Lower Deviation Probabilities for Level Sets of the Branching Random Walk

Shuxiong Zhang

Abstract

Given a supercritical branching random walk \( \{Z_n\}_{n \geq 0} \) on \( \mathbb{R} \), let \( Z_n(A) \) be the number of particles located in a set \( A \subset \mathbb{R} \) at generation \( n \). It is known from Biggins (J Appl Probab 14:630–636, 1977) that under some mild conditions, for \( \theta \in [0, 1) \),

\[
\frac{n^{-1} \log Z_n([\theta x^* n, \infty))}{\log (E[Z_1(\mathbb{R})]) - I(\theta x^* )}
\]

converges almost surely to \( I(\theta x^* ) \) as \( n \to \infty \), where \( x^* \) is the speed of the maximal position of \( \{Z_n\}_{n \geq 0} \) and \( I(\cdot) \) is the large deviation rate function of the underlying random walk. In this work, we investigate its lower deviation probabilities, in other words, the convergence rates of

\[
P\left(\frac{Z_n([\theta x^* n, \infty))}{\log (E[Z_1(\mathbb{R})]) - I(\theta x^* )} < e^{an}\right)
\]

as \( n \to \infty \), where \( a \in [0, \log (E[Z_1(\mathbb{R})]) - I(\theta x^* )) \). Our results complete those in Chen and He (Ann Institut Henri Poincare Probab Stat 56:2507–2539, 2020), Gantert and Höfelsauer (Electron Commun Probab 23(34):1–12, 2018) and Öz (Latin Am J Probab Math Stat 17:711–731, 2020).

Keywords Branching random walk · Level sets · Lower deviation

Mathematics Subject Classification (2020) 60F10 · 60J80 · 60G50

1 Introduction and Main Results

1.1 Introduction

Given a probability distribution \( \{p_k\}_{k \geq 0} \) on \( \mathbb{N} \) and a real-valued random variable \( X \), a branching random walk (BRW) \( \{Z_n\}_{n \geq 0} \) with offspring law \( \{p_k\}_{k \geq 0} \) and step

\( x^* \) is the speed of the maximal position of \( \{Z_n\}_{n \geq 0} \), and \( I(\cdot) \) is the large deviation rate function of the underlying random walk. In this work, we investigate its lower deviation probabilities, in other words, the convergence rates of

\[
P\left(\frac{Z_n([\theta x^* n, \infty))}{\log (E[Z_1(\mathbb{R})]) - I(\theta x^* )} < e^{an}\right)
\]

as \( n \to \infty \), where \( a \in [0, \log (E[Z_1(\mathbb{R})]) - I(\theta x^* )) \). Our results complete those in Chen and He (Ann Institut Henri Poincare Probab Stat 56:2507–2539, 2020), Gantert and Höfelsauer (Electron Commun Probab 23(34):1–12, 2018) and Öz (Latin Am J Probab Math Stat 17:711–731, 2020).

Keywords Branching random walk · Level sets · Lower deviation

Mathematics Subject Classification (2020) 60F10 · 60J80 · 60G50

1 Introduction and Main Results

1.1 Introduction

Given a probability distribution \( \{p_k\}_{k \geq 0} \) on \( \mathbb{N} \) and a real-valued random variable \( X \), a branching random walk (BRW) \( \{Z_n\}_{n \geq 0} \) with offspring law \( \{p_k\}_{k \geq 0} \) and step
size $X$ is defined as follows. At time 0, there is one particle located at the origin (i.e., $Z_0 = \delta_0$). The particle dies and produces progenies according to the offspring distribution $\{p_k\}_{k \geq 0}$. Afterward, the offspring particles move independently according to the law of $X$. This forms a point process at time 1, denoted by $Z_1$. For any $n \geq 2$, we define the point process $Z_n$ by the following iteration:

$$Z_n = \sum_{x \in Z_{n-1}} \tilde{Z}_x^1,$$

where $\tilde{Z}_x^1$ has the same distribution as $Z_1(\cdot - S_x) \cup \{\tilde{Z}_x^1 : x \in Z_{n-1}\}$ (conditioned on $Z_{n-1}$) are independent. Here and later, for a point process $\xi$, $x \in \xi$ means $x$ is an atom of $\xi$ and $S_x$ is the position of $x$ (i.e., $\xi = \sum_{x \in \xi} \delta_{S_x}$).

For $A \subset \mathbb{R}$, let

$$Z_n(A) := \#\{u \in Z_n : S_u \in A\},$$

i.e., $Z_n(A)$ is the number of particles located in the set $A$.

Let $m := \mathbb{E}[Z_1(\mathbb{R})]$ be the average number of particles in the first generation. According to Biggins [5, Theorem 2], if $1 < m < \infty$, $\mathbb{E}[X] = 0$ and $\mathbb{E}[e^{\kappa X}] < \infty$ for some $\kappa > 0$, then for $\theta \in [0, 1)$,

$$\lim_{n \to \infty} \frac{1}{n} \log Z_n(\{\theta x^* n, \infty\}) = \log m - I(\theta x^*), \quad \mathbb{P} - a.s. \text{ on non-extinction},$$

where $I(x) := \sup_{t \in \mathbb{R}} \{tx - \log \mathbb{E}[e^{tX}]\}$ is the rate function in Cramér’s theorem (see [13, Section 2.2]), and $x^* := \sup\{x \geq 0 : I(x) \leq \log m\}$ is the speed of the maximal position of a branching random walk (see [20]). This result tells us the number of particles located in the level set $[\theta x^* n, \infty)$. In this work, we are going to study the lower deviation probabilities of these level sets, i.e., the decay rate of

$$\mathbb{P}(Z_n([\theta x^* n, \infty)) < e^{an})$$

as $n \to \infty$, where $a \in [0, \log m - I(\theta x^*)]$.

In fact, there are some related results for the branching Brownian motion $\{Z_t\}_{t \geq 0}$. ÖZ [27] considered the lower deviation $\mathbb{P}(Z_t(\theta x^* t + B) < e^{at})$, where $B$ is a fixed ball. Aïdékon, Hu and Shi [1] considered the upper deviation $\mathbb{P}(Z_t([\theta x^* t, \infty)) > e^{at})$. We will see that for the branching random walk, the strategy to study this problem and the answers will be very different from theirs.

We also mention here that, since the last few decades, the model BRW has been extensively studied due to its connection to many fields, such as Gaussian multiplicative chaos, random walk in random environment, random polymer, random algorithms and discrete Gaussian free field; see [22], [23], [24], [8] and [1] references therein. One can refer to Shi [29] for a more detailed overview. Particularly, the large deviation probabilities (LDP) for BRW and branching Brownian motion (BBM) on real line have attracted many researcher’s attention. For example, Hu [21], Gantert and Höfelsauer...
[19] and Chen and He [12] considered the LDP and the moderate deviation probabilities of BRW’s maximum (for BBM’s maximum, see Chauvin and Rouault [10] and Derrida and Shi [14–16]). For the LDP of empirical distribution, see Chen, He [11] and Louidor, Perkins [25] and Zhang [30]. Some other related works include Rouault [28], Bhattacharya [4] and Buraczewski, M. Maślanka [9].

1.2 Main Results

Let $|Z_n| := Z_n(\mathbb{R})$. Recall that $\{p_k\}_{k \geq 0}$ is the offspring law, $X$ is the step size and $m = \mathbb{E}[|Z_1|]$. In the sequel of this work, we always need the following assumptions.

**Assumption 1.1**

(i) $\mathbb{E}[X] = 0$, $\mathbb{P}(X = 0) < 1$ and $\mathbb{E}[e^{\kappa X}] < \infty$ for some $\kappa > 0$;  
(ii) $p_0 = 0$, $p_1 < 1$ and $m = \mathbb{E}[|Z_1|] < \infty$;  
(iii) $\theta \in [0, 1)$, $a \in \left[0, \log m - I(\theta x^*)\right]$.

**Remark 1.1** In the above assumption, $\mathbb{E}[X] = 0$ and $p_0 = 0$ are not essential, but simplify the proof. The moment conditions for $X$ and $|Z_1|$ are necessary for the almost sure convergence of $\frac{1}{n} \log Z_n((\theta x^* n, \infty))$.

Usually, we call $\{|Z_n|\}_{n \geq 0}$ the branching process or the Galton–Watson process. $\{|Z_n|\}_{n \geq 0}$ is called a supercritical (critical, subcritical) branching process if $m > 1$ ($= 1$, $< 1$). In supercritical case, the branching process will survive with positive probability, otherwise it will die out almost surely (provided $p_1 \neq 1$). For a more detailed discussion, one can refer to [2]. From Assumption 1.1 (ii), it is easy to know that we always deal with the supercritical case. Whether $p_0 + p_1 > 0$ or $= 0$ will lead a crucial dichotomy for the lower deviation probabilities of a supercritical Galton–Watson process; see [18]. For this reason, some large deviation probabilities about the branching random walk also have phase transitions according to $p_0 + p_1 > 0$ or $= 0$; see [11], [12] and [19]. We say the branching process is in the Schröder case if $p_0 + p_1 > 0$, otherwise it is in the Böttcher case. Now, we are ready to show our main results. The first two theorems consider the Schröder case (i.e., $p_1 > 0$ under Assumption 1.1). Define

$$f(\rho) := \log m - I\left(\frac{\theta x^*}{1 - \rho}\right) - \frac{a}{1 - \rho}, \quad \rho \in [0, 1);$$
$$g_\rho(h) := \log m - I\left(\frac{h + \theta x^*}{1 - \rho}\right) - \frac{a}{1 - \rho}, \quad h \in [0, \infty).$$

**Theorem 1.1** (Schröder case, light tail) Assume $p_1 > 0$ and $\mathbb{E}[e^{\kappa X}] < \infty$ for some $\kappa < 0$. Then,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_n((\theta x^* n, \infty)) < e^{an}) = \phi(\theta, a) \in (-\infty, 0),$$
where
\[
\phi(\theta, a) := -\inf_{\rho \in (0, \tilde{\rho})} \left\{ \rho \log \frac{1}{p_1} + \rho I\left( -\frac{d}{\rho} \right) \right\}
\]
and \( \tilde{\rho} \in (0, 1), \ d \in [0, \infty) \) are defined as follows:
\[
\tilde{\rho} := \tilde{\rho}(\theta, a) = \sup \{ \rho \in (0, 1) : f(\rho) \geq 0 \}
\]
\[
d := d(\rho, \theta, a) = \sup \{ h \in [0, \infty) : g_\rho(h) \geq 0 \}.
\]

**Remark 1.2** More specifically, we have the following characterizations about \( \tilde{\rho} \) and \( d \).

(i) Assume \( \log m - I(x^*) = 0 \). Then, \( \tilde{\rho} \) is the unique solution of
\[
f(\rho) = 0, \ \rho \in (0, 1 - \theta],
\]
and \( d \) is the unique solution of the following equation (w.r.t. \( h \)):
\[
g_\rho(h) = 0, \ h \in (0, (1 - \theta - \rho)x^*].
\]

(ii) Assume \( \log m - I(x^*) > 0 \). Then,
\[
\tilde{\rho} = 1 - \theta \text{ if } \frac{a}{\theta} < \log m - I(x^*),
\]
otherwise \( \tilde{\rho} \) is the unique solution of \( f(\rho) = 0, \ \rho \in (0, 1 - \theta] \). Moreover,
\[
d = (1 - \theta - \rho)x^* \text{ for } \rho \in \left[ 0, 1 - a/(\log m - I(x^*)) \right],
\]
and \( d \) is the unique positive solution of
\[
g_\rho(h) = 0, \ h \in (0, (1 - \theta - \rho)x^*]
\]
for \( \rho \in \left[ 1 - a/(\log m - I(x^*)), \tilde{\rho} \right] \).

**Remark 1.3** If \( a = 0 \), then \( \mathbb{P}(Z_n([\theta x^*n, \infty))) < e^{an} \) = \( \mathbb{P}(M_n < \theta x^*n) \), where \( M_n = \max_{u \in Z_n} S_u \) is the maximal position of the branching random walk at time \( n \). Thus, our results also give the lower deviation probabilities of maximum which are consistent with [19] and [12].

**Remark 1.4** If the step size \( X \) is a standard normal random variable, then \( x^* = \sqrt{2 \log m} \) and
\[
\phi(\theta, a) = -\inf_{\rho \in (0, 1 - a + \sqrt{a^2 + (2\theta \log m)^2}/(2 \log m))} \left\{ \rho \log \frac{1}{p_1} \right. 
\]
\[
+ \left( \sqrt{2(1 - \rho)^2 \log m - 2(1 - \rho)a - \theta \sqrt{2 \log m}}^2/(2\rho) \right) \right\}.
\]
So, if $\theta = 0$, then
\[
\phi(0, a) = \begin{cases} 
-2 \sqrt{(\log m - \log p_1)(\log m - a)} + a - 2 \log m, & a \in \left[0, \frac{-\log p_1 \log m}{\log m - \log p_1}\right]; \\
\frac{\log m - a}{\log m}, & a \in \left(-\log p_1 \log m, \log m\right).
\end{cases}
\]

If $a = 0$, then
\[
\phi(\theta, 0) = -2(1 - \theta) \left(\sqrt{\log m(\log m - \log p_1)} - \log m\right).
\]

If $a$ and $\theta$ are nonzero, since the zero root of a sixth-degree polynomial does not have a formula, the optimization problem above cannot be solved in the general case.

