Gibbs Measures for HC-Model with a Countable Set of Spin Values on a Cayley Tree

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Abstract
In this paper, we study the HC-model with a countable set \( \mathbb{Z} \) of spin values on a Cayley tree of order \( k \geq 2 \). This model is defined by a countable set of parameters (that is, the activity function \( \lambda_i > 0, i \in \mathbb{Z} \)). A functional equation is obtained that provides the consistency condition for finite-dimensional Gibbs distributions. Analyzing this equation, the following results are obtained:

– Let \( \Lambda = \sum_i \lambda_i \). For \( \Lambda = +\infty \) there is no translation-invariant Gibbs measure (TIGM) and no two-periodic Gibbs measure (TPGM);
– For \( \Lambda < +\infty \), the uniqueness of TIGM is proved;
– Let \( \Lambda_{cr}(k) = \frac{k^k}{(k-1)^{k-1}} \). If \( 0 < \Lambda \leq \Lambda_{cr} \), then there is exactly one TPGM that is TIGM;
– For \( \Lambda > \Lambda_{cr} \), there are exactly three TPGMs, one of which is TIGM.

Keywords HC model · Configuration · Cayley tree · Gibbs measure · Boundary law

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1 Introduction

The theory of Gibbs measures is well developed in many classical models from physics (for example, the Ising model, the Potts model, the HC model), when the set of spin values is a finite set. It is known that each Gibbs measure corresponds to one phase of the physical system. Therefore, in the theory of Gibbs measures, one of the important problems is the existence and non-uniqueness of Gibbs measures. The non-uniqueness means that the physical system has a coexistence of its phases (state) at a fixed temperature (see [7], [13], [26], [28], [32]).

In the case of models with a finite set of spin values, the set of all Gibbs measures for a given Hamiltonian forms a non-empty, convex, compact subset in the set of all probability measures. In the setting of gradient interface models in general no Gibbs measure exists [34]. Therefore one often considers gradient Gibbs measures [8].

There are papers devoted to the study of (gradient) Gibbs measures for models with an infinite set of spin values. In particular, [9] shows the uniqueness of the translation-invariant Gibbs measure for the antiferromagnetic Potts model with a countable number of states and a nonzero external field, and [10] describes the Poisson measures, which are Gibbs measures. In [11] the author considers single-spin configuration space as the set of all non-negative integers. With respect to Poisson measure on this space it is proved that the set of all limit Gibbs measures on $\mathbb{Z}^d$ of this model is nonempty.

The work [36] is devoted to the study of Gibbs measures for models of gradient type.

We consider a HC model on Cayley trees. The use of the Cayley tree is motivated (see [29, page 18] and references therein) by the applications, such as information flows and reconstruction algorithms on networks, DNA strands and Holliday junctions, evolution of genetic data and phylogenetics, bacterial growth and fire forest models, or computational complexity on graphs. Moreover, we note that theory of Gibbs measures is mainly developed for systems defined on the lattice $\mathbb{Z}^d$ and on trees. For comparison of methods and results obtained for $\mathbb{Z}^d$ and Cayley tree see [28, Section 1.1] and [29, Chapter 2].

In [14] the existence of several translational invariant gradient Gibbs measures for the SOS model with a countable set of spin values on the Cayley tree is proven. And also the class of 4-periodic gradient Gibbs measures is described. Recently, in [15] (gradient) Gibbs measures for a gradient-type model are studied and the existence of a countable set of ordinary Gibbs measures is shown, and conditions for the existence of gradient Gibbs measures that are not ordinary Gibbs measures are found. In [5] the existence of a relationship between Gibbs measures and Gibbs gradient measures is shown. In [16] the existence of gradient Gibbs measures that are not translation-invariant is proved. In [24] and [25] Gibbs measures of one-dimensional countable state $p$-adic Potts model is studied. The existence of strong phase transition is proved.

Many papers are devoted to the study of limit Gibbs measures for HC-models with a finite set of spin values (see, for example, [19], [28] and the references therein). In this paper, we study HC-model with a countable set of spin values. Our motivation is that there are biological and physical systems involving a countable set of spin values. The prime examples of such spin systems are harmonic oscillators. Another example is the
Ginzburg-Landau interface model; which is obtained from the anharmonic oscillators (see [8], [10], [34]). Moreover, in book [6] populations with infinitely many alleles are studied. In [30] at a population genetic system the countable set is considered as the set of colonies, where each colony contains two alleles.

For HC-model with a countable set of spin values \( \mathbb{Z} \) on a Cayley tree of arbitrary order we will find some conditions for the existence of TIGM, and also prove the uniqueness of such a measure under the existence condition. Besides, for the model under consideration, two-periodic Gibbs measures are studied.

The exact value \( \Lambda_{cr} \) of the parameter \( \Lambda \) is found, (where \( \Lambda \) is the sum of the series obtained from the sequence of parameters \( \{ \lambda_j \}_{j \in \mathbb{Z}} \)), such that for \( 0 < \Lambda \leq \Lambda_{cr} \) there is exactly one periodic Gibbs measure which is translation invariant, and for \( \Lambda > \Lambda_{cr} \) there are exactly three periodic Gibbs measures, one of which is translation invariant.

## 2 Preliminaries

The Cayley tree \( \Gamma^k \) of order \( k \geq 1 \) is an infinite tree, i.e., a graph without cycles, such that exactly \( k + 1 \) edges originate from each vertex. Let \( \Gamma^k = (V, L, i) \), where \( V \) is the set of vertices \( \Gamma^k \), \( L \) is the set of edges and \( i \) is the incidence function setting each edge \( l \in L \) into correspondence with its endpoints \( x, y \in V \). If \( i(l) = \{ x, y \} \), then the vertices \( x \) and \( y \) are called the nearest neighbors, denoted by \( l = \langle x, y \rangle \).

For a fixed point \( x^0 \in V \),

\[
W_n = \{ x \in V \mid d(x, x^0) = n \}, \quad V_n = \bigcup_{m=0}^{n} W_m, \quad L_n = \{ \langle x, y \rangle \in L \mid x, y \in V_n \},
\]

where \( d(x, y) \) is the distance between vertices \( x \) and \( y \) on a Cayley tree, i.e., the number of edges of the shortest path connecting \( x \) and \( y \).

Write \( x < y \), if the path from \( x^0 \) to \( y \) goes through \( x \). Call vertex \( y \) a direct successor of \( x \) if \( y > x \) and \( x, y \) are nearest neighbors. Note that in \( \Gamma^k \) any vertex \( x \neq x^0 \) has \( k \) direct successors and \( x^0 \) has \( k + 1 \) direct successors. Denote by \( S(x) \) the set of direct successors of \( x \), i.e. if \( x \in W_n \), then

\[
S(x) = \{ y_i \in W_{n+1} \mid d(x, y_i) = 1, i = 1, 2, \ldots, k \}.
\]

We consider the Hard-Core (HC) model with a countable set of spin values in which the spin variables take values in the set \( \mathbb{Z} \), and are located at the tree vertices. A configuration \( \sigma = \{ \sigma(x) \mid x \in V \} \) is then defined as a function \( \sigma = \{ \sigma(x) \in \mathbb{Z} : x \in V \} \). In this model, each vertex \( x \) is assigned one of the values \( \sigma(x) \in \mathbb{Z} \), where \( \mathbb{Z} \) is the set of integers. The values \( \sigma(x) \neq 0 \) mean that the vertex \( x \) is ‘occupied’, and \( \sigma(x) = 0 \) means that \( x \) is ‘vacant’.

