Slow-Fast Normal Forms Arising from Piecewise Smooth Vector Fields

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Abstract
We study planar piecewise smooth differential systems of the form

\[ \dot{z} = Z(z) = \frac{1 + \text{sgn}(F)}{2} X(z) + \frac{1 - \text{sgn}(F)}{2} Y(z), \]

where \( F : \mathbb{R}^2 \to \mathbb{R} \) is a smooth map having 0 as a regular value. We consider linear regularizations \( Z^\phi_\varepsilon \) of \( Z \) by replacing \( \text{sgn}(F) \) by \( \phi(F/\varepsilon) \) in the last equation, with \( \varepsilon > 0 \) small and \( \phi \) being a transition function (not necessarily monotonic). Nonlinear regularizations of the vector field \( Z \) whose transition function is monotonic are considered too. It is a well-known fact that the regularized system is a slow-fast system. In this paper, we study typical singularities of slow-fast systems that arise from (linear or nonlinear) regularizations, namely, fold, transcritical and pitchfork singularities. Furthermore, the dependence of the slow-fast system on the graphical properties of the transition function is investigated.

Keywords
Piecewise smooth vector fields · Geometric singular perturbation theory · Regularization of piecewise smooth vector fields · Transition function

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Dedicated to the memory of Jorge Sotomayor Tello
1 Introduction

In real life, there are phenomena whose mathematical models are expressed by piecewise smooth vector fields, which have been studied at least since 1937. These systems are used in many branches of applied sciences, for example, Physics, Control Theory, Economics, Cell Mitosis, etc. For more details see, for instance, [3, 5].

A piecewise smooth vector field (or PSVF for short) is defined as follows: let $\Sigma$ be a closed subset with empty interior of the ambient space (for example, a manifold embedded in $\mathbb{R}^n$). Such subset is called discontinuity locus, and it divides the ambient space in finitely many open subsets $\{U_i\}_{i=1}^k$. In each open subset, $U_i$ is defined a smooth vector field. This paper deals with the case where a smooth curve divides a neighbourhood of $0 \in \mathbb{R}^2$ in two open regions. See Section 2 for a precise definition.

One of the most important questions concerning PSVF’s is how to define the dynamics in $\Sigma$? In other words, how to define the transition between the dynamics defined in two different open sets?

Filippov [5] gave an answer defining the dynamics in $\Sigma$ as the convex combination of two vector fields. This defines the so called sliding vector field. We say that this vector field defined according to Filippov’s ideas follows the Filippov’s convention.

However, for some models, the Filippov’s convention is not sufficient to describe the dynamics. For example, in [15], a model involving friction between an object and a flat surface was studied. The author gave an example that Filippov’s convention takes into account only kinetic friction, while it is possible to consider static friction as well.

Another way to define the dynamics in the discontinuity locus $\Sigma$ is combining two powerful tools: regularizations of PSVF’s and blow-ups. A regularization process that is compatible with the Filippov’s convention is the Sotomayor-Teixeira regularization [22], which consists in obtaining a one-parameter family of smooth vector fields $Z_\varepsilon$ converging to $Z$ when $\varepsilon \to 0$ (see Section 2.2). By using blow-up techniques, the regularized system $\dot{z} = Z_\varepsilon(z)$ becomes a slow-fast system, and therefore, we are able to apply classical results on geometric singular perturbation theory (see Section 2.4) in the study of PSVF’s. Such a link between regularization processes and geometric singular perturbation theory is a recent approach in mathematics, and we refer to [1, 16–20] for further details. A similar approach can also be seen in [10].

Different regularization processes lead to different slow-fast systems, which gives rise to different sliding or sewing regions (see [19–21]). In this paper, we consider linear regularizations and nonlinear regularizations. See Sections 2.2 and 2.3 for precise definitions.

The dynamics of the linearly regularized system depends on the so called transition function $\varphi$, which can be monotonic or non monotonic. In this paper, we highlighted the relation between the properties of the graph of $\varphi$, the properties of the slow-fast system, and the sliding regions of the PSVF’s. See Theorem A below.

The main goal of this paper is to study typical singularities of slow-fast systems that arise from (linear or nonlinear) regularizations. For both linear and nonlinear regularizations are presented examples of PSVF’s such that, after (linear or nonlinear) regularization and directional blow-up, the slow-fast system presents normally hyperbolic, fold, transcritical or pitchfork singularities.

At some point, the reader may think that, after linear regularization and blow-up, it is possible to generate any slow-fast singularity, since it is just a matter of a suitable choice of the transition function. In general, this is not true. Indeed, we show that it does not exist a transition function that generate a pitchfork singularity. However, if we consider nonlinear
regularizations, it is possible to generate such a singularity (see Example 13). This shows that nonlinear regularizations are more general than the linear ones (see also [18, 20]).

Our main results, Theorems A, B and C, are stated and proved in Section 3. In what follows, we briefly describe them.

Firstly, consider linear regularizations. Suppose that we drop the monotonicity condition of the transition function \( \varphi \). In this context, we will prove that the critical points of \( \varphi \) give rise to non normally hyperbolic points of the critical set \( C_0 \) of \( \dot{z} = Z_\varepsilon(z) \). For more details, see Item (a) of Theorem A.

In addition, item (b) of Theorem A assures that we extend the classical Filippov sliding region when the transition function satisfy \( |\varphi(x_0)| > 1 \) for some \( x_0 \) in the open interval \( (-1, 1) \). According to item (c) of the same Theorem, the dynamics in this extended sliding region is naturally defined using the classical Filippov sliding vector field. It is important to emphasize that item (c) of Theorem A was already proved in [21, Theorem 3]. For completeness sake, we incorporated it in the statement of Theorem A and proved it as well.

Finally, item (d) of Theorem A says that there are cases in which it is not possible to apply geometric singular perturbation theory in order to define the sliding dynamics in some points of \( \Sigma \) (see Fig. 1).

