On the spectra of scalar mesons from HQCD models

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Abstract

We determine the holographic spectra of scalar mesons from the fluctuations of the embedding of flavor D-brane probes in HQCD models. The models we consider include a generalization of the Sakai Sugimoto model at zero temperature and at the “high-temperature intermediate phase”, where the system is in a deconfining phase while admitting chiral symmetry breaking and a non-critical 6d model at zero temperature. All these models are based on backgrounds associated with near extremal $N_c$ D4 branes and a set of $N_f << N_c$ flavor probe branes that admit geometrical chiral symmetry breaking. We point out that the spectra of these models include a $0^{--}$ branch which does not show up in nature. At zero temperature we found that the masses of the mesons $M_n$ depend on the “constituent quark mass” parameter $m^c_q$ and on the excitation number $n$ as $M_n^2 \sim m^c_q$ and $M_n^2 \sim n^{1.7}$ for the ten dimensional case and as $M_n \sim m^c_q$ and $M_n \sim n^{0.75}$ in the non-critical case. At the high temperature intermediate phase we detect a decrease of the masses of low spin mesons as a function of the temperature similar to holographic vector mesons and to lattice calculations.

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Whereas realizing confinement in dual holographic models of QCD (HQCD) is easy the incorporation of flavored chiral quarks and in particular chiral symmetry breaking is more difficult. Sakai and Sugimoto \[1\] proposed a model that admits the two phenomena. It is based on placing a set of $N_f$ D8 and anti-D8 probe flavor branes into the gravity model of near extremal D4 branes \[2, 3\].

The mesonic spectra is one of the most important properties of hadron dynamics that can be “measured” in the HQCD laboratory. The low spin mesons are associate with the fluctuations of the fields that reside on the probe flavor branes, the vector mesons with the $U(N_f)$ flavor gauge fields and the scalar mesons with the embedding of the probe branes \[4\]. Here in this paper we focus only on scalar mesons. The motivation behind addressing this problem are the following: (i) To verify that the meson spectrum at zero temperature does not include tachyonic modes. Had there been such modes it would have indicated that the system is unstable. Since the model of \[1\] is based on placing branes and anti-branes one may be worried that the system is unstable and hence the importance of this check. (ii) The spectrum of the scalar mesons has been determined already in \[1\]. However the attempts to derive it in generalizations of the model where the asymptotic separation of the brane anti-brane $L$ is smaller than half of the circumference of the compactified direction $x_4$, namely for $L \leq \pi R$ failed for the symmetric modes \[5, 6\].

(iii) To determine the dependence of the spectrum on the excitation number $n$ and the parameter $m^c_q$ defined in \[12\] that is related to the constituent quark mass. In addition one naturally would like to compare the explicit ratios of meson masses that one deduces from any given HQCD model and the experimental data to get an indication of how well the model describes real hadron physics. (iv) To further examine the differences of physical properties extracted from critical models to non-critical models which were previously discussed in \[5, 7, 8\]. The spectrum of scalar mesons was extracted also in other HQCD models \[9, 10, 11\]. For further reading see \[12\] and references therein.

We can summarize the outcome of the paper as follows

- We were able to choose coordinates that avoid the singularities that were encountered in previous works \[4, 6\] and determine the spectrum of both the anti-symmetric as well as symmetric branches.

- We find that in the models examined and in particular the original model of \[1\] the symmetric solutions correspond to scalar mesons of the form $0^{++}$ whereas the anti-symmetric solutions correspond to $0^{--}$ mesons. This property which seems to be in common to a HQCD models based on probe branes and anti-branes, contradict the low lying spectrum in nature. There are no low lying $0^{--}$ mesons.

\[1\] High spin mesons are naturally described by semi-classical spinning string configurations\[4\].
At zero temperature we found that the masses of the mesons $M_m$ depend on the “constituent quark mass” $m_q$ and on the excitation number $n$ as $M_m^2 \sim m_q^2$ and $M_m^2 \sim n^\alpha$ with $\alpha \sim 1.7$ for the ten dimensional case and as $M_m \sim m_q$ and $M_m \sim n^\beta$ with $\beta \sim 0.75$ when a CS term is incorporated and $\beta \sim 1$ without such a term in the non-critical case. At the high temperature intermediate phase we detect a decrease of the masses of low spin mesons as a function of the temperature similar to holographic vector mesons and to lattice calculations.

The paper is organized as follows. We begin in section 1 with a brief review of the holographic models we investigate. We summarize the main features of the model of Sakai and Sugimoto at zero and finite temperature and an analogous six dimensional non-critical model. In section 2 we describe the extraction of scalar mesons from the fluctuations of the embedding. In particular we point out that in the coordinates introduced in 1 the eigenvalue problem admits a singularity that prevents the numerical determination of the eigenvalues. A different coordinate system is presented in section 3 which evades the problem of the singularity. Using these coordinates, the spectrum of masses as a function of the constituent mass and excitation number is derived. The spectrum of scalar mesons that follows from a non critical model of near extremal D4 branes is analyzed in section 4. Section 5 addresses the issue of parity and charge conjugation of the scalar mesons. It is pointed out that the spectrum includes $0^{--}$ mesons which do not show up in nature. Section 6 is devoted to the spectrum of mesons above the deconfining phase transition in the “intermediate phase”. We summarize the results and raise certain open questions.

