Supersymmetry of the C-metric and the general Plebański-Demiański solution

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Abstract: We derive the necessary and sufficient conditions under which the general Plebański-Demiański (PD) solution of Einstein-Maxwell theory with a negative cosmological constant admits Killing spinors. We consider in detail two different scaling limits of the PD metric. The first of these limits removes the acceleration parameter, and leads to the Carter-Plebański solution. In this case, the integrability conditions for Killing spinors were obtained by Alonso-Alberca, Meessen and Ortín in hep-th/0003071, and we show that these are not only necessary, but also sufficient for the existence of Killing spinors. This fills also a gap in hep-th/9808097, where the integrability conditions for supersymmetry of the Kerr-Newman-AdS black hole were worked out, but the Killing spinor was not constructed explicitly. The second scaling limit eliminates the rotation parameter, and leads to the cosmological C-metric, which describes accelerated black holes in AdS. Also in this case, the supersymmetry conditions are obtained, and it is shown that they follow from the ones of the general PD solution by scaling the parameters appropriately. In all cases, we determine the three-dimensional base space that appears in the classification scheme of hep-th/0307022, and prove that for the 1/2-supersymmetric Reissner-Nordström-AdS spacetime, this base is unique. A Wick-rotation of our results leads to gravitational instantons that generalize the ones constructed recently by Martelli, Passias and Sparks in arXiv:12124618 to $\text{U}(1) \times \text{U}(1)$ symmetry. These instantons are shown to admit an integrable almost complex structure. Finally, our work may open the possibility to systematically construct generalizations of the PD metric that include scalar fields with a potential in matter-coupled gauged supergravity.

Keywords: Black Holes in String Theory, AdS-CFT Correspondence, Superstring Vacua.
1. Introduction

Gravitational backgrounds preserving supersymmetry in supergravity theories are central to the development of string/M-theory, flux compactifications and the AdS/CFT correspondence. Supersymmetric, or simply BPS, solutions are characterized by the presence of Killing spinors $\epsilon$ which are parallel with respect to the supercovariant derivative operator. These Killing spinors define preferred G-structures which provide algebraic and differential constraints on the bilinears constructed from $\epsilon$ [1, 2]. The classification program initiated in [3] made a substantial development and the G-structure enables us to constrain the metric, fluxes and other fields to obey a simpler set of equations [4–15] (see [24] for a recent review).

For 4-dimensional minimal $\mathcal{N} = 2$ ungauged supergravity, a complete list of BPS solutions was obtained [25]. When the vector field constructed as bilinear of a Killing spinor is timelike, it turned out that BPS solutions are completely specified by a complex harmonic function on the 3-dimensional base space $\mathbb{R}^3$. The BPS geometries in minimal $\mathcal{N} = 2$ gauged supergravity

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1 An alternative approach to classifying supergravity solutions consists in expressing spinors in terms of forms and using the gauge symmetry to transform them to a preferred representative of their orbit. This allows to directly solve the Killing spinor equations and goes under the name of spinorial geometry, cf. [16–23] for an incomplete list of references.
were classified by Caldarelli and one of the present authors in [12], and later studied in [18,26]. It was shown that the necessary and sufficient conditions for supersymmetry in the timelike class reduce to the solutions of differential equations on a curved 3-dimensional base space. The striking feature of gauged supergravity is that these differential equations are highly nonlinear, which causes a main difficulty in finding solutions contrary to the ungauged case. The properties of supersymmetric solutions in gauged supergravities are therefore far from understood.

Prior to these studies, Romans analyzed asymptotically AdS static BPS solutions by directly solving a Killing spinor equation [27]. Later on some rotating generalizations have been discussed in [28,29] by investigating the first integrability conditions for the Killing spinor equation. It should be worthwhile to emphasize that integrability conditions are merely the necessary conditions for supersymmetry [30]. Hence, in order to show rigorously that they are also sufficient, one needs to explicitly construct the Killing spinor or to show that these bosonic configurations fit into the classification scheme given in [12]. However, the construction of Killing spinors is difficult in the rotating case since they depend nontrivially on both the radial and angular coordinates. Thus, there remains the possibility that the integrability conditions for supersymmetry obtained in [28,29] may not be sufficient. To fill this gap in [28,29] is one of the purposes of the present paper.

The BPS backgrounds studied in [28,29] are contained in the class of Plebański-Demiański (PD) [31], which represents the general type D spacetime with an aligned non-null electromagnetic field, and describes a rotating, charged and uniformly accelerating mass. In this paper we work out the necessary and sufficient conditions under which the PD solution is supersymmetric in minimal $\mathcal{N} = 2, D = 4$ gauged supergravity. This represents our main result. We shall also consider in detail two different scaling limits of the PD metric. The first of these removes the acceleration parameter, and leads to a solution discovered by Carter and Plebański [32, 33]. In this case, the integrability conditions for Killing spinors were obtained in [29], and we show that these are not only necessary, but also sufficient for supersymmetry. The second scaling limit eliminates the rotation parameter, and leads to the cosmological C-metric, which describes accelerating black holes in AdS. Also in this case, the BPS conditions are obtained. For both scaling limits, it is shown that these constraints follow from the ones of the general PD solution by scaling the parameters appropriately. In all cases, we determine the three-dimensional base space that appears in the classification scheme of [12], and prove that for the 1/2-BPS Reissner-Nordström-AdS spacetime, this base is unique. Finally, a Wick-rotation of our results leads to gravitational instantons with $U(1) \times U(1)$ symmetry that are supersymmetric in Euclidean gauged supergravity. These generalize the $SU(2) \times U(1)$-symmetric ones constructed recently in [34].

The remainder of this paper is organized as follows: In section 2 we review the equations obtained in [12,26] that BPS geometries in minimal $\mathcal{N} = 2, D = 4$ gauged supergravity must satisfy. Section 3 is devoted to the discussion of the above-mentioned subclasses of the general PD solution, which arise in different scaling limits. The BPS conditions for the general PD solution are obtained in 4. In section 5 an analytic continuation to Euclidean signature is
considered. We conclude in section 6 with some final remarks.

2. BPS geometries in minimal $\mathcal{N} = 2, D = 4$ gauged supergravity

The bosonic action of minimal gauged $\mathcal{N} = 2$ supergravity in four dimensions is given by

$$ S = \frac{1}{16\pi G} \int d^4x\sqrt{-g} \left( R - 2\Lambda - \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \right), \quad (2.1) $$

where $\Lambda = -3\ell^{-2}(<0)$ and $\mathcal{F} = dA$. A bosonic configuration is said to be supersymmetric if it admits a Killing spinor $\epsilon$ satisfying

$$ \hat{\nabla}_\mu \epsilon \equiv \left( \nabla_\mu + \frac{i}{4} \mathcal{F}_{\nu\rho}\Gamma_\mu + \frac{1}{2\ell} \Gamma_\mu - \frac{i}{\ell} A_\mu \right) \epsilon = 0. \quad (2.2) $$

