On the viscous Cahn–Hilliard–Oono system with chemotaxis and singular potential

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We analyze a diffuse interface model that couples a viscous Cahn–Hilliard equation for the phase function \( \varphi \) with a diffusion-reaction equation for the nutrient concentration \( \sigma \). The system under consideration also takes into account some important mechanisms like chemotaxis, active transport, and nonlocal interaction of Oono’s type. When the spatial dimension is three, we prove the existence and uniqueness of a global weak solution to the model with singular potentials including the physically relevant logarithmic potential. Then we obtain some regularity properties of the weak solution when \( t > 0 \). In particular, with the aid of the viscous term \( \epsilon \partial_t \varphi \), we prove the so-called instantaneous separation property of the phase variable \( \varphi \) such that it stays away from the pure states \( \pm 1 \) as long as \( t > 0 \). Next, we study longtime behavior of the system, by proving the existence of the global attractor in a suitable phase space and characterizing the \( \omega \)-limit set. Moreover, we show the existence of an exponential attractor, which implies that the global attractor is of finite fractal dimension.

KEYWORDS
Cahn–Hilliard–Oono equation, chemotaxis, longtime behavior, singular potential, well-posedness

MSC CLASSIFICATION
35A01; 35A02; 35K35; 35Q92

1 | INTRODUCTION

In this paper, we consider the following system of partial differential equations:

\[
\begin{align*}
\partial_t \varphi &= \Delta \mu - \alpha (\varphi - c_0), \quad \text{in } \Omega \times (0, +\infty), \\
\mu &= A \Psi'(\varphi) - B \Delta \varphi - \chi \sigma + \epsilon \partial_t \varphi, \quad \text{in } \Omega \times (0, +\infty), \\
\partial_t \sigma &= \Delta (\sigma + \chi (1 - \varphi)), \quad \text{in } \Omega \times (0, +\infty), \\
\partial_n \varphi &= \partial_n \mu = \partial_n \sigma = 0, \quad \text{on } \partial \Omega \times (0, +\infty), \\
\varphi|_{t=0} &= \varphi_0, \quad \sigma|_{t=0} = \sigma_0, \quad \text{in } \Omega.
\end{align*}
\]
Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial \Omega$, and $n = n(x)$ denotes the unit outward normal vector on $\partial \Omega$.

Systems (1.1a)–(1.1c) can be viewed as a simplified, fluid-free version of the general thermodynamically consistent diffuse interface model derived in Lam and Wu\(^1\) for a two-phase incompressible fluid mixture with a chemical species subject to some important mechanisms like diffusion, chemotaxis interaction, and active transport. The order parameter (phase function) $\varphi$ denotes the difference of volume fractions for the two components, while $\sigma$ stands for the concentration of the nutrient. The function $\mu$ defined by (1.1b) is regarded as the chemical potential associated to $(\varphi, \sigma)$. In (1.1b), $A$ and $B$ are two positive constants related to the surface tension and the thickness of the diffuse interface, respectively. The bulk free energy density $\Psi$ considered in this paper enjoys a double well structure that leads to the separation phenomenon of the binary mixture. A typical form is given by

$$\Psi(r) = \frac{\theta}{2}((1 - r) \ln(1 - r) + (1 + r) \ln(1 + r)) + \frac{\theta_0}{2} (1 - r^2), \quad \forall r \in (-1, 1),$$

with $0 < \theta < \theta_0$ (see, e.g., Cahn and Hilliard and Cherfils et al\(^2,3\)). Comparing with the classical quartic potential $\Psi_{\text{reg}}(r) = \frac{1}{4} (r^2 - 1)^2$, the singular nature of its derivative $\Psi'$ at the pure phases $\pm 1$ guarantees that the phase variable $\varphi$ lies in the physical range $[-1, 1]$ (see several studies\(^4-6\)). The nontrivial coupling between Cahn–Hilliard equations (1.1a) and (1.1b) and diffusion equation (1.1c) for the nutrient is characterized by the constant $\chi$, which models some specific mechanisms such as chemotaxis/active transport in the context of tumor growth modeling (see, e.g., Garcke and Lam and Garcke et al\(^7,8\)). Cahn–Hilliard equation (1.1a) also involves some nonlinear interaction that is given by Oono’s type for the sake of simplicity (cf., e.g., Giorgini et al. and Miranville\(^9,10\)), where $a \geq 0$, $c_0 \in (-1, 1)$. Besides, it contains a viscous term $\epsilon \partial_t \varphi$ with $\epsilon > 0$, which represents possible influence of the internal microforces (see, e.g., Gurtin and Novick-Cohen\(^11,12\)). In this sense, one may call (1.1a) and (1.1b) (neglecting the term $\chi \sigma$) a viscous Cahn–Hilliard–Oono system.

The Cahn–Hilliard-type models have been employed as an efficient mathematical tool for the study on dynamics of binary mixtures, in particular, recently for the tumor growth modeling.\(^{13-15}\) Concerning the mathematical analysis of the Cahn–Hilliard equation and its variants, we refer to several studies\(^3,7,10,16-29\) and the references cited therein (see also the recent book\(^30\)).

Our aim in this paper is to perform a first-step study on simplified models (1.1a)–(1.1c) that still maintains some interesting features, for example, the effects of chemotaxis and active transport associated with the nutrient, and certain nonlinear interactions between the two components themselves.

First, under suitable assumptions on the singular potential function $\Psi$ and coefficients of the system, we show the existence and uniqueness of a global weak solution to problems (1.1a)–(1.3) on the whole interval $[0, +\infty)$ (see Theorem 2.1). The proof is based on a suitable Galerkin approximation that mainly follows the argument in He\(^31\) for a more general system with fluid interaction. Thanks to the singular potential, we are allowed to remove certain restricted assumptions on the coefficients $A$ and $\chi$ when a regular potential was adopted (cf. Lam and Wu and Garcke and Lam\(^1,7\)). We recall that the existence of global weak solutions to the Cahn–Hilliard–Oono system with a singular potential has been proven in Giorgini et al\(^9\) without the viscous term (see also Miranville and Temam\(^28\) for the case with fluid interaction, i.e., the Cahn–Hilliard–Oono–Navier–Stokes system). Our result extends the previous works to the case with chemotaxis and active transport, that is, the coupling with Equation (1.1c) and $\chi \neq 0$, in the three-dimensional setting. At the level of weak solutions, the additional viscous term $\epsilon \partial_t \varphi$ does not play an essential role and the conclusion can be easily extended to the non-viscous case with $\epsilon = 0$ (cf. He\(^31\)). For well-posedness results on more general system with further mechanics (like tumor proliferation, apoptosis, nutrient consumption, etc.) and subject to different types of boundary conditions, we refer to Garcke and Lam\(^7,22\) and so on.

Next, we pay our attention to the regularity of global weak solutions to problems (1.1a)–(1.3). After proving the existence and uniqueness of global strong solutions (see Theorem 2.2), we can take advantage of the parabolic nature of the evolution system and show the instantaneous regularizing effect of weak solutions for $t > 0$ (see Corollary 2.1). In particular, thanks to the viscous term $\epsilon \partial_t \varphi$, we obtain the so-called instantaneous strict separation property for the phase function $\varphi$ in the three-dimensional setting; that is, $\varphi$ stays away from the pure phases $\pm 1$ for all $t > 0$, and the separation is uniform when $t \geq \eta$, for an arbitrary but fixed $\eta > 0$. The property of separation from pure states plays an important role in the study of the Cahn–Hilliard-type equations, since the singular potential can thus be regarded as a globally Lipschitz function such that further regularity of the solutions can be gained (see, e.g., several studies\(^3,4,6,9\)). When logarithmic potential (1.4) is considered, the strict separation property for the Cahn–Hilliard equation in two dimensions has been proven in Miranville...
and Zelik. Later on, Giorgini et al. extended the result to the Cahn–Hilliard–Oono equation by using an alternative approach, which can be generalized to more complicated systems with fluid interactions (see, e.g., several studies). However, the situation is less satisfactory when the spatial dimension is three, because the singularity of the logarithmic potential at $\pm 1$ seems not strong enough. On one hand, it was shown in Abels and Wilke that a weak solution of the Cahn–Hilliard equation will stay eventually away from the pure states for sufficiently large time. On the other hand, there were two possible ways in the literature to recover the instantaneous strict separation: one is to impose a stronger singularity on the potential function (Miranville and Zelik; Remark 7.1; see also Londen and Petzeltová for some recent improvements), and the other one is to introduce a viscous term $\varepsilon \partial_t \varphi$ in the chemical potential $\mu$, which brings some regularizing effects on the time derivative of $\varphi$ (see Miranville and Zelik, Section 3). In this paper, we choose to include this viscous term and extend the result in Miranville and Zelik to coupled systems (1.1a)–(1.1c) with chemotaxis, active transport, and Oono's interaction. The argument in Miranville and Zelik is essentially based on a comparison principle for second-order parabolic equations that is only available for $\varepsilon > 0$. In our case, additional efforts have to be made to overcome the difficulties brought by the nonlocal Oono's term (which yields the loss of mass conservation) as well as the coupling with the nutrient $\sigma$ when $\chi \neq 0$.

Finally, we study the longtime behavior of problems (1.1a)–(1.3). Based on some dissipative estimates and the asymptotic compactness of global weak solutions, we prove the existence of a global attractor and an exponential attractor for the corresponding dynamical system generated by global weak solutions of problems (1.1a)–(1.3) in a proper phase space (see Theorems 2.3 and 2.4). In this aspect, we mention Frigeri et al. and Miranville et al. for the study on global attractors of some related diffuse interface models with regular potentials and neglecting the effects due to chemotaxis as well as active transport, while for results on exponential attractors for the Cahn–Hilliard equation, we refer to several studies and the references therein. At last, making use of Lyapunov structure of the system (see (3.40)), we are able to characterize the $\omega$-limit set of an arbitrary initial datum in the finite energy space (see Theorem 2.5). This also provides a dynamical approach for the investigation of steady states to problems (1.1a)–(1.3) even for the non-viscous case $\varepsilon = 0$ (see Corollary 5.1). We remark that there are some further issues worth investigating, such as the convergence of any bounded global weak solution to a single steady state as $t \to +\infty$ (cf. Abels and Wilke and Giorgini et al.) and the asymptotic behavior of solutions as $\varepsilon \to 0$ (cf. Miranville and Zelik and Efendiev et al.). These will be addressed in a forthcoming paper.

The remaining part of this paper is organized as follows. In Section 2, we introduce the functional setting and state the main results. Section 3 is devoted to the existence and uniqueness of global weak solutions. In Section 4, we prove the existence, uniqueness, and the strict separation property of global strong solutions. In Section 5, we first show the instantaneous regularity of global weak solutions for $t > 0$ and, in particular, the separation from pure states. Then we prove the existence of a global attractor as well as an exponential attractor and finally give a characterization of the $\omega$-limit set.

2 | MAIN RESULTS

2.1 | Preliminaries

We assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial \Omega$. For the standard Lebesgue and Sobolev spaces, we use the notations $L^p := L^p(\Omega)$ and $W^{k,p} := W^{k,p}(\Omega)$ for any $p \in [1, +\infty]$, $k \in \mathbb{N} \setminus \{0\}$ equipped with the norms $\| \cdot \|_L^p$ and $\| \cdot \|_{W^{k,p}}$. In the case $p = 2$, we denote $H^k := W^{k,2}(\Omega)$ with the norm $\| \cdot \|_{H^k}$. The norm and inner product on $L^2(\Omega)$ are simply denoted by $\| \cdot \|$ and $\langle \cdot , \cdot \rangle$, respectively. The dual space of a Banach space $X$ is denoted by $X'$, and the duality pairing between $X$ and its dual will be denoted by $\langle \cdot , \cdot \rangle_{X' \times X}$. Given an interval $J \subset \mathbb{R}^+$, we introduce the function space $L^p(J; X)$ with $p \in [1, +\infty]$, which consists of Bochner measurable $p$-integrable functions with values in the Banach space $X$.

For every $f \in H^1(\Omega)'$, we denote by $\bar{f}$ its generalized mean value over $\Omega$ such that $\bar{f} = |\Omega|^{-1} \int_{\Omega} f \, dx$. Besides, in view of the homogeneous Neumann boundary condition (1.2), we also set

$$H^1_N(\Omega) := \{ f \in H^1(\Omega) : \partial_n f = 0 \text{ on } \partial \Omega \}$$

and recall the following Poincaré–Wirtinger inequality:

$$\| f - \bar{f} \| \leq C_p \| \nabla f \|, \forall f \in H^1(\Omega), \tag{2.1}$$
where \( C_F > 0 \) is a constant depending only on the spatial dimension and \( \Omega \). Consider the realization of the operator \(-\Delta\) with homogeneous Neumann boundary condition denoted by \( A_N \). We denote \( A_N \in \mathcal{L}(H^1(\Omega), H^1(\Omega)') \) that is defined by

\[
\langle A_N u, v \rangle_{H^1(\Omega), H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v \, dx, \text{ for } u, v \in H^1(\Omega).
\]

Then for the linear subspaces

\[
V_0 = \{ u \in H^1(\Omega) : \hat{u} = 0 \}, \quad V_0' = \{ u \in H^1(\Omega)' : \hat{u} = 0 \},
\]

the restriction of \( A_N \) from \( V_0 \) onto \( V_0' \) is an isomorphism. In particular, \( A_N \) is positively defined on \( V_0 \) and self-adjoint. We then denote its inverse map by \( \mathcal{N} = A_N^{-1} : V_0' \to V_0 \). Note that for every \( f \in V_0' \), \( u = \mathcal{N} f \in V_0 \) is the unique weak solution of the Neumann problem

\[
-\Delta u = f \quad \text{in } \Omega, \quad \text{with } \partial_\nu u = 0 \quad \text{on } \partial \Omega.
\]

Besides, we have (cf. Giorgini et al\(^9\))

\[
\langle A_N u, \mathcal{N} g \rangle_{V_0', V_0} = \langle g, u \rangle_{H^1(\Omega), H^1(\Omega)} , \quad \forall u \in H^1(\Omega), \forall g \in V_0', \quad \text{(2.2)}
\]

\[
\langle g, \mathcal{N} f \rangle_{V_0', V_0} = \langle f, \mathcal{N} g \rangle_{V_0', V_0} = \int_\Omega \nabla (\mathcal{N} g) \cdot \nabla (\mathcal{N} f) \, dx, \quad \forall g, f \in V_0', \quad \text{(2.3)}
\]

and the chain rule

\[
\langle \partial_t u, \mathcal{N} u(t) \rangle_{V_0', V_0} = \frac{1}{2} \frac{d}{dt} \| \nabla \mathcal{N} u \|^2, \quad \text{a.e. in } (0, T),
\]

for any \( u \in H^1(0, T; V_0') \). For any \( f \in V_0' \), we set \( \| f \|_{V_0'} = \| \nabla \mathcal{N} f \| \). It is well known that \( f \to \| f \|_{V_0'} \) and \( f \to (\| f \|_{V_0'}^2 + |f|^2)^{\frac{1}{2}} \) are equivalent norms on \( V_0' \) and \( H^1(\Omega)' \), respectively. Besides, thanks to (2.1), we see that \( f \to \| \nabla f \|, \quad f \to (\| \nabla f \|^2 + |f|^2)^{\frac{1}{2}} \) are equivalent norms on \( V_0 \) and \( H^1(\Omega) \), and the inequality \( \| f \| \leq \| f \|_{V_0'}^{\frac{1}{2}} \| \nabla f \|^{\frac{1}{2}} \) holds for any \( f \in V_0 \). We also consider the operator \( A_1 := I - \Delta \) with homogeneous Neumann boundary condition that is an unbounded operator on \( L^2(\Omega) \) with domain \( D(A_1) = H^1_0(\Omega) \). It is well known that \( A_1 \) is a positive, unbounded, self-adjoint operator in \( L^2(\Omega) \) with a compact inverse (denoted by \( \mathcal{N}_1 := A_1^{-1} \)) (see, e.g., Temam\(^37\), Chapter II, Section 2.2). Then \( f \to \| \mathcal{N}_1^{\frac{1}{2}} f \| \) is also an equivalent norm on \( H^1(\Omega)' \).

In the sequel, the symbol \( C \) denotes a generic positive constant that may depend on norms of the initial data, the domain \( \Omega \), and parameters of the system. We denote by \( C_T \) if a positive constant depends on the final time \( T \). Specific dependence will be pointed out if necessary.

### 2.2 Main results

We introduce the following hypotheses as in He and He and Wu\(^{31,34}\)

(H1) The singular potential \( \Psi \) belongs to the class of functions \( C([-1, 1]) \cap C^2(-1, 1) \) and can be written into the following form:

\[
\Psi(r) = \Psi_0(r) - \frac{\theta_0}{2} r^2,
\]

such that

\[
\lim_{r \to \pm 1} \Psi_0'(r) = \pm \infty, \quad \text{and} \quad \Psi_0''(r) \geq \theta, \quad \forall r \in (-1, 1),
\]

with some strictly positive constants \( \theta_0, \theta \) satisfying

\[
\theta_0 - \theta := K > 0. \quad (2.4)
\]

We make the extension \( \Psi_0(r) = +\infty \) for any \( r \not\in [-1, 1] \). In addition, there exists a small constant \( a_0 \in (0, 1) \) such that \( \Psi'' \) is nondecreasing in \([1 - a_0, 1]\) and nonincreasing in \((-1, -1 + a_0]\).
(H2) The coefficients $A, B, c, \chi, \alpha$, and $c_0$ are prescribed constants and satisfy
\[ A > 0, B > 0, \epsilon > 0, \alpha > 0, \chi \in \mathbb{R}, c_0 \in (-1, 1). \]

Remark 2.1. It is easy to verify that the logarithmic potential (1.4) fulfills assumption (H1). This physically relevant case is indeed what we are interested in.