1 Review of the holographic models

1.1 The Sakai Sugimoto model

The model of [2], describes the near horizon limit of $N_c$ D4-branes wrapping a circle in the $x_4$ direction with anti periodic boundary condition for the fermions. Into this background a stack of $N_f$ D8 is placed at $x_4 = 0$ and a stack of $N_f$ $\bar{D}8$ at the anti-podal point of the $x_4$ circle [1]. Assuming $N_f << N_c$ one can overlook the modification of the metric and dilaton due to the backreaction of the background by the $N_f$ D8-$\bar{D}8$ systems and continue to use the metric and dilaton associated with the $N_c$ D4 alone. Therefore the metric, dilaton and the RR four form are given by

$$ds^2 = \left(\frac{u}{R_{D4}}\right)^{3/2}\left[-dt^2 + \delta_{ij}dx^i dx^j + f(u)dx_4^2\right] + \left(\frac{R_{D4}}{u}\right)^{3/2} \left[\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right]$$

$$F_4 = \frac{2\pi N_c}{V_4} \epsilon_4, \quad e^\phi = g_s \left(\frac{u}{R_{D4}}\right)^{3/4}, \quad R_{D4}^3 = \pi g_s N_c f_s^3, \quad f(u) = 1 - \left(\frac{u_A}{u}\right)^3$$
Where \( V_4 \) denotes the volume of the unit sphere \( \Omega_4 \) and \( \epsilon_4 \) its corresponding volume form. \( l_s \) is the string length and \( g_s \) a parameter related to the string coupling. The \( x_4 \) is the compactified direction that is asymptotically transverse to the \( D8 \). The manifold spanned by the coordinate \( u, x_4 \) has the topology of a cigar where its tip is at the minimum value of \( u \) which is \( u = u_\Lambda \). The periodicity of this cycle is uniquely determine to be

\[
\delta x_4 = \frac{4\pi}{3} \left( \frac{R_{D4}^3}{u_\Lambda} \right)^{1/2} = 2\pi R
\]

in order to avoid a conical singularity at the tip of the cigar. The classical profile of the \( D8 \) probe brane in this background is given by the classical solution to the e.o.m of the DBI action of that probe brane. The \( D8 \) DBI action is

\[
S_{D8} = T_8 \int dt d^3x du \Omega_4 e^{-\phi} \sqrt{-\text{det} \hat{g}} = \hat{T}_8 \int dt d^3x du u^4 \sqrt{f(u)(\partial_u x_4)^2 + \frac{R_{D4}^3}{u^3 f(u)}}
\]

where \( \hat{g} \) stands for the pullback metric on the \( D8 \) brane. The simplest way of solving this e.o.m is by noting that the action is independent of \( x_4 \) and so its Hamiltonian is conserved.

\[
\frac{u^4 f(u)}{\sqrt{f(u) + \left( \frac{R_{D4}}{u} \right)^3 \frac{u'^2}{f(u)}}} = u_0^4 \sqrt{f(u_0)} = \text{const}
\]

where we assumed that there is a point \( u_0 \) where the curve \( u(x_4) \), which describes the profile of the \( D8 \) brane in the \((u, x_4)\) plane has a minimum. At that point the \( D8 \) brane bends, namely the \( D8-\bar{D}8 \) join together. After some algebra one finds

\[
\left( \frac{\partial x_4}{\partial u} \right)_{\text{cl}} = \frac{1}{f(u) \left( \frac{u}{R_{D4}} \right)^{3/2} \sqrt{\frac{f(u)u^8}{f(u_0)u_0^8} - 1}}
\]

Hence we find that the profile of the \( D8 \) brane probe is

\[
x_4(u) = \int_{u_0}^{u} \frac{du}{f(u) \left( \frac{u}{R_{D4}} \right)^{3/2} \sqrt{\frac{f(u)u^8}{f(u_0)u_0^8} - 1}}
\]

Where \( u_0 \) is a constant of integration setting the lowest value of \( u \) to which the \( D8 \) barne is extending. At that point the \( D8 \) brane join the \( \bar{D}8 \) brane and the brane is extending back into the UV. The value of \( u_0 \) also sets the asymptotic distance \( L \) between the position of the \( D8 \) and \( \bar{D}8 \) brane

\[
L = \int dx_4 = 2 \int_{u_0}^{\infty} \frac{du}{u'} = 2 \left( \frac{R_{D4}^3}{u_0^3} \right)^{1/2} \int_1^{\infty} dy \frac{y^{-3/2}}{\sqrt{f(y) \sqrt{\frac{f(y)}{f'(1)} y^8 - 1}}}
\]
Hence we see

$$L \propto \left( \frac{R_{DA}^3}{u_0} \right)^{1/2}$$  \quad (9)$$

For later use we define

$$\gamma = \frac{u^8}{f(u)u^8 - f(u_0)u_0^8}$$  \quad (10)$$

The DBI action then becomes

$$S = T_8 \int e^{-\phi} \sqrt{|\text{det} \tilde{g}_0|} \sim \int d^4 x du \gamma^{1/2} u^{5/2}$$

1.2 Thermodynamics of the Sakai Sugimoto model

In [13] a study of the thermodynamics of the Sakai Sugimoto model was carried using the conjecture presented in [2].

The conjecture states that the thermodynamics of a field theory with a gravitational dual is determined by taking into account the contribution to the saddle point approximation from all the gravitational backgrounds with the correct 'UV' asymptotic, with compactified Euclidean time direction of period $\beta = \frac{1}{T}$ and with anti-periodic boundary condition for the fermions along this direction. The temperature of the field theory is $T = 1/\beta$ and its properties are read from the manifold responsible for the most dominant contribution to the saddle point approximation namely the one that has the lowest free energy.

When ever one background looses its domination to another background as we vary the temperature, a phase transition occurs in the dual field theory.