When the step size does not have a negative exponential moment, we will see, in the following theorem, the decay scale depends on the left tail of step size. As usual, $f(x) = \Theta(1)g(x)$ as $x \to +\infty$ means there exist constants $C \geq C' > 0$ such that $C' \leq |f(x)/g(x)| \leq C$ for all $x > 1$. $f(x) \sim g(x)$ as $x \to +\infty$ means
\[
\lim_{x \to \infty} f(x)/g(x) = 1.
\]

**Theorem 1.2** (Schröder case, heavy tail) Assume $p_1 > 0$. If $\mathbb{P}(X < -x) = \Theta(1)x^{-\alpha}$ as $x \to +\infty$ for some $\alpha > 0$, then
\[
\lim_{n \to \infty} \frac{1}{\log n} \log \mathbb{P}\left(Z_n([\theta x^* n, \infty)) < e^{\alpha n}\right) = -\alpha.
\]

If $\log \mathbb{P}(X \leq -x) \sim -\lambda x^\alpha$ as $x \to +\infty$ for some $\alpha \in (0, 1)$ and $\lambda > 0$, then
\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P}\left(Z_n([\theta x^* n, \infty)) < e^{\alpha n}\right)
= \begin{cases} 
-\lambda[(1 - \theta)x^*]^\alpha, & a \in [0, \log m - I(x^*)]; \\
-\lambda \hat{c}^\alpha, & a \in (\log m - I(x^*), \log m - I(\theta x^*)),
\end{cases}
\]
where $\hat{c} \in (0, x^* - \theta x^*)$ is the unique positive solution of
\[
\log m - I(\theta x^* + \hat{c}) = a.
\]

**Remark 1.5** In fact, in the case of $\mathbb{P}(X < -x) = \Theta(1)x^{-\alpha}$, $\mathbb{E}[X] = 0$ can be removed in Assumption 1.1.

The next two theorems consider the Böttcher case (i.e., $p_1 = 0$ under Assumption 1.1). As we can see in the following, in this case, the decay rate is very sensitive to the left tail of step size $X$. Let $b := \min\{k \geq 0 : p_k > 0\}$.

**Theorem 1.3** (Böttcher case, bounded tail) Assume $p_1 = 0$ and $\text{ess inf} X = -L \in (-\infty, 0)$. Then,
The following theorem considers the Weibull step size. Recall that

Theorem 1.1 \].

solution of 

\[
\log m \rightarrow \infty
\]

is the minimum population size that 

btn

where 

\[
a^* := \left( \log m - I(x^*) \right) \frac{L + \theta x^*}{L + x^*} + \frac{(1-\theta)x^*}{L + x^*} \log b\right] \wedge \left[ \log m - I(\theta x^*) \right] and \ \hat{c} \ is \ the \ unique \ solution \ of \ the \ following \ equation \ on \ (0, 1):
\]

\[
\log m - I\left( \frac{\theta x^* + \hat{L} \hat{c}}{1 - \hat{c}} \right) - \frac{a - \hat{c} \log b}{1 - \hat{c}} = 0.
\]

Remark 1.6 If \( I(x^*) = \log m \), then we have \( a^* < \log m - I(\theta x^*) \). In fact, since \( I(0) = 0 \) and \( I(x) \) is convex, we have

\[
I(\theta x^*) = I((1-\theta)0 + \theta x^*) \leq (1-\theta)I(0) + \theta I(x^*) = \theta \log m.
\]

Thus,

\[
\frac{(1-\theta)x^*}{L + x^*} \log b + I(\theta x^*) < \frac{(1-\theta)x^*}{x^*} \log m + \theta \log m = \log m,
\]

which implies \( a^* < \log m - I(\theta x^*) \).

Remark 1.7 As we can see, there is a phase transition at \( a = a^* \). This is because \( e^{a^n} \) is the minimum population size that \( b^n \) branching random walks located at \( -Lt_n \) can contribute to \( [\theta x^n, \infty) \) in the remaining \( n - t_n \) time. Thus, when \( a < a^* \), to achieve \( \{Z_n([\theta x^n, \infty)) < e^{a^n}\} \) means there is no particle in \( [\theta x^n, \infty) \) at time \( n \). Hence, in this case, the lower deviation behavior is similar to the maximum of BRW (see [12, Theorem 1.1]).

The following theorem considers the Weibull step size. Recall that \( \hat{c} \) is the unique solution of \( \log m - I(\theta x^* + \hat{c}) = a \) on \( (0, x^* - \theta x^*) \).

Theorem 1.4 (Böttcher case, Weibull tail) Assume \( p_1 = 0 \) and \( \log P(X \leq -x) \sim -\lambda x^\alpha \) as \( x \to +\infty \) for some \( \alpha > 0 \) and \( \lambda > 0 \). Then,

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \log P \left( Z_n([\theta x^n, \infty)) < e^{a^n} \right) = -\lambda C(x^*, \theta, b, \alpha, m, a),
\]

where

\[
C(x^*, \theta, b, \alpha, m, a) = \begin{cases} 
\left( b(x^* - \theta x^*)^\alpha \right), & \alpha \in (0, 1], a \in [0, \log m - I(x^*)]; \\
b\hat{c}^\alpha, & \alpha \in (0, 1], a \in (\log m - I(x^*), \log m - I(\theta x^*)]; \\
\left( b \frac{1}{\alpha - 1} - 1 \right)^{\alpha - 1} (x^* - \theta x^*)^\alpha, & \alpha \in (1, \infty), a \in [0, \log m - I(x^*)]; \\
\left( b \frac{1}{\alpha - 1} - 1 \right)^{\alpha - 1} \hat{c}^\alpha, & \alpha \in (1, \infty), a \in (\log m - I(x^*), \log m - I(\theta x^*)].
\end{cases}
\]
Remark 1.8 In fact, using the same method of proof for Theorem 1.4, one could get the results below for Gumbel step size and Pareto step size.

(i) (Böttcher case, Gumbel tail) If \( p_1 = 0 \) and \( \mathbb{P}(X \leq -x) = \Theta(1) \exp(-e^{x^\alpha}) \) as \( x \to +\infty \) for some \( \alpha > 0 \), then

\[
\lim_{n \to \infty} \frac{1}{n \log n} \log \left[ -\log \mathbb{P} \left( Z_n([\theta x^* n, \infty)) < e^{an} \right) \right] = \begin{cases} 
\frac{\alpha}{1+\alpha} \log b - \frac{\alpha}{1+\alpha} \left( x^* - \theta x^* \right), & a \in [0, \log m - I(x^*)]; \\
\frac{\alpha}{1+\alpha} \log b - c \frac{\alpha}{1+\alpha}, & a \in (\log m - I(x^*), \log m - I(\theta x^*)].
\end{cases}
\]

(ii) (Böttcher case, Pareto tail) If \( p_1 = 0 \) and \( \mathbb{P}(X < -x) = \Theta(1)x^{-\alpha} \) as \( x \to +\infty \) for some \( \alpha > 0 \), then

\[
\lim_{n \to \infty} \frac{1}{n \log n} \log \mathbb{P} \left( Z_n([\theta x^* n, \infty)) < e^{an} \right) = -\alpha b.
\]

The rest of this paper is organized as follows. In Sects. 2 and 3, we study the Schröder case, where Theorems 1.1 and 1.2 are proved. Section 2 considers that the step size has a negative exponential moment. Section 3 is devoted to study the case when the step size has heavy tails. In Sects. 4 and 5, we consider the Böttcher case, where Theorems 1.3 and 1.4 are proved. In Sect. 4, the step size is assumed to be bounded below. Section 5 treats the Weibull step size, and this section is divided into two subsections: In the first section, we study the sub-Weibull case; in the second section, we study the super-Weibull case.

2 Proof of Theorem 1.1: Schröder Case, Light Tail

In this section, we are going to prove Theorem 1.1. We first present several lemmas. The following lemma can be found in [5, Theorem 2].

Lemma 2.1 Let \( \rho \) be a nonnegative constant.

If \( \rho \in [0, 1) \), then

\[
\lim_{n \to \infty} \frac{1}{n} \log Z_n((\rho x^* n, \infty)) = \log m - \log \mathbb{P} - a.s. \text{ on non-extinction}.
\]

If \( \rho \in (1, \infty) \), then

\[
\lim_{n \to \infty} Z_n((\rho x^* n, \infty)) = 0, \mathbb{P} - a.s.
\]

The following remark is a direct consequence of the above lemma.

Corollary 2.2 Let \( \{\rho_n\}_{n \geq 1} \) be a sequence of numbers such that \( \lim_{n \to \infty} \rho_n = \rho \).
If $\rho \in [0, 1)$, then
\[
\lim_{n \to \infty} \frac{1}{n} \log Z_n((\rho_n x^* n, \infty)) = \log m - I(\rho x^*), \quad \mathbb{P} - a.s.
\] (2.1)

If $\rho \in (1, \infty)$, then
\[
\lim_{n \to \infty} Z_n((\rho_n x^* n, \infty)) = 0, \quad \mathbb{P} - a.s.
\] (2.2)

**Remark 2.1** Biggins [6, 7] also studied the almost sure behavior of $Z_n(\theta x^* n + B)$, where $B$ is a bounded measurable set.

The next lemma is the well-known Cramér theorem; see [13, Theorem 2.2.3]. Here and later, for $n \geq 1$, we define $S_n := X_1 + X_2 + \cdots + X_n$, where $X_i, i \geq 1$ are i.i.d. copies of the step size $X$.

**Lemma 2.3** Let $x$ be a positive constant. Then,
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geq nx) = -I(x).
\]

The following lemma gives several properties of $I(x)$. Let
\[
\lambda^* := \sup \left\{ \lambda \geq 0 : \mathbb{E}[e^{\lambda X}] < \infty \right\} \quad \text{and} \quad \Lambda(\lambda) := \log \mathbb{E}[e^{\lambda X}] .
\]

**Lemma 2.4** $I(\cdot)$ is a continuous and strictly increasing function on
\[
\{y \in [0, \infty) : I(y) < \infty\},
\]
and it can be classified into the following three cases.

(i) If $\Lambda'(\lambda) \uparrow +\infty$ as $\lambda \to \lambda^* \in (0, \infty]$, then for any $x > 0$, there exists some $\lambda \in (0, \lambda^*)$ such that
\[
x = \Lambda'(\lambda), \quad I(x) = \lambda \Lambda'(\lambda) - \Lambda(\lambda).
\]
Furthermore, $\lim_{x \to +\infty} \frac{I(x)}{x} = \lambda^*$.

(ii) If $\lambda^* = +\infty$ and $\Lambda'(\lambda)$ converges to some finite limit as $\lambda \to \infty$, then $\Lambda'(\lambda) \uparrow \text{ess sup } X$, and
\[
I(x) = \begin{cases} 
\text{positive finite,} & x \in (0, \text{ess sup } X); \\
- \log \mathbb{P}(X = \text{ess sup } X), & x = \text{ess sup } X; \\
+\infty, & x \in (\text{ess sup } X, \infty).
\end{cases}
\]

Furthermore, as $x \uparrow \text{ess sup } X$, $I(x) \uparrow -\log \mathbb{P}(X = \text{ess sup } X)$.
If $0 < \lambda^* < +\infty$ and $\Lambda'(\lambda)$ converges to some finite limit $T$ as $\lambda \to \lambda^*$, then
\[ \mathbb{E}\left[e^{\lambda^* X}\right] < \infty \]
and
\[ I(x) = \begin{cases} \text{positive finite,} & x \in (0, T]; \\ \lambda^* x - \log \mathbb{E}\left[e^{\lambda^* X}\right], & x \in [T, \infty). \end{cases} \]

**Proof** In fact, all the statements above can be found in [17, Section 2.6] except $\lim_{x \to +\infty} I(x) x = \lambda^*$ in (i). So, we just prove this. Note that by the former part of (i), for any $x > 0$, there exists $\lambda \in (0, \lambda^*)$ such that $x = \Lambda'(\lambda)$. It is easy to see that $x \to +\infty$ implies $\lambda \to \lambda^*$. Furthermore, $\Lambda'(\cdot)$ is infinitely differentiable on $(0, \lambda^*)$ (see [13, Exercise 2.2.24]). Thus, by L'Hospital’s rule,
\[
\lim_{x \to +\infty} I(x) x = \lim_{\lambda \to \lambda^*} \frac{\lambda \Lambda''(\lambda) + \Lambda'(\lambda) - \Lambda'(\lambda)}{\Lambda''(\lambda)} = \lambda^*.
\]

The next lemma considers the lower deviation probabilities of a branching process, which can be found in [19, Theorem 2.1].

**Lemma 2.5** Assume $p_1 > 0$. Let $\{a_n\}_{n \geq 1}$ be a sequence of numbers such that $\lim_{n \to \infty} a_n = +\infty$ and for any $t > 0$, $\lim_{n \to \infty} a_n e^{-tn} = 0$. Then,
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(|Z_n| < a_n) = \log p_1.
\]

Recall that in Assumption 1.1 (iii), we assume $\theta \in (0, 1)$ and $a \in [0, \log m - I(\theta x^*))$ are fixed constants. Recall that
\[
f(\rho) = \log m - I\left(\frac{\theta x^*}{1 - \rho}\right) - \frac{a}{1 - \rho}, \quad \rho \in [0, 1);
\]
\[
\bar{\rho} = \sup\{\rho \in (0, 1) : f(\rho) \geq 0\};
\]
\[
g_\rho(h) = \log m - I\left(\frac{h + \theta x^*}{1 - \rho}\right) - \frac{a}{1 - \rho}, \quad h \in [0, \infty);
\]
\[
d = d(\rho) = \sup\{h \in [0, \infty) : g_\rho(h) \geq 0\}.
\]

The following lemma gives several properties of $\bar{\rho}$ and $d$. Remark 1.2 can be deduced from its proof.

