A CRITERION FOR WEAK CONVERGENCE ON BERKOVICH PROJECTIVE SPACE

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Abstract. We give a criterion for the weak convergence of unit Borel measures on the N-dimensional Berkovich projective space $\mathbf{P}_K^N$ over a complete non-archimedean field $K$. As an application, we give a sufficient condition for a certain type of equidistribution on $\mathbf{P}_K^N$ in terms of a weak Zariski-density property on the scheme-theoretic projective space $\mathbf{P}_K^N$ over the residue field $\bar{K}$. As a second application, in the case of residue characteristic zero we give an ergodic-theoretical equidistribution result for the powers of a point $a$ in the N-dimensional unit torus $\mathbf{T}_K^N$ over $K$. This is a non-archimedean analogue of a well-known result of Weyl over $\mathbb{C}$, and its proof makes essential use of a theorem of Mordell-Lang type for $\mathbb{G}_m^N$ due to Laurent.

1. Introduction

1.1. Let $K$ be a field which is complete with respect to a nontrivial, non-archimedean absolute value. Given an integer $N \geq 1$, the $N$-dimensional projective space $\mathbb{P}^N(K)$ is compact (with respect to its Hausdorff analytic topology) if and only if the field $K$ is locally compact, and this occurs only when $K$ has both a discrete value group and a finite residue field. On the other hand, the $N$-dimensional Berkovich projective space $\mathbf{P}_K^N$ over $K$ is a Hausdorff space which contains the ordinary projective space $\mathbb{P}^N(K)$ as a subspace, and $\mathbf{P}_K^N$ is always compact, regardless of whether or not $K$ is locally compact. Moreover, the Hausdorff topology on $\mathbf{P}_K^N$ is closely related not only to the analytic topology on $\mathbb{P}^N(K)$, but also to the Zariski topology on the scheme-theoretic projective space $\mathbb{P}_K^N$ over the residue field $\bar{K}$. For these and other reasons, there are many situations in which it is preferable to work on the larger space $\mathbf{P}_K^N$ rather than $\mathbb{P}^N(K)$ itself.

An example of an analytic notion which is best studied on compact spaces is that of equidistribution. For each integer $\ell \geq 1$, let $Z_\ell$ be a finite multiset of points in $\mathbf{P}_K^N$ (a multiset is a set whose points occur with multiplicities), and let $\mu$ be a unit Borel measure on $\mathbf{P}_K^N$. The sequence $(Z_\ell)_{\ell=1}^{+\infty}$ is said to be $\mu$-equidistributed if the limit

$$
\lim_{\ell \to +\infty} \frac{1}{|Z_\ell|} \sum_{z \in Z_\ell} \varphi(z) = \int \varphi d\mu
$$

$$
(1)
$$

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holds for all continuous functions $\varphi : \mathbf{P}^N_K \to \mathbb{R}$. Many researchers have been working recently to establish equidistribution results on Berkovich analytic spaces, often for sequences $(Z_\ell)_{\ell=1}^{+\infty}$ arising naturally from questions in arithmetic geometry and dynamical systems; we mention [1], [3], [10], [12], [13], and [17] to name a few examples of such work.

The main result of this paper implies that, in order to establish the equidistribution of a sequence $(Z_\ell)_{\ell=1}^{+\infty}$ of multisets in $\mathbf{P}^N_K$, it suffices to check (1) for a special class of continuous functions $\varphi$ arising naturally from the geometric structure of $\mathbf{P}^N_K$. Since it requires no extra work to do so, we formulate our main result using the more general notion of weak convergence of measures on $\mathbf{P}^N_K$, although our motivating concern is with equidistribution. Moreover, for reasons which we will discuss below, our main result will be stated using nets, rather than sequences; again this requires no extra effort.

We give two applications of our main result, both of which establish equidistribution theorems with respect to the Dirac measure $\delta_\gamma$ supported at the Gauss point $\gamma$ of $\mathbf{P}^N_K$. Letting $\mathbf{P}^N_{\tilde{K}}$ denote the scheme-theoretic projective space over the residue field $\tilde{K}$, there exists a natural reduction map $r : \mathbf{P}^N_K \to \mathbf{P}^N_{\tilde{K}}$, and the Gauss point $\gamma$ can be described as the unique point of $\mathbf{P}^N_K$ reducing to the generic point of $\mathbf{P}^N_{\tilde{K}}$. Our first application gives a useful necessary and sufficient condition for $\delta_\gamma$-equidistribution, and uses this to establish the $\delta_\gamma$-equidistribution of nets whose reduction satisfy a certain weak Zariski-density property.

Our second application is an ergodic-theoretic equidistribution result for the sequence formed by taking the powers of a point in the $N$-dimensional unit torus

$$T^N_K = \{ (a_0 : a_1 : \cdots : a_N) \in \mathbb{P}^N(K) \mid |a_0| = |a_1| = \cdots = |a_N| = 1 \}$$

over $K$. Identifying the group variety $G^N_m$ over the residue field $\tilde{K}$ with the subvariety of $\mathbb{P}^N$ defined by $x_0x_1 \cdots x_N \neq 0$, the reduction map $r : \mathbf{P}^N_K \to \mathbf{P}^N_{\tilde{K}}$ restricts to a map $r : T^N_K \to G^N_m(\tilde{K})$. A point $\tilde{a}$ in $G^N_m(\tilde{K})$ is said to be non-degenerate if it is not contained in any proper algebraic subgroup of $G^N_m$.

**Theorem 1.** Assume that the residue field $\tilde{K}$ has characteristic zero. Let $a \in T^N_K$, and for each integer $\ell \geq 1$, define $Z_\ell = \{a, a^2, \ldots, a^\ell\}$, considered as a multiset in $T^N_K \subset \mathbb{P}^N(K) \subset \mathbf{P}^N_K$ of cardinality $|Z_\ell| = \ell$. The sequence $(Z_\ell)_{\ell=1}^{+\infty}$ is $\delta_\gamma$-equidistributed in $\mathbf{P}^N_K$ if and only if the point $\tilde{a}$ is non-degenerate in $G^N_m(\tilde{K})$.

This theorem is a non-archimedean analogue of a well-known archimedean equidistribution result of Weyl [22]. Given a point $a$ in the compact unit torus $T^N_\mathbb{C}$ over $\mathbb{C}$, Weyl’s result gives necessary and sufficient conditions for the Haar-equidistribution of the sets $Z_\ell = \{a, a^2, \ldots, a^\ell\}$. Usually stated in its additive (rather than multiplicative) form, this theorem is often presented...
as the first nontrivial example of “uniform distribution modulo 1”. We will
give a more detailed discussion of Weyl’s theorem and related results in § 4.

An essential ingredient in our proof of Theorem 1 is a theorem of Mordell-
Lang type on the group variety $\mathbb{G}_m^N$.

This paper is organized as follows:

- In §1.2 we fix some notation and terminology.
- In §2 we review the definitions of the Berkovich affine and projective
  spaces $\mathbb{A}_K^{N+1}$ and $\mathbb{P}_K^N$, and we establish the needed topological
  properties of these spaces.
- In §3 we prove the main result of this paper, namely the criterion
  for weak convergence of unit Borel measures on $\mathbb{P}_K^N$.
- In §4 and §5 we present the two applications.

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1.2. Throughout this paper $K$ denotes a field which is complete with
respect to a nontrivial, non-archimedean absolute value $|\cdot|$. Denote by
$K^0 = \{a \in K \mid |a| \leq 1\}$ the valuation ring of $K$, by $K^{\infty} = \{a \in K \mid |a| < 1\}$
its maximal ideal, and by $\bar{K} = K^0/K^{\infty}$ its residue field. Given an element
$a \in K^0$, we denote by $\bar{a}$ the image of $a$ under the quotient map $K^0 \to \bar{K}$. Let $\overline{K}$ be the completion of an algebraic closure of $K$ with respect to the
unique extension of $|\cdot|$; thus $\overline{K}$ is both complete and algebraically closed
([7], §3.4). Define $K^0$, $K^{\infty}$, and $\overline{K}$ analogously.

