UNIVERSALITY BUT NO RIGIDITY FOR TWO-DIMENSIONAL PERTURBATIONS OF ALMOST COMMUTING PAIRS

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ABSTRACT. In this paper we consider two-dimensional dissipative maps of the annulus which are small perturbations of one-dimensional critical circle maps. It has been shown in [GY20] that such perturbations admit an attractor which is a non-smooth topological circle - a “critical” circle. We study conjugacies of the maps that admit such attractors and show that although the maps exhibit universality - they approach a certain normal form when looked at small scales - two maps in general can not be smoothly conjugate on their critical attractors. This result extends the paradigm of “universality but no rigidity” in two dimensions, discovered in [CLM05], to yet another class of dynamical systems.

1. Preliminaries

1.1. Introduction. Rigidity of conjugacies has always been a central issue in dynamics. In this paper we consider conjugacies of the attractors of dissipative maps of the annulus

\[ A_r = \{(x, y) \in \mathbb{R}^2, \ |y| < r\}/\mathbb{Z} \supset \mathbb{T} \]

obtained as small perturbations of critical circle maps.

We recall that a critical circle map \( f \) is a \( C^3 \)-smooth orientation preserving homeomorphism of the circle \( \mathbb{T} \equiv \mathbb{R}/\mathbb{Z} \) which has a single critical point \( x_0 \in \mathbb{T} \) whose order \( n \) is an odd integer. We will assume that \( x_0 = 0 \), and that \( n = 3 \). An example of a family of such maps is the Arnold’s family

\[ f_\omega(x) = x - 2\pi \sin 2\pi x + \omega. \]

Each \( f_\omega \) commutes with the unit translation, and hence it projects to a well-defined map of the circle \( \mathbb{T} \equiv \mathbb{R}/\mathbb{Z} \), which we denote \( \hat{f}_\omega \).

For a circle homeomorphism \( f \), we will denote \( \rho(f) \in \mathbb{T} \) its rotation number. As was shown by Yoccoz in [Yoc84], every critical circle map \( f \) with \( \rho(f) \notin \mathbb{Q} \) is

Date: April 7, 2022.
topologically conjugate to the rigid rotation
\[ R_{\rho(f)}(x) \equiv x + \rho(f) \mod \mathbb{Z}. \]

Let \( F \) be a real-analytic map \( F : \mathbb{A}_r \to \mathbb{A}_r \).

We let \( \Lambda(F) = \cap_{n \in \mathbb{N}} F^n(\mathbb{A}_r) \), and refer to it as the *attractor* of \( F \). In the case when \( f \) is a map of the circle, we can trivially extend it to the second coordinate, setting \( F_f(x, y) = (f(x), 0) \); in this case, \( \Lambda(F) = \mathbb{T} \). A natural question that can be asked in this regard is whether small two-dimensional perturbations of a map \( F_f(x, y) \) when \( f \) is a critical circle map admit attractors which are “critical” invariant circles: topological circles on which the dynamics is topologically, but not smoothly, conjugate to an irrational rotation.

The work \([\text{GY20}]\) demonstrated that the critical circles exist in typical families, and explained the criticality phenomenon in terms of hyperbolicity of renormalization. Specifically, maps of the annulus with a critical circle with a fixed irrational periodic rotation number of bounded type lie in the stable manifold of the one-dimensional hyperbolic periodic point of renormalization. This also makes it precise how the set of maps with a critical circle is embedded in the typical low-parameter family: the intersection of a transverse family with the stable manifold has a codimension equal to the dimension of the unstable manifold of renormalization.

Of course, renormalization of critical circle maps is a classical subject, and one of the central themes in the development of modern one-dimensional dynamics. We refer the reader to the papers \([\text{Yam03a, Yam03b}]\) in which the main renormalization conjectures, known as Lanford’s Program, were proved. The preceding historical development of the subject is described in \([\text{Yam03a}]\).

Since the space of critical circle maps is ill-suited for defining a renormalization operator (cf \([\text{Yam03a}]\)) approaches using other spaces have been developed from which results about critical circle maps can then be deduced.

The most common definition of renormalization of critical circle maps uses the language of *commuting pairs*, as described below. This approach was pioneered in \([\text{ORSS83}]\) and \([\text{FKS82}]\). Analytic commuting pairs provided the setting for proving the existence of renormalization horseshoe attractor \([\text{dF99, dFdM00, Yam03b}]\). However, there was a conceptual difficulty in proving hyperbolicity in this setting,
as the space of analytic commuting pairs does not possess a natural structure of a Banach manifold.

One of the possible ways to circumvent this difficulty is the so-called cylinder renormalization, introduced in [Yam03a]. Cylinder renormalization acts as an analytic operator on a Banach manifold of analytic circle maps whose domain of analyticity is an annulus in the complex cylinder. Using this setting [Yam03a] proves hyperbolicity of cylinder renormalization. It has been also used to study analytic maps with Siegel disks [Yam08, GY07], and critical circle maps with non-integer critical exponents [GY18].

Another avenue, explored in [GY16], is the setting almost commuting critical pairs. In this approach, one drops the commutation condition for maps in the pair, and requires the commutator to disappear only up to some order. This is equivalent to imposing a finite number of conditions on an appropriate Banach space of analytic circle maps, which, in turn, defines a Banach manifold in this space. [GY16] uses this approach to give a new proof of renormalization hyperbolicity in the setting of almost commuting pairs, and, next, applies renormalization to small two-dimensional perturbations of critical circle maps to prove that for every periodic rotation \( \rho \) number there is an open set \( U_\rho \) of dissipative maps of the annulus which contains a finite codimension manifold \( W_\rho \) of maps with a critical circle attractor.

In this paper we continue the study of dynamics of dissipative maps of the annulus on their critical attractors, using the concept of the average Jacobian, and prove the following result

**Theorem A (No Rigidity).** The restriction of the dynamics of any two commuting pairs with different average Jacobians in \( W_\rho \) to their critical circle attractors is not smoothly conjugate.

At the same time high iterates of these maps assume a universal form.

**Theorem B (Universality).** Let \( Z = (A, B) \in W_\rho \) be a commuting pair, and \( n \) be the period of \( \rho \). Then there is an \( \alpha < 1 \) such that the following holds for the sequence of renormalizations \( R^i Z = (A_i, B_i) \) of the pair:

\[
B_{kn}(x, y) = \left[ \xi_{kn}(x) + b(1 + O(\alpha^n)) q_{kn} f(x) y(1 + O(\alpha^k)) \right]_x
\]

where \( \xi_{kn}(x) = \pi_1 B_{kn}(x, 0) \), is a sequence of critical circle maps converging to a periodic orbit geometrically fast, \( b \) is the average Jacobian of \( Z \), \( q_{kn} \) is the \( kn \)-th first return time of rotation with the rotation number \( \rho \), and \( f(x) \) is a universal function.
which is uniformly bounded away from 0 and \(\infty\) and has uniformly bounded derivative and distortion.

1.2. Commuting pairs. We will begin by recalling the theory of one-dimensional commuting pairs and almost-commuting pairs as well as some of the important theorem about renormalization of almost-commuting pairs. The presentation will closely follow the introduction from [GY20], previous work of the first author.

**Definition 1.1.** A \(C^r\)-smooth (or \(C^\omega\)) critical commuting pair \(\zeta = (\eta, \xi)\) consists of two \(C^r\)-smooth (or \(C^\omega\)) orientation preserving interval homeomorphisms \(\eta : I_\eta \to \eta(I_\eta), \xi : I_\xi \to \xi(I_\xi)\), where

(I) \(I_\eta = [0, \xi(0)], I_\xi = [\eta(0), 0]\);

(II) Both \(\eta\) and \(\xi\) have homeomorphic extensions to interval neighborhoods of their respective domains with the same degree of smoothness, that is \(C^r\) (or \(C^\omega\)), which commute, \(\eta \circ \xi = \xi \circ \eta\);

(III) \(\xi \circ \eta(0) \in I_\eta\);

(IV) \(\eta'(x) \neq 0 \neq \xi'(y)\), for all \(x \in I_\eta \setminus \{0\}\), and all \(y \in I_\xi \setminus \{0\}\);

(V) each of the maps \(\eta\) and \(\xi\) has a cubic critical point at 0:

\[
\eta'(0) = \eta''(0) = \xi'(0) = \xi''(0) = 0, \text{ and } \eta'''(0) \neq 0 \neq \xi'''(0).
\]

We will begin by explaining the connection between critical circle maps and critical commuting pairs. First, let \(\zeta = (\eta, \xi)\) be a critical commuting pair and consider the interval \(I = [\eta(0), \xi \circ \eta(0)]\). We create a circle by identifying the endpoints of \(I\). It can then be shown that the map \(f_\zeta\) given by

\[
f_\zeta(x) = \begin{cases} 
\eta \circ \xi(x) & \text{for } x \in [\eta(0), 0], \\
\eta(x) & \text{for } x \in [0, \xi \circ \eta(0)]
\end{cases}
\]

projects to a well-defined map of the circle up to conjugation by diffeomorphisms of the circle. For more details see, e.g., [GY20].

Conversely, let \(f\) is a critical circle mapping with rotation number

\[
\rho(f) = \frac{1}{r_0 + \frac{1}{r_1 + \frac{1}{r_2 + \cdots}}}
\]

Let \(p_m/q_m = [r_0, \ldots, r_{m-1}]\) and let \(I_m\) be the arc containing 0 and \(f^{q_m}(0)\) but not containing \(f^{q_m+1}(0)\). Let \(\overline{f}\) denote the lift of \(f\) to the real line satisfying \(\overline{f}(0) = 0\)
and $0 < \bar{f}(0) < 1$ and similarly let $T_m$ denote the intervals in $\mathbb{R}$ adjacent to 0 that project back down to $I_m$. Then the pair of maps $(\eta|_{T_m}, \xi|_{T_{m+1}})$ given by

\[
\eta: T_m \to \mathbb{R}, \quad x \mapsto t_1^{-p_{m+1}} \circ \bar{f}^{m+1}(x),
\]

\[
\xi: T_{m+1} \to \mathbb{R}, \quad x \mapsto t_1^{-p_m} \circ \bar{f}^m(x)
\]

where $t_1(x) = x + 1$ denotes translation by 1, constitutes a critical commuting pair. For simplicity a critical commuting pair constructed in this way from a critical circle map is denoted simply by $(f^{q_{m+1}}|_{I_m}, f^{q_m}|_{I_{m+1}})$.

**Definition 1.2.** The *height* $\chi(\zeta)$ of a critical commuting pair $\zeta = (\eta, \xi)$ is the natural number $\chi(\zeta) = r \geq 1$, if it exists, such that $0 \in [\eta^r(\xi(0)), \eta^r(\xi(0))]$. If no such $r$ exists then we say that $\chi(\zeta) = \infty$.

In fact if $\chi(\zeta) = \infty$ then $\eta|_{I_0}$ has a fixed point. The height allows us to make the following definition.

**Definition 1.3.** We say that a critical commuting pair $\zeta = (\eta, \xi)$ is *renormalizable* if $\chi(\zeta) < \infty$. The *renormalization* of a renormalizable commuting pair $\zeta = (\eta, \xi)$ is the commuting pair

\[
\mathcal{R}\zeta = (\eta^\circ \xi|_{\tilde{I}_\eta}, \eta|_{[0, \eta^\circ(\xi(0))]})
\]

where given $\zeta = (\eta, \xi)$ the pair $\tilde{\zeta}$ denotes the pair attained by linearly rescaling $\zeta$ by the factor $\xi(0)$, i.e., $\tilde{\zeta}(x) = (\xi(0))^{-1} \zeta(\xi(0)x)$ and domains $\tilde{I}_\eta = (\xi(0))^{-1} I_\eta = [0, 1]$ and $\tilde{I}_\xi = (\xi(0))^{-1} I_\xi$ respectively. The non-rescaled pair $(\eta^\circ \xi|_{\tilde{I}_\eta}, \eta|_{[0, \eta^\circ(\xi(0))]})$ is called the pre-renormalization and will be denoted $p\mathcal{R}\zeta$.

**Definition 1.4.** Let $\zeta$ be a renormalizable critical commuting pair. The *rotation number* $\rho(\zeta) \in [0, 1]$ of $\zeta$ is given by the continued fraction $[r_0, r_1, \ldots]$ where $r_i = \chi(\mathcal{R}^i \zeta)$ and $1/\infty$ is to be interpreted as 0.

Note that if $\chi(\mathcal{R}^i \zeta) = \infty$ for some $i$ then $\rho(\zeta)$ is rational. Note also that indeed the rotation number of $\zeta$ matches the rotation number of the critical circle map $f_\zeta$ as constructed above.

**Proposition 1.1** (Proposition 1.1 from [GY20]). The rotation number of the mapping $f_\zeta$ is equal to $\rho(\zeta)$.

1.3. **Dynamical partitions.** Renormalizable critical commuting pairs induce a dynamical partition of the interval. To define this partition, first let $\zeta = (\eta, \xi)$ be a $k$-times renormalizable critical commuting pair and let $\tilde{\zeta}_k = (\tilde{\eta}_k|_{I_k}, \tilde{\xi}_k|_{I_k})$ denote its
Then there are multiindices $\bar{s}_k$ and $\bar{t}_k$ such that $\bar{\eta}_k = \zeta^{\bar{s}_k}$ and $\bar{\xi}_k = \zeta^{\bar{t}_k}$, i.e., both $\bar{\eta}_k$ and $\bar{\xi}_k$ can be written as compositions of the original $\eta$ and $\xi$. Thus we consider the set $\mathcal{I}$ of multiindices $\bar{s} = (a_1, b_1, \ldots, a_k, b_k)$ and let $\zeta^{\bar{s}} = \xi^{b_k} \circ \eta^{a_k} \circ \cdots \circ \xi^{b_1} \circ \eta^{a_1}$.

We introduce an ordering $\prec$ on the set $\mathcal{I}$ as by letting $\bar{t} \prec \bar{s}$ for $\bar{s} = (a_1, b_1, \ldots, a_k, b_k)$ if $\bar{t} = (a_1, b_1, \ldots, a_m, b_m, c, d)$ where $m < k$ and either $c < a_{m+1}$ and $d = 0$ or $c = a_{m+1}$ and $d < b_{m+1}$.

**Definition 1.5.** Let $\zeta$ be $k$ times renormalizable and let $\zeta_k = (\eta_k|_{I_k}, \xi_k|_{J_k})$ denote its $k$-th pre-renormalization, where $I_k = [0, \xi_k(0)]$ and $J_k = [0, \eta_k(0)]$. Let the multi-indices $\bar{s}_k$ and $\bar{t}_k$ be defined by $\eta_k = \zeta^{\bar{s}_k}$ and $\xi_k = \zeta^{\bar{t}_k}$ respectively. The *$k$-th dynamical partition* $\mathcal{P}_k$ of $\zeta$ is the set of intervals

$$\mathcal{P}_k = \{\zeta^{\bar{w}}(I_k) \text{ for all } \bar{w} \prec \bar{s}_k \text{ and } \zeta^{\bar{w}}(J_k) \text{ for all } \bar{w} \prec \bar{t}_k\}.$$

Then $\mathcal{P}_k$ is indeed a partition of $[\eta(0), \xi(0)]$:

(a) $\bigcup_{H \in \mathcal{P}_k} H = [\eta(0), \xi(0)]$;

(b) for any two distinct elements $H_1$ and $H_2$ of $\mathcal{P}_k$, the interiors of $H_1$ and $H_2$ are disjoint.

**Figure 1.** The first two dynamical partitions of a pair $\zeta = (\eta, \xi)$ with rotation number $\rho(\zeta) = [2, 1, 1, \ldots]$. The first row shows the first dynamical partition of $\zeta$, the second row shows the first dynamical partition of the pre-renormalization $\zeta_1$, the third row shows the second dynamical partition of $\zeta$.

Dynamical partitions have the following property which is a consequence of the so-called real bounds. The full statement is given in [dFdM99]. The following reformulation is from [GY20].
Proposition 1.2 (Proposition 1.2 from [GY20]). There exists a universal constant $C_0 > 1$ such that the following holds. Let $S$ be a compact set of $C^3$-smooth commuting pairs (note that $S$ could consist of a single pair). Then there exists $N = N(S)$ such that for all $n \geq N$ the following holds. Let $\zeta \in S$ be at least $n$ times renormalizable. Let $I$ and $J$ be two adjacent intervals of the $n$-th dynamical partition of $\zeta$. Then $I$ and $J$ are $C_0$-commensurable:

$$\frac{1}{C_0} |I| < |J| < C_0 |I|.$$ 

In particular, denoting $pR^n\zeta = (\eta', \xi')$, we have

$$\frac{1}{C_0} |I_{\xi'}| < |I_{\eta'}| < C_0 |I_{\xi'}|.$$ 

1.4. Renormalization horseshoe. The final important aspect of renormalization of critical commuting pairs we will mention is the construction of a horseshoe attractor for renormalization of analytic commuting pairs first presented in [Yam01]. Here $\bar{\Sigma}$ refers to the space of bi-infinite sequences

$$(\ldots, r_{-k}, \ldots, r_{-1}, r_0, r_1, \ldots, r_k, \ldots)$$

with $r_i \in \mathbb{N} \cup \{\infty\}$ equipped with the weak topology.

