A known aspect of the Clausius inequality is that an equilibrium system subjected to a squeezing $dS$ of its entropy must release at least an amount $|\bar{d}Q| = T|dS|$ of heat. This serves as a basis for the Landauer principle, which puts a lower bound $T \ln 2$ for the heat generated by erasure of one bit of information. Here we show that in the world of quantum entanglement this law is broken.

A quantum Brownian particle interacting with its thermal bath can either generate less heat or even adsorb heat during an analogous squeezing process, due to entanglement with the bath. The effect exists even for weak but fixed coupling with the bath, provided that temperature is low enough. This invalidates the Landauer bound in the quantum regime, and suggests that quantum carriers of information can be much more efficient than assumed so far.

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1. INTRODUCTION

The laws of thermodynamics are at the basis of our understanding of nature, so it is rather natural that they have many application much beyond their original scopes, e.g. in computing and information processing. The first connection between information storage and thermodynamics was made by von Neumann in the 1950’s. His speculation that each logical operation costs at least an amount of energy $T \ln 2$ proved too pessimistic. Landauer pointed out that reversible “one-to-one” operations can be performed, in principle, without dissipation; only irreversible operations “many-to-one” operations, like erasure, require dissipation of energy, at an amount at least equal to the von Neumann estimate $T \ln 2$ per erased bit. This conclusion is a direct consequence of the Clausius inequality, which connects the change of heat in a given process with the change of entropy. It is perfectly consistent with intuition, since as everybody had a chance to observe, equilibrium substances typically release heat under isothermal compression of their entropy (or volume, which is the same for a good majority of classical cases). Rather recently the base of the effect was finally put on the Clausius inequality, and it was shown that the previous not very strict considerations are particular cases of its application to information processing systems.

The principal importance of erasure among other information-processing operations originates from the fact that it is connected with changes of entropy, and thus cannot be realized in a closed system. One needs to couple the information-carrying system with its environment. Therefore the process is accompanied with changes in heat whose magnitude has to be determined by thermodynamics. It was shown rigorously that all computation can be performed using reversible logical operations only.

Here we will consider thermodynamical aspects of erasure in the context of low temperatures, so low that the quantum effects start to play an important role. We choose the simplest working example: One-dimensional Brownian particle in contact with a thermal bath at temperature $T$, and in the presence of an external confining potential. The main new aspect arising at low temperatures is entanglement of the Brownian particle with the bath. Therefore, even when the total system is in a pure state, the subsystem (Brownian particle) is in a mixed state. Thus its stationary state cannot be given by equilibrium quantum thermodynamics, which would predict for $T \to 0$ that the subsystem goes to its groundstate, the pure vacuum state. The latter can only be reached for not too low temperatures, given a fixed but weak coupling with the bath. In general a new situation arises, for which a generalized thermodynamical interpretation can be given (for analogous situations in glasses and related systems see ). In particular, the classical Clausius inequality is invalid and the classical intuition fails. We stress that this situation is not at all exceptional, since it appears even for a small but generic coupling if temperature is low enough.

Our main result will show that when entropy of the particle is decreased by external agents, namely a part of information carried by it is erased, the particle can adsorb heat in a clear contrast with the classical intuition. Later we shall apply this result to show that there is not anything similar to Landauer bound at low temperatures. Thus in this respect quantum carriers of information can be much more efficient than their classical analogs.
Since we are in a new situation where relations of the standard thermodynamics are possibly broken, we prefer to work with simple exactly-solvable models, where all general relations can be illustrated or disproved explicitly. In analogous situation with the classical theory Szilard used a model with one classical Brownian particle interacting with its thermal bath.

This paper is organized as follows. In section 2 we review the connection between thermodynamics and information erasure. Section 3 is devoted to heat and entropy changes of a quantum Brownian particle in contact with its thermal bath. This model can be considered as an extension to the quantum regime of the seminal model by Szilard. In section 4 our main results on violation of the Landauer principle are presented. In section 5 we analyze the most popular derivation of this principle in order to show where its arguments appear to be inapplicable. Our conclusions are presented in the last section.

2. ERASURE OF INFORMATION AND GIBBSIAN THERMODYNAMICS

1. Source of information

Let us start by briefly recalling what is meant by erasure of information. Since information is carried by physical systems, messages are coded by their states, namely every state (or possibly group of states) corresponds to a “letter”. The simplest example is a two-state system, which carries on one bit of information. The basic model of source of information in Shannonean, probabilistic information theory assumes that the carrier of information can be in different states with certain (so called a priori) probabilities. In other words, the messages of this source appear randomly and the measure of their expectation is given by the corresponding probabilities. For example, in the classical case the carrier of information may occupy a cell in its phase space with volume $d\mathbf{x} d\mathbf{p}/(2\pi\hbar)$ and a priori probability $P(x,p)$. Then different cells will correspond to different messages. In the quantum case the completely analogous situation is described by a density matrix $\rho$:

$$\rho = \sum_n p_n \ket{n}\bra{n},$$

$$\bra{n|m} = \delta_{nm},$$

which means that the carrier occupies a state $\ket{n}$ with the a priori probability $p_n$. Moreover, different quantum states are exclusive as indicated by Eq. (2.2). As in the classical case, the appearance of the carrier in different states will bring different messages.