We consider the set \( \mathbb{Z} \) as the set of vertices of a graph \( G \). We use the graph \( G \) to define a \( G \)-admissible configuration as follows. A configuration \( \sigma \) is called a \( G \)-admissible configuration on the Cayley tree (in a subset \( A \subset V \)), if \( \{ \sigma(x), \sigma(y) \} \) is
one edge of the graph $G$ for any pair of nearest neighbors $x$, $y$ in $V$ (in $A$). We let $\Omega^G$ ($\Omega^G_A$) denote the set of $G$-admissible configurations $\sigma$ (resp. $\sigma_A$).

The activity set [3] for a graph $G$ is the bounded function $\lambda : G \to \mathbb{R}_+$ from the vertices of $G$ to the set of positive real numbers. The value $\lambda_i$ of the function $\lambda$ at the vertex $i \in \mathbb{Z}$ is called the vertex activity.

For given $G$ and $\lambda$ we define the Hamiltonian of the $G$-HC model as

$$H^\lambda_G(\sigma) = \begin{cases} \sum_{x \in V} \ln \lambda_{\sigma(x)}, & \text{if } \sigma \in \Omega^G, \\ +\infty, & \text{if } \sigma \notin \Omega^G. \end{cases} \quad (2.1)$$

For nearest-neighboring interaction potential $\Phi = (\Phi_b)_b$, where $b = (x, y)$ is an edge, define symmetric transfer matrices $Q_b$ by

$$Q_b(\omega_b) = e^{-\left(\Phi_b(\omega_b) + |\partial x|^{-1} \Phi_{\{x\}}(\omega(x)) + |\partial y|^{-1} \Phi_{\{y\}}(\omega(y))\right)}, \quad (2.2)$$

where $\partial x$ is the set of all nearest-neighbors of $x$ and $|S|$ denotes the number of elements of the set $S$. Note that for the Cayley tree of order $k \geq 1$ we have $|\partial x| = |\partial y| = k + 1$.

Define the Markov (Gibbsian) specification as

$$\gamma^\Phi(\sigma_A = \omega_A | \omega) = (Z^\Phi_A)(\omega)^{-1} \prod_{b \cap \Lambda \neq \emptyset} Q_b(\omega_b).$$

Let $L(G)$ be the set of edges of a graph $G$. We let $A \equiv A^G = (a_{ij})_{i,j=0,1,2}$ denote the adjacency matrix of the graph $G$, i.e.,

$$a_{ij} = a^G_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in L(G), \\ 0 & \text{if } \{i, j\} \notin L(G). \end{cases}$$

**Remark 1** Since $i, j \in \mathbb{Z}$, then in cases where $j$ is a specific negative number, instead of $a_{ij}$ we will conventionally write $a_{i,j}$.

**Definition 1** (See [13, Chapter 12], [15])

1. A family of vectors $l = \{l_{xy}\}_{(x,y) \in L}$ with $l_{xy} = \{l_{xy}(i) : i \in \mathbb{Z}\} \in (0, \infty)^\mathbb{Z}$ is called the boundary law for the Hamiltonian (2.1) if for each $(x, y) \in L$ there exists a constant $c_{xy} > 0$ such that the consistency equation

$$l_{xy}(i) = c_{xy} \lambda_i \prod_{z \in \partial x \setminus \{y\}} \sum_{j \in \mathbb{Z}} a_{ij} l_{zx}(j) \quad (2.3)$$

holds for every $i \in \mathbb{Z}$, where $\partial x$ — the set of nearest neighbors of a vertex $x$.

2. A boundary law $l$ is said to be normalisable if and only if

$$\sum_{i \in \mathbb{Z}} \left(\lambda_i \prod_{z \in \partial x} \sum_{j \in \mathbb{Z}} a_{ij} l_{zx}(j)\right) < \infty \quad (2.4)$$

at any $x \in V$.  

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A boundary law is called \emph{q-height-periodic} (or \(q\)-periodic) if \(l_{xy}(i + q) = l_{xy}(i)\) for every oriented edge \(\langle x, y \rangle\) and each \(i \in \mathbb{Z}\).

A boundary law is called \emph{translation invariant} if it does not depend on edges of the tree, i.e., \(l_{xy}(i) = l(i)\) for every oriented edge \(\langle x, y \rangle\) and each \(i \in \mathbb{Z}\).

Assume \(l_{xy}(0) = 1\), for each \(\langle x, y \rangle \in L\) (normalization at 0), then dividing \((2.3)\) to the equality obtained for \(i = 0\) we get

\[
l_{xy}(i) = \frac{\lambda_i}{\lambda_0} \prod_{z \in \partial x \setminus \{y\}} a_{i0} + \sum_{j \in \mathbb{Z}_0} a_{ij} l_{zx}(j) \cdot \frac{a_{00} + \sum_{j \in \mathbb{Z}_0} a_{0j} l_{zx}(j)}{a_{00} + \sum_{j \in \mathbb{Z}_0} a_{0j} l_{zx}(j)}.
\]

(2.5)

\textbf{Remark 2} We note that

(a) There is an one-to-one correspondence between boundary laws and Gibbs measures (i.e., tree-indexed Markov chains) if the boundary laws are normalisable [35] (see [14, Theorem 3.5]).

(b) In [14] it is shown that a translation invariant boundary law \(z \in \mathbb{R}_+^\infty\) satisfies the condition of normalisability, if \(z \in l^{k+1}_k\).

In this paper we consider the nearest-neighboring interaction potential \(\Phi_1 = (\Phi_1)_b\), which corresponds to the HC model \((2.1)\), i.e., for \(G\)-admissible configuration \(\sigma\) and \(A \subset V\):

\[
\Phi_A(\sigma_A) = \begin{cases} 
\lambda_{\sigma(x)}, & \text{if } A = \{x\} \\
0, & \text{if } |A| \geq 2.
\end{cases}
\]

and will study Gibbs measures of this model. By Remark 2 each normalisable boundary law \(l\) defines a Gibbs measure. In this paper our aim is to find 1-height-periodic and two-periodic boundary laws for the HC model for a specially chosen graph \(G\) (see below). We show that these boundary laws will be normalisable and therefore define Gibbs measures.

We consider the graph \(G\) with \(a_{i0} = 1\) for any \(i \in \mathbb{Z}\) and \(a_{im} = 0\) for any \(i, m \in \mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}\) (see Fig. 1). The corresponding admissible configuration satisfies the equality \(\sigma(x)\sigma(y) = 0\) for any \(\langle x, y \rangle \) from \(V\), i.e., if the vertex \(x\) has the spin value \(\sigma(x) = 0\), then on neighboring vertices we can put any value from \(\mathbb{Z}\), and if the vertex \(x\) contains any value from \(\mathbb{Z}_0\), then on neighboring vertices we put only zeros.