Theorem 1.3 (Renormalization horseshoe [Yam01]). There exists an $R$-invariant set $\mathcal{X}$ consisting of analytic commuting pairs with irrational rotation numbers with the following properties. The operator $R$ continuously extends to the closure (in the sense of Carathéodory convergence)

$$\mathcal{A} \equiv \overline{\mathcal{X}}$$

and the action of $R$ on $\mathcal{A}$ is topologically conjugate to the two-sided shift $\sigma : \bar{\Sigma} \to \bar{\Sigma}$:

$$i \circ R \circ i^{-1} = \sigma$$

so that if $\zeta = i^{-1}(\ldots, r_{-k}, \ldots, r_{-1}, r_0, r_1, \ldots, r_k, \ldots)$ then $\rho(\zeta) = [r_0, r_1, \ldots, r_k, \ldots]$. For any analytic commuting pair $\zeta$ with an irrational rotation number we have

$$R^n\zeta \to \mathcal{A}$$

in the Carathéodory sense. Moreover, for any two analytic commuting pairs $\zeta, \zeta'$ with $\rho(\zeta) = \rho(\zeta')$ we have

$$\text{dist}(R^n\zeta, R^n\zeta') \to 0$$

for the uniform distance between analytic extensions of the renormalized pairs on compact sets.
For more information the reader is referred to [McM94] where Carathéodory convergence is introduced in a renormalization context and [Yam03a] which contains a detailed discussion of Carathéodory convergence for the space of analytic commuting pairs. In particular it introduces the corresponding topology on this space.

1.5. Spaces of analytic almost commuting pairs. The commutation condition imposes an infinite number of conditions on the class of commuting pairs, which makes it impossible to define a manifold structure on this space.

[GY16] remedies this by introducing a class of pairs where the commutation conditions is satisfied only approximately, and, further, proves that a space of pairs with an appropriate commutation condition up to a finite order constitutes a Banach manifold.

Approximate commutation has been also used by Mestel in a computer-assisted proof of renormalization hyperbolicity by Mestel [Mes85], however, [Mes85] does not prove that this new class carries a manifold structure.

The following definition has been introduced in [GY16].

**Definition 1.6.** The space $\mathcal{B}$ of analytic almost commuting pairs consists of pairs of non-decreasing interval maps

$$\eta : [0, \xi(0)] \rightarrow [\eta(0), \eta \circ \xi(0)], \quad \xi : [\eta(0), 0] \rightarrow [\xi \circ \eta(0), \xi(0)]$$

which have the following properties:

1. there exists an open neighborhood of the interval $[0, \xi(0)]$ on which the map $\eta$ is analytic, with a single critical point of order 3 at the origin;
2. similarly, there exists an open neighborhood of the interval $[\eta(0), 0]$ on which the map $\xi$ is analytic, with a single critical point of order 3 at the origin;
3. the commutator

$$[\eta, \xi](x) \equiv \eta \circ \xi(x) - \xi \circ \eta(x) = o(x^3) \text{ at } x = 0.$$

The following has been demonstrated in [GY16].

**Proposition 1.4.** The space $\mathcal{B}$ is renormalization invariant: let $\zeta \in \mathcal{B}$ and $\rho(\zeta) \neq 0$. Then $\mathcal{R}(\zeta) \in \mathcal{B}$. Moreover, let $\rho(\zeta) \notin \mathbb{Q}$. Then

$$\mathcal{R}^n(\zeta) \rightarrow A$$

at a geometric rate, where $A$ is the hyperbolic horseshoe attractor of renormalization constructed in Theorem 1.3.
Suppose, $B$ is a complex Banach space whose elements are functions of a complex variable. Let us say that the real slice of $B$ is the real Banach space $B^\mathbb{R}$ consisting of the real-symmetric elements of $B$. If $X$ is a Banach manifold modeled on $B$ with the atlas $\{\Psi_\gamma\}$ we shall say that $X$ is real-symmetric if $\Psi_{\gamma_1}^{-1}(B^\mathbb{R}) \subset B^\mathbb{R}$ for any pair of indices $\gamma_1, \gamma_2$. The real slice of $X$ is then defined as the real Banach manifold $X^\mathbb{R} \subset X$ given by $\Psi_{\gamma}^{-1}(B^\mathbb{R})$ in a local chart $\Psi_\gamma$. An operator $A$ defined on a subset of $X$ is real-symmetric if $A(X^\mathbb{R}) \subset X^\mathbb{R}$.

We will now restrict the definition of the space of almost commuting pairs to a specific choice of domains of analyticity mentioned in (1) and (2) of the definition 1.6 of $B$.

**Definition 1.7.** For a choice of topological disks $D \supset [0, 1]$, $E$, we let $B^{D,E}_0$ consists of pairs in $B$ whose maps $\eta$ and $\xi$ have bounded analytic continuations to $D$ and $E$ correspondingly, such that $[\eta(0), 0] \subset E$, and such that 0 is the only critical point of $\eta$ and $\xi$ on a neighborhood of $I_\eta$, $I_\xi$ respectively. We view it as a subset of the real slice of the complex Banach space $C^\omega(D) \times C^\omega(E)$ where $C^\omega(W)$ denotes the space of bounded holomorphic functions on $W$ with the uniform norm. Finally, denote $B^{D,E}$ the space of pairs in $B^{D,E}_0$ with further normalization conditions $\xi(0) = 1$, and $\frac{1}{2C_0} < |\eta(0)| < 2C_0$, where $C_0$ is as in Proposition 1.2.

As shown in [GY16], this space admits a manifold structure.

**Proposition 1.5.** The space $B^{D,E}$ equipped with the uniform norm is a real Banach manifold, modeled on a finite-codimensional subspace of the real slice of the Banach space $C^\omega(D) \times C^\omega(E)$.

### 1.6. Hyperbolicity and compactness of renormalization in one dimension.

We will now describe two important results that have been proved in [GY16].

The first one is a consequence of the complex a-priori bounds and states that renormalization improves analyticity of almost commuting pairs.

**Theorem 1.6.** There exists a space $B^{D,E}$ and $m \in \mathbb{N}$ such that the following holds. Let $\zeta \in B^{D,E}$ be an $m$-times renormalizable almost commuting pair. There exist larger domains $D' \supset D$, and $E' \supset E$ so that

$$\mathcal{R}^m(\zeta) \in B^{D',E'}.$$

The second is the central result about the renormalization hyperbolcity.

**Theorem 1.7.** Let us fix a periodic point $\zeta_* \in \mathcal{A}$ of $\mathcal{R}$ of period $k$ and let $\rho_* = \rho(\zeta_*)$. There exists a space $B^{D,E}$ and $p = m \cdot k \in \mathbb{N}$ such that the following holds. The pair
ζ∗ is a fixed point of \( R^p \) in the space \( B^{D,E} \). The image
\[
R^p(ζ∗) ∈ B^{D′,E′}
\]
where \( D′ ⊃ D, E′ ⊃ E \).

The linearization
\[
\mathcal{L} ≡ D|_{ζ∗} R^p
\]
in \( B^{D,E} \) is a compact operator with one simple unstable eigenvalue, and the rest of the spectrum is compactly contained in \( \mathbb{D} \). The stable manifold \( \mathcal{W}^s(ζ∗) \) of \( ζ∗ \) contains all pairs in \( B^{D,E} \).

Let \( ζ ∈ \mathcal{W}^s(ζ∗) \) and consider its \( n \)-th pre-renormalization \( ζ_n = (ζ̅_n, ζ̅_{t_n}) \) defined on linear rescalings \( D_n \) and \( E_n \) of the sets \( D \) and \( E \) correspondingly. Consider the collection of topological disks
\[
V_n ≡ \{ ζ̅_w(D_n) \text{ for all } w < s_n \text{ and } ζ̅_w(E_n) \text{ for all } w < t_n \}.
\]
We will refer to this collection of sets the \( n \)-th complex dynamical partition of \( ζ \). It is clear from the construction that the elements \( ζ̅_w(I_n) \) and \( ζ̅_w(J_n) \) of the dynamical partition \( P_n \) are contained in the elements \( ζ̅_w(D_n) \) and \( ζ̅_w(E_n) \), respectively, of the complex dynamical partition \( V_n \). Set \( λ_n = (-1)^n |I_n| \) so that
\[
R^n(ζ(z)) = λ_n^{-1} p R^n(λ_n z).
\]
As a consequence of Theorem 1.7 we have the following:

**Corollary 1.8.** Let \( ζ∗ \) be as in Theorem 1.7. Let \( ζ ∈ \mathcal{W}^s(ζ∗) \). Then there exists \( N = N(ζ), C > 0, C′ > 0, K > 0 \) and \( 0 < γ < 1 \) so that for every \( n > N \) the following holds.

1) If \( Q_n ∈ V_n \) then \( \text{diam}(Q_n) < Cγ^n \).

2) Any two neighboring domains \( Q_n, Q′_n ∈ V_n \) are \( K \)-commensurate.

3) For every \( w < s_n \) (or \( w < t_n \)) set \( ψ_{ζ,w} = ζ̅_w λ_n \). Then \( \|Dψ_{ζ,w}|D\|_∞ < γ^n \) (\( \|Dψ_{ζ,w}|E\|_∞ < γ^n \), respectively).

2. **Renormalization for dissipative two-dimensional pairs**

An extension of renormalization to a space of two-dimensional perturbations of one-dimensional almost commuting pairs has been been implemented in [GY20]. [GY20] takes the route of constructing renormalization for small perturbations of the diagonal embedding \( j \) of almost commuting pairs
\[
j(η, ξ) = \left( \begin{array}{c} η(x) \\ η(x) \end{array} \right), \left( \begin{array}{c} ξ(x) \\ ξ(x) \end{array} \right) \).
Here we will follow another approach and build renormalization with a different embedding which fixes the second component of the second map in the pair to be equal $x$. This is an important point of departure of this paper from [GY20]. Crucially, this allows us to obtain a universality result for the second map of a high renormalization of a pair.

2.1. Functional spaces. Let $\zeta_* = (\eta_*, \xi_*) \in B_{D,E}$ be the hyperbolic fixed point of $R^\rho$ with rotation number $\rho_*$, described in Theorem 1.7. We denote $C_{D,E} \equiv (C^\omega(D) \times C^\omega(E))^R \supset B_{D,E}$.

We set $\Gamma_1 = D \times \tilde{D}, \Gamma_2 = E \times \tilde{E}$, where $D, \tilde{D}, E, \tilde{E} \subset \mathbb{C}$ are domains containing 0, and let $U_{\Gamma_1, \Gamma_2}$ to be the space of pairs of maps $A : \Gamma_1 \to \mathbb{C}^2, B : \Gamma_2 \to \mathbb{C}^2,$ where $A$ and $B$ are both analytic and continuous up to the boundary, equipped with the norm $||(A, B)|| = \frac{1}{2} (||A|| + ||B||)$, where $||.||$ stands for the uniform norm.

For convenience, for a smooth function $F$ from a domain $W \subset \mathbb{C}^2$ to $\mathbb{C}^2$, we will adopt the notation $||F||_y = \sup_{(x,y) \in W} ||\partial_y F(x,y)||$.

We set $A_{\Gamma_1, \Gamma_2} \equiv (U_{\Gamma_1, \Gamma_2})^R$ so that $A_{\Gamma_1, \Gamma_2}$ consists of pairs of real-symmetric two-dimensional maps.

For a two-dimensional map $A(x, y) = (a(x, y), h(x, y))$ we define the projection $\mathcal{L}A(x) \equiv \pi_1(A(x, 0)) = a(x, 0)$

Similarly, for a pair of two-dimensional maps $(A, B)(x, y)$ let us define $\mathcal{L}(A, B)(x) \equiv (\pi_1(A(x, 0)), \pi_1(B(x, 0))) = (a(x, 0), b(x, 0))$.

Next we define the subset of all two-dimensional almost-commuting pairs, i.e. the subset $D_{\Gamma_1, \Gamma_2} \subset A_{\Gamma_1, \Gamma_2}$ consisting of all pairs $(A, B)$ such that

$\begin{align*}
A(x, y) &= (a(x, y), h(x, y)) = (a_y(x), h_y(x)), \\
B(x, y) &= (b(x, y), x) = (b_y(x), x),
\end{align*}$

(2.1)
satisfying the following conditions:

\[(D1) \quad L[A, B](x) = L(A \circ B - B \circ A)(x) = o(|x|^3),\]
\[(D2) \quad LB(0) = 1.\]

Let us define an embedding \(\iota\) of the subset of the twice-renormalizable pairs of maps in the manifold \(B^{D,E}\) into \(D^{\Gamma_1,\Gamma_2}\) as follows. Consider a twice renormalizable pair \((\eta, \xi) \in B^{D,E}\) and let \(\Lambda\) denote the rescaling, so that

\[R^2(\eta, \xi) = \Lambda((\eta^{r_0} \circ \xi)^{r_1} \circ \eta, \eta^{r_0} \circ \xi)\]

where \(r_0\) is the height of the pair \((\eta, \xi)\) and \(r_1\) is the height of the pair \((\eta^{r_0} \circ \xi, \eta)\).

Now define

\[\iota(\eta, \xi)(x, y) = \Lambda\left(\left(\left(\eta^{r_0} \circ \xi\right)^{r_1} \circ \eta(x), \left(\eta^{r_0} \circ \xi\right)^{r_1-1} \circ \eta(x) \right) \right).\]

Here, for a pair \(Z = (A, B)\) the map \(\Lambda\) is defined by

\[\Lambda(Z) = (\Lambda^{-1} \circ A \circ \Lambda, \Lambda^{-1} \circ B \circ \Lambda)\]

where \(\Lambda(x, y) = (\lambda x, \lambda y)\) and \(\lambda = L(B)(0)\), i.e. \(\Lambda\) is the normalization that will ensure Condition \((D2)\). For a pair \(Z\) such that \(\zeta := LZ\) is twice renormalizable we will use \(\lambda_Z\) to denote the rescaling constant of \(\zeta\), i.e.

\[\lambda_Z = \eta^{r_0} \circ \xi(0).\]

By defining the embedding in this way we have

\[L \circ \iota \equiv R^2.\]

Now let \(\zeta_* \in B^{D,E}\) be a periodic point of the one-dimensional renormalization and let \(Z_* = \iota(\zeta_*)\). By \(D^{\Gamma_1,\Gamma_2}(Z_*)\) we will denote the set of pairs \(Z \in D^{\Gamma_1,\Gamma_2}\) satisfying

\[(D3) \quad LZ \in B^{D,E}\]
\[(D4) \quad \|Z - Z_*\| \leq \epsilon.\]

In what follows, we will demonstrate that for each \(Z_*\) there exists \(\epsilon > 0\) and an analytic operator \(R\) defined in \(D^{\Gamma_1,\Gamma_2}(Z_*)\) which has the same hyperbolic properties as the one-dimensional version. In brief the definition consists of two steps:

1. Defining a pre-renormalization mimicking the one-dimensional renormalization in the sense that its action is close to that of \(L \circ L\).
(2) Define a projection from two-dimensional pairs \((A, B)\) onto pairs \((\tilde{A}, \tilde{B})\) such that \(L(\tilde{A}, \tilde{B}) \in B^{D,E}\). This projection is not dynamical, however, crucially for our applications, it does not affect commuting pairs \((A, B)\).

2.2. Defining the pre-renormalization. Unlike in [GY20], where the first step was to define a sufficiently high pre-renormalization as a pair in a neighborhood of \((\eta^{-1}(0), 0)\), we will presently consider only the second pre-renormalization of the pair. Specifically, for a pair \(Z = (A, B) \in D^{\Gamma_1, \Gamma_2}(Z_*)\) such that \(\zeta_0 = L(Z)\) has first two heights \(r_0\) and \(r_1\), we set

\[
p(Z) = ((B \circ A^{r_0})^{r_1} \circ A \circ B \circ A^{r_0}).
\]

Next we introduce the non-linear change of coordinates \(H\) defined by

\[
H(x, y) = (\pi_1 A^{r_0}(\cdot, y))^{-1}(x), y).
\]

The inverse transformation \((\pi_1 A^{r_0}(\cdot, y))^{-1}\) is well defined for each \(y\) if \(\epsilon\) is sufficiently small, i.e. \(\pi_1 A^{r_0}(\cdot, y)\) is sufficiently close to \(\eta r_0 = (L A)^{r_0}\).

Finally we define the new pre-renormalization

\[
\tilde{p}(A, B) = \left(H^{-1} \circ (B \circ A^{r_0})^{r_1} \circ A \circ H, \quad H^{-1} \circ B \circ A^{r_0} \circ H\right),
\]

We remark that \(\tilde{p}A\) is of the form \(\tilde{p}B \circ G\).

**Lemma 2.1.** There exists \(\epsilon > 0\) such that any \(Z \in D^{\Gamma_1, \Gamma_2}(Z_*)\) is at least twice renormalizable, and

\[
\|\Lambda(\tilde{p}Z) - \iota(L(Z))\| = O(\epsilon), \quad \|\tilde{p}Z\|_y = O(\epsilon^2).
\]

**Proof.** The Cauchy estimates for the \(x\)-derivatives and the fact that \(\|Z\|_y = O(\epsilon)\) on compact subsets of \(\Gamma_1\) and \(\Gamma_2\) imply that \(a(x + O(\epsilon), y) = a(x, 0) + O(\epsilon)\) and \(b(x + O(\epsilon), y) = b(x, 0) + O(\epsilon)\) on compact subsets of their domains of definition.