Let $N \geq 0$ be an integer, and let $K[X] = K[X_0, X_1, \ldots, X_N]$ be the
polynomial ring over $K$ in the $N + 1$ variables $X = (X_0, X_1, \ldots, X_N)$. By a
multiplicative seminorm on $K[X]$ extending $|\cdot|$, we mean a nonnegative real-valued function $|\cdot|$ on $K[X]$ satisfying $|a| = |\bar{a}|$ for all constants $a \in K$, and
satisfying $|f + g| \leq \max\{|f|, |g|\}$ and $|fg| = |f||g|$ for all pairs $f, g \in K[X]$.

Given an arbitrary polynomial $f \in K[X]$, denote by $H(f)$ the maximum
absolute value of the coefficients of $f$. Thus $H(f) \leq 1$ if and only if $f$ is
defined over the valuation ring $K^0$; in this case we denote by $\bar{f} \in K[X]$ the
reduction of $f$. If $H(f) = 1$, we say that $f$ is normalized.

If the non-archimedean field $K$ has a countable dense subset, then it is
possible to show using the Urysohn metrization theorem that the space $\mathbb{P}_K^N$
is metrizable; see [2] §1.5. In general, however, $\mathbb{P}_K^N$ is not homeomorphic to
a metric space. Consequently, notions of convergence in $\mathbb{P}_K^N$ are best studied
using nets, rather than sequences.

Briefly, a net in a set $\mathcal{X}$ is a function $\alpha \mapsto x_\alpha$ from a directed set $I$ into $\mathcal{X}$;
it is usually denoted by $\langle x_\alpha \rangle$, suppressing the dependence on $I$. A sequence
in $\mathcal{X}$ is simply a net in $\mathcal{X}$ indexed by the directed set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of
positive integers. In order to distinguish them from arbitrary nets, we will
generally refer to sequences using the notation $\langle x_\ell \rangle_{\ell=1}^{+\infty}$. If the set $\mathcal{X}$ is a
metric space, then many familiar topological concepts can be reformulated
in terms of convergence properties of sequences in \( \mathcal{X} \). These results continue
to hold when \( \mathcal{X} \) is an arbitrary Hausdorff space, but only if one takes care
to properly interpret the statements using nets in place of sequences. We refer
the reader to Folland [13] § 4.3 for a treatment of nets and their convergence
properties.

2. Berkovich Affine and Projective Space

2.1. The Berkovich affine space \( A_K^{N+1} \) over \( K \) is defined to be the set of
multiplicative seminorms on \( K[X] \) extending \( | \cdot | \). As a matter of notation,
we will refer to a point \( \zeta \in A_K^{N+1} \), and denote by \([\cdot]_\zeta\) its corresponding
seminorm. The topology on \( A_K^{N+1} \) is defined to be the weakest topology
with respect to which the real-valued functions \( \zeta \rightarrow [f]_\zeta \) are continuous for
all \( f \in K[X] \). Equivalently, define a family of subsets of \( A_K^{N+1} \) by

\[
U_{s,t}(f) = \{ \zeta \in A_K^{N+1} | s < [f]_\zeta < t \}
\]

for \( f \in K[X] \) and \( s, t \in \mathbb{R} \). By definition, the subsets \( U_{s,t}(f) \) generate a base
of open sets for the topology on \( A_K^{N+1} \).

To see the relation between \( A_K^{N+1} \) and the classical affine space \( K^{N+1} \),
consider a point \( a \in K^{N+1} \). Letting \([\cdot]_a\) denote the multiplicative seminorm
defined by evaluation \([f]_a = |f(a)|\), we obtain a continuous embedding
\( K^{N+1} \hookrightarrow A_K^{N+1} \) given by \( a \mapsto [\cdot]_a \). We regard this embedding as an inclusion
\( K^{N+1} \subset A_K^{N+1} \) by identifying \( K^{N+1} \) with its image in \( A_K^{N+1} \). More
generally, evaluation \( a \mapsto [\cdot]_a \) defines a map \( K^{N+1} \to A_K^{N+1} \) whose image is
homeomorphic to the quotient of \( K^{N+1} \) by the action of \( \text{Gal}(\mathbb{K}/K) \).

The classical affine space \( K^{N+1} \) is always a proper subset of \( A_K^{N+1} \). For
example, an important class of points in \( A_K^{N+1} \) arises by considering poly-
discs

\[
D(c, r) = \{ a \in K^{N+1} | |c_n - a_n| \leq r_n \text{ for all } 0 \leq n \leq N \}
\]

with center \( c \in K^{N+1} \) and polyradius \( r \in |K|^{N+1} \). The function \([\cdot]_{c, r} : K[X] \to \mathbb{R} \)
defined by the supremum \([f]_{c, r} := \sup\{|f(a)| | a \in D(c, r)\}\) is a
multiplicative seminorm on \( K[X] \) extending \( | \cdot | \), and therefore it defines a
point of \( A_K^{N+1} \); it is convenient to denote this point by \( \zeta_{c, r} \).

2.2. Define a function \( \| \cdot \| : A_K^{N+1} \to \mathbb{R} \) by

\[
\|\zeta\| = \max\{|X_0\zeta, [X_1]\zeta, \ldots, [X_N]\zeta\}.
\]

Observe that \( \| \cdot \| \) is continuous, \( \|\zeta\| \geq 0 \) for all \( \zeta \in A_K^{N+1} \), and \( \|\zeta\| = 0 \)
if and only if \( \zeta \) is the point \( 0 = (0, 0, \ldots, 0) \) corresponding to the origin in
\( K^{N+1} \subset A_K^{N+1} \). For each real number \( r > 0 \), define two subsets of \( A_K^{N+1} \) by

\[
E_r = \{ \zeta \in A_K^{N+1} | \|\zeta\| \leq r \} \quad U_r = \{ \zeta \in A_K^{N+1} | \|\zeta\| < r \}.
\]

\( U_r \) is clearly open, and in proving the following proposition we will see that
\( E_r \) is compact.
Proposition 2. $A_K^{N+1}$ is a locally compact Hausdorff space.

Proof. To show that $A_K^{N+1}$ is Hausdorff, let $\zeta, \zeta' \in A_K^{N+1}$ be distinct points. Then $[f]_\zeta \neq [f]_{\zeta'}$ for some $f \in K[X]$. Selecting disjoint open intervals $(s, t)$ and $(s', t')$ containing $[f]_\zeta$ and $[f]_{\zeta'}$, respectively, it follows that $U_{s, t}(f)$ and $U_{s', t'}(f)$ are disjoint open neighborhoods of $\zeta$ and $\zeta'$, respectively.