The fundamental theorem by Shannon states that the information carried by an information source is given by its entropy. Namely it is equal to

$$S = -\int \frac{d\mathbf{x} d\mathbf{p}}{2\pi\hbar} P(x,p) \ln P(x,p)$$

(2.3)

in the classical case, and to

$$S_{vN}(\rho) = -\sum_n p_n \ln p_n = -\text{tr}(\rho \ln \rho)$$

(2.4)

in the quantum situation. Here $S_{vN}(\rho)$ is the von Neumann entropy of the density matrix $\rho$. The physical meaning of this result can be understood as follows. A source which has less entropy occupies less states with higher probability. It can be said to be more known, and therefore the appearance of its results will bring less information. In contrast, a source with higher entropy occupies more states with lower probability. Its messages are less expectable, and therefore bring more information. The rigorous realization of this intuitive arguments appeared to be the most straightforward and fruitful proof of the Shannon theorem. Notice that the entropies (2.3, 2.4) appear here on the information theoretical footing and not as purely thermodynamical quantities.

Needless to mention that the above notion of information source does not exhaust the full meaning of this concept. Here it appears as a model of the probabilistic information theory. Advantages and shortcomings of this approach were nicely reviewed by Kolmogorov.
2. Erasure

Erasure is an operation which is done by an external agent in order to reduce the entropy of the carrier of information. This means that in its final state the carrier brings less information, i.e. some amount of it has been erased. In particular, the complete erasure corresponds to the minimization of entropy. Notice that the erasure is defined as a "blind" operation, which is done independently on the actual state of the information carrier. This is how information processing systems operate, they do not recognize the actual state of a bit before to erase it. Following to standard assumptions \[17,18\] we will model external operations by a time-dependent Hamiltonian $H(t)$ of the carrier, namely some of its parameters will be varied with time according to given trajectories. If the information carrying system is closed, then its dynamics is described by the Liouville equation for $P(x,p)$ in the classical case or by the von Neumann equation

$$\frac{d}{dt} \rho = \frac{i}{\hbar} [\rho(t)H(t) - H(t)\rho(t)]$$ (2.5)

in the quantum situation. As can be shown directly, the entropies (2.3, 2.4) remain constant with time. In order to change them, one has to consider an information carrier, which is an open system. In that case a part of its energy will be controlled (i.e. transferred to or received) by its environment as heat. Indeed, if $U = \text{tr}[H(t)\rho(t)]$ (2.6) is the average energy of the carrier, then its change during time $dt$ reads:

$$dU = dQ + dW = \text{tr}[Hd\rho] + \text{tr}[\rho dH]$$ (2.7)

This is the energetic budget of the system. The last term is the averaged mechanical work $dW$ produced by an external agent $\text{[17,18]}$. The first term in r.h.s. of Eq. (2.7) arises due to the statistical redistribution in phase space. We shall identify it with the change of heat $dQ$, so Eq. (2.7) is just the first law. As can be shown through Eq. (2.5), the heat is explicitly zero for a closed system. All these formulas are valid in the classical case as well. Here $\rho$ should be substituted by $P(x,p)$, and the trace will be changed by the integration over the corresponding phase space.

3. Brownian particle as an information carrier

In order to specify the situation, let us consider a Brownian particle as an information carrying system. A similar simple model was employed by Szilard $\text{[3]}$ in his seminal analysis of the Maxwell’s demon problem. The Brownian particle has a Hamiltonian $H(p,x,t)$, where $p,x$ are coordinate and momentum. A parameter which vary with time can be the mass of the particle or the shape of its potential energy. The environment of the particle will be taken to be a thermal bath. This is a generic situation, in the sense that the bath satisfies to the following generally accepted conditions $\text{[17,18,24]}$, which are exactly the same for both quantum and classical cases.

1) The interaction between the particle and bath is linear. It is assumed to be so weak that the non-linear modes of the bath are not excited, and it can be modeled itself as a collection of harmonic oscillators $\text{[10,23,24]}$. This assumption has been verified rigorously, when starting from rather general microscopic situations.

2) The bath is a macroscopic system, namely the thermodynamical limit is assumed to be taken for it.

3) Before to start to interact with the particle at some initial time the bath was in equilibrium (i.e. in a Gibbsian state) at temperature $T$. This temperature will be refered to as the temperature of the bath. This assumption reflects the typical macroscopic preparation at the initial time.

4) The particle and bath together form a closed system. Thus, the overall system is described by the Schrödinger equation (alternatively Heisenberg equations) in the quantum case, and by the Newton equations in the classical case.

A minimal model, which incorporates all these properties was proposed in $\text{[21,22]}$, and much later became known as the Caldeira-Leggett model $\text{[23,24]}$. The above assumptions ensure that the reduced dynamics of the Brownian particle will be given by the quantum or classical Langevin equations $\text{[24]}$.

As a result of interaction with the macroscopic bath, the Brownian particle will relax with time towards a definite stationary state. In the present paper we will additionally assume that all external operations on the particle are adiabatic, namely they occur on time-scales which are much larger than the characteristic relaxation time. There are several physical reasons for this restriction. First, in many circumstances the adiabatic process can be shown to be optimal, in the sense that it is connected with minimal amount of work done by the external agent $\text{[10,17,18]}$. On the
other hand, this time-scale separation more naturally corresponds to the interaction between a deterministic agent and the microscopic particle.

