Accelerating Polarization via Alphabet Extension

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Abstract

Polarization is an unprecedented coding technique in that it not only achieves channel capacity, but also does so at a faster speed of convergence than any other coding technique. This speed is measured by the “scaling exponent” and its importance is three-fold. Firstly, estimating the scaling exponent is challenging and demands a deeper understanding of the dynamics of communication channels. Secondly, scaling exponents serve as a benchmark for different variants of polar codes that helps us select the proper variant for real-life applications. Thirdly, the need to optimize for the scaling exponent sheds light on how to reinforce the design of polar codes.

In this paper, we generalize the binary erasure channel (BEC), the simplest communication channel and the protagonist of many coding theory studies, to the “tetrahedral erasure channel” (TEC). We then invoke Mori–Tanaka’s $2 \times 2$ matrix over $\mathbb{F}_4$ to construct polar codes over TEC. Our main contribution is showing that the dynamic of TECs converges to an almost–one-parameter family of channels, which then leads to an upper bound of 3.328 on the scaling exponent. This is the first non-binary matrix whose scaling exponent is upper-bounded. It also polarizes BEC faster than all known binary matrices up to $23 \times 23$ in size. Our result indicates that expanding the alphabet is a more effective and practical alternative to enlarging the matrix in order to achieve faster polarization.

Index Terms

Polar code, scaling exponent, non-binary alphabet.

I. INTRODUCTION

A fundamental question at the center of the theory of communication is whether we can fully utilize a noisy channel to transmit information. In modern terminology, can error correcting codes achieve channel capacity? The answer is positive; in fact, multiple code constructions do so. Among them, polar code is a special one as it achieves capacity faster than any other known code.

Polar coding was invented by Arikan around 2008 [2]. During that time, Arikan was experimenting with channel combining and splitting. By treating two independent binary channels as a single quaternary channel (combining) and tasking ourselves with guessing certain linear combinations of the inputs (splitting), he synthesized two channels, denoted by $W^\uparrow$ and $W^\circ$, out of the original channel $W$. Arikan realized that, when combining and splitting is applied recursively, the channels undergo an intriguing dynamic that ultimately results in most synthetic channels being either almost noiseless or extremely noisy. This is channel polarization, the first ingredient underlying polar codes.

The second ingredient of polar codes, also given by Arikan in said seminal paper, is the relation between the dynamic of synthetic channels and the construction and performance of codes. Arikan’s insight was that synthetic channels that become almost noiseless can be used to transmit information bits, and synthetic channels that become extremely noisy can be frozen to some fixed values. The rate at which we communicate meaningful bits is then the proportion of synthetic channels that are almost noiseless. So, whether we can achieve channel capacity becomes a problem of counting the number of good and bad synthetic channels.

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It then became apparent, perhaps even appealing, that one can study the dynamic of synthetic channels by means of stochastic processes. Take the binary erasure channel (BEC) as an example. Let $W$ be BEC$(\varepsilon)$, the BEC with erasure probability $\varepsilon$, where $0 < \varepsilon < 1$. The synthetic channels $W^*$ and $W^\circ$ are BEC$(2\varepsilon - \varepsilon^2)$ and BEC$(\varepsilon^2)$, respectively. Accordingly, a stochastic process $\{H_n\}_n$ is defined by having $H_0 := \varepsilon$ and $H_{n+1} := 2H_n - H_n^2$ or $H_n^2$ with equal probability. It can be shown that if

$$\mathbb{P}\{H_n \leq f(n)\} \geq 1 - H_0 - g(n),$$

where $f$ and $g$ are functions in $n$, then there is a polar code with length $2^n$, miscommunication probability at most $2^n f(n)$, and gap to capacity at most $g(n)$.

It was at this point that the study of polar codes thrived and branched. On the error exponent branch, $g$ is a constant and the asymptotics of $f$ is studied. It was shown that $f(n)$ is roughly $\exp(-e^{3n})$, where $\beta > 0$ is a constant depending on the kernel matrix used in the code construction. The task of determining $\beta$ for each kernel matrix has been fully resolved; interested readers are referred to [3], [4], [5], [6].

On the other branch, called the scaling exponent branch, $f$ is a constant\footnote{Sometimes $f$ is not a constant but converging to 0. For instance $f(n) = \exp(-n^{2/3})$. In this case, $2^n f(n)$, the upper bound on the miscommunication probability, will exceed 1, so the corresponding polar code is useless. Yet the asymptotics of $g$ still helps us study other useful codes.} and the asymptotics of $g$ is examined. For BECs, [7], [8] managed to estimate that $g(n) \approx 2^{-n/3.627}$. For binary memoryless symmetric (BMS) channels, it can be shown that $g(n) < 2^{-n/\mu}$ for some constant $0 < \mu < \infty$ (with a really good decay of error $f(n) = \exp(-2^{-0.49n})$) [9]. This makes polar codes the only code family that is known to achieve capacity at a speed polynomial in the block length. Further estimates of $\mu$ include $3.553 < \mu$ [10], $3.579 < \mu < 6$ [11], $\mu < 5.702$ [12], $\mu < 4.714$ [13], and very recently $\mu < 4.63$ [14]. Now that we know the $\mu$ for polar codes and the optimal value being $\mu \approx 2$ for random codes [15], [16], [17], the discrepancy begs the question: Can one modify polar codes to reach a smaller scaling exponent?

The answer is positive: Arikan used the kernel matrix $[10]$ [11] to combine and split channels. Instead, one can use a larger matrix, for instance

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}$$

(1)

to combine and split channels. In [18], [19], [20], [21], [22], [23], binary matrices ranging from $3 \times 3$ to $64 \times 64$ are studied and the scaling exponents over BECs are estimated. The best scaling exponent up to every matrix size is plotted in Figure 2. There are also meta-asymptotic results stating that $\mu \approx 2$ can be achieved using larger and larger matrices. This statement was proved over $q$-ary erasure channels [24], binary erasure channels [25], all BMS channels [26], and finally discrete memoryless channels [27].

As much as we want to lower polar code’s scaling exponent, there is one caveat that renders large matrices impractical: the smallest matrix whose scaling exponent is strictly better than $[10]$ is the $8 \times 8$ matrix given in (1). Using this matrix takes twice more time to decode (estimate based on the method of [28]), whereas the benefit we gain is that $\mu$ slightly decreases from 3.627 to 3.577. As the matrix gets larger and deviates more from the tensor powers of $[10]$, the time complexity grows drastically. For this reason, it is unlikely that we will ever see polar code based on large matrices (unless it is for other concerns [29]).

Large matrices aside, many other techniques emerge with empirical evidence that they improve polar codes—concatenation, cyclic redundancy check, and list decoder to name a few. But none of them sees a proof of improvements in the scaling exponent; in fact, quite the opposite was reported [30]. So we are back to the drawing board where we want to improve polar codes’ scaling exponent while minimizing the complexity penalty.

One approach that seems promising, albeit very little is known, is to use a non-binary input alphabet. This line of research dates back to low-density parity-check codes [31], [32]. For polar code, it started
from Şaşoğlu [33], [34], [35], wherein the goal was to find at least one way to polarize arbitrary finite alphabets regardless of the speed. In particular, the usual matrix \([1,0]\) is known to polarize prime fields. Later, Sahebi–Pradhan [36] and Park–Barg [37] showed that \([1,1]\) cannot polarize non-prime fields. Then, Mori–Tanaka [6] classified all matrices that can polarize finite fields (i.e., the alphabet size must be a prime power). One step forward, Nasser [38] classified all binary operators (i.e., bivariate functions) that can polarize arbitrary finite alphabets. In [39], [40], the authors showed that, for any polarizing matrix over prime fields, one has \(\mu < \infty\). In [27], the authors showed that \(\mu \approx 2\) is reachable over arbitrary finite alphabets.

Why is a non-binary input alphabet attractive? There are at least three reasons. First, modulation: For quadrature amplitude modulation (QAM) and amplitude and phase-shift keying (APSK), a constellation point is more likely to be confused with constellation points nearer to it. A non-binary channel models this proximity relation more naturally than a series of correlated binary channels do [41], [42]. Second, two-stage polarization: If we weakly-polarize a binary channel with \([1,0]\), treat every two binary channels as one quaternary channel, and strongly-polarize the quaternary channels with the \(4 \times 4\) Reed–Solomon matrix, we can improve the asymptotics of \(f(n)\) from \(\exp(-2^{0.5n})\) to \(\exp(-2^{0.5731n})\) [43] (see also [44], [45]). Third, and most importantly, scaling exponent: Several works have observed that non-binary matrices of the form \([1,0]\) just polarize faster than \([1,1]\) [46], [47], [48]; some reported that even a minuscule amount of permutation can achieve similar effects [49]. Could it be that the non-binary scaling exponents are smaller?

Consider [50]’s technique that uses \([1,0]\) to polarize non-binary channels; their result has an implication that non-binary channels’ scaling exponent is at least as good as binary channels’. In this paper, we aim to answer the question of whether the former is strictly better than the latter. By defining a toy model that contains a pair of BECs as a special case and estimating the scaling exponent of \([1,0]\), we provide a proof of concept result that an expansion in alphabet size does result in an improvement in scaling exponent. Recall that BECs form a one-parameter family and that this property makes its scaling behavior easy to analyze. This paper’s overall strategy is to show that the descendants of a quaternary channel converge to an almost–one–parameter family; we then analyze the scaling behavior of this family and conclude the following.

\textbf{Theorem 1 (main theorem).} Treating a pair of BECs as a quaternary channel, the \(2 \times 2\) matrix \([1,0]\) over \(F_4\) induces a scaling exponent less than \(3.451\). Here, \(\omega^2 + \omega + 1 = 0\).

This paper is organized as follows. Section II reviews polar code. Section III defines tetrahedral erasure channels (TECs) as a generalization of pairs of BECs, defines balanced TECs to be those that possess some symmetry, and defines edge-heavy TECs to be those that will be polarized faster. Section IV defines serial combination and parallel combination that will be used to polarize TECs. Section V shows that unbalanced TECs tend to become very close to balanced TECs, so it suffices to consider the speed of polarization of the latter. Section VI shows that balanced TECs tend to become very close to edge-heavy TECs, so it suffices to consider the speed of polarization of the latter. Section VII estimates the speed of polarization of balanced edge-heavy TECs, which proves the main theorem.

\textbf{II. POLAR CODE}

Readers who are familiar with polar code may skip this section. This section serves a high-level summary of polar code. More details are found in [51, Chapter 2]. We assume BEC throughout the section.

Let \(X \in F_2\) be a random variable following the uniform distribution. Let \(Y \in F_2 \cup \{?\}\) be a random variable with transition probabilities

\[
P\{Y = x \mid X = x\} = 1 - \varepsilon, \quad P\{Y = ? \mid X = x\} = \varepsilon.
\]
Here, $\varepsilon \in [0, 1]$ is called the erasure probability. The pair $(X|Y)$ is called a binary erasure channel (BEC) and denoted by BEC$(\varepsilon)$. The conditional entropy $H(\text{BEC}(\varepsilon)) = H(X|Y) = \varepsilon$ is defined through Shannon’s mean.

Let $(X_1|Y_1)$ and $(X_2|Y_2)$ be two iid copies of BEC$(\varepsilon)$. Define the serial combination BEC$(\varepsilon)^{\cdot \cdot \cdot}$ to be $(X_1 + X_2|Y_1, Y_2)$. That is, what do we know about $X_1 + X_2$ when given $Y_1$ and $Y_2$? One sees that it is information theoretically equivalent to BEC$(2\varepsilon - \varepsilon^2)$. Define the parallel combination BEC$(\varepsilon)^{\otimes}$ to be $(X_1|Y_1, Y_2, X_1 + X_2)$. That is, what do we know about $X_1$ when given $Y_1$, $Y_2$, and $X_1 + X_2$? We can see that it is information theoretically equivalent to BEC$(\varepsilon^2)$.