**Lemma 2.6** The following four properties hold.

(i) $\bar{\rho} \in (0, 1 - \theta)$, $d(\rho) \in (0, (1 - \theta - \rho)x^*)$ for $\rho \in (0, \bar{\rho})$.

(ii) $g_\rho(d + \varepsilon) < 0$ for any $\varepsilon > 0$, and $g_\rho(d - \varepsilon) > 0$ for $\varepsilon \in (0, d)$.

(iii) $d(0) > 0$, $d(\bar{\rho}) = 0$ and $d(\cdot)$ is decreasing and continuous on $[0, \bar{\rho}]$. 

Springer
(iv) Assume \( p_1 > 0 \) and \( \mathbb{E}[e^{\kappa X}] < \infty \) for some \( \kappa < 0 \). Then,

\[
\inf_{\rho \in (0,\bar{\rho})} \left\{ \rho \log \frac{1}{p_1} + \rho I \left( \frac{-d}{\rho} \right) \right\} \in (0, \infty).
\]

**Proof**  We first prove (i). Recall that \( x^* = \sup \{x \geq 0 : I(x) \leq \log m \} \). By Lemma 2.4, \( I(\cdot) \) is continuous on \([0, x^*]\). Write \( f(x+) := \lim_{y \to x^+} f(y) \) and \( f(x-) := \lim_{y \to x^-} f(y) \). By Assumption 1.1 (iii), we have

\[ f(0) = \log m - I(\theta x^*) - a > 0. \]

By the definition of \( x^* \), we have \( \log m - I(x^*) \leq 0 \). Thus,

\[ f((1 - \theta)+) = \log m - I(x^*) - \frac{a}{\theta} \leq -\frac{a}{\theta} \leq 0. \]

From Lemma 2.4, \( I(\cdot) \) must satisfy one of the following:

(a) there exists a constant \( B > x^* \) such that \( I(\cdot) \) is strictly increasing and continuous on \([x^*, B]\);

(b) \( I(x) = +\infty \) for \( x > x^* \) and \( x^* = \text{ess sup} \ x < \infty \).

In case (a), \( f(\rho) \) is continuous and strictly decreasing on \([0, 1 - \theta x^*/B]\), which implies \( f(\rho) < 0 \) for \( \rho \in (1 - \theta, 1 - \theta x^*/B]\). In case (b), we have \( f(\rho) = -\infty \) for \( \rho \in (1 - \theta, 1) \). Putting these together, we always have

\[ \bar{\rho} \in (0, 1 - \theta). \]

Fix \( \rho \in (0, \bar{\rho}) \). By the definition of \( \bar{\rho} \), we have \( f(\bar{\rho} -) \geq 0 \). This, combined with the fact that \( f(\cdot) \) is strictly decreasing on \([0, 1 - \theta]\), implies that

\[ g_\rho(0) = f(\rho) > f(\bar{\rho} -) \geq 0. \quad (2.3) \]

Observe that

\[ g_\rho((1 - \theta - \rho)x^*) = \log m - I(x^*) - \frac{a}{1 - \rho} \leq -\frac{a}{1 - \rho} \leq 0. \]

Similar reason as \( \bar{\rho} \in (0, 1 - \theta) \), we have \( d \in (0, (1 - \theta - \rho)x^*] \).

For (ii), since \( d \in (0, \infty) \) and \( g_\rho(\cdot) \) is decreasing on \([0, \infty) \), the result is a consequence of the definition of \( d \).

For (iii), \( d(0) > 0 \) follows by the fact that \( f(0) = \log m - I(\theta x^*) + a > 0 \) and \( I(\cdot) \) is continuous at \( \theta x^* \). Obviously, \( d(\bar{\rho}) = 0 \) if \( f(\bar{\rho}) = 0 \). If \( f(\bar{\rho}) > 0 \), then \( I(\cdot) \) must satisfy case (b). Therefore, if \( f(\bar{\rho}) > 0 \), then

\[ \frac{\theta x^*}{1 - \rho} = \text{ess sup} \ x = x^* < \infty, \]
and \( g_{\bar{\rho}}(\varepsilon) = -\infty \) for any \( \varepsilon > 0 \), which also implies \( d(\bar{\rho}) = 0 \). The fact that \( d(\rho) \) is decreasing follows from

\[
g_\rho(\cdot) > g_\rho(\cdot) \text{ for } \rho' < \rho.
\]

In the next, we shall prove that \( d(\cdot) \) is continuous on \([0, \bar{\rho}]\). Note that the function

\[
T_\rho(h) := \log m - I \left( \frac{h + \theta x^*}{1 - \rho} \right), \quad h \in [0, (1 - \theta - \rho)x^*]
\]

has its range

\[
\left[ \log m - I(x^*), \log m - I \left( \frac{\theta x^*}{1 - \rho} \right) \right].
\]

Moreover, \( T_\rho(h) < 0 \) if \( h > (1 - \theta - \rho)x^* \). By (2.3), for \( \rho \in [0, \bar{\rho}] \),

\[
\frac{a}{1 - \rho} \leq \log m - I \left( \frac{\theta x^*}{1 - \rho} \right).
\]

Thus, if \( \frac{a}{1 - \rho} \geq \log m - I(x^*) \), then the equation (w.r.t. \( h \))

\[
\log m - I \left( \frac{h + \theta x^*}{1 - \rho} \right) = \frac{a}{1 - \rho}
\]

has a unique solution on \([0, (1 - \theta - \rho)x^*]\), say \( h := d(\rho) \). Furthermore, since \( I(\cdot) \) is continuous and strictly increasing on \([0, x^*]\), \( d(\cdot) \) is continuous on

\[
\left( 1 - \frac{a}{\log m - I(x^*)} \right)^+, \bar{\rho} \right],
\]

where we take \( 0 = (1 - \frac{a}{0})^+ \) by convention. Thus, to complete the proof of (iii), we can further assume that

\[
\log m - I(x^*) > 0 \text{ and } 0 \leq a < \log m - I(x^*).
\]

Observe that if \( \frac{a}{1 - \rho} \leq \log m - I(x^*) \), by the definition of \( d \), then

\[
\frac{d(\rho) + \theta x^*}{1 - \rho} = x^*.
\]

Thus, \( d(\rho) = (1 - \theta - \rho)x^* \) is a continuous function on \([0, 1 - \frac{a}{\log m - I(x^*)}]\). Hence putting above together, we conclude that \( d(\rho) \) is a continuous function on \([0, \bar{\rho}]\).
We proceed to prove (iv). Although Lemma 2.4 considers the case of $x \to +\infty$, one can obtain analogous results for $x \to -\infty$ (since for a random walk $\{S_n\}_{n \geq 0}$, $\{-S_n\}_{n \geq 0}$ is also a random walk). Let

$$\kappa^* := \sup \{-\kappa : \kappa < 0, \, \mathbb{E}[e^{\kappa X}] < \infty\}.$$ 

By the assumption $\mathbb{E}[e^{\kappa X}] < \infty$ for some $\kappa < 0$, we have $\kappa^* \in (0, +\infty]$. So,

$$\lim_{x \to -\infty} \frac{I(x)}{-x} = \kappa^* > 0.$$ 

Note that since $d(0) > 0$ and $d(\cdot)$ is continuous at 0, $\lim_{\rho \to 0} d/\rho = \lim_{\rho \to 0} d(\rho)/\rho = +\infty$. Therefore,

$$\lim_{\rho \to 0} \rho I \left(-\frac{d}{\rho}\right) = \lim_{\rho \to 0} \frac{I(-d/\rho)}{d/\rho} d > 0.$$ 

Thus, there exist constants $\delta > 0$ and $\rho^* \in (0, \tilde{\rho})$ such that for every $0 < \rho \leq \rho^*$,

$$\rho \log \frac{1}{p_1} + \rho I \left(-\frac{d}{\rho}\right) > \delta.$$ 

Furthermore, for $\tilde{\rho} > \rho > \rho^*$,

$$\rho \log \frac{1}{p_1} + \rho I \left(-\frac{d}{\rho}\right) > \rho^* \log \frac{1}{p_1}.$$ 

Thus,

$$\inf_{\rho \in (0, \tilde{\rho})} \left\{ \rho \log \frac{1}{p_1} + \rho I \left(-\frac{d}{\rho}\right) \right\} > \delta \wedge \left( \rho^* \log \frac{1}{p_1} \right) > 0. \quad (2.4)$$

On the other hand, since $\lim_{\rho \to \tilde{\rho}} \frac{d(\rho)}{\rho} = 0$, we have

$$\lim_{\rho \to \tilde{\rho}} \rho \log \frac{1}{p_1} + \rho I \left(-\frac{d}{\rho}\right) = \tilde{\rho} \log \frac{1}{p_1} < \infty. \quad (2.5)$$

As a result, (2.4), together with (2.5), implies (iii). \qed

Now, we are ready to prove Theorem 1.1: If $p_1 > 0$ and $\mathbb{E}[e^{\kappa X}] < \infty$ for some $\kappa < 0$, then

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_n((\theta x^* n, \infty)) < e^{\varphi n}) = -\inf_{\rho \in (0, \tilde{\rho})} \left\{ \rho \log \frac{1}{p_1} + \rho I \left(-\frac{d}{\rho}\right) \right\}.$$ 

For the lower bound, the strategy is to force single births up to time $\rho n$ (where $\rho \in (0, \tilde{\rho})$), with the single particle alive at time $\rho n$ below the position $-dn$ (where $d$ is...
chosen such that $Z_{n-\rho_n}([dn+\theta x^* n, \infty)) \approx e^{an}$. Optimizing for $\rho$ yields the desired lower bound. The proof of the upper bound goes by showing that the above strategy is optimal; thus, we consider a series of intermediate times $[ni/k], 0 < i/k < 1$. To argue that sub-BRWs emanating from time $[ni/k]$ can easily produce $e^{an}$ descendants in $[\theta x^* n, \infty]$ at time $n$, we should insure that: (i) there exist adequate particles at time $[ni/k]$; (ii) each individual at time $[ni/k]$ locates in a not very low position. This motivates the definitions of the following $\rho_n$ and $E_i$.

**Proof Lower bound** Fix $\varepsilon \in (0, 1)$ and $\rho \in (0, \bar{\rho}]$. By Markov property, for $n$ large enough,

$$
\mathbb{P}\left( Z_n([\theta x^* n, \infty)) < e^{an} \right)
\geq \mathbb{P}\left( \left| Z_{\lfloor \rho n \rfloor} \right| = 1, Z_{\lfloor \rho n \rfloor}([-(d + \varepsilon)n, \infty)) = 0, Z_n([\theta x^* n, \infty)) < e^{an} \right)
\geq p_1^{\lfloor \rho n \rfloor} \mathbb{P}\left( S_{\lfloor \rho n \rfloor} \leq -(d + \varepsilon)n \right) \mathbb{P}\left( Z_{n-\lfloor \rho n \rfloor} \left( ((d + \varepsilon \theta x^*)n, \infty) \right) < e^{an} \right).
\tag{2.6}
$$

Since $g_\rho(d + \varepsilon) < 0$, by Corollary 2.2, for $n$ large enough,

$$
\mathbb{P}\left( Z_{n-\lfloor \rho n \rfloor} \left( ((d + \varepsilon \theta x^*)n, \infty) \right) < e^{an} \right)
= \mathbb{P}\left( \frac{1}{n - \lfloor \rho n \rfloor} \log Z_{n-\lfloor \rho n \rfloor} \left( \left\lfloor \frac{(d + \varepsilon \theta x^*)n}{n - \lfloor \rho n \rfloor} (n - \lfloor \rho n \rfloor), \infty) \right) \right)
- \frac{an}{n - \lfloor \rho n \rfloor} < 0 \geq 0.9.
$$

Thus, plugging the above into (2.6) yields that

$$
\mathbb{P}\left( Z_n([\theta x^* n, \infty)) < e^{an} \right) \geq 0.9 p_1^{\lfloor \rho n \rfloor} \mathbb{P}\left( S_{\lfloor \rho n \rfloor} \leq \frac{(d + \varepsilon)n}{\lfloor \rho n \rfloor} \right)
\geq 0.9 p_1^{\lfloor \rho n \rfloor} \exp \left\{ -I \left( \frac{d + 2\varepsilon}{\rho} \right) \lfloor \rho n \rfloor \right\},
$$

where the last inequality comes from the Cramér theorem. As a consequence, for every $\rho \in (0, \bar{\rho}]$ and $\varepsilon \in (0, 1),$

$$
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( Z_n([\theta x^* n, \infty)) < e^{an} \right) \geq - \left\{ \rho \log \frac{1}{p_1} + \rho I \left( \frac{d + 2\varepsilon}{\rho} \right) \right\}.
$$

The desired lower bound follows by letting $\varepsilon \to 0$ and optimizing $\rho$ on $(0, \bar{\rho}]$.

