Thermal evolution of a radiating anisotropic star with shear

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1. Introduction

It has been shown that the role of anisotropy can alter the evolution and subsequently the physical properties of stellar objects. As an example, investigations have shown that the maximal surface redshift for anisotropic stars may differ drastically from isotropic stars. The origin of anisotropy within the stellar core has received widespread attention amongst astrophysicists. A review of the origins and effects of local anisotropy in stellar objects was carried out by Herrera and Santos, and Chan et al. and more recently by Herrera et al. The physical processes that are responsible for deviations from isotropy can be investigated in the high and low density regimes. Hartle et al. have shown that pion condensation at nuclear densities \( (0.2 f^{-3} < \rho < 2.0 f^{-3}, f = 10^{-13} \text{ cm}) \) can drastically reduce the pressure and hence impact on the evolution of the collapsing star. At higher densities the short range repulsion effects dominate which damp out the pionic effects giving rise to significantly different values for the pressure. As pointed out by Martinez viscosity effects due to neutrino trapping at nuclear densities can alter the gravitational
collapse of a radiating, viscous star. Anisotropic velocity distributions and rotation can induce local anisotropy in low density systems.

The boundary of a radiating star divides spacetime into two distinct regions, the interior and the exterior region. In order to fully describe the evolution of such a system, one needs to satisfy the junctions conditions for the smooth matching of the interior and exterior spacetimes. In the case of a shear-free, spherically symmetric star undergoing dissipative gravitational collapse these conditions were first derived by Santos\textsuperscript{6}. Subsequently, many models of shear-free radiative collapse were developed and the physical viability of these models were studied in great detail\textsuperscript{7,8,9,10,11}. These junction conditions were later generalised to include pressure anisotropy\textsuperscript{12} and the electromagnetic field\textsuperscript{13}.

There are a few exact solutions to the Einstein field equations for a bounded shearing matter configuration. There have been numerous attempts to produce models of radiative gravitational collapse which incorporate the effects of shear\textsuperscript{14,15,16}. Most treatments to date are based on numerical results as the resulting temporal evolution equation derived from the junction conditions is highly nonlinear\textsuperscript{15}. It is in this spirit that we seek an exact solution of the Einstein field equations which represents a spherically symmetric radiating star undergoing dissipative gravitational collapse with nonzero shear.

In this treatment, we consider the relaxation effects due to the heat flux and shear separately. We show that earlier assumptions of the relaxation time being proportional to the corresponding collision time only hold for a limited regime of the evolution of the star. These results agree with earlier suggestions by Anile et al\textsuperscript{17}. We are also in a position to integrate the causal heat transport equation and obtain the corresponding causal temperature profile in the interior of the star.

This paper is organised as follows. In section 2, we provide the Einstein field equations for the most general, nonrotating, spherically symmetric line element. The energy momentum tensor for the interior spacetime is that of an imperfect fluid with heat conduction and pressure anisotropy. In section 3, the junction conditions required for the smooth matching of the interior and exterior spacetimes across a time-like hypersurface are presented. The transport equations for the dissipative fluxes within the framework of causal thermodynamics are presented in section 4. The general solution for the special case of constant mean collision time in both the causal and noncausal theories are presented. The junction conditions are used to generate a reasonable model of radiating anisotropic collapse in section 5. When the heat flux vanishes, our model tends towards a Friedmann-like limit, similar to the result obtained for the shear-free case studied by Kolassis et al\textsuperscript{7}. A discussion of some relevant physical parameters is presented in section 6.
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2. Spherically symmetric Spacetimes

The interior spacetime of a non-rotating spherically symmetric collapsing star is most generally described by the line element (in coordinates \((x^a) = (t, r, \theta, \phi)\))

\[
ds^2 = -A^2 dt^2 + B^2 dr^2 + Y^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,
\]

where \(A, B\) and \(Y\) are functions of the coordinates \(t\) and \(r\). The fluid four–velocity \(u^a\) is comoving and is given by

\[
u^a = \frac{1}{A} \delta^a_0 .
\]