The existence of a Killing spinor imposes strong restrictions on the geometry and the Maxwell field, the key role is played by bilinears constructed from a Killing spinor,

$$ f := \bar{\epsilon}\epsilon, \quad g := i\bar{\epsilon}\Gamma_5\epsilon, \quad V_\mu := i\bar{\epsilon}\Gamma_\mu\epsilon, \quad A_\mu := i\bar{\epsilon}\Gamma_\mu\epsilon, \quad \Phi_{\mu\nu} := i\bar{\epsilon}\Gamma_{\mu\nu}\epsilon, \quad (2.3) $$

where $\bar{\epsilon} = i\Gamma^0\epsilon^\dagger$ and $\Gamma_5 = i\Gamma_{0123}$. It turns out that the vector $V_\mu$ is a causal Killing field, so that the general BPS solutions fall into two categories, namely a timelike and a null family. The general timelike supersymmetric solution of this theory was obtained in [12], and reads (cf. also [26])

$$ ds^2 = -\frac{4}{\ell^2FF}(dt + \omega_i dx^i)^2 + \frac{\ell^2FF}{4} \left[ dz^2 + e^{2\phi}(dx^2 + dy^2) \right], \quad (2.4) $$

$$ \mathcal{F} = \frac{\ell^2}{4} \left[ e^{\frac{\phi}{2}} \right] \left[ V \wedge df + * \left( V \wedge \left( dg + \frac{1}{2\ell} dz \right) \right) \right], \quad (2.5) $$

where $i = 1,2$; $x^1 = x$, $x^2 = y$, and we defined $\ell F = 2i/(f - ig)$. The timelike class of solutions preserves at least one quarter of the supersymmetry. In the canonical form (2.4), we have $V^\mu = (\partial_i)^\mu$ and $A_\mu = \nabla_\mu z$. The real function $\phi$ and the complex function $F$, that depend only on $x,y,z$, are determined by the system

$$ \Delta F + e^{2\phi}[F^3 + 3FF' + F''] = 0, \quad (2.6) $$

$$ \Delta \phi + \frac{1}{2} e^{2\phi}[F' + F' + F^2 + F F'] = 0, \quad (2.7) $$

$$ \phi' - \text{Re}F = 0, \quad (2.8) $$

where $\Delta = \partial^2_x + \partial^2_y$, and a prime denotes differentiation with respect to $z$. Eq. (2.6) comes from the Maxwell equations and the Bianchi identity, (2.7) from the integrability condition of a Killing spinor equation and (2.8) from the differential conditions for bilinears. Finally, the one-form $\omega$ is obtained from

$$ \partial_z \omega_i = \frac{\ell^4}{8} (FF)^2 \epsilon_{ij} (f \partial_j g - g \partial_j f), $$

$$ \partial_i \omega_j - \partial_j \omega_i = \frac{\ell^4}{8} (FF) \epsilon_{ij} \left( f \partial_z g - g \partial_z f + \frac{2f}{\ell} \right), \quad (2.9) $$

We have chosen an axial gauge in which $\omega_z = 0$. The integrability of (2.4) is guaranteed by (2.6).
with \( \epsilon_{12} = 1 \). Decomposing \( F \) into its real and imaginary part, \( F = A + iB \), we see that the real part of eqn. (2.7) follows from (2.8), so that the remaining system is

\[
\Delta B + e^{2\phi} [3\phi'^2 B - B^3 + 3\phi'B' + 3B\phi'' + B''] = 0, \tag{2.10}
\]

\[
\Delta \phi + \frac{1}{2} e^{2\phi} [2\phi'' + \phi'^2 - 3B^2] = 0, \tag{2.11}
\]

together with \( A = \phi' \).

A notable feature of supersymmetric solutions in gauged supergravity is that the system obeys the nonlinear set of equations (2.6)–(2.8). This is in sharp contrast with the ungauged theory \([25]\), where the BPS solutions in the timelike class are specified by harmonics on the 3-dimensional base space \( \mathbb{E}^3 \). This is a major obstacle one encounters in attempting to find supersymmetric solutions in the gauged case.

Another difficulty in gauged supergravity is that the Killing vector \( V^\mu = i\bar{\epsilon}\gamma^\mu \epsilon \) is not associated with the natural time translation of AdS space. This fact can be seen by solving the system (2.6)–(2.8) for \( F_{\mu\nu} = 0 \). Setting \( F_{\mu\nu} = 0 \) leads to \( f = \text{constant} \), thereby we can set \( f = 0 \) or 1 without loss of generality. The former case gives AdS in Poincaré coordinates, which do not cover AdS globally, whereas the latter case yields \([12]\)

\[
d s^2 = \frac{d z^2}{1 + z^2/\ell^2} + \left(1 + \frac{z^2}{\ell^2}\right) \left[-(dt + \ell \sinh^2 \frac{\theta}{2} d\varphi)^2 + \frac{\ell^2}{4} (d\theta^2 + \sin^2 \theta d\varphi^2)\right]. \tag{2.12}
\]

The level set \( z = \text{constant} \) represents AdS\(_3\) written in an \( \text{SL}(2,\mathbb{R}) \) invariant form. The embedding into \( \mathbb{E}^{3,2} \) (with signature \((-1,1,1,1,-1)\)) is given by

\[
X^0 = \ell \cos(t/\ell) \cosh(\theta/2) \sqrt{1 + z^2/\ell^2},
\]

\[
X^1 = \ell \cos(\varphi - t/\ell) \sinh(\theta/2) \sqrt{1 + z^2/\ell^2},
\]

\[
X^2 = \ell \sin(\varphi - t/\ell) \sinh(\theta/2) \sqrt{1 + z^2/\ell^2},
\]

\[
X^3 = z,
\]

\[
X^4 = \ell \sin(t/\ell) \cosh(\theta/2) \sqrt{1 + z^2/\ell^2}.
\]

Hence the coordinate system \((t,z,\vartheta,\varphi)\) covers AdS globally and the metric can be brought into the standard form

\[
d s^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \tag{2.14}
\]

by the coordinate transformation

\[
t = \tilde{t}, \quad \varphi = \phi + \frac{\tilde{t}}{\ell}, \quad z = r \cos \theta, \quad \sinh \frac{\theta}{2} = \frac{r \sin \theta}{\ell \sqrt{1 + (r/\ell)^2 \cos^2 \theta}}. \tag{2.15}
\]

Inspection of (2.13) leads to

\[
\frac{\partial}{\partial t} = \frac{1}{\ell} \left( X^0 \frac{\partial}{\partial X^0} - X^4 \frac{\partial}{\partial X^0} \right) = \frac{\partial}{\partial t} + \frac{1}{\ell} \frac{\partial}{\partial \varphi}. \tag{2.16}
\]
This means that an observer following orbits of the Killing vector associated with the Killing spinor is rotating by the constant angular velocity $\ell^{-1}$ with respect to the static observer at AdS infinity. This is in accordance with the fact that the Bogomol’nyi bound for $(N=2)$ gauged supergravity involves the angular momentum $[35, 36]$,

$$M \geq \frac{1}{\ell} |J| + Q, \quad (2.17)$$

where $M$ and $J$ are the Abott-Deser mass and angular momentum respectively [37], and $Q$ denotes the electric charge$^3$. The same remark applies also to the BPS solutions in five dimensions, where for instance AdS$_5$ is represented by a fibration over the Bergmann manifold [4].

The above instance illustrates that well-known static BPS solutions may be expressed in a rotating frame in the formulation of [12]. This raises an additional obstacle to obtain BPS solutions. Bearing these remarks in mind, we shall show below how to derive supersymmetric solutions.

