An algebraic approach to systems with dynamical constraints

e-mail: hanckowiak@wp.pl

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Abstract

Constraints imposed directly on accelerations of the system leading to the relation of constants of motion with appropriate local projectors occurring in the derived equations are considered. In this way a generalization of the Noether’s theorem and a relation of local quantities to global are highlighted. A phenomenon of nonphysical degrees of freedom is also discussed.

Key words: Dynamical and canonical constraints, reaction forces, virtual work, projectors, local and global quantities, Gram-Schmidt process, Cantor’s theorem

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1 Introduction

We consider equations describing discrete or continuous systems with constraints. If no constraints are present, we will assume that the unconstrained system is described by the 'field' equation:

\[ L(\tilde{x}; \varphi) + \lambda N(\tilde{x}; \varphi) + G(\tilde{x}) = 0 \]  

with the main linear functional \( L \) depending on the unknown 'field' (function) \( \varphi(\tilde{x}) \), which necessarily includes differential operations, for example

\[ L(\tilde{x}; \varphi) = (\Box + m^2) \varphi(\tilde{x}) \]

the \( N \) a functional, usually nonlinear (although may also contain additional linear terms), depending on the field \( \varphi \), for example

\[ N(\tilde{x}; \varphi) = \varphi^3(\tilde{x}) \]

and the given function \( G \) usually describing external forces acting on the system. Here and further square brackets mean that a given quantity, except that it is a function, it is also a functional. For discrete systems, such as \( N \) material points, \( \varphi = (q_1, ..., q_{3N}) \) can be a 3N dimensional vector and \( \tilde{x} = (t, i) \) besides the time \( t \) describes the component indexes \( i = 1, ..., 3N \). In this case, we can choose \( L[t, i; \varphi] = \dot{q}_i(t) \). In general, the 'vector' \( \tilde{x} \in \tilde{M} \) has time 'space' components describing 'points', components characterizing the field \( \varphi \) as its tensor type, and the time \( t \). Usually, we will distinguish the time and 'space' components by writing, e.g., \( \tilde{x} = (t, \bar{x}) \). We will assume that all components of the vector \( \tilde{x} \) are discrete variables. In other words, \( \tilde{M} \) is a set defined by a specific properties of the considered system, see also App.7.

We are in good company. Even the space can be described by means of the field \( \varphi(\tilde{x}) \).

The functionals \( L, N \) are also functions depending on the 'vectors' \( \tilde{x} \in \tilde{M} \). The set of functions dependent on the fixed \( \varphi \) will be denoted by \( \tilde{F}_\varphi \).

As usual, we will assume that the freedom of the theory described by Eq(1) is such as the freedom of the theory desribed by the main linear part:

\[ L[\tilde{x}; \varphi] = 0 \]  

(2)

It means that in both cases the same type of initial and boundary conditions can be used to get the unique solution.
We also assume, following an analogy with the classical mechanics, see also \[1\] and App.1, that the system represented by the field \(\varphi\) - subjects to the restrictions of the following type:

\[
\int Q(\tilde{x}, \tilde{y}; \varphi)L(\tilde{y}; \varphi)d\tilde{y} = f(\tilde{x}; \varphi) \tag{3}
\]

where \(Q(\tilde{x}, \tilde{y}; \varphi)\) is a given projector \((Q = Q^2)\) acting in the linear space of functions \(\tilde{F}\) and \(f(\tilde{x}; \varphi)\) is a given function. They both are depending in the linear or nonlinear way on the fixed 'field' \(\varphi\). The restrictions (3) together with additional assumption imposed on the 'reaction forces' \(R(\tilde{x}; \varphi)\), such as (7), are called here the **dynamical constraints** (DC). They can be ideal (IC) or non-ideal constraints (NIC) as well as holonomic or non-holonomic.

In the paper we show how Eq.1 is changing in case of NIDC, Sec.2, and how Eq.3 can be interpreted, Sec.3. In Sec.3 we also show how all formulas and equations are additionally changing in the case of ideal constraints, see also App.3.

In the paper the concept of virtual displacements, typical tool when discussing systems with constraints, is replace by the algebraic concepts such as the projection operators (projectors), see: \[1\], \[3, 4\]. This facilitates the necessary modification of the theory with constraints and illuminates relations of local to global quantities of the theory.

Expecting or demanding a certain conservation law in the theory and treating it as a constraint, the reaction forces can be considered as a sign of a new type of interaction or a necessary modification of already existing interaction. In a sense, it would be a contrary proceeding to the idea of spontaneous symmetry breaking.

What I found interesting in the present study is a connection of the certain constants of motion with the presence of certain projection operators in the considered equations, see Eq.11. It’s like combining constants of motion with a certain symmetry of considered equations resulting from the Noether’s theorem. Equations with projectors as in Eq.11 or Eq.26 mean that certain changes of functionals describing these equations do not change the whole equations. We see in this actually a generalization of symmetries of the equations.

In the case of restrictions (3), Eq.1 has to be changed by

\[
L(\tilde{x}; \varphi) + \lambda N(\tilde{x}; \varphi) + G(\tilde{x}) = R(\tilde{x}; \varphi) \tag{4}
\]

with temporarily unknown 'reaction force' \(R\) (a generalization of Lagrange’s equations of the first kind). In addition, I believe that the emphasis placed here on Lagrange’s equations of the first kind is an expression of a broader approach to the description of the nature including space, see \[5\]. - the opposite of any kind of reductionist approach - inspite of this that they may be acceptable in certain cases, see all arguments behind of Lagrange’s equations of the second kind.