Slow-fast normal forms are well known in the literature (see Section 2.5 and the references therein). In Theorem B, we state conditions that both PSVF and transition function must satisfy in order to generate classical slow-fast normal forms, such as fold and transcritical singularities. Moreover, we prove that there are slow-fast normal forms that can not be generated by linear regularization processes. This is the case of the pitchfork singularity (see Fig. 2).

In order to generate pitchfork singularities, we must consider nonlinear regularization. Theorem C gives the conditions that must satisfy both monotonic transition function and vector field associated with the nonlinearly regularized system to generate this type of singularity.

Fold, transcritical and pitchfork singularities have very interesting dynamical properties. For example, M. Krupa, P. Szmolyan in [11, 12] studied the dynamics of the slow-fast system around this type of singularities for \( \varepsilon > 0 \) and built a map of transition between transversal sections. By applying these results together with Theorems B and C, one can determine the local dynamics of the system regularized around these singularities, and thus, it is possible to make a global study of the dynamics of these systems.

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**Fig. 1** Monotonic transition function (left) and non monotonic transition function (right). The monotonic one generates only normally hyperbolic critical sets, and the sliding region coincides with the one proposed by Filippov. The non monotonic one has a critical point, which generates a non normally hyperbolic point of the critical manifold. Moreover, in this example, such a transition function extends the classical notion of sliding region.
The paper is organized as follows. In Section 2, we present some introductory notions on PSVF, regularization processes, geometric singular perturbation theory and slow-fast normal forms. In Section 3, we state and prove Theorems A, B, and C.

2 Preliminaries on Piecewise Smooth Vector Fields and Geometric Singular Perturbation Theory

This section is devoted to establishing some basic results and notation that will be used throughout the paper.

2.1 Piecewise Smooth Vector Fields

Let $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a sufficiently smooth function and consider $C^r$ vector fields $X, Y : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$. A $C^r$ piecewise smooth vector field $Z : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (or PSVF for short) is given by

$$Z(x) = \frac{1}{2} \left( \left( 1 + \text{sgn} \left(F(x)\right) \right) X(x) + \left( 1 - \text{sgn} \left(F(x)\right) \right) Y(x) \right)$$

where $x \in U$, and we assume that $Z$ is multi-valued in the set

$$\Sigma = \{x \in U; F(x) = 0\},$$

which is called discontinuity locus or discontinuity set. The set of all $C^r$ piecewise smooth vector fields is denoted by $\Omega^r$. A PSVF is also denoted by $Z = (X, Y)$ in order to emphasize the dependency on the smooth vector fields $X$ and $Y$.

The Lie derivative of $F$ with respect to the vector field $X$ is given by $XF = \langle X, \nabla F \rangle$ and $X^i F = \langle X, \nabla X^{i-1} F \rangle$ for all integer $i \geq 2$. This allows us to define the following regions in $\Sigma$:

1. **Filippov sewing region:**

$$\Sigma^W = \{ x \in \Sigma \mid XF(x) \cdot YF(x) > 0 \};$$

2. **Filippov sliding region:**

$$\Sigma^S = \{ x \in \Sigma \mid XF(x) \cdot YF(x) < 0 \}.$$

We emphasize that in the literature, these sets are simply called sewing region and sliding region, respectively. Nevertheless, in [19], the authors presented a new definition of such regions, which depends on the type of regularization adopted (see Definitions 7 and 8). Due
to this fact, we will call these regions as *Filippov regions* in order to stress that we are talking about the classical definition of sewing and sliding (see Fig. 3).

A point $x_0 \in \Sigma$ is a *PS-tangency point* if $XF(x_0) = 0$ or $YF(x_0) = 0$. We say that $x_0$ is a *PS-fold point* of $X$ if $XF(x_0) = 0$ and $X^2F(x_0) \neq 0$. If $X^2F(x_0) > 0$, $x_0$ is a *PS-visible fold* of $X$, and if $X^2F(x_0) < 0$, we say that $x_0$ is an *PS-invisible fold* of $X$. Analogously, we define PS-tangency points and PS-fold points of $Y$. Note that if $Y^2F(x_0) < 0$, $x_0$ is a PS-visible fold of $Y$, and if $Y^2F(x_0) > 0$, the point $x_0$ is a PS-invisible fold of $Y$. If $x_0$ is a PS-fold of both $X$ and $Y$, we say that $x_0$ is a *PS-fold-fold*. Finally, we say that $x_0 \in \Sigma$ is a *PS-cusp point* if $XF(x_0) = X^2F(x_0) = 0$ and $X^3F(x_0) \neq 0$.

Singularities of slow-fast systems will be discussed later. Throughout this paper, a singularity of a PSVF will be called *PS-singularity*, and a singularity of a slow-fast system when $\varepsilon = 0$ will be called *SF-singularity*.

Following Filippov’s convention [5], one can define a vector field in $\Sigma^s \subset \Sigma$. The *Filippov sliding vector field* associated to $Z \in \Omega^r$ is the vector field $Z_{\Sigma} : \Sigma \to T\Sigma$ given by

$$Z_{\Sigma}(x) = \frac{1}{YF - XF}(X \cdot YF - Y \cdot XF), \tag{2}$$

which is the convex combination between $X$ and $Y$.

A regularization of a PSVF, $Z$, is a 1-parameter family of smooth vector fields $Z_{\varepsilon}$, $\varepsilon > 0$, satisfying that $Z_{\varepsilon}$ converges pointwise to $Z$ on $\mathbb{R}^2 \setminus \Sigma$, when $\varepsilon \to 0$ (see [20]). In this paper is considered two types of regularizations: the *linear* and *nonlinear* ones.

### 2.2 Linear Regularization of Piecewise Smooth Vector Fields

The regularization process proposed by Sotomayor and Teixeira in [22] is a powerful tool in the study of piecewise smooth vector fields. With this technique, it is possible to construct a family of smooth vector fields $\{Z_{\varepsilon}\}_{\varepsilon}$ such that $Z_{\varepsilon} \to Z_0 = Z$ when $\varepsilon \to 0$.

