Classical mechanics with lapse

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Abstract

Mechanics is developed over a differentiable manifold as space of possible positions. Time is considered to fill a one-dimensional Riemannian manifold, so having the metric as lapse. Then the system is quantized with covariant instead of partial derivatives in the Schrödinger operator.

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I. INTRODUCTION

In General Relativity, the differential quotient between proper time and coordinate time is called lapse function. In the present article, this notion is used for an arbitrary classical mechanical system. Space is considered as \(n\)-dimensional Riemannian space \(V_n\) and time is considered as 1-dimensional Riemannian space \(V_1\). Then the square of the lapse function turns out to be the metric of this \(V_1\). Possible applications and comparison with other approaches found in the literature will be shown in section VI below.

Let us consider a mechanical system. The space of all possible positions shall be the \(n\)-dimensional differentiable manifold \(M_n\). It is endowed with local coordinates \(q^i, i = 1, \ldots, n\). Most of all mechanical systems have the property that \(M_n\) is a subset of \(\mathbb{R}^m \times (S^1)^{n-m}\), so that the first \(m\) coordinates are Cartesian ones and the remaining are periodic ones (i.e., angles). Here, \(\mathbb{R}\) denotes the space of reals, \(\mathbb{Z}\) the space of integers, and the one-dimensional torus \(S^1\) can be defined as factor space \(S^1 = \mathbb{R}/\mathbb{Z}\). But in general, \(M_n\) cannot be covered by one single coordinate system. The time is denoted by \(t\), and \(\frac{d}{dt}\) will be denoted by a dot. So, \(\dot{q}^i\) is the velocity of a moving particle \(q^i(t)\). Therefore, the velocity at time \(t\) is an element of the tangent space \(T_x M_n\) of \(M_n\) at \(x = q^i(t)\). The tangent bundle \(TM_n\) is the union of all tangent spaces.

Contrarily to the usual procedure we now introduce the lapse function \(N(t)\) which shall be an arbitrary positive function. (Here and below all functions shall have the necessary differentiability properties.) The proper time \(\tau\) is defined by

\[
\tau = \int N(t)dt
\]

(1)

It is uniquely determined up to an integration constant, i.e., without speci-
fying the point where $\tau = 0$. The space of all possible times is a connected oriented one–dimensional Riemannian space $V_1$ with coordinate $x^1 = t$ and metric $g_{11} = N^2(t)$. The orientation is chosen such that increasing time leads into the future. So, Eq. (1) represents the proper time $\tau$ as proper length within this $V_1$.

Remark: The definition is chosen such that proper time does not depend on the velocity, so we do not cover relativistic effects.

Each positive function $N(t)$ defines a gauge, and results should not depend on it. In this manner, we define the following gauge–invariant quantity, the proper velocity $v^i$

$$v^i = \frac{1}{N} \dot{q}^i$$

We have to prove that $v^i$ does not depend on the special choice of $N$; this follows from Eqs. (1, 2) via the equation

$$v^i = \frac{dq^i}{d\tau}$$

The action $I$ is the integral of a Lagrangian $L$

$$I = \int L \, dt$$

and is supposed to be a coordinate–, gauge–, and $T$–invariant quantity. $T$–invariance means that $I$ does not change if the orientation of $V_1$ is reversed. The range of integration in Eq. (3) is a connected subset of $V_1$, i.e., any fixed time–interval; but we do not specify now which kind of interval is used.

We restrict ourselves to first–order Lagrangians, i.e., $L$ is a function

$$L : TM_n \times V_1 \rightarrow \mathbb{R}$$

The next three steps are done by plausible arguments, not by proofs.
First, the explicit $t$–dependence ( $t \in V_1$ ) of $L$, Eq. (4), is compatible with gauge–invariance of $I$ only for the case that the $t$–dependence of $L$ is via $N(t)$ only, i.e.,

$$L = L(q^i, \dot{q}^i, N)$$  \hspace{1cm} (5)

Second, the coordinate– and gauge–invariance of $I$ requires the following form of $L$

$$L = G(q^i, v^i) \cdot N$$  \hspace{1cm} (6)

where $G$ is a certain scalar; this becomes plausible from Eqs. (1, 2, 3).

