DEGENERATION OF SCHUBERT VARIETIES OF $SL_n/B$
TO TORIC VARIETIES

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Abstract. Using the polytopes defined in an earlier paper, we show in this paper the existence of degeneration of a large class of Schubert varieties of $SL_n$ to toric varieties by extending the method used by Goncuilea and Lakshmibai for a miniscule $G/P$ to Schubert varieties in $SL_n$.

Introduction

In this paper, we complete our programme stated in [2] to prove the existence of degenerations of certain Schubert varieties of $SL_n$ into toric varieties, thus generalizing the results of Goncuilea and Lakshmibai [4]. For example, we are able to settle all the Schubert varieties in $SL_3$ here.

The essential idea is that we use the polytopes defined in [2] to construct a distributive lattice, and extend the method used by Goncuilea and Lakshmibai [4] for miniscule $G/P$ to Schubert varieties in $SL_n$. Although they also prove the existence of degenerations for $SL_n/P$ (and also Kempf varieties) in the same paper, their approach is different from the one for a miniscule $G/P$.

Since all the ingredients used here are based on standard monomials, we expect that it can be adapted in the other types. However, the difficult part is to construct a suitable distributive lattice and we shall make it more precise below.

Let $G = SL_{n+1}$, $B$ be a Borel subgroup and $W$ be the Weyl group of $G$ which is the symmetric group of $n + 1$ letters. Let $\alpha_i$, $i = 1, \ldots, n$, be the corresponding set of simple roots so that $\langle \alpha_i, \alpha_j^\vee \rangle = a_{i,j}$ where $(a_{i,j})_{i,j}$ is the Cartan matrix, $s_i$ the corresponding simple reflections in $W$ and let $\omega_i$ be the corresponding fundamental weights. Denote also by $\ell(\cdot)$ and $\preceq$ the length function and the Bruhat order on $W$.

Recall that for $w \in W$, the Demazure module $E_w(\lambda)$ is the $b$-module $U(b)v_{w\lambda}$, where $b$ is the Lie algebra of $B$, $U(b)$ its enveloping algebra and $v_{w\lambda}$ a vector of extremal weight $w\lambda$ of the irreducible representation $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^n k_i\omega_i$, $k_i \geq 0$. Under certain conditions on $w$, in [2] we constructed $n$ polytopes $\Delta_1, \ldots, \Delta_n$, where $n$ is the rank of $G$, such that the number of lattice points in the Minkowski...
Theorem of degeneration of $[4]$ is stated in section 2. Sections 3, 4 and 5 are devoted to showing that the conditions of the theorem are satisfied. Finally in section 6, we discuss briefly which Schubert varieties fall into our context.

We shall use the above notations throughout this paper.

1. Distributive lattice on $\mathcal{W}^w$

For a fundamental weight $\omega_i$, $i = 1, \ldots, n$, let $W_{\omega_i}$ be the subgroup of the Weyl group $W$, stabilizing $\omega_i$, that is $W_{\omega_i} = \{ \tau \in W \mid \tau(\omega_i) = \omega_i \}$. Denote the quotient $W/W_{\omega_i}$ by $W_i$. The set $W_i$ can, on the one hand, be identified with the subset of $W$ consisting of elements $\tau$ such that $\tau \leq \tau s_{\alpha_j}$ for
In [2], we constructed for each fundamental weight ω_i the set of minimal representatives and, on the other hand, with the set of i-tuples (r_1, ..., r_i) such that 0 ≤ r_1 < · · · < r_i ≤ n. The connection between these two identifications is that (r_1, ..., r_i) corresponds to s(r_1,1)s(r_2,2) · · · s(r_i, i) where s(a, b) = s_a s_{a-1} · · · s_b. The induced Bruhat order on W_i, which we shall also denote by ≤, can be expressed under the above identifications by a = (a_1, ..., a_i) ≤ b = (b_1, ..., b_i) if and only if a_k ≤ b_k, 1 ≤ k ≤ i. Furthermore, W_i becomes a distributive lattice (for generalities on distributive lattices, see [5] or section 2 of [4]) under ≤,

\[ \overline{\phi} := (0,1,\ldots ,j-i-1,\bar{r}_j-i+1,\ldots ,\bar{r}_j) \in W^w_j \]

where \( \bar{r}_k = \max \{ k-1, r_{k-j+i} \} \), \( j-i+1 \leq k \leq j \) and for \( \tau = (t_1, \ldots , t_j) \) in \( W^w_j \), let

\[ \bar{\tau} := (t_{j-i+1}, \ldots , t_j) \in W^w_i. \]

We say that \( \phi \preceq_w \tau \) if \( \bar{\phi} \preceq \bar{\tau} \), or equivalently if \( \phi \preceq \bar{\tau} \), and we define \( \tau \vee \phi := \tau \vee \bar{\phi} \in W^w_j \) and \( \tau \wedge \phi := \bar{\tau} \wedge \phi \in W^w_i \) (see Eq. 1.1).