![Fig. 3](image)
We say that \( \varphi : \mathbb{R} \to \mathbb{R} \) is a transition function if the following conditions are satisfied:

1. \( \varphi \) is sufficiently smooth;
2. \( \varphi(t) = -1 \) if \( t \leq -1 \) and \( \varphi(t) = 1 \) if \( t \geq 1 \);
3. \( \varphi'(t) > 0 \) if \( s \in (-1, 1) \). This condition is called monotonicity.

Throughout this paper, it will be clear that, by dropping the monotonicity condition, it is possible to obtain different critical manifolds of the slow-fast system associated to the regularization. Moreover, non monotonic transition functions can expand the Filippov sliding region in \( \Sigma_1 \) (see [19] and Theorem A below).

**Definition 1**

Let \( \varphi \) be a transition function. A \( \varphi \)-linear regularization of a piecewise smooth vector field \( Z = (X, Y) \) is an one-parameter family \( Z^\varphi \) of smooth vector fields given by

\[
Z^\varphi(x) = \left( \frac{1}{2} + \frac{\varphi_\varepsilon(F(x))}{2} \right) X(x) + \left( \frac{1}{2} - \frac{\varphi_\varepsilon(F(x))}{2} \right) Y(x);
\]

with \( \varphi_\varepsilon(s) = \varphi(\frac{s}{\varepsilon}) \) for \( \varepsilon > 0 \). When \( \varphi \) is monotonic, we say that Eq. 3 is the ST-regularization (Sotomayor–Teixeira Regularization) of \( Z \).

Intuitively, regularizing piecewise smooth vector field means to replace the discontinuity set \( \Sigma_1 \) by a stripe (a tubular neighbourhood of \( \Sigma_1 \)) of width \( 2\varepsilon \). Outside this stripe, the vector fields \( Z^\varphi \) and \( Z \) coincide, and inside the stripe, the vector field \( Z^\varphi \) can be seen as the “average” between \( X \) and \( Y \).

### 2.3 Nonlinear Regularization of Piecewise Smooth Vector Fields

In [18, 20], the authors considered another way to generalize the notions of sliding region and sliding vector field by means of nonlinear regularizations.

**Definition 2**

A continuous combination of \( X \) and \( Y \) is a 1-parameter family of smooth vector fields \( \tilde{Z}(\lambda, \cdot) \), with \( \lambda \in [-1, 1] \), such that \( \tilde{Z}(1, p) = X(p) \), \( \tilde{Z}(-1, p) = Y(p) \), \( \forall p \in U \).

Now, we define \( \varphi \)-nonlinear regularization of \( Z = (X, Y) \).

**Definition 3**

Let \( \tilde{Z}(\lambda, p) \) be a continuous combination of \( X \) and \( Y \). A \( \varphi \)-nonlinear regularization of \( Z = (X, Y) \) is the 1-parameter family given by \( \tilde{Z}(\varphi(F_\varepsilon), p) \).

Recall that if \( F > \varepsilon \), then \( \varphi(F_\varepsilon) = 1 \) and \( \tilde{Z}(\varphi(F_\varepsilon), p) = X(p) \); and if \( F < -\varepsilon \), then \( \varphi(F_\varepsilon) = -1 \) and \( \tilde{Z}(\varphi(F_\varepsilon), p) = Y(p) \) (see Fig. 4).

In [18, Theorem 1], it was shown the following result: Let \( \varphi \) be a monotonic transition function and \( \psi \) a non-monotonic transition function. If \( Z^\psi \) is a \( \psi \)-linear regularization, then there exists an unique continuous combination \( \tilde{Z}(\lambda, p) \) such that \( Z^\psi(p) = \tilde{Z}(\varphi(F_\varepsilon)), p \). However, in general, the converse is not true (see Theorems B and C).

### 2.4 Geometric Singular Perturbation Theory

In the 1970s, Neil Fenichel wrote several papers on invariant manifold theory, which allowed a rigorous study of slow-fast systems (i.e., systems of differential equations with multiple time scales). We refer to [8, 9, 23] for a careful introduction on slow–fast systems, as well as
Fig. 4 Linear (red) and nonlinear (blue) regularizations

The system of the form
\[ \varepsilon \dot{x} = f(x, y, \varepsilon); \quad \dot{y} = g(x, y, \varepsilon); \](4)
is called slow-fast system, where \((x, y) \in \mathbb{R}^2, 0 < \varepsilon \ll 1\) and \(f, g : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}\) are sufficiently smooth. The dot \(\cdot\) represents the derivative of the functions \(x(\tau)\) and \(y(\tau)\) with respect to the variable \(\tau\).

If we write \(t = \frac{\tau}{\varepsilon}\), then system Eq. 4 becomes
\[ x' = f(x, y, \varepsilon); \quad y' = \varepsilon g(x, y, \varepsilon); \](5)
in which the apostrophe ' denotes the derivative of the functions \(x(t)\) and \(y(t)\) with respect to the variable \(t\). Observe that the parameter \(\varepsilon = \frac{\tau}{t}\) represents the ratio of the time scales.

Consider Eq. 4 and set \(\varepsilon = 0\). We obtain the so called slow system given by
\[ 0 = f(x, y, 0); \quad \dot{y} = g(x, y, 0). \](6)

This equation is also known in the literature as reduced problem or slow vector field. Note that Eq. 6 is not an ODE, but it is an algebraic differential equation (ADE).

Solutions of Eq. 6 are contained in the set
\[ C_0 = \{(x, y) \in \mathbb{R}^2 \mid f(x, y, 0) = 0\}. \]

**Definition 4** The set \(C_0\) is called critical set. In the case where \(C_0\) is a manifold, \(C_0\) is called critical manifold.

On the other hand, setting \(\varepsilon = 0\) in Eq. 5, we obtain the so called fast system
\[ x' = f(x, y, 0); \quad y' = 0. \](7)

System Eq. 7 is also known in the literature as layer problem, layer equation or fast vector field. Moreover, the system Eq. 7 can be seen as a system of ordinary differential equations, where \(y \in \mathbb{R}\) is a parameter and the critical set \(C_0\) is a set of equilibrium points of Eq. 7.

