ON UNIFORMIZATION OF COMPACT KÄHLER MANIFOLDS

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The aim of the present note is to extend the uniformization theorem of projective manifolds in [9, Introduction, Theorem] to compact Kähler manifolds. In an email to the author (January, 2015), Dennis Sullivan essentially raised the question whether one can generalize the uniformization theorem in [9]. The author would like to thank him for the question.

Let $X$ be a compact complex manifold of dimension $n \geq 2$. We denote its universal covering by $U_X$. We will derive the following theorem from a similar theorem in [9].

**Theorem (uniformization).** Let $X$ be a compact Kähler manifold of dimension $n$ with large and residually finite fundamental group $\pi_1(X)$. If $\pi_1(X)$ is, in addition, nonamenable then $U_X$ is a bounded domain in $\mathbb{C}^n$. Thus $X$ is projective by Poincaré [5, Theorem 5.22].

**Proof of Theorem.** By a theorem of Moishezon [7], it will suffice to establish that $X$ is a Moishezon manifold. In [4, Sect. 3], Gromov uses his notion of Kähler hyperbolicity to obtain holomorphic $L^2$ forms on $U_X$ and prove that $X$ is Moishezon.

A priori, we do not know if there are holomorphic $L^2$ forms on $U_X$. Set $\Gamma := \pi_1(X)$. Let $L$ be an arbitrary complex line bundle on $U_X$. We will consider a section $f \in H^0(L^q, U_X)$ which is not assumed to be $L^p$, where $p < \infty$.

As in Kollár [5, Chap 13.1], we will employ $\ell^p$ sections $f$ on orbits of $\Gamma$ in place of $L^p$ sections. Of course, we need a natural $\Gamma$-invariant Hermitian quasi-metric on $L^q$ (see the definition in the proof of Lemma 3).

Given an arbitrary $\Gamma$-invariant Hermitian metric on $U_X$, we get the induced Riemannian metric on $U_X$ with the volume form $d\mu$. Since $\Gamma$ is nonamenable, we get non-constant bounded harmonic functions on $U_X$ by Lyons and Sullivan [6].

Employing their theorem, Toledo [8] has established that the space of bounded harmonic functions as well as the space generated by bounded positive harmonic functions are infinite dimensional (see [9, Sect. 2.6]). Given $r$ linearly independent functions $g_1, \ldots, g_r$ on $U_X$, clearly there exist $r$ points $u_1, \ldots, u_r \in U_X$ such that the vectors $(g_i(u_1), \ldots, g_i(u_r))$ ($1 \leq i \leq r$) are linearly independent.

Let $H_{ar}(U_X)$ ($H_{ar}^b(U_X)$) be the space of harmonic functions (bounded harmonic functions, respectively) on $U_X$.

In place of the standard $L^2(d\mu)$ integration with the standard Riemannian measure $d\mu$ on $U_X$, we will integrate the *bounded* harmonic functions with respect to the measure

$$dv := p_{U_X}(s, x, Q)d\mu,$$
where \( Q \in U_X \) is a fixed point and \( p_{U_X}(s, x, Q) \) is the heat kernel. Because all bounded harmonic functions are square integrable, i.e. in \( L_2(dv) \), we obtain the pre-Hilbert space of bounded harmonic functions (compare [9, Sect. 2.4 and Sect. 4]). We observe that the latter pre-Hilbert space has a completion in the (real) Hilbert space \( H \) of all harmonic \( L_2(dv) \) functions:

\[
H := \left\{ h \in \text{Har}(U_X) \mid \| h \|_H^2 := \int_{U_X} |h(x)|^2 dv < \infty \right\}.
\]

Let \( H^b \subset H \) be the Hilbert subspace generated by \( \text{Har}^b(U_X) \). These Hilbert spaces are separable infinite dimensional and have reproducing kernels. The group \( \Gamma \) acts isometrically on \( H^b : \psi \mapsto (\psi \circ \gamma) (\gamma \in \Gamma) \).

Let \( \{ \phi_j \} \subset \text{Har}^b(U_X) \) be an orthonormal basis of \( H^b \). We obtain a continuous, even smooth, finite \( \Gamma \)-energy \( \Gamma \)-equivariant mapping

\[
g : U_X \longrightarrow (H^b)^* \quad (u \mapsto (\phi_0(u), \phi_1(u), \ldots)).
\]

Also we get a natural mapping \( g : U_X \longrightarrow \mathbb{P}((H^b)^*) \), \( u \mapsto \psi(u) (\forall \psi \in H^b) \).

We assume \( g \) is harmonic; otherwise, we replace \( g \) by a harmonic mapping homotopic to \( g \). Let \( \mathbf{F}_C(\infty, 0) \) denote the complex flat Fubini space, i.e. a complex Hilbert space.

**Lemma 1.** With assumptions of the theorem, \( g \) will produce a pluriharmonic mapping \( g^{fl} \). There exists a natural holomorphic mapping \( g^h : U_X \longrightarrow \mathbf{F}_C(\infty, 0) \).