Third, we assume that $G$ can be developed into powers of $v^i$

$$G = \sum_{k=0}^{\infty} \alpha^{(k)}_{i_1 \ldots i_k}(q^i) \ v^{i_1} \ldots v^{i_k}$$  \hspace{1cm} (7)

with certain tensors $\alpha^{(k)}_{i_1 \ldots i_k}$. Here, and below, the Einstein sum convention is to be applied. Then it follows from $T$–invariance, that only even values $k$ give a non–vanishing contribution to Eq. (7).

The simplest non–trivial example for Eq. (7) is the case that only $k = 0$ and $k = 2$ give contributions. To meet the usual notation we define

$$V = - \alpha^{(0)}(q^i), \quad h_{ij} = 2 \alpha^{(2)}_{ij}(q^i)$$  \hspace{1cm} (8)

Inserting Eqs. (7, 8) into Eq. (6) we get

$$L = \left( \frac{1}{2} h_{ij} v^i v^j - V \right) \cdot N$$  \hspace{1cm} (9)

Without loss of generality, $h_{ij}$ is assumed to be a symmetric tensor in $M_n$. Here, the coordinate–, gauge–, and $T$–invariance of $I$, Eqs. (3, 9) is immediately seen; so we also could have taken Eq. (9) as a definition of $L$. 

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To give the Lagrangian Eq. (9) the structure defined by Eq. (5) we insert Eq. (2) into Eq. (9) and get

\[ L = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - V \cdot N \]  

(10)

where we used the definition

\[ g_{ij} = \frac{1}{N} \cdot h_{ij} \]  

(11)

Next, we introduce the momentum \( p_i \) by

\[ p_i = \frac{\partial L}{\partial \dot{q}^i} \]  

(12)

From Eq. (10) we get

\[ p_i = g_{ij} \dot{q}^j \]  

(13)

It holds: the momentum is gauge–invariant. This is proven by the fact that from Eqs. (2, 11, 13) one gets

\[ p_i = h_{ij} v^j \]  

(14)

From Eqs. (12, 13) we get

\[ \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = g_{ij} \]  

(15)

where \( g_{ij} \) depends on \( q^i \) and \( N \) only. The analogous gauge–invariant equation to Eq. (15) reads

\[ \frac{\partial p_i}{\partial v^j} = h_{ij} \]  

(16)

and \( h_{ij} \) depends on \( q^i \) only.

Remark: One could use Eqs. (12, 15) also for the general case \( L \), Eqs. (6, 7); but then \( g_{ij} \) would in general depend on the velocities, too. If \( g_{ij} \) is interpreted as metric, then this would be the step from Riemannian to
Finslerian geometry. A typical example of Finslerian geometry appears, if the term with \( k = 4 \) in Eq. (7) is allowed to appear.

Let us introduce the Hamiltonian

\[
H = p_i \dot{q}^i - L
\]  

(17)

The canonical equations make sense only for the case that the velocities can be expressed as functions of the coordinates, momenta, and time. Looking at Eq. (13) one can see that this takes place if and only if \( g_{ij} \) is a regular matrix. So, we assume this to be the case in the following and denote the inverse matrix to \( g_{ij} \) by \( g^{ij} \). From Eq. (11) it follows that also \( h_{ij} \) is invertible. The inverse matrix to \( h_{ij} \) is denoted by \( h^{ij} \). It holds

\[
g_{ij} = N \cdot h^{ij}
\]  

(18)

From Eq. (13) we get

\[
\dot{q}^i = g^{ij} p_j
\]  

(19)

We insert Eqs. (10, 18, 19) into Eq. (17) and get

\[
H = \frac{1}{2} g^{ij} p_i p_j + V \cdot N
\]  

