Rational cuspidal plane curves of type \((d, d - 3)\)*

H. Flenner and M. Zaidenberg

**Abstract**

In the previous paper [FlZa 2] we classified the rational cuspidal plane curves \(C\) with a cusp of multiplicity \(\deg C - 2\). In particular, we showed that any such curve can be transformed into a line by Cremona transformations. Here we do the same for the rational cuspidal plane curves \(C\) with a cusp of multiplicity \(\deg C - 3\).

**Introduction**

Let \(C \subset \mathbb{P}^2\) be a rational cuspidal curve; that is, it has only irreducible singularities (called **cusps**). We say that \(C\) is of type \((d, m)\) if \(d = \deg C\) is the degree and \(m = \max_{P \in \text{Sing} C} \{\text{mult}_P C\}\) is the maximal multiplicity of the singular points of \(C\).

Topologically, \(C\) is a 2-sphere \(S^2\) (non-smoothly) embedded into \(\mathbb{P}^2\). Due to the Poincaré-Lefschetz dualities, the complement \(X := \mathbb{P}^2 \setminus C\) to \(C\) is a \(\mathbb{Q}\)-acyclic affine algebraic surface, i.e. \(H_s(X; \mathbb{Q}) = 0\) (see e.g. [Ra, Fu, Za]). Furthermore, if \(C\) has at least three cusps, then \(X\) is of log-general type, i.e. \(\overline{k}(X) = 2\), where \(\overline{k}\) stands for the logarithmic Kodaira dimension [Wa].

In [FlZa 1] we conjectured that any \(\mathbb{Q}\)-acyclic affine algebraic surface \(X\) of log-general type is rigid in the following sense. Let \(V\) be a minimal smooth projective completion of \(X\) by a simple normal crossing (SNC for short) divisor \(D\). We say that \(X\) is **rigid** (resp. **unobstructed**) if the pair \((V, D)\) has no nontrivial deformations (resp. if the infinitesimal deformations of the pair \((V, D)\) are unobstructed).

In the particular case when \(X = \mathbb{P}^2 \setminus C\) with \(C\) as above, the rigidity conjecture would imply that the curve \(C\) itself is projectively rigid. This means that the only equisingular deformations of \(C\) in \(\mathbb{P}^2\) are those provided by automorphisms of \(\mathbb{P}^2\); in other words, all of them are projectively equivalent to \(C\) (see [FlZa 2 sect. 2]). In turn, this would imply that there is only a finite number of non-equivalent rational cuspidal plane curves of a given degree with at least three cusps. Therefore, one may hope to give a classification of such curves.

In [FlZa 2] we obtained a complete list of rational cuspidal plane curves of type \((d, d - 2)\) with at least three cusps, and showed that all of them are projectively rigid and unobstructed.

---

*Math. Subject Classification: 14H20, 14H10, 14D15, 14N05*
In the theorem below we do the same for rational cuspidal plane curves of type \((d, d-3)\) with at least three cusps.

The principal numerical invariant which characterizes a cusp up to equisingular deformation is its multiplicity sequence. Recall that, if

\[ V_{n+1} \to V_n \to \ldots \to V_1 \to V_0 = \mathbb{C}^2 \]

is a minimal resolution of an irreducible analytic plane curve germ \((C, \mathfrak{m}) \subset (\mathbb{C}^2, \mathfrak{m})\), and \((C_i, P_i)\) denotes the proper transform of \((C, \mathfrak{m})\) in \(V_i\), so that \((C_0, P_0) = (C, \mathfrak{m})\), then \(m = (m^{(i)})_{i=0}^{n+1}\), where \(m^{(i)} = \text{mult}_{P_i} C_i\), is called the multiplicity sequence of the germ \((C, \mathfrak{m})\). Thus, \(m^{(i+1)} \leq m^{(i)}\), \(m^{(n)} \geq 2\) and \(m^{(n+1)} = 1\). A multiplicity sequence has the following characteristic property \([FZa 2] (1.2)\):

for any \(i = 0, \ldots, n-1\) either \(m^{(i)} = m^{(i+1)}\), or there exists \(k > 0\) such that \(i + k \leq n\), and

\[ m^{(i)} = m^{(i+1)} + \ldots + m^{(i+k)} + m^{(i+k+1)} \quad \text{where} \quad m^{(i+1)} = \ldots = m^{(i+k)}. \]

We use the abbreviation \((m_k)\) for a (sub)sequence \(m^{(i+1)} = m^{(i+2)} = \ldots = m^{(i+k)} = m\).

Thus, we present a multiplicity sequence as \((m_1, \ldots, m_s)\) with \(m^{(i+1)} < m^{(i)}\); by abuse of notation, we assume here that \(m^{(s)} \geq 2\). For instance, \((2)\) means an ordinary cusp, and \((2_3) = (2, 2, 2, 1)\) corresponds to a ramphoid cusp. With this notation we can formulate our main result as follows.

**Theorem.** (a) Let \(C \subset \mathbb{P}^2\) be a rational cuspidal plane curve of type \((d, d-3)\), \(d \geq 6\), with at least three cusps. Then \(d = 2k + 3\), where \(k \geq 2\), and \(C\) has exactly three cusps, of types \((2k, 2k)\), \((3k)\), \((2)\), respectively.

(b) For each \(k \geq 1\) there exists a rational cuspidal plane curve \(C_k\) of degree \(d = 2k + 3\) with three cusps of types \((2k, 2k)\), \((3k)\) and \((2)\).

(c) Moreover, the curve \(C_k\) as in (b) is unique up to projective equivalence. It can be defined over \(\mathbb{Q}\).

**Remarks.** (1) A classification of irreducible plane curves up to degree 5 can be found e.g. in \([Nam]\). In particular, there are, up to projective equivalence, only one rational cuspidal plane quartic with three cusps (the Steiner quartic) and only three rational cuspidal plane quintic curves with at least three cusps. Two of them have exactly three cusps, of types \((3)\), \((2_2)\), \((2)\) resp. \((2_2)\), \((2_2)\), \((2)\), and the third one has four cusps of types \((2_3)\), \((2)\), \((2)\) \([Nam] \text{ Thm. } 2.3.10\).

(2) In his construction of \(\mathbb{Q}\)-acyclic surfaces (see e.g. \([D 1], [D 2]\)), T. tom Dieck found certain \((d, d-2)\)- and \((d, d-3)\)-rational cuspidal curves, in particular, those listed in the theorem above, as well as some other series of rational cuspidal plane curves (a private communication\(^1\)). Besides a finite number of sporadic examples, the curves with at least

---

\(^1\)We are grateful to T. tom Dieck for communicating us the list of the multiplicity sequences of the constructed curves.
three cusps in the list of tom Dieck are organized in three series of \((d, d - 2)\), \((d, d - 3)\) and \((d, d - 4)\)-type, respectively. It can be checked that all those curves are rigid and unobstructed.

Following our methods, T. Fenske proved recently that the only possible numerical data of unobstructed rational cuspidal plane curves with at least three cusps and of type \((d, d - 4)\) are those from the list of tom Dieck. He has also classified all rational cuspidal plane curves of degree 6 [Fe]. It turns out that the only examples with at least 3 cusps are those described in [FlZa 2].

(3) For a rational cuspidal plane curve \(C\) of type \((d, m)\) the inequality \(m > d/3\) holds [MaSa]. Recently, S. Orevkov obtained a stronger one\(^1\): If the complement \(\mathbf{P}^2 \setminus C\) has logarithmic Kodaira dimension 2, then \(d < \alpha m + \beta\), where \(\alpha := (3 + \sqrt{5})/2 = 2.6180\ldots\) and \(\beta := \alpha - 1/\sqrt{5} = 2.1708\ldots\).

(4) It was shown in [OrZa 1, OrZa 2] that a rational cuspidal plane curve with at least ten cusps cannot be projectively rigid.

