EXPONENTIAL DECAY FOR QUASILINEAR PARABOLIC EQUATIONS IN ANY DIMENSION

JIAN-WEN SUN
School of Mathematics and Statistics
Lanzhou University
Lanzhou 730000, China

SEONGHAK KIM*
Department of Mathematics, College of Natural Sciences
Kyungpook National University
Daegu 41566, Republic of Korea

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Abstract. We estimate decay rates of solutions to the initial-boundary value problem for a class of quasilinear parabolic equations in any dimension. Such decay rates depend only on the constitutive relations, spatial domain, and range of the initial function.

1. Introduction. The aim of this note is to investigate decay rates of classical solutions to the initial-boundary value problem for a class of quasilinear advection-diffusion equations in any dimension,

\[
\begin{aligned}
  &u_t = \text{div}(a(x)\nabla A(u)) \quad \text{in } \Omega \times (0, \infty), \\
  &u = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
  &u(x, 0) = u_0(x) \quad \text{for } x \in \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^n \) (\( n \geq 1 \)) is a bounded open set, \( a = a(x) : \Omega \to \mathbb{R} \) and \( A = A(s) : \mathbb{R} \to \mathbb{R} \) are given constitutive functions, \( \nabla \) denotes the spatial gradient, \( u_0 : \bar{\Omega} \to \mathbb{R} \) is an initial function, and \( u = u(x, t) : \bar{\Omega} \times [0, \infty) \to \mathbb{R} \) is a solution to the problem. Here, the diffusivity \( a(x)A'(u) \) depends on the spatial variable \( x \) and solution \( u \) in a separated form.

Throughout the note, we assume at least the following.

\[
\begin{aligned}
  &u_0 \in C(\bar{\Omega}), \quad u_0 = 0 \quad \text{on } \partial \Omega, \\
  &a \in C^1(\Omega), \quad \text{and } A \in C^2(\mathbb{R}).
\end{aligned}
\]

For simplicity, we also reserve some notations as follows.

\[
\begin{aligned}
  &m_0 := \min_{\bar{\Omega}} u_0 \leq 0 \leq \max_{\bar{\Omega}} u_0 =: M_0, \\
  &\Omega_\tau := \Omega \times (0, \tau) \quad \forall \tau \in (0, \infty].
\end{aligned}
\]

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* Corresponding author.
Following the pioneering works by Zelenjak [20] and Matano [14] in one space dimension, there have been numerous results on large time behaviors of solutions to various semilinear or quasilinear parabolic problems. In this regard, most studies have focused on one-dimensional problems such as

- convergence or finite-time blow-up for semilinear heat equations under the Dirichlet, Neumann, or periodic boundary conditions [1];
- characterization of blow-up and extinction for general quasilinear heat equations [6];
- stability analysis for quasilinear heat equations under mixed boundary conditions [19, 7];
- convergence for degenerate quasilinear heat equations [4]; and
- exponential decay for quasilinear advection-diffusion equations [11].

On the other hand, there are relatively fewer works on high-dimensional problems dealing with

- convergence for semilinear heat equations on “thin” domains in $\mathbb{R}^2$ [8];
- convergence for semilinear heat equations on balls [10];
- rate of decay for semilinear heat equations in the setup of the Lojasiewicz inequality [9]; and
- decay bounds for various parabolic problems [15, 16, 17, 18].

In this note, we provide an elementary proof on exponential decay rates for the quasilinear advection-diffusion problem (1) in any dimension. It turns out that such decay rates depend only on the constitutive relations $a(x)$ and $A(s)$, spatial domain $\Omega$, and range of the initial function $u_0$. In doing so, we use the quasilinear comparison principle, Lemma 2.2, utilizing super and subsolutions that are constructed from the first eigenvalue and corresponding eigenfunction of the linearized stationary eigenvalue problem (5). Another key fact in the course is a version of the maximum principle, Lemma 2.1, that enables us to “localize” the dependence on $A$ precisely over the interval $[m_0, M_0]$.

