Exact shape of the lowest Landau level in a spin-$\frac{1}{2}$ system with uncorrelated disorders

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Using the path-integral approach developed by Brézin et al. [ Nucl. Phys. B 235, 24 (1984) ], we obtain an analytical expression for the density of states of a spin-$\frac{1}{2}$ disordered two-dimensional electron gas in a strong, perpendicular magnetic field. The density of states of this system illustrates the interplay between the Zeeman splitting of Landau levels and the disorder-induced broadening. We find that the broadening and the band splitting of the Landau bands are enhanced due to the level repulsion from the mixing of two spin orientations caused by the random scatterings. The comparison between the spin-$\frac{1}{2}$ model and the double-layer system is also discussed.

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I. INTRODUCTION

There has been remarkable interest in the problem of a disordered two-dimensional electron gas in a strong, perpendicular magnetic field ever since the discovery of the quantum Hall effect in such systems. Although there have been many studies on this system, however, only a few rigorous results of the density of states (DOS) for the disordered electrons are obtained. The difficulty for obtaining the rigorous expression of DOS comes from the fact that, in the absence of the disorder, the energy spectrum is discrete. Therefore the self-energy of an electron is real in any finite order of the perturbation theory. Hence, to obtain the expected disorder-broadened Landau levels (LL), one needs to sum over the entire diagram expansion. In the strong-field limit, where only the lowest LL is considered, the exact DOS of the spinless electrons was first found by Wegner for the case of a Gaussian white noise distribution of the random potential. Using a functional-integral approach, Brézin, Gross and Itzykson were able to re-derive Wegner’s result and, furthermore, to generalize it to the case in which a non-Gaussian, but still zero-range, distribution of impurities is assumed. The crucial point for these exact derivations is the intrinsic supersymmetry of the problem, by which a mapping from the original two-dimensional system onto a zero-dimensional model becomes possible. Thus, the summation of the entire diagram expansion can be achieved.

However, in the real world, electrons carry spins. As a result, the effects of mixing between LL’s with different spin quantum numbers would become important, when the Zeeman splitting is smaller than the width of each disorder-broadened LL. Hence, the generalization of above results to the spin-$\frac{1}{2}$ case is of practical interest. However, as was pointed out in the last section of Ref. the identity ensuring the existence of the supersymmetry fails to hold in the two-component case and implying that a straightforward generalization to the spin-$\frac{1}{2}$ case appears impossible in general. Nevertheless, we find that an exact expression of DOS can be reached provided that the spin-$\frac{1}{2}$ electrons are influenced by the random potential scattering and the random spin-flip scattering, which are both assumed to be the zero-range white-noise distribution. We find that the energy difference of the splitted bands is larger than the Zeeman splitting energy. Moreover, when the Zeeman splitting is turned off, the width of the disorder-broadened band is larger than, rather than identical to, that of the spinless case. Thus an interesting interplay between the Zeeman splitting of Landau levels and the disorder-induced broadening can be demonstrated in our spin-$\frac{1}{2}$ system. In Section II, we introduce the model Hamiltonian and outline the functional-integral formulation. In Sec.III, the exact results of DOS are presented, which show the enhancement of the splitting and the band width. In Sec.IV, we discuss the asymptotic behavior of DOS when Zeeman splitting is large. Section V is devoted to the conclusion and the comparison between the spin-$\frac{1}{2}$ model and the double-layer system.

II. THE MODEL AND THE FUNCTIONAL-INTEGRAL FORMULATION

We consider the following Hamiltonian,

$$ H = H_0 + U_1(r) + U_2(r)\tau_1, \quad (1) $$

$$ H_0 = \frac{1}{2m}(P - eA)^2 - \frac{1}{2} g \mu_B B \tau_3, \quad (2) $$

where $A(r) = (-B_y/2, B_x/2)$ is the vector potential of the constant magnetic field, $B(r) = B\hat{z}$, in the symmet-
ric gauge, $g$ is the gyromagnetic ratio, $\mu_B$ is the Bohr magneton, $\tau_1$ and $\tau_3$ are Pauli matrices. The random potential scattering and the random spin-flip scattering are denoted by $U_1(r)$ and $U_2(r)\tau_1$, respectively. The latter term can be visualized as a simplified description of the scattering by magnetic impurities, or the spin-orbit scattering. (A similar model has been considered by Wang et al.\(^{1,2}\) to study the effects of Landau level mixing).

