STRING-BASED PERTURBATIVE METHODS
FOR GAUGE THEORIES*

Zvi Bern

Department of Physics
UCLA
Los Angeles, CA 90024

Abstract

Recent progress in the computation of one-loop gluon amplitudes is reviewed. These methods were originally derived from superstring theory and are significantly more efficient than conventional Feynman rules. With these methods, explicit computations can be performed beyond those achieved by traditional methods.

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Recent progress in the computation of one-loop gluon amplitudes is reviewed. These methods were originally derived from superstring theory and are significantly more efficient than conventional Feynman rules. With these methods, explicit computations can be performed beyond those achieved by traditional methods.

1. Introduction and Overview
The ability to uncover new physics at accelerators relies to a large extent on subtracting known physics; in particular, QCD loop corrections provide a significant background but are in general quite formidable to calculate. Intermediate expressions can be many thousands of times larger in size than final expressions. This explosion of terms has been one of the major obstacles preventing computations required by experiment from being performed. In these lectures new techniques which bypass much of the algebra associated with one-loop Feynman diagram computations in gauge theories [1–5] are discussed. Many of the ingredients that make up the new techniques are directly motivated by string theory.

As an example of the power of the new technique, the Ellis and Sexton [6] computation of the next-to-leading order contributions to the $2g \rightarrow 2g$ cross-section required 108 diagrammatic interferences; with the new methods only two relatively simple diagrams are required. Furthermore, with the new string-based techniques the one-loop five-gluon amplitudes have been computed yielding a compact form [4]. These amplitudes have not been obtained with traditional techniques.

Recent years have also seen substantial progress in improving the situation in tree-level calculations. Tree-level matrix elements have been essential for checks of QCD processes and for estimates of QCD backgrounds to new physics searches. Four ideas which have contributed to improvements in calculational ability are the spinor helicity method for gluon polarization vectors [7,8], the color decomposition [9,10], supersymmetry identities [11], and the Berends and Giele recurrence relations [12]. (A review of these ideas can be found in ref. [13].) The tree-level color decomposition [10,14] and recurrence relations [15] emerge quite naturally from string theories. The first three developments have also played an important role at loop level. (Recursion relations of the Berends and Giele type may very well play an important role at loop-level in the future.)
The one-loop method which is discussed in these lecture notes was originally derived from string theory. Although based on string theory it has been summarized in terms of simple rules which require no knowledge of string theory [2,3]; the structure of these rules can also be understood from conventional field theory through a particular gauge choice and organization of the amplitude [16]. The structure of string amplitudes indicates why it might be advantageous to organize a field theory computation according to string theory: a string amplitude contains all field theory diagrams in a single compact ‘master formula’. This may be compared to conventional field theory where the various Feynman diagrams bear little relationship to each other. Since string theories contain gauge theories in the infinite string tension limit [17,18,19] and have a simpler organization of the amplitudes than field theories, one might expect a string-based calculation of the amplitude to be more efficient than a traditional Feynman diagram calculation. The original derivation of the new one-loop method [2] was based on the field theory limit of an appropriate heterotic string [20,21] amplitude. Although the appropriate four-dimensional heterotic string construction turns out to be fairly intricate [22], the consistency of the string guarantees that no extraneous problems enter. Once the correctness of the method is understood much simpler string constructions, such as the bosonic one in reference [23], suffice. By organizing the various contributions which survive in the field theory limit diagrammatic rules for computing amplitudes were derived.

One way to quantify the gain in efficiency over traditional methods is by comparing the calculation of the virtual corrections to one-loop gluon scattering $2g \to 2g$ to the calculation of light-by-light scattering $2\gamma \to 2\gamma$. Using modern spinor helicity methods [7,8] the light-by-light computation is already far simpler than the traditional computation [24]. The power of the string-based methods is such that the gluon calculation is only a bit more difficult than the already much simplified photon computation. This can be contrasted to conventional field theory where the complexity of the non-abelian gluon Feynman vertex as compared to the photon vertex implies that a gluon scattering calculation should be significantly more complicated than a photon scattering calculation. Perhaps even more remarkably, with string-based methods the growth in the complexity of a graviton scattering computation as compared to a photon scattering computation is relatively inconsequential as compared to conventional field theory expectations; in particular, a one-loop graviton-by-graviton scattering computation using string theory is only moderately more complicated than the already much simplified light-by-light scattering computation [25].

Is string theory ‘required’ for field theory calculations? The answer is both yes and no. To develop and extend the methods string theory has been crucial and can be expected to continue to be useful. To actually evaluate amplitudes there is no need to turn to string theory. The main role of string theory is to provide a principle for discovering compact representations for field theory amplitudes. As yet, there is no corresponding principle in conventional field theory. In particular, given the
string-based rules for the one-loop $n$-gluon amplitudes and the understanding of these rules in conventional field theory [16], there does not appear to be a clear way to extend the rules to multi-loops, or gravity without referring back to string theory to at least some extent. It is, however, possible to formulate a conventional field theory framework for obtaining much of the efficiency of the string-based method by working backwards from the string-based rules. At one loop the main field theory ideas which can be used to improve the efficiency of a calculation and are inspired from string theory are the use of the background field gauge for the loop part of diagrams, the non-linear Gervais and Neveu gauge [26] for the tree parts of diagrams, color ordering of vertices, systematic organization of the vertex algebra and a second order formalism for fermions which helps make supersymmetry relations manifest. The spinor helicity method [7,8] is also natural within this framework.

These field theory ideas can be applied more generally to gauge theory amplitude calculations which involve non-abelian vertices [16]. In the future, extensions of the rules to include external fermions, weak interactions and multi-loops can be expected, but in the meantime, at least some of above ideas can be directly applied to any Feynman diagram computation in non-abelian gauge theory.

A complementary field theory approach [27] for understanding the string-based methods in a field theory context is through a first quantized formalism [28]. Its main advantage is that it is simpler than dealing with string theory and is useful for gaining an understanding of the string-based loop substitution rules. With this approach one obtains a description of the one-loop effective action, although as yet it does not provide a satisfactory description of scattering amplitudes nor of the tree parts of diagrams. It might, however, provide an alternative path for extensions to multi-loops.

These lectures are organized as follows: first the basic motivation from experiment for wanting to compute loop diagrams with large multiplicities is explained in Section 2. Such Feynman diagram calculations are generally quite formidable although important for new physics searches at colliders. In Section 3, tree-level techniques which carry over to loop level are reviewed; the three methods are the color decomposition, spinor helicity techniques, and supersymmetry identities. Since the loop-level method is based on string theory, a review of some of the relevant string ideas is given in Section 4. Although lacking complete string consistency, the bosonic string is used as a basis of discussion because of its simplicity as compared to a fully consistent four-dimensional heterotic string. By taking the field theory limit of an appropriately constructed string theory, gauge theory amplitudes can be recovered. These amplitudes are organized in a particularly compact way. In Section 5 the modifications that are needed when applying the tree-level methods of Section 3 to loop level is discussed. One form of string-based rules is presented in Section 6. These rules are then applied, in Section 7, to a specific calculation of a one-loop gluon helicity amplitude that would be rather difficult to evaluate by traditional Feynman diagram methods, but is rather easy in the string-based method. Results
are also presented for one-loop four- and five-gluon helicity amplitudes; these am-
plitudes were first calculated with the string-based methods. An amusing example
of a four-graviton calculation is also presented that would be exceedingly difficult
to evaluate via traditional Feynman diagrams but is rather simple using string the-
ory. The basic structure of the string-based rules can be understood by a particular
organization of field theory; this is explained in Section 8. At tree-level, the non-
linear Gervais-Neveu gauge makes the match to string theory less obscure and its
rather simple diagrammatic structure partly explains how string theory can avoid
many of the large cancellations inherent in conventional field theory. At loop-level
a different gauge choice is required to make the structure of string-based rules less
obscure: background field Feynman gauge. The background field method is briefly
reviewed. The generic structure of the one-loop effective action implied by string
theory is then given followed by a discussion of how one would apply string moti-
vated field theory ideas to more general calculations. Section 8 then concludes with
a discussion of the first quantized approach. Finally in Section 9 a summary and
outlook for the future is given.

2. One-Loop Perturbative QCD

2.1 Requirements by Experiments

The fundamental question of perturbative QCD is whether new physics is hid-
ing in the QCD background. The QCD background generally swamps new physics
signals. As an example, in fig. 1 the number of events is plotted against the two-jet
invariant mass. At approximately 90 GeV one might expect to see peaks from W
and Z production; as seen in the figure these peaks are swamped by the QCD back-
ground. Another example is t quark searches at Fermilab which must deal with
significant QCD backgrounds. In general, to find new physics it is important to
subtract the QCD background. The more precisely the subtraction can be done,
the more likely that new physics can be identified at colliders.

One of the characteristics of many of the interesting events at accelerators are
jets (which are collimated bunches of hadrons heading out from the interaction).
A key ingredient that enters into the theoretical computation of jets are Feynman
diagrams which describe the partons. Other essential ingredients which make up
the computation are the structure functions describing the initial state partons and
the final state hadronization process. Further details can be found in standard
textbooks [29]. (In general the conversion of the matrix elements into physical
scattering processes that can be compared to experiment is nontrivial because of
complexities associated with soft and collinear divergences [30]; Giele and Glover
[31,32] have, however, constructed a convenient formalism for performing that step.)
These lectures will deal with only the Feynman diagram part of the computation.
Fig. 1: An example of the QCD background: the $Z$ and $W$ peaks are completely swamped. (Data from UA2 collaboration.)

In Feynman diagram computations one starts with the Born or tree diagrams, since these give the leading order contribution and are the simplest to compute. The tree level diagrams form the cornerstone for extracting physics from colliders [7,9,13]. The tree-level computations, however, miss essential physics [30,31]. There are three basic problems:

1) The tree-level jet cone angle dependence is wrong. The physical origin of the jet cone angle dependence is that when two jets are nearby they could either be counted as a single jet or as two jets depending on the jet cone definitions.
Depending on how a given jet is counted the apparent cross-section will change. In fig. 2 an example of the difference between the tree-level predictions and loop level predictions is shown for $W+1$ jet production.

\[\text{Cone Size Dependence of } W+1 \text{ jet}\]

\[\begin{array}{c}
0.25 \\
0.20 \\
0.15 \\
0.10 \\
0.05 \\
0.00 \\
\end{array}\]

\[\begin{array}{c}
0.3 \\
0.5 \\
0.7 \\
1.0 \\
\end{array}\]

**Fig. 2:** An example of the dependence of the cross-section on the jet cone-size. The dotted line is the tree-level result and shows no dependence on the cone size. The data points are theoretical results including one-loop corrections and exhibit a significant cone-size dependence. (From ref. [32].)

2) The tree-level renormalization scale dependence is wrong. Physical quantities should not depend on the renormalization scale. At tree level the dependence on the scale enters due to the coupling constant sitting in front of the amplitude; the precise value which should be chosen for the scale (or equivalently, the value of the coupling constant) is not clear, leading to variations in the predicted cross-section of more than fifty percent for sensible choices of the scale, as indicated in fig. 3. This leads to the commonly quoted large theoretical uncertainty. As indicated in fig. 3 the one-loop corrections tend to reduce the scale dependence to about five or ten percent.

3) In QCD there are large infrared logarithms which in general cannot be neglected. Such logarithms are not accounted for by tree calculations and are related to the incorrect cone angle dependence.

To a large extent one-loop corrections fix these problems. This provides the basic
motivation for performing loop level QCD computations.

As one example of a relevant loop computation, experimenters at Fermilab would like to measure the strong coupling constant $\alpha_s$ and its running. At Fermilab one would be able to measure the coupling constant at 250 GeV which is at an energy well beyond what can be currently achieved at LEP. (At these higher energies jets are easier to resolve.) A good way to obtain $\alpha_s$ is from the ratio of the three-jet cross section to the two-jet cross section. Roughly speaking this quantity is proportional to $\alpha_s$. Good data with approximately twenty percent experimental errors exist since 1985 from the UA1 and UA2 experiments at CERN for $\alpha_s$ at various mass scales [33]. Unfortunately, because of the lack of all the required one-loop calculations of the three-jet cross-section, the theoretical uncertainties associated with this quantity are on the order of a hundred percent. This situation may be compared to LEP which quotes $\alpha_s$ at the mass of the $Z$ with a total theoretical and experimental uncertainty of about ten percent [34]. One reason why the relevant theoretical computations have been performed for LEP, but not for hadron machines, is that diagrams involving initial state electrons instead of gluons and

**Fig. 3:** The renormalization scale dependence of a cross-section at tree-level and with one-loop corrections. The dotted line is the tree result which exhibits a strong scale dependence while the solid line includes one-loop corrections and exhibits a much weaker dependence over a wide scale. (From ref. [32].)
other partons are easier to compute; the relevant loop corrections have been given in refs. [35]. (The conversion of the matrix elements into physical quantities is also more complicated [30,31] for hadron scattering.) In terms of the diagrams, the one-loop three-jet calculation for LEP requires box diagrams at worst while the corresponding calculation for hadron colliders requires pentagon diagrams. (Pentagon diagrams are generally many orders of magnitude more complicated to evaluate than box diagrams).

Using the string-based methods discussed in these lectures the first computation of the one-loop gluon matrix elements required for the three-jet cross-section has been performed [4]. The quark contributions have not been computed but progress has been made with string-based methods [36]. From a traditional field theory point of view these are generally easier to compute than the gluon contributions.

Other examples of computations which are relevant for current experiments but have not been performed as yet are:

1) One-loop corrections to $W + n$-jet production at hadron colliders with $n \geq 2$, where $W \rightarrow \ell \nu$. This forms a background to $t$ quark searches at Fermilab.

2) One-loop corrections to $Z + n$-jet production at hadron colliders with $n \geq 2$ where $Z \rightarrow \bar{\nu} \nu$. This forms a background to missing transverse energy searches for new physics at Fermilab.

3) One-loop corrections to $Z \rightarrow 4$ jets [37]. This would be be useful for measurement of $\alpha_s$ from the four- to three-jet ratio at LEP. The calculation is also equivalent to a large extent to the calculation of one-loop corrections to $W, Z \rightarrow 2$ jet production at Fermilab.

4) Two-loop corrections to $Z \rightarrow 3$ jets. LEP is currently sensitive to these corrections.

5) Two-loop corrections to two-jet production at Fermilab. The only way to decisively prove that one-loop corrections are adequate is to calculate the two-loop corrections and show that they are unimportant.

Experimenters need theorists to perform the loop computations associated with these processes, so why haven’t theorists calculated them?

2.2 Difficulty of Loop Computations

Another way to phrase the above question is: why are perturbative QCD computations so complicated? The answer is that there are too many Feynman diagrams and each Feynman diagram is too complicated, especially those diagrams which contain gluons. Such diagrams are important at high energies. An underlying cause of the complexity is that the non-abelian vertices which are given in fig. 4 are relatively complicated. Since the vertices each contain six terms, one encounters a rapidly growing number of terms as one sews together vertices with propagators to form Feynman diagrams.
\[
-k_{\mu} a_{\mu} \left( \eta_{\mu\nu}(k-p)_{\rho} + \eta_{\nu\rho}(p-q)_{\mu} + \eta_{\rho\mu}(q-k)_{\nu} \right)
\]

\[
= -gf^{abc} \left( \eta_{\mu\nu}(k-p)_{\rho} + \eta_{\nu\rho}(p-q)_{\mu} + \eta_{\rho\mu}(q-k)_{\nu} \right)
\]

\[
= \begin{cases} 
-ig^2 f^{abc} f^{ecd} (\eta_{\mu\rho} \eta_{\nu\lambda} - \eta_{\mu\lambda} \eta_{\nu\rho}) \\
+ f^{ade} f^{ebc} (\eta_{\mu\rho} \eta_{\nu\lambda} - \eta_{\mu\lambda} \eta_{\nu\rho}) \\
+ f^{ace} f^{ebd} (\eta_{\mu\rho} \eta_{\nu\lambda} - \eta_{\mu\lambda} \eta_{\nu\rho}) \end{cases}
\]

Fig. 4: The conventional three- and four-point Feynman vertices.

As a simple example consider the pentagon diagram one would encounter in a brute force three-jet computation. A naive count of the number of terms gives about $6^5$ terms. (This count is slightly reduced by the use of on-shell conditions but increased by observing that each internal momentum is a sum of momenta.) Each term is associated with an integral which evaluates to an expression on the order of a page in length. This means that one is faced with about $10^4$ pages of algebra for this single diagram. As bad as this situation might seem, it is actually much worse because of the structure of the results. After evaluating the integrals and summing over diagrams one obtains expressions of the form

\[
\frac{N_1}{D_1} + \frac{N_2}{D_2} + \cdots
\]  

where the $N_i$ and $D_i$ are the numerators and denominators one encounters when performing the integrals. In general the denominators contain spurious singularities which cancel only after putting large numbers of terms on a common denominator; this unfortunately causes an explosion of terms in the numerators. It is therefore not too surprising that the three-jet computation, which involves pentagon diagrams, has not yet been performed with the traditional methods employed, for example, by Ellis and Sexton [6] in their two-jet computation.

The basic observation for being able to improve on conventional computations is that Feynman diagram computations always involve large cancellations amongst the various terms. Anyone who has done a Feynman diagram computation has undoubtedly asked themselves why vasts amounts of algebra are required when answers tend to be quite small. A nice example of this is the four-gluon helicity amplitude

\[
A_{4:1}^{1-loop}(1^-, 2^+, 3^+, 4^+) = -\frac{i}{48\pi^2} \frac{[2 \mid 4 \rangle^2 u}{[1 \mid 2 \rangle \langle 2 \mid 3 \rangle \langle 3 \mid 4 \rangle \langle 4 \mid 1 \rangle}
\]
where the plus and minus signs associated with each leg denote the helicity, the various brackets refer to the spinor helicity notation discussed in the next section, and \( u = 2k_1 \cdot k_3 \) is a Mandelstam variable. (This amplitude has been color decomposed, which will be discussed in Sections 3 and 5.) Although this expression fits on a line, a brute force computation performed in the conventional way would start with expressions containing about \( 10^4 \) terms. Clearly there is considerable room for improving Feynman diagram computations at one loop.

3. Tree Level Methods

The tree-level techniques have been already been reviewed in the article of Mangano and Parke [13] so here only those techniques that have been carried over to loop level will be discussed. The three important tree-level tools which have been carried over to loop level are color decomposition [9], spinor-helicity techniques [7,8], and supersymmetry identities [11].