For the second map in the pair \(\tilde{p}Z\) we then get:

\[
H^{-1} \circ B \circ A^{r_0} \circ H(x, y) = H^{-1} \circ B \circ A^{r_0}((a^{r_0}_y(x), y) = H^{-1} \circ B(x, \tilde{y}) = H^{-1}(b_0(x) + O(\epsilon), x) = (a^{r_0}_x(b_0(x) + O(\epsilon)), x).
\]

We would like to argue now that the composition above is well-defined on \(\Lambda(\Gamma_2)\).

By definition we have that \(LZ = (\eta, \xi)\) is a twice-renormalizable almost commuting one-dimensional pair in \(B^{D,E}\). By continuity, if \(\epsilon > 0\) is sufficiently small, then
\[ \mathcal{L}Z = (\eta, \xi) \] is \( m \)-times renormalizable (since \( \zeta_\ast \) is). By Theorem 1.6, \( \mathcal{R}^n \mathcal{L}Z \) is defined on larger domains \( D' \) and \( E' \) that contain \( D \) and \( E \) compactly. If \( \epsilon > 0 \) is sufficiently small, then one can choose the same domains \( D' \) and \( E' \) for all \( Z \) in \( D^{\Gamma_1, \Gamma_2}(Z_\ast) \). Therefore, all the compositions that enter the definition of \( \nu(\mathcal{L}Z) \) are defined on the larger domains \( \Lambda(D') \) and \( \Lambda(E') \) for all \( Z \) in \( D^{\Gamma_1, \Gamma_2}(Z_\ast) \).

Therefore, if \( \epsilon > 0 \) is sufficiently small, the composition \((a_x^{\Gamma_0}(b_0(x) + O(\epsilon))\) is well-defined on \( \Lambda(D) \). We now have that

\[
H^{-1} \circ B \circ A^{\Gamma_0} \circ H(x, y) = (a_x^{\Gamma_0}(b_0(x) + O(\epsilon)), x)
\]

\[
= (a_0^{\Gamma_0}(b_0(x)) + O(\epsilon), x)
\]

\[
= (\eta^{\Gamma_0} \circ \xi(x) + O(\epsilon), x).
\]  

(2.3)

Similarly, the compositions in the first map in the pair \( \tilde{p}Z \) are well defined on \( \Lambda(\Gamma_1) \) and are given by:

\[
H^{-1} \circ (B \circ A^{\Gamma_0})^{\Gamma_1} \circ A \circ H(x, y) = H^{-1} \circ (B \circ A^{\Gamma_0})^{\Gamma_1} \circ A(a_y^{-\Gamma_0}(x), y)
\]

\[
= H^{-1} \circ (B \circ A^{\Gamma_0})^{\Gamma_1}(a_y^{-\Gamma_0-1}(x), \tilde{y})
\]

\[
= H^{-1} \circ (B \circ A^{\Gamma_0})^{\Gamma_1-1}(b_0(a_0(x)) + O(\epsilon), a(x) + O(\epsilon))
\]

\[
= H^{-1}(b_0(a_0^{\Gamma_0}(\ldots(b_0(a_0(x))\ldots))) + O(\epsilon), a_0^{\Gamma_0}(\ldots(b_0(a_0(x))\ldots)) + O(\epsilon))
\]

\[
= (a_0^{\Gamma_0}(b_0(a_0^{\Gamma_0}(\ldots(b_0(a_0(x))\ldots)))) + O(\epsilon), a_0^{\Gamma_0}(\ldots(b_0(a_0(x))\ldots)) + O(\epsilon))
\]

\[
= ((\eta^{\Gamma_0} \circ \xi)^{\Gamma_1-1} \circ \eta(x) + O(\epsilon), (\eta^{\Gamma_0} \circ \xi)^{\Gamma_1-1} \circ \eta(x) + O(\epsilon))
\]  

(2.4)

To show that \( \|\tilde{p}Z\|_y = O(\epsilon^2) \) we begin by noting that \( \partial_y A(x, y) = O(\epsilon) \) and \( \partial_y B(x, y) = O(\epsilon) \) on compact subsets of their domains of definition since, by definition, \( \|Z - Z_\ast\| \leq \epsilon \). It follows from the chain rule that also \( \partial_y A^r(x, y) = O(\epsilon) \) for any fixed \( r \geq 1 \).

Next we would like to show that \( \partial_y H(x, y) = (O(\epsilon), 1) \). This follows from the fact that the first coordinate is the inverse of \( a_y^{\Gamma_0} \) which has \( y \)-derivative of size \( O(\epsilon) \) by above.

With these two facts we get that

\[
\partial_y (B \circ A^{\Gamma_0} \circ H)(x, y) = \partial_y B(x, \tilde{y}) = \partial_2 B(x, \tilde{y}) \cdot \partial_2 \tilde{y} = O(\epsilon) \cdot O(\epsilon) = O(\epsilon^2)
\]

and hence the same estimate will be true for any further compositions of maps with bounded \( y \)-derivatives, which all involved maps have by definition. This immediately shows that the second map in the pair \( \tilde{p}Z \) has \( y \)-derivative of size \( O(\epsilon^2) \) on \( \Lambda(\Gamma_2) \). Similarly for the first map in the pair \( \tilde{p}Z \). Thus \( \|\tilde{p}Z\|_y = O(\epsilon^2) \) on \( \Lambda(\Gamma_1 \times \Gamma_2) \).
In light of Lemma 2.1 we define a preliminary renormalization according to

\[(2.5) \quad \tilde{R}Z = \Lambda(\tilde{p}Z).\]

Note that if \(Z = (A, B)\) is a commutative pair then so is \((\hat{A}, \hat{B}) = \tilde{R}Z\), hence \(\tilde{R}\) maps commutative pairs in \(D^{\Gamma_1, \Gamma_2}(\mathcal{Z}_*)\) into \(D^{\Gamma_1, \Gamma_2}\). The next step will be to handle non-commutative pairs in \(D^{\Gamma_1, \Gamma_2}(\mathcal{Z}_*)\). For this we will create a projection defined on \(\tilde{R}(D^{\Gamma_1, \Gamma_2}(\mathcal{Z}_*))\) that acts as the identity on commutative pairs and projects other pairs back to \(D^{\Gamma_1, \Gamma_2}\). Recall that the properties that need to be satisfied are

\[(2.6) \quad L[\hat{A}, \hat{B}](x) = o(|x|^3),\]
\[(2.7) \quad L\hat{B}(0) = 1.\]

We remark, that the pair \(\tilde{R}Z\) satisfies Condition (2.7) due to the rescaling \(\Lambda\).

Define a projection

\[
\Pi(A, B)(x, y) = \left(\begin{array}{c} a(x, y) \\ h(x, y) \end{array}\right), \quad \begin{pmatrix} b(x, y) + cx + dx^2 + ex^3 + fx^4 \\ x \end{pmatrix}
\]

where \(c, d, e, f\) are parameters.

**Proposition 2.2.** There exists \(\epsilon > 0\) such that the the following holds. For every pair \(Z \in \tilde{R}(D^{\Gamma_1, \Gamma_2}(\mathcal{Z}_*))\) there exists a unique tuple \((c, d, e, f) \in D_{L^2}(0)^{\otimes 4}\) so that Condition (2.6) holds for \(\Pi Z\). Furthermore, the map

\[
\Pi: Z \mapsto (c, d, e, f)
\]

is analytic.

The proof is a straight-forward application of the Implicit Function Theorem but involves several long computations. It is therefore given in Appendix A.

We define the renormalization of a pair \(Z \in D^{\Gamma_1, \Gamma_2}(\mathcal{Z}_*)\) as

\[(2.8) \quad R(Z) = \Pi \tilde{R}(Z).\]

**Remark 2.1.** If one denotes \(R(Z) = (A_1, B_1)\), then \(A_1 = B_1 \circ G\) where \(G\) is some analytic map from the domain of \(A_1\) into that of \(B_1\) whose particular form depends on the combinatorics of the maps \(Z\).

Suppose \(\zeta_*\) is a one-dimensional almost commuting pair such that \(R^n\zeta_* = \zeta_*\) where \(n \geq 2\) is the smallest such number. We can then define the two-dimensional iterated renormalization operator through pull-backs of the composition of the operators
\( \mathcal{R} : D_{\epsilon_1, \epsilon_2}^{\Gamma_1, \Gamma_2}(\tau(\mathcal{R}v_\epsilon)) \rightarrow D^{\Gamma_1, \Gamma_2} \). This gives us the operator \( \mathcal{R}^n : K_n(Z_*) \rightarrow D^{\Gamma_1, \Gamma_2} \) where \( K_n \) is an open subset of \( D^{\Gamma_1, \Gamma_2}(Z_*) \).

With this definition the statement of Lemma 2.1 carries over a fortiori to \( \mathcal{R}^n \). We are now ready to state and prove hyperbolicity of \( \mathcal{R}^n \).

**Theorem 2.3** (2D renormalization hyperbolicity). Let \( \zeta_* \) be a one-dimensional renormalization fixed point of \( \mathcal{R}^\epsilon \) and let \( Z_\epsilon = \iota(\zeta_\epsilon) \). The pair \( Z_\epsilon \) is the unique fixed point of \( \mathcal{R}^n \) for any \( n = kp, k \in \mathbb{N} \), in \( K_n(Z_*) \). The linearization \( D\vert_{Z_\epsilon} \mathcal{R} \) is a compact operator whose non-zero eigenvalues correspond to the tangent space of the embedded one-dimensional pairs, \( \iota \left( B^{D,E}_\epsilon \right) \), where its spectrum equals that of the one-dimensional renormalization operator \( D\vert_{\zeta_*} \mathcal{R} \).

**Proof.** First we note that

\[
\| \mathcal{R}^n Z \|_y = O(\epsilon^{2n})
\]

by Lemma 2.1 and hence we have that

\[
\lim_{n \to \infty} \| \mathcal{R}^n Z \|_y = 0.
\]

Now let \( Z = (A, B) \) be such that \( \| Z \|_y = 0 \) and write

\[
A(x, y) = (a_0(x), h_0(x)),
\]

\[
B(x, y) = (b_0(x), x).
\]

Using \((A_1, B_1) = \mathcal{R}Z\) we then have

\[
A_1(x, y) = H^{-1} \circ (B \circ A^0)^{\Gamma_1} \circ A \circ H(x, y)
\]

\[
= H^{-1} \circ (B \circ A^0)^{\Gamma_1} \circ A(a_0^{-r_0}, y)
\]

\[
= H^{-1} \circ B \circ (A^0 \circ B)^{\Gamma_1-1} \circ A^{r_0+1}(a_0^{-r_0}(x), h_0(x))
\]

\[
= H^{-1} \circ B \circ (A^0 \circ B)^{\Gamma_1-1}(a_0(x), h_0(x))
\]

\[
= H^{-1} \circ B \left( (a_0^0 \circ b_0)^{\Gamma_1-1}(a_0(x)), \eta \right)
\]

\[
= H^{-1} \left( (b_0 \circ (a_0^0 \circ b_0)^{\Gamma_1-1}(a_0(x)), (a_0^0 \circ b_0)^{\Gamma_1-1}(a_0(x))) \right)
\]

\[
= \left( (a_0^0 \circ b_0)^{\Gamma_1} \circ a_0(x), (a_0^0 \circ b_0)^{\Gamma_1-1} \circ a_0(x) \right)
\]

\[
B_1(x, y) = H^{-1} \circ B \circ A^{r_0} \circ H(x, y)
\]

\[
= H^{-1} \circ B \circ A^{r_0} \left( a_0^{-r_0}(x), y \right)
\]

\[
= H^{-1} \circ B(x, h_0(a_0^{-1}(x)))
\]

\[
= H^{-1}(b_0(x), x)
\]
\[ (a_0^0(b_0(x)), x) \]

and we can see that \((A_1, B_1) = \iota(L(Z)) \) so \( RZ \in \iota(B^{D,E}) \). Hence we can see that \( R^n Z \) converges to \( \iota(B^{D,E}) \) at rate \( O(\epsilon^n) \). It follows that the spectrum of \( R \) in the complement to the tangent space of \( \iota(B^{D,E}) \) at \( Z_* = \iota(\zeta_*) \) is trivial. Using hyperbolicity of renormalization of one-dimensional almost-commuting pairs as found in [GY20] completes the proof. \( \square \)

3. ATTRACTIONS OF DISSIPATIVE MAPS

As before, let \( R^n(\zeta_*) = \zeta_* \). Fix \( \rho_* \equiv \rho(\zeta_*) \in (0, 1) \setminus \mathbb{Q} \). We will work in increments of \( n = kp \) for some \( k \in \mathbb{N} \) to be fixed later: the \( n \)-th iterate of the renormalization operator will be denoted \( R \),

\[ R = R^n. \]

Throughout this Section we will assume that \( Z \) is an infinitely renormalizable commuting pair.

By the definition of the renormalization operator \((2.5)\),

\[ (3.1) \quad RZ = L^{-1}_Z \circ (Z^n_{\tilde{1}}, Z^n_{\tilde{2}}) \circ L_Z, \]

where

\[ (3.2) \quad L_Z = H_Z \circ \Lambda_Z \circ H_{RZ} \circ \Lambda_{RZ} \circ \ldots H_{R^{n-1}Z} \circ \Lambda_{R^{n-1}Z}, \]

where the subscripts in the transformations indicate which maps are used in these transformations. Recall that \( Z \) is a commuting pair, thus the projection \( \Pi \) reduces to the identity.

Given \( Z \in W^s_{loc}(Z_*) \), define the following branches of an iterated function system on \( \Gamma_1 \cup \Gamma_2 \), indexed by multi-indices \( \tilde{w} \):

\[ \psi^{Z}_{\tilde{w}}|_{\Gamma_i} = Z^{\tilde{w}} \circ L_Z, \quad \tilde{w} \prec \tilde{t}_n^i. \]

The following Lemma has been proven in [GY20].

**Lemma 3.1.** There exists a neighborhood \( S \) in \( W^s_{loc}(Z_*) \) of \( Z_* \) such that for every \( \tilde{w} \prec \tilde{t}_n \) and \( Z \in S \)

\[ \| D\psi^{Z}_{\tilde{w}}|_{\Gamma_i} \|_{\infty} < \frac{1}{2}. \]
Suppose that $Z \in \mathcal{S}$. We will say that a collection of indices $\{\bar{w}_m\}_{m=0}^{k-1}$ is admissible on $\Gamma_{i_k}$, $i_k = 1, 2$, if for all $0 \leq m \leq k - 1$, $\bar{w}_m \prec t_{i_m}^{m+1}$ whenever

$$\psi_{\bar{w}_m}^{Z}(\Gamma_{i_{m+1}}) \subset \Gamma_{i_m},$$

where $i_m \in \{1, 2\}$.

We will denote the special collection of indices consisting only of zero sequences by

$$\hat{0}_{k-1} = \{0, \ldots, 0_{k-1}\}.$$

Note that $\psi_0^{Z} = Z^0 \circ L_Z = L_Z$.

Given an admissible collection

$$\hat{w}_{k-1} = \{\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{k-1}\},$$

consider the following renormalization microscope

$$\Psi_{\hat{w}_{k-1},Z}^{k} = \psi_{\bar{w}_0}^{Z} \circ \psi_{\bar{w}_1}^{Z} \circ \ldots \circ \psi_{\bar{w}_{k-1}}^{Z}.$$

By analogy with a dynamical partition of a one-dimensional commuting pair, the collection

$$Q_{2kn} = \{Q_{\hat{w}_{k-1}}^i := \Psi_{\hat{w}_{k-1},Z}^{k}(\Gamma_i) : \hat{w}_{k-1} \text{ is admissible on } \Gamma_i, i = 1, 2\}$$

will be referred to as the $2kn$-th dynamical partition for the pair $Z$.

The following result (as well as its proof which we omit) parallels that of [GY20].

**Lemma 3.2.** Let $Z \in W_{\text{loc}}^s(Z_*)$. Then there exists a constant $C > 0$ such that for any $Z \in W_{\text{loc}}^s(Z_*)$ and any admissible $\hat{w}_{k-1}$

$$\|D\Psi_{\hat{w}_{k-1},Z}^{k}|_{\Gamma_i}\|_{\infty} < C \frac{1}{2^k}.$$

Set $T_a(x) \equiv x + a$, and

$$T_* \equiv (T_{\rho_*}([-1,0]), T_{-1}([0,\rho_*])).$$

The following is straightforward.

**Lemma 3.3.** Let $\rho_*$ be a rotation number with periodic partial fraction expansion of period $n$. Then the commuting pair $T_*$ is a periodic point of renormalization

$$\lambda_n^{-1}(T_*^{\tilde{u}_n}, T_*^{\tilde{v}_n}) \circ \lambda_n = T_*,$$

for some $\lambda_n$, $|\lambda_n| < 1$. 
The intervals \([-1, 0]\) and \([0, \rho_*]\) will be denoted \(I\) and \(J\). The elements of the partition \(P_{kn}\) in the orbit of the intervals \(\lambda_{kn}(I)\) and \(\lambda_{kn}(J)\) will be denoted \(I_{kn}\) and \(J_{kn}\), respectively.