The kinematical quantities for the line element (1) are given by

\[
\dot{u}^a = \left(0, \frac{A'}{AB^2}, 0, 0\right)
\]

\[
\Theta = \frac{1}{A} \left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right),
\]

where \(\dot{u}^a\) is the four–acceleration vector and \(\Theta\) is the expansion scalar. The nonzero components of the shearing tensor are given by\(^2\)

\[
\sigma_{11} = \frac{2B^2}{3A} \left(\frac{\dot{B}}{B} - \frac{\dot{Y}}{Y}\right)
\]

\[
\sigma_{22} = -\frac{2Y^2}{3A} \left(\frac{\dot{B}}{B} - \frac{\dot{Y}}{Y}\right)
\]

\[
\sigma_{33} = \sigma_{22} \sin^2 \theta .
\]

The magnitude of the shear scalar is given by

\[
\sigma_{ab} \sigma^{ab} = \sigma = -\frac{1}{3A} \left(\frac{\dot{B}}{B} - \frac{\dot{Y}}{Y}\right) .
\]

For a relativistic fluid, the kinematical quantities are important for studying the evolution of the system. The interior matter distribution is described by the following energy–momentum tensor

\[
T_{ab} = (\rho + p)u^a u^b + p g_{ab} + \pi_{ab} + q_a u_b + q_b u_a ,
\]

where \(p\) is the isotropic pressure, \(\rho\) is the density of the fluid, \(\pi_{ab}\) is the stress tensor and \(q_a\) is the heat flux vector. The stress tensor takes the form

\[
\pi_{ab} = (p_R - p_T)(n_a n_b - \frac{1}{3} h_{ab}) ,
\]

where \(p_R\) is the radial pressure, \(p_T\) is the tangential pressure, and \(n\) is a unit radial vector given by

\[
n^a = \frac{1}{B} \delta^a_1 .
\]
The isotropic pressure $p$ is related to the radial pressure and the tangential pressure via

$$p = \frac{1}{3}[p_R + 2p_T]. \quad (12)$$

The coupled Einstein field equations for the interior matter distribution become

$$\rho = \frac{2}{A^2} \frac{\dot{B} \dot{Y}}{B Y} + \frac{1}{Y^2} + \frac{1}{A^2} \frac{\dot{Y}^2}{Y^2} \left(-\frac{1}{B^2} \left(2 \frac{Y''}{Y} + \frac{Y'^2}{Y^2} - \frac{2B'Y'}{BY} \right)\right)$$

$$p_R = \frac{1}{A^2} \left(-\frac{\ddot{Y}}{Y} - \frac{\ddot{Y}^2}{Y^2} + \frac{A' \dot{Y}}{AY} \right) + \frac{1}{B^2} \left(\frac{Y'^2}{Y^2} + \frac{2A'Y'}{AY} \right) - \frac{1}{Y^2} \quad (13)$$

$$p_T = -\frac{1}{AB^2} \left(\frac{\dot{B}}{B} - \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{B} \dot{Y}}{BY} - \frac{\dot{A} \dot{Y}}{AY} + \frac{\ddot{Y}}{Y} \right) + \frac{1}{B^2} \left(\frac{A''}{A} - \frac{A'B'}{AB} + \frac{A'Y'}{AY} - \frac{B'Y'}{BY} + \frac{Y''}{Y} \right) \quad (14)$$

$$q = -\frac{2}{AB^2} \left(-\frac{\dot{Y}'}{Y} + \frac{\dot{B} Y'}{BY} + \frac{A' \dot{Y}}{AY} \right). \quad (15)$$

In the above we have defined

$$q = q^1, \quad (17)$$

where $q^a = (0, q^1, 0, 0)$. The field equations (13)–(16) describe the gravitational interaction of a shearing matter distribution with heat flux and anisotropic pressure.

3. **Spherical collapse with heat flow**

The problem of gravitational collapse within the context of general relativity was first investigated by Oppenheimer and Snyder\textsuperscript{18} in which the interior matter distribution was taken to be dust with the exterior spacetime being Schwarzschild. With the discovery of the Vaidya solution\textsuperscript{19}, it became possible to study radiative gravitational collapse where the collapsing core radiated energy to the exterior spacetime\textsuperscript{6,8,16,20}. The Vaidya solution which describes the exterior spacetime of a radiating star is given by

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right)dv^2 - 2dvdr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (18)$$
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in the coordinates \((x^a) = (v, r, \theta, \phi)\). The quantity \(m(v)\) represents the Newtonian mass of the gravitating body as measured by an observer at infinity. The metric (18) is the unique spherically symmetric solution of the Einstein field equations for radiation in the form of a null fluid. The Einstein tensor for the line element (18) is given by

\[
G_{ab} = \frac{-2}{r^2} \frac{dm}{dv} \delta_a^0 \delta_b^0.
\] (19)