We shall not consider the standard issues on the existence, uniqueness, and regularity of a solution to Problem (1) that are well addressed in the classical literatures [5, 12, 13].

**Main result.** For the main result, we assume

$\begin{cases}
  a \in C^\infty(\mathbb{R}^n), \quad a \geq \theta \text{ in } \mathbb{R}^n, \\
  A' > 0 \text{ on } [m_0, M_0],
\end{cases}$  \hspace{1cm} (3)

where $\theta > 0$ is a constant. Let

$\alpha_0 = \min_{[m_0, M_0]} A' > 0.$  \hspace{1cm} (4)

For any bounded open set $\Omega' \subset \mathbb{R}^n$ with $\partial \Omega' \in C^\infty$ and $\Omega \subset \subset \Omega'$, let $\lambda = \lambda_p(\Omega')$ denote the principal eigenvalue of

$\begin{cases}
  -\text{div}(a(x)\nabla \phi) = \lambda \phi \quad \text{in } \Omega', \\
  \phi = 0 \quad \text{on } \partial \Omega',
\end{cases}$  \hspace{1cm} (5)

with its positive eigenfunction $\phi_p(\Omega') = \phi_p(\Omega'; x) \in C^\infty(\Omega')$ satisfying $\|\phi_p(\Omega')\|_{L^\infty(\Omega')} = 1$ (see, e.g., [2, 3]), and set

$N_0(\Omega') = \max_{x \in \Omega} \frac{|A(u_0(x)) - A(0)|}{\phi_p(\Omega'; x)} \geq 0$ and $\gamma_0(\Omega') = \alpha_0 \lambda_p(\Omega') > 0$.  \hspace{1cm} (6)
Note that $N_0(\Omega') = 0$ if and only if $u_0 = 0$ on $\bar{\Omega}$.

The main result of this note is now stated as follows.

**Theorem 1.1** (Exponential decay). Let $u \in C(\bar{\Omega}_\infty) \cap C^{2,1}(\Omega_\infty)$ be a solution to Problem (1). Then for any bounded open set $\Omega' \subset \mathbb{R}^n$ with $\partial\Omega' \in C^\infty$ and $\Omega \subset \subset \Omega'$, one has

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq N_0 e^{-\gamma_0 t}$$

for all $t \geq 0$, where $N_0 := N_0(\Omega')$ and $\gamma_0 := \gamma_0(\Omega')$.

This theorem remains valid when the domain $\mathbb{R}^n$ of the function $a$ is replaced by any open set $\tilde{\Omega} \subset \mathbb{R}^n$ with $\Omega \subset \subset \tilde{\Omega}$. In this case, bounded open sets $\Omega'$ in the statement should be compactly contained in $\tilde{\Omega}$. Also, the $C^\infty$ regularity of $a$ and $\partial\Omega'$ is superfluous. In fact, $a$ and $\partial\Omega'$ need to be smooth as much as one can guarantee that $\phi_p(\Omega') \in C^2(\bar{\Omega'})$ (see [3]).

**Organization of the note.** The remainder of this note is organized as follows. Section 2 prepares suitable maximum and comparison principles for qualitative behaviors of solutions to Problem (1). Based on such principles, a proof of the main result, Theorem 1.1, is provided in Section 3.

2. **Maximum and comparison principles.** In this section, we prepare two useful lemmas on the maximum and comparison principles, related to Problem (1). Other than the basic assumptions (2), we assume in this section that

$$a \geq 0 \text{ in } \Omega.$$  

**2.1. Maximum principle.** We present a version of the maximum principle stating that the value of any classical solution $u$ to Problem (1) goes neither above the maximum value nor below the minimum value of the initial function $u_0$ for all times.