For the systems with the constraints, we can look back in such a way that we want to modify the theory determined by measuring of local entities as
the position of the various parts, taking into account certain global (non-local) entities, for example energy of the system. In this and only this sense, the presented approach to classical mechanics contains some elements of quantum mechanics.

In the case of economics system the local and global entities are important ingredients of many theories. In this case, Adam Smith’s the invisible hand of the market would solve all the problems of capitalism if the constraints imposed by theory would be a result of the primary Eq. 11. Otherwise, the global rules (constraints) can be used to modify the interaction (reaction forces) between the various actors in the market.

As in other papers, author is using integration sign even in the case of discrete variables.

2 The ’reaction force’ $R$ and a principle of virtual works surrogate (PVW(S)); non-ideal constraints

Introducing the complementary projector $P$:

$$P[\tilde{x}, \tilde{y}; \varphi] + Q[\tilde{x}, \tilde{y}; \varphi] = \delta(\tilde{x}, \tilde{y})$$

where $\delta$ is Kronecker or Dirac’s delta, we can express the general solution to Eq. 3 as follows:

$$L[\tilde{x}; \varphi] = f[\tilde{x}; \varphi] + g[\tilde{x}; \varphi]$$

where $f[\varphi] = Q[f[\varphi]] \in Q\tilde{F}_\varphi$ and $g = P g$ is an arbitrary function from $P\tilde{F}_\varphi$, see (18). Here and elsewhere, for example:

$$f[\varphi] = Q[f[\varphi]] f[\varphi] \Leftrightarrow \int d\tilde{y} Q[\tilde{x}, \tilde{y}; \varphi] f[\tilde{y}; \varphi]$$

Equality (18) mean that only certain components of the acceleration of the system are completely expressed by the field $\varphi$.

Assuming that the ’reaction forces’ are such that

$$\int d\tilde{y} P[\tilde{x}, \tilde{y}; \varphi] R[\tilde{y}; \varphi] = 0$$

we get from Eq. 3 that

$$\int d\tilde{y} P[\tilde{x}, \tilde{y}; \varphi] \{L[\tilde{y}; \varphi] + \lambda N[\tilde{y}; \varphi] + G(\tilde{y})\} = 0$$

see (18). Since then, the symbol $d\tilde{y}$ will mean that all the variables with tilda ($) have the same time $t$.

From that, the arbitrary element $g$ in the expression (18):
\[ g[\hat{x}; \varphi] = -\int d\tilde{y}P[\hat{x}, \tilde{y}; \varphi] \{ \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \} \] (9)

and the formula (6) can be described as this:

\[ L[\hat{x}; \varphi] - f[\hat{x}; \varphi] + \int d\tilde{y}P[\hat{x}, \tilde{y}; \varphi] \{ \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \} = 0 \] (10)

Here, \( f[\hat{x}; \varphi] \in Q[\varphi]\tilde{F}\varphi \). Eq.10 substitutes Eq.11 in the case of DCs (3), which are satisfied by any solution to Eq.10.

By comparison with Eq.4, the ‘reaction force’

\[ R[\hat{x}; \varphi] = f[\hat{x}; \varphi] + \int d\tilde{y}Q[\hat{x}, \tilde{y}; \varphi] \{ \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \} \] (11)

Following the analogy with classical mechanics one can say that Eq.7 resembles some surrogate of the \textbf{virtual work principle (VWP)} - a surrogate because the ‘reaction forces’, at the moment \( t \), can not be perpendicular to the surface of the constraints (DC), Eq.10 resembles \textbf{Lagrange’s equations of the first kind}, and Eq.11 is a formula for the ‘\textbf{reaction forces}’ of DC (3). In this analogy, instead of the \textbf{virtual displacements}, we have used appropriate linear projectors depending on the field \( \varphi \). The ‘field’ \( \varphi \) in the simplest case may represent the radius vector. But the main difference of presented approach to constraints and canonical approach lies in the fact that there are explicitly described rather acceleration restrictions caused by the presence of constraints than constraint surfaces. See also \([2]\).

\section{3 Classical constraints. Ideal constraints; A phenomenon of nonphysical degrees of freedom}

We ask now how the restrictions (3) can be derived from the classical constraints (CC) of the dynamical system (1)? To answer this question let us consider classical mechanics with the constraints:

\[ \sum a_{ij}(q, t)\dot{q}_j + g_i(q, t) = 0 \] (12)

where \( \dot{q}_j \) is the \( j \)-th component of the vector \( \dot{q} \). Holonomic constraints can be differentiated once with respect to time to get Eq.12. Differentiating once more with respect to time, in both cases we get equations which, in the matrix-vector form, are:

\[ B(q, t)\ddot{q} = b(\dot{q}, q, t) \] (13)

The matrix \( B \) in this equation has to be a singular. Otherwise, it would be a dynamic equation, which for given initial conditions would describe the problem in an unique way. If we assume that \( B \) is a right invertible matrix, then such a right inverse exists that
\[ B(q, t)B_R^{-1}(q, t) = I \] (14)

and

\[ B_R^{-1}(q, t)B(q, t) = Q(q, t) \] (15)

After multiplication of Eq. 13 by the inverse \( B_R^{-1}(q, t) \) we get analogue of Eq. 3.