We first outline briefly the procedure of calculating DOS developed by Brézin et al.\(^{1,2}\) Assuming that the magnetic field is so strong that the random scatterings cannot induce transitions between different Landau levels, the averaged DOS of the lowest LL can be written as

$$\rho(E) = -\frac{1}{\pi} \text{Im} \text{Tr} \left\{ \frac{1}{E - H + i0} \right\},$$

where $\text{Tr}$ denotes the trace operation over the spin indices and the bar indicates averaging over all configurations of the random potentials. Using the functional-integral approach\(^{1,2}\) the matrix element of the resolvent after random average becomes

$$\text{Tr} \left\{ \frac{1}{E - H + i0} \right\} = -ie^{-\frac{1}{2}\kappa^2|z|^2} \times \prod_{\sigma = \pm} U_{\sigma} D_{\sigma} U_{\sigma}^* D_{\sigma}^{\ast} \sum_{\sigma = \pm} \left( u_{\sigma}^* u_{\sigma} + \bar{v}_{\sigma} v_{\sigma} \right) e^{S},$$

with the “action” $S$ given by

$$S = i \int dzdz^* e^{-\frac{1}{2}\kappa^2|z|^2} \sum_{\sigma = \pm} (\epsilon - \sigma \bar{g}) (u_{\sigma}^* u_{\sigma} + \bar{v}_{\sigma} v_{\sigma})$$

$$+ \int dzdz^* f_1 e^{-\frac{1}{2}\kappa^2|z|^2} \sum_{\sigma = \pm} (u_{\sigma}^* u_{\sigma} + \bar{v}_{\sigma} v_{\sigma}).$$

Consequently, the “action” $S$ can be expressed in terms of the superfields $\Phi_\pm(z, \theta)$ and $\Phi_\pm(\bar{z}^*, \bar{\theta})$ as

$$S = \frac{2\pi i}{\kappa^2} \int dzdz^* d\theta d\bar{\theta} e^{-\frac{1}{2}\kappa^2|z|^2 + \bar{\theta} \theta} \left[ c(\Phi_+ \Phi_+ + \Phi_- \Phi_-) - \bar{g}(\Phi_+ \Phi_+ + \Phi_- \Phi_-) \right]$$

$$- \frac{w}{2} \int dzdz^* \left[ e^{-\frac{1}{2}\kappa^2|z|^2} \frac{2\pi}{\kappa^2} \int d\theta d\bar{\theta} e^{-\frac{1}{2}\kappa^2\bar{\theta} \theta} (\Phi_+ \Phi_+ + \Phi_- \Phi_-) \right]^2$$

$$- \frac{w}{2} \int dzdz^* \left[ e^{-\frac{1}{2}\kappa^2|z|^2} \frac{2\pi}{\kappa^2} \int d\theta d\bar{\theta} e^{-\frac{1}{2}\kappa^2\bar{\theta} \theta} (\Phi_+ \Phi_+ + \Phi_- \Phi_-) \right]^2.$$
spinless electrons. However, as was pointed out in Ref.,
the generalization of Eq.(14) to the two-component case, say, \( \Phi \to (\Phi_1, \Phi_\perp + \Phi_-) \), is not allowed when \( n > 1 \)
as the left-hand side of Eq.(14) contains the four fermion interactions, \( \bar{v}_+ v_+ \bar{v}_+ v_- \), whereas the right-hand side of Eq.(14) does not. Hence, the supersymmetry, which enables one to obtain an exact expression for DOS, does not seem to exist in the spin-\( \frac{1}{2} \) case. However, when the two random distributions \( P_1 \) and \( P_2 \) are both Gaussian and of the same strength, these four fermion interactions cancel each other in the sum of the second and third integrals in Eq.(14). This cancellation can be easily illustrated by the following changes of variables for the superfields:

\[
\begin{align*}
\Phi_1 &= \frac{1}{\sqrt{2}} (\Phi_1 + \Phi_-), \\
\Phi_2 &= \frac{1}{\sqrt{2}} (\Phi_1 - \Phi_-), \\
\Phi_\perp &= \frac{1}{\sqrt{2}} (\Phi_\perp + \Phi_-), \\
\Phi_- &= \frac{1}{\sqrt{2}} (\Phi_\perp - \Phi_-),
\end{align*}
\]

therefore, a supersymmetric “action” is reached,

\[
S = \frac{2\pi i}{\kappa^2} \int \left[ \frac{d\phi_1 d\phi_2 d\phi_\perp d\phi_-}{\kappa^2} e^{-\frac{1}{\kappa^2} \left(|\phi_1|^2 + |\phi_-|^2\right)} \times \left[ \epsilon (\Phi_1 \Phi_2 + \bar{\Phi}_2 \Phi_\perp) - \bar{g}(\Phi_1 \bar{\Phi}_2 + \bar{\Phi}_2 \bar{\Phi}_\perp) \right] \right. \\
\left. \times \left[ \frac{w\pi}{\kappa^2} \int d\phi_1 d\phi_2 d\phi_\perp d\phi_- e^{-\kappa^2 \left(|\phi_1|^2 + |\phi_-|^2\right)} \times \left[ \frac{|\phi_1\phi_2|^2 + |\bar{\Phi}_2\Phi_\perp|^2}{2} \right] \right. \right]
\]

In the derivation of Eq.(17), the identity Eq.(14) is used for each \( \Phi_1 \) and \( \Phi_2 \) in the intermediate step, in which no four fermion term is involved. Because of the dimensional reduction, \( d \to d - 2 \), due to the existence of the supersymmetry in Eq.(17), the calculations of the averaged resolvent, Eq.(18), can be reduced to evaluate the following zero-dimensional integral:

\[
\text{Tr}
\left\{ \frac{1}{E - H + i\epsilon} \right\}
= -i \int d\phi_1 d\phi_2 d\phi_\perp d\phi_- \left( \phi_1^* \phi_2 + \phi_\perp^* \phi_- \right) e^{S_0},
\]

where \( S_0 \) is defined by

\[
S_0 = \frac{2\pi}{\kappa} \epsilon (\phi_1^* \phi_2 + \phi_\perp^* \phi_-) - \frac{2\pi}{\kappa} \bar{g}(\phi_1^* \phi_2 + \phi_\perp^* \phi_-) \\
- \frac{w\pi}{\kappa^2} \left[ (\phi_1^* \phi_2)^2 + (\phi_\perp^* \phi_-)^2 \right].
\]

Combining Eqs.(18), (18) and (19), we have the following exact expression for DOS

\[
\rho(E) = \frac{\kappa^2}{2\pi^2} \text{Im} \frac{\partial}{\partial \epsilon} \ln Z_0
\]

with

\[
Z_0 = \int d\phi_1 d\phi_2 d\phi_\perp d\phi_- e^{S_0}.
\]