(20)

which can also be written as \( H = \left( \frac{1}{2} h_{ij} p_i p_j + V \right) \cdot N \). The canonical equations are

\[
\dot{q}^i = \frac{\partial H}{\partial p_i}
\]  

(21)

and

\[
\dot{p}_i = -\frac{\partial H}{\partial q^i}
\]  

(22)

Eq. (21) is equivalent to Eq. (19), whereas Eq. (22) represents the equation of motion; in the next section we discuss it in more details.
II. THE EQUATION OF MOTION

The acceleration is \( a^i = \ddot{q}^i \). In general, the equation of motion expresses the acceleration as function of coordinates, velocity, and time. To get this structure, we insert Eqs. (13, 20) into Eq. (22). After some calculus we get

\[
a^i = \hat{N} \dot{q}^i - \dot{V}^i_j \cdot N - \dot{q}^j \dot{q}^k \Gamma_{jk}^i \tag{23}
\]

where \( V^i_j = g^{ij} V_j \) and \( \Gamma \) denotes the Christoffel affinity (which is the same both for \( g_{ij} \) and \( h_{ij} \)). As usual, \(<<, i >>\) is an abbreviation for the partial derivative with respect to the coordinate \(<< q^i >>\).

We can give three results immediately: First, for \( N \) and \( V \) being constant, the equation of motion is just the geodesic equation in the \( M_n \) with Riemannian metric \( g_{ij} \). Second, for \( N \) and \( g_{ij} \) being constant, the equation of motion reads \( 0 = a^i + V^i_j \) and equals the classical equation of motion in the potential \( V \). Third, using gauge–invariant quantities, we can write the equation of motion as

\[
0 = \frac{dv^i}{d\tau} + \Gamma_{jk}^i v^j v^k + h^{ij} V_j \tag{24}
\]

The first two terms of the r.h.s. represent the covariant derivative of the proper velocity with respect to proper time.

In the next step we consider, independently of the Hamiltonian, under which condition the action \( I \) Eq. (3) has a stationary value. One should expect that the same equation of motion appears, but this is not fully trivial to show.

The corresponding Euler–Lagrange equation to the action \( I \) reads

\[
0 = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) \tag{25}
\]
With Eq. (12) we get
\[ \dot{p}_i = \frac{\partial L}{\partial q^i} \]  
(26)

Comparing with Eq. (22) we have to show that
\[ \frac{\partial H}{\partial q^i} = -\frac{\partial L}{\partial q^i} \]  
(27)

Looking at Eq. (17) one could get the impression that Eq. (27) can be fulfilled for a constant product \( p_i \dot{q}^i \) only, but this impression is wrong, because in the l.h.s., \( H \) is a function \( H(q^i, p_i, N) \) but in the r.h.s., \( L \) is a function \( L(q^i, \dot{q}^i, N) \). And so, with \( H \) Eq. (20) and \( L \) Eq. (10), the validity of Eq. (27) can be proven.

III. THE LOWER–DIMENSIONAL CASES

Let us consider the simplifications for the lower–dimensional cases. For \( n = 1 \), one knows that the Riemannian space \( V_1 \) is flat, and so the Lagrangian Eq. (9) reduces to \( L = \left[ \frac{m}{2}v^2 - V(x) \right] \cdot N(t) \) with \( q^1 = x, \ v^1 = v \) and \( h_{11} = m = const. \neq 0 \). With \( N = 1 \) this is the usual point particle in a potential \( V \).

For \( n = 2 \), the Riemannian space \( V_2 = (M_2, h_{ij}) \) need not to be flat, but it is always conformally flat. So one can always find local coordinates such that the Lagrangian Eq. (9) can be written as
\[ L = \left[ \frac{m}{2}v^2 + \frac{M}{2}w^2 - W(x, y) \right] \cdot S(x, y) \cdot N(t) \]  
(28)
with \( q^2 = y, v^2 = w \) and \( h_{22} = M = const. \neq 0 \) and \( W \cdot S = V \) as additional relations. \( S \neq 0 \) is the suitably chosen conformal factor.