Recall the Coolidge–Nagata Problem [Co, Nag]:

*Which rational plane curves can be transformed into a line by means of Cremona transformations of \(\mathbf{P}^2\)?*

It can be completed by the following question:

*Is this possible, in particular, for any rational cuspidal plane curve?*

Under certain restrictions, a positive answer was given in [Nag, MKM, MaSa, Ii 2, Ii 3]. It can be verified that the last question has a positive answer for the rational cuspidal plane curves of degree at most five. In [FlZa 2] we showed that any rational cuspidal plane curve of type \((d, d - 2)\) with at least three cusps is rectifiable. Here we extend this result to \((d, d - 3)\)-curves. It will turn out to be an immediate consequence of our construction:

**Corollary.** Any rational cuspidal plane curve of type \((d, d - 3)\) with at least three cusps is rectifiable, i.e. it can be transformed into a line by means of Cremona transformations.

1 **Proofs**

Let \(C \subset \mathbf{P}^2\) be a plane curve, and let \(V \to \mathbf{P}^2\) be the minimal embedded resolution of singularities of \(C\), so that the reduced total transform \(D\) of \(C\) in \(V\) is an SNC–divisor. By [FlZa 1], the cohomology groups \(H^i(\Theta_V(D))\) of the sheaf of germs of holomorphic vector fields on \(V\) tangent to \(D\) control the deformations of the pair \((V, D)\); more precisely, \(H^0(\Theta_V(D))\) is the space of its infinitesimal automorphisms, \(H^1(\Theta_V(D))\) is the space of infinitesimal deformations and \(H^2(\Theta_V(D))\) gives the obstructions for extending infinitesimal deformations.

\(^1\)We are grateful to S. Orevkov for providing us with a preliminary version of his paper.
The surface $X = V \setminus D = \mathbb{P}^2 \setminus C$ being of log-general type, the automorphism group $\text{Aut}X$ is finite\[1\], and hence $h^0(\Theta_V(D)) = 0$. Thus, the holomorphic Euler characteristic of the sheaf $\Theta_V(D)$ is

$$\chi(\Theta_V(D)) = h^2(\Theta_V(D)) - h^1(\Theta_V(D)).$$

**Lemma 1.1.** If $C$ is a rational cuspidal plane curve of type $(d, d - 3)$ with at least three cusps, then $h^2(\Theta_V(D)) = 0$, that is, $C$ is unobstructed\[3\], and so $\chi = \chi(\Theta_V(D)) \leq 0$.

**Proof.** Projecting from the cusp of multiplicity $d - 3$ yields a fibration $V \to \mathbb{P}^1$, which is three-sheeted when restricted to the proper transform of $C$. Now \[FlZa 1\], (6.3)] shows that $h^2(\Theta_V(D)) = 0$. Since $\bar{k}(V \setminus D) = 2$, we also have $h^0(\Theta_V(D)) = 0$. Hence $\chi = -h^1(\Theta_V(D)) \leq 0$. \[\square\]

The next proposition proves part (a) of our main theorem.

**Proposition 1.1.** The only possible rational cuspidal plane curves $C$ of degree $d \geq 6$ with a singular point $Q$ of multiplicity $d - 3$ and at least three cusps are those of degree $d = 2k + 3$, $k = 1, \ldots$, with three cusps of types $(2k, 2k)$, $(3k)$ and $(2)$. Furthermore, these curves are projectively rigid.

**Proof.** By \[FlZa 2\], (2.5)] and Lemma 1.1 above, we have:

$$\chi = -3(d - 3) + \sum_{P \in \text{Sing}C} \chi_P \leq 0,$$

where

$$\chi_P := \eta_P + \omega_P - 1,$$

and where, for a singular point $P \in C$ with the multiplicity sequence $m_P = (m^{(0)}, \ldots, m^{(kP)})$,

$$\eta_P = \sum_{i=0}^{kP} (m^{(i)} - 1) \quad \text{and} \quad \omega_P = \sum_{i=1}^{kP} (\left\lceil \frac{m^{(i-1)}}{m^{(i)}} \right\rceil - 1)$$

(for $a \in \mathbb{R}$, $\left\lceil a \right\rceil$ denotes the smallest integer $\geq a$).

Observe that, by the Bezout theorem, $m_P^{(0)} + m_P^{(1)} \leq d$ and $m_P^{(0)} + m_Q^{(0)} \leq d$. Thus

for $P \neq Q$ we have $m_P^{(0)} \leq 3$; moreover we have $m_Q^{(1)} \leq 2$,

since otherwise the tangent line $TQC$ would have the only point $Q$ in common with $C$, and so, $C \setminus TQC$ would be an affine rational cuspidal plane curve with one point at infinity and with two cusps. But by the Lin-Zaidenberg Theorem \[LiZa\], up to biregular automorphisms

\[3\]i.e. as a plane curve, it has unobstructed equisingular infinitesimal deformations.
of the affine plane $\mathbb{C}^2$, the only irreducible simply connected affine plane curves are the curves $\Gamma_{k,l} = \{x^k - y^l = 0\}$, where $1 \leq k \leq l$, and $(k, l) = 1$. Hence, such a curve cannot have two cusps. Using the above restriction and the characteristic property of a multiplicity sequence cited above we obtain the following possibilities for the multiplicity sequence $m_P$ at a singular point $P$:

$$m_Q = (d - 3) \text{ or } (d - 3, 2),$$
$$m_p = (2_a) \text{ or } (3_a) \text{ or } (3_a, 2) \text{ for } P \neq Q. \quad (R_2)$$

For different possible types of cusps of $C$ we have:

(a) If $Q \in \text{Sing } C$ has the multiplicity sequence $(d - 3)$, then
$$\eta_Q = d - 4, \quad \omega_Q = d - 4 \quad \text{and so} \quad \chi_Q = 2d - 9.$$

(b) If $Q \in \text{Sing } C$ has the multiplicity sequence $(d - 3, 2a)$ then, by the same characteristic property [FlZa 2, (1.2)],

(*) either $d - 3 \leq 2a$ is even or $d - 3 = 2a + 1$.

In any case
$$\eta_Q = d - 4 + a, \quad \omega_Q = \left\lfloor \frac{d - 3}{2} \right\rfloor \quad \text{and so} \quad \chi_Q = d - 5 + a + \left\lfloor \frac{d - 3}{2} \right\rfloor.$$

(c) If $P \in \text{Sing } C$ has the multiplicity sequence $(2_a)$, then
$$\eta_P = a, \quad \omega_P = 1 \quad \text{and so} \quad \chi_P = a.$$

(d) If $P \in \text{Sing } C$ has the multiplicity sequence $(3_a)$, then
$$\eta_P = 2a, \quad \omega_P = 2 \quad \text{and so} \quad \chi_P = 2a + 1.$$

(e) If $P \in \text{Sing } C$ has the multiplicity sequence $(3_a, 2)$, then
$$\eta_P = 2a + 1, \quad \omega_P = 2 \quad \text{and so} \quad \chi_P = 2a + 2.$$

Furthermore, since $C$ rational, by the genus formula, we have

$$\left( \begin{array}{c} d - 1 \\ 2 \end{array} \right) = \sum_{P \in \text{Sing } C} \delta_P \quad \text{where} \quad \delta_P := \sum_{i=1}^{k_P} \left( \begin{array}{c} m_P^{(i)} \\ 2 \end{array} \right).$$

Since $m_Q^{(0)} = d - 3$, we get

$$\left( \begin{array}{c} d - 1 \\ 2 \end{array} \right) = \left( \begin{array}{c} d - 3 \\ 2 \end{array} \right) + \sum_{(P,i) \neq (Q,0)} \left( \begin{array}{c} m_P^{(i)} \\ 2 \end{array} \right),$$

5
or, equivalently,

\[ 2d - 5 = \sum_{(P, i) \neq (Q, 0)} \frac{m_P^{(i)}(m_P^{(i)} - 1)}{2}. \]  \hspace{1cm} (R_3)

At last, consider the projection \( \pi_Q : C \to \mathbf{P}^1 \) from the point \( Q \). By the Riemann-Hurwitz Formula, it has at most four branching points. This gives the restriction (see \([FLZa\ 2, (3.1)]\))

\[ m_Q^{(1)} - 1 + \sum_{P \neq Q} (m_P^{(0)} - 1) \leq 4. \]  \hspace{1cm} (R_4)

Thus, if the curve \( C \) has the numerical data

\[ [(d - 3, 2a_1), (2a_2), \ldots, (2a_k), (3b_1), \ldots, (3b_l), (3c_1, 2), \ldots, (3c_m, 2)], \]

then \( k + 2(l + m) \leq 4 \). Hence, either \( l + m = 0 \) and \( 3 \leq k \leq 4 \), or \( l + m = 1 \) and \( k = 2 \), or \( l + m = 2 \) and \( k = 0 \).

