Levi-flat hypersurface $M = (\mathbb{C} \times \mathbb{R} \setminus \{O\})/T''$ contains a Reeb component.
$R$ of Case (3) for $\lambda$ and $\mu$. It is easy to see that the automorphism $(z, w) \mapsto (a \cdot z + bw^p, c \cdot w)$ of the Hopf surface $W$ restricts to $R$ where $a, c \in \mathbb{C}^*$ and $b \in \mathbb{C}$ are arbitrary constants in the resonant case $\lambda = \mu^p$ and $b = 0$ in the non-resonant case. Therefore any of $\text{Aut} R$ extends to $W$.

On the other hand, a recent work of Koike and Ogawa [KO] seems suggesting that Reeb componets of Case (3) never appers in a Levi-flat hypersurfaces in a complex surface. Our result also mildly suggests that the same might apply to Case (2).

Even in Case (1) it should be still confirmed whether if the automorphism extends to the ambient surface in the case where the Reeb component appears as a part of a Levi-flat hypersurface which bounds a Stein surface.

4 Reeb foliations

The automorphism group of a leafwise complex foliation on a closed 3-manifold which consists of two Reeb components is now easy to compute. In this section we assume that the relevant holonomy is infinitely tangent to the identity at the origin, because it must be so except for two special cases where two Reeb components are pasted in the same direction of the holonomies or exactly in the inverse direction, in both of which cases the pasting yields $S^2 \times S^1$.

Let $R_{\varphi, \lambda}$ be the Reeb component which we dealt with in the previous section. For another pair of a diffeomorphism $\psi \in \text{Diff}_{\infty}([0, \infty))$ which is also expanding and infinitely tangent to the identity at the origin and a constant $\mu \in \mathbb{C}$ with $|\mu| > 1$, take the Reeb component $R_{\psi, \mu}$ and let $\overline{R}_{\varphi, \mu}$ denote the mirror of $R_{\varphi, \mu}$, namely the one which we obtain by reversing the the transverse orientation. It is done by replacing $x$ and $\varphi(x)$ with $-x$ and $-\varphi(-x)$ in the Hopf construction.

For example if $\lambda = \mu$ we can paste $R_{\varphi, \lambda}$ and $\overline{R}_{\varphi, \mu}$ along the common boundary $H = \mathbb{C}^*/\lambda \mathbb{Z}$ by the identity of $H$ to obtain a leafwise complex foliation on $S^2 \times S^1$. In general according to the pasting element $\in SL(2; \mathbb{Z})$ we can choose appropriately $\lambda$ and $\mu$ and paste them. The foliation on $S^3$ obtained in such a way is called the Reeb foliation.

Corollary [KO] yields the following results.

**Theorem 4.1** Let $(M, \mathcal{F}, J)$ be a leafwise complex foliation which is obtained by pasting $R_{\varphi, \lambda}$ and $\overline{R}_{\psi, \mu}$. Then its group of automorphism is naturally isomorphic to the fibre product of $\text{Aut} R_{\varphi, \lambda}$ and $\text{Aut} \overline{R}_{\psi, \mu}$ with respect to $\text{Aut} H$.

If the centralizer $Z_{\psi}$ is isomorphic to $\mathbb{R}$ as an ordered abelian group, then $\text{Aut} R_{\varphi, \lambda}$ is continuously realized as a subgroup in the resulting group of automorphisms.
Theorem 4.2  If $(M^3, F_1, J_1)$ is obtained from $(M^3, F_0, J_0)$ by turbulization along a closed transversal and the resulting Reeb component is isomorphic to $AutR_{\varphi, \lambda}$, the group $Aut(M^3, F_1, J_1)$ naturally contains a subgroup which is isomorphic to $Aut(R_{\varphi, \lambda}, H)$.

Remark 4.3  In both of above theorems, the automorphism group contains an infinite dimensional vector space $Z$ or one more copy. Thus even in the case of closed manifolds, the automorphism group of leafwise complex foliation can be fairly large. This presents a clear contrast between a leafwise complex foliation on a compact manifold and a complex structure on a compact manifold. Due to Meersseman and Verjovsky [MV] in the study of moduli spaces, they present similar features as far as we deal with tame leafwise complex foliation.

Part II : Schröder’s equation on the half line

We study the functional equations on the half line $[0, \infty)$ which appeared in Section 3. The simpler one takes the form

$$\beta \circ \varphi(x) = \lambda \beta(x)$$

for a fixed diffeomorphism $\varphi$ and a constant $\lambda$.

Ernst Schröder started to study a similar (in fact, formally the same) functional equation on the unit disk $D$ in the complex plane $\mathbb{C}$ under the complex analytic setting in [Sch] in the late 19th century. Not only because it is just the natural eigenvalue problem for the pull-back operator to look for $\beta$ and $\lambda$ for a given $\varphi$, also Schröder initiated complex dynamical studies and was interested in the iteration of compositions maybe in the context of Newton’s method. According to the development of the complex dynamics the problems that were treated in these epoch has become fairly well-understood. Recently the studies in this direction seem to be aiming at higher dimensional cases. For the history of an early stage of the complex dynamics and Schröder’s functional equation, we have two nice references [Al1, 2].

Our aim is to solve the equations

**Equation (I) :** $\beta \circ \varphi(x) = \lambda \beta(x)$

**Equation (II) :** $\beta_1 \circ \varphi(x) = \lambda \beta_1(x), \quad \beta_2 \circ \varphi(x) = \lambda \beta_2(x) + \beta_1(x)$

on the half line $[0, \infty)$ for an expanding diffeomorphism $\varphi \in \text{Diff}^\omega([0, \infty))$ and a complex constant $\lambda$ with $|\lambda| > 1$. Equation (II) is generalized to still higher dimensional case (II’) which is expressed by using vector notations as

**Equation (II’) :** $\beta \circ \varphi(x) = A\beta(x)$
where \( \varphi \) is as above, \( A = (a_{ij}) \) is an \( M \times M \) matrix any of its eigenvalues has the absolute value greater than 1, and \( \beta(x) = t(\beta_1(x), \cdots, \beta_M(x)) \in C^\infty([0, \infty); C^N) \) is the unknown function. Of course, the problem is easily reduced to the case where \( A \) is a single non-trivial Jordan block. So we can assume that \( A \) has a unique eigenvalue \( \lambda \) as above and eventually \( a_{ij} = \lambda \) for \( i = j \), \( a_{ij} = 1 \) for \( i = j + 1 \), and \( a_{ij} = 0 \) otherwise.