A simple consequence of the definition is the following lemma.

**Lemma 1.2.** Let w be as in definition 1.4. Then together with the above operations, \( W^w \) is a distributive lattice.

An essential property of this partial order is the following theorem proved in [2].

**Theorem 1.3.** We have \( \phi \preceq_w \tau \) in \( W^w \) if and only if there exist liftings \( \phi' \), \( \tau' \) in \( W \) of \( \phi, \tau \) such that \( \phi' \preceq \tau' \preceq w \).

As we shall see in the next sections, this is used extensively in the proof.

**Remark 1.4.** In [2], we constructed for each fundamental weight \( \omega_i \), a polytope \( \Delta^w_i \) such that the number of lattice points in the Minkowski sum \( \sum_{i=1}^n k_i \Delta^w_i \) is equal to \( E_w(\sum_{i=1}^n k_i \omega_i) \). The set of vertices of the polytope \( \Delta_i \) is indexed by the set \( W^w_i \) and these are the only lattice points of \( \Delta_i \). Moreover considering \( \phi, \tau \in W^w \) as vertices, we have \( \phi + \tau = \phi \vee \tau + \phi \wedge \tau \).

The polytopes \( \Delta_i \) have also the important property that they can be triangulated by simplices of minimal volume so that a lattice point of \( \sum_{i=1}^n k_i \Delta_i \) can be written as the sum of \( k_1 \) lattice points of \( \Delta_1 \) and \( k_2 \) lattice points...
of $\Delta_2$ and so on. This property gives information on the generators of the toric ideal defined by the $\Delta_i$.

We shall end this section by proving certain facts concerning $\tau \lor \phi$ and $\tau \land \phi$ which will be needed throughout the paper. These are generalizations of certain results obtained in [4]. Let us suppose that $w$ is as in definition [4].

Lemma 1.5. Let $j \geq i$ and $\phi \in W_i^w$, $\tau \in W_j^w$ be two non-comparable elements in $W^w$. Let $\sigma = \tau \lor \phi$ and $\kappa = \tau \land \phi$. Then $\tau(\omega_j) + \phi(\omega_i) = \sigma(\omega_j) + \kappa(\omega_i)$.

Proof. This is just a direct consequence of the fact that $\phi + \tau = \phi \lor \tau + \phi \land \tau$ in the polytope described in remark 1.4, see [4].

It is also a straightforward computation by using the fact that if $\tau = (t_1, \ldots, t_j)$, then

$$\tau(\omega_j) = \omega_j - \sum_{k=1}^{j} (\alpha_k + \cdots + \alpha_{t_k}).$$

Lemma 1.6. Let $j \geq i$ and $\phi \in W_i^w$, $\tau \in W_j^w$ with $\sigma = \tau \lor \phi$ and $\kappa = \tau \land \phi$. Then, we have the following

1. if $s_{i_1} \cdots s_{i_k} \tau = \sigma$ and $\ell(\sigma) = \ell(\tau) + k$, then $s_{i_1} \cdots s_{i_k} \tilde{\kappa} = \tilde{\phi}$ with $\ell(\tilde{\kappa}) + k = \ell(\tilde{\phi})$; or equivalently $s_{i_1} \cdots s_{i_k} \kappa = \phi$.
2. if $s_{j_1} \cdots s_{j_l} \phi = \sigma$ with $\ell(\sigma) = \ell(\tilde{\phi}) + l$, then $s_{j_1} \cdots s_{j_l} \tilde{\kappa} = \tau$ with $\ell(\tilde{\kappa}) + l$.
3. The sets $\{\alpha_i\}$ and $\{\alpha_j\}$ has empty intersection and $s_{i_p}, s_{j_q}$ commute.

Proof. Note that as a consequence of Definition 1.1, we have $\sigma = \tau \lor \tilde{\phi}$ and $\tilde{\kappa} = \tau \land \tilde{\phi}$. Using lemmas 7.17 and 7.18 of [4], we conclude that there exist $\alpha_{i_1}, \ldots, \alpha_{i_k}$ and $\alpha_{j_1}, \ldots, \alpha_{j_l}$ all simple enjoying the properties stated above.

2. Theorem on Degeneration

Let us recall some basic facts on standard monomials.