In fact, constraints equations obtained in the above way can have the following structure:

\[ Q'B(q, t)\ddot{q} = b(\dot{q}, q, t) \] (16)

with projected right invertible or invertible operator \( B \), which actually corresponds to a situation in which there are fewer constraints than degrees of freedom, see App. 3. Then, the equivalent equation:

\[ \ddot{q} = B_R^{-1}Q'B(q, t)\ddot{q} = B_R^{-1}b(\dot{q}, q, t) \equiv f \] (17)

has the form 3 with projector \( Q = B_R^{-1}Q'B(q, t) \), \( L = \ddot{q} \) and the functional \( f = B_R^{-1}b(\dot{q}, q, t) \). \( Q \) indeed is a projector because: \( B_R^{-1}Q'B(q, t) \cdot B_R^{-1}Q'B(q, t) = B_R^{-1}Q'B(q, t) \iff Q^2 = Q \). See also App. 3.

Multiplying Eq. 3 with an operator depending on the field \( \varphi \):

\[ A[\varphi] \iff A[\tilde{x}, \tilde{y}; \varphi] \] (18)

we get equation:

\[ A[\varphi]Q[\varphi]L[\varphi] = A[\varphi]f[\varphi] \] (19)

where \( A[\varphi] \) and \( Q[\varphi] \) operate in the space of functions \( \tilde{F}_\varphi \ni L[\varphi], f[\varphi] \). This equation is equivalent to Eq. 3 if, for example, we assume that operator \( A[\varphi] \) is a right invertible:

\[ A[\varphi]A[\varphi]^{-1} = I \] (20)

where \( I \) is the unit operator in space \( \tilde{F}_\varphi \), and that

\[ A_R^{-1}[\varphi]A[\varphi] = Q'' \ni Q \] (21)

where \( Q'', Q \) are projectors.

### 3.1 Ideal constraints

In the case of ideal constraints in which the reaction forces \( R_{\text{ideal}}[\tilde{x}; \varphi] \) are perpendicular to the constraint surfaces and projectors \( P_{\text{ideal}}[\tilde{x}, \tilde{y}; \varphi] \) projecting on the tangent surfaces at ‘points’ \( \varphi(\tilde{x}) \) are known, then we have, of course:

\[ \int P_{\text{ideal}}[\tilde{x}, \tilde{y}; \varphi]R_{\text{ideal}}[\tilde{y}; \varphi]d\tilde{y} = 0 \] (22)
In this case all derived formulas above will not be changed if
\[ Q_{\text{ideal}} Q = Q_{\text{ideal}} \]  
and \( P_{\text{ideal}} = I - Q_{\text{ideal}} \). But then, of course, the projector \( Q \) has to be replaced by \( Q_{\text{ideal}} \).

If (23) is not satisfied, then, starting from the formula (7), we have changes: so that (8) is modified by
\[
\int d\tilde{y} P_{\text{ideal}}[\tilde{x}, \tilde{y}; \varphi] \{ L[\tilde{y}; \varphi] + \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \} = 0
\]  
(24)

with arbitrary element \( P_{\text{ideal}} g[\tilde{x}; \varphi] \). Now, the 'reaction forces' are:
\[
R[\tilde{x}; \varphi] = Q_{\text{ideal}} f[\tilde{x}; \varphi] + \int d\tilde{y} P_{\text{ideal}}[\tilde{x}, \tilde{y}; \varphi] \{ \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \} + Q_{\text{ideal}} g[\tilde{x}; \varphi] \equiv (27)
\]

The condition (3) leads to the restriction of the element \( Q_{\text{ideal}} g[\tilde{x}; \varphi] \):
\[
Q Q_{\text{ideal}} g[\tilde{x}; \varphi] = f[\tilde{x}; \varphi] - Q_{\text{ideal}} f[\tilde{x}; \varphi] + Q \int d\tilde{y} P_{\text{ideal}}[\tilde{x}, \tilde{y}; \varphi] \{ \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \}
\]  
(28)

Substituting the formula (28) in Eq.26 the final equation in which the ideal constraints (22) are incorporated is the following:
\[
L[\tilde{x}; \varphi] - f[\tilde{x}; \varphi] - P Q_{\text{ideal}} f[\tilde{x}; \varphi] + P \int d\tilde{y} P_{\text{ideal}}[\tilde{x}, \tilde{y}; \varphi] \{ \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \} = P Q_{\text{ideal}} g[\tilde{x}; \varphi]
\]  
(29)

3.2 A phenomenon of nonphysical degrees of freedom

Still, the component \( P Q_{\text{ideal}} g[\tilde{x}; \varphi] \) of the projection \( Q_{\text{ideal}} g[\tilde{x}; \varphi] \) is unspecified. However, we claim that Eq.29 with arbitrary element \( P Q_{\text{ideal}} g[\tilde{x}; \varphi] = \)
$PQ_{\text{ideal}}Pq[\tilde{x}; \varphi]$ correctly describes the problems with ideal constraints since in such cases $PQ_{\text{ideal}} \simeq 0$, see App.3. The above indeterminacy in Eq.\ref{eq:29} occurs when we are outside of constraint surfaces. This means that the system has fewer degrees of freedom than the number of variables used in Eq.\ref{eq:29}. In another language we would say that nonphysical degrees of freedom appear, which lead to the presence of ambiguity of used formalism. Such ambiguity may affect the results obtained, if some of the variables (nonphysical variables) will not be expressed by the other, physical variables, according to the constraints. This we call a \textit{phenomenon of nonphysical degrees of freedom}. By the \textbf{physical variables} we understand here any minimal set of variables that are sufficient to uniquely describe the configuration of the system in accordance with the constraints (generalized variables).