Serial and parallel combinations apply recursively. A polar code of block length $2^n$ is specified by a subset of strings $I \subseteq \{\ast, \otimes\}^n$. In this code, a synthetic channel

$$
\left( \cdots ((\text{BEC}(\varepsilon)^{c_1})^{c_2}) \cdots \right)^{c_n}
$$

will be used to transmit useful information iff $(c_1, c_1, \ldots, c_n) \in I$. The code rate of this polar code is $|I|/2^n$. The exact miscommunication probability of this polar code is hard to find, but has an upper bound of

$$
\sum_I H\left( \left( \cdots ((\text{BEC}(\varepsilon)^{c_1})^{c_2}) \cdots \right)^{c_n} \right).
$$

To define a good $I$, choose a function $f(n)$ and collect all strings $(c_1, c_1, \ldots, c_n) \in \{\ast, \otimes\}^n$ such that $H(\text{channel (2)})$ is less than $f(n)$. The fact that the erasure probabilities undergo simple evolutions $\varepsilon \mapsto 2\varepsilon - \varepsilon^2$ and $\varepsilon \mapsto \varepsilon^2$ motivates the following stochastic process: define $\{H_n\}_n$ by initial value $H_0 := \varepsilon$ and evolution rule $H_{n+1} := 2H_n - H_n^2$ or $H_n^2$ with equal probability. Then the code rate $|I|/2^n$ coincides with $\mathbb{P}\{H_n \leq f(n)\}$. The gap to capacity $g(n) := 1 - H_0 - |I|/2^n = 1 - H_0 - \mathbb{P}\{H_n \leq f(n)\}$ can also be expressed using $f$.

In a way, the study of polar code over BEC is the study of the cdf of $H_n$, with emphasis put on the hard threshold at $1 - H_0$. Abusing the same logic, this paper is a study of a stochastic process $\{W_n\}_n$ that lives in the four-dimensional simplex $[0, 1]^5 \cap \{p + q + r + s + t = 1\}$, which happens to have implications in coding theory (the implication being how to use simple instruments to construct low-$\mu$ codes).

### III. An Inspirational Channel Model

We are to define a type of quaternary channels in this section. This should be the smallest possible set of quaternary channels that meet the following two criteria: (a) it should model a pair of BECs as a special case; and (b) it should be closed under pre-processing the input using invertible linear transformations. Allowing such pre-processing is crucial to the improvement of scaling exponent as it helps mixing the information.

#### A. Tetrahedral erasure channel

Let the input alphabet be $\mathbb{F}_2^2$, and we assume the uniform input distribution throughout the paper. For any input $(x_1, x_2) \in \mathbb{F}_2^2$, the output will be in $(\mathbb{F}_2 \cup \{x\})^3$ and assume one of the following five erasure patterns:

- $(x_1, x_1 + x_2, x_2)$ with probability $p$;
- $(x_1, x_2, x_2)$ with probability $q$;
- $(x_1 + x_2, x_1)$ with probability $r$;
- $(x_1, x_2, x)$ with probability $s$;
- $(x, x, x_2)$ with probability $t$.

We call $p$, $q$, $r$, $s$, and $t$ the subspace erasure probabilities and they sum to 1. Such a channel is denoted by TEC$(p, q, r, s, t)$. For brevity, we say a TEC outputs $(x_1, x_2)$, outputs $x_1$, outputs $x_1 + x_2$, outputs $x_2$, and outputs nothing to represent the five erasure patterns.
A TEC can be related to a tetrahedron whose vertices are at $(0, 0, 0), (1, 1, 0), (1, 0, 1)$, and $(0, 1, 1)$. Outputting $(x_1, x_2)$ corresponds to the vertex $(x_1, x_1 + x_2, x_2)$. Outputting $x_1$ corresponds to the edge $(x_1, x_1, 0) - (x_1, 1 - x_1, 1)$. Outputting nothing corresponds to the tetrahedron per se. That is to say, a TEC takes a vertex as an input and outputs the same vertex with probability $p$, outputs one of the edges attached to that vertex with probabilities $q$, $r$, and $s$, respectively, and output the entire tetrahedron with probability $t$.

There is another way to interpret a TEC. Consider $F_4$ and let $\omega$ be a primitive element therein. A TEC takes $x := x_1 + x_2\omega \in F_4$ as an input and outputs $x$, $\text{tr}(x/\omega)$, $\text{tr}(\omega x)$, or $\text{tr}(x)$ or nothing (with a flag that allows the receiver to distinguish these five cases) with probabilities $p$, $q$, $r$, $s$, and $t$, respectively. Here, $\text{tr}: F_4 \to F_2$ is the field trace. It is the matrix trace if we use matrices $[00]$, $[10]$, $[01]$, $[11]$, $[10] \in F_2^{2 \times 2}$ to represent $0, 1, \omega, 1 + \omega \in F_4$.

On top of the the fact that TECs are a natural family of erasure channels, they relate to other channels that have been discussed in literature.

**Proposition 2.** The “$q$-ary erasure channel with erasure probability $\varepsilon$” [52], [24], when $q = 4$, is a TEC of the form $\text{TEC}(1 - \varepsilon, 0, 0, 0, \varepsilon)$.

**Proposition 3.** When transmitting two bits $x_1$ and $x_2$ through $\text{BEC}(\delta)$ and $\text{BEC}(\varepsilon)$, respectively, the outputs can be simulated by $\text{TEC}((1 - \delta)(1 - \varepsilon), (1 - \delta)\varepsilon, 0, \delta(1 - \varepsilon), \delta\varepsilon)$.

Proofs of Propositions 2 and 3 are omitted. The propositions imply that any scaling exponent estimate for TEC immediately generalizes to 4-ary erasure channels and quaternary channels that are pairs of BECs. Note that transmitting information over a pair of $\text{BEC}(\varepsilon)$ is the same matter as transmitting over $\text{BEC}(\varepsilon)$. Hence, an estimate of the scaling exponent over TECs implies the same estimate for BECs.

As H. Pfister pointed out, TEC has been studied in the context of non-binary low-density parity-check code by, among others, Rathi and Urbanke [32], who pointed out that the idea of using non-binary alphabet dates all the way back to Gallager himself. Later in the polar code context, TEC became a very natural toy model for 2-user multiple access channels [53, Section VII-B], as kindly pointed out by one of the anonymous reviewers. More recently, [49] (and its successor [54]) studied an even more flexible generalization of polar coding that, sketchily speaking, uses permutations to shuffle bits and uses erasure patterns to study the performance. When the alphabet is an arbitrary abelian group, one can study erasure...
channels whose erasure patterns are projections onto quotient groups [36], [55], as kindly pointed out by one of the anonymous reviewers.

B. Channel functionals

The conditional entropy (sometimes entropy) of a TEC \( W = \text{TEC}(p,q,r,s,t) \) is defined as

\[ H(W) := \frac{q + r + s}{2} + t. \]

This definition is compatible with Shannon’s definition of conditional entropy in the sense that we lose one out of two bits of information with probability \( q + r + s \) and two out of two bits with probability \( t \). Clearly \( 0 \leq H(W) \leq 1 \).

We define the moment of inertia of a TEC \( W = \text{TEC}(p,q,r,s,t) \) as

\[ A(W) := (q - r)^2 + (r - s)^2 + (s - q)^2. \]

Clearly \( 0 \leq A(W) \leq 3 \). When \( q = r = s \), we say that this TEC is balanced. Put it another way, a TEC is balanced iff the edges of the tetrahedron weigh the same iff its moment of inertia vanishes. See also the “symmetric over the product” condition in [56] and the “equidistance” condition in [33].

We define the edge mass of a TEC \( W = \text{TEC}(p,q,r,s,t) \) as

\[ E(W) := q + r + s. \]

Clearly \( 0 \leq E(W) \leq 1 \). It is not hard to see that \( H \) and \( E \) uniquely determine a balanced TEC by

\[
\begin{align*}
p &= 1 - H(W) - \frac{E(W)}{2}, \\
q &= r = s = \frac{E(W)}{3}, \\
t &= H(W) - \frac{E(W)}{2}.
\end{align*}
\]

This implies that we can use \((H,E)\) to parametrize balanced TECs. While we all agree that \( H \) is a very important parameter of channels, we will see that the \( E \) is also important as it measures the volatility of \( H \): a higher \( E(W) \) means a higher \( H(W^\text{T}) - H(W) \) and a higher \( H(W) - H(W^{\text{rot}}) \).

We define the \( Q \)-index of a TEC \( W \) by

\[ Q(W) := \frac{E(W)}{H(W)(1 - H(W))}. \]

Clearly, \( 0 \leq E(W) \leq 2 \min(H(W),1 - H(W)) \) and hence \( 0 \leq Q(W) \leq 4 \).

We call a TEC \( W = \text{TEC}(p,q,r,s,t) \) edge-positive if \( qrs > 0 \). Edge positivity implies that the descendants of \( W \) are all edge-positive and have positive \( E \), \( H \), and \( 1 - H \), which implies that \( Q \) is always well-defined. We call a TEC \( W \) edge-heavy if \( Q(W) \geq \alpha := 2\sqrt{7} - 4 \). Note that, for a BEC \( V \), \( H(V^\text{tr}) - H(V) = H(V) - H(V^{\text{rot}}) = H(V)(1 - H(V)) \). Hence we treat \( H(1 - H) \) as the standard volatility. And then we compute the quotient of \( E(W) \) by the standard value to get a normalized volatility measure. Why we choose \( \alpha \) as the threshold will be clear later.

IV. CHANNEL SYNTHESIS

TECs can be serially combined or parallely combined as in the theory of density evolution [57]. Simply put, the serial combination of two channels is analogous to a standardized math exercise where, in order to test if students know both \( u \) and \( v \), we ask for \( u + v \).

Parallel combination, on the other hand, is analogous to a generous exercise where we give the true value of \( u + v \) as a hint, and so any student who can compute \( u \) or \( v \) will immediately know both quantities.

In this section, we demonstrate that the serial and parallel combinations of two TECs are again TECs, and we will study how the subspace erasure probabilities evolve under combinations.
A. Serial combination

Let $U$ and $V$ be two independent channels with subspace erasure probabilities $\text{TEC}(p,q,r,s,t)$ and $\text{TEC}(p',q',r',s',t')$, respectively. The serial combination of $U$ and $V$ is defined to be the task of guessing $(u_1 + v_1, u_2 + v_2)$ given the output of inputting $(u_1, u_2)$ into $U$ and the output of inputting $(v_1, v_2)$ into $V$. As $U$ produces five erasure patterns and $V$ also produces five, there are twenty-five erasure patterns in total.

1) With probability $pp'$, $U$ outputs $(u_1, u_2)$ and $V$ outputs $(v_1, v_2)$: In this case, we can infer both $u_1 + v_1$ and $u_2 + v_2$.

2) With probability $pq'$, $U$ outputs $(u_1, u_2)$ and $V$ outputs $v_1$: In this case, we can infer $u_1 + v_1$ but not $u_2 + v_2$. Note that knowing $u_2$ does not reveal any information about $u_2 + v_2$ as $v_2$ is assumed to be uniformly randomly distributed.

3) With probability $pr'$, $U$ outputs $(u_1, u_2)$ and $V$ outputs $v_1 + v_2$: In this case, we can not infer $u_1 + v_1$; nor can we infer $u_2 + v_2$. However, we can still infer their sum $(u_1 + v_1) + (u_2 + v_2)$ because it is equal to $u_1$ (which is known) plus $u_2$ (which is also known) plus $v_1 + v_2$ (which is known as well).

4) With probability $ps'$, $U$ outputs $(u_1, u_2)$ and $V$ outputs $v_2$: In this case, we cannot infer $u_1 + v_1$ but we can infer $u_2 + v_2$.

5) With probability $pt'$, $U$ outputs $(u_1, u_2)$ and $V$ outputs nothing: In this case, we obtain absolutely no information about $u_1 + v_1$ and $u_2 + v_2$ and their sum.

6) With probability $qp'$, $U$ outputs $u_1$ and $V$ outputs $(v_1, v_2)$: In this case, we can infer $u_1 + v_1$ but not $u_2 + v_2$.

7) With probability $qq'$, $U$ outputs $u_1$ and $V$ outputs $v_1$: In this case, we can infer $u_1 + v_1$ but not $u_2 + v_2$.

8) With probability $qr'$, $U$ outputs $u_1$ and $V$ outputs $v_1 + v_2$: In this case, we know nothing about $u_1 + v_1$ because we cannot infer anything about $v_1$ given only $v_1 + v_2$. We know nothing about $(u_1 + v_1) + (u_2 + v_2)$ because $u_2$ is missing. Lastly, we know nothing about $u_2 + v_2$. That is to say, we learn nothing useful.

9) With probability $qs'$, $U$ outputs $u_1$ and $V$ outputs $v_2$: In this case, we know nothing about $u_1 + v_1$ and $u_2 + v_2$ and their sum.

10) With probability $qt'$, $U$ outputs $u_1$ and $V$ outputs nothing: In this case, we know nothing about $u_1 + v_1$ and $u_2 + v_2$ and their sum.

