Rotational splittings for slow to moderate rotators: Latitudinal dependency or higher order effects in $\Omega$?

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Received April 1, 2011; accepted May 16, 2011

ABSTRACT

Context. The unprecedented photometric quality reached by the CoRoT and Kepler space missions opens new prospects for studying stellar rotation. Information about the rotation rate is contained on the one hand in the low frequency part of power spectra, where signatures of nonuniform surface rotation are expected, and on the second hand in the frequency splittings induced by the internal rotation rate.

Aims. We wish to figure out whether the differences between the seismic rotation period as determined by a mean rotational splitting, and the rotation period measured from the low frequency peak in the Fourier spectrum – observed for some of CoRoT’s targets – can provide constraints on the rotation profile.

Methods. For uniform moderate rotators, perturbative corrections to second order and third order in terms of the rotation angular velocity $\Omega$, must not be neglected. These effects, in particular, may mimic differential rotation. We apply our perturbation method to evaluate mode frequencies accurate up to $\Omega^3$ for uniform rotation. Effects of latitudinal dependence are calculated in the linear approximation. Numerical results were obtained for selected models of the upper and lower main sequence. For the latitudinal dependence, we adopted two types of rotation profile: one with rotation uniform in depth, and one with a solar-like tachocline.

Results. In models of \( \beta \) Cephei pulsators, upper main sequence stars, third order effects become comparable to that of a horizontal shear similar to the solar one at rotation rates well below the breakup values. We show how a clean signature of the latitudinal shear may be extracted. Our models of two CoRoT target HD 181906 and HD 181420, which are solar-like pulsators, represent lower main sequence objects. These are slow rotators and nonlinear effects in splittings are accordingly small. We use data on one low frequency peak and one splitting of a dipolar mode to constrain the rotation profile in HD 181420 and HD 181906.

Conclusions. The relative influence of the two effects strongly depends on the type of the oscillation modes at stake in the star and on the magnitude of the rotation rate. Given mean rotational splitting and the frequency of a spot signature, it is possible to distinguish between the two hypothesis, and in the case of differential rotation in latitude, we propose a method to determine the type of rotation profile and a range of values for the shear.

Key words. stellar oscillation – stellar rotation – seismology – differential rotation

1. Introduction

The CoRoT and Kepler space-borne missions with their uninterrupted observations spanning a long-time interval promise a wealth of data suitable for studying stellar rotation. Information about internal rotation is contained in characteristic spacings (known as the rotational splittings) which appear in the power spectra of light curves. Mean values of the rotational splitting have already been determined for a number of stars - HD 181420 (Barban et al. 2009), HD 49933 (Benomar et al. 2009), V 1449 Aql (Belkacem et al. 2009), and HD 181906 (García et al. 2009) - observed by CoRoT. These values yield some information about average rotation rates sampled by modes detected in these objects. Since all the data concern p-modes, the mean values mostly reflect the rotation rate in the outer layers. To probe deeper layers, we need splittings for gravity modes. In three Fourier spectra analyzed so far (HD 181420, HD 181906 and V 1449 Aql), low frequency peaks were found but were attributed to the effects of spots on the rotating stellar surfaces (for a complete review on spot modeling, see Collier Cameron, 2002; Mosser et al. 2009). Rotation periods deduced this way were found to be different from those determined from the splittings. This is not surprising. The spots only give access to the surface rotation rate at the latitude of their location, whereas the splittings yield a mode-dependent mean value of the interior profile.

The linear relation between the splittings and the rotation rate, $\Omega$, follows from the first order perturbative treatment of the Coriolis acceleration (Ledoux, 1994). At moderate rotation rates, the perturbative formalism may still be applicable but we have to go beyond the first order (Reese et al., 2006; Ouazzani et al., 2009; Suárez et al., 2010; Burke et al., 2011). Such corrections arise from the first order perturbative treatment of the Coriolis acceleration and the lowest order corrections in $\Omega$. Such corrections arise from the higher order effects of the Coriolis acceleration and the lowest order effects of the centrifugal acceleration, which causes distortion of the stellar structure. The oscillation frequencies no longer depend on $\Omega$ in a linear way but linear rotational splittings may still be recovered from frequency differences between prograde and retrograde modes of the same degree and order. This simple property is lost when rotation couples modes with close frequencies. The formalism must then be modified (DG92, Soufi et al., 1998) and the recovery is more difficult. The cubic effects in $\Omega$ make it still more complicated. The resulting difference between prograde and retrograde modes becomes dependent on the...
mode’s azimuthal order, $m$, in such a way that it may be misinterpreted as the effects of a latitudinal dependence of the rotation rate.

The question arises whether in the presence of significant nonlinear effects it is still possible to extract the values of linear splittings, which provide integral constraints on differential rotation in the interior. The next question which we ask in this paper is what may be learnt by combining such constraints with data on low frequency peaks that are attributed to spots and yield information on the surface rotation rate. We expect that the answers depend on the type of pulsator, and the characteristics of its observed modes. Here, we specifically consider two very different types of main sequence objects, $\beta$ Cephei and a solar-like pulsators. In the first case, we are dealing with a massive star its observed modes. Here, we specifically consider two very different types of main sequence objects, $\beta$ Cephei and a solar-like pulsators. In the first case, we are dealing with a massive star

The paper is organized as follows. Sect. 2 gives the basic theoretical framework for this study. In Sect. 3, we focus on the effects of cubic order and near degeneracy contributions to pulsation frequencies and present numerical results for a selected model of a $\beta$ Cephei star. Explicit expressions for the rotational splitting in the case of latitude dependent rotation profile are presented. Sect. 6 is devoted to two CoRoT targets, the solar type stars HD 181906 and HD181420, for which we combine the splitting with the low frequency peak and make some inference on the rotation profiles. Sect. 7 is dedicated to conclusions.

### 2. Perturbational treatment of uniform rotation: effects on pulsation frequencies

In the presence of rotation, the centrifugal and Coriolis accelerations come into play. The centrifugal force affects the structure of the star and distorts its shape. The resonant cavity is changed due to rotation (through $g_{\text{eff}}$), which mainly modifies the gravity; on the other hand, $\theta$-dependent perturbations, which is responsible for oblateness. Then all the equilibrium quantities, $X$, are well approximated by:

$$
X(r, \theta) \approx \tilde{X}(r) + X_{22}(r) P_2(\cos \theta),
$$

The spherically symmetric part is then obtained by (see for example [Kippenhahn & Weigert 1994])

$$
\frac{d \rho}{dr} = -\tilde{g}_{\text{eff}} g_{\text{eff}}, \quad \text{where} \quad g_{\text{eff}} = \frac{GM}{r^2} - 2 \frac{r}{3} \Omega^2.
$$

The non-spherically symmetric part is obtained by (see DG92)

$$
p_{22} = -\tilde{p} r^2 \Omega^2 \left( \frac{d \phi_{22}}{d \ln r} \right),
$$

$$
\rho_{22} = \tilde{\rho} r \Omega^2 \left( \frac{d \ln \tilde{r}}{d \ln r} \right) \left( \frac{d \phi_{22}}{d \ln r} \right) + \frac{1}{3} (4 \pi G \rho_{22}).
$$

The boundary conditions can be found in Soufi et al. [1998] (hereafter S98).

### 2.1. Equilibrium configuration

We here consider the case of uniform rotation. The stationary equation of motion in an inertial frame of reference is:

$$
(v_0 \cdot \nabla) v_0 = -\frac{\nabla P}{\rho} - \nabla \phi = F
$$

Where in the left hand side, $v_0 = \Omega \wedge r = \Omega_0 r \sin \theta e_\phi$ in the spherical basis is the velocity field due to rotation at the angular velocity $\Omega_0$, $\theta$ being the colatitude (see e.g. [Unno et al. 1989]), $P$, $\rho$ and $\phi$ are the pressure, density and gravitational potential, respectively. For a rotating star, the left hand side corresponds to the centrifugal acceleration $F = -\Omega \times (\Omega \times r)$, whose effect on the equilibrium structure is twofold: on one hand, a spherically symmetric perturbation, which mainly modifies the gravity; on the other hand, $\theta$-dependent perturbations, which is responsible for oblateness. Then all the equilibrium quantities, $X$, are well approximated by:

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$$

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### 2.2. Oscillation frequencies up to cubic order in $\Omega$

Using expansions of the type given in Eq. (3) for the oscillation quantities, i.e., $p^i = \tilde{p}^i + p_{22}^i$, the oscillation system is then expanded up to the cubic order. According to S98 (see also Karami 2008), the oscillation equation then becomes:

$$
\mathcal{L} \xi = (A + \varepsilon B) \xi + \varepsilon^2 (D + \varepsilon C) \xi + O(\varepsilon^3) = 0
$$

$\varepsilon$ being equal to $\Omega/\Omega_0$, where $\Omega_0 = \sqrt{GM/R^3}$ is the break-up frequency. The operator $A$ represents the basic linear oscillation operator including the spherically symmetric perturbation due to rotation (through $g_{\text{eff}}$). The operators $B$ and $D$, respectively, contain the effects of the Coriolis force and of non-spherically symmetric distortion. The operator $C$ shows that a coupling between the non-spherically symmetric distortion and the Coriolis force exists.