Next, we introduce the notions of weak and strong solutions to initial boundary value problems (1.1a)–(1.3).

**Definition 2.1** (Weak solutions). Suppose that the initial data satisfy $\varphi_0 \in H^1(\Omega)$, $\sigma_0 \in L^2(\Omega)$ with $\|\varphi_0\|_{L^\infty} \leq 1$ and $|\varphi_0| < 1$. A triple $(\varphi, \mu, \sigma)$ is a global weak solution to problems (1.1a)–(1.3), if it fulfills the following regularity properties:
\[
\varphi \in C([0, +\infty); H^1(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^2_N(\Omega)),
\]
\[
\partial_t \varphi \in L^2(0, +\infty; L^2(\Omega)),
\]
\[
\mu \in L^2_{\text{loc}}(0, +\infty; H^1(\Omega)),
\]
\[
\sigma \in C([0, +\infty); L^2(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^1(\Omega)),
\]
\[
\partial_t \sigma \in L^2(0, +\infty; H^1(\Omega)),
\]
with $\varphi \in L^\infty(\Omega \times (0, +\infty))$ and $|\varphi(x, t)| < 1$ a.e. in $\Omega \times (0, +\infty)$, and satisfies
\[
\langle \partial_t \varphi, \xi \rangle_{H^1(\Omega)} = -\langle \nabla \mu, \nabla \xi \rangle - \alpha(\varphi - c_0, \xi), \quad \text{a.e. in } (0, +\infty),
\]
\[
\mu = A\Psi'(\varphi) - B\Delta \varphi - \chi \sigma + e \partial_t \varphi, \quad \text{a.e. in } \Omega \times (0, +\infty),
\]
\[
\langle \partial_t \sigma, \xi \rangle_{H^1(\Omega)} + \langle \nabla \sigma, \nabla \xi \rangle = \chi(\nabla \varphi, \nabla \xi), \quad \text{a.e. in } (0, +\infty),
\]
for all $\xi \in H^1(\Omega)$. Moreover, the initial conditions are fulfilled $\varphi|_{t=0} = \varphi_0$, $\sigma|_{t=0} = \sigma_0$.

**Definition 2.2** (Strong solutions). Suppose that the initial data satisfy $\varphi_0 \in H^2_N(\Omega)$, $\sigma_0 \in H^2_N(\Omega)$ with $\|\varphi_0\|_{L^\infty} \leq 1$ and $|\varphi_0| < 1$. A triple $(\varphi, \mu, \sigma)$ is a global strong solution to problems (1.1a)–(1.3), if it fulfills the following higher-order regularity properties:
\[
\varphi \in C([0, +\infty); H^2_N(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^3(\Omega)),
\]
\[
\partial_t \varphi \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2(0, +\infty; H^1(\Omega)),
\]
\[
\mu \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^1(\Omega)),
\]
\[
\sigma \in C([0, +\infty); H^2_N(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^3(\Omega)),
\]
\[
\partial_t \sigma \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^1(\Omega)),
\]
and satisfies systems (1.1a)–(1.1c) a.e. in $\Omega \times (0, +\infty)$ as well as initial condition (1.3).

Now, we state the main results of this paper.

**Theorem 2.1** (Global weak solutions). Suppose that (H1) and (H2) are satisfied. For any initial data $(\varphi_0, \sigma_0) \in H^1(\Omega) \times L^2(\Omega)$ satisfying $\|\varphi_0\|_{L^\infty} \leq 1$ and $|\varphi_0| < 1$, problems (1.1a)–(1.3) admit a unique global weak solution $(\varphi, \mu, \sigma)$ in the sense of Definition 2.1. Moreover, consider two groups of initial data satisfying $(\varphi_{0i}, \sigma_{0i}) \in H^1(\Omega) \times L^2(\Omega)$ with $\|\varphi_{0i}\|_{L^\infty} \leq 1, |\varphi_{0i}| < 1, i = 1, 2$, and an arbitrary $T > 0$. The global weak solutions $(\varphi_1, \sigma_1)$, $(\varphi_2, \sigma_2)$ to problems (1.1a)–(1.3) on $[0, T]$ with initial data $(\varphi_{0i}, \sigma_{0i})$, $i = 1, 2$, satisfy the following continuous dependence estimate:
\[
\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 + e\|\varphi_1(t) - \varphi_2(t)\|^2 + \|\sigma_1(t) - \sigma_2(t)\|_{H^1(\Omega)}^2
\]
\[
+ \int_0^t \|\varphi_1(\tau) - \varphi_2(\tau)\|_{H^1(\Omega)}^2 d\tau + \int_0^t \|\sigma_1(\tau) - \sigma_2(\tau)\|^2 d\tau
\]
\[
\leq C_T \left( \|\varphi_{01} - \varphi_{02}\|_{H^1(\Omega)}^2 + e\|\varphi_{01} - \varphi_{02}\|^2 + \|\sigma_{01} - \sigma_{02}\|_{H^1(\Omega)}^2 + |\varphi_{01} - \varphi_{02}| \right),
\]
for all $t \in [0, T]$, where the constant $C_T > 0$ may depend on norms of the initial data, $\Omega$, $T$, and coefficients of the system.
Theorem 2.2 (Global strong solutions). Suppose that (H1) and (H2) are satisfied. For any initial data \((\varphi_0, \sigma_0) \in H^2_N(\Omega) \times H^1_N(\Omega)\) satisfying \(|\varphi_0| < 1\) and

\[
\|\varphi_0\|_{L^\infty(\Omega)} \leq 1 - \delta_0, \tag{2.8}
\]

for some \(\delta_0 \in (0, 1)\), problems (1.1a)–(1.3) admit a unique global strong solution in the sense of Definition 2.2. Furthermore, there exists a constant \(\delta \in (0, \delta_0]\) such that

\[
\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall t \in [0, +\infty), \tag{2.9}
\]

where \(\delta\) depends on \(\delta_0\), norms of the initial data, \(\Omega\), and coefficients of the system.

By virtue of the above well-posedness result of global strong solutions, we are able to prove that the global weak solution regularizes instantaneously for \(t > 0\).

Corollary 2.1 (Regularity of weak solutions). Suppose that the assumptions of Theorem 2.1 are satisfied. For any \(\tau > 0\), the global weak solution \((\varphi, \mu, \sigma)\) obtained in Theorem 2.1 becomes a strong one on \([\tau, +\infty)\). Moreover, there exists a constant \(\delta_\tau \in (0, 1)\) such that

\[
\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_\tau, \quad \forall t \in [\tau, +\infty), \tag{2.10}
\]

where \(\delta_\tau\) depends on \(\tau\), \(\|\varphi_0\|_{H^r}\), \(\|\sigma_0\|_{1 - |\varphi_0|}\), \(\Omega\), and coefficients of the system.

Next, we state results on the longtime behavior. For any given constants

\[
m_1 \in [0, 1), \quad c_0 \in [-m_1, m_1], \quad \text{and} \quad m_2 \geq 0,
\]

we introduce the following phase space:

\[
\mathcal{X}_{m_1, m_2} = \{ (\varphi, \sigma) \in H^1(\Omega) \times L^2(\Omega) : \|\varphi\|_{L^\infty} \leq 1, |\varphi| \leq m_1, |\sigma| \leq m_2 \}
\]

endowed with the metric \(d((\varphi_1, \sigma_1), (\varphi_2, \sigma_2)) = \|\varphi_1 - \varphi_2\|_{H^1} + \|\sigma_1 - \sigma_2\|\). It is straightforward to check that \(\mathcal{X}_{m_1, m_2}\) is a complete metric space.

We first prove the existence of a global attractor in \(\mathcal{X}_{m_1, m_2}\):

\[
\text{Theorem 2.3 (Global attractor). Suppose that (H1) and (H2) are satisfied. Problems (1.1a)–(1.3) generate a strongly continuous semigroup } S(t) : \mathcal{X}_{m_1, m_2} \to \mathcal{X}_{m_1, m_2} \text{ such that } S(t)(\varphi_0, \sigma_0) = (\varphi(t), \sigma(t)) \text{ for all } t \geq 0, \text{ where } (\varphi, \sigma) \text{ is the unique global weak solution corresponding to the initial datum } (\varphi_0, \sigma_0) \in \mathcal{X}_{m_1, m_2} \text{ given by Theorem 2.1. Moreover, the dynamical system } (S(t), \mathcal{X}_{m_1, m_2}) \text{ possesses a compact global attractor } A_{m_1, m_2} \subset \mathcal{X}_{m_1, m_2}, \text{ which is bounded in } H^2(\Omega) \times H^1(\Omega).
\]

Next, we prove that the dynamic system \((S(t), \mathcal{X}_{m_1, m_2})\) possesses an exponential attractor \(\mathcal{M}_{m_1, m_2}\).

\[
\text{Theorem 2.4 (Exponential attractor). Suppose that (H1) and (H2) are satisfied, and assume in addition } \Psi \in C^4(-1, 1). \text{ The dynamical system } (S(t), \mathcal{X}_{m_1, m_2}) \text{ possesses an exponential attractor } \mathcal{M}_{m_1, m_2} \subset \mathcal{X}_{m_1, m_2}, \text{ which is bounded in } H^2(\Omega) \times H^1(\Omega) \text{ and satisfies the following properties: }
\]

\begin{enumerate}
\item \(\mathcal{M}_{m_1, m_2}\) is positively invariant such that \(S(t)\mathcal{M}_{m_1, m_2} \subset \mathcal{M}_{m_1, m_2}\) for all \(t \geq 0\).
\item \(\mathcal{M}_{m_1, m_2}\) has finite fractal dimension in \(\mathcal{X}_{m_1, m_2}\).
\item \(\mathcal{M}_{m_1, m_2}\) is an exponentially attracting set; that is, there exists a constant \(\omega > 0\) such that, for every bounded set \(B \subset \mathcal{X}_{m_1, m_2}\), there is a positive constant \(J\) such that

\[
\text{dist}_{H^1 \times L^2}(S(t)B, \mathcal{M}_{m_1, m_2}) \leq Je^{-\omega t}, \quad \forall t \geq 0,
\]

where \(\text{dist}_{H^1 \times L^2}(\cdot, \cdot)\) denotes the Hausdorff semidistance between two sets with respect to the metric of \(H^1(\Omega) \times L^2(\Omega)\).
\end{enumerate}

The last result concerns the sequential convergence of global weak solutions as time goes to infinity. More precisely, we have
Theorem 2.5 (Convergence to equilibrium). Suppose that (H1) and (H2) are satisfied. Let $(\varphi, \sigma)$ be a global weak solution to problems (1.1a)–(1.3) with initial datum $(\varphi_0, \sigma_0)$ as given in Theorem 2.1. Then there exists an equilibrium $(\varphi_\infty, \sigma_\infty) \in H_0^2(\Omega) \times H_0^2(\Omega)$, which is a strong solution to the following stationary problem:

\[-B\Delta \varphi_\infty + A\Psi'(\varphi_\infty) - \chi \sigma_\infty + aN(\varphi_\infty - c_0) = A\Psi'(\varphi_\infty) - \chi \sigma_\infty, \ a.e. \ in \ \Omega, \quad (2.11a)\]

\[\Delta (\sigma_\infty - \chi \varphi_\infty) = 0, \ a.e. \ in \ \Omega, \quad (2.11b)\]

\[\partial_n \varphi_\infty = \partial_n \sigma_\infty = 0, \ on \ \partial \Omega, \quad (2.11c)\]

with $\varphi_\infty = c_0$, $\sigma_\infty = \sigma_0$, (2.11d)

and an increasing unbounded sequence $\{t_n\} \nearrow +\infty$ such that

$(\varphi(t_n), \sigma(t_n)) \to (\varphi_\infty, \sigma_\infty)$ strongly in $H^2(\Omega) \times H^2(\Omega)$, for some $r \in (1/2, 1)$. Furthermore, there exists a constant $\delta_\infty \in (0, 1)$ such that

$\|\varphi_\infty\|_{L^\infty(\Omega)} \leq 1 - \delta_\infty$. (2.12)

Remark 2.2. In light of the abstract theory for infinite-dimensional dynamical systems (see, e.g., Miranville and Zelik38, Section 3), we know that (1) the global attractor $\mathcal{A}_{m_1, m_2}$, if it exists, is unique and (2) the existence of an exponential attractor (see Theorem 2.4) yields that the fractal dimension of the global attractor $\mathcal{A}_{m_1, m_2}$ obtained in Theorem 2.3 is finite. Besides, thanks to the Lyapunov structure of problems (1.1a)–(1.3), we can deduce from Chapter 7, Theorem 4.1 of Temam37 that the global attractor $\mathcal{A}_{m_1, m_2}$ coincides with the unstable manifold of the set of equilibria, that is, $\mathcal{A}_{m_1, m_2} = \mathcal{M} + (\mathcal{E}_{m_1, m_2})$, where $\mathcal{E}_{m_1, m_2}$ consists of equilibria that are solutions to stationary problems (2.11a)–(2.11d) with constraints $m_1, m_2$.

Remark 2.3. (1) Results similar to Theorems 2.1–2.5 can be obtained when the spatial dimension is two. Indeed, in the two-dimensional case, one can even prove the instantaneous separation property for the phase function $\varphi$ when the viscous term vanishes, that is, $\epsilon = 0$, by extending the arguments in Miranville and Zelik and Giorgini et al.9,28 (cf. He and Wu34). (2) In (H2), we have assumed that $\alpha > 0$, that is, including Oono’s interaction in the analysis. Nevertheless, with minor modifications, the case $\alpha = 0$ can be easily treated, keeping in mind that the total mass $\int_\Omega \varphi(x, t)dx$ is now conserved.

3 | GLOBAL WEAK SOLUTIONS

In this section, we prove Theorem 2.1 on the existence and uniqueness of global weak solutions to problems (1.1a)–(1.3).

3.1 | Existence

The existence of global weak solutions can be obtained by adapting the arguments in several studies.9,22,28,31 Since the procedure is standard, we only sketch it here.

3.1.1 | Approximating the singular potential

For the singular potential $\Psi$ satisfying (H1), without loss of generality, we assume that $\Psi_0(0) = 0$. Then we can approximate the singular part $\Psi'_0$ in a standard manner, for instance, as in Giorgini et al. and Miranville and Temam9,28:

$$\Psi'_{0,\kappa}(r) = \begin{cases} 
\Psi'_0(-1 + \kappa) + \Psi'_0(-1 + \kappa)(r + 1 - \kappa), & r < -1 + \kappa, \\
\Psi'_0(r), & |r| \leq 1 - \kappa, \\
\Psi'_0(1 - \kappa) + \Psi'_0(1 - \kappa)(r - 1 + \kappa), & r > 1 - \kappa,
\end{cases}$$ (3.1)
for a sufficiently small parameter \( \kappa \in (0, a_0) \). Define

\[
\Psi_{\kappa}(r) = \int_0^r \Psi_{\kappa}(s) \, ds, \quad \Psi_{\kappa}(r) = \Psi_{\kappa}(r) - \frac{\theta_0}{2} r^2.
\]

We can verify that \( \Psi_{\kappa}(r) \geq \theta > 0 \) and \( \Psi_{\kappa}(r) \geq -L \) for \( r \in \mathbb{R} \), where \( L > 0 \) is a constant independent of \( \kappa \). Moreover, it holds that \( \Psi_{\kappa}(r) \leq \Psi_{\kappa}(0) \) for \( r \in [-1, 1] \) (see, e.g., Giorgini et al. [33]).

Let \( T > 0 \) be an arbitrary but given final time. For any initial data \((\varphi_0, \sigma_0)\) satisfying \( \varphi_0 \in H^1(\Omega) \), \( \sigma_0 \in L^2(\Omega) \) with \( \|\varphi_0\|_{L^\infty} \leq 1 \) and \( |\varphi_0| < 1 \), we consider the following approximate problem: looking for functions \((\varphi^\kappa, \mu^\kappa, \sigma^\kappa)\) satisfying

\[
\begin{align*}
\langle \partial_t \varphi^\kappa, \zeta \rangle_{H^1(\Omega)} &= \langle -(\nabla \mu^\kappa, \nabla \varphi^\kappa) - \alpha(\varphi^\kappa - c_0, \zeta) \rangle, \\
\mu^\kappa &= A\Psi_k(\varphi^\kappa) - B\nabla \varphi^\kappa - \chi \sigma^\kappa + e \partial_t \varphi^\kappa, \\
\langle \partial_t \sigma^\kappa, \zeta \rangle_{H^1(\Omega)} + \langle (\nabla \sigma^\kappa, \nabla \varphi^\kappa) \rangle &= \chi(\nabla \varphi^\kappa, \nabla \zeta), \\
\varphi^\kappa|_{t=0} &= \varphi_0, \quad \sigma^\kappa|_{t=0} = \sigma_0.
\end{align*}
\]

in \( \Omega \times (0, T) \) for all \( \zeta \in H^1(\Omega) \).

