Abstract

Stochastic shortest path (SSP) is a well-known problem in planning and control, in which an agent has to reach a goal state in minimum total expected cost. In this paper we consider adversarial SSPs that also account for adversarial changes in the costs over time, while the dynamics (i.e., transition function) remains unchanged. Formally, an agent interacts with an SSP environment for $K$ episodes, the cost function changes arbitrarily between episodes, and the fixed dynamics are unknown to the agent. We give high probability regret bounds of $\tilde{O}(\sqrt{K})$ assuming all costs are strictly positive, and $\tilde{O}(K^{3/4})$ for the general case. To the best of our knowledge, we are the first to consider this natural setting of adversarial SSP and obtain sub-linear regret for it.

1 Introduction

Stochastic shortest path (SSP) is one of the most basic models in reinforcement learning (RL). In SSP the goal of the agent is to reach a predefined goal state in minimum expected cost, and it captures a wide variety of natural scenarios, such as car navigation and game playing. An important aspect that the SSP model fails to capture is the changes in the environment over time (for example, changes in traffic when navigating a car). This aspect of the environment is theoretically modeled by adversarial Markov decision processes (MDPs), in which the cost function may change arbitrarily over time, while still assuming a fixed transition function.

In this work we present the adversarial SSP model that combines SSPs with adversarial MDPs. In this model, the agent interacts with an SSP environment in $K$ episodes, but the cost function changes between episodes arbitrarily. The agent’s objective is to reach the goal state in every episode while minimizing its total expected cost, and its performance is measured by the regret – the difference between the agent’s total cost and the expected total cost of the best stationary policy in hindsight.

We propose the first algorithms for regret minimization in adversarial SSPs. Our algorithms take recent advances in learning SSP problems [1,2] – that build upon the optimism in face of uncertainty principle, and combine them with the O-REPS framework [3,4,5,6] for adversarial episodic MDPs – which implements the online mirror descent (OMD) algorithm for online convex optimization. We follow the strategy of [1,2] for SSPs – we start by assuming all costs are strictly positive and prove $\tilde{O}(\sqrt{K})$ regret (which is optimal). Then, using a perturbation argument, we remove this assumption and show that our algorithms obtain $\tilde{O}(K^{3/4})$ regret.

First, we consider a simplified case in which the transition function is known to the learner and the regret should be minimized in expectation. For this case, we establish an efficient O-REPS based algorithm and bound its expected regret. Then, we introduce an improvement that ensures the learner will not run too long before reaching the goal, and show that this yields a high probability regret bound. Finally, we remove the known transition function assumption and combine our algorithm with the confidence set framework of UCRL2 [7]. This allows us to prove a high probability regret bound without knowledge of the transition function.

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1.1 Related work

Early work by [8] studied the problem of planning in SSPs, that is, computing the optimal strategy efficiently in a known SSP instance. They established that, under certain assumptions, the optimal strategy is a deterministic stationary policy (a mapping from states to actions) and can be computed efficiently using standard planning algorithms, e.g., Value Iteration or Policy Iteration.

Recently the problem of learning SSPs was addressed by [1] and then improved by [2]. The latter show an efficient algorithm based on optimism in face of uncertainty and prove that it obtains a high probability regret bound of $O(D|S|A/K)$, where $D$ is the diameter, $S$ is the state space and $A$ is the action space. They also prove a nearly matching lower bound of $\Omega(D\sqrt{|S||A|K})$.

Regret minimization in RL has already been extensively studied. However, the literature mainly focuses on the average-cost infinite-horizon model [9, 7] and on the finite-horizon (episodic) model [10, 11, 12, 13, 14, 15].

Adversarial MDPs were first studied in the average-cost infinite-horizon model [16, 17, 18], before focusing on the episodic setting. Early work in the episodic setting by [19] used a reduction to multi-arm bandit [20], but then [3] presented the O-REPS framework. All these works assumed a known transition function, but more recent work [21, 4, 5, 6] considered the more general case where the agent must learn the transition function from experience. Recently, [15, 22] showed that policy optimization methods (that are widely used in practice) can also achieve near-optimal regret bounds in adversarial episodic MDPs.

2 Preliminaries

An adversarial SSP problem is defined by an MDP $M = (S, A, P, s_0, g)$ and a sequence $c_1, \ldots, c_K$ of cost functions, where $c_k : S \times A \to [0, 1]$. We do not make any assumption on the cost functions, i.e., they can be chosen arbitrarily. $S$ and $A$ are finite state and action spaces, and $P$ is a transition function such that $P(s' | s, a)$ is the probability to move to state $s'$ when taking action $a$ in state $s$.

The learner interacts with $M$ in episodes, where $c_k$ is the cost function for episode $k$. However, it is revealed to the learner only in the end of the episode. In every episode, the learner begins at the initial state $s_0 \in S$, and ends the interaction with $M$ by arriving at the goal state $g$ (where $g \notin S$). The full interaction is described in Algorithm 1. To simplify the presentation we denote $S^+ = S \cup \{g\}$ and thus for every $(s, a) \in S \times A$ we have that $\sum_{s' \in S^+} P(s' | s, a) = 1$.

**Algorithm 1 Learner-Environment Interaction**

**Parameters:** MDP $M = (S, A, P, s_0, g)$ and sequence of cost functions $\{c_k\}_{k=1}^K$

for $k = 1$ to $K$

learner starts in state $s^k_1 = s_0$, $i \leftarrow 1$

while $s^k_i \neq g$

learner chooses action $a^k_i \in A$

learner observes state $s^k_{i+1}$ drawn by the environment, i.e., $s^k_{i+1} \sim P(\cdot | s^k_i, a^k_i)$, $i \leftarrow i + 1$

end while

learner observes cost function $c_k$ and suffers cost $\sum_{j=1}^{i-1} c_k(s^k_j, a^k_j)$

end for

2.1 Proper policies

A stationary (stochastic) policy is a mapping $\pi : S \times A \to [0, 1]$, where $\pi(a | s)$ gives the probability that action $a$ is selected in state $s$. Since reaching the goal is one of the main objectives of the learner along with minimizing its cost, we are interested in proper policies.

**Definition 1** (Proper and Improper Policies). A policy $\pi$ is *proper* if playing $\pi$ reaches the goal state with probability 1 when starting from any state. A policy is *improper* if it is not proper.
The set of all proper deterministic policies is denoted by \( \Pi_{\text{proper}} \). In addition, we denote by \( T^\pi(s) \) the expected time it takes for \( \pi \) to reach \( q \) starting at \( s \). In particular, if \( \pi \) is proper then \( T^\pi(s) \) is finite for all \( s \in S \), and if \( \pi \) is improper there must exist some \( s \in S \) such that \( T^\pi(s) = \infty \). We make the basic assumption that the goal is reachable from every state, and formalize it as follows.

**Assumption 1.** There exists at least one proper policy, i.e., \( \Pi_{\text{proper}} \neq \emptyset \).

When paired with a cost function \( c : S \times A \to [0, 1] \), any policy \( \pi \) induces a cost-to-go function \( J^\pi : S \to [0, \infty] \), where \( J^\pi(s) \) is the expected cost when playing policy \( \pi \) and starting at state \( s \), i.e., \( J^\pi(s) = \lim_{T \to \infty} E \left[ \sum_{t=1}^{T} c(s_t, a_t) \mid P, \pi, s_1 = s \right] \). For a proper policy \( \pi \), it follows that \( J^\pi(s) \) is finite for all \( s \in S \). However, note that \( J^\pi(s) \) may be finite even if \( \pi \) is improper.

Under Assumption 1 and the assumption that every improper policy suffers infinite cost from some state, Bertsekas and Tsitsiklis [8] show that the optimal policy is stationary, deterministic and proper; and that every proper policy satisfies the Bellman equations:

\[
J^\pi(s) = \sum_{a \in A} \pi(a \mid s) \left( c(s, a) + \sum_{s' \in S} P(s' \mid s, a) J^\pi(s') \right) \quad \forall s \in S \tag{1}
\]

\[
T^\pi(s) = 1 + \sum_{a \in A} \sum_{s' \in S} \pi(a \mid s) P(s' \mid s, a) T^\pi(s') \quad \forall s \in S \tag{2}
\]

### 2.2 Learning formulation

The success of the learner is measured by the regret, that is the difference between the learner’s total cost over \( K \) episodes and the total expected cost of the best proper policy in hindsight:

\[
R_K = \sum_{k=1}^{K} \sum_{i=1}^{I^k} c_k(s^k_i, a^k_i) - \min_{\pi \in \Pi_{\text{proper}}} \sum_{k=1}^{K} J^\pi_k(s_0) = \sum_{k=1}^{K} \sum_{i=1}^{I^k} c_k(s^k_i, a^k_i) - \sum_{k=1}^{K} J^\pi^*(s_0),
\]

where \( \pi^* \) is the minimizing policy, \( I^k \) is the time it takes the learner to complete episode \( k \) (which may be infinite), \((s^k_i, a^k_i)\) is the \( i \)-th state-action pair at episode \( k \), and \( J^\pi_k \) is the cost-to-go of policy \( \pi \) with respect to (w.r.t) cost function \( c_k \). In the case that \( I^k \) is infinite for some \( k \), we define \( R_K = \infty \). We also define the SSP-diameter [11], \( D = \max_{s \in S} \min_{\pi \in \Pi_{\text{proper}}} T^\pi(s) \), which will appear in our regret bounds but is unknown to the learning algorithms. To simplify presentation we assume \( K \geq |S|^2 |A| \).

To match the conditions that guarantee the Bellman equations hold, throughout the paper we also assume the costs are strictly positive. In Section 6 we relax this assumption.

**Assumption 2.** All costs are positive, i.e., there exists \( c_{\min} > 0 \) such that \( c_k(s, a) \geq c_{\min} \) for every \((s, a) \in S \times A \) and \( k = 1, \ldots, K \).

### 3 Occupancy measures

Every policy \( \pi \) induces an occupancy measure \( q^\pi : S \times A \to [0, \infty] \) such that \( q^\pi(s, a) \) is the expected number of times to visit state \( s \) and take action \( a \) when playing according to \( \pi \), that is,

\[
q^\pi(s, a) = \lim_{T \to \infty} E \left[ \sum_{t=1}^{T} \mathbb{1}\{s_t = s, a_t = a\} \mid P, \pi, s_1 = s_0 \right].
\]

Note that for a proper policy \( \pi \), \( q^\pi(s, a) \) is finite for every state-action pair \((s, a)\). Furthermore, this correspondence between proper policies and finite occupancy measures is 1-to-1, and its inverse for an occupancy measure \( q \) is given by \( \pi^q(a \mid s) = \frac{q(a \mid s)}{q(s)} \), where \( q(s) \defeq \sum_{a \in A} q(s, a) \).\(^1\)

\(^1\)If \( q(s) = 0 \) for some state \( s \) then the inverse mapping is not well-defined. However, since \( s \) will not be reached, we can pick the action there arbitrarily. Alternatively, the correspondence holds when restricting to reachable states.
The aforementioned equivalence between policies and occupancy measures is well-known for MDPs [3], but also holds for SSPs by linear programming formulation [23, 24]. Notice that the expected cost of policy \( \pi \) is linear w.r.t \( q^\pi \),

\[
J_k^\pi(s_0) = \mathbb{E} \left[ \sum_{i=1}^{j_k} c_k(s_i, a_i) | P, \pi_k, s_1 = s_0 \right] = \sum_{s \in S} \sum_{a \in A} q^\pi_k(s, a) c_k(s, a) \overset{\text{def}}{=} \langle q^\pi_k, c_k \rangle.
\]

Thus, minimizing the expected regret can be written as an instance of online linear optimization,

\[
\mathbb{E}[R_K] = \mathbb{E} \left[ \sum_{k=1}^{K} J_k^\pi(s_0) - \sum_{k=1}^{K} J_k^{\pi^*}(s_0) \right] = \mathbb{E} \left[ \sum_{k=1}^{K} \langle q^\pi_k - q^{\pi^*}, c_k \rangle \right].
\]

\section{Known transition function}

Before addressing our main challenge of adversarial SSP with unknown transition function in \cite{section5} we use the simpler case of known dynamics to develop our main techniques. In \cite{section4.1} we establish the implementation of the OMD method in SSP, and then in \cite{section4.2} we develop the technique of converting its expected regret bound into a high probability bound.

\subsection{Online mirror descent for SSP}

Online mirror descent is a popular framework for online convex optimization and its application to occupancy measures yields the O-REPS algorithms [3][4][5][6]. Usually these algorithms operate w.r.t to the set of all occupancy measures (which corresponds to the set of all policies), but here we restrict it to the set \( \Delta(D/c_{\text{min}}) \) – occupancy measures of policies with at most \( D/c_{\text{min}} \) expected time of reaching the goal state from \( s_0 \). Limiting the running time of our policies will be crucial in the regret analysis, and is not a concern in finite-horizon RL. The parameter \( D/c_{\text{min}} \) is chosen because \( q^{\pi^*} \in \Delta(D/c_{\text{min}}) \), where \( q^{\pi^*} \) is the occupancy measure of the best proper policy in hindsight \( \pi^* \) (see Appendix C).

Our algorithm, called SSP-O-REPS, follows the O-REPS framework. In each episode we pick an occupancy measure (and thus a policy) from \( \Delta(D/c_{\text{min}}) \) which minimizes a trade-off between the current cost function and the distance to the previously chosen occupancy measure. Formally,

\[
q_{k+1} = q_{\text{\pi}k+1} = \arg \min_{q \in \Delta(D/c_{\text{min}})} \eta(q, c_k) + \text{KL}(q \parallel q_k),
\]

where \( \text{KL}(\cdot \parallel \cdot) \) is the un-normalized KL-divergence and \( \eta > 0 \) is a learning rate. To compute \( q_{k+1} \) we first find the unconstrained minimizer and then project into \( \Delta(D/c_{\text{min}}) \) (see [3]), i.e.,

\[
q'_{k+1} = \arg \min_{q} \eta(q, c_k) + \text{KL}(q \parallel q_k)
\]

\[
q_{k+1} = \arg \min_{q \in \Delta(D/c_{\text{min}})} \text{KL}(q \parallel q'_{k+1}).
\]

\textbf{Eq. (5)} has a nice closed form of \( q'_{k+1}(s, a) = q_k(s, a)e^{-\eta c_k(s, a)} \), and \textbf{Eq. (6)} can be formalized as a constrained convex optimization problem using the following constraints (together with \( q(s, a) \geq 0 \)):

\[
\sum_{a \in A} q(s, a) - \sum_{s' \in S} \sum_{a' \in A} q(s', a') P(s | s', a') = \mathbb{I}\{s = s_0\} \quad \forall s \in S
\]

\[
\sum_{s \in S} \sum_{a \in A} q(s, a) \leq \frac{D}{c_{\text{min}}},
\]

where \( \mathbb{I} \) is the indicator function, and the second constraint ensures that \( T^{\pi^*}(s_0) \leq D/c_{\text{min}} \). In Appendix A we show that this problem can be solved efficiently and describe in details the implementation of the algorithm. In addition, we
describe how to compute $D$ efficiently by finding the optimal policy w.r.t the constant cost function $c(s, a) = 1$. Full pseudo-code is found in Appendix B.

Following standard OMD analysis we obtain the following regret bound in expectation (proofs in Appendix C). Moreover, we show that all the policies chosen by the algorithm must be proper and therefore the goal state will be reached with probability 1 in all episodes.

**Theorem 4.1.** Under Assumptions 1 and 2 running SSP-O-REPS with known transition function and $\eta = \sqrt{\frac{\log(D|S||A|/\epsilon_{\min})}{K}}$

ensures that

\[
\mathbb{E}[R_K] \leq O\left(\frac{D}{\epsilon_{\min}} \sqrt{K \log \frac{D|S||A|}{\epsilon_{\min}}} \right) = \tilde{O}\left(\frac{D}{\epsilon_{\min}} \sqrt{K} \right).
\]

**4.2 High probability regret bound**

Since the cost suffered in SSP may be unbounded, it is of great importance to bound the regret with high probability and not just in expectation. While O-REPS algorithms yield high probability bounds immediately in episodic MDPs, in our setting there might be some constant probability in which the learner suffers a huge cost (in the finite-horizon setting the cost is always bounded by the horizon $H$).

Thus, we need to bound the deviation of the suffered cost from its expectation. The following lemma shows that bounding the deviation is closely related to bounding the expected time of reaching the goal from any state.

**Lemma 4.2.** Assume the learner plays strategies $\{\sigma_k\}_{k=1}^K$ such that the expected time of reaching the goal from any state when playing $\sigma_k$ is at most $D/\epsilon_{\min}$. Then, with probability at least $1 - \delta$,

\[
\sum_{k=1}^K \sum_{i=1}^I c_k(s_i^k, a_i^k) - \sum_{k=1}^K \mathbb{E}\left[\sum_{i=1}^I c_k(s_i^k, a_i^k) \mid P, \sigma_k, s_0 = s_0 \right] \leq O\left(\frac{D}{\epsilon_{\min}} \sqrt{K \log^3 \frac{K}{\delta}} \right).
\]

Following this lemma, our algorithm SSP-O-REPS2 operates as follows. We start every episode $k$ by playing the policy $\pi_k$ chosen by SSP-O-REPS, i.e., Eq. (4). However, once we reach a state $s$ whose expected time to the goal is too long, i.e., $T_{\pi_k}(s) \geq D/\epsilon_{\min}$, we switch to the fast policy $\pi_f$. The fast policy minimizes the time to the goal from any state and can be computed efficiently similarly to the SSP-diameter $D$ (see Appendix A). Notice that if $q_{\pi_k}(s) > 0$ then $T_{\pi_k}(s)$ must be finite, otherwise $T_{\pi_k}(s_0) = \infty$. Thus we can compute $T_{\pi_k}$ as follows: Ignore states that are not reachable from $s_0$ using $\pi_k$, and solve the (linear) Bellman equations (Eq. (2)). The full pseudo-code is in Appendix D.