### 3.3 One general ideal constraint

We illustrate the above process of thinking in the case of the one general constraint \ref{eq:59} considered in App.3. In this case, we can choose the following \textbf{symmetrical} projectors:

$$Q[\tilde{x}, \tilde{y}; \varphi] = \frac{R_{\text{ideal}}[\tilde{x}, t; \varphi]R_{\text{ideal}}[\tilde{y}, t; \varphi]}{\int R_{\text{ideal}}[\tilde{z}, t; \varphi]R_{\text{ideal}}[\tilde{z}, t; \varphi]d\tilde{z}}, \quad P_{\text{ideal}}[\tilde{x}, \tilde{y}; \varphi] = \frac{\dot{\varphi}(\tilde{x}, t)\dot{\varphi}(\tilde{y}, t)}{\int \dot{\varphi}(\tilde{z}, t)\dot{\varphi}(\tilde{z}, t)d\tilde{z}}$$

see (60) with Eq.\ref{eq:60} and (62). Then, on the constraint surface (59), we have:

$$QP_{\text{ideal}} = P_{\text{ideal}}Q \simeq 0 \quad (30)$$

as pointed by using the symbol $' \simeq '$ instead of the strong equality expressed by the symbol $'= '$ in the last equality. Strong equality in the above formulas is the result of the symmetry of used projectors, while weak equality results from the constraint equations: in this case the vector $\dot{\varphi}$ is tangent and vector $R_{\text{ideal}}$ - normal to the constraint surface (59) - at the point $\varphi$.

Using Eq.\ref{eq:5} and similar identity for projectors with subscript 'ideal', from \ref{eq:30}, we have:

$$PQ_{\text{ideal}} = Q_{\text{ideal}}P \simeq 0 \quad (31)$$

and hence:

$$P \simeq PP_{\text{ideal}} \simeq P_{\text{ideal}}P \quad (32)$$

see App.3.

The first equalities of \ref{eq:31} and \ref{eq:32} simplify equation \ref{eq:29} to the form:

$$L[\tilde{x}; \varphi] - f[\tilde{x}; \varphi] + \int d\tilde{y}P_{\text{ideal}}[\tilde{x}, \tilde{y}; \varphi] \{\lambda N[\tilde{y}; \varphi] + G(\tilde{y})\} = 0 \quad (33)$$

The resulting equation holds for any ideal constraints for which Eqs \ref{eq:30} take place. This means that constraints \ref{eq:3} have to be described by specific, symmetric projectors $Q = I - P$ considered in App.3.
4 Examples of linear dynamical constraints (LDC)

Let us collect the main results:

Eq. (10) is

\[ L[\tilde{x}; \varphi] - f[\tilde{x}; \varphi] + \int d\tilde{y} P[\tilde{x}, \tilde{y}; \varphi] \{ \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \} = 0 \]

DC (3) are

\[ f[\tilde{x}; \varphi] = QL[\tilde{x}; \varphi] \]

with fixed functionals \( L, f, P, Q \) - conjugate projectors (idempotent operators) satisfying Eq. (5):

\[ P[\tilde{x}, \tilde{y}; \varphi] + Q[\tilde{x}, \tilde{y}; \varphi] = \delta(\tilde{x}, \tilde{y}) \iff P + Q = I \]

Because

\[ PQ = QP = 0, \ P = P^2, \ Q = Q^2 \]

we see that DC (3) result immediately from Eq. (10).

Let us take DC (3) with

\[ f[\tilde{x}; \varphi] = \mu QL[\tilde{x}; \varphi] \] (35)

Hence and from Eq. (10)

\[ (I - \mu Q)L[\tilde{x}; \varphi] + \int d\tilde{y} P[\tilde{x}, \tilde{y}; \varphi] \{ \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \} = 0 \] (36)

We can tell immediately that, at \( \mu = 1 \), DC (10), (3) lead only to weakening of the original Eq. (10). What happens, for \( \mu \neq 1 \)? In this case, by inverting the operator \( I - \mu Q \), we get the following equation:

\[ L[\tilde{x}; \varphi] + (I - \mu Q)^{-1} \int d\tilde{y} P[\tilde{x}, \tilde{y}; \varphi] \{ \lambda N[\tilde{y}; \varphi] + G(\tilde{y}) \} = 0 \] (37)

One can understand this result if we take into account that now \( \int Q[\tilde{x}, \tilde{y}; \varphi]L[\tilde{y}; \varphi]d\tilde{y} \).

Another example of the linear DC (3) is given by:

\[ f[\tilde{x}; \varphi] = -QM \varphi(\tilde{x}) \equiv - \int d\tilde{y} d\tilde{z} Q(\tilde{x}, \tilde{y}) M(\tilde{y}, \tilde{z}) \varphi(\tilde{z}) \] (38)
with the linear functional: \( L_M[\ddot{x}; \varphi] = L[\ddot{x}; \varphi] + QM \varphi(\ddot{x}) \). The term \( QM \varphi \) can describe parameters which do not appear in the first term because, e.g., of symmetry in a certain area of considered equations.

One can finally say that any knowledge about the main linear term of Eq.\( \text{(1)} \) expressed in the form of \( \text{(3)} \), allows us to change this equation to the form of Eq.\( \text{(10)} \) if the analoge of the virtual work principle is assumed. In this way, relying more on observation than on the proliferation of some ideas, you can try to understand some phenomena.