### 3.1.2 The Galerkin scheme

For every fixed \( \kappa \in (0, a_0) \), regularized problems (3.2)–(3.5) can be solved via a standard Galerkin scheme like in Garcke and Lam.\(^7,22\) To this end, we take \( \{w_i\} \ (i = 1, 2, \ldots) \) to be eigenfunctions for the Laplacian subject to homogeneous Neumann boundary condition:

\[
-\Delta w_i = \lambda_i w_i \quad \text{in} \ \Omega, \quad \partial_n w_i = 0 \quad \text{on} \ \partial \Omega,
\]

where \( \lambda_i \) is the eigenvalue corresponding to \( w_i \). It is well known that the family of eigenfunctions \( \{w_i\} \) can be chosen as an orthonormal basis of \( L^2(\Omega) \) and then forms an orthogonal basis of \( H^1(\Omega) \). Since constant functions are eigenfunctions, we can take \( w_1 = 1 \) with \( \lambda_1 = 0 \). Let \( W_m := \text{span}\{w_1, \ldots, w_m\} \subset L^2(\Omega) \) denote the finite-dimensional space spanned by the first \( m \) basis functions and \( P_m : L^2(\Omega) \to W_m \) be the projection operator. We look for approximating functions in \( W_m \)

\[
\varphi^m_m(x, t) = \sum_{k=1}^{m} a^m_k(t)w_k(x), \quad \mu^m_m(x, t) = \sum_{k=1}^{m} b^m_k(t)w_k(x), \quad \sigma^m_m(x, t) = \sum_{k=1}^{m} c^m_k(t)w_k(x),
\]

which satisfy

\[
\begin{align*}
\langle \partial_t \varphi^m_m, w_k \rangle &= \langle -(\nabla \mu^m_m, \nabla w_k) - \alpha(\varphi^m_m - c_0, w_k) \rangle, \\
\langle \mu^m_m, w_k \rangle &= A\Psi_k(\varphi^m_m, w_k) + B\nabla \varphi^m_m, \nabla w_k) - \chi(\sigma^m_m, w_k) + e \partial_t \varphi^m_m, w_k), \\
\langle \partial_t \sigma^m_m, w_k \rangle + \langle (\nabla \sigma^m_m, \nabla w_k) \rangle &= \chi(\nabla \varphi^m_m, \nabla w_k), \\
a^m_k(0) &= (\varphi_0, w_k), \quad c^m_k(0) = (\sigma_0, w_k).
\end{align*}
\]

for \( k = 1, \ldots, m \). For each integer \( m \), the initial value problem of nonlinear ODE systems (3.6)–(3.9) can be solved via the classical Cauchy–Lipschitz theorem, such that it has a unique local solution \( (a^m(t), b^m(t), c^m(t)) \) defined on \([0, t_m] \), where \( a^m = (a^m_1, \ldots, a^m_m)^T \), \( b^m = (b^m_1, \ldots, b^m_m)^T \), and \( c^m = (c^m_1, \ldots, c^m_m)^T \). Hence, we obtain the local approximating solution \( (\varphi^m_m, \mu^m_m, \sigma^m_m) \) on \([0, t_m] \).

### 3.1.3 Uniform estimates and passage to the limit

Following the arguments in Appendix of He et al.\(^31\) one can derive uniform estimates for the approximate solutions \( (\varphi^\kappa, \mu^\kappa, \sigma^\kappa) \) with respect to the approximating parameter \( m, \kappa \), and time \( t \) on a given interval \([0, T] \). We note that the additional viscous term \( e \partial_t \varphi^\kappa \) in (3.3) yields some additional regularity on \( \partial_t \varphi^\kappa \) such that \( e \partial_t \varphi^\kappa \in L^2(0, T; L^2(\Omega)) \). Then one can employ a standard compactness argument (see, e.g., Miranville and Temam\(^28\)) to pass to the limit, first as \( m \to +\infty \) and then as \( \kappa \to 0 \), to obtain a global weak solution \( (\varphi, \mu, \sigma) \) of original problems (1.1a)–(1.3) on \([0, T] \) (with \( T > 0 \) being...
arbitrary but fixed). In particular, thanks to the singular potential $\Psi$, we are able to obtain the following uniform bound for $\|\varphi(t)\|_{L^\infty(\Omega)}$ as long as the solution exists (cf. Giorgini et al.\textsuperscript{33, Remark 3.2)}:

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1, \quad \forall t \geq 0. \quad (3.10)$$

### 3.2 Uniform-in-time estimates

To obtain the global weak solution $(\varphi, \mu, \sigma)$ on $[0, T]$, in the procedure described in Section 3.1, actually we did not make use of the above bound of $\|\varphi(t)\|_{L^\infty(\Omega)}$ to handle the chemotactic coupling term in the total energy $\mathcal{E}(t)$ (see (3.15)). However, this will be crucial when we proceed to derive some dissipative estimates that allow us to extend the global weak solution $(\varphi, \sigma)$ from $[0, T]$ to the whole interval $[0, +\infty)$ with uniform-in-time boundedness in $H^1(\Omega) \times L^2(\Omega)$ and thus complete the proof for the existence part of Theorem 2.1. We also note that the dissipative estimates obtained below (see (3.22) and (3.23)) will enable us to study the longtime behavior of problems (1.1a)–(1.3) in Section 5.

#### 3.2.1 First estimate

Testing Equation (2.5a) by 1, we obtain

$$\frac{d}{dt} (\bar{\varphi}(t) - c_0) + \alpha (\bar{\varphi}(t) - c_0) = 0, \quad (3.11)$$

so that

$$\bar{\varphi}(t) = c_0 + e^{-\alpha t} (\bar{\varphi}_0 - c_0), \quad \forall t \geq 0. \quad (3.12)$$

Similarly, choosing the test function $\xi = 1$ in (2.5c), we have

$$\bar{\sigma}(t) = \bar{\sigma}_0, \quad \forall t \geq 0. \quad (3.13)$$

#### 3.2.2 Second estimate

As it has been shown in Giorgini et al.\textsuperscript{9} the convex part of the free energy, that is,

$$\mathcal{E}^*(\varphi) = \frac{B}{2} \|\nabla \varphi\|^2 + A \int_\Omega \Psi(\varphi) dx, \quad \forall \varphi \in H^1(\Omega),$$

is a proper, lower semi-continuous and convex functional. Then for $(\varphi, \mu, \sigma)$, a weak solution in the sense of Definition 2.1, it follows from the standard chain rule that

$$\frac{d}{dt} \mathcal{E}^*(\varphi) = (\partial_t \varphi, \mu + \theta_0 \varphi + \chi \sigma - c \partial_t \varphi).$$

Hence, we can obtain the following energy identity for weak solutions by testing (2.5b) with $\mu$, (2.5a) with $\partial_t \varphi$, and (2.5c) with $\sigma - \chi \varphi$ and adding the resultants together,

$$\frac{d}{dt} \mathcal{E}(t) + D(t) + \int_\Omega \alpha (\varphi(t) - c_0) \mu(t) dx = 0, \quad \forall t \geq 0, \quad (3.14)$$

where

$$\mathcal{E}(t) = \int_\Omega (A \Psi(\varphi(t)) - \chi \sigma(t) \varphi(t)) dx + \frac{B}{2} \|\nabla \varphi(t)\|^2 + \frac{1}{2} \|\sigma(t)\|^2, \quad (3.15)$$

$$D(t) = \|\nabla \mu(t)\|^2 + \|\nabla (\sigma(t) - \chi \varphi(t))\|^2 + \epsilon \|\partial_t \varphi(t)\|^2. \quad (3.16)$$
We deduce from Poincaré–Wirtinger inequality (2.1) and (3.10), (3.12), and (3.13) that
\[
\|\sigma\| \leq \|\sigma - \chi \varphi\| + \|\chi \varphi\| \\
\leq C_\rho \|\nabla(\sigma - \chi \varphi)\| + |\Omega|\|\sigma - \chi \varphi\| + \|\chi \varphi\| \\
\leq C_\rho \|\nabla(\sigma - \chi \varphi)\| + C_1,
\]
where the constant $C_1 > 0$ only depends on $\chi$, $\Omega$, $|\sigma_0|$. Again by (3.10), we have (see He\textsuperscript{31}, (3.32))
\[
\int_\Omega \frac{1}{2}|\sigma|^2 + \chi \sigma(1 - \varphi) \mathrm{d}x \geq \int_\Omega \left( \frac{1}{2}|\sigma|^2 - 2|\chi||\sigma| \right) \mathrm{d}x \\
\geq \frac{1}{4}\|\sigma\|^2 - 4\chi^2|\Omega|,
\]
which implies that $\mathcal{E}(t)$ is indeed bounded from below. Next, we recall the following inequality due to (H1) (see, e.g., Giorgini et al\textsuperscript{9}):
\[
\Psi(r) \leq \Psi(s) + \Psi'(r)(r - s) + \frac{K}{2}(r - s)^2 \quad \forall r, s \in (-1, 1).
\]
Then by (3.10), (3.17), (3.19), and Young’s inequality, we can follow the argument in Giorgini et al. and He\textsuperscript{5,31} to obtain that
\[
\alpha \int_\Omega (\varphi - c_0) \mu \mathrm{d}x \\
= \alpha B \|\nabla \varphi\|^2 + \alpha A \int_\Omega (\varphi - c_0) \Psi'(\varphi) \mathrm{d}x - \alpha \chi \int_\Omega (\varphi - c_0) \sigma \mathrm{d}x \\
+ \alpha \varepsilon \int_\Omega (\varphi - c_0) \partial_t \varphi \mathrm{d}x \\
\geq \alpha B \|\nabla \varphi\|^2 + \alpha A \int_\Omega \left( \Psi(\varphi) - \Psi(c_0) - \frac{K}{2}(\varphi - c_0)^2 \right) \mathrm{d}x \\
- \alpha \chi \int_\Omega (\varphi - c_0) \sigma \mathrm{d}x + \alpha \varepsilon \int_\Omega (\varphi - c_0) \partial_t \varphi \mathrm{d}x \\
\geq \min \left\{ \frac{\alpha}{4C_p^2} \right\} \left( \mathcal{E} + \chi \int_\Omega \sigma \varphi \mathrm{d}x - \frac{1}{2}\|\sigma\|^2 \right) - \frac{1}{4C_p} \|\sigma\|^2 \\
- \left( \frac{\alpha K}{2} + C_p^2 \chi^2 + \frac{\alpha^2 \varepsilon}{2} \right) \|\varphi - c_0\|^2 - \frac{\varepsilon}{2}\|\partial_t \varphi\|^2 - A\varepsilon \|\Psi(c_0)\|\Omega \\
\geq \min \left\{ \frac{\alpha}{4C_p^2} \right\} \left( \mathcal{E} - \frac{\chi^2}{8C_p} \|\sigma\|^2 - \frac{1}{2C_p} \|\sigma\|^2 \\
- \left( \frac{\alpha K}{2} + C_p^2 \chi^2 + \frac{\alpha^2 \varepsilon}{2} \right) \|\varphi - c_0\|^2 - \frac{\varepsilon}{2}\|\partial_t \varphi\|^2 - A\varepsilon \|\Psi(c_0)\|\Omega \\
\geq \min \left\{ \frac{\alpha}{4C_p^2} \right\} \left[ \mathcal{E} - \frac{1}{2}\|\nabla(\sigma - \chi \varphi)\|^2 - \frac{\varepsilon}{2}\|\partial_t \varphi\|^2 - C_2 \right] \\
\geq \min \left\{ \frac{\alpha}{4C_p^2} \right\} \mathcal{E} - \frac{1}{2}D - C_2,
\]
where the constant $C_2 > 0$ depends on $\varepsilon$, $\alpha$, $A$, $B$, $c_0$, $\theta_0$, $\chi$, $\Omega$, and $|\sigma_0|$.

Taking $c_* := \min\{\alpha, 1/(4C_p^2)\} > 0$, we deduce from (3.14) and (3.20) that
\[
\frac{d}{dt} \mathcal{E}(t) + c_* \mathcal{E}(t) + \frac{1}{2} \mathcal{D}(t) \leq C_2, \quad \forall t \geq 0.
\]
By Gronwall’s lemma, we deduce from (3.21) the following dissipative estimates:

\[ \mathcal{E}(t) \leq \mathcal{E}(0)e^{-c_\epsilon t} + \frac{C_2}{c_\epsilon}, \quad \forall t \geq 0, \]  

(3.22)

\[ \int_t^{t+1} D(\tau)d\tau \leq 2\mathcal{E}(0)e^{-c_\epsilon t} + 2C_2(1 + c_\epsilon^{-1}), \quad \forall t \geq 0. \]  

(3.23)

As a consequence, we can infer from (3.10), (3.22), and (3.23) that

\[ \| \varphi \|_{L^\infty(0, +\infty; H^1(\Omega))} + \| \sigma \|_{L^\infty(0, +\infty; L^2(\Omega))} \leq C, \]  

(3.24)

\[ \| \sigma \|_{L^2(0, T; H^1(\Omega))} + \| \partial_t \varphi \|_{L^2(0, T; L^2(\Omega))} + \| \nabla \mu \|_{L^2(0, T; L^2(\Omega))} \leq C_T, \]  

(3.25)

where the constant $C > 0$ is independent of time, and $C_T > 0$ may depend on $T$ (with $T > 0$ being an arbitrary but fixed final time). In particular, the uniform-in-time estimates in (3.24) enable us to extend the weak solutions $(\varphi, \mu, \sigma)$ from an arbitrary interval $[0, T]$ to the whole interval $[0, +\infty)$, being uniformly bounded in the corresponding spaces.

### 3.2.3 Third estimate

Testing (2.5b) by $\dot{\varphi} = 1$, using integration by parts and relations (3.12) and (3.13), we get

\[ |\dot{\varphi}| = |\Omega|^{-1}|A(\Psi'(\varphi), 1) + c(\partial_t \varphi, 1)| \]

\[ \leq |\Omega|^{-1}|A||\Psi'(\varphi)||_{L^1} + |\Omega|^{-1}e\alpha(||\varphi|| + |c_0|). \]  

(3.26)

The term $||\Psi'(\varphi)||_{L^1}$ can be estimated as in Section 3 of Giorgini et al.\(^9\) (see also Miranville and Zelik\(^6\)) with minor modifications such that

\[ ||\Psi'(\varphi)||_{L^1} \leq C(||\nabla \mu|| + ||\nabla \varphi|| + ||\sigma|| + ||\partial_t \varphi||)||\nabla \varphi|| \leq C(||\nabla \mu|| + ||\partial_t \varphi|| + 1). \]  

(3.27)

where the constant $C > 0$ depends on the initial energy \( \mathcal{E}(0) \) and coefficients of the system. As a consequence, from estimates (3.24), (3.25), and (3.26) and inequality (2.1), we obtain

\[ \| \mu \|_{L^2(0, T; H^1(\Omega))} \leq C_T. \]  

(3.28)

### 3.2.4 Fourth estimate

Multiplying (2.5b) with $-\Delta \varphi$ and integrating over $\Omega$, we have (see, e.g., He\(^31\), Section 3.2.1).

\[ A(\Psi''(\varphi) \nabla \varphi, \nabla \varphi) + B||\Delta \varphi||^2 \leq ||\nabla \mu|| ||\nabla \varphi|| + ||\chi|| ||\nabla \sigma|| ||\nabla \varphi|| + c(\partial_t \varphi, \Delta \varphi) \]

\[ \leq C(||\nabla \mu|| + ||\nabla \sigma|| + ||\partial_t \varphi||^2) + \frac{B}{2}||\Delta \varphi||^2. \]  

(3.29)

Thus, it follows from (H1) and (3.29) that

\[ B||\Delta \varphi||^2 \leq C(||\nabla \mu|| + ||\nabla \sigma|| + ||\nabla \varphi||^2 + ||\partial_t \varphi||^2). \]

which together with (3.25) implies

\[ \int_0^T ||\Delta \varphi(t)||^2 dt \leq C_T. \]

Then by the standard elliptic estimates for the Neumann problem, we obtain

\[ \| \varphi \|_{L^2(0, T; H^2(\Omega))} \leq C_T. \]  

(3.30)
where the constant $C_T$ depends on $\mathcal{E}(0), \Omega,$ and coefficients of the system. Next, by comparison in (1.1b) and using estimates (3.24), (3.25), (3.28), and (3.30), we get
\[
\|\Psi'(\varphi)\|_{L^2(0,T;L^2(\Omega))} \leq C_T.
\] (3.31)

### 3.2.5 Fifth estimate

In order to obtain a uniform-in-time estimate for $\partial_t \varphi,$ we rewrite systems (2.5a)–(2.5c) in the following equivalent form:
\[
\langle \partial_t \varphi, \xi \rangle_{H^1, H^1} = -(\nabla \ddot{\mu}, \nabla \xi) - a(\ddot{\varphi} - c_0, \xi),
\] (3.32)
\[
\ddot{\mu} = A \Psi'(\varphi) - B \Delta \varphi - \chi \sigma + C_1,\partial_t \varphi + \alpha \mathcal{N}(\varphi - \ddot{\varphi}),
\] (3.33)
\[
\langle \partial_t \sigma, \xi \rangle_{H^1, H^1} + (\nabla \sigma, \nabla \xi) = \chi(\nabla \varphi, \nabla \xi),
\] (3.34)
\[
\varphi|_{t=0} = \varphi_0, \quad \sigma|_{t=0} = c_0,
\] (3.35)
in $\Omega \times (0, +\infty)$ for all $\xi \in H^1(\Omega).$ Then testing (3.32) with $\ddot{\mu},$ (3.33) with $\partial_t \varphi,$ and (3.34) with $\sigma - \chi \varphi$ and adding the resultants together, we obtain
\[
\frac{d}{dt} F(t) + \mathcal{G}(t) = -a(\ddot{\varphi} - c_0, \ddot{\mu}(t)), \quad \forall t \in [0, +\infty),
\] (3.36)
where
\[
F(t) = \int_{\Omega} (A \Psi(\varphi(t)) - \chi \sigma(t) \varphi(t)) \, dx + \frac{B}{2} \|\nabla \varphi(t)\|^2 + \frac{1}{2} \|\sigma(t)\|^2
\]
\[
+ \frac{a}{2} \|\nabla \mathcal{N}(\varphi(t) - \ddot{\varphi}(t))\|^2,
\]
\[
\mathcal{G}(t) = \|\nabla \ddot{\mu}(t)\|^2 + \|\nabla (\sigma(t) - \chi \varphi(t))\|^2 + \epsilon \|\partial_t \varphi(t)\|^2.
\]

Similar to (3.26) and (3.27), we have
\[
\|\Psi'(\varphi)\|_{L^1} \leq C(1 + \|\nabla \ddot{\mu}\| + \|\partial_t \varphi\|),
\] (3.37)
and thus,
\[
\|\ddot{\mu}\| \leq C(1 + \|\nabla \ddot{\mu}\| + \|\partial_t \varphi\|).
\] (3.38)
By Young’s inequality, we deduce from (3.12) and (3.38) that
\[
-a(\ddot{\varphi} - c_0, \ddot{\mu}) \leq \epsilon e^{-at}\|\ddot{\mu}\| - c_0 \|\ddot{\mu}\|
\]
\[
\leq C e^{-at} + \frac{1}{2} \|\nabla \ddot{\mu}(t)\|^2 + \frac{\epsilon}{2} \|\partial_t \varphi(t)\|^2.
\]
Inserting the above estimate into (3.36), we obtain
\[
\frac{d}{dt} F(t) + \frac{1}{2} \mathcal{G}(t) \leq C e^{-at}, \quad \forall t \in [0, +\infty),
\] (3.39)
which implies
\[
\frac{d}{dt} \left( F(t) + \frac{C}{\alpha} e^{-at} \right) + \frac{1}{2} \mathcal{G}(t) \leq 0, \quad \forall t \in [0, +\infty),
\] (3.40)
such that $F(t) + \frac{C}{\alpha} e^{-at}$ serves as a Lyapunov functional for problems (1.1a)–(1.3).