Let us now consider the classical and quantum situation in separate.

\[ a. \] Classical case

As well known under the above standard assumptions on the thermal bath the classical Brownian particle relax to the Gibbs distribution:

\[ P(p, x) = \frac{1}{Z} \exp[-\frac{1}{T}H(p, x)], \]
\[ Z = \int dp \, dx \exp[-\frac{1}{T}H(p, x)] \quad (2.8) \]

Since the external operation is assumed to be adiabatic, the time-dependent distribution of the particle will be given by Eq. (2.8) with the corresponding time-dependent Hamiltonian \( H(x, p, t) \). The Clausius equality

\[ dQ = TdS, \quad (2.9) \]

which connect the changes of heat and entropy during the process can be derived directly from Eqs. (2.6, 2.3, 2.7). It holds that when compressing the phase space of the particle (\( dS < 0 \)), it releases heat (\( dQ < 0 \)). Since for any non-adiabatic change one has \( |dQ| \leq TdS \) (Clausius inequality), \( |dQ| \) can only increase for not very slow processes. In other words, the minimal amount of the released heat is equal to \( |TdS| \). This is the Landauer principle.

\[ 4. \] Quantum case

Let us now move to the quantum domain, which in the present context just means the domain of low temperatures. We assume that the quantum carrier of information interacts with its thermal bath, but so weakly, that it is described by quantum Gibbs distribution at the bath temperature \( T \):

\[ \rho = \frac{1}{Z} \exp[-\frac{H}{T}], \quad Z = \text{tr} \exp[-\frac{H}{T}]. \quad (2.10) \]

The concrete conditions on the interaction strength will be discussed later. Now it can be easily seen that provided we use entropy as defined in Eq. (2.4) the Clausius inequality (2.3) is still holds and all its consequences including the Landauer principle are generalized automatically. The important difference between the classical and quantum cases has to be noted already here: In the former situation there is no limitation on the interaction strength, and the classical Gibbs distribution appears naturally from the above standard conditions on the thermal bath.

**3. QUANTUM BROWNIAN PARTICLE IN CONTACT WITH ITS THERMAL BATH**

\[ 1. \] Wigner function and effective temperatures

As explained in the introduction, at low temperatures of the bath the Brownian particle is not described by the quantum Gibbs distribution, except very weak interaction with the bath. Therefore, its state at low temperatures has to be found from first principles starting from microscopic description of the bath and the particle. This program was realized in \[ 9,10,24,26 \]. In particular, in \[ 9,10 \] we investigated statistical thermodynamics of the quantum Brownian particle.

Here we consider the simplest example of harmonic oscillator with Hamiltonian

\[ H(p, x) = \frac{p^2}{2m} + \frac{ax^2}{2}, \quad (3.1) \]

where \( m \) is the mass, and \( a \) is the width. The state of this particle can be described through the Wigner function \[ 18 \]. Recall that in quantum theory this object plays nearly the same role as the common distribution of coordinate and momentum in the classical theory. The stationary Wigner function reads \[ 27,24,31,11 \]:
The quantum regime can be understood as follows. For a cubic equation
where \( \gamma \) quantifies the interaction with the thermal bath, and on a large parameter \( \Gamma \) which is the maximal characteristic frequency of the bath. In particular, the Gibbsian limit corresponds to \( \gamma \to 0 \). Then distribution (3.2) tends to the quantum Gibbsian: \( T_p = T_x = \frac{1}{2} \hbar \omega_0 \coth \left( \frac{1}{2} \beta \hbar \omega_0 \right) \), where \( \omega_0 = \sqrt{am} \) (see Eqs. [2,14, 3.16]). In the classical limit, which is realized for \( \hbar \to 0 \) or \( T \to \infty \), the dependence on \( \gamma \) and \( \Gamma \) disappears, and \( T_p, T_x \) go to \( T \), reproducing the classical Gibbsian distribution (2.8). The appearance of the effective temperatures in the quantum regime can be understood as follows. For \( T \to 0 \) quantum Gibbs distribution predicts the pure vacuum state for the particle. Since due to quantum entanglement this is impossible for the non-weakly interacting particle, \( T_p, T_x \) depend on \( \gamma \), and have to be obtained from the first principles, since the state is not any more Gibbsian.

The exact expressions for \( T_p, T_x \) reads [27,24] [10]

\[
T_p = \frac{\hbar \gamma \Gamma^2}{\pi m} \left[ \frac{\psi_1^2 \psi_3}{(\omega_1^2 - \omega_2^2)(\omega_3^2 - \omega_2^2)} \right] + \frac{\hbar \gamma \Gamma^2}{\pi m(\omega_1^2 - \omega_2^2)} \left[ \frac{\psi_1^2 \psi_3}{\omega_3^2 - \omega_2^2} - \frac{\psi_1^2 \psi_3}{\omega_3^2 - \omega_1^2} \right] - T, \tag{3.4}
\]

\[
T_x = -\frac{a \hbar \gamma \Omega^2}{m^2 \pi} \left[ \frac{\psi_3}{(\omega_1^2 - \omega_2^2)(\omega_3^2 - \omega_2^2)} \right] + \frac{a \hbar \gamma \Omega^2}{m^2 \pi(\omega_1^2 - \omega_2^2)} \left[ \frac{\psi_2}{\omega_3^2 - \omega_2^2} - \frac{\psi_1}{\omega_3^2 - \omega_1^2} \right] - T, \tag{3.5}
\]

where \( \psi_k = \psi(\hbar \beta \omega_k/(2\pi)), k = 1, 2, 3, \) and \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is Euler’s psi-function. \( \omega_{1,2,3} \) are roots of the following cubic equation

\[
(\Gamma - \omega)(\omega^2 + \omega_0^2) - \omega \frac{\gamma \Gamma}{m} = 0, \tag{3.6}
\]