Our version of the maximum principle towards the proof of the main result, Theorem 1.1, captures precisely the dependence of $A'$ on the initial function $u_0$ in the sense that the positivity of $A'$ at the end points $M_0$ and $m_0$ of the initial range is sufficient to guarantee the maximum principle.

**Lemma 2.1** (Localized maximum principle). Assume that $A'(M_0) > 0$ and $A'(m_0) > 0$ and that $u \in C(\bar{\Omega}_\infty) \cap C^{2,1}(\Omega_\infty)$ is a solution to Problem (1). Then

$$m_0 \leq u \leq M_0 \text{ on } \bar{\Omega}_\infty.$$  

**Proof.** It suffices to show that

$$m_0 \leq u \leq M_0 \text{ in } \Omega_\infty.$$  

We only prove that $u \geq m_0$ in $\Omega_\infty$ since the other inequality $u \leq M_0$ in $\Omega_\infty$ can be shown in a similar fashion.

To proceed by contradiction, suppose that there exists a point $(x^0, t_0) \in \Omega_\infty$ such that

$$u(x^0, t_0) < m_0.$$  

From this inequality and $A'(m_0) > 0$, we can choose a number $\epsilon_0 > 0$ so small that

$$u(x^0, t_0) < m_0 - 2\epsilon_0 t_0 - \epsilon_0$$

and that

$$A' > 0 \text{ on } [m_0 - 2\epsilon_0 t_0 - \epsilon_0, m_0].$$

(9)

(10)
Define
\[ z(x,t) = m_0 - 2\varepsilon_0 t - \varepsilon_0 - u(x,t) \quad \forall (x,t) \in \bar{\Omega}_\infty. \]
Then from (9), \( u_0 \geq m_0 \) on \( \bar{\Omega} \), and \( u = 0 \) on \( \partial \Omega \times (0, \infty) \), it follows that
\[ \begin{cases} 
  z(x_0, t_0) > 0, & z(x, 0) = m_0 - \varepsilon_0 - u_0(x) \leq -\varepsilon_0 < 0 \quad \text{for } x \in \bar{\Omega}, \\
  z(x, t) = m_0 - 2\varepsilon_0 t - \varepsilon_0 < -\varepsilon_0 < 0 \quad \text{for } (x, t) \in \partial \Omega \times (0, \infty).
\end{cases} \tag{11} \]
Define
\[ \mathcal{T} = \{ t \in [0, \infty) | z < 0 \text{ on } \bar{\Omega} \times [0, t] \} \quad \text{and} \quad t^* = \sup \mathcal{T}. \]
Then from (11), \( \{0\} \subset \mathcal{T} \subset [0, t_0) \), and so \( 0 \leq t^* \leq t_0 \). From the definition of \( t^* \), for each \( j \in \mathbb{N} \),
\[ z(x^j, t_j) \geq 0 \quad \text{for some } (x^j, t_j) \in \bar{\Omega} \times [t^*, t^* + 1/j). \]
Passing to a subsequence if necessary, \( x^j \to x^* \) as \( j \to \infty \), for some \( x^* \in \bar{\Omega} \), and thus, \( z(x^*, t^*) \geq 0 \). This inequality and (11) imply that \( x^* \in \bar{\Omega} \) and \( 0 < t^* \leq t_0 \).
Also, from the definition of \( t^* \), \( z < 0 \) on \( \bar{\Omega} \times [0, t^*) \). In particular, \( z(x, t^*) \leq 0 \) for all \( x \in \bar{\Omega} \). Thus, \( 0 < t^* < t_0 \), and at the point \( (x,t) = (x^*, t^*) \),
\[ z = 0, \quad z_{x_i} = 0, \quad z_{x_i x_i} \leq 0, \quad \text{and} \quad z_t \geq 0, \]
and so
\[ u(x^*, t^*) = m_0 - 2\varepsilon_0 t^* - \varepsilon_0 \in (m_0 - 2\varepsilon_0 t_0 - \varepsilon_0, m_0), \] \[ u_{x_i}(x^*, t^*) = 0, \quad u_{x_i x_i}(x^*, t^*) \geq 0, \quad \text{and} \quad u_t(x^*, t^*) \leq -2\varepsilon_0, \tag{12} \]
where \( i = 1, \ldots, n \).
Finally, observe from (12), (8), and (10) that
\[ 0 > -2\varepsilon_0 \geq u_t(x^*, t^*) = a(x^*) A'(u(x^*, t^*)) \Delta u(x^*, t^*) \geq 0, \]
which is a contradiction. \( \square \)