It is easily seen that in the case of \( \varphi'(0) = 1 \), which is of our main concern, there is no analytic solution but \( \beta(x) \) or \( \beta(x) \equiv 0 \). On the other hand, if \( \varphi'(0) > 1 \), for Equation (I) the space of solution is trivial or of dimension 1, depending on the resonance of \( \lambda \) and \( \varphi'(0) \). These are in fact exactly the same even when working on \( D \in C \). Therefore Schröder’s equation exhibits very distinctive feature when it is considered on the half line with \( \varphi'(0) = 1 \).

The space of solutions to Equation (I) turns out to be an infinite dimensional vector space which is in a sense isomorphic to \( C^\infty(S_1; C) \) whenever \( \varphi'(0) = 1 \). In the subsequent sections, first we describe the space of solutions much clearer, and then we give two different proofs.

It is to be remarked that as \( \varphi \) is expanding, our problem is essentially that on the germs around \( x = 0 \). For a given germ of \( \varphi \), we can extend \( \varphi \) to the whole of \([0, \infty)\) as a realization of the germ as far as \( \varphi(x) > x \) is satisfied for \( x > 0 \). Then the same applies to \( \beta(x) \) because once it is given as a germ around \( x = 0 \), it is automatically and uniquely extended to the whole half line by the equation itself.

The results on Equation (I) are used in Part I of this paper. Those for (II) serve in [Ho] to determine the automorphism groups of Reeb components with complex leaves of complex dimension 2. When we extend our results on the automorphism groups to higher dimensional cases, the results on (II') are necessary.

5 The space of solutions

In this section we give precise statements of our results on the Schröder type functional equations (I), (II), and (II') and describe the spaces of their smooth solutions.

For an expanding diffeomorphism \( \varphi \in \text{Diff}^\infty([0, \infty)) \) and a complex number \( \lambda \) with \( |\lambda| > 1 \), we consider Equation (I), (II), and (II'). First consider these equations (not on the whole \([0, \infty)\) but) on \((0, \infty)\). Then, Equation (I) has a lot of solutions and if we fix any solution \( \beta^*(x) \in C^\infty((0, \infty); C) \) which never vanishes, i.e., \( \beta^*(x) \neq 0 \) for \( x > 0 \), then each solution corresponds to a smooth function on \( S^1 = (0, \infty)/\varphi \mathbb{Z} \) by assigning \( \beta \mapsto \beta/\beta^* \). This gives a bijective correspondence between the space \( \mathcal{Z} = \mathcal{Z}_{\lambda, \varphi} \) of solutions to (I) on \((0, \infty)\) and \( C^\infty(S^1; C) \) as vector space. This correspondence will be more precise in Section \([\mathcal{Z}] \) in fixing the coordinate...
on the circle.

In Section 3 we will make this correspondence more precise, fixing the coordinate on $S^1$ and the choice of $\beta^*$.

Also take the space $S = S_1, \varphi$ of solutions to Equation (II) on $(0, \infty)$. If we assign $\beta_1$ to a solution $(\beta_1, \beta_2) \in S$, we obtain the projection $P_1 : S \to Z$. Here the kernel of $P_2$ is nothing but $Z$. We also see that the projection $P_1$ is surjective because for any $\beta_1 \in Z$

$$\beta_2(x) = \frac{1}{\lambda \log \lambda} \beta_1(x) \log \beta^*(x)$$

gives a solution $(\beta_1, \beta_2) \in S$, where for $\log \lambda$ and $\log \beta^*(x)$ any (smooth) branch can be taken. Therefore, as a vector space, $S$ has a structure such that

$$0 \to Z \to S \to Z \to 0$$

is a short exact sequence.

For Equation (II') we simply repeat this extension. Let $S_M$ denote the set of solutions on $(0, \infty)$. For $1 \leq m < M$, $S_m$ is naturally identified with a quotient $\{ (\beta_1, \cdots, \beta_m) \mid (\beta_1, \cdots, \beta_M) \in S_M \}$ of $S_n$. Each projection $S_m \to S_{m-1}$ is surjective because the multiplication $\frac{1}{\lambda \log \lambda} \beta^*$ is a linear right inverse and its kernel coincides with $Z$. $S_1$ is nothing but $Z$ and $S_2$ the above $S$ as well.

Let $\varphi^*$ denote the pull-back by $\varphi$. Then Equation (I) is expressed as $(\varphi^* - \lambda)\beta = 0$. Now, Equation (II) is nothing but $(\varphi^* - \lambda)^2 \beta_2 = 0$, where we put $\beta_1$ by setting $(\varphi^* - \lambda)\beta_2 = \beta_1$. Inductively we see Equation (II') is nothing but $(\varphi^* - \lambda)M \beta_M = 0$ while $(\varphi^* - \lambda)\beta_m = \beta_{m-1}$ for $m = 2, 3, 4, \cdots, M$ might also be regarded as auxiliary equations.

As in Section 3 of Part I, we divide the situation into the following three cases according to the nature of the jet of $\varphi$ at $x = 0$. For the second and the third cases, the statements of our results are the same. Here, $f^{(i)}$ denotes the $i$-th derivative of $f(x)$ and $j^i f(0)$ denotes the $i$-th jet at $x = 0$.

Case (1) : $\varphi'(0) = \mu > 1$,

Case (2) : for some $n \geq 2$, $j^{n-1} \varphi(0) = j^{n-1} \text{id}(0)$ and $\varphi^{(n)}(0) > 0$,

Case (3) : $j^n \varphi(0) = j^n \text{id}(0)$.

Let us state the results. We start with the easiest case.

**Theorem 5.1** Consider Case (1).

1) (Resonant case) If $\lambda = \mu^n$ is satisfied for some $n \in \mathbb{N}$, then the space of solutions $\mathcal{K}$ to Equation (I) is a complex vector space of dimension 1.

For Equation (II'), $\beta$ satisfies (II') if and only if $\beta_1 = \cdots = \beta_{n-1} \equiv 0$ and $\beta_n \in \mathcal{K}$ hold. Therefore in total the space of solution is also 1-dimensional.

2) (Non-resonant case) If no positive integer $n \in \mathbb{N}$ satisfies $\lambda = \mu^n$, then there exists no solution to (I) but $\beta(x) \equiv 0$, and we have $\mathcal{K} = \{ 0 \}$.
For Equation (II') the same applies. Namely the only smooth solutions on $[0, \infty)$ is $\beta(x) \equiv 0$.

Remark here that if $\varphi(x) = \mu x$, the solution to (I) in resonant case is nothing but $\beta(x) = \text{const} \cdot x^n$. Also remark that accordingly $\lambda$ is a positive real number. This result is so easy that the proof is given here.