3.1 The Color Decomposition

In terms of ordinary Feynman rules the notion of color ordering is fairly simple to implement. The Yang-Mills structure constants are rewritten in terms of fundamental representation matrices

\[
f^{abc} = -\frac{i}{\sqrt{2}} \text{Tr}( [T^a, T^b] T^c )
\]  

(3.1)

where the normalization of the generators is \( \text{Tr}(T^a T^b) = \delta^{ab} \). The color ordered gluon Feynman rules for ordinary Feynman gauge are depicted in fig. 5. These rules are obtained from ordinary Feynman rules given in fig. 4 by restricting attention to a given color ordering. By using eq. 3.1 and extracting the coefficient of \( \text{Tr}(T^a T^b T^c) \) the color ordered three-vertex is obtained. The same type of analysis leads to the color ordered four-vertex which is given by the coefficient of \( \text{Tr}(T^a T^b T^c T^d) \). With these rules one computes a partial amplitude corresponding to a single color trace term; the diagrams should be drawn in a planar fashion with the external legs following the ordering of the color trace under consideration. The full tree-level amplitude can then be reconstructed from the partial amplitudes by multiplying by the associated color trace and summing over all non-cyclic permutations

\[
\mathcal{A}_n(\{k_i, \varepsilon_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) \mathcal{A}_n(k_{\sigma(1)}, \varepsilon_{\sigma(1)}; \ldots; k_{\sigma(n)}, \varepsilon_{\sigma(n)})
\]  

(3.2)

where \( k_i, \varepsilon_i, \) and \( a_i \) are respectively the momentum, polarization vector, and color index of the \( i \)-th external gluon. \( S_n/Z_n \) is the set of non-cyclic permutations of \( \{1,\ldots,n\} \). Note that the partial amplitudes have been defined with the powers of the coupling constant removed.
\[ k_{\mu} p_{\nu} q_{\rho} = \frac{i}{\sqrt{2}} \left( \eta_{\mu \nu} (k - p)_{\rho} + \eta_{\nu \rho} (p - q)_{\mu} + \eta_{\rho \mu} (q - k)_{\nu} \right) \]

\[ a_{\mu \nu} b_{\rho \lambda} = i \eta_{\mu \rho} \eta_{\nu \lambda} - \frac{i}{2} (\eta_{\mu \nu} \eta_{\rho \lambda} + \eta_{\mu \lambda} \eta_{\nu \rho}) \]

**Fig. 5:** The color ordered Feynman gauge vertices for obtaining the partial amplitudes \( A_n \).

The immediate advantage of rewriting Feynman rules in this way is that fewer diagrams contribute. As a simple example with conventional Feynman diagrams one would have a total of four conventional Feynman diagrams, depicted in fig. 6 for the four-point tree amplitude. With color ordered Feynman rules one would compute the partial amplitude \( A_4(1, 2, 3, 4) \) associated with the color trace \( \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \) and would not need to include diagram 6c, since the ordering of the legs do not follow the ordering of the color trace. It is a simple exercise at the four-point level to verify that these color ordered rules reproduce the results obtained from conventional Feynman rules.

**Fig. 6:** The four-point Feynman diagrams. Color ordered Feynman rules do not include diagram (c) for \( A_4(1, 2, 3, 4) \).

The color decomposition (3.2) follows from string theory. In an open string
theory, the full on-shell amplitude for the scattering of \( n \) massless vector mesons can be written as the sum over non-cyclic permutations of the external legs of Chan-Paton factors \([38]\) times Koba-Nielsen partial amplitudes \([39]\)
\[
A^\text{string}_n(\{k_i, \varepsilon_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\ldots T^{a_{\sigma(n)}}) \\
\times A^\text{KN}_n(k_{\sigma(1)}, \varepsilon_{\sigma(1)}; \ldots; k_{\sigma(n)}, \varepsilon_{\sigma(n)}).
\]

By taking the infinite string tension limit, the string amplitudes reduce to field theory amplitudes yielding the field theory color decomposition \((3.2)\). For tree amplitudes where all the external legs are gluons, the matter content is irrelevant, since the matter fields cannot appear as internal lines. Thus one can use the open bosonic string, the simplest of all string constructions, at tree level. The decomposition of the string amplitude leads immediately to the decomposition of on-shell \( n \)-gluon amplitudes \((3.2)\).

This color decomposition is actually quite a bit more useful than just a reduction in the number of diagrams that must be considered. Two additional advantages are the ability to use \( U(N) \) color matrices instead of \( SU(N) \) matrices and certain identities satisfied by the partial amplitudes. The partial amplitudes \( A_n \) possess a number of nice properties that follow immediately from the properties of the Koba-Nielsen amplitudes. Each is gauge invariant on shell, that is invariant under the substitution \( \varepsilon_i \rightarrow \varepsilon_i + \lambda k_i \) for each leg independently. It is also invariant under cyclic permutation of its arguments, and satisfies a reflection identity,
\[
A_n(n, \ldots, 1) = (-1)^n A_n(1, \ldots, n),
\]
where the notation \( A_n(1, \ldots, n) = A_n(k_1, \varepsilon_1; \ldots; k_n, \varepsilon_n) \) is used. The \( U(1) \) gauge boson is an integral part of the string theory (its presence is necessary for unitarity), but in the infinite-tension limit, it must decouple from \( SU(N) \) gauge boson amplitudes; thus in field theory
\[
A_n(\{k_i, \varepsilon_i, a_i\}_{i=1}^{n-1}; k_n, \varepsilon_n, a_{U(1)}) = 0.
\]

This can be used to derive a decoupling identity, simply by extracting the coefficient of \( \text{Tr}(T^{a_1} \ldots T^{a_{n-1}}) \), which is
\[
\sum_{\sigma \in Z_{n-1}} A_n(\sigma(1), \ldots, \sigma(n-1), n) = 0.
\]

(This identity can also be derived starting with the twist operator in open string theory. Mangano, Parke, and Xu \([10,12,14]\) term the identity a dual Ward identity.) Substituting additional photons for gluons leads to equations which are linearly dependent on equation \((3.6)\).
An advantage of using a $U(N)$ gauge group instead of an $SU(N)$ gauge group in the color decomposition is that the $U(N)$ Fierz identities

\[
\begin{align}
\text{Tr}(T^a X) \text{Tr}(T^a Y) &= \text{Tr}(XY) \\
\text{Tr}(T^a XT^a Y) &= \text{Tr}(X) \text{Tr}(Y)
\end{align}
\] (3.7a)

are simpler than their $SU(N)$ counterparts. This is useful when squaring and summing over colors in order to obtain the cross-section.

In summary, the color ordered Feynman rules lead to significant simplifications as compared to conventional Feynman rules. However, the real power of color ordering occurs when coupled with other ideas.

### 3.2 Spinor Helicity Techniques

The spinor helicity method [7,8] involves a rewriting of gluon (or photon) polarization vectors in terms of spinor inner products. At first sight the point of this rewriting may not be clear, but with a few simple examples its power becomes evident. This technique implicitly makes use of clever on-shell gauge transformations in order to make large numbers of terms vanish in a given computation.

In the formalism of Xu, Zhang and Chang a gluon polarization vector is written as

\[
\varepsilon^{(+)}(\mu; k; q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2} \langle q^- | q^+ \rangle}, \quad \varepsilon^{(-)}(\mu; k; q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2} \langle k^+ | q^- \rangle},
\]

(3.8)

where $|k^\pm\rangle$ is a Weyl spinor, with plus and minus helicities, $k$ is the on-shell momentum of the gluon and $q$ is an arbitrary reference momentum satisfying $q^2 = 0$, $k \cdot q \neq 0$. These polarization vectors satisfy the conditions for circular polarization

\[
k \cdot \varepsilon^{(\pm)}(\mu; k; q) = 0, \quad (\varepsilon^{(\pm)})^2 = 0, \quad \varepsilon^{(+)} \cdot \varepsilon^{(-)} = -1
\]

(3.9)

and are therefore sensible definitions for helicities. The convention that all momenta are outgoing is used; the effect of this is to flip helicity notation on an incoming line.

It is convenient to define abbreviations for the various spinor products and the Lorentz product,

\[
\begin{align}
\langle j \ l \rangle &= \langle k_j k_l \rangle = \langle k_j^- | k_l^+ \rangle \\
[j \ l] &= [k_j k_l] = \langle k_j^+ | k_l^- \rangle \\
\langle j \ l \rangle &= \langle j \ l \ \rangle \ [j \ l] = 2k_j \cdot k_l
\end{align}
\] (3.10)

The spinor products are antisymmetric,

\[
\langle j \ l \rangle = -\langle l \ j \rangle, \quad [j \ l] = -[l \ j]
\]

(3.11)
and can be evaluated explicitly using,
\[
\langle k_1 k_2 \rangle = \sqrt{(k^t_1 - k^z_1)(k^t_2 + k^z_2)} \exp(i \text{atan}(k^y_1/k^x_1)) - (1 \leftrightarrow 2)
\]
\[
= \sqrt{\frac{k^t_2 + k^z_2}{k^t_1 + k^z_1}} (k^x_1 + ik^y_1) - (1 \leftrightarrow 2)
\]
\[
[k_1 k_2] = \text{sign}(k^t_1 k^t_2) (\langle k_2 k_1 \rangle)^* .
\]

Gauge-invariant quantities are independent of the choice of reference momentum \(q\), because changing \(q\) just corresponds to a gauge transformation [8]
\[
\varepsilon^{(+)}_\mu(k; q') = \varepsilon^{(+)}_\mu(k; q) + \frac{\sqrt{2}}{\langle q k \rangle \langle q' k \rangle} k_\mu
\]
which follows from the rearrangement or Schouten identity
\[
\langle 1 2 \rangle \langle 3 4 \rangle = \langle 1 4 \rangle \langle 3 2 \rangle + \langle 1 3 \rangle \langle 2 4 \rangle .
\]
The Fierz identity,
\[
\langle 1^- | \gamma^\mu | 2^- \rangle \langle 3^+ | \gamma^\mu | 4^+ \rangle = 2 \langle 1 4 \rangle [3 2]
\]
and the fact that \(\langle 1^- | \gamma^\mu | 2^- \rangle = \langle 2^+ | \gamma^\mu | 1^+ \rangle\) can be used to evaluate dot products of polarization vectors.

Given the reference momenta, the various dot products are simply
\[
k_j \cdot \varepsilon^{(+)}(k_j; q_i) = \frac{\langle q_i j \rangle \langle j l \rangle}{\sqrt{2} \langle q_i l \rangle} , \quad k_j \cdot \varepsilon^{(-)}(k_j; q_i) = \frac{\langle q_i j \rangle \langle j l \rangle}{\sqrt{2} \langle q_i l \rangle} ,
\]
\[
\varepsilon^{(-)}(k_j; q_j) \cdot \varepsilon^{(-)}(k_l; q_l) = \frac{\langle j l \rangle \langle q_i j \rangle \langle j q_j \rangle \langle q_i l \rangle}{\langle j q_j \rangle \langle q_i j \rangle \langle j l \rangle \langle q_i l \rangle} , \quad \varepsilon^{(+)}(k_j; q_j) \cdot \varepsilon^{(+)}(k_l; q_l) = \frac{\langle j l \rangle \langle q_i j \rangle \langle j q_j \rangle \langle q_i l \rangle}{\langle j q_j \rangle \langle q_i j \rangle \langle j l \rangle \langle q_i l \rangle} ,
\]
\[
\varepsilon^{(+)}(k_j; q_j) \cdot \varepsilon^{(-)}(k_l; q_l) = \frac{\langle q_j l \rangle \langle q_j j \rangle \langle q_j j \rangle \langle q_j l \rangle}{\langle q_j j \rangle \langle q_j j \rangle \langle q_j l \rangle \langle q_j l \rangle} .
\]

In making a choice of reference momenta, it is useful to keep the properties noted by Mangano et al. [10] in mind. With the first argument to a polarization vector denoting the momentum of the gluon, and the second its reference momentum, these properties are
\[
q \cdot \varepsilon^{(\pm)}(k; q) = 0
\]
\[
\varepsilon^{(\pm)}(k_j; q) \cdot \varepsilon^{(+)}(k_i; q) = 0
\]
\[
\varepsilon^{(+)}(k_j; q) \cdot \varepsilon^{(\pm)}(k_i; k_j) = 0
\]
so that it is desirable to choose the same reference momenta for all gluons of a given helicity, and to take this momentum to be the momentum of one of the opposite-helicity gluons. This will greatly reduce the number of non-vanishing $\varepsilon_i \cdot \varepsilon_j$ invariants. It also turns out that within the set of choices suggested by these properties, it is preferable to choose a reference momentum that is cyclicly adjacent to the momentum of the gluon.

As one simple example for the amplitude $A(1^-, 2^+, 3^+, 4^+)$, consider reference momenta $(k_4, k_1, k_1, k_1)$ for the legs $(1, 2, 3, 4)$ respectively, leading to the simplifications

$$
\varepsilon_i \cdot \varepsilon_j = 0,
\quad k_4 \cdot \varepsilon_1 = k_1 \cdot \varepsilon_2 = k_1 \cdot \varepsilon_3 = k_1 \cdot \varepsilon_4 = 0
$$

$$
k_3 \cdot \varepsilon_1 = -k_2 \cdot \varepsilon_1,
\quad k_4 \cdot \varepsilon_2 = -k_3 \cdot \varepsilon_2,
\quad k_4 \cdot \varepsilon_3 = -k_2 \cdot \varepsilon_3,
\quad k_3 \cdot \varepsilon_4 = -k_2 \cdot \varepsilon_4.
$$

(3.18)

The reason for using the spinor helicity method is now evident; many of the dot products of polarization vectors amongst themselves and with the external momenta simply vanish. Since an amplitude consists of sums of products of these dot products, with the spinor helicity method many of the terms in an amplitude will also vanish with a judicious choice of the reference momenta.

![Fig. 7: An unreadable form of the five-gluon tree amplitude in terms of dot products of momentum and polarization vectors to illustrate its complexity.](image)

The five-gluon tree amplitude provides a rather clear demonstration of the
power of the spinor helicity method. In fig. 7† the color ordered five-gluon tree amplitude is presented in conventional but unreadable form to illustrate the seeming complexity. The results in this figure are written in terms of dot products of polarization vectors and momenta. Even if this figure were legible it would still be fairly painful to use since one would need to square it and sum over helicities before obtaining the cross-section. Now consider the same expression using the spinor helicity formalism:

\[
A_5^{\text{tree}}(1^+, 2^+, 3^+, 4^+, 5^+) = 0 \\
A_5^{\text{tree}}(1^-, 2^+, 3^+, 4^+, 5^+) = 0 \\
A_5^{\text{tree}}(1^+, \cdots, j^-, \cdots k^-, \cdots 5^+) = i \frac{\langle j \, k \rangle^4}{\langle 1 \, 2 \rangle \langle 2 \, 3 \rangle \langle 3 \, 4 \rangle \langle 4 \, 5 \rangle \langle 5 \, 1 \rangle}
\]

(3.19)

where \( j \) and \( k \) are two negative helicity legs. All other configurations can be obtained from these basic ones by relabelings and parity. From this example it is clear that a basic point behind using a helicity formalism is that results which would otherwise need many pages can be expressed in a few lines.

As another example to illustrate the power of spinor helicity methods consider the gluon helicity amplitudes \( A_n^{\text{tree}}(1^+, 2^+, \cdots, n^+ \) and \( A_n^{\text{tree}}(1^-, 2^+, \cdots, n^+ \). Using spinor helicity it is easy to argue that these amplitudes vanish [40]. In the first case one chooses all the reference momenta equal \( q_i = q \neq k_j \), while in the second case one could choose \( q_1 = k_2 \) and \( q_i \neq 1 = k_1 \). Thus choices exist for \( A_n^{\text{tree}}(1^+, 2^+, \cdots, n^+ \) and \( A_n^{\text{tree}}(1^-, 2^+, \cdots, n^+ \) so that \( \varepsilon_i \cdot \varepsilon_j = 0 \) for all \( i, j \). It is then not difficult to argue that all terms in a tree-level amplitude contain at least one \( \varepsilon_i \cdot \varepsilon_j \). This can be obtained using conventional Feynman diagrams in Feynman gauge. First consider those diagrams with only three-point vertices. These diagrams have \( n \)-legs and \( n - 2 \) vertices. Since each vertex is linear in momenta (and there are no other sources of momenta in the numerator in Feynman gauge) in any given term at most \( n - 2 \) momenta can contract with the polarization vectors to form \( \varepsilon_i \cdot k_j \); this leaves two \( \varepsilon_i \) in every term which contract with one another to yield at least one factor of \( \varepsilon_i \cdot \varepsilon_j \). The inclusion of four-point vertices only helps this argument since a four-point vertex implies an additional \( \varepsilon_i \cdot \varepsilon_j \) in every term. Thus in one swoop infinitely many tree-level Feynman diagrams have been evaluated with the result

\[
A_n^{\text{tree}}(1^\pm, 2^+, 3^+, \cdots, n^+) = 0 .
\]

(3.20)

At loop level this argument does not work because diagrams exist which have \( n \) legs and \( n \) vertices so all polarization vectors can be simultaneously contracted into momenta; indeed, the corresponding one-loop amplitudes do not vanish.

The other tree-level helicity amplitudes do not vanish because reference momenta cannot be chosen to make all \( \varepsilon_i \cdot \varepsilon_j \) vanish simultaneously. There is however,

† I thank D. Kosower for providing this figure.
one other remarkably simple result and that is the Parke-Taylor formula [41]

$$A_{n}^{\text{tree}}(1^{+}, 2^{+}, \ldots, j^{-}, \ldots, k^{-}, \ldots, n^{+}) = i\frac{\langle j k \rangle^{4}}{(1 2) \langle 2 3 \rangle \langle 3 4 \rangle \cdots \langle n 1 \rangle} \tag{3.21}$$

where all the legs are plus helicities except for legs $j$ and $k$ which are minus helicities. This formula has been proven with the Berends-Giele recursion relations [12,15].

3.3 Supersymmetry Identities

Supersymmetry relates bosonic amplitude to fermionic ones. Besides the usual applications to model building, supersymmetry is a useful computational tool in QCD calculations. Although QCD is not a supersymmetric theory, the supersymmetry identities [42] do provide general relationships between bosonic and fermionic amplitudes. Once diagrams containing either bosons or fermions are calculated, information about the other case can be deduced from the identities. These relationships can then be applied to QCD [11] as we briefly review here. We will follow the notation and discussion of ref. [13] in order to obtain identities that will be useful later in these lecture notes.