11) With probability $rp'$, $U$ outputs $u_1 + u_2$ and $V$ outputs $(v_1, v_2)$: In this case, we know $(u_1 + v_1) + (u_2 + v_2)$.

12) With probability $rq'$, $U$ outputs $u_1 + u_2$ and $V$ outputs $v_1$: In this case, we learn nothing useful.

13) With probability $rr'$, $U$ outputs $u_1 + u_2$ and $V$ outputs $v_1 + v_2$: In this case, we learn nothing useful.

14) With probability $rs'$, $U$ outputs $u_1 + u_2$ and $V$ outputs $v_2$: In this case, we learn nothing useful.

15) With probability $rt'$, $U$ outputs $u_1 + u_2$ and $V$ outputs nothing: In this case, we learn nothing useful.

16) With probability $sp'$, $U$ outputs $u_2$ and $V$ outputs $(v_1, v_2)$: In this case, we know $u_2 + v_2$.

17) With probability $sq'$, $U$ outputs $u_2$ and $V$ outputs $v_1$: In this case, we learn nothing useful.

18) With probability $sr'$, $U$ outputs $u_2$ and $V$ outputs $v_1 + v_2$: In this case, we learn nothing useful.

19) With probability $ss'$, $U$ outputs $u_2$ and $V$ outputs $v_2$: In this case, we know $u_2 + v_2$.

20) With probability $st'$, $U$ outputs $u_2$ and $V$ outputs nothing: In this case, we learn nothing useful.

21) With probability $tp'$, $U$ outputs nothing and $V$ outputs $(v_1, v_2)$: In this case, we learn nothing useful.

22) With probability $tq'$, $U$ outputs nothing and $V$ outputs $v_1$: In this case, we learn nothing useful.

23) With probability $tr'$, $U$ outputs nothing and $V$ outputs $v_1 + v_2$: In this case, we learn nothing useful.

24) With probability $ts'$, $U$ outputs nothing and $V$ outputs $v_2$: In this case, we learn nothing useful.

25) With probability $tt'$, $U$ outputs nothing and $V$ outputs nothing: In this case, we learn nothing useful.

Case 1 is when we know both $u_1 + v_1$ and $u_2 + v_2$. Cases 2, 6, and 7 are when we know $u_1 + v_1$ but not the other. Cases 3, 11, and 13 are when we know the sum $(u_1 + v_1) + (u_2 + v_2)$ but not the summands.
Cases 4, 16, 19 are when we know $u_2 + v_2$ but not the other. Cases 5, 8, 9, 10, 12, 14, 15, 17, 18, 20, 21, 22, 23, 24, and 25 are when we know nothing useful. Denote by $U \ast V$ the serial combination of $U$ and $V$; we conclude that it is a TEC with subspace erasure probabilities

$$U \ast V := \text{TEC}(pp',$$

$$pq' + qq' + qp',$$

$$pr' + rr' + rp',$$

$$ps' + ss' + sp',$$

$$1 - \text{the other four terms}).$$

See Table I for a summary.

### B. Parallel combination

The parallel combination of $U$ and $V$ is defined to be the task of guessing $(u_1, u_2)$ given $(u_1 + v_1, u_2 + v_2)$ (the most informative output of $U \oplus V$), the result of feeding $(u_1, u_2)$ into $U$, and the result of feeding $(v_1, v_2)$ into $V$.

Denote by $U \oplus V$ the parallel combination of $U$ and $V$. One can go over its twenty-five erasure patterns similar to what the previous subsection does. For instance, with probability $qr'$, $U$ outputs $u_1$ and $V$ outputs $v_1 + v_2$. In this case, we can infer $v_1$ (using $u_1$ and $u_1 + v_1$), followed by $v_2$ (using $v_1$ and $v_1 + v_2$), and finally $u_2$ (using $v_2$ and $u_2 + v_2$); and hence we can completely recover $u_1$ and $u_2$. Details omitted, it can be shown that $U \oplus V$ is a TEC with subspace erasure probabilities

$$U \oplus V := \text{TEC}(1 - \text{the other four terms},$$

$$tq' + qq' + qt',$$

$$tr' + rr' + rt',$$

$$ts' + ss' + st',$$

$$tt').$$

See also Table I for a summary.

Note that there is a duality between $\text{TEC}(p, q, r, s, t)$ and $\text{TEC}(t, s, r, q, p)$ that respects $A$ and $E$, maps $H$ to $1 - H$, and swaps parallel and serial combinations:

$$\text{Dual}(\text{TEC}(p, q, r, s, t)) := \text{TEC}(t, s, r, q, p),$$

$$\text{Dual}(\text{Dual}(W)) = W,$$

$$H(\text{Dual}(W)) = 1 - H(W),$$

$$A(\text{Dual}(W)) = A(W),$$

$$E(\text{Dual}(W)) = E(W),$$

$$\text{Dual}(U \ast V) = \text{Dual}(U) \oplus \text{Dual}(V),$$

| P | Q | R | S | T |
|---|---|---|---|---|
| P | P | Q | R | S | T |
| Q | Q | Q | T | T | T |
| R | R | T | R | T | T |
| S | S | T | T | T | T |
| T | T | T | T | T | T |

**TABLE I**

_How erasure patterns evolve. We use capital letter $P$ to represent the erasure pattern whose probability is denoted by the lower letter $p$. Same for $Q, R, S, and T._
The kernel \((u,v)\) hints that twisting makes it easier to reduce \(D\). Channel process and channel that multiplies the input by \(\omega\) are denoted by \(W\). The children of \(W\) are the children of \(W\) let \(\mathrm{TEC}(p,q,r,s,t)\) be the linear transformation that reads \((u,v) \mapsto (u,\omega v, v)\) or, equivalently,

\[
[u \quad v] \mapsto [u \quad v] \begin{bmatrix} 1 & 0 \\ \omega & 1 \end{bmatrix}.
\]

This kernel was studied by Mori–Tanaka \cite{6} and is shown to be polarizing. If we treat \(F_4\) as \(F_2^2\) and express \(u\) and \(v\) as \(u = u_1 + u_2\omega\) and \(v = v_1 + v_2\omega\) for \(u_1, u_2, v_1, v_2 \in F_2\), then \(K\) reads \(\left((u_1, u_2), (v_1, v_2)\right) \mapsto (u_1 + v_2, u_2 + v_1 + v_2), (v_1, v_2)\). Equivalently,

\[
[u_1 \quad u_2 \quad v_1 \quad v_2] \mapsto [u_1 \quad u_2 \quad v_1 \quad v_2] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.
\]

The kernel \(K\) combines two TECs \(U\) and \(V\) to synthesize \(U\,\bigoplus\, (V\omega)\) and \(U \,\bigoplus\, (V\omega)\), where \(V\omega\) is the channel that multiplies the input by \(\omega\) before feeding it into \(V\). For brevity, \(W\,\bigoplus\, (W\omega)\) and \(W \,\bigoplus\, (W\omega)\) are denoted by \(W^\dagger\) and \(W^\ominus\), respectively.

Multiplying a TEC by \(\omega\) behaves like a rotation of order 3 (after all, \(\omega^3 = 1\) and it is rotating the tetrahedron). It maps TEC\((p,q,r,s,t)\) to TEC\((p,s,q,r,t)\). If \(W\) is balanced, rotation does not alter it: \(W = W\omega\). If it is not balanced, then the rotation helps to break the alignment of \(q, r,\) and \(s\) so that a large subspace erasure probability is paired with a small one. More precisely, in

\[
\mathrm{TEC}(p,q,r,s,t)^\dagger := \mathrm{TEC}(p^2; ps + sq + qp, pq + qr + rp, pr + rs + sp, 1 - \text{the other 4 terms})
\]

and

\[
\mathrm{TEC}(p,q,r,s,t)^\ominus := \mathrm{TEC}(1 - \text{the other 4 terms}, ts + sq + qt, tq + qr + rt, tr + rs + st, t^2),
\]

we see that the “diagonal terms” \(qq', rr',\) and \(ss'\) are replaced by “misalign terms” \(qr, rs,\) and \(sq\). This hints that twisting makes it easier to reduce \(q, r,\) and \(s\) by redistributing the masses to \(p\) and \(t\).

D. Channel process

For a TEC \(W\), we call \(W^\dagger\) the serial-child of \(W\) and \(W^\ominus\) the parallel-child of \(W\). Together, they are the children of \(W\). The descendants of \(W\) are the children of \(W\) together with the descendants of the children of \(W\). The \(n\)-th-generation descendants of \(W\) are the \((n - 1)\)-th-generation descendants of the children of \(W\); the 0th is \(W\) itself.

When \(W\) is understood from the context, let \(W_0 = W\). For \(n\) a positive integer, let \(W_n\) be a random child of \(W_{n-1}\) with equal probability.
The common strategy used to estimate the scaling exponent concerns a concave function \( \psi: [0, 1] \to \mathbb{R} \) such that \( \psi(0) = \psi(1) = 0 \) and it is positive elsewhere. With \( \psi \), one finds a \( 0 < \mu < \infty \) such that
\[
\frac{\psi(H(W^{\mathbb{N}})) + \psi(H(W^{\mathbb{O}}))}{2\psi(H(W))} \leq 2^{-1/\mu}.
\]
We will treat the preceding formula as an operational definition of the scaling exponent because, with this formula, a routine argument [51, Sections 5.8–5.10] will show that
\[
P \{ H(W_n) < \exp(-n^{1/3}) \} > 1 - H(W_0) - 2^{-n/\mu}.
\]
That is to say, a good choice of \( \psi \) provides a good characterization of the scaling behavior of polar codes.

V. UNBALANCED TEC BECOMES BALANCED

In this section, we argue that TECs undergoing the polarization process tend to become more balanced than before. We do so by showing that the moments of inertia are decreasing. We begin by showing that \( A(W^{\mathbb{Z}}) + A(W^{\mathbb{O}}) \leq A(W) \), which implies that the total amount of moments of inertia is conserved.

**Proposition 4** (conservation of inertia). \( A(W^{\mathbb{Z}}) + A(W^{\mathbb{O}}) \leq A(W) \) for any TEC \( W \).

A proof of Proposition 4 is in Appendix A-A. By the proposition, \( A(W) \geq A(W^{\mathbb{Z}}) + A(W^{\mathbb{O}}) \geq A(W^{\mathbb{Z}}) + A(W^{\mathbb{O}}) + A(W^{\mathbb{Z}}} + A(W^{\mathbb{O}}) \geq A(W^{\mathbb{Z}}} + A(W^{\mathbb{O}}) \cdots \). Hence the expectation of \( A(W_n) \) over all \( W_n \) is at most \( A(W) / 2^n \). By the same idea behind Markov’s inequality, this suggests that for any TEC \( W \), all but a few descendants have exponentially small \( A \)'s: they are almost-balanced.

However, Proposition 4 does not completely rule out the case that some descendants will be highly unbalanced and eventually slow down the polarization. So we present a uniform control on \( A \).

**Proposition 5** (uniform loss of inertia). \( A(W^{\mathbb{Z}}) + A(W^{\mathbb{O}}) \leq A(W) - A(W) / 3 \) for any TEC \( W \).

A proof of Proposition 5 is in Appendix A-B. Now the recurrence relation \( A(W_{n+1}) \leq A(W_n) \times (1 - A(W_n) / 3) \) is equivalent to \( A(W_{n+1}) - A(W_n) \leq -A(W_n)^2 / 3 \) and analogous to the ordinary differential equation \( f'(t) \leq -f(t)^2 / 3 \). Solving it, we get \( f(t) = O(1/t) \). Hence we expect that \( A(W_n) = O(1/n) \).

**Corollary 6** (ultimate loss of inertia). Fix a TEC \( W \), then \( A(W_n) = O(1/n) \) as \( n \to \infty \).

**Proof:** We prove by induction that \( 1/A(W_n) \geq n/3 \). Base case: By the definition of moment of inertia, \( A(W_1) \leq 3 \), hence \( 1/A(W_1) \geq 1/3 \). Induction case:
\[
\frac{1}{A(W_{n+1})} \geq \frac{1}{A(W_n)(1 - A(W_n)/3)} \\
\geq \frac{1}{A(W_n) + A(W_n)/3} \\
= \frac{1}{A(W_n)} + \frac{1}{3} \\
\geq \frac{n}{3} + \frac{1}{3}.
\]
This finishes the induction and the proof that \( A(W_n) = O(1/n) \).