Like in S98 and in Karami (2008), parts of the Coriolis and centrifugal distortion effects are included into the pseudo-zeroth order eigenvalue system. This way, we are able to solve the eigenvalue problem up to cubic order without having to solve the successive equations for the eigenfunctions at each order. The solution yields eigenfrequencies $\sigma_0$, which include parts of the
frequency shifts induced by rotation. To single out various contributions and emphasize the \(m\)-dependence, we write it in the form

\[
\sigma_{0,m} = \sigma_{0}^{(0)} + \sigma_{2}^{(0)} + \sigma_{1,m}^{*} + \sigma_{2,m}^{*} + \sigma_{3,m}^{*} \quad (8)
\]

where \(\sigma_{0}^{(0)}\) is the classical zeroth order frequency ignoring all effects of rotation, \(\sigma_{2}^{(0)}\) is the correction resulting from the spherically-symmetric part of the centrifugal distortion, and the next three terms give the contributions of consecutive orders in \(\Omega\) resulting from the Coriolis acceleration. The linear term, \(\sigma_{1,m} = m\Omega \beta \Omega_k\), is complete. The higher order terms only include the parts resulting from the poloidal component of \(\xi\). The remaining contributions to frequency shifts up to \(\mathcal{O}(\Omega^2)\) are calculated as integrals involving the eigenvectors \(\xi_0\) (see S98). To this accuracy, the complete expression for eigenfrequencies in uniformly rotating star is given by

\[
\sigma_{m} = \sigma_{0,m} + \sigma_{r,m} \quad (9)
\]

with \(\sigma_{r,m} = \sigma_{r,m}^{T} + \sigma_{r,m}^{D} + \sigma_{r,m}^{T} + \sigma_{r,m}^{D} + \sigma_{r,m}^{C}\),

\[
\sigma_{r,m} = \sigma_{r,m}^{T} + \sigma_{r,m}^{D} + \sigma_{r,m}^{T} + \sigma_{r,m}^{D} + \sigma_{r,m}^{C} \quad (10)
\]

where the exponent \(T\) marks contributions from the Coriolis force acting on the toroidal component of \(\xi\), the exponent \(D\) those arising from the non-spherically symmetric distortion, and \(C\) those resulting from coupling of the two effects.

2.3. Near degeneracy

The standard perturbation approach is invalid if rotation couples modes with close frequencies. Treatment of such cases requires modification, which in the context of stellar pulsations was first used by Chandrasekhar & Lebovitz (1962) and developed later by DG92. In the case of latitude-independent rotation profiles, only modes with the same \(m\) and \(\ell\) of the same parity are coupled. In the present work, we study the coupling of two resonant modes denoted \(k_j\) for \((n_j, \ell_j, m)\), \(j = 1, 2\) with frequencies \(\sigma_1 \geq \sigma_2\). For the range of stellar models we are interested in, calculations reveal that near degeneracy occurs for quite a large number of modes.

Near degeneracy is taken into account by searching for solutions of Eq.(7) in the form

\[
\xi = \sum_{k} A_{k} \xi_{0} x_{kj} + \xi_{c} \quad j = 1, 2
\]

with

\[
\xi_{c} = \sum_{k} \alpha_{k} \xi_{0} x_{k}\quad k = 1, N,
\]

where the eigenfunction correction \(\xi_{c}\) is composed of all non-resonant modes. The standard procedure leads to a linear system of equations for \(A_{k}\), and the following condition for a non-zero solution:

\[
(\sigma_{k_1} - \sigma_{m}^{(0)}) (\sigma_{k_2} - \sigma_{m}^{(0)}) - \mathcal{H}_{m}^2 = 0 \quad (12)
\]

where the frequencies \(\sigma_{k_1}\) and \(\sigma_{k_2}\) are given by Eq.(9). The coupling term \(\mathcal{H}_{m}\) corresponds to integrals containing second and third order contributions (see S98 for more details). The solutions of Eq.(12), denoted by \(\sigma_{m}^{(0)}\), provide the desired eigenfrequencies.
Contributions of different approximation orders to \( l = 1, m = 0 \) mode frequencies: \( \sigma_{2\text{eigen}} \) (\( \sigma_{3\text{eigen}} \)) represent the implicit 2\textsuperscript{nd} (3\textsuperscript{rd} respectively) order contribution to the eigenfrequency; \( \sigma_{2T} \) and \( \sigma_{3T} \) denote 2\textsuperscript{nd} and 3\textsuperscript{rd} order frequency corrections due to the Coriolis acceleration; \( \sigma_{2D} \) and \( \sigma_{3D} \) represent 2\textsuperscript{nd} and 3\textsuperscript{rd} order frequency corrections due to the centrifugal distortion; \( \sigma_{3C} \) comes from the coupling of distortion and Coriolis effects. The contributions are plotted as a function of the radial order \( n \) for an 8.5 M\(_\odot\) ZAMS model rotating uniformly at 15\% \( \Omega_k \), i.e. around 95 km s\(^{-1}\) (see stellar parameters in Table 1). The frequencies are scaled by \( \Omega_k = \sqrt{GM/R^3} \); \( \sigma = \omega/\Omega_k \). We recall the use of a negative radial order \( n \) for gravity modes and a positive one for acoustic modes. In this case too, the dominant term grows linearly with the radial order, as noted in Goupil (2009) and Reese et al. (2006) for polytropic models. At the third order, \( \sigma_{3\text{eigen}} \) and \( \sigma_{3D} \) are of the same magnitude but with opposite signs and cancel each other to some extent.

To sum up, as expected, for g-modes the most important contributions are related to the effects of the Coriolis acceleration, whereas for p-modes, we must take into account both the implicit eigenfrequency terms (due to the part of Coriolis force included in the pseudo-zero\textsuperscript{th} eigen-system) and the effects of centrifugal distortion. In Table D.3, numerical values of the different contributions are listed.

### 3.2. Near degeneracy corrections

Here we consider the same sequence of \( l = 1 \) modes as in the previous section but we now take into account the coupling of each mode with the nearest \( l = 3 \) partner. The coupled pairs must be of the same azimuthal order \( m \). We use Eq.(13) here to calculate the frequency shift caused by such a coupling.

In order to compare the magnitude of this near degeneracy effect with the second and third order contributions shown in Fig[1] we depict in Fig[2] the frequency differences between computations with and without near degeneracy being accounted for. Figure 2 shows that near degeneracy primarily affects the p-modes (\( n > 0 \)). This is not surprising as they are more sensitive to the outer regions which are more affected by distortion, a dominant factor in the coupling coefficient \( H \). Moreover, this correction is found to be of the same magnitude as the other second order corrections (see Fig[1]) but of the opposite sign.

Hence the overall effect of the distortion is reduced. However, this is not a universal property. As we may see in Eq.(13), the coupling always causes an increase of frequency separation between modes but the sign of the shift is mode-dependent. In any case, rotational mode coupling is an important effect, especially
for p-modes. Taking it into account (as shown in Suárez et al. 2014),
extends the validity domain of perturbative methods.

Finally, we notice that the sectorial components of the \( \ell = 1 \)
triplet are modified by roughly the same amount which implies that
the rotational splitting should not be strongly affected by
near degeneracy (see Sect. 4.2). This is expected because in this
case it enters as a third order effect.

### 4. Rotational splitting for uniform rotators

The rotational splitting can be defined as:

\[
S_m = (\sigma_m - \sigma_0)/m.
\]

One also uses \( S_m = \sigma_m - \sigma_{m-1} \). In this work, we use a scaled
expression of the rotational splitting (Eq. (1)):

\[
S_m = \sigma_m - \sigma_{-m}/2m
\]  \hspace{1cm} (16)

These various definitions are equivalent only at first order in the
rotation rate, \( \Omega \), and equal to the linear splitting:

\[
S_m = \frac{1}{\Omega k} \int_0^R \int_0^{\pi} K_m(r, \theta) \Omega(r, \theta) \, dr \, d\theta
\]  \hspace{1cm} (17)

where the analytical expression for the kernels \( K_m \) is given in
Goupil (2011) and references therein. At higher orders in terms
of \( \Omega \), the two first definitions are contaminated by the e
asphericity, which introduces an antisymmetric component in
the frequency as a function of \( m \). We choose to remove this sec-
ond order contribution using the splitting expressed in Eq. (17).

### 4.1. Cubic order effects on the splitting

Assuming a uniform rotation \( \Omega = \Omega_0 \), the splitting including
frequency correction due to cubic order effects is given by:

\[
S_{\text{cubic}} = \frac{\Omega_0}{\Omega k} \beta + \frac{\Omega_0}{\sigma_0} \frac{\Omega_0}{\Omega k} T_{3,m}
\]  \hspace{1cm} (18)

\[
T_{3,m} = T_{3,m}^{\text{eigen}} + T_{3,m}^T + T_{3,m}^D + T_{3,m}^C
\]  \hspace{1cm} (19)

where \( \sigma_0 = \omega_0/\Omega k \) is the normalized frequency of the corre-
sponding axisymmetric mode. \( \Omega_0 \) is the uniform rotation rate,
and \( \beta \) is the integral of the kernel \( K_m \) over the star (see Appendix
B, Eq. (B.3)).

\( T_{3,m} \) contains the implicit third order contribution as well as e
ffects due to the Coriolis acceleration, the distortion, and the
coupling of the two. From now on, we define the departure from a
linear splitting as follows:

\[
\delta S_{\text{cubic}}^m \equiv S_{\text{cubic}}^m - \left( \frac{\Omega_0}{\Omega k} \beta + \frac{\Omega_0}{\sigma_0} \frac{\Omega_0}{\Omega k} T_{3,m} \right)
\]  \hspace{1cm} (20)

### 4.2. Near degeneracy correction

According to the formalism explained in Section 2.3, if we con-
sider the coupling of \( \ell = 1 \) and \( \ell = 3 \) modes, let the degenerate
frequency of \( \ell = 1 \) modes be:

\[
\sigma_{\ell=1,m}^{\text{deg}} = \frac{\sigma_{\ell=1,m} + \sigma_{\ell=3,m}}{2} + \frac{\Delta^2_m + \mathcal{H}_m^2}{2} \hspace{1cm} (21)
\]

where \( \sigma_{\ell=1,m} \) and \( \sigma_{\ell=3,m} \) are non degenerate frequencies given
by Eq. (9), and \( \Delta_m \) is defined in Eq. (13). Then the splitting ac-
counting for near degeneracy is given by:

\[
S_{\ell=1,m=1}^{\text{deg}} = S_{\ell=1,m=1}^{\text{ND}} - \frac{1}{2} \left( \Delta_1 - \Delta_{-1} \right) + \frac{1}{2} \left( \sqrt{\Delta_1^2 + \mathcal{H}_1^2} - \sqrt{\Delta_{-1}^2 + \mathcal{H}_{-1}^2} \right)
\]  \hspace{1cm} (22)

where

\[
S_{\ell=1,m=1}^{\text{ND}} = \sigma_{1,1}^{\text{ND}} - \sigma_{1,-1}^{\text{ND}} \hspace{1cm} (23)
\]

with \( \text{ND} \) standing for non-degenerate. Note that \( S_{\ell=1,m=1}^{\text{ND}} \)
contains cubic order contributions mentioned in the previous sec-
tion. The contribution of near degeneracy to the splitting is then
given by \( S_{\ell=1,m=1}^{\text{deg}} = S_{\ell=1,m=1}^{\text{ND}} \).