It is convenient to rewrite \( Z_0 \) in a different form. We first rescale \( \phi_i \) by \( \phi'_i = \sqrt{\frac{2\pi}{\kappa}} \phi_i \), and subsequently decouple the quartic terms in \( S_0 \) with the help of gaussian integral over a pair of auxiliary variables, \( \lambda_1 \) and \( \lambda_2 \). Finally we calculate the remaining integral over \( \phi'_i \) to obtain

\[
Z_0 = \frac{(i\pi)^2}{\pi(2\pi)^{2 \Gamma^2}} \int \int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 e^{-\frac{-(\lambda_1^2 + \lambda_2^2)}{\Gamma^2}} \\
\times \left( e^{-\frac{\epsilon^2}{2\Gamma^2}} \int_{-\infty}^{\infty} dy dx e^{2g^2} - e^{-\frac{\epsilon^2}{2\Gamma^2}} \int_{0}^{\infty} dx \right),
\]

where the magnetic-field-dependent width \( \Gamma = \sqrt{w\kappa^2/\pi} \), \( \nu_4 = \sqrt{2\epsilon/\Gamma} \pm \sqrt{y^2 + \delta^2/\Gamma^2} \), and the changes of variables, \( x = (\lambda_1 + \lambda_2)/\sqrt{2} \) and \( y = (\lambda_1 - \lambda_2)/\sqrt{2} \), are introduced.

III. RESULTS OF VARIOUS DENSITY OF STATES

The general behavior of DOS is shown in Fig.1. Only the part for \( \epsilon \geq 0 \) is shown, since \( \rho(E) \) is symmetric with respect to the band center ( \( \epsilon = 0 \) ). Figure 1(a) shows, as expected, that the splitting of the disordered Landau bands increases as \( g \) increases. To get more insight from our result, we depict in Fig.1(b) by shifting horizontally each curve of Fig.1(a) with the Zeeman energy, \( \bar{g} = \frac{\epsilon}{2} g \mu_B B \). It becomes apparent that, as \( g \) increases, the shifted curves evolve to a single curve having the similar form of DOS in the spinless case, \( \rho_0(E) \). However, two unexpected results emerge: first, the peak values of the splitted bands occur at a larger \( \epsilon \), rather than exactly at \( \epsilon = \bar{g} \) as one expects naively; secondly, if there is no Zeeman splitting (i.e., \( g = 0 \) ), from Eqs.(22), (23) and (24), we obtain

\[
\rho(E)_{\epsilon=0} = \sqrt{2} \frac{\kappa^2}{2\pi^2} \frac{2}{\Gamma_0 \sqrt{\pi}} \left( 1 + \frac{\epsilon^2/2\Gamma_0^2}{\epsilon^2/2\Gamma_0^2} \right)^2.
\]

where \( \Gamma_0 = \sqrt{w\kappa^2/2\pi} = \Gamma/\sqrt{2} \) is the corresponding magnetic-field-dependent width in the spinless case. (See Eqs.(46a) and (46b) in Ref.) Thus the DOS in the spin-\( \frac{1}{2} \) system without Zeeman splitting is not simply twice as large as that in the spinless case. In our case, the peak value at the band center ( \( \epsilon = 0 \) ) of Eq.(23) is merely \( \sqrt{2} \) times of that in the spinless case, and the
width of \( \rho(E) \) is \( \sqrt{2} \) times wider. It is because the random scatterings mix the two spin degrees of freedom, as indicated by the quartic terms in Eq.(1). Therefore, we may expect a level splitting due to the usual repulsion of eigenvalues by these terms, and thus the broadening of the Landau bands is enhanced when \( g = 0 \), while the band splitting becomes larger for \( g \neq 0 \).

On the other hand, their difference can be expressed by

\[
\bar{\rho}(E) = \rho_{\uparrow}(E) - \rho_{\downarrow}(E).
\]

and then we have

\[
\rho_{\uparrow}(E) \xrightarrow{\epsilon \to -\epsilon} \rho_{\uparrow}(E),
\]

\[
\rho_{\downarrow}(E) \xrightarrow{\epsilon \to -\epsilon} \rho_{\downarrow}(E).
\]