Let $\phi \in W_i$ and $\tilde{\phi} = s_{i_1} \cdots s_{i_k}$ be a reduced expression for $\phi$. Then the vector $Q_\phi := X_{-\alpha_{i_1}} \cdots X_{-\alpha_{i_k}} v_{\omega_i}$ is an extremal weight vector in $V(\omega_i)$ of weight $\phi(\omega_i)$. It is shown in [7] that $Q_\phi$ is independent of the choice of reduced expression of $\phi$. Further, we have the following lemmas from [7]:

Lemma 2.1. The set $\{Q_\tau \mid \tau \in W_i, \tau \preceq w\}$ is a $\mathbb{Z}$-basis for $E_{\mathbb{Z}, w}(\omega_i)$.

Let $\{P_\tau \mid \tau \in W_i\}$ be the $\mathbb{Z}$-basis of $V_{\omega_i}^\circ$ dual to $\{Q_\tau \mid \tau \in W_i\}$. Then the set $\{P_\tau \mid \tau \in W_i, \tau \preceq w\}$ is a $\mathbb{Z}$-basis for $H^0(S^r(S^r_{\omega_i}), L_{\omega_i}) = E_{\mathbb{Z}, w}^r(\omega_i)$.

Lemma 2.2. Let $\sigma \succ \kappa \in W_i$ and $\sigma = s_{i_1} \cdots s_{i_k} \kappa$ and $\ell(\sigma) - \ell(\kappa) = r$. Then we have $P_\kappa = (-1)^r X_{-\alpha_{i_1}} \cdots X_{-\alpha_{i_k}} P_\sigma$.  

For a field $k$, let us denote the canonical image of $P_w$ in $H^0(G/P_i, L_{\omega_i})$ by $p_w$, $w \in W_i$.

**Definition 2.3.** A monomial $p_{\tau, k_r} \cdots p_{\tau, 1} p_{\tau, -1, k_r} \cdots p_{\tau, 1}$, where $\tau, j \in W_i^{w_k}$, is called homogeneous of degree $(k_1, \ldots, k_r)$ and of total degree $\sum_{j=1}^r k_j$.

It is called standard on $S(w)$ if for each $i, j$ there exists $\tau, j \in W$, whose class in $W_i$ is $\tau, j$, and $\tau, 1 \preceq \cdots \preceq \tau, k_r \preceq w$ in $W$. In other words, $p_{\tau, k_r} \cdots p_{\tau, 1}$ is standard on $S(w)$ if $\tau, 1 \preceq w \cdots \preceq w \tau, k_r \preceq w$.

**Theorem 2.4.**

1. Let $w \in W$. Then, denoting $\overline{w}$ the representative of $w$ in $W_i$, for $\tau \in W_i$, $p_\tau |_{S(\overline{w})} \neq 0$ if and only if $\tau \preceq \overline{w}$. Furthermore, $\{p_\tau | \tau \in W_i^{w_k}\}$ is a $k$-basis for $H^0(S(\overline{w}), L_{\omega_i})$.

2. The standard monomials on $S(w)$ of degree $(k_1, \ldots, k_n)$ form a basis of $H^0(S(w), \bigotimes_{i=1}^n L^{\omega_i})$.

Let $H$ be a finite distributive lattice. Denote by $P = \langle x_\alpha, \alpha \in H \rangle$ and $I(H) \subset P$ the ideal generated by the binomials $\{x_\alpha x_\beta - x_\alpha \wedge x_\beta, \alpha, \beta \in H\}$ non-comparable.

Let $R = \bigoplus_{\lambda \text{dominant}} H^0(S(w), L_\lambda)$ be the homogeneous coordinate ring of a multicone over $S(\overline{w})$. By the previous theorem, $R$ has a basis indexed by standard monomials on $S(w)$. Thus we have the surjective map $\pi: P \to R$ sending $x_\alpha \mapsto p_\alpha$ where we have $H$ is the set of standard monomials on $S(w)$. Let $I = \ker \pi$ which is an ideal generated by relations in total degree 2 of the form

$$p_\tau p_\phi - \sum C_{\theta \psi} p_{\theta \psi} p_{\phi}$$

where $p_\tau p_\phi$ is non standard and the $p_{\theta \psi} p_{\phi}$'s are standard. These are called straightening relations.

**Theorem 2.5.** Assume that $W^w$ is a distributive lattice such that the ideal $I$ is generated by the straightening relations of the form

$$p_\tau p_\phi - \sum C_{\theta \psi} p_{\theta \psi} p_{\phi}$$

where $\tau, \phi$ are non-comparable and $\theta \succeq \psi$. Further, suppose that we have

1. $c_{\tau \vee \phi, \tau \wedge \phi} = 1$, i.e. $p_{\tau \vee \phi} p_{\tau \wedge \phi}$ occurs on the right-hand side of eq. 2.2 with coefficient 1.