**Upper bound** Fix $\delta \in (0, \bar{\rho}]$ and $k \geq \frac{1}{\rho^3}$ (hence $\lfloor (\bar{\rho} - \delta)k \rfloor \geq 1$). Since $d = d(\rho)$ is decreasing on $(0, \bar{\rho}]$ and $d(\bar{\rho}) = 0$ (see Lemma 2.6), there exists some constant $C_\delta > 0$ such that for every $1 \leq i \leq \lfloor (\bar{\rho} - \delta)k \rfloor$,

$$
d(i/k) > C_\delta.
$$
Fix $\varepsilon \in (0, C_\delta)$. Set

$$\rho_n := \sup\{\rho \in [0, 1] : Z_{\lfloor n \rho \rfloor} \leq n^3\};$$

$$E_i := \left\{ \forall u \in Z_{\lfloor ni/k \rfloor}, \ S_u \in \left[ -(d(i/k) - \varepsilon) n, +\infty \right) \right\}.$$

Let $Z_n^u$ be the $n$th generation of the sub-BRW emanating from particle $u$. By the branching property, we have

$$\mathbb{P}\left( Z_n([\theta x^* n, \infty)) < e^{an} \right)$$

$$= \mathbb{P}\left( Z_n([\theta x^* n, \infty)) < e^{an}, \rho_n \geq \frac{[(\bar{\rho} - \delta)k]}{k} \right)$$

$$+ \sum_{i=1}^{[(\bar{\rho} - \delta)k]} \mathbb{P}\left( Z_n([\theta x^* n, \infty)) < e^{an}, \frac{i-1}{k} \leq \rho_n < \frac{i}{k} \right)$$

$$\leq \sum_{i=1}^{[(\bar{\rho} - \delta)k]} \mathbb{P}\left( \sum_{u \in Z_{\lfloor ni/k \rfloor}} Z_n^u([\theta x^* n, \infty)) < e^{an}, E_i, \frac{i-1}{k} \leq \rho_n < \frac{i}{k} \right)$$

$$+ \mathbb{P}\left( E_i^c, \frac{i-1}{k} \leq \rho_n < \frac{i}{k} \right) + \mathbb{P}\left( \rho_n \geq \frac{[(\bar{\rho} - \delta)k]}{k} \right). \quad (2.7)$$

Let $Z_n^j$, $j \geq 1$ be i.i.d. copies of $Z_n$. For the first term on the r.h.s. of (2.7), by the branching property, there exists $N(k, \varepsilon)$ such that for all $1 \leq i \leq [(\bar{\rho} - \delta)k]$ and $n \geq N(k, \varepsilon)$,

$$\mathbb{P}\left( \sum_{u \in Z_{\lfloor ni/k \rfloor}} Z_n^u([\theta x^* n, \infty)) < e^{an}, E_i, \frac{i-1}{k} \leq \rho_n < \frac{i}{k} \right)$$

$$\leq \mathbb{P}\left( Z_n^{j}[\theta x^* n, \infty)) < e^{an}, 1 \leq j \leq n^3 \right)$$

$$\leq \mathbb{P}\left( \frac{1}{n - \lfloor ni/k \rfloor} \log \left( Z_n^{[ni/k]}([\theta x^* + d(i/k) - \varepsilon) n, \infty)) \right) < \frac{an}{n - \lfloor ni/k \rfloor} \right)$$

$$\leq e^{-n^3}, \quad (2.8)$$

where the last inequality follows by (2.1) and the fact that (see Lemma 2.6 (ii))

$$\log m - I \left( \frac{\theta x^* + d(i/k) - \varepsilon}{1 - i/k} \right) - a > 0.$$
Let $h_{n,i} := \lfloor ni/k \rfloor - \lfloor n(i - 1)/k \rfloor$. By the branching property,

$$
E \left[ |Z|_{ni/k} |1_{|Z|_{ni-1/k}| \leq n^3} \right] = E \left[ \sum_{u \in Z_{ni-1/k}} |Z|^u_{h_{n,i},i} |1_{|Z|_{ni-1/k}| \leq n^3} \right] \\
\leq E \left[ \sum_{k=1}^{n^3} |Z|^k_{h_{n,i},i} |1_{|Z|_{ni-1/k}| \leq n^3} \right] \\
= n^3 m^{h_{n,i}} \mathbb{P} \left( |Z|_{ni-1/k} \leq n^3 \right).
$$

(2.9)

Let $F := \sigma \left( |Z|_l, 1 \leq l \leq h_{n,i} \right)$. For the second term on the r.h.s. of (2.7), by the Markov inequality and (2.9), there exists $N'(k, \varepsilon)$ such that for all $1 \leq i \leq \lfloor (\rho - \delta)k \rfloor$ and $n \geq N'(k, \varepsilon),$

$$
\mathbb{P} \left( E^c_i, \frac{i - 1}{k} \leq \rho_n < \frac{i}{k} \right) \leq \mathbb{P} \left( \exists u \in Z_{ni/k}, \ S_u \leq -(d(i/k) - \varepsilon)n, \ Z_{ni-1/k} \leq n^3 \right) \\
\leq \mathbb{P} \left( \sum_{u \in Z_{ni-1/k}} Z|^u_{h_{n,i},i} \left( (-\infty, -(d(i/k) - \varepsilon)n) \right) > 1; \ Z_{ni-1/k} \leq n^3 \right) \\
\leq \mathbb{E} \left[ \sum_{u \in Z_{ni-1/k}} Z|^u_{h_{n,i},i} \left( (-\infty, -(d(i/k) - \varepsilon)n) \right) 1_{Z_{ni-1/k} \leq n^3} \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[ \sum_{u \in Z_{ni-1/k}} Z|^u_{h_{n,i},i} \left( (-\infty, -(d(i/k) - \varepsilon)n) \right) 1_{Z_{ni-1/k} \leq n^3} | F \right] \right] \\
= \mathbb{E} \left[ |Z|_{ni/k} |1_{Z_{ni-1/k} \leq n^3} \right] \mathbb{P} \left( S_{ni/k} \leq -(d(i/k) - \varepsilon)n \right) \\
\leq n^3 m^{h_{n,i}} \mathbb{P} \left( |Z|_{ni-1/k} \leq n^3 \right) \mathbb{P} \left( S_{ni/k} \leq -(d(i/k) - \varepsilon)n \right) \\
\leq n^3 m^{h_{n,i} + 1} \exp \left\{ -I \left( -\frac{d(i/k) - 2\varepsilon}{i/k} \right) ni/k \right\} \\
\leq n^3 m^{h_{n,i} + 1} \exp \left\{ (\log p_1 + \varepsilon)ni/k - I \left( -\frac{d(i/k) - 2\varepsilon}{i/k} \right) ni/k \right\},
$$

(2.10)

where the second equality follows from the fact that the branching and motion are independent, the second last inequality follows from the Cramér theorem and the last inequality follows from Lemma 2.5.

For the third term on the r.h.s. of (2.7), again by Lemma 2.5, there exists $N(\rho, \delta, \varepsilon) > 0$ such that for $n \geq N(\rho, \delta, \varepsilon)$ and $k \geq 1,
\begin{align*}
\mathbb{P}\left(\rho_n \geq \frac{[\bar{\rho} - \delta)k]}{k}\right) & \leq \mathbb{P}\left(Z_{n[\bar{\rho} - \delta)k]} \leq n^3\right) \\
& \leq \exp\{(\log p_1 + \varepsilon)(\bar{\rho} - \delta)\}. \tag{2.11}
\end{align*}

Plugging (2.8), (2.10) and (2.11) into (2.7) yields that for fixed \(\varepsilon, k, \rho, \delta\) there exists \(N(\varepsilon, k, \rho, \delta)\) such that for \(n > N(\varepsilon, k, \rho, \delta)\),

\begin{align*}
\mathbb{P}\left(Z_n([\theta x^* n, \infty)) < e^{an}\right) & \leq 2n^3m^{(n/k)+1} \sum_{i=1}^{[\bar{\rho} - \delta)k]} \exp\{(\log p_1 + \varepsilon)ni/k - I\left(-\frac{d(i/k) - 2\varepsilon}{i/k}\right)ni/k\} \\
& + \exp\{(\log p_1 + \varepsilon)(\bar{\rho} - \delta)n\}.
\end{align*}

As a consequence,

\begin{align*}
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(Z_n([\theta x^* n, \infty)) < e^{an}\right) & \leq \left[-\min_{1 \leq i \leq [\bar{\rho} - \delta)k]} \left\{(-\varepsilon - \log p_1)\frac{i}{k} - \frac{\log m}{k} + I\left(-\frac{d(i/k) - 2\varepsilon}{i/k}\right)i/k\right\}\right] \\
& \wedge \left[(-\varepsilon - \log p_1)(\bar{\rho} - \delta)\right].
\end{align*}

Letting \(\varepsilon \to 0\) and \(\delta \to 0\), it follows that

\begin{align*}
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(Z_n([\theta x^* n, \infty)) < e^{an}\right) & \leq \left[-\min_{1 \leq i \leq [\bar{\rho} - \delta)k]} \left\{(-\log p_1)\frac{i}{k} - \frac{\log m}{k} + I\left(-\frac{d(i/k)}{i/k}\right)i/k\right\}\right] \\
& \wedge \left[(-\log p_1)\bar{\rho}\right] \\
& = \left[-\frac{\log m}{k} + \min_{1 \leq i \leq [\bar{\rho} - \delta)k]} \left\{(-\log p_1)\frac{i}{k} + I\left(-\frac{d(i/k)}{i/k}\right)i/k\right\}\right] \\
& \wedge \left[(-\log p_1)\bar{\rho}\right] \\
& \leq \left[-\frac{\log m}{k} + \inf_{\rho \in [0, \bar{\rho}]} \left\{(-\log p_1)\rho + I\left(-\frac{d}{\rho}\right)\rho\right\}\right] \\
& \wedge \left[(-\log p_1)\bar{\rho}\right].
\end{align*}

Letting \(k \to \infty\), then

\begin{align*}
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(Z_n([\theta x^* n, \infty)) < e^{an}\right) & \leq \left[\inf_{\rho \in [0, \bar{\rho}]} \left\{(-\log p_1)\rho + I\left(-\frac{d}{\rho}\right)\rho\right\}\right] \\
& \wedge \left[(-\log p_1)\bar{\rho}\right] \\
& = \inf_{\rho \in [0, \bar{\rho}]} \left\{\rho \log \frac{1}{p_1} + \rho I\left(-\frac{d}{\rho}\right)\right\},
\end{align*}

where the last equality comes from the fact that \(I\left(-\frac{d}{\rho}\right) = I(0) = 0\); see Lemma 2.6 (iii). \qed
3 Proof of Theorem 1.2: Schröder Case, Heavy Tail

In this section, we are going to prove Theorem 1.2. We first present several lemmas. The first lemma can be found in [2, p40, Corollary 1], which considers the asymptotic behavior of the branching process’s generating function.

**Lemma 3.1** If $0 < s < 1$, then

$$\lim_{n \to \infty} \frac{\mathbb{E}[s |Z_n|]}{p_1^n} = C_0,$$

where $C_0$ is a positive constant depending only on $s$.

The next lemma concerns large deviation probabilities of sums of i.i.d. Pareto tail random variables.

**Lemma 3.2** Assume $\mathbb{P}(X < -x) = \Theta(1)x^{-\alpha}$ as $x \to +\infty$ for some $\alpha > 0$. Then, there exists some constant $C_\alpha > 0$ such that for all $x > 0$ and $n \geq 1$,

$$\mathbb{P}(S_n \leq -x) \leq C_\alpha n^2x^{-\alpha}.$$

**Proof** According to [26, Corollary 1.5], if $\mathbb{E}[|X|^t1_{\{X \leq 0\}}] < \infty$ for some $t \in (0, 1]$, then for any $x, y > 0$ and $n \geq 1$,

$$\mathbb{P}(S_n \leq -x) \leq n\mathbb{P}(X < -y) + \left(\frac{en\mathbb{E}[|X|^t1_{\{X \leq 0\}}]}{xy^{t-1}}\right)^{\frac{\alpha}{t}}. \quad (3.1)$$

Since $\mathbb{P}(X < x) = \Theta(1)x^{-\alpha}$, there exists some constant $C_1 > 0$ such that for all $x > 0$,

$$\mathbb{P}(X < -x) \leq C_1x^{-\alpha}.$$

Therefore, $\mathbb{E}[|X|^t1_{\{X \leq 0\}}] < \infty$ for $t = \frac{\alpha}{2} \land 1$. In particular, let $y = \frac{t}{\alpha} x$, (3.1) yields

$$\mathbb{P}(S_n \leq x) \leq n\mathbb{P}(X < -\frac{t}{\alpha} x) + \left(\frac{en\mathbb{E}[|X|^t1_{\{X \leq 0\}}]}{(\frac{t}{\alpha})^{t-1}x^t}\right)^{\frac{\alpha}{t}}$$

$$\leq C_1 \left(\frac{t}{\alpha}\right)^{-\alpha} x^{-\alpha} n + \left(\frac{e\mathbb{E}[|X|^t1_{\{X \leq 0\}}]}{(\frac{t}{\alpha})^{t-1}}\right)^{\frac{\alpha}{t}} n^{\frac{\alpha}{t}} x^{-\alpha}$$

$$\leq C_\alpha n^2x^{-\alpha}. \quad \square$$

The following lemma gives an upper bound of large deviation probabilities of sums of i.i.d. Weibull tail random variables, which is a direct consequence of [3, Theorem 2.1].
Lemma 3.3 Assume \( \log \mathbb{P}(X < -y) \sim -\lambda y^\alpha \) as \( y \to +\infty \) for some \( \alpha \in (0, 1) \). Let \( \{t_n\}_{n \geq 1} \) be a sequence of positive integer-valued numbers such that \( t_n \to +\infty \) and \( \lim_{n \to \infty} \frac{t_n}{n} < \infty \). For any given \( \varepsilon \in (0, 1) \) and \( x > 0 \), we have for \( n \) large enough,

\[
\mathbb{P}(S_n \leq -xn) \leq e^{-(1-\varepsilon)\lambda x^\alpha n^\alpha}.
\]

Let \( G(c) := \log m - I(\theta x^* + c) - a, \hat{c} := \inf \{c \geq 0 : G(c) < 0\} \).