In the present paper we will be mostly interested by the so-called quasi-Ohmic limit where \( \Gamma \) is the largest characteristic frequency of the problem. This is the most realistic situation with realistic information storing devices. In this limit one approximately has:

\[
\omega_{1,2} = \frac{\gamma}{2m} \left( 1 \pm \sqrt{1 - 4\xi} \right) + \frac{\gamma^2}{2Gm^2} \left[ 1 \pm \frac{1}{\sqrt{1 - 4\xi}} \right], \tag{3.7}
\]

\[
\omega_3 = \Gamma - \frac{\gamma}{m} - \frac{1}{\Gamma} \left( \frac{\gamma}{m} \right)^2, \tag{3.8}
\]

where an important parameter \( \xi = am/\gamma^2 \) characterizes the relative importance of damping: \( \xi \ll 1 \) corresponds to the overdamped motion, whereas the converse case indicates underdamping. We will basically use the first leading terms in Eqs. (3.7, 3.8). In this way we obtain for the effective temperatures:

\[
T_p = \frac{\hbar}{\pi(\omega_1 - \omega_2)} \left[ (\omega_1^2 - \omega_2^2)\psi(\frac{\beta \hbar \Omega}{2\pi}) - \omega_1^2 \psi(\frac{\beta \hbar \Omega}{2\pi}) \right] - T, \tag{3.9}
\]

\[
T_x = \frac{ha}{m\pi(\omega_1 - \omega_2)} \left[ \psi(\frac{\beta \hbar \Omega}{2\pi}) - \psi(\frac{\beta \hbar \Omega}{2\pi}) \right] - T, \tag{3.10}
\]

Their derivation can be found in [10]. Particular cases can be studied with help of the following approximate values for \( \psi \)-function.

\[
\psi(x) = -\frac{1}{x} - \gamma x + x \frac{\pi^2}{6}, \quad |x| < 1 \tag{3.11}
\]

\[
\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2}, \quad |x| \geq 1 \tag{3.12}
\]
where \( x \) is a complex number, and \( \gamma_E = 0.577216 \) is the Euler constant. In the low-temperature limit \( T \to 0 \) we obtain from Eqs. (3.10, 3.11):

\[
T_p = \frac{\hbar [\omega_1^2 \ln \frac{\omega_1}{\omega_1} - \omega_2^2 \ln \frac{\omega_2}{\omega_2}]}{\pi (\omega_1 - \omega_2)} + \mathcal{O}(T^4),
\]

\[
T_x = \frac{\hbar a}{\pi m (\omega_1 - \omega_2)} \ln \frac{\omega_1}{\omega_2} + \frac{\pi \gamma}{3 \hbar a} T^2 + \mathcal{O}(T^4)
\]

The weak-coupling limit can be obtained from Eqs. (3.10, 3.9) taking \( \xi \gg 1 \) and noticing

\[
\psi(ix) - \psi(-ix) = \frac{1}{ix} + \cot(x \pi),
\]

which is obtained with the reflection formula: \( \Gamma(z) \Gamma(1 - z) = \pi/\sin(\pi z) \). Here one has the following expressions:

\[
T_p = \frac{\hbar \omega_0}{2} \coth \frac{\hbar \omega_0}{2} + \frac{\hbar \gamma}{4 \pi m} G \left( \frac{i \hbar \omega_0}{2} \right) G \left( \frac{i \hbar \omega_0}{2} \right) + \frac{\hbar \gamma}{4 \pi m} \left[ 2 \psi(\beta \hbar) - \psi\left( \frac{i \beta \hbar \omega_0}{2 \pi} \right) - \psi\left( -\frac{i \beta \hbar \omega_0}{2 \pi} \right) \right],
\]

\[
T_x = \frac{\hbar \omega_0}{2} \coth \frac{\hbar \omega_0}{2} + \frac{\hbar \gamma}{4 \pi m} G \left( \frac{i \hbar \omega_0}{2} \right),
\]

\[
G(x) = \frac{e}{x} \left[ \psi'(ix) - \psi'(-ix) \right]
\]

The asymptotic expressions of these quantities read in the opposite, strongly damped region \( \xi \ll 1 \) and for low \( T \):

\[
T_p = \frac{\hbar \gamma}{\pi m} \ln \frac{\Gamma m}{\gamma} + \frac{\hbar a}{\pi \gamma} + \mathcal{O}(T^4),
\]

\[
T_x = \frac{\hbar a}{\pi \gamma} \ln \frac{\gamma^2}{am} + \frac{\pi \gamma}{3 \hbar a} T^2 + \mathcal{O}(T^4),
\]

It is interesting to mention as well the high-temperature (quasi-classical) asymptotic values for \( T_p, T_x \). Applying Eq. (3.11) one gets

\[
T_p = T + \frac{\hbar^2 a (am - \gamma^2 + \Gamma m \gamma)}{12 m^2 T} + \mathcal{O}(\hbar^3 \beta^2),
\]

\[
T_x = T + \frac{\hbar^2 a}{12 m T} + \mathcal{O}(\hbar^3 \beta^2)
\]

2. Energy and partial entropies

The average energy of the Brownian particle

\[
U = \int dx dp W(p, x) H(p, x) = \frac{T_p}{2} + \frac{T_x}{2}
\]

does depend on \( a \) and \( m \), in contrast to its classical value \( T \). We will need entropies of momentum and coordinate distribution

\[
S_p = -\int dp W(p) \ln W(p) = \frac{1}{2} \ln (m T_p),
\]

\[
S_x = -\int dx W(x) \ln W(x) = \frac{1}{2} \ln \frac{T_x}{a}
\]

The ‘Boltzmann’ entropy reads

\[
S_B = -\int dp dx W(p, x) \ln[h W(p, x)] = S_p + S_x - \ln \hbar = \frac{1}{2} \ln \frac{m T_p T_x}{a \hbar^2}
\]

Notice that they all are different from \( S_{\nu N}(\rho) \) defined by (2.4).
3. Heat and work

The expressions for heat and work are generalized from Eqs. (2.7) by simply using the Wigner function $W(p, x)$ instead of $\rho$. This can be easily verified, when using Eq. (3.40). One can prove by a direct calculation that quantities $T_p, T_x$ do deserve their nomenclature, since the classical Clausius equality can be generalized as

$$dU = dQ + dW = T_p dS_p + T_x dS_x + dW$$

(3.27)

for variation of any parameter.