Taking into account the above restrictions \( (R_2) - (R_4) \) and \( (*) \) from \((b)\), the list of all possible data of rational cuspidal plane curves \( C \) of degree \( d \geq 6 \) with a point of multiplicity \( d - 3 \) and at least 3 cusps is as follows, where \( a, b, c, e > 0 \):

\[
\begin{align*}
[(d - 3), (2a), (2b)] & \text{ where } a + b = 2d - 5 & (1) \\
[(d - 3), (2a), (3b)] & \text{ where } a + 3b = 2d - 5 & (2) \\
[(d - 3), (3a), (3b)] & \text{ where } 3a + 3b = 2d - 5 & (3) \\
[(d - 3), (3a, 2), (2b)] & \text{ where } 3a + b = 2d - 6 & (4) \\
[(d - 3), (3a, 2), (3b)] & \text{ where } 3a + 3b = 2d - 6 & (5) \\
[(d - 3), (3a, 2), (3b, 2)] & \text{ where } 3a + 3b = 2d - 7 & (6) \\
[(d - 3, 2a), (2b), (2c)] & \text{ where } a + b + c = 2d - 5 \text{ and } (*) \text{ holds} & (7) \\
[(d - 3, 2a), (3b), (2c)] & \text{ where } a + 3b + c = 2d - 5 \text{ and } (*) \text{ holds} & (8) \\
[(d - 3, 2a), (3b, 2), (2c)] & \text{ where } a + 3b + c = 2d - 6 \text{ and } (*) \text{ holds} & (9) \\
[(d - 3, 2a), (2a), (2b), (2c)] & \text{ where } a + b + c = 2d - 5 & (10) \\
[(d - 3, 3a), (2b), (2c)] & \text{ where } 3a + b + c = 2d - 5 & (11) \\
[(d - 3, 3a, 2), (2b), (2c)] & \text{ where } 3a + b + c = 2d - 6 & (12) \\
[(d - 3, 2a), (2b), (2c, e)] & \text{ where } a + b + c + e = 2d - 5 \text{ and } (*) \text{ holds} & (13) \\
[(d - 3, 2a, 2b), (2c)] & \text{ where } a + b + c + e = 2d - 5 & (14)
\end{align*}
\]

We will examine case by case, computing \( \chi = \chi(\Theta_V \cdot D) \). The genus formula and the restriction \( (R_1) \chi \leq 0 \) provided by Lemma 1.1 will allow to eliminate all the cases but one, namely, a subcase of \((8)\).
Case (1): \([((d - 3), (2_a), (2_b)]\) where \(a + b = 2d - 5\). By \((R_1)\), we have \(\chi = (-3d + 9) + (2d - 9) + (a + b) = d - 5 \leq 0\), a contradiction.

Case (2): \([[(d - 3), (3_b), (2_a)]\) where \(a + 3b = 2d - 5\). We have \(\chi = (-3d + 9) + (2d - 9) + (a + 2b + 1) = d - 4 - b \leq 0\), i.e. \(b \geq d - 4\). On the other hand, \(2d - 5 = a + 3b \geq 3b + 1\), whence \(b \leq \frac{2}{3}d - 2\). Therefore, \(d - 4 \leq \frac{2}{3}d - 2\), i.e. \(d \leq 6\). In the case \(d = 6\) the only possibility would be \([(3), (2), (3_a)]\). Projecting from the cusp with the multiplicity sequence \((3_2)\), we get a contradiction to the Hurwitz formula (see \((R_4)\)).

Case (3): \([[(d - 3), (3_a), (3_b)]\) where \(3a + 3b = 2d - 5\). We have \(\chi = (-3d + 9) + (2d - 9) + (2a + 1 + 2b + 1) = \frac{d - 4}{3} \leq 0\), i.e. \(d \leq 4\), and we are done.

Case (4): \([[(d - 3), (3_a, 2), (2_b)]\) where \(3a + b = 2d - 6\). We have \(\chi = (-3d + 9) + (2d - 9) + (2a + 2 + b) = d - 4 - a \leq 0\), i.e. \(a \geq d - 4\). But \(2d - 6 = 3a + b \geq 3a + 1\), whence \(a \leq \frac{2}{3}d - \frac{2}{3}\), and thus \(d - 4 \leq \frac{2}{3}d - \frac{2}{3}\), or \(d \leq 5\), a contradiction.

Case (5): \([[(d - 3), (3_a, 2), (3_b)]\) where \(3a + 3b = 2d - 6\). We have \(\chi = (-3d + 9) + (2d - 9) + (2a + 2 + 2b + 1) = \frac{d - 4}{3} - 1 \leq 0\), i.e. \(d \leq 3\), which is impossible.

Case (6): \([[(d - 3), (3_a, 2), (3_b)]\) where \(3a + 3b = 2d - 7\). We have \(\chi = (-3d + 9) + (2d - 9) + (2a + 2b + 4) = \frac{d - 4}{3} - \frac{2}{3} \leq 0\), which is impossible.

Case (7): \([[(d - 3), (2_a), (2_b)]\) where \(a + b + c = 2d - 5\) and \((\ast)\) holds. We have \(\chi = (-3d + 9) + (d - 5 + a + \lceil \frac{d - 3}{2} \rceil) + (b + c) = \lceil \frac{d - 3}{2} \rceil - 1 \leq 0\), or \(d \leq 5\), and we are done.

Case (8): \([[(d - 3, 2_a), (3_b), (2_a)]\) where \(a + 3b + c = 2d - 5\) and \((\ast)\) holds. We have \(\chi = (-3d + 9) + (d - 5 + a + \lceil \frac{d - 3}{2} \rceil) + (2b + 1 + c) = \lceil \frac{d - 3}{2} \rceil - b \leq 0\), i.e. \(b \geq \lceil \frac{d - 3}{2} \rceil\).

If \(d - 3\) is odd, then we get \(2d - 5 = a + 3b + c \geq 3b + 1 + \frac{d - 1}{2}\), as \(a = \frac{d - 1}{2}\) by \((\ast)\). Hence, \(b \leq \frac{d}{2} - \frac{4}{3}\). This leads to \(\lceil \frac{d - 3}{2} \rceil = \frac{d - 3}{2} \leq \frac{d - 4}{3} - \frac{4}{3}\), which is a contradiction.

If \(d - 3\) is even, then by \((\ast)\) we get \(2d - 5 = a + 3b + c \geq 3b + 1 + \frac{d - 3}{2}\), hence \(b \leq \frac{d}{2} - \frac{3}{2}\). Thus, \(\lceil \frac{d - 3}{2} \rceil = \frac{d - 3}{2} \leq b \leq \frac{d - 3}{2}\), which is only possible if \(c = 1\), \(a = b = \frac{d - 3}{2}\). With \(k := \frac{d - 3}{2}\), we have \(d = 2k + 3\), \(a = b = k\) and \(c = 1\); that is, \(C\) is as in the proposition. Observe that in this case \(\chi = 0\), and so \(h^1(\Theta_V(\langle D \rangle)) = 0\). Together with Lemma 1.1 this proves that the corresponding curve \(C\) is projectively rigid and unobstructed (see \([FZ 2, \text{Sect. 2}]\)).