The supersymmetry transformation turns bosons into fermions and is given by

$$[Q(\eta), g^{\pm}(p)] = \mp\Gamma^{\pm}(p, \eta)A^{\pm}(p), \quad [Q(\eta), A^{\pm}(p)] = \mp\Gamma^{\mp}(p, \eta)g^{\pm}(p). \tag{3.22}$$

The supercharge is $Q(\eta)$ where $\eta$ is an arbitrary anti-commuting parameter. The gluon field is $g$ while the fermion gluino field is $A$ and the $\pm$ superscripts denote the helicity. The coefficients $\Gamma$ are given by

$$\Gamma^{+}(p, \eta) = [\Gamma^{-}(p, \eta)]^{*} = \bar{\eta}u_{-}(p) \tag{3.23}$$

where $u_{-}(p)$ is a negative helicity spinor satisfying the massless Dirac equation. A convenient choice of the anticommuting parameter is $\bar{\eta} = \theta\bar{u}_{-}(k)$ where $\theta$ is a Grassmann parameter and $k$ is an arbitrary null vector so that

$$\Gamma^{+}(p, \eta) = \theta\langle k^{+}|p^{-}\rangle = \theta[kp]. \tag{3.24}$$

The last expression is in terms of the compact spinor helicity notation.

Supersymmetry identities are obtained by using the fact that in a supersymmetric theory the supercharge $Q$ annihilates the vacuum [43]. The basic supersymmetric identity is then

$$0 = \langle [Q, \prod_{i=1}^{n} \phi_{i}] \rangle_{0} = \sum_{i=1}^{n} \langle \phi_{1} \cdots [Q, \phi_{i}] \cdots \phi_{n} \rangle_{0} \tag{3.25}$$

where $\langle \cdots \rangle_{0}$ means the vacuum expectation value.
From here the specific supersymmetry identities can be derived. Consider for example

$$0 = \langle [Q, A^+_1 g^-_2 g^+_3 \cdots g^+_n] \rangle_0$$

$$= -\Gamma^-(p_1, k) A(g^+_1, g^-_2, g^+_3 \cdots g^+_n) - \Gamma^-(p_2, k) A(A^+_1, A^-_2, g^+_3, \cdots, g^+_n) + \sum_{i=3}^{n} \Gamma^+(p_i, k) A(A^+_1, g^-_2, g^+_3, \cdots, A^+_i, \cdots, g^+_n)$$

(3.26)

where \(k\) is an arbitrary null momentum vector. Since the gluon-fermion vertex conserves fermion helicity, all amplitudes with like helicity (outgoing) fermions vanish so the third term containing the sum drops out. (Note that the notation is such that an incoming ‘+’ has opposite helicity as an outgoing ‘+’.) Thus we obtain

$$0 = \langle [Q, A^+_1 g^-_2 g^+_3 \cdots g^+_n] \rangle_0$$

$$= -\Gamma^-(p_1, k) A(g^+_1, g^-_2, g^+_3 \cdots g^+_n) - \Gamma^-(p_2, k) A(A^+_1, A^-_2, g^+_3, \cdots, g^+_n).$$

(3.27)

By choosing \(k = p_1\) or \(k = p_2\) the two identities

$$A^{\text{susy}}_n(g^+_1, g^-_2, g^+_3, \cdots, g^+_n) = 0$$

(3.28)

and

$$A^{\text{susy}}_n(A^+_1, A^-_2, g^+_3, \cdots, g^+_n) = 0$$

(3.29)

are obtained. Thus, in any space-time supersymmetric theory these amplitudes vanish to all loop orders. Other examples of supersymmetry identities are

$$A^{\text{susy}}_n(g^+_1, g^-_2, \cdots, g^+_n) = 0$$

(3.30)

and

$$A^{\text{susy}}_n(g^-_1, g^-_2, g^+_3, \cdots, g^+_n) = \frac{\langle 1 \rangle}{\langle 1 \ 3 \rangle} A^{\text{susy}}_n(g^-_1, A^-_2, A^+_3, g^+_4, \cdots, g^+_n).$$

(3.31)

These identities can be immediately applied to tree-level QCD computations. At tree level, the \(n\)-gluon amplitudes are completely independent of the matter content of a particular theory since by fermion number conservation, fermion or scalar lines can never appear inside an \(n\)-gluon diagram. This means that the supersymmetry identities (3.28) and (3.30) imply that

$$A^{\text{qcd tree}}_n(g^\pm_1, g^+_2, \cdots, g^+_n) = 0.$$  

(3.32)

This agrees with the result obtained in eq. (3.20) through spinor helicity methods.

A more complete discussion of applications of supersymmetry identities to tree-level QCD computations can be found in ref. [13].
4. Basic String Theory

The infinite tension limit of a string theory is a field theory [17,18,19]. In order to use string theory as a computational tool, control of the massless matter content of the string theory is required, because colored massless matter particles can run around the loops. It is possible to build consistent heterotic string theories [20] whose infinite-tension limit is a non-abelian gauge theory where one of the factors is an $SU(N)$ with no matter fields [22]. The technology needed for such a construction is the one used to construct four-dimensional string models [21]; the formulation of Kawai, Lewellen and Tye is particularly simple, although any of the other formulations can be used depending on one’s taste. In the original derivation of the string-based rules [2,3], it was essential to use a consistent string in order to prevent extraneous problems from entering. Without full string consistency there would be no guarantee that the final results obtained would be correct. A heterotic string was used in the original derivation of the string-based rules because bosonic strings always contain unwanted massless scalars and tachyons, while four-dimensional type II [44,45] and type I [46] superstrings do not have a rich enough variety of fully consistent models.

However, given that a consistent heterotic string derivation as well as a conventional field theory understanding [16] now exists, there is no longer a need to build fully consistent strings as one can verify results either by comparing to the heterotic construction or to field theory. It turns out that any string model will suffice; if the gauge group representation is not correct or the number of flavors is not the desired one this can be fixed by hand in the field theory limit. The important information that string theory supplies is the compact structure of the amplitude.

Bosonic string constructions are generally much simpler than super or heterotic string constructions so that is what will be discussed here. The open bosonic string discussed here is identical to the one used by Metsaev and Tseytlin [47] to obtain the Yang-Mills $\beta$-function from string theory. This string is given by a naive truncation of an oriented open bosonic string to four-dimensions. In this way all massless colored scalars arising from the dimensional compactification are simply thrown away. This string is inconsistent as a fundamental string theory because of the naive truncation of the spectrum. Another technicality is that the string does contain a tachyon, which might be worrisome; however, one can handle this with the prescription that exponentially large terms due to the tachyon should be dropped in the same way that exponentially small terms from the higher mass states are dropped. These potential difficulties are of no concern in the field theory limit where the correctness of the final results can be independently verified. What is important here is the basic structure that emerges from string theory without facing the full technicalities of heterotic string constructions.

In general, an amplitude in string theory is evaluated by performing the Polyakov
where the $V_i \sim \varepsilon_i \cdot \partial X e^{ik \cdot X}$ are the vertex operators for external gluons. At one-loop this path integral is performed on a world-sheet annulus. Since the world-sheet bosons are free, Wick’s theorem can be used to evaluate the string $n$-gluon amplitude in terms of the two-point correlation on the annulus

$$\langle X_\mu(\nu_1)X_\nu(\nu_2) \rangle = \delta_\mu^\nu G_B(\nu_{12}) = -\delta_\mu^\nu \left[ \log |2 \sinh(\nu_{12})| - \frac{(\nu_{12})^2}{\tau} - 4q \sinh^2(\nu_{12}) \right] + \mathcal{O}(q^2)$$

where $\tau = -\log(q)/2$ is the real modular parameter of the annulus, $\nu_i$ represents the location of the vertex operator on the annulus and $\nu_{ij} = \nu_i - \nu_j$. (These parameters are $\pi/i$ times the conventional one in refs. [44,49].) As discussed in ref. [2], in the field theory limit these parameters are proportional to sums of Schwinger proper time parameters. A repeated application of Wick’s theorem to evaluate the surface integral yields the string partial amplitude

$$A_{n;1} = i \left( \frac{4\pi \epsilon/2}{16\pi^2} \right)^n (\alpha')^{n/2-2} \int_0^\infty d\tau \int \prod_{i=1}^{n-1} d\nu_i \theta(\nu_i - \nu_{i+1}) \tau^{-2+\epsilon/2} Z$$

$$\times \prod_{i<j} \exp \left\{ \alpha' k_i \cdot k_j G_B(\nu_{ij}) + \sqrt{\alpha'} (k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i) \tilde{G}_B(\nu_{ij}) \right\} \bigg|_{\text{multi-linear}}$$

where

$$\tilde{G}_B(\nu) = \frac{1}{2} \frac{\partial}{\partial \nu} G_B(\nu) , \quad \tilde{\tilde{G}}_B(\nu) = \frac{1}{4} \frac{\partial^2}{\partial \nu^2} G_B(\nu)$$

and $\nu_n$ is fixed at $\tau$. The ‘multi-linear’ signifies that after expanding the exponential only terms which are linear in all $n$ polarizations vectors are to be kept. The string oscillator contributions to the partition function are

$$Z = q^{-1} \prod_{n=1}^\infty (1 - q^n)^{-2(1-\delta_R \epsilon/2)} .$$

Full consistency of the string demands that the dimension $D = 26$ [50], but for the purposes of obtaining field theory amplitudes $D = 4 - \epsilon$ where $\epsilon$ is the dimensional
regularization parameter necessary to handle infrared divergences; the regularization parameter $\delta R$, included in the string partition function, determines the precise form of the regularization [2]. (The regularization issues will be discussed further in Sections 5 and 8.) In order to obtain a sensible field theory limit, the leading $q^{-1}$ has been maintained by hand independent of the number of dimensions. (A fully consistent heterotic string such as the one used in ref. [2] does not require any adjustments, such as this one.) The field theory limit of the amplitude (4.3) yields the pure Yang-Mills contributions to the amplitude including Faddeev-Popov ghosts. The conventions have been adjusted so that in the field theory limit the number of $\pi$'s and 2's which need to be shuffled around are minimized.

Partial amplitudes associated with two color traces are a bit different since the string vertex operators are located on both boundaries of the annulus; examples can be found in chapter 8 of ref. [49].

In order to take the infinite string tension limit $\alpha' \rightarrow 0$ of the string amplitude (4.3), it is convenient to first integrate by parts on the string world-sheet in order to remove all $\dot{G}_B$ from the kinematic factor [2,3]. (The analysis of the field theory limit can also be performed without the integration-by-parts step [5] so it should not be taken as an essential ingredient to the string-based method.) In open string theory there are potential boundary terms, but these can be removed by an appropriate analytic continuation in external momenta since all the boundary terms contain a factor of $|\nu_i - \nu_j|^{n-\alpha' k_i \cdot k_j} \nu_i \rightarrow \nu_j = 0$. (One technicality is that the periodicity on the annulus under $\nu \rightarrow \nu + \tau$ must be used to remove some of the surface terms.)

As an example of the integration by parts procedure consider the following term in three-point function

$$
\int \prod_i d\nu_i \dot{G}_B(\nu_{12}) \dot{G}_B(\nu_{23}) \exp \left[ \alpha' (k_1 \cdot k_2 G_B(\nu_{12}) + k_2 \cdot k_3 G_B(\nu_{23}) + k_1 \cdot k_3 G_B(\nu_{13})) \right] 
$$

$$
\rightarrow -\alpha' \int \prod_i d\nu_i \dot{G}_B(\nu_{12}) \dot{G}_B(\nu_{23}) \left( k_1 \cdot k_2 \dot{G}_B(\nu_{12}) + k_1 \cdot k_3 \dot{G}_B(\nu_{13}) \right) 
\times \exp \left[ \alpha' (k_1 \cdot k_2 G_B(\nu_{12}) + k_2 \cdot k_3 G_B(\nu_{23}) + k_1 \cdot k_3 G_B(\nu_{13})) \right]
$$

(4.6)

where the integration by parts was performed with respect to $\nu_1$. In appendix B of ref. [51] it was proven that all $\dot{G}_B$’s can always be eliminated from the kinematic function, by appropriate integration by parts.

In the field theory limit, the contributions to an integrated-by-parts one-loop amplitude can be classified in terms of tree and loop parts. The tree parts are obtained by first extracting the massless poles in the $S$-matrix before taking the field theory limit of the loop. Examples of these kinematic poles are found in the regions where $\nu_i \rightarrow \nu_j$ and are of the form

$$
\int d\nu_i \frac{1}{\nu_{ij}^{1+\alpha' k_i \cdot k_j}} \rightarrow -\frac{1}{\alpha' k_i \cdot k_j} \quad (\alpha' \rightarrow 0).
$$

(4.7)
In general, the kinematic poles extracted in this way correspond to the poles of a scalar $\phi^3$ diagram.

After kinematic poles have been extracted, the field theory limit of the loop is needed. This is obtained by taking $\tau, |\nu| \to \infty$ which corresponds to squeezing the annulus down to a field theory loop. The values of the Green functions in this limit are

$$\exp(G_B(\nu)) \to \exp\left(\frac{\nu^2}{\tau} - |\nu|\right) \times \text{constant}$$

$$\dot{G}_B(\nu) \to \frac{\nu}{\tau} - \text{sign}(\nu)(\frac{1}{2} + e^{-2|\nu|} - qe^{2|\nu|}) .$$

The exponentiated bosonic Green function was not expanded beyond $O(q^0)$; after carrying out the integration by parts procedure the higher order terms do not contribute since they carry too many explicit powers of $\alpha'$. For $\dot{G}_B$, terms through $O(q)$ should be kept due to the presence of the overall $q^{-1}$ in the string amplitude (4.3).

In the field theory limit two types of loop contributions are obtained depending on whether a power of $q$ is extracted from the string partition function or from the Green functions. For the former contribution one simply keeps the leading order contributions from the bosonic Green functions. This type of contribution is described by the bosonic zero-mode [49] or loop momentum integral of the string [16]. A product of $\dot{G}_B$’s contains exponentially growing and decaying terms as well as terms which are constant. In general, when terms proportional to $q = e^{-2\tau}$ are extracted from a product of $\dot{G}_B$ in order to cancel the overall $q^{-1}$, a factor of the form

$$\exp\left[\left(|x_k - x_l| - \sum |x_i - x_j|\right)\tau\right]$$

is obtained where $x_i \equiv \nu_i/\tau$. In order to avoid exponential suppression or growth as $\alpha' \to 0$ the sum must add up to cancel within the exponential exactly. This will happen only if each $x_i$ which appears with a positive sign also appears with a negative sign after expressing the absolute values in terms of the $x_i$'s directly. The correct prescription for dealing with exponentially growing terms due to the tachyon is to simply drop them in the same way that exponentially decaying terms are dropped. (The exponential growth is an artifact of the Schwinger proper time representation of tachyonic propagators.)

The result of collecting those terms where the exponential terms completely cancel is that only those which form a cycle of $\dot{G}_B$’s, defined to be a product of $\dot{G}_B$’s with indices arranged in the form

$$\dot{G}_B(\nu_{i_1 i_2})\dot{G}_B(\nu_{i_2 i_3})\cdots\dot{G}_B(\nu_{i_m i_1}) ,$$

will not vanish. Furthermore, the cyclic ordering of the indices must follow the same ordering of the corresponding legs in the partial amplitude.

The superstring works in pretty much the same manner, except there are now fermionic fields on the world-sheet. A superstring is essential in order to be able
to include space-time fermions into the string formalism. Although all superstrings must maintain world-sheet supersymmetry for their consistency, such strings are not necessarily space-time supersymmetric. In particular, the string models of interest which have QCD-like spectra are not space-time supersymmetric.

In the superstring the vertex operator is of the form

$$V \sim \varepsilon \cdot (\partial X + i\psi k \cdot \psi)e^{ik \cdot X} \quad (4.11)$$

where $X$ is the same world sheet bosonic field as in the bosonic string and $\psi$ is a free fermionic field. As for the bosonic string these vertex operators are inserted into the Polyakov path integral

$$A_n^{\text{superstring}}(\{k_i, \varepsilon_i\}) \sim \int [DX][D\psi] \exp[-S] V(k_1, \varepsilon_1) \cdots V(k_n, \varepsilon_n) \quad (4.12)$$

in order to obtain amplitudes. The contributions of the world-sheet fermions can be computed by noting that they are free fields so that Wick’s theorem can be used. In this way, any product of fermion fields can be evaluated from the basic two-point correlation function

$$\langle \psi^\mu(\nu_1) \psi^\sigma(\nu_2) \rangle^\alpha_\beta = \delta^{\mu\sigma} G_F^\alpha_\beta (\nu_1 - \nu_2) \quad (4.13)$$

where $\alpha$ and $\beta$ refer to the particular world sheet boundary conditions. One of the features controlled by these boundary conditions is whether the particles in the loop are bosons or fermions [49].

It is a simple matter to verify from Wick’s theorem and the vertex operator (4.11) that the fermionic Green functions always arrange themselves into cycles. For example

$$\langle \psi^{\mu_1}(\nu_1) \psi^{\mu_2}(\nu_2) \psi^{\sigma_2}(\nu_2) \psi^{\mu_3}(\nu_3) \psi^{\sigma_3}(\nu_3) \rangle \sim G_F(\nu_{12})G_F(\nu_{23})G_F(\nu_{31}) \quad (4.14)$$

exhibits the cycle structure. These cycles are analogous to the cycles of bosonic Green functions discussed above.

Since world-sheet supersymmetry in a superstring relates $\psi^\mu$ to $X^\mu$ it turns out that it is possible to obtain the contributions from the world-sheet fermions from the world-sheet bosons [23]. For an appropriate integration by parts the fermion Green functions $G_F$ of a superstring satisfy the constraint that after an appropriate integration by parts the superstring kinematic expression vanishes after substituting $G_F^{\alpha_j} \rightarrow -\hat{G}_B^{i_j}$. In this way a precise match between the $G_F$ cycles and $\hat{G}_B$ cycles can be made. A relation between the bosonic and fermionic Green function contributions to the amplitude is, of course, no surprise since this is precisely the role of world-sheet supersymmetry. (The relationship between the $\hat{G}_B$ and $G_F$ terms
follows from the cancellation of spurious $F_1$ formalism \cite{49} tachyon poles; in particular the tadpole diagram with all legs pinched together should not have a tachyon pole, so the pole contribution generally is the form of a total derivative. An appropriate integration-by-parts to remove the total derivative then makes the matchup between $G_F$ and $\dot{G}_B$ terms manifest.) By matching up the world-sheet fermions to the world-sheet bosons in this way one can show that the results obtained from a bosonic string match those of a superstring. This is of particular interest for the case where space-time fermions circulate in the loop since this cannot be obtained directly from a bosonic string. The trick is thus to include $G_F$ superstring contributions as additional $\dot{G}_B$ contributions. In this way rules can be constructed which contain space-time fermions in the loop, but are based on the simpler bosonic string kinematic master formula. Rules obtained in this way are presented in Section 6.