Lemma 3.4 (i) If \( a \in [0, \log m - I(x^*)] \), then \( \hat{c} = (1 - \theta)x^* \). If \( a \in (\log m - I(x^*), \log m - I(\theta x^*)) \), then \( \hat{c} \) is the unique solution of \( G(c) = 0 \) on \((0, (1 - \theta)x^*)\).

(ii) If \( \varepsilon > 0 \), then \( G(\hat{c} + \varepsilon) < 0 \). If \( \varepsilon \in (0, \hat{c}) \), then \( G(\hat{c} - \varepsilon) > 0 \).

Proof If \( a \in [0, \log m - I(x^*)] \), without loss of generality, we assume \( \log m - I(x^*) > 0 \). By the definition of \( x^* \), we have \( x^* = \text{ess sup} X \in (0, \infty) \). Thus, by Lemma 2.4 (ii), \( G(c) = -\infty \) for \( c > (1 - \theta)x^* \). This, together with the fact that \( G(\cdot) \) is decreasing on \([0, (1 - \theta)x^*)\), implies \( \hat{c} = (1 - \theta)x^* \).

If \( a \in [\log m - I(x^*), \log m - I(\theta x^*)) \), then

\[
G(0) = \log m - I(\theta x^*) - a < 0, \quad G((1 - \theta)x^*) = \log m - I(x^*) - a > 0.
\]

Since \( G(\cdot) \) is continuous and strictly decreasing on \((0, (1 - \theta)x^*)\), \( \hat{c} \) is the unique solution of \( G(c) = 0 \). Thus (i) holds. By the above arguments, we can obtain (ii) easily.

Now, we are ready to prove Theorem 1.2:

(i) If \( p_1 > 0 \) and \( \mathbb{P}(X < -x) = \Theta(1)x^{-\alpha} \) as \( x \to \infty \) for some \( \alpha > 0 \), then

\[
\lim_{n \to \infty} \frac{1}{\log n} \log \mathbb{P}
\left(Z_n(\{\theta x^* n, \infty\}) < e^{an}\right) = -\alpha.
\]

(ii) If \( p_1 > 0 \) and \( \log \mathbb{P}(X \leq -x) \sim -\lambda x^\alpha \) as \( x \to \infty \) for some \( \alpha \in (0, 1) \) and \( \lambda > 0 \), then

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P}
\left(Z_n(\{\theta x^* n, \infty\}) < e^{an}\right) = \begin{cases} -\lambda[(1 - \theta)x^*)^\alpha, & a \in [0, \log m - I(x^*)]; \\
-\lambda\hat{c}^\alpha, & a \in (\log m - I(x^*), \log m - I(\theta x^*)). \end{cases}
\]

For the lower bound, the strategy is to let the initial particle produces exactly one child and force its child to reach below some \( -\hat{c}n \) (where \( \hat{c} \) is chosen such that \( Z_{n-1}(\{(\theta x^* n + \hat{c}n, \infty)\}) \approx e^{an} \)). Let \( t_n := \Theta(1) \log n \). For the upper bound, we first prove that under the event \( \{Z_{t_n}(\{-\hat{c}n, \infty)\}/|Z_{t_n}| \geq 2/3\} \), it is very hard for \( \{Z_n(\{\theta x^* n, \infty\}) < e^{an}\} \) to happen. Thus, the desired upper bound comes from the probability \( \hat{c} \) Springer
which is very easy to handle by a Markov inequality.

**Proof** Lower bound  
We first consider the case $P(X < -x) = \Theta(1)x^{-\alpha}$ as $x \to \infty$. There exists a constant $C_2 > 0$ such that for $x > 1$,

$$P(X < -x) \geq C_2 x^{-\alpha}.$$  

Fix $\varepsilon > 0$. Applying the Markov property at time $n = 1$, for $n$ large enough,

$$P\left(Z_n\left(\left[\theta x^* n, \infty\right)\right) < e^{an}\right) \geq P\left(|Z_1| = 1, S_1 \leq -(\hat{c} + \varepsilon)n\right) P\left(Z_{n-1}\left(\left[(\theta x^* + \hat{c} + \varepsilon)n, \infty\right)\right) < e^{an}\right) \geq 0.9 p_1 C_2 (\hat{c} + \varepsilon) n^{-\alpha},$$ (3.2)  

where the last inequality follows from Corollary 2.2 and the fact that $G(\hat{c} + \varepsilon) < 0$ (see Lemma 3.4). As a result,

$$\lim \inf_{n \to \infty} \frac{1}{\log n} \log P\left(Z_n\left(\left[\theta x^* n, \infty\right)\right) < e^{an}\right) \geq -\alpha.$$  

Now, we consider the Weibull case. For any $\delta > 0$, we have for $x$ large enough,

$$P(X < -x) \geq e^{-(\lambda + \delta)x^\alpha}.$$  

Similarly to (3.2), we have

$$P\left(Z_n\left(\left[\theta x^* n, \infty\right)\right) < e^{an}\right) \geq 0.9 p_1 e^{-(\lambda + \delta)(\hat{c} + \varepsilon)x^\alpha}.$$  

By Lemma 3.4 (i), we have

$$\lim \inf_{n \to \infty} \frac{1}{n^\alpha} \log P\left(Z_n\left(\left[\theta x^* n, \infty\right)\right) < e^{an}\right) \geq \begin{cases} - (\lambda + \delta)(1 - \theta)x^* + \varepsilon)^{\alpha}, & a \in [0, \log m - I(x^*)]; \\ - (\lambda + \delta)(\hat{c} + \varepsilon)^{\alpha}, & a \in (\log m - I(x^*), \log m - I(\theta x^*)). \end{cases}$$  

The desired lower bound follows by letting $\varepsilon \to 0$ and $\delta \to 0$.

**Upper bound**  
Again, we first consider the case $P(X < -x) = \Theta(1)x^{-\alpha}$ as $x \to \infty$. Let $t_n = \lfloor h \log n \rfloor$, where $h > \frac{\alpha}{\log p_1}$. Recall that $Z^u_n$ is the $n$th generation of the sub-BRW emanating from particle $u$, and $Z^i_n, i \geq 1$ are i.i.d. copies of $Z_n$. Fix $\varepsilon \in (0, \hat{c})$.

Observe that...
\[
P(Z_n([\theta^* n, \infty)) < e^{an})
\]
\[
\leq P \left( Z_{tn} \left( [-(\hat{c} - \varepsilon)n, \infty) \right) > \frac{2}{3}|Z_{tn}|, \sum_{u \in Z_{tn}} Z^u_{n-t_n} ([\theta x^* n, \infty)) < e^{an} \right)
\]
\[
+ P \left( \frac{1}{|Z_{tn}|} Z_{tn} \left( [-(\hat{c} - \varepsilon)n, \infty) \right) \leq \frac{2}{3} \right)
\]
\[
\leq P \left( \forall 1 \leq i < \frac{2}{3}|Z_{tn}|, Z^i_{n-t_n} ([\theta x^* + \hat{c} - \varepsilon)n, \infty)) < e^{an} \right)
\]
\[
+ P \left( \frac{1}{|Z_{tn}|} Z_{tn} \left( (-\infty, -(\hat{c} - \varepsilon)n) \right) > \frac{1}{3} \right). \tag{3.3}
\]

Note that since \(G(\hat{c} - \varepsilon) < 0\) (see Lemma 3.4), by (2.1), for \(n\) large enough,
\[
P(Z_{n-t_n} \left( [((\theta x^* + \hat{c} - \varepsilon)n, \infty)) < e^{an} \right) < e^{-1}. \tag{3.4}
\]

Let \(G = \sigma(|Z_{tn}|)\). For the first term on the r.h.s. of (3.3), since \(Z_{tn}\) and \(\{Z^i_{n}\}_{n \geq 1}, i \geq 1\) are independent, we have for \(n\) large enough,
\[
P \left( \forall 1 \leq i < \frac{2}{3}|Z_{tn}|, Z^i_{n-t_n} ([\theta x^* + \hat{c} - \varepsilon)n, \infty)) < e^{an} \right)
\]
\[
= \mathbb{E} \left[ \prod_{1 \leq i < \frac{2}{3}|Z_{tn}|} P \left( Z^i_{n-t_n} \left( [((\theta x^* + \hat{c} - \varepsilon)n, \infty)) < e^{an} \right| G \right) \right]
\]
\[
\leq \mathbb{E} \left[ P \left( Z_{n-t_n} \left( [((\theta x^* + \hat{c} - \varepsilon)n, \infty)) < e^{an} \right) \right)^{\frac{1}{3}|Z_{tn}|} \right]
\]
\[
\leq \mathbb{E} \left[ e^{-\frac{1}{3}|Z_{tn}|} \right]
\]
\[
\leq 2C_0 p_1^{h \log n}
\]
\[
\leq 2C_0 n^{-\alpha}, \tag{3.5}
\]

where the second, third and last inequalities follow from (3.4), Lemma 3.1 and the fact that \(h > \frac{\alpha}{-\log p_1}\), respectively. For the second term on the r.h.s. of (3.3), by Markov inequality,
\[
P \left( Z_{tn} \left( (-\infty, -(\hat{c} - \varepsilon)n) \right) > \frac{1}{3} \right)
\]
\[
\leq 3 \mathbb{E} \left[ \frac{1}{|Z_{tn}|} Z_{tn} \left( (-\infty, -(\hat{c} - \varepsilon)n) \right) \right]
\]
\[
= 3 \mathbb{E} \left[ \mathbb{E} \left[ \frac{1}{|Z_{tn}|} \sum_{u \in Z_{tn}} 1_{((-\infty, -(\hat{c} - \varepsilon)n)(S_u) \right| G} \right] \right].
\]
\[ P\left( S_{\lfloor h \log n \rfloor} < - (\hat{c} - \epsilon)n \right) \leq 3 C_{\alpha}(h \log n)^2 (\hat{c} - \epsilon)^{-\alpha} n^{-\alpha}, \quad (3.6) \]

where the second equality follows by the fact that the branching and motion are independent and the last inequality comes from Lemma 3.2. Plugging (3.5) and (3.6) into (3.3) yields that

\[ P\left( Z_n([\theta x^* n, \infty)) < e^{an} \right) \leq 2 C_0 n^{-\alpha} + 3 C_{\alpha}(h \log n)^2 (\hat{c} - \epsilon)^{-\alpha} n^{-\alpha}. \quad (3.7) \]

As a result,

\[ \limsup_{n \to \infty} \frac{1}{\log n} \log P\left( Z_n([\theta x^* n, \infty)) < e^{an} \right) \leq -\alpha. \]

For the Weibull case, the proof is similar, the main changes are to let \( t_n = \lfloor h^{\alpha} n \rfloor \) in the upper bound (where \( h > \frac{L_{\alpha^*}}{-\log P_1} \)) and use Lemma 3.3. We feel free to omit its proof here. \( \square \)

4 Proof of Theorem 1.3: Böttcher Case, Bounded Tail

In this section, we are going to prove Theorem 1.3. We first present a lemma. Recall that

\[ L = -\text{ess inf } X \in (0, \infty), \quad b = \min\{ k \geq 0 : p_k > 0 \}, \]

\[ a^* = \left( \log m - I(x^*) \right) \frac{L + \theta x^*}{L + x^*} + \frac{(1 - \theta) x^*}{L + x^*} \log b \] \( \land \left[ \log m - I(\theta x^*) \right]. \]

Let

\[ F_L(c) := \log m - I \left( \frac{\theta x^* + Lc}{1 - c} \right) - \frac{a - c \log b}{1 - c}, \quad c \in \left( 0, \frac{(1 - \theta) x^*}{L + x^*} \right]. \quad (4.1) \]

Lemma 4.1 Assume \( a^* < \log m - I(\theta x^*). \) Then, for any \( a \in (a^*, \log m - I(\theta x^*)), \)

(i) the equation \( F_L(c) = 0 \) has a unique solution on \( 0, \left( \frac{(1 - \theta) x^*}{L + x^*} \right), \) denoted by \( \tilde{c}(L) \) (or \( \hat{c} \));

(ii) \( \tilde{c}(L) \) is continuous w.r.t. \( L, \) and

\[ F_L(\tilde{c}(L) - \delta) > 0 \text{ for } \delta \in (0, \tilde{c}(L)); \quad F_L(\tilde{c}(L) + \delta) < 0 \text{ for } \delta > 0. \quad (4.2) \]

Proof It is easy to see that

\[ F_L(c) = \log m - \log b - I \left( \frac{\theta x^* + Lc}{1 - c} \right) + \frac{\log b - a}{1 - c}, \]
and \( F_L(\cdot) \) is differentiable on \((0, (1-\theta)x^*/L+x^*)\) (since \( \frac{\theta x^* + Lc}{1-c} < x^* \)). Thus,

\[
F_L'(c) = -\frac{1}{(1-c)^2} \left[ I'(\frac{\theta x^* + Lc}{1-c}) (L + \theta x^*) + a - \log b \right].
\]

Since \( I(\cdot) \) is strictly convex and differentiable on \((0, x^*)\) (see [13, Exercise 2.2.24]), \( I'(x) \) is strictly increasing on \((0, x^*)\). Therefore, \( F_L'(c) = 0 \) admits at most one solution on \((0, (1-\theta)x^*/L+x^*)\).