The following theorem is analogous to a result proved \([GY20]\) using the idea of the renormalization microscope. We will include its short proof which will make clear our normalization for the conjugacy map \(\phi\).

**Theorem 3.4.** Let \(\zeta_s = \mathcal{R}^n(\zeta_s)\) be as above and let
\[
Z_s = (A_s, B_s) = \iota(\zeta_s) \in \mathcal{D}_{\epsilon_n}^{\Gamma_1, \Gamma_2}.
\]
Suppose \(Z = (A, B) \in W^s_{\text{loc}}(Z_s) \subset K_n\), where \(K_n\) is as in Theorem 2.3, and suppose that maps \(A\) and \(B\) commute, that is \(A \circ B = B \circ A\), where defined.

Then \(Z\) has a minimal attractor \(\Sigma_Z\) in \(((D \cup E) \cap \mathbb{R}) \times \mathbb{R}\). \(\Sigma_Z\) is a Jordan arc, and the restriction \(Z|_{\Sigma_Z}\) is topologically but not smoothly conjugate to \(T_*\).

**Proof.** Select a distinct point \((x_{\hat{w}_{k-1}}^i, y_{\hat{w}_{k-1}}^i)\) in each of the sets \(Q_{\hat{w}_{k-1}}^i \in Q_{2kn}\). Consider the \(2kn\)-th dynamical partition \(P_{2kn}\) for the pair \(T_*\) as defined in Section 1.3. Consider a piecewise-constant map \(\phi_k\) sending the element
\[
I_{\hat{w}_{k-1}} = \Psi_{\hat{w}_{k-1}, T_*}^k(I)
\]
to \((x_{\hat{w}_{k-1}}^i, y_{\hat{w}_{k-1}}^i)\), if \(i = 1\), or
\[
J_{\hat{w}_{k-1}} = \Psi_{\hat{w}_{k-1}, T_*}^k(J)
\]
to \((x_{\hat{w}_{k-1}}^i, y_{\hat{w}_{k-1}}^i)\), if \(i = 2\). By (3.4), the diameters of the sets \(Q_{\hat{w}_{k-1}}^i\) decrease at a geometric rate. Thus, the maps \(\phi_k\) converge uniformly to a continuous map \(\phi_Z\) of the interval \([-1, \rho_*]\) which is a homeomorphism onto the image. Set
\[
\Sigma_Z \equiv \phi_Z([-1, \rho_*]).
\]
By construction,
\[
\phi_Z \circ T_* = Z \circ \phi_Z,
\]
and the curve \(\Sigma_Z\) is the attractor for the pair \(Z\). The conjugacy \(\phi_Z\) cannot be \(C^1\)-smooth, since, the limiting pair \(\zeta_s\) has a critical point at 0. \(\square\)

### 4. The average Jacobian

Let \(Z \in W^s_{\text{loc}}(Z_s)\), and suppose \(\phi_Z : [-1, \rho_*] \mapsto \Sigma_Z\) is the topological conjugacy introduced in Section 3. Since \(Z|_{\Sigma_Z}\) is topologically conjugate to \(T_*|_{[-1, \rho_*]}\), it admits the unique ergodic invariant measure \(\mu_Z = \mu_{\text{Leb}} \circ \phi_Z^{-1}\), where \(\mu_{\text{Leb}}\) is the Lebesgue measure on \([-1, \rho_*]\). \(\mu_Z\) is also the unique ergodic invariant measure for \(Z|_{\Gamma_1 \cup \Gamma_2}\).
For iterated renormalizations and prerenormalizations we will use the notation
\[ Z_i = (A_i, B_i) := R^i Z, \]
\[ pZ_{kn} = (pA_{kn}, pB_{kn}) = (Z_{kn}^1, Z_{kn}^2), \]
for \( k \geq 1 \). Here, the pre-renormalized pair \((Z_{kn}^1, Z_{kn}^2)\) is analytically conjugate to \( R^k Z \) by \( L_z \circ L_{RZ} \circ \ldots \circ L_{R^{k-1} Z} \), similarly to (3.1).

We would also like to emphasize the difference between the two sets of indices in use: on one hand, \( \bar{u}_{kn} \) and \( \bar{v}_{kn} \), used in prerenormalizations for the irrational rotation \( T^* \):
\[ R^{kn} T^* = \lambda^{-1}_{kn} (T^{\bar{u}_{kn}}, T^{\bar{v}_{kn}}) \lambda_{kn}, \]
and, on the other hand, \( \bar{t}_{kn}^1 \) and \( \bar{t}_{kn}^2 \) as defined in (4.2). Indices \( \bar{u}_{kn} \) and \( \bar{v}_{kn} \) encode the sequence of iterates of \( T^* \) under which the central intervals \( I_{kn} \) and \( J_{kn} \) return to \( I_{kn} \cup J_{kn} \), while, according to (4.2), indices \( \bar{t}_{kn}^1 \) and \( \bar{t}_{kn}^2 \) encode the returns of the non-central arcs
\[ I_{\bar{t}_{kn} - 1} = L_z \circ \ldots \circ L_{R^{n-1} Z} \phi Z(I) = \Psi_{\bar{t}_{kn} - 1}(\phi Z(I)) \subset Q_{\bar{t}_{kn} - 1}^1; \]
\[ J_{\bar{t}_{kn} - 1} = L_z \circ \ldots \circ L_{R^{n-1} Z} \phi Z(J) = \Psi_{\bar{t}_{kn} - 1}(\phi Z(J)) \subset Q_{\bar{t}_{kn} - 1}^2 \]
to \( I_{\bar{t}_{kn} - 1} \cup J_{\bar{t}_{kn} - 1} \).

Let \( w := |\bar{w}| \) denote the \( l_1 \)-norm of the multi-index \( \bar{w} \).

Clearly, the return times of \( T_{\bar{u}_{kn}} \cup J_{kn} \) and \( Z_{\bar{t}_{kn}^1} \cup J_{\bar{t}_{kn}^2} \) are the same, that is \( |\bar{u}_{kn}| = |\bar{t}_{kn}^1| \) and \( |\bar{v}_{kn}| = |\bar{t}_{kn}^2| \), and are equal to the cardinalities of the partitions \( I_{kn} \) and \( J_{kn} \), respectively.

Consider the average Jacobian with respect to the measure \( \mu_Z \),
\[ b = \exp \left( \frac{1}{1 + \rho_*} \int_{\Sigma Z} \ln \text{Jac} Z |_{\Sigma Z} \, d\mu_Z \right). \]

We will require a result about the dominance of the geometric convergence rates as compared to the growth of the return times.

**Lemma 4.1.** Let \( \rho_* \) have a periodic partial fraction expansion with period \( n \). Suppose that \( \bar{u}_{kn} \) and \( \bar{v}_{kn} \) are such that
\[ \lambda^{-1}_{kn}(T^{\bar{u}_{kn}}, T^{\bar{v}_{kn}}) \lambda_{kn} = T_* \].
Then there exists constant $A_{m,n} = A_{m,n}(\rho_\ast)$, such that
\begin{align}
|\bar{u}_{(k+m)n}|\lambda_{kn}^2 &< A_{m,n}\alpha^{kn}, \\
|\bar{v}_{(k+m)n}|\lambda_{kn}^2 &< A_{m,n}\alpha^{kn},
\end{align}
where
\[
\alpha = \sqrt{1 + \theta}, \quad \theta = (\sqrt{5} - 1)/2.
\]

Proof. By a classical result (cf [McM96]), for a periodic
\[
\rho_\ast = [r_0, r_1, \ldots, r_{n-1}, r_0, r_1, \ldots, r_{n-1}, \ldots],
\]
the scaling ratios are given by
\[
|\lambda_{kn}| = (\theta_0\theta_2\cdots\theta_{n-1})^k,
\]
where
\[
\theta_i = [r_i, r_{i+1}, r_{i+2}, \ldots].
\]
With the standard convention $p_{-1} = 1, q_{-1} = 0, p_0 = 0, q_0 = 1$, it is straightforward
that for $m \geq 0$
\[
q_m + p_m = |\bar{u}_m|, \\
q_{m-1} + p_{m-1} = |\bar{v}_m|,
\]
where, as before,
\[
\frac{p_m}{q_m} = [r_0, r_1, \ldots, r_{m-1}].
\]
By the recursion formula for $q_m$,
\[
\frac{q_{m+1}}{q_m} = r_m + \frac{q_{m-1}}{q_m}
\]
Since at least one of the sequential numbers $q_{m-1}/q_{m-2}$ and $q_m/q_{m-1}$ is larger than
$\theta^{-1}$, the inverse golden mean (cf [Kar13], Lemma 1.22), we have that either
\begin{align}
\frac{q_{m+1}}{q_m} &\leq r_m + \theta, \quad \frac{q_m}{q_{m-1}} \leq r_{m-1} + 1, \text{ or} \\
\frac{q_{m+1}}{q_m} &\leq r_m + 1, \quad \frac{q_m}{q_{m-1}} \leq r_{m-1} + \theta.
\end{align}
We first consider the case when $kn$ is even. Then
\[
q_{(k+m)n}|\lambda_{kn}^2 | \leq \prod_{i=1}^{(k+m)n} \frac{q_i}{q_{i-1}} \prod_{l=0}^{kn-1} \theta_i^2.
\]
\[
\leq \left( \prod_{i=kn+1}^{(k+m)n} (r_{i-1} + 1) \right)^{kn} \prod_{j=1}^{kn} q_j \theta_{j-1}^2
\]
\[
\leq A_{m,n} \prod_{j=1}^{kn} \frac{q_j}{q_j-1} \theta_{j-1}^2,
\]
(4.10)
where \(A_{m,n}\) denotes the product in parenthesis and depends on \(m, n\) and \(\rho_*\). Each pair of factors in the product is bounded as follows:

\[
\frac{q_j}{q_j-1} \frac{q_{j+1}}{q_j} \theta_j^2 \leq \max \left\{ (r_{j-1} + \theta) \theta_{j-1}^2 (r_j + 1) \theta_j^2, (r_{j-1} + 1) \theta_{j-1}^2 (r_j + \theta) \theta_j^2 \right\}.
\]

Consider the following product of two consecutive factors:

\[
(r_j + 1) \theta_j^2 (r_{j+1} + \theta) \theta_{j+1}^2 \leq \frac{2}{(1 + \theta_{j+1})^2} (r_{j+1} + \theta) \theta_{j+1}^2
\]
\[
\leq \frac{2 (r_{j+1} + \theta)}{(1 + \theta_{j+1})^2}
\]
\[
\leq \frac{2 (r_{j+1} + \theta)}{(r_{j+1} + 1)^2}
\]
\[
\leq \frac{(1 + \theta)}{2} =: \alpha^2 < 1.
\]
(4.11)
Similarly for the product \((r_j + \theta) \theta_j^2 (r_{j+1} + 1) \theta_{j+1}^2\). Therefore,

\[
q_{(k+m)n} |\lambda_{kn}^2| \leq A_{m,n} \alpha^{kn}.
\]
If \(kn\) is odd, then the extra factor in the product (4.10) can be included in the constant \(A_{m,n}\).

A similar computation holds for \(p_{(k+m)n}\). The conclusion follows. \(\square\)

We recall that \(\mathcal{P}_n\) denotes the dynamical partition of level \(n\), while \(\mathcal{I}_n\) and \(\mathcal{J}_n\) denote the orbits of the central intervals \(I_n\) and \(J_n\) in \(\mathcal{P}_n\), respectively.

The second result that we will need is the following

**Lemma 4.2.** Let \(\rho_*\) have a periodic partial fraction expansion with period \(n\). Then there exist constants \(C = C(\rho_*) > 0\), \(0 < d < 1\) and a universal \(\alpha < 1\), independent of \(\rho_*\), such that the following holds for any \(l > m\), \(l, m \in \mathbb{N}\), and any interval \(L\) of the partition \(\mathcal{P}_{mn}\):

\[
\left| \frac{|\mathcal{I}_m \cap L|}{|L|} - d \right| < C \alpha^{(l-m)n}.
\]
(4.12)
Similarly,
\begin{equation}
|\mathcal{J}_{ln} \cap L| \left| 1 - d \right| < C \alpha^{t-m}\n.
\end{equation}

Proof. The result is a consequence of the unique ergodicity of $T_*$.

Consider the length of the interval $I_{kn}$,
\begin{equation}
|I_{kn}| = \left| p_{kn-1} - \rho_* q_{kn-1} \right| = \frac{1}{R_{kn} q_{kn-1} + q_{kn-2}}.
\end{equation}
(cf [Kar13], equation (1.3)) where
\[ R_i = r_i - 1 + [r_i, \ldots]. \]

In particular
\begin{equation}
|I_{kn}| = |I_{kn}|(q_{kn} + p_{kn}) = q_{kn} + p_{kn} 
\end{equation}

Let $p$ be a point in the interval $I_{kn}$. Then
\begin{equation}
u_{kn} = \sum_{i=0}^{\lfloor \bar{u}_{kn} \rfloor - 1} \chi_i(T^i(p))
\end{equation}

is the cardinality of $I_{kn} \cap I$. By the ergodicity of the irrational rotation
\begin{equation}
\left| \frac{\nu_{kn}}{|\bar{u}_{kn}|} - \frac{1}{1 + \rho_*} \right| = \left| \frac{1}{|\bar{u}_{kn}|} \sum_{i=0}^{\lfloor \bar{u}_{kn} \rfloor - 1} \chi_i(T^i(p)) - \frac{1}{1 + \rho_*} \int_{-1}^{1} \chi_1(x)dx \right| \leq D_{kn},
\end{equation}
where the discrepancy $D_{kn}$ can be bounded through the Erdős-Turan inequality: given an arbitrary $K$,
\[ D_{kn} \leq \frac{\log 2}{\pi(K + 1)} + \frac{1}{\pi |\bar{u}_{kn}|} \sum_{k=1}^{K} \frac{1}{k} \min \left( |\bar{u}_{kn}|, \frac{1}{\{k \rho_*\}} \right), \]
where $\{ \cdot \}$ denotes the fractional part. Since $\rho_*$ is periodic, and thus, bounded, $\{k \rho_*\} > \frac{1}{K}$, and
\begin{equation}
D_{kn} \leq \frac{\log 2}{\pi K} + \frac{1}{\pi |\bar{u}_{kn}|} \sum_{k=1}^{K} \frac{1}{c} = \frac{\log 2}{\pi K} + \frac{K}{\pi |\bar{u}_{kn}| c}.
\end{equation}

