THIN TRIANGLES AND A MULTIPLICATIVE ERGODIC THEOREM FOR TEICHMÜLLER GEOMETRY

MOON DUCHIN

1. Introduction

1.1. Overview. In this paper, we prove a curvature-type result about Teichmüller space, in the style of synthetic geometry. We show that, in the Teichmüller metric, “thin-framed triangles are thin”—that is, under suitable hypotheses, the variation of geodesics obeys a hyperbolic-like inequality. This theorem has applications to the study of random walks on Teichmüller space. In particular, an application is worked out for the action of the mapping class group: we show that geodesics track random walks sublinearly.

Recall that the Teichmüller space $T_{g,n}$ is a parameter space for marked metrics on oriented surfaces of a fixed topological type (the marking is a choice of generators for $\pi_1$; the type $(g,n)$ is the genus and number of punctures or boundary components, chosen so that $\Sigma_{g,n}$ is a hyperbolic surface). The points of $T_{g,n}$ are conformal classes of marked metrics—equivalently, since there is exactly one metric of constant curvature in each conformal class, each point can be identified with a (marked) Poincaré metric on the surface $\Sigma_{g,n}$. The mapping class group $\text{Mod}(g,n)$ is the collection of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,n}$. For a small value $\epsilon$, the cusps of $T_{g,n}$ are the regions containing metrics on $\Sigma_{g,n}$ with some nontrivial curve shorter than $\epsilon$; the complement of the cusps is called the thick part.

For any two points of Teichmüller space, there are quasiconformal maps between them; Teichmüller showed that there is a unique quasiconformal map of minimal dilatation (the eccentricity of its ellipse field). He defined a distance function accordingly, though it yields only a Finsler—not a Riemannian—metric. This Teichmüller metric is one of several natural metrics on $T_{g,n}$. Here, we will restrict attention to this choice of metric, and to Teichmüller spaces $T_g$ of compact hyperbolic surfaces.

There is a long and involved history of studying the metric geometry of Teichmüller space since the introduction of the Teichmüller metric in the late 1930s. In 1959, Kravetz had a much-cited result purporting to show that the Teichmüller metric was Busemann non-positively curved (that is, that it had a convex distance function). Linch showed that this argument was incorrect in her thesis of 1971, and Masur proved that Teichmüller space was in fact not Busemann non-positively curved and not CAT(0) (a slightly stronger condition) in his own thesis of 1975.

\footnote{Synthetic geometry encompasses metric geometry as well as a range of axiomatic approaches to the study of geometric objects, as opposed to the coordinatized or tensorial approaches of classical differential geometry. Active areas in the synthetic tradition include CAT(0) spaces, $\delta$-hyperbolicity, and geometric group theory in general. For a foundational text of modern synthetic geometry, see Busemann.}
Since then, Teichmüller space has been shown to have quite a few properties
which are suggestive of negative/non-positive curvature:

1. every finite group of isometries has a fixed point, as is the case for CAT(0)
spaces. For \( T_g \), the fixed point property is equivalent to the Nielsen
realization problem and was settled affirmatively by Kerckhoff in 1983 [16].

2. \( T_g \) admits a boundary at infinity similar to hyperbolic space—namely, the
visual and Thurston compactifications each give a sphere at infinity. Kerckhoff
showed those two to be slightly different [15], and the boundary
theory has since been extensively developed by Masur and Kaimanovich-
Masur [22].

3. the geodesic flow on moduli space (the quotient of Teichmüller space by
its isometry group, the mapping class group) is ergodic [21], as is the case
with quotients of hyperbolic space; and geodesics in moduli space obey a
logarithm law [23] governing their rate of escape which is a direct analog
of Sullivan’s logarithm law for geodesics on hyperbolic manifolds [32].

4. Teichmüller space is relatively hyperbolic with respect to the cusps (that
is, the electric Teichmüller space – obtained from \( T_g \) by coning off at the
cusps – is \( \delta \)-hyperbolic) [5], [26].

However, other features undercut the usual ways of asserting negative/non-
positive curvature:

1. there are families of geodesic rays from every point of \( T_g \) which stay a
bounded distance apart, so \( T_g \) is not CAT(0) [15].

2. there are arbitrarily large geodesic triangles for which one edge stays far
from the opposing vertex, so \( T_g \) is not \( \delta \)-hyperbolic [25].

3. there is sup behavior in the cusps: the Teichmüller metric in the cusps
differs only by an additive constant from the sup metric on a product of
lower-dimensional Teichmüller metrics and some hyperbolic metrics [29].
The sup metric ensures some positive-curvature characteristics, such as the
existence of many pairs of conjugate points. (That is, geodesics are not
unique.)

This state of affairs—negative curvature in the thick part, positive curvature in
the cusps—calls for a result accounting for the geodesic interactions between those
two parts of the thick-thin decomposition, since “most” geodesic rays are missed
by restricting attention to one or the other.\(^2\) This paper provides a result in that
direction (Theorem A), establishing a metric comparison condition for Teichmüller
space, expressed in terms of geodesic triangles, which is not restricted to geodesics
in the thick part. (See §2.1.)

Using this result, we deduce a multiplicative ergodic theorem for Teichmüller
space (Theorem B), furthering the parallels between Teichmüller space and the
theory of symmetric spaces and nonpositively-curved metric spaces. This
application can be interpreted as evidence that Theorem A is the right kind of condition
to explain some of the non-positive curvature properties of Teichmüller space.

1.2. **Ergodic theory and the theory of random walks.** The Oseledec Theorem
(or Multiplicative Ergodic Theorem) describes the asymptotic behavior of a
broad class of ergodic random walks, and in particular is often quoted for products

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\(^2\)That is, for a generic point and direction, the associated geodesic winds deeply in and out of
the cusps. This is a consequence of Masur’s logarithm law, already mentioned.
of random matrices \[^3\]. If \( G = SL_n(\mathbb{R}) \), say, then we can sample matrices by a probability measure \( \mu \) on \( G \) and consider their product. A random walk is formed on the associated symmetric space \( G/K \) by applying the successive products to a basepoint. The theorem provides, in part, that under mild conditions on \( \mu \) (finite first logarithmic moment), the rates of exponential growth of the eigenvalues of the product are almost everywhere constant. In terms of these invariants, called Lyapunov exponents, the M.E.T. also contains a convergence statement on eigenspaces for the random walk. This powerful theorem, first proved in the 1960s, has been foundational in modern dynamics.

The same result can be usefully restated in geometric language in the case of \( SL_n(\mathbb{R}) \), for instance: the theorem asserts that almost every sample path (for a random walk satisfying the hypotheses) leaves its basepoint with a speed which is an invariant of the dynamical system \((G, \mu)\). Then when that speed – the first Lyapunov exponent – is positive, almost every sample path is sublinearly approximated by constant-speed travel along a ray in a flat.

Karlsson and Margulis proved a version of the multiplicative ergodic theorem for nonpositively curved spaces in \[^{13}\].

**Theorem 1** (Karlsson-Margulis). Let \((Y, d)\) be a uniformly convex, complete metric space that is Busemann non-positively curved. Let \( \Gamma \) be a semigroup of semicontractions \( D \to D \), where \( D \) is a nonempty subset of \( Y \), and fix a point \( y \in D \). Let \((\Omega, P)\) be a measure space with \( P(\Omega) = 1 \) and let \( L : \Omega \to \Omega \) be an ergodic, measure-preserving transformation. For a measurable map \( \pi : \Omega \to \Gamma \), form the cocycle

\[
u(n, \omega) := \pi(\omega) \cdot \pi(L\omega) \cdots \pi(L^{n-1}\omega),
\]

and let \( y_n = u(n)y \). Then there is a value \( A \) such that for \( P\)-a.e. \( \omega \), the limit\[
\lim_{n \to \infty} \frac{1}{n} d(y, y_n(\omega)) = A.
\]
If \( P \) has finite first moment (\( \int_{\Omega} d(y, y_1) \, dP < \infty \)) and \( A > 0 \), then for \( P\)-a.e. \( \omega \), there is a unique geodesic \( \gamma \) starting at \( y \) and such that

\[
\lim_{n \to \infty} \frac{1}{n} d(y_n, \gamma(An)) = 0.
\]

In other words, consider a random walk by semicontractions (distance non-increasing maps) on a space satisfying the geometric hypotheses. If the average jump size is not too large (finite first moment), then there is a well-defined dominant speed of deviation from the basepoint (namely \( A \)). When that speed is positive, it follows that for almost every sample path, there is a unique geodesic ray starting at the basepoint and deviating sublinearly from the path.

