Ferromagnetism and superconductivity in twisted double bilayer graphene

Fengcheng Wu\textsuperscript{1} and Sankar Das Sarma\textsuperscript{1}

\textsuperscript{1}Condensed Matter Theory Center and Joint Quantum Institute, Department of Physics, University of Maryland, College Park, Maryland 20742, USA

(Dated: June 19, 2019)

We present a theory of competing ferromagnetic and superconducting orders in twisted double bilayer graphene. In our theory, ferromagnetism is induced by Coulomb repulsion, while superconductivity with intervalley equal-spin pairing is mediated by electron-acoustic phonon interactions. We calculate the ferromagnetic and superconducting transition temperature as a function of moiré band filling factor, and find that superconducting domes can appear on both the electron and hole sides of the ferromagnetic insulator at half filling. We also show that the ferromagnetic insulating gap and transition temperature can be tuned by controlling the electrostatic environment.

Introduction.— Moiré superlattices form in van der Waals bilayers with a small orientation misalignment and/or lattice constant mismatch. Recently moiré bilayers have emerged as a platform to study fundamental physics of strongly interacting systems, in view of the discovery of correlated insulator (CI) and superconducting (SC) state in twisted bilayer graphene\cite{11,12}. Moiré superlattices often generate spatial confinement for low-energy electrons, suppress electron kinetic energy, and therefore effectively enhance interaction effects. Evidence of CI and SC states has so far been reported in three graphene-based moiré systems, including twisted bilayer graphene (TBG)\cite{11,13}, twisted double bilayer graphene (TDBG)\cite{12,14}, and ABC trialyer graphene on hexagonal boron nitride\cite{15–17}. TBG is a subject under intense theoretical study\cite{18–60}, but the exact nature of the CI and SC state in TBG remains elusive. The half-filled CI in TBG crosses over to a metallic state by a strong perpendicular or parallel magnetic field\cite{11,12}, which possibly rules out spin-polarized ferromagnetic (FM) states, but leaves a large number of non-FM states as candidates, e.g., valley polarized state, and charge/spin/valley density wave states to name a few. By contrast, there appears to be clear experimental evidence that the half-filled CI in TDBG with a twist angle $\theta$ around 1.3° is spin polarized\cite{12,14}. Moiré bands in TDBG can be effectively tuned by an out-of-plane electric displacement field. When the first moiré conduction band in TDBG ($\theta \sim 1.3^\circ$) is half filled, a CI develops for a certain range of displacement field, and the insulating gap is enhanced by an in-plane magnetic field\cite{12,14} with a g-factor close to 2\cite{12,13}, strongly suggesting a spin-polarized FM insulating state. SC domes also appear in TDBG when the half-filled CI is doped with electrons\cite{12,13}. From a theoretical point of view, TDBG represents a simpler system compared to TBG, because the first moiré conduction band in TDBG can be isolated in energy from other bands, whereas the first moiré valence and conduction bands in TBG are connected via Dirac points enforced by symmetry\cite{19}.

In this Letter, we theoretically study FM and SC orders in the first moiré conduction band of TDBG. In our theory, ferromagnetism is driven by Coulomb repulsion as in Stoner model, but superconductivity is mediated by electron-phonon interactions. Ferromagnetic correlated insulator (FMCI) can occur at half filling when spin majority and minority bands are separated in energy by Coulomb exchange interaction. Away from half filling, the state is generally metallic, which can be susceptible to SC instability at low temperature. Because electron-acoustic phonon interactions in graphene mediate both spin singlet and spin triplet intervalley Cooper pairing\cite{58}, superconductivity can take place even in the presence of FM order, as long as the spinless time-reversal symmetry is preserved. We estimate the FM and SC transition temperatures as a function of filling factor, and find SC domes on both sides of the half-filled FMCI.