\textbf{Remark 3} Our choice of the graph \(G\) can be motivated as

- it is a natural generalization of the classical HC-model, with two spin values 0, 1 for which the admissibility condition is the same: \(\sigma(x)\sigma(y) = 0\) for any \(\langle x, y \rangle\) from \(V\).
- the difficulty to solve equation \((2.5)\) depends on graph \(G\). For the graph of Fig. 1 we are able to explicitly solve the corresponding equation. For another graph which has additional edges (or loops) the equation becomes more difficult to solve.
- We will show that in the HC-model corresponding to the graph Fig. 1 phase transition (non-uniqueness of Gibbs measure) occurs.
For $x \in S(y)$, $y \in V$, introduce new variables as $z_{i,x} = l_{xy}(i)$, then in case of $G$ (given in Fig. 1), from (2.5) (see [13] and [2]) we obtain

$$z_{i,x} = \lambda_i \prod_{j \in S(x)} \frac{1}{1 + \sum_{j \in \mathbb{Z}_0} z_{j,y}} \cdot \quad i \in \mathbb{Z}_0.$$  

(2.6)

3 Translation Invariant Measures

The problem of the finding of the general form of solutions of the equation (2.6) seems to be very difficult. In this subsection, we consider translation-invariant solutions, i.e., $z_{x} = z = (z_i)_{i \in \mathbb{Z}_0}$, with $z_i \in \mathbb{R}_+$. In this case the equation (2.6) has the following form

$$z_i = \lambda_i \cdot \left( \frac{1}{1 + \sum_{j \in \mathbb{Z}_0} z_j} \right)^k, \quad i \in \mathbb{Z}_0.$$  

(3.1)

Here $\lambda_i > 0$, $z_i > 0$.

**Lemma 1** Let $k \geq 2$. If there is a positive solution $\{z_j\}_{j \in \mathbb{Z}_0}$ of the system of equations (3.1) for some sequence of parameters $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ then series $\sum_{j \in \mathbb{Z}_0} z_j$ and $\sum_{j \in \mathbb{Z}_0} \lambda_j$ obtained respectively from $\{z_i\}_{i \in \mathbb{Z}_0}$ and $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ converge.

**Proof** Let $\{z_j\}_{j \in \mathbb{Z}_0}$ be a solution of the system of equations (3.1). We assume that the series $\sum_{j \in \mathbb{Z}_0} z_j$ diverges. Then, since $z_j > 0$, it is obvious that $\sum_{j \in \mathbb{Z}_0} z_j = +\infty$. Hence due to (3.1) we get $z_i = 0$, $i \in \mathbb{Z}_0$, i.e., $\sum_{i \in \mathbb{Z}_0} z_i < +\infty$. This is a contradiction. So under the conditions of lemma the series $\sum_{j \in \mathbb{Z}_0} z_j$ converges.

Let $\sum_{j \in \mathbb{Z}_0} z_j = A$. Then from (3.1) we obtain $z_i = \lambda_i \cdot \left( \frac{1}{1 + A} \right)^k$. Thence $\sum_{i \in \mathbb{Z}_0} \lambda_i = A(1 + A)^k$, i.e., the series $\sum_{i \in \mathbb{Z}_0} \lambda_i$ converges. Lemma is proved. $\square$

By Lemma 1 it follows that there is no positive solution of the system of equations (3.1) for which the series $\sum_{j \in \mathbb{Z}_0} z_j$ and $\sum_{j \in \mathbb{Z}_0} \lambda_j$ diverge, i.e., these conditions are necessary for the existence of a solution (3.1).

**Proposition 1** Let $k \geq 2$. If the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$ obtained from a sequence of parameters $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ converges then for the sequence $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ there exists a unique positive solution $\{z_j\}_{j \in \mathbb{Z}_0}$ of the system of equations (3.1).
Proof. Let the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$ converge and its sum is $\sum_{j \in \mathbb{Z}_0} \lambda_j = \Lambda$. We will prove that for the sequence $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ there is a unique solution of the system of equations (3.1). By Lemma 1 it follows that for the existence of a solution $\{z_j\}_{j \in \mathbb{Z}_0}$ of the system of equations (3.1) the convergence of the series $\sum_{j \in \mathbb{Z}_0} z_j$ is necessary.

Let $\sum_{j \in \mathbb{Z}_0} z_j = A$. Then due to (3.1) we get $z_i = \frac{\lambda_i}{(1 + A)^k}$, $i \in \mathbb{Z}_0$. Hence

$$\sum_{j \in \mathbb{Z}_0} z_j = \frac{\sum_{j \in \mathbb{Z}_0} \lambda_j}{(1 + A)^k}, \quad (3.2)$$

i.e.,

$$A(1 + A)^k - \Lambda = 0, \quad A > 0.$$ 

By the Descartes rule of signs, the last equation has only one positive root $A = A_0$. There are infinitely many sequences $\{z_j\}$ for which $\sum_{j \in \mathbb{Z}_0} z_j = A_0$. Among them the sequence $\{z_j\}$ for which there exists $\{\lambda_j\}$ such that $\{z_j\}$ satisfies (3.1) is unique. This follows from the equality $z_i = \frac{\lambda_i}{(1 + A_0)^k}$ because $\lambda_i$ are fixed and $A_0$ is unique. \qed

Remark 4. We note that the solution $\{z_j\}_{j \in \mathbb{Z}_0}$ in Proposition 1 is normalisable because the convergence of series $\sum_{j \in \mathbb{Z}_0} z_j^{k+1}$ follows from the convergence of $\sum_{j \in \mathbb{Z}_0} z_j$. Then by Remark 2 the Gibbs measure (denoted by $\mu_0$) corresponding to this solution exists.

3.1 Markov chain corresponding to the translation-invariant Gibbs measure

Below for the Gibbs measure $\mu_0$ (of the unique solution of the system of equations (3.1)) using the unique solution we define matrix $P$ of transition probabilities. Thus by a solution of the system of equations one defines a Gibbs measure and a Markov chain, which is called a Markov chain corresponding to the Gibbs measure. We will check the existence of a stationary distribution of the Markov chain corresponding to the measure $\mu_0$.

Consider the matrix $P$ of transition probabilities corresponding to the measure $\mu_0$:

$$P = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & p_{-2-2} & p_{-2-1} & p_{-20} & p_{-21} & p_{-22} & \cdots \\
\cdots & p_{-1-2} & p_{-1-1} & p_{-10} & p_{-11} & p_{-12} & \cdots \\
\cdots & p_{0-2} & p_{0-1} & p_{00} & p_{01} & p_{02} & \cdots \\
\cdots & p_{1-2} & p_{1-1} & p_{10} & p_{11} & p_{12} & \cdots \\
\cdots & p_{2-2} & p_{2-1} & p_{20} & p_{21} & p_{22} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}.$$
Here (depending on solution \(z = (z_i)_{i \in \mathbb{Z}_0}\))

\[
p(x)\sigma(y) = \frac{a_{\sigma(x)\sigma(y)} \lambda_{\sigma(y)} \xi_{\sigma(y)}}{\sum_{\sigma(y) \in \mathbb{Z}} a_{\sigma(x)\sigma(y)} \lambda_{\sigma(y)} \xi_{\sigma(y)}} \Rightarrow p_{ij} = \frac{a_{ij} \lambda_{j} z_{j}}{\sum_{l \in \mathbb{Z}} a_{il} \lambda_{i} z_{l}}.
\]