Case (9): \([[(d - 3, 2_a), (3_b, 2), (2_c)]\) where \(a + 3b + c = 2d - 6\) and \((\ast)\) holds. We have \(\chi = (-3d + 9) + (d - 5 + a + \lceil \frac{d - 3}{2} \rceil) + (2b + 2 + c) = \lceil \frac{d - 3}{2} \rceil - b \leq 0\), which gives \(b \geq \lceil \frac{d - 3}{2} \rceil\).

If \(d - 3\) is odd, then we get \(2d - 6 = a + 3b + c \geq 3b + 1 + \frac{d - 4}{2}\), as \(a = \frac{d - 4}{2}\) by \((\ast)\). Thus, \(b \leq \frac{d}{2} - \frac{5}{3}\), and so we have \(\frac{d - 2}{2} \leq \frac{d}{2} - \frac{5}{3}\), which is a contradiction.

If \(d - 3\) is even, then we get \(2d - 6 = a + 3b + c \geq 3b + 1 + \frac{d - 3}{2}\). Hence, \(b \leq \frac{d}{2} - \frac{11}{6}\). This yields \(\frac{d - 2}{2} \leq \frac{d}{2} - \frac{11}{6}\), which again gives a contradiction.

Case (10): \([[(d - 3), (2_a), (2_b), (2_c)]\) where \(a + b + c = 2d - 5\). We have \(\chi = (-3d + 9) + (2d - 9) + (a + b + c) = d - 5 \leq 0\), and we are done.

Case (11) resp. (12), (13), (14) can be ruled out by the same computations as in case (2) resp. (4), (7), (10). This completes the proof of Proposition 1.1.

For the proof of part (b) and (c) the main theorem we need the following facts.

\[\square\]
Lemma 1.2. Let \((C, \mathfrak{u}), (D, \mathfrak{v}) \subseteq (\mathbb{C}^2, \mathfrak{u})\) be two curve singularities which have no component in common. Then the following hold.

(a) \((CD)_{\mathfrak{u}} = \sum_P \text{mult}_P C \text{mult}_P D\), where the sum is taken over \(\mathfrak{u}\) and all its infinitesimally near points.

(b) Assume that \((D, \mathfrak{v})\) is a smooth germ and \((C, \mathfrak{u} = )\) is a cusp with the multiplicity sequence \(m = (m^{(0)}, \ldots, m^{(s)})\). Then \((CD)_{\mathfrak{u}} = m^{(0)} + \ldots + m^{(s)}\) for some \(s \geq 0\), where \(m^{(0)} = \ldots = m^{(s-1)}\).

(c) Let \(\pi : X \to \mathbb{C}^2\) be the blow up at \(\mathfrak{u}\). Denote by \(E \subseteq X\) the exceptional curve, and by \(C'\) the proper transform of \(C\). Then

\[
\text{mult}_{\mathfrak{u}} C = \sum_{P \in E} (EC')_P.
\]

Proof. The statements (a) and (c) are well known (see e.g. [Co]), whereas (b) is shown in [FlZa 2, (1.4)]. \(\square\)

The next result proves part (b) and (c) of the main theorem as well as the corollary from the introduction.

Proposition 1.2. (a) For each \(k \geq 1\) there exists a rational cuspidal plane curve \(C_k\) of degree \(d = 2k + 3\) with three cusps \(Q_k, P_k, R_k\) of types \((2k, 2k), (3k)\) and \(2\), respectively.

(b) \(C_k\) is unique up to a projective transformation of the plane.

(c) \(C_k\) is defined over \(\mathbb{Q}\).

(d) \(C_k\) is rectifiable.

Proof. We proceed by induction on \(k\). Namely, given a curve \(C_k\) as in (a), we find a Cremona transformation \(\psi_k : \mathbb{P}^2 \to \mathbb{P}^2\) such that the proper transform \(C_{k+1} = \psi_k(C_k)\) of \(C_k\) under \(\psi_k\) is a cuspidal curve of degree \(2k + 5\) with three cusps of type \(2k + 2, 2k + 1, 3k + 1\), (2). Hence the existence follows. This construction will also show that (b)–(d) hold.

We start with the rational cuspidal cubic \(C_0 \subseteq \mathbb{P}^2\) given by the equation \(x^2z = y^3\). Observe that \(C_0\) is rectifiable. It has a simple cusp at \(P_0 := (0 : 0 : 1)\) and the only inflectional tangent line \(\ell_0\) at \(P_0 := (1 : 0 : 0)\); that is, \(\ell_0 \cdot C_0 = 3P_0\). Fix an arbitrary point \(Q_0 \in C_0 \setminus \{P_0, R_0\}\). Let \(t_0\) be the tangent line to \(C_0\) at \(Q_0\); then we have \(t_0 \cdot C_0 = 2Q_0 + S_0\), where, as it is easily seen, \(S_0 \in C_{-0}\) is different from \(P_0, Q_0\) and \(R_0\). Let \(Q_0^*\) denote the intersection point \(l_0 \cap t_0\); clearly, \(Q_0^* \notin C_0\).

Let, for a given \(k > 0\), \(C_k\) denotes a curve with the cusps \(Q_k, P_k, R_k\) as in the proposition, and let \(C_0\) be the rational cubic with the distinguished points \(Q_0, P_0, R_0, S_0\) as described above. For \(k > 0\) let \(t_k\) be the tangent line of \(C_k\) at \(Q_k\), and \(\ell_k\) be the line \(P_kQ_k\), whereas for \(k = 0\) we choose \(t_0\) and \(\ell_0\) as above. In any case, using Bezout’s Theorem and Lemma 1.2, we have

\[
\ell_k \cdot C_k = (d - 3)Q_k + 3P_k, \quad \text{and} \quad t_k \cdot C_k = (d - 1)Q_k + S_k.
\]

\[\text{Observe that the projective transformation group } (x : y : z) \mapsto (t^3x : t^2y : t^3z), \ t \in \mathbb{C}^*, \text{ acts transitively in } C_0 \setminus \{P_0, R_0\}.\]
where $S_k \in C_k$ is different from $P_k, Q_k$ and $R_k$. Indeed, the line $t_k$ intersects $C_k$ at the point $Q_k$ with multiplicity $d - 1$ if $k > 1$ (see Lemma 1.2 (b)) or $k = 0$. To show that this is also true for $k = 1$, assume that $t_1$ and $C_1$ only intersect in $Q_1$ with $(t_1C_1)_{Q_1} = d = 5$. The linear projection from $Q_1$ yields a 3-sheeted covering of the normalization of $C_1$ onto $\mathbb{P}^1$. By the Riemann-Hurwitz formula, it must have four ramification points. But since $(t_1C_1)_{Q_1} = d = 5$, the point $Q_1$ would be a ramification point of index $\geq 2$ (see Lemma 1.2(a)), and so we would have three ramification points $Q_1, P_1, R_1$ of indices 2, 2, 1, respectively, which is a contradiction.

Hence, for any $k \geq 0$ there is exactly one further intersection point $S_k \in C_k \cap t_k$ with $(t_kC_k)_{S_k} = 1$.