Proof. From Sternberg’s linearization theorem [St], there exists a diffeomorphism $h \in \text{Diff}^\infty([0, \infty))$ which conjugates $\varphi$ into the linear one $h^{-1} \circ \varphi \circ h(x) = \mu x$. Therefore in solving the equations, from the first we can assume $\varphi(x) = \mu x$. Therefore the equation (I) takes the following form.

$$\beta(\mu x) = \lambda \beta(x) \quad \text{for } x \in [0, \infty).$$

In both cases, by differentiating Equation (I) for arbitrary many times at $x = 0$, we see that the Taylor expansion at $x = 0$ can be non-trivial only at the degree $n = \log \lambda / \log \mu$. Therefore in the resonant case, $\beta(x)$ must be in the form $\beta(x) = c \cdot x^n + f(x)$ where $f(x)$ is a flat function, and in the non-resonant case, the same form with $c = 0$. Then, in the resonant case, as $c \cdot x^n$ is a solution to (I), so is $f(x)$. Therefore in both cases, it is enough to check that any non-trivial flat function can no be a solution.

If we had a non-trivial flat solution $f(x)$, it would contradict as follows. Take $x_0 \in (0, \infty)$ with $f(x_0) \neq 0$ and look at $f(\mu^{-k}x_0) = \lambda^{-k}f(x_0)$ for $k \in \mathbb{N}$. On the other hand, as $f$ is flat we have $\lim_{x \to 0} f(x)/x^l = 0$ for any $l \in \mathbb{N}$. So large enough $l (\geq |\log \lambda / \log \mu|)$ gives rise to a contradiction.

This is well-known also as a fact (even for higher dimensional case) that a weighted-homogeneous function is smooth at the origin only when it is a polynomial.

For Equation (II), from the above result, we assume $\beta_1(x) = cx^n$ in the resonant case. Then a similar computation for $\beta_2$ implies $c = 0$. Therefore we have $\beta_1 = 0$ and thus $\beta_2 = c'x^n$ for some $c' \in \mathbb{C}$. In the non-resonant case, the argument for (I) suffices. For (II'), the argument for (II) works as an inductive step.

The results for Case (2) and (3) can be stated together.

**Theorem 5.2** For both of Case (2) and Case (3), and for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, the followings hold.

1) Any solution $\beta \in Z_{\lambda, \varphi}$ to Equation (I) on $(0, \infty)$ extends to $[0, \infty)$ so as to be a smooth function which is flat at $x = 0$, i.e., the $k$-th jet satisfies $j^k \beta(0) = 0$ for any $k = 0, 1, 2, \cdots$. In other words, the space $K$ of all solutions to (I) considered on $[0, \infty)$ coincides with $Z_{\lambda, \varphi}$.

2) The same applies to Equation (II) and (II'). Namely by putting $\beta(0) = 0$, any $\beta \in S_M$ is smooth and flat at $x = 0$.

In the next section we give a proof for Case (3), which is more direct and simpler than one given in the final section. In this proof, we directly
estimate the derivatives of any order of $\beta \circ \varphi$ for $\beta \in \mathbb{Z}$. The higher derivatives of a composite function is complicated and described in the formula of Faà di Bruno. We do not need its full length.

Unfortunately this method does not work for Case (2).

A proof by a different approach for Case (2) is given in the following section. It is based on two theories. One is Takens’ normal forms [Ta] for germs around the origin of $\text{Diff}^\infty(0,\infty)$ and of vector fields on $[0,\infty)$ with non-trivial finite order jets. The other one is a classical theory of Fourier expansion of $\mathcal{C}^\infty(S^1;\mathbb{C})$. Unfortunately again, this method seems difficult to apply to Case (3).

Then in the final section we give a proof relying on the center manifold theory, which covers both of Case (2) and (3). This proof is suggested by Masayuki Asaoka. It might be worth remarking that when a proposition is proved in the framework of hyperbolic dynamical systems, quite often it is also proved in Fourier analysis, and vice versa. Here we might observe a similar phenomenon.

6 Direct proof for Case (3)

We prove Theorem 5.2 for Case (3), namely, in the case where $\varphi$ is flat to the identity, by a direct estimate of the derivatives of $\beta(x)$ of an arbitrary order when $x \to 0$.

In order to clarify the strategy it might be suggested to the readers to check $\lim_{x \to +0} \beta(x) = 0$ and $\lim_{x \to +0} \beta'(x) = 0$, i.e., for $k = 0, 1$, which are easy and reviewed in Proposition 8.3 and then the the second jet $k = 2$). Looking at up to the case $k = 3$ might make the roll of the following lemma clearer.

Lemma 6.1 The $n$-th derivative $\{\beta(\varphi(x))\}^{(n)}$ is written in the following form for $n \in \mathbb{N}$.

$$\{\beta(\varphi(x))\}^{(n)} = (\varphi'(x))^n \cdot \beta^{(n)}(\varphi(x)) + \sum_{k=1}^{n-1} \Phi_{n,k} \cdot \beta^{(k)}(\varphi(x)).$$

Here, $\Phi_{n,k}$ is an integral polynomial in $\varphi'(x), \varphi''(x), \cdots, \varphi^{(n)}(x)$, without constant term and no term is of monomial only in $\varphi'(x)$.

This lemma is easily seen by the induction, but in fact it is a corollary to the well-known formula of Faà di Bruno (e.g., see [Ri], [Ro], or textbooks on calculus). It is independent of our assumption on $\varphi$ and is valid for any composite functions. On the other hand the flatness of $\varphi$ at the origin implies $(\varphi'(x))^n \to 1$ and $\Phi_{n,k} \to 0$ when $x \to 0 + 0$.

Now let us prove 1) of Theorem 5.2. Let $\beta$ be a solution to (I) on $(0, \infty)$. From the equation it is easy to see that $\beta(x) \to 0$ when $x \to 0 + 0$. 

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Now fix any integer $N$. $\beta'(x) \to 0$ is also easy to see, but for higher derivatives, in a natural estimate the lower derivatives are involved. Thus the basic strategy is not to estimate the higher derivatives by induction on the order, but to estimate them all together up to the fixed order $N$.