Finally, setting $K = \sqrt{|\bar{u}_{kn}|}$ and using (4.15), we get that
\[ \left| \frac{\nu_{kn}}{|\bar{u}_{kn}|} - \frac{1}{1 + \rho_*} \right| \leq \frac{C}{\sqrt{|\bar{u}_{kn}|}} \]
for some constant \(C\) which depends on \(c\) in (4.18), and, therefore, on \(\rho_*\). Similarly
\[
\left| \frac{v_{kn}^I}{\tilde{v}_{kn}} - \frac{\rho_*}{1 + \rho_*} \right| \leq \frac{C}{\sqrt{|v_{kn}|}}.
\]
We, therefore, have
\[
\frac{|I_{kn} \cap I|}{|I|} = \frac{|I_{kn}| |\tilde{u}_{kn}|}{|I|} \left( 1 + O \left( \frac{1}{\sqrt{|u_{kn}|}} \right) \right) = \frac{B_{kn}}{1 + \rho_*} \left( 1 + O \left( \frac{1}{\sqrt{|u_{kn}|}} \right) \right).
\]
According to (4.15)
\[
\frac{B_{(k+1)n}}{B_{kn}} = \frac{R_{(k+1)n} q_{kn-1} + q_{kn-2}}{R_{(k+1)n} q_{(k+1)n-1} + q_{(k+1)n-2}} \frac{q_{(k+1)n} + p_{(k+1)n}}{q_{kn} + p_{kn}}
\]
\[
= \frac{[r_{kn-1} + \ldots] q_{kn-1} + q_{kn-2}}{[r_{(k+1)n-1} + \ldots] q_{(k+1)n-1} + q_{(k+1)n-2}} \frac{q_{(k+1)n} + p_{(k+1)n}}{q_{kn} + p_{kn}}
\]
\[
= \frac{[r_{kn-1} + \ldots] q_{kn-1} + q_{kn}}{[r_{(k+1)n-1} + \ldots] q_{(k+1)n-1} + q_{(k+1)n}} \frac{1 + \frac{p_{(k+1)n}}{q_{(k+1)n}}}{1 + \frac{p_{kn}}{q_{kn}}}
\]
\[
= \frac{[r_{kn-1} + \ldots] q_{kn-1} + q_{kn}}{[r_{(k+1)n-1} + \ldots] q_{(k+1)n-1} + q_{(k+1)n}} \frac{1 + \frac{p_{(k+1)n}}{q_{(k+1)n}}}{1 + \frac{p_{kn}}{q_{kn}}}
\]
\[
= \frac{[r_{kn-1} + \ldots] q_{kn-1} + q_{kn}}{[r_{(k+1)n-1} + \ldots] q_{(k+1)n-1} + q_{(k+1)n}} \frac{1 + \frac{p_{(k+1)n}}{q_{(k+1)n}}}{1 + \frac{p_{kn}}{q_{kn}}}
\]
\[
= \frac{[r_{kn-1} + \ldots] q_{kn-1} + q_{kn}}{[r_{(k+1)n-1} + \ldots] q_{(k+1)n-1} + q_{(k+1)n}} \frac{1 + \frac{p_{(k+1)n}}{q_{(k+1)n}}}{1 + \frac{p_{kn}}{q_{kn}}}
\]
\[
= \frac{[r_{kn-1} + \ldots] q_{kn-1} + q_{kn}}{[r_{(k+1)n-1} + \ldots] q_{(k+1)n-1} + q_{(k+1)n}} \frac{1 + \frac{p_{(k+1)n}}{q_{(k+1)n}}}{1 + \frac{p_{kn}}{q_{kn}}}
\]
Notice that by the periodicity of \(\rho_*\) we have \([r_{(k+1)n}, \ldots] = [r_{kn}, \ldots] = \rho_*\) and \([r_{kn-1}, r_{kn-2}, \ldots r_0]\) is the truncation of \([r_{(k+1)n-1}, r_{(k+1)n-2}, \ldots r_0]\) of order \(kn\).
\[
[r_{0}, r_{1}, \ldots, r_{(k+1)n+1}] - [r_{0}, r_{1}, \ldots, r_{n+1}] < C \sum_{i=kn}^{(k+1)n} \frac{1}{\text{Fib}_i \text{Fib}_{i+1}} < C \theta^{2kn},
\]
\[
[r_{(k+1)n-1}, r_{(k+1)n-2}, \ldots r_0] - [r_{kn-1}, r_{kn-2}, \ldots r_0] < \frac{1}{\text{Fib}_n \text{Fib}_{n+1}} < C \theta^{2kn},
\]
where \(\text{Fib}_i\) are the Fibonacci numbers, we get
\[
\frac{B_{(k+1)n}}{B_{kn}} = \frac{1 + \rho_* [r_{kn-1}, r_{kn-2}, \ldots, r_0]}{1 + \rho_* [r_{(k+1)n-1}, r_{(k+1)n-2}, \ldots, r_0]} \frac{1 + \frac{p_{(k+1)n}}{q_{(k+1)n}}}{1 + \frac{p_{kn}}{q_{kn}}} \leq (1 + C \theta^{2kn}).
\]
Therefore, $B_{kn}$ converges as $k \to \infty$, and
\[
\frac{|I_{kn} \cap I|}{|I|} = d \left( 1 + O \left( \frac{1}{\sqrt{u_{kn}}} \right) \right).
\]
Since $|u_{kn}|$ grows at least as fast as the Fibonacci numbers, i.e. as $\theta^{-kn}$,
\[
\frac{|I_{kn} \cap I|}{|I|} = d \left( 1 + O \left( \theta^{\frac{2n}{k}} \right) \right).
\]
Claim (4.12) follows since $T^*$ is a renormalization fixed point of period $n$. The argument for the relative measure of $J_{kn}$ is similar. □

The next result describes the Jacobians of the prerenormalizations of a pair.

**Lemma 4.3.** Suppose $Z \in W_{\text{loc}}^s(Z_*)$. There exists $0 < \alpha < 1$ such that for any pair $l, m \in \mathbb{N}$, $l > m,$
\[
\begin{align*}
\text{Jac } & pA|_{Q^l_{0l-1}} = b^{(1+O(\lambda_{mn}))(1+O(\alpha^{(l-m)n}))} (1 + C A_{l-m,n} \alpha^{mn}) \cdot \\
\text{Jac } & pB|_{Q^2_{0l-1}} = b^{(1+O(\lambda_{mn}))(1+O(\alpha^{(l-m)n}))} (1 + C A_{l-m,n} \alpha^{mn}) \cdot \\
\end{align*}
\]
and for any two points $z_1$ and $z_2$ in $Q^l_{0l-1}$,
\[
\ln \left| \frac{\text{Jac } pZ_{ln}(z_1)}{\text{Jac } pZ_{ln}(z_2)} \right| = O(\alpha^{ln}).
\]

**Proof.** Consider the dynamical partition $\mathcal{P}_{mn}$, $m \in \mathbb{N}$, of $[-\rho_*, 1]$ for the pair $T_*$. Since Jac $Z$ is a piecewise smooth function on $\Sigma_Z$ with one point of discontinuity at 0, the standard estimates for midpoint Riemann sums $A_{mid}^{mn}$ over the intervals of the partition $\mathcal{P}_{mn}$ give
\[
|\ln b - A_{mid}^{mn}| = \left| \frac{1}{1 + \rho_*} \int_{\Sigma_Z} \ln \text{Jac } Z \, d\mu_Z - A_{mid}^{mn} \right| \\
\leq \frac{M_2}{24} \max\{|I_{mn}|, |J_{mn}|\} \leq C \lambda_{mn}^2,
\]
where $M_2$ is a bound on the absolute value of the second derivative of $\ln \text{Jac } A|_{\phi_2(I)}$ and $\ln \text{Jac } B|_{\phi_2(I)}$.

Let $m(P)$ denote the midpoint of an interval $P$. Define a piece-wise constant function $F_l$ on $[-1, \rho_*]$ by settting
\[
F_l(x) = (\ln \text{Jac } Z) \phi_Z(m(P))
\]
whenever $x$ is in an interval $P$ in $P_l$. $F_l|_P$ denotes the restriction of $F_l$ to an interval of $P \in P_l$, a constant function. Then, for any $l > m$,

$$A_{mn}^{mid} = \frac{1}{1 + \rho_*} \int_{-\rho_*}^{1} F_{mn}(x) dx$$

$$= \frac{1}{1 + \rho_*} \sum_{P \in P_{mn}^l} |P| |F_{mn}|_P.$$

Since the value of $F_{ln}|_L$ differs from that of $F_{mn}|_P$ for every $L \subset P$ by a factor of $(1 + O(|\lambda_{mn}|))$, we get

$$A_{mn}^{mid} = \frac{1}{1 + \rho_*} (1 + O(|\lambda_{mn}|)) \sum_{P \in P_{mn}^l} |P| |\mathcal{I}_{ln} \cap P| \sum_{L \in \mathcal{I}_{ln} \cap P} |L| |F_{ln}|_L.$$

By Lemma 4.2,

$$\frac{|P|}{|\mathcal{I}_{ln} \cap P|} = \frac{1}{d} \left(1 + O\left(\alpha^{(l-m)n}\right)\right),$$

whenever $P \in P_{mn}^l$; therefore,

$$A_{mn}^{mid} = \frac{1}{d(1 + \rho_*)} (1 + O(|\lambda_{mn}|)) \left(1 + O\left(\alpha^{(l-m)n}\right)\right) \sum_{L \in \mathcal{I}_{ln}} |L| |F_{ln}|_L.$$

Since $|L| = |I_{ln}|$ for every $L \in \mathcal{I}_{ln}$,

$$A_{mn}^{ln} = \frac{|I_{ln}|}{d(1 + \rho_*)} (1 + O(|\lambda_{mn}|)) \left(1 + O\left(\alpha^{(l-m)n}\right)\right) \sum_{L \in \mathcal{I}_{ln}} (\ln \text{Jac} Z)(\phi_Z(Z(m(L))))$$

$$= \frac{|I_{ln}|}{d(1 + \rho_*) |\bar{u}_{ln}|} \left(1 + O(|\lambda_{mn}|)\right) \left(1 + O\left(\alpha^{(l-m)n}\right)\right) \sum_{i=0}^{\bar{u}_{ln} - 1} (\ln \text{Jac} Z)(Z^i(\phi_Z(Z(m(I_{ln})))))$$

$$= (1 + O(|\lambda_{mn}|)) \left(1 + O\left(\alpha^{(l-m)n}\right)\right) \frac{1}{|\bar{u}_{ln}|} \ln \text{Jac} pA_{ln}(\phi_Z(Z(m(I_{ln})))),$$

and, therefore, obtain that

$$(4.19) \quad \text{Jac} pA_{ln}(\phi_Z(Z(m(I_{ln})))) = b^{\bar{u}_{ln} - 1} (1 + O(|\lambda_{mn}|) (1 + O\left(\alpha^{(l-m)n}\right))) (1 + O(|\bar{u}_{ln}| |\lambda_{mn}|^2)).$$

By Lemma 4.1, $O(|u_{mn}| |\lambda_{mn}|^2) \to 0$ as $mn \to \infty$ geometrically under $l - m$ fixed.

Since $Q_0^{i-1}$ contains a point $\phi_Z(Z(m(I_{ln})))$, the last claim follows from the standard distortion bounds on $Q_0^{i-1}$.

By the Oseledets Multiplicative Ergodic Theorem, the dynamical system $(Z, \Sigma_{\mathbb{R}^k Z}, \mu_Z)$ admits a set of (one or two) characteristic exponents. In fact characteristic exponents can be computed exactly.
Lemma 4.4. Suppose \((Z, \Sigma_Z, \mu_Z)\) is a commuting pair. Then it admits two characteristic exponents \(\chi_0 = 0\) and \(\chi_- = \ln b\).

Proof. Let \(\phi_Z\) be the topological conjugacy between \((T_* [-1, \rho_*])\) and \((Z, \Sigma_Z)\). The \(k\)-th renormalization \(R^k Z\) is analytically conjugate to \(pR^k Z\) on \(\phi_Z(I_{kn}) \cup \phi_Z(J_{kn})\) by \(\Psi_{\hat{0}_{k-1},Z}^k\). Define \(\mu_{kn}^Z\) as 
\[
\mu_{kn}^Z := (1 + \rho_*) \frac{\mu_Z}{\mu_Z(\phi_Z(I_{kn}) \cup \phi_Z(J_{kn}))}
\]
on \(\phi_Z(I_{kn}) \cup \phi_Z(J_{kn})\) and zero otherwise. Consider the largest Lyapunov exponent
\[
\chi_0 \left( \frac{pR^k Z}{\phi_Z(I_{kn}) \cup \phi_Z(J_{kn})}, \mu_{kn}^Z \right) = \chi_0 (R^k Z|_{\Sigma_{R^k Z}}, \mu_{R^k Z}) \leq \frac{1}{1 + \rho_*} \int_{\Sigma_{R^k Z}} \ln \|D R^k Z\| \, d\mu_{R^k Z}.
\]
The last integral is bounded by some constant \(C\). We have
\[
\min \{ |\bar{u}_{kn}|, |\bar{v}_{kn}| \} \chi_0 (Z|_{\Sigma_Z}, \mu_Z) \leq \chi_0 \left( \frac{pR^k Z}{\phi_Z(I_{kn}) \cup \phi_Z(J_{kn})}, \mu_{kn}^Z \right) \rightarrow \chi_0 (Z|_{\Sigma_Z}, \mu_Z) \leq \frac{C}{\min \{ |\bar{u}_{kn}|, |\bar{v}_{kn}| \}},
\]
and \(|\bar{u}_{kn}|\) and \(|\bar{v}_{kn}|\) increase geometrically with \(k \in \mathbb{N}\), therefore, \(\chi_0 \leq 0\).

Now, consider \(Z\) on its domain \(\Gamma_1 \cup \Gamma_2\), and suppose that \(\chi_0 (Z|_{\Sigma_Z}, \mu_Z)\) is negative. Then the local stable manifold of the derivative cocycle has dimension 2. By the transitivity of \(Z|_{\Sigma_Z}\), for any neighborhood \(O \subset \mathbb{C}^2\) of \(z \in \Sigma_Z\) there exists a sufficiently large iterate of \(Z\) which maps \(O\) compactly into itself. \(O\) contains an attracting periodic orbit, which contradict uniqueness of the attractor in \(\Gamma_1 \cup \Gamma_2\). We conclude that \(\chi_0 = 0\).

The claim follows from the fact that the sum of the characteristic exponents is equal to \(\ln b\). \(\Box\)

5. Universality

Throughout this section \(Z = (A, B)\) will denote a commuting pair in the stable manifold of a periodic point of renormalization \(R^n Z_* = RZ_* = Z_*\).

Lemma 5.1. The point \((1, 0)\) is an attracting fixed point of the zero branch of the renormalization microscope of \(Z_*\).

Proof. Denote \(Z_* = (A_*, B_*)\) and \(\zeta_* := L(Z_*) = (\eta_*, \xi_*)\). Then we have \(H(x, y) = (\eta_*^{-\gamma_0}(x), y)\). Next we note that \(\lambda_{Z_*} = \eta_*^{\gamma_0}(\xi_*(0)) = \eta_*^{\gamma_0}(1)\) by the normalization
\( \mathcal{L}B(0) = 1 \). Thus
\[
H_Z \circ \Lambda_Z, (1, 0) = H_Z, (\lambda_Z, 0) \\
= (\eta_s^{-\tau_0}(\lambda_Z), 0) \\
= (\eta_s^{-\tau_0}(\eta_s^0(1)), 0) \\
= (1, 0)
\]
so \((1, 0)\) is a fixed point of the map \( H_Z \circ \Lambda_Z \), hence it is a fixed point of \( L_Z \). It follows that it is the fixed point of the zero branch of the microscope. \( \square \)

Motivated by the above lemma and letting, as before, \( T_1(x, y) = (x + 1, y) \) be the translation by 1 in the \( x \)-coordinate we define the maps \( \ell_Z \) and \( l_Z \) as follows
\[
T^{-1}_1 \circ L_Z \circ T_1(x, y) = (\ell_Z, n(x, y), \lambda_Z, n, y), \\
l_Z, n(x) = \ell_Z, n(x, 0),
\]
where
\[
(5.1) \quad \lambda_{Z, n} = \lambda_Z \cdot \lambda_{RZ} \cdot \ldots \cdot \lambda_{R^{n-1}Z}.
\]
Then \( l_{Z, k} \) has an attracting fixed point at \( x = 0 \). Denote \( \sigma_Z \) one of the two eigenvalues of the transformation \( H_Z \circ \Lambda_Z \) at \((1, 0)\) (the other being \( \lambda_Z \)) then
\[
(5.2) \quad \prod_{i=0}^{n-1} \sigma_{R^iZ} = l_{Z, n}^\prime(0).
\]
It follows by a standard argument (see, for example, Lemma 4.3 in [Yan20]) that
\[
(5.3) \quad \lim_{m \to \infty} \left| \frac{1}{\prod_{k=1}^{(m+1)n} \sigma_{R^{k-1}Z}} l_{Z, n} \circ l_{RZ, n} \ldots l_{R^mZ, n} - \frac{1}{\prod_{k=0}^{n-1} \sigma_{R^{k}Z}} l_{Z, n}^{(m+1)} \right| = 0
\]
uniformly on compact sets, while both sequences converge to the linearizing map \( v_Z \), solving the equation
\[
(5.4) \quad \prod_{k=0}^{n-1} \sigma_{R^kZ} v_{Z*} = v_{Z*} \circ l_{Z*, n}.
\]
To identify the eigenvalues \( \sigma_{R^kZ*} \), we recall (2.3), and take the third derivative at zero of the relation \( RZ^k = Z^{k+1} \) for the first component of the second map in the pair. Here \( Z^k \) are the pairs in the \( n \)-periodic orbit \( \{Z^0, Z^1, \ldots, Z^{n-1}\} \) under \( R \):
\[
(5.5) \quad (\eta_k^{r_k})(\xi_k(0)) \cdot \xi_k''(0) \cdot \lambda_{Z*}^2 = \xi_k''(0),
\]
where \( r_k \) is the height of the pair \((\eta_k, \xi_k)\). Therefore,

\[
(5.6) \quad l'_{Z,n}(0) = \lambda_{Z,n}^3 \prod_{i=0}^{n-1} \xi_{k+1}^m(0) = \lambda_{Z,n}^3 \xi_n^m(0) = \lambda_{Z,n}^3.
\]

Additionally, by Lemma 3.1, \( \prod_{i=0}^{n-1} \xi_{k+1}^m(0) \) contracts its domains at a rate \( K2^{-m+i} \), therefore,

\[
(5.7) \quad \left\| \prod_{i=0}^{n-1} \frac{\partial_x}{\partial_y} \frac{\partial_y}{\partial_x} (\pi_1 A_{l_{Z,n+1}}^m(\cdot, y))^{-1} \right\| \leq C \prod_{k=0}^{n-1} |\sigma_{R_k Z,n}|^m
\]

(throughout this Section \( C, K, R, D, c, d \) etc., will denote immaterial constants). As a consequence of hyperbolicity we have that \( T_1^{-1} \circ L_{Z,n} \circ T_1 \) converges geometrically to \((l_{Z,n}(x), \lambda_{Z,n}y)\). Furthermore, using the notation (4.1),

\[
D_{\nu_0}^{Z_i} = \prod_{k=0}^{n-1} \Lambda_{Z_{l_{k+1}}} \left[ \partial_x \left( \eta_{l_{k+1}} \circ \lambda_{Z_{l_{k+1}}} \right) \circ H_{Z_{l_{k+1}}} \circ \Lambda_{Z_{l_{k+1}}} \right]
\]

\[
(5.8) \quad = \prod_{k=0}^{n-1} \left[ \partial_x \left( \eta_{l_{k+1}} \circ \lambda_{Z_{l_{k+1}}} \right) \circ l_{Z_{l_{k+1}}, n-k-1} + \xi_{Z_{l_{k+1}}} \right]
\]

where \( r_i \) is the height of the pair \( L Z_i \), and \( \eta_{l_{k+1}}(x) \) is a sequence of holomorphic functions converging to the orbit of \( \eta_x \) under \( R \), and

\[ ||\xi_{Z_{l_{k+1}}}|| < Ce^{2l_{k+1}} \quad \text{and} \quad ||\xi_{Z_{l_{k+1}}}|| < Ce^{2l_{k+1}}.\]