5. Tree Level Methods at Loop Level

The tree-level methods discussed in Section 3 carry over to loop level; there are, however, a number of differences between the situation at loop and tree level as we discuss in this section.

5.1 The Color Decomposition

A detailed discussion of the one-loop color decomposition has been given in ref. \cite{51}. The major difference between the tree color decomposition and one-loop decomposition is that at one loop up to two color traces can appear in a given term. In particular, the $SU(N)$ four-point gluon amplitude can be written in the form,

$$A_{\text{one-loop}}^4 = g^4 \sum_{\sigma \in S_4/Z_4} N \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) A_{4;1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$$

$$+ \sum_{\sigma \in S_4/Z_2^3} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}}) \text{Tr}(T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) A_{4;3}(\sigma(1), \sigma(2); \sigma(3), \sigma(4)).$$

The notation ‘$S_4/Z_4$’ denotes the set of all permutations $S_4$ of four objects, omitting the purely cyclic transformations $(1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 4, 4 \rightarrow 1)$, etc. The notation ‘$S_4/Z_2^3$’ refers again to the set of permutations of four objects but with permutations considered equivalent (and only one representative picked) if they exchange labels within a single trace or exchange the two traces: $S_4/Z_2^3 = \{(1\,2\,3\,4), (1\,3\,2\,4), (1\,4\,2\,3)\}$.

More generally, the one-loop color decomposition for adjoint representation states is given by

$$A_{n;\text{loop}}^n = g^n \sum_{j=1}^{[n/2]+1} \sum_{\rho \in S_n/S_{n;j}} \text{Gr}_{n;\rho} (\rho(1), \ldots, \rho(n)) A_{n;j}(k_{\rho(1)}, \varepsilon_{\rho(1)}; \ldots; k_{\rho(n)}, \varepsilon_{\rho(n)})$$

(5.2)
where $S_n/Z_n$ is the set of non-cyclic permutations of $\{1, \ldots, n\}$; $\text{Gr}_{n:j}$ denote the double-trace structures

$$
\begin{align*}
\text{Gr}_{n:1} (1, \ldots, n) &= \text{Tr}(1) \text{Tr}(T^{a_1} \cdots T^{a_n}) \\
&= N \text{Tr}(T^{a_1} \cdots T^{a_n}) \\
\text{Gr}_{n:j} (1, \ldots, n) &= \text{Tr}(T^{a_1} \cdots T^{a_{j-1}}) \text{Tr}(T^{a_j} \cdots T^{a_n}),
\end{align*}
$$

and $S_{n:j}$ is the subset of the permutation group $S_n$ that leaves the trace structure $\text{Gr}_{n:j}$ invariant. ($S_{n:1}$ is just the set of cyclic permutations of $n$ objects, $Z_n$.) For pure-glue amplitudes in $SU(N)$, the partial amplitude $A_{n:2}$ drops out since its coefficient includes a trace over a single $SU(N)$ generator, which vanishes identically.

The contribution from fundamental representation states is a bit simpler and can be obtained from the same partial amplitudes which were used for adjoint states and is given by

$$
A_n^{\text{fund}}(\{a_i, k_i, \epsilon_i\}) = g^n \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_{n:1}(k_{\sigma(1)}, \epsilon_{\sigma(1)}; \ldots; k_{\sigma(n)}, \epsilon_{\sigma(n)}).
$$

The color decomposition (5.2) can be heuristically understood from the open bosonic string [38]. At one loop, the schematic form of the $n$-point amplitude is

$$
A_n^{\text{string}}(\{a_i, k_i, \epsilon_i\}) =
\sum_{\sigma \in S_n/Z_n} N \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_1^{\text{string}}(k_{\sigma(1)}, \epsilon_{\sigma(1)}; \ldots; k_{\sigma(n)}, \epsilon_{\sigma(n)}) \\
+ \sum_{m} \sum_{\sigma \in S_n/Z_m \times Z_{n-m}} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(m)}}) \text{Tr}(T^{a_{\sigma(m+1)}} \cdots T^{a_{\sigma(n)}}) \\
\times A_2^{\text{string}}(k_{\sigma(1)}, \epsilon_{\sigma(1)}; \ldots; k_{\sigma(n)}, \epsilon_{\sigma(n)}) + O(\alpha')
$$

where the first term appears when all gluons are attached to a single string boundary as depicted in fig. 8a, while the second term appears when gluons are attached to both string boundaries as depicted in fig. 8b. The higher order corrections in the inverse string tension $\alpha'$ (which do contain terms with three or more non-trivial traces) arise from graviton exchange. Such contributions disappear in the gauge theory (or infinite-tension) limit where the coupling to gravitons and other colorless states vanishes.
Fig. 8: The open string diagrams. When the vertex operators are all attached to the same boundary as in (a) a single color trace is obtained while if the vertex operators are attached to both boundaries as in (b) a product of two color traces is obtained.

The one-loop partial amplitudes $A_{n;j}$ have properties analogous to those of their tree-level counterparts: they are gauge-invariant on-shell, satisfy a symmetry equation,

$$\forall \sigma \in S_{n;j}, \quad A_{n;j}(\sigma(1), \ldots, \sigma(n)) = A_{n;j}(1, \ldots, n) \quad (5.6)$$

and a reflection identity,

$$A_{n;j}(R_{n;j}(1, \ldots, n)) = (-1)^n A_{n;j}(1, \ldots, n) \quad (5.7)$$

where

$$R_{n;j}(i_1, \ldots, i_n) = (i_{j-1}, \ldots, i_1, i_n, \ldots, i_j) \quad (5.8)$$

In addition the partial amplitudes satisfy a set of $U(1)$ decoupling equations which were discussed in detail in ref. [51]. In the case of the four-point function, these take the form

$$A_{4;3}(1, 2, 3, 4) = \sum_{\sigma \in S_4/Z_4} A_{4;1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$$

$$A_{4;2}(1, 2, 3, 4) = -\sum_{\sigma \in Z_3\{2,3,4\}} A_{4;1}(1, \sigma(2), \sigma(3), \sigma(4)) \quad (5.9)$$

so all information about the four-point amplitude is contained in $A_{4;1}$. It turns out that the decoupling equations also imply that $A_{5;1}$ contains all information, but for larger numbers of legs other $A_{n;j}$ besides $A_{n;1}$ are required.

Using the decoupling equations, one can simplify the color-summed next-to-leading correction to the four-gluon process,

$$\sum_{\text{colors}} [A_4^\dagger A_4]_{\text{NLO}} = 2g^6 N^3 \left(N^2 - 1\right) \text{Re} \sum_{\sigma \in S_4/Z_4} A_{4;1}(\sigma) A_{4;1}(\sigma) \quad (5.10)$$
where we have abbreviated $A_{n;j}(\sigma(1), \ldots, \sigma(n))$ by $A_{n;j}(\sigma)$. The leading-order result has the form

$$\sum_{\text{colors}} [A_4^* A_4]_{\text{LO}} = g^4 N^2 (N^2 - 1) \sum_{\sigma \in S_4/Z_4} |A_{4 \text{tree}}(\sigma)|^2. \quad (5.11)$$

The explicit form of the five-point result can be found in ref. [51].

5.2 Spinor Helicity Techniques

One difference between the tree-level and loop-level use of spinor helicity methods is the appearance of loop momentum. Since spinor helicity methods require on-shell momenta, and loop momentum is an off-shell quantity, until it is integrated out the spinor helicity method cannot be used to its full power. As discussed in Section 8, when following the string-based organization, integrating out the loop momentum in the string reorganized form of the $n$-gluon amplitude is an easy step, just as it is in string theory. For other amplitudes a direct term-by-term integration of the loop momentum to obtain a Feynman parametrized form can be performed. One systematic approach for integrating out loop momentum is with the electric circuit analogy [52,53]. Once the loop momentum is integrated out the full power of the spinor helicity method can then be used to simplify expressions.

A more significant difference between tree- and loop-level use of spinor helicity is the apparent incompatibility of spinor helicity with conventional dimensional regularization (CDR) [54,2]. Dimensional regularization is by far the most convenient regularization scheme in practical calculations so this issue must be addressed before spinor helicity can be used at loop level. The problem is that spinor helicity inherently assumes that the gluon polarization vectors are in four dimensions. This is in conflict with the CDR scheme where the external polarization vectors are continued to $4 - \epsilon$ dimensions. However, this can be repaired by introducing the notion of $\epsilon'$-helicity [55]. To do this in the framework of the spinor helicity basis, one introduces an additional $\epsilon$-helicity, with the following rules in $4 - \epsilon$ dimensions,

$$k \cdot \varepsilon^{(\epsilon)}(k'; q) = 0$$

$$\varepsilon^{(\pm)}(k; q) \cdot \varepsilon^{(\epsilon)}(k'; q') = 0$$

$$\varepsilon_1^{(\epsilon)}(k; q) \cdot \varepsilon_2^{(\epsilon)}(k'; q) = -\delta_{i_1 i_2}^{(4-\epsilon)}. \quad (5.12)$$

In the last expression, $i_1$ and $i_2$ run over the $-\epsilon$ additional dimensions; in squaring an amplitude (or forming an interference), one must sum over these additional indices:

$$\delta^{i_1 i_2}_{(4-\epsilon)} \delta^{i_2 i_3}_{(4-\epsilon)} = \delta^{i_1 i_3}_{(4-\epsilon)}, \quad \delta^{i_1 i_2}_{(4-\epsilon)} \delta^{i_1 i_2}_{(4-\epsilon)} = -\epsilon. \quad (5.13)$$

It is convenient to abbreviate $\delta^{i_1 i_2}_{(4-\epsilon)}$ to $\delta^{(4-\epsilon)}$. Although this scheme allows one to continue using the conventional scheme (which is quite useful when comparing to
previous results obtained by more traditional methods), the introduction of additional helicities causes a severe computational penalty in practical calculations since there is a significant amount of work in calculating the additional $\epsilon$-helicities.

A more efficient approach is instead to modify the dimensional regularization scheme so that only plus and minus ‘observed’ helicities are considered, as advocated in ref. [2]. ‘Observed’ gluons are those which are neither virtual, collinear, nor soft. The basic idea of these schemes is to retain the polarization vectors in four-dimensions. All dimensional regularization schemes entail continuing the momentum integrals (both the loop integrals and the integrals over soft and collinear phase space) to $4 - \epsilon$ dimensions in order to render them finite. There are, however, a number of versions of dimensional regularization, which differ in their treatment of the polarization vectors (or helicities) of the observed and unobserved particles:

(a) The ‘conventional’ dimensional regularization (CDR) used, for example, by Ellis and Sexton [6] in which both observed and unobserved gluon polarization vectors are continued to $4 - \epsilon$ dimensions (so that all gluons have $2 - \epsilon$ helicity states); and

(b) the ’t Hooft and Veltman scheme [56] in which all polarization vectors of unobserved gluons are continued to $4 - \epsilon$ dimensions (so that unobserved gluons have $2 - \epsilon$ helicity states), but observed gluon polarizations are kept in four dimensions (so that observed gluons have 2 helicity states);

(c) a four-dimensional helicity scheme (FDH) which naturally arises when using the spinor helicity formalism. In this scheme all helicities (of both observed and unobserved particles) are treated in four dimensions (so that all gluons have 2 helicity states).

The defining properties of the various regularization schemes are summarized in Table 1.

| Momentum components | CDR          | ’t Hooft-Veltman | FDH          |
|---------------------|--------------|-----------------|--------------|
| Unobserved particles| $4 - \epsilon$ | $4 - \epsilon$ | $4 - \epsilon$ |
| Observed particles  | $4 - \epsilon$ | 4               | 4            |

| Helicities          | CDR          | ’t Hooft-Veltman | FDH          |
|---------------------|--------------|-----------------|--------------|
| Unobserved particles| $2 - \epsilon$ | $2 - \epsilon$ | 2            |
| Observed particles  | $2 - \epsilon$ | 2               | 2            |

**Table 1:** Defining properties of the various dimensional regularization schemes.

The CDR scheme is conceptually the simplest one as all quantities are uniformly continued to $4 - \epsilon$ dimensions. (Actually, as a practical matter the observable external momenta can be effectively taken to be four-dimensional by taking the momentum components in the $\epsilon$ dimensions to vanish in a given scattering process.)
the ’t Hooft-Veltman scheme all observed polarizations are kept in four-dimensions but the unobserved ones such as the virtual ones in the loop are continued to $4 - \epsilon$ dimensions. In the ’t Hooft-Veltman scheme one must carefully distinguish between the observed and unobserved particles (virtual, soft and collinear) in order to prevent violations of the optical theorem. The four-dimensional helicity scheme involves retaining all states uniformly in four-dimensions and therefore is a conceptually simpler scheme to work with. However, this scheme has not been well studied beyond the one-loop four-gluon amplitude. In terms of computational complexity the CDR scheme is by far the most complicated to work with when using spinor helicity because of the additional $[\epsilon]$-helcities while the ’t Hooft-Veltman and FDH schemes are of comparable complexity. The calculational differences between the three schemes are summarized in Table 2.

|                        | FDH | ’t Hooft-Veltman | CDR |
|------------------------|-----|-----------------|-----|
| Continue loop momentum | Yes | Yes             | Yes |
| Remove $\epsilon$ bosonic states | No  | Yes             | Yes |
| $\epsilon_i^{(4)} \rightarrow \epsilon_i^{(4-\epsilon)}$ | No  | No              | Yes |

**Table 2:** Modifications needed to construct various versions of dimensional regularization from the unregularized amplitude.

In field theory, with all these schemes one continues the loop momentum integral from $D = 4$ to $D = 4 - \epsilon$; this renders the integrals finite. The string theory equivalent of this analytic continuation is obtained by shifting the overall factor in the integrand of $\tau^{-2}$ to $\tau^{-2+\epsilon/2}$ as was done in eq. (4.5). This change is of the same type as one would obtain in field theory in a dimensionally regularized Schwinger proper time formalism after integrating out the loop momentum.

The FDH scheme is similar, but not identical, to Siegel’s regularization by dimensional reduction [57]; in Siegel’s scheme the polarization vectors of the gluons are taken to be in $D = 4 - \epsilon$ dimensions so they represent a total of $2 - \epsilon$ states, but there are additional $\epsilon$-scalars that then brings the total back to 2 states. The dimensional reduction scheme has been used together with spinor helicity methods in ref. [54]. One nice property of dimensional reduction (which is the purpose of the scheme) is that it maintains space-time supersymmetry. In general, the FDH scheme can also be expected to maintain supersymmetry, since it leaves the number of states at their four-dimensional value [5].

A fundamental requirement on any of these schemes is that they preserve gauge invariance. String theory provides a useful tool for quickly verifying the gauge invariance of the diagrams after regularization. In field theory the amplitude is described in terms of a variety of diagrams; it is only their sum which is gauge
invariant. In string theory (before the field theory limit is taken) each partial amplitude is described in terms of a single diagram; under the shift \( \varepsilon_i \rightarrow \varepsilon_i + k_i \) this single diagram should be invariant. This makes the proof of gauge invariance in string theory much simpler than in field theory.

In string theory, the substitution of the external momentum \( k_i \) for the corresponding polarization vector \( \varepsilon_i \) formally leads to the vanishing of the unregulated amplitude, because one obtains the integral of a total derivative with vanishing boundary terms. Start with the gluon string vertex operator

\[
V \sim : \varepsilon \cdot \partial_{\nu} X e^{ik \cdot X(\nu)} : \quad (5.14)
\]

and set \( \varepsilon = k \); the vertex operator becomes

\[
V|_{\varepsilon = k} \sim : \partial_{\nu} e^{ik \cdot X(\nu)} : \quad (5.15)
\]

If we now compute expectation values using this vertex operator instead of the usual one for the first external gluon, we obtain an integrand which is a total derivative; that is with \( \varepsilon_1 \) replaced by \( k_1 \), the integrand of the amplitude is a total derivative in \( \nu_1 \). As discussed above, the various dimensional regularization schemes modify only the overall factor of \( \tau \), the number of states in the string partition function and possibly the external polarization vectors; the important point is that none of these changes alter the fact that the integrand is a total derivative in \( \nu_1 \), because they do not affect the structure of the Green functions. As a result, the dimensional regularization schemes do not alter the formal argument. (There are a number of subtleties regarding the vanishing of boundary terms, but a more careful argument shows there are no difficulties [2].)

Although the FDH scheme maintains gauge invariance and has explicitly been shown to give identical final results as CDR for the unpolarized four-gluon amplitude, a complete proof of its consistency, especially for the case of fermions is lacking. However, because of the enormous computational advantage obtained with spinor helicity methods there is little doubt that regularization schemes (such as the 't Hooft-Veltman or FDH schemes) which are compatible with spinor helicity will be used in many future calculations.

### 5.3 Space-Time Supersymmetry

The supersymmetry identities hold to all orders of perturbation theory. However, at loop-level the various states present in a supersymmetric theory can circulate in the loop modifying the implication that can be extracted for QCD. In particular, the identities (3.28) and (3.30) no longer imply that the corresponding QCD amplitudes vanish. What it does imply are relationships between the bosonic and fermionic loop contributions.

Since the states in an \( N = 1 \) supersymmetric Yang-Mills theory are a gluon and a fermion gluino, each of which can circulate in the loop, the supersymmetric
\( n \)-gluon amplitude is

\[
A_{n;j}^{N=1 \text{ susy}}(1^\pm, 2^+, 3^+, \ldots, n^+) = A_{n;j}^{\text{gluon}}(1^\pm, 2^+, 3^+, \ldots, n^+)
\]

\[
+ A_{n;j}^{\text{fermion}}(1^\pm, 2^+, 3^+, \ldots, n^+)
\]

\( = 0 \)  

where the particle labels refer to the states circulating in the loop and eqs. (3.28) and (3.30) were used. Thus the contribution to these gluon helicity amplitudes of a fermion loop is minus that of a gluon loop. This means that once we have computed either the fermion or gluon loop for these helicities there is no need to explicitly compute the other. In the supersymmetric theory the fermions are in the adjoint representation, but in QCD the fermions are in the fundamental representation; this difference is rather minor since the partial amplitudes are identical in either case but one would use either eq. (5.2) or (5.4) to construct the full amplitudes depending on the color representation.

