Cubes and Their Centers

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February 10, 2015

Abstract

We study the relationship between the sizes of sets $B, S \subseteq \mathbb{R}^n$ where $B$ contains the $k$-skeleton of an axes-parallel cube around each point in $S$, generalizing the results of Keleti, Nagy, and Shmerkin [5] about such sets in the plane. We find sharp estimates for the possible packing and box counting dimensions of $B$ and $S$. These estimates follow from related cardinality bounds for sets containing the discrete skeleton of cubes around a finite set of a given size. The Katona-Kruskal theorem from hypergraph theory plays an important role. We also find partial results for the Hausdorff dimension and settle an analogous question for the dual polytope of the cube, the orthoplex.

1 Introduction and Statements of Results

1.1 Introduction

In [5] the authors find sharp bounds for the Hausdorff, box counting, and packing dimensions of sets $S, B \subseteq \mathbb{R}^2$ where $B$ contains either the vertices or boundary of an axes-parallel square around every point in $S$, and cardinality bounds for finite sets satisfying discrete versions of these conditions. Their results are summarized in the following table. If $S$ has size $s$ for the given notion of size, then a sharp lower bound for the size of $B$ is given in terms of $s$:

| Notion of Size | Vertex problem (0-skeleton of a 2-cube) | Boundary problem (1-skeleton of a 2-cube) |
|----------------|----------------------------------------|------------------------------------------|
| $\dim_P$       | $\frac{3}{4}s$                         | $1 + \frac{3}{8}s$                      |
| $\overline{\dim}_B$ | $\frac{3}{4}s$                         | max$\{1, \frac{7}{8}s\}$               |
| $\underline{\dim}_B$ | $\frac{5}{8}s$                         | max$\{1, \frac{7}{8}s\}$               |
| $\dim_H$       | max$\{0, s - 1\}$                      | $1$                                      |
| $| \cdot |$         | $\Omega(s^{\frac{3}{2}})$              | $\Omega(s^{\frac{5}{2} - \epsilon})$   |
The main results are the generalizations of the bounds for box counting and packing dimensions in $[5]$. We want to find similar bounds for sets $B, S \subseteq \mathbb{R}^n$ where $B$ contains the $k$-skeleton of a $n$-cube around each point in $S$. That is, the goal of this paper is to extend each line of the above table into an $\mathbb{N} \times \mathbb{N}$ array. We manage this in every case except for the Hausdorff dimension.

The authors in $[5]$ were inspired by Bourgain $[1]$ and Marstrand’s $[8]$ results about packing circles, which in turn completed work done by E. Stein on $n$-spheres for $n \geq 3$ $[10]$. The problem we study, the original problem in $[5]$, and the problems settled by Stein, and (independently) by Bourgain and Marstrand are members of the abundant family of “Kakeya type” problems. The problem of minimizing size, in the sense of some measure or fractal dimension, over some family of sets in $\mathbb{R}^d$ occurs commonly in geometric measure theory and other areas of analysis. The discrete analogue of these problems is the domain of the well-studied field of extremal combinatorics. This paper and $[5]$ illuminate some relationships between these two areas of research.

1.2 Notation

Throughout, lowercase latinate letter stand for integers unless otherwise specified. The expression $[a, b]$ stands for the discrete interval $\{a, a+1, ..., b-1, b\}$ and $\binom{n}{k}$ for the set of $k$-element subsets of $[1,n]$. For $x \in \mathbb{R}^n, I \in \binom{n}{k}$, $xI$ is the vector in $\mathbb{R}^k$ formed by taking the entries of $x$ indexed by $I$.

We use the convention that $0 \notin \mathbb{N}$. A cube will alway mean a cube with all sides parallel to the axes unless otherwise specified, that is, a cube is a set of the form $x + \prod_{i=1}^n [a, b]$ for some $x, a < b \in \mathbb{R}$. The $k$-skeleton of a cube $x + [a, b]^n$ is the set $x + \bigcup_{I \in \binom{n}{k}} \prod_{i=1}^n A_{I,i}$, where $A_{I,i} = [a, b]$ if $i \in I$ and $\{a, b\}$ otherwise.

A discrete cube is a set of the form $x + \lfloor [0, r] \rfloor^n$ for some $x \in \mathbb{R}$, and $r \in \mathbb{N}$. The discrete $k$-skeleton of a cube $x + \lfloor [0, r] \rfloor^n$ is then $x + \bigcup_{I \in \binom{n}{k}} \prod_{i=1}^n A_{I,i}$ where $A_{I,i} = \lfloor [0, r] \rfloor$ if $i \in I$ and $\{0, r\}$ otherwise.

1.3 Main Results

The main results are the generalizations of the bounds for box counting and packing dimensions in $[5]$:  

**Theorem 1.1.** For any $0 \leq k < n$ and any sets $B, S \subseteq \mathbb{R}^n$ such that $B$ contains the $k$-skeleton of a cube around every point in $S$,

1. $\dim_P(B) \geq k + \frac{(n-k)(2n-1)}{2n^2} \dim_P(S)$, and 
2. $\dim_B(B) \geq \max \left\{ k, \left( k + \frac{(n-k)(2n-1)}{2n^2} \right) \dim_B(S) \right\} = \max \left\{ k, (1 - \frac{n-k}{2n^2}) \dim_B(S) \right\}$.
We also have constructions showing that these bounds are sharp in the sense of the following theorem:

**Theorem 1.2.** Given any $0 \leq k < n$, $s \in [0, n]$, there are compact $B, S, B', S' \subseteq \mathbb{R}^n$ where $\dim_P(S) = \dim_B(S') = s$, $B$ and $B'$ contain the $k$-skeleton of a cube around every point in $S$ and $S'$ respectively, and

1. $\dim_P(B) = k + \frac{(n-k)(2n-1)}{2n^2}s$, and
2. $\dim_B(B') = \max \{ (1 - \frac{n-k}{2n^2})s, k \}$.

The above results are analytic extensions of the following four theorems for discrete cubes:

**Theorem 1.3.** If $B, S \subseteq \mathbb{Z}^n$, and $B$ contains the discrete $k$-skeleton of a cube around every point in $S$, then for every $\alpha < 1 - \frac{n-k}{2n^2}$

$$|B| \geq \Omega(|S|^\alpha).$$

**Theorem 1.4.** If $0 < \ell \leq n$, $A \subseteq \mathbb{R}^\ell$, $S \subseteq \mathbb{R}^n$ are finite sets such that

$$\forall x \in S \exists r \in \mathbb{R}^+ \forall I \in \left[\begin{array}{c} \ell \\ \ell \end{array}\right] \forall \sigma \in \{-1, 1\}^\ell : x_I + r\sigma \in A,$$

then $|A| \geq \Omega(|S|^{(2n-1)/2n^2})$.

Geometrically above means that $A$ contains the vertices of an $\ell$-cube of the same size around each point in any projection of $S$ onto an axis-spanned plane. Intuitively, $A$ collects the non-(n-k)-face cofactors of cubes around each point in $S$. We also have constructions showing these are sharp:

**Theorem 1.5.** For every $0 \leq k < n$ and every $p \geq 0$, there are $B, S \subseteq \mathbb{Z}^n$ such that $B$ contains the discrete $k$-skeleton of a cube around every point in $S$, $|S| = p$, and $|B| \leq O \left( |S|^{1 - \frac{n-k}{2n^2}} \right)$.

**Theorem 1.6.** For every $0 < \ell \leq n$ and every $p \geq 0$, there are $B \subseteq \mathbb{Z}^\ell$, $S \subseteq \mathbb{Z}^n$ such that $|B| \leq O(|S|^{(2n-1)/(2n^2)})$, $|S| = p$, and

$$\forall x \in S \exists r \in \mathbb{R}^+ \forall I \in \left[\begin{array}{c} \ell \\ \ell \end{array}\right] \forall \sigma \in \{-1, 1\}^\ell : x_I + r\sigma \in B.$$

We have found a general bound for the Hausdorff dimension and shown its sharpness in several cases.

**Theorem 1.7.** If $B, S \subseteq \mathbb{R}^n$ and $B$ contains the $k$-skeleton of a cube around every point in $S$, then $\dim_H(B) \geq \max\{\dim_H(S) - 1, k\}$.

**Theorem 1.8.** For $0 \leq k < n$, $s \in [n-k, n]$, there are $G_\delta$ sets $B, S \subseteq \mathbb{R}^n$ where $B$ contains the $k$-skeleton of an $n$-cube around each point in $S$, $\dim_H(B) = \max\{k, s - 1\}$ and $\dim_H(S) = s$.

Further, if $k = 0$, there are $B, S$ as above for $s \in [n - 1, n]$. 