3. The Plebański-Demiański solution

The complete family of type-D spacetimes with a non-null electromagnetic field, whose two principal null congruences are aligned with the two repeated principal null congruences of the Weyl tensor, was given by Plebański and Demiański [31]$^4$. It solves the field equations of Einstein-Maxwell-(A)dS gravity and describes a rotating, charged and uniformly accelerating mass. The metric and field strength read respectively

$$ds^2 = \frac{1}{(1 - pq)^2} \left\{-\frac{Q(q)}{p^2 + q^2} (d\tau - p^2 d\sigma)^2 + \frac{p^2 + q^2}{Q(q)} dq^2 + \frac{p^2 + q^2}{P(p)} dp^2 + \frac{P(p)}{p^2 + q^2} (d\tau + q^2 d\sigma)^2 \right\}, \quad (3.1)$$

$$F = \frac{Q(p^2 - q^2) + 2Ppq}{(p^2 + q^2)^2} dq \wedge (d\tau - p^2 d\sigma) + \frac{P(p^2 - q^2) - 2Qpq}{(p^2 + q^2)^2} dp \wedge (d\tau + q^2 d\sigma), \quad (3.2)$$

where the structure functions are given by

$$P(p) = (-\Lambda/6 - P^2 + \alpha) + 2np - \varepsilon p^2 + 2mp^3 + (-\Lambda/6 - Q^2 - \alpha)p^4,$$

$$Q(q) = (-\Lambda/6 + P^2 + \alpha) - 2mq + \varepsilon q^2 - 2nq^3 + (-\Lambda/6 + P^2 - \alpha)q^4. \quad (3.3)$$

Here, $\alpha, \varepsilon, m, n, P, Q$ are arbitrary parameters, with $P$ and $Q$ representing the magnetic and electric charges respectively. Eqn. (3.1) together with (3.2) solve the equations of motion following from (2.1).

The main purpose of this section is to determine the condition under which the general PD solution preserves supersymmetry.

$^3$Note that the magnetic charge does not enter the osp(4|2) superalgebra since it breaks SO(3,2) covariance [38]. The issue of BPS bounds in minimal $\mathcal{N} = 2$, $D = 4$ gauged supergravity has recently been studied and clarified in [39].

$^4$For a more recent review cf. [40].
3.1 First scaling limit: The Carter-Plebański metric

A subclass of solutions can be obtained by scaling the coordinates according to
\[ p \to l^{-1}p, \quad q \to l^{-1}q, \quad \tau \to l\tau, \quad \sigma \to l^3\sigma, \quad (3.4) \]
and simultaneously adjusting the constants
\[ P \to l^{-2}P, \quad Q \to l^{-2}Q, \quad m \to l^{-3}m, \quad n \to l^{-3}n, \quad \varepsilon \to l^{-2}\varepsilon, \quad \alpha \to l^{-4}\alpha + \Lambda/6, \quad (3.5) \]
and taking the limit \( l \to \infty \). This removes the acceleration parameter\(^5\) and leads to [31]
\[ ds^2 = -\frac{Q(q)}{p^2 + q^2}(d\tau - p^2d\sigma)^2 + \frac{p^2 + q^2}{Q(q)}dq^2 + \frac{p^2 + q^2}{P(p)}dp^2 + \frac{P(p)}{p^2 + q^2}(d\tau + q^2d\sigma)^2, \quad (3.6) \]
\[ P(p) = \alpha - P^2 + 2np - \varepsilon p^2 + (-\Lambda/3)p^4, \]
\[ Q(q) = \alpha + Q^2 - 2mq + \varepsilon q^2 + (-\Lambda/3)q^4. \quad (3.7) \]
The electromagnetic field is still given by (3.2). In what follows, we shall refer to (3.6) as the Carter-Plebański solution, since it was derived and studied already by Carter [32] and later by Plebański [33]. Notice that one can take a different scaling limit (after the inversion \( q \to -1/q \)), leading to the cosmological C-metric, which will be considered in the next subsection.

The first integrability conditions for (3.6) to admit Killing spinors as a solution to minimal gauged \( N = 2 \) supergravity were analyzed in [29]. There it was found that they are equivalent to
\[ \frac{1}{\ell}[mP + nQ] = 0, \quad B_+B_- = 0, \quad (3.8) \]
where
\[ B_\pm \equiv m^2 + n^2 - (\varepsilon \pm 2\alpha^{1/2}/\ell)(P^2 + Q^2). \quad (3.9) \]
As is well-known [30], in general the integrability conditions are necessary but not sufficient for a solution to admit Killing spinors. (An explicit counterexample was given in [26]). However, for the case of the Carter-Plebański solution (3.6), we will show below that the conditions (3.8) are not only necessary but also sufficient.

To this aim, we show how to obtain the (supersymmetric) Carter-Plebański solution from the equations (2.10), (2.11). It turns out that the correct way to do this is to define new coordinates \( q, p \) by
\[ x = \ell[\alpha(q) + \beta(p)], \quad z = \ell\gamma(q)\delta(p), \quad (3.10) \]
where \( \gamma(q) = \gamma_0 + \gamma_1q, \delta(p) = \delta_0 + \delta_1p, \) and \( \gamma_0, \gamma_1, \delta_0, \delta_1 \) are real constants. By rescaling \( q \) and \( p \) one can always set \( \gamma_1 = \delta_1 = \ell^{-1} \). The function \( \phi \) is assumed to be separable,
\[ e^{2\phi} = \rho(q)\psi(p), \quad (3.11) \]
\(^5\)The acceleration parameter is essentially given by \( l^{-2} \), as can be seen by comparing (3.4) and (3.5) with eqns. (3) and (4) of [40].
where $\rho$ and $\psi$ are both fourth-order polynomials,

$$
\rho(q) = \sum_{n=0}^{4} \rho_n q^n, \quad \psi(p) = \sum_{n=0}^{4} \psi_n p^n.
$$

(3.12)

Finally, $\alpha, \beta$ are determined by requiring that there be no mixed terms $\sim dpdq$ in the base space metric

$$
ds_3^2 = dz^2 + e^{2\phi}(dx^2 + dy^2).
$$

(3.13)

This yields

$$
\alpha'(q) = -\frac{\gamma}{\ell\rho}, \quad \beta'(p) = \frac{\delta}{\ell\psi},
$$

(3.14)

where a prime denotes differentiation w.r.t. the corresponding argument. Then, eq. (2.11) allows to compute $B$, with the result

$$
3B^2 = \left(\gamma^2 \psi + \delta^2 \rho\right)^{-2} \left[\left(\rho'' + \psi''\right)(\gamma^2 \psi + \delta^2 \rho) - \frac{3}{4}\left(\gamma \psi' + \delta \rho'\right)^2\right].
$$

(3.15)

Obviously, the final metric will be rather complicated unless the expression on the rhs of (3.15) is a perfect square. It can be checked that this is the case if and only if the following relations for the coefficients hold:

$$
\psi_2 = -\rho_2, \quad \psi_3 = \rho_3 = 0, \quad \psi_4 = \rho_4, \quad \delta_0 = -\frac{\gamma_0 \psi_1}{\rho_1},
$$