From Equation (I) and the above lemma we have the following computation.

$$
\sum_{n=1}^{N} |\beta^{(n)}(x)| = \frac{1}{|\lambda|} \sum_{n=1}^{N} |\{\beta(\varphi(x))\}^{(n)}| \\
\leq \frac{1}{|\lambda|} \sum_{n=1}^{N} \left\{ (\varphi'(x))^n \cdot |\beta^{(n)}(\varphi(x))| + \sum_{k=1}^{n-1} |\Phi_{n,k}| \cdot |\beta^{(k)}(\varphi(x))| \right\} \\
\leq \frac{1}{|\lambda|} \sum_{k=1}^{N} \left( (\varphi'(x))^k + \sum_{n=k+1}^{N} |\Phi_{n,k}| \right) \cdot |\beta^{(k)}(\varphi(x))| 
$$

As is remarked above, we know $(\varphi'(x))^k \to 1$ and $\sum_{n=k+1}^{N} |\Phi_{n,k}| \to 0$ when $x \to 0$. Therefore there exists $b_N > 0$ such that for $x \in (0, b_N]$ we have

$$(\varphi'(x))^k + \sum_{n=k+1}^{N} |\Phi_{n,k}| \leq \sqrt{|\lambda|} \quad \text{for } k = 1, 2, \cdots, N.$$ 

This implies for any $x \in (0, b_N]$

$$
\sum_{n=1}^{N} |\beta^{(n)}(x)| \leq \frac{1}{\sqrt{|\lambda|}} \sum_{n=1}^{N} |\beta^{(n)}(\varphi(x))|.
$$

Put $M = \max\{\sum_{n=1}^{N} |\beta^{(n)}(x)| ; x \in [b_N, \varphi(b_N)]\}$ and define $m(x) \in \mathbb{N}$ for $x \in (0, b_N)$ so that $\varphi^{m(x)} \in [b_N, \varphi(b_N))$. Then, the above inequality implies

$$
\sum_{n=1}^{N} |\beta^{(n)}(x)| \leq M \cdot \sqrt{|\lambda|}^{-m(x)}
$$

for $x \in (0, b_N)$. Because ‘$x \to 0 + 0$’ is equivalent to ‘$m(x) \to \infty$’, we obtained the convergence

$$
\beta^{(n)}(x) \to 0 \quad (x \to 0 + 0) \quad \text{for } n = 1, \cdots, N.
$$

This completes the proof of 1).

Let us outline the proof of 2) for $M = 2$. We extend the basic strategy of the proof of 1) in the following sense. When we estimate the derivatives of $\beta_2$, naturally those of $\beta_1$ are involved. Therefore we will estimate the derivatives of $\beta_2$ and $\beta_1$ all together up to a fixed order $N$, even though the flatness of $\beta_1$ is already proved in 1).

First we fix $\varepsilon > 0$ so small that $\varepsilon \leq \frac{1}{2} |\lambda|^{\frac{3}{2}} - |\lambda|$ is satisfied. Now take any solution $(\beta_1, \beta_2) \in S$ and put $\tilde{\beta}_1 = \varepsilon^{-1} \beta_1$. Then instead of Equation (II),

$$
\sum_{n=1}^{N} |\beta^{(n)}(x)| \leq \frac{1}{\sqrt{|\lambda|}} \sum_{n=1}^{N} |\beta^{(n)}(\varphi(x))|.
$$

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\( \tilde{\beta}_1 \) and \( \beta_2 \) satisfy

Equation (I\( \tilde{I} \)) : \( \tilde{\beta}_1(\varphi(x)) = \lambda \tilde{\beta}_1(x), \quad \beta_2(\varphi(x)) = \lambda \beta_2(x) + \epsilon \tilde{\beta}_1(x) \).

Then, from (I\( \tilde{I} \)) we have

\[
\frac{e^{i\theta \lambda} - \epsilon}{\lambda} \tilde{\beta}_1(\varphi(x)) + \beta_2(\varphi(x)) = e^{i\theta \lambda} \tilde{\beta}_1(x) + \lambda \beta_2(x)
\]

and consider the \( n \)-th derivatives of both sides. For any \( \theta \in \mathbb{R} \) and \( n = 1, \cdots, N \), we have

\[
|e^{i\theta \tilde{\beta}_1^{(n)}(x)} + \beta_2^{(n)}(x)| \leq \frac{1}{|\lambda|^n} \left( |\lambda| + |\{\tilde{\beta}_1(\varphi(x))\}^{(n)}| + |\{\beta_2(\varphi(x))\}^{(n)}| \right)
\]

Because the right hand side is independent of \( \theta \), using the inequality

\[
|e^{i\theta \lambda} - \epsilon| \leq |\lambda| + \epsilon |\lambda| \quad \text{for any } \theta \in \mathbb{R}
\]

we obtain

\[
|\tilde{\beta}_1^{(n)}(x)| + |\beta_2^{(n)}(x)| \leq \frac{1}{|\lambda|^n} \left( |\{\tilde{\beta}_1(\varphi(x))\}^{(n)}| + |\{\beta_2(\varphi(x))\}^{(n)}| \right).
\]

Applying Lemma 6.1 to \( \tilde{\beta}_1(\varphi(x)) \) and to \( \beta_2(\varphi(x)) \) for \( n = 1, \cdots, N \), from the same argument as in 1) we obtain

\[
\sum_{n=1}^{N} \left( |\tilde{\beta}_1^{(n)}(x)| + |\beta_2^{(n)}(x)| \right) \leq \frac{1}{|\lambda|^n} \sum_{n=1}^{N} \left( |\tilde{\beta}_1^{(n)}(\varphi(x))| + |\beta_2^{(n)}(\varphi(x))| \right)
\]

for \( x \in (0, b_N] \), where \( b_N \) is exactly the same as in the proof of 1). \( \square \)

Now it is almost straightforward to further generalize this proof for Equation (II\( \prime \)).

7 Proof by Fourier series for Case (2)

A proof of Theorem 5.2 for Case (2) is given here. It relies on two big tools. The first one is Takens’ normal form which plays a similar role as Sternberg’s linearization in the proof of Theorem 5.1. The second one is a classical Fourier expansion/series of smooth functions on the circle. Takens’ normal form enables us to consider (countably many) simple linear homogeneous ordinary differential equations instead of considering Equation (I). Equation (II) and (II\( \prime \)) correspond to inhomogeneous or vector valued case. As we will see below, we have an ODE for each choice of the value of \( \log \lambda \), Our functional equation (I) and countably many ODE’s are related by Fourier expansion and series.