Below, we fashion a multiplicative ergodic theorem for Teichmüller space. A metric comparison result (Theorem A) is proved and used in the proof in place of the Busemann nonpositive curvature assumption (which fails for Teichmüller space, as noted above). The mapping class group acts by isometries (which are, in particular, semicontractions), and we consider a measure \( \mu \) on \( \text{Mod}(g) \).

**Theorem B.** Let \((Y, d) = (T_g, d_T)\) be Teichmüller space with the Teichmüller metric. Let \( K \) be some thick part of \( T_g \) and fix a point \( y \in K \). Suppose \( \mu \) is a probability measure on the mapping class group \( \Gamma = \text{Mod}(g) \) such that the group generated by its support is non-elementary (has no fixed points on the boundary). Let \( P \) be the product measure on \( \Omega = \Gamma^\mathbb{Z} \), let \( L : \Omega \to \Omega \) be the left-shift, and let \( \pi(\omega) = \omega_0 \) read off the first component. Form the cocycle

\[
u(n, \omega) := \pi(\omega) \cdot \pi(L\omega) \cdots \pi(L^{n-1}\omega),
\]
and let \( y_n = u(n)y \). Then there is a value \( A \) such that for \( P \)-a.e. \( \omega \), the limit
\[
\lim_{n \to \infty} \frac{1}{n} d(y, y_n(\omega)) = A.
\]
If \( P \) has finite first moment and \( A > 0 \), then for \( P \)-a.e. \( \omega \), there is a unique geodesic \( \gamma \) starting at \( y \) and such that
\[
\lim_{n \to \infty} \frac{1}{n} d(y_n, \gamma(An)) = 0
\]
for \( p_n = \gamma(d(y, y_n)) \).

This says, in other words, that almost every sample path of the random walk by mapping classes has an associated Teichmüller geodesic that tracks it sublinearly while it travels in the thick part.

In fact, as long as the geodesic \( \gamma \) associated to \( \{y_n\} \) leaves the thick part \( K \) sublinearly, the \( \chi_K \) term can be dropped.

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2. Triangle comparison in the Teichmüller metric

2.1. The “thin-framed triangles are thin” condition. A natural way to think about the curvature of a metric space is to ask about divergence of geodesics; flat spaces are characterized by the linear spread of their geodesics, while positive and negative curvature mean slower and faster divergence, respectively. A related question one could pose of a fixed geodesic segment, say with endpoints \( y \) and \( z \), would be to take a point \( w \) on \( yz \) and ask how much more distance is required to pass from \( y \) to \( z \) through a perturbation \( x \) of \( w \) rather than through \( w \) itself. (See Figure 1.) The property proposed in this section will compare the extra distance needed (namely \( d(y, x) + d(x, z) - d(y, z) \)) with the size of the perturbation (namely \( d(x, w) \)).

Here, we will formulate this measurement in terms of triangles in geodesic spaces. Consider an arbitrary triangle \( \triangle xyz \) in a geodesic space \( Y \). Relabeling the vertices if necessary, say that the longest of the three pairwise distances between the points is \( d(y, z) \). Let \( w \) be the point along the geodesic from \( y \) to \( z \) such that \( d(w, y) = d(x, y) \). For notational convenience, we will use the letters \( a, b, c, d \) for important distances: \( c = d(y, z) \) is the length of the longest side, \( a = d(x, y) = d(w, y) \) and \( b = d(x, z) \) are the other two sidelengths, and \( d = d(w, x) \) is the distance from \( x \) to the specified point on the opposite side.

The following property was suggested by the work of Karlsson and Margulis in [13].

Definition. A collection \( \mathcal{T} \subset Y^3 \) of geodesic triangles in the space \( Y \) has the property that thin-framed triangles are thin if
\[
(*) \text{ there is some function } f(t) \text{ tending to zero as } t \to 0 \text{ such that}
\]
\[
\forall \rho > 0, \quad a + b - c < a \cdot \rho \quad \implies \quad d < a \cdot f(\rho)
\]
for every triangle in \( \mathcal{T} \). (\( \mathcal{T} \) will be taken to be all of \( Y^3 \) when not specified.)

The idea behind using this definition is that the rate at which \( f(t) \) approaches zero (as \( t \) goes to zero) detects the curvature of the space.

\(^3\)Note that in terms of visual measure on geodesics, a logarithmic rate of escape is generic, so this might be a reasonable hope. At the moment, however, the hitting measure \( \nu \) of sample paths on the boundary has few properties known; whether it is absolutely continuous with respect to visual measure is open.
Figure 1. Can the point $x$ be far from the geodesic segment $\overline{yz}$ while the distance $d(y, x) + d(x, z)$ remains close to $d(y, z)$?

**Proposition 2.**

1. All triangles in trees (including both simplicial and $\mathbb{R}$-trees) satisfy ($\star$) with bounding function $f(t) = t$, and no space has a bounding function going to zero faster than linearly.
2. All triangles in $\mathbb{R}^2$ satisfy ($\star$) with bounding function $f(t) = \sqrt{2t}$, and this function gives a sharp bound for the plane.
3. In $\delta$-hyperbolic spaces, if $\mathcal{T}$ is taken to be any collection of triangles with sidelengths bounded away from zero, then ($\star$) is satisfied with linear rate.
4. Generally, for a space $Y$,

$$Y \text{ is } CAT(0) \implies \text{thin-framed triangles are thin } \implies \text{geodesics are unique.}$$

In particular, spheres do not satisfy ($\star$) for any bounding function.

**Proof.**

1. Graphs can be regarded as metric spaces with the path metric, assigning distance one to each edge; non-degenerate triangles in trees are tripods (this holds also for $\mathbb{R}$-trees). In a tripod, $a + b - c = d$, so $f(t) = t$ is sharp.

In general, in any metric space, $c + 2d$ is the length of one path from $y$ to $x$ to $z$ (going through $w$) and $a + b$ is the length of the most efficient path from $y$ to $x$ to $z$, so $d \geq \frac{1}{2}(a + b - c)$ for any triangle in any metric space. Thus a linear bound is best-possible (for any bounding function $f(t)$ for any metric space, $\lim_{t \to 0} \frac{1}{t}f(t)$ is finite).
(2) In a planar triangle, consider the angle $\theta$ at vertex $y$. By the law of cosines,

$$\frac{2a^2 - d^2}{2a^2} = \cos \theta = \frac{a^2 + c^2 - b^2}{2ac}$$

so that

$$d^2 = \frac{a}{c}(b^2 - (c - a)^2) = a(a + b - c)\frac{c + b - a}{c}.$$ 

Now, using the hypothesis $(a + b - c < a\rho)$ and the fact that $b - a$ is less than $c$ but can get arbitrarily close when $b \approx c$, we get that

$$d^2 < 2\rho a^2$$

and conclude that $d < \sqrt{2\rho} \cdot a$.

(3) In any $\delta$-hyperbolic space, such as the hyperbolic plane $\mathbb{H}^2$, the metric is only boundedly far from that in a tree. For every nondegenerate triangle in any metric space, there is an associated tree with three endpoints (a tripod) whose edge lengths match the triangle. One definition of $\delta$-hyperbolicity for a space $Y$ is that $\delta$ gives a uniform bound for the insize of geodesic triangles from $Y$ (the diameter of the set of three points in the triangle which map to the central vertex in the tripod).

But this means that, if $r, s, t$ are chosen to be the lengths of the legs of the tripod (that is, $a = r + s$, $b = s + t$, $c = r + t$), then $d(w, x) \leq 2s + \delta$. Since $a + b - c = 2s$, we have $d \leq (a + b - c) + \delta$, and under the hypothesis that $a + b - c < a\rho$, this yields $d \leq a\rho + \delta$.

If we restrict attention to triangles with sidelengths bounded away from zero, say $a \geq 1$, then $d \leq a(\rho + \delta)$, so the function $f(t) = t + \delta$ provides a bound as needed for $(\star)$.

(4) The definition of CAT(0) provides that distances within a geodesic triangle are less than or equal to those in a corresponding planar triangle. So it is clear that CAT(0) spaces satisfy $(\star)$ with the same bound as we found for $\mathbb{R}^2$.