2. $\tau, \phi \in \psi, \theta = \{\gamma \in W_w | \psi \leq \gamma \leq \theta\}$ for every pair $(\theta, \psi)$ appearing on the right-hand side of eq. 2.2.

3. There exist integers $n_1, \ldots, n_d \geq 1$ and an embedding of distributive lattices

$$\iota: W^w \hookrightarrow \bigcup_{d=1}^n C(n_1, \ldots, n_d)$$

where $C(n_1, \ldots, n_d)$ is the set of $d$-tuples $(i_1, \ldots, i_d)$ with $1 \leq i_j \leq n_j$, such that for every pair $(\theta, \psi)$ appearing on the right-hand side of eq. 2.2, $\iota(\tau) \cup \iota(\phi) = \iota(\theta) \cup \iota(\psi)$ where $\cup$ denotes the disjoint union.
Then there exists a flat deformation whose special fiber is \( P/I(W^w) \) and whose general fiber is \( R \).

By lemma 3.2, if \( w \) is as in definition 3.1, then \( W^w \) is a distributive lattice. In the next sections, we shall prove that all the conditions of the theorem are satisfied. Let us assume in the next sections that \( w \) is as in definition 3.1.

3. Condition (2) of Theorem 2.5

**Theorem 3.1.** ([8], [4]) Let \( i \leq j, \tau \in W^w_j, \phi \in W^w_i \) and \( p_\tau p_\phi \) be a non standard monomial on \( S(w) \). Let the corresponding straightening relation be given by

\[
p_\tau p_\phi = \sum_{l=1}^{N} c_l p_{\theta_l} p_{\psi_l}.
\]

Then \( \tau, \phi \prec_w \theta_1, \psi \prec_w \tau, \phi \) for all \( l \) such that \( c_l \neq 0 \).

**Proof.** The proof given here is just a generalization of the proof of proposition 2.5 of [3]. Among the \( \theta_l \), choose a minimal one, which we denote by \( \theta \). Let us reindex the \( \theta_l \) so that \( \theta = \theta_l \) for \( 1 \leq l \leq s \). Note that since \( \theta \) is minimal we have \( \theta_l \nmid \theta \) for \( s < l \leq N \). Since \( p_\theta p_\psi \) is standard, we can choose \( \kappa_1^{(l)}, \kappa_2^{(l)} \in W \) such that \( \kappa_2^{(l)} \leq \kappa_1^{(l)} \leq w \), the class of \( \kappa_1^{(l)} \) in \( W \) is \( \theta_l \) and the class of \( \kappa_1^{(l)} \) in \( W_i \) is \( \psi_l \). Let \( Z_1 = \bigcup_{l=1}^{s} S(\kappa_1^{(l)}) \) and restrict eq. 3.1 to \( Z_1 \). Then \( p_{\theta_l} p_{\psi_l} |_{Z_1} \) is standard on \( Z_1 \) for \( 1 \leq l \leq s \) and \( p_{\theta_l} p_{\psi_l} |_{Z_1} = 0 \) for \( s < l \leq N \). By the linear independence of standard monomials, eq. 3.1 restricted to \( Z_1 \) is not zero. Hence \( p_\tau p_\phi |_{Z_1} \neq 0 \). This implies that \( \tau, \phi \prec_{w} \kappa_1^{(l)} \).

According to Theorem 1.3 (or Lemma 8.12 of [2]) we have \( \tau, \phi \prec_w \theta \); note that \( \tau \) (or \( \phi \)) can not be equal to \( \theta \), because \( p_\tau p_\phi \) is non standard. From this argument we deduce that \( \tau, \phi \prec_{w} \theta_l \) for all \( l \).

Let \( \sigma = \tau \lor \phi \in W^w_j \) and \( \kappa = \tau \land \phi \in W^w_i \). Now \( \theta_l \in W^w_j \) and \( \psi_l \in W^w_i \). By weight consideration, we have \( \sigma(\omega_j) + \kappa(\omega_i) = \theta_l(\omega_j) + \psi_l(\omega_i) \). Furthermore \( \tau, \phi \prec_{w} \theta_l \) implies that \( \sigma \leq_{w} \theta_l \), or equivalently \( \sigma \leq \theta_l \) since both belong to \( W_j \). Therefore \( \theta_l(\omega_j) \leq \sigma(\omega_j) \), which implies that \( \kappa(\omega_i) \leq \psi_l(\omega_i) \). Therefore \( \psi_l \preceq \kappa \). In other words \( \psi_l \preceq_w \kappa \preceq_w \tau, \phi \). 