From Proposition 4 to Corollary 6, these results all lead to the same conclusion that any unbalanced TEC will quickly become very similar to a balanced one. Therefore, we expect that the speed of polarization of unbalanced TECs is dominated by that of balanced TECs.

We invite readers to assume that it suffices to consider balanced TECs when estimating the scaling exponent over TECs. These readers may jump to the next section, where we will be studying the evolution...
of \( \{H(W_n)\}_n \) and \( \{E(W_n)\}_n \) for balanced \( W \)'s. For readers who want to see a complete, rigorous proof, we introduce the following technical results that will be used later.

**Proposition 7** (continuity in inertia). For any TEC \( W = \text{TEC}(p, q, r, s, t) \) and its balanced version

\[
\bar{W} := \text{TEC} \left( p, \frac{q + r + s}{3}, \frac{q + r + s}{3}, \frac{q + r + s}{3}, t \right).
\]

we have

\[
H(W) = H(\bar{W}), \\
E(W) = E(\bar{W}), \\
H(W^{\overline{1}}) = H(\bar{W}^{\overline{1}}) + \frac{A(W)}{12}, \\
E(W^{*1}) = E(\bar{W}^{*1}) - \frac{A(W)}{6}, \\
H(W^{\odot}) = H(\bar{W}^{\odot}) - \frac{A(W)}{12}, \\
E(W^{\odot}) = E(\bar{W}^{\odot}) - \frac{A(W)}{6}.
\]

**Theorem 8** (monotonicity of \( A/E \)). For any TEC \( W \),

\[
\frac{A(W^{\overline{1}})}{E(W^{\overline{1}})} \leq \frac{A(W)}{E(W)} \leq 2E(W).
\]

**Theorem 9** (fast loss of inertia). For any TEC \( W \),

\[
A(W^{\odot}) \leq A(W)H(W)^2, \\
A(W^{*}) \leq A(W)(1 - H(W))^2.
\]

A proof of Proposition 7 is in Appendix A-C. A proof of Theorem 8 is in Appendix A-D. A proof of Theorem 9 is in Appendix A-E. Note that Theorem 9 generalizes Proposition 4 as \( H(W)^2 + (1 - H(W))^2 \leq 1 \).

### VI. Balanced TECs Hoard Edge Mass

In this section, we want to show that the Q-index \( Q(W_n) = E(W_n)/H(W_n)(1 - H(W_n)) \) of a sufficiently deep descendant is about 1.6. Put another way, there is a “trap” that constrains the relation between \( E(W_n) \) and \( H(W_n) \). Our main strategy is to show that, if \( Q(W_n) \) is too low, \( Q(W_{n+1}) \) will become higher, and vice versa.

Recall that \( W = \text{TEC}(p, q, r, s, t) \) is said to be edge-positive if \( qrs > 0 \), which implies that \( E, H, 1 - H \) are all positive for all descendants of \( W \), which then implies that \( Q \) are always defined. Recall also that \( W \) is said to be edge-heavy if \( Q(W) \geq \alpha := 2\sqrt{7} - 4 \). Note that \( \alpha \approx 1.3 \).

**Theorem 10** (trapping region). If \( W \) is balanced and edge-heavy, then its children are balanced and edge-heavy.

A proof of Theorem 10 is in Appendix B-A. The theorem implies that all descendants of an edge-heavy TEC are edge-heavy. For a TEC that is not edge-heavy, its descendants will become “edge-heavier” by the following lemma.

**Lemma 11** (attraction toward the trap). Fix any \( \varepsilon > 0 \); choose \( \delta := 3\varepsilon/8 \). Let \( W \) be balanced and edge-positive. We have that \( Q(W) \leq \alpha - \varepsilon \) implies

\[
Q(W^{*1}) \geq Q(W)(1 + H(W)\delta),
\]
\[ Q(W^\ominus) \geq Q(W)(1 + (1 - H(W))\delta). \]

A proof of Lemma 11 is in Appendix B-B. It is unfortunate that the factors \( H(W) \) and \( 1 - H(W) \) before \( \delta \) slow down the rate at which \( Q(W_n) \) approaches \( \alpha = 2\sqrt{7} - 4 \), especially when \( H(W) \) is close to 0 or 1, respectively. These factors cannot be optimized away. To see why, suppose that \( H(W) = x \approx 1 \) and \( E(W) = y \approx 0 \). Then \( H(W^\ominus) \) is about \( x^2 + O(y^2) \) and \( E(W^\ominus) \) is about \( 2xy + O(y^2) \). Hence \( Q(W^\ominus) \) is about \( 2xy/x^2(1 - x^2) \approx y/x(1 - x) = Q(W) \). That being the case, we can see from this example that TECs whose Q-indices can hardly be improved are already polarized, so we shall not worry about them. Besides, we can prove uniform attraction.

**Theorem 12** (uniform attraction). Fix any \( \varepsilon > 0 \). For any balanced and edge-positive TEC \( W \) such that \( Q(W) \leq \alpha - \varepsilon \), there exists an integer \( m > 0 \) such that \( Q(W_n) \geq Q(W)(1 + \varepsilon/8) \) for all \( n \geq m \).

A proof of Theorem 12 is in Appendix B-C. Uniform attraction means that every child is at least making some positive progress toward the trap. Small steps of the descendants accumulate to a giant leap of the family.

**Corollary 13** (ultimate attraction). For any \( \varepsilon > 0 \) and any balanced and edge-positive TEC \( W \), there exists an integer \( m > 0 \) such that \( Q(W_n) \geq \alpha - \varepsilon \) for all \( n \geq m \).

**Proof:** Apply uniform attraction (Theorem 12) repeatedly. Every application improves the Q-index by a factor of \( 1 + \varepsilon/8 \). So after a finite number of applications the Q-index can be made \( \geq \alpha - \varepsilon \). \( \blacksquare \)

To summarize this and the previous section, we have two trends: unbalanced TECs tend to become balanced; and “edge-light” TECs tend to become edge-heavy.

The following proposition is a bound on the Q-indices in the opposite direction.

**Proposition 14** (trap on the other side). Let \( W \) be a balanced, edge-positive TEC with \( Q(W) \leq 2 \). Then \( Q(W^\ominus) \leq 2 \) and \( Q(W^{\ominus}) \leq 2 \).

A proof of Proposition 14 is in Appendix B-D. This proposition is not required for the proof of the main theorem, but it helps us understand the dynamic of TECs. Moreover, we anticipate that the counterparts of Lemma 11, Theorem 12, and Corollary 13 can all be stated and proved for bounding \( Q \) from above.

The following proposition gives a tighter (but more complex) trapping region than Theorem 10 and Proposition 14 do. A proof is omitted but the ideas are the same. But even these are not the optimal trapping region. For the optimal one, see the discussion in Appendix D.

**Proposition 15** (tighter trap). Let \( f(x) := x(1 - x)(1.66 - 0.38x(1 - x)) \). Then \( E(W) \leq f(H(W)) \) implies \( E(W^\ominus) \leq f(H(W^\ominus)) \) and \( E(W^{\ominus}) \leq f(H(W^{\ominus})) \). Let \( g(x) := x(1 - x)(2 - 2x(1 - x)/3) \). Then \( E(W) \geq g(H(W)) \) implies \( E(W^\ominus) \geq g(H(W^\ominus)) \) and \( E(W^{\ominus}) \geq g(H(W^{\ominus})) \).

At the end of Section V, we argue that since TECs will become practically indistinguishable from balanced TECs, all we need is to compute the scaling exponent for balanced TECs. To convert that idea to rigorous proofs, one way is to generalize Theorem 10–Corollary 13 to the case where \( W \) is unbalanced. We begin with the counterpart of Theorem 10.

**Theorem 16** (weak trap for the unbalanced). There exists a small number \( \varepsilon > 0 \) such that, for any edge-positive \( W \) (note: not necessarily balanced), we have \( Q(W^\ominus), Q(W^{\ominus}) \geq \min(Q(W), \varepsilon) \).

A proof of Theorem 16 is in Appendix B-E. This theorem is the one of the two puzzle pieces of the following (somewhat topological) argument that addresses how to handle almost but not strictly-speaking balanced TECs.

Suppose that the process \( \{H(W_n)\} \) always stays in the interval \([a, d]\) for some constants \( 0 < a < d < 1 \) such that \( d - a \approx 1 \), then \( H(W_n)(1 - H(W_n)) \geq a(1 - d) \) is bounded from below. Since \( Q(W_n) \) is bounded from below by \( \min(Q(W_0), \varepsilon) \) by Theorem 16, \( E(W_n) = Q(W_n)H(W_n)(1 - H(W_n)) \) is also bounded.
from below by \(a(1-d) \min(Q(W_0), \varepsilon)\). Now we apply Proposition 7 to \(W_n\) and its balanced version \(\overline{W}_{n}\). This allows us to predict \(H(W_{n+1})\) and \(E(W_{n+1})\) by

\[
H(W_n^\pm) = H(\overline{W}_{n}^\pm) + \frac{A(W_n)}{12},
\]

\[
E(W_n^\pm) = E(\overline{W}_{n}^\pm) - \frac{A(W_n)}{6},
\]

\[
H(W_n^\circ) = H(\overline{W}_{n}^\circ) - \frac{A(W_n)}{12},
\]

\[
E(W_n^\circ) = E(\overline{W}_{n}^\circ) - \frac{A(W_n)}{6}.
\]

But by Corollary 6, \(A(W_n)\) will keep decreasing as \(n \to \infty\) to the point that it becomes negligible compared to \(H(W_{n+1}), 1-H(W_{n+1}),\) and \(E(W_{n+1})\) (which we know are bounded from below). At that point, the additive error \(O(A(W_n))\) can be seen as a multiplicative error of \(1 \pm \) (some small number). This means that we can apply Lemma 11–Corollary 13 up to some multiplicative error to the case where \(W\) is unbalanced, instead of having to re-prove them for the unbalanced case.

On the other hand, suppose that the process \(\{H(W_n)\}_n\) escapes the interval \([b, c]\) at \(n = m\), where \(0 < a < b < c < d < 1\) and \(c - b \approx 1\), and never reenters \([b, c]\). Then either \(H(W_n)\) or \(1 - H(W_n)\) is very small. Without loss of generality, let us discuss the case that \(H(W_n)\) is small, where we have

\[
H(W_n^\circ) \leq O(H(W_n)^2),
\]

\[
H(W_n^\pm) \geq 2H(W_n) - O(H(W_n)^2)
\]

for all \(n \geq m\). This is the classical square-or-double behavior of Ankan’s martingale. The precise factor behind the big-O do not affect the estimate of the scaling exponent if \(b\) is taken to be small enough.\(^3\) To summarize this case, we do not have \(Q(W_n) \geq \alpha\) but we do not need it anyways.

There is one possibility left, namely the process \(\{H(W_n)\}_n\) escapes the interval \([a, d]\), but reenters \([b, c]\) later. In this case, we cannot apply the square-or-double argument because \(H(W_n)(1-H(W_n))\) is not eventually small; nor can we apply Lemma 11–Corollary 13 because \(H(W_n)(1-H(W_n))\) is sometimes too small. That being said, we claim that we can choose a tiny \(b\) (and \(c := 1-b\)) and an \(a\) that is very tiny compared to \(b\) (and \(d := 1-a\)) such that when \(H(W_n)\) reenters \([b, c]\), \(A(W_n)\) will be negligibly small and \(Q(W_n)\) extremely close to 2.

**Theorem 17** (becomes extreme and then mediocre). Let \(\gamma > 0\) be fixed. It is possible to select \(0 < a < b < c < d < 1\) such that, for any realization of \(\{H(W_n)\}_n\) that escapes \([a, d]\) and then reenters \([b, c]\), \(A(W_n)/E(W_{n+1}) < \gamma\) \(\text{and}\) \(Q(W_n) > 2 - \gamma\) at the \(n\) where the reentrance takes place.

A proof of Theorem 17 is in Appendix B-F. This theorem is the last puzzle pieces of the topological argument that addresses how to handle almost but not strictly-speaking balanced TECs.

The topological argument presented between Theorem 16 and Theorem 17 will be our strategy to prove the main theorem, which is done in the next section. While we, the authors, did our best to simplify the argument, it is understandable that some readers might find it pleasant to read. For those readers, we prepared a coding technique that immediately turns any unbalanced TEC into a perfectly balanced one. See Appendix E.