For \( n \geq 3 \), however, a \( V_n \) need not to be conformally flat, and so, in general, the usual kinetic term with constant masses can be reached neither by a coordinate nor by a conformal transformation.
IV. QUANTIZATION

The usual quantization procedure is to substitute $p_k$ by $i\hbar \frac{\partial}{\partial q^k}$ in the Hamiltonian to come from the function to the operator. If we make this in our approach, then gauge–invariance is automatically ensured, because both $q^k$ and $p_k$ are gauge–invariant quantities. (To prevent misunderstandings, we explicitly say: $i$ is an index $\in \{1, \ldots, n\}$ if written in index position, and it is the imaginary unit otherwise.) But to ensure coordinate–invariance, the partial derivative is not sufficient. The most natural way to circumvent this difficulty is to use the covariant derivative with the same $\Gamma$ as before. Then $\nabla_k$ denotes the covariant derivative with respect to $q^k$.

The world function is denoted by $\psi$, it is a function

$$\psi : M_n \rightarrow \mathbb{C}$$  \hspace{1cm} (29)

where $\mathbb{C}$ denotes the set of complex numbers.

The energy of the system is $E = H/N$. It is a gauge–invariant scalar, and it is constant along classical trajectories: $\frac{dE}{dt} = 0$ which follows from Eqs. (20, 21, 22).

So we get the Schrödinger equation $\hat{H}\psi = E \cdot N \cdot \psi$ with $\psi = \psi(q^i)$ and

$$\hat{H} = -\frac{1}{2}\hbar^2 g^{ij} \nabla_i \nabla_j + V \cdot N$$  \hspace{1cm} (30)

The zero energy Schrödinger equation simply reads

$$\hbar^2 \Box \psi = 2V \psi$$  \hspace{1cm} (31)

where $\Box$ denotes the D’Alembertian with respect to the metric $h_{ij}$, i.e., $\Box = h^{ij} \nabla_i \nabla_j$, whereas the general Schrödinger equation can be obtained from this one by a suitable redefinition of $V$. 

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To circumvent the explicit calculation of the Christoffel affinities we apply the following formula
\[ \Box = \frac{1}{\sqrt{h}} \partial_i \sqrt{h} h^{ij} \partial_j \] (32)
where \( h = |\text{det} h_{ij}| \neq 0 \).

*Remark:* One should observe that the form used here is surely the simplest possible way to get a coordinate–invariant Schrödinger equation; however, it is not the only possible one which goes over to the classical Schrödinger equation (i.e., that one with partial derivatives) if \( h_{ij} \) becomes constant. Indeed, one could use the conformally invariant operator \( \Box_c = \Box - \xi R \) instead of \( \Box \), where \( R \) is the curvature scalar of the metric \( h_{ij} \) and \( \xi = \frac{n-2}{4(n-1)} \). Only for \( n \leq 2 \) one has \( \Box_c = \Box \); for \( n = 2 \) because of \( \xi = 0 \), and for \( n = 1 \) because of \( R = 0 \). But even for \( n \geq 3 \) one can cover this variant by a suitable redefinition of \( V \).

Let us shortly say what happens for the lower–dimensional cases. For \( n = 1 \), one simply uses coordinates such that \( h_{11} = 1 \) and one gets the usual equation. For \( n = 2 \), however, it is a little more involved. We employ the fact that \( h_{ij} \) is conformally flat and so it can be written as \( h_{ij} = \sqrt{h} \eta_{ij} \) where \( \eta_{ij} \) is a matrix in diagonal form where all diagonal elements are \( \in \{+1, -1\} \). \( \eta^{ij} \) is the inverse to \( \eta_{ij} \); and, by construction, they coincide. Then we insert Eq. (32) into Eq. (31) and get
\[ h^2 \eta^{ij} \partial_i \partial_j \psi = 2 \sqrt{h} V \psi \] (33)
The l.h.s. represents the flat–space D’Alembertian, and the factor \( \sqrt{h} \) in the r.h.s. can be absorbed by a redefinition of \( V \).