For the considered model, we have \(\sigma(x)\sigma(y) = 0\) for any nearest neighbors \(\{x, y\}\). If \(i \in \mathbb{Z}_0\) and \(j \in \mathbb{Z}_0\) then \(a_{ij} = 0\), \(a_{i0} = 1\) and \(a_{0j} = 1\). Hence

\[
p_{00} = \frac{\lambda_0 z_0}{\lambda_0 z_0 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l}},
\]

\[
p_{01} = \frac{\lambda_1 z_0}{\lambda_0 z_0 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l}}, \quad p_{0i} = \frac{\lambda_i z_i}{1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l}},
\]

\[
p_{10} = \frac{\lambda_1 z_0}{\lambda_1 z_0 + \sum_{l \in \mathbb{Z}_0} a_{1l} \lambda_{l} z_{l}} = 1, \quad p_{i0} = \frac{\lambda_i z_0}{\lambda_i z_0 + \sum_{l \in \mathbb{Z}_0} a_{il} \lambda_{l} z_{l}} = 1.
\]

Therefore \(\mathbb{P}\) has the following form (for \(z_0 = 1\)):

\[
\mathbb{P} = \left(\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\lambda_{z_2} & \lambda_{z_1} & & & & & \\
1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l} & 1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l} & 1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l} & 1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l} & 1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l} & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}\right).
\]

We consider the vector \(X = (\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots), \sum_{j \in \mathbb{Z}} x_j = 1\). If the system of equations \(X \cdot \mathbb{P} = X\) has a solution then there exists a stationary distribution of the Markov chain corresponding to the measure \(\mu_0\). So we solve the equation \(X \cdot \mathbb{P} = X\). We have

\[
X \cdot \mathbb{P} = \left(\begin{array}{ccccccc}
\cdots & \frac{x_0 \lambda_{z_2} - 1}{1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l}} & \cdots & \end{array}\right).
\]

From the equality \(X \cdot \mathbb{P} = X\) we get

\[
x_0 = 1 - \frac{x_0 \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l}}{1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l}}, \quad x_j = \frac{x_0 \lambda_j z_j}{1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l}}, \quad j \in \mathbb{Z}_0.
\]

From the first equality of (3.3) we find \(x_0\):

\[
x_0 = \frac{1 + \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l}}{1 + 2 \sum_{l \in \mathbb{Z}_0} \lambda_{l} z_{l}}.
\]
Using the expression for $x_0$ from the second equality of (3.3) we find $x_j$:

$$x_j = \frac{\lambda_j z_j}{1 + 2 \sum_{l \in \mathbb{Z}_0} \lambda_l z_l}, \quad j \in \mathbb{Z}_0.$$  

It is easy to see that the resulting vector is stochastic:

$$\sum_{j \in \mathbb{Z}_0} x_j = \frac{\sum_{l \in \mathbb{Z}_0} \lambda_l z_l}{1 + 2 \sum_{l \in \mathbb{Z}_0} \lambda_l z_l} + \frac{1 + \sum_{l \in \mathbb{Z}_0} \lambda_l z_l}{1 + 2 \sum_{l \in \mathbb{Z}_0} \lambda_l z_l} = 1.$$  

Hence there exists a stationary distribution of the Markov chain corresponding to the measure $\mu_0$.

Due to the uniqueness of the stationary distribution from Theorem 2 in [31], (p.612) we get

**Corollary 1** In the set of states $\mathbb{Z}$ of a Markov chain with the transition probabilities matrix $P$, there is exactly one positive recurrent class of essential communicating states (for definitions, see Chapter VIII in [31]).

Summarizing we the following

**Theorem 1** Let $k \geq 2$. Then for the HC model with a countable set of states (corresponding to the graph from Fig. 1) the following statements are true:

1. If the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$ obtained from a sequence of parameters $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ converges then there exists a unique translation-invariant Gibbs measure.
2. If the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$ diverges there is no translation-invariant Gibbs measure.

Now we give examples to illustrate Theorem 1. The Example 1 (and Example 4) is chosen just to satisfy conditions of theorem. But remaining examples are related to very known Poisson and geometric distributions.

**Example 1** Let $k = 2$. Find $\{z_j\}_{j \in \mathbb{Z}_0}$ satisfying the system of equations (3.1) for the sequence

$$\lambda_j = \begin{cases} 
\frac{9}{4(4j-3)(4j-1)}, & j \in \mathbb{N}, \\
\frac{9}{4(4j-1)(4j+1)}, & j \in \mathbb{Z}_0 \setminus \mathbb{N}.
\end{cases}$$

**Solution** First we find the sum of the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$.

$$\Lambda = \sum_{j \in \mathbb{Z}_0} \lambda_j = \frac{9}{4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{9}{8}.$$  

For $k = 2$ and $\Lambda = \frac{9}{8}$ from the equality $\Lambda = A(1+A)^2$ we find $A_0 = \frac{1}{2}$. From the equality $z_i = \frac{\lambda_i}{(1+A_0)^2}$ we find terms of the series:

$$z_n = \frac{1}{(4n-3)(4n-1)}, \quad z_{-n} = \frac{1}{(4n-1)(4n+1)}.$$
Hence the series $\sum_{j \in \mathbb{Z}_0} z_j$ has the following form:

$$
\sum_{j \in \mathbb{Z}_0} z_j = \sum_{n=-1}^{-\infty} \frac{1}{(4n-1)(4n+1)} + \sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-1)} = \frac{1}{2}.
$$

**Example 2** For $k = 2$ and for a sequence of parameters given by the Poisson distribution

$$
\lambda_j = \begin{cases}
\frac{2.4^j}{j!} \cdot e^{-2.4}, & j \in \mathbb{N}, \\
\frac{8^{-j}}{|j|!} \cdot e^{-8}, & j \in \mathbb{Z}_0 \setminus \mathbb{N},
\end{cases}
$$

find a series $\sum_{j \in \mathbb{Z}_0} z_j$ for which $\{z_j\}_{j \in \mathbb{Z}_0}$ is the solution of the system of equations (3.1).

**Solution** First we find the sum of the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$.

$$
\Lambda = \sum_{j \in \mathbb{Z}_0} \lambda_j = \sum_{j=1}^{\infty} \frac{2.4^j}{j!} \cdot e^{-2.4} + \sum_{j=1}^{\infty} \frac{8^j}{j!} \cdot e^{-8} = \Pi(2.4) + \Pi(8) = 0.9003 + 0.9997 = 1.9.
$$

For $k = 2$ and $\Lambda = 1.9$ from the equality $\Lambda = A(1 + A)^2$ we find $A_0 \approx 0.676223$.