We conjecture that constructions as in the theorem above can be found for all $s$. These results are summarized in the following table. For $S$ of size $s$, a sharp lower bound for the size of $B$ is given in terms of $s$:

| Notion of Size | $k$-skeleton of an $n$-cube |
|----------------|--------------------------------|
| $\dim_P$      | $k + \frac{(n-k)(2n-1)}{2n^2}s$ |
| $\dim_B$      | $\max\{k, (1 - \frac{n-k}{2n^2})s\}$ |
| $\dim_H$      | $\max\{k, (1 - \frac{n-k}{2n^2})s\}$ |
| $|\cdot|$      | $\Omega\left(s^{1-\frac{n-k}{2n^2}-\epsilon}\right)$ |

We also have bounds for sets $B$ in $\mathbb{R}^n$ containing the vertices of the dual polytope of the cube, the orthoplex, around every point in a set $S$ of a given dimension:

**Theorem 1.9.** Let $B, S \subseteq \mathbb{R}^n$ such that for all $x \in S$ there is some $r \in \mathbb{R}^+$ such that $x \pm re_i \in B$, where $e_i$ is the $i^{th}$ standard basis vector. Then the following hold:

1. $\dim_H(B) \geq \dim_H(S) - 1$,
2. if $B, S$ are finite, then $|B| \geq \Omega(|S|^{\frac{2n-1}{2n}})$,
3. $\dim_B(B) \geq \frac{2n-1}{2n} \dim_B(S)$, and
4. $\dim_P(B) \geq \frac{2n-1}{2n} \dim_P(S)$.

And, these bounds are sharp:

**Theorem 1.10.** For any $n, p \in \mathbb{N}$ and $s \in [0, n]$, we can find compact sets $B_H, B_B, B_P, B_f, S_H, S_B, S_P, S_f \subseteq \mathbb{R}^n$ such that $B_X$ contains the vertices of an orthoplex around each point in $S_X$ for any $X \in \{H, B, P, f\}$, $\dim_X(S_X) = s$ for $X \in \{H, B, P\}$, $|S_f| = p$, and

1. $\dim_H(B_H) = \max\{0, \dim_H(S_H) - 1\}$,
2. $|B_f| = O(|S_f|^{\frac{2n-1}{2n}})$ (in particular, $S_f, B_f$ are finite),
3. $\dim_B(B_B) = \frac{2n-1}{2n} \dim_B(S_B)$, and
4. $\dim_P(B_P) = \frac{2n-1}{2n} \dim_P(S_P)$.
1.4 Structure of the Paper

The rest of this paper is organized as follows. In section 2, we prove the discrete results. In section 3, we collect results in dimension theory which we will require. In section 4, we give constructions showing the sharpness of the bounds for packing dimension and of the bound for box counting dimension in the case $k = 0$ (the vertex case). In section 5, we give a construction showing the sharpness of the box counting bound. In section 6, we give a construction showing the sharpness of Hausdorff dimension and pose a conjecture about this case. In section 7, we cover what is known about Hausdorff dimension and pose a conjecture about this case. In section 8, we settle the vertex problem for the orthoplex.

2 Discrete Results

The discrete results all follow more or less as corollaries from two lemmas: a construction based on $i$-ary expansions and a bound relating sets in dimension $n$ and $\ell$ for any $\ell < n$. We give constructions first. The lemma below generalizes [5, Lemma 4.3]

**Lemma 2.1 (Digit Construction)**. For every $n, i$, there is a $D_{i,n} \subseteq \left[-(2i)^{2n}, (2i)^{2n}\right]$ of size $O(i^{2n})$ such that, for every $x_1, \ldots, x_n \in \left[1, i^{2n} - 1\right]$, there is some positive $r$ such that, for every $0 < j \leq n$, $x_j \pm r \in D_{i,n}$.

**Proof.** Note that it is enough to guarantee a nonzero $r$ with each $x_i \pm r \in D_{i,n}$.

The discrete interval $\left[1, i^{2n} - 1\right]$ is the set of integers which can be written in base $i$ with $2n$ digits and at least 1 nonzero digit. Informally, $D_{i,n}$ is the set of numbers with at least one 0 in their base $i$ expansion if we allow for negative digits.

$$D_{i,n} = \left\{ \sum_{j=0}^{2n-1} a_j i^j : a_j \in [2(1-i), 2(i-1)], \prod_j a_j = 0 \right\}.$$ 

Note that $|D_{i,n}| \leq (4i-4)^{2n} - (4i-5)^{2n} = O(i^{2n-1})$. Given $n$ numbers $x_1, \ldots, x_n \in \left[1, i^{2n} - 1\right]$, denote the terms of their $i$-expansions by $x_j = x_{j,2n-1} \ldots x_{j,0}$, that is

$$x_j = \sum_{m=0}^{2n-1} x_{j,m} i^m \text{ with } 0 \leq x_{j,m} < i,$$

and let $r = x_{n-1,2n-2}0 \ldots 0x_{0,0} - x_{n-1,2n-1} \ldots 0x_{0,1}0$, that is

$$r = \left( \sum_{m=0}^{n-1} x_{m,2m} i^{2m} \right) - \left( \sum_{m=0}^{n-1} x_{m,2m+1} i^{2m+1} \right).$$
By permuting the $x_i$, we may assume that at least one $x_i, 2i$ or $x_{i, 2i+1}$ is nonzero, and so $r$ is nonzero. For any $j$, $x_j + r = \left(\sum_{m=0}^{n-1}(x_{j, 2m} + x_{m, 2m})i^{2m}\right) + \left(\sum_{m=0}^{n-1}(x_{j, 2m+1} - x_{m, 2m+1})i^{2m+1}\right)$. We have that $|x_{j, m} \pm x_{\lfloor m/2, \rfloor, m}| \leq 2(i - 1)$, and $x_{j, 2j+1} - x_{j, 2j+1} = 0$, so $x_j + r \in D_{i, n}$. And similarly, $x_j - r \in D_{i, n}$.

**Corollary 2.2** (Theorem 1.6 in the Introduction). For every $n \geq \ell > 0$ and $p \geq 0$, there are $B \subseteq \mathbb{Z}^\ell$, $S \subseteq \mathbb{Z}^n$ such that $|B| \leq O(|S|^{(2n-1)/(2n^2)})$, $|S| = p$, and

$$\forall x \in S \exists r \in \mathbb{R}^\ell \forall I \in \left[\begin{array}{c} n \\ \ell \end{array}\right] \forall \sigma \in \{-1, 1\}^\ell : x_I + r\sigma \in B.$$ 

**Proof.** If $p = (i^{2n-1})^n$ for some $i$, take $B = (D_{i, n})^\ell$ and $S = [[1, i^{2n} - 1]]^n$. For intermediate values of $p$, interpolate by taking $i$ to be the smallest such that $p \leq (i^{2n} - 1)^n$ and let $S$ be a subset of $[[1, i^{2n} - 1]]^n$ of size $p$, and $B = (D_{i, n})^\ell$. Since $(i^{2n} - 1)^n - (i^{2n} - 1)^n \leq 2n^2 i^{2n}$, we have $(i^{2n} - 1)^n \leq O(p)$. The desired inequality follows directly.

**Corollary 2.3** (Theorem 1.5 in the Introduction). For every $n > k \geq 0$ and $p \geq 0$, there are $B, S \subseteq \mathbb{Z}^n$ such that $B$ contains the discrete $k$-skeleton of a cube around every point in $S$, $|S| = p$, and $|B| \leq O\left(|S|^{1 - \frac{2k^2}{m}}\right)$.

**Proof.** If $p = i^{2n^2}$, take $B = \bigcup_{J \subseteq \left[\begin{array}{c} n \\ \ell \end{array}\right]} \prod_{j=1}^{n} A_j$, where

$$A_j = \begin{cases} D_{i, n} & j \notin J \\ [[0, i^{2n} - 1]] & j \in J \end{cases},$$

and $S = [[0, i^{2n} - 1]]^n$. Otherwise, interpolate as above.

To get bounds showing that the above constructions are optimal, we’ll need a theorem comparing sets in $\mathbb{R}^\ell$ to sets in $\mathbb{R}^n$, the $(n, \ell)$-dimensional lemma, Lemma 2.4. To get this, we start by proving the case where $n = \ell$, the $n$-dimensional lemma, Lemma 2.4. This is a generalization of \cite{5} Theorem 4.1. The idea of the proof there is decompose the square into two intersecting lines and use a bound in $\mathbb{R}$ to get a bound in $\mathbb{R}^2$. The argument below similarly decomposes the $n$-cube into a line and a $(n-1)$-plane and proceeds by induction induction.

We make a slightly more opaque statement of the lemma because this will be useful for resolving questions about the orthoplex and because the proof is more natural.