We denote by $\sigma_k$ the strategy of playing $\pi_k$ until reaching a “bad” state and then switching to the fast policy. Now, by Lemma 4.2 we can bound the deviation of our suffered cost from its expectation. Next, we again turn to bounding the expected regret. We cannot apply OMD analysis immediately since we did not play $\pi_k$ all through episode $k$. However, Lemma 4.3 shows that our mid episode policy switch only improves the expected cost. Thus, together with Lemma 4.2 it yields the high probability regret bound in Theorem 4.4 (proofs in Appendix D).

**Lemma 4.3.** For every $k = 1, \ldots, K$ it holds that

\[
\mathbb{E}\left[\sum_{i=1}^I c_k(s_i^k, a_i^k) \mid P, \pi_k, s_1^k = s_0 \right] \leq \mathbb{E}\left[\sum_{i=1}^I c_k(s_i^k, a_i^k) \mid P, \pi_k, s_1^k = s_0 \right] = J_k^{\pi_k}(s_0).
\]

**Theorem 4.4.** Under Assumptions 1 and 2 running SSP-O-REPS2 with known transition function and $\eta = \sqrt{\frac{\log(D|S||A|/\epsilon_{\min})}{K}}$

ensures that, with probability at least $1 - \delta$,

\[
R_K \leq O\left(\frac{D}{\epsilon_{\min}} \sqrt{K \log^3 \frac{KD|S||A|}{\delta\epsilon_{\min}}} \right) = \tilde{O}\left(\frac{D}{\epsilon_{\min}} \sqrt{K} \right).
\]
5 Unknown transition function

When the transition function is unknown the learner must estimate it from experience. Thus, we keep confidence sets that contain $P$ with high probability, similarly to UCLRL2 [7]. Our algorithm SSP-O-REPS3 proceeds in intervals and updates the confidence set at the beginning of every interval. The first interval begins at the first time step, and an interval ends once an episode ends, the number of visits to some state-action pair is doubled, a “bad” state is reached (similarly to SSP-O-REPS2) or an unknown state is reached (will be defined later). Denote by $N^m(s, a)$ the number of visits to $(s, a)$ up to (and not including) interval $m$, and similarly $N^m(s, a, s')$. The empirical transition function for interval $m$ will be $\tilde{P}_m(s' | s, a) = \frac{N^m(s, a, s')}{N^m(s, a)}$, where $N^m(s, a) = \max\{N^m(s, a), 1\}$, and the confidence set for interval $m$ contains all transition functions $P'$ such that for every $(s, a, s') \in S \times A \times S^+$,

$$|P'(s' | s, a) - \tilde{P}_m(s' | s, a)| \leq \epsilon_m(s' | s, a) = 4\sqrt{\tilde{P}_m(s' | s, a)A^m(s, a)} + 28A^m(s, a),$$

where $A^m(s, a) = \log(|S|A|N^m(s, a)/\delta)|N^m(s, a)$ and $\epsilon_m(s' | s, a)$ gives the size of the confidence set.

In order to run OMD without knowledge of the transition function, we must extend the definition of occupancy measures to state-action-state triplets $[4]$ as follows,

$$q^{P, \pi}(s, a, s') = \lim_{T \to \infty} E\left[ \sum_{t=1}^{T} I\{s_t = s, a_t = a, s_{t+1} = s'\} | P, \pi, s_1 = s_0 \right].$$

Now an occupancy measure corresponds to a policy-transition function pair. The inverse mapping is given by $\pi^q(a | s) = \frac{q(s, a)q(s)}{q(s)}$ and $P^q(s' | s, a) = \frac{q(s, a, s')}{q(s, a)}$, where $q(s, a) = \sum_{s' \in S^+} q(s, a, s')$. Thus, we can use the OMD update of Eq. (4) over the set $\Delta_m(D/c_{\min})$—occupancy measures $q$ whose induced transition function $P^q$ is in the confidence set of interval $m$ and the expected time of $\pi^q$ (w.r.t $P^q$) from $s_0$ to the goal state is at most $D/c_{\min}$, i.e.,

$$q_{k+1} = q^{P_{k+1}, \pi_{k+1}} = \arg\min_{q \in \Delta_m(k+1)(D/c_{\min})} q KL(q \| q_k),$$

where $m(k)$ denotes the interval at the beginning of episode $k$. As in Section 4.1 this update can be performed in two steps, where the unconstrained minimization step is identical to Eq. (5) and the projection step is implemented similarly to Eq. (6) but with different constraints. Specifically, we accommodate the constraints of Eqs. (7) and (8) for extended occupancy measures (see [4]), and show that a set of linear constraints can express the condition that $P^q$ is in the confidence set (see details in Appendix 1). Note that $D$ can not be computed without knowing the transition function. Here we assume $D$ is known, and in Section 6 we remove this assumption.

Similarly to SSP-O-REPS2, once we reach a state whose expected time to the goal is too long, we want to switch to the fast policy. However, since $P$ is unknown we cannot compute $T^\pi_k$ or the fast policy. Instead, we use the expected time of $\pi_k$ w.r.t $P_k$ which we denote by $T^\pi_k$, and the optimistic fast policy $\pi^m_k$. This policy (together with the optimistic fast transition function) minimizes the expected time to the goal out of all pairs of policies and transition functions from the confidence set. Its computation is done similarly to [4], and we describe it in details in Appendix 1.

To ensure that the algorithm reaches the goal state in every episode, we distinguish between known and unknown states. Every state $s \in S$ starts as unknown, and becomes known once the number of visits to $(s, a)$ reaches $\alpha \frac{D|S|}{c_{\max}} \log \frac{D|S|A}{\delta c_{\max}}$ for all actions $a \in A$ (for some constant $\alpha > 0$). Once an unknown state is reached we play the action that was played the least up to this point in order to advance this state to become known. Then, we update the confidence set and switch to the optimistic fast policy which we recompute. The definition of known states is important because once all states are known, the optimistic fast policy will be proper with high probability. We do not show this directly, but take advantage of this fact through the intervals. We also recompute the optimistic fast policy once the number of visits to some state-action pair is doubled, similarly to UCRL2.

To summarize, we start each episode $k$ by playing $\pi_0$ computed in Eq. (9). When we reach a state $s$ such that $\tilde{T}_k^\pi(s) \geq D/c_{\min}$, we switch to the optimistic fast policy. In addition, when an unknown state is reached we play the least played action, recompute the optimistic fast policy and play it. Finally, once the number of visits to some
state-action pair is doubled we also recompute the optimistic fast policy (see full pseudo-code in Appendix G). Next, we give an overview of the regret analysis for SSP-O-REPS3, which yields the following regret bound (full proof in Appendix H).

**Theorem 5.1.** Under Assumptions 7 and 2 running SSP-O-REPS3 with known SSP-diameter $D$ and \( \eta = \sqrt{\frac{6 \log(D|S||A|/c_{\text{min}})}{K}} \) ensures that, with probability at least \( 1 - \delta \),

\[
R_K \leq O\left(\frac{D|S|\sqrt{|A|K} \log K D|S||A|}{\delta c_{\text{min}}} + \frac{D^2|S|^2|A|}{c_{\text{min}}^2} \log^2 \frac{K D|S||A|}{\delta c_{\text{min}}}\right) = \tilde{O}\left(\frac{D|S|\sqrt{|A|K}}{c_{\text{min}}\sqrt{|A|K}}\right),
\]

where the last equality holds for \( K \geq D^2|S|^2|A|/c_{\text{min}}^2 \).

We have two objectives in our analysis: bounding the number of steps \( T \) taken by the algorithm (to show that we reach the goal in every episode) and bounding the regret. A natural approach to bound \( T \) is to bound the time of every interval (since there is a finite number of intervals). However, we bound the regret directly and this yields a finite number of steps (since the costs are strictly positive). We start by showing that the confidence sets contain \( P \) with high probability, which is a common result (see, e.g., [2, 13, 14]). Define \( \Omega^m \) the event that \( P \) is in the confidence set of interval \( m \).

**Lemma 5.2 (Lemma 4.2).** With probability at least \( 1 - \delta/2 \), the event \( \Omega^m \) holds for all intervals \( m \) simultaneously.

There are two dependant probabilistic events that are important for the analysis. The first are the events \( \Omega^m \), and the second is that the deviation in the cost of a given policy from its expected value is not large. To disentangle these events we define an alternative regret for every \( M = 1, 2, \ldots \),

\[
\tilde{R}_M = \sum_{m=1}^M \sum_{h=1}^{H_m} \tilde{\pi}_m(a \mid s^m_h) c_m(s^m_h, a) \|\Omega^m\} - \sum_{k=1}^K J^*_{k}(s_0),
\]

where \( c_m = c_k \) for the episode \( k \) that interval \( m \) belongs to, \( \tilde{\pi}_m \) is the policy followed by the learner in interval \( m \) and \( U^m = (s^m_1, a^m_1, \ldots, s^m_{H_m}, a^m_{H_m}, s^m_{H_m+1}) \) is the trajectory visited in interval \( m \).

A bound on \( \tilde{R}_M \) yields a bound on \( R_K \) by Lemma 5.2 and an application of Azuma inequality, when \( M \) is the number of intervals in which the first \( K \) episodes elapse (we show that the learner indeed completes these \( K \) episodes).

Instead of bounding the length of each interval we introduce artificial intervals. That is, an interval \( m \) also ends at the first time step \( \tilde{H} \) such that \( \sum_{h=1}^{H_m} \sum_{a \in A} \tilde{\pi}_m(a \mid s^m_h) c_m(s^m_h, a) \geq D/c_{\text{min}} \). The artificial intervals are only introduced for the analysis and do not affect the algorithm. Now the length of each interval is bounded by \( 2D/c_{\text{min}}^2 \) and we can bound the number of intervals as follows.

**Observation 5.3.** Denote \( \tilde{C}_M = \sum_{m=1}^M \sum_{h=1}^{H_m} \tilde{\pi}_m(a \mid s^m_h) c_m(s^m_h, a) \). The total time satisfies \( T \leq \tilde{C}_M / c_{\text{min}} \) and the total number of intervals satisfies

\[
M \leq \frac{c_{\text{min}} \tilde{C}_M}{D} + 2|S||A| \log T + 2K + \frac{D|S|^2|A|}{c_{\text{min}}^2} \log \left(\frac{D|S||A|}{\delta c_{\text{min}}}\right).
\]

Note that for an artificial interval \( m \), \( \Omega^m = \Omega^{m-1} \) since there is no update to the confidence set. Next we bound \( \tilde{C}_M \) as a function of the number of intervals \( M \). Through summation of our confidence bounds, and by showing that the variance in each interval is bounded by \( D/c_{\text{min}} \) we are able to obtain the following bound when Lemma 5.2 holds,

\[
\tilde{C}_M \leq \sum_{k=1}^K \langle q_k, c_k \rangle \mathbb{I}\{\Omega^{m(k)}\} + \tilde{O}\left(\frac{D|S|\sqrt{|A|K}}{c_{\text{min}}\sqrt{|A|K}} + \frac{D^2|S|^2|A|}{c_{\text{min}}^2}\right).
\]

Substituting in Observation 5.3 and solving for \( \tilde{C}_M \) we get

\[
\tilde{R}_M = \tilde{C}_M - \sum_{k=1}^K J^*_{k}(s_0) \leq \sum_{k=1}^K \langle q_k - q^{P^*, c_k} \rangle \mathbb{I}\{\Omega^{m(k)}\} + \tilde{O}\left(\frac{D|S|\sqrt{|A|K}}{c_{\text{min}}\sqrt{|A|K}} + \frac{D^2|S|^2|A|}{c_{\text{min}}^2}\right),
\]

and the final bound is obtained by an OMD analysis for the first term.
5.1 Adversarial vs. stochastic costs in SSP

The regret bound for SSP with stochastic costs is obtained by the optimism in face of uncertainty principle [1][2]. It highly builds on the fact that the cost function is either known in advance or can be estimated faster than the transition function. Then, the optimism is used as a mechanism in the estimation of the transition function, and is with respect to the known cost function.

In adversarial SSP our main mechanism must be OMD (or similar methods from online learning) to handle the arbitrarily changing cost functions. The optimism is used as a secondary mechanism, as we still need to estimate the fixed transition function and make sure that the learner reaches the goal state in every episode. The main challenge is accommodating the optimistic framework and known states tracking [2], to the main method in which we pick policies — online mirror descent.

On a technical level, some challenges arise when incorporating OMD in our policy selection. For the length of every interval \( m \) to be bounded, we need the expected time of \( \tilde{m} \) w.r.t the transition function that was involved in the computation of \( \tilde{m} \) (e.g., \( P_k \) in the first interval of episode \( k \)) to be bounded. When the cost function is stochastic this is guaranteed by optimism, but here we force it by switching to the optimistic fast policy once a “bad” state is reached. However, it means that we must know (or estimate) the SSP-diameter. This is guaranteed by optimism, but here we force it by switching to the optimistic fast policy once a “bad” state is reached. However, it means that we must know (or estimate) the SSP-diameter. Finally, the fact that we must use stochastic policies (otherwise the adversary can take advantage of our determinism) forces us to make the known states tracking explicit, i.e., recompute the optimistic fast policy every time an unknown state is reached to make sure the visits count advances for all actions. In SSP with stochastic costs, this can be avoided by recomputing the optimistic policy only when the number of visits to some state-action pair is doubled, since all the policies used by the learner are deterministic.

6 Relaxation of assumptions

**Estimating the SSP-diameter.** A key point in the analysis is that the sets on which we perform OMD \( \Delta_m(D/c_{\min}) \) contain the occupancy measure of the best policy in hindsight \( q^{\pi^*} \) (with high probability). To that end, we chose \( D/c_{\min} \) as an upper bound on \( T^{\pi^*}(s_0) \) (see Appendix E). Once \( D \) is unknown to the learner, we compute an upper bound \( \tilde{D} \) on the expected time of the fast policy \( T^{\pi^*}(s_0) \), and then \( \tilde{D}/c_{\min} \) will be an upper bound on \( T^{\pi^*}(s_0) \).

We dedicate the first \( L \) episodes to estimating this upper bound \( \tilde{D} \), before running SSP-O-REPS3. Notice that \( \pi^f \) is the optimal policy w.r.t the constant cost function \( c(s, a) = 1 \), and its expected cost is \( T^{\pi^f}(s_0) \). Thus, to compute \( \tilde{D} \) we run an algorithm for regret minimization in regular SSPs for \( L \) episodes with this cost function, and set \( \tilde{D} \) to be the average cost per episode times 10.

By the regret bound of [2], we can set \( L = \Theta(\max\{|S|\sqrt{|A|/K/c_{\min}}, |S|^2|A|/c_{\min}^2\}) \) without suffering additional regret. In Appendix I we show that this yields the two properties we desire for large enough \( K \). First, \( \tilde{D} \) is an upper bound on \( T^{\pi^*}(s_0) \) with probability at least \( 1 - \delta \). Second, \( \tilde{D} \leq O(D) \) with probability at least \( 1 - \delta \). Therefore, we get the same regret bound as in Theorem 5.1.

**Zero costs.** Similarly to [1][2], we can eliminate Assumption 2 by applying a perturbation to the instantaneous costs. That is, instead of \( c_k \) we use the cost function \( \tilde{c}_k(s, a) = \max\{c_k(s, a), \epsilon\} \) for some \( \epsilon > 0 \). Thus, Assumption 2 holds with \( c_{\min} = \epsilon \), but we introduced additional bias into the model. Choosing \( \epsilon = \Theta(K^{-1/4}) \) ensures that all our algorithms obtain regret bounds of \( \tilde{O}(K^{3/4}) \) for the general case (see details in Appendix I).

7 Conclusions and future work

In this paper we present the first algorithms to achieve sub-linear regret for stochastic shortest path with arbitrarily changing cost functions. However, our regret bounds are still far from optimal. For the general case where costs might be zero, our algorithms obtain only \( \tilde{O}(K^{3/4}) \) regret, but this gap is a consequence of our sub-optimal regret when assuming strictly positive costs. Specifically, it stems from bounding the expected time of the best policy in hindsight.
by $D/c_{\min}$. Other approaches may try to bound the running time of the learner’s policies only implicitly, e.g., in regular SSP this is done with the use of optimism.

When the transition function is known, our regret bound is worse by only a factor of $1/c_{\min}$ from the lower bound of $\Omega(D\sqrt{K})$ that follows from the episodic case (see, e.g., [3]). For unknown transitions our regret bound matches, again up to a factor of $1/c_{\min}$, the regret of [2]. For this case, the lower bound is $\Omega(D\sqrt{|S||A|K})$ [2], so the gap is by a factor of $\sqrt{|S|}/c_{\min}$.

