CFAR multi-target detection based on non-central Chi-square distribution for FMCW

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Abstract. The results of synthesis and analysis of the effectiveness of the optimal location-based system for joint detection and estimation of informative parameters of quasi-determined radar signals with frequency modulated continuous wave are presented. A method has been developed for calculating the noise immunity indices of frequency modulated continuous wave radars and the detection threshold of signals in the quadrature channels of the optimal radar receiver based on off-center $\chi^2$-distribution for given technical characteristics of the location system, including the mode of operation for detecting and evaluating parameters in multi-target mode.

1. Introduction

For the modern “smart city” with the most comfortable and safe control of the movement of vehicles, especially in the autonomous (unmanned) mode, solving the problem of remote monitoring of the location and speed of surrounding objects in real time and with high resolution is of particular relevance. Today, the most effective systems for the remote determination of parameters such as radial velocity $V$ and target range $R$ of land vehicles and unmanned aerial vehicles are frequency-modulated continuous-wave microwave radars (FMCW) [1-4]. FMCW radars are also widely used in tasks of near active locations along with lidars [5], especially in conditions of a low level of reflected signal power and the presence of multiple reflection objects (targets).

The main indicators of the effectiveness of radar systems under the influence of interference include the probability of false alarms or type I error $\alpha$ (Type I error - false alarm ratio - FAR) and missed targets or type II error $\beta$ (Type II error - missed detection) [6]. The task of synthesizing the optimal radar receiver-detector is reduced to the task of testing statistical hypotheses [6], the solution of which is determined by the completeness of a priori statistical data. The choice of strategy and the development of an appropriate optimal algorithm for processing the input FMCW radar signal is reduced to the non-parametric task of detecting informative signals and evaluating parameters such as amplitudes, frequencies and phases of harmonic signals against the background of additive noise. A universal signal detection strategy is to compute sufficient statistics and compare them with a threshold. The statistical estimation of the amplitude, frequency, and phase of harmonic signals against additive Gaussian noises is based on the maximum likelihood method and uses the discrete-time Fourier transform (DFT), including the fast Fourier transform (FFT) and the calculation of the periodogram of the signal sample vector [7-9]. For this reason, signal processing algorithms by digital processors of modern FMCW vehicle radars are based on fast DFT, despite the difference in sounding signals used in location systems...
[1–4, 10–12].

The calculation of indicators α and β of the noise immunity of FMCW vehicle radars and the detection threshold was carried out by many authors, the results are presented in [13–19]. The calculation of sufficient statistics and the detection threshold was carried out by the authors of these works either under the assumption of an exponential distribution of a random signal level [13–17], the double exponential distribution of the Gumbel [18], or under the assumption of the Rayleigh-Rice distribution [19].

The exponential distribution, as a special case of centered χ²-distribution with two degrees of freedom, involves processing signals of quadrature channels that do not contain a nonrandom component. Thus, the use of this distribution is not correct if there is an negligible power level at the input of the radar reflected from the target. The task of detecting a signal against a background of noise under the Rayleigh-Rice distribution aims at calculating the statistical properties of a random variable equal to the square root of the periodogram at the frequency of the spectral channel, i.e. the square root of the squared module of the complex DFT of quadrature channel signals. To optimally estimate the distance, velocity, and effective scattering area (EPR) of a target, statistics of the periodogram itself at the frequency of the spectral channel are necessary [7–9].

In the above studies, the strategy of joint detection and estimation of parameters of echo signals in the quadrature channels of the optimal FMCW radar receiver, including in the mode of operation of multiple targets, was not considered. The purpose of this article is to synthesize and analyze the effectiveness of an optimal system for joint detection and estimation of parameters of location-based FMCW radar signals.

2. The mathematical model of the FMCW radar signal
To a first approximation, the mixer output signals \( u(t_k, k) \) and \( u_Q(t_k, k) \) in common-mode I and quadrature Q channels of the radar receiver intermediate frequency (IF) as a function of “fast” time \( t_k \) for each \( k \)-th frame of duration \( T \) and any of the \( m \) observed targets represent a harmonic signal [3,4] with amplitude \( U_m \), frequency \( f_m \) and phase \( \Phi_m \)

\[
f_m = \frac{\Delta f}{T} \tau_m + f_{1_m}; \quad \Phi_m = 2\pi(f_{1_m} T \cdot k + f_1 \tau_m) + \Delta \varphi_m(k),
\]

where \( f_1 \) is the initial frequency of the microwave chirp signal; \( \Delta f \) – frequency deviation during chirp; \( \tau_m \) – delay of the radar input radio signal, reflected from the \( m \)-th target, in relation to the probe signal of the transmitter; \( f_{1_m} \) – Doppler frequency shift when reflected from a moving target.