Hence, integrating (3.40) with respect to time, we get
\[
F(t) + \frac{1}{2} \int_0^t \mathcal{G}(\tau) \, d\tau \leq F(0) + C e^{-at}, \quad \forall t \in [0, +\infty).
\] (3.41)
As a consequence,
\[
\|\nabla \ddot{\mu}\|_{L^2(0, +\infty;L^2(\Omega))} + \epsilon \|\partial_t \varphi\|_{L^2(0, +\infty;L^2(\Omega))} \leq C,
\] (3.42)
and
\[
\|\nabla(\sigma - \chi \varphi)\|_{L^2(0, +\infty; L^2(\Omega))} \leq C, \tag{3.43}
\]
where the constant $C$ may depend on $\alpha, A, B, \Psi_0, \theta_0, \chi, \Omega, \|\varphi_0\|_{H^1}$, and $\|\sigma_0\|$. In view of (2.5c) and (3.43), we also obtain
\[
\|\partial_t \sigma\|^2_{L^2(0, +\infty; H^1(\Omega))} \leq C. \tag{3.44}
\]

### 3.3 Continuous dependence

We now prove the continuous dependence estimate that yields the uniqueness of global weak solutions. To this end, let $(\varphi_1, \mu_1, \sigma_1)$ and $(\varphi_2, \mu_2, \sigma_2)$ be two weak solutions of problems (1.1a)–(1.3) given by Theorem 2.1 corresponding to the initial data $(\varphi_{01}, \sigma_{01})$ and $(\varphi_{02}, \sigma_{02})$, respectively. Denote their differences by

\[
(\varphi, \mu, \sigma) = (\varphi_1 - \varphi_2, \mu_1 - \mu_2, \sigma_1 - \sigma_2),
(\varphi_0, \sigma_0) = (\varphi_{01} - \varphi_{02}, \sigma_{01} - \sigma_{02}).
\]

Then we have
\[
\langle \partial_t \varphi, \xi \rangle_{H^1, H^1} = - (\nabla \mu, \nabla \xi) - \alpha(\varphi, \xi), \tag{3.45a}
\]
\[
\mu = A\Psi'(\varphi_1) - A\Psi'(\varphi_2) - B\Delta \varphi - \chi \sigma + \epsilon \partial_t \varphi, \tag{3.45b}
\]
\[
\langle \partial_t \sigma, \xi \rangle_{H^1, H^1} + (\nabla \sigma, \nabla \xi) = \chi (\nabla \varphi, \nabla \xi), \tag{3.45c}
\]
for almost every $t \in [0, +\infty)$ and any $\xi \in H^1(\Omega)$. From (3.45a), we infer that
\[
\frac{d}{dt} \tilde{\varphi} + \alpha \tilde{\varphi} = 0 \text{ as well as } \frac{1}{2} \frac{d}{dt} \tilde{\varphi}^2 + \alpha \tilde{\varphi}^2 = 0. \tag{3.46}
\]

Next, taking the test function $\tilde{\xi} = \mathcal{N}(\varphi - \tilde{\varphi})$ in (3.45a), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\varphi - \tilde{\varphi}\|^2_{V_0} + (\mu, \varphi - \tilde{\varphi}) + \alpha \|\varphi - \tilde{\varphi}\|^2_{V_0} = 0. \tag{3.47}
\]

Following the arguments in Giorgini et al.\(^9\) (see also He\(^31\), Section 4), we can deduce that
\[
(\mu, \varphi - \tilde{\varphi}) \geq B\|\nabla \varphi\|^2 - (A|\theta_0 - \theta| + \chi^2)\|\varphi\|^2 - \frac{1}{2} \|\sigma\|^2 - \chi^2|\Omega|\tilde{\varphi}^2 - A(\Psi'(\varphi_1) - \Psi'(\varphi_2), \tilde{\varphi}) + \frac{\epsilon}{2} \frac{d}{dt} \|\varphi - \tilde{\varphi}\|^2,
\]
with
\[
(\mathcal{A}(\theta_0 - \theta| + \chi^2)\|\varphi\|^2 \leq \frac{B}{2} \|\nabla \varphi\|^2 + C\|\varphi\|^2_{H^1}. \tag{3.48}
\]

Thus, from (3.46) and (3.47) and using the equivalent norm on $(H^1)'$, we get
\[
\frac{1}{2} \frac{d}{dt} (\|\varphi\|^2_{H^1} + \epsilon \|\varphi - \tilde{\varphi}\|^2) + \frac{B}{2} \|\nabla \varphi\|^2 + \alpha \|\tilde{\varphi}\|^2 \leq \frac{1}{2} \|\sigma\|^2 + C\|\varphi\|^2_{H^1} + A \left(\|\Psi'(\varphi_1)\|_{L^1} + \|\Psi'(\varphi_2)\|_{L^1}\right) |\tilde{\varphi}|. \tag{3.48}
\]

On the other hand, taking the test function $\xi = \mathcal{N}_1 \sigma$ in (3.45c), we get
\[
\frac{1}{2} \frac{d}{dt} \|\sigma\|^2_{H^1} + \|\sigma\|^2 = (\sigma, \mathcal{N}_1 \sigma) + \chi (\nabla \varphi, \nabla \mathcal{N}_1 \sigma), \tag{3.49}
\]
where
\[
(\sigma, \mathcal{N}_1 \sigma) + \chi(\nabla \varphi, \nabla \mathcal{N}_1 \sigma) \leq \|\sigma\|^2_{(H^1)'_Y} + |\chi|\|\nabla \varphi\|\|\nabla \mathcal{N}_1 \sigma\|
\leq \frac{B}{4}\|\nabla \varphi\|^2 + C\|\sigma\|^2_{(H^1)'_Y}. \tag{3.50}
\]

Collecting the above estimates, we deduce from (3.48)–(3.50) that
\[
\frac{1}{2} \frac{d}{dt}\|\varphi\|^2_{(H^1)'_Y} + e\|\varphi - \bar{\varphi}\|^2 + \|\sigma\|^2_{(H^1)'_Y} + \int_t^T \|\varphi(\tau)\|^2_{H^1_Y} d\tau + \int_t^T \|\sigma(\tau)\|^2 d\tau
\leq C\|\varphi_0\|^2_{(H^1)'_Y} + e\|\varphi_0 - \bar{\varphi}_0\|^2 + \|\sigma_0\|^2_{(H^1)'_Y} + |\bar{\varphi}_0|, \quad \forall t \in [0, T]. \tag{3.54}
\]

where the constant $C > 0$ depends on norms of the initial data, $\Omega$, and the coefficients of the system, and the constant $C_T$ also depends on $T$. Combining (3.54) with (3.52), we arrive at the conclusion of Theorem 2.1.

4 | GLOBAL STRONG SOLUTIONS

In this section, we prove Theorem 2.2 on the existence and uniqueness of global strong solutions to problems (1.1a)–(1.3). Since a strong solution is indeed a weak solution, then its uniqueness is a direct consequence of continuous dependence estimate (3.54). Next, concerning the existence part, recalling the arguments in Corollary 4.1 of Miranville and Zelik\(^6\) for the viscous Cahn–Hilliard equation, we only need to derive sufficient a priori estimates. Differently from Miranville and Zelik\(^6\) here we have to overcome several additional difficulties due to Oono’s term (i.e., the loss of mass conservation) and the coupling with the nutrient equation.

Hence, in what follows, we provide a formal derivation of a priori estimates for $(\varphi(t), \sigma(t))$, assuming that they are sufficiently regular functions satisfying the additional assumption
\[
\|\varphi(t)\|_{L^\infty(\Omega)} < 1, \quad \forall t \geq 0. \tag{4.1}
\]

It is obvious that the lower-order estimates that we have obtained in Section 3.2 still hold for $(\varphi, \sigma)$. Below, we provide a priori higher-order estimates.

4.1 | First estimate

Similar to Section 3 of Miranville and Zelik\(^6\) we can rewrite problems (1.1a) and (1.1b) in the following equivalent form by using (3.11):
\[
\begin{align*}
\varepsilon(\partial_t \varphi - \bar{\partial}_t \varphi) + \mathcal{N}'(\partial_t \varphi - \bar{\partial}_t \varphi) + \alpha \mathcal{N}'(\varphi - \bar{\varphi}) = B \Delta \varphi - A \Psi'(\varphi) + \chi \sigma + A \bar{\Psi}'(\varphi) - \chi \bar{\sigma}.
\end{align*}
\tag{4.2}
\]
Differentiating (4.2) with respect to $t$, we get

$$
\begin{align*}
\partial_t (\partial_t \varphi - \overline{\partial_t \varphi}) + \partial_t N (\partial_t \varphi - \overline{\partial_t \varphi}) + \alpha N (\partial_t \varphi - \overline{\partial_t \varphi})
= B \Delta \partial_t \varphi - A \Psi' (\varphi) \partial_t \varphi + \chi \partial_t \sigma + A \overline{\Psi'} (\varphi) \overline{\partial_t \varphi},
\end{align*}
$$

where we have used the fact (3.13). Multiplying the above equation by $\partial_t \varphi - \overline{\partial_t \varphi}$, and integrating over $\Omega$, we obtain

$$
\frac{d}{dt} H(t) + B \|\nabla \partial_t \varphi(t)\|^2 + \alpha \|\nabla N (\partial_t \varphi(t) - \overline{\partial_t \varphi(t)})\|^2 = J_1(t) + J_2(t), \forall t \geq 0,
$$

(4.3)

where

$$
\begin{align*}
H(t) &= \frac{c}{2} \|\partial_t \varphi(t) - \overline{\partial_t \varphi(t)}\|^2 + \frac{1}{2} \|\nabla N (\partial_t \varphi(t) - \overline{\partial_t \varphi(t)})\|^2, \\
J_1(t) &= (\chi \partial_t \sigma(t), \partial_t \varphi(t) - \overline{\partial_t \varphi(t)}), \\
J_2(t) &= (-A \Psi' (\varphi(t)) \partial_t \varphi(t) + A \overline{\Psi'} (\varphi) \overline{\partial_t \varphi(t)}, \partial_t \varphi(t) - \overline{\partial_t \varphi(t)}).
\end{align*}
$$

The two terms on the right-hand side of (4.3) can be estimated as follows:

$$
J_1 = -(\chi \Delta N \partial_t \sigma, \partial_t \varphi - \overline{\partial_t \varphi}) = \chi (\nabla N \partial_t \sigma, \nabla \partial_t \varphi) \leq \frac{B}{2} \|\nabla \partial_t \varphi\|^2 + \frac{\chi^2}{2B} \|\nabla N \partial_t \sigma\|^2,
$$

while for $J_2$, we infer from the fact $\Psi' (\varphi) \geq -K$ (thanks to (H1) and (4.1)) and (3.12) that

$$
\begin{align*}
(-A \Psi' (\varphi) \partial_t \varphi + A \overline{\Psi'} (\varphi) \overline{\partial_t \varphi}) &\leq AK \|\partial_t \varphi\|^2 + A |\Omega| \Psi' (\varphi) \partial_t \varphi \overline{\partial_t \varphi} \\
&= AK \|\partial_t \varphi\|^2 + A |\Omega| \frac{d}{dt} \left( \Psi' (\varphi) \partial_t \varphi \right) - A |\Omega| \Psi' (\varphi) \frac{d}{dt} \partial_t \varphi \\
&\leq AK \|\partial_t \varphi\|^2 + A |\Omega| \frac{d}{dt} \left( \Psi' (\varphi) \partial_t \varphi \right) + A \alpha^2 e^{-\alpha t} \|\Psi' (\varphi)\|_{L^1 (\varphi_0 - \sigma_0)}.
\end{align*}
$$

Then it follows from (4.3) and the above estimates that

$$
\frac{d}{dt} H(t) + B \|\nabla \partial_t \varphi(t)\|^2 + \alpha \|\nabla N (\partial_t \varphi(t) - \overline{\partial_t \varphi(t)})\|^2 \leq C (e^{-\alpha t} \|\Psi' (\varphi)\|_{L^1} + \|\partial_t \varphi\|^2 + \|\nabla N \partial_t \sigma\|^2) + A |\Omega| \frac{d}{dt} \left( \Psi' (\varphi) \partial_t \varphi \right), \forall t \geq 0.
$$

(4.4)

Then it follows from (4.3) and the above estimates that

$$
H(0) = \frac{c}{2} \|\partial_t \varphi_0 - \overline{\partial_t \varphi_0}\|^2 + \frac{1}{2} \|\nabla N (\partial_t \varphi_0 - \overline{\partial_t \varphi_0})\|^2 =: H_0,
$$

(4.5)

where in view of (4.2), the initial value for the time derivative of $\varphi$ denoted by $\partial_t \varphi_0$ is understood as

$$
\begin{align*}
\partial_t \varphi_0 &= (\partial_t \varphi(t))|_{t=0} \\
&= (\epsilon + N)^{-1} \left( B \Delta \varphi_0 - A \Psi' (\varphi_0) + \chi \sigma_0 + A \overline{\Psi'} (\varphi_0) - \chi \sigma_0 - \alpha N (\varphi_0 - \sigma_0) \right) - \alpha (\varphi_0 - \sigma_0).
\end{align*}
$$

(4.6)

From our assumptions on the initial data, it follows that $\|\partial_t \varphi_0\| \leq C$, where $C > 0$ may depend on $\|\varphi_0\|_{L^1}$, $\|\sigma_0\|$, $\Delta \sigma_0$, $\Omega$, and coefficients of the system. As a result, $H(0) \leq C$. On the other hand, recalling relation (3.27) and dissipative
estimate (3.23), we see that

\[
\int_0^{+\infty} e^{-at} \|\Psi'(\phi(t))\|_{L^1} \, dt = \sum_{n=0}^{\infty} \int_n^{n+1} e^{-at} \|\Psi'(\phi(t))\|_{L^1} \, dt \\
\leq \sum_{n=0}^{\infty} e^{-an} \int_n^{n+1} \|\Psi'(\phi(t))\|_{L^1} \, dt \leq C,
\]  
(4.7)

where \( C > 0 \) is a constant depending on norms of the initial data, \( \Omega \), and the coefficients of the system. Thus, integrating (4.4) with respect to time and using estimates (3.42), (3.44), and (4.7), we obtain

\[
\mathcal{H}(t) + \frac{B}{2} \int_0^t \|\nabla \phi(\tau)\|^2 \, d\tau \\
\leq C + A|\Omega| \left( \Psi'(\phi(t)) \delta_{t} \phi(t) \right) - A|\Omega| \left( \Psi'(\phi_0) \delta_{t} \phi_0 \right) \\
\leq C + A|\Omega| \left( \Psi'(\phi(t)) \delta_{t} \phi(t) \right), \quad \forall t \geq 0.
\]  
(4.8)

From the definition of \( \mathcal{H}(t) \), it follows that

\[
\|\delta_{t} \phi(t) \delta_{t} \phi(t) \|^2 \leq 2e^{-1}A|\Omega| \|\Psi'(\phi(t)) \delta_{t} \phi(t) \|^2 + C, \quad \forall t \geq 0,
\]  
(4.9)

which further implies

\[
\|\delta_{t} \phi(t) \|^2 \leq 4e^{-1}A|\Omega| \|\Psi'(\phi(t)) \delta_{t} \phi(t) \|^2 + C, \quad \forall t \geq 0.
\]  
(4.10)