5 Examples of nonlinear dynamical constraints (NDC) and linear original dynamics (LOD)

We assume that, for example:

\[
f[\ddot{x}; \varphi] = \mu QN[\ddot{x}; \varphi] \iff f[\ddot{x}; \varphi] = \mu \int Q[\ddot{x}, \ddot{y}; \varphi]N[\ddot{y}; \varphi]d\ddot{y}
\]

This effectively means that, for \( G = 0 \), we modify the nonlinear part of Eq.\( \text{(1)} \). In this case, Eq.\( \text{(10)} \) takes the form:

\[
L[\ddot{x}; \varphi] - \mu \int Q[\ddot{x}, \ddot{y}; \varphi]N[\ddot{y}; \varphi]d\ddot{y} + \int d\ddot{y}P[\ddot{x}, \ddot{y}; \varphi] \{ \lambda N[\ddot{y}; \varphi] + G(\ddot{y}) \} = 0
\]

or in short as

\[
L + (\lambda P - \mu Q) N + PG = 0
\]

Hence, the equivalent,

\[
(\lambda P - \mu Q)^{-1} (L + PG) + N = 0
\]

where \((\lambda P - \mu Q)^{-1} = \lambda^{-1}P - \mu^{-1}Q\), see (18). In other words, all the modification of the theory can be transferred to linear terms, although without non-linear terms the above modification disappears!

In all these examples one can treat some constant of motions as constraints related to a kind of material surfaces and some, as energy, as purely dynamical quantities, see also App.1.

5.1 Linear original dynamics

In this case, the starting equation is:

\[
L[\ddot{x}; \varphi] + G(\ddot{x}) = 0
\]
It is an inhomogeneous linear equation which should be changed to satisfy the constraints. In fact, we only have to modify previously derived formulas putting the nonlinear term $N \equiv 0$. It results that, for the ideal constraints, the modified formulas are nonlinear even for the linear constraints, see \eqref{62} and Eq.\eqref{29}.

6 Appendix

6.1 About analogy with classical mechanics and the essence of the constraints

In classical mechanics the main quantity around which everything revolves is an acceleration of objects either extended or point like particles. The accelerations in the dynamical equations appear in the linear way. Moreover, if the constraints are proper times differentiated with respect to time (once or twice), then accelerations also appear in the linear way, see \cite{1}. Such quantities, which describe changes, or changes of changes as in the case of acceleration, also appear in a linear way in the case of ‘physical’ fields describing extended systems. They are responsible for the additional conditions as the initial and boundary conditions, which must be taken into account to get an unique solution to the considered equations.

When we look at constraints as constants of motion, the question naturally arises, what is the difference? The difference lies in the fact that other constants of motion are not carried out by the physical surfaces as the constants of motion interpreted as the constraints.

Constants of motion related to constraints are explicitly present in the theory. Constrains, however, limit the initial conditions of theory and failure to do so push the system outside the surface of the constraints. They also are related to a global description of systems. Constrains have an effect on local interaction of individuals composing complex systems like economic systems.

6.2 About the classical and dynamical constraints ((CC) and(DC))

By CC we understand mathematical or physical restrictions which descriptions does not require 'accelerations' or their analogues. In classical mechanics they are called holonomic and nonholonomic constraints. From their definitions results that for systems with CC the initial and boundary conditions can not be arbitrary. It is result of fact that CC eliminate some number of degrees of freedom like in the case of pendulum or incompressible liquid.

Main difference with DC is such that CC are automatically realized by there equations: 'surfaces' which realizes such constraints. This is not the case of DC which are realized by the extra forces calculated with the help of dynamical equations!
6.3 Spherical and more general ideal constraints

Let us assume that we have the following spherical constraint:

\[ \int d\bar{y} \varphi^2(\bar{y}, t) = \text{constant} = R^2 \]  

(45)

Hence,

\[ \int d\bar{y} \varphi(\bar{y}, t) \dot{\varphi}(\bar{y}, t) = 0 \]  

(46)

and

\[ \int d\bar{y} \varphi(\bar{y}, t) \ddot{\varphi}(\bar{y}, t) + \int d\bar{y} \dot{\varphi}(\bar{y}, t) \dot{\varphi}(\bar{y}, t) = 0 \]  

(47)

In this case, to get an analogue of formula (13), or rather (16), we can choose:

\[ B[\bar{x}, \bar{y}, t; \varphi] = \delta(\bar{x} - \bar{y}) \varphi(\bar{x}, t) \]  

(48)

where \( B \) is a nonsingular operator at least for \( t \) for which \( \varphi \neq 0 \);

\[ B_R^{-1}[\bar{y}, \bar{z}, t; \varphi] = B^{-1}[\bar{y}, \bar{z}, t; \varphi] = \delta(\bar{y} - \bar{z}) \frac{1}{\varphi(\bar{y}, t)} \]  

(49)

\[ Q'(\bar{x}, \bar{y}) = \frac{1}{V} \int d\bar{x} \delta(\bar{x} - \bar{y}) = \frac{1}{V} \]  

(50)

and

\[ b[t; \varphi] = - \int d\bar{y} \dot{\varphi}(\bar{y}, t) \dot{\varphi}(\bar{y}, t) \]  

(51)