We will be especially interested in variation of the mass and the width of the potential. The corresponding changes of heat read

$$d_a Q = \frac{1}{2} \left( \frac{\partial T_p}{\partial a} + \frac{\partial T_x}{\partial a} - \frac{T_x}{a} \right) da,$$

(3.28)

and

$$d_m Q = \frac{1}{2} \left( \frac{\partial T_p}{\partial m} + \frac{\partial T_x}{\partial m} + \frac{T_p}{m} \right) dm,$$

(3.29)

Using Eqs. (3.10, 3.9) one can get general formulas for heat. Let us first introduce the following notations

$$z = \sqrt{1 - \xi}, \quad \alpha_1 = \frac{\hbar \gamma}{4\pi m T}, \quad \alpha_2 = \frac{a \hbar}{\pi \gamma T},$$

(3.30)

and get

$$\frac{\partial Q}{\partial a} = \frac{2mT}{\gamma^2} \left( \frac{1}{1-z^2} + \frac{\alpha_1^2}{z} \{ (1+z)\psi'(\alpha_1[1+z]) - (1-z)\psi'(\alpha_1[1-z]) \} \right),$$

(3.31)

and

$$\frac{\partial Q}{\partial m} = \frac{2aT}{\gamma^2} \left( \frac{1}{z^2 - 1} + \frac{\alpha_2^2}{z} \{ \frac{1}{(1-z)^3} \psi'(\alpha_2[1-z]) - \frac{1}{(1+z)^3} \psi'(\alpha_2[1+z]) \} \right).$$

(3.32)

Following to asymptotic expressions of $\psi'(x)$ given by Eqs. (3.11, 3.12) one derives

$$\frac{\partial Q}{\partial a} = -\frac{T}{2a} + \frac{\hbar^2}{24mT}, \quad \alpha_1 \ll 1,$$

(3.33)

$$\frac{\partial Q}{\partial a} = -\frac{\pi \gamma T^2}{3a^2}, \quad \alpha_1 \geq 1,$$

(3.34)

$$\frac{\partial Q}{\partial m} = \frac{T}{2m} + \frac{\hbar^2 \gamma^2 (z^2 + 1)}{66m^3 T}, \quad \alpha_2 \ll 1,$$

(3.35)

$$\frac{\partial Q}{\partial m} = \frac{\hbar \gamma}{2\pi m^2}, \quad \alpha_2 \geq 1.$$  

(3.36)

Notice that the last equation applies not only for low temperatures, but also for weak coupling (see Eq. (3.30)).

Using these results one can show that

$$\frac{\partial Q}{\partial a} \leq 0, \quad \frac{\partial Q}{\partial m} \geq 0$$

(3.37)

for all values of parameters including, of course, the classical limit. For the work done in this processes one obtains

$$\frac{\partial W}{\partial a} = \frac{1}{2} \langle x^2 \rangle = \frac{1}{2} \frac{T_x}{a} \geq 0,$$

(3.38)

$$\frac{\partial W}{\partial m} = -\frac{1}{2} \langle p^2 \rangle = -\frac{1}{2} \frac{T_p}{m} \leq 0,$$

(3.39)

It is interesting to mention that the signs of $\partial Q$ and $\partial W$ in Eqs. (3.37, 3.38, 3.39) are the same as in the classical case, where $T_x = T_p = T$. 

7
4. Density matrix

To investigate von Neumann entropy \( S_{vN} \) one needs density matrix corresponding to the Wigner function \( W \). Applying the standard relation:

\[
\langle x + \frac{u}{2} \mid \rho \mid x - \frac{u}{2} \rangle = \int dp \ e^{-ipu/\hbar} W(p, x) \tag{3.40}
\]

which connects the density matrix in coordinate representation with the Wigner function, one gets the following expression

\[
\langle x \mid \rho \rangle \langle x' \rangle = \frac{1}{\sqrt{2\pi}(x^2)} \exp \left[ \frac{(x + x')^2}{8(x^2)} - \frac{(x - x')^2}{2\hbar^2/\langle p^2 \rangle} \right] \tag{3.41}
\]

The physical meaning of Eq. (3.41) is clear: The diagonal elements \( (x = x') \) are distributed at the scale \( \sqrt{(x^2)} \), while the maximally off-diagonal elements \( (x = -x') \), which characterize coherence, are distributed with the characteristic scale \( \hbar/\sqrt{(p^2)} \).