Since \( n \) is a fixed integer,

\[
D_{\nu_0}^{Z_i} = \left[ \prod_{k=0}^{n-1} \xi_{Z_{l_{k+1}}, 0} \circ l_{Z_{l_{k+1}}, n-k-2} + \xi_{Z_{l_{k+1}}} \left( \lambda_{Z_{l_{k+1}}} \right) \right]
\]

\[
(5.9) \quad = \left[ \xi_{Z_{l_{k+1}}} + \xi_{Z_{l_{k+1}}} \left( \lambda_{Z_{l_{k+1}}} \right) \right]
\]

where \( \lambda_{Z,n} \) has been defined in (5.1), and

\[ ||\xi_{Z_{l_{k+1}}}|| < Re^{2l_{k+1}} \quad \text{and} \quad ||\xi_{Z_{l_{k+1}}}|| < Re^{2l_{k+1}}.\]

We can now estimate the derivative of

\[
(5.10) \quad \Phi_{Z}^m \equiv \Psi_{0m-1}^m,\]
defined in (3.3):
\[ D\Phi^m_{\mathcal{R}^jZ} = \prod_{i=0}^{m-1} \left[ l^i_{\mathcal{R}^jZ,n} + \varepsilon_{\mathcal{R}^i+jZ} \frac{\varepsilon_{\mathcal{R}^i+jZ}}{\lambda_{\mathcal{R}^i+jZ,n}} \right] \circ \prod_{i=l+1}^{m-1} \psi^{i+jZ}_0. \]

Assume momentarily, towards a proof by induction, that there exist \( \alpha_1 < \alpha_2 < 1 \) and \( \gamma_1 < \gamma_2 < 1 \) such that

\[ D\Phi^m_{\mathcal{R}^jZ} = \left[ \prod_{i=0}^{m-1} l^i_{\mathcal{R}^jZ,n} \circ \prod_{i=l+1}^{m-1} l_{\mathcal{R}^i+jZ,n} + E_{m,\mathcal{R}^jZ} \frac{E_{m,\mathcal{R}^jZ}}{\prod_{i=0}^{m-1} \lambda_{\mathcal{R}^i+jZ,n}} \right]. \]

where

\[ C \varepsilon^{2jn} \alpha_1 \leq \|E_{m,\mathcal{R}^jZ}\| \leq C \varepsilon^{2jn} \alpha_2, \quad C \varepsilon^{2jn} \gamma_1 \leq \|E_{m,\mathcal{R}^jZ}\| \leq C \varepsilon^{2jn} \gamma_2. \]

Then

\[ D\Phi^{m+1}_{\mathcal{R}^jZ} = \left[ \prod_{i=0}^{m} l^i_{\mathcal{R}^jZ,n} \circ \prod_{i=l+1}^{m} l_{\mathcal{R}^i+jZ,n} + E_{m+1,\mathcal{R}^jZ} \frac{E_{m+1,\mathcal{R}^jZ}}{\prod_{i=0}^{m} \lambda_{\mathcal{R}^i+jZ,n}} \right], \]

where the function

\[ E_{m+1,\mathcal{R}^jZ} = \varepsilon_{\mathcal{R}^j+mZ} \cdot \prod_{i=0}^{m} l^i_{\mathcal{R}^jZ,n} \circ \prod_{i=l+1}^{m} l_{\mathcal{R}^i+jZ,n} + E_{m,\mathcal{R}^jZ} \cdot (l^m_{\mathcal{R}^j+mZ} + \varepsilon_{\mathcal{R}^j+mZ}) \]

satisfies for sufficiently large \( m \) and \( n \):

\[ \|E_{m+1,\mathcal{R}^jZ}\| \leq C \varepsilon^{2(m+j)n} C \prod_{k=0}^{n-1} |\sigma_{\mathcal{R}^kZ_*}|^m + C \varepsilon^{2jn} \alpha_2 \left( D \prod_{k=0}^{n-1} |\sigma_{\mathcal{R}^kZ_*}| + R \varepsilon^{2(m+j)n} \right) \]

(5.12)

\[ = C \varepsilon^{2jn} \alpha_2 \alpha_1 \left( D \varepsilon^{2mn} C \prod_{k=0}^{n-1} |\sigma_{\mathcal{R}^kZ_*}|^m + D \prod_{k=0}^{n-1} |\sigma_{\mathcal{R}^kZ_*}| + R \varepsilon^{2(m+j)n} \right), \]

where we have used (5.7). Here, \( D \approx 1 \). Again, if \( m \) and \( n \) are sufficiently large, then there is an

\[ \alpha_2 \approx \prod_{k=0}^{n-1} |\sigma_{\mathcal{R}^kZ_*}| \approx |\lambda_{Z,n}|^3 < 1 \]

(5.13)

for which the expression in the parenthesis is less than 1. On the other hand

\[ \|E_{m+1,\mathcal{R}^jZ}\| \geq C \varepsilon^{2jn} \alpha_1 \prod_{k=0}^{n-1} |\sigma_{\mathcal{R}^kZ_*}| - C \varepsilon^{2(m+j)n} C \prod_{k=0}^{n-1} |\sigma_{\mathcal{R}^kZ_*}|^m - C \varepsilon^{2jn} \alpha_2 R \varepsilon^{2(m+j)n} \]

(5.14)

\[ = C \varepsilon^{2jn} \alpha_1^{m+1} \left( D \prod_{k=0}^{n-1} |\sigma_{\mathcal{R}^kZ_*}| + D \prod_{k=0}^{n-1} |\sigma_{\mathcal{R}^kZ_*}| + R \varepsilon^{2(m+j)n} \right). \]
Here, \( d \approx 1 \). Again, if \( m \) and \( n \) are sufficiently large, then there is an
\[
\alpha_1 \asymp \prod_{k=0}^{n-1} |\sigma_{R^kZ_*}| \asymp |\lambda_{Z_*,n}|^3 < 1
\]
for which the expression in parenthesis is larger than 1. Similarly,
\[
\mathcal{E}_{m+1, R^jZ} = \varepsilon_{R^{j+m}Z} \cdot \left( \prod_{l=0}^{m-1} l_{R^{l+j}, n} \circ \prod_{i=l+1}^m l_{R^{i+j}Z, n} + E_{m, R^jZ} \right) + \mathcal{E}_{m, R^jZ} \cdot \lambda_{R^{m+j}Z, n}
\]
which implies
\[
\|\mathcal{E}_{m+1, R^jZ}\| \leq K \varepsilon^{2(m+j)n} \left( C \prod_{k=0}^{n-1} |\sigma_{R^kZ_*}|^m + C \alpha_2^{m+j} \right) + C \varepsilon^{2jn} \gamma_2^m |\lambda_{R^{m+j}Z, n}|
\]
\[
= C \varepsilon^{2jn} \gamma_2^{m+1} \left( K \varepsilon^{2(m+j)n} \left( \prod_{k=0}^{n-1} |\sigma_{R^kZ_*}|^m \right) \left( \frac{\prod_{k=0}^{n-1} |\sigma_{R^kZ_*}|^m}{\gamma_2^{m+1}} + \frac{\alpha_2^{m+j}}{\gamma_2^{m+1}} \right) \right) |\lambda_{R^{m+j}Z, n}|
\]
The expression in parenthesis is smaller than 1 for large \( m \) and \( n \) and
\[
\gamma_2 \asymp |\lambda_{Z_{(m+j)n, n}}|, \quad |\lambda_{Z_{(m+j)n, n}}| < \gamma_2 < 1
\]
We obtain that
\[
\|\mathcal{E}_{m+1, R^jZ}\| \leq C \varepsilon^{2jn} \gamma_2^{m+1}.
\]
Finally, similarly to (5.14), one obtains that there exists
\[
\gamma_1 \asymp |\lambda_{Z_{(m+j)n, n}}|, \quad |\lambda_{Z_{(m+j)n, n}}| > \gamma_1
\]
such that
\[
\|\mathcal{E}_{m+1, R^jZ}\| \geq c \varepsilon^{2jn} \gamma_1^{m+1}.
\]
We are now ready to prove the main result of this section.

**Theorem 5.2.** Let \( Z = (A, B) \in W^s(Z_*) \) be a commuting pair, where \( R^nZ_* = Z_* \). Then there is an \( \alpha < 1 \) such that
\[
B_{kn}(x, y) = \left[ \xi_{kn}(x) + b(1+O(\alpha^n))|\tilde{v}_{kn}| f(x)y(1 + O(\alpha^n)) \right]
\]
where \( \xi_{kn} = \mathcal{L}(B_{kn}) \), \( b \) is the average Jacobian of \( Z \), \( \tilde{v}_{kn} \) is given by Equation 4.3 and \( f \) is a universal function which is uniformly bounded away from 0 and \( \infty \) and has uniformly bounded derivative and distortion.
Proof. By definition we have that
\[ B_{kn} = (\Phi_Z^k)^{-1} \circ pB_{kn} \circ \Phi_Z^k \]
and therefore we have that
\[ \text{Jac} B_{kn}(x, y) = \text{Jac} pB_{kn}(\Phi_Z^k(x, y)) \frac{\text{Jac} \Phi_Z^k(x, y)}{\text{Jac} \Phi_Z^k(B_{kn}(x, y))}. \]
Using (5.3), (5.4) and (5.11) we get that
\[ \text{Jac} \Phi_Z^k(x, y) \xrightarrow{k \to \infty} v'_Z(x) \]
as \( k \to \infty \). Denote this limit by \( f(x) \).

Writing
\[ B_{kn}(x, y) = \begin{bmatrix} \xi_{kn}(x) + E_{kn}(x, y) \\ x \end{bmatrix} \]
for some yet undetermined \( E_{kn} \) we get that
\[ \partial_y E_{kn}(x, y) = \text{Jac} B_{kn}(x, y). \]
Using the above, Lemma 4.3 with \( l = k \) and a fixed difference \( l - m \), and integrating with respect to \( y \) yields the required form. \( \square \)

6. No rigidity

We will now apply the universality result to show that any two pairs of commuting maps on the renormalization stable manifold, whose average Jacobians differ, cannot be smoothly conjugate on their attractors. The proof is similar to that of the pioneering work [CLM05].

Consider the derivatives of the non-linear transformation \( H_{R^k R^l Z} \circ \Lambda_{R^k R^l Z}, k = 0, \ldots, n - 1 \), that enters definition (3.2) of \( L_{R^l Z} \) in the case of a commuting pair \( Z \in W^s_{loc}(Z^*_*) \) (we will use notations \( R^k R^l Z = Z_{ln+k} \) interchangeably):
\[
DH_{R^k R^l Z} \circ \Lambda_{R^k R^l Z} = \begin{bmatrix}
\partial_x (\pi_1 A_{ln+k}^l (\cdot, y))^{-1} & \partial_y (\pi_1 A_{ln+k}^l (\cdot, y))^{-1} \\
1 & 1
\end{bmatrix} \circ \Lambda Z_{ln+k} \cdot \Lambda Z_{ln+k} = \\
\partial_x (\eta_{ln+k} \circ \lambda Z_{ln+k}) + \delta Z_{ln+k} + \epsilon Z_{ln+k} \quad \bar{\epsilon} Z_{ln+k} \\
0 & \lambda Z_{ln+k}
\]
where \( \sigma_{Z^*_*} \) is as in (5.4) and \( \bar{\epsilon} Z_{ln+k} = O(\epsilon_{ln+k}^2) \), \( \bar{\epsilon} Z_{ln+k} = O(\epsilon_{ln+k}^2) \) and \( \delta Z_{ln+k} = O(\nu) \), \( \nu \) being the largest eigenvalue of \( D\mathcal{R}(Z^*_*) \) on the tangent space to \( W^s_{loc}(Z^*_*) \).
We will momentarily concentrate on estimating the order of the function $\tilde{\varepsilon}_{Z_{m+k}}$ more carefully. To that end we consider the derivative of the transformation $\Lambda^{-1}_{R^k,R'|Z} \circ H_{R^k,R'|Z}$

$$D\Lambda^{-1}_{R^k,R'|Z} \circ H^{-1}_{R^k,R'|Z} = \Lambda^{-1}_{Z_{m+k}} \cdot \begin{bmatrix} \partial_x \pi_1 A^{r_{m+k}}_{m+k} & \partial_y \pi_1 A^{r_{m+k}}_{m+k} \\ 0 & 1 \end{bmatrix}$$

(6.1)

$$= \Lambda^{-1}_{Z_{m+k}} \cdot \begin{bmatrix} \partial_x \eta_{1}^{r_{m+k}} + \hat{\varepsilon}_{Z_{m+k}} + \hat{\eta}_{Z_{m+k}} & \partial_y \pi_1 A^{r_{m+k}}_{m+k} \\ 0 & 1 \end{bmatrix},$$

where $\tilde{\varepsilon}_{Z_{m+k}} = O(\varepsilon^{2^{m+k}})$ and $\hat{\varepsilon}_{Z_{m+k}} = O(\nu^{r_{m+k}})$. Additionally, since maps $B$ and $A$ commute, we have

$A^{r_{m+k}}_{m+k} = (\Lambda^{-1}_{m+k-1} \circ H^{-1}_{m+k-1} \circ (B_{m+k-1} \circ A^{r_{m+k}}_{m+k-1})^{r_{m+k}+1} \circ \Lambda_{m+k-1} \circ H_{m+k-1} \circ \Lambda_{m+k-1})^{r_{m+k}}$

$$= (\Lambda^{-1}_{m+k-1} \circ H^{-1}_{m+k-1} \circ (B_{m+k-1} \circ A^{r_{m+k}}_{m+k-1})^{r_{m+k}+1} \circ \Lambda_{m+k-1} \circ H_{m+k-1} \circ \Lambda_{m+k-1})$$

$$= G_{m+k} \circ B_{m+k},$$

where $G_{m+k}$ is some analytic map defined on the range of $B_{m+k}$ which depends on the combinatorics of $Z$. We have therefore,

$$\partial_y \pi_1 A^{r_{m+k}}_{m+k}(x, y) = \nabla \pi_1 G_{m+k} \circ B_{m+k}(x, y) \cdot (\partial_y \pi_1 B_{m+k}(x, y), \partial_y \pi_2 B_{m+k}(x, y)).$$

We can now use Theorem 5.2 to get

$$\partial_y \pi_1 A^{r_{m+k}}_{m+k}(x, y) = \partial_x \pi_1 G_{m+k} \circ B_{m+k}(x, y) b^{(1+O(\alpha^n))|\bar{v}_m|} f(x)(1 + O(\alpha^l)),$$

and since the derivatives of the analytic map $G_{m+k}$ are uniformly bounded on the image of the domain of $A_{m+k}$ under $B_{m+k}$, we get that

(6.2) $$\partial_y \pi_1 A^{r_{m+k}}_{m+k} = O(b^{(1+O(\alpha^n))|\bar{v}_m|}),$$

where $b$ is the average Jacobian of the pair $Z$. Inverting the matrix (6.1), we obtain a similar result for $\tilde{\varepsilon}_{Z_{m+k}}$ introduced in (5.8):

(6.3) $$\tilde{\varepsilon}_{Z_{m+k}} = O\left(b^{(1+O(\alpha^n))|\bar{v}_m|}\right).$$

Repeating the calculations of Section 5 leading to (5.11), (5.14), (5.12), (5.19) and (5.17), we get the following bounds for the functions entering the expression (5.11) for the derivative of $\Phi^m_{R^k,R'|Z}$

(6.4) $$c\varepsilon^{2^m} \alpha_1^m \leq \|E_{m,R^k,R'|Z}\| \leq C\varepsilon^{2^m} \alpha_2^n,$$
\[ cb^{(1+O(\alpha^n))}|\bar{v}_{kn}|_{g_1}^m \leq \left\| E_{m,\bar{R}Z} \right\| \leq Cb^{(1+O(\alpha^n))}|\bar{v}_{kn}|_{g_2}^m \]

with \( \alpha_1, \alpha_2, \gamma_1 \) and \( \gamma_2 \) as in (5.15), (5.13), (5.18) and (5.16).

**Theorem 6.1.** Let \( Z \) and \( \tilde{Z} \) be two diffeomorphisms in \( W^{s}_{\text{loc}}(Z_*) \) with average Jacobians \( b > \tilde{b} \). Let \( \phi \) be the homeomorphism that conjugates \( Z|_{\Sigma Z} \) to \( Z|_{\Sigma \tilde{Z}} \) normalized so that \( \phi(\tau) = \tau \). If \( \phi \) is \( \kappa \)-Hölder, then

\[ \kappa \leq \frac{1}{3} + \frac{2 \ln b}{3 \ln \tilde{b}} \]

**Proof.** Throughout the proof we will use \( C, K, P, R \) for constants whose specific values are irrelevant to the proof.