In [13] two manifolds where found to have the same 'UV' asymptotic as the one of Sakai and Sugimoto model, the background \([11]\), and the same configuration only with the time and $x_4$ directions interchange.

$$ds^2 = \left( \frac{u}{R_{DA}} \right)^{3/2} \left[ -f(u)dt^2 + \delta_{ij} dx^i dx^j + dx_4^2 \right] + \left( \frac{R_{DA}}{u} \right)^{3/2} \left[ \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right]$$  \quad (11)$$

with

$$f(u) = 1 - \left( \frac{u_T}{u} \right)^3$$  \quad (12)$$

and the temperature is given by

$$\delta t = \frac{4\pi}{3} \left( \frac{R_{DA}^3}{u_T} \right)^{1/2} = \beta$$  \quad (13)$$

\[\text{see also [14].}\]
The difference between the free energy densities of the two backgrounds is proportional to $N_c^2\left[(2\pi T)^6 - 1/R^6\right]^2$. This means that when the circumference of the $x_4$ cycle is smaller than that of the time direction namely when $T < 1/2\pi R$ the background $\text{(1)}$ is the dominant one, while when the opposite occurs and $T > 2\pi R$ the action of $\text{(11)}$ will dominates. At the temperature $T = 1/2\pi R$ the two actions are the same since the two backgrounds are different by the labeling of the coordinates, so at $T = T_c = 1/2\pi R$ the system has a first order phase transition. In $[13]$ it was argue that in the dual field theory, the physical interpretation to this phase transition is a transition from confined phase at $T < 1/2\pi R$ to deconfined at $T > 1/2\pi R$. This can be seen via a computation of the quark anti-quark potential $[15]$ in the two backgrounds. Another indication to this interpretation is that the renormalized free energy of the low temperature phase shows a $N_c^0$ behavior while that of the high temperature phase shows a $N_c^2$ one. Hence from now on we will denote $T_c = T_d$.

At the high temperature phase there is another possible classical solution to the profile of the $D_8$ brane which is a configuration with constant $x_4$ namely, $x_4(u) = 0, L$. Now since the bulk free energy is the same for the two configurations of the $D_8$ branes, the difference of the free energy of the $D_8$ probes determines which of the two configurations is the preferable one for a given temperature. It turns out that the transition between the two configuration depends on the parameter $y_T = u_0/u_T$, its value at the phase transition turns out to be $y_T^c \sim 0.73572$.

Using eq. (8) we find $L_c = 0.751\left(R_{D4}^3/ u_0\right)^{1/2}$, hence at the critical point $y_T = y_T^c$ the critical temperature is set by the asymptotic distance between the branes (setting $R_{D4} = 1$)

$$T_c = \frac{3}{4\pi} u_T^{1/2} = \frac{3}{4\pi} (y_T^c u_0)^{1/2} = 0.154/L$$

(14)

The field theory sees this transition as chiral symmetry restoration at high temperature, this interpretation is natural since the $D_8$ branes are now disconnected and there is an $U(N_f) \times U(N_f)$ global symmetry.

Hence we will denote this critical temperature as $T_{\chi SB}$. Note that this only happens at the high temperature phase so there is still the condition $T_{\chi SB} = 0.154/L > 1/2\pi R$.

So if $L > 0.97 R$, we find that $T_d$ is always higher than $T_{\chi SB}$, and so deconfinement and chiral symmetry restoration phase transition happen together. We see that in this model $\chi SB$ and confinement appear independently of one another as a result of the existence

3 Of course in our model there are also $D_8$ brane which their DBI action will contribute to the total free energy of the configuration as well, but this is sub-leading to the bulk action since the bulk action is of order $N_c^2$ and the contribution of the $D_8$ is of order $N_c \cdot N_f$ which is negligible in the probe approximation.

4 This configuration was not possible in the low temperature, but in the high temperature phase the time circle shrink to zero at $u = u_A$ and so the $D_8$ brane can just smoothly end there.
of the free parameter $L$ coming from the 5d nature of the field theory.

1.3 Non critical holographic model

A non critical model with a very similar properties to Sakai-Sugimoto model was presented in [5, 7], this model consists of non-extremal configuration of $N_c$ D4 branes placed in a six dimension space-time with one of the D4 coordinates taken to be periodic with anti periodic boundary condition for the fermions.

The metric, dilaton and RR six-form field take the form [7]

$$
\begin{align*}
    ds^2 &= \left( \frac{u}{R_{AdS}} \right)^2 dx_{1,2}^2 + \left( \frac{R_{AdS}}{u} \right)^2 du^2 + \left( \frac{u}{R_{AdS}} \right)^2 f(u) dx_4^2 \\
    F_6 &= Q_c \left( \frac{u}{R_{AdS}} \right)^4 d\sigma_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge du \wedge dx_4 \\
    e^\phi &= \frac{2\sqrt{2}}{\sqrt{3}Q_c}; \quad R_{AdS}^2 = \frac{15}{2}
\end{align*}
$$

with

$$
f(u) = 1 - \left( \frac{u_\Lambda}{u} \right)^5
$$

and where $Q_c$ is proportional to $N_c$, the number of color D4 branes. In order to avoid conical singularity the periodicity of the cycle of $x_4$ is set to

$$
x_4 \sim x_4 + \delta x_4 \quad ; \quad \delta x_4 = \frac{4\pi R_{AdS}^2}{5u_\Lambda}
$$

Of course the curvature of order one of this background makes the leading order supergravity an un justified approximation to string theory on this background. Nevertheless its believed that at least the extremal model due to its symmetries, is indeed a good background for the study of non-critical string theory [16]. Now we place $N_f$ D4 branes which are transverse to the $S^1$ cycle and extend up to infinity in the $u$ direction. The properties of the four dimensional low energy effective field theory living on the intersection of these color and flavor D4 is then seem to be very similar to those found at the Sakai Sugimoto model. Thus we would like to study its spectrum of scalar excitations and check if there is no tachyon in the model.

Just like in the critical model the D4 brane may bend on the $(u, x_4)$ cigar and in order to find its profile one must solve the e.o.m of the $x_4$ coordinate. This e.o.m is derived

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5For other non-critical SUGRA models with flavor see [16, 17, 18, 19, 20].
from the action of the flavor $D4$ brans namely the DBI action plus the CS term which are are given by

$$S_{D4} = -T_4 \int d^5x e^{-\phi} \sqrt{-\text{det}(\hat{g})} + T_3 \tilde{a} \int P(C_{(5)})$$ (18)

Following similar steps to does taken in the previous section we find

$$x_{4,\text{cl}}(u) = \int_{u_0}^{u} \frac{(u_0^5 f^{1/2}(u_0) - au_0^5 + au'^5)du'}{(\frac{u'}{R_{\text{AdS}}})^2 f(u') \sqrt{u'^{10} f(u') - (u_0^5 f^{1/2}(u_0) - au_0^5 + au'^5)}}$$ (19)

where $a = \frac{2}{\sqrt{3}}$.