It is worth recalling that neglecting all cubic order contribu-
tions in Eqs. (10) and (13) results in \( \Delta_1 = \Delta_{-1} \) and \( \mathcal{H}_1 = \mathcal{H}_{-1} \).
In that particular case, degeneracy does not con-
tribute to the rotational splitting, and the rotational splitting is
linear in \( \Omega \) up to second order and satisfies:

\[
S_m = \frac{\Omega_0}{\Omega k} \beta
\]  \hspace{1cm} (24)

### 4.3. Sensitivity to the nature of the eigenmode

Figure 3 displays the near degenerate contributions to the split-
ting for p- and g-modes together with the cubic ones for a ZAMS
stellar model (Table I). The values of these different contribu-
tions are given by mode by mode in Table D.2.

In Fig. 3 (left), the Coriolis correction \( T_{3,m}^T \) dominates for g-
modes (by roughly a factor 10^2 over other contributions) and
decreases with the radial order to a roughly constant value for low \( |n| \).
The scale is too large in this figure to see the behavior of \( T_{3,m}^{\text{eigen}}, T_{3,m}^D \), and the near degeneracy contribution to the split-
ting, but we refer to Table D.1 where it is shown that there is no
asymptotic behavior for these four contributions. Near degener-
cy is fully negligible for all g-modes except for the \( n = -1 \) mode
that actually is a mixed mode. The g-mode spectrum is much denser than the p-mode one but as shown in Fig. 3 (right),
the coupling term \( H_m \) is much smaller than \( \Delta_m \). This is due to the fact that distortion effects are small for g-
modes and therefore the overall (second and third order) contribu-
tion to \( H_m \) remains small.

In order to emphasize possible asymptotic behavior, the con-
tributions to p-modes have been divided by \( \sigma_0^2 \). In Fig. 3 (right),
the implicit cubic order and the near-degenerate contributions
dominate for p-modes but with opposite signs and therefore
roughly compensate each other. Hence, as was the case for pul-
sation frequencies, the near degeneracy correction tends to re-
duce over-estimated contributions to rotational splittings.
The contributions dominated by centrifugal distortion, and in par-
sicular the near degeneracy one, scale as \( \sigma_0^2 \). In Appendix B
it is shown that near degeneracy contributes to the rotational split-
ting only if third order effects are taken into account (see Eq. 4.5).
However it is also shown that second order effects (\( \mathcal{H}_2 \)) – dom-
ninated by distortion (\( \mathcal{H}_2 \)) for p modes – are involved. This
explains why the near degeneracy frequency variation follows a \( \sigma_0^2 \)
behavior for p-modes.

Similar conclusions for more evolved models with more
complex structures are found for pure p-modes and pure g-
modes. However these complex structures also give rise to mixed
modes for which all effects contribute equally, and a precise in-
vestigation, mode by mode, has to be done for each equilibrium
model. This will be investigated in details in Sect. 5.
Integrals

5. Effects of latitudinal shear on the splitting

Hansen et al. (1977) derived the expression for the rotational splitting of adiabatic nonradial oscillations for slow differential (steady, axially symmetric) rotation $\Omega(r, \theta)$ and applied it to numerical models of white dwarfs and of massive main sequence stars assuming a cylindrically symmetric rotation law. In the solar case, the effects of latitudinal differential rotation on theoretical frequencies were investigated by Gough & Thompson (1990), Dziembowski & Goode (1991) and Dziembowski & Goode (1992).

In order to be able to compute the splittings from Eq. (17), one must specify a rotation law. It is convenient to assume the following form:

$$\Omega(r, \theta) = \sum_{s=0}^{\text{max}} \Omega_2(r) (\cos \theta)^{2s}$$  \hspace{1cm} (25)

where $\theta$ is the colatitude. The surface rotation rate at the equator is $\Omega(r = R, \theta = \pi/2) = \Omega_2(r = R)$.

Note that in the solar case, $\Omega_2$ and $\Omega_4$ are negative and the equator rotates faster than the poles.

Inserting Eq. (25) into Eq. (17) yields the following expression for the splitting:

$$S_m = \frac{1}{\Omega_k} \int_0^R \Omega_0(r) \frac{1}{\Omega_k} \sum_{s=0}^{\text{max}} m^{2s} H_m(s, \Omega) \mathrm{d}r + \frac{1}{\Omega_k} \sum_{s=0}^{\text{max}} m^{2s} H_m(s, \Omega) \gamma$$  \hspace{1cm} (26)

The expression for $H_m(s, \Omega)$ can be found in Appendix [B].

5.1. Latitudinally differential rotation only $\Omega(\theta)$

In this case, for which rotation is assumed to be uniform in depth, the splitting becomes: (Goupil, 2011)

$$S_m = \frac{\Omega_0}{\Omega_k} \beta + \frac{1}{\Omega_k} \sum_{s=0}^{\text{max}} m^{2s} (R_m(\Omega) \beta + Q_m(\Omega) \gamma)$$  \hspace{1cm} (27)

Fig. 3. Scaled contributions to the splittings due to implicit third order terms in the eigenfrequencies ($T_{3}^{\text{eigen}}$), the Coriolis effect ($T_{3}^{\text{c}}$), distortion ($T_{3}^{\text{d}}$), the coupling of the two ($T_{3}^{\text{c}}$) and near degeneracy. **Left:** for $\ell = 1$ g-modes, **Right:** for $\ell = 1$ p-modes, divided by the square of the central frequency. Computations were made for an 8.5 $M_\odot$ ZAMS model uniformly rotating at 15% $\Omega_0$, i.e. 95 $km.s^{-1}$ (see stellar parameters in Table [I]).

Fig. 4. Behavior of **Left:** the scaled integrals $\beta$ and $\gamma$ (Eq. (28)), **Right:** the scaled contributions to the splitting, as a function of the radial order $n$ (Eq. (25) to (24)). Computations were made for an 8.5 $M_\odot$ ZAMS model uniformly rotating at 15% $\Omega_0$, i.e. 95 $km.s^{-1}$ (see stellar parameters in Table [I]).
Expressions for $R_s$ and $Q_s$ can be found in Appendix B. $\beta$ and $\gamma$ depend on the radial and horizontal components of the mode:

$$
\beta = \frac{1}{\Omega} \int_0^R \left[ \xi_t^2 - 2 \xi_t \xi_\theta + (\Lambda - 1) \xi_t^2 \right] \rho_0 r^2 dr
$$

(28)

$$
\gamma = -1/1 \int_0^R \xi_t^2 \rho_0 r^2 dr
$$

(29)

I being the inertia of the mode:

$$
I = \int_0^R (\xi_t^2 + \Lambda \xi_t^2) \rho_0 r^2 dr
$$

(30)

For the sake of simplicity, we restrict our study to $s = 1$ in Eq. (26). The rotation law can then be expressed as:

$$
\Omega(\theta) = \Omega_0 - \Delta \Omega \cos^2 \theta
$$

(31)

Where $\Omega_0 = \Omega(\theta = 0, \theta = \pi/2)$, $\Omega_2 = -\Delta \Omega$ and $\Omega_4 = 0$. After some calculations provided in Appendix B, we are able to express the splittings of several $(l, m)$ combinations:

a. $(l, m) = (1, \pm 1)$ modes:

$$
S_{l=1,m=1}^{\text{lat}} = \frac{\Omega_0}{\Omega_k} \beta \left( 1 - \frac{1}{5} \frac{\Delta \Omega}{\Omega_0} \right)
$$

(32)

b. $(l, m) = (2, \pm 1)$ modes:

$$
S_{l=2,m=\pm 1}^{\text{lat}} = \frac{\Omega_0}{\Omega_k} \left( 2 \beta + \frac{1}{7} \frac{\Delta \Omega}{\Omega_0} (8 \gamma - 3 \beta) \right)
$$

(33)

c. $(l, m) = (2, \pm 2)$ modes:

$$
S_{l=2,m=\pm 2}^{\text{lat}} = \frac{\Omega_0}{\Omega_k} \left( 2 \beta - \frac{1}{7} \frac{\Delta \Omega}{\Omega_0} (2 \gamma + \beta) \right)
$$

(34)

In the solar case, $\beta \sim 1$ and $\beta >> |\gamma|$ for the excited high frequency $p$-modes. Then the $(l = 1, m = 1)$ splitting is:

$$
S_{l=1,m=1}^{\text{lat}} = \frac{\Omega_0}{\Omega_k} \left[ 1 + \frac{1}{5} (\Omega_2/\Omega_0 + 3/7 \Omega_4/\Omega_0) \right]
$$

(35)

With $\Omega_2/\Omega_0 = 0.127, \Omega_4/\Omega_0 = -0.159$, one obtains a departure of $|S_{l=1,m=1}| = 0.04$ from a linear splitting, i.e. a 4% change in the solar case.