2.2. Comparison principle. Concerning the equation in (1)
\[ u_t = \text{div}(a(x) \nabla A(u)), \tag{13} \]
for a function \( u \in C^{2,1}(\Omega_\infty) \), we shall say that \( u \) is a supersolution of Equation (13) in \( \Omega_\infty \) if
\[ u_t \geq \text{div}(a(x) \nabla A(u)) \quad \text{in } \Omega_\infty \]
and that it is a strict supersolution of (13) in \( \Omega_\infty \) if
\[ u_t > \text{div}(a(x) \nabla A(u)) \quad \text{in } \Omega_\infty. \]
Similarly, we define (strict) subsolutions of (13) with the (strict) inequality reversed.

As another main ingredient for the proof of Theorem 1.1, we present a quasilinear comparison principle as follows.

Lemma 2.2 (Comparison principle). Assume that \( A' \geq 0 \) in \( \mathbb{R} \). Let \( v, w \in C(\bar{\Omega}_\infty) \cap C^{2,1}(\Omega_\infty) \) be a supersolution and a subsolution of Equation (13) in \( \Omega_\infty \), respectively, such that at least one of \( v \) and \( w \) is strict and that
\[ v > w \quad \text{on } \partial \Omega_\infty. \]
Then \( v > w \) in \( \Omega_\infty \).
Proof. To prove by contradiction, suppose that
\[ v(x^0, t_0) \leq w(x^0, t_0) \text{ for some } (x^0, t_0) \in \Omega_\infty. \]  
(14)

Define
\[ z(x, t) = e^{-t}(v(x, t) - w(x, t)) \quad \forall (x, t) \in \bar{\Omega}_\infty; \]
then
\[ z > 0 \text{ on } \partial \Omega_\infty. \]  
(15)

Define
\[ \mathcal{T} = \{ t \in [0, \infty) \mid z > 0 \text{ on } \bar{\Omega} \times [0, t] \} \text{ and } t^* = \sup \mathcal{T}. \]

From (14) and (15), \( \{ 0 \} \subset \mathcal{T} \subset [0, t_0) \), and so \( 0 \leq t^* \leq t_0 \). From the definition of \( t^* \), for each \( j \in \mathbb{N} \),
\[ z(x^j, t_0) = 0 \text{ for some } (x^j, t_0) \in \bar{\Omega} \times [t^*, t^* + 1/j). \]

Passing to a subsequence if necessary, we have \( x^j \to x^* \) as \( j \to \infty \), for some \( x^* \in \bar{\Omega} \); then \( z(x^*, t^*) \leq 0 \). From this and (15), we see that \( x^* \in \Omega \) and \( 0 < t^* \leq t_0 \). Also, from the definition of \( t^* \), we have \( z > 0 \) on \( \bar{\Omega} \times [0, t^*) \). In particular, \( z \geq 0 \) on \( \bar{\Omega} \times \{ t^* \} \). So we easily deduce that at the point \( (x, t) = (x^*, t^*) \),
\[ z = 0, \quad z_{x_i} = 0, \quad z_{x_i x_i} \geq 0, \quad \text{and } z_t \leq 0. \]  
(16)

Thus, at \( (x, t) = (x^*, t^*) \),
\[ v = w, \quad v_{x_i} = w_{x_i}, \quad \text{and } v_{x_i x_i} \geq w_{x_i x_i}, \]  
(17)
where \( i = 1, \ldots, n \).