### 2 Fluctuation of the embedding and scalar mesons.

We now turn our attention to the study of the fluctuation of the $D8$ brane around its classical profile. As was mentioned in the introduction, one has a two fold interest in those fluctuations: (i) They correspond to scalar mesons in the dual gauge theory. (ii) Tachyonic modes of the fluctuation signals an instability of the system.

We start by expanding the $x_4$ coordinate around its classical value and define the fluctuation $\xi(u, x^\mu)$ as follows:

$$x_{4}(u, x^\mu) = x_{4}(u)_{\text{cl}} + \xi(u, x^\mu)$$ (20)

Substituting this into the action (3) and expanding to quadratic order in $\xi$ we find the following action for the fluctuations

$$S \propto \frac{1}{2} \int d^4x du \left\{ u^{5/2} R^3_{D4} \gamma^{-1/2} \eta^{\mu\nu} \partial_\mu \xi \partial_\nu \xi + u^{11/2} \gamma^{-3/2} (\partial_\mu \xi)^2 \right\}$$ (21)

where $\gamma$ is defined in (10). We now introduce the following mode expansion

$$\xi(u, x^\mu) = \sum_{n=0}^{\infty} f_n(x^\mu) \xi_n(u)$$ (22)

Using the symmetries along the $x^\mu$ directions we have

$$\eta^{\mu\nu} \partial_\mu \partial_\nu f_n = -m_n^2 f_n$$ (23)

The e.o.m for the $\xi_n$ modes reads

$$\partial_u [(u^{11/2} \gamma^{-3/2}) \partial_u \xi_n] = -m_n^2 R^3_{D4} u^{5/2} \gamma^{-1/2} \xi_n$$ (24)
or in its canonical form
\[
\left\{ \frac{\partial^2}{\partial u} + \left[ \frac{12}{u} - \frac{15}{2u^4} \gamma - \frac{13}{2u} \right] \frac{\partial}{\partial u} \right\} \xi_n = -\frac{m_n^2 R_D^3 \gamma}{u^3} \xi_n
\]  
(25)

For \( u_0 >> u_A \), \( f(u) \rightarrow 1 \), the e.o.m simplifies and the qualitative behavior of \( m_n \) can be determined by using dimensional arguments [13]. Define the dimensionless parameter \( v = \frac{u}{u_0} \) then for the limit \( u_0 >> u_A \) where \( f \rightarrow 1 \)

\[
\gamma \rightarrow \frac{1}{1 - \frac{v^3}{\gamma}}
\]  
(26)

The e.o.m in terms of \( v \) reads
\[
\partial_v \left( v^{11/2} \gamma^{-3/2} \right) \partial_v \xi_n = -m_n^2 R_D^3 u_0^5 \gamma^{-1/2} \xi_n
\]  
(27)

Since the L.H.S is dimensionless so must be the R.H.S and hence

\[
m_n^2 \propto \frac{u_0}{R_D^3}
\]  
(28)

Using the relation (9) between \( u_0 \) and \( L \) we find

\[
m_n \propto \frac{1}{L}
\]  
(29)

while the mass of the glueball is related to \( m_{gb} \sim \frac{1}{R} \). For the case \( u_A = u_0 \), \( L = \pi R \) so the glueball and mesons masses have the same scale. However in the general case where \( u_0 > u_A \) there are two different scales \( m_n \sim \frac{1}{L} > \frac{1}{R} \sim m_{gb} \).

In order to find the exact spectrum of the eigenvalues of (25) one can use the shooting technic which is implemented by demanding symmetric or anti-symmetric boundary condition to \( \xi_n \) at \( u = u_0 \) and integrating the equation up to the \( u >> u_0 \) region where the solution could be matched to its normalizable asymptotic expansion. Of course this matching is only possible when the correct eigenvalues are being used and so one shoots with different eigenvalues until a matching is obtained. However there is a problem with these coordinates at \( u = u_0 \) since, \( \frac{d \xi_n}{du} |_{u=u_0} \rightarrow \infty \) (see eq. (6)). An odd perturbation to the classical configuration will cause no change in the shape of this singularity but an even one will, and so will also have a singular derivative.

This problem is reflected in the singularity of the e.o.m (25) at \( u \rightarrow u_0 \). To see this behavior explicitly we change coordinate to a dimensionless parameter \( z \) as follows

\[
u^3 = u_0^3 + u_A^3 z^2
\]  
(30)

the eigenvalue problem (25) then becomes
\[
\left\{ \frac{\partial^2}{\partial z^2} + \left[ \frac{5 u_A^3 z}{u_0^3 + u_A^3 z^2} - \frac{1}{z} - \frac{\gamma u_A^3 z}{(u_0^3 + u_A^3 z^2)^{2/3} \gamma} \right] \frac{\partial}{\partial z} \right\} \xi_n = -\frac{m_n^2 R_D^3 u_0^6 \gamma z^2}{(u_0^3 + u_A^3 z^2)^{4/3}} \xi_n
\]  
(31)
where $\gamma'$ stands for the derivative of $\gamma$ with respect to $u$. Since

$$\gamma_{z\to 0} = \frac{3u_0^6 u_0^{10} - 5u_0^6}{z^2}; \quad \gamma'_{z\to 0} = -\frac{9u_0^6 u_0^{10} - 5u_0^6}{z^4}$$

we find that this equation has a regular singularity at $z = 0$!