**Lemma 2.4** (n-dimensional Lemma). For any $n > 0$, $B, S \subseteq \mathbb{R}^n$, if there are $v_1, \ldots, v_n$ linearly independent vectors such that for any $x \in S$ and $0 \leq i \leq n$, there is some $r \in \mathbb{R}^+$ such that $x \pm r v_i \in B$ (and in particular, if $B$ contains the vertices of an $n$-cube around each point in $S$), then $|B| \geq \Omega\left(|S|^{(2n-1)/(2n)}\right)$. 

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We will prove that, in fact, \(|B| \geq \frac{1}{2^n} |S|^{(2n-1)/(2n)}\). We proceed by induction.

If \(n = 1\), then any point in \(S\) is the midpoint of two points in \(B\), so \(|S| \leq (|B|)^2 \leq |B|^2\).

Suppose that the bound holds for \(\mathbb{R}^n\); we want to verify it for \(\mathbb{R}^{n+1}\). Let \(c = \frac{1}{2^n}\). Consider the planes \(P_1, ..., P_k\) and lines \(\ell_1, ..., \ell_m\) through points in \(S\) with each \(P_i\) parallel to \(\text{span}(v_1, ..., v_n)\) and \(\ell_i\) parallel to \(v_{n+1}\). Let \(p_i = |S \cap P_i|\) and \(q_i = |S \cap \ell_i|\).

Note that for each \(s \in P_i \cap S\), there is some \(r\) such that \(P_i \cap S\) contains \(s \pm rv_i\) for \(1 \leq i \leq n\). So, by the inductive hypothesis:

\[
|B \cap P_i| \geq 2c p_i^{(2n-1)/(2n)} \quad \text{and} \quad |B \cap \ell_i| \geq q_i^{1/2},
\]

so that

\[
|B| \geq 2c \sum_{i \leq k} p_i^{(2n-1)/(2n)} \quad \text{and} \quad |B| \geq \sum_{i \leq m} q_i^{1/2}.
\]

Let \(a_1, ..., a_w\) be the \(p_i\) which are less than \(|S|^{n/(n+1)}\). There are two cases we need to consider.

Case 1: \(a_1 + ... + a_w \geq \frac{1}{2} |S|\). Here we have the following:

\[
|B| \geq 2c \sum_{i \leq k} a_i^{2n-1} \geq 2c \sum_{i \leq w} a_i^{2n-1} \geq 2c \sum_{i \leq w} \frac{a_i}{|S|^{2/(n+1)}} \geq c |S|^{2/(n+1)},
\]

Case 2: \(a_1 + ... + a_w < \frac{1}{2} |S|\). Here, for each plane \(P_i\) where \(p_i > |S|^{n/(n+1)}\) color \(P_i \cap S\) blue. Note that there are at most \(|S|^{1/(n+1)}\) planes with blue points. Let \(q'_i\) be the number of blue points on \(\ell_i\). Since the \(\ell_i\) don’t lie in any \(P_i\) we have

\[
q'_i \leq |S|^{1/(n+1)} \quad \text{and} \quad q'_i \leq q_i.
\]

Also, since every point in \(S\) has some line \(\ell_i\) going through it

\[
\sum_{i \leq m} q'_i \geq |S| - a_1 - ... - a_w \geq \frac{1}{2} |S|,
\]

so that

\[
|B| \geq \sum_{i \leq m} \sqrt{q'_i} \geq \sum_{i \leq m} \frac{q'_i}{|S|^{1/(n+1)}} \geq \frac{1}{2} |S|^{2/(n+1)} \geq c |S|^{2/(n+1)}.
\]

To deduce the \(n, \ell\)-lemma, the generalization of theorem 4.2 in [5], we’ll appeal to Lovasz’s corollary of the Katona-Kruskal theorem:
Theorem 2.5 (Katona [4], Kruskal [8]). Let $Y$ be a subset of $\binom{\ast}{\ast}$ for some $a, b$, let $X$ be the $(b - c)$ element subsets of sets in $Y$, and let
\[ |Y| = \binom{n_1}{k} + \binom{n_2}{k - 1} + \cdots + \binom{n_j}{k - j}, \]
where the $n_i$ are a sequence of nonnegative decreasing integers. Then,
\[ |X| \geq \binom{n_1}{k - c} + \binom{n_2}{k - c - 1} + \cdots + \binom{n_j}{k - c - j}. \]

Corollary 2.6 (Lovasz [7]). Let $X, Y, a, b, c$ be as above, and let $x \in \mathbb{R}$ be such that $|Y| = \binom{x}{a}$, then $|X| \geq \binom{x}{b}$, where $\binom{x}{k} = \frac{2(x - 1)(x - k + 1)}{x} - 1$.

A short proof of these is given in [3].

Theorem 2.7 ($n, \ell$-dimensional Main Lemma). If $A \subseteq \mathbb{R}^\ell$, $S \subseteq \mathbb{R}^n$, and
\[ \forall x \in S \, \exists r \in \mathbb{R}^+ \, \forall I \in \binom{\ast}{\ast} \, \forall \sigma \in \{-1, 1\}^\ell : \, x_I + r \sigma \in A, \quad (2) \]
then $|A| \geq \Omega\left(|S|^{\ell(2n - 1)/(2n)}\right)$.

Proof. The condition (2) implies that $B = \{ x \in \mathbb{R}^n : \forall I \in \binom{\ast}{\ast} \, x_I \in A \}$ and $S$ satisfy the hypotheses of the $n$-dimensional lemma, Lemma 2.4; indeed, let $x \in S$, then by linearity of projections
\[
\exists r \in \mathbb{R}^+ \forall \sigma \in \{-1, 1\}^n \forall I \in \binom{\ast}{\ast} : \, x_I + r \sigma \in A \quad \iff \quad \exists r \in \mathbb{R}^+ \forall \sigma \in \{-1, 1\}^n \forall I : \, (x + r \sigma)_I \in A \quad \iff \quad \exists r \in \mathbb{R}^+ \forall \sigma \in \{-1, 1\}^n : \, x + r \sigma \in B.
\]
So, by the previous lemma we have
\[ |B| \geq \Omega(|S|^{\ell(2n - 1)/(2n)}). \quad (3) \]
To compare $B$ and $A$, we can make the following simplifications. By translating $A$ appropriately we may assume that for any $x \in A$, the $x_i$ are distinct. Let the symmetric group $S_\ell$ act on $\mathbb{R}^n$ by $\pi(x_1, \ldots, x_n) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. Taking the orbit of $A$ under $S_\ell$ only increase size by a factor of $\ell!$, so we may assume that if $x \in A$, then $(x_{\pi(1)}, \ldots, x_{\pi(\ell)}) \in A$ for any $\pi \in S_\ell$. Now let
\[ \hat{A} = \{ \{x_1, \ldots, x_\ell\} : x \in A \} \quad \hat{B} = \{ \{x_1, \ldots, x_n\} : x \in B \} \]
\[ |A| \geq |\hat{A}| \quad |B| \leq n!|\hat{B}|. \]
Note that $|A| \geq |\hat{A}|$ and $|B| \leq n!|\hat{B}|$. If $C \in \hat{B}$ and $C' \subset C$ with $|C'| = \ell$, then $C' \in A$, so we may apply the Katona-Kruskal-Lovasz theorem to $\hat{B}$ and $\hat{A}$.\footnote{It is not difficult to see that any integer has a unique such representation. This is not needed in the following.}
Let \( x \in \mathbb{R} \) be such that \( \binom{x}{n} \geq |\hat{B}| \). Then \( |A| \geq \binom{x}{\ell} \geq cx^{\ell} \).

\[
|B| \leq n!|\hat{B}| = n! \binom{x}{n} \leq x^n.
\]

Combining this with \( \mathfrak{U} \), we get
\[
|A| \geq \Omega \left( |B|^\frac{1}{n} \right) \geq \Omega \left( |S|^{\frac{(2n-1)(n-k)}{2n^2}} \right).
\]

We will see that the packing dimension estimate reduces to almost exactly the above theorem. The next theorem finishes off the discrete problem and will later be used to give a bound for box counting dimension. The argument given below generalizes an unpublished proof by Dániel T. Nagy.