The main goal of geometric singular perturbation theory is to study systems Eqs. 6 and 7 in order to obtain information of the full system Eq. 4. Observe that the systems Eqs. 4 and 5 are equivalent when \(\varepsilon > 0\), since they only differ by time scale.
Definition 5 We say that $x_0 \in C_0$ is normally hyperbolic if $f_x(x_0) \neq 0$. The set of all normally hyperbolic points of $C_0$ will be denoted by $NH(C_0)$.

Recall that the nomenclature $PS$-singularity and $SF$-singularity is adopted in order to emphasize when $p$ is a singularity of the piecewise smooth vector field Eq. 1 or a singularity of the slow system Eq. 6.

2.5 Normal Forms of Slow-Fast Systems

In what follows, we briefly recall some normal forms of slow-fast systems. An overview on this subject can be found in Chapter 4 of [13], and the reader can see the references therein for further details of the proofs. The normal forms of planar SF-generic transcritical and SF-generic pitchfork singularities were given in [12].

We say that the critical manifold $C_0 = \{ f(x, y, 0) = 0 \}$ has a planar SF-generic fold (or SF-fold for short) at the origin if

$$f_x(0, 0, 0) = 0; \quad f_{xx}(0, 0, 0) \neq 0; \quad f_y(0, 0, 0) \neq 0 \quad \text{and} \quad g(0, 0, 0) \neq 0.$$  \hspace{1cm} (8)

In order to obtain a SF-generic transcritical singularity at the origin, the planar slow-fast system Eq. 4 must satisfy the following conditions:

$$f(0, 0, 0) = f_x(0, 0, 0) = f_y(0, 0, 0) = 0; \quad \det \text{Hes}(f) < 0; \quad f_{xx}(0, 0, 0) \neq 0 \neq g(0, 0, 0);$$  \hspace{1cm} (9)

where $\text{Hes}(f)$ denotes the Hessian matrix of $f$.

On the other hand, in order to obtain a SF-generic pitchfork singularity at the origin, we must require the following conditions:

$$f(0, 0, 0) = f_x(0, 0, 0) = f_{xx}(0, 0, 0) = f_y(0, 0, 0) = 0; \quad f_{xxx}(0, 0, 0) \neq 0, f_{xyy}(0, 0, 0) \neq 0, g(0, 0, 0) \neq 0.$$  \hspace{1cm} (10)

The normal forms of planar SF-generic fold, SF-generic transcritical and SF-generic pitchfork singularities were given in [11, 12]. The normal form of a normally hyperbolic point can be found in [13].

Theorem 6 gathers the results mentioned in the previous paragraph. The notation $\mathcal{O}$ denotes the higher order terms of a function. Moreover, in each case, $\lambda$ denotes a constant that depends on the conditions of non-degeneracy of each SF-singularity (see [11–13] for details).

Theorem 6 There exists a smooth change of coordinates such that for $(x, y)$ sufficiently small the System Eq. 5 can be written as

(a): If the slow-fast system Eq. 5 satisfies the non-degeneracy conditions Eq. 8 of a planar SF-generic fold:

$$x' = y + x^2 + \mathcal{O}(x^3, xy, y^2, \varepsilon); \quad y' = \varepsilon \left( \pm 1 + \mathcal{O}(x, y, \varepsilon) \right);$$  \hspace{1cm} (11)

(b): If the slow-fast system Eq. 5 satisfies the non-degeneracy conditions Eq. 9 of a SF-generic transcritical singularity:

$$x' = x^2 - y^2 + \lambda \varepsilon + \mathcal{O}(x^3, x^2y, xy^2, y^3, \varepsilon x, \varepsilon y, \varepsilon^2); \quad y' = \varepsilon \left( 1 + \mathcal{O}(x, y, \varepsilon) \right);$$  \hspace{1cm} (12)
(c): If the slow-fast system Eq. 5 satisfies the non-degeneracy conditions Eq. 10 of a SF-
pitchfork singularity:

\[ x' = (y - x^2) + \lambda \varepsilon + O(x^2 y, x y^2, y^3, \varepsilon x, \varepsilon y, \varepsilon^2); \quad y' = \varepsilon \left( \pm 1 + O(x, y, \varepsilon) \right). \]  

(d): If \( 0 \in C_0 \) is a normally hyperbolic point:

\[
\begin{cases}
  x' = \Lambda(x, y, \varepsilon)x; \\
  y' = \varepsilon \left( h(y, \varepsilon) + H(x, y, \varepsilon)(x) \right);
\end{cases}
\]

where \( x \) is sufficiently small, \( \Lambda, h \) and \( H \) are \( C^{r-1} \) in all arguments. Moreover, \( \Lambda(x, y, \varepsilon) \) is non-zero, and \( H(x, y, \varepsilon) \) is linear when applied to \( x \).

### 3. Regularizations and Typical SF-Singularities

The relation between (linear) regularization of piecewise smooth vector fields and slow-fast systems had led mathematicians in a new direction in the research in qualitative theory of ordinary differential equations. By applying a directional blow-up, it is possible to transform a (linearly) regularized vector field into a slow-fast system. This approach was used for the first time in [1] in the context of planar piecewise smooth vector fields, and lately by [16] in the 3-dimensional case. The \( n \)-dimensional case was discussed in [17].

This study starts considering a planar piecewise smooth vector field whose discontinuity set is a smooth curve and linear regularizations. Without loss of generality, we adopt a coordinate system such that \( Z = (X, Y) \) is written as

\[ \dot{z} = Z(z) = \frac{1 + \text{sgn}(x)}{2} X(z) + \frac{1 - \text{sgn}(x)}{2} Y(z), \quad z = (x, y) \]  

that is, the discontinuity set is a straight line. A linear regularization of Eq. 15 is the family

\[ \dot{z} = Z^\varepsilon(z) = \frac{1 + \varphi(x/\varepsilon)}{2} X(z) + \frac{1 - \varphi(x/\varepsilon)}{2} Y(z), \]

where \( X = (f_1, f_2), Y = (g_1, g_2) \) are applied in \( z = (x, y) \). We emphasize that in this study, the transition function \( \varphi \) is not necessarily monotonic.