**Corollary 3.2.** Let the notations be as in Lemma 3.1. Then in the straightening relation \( p_\tau p_\phi = \sum c_{\theta \phi} p_{\theta} p_{\psi} \), either \( \sigma \preceq_w \theta \) or \( \theta = \sigma, \psi = \kappa \).

**Proof.** From Theorem 3.1, we know that for any pair \( (\theta, \psi) \) on the right-hand side, \( \sigma \preceq_w \theta \) and \( \psi \preceq_w \kappa \). Moreover if \( \sigma = \theta \), then due to weight considerations, i.e. \( \theta(\omega_j) + \psi(\omega_i) = \sigma(\omega_j) + \kappa(\omega_i) \), we see that \( \kappa = \psi \). 

4. Condition (3) of Theorem 2.5

Considering the set \( W^w := \prod_{i=1}^{n} W^w_i \), we noted at the beginning of Section 3 that each set \( W^w_i \) can be identified with the subset of \( i \)-tuples \( (a_1, \ldots, a_i) \) where \( 0 \leq a_1 < \cdots < a_i \leq n \) and \( (a_1, \ldots, a_i) \) is smaller than
the representative of \( w \) in \( W_i \). Hence we have \( \iota : W^n \hookrightarrow \bigcup_{k=1}^{n_d} C(n_1, \ldots, n_d) \). For simplicity, we shall denote \( \iota(\tau) \) also by \( \tau \). We want to prove the following lemma:

**Lemma 4.1.** Let \( \tau, \phi \) be two non-comparable elements in \( W^n \). Then for any \((\theta, \psi)\) appearing on the right-hand side of the straightening relation \[2.2\], \( \theta \cup \psi = \tau \cup \phi \).

**Proof.** Let \( \tau = (t_1, \ldots, t_j), \phi = (r_1, \ldots, r_i), \theta = (a_1, \ldots, a_j) \) and \( \psi = (b_1, \ldots, b_i) \). A necessary condition for \( \rho \rho \varphi \) to appear on the right-hand side of eq. \[2.2\] is \( \tau(\omega_j) + \phi(\omega_i) = \theta(\omega_j) + \psi(\omega_i) \). Here, we shall prove that this condition immediately implies the assertion, i.e. \( \{t_1, \ldots, t_j\} \cup \{r_1, \ldots, r_i\} = \{a_1, \ldots, a_j, b_1, \ldots, b_i\} \). The proof is by induction on \( i + j \).

The fact that \( \tau(\omega_j) + \phi(\omega_i) = \theta(\omega_j) + \psi(\omega_i) \) implies (see the proof of lemma \[1.5\])

\[
\sum_{k=1}^{j} (\alpha_k + \cdots + \alpha_{t_k}) + \sum_{l=1}^{i} (\alpha_l + \cdots + \alpha_{r_l}) = \\
\sum_{k=1}^{j} (\alpha_k + \cdots + \alpha_{a_k}) + \sum_{l=1}^{i} (\alpha_l + \cdots + \alpha_{b_l})
\]

(4.1)

Note that

\[
\max\{t_1, \ldots, t_j, r_1, \ldots, r_i\} = \max\{t_j, r_i\}
\]

and that

\[
\max\{a_1, \ldots, a_j, b_1, \ldots, b_i\} = \max\{a_j, b_i\}.\]

Then due to the equality in eq. \[4.1\], we must have \( \max\{t_j, r_i\} = \max\{a_j, b_i\} \).

There are four cases to consider.

- **Case (1)** \( t_j = a_j \geq b_i \). This implies that \( \alpha_j + \cdots + \alpha_{t_j} = \alpha_j + \cdots + \alpha_{a_j} \).

Hence denoting \( \tau' = (t_1, \ldots, t_{j-1}) \) and \( \theta' = (a_1, \ldots, a_{j-1}) \), eq. \[1.1\] implies that \( \tau'(\omega_{j-1}) + \phi(\omega_i) = \theta'(\omega_{j-1}) + \psi(\omega_i) \). By induction we are done.

- **Case (2)** \( t_j = b_j > a_j \). Let \( m \) be the smallest number such that \( a_{j-m} > b_{i-m} \) (if such an \( m \) less than \( i-1 \) does not exist, let \( m = i \)).