Closing this gap is the natural open problem that arises from this paper. The second direction that should be studied is bandit feedback. In this work we assumed that the entire cost function is revealed to the learner in the end of the episode, i.e., full information feedback. However, in many natural applications, the learner only observes the costs associated with the actions it took – this is called bandit feedback. Extending our results to bandit feedback is not trivial, even when the transition function is known, and is left for future work. Finally, it is of great importance to see if policy optimization methods can also obtain regret bounds in adversarial SSPs as done in adversarial MDPs recently [22] [15], since they are widely used in practice.
Broader Impact

This work does not present any foreseeable societal consequence.

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A Implementation details for SSP-O-REPS

A.1 Computing $q_k$

Before describing the algorithm, some more definitions are in order. First, define $\text{KL}(q \parallel q')$ as the unnormalized Kullback–Leibler divergence between two occupancy measures $q$ and $q'$:

$$\text{KL}(q \parallel q') = \sum_{s \in S} \sum_{a \in A} q(s, a) \log \frac{q(s, a)}{q'(s, a)} + q'(s, a) - q(s, a).$$

Furthermore, let $R(q)$ define the unnormalized negative entropy of the occupancy measure $q$:

$$R(q) = \sum_{s \in S} \sum_{a \in A} q(s, a) \log q(s, a) - q(s, a).$$

SSP-O-REPS chooses its occupancy measures as follows:

$$q_1 = q^{\pi_1} = \arg\min_{q \in \Delta(D/c_{\text{min}})} R(q),$$

$$q_{k+1} = q^{\pi_{k+1}} = \arg\min_{q \in \Delta(D/c_{\text{min}})} \eta (q, c_k) + \text{KL}(q \parallel q_k).$$

As shown in [3], each of these steps can be split into an unconstrained minimization step, and a projection step. Thus, $q_1$ can be computed as follows:

$$q'_1 = \arg\min_q R(q),$$

$$q_1 = \arg\min_{q \in \Delta(D/c_{\text{min}})} \text{KL}(q \parallel q'_1),$$

where $q'_1$ has a closed-form solution $q'_1(s, a) = 1$ for every $s \in S$ and $a \in A$. Similarly, $q_{k+1}$ is computed as follows for every $k = 1, \ldots, K - 1$:

$$q'_{k+1} = \arg\min_q \eta (q, c_k) + \text{KL}(q \parallel q_k),$$

$$q_{k+1} = \arg\min_{q \in \Delta(D/c_{\text{min}})} \text{KL}(q \parallel q'_{k+1}),$$

where again $q_{k+1}$ has a closed-form solution $q_{k+1}(s, a) = q_k(s, a)e^{-\eta c_k(s, a)}$ for every $s \in S$ and $a \in A$.

Therefore, we just need to show that the projection step can be computed efficiently (the implementation follows [3]). We start by formulating the projection step as a constrained convex optimization problem:

$$\begin{align*}
\min_q \quad & \text{KL}(q \parallel q_{k+1}) \\
\text{s.t.} \quad & \sum_{a \in A} q(s, a) - \sum_{s' \in S} \sum_{a' \in A} P(s \mid s', a')q(s', a') = \mathbb{I}\{s = s_0\} \quad \forall s \in S \\
& \sum_{s \in S} \sum_{a \in A} q(s, a) \leq \frac{D}{c_{\text{min}}} \\
& q(s, a) \geq 0 \quad \forall (s, a) \in S \times A
\end{align*}$$

\[12\]
To solve the problem, consider the Lagrangian:

\[
L(q, \lambda, v) = \text{KL}(q \parallel q_{k+1}') + \lambda \left( \sum_{s \in S} \sum_{a \in A} q(s, a) - \frac{D}{c_{\min}} \right) + \sum_{s \in S} v(s) \left( \sum_{s' \in S} \sum_{a' \in A} P(s'\mid s', a')q(s, a') + \mathbb{I}\{s = s_0\} - \sum_{a \in A} q(s, a) \right)
\]

\[
= \text{KL}(q \parallel q_{k+1}') + \sum_{s \in S} \sum_{a \in A} q(s, a) \left( \lambda + \sum_{s' \in S} P(s' \mid s, a)v(s') - v(s) \right) + v(s_0) - \frac{\lambda D}{c_{\min}}
\]

where \(\lambda\) and \(\{v(s)\}_{s \in S}\) are Lagrange multipliers. Differentiating the Lagrangian with respect to any \(q(s, a)\), we get

\[
\frac{\partial L(q, \lambda, v)}{\partial q(s, a)} = \log q(s, a) - \log q_{k+1}'(s, a) + \lambda + \sum_{s' \in S} P(s' \mid s, a)v(s') - v(s).
\]

Hence, setting the gradient to zero, we obtain the formula for \(q_{k+1}(s, a)\):

\[
q_{k+1}(s, a) = q_{k+1}'(s, a)e^{-\lambda - \sum_{s' \in S} P(s'\mid s, a)v(s')} + v(s)
\]

\[
= q_k(s, a)e^{-\lambda - \eta c_k(s, a) - \sum_{s' \in S} P(s'\mid s, a)v(s')} + v(s)
\]

\[
= q_k(s, a)e^{-\lambda + \eta c_k(s, a)}, \tag{10}
\]

where the second equality follows from the formula of \(q_{k+1}'(s, a)\), and setting \(c_0(s, a) = 0\) and \(q_0(s, a) = 1\) for every \(s \in S\) and \(a \in A\). The last equality follows by defining \(B_k^c(s, a) = v(s) - \eta c_k(s, a) - \sum_{s' \in S} P(s'\mid s, a)v(s')\).

We now need to compute the value of \(\lambda\) and \(v\) at the optimum. To that end, we write the dual problem \(D(\lambda, v) = \min_q L(q, \lambda, v)\) by substituting \(q_{k+1}\) back into \(L\):

\[
D(\lambda, v) = \sum_{s \in S} \sum_{a \in A} q_{k+1}'(s, a) - \sum_{s \in S} \sum_{a \in A} q_{k+1}(s, a) + v(s_0) - \frac{\lambda D}{c_{\min}}
\]

\[
= - \sum_{s \in S} \sum_{a \in A} q_k(s, a)e^{-\lambda + \eta c_k(s, a) + v(s_0)} - \frac{\lambda D}{c_{\min}} + \sum_{s \in S} \sum_{a \in A} q_{k+1}'(s, a).
\]

Now we obtain \(\lambda\) and \(v\) by maximizing the dual. Equivalently, we can minimize the negation of the dual (and ignore the term \(\sum_{s \in S} \sum_{a \in A} q_{k+1}'(s, a)\)), that is:

\[
\lambda_{k+1}, v_{k+1} = \arg \min_{\lambda \geq 0, v} \sum_{s \in S} \sum_{a \in A} q_k(s, a)e^{-\lambda + \eta c_k(s, a) + v(s_0)} + \frac{\lambda D}{c_{\min}} - v(s_0).
\]

This is a convex optimization problem with only non-negativity constraints (and no constraints about the relations between the variables), which can be solved efficiently using iterative methods like gradient descent [25].

### A.2 Computing the SSP-diameter and the fast policy

The fast policy \(\pi^f\) is a deterministic stationary policy that minimizes the time to the goal state from all states simultaneously (its existence is similar to regular MDPs, for a detailed proof see [8]). Thus, \(\pi^f\) is the optimal policy w.r.t the constant cost function \(c(s, a) = 1\) for every \(s \in S\) and \(a \in A\).

Finding the optimal policy of an SSP instance is known as the planning problem. By [8], this problem can be solved efficiently using Linear Programming (LP), Value Iteration (VI) or Policy Iteration (PI).

The SSP-diameter \(D\) is an upper bound on the expected time it takes to reach the goal from some state, and therefore \(D = \max_{s \in S} T^{\pi^f}(s)\). Thus, in order to compute \(\pi^f\) and \(D\) we need to perform the following steps:
1. Compute the optimal policy $\pi^f$ w.r.t the constant cost function $c(s, a) = 1$, using LP or VI.

2. Compute $T^{\pi^f}(s)$ for every $s \in S$ by solving the linear Bellman equations:

$$T^{\pi^f}(s) = 1 + \sum_{a \in A} \sum_{s' \in S} \pi^f(a | s) P(s' | s, a) T^{\pi^f}(s') \quad \forall s \in S.$$ 

3. Set $D = \max_{s \in S} T^{\pi^f}(s)$.
Algorithm 2 SSP-O-REPS

**input:** state space $S$, action space $A$, minimal cost $c_{\text{min}}$, optimization parameter $\eta$.

**initialization:**
compute the SSP-diameter $D$ (see Appendix A.2).
set $q_0(s,a) = 1$ and $c_0(s,a) = 0$ for every $(s,a) \in S \times A$.

for $k = 1, 2, \ldots$ do
compute $\lambda_k, v_k$ as follows (using, e.g., gradient descent):

$$
\lambda_k, v_k = \arg \min_{\lambda \geq 0, v} \sum_{s \in S} \sum_{a \in A} q_{k-1}(s,a) e^{-\lambda + B_k^{v}(s,a)} + \lambda \frac{D}{c_{\text{min}}} - v(s_0),
$$

where $B_k^{v}(s,a) = v(s) - \eta c_k(s,a) - \sum_{s' \in S} P(s' | s,a) v(s')$.
compute $q_k$ as follows for every $(s,a) \in S \times A$:

$$
q_k(s,a) = q_{k-1}(s,a) e^{-\lambda_k + B_k^{v_k}(s,a)},
$$

compute $\pi_k$ as follows for every $(s,a) \in S \times A$:

$$
\pi_k(a | s) = \frac{q_k(s,a)}{\sum_{a' \in A} q_k(s,a')},
$$

set $s^k_1 \leftarrow s_0$, $i \leftarrow 1$.

while $s^k_i \neq g$ do
play action according to $\pi_k$, i.e., $a^k_i \sim \pi_k(\cdot | s^k_i)$.
observe next state $s^k_{i+1} \sim P(\cdot | s^k_i, a^k_i)$, $i \leftarrow i + 1$.
end while

set $I^k \leftarrow i - 1$.
observe cost function $c_k$, and suffer cost $\sum_{j=1}^{I_k} c_k(s^k_j, a^k_j)$.
end for
\section{Proofs for Section 4.1}

\begin{Lemma}[C.1] It holds that $q^\pi \in \Delta\left(\frac{D}{c_{\min}}\right)$. \end{Lemma}

\begin{Proof} Denote by $\pi^f$ the fast policy, i.e., $\pi^f = \arg\min_{\pi \in \Pi_{\text{proper}}} T^\pi(s_0)$. By definition of the SSP-diameter we have that $T^\pi(s_0) \leq D$. Now, recall that $\pi^*$ is the best policy in hindsight and therefore

\begin{align}
\frac{1}{K} \sum_{k=1}^{K} J^\pi_k(s_0) \leq \frac{1}{K} \sum_{k=1}^{K} J^\pi_k(s_0) \leq \frac{1}{K} \sum_{k=1}^{K} T^\pi(s_0) \leq D, \tag{11}
\end{align}

where the second inequality follows because $c_k(s, a) \leq 1$. However, we also have that $c_k(s, a) \geq c_{\min}$ and therefore $J^\pi_k(s_0) \geq c_{\min} T^\pi_k(s_0)$. Thus, combining with Eq. (11) we obtain

$$c_{\min} T^\pi_k(s_0) \leq \frac{1}{K} \sum_{k=1}^{K} J^\pi_k(s_0) \leq D.$$  

This finishes the proof since $T^\pi_k(s_0) \leq \frac{D}{c_{\min}}$. \end{Proof}

\subsection{Proof of Theorem 4.1}

\begin{Lemma} Let $\tau \geq 1$. For every $q \in \Delta(\tau)$ it holds that $R(q) \leq \tau \log \tau$. \end{Lemma}

\begin{Proof} \begin{align*}
R(q) & = \sum_{s \in S} \sum_{a \in A} q(s, a) \log q(s, a) - \sum_{s \in S} \sum_{a \in A} q(s, a) \\
& \leq \sum_{s \in S} \sum_{a \in A} q(s, a) \log q(s, a) \\
& = \sum_{s \in S} \sum_{a \in A} q(s, a) \log \frac{q(s, a)}{\tau} + \sum_{s \in S} \sum_{a \in A} q(s, a) \log \tau \\
& \leq \sum_{s \in S} \sum_{a \in A} q(s, a) \log \tau \leq \tau \log \tau
\end{align*}

where the first two inequalities follow from non-positivity, and the last one from the definition of $\Delta(\tau)$. \end{Proof}

\begin{Lemma} Let $\tau \geq 1$. For every $q \in \Delta(\tau)$ it holds that $-R(q) \leq \tau(1 + \log(|S||A|))$. \end{Lemma}

\begin{Proof} Similarly to Lemma C.2 we have that

$$-R(q) = -\sum_{s \in S} \sum_{a \in A} q(s, a) \log \frac{q(s, a)}{\tau} + \sum_{s \in S} \sum_{a \in A} q(s, a) - \sum_{s \in S} \sum_{a \in A} q(s, a) \log \tau$$

$$\leq -\tau \sum_{s \in S} \sum_{a \in A} \frac{q(s, a)}{\tau} \log \frac{q(s, a)}{\tau} + \tau \leq \tau \log(|S||A|) + \tau,$$

where the first inequality follows because the last term is non-positive and from the definition of $\Delta(\tau)$, and the last inequality follows from properties of Shannon’s entropy. \end{Proof}

\begin{Proof}[Theorem 4.1] We start with a fundamental inequality of OMD (see, e.g., [3]) that holds for every $q \in \Delta(D/c_{\min})$ by Lemma C.1, it also holds for $q^\pi$,

$$\sum_{k=1}^{K} \langle q_k - q^\pi, c_k \rangle \leq \sum_{k=1}^{K} \langle q_k - q_{k+1}^l, c_k \rangle + \frac{\text{KL}(q^\pi || q_1)}{\eta}. \tag{12}$$

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For the first term we use the exact form of $q'_{k+1}$ and the inequality $e^x \geq 1 + x$ to obtain

$$q'_{k+1}(s, a) = q_k(s, a)e^{-\eta c_k(s, a)} \geq q_k(s, a) - \eta q_k(s, a)c_k(s, a).$$

We substitute this back and obtain

$$\sum_{k=1}^{K} \langle q_k - q'_{k+1}, c_k \rangle \leq \eta \sum_{k=1}^{K} \sum_{s \in S} \sum_{a \in A} q_k(s, a)c_k(s, a)^2 \leq \eta \sum_{k=1}^{K} \sum_{s \in S} \sum_{a \in A} q_k(s, a).$$

where the last inequality follows from the definition of $\Delta(D/c_{min})$.

Next we use Lemmas C.2 and C.3 to bound the second term of Eq. (12). Recall that $q_1$ minimizes $R$ in $\Delta(D/c_{min})$, this implies that $\langle \nabla R(q_1), q^\pi - q_1 \rangle \geq 0$ because otherwise we could decrease $R$ by taking small step in the direction $q^\pi - q_1$. Thus we obtain

$$KL(q^\pi \parallel q_1) = R(q^\pi) - R(q_1) - \langle \nabla R(q_1), q^\pi - q_1 \rangle \leq R(q^\pi) - R(q_1) \leq \frac{D}{c_{min}} \log \frac{D}{c_{min}} + \frac{D}{c_{min}}(1 + \log(|S||A|)) \leq \frac{3D}{c_{min}} \log \frac{D|S||A|}{c_{min}}. \quad (14)$$

By substituting Eqs. (13) and (14) into Eq. (12) and using the choice of $\eta$, we obtain,

$$\sum_{k=1}^{K} \langle q_k - q^\pi, c_k \rangle \leq \eta K \frac{D}{c_{min}} + \frac{3D}{c_{min} \eta} \log \frac{D|S||A|}{c_{min}} \leq \frac{2D}{c_{min}} \sqrt{3K \log \frac{D|S||A|}{c_{min}}}. \quad (15)$$

This finishes the proof since

$$\mathbb{E}[R_K] = \mathbb{E} \left[ \sum_{k=1}^{K} \langle q_k - q^\pi, c_k \rangle \right].$$

C.2 SSP-O-REPS picks proper policies

For every policy $\pi_k$ chosen by SSP-O-REPS it holds that $T^\pi_k(s_0) \leq D/c_{min}$. If there exists some state $s \in S$ such that $T^\pi_k(s) = \infty$, then the probability to reach it must be zero, since otherwise $T^\pi_k(s_0) = \infty$. Thus there exists $B > 0$ such that if $s$ is reachable from $s_0$ using $\pi_k$ then $T^\pi_k(s) \leq B$. By Lemma E.1 this implies that the goal state will be reached in every episode with probability 1. Thus, all policies chosen by SSP-O-REPS are proper.
D Pseudo-code for SSP-O-REPS2
Algorithm 3 SSP-O-REPS2

**input:** state space $S$, action space $A$, minimal cost $c_{\text{min}}$, optimization parameter $\eta$.

**initialization:**
compute the SSP-diameter $D$ and the fast policy $\pi^f$ (see Appendix A.2).
set $q_0(s, a) = 1$ and $c_0(s, a) = 0$ for every $(s, a) \in S \times A$.

for $k = 1, 2, \ldots$ do
compute $\lambda_k, v_k$ as follows (using, e.g., gradient descent):

$$\lambda_k, v_k = \arg \min_{\lambda \geq 0, v} \sum_{s \in S} \sum_{a \in A} q_{k-1}(s, a) e^{-\lambda + B_{k-1}^v(s, a)} + \lambda \frac{D}{c_{\text{min}}} - v(s_0),$$

where $B_{k}^v(s, a) = v(s) - \eta v_k(s, a) - \sum_{s' \in S} P(s' | s, a) v(s')$.
compute $q_k$ as follows for every $(s, a) \in S \times A$:

$$q_k(s, a) = q_{k-1}(s, a) e^{-\lambda_k + B_{k-1}^v(s, a)}.$$
compute $\pi_k$ as follows for every $(s, a) \in S \times A$:

$$\pi_k(a \mid s) = \frac{q_k(s, a)}{\sum_{a' \in A} q_k(s, a')}.$$
set $T^{\pi_k}(s) \leftarrow \frac{D}{c_{\text{min}}}$ for every $s \in S$ such that $\sum_{a \in A} q^{\pi_k}(s, a) = 0$.
compute $T^{\pi_k}$ by solving the following linear equations:

$$T^{\pi_k}(s) = 1 + \sum_{a \in A} \sum_{s' \in S} \pi_k(a \mid s) P(s' \mid s, a) T^{\pi_k}(s') \quad \forall s \in \{s \in S : \sum_{a \in A} q^{\pi_k}(s, a) > 0\}.$$
set $s_k^1 \leftarrow s_0, i \leftarrow 1$.
while $s_k^i \neq g$ and $T^{\pi_k}(s_k^i) < \frac{D}{c_{\text{min}}}$ do
play action according to $\pi_k$, i.e., $a_k^i \sim \pi_k(\cdot \mid s_k^i)$.
obs next state $s_{k+1}^i \sim P(\cdot \mid s_k^i, a_k^i), i \leftarrow i + 1$.
end while
while $s_k^i \neq g$ do
play action according to $\pi^f$, i.e., $a_k^i \sim \pi^f(\cdot \mid s_k^i)$.
obs next state $s_{k+1}^i \sim P(\cdot \mid s_k^i, a_k^i), i \leftarrow i + 1$.
end while
set $k \leftarrow i - 1$.
obs cost function $c_k$, and suffer cost $\sum_{j=1}^k c_k(s_j^i, a_j^i)$.
end for
E Proofs for Section 4.2

E.1 Proof of Lemma 4.2

Lemma E.1. Let $\sigma$ be a strategy such that the expected time of reaching the goal state when starting at state $s$ is at most $\tau$ for every $s \in S$. Then, the probability that $\sigma$ takes more than $m$ steps to reach the goal state is at most $2e^{-\frac{m}{\tau}}$.