**VII. Edge-heavy TECs Polarize Faster**

Let \(W\) be any balanced TEC with \(H(W) = x\) and \(E(W) = y\). Recall that \(x\) and \(y\) uniquely determine a balanced TEC via (3), (4), and (5), hence we can compute and see that \(H(W^\circ) = 2x - x^2 + y^2/12\) and \(H(W^\circ) = x^2 - y^2/12\)

\(^3\)In fact, the square-or-double behavior is even compatible with the optimal scaling exponent 2 \cite{26}
Note that $H(W_{1\Sigma})$ is increasing in $y$ and $H(W_{\ominus})$ decreasing in $y$. The monotonicity has two applications. Application one: If we know too little to lower bound $Q(W)$, we will upper bound $H(W_{\ominus})$ using $x^2$. In this case, the speed of polarization is at least $\mu \approx 3.627$, the number associated to BECs.

Application two: If we know $Q(W) \geq \alpha$, we will upper bound $H(W_{\ominus})$ using $x^2 - (\alpha x(1-x))^2/12$. This time, $H(W_{\ominus})$ and $H(W^*)$ are more separated so the speed of polarization is strictly better than $\mu \approx 3.627$. Any positive $\alpha$, not necessarily $2\sqrt{7} - 4$, can improve the scaling. This is demonstrated by the following lemma that uses $9/7$ in place of $\alpha$.

**Lemma 18** (eigenfunction and eigenvalue). Let $\psi(x) := (x(1-x))^{0.697}(5 - \sqrt{x(1-x)})$. For balanced TECs with $Q(W) \geq 9/7$,

$$\frac{\psi(H(W_{1\Sigma})) + \psi(H(W_{\ominus}))}{2\psi(H(W))} < 0.818.$$  

Comments on how to verify the lemma are in Appendix C. We are now ready for the main theorem.

**Theorem 19** (main theorem). Consider a pair of BECs treated as a TEC, or consider any edge-positive TEC. The $2 \times 2$ matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ over $\mathbb{F}_4$ induces a scaling exponent less than $3.451$.

**Proof:** Two iid copies of $\text{BEC}(\varepsilon)$ can be seen as $W := \text{TEC}((1-\varepsilon)^2, (1-\varepsilon)\varepsilon, 0, \varepsilon(1-\varepsilon), \varepsilon^2)$. If $\varepsilon$ is 0 or 1, there is nothing to prove. Suppose $0 < \varepsilon < 1$, then both $W_{1\Sigma}$ and $W_{\ominus}$ have five positive subspace erasure probabilities. (That is, their “p, q, r, s, t” are all positive). They are edge-positive and so are the descendants. Hence, the $Q$-index is always well-defined.

By Theorem 16 and Theorem 17 and the discussion in between, we know that the behavior process $\{H(W_n)\}_n$ can be classified in to three categories (A), (B), and (C).

(A) $H(W_n) \in [a, d]$ for all $n$ for some carefully selected $0 < a < d < 1$. In this case, $A(W_n)$ converges to 0 while, thanks to Theorem 16, $H(W_n), 1 - H(W_n), Q(W_n)$, and $E(W_n)$ are all bounded from below. Therefore, Corollary 13 applies. We infer that $W_n$ are almost-balanced and almost-edge-heavy for all $n \geq m$ for some large number $m$ depending on $a, d, \text{and } W$. In particular, we will have $Q(W_n) \geq 9/7$ for all $n \geq m$ (because $9/7 \approx 1.286 < 1.291 \approx 2\sqrt{7} - 4$); so the eigenvalues of the form

$$\frac{\psi(H(W_{1\Sigma})) + \psi(H(W_{\ominus}))}{2\psi(H(W_n))}$$

are less than $0.818 < 2^{-1/3.451}$, where the value is estimated in Lemma 18.

(B) $H(W_n) \in [0, 1] \setminus [b, c]$ for some carefully selected $0 < a < b < c < d < 1$ after $n = m'$ ($m'$ might be greater or less than $m$; it does not matter). In this case, the $H$’s are in the square-or-double region where, regardless of $A$ and $Q$,

$$\psi(H(W_{n1\Sigma})) \approx \psi(2H(W_n) - O(H(W_n)^2)) \approx \psi(2H(W_n))$$

and

$$\psi(H(W_{n\ominus})) \approx \psi(O(H(W_n)^2)) \approx 0$$

Hence the eigenvalues are still less than $0.818 < 2^{-1/3.451}$. Before $n = \min(m, m')$, the eigenvalues can be upper bounded by 1 due to the convexity of $\psi$. Since only $\min(m, m')$ generations assume the bad eigenvalue, the overall scaling exponent is dominated by the improved value 3.451.

(C) For any realization of $\{W_n\}_n$ that does not fall into categories (A) and (B), it must be the case that $H(W_n)$, at some $n = m'$, escapes $[a, d]$ but reenters $[b, c]$ at some greater $n = m''$. Before $n = m''$, category (B)’s argument applies: the eigenvalue can be upper bounded by the improved value $0.818 < 2^{-1/3.451}$ for all but $\min(m, m')$ generations. So it remains to study what happens after $H(W_n)$ reenters $[b, c]$. By the discussion from Theorem 16 to Theorem 17, we can select $a, b, c, d$ very carefully such that it takes sufficiently many generations to escape and reenter. When the process is doing that, we see that it undergoes an unbalanced version of the square-or-double behavior, which makes it such that when $H(W_n)$ reenters $[b, c]$, $A(W_n)$ is low enough that we can pretend that $W_n$ is balanced and $Q(W_n)$ is high enough.
to enjoy the good scaling exponent. The process \( \{W_n\} \) may escape \([a,d]\) and reenter \([b,c]\) multiple times; but the same argument applies every time: When \( H(W_n) \) is outside \([b,c]\), it assumes the good eigenvalue for the same reason as category (B). When \( H(W_n) \) is in \([b,c]\), it assumes the good eigenvalue for the same reason as category (A). The key idea is that, every time \( H(W_n) \) reenters \([b,c]\) from outside \([a,d]\), it does not have to wait for \( m \) generations to raise its \( Q \); it starts enjoying the good eigenvalue right away by Theorem 17.

From what we have discussed above, only the first \( m \) generations will assume the bad eigenvalue and all other generations will assume the good eigenvalue. By considering polar code whose block length is large enough, the geometric mean of the eigenvalues approaches the good one. Hence \( W \), and hence any BEC, enjoys a scaling exponent less than 3.451. \( \blacksquare \)

In the abstract, we claim that the scaling exponent of \( [\omega_0] \) over TECs (and hence BECs) is \( < 3.328 \). This number will be derived in Appendix D with more intense numerical calculations. In particular, there is a new trapping region that is bounded by two linear splines and is significantly smaller than the region bounded by \( ax(1-x) \) for \( a = 2\sqrt{7} - 4 \) and 2; the attraction toward the new trap is witnessed by sampling TECs with low edge-mass. In Appendix F, we also examine the actual values of \( H(W_n) \) and its asymptotic behavior aligns with the estimate 3.328.

VIII. CONCLUSIONS

In this paper, we argue that \( [\omega_0] \) polarizes BECs faster than \( [\omega_1] \) does. We first show that a pair of BECs will be transformed into balanced TECs. We then show that balanced TECs will be transformed into edge-heavy TECs. Finally, we show that edge-heavy TECs assume a better scaling exponent.

Our rigorous overestimate of the scaling exponent is 3.451; there is another overestimate of 3.328 with strong numerical evidence. Compared to Arikan’s \( 2 \times 2 \) matrix with \( \mu \approx 3.627 \), Fazeli–Vardy’s \( 8 \times 8 \) matrix with \( \mu \approx 3.577 \) [18], Trofimiuk–Trifonov’s \( 16 \times 16 \) matrix with \( \mu \approx 3.346 \) [20], and Yao–Fazeli–Vardy’s \( 32 \times 32 \) matrix with \( \mu \approx 3.122 \) [19], our result suggests that one should consider expanding the alphabet size prior to enlarging the matrix size. More precisely, the rigorous estimate is analogous to a \( 15 \times 15 \) binary matrix; the more accurate estimate is analogous to a \( 20 \times 20 \) binary matrix (see Figure 2).

APPENDIX A
PROOFS FOR BALANCING CHANNELS

For any TEC \( W = TEC(p,q,r,s,t) \),
\[
A(W) = (q - r)^2 + (r - s)^2 + (s - q)^2,
\]
A(W^{\uparrow}) = (q - r)^2(s + p)^2 \\
+ (r - s)^2(q + p)^2 \\
+ (s - q)^2(r + p)^2,
A(W^{\circ}) = (q - r)^2(s + t)^2 \\
+ (r - s)^2(q + t)^2 \\
+ (s - q)^2(r + t)^2.

The first line is the definition and the two lines below are easily-verifiable algebraic identities. They will be useful for this appendix.

A. Average loss of inertia (Proposition 4)

We want to prove \( A(W^*) + A(W^{\circ}) \leq A(W) \).

Proof: First, note that \((q - r)^2(r - s)(s - q) + (r - s)^2(s - q)(q - r) + (s - q)^2(q - r)(r - s)\) is zero by factoring out \((q - r)(r - s)(s - q)\). Next, we want to prove that the following is nonnegative:

\[
A(W) - A(W^{\downarrow}) - A(W^{\circ}) \\
= (q - r)^2(1 - (s + p)^2 - (s + t)^2) \\
+ (r - s)^2(1 - (q + p)^2 - (q + t)^2) \\
+ (s - q)^2(1 - (r + p)^2 - (r + t)^2) \\
= (q - r)^2(1 - (s + p)^2 - (s + t)^2 - (r - s)(s - q)) \\
+ (r - s)^2(1 - (q + p)^2 - (q + t)^2 - (s - q)(q - r)) \\
+ (s - q)^2(1 - (r + p)^2 - (r + t)^2 - (q - r)(r - s)).
\]

It remains to show that \(1 - (s + p)^2 - (s + t)^2 - (r - s)(s - q)\) is nonnegative, as the rest will follow by symmetry. To show so, replace 1 with \((p + q + r + s + t)^2\). One sees that \((p + q + r + s + t)^2 - (s + p)^2 - (s + t)^2 - (r - s)(s - q) = (2p + q + r + s + 2t)(q + r) + qr + 2pt\) is nonnegative.

B. Uniform loss of inertia (Proposition 5)

It suffices to prove \( A(W^{\circ}) \leq A(W)(1 - A(W)/3) \) for any TEC W because \( A(W^{\downarrow}) \leq A(W)(1 - A(W)/3) \) will follow by duality.

Proof: We have \((s + t)^2 = (1 - p - q - r)^2 \leq (1 - q - r)^2 \leq (1 - q + r)(1 + q - r) = 1 - (q - r)^2\). So we see that the three terms that sum to \( A(W^{\circ}) \) can be bounded by

\[
(q - r)^2(s + t)^2 \leq (q - r)^2(1 - (q - r)^2), \\
(r - s)^2(q + t)^2 \leq (r - s)^2(1 - (r - s)^2), \\
(s - q)^2(r + t)^2 \leq (s - q)^2(1 - (s - q)^2).
\]

The average of these three terms is no greater than \((A(W)/3)(1 - A(W)/3)\) because \(A(W)/3\) is the average of \((q - r)^2\), \((r - s)^2\), and \((s - q)^2\) and because \(x(1 - x)\) is concave in \(x\). We then conclude that \( A(W^{\circ}) \leq A(W)(1 - A(W)/3) \). The upper bound on \( A(W^{\uparrow}) \) follows by duality.

C. Continuity in inertia (Proposition 7)

Recall how \( \tilde{W} \), the balanced version of \( W \), is defined: they have the same “\( p \)” and “\( t \)” but \( \tilde{W} \)’s “\( q \)”, “\( r \)”, and “\( s \)” are set to be the average. It is clear that \( H(W) = H(\tilde{W}) \) and \( E(W) = E(\tilde{W}) \). It remains to compare \( E(W^{\downarrow}) \) and \( E(W^{\circ}) \) because the other identities will follow easily.