Note that

\[
b_{i-m+1} \geq a_{j-m+1} > a_{j-m} > b_{i-m}.
\]

Set

\[
\tau' = (t_1, \ldots, t_{j-1}) \in W_{j-1},
\]

\[
\theta' = (a_1, \ldots, a_{j-m}, b_{i-m+1}, b_{i-m+2}, \ldots, b_{i-1}) \in W_{j-1},
\]

\[
\psi' = (b_1, \ldots, b_{i-m}, a_{j-m+1}, a_{j-m+2}, \ldots, a_j) \in W_i \text{ if } m \neq i \text{ and }
\]

\[
\psi' = (a_{j-i+1}, \ldots, a_j) \text{ if } m = i.
\]

Since \( i \leq j \), we have \( i - k - 1 \leq j - k - 1 \leq a_j - k \) for \( 0 \leq k < i \). Therefore \( \psi' \in W_i \). Using the fact that for \( 0 \leq k < m \), we have \( i - k - 1 \leq j - k - 1 \leq a_j - k \leq b_{i-k} \), then

\[
(\alpha_{j-k} + \cdots + \alpha_{a_{j-k}}) + (\alpha_{i-k} + \cdots + \alpha_{b_{i-k}}) = \\
(\alpha_{j-k} + \cdots + \alpha_{a_{j-k}} + \alpha_{a_{j-k+1}} + \cdots + \alpha_{b_{i-k}}) + (\alpha_{i-k} + \cdots + \alpha_{a_{j-k}})
\]

(4.2)
From eqs. 1.1 and 4.2, we can conclude that \(\tau'(\omega_j - 1) + \phi(\omega_i) = \theta(\omega_j) + \psi(\omega_i)\). The rest follows by induction.

- Case (3) \(r_i = b_i \geq a_j\) is similar to case (1).
- Case (4) \(r_i = a_j > b_i\) is similar to case (2). \(\square\)

In fact, we have proved:

**Lemma 4.2.** Let \(j \geq i, \tau, \theta \in W_j, \phi, \psi \in W_i\) be such that \(\tau(\omega_j) + \phi(\omega_i) = \theta(\omega_j) + \psi(\omega_i)\). Then \(\theta \cup \psi = \tau \cup \phi\).

5. **Condition (1) of Theorem 2.5**

**Proposition 5.1.** Let \(\tau, \phi \in W^w\) be two non-comparable elements. Then in the straightening relation \([\tau, \phi]\), \(p_{\tau \cup \phi} p_{\tau \cap \phi}\) occurs with coefficient \(\pm 1\).

**Proof.** As before, denote \(\sigma = \tau \vee \phi, \chi = \tau \wedge \phi\). Note that \(\tau, \phi \prec_w \sigma\) (that is there exist liftings \(\tilde{\tau}, \tilde{\phi}, \tilde{\sigma}\) in \(W\) such that \(\tilde{\tau}, \tilde{\phi} \preceq \tilde{\sigma} \preceq w\)). Corollary 3.3 implies that the restriction of eq. 3.2 to the Schubert variety \(S(\tilde{\sigma})\) is \(p_{\tau} p_{\phi} = a p_{\sigma} p_{\kappa}\), where \(a \neq 0\). Since standard monomial basis is characteristic free, this holds in any characteristics. Hence \(a = \pm 1\). \(\square\)

So far we have to prove that \(a = 1\). Since the irreducible representation \(V(\omega_i + \omega_j)\), appears as a direct sum in the decomposition in \(V(\omega_j) \otimes V(\omega_i)\) into irreducible representations, we have an imbedding \(V(\omega_i + \omega_j) \hookrightarrow V(\omega_i) \otimes V(\omega_i)\). Note that since the weight space of weight \(\omega_i + \omega_j\) is one-dimensional, the element \(u_{\omega_i} \otimes v_{\omega_j}\) belongs to \(V(\omega_i + \omega_j)\). The imbedding above induces a projection \(H^0(G/B, L_{\omega_i}) \otimes H^0(G/B, L_{\omega_j}) \to H^0(G/B, L_{\omega_i} \otimes L_{\omega_j})\). For simplicity we shall denote the image of \(f \otimes g\) under this projection by \(fg\).

We shall construct a basis for \(E_{\varphi, w}(\omega_i + \omega_j)\) which is a “rank two” version of the one given in [5].

In the following let \(i \leq j\) (that is no element of \(W_i^w\) can be bigger than an element of \(W_i^w\) and recall from lemma 2.1 that, for \(\phi \in W_i\), we have denoted by \(Q_{\phi}\) an extremal weight vector in \(V_\varphi(\omega_i)\) of weight \(\phi(\omega_i)\).