Proof. By Markov inequality, the probability that $\sigma$ takes more than $2\tau$ steps before reaching the goal state is at most $1/2$. Iterating this argument, we get that the probability that $\sigma$ takes more than $2k\tau$ steps before reaching the goal state is at most $2^{-k}$ for every integer $k \geq 0$. In general, for any $m \geq 0$, the probability that $\sigma$ takes more than $m$ steps before reaching the goal state is at most $2^{-\frac{m}{\tau}} \leq 2e^{-\frac{m}{\tau}}$. □

Lemma E.2. For every $k = 1, \ldots, K$, the strategy $\sigma_k$ of the learner ensures that the expected time to the goal state from any initial state is at most $D/c_{\min}$.

Proof. Define $X_k = \sum_{i=1}^{I_k} c_k(s_i^k, a_i^k) - \mathbb{E} \left[ \sum_{i=1}^{I_k} c_k(s_i^k, a_i^k) \mid P, \sigma_k, s_1^k = s_0 \right]$. This is a martingale difference sequence, and in order to use Theorem 1.5 we need to show that $\Pr[|X_k| > m] \leq 2e^{-\frac{m}{\tau}}$ for every $k = 1, 2, \ldots$ and $m \geq 0$. This follows immediately from Lemmas E.1 and E.2 since the total cost is bounded by the total time.

By Theorem 1.5 $\sum_{k=1}^{K} X_k \leq \frac{4D}{c_{\min}} \sqrt{K \log \frac{4K}{\delta}}$ with probability $1 - \delta$, which gives the Lemma’s statement. □

E.2 Proof of Lemma 4.3

Proof of Lemma 4.3. Until a state $s \in S$ with $T^{\pi_k}(s) \geq D/c_{\min}$ is reached, the strategy $\sigma_k$ is the same as the policy $\pi_k$. If such a state is reached then $J_k^{\pi_k}(s) \geq c_{\min}T^{\pi_k}(s) \geq c_{\min} \frac{D}{c_{\min}} = D$, where the first inequality is because all costs are bounded from below by $c_{\min}$. On the other hand, $J_k^{\pi_k}(s) \leq T^{\pi_k}(s) \leq D$, where the last inequality follows by the definition of the fast policy and the SSP-diameter. Therefore, $J_k^{\pi_k}(s) \leq J_k^{\pi_k}(s)$. □

E.3 Proof of Theorem 4.4

Proof of Theorem 4.4. We decompose the regret into two terms as follows,

$$R_K = \sum_{k=1}^{K} \sum_{i=1}^{I_k} c_k(s_i^k, a_i^k) - \sum_{k=1}^{K} J_k^{\pi_k}(s_0)$$

$$= \sum_{k=1}^{K} \sum_{i=1}^{I_k} c_k(s_i^k, a_i^k) - \mathbb{E} \left[ \sum_{i=1}^{I_k} c_k(s_i^k, a_i^k) \mid P, \sigma_k, s_1^k = s_0 \right]$$

$$+ \sum_{k=1}^{K} \mathbb{E} \left[ \sum_{i=1}^{I_k} c_k(s_i^k, a_i^k) \mid P, \sigma_k, s_1^k = s_0 \right] - \sum_{k=1}^{K} J_k^{\pi_k}(s_0).$$

The first term accounts for the deviations in the performance of the learner’s strategies from their expected value, and is bounded with high probability using Lemma 4.2.
The second term is the difference between the expected performance of the learner’s strategies and the best policy in hindsight. Using Lemma 4.3 we can bound it as follows,

\[
\sum_{k=1}^{K} \mathbb{E} \left[ \sum_{i=1}^{T_k} c_k(s^k_i, a^k_i) \mid P, \sigma_k, s^k_1 = s_0 \right] - \sum_{k=1}^{K} J^\pi_k(s_0) \leq \sum_{k=1}^{K} J^\pi_k(s_0) - \sum_{k=1}^{K} J^\pi_k(s_0) = \sum_{k=1}^{K} (q^\pi_k - q^\pi, c_k) \\
\leq \frac{2D}{\epsilon_{\min}} \sqrt{3K \log \frac{D|S||A|}{\epsilon_{\min}}},
\]

where the equality follows from Eq. (3) and the last inequality follows from Eq. (15). □
F Implementation details for SSP-O-REPS3

F.1 Computing \( q_k \)

After extending the occupancy measures, we must extend our additional definitions. Define \( KL(q \parallel q') \) as the unnormalized Kullback–Leibler divergence between two occupancy measures \( q \) and \( q' \):

\[
KL(q \parallel q') = \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \log \frac{q(s, a, s')}{q'(s, a, s')} + q'(s, a, s') - q(s, a, s),
\]

where \( S^+ = S \cup \{g\} \). Furthermore, let \( R(q) \) define the unnormalized negative entropy of the occupancy measure \( q \):

\[
R(q) = \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \log q(s, a, s') - q(s, a, s').
\]

SSP-O-REPS3 chooses its occupancy measures as follows:

\[
q_1 = q_{P_1, \pi_1} = \arg \min_{q \in \Delta_{m(1)}(D/c_{\text{num}})} R(q)
\]

\[
q_{k+1} = q_{P_{k+1}, \pi_{k+1}} = \arg \min_{q \in \Delta_{m(k+1)}(D/c_{\text{num}})} \eta(q, c_k) + KL(q \parallel q_k).
\]

As shown in [4], each of these steps can be split into an unconstrained minimization step, and a projection step. Thus, \( q_1 \) can be computed as follows:

\[
q_1' = \arg \min_q R(q)
\]

\[
q_1 = \arg \min_{q \in \Delta_{m(1)}(D/c_{\text{num}})} KL(q \parallel q_1'),
\]

where \( q_1' \) has a closed-from solution \( q_1'(s, a, s') = 1 \) for every \( (s, a, s') \in S \times A \times S^+ \). Similarly, \( q_{k+1} \) is computed as follows for every \( k = 1, \ldots, K - 1 \):

\[
q_{k+1} = \arg \min_{q \in \Delta_{m(k+1)}(D/c_{\text{num}})} \eta(q, c_k) + KL(q \parallel q_k)
\]

\[
q_{k+1} = \arg \min_{q \in \Delta_{m(k+1)}(D/c_{\text{num}})} KL(q \parallel q_{k+1}'),
\]

where again \( q_{k+1}' \) has a closed-from solution \( q_{k+1}'(s, a, s') = q_k(s, a, s')e^{-\eta c_k(s, a)} \) for every \( (s, a, s') \in S \times A \times S^+ \).

Therefore, we just need to show that the projection step can be computed efficiently (the implementation follows [4] [5]). We start by formulating the projection step as a constrained convex optimization problem (where \( m = m(k + 1) \)):

\[
\min_q \left\{ KL(q \parallel q_{k+1}') \right\}
\]

\[
s.t.
\sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') - \sum_{s' \in S^+} q(s', a', s) = 1 \{s = s_0\} \\forall s \in S
\]

\[
q(s, a, s') \leq \left( \bar{P}_m(s' | s, a) + \epsilon_m(s' | s, a) \right) \sum_{s'' \in S^+} q(s, a, s'') \quad \forall(s, a, s') \in S \times A \times S^+
\]

\[
q(s, a, s') \geq \left( \bar{P}_m(s' | s, a) - \epsilon_m(s' | s, a) \right) \sum_{s'' \in S^+} q(s, a, s'') \quad \forall(s, a, s') \in S \times A \times S^+
\]

\[
\sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \leq \frac{D}{c_{\text{min}}} \\
q(s, a, s') \geq 0 \quad \forall(s, a, s') \in S \times A \times S^+
\]
To solve the problem, consider the Lagrangian:

\[
\mathcal{L}(q, \lambda, v, \mu) = \text{KL}(q \parallel q'_{k+1}) + \lambda \left( \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') - \frac{D}{t_{\text{min}}} \right) \\
+ \sum_{s \in S} v(s) \left( \sum_{s' \in S} q(s', a', s) + \mathbb{1}\{s = s_0\} - \sum_{a \in A, s' \in S^+} q(s, a, s') \right) \\
+ \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} \mu^+(s, a, s') \left( q(s, a, s') - (\bar{P}_m(s' | s, a) + \epsilon_m(s' | s, a)) \sum_{s'' \in S^+} q(s, a, s'') \right) \\
+ \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} \mu^- (s, a, s') \left( (\bar{P}_m(s' | s, a) - \epsilon_m(s' | s, a)) \sum_{s'' \in S^+} q(s, a, s'') - q(s, a, s') \right) \\
= \text{KL}(q \parallel q'_{k+1}) + v(s_0) - \lambda \frac{D}{t_{\text{min}}} \\
+ \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \left( \lambda + v(s') - v(s) + \mu^+(s, a, s') - \mu^-(s, a, s') \right) \\
- \sum_{s'' \in S^+} \bar{P}_m(s'' | s, a) (\mu^+(s, a, s'') - \mu^-(s, a, s'')) \\
- \sum_{s'' \in S^+} \epsilon_m(s'' | s, a) (\mu^+(s, a, s'') + \mu^-(s, a, s''))
\]

where \(\lambda, \{v(s)\}_{s \in S}, \{\mu^+(s, a, s')\}_{(s, a, s') \in S \times A \times S^+} \) and \(\{\mu^-(s, a, s')\}_{(s, a, s') \in S \times A \times S^+}\) are Lagrange multipliers, and we set \(v(g) = 0\) for convenience. Differentiating the Lagrangian with respect to any \(q(s, a, s')\), we get

\[
\frac{\partial \mathcal{L}(q, \lambda, v, \mu)}{\partial q(s, a, s')} = \log \frac{q(s, a, s')}{q_{k+1}(s, a, s')} + \lambda + v(s') - v(s) + \mu^+(s, a, s') - \mu^-(s, a, s') \\
- \sum_{s'' \in S^+} \bar{P}_m(s'' | s, a) (\mu^+(s, a, s'') - \mu^-(s, a, s'')) \\
- \sum_{s'' \in S^+} \epsilon_m(s'' | s, a) (\mu^+(s, a, s'') + \mu^-(s, a, s''))
\]

Next we define

\[
B_k^{v, \mu}(s, a, s') = v(s) - v(s') + \mu^-(s, a, s') - \mu^+(s, a, s') - \eta \epsilon m(s, a) \\
+ \sum_{s'' \in S^+} \bar{P}_m(s'' | s, a) (\mu^+(s, a, s'') - \mu^-(s, a, s'')) \\
+ \sum_{s'' \in S^+} \epsilon_m(s'' | s, a) (\mu^+(s, a, s'') + \mu^-(s, a, s'')). \tag{16}
\]

Hence, setting the gradient to zero, we obtain the formula for \(q_{k+1}(s, a)\):

\[
q_{k+1}(s, a, s') = q'_{k+1}(s, a, s') e^{-\lambda + \eta \epsilon m(s, a) + B_k^{v, \mu}(s, a, s')} \\
= q_k(s, a, s') e^{-\lambda + B_k^{v, \mu}(s, a, s')}, \tag{17}
\]

where the last equality follows from the formula of \(q'_{k+1}(s, a, s')\), and setting \(c_0(s, a) = 0\) and \(q_0(s, a, s') = 1\) for every \((s, a, s') \in S \times A \times S^+\).
We now need to compute the value of \( \lambda, v, \mu \) at the optimum. To that end, we write the dual problem \( D(\lambda, v, \mu) = \min_q L(q, \lambda, v, \mu) \) by substituting \( q_{k+1} \) back into \( L \):

\[
D(\lambda, v, \mu) = \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q_{k+1}'(s, a, s') - \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q_{k+1}(s, a, s') + v(s_0) - \lambda \frac{D}{c_{\text{min}}}
\]

\[
= - \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q_k(s, a, s') e^{-\lambda + B^{v,\mu}_{k}(s, a, s')} + v(s_0) - \lambda \frac{D}{c_{\text{min}}} + \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q_{k+1}'(s, a, s').
\]

Now we obtain \( \lambda, v, \mu \) by maximizing the dual. Equivalently, we can minimize the negation of the dual (and ignore the term \( \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q_{k+1}'(s, a, s') \)), that is:

\[
\lambda_{k+1}, v_{k+1}, \mu_{k+1} = \arg \min_{\lambda \geq 0, v \geq 0, \mu \geq 0} \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q_k(s, a, s') e^{-\lambda + B^{v,\mu}_{k}(s, a, s')} + \lambda \frac{D}{c_{\text{min}}} - v(s_0).
\]

This is a convex optimization problem with only non-negativity constraints (and no constraints about the relations between the variables), which can be solved efficiently using iterative methods like gradient descent [25].

**F.2 Computing the optimistic fast policy**

The optimistic fast policy \( \tilde{\pi}_m \) is a deterministic stationary policy that together with the optimistic fast transition function from the confidence set of interval \( m \), minimizes the time to the goal state from all states simultaneously out of all pairs of policies and transition functions from the confidence set. Essentially, this is the optimal pair of policy and transition function from the confidence set w.r.t. the constant cost function \( c(s, a) = 1 \) for every \( s \in S \) and \( a \in A \).

The existence of the optimistic fast policy is proven in [1], and there they also show that it can be computed efficiently with Extended Value Iteration. In [2], the authors compute the following optimistic fast transition function for every \((s, a, s') \in S \times A \times S\):

\[
\tilde{P}_m(s' \mid s, a) = \max \left\{ 0, \tilde{P}_m(s' \mid s, a) - 28 A^m(s, a) - 4 \sqrt{\tilde{P}_m(s' \mid s, a) A^m(s, a)} \right\},
\]

where the remaining probability mass goes to \( \tilde{P}_m(g \mid s, a) \). Then, \( \tilde{\pi}_m \) is computed by finding the fast policy w.r.t \( \tilde{P}_m \) (see Appendix A.2).

While this method is simpler and more efficient than Extended Value Iteration, the authors do not prove that this is indeed the optimistic fast policy. However, this policy is sufficient for their analysis and for our analysis as well. For simplicity, throughout the analysis we assume that \( \tilde{\pi}_m \) is the optimistic fast policy, but every step of the proof works with this computation as well.
G Pseudo-code for SSP-O-REPS3

Algorithm 4 SSP-O-REPS3

**input:** state and space $S$, action space $A$, minimal cost $c_{\text{min}}$, optimization parameter $\eta$ and confidence parameter $\delta$.