We will now show that the sets $E_r$ are compact; since $A_K^{N+1} = \bigcup_{r>0} U_r$, it will follow at once that $A_K^{N+1}$ is locally compact. In order to show that $E_r$ is compact it suffices to show that every net $\langle \zeta_\alpha \rangle$ in $E_r$ has a subnet converging to a limit in $E_r$. For each $f \in K[X]$, there exists a constant $C_{f,r} > 0$ such that $[f]_r \leq C_{f,r}$ for all $\zeta \in E_r$. (Each seminorm $[\cdot]_r$ satisfies the ultrametric inequality, so one could take $C_{f,r} = H(f) r^{\deg(f)}$.) Therefore the association $\zeta \mapsto [f]_\zeta$ defines a continuous map $E_r \to [0, C_{f,r}]$. Letting $\Pi$ denote the product $\prod_{f \in K[X]} [0, C_{f,r}]$, we obtain a continuous map $\iota : E_r \to \Pi$, which is injective by the definition of $A_K^{N+1}$. Since $\Pi$ is compact (Tychonoff’s theorem, [14], §4.6), the net $\langle \iota(\zeta_\alpha) \rangle$ has a subnet $\langle \iota(\zeta_\beta) \rangle$ converging to some point $\langle \xi(f) \rangle_{f \in K[X]} \in \Pi$. Define a function $\lceil \cdot \rceil : K[X] \to \mathbb{R}$ by $[f]_\xi = \xi(f)$. Then $\lceil \cdot \rceil_\xi$ is a multiplicative seminorm on $K[X]$ restricting to $|[\cdot]|$ on $K$, and thus it corresponds to a point $\xi \in A_K^{N+1}$. Moreover, $\xi \in E_r$ and $\zeta_\beta \to \xi$, as desired, completing the proof that $E_r$ is compact. \hfill $\square$

2.3. Assume now that $N \geq 1$. Define an equivalence relation $\sim$ on $A_K^{N+1} \setminus \{0\}$ by declaring that $\zeta \sim \xi$ if and only if there exists a constant $\lambda > 0$ such that $[f]_\zeta = \lambda^{\deg(f)} [f]_\xi$ for all homogeneous polynomials $f \in K[X]$. The Berkovich projective space $P_K^N$ is defined to be the quotient of $A_K^{N+1} \setminus \{0\}$ modulo $\sim$, endowed with the quotient topology; denote by $\pi : A_K^{N+1} \setminus \{0\} \to P_K^N$ the quotient map.

The embedding $K^{N+1} \hookrightarrow A_K^{N+1}$ discussed in §2 restricts to a map $K^{N+1} \setminus \{0\} \to A_K^{N+1} \setminus \{0\}$, which descends modulo $\sim$ to an embedding $\mathbb{P}^N(K) \hookrightarrow P_K^N$. We again regard this map as an inclusion $\mathbb{P}^N(K) \subset P_K^N$ by identifying $\mathbb{P}^N(K)$ with its image in $P_K^N$. Similarly, the map $K^{N+1} \to A_K^{N+1}$ descends modulo $\sim$ to a map $\mathbb{P}^N(K) \hookrightarrow P_K^N$ whose image is homeomorphic to the quotient of $\mathbb{P}^N(K)$ by the action of $\text{Gal}(K/K)$.

Consider the subset of $A_K^{N+1}$ defined by

$$S_K^N = \{ \zeta \in A_K^{N+1} | \|\zeta\| = 1 \}.$$ 

Note that $S_K^N$ is compact, since $S_K^N = E_1 \setminus U_1$ for the compact set $E_1$ and the open set $U_1$ defined in §2.2. The following lemma shows that the quotient map $\pi : A_K^{N+1} \setminus \{0\} \to P_K^N$ remains surjective when restricted to $S_K^N$; we will still use the notation $\pi : S_K^N \to P_K^N$ to refer to this restricted map.

Lemma 3. Given a point $z \in P_K^N$, there exists a point $\zeta_z \in S_K^N$ such that $\pi(\zeta_z) = z$. 
Proposition 4. \( P^N_K \) is a compact Hausdorff space.

Proof. The following is a standard result of general topology (S § 10.2): if \( S \) is a compact Hausdorff space and \( f : S \to S' \) is a surjective quotient map onto a topological space \( S' \), then \( S' \) is Hausdorff if and only if the set \( \{(x, y) \mid f(x) = f(y)\} \) is closed in \( S \times S \).

Applying this result to the map \( \pi : S_K^N \to P_K^N \), in order to show that \( P_K^N \) is Hausdorff it suffices to show that the set \( R = \{(\zeta, \xi) \in S_K^N \times S_K^N \mid \zeta \sim \xi\} \) is closed in \( S_K^N \times S_K^N \). To show that \( R \) is closed, consider a convergent net \( \langle(\zeta_\alpha, \xi_\alpha)\rangle \) in \( S_K^N \times S_K^N \) with \( \zeta_\alpha \sim \zeta_\alpha \) for all \( \alpha \in A \), and with \( (\zeta_\alpha, \xi_\alpha) \to (\zeta, \xi) \in S_K^N \times S_K^N \); we must show that \( \zeta \sim \xi \). By the definition of \( \sim \), there exists a net of positive real numbers \( \langle\lambda_\alpha\rangle \) such that \( [f]_{\zeta_\alpha} = \lambda_\alpha^{\deg(f)} [f]_{\xi_\alpha} \) for all homogeneous \( f \in K[X] \) and all \( \alpha \in A \). Since \( \zeta_\alpha \to \zeta \) and \( \xi_\alpha \to \xi \), and since the maps \( \zeta \mapsto [f]_{\zeta} \) are continuous, we deduce that \( [f]_{\zeta_\alpha} \to [f]_{\zeta} \) and \( [f]_{\xi_\alpha} \to [f]_{\xi} \) for all \( f \in K[X] \). It follows that the net \( \langle\lambda_\alpha\rangle \) converges to the number \( \lambda := ([f]_{\zeta}/[f]_{\xi})^{1/\deg(f)} \) for all homogeneous \( f \in K[X] \). Since \( \mathbb{R} \) is Hausdorff, the limit \( \lambda \) is unique and therefore independent of \( f \). Since \( [f]_{\zeta} = \lambda^{\deg(f)} [f]_{\xi} \), we deduce that \( \zeta \sim \xi \). This concludes the proof that \( R \) is closed in \( S_K^N \times S_K^N \), and therefore that \( P_K^N \) is Hausdorff. Since \( S_K^N \) is compact and \( \pi : S_K^N \to P_K^N \) is continuous and surjective, \( P_K^N \) must also be compact. \( \square \)

2.4. For background on Berkovich analytic spaces, see Berkovich’s original monograph [1], especially § 1.5 for a discussion of affine space \( A_K^{N+1} \). The construction of \( P_K^N \) via the equivalence relation \( \sim \) on \( A_K^{N+1} \), which is similar to the scheme-theoretic \( \text{Proj} \) construction, is due to Berkovich himself [5]. Baker-Rumely (2 § 2.2) have treated the case \( N = 1 \) at length, but for general \( N \geq 1 \) the construction and basic topological properties of \( P_K^N \) do not seem to have been written out in detail before now.
The fundamental compactness argument, used here in the proof of Proposition 2 is due to Baker-Rumely (2 Thm. C.3); it is slightly different than Berkovich’s original argument (1 Thm. 1.2.1). Naturally, both proofs use Tychonoff’s theorem.

3. A Criterion for Weak Convergence

3.1. Let \( C(P_K^N) \) denote the space of continuous functions \( P_K^N \to \mathbb{R} \). Thus \( C(P_K^N) \) is a Banach algebra (with respect to the supremum norm). By a Borel measure \( \mu \) on \( P_K^N \) we mean a positive measure on the Borel \( \sigma \)-algebra of \( P_K^N \); we say \( \mu \) is a unit Borel measure if \( \mu(P_K^N) = 1 \). Given a net \( \langle \mu_\alpha \rangle \) of Borel measures on \( P_K^N \), and and another Borel measure \( \mu \) on \( P_K^N \), we say that \( \mu_\alpha \to \mu \) weakly if \( \int \varphi d\mu_\alpha \to \int \varphi d\mu \) for all \( \varphi \in C(P_K^N) \).