One can actually do even better by appealing to \( N = 2 \) supersymmetry [4,5]. The basic supersymmetry identities (3.28) and (3.30) are still the same, but the spectrum now consists of two real scalars, two fermions and one gluon [43]. This gives for the one-loop \( n \)-gluon amplitude

\[
A_{n;j}^{N=2 \text{ susy}}(1^\pm, 2^+, 3^+, \ldots, n^+)
\]

\[
= A_{n;j}^{\text{gluon}}(1^\pm, 2^+, 3^+, \ldots, n^+)
\]

\[
+ 2A_{n;j}^{\text{fermion}}(1^\pm, 2^+, 3^+, \ldots, n^+)
\]

\[
+ 2A_{n;j}^{\text{scalar}}(1^\pm, 2^+, 3^+, \ldots, n^+)
\]

\( = 0 \).

By combining the \( N = 1 \) identity (5.16) with the \( N = 2 \) identity (5.17) we then obtain the result that

\[
A_{n;j}^{\text{gluon}}(1^\pm, 2^+, 3^+, \ldots, n^+) = -A_{n;j}^{\text{fermion}}(1^\pm, 2^+, 3^+, \ldots, n^+)
\]

\( = 2A_{n;j}^{\text{scalar}}(1^\pm, 2^+, 3^+, \ldots, n^+) \).

Thus, for these particular gluon helicities, once the scalar contribution to the \( n \)-gluon amplitude is computed we also have the fermion and gluon contributions to the loop. The supersymmetry identities for other helicity amplitudes, such as in eq. (3.31), relate the gluon amplitudes to ones with external fermions and provide useful checks on QCD calculations with external fermions.

It turns out that for \( A_{4;j}(1^\pm, 2^+, 3^+, 5^+) \) and \( A_{5;j}(1^\pm, 2^+, 3^+, 4^+, 5^+) \) (and most likely for any number of legs) the integrands of each diagram, whether gluons, fermions or scalars circulate in the loop, are equal up to the overall constants in eq. (5.18). In this way the information contained in the supersymmetry identities
is already encoded in the string-based methods. It also turns out that the integrands of diagrams in the string-based methods also exhibit simplifications implicit in $N = 4$ supersymmetric theories which go beyond the above type of supersymmetry identities. This will be discussed in the next section where one form of the string-based rules are presented.

6. Perturbative Rules

The rules which are presented here are similar to the ones presented in refs. [2,3] except that the kinematic coefficient is based on the simpler bosonic string. These rules are the ones presented in ref. [23]. (Other forms exist which avoid the integration-by-parts step discussed near eq. (4.6) [5]. This makes more complicated rules but simpler Feynman parameter polynomials.)

The starting point of these rules are labeled $\phi^3$ diagrams excluding tadpoles. The cyclic labeling of legs of the diagrams must follow the cyclic ordering of the associated color trace structure. For partial amplitudes associated with two color traces, the labels corresponding to the first trace must follow a counterclockwise ordering while the second trace must follow a clockwise ordering although the two sets can be ordered arbitrarily with respect to each other. The labeling of inner lines of a tree attached to a loop is determined according to the rule that as one moves from the outer lines toward the inner lines, one chooses the label of the most clockwise of the two outer lines at a vertex to label the inner line as depicted in fig. 9.

Fig. 9: The labeling of the lines making up the tree parts of the diagrams.

For the partial amplitude $A_{n;1}$, the rules presented below for evaluating a given $\phi^3$ diagram follow directly from the open string amplitude (4.3). For $A_{n;j>1}$ the form of the rules follows the closed string form of the rules which is what one would obtain by a comparison to the heterotic string (which is a closed string) form of the
rules. (This distinction between open and closed strings is unimportant up to the five-gluon calculation since one would not bother computing $A_{n;j>1}$ directly, but would use the $U(1)$ decoupling equations to obtain those from $A_{n;1}$.)

According to the rules, each labeled $\phi^3$-like diagram evaluates to

$$D = \left(\frac{4\pi}{16\pi^2}\right)^{\epsilon/2} \Gamma(n_\ell - 2 + \epsilon/2) \int_0^1 \prod_{j=1}^{n_\ell} dx_{i_1} \delta\left(\sum_j a_j - 1\right) \frac{K_{\text{red}}}{\left(\sum_{l<m} P_{i_l} \cdot P_{i_m} x_{i_m,i_l} (1 - x_{i_m,i_l})\right)^{n_\ell-2+\epsilon/2}}$$

where the ordering of the loop parameter integrals corresponds to the ordering of the $n_\ell$ legs attached to the loop, $x_{ij} \equiv x_i - x_j$, and $K_{\text{red}}$ is the reduced kinematic factor. With the string-based rules $K_{\text{red}}$ can be obtained in a compact and efficient manner. The lines attached to the loop carry momenta $P_i$ which need not be on-shell as there may be trees attached to the loop. For $K_{\text{red}} = (-1)^{n_\ell}$, $D$ is an $n_\ell$-point loop in massless $\phi^3$ theory. The dimensional regularization parameter $\epsilon = 4 - D$ handles all ultra-violet and infrared divergences. The $x_{i_m}$ are related to ordinary Feynman parameters by

$$x_{i_m} = \sum_{j=1}^m a_j$$

so that the loop parameter integral can alternatively be written as

$$D = \left(\frac{4\pi}{16\pi^2}\right)^{\epsilon/2} \Gamma(n_\ell - 2 + \epsilon/2) \int_0^1 \prod_{j=1}^{n_\ell} da_j \delta\left(\sum_j a_j - 1\right) \frac{K_{\text{red}}}{\left(\sum_{l<m} P_{i_l} \cdot P_{i_m} \left(\sum_{j=l+1}^m a_j\right) \left(\sum_{j=1}^{l-1} a_j + \sum_{j=m+1}^{n_\ell} a_j\right)\right)^{n_\ell-2+\epsilon/2}}.$$  

(6.1)

The partial amplitudes are then given by the sum over all diagrams whose legs follow the ordering of the color trace so that

$$A_{n;j}(1, 2, \cdots, j - 1; j, j + 1, \cdots, n) = i(\sqrt{2})^n \mu^\epsilon \sum_{\text{diagrams}} (-1)^{j_\ell-1} D$$

(6.4)

where $j_\ell - 1$ is the number of legs attached to the loop associated with the first of the two color traces; for $A_{n;1}$ this is always zero. An additional color combinatoric factor of 2 is required for $A_{4;3}$; no other combinatoric factors appear. The parameter $\mu$ is the usual renormalization scale parameter that appears in dimensional regularization.
The starting point for evaluating $K_{\text{red}}$ for any given diagram is the full kinematic expression given by

$$K = \int \prod_{i=1}^{n} dx_i \prod_{i<j} \exp \left( k_i \cdot k_j G_{B}^{i,j} + (k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i) \dot{G}_{B}^{i,j} - \varepsilon_i \cdot \varepsilon_j \ddot{G}_{B}^{i,j} \right)_{\text{multi-linear}}$$

(6.5)

where the exponential should be taylor-expanded to obtain those terms which are linear in all $n$ polarization vectors. (Since all powers of the inverse string tension cancel at the end, all powers of $\alpha'$ have been dropped.) Although simpler than the kinematic expression obtained from the heterotic string [2,3], this kinematic expression contains identical information. This kinematic expression represents all information contained in a one-loop $n$-gluon amplitude; the value of all diagrams is encoded in this kinematic factor. Although a string theorist may recognize $\dot{G}_B$ and $\ddot{G}_B$ as derivatives of the bosonic Green function on the world sheet, a field theorist should view these functions as ‘Feynman parameter functions’. From a conventional Feynman diagram point of view the existence of a universal kinematic function is strange as there is apparently no simple relationship between the various Feynman diagrams contributing to a given process. As discussed in refs. [2,3], the fact that no off-shell momenta or polarization vectors appear in kinematic expressions of the type (6.5) allows one to use the full power of the spinor helicity basis [7,8] on the first line of an explicit computation.

The first step in applying the rules presented here is to integrate by parts in the kinematic expression (6.5) (ignoring surface terms) so as to remove all $\ddot{G}_{B}^{i,j}$; this is always possible as was proven in appendix B of ref. [51]. After the integration by parts has been performed the integrals sitting in front of the kinematic expression along with the $\prod_{i<j} \exp(k_i \cdot k_j G_{B}^{i,j})$ factor should simply be dropped since the rules include the appropriate factors. The integration-by-parts step is a matter of convenience as it simplifies the form of the rules. (It turns out that an alternative form of string-based rules exists which avoids the integration-by-parts step and is of practical significance for the computations of five-point amplitudes and beyond. These rules will be presented elsewhere [5].)

Fig. 10: The tree substitution rules.
Given the integrated-by-parts kinematic factor and a particular labeled diagram one then applies the tree rules of fig. 10. If a given tree contains all the legs associated with a single color trace the diagram vanishes. (This rule follows from the closed string color organization of the amplitude and does not play a role in the calculation of $A_{n;1}$.) In particular, for a two-point tree with legs labeled by $i$ and $j$ belonging to a subset of the same color trace and $i$ appearing before $j$ in the clockwise ordering, a $\hat{G}_{iB}^{i,j}$ yields a factor of $(-2k_i \cdot k_j)^{-1}$. These tree rules do not depend on the whether gluons, scalars or fermions circulate in the loop.

The $n$-gluon diagram which has a tree with $(n - 1)$-legs so that the loop is isolated on the remaining leg might seem to be ill-defined as it contains a ‘$0/0$’ [19,58,59] after applying the tree rules. A four-point example of this type of diagram is given in fig. 11. However, in dimensional regularization there is an additional factor of zero in such diagrams of the form $(p^2)^\epsilon/\epsilon$ with $p^2 = 0$ (since the leg is on-shell). This is interpreted as a complete cancellation of ultraviolet and infrared divergences [60,2]. (If one wishes to distinguish between ultraviolet and infrared divergences then one should resolve the $0/0$ according to a prescription such as the ones given in refs. [19,59].)

![Fig. 11: A four-point example of a diagram with a bubble on an external leg containing a potential $0/0$ ambiguity.](image)

After the tree rules have been applied to the diagram the loop substitution rules are then applied. A summary of the loop rules is provided in fig. 12. For the case of gluons (and the associated ghosts) circulating in the loop, in general every term generates two types of contributions.

The first type of contribution is obtained by multiplying the kinematic expression by a factor of $2(1 - \delta_R \epsilon/2)$ and substituting

$$\hat{G}_{iB}^{i,j} \longrightarrow \frac{1}{2}(-\text{sign}(x_{ij}) + 2x_{ij}).$$

The regularization parameter is $\delta_R = 1$ in a conventional or ’t Hooft-Veltman type dimensional regularization scheme while $\delta_R = 0$ in the four-dimensional helicity scheme [2].

The second type of contribution for gluons arises if a particular term contains a cycle of $\hat{G}_B$ which follows the ordering of integration parameters $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_{n-1}}$. A cycle of $\hat{G}_B$’s is defined by

$$\hat{G}_{i_1}^{i_1,i_2} \hat{G}_{i_2}^{i_2,i_3} \cdots \hat{G}_{i_{n-1}}^{i_{n-1},i_1}.$$
Since the $\hat{G}_B^{i,j}$ are antisymmetric, the indices can be put into this canonical ordering at the cost of some signs. For each cycle of $\hat{G}_B$, the loop substitution rules are

\[
\hat{G}_B^{i_1,i_2} \rightarrow 2 \quad (6.8)
\]

and

\[
\hat{G}_B^{i_1,i_2} \dot{G}_B^{i_2,i_3} \ldots \dot{G}_B^{i_{m-1},i_m} \dot{G}_B^{i_m,i_1} \rightarrow 1 \quad (m > 2) \quad (6.9)
\]

Only one cycle at a time may contribute to any given term. If the cycle does not follow the ordering of the legs then there is no contribution. After these substitution rules have been applied to a given cycle the substitution rule (6.6) is applied to all remaining factors in the term of interest. One then sums over all cycles in a given term.

Gluon in Loop:

(a) Overall $2(1 - \delta_R \epsilon/2)$, $\hat{G}_B^{i,j} \rightarrow \frac{1}{2} (- \text{sign}(x_{ij}) + 2x_{ij})$,

(b) $\hat{G}_B^{i_1,i_2} \dot{G}_B^{i_2,i_3} \ldots \dot{G}_B^{i_{m-1},i_m} \dot{G}_B^{i_m,i_1} \rightarrow 1 \quad (m > 2)$, cycle follows leg ordering

Real Scalar in Loop:

Overall $N_s$, $\hat{G}_B^{i,j} \rightarrow \frac{1}{2} (- \text{sign}(x_{ij}) + 2x_{ij})$,

Fermion in Loop:

Overall $-4N_d$ for Dirac and $-2N_w$ for Weyl,

$\hat{G}_B^{i_1,i_2} \dot{G}_B^{i_2,i_3} \ldots \dot{G}_B^{i_{m-1},i_m} \dot{G}_B^{i_m,i_1} \rightarrow \left(\frac{1}{2}\right)^m \prod_{k=1}^{m} (- \text{sign}(x_{i_k,i_{k+1}}) + 2x_{i_k,i_{k+1}}) - (-1)^m \prod_{k=1}^{m} \text{sign}(x_{i_k,i_{k+1}})$.

Fig. 12: The loop substitution rules for various particle contents.
As an example of the loop substitution rules for gluons in the loop, consider the term
\[
\left( \dot{G}_{B}^{1,2} \dot{G}_{B}^{2,1} \right) \left( \dot{G}_{B}^{3,4} \dot{G}_{B}^{4,5} \dot{G}_{B}^{5,3} \right)
\]
with the ordering \( x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \). After applying the loop rules a total of three terms are generated (since it contains two cycles) yielding
\[
2(1 - \epsilon \delta_R/2)(\tfrac{1}{2} + x_{12})(\tfrac{1}{2} + x_{21})(\tfrac{1}{2} + x_{34})(\tfrac{1}{2} + x_{45})(\tfrac{1}{2} + x_{53})
+ 2(\tfrac{1}{2} + x_{34})(\tfrac{1}{2} + x_{45})(\tfrac{1}{2} + x_{53})
+ (\tfrac{1}{2} + x_{12})(\tfrac{1}{2} + x_{21}).
\]

The case of scalars circulating in the loop is simpler and can be obtained by modifying the string to contain scalars; the resulting rule for scalars in the loop is to multiply the kinematic expression by an overall factor of \( N_s \) representing the number of real scalars and then apply the substitution rule (6.6). There are no further contributions in this case.

Since a bosonic string does not contain space-time fermions in the spectrum it is not possible to obtain the fermion’s contributions directly from a bosonic string. However, as was noted in Section 4, there is a close relationship of the world-sheet fermions to the world-sheet bosons. A practical consequence is that all information about the amplitude for any particle content can actually be extracted from the bosonic Green functions. In this way rules for space-time fermions can be constructed; these reproduce the results of the heterotic string but are based on the simpler bosonic string kinematic expression.

The first step for fermions circulating in the loop is to multiply by an overall factor of \(-4N_d\) where \( N_d \) is the number of flavors of Dirac fermions circulating in the loop. For \( N_w \) Weyl fermions the appropriate factor is \(-2N_w\). Once again cycles must be identified; the main difference is that in this case all cycles, independent of the ordering, lead to additional contributions. The substitution rule for space-time fermions circulating in the loop is
\[
\left( \dot{G}_{B}^{i_1,i_2} \dot{G}_{B}^{i_2,i_3} \ldots \dot{G}_{B}^{i_{m-1},i_m} \dot{G}_{B}^{i_m,i_1} \right) \rightarrow \left( \frac{1}{2} \right)^m \left[ \prod_{k=1}^{m} \left( -\text{sign}(x_{ik_{k+1}}) + 2x_{ik_{k+1}} \right) - (-1)^m \prod_{k=1}^{m} \text{sign}(x_{ik_{k+1}}) \right]
\]
where \( x_{m+1} \equiv x_1 \) and the ordering of legs does not matter. The first term on the right hand side is the same one as one would obtain by either the scalar loop rules or by the no-cycle gluon loop rule (6.6), up to the overall constant; this constitutes the ‘no-cycle’ fermion loop contribution. The second term is the additional cycle contribution for a fermion in the loop. Remaining \( \dot{G}_{B} \) which do not belong to any cycle should have the substitution rule (6.6) applied.
For example, consider the term $G^1_2 G^2_3 G^3_4 G^4_B$ which contains a (1,2,3) cycle. Applying the internal fermion loop rules generates the terms (with $x_1 \leq x_2 \leq x_3 \leq x_4$)

$$-4N_d \left( \frac{1}{2} \right)^4 \left[ (1 + 2x_{12})(1 + 2x_{23})(1 + 2x_{13}) - 1 \right] (1 + 2x_{14}).$$  \hspace{1cm} (6.13)

As mentioned in the previous section the string-based rules go beyond the standard supersymmetry identities discussed in that section for simplifying calculations when bosons and fermions are present. According to the rules, the no-cycle contributions for any particle in the loop are all equal up to an overall constant. Furthermore, it is easy to verify that the two- and three-cycle contributions for a gluon are $-4$ times that of a Weyl fermion. This holds for the Feynman parameter polynomials of all diagrams in the string-based rules. For the four-cycle and beyond or products of cycles there is no longer as simple a relationship between fermion and gluon loop contributions. However, the most technically complicated terms are the zero-, two- and three-cycle terms since those generate the most complicated Feynman parameter polynomials. This structure then implies that the contributions of a real scalar, a Weyl fermion and a gluon to the one-loop $n$-gluon amplitude are given by the generic formulas [4,5]

$$A_{n; j}^{\text{scalar}} = S$$

$$A_{n; j}^{\text{fermion}} = -2S - F$$

$$A_{n; j}^{\text{gluon}} = 2(1 - \delta_{R^2}/2)S + 4F + G$$ \hspace{1cm} (6.14)

where $S$ is the no-cycle contribution, $-F$ the terms containing contributions from cycles for a space-time fermion loop and $4F + G$ are the contributions containing cycles for a gluon loop. As before, the particle labels refer to the states circulating in the loop.