Proof: Compare

\[
E(W^{\circ}) = sq + qr + rs + 2t(q + r + s),
\]
\[ E(\bar{W}^\circ) = \frac{(q + r + s)^2}{3} + 2t(q + r + s). \]

We infer that

\[
E(\bar{W}^\circ) - E(W^\circ) \\
= \frac{2(q + r + s)^2 - (6sq + 6qr + 6rs)}{6} \\
= \frac{A(W)}{6}.
\]

For \( H(\bar{W}^\circ) - H(W^\circ) \), notice that \( H(W^\circ) - E(W^\circ)/2 = H(\bar{W}^\circ) - E(\bar{W}^\circ)/2 = t^2 \) and so the \( t^2 \) cancels.

\[ \]

**D. Monotonicity of \( A/E \) (Theorem 8)**

One way to relate \( A \) to \( E \) is via \( A(W) = 2E(W) - 6(qr + rs + sq) \leq 2E(W)^2 \), which proves the second inequality in the theorem statement. We now want to show that \( A(W^\circ)^2/E(W^\circ) \leq A(W)/E(W) \); this is enough for proving the first inequality in the theorem statement because \( A(W^\circ)/E(W^\circ) \leq A(W)/E(W) \) will also hold by symmetry.

**Proof:** It suffices to prove

\[ A(W)E(W^\circ)(p + q + r + s + t) - A(W^\circ)E(W) \geq 0. \]

For that, sum the following eleven terms that are obviously positive:

\[
\begin{align*}
p^2(q + r + s)((q - r)^2 + (r - s)^2 + (s - q)^2), \\
2p((q^2 - r^2)^2 + (r^2 - s^2)^2 + (s^2 - q^2)^2), \\
4p(qr(q - r)^2 + rs(r - s)^2 + sq(s - q)^2), \\
3p(q^2(r - s)^2 + r^2(s - q)^2 + s^2(q - r)^2), \\
2pt(q + r + s)((q - r)^2 + (r - s)^2 + (s - q)^2), \\
2t(qr(q - r)^2 + rs(r - s)^2 + sq(s - q)^2), \\
t(q^2(r - s)^2 + r^2(s - q)^2 + s^2(q - r)^2), \\
2qr(q + r)(q - r)^2, \\
2rs(r + s)(r - s)^2, \\
2sq(s + q)(s - q)^2, \\
2qrs((q - r)^2 + (r - s)^2 + (s - q)^2)
\end{align*}
\]

and we are done. Remark: This is just Muirhead’s inequality.

\[ \]

**E. Fast loss of inertia (Theorem 9)**

We want to show \( A(W^\circ) \leq A(W)H(W)^2 \); this is enough for proving the theorem because \( A(W^\circ)^2 \leq A(W)(1 - H(W))^2 \) will also hold by symmetry.

**Proof:** It suffices to demonstrate

\[ A(W^\circ) - A(W)H(W)^2 \geq 0. \]

To do so, expand the left-hand side into a polynomial in \( p, q, r, s, t \). Then, observe that there is no monomial with \( t^2, t^3 \), or higher power. The monomials that contain \( t \) are

\[ + 2q^3 t + 2r^3 t + 2s^3 t + 6qrst \]
\[ -2q^2rt - 2qr^2t - 2r^2st - 2rs^2t - 2s^2qt - 2sq^2t. \]

We know that the sum of these monomials is nonnegative by Schur’s inequality. The monomials that do not contain \( t \) are

\[ + q^4 + r^4 + s^4 \]
\[ + q^3r + qr^3 + r^3s + rs^3 + s^3q + q^3s \]
\[ + q^2rs + qr^2s + qrs^2 \]
\[ - 4q^2r^2 - 4r^2s^2 - 4s^2q^2 \]

divided by 2. These monomials sum to

\[(q + r + s) \cdot \text{(Schur’s inequality)} \]
\[ + qr(q - r)^2 + rs(r - s)^2 + sq(s - q)^2, \]

which we can see is nonnegative. Since the coefficients of \( t^1 \) and \( t^0 \) are nonnegative, the polynomial \( A(W^\circ) - A(W)H(W)^2 \) is nonnegative.

\[ \square \]

**APPENDIX B**

**PROOFS FOR TRAPPING CHANNELS**

Recall that, if \( W = \text{TEC}(p, r, r, r, t) \) is a balanced TEC with \( H(W) = x \) and \( E(W) = y \), then \( x \) and \( y \) uniquely determine \( W \) by

\[ p = 1 - x - \frac{y}{2}, \]
\[ r = \frac{y}{3}, \]
\[ t = x - \frac{y}{2}. \]

Using these, we can easily derive that

\[ H(W^\circ) = x^2 - \frac{y^2}{12}, \]
\[ E(W^\circ) = 2xy - \frac{2y^2}{3}, \]
\[ H(W^\updownarrow) = 2x - x^2 + \frac{y^2}{12}, \]
\[ E(W^\updownarrow) = 2y - 2xy - \frac{2y^2}{3}. \]

These expressions will be useful for this appendix. Note that we use the balanced-ness condition here to express \( H(W^\circ), E(W^\circ), H(W^\updownarrow) \), and \( E(W^\updownarrow) \) using two parameters.

**A. Trapping region (Theorem 10)**

Given \( Q(W) \geq \alpha \equiv 2\sqrt{7} - 4 \), now we want to prove \( Q(W^\circ) \geq \alpha \) as \( Q(W^\updownarrow) \geq \alpha \) will follow by duality.

**Proof:** Consider a balanced TEC with entropy \( H(W) = x \) and edge mass \( E(W) = y \). We first prove that \( Q(W^\circ) \geq \alpha \) whenever \( Q(W) = \alpha \), that is, whenever \( y = \alpha x(1 - x) \). To do so, we want \( E(W^\circ) \geq \alpha H(W^\circ)(1 - H(W^\circ)) \); so we want the following difference to be nonnegative:

\[ E(W^\circ) - \alpha H(W^\circ)(1 - H(W^\circ)) \]
\[ = 2xy - \frac{2y^2}{3} - \alpha \left( x^2 - \frac{y^2}{12} \right) \left( 1 - x^2 + \frac{y^2}{12} \right) \]
\[ = ax^2(1 - x)^2 \cdot (X - Y) \]

where 144X is
\[ ax^2(1 - x)(24 + 24x + \alpha^2x^2 - \alpha^2x^3) \]

and 12Y is
\[ \alpha^2 + 8\alpha - 12. \]

Y is zero because \( 2\sqrt{7} - 4 \) is a root. \( X \) is clearly nonnegative because \( 0 \leq x \leq 1 \). This confirms that \( E(W^\circ) \geq H(W^\circ)(1 - H(W^\circ)) \) and that \( Q(W^\circ) \geq \alpha \) given that \( y = ax(1 - x) \).

Now that we finished proving \( Q(W^\circ) \geq \alpha \) when \( Q(W) = \alpha \), it remains to consider the case when \( Q(W) > \alpha \), that is, when \( y > ax(1 - x) \). Since the treatment for this case is lengthy and somewhat unrelated to the techniques used above, we put remaining of the proof in the next lemma.

\textbf{Lemma 20} (already trapped). \textit{If} \( Q(W_0) > \alpha \text{ for some balanced TEC } W_0 \), \textit{there exists a balanced TEC } \( W_1 \) \textit{such that } \( Q(W_0^\circ) \geq Q(W_1^\circ) \geq \alpha \), \textit{which witnesses } \( Q(W_0^\circ) \geq \alpha \) \textit{and finishes the proof of Theorem 10.}

\textit{Proof:} Consider this map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \)
\[ \pi(x, y) := \left( x^2 - \frac{y^2}{12}, 2xy - \frac{2y^2}{3} \right) \]

that encodes the evolution of \((H, E)\) under parallel combination. We claim that, for any \((x_0, y_0)\)-pair such that \( y_0 > \alpha x_0(1 - x_0) \), there exist a pair \((x_1, y_1)\) such that

- \((x_0, y_0)\) lies above the parabola \( y = \alpha x(1 - x) \),
- \((x_1, y_1)\) lies on the parabola \( y = \alpha x(1 - x) \),
- \((x_0, y_0)\) and \((x_1, y_1)\) lie on the same hyperbola of the form \( x^2 - y^2/12 = \text{const} \).
- \( \pi(x_0, y_0) \) and \( \pi(x_1, y_1) \) lie on the same vertical line, and
- \( \pi(x_0, y_0) \) lies above \( \pi(x_1, y_1) \).

That is to say, there is a TEC \( W_1 \) that has the “correct index” \( Q(W_1) = \alpha \) we know how to deal with; and we will use \( W_1 \) as a reference point to bound \( Q(W_0^\circ) \).

To prove the claim, consider the ordinary differential equation (ODE):
\[ f(0) = x_0, \quad f'(t) = -1/f(t), \]
\[ g(0) = y_0, \quad g'(t) = -12/g(t). \]

The intuition behind this ODE is to imagine a particle starting from \((x_0, y_0)\) and traveling downward along a hyperbola \( x^2 - y^2/12 = \text{const} \) until \((f, g)\) reaches the parabola \( y = \alpha x(1 - x) \). The moment it reaches the parabola, the position this particle is at is the \((x_1, y_1)\) we want. To make this argument sound, there are several details we need to verify.

(A) We need that the solution to the ODE exists til \( g \) become 0. This is true because \( E(W) \leq 2H(W) \) for any TEC \( W \) and so we only need to consider the initial points \((x_0, y_0)\) that satisfy \( y_0 \leq 2x_0 \). Now that \( g \) is initialized as a value lower than \( 2f' \)'s and it decreases faster than \( 2f \) does, we conclude that \( g \leq 2f \) and that \( g \) will reach zero first, and before that happens the solution to the ODE is well-defined.

(B) We need that the particle does travel along a hyperbola \( x^2 - y^2/12 = \text{const} \). This is true as the derivative
\[ \frac{(f^2 - \frac{g^2}{12})'}{2f} = \frac{2fg'}{6} = -\frac{2f}{f} + \frac{12g}{6g} \]

\[ = 0 \]

does vanish.
(C) We need that as the particle travels, \( g/f(1-f) \) is decreasing. This is true because we can compute the derivative
\[
\left( \frac{g}{f(1-f)} \right)' = \frac{g'f(1-f) - gf'(1-f) + gf(1-f')}{f^2(1-f)^2} = \frac{-12f^2(1-f) + g^2(1-f) + g^2f(f+1)}{gf^2(1-f)^2} = \frac{12f^3 - 12f^2 - 2fg^2 + g^2}{gf^2(1-f)^2} = \frac{-12f^2(1-f) + g^2(1-2f)}{gf^2(1-f)^2}.
\]
The derivative is negative (or zero) because \( 12f^2 \geq g^2 \) and \( 1-f \geq 1-2f \).

(D) We need that the image of the particle is traveling downward. This is true because the derivative of the vertical coordinate of \( \pi(f,g) \):
\[
\left( 2fg - \frac{2g^2}{3} \right)' = 2f'g + 2fg' - \frac{4gg'}{3} = -\frac{2g}{f} - \frac{24f}{g} + 16 = \frac{-2(6f-g)(2f-g)}{fg} \leq 0.
\]
is indeed negative or zero.

Combining (A), (B), (C), and (D) proves the lemma.

Remark: we apologize for the lengthy proof of Lemma 20. As it turns out, moving the particle \((f,g)\) along a vertical line does not work. Instead, we have to move \(\pi(f,g)\) along a vertical line.

**B. Attraction toward the trap (Theorem 11)**

We want to prove that \( Q(W) \leq \alpha - \varepsilon \) implies \( Q(W^\circ) \geq Q(W)(1 + (1 - H(W))\delta) \). And then \( Q(W^\square) \geq Q(W)(1 + H(W)\delta) \) will follow by duality.

**Proof:** Let \( W \) be a balanced TEC with entropy \( H(W) = x \), edge mass \( E(W) = y \), and Q-index \( Q(W) = y/x(1-x) = b \). Suppose \( b < \alpha - \varepsilon \), then
\[
E(W^\circ) - bH(W^\circ)(1-H(W^\circ)) = bx^2(1-x)^2 \cdot (X-Y)
\]
where \( 144X = b^2(1-x)(24 + 24x + b^2x^2 - b^2x^3) \) and \( 12Y = b^2 + 8b - 12 \). We have seen that \( 144X \) is nonnegative. For \( Y \), we have \( Y = (b-\alpha)(b - (-2\sqrt{7} - 4)) < -12\varepsilon(2\sqrt{7} + 4) < -9\varepsilon \). Therefore,
\[
Q(W^\circ) = \frac{E(W^\circ)}{H(W^\circ)(1-H(W^\circ))} = b + \frac{bx^2(1-x)^2(X-Y)}{H(W^\circ)(1-H(W^\circ))} \geq b + \frac{bx^2(1-x)^2(X-Y)}{x^2(1-x^2)}
\]
\[
\begin{align*}
&\geq b + \frac{b(1 - x)(X - Y)}{1 + x} \\
&\geq b + \frac{9b(1 - x)\varepsilon}{12}.
\end{align*}
\]

So \(3\varepsilon/8\), our choice of \(\delta\), is valid.

\[\Box\]

C. Uniform attraction (Theorem 12)

We want to prove that if \(Q(W) \leq \alpha - \varepsilon\), there exists an integer \(m > 0\) such that \(Q(W_n) \geq Q(W)(1 + \varepsilon/8)\) for all \(n \geq m\).