Indeed it was already noticed in [6] that by employing the 'shooting' technique only half of the spectrum could be found, namely only the odd modes where seen while the even ones could not be obtained, these modes that should have been obtained by setting the boundary conditions to

$$\xi_n(z = 0) = 1; \quad \partial_z \xi_n(z = 0) = 0.$$  

(33)

turned to be singular and could not be integrated. In [1] only the special case of $u_0 = u_\Lambda$ was analyzed, in this case since $\lim_{u_0 \to u_\Lambda} \partial_u x_{cl} = 0$, a smooth and nonsingular transformation into cartesian coordinates is allowed via

$$u^3 = u_\Lambda^3 + u_\Lambda^3 (z^2 + y^2); \quad x_4 = R \arctan \left( \frac{y}{z} \right)$$

(34)

The corresponding action for $y$ is (after setting $u_\Lambda = 1$)

$$S \sim \int d^4 x dz \left[ \frac{\left( \partial_{\mu} y \right)^2}{u(z)} + u(z)^3 (\partial_z y)^2 + 2y^2 \right]$$

(35)

inserting the expansion $y = \sum_{n=1} \varphi_n(x^\mu) y_n(z)$ the e.o.m for $y_n$ is

$$\partial_z^2 y_n + \frac{2z}{1 + z^2} \partial_z y_n - \frac{2y_n}{1 + z^2} = \frac{m_n^2}{(1 + z^2)^{4/3}} y_n$$

(36)

which is non-singular. For the more general case of $u_0 > u_\Lambda$ we were not able to find a similar coordinate transformation and hence we follow a different approach described in the next section.

### 3 A regular e.o.m for the scalar fluctuation at the low temperature phase

As we have seen above, we could not obtain the even modes of the fluctuation around the classical curve because $\frac{dx_{4,cl}}{du}$ diverges at $u = u_0$. The issue of choosing a direction along which one should analyze the fluctuations, has been discussed in the context of the stringy description of the Wilson like [21]. It was found that the safest approach is to use

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6If the classical curve $x_{4,cl}$ was odd, then the odd mode would become singular.
the fluctuation in the direction which is normal to the classical configuration. For our case the normal to the classical configuration at the tip \( u = u_0 \) is along the \( u \) direction. Thus from here on we study the fluctuation in the \( u \) direction, that is

\[
u(x_4, x^\mu) = u_{cl}(x_4) + \xi(x_4, x^\mu)
\]

(37)

our classical curve would be \( u_{cl}(x_4) \) and as can be seen from (6) we have \( \frac{du_{cl}}{dx_4}|_{x=0} = 0 \) so the point \( u(x_4 = 0) = u_0 \) pause no problem now! The quadratic action for these fluctuation is (after setting \( u_\Lambda = 1 \))

\[
S = \frac{1}{2} \int dx_4 \left\{ \frac{a_0}{u^{12}f^3} (\partial_{x_4} \xi)^2 + \frac{1}{u^{3}f} (\partial_{\mu} \xi)^2 - \frac{(11u^{14} + 18a_0 + 3u^{11} - 12u^8 - 27a_0(u^3 + u^6) - 2u^5)}{2u^{16}f^3} \xi^2 \right\}
\]

(38)

where \( a_0 = u_0^8 f(u_0) \) and it should be understood that \( u = u_{cl}(x_4) \) and its formal expression is

\[
u(x_4) = \int_0^{x_4} dx_4 f(u) \left( \frac{u}{R_D} \right)^{3/2} \sqrt{\frac{f(u)u^8}{f(u_0)u_0^8} - 1}
\]

(39)

after plugging a mode expansion the e.o.m in its canonical form is

\[
\partial^2_{\xi_n} - \left( \frac{11}{u} + \frac{9}{u^f} \right)u_x \partial_x \xi_n - \frac{f^2 u^8 m_n^2}{a_0} \xi_n + \frac{(11u^{14} + 18a_0 + 6u^{11} - 12u^8 - 27a_0(u^3 + u^6) - 2u^5)}{2a_0 u^5} \xi_n = 0
\]

(40)

where \( u_x = \partial_{x_4} u_{cl} \). Since there is no analytic expression for the integral in (39), we obtained \( u(x_4) \) numerically during the integration of eq. (40) when ’shooting’ to find the eigenvalues of (40).

The resulted spectra are summarized in figures (1), (2) and (3). The following properties characterize these spectra

- The first observation one can make is that for \( u_0 = u_\Lambda \) our results for the symmetric and anti-symmetric lowest lying states match those of [1].

\[
m_s^2 = 3.3 \quad ; \quad m_{as}^2 = 5.3
\]

(41)

- The figures (1) and (2) describe the dependence of the squared mass of the first excited symmetric and anti-symmetric states as a function of the “constituent quark mass” defined in [5] and [4], as follows

\[
m_q^c = \frac{1}{2\pi \alpha'} \int_{u_\Lambda}^{u_0} \sqrt{-g_{tt} g_{uu}} du = \frac{1}{2\pi \alpha'} \int_{u_\Lambda}^{u_0} f^{-1/2}(u) du
\]

(42)
This parameter relates to the constituent quark mass and not to the current algebra (QCD) mass, since even when it is turned on the fluctuations that correspond to the pions are massless. In fact the quantity dual of the constituent quark mass should associate with $m^c_q$ plus a constant term which is independent of $u_0$ since already for $u_0 = u_\Lambda$ the mesons are massive and hence there is a non-trivial constituent mass. This assignment is also in accordance with the semi-classical description of high spin mesons [4] and their stringy split into two lower mass mesons [22]. From these figures we see that indeed for for $u_0 > u_\Lambda$ the square of the mass of the scalars grows linearly with $m^c_q$. This is to be contrasted with the results found in [5] for vector mesons of non-critical models where the mass itself is found to be linear with the $m^c_q$ (see also down in section (5).)