**Theorem 2.8** (Theorem 1.3 in the Introduction). If \( B, S \subseteq \mathbb{Z}^n \), and \( B \) contains the discrete \( k \)-skeleton of a cube around each point in \( S \), then, for every \( \alpha < 1 - \frac{n-k}{2n^2} \)

\[
|B| \geq \Omega(|S|^\alpha).
\]

**Proof.** Let \( R(\alpha) = \frac{2n^2-(2n-1)(n-k)}{(2n)! (k+n(1-\alpha))} \), \( f(\alpha) = R(\alpha)k + \frac{(2n-1)(n-k)}{2n^2} \), and \( \beta = 1 - \frac{n-k}{2n^2} \). Call \( \alpha \) good if, whenever \( B \) contains the discrete \( k \)-skeleton of a cube around each point in \( S \), then \( |B| \geq \Omega(|S|^\alpha) \). We will show that if \( \alpha \) is good, so is \( f(\alpha) \) and that the limit of \( f^n(0) = \beta \). One can check that \( R(\alpha) \) has been chosen so that

\[
f(\alpha) = 1 - (R(\alpha)n(1-\alpha)).
\]

Given sets \( S, B \) as in the statement, call a cube in \( B \) large if it has side length at least \( |S|^{\frac{R(\alpha)}{2}} \) and small otherwise. Note that the discrete construction, Corollary \( \mathfrak{U} \), shows that any good \( \alpha \) is at most \( \beta \). So, let \( \alpha \) be good.

**Case 1:** Suppose at least \( \frac{|S|}{2} \) points of \( S \) are centers of large cubes. For any set \( X \subseteq \mathbb{R}^n \) and \( I \in \left\lfloor \frac{n}{\ell} \right\rfloor \), let \( X_I = \{x_I : x \in X\} \). Let \( V \) be the of vertices of large cubes in \( B \) and \( A = \bigcup_{I \in \left\lfloor \frac{n}{\ell} \right\rfloor} V_I \). Let \( S_I \) be the set of centers of the large cubes. Then, \( A, S_I \) satisfy the \( n, (n-k) \)-dimensional lemma (Lemma \( \mathfrak{U} \) with \( l = n - k \)), so

\[
|A| \geq \Omega \left( |S_I|^{\frac{(n-k)(2n-1)}{2n^2}} \right) \geq \Omega \left( |S|^{\frac{(2n-1)(n-k)}{2n^2}} \right).
\]

At least one \( V_I \) will contain \( \binom{n}{k}^{-1} |A| \geq \Omega \left( |S|^{\frac{(2n-1)(n-k)}{2n^2}} \right) \) points. The set \( B \) contains \( \Omega(|V_I|) k \)-faces of large cubes whence
\[ |B| \geq \Omega(\lambda_1(|S|^{kR(\alpha)})) \geq \Omega\left(|S|^{R(\alpha)k + \frac{2^{n-1}(\alpha-1)}{2^{n-1}}}ight) = \Omega(|S|^{f(\alpha)}). \]

**Case 2:** Suppose that \( \frac{|S|}{2} \) points of \( |S| \) are centers of small cubes in \( B \). Denote the set of these centers by \( S_2 \). Divide \( \mathbb{Z}^n \) into cubes of side length \( |S|^{R(\alpha)} \). Assume that these partitions contain \( x_1, ..., x_m \) points of \( S_2 \) (ignoring the empty partitions):

\[ 1 \leq x_i \leq |S|^{nR(\alpha)}, \quad \sum x_i = |S_2| \geq \frac{|S|}{2}. \]

For each point \( b \) of \( B \) consider the large cubes containing \( b \). The centers of these \( n \)-cubes cannot be in more than \( 2^n \) partitions. Let \( Y_i \) be the union of large cubes in \( B \) with centers counted in \( x_i \). We have \( \sum |Y_i| < 2^n |B| \), and \( x_1^n < O(|Y_i|) \) since \( \alpha \) is good. So,

\[ 2^n |B| \geq \sum \Omega(x_i^n) \geq \Omega\left(\frac{\sum x_i}{|S|^{R(\alpha)n(1-\alpha)}}\right) \geq \Omega(|S|^{1-(R(\alpha)n(1-\alpha))}) = \Omega(|S|^{f(\alpha)}). \]

Algebra shows that \( f \) has two fixpoints, 1 and \( \beta \). By inspection \( f \) is monotone on the interval \([0, \beta] \subset [0, 1]\). We then have

\[ 0 \leq f(0) \Rightarrow f^n(0) < f^{n+1}(0) \]

and

\[ 0 < \beta \Rightarrow f^n(0) \leq f^n(\beta) = \beta. \]

So, the sequence \( f^n(0) \) must converge to \( \beta \).

### 3 Dimension Theory Primer

Before proving results about dimension, we collect results from the general theory that we will use. The first is an equivalent characterization of packing dimension.

**Theorem 3.1** (Packing Dimension Equivalents). The packing dimension and modified box counting dimension are equivalent. That is, for \( A \subseteq \mathbb{R}^n \)

\[ \dim_P(A) = \underline{\dim}_{MB}(A) = \inf \left\{ \sup_{i \in \mathbb{N}} \underline{\dim}_B(A_i) : A \subseteq \bigcup_{i \in \mathbb{N}} A_i \right\}. \]

See, for instance, [2, Proposition 3.8] for a proof. Note that, since box counting dimension is finitely stable (see, for instance [2, Section 3.2]), we can require ascending unions:

\[ \dim_P(A) = \inf \left\{ \sup_{i \in \mathbb{N}} \underline{\dim}_B(A_i) : A \subseteq \bigcup_{i \in \mathbb{N}} A_i, A_i \subseteq A_{i+1} \right\}. \]

It is helpful to have comparisons between the different notions of dimensions.
**Theorem 3.2** (Dimension Inequalities). For $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^\ell$, the following inequalities hold

$$\dim_H(A) \leq \dim_P(A), \quad \dim_B(A) \leq \dim_B(A).$$

In general, $\dim_P(A)$ and $\dim_B(A)$ are not comparable. For proof, see e.g., [9, Theorem 8.10]. Finally, we will want to compare the dimensions of products of sets.

**Theorem 3.3** (Product Rules). For $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^\ell$, the following inequalities hold:

1. $\dim_H(A) + \dim_H(B) \leq \dim_H(A \times B) \leq \dim_H(A) + \dim_B(B)$
2. $\dim_P(A) + \dim_H(B) \leq \dim_P(A \times B) \leq \dim_P(A) + \dim_P(B)$
3. $\dim_B(A \times B) \leq \dim_B(A) + \dim_B(B)$

See [11] for a proof (2) and [2, Section 7.1] for proofs of (1) and (3). All of the inequalities in this section can be strict.

### 4 Packing and Box Counting Estimates

The $(n, \ell)$-dimensional lemma, and discrete bounds give rise to continuous analogues. The proofs sketched in [5] apply almost directly. For completeness, we fill in some details below.

**Theorem 4.1** (Theorem 1.1 part 2 in the Introduction). If $B, S \subseteq \mathbb{R}^n$ and $B$ contains the $k$-skeleton of an $n$-cube around every point in $S$, then

$$\dim_B(B) \geq \max \left\{ \left( \frac{k}{n} + \frac{(n-k)(2n-1)}{2n^2} \right) \dim_B(S), k \right\}$$

**Proof.** Since $B$ contains a $k$-cube, $\dim_B(B) \geq k$.

If $x \in \mathbb{R}^d$, for some $d$, let $(\bar{x})_m$ be the center of the half-open dyadic cube of side length $2^{-m}$ containing $x$. Define $\bar{S}_m := \{ (\bar{x})_m : x \in S \}$. Without loss of generality, $B$ is the union of $k$-skeletons of cubes with centers in $S$. That is, $B = \bigcup_{x \in S} C(x, r(x))$, where $C(x, r(x))$ is the cube centered at $x$ with side length $2r(x)$ depending on $x$. Let $\bar{B}_m = \bigcup_{x \in S} C((\bar{x})_m, (\bar{r}(x))_m)$. It is clear that $B$ meets $\Omega(|\bar{B}_m|)$ cubes in $2^{-m}\mathbb{Z}$, and from the discrete bound, Theorem 2.5

$$|\bar{B}_m| \geq \Omega(|S|^{1 - \frac{n-k}{2n^2}}).$$

So, from some $c_1, c_2$ depending only on $n, k$,

$$\dim_B(B) \geq \lim_{m} \frac{\log(|\bar{B}_m|)}{m} + c_1 \geq \lim_{m} \frac{\log(|\bar{S}_m|)}{m} \left( 1 - \frac{n-k}{2n^2} \right) + c_2 \geq \dim_B(S) \left( 1 - \frac{n-k}{2n^2} \right).$$

$\square$
Lemma 4.2. If $A \subseteq \mathbb{R}^\ell$, $S \subseteq \mathbb{R}^n$, and
$$\forall x \in S \ \exists r \in \mathbb{R}^+ \ \forall I \in \left[0, 1\right]^\ell : \forall \sigma \in \{-1, 1\}^\ell : \ x_I + \sigma r \in A$$
then,
$$\dim_B(A) \geq \left(\frac{\ell(2n-1)}{2n^2}\right) \dim_B(S).$$