For \( n \geq 3 \), however, it requires special circumstances to get the Schrödinger equation in the form of a flat–space D’Alembertian.
V. SOLUTIONS OF THE SCHRÖDINGER EQUATION

From the full set of solutions of the Schrödinger equation (31) we are essentially interested in those solutions which correspond to the classical solutions of the system (21, 22). To this end we apply the WKB-approximation and insert the ansatz

\[ \psi = a \cdot \exp(iS/\hbar) \]  

into Eq. (31) and get

\[ \hbar^2 \Box a + i\hbar (2a_{,k} S^{,k} + a \Box S) - a S_{,k} S^{,k} = 2aV \]  

where \( S^{,k} = h^{jk} S_{,j} \). From Eq. (34) we have the situation that now two functions (\( a, S \)) represent one function (\( \psi \)). So we are free to put an additional relation as calibration. It turns out that the following calibration is useful: we set for a moment \( \hbar = 0 \), insert this into Eq. (35) and use the resulting equation

\[ S_{,k} S^{,k} + 2V = 0 \]  

as natural calibration. This is the usual classical limit.

Before we proceed we must be sure that Eq. (36) possesses solutions. If the metric \( h_{ij} \) has indefinite signature, then this is trivial. Let \( h_{ij} \) be of definite signature; without loss of generality it shall be positively definite, for, otherwise, simply \( V \) has to change its sign. In regions where \( V \leq 0 \), Eq. (36) has solutions, but in regions with \( V > 0 \) it does not have any solutions. One should remember here, that we have redefined \( V \) such that the whole system has zero energy. So, \( V > 0 \) corresponds to a negative kinetic energy; the latter is impossible for a positively definite metric \( h_{ij} \). We get as result:
the calibration Eq. (36) is possible if and only if classical motion takes place there.

Now we insert Eq. (36) into Eq. (35) and get

\[ 0 = \hbar \Box a + 2i a_k S^k + i a \Box S \]  

(37)

To proceed, there exist different possibilities: first, one again neglects

the term with \( \hbar \), second, one requires \( a \) to be a slowly varying amplitude such

that \( \Box a \) is negligible in comparison with \( \Box S \), or, third, one thinks of \( a \) and

\( S \) as real functions and so Eq. (37) splits into real and imaginary parts. It

is not so essential which of these three arguments are applied, because all of

them give rise to the equation

\[ 0 = 2a_k S^k + a \Box S \]  

(38)

Eq. (38) can be solved as follows: let \( S(q^i) \) be a solution of Eq. (36) with

\( S_k \neq 0 \). There exists no time in the system, but we can introduce a time \( T \)

by requiring that \( \frac{d}{dT} = S^k \partial_k \). With \( b = \ln a^2 \), Eq. (38) now reads

\[ \frac{db}{dT} = -\Box S \]  

(39)

which can be integrated along the trajectories of \( T \). In an afterwards–

interpretation one can identify \( T \) with \( \tau \), \( S^k \) with \( v^k \) and \( S_k \) with \( p_k \); this

turns out to be compatible with the classical (= non–quantum) equations.

But this alone does not suffice: from Eq. (39) one calculates the function

\( a(q^i) \) and inserts it together with \( S(q^i) \) into Eq. (37). Then the WKB–

approximation turns out to yield results close to the exact solution only for

the case that indeed, \( |\hbar \Box a| \) is negligible in comparison to \( |a \Box S| \). So one can

check in which region the semiclassical approach makes sense.
VI. CONCLUSION

Classical mechanics, as is usually presented, e.g. in Refs. 1 and 2, uses essentially vector spaces as space of possible positions. Then one has the duality between coordinates and momenta (which we do loose here) and can build a symplectic manifold. Furthermore, one has usually a constant mass tensor (which means a constant matrix $h_{ij}$ in our notation).