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7.1 Takens’ normal form and Fourier basis

**Theorem 7.1** (Takens, [Ta]) Let \( \varphi \in \text{Diff}^\infty([0, \infty)) \) be in Case (2).

1) There exists a diffeomorphism \( h \in \text{Diff}^\infty([0, \infty)) \) which conjugates \( \varphi \) into a diffeomorphism \( \psi \in \text{Diff}^\infty([0, \infty)) \) of the following polynomial type on \([0, x_1] \) (\( \exists x_1 > 0 \))

\[
\psi(x) = x + x^n + ax^{2n-1} \quad \text{and} \quad \psi = h^{-1} \circ \varphi \circ h.
\]

The coefficient \( a \in \mathbb{R} \) is determined by the \((2n-1)\)-jet of \( \varphi \) at \( x = 0 \).

2) There also exists a diffeomorphism \( k \in \text{Diff}^\infty([0, \infty)) \) which conjugates \( \varphi \) in a neighborhood of the origin into the exponential map

\[
\exp X = k^{-1} \circ \varphi \circ k.
\]

of the vector field

\[
X = \rho(x) \frac{d}{dx}, \quad \rho(x) = x^n + ax^{2n-1}.
\]

The the coefficients \( a \) in 1) and \( a \) here are related by \( a = \alpha - n/2 \).

**Remark 7.2**

1) The second statement follows from the first one, because a direct computation shows

\[
\exp(x^n + ax^{2n-1}) \frac{d}{dx} = x + x^n + \left(a + \frac{n}{2}\right)x^{2n-1} + [\text{higher order terms}].
\]

2) Takens also gave a normal form for vector fields. It takes almost the same form but we do not need it here.

Thanks to Takens’ theorem, we can conjugate our equations by a smooth diffeomorphism and are allowed to assume that the holonomy \( \varphi \) is of the form

\[
\varphi = \exp X, \quad X = \rho(x)\frac{d}{dx}, \quad \rho(x) = x^n + ax^{2n-1} \text{ on } [0, x_0]
\]

for some \( n \geq 2, a \in \mathbb{R} \), and \( x_0 > 0 \). We also assume that \( \rho(x) > 0 \) on \((0, \infty)\) and \( \rho(x) \equiv 1 \) on \((x_1, \infty)\) for some \( x_1 > x_0 \).

We consider the following ordinary differential equation on \((0, \infty)\)

Equation (I-\( \Lambda \)) : 

\[
\beta'(x) = \frac{\Delta}{\rho(x)} \beta(x).
\]

This is of course equivalent to the following ODE in the variable \( t \).

\[
\frac{d}{dt} \bigg|_{t=0} \beta(\exp(tX)(x)) = \Delta \cdot \beta(x)
\]
Therefore any solution $\beta$ is presented as $\beta(\exp(tX)(x_0)) = e^{\Lambda t} \cdot \beta(x_0) = C \cdot e^{\Lambda t}$ for a constant $C \in \mathbb{C}$. It is also clear that $\beta$ satisfies Equation (I) on $(0, \infty)$. In the variable $x$, $\beta(x)$ is presented as $\beta(x) = C \cdot \exp \left( \Lambda \int_{x_0}^{x} \frac{1}{\rho(u)} \, du \right)$.

In particular on $(0, x_0)$, we have

$$\beta(x) = C \cdot \exp \left( (R + i \text{Im}\Lambda) \int_{x_0}^{x} \frac{1}{u^n(1 + au^{n-1})} \, du \right)$$

where the real part $R = \log |\lambda|$ is positive.

Now we choose $\Lambda_{\langle 0 \rangle}$ as a value of $\log \lambda$ and fix it. Then other general values of $\log \lambda$ are given as $\Lambda_{\langle l \rangle} = R + i \theta_l$, $\theta_l = \theta_0 + 2l\pi$ for $l \in \mathbb{Z}$.

Here we make the correspondence between the space $\mathcal{Z}_{\lambda, \varphi}$ of solutions of (I) considered on $(0, \infty)$ and $C^\infty(S^1; \mathbb{C})$ more precise. As a coordinate on the circle $S^1$ we take $\theta = t \mod 2\pi$ where $x(t) = \beta(\exp(tX)(x_0))$ is assumed. Now for each $l \in \mathbb{Z}$, let $\tilde{\beta}_{\langle l \rangle}$ denote the solution to the ODE $(I - \Lambda_{\langle l \rangle})$ which satisfies $\tilde{\beta}_{\langle l \rangle}(x_0) = 1$. Therefore we easily know that $\tilde{\beta}_{\langle l \rangle}(x(t)) = e^{2\pi ilt \cdot i} \cdot \beta_{\langle 0 \rangle}(x(t))$.

Take $\beta_{\langle 0 \rangle}$ as $\beta^*$ in defining the correspondence. Then $\beta_{\langle l \rangle}$ corresponds to the constant function $1$ on $S^1$ and in general $\beta_{\langle l \rangle}$ corresponds to $\tilde{\beta}_{\langle l \rangle} \in C^\infty(S^1; \mathbb{C})$, namely,

$$\tilde{\beta}_{\langle l \rangle}(\theta) = e^{2\pi il\theta} \quad \text{for } l \in \mathbb{Z},$$

so that $\tilde{\beta}_{\langle l \rangle}$'s ($l \in \mathbb{Z}$) form the standard Fourier basis for $C^\infty(S^1; \mathbb{C})$. The following is well-known and well fits into our situation.

**Theorem 7.3** (see e.g., [Ka]) The infinite sum with coefficients $c_l \in \mathbb{C}$

$$\sum_{l=-\infty}^{\infty} c_l \cdot e^{i\theta}$$

defines a smooth function on $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ if and only if the sequence of coefficients $\{c_k\}_{k \in \mathbb{Z}}$ is rapidly decreasing, namely it satisfies

$$\sum_{l=-\infty}^{\infty} |l|^d |c_l| < \infty \quad \text{for any } d \in \mathbb{N}.$$
7.2 Proof of Main Theorem

What we have to prove in this section is stated as follows.

**Theorem 7.4** For any rapidly decreasing \( \{c_k\}_{k \in \mathbb{Z}} \), \( \hat{\beta} = \sum_{l=-\infty}^{\infty} c_l \cdot \hat{\beta}(l) \) is extended to \([0, \infty)\) as \( \hat{\beta}(0) = 0 \) and is smooth and flat at \( x = 0 \).

Let us verify this for each base.

**Proposition 7.5** The solution \( \beta(x) \) to \((I-\Lambda)\) is extended to \([0, \infty)\) as \( \beta(0) = 0 \), and then \( \beta(x) \) is smooth and flat at \( x = 0 \).