Proof. The proof is exactly as above, except appealing to the $(n, \ell)$-dimensional lemma, Lemma 2.7 instead of Theorem 2.8. \qed

Lemma 4.3. If $A \subset \mathbb{R}^\ell$, $S \subset \mathbb{R}^n$, and
$$\forall x \in S \ \exists r \in \mathbb{R}^+ \ \forall I \in \left[0, 1\right]^\ell : \forall \sigma \in \{-1, 1\}^\ell : \ x_I + \sigma r \in A$$
then
$$\dim_P(A) \geq \left(\frac{\ell(2n-1)}{2n^2}\right) \dim_P(S).$$

Proof. We use the equivalence of modified box counting dimension and packing dimension, Theorem 3.1 and the remarks following. Write $A$ as an ascending union $A = \bigcup_{i \in \mathbb{N}} A_i$. We want to show that $\lim_i \dim_B(A_i) \geq \dim_P(S) \left(\frac{\ell(2n-1)}{2n^2}\right).$

Let $S_i$ be the set of centers of cubes with vertices whose projections are in $A_i$. By the box counting bound in the previous lemma, $\dim_B(A_i) \geq \left(\frac{\ell(2n-1)}{2n^2}\right) \dim_B(S_i)$, whence
$$\lim_i \dim_B(B_i) \geq \lim_i \dim_B(S_i) \left(\frac{\ell(2n-1)}{2n^2}\right).$$

Note that $S$ is the ascending union $\bigcup_{i \in \mathbb{N}} S_i$. So, $\lim_i \dim_B(S_i) \geq \dim_P(S)$. This completes the proof. \qed

To get a packing dimension bound for sets $B$ containing skeleta of cubes around points in some set $S$, we can take rational translates of the set $B$ to ensure it is in a product form, then apply the above continuous analogue of the $(n, \ell)$-dimensional lemma to the factors.

Theorem 4.4 (Theorem 1.1 part 1 in the Introduction). If $B, S \subseteq \mathbb{R}^n$ and $B$ contains the $k$-skeleton of an $n$-cube around every point in $S$, then
$$\dim_P(B) \geq \dim_P(S) \left(\frac{2n-1)(n-k)}{2n^2}\right) + k.$$
\[ B' \supseteq \bigcup_{\pi \in S_n} \pi(A_{\pi} \times \mathbb{R}^k), \]

where every \( P_I \) is contained in some \( A_{\pi} \) when \( P \) is a \( k \)-face of a cube in \( B \) and \( P_I \) is a singleton, and where \( S_n \) acts on \( \mathbb{R}^n \) by \( \pi(x_1, ..., x_n) = (x_{\pi(1)}, ..., x_{\pi(n)}) \). By the product rule, Theorem 3.3 part 2, and the fact that \( \dim_H(\mathbb{R}) = \dim_P(\mathbb{R}) = 1 \), we have

\[
\dim_P(B') \geq \max_{\pi \in S_n} \dim_P(A_{\pi}) + k = \dim_P \left( \bigcup_{\pi \in S_n} A_{\pi} \right) + k. \tag{4}
\]

Note that, if \( x \in S \), there is some \( r \in \mathbb{R}^+ \) such that there is a \( k \)-face of a cube in \( B \) at distance \( r \) from \( x \) in every direction. That is, for \( I \in \mathcal{B}_n \), \( x_I \) is the center of a cube in \( \bigcup_{\pi \in S_n} A_{\pi} \). This means \( A = \bigcup_{\pi \in S_n} A_{\pi} \) and \( S \) satisfy the conditions of the continuous analogue \( n, (n-k) \)-dimensional lemma, Lemma 4.3. We then have

\[
\dim_P(A) \geq \frac{(n-k)(2n-1)}{2n^2} \dim_P(S).
\]

Combining this with (4), we get

\[
\dim_P(B) = \dim_P(B') \geq \dim_P(A) + k \geq \frac{(n-k)(2n-1)}{2n^2} \dim_P(S) + k.
\]

\[ \square \]

5 Packing and Vertex Constructions

The constructions for packing dimension and the vertex case of box counting dimension are completely analogous to those in [5]. The key is a lemma generalizing the construction of the Cantor set.

**Lemma 5.1** ( [5] Lemma 6.1 ). Let \( \{Q_i\}_{i \in \mathbb{N}} \) be a sequence of finite sets in \( \mathbb{R} \) such that \( \text{diam} Q_i \leq d_i \), \( Q_i \) is \( \delta_i \)-separated (if \( x, y \) are in \( Q_i \) and distinct, then \( |x - y| > \delta_i \)), \( |Q_i| = \ell_i \), \( \sum_{i \in \mathbb{N}} \min Q_i > -\infty \) and \( \sum_{i \in \mathbb{N}} \max Q_i < \infty \). Let

\[
P = \sum_{i \in \mathbb{N}} Q_i = \left\{ \sum_{i \in \mathbb{N}} q_i : q_i \in Q_i \right\}.
\]

1. If there is some \( c < 1 \) such that \( d_i \leq c d_{i-1} \) for every \( i \in \mathbb{N} \), then

\[
\overline{\dim_B} P \leq \limsup_{j \to \infty} \frac{\log(\ell_1...\ell_j)}{-\log(d_j)}.
\]
2. If \( d_i + \delta_i \leq \delta_{i-1} \) then

\[
\text{dim}_H(P) \geq \liminf_{j \to \infty} \frac{\log(\ell_1 \ldots \ell_j)}{-\log(d_{j+1} \ell_{j+1})}.
\]

**Theorem 5.2** (Vertex Constructions). For any positive integer \( n \) and any \( t \in [0,1] \) there are compact sets \( A, T \subseteq \mathbb{R} \) such that

\[
\text{dim}_H(T) = \text{dim}_B(T) = \text{dim}_P(T) = t
\]  

(5)

and for all \( x_1, x_2, ..., x_n \in T \), there is an \( r \in \mathbb{R}^+ \) such that

\[
x_1 \pm r, x_2 \pm r, ..., x_n \pm r \in A
\]  

(6)

and

\[
\text{dim}_P(A) = \text{dim}_B(A) = \text{dim}_H(A) = \frac{2n-1}{2n}t.
\]  

(7)

**Proof.** Let \( \beta_i = ((i-1)!)^{\frac{2n}{i}} \), \( A_i = \frac{\beta_i}{i^n} D_{i,n} \), and \( T_i = \frac{\beta_i}{i^n}[0,i^{2n}-1] \), where \( D_{i,n} \) is as in Lemma 2.1. Note that \( A = \sum_{i \in \mathbb{N}} A_i \) and \( T = \sum_{i \in \mathbb{N}} T_i \) satisfy (5). It remains to verify the dimension conditions (5) and (7). One inequality is simply Theorem 4.1 (for box counting dimension) and Lemma 4.3 (for packing dimension) with \( k = 0, \ell = n \). The other inequality follows from the dimension inequalities (Theorem 3.2) and the fact that the \( A_i \) and \( T_i \) satisfy the hypotheses of the previous lemma with \( \delta_i = \frac{\beta_i}{i^n} \), \( d_i = \frac{\beta_i}{i^n}(i^{2n}-1) \), \( \ell_i = i^{2n} \) for the \( T_i \) and \( \delta = \frac{\beta_i}{i^n} \), \( d_i = 3\frac{\beta_i}{i^n}(i^{2n}-1) \), \( \ell_i = |D_k| \leq O(i^{2n-1}) \) for the \( A_i \) (one can check this).

\[\square\]

**Corollary 5.3.** For every \( n, s \in [0,n] \), there are compact sets \( B, S \subseteq \mathbb{R}^n \) such that \( B \) contains the vertices of an \( n \)-cube around every point in \( S \), \( \text{dim}(S) = s \), and \( \text{dim}(B) = \frac{2n-1}{2n}s \), where \( \text{dim} \) is either box counting or packing dimension.