We define terms of the series from the equality $z_i = \frac{\lambda_i}{(1 + A_0)^2}$:

$$
z_n = \frac{2.4^n}{e^{2.4n}} \cdot \frac{1}{(1 + A_0)^2}, \quad z_{-n} = \frac{8^n}{e^{8n}} \cdot \frac{1}{(1 + A_0)^2}.
$$

Hence the desired series $\sum_{j \in \mathbb{Z}_0} z_j$ has the following form:

$$
\sum_{j \in \mathbb{Z}_0} z_j = \frac{1}{(1 + A_0)^2} \cdot \sum_{n=-1}^{-\infty} \frac{8^{-n}}{|n|!} + \frac{1}{(1 + A_0)^2} \cdot \sum_{n=1}^{\infty} \frac{2.4^n}{e^{2.4n}} = A_0.
$$

**Example 3** Let $k = 2$. For a sequence of parameters given by a geometric distribution:

$$
\lambda_j = \begin{cases}
\alpha(1 - \alpha)^j, & \alpha \in (0; 1), \quad j \in \mathbb{N}, \\
\beta(1 - \beta)^{-j}, & \beta \in (0; 1), \quad \alpha + \beta = 0.875, \quad j \in \mathbb{Z}_0 \setminus \mathbb{N},
\end{cases}
$$

find a series $\sum_{j \in \mathbb{Z}_0} z_j$ for which $\{z_j\}_{j \in \mathbb{Z}_0}$ is the solution of the system of equations (3.1).

**Solution** First we find the sum of the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$.

$$
\Lambda = \sum_{j \in \mathbb{Z}_0} \lambda_j = \sum_{j=1}^{\infty} \alpha(1 - \alpha)^j + \sum_{j=1}^{\infty} \beta(1 - \beta)^j = 1 - \alpha + 1 - \beta = 2 - (\alpha + \beta) = 1.125.
$$
For $k = 2$ and $\Lambda = 1.125$ from the equality $\Lambda = A(1 + A)^2$ we find $A_0 = 0.5$. From the equality $z_i = \frac{\lambda_i}{(1 + \Lambda A_0)^2}$ we define terms of the series:

$$z_n = \frac{4\alpha(1 - \alpha)^n}{9}, \quad z_{-n} = \frac{4\beta(1 - \beta)^n}{9}.$$

So the desired series $\sum_{j \in \mathbb{Z}_0} z_j$ has the following form:

$$\sum_{j \in \mathbb{Z}_0} z_j = -\sum_{n=-1}^{\infty} \frac{4\beta(1 - \beta)^{-n}}{9} + \sum_{n=1}^{\infty} \frac{4\alpha(1 - \alpha)^n}{9} = \frac{1}{2}.$$

### 4 Periodic Gibbs Measures

It is known that there exists one-to-one correspondence between the set $V$ of vertices of a Cayley tree of order $k \geq 1$ and the group $G_k$ that is the free product of $k + 1$ cyclic groups of second order with the corresponding generators $a_1, a_2, \ldots, a_{k+1}$. Therefore, the set $V$ can be identified with the set $G_k$.

Let $G_k/\hat{G}_k = \{H_1, \ldots, H_r\}$ be the quotient group, where $\hat{G}_k$ is a normal subgroup of index $r \geq 1$.

**Definition 2** The set of vectors $z = \{z_x, x \in G_k\}$ is said to be $\hat{G}_k$-periodic if $z_{xy} = z_x$ for all $\forall x \in G_k, y \in \hat{G}_k$.

$G_k$-periodic sets are said to be translation-invariant.

**Definition 3** A measure $\mu$ is said to be $\hat{G}_k$-periodic if it corresponds to the $\hat{G}_k$-periodic set of vectors $z$.

Let $G^{(2)}_k$ be the subgroup of $G_k$ consisting the words of even length.

Consider the $G^{(2)}_k$-periodic Gibbs measures that correspond to set of vectors $z = \{z_x \in \mathbb{R}^\infty_+: x \in G_k\}$ of the form

$$z_x = \begin{cases} z, & \text{if } |x| \text{ is even}, \\ \tilde{z}, & \text{if } |x| \text{ is odd}. \end{cases}$$

Here $z = (\ldots, z_{-1}, z_0, z_1, \ldots), \tilde{z} = (\ldots, \tilde{z}_{-1}, \tilde{z}_0, \tilde{z}_1, \ldots)$.

Then due to (2.6) for $G^{(2)}_k$-periodic Gibbs measures we have:

$$\begin{align*}
    z_i &= \lambda_i \cdot \left(1 + \sum_{j \in \mathbb{Z}_0} \tilde{z}_j\right)^{-k}, \quad i \in \mathbb{Z}_0, \\
    \tilde{z}_i &= \lambda_i \cdot \left(1 + \sum_{j \in \mathbb{Z}_0} z_j\right)^{-k}, \quad i \in \mathbb{Z}_0.
\end{align*}$$

**Remark 5** We note that the solution $(z, \tilde{z})$ (i.e., $z = \tilde{z}$) in (4.1) corresponds to the unique translation-invariant Gibbs measure. Therefore, we are interested in solutions of the form $(z, \tilde{z}), z \neq \tilde{z}$.
Lemma 2 Let $k \geq 2$. If there is a positive solution $(z_j, \tilde{z}_j), \ j \in \mathbb{Z}_0$ of the system of equations (4.1) for some sequence of parameters $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ then the series $\sum_{j \in \mathbb{Z}_0} z_j$, $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j$ and $\sum_{j \in \mathbb{Z}_0} \lambda_j$ converge.

Proof Let $k \geq 2$ and there is a positive solution $(z_j, \tilde{z}_j), \ j \in \mathbb{Z}_0$ of the system of equations (4.1). Let us first show that the series $\sum_{j \in \mathbb{Z}_0} z_j$ and $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j$ converge. Assume the opposite, let one of the series, for example, the series $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j$ diverge, i.e., $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j = +\infty$. Then from the second equation of (4.1) we obtain $\tilde{z}_j = 0$, i.e., the system of equations (4.1) has no positive solutions. In the case when both series $\sum_{j \in \mathbb{Z}_0} z_j, \sum_{j \in \mathbb{Z}_0} \tilde{z}_j$ converge, it is obvious that the system of equations (4.1) has no solutions. Hence if there is a solution (4.1) then the series $\sum_{j \in \mathbb{Z}_0} z_j$ and $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j$ converge.

Now let is prove that the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$ converge. For this we assume $\sum_{j \in \mathbb{Z}_0} z_j = A$ and $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j = B$. Then from (4.1) we obtain $\sum_{i \in \mathbb{Z}_0} \lambda_i = A(1 + B)^k$. Lemma is proved. \qed

Remark 6 It follows from Lemma 2 that if one of the series $\sum_{j \in \mathbb{Z}_0} z_j, \sum_{j \in \mathbb{Z}_0} \tilde{z}_j$ and $\sum_{j \in \mathbb{Z}_0} \lambda_j$ diverges, then the system of equations (4.1) does not have a positive solution.