On one hand, we have from (16) and (17) that
\[ 0 \geq z_t(x^*, t^*) = e^{-t^*}((v - w)_t(x^*, t^*) - (v - w)(x^*, t^*)) \]
\[ = e^{-t^*}(v - w)_t(x^*, t^*). \]

Thus,
\[ (v - w)_t(x^*, t^*) \leq 0. \]

On the other hand, since \( v \) and \( w \) are a supersolution and a subsolution of Equation (13) in \( \Omega_\infty \), respectively, with one of them being strict, it follows from (17), (8), and \( A' \geq 0 \) in \( \mathbb{R} \) that
\[ (v - w)_t(x^*, t^*) \]
\[ > a(x^*) \left[ A'(v(x^*, t^*)) \Delta v(x^*, t^*) - A'(w(x^*, t^*)) \Delta w(x^*, t^*) \right] \]
\[ + A'(v(x^*, t^*)) \sum_{i=1}^n a_{x_i}(x^*) v_{x_i}(x^*, t^*) - A'(w(x^*, t^*)) \sum_{i=1}^n a_{x_i}(x^*) w_{x_i}(x^*, t^*) \]
\[ + a(x^*) \left[ A''(v(x^*, t^*)) \sum_{i=1}^n v_{x_i}^2(x^*, t^*) - A''(w(x^*, t^*)) \sum_{i=1}^n w_{x_i}^2(x^*, t^*) \right] \]
\[ = a(x^*) A'(v(x^*, t^*)) \Delta (v - w)(x^*, t^*) \geq 0; \]
that is, \( (v - w)_t(x^*, t^*) > 0 \). We thus have a contradiction. \( \square \)
3. Exponential decay. In this section, we prove the main result of this note, Theorem 1.1, on the exponential decay rates for classical solutions to Problem (1).

Let \( 0 < \epsilon < \alpha_0 \). In the appendix below, we construct a function \( A_\epsilon \in C^2(\mathbb{R}) \) such that

\[
\begin{align*}
A_\epsilon' &> 0 \text{ in } \mathbb{R}, \quad A_\epsilon(s) = A(s) - A(0) \quad \forall s \in [m_\alpha, M_0], \\
A_\epsilon'(s) &\geq A_\epsilon'(M_0) - \epsilon \quad \forall s \in [M_0, \infty), \\
A_\epsilon'(s) &\geq A_\epsilon'(m_0) - \epsilon \quad \forall s \in (-\infty, m_0], \\
\lim_{s \to -\infty} A_\epsilon(s) &= \infty, \quad \text{and } \lim_{s \to -\infty} A_\epsilon(s) = -\infty.
\end{align*}
\]

Let \( B_\epsilon \in C^2(\mathbb{R}) \) denote the inverse function of \( A_\epsilon \). Then from (4) and (18), we see that \( B_\epsilon' > 0 \) in \( \mathbb{R} \), \( B_\epsilon(0) = 0 \), and

\[
0 < \alpha_0 - \epsilon = \min_{[m_\alpha, M_0]} A_\epsilon - \epsilon \leq \inf_{\mathbb{R}} A_\epsilon' = \left( \sup_{\mathbb{R}} B_\epsilon' \right)^{-1} =: \alpha \leq \alpha_0.
\]