We can assume that \( n \) is large, possibly after taking its integer multiple. For such fixed \( n \), choose a large \( k \) and \( l = l(k) \) so that

\[ \prod_{i=0}^{n-1} |\sigma_{\mathcal{R}^i Z}|^{i+1} \ll \tilde{b}^{(1+O(\alpha^n))}|\bar{v}_{kn}|_{\lambda Z, n}^l \ll \prod_{i=0}^{n-1} |\sigma_{\mathcal{R}^i \tilde{Z}}|^{i} \ll b^{(1+O(\alpha^n))}|\bar{v}_{kn}|_{\gamma_1}^l. \]

Such a choice is possible, since by (5.2), (5.6) and (5.18) the first, the third and the fourth expressions in (6.6) are commensurate with \( \lambda Z_{*, n}^{3(l+1)} \), \( \lambda Z_{*, n}^{3l} \) and \( b^{(1+O(\alpha^n))}|\bar{v}_{kn}|_{\lambda Z, n}^l \), respectively, and, therefore, (6.6) is implied by

\[ \lambda Z_{*, n}^{3(l+1)} \ll |\bar{v}_{kn}|_{\lambda Z, n} l \ll |\lambda Z_{*, n}^{3l} \ll \tilde{b}^{l}|\bar{v}_{kn}|_{\lambda Z, n}^l; \]

which, in turn, is implied by

\[ 2l + 3 > |\bar{v}_{kn}| \frac{\ln \tilde{b}}{\ln |\lambda Z_{*, n}|} > 2l > |\bar{v}_{kn}| \frac{\ln b}{\ln |\lambda Z_{*, n}|}, \quad |\lambda Z_{*, n}| \ll 1. \]

The last inequality can be ensured by fixing a large \( n \). Take a large \( k \), so that the interval

\[ \left( |\bar{v}_{kn}| \frac{\ln b}{\ln |\lambda Z_{*, n}|}, |\bar{v}_{kn}| \frac{\ln \tilde{b}}{\ln |\lambda Z_{*, n}|} \right) \]

contains an even integer. Now, \( l \) can be chosen so that the right endpoint of the interval lies between \( 2l \) and \( 2l + 3 \).

Next, consider the action of the \( k \)-th renormalization on the tip \( \tau \):

\[ c_k = \mathcal{R}^k Z(\tau) = A_{kn}(\tau), \]
\[ \tilde{c}_k = \mathcal{R}^k \tilde{Z}(\tau) = \tilde{A}_{kn}(\tau), \]

and denote

\[ X_k = |\pi_1 \tau - \pi_1 c_k|. \]
\[ Y_k = |\pi_2 - \pi_2 c_k| = |\pi_2 c_k|. \]

Clearly, since distances \( X_k \) and \( Y_k \) are those between images of points under one iterate of a map \( A_{kn} \), which is close to the embedding \( \iota(\zeta_s) \) of a one-dimensional pair whose both components are non-trivial functions of \( x \), we have that \( X_k \asymp Y_k \).

We will consider points at three levels of renormalization \( \mathcal{R} \): level \( l + k \), level \( k \) and level 0.

Let \( \Phi_{\mathcal{R}^k Z}^i \) be the map defined in (5.10). Consider the following points in \( \Sigma_{\mathcal{R}^k Z} \) and \( \Sigma_{\mathcal{R}^k \tilde{Z}} \) at level \( k \):

\[ \tau = \Phi_{\mathcal{R}^k Z}^i(\tau), \quad c_k^{l+k} = \Phi_{\mathcal{R}^k Z}^i(c_{l+k}) \quad \text{and} \quad \tilde{c}_k^{l+k} = \Phi_{\mathcal{R}^k \tilde{Z}}^i(\tilde{c}_{l+k}). \]

We have by (5.6) and (5.13), the following inequalities for \( k \) large:

\[ \left| \prod_{j=0}^{l-1} l'_{\mathcal{R}^j Z,n} \circ \prod_{i=j+1}^{l-1} l_{\mathcal{R}^j Z,n} \right| \asymp |\lambda_{Z,n}|^{3l} \gg |\lambda_{Z,n}|^{3(l+k)}, \quad C\alpha_2 \gg \|E_{l,\mathcal{R}^k Z}\|. \]

Therefore, using (5.11), (6.6), (6.4) and (6.5):

\[ |\pi_1 - \pi_1 c_{k}^{l+k}| > |c b^{(1 + O(a^n))}|k_{n}^{+1} |\gamma_1 Y_{k+l} - R \prod_{i=0}^{n-1} \sigma^i_{\mathcal{R}^k Z} X_{k+l} - C\epsilon c_{k}^{l+k}| \]

\[ \gg K b^{(1 + O(a^n))} |\kappa_{n}|^{1}, \]

\[ \gg C |\lambda_{Z,n}|^{l} Y_{l+k} > K |\lambda_{Z,n}|^{l}. \]

Next, consider \( c_k \) and

\[ \zeta_k^{l+k} = A_{kn}(c_k^{l+k}). \]

We notice that

\[ A_{kn} = G_{kn} \circ B_{kn}, \]

and since \( A_{kn} \) converges to the first map in the pair \( Z_s \), given by

\[ Z_s(x, y) = \Lambda \left( \left( (\eta^{r_0}_s \circ \xi_s)^{r_1} \circ \eta_s(x) \right), \left( (\eta^{r_0}_s \circ \xi_s)^{r_1-1} \circ \eta_s(x) \right), x \right). \]

Therefore,

\[ G_{kn}(x, y) = \left( g_{kn}(x) + \delta_1(x, y), g_{kn}(y) + \delta_2(x, y) \right), \]

where \( g_{kn} \) converges geometrically fast to \( (\eta^{r_0}_s \circ \xi_s)^{r_1} \circ \eta_s(x) \), and the norms of \( \delta_1 \) and \( \delta_2 \) are of order \( c_{kn}^{l+k} \). We, therefore, obtain that

\[ \partial x \pi_2 A_{kn} = \nabla \pi_2 G_{kn} \circ B_{kn} \cdot (\partial x \pi_1 B_{kn}, \partial x \pi_2 B_{kn}) \]
Next, consider the distance between the points $\tau$ and $(6.13)$.

Finally, consider

We will now consider similar three pairs of points for the map $\tilde{\tau}$. Crucially, the order of the distance between the points $\tau$ and $\tilde{\tau}$ is different from that for $Z$. By (6.6),

We will now consider similar three pairs of points for the map $Z$. Crucially, the order of the distance between the points $\tau$ and $\tilde{\tau}$ is different from that for $Z$. By (6.6),

Next,

and

\[
\partial_x\pi_1\tilde{A}_{kn} = \nabla \pi_1 \tilde{G}_{kn} \circ \tilde{B}_{kn} \cdot \left( \partial_x\pi_1 \tilde{B}_{kn}, \partial_x\pi_2 \tilde{B}_{kn} \right)
\]
\[ \partial_y \pi_1 \tilde{A}_{kn} = \nabla \pi_1 \tilde{G}_{kn} \circ \tilde{B}_{kn} \cdot (\partial_y \pi_1 \tilde{B}_{kn}, \partial_y \pi_2 \tilde{B}_{kn}) = (O(1), e^{\bar{c}2k^2}) \cdot (O(1), O(1)) = O(1), \]

Therefore, by (6.6),

\[ |\pi_1 \tilde{c}_k - \pi_1 \tilde{c}_k^{l+k}| \leq D|\lambda_{Z,n}|^{3l} + K\tilde{b}^{(1+O(a^0))} |\bar{v}_{kn}| |\lambda_{Z,n}|^l \leq K|\lambda_{Z,n}|^{3l}, \]

\[ |\pi_2 \tilde{c}_k - \pi_2 \tilde{c}_k^{l+k}| \leq D|\lambda_{Z,n}|^{3l} + K\epsilon^{2k^2} \tilde{b}^{(1+O(a^0))} |\bar{v}_{kn}| |\lambda_{Z,n}|^l \leq K|\lambda_{Z,n}|^{3l}. \]

Finally,

\[ |\pi_1 \tilde{z}_k^{l+k} - \pi_1 \tilde{\zeta}_k| < K|\lambda_{Z,n}|^{3l+3k} + K\epsilon^{2k^2} |\lambda_{Z,n}|^{3l} + C\tilde{b}^{(1+O(a^0))} |\bar{v}_{kn}| |\lambda_{Z,n}|^{3l} \gamma_2^k, \]

\[ |\pi_2 \tilde{z}_k^{l+k} - \pi_2 \tilde{\zeta}_k| < K|\lambda_{Z,n}|^{3l+k}, \]

and, by (6.6),

\[ |\tilde{z}_k^{l+k} - \tilde{\zeta}_k| \leq K|\lambda_{Z,n}|^{3l+k}, \]

which, again, by the last inequality of (6.6) implies that distances (6.12) and (6.14) between two identical pairs points in the attractors \( \Sigma \) and \( \tilde{\Sigma} \) are not commensurate.

To complete the proof, assume that the conjugacy \( \phi \) is \( \kappa \)-Hölder. Then, necessarily,

\[ |\tilde{z}_k^{l+k} - \tilde{\zeta}_k| \leq C |\lambda_{Z,n}|^{3l+k}, \]

for some \( C \). By (6.12) and (6.14) this implies that

\[ b^{\bar{v}_{kn}} |\lambda_{Z,n}|^{k \gamma_1} \leq C|\lambda_{Z,n}|^{\kappa(3l+k)}. \]

Since \( \gamma_1 \leq |\lambda_{Z,n}| \), (6.15) is equivalent to

\[ b^{\bar{v}_{kn}} |\lambda_{Z,n}|^{l+k} \leq C|\lambda_{Z,n}|^{\kappa(3l+k)}. \]

We will now bound powers of \( |\lambda_{Z,n}| \) in (6.16) from below and above in an optimal way. By (6.7) we can bound the powers of \( |\lambda_{Z,n}| \) on the right hand side of (6.16) using the first inequality in (6.7):

\[ |\lambda_{Z,n}|^{3l} \leq C \tilde{b}^{\frac{3}{2} |\bar{v}_{kn}|}. \]

The power \( |\lambda_{Z,n}|^l \) on the left hand side will be bounded from below through the second inequality in (6.7):

\[ |\lambda_{Z,n}|^l \geq \tilde{b}^{\frac{1}{2} |\bar{v}_{kn}|}. \]
Therefore, (6.16) can be weakened as

$$\tilde{b}^{|\tilde{v}_{kn}|^\frac{1}{2}} \leq C|\lambda_{Z,n}|^{(\kappa-1)k\tilde{b}^{|\tilde{v}_{kn}|}},$$

or, equivalently,

$$|\tilde{v}_{kn}| \left( \ln b + \frac{1}{2} \ln \tilde{b} - \frac{3}{2} \kappa \ln \tilde{b} \right) + (1 - \kappa)k \ln |\lambda_{Z,n}| \leq C.$$  

For a fixed $n$, the first term $|\tilde{v}_{kn}| \left( \ln b + (1/2) \ln \tilde{b} - (3/2)\kappa \ln \tilde{b} \right)$ in (6.18) grows faster in the absolute value than the second term as $k \to \infty$. For this term not to grow arbitrarily large, it is necessary that

$$\ln b + \frac{1}{2} \ln \tilde{b} - \frac{3}{2} \kappa \ln \tilde{b} \leq 0,$$

which implies the claim. \hfill \Box

**Appendix A. Proof of Proposition 2.2**

Using $q(x) = B(x,0)$ and $p(x) = A(x,0)$ we have

$$\mathcal{L}([\tilde{A}, \tilde{B}]) (x) = a_x(q(x))$$

$$- b(p(x)) - ca_0(x) - da_0(x)^2 - ea_0(x)^3 - fa_0(x)^4.$$

$$\mathcal{L}([\tilde{A}, \tilde{B}])^{(1)} (x) = \partial_1 a_x(q(x)) \partial_1 q(x) + \partial_2 a_x(q(x))$$

$$- \partial_1 b(p(x)) a_0^{(1)}(x) - \partial_2 b(p(x)) h_0^{(1)}(x)$$

$$- a_0^{(1)}(x) \left( c + 2da_0(x) + 3ea_0(x)^2 + 4fa_0(x)^3 \right)$$

$$\mathcal{L}([\tilde{A}, \tilde{B}])^{(2)} (x) = \partial_1^2 a_x(q(x)) \partial_1 q(x)^2 + 2\partial_2 \partial_1 a_x(q(x)) \partial_1 q(x) + \partial_1 a_x(q(x)) \partial_2^2 q(x)$$

$$+ \partial_2^2 a_x(q(x)) - \left[ \partial_1^2 b(p(x)) a_0^{(1)}(x)^2 + 2\partial_2 \partial_1 b(p(x)) a_0^{(1)}(x) h_0^{(1)}(x) \right.$$

$$+ \partial_2^2 b(p(x)) h_0^{(1)}(x)^2 + \partial_1 b(p(x)) a_0^{(2)}(x) + \partial_2 b(p(x)) h_0^{(2)}(x)$$

$$+ a_0^{(2)}(x) \left( c + 2da_0(x) + 3ea_0(x)^2 + 4fa_0(x)^3 \right)$$

$$+ a_0^{(1)}(x)^2 \left( 2c + 6da_0(x) + 12fa_0(x)^2 \right) \right]$$

$$\mathcal{L}([\tilde{A}, \tilde{B}])^{(3)} (x) = \partial_1^3 a_x(q(x)) q^{(1)}(x)^3 + 3\partial_2 \partial_1^2 a_x(q(x)) q^{(1)}(x)^2 + 3\partial_2^2 \partial_1 a_x(q(x)) q^{(1)}(x)$$

$$+ \partial_3^2 a_x(q(x)) + 3\partial_2^2 a_x(q(x)) q^{(1)}(x) q^{(2)}(x)$$

$$+ 3\partial_2 \partial_1 a_x(q(x)) q^{(2)}(x) + \partial_1 a_x(q(x)) q^{(3)}(x)$$

$$- \left[ \partial_1^2 b(p(x)) a_0^{(1)}(x)^3 + 3\partial_2 \partial_1 b(p(x)) a_0^{(1)}(x)^2 h_0^{(1)}(x) \right.$$

$$+ \partial_2^2 b(p(x)) h_0^{(1)}(x)^3 + \partial_1 b(p(x)) a_0^{(2)}(x) + \partial_2 b(p(x)) h_0^{(2)}(x)$$

$$+ a_0^{(2)}(x) \left( c + 2da_0(x) + 3ea_0(x)^2 + 4fa_0(x)^3 \right)$$

$$+ a_0^{(1)}(x)^2 \left( 2c + 6da_0(x) + 12fa_0(x)^2 \right) \right]$$

$$+ a_0^{(1)}(x)^2 \left( 2d + 6ea_0(x)^2 + 12fa_0(x)^3 \right) \right].$$
which gives us:

\[
\begin{align*}
\mathcal{L}([\hat{A}, \hat{B}])^{(0)}(0) &= a_0(b_0(0)) - b(p(0)) - ca_0(0) - da_0(0)^2 - ca_0(0)^3 - fa_0(0)^4, \\
\mathcal{L}([\hat{A}, \hat{B}])^{(1)}(0) &= a_0^{(1)}(b_0(0)) \left(b_0^{(1)}(0) + c\right) + \partial_2 a_0(b_0(0)) \\
&- \partial_1 b(p(0)) a_0^{(1)}(0) - \partial_2 b(p(0)) h_0^{(1)}(0) \\
&- a_0^{(1)}(0) \left(c + 2da_0(0) + 3ea_0(0)^2 + 4fa_0(0)^3\right), \\
\mathcal{L}([\hat{A}, \hat{B}])^{(2)}(0) &= a_0^{(2)}(b_0(0)) \left(b_0^{(1)}(0) + c\right)^2 + 2\partial_2 a_0^{(1)}(b_0(0)) \left(b_0^{(1)}(0) + c\right) \\
&+ a_0^{(1)}(b_0(0)) \left(b_0^{(2)}(0) + 2d\right) + \partial_2^2 a(b_0(0)) \\
&- \left[\partial_1^2 b(p(0)) a_0^{(1)}(0)^2 + 2\partial_2 \partial_1 b(p(0)) a_0^{(1)}(0) h_0^{(1)}(0)\right] \\
&+ \partial_2^2 b(p(0)) h_0^{(1)}(0)^2 + \partial_1 b(p(0)) a_0^{(2)}(0) + \partial_2 b(p(0)) h_0^{(2)}(0) \\
&+ a_0^{(2)}(0) \left(c + 2da_0(0) + 3ea_0(0)^2 + 4fa_0(0)^3\right) \\
&+ a_0^{(1)}(0)^2 \left(2d + 6ea_0(0) + 12fa_0(0)^2\right), \\
\mathcal{L}([\hat{A}, \hat{B}])^{(3)}(0) &= a_0^{(3)}(b_0(0)) \left(b_0^{(1)}(0) + c\right)^3 + 3\partial_2 a_0^{(2)}(b_0(0)) (b_0^{(1)}(0) + c)^2 \\
&+ 3\partial_2^2 a_0^{(1)}(b_0(0)) (b_0^{(1)}(0) + c) + \partial_2 a_0(b_0(0)) \\
&+ 3a_0^{(2)}(b_0(0)) (b_0^{(1)}(0) + c)(b_0^{(2)} + 2d) + 3\partial_2 a_0^{(1)}(b_0(0)) (b_0^{(2)} + 2d) \\
&+ a_0^{(1)}(b_0(0)) (b_0^{(3)}(0) + 6e) - \left[\partial_1^2 b(p(0)) a_0^{(1)}(0)^3\right] \\
&+ 3\partial_2 \partial_1^2 b(p(0)) a_0^{(1)}(0)^2 h_0^{(1)}(0) + 3\partial_2^2 \partial_1 b(p(0)) a_0^{(1)}(0) h_0^{(1)}(0)^2 \\
&+ \partial_2^3 b(p(0)) h_0^{(1)}(0)^3 + 3\partial_2 b(p(0)) a_0^{(1)}(0) a_0^{(2)}(0) \\
&+ 3\partial_2 \partial_1 b(p(0)) a_0^{(2)}(0) h_0^{(1)}(0) + 3\partial_2 \partial_1 b(p(0)) a_0^{(1)}(0) h_0^{(2)}(0)
\end{align*}
\]

\[\begin{align*}
&+ 3\partial_2^2 b(p(0)) h_0^{(1)}(0) h_0^{(2)}(0) + \partial_1 b(p(0)) a_0^{(3)}(0) + \partial_2 b(p(0)) h_0^{(3)}(0) \\
&+ a_0^{(3)}(0) \left( c + 2da_0(0) + 3ea_0(0)^2 + 4fa_0(0)^3 \right) \\
&+ 3a_0^{(1)}(0) a_0^{(2)}(0) \left( 2d + 6ea_0(0) + 12fa_0(0)^2 \right) \\
&+ a_0^{(1)}(0)^3 \left( 6e + 24fa_0(0) \right) \]
\]