For upper main sequence stars, excited modes are around the fundamental radial mode and may be mixed modes with $|\beta| \sim |\gamma| \sim 1/2$. This leads for instance to $|S_{l=1,m=1}| \sim 1$ for $\Omega_2/\Omega_0$ and $\Omega_4/\Omega_0$ equal to a half of the solar values. We recall that for the stars treated in this article, we have taken $\Omega_2 = -\Delta \Omega$ and $\Omega_4 = 0$.

As shown in Eqs. (27) to Eq. (30), the additional term due to the latitudinal shear strongly depends on the eigenfunction of the mode, through the radial and horizontal components of the displacement. Therefore, we investigated the contributions for different types of modes. Again, the plots presented in Fig. 4 have been performed for an 8.5 $M_\odot$, 4 $R_\odot$ ZAMS model, rotating at $\Omega = 15$% $\Omega_\odot$. Since once, computations for more evolved stellar models give similar results – except for mixed modes around $n = 1$ – even with stronger rotation rates.

Figure 4 shows the dependency of the $\beta$ and $\gamma$ integrals on the $p$ or $g$ nature of the mode. Globally, for $g$-modes, at a given rotation rate, we expect a small contribution of the latitudinal shear to the splittings, smaller for $S_{1,1}$ than for $S_{2,1}$ and $S_{2,2}$, whereas for $p$-modes, the contribution is quite important for $S_{2,1}$.

For the sake of simplicity we will focus on $l = 1$ splittings from now on. Note that the investigations presented in the next Sections have been addressed for $l = 2$ splittings as well and had lead to the same conclusions.

5.2. A tachocline-like rotation profile: $\Omega(r, \theta)$

Let us refine the modeling of the rotation profile, assuming a rotation profile in depth as in the solar case. Rotation is uniform $(\Omega = \Omega_0)$ in the inner layers and differential latitudinally-as expressed in Eq. (11) in the convective envelope. $r_c$ being the radius of the inner stable layer:

$$
\begin{align*}
&\text{for } r < r_c, \quad \Omega(r, \theta) = \Omega_0 \\
&\text{for } r \geq r_c, \quad \Omega(r, \theta) = \Omega_0 - \Delta \Omega \cos^2 \theta
\end{align*}
$$

(36)

Therefore, after some calculations similar to those presented in Appendix B, equation (41) is no longer relevant, and should be replaced by:

$$
S_{l=1,m=1}^{\text{lat}} = \frac{\Omega_0}{\Omega_k} \left( \beta - \frac{\Delta \Omega}{\Omega_0} \frac{1}{5} \beta_c \right)
$$

(37)

where

$$
\beta_c = \frac{1}{1} \int r_c^R \left[ \xi_t^2 - 2 \xi_t \xi_\theta + (\Lambda - 1) \xi_t^2 \right] \rho_0 r^2 dr
$$

(38)

Note that for $p$ modes for instance, $\beta_c$ is smaller than $\beta$ in the whole star - $\beta_c \approx 0.45$ whereas $\beta \sim 1$. Therefore in the case of a tachocline-like profile, the effect of a same latitudinal shear $\Delta \Omega$ is smaller than in the case where rotation is uniform in depth.

6. The case of a $\beta$ Cephei on the main sequence

Massive stars on the main sequence are usually fast rotators and their fast rotation affects their internal structure as well as their evolution. Rotation of $\beta$ Cephei stars extends from slow rotational velocity ($v < 50$ km/s) to extremely rapid ones ($v > 250$ km/s) [Stankov & Handler 2005]. These stars usually have a radiative envelope which may or may not be in latitudinal differential rotation. For these fast rotators, one can wonder whether cubic order or near degeneracy contributions dominate over the effects of latitudinal shear.

Here we investigate the importance of deviations from a linear splitting for an evolved main sequence model of an 8.5 $M_\odot$ star with a 5.07 $R_\odot$ radius (see Table 2 for the stellar parameters of the model).

| Parameter | Value |
|-----------|-------|
| $M$       | 8.50 $M_\odot$ |
| $R$       | 5.07 $R_\odot$ |
| $L$       | $2 \times 10^{37}$ erg s$^{-1}$ |
| age       | 50 My |
| $X_0$     | 0.713 |
| $Z_0$     | 0.014 |
| $\alpha$  | 1.76 |

6.1. Departure from a linear splitting as a function of the frequency

According to the region to which a mode belongs, very different types of behaviour are observed for the different contributions to the splitting, as illustrated in Fig. 5.

Figure 5 displays the departure from a linear splitting due to cubic order effects with and without near degeneracy corrections, and to latitudinal differential rotation (uniform in depth). It has been computed for $l = 1$ modes with radial orders ranging from 10 to 5. The parameters of the stellar model considered here are given Table 2. It is rotating at 20%$\Omega_0$ (left) and 30%$\Omega_0$
Fig. 5. Departure from a linear splitting for \((\ell = 1, m = 1)\) triplets, Left: for \(\Omega = 20\% \Omega_\odot\), Right: for \(\Omega = 30\% \Omega_\odot\), as a function of the radial order. Different contributions result from: cubic order effects (dark blue), cubic order effects with near degeneracy (purple, these two contributions overlap for all g-modes), and latitudinally differential rotation with \(\Delta \Omega / \Omega = 0.127\) (light blue), 0.2 (in green), 0.3 (in orange), 0.4 (in red). These results were computed for the uniformly rotating evolved 8.5 M_⊙ β Cephei model described in Sect. 5.1 (Table 2).

Fig. 6. Departure from a linear splitting for \((\ell = 1, m = 1)\) triplets due to cubic order effects (dark blue), to cubic order effects including near degeneracy corrections (purple, these two contributions overlap for g_5 and g_2) and to latitudinally differential rotation with \(\Delta \Omega / \Omega = 0.127\) (light blue), 0.2 (green), 0.3 (orange), and 0.4 (red). These departures are plotted as a function of the rotation rate for a g_5 mode (left), a g_2 mode (center) and a p_1 mode (right). These results were computed for the uniformly rotating evolved 8.5 M_⊙ β Cephei model described in Sect. 5.1 (Table 2).

(right). On these plots we observe three regions where the behavior of \(\delta S_{\ell,m}\) differs.

In the g-mode region \((n \leq -2)\), as the cubic order effect is proportional to \(n^2 / (\ell + 1)^2 \delta \Omega\) (see Eq. (20)), the smaller the frequency is, the higher cubic order terms are, and this effect increases with the rotation rate. In this region, cubic order effects overtake those from latitudinally differential rotation.

In the p-mode region \((n > 2)\), latitudinal shear effects are larger than for g-modes. Cubic order effects are of same order of magnitude as the contribution from latitudinally differential rotation with a shear of only 12.7%.

In between, in the mixed mode region, cubic order effects with degeneracy corrections are of same order of magnitude as effects from latitudinally differential rotation for all the considered shears.

In order to get a better understanding of these different types of behavior, one can look at Fig C.1 in Appendix C. Figure C.1 displays mode inertia (defined in Eq. (30)) as a function of the radial order, for the model described in Table 2 uniformly rotating at 30 \% \(\Omega_\odot\). From this figure, the three frequency domains related to the nature of modes are clearly visible: below g_3 \((n \in [-10, -3])\) are pure g-modes, above p_2 are pure p-modes, and in between is located the mixed mode region.

In the p-mode domain, \(\Delta m\) decreases, while \(\mathcal{H}_m\) increases causing the near degeneracy contribution to the splitting to increase. Therefore, the near degeneracy effect is larger in the pure p-mode region. This causes the departure from a linear splitting due to cubic order effects including near degeneracy to be of the same order of magnitude as the latitudinal shear contribution (Fig 5).
6.2. Departure from a linear splitting as a function of the rotation rate

We studied the relative values of third order, near degeneracy and latitudinal shear contributions to the splitting as a function of the rotation rate mode by mode. Figure 6 shows three different cases:

- the $g_2$ mode illustrates the case of high order g-modes (or pure g-modes) where the cubic order contribution overtakes the latitudinal shear contribution as soon as the mean rotation rate exceeds 15% $\Omega_0$ (which corresponds to a rotation velocity of around 80 km$s^{-1}$).
- the $g_2$ mode is located in the mixed mode region where cubic order effects are larger than latitudinal differential rotation ones.
- the $p_1$ mode is still a mixed mode but with a nature closer to a pure p-mode for which near degeneracy is no longer negligible. As a result, the total cubic contribution including degeneracy is of same order as a latitudinal shear of 12.7%.

6.3. How do we disentangle the two effects?

In the previous subsection, we have shown that for a massive star on the main sequence, in the frequency range where we expect pulsation modes, it is not easy to conclude whether a departure from a linear splitting is due to latitudinally differential rotation or to cubic order effects. Here we suggest a method to disentangle the two effects.

Let us consider two modes: $g_2$ and $p_1$ as representative of a mixed mode close to g-modes and a mixed mode close to p-modes, respectively, as seen in the previous section. In Fig. 7 $S_l(p_1)$ versus $S_g(g_2)$ is plotted for the two different assumptions (latitudinal shear or cubic order effects with near degeneracy corrections) for different rotation rates ($\Omega \leq 35%\Omega_0$). Note that for g-modes, $\beta$ is roughly equal to 1/2, and for p-modes $\beta$ approaches 1. Accordingly, the ratio between $S_l(p_1)$ and $S_l(g_2)$ does not depend on $\Delta\Omega$ and is approximatively equal to 2. On the other hand, the curve $S_l(p_1)$ as a function of $S_l(g_2)$ for splittings including cubic order effects with near degeneracy corrections starts to deviate from a slope of 2 when the rotation rate is large enough. The deviation increases with the rotation rate as expected.