With these relations in mind, we show our results of \( \rho_{\uparrow}(E) \) and \( \rho_{\downarrow}(E) \) only for \( \epsilon \geq 0 \) in Figs.2(a) and (b). We find that, as \( g \) increases, \( \rho_{\downarrow}(E) \) diminishes, while \( \rho_{\uparrow}(E) \) evolves to a similar form of \( \rho_{0}(E) \). The latter can be seen more clearly, if we shift horizontally each curve in Fig.2(a) with \( \bar{g} \) as shown in Fig.3. The interesting points are that : first, the peak value of \( \rho_{\uparrow}(E) \) ( \( \rho_{\downarrow}(E) \) ) does not occur at \( \epsilon = \bar{g} \) ( \( \epsilon = -\bar{g} \) ); secondly, \( \rho_{\uparrow}(E) \) and \( \rho_{\downarrow}(E) \) at the band center diminish very fast so that a small bump of \( \rho_{\downarrow}(E) \) ( \( \rho_{\uparrow}(E) \) ) is developed around \( \epsilon \sim \bar{g} \) ( \( \epsilon \sim -\bar{g} \) ). Again, these results can be explained by the level repulsion from the mixing of two spin degrees of freedom.

Moreover, the exact expression of the disorder-averaged DOS for each spin component, which is denoted by \( \rho_{\uparrow}(E) \) ( \( \rho_{\downarrow}(E) \) ) for up (down) spin, can be obtained for the present model. Parallel to the discussion for the derivation of the sum of DOS’s of two spin components, we have

\[
\rho(E) = \rho_{\uparrow}(E) + \rho_{\downarrow}(E).
\]

On the other hand, their difference can be expressed by (c.f. Eq.(20))

\[
\bar{\rho}(E) = \rho_{\uparrow}(E) - \rho_{\downarrow}(E)
\]

\[
= -\frac{\kappa^2}{2\pi^2} \text{Im} \frac{\partial}{\partial \bar{g}} \ln Z_{0}.
\]

Consequently, we have

\[
\rho_{\uparrow}(E) = \frac{1}{2} (\rho(E) + \bar{\rho}(E)),
\]

\[
\rho_{\downarrow}(E) = \frac{1}{2} (\rho(E) - \bar{\rho}(E)).
\]

By a similar reasoning of the mirror symmetry about the band center ( \( \epsilon = 0 \) ) for \( \rho(E) \), one can easily show that

\[
\bar{\rho}(E) \xrightarrow{\epsilon \to -\epsilon} -\bar{\rho}(E),
\]

Fig.1 : (a) The averaged DOS for a spin-\( \frac{1}{2} \) system in units of the peak value of the spinless case, \( \rho_{0}(E)|_{\epsilon=0} = \sqrt{2} \kappa^2/\pi^2/\Gamma \), for values of \( \bar{g}/\Gamma = 0.0 \) (points), 0.4 (dotted line), 0.8 (dashdotted line), 1.2 (dashed line), and 2.0 (solid line). (b) The curves in (a) are shifted horizontally by the Zeeman splitting energy, \( \bar{g} = \frac{1}{2} g \mu_{B} B \).

Fig.2 : The averaged DOS for (a) the spin-up component, \( \rho_{\uparrow}(E) \), and (b) the spin-down component, \( \rho_{\downarrow}(E) \), in units of the peak value of the spinless case, \( \rho_{0}(E)|_{\epsilon=0} = \sqrt{2} \kappa^2/\pi^2/\Gamma \), for values of \( \bar{g}/\Gamma = 0.0 \) (points), 0.4 (dotted line), 0.8 (dashdotted line), 1.2 (dashed line), and 2.0 (solid line).
IV. ASYMMETRIC BEHAVIOR OF DENSITY OF STATES FOR LARGE ZEEMAN SPLITTING

In the following, we consider the asymmetric behavior of DOS in large-\( \bar{g} \) limit (\( \bar{g}/\Gamma \gg 1 \)) and give the analytical descriptions of the features mentioned above. Especially, we will focus our attention on the three regions of energy: (1) near the location of the peak; (2) at the band center; (3) toward the band tails.