Moiré bands.— We consider TDBG with a small twist angle $\theta$ (Fig. 1), and calculate the moiré band structure using a continuum Hamiltonian generalized from TBG\cite{61} to TDBG\cite{62,63}, with details given in the Supplemental Material\cite{66}. A representative moiré band structure is shown in Fig. 2 for $\theta = 1.24^\circ$ and $U = 45$ meV, where $U$ parametrizes the layer-dependent potential [Fig. 1(b)] generated by the displacement field. In this work, we focus on the first conduction band in Fig. 2(a), which is isolated in energy from other bands, narrow in bandwidth ($\sim 13$ meV), and topologically nontrivial with a Chern number of $+2$ in $+K$ valley.
Ferromagnetism.— We study flatband FM order driven by Coulomb repulsion using a momentum-space approach [17, 62, 65], and only retain the first conduction band plotted in Fig. 2(a). The projected single-particle Hamiltonian has a simple form as follows

$$H_0 = \sum_{\mathbf{k}, \tau, s} \varepsilon_{\mathbf{k}, \tau} c_{\mathbf{k}, \tau, s}^\dagger c_{\mathbf{k}, \tau, s},$$

(1)

where $\mathbf{k}$ is momentum measured relative to the center of the moiré Brillouin zone, $\tau = \pm$ is the valley index and $s$ represents spin ($\uparrow, \downarrow$). $\varepsilon_{\mathbf{k}, \tau}$ is the spin independent moiré band energy, and its valley dependence is constrained by time reversal symmetry with $\varepsilon_{\mathbf{k}, \tau} = \varepsilon_{-\mathbf{k}, -\tau}$.

We project Coulomb interaction onto the first conduction band, and the interacting Hamiltonian has the form

$$H_1 = \frac{1}{24} \sum_{\mathbf{k}, \mathbf{k}', s} V_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}(\mathbf{r}) c_{\mathbf{k}_1, \tau, s}^\dagger c_{\mathbf{k}_2, \tau', s'}^\dagger c_{\mathbf{k}_3, \tau', s'} c_{\mathbf{k}_4, \tau, s},$$

$$V_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}(\mathbf{r}) = \sum_q V(q) O_{\mathbf{k}_1 \mathbf{k}_4}(q) O_{\mathbf{k}_2 \mathbf{k}_3}(-q),$$

(2)

where $A$ is the system area and $\Phi_{\tau, \mathbf{k}}(r)$ is the Bloch wave function for the first conduction band in valley $\tau \mathbf{k}$ and at momentum $\mathbf{k}$. The indices $\sigma$ and $\ell$ respectively label sublattices and layers. By time-reversal symmetry, $\Phi_{\tau, \mathbf{k}}(r) = \Phi_{-\tau, -\mathbf{k}}^*(r)$. Hamiltonian $H_1$ represents density-density interaction, and preserves spin $\text{SU}(2)$ and valley $\text{U}(1)$ symmetry. In fact, $H_1$ has an enlarged $\text{SU}(2) \times \text{SU}(2)$ symmetry, which stands for an independent spin rotational symmetry within each valley. Short-range interactions (e.g., atomic scale on-site Hubbard repulsion), which we do not study explicitly, breaks the $\text{SU}(2) \times \text{SU}(2)$ symmetry down to spin $\text{SU}(2)$ symmetry.

We use Coulomb interaction screened by dielectric environment and nearby metallic gates. To mimic the experimental setup [12, 14], we assume that TDBG is encapsulated by an insulator (typically boron nitride), and is in the middle of two metallic gates, which generates an infinite series of equally spaced image charges with alternating signs. Under this image charge approximation, the screened Coulomb potential in momentum space is

$$V(q) = \frac{2 \pi \epsilon^2}{eq} \frac{2}{1 + \exp(qd)},$$

(3)

where $\epsilon$ is the dielectric constant of the encapsulating insulator, and $d/2$ is the vertical distance between the top (bottom) metallic gate and TDBG. We take $\epsilon$ and $d$ as free parameters that control the strength of Coulomb interaction. The Coulomb interaction energy scale is set by $E_C = \epsilon^2/(\epsilon a_M)$. At $\theta = 1.24^\circ$, $a_M \approx 11.4$nm, and $E_C$ is about 25 meV if we use $\epsilon = 5$. Since the typical Coulomb interaction energy scale is sizable compared to the bandwidth (~10 meV), there is a strong tendency towards symmetry-breaking phases driven by interactions. The system is characterized by almost-flat narrow bands with large Coulomb energy, a classic situation for the manifestation of strong correlation physics.

We use Hartree-Fock (HF) approximation and assume that both moiré periodicity and valley $\text{U}(1)$ symmetry
can be linearized as follows

\[
\Sigma_{k,\tau}^{(\tau)} = \sum_{k'} M_{kk'}^{(\tau)} \Sigma_{k',\tau}^{(\tau)},
\]

\[
M_{kk'}^{(\tau)} = -\frac{1}{A} \sum_{\tau\tau'} V_{kk'kk'} n_F (\epsilon_{k',\tau} + \Sigma_{k',\tau}),
\]

from which \( T_{\text{FM}} \) can be obtained by requiring the largest eigenvalue of the matrix \( M^{(\tau)} \) to be 1.

