UPPER-BOUND FOR THE NUMBER OF ROBUST PARABOLIC CURVES FOR A
CLASS OF MAPS TANGENT TO IDENTITY

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1. Introduction

The Leau-Fatou flower theorem [8] completely describes the dynamic behavior of 1–dimen-
sional maps tangent to the identity. In dimension two Hakim [12] and Abate [1] proved that
if \( f \) is a holomorphic map tangent to the identity in \( \mathbb{C}^2 \) and \( \nu(f) \) is the degree of the first non
vanishing jet of \( f - Id \) then there exist \( \nu(f) - 1 \) robust parabolic curves (RP curves for short),
namely attractive petals at the origin which survive under by blow-up (see [3] and Section 3).

The set of the exponential of holomorphic vector fields (of order greater than or equal to two),
\( \Phi_{\geq 2}(\mathbb{C}^2, 0) \), is dense in the space of germs of maps tangent to the identity.

In this paper we give an upper-bound for the number of robust parabolic curves of
\( f \in \Phi_{\geq 2}(\mathbb{C}^2, 0) \).

**Theorem 1.1.** Let \( f = (f_1, f_2) \in \Phi_{\geq 2}(\mathbb{C}^2, 0) \) be a non-dicritical holomorphic map. Set \( \eta(f) := \max\{\text{ord}(f_1 - \text{Id}), \text{ord}(f_2 - \text{Id})\} \) and \( \mu(f) \) the Milnor number of \( f \). Then the number of RP curves is at most

\[
(\mu(f) + 1)(\eta^2(f) - \eta(f)).
\]

In the dicritical case in [6] Bracci proved that \( f \) is dicritical if and only if it has infinitely
many parabolic curves. Here we show that the parabolic curves at a dicritical point are indeed
robust ones:

**Proposition 1.2.** Let \( f = (f_1, f_2) \in \Phi_{\geq 2}(\mathbb{C}^2, 0) \) be a dicritical holomorphic map. Then there
exist infinitely many RP curves.

A new approach to study dynamics of germs of mappings tangent to the identity has been
introduced in [6], [2] by Abate, Bracci and Tovena. Their idea is to study discrete dynamics
using families of vector fields whose flows are approximations at the first order of \( f \). This
technique turns out to be is very useful.

In case \( f \) is the exponential of an holomorphic vector field we notice a very strict relation
between the dynamics of the map and the dynamics of the vector field. So, if the diffeomorphism
\( f \), is such that there exists a vector field \( X \) such that \( \exp(X) = f \), then it turns out that the RP
curves are "geometrically" determined by the fact they lay in a separatrix of \( X \). Thus we can
use a result of Corral and Fernandez Sanchez [10] concerning the upper-bound of the number
of separatrices of \( X \) to estimate the number of RP curves.

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2. EXPONENTIAL OF A VECTOR FIELD

Let \( f \in \Phi_{\geq 2}(\mathbb{C}^2, 0) \) be a holomorphic map tangent to the identity in \( \mathbb{C}^2 \) then there exists a vector field \( X \) such that \( \exp(X) = F \). The two objects, \( f \) and \( X \), are related by the following classical result:

**Proposition 2.1.** Let \( X \) be a germ of holomorphic vector in \((\mathbb{C}^2, 0)\). Then its time one flow can be written as:

\[
    f^t(x, y) = \left( x + \sum_{n=1}^{\infty} \frac{t^n}{n!} X^n.x, y + \sum_{n=1}^{\infty} \frac{t^n}{n!} X^n.y \right),
\]

where \( X^n.x \) is defined by \( X \) applied to \( X^{n-1}.x \) and \( X.x \) is the application of \( X \) to \( x \).

**Remark 2.2.** If \( X \) is an holomorphic vector field in \((\mathbb{C}^2, 0)\) then its time one map is given by:

\[
    \exp(\hat{X}) = \left( x + \sum_{n=1}^{\infty} \frac{1}{n!} \hat{X}^n.x, y + \sum_{n=1}^{\infty} \frac{1}{n!} \hat{X}^n.y \right).
\]

**Remark 2.3.** If \( f \) is a holomorphic map tangent to the identity in \( \mathbb{C}^2 \) at the origin then the associated vector field has order of singularity at \( 0 \) greater or equal to \( 2 \).