We will now state and prove the main result of this paper. Given a homogeneous polynomial \( f \in K[X] \), it follows from the definition of the equivalence relation \( \sim \) that the real-valued function \( \zeta \mapsto [f]_\zeta / ||\zeta||^{\deg(f)} \) on \( A_K^{N+1} \) is constant on \( \sim \)-equivalence classes. We may therefore define the function

\[
\lambda_f : P_K^N \to \mathbb{R} \quad \lambda_f(\pi(\zeta)) = \frac{[f]_\zeta}{||\zeta||^{\deg(f)}}.
\]

**Theorem 5.** Let \( \langle \mu_\alpha \rangle \) be a net of unit Borel measures on \( P_K^N \), and let \( \mu \) be another unit Borel measure on \( P_K^N \). Then \( \mu_\alpha \to \mu \) weakly if and only if \( \int \lambda_f d\mu_\alpha \to \int \lambda_f d\mu \) for all normalized homogeneous polynomials \( f \in K[X] \).

**Proof.** The “only if” direction is trivial since each function \( \lambda_f \) is continuous.

To prove the “if” direction, assume that \( \int \lambda_f d\mu_\alpha \to \int \lambda_f d\mu \) for all normalized homogeneous polynomials \( f \in K[X] \). Then in fact this limit must hold for arbitrary nonzero homogeneous \( f \in K[X] \), which is easy to see by scaling \( f \) and using the fact that \( \lambda_c f = |c| \lambda_f \) for all nonzero \( c \in K \).

Denote by \( \mathcal{A}(P_K^N) \) the subspace of \( C(P_K^N) \) generated over \( \mathbb{R} \) by the functions of the form \( \lambda_f : P_K^N \to \mathbb{R} \) for homogeneous \( f \in K[X] \). Then \( \mathcal{A}(P_K^N) \) is a dense subalgebra of \( C(P_K^N) \). To see this, note that \( \mathcal{A}(P_K^N) \) is closed under multiplication, since \( \lambda_f \lambda_g = \lambda_{fg} \), and it is therefore a subalgebra. In order to show that \( \mathcal{A}(P_K^N) \) is dense in \( C(P_K^N) \), it suffices by the Stone-Weierstrass theorem (11 § 4.7) to show that \( \mathcal{A}(P_K^N) \) separates the points of \( P_K^N \). Consider two points \( z, w \in P_K^N \) such that \( \lambda_f(z) = \lambda_f(w) \) for all homogeneous \( f \in K[X] \). Taking \( \zeta \in \pi^{-1}(z) \) and \( \xi \in \pi^{-1}(w) \), we have \( [f]_\zeta = (||\zeta||/||\xi||)^{\deg(f)} [f]_\xi \) for all homogeneous \( f \in K[X] \), which means that \( \zeta \sim \xi \), whereby \( z = w \). In other words, \( z \neq w \) implies that \( \lambda_f(z) \neq \lambda_f(w) \) for some homogeneous \( f \in K[X] \), showing that \( \mathcal{A}(P_K^N) \) separates the points of \( P_K^N \), and completing the proof that \( \mathcal{A}(P_K^N) \) is dense in \( C(P_K^N) \).

To show that \( \int \varphi d\mu_\alpha \to \int \varphi d\mu \) for any \( \varphi \in C(P_K^N) \), it suffices by a standard approximation argument verify it for \( \varphi \) in a dense subspace of \( C(P_K^N) \). By linearity and what we have already shown, one only needs to
Corollary 6. Let $\varphi = \lambda_f$ for an arbitrary normalized homogeneous polynomial $f \in K[X]$, which holds by hypothesis.

Remark. The main ingredient in the proof of Theorem 5, namely the density of the subalgebra $\mathcal{A}(\mathbb{P}_K^N)$ in $\mathcal{C}(\mathbb{P}_K^N)$, is similar in spirit to a density result of Gubler [16] Thm. 7.12. Specifically, working over an arbitrary compact Berkovich analytic space $X$, Gubler considers the space of functions $z \mapsto -\log \|1(z)\|^{1/m}$, for integers $m \geq 1$ and a certain class of algebraic metrics $\|\cdot\|$ defined on the trivial line bundle $O_X$. After showing that the space of all such functions is point-separating and closed under taking maximums and minimums, he appeals to the lattice form of the Stone-Weierstrass theorem to show that this space is dense in the space $\mathcal{C}(X)$ of continuous functions.

Working only over $\mathbb{P}_K^N$, our density result is rather simpler than Gubler’s. Given a normalized homogeneous polynomial $f \in K[X]$ of degree $d$, we may view it as a section $f \in \Gamma(\mathbb{P}^N, \mathcal{O}(d))$, and we may therefore write $\lambda_f(z) = \|f(z)\|_{\text{sup}}$ where $\|\cdot\|_{\text{sup}}$ is the sup-metric on $\mathcal{O}(d)$. Taking advantage of the identity $\lambda_f \lambda_g = \lambda_{fg}$, we use the multiplicative form of the Stone-Weierstrass theorem to obtain the density of the algebra $\mathcal{A}(\mathbb{P}_K^N)$ generated by the functions $\lambda_f$.

3.2. Theorem 5 was stated in terms of the weak convergence of nets of arbitrary unit Borel measures on $\mathbb{P}_K^N$, but our principal concern is with the more specific notion of equidistribution. Given a finite multiset $Z$ of points in $\mathbb{P}_K^N$, define a unit Borel measure $\delta_Z$ on $\mathbb{P}_K^N$ by

$$\delta_Z = \frac{1}{|Z|} \sum_{z \in Z} \delta_z.$$ 

Here $\delta_z$ is the unit Dirac measure at $z$, characterized by the formula $\int \varphi \, d\delta_z = \varphi(z)$ for all $\varphi \in \mathcal{C}(\mathbb{P}_K^N)$. Since $Z$ is a multiset, we understand the cardinality $|Z|$ and the sum over $z \in Z$ to be computed according to multiplicity. Given a net $\langle Z_\alpha \rangle$ of finite multisets in $\mathbb{P}_K^N$, and a unit Borel measure on $\mathbb{P}_K$, we say that the net $\langle Z_\alpha \rangle$ is $\mu$-equidistributed if $\delta_{Z_\alpha} \to \mu$ weakly.

Corollary 6. Let $\langle Z_\alpha \rangle$ be a net of finite multisets in $\mathbb{P}_K^N$, and let $\mu$ be a unit Borel measure on $\mathbb{P}_K^N$. Then $\langle Z_\alpha \rangle$ is $\mu$-equidistributed if and only if $\frac{1}{|Z_\alpha|} \sum_{z \in Z_\alpha} \lambda_f(z) \to \int \lambda_f \, d\mu$ for all normalized homogeneous polynomials $f \in K[X]$.

4. Equidistribution and Reduction

4.1. Let $r : \mathbb{P}^N(K) \to \mathbb{P}^N(\bar{K})$ be the usual reduction map on the ordinary projective space; thus $r(a_0 : a_1 : \cdots : a_N) = (\bar{a}_0 : \bar{a}_1 : \cdots : \bar{a}_N)$, where homogeneous coordinates have been chosen so that $\max\{|a_0|, |a_1|, \ldots, |a_N|\} = 1$. In §4.2 we will discuss how this map extends naturally to a reduction map $r : \mathbb{P}_K^N \to \mathbb{P}_K^N$ from the Berkovich projective space $\mathbb{P}_K^N$ onto the scheme-theoretic projective space $\mathbb{P}_K^N$ over the residue field $\bar{K}$.
Recall that $\zeta_{0,1}$ denotes the point of $A_K^{N+1}$ corresponding to the supremum norm on the polydisc $D(0,1)$ in $K^{N+1}$ with center $0 = (0,\ldots,0)$ and polyradius $1 = (1,\ldots,1)$, as discussed in §2.1. Let $\gamma$ be the point of $P_K^N$ defined by $\gamma = \pi(\zeta_{0,1})$, and let $\delta_\gamma$ be the unit Dirac measure supported at $\gamma$. In this section we will give a useful necessary and sufficient condition for $\delta_\gamma$-equidistribution in terms of the functions $\lambda_f$. In §4.3 we will use this result to establish the $\delta_\gamma$-equidistribution of a net $\langle Z_\alpha \rangle$ in $P_K^N$, provided the image of $\langle Z_\alpha \rangle$ under the reduction map $r : P_K^N \to P_K^N$ satisfies a certain weak Zariski-density property.