**Proof.** Apply the above theorem with \( t = \frac{1}{n}s \) and \( k = n \) and let \( S = T^n \), \( B = A^n \). By the product rules, Theorem 5.2 we get \( \text{dim}(B) \leq \frac{2n-1}{2n}s \). And by the dimension bounds, Theorem 5.1 and Lemma 4.3 we get \( \text{dim}(B) \geq \frac{2n-1}{2n}s \). \[\square\]

**Theorem 5.4** (Theorem 1.2 part 1 in the Introduction). For every \( n, k \) with \( 0 \leq k < n \) and every \( s \in [0,n] \), there are compact sets \( B, S \subseteq \mathbb{R}^n \) where \( B \) contains the \( k \)-skeleton of an \( n \)-cube around every point in \( S \), and

\[
\text{dim}_P(S) = s \quad \text{and} \quad \text{dim}_P(B) = k + \text{dim}_P(S)\frac{(n-k)(2n-1)}{2n^2}.
\]
Proof. First, we want an $S$ of the appropriate dimension and an $A$ with
\[ \forall x_1, \ldots, x_n \in S \exists r \in \mathbb{R}^+ \forall I \subseteq \left[ \begin{array}{c} n \\ n \end{array} \right] \forall i \in I: \ x_i \pm r \in A. \] (8)
(We will get $B$ by taking a power of $A$ and interleaving copies of $[0,1]$. The condition (8) is clearly satisfied if there is some $r$ such that $x_i \pm r \in A$ for every $1 \leq i \leq n$.)

Apply the vertex construction, Lemma 5.2, to get compact $T \subseteq [0,1]$ of dimension $s$ and $A \subseteq$ of dimension $2n^2 - 1$ satisfying
\[ \forall x_1, \ldots, x_n \in T \exists r \in \mathbb{R}^+ \forall 1 \leq i \leq n : \ x_i \pm r \in A. \]

Take $S = T^n$ and $B = \bigcup_{I \subseteq \left[ \begin{array}{c} n \\ n \end{array} \right]} \prod_{i=1}^n A_{I,i}$ where $A_{I,i} = \begin{cases} [0,1] & i \in I \\ A & \text{otherwise} \end{cases}$.

Then,
\[ \dim_P(S) = \dim_P(T)n = s \]
and
\[ \dim_P(B) \leq \dim_P(A)(n-k) + k = \frac{(n-k)(2n-1)}{2n^2}s + k. \]

For any $x \in S$, there is some $r \in \mathbb{R}^+$ such that, for any $I \subseteq \left[ \begin{array}{c} n \\ n \end{array} \right]$ and $\sigma \in \{-1,1\}^n$, $x_I + r \sigma I \in A^k$, so $x + r \sigma \in \prod_{i=1}^n A_{I,i} \subseteq B$. So, by the packing dimension bound, Theorem 4.4
\[ \dim_P(B) \geq \frac{(n-k)(2n-1)}{2n^2}s + k. \]

Note that in the constructions given, the set $S$ is exactly the unit cube when $s$ is taken to be $n$.

6 Box Counting Constructions

The box counting construction is again completely analogous to the construction in [5, Section 6.2], however there are many more details to keep track of. For completeness we provide the entire argument below:

**Theorem 6.1** (Theorem 1.2 part 2 in the Introduction). For every $n, k$ with $0 \leq k < n$ and every $s \in [0,n]$, there are compact sets $B, S \subseteq \mathbb{R}^n$ where $B$ contains the $k$-skeleton of an $n$-cube around every point in $S$, and
\[ \dim_B(S) = s \quad \text{and} \quad \dim_B(B) = \max \left\{ k, \left( 1 - \frac{(n-k)}{2n^2} \right)s \right\} \]
The set $A$ will be stitched together from analogues of the $D_{i,n}$ of Lemma 2.1. We first need to get better control of how difficult our $D_{i,n}$ analogues are to cover with intervals of a given length.

**Lemma 6.2.** For any positive integer $n$, there is a sequence of sets $\{A_N\}_{N \in \mathbb{N}}$ of integers such that the following hold:

1. For every $N$ and every $x_1, \ldots, x_n \in [1, N]$ there is $r > 0$ such that $\forall 1 \leq i \leq n$ $x_i \pm r \in A_N$

2. For every $\delta > 0$ there is $C = C(\delta) \in \mathbb{R}^+$ such that for every $R \in [1, N]$, the set $A_N$ can be covered by $CN^\delta(N/R)^{(2n-1)/(2n)}$ intervals of length $R$.

**Proof of Lemma.** We first consider $N$ of the form $(p!)^{2n}$ for some $p$. Here let

$$A_N = (p!)^{2n} \sum_{i=1}^{p} D_{i+1,n}$$

where $D_{i,n}$ is as in Lemma 2.1. One can check that this set satisfies (1) similarly to the set defined in the proof of Lemma 6.2. We now need to verify the covering property (2). For $1 \leq j \leq p$, $A_N$ can be covered in

$$|D_{2,n}|D_{3,n}|\ldots|D_{j+1,n}| = O(|D_{1,n}|\ldots|D_{j,n}|) = O((1)^j(j!)^{2n-1})$$

intervals of length

$$(p!)^{2n} \sum_{m=j+1}^{p} \frac{(3m)^{2n}}{(m!)^{2n}} \leq \frac{3^{2n+1}(p!)^{2n}}{(j!)^{2n}}.$$

Define $R_j$ as $\frac{3^{2n+1}(p!)^{2n}}{(j!)^{2n}}$. We have just shown that $A_N$ can be covered by $O(1)^j(N/R_j)^{(2n-1)/(2n)}$ intervals of length $R_j$ (independent of $\delta$). Note that the $R_j$ are increasing and $R_1 = 3^{2n+1}$. For general $R$ we interpolate as follows. By making $C$ large enough, we may assume $R \in [3^{2n+1}, N]$. Select $j$ such that $R_{j+1} < R < R_j$; $A_N$ can be covered by $O((N/R_{j+1})^{2n-1}/(2n))$ intervals of length $R$. Since $\frac{\log(j+1)!}{\log(p)!} \to 1$ as $j \to \infty$, for every $\delta > 0$, there is some $C > 0$ such that $R_j < CR_{j+1}^{1+\delta} \leq O(N^\delta)R_{j+1}$. This tells us $A_N$ can be covered by $O(N^{2\delta})(N/R)^{(2n-1)/(2n)}$ intervals of length $R$.

For general $N$ we again interpolate between values of $(p!)^{2n}$ by using the fact that $\lim_{n \to \infty} \frac{\log(p+1)!}{\log(p)!} = 1$.

**Proof of Lemma 6.1.** Let $\alpha > 0$ be such that $s = \frac{\alpha n}{1+\alpha}$. For each $i \in \mathbb{N}$, let $N_i = [2^{\alpha i}]$ and define

$$S_i = [1, N_i] \quad \text{and} \quad B_i = \bigcup_{j \in \left[\frac{\alpha}{s-\alpha}\right]} \left[1 \leq i \leq n\right] A_i' \quad \text{for each} \quad i \in \mathbb{N}.$$
where \( A_i' = \begin{cases} A_N, & i \in I \\ [-3N, 4N] & \text{otherwise} \end{cases} \) and \( A_N \) is as in the above lemma.

Let \( \varepsilon_i = 2^{-(1+\alpha)i} \) and define

\[
S = \{(0, ..., 0) \} \cup \bigcup_{i \in \mathbb{N}} \left( (2^{-i}, 0, 0) + \varepsilon_i S_i \right)
\]

and

\[
B = C_0 \cup \bigcup_{i \in \mathbb{N}} \left( (2^{-i}, 0, 0) + \varepsilon_i B_i \right),
\]

where \( C_0 \) is the \( k \)-skeleton of a cube of unit side length centered at the origin.

The set \( S \) then consists of a sequence of translated shrinking copies of the discrete cube and \( B \) consists of a series of \( k \)-skeletons around these points, since \( B_i \) contains the \( k \)-skeleton of an \( n \)-cube around each point in \( S_i \). We then have that \( B \) and \( S \) satisfy (1) in the statement of the theorem. It remains to show that these sets have the correct dimension.

We first verify that \( \dim (S) \geq s \). \( S \) contains a translate of \( \varepsilon_i S_i \), so contains \( |N_i|^{\|} = \Omega(2^{n\alpha j}) = \Omega(\varepsilon_i^{-s}) \) points at pairwise distance at least \( \varepsilon_i \). Interpolating an arbitrary \( \varepsilon \in (0, 1) \) between consecutive values of \( \varepsilon_i \), we deduce that \( \dim (S) \geq s \).