**Proof.** It is easy to compute the integration but we only need to remark that for some \( \delta > 0 \) and any \( x \in (0, \delta) \) we have

\[
\left| \int_{x_0}^{x} \frac{1}{u^n(1 + au^{n-1})} \, du \right| \geq \left| \frac{1}{2x} \right|.
\]

The derivative of \( \beta \) of order \( k \in \mathbb{N} \) is a multiplication of \( \beta \) and some rational function in the variable \( x \). Therefore for any \( k \in \mathbb{N} \) we have

\[
|\beta^{(k)}(x)| \leq |\text{a rational function}| \times \exp\left(-\frac{1}{2x}\right) \rightarrow 0 \quad (x \rightarrow 0)
\]

which suffices to show the smoothness and flatness of \( \beta \) at \( x = 0 \). \( \square \)

In order to proceed further, we need to take a slightly closer look at those rational functions. Recall that \( n, a, \) and \( \lambda \) are already fixed.

**Lemma 7.6** For \( k \in \mathbb{N} \) and \( j = 1, \ldots, k \), there exists a fixed polynomial \( Q_{k,j}(x) \) which satisfies on \((0, x_0)\)

\[
\hat{\beta}_{(l)}^{(k)}(x) = \left\{ \frac{1}{P(x)^k} \sum_{j=1}^{k} Q_{k,j}(x)(R + i\theta_l)^j \right\} \beta_{(l)}(x), \quad P(x) = x^n + ax^{2n-1}
\]

and \( Q_{k,j}(x) \) is a linear combination of multiplications of \((k-j)\)-many of \( P(x), P'(x), \ldots, P^{(k-j)}(x) \), with total degree of differentiation \((k-j)\).

For example, \( Q_{k,k}(x) = 1, Q_{k,k-1}(x) = k(1-k)P'(x)/2, \) and so on. The lemma is easily proved by the induction on \( k \).

Let us develop \((R + i\theta_l)^j\) into a polynomial of \( l \) as follows.

\[
(R + i\theta_l)^j = (R + i(\theta_0 + 2\pi l))^j = \sum_{d=0}^{j} R_{j,d} l^d
\]

Here the constants \( R_{j,d} (j \in \mathbb{N}, d = 0, \ldots, j) \) are determined by \( R \) and \( \theta_0 \).
Proof of Theorem 7.4. For a rapidly decreasing sequence

\[ \{c_k\}_{k \in \mathbb{Z}} \text{ with } \sum_{l=-\infty}^{\infty} |l|^d |c_l| = M_d < \infty \text{ for } \forall d \in \mathbb{N} \cup \{0\} \]

take \( \beta(x) = \sum_{l=-\infty}^{\infty} c_l \cdot \beta_{(l)}(x) \). Then we have the following estimate;

\[
|\beta^{(k)}(x)| = \left| \sum_{l=-\infty}^{\infty} c_l \cdot \beta^{(k)}_{(l)}(x) \right| \\
\leq \frac{1}{P(x)^k} \left| \sum_{j=1}^{k} Q_{k,j}(x) \left( \sum_{l=-\infty}^{\infty} c_l \cdot \left( \sum_{d=0}^j R_{j,d} l^d \right) \right) \beta_{(l)}(x) \right| \\
\leq \left\{ \frac{1}{P(x)^k} \sum_{j=1}^{k} \sum_{d=0}^j |Q_{k,j}(x)||R_{j,d}||M_d| \right\} |\beta_{(0)}(x)| \rightarrow 0 \quad (x \rightarrow 0 + 0)
\]

because the last \( \cdots \) is a rational function when \( x \) is close to 0. Also this computation shows the validity of the first equality.

\[ \square \]

Remark 7.7. For the equation (II), the smoothness and flatness of \( \beta \log \beta_{(0)} \) for a solution \( \beta \) to (I) follow from more or less the same arguments, because \( \log \beta_{(0)} = (R + i\theta_0) \int_{x_0}^{x} \frac{1}{P(u)} du \).

8 Unified proof for Case (2) and Case (3)

The proof of Theorem 5.2 given in this section relies on the theory of center manifolds and the idea of graph transformation. For this theory, refer to a nice book by Shub [Sh], in particular, Appendix III to Chapter 5. First we review the center manifold theorem in a form which is suitable in and focused to our context.

Theorem 8.1 (Theorem III. 2, [Sh], modified) Let \( T : E \rightarrow E \) be a continuous linear endomorphism on a Banach space \( E \) with a \( T \)-invariant decomposition \( E = E_1 \oplus E_2 \) into closed subspaces. For the restrictions \( T_i : E_i \rightarrow E_i \) \((i = 1, 2)\) of \( T \) we assume that \( T_i \) is an isomorphism and there exist positive constants \( 0 < \mu^* < \lambda^* \) satisfying the following conditions.

\[
\|T_1(v)\| > \mu^* \|v\| \quad \text{for all } v \neq 0 \in E_1, \\
\|T_2(v)\| < \lambda^* \|v\| \quad \text{for all } v \neq 0 \in E_2.
\]
Then there exists a real number $\varepsilon^* > 0$ such that if a $C^r$-map $\Phi : E \to E$ ($r \geq 1$) satisfies $\Phi(0) = 0$ and $\text{Lip}(\Phi - T) < \varepsilon^*$, then we have the followings. 

1) The set $W_1 = \cap_{n \geq 0} \Phi^n(S_1)$ where $S_1 = \{(v_1, v_2) \in E_1 \times E_2; \|v_1\| \geq \|v_2\|\}$ is the graph of a $C^1$-map $g : E_1 \to E_2$ with $\text{Lip}(g) \leq 1$ and is invariant by $\Phi$, namely, $\Phi(W_1) = W_1$.

2) If $\lambda^* < (\mu^*)^r$ holds, then the map $g$ is of $C^r$.

In this theorem $\text{Lip}$ denotes the Lipschitz constant of a Lipschitz map, i.e., $\text{Lip}(f) = \sup\{\|f(v_2) - f(v_1)\|/\|v_2 - v_1\|; v_2 \neq v_1, v_1, v_2 \in E\}$.

In order to prove Theorem 5.2 concerning Equation (I), we take $E_1 = \mathbb{R}$, $E_2 = \mathbb{C}$, $T_1 = \text{id}_{\mathbb{R}}$, and $T_2$ is a scalar multiplication by $\lambda^{-1}$. Thereofre the real number $\varepsilon^*$ in the theorem is determined by $\lambda$.

The very virtue of this theorem is that higher order regularities are assured only by estimates on 1-jets.