Let $\sum_{j \in \mathbb{Z}_0} z_j = A$ and $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j = B$. Then from (4.1) we have

$$z_i = \frac{\lambda_i}{(1 + B)^k}, \quad \tilde{z}_i = \frac{\lambda_i}{(1 + A)^k}.$$ 

Consequently,

$$\sum_{j \in \mathbb{Z}_0} z_j = \frac{\sum_{j \in \mathbb{Z}_0} \lambda_j}{(1 + B)^k}, \quad \sum_{j \in \mathbb{Z}_0} \tilde{z}_j = \frac{\sum_{j \in \mathbb{Z}_0} \lambda_j}{(1 + A)^k}$$

or

$$A(1 + B)^k = \sum_{j \in \mathbb{Z}_0} \lambda_j, \quad B(1 + A)^k = \sum_{j \in \mathbb{Z}_0} \lambda_j. \quad (4.2)$$

Proposition 2 Let $k \geq 2$. If the series $\sum_{j \in \mathbb{Z}_0} z_j$ converge and its sum is $\sum_{j \in \mathbb{Z}_0} z_j = A \neq \frac{1}{k-1}$. Then there exists a number $B$ which is the sum of a unique series: $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j$ $(z \neq \tilde{z})$ under the condition $A(1 + B)^k = B(1 + A)^k$ and there is a unique sequence of parameters $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ such that $(z, \tilde{z})$ and $(\tilde{z}, z)$ are solutions of (4.1), where $z = \{z_j\}_{j \in \mathbb{Z}_0}, \tilde{z} = \{\tilde{z}_j\}_{j \in \mathbb{Z}_0}$.

Proof Let $\sum_{j \in \mathbb{Z}_0} z_j = A$. First we prove the existence of the number $B$ under the conditions $A(1 + B)^k = B(1 + A)^k, z \neq \tilde{z}, (A \neq B)$ and $A \neq \frac{1}{k-1}$.

We note that for $A = B$ by (4.1) it follows that $z_j = \tilde{z}_j$ for any $j \in \mathbb{Z}_0$, i.e., the solution of this form corresponds to the translation-invariant Gibbs measure. Therefore, we will consider the case $A \neq B.$
In the equality

$$A(1 + B)^k - B(1 + A)^k = 0 \quad (4.3)$$

we introduce the notation $B = x, A = y$ and (for a fixed $y$) consider the following function:

$$f(x) = y(1 + x)^k - x(1 + y)^k.$$  

It is clear that $f(y) = 0$, i.e., $x = y$ is the root of the equation $f(x) = 0$.

We consider the case $x \neq y$. We rewrite the function $f(x)$ as follows:

$$f(x) = y(x^k + kx^{k-1} + \frac{k(k - 1)}{2}x^{k-2} + \ldots + \frac{k(k - 1)}{2}x^2) - ((1 + y)^k - ky)x + y.$$  

From the Bernoulli inequality we have $(1 + y)^k > ky, \; (y > 0)$. Then on the RHS of the last equality the signs change twice, i.e., by the Descartes rule of signs, the equation $f(x) = 0$ has two positive root or this equation has no positive solutions at all. But we have the solution $x = y$. So, the equation $f(x) = 0$ has one more positive solution different from $x = y$.

Next we check the multiplicity of the root $x = y$. For this we use the theorem on zeros of the holomorphic function, i.e., $x = y$ is not a multiple root of the equation $f(x) = 0$, if $f(y) = 0$ and $f'(y) \neq 0$.

We have $f(y) = 0$ and $f'(x) = ky(1 + x)^{k-1} - (1 + y)^k$. Therefore, from

$$f'(y) = (1 + y)^{k-1}(k - 1)(y - 1) = 0$$

we get

$$y = \frac{1}{k - 1}.$$  

Hence $x = y$ is not a multiple root, if $y \neq \frac{1}{k - 1}$.

Let $\widetilde{B}$ be a solution of $(4.3)$ different from $A$. Then the existence and uniqueness of $\lambda_i$ follows from the equality $\lambda_i = z_i(1 + \widetilde{B})^k, \; i \in \mathbb{Z}_0$.

There are infinitely many sequences $\{\tilde{z}_j\}_{j \in \mathbb{Z}_0}$ for which $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j = \widetilde{B}$. Among them the sequence $\{\tilde{z}_j\}_{j \in \mathbb{Z}_0}$ for which there exists $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ such that $(\tilde{z}_j, j \in \mathbb{Z}_0)$ satisfies $(4.1)$ is unique. This follows from the equality $\tilde{z}_i = \frac{\lambda_i}{(1 + A)^k}$.

From the above it follows that there is a sequence of parameters $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ for which $(z, \tilde{z})$ is the solution of $(4.1)$, where $z = \{z_j\}_{j \in \mathbb{Z}_0}$ and $\tilde{z} = \{\tilde{z}_j\}_{j \in \mathbb{Z}_0}$. We note that due to the symmetry $(\tilde{z}, z)$ is also a solution of $(4.1)$. The pf is complete. \hfill $\square$

Now let us determine under what conditions on the parameters $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ there are solutions $(z, \tilde{z})$ and $(\tilde{z}, z)$. We introduce the following definition.

**Definition 4** [12] A twice continuously differentiable function $f : [0, \infty) \mapsto [0, \infty)$ is said to be $S$-shaped if it has the following properties:
There exists \( \tilde{x} \in (0, \infty) \) such that the derivative \( f' \) is monotone increasing in the interval \((0, \tilde{x})\) and monotone decreasing in the interval \((\tilde{x}, \infty)\); in other words, \( \tilde{x} \) satisfies \( f''(\tilde{x}) = 0 \) and is the unique inflection point of \( f(x) \).

It is known [12] that any \( S \)-shaped function has at most three fixed points in the interval \([0, \infty)\).

**Lemma 3** [18] Let \( f : [0, 1] \rightarrow [0, 1] \) be a continuous function with a fixed point \( \xi \in (0, 1) \). We assume that \( f \) is differentiable at \( \xi \) and \( f'(\xi) < -1 \). Then there exist points \( x_0 \) and \( x_1 \), \( 0 \leq x_0 < \xi < x_1 < 1 \), such that \( f(x_0) = x_1 \) and \( f(x_1) = x_0 \).

We rewrite the system of equations (4.2):

\[
\begin{align*}
A &= f(B); \\
B &= f(A). 
\end{align*}
\]  

(4.4)

Here \( f(x) = \frac{\Lambda}{(1+x)^k} \).

**Remark 7** The system of equations (4.4) is equivalent to the equation for 2-periodic points of the function \( f \). Such systems of equations (with different functions) already studied for description of periodic Gibbs measures for Ising, Potts and HC models on Cayley trees (see corresponding chapters in [28] and [29]). We note that in case of uniqueness of TIGM, the 2-periodic points are sufficient to prove the existence of a phase transition.

The system of equations (4.4) is well studied in [12] (sec. 2.2, p. 904) and [22] (sec. 5.2, p. 153). It is shown that the function \( h(x) = f(f(x)) \) is an \( S \)-shaped function. The following properties of the function \( h(x) \) are obvious:

- \( h(x) \) is an \( S \)-shaped function with \( h(0) = \frac{\Lambda}{(1+\Lambda)^k} \) and \( \sup_x h(x) = \Lambda \).
- \( f(x) \) has a unique fixed point, \( x_0 \), which is also a fixed point of \( h(x) \).
- There exists \( \Lambda_{\text{cr}} > 0 \) such that if \( \Lambda \leq \Lambda_{\text{cr}} \) then \( h'(x) \leq 1 \) for any \( x \geq 0 \) and \( x_0 \) is the only fixed point for \( h(x) \).
- If \( \Lambda > \Lambda_{\text{cr}} \), then \( h(x) \) has three fixed points \( x_1 < x_0 < x_2 \), where \( f(x_1) = x_2 \) and \( f(x_2) = x_1 \). Moreover \( h'(x_0) > 1 \), \( h'(x) < 1 \) for \( x \in [0; \Lambda_{\text{cr}}] \cup [x_2; \infty) \) and the three fixed points converge to \( x_0(\Lambda_{\text{cr}}) \) as \( \Lambda \rightarrow \Lambda_{\text{cr}} \).