### 4.2 Second estimate

Testing (4.2) with \( \Delta \phi \) and using the facts \( \int_{\Omega} \Delta \phi dx = 0 \) and \( \Psi''(\phi) \geq -K \), we obtain

\[
B\|\Delta \phi\|^2 + a\|\phi - \bar{\phi}\|^2 \\
= (N'(\delta_{t} \phi - \delta_{t} \phi_0), \Delta \phi) + A(\Psi'(\phi), \Delta \phi) - \chi(\sigma, \Delta \phi) + c(\delta_{t} \phi, \Delta \phi) \\
= -(\delta_{t} \phi - \delta_{t} \phi_0, \delta_{t} \phi_0) - A(\Psi'(\phi) \nabla \phi, \nabla \phi) - \chi(\sigma, \Delta \phi) + (c\delta_{t} \phi, \Delta \phi) \\
\leq \frac{1}{2}\|\delta_{t} \phi\|^2 + \frac{1}{2}\| \phi \|^2 + |\Omega| |\bar{\phi}| |\delta_{t} \phi| + AK||\nabla \phi||^2 + \frac{B}{2}\|\Delta \phi\|^2 \\
+ \frac{\chi^2}{B}\|\sigma\|^2 + \frac{c^2}{B}\|\delta_{t} \phi\|^2.
\]  
(4.11)

From (3.12), (3.24), and (4.11), we deduce that

\[
\frac{B}{2}\|\Delta \phi\|^2 + a\|\phi - \bar{\phi}\|^2 \leq \left( \frac{c^2}{B} + \frac{1}{2} \right)\|\delta_{t} \phi\|^2 + C.
\]  
(4.12)

where \( C > 0 \) is a constant depending on norms of the initial data, \( \Omega \), and coefficients of the system.
4.3 | Third estimate

It follows from Equation (4.2) and estimates (3.24), (4.10), and (4.12) that

\[ \| \Psi'(\varphi) - \overline{\Psi'(\varphi)} \|^2 \]
\[ = \frac{1}{A^2} \| B \Delta \varphi + \chi(\sigma - \tilde{\sigma}) - \epsilon (\partial_t \varphi - \overline{\partial_t \varphi}) - \mathcal{N}(\partial_t \varphi - \overline{\partial_t \varphi}) - \alpha \mathcal{N}(\varphi - \overline{\varphi}) \|^2 \]
\[ \leq \frac{C}{A^2} (B^2 \| \Delta \varphi \|^2 + (\epsilon^2 + C \| (\partial_t \varphi - \overline{\partial_t \varphi}) \|^2 + \alpha^2 \| \mathcal{N}(\varphi - \overline{\varphi}) \|^2 + \chi^2 \| \sigma - \tilde{\sigma} \|^2) \]
\[ \leq C \left( \frac{B}{2} \| \Delta \varphi \|^2 + \alpha \| \varphi - \overline{\varphi} \|^2 \right) + C \| \partial_t \varphi \|^2 + C \]
\[ \leq C (1 + \epsilon^2) \| \varphi - \overline{\varphi} \|^2 + C. \]  

From the assumptions on the initial data, we have \( \delta_0 := 1 - |\overline{\varphi_0}| \in (0, 1) \). On the other hand, by \( c_0 \in (-1, 1) \) and (3.12), we have \( \varphi(t) \in [c_0, \overline{\varphi_0}] \subset [c_0, 1 - \delta_0] \) if \( \varphi_0 \geq c_0 \), or \( \varphi(t) \in [\overline{\varphi_0}, c_0] \subset [-1 + \delta_0, c_0] \) if \( \varphi_0 < c_0 \). Hence, we have

\[ |\varphi(t)| \leq 1 - \delta_1, \quad \forall t \geq 0, \text{ with } \delta_1 = \min \{1 - c_0, c_0 + 1, \delta_0 \} > 0. \]  

Thanks to (4.14), we can apply Proposition A.2 of Miranville and Zelik\(^6\) to obtain

\[ |\Psi'(\varphi)| \leq C \| \Psi'(\varphi) - \overline{\Psi'(\varphi)} \|_{L^1} + C, \]  

where the constant \( C > 0 \) depends on \( \delta_1 \). From Young’s inequality, we have

\[ |\Psi'(\varphi)| \leq \frac{1}{2C_2(1 + \epsilon^2)} \| \Psi'(\varphi) - \overline{\Psi'(\varphi)} \|^2 + C. \]  

In light of (4.13) and (4.16), we have

\[ |\Psi'(\varphi(t))| \leq C, \quad \forall t \geq 0, \]  

where \( C \) is a constant depending on norms of the initial data, \( \Omega \), and coefficients of the system. By (4.8), (4.12), (4.13), and (4.17), we obtain that

\[ \| \partial_t \varphi(t) \|^2 + \| \Delta \varphi(t) \|^2 + \| \Psi'(\varphi(t)) \|^2 + \int_0^t \| \nabla \partial_t \varphi(\tau) \|^2 d\tau \leq C, \quad \forall t \geq 0. \]  

The constant \( C \) may depend on the norm of the initial data, \( \Omega \), and coefficients of the system but is independent on time. Thus, the above estimate together with (3.24) and (3.42) yields

\[ \| \partial_t \varphi \|_{L^\infty(0, +\infty; L^2(\Omega))} + \| \partial_t \varphi \|_{L^2(0, +\infty; H^1(\Omega))} + \| \varphi \|_{L^\infty(0, +\infty; H^1(\Omega))} \leq C. \]  

From (4.13) and (4.17), we have

\[ \| \Psi'(\varphi) \|_{L^\infty(0, +\infty; L^2(\Omega))} \leq C. \]  

Then by estimates (3.24), (4.19), and (4.20), we infer from Equation (1.1b) that

\[ \| \mu \|_{L^\infty(0, +\infty; L^2(\Omega))} \leq C. \]  

4.4 | Fourth estimate

Multiplying (1.1c) with \( -\Delta \sigma \) and integrating over \( \Omega \), we get

\[ \frac{1}{2} \frac{d}{dt} \| \nabla \sigma \|^2 + \| \Delta \sigma \|^2 = \chi(\Delta \varphi, \Delta \sigma). \]
An application of Young's inequality leads to
\[
\chi(\Delta \varphi, \Delta \sigma) \leq \frac{1}{2} \| \Delta \sigma \|^2 + \frac{\chi^2}{2} \| \Delta \varphi \|^2.
\]
\[
\| \nabla \sigma \|^2 = -(\sigma, \Delta \sigma) \leq \frac{1}{4} \| \Delta \sigma \|^2 + \| \sigma \|^2.
\]
which together with estimates (3.24) and (4.19) implies that
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \sigma \|^2 + \| \nabla \sigma \|^2 + \frac{1}{4} \| \Delta \sigma \|^2 \leq C.
\]
From Gronwall's lemma and (3.24) and (3.25), we obtain
\[
\| \sigma \|_{L^\infty(0, +\infty; H^1(\Omega))} \leq C, \quad \| \sigma \|_{L^2(0,T; H^2(\Omega))} \leq C_T.
\]
By comparison with (1.1c), we infer from (4.19) and the above estimate that
\[
\partial_t \sigma \in L^2_{\text{loc}}(0, +\infty; L^2(\Omega)).
\]
Next, differentiating (1.1c) with respect to time, multiplying the resultant by \( \partial_t \sigma \) and integrating over \( \Omega \), we get
\[
\frac{1}{2} \frac{d}{dt} \| \partial_t \sigma \|^2 + \| \nabla \partial_t \sigma \|^2 \leq \frac{1}{2} \| \nabla \partial_t \sigma \|^2 + \frac{\chi^2}{2} \| \nabla \partial_t \varphi \|^2.
\]
Noticing that
\[
\| \partial_t \sigma(0) \| = \| \Delta \sigma_0 - \chi \Delta \varphi_0 \| \leq \| \sigma_0 \|_{H^1} + |\chi| \| \varphi_0 \|_{H^1}
\]
and invoking the estimate (4.19), we conclude
\[
\| \partial_t \sigma(t) \|^2 + \int_0^t \| \nabla \partial_t \sigma(\tau) \|^2 d\tau \leq C, \quad \forall t \geq 0.
\]
Since \( \overline{\partial_t \sigma} = 0 \), we deduce from the Poincaré–Wirtinger inequality and (4.23) that
\[
\| \partial_t \sigma \|_{L^2(0, +\infty; H^1(\Omega))} \leq C.
\]
By the elliptic estimate for the Neumann problem and (4.19) and (4.23), we have
\[
\| \sigma \|_{L^\infty(0, +\infty; H^1(\Omega))} \leq C.
\]
which together with the Sobolev embedding theorem yields
\[
\| \sigma \|_{L^\infty(0, +\infty; L^\infty(\Omega))} \leq C.
\]

### 4.5 Fifth estimate

As in Miranville and Zelik, we rewrite (4.2) as
\[
\epsilon \partial_t \varphi - B \Delta \varphi + A \Psi'(\varphi) = \tilde{h},
\]
where the right-hand side is given by
\[
\tilde{h} = \chi \sigma - \chi \overline{\sigma} + A \Psi'(\varphi) + \epsilon \overline{\partial_t \varphi} - \mathcal{N}(\partial_t \varphi - \overline{\partial_t \varphi}) - \alpha \mathcal{N}(\varphi - \overline{\varphi}) + A \partial_0 \varphi.
\]
Then, thanks to estimates (4.18), (4.19), and (4.25), we have
\[ \| \tilde{h}(t) \|_{L^\infty(\Omega)} \leq C_h, \quad \forall t \geq 0, \]
where the positive constant \( C_h \) may depend on the norm of the initial data, \( \Omega \), and coefficients of the system. The above estimate enables us to invoke the following auxiliary ODE systems as in Section 3 of Miranville and Zelik:
\[
\begin{align*}
\epsilon \frac{d}{dt} y_{\pm}(t) + A \Psi'(y_{\pm}(t)) &= \pm C_h, \quad \forall t \geq 0, \\
y_{\pm}(0) &= \pm (1 - \delta_0).
\end{align*}
\] (4.27)

Here, \( \delta_0 \in (0, 1) \) is the constant in (2.8). By the Picard–Lindelöf theorem and the comparison principle, we see that the solutions \( y_+ \) and \( y_- \) are well defined for \( t \geq 0 \) and there exists a constant \( \delta = \delta(\delta_0, C_h) \in (0, \delta_0) \) satisfying (cf. Miranville and Zelik, Proposition A.3)
\[ y_+(t) \leq 1 - \delta, \quad y_-(t) \geq -1 + \delta, \quad \forall t \geq 0. \]

Due to the comparison principle for second-order parabolic equations, for (4.26), we see that
\[ -1 + \delta \leq y_-(t) \leq \varphi(t, x) \leq y_+(t) \leq 1 - \delta, \quad \forall t \geq 0, \quad x \in \Omega. \]

As a consequence, it holds
\[ \| \varphi(t) \|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall t \geq 0, \] (4.28)
which implies that, if the initial value of \( \varphi(t) \) is strictly separated from the pure states \( \pm 1 \), then it is uniformly separated from \( \pm 1 \) for all time.

### 4.6 Sixth estimate

Consider the elliptic problem
\[
\begin{align*}
- B \Delta \varphi &= \mu - A \Psi'(\varphi) + \chi \sigma - \epsilon \partial_t \varphi, \quad \text{in } \Omega, \\
\partial_n \varphi &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]
From (H1), (3.28), (4.19), (4.28), and the standard elliptic estimate, we infer that
\[ \| \varphi \|_{L^2(0, T; H^1(\Omega))} \leq C_T. \] (4.29)

In a similar manner, invoking (1.1c), estimate (4.29) together with (4.23) yields
\[ \| \sigma \|_{L^2(0, T; H^1(\Omega))} \leq C_T. \] (4.30)

Finally, we present the following results, which guarantee that the above a priori estimates can be made rigorously.

**Lemma 4.1.** Suppose that the assumptions in Theorem 2.2 are satisfied and \( T > 0 \) is an arbitrary but fixed final time. For any \( \kappa \in (0, \min \{ a_0, \delta_0 \}) \), auxiliary systems (3.2)–(3.5) admit a unique global strong solution \((\varphi^\kappa, \mu^\kappa, \sigma^\kappa)\) satisfying
\[
\begin{align*}
\| \partial_t \varphi^\kappa \|_{L^\infty(0, T; L^2(\Omega))} + \| \varphi^\kappa \|_{L^2(0, T; H^1(\Omega))} &\leq C_T, \\
\| \mu^\kappa \|_{L^2(0, T; L^2(\Omega))} &\leq C_T, \\
\| \sigma^\kappa \|_{L^2(0, T; H^1(\Omega))} &\leq C_T.
\end{align*}
\] (4.31)

where the constant \( C_T \) may depend on \( T \), norms of the initial data, \( \Omega \), as well as coefficients of the system, but is independent of \( \kappa \).

Lemma 4.1 can be proved by employing the Galerkin scheme in the previous section together with a standard compactness argument (cf. Miranville and Zelik). The lower-order estimates can be found in Appendix of He, and the higher-order estimates are similar to (4.2)–(4.25), we omit the details here. Based on Lemma 4.1, we obtain the existence and uniqueness of strong solutions to systems (1.1a)–(1.3) on \([0, T]\) as follows:
**Lemma 4.2.** Suppose that the assumptions in Theorem 2.2 are satisfied and $T > 0$ is an arbitrary but fixed final time. Problems (1.1a)–(1.3) admit a unique global strong solution on $[0, T]$ satisfying

$$\begin{aligned}
\| \partial_t \varphi \|_{L^\infty(0,T; L^2(\Omega))} &+ \| \varphi \|_{L^\infty(0,T; H^1(\Omega))} \leq C_T, \\
\| \mu \|_{L^\infty(0,T; L^2(\Omega))} &+ \| \varphi \|_{L^2(\Omega; H^1(\Omega))} \leq C_T, \\
\| \partial_t \sigma \|_{L^2(0,T; H^1(\Omega))} &+ \| \sigma \|_{L^2(0,T; H^1(\Omega))} \leq C_T,
\end{aligned}$$

(4.32)

where the constant $C_T$ may depend on $T$, norms of the initial data, $\Omega$, and coefficients of the system. Furthermore, there exists a constant $\delta_T \in (0, \delta_0)$ such that

$$\| \varphi(t) \|_{L^\infty(\Omega)} \leq 1 - \delta_T, \quad \forall t \in [0, T],$$

(4.33)

where $\delta_T$ depends on $T$, $\delta_0$, norms of the initial data, $\Omega$, and coefficients of the system.

**Proof.** Based on Lemma 4.1, we first obtain the uniform separation property with respect to $\kappa$ for $\varphi^\kappa$, and then we take $\kappa$ to be sufficiently small to show that $(\varphi^\kappa, \mu^\kappa, \sigma^\kappa)$ is just the (unique) global strong solution to problems (1.1a)–(1.3) on $[0, T]$.

For the approximate solution $(\varphi^\kappa, \mu^\kappa, \sigma^\kappa)$ obtained in Lemma 4.1, we denote

$$\tilde{h}^\kappa = \chi \sigma^\kappa - \chi \sigma^\kappa + A H^\kappa(\varphi^\kappa) + \varepsilon \partial_t \varphi^\kappa - \mathcal{N}(\partial_t \varphi^\kappa - \partial_t \varphi^\kappa) - a \mathcal{N}(\varphi^\kappa - \varphi^\kappa) + A \theta_0 \varphi^\kappa.$$ 

Then, thanks to the estimates in (4.31), we have

$$\| \tilde{h}^\kappa(t) \|_{L^\infty(\Omega)} \leq C_{h,T}, \quad \forall t \in [0, T],$$

where the positive constant $C_{h,T}$ may depend on $T$, norms of the initial data, $\Omega$, and coefficients of the system but is independent of $\kappa$.

Since $\Psi_0'$ is monotone, there exists $\kappa_0 \in (0, \min\{a_0, \delta_0\})$ such that for all $\kappa \in (0, \kappa_0]$, it holds

$$\Psi_0'(1 - \kappa_0) = \Psi_0'(1 - \kappa_0) \geq A^{-1} C_{h,T}, \quad \Psi_0'(-1 + \kappa_0) = \Psi_0'(-1 + \kappa_0) \leq -A^{-1} C_{h,T}. \quad (4.34)$$

Then we can apply a similar argument as (4.27) and (4.28) to derive the separation property for $\varphi^\kappa$. This process is rigorous because of the regularity of the approximate potential $\Psi_{0,\kappa}'$. Hence, there exists a constant $\delta_T \in (0, \delta_0]$ such that

$$\| \varphi^\kappa(t) \|_{L^\infty(\Omega)} \leq 1 - \delta_T, \quad \forall t \in [0, T],$$

(4.35)

where $\delta_T$ depends on $T$, $\delta_0$, norms of the initial data, $\Omega$, coefficients of the system, and $\kappa_0$ but is independent of $\kappa \in (0, \kappa_0]$. Then we take $\kappa$ to be sufficiently small satisfying (4.34) and $\kappa \in (0, \min\{\kappa_0, \delta_T\})$, the separation property (4.35) still holds, and in particular, it implies

$$\Psi_0'(\varphi^\kappa(t)) = \Psi_0'(\varphi^\kappa(t)), \quad \forall t \in [0, T].$$

(4.36)

Therefore, it is easy to see that $(\varphi^\kappa, \mu^\kappa, \sigma^\kappa)$ is indeed a unique global strong solution to original problems (1.1a)–(1.3) on $[0, T]$. The proof of Lemma 4.2 is complete. \qed

**Proof of Theorem 2.2.** For any given $T > 0$, the a priori estimates (4.2)–(4.30) can be performed rigorously for the global strong solution $(\varphi, \mu, \sigma)$ on $[0, T]$ obtained in Lemma 4.2 (note that (4.1) is fulfilled). Since the estimates are uniform with respect to time, we are able to extend the unique global strong solution from the finite interval $[0, T]$ to $[0, +\infty)$. This completes the proof of Theorem 2.2. \qed
5 | LONGTIME BEHAVIOR

In this section, we aim to study the long-time behavior of global weak solutions to problems (1.1a)–(1.3).

5.1 | Instantaneous regularity

First, we show the instantaneous regularity of weak solutions for $t > 0$, in particular, the instantaneous separation from pure states $\pm 1$.