Here \( V \) denotes the volume of an integration region, \( \bar{x} \in V \). Of course, (50) is a projector, which action on a function is reduced to integration and multiplication by the factor \( 1/V \) to get in result a constant. Now, we can use the formula (16) and (17) to describe CC (32) in the form of Eq. 3 of DC with

\[ Q[\bar{x}, \bar{y}, t; \varphi] = B_R^{-1} Q' B[\bar{x}, \bar{y}, t; \varphi] = \int B_R^{-1}[\bar{x}, \bar{z}, t; \varphi] Q'[\bar{z}, \bar{w}] B[\bar{w}, \bar{y}, t; \varphi] = \int d\bar{z} \delta(\bar{z} - \bar{x}) \frac{1}{\varphi(\bar{x}, t)} \int d\bar{w} \delta(\bar{w} - \bar{y}) \varphi(\bar{w}, t) = \frac{1}{V} \frac{\varphi(\bar{y}, t)}{\varphi(\bar{x}, t)} \]  

(52)

Hence, in the DC (3),

\[ f[\bar{x}, t; \varphi] = - \frac{1}{\varphi(\bar{x}, t)} \int d\bar{y} \dot{\varphi}(\bar{y}, t) \dot{\varphi}(\bar{y}, t) \]  

(53)

It is worth noting here that \( Q \) is a projector, but it is a symmetric projector only for all field variables equal to each other:

\[ \varphi(\bar{x}, t) = \varphi(\bar{y}, t), \text{ for } \bar{x}, \bar{y} \in V \]  

(54)
In other cases, (3) and (7), with (52), can describe the non-ideal constraints described by a surrogate of virtual work principle:

$$PR[\bar{x}, t; \varphi] = R[\bar{x}, t; \varphi] - \frac{1}{V(\varphi(\bar{x}, t))} \int d\bar{y} \varphi(\bar{y}, t) R[\bar{y}, t; \varphi] = 0$$  \hspace{1cm} (55)

where the projector $P$ was chosen as:

$$P[\bar{x}, \bar{y}, t; \varphi] = \delta(\bar{x} - \bar{y}) - Q[\bar{x}, \bar{y}, t; \varphi] = \delta(\bar{x} - \bar{y}) - \frac{1}{V(\varphi(\bar{x}, t))} \varphi(\bar{y}, t)$$  \hspace{1cm} (56)

This projector reflects circular symmetry in the case of non-ideal constraints (32).

From (55),

$$R[\bar{y}, t; \varphi] = \frac{G[t; \varphi]}{\varphi(\bar{y}, t)}$$  \hspace{1cm} (57)

where a functional $G$ does not depend on the variable $\bar{y}$. The values of 'field' in the denominator should not necessarily worry us, because the infinity of the expression $1/\varphi(\bar{y}, t)$, for $t \to t'$, can be simultaneously neutralized by $G \to 0$.

Once more, for the ideal spherical constraints, where a sphere is considered in the space $F$ of functions $\varphi$, we should have:

$$R[\bar{y}, t; \varphi] = H[t; \varphi] \varphi(\bar{y}, t)$$  \hspace{1cm} (58)

with a functional $H[t; \varphi]$ which do not depend on variable $\bar{y}$. From (57) we get

$$\varphi(\bar{y}, t)^2 = \frac{G[\varphi]}{H[\varphi]}$$

but this would mean that $\varphi$ does not depend on $\bar{y}$ in a continuous way. It also means that in this case the conditions (55) and (3) can not describe ideal constraints.

Spherical constraints describe the simplest nonlinear, holonomic constraints in physics. They contain the symmetry of the circle, which throughout human history has been synonymous with - excellence. So would not be strange if they would be found in some basic field theory describing the Universe. A sphere in the configuration space of such system as the universe is the favorite model in cosmology. It was also considered by Henri Poincare, see Wikipedia. In fact, the constraints (32) do not mean that all particles are located on the sphere with radius $R$ but only that the sum of all squares of their radius vectors is equal to $R^2$. They can describe a fancy model of particles in which location of one particle at the extreme distance equal to $R$ leads to locations of other particles at the center of the sphere with radius $R$! In other words, in this model the influence of the global quantity represented by Eq. (32) on the local inter-particle interaction can be traced.
6.3.1 A single scleronomic ideal constraint

\[ H[\varphi] = \text{constant} \]  

(59)

In this case the reaction forces are proportional to the gradient of the functional \(H\):

\[ R_{\text{ideal}}[\bar{y}; \varphi] \propto \frac{\delta H[\varphi]}{\delta \varphi(\bar{y},)} \equiv V[\bar{y}; \varphi] \equiv V \]  

(60)

describes a given constraint surface. \(H\) may depend on the 'space' variable \(\bar{y}\) of the function \(\varphi\) in the non-local way, see Eq.(32). Like in classical mechanics we assume that 'all' \(\varphi\) are taken at the same time \(t\). We also assume that the functional derivative \(\delta / \delta \varphi(\bar{y})\) is defined in such a way that \(\delta \varphi(\bar{x})/\delta \varphi(\bar{y}) = \delta(\bar{x} - \bar{y})\). Let us also notice that from (59)

\[ \frac{d}{dt}H[\varphi(t)] = \int d\bar{y}V[\bar{y}; \varphi(t)]\dot{\varphi}(\bar{y}, t) = 0 \]  

(61)

The above equation resulting from the observation that \(H\) is also a constant of motion, see (59), shows to us that \(R_{\text{ideal}}\) defined by Eq.(60) is perpendicular to the surface \(H\) at the 'point' \(\varphi\).