Let us now define $S^{\text{obs}}_l$ as the observed splittings for $l = 1$ modes for a fast rotator with a uniform rotation profile $\Omega_{\text{true}}$. Their values are then assumed to be given by $S^{\text{obs}}_l = S_l^0(\Omega_{\text{true}})$ (Eq. 22) i.e. rotational splittings computed up to cubic order, including near degeneracy corrections.

If one misinterprets $S^{\text{obs}}_l$ as due to a latitudinal shear, $S^{\text{obs}}_l$ is assumed to obey Eq. 23. The point representing $S^{\text{obs}}_l(p_1)$ as a function of $S^{\text{obs}}_l(g_2)$ ought to lie on the straight line with slope 2 in Fig. 7. We represent this point with the splittings $S^{\text{obs}}_l(p_1)$ and $S^{\text{obs}}_l(g_2)$ computed according to Eq. 22 assuming a rotation rate of 15%$\Omega_0$, $v = 80 \text{ km.s}^{-1}$. As the observed point is not located on the straight line given by $S^{\text{lat}}_l(p_1) = 2S^{\text{lat}}_l(g_2)$, we are then able to conclude that the deviation from a linear splitting is not due to a latitudinal shear.

In the case of massive stars on the main sequence, the effects of latitudinal differential rotation and cubic order or near degeneracy are of the same order of magnitude in the frequency domain where we expect to observe oscillation modes. This is mostly due to the mixed nature of modes around the fundamental frequency, and should therefore depend on the evolutionary stage of the star. But if two individual rotational splittings are available (one g-mode under the mixed mode region, and one p-mode above it), this method helps to disentangle whether a departure from a linear splitting is due to cubic order contributions including accidental degeneracy, or latitudinal shear.

7. The case of solar type stars

Low mass main sequence stars are known to be slow rotators. Indeed due to their outer convection zone, they undergo magnetic braking during their evolution (Kawaler 1988). Observational evidence exists for surface latitudinal differential rotation (coming from stellar spots due again to their outer convection zone). Hence, for these stars, the averaged surface rotation rate $\Omega$ is much smaller compared to that of more massive stars, such as $\beta$ Cephei stars, and $\Delta\Omega = \Omega_{\text{equi}} - \Omega_{\text{pole}}$, the difference between the rotation rates at the equator and the poles, can be large (25% for the Sun, between 1% and 45% for a star like Procyon, Bonanno et al., 2007). One therefore expects that latitudinal corrections to the splittings will dominate over cubic order effects which are negligible. As illustrative examples, we studied the case of HD 181906 and HD 181420, which are two solar like stars observed by CoRoT during the first long run, and which lightcurves have been analysed by García et al. (2009) and Burban et al. (2009) respectively.

Before comparing the different contributions to the rotational splitting, one should wonder whether a perturbative approach up to the cubic order accounting for near degeneracy effects is valid for seismic interpretation purpose for this type of star. To answer this issue, we rely on the validity study done in Suárez et al. (2010). This study has been performed for a polytropic model of 1.3 $\text{M}_\odot$, that can be representative of HD 181420. Therefore we consider this study as well adapted in order to determine the validity of our approach for computing high order pressure pulsation modes which are propagating in the outer layers of a solar like star. In this work Suárez et al. (2010) show that for rotational velocities under approximately 40km.s$^{-1}$, the structures of the frequency spectra computed with a non perturbative and their perturbative method are very similar. Here, we study the rotational splitting (that is a frequency difference which removes
part of second order effects), with a cubic order perturbative approach accounting for the effect of near degeneracy. We then consider our approach as valid for rotation velocities at stake in the stars we study in this article.

7.1. Competition between the three effects

Here we investigate the order of magnitude of deviations from Eq. (24) for an $M = 1.36 \, M_\odot$, $R = 1.63 \, R_\odot$, main sequence stellar model (see Table 3 for the stellar parameters taken as a model representative of HD 181420).

Table 3. Stellar parameters of the model of HD181420 (Sect. 6)

| Parameter | Value |
|-----------|-------|
| $M$       | $1.36 \, M_\odot$ |
| $L$       | $4.41 \, L_\odot$ |
| $P_c$     | $183 \, \text{dyn cm}^{-2}$ |
| $\rho_c$  | $5.1 \times 10^{-10} \, \text{g cm}^{-3}$ |

In Fig. 8 we plot the three kinds of contributions – i.e. cubic order, cubic order with near degeneracy, and latitudinal shear – to ($\ell = 1$, $m = 1$) splittings as a function of the central mode frequency for two different rotational angular velocities (3.7% $\Omega_0$ and 7.4% $\Omega_0$), which correspond to 15 km s$^{-1}$ and 30 km s$^{-1}$, respectively, for the model described in Table 3. Once again, these contributions show a peculiar behavior in the mixed mode region, where the two types of contributions can be of the same order. We focus here on higher frequencies, since the oscillation modes of HD 181420 are detected in the 1.5 – 2 mHz frequency range (Barban et al. 2009). Therefore, these plots show that even for rotation rates which are high for this type of star (7.4% $\Omega_0$), cubic order contributions (with or without near degeneracy) can be neglected compared to additional terms due to latitudinal shear. This leads to the conclusion that for HD 181420 in particular – as we expect for solar-like stars in general – the cubic order contributions to the splitting can safely be neglected in comparison with potential latitudinal shear contributions. The same computations have been performed for a 1.2 $M_\odot$, 1.4 $R_\odot$ stellar model representative of HD 181906, and lead to the same conclusions.

7.2. Determination of a latitudinal shear from seismic observations:

The results of the seismic analysis provided in Barban et al. (2009) and García et al. (2009) both find significantly different values for the low frequency peak in the Fourier spectrum and what is interpreted as the rotational splitting. If a uniform surface rotation is assumed, it is not possible to reproduce such differences between these observables. Let us assume a rotation profile of the form:

$$\Omega(\theta) = \Omega_0 - \Delta \Omega \cos^2 \theta$$  \hspace{1cm} (39)

where $\Omega_0$ is the rotation surface angular velocity at the equator, $\Omega_{\text{equator}}$, and $\Delta \Omega = \Omega_{\text{equator}} - \Omega_{\text{pole}}$. The rotation frequency is then given by:

$$\nu_{\text{rot}}(\theta) = \nu_{\text{eq}} \left(1 - \frac{\Delta \Omega}{\Omega_0} \cos^2 \theta \right)$$  \hspace{1cm} (40)

where $\nu_{\text{eq}}$ and $\nu_{\text{rot}}$ correspond to the equatorial rotation rate (i.e. $\Omega_0$) and the rotation frequency at the colatitude $\theta$ in mHz respectively.

Using again the rotation law Eq. (39) to derive the rotational splitting from Eq. (27), we obtain for $\ell = 1$, $m = \pm 1$:

$$S_{\text{lat}}^{\ell = 1} = \frac{\Omega_0}{\Omega_k} \beta \left(1 - \frac{1}{5} \frac{\Delta \Omega}{\Omega_0} \right)$$  \hspace{1cm} (41)

which can be written under the following form:

$$v_{\text{split}} = v_{\text{eq}} \beta \left(1 - \frac{1}{5} \frac{\Delta \Omega}{\Omega_0} \right)$$  \hspace{1cm} (42)

where $v_{\text{split}}$ corresponds to $S_{1,1}$. Then Eq. (40) and Eq. (42) are two equations with three unknowns: $v_{\text{eq}}$, $\cos \theta$ and $\Delta \Omega/\Omega_0$. Dividing Eq. (42) by Eq. (40) leads to:

$$\frac{\Delta \Omega}{\Omega_0} = \frac{1 - \beta d}{\cos \theta^2 - \frac{1}{5} \beta d}$$  \hspace{1cm} (43)

Where we have introduced the parameter:

$$d = \frac{v_{\text{rot}}}{v_{\text{split}}}$$  \hspace{1cm} (44)
Fig. 9. Possible values for $\Delta \Omega/\Omega_0$, depending on the parameter $d$ (Eq. 44) and the value of the integral $\beta$ (Eq. 55). The line colors correspond to the value of $(d \times \beta)$ which ranges here from 0.3 (in blue) to 1.5 (in red). In grey is the acceptable domain of values for the latitudinal gradient (Eq. 46). This plot has been performed for high order $p$ modes.

We then determine $\Delta \Omega/\Omega_0$ as a function of $\cos \theta$, along with the following constraints:

$$\theta \in \left[0, \frac{\pi}{2}\right] \Rightarrow \cos \theta \in [0; 1]$$

$$\frac{\Delta \Omega}{\Omega_0} \in [-1; 1]$$

As illustrated in Fig. 9 the latitudinal shear is a hyperbolic function of $\cos \theta$, centered in $\sqrt{\beta d}/5$. Depending of the values of the product $\beta d$ of the observational parameter $d$ and of the integral $\beta$, graphically we find different possible ranges of values for $\Delta \Omega/\Omega_0$ and $\cos \theta$.

7.3. The case of HD 181906

HD 181906 is an F8 dwarf for which fundamental parameters have been determined by Bruntt (2009), considering the presence of a background star: $m_v = 7.65$, $L/L_\odot = 3.32 \pm 0.45$, $T_{eff} = (6300 \pm 150)$K, $[\text{Fe}/\text{H}] = -0.11 \pm 0.14$, $M/M_\odot = 1.144 \pm 0.119$, $v \sin i = (10 \pm 1)$km.s$^{-1}$. Note that Nordström et al. (2004) had found $v \sin i = (16 \pm 1)$km.s$^{-1}$ considering HD 181906 as single, with no background star. It has been observed during the first long run of CoRoT, and its light curve has been analysed by Garcia et al. (2009). They find two possible interpretations for the mode identification, as listed in Table 4.