(3.16)

$$
\rho_0 = \frac{\ell \gamma_0 \rho_1}{2} + \frac{\lambda^2}{4\rho_4}, \quad \psi_0 = -\frac{\ell \gamma_0 \psi_1^2}{2\rho_1} + \frac{\lambda^2}{4\rho_4},
$$

(3.17)

where we defined

$$
\lambda = \frac{\rho_1}{2\gamma_0 \ell} - \rho_2.
$$

(3.18)

In this case,

$$
(\rho'' + \psi'')(\gamma^2 \psi + \delta^2 \rho) - \frac{3}{4}\left(\gamma \psi' + \delta \rho'\right)^2 = 3 \left[a_1 p + a_2 p^2 + q(b_0 + b_2 p^2) + q^2(c_0 + c_1 p)\right]^2,
$$

with

$$
a_1 = -\frac{\psi_1 \gamma_0 \lambda}{\rho_1}, \quad a_2 = c_0 = \frac{\lambda}{\ell}, \quad b_0 = \gamma_0 \lambda, \quad b_2 = 2\gamma_0 \rho_4, \quad c_1 = \frac{2\gamma_0 \psi_1 \rho_4}{\rho_1},
$$

and thus

$$
B = \frac{1}{\gamma^2 \psi + \delta^2 \rho} \left[-\frac{\psi_1 \gamma_0 \lambda}{\rho_1} p + \frac{\lambda}{\ell} p^2 + q \gamma_0 (\lambda + 2\rho_4 p^2) + q^2 \left(\frac{\lambda}{\ell} + \frac{2\gamma_0 \psi_1 \rho_4}{\rho_1}\right) p\right].
$$

(3.19)

Remarkably, one finds that then eqn. (2.10) is automatically satisfied. It would be very interesting to understand if there is a deeper reason for this.
Given $\phi$ and $B$, the function $F$ can be computed from $F = \partial_z \phi + iB$. Finally, the one-form $\omega$ is obtained by integrating (2.9), with the result $\omega = \omega_y dy$, where

$$
\omega_y = -\frac{\ell^2 \rho_1}{2\gamma_0} \left[ \rho \left( p^2 + \frac{\lambda}{2p^2} \right) + \psi \left( q^2 - \frac{\lambda}{2q^2} \right) \right] + c \sqrt{\frac{2\gamma_0 \ell\rho_1}{\ell\rho_1}}.
$$

(3.20)

Note that the integration constant in (3.20) was chosen for later convenience and the dimensionless constant $c$ was inserted to take limits in the subsequent sections. If we introduce new coordinates $\tau, \sigma$ according to

$$
(t, y) = \left( \frac{1}{\rho_4} \sqrt{\frac{\ell\rho_4}{2\gamma_0 \rho_1}} \sqrt{\frac{2\gamma_0 \rho_4}{\rho_1}} \right) \left( \tau, \sigma \right),
$$

(3.21)

the metric (2.4) becomes

$$
ds^2 = -\frac{Q(q)}{p^2 + q^2}(d\tau - p^2 d\sigma)^2 + \frac{p^2 + q^2}{Q(q)} dq^2 + \frac{p^2 + q^2}{P(p)} dp^2 + \frac{P(p)}{p^2 + q^2}(d\tau + q^2 d\sigma)^2,
$$

(3.22)

with the structure functions

$$
Q = \frac{\rho}{\ell^2 \rho_4}, \quad P = \frac{\psi}{\ell^2 \rho_4}.
$$

(3.23)

The fluxes can be computed from (2.5), which yields

$$
\mathcal{F}_{01} = \frac{Q(q^2 - p^2) - 2Ppq}{(p^2 + q^2)^2}, \quad \mathcal{F}_{23} = -\frac{P(q^2 - p^2) + 2Qpq}{(p^2 + q^2)^2},
$$

(3.24)

where the electric and magnetic charges are given respectively by

$$
Q = -\frac{\gamma_0}{\ell} \sqrt{\frac{\rho_4}{2\gamma_0 \rho_1}}, \quad P = -\frac{\gamma_0 \psi}{\ell \rho_1} \sqrt{\frac{\rho_1}{2\gamma_0 \rho_4}},
$$

(3.25)

and we have chosen the tetrad

$$
e^0 = \left( \frac{Q}{p^2 + q^2} \right)^{1/2} (d\tau - p^2 d\sigma), \quad e^1 = \left( \frac{p^2 + q^2}{Q} \right)^{1/2} dq,
$$

$$
e^2 = \left( \frac{p^2 + q^2}{P} \right)^{1/2} dp, \quad e^3 = \left( \frac{P}{p^2 + q^2} \right)^{1/2} (d\tau + q^2 d\sigma).
$$

Eqns. (3.22) and (3.24) coincide precisely with (3.6) and (3.2), apart from the obvious fact that the structure functions $Q, P$ in (3.23) are more restricted than (3.7) due to supersymmetry. Comparing (3.23) with (3.7), we obtain for the parameters $\varepsilon, \alpha, m, n$

$$
\varepsilon = \frac{\rho_2}{\ell^2 \rho_4}, \quad \alpha = \frac{1}{4\ell^2 \rho_4^2} \left( \frac{\rho_1}{2\gamma_0 \ell} - \rho_2 \right)^2, \quad m = -\frac{\rho_1}{2\ell^2 \rho_4}, \quad n = \frac{\psi}{2\ell^2 \rho_4}.
$$

(3.26)
The BPS solution is therefore completely specified by the four constants $\gamma_0$, $\rho_1/(\ell^2\rho_4)$, $\rho_2/(\ell^2\rho_4)$ and $\psi_1/(\ell^2\rho_4)$. One easily verifies that the charges (3.25), together with the parameters (3.26), do indeed satisfy the conditions (3.8) found in [29], where $B_+ = 0$ if $\rho_1/(2\gamma_0\ell) - \rho_2 > 0$ and $B_- = 0$ if $\rho_1/(2\gamma_0\ell) - \rho_2 < 0$. We have thus confirmed that the supersymmetric Carter-Plebański spacetime must obey (3.8), but the converse is also true: The two constraints (3.8) leave four free constants out of the six $Q, P, \varepsilon, \alpha, m, n$. Since the BPS solution that we obtained here contains also four parameters, the integrability conditions (3.8) are not only necessary, but also sufficient for the existence of a Killing spinor.

### 3.1.1 Kerr-Newman-AdS

The Kerr-Newman-AdS (KNAdS) spacetime can be obtained from the Carter-Plebański solution (3.6) by setting

$$p = a \cos \theta, \quad q = r, \quad \tau = \tilde{t} - \frac{a}{\Xi} \phi, \quad \sigma = -\frac{\phi}{a\Xi},$$

$$n = 0, \quad \varepsilon = 1 + \frac{a^2}{\ell^2}, \quad \alpha = a^2 + P^2,$$

which yields the KNAdS metric in Boyer-Lindquist coordinates,

$$ds^2 = -\frac{\Delta}{\Sigma^2} \left( d\tilde{t} - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\Sigma^2}{\Delta} dr^2 + \frac{\Sigma^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\Sigma^2} \left( a d\tilde{t} - \frac{r^2 + a^2}{\Xi} d\phi \right)^2,$$

where

$$\Sigma^2 = r^2 + a^2 \cos^2 \theta, \quad \Xi = 1 - a^2 \ell^{-2},$$

$$\Delta = (r^2 + a^2)(1 + \ell^{-2}r^2) - 2mr + Q^2 + P^2, \quad \Delta_\theta = 1 - a^2 \ell^{-2} \cos^2 \theta.$$