Let $\sigma_k : X_k \to \mathbb{P}^2$ be the blow up at the point $t_k \cap E_k$, which is $Q_k$ for $k > 0$ and $Q_k'$ for $k = 0$. Denote by $C'_k, t'_k, \ell'_k$ the proper transforms in $X_k$ of the curves $C_k, t_k, E_k$, respectively. Then $X_k \simeq \Sigma_1$ is a Hirzebruch surface with a ruling $\pi_k : X_k \to \mathbb{P}^1$ given by the pencil of lines through $k$ true for $k > 0$, and $\mathbb{P}^1$ for $k = 0$. Let $\ell_0 = (\pi_k)_{E_0}$, with the exceptional section $E_k = \sigma_k^{-1}(Q_k), k > 0, \text{resp.} E_0 = \sigma_0^{-1}(Q_0^*)$, where $E_k^* = -1$. Thus, $t'_k, t_k = t'$ are fibres of this ruling. By construction, the restriction $\pi_k|_{C'_k} : C'_k \to \mathbb{P}^1$ is 3-sheeted, and we have

$$\ell'_k \cdot C'_k = 3P'_k, \quad t'_k \cdot C'_k = 2Q'_k + S'_k, \quad \text{and} \quad E'_k \cdot C'_k = (d - 3)Q'_k = 2kQ'_k,$$

where $P'_k, Q'_k, R'_k$ and $S'_k$ are the points of $C'_k$ infinitesimally near to $P_k, Q_k, R_k$ and $S_k \in C_k$, respectively (indeed, by Lemma 1.2(c), we have $(E'_kC'_k)_{Q'_k} = \text{mult}_{Q'_k}C'_k = d - 3$, where for $k > 0$ we set $Q'_k = Q_k$). Clearly, for $k > 0$ $P'_k, Q'_k$ and $R'_k$ are cusps of $C'_k$ of types $(3_k), (2_k)$ and $(2)$, respectively, whereas $S'_k$ is a smooth point.

Next we perform two elementary transformations of $X_k$, one at the point $S'_k$ and the other one at the intersection point $T'_k := \{E_k \cap \ell'_k\}$. We arrive at a new Hirzebruch surface $X_{k+1} \simeq \Sigma_1$, with the exceptional section $E_{k+1}^*$ being the proper transform of $E_k$ (indeed, since we perform elementary transformations at the points $S_k \notin E_k$ and $T_k \notin E_k$, we have $E_0^* = E_k^* = -1$). Denote by $C_{k+1}'$ the proper transform of $C_k'$, and by $t'_{k+1}, \ell'_{k+1}$ the fibres of the ruling $\pi_{k+1} : X_{k+1} \to \mathbb{P}^1$ which replace $t'_k$ resp. $\ell'_k$. Using formal properties of the blowing up/down process we obtain, once again, the relations

$$\ell'_{k+1} \cdot C'_{k+1} = 3P'_{k+1}, \quad t'_{k+1} \cdot C'_{k+1} = 2Q'_{k+1} + S'_{k+1}, \quad \text{and} \quad E'_{k+1} \cdot C'_{k+1} = 2(k + 1)Q'_{k+1},$$

where $P'_{k+1}, Q'_{k+1}, R'_{k+1}$ and $S'_{k+1}$ are the points of $C'_{k+1}$ infinitesimally near to $P'_k, Q'_k, R'_k$ and $S'_k \in C'_k$, respectively. It is easily seen that $P'_{k+1}$ resp. $Q'_{k+1}, R'_{k+1}$ are cusps of $C'_{k+1}$ of types $(3_{k+1}), (2_{k+1})$ and $(2)$, respectively, whereas $S'_{k+1}$ is a smooth point.

Blowing down the exceptional curve $E'_{k+1} \subset X_{k+1}$ we arrive again at $\mathbb{P}^2$. Denote the images of $C''_{k+1}, Q''_{k+1}, P''_{k+1}, R''_{k+1}$ resp. by $C_{k+1}, Q_{k+1}, P_{k+1}, R_{k+1}$. We have constructed a rational cuspidal plane curve $C_{k+1}$ which has cusps at $Q_{k+1}, P_{k+1}, R_{k+1}$ with multiplicity sequences $(2(k + 1), 2_{k+1}), (3_{k+1}), (2)$, respectively (see Lemma 1.2(c)). This completes the proof of existence.

5Recall that an elementary transformation of a ruled surface consists in blowing up at a point of a given irreducible fibre followed by the contraction of the proper transform of this fibre.
Note that the birational transformation \( \psi_k : \mathbb{P}^2 \to \mathbb{P}^2 \), by which we obtained \( C_{k+1} = \psi_k(C_k) \) from \( C_k \), is just the Cremona transformation in the points \( S_k, Q_k \) and the intersection point \( E_k \cap \ell_k \), which is infinitesimally near to \( Q_k \). This transformation only depends upon \( Q_k, S_k \) and the line \( \ell_k \); we denote it by \( \psi(S_k, Q_k, \ell_k) := \psi_k \). The inverse \( \psi_k^{-1} \) is the transformation \( \varphi_k = \psi(P_{k+1}, Q_{k+1}, t_{k+1}) \). Therefore, the curve \( C_k \) is always transformable into the cuspidal cubic, and thus also into a line, by means of Cremona transformations, proving (d). In order to show (c) we note that, moreover, so constructed \( C_k \), as well as \( P_k, Q_k, R_k \) and \( S_k \), are defined over \( Q \), as follows by an easy induction.

Finally, let us show that the curve \( C_k \) is uniquely determined up to a projective transformation of the plane. We will again proceed by induction on \( k \). Clearly, the cuspidal cubic is uniquely determined up to a projective transformation. Assume that uniqueness is shown for the curve \( C_k \), and consider two curves \( C_{k+1}, \tilde{C}_{k+1} \) as in (a). Let \( P_{k+1} \in C_{k+1}, Q_{k+1} \in C_{k+1} \) and the tangent line \( t_{k+1} \) of \( C_{k+1} \) at \( Q_{k+1} \), be as above; denote the corresponding data for \( C_{k+1} \) by \( \tilde{P}_{k+1}, \tilde{Q}_{k+1} \) and \( \tilde{t}_{k+1} \). Consider the Cremona transformations \( \varphi_k := \psi(P_{k+1}, Q_{k+1}, t_{k+1}) \) and \( \tilde{\varphi}_k := \psi(P_{k+1}, Q_{k+1}, \tilde{t}_{k+1}) \), and also the proper transforms \( \tilde{C}_k := \varphi_k(C_{k+1}) \) and \( \tilde{C}_k := \tilde{\varphi}_k(\tilde{C}_{k+1}) \). Reversing the above arguments it is easily seen that the both curves \( C_k, \tilde{C}_k \) are as in (a). By the induction hypothesis, they differ by a projective transformation \( f : \mathbb{P}^2 \to \mathbb{P}^2 \), i.e. \( f(C_k) = \tilde{C}_k \). For \( k > 0 \) the points \( Q_k \in C_k, S_k \in C_k \) and the line \( \ell_k \) are intrinsically defined by the curve \( C_k \), and so, \( f \) maps these data onto the corresponding data \( \tilde{Q}_k, \tilde{S}_k \) and \( \tilde{\ell}_k \) for the curve \( \tilde{C}_k \). Moreover, in the case \( k = 0 \) it is easily seen that one can choose \( f \) in such a way that \( f(Q_0) = \tilde{Q}_0 \). Then again \( f(S_k) = f(\tilde{S}_k) \) and \( f(\ell_k) = \tilde{\ell}_k \). Hence, the map \( f \) is compatible with the Cremona transformations \( \varphi_k^{-1} = \psi(S_k, Q_k, \ell_k) \) and \( \tilde{\varphi}_k^{-1} = \psi(\tilde{S}_k, \tilde{Q}_k, \tilde{\ell}_k) \), i.e. there is a linear transformation \( g \) of \( \mathbb{P}^2 \) such that \( \varphi_k \circ g = f \circ \tilde{\varphi}_k \). Clearly, \( g \) transforms \( C_{k+1} \) into \( \tilde{C}_{k+1} \).

\[ \square \]

Remarks. (1) By the same approach as in the proof of Proposition 1.2, it is possible to show the existence and uniqueness of the rational cuspidal curves of type \((d, d - 2)\) with at least three cusps, which was done by a different method in [FlZa 2]. By the result of loc.cit such a curve \( C \) has exactly three cusps, say \( Q, P, R \), with the multiplicity sequences \((d - 2), (2_a), (2_b)\), respectively, where \( a + b = d - 2 \). Set \( \ell_P := QP, \ell_R := QR \) and denote by \( t_Q \) the tangent line at \( Q \). By Bezout’s Theorem, \( t_Q \) intersects \( C \) in one further point \( S \) different from \( Q \). Performing the Cremona transformation \( \psi(S, Q, \ell_P) \) to the curve \( C \), we obtain a curve of degree \( d + 1 \) with the multiplicity sequences \((d - 1), (2_{a+1}), (2_b)\) at the cusps. Similarly, under the Cremona transformation \( \psi(P, Q, \ell_R) \) the curve \( C \) is transformed into a cuspidal curve of the same degree \( d \) with the multiplicity sequences \((d - 2), (2_{a+1}), (2_{b-1})\). Thus, starting from the rational cuspidal quartic with three cusps, we can construct all such curves. It follows from this construction that these curves are rectifiable.