On the other hand, spaces with non-unique geodesics, and in particular spheres, do not satisfy $(\star)$ for any bounding function: construct a triangle by starting with a pair of points that are joined by two distinct geodesics, and take the third vertex to be any interior point on one of the geodesics (but not both). For such a triangle, $c = b + a$, so $\rho$ can be taken arbitrarily close to zero, but $d/a$ is non-zero.

\[\square\]

Remark. In the case of the hyperbolic plane $\mathbb{H}^2$, we will not be able to improve on the $\mathbb{R}^2$ bound (where $f(t) \to 0$ at square-root rate) overall, since the metric on $\mathbb{H}^2$ degenerates to the Euclidean metric at small scale. The proposition above, though, says that we can do better in the large: large-scale triangles in negative curvature admit a linear bound for $(\star)$. This is different from the Euclidean case, where passing to the large scale does not change the bounding function: the square-root bound is still sharp.

Below (§2.3), a large-scale statement that thin-framed triangles are thin will be proved for Teichmüller space.
2.2. **Background.** General references for Teichmüller theory, quadratic differentials, and polarized flat structures can be found in [1], [8], [31], [6]/[7].

One of many equivalent definitions of *Teichmüller space* is

\[ T_g = \{ (X, f) | f \text{ quasiconformal } X_0 \to X \} / \sim \]

where \( X \) is a Riemann surface with genus \( g \), \( X_0 \) is a fixed Riemann surface serving as a basepoint, and the equivalence relation is given by

\[ (X_1, f_1) \sim (X_2, f_2) \iff \exists \text{ conformal } h \text{ such that } h \simeq f_2 \circ f_1^{-1}. \]

Accordingly, the Teichmüller metric is given by

\[ d_T((X_1, f_1), (X_2, f_2)) = \frac{1}{2} \ln \inf_h K(h), \]

infimized over quasiconformal \( h \simeq f_2 \circ f_1^{-1} \). It is well-known that with this distance function, tangent spaces are equipped with a notion of length (it is a Finsler metric), but no notion of angle via an inner-product (it is not Riemannian).

The mapping class group, or modular group, is defined as

\[ \text{Mod}(g) = \pi_0(\text{Diff}^+(\Sigma_g)) = \text{Diff}^+(\Sigma_g)/\text{Diff}_0(\Sigma_g). \]

The mapping class group acts on \( T_g \) by changing markings and it is the full (orientation-preserving) isometry group; the quotient \( \text{Mod}(g) \backslash T_g \) is the moduli space \( \mathcal{M}_g \) of (unmarked) conformal classes of metrics. Because the action has some fixed points, there is an orbifold structure on the quotient.

Among the metrics in each conformal class is a unique Poincaré metric (that is, a metric of constant curvature \(-1\)) by uniformization. Besides this metric, the conformal class contains many singular flat structures: metrics that are Euclidean away from a finite number of points, in which all the negative curvature is concentrated.

A *quadratic differential* \( \phi \) on a Riemann surface is a holomorphic 2-tensor given in local coordinates by \( \phi(z)dz^2 \). If the atlas for the Riemann surface is \( (U_\alpha, z_\alpha) \), then the quadratic differential is a system \( (U_\alpha, z_\alpha, \phi_\alpha) \) of holomorphic functions on each coordinate patch. Given \( \alpha \) and \( \beta \), the functions transform by the rule

\[ \phi_\alpha(z_\alpha) \left( \frac{dz}{dz_\beta} \right)^2 = \phi_\beta(z_\beta). \]

The collection of quadratic differentials at a point \( p \in T_g \) will be denoted \( \mathcal{QD}(p) \). While the value of \( \phi \) is not well-defined at a point on the Riemann surface, its zeros (and their orders) are, because they are preserved by the transformation rule.

It follows that the transition functions are semi-translations \( (z \mapsto \pm z + c) \), which means that the quadratic differential preserves a pair of line fields corresponding to vertical and horizontal directions. Note, however, that the directions are not oriented. If instead we considered an abelian differential \( f(z)dz \), we would obtain translation surfaces with oriented vertical and horizontal directions.

Since a Euclidean coordinate can be obtained in each chart by \( \left( \phi(z) \right)^{1/2} \), a quadratic differential gives rise to a *polarized flat structure* (the pair of line fields is said to be a polarization) with singularities at the zeros. In fact, over an oriented topological surface of fixed type, there is a bijective correspondence between three sets of structures: quadratic differentials, polarized flat structures, and Euclidean polygons with semi-translation gluings and cone angles which are multiples of \( \pi \). Because of this correspondence, the three notions will be used interchangeably.
Here, a flat structure will be written $X = (p, \phi)$ for some quadratic differential $\phi \in QD(p)$, where $p \in T_g$. The length measured in that flat metric may be denoted $\ell_X(\ell_{(p,\phi)})$, or simply $\ell$, according to context.

A saddle connection is a curve on $X$ whose endpoints are at singularities and which has no singularities in its interior. A cylinder in a flat structure is an isometrically embedded Euclidean cylinder. This occurs whenever there is a closed nonsingular curve because it must have a family of homotopic parallel translates. The boundary of a cylinder must consist of a union of saddle connections.

For a closed curve $\gamma$ on a flat structure $X = (p, \phi)$, write $\mathcal{T}_X$ for its straightening—the geodesic in its homotopy class—which must take the form of a sequence of straight lines $\gamma_1, \ldots, \gamma_k$. Either $\gamma$ is part of a cylinder, in the case that $\mathcal{T} = \gamma_1$ is a single nonsingular straight-line curve, or its geodesic representative is a sequence of saddle connections $\gamma_1, \ldots, \gamma_k$ between not-necessarily-distinct singularities (zeros of the quadratic differential $\phi$).

Where the flat structure $X = (p, \phi)$ is understood, let $h_i$ and $v_i$ denote the components of the affine holonomy of each segment:

$$\text{hol}(\gamma_i) = (h_i, v_i) = (\Re \int_{\gamma_i} \phi^{1/2} dz, \Im \int_{\gamma_i} \phi^{1/2} dz).$$

Define the unsigned holonomy by $|h|(\gamma) = \sum_i |h_i|$, $|v|(\gamma) = \sum_i |v_i|$, and

$$|\text{hol}|(\gamma) = (|h|(\gamma), |v|(\gamma)).$$

A curve will be called vertical if $|h| = 0$ and horizontal if $|v| = 0$.

Remark. $|h|, |v| \leq (|h|^2 + |v|^2)^{1/2} \leq \ell_X(\gamma) \leq |h| + |v|$.

Fact 1. (Teichmüller’s Theorem) Any two points of $T_g$ are connected by a unique geodesic.

The Teichmüller geodesic flow is given by the multiplicative action of $g_t = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}$ on quadratic differentials, so that if a curve has piecewise holonomy coordinates $(h_i, v_i)$ in the flat structure given by the pair $(x, \phi)$, then its piecewise coordinates in the push-forward by distance $t$ are given by $(e^{t}h_i, e^{-t}v_i)$ on the flat structure $g_t(X) = g_t(x, \phi)$ or $(x', g_t\phi)$ for the appropriate point $x' \in T_g$.

That is, each direction in the visual sphere is associated to a quadratic differential, and geodesic flow is given by the diagonal action of $SL_2(\mathbb{R})$ on the corresponding flat structure. There is an appropriate normalization of quadratic differentials so that the map of the open ball into Teichmüller space, $B_1 \mapsto T_g$, is a homeomorphism. This is called the visual embedding.

Remark. Unsigned holonomy behaves well under the Teichmüller map:

$$|\text{hol}|_{g_t} = (e^{t}|h|, e^{-t}|v|).$$

We continue with key definitions. For $x \in T_g$ and $\gamma \in SS$ (the simple closed curves on $\Sigma_g$, up to homotopy), the extremal length of $\gamma$ is

$$\text{ext}_x(\gamma) = \sup_{\ell} \inf_{\gamma_0 \sim \gamma} \ell(\gamma)^2,$$

where the sup is over metrics $\ell$ in the conformal class $x$. 
Fact 2. $\text{ext}_x(\alpha)$ is achieved in a flat structure $X$ (called the Jenkins-Strebel differential) which is composed of a single cylinder, having $\alpha$ as the core curve. This is discussed in [31].

Fact 3. Knowing how much curves are distorted is enough to compute Teichmüller distance:

$$d_T(x, y) = \frac{1}{2} \ln \sup_{\alpha} \left( \frac{\text{ext}_x(\alpha)}{\text{ext}_y(\alpha)} \right).$$

This formula is due to Kerckhoff [15].