We have to find eigenfunctions and eigenvectors of this density matrix

\[
\int dx' \langle x \mid \rho \rangle \langle x' \rangle f_n(x') = p_n f_n(x) \tag{3.42}
\]

The solution of this problem uses some tabulated formulas for Hermite polynomials, and results in

\[
p_n = \frac{1}{w + \frac{1}{2}} \left[ \frac{w - \frac{1}{2}}{w + \frac{1}{2}} \right]^n, \tag{3.43}
\]

\[
f_n(x) = c H_n(c x) e^{-x^2/2}, \quad c = \left( \frac{\langle p^2 \rangle}{\hbar^2 \langle x^2 \rangle} \right)^{1/4} \tag{3.44}
\]

\[
w = \frac{\Delta p \Delta x}{\hbar} = \sqrt{\frac{\langle p^2 \rangle \langle x^2 \rangle}{\hbar^2}} = \sqrt{\frac{m T_p T_x}{\hbar^2 a}} \tag{3.45}
\]

where \( H_n \) are Hermite polynomials, and it holds that \( w \geq \frac{1}{2} \) due to the Heisenberg uncertainty relation. The result for the von Neumann entropy \( S_{vN} \) now reads \[24\]

\[
S_{vN} = (w + \frac{1}{2}) \ln(w + \frac{1}{2}) - (w - \frac{1}{2}) \ln(w - \frac{1}{2}) \tag{3.46}
\]

The first terms in its large \( w \)-expansion are

\[
S_{vN} = \ln w + 1 - \frac{1}{24 w^2} - \frac{1}{320 w^4} - \frac{1}{2088 w^6} \tag{3.47}
\]

Notice that the same quantity \( w \) governs the Boltzmann entropy

\[
S_B = S_p + S_x - \ln \hbar = \ln w + 1 \tag{3.48}
\]

This appears to coincide with the leading terms of (3.47). It is known to be larger than the Von Neumann entropy, and this is obvious from the sign of the correction terms.

If some parameter \( (a \text{ or } m) \) is varied, then the derivative of \( S_{vN} \) with respect to it reads:

\[
dS_{vN} = \frac{\ln w + \frac{1}{w - \frac{1}{2}}}{w - \frac{1}{2}} dw \tag{3.49}
\]

In other words, the sign of the change in \( S_{vN} \) is determined by the sign of the change in \( w \). This holds as well for the change in \( S_B \), so qualitatively they carry the same information.

In this context let us stress again that von Neumann entropy \( S_{vN}(\rho) \) is the unique quantum measure of localization and information, whereas the entropies \( S_p, S_x \) characterize localizations of momenta and coordinate separately. Differences between \( S_p + S_x \) and \( S_{vN} \) are due to the fact that in quantum theory momentum and coordinate cannot be measured simultaneously; in this sense \( S_p + S_x \) characterize two different measurement setups. Nevertheless, for the harmonic particle if \( S_{vN} \) increases (decreases), then \( S_p + S_x \) increases (decreases) as well. Notice that the real importance of \( S_p, S_x \) becomes clear when they have to be used to generalize the Clausius inequality. The von Neumann entropy \( S_{vN} \) cannot be used for this purpose if \( T_x \neq T_p \).
4. ENTROPY DECREASE WITH HEAT ADSORPTION

Now we will show that there are erasure processes, namely processes where \( \Delta S_vN \leq 0 \), which are accompanied by adsorption of heat. As we know heat is always adsorbed, when the mass is increased (see Eq. (3.36)). It will be shown that there is a mass-increasing process, where \( \Delta S_vN \leq 0 \). Using Eqs. (3.20, 3.19) one has for \( \partial_m(\langle x^2 \rangle \langle p^2 \rangle) \) at very low temperatures:

\[
\frac{\partial w^2}{\partial m} = \frac{\partial}{\partial m} \left[ \frac{1}{\hbar^2} \langle x^2 \rangle \langle p^2 \rangle \right] \\
= \frac{a}{\pi^2 \gamma^2} \left[ -1 - \frac{\gamma^2}{am} \ln \frac{\Gamma m}{\gamma} + (1 + \frac{\gamma^2}{am}) \ln \frac{\gamma^2}{am} \right]
\]

(4.1)

This expression is negative in its range of applicability.

FIG. 4.1. Dimensionless phase space volume \( w = \Delta p \Delta x / \hbar = \sqrt{\langle x^2 \rangle \langle p^2 \rangle / \hbar^2} \), versus mass \( m \). The other parameters are \( a = \gamma = 1 \), \( \Gamma = 500 \), \( \hbar = 1 \) and \( T = 0 \). It is seen that the volume decays monotonically towards its minimal value \( 1/2 \), set by the uncertainty relation.

An analogous argument can be brought about in the weak-coupling case. Having started from Eqs. (3.13, 3.14) or alternatively from Eqs. (3.16, 3.17) one derived the following expressions for the effective temperatures in the weak-coupling \( \gamma \to 0 \) and low-temperature limit:

\[
T_p = \frac{\hbar \omega_0}{2} + \frac{\hbar \gamma}{\pi m} \ln \frac{\Gamma}{\omega_0 \sqrt{e}},
\]

(4.2)

\[
T_x = \frac{\hbar \omega_0}{2} - \frac{\hbar \gamma}{2 \pi m},
\]

(4.3)

This implies

\[
\frac{\partial w^2}{\partial m} = -\frac{\gamma}{4m \sqrt{am}} \ln \frac{\Gamma}{\omega_0 e^2},
\]

(4.4)

which is again negative in its range of applicability \( \Gamma \gg \omega_0 \). The general situation at low temperatures is illustrated by Fig. 4.1, where it is seen that at low temperatures the dimensionless phase-space volume \( w = \sqrt{\langle x^2 \rangle \langle p^2 \rangle / \hbar^2} \) monotonically decreases when increasing the mass. In the limit \( m \to \infty \) it tends to its corresponding gibbsian value. This can be understood noticing that the stationary state of a very heavy Brownian particle will not be influenced much by the bath. Indeed, as seen from Eqs. (3.7, 3.8, 3.9, 3.10) the dimensionless parameter which controls transition from the weakly damped to the strongly-damped regime is \( \xi = \alpha m / \gamma^2 \). So to increase the mass for all other parameters are being fixed produces the same effect as decrease of the coupling constant \( \gamma \).