**Proof of Corollary 2.1.** For any given initial data $(\varphi_0, \sigma_0)$ satisfying $\varphi_0 \in H^1(\Omega)$, $\sigma_0 \in L^2(\Omega)$ with $\|\varphi_0\|_{L^\infty} \leq 1$ and $|\varphi_0| < 1$, let $(\varphi, \sigma)$ be the corresponding unique global weak solution to problems (1.1a)–(1.3) defined by Theorem 2.1. For any $\tau \in (0, 1]$, we deduce from Theorem 2.1, in particular estimates (3.25), (3.30), and (3.31), that there exists $\tau_0 \in \left( \frac{\tau}{\delta}, \frac{1}{\delta} \right)$ such that

\[
\varphi(\tau_0) \in H^2_N(\Omega), \quad \sigma(\tau_0) \in H^1(\Omega), \quad \Psi'(\varphi(\tau_0)) \in L^2(\Omega), \quad \partial_t \varphi(\tau_0) \in L^2(\Omega),
\]

where

\[
\partial_t \varphi(\tau_0) = (\varepsilon + \mathcal{N})^{-1} \left( B\Delta \varphi(\tau_0) - A \Psi'(\varphi(\tau_0)) + A \overline{\Psi'(\varphi(\tau_0))} + \chi \sigma(\tau_0) - \overline{\chi \sigma(\tau_0)} \right) - \alpha(\varepsilon + \mathcal{N})^{-1} \mathcal{N}(\varphi(\tau_0) - \overline{\varphi(\tau_0)}) - \alpha(\varphi(\tau_0) - \sigma(\tau_0) - c_0).
\]

Step 1. For $(\varphi(\tau_0), \sigma(\tau_0))$ given by (5.1), we can find a sequence of approximating functions $(\varphi^{(n)}_0, \sigma^{(n)}_0)$ with $n \in \mathbb{N}^+$, such that $\varphi^{(n)}_0$ stays away from the pure states $\pm 1$ and $\sigma^{(n)}_0 \in H^2_N(\Omega)$. To this end, we take

\[
\varphi^{(n)}_0 = \left( 1 - \frac{1}{2n} \right) \varphi(\tau_0), \quad \sigma^{(n)}_0 = \left( 1 + \frac{1}{2n} A_N \right)^{-1} \sigma(\tau_0).
\]

It easily follows that $\sigma^{(n)}_0 \in H^2_N(\Omega)$ satisfies $\|\sigma^{(n)}_0\|_{H^2} \to \|\sigma(\tau_0)\|_{H^2}$ as $n \to +\infty$. Next, as in Miranville and Zelik, the regularity stated in (5.1) is not sufficient for the application of Theorem 2.2. To achieve the goal, we extend the approximating procedure in Miranville and Zelik for the viscous Cahn–Hilliard equation. The proof consists of several steps.

Step 2. Taking $(\varphi^{(n)}_0, \sigma^{(n)}_0)$ as the initial data, we can now apply Theorem 2.2 to conclude that there exists a unique strong solution $(\varphi^{(n)}, \mu^{(n)}, \sigma^{(n)})$ to problems (1.1a)–(1.3), satisfying

\[
\|\partial_t \varphi^{(n)}\|_{L^\infty(\tau_0, +\infty; L^2(\Omega))} + \|\varphi^{(n)}\|_{L^\infty(\tau_0, +\infty; H^2(\Omega))} + \|\mu^{(n)}\|_{L^\infty(\tau_0, +\infty; L^2(\Omega))} + \|\Psi'(\varphi^{(n)})\|_{L^\infty(\tau_0, +\infty; L^2(\Omega))} + \|\sigma^{(n)}\|_{L^\infty(\tau_0, +\infty; H^2(\Omega))} \leq C.
\]
and
\[
\|\mu^{(n)}\|_{L^2(\tau_0, T; H^1(\Omega))} + \|\partial_t \sigma^{(n)}\|_{L^2(\tau_0, T; L^2(\Omega))} + \|\sigma^{(n)}\|_{L^2(\tau_0, T; H^2(\Omega))} \leq C_T. \tag{5.4}
\]

The positive constants $C$ and $C_T$ in the above estimates may depend on $\|\phi_0\|_{H^1}$, $|\sigma_0|$, $\bar{\mu}_0$, $\Omega$, coefficients of the system, and $\tau_0$ (and thus on $\tau$) but is independent of $n$ according the approximating property of the initial data shown in Step 1. Of course, the constant $C_T$ may also depend on $T$ with $T \geq \tau_0$.

Now, for the approximate solution $\sigma^{(n)}$, we infer from (5.4) that there exists $\tau_1 \in \left(\tau_0, \frac{2\tau}{3}\right)$ such that
\[
\|\partial_t \sigma^{(n)}(\tau_1)\| \leq C(\tau, \tau_1), \tag{5.5}
\]
where the constant $C(\tau, \tau_1)$ depends on $\tau$ and $\tau_1$ but is independent of $n$. We note that $\tau_1$ can be chosen arbitrary close to $\tau_0$. For any given $h > 0$, we denote the difference quotient of a function $f$ by $\partial_t^h f(t) = h^{-1}(f(t+h) - f(t))$.

Applying it to (1.1c), we get
\[
\frac{d}{dt} \partial_t^h \sigma^{(n)} - \Delta \partial_t^h \sigma^{(n)} = -\chi \Delta \partial_t^h \phi^{(n)}. \tag{5.6}
\]

Multiplying (5.6) by $\partial_t^h \sigma^{(n)}$ and integrating over $\Omega$, it follows that
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t^h \sigma^{(n)}\|^2 + \frac{1}{2} \|\nabla \partial_t^h \sigma^{(n)}\|^2 = -\frac{\chi^2}{2} \|\nabla \partial_t^h \phi^{(n)}\|^2. \tag{5.7}
\]

Like in Section 4, from Young’s inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t^h \sigma^{(n)}\|^2 + \frac{1}{2} \|\nabla \partial_t^h \sigma^{(n)}\|^2 \leq \frac{\chi^2}{2} \|\nabla \partial_t^h \phi^{(n)}\|^2. \tag{5.8}
\]

Recalling estimates (5.3) and (5.5), we can integrate (5.8) with respect to time on $(\tau_1, +\infty)$ and then pass to the limit as $h \to 0$ to get
\[
\|\partial_t \sigma^{(n)}\|_{L^\infty(\tau_1, +\infty; L^2(\Omega))} + \|\nabla \partial_t \sigma^{(n)}\|_{L^2(\tau_1, +\infty; L^2(\Omega))} \leq C.
\]

Hence, it follows from the Poincaré–Wirtinger inequality and the fact $\langle \partial_t \sigma^{(n)} \rangle = 0$ that
\[
\|\partial_t \sigma^{(n)}\|_{L^2(\tau_1, +\infty; H^1(\Omega))} \leq C. \tag{5.9}
\]

Besides, from Equation (1.1c) and the elliptic estimate for Neumann problem, we have
\[
\|\sigma^{(n)}\|_{L^\infty(\tau_1, +\infty; H^1(\Omega))} \leq C, \tag{5.10}
\]
which implies
\[
\|\sigma^{(n)}\|_{L^\infty(\tau_1, +\infty; L^\infty(\Omega))} \leq C. \tag{5.11}
\]

The constant $C > 0$ in (5.9)–(5.11) depends on $\tau$ and $\tau_1$ but is independent of $n$.

Step 3. We consider this equation
\[
\partial_t^h \varphi^{(n)} - B \Delta \varphi^{(n)} + A \Psi_0(\varphi^{(n)}) = \tilde{h}^{(n)}, \tag{5.12}
\]
with its right-hand side given by
\[
\tilde{h}^{(n)} = \chi \varphi^{(n)} - \chi \varphi^{(n)} + A \Psi_0(\varphi^{(n)}) + e \partial_t \varphi^{(n)} - N(\partial_t \varphi^{(n)} - \partial_t \varphi^{(n)}) - a \mathcal{N}(\varphi^{(n)} - \varphi^{(n)}) + Ah_0 \varphi^{(n)}. \tag{5.13}
\]

Thanks to estimates (5.3) and (5.11), we have
\[
\|\tilde{h}^{(n)}(t)\|_{L^\infty(\Omega)} \leq \tilde{C}_h, \quad \forall t \geq \tau_1, \tag{5.14}
\]
where the constant $\tilde{C}_h > 0$ is independent of $n$. Consider the initial value problem of ODEs (cf. (4.27))

$$\begin{align*}
&\frac{d}{dt} y^{(n)}_\pm(t) + A \Psi_0(y^{(n)}_\pm(t)) = \pm \tilde{C}_h, \quad \forall t \in [\tau_1, +\infty), \\
&y^{(n)}_\pm(\tau_1) = \pm \left(1 - \frac{1}{m}\right).
\end{align*}$$

We infer from Corollary A.1 of Miranville and Zelik\(^6\) that for any $\tau_2 > \tau_1$, there exists a constant $\tilde{\delta} \in (0, 1)$ depending on $\tau$, $\tau_1$, $\tilde{C}_h$, and $\tau_2$ but is independent of $n$, such that

$$|y^{(n)}_\pm(t)| \leq 1 - \tilde{\delta}, \quad \forall t \geq \tau_2.$$  

Then by the comparison principle for second-order parabolic equations, we obtain

$$-1 + \delta \leq y^{(n)}_\pm(t) \leq \varphi^{(n)}(t, x) \leq y^{(n)}_\pm(t) \leq 1 - \tilde{\delta}, \quad \forall t \geq \tau_2, \quad x \in \Omega,$$

which implies the uniform separation of $\varphi^{(n)}$ from the pure states $\pm 1$ for $t \geq \tau_2$. We note that $\delta$ may depend on $\tau$, $\tau_1$, and $\tau_2$ but is independent of $n$. From (5.16) and by a similar argument in the sixth estimate in Section 4, we also obtain

$$\|\varphi^{(n)}\|_{L^2(\tau_2, T; H^1(\Omega))} + \|\sigma^{(n)}\|_{L^2(\tau_2, T; H^1(\Omega))} \leq C_T.$$  

Step 4. Since estimates (5.3), (5.4), (5.9)–(5.11), (5.16), and (5.17) obtained in the previous steps are independent of $n$, then by using a similar compactness argument like in Section 4 of Miranville and Temam,\(^{28}\) we are able to pass to the limit as $n \to +\infty$ and find a convergent subsequence of $(\varphi^{(n)}, \sigma^{(n)}, \mu^{(n)})$ such that the limit function denoted by $(\hat{\varphi}, \hat{\sigma}, \hat{\mu})$ is indeed a global weak solution to problems (1.1a)–(1.3) on $[\tau_0, +\infty)$ subject to the initial data $(\varphi(\tau_0), \sigma(\tau_0))$, with the following regularity properties:

$$\begin{align*}
\hat{\varphi} &\in L^\infty(\tau_0, +\infty; H^2_N(\Omega)), \\
\partial_t \hat{\varphi} &\in L^\infty(\tau_0, +\infty; L^2(\Omega)) \cap L^2(\tau_0, +\infty; H^1(\Omega)), \\
\mu &\in L^\infty(\tau_0, +\infty; L^2(\Omega)) \cap L^2_{loc}(\tau_0, +\infty; H^1(\Omega)), \\
\hat{\sigma} &\in L^\infty(\tau_0, +\infty; H^1(\Omega)) \cap L^2_{loc}(\tau_0, +\infty; H^2(\Omega)), \\
\partial_t \hat{\sigma} &\in L^2_{loc}(\tau_0, +\infty; H^1(\Omega)),
\end{align*}$$

as well as

$$\begin{align*}
\hat{\mu} &\in L^\infty(\tau_1, +\infty; H^2_N(\Omega)), \\
\partial_t \hat{\mu} &\in L^2(\tau_1, +\infty; H^1(\Omega)), \\
\hat{\varphi} &\in L^2_{loc}(\tau_2, +\infty; H^1(\Omega)), \\
\hat{\sigma} &\in L^2_{loc}(\tau_2, +\infty; H^2(\Omega)),
\end{align*}$$

for $\tau_0 < \tau_1 < \tau_2 < \tau$, and in particular,

$$\|\hat{\varphi}(t)\|_{L^\infty(\Omega)} \leq 1 - \tilde{\delta}, \quad \forall t \geq \tau_2.$$  

Hence, we are now in a position to complete the proof of Corollary 2.1. Thanks to the uniqueness of weak solutions of problems (1.1a)–(1.3), it follows that $(\varphi, \mu, \sigma)(t) = (\hat{\varphi}, \hat{\sigma}, \hat{\mu})(t)$ for all $t \geq \tau_0$. As a consequence, the global weak solution $(\varphi, \mu, \sigma)$ becomes a strong one on the interval $[\tau, +\infty)$. Besides, fixing $\tau_1 \in \left(\frac{\tau_2}{2}, \frac{3\tau_2}{2}\right)$ and $\tau_2 = \frac{5\tau}{6}$ in the estimates of previous steps, we can obtain the uniform separation property

$$\|\varphi(t)\|_{L^\infty} \leq 1 - \delta, \quad \forall t \geq \tau,$$  

where the constant $\delta \in (0, 1)$ depends on $\tau$, $\|\varphi_0\|_{H^1}$, $\|\sigma_0\|$, $1 - |\varphi_0|$, $\Omega$, and coefficients of the system. The proof of Corollary 2.1 is complete.
5.2 Existence of the global attractor and exponential attractors

We first prove the existence of a global attractor for problems (1.1a)–(1.3), which is the unique compact set in a suitable phase space, being invariant under the semigroup generated by the evolution system and attracting all bounded sets as time goes to infinity (see Temam, Chapter 1, Definition 1.3).

Proof of Theorem 2.3. First, it follows from Theorem 2.1 that the global weak solutions to problems (1.1a)–(1.3) define a closed semigroup (in the sense of Pata and Zelik) $S(t)$ on the phase space $\mathcal{X}_{m_1,m_2}$. Next, we observe that Corollary 2.1 implies the asymptotic compactness of $S(t)$. In particular, letting $(\varphi, \sigma)$ be a global weak solution to problems (1.1a)–(1.3), we infer that for any $\tau > 0$,

$$\|\varphi(t)\|_{H^2} + \|\sigma(t)\|_{H^2} \leq C_\tau, \forall t \geq \tau,$$

where the constant $C_\tau > 0$ depends on $\|\varphi_0\|_{H^2}$, $\|\sigma_0\|$, $\Omega$, coefficients of the system, and $\tau$ but is independent of $t$. From the above estimate, continuous dependence estimate (2.7), and a standard interpolation argument (cf., e.g., Giorgini et al., Proposition 6.1), we see that $S(t) \in C(\mathcal{X}_{m_1,m_2}, \mathcal{X}_{m_1,m_2})$ for every $t \geq 0$. As a consequence, $S(t)$ is actually a strongly continuous semigroup on $\mathcal{X}_{m_1,m_2}$.

On the other hand, we note that estimates (3.10), (3.12), and (3.13) together with dissipative estimate (3.22) imply the dissipativity property of $S(t)$ in $\mathcal{X}_{m_1,m_2}$. Namely, $S(t)$ possesses a bounded absorbing ball

$$B_0 = \{ (\varphi, \sigma) \in \mathcal{X}_{m_1,m_2} : \|\varphi\|_{H^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} \leq R_0 \},$$

whose radius $R_0 > 0$ depends on $\Omega$, $m_1$, $m_2$, and coefficients of the system but is independent of the initial data. For every bounded set $B \subset \mathcal{X}_{m_1,m_2}$, there exists a time $t_0 = t_0(B) > 0$ such that

$$S(t)B \subset B_0, \forall t \geq t_0. \tag{5.19}$$

Thanks to the above facts, by applying the abstract theory of infinite-dimensional dynamical systems (see, e.g., Temam, Chapter 1, Theorem 1.1), we can conclude that the dynamical system $(S(t), \mathcal{X}_{m_1,m_2})$ defined by problems (1.1a)–(1.3) admits a global attractor $A_{m_1,m_2} \subset \mathcal{X}_{m_1,m_2}$ that is bounded in $H^2(\Omega) \times H^2(\Omega)$.

The proof of Theorem 2.3 is complete.

Next, concerning the existence of an exponential attractor, the crucial point is the validity of the separation property (2.10) obtained in Corollary 2.1, which rules out the possible singularity due to the derivatives of the potential $\Psi$.

Proof of Theorem 2.4. We prove Theorem 2.4 by adapting the general framework in Eden et al. (see also Miranville and Zelik and Efendiev et al.) and applying the abstract results in Berti and Gatti and Gatti et al.

Step 1. We derive an improved dissipativity of the semigroup $S(t)$ such that it admits a compact absorbing set belonging to $H^2(\Omega) \times H^2(\Omega)$.

Lemma 5.1. There exist $R_1 > 0$ and $\delta_1 \in (0, 1)$ such that the ball

$$B_1 = \{ (\varphi, \sigma) \in \mathcal{X}_{m_1,m_2} \cap (H^2(\Omega) \times H^2(\Omega)) : \|\varphi\|_{H^2(\Omega)} + \|\sigma\|_{H^2(\Omega)} \leq R_1, \|\varphi\|_{L^\infty(\Omega)} \leq 1 - \delta_1 \}$$

is a compact absorbing set for $S(t)$ in $\mathcal{X}_{m_1,m_2}$. Namely, for every bounded set $B \subset \mathcal{X}_{m_1,m_2}$, there exists a time $t_1 = t_1(B) > 0$ such that $S(t)B \subset B_1$, for all $t \geq t_1$.