Both points are generalized in the present paper. The present approach is inspired by work on Hamiltonian quantum cosmology, e.g. Ref. 3, where the space of possible positions is the set of all possible spatial geometries (called superspace). The set of all possible spatial geometries turns out to be neither a vector space nor is the matrix $h_{ij}$ a constant one. Even, if one restricts to the minisuperspace which corresponds to homogeneous spatial geometries, one does not get a vector space. Example: The set of all homogeneous 3–spaces of Bianchi–type IX [i.e., there exists a transitive subgroup of the isometry group isomorphic to $SO(3)$] which is a manifold with boundary, the interior is composed of points corresponding to spaces whose isometry group is 3–dimensional, and the boundary points are formed by spaces with 4–dimensional isometry group (i.e., the axially symmetric Bianchi–type IX models), and the edge (the boundary of the boundary) consists of one line which itself corresponds to the isotropic 3–spheres with 6–dimensional isometry group. (Concerning details to this point see e.g. Ref. 4).

If $M_n$ is such a manifold with boundary, then a trajectory is simply mirrored at the boundary.

Here we carefully distinguish between co– and contravariant tensor indices, and the Einstein sum convention is used in its strong version: summa-
tion over double indices takes place only for the case that one of them is in upper (= contravariant) and the other in lower (= covariant) position. It is a nice additional check of the formulae that the necessity to write the $\Sigma$–sign never appeared.

The essential result of the present paper is to show that up to dimension two, the Schrödinger equation comes out with the flat–space D’Alembertian whereas for higher dimensions, it requires a special structure of the action to have this property. This has the following consequence for quantum cosmology: All models with one– or two–dimensional minisuperspace can be written with the flat–space D’Alembertian in the Schrödinger equation (which is called Wheeler de Witt equation here), whereas for higher–dimensional minisuperspace models, e.g. Ref. 5, this property requires a special structure of the underlying system.

Reformulated for the classical (i.e., non–quantized) system one can state: A system with one or two degrees of freedom has always a kinetic energy which can be written as sum of terms of the type $\pm m^2 v^2$ with positive constant values $m$, whereas for three or higher dimension this need not to be the case.

The kind of introducing the covariant derivative in Eq. (30) instead of the partial one is the mathematical background of the (today widely accepted) solution of the so–called factor–ordering problem, which filled many papers on quantum cosmology in the eighties, see Ref. 6 which is a bibliography of papers on the topic.

We always wrote velocities with upper (contravariant) and momenta with lower (covariant) index; this is more than a purely notational arbitrariness, moreover, it is the only adequate form from the differential geometric point
of view.

A geometric description of non–relativistic quantum mechanics has already carried out by Kuchař [7] in 1980. He uses a degenerate metric (i.e., a metric with vanishing determinant), so that he needs additional considerations to relate the co– and the contravariant components of it. He solves the factor–ordering problem by writing the Laplacian covariant with respect to this degenerate metric. Contrary to our approach (see also Ref. [5] for more details), he uses Dirac’s constraint quantization.

Section 7.2 of Ref. [8] develops classical mechanics in parametrized form. In this form, it becomes time–reparametrization invariant just as General Relativity is coordinate–invariant. Their approach takes velocities and momenta on the same footing (both are covariantly written vectors).

The book [9] by Zeh reviews many aspects of the direction of time. In subsection 5.2.1 of that book, also the reparametrization invariance of time is mentioned, Zeh relates this property to Mach’s principle (regarding time). Ref. [10] presents a geometrization of classical mechanics by use of a symplectic structure. Refs. [11] discuss the recovering of time and the deduction of the Wheeler de Witt equation in quantum cosmology.

An application of the present approach (the present article is a revised version of the unpublished Potsdam–Report No. 93/10 from January 1993) can be found in section V A of Ref. [12], where it is used to deduce the Wheeler de Witt equation for the Starobinsky cosmological model. For further generalizations see e.g. Ref. [13].
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