Now we show \( \dim (B) \leq \max\{k, (1 - \frac{n-k}{2n})s\} + O(\delta) \) for any \( \delta > 0 \). To get the desired estimate, we will count the number of cubes of side length \( \varepsilon_i \) are needed to cover \( B \). First fix \( i \) and and decompose \( B \) as \( B'_i \cup B''_i \cup B'''_i \), where

\[
B'_i = C_0 \cup \bigcup_{j=1}^{i-1} \left( (2^{-j}, 0, 0) + \varepsilon_j B_j \right)
\]

\[
B''_i = \bigcup_{j: \varepsilon_j \leq \varepsilon_i \leq N_i \varepsilon_j} \left( (2^{-j}, 0, 0) + \varepsilon_j B_j \right)
\]

\[
B'''_i = \bigcup_{j: n \varepsilon_j < \varepsilon_i} \left( (2^{-j}, 0, 0) + \varepsilon_j B_j \right),
\]

and where \( C_0 \) is the \( k \)-skeleton of the unit cube. We first count the \( \varepsilon_i \) balls needed to cover \( B'_i \). Note that for \( j < i \), \( \varepsilon_j B_j \) consists of \( \binom{k}{i} |A_N|^{n-k} \) \( k \)-cubes of side length \( O(\varepsilon_j N_j) \). Since, part 2 of the above lemma applied to \( R = 1, |A_N| = O \left( 2^{n i (\frac{2n-1}{2n}) j + \frac{1}{2n} + \delta} \right) \), \( \varepsilon_j B_j \) can be covered by

\[
O(1)|A_{N_j}|^{n-k} \left( \frac{\varepsilon_j N_j}{\varepsilon_i} \right)^k = O(1)2^{j \alpha i (\frac{2n-1}{2n}) (n-k) j + \delta (n-k) \alpha j - k} \varepsilon_i^{-k} \leq 0
\]

balls of radius \( \varepsilon_i \). Straightforward calculation shows

\[
\left( \frac{(2n-1)(n-k)}{2n} \right) \alpha - k \leq 0
\]
exactly when
\[ s \leq \frac{2n^2 k}{2n^2 - (n - k)} = \frac{k}{1 - \frac{(n-k)}{2n^2}}, \]

and in this case (shrinking \( \delta \) by a factor depending on \( s \)), \( B'_k \) can be covered by \( O(\varepsilon_i^{-k}) \) balls of radius \( \varepsilon_i \) (this means, more or less, that the minimum dimension \( k \) achieved when \( s \) is small enough). Otherwise, \( s > \frac{2n^2}{2n^2 - (n-k)} \) and \( B'_k \) can be covered by

\[ \sum_{j=1}^{i-1} O(1)\varepsilon_i^{-k}2^{(\frac{(2n-1)(n-k)}{2n} - j)\alpha + \delta(n-k)\alpha j} \leq \]

\[ O(1)\varepsilon_i^{-k}2^{(\frac{(2n-1)(n-k)}{2n} - k)\alpha + \delta(n-k)\alpha i} = \]

\[ O(1)2^{(\frac{(2n-1)(n-k)}{2n} - 1)\alpha + \delta(n-k)\alpha i} \equiv O\left(\varepsilon_i\frac{(\frac{(2n-1)(n-k)}{2n} + k)}{2n^2} s - O(\delta)\right). \]

We next count the \( \varepsilon_i \) balls needed to cover \( B''_i \). Suppose \( \varepsilon_j < \varepsilon_i < \varepsilon_j N_j \). Again using the second part of the above lemma, \( A_{N_j} \) can be covered by

\[ O(n_j^\delta)(N_j/R)^{(\frac{2(n-1)}{2n} + k)} \]

balls of radius \( R \), and \( \varepsilon_j B_j \) can be covered by the same number of balls of radius \( \varepsilon_j R \). Applying this to \( R = \varepsilon_i/\varepsilon_j \) (which is in \([1, N_j]\) by assumption), we get \( \varepsilon_j B_j \) can be covered by

\[ O(n_j^\delta)\left(\frac{N_j\varepsilon_j}{\varepsilon_i}\right)^{(\frac{(2n-1)(n-k)}{2n} + k)} = O(1)(2^{(1 - O(\delta))\varepsilon_i} - (\frac{(2n-1)(n-k)}{2n} + k)) \]

balls of radius \( \varepsilon_i \). Taking \( \delta \) small enough and summing over \( j \geq i \) (via the geometric sum formula,) we can cover \( B''_i \) by

\[ O(1)(2^{(1 - O(\delta))\varepsilon_i} - (\frac{(2n-1)(n-k)}{2n} + k)) = O\left(\varepsilon_i\frac{(\frac{(2n-1)(n-k)}{2n} + k)}{2n^2} s - O(\delta)\right) \]

balls of radius \( \varepsilon_i \).

Finally, we count the number of balls needed to cover \( B''_i \). The smallest \( j \) such that \( \varepsilon_j N_j < \varepsilon_i \) satisfies \( 2^{-j} \leq O(\varepsilon_j) \), so \( B''_i \) has diameter \( O(\varepsilon_i) \), and can be covered by \( O(1) \) balls of radius \( \varepsilon_i \).

Putting the above estimates together, we get that \( B \) can be covered by

\( O(\varepsilon_i^{-k}) \) balls of radius \( \varepsilon_i \) when \( s(1 - \frac{n-k}{2n^2}) < k \) and by \( O(1)\varepsilon_i^{(1 - \frac{n-k}{2n^2})s + O(\delta)} \)

balls of radius \( \varepsilon_i \) otherwise. This implies that \( \text{dim}_B(B) \leq \max\{k, (1 - \frac{n-k}{2n^2})s + O(\delta)\} \)

for small enough \( \delta \). Letting \( \delta \) tend to 0 gives the desired result.
Again note that the above construction makes \( S \) a cube when \( s = n \).

7 Hausdorff Results

The results for Hausdorff dimension are mostly trivial generalizations of the results in [5]. We will formulate a conjecture based on these partial results.

**Theorem 7.1** (Theorem 1.7 in the Introduction). If \( B, S \subseteq \mathbb{R}^n \) and \( B \) contains the \( k \)-skeleton of a cube around every point in \( S \), then \( \dim_H(B) \geq \max\{\dim_H(S) - 1, k\} \).

**Proof.** Let \( P \) be the projection on the plane normal to \((1, ..., 1)\). Since \( B \) contains the vertices of a cube around each point in \( S \), \( P(S) \subseteq P(B) \). So, \( \dim_H(S) \leq \dim_H(P(B)) + 1 \leq \dim_H(B) + 1 \). \( \square \)

This bound seems quite weak, but it is sharp in the case when \( n = 2 \), as shown in [5]. We can extend this a bit further.

**Theorem 7.2.** There is a \( G_\delta \) set \( B \subseteq \mathbb{R}^n \) which contains the boundary, i.e. the \((n - 1)\)-skeleton, of an \( n \)-cube around each point in \( \mathbb{R}^n \) such that \( \dim_H(B) = k \).

**Proof.** Let \( A \subseteq \mathbb{R}^k \) be a comeager \( G_\delta \) set of Hausdorff dimension 0. Since \( \bigcap_{1 \leq i \leq n} f_i(A) \) is comeager for any finite family of affine maps \( \{f_i\} \), we have, for any \( x_1, ..., x_n \in \mathbb{R} \)

\[
(A - x_1) \cap (A - x_2) \cap (A - x_3) \cap ... \cap (A - x_n) \subseteq \{0\}.
\]

This means there is some \( r \in \mathbb{R}^+ \) such that \( x_i \pm r \in A \) for all \( 1 \leq i \leq n \). It then follows that \( B = \bigcup_{1 \leq i \leq n} \mathbb{R}^{i-1} \times A \times \mathbb{R}^{n-i} \) is as desired. \( \square \)

Using the methods in [5] and in Section 8 one can make compact sets \( B \) containing the boundary of a skeleton of a cube around each point in \([0, 1] \) of Hausdorff dimension \( n - 1 \).

**Corollary 7.3** (Theorem 1.8 in the Introduction). For \( 0 \leq k < n, s \in [n - k, n] \), there are \( B, S \subseteq \mathbb{R}^n \) where \( B \) contains the \( k \)-skeleton of an \( n \)-cube around each point in \( S \), \( \dim_H(B) = \max\{k, s - 1\} \) and \( \dim_H(S) = s \).

Further, if \( k = 0 \), there are \( B, S \) as above for \( s \in [n - 1, n] \).

**Proof.** The case where \( n - 1 = k \neq 0 \) is implied by the previous theorem. The case where \( n = 2, k = 0 \) follows from [5] Theorem 3.5]. We can extend these results to constructions in higher dimensions simply by taking a product with \( \mathbb{R} \). Let \( n \) be an arbitrary natural number. The constructions for each \( k \) are nearly identical, but it is clearer to separate the case where \( k = 0 \).