Let \( \Omega' \subset \mathbb{R}^n \) be any bounded open set with \( \partial \Omega' \subset C^\infty \) and \( \Omega \subset \subset \Omega' \). For any two fixed reals \( N > N_0 := N_0(\Omega') \) and \( 0 < \gamma < \gamma_\epsilon := \alpha_\epsilon \lambda_p(\Omega') \), define

\[
\begin{align*}
v(x, t) &= v_{N, \gamma}(x, t) = B_\epsilon(N e^{-\gamma t} \phi(x)), \\
w(x, t) &= w_{N, \gamma}(x, t) = B_\epsilon(-N e^{-\gamma t} \phi(x))
\end{align*}
\]

for all \((x, t) \in \bar{\Omega}_\infty\), where \( \phi := \phi_p(\Omega') \in C^\infty(\bar{\Omega}') \). Then for every \( x \in \bar{\Omega} \), since \( \phi(x) > 0 \), \( \frac{|A([u_0(x)] - A(0))|}{\phi(x)} \leq N_0 \) (from (6)), and \( u_0(x) \in [m_\alpha, M_0] \), it follows from (18) that

\[
v(x, 0) = B_\epsilon(N \phi(x)) > B_\epsilon(N_0 \phi(x)) \geq B_\epsilon(|A_\epsilon(u_0(x))|) \geq u_0(x),
\]

\[
w(x, 0) = B_\epsilon(-N \phi(x)) < B_\epsilon(-N_0 \phi(x)) \leq B_\epsilon(-|A_\epsilon(u_0(x))|) \leq u_0(x).
\]

If \((x, t) \in \partial \Omega \times (0, \infty)\), then \( Ne^{-\gamma t} \phi(x) > 0 \) so that

\[
v(x, t) = B_\epsilon(N e^{-\gamma t} \phi(x)) > 0 > B_\epsilon(-N e^{-\gamma t} \phi(x)) = w(x, t).
\]

If \((x, t) \in \Omega_\infty\), then from (5) and the definition of \( \alpha_\epsilon \) and \( \gamma_\epsilon \),

\[
\begin{align*}
u_t(x, t) &= -\gamma B_\epsilon'(N e^{-\gamma t} \phi(x)) Ne^{-\gamma t} \phi(x) > -\gamma_\epsilon B_\epsilon'(N e^{-\gamma t} \phi(x)) Ne^{-\gamma t} \phi(x) \\
&= -\alpha_\epsilon B_\epsilon'(N e^{-\gamma t} \phi(x)) Ne^{-\gamma t} \lambda_p(\Omega') \phi(x) \\
&\geq -N e^{-\gamma t} \lambda_p(\Omega') \phi(x) = \text{div}(\alpha(x) \nabla A_\epsilon(w(x, t))).
\end{align*}
\]

and similarly,

\[
w_t(x, t) < \text{div}(\alpha(x) \nabla A_\epsilon(w(x, t))).
\]

Now, let \( u \in C(\bar{\Omega}_\infty) \cap C^{2,1}(\Omega_\infty) \) be a solution to Problem (1). Then from (18) and Lemma 2.1, \( u \) is also a solution to (1) with \( A \) replaced by \( A_\epsilon \). Moreover, from the above, we see that \( v, w \in C(\bar{\Omega}_\infty) \subset C^{2,1}(\Omega_\infty) \) are a strict supersolution and a strict subsolution, respectively, of Equation (13), with \( A \) replaced by \( A_\epsilon \), such that

\[
v > u > w \quad \text{on } \partial \Omega_\infty.
\]

Applying Lemma 2.2 to the pairs \((v, u)\) and \((u, w)\) with \( A \) replaced by \( A_\epsilon \), we obtain that for all \((x, t) \in \Omega_\infty\),

\[
B_\epsilon(-N e^{-\gamma t} \phi(x)) = w(x, t) < u(x, t) < v(x, t) = B_\epsilon(N e^{-\gamma t} \phi(x)).
\]

Letting \( N \searrow N_0 \) and \( \gamma \nearrow \gamma_\epsilon \), for all \((x, t) \in \Omega_\infty\),

\[
B_\epsilon(-N_0 e^{-\gamma t} \phi(x)) \leq u(x, t) \leq B_\epsilon(N_0 e^{-\gamma t} \phi(x)),
\]

such that
and thus,
\[ |u(x,t)| \leq \alpha^{-1}_t N_0 e^{-\gamma t} \phi(x). \]
Therefore, for all \( t \geq 0 \),
\[ \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \alpha^{-1}_t N_0 e^{-\gamma t} \]
as \( \|\phi\|_{L^\infty(\Omega')} = \|\phi_0(\Omega')\|_{L^\infty(\Omega')} = 1 \). Finally, taking \( \epsilon \searrow 0 \), we get the desired decay estimate (7).