(2) Using the above arguments, it is also possible to classify the rational cuspidal curves of degree five with at least three cusps, which was done by M. Namba by a different method, see [Nam, Thm.2.3.10]. Indeed, if the largest multiplicity of a cusp is 3, then projecting \( C \) from this point, say \( Q \), gives a two-sheeted covering \( C \to \mathbb{P}^1 \) with two ramification points. Hence, in this case \( C \).
has three cusps, with multiplicity sequences \((3)\) (at \(Q\)), \((2)\), \((2)\), respectively.

If all the cusps are of multiplicity 2, then \(C\) has singular points \(P, Q, R, \ldots\) with multiplicity sequences \((2)\), \((2)\), \((2)\), \ldots, where \(p + q + r + \ldots = 6\). We may assume that \(p \geq q \geq r \ldots\). Projecting from \(P\) gives a three-sheeted covering \(C \to \mathbb{P}^1\) with four ramification points. Hence, \(C\) has at most four cusps. The possibilities are as follows:

1. \(C\) has 3 cusps of type \(P = (2)\), \(Q = (2)\), \(R = (2)\).
2. \(C\) has 3 cusps of type \(P = (2)\), \(Q = (2)\), \(R = (2)\).
3. \(C\) has 3 cusps of type \(P = (2)\), \(Q = (2)\), \(R = (2)\).
4. \(C\) has 4 cusps of type \(P = (2)\), \(Q = (2)\), \(R = (2)\), \(S = (2)\).
5. \(C\) has 4 cusps of type \(P = (2)\), \(Q = (2)\), \(R = (2)\), \(S = (2)\).

Curves as in (1) and (4) do exist and can be constructed by Cremona transformations. The other cases are not possible, as can be seen by the following arguments.

(5) can be excluded since the dual curve would be a cubic with two cusps, which is impossible.

To exclude (3), denote by \(t_P\) the tangent line of \(C\) at \(P\). By the Cremona transformation \(\psi := \psi(Q, P, t_P)\) a curve \(C\) as in (3) is transformed into a quartic \(C'\) with three simple cusps \(P', Q', R'\). It can be seen that there is a tangent line at a smooth point \(S'\) of \(C'\) passing through one of the cusps, say \(Q'\). Projecting from \(Q'\) gives a two-sheeted covering \(C'' \to \mathbb{P}^1\) with three ramification points, namely \(P', R'\) and \(S'\). This contradicts the Hurwitz formula.

In the case (2), consider the blow up at \(P\), and perform an elementary transformation at the point of the proper transform of \(C\) over \(P\). Then the image of \(P\) will be a point with the multiplicity sequence \((2)\). Performing at this point another elementary transformation and blowing down to \(\mathbb{P}^2\), we arrive at the same configuration as above. Hence, also (2) is impossible. (This last transformation may also be considered as a Cremona transformation, namely in the points \(P, P'\) and \(P''\), where \(P'\) is infinitesimally near to \(P\) and \(P''\) is infinitesimally near to \(P'\).)

Similarly, using Cremona transformations for the cases 1 and 4, one can construct these curves and show that they are rectifiable and projectively unique. It is also possible to treat in the same way the rational cuspidal quintics with one or two cusps.

Finally, we give an alternative proof for the existence and uniqueness statements of Proposition 1.2. It provides a way of computing an explicit parameterization for these curves.

**Alternative proof of Proposition 1.2 (a)-(c).** For \(k = 1\) the result is known (see e.g. [Nam]). Let \(C_k\) \((k > 1)\) be a rational cuspidal plane curve of degree \(d = 2k + 3\) with three cusps \(P, Q, R\) of types \((3)\), \((2k)\), \((2k)\), and \((2)\), respectively. Since, by Bezout’s Theorem, they are not at the same line, we may chose them as \(Q(0 : 0 : 1)\), \(P(0 : 1 : 0)\), \(R(1 : 0 : 0)\). We may also chose a parameterization \(n : \mathbb{P}^1 \to C_k\) of \(C_k\) such that \((0 : 1) \mapsto Q(1 : 0) \mapsto P(1 : 1) \mapsto R\). Then, up to constant factors, this parameterization can be written as

\[
(x, y, z) = (s^{2k}t^3, s^{2k}(s-t)^2(as+bt), t^3(s-t)^2g_k(s, t)),
\]

where \(g_k(s, t)\) is a polynomial of degree \(2k\).
where \( q_k \in \mathbb{C}[s, t] \) is a homogeneous polynomial of degree \( 2k - 2 \). Let \( \Gamma \) denotes a curve parameterized as above (with \( q \) instead of \( q_k \)). It is enough to prove the following

Claim. There exist unique polynomials \( as + bt \) and \( q \) with rational coefficients, where \( q(1, 0) = 1 \), such that the multiplicity sequences of \( \Gamma \) at the points \( P, Q, R \in \Gamma = \text{start}, \) respectively, with \((3_k), (2k, 2k)\) and \((2)\).

Indeed, if this is the case, then, by the genus formula, these multiplicity sequences actually coincide resp. with \((3_k), (2k, 2k)\) and \((2)\), and so, \( C_k = \Gamma \) up to projective equivalence. This will prove the existence of the curves \( C_k \) defined over \( \mathbb{Q} \) for all \( k > 1 \), as well as their uniqueness, up to projective equivalence.

Proof of the claim. It is easily seen that, after blowing up at \( Q \), the infinitesimally near point \( Q' \) to \( Q \) at the proper transform \( \Gamma' \) of \( \Gamma \) will be a singular point of multiplicity 2 iff \( as + bt = 2s + t \). By [FlZa 2, (1.2)], under this condition the multiplicity sequence of \( \Gamma \) at \( Q \) starts with \((2k, 2k)\).

In the affine chart \((\bar{x}, \bar{z}) := (x/y, z/y)\) centered at \( P \) we have

\[
\bar{x} = \frac{t^3}{(s-t)^2(2s+t)}, \quad \bar{z} = \frac{t^3q(s, t)}{s^{2k}(2s+t)}.
\]

In the sequel we denote by the same letter \( t \) the affine coordinate \( t/s \) in \( \mathbb{P}^1 \setminus \{(0 : 1)\} \). Thus, in this affine chart in \( \mathbb{P}^1 \) centered at \((1 : 0)\) we have

\[
(\bar{x}, \bar{z}) = \left( \frac{t^3}{(t-1)^2(t+2)}, \frac{t^3}{(t+2)} \bar{q}(t) \right),
\]

where \( \bar{q}(t) = \sum_{i=0}^{2k-2} c_i t^i \) and where, by the above assumption, \( c_0 = 1 \).