For a small value of $\epsilon > 0$, the set $K = K(\epsilon) = \{ x \in T_g : \text{injrad} \geq \epsilon \}$ of metrics with no curve shorter than $\epsilon$ will be called a thick part of Teichmüller space. A thick part $K(\epsilon)$ projects to a compact set $K_0 = \pi(K)$ in moduli space but $K$ itself is not compact, since it is $\text{Mod}(g)$-invariant but a sequence of mapping classes can send a point $x \in T_g$ to infinity by changing the marking.

Fact 4. For a thick part $K(\epsilon)$ of $T_g$, there is a constant $F = F(\epsilon)$ (growing as $\epsilon \to 0$) such that $\forall \gamma \in S$,

$$\frac{1}{F} \ell_\phi(\gamma) \leq \ell_\psi(\gamma) \leq F \ell_\phi(\gamma)$$

for all $\phi, \psi \in QD(x)$. In particular, this means that the length-squared of any homotopy representative of $\gamma$ in any flat structure over the point $x$ is within a factor $F^2$ of the extremal length $\text{ext}_x(\gamma)$. The proof is straightforward: at each point in moduli space, there is some bound on the greatest ratio of lengths of curves between metrics; this varies continuously over the compact set $K_0$.

Lemma 3. There is some constant $R$ depending only on $\epsilon$ such that, for every point $x$ in the thick part $K(\epsilon)$ of $T_g$, there is a collection of closed curves of length $\leq R$ whose complementary regions contain no geodesic segment of length more than $3R$.

Proof. Let

$$R_1 = \sup \inf_{x} \max_{P \in \mathcal{P}, \gamma \in P} \max_{x} [\text{ext}_x(\gamma)]^{1/2}$$

for $x \in K = K(\epsilon)$ and $\gamma$ ranges over curves in the pants decomposition $P$, and $P$ ranges over the pants complex $\mathcal{P}$. The value $R_1$ exists because $\inf_{P \in \mathcal{P}, \gamma \in P} [\text{ext}_x(\gamma)]^{1/2}$ varies continuously as $\pi(x)$ varies in the compact set $K_0$. This means that there is some uniform constant $(F \cdot R_1)$, for $F$ as in Fact 4 such that every flat structure on every point over $K$ has some pants decomposition with all pants curves shorter than that length. Such a pants decomposition, $P$, divides the surface into three-punctured spheres which, in the Poincaré metric, are doubles of hyperbolic hexagons. If the $3g − 3$ curves of $P$ are called cuffs, then these uniquely determine $3g − 3$ (not closed) curves called inseams which orthogonally connect pairs of cuffs. Call the cuffs $\gamma_i$ and the inseams $\delta_{ij}$ (for all $i, j$ such that $\gamma_i, \gamma_j$ bound the same pair of pants, including $\delta_{ii}$ when $\gamma_i$ is separating). In particular, the cuff lengths—here no longer than $R_1$ and no shorter than $\epsilon$—determine the inseam lengths, so let $R_2$ be largest possible inseam length. Each $\delta_{ii}$ can be completed to a closed loop of length at most $R_1 + R_2$ by adding part of $\gamma_i$. If there are indices $i_1, \ldots, i_k$ such that

$$\delta_{i_1i_2}, \delta_{i_2i_3}, \ldots, \delta_{i_ki_1}$$

is a sequence of inseams, then there is a closed loop of length at most $(3g − 3)(R_1 + R_2)$ containing those $k$ inseams as pieces. Continuing until every inseam curve is
included in some closed curve—a finite process—let $R$ be the maximum of $R_1$ and any of the curves so obtained. These closed curves, taken together with the pants curves, divide the surface into simply connected pieces with six sides of length $\leq R$, so no geodesic segment on a complementary region can have length more than $3R$. This implies that, if $\mathcal{M}$ is the multicurve given by the pants curves and closed-up inseam curves, any geodesic $\gamma$ on $\Sigma$ satisfies the intersection number inequality $i(\gamma, \mathcal{M}) \geq \ell(\gamma)/3R$. Finally, the geodesic representatives of these new curves are shorter or the same length, and satisfy the same inequality. □

**Lemma 4.** For any two simple closed curves $\gamma_1, \gamma_2$ on $\Sigma_g$ and any point $x \in K(\epsilon) \subset T_g$, the intersection number of the curves is bounded above with respect to the length in any metric in the conformal class $x$:

$$i(\gamma_1, \gamma_2) \leq \left(\frac{4}{\epsilon^2}\right) \ell(\gamma_1)\ell(\gamma_2).$$

**Proof.** Let $\gamma_1(j)$ be a small piece of $\gamma_1$ having length less than $\epsilon/2$, half the injectivity radius bound. Between each two intersections of $\gamma_2$ with $\gamma_1(j)$, the length along $\gamma_2$ is at least $\epsilon/2$ (because otherwise a loop of length $< \epsilon$ would be formed). Thus if $i(\gamma_1(j), \gamma_2) = n_j$, it follows that $n_j \frac{\epsilon}{2} \leq \ell(\gamma_2)$. Then cover $\gamma_1$ with such pieces, of which at most $\frac{\ell(\gamma_1)}{\epsilon/2}$ are needed, and let $n = \max\{n_j\}$. This completes the proof:

$$i(\gamma_1, \gamma_2) \leq n \cdot \frac{\ell(\gamma_1)}{\epsilon/2} \leq \left(\frac{4}{\epsilon^2}\right) \ell(\gamma_1)\ell(\gamma_2)$$

□

**Fact 5.** Generically, geodesics wind in and out of cusps. This is discussed quantitatively in [23], where Masur shows that generically (relative to the visual measure on directions) a geodesic returns to the thick part between progressively deeper sojourns to various cusps. He finds that

$$\limsup_{t \to \infty} \frac{d_t}{\log t} = 1/2,$$

where $d_t$ computed by following the geodesic for time $t$, then measuring the distance in the moduli space between the projections of the starting and ending points.

**Fact 6.** For a flat structure $X$, the minimal components under the flow $F_\theta$ (closed, flow-invariant sets on which the flow is minimal) are bounded by saddle connections in the direction $\theta$. Infinite leaves are dense in subsurfaces which are minimal components for the flow, so it follows that infinite leaves get arbitrarily close to saddle connections. This is discussed in the Luminy lectures of Masur and Hubert-Schmidt ([9],[24]).

**Fact 7.** For a flat structure $X$, let

$$V(X) = \{v \in \mathbb{R}^2 \mid \exists \text{ cylinder in } X \text{ with holonomy } v\}$$

and let $V_{sc}(X)$ be defined similarly as the holonomies of saddle connections. Then both are discrete as subsets of $\mathbb{R}^2$. The Luminy lectures ([3],[9],[23]) contain a proof of this fact and a discussion of related counting problems.

**Fact 8.** Teichmüller space is not $\delta$-hyperbolic. Masur and Wolf prove this in [24], by considering triangles given by a point $x \in T_g$ and its Dehn twists about disjoint curves. For large $n$, the triangle $\triangle x\tau_\alpha^n(x)\tau_\beta^{-n}(x)$ does not have a uniform bound.
on the distance from the vertex $x$ to the opposite side. (Ivanov later gave a simpler proof in [10] using the observation that some geodesics spread slower than linearly, which was the main finding of the much earlier paper [19].)

**Fact 9.** In a flat structure $X$ with a minimal flow $F_\theta$, any curve transverse to the flow (called a section) yields a zippered rectangle configuration by considering the first return of the flow to the section (from either side): this is a collection of rectangles with bases on the section which recover the surface under gluings. If the flat structure is a translation surface (the quadratic differential is the square of some abelian differential; equivalently, the transition maps are translations) then the flow is oriented and the rectangles are all distinct. In general, the flow can return to the same side of the section that it left from. Zippered rectangles are a construction of Veech [37].

### 2.3. A thin triangle theorem for Teichmüller space.

The main theorem of this section states that, among large-scale triangles in Teichmüller space whose four comparison points $(x, y, z, w)$ are in some common thick part, thin-framed triangles are thin. In fact, the bounding function goes to zero not with the $t^{1/2}$ rate found in the Euclidean case, but with the linear rate associated with negative curvature.