Let \(\Sigma(w) := \{(\tau, \sigma) \in W_j \times W_i \mid \text{there exist liftings } \tilde{\tau}, \tilde{\sigma} \in W \text{ such that } \tilde{\sigma} \preceq \tilde{\tau} \preceq w\}\).

**Definition 5.2.** Let \(w\) be as in Definition 1.1. Let \(\kappa \in W_j^w, \sigma \in W_i^w\) be such that \((\sigma, \kappa) \in \Sigma(w)\) and let \(\sigma = s_i \cdots s_i \tilde{k}\) where \(r = \ell(\sigma) - \ell(\kappa)\). Define \(E_{\bar{k}, \kappa} := Q_{\kappa} \otimes Q_{\kappa} \in V_\varphi(\omega_j) \otimes V_\varphi(\omega_i)\) and define \(E_{\sigma, \kappa} := X_{-\alpha_i} \cdots X_{-\alpha_i} E_{\bar{k}, \kappa}\).

Note that \(E_{\bar{k}, \kappa}\) is an extremal weight vector since \(\tilde{k}\) is the image of \(\kappa\) (the minimal representative in \(W\)) in \(W_j\). It is also clear that \(E_{\sigma, \kappa}\) is a weight vector of weight \(\kappa(\omega_i) + \sigma(\omega_j)\).

**Proposition 5.3.** Let \(w \in W\) be as in definition 1.1. Then \(E_{\sigma, \kappa}\) does not depend on the choice of reduced expression and the set \(\{E_{\sigma, \kappa} \mid \kappa \in W_j^w, \sigma \in W_i^w, \kappa \preceq_w \sigma\}\) is a \(\mathbb{Z}\)-basis for the Demazure module \(E_{\varphi, w}(\omega_i + \omega_j)\).
Proof. Let $\sigma = s_{i_r} \cdots s_{i_1} \bar{k} = s_{j_r} \cdots s_{j_1} \bar{k}$. Denote by $\phi = s_{j_{r-1}} \cdots s_{j_1} \bar{k}$. Then we have $\sigma = s_{j_r} \phi$. Now if $i_r = j_r$, then we proceed by induction on the length of $\sigma$. Otherwise, let $k$ be the largest integer such that $j_r = i_k$. We have $\phi \vee \bar{k} = \sigma$, thus by lemma 1.8, we have that $s_{j_r}$ commute with $s_{i_l}$ for $l \geq k$. Thus

$$X_{-\alpha_{i_r}} \cdots X_{-\alpha_1} E_{\bar{k}, \bar{\kappa}} = X_{-\alpha_{i_k}} X_{-\alpha_{i_{r-1}}} \cdots X_{-\alpha_{i_{k+1}}} X_{-\alpha_{i_{k-1}}} \cdots X_{-\alpha_1} E_{\bar{k}, \bar{\kappa}}$$

By induction, $E_{\phi, \kappa} = X_{-\alpha_{i_{r-1}}} \cdots X_{-\alpha_1} E_{\bar{k}, \bar{\kappa}}$. Therefore the right-hand side is $X_{-\alpha_{j_r}} E_{\phi, \kappa}$ and we have proved that the definition of $E_{\sigma, \kappa}$ does not depend on the choice of the reduced expression.

We are left to show that these elements form a basis for $E_{Z, w}(\omega_i + \omega_j)$.

We claim that $E_{\sigma, \kappa} \in E_{Z, w}(\omega_i + \omega_j)$. It is clear that $E_{\bar{k}, \bar{\kappa}} \in E_{Z, \kappa}(\omega_i + \omega_j)$. Now, since $w$ satisfies the condition of definition 1.1, we have $w \geq s_{i_r} \cdots s_{i_1} \kappa$, thus

$$E_{\sigma, \kappa} \in X_{-\alpha_{i_r}} \cdots X_{-\alpha_1} E_{Z, \kappa}(\omega_i + \omega_j) \subset E_{Z, w}(\omega_i + \omega_j).$$

We have therefore our claim.

Now by the definition of $E_{\sigma, \kappa}$, we have

$$E_{\sigma, \kappa} = Q_\sigma \otimes Q_\kappa + \sum_{(u,v) \in I} Q_u \otimes Q_v$$

where $I \subset W_j \times W_i$ and for each $(u,v) \in I$, we have $u < \sigma$ in $W_j$, $v \succ \kappa$ in $W_i$ and $\sigma(\omega_j) + \kappa(\omega_i) = u(\omega_j) + v(\omega_i)$. It is now clear that the $E_{\sigma, \kappa}$'s are independant.