3. ROBUST PARABOLIC CURVES

In this section we give the definition of parabolic and robust parabolic curves for a germ of holomorphic map tangent to the identity at the origin in \( \mathbb{C}^2 \) and we study the relationship between the separatrices of the vector field associated to \( f \) and these curves.

**Definition 3.1.** A **parabolic curve** for \( f \in \text{Diff}(\mathbb{C}^2, 0) \) at the origin is an injective holomorphic map \( \varphi : \Delta \to \mathbb{C}^2 \) satisfying the following properties:

1. \( \Delta \) is a simply connected domain in \( \mathbb{C} \) with \( 0 \in \partial \Delta \);
2. \( \varphi \) is continuous at the origin, and \( \varphi(0) = 0 \);
3. \( \varphi(\Delta) \) is invariant under \( f \), and \( (f|_{\varphi(\Delta)})^n \to 0 \) as \( n \to \infty \).

The idea of robust parabolic curve was first introduced by Abate and Tovena in [3]:

**Definition 3.2.** A **robust parabolic curve** is a parabolic curve that satisfies the following conditions:

1. we can blow-up \( \varphi \) at level \( h \) for any \( h \geq 1 \),
2. there is a formal power series \( Q \in (\mathbb{C}[\![x]\!] )^2 \) such that for every \( h \geq 1 \) there is \( r_h > 0 \) such that \( \varphi - Q_h = O(\zeta^{h+1}) \) in \( \Delta_{r_h} \), where \( Q_h \) denotes the truncation at degree \( h \) of \( Q \).

**Remark 3.3.** Essentially when we say condition (1) we mean that the strict transform of the parabolic curve is also a parabolic curve, but for a clear definition of "blow-up at level \( h \)" we refer to [3].

The geometric meaning of Definition [3.2] is clarified by the following proposition:

**Proposition 3.4.** Let \( f \in \Phi_{\geq 2}(\mathbb{C}^2, 0) \) be a holomorphic map and let \( X \) be vector field such that \( \exp(X) = f \). Let \( \varphi \) be a robust parabolic curve. Then \( \varphi \) is contained in a separatrix of \( X \). Conversely in every separatrix of \( X \) there exists at least one RP curve for \( f \).
Proof. Let be \( p \in \varphi(\Delta) \). Since \[ \exp(X) = f. \] then the orbit \{f^n(p)\} is contained in a separatrix, \( S \), of \( X \). We have only to prove that every orbit generated by a generic point \( q \in \varphi(\Delta) \) stays inside \( S \). By contradiction we can find two orbits that converge to zero living in two different separatrices, say \( S_1 \) and \( S_2 \) and let \( h \) be the order of the first non-zero jet of \( l_1 - l_2 \). If we blow-up the vector field \( h \) times then, by property (1) of the definition of RP curves, we have that the two orbits converge to zero with two different directions and this contradicts property (2) of Definition 3.2.

Let prove the converse. Let be \( S \) a separatrix and let \( y = \varphi(x) = x^{\frac{p}{q}} + \cdots \) be its expression in Puiseux series.

**Remark 3.5.** We can suppose \( p \) and \( q \) prime each other and that \( \frac{p}{q} \geq 1 \), if this is not true we can choose the parametrization of the separatrix in the form \( x = \psi(y) \) which satisfies the required condition.