**initialization:**
obtain SSP-diameter $D$ from user or estimate it (see Appendix I).
set $q_0(s, a, s') = 1$ and $c_0(s, a) = 0$ for every $(s, a, s') \in S \times A \times S^+$.
set $m \leftarrow 0$ and for every $(s, a, s') \in S \times A \times S^+$: $N^0(s, a) \leftarrow 0$, $N^0(s, a, s') \leftarrow 0$, $n^0(s, a) \leftarrow 0$, $n^0(s, a, s') \leftarrow 0$.

for $k = 1, 2, \ldots$ do
  $m \leftarrow m + 1$, start new interval \((\text{Algorithm 5})\).
  set $s^k_i \leftarrow s_0$, $i \leftarrow 1$.
  while $s^k_i \neq g$ and $\bar{T}^k_i(s^k_i) < \frac{D}{c_{\text{min}}}$ do
    play action according to $\pi_k$, i.e., $a^k_i \sim \pi_k(\cdot | s^k_i)$.
    observe next state $s^k_{i+1} \sim P(\cdot | s^k_i, a^k_i)$.
    update counters: $n^m(s^k_i, a^k_i) \leftarrow n^m(s^k_i, a^k_i) + 1$, $n^m(s^k_i, a^k_i, s^k_{i+1}) \leftarrow n^m(s^k_i, a^k_i, s^k_{i+1}) + 1$.
    $i \leftarrow i + 1$.
  end while
  if $n^m(s^k_{i-1}, a^k_{i-1}) \geq N^m(s^k_{i-1}, a^k_{i-1})$ or $\exists a. n^m(s^k_i, a) + N^m(s^k_i, a) \leq \alpha \frac{D|S|}{c_{\text{min}}} \log \frac{D|S||A|}{\delta c_{\text{min}}}$ then
    break
  end if
  $m \leftarrow m + 1$, start new interval \((\text{Algorithm 5})\).
  while $s^k_i \neq g$ do
    if $\exists a \in A. n^m(s^k_i, a) + N^m(s^k_i, a) \leq \alpha \frac{D|S|}{c_{\text{min}}} \log \frac{D|S||A|}{\delta c_{\text{min}}}$ then
      play the least played action $a^k_i = \arg \min_{a \in A} n^m(s^k_i, a) + N^m(s^k_i, a)$.
    else
      play according to $\tilde{\pi}^f_i$, i.e., $a^k_i \sim \tilde{\pi}^f_i(\cdot | s^k_i)$.
    end if
    observe next state $s^k_{i+1} \sim P(\cdot | s^k_i, a^k_i)$.
    update counters: $n^m(s^k_i, a^k_i) \leftarrow n^m(s^k_i, a^k_i) + 1$, $n^m(s^k_i, a^k_i, s^k_{i+1}) \leftarrow n^m(s^k_i, a^k_i, s^k_{i+1}) + 1$.
    $i \leftarrow i + 1$.
    if $n^m(s^k_{i-1}, a^k_{i-1}) \geq N^m(s^k_{i-1}, a^k_{i-1})$ or $\exists a. n^m(s^k_{i-1}, a) + N^m(s^k_{i-1}, a) \leq \alpha \frac{D|S|}{c_{\text{min}}} \log \frac{D|S||A|}{\delta c_{\text{min}}}$ then
      $m \leftarrow m + 1$, start new interval \((\text{Algorithm 5})\).
    end if
  end while
  set $I^k_i \leftarrow i - 1$.
  observe cost function $c_k$, and suffer cost $\sum_{j=1}^{I^k_i} c_k(s^k_j, a^k_j)$.
end for
Algorithm 5 START NEW INTERVAL

update counters for every \((s, a, s') \in S \times A \times S^+\):

\[
N_m(s, a) \leftarrow N_{m-1}(s, a) + n_m^{-1}(s, a) \quad ; \quad n_m(s, a) \leftarrow 0
\]

\[
N_m(s, a, s') \leftarrow N_{m-1}(s, a, s') + n_m^{-1}(s, a, s') \quad ; \quad n_m(s, a, s') \leftarrow 0
\]

update confidence set for every \((s, a, s') \in S \times A \times S^+\):

\[
\bar{P}_m(s' \mid s, a) = \frac{N_m(s, a, s')}{N_m(s, a)}
\]

\[
\epsilon_m(s' \mid s, a) = 4 \sqrt{\bar{P}_m(s' \mid s, a) A_m(s, a) + 28 A^m(s, a)}
\]

where \(A^m(s, a) = \log(|S||A|N_m(s, a)/\delta)/N_m(s, a)\).

if \(m\) is the first interval of episode \(k\) then

compute \(\lambda_k, v_k, \mu_k\) as follows (using, e.g., gradient descent):

\[
\lambda_k, v_k, \mu_k = \arg \min_{\lambda \geq 0, v, \mu \geq 0} \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q_{k-1}(s, a, s') e^{-\lambda + B_{k-1}^{v, \mu}(s, a, s')} + \lambda \frac{D}{c_{\min}} - v(s_0),
\]

where \(B_k^{v, \mu}(s, a, s')\) is defined in Eq. (16).

compute \(q_k\) as follows for every \((s, a, s') \in S \times A \times S^+\):

\[
q_k(s, a, s') = q_{k-1}(s, a, s') e^{-\lambda_k + B_k^{v_k, \mu_k}(s, a, s')}.
\]

compute \(\pi_k\) and \(P_k\) as follows for every \((s, a, s') \in S \times A \times S^+\):

\[
\pi_k(a \mid s) = \frac{\sum_{s' \in S^+} q_k(s, a, s')}{\sum_{a' \in A} \sum_{s' \in S^+} q_k(s, a', s')}, \quad P_k(s' \mid s, a) = \frac{q_k(s, a, s')}{\sum_{s'' \in S^+} q_k(s, a, s'')}.
\]

set \(\hat{T}_k(s) \leftarrow \frac{D}{c_{\min}}\) for every \(s \in S\) such that \(\sum_{a \in A} \sum_{s' \in S^+} q_k(s, a, s') = 0\).

compute \(\hat{T}^\pi_k\) by solving the following linear equations:

\[
\hat{T}^\pi_k(s) = 1 + \sum_{a \in A} \sum_{s' \in S} \pi_k(a \mid s) P_k(s' \mid s, a) \hat{T}^\pi_k(s') \quad \forall s \in \{s \in S : \sum_{a \in A} \sum_{s' \in S^+} q_k(s, a, s') > 0\}
\]

else

compute the optimistic fast policy \(\pi^f_m\) (see Appendix F.2).

end if
H Analysis of SSP-O-REPS3 (proofs for Section 5)

H.1 Notations
Denote the trajectory visited in interval \(m\) by \(U^m = (s^m_0, a^m_1, \ldots, s^m_{H^m}, a^m_{H^m}, s^m_{H^m+1})\), where \(a^m_i\) is the action taken in \(s^m_i\), and \(H^m\) is the length of the interval. In addition, the concatenation of trajectories in the intervals up to and including interval \(m\) is denoted by \(U^m = \cup_{m' \leq m} U^{m'}\).

The policy that the learner follows in interval \(m\) is denoted by \(\tilde{\pi}_m\), and the transition function that was involved in the choice of \(\tilde{\pi}_m\) is denoted by \(\tilde{P}_m\). For the first interval of every episode these are chosen by OMD, i.e., \(\pi_k\) and \(P_k\), and for other intervals these are the optimistic fast policy \(\pi_m\) and the transition function chosen from the confidence set together with it. For intervals of unknown states (of length 1) there is no policy since only one action is performed – we ignore visits to unknown states and we suffer their cost directly in Lemma H.3.

The expected cost of \(\tilde{\pi}_m\) w.r.t \(\tilde{P}_m\) is denoted by \(J^m\), and the expected time to the goal is denoted by \(\tilde{T}^m\). For intervals in which we follow the optimistic fast policy, we will show that \(\tilde{T}^m(s) \leq D/c_{\min}\) for every \(s \in S\) when \(\Omega^m\) holds. We would like to have a similar property for intervals in which we follow the OMD policy, i.e., the first interval of every episode.

Note that for the first interval \(m\) of episode \(k\), we have that \(\tilde{T}^k\) is \(\tilde{T}^m\), and recall that reaching a state \(s \in S\) such that \(\tilde{T}^k(s) \geq D/c_{\min}\) ends the current interval. We would like to take advantage of this fact in order to make sure that \(\tilde{T}^m\) is always bounded by \(D/c_{\min}\). Similarly to Section 4.2, we compute \(\tilde{T}^k(s)\) only for states \(s\) that are reachable from \(s_0\) w.r.t \(P_k\). Since reaching a state \(s\) with \(\tilde{T}^k(s) \geq D/c_{\min}\) yields the start of a new interval for which we use the optimistic fast policy, we can set \(\tilde{T}^k(s) = D/c_{\min}\) for states that are not reachable from \(s_0\) without affecting the algorithm’s choices.

We make another change to \(\tilde{P}_m\) for interval \(m\) that is the first interval of episode \(k\). Since reaching a state \(s \in S\) such that \(\tilde{T}^k(s) \geq D/c_{\min}\) ends the interval, we tweak \(\tilde{P}_m\) such that from such a state it goes directly to the goal with expected time of \(D/c_{\min}\) and expected cost of \(D\) (can be done with a self-loop that has \(c_{\min}/D\) probability to go to \(g\)). Thus, when we consider the expected cost of \(\tilde{\pi}_m\) w.r.t \(\tilde{P}_m\), we have that \(\tilde{T}^m(s_0) \leq \tilde{T}^k(s_0)\) because we only decreased the cost from some states. However, notice that now \(\tilde{P}_m\) is in the confidence set only for states that we did not tweak. We show that this does not affect the analysis, since reaching those states ends the interval.

We would like to emphasize that tweaking \(\tilde{P}_m\) is only done in hindsight as a part of the analysis, and does not change the algorithm.

H.2 Properties of the learner’s policies

**Lemma H.1.** Let \(m\) be an interval. If \(m\) is the first interval of episode \(k\) then \(\tilde{T}^m(s) \leq D/c_{\min}\) for every \(s \in S\). Otherwise, if \(\Omega^m\) holds then \(\tilde{T}^m(s) \leq D\) for every \(s \in S\).

**Proof.** The first case holds by definition of \(\tilde{P}_m\) for intervals that are in the beginning of some episode (see discussion in Appendix H.1). The second case follows by optimism and the fact that \(\tilde{P}\) is in the confidence set (see [2], Lemma B.2).

**Lemma H.2.** Let \(m\) be an interval and let \(1 \leq h \leq H^m\). If \(\Omega^m\) holds then the following Bellman equations hold:

\[
\tilde{T}^m(s_h^m) = 1 + \sum_{a \in A} \tilde{\pi}_m(a \mid s_h^m) c_{\min}(s_h^m, a) + \sum_{a \in A} \sum_{s' \in S} \tilde{\pi}_m(a \mid s_h^m) \tilde{P}_m(s' \mid s_h^m, a) \tilde{T}^m(s')
\]

\[
\tilde{T}^m(s_h^m) = 1 + \sum_{a \in A} \sum_{s' \in S} \tilde{\pi}_m(a \mid s_h^m) \tilde{P}_m(s' \mid s_h^m, a) \tilde{T}^m(s').
\]

**Proof.** For the optimistic fast policy \(\tilde{\pi}_m\) the Bellman equations hold for every \(s \in S\) since it is proper w.r.t \(\tilde{P}_m\) (see [2], Lemma B.11). When \(\tilde{\pi}_m\) is the policy chosen by OMD \(\pi_k\), reaching a state \(s\) such that \(q^{P_k, \pi_k}(s) = 0\) will end the interval (since we set \(\tilde{T}^k(s) = D/c_{\min}\) for these states). Thus, it suffices to show that the Bellman equations hold for all states in \(\{s \in S : q^{P_k, \pi_k}(s) > 0\}\).
We now write $$\hat{T}^m$$ is bounded by $$D/c_{\min}$$ and therefore $$\hat{\pi}_m$$ is proper w.r.t $$\bar{P}_m$$ and the Bellman equations hold. Note that we did not make changes to $$\bar{P}_m$$ or $$c_m$$ in states that can be visited during the interval. 

**H.3 Regret decomposition**

**Lemma H.3.** It holds that

$$\bar{R}_M \leq \sum_{m=1}^{M} \bar{R}_m^1 + \sum_{m=1}^{M} \bar{R}_m^2 - \sum_{k=1}^{K} J^*_k(s_0) + \alpha \frac{D|S|^2|A|}{c_{\min}^2} \log \frac{D|S||A|}{\delta_{\min}},$$

where

$$\bar{R}_m^1 = (\bar{J}m(s^m_0) - \bar{J}m(s^m_{H+1}))\{\Omega^m\}$$

$$\bar{R}_m^2 = \sum_{h=1}^{H_m} \left( \bar{J}m(s^m_{h+1}) - \sum_{a \in A} \sum_{s' \in S} \hat{\pi}_m(a \mid s^m_h)\bar{P}_m(s' \mid s^m_h, a)\bar{J}m(s') \right) \{\Omega^m\}.$$

**Proof.** First we have a cost of at most 1 every time we visit an unknown state. Each state becomes known after $$\alpha|A|c_{\max} \log \frac{D|S||A|}{\delta_{\min}}$$ visits, and therefore the total cost from these visits is at most $$\alpha|S||A|c_{\max} \log \frac{D|S||A|}{\delta_{\min}}$$. From now on we will ignore visits to unknown states throughout the analysis because we calculated their contribution to the total cost.

We can use the Bellman equations w.r.t $$\bar{P}_m$$ (Lemma H.2) to have the following interpretation of the costs for every interval $$m$$ and time $$h$$:

$$\sum_{a \in A} \hat{\pi}_m(a \mid s^m_h)c_m(s^m_h, a)\{\Omega^m\} =$$

$$= \left( \bar{J}m(s^m_0) - \sum_{a \in A} \sum_{s' \in S} \hat{\pi}_m(a \mid s^m_h)\bar{P}_m(s' \mid s^m_h, a)\bar{J}m(s') \right) \{\Omega^m\}$$

$$= \left( \bar{J}m(s^m_0) - \bar{J}m(s^m_{H+1}) \right) \{\Omega^m\}$$

$$+ \left( \bar{J}m(s^m_{H+1}) - \sum_{a \in A} \sum_{s' \in S} \hat{\pi}_m(a \mid s^m_h)\bar{P}_m(s' \mid s^m_h, a)\bar{J}m(s') \right) \{\Omega^m\}. \quad (18)$$

We now write $$\bar{R}_M = \sum_{m=1}^{M} \sum_{h=1}^{H_m} \sum_{a \in A} \hat{\pi}_m(a \mid s^m_h)c_m(s^m_h, a)\{\Omega^m\} - \sum_{k=1}^{K} J^*_k(s_0),$$ and substitute for each cost using Eq. (18) to get the lemma, noting that the first term telescopes within the interval. 

**Lemma H.4.** It holds that

$$\sum_{m=1}^{M} \bar{R}_m^1 \leq 2D|S||A| \log T + \alpha \frac{D^2|S|^2|A|}{c_{\min}^2} \log \frac{D|S||A|}{\delta_{\min}} + \sum_{k=1}^{K} J^*_k(s_0)\{\Omega^m(k)\},$$

where $$m(k)$$ is the first interval of episode $$k$$.

**Proof.** For every two consecutive intervals $$m, m+1$$ we have one of the following:

(i) If interval $$m$$ ended in the goal state then $$\bar{J}m(s^m_{H+1}) = \bar{J}m(g) = 0$$ and $$\bar{J}m+1(s^m_{H+1}) = \bar{J}m(k)(s_0) \leq J^*_k(s_0),$$ where $$m+1$$ is the first interval of episode $$k$$. Therefore,

$$\bar{J}m+1(s^m_{H+1})\{\Omega^m+1\} - \bar{J}m(s^m_{H+1})\{\Omega^m\} \leq J^*_k(s_0)\{\Omega^m(k)\}.$$ 

This happens at most $$K$$ times, once for every value $$k$$. 

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where for the first inequality we used Lemmas H.1 and H.2, and the last inequality follows because the cost in every 
\[ \tilde{E}(\tilde{W}) \] 
With probability at least Lemma H.5.

Consider the following martingale difference sequence

\[ m \]

(iii) If interval \( m \) ended by reaching an unknown state, then we switch policy. Thus,

\[ \tilde{J}^{m+1}(s_{1}^{m+1}) \mathbb{P}(\Omega^{m+1}) - \tilde{J}^{m}(s_{H}^{m+1}) \mathbb{P}(\Omega^{m}) \leq \tilde{J}^{m+1}(s_{1}^{m+1}) \mathbb{P}(\Omega^{m+1}) \leq D, \]

where the last inequality follows because we switched to the optimistic fast policy and thus its expected time will be bounded by \( D \) if \( P \) is in the confidence set (see Lemma H.1). This happens at most \( |S| |A| \frac{D|S|}{c_{\min}} \log \frac{|S||A|}{\delta c_{\min}} \) times.

(iv) If interval \( m \) ended with doubling the visits to some state-action pair, then similarly to the previous article,

\[ \tilde{J}^{m+1}(s_{1}^{m+1}) \mathbb{P}(\Omega^{m+1}) - \tilde{J}^{m}(s_{H}^{m+1}) \mathbb{P}(\Omega^{m}) \leq \tilde{J}^{m+1}(s_{1}^{m+1}) \mathbb{P}(\Omega^{m+1}) \leq D. \]

This happens at most \( 2|S||A| \log T \).