We begin with a well-known lemma which records a basic property of the seminorm $[|.|]_{\zeta_{0,1}}$. Given a polynomial $f \in K[X]$, recall that $H(f)$ denotes the maximum absolute value of the coefficients of $f$.

**Lemma 7.** The identity $[f]_{\zeta_{0,1}} = H(f)$ holds for all $f \in K[X]$.

**Proof.** This is trivial if $f = 0$, so we may assume $f \neq 0$. Scaling $f$ by an appropriate element of $K$, we may assume without loss of generality that $f$ is normalized, and thus we must show that $[f]_{\zeta_{0,1}} = 1$. Plainly $|f(a)| \leq 1$ for all $a \in D(0,1)$ by the ultrametric inequality, whereby $[f]_{\zeta_{0,1}} \leq 1$. Since $f$ has coefficients in $K^\circ$, and thus in $K$, it reduces to a polynomial $\tilde{f}(X) \in \hat{K}[X]$. Since $H(f) = 1$, the reduced polynomial $\tilde{f}(X)$ is nonzero and therefore is nonvanishing on a nonempty Zariski-open subset of $\hat{K}^{N+1}$ (note that $\hat{K}$ is algebraically closed). Select some $\tilde{a}_0 \in \hat{K}^{N+1}$ such that $\tilde{f}(\tilde{a}_0) \neq 0$, and let $a_0 \in D(0,1)$ be a point which reduces to $\tilde{a}_0$. Then $1 = |f(a_0)| \leq [f]_{\zeta_{0,1}}$, completing the proof that $[f]_{\zeta_{0,1}} = 1$. \hfill $\square$

**Remark.** It follows from Lemma 7 and the multiplicativity of the seminorm $[|.|]_{\zeta_{0,1}}$ that $H(fg) = H(f)H(g)$ for any two polynomials $f, g \in K[X]$; this fact is essentially equivalent to Gauss’s lemma from algebraic number theory. Consequently, $\zeta_{0,1}$ is commonly called the Gauss point of $A_K^{N+1}$, and likewise $\gamma = \pi(\zeta_{0,1})$ the Gauss point of $P_K^N$.

**Theorem 8.** Let $\langle Z_\alpha \rangle$ be a net of nonempty finite multisets in $P_K^N$. Then $\langle Z_\alpha \rangle$ is $\delta_\gamma$-equidistributed if and only if the limit

$$\lim_{\alpha} \frac{|\{z \in Z_\alpha \mid \lambda_f(z) < t\}|}{|Z_\alpha|} = 0$$

holds for each normalized homogeneous polynomial $f \in K[X]$ and each real number $0 < t < 1$.

**Proof.** Given a normalized homogeneous polynomial $f \in K[X]$, we have $0 \leq \lambda_f(z) \leq 1$ for all $z \in P_K^N$ (the upper bound following from the ultrametric inequality), and $\lambda_f(\gamma) = [f]_{\zeta_{0,1}} \cdot \|\zeta_{0,1}\|^{|\deg(f)|} = 1$ by Lemma 7. For each multiset $Z_\alpha$, define the sum

$$S_\alpha(f) = \frac{1}{|Z_\alpha|} \sum_{z \in Z_\alpha} \lambda_f(z).$$
Plainly $0 \leq S_\alpha(f) \leq 1$, and it follows from the above observations and Corollary~\ref{equidistr} that $\langle Z_\alpha \rangle$ is \(\delta_\gamma\)-equidistributed if and only if $S_\alpha(f) \to 1$ for all normalized homogeneous polynomials $f \in K[X]$. In order to prove the theorem, it therefore suffices to show that $S_\alpha(f) \to 1$ if and only if the limit \eqref{limit} holds for all $0 < t < 1$.

Assuming that the limit \eqref{limit} holds for all $0 < t < 1$, we have

$$S_\alpha(f) \geq t \frac{|\{z \in Z_\alpha \mid \lambda_f(z) \geq t\}|}{|Z_\alpha|} = t \left( 1 - \frac{|\{z \in Z_\alpha \mid \lambda_f(z) < t\}|}{|Z_\alpha|} \right) \to t.$$

As $0 < t < 1$ is arbitrary and $S_\alpha(f) \leq 1$, we deduce that $S_\alpha(f) \to 1$.

Conversely, suppose that

$$\limsup_{\alpha} \frac{|\{z \in Z_\alpha \mid \lambda_f(z) < t\}|}{|Z_\alpha|} = \epsilon > 0,$$

for some $0 < t < 1$. Passing to a subnet, we may assume that this limsup is actually a limit. Using the fact that $0 \leq \lambda_f(z) \leq 1$ for all $z \in \mathbb{P}_K^N$, we have

$$S_\alpha(f) = \frac{1}{|Z_\alpha|} \sum_{\lambda_f(z) < t} \lambda_f(z) + \frac{1}{|Z_\alpha|} \sum_{\lambda_f(z) \geq t} \lambda_f(z) \leq t \frac{|\{z \in Z_\alpha \mid \lambda_f(z) < t\}|}{|Z_\alpha|} + \frac{|\{z \in Z_\alpha \mid \lambda_f(z) \geq t\}|}{|Z_\alpha|} \to t \epsilon + (1 - \epsilon) < 1,$$

which means that $S_\alpha(f) \not\to 1$ in this case. \hfill \qed

4.2. Let $A_K^{N+1} = \text{Spec}(\tilde{K}[X])$ and $\mathbb{P}_K^N = \text{Proj}(\tilde{K}[X])$ be the usual scheme-theoretic affine and projective spaces over the residue field $\tilde{K}$, as defined say in \textit{[LS]}} § II.2. Let $\pi_{\text{sch}} : A_K^{N+1} \setminus \{p_0\} \to \mathbb{P}_K^N$ denote the standard projection map, where $p_0$ denotes the ideal of polynomials in $\tilde{K}[X]$ vanishing at $0 = (0, 0, \ldots, 0)$.

Recall from § 2.3 that the set $S_K^N = \{\zeta \in A_K^{N+1} \mid \|\zeta\| = 1\}$ can be taken as a domain for the quotient map $\pi : S_K^N \to \mathbb{P}_K^N$. Given a point $\zeta \in S_K^N$, the ultrametric inequality implies that $[f]_\zeta \leq 1$ for all $f \in K^\circ[X]$. Define

$$\mathcal{G}_\zeta = \{f \in K^\circ[X] \mid [f]_\zeta < 1\},$$

and define $\tilde{\mathcal{G}}_\zeta$ to be the image of $\mathcal{G}_\zeta$ under the reduction map $K^\circ[X] \to \tilde{K}[X]$. Then $\mathcal{G}_\zeta$ is a prime ideal of the ring $K^\circ[X]$ (since $[\cdot]_\zeta$ is multiplicative), and $\tilde{\mathcal{G}}_\zeta$ is a prime ideal of the ring $\tilde{K}[X]$ (since $\tilde{K}[X] \simeq K^\circ[X]/K^{\circ \circ}$). We obtain affine and projective reduction maps

$$
\begin{array}{ccc}
S_K^N & \longrightarrow & A_K^{N+1} \setminus \{p_0\} \\
\pi & \downarrow & \downarrow \pi_{\text{sch}} \\
\mathbb{P}_K^N & \longrightarrow & \mathbb{P}_K^N
\end{array}
$$

(4)
Here the affine reduction map $S_K^N \rightarrow A_K^{N+1} \setminus \{0\}$ is given by $\zeta \mapsto \bar{\zeta}$, and to see that there exists a unique map $r : P_K^N \rightarrow P_K^N$ completing the commutative diagram (4), it suffices to show that $\pi_{\text{sch}}(\bar{\zeta}) = \pi_{\text{sch}}(\bar{\zeta})$ whenever $\pi(\zeta) = \pi(\xi)$; this is straightforward to check using the definitions of $\pi$ and $\pi_{\text{sch}}$. It is also a standard exercise in the definitions to show that the reduction map $r$ is surjective (see [4] § 2.4).