Let now make the following change of variables:
\[
\begin{cases}
  u = x \\
  v = y - \varphi(x)
\end{cases}
\]
The vector field in the new coordinates is:
\[
\begin{aligned}
  \dot{u} &= A(u, v + \varphi(u)) \\
  \dot{v} &= B(u, v + \varphi(u)) - \dot{\varphi}(u)A(u, v + \varphi(u))
\end{aligned}
\]
If we compute the exponential of this new vector field restricted to the separatrix \( \{v = 0\} \) we find:
\[ \exp(A(u, \varphi(u)) \frac{\partial}{\partial u}) \]
Let us make the change of variables:
\[ u = z^q \]
and then the first component of the vector field is:
\[ \dot{z} = A(z^q, \varphi(z^q)) \frac{1}{q z^{q-1}}. \]
By Remark 3.5 the left-hand side of (3) is expressed as a power series. Now if we take the exponential of this new vector field we find a map tangent to the identity conjugated to the original one given by \( (z, w) \mapsto (z + z^h + \cdots, w) \). By the Leau-Fatou Theorem ([8]) we get the assertion. \( \square \)

As a consequence of this last result we easily prove the existence of RP curves for map in \( \Phi_{\geq2}(\mathbb{C}^2, 0) \) [1], [12], [3].

**Proposition 3.6.** Let \( f \in \Phi_{\geq2}(\mathbb{C}^2, 0) \) be a holomorphic map tangent to the identity in \( \mathbb{C}^2 \). Then there exists at least one RP curve.
4. NON-DICRITICAL CASE

Proposition 3.4 shows that the RP curves live inside separatrices of the associated vector field. So the idea is to estimate the number of separatrices of the vector field and then the number of RP curves inside a separatrix. In [10] Corral and Fernandez Sanchez find the optimal estimation of the number of separatrices by the Milnor number of $X$ [5].

Proposition 4.1. [10] Let $X$ be an holomorphic vector field in $\mathbb{C}^2$, singular at the origin. Let $S$ be the curve determined by all the separatrices through the origin. Let $r_0(S)$ be the number of the irreducible components of $S$. Then:

$$r_0(S) \leq \mu_0(X) + 1,$$

where $\mu_0$ is the Milnor number of $X$ at the origin.

The proof of this proposition can be found in [10]. In order to express the previous estimation in terms of invariants of $f$ we introduce the intersection multiplicity [1].

Proposition 4.2. Let $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a holomorphic map tangent to the identity in $\mathbb{C}^2$ and let $X$ be the associated vector field. Then the Milnor number of $X$, at the origin, is equal to the intersection multiplicity of $f - \text{Id}$.

Proof. Let observe that, if

$$X = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y},$$

then the Milnor number of $X$ is equal to the intersection multiplicity at the origin [5]. Now observe that the intersection multiplicity of two function $g(x, y)$ and $h(x, y)$ depends only on the first non zero jet of $g$ and on the lowest exponent of the Puiseux parametrization of $h$ [11]. The Newton-Puiseux polygon shows that the lowest exponent of the parametrization depends only on the part of the polygon determined by the first non zero jet of the function [9]. This concludes the proof because the lowest non zero jets of $f - \text{Id}$, $A$ and $B$ are the same. \hfill \Box

Now we can start with the second part of the proof i.e. the estimation of the number of RP curves that are in a separatrix. Proceeding as in Proposition 3.4 we can conjugate the restriction of $f$ to the separatrix to a map of the kind $(z, w) \mapsto (z + zg + \cdots, w)$.

We have now to estimate the exponent $h$. An easy computation shows that the exponent $h$ is the lowest degree of the expression of $\frac{A(x^q, y^q)}{q^{p+1}}$. The same computation proves that the order of $A(x, y)$ is the same as the order, $\nu_1$, of $f_1 - \text{Id}$. Then

$$\frac{A_{\nu_1}(z^q, \varphi(z^q))}{q^{p+1}} = \sum_{i+j=\nu_1} z^{q(i+j)} \varphi(z^q)^j z^{1-q},$$

so the lowest degree is:

$$q \iota + pj + 1 - j,$$

for $0 \leq \iota, j \leq \nu_1$. Let us maximize the quantity (5). According to the cases $p, q > 0$ and $p, q < 0$ and by the assumption $\frac{p}{q} \geq 1$ we have that:

$$q \iota + pj + 1 - j \leq \nu_1 p + 1 - q \leq \nu_1 p,$$
where the last inequality holds because \( q \geq 1 \). By Remark 3.5 the number of RP curves in \( S \) is estimated from above by:

\[
\max\{\nu_1, \nu_2\} p.
\]