The transition temperature \( T_{\text{FM}} \) as a function of band filling factor is shown in Fig. 3(a) and 3(b) for \( \epsilon = 5 \) and \( \epsilon = 10 \), respectively, with the same \( d \) value of 6 nm. In Fig. 3 there is a large range of filling factors where ferromagnetism develops with \( T_{\text{FM}} \) up to few tens of Kelvin. We note that, as usual, our mean-field theory overestimates the tendency towards ordering, as fluctuations like spin waves are neglected in our theory. In addition, even within the HF theory, our neglect of the higher bands leads to an over-estimation of the \( T_{\text{FM}} \). Therefore, we focus on the qualitative aspect of our theory. The FM state at half filling can be an insulator at zero temperature, when the spin majority bands are fully filled and separated from the empty spin minority bands by an energy gap \( \Delta_{\text{FM}} \). As illustrated in Fig. 4(a), \( \Delta_{\text{FM}} \) is tunable by varying \( d \) and \( \epsilon \), providing an important experimental knob to control the phase diagram. At small enough \( d \) and large enough \( \epsilon \) [region (II) of Fig. 4(a)], spin majority and minority bands overlap in energy and the half-filled FM state is no longer insulating, which is the case for Fig. 3(b). In contrast, the half-filled FM state in Fig. 3(a) has an insulating gap \( \Delta_{\text{FM}} \) that persists up to \( T_{\text{FM}} \), which is smaller than \( T_{\text{FM}} \), as the insulator is destroyed before spin polarization vanishes [Fig. 4(b)]. Thus, we have a clear prediction on how to tune the TDBG FMI state by controlling the electrostatic environment either by changing the effective dielectric constant and/or by using metallic gates to suitably suppress the Coulomb interaction, thus suppressing the FM phase. Next we consider possible SC ordering in TDBG, which in our theory cannot arise from Coulomb repulsion.

Superconductivity. — While the state at half filling can be a correlation–driven FM insulator, it is generally metallic away from half filling and can be susceptible to SC instability due to enhanced electron-phonon interaction in moiré flatband systems. Here we consider electron-acoustic phonon interactions [20], which mediate effective electron attraction as follows

\[
H_{\text{att}} = -g_0 \sum_{\sigma,\sigma',\ell,s} \int d\mathbf{r} \hat{\psi}_{\sigma,\ell,s}(\mathbf{r}) \hat{\psi}_{\sigma',\ell,s} \hat{\psi}_{-\sigma',\ell,s} \hat{\psi}_{-\sigma,\ell,s},
\]

where \( \hat{\psi}_{\tau,\ell,s}(\mathbf{r}) \) is the electron field operator at the coarse-grained position \( \mathbf{r} \) associated with valley \( \tau K \), sublattice \( \sigma = A, B \), layer \( \ell = 1, 2, 3, 4 \) and spin \( s = \uparrow, \downarrow \). In Eq. (6), we only retain attractive interactions that pair electrons from opposite valleys. The coupling constant \( g_0 \) is given by \( D^2/(\rho_m \overline{v}_F^2) \), where \( D \) is the deformation potential, \( \rho_m \)
is the mass density of monolayer graphene, and \( v_s \) is the velocity of acoustic longitudinal phonon. Using \( D = 30 \text{ eV}, \rho_m = 7.6 \times 10^{-8} \text{ g/cm}^2, v_s = 2 \times 10^6 \text{ cm/s}, \) we estimate \( g_0 \) to be 474 meV nm\(^2\). Here we neglect retardation effects in the phonon mediated electron attraction for simplicity [67].