The Gauss point $\gamma = \pi(\zeta_{0,1})$ can be characterized as the unique point of $P_K^N$ which reduces to the generic point of $P_K^N$. To see this, note that $\bar{\zeta}_{0,1}$ is the zero ideal of $K[X]$ by Lemma 4.4, thus $r(\gamma) = \pi_{\text{sch}}(\bar{\zeta}_{0,1})$ is the generic point of $P_K^N$. Conversely, if $\zeta \in S_K^N$ and $\pi_{\text{sch}}(\bar{\zeta})$ is the generic point of $P_K^N$, then the ideal $\bar{\zeta}$ contains no nonzero homogeneous polynomials in $K[X]$. In other words, $\lambda_f(\pi(\zeta)) = [f]_{\zeta} = 1$ for all normalized homogeneous polynomials $f \in K[X]$. Since $\lambda_f(\gamma) = 1$ for all such $f$ as well, and since the functions $\lambda_f$ separate the points of $P_K^N$, we conclude that $\pi(\zeta) = \gamma$.

4.3. Given a finite multiset $Z$ of points in $P_K^N$, define its reduction $\tilde{Z}$ to be the finite multiset in $P_K^N$ where the multiplicity of a point $\tilde{z}$ in $\tilde{Z}$ is the sum of the multiplicities of the points $z \in r^{-1}(\tilde{z})$ in $Z$. Thus $|Z| = |\tilde{Z}|$.

Let $\langle \tilde{Z}_\alpha \rangle$ be a net of nonempty finite multisets in $P_K^N$. We say that the net $\langle \tilde{Z}_\alpha \rangle$ is generic if, given any subnet $\langle \tilde{Z}_\beta \rangle$ of $\langle \tilde{Z}_\alpha \rangle$ and any proper Zariski-closed subset $W \subset P_K^N$, there exists $\beta_0$ such that $\tilde{Z}_\beta \cap W = \emptyset$ for all $\beta \geq \beta_0$. We say that the net $\langle \tilde{Z}_\alpha \rangle$ is weakly generic if the limit

\begin{equation}
\lim_{\alpha} \frac{|\tilde{Z}_\alpha \cap W|}{|\tilde{Z}_\alpha|} = 0
\end{equation}

holds for all proper Zariski-closed subsets $W \subset P_K^N$. Note that a generic net is weakly generic: if $\langle \tilde{Z}_\alpha \rangle$ is generic, then $|\tilde{Z}_\alpha \cap W| = 0$ for all sufficiently large $\alpha$, whereby $|\tilde{Z}_\alpha \cap W|/|\tilde{Z}_\alpha| \rightarrow 0$.

**Theorem 9.** Let $\langle Z_\alpha \rangle$ be a net of nonempty finite multisets in $P_K^N$. If the reduction $\langle \tilde{Z}_\alpha \rangle$ is weakly generic in $P_K^N$, then $\langle Z_\alpha \rangle$ is $\delta_g$-equidistributed.

**Proof.** Given a normalized homogeneous polynomials $f \in K[X]$, let $\tilde{f} \in K[X]$ denote its reduction, and define

\[ V(\tilde{f}) = \{ p \in P_K^N \mid (\tilde{f}) \subseteq p \} \]

to be the hypersurface in $P_K^N$ associated to $\tilde{f}$. Given a multiset $Z_\alpha$ and a point $z \in Z_\alpha$, by Lemma 4.3 we may select $\zeta_z \in A_K^{N+1}$ such that $\pi(\zeta_z) = z$ and $||\zeta_z|| = 1$. It follows from the assumptions $H(f) = 1$ and $||\zeta_z|| = 1$, along with the ultrametric inequality, that $\lambda_f(z) = [f]_{\zeta_z} \leq 1$. Moreover, it is easy to check using the definition of the reduction map that $\lambda_f(z) = [f]_{\zeta_z} < 1$ if
and only if \( \tilde{z} = r(z) \) is contained in the hypersurface \( V(\tilde{f}) \). Thus

\[
\frac{|\{z \in Z_\alpha \mid \lambda_f(z) < 1\}|}{|Z_\alpha|} = \frac{|V(\tilde{f}) \cap Z_\alpha|}{|Z_\alpha|}.
\]

Since \( \langle \tilde{Z}_\alpha \rangle \) is weakly generic in \( \mathbb{P}^N_k \), the right-hand-side of (6) goes to zero in the limit. It follows that the limit (2) holds for all normalized homogeneous polynomials \( f \in K[X] \) and each real number \( 0 < t < 1 \), and we conclude that \( \langle Z_\alpha \rangle \) is \( \delta_T \)-equidistributed using Theorem 8.

Remark. The converse of Theorem 9 is false. For example, let \( \langle x_\alpha \rangle \) be a net in \( K \) such that \( |x_\alpha| < 1 \) for all \( \alpha \), but such that \( |x_\alpha| \rightarrow 1 \). Theorem 8 implies that the net \( \langle Z_\alpha \rangle \) of singleton sets \( Z_\alpha = \{(1 : x_\alpha)\} \) is \( \delta_T \)-equidistributed. But each point \( (1 : x_\alpha) \) reduces to the same point \( (1 : 0) \) in \( \mathbb{P}^1(K) \), so \( \langle \tilde{Z}_\alpha \rangle \) is not weakly generic in \( \mathbb{P}^1_k \).

In practice, however, the converse of Theorem 9 may hold for certain classes of nets \( \langle Z_\alpha \rangle \) of particular interest. We will see an example of this in the proof of Theorem 11.

5. A Ergodic Equidistribution Theorem

5.1. Define the unit torus \( T_K^N \) in \( \mathbb{P}^N(K) \) by

\[
T_K^N = \{(a_0 : a_1 : \cdots : a_N) \in \mathbb{P}^N(K) \mid |a_0| = |a_1| = \cdots = |a_N| = 1\}.
\]

Note that \( T_K^N \) is a group under coordinate multiplication, with neutral element \( 1 = (1 : \cdots : 1) \). In this section we will prove an equidistribution result, in the case of residue characteristic zero, for the sequence \( \langle a^\ell \rangle_{\ell=1}^{+\infty} \) formed by taking the powers of a point \( a \in T_K^N \).

5.2. We begin with some algebraic preliminaries. Let \( k \) be an arbitrary field of characteristic zero, and fix homogeneous coordinates \( (x_0 : x_1 : \cdots : x_N) \) on \( \mathbb{P}^N \) over \( k \). We identify the group variety \( \mathbb{G}_m^N \) over \( k \) with the subvariety of \( \mathbb{P}^N \) defined by \( x_0x_1 \cdots x_N \neq 0 \); the group law on \( \mathbb{G}_m^N(k) \) is given by coordinate multiplication, with neutral element \( 1 = (1 : \cdots : 1) \).