In this subcase, the necessary and sufficient conditions for supersymmetry (3.8) boil down to

$$mP = 0, \quad B_+ B_- = 0,$$

with

$$B_\pm = m^2 - \left[ 1 + \frac{a^2}{\ell^2} \pm \frac{2}{\ell} (a^2 + P^2)^{1/2} \right] (P^2 + Q^2).$$

For $m = 0$ we have thus the Dirac-type condition

$$\left( 1 - \frac{a^2}{\ell^2} \right)^2 = \frac{4}{\ell^2} P^2,$$

which is precisely eqn. (98) of [28], while for $P = 0$ we get

$$m^2 = \left( 1 \pm \frac{a}{\ell} \right)^2 Q^2,$$

i.e., eqn. (93) of [28]. This fills a gap in [28], where only the first integrability conditions for Killing spinors of the KNAdS black hole were considered. In a similar way one can easily verify
that the integrability conditions for Killing spinors of the rotating cylindrical or hyperbolic black holes (which also arise as subcases of the general metric (3.6)) given in [28] are sufficient as well.

It is obvious that (3.29) implies no event horizon, whereas eqn. (3.30) leads to

$$\Delta = \left(\frac{r^2}{\ell} \mp a \right)^2 + \left[ Q - \left( 1 \pm \frac{a}{\ell} \right) r \right]^2. \quad (3.31)$$

Therefore, the supersymmetric KNAdS metric describes a naked singularity unless $Q = \sqrt{|a\ell|(1 \pm a/\ell)}$, which provides a degenerate horizon.

### 3.1.2 Reissner-Nordström-AdS

It is also enlightening to examine the Reissner-Nordström-AdS (RNAdS) limit. In [27], it was shown by direct integration of the Killing spinor equations that the RNAdS solution has a supersymmetric limit. The metric of the supersymmetric RNAdS space-time is given by [27]

$$\begin{align*}
\text{d}s^2 &= -U^2(r)\text{d}t^2 + U^{-2}(r)\text{d}r^2 + r^2(\text{d}\theta^2 + \sin^2\theta \text{d}\phi^2),
\end{align*} \quad (3.32)$$

where

$$U(r) = \left[ \left(1 - \frac{M}{r} \right)^2 + \frac{r^2}{\ell^2} \right]^{1/2}. \quad (3.33)$$

It admits the Killing spinor [27]

$$\epsilon = \exp \left( \frac{it}{2\ell} \right) \left( \cos \frac{\theta}{2} + i\Gamma_{012} \sin \frac{\theta}{2} \right) \left( \cos \frac{\phi}{2} + \Gamma_{23} \sin \frac{\phi}{2} \right) \tilde{\epsilon}(r), \quad (3.34)$$

with

$$\tilde{\epsilon}(r) = \left( \frac{1}{\sqrt{U(r) + r/\ell}} + i\Gamma_0 \frac{\sqrt{U(r) - r/\ell}}{U(r)} \right) P(\Gamma_1)\epsilon_0. \quad (3.35)$$

Here, $\epsilon_0$ is a constant Dirac spinor, and $P(\Gamma_1) = (1 - \Gamma_1)/2$ is a projector that reduces the solution space from four to two complex dimensions, i.e., (3.32) preserves half of the supersymmetries. We have thus two linearly independent Killing spinors, each of which leading to a set of bilinears (2.3) that determines a fibration (2.4) over a three-dimensional base space. In order to determine the most general three-dimensional base, let us compute these bilinears.

Taking into account that $\Gamma^{0\dagger} = -\Gamma_0$, $\Gamma^{i\dagger} = \Gamma_i$ ($i = 1, 2, 3$), $\bar{\epsilon} = i\epsilon^{\dagger}\Gamma^0$, we obtain

$$f = -2 \left( 1 - \frac{M}{r} \right) \bar{\epsilon}^{\dagger}\zeta, \quad (3.36)$$

where we defined $\zeta \equiv P(\Gamma_1)\epsilon_0$, and

$$g = \frac{2ir}{\ell} \bar{\zeta}^{\dagger} \left( \Gamma_{23} \cos \theta - i\Gamma_{03} \sin \theta \cos \phi + i\Gamma_{02} \sin \theta \sin \phi \right) \zeta. \quad (3.37)$$
In order to simplify (3.37) further, decompose
\[ \zeta = \zeta_+ + \zeta_-, \quad \zeta_\pm = \frac{1}{2}(1 \pm i\Gamma_{23})\zeta \equiv P(\pm i\Gamma_{23})\zeta. \]

Note that \( P(\pm i\Gamma_{23}) \) are projectors that commute with \( P(-\Gamma_1) \). The spinors \( \zeta_\pm \) have thus each one independent complex component. This yields
\[ g = \frac{2r}{\ell} \left[ (\zeta_+^\dagger \zeta_+ - \zeta_-^\dagger \zeta_-) \cos \theta + i(\zeta_+^\dagger \Gamma_{02} \zeta_- e^{i\phi} - \zeta_-^\dagger \Gamma_{02} \zeta_+ e^{-i\phi}) \sin \theta \right]. \] (3.38)

For the timelike Killing vector \( V \) and the closed one-form \( A \) one gets
\[ V^0 = 2U \zeta^\dagger \zeta, \quad V^1 = 0, \quad V^2 = \frac{2r}{\ell}(e^{-i\phi} \zeta_+^\dagger \Gamma_{02} \zeta_+ + e^{i\phi} \zeta_-^\dagger \Gamma_{02} \zeta_-), \]
\[ V^3 = -\frac{2r}{\ell} \left[ \sin \theta(\zeta_+^\dagger \zeta_+ - \zeta_-^\dagger \zeta_-) + i \cos \theta(e^{-i\phi} \zeta_+^\dagger \Gamma_{02} \zeta_+ - e^{i\phi} \zeta_-^\dagger \Gamma_{02} \zeta_-) \right], \]
\[ A^0 = 0, \quad A^1 = 2U \left[ -\cos \theta(\zeta_+^\dagger \zeta_+ - \zeta_-^\dagger \zeta_-) + i \sin \theta(e^{-i\phi} \zeta_+^\dagger \Gamma_{02} \zeta_+ - e^{i\phi} \zeta_-^\dagger \Gamma_{02} \zeta_-) \right], \]
\[ A^2 = 2 \left( 1 - \frac{M}{r} \right) \left[ \sin \theta(\zeta_+^\dagger \zeta_+ - \zeta_-^\dagger \zeta_-) + i \cos \theta(e^{-i\phi} \zeta_+^\dagger \Gamma_{02} \zeta_+ - e^{i\phi} \zeta_-^\dagger \Gamma_{02} \zeta_-) \right], \]
\[ A^3 = 2 \left( 1 - \frac{M}{r} \right) (e^{-i\phi} \zeta_+^\dagger \Gamma_{02} \zeta_+ + e^{i\phi} \zeta_-^\dagger \Gamma_{02} \zeta_-), \] (3.39)

where we have defined the tetrad frame as
\[ e^0 = Ud\tilde{t}, \quad e^1 = U^{-1}dr, \quad e^2 = r d\theta, \quad e^3 = r \sin \theta d\phi. \] (3.40)