The energy–momentum tensor for null radiation assumes the form

\[
T_{ab} = \epsilon w_a w_b,
\] (20)

where the null four–vector \(w\) is given by \(w_a = (1, 0, 0, 0)\). Thus from (19) and (20) we have that

\[
\epsilon = \frac{-2}{r^2} \frac{dm}{dv}.
\] (21)

for the energy density of the null radiation. Since the star is radiating energy to the exterior spacetime we must have \(\frac{dm}{dv} \leq 0\).

In deriving the junction conditions we employ the Darmois conditions as these were shown to be the most convenient and reliable\(^{21}\). Here we merely state the results of the matching of the line elements (1) and (18) since these conditions have been extensively investigated in the past. For a more comprehensive treatment of the junction conditions the reader is referred to the works of Glass\(^{22}\), Chan\(^{12,15}\) and Govender et al\(^{11}\).

We require that the metrics (1) and (18) match smoothly across the boundary \(\Sigma\). This generates the first junction condition

\[
(ds^2)_\Sigma = (ds^2)_{\Sigma} = ds^2_{\Sigma}.
\] (22)

(We use the notation \((\ )\)\(\Sigma\) to represent the value of \((\ )\) on \(\Sigma\).) The second junction condition is obtained by requiring continuity of the extrinsic curvature across the boundary. This gives

\[
K^+_{ij} = K^-_{ij},
\] (23)

where

\[
K^\pm_{ij} \equiv -n^a_{\pm} \frac{\partial^2 X^a_{\pm}}{\partial \xi^i \partial x^j} - n^a_{\pm} \Gamma^a_{cd} \frac{\partial X^c_{\pm}}{\partial \xi^i} \frac{\partial X^d_{\pm}}{\partial \xi^j},
\] (24)

and \(n^a_{\pm}(X^b_{\pm})\) are the components of the vector normal to \(\Sigma\). We find that the necessary and sufficient conditions on the spacetimes for the first junction condition to be valid are that

\[
A(r_\Sigma, t)dt = \left(1 - \frac{2m}{r_\Sigma} + 2 \frac{dr_\Sigma}{dv}\right)^{\frac{1}{2}} dv
\] (25)

\[
Y(r_\Sigma, t) = r_\Sigma(v).
\] (26)
By equating the appropriate extrinsic curvature components for the interior and exterior spacetimes we generate the second set of junction conditions which are given by

\[ m(v) = \left[ \frac{Y}{2} \left( 1 + \frac{Y_i^2}{A^2} - \frac{Y_r^2}{B^2} \right) \right]_\Sigma \]

(27)

\[ (p_R)_{\Sigma} = (qB)_{\Sigma}, \]

(28)

where we may interpret \( m(v) \) as representing the total gravitational mass within the surface \( \Sigma \). The expression (27) corresponds to the mass function of Cahill and McVittie\(^{23}\) representing spheres of radius \( r \) inside \( \Sigma \).

The important result \( (p_R)_{\Sigma} = (qB)_{\Sigma} \), relating the radial pressure \( p_R \) to the heat flow \( q \), was first established by Santos\(^6\) for shear–free spacetimes. The first attempt to generalise the above junction conditions to include shear for neutral matter was carried out by Glass\(^{22}\).

The equations (25)–(28) are the most general matching conditions for the spherically symmetric spacetimes \( \mathcal{M}^+ \) and \( \mathcal{M}^- \). Relation (28) implies that the radial pressure \( p_R \) is proportional to the magnitude of the heat flow \( q \) which is non-vanishing in general.