For the projector

\[ P_{\text{ideal}} = \frac{\dot{\varphi}(\bar{x}, t)\dot{\varphi}(\bar{y}, t)}{\int \varphi(\bar{z}, t)\varphi(\bar{z}, t)d\bar{z}} \]  

(62)

where \(\dot{\varphi}(\bar{x}, t) = \frac{\partial}{\partial t} \varphi(\bar{x}, t)\), we have

\[ \int P_{\text{ideal}}[\bar{x}, \bar{y}; \varphi]R_{\text{ideal}}[\bar{y}; \varphi]d\bar{y} = 0 \]  

(63)

Hence, we can interpret the projector \(P_{\text{ideal}} \equiv P_{\text{ideal}}[\bar{x}, \bar{y}, t; \varphi]\) as an operator projecting on the tangent space of the surface (59) at the 'point' \(\varphi\).

We have to remind you that in the all above formulas, the symbol \(d\bar{y}\) means that in vectors \(\bar{x}, \bar{y}, \bar{z}\) all time components are equal to \(t\). In the case of general, scleronomic (explicitly independent of time) ideal constraints described by the Eq.(59) by double differentiations, we get an equation similar to Eq.(13):

\[ \hat{\delta}V[\bar{y}; \varphi(t)]\ddot{\varphi}(\bar{y}, t) = b[\dot{\varphi}, \varphi, t] \]  

(64)

which can be described in an equivalent form as follows:

\[ V[\bar{x}; \varphi(t)] \int d\bar{y}V[\bar{y}; \varphi(t)]\dot{\varphi}(\bar{y}, t) = V[\bar{x}; \varphi(t)]b[\dot{\varphi}, \varphi, t] \equiv f[\bar{x}; \varphi] \]  

(65)

This is a rather peculiar equivalent form of Eq.(64) but thanks to the above substitution the constraints (59) can be described in the form of Eq.(3). This can be seen if the symmetric projector
\[ Q[\bar{x}, \bar{y}; \varphi] = \frac{V[\bar{x}; \varphi(t)]V[\bar{y}; \varphi(t)]}{\int V[\bar{z}; \varphi(t)]V[\bar{z}; \varphi(t)]} \quad (66) \]

is introduced \( \clubsuit \). For definition of \( V[\bar{y}; \varphi] \), see Eq.(60). Further generalization of this topic see just below.

6.3.2 A few sclerenemic ideal constraints

In such cases, instead of a single Eq.(59) we have more equations:

\[ H_i[\varphi] = constant, \quad \text{for } i = 1, 2, ..., k \quad (67) \]

To each of them one can write:

\[ \frac{d}{dt}H_i[\varphi(t)] = \int d\bar{y} \frac{\delta H_i[\varphi(t)]}{\delta \varphi(\bar{y}, t)} \dot{\varphi}(\bar{y}, t) \equiv \int d\bar{y}V_i[\bar{y}; \varphi]\dot{\varphi}(\bar{y}) \approx 0 \quad (68) \]

where the symbol \( \approx' \) means that Eqs (68) are satisfied only if \( \varphi \) fulfills constrain equations (67).

Acting again with the time derivative on the (68), we get

\[ \int d\bar{y}V_i[\bar{y}; \varphi] \dddot{\varphi}(\bar{y}) \approx f_i[\varphi, \dot{\varphi}] \quad (69) \]

for \( i = 1, ..., k \). Goal that we set now is: How constraints (67) described in the form (69) can be written in the form of the Eq.3 with symmetrical projector \( Q \)? For the sake of simplicity, we assume that

\[ L[\bar{x}; \varphi] \equiv \dddot{\varphi}(\bar{x}) = \dddot{\varphi}(t, \bar{x}, \alpha, \beta, \gamma, \ldots) \quad (70) \]

Let us treat these \( k \) f.f. \( V_i[\bar{y}; \varphi] \) as \( k \) vectors denoted by \( V_i \):

\[ V_i \iff V_i[\bar{y}; \varphi] = \frac{\delta H_i[\varphi]}{\delta \varphi(\bar{y})} \quad (71) \]

Then, (68) is:

\[ < V_i, \dot{\varphi} > \approx 0 \quad (72) \]

for \( i = 1, ..., k \). These equations tell us that at the 'point' \( \varphi \) the vectors \( V_i \) are perpendicular to the vectors \( \dot{\varphi} \), the infinitesimal change of which to the infinitesimal change of time is tangent to the constraints surface. If vectors \( V_i \), which also enter Eq.(68), are linear independent, then by means of Gram-Schmidt process one can construct \( k \) orthonormal vectors \( U_i \) which are also perpendicular to the velocity vectors \( \dot{\varphi} \). With the help of them and Dirac’s notation one can express the projector \( Q \) of Eq.3 as follows:

\[ Q = \sum_{i=1}^{k} |U_i > < U_i| \iff Q[\bar{x}, \bar{y}; \varphi] = \sum_{i=1}^{k} U_i[\bar{x}; \varphi]U_i[\bar{y}; \varphi] \quad (73) \]
where we have assumed that all values of functions $U_i$ are real. This is a generalization of formula (66). The projector $P = I - Q$ projects on space of vectors tangent to the constraint surface at the point $\varphi$. One can also show that the above projector $Q$ constructed by means of the Gram-Schmidt process satisfies

$$QP_{\text{ideal}} = P_{\text{ideal}}Q \simeq 0$$

Hence, introducing the pair of projectors $P_{\text{ideal}} + Q_{\text{ideal}} = I$, where $P_{\text{ideal}}$ is given by the formula (62), one can derive the following equalities:

$$Q \simeq Q_{\text{ideal}}Q_{\text{ideal}}$$

and

$$P_{\text{ideal}} \simeq PP_{\text{ideal}}$$

which allows to make the following identification of projectors:

$$Q \simeq Q_{\text{ideal}}, \quad P \simeq P_{\text{ideal}}$$

The above identifications allow us to satisfy Eqs (75) as well Eqs (74). However, to describe Eq (29) as an equation (33) the scleronomic constraints (67) should be described in the form (3) with the help of the projector (76) constructed with Gram-Schmidt process. See Sec.3 (A phenomenon of nonphysical degrees of freedom).