Thus a good strategy for computing the gluon in the loop is to first calculate the scalar in the loop to obtain $S$. This is a universal contribution which appears for all states circulating in the loop. Then the cycle parts of fermion in the loop computation can be computed to determine $F$. Finally, for the gluon loop contributions $G$ can be obtained by computing

$$G = 4(\text{cycle contributions for Weyl fermions}) + (\text{cycle contributions for gluons})$$ \hspace{1cm} (6.15)

for each diagram and then summing over diagrams. In each diagram this quantity vanishes for all two- and three-cycles leaving behind a much simpler Feynman parameter polynomial. In this way $G$ can be directly computed. Observe that $S$, $G$ and $F$ are gauge invariant since the scalar, fermion and gluon loop contributions are individually gauge invariant.

The underlying reason for the simple relationship of space-time boson loops to fermion loops can be traced back to the essentially equal treatment of either boson
(Neveu-Schwarz) or fermion (Ramond) loops in string theory; only the world-sheet boundary conditions differ between the two cases.

Observe that from the general structure (6.14), for $N = 4$ super Yang-Mills which contains one gluon, four Weyl fermions and 6 real scalars, the gluon amplitude satisfies [4] (with $\delta R = 0$)

\[ A_{N=4 \text{ susy}}^{(1, 2, \cdots, n)} = G = \text{simple} \quad (6.16) \]

independent of the helicity choices. The string-based rules make this simple supersymmetry structure evident at the level of the integrands of each diagram.

Because of this structure once the fermion loop contributions have been computed, obtaining the gluon loop contributions represents a small fraction of the work required to obtain the fermion loop contributions. It is amusing to make a comparison of the one-loop diagrams in the gluon-by-gluon scattering computation to the diagrams of a QED photon-by-photon scattering computation (which makes use of modern spinor helicity techniques). (In QCD the conversion of the matrix elements into quantities which may be compared to experiment is significantly more complicated but here we are interested in the comparison of the virtual diagrams, which traditionally are also far more difficult in QCD.) There are three main differences between a one-loop QCD and QED diagram computation. In QED one only has massive fermions circulating in the loop while in QCD one can have gluons, ghosts, and fermions. Two other differences are that in QCD there are additional diagrams with gluon trees and that in QCD masses are generally negligible but not in QED. In field theory, the complexity of the non-abelian vertex indicates that a gluon loop should be significantly more complicated than a fermion loop. With the string-based methods, the computational difference between a gluon loop and a fermion loop given by $G$ is relatively small. Furthermore, the diagrams with lower point loops are generally much simpler to evaluate since the associated loop integrals are simpler. The appearance of masses in QED also complicates the integrals as compared to QCD but lead to less severe infrared problems. This leads to the result that the gluon one-loop diagrams are only moderately more difficult to compute than photon diagrams, contrary to traditional field theory expectations.

After the partial amplitudes have been computed, the full amplitude can then be obtained by summing the partial amplitudes with appropriate color trace factors; for adjoint representation states circulating in the loop the appropriate sum is given in eq. (5.2), while for fundamental states in the loop the appropriate sum is given in eq. (5.4).

Modifying these rules to include masses for the internal fermions or scalars is simple; the only change that needs to be made is in the denominator in eq. (6.1) where the massless Feynman denominator is replaced with one corresponding to massive states circulating in the loop.
7. Explicit Examples

In this section a number of explicit examples are presented of computations which would be exceedingly difficult with traditional Feynman diagram techniques but are much simpler with string-based methods.

7.1 Four-gluon Amplitudes

The first example is the computation of $A(1^-, 2^+, 3^+, 4^+)$). This example is nice because of its simplicity. Since the amplitude turns out to be finite on a diagram-by-diagram basis there is no need to use dimensional regularization.

The first step is to insert the spinor helicity simplifications, which for this case are given in eq. (3.18), into the kinematic expression (6.5). From (3.18) we can read off the kinematic expression for this helicity choice as

\[
K = \varepsilon_1 \cdot k_3 (-\hat{G}_B^{1,3} + \hat{G}_B^{1,2}) \varepsilon_2 \cdot k_4 (-\hat{G}_B^{2,4} + \hat{G}_B^{2,3}) \varepsilon_3 \cdot k_4 (-\hat{G}_B^{3,4} + \hat{G}_B^{3,2}) \\
\times \varepsilon_4 \cdot k_3 (-\hat{G}_B^{4,3} + \hat{G}_B^{4,2})
\]

\[
= C(\hat{G}_B^{1,2} - \hat{G}_B^{1,3})(\hat{G}_B^{2,4} - \hat{G}_B^{2,3})(\hat{G}_B^{3,4} + \hat{G}_B^{3,3})(\hat{G}_B^{4,4} - \hat{G}_B^{4,4})
\]

where

\[
C = -\varepsilon_1 \cdot k_3 \varepsilon_2 \cdot k_4 \varepsilon_3 \cdot k_4 \varepsilon_4 \cdot k_3 \\
= \frac{1}{4} s^2 t \frac{[2 4]^2}{[1 2] [2 3] [3 4] [4 1]}
\]

(7.1)

(7.2)

(7.3)

Since there are no $\hat{G}_B$ factors there is no need to perform the integration-by-parts step in this particular case. For simplicity of notation the Mandelstam variables

\[
s = 2k_1 \cdot k_2 \\
t = 2k_1 \cdot k_4 \\
u = 2k_1 \cdot k_3
\]

are used.

There are a total of seven diagrams with potential contributions as depicted in fig. 13. Of these only two diagrams (a) and (b) are non-vanishing after applying the tree rules. For example, since there is no $\hat{G}_B^{14}$ present diagram (c) vanishes. Diagram (d) vanishes because one of the factors vanishes when $x_2 \to x_3$. Similarly all other diagrams vanish.
Fig. 13: Diagrams which potentially contribute. Diagrams (c)-(g) trivially vanish after applying the tree rules.

The box diagram 13a is computed by applying the loop substitution rules. With these rules, the first type of contribution is obtained by multiplying by 2 (for no dimensional regularization) and applying the substitution (6.6). This gives a contribution to the Feynman parameter polynomial given by

$$T_1 = 2C(x_{12} - x_{13})(x_{23} - x_{24})(1 + x_{34} + x_{23})(x_{34} - x_{24}) = 2C(x_3 - x_2)^2(1 - x_3)x_2.$$ \hfill (7.4)

The second type of contribution is given from cycles. In order to exhibit the cycles it is best to expand out the kinematic coefficient

$$K = C(\hat{\mathcal{G}}_{B}^{1,2} - \hat{\mathcal{G}}_{B}^{1,3})[\hat{\mathcal{G}}_{B}^{2,3}(\hat{\mathcal{G}}_{B}^{3,4})^2 - \hat{\mathcal{G}}_{B}^{2,4}(\hat{\mathcal{G}}_{B}^{3,4})^2 + (\hat{\mathcal{G}}_{B}^{2,3})^2\hat{\mathcal{G}}_{B}^{3,4} - \hat{\mathcal{G}}_{B}^{2,4}\hat{\mathcal{G}}_{B}^{3,4}\hat{\mathcal{G}}_{B}^{2,3} + (\hat{\mathcal{G}}_{B}^{2,4})^2\hat{\mathcal{G}}_{B}^{3,4} - (\hat{\mathcal{G}}_{B}^{2,4})^2\hat{\mathcal{G}}_{B}^{2,3} + (\hat{\mathcal{G}}_{B}^{2,3})^2\hat{\mathcal{G}}_{B}^{2,4} - (\hat{\mathcal{G}}_{B}^{2,4})^2\hat{\mathcal{G}}_{B}^{2,3}]$$ \hfill (7.5)

The first factor is not expanded out since it is not part of any cycles. The cycle substitution rules to be applied to this are

$$(\hat{\mathcal{G}}_{B}^{i,j})^2 \to -2,$$

$$(\hat{\mathcal{G}}_{B}^{2,3})^2 \hat{\mathcal{G}}_{B}^{3,4} \hat{\mathcal{G}}_{B}^{2,4} \to -1$$ \hfill (7.6)

after which the substitution (6.6) is performed in remaining factors. This gives the cycle contributions

$$T_2 = C\frac{1}{2}(1 + x_{12} - 1 - x_{13})[-2(1 + 2x_{23}) + 2(1 + 2x_{24}) - 2(1 + 2x_{34}) + 2] = 0.$$ \hfill (7.7)

This vanishing of contributions with cycles can be understood from the space-time supersymmetry identities (5.18), as explained below. Observe that this cancellation takes place at the level of the integrand.
Summing over the two types of contributions and inserting into the scalar parameter integral gives the value of the first diagram

\[ D_a = \frac{i C}{4\pi^2} \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \frac{2(x_3 - x_2)^2(1 - x_3)x_2}{[sx_1(x_3 - x_2) + t(x_2 - x_1)(1 - x_3)]^2}. \]  

(7.8)

(This can also be rewritten in terms of conventional Feynman parameters using (6.2) if one prefers.) Since

\[ \int_0^{x_2} dx_1 \frac{2(x_3 - x_2)^2(1 - x_3)x_2}{[sx_1(x_3 - x_2) + t(x_2 - x_1)(1 - x_3)]^2} = 2 \frac{(x_3 - x_2)}{st} \]

(7.9)

the remaining integrals are trivial, yielding

\[ D_a = \frac{i}{12\pi^2} C \frac{1}{st}. \]  

(7.10)

Now we must evaluate the second non-vanishing diagram given in fig. 13b with the 1–2 tree. According to the tree rules extract the coefficient of \( \dot{G}_{B}^{12} \) and multiply by \(-1/k_1 \cdot k_2\), obtaining the reduced kinematic expression

\[ K_{12}^{\text{red}} = -\frac{C}{2k_1 \cdot k_2} (\dot{G}_{B}^{2,3} - \dot{G}_{B}^{2,4})(\dot{G}_{B}^{3,4} + \dot{G}_{B}^{2,3})(\dot{G}_{B}^{3,4} - \dot{G}_{B}^{2,4}). \]  

(7.11)

Note that the last three factors are the same ones as in the box diagram. This means that we can read off the results of applying the loop rules from the box diagram yielding the Feynman parameter polynomial

\[ -2C \frac{s}{s} x_2(1 - x_3)(x_3 - x_2). \]  

(7.12)

Inserting this into the loop parameter integral yields

\[ D_b = -iC \frac{1}{4\pi^2 s} \int_0^1 dx_3 \int_0^{x_3} dx_2 x_2(1 - x_3)(x_3 - x_2) \frac{s x_2(x_3 - x_2)}{-s x_2(x_3 - x_2)}. \]  

(7.13)

Since the denominator cancels against the numerator the integrals are trivial, yielding

\[ D_b = \frac{i}{12\pi^2} C \frac{1}{s^2}. \]  

(7.14)

Summing over the contributions of the two non-vanishing diagrams and using eq. (7.2) yields the amplitude

\[ \mathcal{A}_{4;1}^{\text{gluon}}(1^{-}, 2^{+}, 3^{+}, 4^{+}) = \frac{i}{48\pi^2} \frac{(s + t)[2 \ 4]^2}{[1 \ 2 \ 2 \ 3 \ 3 \ 4 \ 4 \ 1]}. \]  

(7.15)
which is a result that was first computed with string-based techniques.

Since the contributions with cycles drop out we immediately have that the contribution of a real scalar in the loop to the four-gluon amplitude is given by

\[ A_{4:1}^{\text{scalar}}(1^-, 2^+, 3^+, 4^+) = \frac{N_s}{2} A_{4:1}^{\text{gluon}}(1^-, 2^+, 3^+, 4^+) \] (7.16)

It is also not difficult to verify that cycle contributions also drop out for fermions in the loop. This then yields

\[ A_{4:1}^{\text{fermion}}(1^-, 2^+, 3^+, 4^+) = -N_w A_{4:1}^{\text{gluon}}(1^-, 2^+, 3^+, 4^+) \] (7.17)

where \( N_w \) is the number of Weyl fermions. Summing over the various contributions yields

\[ A_{4:1}(1^-, 2^+, 3^+, 4^+) = (2 + N_s - 2 N_w) \frac{i}{96 \pi^2} \frac{(s + t) [24]^2}{[12] [23] [34] [41]} \] (7.18)

where all states are in the adjoint representation. Since this expression vanishes when the number of bosonic states equals the number of fermionic states, we have that

\[ A_{4:1}^{\text{susy}}(1^-, 2^+, 3^+, 4^+) = 0 \] (7.19)

in agreement with the supersymmetry identity (3.28). Observe that in the string-based formalism there is no need to explicitly perform the integrals to obtain this identity as the supersymmetry identities hold in the integrands of each diagram. In particular, the vanishing of the cycle contributions in (7.7) is a direct manifestation of the implicit inclusion of supersymmetry identities in the string-based methods.

The other helicity amplitudes can be evaluated in pretty much the same manner, except that dimensional regularization is required. The four-point partial amplitudes which contribute to the next-to-leading order cross-section are those with two minus and two plus helicities, since those are the only tree amplitudes which do not vanish. (The next-to-leading order correction to the cross-section is obtained from an interference of tree and one-loop amplitudes.) By making use of the string-based rules and following the same type of calculation as discussed above one can obtain [2] the dispersive parts of the one-loop partial amplitudes needed for the
next-to-leading corrections (dropping all terms of $O(\epsilon)$),

$$A_{4;1}(1^-, 2^-, 3^+, 4^+) = i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{\Gamma^2(1-\epsilon/2)\Gamma(1+\epsilon/2)}{8\pi^2\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^{\epsilon/2} \times \left(-8 - \frac{22}{3\epsilon} + \frac{11}{6} l_Q(\mu^2) + \frac{2}{\epsilon} (l_Q(s) + l_Q(t)) - \frac{2}{3} (s st - \frac{11}{6} u^2 + \frac{3}{4} st\Theta(1) - \frac{1}{2} st\tilde{\Theta}(1)\frac{u^2 - st}{u^4} - \frac{32}{9} - \delta_R \right) \right)$$

$$A_{4;1}(1^-, 2^+, 3^-, 4^+) = i \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{\Gamma^2(1-\epsilon/2)\Gamma(1+\epsilon/2)}{8\pi^2\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^{\epsilon/2} \times \left(-8 - \frac{22}{3\epsilon} + \frac{11}{6} l_Q(\mu^2) + \frac{2}{\epsilon} (l_Q(s) + l_Q(t)) - \frac{2}{3} (s st - \frac{11}{6} u^2 + \frac{3}{4} st\Theta(1) - \frac{1}{2} st\tilde{\Theta}(1)\frac{u^2 - st}{u^4} - \frac{32}{9} - \delta_R \right) \right)$$

(7.20)

where $\mu^2$ is the renormalization scale, $Q^2$ is a completely arbitrary scale introduced in order to simplify the comparison to the cross-section of Ellis and Sexton [6], $l_Q(x) = \ln |x/Q^2|$, $\Theta(x > 0) = 1$, $\tilde{\Theta}(x < 0) = 0$,

$$\delta_R = \begin{cases} 0, \quad \text{four dimensional helicity scheme}, \\ 1, \quad \text{'t Hooft-Veltman or conventional scheme}, \end{cases}$$

(7.21)

and where evaluation in the physical region is assumed (that is, only one of the Mandelstam variables (7.3) $s$, $t$, or $u$ may be positive). The absorptive parts are not included in the above equations but can be calculated using the appropriate $i\epsilon$ prescription. A modified minimal subtraction was performed on these amplitudes to subtract out the ultra-violet divergence; the remaining divergences are soft and collinear and cancel against contributions from the five-gluon tree amplitude.

How do we know that the results are correct? In ref. [2] a number of checks were performed on these expressions including checks on gauge invariance, unitarity and, best of all, a comparison to a previous calculation [6] of the next-to-leading order corrections to the unpolarized cross-section. Additionally, a mapping to conventional field theory has been found which verifies that the string-based methods give the same results for this calculation as a field theory calculation would.

The checks on gauge invariance were performed in two ways. One way was by verifying that a change in the spinor helicity basis does not modify the result. From eq. (3.13), a change in the spinor basis is equivalent to an on-shell gauge transformation which should leave an on-shell amplitude unchanged. This indeed works as expected. An alternative check is to simply replace a polarization vector with the momentum of that external line. When the remaining legs are all on shell this longitudinal amplitude should just vanish as indeed it does.
The check on unitarity which was performed was a verification of the optical theorem. The optical theorem says that the imaginary or dispersive part of an amplitude is proportional to a tree-level cross-section. In ref. [2], the amplitudes were explicitly shown to satisfy this property.

The best check was against the previous calculation of Ellis and Sexton [6] who calculated the one-loop corrections to unpolarized cross-section into two jets. After including the \( \epsilon \)-helicities discussed in Section 5 in order to obtain the conventional version of dimensional regularization, the unpolarized cross-section is in complete agreement with their result. This provides the first complete check of the Ellis and Sexton cross-section and verifies that the conventional dimensional regularization prescription used in string theory is identical to the one of field theory.

7.2 Five-gluon Amplitudes

Using the methods discussed above a computation of the one-loop five gluon amplitudes has been performed [4]. Additional ingredients, beyond those discussed in these lectures, which enter into this calculation are a simple integral table for the pentagon parameter integrals [61] and improvements in the spinor helicity method [5]. These amplitudes will enter into the theoretical analysis needed for measurement of \( \alpha_s \) at hadron colliders from jets.

The finite helicity amplitudes are

\[
A_{5:1}^{1-\text{loop}}(1^+, 2^+, 3^+, 4^+, 5^+) = \left(1 + \frac{1}{2}N_{s\text{adj}} - N_{w\text{adj}}\right) \\
\times \frac{i}{48\pi^2} \frac{\langle 12 \rangle \langle 12 \rangle \langle 23 \rangle \langle 23 \rangle + \langle 45 \rangle \langle 45 \rangle \langle 51 \rangle \langle 51 \rangle + \langle 23 \rangle \langle 45 \rangle \langle 25 \rangle \langle 34 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}
\]

\[
A_{5:1}^{1-\text{loop}}(1^-, 2^+, 3^+, 4^+, 5^+) = \left(1 + \frac{1}{2}N_{s\text{adj}} - N_{w\text{adj}}\right) \\
\times \frac{i}{48\pi^2} \frac{1}{\langle 34 \rangle^2} \left[ -\frac{\langle 25 \rangle^3}{\langle 12 \rangle \langle 51 \rangle} + \frac{\langle 14 \rangle^3 \langle 45 \rangle \langle 35 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 45 \rangle^2} - \frac{\langle 13 \rangle^3 \langle 32 \rangle \langle 42 \rangle}{\langle 15 \rangle \langle 54 \rangle \langle 32 \rangle^2} \right]
\]

where \( N_{s\text{adj}} \) and \( N_{w\text{adj}} \) are the number of adjoint massless real scalars and Weyl fermions. (For fundamental representation fermions one would use eq. (5.4) to construct the full amplitude.) The double trace \( A_{5:2} \) and \( A_{5:3} \) partial amplitudes follow from the formulae in ref. [51]. In the string-based formalism the supersymmetry identities for these amplitudes (5.18) are satisfied trivially because they hold for the integrand of each string-based diagram; all cycle contributions cancel out from the integrands so that the contribution from any state is identical up to an overall sign determined by statistics.