Due to the exponential factor, \( \exp(-y^2) \), in the integrands of Eqs.(23) and (24), the characteristic value of \( y \), being of order 1, is much smaller than \( \bar{g}/\Gamma \) when \( \bar{g}/\Gamma \gg 1 \). Thus we can neglect \( y^2 \) in \( \sqrt{y^2+2g^2}/\Gamma \), and then \( \nu_{\pm} \approx \sqrt{2|y|} \), where \( x_\pm = (\epsilon \pm \bar{g})/\Gamma \) is independent of \( y \). In this case, the integrations over \( y \) in Eqs.(23) and (24) can merely contribute constant factors. Therefore, the real part and the imaginary part of \( Z_0 \) become

\[
Z_0^R \approx -\frac{\kappa^4}{2\sqrt{2}|\bar{g}|} (F_- - F_+),
\]

\[
Z_0^I \approx \frac{\kappa^4}{\sqrt{2}} \frac{\sqrt{\Gamma}}{2} (F_- - f_+),
\]

where \( f_\pm = e^{-2x^2} \), and \( F_\pm = e^{-2x^2} f_0^{\sqrt{x^2+2}} \). From Eqs.(29) and (27), we have

\[
\rho(E) \approx -\frac{\sqrt{2}\kappa^2}{\pi^{5/2}\Gamma} \frac{4\sqrt{2}|\bar{g}|}{2} \frac{\sqrt{\Gamma}}{2} (F_- - F_+) + (F_- - f_+)^2, \]

\[
\bar{\rho}(E) \approx \frac{\sqrt{2}\kappa^2}{\pi^{5/2}\Gamma} \frac{4\sqrt{2}\sqrt{|\bar{g}|}}{\pi} (F_- - F_+) + \frac{\sqrt{\Gamma}}{2} (F_- - f_+)^2. \]

Consequently, the spin-up and spin-down components of DOS become

\[
\rho_{\uparrow}(E) \approx \frac{\sqrt{2}\kappa^2}{\pi^{5/2}\Gamma} \frac{2\sqrt{2}|\bar{g}| \sqrt{\Gamma}}{\pi} (F_- - F_+) + \frac{\sqrt{\Gamma}}{2} (F_- - f_+)^2 + \frac{\sqrt{\Gamma}}{2} (F_- - f_+ - F_+)^2 + (f_- - f_+)^2, \]

\[
\rho_{\downarrow}(E) \approx -\frac{\sqrt{2}\kappa^2}{\pi^{5/2}\Gamma} \frac{2\sqrt{2}|\bar{g}| \sqrt{\Gamma}}{\pi} (F_- - F_+) + \frac{\sqrt{\Gamma}}{2} (F_- - f_+)^2 + (f_- - f_+)^2. \]

It can be easily shown that the mirror symmetry about \( \epsilon = 0 \) still remains in the large-\( \bar{g} \) limit.

Let us first consider the behavior of \( \rho_{\uparrow}(E) \) near the location of its peak value. From Fig. 3, we find that the peak of \( \rho_{\uparrow}(E) \) occurs at \( \epsilon = \bar{g}_0 \), which is somewhat larger than \( \bar{g} \), and the deviation of \( \bar{g}_0 \) from \( \bar{g} \) decreases as \( \bar{g} \) increases. Hence, \( \rho_{\uparrow}(E) \) can be approximated to its second-order Taylor polynomial about \( \epsilon = \bar{g} \), and then we have

\[
\rho_{\uparrow}(E) \approx A \left( \epsilon - \bar{g}_0 \right) + B \left( \epsilon - \bar{g}_0 \right)^2 + C, \]

with

\[
A = \frac{\sqrt{2}\kappa^2}{\pi^{5/2}\Gamma} 2(1 - \frac{\epsilon}{\bar{g}_0} - \frac{1}{\bar{g}_0^2}) + O\left( \frac{1}{\bar{g}_0^3} \right),
\]

\[
B = \frac{1}{8} \frac{\bar{g}_0}{\Gamma} + O\left( \frac{1}{\bar{g}_0^3} \right),
\]