Consider a compact hyperbolic surface $\Sigma_g$ and points $x, y, z$ in $T_g$, which all lie in some thick part $K(\epsilon)$, so that there are no very short closed curves (shorter than $\epsilon$) on the corresponding Riemann surfaces. The Teichmüller geodesics between such points, on the other hand, can wind deeply in and out of cusps (Fact 5)—that is, the shortest path between two surfaces without short curves may certainly pass through surfaces with very short curves.

Choose quadratic differentials $\phi, \psi$ (co-)tangent to the geodesics $yx$ and $yz$. Then $(y, \phi)$ and $(y, \psi)$ are two polarized flat structures with the same underlying Riemann surface. See Figure 2.

The goal for this section is

**Theorem A.** Fix $K \subset T_g$, a thick part of Teichmüller space. Consider the collection of triangles $T_M = \{ (x, y, z) \in K^3 \mid w \in K, a, b, c > M \}$. For sufficiently large $M$, $T_M$ has the property that thin-framed triangles are thin, satisfying (*) with bounding function $f(t) = kt$, where $k$ depends on $K$.

**Proof of Theorem.** We will consider a sequence of Teichmüller geodesic triangles $\{\triangle_i\}$ with sidelengths $a = a(\triangle_i), b = b(\triangle_i), c = c(\triangle_i)$ as above. We will assume that $a, b, c \to \infty$ and $\frac{a + b - c}{2} \to 0$ as $i \to \infty$.

To get an estimate of $d$, we define two families of curves for use in the analysis of the flat structures of interest in the triangle. We will consider moderate-length curves at the vertices $y, z$. An earlier lemma allows us to choose a length cutoff which is large enough to assure that the families contain enough curves to cut up the surface into simply connected pieces.

**Definition.** Where $SS$ is the collection of simple closed curves on the surface, up to homotopy, and $R$ is as in Lemma 9 above, let

$$C_1 := \{ \alpha \in SS : \ell_\phi(\alpha) < R, \frac{|v|_\phi}{|h|_\phi} < R \}$$

$$C_2 := \{ \beta \in SS : \ell_{g, \phi}(\beta) < R, \frac{|h|_{g, \phi}}{|v|_{g, \phi}} < R \}$$
Remarks.

- These sets are nonempty; indeed, each contains a pants decomposition as well as a collection of inseam-curves, as seen in the proof of Lemma 3.
- These sets are finite: there are only finitely many homotopy classes of curves such that any representative has length at most $R$ on a given flat structure $X$, by the discreteness of the sets $V(X)$ and $V_{sc}(X)$ (Fact 7).
- The slope bounds will be used to control the way the length changes under Teichmüller geodesic flow in the following lemma.

We want to show $d/a \to 0$ as $\rho \to 0$. Assume for contradiction that $d \gg a\rho$ for some $\rho$ such that $a + b - c < a\rho$. In the following sequence of lemmas, we show that curves from the families $C_1$ and $C_2$ are not changed too greatly in length between metrics based at $w$ or at $x$ (Lemma 5): that they are long and nearly perpendicular in the flat structure $(w, g_w \phi)$, so have high intersection number (Lemma 6); that they would need to be nearly parallel in $(w, \chi)$ in order to have $d \gg a\rho$ (Lemma 7);
and finally, that this would imply low intersection number (Lemmas 8 and 9), producing a contradiction.

**Lemma 5.** There is a constant \( k_5 = k_5(\epsilon) \) so that for all curves \( \gamma \) in \( C_1 \cup C_2 \),
\[
\frac{1}{k_5} \cdot e^{-\rho} < \frac{\text{ext}_w(\gamma)}{\text{ext}_x(\gamma)} < k_5 \cdot e^{\rho}.
\]

Proof. Suppose \( \alpha \in C_1 \) has unsigned holonomy coordinates \( |\text{hol}|_{(y,\phi)}(\alpha) = (L, L') \), where \( L'/L < R \) by definition of \( C_1 \).

Then, after applying geodesic flow for time \( a \), we have \( |\text{hol}|_{(w,g_n\phi)}(\alpha) = (e^a L, e^{-a} L') \).

The idea of the rest of this calculation is straightforward: knowing that the length of \( \alpha \) measured at \( w \) is on the order of \( e^a L \), we will find the largest and smallest possible length of \( \alpha \) measured at \( x \) by going around the triangle both ways from \( y \).

On one hand, the distance from \( y \) to \( x \) is \( a \), so the longest \( \alpha \) could measure at \( x \) is \( \sim e^a L \). On the other hand, the length at \( z \) is roughly \( e^c L \) and the distance from \( z \) to \( x \) is \( b \), so the shortest possible length at \( x \) is on the order of \( e^{c-b} L \). But then the biggest ratio of lengths is \( e^{a+b-c} \); then the “thin frame” hypothesis and Kerckhoff’s formula will complete the proof.

Now we will formalize this idea. For large enough \( a \), by repeated use of the observation that the length in any flat structure is bounded by \( |h| \leq \ell \leq |h| + |v| \), we deduce the following collection of equalities and inequalities.

\[
e^a L \leq \ell_{(w,g_n\phi)}(\alpha) \leq 2e^a L,
\]
\[
(e^a L)^2 \leq \text{ext}_w(\alpha) \leq (2F e^a L)^2,
\]
\[
|\text{hol}|_{(z,g_n\phi)}(\alpha) = (e^c L, e^{-c} L'),
\]
\[
F^{-1} e^c L \leq \ell_{(z,g_n\xi)}(\alpha) \leq 2Fe^c L,
\]
\[
F^{-1} e^{c-b} L \leq \ell_{(x,\xi)}(\alpha),
\]
\[
(F^{-1} e^{c-b} L)^2 \leq \text{ext}_x(\alpha),
\]
\[
\ell_{(y,\phi)}(\alpha) \leq L + L',
\]
\[
\ell_{(y,\psi)}(\alpha) \leq F(L + L'),
\]
\[
\ell_{(x,g_n\psi)}(\alpha) \leq e^a F(L + L') \leq e^a F(R + 1)L,
\]
\[
\text{ext}_x(\alpha) \leq (e^a F^2(R + 1)L)^2.
\]

Thus the length of \( \alpha \) is not expanded by more than a constant factor when passing from \( (w,\chi) \) to \( (x, g_d\chi) \) and it may be contracted:
\[
F^{-4}(R + 1)^{-2} \leq \frac{\text{ext}_w(\alpha)}{\text{ext}_x(\alpha)} \leq (2F^2 e^{a+b-c})^2 < (2F^2 e^{a+\rho})^2.
\]

In an exactly similar way, for \( \beta \in C_2 \),
\[
(F^{-2}(R + 1)^{-1} e^{-a-\rho})^2 < (F^{-2}(R + 1)^{-1} e^{c-a-b})^2 \leq \frac{\text{ext}_w(\beta)}{\text{ext}_x(\beta)} \leq 4F^{-2}
\]

and both constants, \( R \) and \( F \), depend only on the choice of the compact part of moduli space (that is, on \( \epsilon \)). \( \square \)
Lemma 6. There is a constant \( k_0 \) (depending on the genus \( g \) and on \( \epsilon \)) so that for any \( \alpha \) in \( C_1 \), there is some \( \beta \) in \( C_2 \) such that

\[
i(\alpha, \beta) \geq k_0 \text{ext}_w(\alpha) \text{ext}_w(\beta).
\]

Proof. We will consider the flat structure \((z, g_\phi)\), where the \( C_1 \) curves are long and near-horizontal and the \( C_2 \) curves have moderate length. By Lemma 4, there is a collection of curves from \( C_2 \) that decompose the surface into simply connected pieces, and the curves are frequently intersected by long geodesics. Call this collection of simple closed curves \( B \).

Fix any curve \( \alpha \in C_1 \). For sufficiently large \( c \), we have \( \ell_{(z, g_\phi)}(\alpha) \sim e^c A \).