Additive noise \( n(t_k, k) \) is described by a model of white Gaussian noise with zero mean, power spectral density (PSD) equal to \( N_0/2 \) and dispersion (power in the receiver frequency band) equal to \( \sigma^2 \). It is assumed that the amplitude-frequency and phase-frequency characteristics of the quadrature channels are identical.

The IF signal parameters described by formula (1) depend on the type of the probing signal as a function of time. However, as a rule, FMCW radars use probing signals with linear frequency modulation [1–4, 10], although with a different method of modulating parameters such as the duration of an individual chirp \( T_k \), its frequency deviation \( \Delta f_k \), frequency offset \( \delta f \) in the frame sequence sounding signal. In this case, according to (1), the IF signals at the output of the mixer are quasi-harmonic signals with unknown (nonrandom) parameters: amplitudes, frequencies, and phases, the evaluation of which according to the strategy allows us to calculate the main parameters of the targeted targets: radial velocity \( V_m \), target range \( R_m(t) \) and the intensity of the reflected signal \( U_m^2 \) (target ESR). Under these assumptions, the signal model (1) is invariant in the class of probing signals with linear frequency modulation.

A feature of the synthesis problem under consideration is the one-step hypothesis testing task during observation time \( T \). This means that the duration of the optimal processing for detecting and evaluating signal parameters is equal to the duration of the current chirp \( T_k \), i.e. on each interval \( T_k \), a conclusion is
drawn about the presence of a useful signal or noise and, in the case of the presence of a signal, the determination of its parameters. In this sense, the mathematical model of the IF signal is independent of \( k \), although the estimates obtained during the estimation are a function of the “slow” time \( k \). In the future, for brevity, writing mathematical expressions, the dependence on \( k \) will be omitted. The mathematical model of the digital IF signal when it is processed in the digital radar signal processor can be represented as a complex sample vector \( Z \in \mathbb{C}^N \), the real part of which is the digital samples of the common-mode channel I signal, and the imaginary part is the digital samples of the quadrature channel signal Q in times

\[
\bar{Z} = \bar{X}_I + j \bar{Y}_Q; \quad t_i \in [0, T]
\]

\[
\bar{Z} = (z_0, z_1, \ldots, z_{N-1})^T; \quad \bar{X}_I = (x_0, x_1, \ldots, x_{N-1})^T; \quad \bar{Y}_Q = (y_0, y_1, \ldots, y_{N-1})^T;
\]

\[
z_i = z(t_i) = z(i \cdot \Delta t); \quad x_i = u_I(t_i) = u_I(i \cdot \Delta t); \quad y_i = u_Q(t_i) = u_Q(i \cdot \Delta t);
\]

where the sign “ * ” means the transposition of the vector; \( i \in [0, N - 1] \), and the sampling time interval (step) \( \Delta t = T/N \) is determined by the number of frequency samples \( N \) during the Fourier transform of the intermediate frequency signal and the bandwidth of the IF path of the radar receiver \( \Delta f_{R} = N/T \).

In the general case, the digital samples of the in-phase I signals and the quadrature channels Q are not connected by the Hilbert transform, as is assumed in [8]. The components \( s_I(t_i) \) and \( s_Q(t_i) \) of the informative narrowband signal are related by the Hilbert transform in a first approximation, for the components \( n_I(t_i) \) and \( n_Q(t_i) \) there is no such connection. The Fourier transform of the complex signal \( Z \), detects the ability to determine the removal or approximation of the target by the sign of the Doppler frequency.

The mathematical model of the received radar signal, defined by relations (1) and (2), allows us to develop a method for calculating the \( \alpha \) and \( \beta \) noise immunity indices of FMCW vehicle radars and the FAR detection threshold.

3. Detection Performance for FMCW Radar

According to the assumptions made and the mathematical model of the IF signal, the above-defined vector sample \( Z \) of volume (dimension) \( N \) belongs to the class of complex vector random variables, the real and imaginary parts of which have a joint multidimensional normal distribution. In this case, we can say that the random vector \( Z \) has a complex normal distribution \( \mathcal{N}_c(\mu, K_{zz}) \) [20], where \( \mu \in \mathbb{C}^N \) - component complex vector of mean values, and \( K_{zz} \) - Hermitian non-negative definite \( N \times N \) matrix

\[
\operatorname{Re} \mu_I = s_I(t_i); \quad K_{zz,il} = \frac{1}{2} m_I \left\{ (n_I(t_i) + j n_Q(t_i)) \cdot (n_I(t_i) - j n_Q(t_i)) \right\}; \quad \operatorname{Im} \mu_I = s_Q(t_i), \quad K_{zz} = \sigma_z^2 \cdot I,
\]

where \( m_I \{ \cdot \} \) – random averaging operator, and covariance matrix \( K_{zz} \) is diagonal, whose elements are different from 0 for \( i \neq l \), and, therefore, the components of the random vector \( Z \) are statistically independent, \( I \) – the identity matrix.