Suppose \( k = 0 \). For any \( s \in [n - 1, n] \), there are sets \( S', B' \subseteq \mathbb{R}^2 \) such that \( \dim_H(S') = s - (n - 2) \), \( \dim_H(B') = s - (n - 1) \) and \( B' \) contains the vertices of a square around every point in \( S' \) (such sets are guaranteed by case when \( n = 2 \)). Take \( B = B' \times \mathbb{R}^{n-2} \) and \( S = S' \times \mathbb{R}^{n-2} \). Clearly \( B \) contains the
vertices of an \(n\)-cube around each point in \(S\). By the product rules, Theorem 3.3
\[
\dim_H(S) = \dim_H(S') + (n - 2) = s \quad \text{and} \quad \dim_H(B) = \dim_H(B') + (n - 2) = s - 1.
\]
Suppose \(k > 0\). For any \(s \in [n - k, n]\), there are \(S', B' \subseteq \mathbb{R}^{k+1}\) such that
\[
\dim_H(S') = s - (n - k - 1), \quad \dim_H(B') = s - (n - k), \quad \text{and} \quad B' \text{ contains the } k\text{-skeleton of a cube around each point in } S.
\]
and take \(S = S' \times \mathbb{R}^{n-k-1}\) and \(B = B' \times \mathbb{R}^{n-k-1}\). \(\square\)

On the basis of these results, we conjecture that the bound \(\dim_H(B) \geq \dim_H(S) - 1\) is sharp in all cases.

### 8 Orthoplex Vertex Problem

It is fairly natural to consider the vertex problem for the dual polytope of the cube, the orthoplex. That is, how small can \(B\) be and still contain the vertices of an orthoplex around each point in a set \(S\) of a given size? We can find sharp bounds for each notion of dimension discussed in the above.

**Theorem 8.1** (Theorem 1.9 in the Introduction). Let \(B, S \subseteq \mathbb{R}^n\) such that for all \(x \in S\) there is some \(r \in \mathbb{R}^+\) such that \(x \pm re_i \in B\), where \(e_i\) is the \(i\)th standard basis vector. Then,

1. \(\dim_H(B) \geq \dim_H(S) - 1\)
2. If \(B, S\) are finite, then \(|B| \geq \Omega(|S|^\frac{2n-1}{2n})\)
3. \(\dim_B(B) \geq \frac{2n-1}{2n} \dim_B(S)\)
4. \(\dim_P(B) \geq \frac{2n-1}{2n} \dim_P(S)\)

**Proof.** Precisely the same argument as for the hausdorff bound of the cube problem (Theorem 7.1) gives (1). The \(n\)-dimensional lemma, Lemma 2.4 gives (2). And, (2) gives (3) and (4) exactly as Theorem 2.7 gives Lemma 4.1 and Corollary 4.3. \(\square\)

We can show that all of these bounds are sharp.

**Theorem 8.2** (Theorem 1.10 in the Introduction). For any \(n, p \in \mathbb{N}\) and \(s \in [0, n]\), we can find compact sets \(B_H, B_B, B_P, B_f, S_H, S_B, S_P, \) and \(S_f \subseteq \mathbb{R}^n\) such that \(B_X\) contains the vertices of an orthoplex around each point in \(S_X\) for any \(X \in \{H, B, P, f,\}\), \(\dim_X(S_X) = s\) for \(X \in \{H, B, P\}\), \(|S_f| = p\), and

1. \(\dim_H(B_H) = \max\{0, \dim_H(S_H) - 1\}\)
2. \(|B_f| = O(|S_f|^\frac{2n-1}{2n})\)

---

2The simplex is not as natural as it doesn’t have as obvious an additive combinatorial structure as the cube or orthoplex. The \(k\)-faces of the orthoplex, in general, are not orthoplexes or cubes, so the general \(n,k\)-dimensional problem for the orthoplex is again somewhat unnatural. Of course, these problems may turn out to have interesting solutions.
3. \( \dim_B(B_B) = \frac{2n-1}{2^n} \dim_B(S_B) \)

4. \( \dim_P(B_P) = \frac{2n-1}{2^n} \dim_P(S_P) \).

**Proofs for discrete, box counting, and packing dimension.** A cube has vertices in at least \( n \) independent directions from the center, so we may take the vertex constructions for the cube and apply a linear map to get constructions for (2), (3), and (4). More explicitly, there are \( B_X \) and \( S_X \) for \( X \in \{ P, B, f \} \) satisfying the conditions (1), (2), and (3) above where \( B_X' \) contains the vertices of a cube around each point in \( S_X' \) (Theorems 5.4, 6.1 and 2.3 respectively). There is a basis \( b_1, ..., b_n \) for \( \mathbb{R}^n \) such that \( x \pm b_i \in B_X' \) for every \( x \in S_X' \) and \( 1 \leq i \leq n \). Let \( g \) be the invertible linear function defined by \( g(b_i) = e_i \). Then clearly \( S_X = g(S_X') \) and \( B_X = g(B_X') \) is as desired. 

\( \square \)

To get a construction for the Hausdorff dimension, we will generalize an argument from [5]. The key is the splicing operation \( SPL \) (defined in [5]) which takes a sequence of sets in some \( \mathbb{R}^m \) and returns a set in \( \mathbb{R}^m \). Most of the key properties of this set are established in lemma 3.7 and proposition 3.6 in [5]. We will need the following:

**Lemma 8.3.** Let \( X_i, Z_i, Y_i \subseteq \mathbb{R}^m \) for some \( m \) and all \( i \in \mathbb{N} \).

1. \( \min_{n \in \mathbb{N}} \dim_H((Z_i)) \leq \dim_H((SPL((Z_i)_{i \in \mathbb{N}}))) \leq \min_{n \in \mathbb{N}} \dim_B(Z_i) \), provided there are only finitely many distinct \( Z_i \) and each occurs infinitely often.

2. \( SPL((X_i \times Y_i)_{i \in \mathbb{N}}) = SPL((X_i)_{i \in \mathbb{N}}) \times SPL((Y_i)_{i \in \mathbb{N}}) \), and

3. For any compact set of affine maps, \( F \), there are invertible affine maps \( g_p, 0 < p \leq M \), such that \( A = \cup_{0 < p \leq M} g_p(K_p) \) intersects every \( \cap_{0 \leq i \leq m} f_i(A) \) where \( f_i \in F \), and where \( K_p \cap [0, 1] = SPL((X_i^{(p)})_{i \in \mathbb{N}}) \) where

\[
X_i^{(p)} = \begin{cases} 
\{0\} & \exists j : i = (2j - 1)2^p \\
[0, 1] & \text{otherwise}
\end{cases}
\]

**Proof for Hausdorff dimension.** We may assume \( s \in [1, n] \). Let \( B \) be any compact subset of \([0, 1]\) such that \( \dim_H(B) = \dim_B(B) = \frac{s}{n-1} \). Let \( F \) be the set of translations by elements of \( B \), that is, \( F = \{ x \mapsto y - x : y \in B \} \cup \{ x \mapsto x - y : y \in B \} \), and let \( A \) be the corresponding set given by part (3) of the previous lemma. For \( 0 < q \leq n \), let

\[
C_i^{(q)} = \begin{cases} 
[0, 1] & \exists j, p : 2^pj = i \text{ and } j = 2q - 1 \text{ (mod 2n)} \\
B & \text{otherwise}
\end{cases}
\]

Let \( C^{(q)} = SPL((C_i^{(q)})_{i \in \mathbb{N}}) \). We have that \( \prod_{1 \leq p \leq n} C_i^{(p)} \) is (up to a permutation) \( B^m \times [0, 1]^{n-m} \) for some \( m < n \), and \( B^{n-1} \times [0, 1] \) occurs infinitely often. So,
by properties (1) and (2) of SPL, \( \dim_H \left( \prod_{1 \leq p \leq n} C^{(p)} \right) = (n-1) \dim_H(B) + 1 = s \)

We also have that, for any \( I \in \left[ \frac{n}{n-1} \right] \), \( X_i^{(p)} \times \prod_{q \in I} C_{i/q} \) is (up to a permutation) one of \( \{0\} \times B_{n-1} \), \( \{0\} \times [0,1] \times B_{n-2} \), \( [0,1] \times B_{n-1} \), or \( [0,1]^2 \times B_{n-2} \), and each of these shows up infinitely often. So \( \dim_H(A \times \prod_{q \in I} C^{(q)}) = (n-1) \dim_H(B) = s - 1 \).

Since, for any \( x_1, \ldots, x_n \in \bigcup_{1 \leq q \leq n} C^{(q)} \),

\[
(A - x_1) \cap (x_1 - A) \cap (A - x_2) \cap (x_2 - A) \cap \ldots \cap (A - x_n) \cap (x_n - A) \nsubseteq \{0\},
\]

if \( x \in \prod_{1 \leq q \leq n} C^{(q)} \), \( 1 \leq i \leq n \), there is some \( r \in \mathbb{R}^+ \) such that \( x \pm re_i \in \bigcup_{\pi \in S_n} \bigcup_{I \in \left[ \frac{n}{n-1} \right]} \pi \left( A \times \prod_{q \in I} C^{(q)} \right) \), as desired. \( \square \)

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