Considering the uncertainties attached to the seismic observables, the result obtained for the latitudinal shear is presented in Tab. 5. These results show that the hypothesis of latitudinal differential rotation is consistent with the available observables. Therefore, the latitudinal differential rotation profile is a reliable assumption concerning HD 181906. Moreover, The results define a range of possible values for different characteristics of the rotation profile: the latitudinal shear, the rotation surface velocity at the equator, as well the inclination angle. The correspondent error bars are discussed in Sect. 7.4. Both scenarii give the same conclusions very similar results.

Table 5. Latitudinal shear, equatorial velocity and inclination angle obtained for HD 181906, using models with rotation uniform in depth. The two last lines correspond to $i = \arcsin (v \sin i/v_{eq})$ with $v \sin i = (10 \pm 1)$km.s$^{-1}$ (Bruntt, 2009), or with $v \sin i = (16 \pm 1)$km.s$^{-1}$ (Nordström et al. 2004). Note that the two spots give hardly the same results, we present here mean values computed for the two surface rotation velocities.

| $\Delta \Omega/\Omega_0$ | Scenario A | Scenario B |
|------------------------|------------|------------|
| $v_{eq}$ (km.s$^{-1}$)  | $0.35 \pm 0.31$ | $0.39 \pm 0.18$ |
| $i$ (Bruntt 2009) (°)   | 41.3 ± 2.5  | 43.8 ± 1.6 |
| $i$ (Nordström 2004) (°) | 26 ± 2      | 24 ± 2     |

7.4. Solar or anti-solar latitudinal shear for HD 181420?

HD 181420 is an F2 main sequence star whose fundamental parameters have been determined by Bruntt (2009): $m_v = 6.57$, $L/L_\odot = 4.28 \pm 0.28$, $T_{eff} = (6580 \pm 105)$K, $[\text{Fe}/\text{H}] = 0.00 \pm 0.06$, $M/M_\odot = 1.31 \pm 0.063$, $v \sin i = (18 \pm 3)$km.s$^{-1}$. It has been observed during the first long run of CoRoT, and its light curve has been analysed by Barban et al. (2009). They find two possible interpretations for the mode identification, as listed in Table 6. Later Bedding & Kjeldsen (2010) used some empirical scaling method that seemed to favor scenario 1.

Table 6. Results concerning rotation from the analysis of HD 181420 performed in Barban et al. (2009).

| $\nu_{rot}$ (µHz) | Scenario 1 | Scenario 2 |
|-------------------|------------|------------|
| $\nu_{split}$ (µHz) | 5.8 ± 0.14 | 6.1 ± 0.14 |
| $\nu_{split}$ (µHz) | 5.8 ± 0.14 | 6.1 ± 0.14 |

Concerning HD 181420, our model gives unexpected results: in both scenarii, the latitudinal shear is found negative, i.e. the pole rotate faster than the equator, which is the opposite of what is known for the Sun. In order to appreciate the reliability of this result, one should refer to the work of Käpylä et al. (2011), who study the impact of rotation on turbulent angular momentum and heat transport in solar like stars convective zone, by the mean of direct numerical simulations of turbulent convection in spherical geometry. According to the authors, the rotation profile varies from anti-solar (equator rotates slower than poles) for low Coriolis number to solar (equator rotates faster), with a transition around Co = 3. The Coriolis number which measures the impact of rotation on turbulent motion is given by:

$$Co = 2 \Omega \tau_c$$

(47)
Where \( \tau \) is the convective overturning time scale -computed in the stellar evolution code CESAM (Morel & Lebreton, 2008)-, and \( \Omega \) is taken as the angular velocity at the equator. For HD 181420, when rotation is supposed to be uniform in depth, the Coriolis number is \((3.2 \pm 0.5)\) for scenario 1 and \((4.0 \pm 0.3)\), i.e. both values are only slightly above the transition threshold between anti-solar and solar surface rotation (see Fig. 17 of Kapyla et al. [2011]). Therefore, it is difficult to confirm that the physics at stake in the convective envelope of HD 181420 can lead to anti-solar rotation profile for HD 181420.

Let us refine the modeling of HD 181420, taking a tachocline-like profile. As explained in Sect. 5.2 it consists in assuming uniform rotation \((\Omega = \Omega_0)\) in the inner layers, and latitudinal differential rotation \((\Omega(\theta))\) in the convective envelop. The rotational splitting given by Eq. (41) is no longer relevant, and should be replaced by Eq. (47).

In this case, we found different values for \(\Omega_0\) than in the former case, as a consequence, the Coriolis number reaches \((8.9 \pm 1.7)\) for scenario 1 and \((11.8 \pm 0.7)\) for scenario 2, i.e. high above the transition limit between anti-solar and solar rotational shear. The integral \(\beta_c\) reaches a constant value of 0.45 for high order \(p\)-modes, which changes the relationship between the colatitude of the spot and the latitudinal shear \((\beta_s)\), and as a consequence changes the domain of possible shear. Positive latitudinal shear corresponding to solar type latitudinal rotation are obtained for the two scenarios, which is this time fully consistent with the Coriolis number values. The results obtained with this tachocline-like model are listed in Tab. 4.

Table 7. Latitudinal shear, equatorial velocity and inclination angle obtained for HD 181420, using models with tachocline-like rotation profile. The last line corresponds to \(i = \arcsin \left(\frac{\sin i}{\nu_{eq}}\right)\) with \(\nu_{eq} = (18 \pm 1)\) km s\(^{-1}\) (Brunt, 2009).

| Scenario 1 | Scenario 2 |
|---|---|
| \(\Delta \Omega/\Omega_0 = \) | 0.50 ± 0.45 | 0.66 ± 0.11 |
| \(\nu_{eq} (\mu Hz) = \) | 1.0 ± 0.2 | 1.3 ± 0.1 |
| \(\nu_{eq} (\text{km.s}^{-1}) = \) | 45.6 ± 8.7 | 60.1 ± 3.5 |
| \(i (\text{Brunt 2009}) (\degree) = \) | 23.2 ± 7.4 | 17.4 ± 2.0 |

The case of HD 181420 is particularly interesting as a simple model of rotation uniform in depth and differential in latitude leads to non-physical latitudinal shear. Only a little more sophisticated model, where rotation varies in depth according to a tachocline-like profile, is fully coherent with the physics of the convective zone as well as the observables. We also give a range of possible values for the rotation profile parameters: the latitudinal shear in the convective zone, the rotation surface velocity at the equator and the inclination angle. To conclude, for this star, we find that rotation profile inside the star should rather be a tachocline like profile, with a solar type latitudinal rotation. Moreover, these results, when compared to those of Mosser et al. (2009), seem to favor scenario 1 since the rotational frequency at the equator in the hypothesis of scenario 1 is closest to the one found in Mosser et al. (2009) by spot modelling -\((5.14 \pm 0.07) \times 10^{-6} \text{rad.s}^{-1}\).

7.5. Discussion

As already mentioned in the beginning of this section, Suárez et al. (2014) have found that second order perturbative methods including near-degeneracy corrections are valid for rotation velocity up to approximately 40 km s\(^{-1}\). In this study, third order perturbative methods have been used for the computation of rotational splittings in Sect. 7.1. In Sect. 7.2, is given a simple recipe which allows to compute the latitudinal shear given seismic observables. This formulation rely on the validity of the third order perturbative method, but the only quantity provided by the modeling are \(\beta\) (Eq. B.5) and \(\beta_c\), i.e. quantities computed from first order eigenfunctions. The question is then: are eigenfunctions sufficiently varying with rotation rate (under 60 km s\(^{-1}\) i.e. 15 \(\%\) \(\Omega_0\)), to impact the value of \(\beta\)? After verifications, it appears that for high order acoustic modes in moderately rotating stars (under 15 \(\%\) \(\Omega_0\)) \(\beta \approx 1\), and \(\beta_c \approx 0.45\) are satisfying evaluations. Therefore the recipe is valid for rotation rates at stake in Sect. 7.3 and 7.4. Note the convenience of the proposed recipe which only need for seismic observables, and the values of \(\beta\).

The large error bars obtained in particular for the values of latitudinal shears (Tab. 5, 7) is not only due to the observational uncertainties, but it can also be attributed to the simplicity of the spot model used. This model does not account for spot parameters such as for example a spot lifetime, or the spot distribution on the observed stellar disc (for more sophisticated modelling see Mosser et al. 2009). We only consider a unique spot of infinite lifetime. Note that the use of a more complicated model would require more observational constraints than only one spot rotation signature. When only mean values of rotational splittings are available, and the observational error bars on the low frequency signature of a spot rotation are large (Barban et al. 2009), we are able to give general conclusions concerning the rotation profile -i.e. uniform in depth or tachocline like, solar or anti-solar-, but no reliable numerical values caracterising it.

8. Conclusions

With the help of the perturbative approach established in Soufi et al. (1998), we investigated second and third order contributions of the Coriolis and the centrifugal accelerations to \(p\) and \(g\)-modes frequencies, as well as near degeneracy effects on the rotational splittings. Their effects were then compared with those of a latitudinal shear. We studied two types of oscillating stars.

For an evolved model of a \(\beta\) Cephei, the effects of near degeneracy, cubic order perturbations and latitudinal shear are of the same order of magnitude in the frequency range, relevant to such stars – i.e. low order \(p\)- and \(g\)-modes – and for reasonable rotation rates ranging from 15\(\%\) \(\Omega_0\) to 30\(\%\) \(\Omega_0\). Nevertheless, when two individual splittings for modes of a different nature (a pure \(g\) and a mixed mode for example) are available, a method is proposed to distinguish between a latitudinal shear and the other effects.