This estimation depends on \( p \) and \( q \) and then on the particular separatrix. It is possible to improve this result removing the dependence on the separatrix in the following way. Since:

\[
\frac{dy}{dx} = \frac{B(x, y)}{A(x, y)}
\]

and replacing \( y = \varphi(x) = x^k + \cdots \), we find that:

\[
k = \frac{i_1 - i_1 + 1}{j_2 - j_1 + 1},
\]

with \( i_1 + j_1 = \nu_1 \) and \( i_2 + j_2 = \nu_2 \). So we easily find:

\[
k \leq \frac{\nu_2 - 1}{\nu_1 - 1}.
\]

Then \( p \leq (\nu_2 - 1) \) and \( q \leq (\nu_1 - 1) \) because \( p \) and \( q \) are prime each other.

This proves the main Theorem 1.1.

5. Dicritical case

In this section we estimate the number of RP curves in the dicritical case. We briefly recall the definition of dicritical singularity for maps and for vector fields.

**Definition 5.1.** Let \( X \) be an holomorphic vector field and let \( p \) be a singularity. We say that \( p \) is a dicritical singularity if the exceptional divisor is not invariant for the strict transform of \( X \).

For the case of maps tangent to the identity in \( \mathbb{C}^2 \) we have the following definitions:

**Definition 5.2.** \([6]\) Let \( S \) be an irreducible curve in \( \mathbb{C}^2 \) and \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) be an holomorphic map such that \( f|_S = Id_S \) and \( f \neq Id \). Let \( \mathcal{I}(S)_p \subset O_p \) be the ideal of germs vanishing on \( S \). We say that \( f \) is tangential on \( S \) at \( p \) if for a defining function \( l \) of \( S \) at \( p \):

\[
\frac{l \circ f - l}{l^T} \equiv 0 \mod \mathcal{I}(S)_p,
\]

where \( T \) is the order of \( f \) on \( S \) at \( p \).

**Definition 5.3.** \([6][1][7]\) Let \( f \) be a holomorphic map tangent to the identity in \( \mathbb{C}^2 \). We say that \( 0 \) is dicritical for \( f \) if the blow-up of \( f \) is non-tangential on the exceptional divisor.

By the theorem on existence and uniqueness of solutions of ordinary differential equations \([14]\) we have:

**Proposition 5.4.** Let \( X \) be an holomorphic vector field and \( p \) a dicritical singularity, then there exist infinitely many separatrices trough \( p \).

In the discrete case the following result holds:

**Proposition 5.5.** \([1][6]\) Let \( f \) be a map tangent to the identity in \( \mathbb{C}^2 \). If \( 0 \) is dicritical then there exists infinitely many parabolic curves for \( f \).
The last two propositions suggest that there exist a strict relationship between the two notions of dicriticity:

**Proposition 5.6.** Let \( f \in \Phi_{\geq 2}(\mathbb{C}^2, 0) \) be a map tangent to the identity in \( \mathbb{C}^2 \) and \( X \) the vector field such that \( \exp(X) = f \). Then \( f \) is dicritical in 0 if and only if \( X \) is dicritical in 0.

Before proving the previous proposition we need the following lemma, that is a direct application of Proposition 4.2.4 (pg. 267) of [4]:

**Lemma 5.7.** Let \( f \in \Phi_{\geq 2}(\mathbb{C}^2, 0) \) be a map tangent to the identity in \( \mathbb{C}^2 \) and set \( \tilde{f} \) the blow-up of \( f \). Let \( X \) be the vector field associated to \( f \). We have that:

\[
\tilde{f} = \exp(\pi^*X),
\]

where \( \pi^*X \) is the pull-back of \( X \) by the blow-up map \( \pi \).

**Remark 5.8.** This lemma says that the relationship between maps and vector fields is preserved under blow-up if, instead of saturating the vector field, we divide it only by the first power of a local expression of the exceptional divisor.