The attraction in Eq. (6) can be decomposed into four different pairing channels \([58]\) that are distinguished by their orbital and spin characters: (1) intrasublattice spin-singlet \( s \)-wave pairing, i.e., \((s_y)_{ss'Fss'F}^\dagger \psi_+^\dagger \psi_- \); (2) intrasublattice spin-triplet \( p \)-wave pairing, e.g., \( F_{ss'}^\dagger \psi_+^\dagger_{ss'} \psi_- \); where \( F \) can be any one of the three symmetric tensors \([s_0 \pm s_z]/2 \) and \( s_z \); (3) intersublattice spin-singlet \( d \)-wave pairing, e.g., \((s_y)_{ss'}F_{ss'}^\dagger \psi_+^\dagger \psi_+^\dagger \); and (4) intrasublattice spin-triplet \( f \)-wave pairing, i.e., \( F_{ss'}^\dagger \psi_+^\dagger_{ss'} \psi_- \). The \( s \)-wave and \( f \)-wave pairings are only distinguished by their spin characters, and the same is true for \( p \) and \( d \) pairings. The angular momenta of intersublattice Cooper pairs arise from the valley-contrast sublattice chirality under threefold rotation [57, 58]. In AB bilayer graphene, one of the sublattices in each layer \([A_1 \text{ and } B_2 \text{ sites in Fig. 1(b)}]\) is pushed to higher energy by interlayer tunneling. Therefore, intersublattice pairing is energetically less favorable compared to intrasublattice pairing in TDBG. In the following, we only consider interactions that pair electrons on the same sublattice, and project such interactions onto the first moiré conduction band. The projected pairing Hamiltonian is

\[
H_p = -\frac{1}{2} \sum_{\kappa,\sigma} g_{\kappa\kappa'} c_{\kappa,+,\sigma}^\dagger c_{\kappa',-,\sigma}^\dagger + \text{H.c.},
\]

\[
g_{\kappa\kappa'} = g_0 A \sum_{\alpha,\ell} \int d^2r |\Phi_{+,\kappa,\sigma,\ell}(r)|^2 |\Phi_{+,\kappa',\sigma,\ell}(r)|^2,
\]

where we only keep interactions that pair electrons with opposite momenta, i.e., momentum \( \kappa \) in \(+K\) valley and momentum \(-\kappa\) in \(-K\) valley. The pairing Hamiltonian \( H_p \) also has the SU(2)\( \times \)SU(2) symmetry, and supports both spin singlet \( s \)-wave and spin triplet \( f \)-wave pairings. Because of the ferromagnetism induced by Coulomb repulsion, equal spin pairing is more favored compared to spin singlet pairing. Therefore, we consider intervalley pairing between electrons with the same spin, which leads to the following Bardeen-Cooper-Schrieffer (BCS) mean-field Hamiltonian

\[
H_{BCS} = -\sum_{\kappa,\sigma} (\Delta_{\kappa,\sigma} c_{\kappa,+,\sigma}^\dagger c_{\kappa,-,\sigma} + \text{H.c.}),
\]

\[
\Delta_{\kappa,\sigma} = \frac{1}{2} \sum_{\kappa'} g_{\kappa\kappa'} \langle c_{\kappa',-,\sigma}^\dagger c_{\kappa',+,\sigma}^\dagger \rangle.
\]

By combining the BCS Hamiltonian \( H_{BCS} \) and the effective single-particle Hamiltonian \( H_{FM} \) [Eq. (4)] that is renormalized by the Coulomb interaction, we obtain the SC linearized gap equation

\[
\Delta_{\kappa,\sigma} = \sum_{\kappa'} \chi_{\kappa\kappa'}(\kappa') \Delta_{\kappa',\sigma},
\]

\[
\chi_{\kappa\kappa'} = \frac{g_{\kappa\kappa'}^2}{2E_{\kappa',+,\sigma}^\dagger - E_{\kappa',+,\sigma} - \mu},
\]

where \( \mu \) is the chemical potential, and \( E_{\kappa,\tau,\sigma} = \varepsilon_{\kappa,\tau} + \Sigma_{\kappa,\tau,\sigma} \) is the effective band energy including the self energy. We have used the spinless time-reversal symmetry, which implies \( E_{\kappa,\tau,\sigma} = E_{-\kappa,\tau,\sigma} \), to simplify the SC susceptibility \( \chi(s) \). Because of this symmetry, FM order does not lead to depairing effect for superconductivity with intervalley equal-spin pairing. In Eq. (9), spin up and down channels have independent gap equations. The SC transition temperature \( T_{SC} \) is reached when the largest eigenvalue of \( \chi(s) \) is 1. Fig. 3 plots \( T_{SC} \) as a function of filling factor, and shows two SC domes respectively on the two sides of the half-filled FM state. In Fig. 3 maximum \( T_{SC} \) reaches about 1.5 K, which depends crucially on the flatness of the moiré band. The experiments in Refs. 12 and 13 so far only observed the SC dome on the electron side of the half-filled FM insulator, possibly because \( T_{SC} \) for the dome on the hole side can be very low [Fig. 3(b)].