**Appendix.** In this appendix, given a number \( \epsilon \in (0,\alpha_0) \), we construct a function \( A_\epsilon \in C^2(\mathbb{R}) \) satisfying (18). We only construct such an \( A_\epsilon \) on \([m_0,\infty)\) as it can be extended to the left of \( m_0 \) in a similar way.

First, let \( A_0(s) = A(s) - A(0) \) for all \( s \in \mathbb{R} \), and define
\[ A_\epsilon(s) = A_0(s) \quad \forall s \in [m_0, M_0]. \tag{19} \]
Next, define
\[ A_\epsilon(s) = \beta_\epsilon \frac{1}{6} (s - M_0)^3 + \frac{A_0''(M_0)}{2} (s - M_0)^2 + A_0'(M_0)(s - M_0) + A_0(M_0) \]
for all \( s \in [M_0, \infty) \), where \( \beta_\epsilon \geq 0 \) is a constant to be specified below. At this stage, we easily see that \( A_\epsilon \in C^2([m_0, \infty)) \) and that \( A_\epsilon = A(s) - A(0) \) for all \( s \in [m_0, M_0] \).

Consider the two cases (i): \( A_0''(M_0) \geq 0 \) and (ii): \( A_0''(M_0) < 0 \). In the case that (i) holds, take \( \beta_\epsilon = 0 \). Since \( A_0''(M_0) = A''(M_0) \geq \alpha_0 > \epsilon \) and \( A_0''(M_0) \geq 0 \), we have
\[ A_\epsilon'(s) = A_0''(M_0)(s - M_0) + A_0'(M_0) \geq A_0'(M_0) \geq A_\epsilon'(M_0) - \epsilon > 0 \]
for all \( s \in [M_0, \infty) \), and \( \lim_{s \to \infty} A_\epsilon(s) = \infty \). From this, (3), and (19), it follows that \( A_\epsilon' > 0 \) on \([m_0, \infty) \).

If (ii) is true, select
\[ \beta_\epsilon > \frac{|A_0''(M_0)|^2}{2\epsilon} > 0. \tag{20} \]
Note that for all \( s \in [M_0, \infty) \),
\[ A_\epsilon'(s) = \frac{\beta_\epsilon}{2} (s - M_0)^2 + A_0''(M_0)(s - M_0) + A_0'(M_0) =: f_\epsilon(s) + A_0'(M_0). \]
Since \( A_0''(M_0) < 0 \) and
\[ f_\epsilon'(s) = \beta_\epsilon (s - M_0) + A_0''(M_0) \quad (s \in [M_0, \infty)), \]
it follows from (20) that
\[ f_\epsilon(s) \geq f_\epsilon \left( - \frac{A_0''(M_0)}{\beta_\epsilon} + M_0 \right) = - \frac{|A_0''(M_0)|^2}{2\beta_\epsilon} > -\epsilon \]
for all \( s \in [M_0, \infty) \). This implies that
\[ A_\epsilon'(s) > A_\epsilon'(M_0) - \epsilon = A_\epsilon'(M_0) - \epsilon > 0 \]
for all \( s \in [M_0, \infty) \). From this, (3), and (19), we have \( A_\epsilon' > 0 \) on \([m_0, \infty) \). Since \( \beta_\epsilon > 0 \), \( \lim_{s \to \infty} A_\epsilon(s) = \infty \).
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E-mail address: jianwensun@lzu.edu.cn
E-mail address: shkim17@knu.ac.kr