For reference we have:

\[\begin{align*}
\mathcal{L}([A, B])(x) &= a_x(b_0(x)) - b(p(x)) \\
\mathcal{L}([A, B])^{(1)}(x) &= \partial_1 a_x(b_0(x)) h_0^{(1)}(x) + \partial_2 a_x(b_0(x)) \\
&- \partial_1 b(p(x)) a_0^{(1)}(x) - \partial_2 b(p(x)) h_0^{(1)}(x) \\
\mathcal{L}([A, B])^{(2)}(x) &= \partial_1^2 a_x(b_0(x)) h_0^{(1)}(x)^2 + 2\partial_2 \partial_1 a_x(b_0(x)) b_0^{(1)}(x) \\
&+ \partial_2^2 a_x(b_0(x)) + \partial_1 a_x(b_0(x)) b_0^{(2)}(x) \\
&- \left[ \partial_1^2 b(p(x)) a_0^{(1)}(x)^2 + 2\partial_2 \partial_1 b(p(x)) a_0^{(1)}(x) h_0^{(1)}(x) \right] \\
&+ \partial_2^2 b(p(x)) h_0^{(1)}(x)^2 + \partial_1 b(p(x)) a_0^{(2)}(x) \\
&+ \partial_2 b(p(x)) h_0^{(2)}(x) \right] \\
\mathcal{L}([A, B])^{(3)}(x) &= \partial_1^3 a_x(b_0(x)) h_0^{(1)}(x)^3 + 3\partial_2 \partial_1^2 a_x(b_0(x)) b_0^{(1)}(x)^2 \\
&+ 3\partial_2^2 \partial_1 a_x(b_0(x)) b_0^{(1)}(x) + \partial_2^3 a_x(b_0(x)) + 3\partial_1^2 a_x(b_0(x)) b_0^{(1)}(x) b_0^{(2)}(x) \\
&+ 3\partial_2 \partial_1 a_x(b_0(x)) b_0^{(2)}(x) + \partial_1 a_x(b_0(x)) h_0^{(3)}(x) \\
&- \left[ \partial_1^3 b(p(x)) a_0^{(1)}(x)^3 + 3\partial_2 \partial_1^2 b(p(x)) a_0^{(1)}(x)^2 h_0^{(1)}(x) \right] \\
&+ 3\partial_2^2 \partial_1 b(p(x)) a_0^{(1)}(x) h_0^{(1)}(x)^2 + \partial_2^3 b(p(x)) h_0^{(1)}(x)^3 \\
&+ 3\partial_2^2 b(p(x)) a_0^{(1)}(x) a_0^{(2)}(x) + 3\partial_2 \partial_1 b(p(x)) a_0^{(2)}(x) h_0^{(1)}(x) \\
&+ 3\partial_2 \partial_1 b(p(x)) a_0^{(1)}(x) h_0^{(2)}(x) + 3\partial_2^2 b(p(x)) h_0^{(1)}(x) h_0^{(2)}(x) \\
&+ \partial_1 b(p(x)) a_0^{(3)}(x) + \partial_2^2 b(p(x)) h_0^{(3)}(x) \right]
\end{align*}\]

which gives us:

\[\begin{align*}
\mathcal{L}([A, B])(0) &= a_0(b_0(0)) - b(a_0(0), h_0(0)) \\
\mathcal{L}([A, B])^{(1)}(0) &= a_0^{(1)}(b_0(0)) h_0^{(1)}(0) + \partial_2 a_0(b_0(0)) \\
&- \partial_1 b(a_0(0), h_0(0)) a_0^{(1)}(0) - \partial_2 b(a_0(0), h_0(0)) h_0^{(1)}(0) \]

\end{align*}\]
\[ \mathcal{L}(\{A, B\})^{(2)}(0) = a_0^{(2)}(b_0(0))b_0^{(1)}(0)^2 + 2\partial_2 a_0^{(1)}(b_0(0))b_0^{(1)}(0) \\
\quad + \partial_2^2 a_0(b_0(0)) + a_0^{(1)}(b_0(0))b_0^{(2)}(0) \\
- \left[ \partial_2^2 b(a_0(0), h_0(0))a_0^{(1)}(0)^2 + 2\partial_2 \partial_1 b(a_0(0), h_0(0))a_0^{(1)}(0)h_0^{(1)}(0) \\
+ \partial_2^2 b(a_0(0), h_0(0))h_0^{(1)}(0)^2 + \partial_1 b(a_0(0), h_0(0))a_0^{(2)}(0) \\
+ \partial_2 b(a_0(0), h_0(0))h_0^{(2)}(0) \right] \]

\[ \mathcal{L}(\{A, B\})^{(3)}(0) = a_0^{(3)}(b_0(0))b_0^{(1)}(0)^3 + 3\partial_2 a_0^{(2)}(b_0(0))b_0^{(1)}(0)^2 \\
+ 3\partial_2^2 a_0^{(1)}(b_0(0))b_0^{(1)}(0) + \partial_2^3 a_0(b_0(0)) + 3a_0^{(2)}(b_0(0))b_0^{(1)}(0)h_0^{(2)}(0) \\
+ 3\partial_2 a_0^{(1)}(b_0(0))b_0^{(2)}(0) + a_0^{(1)}(b_0(0))b_0^{(3)}(0) \\
- \left[ \partial_2^3 b(p(0))a_0^{(1)}(0)^3 + 3\partial_2 \partial_1^2 b(p(0))a_0^{(1)}(0)^2h_0^{(1)}(0) \\
+ 3\partial_2^2 \partial_1 b(p(0))a_0^{(1)}(0)h_0^{(1)}(0)^2 + \partial_2^2 b(p(0))h_0^{(1)}(0)^3 \\
+ 3\partial_2^2 b(p(0))a_0^{(1)}(0)a_0^{(2)}(0) + 3\partial_2 \partial_1 b(p(0))a_0^{(2)}(0)h_0^{(1)}(0) \\
+ 3\partial_2 \partial_1 b(p(0))a_0^{(1)}(0)h_0^{(2)}(0) + 3\partial_2^2 b(p(0))h_0^{(1)}(0)h_0^{(2)}(0) \\
+ \partial_1 b(p(0))a_0^{(3)}(0) + \partial_2 b(p(0))h_0^{(3)}(0) \right] \]

The commutation condition of Equation 2.6 can then be translated to the following conditions:

\[ 0 = \mathcal{L}(\{\tilde{A}, \tilde{B}\})(0) = \mathcal{L}(\{A, B\})(0) - ca_0(0) - da_0(0)^2 - ea_0(0)^3 - fa_0(0)^4 \]

\[ 0 = \mathcal{L}(\{\tilde{A}, \tilde{B}\})^{(1)}(0) = \mathcal{L}(\{A, B\})^{(1)}(0) + ca_0^{(1)}(b_0(0)) \\
\quad - a_0^{(1)}(0) \left( c + 2da_0(0) + 3ea_0(0)^2 + 4fa_0(0)^3 \right) \]

\[ 0 = \mathcal{L}(\{\tilde{A}, \tilde{B}\})^{(2)}(0) = \mathcal{L}(\{A, B\})^{(2)}(0) + 2ca_0^{(2)}(b_0(0))b_0^{(1)}(0) + c^2 a_0^{(2)}(b_0(0)) \\
\quad + 2c\partial_2 a_0^{(1)}(b_0(0)) + 2da_0^{(1)}(b_0(0)) \\
\quad - a_0^{(2)}(0) \left( c + 2da_0(0) + 3ea_0(0)^2 + 4fa_0(0)^3 \right) \\
\quad - a_0^{(1)}(0)^2 \left( 2d + 6ea_0(0) + 12fa_0(0)^2 \right) \]

\[ 0 = \mathcal{L}(\{\tilde{A}, \tilde{B}\})^{(3)}(0) = \mathcal{L}(\{A, B\})^{(3)}(0) + a_0^{(3)}(b_0(0))(3cb_0^{(1)}(0)^2 + 3c^2 b_0^{(1)}(0) + c^3) \\
\quad + 3\partial_2 a_0^{(2)}(b_0(0))(2cb_0^{(1)}(0) + c^2) + 3c\partial_2^2 a_0^{(1)}(b_0(0)) \\
\quad + 3a_0^{(2)}(b_0(0))(2db_0^{(1)}(0) + cb_0^{(2)}(0) + 2cd) \]
\[ + 6d \partial_2a_0^{(1)}(0) + 6ea_0^{(1)}(0) \]
\[ - \left[ a_0^{(3)}(0) \left( c + 2da_0(0) + 3ea_0(0)^2 + 4fa_0(0)^3 \right) \right] \]
\[ + 3a_0^{(1)}(0)a_0^{(2)}(0) \left( 2d + 6ea_0(0) + 12fa_0(0)^2 \right) \]
\[ + a_0^{(1)}(0)^3(6e + 24fa_0(0)) \]

We can clearly see that all the equations are satisfied if \( L([A, B]) = o(\|x\|^3) \) and \( c = d = e = f = 0 \). This shows that the projection acts as the identity on the set of almost commuting pairs and proves the existence of solutions.

To show that the projection is analytic and uniqueness of solutions we will use the Implicit Function Theorem. Therefore, consider the map

\[(c, d, e, f) \mapsto \left( L([A, B])(0), L([A, B])^{(1)}(0), L([A, B])^{(2)}(0), L([A, B])^{(3)}(0) \right)\]

for a fixed pair \((\tilde{A}, \tilde{B})\). The differential of this map is then given by

\[
\begin{pmatrix}
-a_0(0) \\
-a_0^{(1)}(0) - a_0(0)^2 \\
-2a_0(0)a_0^{(1)}(0) - 3a_0(0)^2a_0^{(1)}(0) - 4a_0(0)^3a_0^{(1)}(0) \\
-a_0^{(1)}(b_0(0)) - a_0^{(2)}(b_0(0)) - 2a_0^{(1)}(0) \\
a_0^{(1)}(0)a_0^{(2)}(0) + 2a_0^{(1)}(0)^2 \\
-a_0(0) - 2a_0(0)a_0^{(2)}(0) + 3a_0(0)^2a_0^{(1)}(0) \\
-4a_0(0)(0) - 4a_0(0)^3a_0^{(1)}(0) \\
\end{pmatrix}
\]

where we have

\[
A = 2 \left( c + b_0^{(1)}(0) \right) a_0^{(2)}(b_0(0)) + 2\partial_2a_0^{(1)}(b_0(0)) - a_0^{(2)}(0)
\]
\[
B = 2a_0^{(1)}(b_0(0)) - 2a_0^{(2)}(0)a_0(0) - 2a_0^{(1)}(0)^2
\]
\[
C = -3a_0(0) \left( a_0(0)a_0^{(2)}(0) + 2a_0^{(1)}(0)^2 \right)
\]
\[
D = -4a_0(0)(0) \left( a_0(0)a_0^{(2)}(0) + 3a_0^{(1)}(0)^2 \right)
\]
\[
E = a_0^{(3)}(b_0(0)) \left( 3b_0^{(1)}(0)^2 + 6cb_0^{(1)}(0) + 3c^2 \right) + 3\partial_2a_0^{(2)}(b_0(0)) \left( 2b_0^{(1)}(0) + 2c \right)
\]
\[
+ 3\partial_2a_0^{(1)}(b_0(0)) + 3a_0^{(2)}(b_0(0)) \left( b_0^{(2)}(0) + 2d \right) - a_0^{(3)}(0)
\]
\[
F = 3a_0^{(2)}(b_0(0)) \left( 2b_0^{(1)}(0) + 2c \right) + 6\partial_2a_0^{(1)}(b_0(0)) - 2a_0(0)a_0^{(3)}(0) - 6a_0^{(1)}(0)a_0^{(2)}(0)
\]
\[
G = 6a_0^{(1)}(b_0(0)) - 3a_0(0)^2a_0^{(3)}(0) - 18a_0(0)a_0^{(1)}(0)a_0^{(2)}(0) - 6a_0^{(1)}(0)^3
\]
\[
H = -4a_0(0)^3a_0^{(1)}(0) - 36a_0(0)^2a_0^{(1)}(0)a_0^{(2)}(0) - 24a_0(0)a_0^{(1)}(0)^3
\]
Letting $c = d = e = f = 0$ the first two rows stay the same. The last row simplifies as follows:

\[
A = 2a_0(2)(b_0(0))b_0(1)(0) + 2\partial_2 a_0(1)(b_0(0)) - a_0(2)(0)
\]
\[
B = 2a_0(1)(b_0(0)) - 2a_0(2)(0)a_0(0) - 2a_0(1)(0)^2
\]
\[
C = -3a_0(0)\left(a_0(0)a_0(2)(0) + 2a_0(1)(0)^2\right)
\]

First note that all entries on the first row are bounded away from zero.

For the second row, the first term $a_0(1)(b_0(0))$ is bounded away from 0 and the rest of the terms in the second row are $O(\epsilon)$ by the Cauchy integral formula since they are at least $O(\epsilon)$ close to the embedding of the 1D renormalized pair (which by definition is an almost commuting pair). Thus the entry in the first column is bounded away from 0 and the rest are small.

For the third row we have:

\[
A = 2a_0(2)(b_0(0))b_0(1)(0) + 2\partial_2 a_0(1)(b_0(0)) - a_0(2)(0)
\]
\[
B = 2a_0(1)(b_0(0)) - 2a_0(2)(0)a_0(0) - 2a_0(1)(0)^2
\]
\[
C = -3a_0(0)\left(a_0(0)a_0(2)(0) + 2a_0(1)(0)^2\right)
\]
\[
D = -4a_0(0)\left(a_0(0)a_0(2)(0) + 3a_0(1)(0)^2\right)
\]
\[
E = 3a_0(3)(b_0(0))b_0(1)(0)^2 + 6\partial_2 a_0(2)(b_0(0))b_0(1)(0)
\]
\[
\quad + 3\partial_2^2 a_0(1)(b_0(0)) + 3a_0(2)(b_0(0))b_0(2)(0) - a_0(3)(0)
\]
\[
F = 6a_0(2)(b_0(0))b_0(1)(0) + 6\partial_2 a_0(1)(b_0(0)) - 2a_0(0)a_0(3)(0) - 6a_0(1)(0)a_0(2)(0)
\]
\[
G = 6a_0(1)(b_0(0)) - 3a_0(0)^2a_0(3)(0) - 18a_0(0)a_0(1)(0)a_0(2)(0) - 6a_0(1)(0)^3
\]
\[
H = -4a_0(0)^3a_0(3)(0) - 36a_0(0)^2a_0(1)(0)a_0(2)(0) - 24a_0(0)a_0(3)(0)^2
\]

So the entry in the second column is bounded away from zero and the rest are small.

For the fourth row we have:

\[
E = O(\epsilon)
\]
\[
F = O(\epsilon)
\]
\[
G = 6a_0(1)(b_0(0)) + O(\epsilon)
\]
\[
H = O(\epsilon)
\]

So the entry in the third column is bounded away from zero and the rest are small.
It follows that the Jacobian matrix at \((c,d,e,f) = (0,0,0,0)\) is invertible for any commuting \((A,B)\) for small enough \(\epsilon\). Using the Implicit Function Theorem applied to the map

\[
((A,B), c,d,e,f) \mapsto (\mathcal{L}[\tilde{A}, \tilde{B}](0), \mathcal{L}([\tilde{A}, \tilde{B}])^{(1)}(0), \mathcal{L}([\tilde{A}, \tilde{B}])^{(2)}(0), \mathcal{L}([\tilde{A}, \tilde{B}])^{(3)}(0))
\]

finishes the proof.
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