The total luminosity for an observer at rest at infinity is given by

\[ L_\infty(v) = - \left( \frac{dm}{dv} \right)_\Sigma = \frac{(p_R)_{\Sigma}}{2} \left[ Y^2 \left( \frac{Y'}{B} + \dot{Y} \right)^2 \right]_\Sigma, \]

(29)

where \( \frac{dm}{dv} \leq 0 \) since \( L_\infty \) is positive. An observer with four–velocity \( e^a = (\dot{v}, \dot{r}, 0, 0) \) located on \( \Sigma \) has proper time \( \eta \) related to the time \( t \) by \( d\eta = Adt \). The radiation energy density that this observer measures on \( \Sigma \) is

\[ \epsilon_\Sigma = \frac{1}{4\pi} \left( - \frac{\dot{v}^2 dm}{r^2 dv} \right)_\Sigma, \]

(30)

and the luminosity observed on \( \Sigma \) can be written as

\[ L_\Sigma = 4\pi r^2 \epsilon_\Sigma. \]

(31)

The boundary redshift \( z_\Sigma \) of the radiation emitted by the star is given by

\[ 1 + z_\Sigma = \frac{dv}{d\eta} = \left( \frac{Y'}{B} + \dot{Y} \right)^{-1}_\Sigma \]

(32)

which can be used to determine the time of formation of the horizon. The above expressions allow us to write

\[ 1 + z_\Sigma = \left( \frac{L_\Sigma}{L_\infty} \right)^\frac{1}{2} \]

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which relates the luminosities \( L_\Sigma \) to \( L_\infty \). The redshift for an observer at infinity diverges at the time of formation of the horizon which is determined from

\[
\frac{Y'}{B} + \dot{Y} = 0 .
\]  

(34)

4. Thermodynamics

Previous treatments of heat flow in relativistic stellar models have shown that relaxational effects play a significant role in the evolution of the temperature and luminosity profiles during late stages of collapse\(^5,10,11,24\). Since we are interested in the effects of shear within the stellar core, we employ the causal transport equations for the heat flux and the shear stress. Assuming there is no viscous/heat coupling, we have the following relevant transport equations for the dissipative fluxes\(^25\)

\[
\tau \eta_{ab} + q_a = -\kappa(D_aT + \dot{T} \dot{u}^a) - \left[ \frac{1}{2} \kappa T^2 \left( \frac{\tau_1}{\kappa T^2} \right)_{,a} q_a \right]
\]  

(35)

\[
\tau_1 \eta_{ab} \dddot{\pi}_{cd} + \pi_{ab} = -2\eta \sigma_{ab} - \left[ \eta T \left( \frac{\tau_2}{2\eta T} \right)_{,d} \pi_{ab} \right],
\]  

(36)

where \( \tau \) and \( \tau_1 \) are the relaxation times of thermal and shear viscous signals respectively. The truncated transport equations, together with the no–coupling assumption are differential equations of Maxwell–Catteneo form

\[
\tau \eta_{ab} + q_a = -\kappa(D_aT + \dot{T} \dot{u}^a)
\]  

(37)

\[
\tau_1 \eta_{ab} \dddot{\pi}_{cd} + \pi_{ab} = -2\eta \sigma_{ab}.
\]  

(38)

Setting \( \tau = \tau_1 = 0 \) in the above, we regain the so-called Eckart transport equations which predict infinite propagation velocities for the dissipative fluxes.

In this paper we are primarily interested in heat transport in relativistic astrophysics and hence (37) plays a significant role in determining the evolution of the causal temperature profile of our models. For the line element (1) the causal transport equation (37) becomes

\[
\tau(qB)_{,a} + A(qB) = -\kappa \frac{(AT)_{,a}}{B}
\]  

(39)

which governs the behaviour of the temperature. Setting \( \tau = 0 \) in (39) we obtain the familiar Fourier heat transport equation

\[
A(qB) = -\kappa \frac{(AT)_{,a}}{B}
\]  

(40)

which predicts reasonable temperatures when the fluid is close to quasi–stationary equilibrium.

For a physically reasonable model, we use the thermodynamic coefficients for radiative transfer outlined in Martínez\(^5\). We consider the situation where energy is...
carried away from the stellar core by massless particles that are thermally generated with energies of the order of $kT$. The thermal conductivity takes the form

$$\kappa = \gamma T^3 \tau_c, \quad (41)$$

where $\gamma (\geq 0)$ is a constant and $\tau_c$ is the mean collision time between the massless and massive particles. Based on this treatment we assume the power–law behaviour

$$\tau_c = \left( \frac{\alpha}{\gamma} \right) T^{-\omega}, \quad (42)$$

where $\alpha (\geq 0)$ and $\omega (\geq 0)$ are constants. With $\omega = \frac{3}{2}$ we regain the case of thermally generated neutrinos in neutron stars. The mean collision time decreases with growing temperature, as expected, except for the special case $\omega = 0$, when it is constant. This special case can only give a reasonable model for a limited range of temperature. Following Martínez\textsuperscript{5}, we assume that the velocity of thermal dissipative signals is comparable to the adiabatic sound speed which is satisfied if the relaxation time is proportional to the collision time:

$$\tau = \left( \frac{\beta \gamma}{\alpha} \right) \tau_c, \quad (43)$$

where $\tau (\geq 0)$ is a constant. We can think of $\tau$ as the ‘causality’ index, measuring the strength of relaxational effects, with $\tau = 0$ giving the noncausal case.