The infrared divergent ones (which are the ones which interfere with the tree diagrams to produce the next-to-leading order corrections to the cross-section) are given in ref. [4]. These amplitudes, have not been obtained with traditional techniques.
The matrix elements with external quark lines also need to be calculated. These have not been calculated yet, although progress has been made in a string-based approach [36].

One must then combine the virtual corrections with the singular terms in the six-gluon tree-level matrix elements arising from the phase space integration in soft and collinear regions. One expects all these divergences to cancel in physical quantities [62]. The Giele-Glover formalism [31,32] makes use of the color ordering in construction of universal functions representing the results of the soft and collinear integrations, and is the most convenient one for evaluating physical scattering.

7.3 A Gravity Example

Another application of the string-based technique is to gravity. Roughly speaking the structure of string theory implies that

\[(\text{Closed String}) \sim (\text{Open String})^2.\]  \hspace{1cm} (7.23)

Since closed strings contain gravity and open strings contain gauge theory one might expect that

\[(\text{Gravity}) \sim (\text{Yang-Mills})^2.\]  \hspace{1cm} (7.24)

This relationship can be made precise and turned into an extremely efficient computational tool for perturbative gravity amplitudes. At tree-level Berends, Giele and Kuijf [63] have made use of this relationship, as formulated by Kawai, Lewellen and Tye [64], in order to calculate tree-level gravity amplitudes from known Yang-Mills amplitudes. At one-loop this relationship can also be made precise [25]; in particular, the calculation of the one-loop four-graviton amplitude with one minus and three plus helicities is rather easy by making use of string-based rules modified for the case of gravity. The result of such a calculation is given by

\[A(1^-, 2^+, 3^+, 4^+) = \frac{i\kappa^4}{(4\pi)^2} \frac{1}{5760} (N_b - N_f) \frac{s^2t^2}{u^2} (u^2 - st) \left( \frac{[24]^2}{[12][23][34][41]} \right)^2 \]  \hspace{1cm} (7.25)

where \(\kappa\) is the gravitational coupling, \(N_b\) is the number of physical bosonic states and \(N_f\) is the number of fermionic states in the particular theory of gravity under consideration. The reason why any state gives an identical contribution up to a sign is in agreement with the supersymmetry identities [42] and manifests itself in the string-based formalism as a vanishing of all cycle contributions. This is similar to the vanishing of the cycle contributions in the four-gluon amplitude with the same helicities.

This type of calculation would be exceedingly difficult with conventional techniques, given the complexity of the gravity three- and four-point field theory vertices. This may be compared to the string-based technique where the calculation of the above helicity amplitude is reduced to an elementary algebraic exercise. It is amusing that the string-based gravity calculation is only slightly more difficult than
the gluon calculation which is in turn only slightly more difficult than a modern light-by-light scattering calculation. It is intriguing that in terms of conventional field theory the required reorganization is fairly difficult to guess without some input from string theory. The case of gravity will be discussed more fully elsewhere [25].

8. Field Theory Understanding

An understanding of the string-based rules in terms of conventional field theory is important for a number of reasons. Phenomenologists tend to be uninterested in string theory, so it is important to be able to explain the computational advances implied by string theory in terms of a more conventional field theory language. With a mapping to conventional field theory certain string theory subtleties are also no longer a problem. In particular, in order to guarantee that the string-based dimensional regularization scheme is identical to the field theory scheme a mapping between field theory and string theory is necessary. The mapping can also be used to explicitly demonstrate how the string-based methods bypass many of the large cancellations inherent in conventional field theory calculations since it is easy to make comparisons when using the same type of formalism.

As we shall see, the interpretation of the string-based method in terms of conventional field theory is a collection of ideas combined in a particular way. String theory provides the unifying principle for applying these ideas to a field theory calculation. Before turning to the case of loop level we first discuss the tree-level case.

8.1 Tree-Level Mapping

The string reorganization at tree level can be understood in terms of three basic ideas [16]: color ordering (which was discussed in Section 3), a non-linear gauge choice discovered by Gervais and Neveu [26] and a systematic evaluation of the Lorentz contraction algebra generated by the vertices and propagators.

The non-linear Gervais-Neveu gauge makes the comparison of field theory and string theory tree-level results relatively simple. Other gauge choices generally lead to complicated reshuffling of terms between diagrams, obscuring the connection between field theory and string theory. This gauge was originally obtained by Gervais and Neveu by analyzing the field theory limit of open string theory. The terms generated by the Feynman rules in this gauge are in fairly close correspondence to the terms generated by tree-level string theory; the main difference is that with string theory all algebra associated with contracting momenta is bypassed.

The action in the Gervais-Neveu gauge is given by

\[ S_{\text{GN}} = \int d^4x \left( -\frac{1}{4} \text{Tr}[F^2] - \frac{1}{2} \text{Tr}[(\partial \cdot A + igA^2/\sqrt{2})^2] \right) \] (8.1)
where
\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c \]  
(8.2)
and we are ignoring ghosts because here we are only interested in tree level. The peculiar normalization of the terms in the action (8.1) is due to the unconventional normalization of the group generators \( \text{Tr}(T^a T^b) = \delta^{ab} \).

\[ k = i\sqrt{2} \left( \eta_{\mu\nu} k_\rho + \eta_{\nu\rho} p_\mu + \eta_{\rho\mu} q_\nu \right) \]

\[ \mu \nu \lambda \rho = i\eta_{\mu\rho} \eta_{\nu\lambda} \]

**Fig. 14:** The color ordered Gervais-Neveu gauge vertices.

The color ordered three- and four-vertices generated by the action (8.1) are depicted in fig. 14 corresponding to the color traces \( \text{Tr}(T^{a_1} T^{a_2} T^{a_3}) \) and \( \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \). As before, the convention is to take momenta to be outgoing in three-vertices. The propagator in this gauge is also the same simple one as in conventional Feynman gauge
\[ P_{\mu\nu} = -i \frac{\eta_{\mu\nu}}{p^2 + i\epsilon}. \]  
(8.3)

These vertices may be compared to the color ordered form of the conventional Feynman gauge vertices given in fig. 5. The computational simplicity when using the Gervais-Neveu gauge for tree level calculations is clear; the three- and four-point Gervais-Neveu vertices have half and a third as many terms as the corresponding Feynman gauge vertices. Thus, a diagram in the Gervais-Neveu gauge has a factor of approximately \( 2^{n_3} 3^{n_4} \) fewer terms than a corresponding diagram in color ordered Feynman gauge, where \( n_3 \) is the number of three-point vertices and \( n_4 \) is the number of four-point vertices. (We have ignored simplifications from the on-shell conditions which decrease the count and the fact that internal line momenta are sums of external momenta which increase the count.) This provides an explanation of how string theory avoids many of the large cancellations inherent in conventional computations of tree-level amplitudes; at tree level, in Feynman gauge most of the terms must cancel to reproduce the simplicity of the Gervais-Neveu gauge. The use of
Gervais-Neveu gauge provides only a partial explanation of the string-based rules; by using string-based rules the algebra associated with sewing the vertices together is completely bypassed. The Gervais-Neveu gauge is thus useful for allowing a relatively simple understanding of the string organization of the tree level amplitude, although for practical computations it is advantageous to use recursive methods [12,15].

8.2 Loop Level

Given the tree level understanding of the string reorganization of the amplitude, one might think that this can be directly carried over to one loop. However, string theory does not work this way; the field theory descriptions which most closely resemble the string reorganizations at tree level and at one loop are rather different. In particular, a different set of gauge choices are needed at tree and loop level. This provides the general notion that one should use different gauge choices or more generally different field variables at each order of perturbation theory in order to maximize efficiency. String theory provides a particularly efficient way to accomplish this.

The required field theory ideas which are needed to reproduce much of the simplicity of the string-based rules for one-loop $n$-gluons amplitudes are [16]: background field Feynman gauge, color ordering (which was discussed in Section 5), systematic organization of the vertex algebra and a second order formalism for fermions. The most important new ingredient is the background field method which we now review.

8.3 Review of Background Field Method

The background field method [65] is a popular technique for computing effective actions and $\beta$-functions in field theories. Its distinguishing feature is manifest gauge invariance of the effective action which is a property that does not hold for conventional Lorentz gauges.

The basic idea of the background field method is to split the gauge field into quantum and background fields, $A = Q + B$. The quantum field is then gauge fixed in such a way as to maintain the gauge invariance of the background field. In the background field method the effective action is computed by considering one-particle irreducible diagrams with external background $B$ fields.

Although this procedure generates an ‘effective action’, before it can be used in an amplitude calculation its connection to the usual effective action which is the Legendre transformation of the connected diagrams must be understood. Consider the background field generating function

$$Z[B] = \int DQ \Delta_{FP} \exp \left[ i \int d^4x \left( \mathcal{L}(Q + B) - \frac{1}{2\alpha} \text{Tr}[(DB \cdot Q)^2] \right) \right]$$  \hspace{1cm} (8.4)$$

where $\Delta_{FP}$ is the Faddeev-Popov determinant and $\mathcal{L}$ is the lagrangian for the gauge
theory under consideration. For pure Yang-Mills theory

\[ \mathcal{L} = -\frac{1}{4} \text{Tr}[F^2] \]  

(8.5)

where \( F \) is the field strength. The generating function (8.4) is gauge invariant under the background field gauge transformation

\[ \delta B = D^B \lambda, \quad \delta Q = i[\lambda, Q] \]  

(8.6)

where \( D^B = \partial_\mu + igT^a B^a_\mu \) is the derivative covariant with respect to the background field. Since \( Q \) is a dummy integration field, \( Z[B] \) is gauge invariant with respect to \( B \).

But what is \( Z[B] \) in terms of the more conventional effective action? To answer this look at

\[ \tilde{Z}[J, B] = \int DQ \Delta_{FP} \exp \left[ i \int d^4 x \left( \mathcal{L}(Q + B) - \frac{1}{2\alpha} \text{Tr}[(D^B \cdot Q)^2 + \text{Tr}(J \cdot Q)] \right) \right] \]  

(8.7)

which is a generating functional, but with an additional background field and a peculiar gauge fixing. By performing a Legendre transformation we obtain an effective action defined by

\[ \tilde{\Gamma}[\tilde{Q}_{\text{cl}}, B] = \tilde{W}[J, B] - \int d^4 x J^a \cdot \tilde{Q}_{\text{cl}}^a \]  

(8.8)

where

\[ \tilde{W} = -i \ln \tilde{Z}, \quad \tilde{Q}_{\text{cl}} = \frac{\delta \tilde{W}}{\delta J}. \]  

(8.9)

In order to connect this object to the more usual effective action consider the alternative way of evaluating the path integral (8.7) by making the change of variables \( Q \to Q - B \). This yields

\[ \tilde{Z}[J, B] = \exp \left[ -i \int d^4 x \text{Tr}(J \cdot B) \right] \mathcal{Z}[J] \]

\[ = \exp \left[ -i \int d^4 x \text{Tr}(J \cdot B) \right] \times \int DQ \Delta_{FP} \exp \left[ i \int d^4 x \left( \mathcal{L}(Q) - \frac{1}{2\alpha} \text{Tr}(D^B \cdot (Q - B))^2 + \text{Tr}(J \cdot Q) \right) \right] \]  

(8.10)

where \( \mathcal{Z}[J] \) is a conventional generating functional, but with a peculiar gauge fixing which depends on the arbitrary field \( B \).

By comparing the two forms of the path integral we have

\[ \tilde{W} \equiv -i \ln \tilde{Z} = -i \ln Z - \int d^4 x J^a \cdot B^a \equiv W - \int d^4 x J^a \cdot B^a. \]  

(8.11)
This implies that
\[
\tilde{\Gamma} = \tilde{W} - \int d^4x \, J^a \cdot \tilde{Q}_{cl}^a = W - \int d^4x \, J^a \cdot (\tilde{Q}_{cl}^a + B^a) .
\] (8.12)

Thus
\[
\tilde{\Gamma}[\tilde{Q}_{cl}, B] = \Gamma[\tilde{Q}_{cl} + B, B]
\] (8.13)
which relates the background field effective action \( \tilde{\Gamma} \) to the conventional effective action \( \Gamma \). The \( B \) in the second argument of \( \Gamma \) refers to the dependence of the gauge-fixing on the background field. For \( \tilde{Q}_{cl} = 0 \) we then obtain the fundamental equation of the background field method
\[
\tilde{\Gamma}[0, B] = \Gamma[B, B] .
\] (8.14)

In this equation the object on the left hand side is the background field effective action (with no other external sources) while the quantity on the right hand side is the usual effective action, but with a \( B \) dependent gauge fixing.

In an actual calculation with the background field method there is no need to perform a Legendre transformation since \( \tilde{\Gamma} \) can be directly computed from the background field Feynman vertices generated by the path integral (8.4). The basic background field formula then ensures that one-particle irreducible diagrams (i.e., the diagrams of the effective action) which have only external background \( B \) fields are meaningful quantities.

The background field method has traditionally been used for effective action calculations because of its natural interpretation in terms of effective actions. Here we are interested in the scattering amplitudes and not in the effective action. In order to obtain the scattering amplitudes from the effective action we need to sew trees onto the one-particle irreducible diagrams in order to form the connected diagrams. What gauge is this sewing to be performed in? The fact that the effective action is gauge invariant leads one to the notion that it really does not matter what gauge is chosen for the sewing. Since the background field effective action differs slightly from the conventional effective action because of the dependence of the gauge fixing on the background field, one needs to prove that the \( B \) field in eq. (8.14) which comes from the gauge fixing does not interfere with the sewing procedure. A proof has been given by Abbott, Grisaru and Schaefer [66]. Thus, the procedure for constructing scattering amplitudes out of the background field effective action is to simply sew tree diagrams onto the one-particle irreducible diagrams of the effective action using some other gauge, such as ordinary Feynman gauge or Gervais–Neveu gauge. For practical calculations in field theory the Gervais–Neveu gauge is advantageous because of its simpler vertices.

8.4 String Organization of Loop

The vertices can be generated from the background field generating function (8.4). Since we wish to evaluate the term in the amplitude associated with the
color trace $\text{Tr}(T^{a_1}T^{a_2} \cdots T^{a_n})$ it is convenient to color order the background field gauge vertices. These background field three- and four-point vertices are depicted in fig. 15. There are also ghost vertices given by

$$V^G_\mu(p,q) = \frac{i}{\sqrt{2}}(p-q)_\mu$$ \hspace{1cm} (8.15)

for the coupling of a ghost and anti-ghost to a single background field $B$ and

$$V^G_{\mu\nu} = -\frac{i}{2} \eta_{\mu\nu}$$ \hspace{1cm} (8.16)

for the coupling of a ghost and anti-ghost to two background fields. Observe that the ghosts of one-loop background field gauge couple precisely the same way as complex scalars. Indeed, at one-loop in background field gauge the ghost contribution is minus that of a complex scalar.

In order to arrange the computation according to the string organization it is desirable to break the three-point vertex given in fig. 15 into three pieces

$$V_B = \frac{i}{\sqrt{2}} \eta_{\nu\rho}(p-q)_\mu; \quad V_F^+ = i\sqrt{2} \eta_{\mu\nu} k_\rho; \quad V_F^- = -i\sqrt{2} \eta_{\mu\rho} k_\nu$$ \hspace{1cm} (8.17)

where $\mu$ is to be contracted against an external polarization and $k$ is the external background field momentum as depicted in fig. 15. With this breakup, the loops containing only three-vertices can be arranged to follow the string organization. Observe that the $V_B$ vertex is identical to the ghost-background field vertex (8.15) except that it has no $\eta_{\nu\rho}$.

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First consider the case with either only $V_B$ or $V_G$ vertices in the $\phi^3$-like loop depicted in fig. 16. Sewing together the vertices into a loop yields

\[
(\sqrt{2})^{n^2}(\delta^\mu_{\mu} - 2) \int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}} \frac{\prod_{i=1}^{n} \varepsilon_i \cdot (p + q_i)}{\prod_{i=1}^{n} (p + q_i)^2}
\]  

(8.18)

where $\delta^\mu_{\mu}$ arises from sewing the $\eta_{\nu\rho}$ of the $V_B$ vertices around the loop and $-2$ is the statistical and combinatoric factor associated with the ghost loop. Rewriting this in terms of Schwinger proper-time variables yields

\[
(\sqrt{2})^{n^2}2R \int_0^\infty dt_1 \int_0^\infty dt_2 \cdots \int_0^\infty dt_{n-1} \int_0^\infty dt_n \int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}} \\
\times \exp \left( -\sum_{i=1}^{n} t_i (p + q_i)^2 + \sum_{i=1}^{n} \varepsilon_i \cdot (p + q_i) \right)
\]

(8.19)

where

\[ q_i = k_i + \cdots + k_n = -k_1 - k_2 - \cdots - k_{i-1} \]  

(8.20)

and we have observed that

\[ \delta^\mu_{\mu} - 2 = 2 - \delta_R \epsilon \equiv 2R . \]  

(8.21)

The parameter $\delta_R$ controls the precise form of dimensional regularization, as discussed in Section 5. This form of the loop momentum integral may be recognized as the bosonic zero-mode of the string (see Chapter 8 of ref. [49]) providing the connection to string theory.

\[ \text{Fig. 16: A } \phi^3\text{-like loop diagram with no attached trees.} \]

The loop momentum integral in eq. (8.19) can be performed by completing the
square in the usual fashion (see for example eq. (8.1.56) of ref. [49]) yielding

\[
A_B(\varepsilon_1, k_1; \cdots; \varepsilon_n, k_n) =
\]

\[
i(\sqrt{2})^n 2^R (4\pi)^{\varepsilon/2} \frac{1}{16\pi^2} \int \prod_{i=1}^{n-1} dx_i \int_0^\infty dT \ T^{n-3+\varepsilon/2}
\]

\[
\times \prod_{i<j}^{n} \exp\left\{ \int \prod_{i=1}^{n-1} dx_i \int_0^\infty dT \ T^{n-3+\varepsilon/2}
\]

\[
\times \prod_{i<j}^{n} \exp\left\{ k_i \cdot k_j \tilde{G}^{ij}_B + (k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i) \tilde{G}^{ij}_B - \varepsilon_i \cdot \varepsilon_j \tilde{G}^{ij}_B \right\}_{\text{multi-linear}}
\]

where \( x_i = \sum_{j=1}^{i} t_j / T \), \( T = \sum_{i=1}^{n} t_i \), \( x_n = 1 \) and the \( \tilde{G}^{ij}_B \)'s are given by

\[
\tilde{G}^{ij}_B = T x_{ij} (1 + x_{ij}) \text{,} \quad \tilde{G}^{ij}_B = \frac{1}{2} + x_{ij} \text{,} \quad \tilde{G}^{ij}_B = \frac{1}{2T} \text{.} \quad (8.22)
\]

As the notation suggests these functions are the same ones as one obtains from the loop rules applied to the string master formula. Of course, in this case the quantities are purely field theory expressions. In this way the background field \( V_B \) vertices reproduce the pure \( G_B \) parts of the string kinematic expression evaluated with loop substitution rules without integration by parts for diagrams of the form in fig. 16. The integration by parts discussed in Sections 4 and 6 obscures the connection between the string-based rules and field theory. One can also integrate by parts in field theory [16] but care must be taken to keep track of surface terms. This complicates the match between field theory and string theory.