**Example 5.9.** This example shows that the equality \( \tilde{f} = \exp(\tilde{X}) \) is not generally true. Let:

\[
f(x, y) = (x + \sum_{k \geq 1} x^{k+1}, y + \sum_{k \geq 1} y^{k+1}).
\]

The associated vector field \( X \) is:

\[
X = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y},
\]

i.e.

\[
\exp(X) = f.
\]

The blow-up of \( f \) and \( X \), in the chart for which \( \pi(u, v) = (u, uv) \), are:

\[
\tilde{f} = (u + \sum_{k \geq 1} u^{k+1}, v + \sum_{k \geq 1} \frac{u^k v^{k+1}}{1 + \sum_{k \geq 1} u^k});
\]

\[
\tilde{X} = u \frac{\partial}{\partial u} + (v^2 - v) \frac{\partial}{\partial v}.
\]

So we easily prove that:

\[
\exp(\tilde{X}) = (eu, \cdots) \neq \tilde{f}.
\]

But if we take:

\[
\tilde{X} = \pi^*(X) = u^2 \frac{\partial}{\partial u} + u(v^2 - v) \frac{\partial}{\partial v},
\]

we see that \( \exp(\tilde{X}) = \tilde{f} \). We can observe that generally \( \exp(\tilde{X}) \) is not tangent to the identity because the vector field has linear part.

**Proof.** If \( X \) is dicritical then by Proposition 5.4 there exists infinitely many separatrices and then, by Proposition 3.4, \( f \) admits infinitely many RP curves. So \( f \), by Theorem 1.1 has to be dicritical. Let prove the converse. Let \( X \) be not dicritical and let prove that \( f \) is not dicritical. Let denote by \( \tilde{f} \) and \( \tilde{X} \) (respectively) the blow-up of the map \( f \) and of the field \( X \). We have
to prove that if the exceptional divisor $D$ is invariant by $\tilde{X}$ then $\tilde{f}$ is tangential on $D$. We can suppose that:

$$\tilde{X} = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y},$$
and $D = \{l(x, y) = x = 0\}$. The invariance of $D$ is equivalent to the fact that $x$ divides $A(x, y)$. Let be

$$T = \max\{s \in \mathbb{N} \mid x^s \mid A(x, y)\},$$
i.e. $A(x, y) = x^T a(x, y)$. So we have:

$$X = x^T a(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y},$$
with $x \nmid B(x, y)$ i.e. $B(0, y) = y^k + \cdots$. Now let be $\tilde{X} := \pi^*(X)$ and observe that this field has the following structure:

$$\tilde{X} = x^\alpha a(x, y) \frac{\partial}{\partial x} + x^\beta B(x, y) \frac{\partial}{\partial y},$$
where $a(x, y)$ and $B(x, y)$ are the previous ones and $\alpha > \beta$. By Lemma 5.7 we know that $\exp(\tilde{X}) = \tilde{f}$ and so we can reconstruct the map by formula (2).

We find $\tilde{X}^j.x = x^\alpha(\cdots)$ for all $j$. On the other hand when we compute $\tilde{X}^i.y$, we find a structure of the type $\tilde{X}^i.y = x^\alpha(\cdots) + x^{i\beta}(\cdots)$. Then the lowest power of $x$ appears in the term $\tilde{X}.y$ and it is $x^\beta y^k$. So the order of $\tilde{f}$ on $D$ is $\min(\alpha, \beta) = \beta$ and then

$$\frac{l \circ \tilde{f} - l}{\tilde{f}} = \frac{\tilde{f}_1 - x^{\beta}}{x^\beta} = x^{\alpha-\beta}(\cdots) \equiv 0 \mod \mathcal{I}(S)_p.$$

In this setting we have

**Proposition 5.10.** Let $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a dicritical holomorphic map tangent to the identity in $\mathbb{C}^2$. Then there exist infinitely many RP curves.

**Proof.** Since $f$ is dicritical then, by Proposition 5.6 $X$ is dicritical. By Proposition 5.4 there exist infinite separatrices and then by Proposition 5.4 we get the assertion. □

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