Given an arbitrary point \( a \in \mathbb{G}_m^N(k) \), we denote by \( a_1, \ldots, a_N \) the unique elements of \( k^\times \) such that \( a = (1 : a_1 : \cdots : a_N) \). We say that \( a \) is degenerate if the elements \( a_1, \ldots, a_N \) are multiplicatively dependent in \( k^\times \); otherwise we say that \( a \) is non-degenerate.

Consider a subgroup \( \Lambda \) of the integer lattice \( \mathbb{Z}^N \). The group \( \Lambda \) gives rise to an algebraic subgroup \( \mathbb{G}_\Lambda \) of \( \mathbb{G}_m^N \) by

\[
\mathbb{G}_\Lambda(k) = \{a \in \mathbb{G}_m^N(k) \mid a_1^{\ell_1} \cdots a_N^{\ell_N} = 1 \text{ for all } (\ell_1, \ldots, \ell_N) \in \Lambda\}.
\]

Conversely, all algebraic subgroups of \( \mathbb{G}_m^N \) arise in this way (\( \S \)3.2). It follows from this correspondence that a point \( a \in \mathbb{G}_m^N(k) \) is degenerate if and only if \( a \) is contained in some proper algebraic subgroup of \( \mathbb{G}_m^N \). Moreover, it is clear that \( a \) is degenerate if and only if \( a^\ell \) is degenerate for all nonzero \( \ell \in \mathbb{Z} \).
Proposition 10. A point $a \in \Gamma^N_m(k)$ is non-degenerate if and only if, for each proper Zariski-closed subset $W$ of $\Gamma^N_m(k)$, there exist only finitely many integers $\ell$ such that $a^\ell \in W$.

Proof. The “if” direction is trivial. For if $a$ is degenerate then $a \in G(k)$ for some proper algebraic subgroup $G$ of $\Gamma^N_m$, and therefore $a^\ell \in G(k)$ for all $\ell \in \mathbb{Z}$.

To prove the “only if” direction, assume that $a$ is non-degenerate, and consider a proper Zariski-closed subset $W$ of $\Gamma^N_m(k)$. Denote by $a^\mathbb{Z}$ the cyclic subgroup of $\Gamma^N_m(k)$ generated by $a$. Since $a$ is non-degenerate, it is non-torsion, and therefore in order to complete the proof it is enough to show that $a^\mathbb{Z} \cap W$ is finite. Replacing $W$ with the Zariski-closure of $a^\mathbb{Z} \cap W$, we may assume without loss of generality that $a^\mathbb{Z} \cap W$ is Zariski-dense in $W$. A result of Laurent (see also [3] Thm. 7.4.7) implies that, since $a^\mathbb{Z} \cap W$ is Zariski-dense in $W$, we must have $W = \cup_{j=1}^J y_j G_j$ for some finite set $y_1, \ldots, y_J \in \Gamma^N_m(k)$ and some finite collection $G_1, \ldots, G_J$ of proper algebraic subgroups of $\Gamma^N_m$. Therefore, in order to show that $a^\mathbb{Z} \cap W$ is finite, it suffices to show that $a^\mathbb{Z} \cap yG(k)$ is finite for an arbitrary $y \in \Gamma^N_m(k)$ and an arbitrary proper algebraic subgroup $G$ of $\Gamma^N_m$. In fact, $a^\mathbb{Z} \cap yG(k)$ can contain at most one point. For if $a^\ell$ and $a^{\ell'}$ are elements of $yG(k)$ for some distinct integers $\ell, \ell' \in \mathbb{Z}$, then $a^{\ell - \ell'} \in G(k)$, implying that $a^{\ell - \ell'}$ is degenerate. This contradicts the assumption that $a$ is non-degenerate. □

Remark. The result of Laurent used in this proof is the $\Gamma^N_m$ case of what is commonly called the Lang (sometimes Mordell-Lang) conjecture. It is now a much more general theorem, holding for finite-rank subgroups of semi-abelian varieties, due to Laurent, Faltings, Vojta, and McQuillen; see [19] § F.1.1 for a survey.

5.3. We return to our complete non-archimedean field $K$. Observe that, given a point $a \in \mathbb{P}^N(K)$, we have $a \in T^N_K$ if and only if $\tilde{a} \in \Gamma^N_m(\tilde{K})$.

Theorem 11. Assume that the residue field $\tilde{K}$ has characteristic zero. Let $a \in T^N_K$, and for each integer $\ell \geq 1$, define $Z_\ell = \{a, a^2, \ldots, a^\ell\}$, considered as a multiset in $T^N_K \subset \mathbb{P}^N(K) \subset \mathbb{P}^N_K$ of cardinality $|Z_\ell| = \ell$. The sequence $(Z_\ell)_{\ell=1}^\infty$ is $\delta_\gamma$-equidistributed in $\mathbb{P}^N_K$ if and only if the point $\tilde{a}$ is non-degenerate in $\Gamma^N_m(\tilde{K})$.

Proof. Assume that $\tilde{a}$ is non-degenerate in $\Gamma^N_m(\tilde{K})$. By Proposition [10] $a^\mathbb{Z} \cap W$ is finite for any proper Zariski-closed subset $W$ of $\mathbb{P}^N(\tilde{K})$, which implies that the sequence $(Z_\ell)_{\ell=1}^\infty$ is weakly generic in $\mathbb{P}^N(\tilde{K})$. By Theorem [9] we conclude that $(Z_\ell)_{\ell=1}^\infty$ is $\delta_\gamma$-equidistributed.

Conversely, suppose that $\tilde{a}$ is degenerate in $\Gamma^N_m(\tilde{K})$. Writing $a = (1 : a_1 : \cdots : a_N) \in T^N_K \subset \mathbb{P}^N(K)$ the fact that $\tilde{a}$ is degenerate means that $a_1^{\ell_1} \cdots a_N^{\ell_N} = 1$ in $\tilde{K}$ for a nonzero vector $(\ell_1, \ldots, \ell_N) \in \mathbb{Z}^N$. Therefore the element $A := a_1^{\ell_1} \cdots a_N^{\ell_N} - 1 \in K$
satisfies $|A| < 1$. Select an integer $r \geq 0$ such that $\ell_n + r \geq 0$ for all $1 \leq n \leq N$, let $\ell = \sum_{n=1}^{N} \ell_n$, and define $f \in K[X]$ by

$$f(X) = X_1^{\ell_1 + r} \cdots X_N^{\ell_N + r} - X_0^{\ell}(X_1 \cdots X_N)^r.$$ 

Note that $f$ is nonzero, homogeneous, and satisfies $H(f) = 1$. In particular, 

$$\int_\gamma \lambda_f d\delta_\gamma = \lambda_f(\gamma) = [f]_{\delta_{0,1}} = H(f) = 1. \tag{7}$$ 

On the other hand, it is easy to check that $|f(1, a_1, \ldots, a_N)| = |A|$, and that more generally $|f(1, a_1^j, \ldots, a_N^j)| \leq |A|$ for all integers $j \geq 1$. Therefore 

$$\frac{1}{|Z_\ell|} \sum_{z \in Z_\ell} \lambda_f(z) = \frac{1}{\ell} \sum_{j=1}^{\ell} |f(1, a_1^j, \ldots, a_N^j)| \leq |A| < 1. \tag{8}$$ 

Letting $\ell \to +\infty$, (7) and (8) together show that the sequence $(Z_\ell)_{\ell=1}^{\infty}$ fails the criterion for $\delta_\gamma$-equidistribution stated in Corollary 8. \hfill \Box 

5.4. Theorem 11 is a non-archimedean analogue of the following classical equidistribution result of Weyl [22]. As in the non-archimedean case, define the unit torus $T_K^N$ in $\mathbb{P}^N(\mathbb{C})$ by

$$T_K^N = \{(a_0 : a_1 : \cdots : a_N) \in \mathbb{P}^N(\mathbb{C}) \mid |a_0| = |a_1| = \cdots = |a_N| = 1\}.$$ 

Then $T_K^N$ is a compact topological group, and as such it carries a unique normalized Haar measure.