After a directional blow-up of the form \( x = \varepsilon \tilde{x} \), one obtains the slow-fast system (dropping the tilde in order to simplify the notation)

\[ \varepsilon \tilde{x} = \frac{f_1 + g_1}{2} + \varphi(x) \left( \frac{f_1 - g_1}{2} \right); \quad \tilde{y} = \frac{f_2 + g_2}{2} + \varphi(x) \left( \frac{f_2 - g_2}{2} \right); \]

where \( f_1, f_2, g_1, g_2 \) are applied in \( (\varepsilon x, y) \). Denote the critical set of Eq. 17 by \( C_0 \), which is given by

\[ C_0 = \left\{ (x, y) \left| \frac{f_1(0, y) + g_1(0, y)}{2} + \varphi(x) \left( \frac{f_1(0, y) - g_1(0, y)}{2} \right) = 0 \right. \right\}. \]

Now, we recall the definitions of sliding and sewing points presented in [19]. Observe that such notions are local.

**Definition 7** A point \( p \in \Sigma \) is a sliding point if there is an open set \( U \owns p \) and a family of smooth manifolds \( S_\varepsilon \subset U \) such that

1. For each \( \varepsilon \), \( S_\varepsilon \) is invariant by the regularized system Eq. 16;
For each compact subset $K \subset U$, the sequence $S_\varepsilon \cap K$ converges to $\Sigma \cap K$ as $\varepsilon \to 0$ according to Hausdorff distance.

**Definition 8** We say that $p \in \Sigma$ is a sewing point if $XF(p) \cdot YF(p) \neq 0$ and there is an open set $U \ni p$ and local coordinates defined in $U$ such that

1. $\Sigma = \{x = 0\}$;
2. For each $\varepsilon > 0$, the horizontal vector field $v(x, y) = (1, 0)$ is a generator of the regularized system Eq. 16 in $U$.

Intuitively, a point $p$ is a sewing point if the flow of Eq. 16 around $p$ is transversal to $\Sigma$.

Concerning linear regularizations, if the transition function is monotonic and the discontinuity set is smooth, the dynamics of the sliding vector field according to Filippov’s convention is equivalent to the dynamics of the slow system associated (see [16, Theorem 1.1]). However, if we do not consider monotonic transition functions, one can obtain different dynamics of the (linearly) regularized vector field and consequently different singular perturbation problems, which can lead us to different definitions of sliding or sewing regions. See [19, 20] and Theorem A below. Nonlinear regularizations also lead us to different notions of sewing and sliding. See [18, 20].

Before we state Theorem A, let us introduce some notation. In what follows, $\Pi : \mathbb{R}^2 \to \Sigma$ is the canonical projection $\Pi(x, y) = (0, y)$.

From the definitions discussed previously, it is clear that different (linear or nonlinear) regularizations lead to different slow-fast systems, which gives rise to different sliding or sewing regions. In order to emphasize the dependency of the regularization adopted, we will call these sets as the $r$-Sliding and $r$-Sewing regions, and we will denote them as $\Sigma^r_\varepsilon$ and $\Sigma^w_\varepsilon$, respectively. It can be shown that $\Sigma^r_\varepsilon \cap \Sigma^w_\varepsilon = \emptyset$ (see [19, Remark 5]).

Consider the Filippov sliding vector field $Z^\Sigma$ associated to the PSVF Eq. 15. Although in the literature it is only considered the dynamics of $Z^\Sigma$ in the Filippov sliding or escaping regions, the domain $D(Z^\Sigma) \subset \Sigma$ of $Z^\Sigma$ may be greater than $\Sigma^s$. In this sense, for our purposes, the domain $D(Z^\Sigma)$ of $Z^\Sigma$ is the subset of $\Sigma$ in which $Z^\Sigma$ is well defined, and not only the Filippov sliding region $\Sigma^s$.

**Theorem A** Consider the PSVF Eq. 15 and denote its Filippov sliding vector field by $Z^\Sigma$, which domain is the set $D(Z^\Sigma) \subset \Sigma$. Consider linear regularization of $Z$ and let $\varphi$ be a transition function, not necessarily monotonic. Let $\Pi : \mathbb{R}^2 \to \Sigma$ be the canonical projection and $x_0 \in (-1, 1)$. Then the following hold:

(a): If $\varphi'(x_0) = 0$, then the set of points $(x_0, y)$ such that

$$f_1(x_0, y) + g_1(x_0, y) + \varphi(x_0)\left(f_1(x_0, y) - g_1(x_0, y)\right) = 0$$

is contained in $C_0 \setminus NH(C_0)$. In other words, critical points of $\varphi$ gives rise to non normally hyperbolic points of the critical set $C_0$ of Eq. 17.

(b): Suppose that $(x_0, y_0) \in NH(C_0)$ and $|\varphi(x_0)| > 1$. Then $\Pi(NH(C_0)) \cap \Sigma^w \neq \emptyset$.

Moreover, $\Sigma^s \subset \neq \Sigma^r_\varepsilon$. In other words, if $|\varphi(x_0)| > 1$, then the $r$-sliding region is greater than the classical Filippov sliding region.

(c): If $(0, y_0) \in \Sigma^s_\varepsilon$ satisfies $f_1(0, y_0) \neq g_1(0, y_0)$, near such a point the dynamics in the $r$-sliding region $\Sigma^r_\varepsilon$ is given by the classical Filippov sliding vector field $Z^\Sigma$. In other words, even if we extend $\Sigma^s$ to $\Sigma^r_\varepsilon$, the dynamics in such a set is given by $Z^\Sigma$. In particular,
(x₀, y₀) is an SF-equilibrium point of Eq. 17 if, and only if, (0, y₀) is an equilibrium point of ZΣ.