Recall that the corresponding expression (3.34) for \( \partial Q / \partial m \) was positive. This just means that for the variation of \( m \) we have an interesting case where heat is adsorbed when entropy is decreasing. This is a counterexample for the general validity of the Landauer principle.
1. Where classical intuition is correct and where it fails

In the classical case one has an intuitively clear result: Upon increasing the mass of the particle its entropy increases and it releases heat. At the same time it does work against the external agent \( \partial W / \partial m < 0 \).

The first part of this result holds as well in the quantum case, as follows from Eq. (3.39). However in the first part a unusual point appears: The quantum particle decreases its entropy when the mass is increased. Simultaneously, it adsorbs heat. To understand this point we notice that at low temperatures of the bath the particle has an appreciable entropy due to entanglement. When its mass is increased, its state moves towards the gibbsian limit, and the entropy is reduced just because in the zero-temperature gibbsian case the entropy is just zero. This will take also for low but finite temperatures of the bath, as far as entanglement appreciable contribute to the entropy. So it is entanglement which leads to such a counterintuitive result.

Notice that this effect does not imply a violation of the second law in Thomson's formulation, which speaks about the impossibility to extract work by a cyclic variation of a system parameter \( [9,10] \). Indeed, if, after increasing the mass, one decrease it in order to complete the cycle, the external agent will do work on the particle, and it will release heat, thereby nullifying the overall work and heat (as expected, the overall work is positive if non-adiabatic variations are considered \( [9,10] \)).

2. Finite temperatures

![FIG. 4.2. Dimensionless phase space volume \( w = \Delta p \Delta x / \hbar = \sqrt{\langle x^2 \rangle \langle p^2 \rangle / \hbar^2} \), versus mass \( m \). From the top to the bottom: \( T = 0.25, 0.20, 0.15, 0.10 \). The other parameters are the same as in Fig. 4.1: \( a = \gamma = 1, \Gamma = 500, \bar{\hbar} = 1 \). It is seen that there is a region of \( m \)'s where the volume decays. This region is completely shrunk for \( T = 0.47553 \). For higher temperatures the phase space volume monotonically increases with \( m \).

The above effect \( \partial w / \partial m < 0 \) was analytically illustrated for \( T \to 0 \). However, it persists as well at finite, but sufficiently small temperatures. This situation is illustrated in Fig. (4.2). Since in the classical case, namely with high temperatures, one always has \( \partial_m w = T / (2 \sqrt{ma}) > 0 \), we expect that the region with \( \partial_m w < 0 \) will completely disappear at some finite temperature. This is indeed the case as Fig. (4.2) shows.

3. Variation of the spring constant

The analogous variation of \( a \) does not lead to such an usual result. Here instead of Eq. (4.1) one has

\[
\frac{\partial}{\partial a} \left[ \frac{1}{\hbar^2} \langle x^2 \rangle \langle p^2 \rangle \right] = \frac{m}{\pi \hbar \gamma^2} \left[ -1 - \frac{\gamma^2}{am} \ln \frac{\Gamma m}{\gamma} + \frac{\gamma^2}{am} \right] \leq 0,
\]

but the corresponding expression (3.31) for \( \partial Q / \partial a \) is negative as well. An expression analogous to Eq. (4.1) can be gotten also in the weak-coupling limit. Using Eqs. (4.2, 4.3) one gets:
\[
\frac{\partial w^2}{\partial a} = -\frac{\gamma}{a \sqrt{am}} \ln \frac{\Gamma}{\omega_0} < 0 \quad (4.6)
\]

An important fact should be mentioned here. Although the particle just releases heat during localization, this heat scales at low temperatures as \(|dQ| \sim T^2 da\) (see Eq. (3.34)). This is already invalidating the Landauer bound \(|dQ| \geq T|dS| \sim Tda\).

In this context the parameters of a statistical system can be distinguished as active and passive. In our concrete case the width of the potential \(a\) is a passive parameter, in a sense that its variations result in effects, which for any temperature are qualitatively (but not quantitatively) similar to the classical case. In contrast, the active parameters (in our case it is the mass \(m\)) invert their behavior at low temperatures. If one needs to increase entropy with adsorption of heat he/she is advised to vary a passive parameter. In contrast, being aimed to adsorb heat during contraction of entropy, one varies an active parameter.

### 4. Weak coupling limit

Here we will especially point out on applicability of our result in the weak-coupling limit. First we will make an obvious remark that the precise meaning of this limit must not be understood in the sense \(\gamma = 0\), since the damping constant \(\gamma\) is never explicitly zero in practice, and having put it zero one will not have at all a possibility to change entropy of the particle. The weak-coupling limit is understood in a sense that the interaction energy of the particle and the bath happened to be sufficiently small compared to the energy of the particle itself \[18\] (since the energy of the bath is infinite there is no need to involve it here). For low temperatures and \(\gamma \rightarrow 0\) the energy of the particle is given by its zero-point value which is \(\frac{\hbar}{2} \sqrt{a/m}\). Because of the interaction energy is explicitly zero for \(\gamma = 0\), it will be enough to choose \(\gamma\) sufficiently small to ensure the above condition of the weak-coupling limit.