**Proof of Lemma 1.** We define a harmonic \( \Gamma \)-equivariant mapping

\[
g^{fl} := S_{g(Q)} \cdot g : U_X \longrightarrow (H^b)^*.
\]

We have applied the mapping \( g \) followed by the Calabi flattening out \( S_{g(Q)} \) (a generalized stereographic projection from \( g(Q) \)) of the real projective space \( \mathbf{F}_R(\infty, 1) \) into the Hilbert space \( \mathbf{F}_R(\infty, 1) \). By \([2, \text{Chap. 4, Cor. 1, p. 20}]\), the whole \( \mathbf{F}_R(\infty, 1) \), except the antipolar hyperplane \( A \) of \( g(Q) \), can be flatten out into \( \mathbf{F}_R(\infty, 0) \). The image of \( g \) does not intersect the antipolar hyperplane \( A \) of \( g(Q) \). Thus we have introduced a flat metric in a large (i.e. outside \( A \)) neighborhood of \( g(Q) \) in \( \mathbf{P}((H^b)^*) \).

Since the mapping \( g^{fl} \) has finite \( \Gamma \)-energy, it is pluriharmonic; this is a special case of a theorem of Sin (see, e.g., [1]). Since \( U_X \) is simply connected, we obtain the natural holomorphic mappings

\[
g^h : U_X \longrightarrow \mathbf{F}_C(\infty, 0) (\hookrightarrow \mathbf{P}_C((H^b)^*) = \mathbf{F}_C(\infty, 1)).
\]

**Lemma 2.** Construction of a complex line bundle \( \mathcal{L}_X \) on \( X \) and its pullback on \( U_X \), denoted by \( \mathcal{L} \).

**Proof of Lemma 2.** We take a point \( u \in U_X \). Let \( v := g^h(u) \in \mathbf{F}_C(\infty, 1) \), where \( \mathbf{F}_C(\infty, 1) \) is the complex projective space. We consider the linear system of hyperplanes in \( \mathbf{F}_C(\infty, 1) \) through \( v \) and its proper transform on \( U_X \). We consider only the moving part. The projection on \( X \) of the latter linear system on \( U_X \) will produce a linear system on \( X \).

A connected component of a general member of the latter linear system on \( X \) will be an irreducible divisor \( D \) on \( X \) by Bertini’s theorem. The corresponding line bundle will be the desired \( \mathcal{L}_X := \mathcal{O}_X(D) \) on \( X \).
Lemma 3. Conclusion of the proof of theorem by induction on \( \dim X \).

Proof of Lemma 3. By the Campana-Deligne theorem [5, Theorem 2.14], \( \pi_1(D) \) will be nonamenable. We proceed by induction on \( \dim X \), the case \( \dim X = 1 \) being trivial. Let \( q = q(n) \) be an appropriate integer.

We get a global holomorphic function-section \( f \) of \( \mathcal{L}^g \) corresponding to a bounded pluriharmonic function (see Lemma 1 and [9, Sect. 4]). We will define a \( \Gamma \)-invariant Hermitian quasi-metric on sections of \( \mathcal{L}^g \) below. Furthermore, \( f \) is \( \ell^2 \) on orbits of \( \Gamma \), and it is not identically zero on any orbit because, otherwise, we could have replaced \( U_X \) by \( U_X \setminus B \), where the closed analytic subset \( B \subset U_X \) is the union of those orbits on which \( f \) had vanished [5, Theorem 13.2, Proof of Theorem 13.9].

One can show that \( f \) satisfies the above conditions by taking linear systems of curvilinear sections of \( U_X \) through \( u \in U_X \) and their projections on \( X \) (see the proof of Lemma 2 above), since the statements are trivial in dimension one. The required Hermitian quasi-metric on \( \mathcal{L}_X^g \) is also defined by induction on dimension with the help of the Poincaré residue map [3, pp. 147-148].

The condition \( \ell^2 \) on orbits of \( \Gamma \) is a local property on \( X \). We get only a Hermitian quasi-metric on \( \mathcal{L}_X^g \) (instead of a Hermitian metric). Precisely, we get Hermitian metrics over small neighborhoods of points of \( X \), and on the intersections of neighborhoods, they will differ by constant multiples (see [5, Chap. 5.13]).

For \( \forall k > N \gg 0 \), the Poincaré series are continuous sections

\[
P(f^k)(u) := \sum_{\gamma \in \Gamma} \gamma^* f^k(\gamma u),
\]

and they do not vanish for infinitely many \( k \) (see [5, Sect. 13.1, Theorem 13.2]).

Finally, we can apply Gromov’s theorem, precisely, its generalization by Kollár (see [4, Corollary 3.2.B, Remark 3.2.B’] and [5, Theorem 13.8, Corollary 13.8.2, Theorem 13.9, Theorem 13.10]). So, \( X \) is a Moishezon manifold.

The Lemma 3 and Theorem are established.

Remarks. i) The theorem of the present note provides an alternative proof of a conjecture of H. Wu provided \( \pi_1(X) \) is residually finite (see [10]).

ii) A generalization of the theorem to singular spaces will appear elsewhere.

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