It follows that \( i(\alpha, B) \geq e^c \frac{\text{AR}}{18g-18} \). But there are no more than \( 6g - 6 \) curves in \( B \), so \( \exists \beta \in B \) such that

\[
i(\alpha, \beta) \geq e^c \frac{\text{AR}}{(18g-18)R}.
\]

However, \( \text{ext}_w(\beta) \leq e^{c-a} R \) and \( \text{ext}_w(\alpha) \leq e^a A \), so

\[
i(\alpha, \beta) \geq \frac{e^c AR}{(18g-18)R^2} \geq \frac{1}{18g-18} \text{ext}_w(\alpha) \text{ext}_w(\beta).
\]

\[\square\]

Lemma 7. There are constants \( k_7, k_7' \) (depending on \( \epsilon \)) so that for any \( \gamma \in C_1 \cup C_2 \), if we write \( |\text{hol}|_{(w, \chi)}(\gamma) = (|h|, |v|) \), then

\[
k_7 \cdot e^{d-a\rho} < \frac{|v|}{|h|} < k_7' \cdot e^{d+a\rho}.
\]

Proof. Let \( X = (w, \chi) \). First, note that \( |v| \) is greater than \( |h| \); otherwise, \( \ell_X(\gamma) \sim |h| \) and \( \ell_{g_\phi X}(\gamma) \sim e^d |h| \), contradicting Lemma 4 since \( d \) is assumed much greater than \( a\rho \). Thus, we can write \( |\text{hol}|_{X}(\gamma) = (|h|, e^m|h|) \) where \( m > 0 \), so that \( \ell_X(\gamma) \sim e^m |h| \). Observe that \( m < 2d \); if not, then \( \ell_{g_\phi X}(\gamma) \sim e^{m-d} |h| \), so the ratio of the lengths again contradicts Lemma 4. Therefore, we have \( \ell_{g_\phi X}(\gamma) \sim e^d |h| \), so \( \text{ext}_w(\gamma) / \text{ext}_x(\gamma) \sim e^{m-d} \). But \( m - d < a\rho \implies m < d + a\rho \) and \( d - m < a\rho \implies m > d - a\rho \). \[\square\]

The previous lemma says that the curves from the two principal families are near-vertical considered in the flat structure \( X = (w, \chi) \); they have slope close to \( e^d \).

(Which isn’t uniformly bounded because Teichmüller space is not \( \delta \)-hyperbolic—Fact 3—and the triangles that witness the failure of \( \delta \)-hyperbolicity can be arbitrarily large.)

For any flat structure \( X \), the surface decomposes in the vertical direction into a certain number of Euclidean cylinders and a certain number of minimal components. Fix a minimal component \( X_0 \) and a section \( \Gamma : [0, l] \to X_0 \) transverse to the vertical foliation on \( X \) and parameterized by arclength. Let \( \Gamma^{\ell}(l) : [0, l] \to X_0 \) be a subsection of length \( l \). Let \( h(l) \) be the shortest return time to the subsection and \( w(l) \) be the minimum length of a subinterval of \( [0, l] \) on which the first return map is continuous. (That is, \( h(l) \) is the smallest height of a rectangle and \( w(l) \) is the smallest width of a rectangle in the zippered rectangle configuration over \( \Gamma^{\ell}(l) \).)

Lemma 8. If \( X_0 \) is a minimal component of \( X \) for the vertical flow, and \( H > 0 \) is chosen arbitrarily, then for sufficiently small \( l = l(H) \), the vertical leaves over \( \Gamma^{\ell}(l) \) neither hit singularities nor return to \( \Gamma^{\ell}(l) \) for at least time \( H \).
Proof. Let $T$ be the interval exchange given by the first return to $\Gamma$ of the vertical foliation.

Consider the points $T(0), T^2(0), \ldots, T^K(0)$ where $K = \lceil H/h(1) \rceil$. Then for $l_0$ less than the minimum of these values, the first rectangle over $[0, l_0]$ has height at least $H$. (It is possible to choose such an $l_0$ because none of the values $T^i(0)$ can be equal to 0 in a minimal component—this would imply a closed orbit.) Its width is at most $w(l_0)$, so by choosing $l_1 < w(l_0)$ we can be certain that all of the return times to the $l_1$-subsection are at least $H$.

Finally, by choosing $l_2 < w(l_1)$, we ensure that the singularities are not encountered until past height $H$ over the $l_2$-subsection. \hfill \Box

In particular, this lemma allows the construction of a zippered rectangle configuration on $X_0$ such that all rectangles are as tall as we like.

Lemma 9. Fix a constant $H > 0$ and a flat structure $X$ in a thick part $K(\epsilon)$ of Teichmüller space. Let $C_1, \ldots, C_r$ be the maximal vertical cylinders in $X$ and let $X_1, \ldots, X_s$ be the minimal components for the vertical flow. Let $\Gamma^{(i)}_j : [0, l_i] \rightarrow X_i$ be sections in each minimal component chosen relative to $H$ as in the previous lemma. Then there is a constant $M = M(H, \{C_i\}, \{\Gamma_i\}, \epsilon)$ and a constant $k_0$ such that for any two curves $\alpha$ and $\beta$ with slope $|v|/|h| > M$,

$$i(\alpha, \beta) < k_0 \frac{\ell_X(\alpha) \ell_X(\beta)}{H}.$$ 

Proof. The intersection number is the minimum over homotopy representatives of the curves. To compute it, it suffices to consider the geodesic straightening of $\alpha$ and $\beta$ in the flat structure $X$. We will denote by $i_C(\alpha, \beta)$ the number of intersections of the curves in the interiors or on the boundary of any of the cylinders $C_i$ and by $i_Z(\alpha, \beta)$ the number of intersections in or on the boundary of any of the minimal components, so that $i(\alpha, \beta) \leq i_C(\alpha, \beta) + i_Z(\alpha, \beta)$.

In the Euclidean picture $C_1 \sqcup C_r \sqcup X_1 \sqcup X_s$, each curve decomposes into a certain number of maximal connected line segments, which we call simply segments. See Figure 9.

Let $w_0$ be the smallest of any of the widths of the rectangles or cylinders; this depends only on the $C_i$ and $\Gamma_i$. Let $M_1 = H/w_0$. We will show that if every segment of the curves $\alpha$ and $\beta$ has slope $\geq M_1$, then $i(\alpha, \beta) < k_0 \frac{\ell_X(\alpha) \ell_X(\beta)}{H}$. In what follows, the length in the flat structure $X$ will be denoted simply $\ell$.

First note that each of the cylinders must be of (vertical) height at least $\epsilon$, since the surface has no closed curves shorter than that. Suppose $\alpha$ and $\beta$ each pass through cylinder $C_k$. In fact, each can pass through the cylinder several times; let $\alpha_1, \ldots, \alpha_p$ be the sojourns of $\alpha$ to $C_k$ (so that each is made up of many segments, one of which intersects the left side of the cylinder and one of which intersects the right) and likewise write $\beta_1, \ldots, \beta_q$. Each of these $pq$ curves has a fixed slope for all of its segments because there are no singularities in the cylinder, so let $m_i$ be the slope of $\alpha_i$ and likewise $n_j$ of $\beta_j$ so that $m_i, n_j > M_1$ for all $i, j$.

Suppose $n_j > m_i$ for all $i, j$. (If not, this estimate is similar but messier.)

Consider $i(\alpha_i, \beta_j)$. Because $\beta_j$ has the greater slope, at most two segments of $\alpha_i$ can intersect each of its segments. Thus $i(\alpha_i, \beta_j) \leq 2(w/h)n_j$, twice the number of segments of $\beta_j$ in the cylinder with width $w$ and (vertical) height $h$. But the larger the slope, the greater the length of $\beta_j$ must be: $\ell(\beta_j) > n_j w$. This means $i(\alpha_i, \beta_j) \leq (2/h)\ell(\beta_j)$. Repeating this calculation for each of the pieces of $\alpha$, we
Figure 3. This figure shows one cylinder, one minimal component made up of three rectangles, and a total of eight segments of a single straightened curve. Within the zippered rectangle figure, sides of adjacent rectangles are glued together below the singularities. The straightened curve can change slope only at singularities, as pictured. The three segments that are shown in the cylinder are all part of a single sojourn of the curve to the cylinder.

\[ i(\alpha, \beta_j) \leq \frac{2p}{h} \ell(\beta_j) \Rightarrow i(\alpha, \beta) \leq \frac{2p}{h} \ell(\beta). \]

On the other hand, \( \ell(\alpha_1) > |wm_1| > w_0M_1 > H \), so \( \ell(\alpha) > pH \), and thus

\[ i_C(\alpha, \beta) \leq \frac{2}{Hh} \ell(\alpha)\ell(\beta). \]

Let \( A \) be the number of segments of \( \alpha \) which intersect the sections \( \Gamma_i \) (that is, the bottom edges of rectangles in zippered rectangle configurations).\(^4\) Since every such segment must have length greater than \( H \), it follows that \( A < \ell(\alpha)/H \).