Let us now turn to Eq. (3.36) which represents the amount of heat obtained by the particle when changing the mass at low temperatures. It is seen that in the leading order this quantity is proportional to \(\gamma\). Thus, although the particle is in the weak-coupling regime, it still gets a positive (though small) amount of heat during variation of its mass.

### 5. High temperatures

Finally we wish in more details that the Landauer principle does hold in our model for sufficiently high temperatures. This follows from the fact that in this limit \(T_p, T_x \rightarrow T\) as seen from Eqs. (3.21, 3.22). A more elaborated discussion goes as follows. One has the following exact relation:

\[
dQ - TdS_{vN} = (T_x - T)dS_x + (T_p - T)dS_p - Td(S_{vN} - S_x - S_p) \quad (4.7)
\]

One applies here Eqs. (3.21, 3.22, 3.47) to get for variation of \(m\)

\[
\frac{\partial Q}{\partial m} - T \frac{\partial S_{vN}}{\partial m} = -\frac{\hbar^2 \gamma^2}{24m^3T} + O(\hbar^3 \beta^2) \quad (4.8)
\]

Now it is seen that the deviation from the Clausius equality, and thus from the Landauer principle, will disappear for high temperatures or for \(\gamma \rightarrow 0\) and/or \(\hbar \rightarrow 0\) as should be. It is seen as well that the correction has the opposite sign to the main effect: When increasing mass the particle releases heat as the main standard thermodynamical effect, but the small correction in r.h.s. of Eq. (4.8), which appears due to the common influence of the quantum effects and interaction with the bath, tends to do this released heat slightly smaller.

### 5. ON A POPULAR DERIVATION OF THE LANDAUER BOUND

Let us discuss in a more general perspective the obtained result on the violation of the Landauer principle. For this purpose we will analyze one of the simplest derivations of this principle \[4–6\], in order to understand what essentially goes into it and where its argument may be inapplicable. The derivation goes as follows. Erasure is accompanied by reduction of entropy of the information-carrying system. Since entropy of the overall system, which is the carrier plus bath, cannot decrease, one quickly concludes that entropy of the bath should increase thereby
producing heat. This argument seems to be rather solid, because, instead involving any derivation, it just directly refers to the second law. However, there are three assumptions in it, which are rather restrictive and need not be valid in situations of physical interest. The first assumption is that the total entropy $S$ of the overall system is sum of partial entropies of system and bath, $S = S_S + S_B$. The second is quick thermalization in the bath, implying $dQ_B = TdS_B$. The third assumption is smallness of the interaction energy $dQ_I$, allowing to conclude from the energy conservation $dQ_S + dQ_B + dQ_I = 0$ that $dS_B = dQ_B/T = -dQ_S/T$. With these assumptions it follows immediately that $0 \leq dS = dS_S + dS_B = dS_S - dQ_S/T$. These assumptions are strictly valid only for non-interacting information carrier and its bath. However, without interaction there is no reason to speak about erasure. These assumptions may be valid as certain approximations in the weak coupling case plus several additional conditions [18]. Their validity is especially endangered in the quantum regime where the complete entropy, which is the subject of the second law applied to the complete system, is not equal to the sum of the separate entropies if there occurs quantum entanglement. So the above simple derivation is actually rather restricted, as was noted already in the context of rather different physical arguments [28,29]. It need not be invoked at all, since the Landauer principle can be completely embedded in the Clausius inequality. The latter is typically valid without any weak-coupling limit, as it happens for classical Brownian motion. Even in situations very far from the equilibrium thermodynamics, one can figure out regions of its validity [11,12]. The conclusion of this brief consideration is that in situations where the weak-coupling assumption is valid, the argument that the total entropy cannot decrease need not be invoked, since it amounts to rederiving the Clausius inequality. It can, of course, still be kept as a useful explanation. But the general validity of the Landauer principle must be completely put on the the Clausius inequality; it is just a direct consequence of this inequality, which has a substantially larger validity than weak coupling.

Nevertheless, it was discussed here that quantum entanglement limits the validity of the Clausius inequality and, consequently, Landauer’s bound. It can be checked explicitly that in that regime all three above assumptions are invalid [30,31]. Recently violations of other formulations of the second law were noticed and investigated in [13,14].

6. CONCLUSION

The Landauer principle requires dissipation (release) of $T|dS|$ units of energy as a consequence of erasure of $|dS|$ units of information. This was believed to be the only fundamental energy cost of computational processes [4,5]. Though in practice computers dissipate much more energy, the Landauer principle was considered to put a general physical bound to which every computational device interacting with its thermal environment must satisfy. Indeed, in several physical situations the Landauer principle can be proved explicitly [8].

The main purpose of the present paper was to provide a counterexample of this principle, and thus to question its universal validity. In the reported case all general requirements on the information carrier and its interaction with the bath are met. The only new point of our approach is that we were interested by sufficiently low temperatures, where quantum effects are relevant. The Landauer principle appeared to be violated by these effects (in particular, by entanglement). At high temperatures we reproduce its validity. In fact, in this limit our model is equivalent to that considered in Ref. [8], where the classical Landauer principle was derived in a quite general ground.

Recently the Landauer bound attracted a serious attention by workers in the field of applied information science [30]. There is a definite belief that this bound can be approached by further miniaturization of computational devices. It is hoped that the present paper will help to understand limitations of the Landauer principle itself, which may lead to unexpected mechanisms for computing in the quantum regime.

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