Consider a non-Bayesian strategy for simultaneously testing a simple hypothesis \( H_0 \) against a complex two-way alternative \( H_1 \) and estimates of nonrandom vector parameters of IF signal \( \theta \)

\[
H_0 : \mu = 0; \quad \sigma_z^2 \text{ - known;} \quad H_1 : \mu \neq 0; \quad \bar{\theta} = (U_m^2, f_m, \Phi_m)^T; \quad U_m, f_m, \Phi_m \text{ - unknown.}
\]

The radar picture of the radar displays the EPR, the observed targets and their location in speed-distance coordinates, and, therefore, the following parameters are subject to evaluation: signal power \( U_m^2 \), beat frequency \( f_m \) and phase \( \Phi_m \). We believe that it is not difficult for a hardware developer to a priori measure or calculate the level of the receiver noise \( \sigma_z^2 \) (or PSD noise for an analog signal). Otherwise, to estimate the unknown parameters \( \sigma_z^2 \) and \( \bar{\theta} \) it is productive to use the method of complete sufficient statistics [21, 22].
We consider that the estimated parameter vector \( \theta \) is nonrandom belonging to the set \( \Theta^{(3\times M)} \). In this case, for the synthesis of the optimal detector, it is advisable to use the generalized likelihood ratio criterion. According to this criterion, the optimal detection rule has the form

\[
\Lambda(\hat{z}) = \max \frac{w(\hat{z} \mid \hat{\theta}, \sigma^2, \mu \neq 0)}{w(\hat{z} \mid \hat{\theta}, \sigma^2, \mu = 0)} = \max \Lambda(\hat{z} \mid \hat{\theta}, \sigma^2) = \Lambda_M(\hat{z} \mid \hat{\theta}, \sigma^2) > \gamma ,
\]

i.e., statistics (5) represents the maximum value of the conditional likelihood ratio, the threshold value \( \gamma \) is selected according to the given probability of error of the first kind \( \alpha_0 \).

Relation (5) determines the method of joint detection and estimation of the parameters of the received signal. At the first stage of the method, the parameter is estimated \( \hat{\theta} \) as a value maximizing conditional probability density \( w(z \mid \theta, \sigma^2, \mu) \) in the entire area of parameter variation \( \theta \in \Theta^{(3\times M)} \). At the second stage, the found parameter value \( \hat{\theta} \) arrives at the threshold device and at the logic element that implements the conjunction function (coincidence scheme). The threshold device generates a conditional likelihood ratio \( \Lambda_M \). If \( \Lambda_M \) exceeds the threshold, then a decision is made in favor of an alternative to the presence of an informative signal, and the “true” command is sent to the match circuit and at its output we have an estimate of the parameter \( \hat{\theta} \). The detector of the joint detection and estimation system under consideration is optimal according to the maximum likelihood ratio criterion, and the meter is optimal in the sense that it forms the maximum likelihood estimate.

The maximum likelihood method leads to the following relations [7–9, 13] for an optimal estimate of the power level from the signal periodogram \( P_\ell(n \cdot \Delta f) \) in \( n \)-th FFT Spectral Channel:

\[
\max \Lambda(\hat{z} \mid \hat{\theta}, \sigma^2) = U^2_n = P^2_\ell(n \cdot \Delta f) = \left| \Re \{z(t)\} \right|^2 + \left| \Im \{z(t)\} \right|^2 = \frac{2}{N} \sum_{i=0}^{N-1} z(t_i) \exp(-j2\pi \frac{mi}{N})^2 = \frac{2}{N} \sum_{i=0}^{N-1} z(t_i) \exp(-j2\pi \frac{mi}{N})^2 = \left[ \Re \{z(t)\} \right]^2 + \left[ \Im \{z(t)\} \right]^2 = X^2_1 + X^2_2 ,
\]

where \( n = 0, \pm 1, \pm 2 \ldots N/2, \Delta f = 1/T \) – frequency sampling step, \( \Re \{ \cdot \} \) – direct FFT of a complex signal \( z(t) \). According to criterion (5), at the first stage of detecting a useful signal, sufficient statistics - power level \( U^2_n \), defined by relation (6) is compared with a threshold \( \gamma \). If it is decided that the spectral channel \( m \) contains the spectral component of the useful signal, then the coincidence circuit calculates an estimate of the beat frequency of the \( m \)-th target equal to \( \hat{f}_m = m \Delta f \). At the next stage, the remaining parameters are evaluated \( U^2_m \) and \( \Phi_m \) in accordance with the following ratios [7, 9]