For solar-like stars such as HD 181420 and HD 181906, which are mostly moderate rotators and oscillate with high order \(p\) modes, cubic order effects on frequency splittings are shown to be small in front of the effects of latitudinally differential rotation. Therefore, given a splitting and a rotation period signature, it is possible to infer a range for the latitudinal shear coefficient \(\Delta \Omega/\Omega_0\). Although no precise determination of latitudinal shear is possible unless the spot latitude is fully determined, we have been able to determine the most reliable rotation profile for each of the two stars. More over the determination of the latitudinal shear by our seismic method can be taken as a constraint to be added to other observational constraints, such as those coming from spectropolarimetric results (e.g. Donati et al. 2010) or spot modelling (e.g. Mosser et al. 2009).
Appendix A: Near degeneracy corrections to the splittings of high order p-modes

This section is dedicated to a qualitative estimation of the near degeneracy corrections to rotational splittings:

$$S_{\ell=1, m=1}^{\text{deg}} - S_{\ell=1, m=1}^{\text{ND}} = \frac{\sqrt{\Delta_1^2 + \mathcal{H}_1^2} - \sqrt{\Delta_{-1}^2 + \mathcal{H}_{-1}^2}}{2} - \frac{1}{2}(\Delta_1 - \Delta_{-1})$$  \hspace{1cm} (A.1)

where $\Delta_1$ and $\mathcal{H}_1$ are normalized by the break-up frequency $\Omega$. The oscillation frequency given in Eq. (8) can be rewritten in the following the generic form:

$$\sigma_{n',m} = \sigma_{0,n',m}^0 + \frac{\Omega}{\Omega_k} \beta_{n',3} + (d_{n,1,1} - d_{n,3,1}) + (t_{n,1,1} - t_{n,3,1})$$  \hspace{1cm} (A.2)

where $\Delta_0^0$ stands for $\sigma_{0,n',1} - \sigma_{0,n',3}$, the difference between the eigenfrequencies without rotation.

For high frequency p-modes, the radial contributions are concentrated in the outer layers and are then nearly the same for $(n, \ell = 1)$ and $(n', \ell' = 3)$. It is then legitimate to neglect the differences in their radial contributions. We then omit the radial order subscripts $n, n'$. For the same reason, the Ledoux constants are also quite similar $\beta_1 \sim \beta_3$. Then

$$\Delta_1 \approx \Delta_0^0 + (d_{1,1} - d_{3,1}) + (t_{1,1} - t_{3,1})$$

Similarly

$$\Delta_{-1} \approx \Delta_0^0 + (d_{1,1} - d_{3,1}) - (t_{1,1} - t_{3,1})$$

Hence, the quantity $(\Delta_1 - \Delta_{-1})$ in Eq. (A.1):

$$\Delta_1 - \Delta_{-1} \approx 2(t_{1,1} - t_{3,1})$$  \hspace{1cm} (A.3)

is of third order. One can then approximate $\Delta_{-1} \sim \Delta_1$ in the square root of Eq. (A.1), and the correction due to near degeneracy is given by:

$$S_{\ell=1}^{\text{deg}} - S_{\ell=1}^{\text{ND}} \approx (t_{1,1} - t_{3,1})$$

$$\frac{\sqrt{\Delta_1^2 + \mathcal{H}_1^2} - \sqrt{\Delta_{-1}^2 + \mathcal{H}_{-1}^2}}{2}$$  \hspace{1cm} (A.4)

For the coupling term, one can separate the second $\mathcal{H}_2$ and third $\mathcal{H}_3$ order terms as $\mathcal{H}_m = \mathcal{H}_2 + m\mathcal{H}_3$. Thus $\mathcal{H}_1 = \mathcal{H}_2 + \mathcal{H}_3$, and $\mathcal{H}_{-1} = \mathcal{H}_2 - \mathcal{H}_3$. With $|\mathcal{H}_3| << |\mathcal{H}_2|$, the near degeneracy correction to the splitting becomes:

$$S_{\ell=1}^{\text{deg}} - S_{\ell=1}^{\text{ND}} \approx (t_{1,1} - t_{3,1}) + \frac{2\mathcal{H}_3\mathcal{H}_5}{\sqrt{\Delta_1^2 + \mathcal{H}_1^2}}$$  \hspace{1cm} (A.5)

which shows that for slow rotators, such as HD 181420, if cubic order effects are neglected, then the near degeneracy contribution is zero, and the splitting is a linear function of rotation. If cubic order effects are included, then near degeneracy corrections affect the splitting, and the departure from a linear splitting is dominated by distortion (predominantly in $\mathcal{H}_2$).
Appendix B: Contribution of the latitudinal shear to the splittings

In order to be able to compute the splittings from Eq. (1) and Eq. (17), one must specify a rotation law. It is convenient to assume the following form:

$$\Omega(r, \theta) = \sum_{n=0}^{n_{max}} \Omega_n(r) (\cos \theta)^{2n}$$

(B.1)

where $\theta$ is the colatitude and we have limited our investigation $s_{max} = 1$. The surface rotation rate at the equator is $\Omega(r = R, \theta = \pi/2) = \Omega_0(r = R)$.

Inserting Eq. (25) into Eq. (17) yields the following expression for the generalized splitting:

$$S_m = \frac{1}{\Omega_k} \int_0^R \frac{\Omega_0(r) K(r)}{\Omega_0} \, dr + \frac{1}{\Omega_k} \int_0^R m_s^2 H_{n,s}(\Omega) \, dr$$

(B.2)

with (see Goupil [2011]),

$$H_{n,s}(\Omega) = -1/1 \int_0^R R_s \left( \xi_s^2 - 2 \xi_s \xi_h + \xi_h^2 (\Lambda - 1) \right) dr$$

where $R_s$ and $Q_s$ depend on $\Omega_2$, $\Omega_4$ and $\Lambda = \ell(\ell + 1)$. For example, if shellular rotation is expected, then $\Omega(r, \theta) = \Omega_0(r)$ and $s_{max} = 0$. Moreover, $\Omega_2$, $\Omega_4$ = 0, i.e. $R_s = Q_s = 0$ for $j = 0, 2$, and:

$$S_m = \frac{1}{\Omega_k} \int_0^R \frac{\Omega_0(r) K(r)}{\Omega_0} \left[ \xi_s^2 - 2 \xi_s \xi_h + \Lambda \xi_h^2 \right] \rho_0 r^2 \, dr$$

(B.3)

If we consider latitudinally differential rotation only, $\Omega_{2,j}$ are depth independent, $\Omega_0 = \Omega_{equator}$, $\Omega_2 = -\Delta \Omega$, and $\Omega_4 = 0$. $R_s$ and $Q_s$ are constant and:

$$S_m = \frac{\Omega_0}{\Omega_k} \beta + \frac{1}{\Omega_k} \sum_{s=0}^{s_{max}} m_s^2 \left( R_s(\Omega) \beta + Q_s(\Omega) \gamma \right)$$

(B.4)

with $\beta$ and $\gamma$ defined as:

$$\beta = \frac{1}{1} \int_0^R \left[ \xi_s^2 - 2 \xi_s \xi_h + (\Lambda - 1) \xi_h^2 \right] \rho_0 r^2 \, dr$$

(B.5)

$$\gamma = \frac{1}{1} \int_0^R \xi_h^2 \rho_0 r^2 \, dr$$

(B.6)

and $I$ being the inertia of the mode:

$$I = \int_0^R \left( \xi_s^2 + \Lambda \xi_h^2 \right) \rho_0 r^2 \, dr$$

(B.7)

Then the rotational splitting of a $[n, \ell, m]$ mode is given by

$$S_{\ell,m} = \frac{1}{\Omega_k} \left[ \Omega_0 \beta + R_{\ell}^\prime \beta + Q_{\ell}^\prime \gamma + m^2 \left( R_{\ell}^\prime \beta + Q_{\ell}^\prime \gamma \right) \right]$$

(B.8)

In Goupil [2011], $Q_s$ and $R_s$ are given for $s = 0, 1, 2$:

$$R_0^\prime = \frac{\Delta \Omega}{\Omega_0} \frac{2 \Lambda - 1}{4 \Lambda - 3}$$

(B.9)

$$R_1^\prime = \frac{\Delta \Omega}{\Omega_0} \frac{2}{4 \Lambda - 3}$$

(B.10)

$$Q_0^\prime = \frac{\Delta \Omega}{\Omega_0} \frac{2(1 - 3\Lambda)}{4 \Lambda - 3}$$

(B.11)

$$Q_1^\prime = -\frac{\Delta \Omega}{\Omega_0} \frac{10}{4 \Lambda - 3}$$

(B.12)

For $\ell = 1$ and $\ell = 2$ modes, this yields:

a. for $\ell = 1, m = \pm 1$ (i.e. $\Lambda = 2$):

$$S_{1,1} = \frac{\Omega_0}{\Omega_k} \beta \left[ 1 - \frac{1}{5} \frac{\Delta \Omega}{\Omega_0} \right]$$

(B.13)

b. for $\ell = 2, m = \pm 1$ (i.e. $\Lambda = 6$):

$$S_{2,1} = \frac{\Omega_0}{\Omega_k} \left[ \beta + \frac{1}{7} \frac{\Delta \Omega}{\Omega_0} (8 \gamma - 3 \beta) \right]$$

(B.14)

c. for $\ell = 2, m = \pm 2$ (i.e. $\Lambda = 6$):

$$S_{2,2} = \frac{\Omega_0}{\Omega_k} \left[ 2 \beta - \frac{1}{7} \frac{\Delta \Omega}{\Omega_0} (2 \gamma + \beta) \right]$$

(B.15)

Appendix C: Conditions for significant near degeneracy

As already mentioned, near degeneracy between two modes occurs whenever their frequencies are close, their azimuthal numbers are equal, and their angular degrees differ by 2. However the magnitude of the near-degenerate corrections to both frequencies also depends on the magnitude of the coupling term $H_m$. In turn the magnitude of $H_m$ depends on the nature of the involved modes whether they are g-modes, p-modes or mixed modes with a dominant p or g nature.