Let us explain a rough idea before getting into the details. As $\Phi$, the map $(x, z) \mapsto (\varphi^{-1}(x), \lambda^{-1}z)$ or its modification will be taken. Here $\varphi(x) = \Phi(x)$ for $x \geq 0$. If we apply this theorem by taking $\Phi(x, z) = (\varphi^{-1}(x), \lambda^{-1}z)$, we just obtain $g(x) \equiv 0$ and nothing more.

The basic strategy is, not exactly but roughly; for any $\beta \in \mathcal{Z}$ and for any $\varepsilon > 0$, we look for an appropriate $\Phi$ with $\text{Lip}(\Phi - T) < \varepsilon$, so that the resultant $g$ coincides with $\tilde{\beta}$ for $x < \delta$ for some $\delta > 0$. Here $\tilde{\beta}$ is an extension of $\beta$ to $\mathbb{R}$ by taking $\tilde{\beta}|_{(-\infty, 0]} \equiv 0$. Before these arguments, we need to take appropriate modifications of $\varphi$, and for $\beta$ we choose $\Phi$ in a suitable way.

Let us start the proof of the theorem for Equation (I). We fix a smooth function $h \in C^\infty(\mathbb{R}; [0, 1])$ satisfying

$$h(x) \equiv 0 \text{ on } (-\infty, 1/3] \text{ and } h(x) \equiv 1 \text{ on } [2/3, \infty).$$

Now take $\varphi \in \text{Diff}^\infty([-0, \infty))$ in either of Case (2) or (3) and $\lambda \in \mathbb{C}$ as well. Take any extension of $\varphi$ in $\text{Diff}^\infty(\mathbb{R})$. For abuse of notation, it is denoted by $\varphi$ again. In Case (3), of course we can take $\varphi$ so as to be the identity on the negative side $(-\infty, 0]$.

First, by the following lemma, we modify it away from the origin so as to be suitable for the center manifold theory while its germ is not changed.

**Lemma 8.2** For $\delta > 0$ define $\varphi_\delta$ as follows.

$$\varphi_\delta(x) = h \left( \frac{x}{\delta} \right) \cdot x + \left( 1 - h \left( \frac{x}{\delta} \right) \right) \cdot \varphi(x) \text{ for } x \leq 0,$$

$$\varphi_\delta(x) = h \left( \frac{x}{\delta} \right) \cdot (x + \delta^2) + \left( 1 - h \left( \frac{x}{\delta} \right) \right) \cdot \varphi(x) \text{ for } x \geq 0.$$

Then we have

$$\lim_{\delta \to 0^+} \text{Lip}(\varphi_\delta - \text{id}_\mathbb{R}) = 0.$$
In particular, we have the followings.

1) For small enough $\delta$, $\tilde{\phi}_\delta$ is in $\text{Diff}^\infty(\mathbb{R})$ and expanding on $[0, \infty)$.
2) The germ of $\tilde{\phi}_\delta$ around $x=0$ is the same as that of $\phi$.
3) $\tilde{\phi}_\delta|(-\infty,-\delta[ = \text{id}|(-\infty,-\delta]$ and $\tilde{\phi}_\delta|\delta,\infty) = \text{id}|\delta,\infty) + \delta^2$.
4) $\lim_{\delta \to 0^+} \text{Lip}(\tilde{\phi}_\delta^{-1} - \text{id}_\mathbb{R}) = 0$.

Proof. From a direct computation using $\phi'(0) = 1$, the uniform convergence $\tilde{\phi}_\delta'(x) \to 0$ when $\delta \to 0^+$ is easily obtained. Then, because the support of $\tilde{\phi}_\delta - 1$ is contained in $[-\delta,\delta]$ and $\tilde{\phi}_\delta$ converges to $\text{id}_\mathbb{R}$ uniformly, we obtain the above estimate for the Lipschitz constant. The statements 1) - 4) follow naturally. $\square$

For each $\tilde{\phi}_\delta|_{[0,\infty)} \in \text{Diff}^\infty([0,\infty))$, take any solution $\beta \in Z_{\lambda,\tilde{\phi}_\delta}$ and the extension $\tilde{\beta}$ to $\mathbb{R}$ as explained above. Our objective is to prove that $\tilde{\beta}$ is smooth on $\mathbb{R}$. At least for $\tilde{\beta}'(0)$, not only we see it easily but we need it for our proof. This fact is true even for the case $\phi'(0) > 1$ as far as $\phi'(0) < |\lambda|$.

Proposition 8.3 $\tilde{\beta}$ is of $C^1$, namely, $\lim_{x \to 0^+} \beta(x) = \lim_{x \to 0^+} \beta'(x) = 0$ holds.

Proof. Only $\lim_{x \to 0^+} \beta'(x) = 0$ is verified. From Equation (I), we have

$$\beta'(\phi(x)) = (\phi'(x))^{-1} \beta'(x).$$

From the condition there exist $x_1 > 0$ and $\nu > 1$ such that $\phi^{-1}|\lambda| < \nu$ holds on $[0, x_1]$. Take $M = \max |\beta'(x)|$ on the fundamental domain $[\phi^{-1}(x_1), x_1]$ of the action of $\phi$ on $(0, \infty)$, we see that when $x$ approaches to 0 in $(0, x_1)$, each time it passes through a smaller fundamental domain, $|\beta'(x)|$ shrinks by $\nu^{-1}$. $\square$

Next step is to look for a suitable $\Phi : \mathbb{R} \times \mathbb{C} \to \mathbb{R} \times \mathbb{C}$. First put $\Phi_0(x,z) = (\phi^{-1}_\delta(x), \lambda^{-1}z)$. The graph of $\tilde{\beta}$ is invariant under $\Phi_0$, while the only invariant one contained in $W_1$ in the center manifold theorem is the real axis $\mathbb{R} \times \{0\}$, because any non-trivial solution $\beta$ grows exponentially. To avoid this inconvenience, consider a diffeomorphism

$$H_c(x, z) = \left( x, z + h \left( \frac{x}{c} \right) \tilde{\beta}(x) \right)$$

of $\mathbb{R} \times \mathbb{C}$ depending on the parameter $c > 0$. $H_c$ is the identity on $\{x \leq c/3\}$. Then by $H_c$ we take the conjugate

$$\Phi_c = H_c^{-1} \circ \Phi_0 \circ H_c.$$
Lemma 8.4  For any \( \varphi \), small enough \( \delta \), and \( \beta \), the followings hold.

1) \( \lim_{c \to 0^+} \text{Lip}(\Phi_c - \Phi_0) = 0 \).

2) The graph of \( (1 - h(\tilde{z})) \tilde{\beta}(x) \) is invariant under \( \Phi_c \).