Let us normalize \( \zeta^\dagger \zeta = 1/2 \) and define the constants
\[ c_1 = 2(i \zeta_+^\dagger \Gamma_{02} \zeta_+ + c.c), \quad c_2 = -2(\zeta_+^\dagger \Gamma_{02} \zeta_+ + c.c), \quad c_3 = 2(\zeta_+^\dagger \zeta_+ - \zeta_-^\dagger \zeta_-). \] (3.41)

It then follows that \( \sum_i c_i^2 = 1 \) and
\[ f = - \left( 1 - \frac{M}{r} \right), \quad g = \frac{r}{\ell} \left[ \sin \theta(-c_1 \cos \phi + c_2 \sin \phi) + c_3 \cos \theta \right], \]
\[ V = \partial_t - \frac{1}{\ell} \sum_i c_i \xi_i, \quad A = d \left[ (r - M) \{(c_1 \cos \phi - c_2 \sin \phi) \sin \theta - c_3 \cos \theta \} \right]. \] (3.42)

where the \( \xi_i \) denote \( \text{SO}(3) \) Killing vectors,
\[ \xi_1 = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi, \quad \xi_2 = \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, \quad \xi_3 = \partial_\phi, \] (3.43)

satisfying \( [\xi_i, \xi_j] = \epsilon_{ijk} \xi_k \).

We now have two independent constants which specify the 3-dimensional base space. In appearance, this would give two different bases. However, we can always achieve \( c_1 = c_2 = 0 \) by a rotation of the \( S^2 \), implying that the base space is unique up to isometry. A similar situation occurs for five-dimensional minimal gauged supergravity for which the only way to describe \( \text{AdS}_5 \) in the timelike canonical form is the fibration over the Bergmann space [42].
The base space with \( c_3 = 1 \) corresponds to the one obtained by the \( a \to 0 \) limit of Kerr-Newman-AdS (to take this limit, the choice \( c = -\ell/a \) in (3.20) is convenient). In this case, the metric in the canonical form is given by

\[
\text{d}s^2 = -N \left( \text{d}t + \frac{r^2 \sin^2 \theta}{N \ell^2} \text{d}y \right)^2 + \frac{1}{N} \left[ \text{d}z^2 + \frac{r^2}{\ell^2} U^2 \sin^2 \theta (\text{d}x^2 + \text{d}y^2) \right],
\]

where

\[
t = \tilde{t}, \quad y = \ell \phi + \tilde{t}, \quad z = (r - M) \cos \theta, \quad x = \ell [\alpha(r) + \beta(\theta)],
\]

with

\[
N = \left( 1 - \frac{M}{r} \right)^2 + \frac{r^2}{\ell^2} \cos^2 \theta, \quad \alpha'(r) = \frac{r - M r^2 U^2}{r^2 U^2}, \quad \beta'(\theta) = \cot \theta.
\]

This solution exemplifies that the static BPS metric is rotating in the canonical form.

### 3.2 Second scaling limit: The C-metric

The PD solution contains the C-metric as another subclass. After the inversion \( q \to -1/q \), we perform the rescaling \((q, p, \sigma, \tau) \to l^{-1}(q, p, \sigma, \tau), \) accompanied by

\[
n \to ln, \quad \varepsilon \to l^2 \varepsilon, \quad m \to l^3 m, \quad Q + iP \to l^2(Q + iP), \quad \alpha \to \alpha + l^4 P^2.
\]

The \( l \to \infty \) limit removes the rotation parameter and gives the C-metric

\[
\text{d}s^2 = \frac{1}{(p + q)^2} \left[ -Q(q) \text{d}\tau^2 + P(p) \text{d}\sigma^2 + \frac{dp^2}{P(p)} + \frac{dq^2}{Q(q)} \right], \quad A = Qq d\tau + Ppd\sigma,
\]

where

\[
P(p) = (\alpha - \Lambda/6) + 2np - \varepsilon p^2 + 2mp^3 - (Q^2 + P^2)p^4,
\]

\[
Q(q) = (-\alpha - \Lambda/6) + 2nq + \varepsilon q^2 + 2mq^3 + (Q^2 + P^2)q^4.
\]

The solution (3.48) with cosmological constant appeared for the first time in [31]. For \( \Lambda < 0 \) (the AdS C-metric, which is the case considered here), it describes either a pair of accelerated black holes (with the acceleration provided by the pressure exerted by a strut), or a single accelerated black hole, depending on the value of the acceleration parameter. A detailed discussion of the physics described by the AdS C-metric can be found in [41].

We now wish to obtain the conditions under which the solution (3.48) admits Killing spinors. The integrability conditions for (2.2) yield

\[
\hat{\nabla}_{[\mu} \hat{\nabla}_{\nu]} \epsilon = \left[ \frac{1}{8} C_{\mu\nu\rho\sigma} \Gamma^\rho_{\sigma} + \frac{i}{4} (\nabla_{\rho} F_{\mu\nu} + i \Gamma_5 \nabla_{\rho} \star F_{\mu\nu}) \Gamma^\rho \right.
\]

\[
-\frac{i}{2\ell} (F_{\mu\nu} + i \Gamma_5 \star F_{\mu\nu} + \Gamma_{[\mu} \rho F_{\nu]\rho})
\]

\[
+ \frac{1}{4} \left( E_{p[\mu} - \frac{1}{6} E_\sigma \sigma_{p[\mu]} \right) \Gamma^\rho_{\nu]} - \frac{3i}{4} \left( \nabla_{[\mu} F_{\nu]\rho] + i \Gamma_5 \nabla_{[\mu} \star F_{\nu]\rho]} \right) \Gamma^\rho \epsilon,
\]

\( - 12 - \)
where \( \star F_{\mu \nu} = (1/2) \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma} \), and

\[
E_{\mu \nu} \equiv R_{\mu \nu} - 2 \left( F_{\mu \nu} F^\rho - \frac{1}{4} g_{\mu \nu} F_{\rho \sigma} F^{\rho \sigma} \right) + \frac{3}{\ell^2} g_{\mu \nu}.
\] (3.52)

When the bosonic equations of motion are satisfied, the last line of (3.51) drops out. For the C-metric (3.48), \( \det([\hat{\nabla}_\mu, \hat{\nabla}_\nu]) = 0 \) is equivalent to \( \det \Pi = 0 \), where

\[
\Pi \equiv (Q - i \Gamma_5 P)(\ell^{-1} - \sqrt{Q} \Gamma_1 - \sqrt{P} \Gamma_2) + \frac{i}{12} (P'' + Q'') \Gamma_0 \Gamma_1.
\] (3.53)

Here we have employed the frame

\[
e^0 = \frac{\sqrt{Q} dt}{p + q}, \quad e^1 = \frac{d q}{(p + q) \sqrt{Q}}, \quad e^2 = \frac{d p}{(p + q) \sqrt{P}}, \quad e^3 = \frac{\sqrt{P} d \sigma}{p + q}.
\] (3.54)

The condition \( \det \Pi = 0 \) boils down to

\[
m^2 m^2 - (Q^2 + P^2) \varepsilon + 2 n (Q^2 + P^2)^2 = 0, \quad m^2 \left[ \frac{Q^2 - P^2}{2 \ell^2} + (Q^2 + P^2) \alpha \right] + n^2 (Q^2 + P^2)^2 = 0.
\] (3.55, 3.56)