\[
C = \frac{\sqrt{2}\kappa^2}{\pi^{5/2}\Gamma} \left( 1 - \frac{1}{32\bar{g}_0^2} + O\left( \frac{1}{\bar{g}_0^3} \right) \right),
\]

where the asymptotic expansion

\[
\int_0^z dx e^{x^2} \approx \frac{e^{z^2}}{2z} \left( 1 + \frac{1}{2z^2} \right),
\]

for large \( z \) is used for \( F_\pm \). From Eq.(35), we find that the peak of \( \rho_{\uparrow}(E) \) occurs at \( \epsilon = \bar{g}_0 \). Therefore, in the large-\( \bar{g}_0 \) limit, the DOS of each spin component in our spin-selective system seems to behave as the shifted \( \rho_{\downarrow}(E) \). However, we will show below that this is not the case.
which is only one-half of the value of \( \rho_0(E) \big|_{k=\pm \bar{g}} \) for \( \bar{g}/\Gamma_0 \gg 1 \) (cf. Eq.(47) in Ref.\textsuperscript{1}). It indicates that the behavior of \( \rho_\uparrow(E) \) (and \( \rho_\downarrow(E) \), due to the mirror symmetry) far from the peak is not identical to that of \( \rho_0(E) \) in the band tail.

We can show the difference in the asymptotic behavior between \( \rho_0(E) \) and \( \rho_\uparrow(E) \) (or \( \rho_\downarrow(E) \)) by considering the regions around the band tails directly. For \((\epsilon - \bar{g})/\Gamma \geq \bar{g}/\Gamma \gg 1\), namely, \( x_+ \gg x_- \gg 1 \), from Eqs.(\textsuperscript{17}) and (\textsuperscript{43}), we obtain

\[
\rho_\uparrow(E) \approx \frac{\kappa^2}{\pi^{5/2}\Gamma_0} \frac{\pi}{2} \left( \frac{\epsilon'}{\Gamma_0} \right)^2 \left( \frac{\epsilon' + \bar{g}}{\bar{g}} \right)^2 e^{-\left( \epsilon'/\Gamma_0 \right)^2}, \quad \epsilon' \to \infty. \tag{46}
\]

Due to the mirror symmetry about \( \epsilon = 0 \), the behavior of \( \rho_\uparrow(E) \) in the limit \( \epsilon' \to -\infty \) (or equivalently, \( \epsilon \to -\infty \)) can be inferred by that of \( \rho_\downarrow(E) \) in the limit \( \epsilon' \to \infty \). Therefore, by Eqs.(\textsuperscript{48}) and (\textsuperscript{43}), we have

\[
\rho_\downarrow(E) \approx \frac{\kappa^2}{\pi^{5/2}\Gamma_0} \frac{\pi}{8} \left( \frac{\epsilon' + \bar{g}}{\bar{g}} \right)^2 e^{-\left( \epsilon'/\Gamma_0 \right)^2}, \quad \epsilon' \to -\infty, \tag{47}
\]

which diminishes faster than that in the opposite side of the band tail. Hence, the asymptotic behavior in both limits, \( \epsilon' \to \pm \infty \), does not reduce to that of the shifted \( \rho_0(E) \) in the band tails (cf. Eq.(47) of Ref.\textsuperscript{1}), as indicated in the discussion for \( \epsilon = 0 \). Moreover, we find that the asymptotic behavior of \( \rho_\uparrow(E) \) (or \( \rho_\downarrow(E) \)) is not symmetric with respect to the position of its peak. Thus \( \rho_\uparrow(E) \) and \( \rho_\downarrow(E) \) do not evolve to the identical form of the shifted \( \rho_0(E) \) in all regions of energy, although they do so near the location of their peaks.

**V. CONCLUSION AND DISCUSSION**

In summary, we extend the exact solution obtained by Brézin et al. for the DOS of spinless electrons at the lowest Landau level with a short-range disorder to the cases of a spin-\( \frac{1}{2} \) system. In this generalization, we obtain an analytical expression for DOS of spin-\( \frac{1}{2} \) system, illustrating the interplay between the Zeeman splitting of Landau levels and the disorder-induced broadening. We also find that the broadening and the band splitting of the Landau bands are enhanced due to the level repulsion from the mixing of two spin degrees of freedom by the random scatterings.