Proof of Theorem 2.4. Since we already know the existence of an absorbing set $B_0 \subset \mathcal{X}_{m_1,m_2}$ (see (5.19) and cf. (3.22) and (3.23)), for every bounded set $B \subset \mathcal{X}_{m_1,m_2}$, we only need to consider the evolution after the time $t_0 = t_0(B)$, that is, starting from the set $B_0$. For any initial data $(\varphi_0, \sigma_0) \in B_0$, with $\|\varphi_0\|_{H^2} + \|\sigma_0\| \leq R_0$, the estimates obtained in Section 5.1 indicate that the corresponding unique global weak solution $(\varphi, \sigma)$ satisfies

$$\|\varphi(t)\|_{H^2} + \|\sigma(t)\|_{H^2} \leq R_1 \text{ and } \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_1, \forall t \geq 1,$$
where the constants $R_1 > 0$ and $\delta_1 \in (0, 1)$ depend on $R_0$, $m_1$, $m_2$, $\Omega$, and coefficients of the system. Hence, for every bounded set $B \subset X_{m_1, m_2}$, we can take $t_1(B) = t_0(B) + 1$, then for the ball $B_1$ defined above, it holds $S(t)B \subset B_1$ for all $t \geq t_1$.

**Step 2.** From the construction of the absorbing set $B_1$, we see that there exists some $t_1(B_1) > 0$ such that $S(t)B_1 \subset B_1$, for all $t \geq t_1(B_1)$. Take

$$B^* = \bigcup_{t \geq t_1(B_1)} S(t)B_1.$$  

It easily follows that $B^* \subset B_1$. To see this, for any $(\varphi^*, \sigma^*) \in B^*$, there exist certain $(\varphi_0, \sigma_0) \in B_1$ and $t^* \geq t_1(B_1)$ such that $(\varphi^*, \sigma^*) = S(t^*)(\varphi_0, \sigma_0) \in B_1$. Besides, we have $S(t)B^* \subset B^*$ for all $t \geq 0$. This is because, for any $(\varphi^*, \sigma^*) \in B^*$, $S(t)(\varphi^*, \sigma^*) = S(t)S(t^*)(\varphi_0, \sigma_0) = S(t + t^*)(\varphi_0, \sigma_0) \in B^*$. Therefore, $B^*$ is positive invariant under $S(t)$ and relatively compact in $X_{m_1, m_2}$.

Now, we restrict the dynamics on the set

$$\tilde{B}_1 = B^{t \in [0, T]}.$$  

It is obvious that $\tilde{B}_1$ is a closed subset of $B_1$ and thus compact in $X_{m_1, m_2}$. Besides, $\tilde{B}_1$ is also positively invariant under $S(t)$. This follows from the continuity of $\tilde{S}(t)$ such that for all $t \geq 0$, $\tilde{S}(t)\tilde{B}_1 \subset \tilde{S}(t)\tilde{B}^* \subset \tilde{B}^* = \tilde{B}_1$ (with closure taken in $H^1(\Omega) \times L^2(\Omega)$). Since $S(t)\tilde{B}_1 \subset \tilde{B}_1$ for all $t \geq t_1(B_1)$, then $\tilde{B}_1$ is indeed a compact absorbing set for $S(t)$.

We observe that the elements in $\tilde{B}_1$ fulfill the same bounds in terms of $R_1$ and $d_1$, given by $B_1$. In particular, for any $(\varphi, \sigma) \in \tilde{B}_1$, the uniform separation property for $\varphi$ guarantees that the singular potential $\Psi(\varphi)$ can be regarded as a globally Lipschitz smooth potential function. This fact enables us to obtain the following properties:

**Lemma 5.2.** The mapping $(t, (\varphi_0, \sigma_0)) \mapsto S(t)(\varphi_0, \sigma_0) : [0, T] \times \tilde{B}_1 \to \tilde{B}_1$ is $\frac{1}{2}$-Hölder continuous in time and Lipschitz continuous with respect to the initial data, when $\tilde{B}_1$ is endowed with the $H^1(\Omega) \times L^2(\Omega)$-topology. Moreover, let $(\varphi_i, \sigma_i), i = 1, 2$ be two global weak solutions to problems (1.1a)-(1.3) corresponding to the initial data $(\varphi_{0i}, \sigma_{0i}) \in \tilde{B}_1$, respectively. The following smoothing estimate for the difference of solutions holds:

$$\|\varphi_1(t) - \varphi_2(t)\|_{H^1} + \|\sigma_1(t) - \sigma_2(t)\|_{H^1} \leq C_T \left(\frac{1 + T}{t}\right) (\|\varphi_{01} - \varphi_{02}\|_{H^1} + \|\sigma_{01} - \sigma_{02}\|),$$  

for any $t \in (0, T]$.

**Proof of Theorem 2.4.** The $\frac{1}{2}$-Hölder continuity in time for $S(t)$ in the $H^1(\Omega) \times L^2(\Omega)$-topology easily follows from (4.18), (4.23), and the following estimates:

$$\|\varphi(t_1) - \varphi(t_2)\|_{H^1} \leq |t_1 - t_2| \frac{1}{2} \left(\int_{t_1}^{t_2} \|\partial_t \varphi(r)\|^2_{H^1} dr\right)^{\frac{1}{2}},$$

$$\|\sigma(t_1) - \sigma(t_2)\| \leq |t_1 - t_2| \frac{1}{2} \left(\int_{t_1}^{t_2} \|\partial_t \sigma(r)\|^2 dr\right)^{\frac{1}{2}},$$

for any $0 \leq t_1 < t_2 \leq T$.

Now, for two global (strong) solutions with their initial data in $\tilde{B}_1$, we denote the differences by

$$(\varphi, \mu, \sigma) = (\varphi_1 - \varphi_2, \mu_1 - \mu_2, \sigma_1 - \sigma_2),$$

$$(\varphi_0, \sigma_0) = (\varphi_{01} - \varphi_{02}, \sigma_{01} - \sigma_{02}),$$

which satisfy

$$\partial_t \varphi = \Delta \mu - \alpha \varphi, \text{ a.e. in } \Omega \times (0, T), \quad (5.21a)$$

$$\mu = A \Psi'(\varphi_1) - A \Psi'(\varphi_2) - B \Delta \varphi - \chi \varphi + \epsilon \partial_t \varphi, \text{ a.e. in } \Omega \times (0, T), \quad (5.21b)$$

$$\partial_t \sigma - \Delta \sigma = -\chi \Delta \varphi, \text{ a.e. in } \Omega \times (0, T), \quad (5.21c)$$

$$\partial_n \varphi = \partial_n \mu = \partial_n \sigma = 0, \text{ on } \partial \Omega \times (0, T), \quad (5.21d)$$
\[\varphi|_{t=0} = \varphi_0, \sigma|_{t=0} = \sigma_0, \quad \text{in } \Omega. \quad \text{(5.21e)}\]

We first show the Lipschitz continuity of \(S(t)\) with respect to the initial data in \(H^1(\Omega) \times L^2(\Omega)\) (cf. the weaker continuous dependence estimate (2.7) for global weak solutions). To this end, multiplying (5.21a) with \(N'(\partial_t \varphi - \overline{\partial_t \varphi})\) and (5.21c) with \(\sigma\), integrating over \(\Omega\), and adding the resultant terms together, we get

\[
\frac{1}{2} \frac{d}{dt} \left( B\|\nabla \varphi\|^2 + \alpha \|\varphi - \overline{\varphi}\|_{H^{1, \gamma}}^2 + \|\sigma\|^2 \right) + \|\partial_t \varphi - \overline{\partial_t \varphi}\|_{H^{1, \gamma}}^2 + \epsilon \|\partial_t \varphi\|^2 + \|\nabla \sigma\|^2 \\
= (\chi \sigma, \partial_t \varphi - \overline{\partial_t \varphi}) + (\chi \nabla \varphi, \nabla \sigma) + c|\Omega| (\partial_t \varphi) - \left( A\Psi'(\varphi_1) - A\Psi'(\varphi_2), \partial_t \varphi - \overline{\partial_t \varphi} \right) \\
\leq |\chi| \|\sigma\| (\|\partial_t \varphi\| + |\Omega|^2 \frac{1}{2}(\partial_t \varphi)) + |\chi| \|\nabla \varphi\| \|\nabla \sigma\| + c|\Omega| (\partial_t \varphi)^2 \\
+ A\|\Psi'(\varphi_1) - \Psi'(\varphi_2)\| (\|\partial_t \varphi\| + |\Omega| \frac{1}{2}(\partial_t \varphi)) \\
\leq \frac{c}{2} \|\partial_t \varphi\|^2 + \frac{1}{2} \|\nabla \sigma\|^2 + C \left( \|\nabla \varphi\|^2 + \|\sigma\|^2 \right) + \| \int_0^1 \Psi'(s\varphi_1 + (1-s)\varphi_2) \varphi \, ds \| \|_2^2 \\
+ 2c|\Omega| (\partial_t \varphi)^2 \\
\leq \frac{c}{2} \|\partial_t \varphi\|^2 + \frac{1}{2} \|\nabla \sigma\|^2 + C \left( \|\nabla \varphi\|^2 + \|\sigma\|^2 \right) + \sup_{s \in [0,1]} \|\Psi'(s\varphi_1 + (1-s)\varphi_2)\|_2^2 \|\varphi\|^2 \\
+ 2c|\Omega| (\partial_t \varphi)^2, \quad \text{(5.22)}
\]

where the constant \(C > 0\) depends on coefficients of the system and \(\Omega\). Using the strict separation of solutions from pure states \(\pm 1\) in \(\tilde{B}_1\), the Poincaré–Wirtinger inequality, and estimates (3.46) and (3.52), we obtain the following inequality from (5.22):

\[
\frac{1}{2} \frac{d}{dt} \left( B\|\nabla \varphi(t)\|^2 + \alpha \|\varphi(t) - \overline{\varphi(t)}\|_{H^{1, \gamma}}^2 + \|\sigma(t)\|^2 \right) + \|\partial_t \varphi(t) - \overline{\partial_t \varphi(t)}\|_{H^{1, \gamma}}^2 + \frac{c}{2} \|\partial_t \varphi(t)\|^2 + \frac{1}{2} \|\nabla \sigma(t)\|^2 \\
\leq C(\|\nabla \varphi\|^2 + \|\sigma\|^2) + C(\overline{\varphi}^2). \quad \text{(5.23)}
\]

Then it follows from Gronwall’s lemma that

\[
\frac{1}{2} \frac{d}{dt} \left( B\|\nabla \varphi(t)\|^2 + \alpha \|\varphi(t) - \overline{\varphi(t)}\|_{H^{1, \gamma}}^2 + \|\sigma(t)\|^2 \right) \\
+ \int_0^t \|\partial_t \varphi(s) - \overline{\partial_t \varphi(s)}\|_{H^{1, \gamma}}^2 + \epsilon \|\partial_t \varphi(s)\|^2 + \|\nabla \sigma(s)\|^2 \, ds \\
\leq C_T (\|\nabla \varphi_0\|^2 + \alpha \|\varphi_0 - \overline{\varphi_0}\|_{H^{1, \gamma}}^2 + \|\sigma_0\|^2 + |\overline{\varphi_0}|^2), \quad \forall t \in [0, T], \quad \text{(5.24)}
\]

which together with (3.52) and the Poincaré–Wirtinger inequality yields the Lipschitz continuity of \(S(t)\) with respect to the initial data in \(\tilde{B}_1\) endowed with the \(H^1(\Omega) \times L^2(\Omega)\)-topology.

Next, we prove the smoothing estimate for the difference of solutions. Multiplying (5.21c) by \(\partial_t \sigma\) and integrating over \(\Omega\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \sigma\|^2 + \|\partial_t \sigma\|^2 = -\chi \int_\Omega \Delta \varphi \partial_t \sigma \, dx \\
\leq \frac{1}{2} \|\partial_t \sigma\|^2 + \frac{\chi^2}{2} \|\Delta \varphi\|^2. \quad \text{(5.25)}
\]

Multiplying the above equality by \(t\) and integrating with respect to time, we obtain

\[
t\|\nabla \sigma(t)\|^2 + \int_0^t \|\partial_t \sigma(t)\|^2 \, dt \\
\leq \int_0^t \|\nabla \sigma(t)\|^2 \, dt + \chi^2 \int_0^t \|\Delta \varphi(t)\|^2 \, dt. \quad \text{(5.26)}
\]

Similar to (4.2), we rewrite (5.21a) as

\[
c(\partial_t \varphi - \overline{\partial_t \varphi}) + N(\partial_t \varphi - \overline{\partial_t \varphi}) + \alpha N(\varphi - \overline{\varphi}) \\
= B\Delta \varphi - (A \Psi'(\varphi_1) - A \Psi'(\varphi_2)) + \chi \sigma + A \Psi'(\varphi_1) - \Psi'(\varphi_2) - \chi \bar{\sigma}. \quad \text{(5.27)}
\]
Multiplying (5.27) by \(\Delta \varphi\) and integrating over \(\Omega\), we get

\[
B\|\Delta \varphi\|^2 = \int_{\Omega} e(\partial_t \varphi - \overline{\partial_t \varphi})\Delta \varphi + N'(\partial_t \varphi - \overline{\partial_t \varphi})\Delta \varphi + \alpha N(\varphi - \overline{\varphi})\Delta \varphi \, dx \\
+ \int_{\Omega} (A\Psi'(\varphi_1) - A\Psi'(\varphi_2))\Delta \varphi - \chi \sigma \Delta \varphi \, dx
\]

\[
\leq C\|\Delta \varphi\| \left(\|\partial_t \varphi\| + \|\overline{\partial_t \varphi}\| + \|\varphi\| + \left\|\int_{0}^{1} \Psi''(s\varphi_1 + (1-s)\varphi_2) \varphi \, ds\right\| + \|\sigma\|\right)
\]

\[
\leq \frac{B}{2}\|\Delta \varphi\|^2 + C(\|\partial_t \varphi\|^2 + \|\overline{\partial_t \varphi}\|^2 + \|\varphi\|^2_{H^1} + \|\sigma\|^2).
\]

which implies

\[
\|\Delta \varphi\|^2 \leq C(\|\partial_t \varphi\|^2 + \|\overline{\partial_t \varphi}\|^2 + \|\varphi\|^2_{H^1} + \|\sigma\|^2).
\] (5.28)

In the above estimates, we have used the uniform bound of \(\varphi_1, \varphi_2\) in \(H^2\) (in terms of \(R_1\)) and the property of strict separation from \(\pm 1\) (in terms of \(\delta_1\)) in \(\tilde{B}_1\). Then it follows from (5.24), (5.26), and (5.28) that

\[
l\|\nabla \sigma(t)\|^2 + \int_{0}^{t} \tau\|\partial_t \sigma(\tau)\|^2 \, d\tau \leq C_T(\|\varphi_0\|_{H^2}^2 + \|\sigma_0\|_{H^2}^2), \forall t \in (0, T].
\] (5.29)

Differentiating (5.27) with respect to time, multiplying the resultant by \(\partial_t \varphi - \overline{\partial_t \varphi}\), and integrating over \(\Omega\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left(\|\partial_t \varphi - \overline{\partial_t \varphi}\|^2 + \|\partial_t \varphi - \overline{\partial_t \varphi}\|^2_{L^2(\Omega)}\right) + B\|\nabla \partial_t \varphi\|^2 + \alpha\|\partial_t \varphi - \overline{\partial_t \varphi}\|^2_{(H^1)^*}
\]

\[
= -A\int_{\Omega} (\Psi''(\varphi_1)\partial_t \varphi_1 - \Psi''(\varphi_2)\partial_t \varphi_2) (\partial_t \varphi - \overline{\partial_t \varphi}) \, dx + \chi \int_{\Omega} \partial_t \sigma(\partial_t \varphi - \overline{\partial_t \varphi}) \, dx.
\] (5.30)

As \(\Psi \in C^3(\pm 1, 1)\), the first term on the right-hand side of (5.30) can be estimated as follows:

\[
-A\int_{\Omega} (\Psi''(\varphi_1)\partial_t \varphi_1 - \Psi''(\varphi_2)\partial_t \varphi_2) (\partial_t \varphi - \overline{\partial_t \varphi}) \, dx
\]

\[
\leq A\|\Psi''(\varphi_1)\|_{L^\infty(\Omega)}\|\partial_t \varphi - \overline{\partial_t \varphi}\| + A\|\Psi''(\varphi_2)\|_{L^\infty(\Omega)}\|\partial_t \varphi - \overline{\partial_t \varphi}\| + A\|\Psi''(\varphi)\|_{L^\infty(\Omega)}\|\partial_t \varphi - \overline{\partial_t \varphi}\|
\]

\[
\leq C\|\partial_t \varphi - \overline{\partial_t \varphi}\|^2 + C\|\partial_t \varphi\|^2 + C\|\partial_t \varphi\|^2_{H^1} + \|\varphi\|^2_{H^1},
\] (5.31)

where we have used again the strict separation properties of \(\varphi_1\) and \(\varphi_2\) in \(\tilde{B}_1\). The second term on the right-hand side of (5.30) can be simply estimated by

\[
\chi \int_{\Omega} \partial_t \sigma(\partial_t \varphi - \overline{\partial_t \varphi}) \, dx \leq \frac{1}{2}\|\partial_t \varphi - \overline{\partial_t \varphi}\|^2 + \frac{\chi^2}{2}\|\partial_t \sigma\|^2
\]

\[
\leq C\|\partial_t \varphi\|^2 + \frac{\chi^2}{2}\|\partial_t \varphi\|^2.
\] (5.32)