After blowing up at \( P \), in the affine chart with the coordinates \((u, v)\), where \((\bar{x}, \bar{z}) = (u, uv)\), we will have

\[
(u, v) = (\bar{x}, \bar{z}/\bar{x}) = \left( \frac{t^3}{(t-1)^2(t+2)}, \bar{q}(t)(t-1)^2 \right).
\]

To move the origin to the infinitesimally near point \( P' \in \Gamma' \) of \( P \), we set

\[
(\bar{u}, \bar{v}) = (u, v - 1) = \left( \frac{t^3}{(t-1)^2(t+2)}, \bar{q}(t)(t-1)^2 - 1 \right).
\]

The following conditions guarantee that the multiplicity of the curve \( \Gamma' = at P' \) is at least 3:

\[
t^3 \mid [\bar{q}(t)(t-1)^2 - 1] \iff [\bar{q}(t)(t-1)^2 - 1]_0^\prime = [\bar{q}(t)(t-1)^2 - 1]_0^\prime = 0 \iff \bar{q}'(0) = 2, \bar{q}''(0) = 6 \iff c_1 = 2, c_2 = 3.
\]
In the case when $k = 2$ this uniquely determines the polynomial $q$:

$$q(s, t) = s^2 + 2st + 3t^2.$$ 

In what follows we suppose that $k > 2$. Assume that the conditions (15) are fulfilled. Then we have the following coordinate presentation of $\Gamma'$:

$$(\hat{u}, \hat{v}) = \left( \frac{t^3}{(t-1)^2(t+2)}, \ t^3 h(t) \right),$$

where $h(t) := \hat{q}(t)(t-1)^2 - 1)/t^3$ is a polynomial of degree $2k - 3$, which satisfies the conditions

$$(t-1)^2 | [t^3 h(t) + 1] \iff h(1) = -1, h'(1) = 3. \tag{15'}$$

Once (15') are fulfilled, one can find $\hat{q}$ as $\hat{q} = [t^3 h(t) + 1]/(t-1)^2$, and we have $\hat{q} \in \mathbb{Q}[t]$ iff $h \in \mathbb{Q}[t]$. Assume that the conditions (15) are fulfilled. Then

$$h \in \mathbb{Q}[t], \ h = \frac{t^3}{(t-1)^2(t+2)}. \ By \ [FlZa \ 2, (3.4)], \ the \ multiplicity \ sequence \ of \ \Gamma' \ at \ P' \ starts \ with \ (3)_{k-1} \ iff \ h \in \mathbb{Q}[t].$$

Let $\xi \in \mathbb{C}[[t]]$ be such that $\xi^3 = \frac{t^3}{(t-1)^2(t+2)}$. By $[FlZa \ 2, (3.4)]$, the multiplicity sequence of $\Gamma'$ at $P'$ starts with $(3)_{k-1}$ iff

$$t^3 h(t) \equiv \hat{f}(\xi^3) \mod \xi^{3(k-1)},$$

where $\hat{f} = \sum_{i=0}^{k-1} \hat{a}_i x^i \in \mathbb{C}[x]$ is a polynomial of degree $\leq k - 1$. Multiplying both sides by the unit $[(t-1)^2(t+2)]^{k-1} \in \mathbb{C}[[t]]$, we will get

$$[(t-1)^2(t+2)]^{k-1} t^3 h(t) \equiv [(t-1)^2(t+2)]^{k-1} \sum_{i=0}^{k-1} \hat{a}_i \xi^{3i} \equiv \sum_{i=0}^{k-1} \hat{a}_i t^{3i} [(t-1)^2(t+2)]^{k-1-i} \mod t^{3(k-1)}.$$ 

Since, by our assumption, $k > 1$, we should have $\hat{a}_0 = 0$, and after dividing out the factor $t^3$, we get

$$([(t-1)^2(t+2)]^{k-1} h(t) \equiv \sum_{i=0}^{k-2} \hat{a}_i' t^{3i} [(t-1)^2(t+2)]^{k-2-i} \mod t^{3(k-2)},$$

where $\hat{a}_i' = \hat{a}_{i-1}$, $i = 1, \ldots, k - 2$. In other words, we have

$$([(t-1)^2(t+2)]^{k-1} h(t) = \hat{f}(t^3, (t-1)^2(t+2)) + \hat{g}(t) t^{3(k-2)},$$

where $\hat{f}(x, y) = \hat{f}_k(x, y) := \sum_{i=0}^{k-2} \hat{a}_i' x^i y^{k-2-i}$ is a homogeneous polynomial of degree $k - 2$, and hence $\hat{g}(t) = \hat{g}_k(t) = \sum_{i=0}^{2k} \hat{b}_i t^i$ should be a polynomial of degree $2k$. Denoting $\tau = t^3$ and $\lambda = (t-1)^2(t+2) = t^3 - 3t + 2$, we have

$$\lambda^{k-1} h = \hat{f}(\tau, \lambda) + \tau^{k-2} \hat{g}.$$ 

Observe that $\hat{f}(\tau, \lambda)$ (resp. $\tau^{k-2} \hat{g}$) contains the monomial $\hat{a}_0' \tau^{k+2}$ (resp. $\hat{b}_0 \tau^{k+2}$). To avoid indeterminacy, we may assume, for instance, that $\hat{a}_0' = 0$. Then $\hat{f} = \lambda f$, where $f(x, y) := \sum_{i=0}^{k-2} a_i x^i y^{k-3-i}$, $a_i := \hat{a}_{i-1}'$, $i = 0, \ldots, k - 3$, and so

$$\lambda^{k-1} h = \lambda f(\tau, \lambda) + \tau^{k-2} \hat{g}.$$ 

13
Since \((\tau, \lambda) = 1\), we have \(\lambda | \tilde{g}\), that is, \(\tilde{g} = \lambda g\), where \(g(t) := \sum_{i=0}^{2k-3} b_i t^i\). Finally, we arrive at the relation
\[
\lambda^{k-2} h(t) = f(\tau, \lambda) + \tau^{k-2} g(t),
\]
where \(\deg f = k - 3\), \(\deg h = \deg g = 2k - 3\), and \(h\) should satisfy the conditions \((15')\). It follows that
\[
\lambda^{k-2} | [f(\tau, \lambda) + \tau^{k-2} g],
\]
and
\[
\tau^{k-2} | [f(\tau, \lambda) - \lambda^{k-2} h].
\]
Each of these conditions together with \((15')\) determines the triple of polynomials \(f, g, h\) as above in a unique way. Indeed, once \(f\) and \(g\) satisfy \((15')\) and \((16)\), we can find \(h\) as \(h = [f(\tau, \lambda) + \tau^{k-2} g]/\lambda^{k-2}\). Actually, \((16)\) is equivalent to the vanishing of derivatives of the function \(f(\tau, \lambda) + \tau^{k-2} g \in \mathbb{C}[t]\) at the point \(t = 1\) up to order \(2k - 5\) and at the point \(t = -2\) up to order \(k - 3\). This yields a system of \(3k - 6\) linear equations in the \(3k - 4\) unknown coefficients of \(f\) and \(g\); \((15')\) provides another two linear equations. That is, we have the following system:
\[
\left( f(\tau, \lambda) + \tau^{k-2} g \right)_{t=-2}^{(m)} = 0, \quad m = 0, \ldots, k - 3
\]
\[
\left( f(\tau, \lambda) + \tau^{k-2} g \right)_{t=1}^{(m)} = 0, \quad m = 0, \ldots, 2k - 5
\]
\[
\left( f(\tau, \lambda) + \tau^{k-2} g \right)_{t=1}^{(2k-4)} = -3^{k-2}(2k - 2)!
\]
\[
\left( f(\tau, \lambda) + \tau^{k-2} g \right)_{t=1}^{(2k-3)} = -3^{k-3}(k - 11)(2k - 1)!
\]
(Indeed, put \(u = t - 1\); in view of \((15')\) we have
\[
\lambda = (t - 1)^2(t + 2) = u^2(u + 3), \quad h(t) = -1 + 3u + \ldots,
\]
and hence
\[
f(\tau, \lambda) + \tau^{k-2} g(t) = \lambda^{k-2} h(t) = [u^2(u + 3)]^{k-2} h(t) =
\]
\[
u^{2k-4}(3^{k-2} + (k - 2)3^{k-3} u + \ldots)(-1 + 3u + \ldots) = u^{2k-4}(-3^{k-2} - 3^{k-3}(k - 11)u + \ldots).\]
The system \((S)\) has a unique solution iff it is so for the associated homogeneous system, say, \((S_0)\). Passing from \((S)\) to \((S_0)\) actually corresponds to passing from \(h\) to a polynomial \(h_0\) of degree \(\leq 2k - 3\) which satisfies, instead of \((15')\), the conditions
\[
h_0(1) = h_0'(1) = 0 \iff (t - 1)^2 | h_0(t) \iff h_0(t) = (t - 1)^2 \tilde{h}(t), \quad \deg \tilde{h} \leq 2k - 5. \quad (15'')\]
Thus, we have to prove that the equality
\[
\lambda^{k-2}(t - 1)^2 \tilde{h}(t) = f(\tau, \lambda) + \tau^{k-2} g(t),
\]
where $f = 0$ or $\deg f = k - 3$, $\deg g \leq 2k - 3$, and $\deg h \leq 2k - 5$, is only possible for $f = g = h = 0$. Or, equivalently, we have to show that the $5k - 8$ polynomials in $t$ in the union $T$ of the three systems:

$$T_1 := \left\{ \tau^i \lambda^{k-3-i} \right\}_{i=0,\ldots,k-3}, \quad T_2 := \left\{ t^i(t-1)^2\lambda^{k-2} \right\}_{i=0,\ldots,2k-5}, \quad T_3 := \left\{ t^i t^{k-2} \right\}_{i=0,\ldots,2k-3}$$

are linearly independent. After replacing the system $T_2$ by the equivalent one:

$$T_2' := \left\{ (t-1)^{2k-2}(t+2)^{k-2+i} \right\}_{i=0,\ldots,2k-5},$$

we will present these three systems as follows:

$$T_1 = \left\{ p_i := \tau^{k-3-i}\lambda^i = t^{3(k-2-i)}(t-1)^2(t+2)^i, \quad i = 0,\ldots,k-3 \right\}$$

$$T_2' = \left\{ p_i := (t-1)^{2k-2}(t+2)^i, \quad i = k-2,\ldots,3k-7 \right\}$$

$$T_3 = \left\{ p_i := t^i, \quad i = 3k-6,\ldots,5k-9 \right\}.$$

Denote $P = \text{span}(T_1, T_2, T_3) = \text{span}(T_1, T_2', T_3)$. Note that $\deg p \leq 5k - 9$ for all $p \in P$, that is, $\dim P \leq 5k - 8$. Consider the following system of $5k - 8$ linear functionals on $P$:

$$\varphi_i : p \mapsto p^{(i)}(-2), \quad i = 0,\ldots,3k-7,$$

$$\varphi_i : p \mapsto p^{(i)}(0), \quad i = 3k-6,\ldots,5k-9.$$  

It is easily seen that the matrix $M := (\varphi_i(p_j))_{i,j=0,\ldots,5k-9}$ is triangular with non-zero diagonal entries. This proves that, indeed, $\text{rang } T = \dim P = 5k - 8$, as stated.

The coefficients of the system $(S)$ being integers, its unique solution is rational, i.e. the polynomials $f$ and $g$ are defined over $\mathbb{Q}$. It follows as above that the polynomials $h$ and $q$ are also defined over $\mathbb{Q}$. This completes the alternative proof of Proposition 1.2.

\begin{itemize}
\item Remarks. (1) In principle, the method used in the proof allows to compute explicitly parameterizations of the curves $C_k$. For instance, we saw above that for $k = 2$ a parameterization of $C_2$ is given by the choice

$$q_2(s, t) := s^2 + 2st + 3t^2, \quad a := 2, \quad b := 1.$$  

(2) We have to apologize for a pity mistake in Lemma 4.1(b) [FlZa 2, Miscellaneous] (this does not affect the other results of [FlZa 2], besides only the immediate Corollary 4.2).
\end{itemize}

\section*{References}

[Co] J. L. Coolidge. A Treatise on Algebraic Plane Curves. Oxford Univ. Press, Oxford, 1928
[Fe] T. Fenske. Rationale kuspidale Kurven im $\mathbb{P}^2$. Diplomarbeit Bochum 1996

[FlZa 1] H. Flenner, M. Zaidenberg. $\mathbb{Q}$–acyclic surfaces and their deformations. Contemporary Mathem. 162 (1994), 143–208

[FlZa 2] H. Flenner, M. Zaidenberg. On a class of rational cuspidal plane curves. Manuscr. Mathem. 89 (1996), 439-460; E-print [alg-geom/9507004]

[Fu] T. Fujita. On the topology of non complete algebraic surfaces. J. Fac. Sci. Univ. Tokyo, Sect.IA 29 (1982), 503–566

[Ii 1] Sh. Iitaka. On logarithmic Kodaira dimension of algebraic varieties. In: Complex Analysis and Algebraic Geometry, Cambridge Univ. Press, Cambridge e.a., 1977, 175–190

[Ii 2] Sh. Iitaka. Theory of curves, new and old. In: Proc. of the Algebraic Geometry Seminar, Singapore 1987, 5–13

[Ii 3] Sh. Iitaka. Noether inequality and minimal plane curves. In: Algebraic Geometry. Proc. Conf., Saitama Univ., March 15–17, 1995, Urawa, 1995, 26–37

[LiZa] V. Lin, M. Zaidenberg. An irreducible simply connected curve in $\mathbb{C}^2$ is equivalent to a quasihomogeneous curve. Soviet Math. Dokl. 28 (1983), 200-204

[MaSa] T. Matsuoka, F. Sakai. The degree of rational cuspidal curves. Math. Ann. 285 (1989), 233–247

[MKM] N. Mohan Kumar, M. P. Murthy. Curves with negative selfintersection on rational surfaces. J. Math. Kyoto Univ. 22 (1983), 767–777

[Nag] M. Nagata. On rational surfaces. I. Mem. Col. Sci. Univ. Kyoto 32 (1960), 351–370

[Nam] M. Namba. Geometry of projective algebraic curves. Marcel Dekker, N.Y. a.e., 1984

[OrZa 1] S. Orevkov, M. Zaidenberg. On the number of singular points of plane curves. In: Algebraic Geometry. Proc. Conf., Saitama Univ., March 15–17, 1995, Urawa, 1995, 156–177

[OrZa 2] S.Y. Orevkov, M.G. Zaidenberg. On rigid rational cuspidal plane curves. Russian Math. Survey 51:1 (1996), 149–150

[Ra] C.P. Ramanujam. A topological characterization of the affine plane as an algebraic variety. Ann. Math. 94 (1971), 69-88

[Sa] F. Sakai. Singularities of plane curves. In: Geometry of Complex Projective Varieties, Seminars and Conferences 9, Mediterranean Press, Rende, 1993, 257-273

[tD 1] T. tom Dieck. Symmetric homology planes. Math. Ann. 286 (1990), 143–152
[tD 2] T. tom Dieck. Optimal rational curves and homology planes. Preprint Mathematica G=96ttingensis 9 (1992), 1–22

[Wa] I. Wakabayashi. On the logarithmic Kodaira dimension of the complement of a curve in $\mathbb{P}^2$. Proc. Japan Acad. 54(A) (1978), 157–162

[Yo] H. Yoshihara. Plane curves whose singular points are cusps and triple coverings of $\mathbb{P}^2$. Manuscr. Math. 64 (1989), 169-187

[Za] M. Zaidenberg. On Ramanujam surfaces, $\mathbb{C}^{**}$-families and exotic algebraic structures on $\mathbb{C}^n$, $n \geq 3$. Trans. Moscow Math. Soc. 55 (1994), 1–56

Hubert Flenner
Fakultät für Mathematik
Ruhr Universität Bochum
Geb. NA 2/72
Universitätsstr. 150
44780 BOCHUM
Germany
e-mail: Hubert.Flenner@rz.ruhr-uni-bochum.de

Mikhail Zaidenberg
Université Grenoble I
Institut Fourier
UMR 5582 CNRS-UJF
BP 74
38402 St. Martin d’Hères–cédex
France
e-mail: zaidenbe@ujf-grenoble.fr