(d): If

\[ \Pi(C₀) \cap \left( \Sigma \setminus D\left( Z\Sigma \right) \right) \neq \emptyset, \]

then (0, y₀) ∈ \( \Pi(C₀) \cap \left( \Sigma \setminus D\left( Z\Sigma \right) \right) \) is a tangency point for both vector fields X and Y simultaneously, and the line \( \{ y = y₀ \} \) is a component of \( C₀ \). See Fig. 5.

Proof (a): Without loss of generality, we suppose that \( x₀ = 0 \). Expanding the first equation of Eq. 17 in Taylor series, one obtains

\[ x' = \frac{1}{2} \left( (f₁ + g₁) + \varphi(0)(f₁ - g₁) \right) + \frac{1}{2} \left( \varphi'(0)(f₁ - g₁) \right)x + \ldots \]

A point of the form (0, y, 0) is normally hyperbolic if, and only if, the following conditions are satisfied:

\[ (f₁ + g₁) + \varphi(0)(f₁ - g₁) = 0, \quad \varphi'(0)(f₁ - g₁) \neq 0. \quad (18) \]

Therefore, if \( \varphi'(0) = 0 \) (that is, 0 is a critical point of the transition function), then (0, y, 0) is not normally hyperbolic.

(b): We already know that \( \Sigma² \subset \Sigma¹ \) (see [21, Theorem 3]). Now, we prove that \( \Sigma¹ \) contains points that do not belong to \( \Sigma² \). Since (0, y₀) ∈ \( N\mathcal{H}(C₀) \) and \( |\varphi(0)| > 1 \), define the constant \( a \) as

\[ a = \frac{\varphi(0) + 1}{\varphi(0) - 1} \iff \varphi(0) = \frac{a + 1}{a - 1}. \]

Then, the conditions Eq. 18 can be rewritten as

\[ g₁ = af₁, \quad \varphi'(0) \neq 0, \quad (19) \]

![Fig. 5](image-url) Level \( \varepsilon = 0 \) of the regularized vector field. The semi-cylinder represents the blowing up locus and the flows with simple arrow and with double arrow represent the slow and the fast system, respectively. Statement (d) says that, if the projection \( \Pi(C₀) \) on \( \Sigma \) contains a point \( p = (0, y₀) \) that do not belong to the domain \( D\left( Z\Sigma \right) \) of the Filippov sliding vector field \( Z\Sigma \), then \( p \) is a tangency point for both \( X \) and \( Y \), that is, \( g₁(p) = f₁(p) = 0 \). Moreover, the critical manifold \( C₀ \) (highlighted in green) contains a line \( y = y₀ \). It is not possible to define dynamics in \( \Sigma \) through \( p \) using geometric singular perturbation theory.
where \( f_1 \) and \( g_1 \) are applied in \((0, y)\) and \( a \neq 1 \). Note that the condition \( a \neq 1 \) is naturally satisfied with the assumptions above. Observe that \( a < 0 \) if, and only if, \( |\varphi(0)| < 1 \). Analogously, it can be checked that \( a > 0 \) if, and only if, \( |\varphi(0)| > 1 \).

Since \( |\varphi(0)| > 1 \), then \( a > 0 \) and points of \( \mathcal{N}H(C_0) \) of the form \((0, y)\) such that \( g_1(0, y) = af_1(0, y) \) are projected in the Filippov sewing region \( \Sigma^u \) by \( \Pi \).

By [19, Theorem 4.2], we have the inclusion \( \Pi(\mathcal{N}H(C_0)) \subset \Sigma^r_\varepsilon \). This means that \((0, y) \notin \Sigma^r \) is a sliding point, which implies that \( \Sigma^r \subset \Sigma^s \).

(c): Setting \( \varepsilon = 0 \) in the first equation of Eq. 17, we have

\[
\varphi(x) = \frac{g_1(0, y) + f_1(0, y)}{g_1(0, y) - f_1(0, y)}.
\]

Combining this expression with the second equation of Eq. 17, we obtain

\[
\dot{y} = \frac{f_2 + g_2}{2} + \left( \frac{g_1 + f_1}{g_1 - f_1} \right) \left( \frac{f_2 - g_2}{2} \right)
\]

\[
= \frac{(g_1 - f_1)(f_2 + g_2) + (g_1 + f_1)(f_2 - g_2)}{2(g_1 - f_1)}
\]

\[
= \frac{g_1(0, y) f_2(0, y) - f_1(0, y) g_2(0, y)}{g_1(0, y) - f_1(0, y)}
\]

which is exactly the expression of \( Z^\Sigma \). Therefore, the dynamics in the \( r \)-sliding region \( \Sigma^r_\varepsilon \) is given by the classical Filippov sliding vector field \( Z^\Sigma \).

(d): The domain of \( Z^\Sigma \) is precisely the set

\[
D\left(Z^\Sigma\right) = \{(0, y) \in \Sigma \; g_1(0, y) \neq f_1(0, y)\}.
\]

If \((0, y_0) \notin D\left(Z^\Sigma\right) \) and \((0, y_0) \in \Pi(\mathcal{N}H(C_0)) \), then \( g_1(0, y_0) = f_1(0, y_0) \). From the expression of \( C_0 \), \((0, y_0)\) must be a tangency point for both \( X \) and \( Y \). Moreover, the equation \( f_1(0, y_0) = 0 \) assures that the horizontal line \( \{y = y_0\} \) is a component of the critical manifold \( C_0 \) (see Fig. 5).

\[\square\]

Item (a) of Theorem A assures that, in order to generate SF–singularities with linear regularizations, we may drop the monotonicity of the transition function \( \varphi \) (see also Theorem B). Moreover, \( \varphi(0) = 1 \) implies \( f_1(0, y) = 0 \), that is, there is a PS-tangency point between \( X \) and \( \Sigma \). Analogously, \( \varphi(0) = -1 \) implies \( g_1(0, y) = 0 \), that is, there is a PS-tangency point between \( Y \) and \( \Sigma \).