**Theorem 12 (Weyl).** Let $a \in T_K^N$, and for each integer $\ell \geq 1$, define $Z_\ell = \{a, a^2, \ldots, a^\ell\}$, considered as a multiset in $T_K^N$ of cardinality $|Z_\ell| = \ell$. The sequence $(Z_\ell)_{\ell=1}^{\infty}$ is Haar-equidistributed in $T_K^N$ if and only if $a$ is non-degenerate in $G_m(\mathbb{C})$.

An important difference between Theorems 11 and 12 stems from the fact that, in the non-archimedean case, the assumption that char($\hat{K}$) = 0 ensures that $\hat{K}$ has infinitely many elements, which implies that the field $K$ is not locally compact. Consequently, the unit torus $T_K^N$ is noncompact and thus has no Haar measure in the traditional sense. On the other hand, observe that $T_K^N$ is contained in the compact Berkovich unit torus 

$$T_K^N = \{\pi(\zeta) \mid \zeta \in A_K^{N+1} \text{ and } [X_0]_\zeta = [X_1]_\zeta = \cdots = [X_N]_\zeta = 1\},$$ 

and that the Dirac measure $\delta_\gamma$ supported on the Gauss point $\gamma \in T_K^N$ is invariant under the translation action of the group $T_K^N$ on $T_K^N$. Thus $\delta_\gamma$ is a natural substitute for Haar measure in this setting.

Analogues of Weyl’s Haar-equidistribution result have been investigated over the locally compact non-archimedean field $\mathbb{Q}_p$, at least in the case $N = 1$; see Bryk-Silva [9] and Coelho-Parry [11].

As pointed out by the referee, several things can be said in the direction of Theorem 11 when the residue field $\hat{K}$ has characteristic $p \neq 0$. First, the “only if” direction of the theorem continues to hold, with the same proof. If
\(\tilde{K}\) is algebraic over its prime field \(\mathbb{F}_p\), then the statement of Theorem 11 holds trivially, since all points of \(\mathbb{G}_m^N(\tilde{\mathbb{F}_p})\) are torsion and therefore degenerate. Finally, the statement of Theorem 11 continues to hold for an arbitrary field \(K\) of residue characteristic \(p \neq 0\) in the one-dimensional case. For observe that an element \(\tilde{a}\) in \(\mathbb{G}_m^K(\tilde{K})\) is degenerate if and only if it is torsion. Using the trivial fact that the cyclic subgroup \(\tilde{a}\mathbb{Z}\) of \(\mathbb{G}_m^K(\tilde{K})\) is either finite or Zariski-dense, there is no need for Laurent’s theorem, and therefore no need to assume that \(\text{char}(\tilde{K}) = 0\). We do not know whether the statement of Theorem 11 holds for residue characteristic \(p \neq 0\) and \(N \geq 2\).

Finally, we point out that Theorems 11 and 12 can be viewed as examples of a more general class of equidistribution results arising naturally in ergodic theory. Let \(\mathcal{X}\) be a compact Hausdorff space, let \(T : \mathcal{X} \to \mathcal{X}\) be an automorphism, let \(\mu\) be a \(T\)-invariant unit Borel measure on \(\mathcal{X}\), and let \(x \in \mathcal{X}\) be a point. One of the basic goals of ergodic theory is to establish conditions under which the sequence \(\langle Z_\ell \rangle_{\ell=1}^{+\infty}\) of multisets \(Z_\ell = \{T^\ell(x), T^{2\ell}(x), \ldots, T^{\ell^2(x)}\}\) is \(\mu\)-equidistributed; see for example Furstenberg [15] and Lindenstrauss [21] for discussions of such results with a particular eye toward arithmetic applications.

Weyl’s original proof of Theorem 12 uses Fourier analysis, but there exists an alternate, ergodic-theoretic proof, see Furstenberg [15] Ch. 3. It would be interesting to pursue non-archimedean equidistribution results such as Theorem 11 from an ergodic-theoretic angle. In view of Proposition 10 and its reliance on Laurent’s theorem [20], it is especially intriguing to consider the possibility of deeper connections between ergodic theory and questions of Mordell-Lang type.

**References**

[1] M. Baker and C. Petsche. Global discrepancy and small points on elliptic curves. *Int. Math. Res. Not.* 61 (2005) 3791-3834.
[2] M. Baker and R. Rumely. *Potential Theory and Dynamics on the Berkovich Projective Line*. AMS Surveys and Mathematical Monographs series, to appear.
[3] M. Baker and R. Rumely, Equidistribution of small points, rational dynamics, and potential theory, *Ann. Inst. Fourier (Grenoble)* 56 no. 3 (2006) 625-688.
[4] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, AMS Mathematical Surveys and Monographs 33 (AMS, Providence, 1990).
[5] V. Berkovich, The automorphism group of the Drinfeld half-plane, *C. R. Acad. Sci. Paris Sér. I Math.* 321 (1995), no. 9, 1127-1132.
[6] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry*, Cambridge University Press, New York, 2006.
[7] S. Bosch, U. Gänzer, and R. Remmert. *Non-Archimedean Analysis*, Springer-Verlag, Berlin, 1984.
[8] N. Bourbaki. *Elements of Mathematics: General Topology Part I*. Hermann, Paris, 1966.
[9] J. Bryk and C. Silva, Measurable dynamics of simple \(p\)-adic polynomials, *Amer. Math. Monthly* Vol. 112 (2005), no. 3, 212-232.
[10] A. Chambert-Loir, Mesures et équidistribution sur les espaces de Berkovich, *J. für die reine und angewandte Mathematik* 595 (2006) 215-235.
[11] Z. Coelho and W. Parry, Ergodicity of $p$-adic multiplications and the distribution of Fibonacci numbers, in Topology, Ergodic Theory, Real Algebraic Geometry, Amer. Math. Soc. Transl. Ser. 2, no. 202, Providence (2001) pp. 51-70.

[12] X.W.C. Faber. Equidistribution of dynamically small subvarieties over the function field of a curve. Acta Arith. 137 (2009) 4, 345-389.

[13] C. Favre and J. Rivera-Letelier, Equidistribution quantitative des points de petite hauteur sur la droite projective, Math. Ann. (2) 335 (2006) 311361.

[14] G. Folland. Real Analysis: Modern Techniques and their Applications (2nd ed.) John Wiley, 1999.

[15] H. Furstenberg. Recurrence in Ergodic Theory and Combinatorial Number Theory Vol. 201 of Graduate Texts in Mathematics. Princeton University Press, 1981.

[16] W. Gubler. Local heights of subvarieties over non-archimedean fields. J. reine agnew. Math. 498 (1998) 61-113.

[17] W. Gubler. Equidistribution over function fields. Manuscripta Math. 127 (2008) 4, 485-510.

[18] R. Hartshorne. Algebraic Geometry. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[19] M. Hindry and J. H. Silverman. Diophantine Geometry: an Introduction Vol. 201 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2000.

[20] M. Laurent, Équations diophantiennes exponentielles, Invent. Math. 78 (1984), no. 2, 299-327.

[21] E. Lindenstrauss. Some examples how to use measure classification in number theory, in Equidistribution in number theory, an introduction, NATO Sci. Ser. II Math. Phys. Chem., v. 237, pp. 261-303. Springer, Dordrecht, 2007.

[22] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann. 77 (1916) 313-352.