(v) If \( m \) is the first interval of an episode \( k \) and it ended because we reached a "bad" state then \( \tilde{J}^{m}(s_{H}^{m+1}) = D \) and \( \tilde{J}^{m+1}(s_{1}^{m+1}) \leq D \) since this is the optimistic fast policy. Thus,

\[ \tilde{J}^{m+1}(s_{1}^{m+1}) \mathbb{P}(\Omega^{m+1}) - \tilde{J}^{m}(s_{H}^{m+1}) \mathbb{P}(\Omega^{m}) \leq 0. \]

\[ \square \]

**Lemma H.5.** With probability at least \( 1 - \frac{\delta}{6} \), the following holds for all \( M = 1, 2, \ldots \) simultaneously.

\[ \sum_{m=1}^{M} \tilde{R}_{m}^{2} \leq \sum_{m=1}^{M} \mathbb{E}[\tilde{R}_{m}^{2} | \tilde{U}^{m-1}] + \frac{6D}{c_{\min}} \sqrt{M \log \frac{4M}{\delta}}, \]

where \( \mathbb{E}[- | \tilde{U}^{m-1}] \) is the expectation conditioned on the trajectory up to interval \( m \).

**Proof.** Consider the following martingale difference sequence \( (X_{m})_{m=1}^{\infty} \) defined by

\[ X_{m} = \sum_{h=1}^{H_{m}} \left( \tilde{J}^{m}(s_{h+1}^{m}) - \sum_{a \in A} \sum_{s' \in S} \tilde{\pi}_{m}(a | s_{h}^{m}) \tilde{P}_{m}(s' | s_{h}^{m}, a) \tilde{J}^{m}(s') \right) \mathbb{P}(\Omega^{m}). \]

The Bellman equations of \( \tilde{\pi}_{m} \) w.r.t. \( \tilde{P}_{m} \) (Lemma H.2) obtain

\[ |X_{m}| = \left| \left( \tilde{J}^{m}(s_{H}^{m+1}) - \tilde{J}^{m}(s_{1}^{m}) \right) \right| \leq \frac{D}{c_{\min}} \]

\[ + \sum_{h=1}^{H_{m}} \tilde{J}^{m}(s_{h}^{m}) - \sum_{a \in A} \sum_{s' \in S} \tilde{\pi}_{m}(a | s_{h}^{m}) \tilde{P}_{m}(s' | s_{h}^{m}, a) \tilde{J}^{m}(s') \mathbb{P}(\Omega^{m}) \]

\[ \leq \frac{D}{c_{\min}} \sum_{h=1}^{H_{m}} \sum_{a \in A} \tilde{\pi}_{m}(a | s_{h}^{m}) c_{m}(s_{h}^{m}, a) \sum_{h=1}^{H_{m}} \sum_{a \in A} \tilde{\pi}_{m}(a | s_{h}^{m}) c_{m}(s_{h}^{m}, a) \]

\[ \leq \frac{D}{c_{\min}} \sum_{h=1}^{H_{m}} \sum_{a \in A} \tilde{\pi}_{m}(a | s_{h}^{m}) c_{m}(s_{h}^{m}, a) \leq \frac{3D}{c_{\min}} \]

where for the first inequality we used Lemmas H.1 and H.2 and the last inequality follows because the cost in every interval is at most \( 2D/c_{\min} \).

Therefore, we use anytime Azuma inequality (Theorem J.1) to obtain that with probability at least \( 1 - \delta/6 \):

\[ \sum_{m=1}^{M} X_{m} \leq \sum_{m=1}^{M} \mathbb{E}[X_{m} | \tilde{U}^{m-1}] + \frac{6D}{c_{\min}} \sqrt{M \log \frac{4M}{\delta}}. \]

\[ \square \]
H.4 Bounding the variance within an interval

Lemma H.6 ([3], Lemma B.13). Denote \( A^m(s, a) = \frac{\log(|S||A|^m(s, a)/\delta)}{n^m(s, a)} \). When \( \Omega^m \) holds we have for any \((s, a, s') \in S \times A \times S^+\):

\[
|P(s' | s, a) - \tilde{P}_m(s' | s, a)| \leq 8\sqrt{P(s' | s, a)A^m(s, a)} + 136A^m(s, a).
\]

Lemma H.7. Denote \( A^m_h = A^m(s^m_h, a^m_h) \). For every interval \( m \) it holds that,

\[
\mathbb{E}[\tilde{R}_m^2 | \tilde{U}^{m-1}] \leq 16\mathbb{E}\left[\sum_{h=1}^{H^m} \sqrt{|S|\mathbb{V}_m^h} \mathbb{I}\{\Omega^m\} | \tilde{U}^{m-1}\right] + 272\mathbb{E}\left[\sum_{h=1}^{H^m} \frac{D}{\epsilon_{\min}} |S|A^m_h \mathbb{I}\{\Omega^m\} | \tilde{U}^{m-1}\right],
\]

where \( \mathbb{V}_m^h \) is the empirical variance defined as

\[
\mathbb{V}_m^h = \sum_{s' \in S^+} P(s' | s^m_h, a^m_h) \left( \tilde{J}^m(s') - \mu_m^h \right)^2,
\]

and \( \mu_m^h = \sum_{a \in A} \sum_{s' \in S^+} \tilde{\pi}_m(a | s^m_h)P(s' | s^m_h, a)\tilde{J}^m(s') \).

Proof. Denote

\[
X^m = \sum_{h=1}^{H^m} \left( \tilde{J}^m(s^m_{h+1}) - \sum_{a \in A} \sum_{s' \in S} \tilde{\pi}_m(a | s^m_h)\tilde{P}_m(s' | s^m_h, a)\tilde{J}^m(s') \right) \mathbb{I}\{\Omega^m\} | \tilde{U}^{m-1},
\]

\[
Z^m_h = \left( \tilde{J}^m(s^m_{h+1}) - \sum_{a \in A} \sum_{s' \in S} \tilde{\pi}_m(a | s^m_h)P(s' | s^m_h, a)\tilde{J}^m(s') \right) \mathbb{I}\{\Omega^m\}.
\]

Think of the interval as an infinite stochastic process, and note that, conditioned on \( \tilde{U}^{m-1} \), \( (Z^m_h)_{h=1}^{\infty} \) is a martingale difference sequence w.r.t \( (U^h)_{h=1}^{\infty} \), where \( U^h \) is the trajectory of the learner from the beginning of the interval and up to and including time \( h \). This holds since, by conditioning on \( \tilde{U}^{m-1} \), \( \Omega^m \) is determined and is independent of the randomness generated during the interval.

Note that \( H^m \) is a stopping time with respect to \( (Z^m_h)_{h=1}^{\infty} \) which is bounded by \( 2D/\epsilon_{\min}^2 \). Hence by the optional stopping theorem \( \mathbb{E}[\sum_{h=1}^{H^m} Z^m_h | \tilde{U}^{m-1}] = 0 \), which gets us

\[
\mathbb{E}[X^m | \tilde{U}^{m-1}] = \mathbb{E}\left[\sum_{h=1}^{H^m} \left( \tilde{J}^m(s^m_{h+1}) - \sum_{a \in A} \sum_{s' \in S} \tilde{\pi}_m(a | s^m_h)\tilde{P}_m(s' | s^m_h, a)\tilde{J}^m(s') \right) \mathbb{I}\{\Omega^m\} | \tilde{U}^{m-1}\right] = 0.
\]
Furthermore, we have

$$
\mathbb{E} \left[ \sum_{h=1}^{H^m} \sum_{a \in A} \sum_{s' \in S^+} (P(s' \mid s_h^m, a) - \tilde{P}_m(s' \mid s_h^m, a)) \tilde{\pi}_m(a \mid s_h^m) \tilde{\mathcal{J}}^m(s') \mathbb{I}\{\Omega^m\} \mid \bar{U}^{m-1} \right] = \\
= \mathbb{E} \left[ \sum_{h=1}^{H^m} \sum_{a \in A} \sum_{s' \in S^+} (P(s' \mid s_h^m, a) - \tilde{P}_m(s' \mid s_h^m, a)) \tilde{\pi}_m(a \mid s_h^m) \tilde{\mathcal{J}}^m(s') \mathbb{I}\{\Omega^m\} \mid \bar{U}^{m-1} \right] = \\
= \mathbb{E} \left[ \sum_{h=1}^{H^m} \sum_{s' \in S^+} (P(s' \mid s_h^m, a_h^m) - \tilde{P}_m(s' \mid s_h^m, a_h^m)) \tilde{\mathcal{J}}^m(s') \mathbb{I}\{\Omega^m\} \mid \bar{U}^{m-1} \right] = \\
\leq \mathbb{E} \left[ \sum_{h=1}^{H^m} \sum_{s' \in S^+} \left( \tilde{A}_h^m P(s' \mid s_h^m, a_h^m) \left( \tilde{\mathcal{J}}^m(s') - \mu_h^m \right)^2 \mathbb{I}\{\Omega^m\} \mid \bar{U}^{m-1} \right) \\
+ \mathbb{E} \left[ \sum_{h=1}^{H^m} \sum_{s' \in S^+} \tilde{A}_h^m \tilde{\mathcal{J}}^m(s') - \mu_h^m \mathbb{I}\{\Omega^m\} \mid \bar{U}^{m-1} \right] \right] \leq \mathbb{E} \left[ \sum_{h=1}^{H^m} \sum_{s' \in S^+} \left( \tilde{A}_h^m \tilde{\mathcal{J}}^m(s') - \mu_h^m \right)^2 \mathbb{I}\{\Omega^m\} \mid \bar{U}^{m-1} \right],
$$

where the first equality follows because \( \tilde{J}^m(g) = 0 \) and the second by the definition of \( a_h^m \). The third equality follows since \( P(\cdot \mid s_h^m, a_h^m) \) and \( \tilde{P}_m(\cdot \mid s_h^m, a_h^m) \) are probability distributions over \( S^+ \) whence \( \mu_h^m \) does not depend on \( s' \). The first inequality follows from Lemma H.6 and the second inequality from Jensen’s inequality, Lemma H.1

The following lemma will help us bound the variance within an interval, and it follows by the fact that known states were visited many times so our estimation of the transition function in these states is relatively accurate.

**Lemma H.8** ([2], Lemma B.14). Let \( m \) be an interval and \( s \) be a known state. If \( \Omega^m \) holds then for every \( a \in A \) and \( s' \in S^+ \),

$$
| \tilde{P}_m(s' \mid s, a) - P(s' \mid s, a) | \leq \frac{1}{8} \sqrt{ \frac{\varepsilon_{\min} \cdot P(s' \mid s, a)}{|S|D} } + \frac{\varepsilon_{\min}^2}{4 |S|D}.
$$

Define \( \mu^m(s) = \sum_{a \in A} \sum_{s' \in S^+} \tilde{\pi}_m(a \mid s) P(s' \mid s, a) \tilde{\mathcal{J}}^m(s') \) and therefore \( \mu_h^m = \mu^m(s_h^m) \). Similarly, define \( \mathcal{V}^m(s, a) = \sum_{s' \in S^+} P(s' \mid s, a) \left( \tilde{\mathcal{J}}^m(s') - \mu^m(s) \right)^2 \) and therefore \( \mathcal{V}_h^m = \mathcal{V}^m(s_h^m, a_h^m) \). The next lemma bounds the variance within a single interval.

**Lemma H.9.** For any interval \( m \) it holds that \( \mathbb{E} \left[ \sum_{h=1}^{H^m} \mathcal{V}_h^m \mathbb{I}\{\Omega^m\} \mid \bar{U}^{m-1} \right] \leq 64 \frac{\varepsilon_{\min}^2}{\varepsilon_{\min}^2}.

**Proof.** Denote

$$
Z_h^m = \left( \tilde{\mathcal{J}}^m(s_h^m) - \sum_{a \in A} \sum_{s' \in S} \tilde{\pi}_m(a \mid s_h^m) P(s' \mid s_h^m, a) \tilde{\mathcal{J}}^m(s') \right) \mathbb{I}\{\Omega^m\},
$$

and think of the interval as an infinite stochastic process. Note that, conditioned on \( \bar{U}^{m-1} \), \( (Z_h^m)_{h=1}^{\infty} \) is a martingale difference sequence w.r.t \( (U^h)_{h=1}^{\infty} \), where \( U^h \) is the trajectory of the learner from the beginning of the interval and up to time \( h \) and including. This holds since, by conditioning on \( \bar{U}^{m-1} \), \( \Omega^m \) is determined and is independent of the
randomness generated during the interval. Note that $H^m$ is a stopping time with respect to $(Z^m_h)_{h=1}^∞$ which is bounded by $2D/ε_{min}^2$. Therefore, applying Lemma 1.2 obtains

$$Ε\left[\sum_{h=1}^{H^m} π^m_h | Ω^m \right] = Ε\left[\sum_{h=1}^{H^m} Z^m_h | Ω^m \right]^2 | U^{-m-1}.$$ (19)

We now proceed by bounding $|\sum_{h=1}^{H^m} Z^m_h |$ when $Ω^m$ occurs. Therefore,

$$\left| \sum_{h=1}^{H^m} Z^m_h \right| = \left| \sum_{h=1}^{H^m} J^m(s^m_{h+1} | a, s^m_{h}) P(s' | s^m_{h}, a) J^m(s') \right|$$

$$\leq \sum_{h=1}^{H^m} J^m(s^m_{h+1} | a, s^m_{h}) P(s' | s^m_{h}, a) J^m(s')$$

$$+ \sum_{a ∈ A \ s' ∈ S} \left( π^m(a | s^m_{h}) P_m(s' | s^m_{h}, a) - P(s' | s^m_{h}, a) \right) \left( J^m(s') - μ^m_h \right),$$ (22)

where Eq. (22) is given as $P(· | s^m_{h}, a)$ and $P_m(· | s^m_{h}, a)$ are probability distributions over $S^+, μ^m_h$ is constant w.r.t $s'_h$, and $J^m(g) = 0$.

We now bound each of the three terms above individually. Eq. (20) is a telescopic sum that is at most $D/ε_{min}$ on $Ω^m$ (Lemma H.1). For Eq. (21) we use the Bellman equations for $π^m, P_m$ (Lemma H.2) thus it is at most $2D/ε_{min}$ (see proof of Lemma H.5). For Eq. (22) recall that all states at times $h = 1, \ldots, H^m$ are known by definition of $H^m$. Hence by Lemma H.8,

$$\left| \sum_{s' ∈ S^+} \left( P(s' | s^m_{h}, a) - P_m(s' | s^m_{h}, a) \right) \left( J^m(s') - μ^m_h \right) \right| \leq$$

$$\leq \frac{1}{8} \sum_{s' ∈ S^+} \left| ε_{min}^2 \frac{P(s' | s^m_{h}, a) \left( J^m(s') - μ^m_h \right)^2}{|S|D} \right|$$

$$+ \sum_{s' ∈ S^+} \left( ε_{min}^2 \frac{4|S|D \left| J^m(s') - μ^m_h \right|}{\leq D/ε_{min}} \right)$$

$$\leq \frac{1}{4} \sqrt{\frac{ε_{min}^2 |s^m_h | a}{D}} + \frac{ε_{min}}{2},$$

where the last inequality follows from Jensen’s inequality and because $|S^+| ≤ 2|S|$. Therefore,

$$\sum_{a ∈ A} \sum_{s' ∈ S^+} π^m(a | s^m_{h}) \left( P(s' | s^m_{h}, a) - P_m(s' | s^m_{h}, a) \right) \left( J^m(s') - μ^m_h \right) \leq$$

$$\leq \frac{1}{4} \sum_{a ∈ A} \sum_{s' ∈ S^+} π^m(a | s^m_{h}) \sqrt{\frac{ε_{min}^2 |s^m_h | a}{D}} + \frac{ε_{min}}{2}$$

$$\leq \frac{1}{4} \sqrt{\frac{ε_{min}^2 \sum_{a ∈ A} π^m(a | s^m_{h}) |s^m_h | a}{D}} + \frac{ε_{min}}{2},$$
where the last inequality follows again from Jensen’s inequality. We use Jensen’s inequality one last time to obtain

\[
\sum_{h=1}^{H^m} \frac{1}{4} \sqrt{\frac{c_{\min}^2 \sum_{a \in A} \tilde{p}_m(a | s_h^m) \mathcal{V}_m(s_h^m, a)}{D}} + \sum_{h=1}^{H^m} \frac{c_{\min}}{2} \leq
\]

\[
= \frac{1}{4} \sqrt{\frac{H^m c_{\min}^2 \sum_{a \in A} \tilde{p}_m(a | s_h^m) \mathcal{V}_m(s_h^m, a)}{D}} + \frac{c_{\min} H^m}{2},
\]

where we used the fact that \(H^m \leq 2D/c_{\min}^2\).

Plugging these bounds back into Eq. (19) gets us

\[
\mathbb{E} \left[ \sum_{h=1}^{H^m} \mathcal{V}_h^m \{ \Omega^m \} \mid \bar{U}^{m-1} \right] \leq \mathbb{E} \left[ \left( \frac{4D}{c_{\min}} + \frac{1}{2} \sum_{h=1}^{H^m} \sum_{a \in A} \tilde{p}_m(a | s_h^m) \mathcal{V}_m(s_h^m, a) \| \{ \Omega^m \} \| \right)^2 \mid \bar{U}^{m-1} \right]
\]

\[
\leq \frac{32D^2}{c_{\min}^2} + \frac{1}{2} \mathbb{E} \sum_{h=1}^{H^m} \sum_{a \in A} \tilde{p}_m(a | s_h^m) \mathcal{V}_m(s_h^m, a) \| \{ \Omega^m \} \| \mid \bar{U}^{m-1} \right],
\]

where the second inequality is by the elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\), and the last equality is by definition of \(a_h^m\) and \(\mathcal{V}_m^h\). Rearranging gets us \(\mathbb{E} \left[ \sum_{h=1}^{H^m} \mathcal{V}_h^m \{ \Omega^m \} \mid \bar{U}^{m-1} \right] \leq 64D^2/c_{\min}^2\), and the lemma follows.