Further, one deduces from the expression for $E_{\sigma, \kappa}$ above that the $Z$-submodule generated by the $E_{\sigma, \kappa}$'s is a direct summand of the tensor product $V_Z(\omega_j) \otimes V_Z(\omega_i)$. Finally, by standard monomial theory, the cardinal of $\Sigma(w)$ is the rank of $E_{Z, w}(\omega_i + \omega_j)$. So the result follows. \hfill \Box

We can now prove that $a = 1$.

Corollary 5.4. Let the notations be as in Lemma 1.3, then in the straightening relation $p_{\tau} p_{\phi} = \sum_{l=1}^N \alpha_l p_{\theta_l} p_{\psi_l}$, the term $p_{\tau} p_{\phi}$ occurs on the right hand side with coefficient 1.

Proof. Recall from the proof of 5.3 that

$$E_{\sigma, \kappa} = Q_\sigma \otimes Q_\kappa + \sum_{(u,v) \in I} Q_u \otimes Q_v$$

where $I \subset W_j \times W_i$ and for each $(u,v) \in I$, we have $u < \sigma$ in $W_j$, $v \succ \kappa$ in $W_i$ and $\sigma(\omega_j) + \kappa(\omega_i) = u(\omega_j) + v(\omega_i)$.

Let us apply $p_{\tau} p_{\phi}$ to $E_{\sigma, \kappa}$. Then from the explicite expression of $E_{\sigma, \kappa}$ above, this is either 0 or 1 depending if $Q_{\tau} \otimes Q_{\phi}$ appears in the right hand side or not.

On the other hand, if we replace $p_{\tau} p_{\phi}$ by the right hand side of the straightening relation, then it is clear from Theorem 3.1 that the same evaluation yields $a_{\sigma, \kappa}$. But this is non zero from Proposition 5.1. So it must be 1. \hfill \Box
6. Consequence

As an immediate consequence, we have:

**Theorem 6.1.** Let \( w \) be as in definition 1.1. Then there exists a flat deformation whose special fiber is a toric variety and whose general fiber is \( S(w) \).

**Proof.** By theorem 2.4, there exists a flat deformation whose general fiber is \( S(w) \) and whose special fiber is a variety defined by a binomial ideal associated to a distributive lattice. This latter is toric as shown in [3].

**Remark 6.2.** If we look closely at the proofs, then we realize that theorem 2.4 can be replaced by the following.

Suppose that \( W^w \) admits a structure of distributive lattice such that

1. the partial order corresponds to standardness, cf theorem 1.3;
2. weights are preserved, cf lemma 1.5;
3. lemma 1.6 is satisfied.

Then there exists a flat deformation whose special fiber is a toric variety and whose general fiber is \( S(w) \).

In particular, consider the bijection \( \Theta \) of \( W \) defined by \( s_i \mapsto s_{n+1-i} \) induced by the non-trivial Dynkin diagram automorphism. This induces a bijection between \( W \) and \( W_{n+1-i} \) which preserves the induced Bruhat order.

Now let \( w \) be as in Definition 1.1, then \( \Theta \) induces a structure of distributive lattice on \( W^\Theta(w) \). It is easy to check that the same proof works. Thus we have,

**Theorem 6.3.** Let \( w \) or \( \Theta(w) \) be as in definition 1.1. Then there exists a flat deformation whose special fiber is a toric variety and whose general fiber is \( S(w) \).

**Remark 6.4.** As noticed in [2], we can extend our results to the following elements. Let \( 0 \leq k_1 < k_2 < \cdots < k_{r+1} \leq n+1 \), and for \( 1 \leq i \leq r \), let \( S_i \) be the subgroup of \( W \) generated by the reflections \( s_{k_{i+1}}, \ldots, s_{k_{i+1}+1-1} \).

Now suppose that \( w = w_1 \cdots w_r \) where \( w_i \in S_i \). Then it is clear that \( w_i \) and \( w_j \) commute if \( i \neq j \) and it follows easily that if each \( w_i \) satisfies the condition of theorem 6.3, i.e. either \( w_i \) or \( \Theta(w_i) \) is as in definition 1.1, then the conclusion of the same theorem holds for \( w \).

For example, the element \( s_1 s_2 s_5 s_4 \) satisfies the above conditions.

Our results apply to all the elements of \( W \) in the case of \( SL_3 \) thus giving a more general proof to [3]. However, in the case of \( SL_4 \), there are precisely 4 elements for which the condition of the theorem is not satisfied. Namely, they are \( s_1 s_2 s_8, s_2 s_1 s_3, s_2 s_1 s_8 s_2 \) and \( s_1 s_2 s_8 s_2 s_1 \). The main problem in these cases is that standardness is not transitive in all the obvious “orderings”.
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