- We have also determined the spectrum of the higher excited mesons both the symmetric as well as the anti-symmetric ones. The dependence of the squared masses on the excitation number for various values of $m^c_q$ is drawn in figure (3). The linear fit to this curves are given by

$$
m_n^2 = 3.3 + 1.6n^{1.7} \quad m^c_q = 0
$$

$$
m_n^2 = 10.5 + 6.5n^{1.789} \quad m^c_q = 9.3
$$

$$
m_n^2 = 15.8 + 9.5n^{1.818} \quad m^c_q = 14.3
$$

Stringy modes are characterized by the well known $m^2 \sim n$ behavior. We thus see that the scalar meson spectra that follow from the model of [1] do not correspond to stringy modes. This is of course of no surprise since it follows from a low energy effective field theory and not from a semi-classical treatment.

- Last by not least we see from figure (1) that the lowest scalar excitation remain non-tachyonic for all values of $u_0$ which serves as an partial evidence for the stability of the Sakai Sugimoto model.

## 4 Scalar mesons in a non critical holographic model

We would like now to find the masses of the scalar modes associated with the fluctuations of the probe brane around the classical profile in the non-critical gravity background of [7]. Using the background (15) in an effective action that includes only the DBI. The CS term

$$S_{CS} \sim \int_{D4} C_5 \sim \int_{D4} \frac{u^5}{R^4_{AdS}}$$

(44)
Figure 1: (A) The mass squared $m_1^2$ of the lowest exited symmetric mode as a function of $m_q^c$ ($R_{D4} = u_A = 1$)

Figure 2: (B) The mass squared $m_2^2$ of the lowest exited antisymmetric mode as a function of $m_q^c$ ($R_{D4} = u_A = 1$)

Figure 3: (A) The tower of the mesons squared mass $m_n^2$ in the low temperature phase ($R_{D4} = u_T = 1$)
would contribute upon a substitution of the form \( (37) \) for \( u \) and expanding a linear and quadratic term to the action, thus affecting the spectrum. However, it was found in [8] that including this CS does not yield a sensible thermal phase diagram and hence we discuss separately an effective action that includes only a DBI action and one with both the DBI and CS terms. We start first with the former case: Analyzing the spectrum in a similar manner to the analysis of section (3) we find that the fluctuations are subjected to the following eigenvalue equation

\[
\partial_u (u^4 \gamma^{-3/2}) \partial_u \xi_n = - \frac{R_{AdS}^4 m_n^2 u^2}{u^{1/2}} \xi_n
\]  

(45)

Like in the critical case, for \( u_0 \gg u_A \) the qualitative behavior of \( m'_n \) can be seen by changing the variable \( u \) into the dimensionless parameter \( y = \frac{u}{u_0} \). At the limit \( u_0 \gg u_A \) we find that \( f(u) \rightarrow 1 \) and so

\[
\gamma \rightarrow \frac{1}{u_0^2 (y^2 - \frac{1}{y^2})}
\]  

(46)

and find that in terms of the dimensionless parameter \( y \) the e.o.m is now

\[
\partial_y (y^4 \gamma^{-3/2}) \partial_y \xi_n = - \frac{R_{AdS}^4 m_n^2 y^2}{u_0^{1/2} \gamma^{1/2}} \xi_n
\]  

(47)

Since the L.H.S is dimensionless so is the R.H.S and we find

\[
m_n^2 \propto \frac{u_0^2}{R_{AdS}}
\]  

(48)

Note that due to the different background now \( L \sim \frac{R_{AdS}^2 \eta}{u_0} \) and hence again we get that \( m'_n \sim \frac{1}{L} \). However in terms of \( m^c \) the asymptotic behavior is that \( m'_n \sim m^c \) and not \( m_n^2 \sim m_q^c \) as was the case for the mesons of the critical model.

Repeating the exact same steps as for the critical case we find that the quadratic action for fluctuation in the \( x_4 \) direction around the classical curve leads to an e.o.m which is singular at \( u = u_0 \) and as a consequence the attempt carried in [5] to obtain the spectrum of the even modes had indeed failed. And so like in the critical case we turn to study the fluctuation in the \( u \) direction instead. The action for the fluctuation is then

\[
S = \frac{1}{2} \int dx_4 \left\{ \frac{a_0^{3/2}}{u^{1/4} f^3} (\partial_{x_4} \xi)^2 + \frac{a_0^{1/2} R_{AdS}^4}{u^4 f} (\partial_\mu \xi)^2 - \frac{a_0^{1/2}}{u^{1/2}} (u^5 + 36a_0 - 63u^{10} + 14u^{20} + 48u^{15} - 92a_0 u^5 - 44a_0^2) \right\}
\]  

(49)

and indeed this action leads to a regular e.o.m at \( u(x_4 = 0) = u_0 \).

\[
\partial_x^2 \xi - \left( \frac{14}{u} + \frac{15}{u^6 f} \right) u_x \partial_x \xi + \frac{u^{10} f^2 R_{AdS}^4}{a_0} \eta_{\mu \nu} \partial_\mu \partial_\nu \xi
\]
Using the shooting technic we found the eigenvalues of different modes of the fluctuation for various values of $m_q$, our finding are summarize in figures (4),(5). One can see that the masses $m_1'$ and $m_2'$ grow linearly with $m_q$ as expected from (48). At $u_0 = u_\Lambda = 1$ we find\footnote{Our results are for $R_{AdS} = 1.$}

$$m_2'^2 = 1.51 \quad ; \quad m_{as}^2 = 2.07.$$ \hspace{1cm} (51)

which is in agreement with \cite{5}.\footnote{To keep contact with the results in \cite{5} we had renormalized the masses by the factor $\frac{2}{5}$ coming from the change of variables $u \rightarrow z.$}

Again we also studied the dependence of the mass on the excitation number, the results are summarized in figure (6) and are;

$$m_n = 1.51 + 2.32n^{1.04} \quad m_q = 0$$ \hspace{1cm} (52)

$$m_n = 13.5 + 4.95n^{1.04} \quad m_q = 9.3$$

Thus we see that both in terms of the dependence on $n$ as well as the dependence on $m_q$ the scalar meson spectra admits a different behavior than that of the critical model of \cite{11}. A similar behavior has been observed for the vector mesons in \cite{5}.