**Lemma H.10.** With probability at least \(1 - \frac{\delta}{M}\), the following holds for all \(M = 1, 2, \ldots\) simultaneously.

\[
\sum_{m=1}^{M} \mathbb{E} [\bar{R}_m^2 \mid \bar{U}^{m-1}] \leq 573 D |S| \sqrt{\frac{M|A| \log 2 |S| |A|}{\delta}} + 5440 \frac{D}{c_{\min}} |S| |A| \log^2 \frac{T |S| |A|}{\delta}.
\]

**Proof.** From Lemma H.7 we have that

\[
\mathbb{E} [\bar{R}_m^2 \mid \bar{U}^{m-1}] \leq 16 \mathbb{E} \left[ \sum_{h=1}^{H^m} \sqrt{|S| \mathcal{V}_h^m \| A_h^m \| \{ \Omega^m \}} \mid \bar{U}^{m-1} \right]
\]

\[
+ 272 \mathbb{E} \left[ \sum_{h=1}^{H^m} \frac{D}{c_{\min}} |S| A_h^m \| \{ \Omega^m \} \mid \bar{U}^{m-1} \right],
\]

Moreover, by applying the Cauchy-Schwartz inequality twice, we get that

\[
\mathbb{E} \left[ \sum_{h=1}^{H^m} \sqrt{|S| \mathcal{V}_h^m \| A_h^m \| \{ \Omega^m \}} \mid \bar{U}^{m-1} \right] \leq \mathbb{E} \left[ \sum_{h=1}^{H^m} \mathcal{V}_h^m \| \{ \Omega^m \} \mid \bar{U}^{m-1} \right] \cdot \mathbb{E} \left[ \sum_{h=1}^{H^m} A_h^m \| \{ \Omega^m \} \mid \bar{U}^{m-1} \right]
\]

\[
\leq \sqrt{\mathbb{E} \left[ \sum_{h=1}^{H^m} A_h^m \| \{ \Omega^m \} \mid \bar{U}^{m-1} \right] \cdot \mathbb{E} \left[ \sum_{h=1}^{H^m} \mathcal{V}_h^m \| \{ \Omega^m \} \mid \bar{U}^{m-1} \right]}
\]

\[
\leq 8D \sqrt{\frac{c_{\min} \mathbb{E}}{\sum_{h=1}^{H^m} A_h^m \| \{ \Omega^m \} \mid \bar{U}^{m-1} \right]},
\]

\[
\leq 8D \sqrt{\frac{c_{\min} M|A| \log 2 |S| |A|}{\delta}}.
\]
where the last inequality is by Lemma H.9. We sum over all intervals to obtain

\[
\sum_{m=1}^{M} \mathbb{E}[\bar{R}_m \mid \bar{U}^{m-1}] \leq \frac{128D}{c_{\text{min}}} \sum_{m=1}^{M} \left| S \right| \sum_{h=1}^{H} A_h^m \sum_{\Omega_m} \left| \Omega_m \right| \left| \bar{U}^{m-1} \right|
\]

\[
+ \frac{272D |S|}{c_{\text{min}}} \sum_{m=1}^{M} \sum_{h=1}^{H} A_h^m \sum_{\Omega_m} \left| \Omega_m \right| \left| \bar{U}^{m-1} \right|
\]

\[
\leq \frac{128D}{c_{\text{min}}} \left| S \right| \sum_{m=1}^{M} \sum_{h=1}^{H} A_h^m \sum_{\Omega_m} \left| \Omega_m \right| \left| \bar{U}^{m-1} \right|
\]

\[
+ \frac{272D |S|}{c_{\text{min}}} \sum_{m=1}^{M} \sum_{h=1}^{H} A_h^m \sum_{\Omega_m} \left| \Omega_m \right| \left| \bar{U}^{m-1} \right|
\]

where the last inequality follows from Jensen’s inequality. We finish the proof using Lemma H.11 below.

**Lemma H.11.** With probability at least \(1 − \delta/6\), the following holds for \(M = 1, 2, \ldots\) simultaneously.

\[
\sum_{m=1}^{M} \mathbb{E} \left[ \sum_{h=1}^{H} A_h^m \sum_{\Omega_m} \left| \Omega_m \right| \left| \bar{U}^{m-1} \right| \right] \leq 20 |S||A| \log^2 \frac{T|S||A|}{\delta}.
\]

**Proof.** Define the infinite sequence of random variables: \(X^m = \sum_{h=1}^{H} A_h^m \sum_{\Omega_m} \left| \Omega_m \right| \left| \bar{U}^{m-1} \right|\) for which \(|X^m| \leq 2\) due to Lemma H.12 below. We apply Eq. (30) of Lemma J.3 to obtain with probability at least \(1 − \delta/6\), for all \(M = 1, 2, \ldots\) simultaneously

\[
\sum_{m=1}^{M} \mathbb{E}[X^m \mid \bar{U}^{m-1}] \leq 2 \sum_{m=1}^{M} X^m + 8 \log \frac{12M}{\delta}.
\]

Now, we bound the sum over \(X^m\) by rewriting it as a sum over intervals:

\[
\sum_{m=1}^{M} X^m \leq \sum_{m=1}^{M} \sum_{h=1}^{H} \frac{\log(|S||A|N^m(s^m_h, a^m_h)/\delta)}{N^m(s^m_h, a^m_h)} \leq \log \frac{|S||A|T}{\delta} \sum_{s \in S} \sum_{a \in A} \sum_{m=1}^{M} n^m(s, a)
\]

where \(n^m(s, a)\) is the number of visits to \((s, a)\) during interval \(m\). Here we ignore artificial intervals since the confidence set does not update in them. This means that in this sum the length of each interval is not bounded by \(2D/c_{\text{min}}\). Note that \(n^m(s, a) \leq N^m(s, a)\) by definition of our intervals (specifically, because a new interval starts when the number of visits to some state-action pair is doubled). From Lemma H.13 below we have that for every \((s, a) \in S \times A\),

\[
\sum_{m=1}^{M} \frac{n^m(s, a)}{N^m(s, a)} \leq 2 \log N_{M+1}(s, a) \leq 2 \log T.
\]

We now plug the resulting bound for \(\sum_{m=1}^{M} X^m\) and simplify the acquired expression by using \(M \leq T\).

**Lemma H.12.** For any interval \(m\), \(| \sum_{h=1}^{H} A_h^m \mid \leq 2\).

**Proof.** Note that all states during the interval are known. Hence, \(N^m(s^m_h, a^m_h) \geq \alpha \cdot \frac{D|S|}{c_{\text{min}}} \log \frac{D|S||A|}{n_{\text{min}}}\). Therefore, since \(\log(x)/x\) is decreasing and since \(|A| \geq 2\) (otherwise the learner has no choices),

\[
\sum_{h=1}^{H} \frac{\log(|S||A|N^m(s^m_h, a^m_h)/\delta)}{N^m(s^m_h, a^m_h)} \leq \frac{c_{\text{min}}^{2} H^m}{D} \leq 2.
\]
We start by bounding \( \tilde{M} \).

Recall that \( Z \) and \( Z_0 = 1 \), it holds that

\[
\sum_{k=1}^{n} \frac{z_k}{Z_{k-1}} \leq 2 \log Z_n.
\]

H.5 Proof of Theorem 5.1

Proof of Theorem 5.1. With probability at least \( 1 - \delta \), via a union bound, we have that Lemmas 5.2, H.5, and H.10 hold and the following holds by Azuma inequality for every \( T = 1, 2, \ldots \) simultaneously,

\[
\sum_{m=1}^{M} \sum_{h=1}^{H^m} c_m(s_h^m, a_h^m) \leq \sum_{m=1}^{M} \sum_{h=1}^{H^m} \pi_m(a \mid s_h^m) c_m(s_h^m, a) + 4 \sqrt{T \log \frac{T}{\delta}}. \tag{23}
\]

We start by bounding \( \tilde{R}_M \) and in the end we explain how this yields a bound on \( R_K \).

Plugging in the bounds of Lemmas H.4, H.5, and H.10 into Lemma H.3, we have that for any number of intervals \( S \),

\[
\tilde{C}_M \leq \sum_{k=1}^{K} \tilde{J}^{\pi_k}(s_0) \mathbb{I}\{\Omega^m(k)\} + O \left( \frac{D|S|}{c_{\min}} \sqrt{M|A|} \log \frac{T|S| |A|}{\delta} \right).
\]

We now plug in the bound on \( M \) from Observation 5.3 into the bound above. After simplifying this gets us

\[
\tilde{C}_M \leq \sum_{k=1}^{K} \tilde{J}^{\pi_k}(s_0) \mathbb{I}\{\Omega^m(k)\} + O \left( \frac{D^2|S|^2|A|}{c_{\min}^2} \log^2 \frac{T D|S| |A|}{\delta c_{\min}} \right).
\]

From which, by solving for \( \tilde{C}_M \) (using that \( x \leq a \sqrt{x} + b \) implies \( x \leq (a + \sqrt{b})^2 \) for \( a \geq 0 \) and \( b \geq 0 \)), and simplifying the resulting expression by applying \( \tilde{J}^{\pi_k}(s_0) \leq D/c_{\min} \) and our assumptions that \( K \geq |S|^2 |A|, |A| \geq 2 \), we get that

\[
\tilde{C}_M \leq \sum_{k=1}^{K} \tilde{J}^{\pi_k}(s_0) \mathbb{I}\{\Omega^m(k)\} \tag{24}
\]

\[
+ O \left( \frac{D|S|}{c_{\min}} \sqrt{|A| K} \log \frac{T D |S| |A|}{\delta c_{\min}} + \frac{D^2 |S|^2 |A|}{c_{\min}^2} \log^2 \frac{T D |S| |A|}{\delta c_{\min}} \right).
\]

Note that in particular, by simplifying the bound above, we obtain a polynomial bound on the total cost: \( \tilde{C}_M = O \left( \sqrt{D^3 |S|^4 |A|^2 K T / c_{\min}^2 \delta} \right) \). Next we combine this with the fact, stated in Observation 5.3 that \( T \leq \tilde{C}_M / c_{\min} \).

Isolating \( T \) gets \( T = O \left( \frac{D^3 |S|^4 |A|^2 K}{c_{\min}^2 \delta} \right) \), and plugging this bound back into Eq. (24) and simplifying gets us

\[
\tilde{C}_M \leq \sum_{k=1}^{K} \tilde{J}^{\pi_k}(s_0) \mathbb{I}\{\Omega^m(k)\} \tag{25}
\]

Recall that

\[
\sum_{k=1}^{K} \tilde{J}^{\pi_k}(s_0) - J^{\pi_k}(s_0) = \sum_{k=1}^{K} (q_k - q^{\pi_k}, c_k),
\]

and thus applying OMD analysis (see Appendix H.6) we obtain

\[
\tilde{R}_M \leq O \left( \frac{D|S|}{c_{\min}} \sqrt{|A| K} \log \frac{K D |S| |A|}{\delta c_{\min}} + \frac{D^2 |S|^2 |A|}{c_{\min}^2} \log^2 \frac{K D |S| |A|}{\delta c_{\min}} \right).
\]
Now, as $\Omega^m$ hold for all intervals, we use Eq. (23) to bound the actual regret (together with $T \leq C_M/e_{\min}$) for any number of intervals $M$, with the bound we have for $R_M$.

we note that the bound above holds for any number of intervals $M$ as long as $K$ episodes do not elapse. As the instantaneous costs in the model are positive, this means that the learner must eventually finish the $K$ episodes from which we derive the bound for $R_K$ claimed by the theorem.

H.6 OMD analysis

This analysis follows the lines of Appendix C, but it is adjusted to extended occupancy measures (see [4]).

**Lemma H.14.** Let $\tau \geq 1$. For every $q \in \tilde{\Delta}_m(\tau)$ it holds that $R(q) \leq \tau \log \tau$.

**Proof.**

\[
R(q) = \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \log q(s, a, s') - \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \\
\leq \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \log q(s, a, s') \\
= \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \log \frac{q(s, a, s')}{\tau} + \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \log \tau \\
\leq \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \log \tau \leq \tau \log \tau
\]

where the first two inequalities follow from non-positivity, and the last one from the definition of $\tilde{\Delta}_m(\tau)$.

**Lemma H.15.** Let $\tau \geq 1$. For every $q \in \tilde{\Delta}_m(\tau)$ it holds that $-R(q) \leq \tau(1 + \log(|S|^2|A|))$.

**Proof.** Similarly to Lemma C.2 we have that

\[
-R(q) = -\sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \log \frac{q(s, a, s')}{\tau} + \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q(s, a, s') \\
\leq -\tau \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} \frac{q(s, a, s')}{\tau} \log \frac{q(s, a, s')}{\tau} + \tau \leq \tau \log(|S|^2|A|) + \tau,
\]

where the first inequality follows because the last term is non-positive and from the definition of $\tilde{\Delta}_m(\tau)$, and the last inequality follows from properties of Shannon’s entropy.

**Lemma H.16.** If $\Omega^m$ holds for all intervals $m$, then

\[
\sum_{k=1}^{K} \langle q_k - q_{\pi^*}, c_k \rangle \leq 2D/e_{\min} \sqrt{6K \log \frac{D|S||A|}{e_{\min}}}.
\]

**Proof.** We start with a fundamental inequality of OMD (see, e.g., [4]) that holds for every $q \in \Delta_m(D/e_{\min})$ for every $m$ (since $\Omega^m$ holds it also holds for $q_{\pi^*}$),

\[
\sum_{k=1}^{K} \langle q_k - q_{\pi^*}, c_k \rangle \leq \sum_{k=1}^{K} \langle q_k - q_{k+1}, c_k \rangle + \frac{\text{KL}(q_{\pi^*} || q_1)}{\eta}.
\]

(26)
For the first term we use the exact form of $q_{k+1}'$ and the inequality $e^x \geq 1 + x$ to obtain

$$q_{k+1}'(s, a, s') = q_k(s, a, s')e^{-\eta c_k(s, a)} \geq q_k(s, a, s') - \eta q_k(s, a, s')c_k(s, a).$$

We substitute this back and obtain

$$\sum_{k=1}^{K} \langle q_k - q_{k+1}', c_k \rangle \leq \eta \sum_{k=1}^{K} \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q_k(s, a, s')c_k(s, a) \leq \eta \sum_{k=1}^{K} \sum_{s \in S} \sum_{a \in A} \sum_{s' \in S^+} q_k(s, a, s')$$

$$= \eta \sum_{k=1}^{K} \tilde{T}^\pi_k(s_0) \leq \eta K \frac{D}{c_{\min}}, \quad (27)$$

where the last inequality follows from the definition of $\tilde{\Delta}_{m(k)}(D/c_{\min}).$

Next we use Lemmas H.14 and H.15 to bound the second term of Eq. (26). Recall that $q_1$ minimizes $R$ in $\Delta_1(D/c_{\min}),$ this implies that $\langle \nabla R(q_1), q^P,\pi^* - q_1 \rangle \geq 0$ because otherwise we could decrease $R$ by taking small step in the direction $q^P,\pi^* - q_1.$ Thus we obtain

$$\text{KL}(q^P,\pi^* \parallel q_1) = R(q^P,\pi^*) - R(q_1) - \langle \nabla R(q_1), q^P,\pi^* - q_1 \rangle \leq R(q^P,\pi^*) - R(q_1)$$

$$\leq \frac{D}{c_{\min}} \log \frac{D}{c_{\min}} + \frac{D}{c_{\min}} (1 + \log(|S|^2|A|)) \leq \frac{6D}{c_{\min}} \log \frac{D|S||A|}{c_{\min}}. \quad (28)$$

By substituting Eqs. (27) and (28) into Eq. (26) and using the choice of $\eta,$ we obtain,

$$\sum_{k=1}^{K} \langle q_k - q^P,\pi^*, c_k \rangle \leq \eta K \frac{D}{c_{\min}} + \frac{6D}{c_{\min}} \log \frac{D|S||A|}{c_{\min}} \leq 2 \frac{D}{c_{\min}} \sqrt{6K \log \frac{D|S||A|}{c_{\min}}}.$$
I  Proofs for Section 6

I.1 Estimating the SSP-diameter

When D is given, we use it to get the upper bound $D/c_{\min}$ on the expected time of the best policy in hindsight $T^\pi_\star(s_0)$. The reason that $T^\pi_\star(s_0) \leq D/c_{\min}$ is that $D$ is an upper bound on the expected time of the fast policy, i.e., $T^\pi_\star(s_0) \leq D$ (see Lemma C.1).

We would like to use the first $L$ episodes in order to estimate an upper bound $\tilde{D}$ on the expected time of the fast policy, and then we can run SSP-O-REPS3 and obtain the same regret bound as in Theorem 5.1 but with $\tilde{D}$ replacing $D$.

Notice that $\pi^f$ is the optimal policy w.r.t the constant cost function $c(s,a) = 1$, and its expected cost is $T^\pi_\star(s_0)$. Thus, we run the SSP regret minimization algorithm of [2] with the cost function $c(s,a) = 1$ for $L$ episodes. Then, we set $\tilde{D}$ to be the average cost per episode times 10, that is,

$$\tilde{D} = \frac{10}{L} \sum_{k=1}^{L} \sum_{i=1}^{I_k} c(s_k^i, a_k^i) = \frac{10}{L} \sum_{k=1}^{L} I_k.$$

We start by showing that $\tilde{D}$ is indeed an upper bound on $T^\pi_\star(s_0)$, given $L$ is large enough.