To see that conditions (72) are also satisfied by orthogonal vectors $U_i'$ obtained from vectors $V_j$ in the Gram-Schmidt process I will write them here with the help of function:

$$\text{proj}_U(V) = \frac{<U, V>}{<U, U>} U$$

and the reccurent formula

$$U'_1 = V_1,$$

$$U'_j[\bar{x}; \varphi] \iff U'_j = V_j + \sum_{i=1}^{j-1} \text{proj}_{U'_i}(V_j)$$

for $j=2,...,k$. It is easy to see that the projector $Q$ constructed by means of normalized vectors, $U_j = U'_j/\sqrt{<U'_j, U'_j>}$, via the formula (74), satisfies conditions (74).

### 6.4 About one-sided constraints (CC) in classical mechanics; short-range forces

On this subject I speak of the following reasons: First, in the Internet, I found the discussion of such constraints by means of advanced means or complicated cases including solid mechanics. Secondly, as previously discussed, I am focusing not on the elimination of redundant degrees of freedom, but on the forces that are doing it.
In the case of \( n \) material points, the one-sided constraints are characterized not by equations but by inequalities. Thus, in the case of holonomic constraints we have:

\[
f_i(\vec{r}_1, ..., \vec{r}_n; t) \leq 0, \text{ for } i = 1, ..., k < 3n
\]

where \( \vec{r}_i \) means the radius vector of the \( i \)-th particle. Inequalities mean a drastic loosening of restrictions: only if there is 'threat' of their failure, the system 'suffers' of reaction forces. Such situation can be described by short-range forces, whose centers satisfy the equations

\[
f_i(\vec{r}_1, ..., \vec{r}_n; t) = 0, \text{ for } i = 1, ..., k < 3n
\]

Usually, the surfaces satisfying the above equations are calle the walls. Short-range reaction forces should be a priori chosen in such a way that an energy, which is available for individual particles is not enough to cross the walls. In this way we avoid tracking, when the particles are hitting in to the walls, nor the need for discontinuous changes in their momenta. Everything is encoded in the dynamical equations.

### 6.5 About one-sided invertible operators

A right invertible operator \( A \) is defined as an operator for which one can write the following equation:

\[
AA_R^{-1} = I
\]

with not uniquely chosen a right inverse operator \( A_R^{-1} \) and the unite operator \( I \) in a considered linear space. For a left invertible operator, we would have a similar definition, but the operator \( A_R^{-1} \) is substituted by an operator \( A_L^{-1} \) standing at the l.h.s. of the operator \( A \):

\[
A_L^{-1}A = I
\]

Occurring here operators \( A_R^{-1}, A_L^{-1} \) satisfy the first two demands of the Moore-Penrose definition of the generalized inverse (pseudoinverse) denoted by \( A^+ \):

\[
(1) \quad AA^+A = A
\]

\[
(2) \quad A^+AA^+ = A^+
\]

and often, in considered examples, are satisfied the second two demands of the Moore-Penrose definition:

\[
(3) \quad (AA^+)^* = AA^+
\]

\[
(4) \quad (A^+A)^* = A^+A
\]
see [4], what guarantees of getting a least squares solution to the considered system of equations.

We think however that one-sided invertible operators in the sense of Eqs(77,78), are more simple and therefore more useful for basic description of nature, and except that, the request: 'least squares solution' is not always necessary. see [3, 4] and other author’s 'recent' papers.

6.6 About strange behavior of some objects

Let us assume that we consider a discrete system such that from Eq.4 $R$, the reaction force, has to be a vector. In this case Eq.57 means that $G$ is not a scalar but must behave so that $R$, at the transformation of the coordinate system, is the vector. Taking, however, the scalar product of the two vectors $\varphi, R$ :

$$(\varphi(-, t), R[\cdot, t; \varphi]) = VG[t; \varphi]$$

we should get, in the r.h.s., the scalar. This explicit contradiction, we can probably explained by the fact that $G$ behaves as a scalar on the subset of vectors $\varphi$ satisfying Eq.32.

6.7 About space $\tilde{M}$, Cantor’s theorem and evolution theory

In Sec.1 we said that the set $\tilde{M}$ consists of elements (vectors) reflecting specific properties of the considered system. This is only partly true because in these elements are also included certain properties of the observer as the experience of one, two or three dimensional spaces. As we know from the Cantor’s theorem, there is 1-1 correspondence between the points of the plane or of n-dimensional space and of the straight line. It seems, however, that the identification of objects with a higher dimensional space is much simpler and effective than using the one-dimensional, and this was used at least by some organisms, see also [5], page 20, where other opinions are presented.

Higher dimensional spaces particularly preferred by quantum field theory to get meaningful theory appear to be evidence of the fact that even in the field of logic a similar phenomenon can be observed. By means of constraints certain dimensions can be roll up. By means of them also some constants can be introduced into considered equations.

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