This implies the monotonicity of \( F_L(c) \) has the following three cases:

(a) increasing on \((0, (1-\theta)x^*/L+x^*)\);
(b) decreasing on \((0, (1-\theta)x^*/L+x^*)\);
(c) increasing on some \((0, h)\), then decreasing on \([h, (1-\theta)x^*/L+x^*)\).

Since \( F(0) = \log m - I(\theta x^*) - a > 0 \), the monotonicity implies that \( F_L(c) = 0 \) has at most one solution in all three cases. On the other hand, since \( a > a^* \), we have

\[
F_L(\frac{(1-\theta)x^*}{L+x^*}) = \log m - I(x^*) - \frac{a - \log b}{1 - \frac{(1-\theta)x^*}{L+x^*}} < 0.
\]

Hence, \( F_L(c) = 0 \) has a unique solution on \((0, (1-\theta)x^*/L+x^*)\). This concludes (i). For (ii), by implicit function theorem, it is easy to know that \( \tilde{c}(L) \) is continuous w.r.t. \( L \). (4.2) follows by the definition of \( \tilde{c}(L) \).

Now, we are ready to prove Theorem 1.3: If \( p_1 = 0 \) and \( \text{ess inf } X = -L \in (-\infty, 0) \), then

\[
\lim_{n \to \infty} \frac{1}{n} \log \left[ -\log P \left( Z_n \left( (\theta x^* n, \infty) \right) < e^{an} \right) \right] = \begin{cases} 
\frac{(1-\theta)x^*}{L+x^*} \log b, & a \in [0, a^*);
\tilde{c}(L) \log b, & a \in [a^*, \log m - I(\theta x^*)].
\end{cases}
\]

For the lower bound, the strategy is to force every particle that produces \( b \) children up to time \( c^*(L)n \). Then, let the displacement of every individual before time \( c^*(L)n \) be close enough to \(-L\). Optimizing for \( c^*(L) \) yields the desired lower bound. For the upper bound, our inspiration comes from the proof of Cramér’s theorem. By the branching property, \( Z_n((\theta x^* n, \infty)) \) can be bounded below by \( \sum_{i=1}^{b_n} Z_{n-i} \) (sums of i.i.d. copies of BRW). Thus, by Chebyshev’s inequality, one can get an upper bound. To let this bound tend to zero, we use the dominated convergence theorem (which motivates us to define the following event \( E_n \) and \( K_n \) to satisfy the dominated convergence theorem’s condition).

**Proof Lower bound** Let

\[
c^*(L) = \begin{cases} 
\tilde{c}(L), & a^* < \log m - I(\theta x^*) \text{ and } a \in (a^*, \log m - I(\theta x^*));
\frac{(1-\theta)x^*}{L+x^*}, & \text{else}.
\end{cases}
\]

\( \square \) Springer
For every $L' \in (0, L)$ and $\delta \in (0, 1 - c^*(L'))$, let $t_n := \lfloor (c^*(L') + \delta)n \rfloor$. Observe that
\[
\mathbb{P} \left( Z_n \left( (\theta x^* n, \infty) \right) < e^{an} \right) \\
\quad \geq \mathbb{P} \left( \left| Z_{t_n} \right| = b^{j_n}, \forall u \in Z_{t_n}, S_u \leq -L't_n, Z_n \left( (\theta x^* n, \infty) \right) < e^{an} \right) \\
\quad \geq p_b \sum_{k=0}^{t_n-1} b^k \mathbb{P} (X \leq -L') \mathbb{P} \left( Z_{n-t_n} \left( (\theta x^* n + L't_n, \infty) \right) < e^{an} \right) \frac{b^n}{b^{j_n}}. \tag{4.3}
\]

We first consider the case that $a^* < \log m - I (\theta x^*)$ and $a \in (a^*, \log m - I (\theta x^*))$. In this case, since $c^*(L)$ is continuous w.r.t. $L$ and $c^*(L) < \frac{(1-\theta)x^*}{L+x^*}$ (see Lemma 4.1), we have
\[
\lim_{L' \to L \delta \to 0} \frac{\theta x^* + L'(c^*(L') + \delta)}{1 - (c^*(L') + \delta)} = \frac{\theta x^* + Lc^*(L)}{1 - c^*(L)} < x^*.
\]

Hence, for $L'$ close enough to $L$ and $\delta > 0$ small enough,
\[
\frac{\theta x^* + L'(c^*(L') + \delta)}{1 - (c^*(L') + \delta)} < x^*.
\]

As a consequence, applying (2.1), for $n$ large enough,
\[
\mathbb{P} \left( Z_{n-t_n} \left( (\theta x^* n + L't_n, \infty) \right) < e^{an} \right) \\
\quad \geq \mathbb{P} \left( \frac{1}{n-t_n} \log Z_{n-t_n} \left( (\theta x^* n + L't_n, \infty) \right) < \frac{an - t_n \log b}{n-t_n} \right) \geq 0.9, \tag{4.4}
\]

where the last inequality follows by Corollary 2.2 and the fact $F_{L'} (c^*(L') + \delta) < 0$ (see Lemma 4.1).

In the next, we consider the case of $a^* \geq \log m - I (\theta x^*)$ or $a \in [0, a^*)$. Note that in this case $c^*(L) = \frac{(1-\theta)x^*}{L+x^*}$. Since $\frac{\theta x^* + L(c^*(L) + \delta)}{1 - (c^*(L) + \delta)} > \frac{\theta x^* + Lc^*(L)}{1 - c^*(L)} = x^*$, there exists some $L'$ close enough to $L$ such that
\[
\frac{\theta x^* + L'(c^*(L') + \delta)}{1 - (c^*(L') + \delta)} > x^*.
\]

Thus, by (2.2), for $n$ large enough,
\[
\mathbb{P} \left( Z_{n-t_n} \left( (\theta x^* n + L't_n, \infty) \right) < e^{an} \right) \geq 0.9. \tag{4.5}
\]

Plugging (4.4) or (4.5) into (4.3) yields that for $n$ large enough,
\begin{align*}
P\left( Z_n([\theta x^n, \infty)) < e^{an} \right) & \geq \frac{1}{b^n} \sum_{k=0}^{t_n-1} b^k \frac{\log \left( \frac{t_n}{k+1} \right)}{\log b} \\
& \geq \left[ 0.9 \frac{\log \left( \frac{t_n}{k+1} \right)}{\log b} \right]^{b^n},
\end{align*}

where the last inequality follows by the fact that \( \sum_{k=0}^{t_n-1} b^k < b^{t_n} \). As a consequence, for \( L' \) close enough to \( L \) and \( \delta \) small enough, we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log \left[ -\log P\left( Z_n([\theta x^n, \infty)) < e^{an} \right) \right] \leq (c^*(L') + \delta) \log b.
\]

So, the desired lower bound follows by letting \( L' \to L \) and \( \delta \to 0 \).

**Upper bound** Let \( t_n := \lfloor (c^*(L) - \delta)n \rfloor \) for \( \delta \in (0, c^*(L)) \). Fix \( \epsilon \in (0, 1 - \log 2) \). By the branching property,

\[
P\left( Z_n([\theta x^n, \infty)) < e^{an} \right) \leq P\left( Z_{t_n}([-L, \infty)) \geq b^{t_n}, Z_n([\theta x^n, \infty)) < e^{an} \right) + P\left( |E_n| \geq \lfloor \epsilon b^{t_n} \rfloor \right),
\]

where \( Z^{(i)}_{t_n-t} \), \( 1 \leq i \leq b^{t_n} \) are i.i.d. copies of \( Z_{t_n-t} \) and

\[
E_n := \left\{ 1 \leq i \leq b^{t_n} : Z^{(i)}_{t_n-t}([\theta x^n + L, \infty)) \geq 1 \right\}.
\]

Set

\[
K_n := \left\{ \forall 1 \leq j \leq \lfloor \epsilon b^{t_n} \rfloor, Z^{(j)}_{t_n-t}([\theta x^n + L, \infty)) \geq 1 \right\}.
\]

Observe that, by Stirling’s formula \( \lim_{n \to \infty} \frac{n^\gamma}{\sqrt{2\pi n^{(3/2)}}} = 1 \), we have the following upper bound for combinatorial number: For \( n \) large enough,

\[
\left( \frac{b^{t_n}}{\lfloor \epsilon b^{t_n} \rfloor} \right) \leq \exp \left\{ - \left[ (1 - \epsilon) \log(1 - \epsilon) + \epsilon \log \epsilon \right]^{b^{t_n}} \right\} \leq e^{(\log 2)b^{t_n}}.
\]

Note that \( F_L(c^*(L) - \delta) > 0 \). (If \( c^*(L) = \frac{(1-\theta)x^*}{L+x^*} \) this comes from the definition of \( a^* \); if \( c^*(L) = \tilde{c}(L) \) this follows by Lemma 4.1 (ii).) Write

\[
T_\epsilon(n) := \frac{an - \log \lfloor \epsilon b^{t_n} \rfloor}{n - t_n}.
\]
As usual, write $P^n (\cdot) := (P (\cdot))^n$ and $E^n [\cdot] := (E [\cdot])^n$. For the first term on the r.h.s. of (4.6), for $n$ large enough,

$$
P \left( \sum_{i=1}^{b^n} Z_{n-t_n}^{(i)} ([\theta x^* n + L t_n, \infty)) < e^{a n}, |E_n| \geq [ \epsilon b^n ] \right)
$$

\begin{align*}
& \leq ( [ \epsilon b^n ] ) \mathbb{P} \left( \sum_{j=1}^{[ \epsilon b^n ]} \frac{1}{n-t_n} \log Z_{n-t_n}^{(j)} ([\theta x^* n + L t_n, \infty)) < T_\epsilon (n); K_n \right) \\
& \leq ( [ \epsilon b^n ] ) \mathbb{E} \left[ \exp \left\{ \frac{2}{\epsilon F_L (c^*(L) - \delta)} [T_\epsilon (n); [ \epsilon b^n ] \\
& \quad - \sum_{j=1}^{[ \epsilon b^n ]} \frac{1}{n-t_n} \log Z_{n-t_n}^{(j)} ([\theta x^* n + L t_n, \infty)) \right\}; K_n \right] \\
& = ( [ \epsilon b^n ] ) \mathbb{E} \left[ \exp \left\{ \frac{2}{\epsilon F_L (c^*(L) - \delta)} [T_\epsilon (n) \\
& \quad - \frac{1}{n-t_n} \log Z_{n-t_n} ([\theta x^* n + L t_n, \infty)) \right\}; Z_{n-t_n} ([\theta x^* n + L t_n, \infty)) \geq 1 \right],
\end{align*}

(4.8)

where the second inequality follows by the concavity of $g(x) = \log x$ and the third inequality comes from the exponential Chebyshev’s inequality. Note that

$$
\frac{\theta x^* + L (c^*(L) - \delta)}{1 - (c^*(L) - \delta)} < x^*.
$$

Thus, by (2.1), almost surely,

$$
\lim_{n \to \infty} \frac{2}{\epsilon F_L (c^*(L) - \delta)} \left[ T_\epsilon (n) - \frac{1}{n-t_n} \log Z_{n-t_n} ([\theta x^* n + L t_n, \infty)) \right] \\
= \frac{2}{\epsilon F_L (c^*(L) - \delta)} \left[ a - (c^*(L) - \delta) \log b \frac{1}{1 - (c^*(L) - \delta)} - \log m + I \left( \frac{\theta x^* + L (c^*(L) - \delta)}{1 - (c^*(L) - \delta)} \right) \right] \\
= - \frac{2}{\epsilon},
$$

(4.9)

where the last equality follows from (4.1). Therefore, by the dominated convergence theorem, for $n$ large enough, the expectation term in (4.8) is bounded above by $e^{-b^n}$. This, together with (4.7), yields that
\[
\mathbb{P}\left( \sum_{i=1}^{b^{|n|}} Z_{n-1}^{(i)} ([\theta x^* n + L_{t_n}, \infty)) < e^{a n}, |E_n| \geq |e b^{\alpha n}| \right) \leq e^{-(1-\log 2) b^{\alpha n}}. \quad (4.10)
\]

For the second term on the r.h.s. of (4.6), for \( n \) large enough,
\[
\mathbb{P}\left( |E_n| < |e b^{\alpha n}| \right) \leq \left( \frac{b^{\alpha n}}{1-\log 2} \right) \mathbb{P}\left( Z_{n-1}^{(i)} ([\theta x^* n + L_{t_n}, \infty)) = 0 \right) \leq e^{(1-\log 2) b^{\alpha n}} \leq e^{-(1-\log 2) b^{\alpha n}}. \quad (4.11)
\]

where the second inequality follows from (4.7) and the third inequality follows from (2.1). Plugging (4.10) and (4.11) into (4.6) yields that for \( \epsilon \in (0, 1-\log 2) \) and \( n \) large enough,
\[
\mathbb{P}\left( Z_n ([\theta x^* n, \infty)) < e^{a n} \right) \leq e^{-(1-\log 2) b^{\alpha n}} + e^{-[(1-\epsilon)-\log 2] b^{\alpha n}} \leq 2 e^{-[(1-\epsilon)-\log 2] b^{\alpha n}}.
\]

Therefore,
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( Z_n ([\theta x^* n, \infty)) < e^{a_n} \right) \geq (c^*(L) - \delta) \log b.
\]

So, the desired upper bound follows by letting \( \delta \to 0. \)

5 Proof of Theorem 1.4: Böttcher Case, Weibull Tail

In this section, we assume \( p_1 = 0 \) and \( \log \mathbb{P}(X \leq -z) \sim -\lambda z^\alpha \) as \( z \to \infty \) for \( \alpha, \lambda > 0 \). This section will be divided into two subsections. In the first subsection, we consider the sub-Weibull case (i.e., \( \alpha \in (0, 1) \)). In the second subsection, we consider the super-Weibull case (i.e., \( \alpha > 1 \)).