Using the above definitions for $\tau$ and $\kappa$, (39) takes the form

$$\beta(qB)_t T^{-\omega} + A(qB) = -\alpha \frac{T^{3-\omega}(AT)_t}{B}. \quad (44)$$

When $\beta = 0$, we can find all noncausal solutions of (44), viz.

$$(AT)^{4-\omega} = \frac{\omega - 4}{\alpha} \int A^{4-\omega} qB^2 dr + F(t) \quad \omega \neq 4$$

$$\ln(AT) = -\frac{1}{\alpha} \int qB^2 dr + F(t) \quad \omega = 4, \quad (45)$$

where $F(t)$ is an arbitrary function of integration. This is fixed by the expression for the temperature of the star at its surface $\Sigma$.

In the case of constant mean collision time, i.e. $\omega = 0$, the causal transport equation (44) is simply integrated to yield

$$(AT)^4 = -\frac{4}{\alpha} \left[ \beta \int A^3 B(qB)_t dr + \int A^4 qB^2 dr \right] + F(t) \quad (46)$$

while one solution valid for a less limited range of temperature can be found for $\omega = 4$, which corresponds to nonconstant collision time\textsuperscript{26}:

$$(AT)^4 = -\frac{4\beta}{\alpha} \exp \left( -\int \frac{4qB^2}{\alpha} dr \right) \int A^3 B(qB)_t \exp \left( \int \frac{4qB^2}{\alpha} dr \right) dr + F(t) \exp \left( -\int \frac{4qB^2}{\alpha} dr \right). \quad (47)$$
In order to investigate the relaxational effects due to shear we utilise (38) as a definition for the relaxation time for the shear stress. For the metric (1) the shear transport equation (38) reduces to

\[ \tau_1 = \frac{-P}{P + \frac{8}{15} r_0 T^4}, \] (48)

where we have used the coefficient of shear viscosity for a radiative fluid

\[ \eta = \frac{4}{15} r_0 T^4 \tau_1, \] (49)

\[ P = \frac{1}{3} (p_T - p_R) \] and \( r_0 \) is the radiation constant for photons. We have further assumed that \( \tau_1 = \beta_1 \tau_c \).

5. Radiating anisotropic collapse

The junction condition \((p_R)_\Sigma = (qB)_\Sigma\) yields

\[ \left[ 2 Y \ddot{Y} + \dot{Y}^2 - \frac{Y'}{B^2} + \frac{2}{B} Y \dot{Y}' - \frac{2}{B^2} Y' Y' + 1 \right]_\Sigma = 0, \] (50)

where we have set \( A(r, t) = 1 \). A particular solution of (50) is given by

\[ Y = rt^{2/3} \] (51)

\[ B = \left( \frac{1 + c_1(r)e^{4t^{1/3}}}{1 - c_1(r)e^{4t^{1/3}}} \right) t^{2/3} \] (52)

which yields the line element

\[ ds^2 = -dt^2 + t^{4/3} \left[ \left( \frac{1 + c_1(r)e^{4t^{1/3}}}{1 - c_1(r)e^{4t^{1/3}}} \right)^2 dr^2 + r^2 d\Omega^2 \right]. \] (53)