Next we consider the case where the effective action includes both the DBI and CS terms. Including now the CS term (with its full strengh $\tilde{a} = 1$) the quadratic action for the fluctuation becomes

$$S = \frac{1}{2} \int dx^4 \left\{ \frac{B^{3/2}}{u^{14}f^3}(\partial_x \xi)^2 + \frac{B^{1/2}R_{AdS}}{u^4f^2}(\partial_x \xi)^2 \right\}$$

$$\quad - \frac{B^{1/2}(u^5 + 36B - 63u^{10} + 14u^{20} + 48u^{15} - 92Bu^5 - 44B^{10})}{2u^8f^3} - \frac{20u^3}{\sqrt{5}}u^3\xi^2 \right\}$$

where $B = (u_0^5f^{1/2}(u_0) - u_0^5 + u^5)^2$ and the e.o.m is then

$$\partial_x^2 \xi - \left( \frac{14}{u} + \frac{15u^4}{B^{1/2}} \right)u_x \partial_x \xi + \frac{u^{10}f^2R_{AdS}}{B} \eta^{\mu\nu} \partial_\mu \partial_\nu \xi$$

$$+ \frac{(u^5 + 36B - 63u^{10} + 14u^{20} + 48u^{15} - 92Bu^5 - 44B^{10})}{2u^8B} \xi + \frac{20u^3}{\sqrt{5}B^{3/2}}\xi = 0 \hspace{1cm} (54)$$

With the Chers Simon taking into account the dependence of the mass squared on the excitation number is now to be read from figure (6) to be:

$$m_n = 2.07 + 5.42n^{0.75} \quad m_q = 0$$ \hspace{1cm} (55)

$$m_n = 16.49 + 1.01n^{0.75} \quad m_q = 9.3$$

The dependence on $m_q$ is described in figures (7) and (8).
Figure 4: (A) The mass $m'_1$ of the lowest exited symmetric mode of the non-critical model as a function of $m'_q$ ($R_{AdS} = u_\Lambda = 1$).

Figure 5: (B) The mass $m'_2$ of the lowest exited antisymmetric mode of the non-critical model as a function of $m'_q$ ($R_{AdS} = u_\Lambda = 1$).

Figure 6: The tower of mesons masses $m'_n$ in the non-critical model.
Non critical model with CS term (a=1): The mass of the first excited symmetric mode vs. $m_q$.

Figure 7: (A) The mass $m'_1$ of the lowest exited symmetric mode of the non-critical model with CS term included as a function of $m_q$.

Figure 8: (B) The mass $m'_2$ of the lowest exited antisymmetric mode of the non-critical model with CS term included as a function of $m_q$.

Non critical model with CS term (a=1): The mass of the first excited anti-symmetric mode vs. $m_q$.

Figure 9: The tower of mesons masses $m'_n$ in the non-critical model with CS term included.
5 Parity and charge conjugation

In order to compare the resulting spectra from both the critical and non-critical models, we first have to identify the “quantum numbers” of the states that correspond to the fluctuations. More explicitly we have to determine the operations in the gravity models which correspond to charge conjugation and parity transformations. In the model of \[1\] they were defined as follows: The charge conjugation operation associates with exchanging the left and right handed quarks which maps into the interchange of a $D8$ and an anti $D8$ or differently transforming $z \rightarrow -z$. Parity transformation in the five-dimensional space-time spanned by $x_i, z$ where $i = 1, 2, 3$ means the following transformation $(x_i, z) \rightarrow (-x_i, -z)$.

For the generalized set up with $u_0 > u_\Lambda$ we can still define the coordinate $z$ as follows

$$u^3 = u_0^3 + u_\Lambda z^2$$

(56)

Note the difference with respect to (30) since here we take $z$ to have dimension of length. With this definition of the $z$ coordinate the discrete transformations of \[1\] remain in tact. The effective action on the probe brane has to be invariant under both parity and charge conjugation. The DBI part (21) is quadratic in $\xi$ and hence cannot determine the right transformation of the fluctuation modes. The situation with the CS term is different. Recall that the CS term has the form

$$S_{CS} \sim \int_{D8} F \wedge F \wedge C_5 = \int_{S^4} F \wedge F \wedge \int d^4x dz C_5 = \int_{S^4} F \wedge F \int d^4x dz \zeta(x^\mu, z)$$

(57)

The last part we have used the explicit form of the $C_5$

$$C_5 = \zeta(x^\mu, z) dx^0 \wedge \ldots dx^3 \wedge dz$$

(58)

In order for this term in the action to be invariant under parity and charge conjugation it is clear that $\zeta(x^\mu, z)$ has to be even under both charge conjugation and parity transformation. Now since $\zeta(x, z) = \sum_n f_n(x^\mu) \xi_n(z)$ we conclude that the map between the fluctuation modes and scalar particles is the following

$$\begin{align*}
\text{symmetric} & \quad \xi_n \quad \rightarrow \quad 0^{++} \text{ mesons} \\
\text{antisymmetric} & \quad \zeta_n \quad \rightarrow \quad 0^{--} \text{ mesons}
\end{align*}$$

(59)

For the non-critical model again the DBI action does not determine the transformations of $\xi$ under parity and charge conjugation. We have argued above based on \[8\] that a CS term of the form (44) should not be incorporated. Thus there is no way to this order to determine the transformation of $\xi$. 

17
Without the constraint from the CS term we may have that $\xi$ is even or odd under charge conjugation and parity transformations. In the latter case the assignments of (59) have to be reversed, namely symmetric $\xi$ corresponds to $0^{--}$ and antisymmetric $\xi$ to $0^{++}$.

Next we want to compare the spectra to mesons observed in nature. It is well known that scalar mesons in nature are either $0^{++}$ or pseudo scalars of the form $0^{--}$ and there are no observed low lying mesons of the form $0^{--}$. Thus there is a serious mismatch between the holographic scalar mesons extracted from models with flavor branes anti-branes of critical models and with the observed mesons in nature. We will come back to this issue in the conclusions.