5.1 Sub-Weibull Tail: \( \alpha \in (0, 1) \)

In this subsection, we consider the sub-Weibull tail case, i.e., \( \log \mathbb{P}(X \leq -z) \sim -\lambda z^\alpha \), where \( \alpha \in (0, 1] \). We are going to prove the following:

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P}\left( Z_n ([\theta x^* n, \infty)) < e^{a_n} \right) = \begin{cases} 
-\lambda b (x^* - \theta x^*)^\alpha, & \alpha \in (0, 1], \ a \in [0, \log m - I(x^*)]; \\
-\lambda b \hat{c}^\alpha, & \alpha \in (0, 1], \ a \in (\log m - I(x^*), \log m - I(\theta x^*)),
\end{cases}
\]

where \( \hat{c} \) is the unique solution of \( \log m - I(\theta x^* + \hat{c}) = a \).

For the lower bound, the strategy is to let the initial particle produce exactly \( b \) children and force these children to reach below \(-\hat{c} n \) (where \( \hat{c} \) is chosen
such that \( Z_{n-1}(\theta x^n + \hat{c}, \infty)) \approx e^{an} \). Let \( t_n := \lfloor \frac{2a}{\log b} \log n \rfloor \). For the upper bound, we first prove that under the event \( \{ Z_{t_n}([-\hat{c}, \infty)) \geq b^{l_{n-1}} \} \), it is very hard for \( \{ Z_n(\theta x^n, \infty)) < e^{an} \) to happen. Thus, the leading order of \( \mathbb{P}(Z_n(\theta x^n, \infty)) < e^{an} \) comes from the probability \( \mathbb{P}(Z_{t_n}([-\hat{c}, \infty)) < b^{l_{n-1}}) \), which has been already considered in \([12, (3.11)-(3.12)]\).

**Proof Lower bound** Recall that \( G(c) = \log m - I(\theta x^* + c) - a, \hat{c} = \inf \{ c \geq 0 : G(c) < 0 \} \). We first consider the case \( a > \log m - I(x^*) \). In this case, we have shown (see Lemma 3.4):

\( \hat{c} \) is the unique solution of \( G(c) = 0 \) on \((0, (1 - \theta)x^*)\); \( G(\hat{c} + \delta) < 0 \) for \( \delta > 0 \).

Since \( \log \mathbb{P}(X \leq -z) \sim -\lambda z^\alpha \), for \( z \) large enough,

\[
\mathbb{P}(X \leq -z) \geq e^{-(\lambda + \delta)z^\alpha}. 
\] (5.1)

The branching property gives, for any \( \delta > 0 \) and \( n \) large enough,

\[
\mathbb{P}(Z_n(\theta x^n, \infty)) < e^{an})
\]
\[
\geq \mathbb{P}\left(Z_1 = b, \forall u \in Z_1, S_u \leq -(\hat{c} + \delta)n, \sum_{i=1}^b Z_{n-1}^{(i)}(\theta x^n + \hat{c} + \delta)n) < e^{an}\right)
\]
\[
\geq \mathbb{P}(Z_1 = b, \forall u \in Z_1, S_u \leq -(\hat{c} + \delta)n)
\]
\[
\times \mathbb{P}\left(Z_{n-1}^{(i)}(\theta x^n + \hat{c} + \delta)n, \infty) < e^{an}/b, \forall 1 \leq i \leq b\right)
\]
\[
\geq p_b e^{-(\lambda + \delta)(\hat{c} + \delta)n^\alpha b} \mathbb{P}\left(Z_{n-1}(\theta x^n + \hat{c} + \delta)n, \infty) < e^{an}/b\right)^b
\]
\[
\geq p_b e^{-(\lambda + \delta)(\hat{c} + \delta)n^\alpha b} \left(\frac{\log Z_{n-1}(\theta x^n + \hat{c} + \delta)n, \infty) \approx \frac{an - \log b}{n - 1}\right)^b
\]
\[
\geq 0.9 p_b e^{-(\lambda + \delta)(\hat{c} + \delta)n^\alpha b},
\] (5.2)

where the last inequality follows by (2.1) and the fact \( G(\hat{c} + \delta) < 0 \). Thus, for any \( \delta > 0 \),

\[
\liminf_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P}(Z_n(\theta x^n, \infty)) < e^{an}) \geq -(\lambda + \delta)(\hat{c} + \delta)n^\alpha b.
\]

So, the desired lower bound follows by letting \( \delta \to 0 \).

**Upper bound** Let \( t_n := \lfloor \frac{2a}{\log b} \log n \rfloor \). Fix \( \delta \in (0, \hat{c}) \). Note that

\[
\mathbb{P}(Z_n(\theta x^n, \infty)) < e^{an})
\]
\[
\leq \mathbb{P}(Z_{t_n}([-\hat{c} - \delta n, \infty)) \geq b^{l_{n-1}}, Z_n(\theta x^n, \infty)) < e^{an})
\]
\[
+ \mathbb{P}(Z_{t_n}([-\hat{c} - \delta n, \infty)) < b^{l_{n-1}}).
\]
For the first term on the right-hand side above, note that by the branching property, for \( n \) large enough,

\[
\begin{align*}
\mathbb{P}
\left(Z_{t_n} \left(\{-(\hat{c} - \delta)n, \infty\}\right) &\geq b^{n-1}, \quad Z_n(\{\theta x^* n, \infty\}) < e^{an} \right) \\
&\leq \mathbb{P}
\left(\sum_{k=1}^{b^{n-1}} Z_{n-t_n}^{(i)}(\{((\theta x^* + \hat{c} - \delta)n, \infty\}) < e^{an} \right) \\
&\leq \mathbb{P}
\left(\frac{\log Z_{n-t_n}(\{((\theta x^* + \hat{c} - \delta)n, \infty\})}{n-t_n} < \frac{an}{n-t_n} \right) \\
&\leq 0.9^{b^{n-1}},
\end{align*}
\]

where the last inequality follows by (2.1) and the fact that \( G(\hat{c} - \delta) > 0 \) (see Lemma 3.4).

For the second term, since \(|Z_1| \geq b\), for any \( \epsilon \in (0, 1) \) and \( n \) large enough, we have

\[
\begin{align*}
\mathbb{P}
\left(Z_{t_n} \left(\{-(\hat{c} - \delta)n, \infty\}\right) &< b^{n-1} \right) \\
&\leq \mathbb{P}
\left(\forall u \in Z_1, Z_{t_n-1}^{u} \left(\{-(\infty, -(\hat{c} - \delta)n)\} \geq 1 \right) \\
&\leq \mathbb{P}
\left(Z_{t_n-1} \left(\{-(\infty, -(\hat{c} - \delta)n - X')\} \geq 1 \right) \\
&\leq \mathbb{E}\left[Z_{t_n-1} \left(\{-(\infty, -(\hat{c} - \delta)n - X')\}\right)\right] \\
&= \left[m^{(t_n-1)}\mathbb{P}\left(S_{t_n} \leq -(\hat{c} - \delta)n\right)\right]^b \\
&\leq m^{b(t_n-1)} e^{-\lambda(1-\epsilon)(\hat{c}-\delta)\alpha n^b}, \quad (5.3)
\end{align*}
\]

where \( X' \) is a copy of \( X \) and independent of \( Z_{t_n-1} \) and the last inequality follows from Lemma 3.3. Hence, for \( \delta \in (0, \hat{c}) \) and \( \epsilon \in (0, 1) \),

\[
\mathbb{P}
\left(Z_n(\{\theta x^* n, \infty\}) < e^{an} \right) \leq m^{b(t_n-1)} e^{-\lambda(1-\epsilon)(\hat{c}-\delta)\alpha n^b} + 0.9^{b^{n}} \\
\leq 2m^{b(t_n-1)} e^{-\lambda(1-\epsilon)(\hat{c}-\delta)\alpha n^b},
\]

which implies the desired upper bound.

Now, it remains to deal with the case that \( \log m - I(x^*) > 0 \) and \( a \in [0, \log m - I(x^*)] \). In this case, \( \hat{c} = (1 - \theta)x^* \). The proof is basically the same as above. The only difference is to replace (2.1) with (2.2). \( \square \)

### 5.2 Super-Weibull Tail: \( \alpha > 1 \)

In this subsection, we consider the super-Weibull tail case, i.e., \( \log \mathbb{P}(X \leq -z) \sim -\lambda z^\alpha \) for some \( \alpha \in (1, \infty) \). We are going to prove the following:

(i) If \( a \in [0, \log m - I(x^*)] \), then

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P}(Z_n(\{\theta x^* n, \infty\}) < e^{an}) = -\lambda \left(\frac{1}{b^{\alpha-1}} - 1\right)^{\alpha-1} (x^* - \theta x^*)^\alpha;
\]

\( \square \) Springer
(ii) If \( a \in (\log m - I(x^*), \log m - I(\theta x^*)) \), then

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P}(Z_n([\theta x^* n, \infty)) < e^{\alpha n}) = -\lambda \left( b^{\frac{1}{\alpha} - 1} \right)^{\alpha - 1} \hat{c}^\alpha.
\]

The upper bound uses the same method as sub-Weibull case. The proof of the lower bound that is given below uses a method similar to that of [12, Theorem 1.3]. We force the branching random walk to behave a \( b \)-regular tree up to some time \( t_n = \Theta(1) \log n \). Then, we let each individual on this regular tree to make displacement smaller than some \(-a_k\) with \( \sum_{k=1}^{t_n} a_k = \hat{c} n \) (where \( k \) is the generation of the individual and \( \hat{c} \) is chosen such that \( \sum_{i=1}^{b^{t_n}} Z_n^{(i)}[\theta x^* n + \hat{c} n, \infty) \approx e^{\alpha n} \)). Obviously, to make the lower bound as large as possible, \( \{a_k\}_{1 \leq k \leq t_n} \) should be the solution of

\[
\inf_{\sum_{k=1}^{t_n} a_k = \hat{c} n, a_k \geq 0, 1 \leq k \leq t_n} \sum_{k=1}^{t_n} a_k^{\alpha} b^k.
\]

In the next, we are going to solve this optimization problem. Let \( u_k := b^{\frac{1}{\alpha} - 1} \), \( 1 \leq k \leq t_n \). It is simple to see that

\[
u_k^{1-\alpha} b^k = 1 \quad \text{for} \quad 1 \leq k \leq t_n.
\]

By the Jensen inequality,

\[
\sum_{k=1}^{t_n} a_k^{\alpha} b^k = \left( \sum_{i=1}^{t_n} u_i^{1-\alpha} b^i \right)^\alpha \sum_{k=1}^{t_n} (a_k u_k)^\alpha \frac{u_k^{1-\alpha} b^k}{\sum_{i=1}^{t_n} u_i^{1-\alpha} b^i} \geq \left( \sum_{i=1}^{t_n} u_i^{1-\alpha} b^i \right)^1 \left[ \sum_{k=1}^{t_n} a_k u_k^{1-\alpha} b^k \right]^\alpha = \left[ \sum_{i=1}^{t_n} (b^{\frac{1}{\alpha} - 1} b^i) \right]^{1-\alpha} \left[ \sum_{k=1}^{t_n} a_k \right]^\alpha \geq \left[ \sum_{i=1}^{t_n} (b^{\frac{1}{\alpha} - 1} b^i) \right]^{1-\alpha} \left( \hat{c} n \right)^\alpha,
\]

(5.5)
where the third equality follows from (5.4). Through simple calculations,

\[
\sum_{i=1}^{t_n} \left( b^{\frac{1}{a-1}} \right)^i = \frac{b^{\frac{1}{a-1}} \left( 1 - b^{\frac{a-n}{a-1}} \right)}{1 - b^{\frac{1}{a-1}}} \sim \frac{1}{b^{\frac{1}{a-1}} - 1} = \frac{1}{b^\alpha - 1},
\]

where \( a_n \sim b_n \) means \( \lim_{n \to \infty} a_n / b_n = 1 \) and \( b^\alpha := b^{\frac{1}{a-1}} \). Note that the right-hand side of (5.5) is fixed (independent of \( \{a_k\}_{1 \leq k \leq t_n} \)). Thus, to achieve the lower bound, we must choose \( \{a_k\}_{1 \leq k \leq t_n} \) such that \( \{a_k u_k\}_{1 \leq k \leq t_n} \) are independent of \( k \).