Following our notation, [21, Theorem 3] assures that \( \Sigma^s \subset \Sigma^r_\varepsilon \). However, in our statement, we give a condition such that \( \Sigma^s \subset \Sigma^r_\varepsilon \). In other words, if \((x_0, y_0) \in \mathcal{N}H(C_0) \) and \( |\varphi(x_0)| > 1 \) for \( x_0 \in (-1, 1) \), then there exists a point \((0, y_0) \in \Sigma^r_\varepsilon \) that does not belong to \( \Sigma^r \).

According to item (c), the dynamics in \( r \)-sliding region \( \Sigma^r_\varepsilon \) is naturally extended using the classical Filippov sliding vector field. Finally, item (d) says that \( \Pi(C_0) \) is entirely contained in \( D(Z^\Sigma) \), unless \( C_0 \) contains horizontal lines. This means that we can not define a sliding dynamics in \( \Sigma \setminus D(Z^\Sigma) \) using geometric singular perturbation theory.

Now, we are concerned in establishing conditions that both piecewise smooth vector field and transition function must satisfy in order to generate SF-singularities.
Theorem B Consider the PSVF Eq. 15 and let $\varphi$ be a transition function, not necessarily monotonic. After linear regularization and directional blow-up, it is possible to generate normally hyperbolic points, SF-fold singularities and SF-transcritical singularities. However, it is not possible to generate SF-pitchfork singularities.

Proof Let $\varphi$ be a transition function (not necessarily monotonic) and $Z = (X, Y)$ be a PSVF, in which $X = (f_1, f_2)$ and $Y = (g_1, g_2)$.

The proof is given by direct computations. The idea is to compare the coefficients of the Taylor expansion at the origin of the function that defines the critical set $C_0$ of Eq. 17 with the expressions of the normal forms given in Section 2.5. With this procedure, we obtain that such coefficients must satisfy the following conditions in order to generate SF-singularities:

(a): Fenichel normal form (normally hyperbolic point):
\[
f_1(0, 0) - g_1(0, 0) \neq 0, \quad \varphi'(0) \neq 0;
\]
\[
\varphi(0) = g_1(0, 0) + f_1(0, 0); \quad (f_{1,x}(0, 0) + g_{1,x}(0, 0)) + \varphi(0) \left( f_{1,y}(0, 0) - g_{1,y}(0, 0) \right) \neq 0. \tag{21}
\]

(b): SF-generic fold:
\[
f_1(0, 0) - g_1(0, 0) \neq 0, \quad \varphi'(0) = 0, \quad \varphi''(0) \neq 0;
\]
\[
\varphi(0) = g_1(0, 0) + f_1(0, 0); \quad (f_{1,x}(0, 0) + g_{1,x}(0, 0)) + \varphi(0) \left( f_{1,y}(0, 0) - g_{1,y}(0, 0) \right) = 0;
\]
\[
\left[ \frac{1}{2} \frac{1}{f_1-g_1}(\varphi'(0)) \quad 0 \right] \begin{bmatrix} 0 \\ 0 \left[ (1 + \varphi(0)) f_{1,y} + (1 - \varphi(0)) g_{1,y} \right] \end{bmatrix} < 0; \tag{22}
\]

where $f_{1,x}$, $g_1$, $f_{1,yy}$ and $g_{1,yy}$ are computed at $(0, 0)$.

(c): SF-transcritical singularity:
\[
f_1(0, 0) - g_1(0, 0) \neq 0, \quad \varphi'(0) = 0, \quad \varphi''(0) \neq 0;
\]
\[
\varphi(0) = g_1(0, 0) + f_1(0, 0); \quad (f_{1,x}(0, 0) + g_{1,x}(0, 0)) + \varphi(0) \left( f_{1,y}(0, 0) - g_{1,y}(0, 0) \right) = 0;
\]
\[
\left[ \frac{1}{2} \frac{1}{f_1-g_1}(\varphi''(0)) \quad 0 \right] \begin{bmatrix} 0 \\ 0 \left[ (1 + \varphi(0)) f_{1,y} + (1 - \varphi(0)) g_{1,y} \right] \end{bmatrix} < 0; \tag{23}
\]

and therefore, the transition function would satisfy $\varphi'(0) = 0$ and $\varphi'(0) \neq 0$ simultaneously, which is a contradiction.

Remark 9 Notice that the SF-fold, SF-transcritical, and SF-pitchfork singularities are non normally hyperbolic points.

Due to Theorem B, one can start to search for examples of regularized systems that possess normally hyperbolic points, SF-fold singularities and SF-transcritical singularities. In the following example, we present a regularized system that has a SF-transcritical singularity at the origin.
Example 10  Consider the normal form of a PS-cusp singularity

\[ Z(x, y) = \begin{cases} 
X(x, y) = (-y^2, 1), & \text{if } x > 0; \\
Y(x, y) = (1, 1), & \text{if } x < 0.
\end{cases} \tag{24} \]

Recall that the origin is a PS-cusp singularity and \( \Sigma^x = \Sigma \setminus \{0\} \). Now, consider the transition function \( \varphi \) given by

\[ \varphi(t) = \begin{cases} 
-1, & \text{if } t \leq -1; \\
-3t^3 + t^4 + 5t^3 - 2t^2 + 1, & \text{if } -1 \leq t \leq 1; \\
1, & \text{if } t \geq 1;
\end{cases} \tag{25} \]

in which \( t_0 = 0 \) and \( t_1 = \frac{8}{15} \) are local maximum and minimum, respectively (see Fig. 6).

After regularization and blow-up, one obtains the slow-fast system

\[ \begin{cases} 
\varepsilon \dot{x} = \frac{1}{4} (x^2(x - 1)^2(3x + 4)(y^2 + 1) - 4y^2) \\
\dot{y} = 1.
\end{cases} \tag{26} \]

Observe that for \( x = 0 \) and \( x = \frac{8}{15} \), the critical manifold presents non normally hyperbolic points. In particular, the origin is a transcritical singularity.

It is important to note that this example can be generalized as follows.