Consider now the case with pure \( V_F^+ \) vertices. In this case loop momentum does not appear in the vertices and one obtains

\[
(-1)^n \varepsilon_1 \cdot k_2 \varepsilon_2 \cdot k_3 \cdots \varepsilon_{n-1} \cdot k_n \varepsilon_n \cdot k_1 \int \frac{d^{4-\varepsilon} p}{(2\pi)^{4-\varepsilon}} \frac{1}{\prod_{i=1}^{n} (p + q_i)^2} \quad (8.24)
\]

where the integral is just a scalar integral. Replacing any \( V_F^+ \) with a \( V_F^- \) interchanges \( \varepsilon_i \leftrightarrow k_i \) and gives a minus sign. In this way the pure \( V_F^\pm \) terms reproduce all the pure cycles given by applying the cycle rules to the string kinematic factor.

But what about mixed terms? Since the \( V_B \) vertex contains an \( \eta_{\nu \rho} \) which contracts the two internal indices, there is no complicated mixing between the \( V_B \) and \( V_F \) terms. In this way all the remaining mixed terms can be reproduced.

At the loop level, the background Feynman gauge three-vertex in fig. 15 exhibits a considerable simplicity when compared to the Feynman gauge three-vertex in fig. 5. Firstly, there are only three terms in the background field vertex in fig. 15 as compared to five terms in the Feynman gauge vertex in fig. 5 (where we have used
the on-shell condition to eliminate one term in each case). This reduces the number of terms encountered in a Feynman diagram loop computation, eliminating many of the cancellations between terms in much the same way as the Gervais-Neveu gauge does for tree level Feynman diagrams. At loop level the background field vertex has the additional advantage over the Gervais-Neveu gauge since it contains loop momentum in only one of the three terms while the Gervais-Neveu vertex contains loop momentum in two terms. The simple way that the loop momentum appears in the vertex allows for the simple structure of the string-based loop rules and minimizes cancellations between terms. The background field gauge partially explains the efficiency of the string organization; one must also organize the algebra in the particular way discussed above in order to mimic the simple structure of the string organization of the amplitude. There is also no need to perform the usual steps of carrying out the Lorentz contractions, Feynman parameterizations and loop momentum integrals, as the results of these operations are contained in the string-based loop rules.

The way that four-point contact terms are matched in string theory is more complicated because some of these terms are tied up with the string trees. In general, the field theory description of the string tree rules is much more obscure than the loop rules. This is especially true when trees become complicated as contributions can be scattered between diagrams depending on the precise integration by parts used. However, there is no problem in principle to proceed with the match although it becomes increasingly tedious as pointed out in ref. [16]. The $\phi^3$ nature of the integrated-by-parts form of the string-based rules means that some of the four-point contact terms are to be found in the collapse of a tree pole by a numerator factor in the kinematic coefficient as depicted in fig. 17. One amusing result of integrating by parts in field theory is that surface terms cancel against remaining four-point contact interaction diagrams that are not of the form of the collapsed tree poles [16,53] leaving behind the $\phi^3$-like structure of the string-based rules.

![Fig. 17: Contact terms can be generated when a propagator is cancelled by a momentum invariant in the numerator.](image)

It is amusing that other gauge choices yield the same one-loop Feynman rules; in particular, a background field gauge version of the Gervais-Neveu gauge yields precisely the same one-loop effective action as the background field Feynman gauge. The gauge fixing term, $\text{Tr}[\left( (D^A \cdot Q + iQ^2/\sqrt{2})^2 \right) / 2$, generates extra terms only in the vertices with at least three quantum $Q$-fields. The only vertices contributing to
the one-loop effective action have two \( Q \)-fields so these extra terms are irrelevant at one loop as are the extra terms in the Faddeev-Popov determinant. The additional terms in the Gervais-Neveu background field gauge, however, might be of interest in understanding the multi-loop string organization of the amplitude, but this has not been investigated.

Having identified the mapping between the loops of the string-based rules and the loops of the background field gauge, one can verify that the identification of string theory dimensional regularization schemes in terms of field theory schemes included in the rules of Section 6 is correct. Although dimensional regularization schemes can be constructed in string theories \([18,2]\), in practical calculations it is essential to precisely identify the corresponding field theory regularization scheme. With the term-by-term mapping of string loops into background field gauge loops a direct comparison of regularizations schemes in string theory and field theory can be made. In field theory one can choose variations of dimensional regularization depending on how the external and internal states are handled. The match between external state prescriptions is rather simple; one simply uses the same prescription as one would use in field theory. For internal states, in field theory one must choose the value of \( \eta^I_{\mu\nu} \) resulting from a trace around a loop. As discussed above, such traces arise in background field Feynman gauge when only \( V_B \) vertices are used on all legs. Because of the similarity of the ghost \( V_G \) and gluon \( V_B \) vertices this quantity always appears in the combination \( \eta^I_{\mu\nu} - 2 \equiv 2R \). In a conventional scheme or \`t Hooft-Veltman scheme where \( \eta^I_{\mu\nu} = 4 - \epsilon \) we have \( R = 1 - \epsilon/2 \), while in a four-dimensional helicity scheme \([2]\) \( \eta^I_{\mu\nu} = 4 \) so that \( R = 1 \), in agreement with the string-based rules of Section 6.

A consequence of the mapping is that the string reorganization can now be applied to the fully off-shell one-loop effective action. This could be of practical interest for higher point amplitudes, since it allows for use of tree-level recursive techniques \([12,15]\) for the trees sewn onto the loop.

### 8.5 String Form of Effective Action

In the string-based rules there is a single master formula which can be used to describe any particle circulating in the loop. Can we mimic this in field theory? Normally in field theory, the diagrammatic structure of a scalar, fermion or gluon circulating in a loop is rather different. Here we explain how one can reorganize field theory so that these three cases look pretty much the same. A practical consequence is that the space-time supersymmetry relations between the various amplitudes become much more apparent, just as they do in string theory.

As we already discussed, the background field method plays a central role in the one-loop understanding of the string-based rules. At one-loop the background field Feynman gauge can be summarized by the one-loop determinant

\[
\Gamma_{\text{gluon}}[A] = \ln \det^{-1/2} \left[ D^2 \eta_{\mu\nu} - ig (\Sigma_{\mu\nu})_{\rho\sigma} F^{\rho\sigma} \right] \ln \det[D^2] \tag{8.25}
\]
where

\[(\Sigma_{\mu\nu})_{\rho\sigma} = i(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})\]  \hspace{1cm} (8.26)

is the generator of Lorentz transformations for a vector field and \(F_{\mu\nu}\) is the adjoint representation field strength matrix \((F_{\mu\nu})^{bc} = f^{abc}F_{\mu\nu}^a\). The first determinant in eq. (8.25) arises from the gaussian integral over the gauge fields while the second determinant is the one-loop Faddeev-Popov determinant. The \(D^2\) terms in the determinants generate terms corresponding to the terms generated by the substitution rule (6.6), while the \(F^{\rho\sigma}\) term generates terms corresponding to the cycle rules.

The scalar contribution, which matches the string form, is the usual effective action given by

\[\Gamma_{\text{scalar}}[A] = \ln \det^{-1/2}D^2 = -\frac{1}{2} \text{Tr} \ln D^2\]

(8.27)

where \(D_\mu = \partial_\mu + igT^aA^a_\mu\). By expanding out the trace, the one-loop diagrams representing a scalar in the loop are reproduced.

What about internal fermions? Once again the analysis for the non-cycle contributions of the \(\hat{G}_B\) terms are identical to the case of gluons in the loop. This indicates that the fermion determinant should also contain a \(D^2\). Thus, the expected form of the fermion determinant is the second order form

\[\Gamma_{\text{fermion}}[A] = \ln \det^{1/2}(D^2 - \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} + m^2)\]

(8.28)

where \(\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]\) is the generator of spinor Lorentz transformations. This amounts to a simple rewriting of the usual fermion determinant \(\det[\not{D} + im]\) as \(\det^{1/2}[\not{D}^2 + m^2]\). It is then not difficult to check that additional cycle contribution in the fermion substitution rule (6.12) applied to the kinematic coefficient (8.22) exactly reproduces those terms generated by the \(\sigma_{\mu\nu}F^{\mu\nu}\) term in the fermion determinant (8.28) after carrying out the trace over the \(\gamma\)-matrices.

The string organization of gluon, scalar and fermion contributions to the exponentiated one-loop effective action can then be summarized by the generic formula

\[\Gamma_{\text{state}}[A] \sim \ln \det^{\mp 1/2}[D^2 - \Sigma_{\rho\sigma}F^{\rho\sigma}]\]

(8.29)

where the operator \(\Sigma_{\rho\sigma}\) acts in the representation of the Lorentz algebra of the state. The universality of the \(D^2\) term in the determinant is a direct consequence of the fact that the string loop momentum integral for the \(n\)-gluon amplitude does not depend on the choice of states circulating in the loop. The structure of the remaining term follows from Lorentz and manifest gauge invariance of the effective action and the requirement that the three-vertex contains only a single power of momentum.

8.6 Applications to General Gauge Theories

The above field theory ideas can be applied to any gauge theory calculation which involve non-abelian vertices [16]. In the future extensions of the rules to
include external fermions, weak interactions and multi-loops can be expected, but in the meantime, the above ideas can be directly used by anyone wishing to do loop level Feynman diagram computations in non-abelian gauge theory.

The following strategy incorporates the ideas which were extracted from the mapping between field theory and string theory greatly improving the calculational efficiency over traditional Feynman diagram computations. First background field Feynman gauge [65,66] should be used in calculations where a non-abelian vertex appears in the loop. In this way a gauge invariant effective action is produced. For sewing trees onto the one-particle irreducible loop diagrams the Gervais-Neveu gauge [26] is a particularly efficient gauge because of the simplicity of the three- and four-point vertices. One should use color ordered [9] vertices in order to minimize the number of diagrams which must be explicitly computed. For internal fermions it is best to use the second order formalism because then the gauge boson and fermion contributions are quite similar so that a good fraction of the work does not have to be duplicated.

Spinor helicity methods [7,8] are also important to help minimize the amount of required algebra. Since spinor helicity methods do not handle off-shell loop momentum efficiently it should be integrated out early in the calculation to obtain a representation in terms of Feynman parameters. In order to minimize the number of terms which appear, spinor helicity should be applied on a term-by-term basis in the numerator as one integrates out loop momentum. An alternative approach which implicitly and systematically integrates out the loop momentum is the electric circuit analogy discussed by Lam [52,53]. In order to maintain the gains in efficiency obtained with the spinor helicity method one should use either the ’t Hooft-Veltman or four-dimensional helicity scheme, as the conventional scheme would undo much of the gain implicit in the spinor helicity method due to the extra \( \epsilon \)-helicities. The resulting Feynman parameter integrals can then be conveniently integrated using the method of ref. [61].

Although, in this way one can expect to greatly improve the efficiency over a traditional Feynman diagram computation, in general this type of approach cannot be expected to be as efficient as a more direct string approach. Gravity is a concrete example where further input from string theory provides further large improvements in computational efficiency [25]. It is also difficult to understand within a conventional field theory context the relatively compact multi-loop structure implied by string theory.

8.7 First Quantized Formalism

Starting from the one-loop determinants (8.25), (8.27) and (8.28), a one-loop first quantized formalism can be obtained [28,27] which mimics the structure of string theory. This provides a complementary description of the loop parts of the rules within a field theory context.
For the case of the scalar in the loop this is given by

\[ \Gamma_{\text{scalar}}[A] = \ln \det^{-1/2} D^2 = \mathcal{N} \int_0^\infty \frac{dT}{T} \int DX \exp \left[ - \int_0^T d\tau \left( \frac{1}{2} \dot{X}^2(\tau) - igA(\tau) \cdot \dot{X}(\tau) \right) \right] \]

where \( A \) is the gauge field and \( \mathcal{N} \) is an appropriate normalization. This path integral looks very much like the Polyakov string path integral, except that here the path integral is over world-lines instead of world-sheets.

By functionally differentiating with respect to \( A \) \( n \)-times, setting \( A \to 0 \) and then Fourier transforming the vertex operator form can be recovered

\[ \Gamma_n = \mathcal{N} \int_0^\infty \frac{dT}{T} \int DX \exp \left[ - \int_0^T d\tau \frac{1}{2} \dot{X}^2(\tau) \right] V_1 V_2 \cdots V_n \]

where the vertex operator is

\[ V_j = \int d\tau_j \varepsilon_j \cdot \dot{X}(\tau_j) e^{ik_j \cdot X(\tau_j)} \]

and \( \varepsilon_j \) is the polarization vector. One can then construct field theory Green functions on the circle which reproduce the field theory loop substitution rules.

The structure of space-time fermion and gluon loop contributions has also been worked out by Strassler in a first quantized formalism \[27\] based on the superstring construction.

The main advantage of this first quantized formalism is its simplicity in deriving the loop part of the rules. However, the non-abelian contact terms are not conveniently described by the master formula as they are in the full string-based formalism. Furthermore, tree parts of the rules have as yet not been obtained with first quantized methods. This makes the first quantized formalism useful for studying effective actions but not amplitudes. It may also provide a possible alternative path for extensions to multi-loops.

9. Summary and Conclusions

In these lectures a new method, based on superstring theory, for evaluating one-loop \( n \)-gluon amplitudes in perturbative QCD was discussed \[2,3\]. The method was originally derived by taking the field theory limit of an appropriately constructed \[22\] four-dimensional string theory \[21\]. It was first applied to reproduce, in a relatively simple way, the two-jet cross-section of Ellis and Sexton \[6\]. As a by-product a first calculation of all the four-gluon one-loop helicity amplitudes was also performed. More recently, using the string-based method, together with a simple integral table \[61\] and improvements in the spinor helicity method, a first calculation
of all five-gluon helicity amplitudes has been performed [4]. These amplitudes are required for the analysis of three-jet events at Fermilab.

The string-based method meshes naturally with spinor helicity methods since loop momentum does not appear in the initial expression, but to take full advantage of the gains one should use a version of dimensional regularization which leaves observed polarization vectors in four-dimensions, such as the ’t Hooft-Veltman or four-dimensional helicity schemes [2]. Another important ingredient is the color decomposition of the amplitude into smaller gauge-invariant partial amplitudes. The string also provides a systematic and compact expression for the n-point partial amplitude, eliminating many of the large cancellations inherent in traditional Feynman diagram computations.

In the string-based method the diagrams are obtained by applying certain substitution rules to the string ‘master formula’. In this way one obtains a set of Feynman parameter polynomials which are far more compact than one would obtain by traditional Feynman diagram methods. The master formula is the usual kinematic expression one obtains in string theory for the n-gluon amplitude. In the string-based method the master formula contains all information about all field theory diagrams and particle contents. Because the contribution of any type of particle is contained in the master formula, relationships between fermion and boson contributions become apparent within the integrands of each diagram. This can be used to obtain even further simplifications; once the fermion loop contribution to the n-gluon amplitude has been computed, calculating the gluon loop contribution is relatively simple [4,5].

The collection of conventional field theory ideas which describe many of simplifications of the string-based method are [16] color ordering [9,13,51], use of background field Feynman gauge [65] for the one-particle irreducible parts of diagrams, systematic organizations of the vertex algebra and a second order formalism for fermions. One can also use the Gervais-Neveu gauge [26] for the tree parts of calculations since this gauge has particularly simple vertices. (Within the background field method the different choices of gauge are made for the loop and tree parts of a diagram [66].) The spinor helicity method can also be used [7,8] to provide further simplifications. These ideas can be applied to more general calculations within a conventional field theory framework [16], such as ones which include external fermions, massive gauge bosons, or multi-loops.

Although one can obtain improvements in computational efficiency in field theory in this way, string theory goes beyond this naive application of known field theory ideas. String theory provides a guiding principle for finding compact organizations; as yet, no corresponding principle has been found within conventional field theory. In particular, obtaining a compact string-like organization for gravity is fairly straightforward by directly using string theory, but rather obscure in field theory due to the non-trivial field redefinitions needed to mimic the simple string reorganization [25]. String theory also implies that the compact structure of the
one-loop ‘master formula’ should continue to hold to multi-loops; it is not clear how
one would obtain this in conventional field theory (without looking at string theory
to some extent).

There are a number of other extensions of the string motivated techniques.
One important area which was not discussed in these lectures is external fermions.
In string theory, external fermions are technically more involved than the case of
external bosons. Progress has however been made for this case and an explicit test
calculation of $q\bar{q} \rightarrow gg$ has been performed based on string theory [36]. The methods
can also be applied to certain weak interaction processes [67]. Some progress has
also been made on the extension of these methods to multiloops [68].

In summary, there is every reason to believe that string theory will continue to
be as helpful for gauge theory perturbative computations as it has been in the past.

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