**Proof:** In this proof, we call a descendant \(W'\) of \(W\) **good** if \(Q(W') \geq Q(W)(1 + \varepsilon/8)\). Being good is hereditary: if a balanced TEC is good, its descendants are all good because their \(Q\)'s, as long as \(Q < \alpha\), are non-decreasing (Lemma 11). Now imagine the family tree consisting of the root \(W\) and all the descendants that are not good. The goal of this theorem is to show that this tree is finite.

Fix a balanced TEC \(W\), we know either \(H(W) \geq 1/3\) or \(H(W) \leq 2/3\). For the former case,

\[
Q(W^\ominus) \geq Q(W)(1 + H(W)\delta) \geq Q(W)(1 + \frac{1}{3} \cdot \frac{3\varepsilon}{8}) \geq Q(W)(1 + \frac{\varepsilon}{8}).
\]

For the latter case,

\[
Q(W^\ominus) \geq Q(W)(1 + (1 - H(W))\delta) \geq Q(W)(1 + \frac{1}{3} \cdot \frac{3\varepsilon}{8}) \geq Q(W)(1 + \frac{\varepsilon}{8}).
\]

We infer that \(H(W) \geq 1/3\) implies \(W^\ominus\) good and \(H(W) \leq 2/3\) implies \(W^\ominus\) good. We see that the family tree of the bad descendants is uniparous—every node in this tree has at most one child.

At this point, the only concern is whether there exists an infinite path of TECs \(W, W_1, W_2, W_3, \ldots\) such that each is a child of the previous TEC, and none of them has entropy lying in \([1/3, 2/3]\). To see why this cannot happen, suppose we begin with \(x := H(W) > 2/3\). We know \(W^\ominus\) is definitely good so \(W_1\) must be \(W^\ominus\). Given that \(H(W^\ominus) = x^2 - y^2/12\) and \(0 \leq y \leq 2 - 2x\), we see

\[
H(W_1) = x^2 - \frac{y^2}{12} \geq x^2 - \frac{(1 - x)^2}{3} \geq \frac{11}{27} \geq \frac{1}{3}.
\]

This implies that the “gap” \([1/3, 2/3]\) is too large and that the path of TECs \(W, W_1, W_2, \ldots\) cannot cross this gap—it must stay within \((2/3, 1)\) if \(H(W)\) began there.

According to the last few paragraphs, what will contradict the theorem is a path of TECs \(W, W_1, \ldots\) such that each is the parallel-child of the previous and all of them have entropy \(> 2/3\). But this cannot happen because the \(H\) of the parallel-child is at most the square of the previous \(H\), and squaring a number in \((2/3, 1)\) will eventually make it less than \(2/3\). By duality, there cannot be a path of TECs such that each is the serial-child of the previous and all of them have entropy \(< 1/3\). This finishes the proof.

\[\Box\]

D. Attraction on the other side (Proposition 14)

Given \(Q(W) \leq 2\), we want to prove \(Q(W^\ominus) \leq 2\) as \(Q(W^\ominus) \leq 2\) will follow by duality.

**Proof:** Consider a balanced TEC with entropy \(H(W) = x\) and edge mass \(E(W) = y = cx(1-x)\) for some \(c \leq 2\). We prove \(Q(W^\ominus) \leq 2\) by considering the positivity of the following quantity:

\[
2H(W^\ominus)(1 - H(W^\ominus)) - E(W^\ominus)
\]
After proving (A) and (B), it is left to readers to check that 
\[ \min(t, \epsilon H) \]
is a small number that meets the requirement of the theorem statement: 
\[ Q(W^\circ), Q(W) \geq \min(Q(W), \min(\epsilon, \delta)) \]

**Proof:** We first discuss (A). If \( Q(W) < \epsilon \) for some very small \( \epsilon > 0 \), then \( E(W) < \epsilon H(W)(1 - H(W)) < \epsilon H(W)(1 - t) = \epsilon(E(W)/2 + t)(1 - t) \). This implies that 
\[ E(W) = \frac{\epsilon}{1 - \epsilon/2} \cdot t(1 - t) < O(\epsilon')t, \]
where \( \epsilon' := \epsilon(1 - t) \). In other words, \( t \) is very large compared to \( q, r, s \).

We now see 
\[ E(W^\circ) = qr + rs + sq + 2tE(W) = 2tE(W)(1 \pm O(\epsilon')). \]
and 
\[ H(W) = \frac{E(W)}{2} + t = t(1 + O(\epsilon')) \]
and 
\[ H(W^\circ) = \frac{E(W)}{2} + t^2 = t^2(1 \pm O(\epsilon')) = tH(W)(1 \pm O(\epsilon')) \]
and 
\[ 1 - H(W^\circ) = 1 - t^2(1 \pm O(\epsilon')) = (1 - t)(1 + t)(1 \pm O(\epsilon')) = (1 - H(W))(1 + t)(1 \pm O(\epsilon')). \]
So, 
\[ Q(W^\circ) = \frac{E(W^\circ)}{H(W^\circ)(1 - H(W^\circ))} = \frac{2tE(W)(1 \pm O(\epsilon'))}{tH(W)(1 - H(W))(1 + t)(1 \pm O(\epsilon'))} = \frac{2(1 \pm O(\epsilon'))}{1 + t} \cdot Q(W), \]
which is greater than \( Q(W) \). This finishes the proof of (A).

We next discuss (B). Suppose \( Q(W^\circ) \leq \delta \) for some very small \( \delta > 0 \). Then by definition, 
\[ \delta \geq Q(W^\circ) = E(W^\circ)/H(W^\circ)(1 - H(W^\circ)) \geq E(W^\circ)/H(W^\circ) = E(W^\circ)/(E(W^\circ)/2 + t^2). \]
This inequality suggests that \( t^2 \) is very large compared to \( E(W^\circ) = O(E(W))^2 + 2tE(W) \). We then infer that \( q, r, s \) is small compared to \( t \) and that implies that \( Q(W) \) is sufficiently small. This finishes the proof of (B).
F. Become extreme and then mediocre (Theorem 17)

We focus on how to select $0 < a < b$ to control the behavior of $\{H(W_n)\}_n$ after it gets below $a$ and climbs back to $b$. The same argument applies, by symmetry, to the selection of $c < d < 1$.

Proof: Let’s first discuss how $A(W_n)$ evolves. First, by the second inequality of Theorem 8, we know $A(W_n)/E(W_n) \leq 2E(W_n) \leq 4H(W_n) < 4a$ when $H(W_n)$ escapes $[a, d]$. By the first inequality of Theorem 8, we also know that the $A/E$-ratio only decreases, so all $A(W_n)/E(W_n)$ afterward are $< 4a$. Hence, when $H(W_n)$ increases and crosses $\geq b$, we have $A(W_n) < 4aE(W_n) \leq 4a$. We can make $A(W_n)/E(W_{n+1}) = A(W_n)/Q(W_{n+1})H(W_{n+1})(1 - H(W_{n+1})) < 4a/\varepsilon b(1 - b)$ smaller than $\gamma$ by choosing an $a$ smaller than $\gamma\varepsilon b(1 - b)/4$. Here, this $\varepsilon$ is the $\varepsilon$ in the statement of Theorem 16.

Let’s next discuss how $Q(W_n)$ evolves. For simplicity, let’s first assume that $\{W_n\}_n$ undergoes $\cong \cdots \cong \star 1 \cdots 1$ to escape $[a, d]$ and then reenter $[b, c]$.

When the channels are undergoing parallel combinations, we claim that $t$ will decay faster than $E$ will. To see why, observe that $\circ$ evolves $q$ into $ts + sq + qt \geq (q + s)t$, which means that $\circ$ evolves $E$ into $\geq 2Et$. But $\circ$ evolves $t$ into $t^2$ only. In other words, the $E$-to-$t$ ratio doubles every parallel combination. Hence $Q(W_n)$, which is $E/H(1 - H) \geq E/H = E/(E/2 + t) = 2 \pm O(t/E)$, will converge to 2 quickly. (Note that this is similar to Lemma 11, where $Q(W^{(\circ)})$ increases by a larger amount when $H(W)$ is smaller.) To be more precise, either of the following happens when $H(W_n)$ is still $\geq b$: (A) The $E$-to-$t$ ratio is big enough such that $Q > 2 - \gamma$. Since $\circ$ only increases $E/t$, $Q$ always stays $> 2 - \gamma$. (B) The $E$-to-$t$ ratio is not big enough, in which case $E$ and $t$ satisfies $b < H = E/2 + t = (E/2t + 1)t$ and hence $t$ is bounded from below. Now $\circ$ evolves $t$ into $t, t^2, t^4$, etc, and multiplies $E$ by $2t$, then by $2t^2$, then by $2t^4$, etc, which are all lower-bounded. It is possible to choose a small $a/b$ so that, by the time $H(W_n)$ becomes $< a$, the $E/t$-ratio is doubled by a certain amount of times such that $Q > 2 - \gamma$.

When the channels are undergoing serial combinations, we claim that $E$ and $t$ will evolve into $2E$ and $2t$, respectively, up to some multiplicative errors $1 \pm O(H)$. To see why, observe that $\ast$ evolves $q$ into $p + sq + qp \geq p(q + s)$, where $p \geq 1 - 2H \approx 1$; similar estimates apply to $r, s,$ and $t$. (Note that this can be seen as the unbalanced version of the square-or-double behavior.) From that, we see that the $E/t$-ratio remains somewhat unchanged, which implies that $Q$ remains somewhat unchanged and is $> 2 - O(\gamma)$. (And then can make $Q > 2 - \gamma$ by replacing $\gamma$ with $\Omega(\gamma)$.)

It remains to show that the order of parallel and serial combinations does not invalidate the claim that $Q$ will converge to $2 - \gamma$. For the case (A) above, nothing will change. For the case (B) above, if there is any $\sqsupset$ inserted among $\circ \ast \cdots \circ$, they only replace $t, t^2, t^4, \ldots$ and $2t, 2t^2, 2t^4$ by greater numbers, so the argument for (B) still holds. To see that the error term $1 \pm O(H)$ will not alter the $E/t$-ratio by too much, we note that for any $\sqsupset 1 \cdots \sqsupset$ inserted between two $\circ$’s, $H(W_{n+1}) \approx 2H(W_n)$ and hence the error terms $1 \pm O(H) = \exp(\pm O(H))$ accumulate to $\exp(\pm O(H))$; $\exp(\pm O(H))$. Considering that a single $\circ$ will double $E/t$ and will shadow the effect of $\exp(\pm O(b))$, we conclude that the order of $\circ$’s and $\sqsupset$ does not matter.

To summarize the proof of Theorem 17, we choose $b$ (and $c := 1 - b$) by making sure that $\exp(\pm O(b))$ is close to 1. We then choose $a$ (and $d := 1 - a$) by making sure that $A$ is small enough and $E/t$ doubles sufficiently many times so that $Q$ is close enough to 2. ■

APPENDIX C

EIGENVALUE AND EIGENVECTOR (LEMMA 18)

Lemma 18 has $\psi(x) := (x(1 - x))^{0.697}(5 - \sqrt{x(1 - x)})$ and $W$ balanced and $Q(W) \geq 9/7$. We want to verify that

$$\frac{\psi(H(W^{(\ell)})) + \psi(H(W^{(\circ)}))}{2\psi(H(W))} < 0.818.$$
Let $H(W) = x$ and $E(W) = y = 9x(1 - x)/7$. Then $H(W^\odot) = x^2 - y^2/12 = (169x^2 + 54x^3 - 27x^4)/196$, while $H(W^\perp) = 2x - x^2 + y^2/12 = (392x - 169x^2 - 54x^3 + 27x^4)/196$. The statement we want to prove boils down to showing

\[
\psi \left( \frac{169x^2 + 54x^3 - 27x^4}{196} \right) + \psi \left( \frac{392x - 169x^2 - 54x^3 + 27x^4}{196} \right) < 0.818 \quad \text{for} \quad 0 < x < 1.
\]

One can verify this numerically.