6 Scalar mesons in the intermediate temperature phase

The background that corresponds to the deconfined phase namely $T > 1/2\pi R$ is given in (11). As was shown in [13] this deconfined background can admit also a phase where chiral symmetry is broken, the so called “intermediate phase” We now analyze the spectrum of the scalar mesons in this phase. Since the procedure of extracting the scalar meson is identical to that of the low temperature analysis of the previous sections we present the final results for the spectra of masses. The spectra are presented in figures (10), (11) and (12). The main features that these spectra admit are the following

• As can be seen, at the phase transition $T = T_d$ the value of the masses are (for the values $u_T = 1, u_0 = 8$)

\[ m_s^2(T = T_d) = 8.36 \quad ; \quad m_{as}^2(T = T_d) = 45.96 \] (60)

while in the low temperature phase at the point of phase transition with $u_\Lambda \rightarrow u_T = 1, u_0 = 8$ the masses are

\[ m_s^2(T = T_d) = 8.40 \quad ; \quad m_{as}^2(T = T_d) = 46.00 \] (61)

We see a very small jump in the masses at the transition point, the same as was seen for the vectors in [6].

• While in the low temperature confining phase the masses of the mesons are temperature independent since the background in this phase does not depend on the temperature, the masses of the mesons do depend on the temperature in the intermediate deconfined phase. As was observed in Lattice simulations and was found also for holographic vector mesons [3], the masses decrease as a function of the temperature. The symmetric mesons decrease at the chiral symmetry phase transition temperature $T = T_{\chi_{SB}}$ to \( \sim 60\% \) percent of their values whereas the antisymmetric
ones to $\sim 80\%$. This drop off is much more significant than for the vector mesons of the critical model \[6\].

- Note that it is only consistent to increase the temperature up to where the next phase transition occurs and chiral symmetry is restored. This happens at $T = T_{\chi SB}$ (for the choice $u_0 = 8$ we found that $T_{\chi SB} = 2.44T_d$), then the merged together $D8-\bar{D}8$ breaks into a separate pair of $D8-\bar{D}8$. We can also see from figure (10) that if we continue to increase the temperature beyond $T_{\chi SB}$ then at some point the scalar mode becomes tachyonic, signaling that this background is no longer stable at this temperature as indeed we know.

- Like in the low temperature we had also checked the squared masses dependence on the excitation number see figure (12), this was found to be:

\[
\begin{align*}
m_n^2 &= 8.3 + 6.4n^{1.7} & T = T_d \\
m_n^2 &= 7.6 + 6.9n^{1.65} & T = 2T_d
\end{align*}
\]

7 Conclusions

In this paper we had overcome technical problems faced in \[22,5\] and succeeded to obtain the holographic mass spectra of the scalars in the low and intermediate phases of the chiral symmetry broken phase of the critical model and also of those of the non-critical. Let us summarize the results of this work and mention certain open directions.

- There is a difference between the dependence of the mass of the scalar mesons on the “constituent mass parameter” $m_q^c$. In the ten dimensional models one finds a $m^2 \propto m_q^c$ relation (see figures (1),(2) for the first two excited modes), whereas for the non-critical model the relation is $m \propto m_q^c$ (see figures (4),(5) and (7),(8)).

- Both the critical models and the non-critical one do not admit a Regge/stringy behavior of $M_n^2 \sim n$. This is not unexpected since the stringy excitations is not visible in the low energy effective field theory.

- One can compare the ratio of the low lying mesons both vector and scalar mesons to those observed in nature. Table (1) present such a comparison. It is interesting to note that turning on a constituent mass $m_q^c$ improves the ratios with respect to those for zero $m_q^c$. 

19
Figure 10: (A) The mass squared $m^2(T)$ of the lowest exited symmetric mode as a function of $T/T_d$ ($u_0 = 8$, $R_{D4} = 1$ and $R = 2/3$)

Figure 11: (B) The mass squared $m^2(T)$ of the lowest exited antisymmetric mode as a function of $T/T_d$ ($u_0 = 8$, $R_{D4} = 1$ and $R = 2/3$)

Figure 12: The tower of mesons squared mass $m_n^2$ in the intermediate phase
Table 1: A comparison to the experimental data where the best fitted $m_q^c$ is presented vs. $m_q^c = 0$ (for the critical case we have found that there is no improvement in ratios of the vectors so we have left these entries empty.).

| Experiment | D4-D8 at $m_q^c = 0$ / 0.38 | Non-critical at $m_q^c = 0$ / 0.16 |
|------------|-----------------------------|-----------------------------------|
| $m_{v,2}/m_{v,1}$ | 2.51 | 2.4 / - |
| $m_{v,3}/m_{v,1}$ | 3.56 | 4.3 / - |
| $m_{s}/m_{v,1}$ | 3.61 | 4.9 / 3.63 |
| $m_{s}/m_{s}$ | 0.7 | 0.49 / 0.62 |

- The holographic spectra of the critical models admit a branch of scalar mesons of the type $0^{-+}$. These does not exist in nature. It seems to be a severe shortcoming of these holographic models. This difference cannot be attributed to the fact that we consider large $N_c$. It will be interesting to investigate the question of how generic is this situation and whether one can construct a mechanism to project it out from the low lying spectra.

- The behavior of the scalar mesons at finite temperature in the intermediate phase is similar to that for the vector meson in the model of [6]. However the decrease of the mass with increasing temperature is more dramatic for the scalar mesons. It is interesting to check if a similar phenomenon occurs also in lattice simulations.

Acknowledgments

We would like to thank Kasper Peeters, Tadakatsu Sakai and Marija Zamaklar for useful discussions, and specially to Ofer Aharony for many insightful conversations. This work was supported in part by a center of excellence supported by the Israel Science Foundation (grant number 1468/06), by a grant (DIP H52) of the German Israel Project Cooperation, by a BSF grant and by the European Network MRTN-CT-2004-512194.

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