In this case, it is straightforward to check that with

\[
f = \frac{Q P}{Q^2 + P^2} \left[m - (p - q)(Q^2 + P^2)\right],
\] (3.57a)

\[
g = \frac{(Q^2 + P^2) \varepsilon - m^2 + 2 m (P^2 q - Q^2 p) + 2 (Q^2 + P^2) (p^2 Q^2 + q^2 P^2)}{2 (Q^2 + P^2) (p + q)},
\] (3.57b)

\[
V = P \partial_t - Q \partial_\sigma,
\] (3.57c)

\[
z = \ell \frac{m^2 + (Q^2 + P^2) [m(p - q) + 2 pq (Q^2 + P^2) - \varepsilon]}{2 (Q^2 + P^2) (p + q)},
\] (3.57d)

all the algebraic and differential bilinear equations (2.13)-(2.18) and (2.24)-(2.28) in [12] are satisfied. The conditions (3.55), (3.56) are thus not only necessary, but also sufficient for supersymmetry of the AdS C-metric\(^6\).

The canonical form (2.4) of the AdS C-metric in the BPS limit is given by

\[
d s^2 = - \frac{P^2 Q(q) - Q^2 P(p)}{(p + q)^2} \left[ dt + \frac{P^2 Q(q) + Q^2 P(p)}{P^2 Q(q) - Q^2 P(p)} dy \right]^2 + \frac{(p + q)^2}{P^2 Q(q) - Q^2 P(p)} \left[ dz^2 + \frac{4 P^2 Q^2 Q(q) P(p)}{(p + q)^4} (dx^2 + dy^2) \right],
\] (3.58)

\(^6\)One might be concerned with equation (2.7), which actually does not result from algebraic or differential constraints on the bilinears, but from the additional condition (3.25) of [12]. However, this equation is satisfied since one can verify that it is equivalent to the trace part of Einstein’s equations, provided (2.6), (2.8) and (2.9) hold.
where
\[ t = \frac{\tau}{2P} - \frac{\sigma}{2Q}, \quad y = \frac{\tau}{2P} + \frac{\sigma}{2Q}, \quad x = \alpha(q) + \beta(p), \]
with
\[ \alpha'(q) = \frac{m\ell q + \ell(P^2 + Q^2)q^2 - \frac{\ell m^2}{2(P^2 + Q^2)} + \frac{\ell\varepsilon}{2}}{2PQQ(q)}, \]
\[ \beta'(p) = \frac{m\ell p - \ell(P^2 + Q^2)p^2 + \frac{\ell m^2}{2(P^2 + Q^2)} - \frac{\ell\varepsilon}{2}}{2PQP'(p)}. \]

In particular, we see that
\[ e^{2\phi} = \frac{4P^2Q^2Q(q)P(p)}{(p + q)^4}, \quad (3.59) \]
which is very similar to (3.11), but the product of two quartic functions is now dressed with a factor \((p + q)^{-4}\).

4. Supersymmetry of the general PD solution

After having studied the two different scaling limits which remove either the acceleration or the rotation parameter, we come now to the general PD solution (3.1), (3.2), with the aim to work out the necessary and sufficient constraints imposed by the existence of Killing spinors. It turns out that the first integrability condition \(\det([\hat{\nabla}_\mu, \hat{\nabla}_\nu]) = 0\) reduces again to the single equation \(\det \Pi = 0\). The exact form of \(\Pi\) is not illuminating so we do not display it here and only show the final result. We find that \(\det \Pi = 0\) is equivalent to the two conditions
\[ \begin{align*}
\left[ n[m^2 + n^2 - (P^2 + Q^2)\varepsilon] + 2m(P^2 + Q^2)(P^2 - \alpha) \\
+ \frac{1}{\ell^2} \left[ 2nPQ + m(P^2 - Q^2) \right] \right] &= 0, \quad (4.1) \\
(P^2 + Q^2)[m^2P^2 - n^2Q^2 - (m^2 + n^2)\alpha] \\
+ \frac{1}{\ell^2} \left[ 2mnPQ + \frac{1}{2}(P^2 - Q^2)(m^2 - n^2) \right] &= 0. \quad (4.2)
\end{align*} \]

These equations constrain the parameters \(\alpha\) and \(\varepsilon\) to be functions of \(m, n, P, Q\).

In order to recover the integrability conditions for the two limiting cases, we have to be careful since eqn. (4.1) does not survive in these limits. For the Carter-Plebański metric, we use the following relation instead of (4.1),
\[ [(m^2 + n^2) - (P^2 + Q^2)\varepsilon]^2 - (P^2 + Q^2)^2 \left[ \frac{1}{\ell^4} + \frac{2}{\ell^2}(P^2 - Q^2) + 4(P^2 - \alpha)(Q^2 + \alpha) \right] = 0, \]
which is obtained from (4.1) and (4.2). Then the limit (3.5) gives precisely eqn. (3.8). For the C-metric, we need to use the equation which eliminates \(\alpha\) from (4.1) by using (4.2). Then the limit (3.47) recovers (3.55) and (3.56).
Provided the eqns. (4.1) and (4.2) hold, a long but straightforward calculation shows that

\[
f = \frac{(P^2 + Q^2)[c_-pq(pQ + Pq) - c_+(pP - qQ)] - c_+c_-(p^2 + q^2)}{(p^2 + q^2)(P^2 + Q^2)}, \tag{4.3a}
\]

\[
g = -\frac{c_2pq + c_2^2}{(1 - pq)(P^2 + Q^2)} + \frac{c_+[P(p^3 + q) + Q(p + q^3)] - c_-[Pq^2(p^3 + q) - Qp^2(p + q^3)]}{(1 - pq)(p^2 + q^2)}, \tag{4.3b}
\]

\[
V = c_+\partial_\tau - c_-\partial_\sigma, \tag{4.3c}
\]

\[
z = \ell_m^2 + n^2 - (P^2 + Q^2)(mp + nq) \frac{1}{1 - pq}, \tag{4.3d}
\]

where

\[
c_+ = mP + nQ, \quad c_- = mQ - nP, \tag{4.4}
\]

all the bilinear equations in [12] are satisfied. Eqs. (4.1) and (4.2) are thus necessary and sufficient for supersymmetry of the general PD solution.

5. Euclidean case

Euclidean supersymmetric solutions and gravitational instantons are of importance due to their relevance for non-perturbative effects in quantum gravity [43–45]. Moreover, the boundaries of BPS geometries that asymptote to Euclidean AdS admit conformal Killing spinors [46], and provide thus possible backgrounds on which Euclidean superconformal field theories can be defined. Such theories have been attracting much attention recently in the context of localization techniques [47–50].