Every segment that intersects the top edge of a rectangle is naturally paired (by the gluing pattern) with one that intersects a bottom edge, so the number of such segments is also \( A \).

Finally, consider the segments which cross rectangles completely, intersecting both sides. The horizontal component is at least \( w_0 \), so the vertical component—and therefore the length of the segment—is at least \( H \). Thus there are no more than \( A \) such segments.

Therefore, \( \alpha \) has at most \( 3A \) Euclidean segments in the zippered rectangles \( X_1, \ldots, X_s \), and the same bound holds for \( \beta \). Thus,

\[ i_Z(\alpha, \beta) < 9A^2 < 9\frac{\ell(\alpha)\ell(\beta)}{H^2}. \]

To complete the proof, we need to account for segments whose slope may be less than \( M_1 \). We will choose a value of \( M \) large enough that if the curve \( \alpha \) has overall

\(^4\)Since the flow is not oriented, rectangles may be repeated and the “top” of one rectangle may be the “bottom” of another. But then treating them as distinct will if anything overcount the number of intersections of the two curves.
slope $M$, its segments of slope less than $M_1$ must be very short. Set $M > H(M_1 + 1)$. Then segments of slope less than $M_1$ have length $\ell_1 \leq |h|(M_1 + 1)$ (see Figure 4).

$$|h| \quad M_1|h| \quad M|h|$$

**Figure 4.** Having $|v|/|h|$ greater than $M$ means that the pieces of slope less than $M_1$ are no more than $1/H$ of the total length.

Since the entire curve has length at least $M|h|$, this implies $\ell_1 < \ell/H$. But then, for the low-slope pieces, Lemma 4 implies that

$$i(\alpha, \beta) \leq \left( \frac{4}{\epsilon^2} \right) \frac{\ell_X(\alpha)\ell_X(\beta)}{H}$$

This shows that the bound in the statement of the lemma holds for all cases, completing the proof.

Thus, by making $H$ (and thereby $M$) big enough, the intersection number can be made less than an arbitrarily small proportion of the product of the lengths of the curves.

**Remark.** This result is stable under perturbation; the same bound $M$ will work for an open neighborhood of the flat structure $X$. This is because a small perturbation (by say $\zeta$) will alter the structures only by changing the widths of the rectangles and maybe shifting the gluings (thus ruining the cylinders). If these changes are bounded by $\zeta$, which is taken to be very small relative to the other parameters, the estimate is robust. In particular, considering $i(\alpha_i, \beta_j)$ and supposing $n_j > m_i$, the fact that a section of $\beta_j$ intersects at most two sections of $\alpha_i$ is preserved as long as $\zeta < \frac{1}{2} \left( \frac{m_j}{n_j} - \frac{n_i}{m_i} \right)$. But there are only finitely many values of the slopes, so $\zeta$ can be chosen sufficiently small to preserve the estimates.

To finish the proof, we assemble the above information: by Lemma 4, curves in $\mathcal{C}_i$ would eventually have slope greater than any $M$. But then Lemmas 9 and 6 give contradictory inequalities for the intersection numbers.

Along the same lines, other triangle invariants can be bounded relative to the framing defect $a + b - c$. For instance, as a “corollary of the proof,” we can show that for large-scale triangles, the insize is bounded above by $k(a + b - c)$ just as the
value $d$ is. Such a bound (of the insize relative to the framing defect) is obtained with a different formulation by Kent and Leininger in a recent paper [14].

3. Random Walk Application

In this application, we consider random walks by mapping classes on Teichmüller space. The triangle comparison property from the previous section is used to argue that most sample paths are well-approximated by Teichmüller geodesics, under mild assumptions given below.

First, we use Teichmüller-theoretic results of Masur and Kaimanovich-Masur to associate to each (a.e.) sample path a geodesic that is a candidate to track the path. Then, we invoke an ergodic argument from Karlsson-Margulis to show that there are large, thin-framed triangles whose vertices are points in the sample path. Finally, we supply a geometric argument to complete the estimate of the deviation of the sample path from its associated geodesic.

3.1. Setup and ergodic ingredients. The theorem of Karlsson and Margulis concerning ergodic transformations on nonpositively curved spaces is stated in [13] in a great deal of generality. We will present their results of interest in full generality, then indicate the chief and guiding example for this application, which will provide an opportunity for concrete interpretation.

For their setup, let $(\Omega, \mu)$ be a measure space with $\mu(\Omega) = 1$ and let $L : \Omega \to \Omega$ be a measure-preserving transformation. Let $a : \mathbb{N} \times \Omega \to \mathbb{R}$ be a subadditive (measurable) cocycle, that is

$$a(n + m, \omega) \leq a(n, L^m \omega) + a(m, \omega)$$

for $n, m \in \mathbb{N}, \omega \in \Omega$ (adopting the convention that $a(0, \omega) = 0$). We will assume that the following integrability condition is satisfied:

$$\int_{\Omega} a^+(1, \omega) \, d\mu(\omega) < \infty,$$

where $a^+(1, \omega) = \max\{0, a(1, \omega)\}$. If $\mu$ satisfies this integrability condition, then it is said to have finite first moment. For each $n$, let

$$a_n = \int_{\Omega} a(n, \omega) \, d\mu(\omega).$$

Then $A := \lim_{n \to \infty} \frac{1}{n} a_n$ exists and $A < \infty$. (See [13].)

Lemma 10 (Karlsson-Margulis [13], Prop 4.2). Suppose that $L$ is ergodic and $A > -\infty$. For any $\delta > 0$, let $E_\delta$ be the set of $\omega \in \Omega$ such that there exist an integer $N = N(\omega)$ and infinitely many $n$ such that

$$a(n, \omega) - a(n - k, L^k \omega) \geq (A - \delta)k$$

for all $k$, $N \leq k \leq n$. If $E = \bigcup_{\delta > 0} E_\delta$, then $\mu(E) = 1$.

Corollary 11 (Kingmann Subadditive Ergodic Theorem, [13] Cor 4.3). Under the same assumptions,

$$\lim_{n \to \infty} \frac{1}{n} a(n, \omega) = A$$

for $\mu$-a.e. $\omega$. 
A geometric interpretation is obtained by restricting attention to the following case: Let $\Gamma = \text{Mod}(g)$ be the mapping class group of a genus $g$ surface and let $\Omega$ be the state space $\Omega = \Gamma^\mathbb{Z}$ so that an element $\omega \in \Omega$ is a bi-infinite sequence of mapping classes $\omega = (\cdots, \omega_0, \omega_1, \omega_2, \cdots)$. Let $P$ be the product measure on $\Omega$. Define the projection $\pi : \Omega \to \Gamma$ by $\pi(\omega) = \omega_0$ and let the shift $L$ re-index by moving one position to the left, i.e., $\pi(L^k\omega) = \omega_k$. (This is a Bernoulli shift, so an ergodic transformation.) Let $Y = T_g$ be the Teichmüller space, fix a basepoint $y \in Y$, and let
\begin{align*}
y_n = y_n(\omega) = \pi(\omega)\pi(L\omega)\cdots\pi(L^{n-1}\omega)y = \omega_0\omega_1\cdots\omega_{n-1}y.\end{align*}
This is the random walk being studied in this work: starting at the basepoint $y$, the subsequent points in the sample path are obtained by applying products of mapping classes to $y$. Note the order of composition of the mapping classes is such that the first chosen is applied last. As shorthand for this dynamical system, we will write the triple $(Y, \Gamma, \mu)$.

We will consider the cocycle $a(n, \omega) = d(y, y_n(\omega))$, where $d$ is the Teichmüller distance. Subadditivity is an immediate consequence of the triangle inequality, because
\begin{align*}
a(n, L^m\omega) = d(y, \pi(L^m\omega)\cdots\pi(L^{m+n-1}\omega)y) = d(y_m, y_{m+n}).
\end{align*}
The integrability assumption (finite first moment) for this distance cocycle is just the assumption that the average jump size is finite. Then $a_n$ can be interpreted as an average distance from the basepoint after $n$ steps, so that $A$ should be thought of as a dominant rate of escape for the random walk—an invariant of the dynamical system $(Y, \Gamma, \mu)$.