Since \( h'(x_0) = (f'(x_0))^2 \), it is easy to see that \( x_0 \) is the unique fixed point of the function \( h(x) \) if and only if \( f'(x_0) \geq -1 \).

Let \( A_0 \) be the unique solution of the equation \( A = \frac{\Lambda}{(1+A)^k} \). We calculate the derivative \( f'(A_0) \):

\[
\begin{align*}
   f'(A) &= -\frac{k\Lambda}{(1+A)^k+1}, \quad f'(A_0) = -\frac{kA_0}{1+A_0}.
\end{align*}
\]
We solve the inequality $f'(A_0) \geq -1$, and its solution has the form $A_0 \leq \frac{1}{k-1}$. Then from $A_0 = \frac{\Lambda}{(1+A_0)^k}$ we obtain that the equation $h(x) = x$ has only one fixed point for

$$\Lambda \leq \frac{k^k}{(k-1)^{k+1}} = \Lambda_{cr}(k).$$

From the inequality $f'(A_0) < -1$ we can get $\Lambda > \Lambda_{cr}$. Then by Lemma 3, there are at least three fixed points. On the other side under this condition by the property of an S-shaped function we have at most three fixed points for the function $h(x)$. Hence there are exactly three fixed points of the equation $h(x) = x$ for $\Lambda > \Lambda_{cr}$.

So the system of equations (4.4) has a unique solution of the form $(A; A)$, i.e., $A = B$ for $\Lambda \leq \Lambda_{cr}$, and for $\Lambda > \Lambda_{cr}$ it has two positive solutions $(A_0; B_0)$ and $(B_0; A_0)$ besides the solution $(A, A)$.

**Remark 8** Solutions of the system of equations (4.1) corresponding to $G^{(2)}_k$-periodic Gibbs measures are sometimes called two-periodic. Similarly to work [14] it is easy to obtain that two-periodic solutions $(z, \tilde{z})$ and $(\tilde{z}, z)$ of the system of equations (4.1) are normalizable if the series $\sum_{i \in \mathbb{Z}} z_i^{k+1}$ and $\sum_{i \in \mathbb{Z}} \tilde{z}_i^{k+1}$ converge. These series by virtue of Lemma 2 (a necessary condition for the existence of a solution), converge. Then in the case of two-periodic solutions there exist Gibbs measures $\mu_1$ and $\mu_2$ corresponding to the solutions $(z, \tilde{z})$ and $(\tilde{z}, z)$, respectively.

### 4.1 Markov chain corresponding to the periodic Gibbs measure

Similarly to the case of the translation-invariant Gibbs measure, we check the existence of a stationary distribution of the Markov chain corresponding to the measures $\mu_1$ and $\mu_2$. Using the method from [27] ($\mathbb{P} = \mathbb{P}_{\mu_1} \cdot \mathbb{P}_{\mu_2}$) we construct the probability transition matrix $\mathbb{P}$ corresponding to the measure $\mu_i$ ($i = 1, 2$):

$$\mathbb{P} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \vdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \times$$

$$\begin{pmatrix} \lambda_{-2z-2} & \lambda_{-1z-1} & \frac{1}{\nu_1} & \lambda_{1z1} & \lambda_{2z2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$
It is easy to see that the resulting vector is stochastic: 

$$
\sum_{j \in \mathbb{Z}} x_j = \frac{\sum_{l \in \mathbb{Z}_0} \lambda_j z_l}{1 + \sum_{l \in \mathbb{Z}_0} \lambda_j z_l + \sum_{l \in \mathbb{Z}_0} \lambda_l z_l} + \frac{1}{1 + \sum_{l \in \mathbb{Z}_0} \lambda_j z_l + \sum_{l \in \mathbb{Z}_0} \lambda_l z_l} = 1.
$$
Hence there exists a stationary distribution of the Markov chain corresponding to the measure $\mu_i$ ($i = 1, 2$).

Thus, the following theorem is true.

**Theorem 2** Let $k \geq 2$ and $\Lambda_{\text{cr}}(k) = \frac{k}{(k-1)^{k-1}}$. Then for the HC model with a countable set of states (corresponding to the graph from Fig. 1) the following statements are true:

1. If the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$ obtained from a sequence of parameters $\{\lambda_j\}_{j \in \mathbb{Z}_0}$ converges and its sum is $\sum_{j \in \mathbb{Z}_0} \lambda_j = \Lambda$, then for $0 < \Lambda \leq \Lambda_{\text{cr}}$ there exists unique $G_k^{(2)}$-periodic Gibbs measure $\mu_k$ that is translation-invariant, and for $\Lambda > \Lambda_{\text{cr}}$ there are exactly three $G_k^{(2)}$-periodic Gibbs measures $\mu_0, \mu_1, \mu_2$, where measures $\mu_1$ and $\mu_2$ are $G_k^{(2)}$-periodic (non-translation-invariant) Gibbs measures.

2. If the series $\sum_{j \in \mathbb{Z}_0} \lambda_j$ diverges then there is no $G_k^{(2)}$-periodic Gibbs measure.

Let us give several remarks related to some comparisons between classical HC-model and models with countable set of spin values.

**Remark 9** For the classic HC-model with spin values 0, 1 and the activity parameter $\lambda > 0$ it is well known (see [4], [33] and the references therein) that the occurrence of multiple Gibbs measures is increasing in the parameter $\lambda$ (i.e., monotonic behavior), namely, the model has a unique Gibbs measure if $\lambda \leq \lambda_{\text{cr}} := \frac{1}{k-1} \left( \frac{k}{k-1} \right)^{k-1}$ and multiple Gibbs measures if $\lambda > \lambda_{\text{cr}}$. Since our model is defined by a sequence of parameters $\{\lambda_j\}_{j \in \mathbb{Z}_0}$, by Theorem 2, we say that similar monotonic behavior holds with respect to parameter $\Lambda = \sum_{j \in \mathbb{Z}_0} \lambda_j$.

In [4] it was shown that the monotonicity fails for some general graphs (different from the Cayley tree). We conjecture that such a result should also be true for our HC-model with countable set of spin values. This will be proved in a separate paper.

**Remark 10** Usually checking extremality of a Gibbs measure is a difficult problem. But for finite set of spin values there are some methods (see [21] for details):

(a) Kesten-Stigum conditions of non-extremality and
(b) Martinelli-Sinclair-Weitz condition of extremality.

Both these conditions are given on the finite-dimensional transition matrix (Markov chain) corresponding to the Gibbs measure. For our countable spin values case the matrix becomes infinite-dimensional (see matrix $P$ given above) and we do not know how to extend the conditions a) and b) to the infinite-dimensional case. Therefore extremality of Gibbs measure $\mu_i, i = 0, 1, 2$ (mentioned in Theorem 2) is an open problem. We conjecture that $G_k^{(2)}$-periodic (non-translation invariant) Gibbs measures are extreme in the region where they exist.