In order to analytically continue the general PD solution (3.1), (3.2) to the Euclidean case, we first proceed as in [40], and explicitly include a parameter \(\omega\) which represents the twist of the repeated principal null congruences. This is done by rescaling

\[
p \to \omega^{1/2}p, \quad q \to \omega^{-1/2}q, \quad \tau \to \omega^{1/2}\tau, \quad \sigma \to \omega^{1/2}\sigma,
\]

\[
m + in \to \omega^{-3/2}(m + in), \quad Q + iP \to \omega^{-1}(Q + iP), \quad \varepsilon \to \omega^{-1}\varepsilon, \quad k \to k,
\]

where the parameter \(k\) is defined by \(k = -\Lambda/6 - P^2 + \alpha\). The resulting solution can then be Wick-rotated by taking

\[
\tau \to i\tau, \quad \omega \to i\omega, \quad n \to in, \quad Q \to iQ, \tag{5.1}
\]

which leads to

\[
ds^2 = \frac{1}{(1 - pq)^2} \left\{ \frac{Q(q)}{q^2 - \omega^2p^2}(d\tau - \omega p^2 d\sigma)^2 + \frac{q^2 - \omega^2p^2}{Q(q)} dq^2 \right. \\
+ \frac{q^2 - \omega^2p^2}{P(p)} dp^2 + \left. \frac{P(p)}{q^2 - \omega^2p^2}(-\omega d\tau + q^2 d\sigma)^2 \right\}, \tag{5.2}
\]
\[ F = \frac{Q(q^2 + \omega^2 p^2) - 2Ppq \omega}{(q^2 - \omega^2 p^2)^2} \, dq \wedge (d\tau - \omega^2 d\sigma) + \frac{2Ppq \omega - P(q^2 + \omega^2 p^2)}{(q^2 - \omega^2 p^2)^2} \, dp \wedge (q^2 \, d\sigma - \omega d\tau), \tag{5.3} \]

where the structure functions are given by

\[ P(p) = k + 2\omega^{-1} np - \varepsilon p^2 + 2mp^3 + (\omega^2 k - P^2 + Q^2 + \omega^2 \Lambda/3)p^4, \]

\[ Q(q) = (-\omega^2 k + P^2 - Q^2) - 2mq + \varepsilon q^2 - 2\omega^{-1} nq^3 - (k + \Lambda/3)q^4. \tag{5.4} \]

Taking the tetrad frame

\[ e^1 = \sqrt{\frac{Q}{q^2 - \omega^2 p^2}} \frac{(d\tau - \omega p^2 d\sigma)}{1 - pq}, \quad e^2 = \sqrt{\frac{q^2 - \omega^2 p^2}{Q}} \frac{dq}{1 - pq}, \]

\[ e^3 = \sqrt{\frac{q^2 - \omega^2 p^2}{P}} \frac{dp}{1 - pq}, \quad e^4 = \sqrt{\frac{P}{q^2 - \omega^2 p^2}} \frac{(-\omega d\tau + q^2 d\sigma)}{1 - pq}, \tag{5.5} \]

one can define a self-dual two-form \( \Omega = e^1 \wedge e^2 + e^3 \wedge e^4 \) and \( J\nu : = g^{\nu\rho} \Omega_{\rho\nu} \). It can then be shown that \( J \cdot J = -1 \) and the Nijenhuis tensor for \( J \) vanishes, i.e., the almost complex structure \( J \) is integrable\(^7\) (see [51] for a discussion of the Carter-Plebański family). It should be noted that the two-form \( \Omega \) fails to be closed so that it does not correspond to a Kähler structure.

Taking into account the above rescaling and subsequent analytic continuation, the BPS conditions (4.1), (4.2) become

\[ n[m^2 - n^2 - (P^2 - Q^2)\varepsilon] + 2\omega m(Q^2 - P^2)(\Lambda/6 + k) \]

\[ + \frac{\Lambda \omega}{3} \left[ 2nPQ - m(P^2 + Q^2) \right] = 0, \tag{5.6} \]

\[ (P^2 - Q^2)(m^2 P^2 - n^2 Q^2 + (m^2 - n^2)(\omega^2 k + \Lambda \omega^2/6 - P^2)) \]

\[ + \frac{\Lambda \omega^2}{3} \left[ -2mnPQ + \frac{1}{2}(P^2 + Q^2)(m^2 + n^2) \right] = 0. \tag{5.7} \]

It would be interesting to explicitly check from first principles, using the results of [45], that the Euclidean solution (5.2), (5.3), with the parameters given by (5.6), (5.7), is indeed supersymmetric in Euclidean gauged supergravity.

One can easily verify that the Euclidean PD solution is (anti-)self-dual if

\[ m = \pm n, \quad Q = \pm P, \tag{5.8} \]

for which \( C_{\mu\nu\rho} = \pm (1/2)\varepsilon_{\mu\nu\lambda\tau} C^{\lambda\tau}_{\rho\delta} \) and \( F_{\mu\nu} = \pm (1/2)\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \) are satisfied. When the (anti-)self-duality condition (5.8) holds, the stress-energy tensor of the Maxwell field vanishes identically (hence the metric is Einstein), and the BPS equations (5.4) and (5.7) follow automatically.

\(^7\)This argument does not ensure the global existence of the complex structure on the whole space. For instance, the manifold with \( m = n = |P| - |Q| = 0 \) and \( \Lambda > 0 \) describes \( S^4 \), in which a global complex structure cannot be defined. We thank Yukinori Yasui for pointing this out.
Notice that the gravitational instantons (5.2), (5.3), which have $U(1) \times U(1)$ symmetry, generalize the ones with $SU(2) \times U(1)$ symmetry constructed recently by Martelli, Passias and Sparks in [34]. It should be straightforward to recover the latter by a reasoning similar to that in section 3 of [40]. We shall not attempt to do this here.

Note finally that in the subcase of the C-metric, a Euclidean continuation can be done trivially by taking $\tau \to i\tau$, $Q \to iQ$.

6. Final remarks

Our main result in this paper are the necessary and sufficient conditions for supersymmetry of the general Plebański-Demiański solution (3.1), (3.2). We also considered two different scaling limits of this geometry, that lead to the Carter-Plebański solution or the C-metric. For the former, the first integrability conditions for Killing spinors were worked out in [29], and we showed that these are also sufficient for supersymmetry. The results obtained in our paper resolve thus also some issues that remained open in the literature.

For these classes of solutions, we also revealed the general structure of the three-dimensional base space over which a BPS geometry in $\mathcal{N} = 2$ gauged supergravity is fibered, and showed that for Reissner-Nordström-AdS, this base space is unique up to isometry, in spite of the existence of two linearly independent Killing spinors.

Generically, the BPS solutions considered here are written in a rotating frame in the canonical form (2.4), even when they are static. This feature appears also in AdS$_5$ [4], and it would be clearly desirable to understand this better.

The analytical continuation of our results to Euclidean signature yields gravitational instantons that are supersymmetric in Euclidean gauged supergravity. The conformal boundaries of these backgrounds provide new three-dimensional geometries on which Euclidean supersymmetric field theories can be defined, since they admit conformal Killing spinors [46].

Finally, our work may open the possibility to systematically construct generalizations of the PD metric in matter-coupled gauged supergravity, where nontrivial scalar fields with a potential are turned on, by using the recipe of [21]. Important examples of such solutions that have been constructed so far include the rotating or NUT-charged black holes of [52,53], the dilatonic C-metric without potential [54], the PD solution conformally coupled to a scalar in presence of a cosmological constant and a $\phi^4$ potential [55,56], as well as the solutions of [57]. For obvious reasons, geometries of this type (in particular with a Liouville potential for the dilaton) may be instrumental for the construction of black rings [58] in AdS$_5$ [59]. We shall come back to these points in a forthcoming publication.

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