Proof. Even though \( H \) is exponentially away from the identity according to \( \tilde{\beta} \) growing when \( x \to \infty \), thanks to the fact that \( H \) and \( \Phi_0 \) commute to each other on \( \{ x \geq \tilde{\varphi}_\delta(c) \} \), the result of conjugation does not go away. Precisely the lemma is proved by direct computations as follows.

From the definition and Equation (I) we have

\[
H_{c^{-1}} \circ \Phi_c \circ H_c(x, z) = \Phi_0(x, z)
\]

\[
= \left( 0, \lambda^{-1} h \left( \frac{x}{c} \right) \tilde{\beta}(x) - h \left( \frac{\tilde{\varphi}_\delta^{-1}(x)}{c} \right) \tilde{\beta}(\tilde{\varphi}_\delta^{-1}(x)) \right)
\]

\[
= \left( 0, \lambda^{-1} \tilde{\beta}(x) \left( h \left( \frac{x}{c} \right) - h \left( \frac{\tilde{\varphi}_\delta^{-1}(x)}{c} \right) \right) \right).
\]

Therefore in order to conclude \( \lim_{c \to 0^+} \text{Lip}(\Phi_c - \Phi_0) = 0 \), it is enough to show the uniform convergence of the derivative of the second component with respect to \( x \) to 0 when \( c \) tends to 0.

The estimates concerning \( c \to 0 \) which appear below are uniform in \( x \). Let us make this point clearer. The second component of the above has the support contained in \( [0, \tilde{\varphi}_\delta(c)] \) as a function on \( x \). As we assumed that \( \lim_{x \to 0^+} \tilde{\varphi}_\delta(x) \to 1 \), taking \( c > 0 \) small enough, we can also assume that \( \tilde{\varphi}_\delta(c) \leq 2c \) and it is enough to verify the estimates on \( [0, 2c] \). Now for example, as we remarked in the above proposition we know \( |\tilde{\beta}(x)| = o(|x|) \) and hence we have \( \max\{|\tilde{\beta}(x)|; 0 \leq x \leq 2c\} = o(c) \).

Now we show the derivative of the second component with respect to \( x \) uniformly converges to 0 (namely \( o(1) \)) when \( c \to 0 \). We can forget about \( \lambda^{-1} \) because it is just a constant.

\[
\frac{d}{dx} \left\{ \tilde{\beta}(x) \left( h \left( \frac{x}{c} \right) - h \left( \frac{\tilde{\varphi}_\delta^{-1}(x)}{c} \right) \right) \right\}
\]

\[
= \tilde{\beta}'(x) \left( h \left( \frac{x}{c} \right) - h \left( \frac{\tilde{\varphi}_\delta^{-1}(x)}{c} \right) \right) + \tilde{\beta}(x) \left( \frac{h'}{c} \left( \frac{x}{c} \right) - (\tilde{\varphi}_\delta^{-1})'(x) h' \left( \frac{\tilde{\varphi}_\delta^{-1}(x)}{c} \right) \right).\]

The first term is of \( o(1) \) because \( \tilde{\beta}'(x) = o(1) \) and \( |h(x) - h(\tilde{\varphi}_\delta^{-1}(x))| \leq 1 \). As to the second term, \( h' = O(1) \), \( \tilde{\beta}' = O(1) \), and \( \tilde{\beta}(x) = o(c) \) as remarked above. Therefore 1) is proved.
A direct computation verifies 2). It is also understood from the arrangement of $\Phi_c$, because $H_c$ sends the graph of $(1 - h(\frac{x}{\epsilon}))\tilde{\beta}(x)$ to that of $\tilde{\beta}(x)$ which is invariant under $\Phi_0$.

Now we are ready to apply the center manifold theorem to proof of Theorem 5.2 1). Recall that our $T$ was fixed as $T(x,z) = (x, \lambda z)$. From our settings, we can take such $\mu^*$ and $\lambda^*$ that

- $\mu^*$ is as close to 1 as we want as far as $\mu^* < 1$ is satisfied and
- $\lambda^*$ is as close to $\lambda - 1$ as we want as far as $\mu^* > \lambda - 1$ is satisfied.

Therefore for an arbitrary fixed $r \in \mathbb{N}$ we take $\mu^*$ and $\lambda^*$ so that $\lambda^* < (\mu^*)^r$ is also satisfied. Then by Lemma 8.2, we can find $\delta > 0$ so that $\text{Lip}(\tilde{\phi}_\delta - \text{id}_R) < \epsilon^*/2$. This means $\text{Lip}(\Phi_0 - T) < \epsilon^*/2$ because $\text{Lip}(\Phi_0 - T) = \text{Lip}(\tilde{\phi}_\delta - \text{id}_R)$. For this $\delta$, by Lemma 8.4, we can find $c > 0$ so that $\text{Lip}(\Phi_c - \Phi_0) < \epsilon^*/2$. Therefore the center manifold theorem is applicable to $T$ and to our $\Phi_c$ as $\Phi$ in the theorem.

The second statement of Lemma 8.4 tells that the graph of $(1 - h(\frac{x}{\epsilon}))\tilde{\beta}(x)$ is invariant by $\Phi_c$. From the arrangement it is also clear that the graph is contained in the sector $S_1$ in the center manifold theorem. Therefore the graph is nothing but $W_1$ in the center manifold theorem and $g$ in the theorem turns out to be $(1 - h(\frac{x}{\epsilon}))\tilde{\beta}(x)$ in our case. Therefore it is concluded that this function is of $C^r$ and so is $\tilde{\beta}$.

We are free to improve the choice of $\mu^*$ and $\lambda^*$ to obtain another arbitrary $r \in \mathbb{N}$. As a conclusion, $\tilde{\beta}$ is of $C^\infty$. This implies nothing but the fact that $\beta$ is smooth and flat at $x = 0$ and completes the proof.

It is easy to arrange the proof for Equation (II'). The space $E_2$ is now taken to be $C^M$ and the operator $T$ to be $A^{-1}$. We should remark here that by change of basis, $A$ can be conjugate to one which is arbitrarily close to $\lambda \cdot E$ where $E$ denotes the identity matrix. This enables us to choose $\mu^*$ and $\lambda^*$ in the same way as in the above proof.

**Remark 8.5** Instead of using the center manifold theorem, we can also arrange the proof so as to rely on the $C^r$ section theorem due to Hirsch-Pugh-Shub (cf. [Sh]), which even proves the center manifold theorem. We also need the conjugation by $H_c$ to obtain $\Phi_c$ and then we may look at the space of functions which are supported on a large enough interval $[-R, R]$.

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