**Remark 11** To obtain non-periodic Gibbs measures one can look for weakly periodic Gibbs measures. In [19] this problem is solved for the case of classical HC-model. One can study weakly periodic measures of our model too. But in this case the problem will be more difficult (compared with [19]), because the systems which are needed to solve will be infinite-dimensional. A separate paper will be devoted to this problem and to extension of Bleher-Ganikhodjaev’s construction (see [1]) of non-weakly periodic Gibbs measures.
By virtue of (4.1) from equalities 
\[ z_i = \frac{\lambda_i}{(1 + B_0)^k} \quad \text{and} \quad \tilde{z}_i = \frac{\lambda_i}{(1 + A_0)^k} \]
we can get
\[ \sum_{j \in \mathbb{Z}_0} \frac{\lambda_j}{(1 + B_0)^k} = A_0, \quad \sum_{j \in \mathbb{Z}_0} \frac{\lambda_j}{(1 + A_0)^k} = B_0. \]

**Remark 12** We have \( \Lambda_{cr}(2) = 4 \) for \( k = 2 \) and the sums of the series \( \sum_{j \in \mathbb{Z}_0} z_j \) and \( \sum_{j \in \mathbb{Z}_0} \tilde{z}_j \) obtained from the solution \((z, \tilde{z})\) have the following sum
\[ A_0(\Lambda) = \frac{\Lambda - 2 \pm \sqrt{\Lambda(\Lambda - 4)}}{2}, \quad B_0(\Lambda) = \frac{\Lambda - 2 \mp \sqrt{\Lambda(\Lambda - 4)}}{2}. \]  

(4.6)

**Example 4** Let \( k = 2 \). Find solutions of the system of equations (4.1) corresponding to \( G^{(2)}_k \)-periodic (not translation-invariant) Gibbs measures for the sequence
\[ \lambda_j = \begin{cases} 
\frac{9}{(4j - 3)(4j - 1)}, & j \in \mathbb{N}, \\
\frac{9}{(4j - 1)(4j + 1)}, & j \in \mathbb{Z}_0 \setminus \mathbb{N}.
\end{cases} \]

**Solution** First we find the sum of the series \( \sum_{j \in \mathbb{Z}_0} \lambda_j \):
\[ \Lambda = \sum_{j \in \mathbb{Z}_0} \lambda_j = 9 \cdot \sum_{n=1}^{\infty} \frac{1}{(2n - 1)(2n + 1)} = \frac{9}{2}. \]

There exist solutions (with condition \( A \neq B \)) of the system of equations (4.1) corresponding to \( G^{(2)}_k \)-periodic (not translation-invariant) Gibbs measures for \( k = 2 \) and \( \Lambda > \Lambda_{cr}(2) = 4 \). Using the formulas (4.6) we solve the following system of equations
\[ \begin{cases} 
A(1 + B)^2 = \frac{9}{2}; \\
B(1 + A)^2 = \frac{9}{2}.
\end{cases} \]

Solutions have the following forms: \( \left( \frac{1}{2}, 2 \right) \) and \( \left( 2, \frac{1}{2} \right) \).

Let \( A_0 = \frac{1}{2} \) and \( B_0 = 2 \). In this case, we determine the terms of the series from the equality \( z_i = \frac{\lambda_i}{(1 + B_0)^2} \): 
\[ z_x = \left( \ldots, \frac{1}{(4n - 1)(4n + 1)}, \ldots, \frac{1}{63}, \frac{1}{15}, \frac{1}{5}, \frac{1}{3}, \frac{1}{35}, \ldots, \frac{1}{(4n - 3)(4n - 1)}, \ldots \right) \]

Hence, the desired series \( \sum_{j \in \mathbb{Z}_0} z_j \) has the following form
\[ \sum_{j \in \mathbb{Z}_0} z_j = \sum_{n=-1}^{-\infty} \frac{1}{(4n + 1)(4n - 1)} + \sum_{n=1}^{\infty} \frac{1}{(4n - 3)(4n - 1)} = \frac{1}{2}. \]
We determine the values of $\tilde{z}_j$ from the equality $\tilde{z}_j = \frac{\lambda_j}{(1 + A_0)^j}$:

$$\tilde{z}_x = \left( \ldots, \frac{4}{(4n - 1)(4n + 1)}, \ldots, \frac{4}{63}, \frac{4}{15}, 1, \frac{4}{3}, \frac{4}{35}, \ldots, \frac{4}{(4n - 3)(4n - 1)}, \ldots \right)$$

From here the series $\sum_{j \in \mathbb{Z}_0} \tilde{z}_j$ has the form

$$\sum_{j \in \mathbb{Z}_0} \tilde{z}_j = -\infty \sum_{n = 1}^{\infty} \frac{4}{(4n + 1)(4n - 1)} + \sum_{n = 1}^{\infty} \frac{4}{(4n - 3)(4n - 1)} = 2.$$
Solutions have the following forms: \((\frac{1}{3}, 3)\) and \((3, \frac{1}{3})\).

Let \(A_0 = \frac{1}{3}\) and \(B_0 = 3\). In this case, we determine the terms of the series from the equality \(z_i = \frac{\lambda_i}{(1+B_0)^2}\). Hence, the desired series \(\sum_{j \in \mathbb{Z}_0} z_j\) has the following form:

\[
\sum_{j \in \mathbb{Z}_0} z_j = \sum_{j=1}^{\infty} \frac{\alpha(1-\alpha)^j}{4} + \sum_{j=-\infty}^{1} \frac{\beta(1-\beta)^{-j}}{4} = \frac{1}{3}.
\]

We determine the values of \(\tilde{z}_j\) from the equality \(\tilde{z}_j = \frac{\lambda_j}{(1+A_0)^2}\). From here the series \(\sum_{j \in \mathbb{Z}_0} \tilde{z}_j\) has the form:

\[
\sum_{j \in \mathbb{Z}_0} \tilde{z}_j = \sum_{j=1}^{\infty} \frac{9\alpha(1-\alpha)^j}{4} + \sum_{j=-\infty}^{1} \frac{\beta(1-\beta)^{-j}}{4} = 3.
\]

In the case \(A_0 = 3, B_0 = \frac{1}{3}\) we can similarly obtain as solutions

\[
\sum_{j \in \mathbb{Z}_0} z_j = \sum_{j=1}^{\infty} \frac{9\alpha(1-\alpha)^j}{4} + \sum_{j=-\infty}^{1} \frac{\beta(1-\beta)^{-j}}{4} = 3,
\]

\[
\sum_{j \in \mathbb{Z}_0} \tilde{z}_j = \sum_{j=1}^{\infty} \frac{\alpha(1-\alpha)^j}{4} + \sum_{j=-\infty}^{1} \frac{\beta(1-\beta)^{-j}}{4} = \frac{1}{3}.
\]

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Data Availability Statement The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest statement On behalf of all authors, the corresponding author (U.A.Rozikov) states that there is no conflict of interest.

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