We discuss the effect of an in-plane magnetic field \( B_\parallel \) on \( T_{SC} \) in the \( f \)-wave channel. If the parent state for superconductivity is spin unpolarized, then \( T_{SC} \) can be slightly enhanced by \( B_\parallel \) in the low-field regime, because Zeeman energy leads to an effective spin dependent chemical potential shift [60, 65]. On the other hand, if the parent state already has maximum spin polarization allowed by a given filling factor, then an externally applied \( B_\parallel \) field can no longer change the amount of spin polarization, and \( T_{SC} \) is reduced by \( B_\parallel \) due to orbital effect [60, 65]. In Ref. 13 \( T_{SC} \) is found to be slightly enhanced by weak \( B_\parallel \) field, indicating that the SC state has spin triplet pairing but with no spin polarization. Our mean-field phase diagram in Fig. 3 likely overestimates the filling factor regime of stability for ferromagnetism. We emphasize that there is always a SC instability in a partially filled band regardless of the presence or absence of ferromagnetism in our theory, where the superconductivity is mediated by electron-phonon interactions and ferromagnetism is driven by Coulomb repulsion.

Ferromagnetism and superconductivity are two prototypical orders that can occur in moiré flat bands, while there are many other possible competing and/or intertwined orders, such as nematicity that breaks rotational symmetry and density wave state that breaks moiré translation symmetry. In TDBG, there is experimental evidence that states with both spin and valley polarization are possibly stabilized at 1/4 and 3/4 fillings by a finite in-plane magnetic field [13, 14]. Our work should be viewed as a step towards a full quantitative theory of the potentially very rich TDBG phase.
diagram. The most noteworthy qualitative feature of the current work is the possibility, already apparent at the mean field level, that SC and FMI phases, although they arise from different interactions (electron-phonon for SC and electron-electron for FMI), could compete with each other in TDBG moiré flatband with the FM phase centered around half-filling and the SC domes manifesting on both electron- and hole-doped sides of half-filling. The fact that this indeed appears to be the experimental TDBG situation may indicate that our theory captures some essential qualitative aspect of moiré interaction physics although our use of mean field theory (and many other approximations, e.g., neglect of higher bands) exaggerates the quantitative stability of the symmetry-broken phases compared with experiments.

F. W. thanks Y.-T. Hsu, X. Li, and R.-X. Zhang for discussions. This work is supported by Laboratory for Physical Sciences.

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which is in the basis of $A_1$, $B_1$, $A_2$ and $B_2$ sites [Fig. 1(a)] from one AB bilayer graphene. $k_\pm$ stands for $k_x \pm ik_y$. Parameter values are taken as

\[
(v_1, v_2, v_3) = (0.844, -0.045, -0.091) \times 10^6 \text{ m/s,} \\
\gamma_1 = 361 \text{ meV,} \Delta_0 = 15 \text{ meV},
\]

which are extracted from \textit{ab initio} results of Ref. 68. The moiré Hamiltonian in $+K$ valley is given by

\[
\mathcal{H}_{+K} = \begin{pmatrix}
    h_b(k) & \tilde{T}(r) \\
    \tilde{T}^\dagger(r) & h_4(k)
\end{pmatrix},
\]

where $h_b(k)$ and $h_4(k)$ are $k.p$ Hamiltonians for bottom and top bilayer graphene, and are equal to $\mathcal{H}_0[R(+\pi/2)(k - \kappa_+)]$ and $\mathcal{H}_0[R(-\pi/2)(k - \kappa_-)]$, respectively. Here $R(\pm\pi/2)$ are rotation matrices and $\kappa_\pm = [4\pi/(3a_M)](-\sqrt{3}/2, \mp1/2)$. The moiré period $a_M$ is approximately $a_0/\theta$, where $a_0$ is the monolayer graphene lattice constant. $\tilde{T}(r)$ is the tunneling between bottom and top bilayer graphene, which varies spatially with the moiré period as specified by

\[
\tilde{T}(r) = \begin{pmatrix}
    0 & T(r) \\
    0 & 0
\end{pmatrix},
\]