Differentiating (5.21c) with respect to time, multiplying the resultant by \(\partial_t \sigma\), and integrating over \(\Omega\), we obtain

\[
\frac{1}{2} \frac{d}{dt}(\|\partial_t \sigma\|^2 + \|\nabla \partial_t \sigma\|^2) = \chi \int_{\Omega} \nabla \partial_t \varphi \cdot \nabla \partial_t \sigma \, dx
\]

\[
\leq \frac{1}{2}\|\nabla \partial_t \sigma\|^2 + \frac{\chi^2}{2}\|\nabla \partial_t \varphi\|^2.
\] (5.33)
Multiplying (5.33) by $\frac{B}{\chi^2}$ (here we only consider $\chi \neq 0$ since the decoupling case $\chi = 0$ is indeed easier), adding the resultant with (5.30), and using (5.31) and (5.33), we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|\partial_t \varphi - \overline{\partial_t \varphi}\|^2 + \|\partial_t \varphi - \overline{\partial_t \varphi}\|^2_{(H^1)^N} + \frac{B}{\chi^2} \|\partial_t \sigma\|^2 \right) + \frac{B}{2} \|\nabla \partial_t \varphi\|^2 + a\|\partial_t \varphi - \overline{\partial_t \varphi}\|^2_{(H^1)^N} + \frac{B}{2\chi^2} \|\nabla \partial_t \sigma\|^2 \\
\leq C\|\partial_t \varphi\|^2 + \frac{\chi^2}{2} \|\partial_t \sigma\|^2 + C\|\partial_t \varphi_2\|^2_{H^1} \|\varphi\|^2_{H^1}.
\end{equation}

Multiplying the above inequality by $t^2$ and integrating with respect to time, we get
\begin{equation}
\frac{1}{2} t^2 \left( \|\partial_t \varphi(t) - \overline{\partial_t \varphi(t)}\|^2 + \|\partial_t \varphi(t) - \overline{\partial_t \varphi(t)}\|^2_{(H^1)^N} + \frac{B}{\chi^2} \|\partial_t \sigma(t)\|^2 \right) \\
\leq \int_0^t \tau \left( \|\partial_t \varphi(\tau) - \overline{\partial_t \varphi(\tau)}\|^2 + \|\partial_t \varphi(\tau) - \overline{\partial_t \varphi(\tau)}\|^2_{(H^1)^N} + \frac{B}{\chi^2} \|\partial_t \sigma(\tau)\|^2 \right) d\tau \\
+ C \int_0^t \tau^2 \left( \|\partial_t \varphi(\tau)\|^2 + \|\partial_t \sigma(\tau)\|^2 + \|\partial_t \varphi_2(\tau)\|^2 \|\varphi(\tau)\|^2_{H^1} \right) d\tau \\
\leq C(t + t^2) \int_0^t \|\partial_t \varphi(\tau)\|^2 d\tau + C(1 + t) \int_0^t \tau \|\partial_t \sigma(\tau)\|^2 d\tau \\
+ Ct^2 \sup_{\tau \in [0,t]} \|\varphi(\tau)\|^2_{H^1} \int_0^t \|\partial_t \varphi_2(\tau)\|^2_{H^1} d\tau.
\end{equation}

The above inequality together with (5.24), (5.29), (3.46), and (3.52) yields that
\begin{equation}
t^2 \left( \|\partial_t \varphi(t)\|^2 + \|\partial_t \sigma(t)\|^2 \right) \leq C_T (1 + t^2) (\|\varphi_0\|^2_{H^1} + \|\sigma_0\|^2), \quad \forall t \in (0, T].
\end{equation}

From the above estimate and (5.21c) and (5.28), we then deduce that
\begin{equation}
\|\Delta \varphi(t)\|^2 + \|\Delta \sigma(t)\|^2 \leq C_T \left( \frac{1 + t^2}{t^2} \right) (\|\varphi_0\|^2_{H^1} + \|\sigma_0\|^2), \quad \forall t \in (0, T],
\end{equation}

which combined with (5.24) leads to smooth estimate (5.20) for the difference of solutions. 

The proof is complete.

\begin{proof}

\end{proof}

Step 3. Applying the abstract result of Berti and Gatti\textsuperscript{42, Lemma 5.3} (see also Gatti et al\textsuperscript{43, Section 2} for the more general version), we can deduce from Lemmas 5.1 and 5.2 that there exists a closed bounded set $\mathcal{M}_{m_1,m_2} \subset \overline{B}_1$, which is of finite fractal dimension in $\mathcal{X}_{m_1,m_2}$ and positively invariant under $S(t)$. Moreover, for some constants $\omega_0 > 0$ and $J_0 > 0$, it holds
\begin{equation}
\text{dist}_{H^1 \times L^2}(S(t) \overline{B}_1, \mathcal{M}_{m_1,m_2}) \leq J_0 e^{-\omega_0 t}, \quad \forall t \geq 0.
\end{equation}

Since $\overline{B}_1$ itself is a compact absorbing set, for every bounded set $B \subset \mathcal{X}_{m_1,m_2}$, there exists $t_2 = t_2(B) > 0$ such that $S(t)B \subset \overline{B}_1$ for all $t \geq t_2(B)$. Hence, we have
\begin{equation}
\text{dist}_{H^1 \times L^2}(S(t)B, \mathcal{M}_{m_1,m_2}) \leq J_0 e^{-\omega_0 [t - t_2(B)]}, \quad \forall t \geq t_2(B).
\end{equation}

Besides, it follows from estimates (3.10), (3.12), (3.13), and (3.22) and the boundedness of $\mathcal{M}_{m_1,m_2}$ that
\begin{equation}
\text{dist}_{H^1 \times L^2}(S(t)B, \mathcal{M}_{m_1,m_2}) \leq \sup_{(\varphi,\sigma) \in B} \sup_{t \in [0,t_2(B)]} \|S(t)(\varphi,\sigma)\|_{H^1 \times L^2} + \sup_{(\varphi,\sigma) \in \mathcal{M}_{m_1,m_2}} (\|\varphi\|_{H^1} + \|\sigma\|_{L^2}) \\
\leq C_{B_1}, \quad \forall t \in [0,t_2(B)).
\end{equation}
where \( C_B > 0 \) depends on the size of \( B, R_1, \Omega, m_1, m_2 \), and coefficients of the system. As a consequence, for every bounded set \( B \subset X_{m_1,m_2} \), we have
\[
\text{dist}_{H^{1/2};L^2}(S(t)B, \mathcal{M}_{m_1,m_2}) \leq \left(\max\{J_0, C_B\}\right) e^{\omega_0 t(B)} e^{-m_2 t}. \forall t \geq 0.
\]
This implies that \( \mathcal{M}_{m_1,m_2} \) attracts exponentially fast all bounded subsets \( B \subset X_{m_1,m_2} \), that is, its basin of exponential attraction is the whole phase space \( X_{m_1,m_2} \).

The proof of Theorem 2.4 is complete.

\[\square\]

5.3 \ The \( \omega \)-limit set

Finally, we proceed to characterize the \( \omega \)-limit set of an arbitrary given initial datum \((\varphi_0, \sigma_0)\) belonging to the set
\[
\mathcal{Z} = \{(z_1, z_2) \in H^1(\Omega) \times L^2(\Omega) : \|z_1\|_{L^\infty} \leq 1, |z_2| < 1\}.
\]

Denoting \( H = H^{2r}(\Omega) \times H^{2r}(\Omega) \) with \( r \in (1/2, 1) \), we define the \( \omega \)-limit set of \((\varphi_0, \sigma_0)\) as
\[
\omega(\varphi_0, \sigma_0) = \{(z_1, z_2) \in \mathcal{Z} : \exists \{t_n\} \nearrow +\infty \text{ s.t. } (\varphi(t_n), \sigma(t_n)) \to (z_1, z_2) \text{ strongly in } H\}.
\]

where \((\varphi, \sigma)\) is the unique global weak solution corresponding to \((\varphi_0, \sigma_0)\). Then we have

**Proposition 5.1.** Suppose that (H1) and (H2) are satisfied. For any initial datum \((\varphi_0, \sigma_0) \in \mathcal{Z}\), its \( \omega \)-limit set \( \omega(\varphi_0, \sigma_0) \) is nonempty and compact in \( H \). Moreover,

1. \( \omega(\varphi_0, \sigma_0) \) consists of stationary points \((\varphi_\infty, \sigma_\infty)\) only, where \((\varphi_\infty, \sigma_\infty) \in H^2_N(\Omega) \times H^2_N(\Omega) \) is a strong solution to stationary problems (2.11a)–(2.11d), satisfying \((\varphi_\infty, \sigma_\infty) = (c_0, \sigma_0)\). In particular, there exists a constant \( \delta \in (0, 1) \) such that
\[
\|\varphi_\infty\|_{L^\infty} \leq 1 - \delta,
\]

where \( \delta \) is independent of \( \varphi_\infty \).

2. \( \omega(\varphi_0, \sigma_0) \) is invariant under the semigroup \( S(t) \) defined by the global weak solution to problems (1.1a)–(1.3), that is, \( S(t)\omega(\varphi_0, \sigma_0) = \omega(\varphi_0, \sigma_0) \) for all \( t \geq 0 \).

**Proof.** For any initial datum \((\varphi_0, \sigma_0) \in \mathcal{Z}\), we denote the associated unique global weak solution by \((\varphi, \sigma)\). From Corollary 2.1, we see that for any fixed \( t > 0 \), it holds \( \varphi \in C([r, +\infty); H^2(\Omega)), \sigma \in C([r, +\infty); H^2(\Omega)) \), being uniformly bounded for all \( t \geq r \). Hence, it follows from the Sobolev embedding theorem that the orbit \( \{(\varphi(t), \sigma(t))\}_{t \geq r} \) is relatively compact in \( H \), which implies that \( \omega(\varphi_0, \sigma_0) \) is a nonempty, bounded subset in \( H^2(\Omega) \times H^2(\Omega) \) and thus compact in \( H \).

Then for any cluster point \((\varphi_\infty, \sigma_\infty) \in \omega(\varphi_0, \sigma_0)\), we see that \((\varphi_\infty, \sigma_\infty) \in H^2(\Omega) \times H^2(\Omega) \) and \( \varphi_\infty = c_0, \sigma_\infty = \sigma_0 \). Following the argument in Cavaterra et al. and Huang and Takáč, we are able to show that \((\varphi_\infty, \sigma_\infty) \) is indeed a solution to stationary problems (2.11a)–(2.11d). To this end, we recall (3.40), which implies that \( F(t) + \frac{C}{a} e^{-at} \) is indeed a strict Lyapunov function for problems (1.1a)–(1.3). Since it is nonincreasing in time and bounded from below, there exists some constant \( F_\infty \in \mathbb{R} \) such that
\[
\lim_{t \to +\infty} F(t) = \lim_{t \to +\infty} \left(F(t) + \frac{C}{a} e^{-at}\right) = F_\infty.
\]

Let \( \{t_n\} \) be an unbounded increasing sequence such that
\[
(\varphi(t_n), \sigma(t_n)) \to (\varphi_\infty, \sigma_\infty) \text{ strongly in } H,
\]
\[
(\varphi(t_n), \sigma(t_n)) \to (\varphi_\infty, \sigma_\infty) \text{ weakly in } H^2(\Omega) \times H^2(\Omega).
\]

For any \( r \in (0, +\infty) \), we recall Corollary 2.1 such that
\[
\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta(r), \forall t \in [r, +\infty),
\]

\[\square\]
which together with the Sobolev embedding theorem $H^2(\Omega) \hookrightarrow L^6(\Omega)$ (for $r \in (3/4, 1)$) yields conclusion (5.37). Without loss of generality, we assume $t_{n+1} \geq t_n + 1$ for $n \in \mathbb{N}$. Integrating inequality (3.40) on the time interval $[t_n, t_{n+1}]$, we obtain

$$F(t_{n+1}) - F(t_n) + \frac{1}{2} \int_{t_n}^{t_{n+1}} \| \nabla \bar{\mu}(s) \|^2 + \| \nabla (\sigma(s) - \chi \varphi(s)) \|^2 + e \| \partial_t \varphi(s) \|^2 ds$$

which implies

$$\lim_{n \to +\infty} \int_{t_n}^{t_{n+1}} \| \nabla \bar{\mu}(s) \|^2 + \| \partial_t \sigma(s) \|_{V^o}^2 + e \| \partial_t \varphi(s) \|^2 ds = 0.$$  

As a result, we have for $n \to \infty$

$$\| \varphi(t_n + s_1) - \varphi(t_n + s_2) \| \to 0, \text{ uniformly for all } s_1, s_2 \in [0, 1],$$

$$\| \sigma(t_n + s_1) - \sigma(t_n + s_2) \|_{V^o} \to 0, \text{ uniformly for all } s_1, s_2 \in [0, 1].$$

Combining the above facts with the boundedness of $(\varphi(t), \sigma(t))$ in $H^2(\Omega) \times H^2(\Omega)$, by a standard interpolation, we infer that

$$\| (\varphi(t_n + s), \sigma(t_n + s)) - (\varphi_\infty, \sigma_\infty) \|_H \to 0, \text{ uniformly for all } s \in [0, 1].$$  

Moreover, strict separation property (5.39) enables us to regard the singular potential $\Psi$ as a globally Lipschitz function when $n$ is large. Denote $\bar{t}_n := t_n + s$ with $s \in [0, 1]$. Then for any $\xi \in H^1(\Omega)$, we deduce from Lebesgue's dominated convergence theorem that

$$\int_{\Omega} B \nabla \varphi_\infty \cdot \nabla \xi + A \Psi'(\varphi_\infty) \cdot \xi - A \Psi'(\varphi_\infty) \cdot \chi \sigma_\infty \xi + \chi \sigma_\infty \xi + \chi \sigma_\infty \xi + \alpha N(\varphi_\infty - \varphi_\infty) \xi \, dx$$

$$= \lim_{n \to +\infty} \int_{0}^{1} \int_{\Omega} B \nabla \bar{\varphi}(\bar{t}_n) \cdot \nabla \xi + A \Psi'(\bar{\varphi}(\bar{t}_n)) \cdot \xi - A \Psi'(\bar{\varphi}(\bar{t}_n)) \xi$$

$$- \chi \sigma(\bar{t}_n) \xi + \chi \sigma(\bar{t}_n) \xi + \alpha N(\bar{\varphi}(\bar{t}_n) - \varphi(\bar{t}_n)) \xi \, dx \, ds$$

$$\leq \lim_{n \to +\infty} \int_{0}^{1} \int_{\Omega} \left( \bar{\mu}(\bar{t}_n) - \mu(\bar{t}_n) - e \partial_t \varphi(\bar{t}_n) + e \partial_t \varphi(\bar{t}_n) \right) \xi \, dx \, ds$$

$$\leq \int_{0}^{1} \| \bar{\mu}(\bar{t}_n) - \mu(\bar{t}_n) \| \| \xi \| \, ds + e \int_{0}^{1} \| \partial_t \varphi(\bar{t}_n) - \partial_t \varphi(\bar{t}_n) \| \| \xi \| \, ds$$

$$\leq C \lim_{n \to +\infty} \left( \int_{0}^{1} \| \nabla \bar{\mu}(\bar{t}_n) \|^2 \, ds \right)^{\frac{1}{2}} \| \xi \| + e C \left( \int_{0}^{1} \| \partial_t \varphi(\bar{t}_n) \|^2 \, ds \right)^{\frac{1}{2}} \| \xi \|$$

$$= 0.$$

From (5.40), we also infer that

$$\int_{\Omega} \nabla \sigma_\infty \cdot \nabla \xi \, dx = 0, \forall \xi \in H^1(\Omega).$$

As a consequence, $(\varphi_\infty, \sigma_\infty)$ is a weak solution to stationary problems (2.11a)–(2.11d) with $\mu_\infty = A \Psi'(\varphi_\infty) - \chi \sigma_\infty$. Since we already know that $(\varphi_\infty, \sigma_\infty) \in H^2(\Omega) \times H^2(\Omega)$, then it is also a strong solution.

Finally, since every $(\varphi_\infty, \sigma_\infty) \in o(\varphi_\infty, \sigma_\infty)$ is a stationary point, that is, $S(t)(\varphi_\infty, \sigma_\infty) = (\varphi_\infty, \sigma_\infty)$ for $t \geq 0$, this yields the invariance of $o(\varphi_\infty, \sigma_\infty)$ under $S(t)$. The proof is complete.

**Proof of Theorem 2.5.** It is straightforward to check that the conclusion of Theorem 2.5 is an immediate consequence of Proposition 5.1. ♠

We note that Proposition 5.1 also provides a dynamical approach for the study of stationary problems (2.11a)–(2.11d), which is independent of the viscous parameter $\epsilon$.

**Corollary 5.1.** Let $(\varphi_*, \sigma_*) \in Z$ be a weak solution of stationary problems (2.11a)–(2.11d). Then $(\varphi_*, \sigma_*) \in H^2(\Omega) \times H^2(\Omega)$, and there exists a positive constant $\delta \in (0, 1)$ such that the strict separation property (5.37) holds for $\varphi_*$. ♠
Proof. It is obvious that \((\varphi_*, \sigma_*)\) can be viewed as a global weak solution to evolution problems (1.1a)–(1.3) with the particular choice of initial datum \((\varphi_0, \sigma_0) = (\varphi_*, \sigma_*)\) (cf. Miranville and Zelik\(^6\) for the viscous Cahn–Hilliard equation). Thanks to the uniqueness of weak solution (recall (2.7)), we have \(S(t)(\varphi_*, \sigma_*) = (\varphi_*, \sigma_*)\) for all \(t \geq 0\). Therefore, we can infer that \((\varphi_*, \sigma_*) \in \omega(\varphi_*, \sigma_*)\) and then the conclusion follows from Proposition 5.1.

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CONFLICT OF INTEREST

The author states that this work does not have any conflicts of interest.

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