With this form of the line element, the field equations (13)–(16) become

\[ \rho = \frac{4}{3 t^2} \left( 1 + \frac{1}{r^3} \left( \frac{r^2 t^{2/3}}{1 - e^{4t^{1/3}/r} c_1(r)} + \frac{18 t}{1 + e^{4t^{1/3}/r} c_1(r)} \right)^3 - \frac{3 \left( r t^{2/3} + 9 t \right)}{\left( 1 + e^{4t^{1/3}/r} c_1(r) \right)^2} + \frac{-\left( r^2 t^{2/3} + 3 r t^{2/3} + 9 t \right)}{1 + e^{4t^{1/3}/r} c_1(r)} \right) \]

\[ - \frac{3 e^{4t^{1/3}/r} t^{2/3}}{r^3} \left( -1 + e^{4t^{1/3}/r} c_1(r) \right) c'_1(r) \]

\[ - \frac{r \left( 1 + e^{4t^{1/3}/r} c_1(r) \right)^3}{r \left( 1 + e^{4t^{1/3}/r} c_1(r) \right)^3} \] (54)
\[ p_R = \frac{-4e^{\frac{3t}{r}} c_1(r)}{t^\frac{2}{3} \left(r + e^{\frac{3t}{r}} r c_1(r)\right)^2} \]  
\[ (55) \]

\[ p_T = \frac{2}{3r^3 t^\frac{2}{3}} \left(\frac{-3r t^\frac{1}{2}}{-1 + e^{\frac{3t}{r}} c_1(r)}\right)^2 + \frac{r \left(2r - 3t^\frac{1}{2}\right)}{-1 + e^{\frac{3t}{r}} c_1(r)} + \frac{9t^\frac{2}{3} \left(3t^\frac{1}{2} c_1(r) - r^2 c_1'(r)\right)}{c_1(r) \left(1 + e^{\frac{3t}{r}} c_1(r)\right)^2} + \frac{6t^\frac{2}{3} \left(-3t^\frac{1}{2} c_1(r) + r^2 c_1'(r)\right)}{c_1(r) \left(1 + e^{\frac{3t}{r}} c_1(r)\right)^3} \]
\[ + \frac{2r^2 c_1(r) - 9t^\frac{2}{3} c_1(r) + 3r^2 t^\frac{1}{2} c_1'(r)}{c_1(r) + e^{\frac{3t}{r}} c_1(r)^2} \]
\[ q = \frac{4e^{\frac{3t}{r}} c_1(r) \left(-1 + e^{\frac{3t}{r}} c_1(r)\right)}{r^2 t^2 \left(1 + e^{\frac{3t}{r}} c_1(r)\right)^3}. \]  
\[ (56) \]

In the absence of heat flux \((c_1 = 0)\) our model yields

\[ Y = rt^{2/3} \]  
\[ (58) \]

\[ B = t^{2/3} \]  
\[ (59) \]

\[ \rho = \frac{4}{3t^2} \]  
\[ (60) \]

\[ p_R = p_T = 0. \]  
\[ (61) \]

(Note that the solution presented above is not simply obtained by substituting \(c_1 = 0\) into (54)–(57). Rather, one must go back to the original metric and then derive the field equations \textit{ab initio}.) The above solution represents a dust sphere and the metric is described by the Einstein–de Sitter solution. We note that when \(q = 0\) the pressure must vanish which allows for the matter to have free-fall motion. For \(q \neq 0\), the pressure is non-vanishing so that it compensates for the outgoing heat flux thus allowing for free-fall motion. With this in mind we expect that the luminosity radius of the star in both the radiative and non-radiative cases have the same temporal dependence. Calculating the luminosity radius for our radiating model, we obtain

\[ Y_\Sigma = b t^{2/3} \]  
\[ (62) \]
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which is independent of \( c_1 \). Hence the case \( c_1 = 0 \) reduces to the model investigated by Oppenheimer and Snyder\(^{18} \).

6. Physical considerations

In order to check the physical viability of our model, we investigate the evolution of the temperature profile within the framework of extended irreversible thermodynamics.

Utilising (46) and (53) we obtain

\[
T^4 = \frac{L_\infty}{(4\pi\delta Y^2)\Sigma} + \frac{16\beta c_1}{3a t^{5/3}} \left[ \frac{e^{3t^{1/3}/b}}{b(1 + c_1 e^{3t^{1/3}/b})} - \frac{e^{3t^{1/3}/r}}{r(1 + c_1 e^{3t^{1/3}/r})} \right]
\]

\[+ \frac{16}{9a t^2} \left\{ \log \left[ \left( \frac{-1 + c_1 e^{3t^{1/3}/b}}{-1 + c_1 e^{3t^{1/3}/r}} \right)^2 \left( \frac{1 + c_1 e^{3t^{1/3}/r}}{1 + c_1 e^{3t^{1/3}/b}} \right)^3 \right] \right\}
\]

\[+ \frac{16}{3a t} \left[ \tanh^{-1}(c_1 e^{2t^{1/3}/b}) - \tanh^{-1}(c_1 e^{2t^{1/3}/r}) \right] . \tag{63}\]

where \( L_\infty \) is given by (29). We note that the causal and the noncausal (\( \beta = 0 \)) temperatures coincide at the boundary (\( r = b \)):

\[T(t, b) = \tilde{T}(t, b) . \tag{64}\]