\[
U^2_m = P_\ell(m \Delta f) = \frac{2}{N} \sum_{i=0}^{N-1} z(t_i) \exp(-j2\pi \frac{mi}{N})^2 = \hat{\Phi}_m = \arg \left\{ \Re \{z(t_i)\}_{j = 0} \right\} = \arg \left\{ \sum_{i=0}^{N-1} z(t_i) \exp(-j2\pi \frac{mi}{N}) \right\} ,
\]

where \( \arg \{ \cdot \} \) – function to calculate the argument of a complex number.

Random value distribution \( U^2_n \) as the sum of the squares of two independent Gaussian random variables with parameters \( N \chi^2(\mu_1, \sigma^2) \) and \( N \chi^2(\mu_2, \sigma^2) \) respectively, it is off-center \( \chi^2 \) – distribution with two degrees of freedom and noncentrality parameter

\[
\mu = \frac{1}{\sigma_0^2} (\mu_1 + \mu_2) = \frac{T_R}{N_0} S^2 [\Re \{W(t_L)\}^2]_{\hat{f}_m} = \frac{N}{2} \left[ \Re \{W(t_L)\}^2 \cdot \text{SNR}_Z \right] ; \mu = \frac{1}{2} \left[ \Re \{W(t_L)\}^2 \cdot \text{SNR}_1 \right] .
\]

Here \( \Re \{ W \} \) – Fourier transform of the window function at the point of the spectral channel, \( \text{SNR}_Z \) – the ratio of the power of the informative microwave signal to the integrated noise power in the entire working band of the receiver, \( \text{SNR}_1 \) – noise power reduced to the band of one spectral channel.
The probability density is described by the following relation [20]: \[ w(x, \mu) = \frac{1}{2} \exp \left( -\frac{x + \mu}{2} \right) \sum_{i=0}^{\infty} \frac{(\frac{x + \mu}{\sqrt{2}})^i}{(i!)^2} = \frac{1}{2} \exp \left( -\frac{x + \mu}{2} \right) I_0 \left( \sqrt{2}x \right), \quad x \geq 0. \] (8)

where \( I_0(\cdot) \) – modified zero-order Bessel function of a complex argument.

Distribution moments expectation \( m_1(\cdot) \) and variance \( M_2(\cdot) \)
\[
m_1(\xi) = \mu + 2, \quad M_2(\xi) = 4\mu + 4. \] (9)

In figure 1 a graph of an off-center \( \chi^2 \)-distribution with two degrees of freedom and an off-center parameter \( \mu = 0, 1, 7 \) is presented. For \( \mu = 0 \), the \( \chi^2 \)-distribution degenerates into an exponential distribution.

The calculation of the signal detection characteristic – the dependence of the probability of correct detection \( 1 - \beta \) on the signal-to-noise ratio SNR is carried out at a fixed probability of false alarm \( \alpha \). For a given value of the parameter \( \alpha \), the detection threshold was calculated using relation (8) for the probability distribution density \( w(x, \mu = 0) \). Then a numerical calculation of the power of the \( 1 - \beta \) algorithm was performed as a function of SNR taking into account the number of spectral channels. Figure 2 shows the family of FMCW radar signal detection characteristics for two parameter values \( \alpha = 10^{-2} \) and \( 10^{-4} \) for the case of 1024 FFT spectral channels. An analysis of the calculation results shows that for practical values of the probability of false alarm \( \alpha \) and the probability of correct detection of \( 1 - \beta \), the signal-to-noise ratio SNR should reach at least 17 dB, which determines the range of the vehicle’s FMCW radar at a fixed transmitter microwave power level.

**4. Conclusion**

The presented method for calculating the FMCW noise immunity indicators of a vehicle’s radar and the signal detection threshold based on the off-center \( \chi^2 \)-distribution allows us to estimate the value of the input signal-to-noise ratio for a given probability of correct detection of \( 1 - \beta \) and the probability of false alarm \( \alpha \). The developed processing algorithm solves the problem of joint detection and estimation of parameters of radar FMCW radar signals taking into account the modern architecture of the microwave receiver with quadrature channels and the further application of the FFT conversion to the intermediate frequency signal. For the values of the probability of false alarm and missed target...
α=β=10^4 and the number of FFT spectral channels equal to 1024, the signal-to-noise ratio SNR should reach at least 17 dB, which determines the range of the vehicle’s FMCW radar at a fixed power level Microwave signal transmitter.

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