Say,

\[
a_k u_k = (b^\alpha - 1) \hat{c} n \quad \text{for} \quad 1 \leq k \leq t_n.
\]

Hence,

\[
a_k = \frac{b^\alpha - 1}{b^{\alpha} \hat{c} n}, \quad 1 \leq k \leq t_n.
\]

By this means, we can replace \( \geq \) in (5.5) with \( = \). Plugging these into (5.5) yields that

\[
\sum_{k=1}^{t_n} a_k^\alpha b^k \sim \left( b^{1/(\alpha-1)} - 1 \right)^{\alpha-1} (\hat{c} n)^\alpha.
\]

Therefore, we conclude that

\[
\inf_{\sum_{k=1}^{t_n} a_k \geq \hat{c} n} \sum_{k=1}^{t_n} a_k^\alpha b^k \sim \left( b^{1/(\alpha-1)} - 1 \right)^{\alpha-1} (\hat{c} n)^\alpha.
\]

Note that the right-hand side of the above appears in the theorem. Now, we are ready to prove the theorem.

**Proof Lower bound** Recall that \( G(c) = \log m - I(\theta \bar{x}^* + c) - a \) and \( \hat{c} = \inf \{c \geq 0 : G(c) < 0\} \). Again, we first consider the case \( a > \log m - I(\bar{x}^*) \). Set \( t_n := \lfloor \frac{\alpha}{2 \log b} \log n \rfloor \) and \( a_k := \frac{b^{\alpha-n}}{b^{\alpha}} (\hat{c} + \delta) n \) for \( 1 \leq k \leq t_n \), where \( \delta \) is a positive constant. By a simple calculation, for \( n \) large enough,

\[
(\hat{c} + \delta / 2)n \leq \sum_{k=1}^{t_n} a_k \leq (\hat{c} + \delta) n, \quad \sum_{k=1}^{t_n} a_k^\alpha b^k \leq \left( b^{\frac{1}{a-1}} - 1 \right)^{\alpha-1} (\hat{c} + \delta)^\alpha n^\alpha.
\]

(5.7)

Denote by \( X_u \) the displacement of particle \( u \). By the branching property, for \( n \) large enough,

\[ \hat{c} \] Springer
\[
\mathbb{P}\left(Z_n((\theta x^* n, \infty)) < e^{an}\right) \\
\geq \mathbb{P}\left(|Z_{tn}| = b^{tn}; \forall u \in Z_i, 1 \leq i \leq t_n, X_u \leq -a|u|\right) \\
\times \sum_{i=1}^{b^{tn}} \mathbb{P}\left(Z_{n-tn}^{(i)}\left((\theta x^* n + \sum_{k=1}^{t_n} a_k, \infty)\right) < e^{an}\right) \\
\geq \prod_{k=0}^{t_n-1} b^k \mathbb{P}(X_k \leq -a) b^k \mathbb{P}^{b^n} \left(Z_{n-tn}^{(i)}\left((\theta x^* n + \sum_{k=1}^{t_n} a_k, \infty)\right) < e^{an}\right) \\
\geq p_b^{\sum_{k=0}^{t_n-1} b^k} \exp\left\{-\left(\lambda + \delta\right) \sum_{k=1}^{t_n} a^\alpha b^k\right\} \\
\times \mathbb{P}^{b^n} \left(\frac{1}{n-t_n} \log \mathbb{P}(Z_{n-tn}^{(i)}\left((\theta x^* n + \hat{c} + \delta/2)n, \infty)\right) < \frac{an - tn \log b}{n-t_n}\right), (5.8)
\]

where the last inequality follows from (5.1) and (5.7). Since (2.1) and \( G(\hat{c} + \delta/2) < 0 \) (see Lemma 3.4), for \( n \) large enough, we have

\[
\mathbb{P}\left(\frac{\log \mathbb{P}(Z_{n-tn}^{(i)}\left((\theta x^* n + \hat{c} + \delta/2)n, \infty)\right) < \frac{an - tn \log b}{n-t_n}}{n-t_n}\right) \geq 0.9. \tag{5.9}
\]

So, plugging (5.7) and (5.9) into (5.8) yields that

\[
\mathbb{P}\left(Z_n(\theta x^* n + B) < e^{an}\right) \\
\geq p_b^{\sum_{k=0}^{t_n-1} b^k} \left(0.9\right)^{b_n} \exp\left\{-\left(\lambda + \delta\right) \sum_{k=1}^{t_n} a^\alpha b^k\right\} \\
\geq (0.9 p_b)^{b_n+1} \exp\left\{-\left(\lambda + \delta\right) \left(b \frac{1}{\alpha-1} - 1\right)^{\alpha-1} (\hat{c} + \delta)^\alpha n^\alpha\right\} \\
\geq (0.9 p_b)^{bn^{\alpha/2}} \exp\left\{-\left(\lambda + \delta\right) \left(b \frac{1}{\alpha-1} - 1\right)^{\alpha-1} (\hat{c} + \delta)^\alpha n^\alpha\right\}, \tag{5.10}
\]

where the second inequality follows by the fact that \( \sum_{k=0}^{t_n} b^k < b^{n+1} \). Thus, for every small \( \delta > 0 \), we have

\[
\liminf_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P}\left(Z_n((\theta x^* n, \infty) < e^{an}\right) \geq -\left(\lambda + \delta\right) \left(b \frac{1}{\alpha-1} - 1\right)^{\alpha-1} (\hat{c} + \delta)^\alpha.
\]

The desired lower bound finally follows by letting \( \delta \to 0 \).
Upper bound The upper bound is similar to the sub-Weibull tail case. Set \( t_n := \lfloor \frac{2\alpha}{\log b} \log n \rfloor \) and \( \delta_n := \lfloor \frac{\alpha}{2 \log b} \log n \rfloor \). For \( \delta \in (0, \hat{c}) \), we have

\[
\mathbb{P}(Z_n((\theta x^* n, \infty)) < e^{\alpha n}) \\
\leq \mathbb{P}(Z_{tn}([-(\hat{c} - \delta) n, \infty)) \geq b^{t_n - \delta n}, Z_n((\theta x^* n, \infty)) < e^{\alpha n}) \\
+ \mathbb{P}(Z_{tn}([-(\hat{c} - \delta) n, \infty)) < b^{t_n - \delta n}).
\]

For the first term on the r.h.s. above, note that by the branching property and (2.1), we have

\[
\mathbb{P}(Z_{tn}([-(\hat{c} - \delta) n, \infty)) \geq b^{t_n - \delta n}, Z_n((\theta x^* n, \infty)) < e^{\alpha n}) \leq (0.9)b^{t_n - \delta n}.
\]

For the second term, [12, (3.36)] gives that for any \( \epsilon > 0 \) and \( n \) large enough,

\[
\mathbb{P}(Z_{tn}([-(\hat{c} - \delta) n, \infty)) < b^{t_n - \delta n}) \\
\leq \exp \left\{ -\lambda(1 - \epsilon) \left( b^{\frac{1}{\alpha - 1}} - 1 \right)^{\alpha - 1} (1 - b^{-\delta n})^{1+\alpha} (\hat{c} - \delta)^{\alpha} n^{\alpha} \right\}.
\]

Thus, for any small \( \epsilon > 0 \) and \( \delta > 0 \), we have

\[
\limsup_{n \to \infty} \frac{1}{n^{\alpha}} \log \mathbb{P}(Z_n((\theta x^* n, \infty)) < e^{\alpha n}) \geq -\lambda(1 - \epsilon) \left( b^{\frac{1}{\alpha - 1}} - 1 \right)^{\alpha - 1} (\hat{c} - \delta)^{\alpha},
\]

which implies the result.

It remains to treat the case of \( \log m - I(x^*) > 0 \) and \( a \in [0, \log m - I(x^*)] \). In this case, one can show that \( \hat{c} = (1 - \theta)x^* \). The proof is basically the same as above. The only difference is to replace (2.1) with (2.2).

Acknowledgements This paper is part of my Ph.D. thesis at Beijing Normal University. I want to thank my supervisor Prof. Hui He for his help of weakening the assumption in Theorem 1.1. I also would like to thank Lianghui Luo for interesting discussions and pointing out a mistake in an earlier version of this manuscript. Thanks are also due to the two referees for their valuable suggestions and careful reading.

Funding There are no funding bodies to thank relating to this creation of this article.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest There are no competing interests to declare which arose during the preparation or publication process of this article.
References

1. Aidékon, E., Hu, Y., Shi, Z.: Large deviations for level sets of branching Brownian motion and Gaussian free fields. J. Math. Sci. 238(4), 348–365 (2019)
2. Athreya, K.B., Ney, P.E.: Branching Processes. Springer, Berlin (1972)
3. Aurzada, F.: Large deviations for infinite weighted sums of stretched exponential random variables. J. Math. Anal. Appl. 485(2), 123814 (2020)
4. Bhattacharya, A.: Large deviation for extremes in branching random walk with regularly varying displacements. arXiv:1802.05938
5. Biggins, J.D.: Chernoffs theorem in the branching random walk. J. Appl. Probab. 14, 630–636 (1977)
6. Biggins, J. D.: Growth rates in the branching random walk. Z. Wahrscheinlichkeitstheorieverw. Gebiete 48, 17–34 (1979)
7. Biggins, J.D.: Uniform convergence in the branching random walk. Ann. Probab. 20, 137–151 (1992)
8. Bramson, M.D., Ding, J., Zeitouni, O.: Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. Commun. Pure Appl. Math. 69, 62–123 (2015)
9. Buraczewski, D., Masmalka, M.: Large deviation estimates for branching random walks. ESAIM. Probab. Stat. 23, 823–840 (2019)
10. Chauvin, B., Rouault, A.: KPP equation and supercritical branching Brownian motion in the subcritical speed area: application to spatial trees. Probab. Theory Relat. Fields 80, 299–314 (1988)
11. Chen, X., He, H.: On large deviation probabilities for empirical distribution of supercritical branching random walks with bounded displacements. Probab. Theory Relat. Fields 175, 255–307 (2019)
12. Chen, X., He, H.: Lower deviation and moderate deviation probabilities for maximum of a branching random walk. Ann. Institut Henri Poincare Probab. Stat. 56, 2507–2539 (2020)
13. Dembo, A., Zeitouni, O.: Large Deviation Techniques and Applications, 2nd edn. Springer, Berlin (1998)
14. Derrida, B., Shi, Z.: Large deviations for the branching Brownian motion in presence of selection or coalescence. J. Stat. Phys. 163(6), 1285–1311 (2016)
15. Derrida, B., Shi, Z.: Large deviations for the rightmost position in a branching Brownian motion. In: Panov V. (eds) Modern Problems of Stochastic Analysis and Statistics. MPSAS 2016. Springer Proceedings in Mathematics and Statistics, vol 208. Springer, Cham (2016)
16. Derrida, B., Shi, Z.: Slower deviations of the branching Brownian motion and of branching random walks. J. Phys. A Math. Theor. 50, 344001 (2017)
17. Durrett, R.: Probability: Theory and Examples, 4th edn. Cambridge University Press, Cambridge (2010)
18. Fleischmann, K., Wachtel, V.: Lower deviation probabilities for supercritical Galton–Watson processes. Ann. Institut Henri Poincare Probab. Stat. 43, 233–255 (2007)
19. Gantert, N., Höfelsauer, T.: Large deviations for the maximum of a branching random walk. Electron. Commun. Probab. 23(34), 1–12 (2018)
20. Hammersley, J.M.: Postulates for subadditive processes. Ann. Probab. 2, 652–680 (1974)
21. Hu, Y.: How big is the minimum of a branching random walk? Ann. Inst. Henri Poincaré Probab. Stat. 52(1), 233–260 (2016)
22. Hu, Y., Shi, Z.: A subdiffusive behaviour of recurrent random walk in random environment on a regular tree. Probab. Theory Relat. Fields 138, 521–549 (2007)
23. Liu, Q.: Fixed points of a generalised smoothing transformation and applications to branching processes. Adv. Appl. Probab. 30(1), 85–112 (1995)
24. Liu, Q.: On generalised multiplicative cascades. Stoch. Process. Appl. 86(2), 263–286 (2006)
25. Louidor, O., Perkins, W.: Large deviations for the empirical distribution in the branching random walk. Electron. J. Probab. 18, 1–19 (2015)
26. Nagaev, S.V.: Large deviations of sums of independent random variables. Ann. Probab. 7, 745–789 (1979)
27. Öz, M.: Large deviations for local mass of branching Brownian motion. ALEA. Latin Am. J. Probab. Math. Stat. 17, 711–731 (2020)
28. Rouault, A.: Precise estimates of presence probabilities in the branching random walk. Stoch. Process. Appl. 44(1), 27–39 (1993)
29. Shi, Z.: Branching Random Walks. École d’Été de Probabilités de Saint-Flour XLII-2012. Lecture Notes in Mathematics 2151. Springer, Berlin (2015)
30. Zhang, S.: On large deviation probabilities for empirical distribution of branching random walks with heavy tails. J. Appl. Probab. 59(2), 1–24 (2022)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.