However, Figure 1 shows that at all interior points, the causal and non-causal temperatures differ. In particular, we observe that the causal temperature is greater than the non-causal temperature at each interior point of the star. For small values of \( \beta \), the temperature profile is similar to that of the non-causal theory; but as \( \beta \) is increased, i.e. as relaxational effects grow, it is clear from Figure 1 that the temperature profile can deviate substantially from that of the non-causal theory. Similar results were obtained in the shear-free models studied by Herrera and Santos\(^1 \) and Govender et al\(^{10,11} \). Also, from the plots in Figure 2 the relaxation
time for the shear stress exhibits substantially different behaviour when the fluid is close to hydrostatic equilibrium as opposed to late-time collapse. In particular, we find that

$$\frac{(\tau_1)_{\text{early}}}{(\tau_1)_{\text{late}}} \approx 100,$$

emphasizing the importance of relaxational effects during the different stages of collapse. We further note that while the relaxation time for the heat flux is taken to be constant, the relaxation time for the shear stress increases as the collapse proceeds.
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Making use of (1) and (52) the proper radius can be written as

\[ R = \int_0^{r_0} B dr = \frac{t^{2/3}}{3} \int_0^{r_0} \left( \frac{1 + c_1 e^{\frac{3t^{1/3}}{r}}}{1 - c_1 e^{\frac{3t^{1/3}}{r}}} \right) dr. \]  

(66)

Numerical integration of (66) (with \( c_1(r) = -1 \)) shows that the proper radius is a decreasing function of time. This is expected as the star is contracting and losing mass. From Figure 3 we note that the proper radius decreases at a faster rate during the latter stages of collapse.

7. Discussion

We have successfully provided an analytical model of a radiating star undergoing gravitational collapse with non-vanishing shear. This model has a Friedmann-like limit when the heat flux vanishes. We further showed that the causal temperature (representing the stellar fluid out of hydrostatic equilibrium) is higher than the noncausal temperature at all points of the star. Further analysis revealed that the relaxation time for the shear stress (taken to be proportional to the mean collision time) increases radially outwards, towards the surface of the star. This is expected, as the outer layers of the fluid are cooler than the central regions. Of particular significance is the result that the relaxation time for the heat flux (in our case taken to be constant) differs from the relaxation time for the shear stress. This is contrary to earlier treatments where it was assumed that \((\tau_r)_{\text{heat}} \approx (\tau_r)_{\text{shear}}\)\(^5,16\).

As presented above, the solution (53) together with (54)–(57) admits singularities at \( t = 0, r = 0 \) and \( 1 \pm c_1(r) \exp(3t^{1/3}/r) = 0 \). The first singularity is avoided by noting that the life of the star is usually taken to start at \( t = -\infty \) and end at \( t = 0 \). The model we present here is really suitable for the early life of a star. In fact, one can show that a black hole arises as \( t \to 0 \). This consideration also takes care of the third case. For early times (large negative values of \( t \) in which we only take real roots) \( \exp(3t^{1/3}/r) \ll 1 \) and so the expression does not evaluate to zero regardless of the sign taken.

The singularity at \( r = 0 \) can be avoided if the solution presented is viewed as an “envelope” with a core represented by a metric of the form

\[ ds^2 = -A_0(r)^2 dt^2 + B_0(r)^2 (f(t)^2 dr^2 + r^2 g(t)^2 d\Omega^2), \]  

(67)

where \( A_0 \) and \( B_0 \) represent a known static solution (See Govender et al\(^{27}\) for a further discussion of this approach.). It is a simple matter\(^{28}\) to match (67) to (53) at some inner boundary \( r_0 \). Since (67) is not singular at \( r = 0 \), this singularity is avoided.

To make a more realistic comparison of the relaxation times, one requires an analytic solution of the causal temperature equation for non-constant relaxation times. Future work in this direction will also require the comparison of the various relaxation times using the truncated and full transport equations.
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