3.2. Teichmüller ingredients. Kaimanovich and Masur have showed that under mild conditions on the measure $\mu$ (see below), almost every sample path $\{y_n\}$ of $(Y, \Gamma, \mu)$ converges to a uniquely ergodic foliation—call it $F$—on the boundary of Teichmüller space. Thus, the unique geodesic from the basepoint $y$ to the limit foliation $F$ is a natural candidate to approximate the sample path.

**Theorem 12** (Kaimanovich-Masur). If $\mu$ is a probability measure on the mapping class group $\Gamma = \text{Mod}(g)$ such that the group generated by its support is non-elementary, then for any $y \in T_g$ and $P$-a.e. sample path $\omega = \{\omega_n\}$ of the random walk $(\Gamma, \mu)$, the sequence $y_n$ converges in the Thurston compactification to a limit $F = F(\omega) \in UE \subset PMF$.

This originally appears as Kaimanovich-Masur [11], Thm 2.2.4, where more is proved: they show that this hitting measure $\nu$ on $PMF$ (concentrated on $UE \subset PMF$, as above) is the unique $\mu$-stationary probability measure $\nu$ on $PMF$, $\nu$ is purely non-atomic, and the measure space $(UE, \nu)$ is the Poisson boundary of $(\Gamma, \mu)$.

We will also use a result of Masur.

**Theorem 13.** Suppose $F \in UE \subset PMF$ is a uniquely ergodic foliation. Suppose further that there is a sequence of points $y_i \in T_g$ that converges to $F$ in the Thurston compactification. Then, letting $y = y_0$ be a basepoint, there is a sequence of quadratic differentials $\varphi_i \in QD(y)$ so that for each $i$, $\exists t > 0$ such that $y_i = g_i(y, \varphi_i)$, and those converge in $QD(y)$ to a quadratic differential $\varphi$ such that $F = \cdots$
\lim_{t \to \infty} g_t(y, \varphi). Therefore, for any fixed \( m > 0 \), the points \( g_m(y, \varphi_i) \) converge to \( g_m(y, \varphi) \) in \( T_g \).

This follows from Masur [22], p.184, a result comparing the visual and Thurston compactifications of \( T_g \), which were known by work of Kerckhoff not to be homeomorphic. The boundary spheres are the unit sphere of \( QD \) and the sphere \( \mathcal{PMF} \) of projective measured foliations, respectively; Masur showed that \( B_1 \cup Q_1^{\text{ae}} \cong T_g \cup \mathcal{UE} \), where \( Q_1^{\text{ae}} \) is the set of quadratic differentials in the unit sphere which have uniquely ergodic vertical foliation. In other words, the natural maps between \( B_1 \) and \( T_g \) extend to the boundary spheres at uniquely ergodic points.

3.3. Assembling the ingredients. Assume for this part that \( A > 0 \) and fix an \( \omega \) from the co-null subset of \( \Omega \) such that \( \lim_{n \to \infty} \frac{1}{n} d(y, y_n(\omega)) = A \). Write \( \gamma_k \) for the geodesic ray from \( y \) through \( y_k \) and \( \gamma_\infty \) for the geodesic ray from \( y \) to \( F = \lim y_n \). For any \( k \), let \( r_k = d(y, y_k) \) and \( p_k = \gamma_\infty(r_k) \).

Lemma 14. For any \( \delta > 0 \), there exists \( M_\delta \) with the property that whenever \( n \) is greater than \( M_\delta \) there is some \( m \) such that

\[
\begin{align*}
(1) & \quad d(y, y_n) + d(y_n, y_m) - d(y, y_m) \leq \left( \frac{2\delta}{A - \delta} \right) d(y, y_n) \\
(2) & \quad d(p_n, \gamma_m(r_n)) < \delta.
\end{align*}
\]

Proof. We know that for any \( \delta > 0 \), there is an integer \( N_\delta \) and infinitely many \( m \) so that \( d(y, y_m) - d(y_n, y_m) \geq (A - \delta)n \) (Lemma 10) and \( (A - \delta)n \leq d(y, y_n) \leq (A + \delta)n \) (Cor 11) for all \( N_\delta \leq n \leq m \) (see Figure 5).

Then \( d(y, y_k) + d(y_n, y_m) - d(y, y_m) \leq 2\delta n \). Since \( (A - \delta)n \leq d(y, y_n) \), we finally get (1), which meets the “thin frame” hypothesis from the comparison triangle criterion discussed above with respect to the triangle \( \Delta y y_n y_m \).

For (2), recall that the points \( \{ y_i \} \) converge in the Thurston compactification to a foliation \( F \) (Thm 12). This means that the associated quadratic differentials converge in \( QD \) (Thm 13). But by the visual embedding \( B_1 \to T_g \), convergence of quadratic differentials implies convergence in Teichmüller space for points on the associated geodesics at a fixed distance from the basepoint. This means that the points \( \{ \gamma_i(r_n) \} \) converge to \( \gamma_\infty(r_n) = p_n \); choose sufficiently large \( M_\delta \) so that \( d(\gamma_m(r_n), p_n) < \delta \) for all \( m > M_\delta \).

\[ \square \]

Lemma 15. Suppose \( \epsilon \) is fixed such that \( K(\epsilon) \) contains \( y \). Then there is a constant \( k_{15} \) (depending only on \( \epsilon \), \( A \), and the genus \( g \)) so that for any \( \rho > 0 \) there exists \( Q_\rho \) such that

\[ n > Q_\rho, \quad p_n \in K \implies \frac{1}{n} d(y_n, \gamma_\infty(A n)) < k_{15} \cdot \rho. \]

Proof. Since \( K = K(\epsilon) \) is the lift of a compact set in moduli space, \( y \in K \) implies that \( y_i \in K \) for all \( i \), since mapping classes project to the identity on moduli space.

Choose \( \delta \) small enough that \( \delta, \frac{2\delta}{A - \delta} \) are both less than \( \rho \).

Choose an \( \epsilon' \) small enough that \( K(\epsilon') \) contains a 1-neighborhood of \( K(\epsilon) \). (Thus \( \epsilon' \) depends on \( \epsilon \) and the genus.) Then for \( n \) bigger than \( M_\delta \), there is an \( m \) as in Lemma 14 such that \( d(p_n, \gamma_m(r_n)) < \delta \), which is less than 1 without loss of generality. For this value \( m \), it follows from \( p_n \in K \) that \( \gamma_m(r_n) \in K(\epsilon') \).
This means all four comparison points of △ynnym are in some common thick part, and \( d(y, y_n) + d(y_n, y_m) - d(y, y_m) \leq \rho \cdot d(y, y_n) \) (by Lemma \ref{lem:lemma_4.1} and the choice of \( \delta \)). We can then apply Theorem A, obtaining the estimate
\[
d(y_n, \gamma_m(r_n)) \leq k_0 \cdot \rho \cdot r_n,
\]
where the constant \( k_0 \) depends on the value \( \epsilon' \).

Putting this together, we find that
\[
d(y_n, \gamma_\infty(An)) \leq d(y_n, \gamma_m(r_n)) + d(\gamma_m(r_n), p_n) + d(p_n, \gamma_\infty(An)) \leq k_0 \rho \cdot r_n + \rho + \rho n
\]
\[
\text{so } \frac{d(y_n, \gamma_\infty(An))}{n} \leq \rho \left( \frac{k_0 r_n}{n} + \frac{1}{n} + 1 \right).
\]

Set a new constant \( k = 2(k_0 A + 1) \)—this depends only on the genus \( g \) and the values \( A \) and \( \epsilon \).

Let \( Q_\rho \) be the maximum of \( M_\delta \) and a value large enough that \( \frac{k_0 r_n}{n} + \frac{1}{n} + 1 < k \) for all \( n > Q_\rho \). This is possible because, once again applying Kingman,
\[
\left( \frac{k_0 r_n}{n} + \frac{1}{n} + 1 \right) \to (k_0 A + 1) \text{ as } n \to \infty.
\]
Together, these prove:

**Theorem B.** Suppose $\mu$ is a probability measure on the mapping class group $\Gamma = \text{Mod}(g)$ such that the group generated by its support is non-elementary. Then if the measure has finite first moment and $A > 0$, it follows that for any $y \in T_g$, any thick part $K$ of $T_g$ containing $y$, and $P$-a.e. $\omega$,

$$\lim_{n \to \infty} \left[ \frac{1}{n} d(y_n, \gamma_\infty(An)) \cdot \chi_K(p_n) \right] = 0$$

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