We notice that a similar expression of DOS is recently reached by Shahbazyan and Raikh\textsuperscript{22} in the context of the double-layer system with a tunneling coupling constant \( t \). Alternatively, we can compare these two systems explicitly by calculating their DOS using the path-integral approach, rather than summing up the entire diagrams in Ref.\textsuperscript{22}. After introducing the holomorphic superfields \( \Psi_i \) and their adjoints \( \bar{\Psi}_i \) (\( i = 1, 2 \)) for electrons in each layer, the “action” of the double-layer system, \( S_D \), becomes (cf. Eq.(\textsuperscript{1}))

\[
S_D = \frac{2\pi i}{\kappa^2} \int dz dz^* d\theta d\bar{\theta} e^{-\frac{i}{2}\kappa^2 (|z|^2 + \theta \bar{\theta})} \left[ e(\bar{\Psi}_1 \Psi_1 + \bar{\Psi}_2 \Psi_2) - t(\bar{\Psi}_1 \Psi_2 + \bar{\Psi}_2 \Psi_1) \right] + \int dz dz^* g_1 \left[ e^{-\frac{\kappa^2}{4} |z|^2} \frac{2\pi}{\kappa^2} \int d\theta d\bar{\theta} e^{-\frac{i}{2}\kappa^2 \theta \bar{\theta}} \Psi_1 \bar{\Psi}_1 \right] + \int dz dz^* g_2 \left[ e^{-\frac{\kappa^2}{4} |z|^2} \frac{2\pi}{\kappa^2} \int d\theta d\bar{\theta} e^{-\frac{i}{2}\kappa^2 \theta \bar{\theta}} \bar{\Psi}_2 \Psi_2 \right], \tag{48}
\]

where \( g_1(\alpha) \) and \( g_2(\alpha) \) are the corresponding Fourier transforms of the distribution functions for the random potentials in each layer. For the white-noise distributions of the same strength, by Eq.(\textsuperscript{43}), \( S_D \) reduces to the same form as our supersymmetric “action” in Eq.(\textsuperscript{15}), and then leads to a similar expression of DOS. Thus, by the change of variables in Eqs.(\textsuperscript{15}) and (\textsuperscript{16}), our model can be related to the double-layer one. However, the equivalence between these two models is not held in general. If the random distributions are not white-noise type of the same strength, the cancellation of the four fermion interactions will fail, and then there is no supersymmetry for the spin-\( \frac{1}{2} \) case. However, for the double-layer system, a supersymmetric “action” for \( S_D \) can still be obtained (because there is no four fermion interaction in \( S_D \)), and an exact expression of DOS can be formulated for any zero-range random distributions by a straightforward generalization of the work of Brézin et al.\textsuperscript{1} That is, once the disorder is included, these two systems are not exactly the same and, moreover, for the spin-\( \frac{1}{2} \) model, it is quite limited to have an exact expression of DOS.

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If we set $\epsilon \rightarrow -\epsilon$, then $\nu_{\pm} \rightarrow -\nu_{\mp}$, and, by Eqs. (24) and (25), we have $Z_0^R \rightarrow Z_0^R, Z_0^I \rightarrow -Z_0^I,$ and $\frac{\partial}{\partial \epsilon}(Z_0^I/Z_0^R) \rightarrow -\frac{\partial}{\partial \epsilon}(Z_0^I/Z_0^R)$. Thus, after rewriting Eq. (26) as $\rho(E) = \frac{2\pi Z_0^I/Z_0^R}{Z_0^R/Z_0^I[1 + (Z_0^I/Z_0^R)^2]}$, the symmetry with respect to $\epsilon \rightarrow -\epsilon$ becomes obvious.