The g-modes spectrum is much denser than the p-mode one. Hence, as shown in Fig. C.2, $\Delta m$ is much smaller than for p-modes. This ought to favor near degeneracy. However, the coupling term $H_m$ for g-modes is much smaller than for p-modes. This is due to the fact that distortion effects are small for g-modes and therefore the overall (second and third order) contribution to $H_m$ remains small. As a result, the coupling term is much smaller than $\Delta m$ for g-modes, which are then hardly coupled.

Fig. C.1. Inertia of axisymmetric $\ell = 1$ modes as a function of the radial order, $n$, for an evolved 8.5 M_⊙ β Cephei model, uniformly rotating at 30% $\Omega_k$, i.e. 160 km s^{-1} (see Table II Sect. 4.1).
Fig. C.2. The terms $\Delta_m$ and $\mathcal{H}_m$ which are involved in near degeneracy corrections (see Eq. (13)) for $\ell = 1$ modes computed for the same $\beta$ Cephei model as in Fig. C.1. A logscale is used for the y axis.
Appendix D: Values for third order contributions to the frequencies and to the splittings

Table D.1. Scaled contributions to the splittings of \( \ell = 1 \) g-modes due to: implicit third order terms in the eigenfrequency \( (T^{\text{r1con}}) \), Coriolis effects \( (T^C) \), distorsion \( (T^D) \), and coupling of the two \( (T^E) \). The impact of near degeneracy is so small that its contribution is fully negligible and is therefore not listed. The first row lists the radial order and the second row the centroid mode \( m = 0 \) frequency.

| n  | \( \sigma_n \) | \( T^{\text{r1con}}/\sigma_n^2 \) | \( T^C/\sigma_n^2 \) | \( T^D/\sigma_n^2 \) | \( T^E/\sigma_n^2 \) |
|----|----------------|---------------------------------|-----------------|-----------------|-----------------|
| −8 | −0.29 0.705E + 02 | 0.391E + 04 | −0.214E + 02 | −0.424E + 03 | −0.354E + 03 |
| −6 | 0.34 0.770E + 02 | 0.379E + 02 | −0.362E + 03 | −0.362E + 03 | −0.362E + 03 |
| −5 | 0.41 0.943E + 02 | 0.518E + 02 | −0.354E + 03 | −0.354E + 03 | −0.354E + 03 |
| −4 | 0.52 0.123E + 03 | 0.683E + 02 | −0.371E + 03 | −0.371E + 03 | −0.371E + 03 |
| −3 | 0.70 0.155E + 03 | 0.905E + 02 | −0.374E + 03 | −0.374E + 03 | −0.374E + 03 |
| −2 | 1.06 0.850E + 02 | 0.776E + 03 | 0.116E + 03 | −0.318E + 03 | −0.318E + 03 |
| −1 | 3.52 −0.269E + 04 | 0.833E + 03 | −0.439E + 03 | 0.851E + 03 | 0.851E + 03 |

Table D.2. Scaled contributions to the splittings of \( \ell = 1 \) p-modes due to third order effects divided by the square of the central mode frequency. The first row lists the radial order and the second row the centroid mode \( m = 0 \) frequency.

| n  | \( \sigma_n \) | \( T^{\text{r1con}}/\sigma_n^2 \) | \( T^C/\sigma_n^2 \) | \( T^D/\sigma_n^2 \) | \( T^E/\sigma_n^2 \) |
|----|----------------|---------------------------------|-----------------|-----------------|-----------------|
| −1 | 4.73 −0.140E + 03 | 0.409E + 02 | −0.181E + 02 | 0.433E + 02 | 0.697E + 00 |
| 1  | 5.78 −0.106E + 03 | 0.285E + 02 | −0.115E + 02 | 0.343E + 02 | 0.174E + 02 |
| 3  | 6.85 −0.862E + 02 | 0.181E + 02 | −0.554E + 01 | 0.264E + 02 | 0.318E + 02 |
| 4  | 7.96 −0.727E + 02 | 0.112E + 02 | −0.666E + 00 | 0.192E + 02 | 0.380E + 02 |
| 5  | 9.11 −0.636E + 02 | 0.767E + 01 | 0.217E + 01 | 0.145E + 02 | 0.435E + 02 |
| 6  | 10.30 −0.566E + 02 | 0.217E + 01 | 0.110E + 02 | 0.447E + 02 | 0.447E + 02 |
| 7  | 11.49 −0.514E + 02 | 0.494E + 01 | 0.456E + 01 | 0.884E + 01 | 0.472E + 02 |
| 8  | 12.67 −0.477E + 02 | 0.440E + 01 | 0.472E + 01 | 0.762E + 01 | 0.503E + 02 |
| 9  | 13.82 −0.447E + 02 | 0.374E + 01 | 0.471E + 01 | 0.688E + 01 | 0.540E + 02 |
| 10 | 14.96 −0.417E + 02 | 0.290E + 01 | 0.489E + 01 | 0.601E + 01 | 0.555E + 02 |
| 11 | 16.11 −0.387E + 02 | 0.221E + 01 | 0.523E + 01 | 0.494E + 01 | 0.543E + 02 |
| 12 | 17.28 −0.360E + 02 | 0.181E + 01 | 0.552E + 01 | 0.395E + 01 | 0.527E + 02 |
| 13 | 18.46 −0.338E + 02 | 0.161E + 01 | 0.570E + 01 | 0.316E + 01 | 0.516E + 02 |
| 14 | 19.63 −0.318E + 02 | 0.149E + 01 | 0.580E + 01 | 0.252E + 01 | 0.508E + 02 |
| 15 | 20.81 −0.301E + 02 | 0.140E + 01 | 0.585E + 01 | 0.196E + 01 | 0.498E + 02 |
### Table D.3. Different order contributions to the mode frequencies for various radial orders, n. All contributions are scaled by $\Omega_\kappa$.

| n  | $\sigma_0$ | $\sigma_1$ | $\sigma_{\text{geom}}^{\text{mod}}$ | $\sigma_2^{\text{mod}}$ | $\sigma_3^{\text{mod}}$ | $\sigma_4^{\text{mod}} - \sigma_{\text{geom}}^{\text{mod}}$ |
|----|-----------|-----------|---------------------------------|----------------|----------------|----------------------------------|
| -8 | 0.25      | 0.77E-01  | 0.989E-02                       | 0.25E+01       | -1.06E+03      | -1.06E+03                       |
| -7 | 0.29      | 0.782E-01 | 0.857E-02                       | 0.193E-01      | -0.993E-03     | -0.193E-03                      |
| -6 | 0.34      | 0.790E-01 | 0.713E-02                       | 0.136E-01      | -0.773E-03     | -0.773E-03                      |
| -5 | 0.41      | 0.802E-01 | 0.562E-02                       | 0.088E-01      | -0.392E-03     | -0.392E-03                      |
| -4 | 0.52      | 0.819E-01 | 0.407E-02                       | 0.041E-01      | -0.052E-03     | -0.052E-03                      |
| -3 | 0.70      | 0.839E-01 | 0.252E-02                       | 0.007E-01      | -0.007E-03     | -0.007E-03                      |
| -2 | 1.06      | 0.859E-01 | 0.954E-02                       | 0.037E-01      | -0.037E-03     | -0.037E-03                      |
| -1 | 3.52      | 0.142E+00 | 0.677E-02                       | 0.312E-02      | -0.312E-02     | -0.312E-02                      |
| 1  | 4.73      | 0.141E+00 | 0.497E-02                       | 0.143E-02      | -0.143E-02     | -0.143E-02                      |
| 2  | 5.78      | 0.141E+00 | 0.401E-02                       | 0.118E-02      | -0.118E-02     | -0.118E-02                      |
| 3  | 6.85      | 0.142E+00 | 0.337E-02                       | 0.117E-02      | -0.117E-02     | -0.117E-02                      |
| 4  | 7.96      | 0.142E+00 | 0.288E-02                       | 0.107E-02      | -0.107E-02     | -0.107E-02                      |
| 5  | 9.11      | 0.143E+00 | 0.251E-02                       | 0.097E-02      | -0.097E-02     | -0.097E-02                      |
| 6  | 10.30     | 0.144E+00 | 0.221E-02                       | 0.089E-02      | -0.089E-02     | -0.089E-02                      |
| 7  | 11.49     | 0.145E+00 | 0.197E-02                       | 0.082E-02      | -0.082E-02     | -0.082E-02                      |
| 8  | 12.67     | 0.145E+00 | 0.178E-02                       | 0.075E-02      | -0.075E-02     | -0.075E-02                      |
| 9  | 13.82     | 0.146E+00 | 0.163E-02                       | 0.069E-02      | -0.069E-02     | -0.069E-02                      |
| 10 | 14.96     | 0.146E+00 | 0.150E-02                       | 0.063E-02      | -0.063E-02     | -0.063E-02                      |
| 11 | 16.11     | 0.146E+00 | 0.139E-02                       | 0.057E-02      | -0.057E-02     | -0.057E-02                      |
| 12 | 17.28     | 0.147E+00 | 0.129E-02                       | 0.052E-02      | -0.052E-02     | -0.052E-02                      |
| 13 | 18.46     | 0.147E+00 | 0.121E-02                       | 0.044E-02      | -0.044E-02     | -0.044E-02                      |
| 14 | 19.63     | 0.147E+00 | 0.114E-02                       | 0.037E-02      | -0.037E-02     | -0.037E-02                      |
| 15 | 20.81     | 0.147E+00 | 0.107E-02                       | 0.030E-02      | -0.030E-02     | -0.030E-02                      |