If $y > 9x(1 - x)/7$, then $H(W^\odot)$ and $H(W^\perp)$ are more separated than when $y = 9x(1 - x)/7$. Hence $\psi(H(W^\perp)) + \psi(H(W^\odot))$ is smaller than the numerator of formula (6) as $\psi$ is convex. Hence the quotient is still smaller than 0.818.

### A. Suboptimality of trapping region

The eigenvalue 0.818 is not optimal in two aspects. For one, the eigenfunction $\psi$ is not optimal. A common practice is to run power iteration

\[
\psi_0(x) := (x(1 - x))^{0.7},
\]

\[
\psi_{k+1}(x) := \frac{\psi_k \left( \frac{169x^2 + 54x^3 - 27x^4}{196} \right) + \psi_k \left( \frac{392x - 169x^2 - 54x^3 + 27x^4}{196} \right)}{2 \max \psi_k}
\]

until $\psi_k$ converges (note: use spline). Then the limit of $\psi_k$ will induce a smaller eigenvalue. See [14, Section III.C] for more details.

For another, the trapping region we used is $Q(W) \geq 9/7$. A better value is $\alpha = 2\sqrt{7} - 4$ itself. But even $Q(W) \geq 2\sqrt{7} - 4$ is not optimal. A smaller trapping region is $y \geq x(1 - x)(1.66 - 0.38x(1 - x))$. But even that is not optimal. So in the next appendix, we will use power iteration to find the optimal trapping region.

### Appendix D

**Numerical Trapping Region**

In this appendix, we want to find the optimal (smallest) trapping region. In this appendix, $W$ is a balanced TEC and $x = H(W)$ and $y = E(W)$. Recall that $x$ and $y$ determine $W$ uniquely; define

\[
h_\odot(x, y) := H(W^\odot) = x^2 - \frac{y^2}{12},
\]

\[
e_\odot(x, y) := E(W^\odot) = 2xy - \frac{2y^2}{3},
\]

\[
h_\perp(x, y) := H(W^\perp) = 2x - x^2 + \frac{y^2}{12},
\]

\[
e_\perp(x, y) := E(W^\perp) = 2y - 2xy - \frac{2y^2}{3}.
\]

We call the lower boundary of a trapping region the *inner bound* and the upper boundary of a trapping region the *outer bound*. For instance, $y = (2\sqrt{7} - 4)x(1 - x)$ and $y = x(1 - x)(1.66 - 0.38x(1 - x))$ are inner bounds; $y = 2x(1 - x)$ and $y = x(1 - x)(2 - 2x(1 - x)/3)$ are outer bounds. Paraphrased, the mission of this appendix is to find the optimal inner and outer bounds.
A. Numerical inner bound

Suppose \( y = \varphi(x) \) is an inner bound, i.e., \( \varphi \) is such that the children of a TEC above the curve \( y = \varphi(x) \) will still be above the same curve. Then the definition translates into:
\[
\begin{align*}
\epsilon_{\odot}(x, \varphi(x)) &< \varphi(h_{\odot}(x, \varphi(x))), \\
\epsilon_{\mathcal{T}}(x, \varphi(x)) &< \varphi(h_{\mathcal{T}}(x, \varphi(x))).
\end{align*}
\]
In other words,
\[
\begin{align*}
\varphi(x) &< \epsilon_{\odot}(h_{\odot}^{-1}(x), \varphi(h_{\odot}^{-1}(x))), \\
\varphi(x) &< \epsilon_{\mathcal{T}}(h_{\mathcal{T}}^{-1}(x), \varphi(h_{\mathcal{T}}^{-1}(x))).
\end{align*}
\]
where \( h_{\odot}^{-1} \) and \( h_{\mathcal{T}}^{-1} \) are inverse functions of \( x \mapsto h_{\odot}(x, \varphi(x)) \) and \( x \mapsto h_{\mathcal{T}}(x, \varphi(x)) \), respectively.

Suppose there exists an optimal inner bound and it is of the form \( y = \varphi(x) \), i.e., \( \varphi \) is the greatest function such that the children of a TEC above the curve \( y = \varphi(x) \) will still be above the same curve. Then the optimality of \( \varphi \) translates into
\[
\varphi(x) = \min \left( \epsilon_{\odot}(h_{\odot}^{-1}(x), \varphi(h_{\odot}^{-1}(x))), \epsilon_{\mathcal{T}}(h_{\mathcal{T}}^{-1}(x), \varphi(h_{\mathcal{T}}^{-1}(x))) \right).
\]

We do not know a priori if the optimal inner bound exists. But we can consider the following inductive definition
\[
\begin{align*}
\varphi_0(x) &:= 2x(1 - x), \\
h_{\odot k}(x) &:= h_{\odot}(x, \varphi_k(x)), \\
h_{\mathcal{T} k}(x) &:= h_{\mathcal{T}}(x, \varphi_k(x)), \\
\epsilon_{\odot k}(x) &:= \epsilon_{\odot}(h_{\odot k}^{-1}(x), \varphi_k(h_{\odot k}^{-1}(x))), \\
\epsilon_{\mathcal{T} k}(x) &:= \epsilon_{\mathcal{T}}(h_{\mathcal{T} k}^{-1}(x), \varphi_k(h_{\mathcal{T} k}^{-1}(x))), \\
\varphi_{k+1}(x) &:= \min(\epsilon_{\odot k}(x), \epsilon_{\mathcal{T} k}(x)).
\end{align*}
\]
These functions can be and were implemented by linear splines with \( 10^5 \) nodes. The advantage of linear splines is that the inverse function of a linear spline is still a linear spline. Per our computation, \( \varphi_k \) converges as \( k \to \infty \). So we suspect that the limit of \( \varphi_k \) is the optimal inner bound.

A plot of the limit of \( \varphi_k \) is in Figure 4.

B. Numerical outer bound

An argument similar to the previous sub-appendix applies to outer bound. Suppose \( y = \chi(x) \) is an outer bound, i.e., \( \chi \) is such that the children of a TEC below the curve \( y = \chi(x) \) will still be below the curve. The definition translates into
\[
\begin{align*}
\epsilon_{\odot}(x, \chi(x)) &< \chi(h_{\odot}(x, \chi(x))), \\
\epsilon_{\mathcal{T}}(x, \chi(x)) &< \chi(h_{\mathcal{T}}(x, \chi(x))).
\end{align*}
\]
In other words,
\[
\begin{align*}
\chi(x) &> \epsilon_{\odot}(h_{\odot}^{-1}(x), \chi(h_{\odot}^{-1}(x))), \\
\chi(x) &> \epsilon_{\mathcal{T}}(h_{\mathcal{T}}^{-1}(x), \chi(h_{\mathcal{T}}^{-1}(x))).
\end{align*}
\]
Suppose \( y = \chi(x) \) is the optimal outer bound, then the optimality implies
\[
\chi(x) = \max \left( \epsilon_{\odot}(h_{\odot}^{-1}(x), \chi(h_{\odot}^{-1}(x))), \epsilon_{\mathcal{T}}(h_{\mathcal{T}}^{-1}(x), \chi(h_{\mathcal{T}}^{-1}(x))) \right).
\]
Fig. 3. Horizontal axis: $H(W_n)$; vertical axis: $\log_2(A(W_n))$. Tree: a pair of BEC(0.55) and its descendants up to the 7th-generation.

$e_\star(h^{-1}(x), \chi(h^{-1}(x)))$.

This means that we can setup an inductive definition almost identical to the one above except that the last line will be with $\max$:

$\chi_0(x) := 2x(1 - x),
\quad h_{\otimes k}(x) := h_{\otimes}(x, \chi_k(x)),
\quad h_{\tau k}(x) := h_{\tau}(x, \chi_k(x)),
\quad e_{\otimes k}(x) := e_{\otimes}(h^{-1}_{\otimes k}(x), \chi_k(h^{-1}_{\otimes k}(x))),
\quad e_{\tau k}(x) := e_{\tau}(h^{-1}_{\tau k}(x), \chi_k(h^{-1}_{\tau k}(x))),
\quad \chi_{k+1}(x) := \max(e_{\otimes k}(x), e_{\tau k}(x)).$

We did the computation and the end result of $\chi_k$ is plotted in Figure 4.

C. Improved estimate of scaling exponent

Using the numerical limit of $\varphi_k(x)$ in place of $(2\sqrt{7} - 4)x(1 - x)$, we can strengthen Lemma 18 and Theorem 19. Details omitted, our final number is $\mu < 3.328$.

**Theorem 21** (main theorem with optimized constants). Consider a pair of BECs treated as a TEC, or consider any TEC where $pqrs > 0$. The $2 \times 2$ matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ over $F_4$ induces a scaling exponent less than 3.328.

**Appendix E**

**Immediate Balance**

In Appendix A, we spend several paragraphs to explain how $A(W_n)$ will decay to 0 as $n$ goes to infinity. In this appendix, we present a coding technique that will balance any TEC at $n = 0$.

Suppose $W = TEC(p, q, r, s, t)$ is an unbalanced TEC that we want to send message over. We build a new channel $\tilde{W}$ in the following way: Upon receiving the input $(x_1, x_2) \in F_2^2$, pick a random matrix $M \in \{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$ independently and uniformly; input $xM$ into $W$; and then output $M$ and the output of $W$.

Since $\tilde{W}$ is a derivation of $W$, it is a channel that is at most as good as $W$. Meanwhile, we permute the subspace erasure probabilities $q$, $r$, and $s$ by a random power of the permutation $q \rightarrow r \rightarrow s \rightarrow q.$
Therefore, the probability that the receiver learns \( x_1 \) but not \( x_2 \) is the average \((q + r + s)/3\). The same logic applies to other erasure patterns. This implies that \( \tilde{W} \) is at least as good as
\[
\bar{W} := \text{TEC}\left(p, \frac{q + r + s}{3}, \frac{q + r + s}{3}, \frac{q + r + s}{3}, t\right).
\]
One then computes and sees that \( H(\tilde{W}) = H(\bar{W}) = H(W) \). Thus, we did not lose any capacity by transmitting information over \( \tilde{W} \), which we know is equivalent to a balanced TEC.

**APPENDIX F**

**SIMULATIONS**

In Figure 3, we take a pair of BEC(0.55) and treat them as a TEC \( W \). We then compute \( A(W_n) \) for all possible \( W_n \) for \( n \leq 7 \). In this figure, we see that \( A \) decreases by a lot if \( H \) is going toward the closer end of 0 or 1, which is correctly predicted by Theorem 9.

Note that, on paper, we can only prove that the worst-case decay of \( A \) is about \( O(1/n) \) (Corollary 6). But in practice, we see that even the slowest-decaying branch decays exponentially fast. Per our simulation, the sum of all \( A(W_n) \) for \( n \) fixed decays as fast as \( C \cdot 1.95^{-n} \).

In Figure 4, we compute \( H(W_n) \) and \( E(W_n) \) for all possible \( W_n \) for \( n = 10 \), and compare it with trapping regions. As can be seen in the figure, not only do points fall inside the provable trapping region (blue), they all fall inside the numerical trapping region (gold).

Note that, on paper, we have to wait for \( W_n \) to become almost-balanced and only when TECs are balanced enough can we infer that the Q-index will converge to the trapping region. But in practice, we see that TECs become balanced and edge-heavy at the same time.

In Figure 5, we compute the expectation of \( \psi(H(W_n)) \) for \( W_n \) defined by \( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) and that defined by \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), and compare them against lines with slopes \( 1/3.328 \) and \( 1/3.627 \). The result shows a clear advantage of \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), which assumes a higher slope, over \( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \).

Note that, on paper, we have to wait for \( W_n \) to become almost-balanced and only then the Q-index will converge to the trapping region, and then we have to wait for \( W_n \) to become edge-heavy and only then we will see the improved scaling exponent. But in practice, all the above happen at the same time.

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Fig. 5. Horizontal axis: generation (that is, \(n\)); vertical axis: \(-\log_2 \psi(\psi(H(W_n))/\psi(H(W_0))\), where \(\psi(x) := (x(1−x))^{0.7}\). Triangle marks: BEC(0.55) polarized by \([\frac{10}{11}]\); circle marks: BEC(0.55) polarized by \([\frac{11}{10}]\). Dotted lines: the lines of slopes \(1/3.328\) (above) and \(1/3.627\) (below). This reaffirms that \(\mu \approx 3.627\) is accurate for \([\frac{11}{10}]\) and \(\mu < 3.328\) is an overestimate (but probably very close to the true value) for \([\frac{10}{11}]\).
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