**Lemma I.1.** If $L \geq \frac{2400D^2}{T^\pi_\star(s_0)} \log^3 \frac{4K}{\delta}$ then, with probability at least $1 - \delta$, $T^\pi_\star(s_0) \leq \tilde{D}$.

**Proof.** Notice that playing $\pi^f$ during the first $L$ episodes will result in smaller total cost than running the regret minimization algorithm. Thus, it suffices to prove the Lemma as if we are playing the fast policy. Define

$$X_k = \sum_{i=1}^{I_k} c(s_k^i, a_k^i) - E\left[\sum_{i=1}^{I_k} c(s_k^i, a_k^i) \mid P, \pi^f, s_k^i = s_0\right] = \sum_{i=1}^{I_k} c(s_k^i, a_k^i) - T^\pi_\star(s_0).$$

This is a martingale difference sequence, and in order to use Theorem J.5 we need to show that $Pr[|X_k| > m] \leq 2e^{-m^2}$ for every $k = 1, 2, \ldots$ and $m \geq 0$. This follows immediately from Lemma E.1 since the total cost is equal to the total time for the cost function $c(s,a) = 1$.

By Theorem J.5 $\sum_{k=1}^{L} X_k \leq 44D \sqrt{L \log^3 \frac{4L}{\delta}}$ with probability $1 - \delta$. Therefore we have,

$$\sum_{k=1}^{L} \sum_{i=1}^{I_k} c(s_k^i, a_k^i) \geq L T^\pi_\star(s_0) - 44D \sqrt{L \log^3 \frac{4L}{\delta}},$$

and thus,

$$\tilde{D} = \frac{1}{L} \sum_{k=1}^{L} \sum_{i=1}^{I_k} c(s_k^i, a_k^i) \geq T^\pi_\star(s_0) - 44D \sqrt{\frac{\log^3 \frac{4L}{\delta}}{L}}. \quad (29)$$

Since $L \geq \frac{2400D^2}{T^\pi_\star(s_0)} \log^3 \frac{4K}{\delta}$, we have that $44D \sqrt{\frac{\log^3 \frac{4L}{\delta}}{L}} \leq \frac{9}{10} T^\pi_\star(s_0)$ and therefore we obtain from Eq. (29) that $T^\pi_\star(s_0) \leq \tilde{D}$. \hfill \Box

Next, we show that $\tilde{D}$ is a good estimation of $T^\pi_\star(s_0)$, given $L$ is large enough.

**Lemma I.2.** If $L \geq |S|^2 |A| \sqrt{D} \log^2 \frac{K |D| |S| |A|}{\delta}$ then, with probability at least $1 - \delta$, $\tilde{D} \leq O(D)$.

**Proof.** By the regret bound of the SSP regret minimization algorithm we have, with probability at least $1 - \delta$,

$$\frac{1}{L} \sum_{k=1}^{L} \sum_{i=1}^{I_k} c(s_k^i, a_k^i) - T^\pi_\star(s_0) \leq O \left( \frac{|D| |S| |A| \sqrt{L} \log \frac{L |D| |S| |A|}{\delta}}{\sqrt{L}} + \frac{D^{3/2} |S|^2 |A| \log^2 \frac{L |D| |S| |A|}{\delta}}{L} \right).$$

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Since $T^{\pi^f}(s_0) \leq D$ we obtain
\[ \tilde{D} \leq O \left( D + \frac{D|S|\sqrt{|A|} \log \frac{LD|S||A|}{\delta}}{\sqrt{L}} + \frac{D^{3/2}|S|^2|A| \log^2 \frac{LD|S||A|}{\delta}}{L} \right) \leq O(D), \]
where the last inequality follows because $L \geq |S|^2|A|\sqrt{D} \log^2 \frac{KD|S||A|}{\delta}$.

Combining Lemmas 1.1 and 1.2 together with the regret bound for SSP-O-REPS3 in Theorem 5.1 gives the following regret bound.

**Theorem I.3.** Under Assumptions 1 and 2, running SSP-O-REPS3 with $\eta = \sqrt{\frac{3 \log(\tilde{D}|S||A|/c_{\min})}{K}}$ and $L = \max\left\{ \frac{2400|S|^2|A|}{c_{\min}} \log^3 \frac{K|S||A|}{\delta}, \frac{2400|S|^2}{c_{\min}} \sqrt{|A|K \log \frac{K|S||A|}{\delta}} \right\}$ ensures that, with probability at least $1 - \delta$,
\[ R_K \leq O \left( \frac{D|S|\sqrt{|A|} K \log \frac{KD|S||A|}{\delta c_{\min}}}{c_{\min}} + \frac{D^2|S|^2|A| \log^3 \frac{KD|S||A|}{\delta c_{\min}}}{c_{\min}^2} \right), \]
for
\[ K \geq \max\left\{ \frac{c_{\min}^2 D|S|^2|A| \log^2 \frac{|S||A|}{\delta}}{c_{\min}}, \frac{c_{\min}^2 D^4 \log^4 \frac{|S||A|}{\delta}}{|S|^2|A| T^{\pi^f}(s_0)^4} \right\}. \]

Notice that $K \geq D^3|S|^2|A|$ suffices but it may be much smaller, especially if $s_0$ is one of the furthest states from the goal (i.e., $T^{\pi^f}(s_0)$ is close to $D$) or $c_{\min}$ is very small.

**Proof.** By a union bound, Lemmas 1.1 and 1.2 and the regret bound of SSP-O-REPS3 all hold with probability at least $1 - 3\delta$ (because of the $O(\cdot)$ notation it is the same as $1 - \delta$). Therefore, $T^{\pi^f}(s_0) \leq \tilde{D} \leq O(D)$. During the first $L$ episodes our cost is bounded as follows,
\[ \sum_{k=1}^{L} \sum_{i=1}^{T_k} c_k(s_i, a_i^f) \leq LD + O \left( D|S|\sqrt{|A|L} \log \frac{LD|S||A|}{\delta} + D^{3/2}|S|^2|A| \log^2 \frac{LD|S||A|}{\delta} \right) \]
\[ \leq O \left( \frac{D|S|\sqrt{|A|} K \log \frac{KD|S||A|}{\delta c_{\min}}}{c_{\min}} + \frac{D^2|S|^2|A| \log^3 \frac{KD|S||A|}{\delta c_{\min}}}{c_{\min}^2} \right), \]
and then we bound the regret as in Theorem 5.1 to get the final result.

### I.2 Zero costs

We can artificially fulfill Assumption 2 by adding a small $\epsilon > 0$ perturbation to the costs. That is, when $c_k$ is revealed, we pass to the learner the perturbed cost function $\tilde{c}_k(s, a) = \max\{c_k(s, a), \epsilon\}$ for every $s \in S$ and $a \in A$.

Notice that changing the cost function does not change the transition function or the SSP-diameter. However, the bias introduced by our perturbation adds an additional $\epsilon D^* K$ term to the regret, where $D^*$ is the expected time it takes the best policy in hindsight to reach the goal state.

Choosing $\epsilon$ to balance the algorithms’ regret with the new term yields the following regret bounds for the general case. Theorem 4.1 matches Theorem 4.1 Theorem I.4 matches Theorem 4.4, Theorem I.5 matches Theorem 5.1, and Theorem I.6 matches Theorem 5.1 and Theorem I.7 matches Theorem I.3.

**Theorem I.4.** Under Assumption 1 running SSP-O-REPS with known transition function, $\eta = \sqrt{\frac{3 \log(\tilde{D}|S||A|/c_{\min})}{K}}$ and $\epsilon = K^{-1/4}$ ensures that
\[ \mathbb{E}[R_K] \leq O \left( D^* K^{3/4} \sqrt{\log(KD|S||A|)} \right). \]

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Theorem I.5. Under Assumption I, running SSP-O-REPS2 with known transition function, \( \eta = \sqrt{3 \log(\frac{K D|S||A|}{c_{\text{min}}})} \)
and \( \epsilon = K^{-1/4} \sqrt{\log \frac{K D|S||A|}{\delta}} \) ensures that, with probability \( 1 - \delta \),
\[
R_K \leq O \left( D^* K^{3/4} \log \frac{K D|S||A|}{\delta} \right).
\]

Theorem I.6. Under Assumption I, running SSP-O-REPS3 with known SSP-diameter \( D \), \( \eta = \sqrt{3 \log(\frac{D|S||A|}{c_{\text{min}}})} \)
and \( \epsilon = K^{-1/4} |S| \sqrt{|A| \log \frac{K D|S||A|}{\delta}} \) ensures that, with probability \( 1 - \delta \),
\[
R_K \leq O \left( D^* |S| |A| K^{3/4} \log \frac{K D|S||A|}{\delta} + D^2 \sqrt{K \log \frac{K D|S||A|}{\delta}} \right).
\]

Theorem I.7. Under Assumption I, running SSP-O-REPS3 with \( \epsilon = K^{-1/4} |S| \sqrt{|A| \log \frac{K D|S||A|}{\delta}} \), \( \eta = \sqrt{3 \log(\frac{D|S||A|}{\epsilon})} \)
and \( L = \max\left\{ \frac{2400 |S|^2 |A|}{\epsilon^2} \log^3 \frac{K |S||A|}{\delta}, \frac{2400 |S|}{\epsilon^4} \sqrt{|A| K \log \frac{K D|S||A|}{\delta}} \right\} \) ensures that, with probability at least \( 1 - \delta \),
\[
R_K \leq O \left( D^* |S| \sqrt{|A| K^{3/4} \log \frac{K D|S||A|}{\delta}} + D^2 \sqrt{K \log^2 \frac{K D|S||A|}{\delta}} \right),
\]
for
\[
K \geq \max\left\{ D^{2/3} |S|^{8/3} |A|^{4/3} \log^2 \frac{|S||A|}{\delta}, \frac{D^{8/3} \log^{10/3} |S||A|}{T_{\pi_f}(s_0)^{8/3}} \right\}.
\]

Notice that \( K \geq D^3 |S|^3 |A|^2 \) suffices but it may be much smaller, especially if \( s_0 \) is one of the furthest states from the
goal (i.e., \( T_{\pi_f}(s_0) \) is close to \( D \)) or \( D \) is much larger than \( |S||A| \).

Note that for \( \epsilon \leq 1 \) in Theorems I.6 and I.7, we need \( K \geq |S|^4 |A|^2 \).
J Concentration inequalities

Theorem J.1 (Anytime Azuma). Let \((X_n)_{n=1}^\infty\) be a martingale difference sequence such that \(|X_n| \leq B_n\) almost surely. Then with probability at least \(1 - \delta\),

\[
\sum_{n=1}^N X_n \leq 4 \sqrt{\sum_{n=1}^N B_n^2 \log \frac{N}{\delta}} \quad \forall N \geq 1.
\]

Lemma J.2 (\cite{2}, Lemma B.15). Let \((X_i)_{i=1}^\infty\) be a martingale difference sequence adapted to the filtration \((\mathcal{F}_t)_{t=0}^\infty\). Let 
\[Y_n = (\sum_{i=1}^n X_i)^2 - \sum_{i=1}^n \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}].\]
Then \((Y_n)_{n=0}^\infty\) is a martingale, and in particular if \(\tau\) is a stopping time such that \(\tau \leq \infty\) almost surely, then \(\mathbb{E}[Y_\tau] = 0\).

Lemma J.3 (\cite{2}, Lemma D.4). Let \((X_n)_{n=1}^\infty\) be a sequence of random variables with expectation adapted to the filtration \((\mathcal{F}_n)_{n=0}^\infty\). Suppose that \(0 \leq X_n \leq B\) almost surely. Then with probability at least \(1 - \delta\), the following holds for all \(n \geq 1\) simultaneously:

\[
\sum_{i=1}^n \mathbb{E}[X_i \mid \mathcal{F}_{i-1}] \leq 2 \sum_{i=1}^n X_i + 4B \log \frac{2n}{\delta}.
\]  

Lemma J.4. Let \(X\) be a non-negative random variable such that \(\mathbb{P}[[X] > m] \leq ae^{-m/b} (a \geq 1)\) for all \(m \geq 0\). Then, \(\mathbb{E}[X \mathbb{I}\{X > r\}] \leq a(r + b)e^{-r/b}\).

Proof. We have that,

\[
\mathbb{E}[XI\{X > r\}] = r \mathbb{P}[X > r] + \mathbb{E}[(X - r)I\{X > r \geq 0\}],
\]

and

\[
\mathbb{E}[(X - r)I\{X > r \geq 0\}] = \int_{m=0}^\infty \mathbb{P}[X - r > m] \mathbb{d}m = \int_{m=r}^\infty \mathbb{P}[X > m] \mathbb{d}m \leq \int_{m=r}^\infty ae^{-m/b} \mathbb{d}m = abe^{-r/b}.
\]

Hence \(\mathbb{E}[X \mathbb{I}\{X > r\}] \leq a(r + b)e^{-r/b}\) as required. \(\square\)

Theorem J.5 (Anytime Azuma for Unbounded Martingales). Let \((X_n)_{n=1}^\infty\) be a non-negative martingale difference sequence adapted to the filtration \((\mathcal{F}_n)_{n=1}^\infty\) such that \(\mathbb{P}[[X_n] > m] \leq ae^{-m/b} (a \geq 1)\) for all \(n \geq 1\) and \(m \geq 0\). Then, with probability at least \(1 - \delta\),

\[
\sum_{n=1}^N X_n \leq 11b \sqrt{N \log^3 \frac{2aN}{\delta}} \quad \forall N \geq 1.
\]

Proof. Define \(r_n = 2b \log \frac{2an}{\delta}\), and note that \(\mathbb{P}[[X_n] > r_n] \leq \frac{\delta}{4n^2}\).

Additionally define \(Y_n = X_n \mathbb{I}\{X_n \leq r_n\} - \mathbb{E}[X_n \mathbb{I}\{X_n \leq r_n\} \mid \mathcal{F}_{n-1}]\). \((Y_n)_{n=1}^\infty\) is a bounded martingale difference sequence, and by \(\text{Theorem J.1}\) we have that with probability at least \(1 - \frac{\delta}{2}\),

\[
\sum_{n=1}^N Y_n \leq 4 \sqrt{\sum_{n=1}^N r_n^2 \log \frac{N}{\delta}} \quad \forall N \geq 1.
\]
Therefore, by a union bound, both the above holds and $|X_n| \leq r_n$ for all $n \geq 1$ with probability at least $1 - \delta$. We get that
\[
\left| \sum_{n=1}^{N} X_n I\{|X_n| \leq r_n\} - \mathbb{E}[X_n I\{|X_n| \leq r_n\} | \mathcal{F}_{n-1}] \right| \leq 4 \sqrt{\sum_{n=1}^{N} r_n^2 \log \frac{N}{\delta}},
\]
and simplifying using the definition of $r_n$ gets
\[
\left| \sum_{n=1}^{N} X_n I\{|X_n| \leq r_n\} \right| \leq \left| \sum_{n=1}^{N} \mathbb{E}[X_n I\{|X_n| \leq r_n\} | \mathcal{F}_{n-1}] \right| + 8b \sqrt{N \log \frac{3}{2}} \frac{aN}{\delta}.
\]
It thus remains to upper bound $\left| \sum_{n=1}^{N} \mathbb{E}[X_n I\{|X_n| \leq r_n\} | \mathcal{F}_{n-1}] \right|$. First note that (since $X_n$ is a martingale difference sequence)
\[
\mathbb{E}[X_n I\{|X_n| \leq r_n\} | \mathcal{F}_{n-1}] = \mathbb{E}[X_n | \mathcal{F}_{n-1}] - \mathbb{E}[X_n I\{|X_n| > r_n\} | \mathcal{F}_{n-1}]
\]
\[
= -\mathbb{E}[X_n I\{|X_n| > r_n\} | \mathcal{F}_{n-1}],
\]
from which
\[
\left| \sum_{n=1}^{N} \mathbb{E}[X_n I\{|X_n| \leq r_n\} | \mathcal{F}_{n-1}] \right| = \left| \sum_{n=1}^{N} \mathbb{E}[X_n I\{|X_n| > r_n\} | \mathcal{F}_{n-1}] \right|
\]
\[
\leq \sum_{n=1}^{N} \mathbb{E}\left[ |X_n| I\{|X_n| > r_n\} | \mathcal{F}_{n-1} \right]
\]
\[
\leq \sum_{n=1}^{N} a(r_n + b)e^{-r_n/b}
\]
\[
\leq \sum_{n=1}^{N} 3ab \left( \frac{\delta}{2an} \right)^2 \log \frac{2an}{\delta}
\]
\[
\leq \sum_{n=1}^{N} 6ab \left( \frac{\delta}{2an} \right)^2 \left( \frac{2an}{\delta} \right)^{1/2}
\]
\[
= \sum_{n=1}^{N} 6ab \left( \frac{\delta}{2an} \right)^{3/2}
\]
\[
\leq \sum_{n=1}^{N} 3b a \frac{3b}{n^{3/2}} \leq 3b \log(N + 1) \leq 3b \log(2N),
\]
where the second inequality follows from Lemma J.4 and and the third inequality follows because $\log x \leq 2\sqrt{x}$. \qed