\[
T(r) = w_0 T_0 + w_1 (e^{-ib} - e^{-ib}, e^{-ib} - e^{-ib}),
\]

where we only keep tunneling terms between adjacent layers, and $b_\pm$ are moiré reciprocal lattice vectors given by $[4\pi/(\sqrt{3}a_M)](\pm1/2, \mp\sqrt{3}/2)$. $w_0$ and $w_1$ are two tunneling parameters, which in general have different numerical values due to layer corrugation in the moiré pattern 20 [69]. We take $w_0 = 88 \text{ meV}$ and and $w_1 = 100 \text{ meV}$. An out-of-plane electric displacement field generates a layer dependent potential, which can be characterized using a single parameter $U$ as illustrated in Fig. 1(b). The point group symmetry of TDBG is $D_3$ in the absence of the displacement field ($U = 0$), and is broken down to $C_3$ when $U$ is finite.

In one valley, the conduction band minima in Fig. 2(b) are located at three inequivalent momenta related by $C_3$, which can explain the degeneracy of 3 for the first Landau fan on the conduction band side 13. Such a moiré band with multiple degenerate band extrema can provide a system to study quantum Hall nematic and ferroelectric states in a strong magnetic field 70 [72]. We note that our parameter values used in the moiré Hamiltonian are the same as Ref. 65, where a systematic study of the moiré band structure as a function of $\theta$ and $U$ can be found.

**Supplemental Material**

In this supplemental material, we present details of the TDBG continuum moiré Hamiltonian. Within the continuum approximation, $\pm K$ valleys are treated separately. For each AB bilayer graphene, we use the following $k.p$ Hamiltonian in $+K$ valley

\[
\mathcal{H}_0(k) = \begin{pmatrix}
    \Delta_0 & h v_1 k_- & h v_2 k_+ & \gamma_1 \\
    h v_1 k_+ & 0 & h v_2 k_+ & 0 \\
    h v_2 k_- & h v_3 k_+ & 0 & h v_1 k_- \\
    \gamma_1 & h v_2 k_- & h v_1 k_+ & \Delta_0
\end{pmatrix},
\]

where $h_b(k)$ and $h_4(k)$ are $k.p$ Hamiltonians for bottom and top bilayer graphene, and are equal to $\mathcal{H}_0[R(+\pi/2)(k - \kappa_+)]$ and $\mathcal{H}_0[R(-\pi/2)(k - \kappa_-)]$, respectively. Here $R(\pm\pi/2)$ are rotation matrices and $\kappa_\pm = [4\pi/(3a_M)](-\sqrt{3}/2, \mp1/2)$. The moiré period $a_M$ is approximately $a_0/\theta$, where $a_0$ is the monolayer graphene lattice constant. $\tilde{T}(r)$ is the tunneling between bottom and top bilayer graphene, which varies spatially with the moiré period as specified by

\[
\tilde{T}(r) = \begin{pmatrix}
    0 & T(r) \\
    0 & 0
\end{pmatrix},
\]

\[
T(r) = w_0 T_0 + w_1 (e^{-ib} - e^{-ib}, e^{-ib} - e^{-ib}),
\]

where we only keep tunneling terms between adjacent layers, and $b_\pm$ are moiré reciprocal lattice vectors given by $[4\pi/(\sqrt{3}a_M)](\pm1/2, \mp\sqrt{3}/2)$. $w_0$ and $w_1$ are two tunneling parameters, which in general have different numerical values due to layer corrugation in the moiré pattern 20 [69]. We take $w_0 = 88 \text{ meV}$ and and $w_1 = 100 \text{ meV}$. An out-of-plane electric displacement field generates a layer dependent potential, which can be characterized using a single parameter $U$ as illustrated in Fig. 1(b). The point group symmetry of TDBG is $D_3$ in the absence of the displacement field ($U = 0$), and is broken down to $C_3$ when $U$ is finite.

In one valley, the conduction band minima in Fig. 2(b) are located at three inequivalent momenta related by $C_3$, which can explain the degeneracy of 3 for the first Landau fan on the conduction band side 13. Such a moiré band with multiple degenerate band extrema can provide a system to study quantum Hall nematic and ferroelectric states in a strong magnetic field 70 [72]. We note that our parameter values used in the moiré Hamiltonian are the same as Ref. 65, where a systematic study of the moiré band structure as a function of $\theta$ and $U$ can be found.