**Corollary 11**  Suppose that the origin is a PS-cusp singularity of the PSVF Eq. 15 and let \( \varphi \) be a non-monotonic transition function such that \( \varphi(0) = 1, \varphi'(0) = 0, \) and \( \varphi''(0) \neq 0 \). If \( g_1(0, 0)\varphi''(0)f_{1,yy}(0, 0) > 0 \), then the regularized system associated with \( Z \) has a SF-transcritical singularity at origin.

**Proof**  Suppose that the origin is a regular-cusp singularity of the PSVF Eq. 15, that is,

- \( XF(0, 0) = f_1(0, 0) = 0; \)
- \( X^2F(0, 0) = f_{1,y}(0, 0)f_2(0, 0) = 0, \) thus \( f_{1,y}(0, 0) = 0; \)

![Fig. 6](https://via.placeholder.com/150)  Graphic of the monotone transition function \( \varphi \) (left) and the linear regularized system Eq. 26 (right). The critical manifold is highlighted in green. 
\[ X^3 F(0,0) = f_{1,yy}(0,0)(f_2(0,0))^2 \neq 0, \text{ hence } f_{1,yy}(0,0) \neq 0; \]
\[ g_1(0,0) \neq 0; \]

where \( F(x, y) = x \) and \( XF \) is the Lie derivative of \( F \) with respect to the vector field \( X \). Then, we get that

\[ (f_1 - g_1)(0,0) = -g_1(0,0) \neq 0; \]
\[ \varphi(0) = 1; \]
\[ \left| \frac{1}{4}(f_1 - g_1)\varphi''(0) \right| = -\frac{g_1\varphi''(0)f_{1,yy}}{8}. \]

Since \( \varphi'(0) = 0 \) and \( g_1(0,0)\varphi''(0)f_{1,yy}(0,0) > 0 \), then the conditions obtained in the proof of Theorem B imply that the origin is a SF-transcritical singularity.

Using the definition of a PS-fold singularity of the PSVF Eq. 15 and Theorem B, we obtain the following result.

**Corollary 12** Suppose that the origin is a PS-fold singularity of the PSVF Eq. 15 and let \( \varphi \) be a non-monotonic transition function such that \( \varphi(0) = 1, \varphi'(0) = 0, \) and \( \varphi''(0) \neq 0. \)

Then, the regularized system associated with \( Z \) has a SF-fold singularity at origin.

At some point, the reader may think that, after non monotonic linear regularization and blow-up, it is possible to generate any SF-singularity, since it is just a matter of a suitable choice of the transition function. In general, this is not true. Indeed, Theorem B assures that it does not exist a transition function that generates a SF-pitchfork singularity. This leads us to consider nonlinear regularizations.

### 3.1 Nonlinear Regularization and SF-Singularities

In what follows, we present a version of Theorem B for nonlinear regularization.

**Theorem C** Consider the PSVF Eq. 15 and let \( \varphi \) be a monotonic transition function. After \( \varphi \)-nonlinear regularization \( \tilde{Z}(\varphi(\frac{x}{\varepsilon}), x, y) \) and directional blow-up, it is possible to generate normally hyperbolic points, SF-fold singularities, SF-transcritical singularities and SF-pitchfork singularities.

**Proof** Let \( \varphi \) be a monotonic transition function and \( Z = (X, Y) \) be a PSVF. Consider the \( \varphi \)-nonlinear regularization \( \tilde{Z}(\varphi(\frac{x}{\varepsilon}), x, y) \) of \( Z \), where \( \tilde{Z} = (\tilde{Z}^1, \tilde{Z}^2) \). The proof is given by direct computations. The idea is to compare the coefficients of the Taylor expansion of the function \( \tilde{Z}^1(\varphi(\frac{x}{\varepsilon}), \varepsilon x, y) \) near \((0, 0, 0)\) with the expressions of the normal forms given in Section 2.5 and use that \( \varphi'(t) \neq 0 \) for all \( t \in (-1, 1) \). With this procedure, we obtain that such coefficients must satisfy the following conditions in order to generate SF-singularities:

(a): **Fenichel normal form (normally hyperbolic point):**

\[ \tilde{Z}^1_\lambda(\varphi(0), 0, 0) \neq 0; \]

(b): **SF-generic fold:**

\[ \tilde{Z}^1(\varphi(0), 0, 0) = 0; \quad \tilde{Z}^1_\lambda(\varphi(0), 0, 0) = 0; \quad \tilde{Z}^1_{\lambda\lambda}(\varphi(0), 0, 0) \neq 0; \]
\[ \tilde{Z}^1_\gamma(\varphi(0), 0, 0) \neq 0; \quad \tilde{Z}^2(\varphi(0), 0, 0) \neq 0. \]
To end this section, we present an example of a nonlinear regularized system with SF-pitchfork singularity.

Example 13 Let \( Z = (X, Y) \) be a PSVF defined on \( \mathbb{R}^2 \) with \( F(x, y) = x, X(x, y) = ((x + 1)y + 1, -1), Y(x, y) = ((x - 1)y - 1, -1) \). Consider the continuous combination of \( X \) and \( Y \) given by

\[
\tilde{Z}(\lambda, x, y) = (x + \lambda)y + \lambda^3, -1).
\]

Assume that the monotonic transition function \( \varphi \) satisfies \( \varphi(0) = 0 \) and \( \varphi'(0) \neq 0 \) (for example, \( \varphi(t) = -\frac{t^5}{2} + \frac{t^3}{2} + t, \) for all \( t \in (-1, 1) \)). Thus, after nonlinear regularization and directional blow-up, we obtain

\[
\varepsilon \dot{\hat{x}} = (\varepsilon \hat{x} + \varphi(\hat{x}))y + \varphi(\hat{x})^3; \quad \dot{y} = -1;
\]

where \( \hat{x} = \frac{x}{\varepsilon} \). Notice that Eq. 31 satisfies conditions Eq. 30, and therefore, the origin is a SF-pitchfork singularity (see Fig. 7).

![Graphic of the monotone transition function \( \varphi \) (left) and the \( \varphi \)-nonlinear regularization Eq. 31 of \( f \) and \( g \) (right). The critical manifold is highlighted in green](image-url)
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